Consistency of Quine’s New Foundations

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We prove consistency of Quine’s New Foundations by an ultrafilter completion construction based on a combination of iterated powersets models and a Cut-elimination proof.

Additional Key Words and Phrases: Set theory, Quine’s New Foundations, Consistency

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1. INTRODUCTION

1.1. New foundations

Consider the following false reasoning: define \( x = \{ a \mid a \notin a \} \). It is easy to check that \( x \in x \) if and only if \( x \notin x \). This is Russell’s paradox and is one of the central paradoxes of (naive) set theory.

Zermelo-Fraenkel set theory (ZF) avoids paradox by insisting instead that \( a \) be guarded; we can only form \( \{ a \in y \mid a \notin a \} \) where \( y \) is already known to be a set. The price we pay for this is that we cannot form ‘reasonable’ sets such as the universal set \( \{ a \mid \top \} \) (the set of all sets) or the set of ‘all sets with 2 elements’, and so on. In ZF, these are proper classes.

Quine’s New Foundations (NF) avoids paradox by insisting on a stratifiable language [Qui37]. Every variable and term can be assigned a level, such that we only form \( t \in s \) provided that \( \text{lev}(s) = \text{lev}(t) + 1 \). So \( a \in a \) and \( a \notin a \) are outlawed because no matter what level \( i \) we assign to \( a \), we cannot make \( i \) be equal to \( i + 1 \).

We can stratify \( \top \) so we can still form the universal set in NF (and ‘has 2 elements’ is also stratifiable). Excellent discussions are in [For95] and [Hol98], and a clear summary with a brief but well-chosen bibliography is in [For97]. At the time of writing there is no canonical proof of consistency for NF.\(^2\) This has been the situation since NF was introduced in 1937 in [Qui37] — though a related theory NFU (NF with urelemente) is known consistent by a proof of Jensen [Jen69; Hol98].

Remark 1.1 (Structure of the proof). At a high level, we take an ultrafilter completion of the set of TST assertions that are true at all levels in the standard powersets model \( P = (\text{pow}^i(\mathbb{N}) \mid i \geq 0) \) of typed set theory (we detail the close relation between TST and NF in Remark 2.1).

That is: we take \( C \) a consistent set of closed \( \phi \) that are true at every level in \( P \), and then saturate to a maximally consistent set of possibly open predicates \( Q \) that contains \( C \) and has good properties, such as witnessing existentials and disjuncts. Then, we use this \( Q \) to build an extensional sets model that just so happens to satisfy the axioms of a theory called TST+, which is known equiconsistent with NF [Spe62].

There are details: we work a lot with a notion of prenex normal forms; we saturate with respect to a bespoke (though simple) cut-eliminating sequent system; and the ultrafilter construction proceeds in two stages — and all these components are designed to fit together just so — but that is the outline.

Remark 1.2 (Map of the paper).

1. This Introduction is Section 1: You are Here.
2. In Section 2 we introduce stratified syntax (the syntax of Typed Set Theory) and define the derivation system TST+ and its axioms. We also introduce notions of sets model, prenex normal form, and de Morgan dual.
3. In Section 3 we show that if there exists an ECT theory in TST syntax (meaning: a consistent set of TST predicates with certain closure properties which we label ‘ECT’ for extensional, comprehensive, and typically ambiguous), then TST+ has a concrete sets model and so is consistent. Since TST+ is known equiconsistent with NF, this reduces the proof of consistency to the construction of an ECT theory.
4. In Section 4 we introduce the (standard) iterated powersets model of TST syntax, and (novel) notions of validity \( \models \) and \( \models^\Box \) which roughly speaking correspond to ‘is possible at every level’ and ‘is necessary at every level’. Subsection 4.2 in particular collects some technical lemmas which clearly have to do with soundness of a derivation system which we define next.
5. In Section 5 we introduce a bespoke derivation system \( \vdash \). Its definition is sound for \( \models \), by the technical lemmas already proved.
6. In Section 6 we introduce a Cut rule and prove a partial Cut-admissibility result.

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\(^1\) A nice historical account of Russell’s paradox is in [Gri04]. For ZF set theory, see e.g. [Jec06].

\(^2\) Holmes has a claimed proof which is under review.
2. THE SYNTAX AND ENTAILMENT OF TST+

2.1. Syntax and entailment

Remark 2.1. Intuitively, Typed Set Theory TST+ replaces the stratifiability condition of NF, with a stratification condition on syntax — so levels are fixed — along with a typical ambiguity axiom which relates truth at one level with truth at another. TST+ and NF are known equiconsistent [Spe62] (we reproduce the core of the argument in Appendix A for reference).

Definition 2.2(1) For each integer \( i \in \mathbb{N} \) fix a disjoint countably infinite set

\[ \text{Var}_i = \{a, b, c, \ldots\} \]

of variable symbols \( a, b, c, \ldots \) of level \( i \). Write \( \text{lev}(a) \) for the level of \( a \).

(2) Define raw predicates by the following BNF grammar:\(^3\)

\[ \phi, \psi ::= \bot | \top | \neg \phi | \phi \land \phi | \phi \lor \phi | \forall a. \phi | \exists a. \phi | b \in a \]

This syntax has some redundancy; for instance we have \( \bot, \top, \neg \bot \). This does no harm and will be convenient later (for example, to express \( \sim \) in Figure 4). We may call a predicate of the form \( b \in a \) atomic.

(3) Then define TST predicates (also called stratified predicates) to be those raw predicates such that \( b \in a \) only occurs when

\[ \text{lev}(a) = \text{lev}(b) + 1. \]

(4) We will only be concerned with TST predicates and terms henceforth, so all syntax is assumed stratified. We will not mention raw syntax again, except for Appendix B in which we show how non-normalising rewrites can appear if we relax the stratification condition.

We may omit mention of levels, where this is irrelevant or understood.

Example 2.3(1) \( \exists b. b \in a \) is stratified if and only if \( \text{lev}(a) = \text{lev}(b) + 1. \)

(2) \( \neg(a \in a) \) and \( \exists a. a \in a \) are never stratified, because \( \text{lev}(a) = \text{lev}(a) + 1 \) is impossible.

More information can be found e.g. in [Hol98; sep16].

Notation 2.4(1) Write \( \text{Pred} \) for the set of all predicates (the `stratified’ here is understood now and henceforth).

(2) Write \( \text{fv}(\phi) \) for the free variables of \( \phi \) as standard for first-order logic: \( \text{fv}(\bot) = \text{fv}(\top) = \emptyset \) and \( \text{fv}(\neg \phi) = \text{fv}(\phi) \) and \( \text{fv}(b \in a) = \{b, a\} \) and \( \text{fv}(\phi \land \psi) = \text{fv}(\phi) \cup \text{fv}(\psi) \) and \( \text{fv}(\forall a. \phi) = \text{fv}(\exists a. \phi) = \text{fv}(\phi) \setminus \{a\} \).

Call a predicate \( \phi \in \text{Pred} \) closed when it has no free variables, and open when it does have free variables.

(3) Given a closed predicate \( \phi \), write \( \phi^+ \) (read as “shift \( \phi \)”) for the predicate obtained by uniformly shifting all variable symbols up by one level. So for instance:

\[ (\forall b. \exists a. b \in a)^+ = \forall b'. \exists a'. b' \in a' \]

where \( \text{lev}(a') = \text{lev}(a) + 1 \) and \( \text{lev}(b') = \text{lev}(a') + 1 \). An inductive definition is not hard to write.

Definition 2.5(1) Let TST entailment be the usual derivation relation of first-order logic, along with the axiom (Ext) in Figure 1.

---

\(^3\) `\( \phi \land \phi \)` in this grammar is not a typo: as standard in BNF grammar notation, instances of \( \phi \) represent any (not necessarily equal) predicate, while also indicating the typical root name(s) of variables typically ranging over the datatype. The \( \phi \) and also \( \psi \) on the left indicates that \( \phi \) and also \( \psi \) will range over predicates.
Let \( \text{TST+} \) entailment be the derivation relation of TST, along with axioms

1. \((\text{Ext}a,a',c)\) for every \(a\) and \(a'\), and
2. \((\text{Cmp}_b,\phi)\) for every \(b\) and \(\phi\), and
3. \((\text{TA}_\phi)\) in Figure 1 for every closed \(\phi\).

Remark 2.6(1) \((\text{Ext})\) asserts extensionality: that extensionally equal elements are equal. Although we may call this an axiom, it is an axiom-scheme, with one axiom at each level.

Remark 2.6(2) \((\text{Cmp}_b,\phi)\) asserts comprehension: each sets comprehension \(\{b|\phi\}\) “the set of \(b\) such that \(\phi\)” is indeed realised by some element.

Remark 2.6(3) The \(+\) in \(\text{TST+}\) refers to its characteristic typical ambiguity axiom \((\text{TA}_\phi)\), which asserts that validity (for closed predicates) is invariant under shifts of levels. There is one \((\text{TA}_\phi)\) axiom for each choice of closed \(\phi\).

### 2.2. Sets models of TST

It will be useful to recall a standard method of building models of TST.

**Definition 2.7** (Sets models). Choose the following data:

1. Make a choice of base set \(V_0\) (\(\mathbb{N}\) or \(\emptyset\) would be standard choices).
2. For each \(i \geq 0\), make a choice of level \(i+1\) universe \(V_{i+1} \subseteq \text{pow}(V_i)\)

Call the sequence \(V = (V_i | i \geq 0)\)
a sets model of TST.

**Definition 2.8**. Fix a sets model \(V\). Then:

1. Let a valuation \(\varsigma\) be a function mapping each \(a \in \text{Var}_i\) to some element \(\varsigma(a) \in V_i\), for every \(i \geq 0\).
2. Given a valuation \(\varsigma\) and \(a \in \text{Var}_i\) and \(x \in V_i\), write \(\varsigma[a:=x]\) for the valuation mapping \(a\) to \(x\) and all other \(b\) to \(\varsigma(b)\).
3. Define a notion of validity

\[ [\phi]_\varsigma \in \{\bot, \top\} \]

as in Figure 2.

Remark 2.9. We may write “\([\phi]_\varsigma\)” or “\([\phi]_\varsigma\) holds” for the judgement “\([\phi]_\varsigma = \top\)”. It will always be clear what is intended and (where it matters) it will always be specified with respect to which sets model \([\phi]_\varsigma\) is intended.

Definition 2.10 will be useful in Lemma 3.15 to help measure the size of the model which we build:

**Definition 2.10** (Countably infinite sets models). Call a sets model \((V_i | i \geq 0)\) countably infinite when each \(V_i \subseteq \text{pow}^i(V_0)\) is countably infinite.

### 2.3. Useful preliminaries: extensionality, prenex normal form, and the de Morgan dual

We collect here some familiar observations about logic and sets. The reader is welcome to skip this section and use it for reference when the ideas are used later:
2.3.1. Sets models are extensional, because sets are extensional

**Lemma 2.11.** The sets model in Definition 2.8 is extensional by construction, for any choice of $\mathcal{V} = (V_i \mid i \geq 0)$. That is:

$$\langle (\text{Ext}) \rangle_c \text{ always.}$$

**Proof.** Extensionally equal elements are equal, because sets are extensional. \qed

**Remark 2.12.** So every sets model is extensional and satisfies (Ext).

Not every sets model is comprehensive: it depends on whether the sets model $\mathcal{V}$ has 'enough elements'. If we take $\mathcal{V} = (\text{pow}^i(X) \mid i \geq 0)$ for some base set $X$, then having enough elements is guaranteed by the fact that we include every possible set. We will use this fact later, in Section 4 (see in particular Lemma 4.16(4)).

2.3.2. Prenex normal form $\text{pnf}(\phi)$

**Definition 2.13.** Suppose $\phi$ is a TST predicate (Definition 2.2(3)). Say that $\phi$ is in prenex normal form, and call it a **PNF predicate**, when it has the form $Q.\phi'$ where:

1. $Q$ is a **quantifier prefix** (a possibly empty sequence of $\forall$ and $\exists$ quantifiers) and
2. $\phi'$ is a quantifier-free purely propositional predicate (usually called the **matrix**), and
3. in addition, we insist on a **non-triviality** condition that every quantifier in the prefix $Q$ binds some variable in the matrix $\phi'$.

**Remark 2.14.** The nontriviality condition (condition 3 of Definition 2.13) is not usually taken as part of the PNF conditions, so that the notion of “PNF” used in this paper might more properly be written as “PNF + non-triviality”, or “PNFnt” for short. The reader can rewrite “PNF” to “PNFnt” in the rest of this paper, if desired, and no harm will come of it.

Whichever terminology we prefer, the non-triviality condition itself is very natural. We use it in the proofs of Lemmas 4.11 and 4.12 (final case of each proof; search for “is excluded by condition 3”).

**Example 2.15.** Examples of predicates in prenex normal form are:

1. $\bot$ and $\top$ and $\neg \bot \land \top$.
2. $\forall a.\exists b. b \in a$.
3. $\forall a.\exists b. \forall a'. \exists b'. (b \in a \land b' \in a')$.

Examples of predicates that are not in prenex normal form are:

1. $\forall a. \bot$, because the quantification is vacuous, i.e. the $\forall a$ does not bind an $a$ in $\bot$.
2. $(\forall a. \exists b. b \in a) \land (\forall a. \exists b. b \in a)$, because this does not consist of a quantifier prefix on a propositional predicate.

Furthermore, $\forall a. \exists b. (b \in a) \land (a \in b)$ is not even a predicate because (no matter what the levels of $a$ and $b$) the stratification conditions cannot be satisfied so it cannot be TST syntax.

**Definition 2.16.** Suppose $\phi$ is a predicate. Compute a predicate $\text{pnf}(\phi)$
makes the familiar observation that semantically, the de Morgan dual of

observes that taking a de Morgan dual has an advantage over adding a negation

and note that they do not affect validity.

Note that our notion of prenex normal form is garbage-collecting (no vacuous quantifier bindings),
as per the final rules in Figure 3. For example if \( \phi = (\forall b. b \in a) \land (\exists ! b'. b' \in a) \) then

\[
\text{pnf}(\phi) = \forall b. \exists b'. (b \in a \land b' \in a).
\]

Lemma 2.18. Suppose \( \phi \) is a predicate. Then

\[
[\phi]_\|= = [\text{pnf}(\phi)]_\|.
\]

Proof. We check each of the rewrites in Definition 2.16 and note that they do not affect validity. □

2.3.3. The de Morgan dual \( \sim \phi \)

Definition 2.19. Suppose \( \phi \) is a predicate. Then we define its (de Morgan) dual \( \sim \phi \) inductively as in Figure 4.

\[
\text{Lemma 2.20 makes the familiar observation that semantically, the de Morgan dual of } \phi \text{ behaves like } \sim \phi:
\]

\[
\text{Lemma 2.20. Suppose } \phi \text{ is a TST predicate. Then}
\]

\[
[\sim \phi]_\|= = [\sim \phi]_\|= = [\phi]_\|
\]

Proof. A fact of sets models. □

\[
\text{Lemma 2.21 observes that taking a de Morgan dual has an advantage over adding a negation symbol: that it preserves the property of being a prenex normal form. This result is implicit in much of the manipulations that follow: if we form } \sim \phi \text{ for } \phi \text{ a PNF predicate and assume that } \sim \phi \text{ is also a PNF predicate, this is by (folklore or) Lemma 2.21.}
\]

\[
\text{It is easy to check that } \phi \text{ is in prenex normal form in the sense of Definition 2.13, if and only if } \phi = \text{pnf}(\phi).
\]
In this paper we may write:

\[
\begin{align*}
\phi & \Rightarrow \psi \quad \text{for} \quad (\neg \phi) \lor \psi \\
\phi & \Leftrightarrow \psi \quad \text{for} \quad (\phi \Rightarrow \psi) \land (\psi \Rightarrow \phi) \\
a & \equiv a' \quad \text{for} \quad \forall b. (b \in a \iff b \in a') \quad (\text{lev}(a) = \text{lev}(a') = \text{lev}(b) + 1)
\end{align*}
\]

Fig. 5: Some notation

| (1) | \( \bot \notin \mathcal{U} \) and \( \top \in \mathcal{U} \). |
| (2) | \( \neg \phi \in \mathcal{U} \) if and only if \( \phi \notin \mathcal{U} \). |
| (3) | \( \phi \land \phi' \in \mathcal{U} \) if and only if \( \phi \in \mathcal{U} \land \phi' \in \mathcal{U} \). |
| (4) | \( \phi \lor \phi' \in \mathcal{U} \) if and only if \( \phi \in \mathcal{U} \lor \phi' \in \mathcal{U} \). |
| (5) | \( \forall a. \phi \in \mathcal{U} \) if and only if \( \forall a' \in \text{Var}_{\text{lev}(a)}.(\phi[a:=a'] \in \mathcal{U}) \). |
| (6) | \( \exists a. \phi \in \mathcal{U} \) if and only if \( \exists a' \in \text{Var}_{\text{lev}(a)}.(\phi[a:=a'] \in \mathcal{U}) \). |

Fig. 6: Properties of a theory (Definition 3.2)

**Lemma 2.21** (1) It is not in general the case that \( \phi \) is a PNF predicate if and only if \( \neg \phi \) is.

(2) It is the case that \( \phi \) is a PNF predicate if and only if \( \sim \phi \) is.

**Proof.** (1) A counterexample is \( \forall a. a \in b \): this is in prenex normal form but \( \neg \forall a. a \in b \) is not.

(2) A fact of Figure 4. For example, \( \sim \forall a. a \in b = \exists a. \neg (a \in b) \). 

\[\square\]

### 3. CONSTRUCTION OF THE ECT SETS MODEL, ASSUMING AN ECT THEORY \( \mathcal{U} \)

Assume for the sake of argument that an ECT theory \( \mathcal{U} \) of TST+ in the sense of Definition 3.3 below is given. We will return to discuss what this assumption means in Remark 3.16.

In this Section we will use \( \mathcal{U} \) to explicitly construct a sets model \( \mathcal{T} \) of TST+ (in the sense of Definition 2.8) which will be:

— *sets-extensional* (extensionally equal elements are also identical sets),
— *comprehensive* (every sets comprehension \( \{b|\phi\} \) is witnessed by some element), and
— *typically ambiguous*.

This culminates at the end of this Section with Corollary 3.14.

This Corollary in itself does not prove consistency of TST+, but it does reduce (or at least rephrase) the problem of the consistency of TST+ to the problem of exhibiting some ECT theory \( \mathcal{U} \).

#### 3.1. Definition of an ECT theory

**Notation 3.1.** We may use the notation in Figure 5 without comment.

Recall also the notion of a PNF predicate from Definition 2.13:

**Definition 3.2.** Define a (PNF) theory to be a set \( \mathcal{U} \) of PNF predicates such that the properties in Figure 6 hold.

**Definition 3.3.** Suppose \( \mathcal{U} \) is a theory (Definition 3.2 and Figure 6) and note the axioms in Figure 1. Then:

1. Call \( \mathcal{U} \) **extensional** when

\[
(\text{Ext}) \in \mathcal{U}.
\]

2. Call \( \mathcal{U} \) **comprehensive** when

\[
\text{pnf}(\text{Cmp}_{b,\phi}) \in \mathcal{U}
\]
$T_0 = \text{Var}_0$

$T_{i+1} = \{ t(a) \mid a \in \text{Var}_{i+1} \} \subseteq \text{pow}(T_i)$

$t(a) = a \quad a \in \text{Var}_0$

$t(a) = \{ t(b) \in T_i \mid U \models b \in a, b \in \text{Var}_i \} \quad a \in \text{Var}_{i+1}$

Fig. 7: The ECT sets model (Definition 3.6)

for every variable $b$ and predicate $\phi$.

(3) Call $U$ **typically ambiguous** when

$\phi \in U \quad \text{if and only if} \quad \phi^+ \in U$

for every closed PNF predicate $\phi$.

(4) We may call a PNF theory that is

— extensional,

— comprehensive, and

— typically ambiguous,

an **ECT theory**.

For the rest of this Section, we assume that there exists an ECT theory, and we call it $U$.

### 3.2. A concrete sets model

**Notation 3.4.** We call the $U$ fixed above a **theory**, so we may write

$\phi \in U \quad \text{as} \quad U \models \phi$.

**Remark 3.5.** For instance in the final line of Figure 7

$\{ t(b) \in T_i \mid U \models b \in a, b \in \text{Var}_i \}$

means $\{ t(b) \in T_i \mid (b \in a) \in U, b \in \text{Var}_i \}$.

We take this opportunity to pause and read through the four uses of the $\in$ symbol above right:

— $b \in \text{Var}_i$ is part of a bounded sets comprehension $\{ \cdots \mid b \in \text{Var}_i \}$.

— $b \in a$ is a proposition in the syntax from Definition 2.2.

— $t(b) \in T_i$ asserts ‘$t(b)$ is an element of $T_i$’ in the usual way, where $T_i$ is a set which in Figure 7 was defined at an earlier stage of an inductive definition.

— $(b \in a) \in U$ also just asserts ‘$(b \in a)$ is an element of $U$’, but $U$ is a logical theory which we fixed in advance of the definition at the start of this Section, so we prefer to use a denotational entailment symbol $\models$ to reflect this, and also for readability since three uses of $\in$ is already plenty.

**Definition 3.6.** Define the **ECT sets model** (Definition 2.8)

$T = (T_i \mid i \geq 0)$

as in Figure 7.

We will use this model for the rest of this Section (up to and not including Section 4), so when we write $[\phi]_i$ in this Section then this is defined relative to $T$ from this definition ... and of course $T$ is defined relative to the ECT theory which we assumed at the end of Definition 3.3.

**Lemma 3.7.** Suppose $i \geq 0$ and $a_1, a_2 \in \text{Var}_i$. Then:

(1) If $i = 0$ then

$t(a_1) = t(a_2) \quad \text{if and only if} \quad a_1 = a_2$.

---

5The use of $\text{pnf}$ here is an artefact of the fact that $(\text{Cmp}_{b,\phi})$ from Figure 1 is not necessarily in prenex normal form. However, for each $\phi$ it can easily be rewritten to be so, using the rewrites in Definition 2.16.
(2) If \( i > 0 \) then

\[ t(a_1) = t(a_2) \quad \text{if and only if} \quad \mathcal{U} \models a_1 \approx a_2. \]

(Recall that \( \approx \) is from Figure 5.)

Proof. First we consider the right-to-left implications of both parts. The case \( i = 0 \) is direct from Figure 7, since \( T_0 = \text{Var}_0 \) and \( t(a) = a \). For the case \( i > 0 \), if \( \mathcal{U} \models a_1 \approx a_2 \) then \( \mathcal{U} \models b \in a_1 \) if and only if \( \mathcal{U} \models b \in a_2 \) and it follows from Figure 7 that \( t(a_1) = t(a_2) \).

For the left-to-right implication we work by induction on the level \( i \):

— Suppose \( i = 0 \).
  If \( t(a_1) = t(a_2) \) then by Definition 3.6 \( a_1 = a_2 \). The result follows.

— Suppose \( i > 0 \). Suppose that

\[ t(a_1) = t(a_2). \]

By Definition 3.6 this means that

\[ \forall b \in \text{Var}_{i-1}. t(b) \in t(a_1) \iff t(b) \in t(a_2). \]

Unpacking Definition 3.6 this means that

\[ \forall b \in \text{Var}_{i-1}. (\exists b_1 \in \text{Var}_{i-1}. (t(b_1) = t(b) \land \mathcal{U} \models b_1 \in a_1)) \iff (\exists b_2 \in \text{Var}_{i-1}. (t(b_2) = t(b) \land \mathcal{U} \models b_2 \in a_2)). \]

By inductive hypothesis in the equation above:

— \( t(b_1) = t(b) \) is equivalent to \( \mathcal{U} \models b_1 \approx b \) (or to \( b_1 = b \) if \( i = 1 \)), and

— \( t(b_2) = t(b) \) is equivalent to \( \mathcal{U} \models b_2 \approx b \) (or to \( b_2 = b \) if \( i = 1 \)).

From Extensionality of \( \mathcal{U} \) (Definition 3.3(1)) and Figure 6 also

— \( \mathcal{U} \models b_1 \in a_1 \) if and only if \( \mathcal{U} \models b \in a_1 \), and

— \( \mathcal{U} \models b_2 \in a_2 \) if and only if \( \mathcal{U} \models b \in a_2 \).

It follows that the equation above is equivalent to

\[ \forall b \in \text{Var}_{i-1}. (\mathcal{U} \models b \in a_1 \iff \mathcal{U} \models b \in a_2) \]

and so using Figure 6

\[ \mathcal{U} \models a_1 \approx a_2 \]

as required.

\[ \square \]

Corollary 3.8. Suppose \( i \geq 0 \) and \( b \in \text{Var}_i \) and \( a \in \text{Var}_{i+1} \). Then:

\[ t(b) \in t(a) \quad \text{if and only if} \quad \mathcal{U} \models b \in a. \]

Proof. If \( i > 0 \) then we reason as follows:

\[ t(b) \in t(a) \iff \exists b' \in \text{Var}_i. (t(b') = t(b) \land \mathcal{U} \models b' \in a) \]

Figure 7

\[ \iff \exists b' \in \text{Var}_i. (\mathcal{U} \models b' \approx b \land \mathcal{U} \models b' \in a) \]

Lemma 3.7

\[ \iff \mathcal{U} \models b \in a \]

\[ \mathcal{U} \text{ extensional (Def. 3.3(1))} \]

The case that \( i = 0 \) is similar but simpler, replacing \( \mathcal{U} \models b' \approx b \) with \( b' = b \) (i.e. \( b' \) and \( b \) are the same variable symbol).

\[ \square \]
3.3. Validity with respect to the ECT sets model

**Definition 3.9.** Recall that the sets model from Definition 3.9 yields a notion of validity

\[ [\phi]_\varsigma \]

for \( \phi \) a possibly open predicate (which need not necessarily be in prenex normal form), and \( \varsigma \) a valuation to \( T \) (Definition 2.8(1) and Figure 7).

**Remark 3.10.** Note from the construction in Figure 7 that any \( x \in T_{lev(a)} \) is equal to \( t(a) \) for some \( a \in \text{Var}_{lev(a)} \). It follows that equivalent formulations of the rules for binders in Figure 2, for the specific case of \( T \) from Figure 7, are:

\[ [\forall a.\phi]_\varsigma \iff \forall a \in \text{Var}_{lev(a)}. [\phi]_{[a:=t(a)]}\varsigma \]

\[ [\exists a.\phi]_\varsigma \iff \exists a \in \text{Var}_{lev(a)}. [\phi]_{[a:=t(a)]}\varsigma \]

**Definition 3.11.**

1. A **substitution** \( \sigma \) is a function mapping \( \text{Var}_i \) to \( \text{Var}_i \) for each \( i \geq 0 \).
2. Write \( \text{id} \) for the **identity substitution** mapping each variable symbol \( a \in \text{Var}_i \) to itself.
3. If \( a, a' \in \text{Var}_i \) for some \( i \geq 0 \), then write \( [a:=a'] \) for the substitution that maps \( a \) to \( a' \) and maps every other \( b \) to itself.
4. If \( \sigma \) is a substitution and \( a \in \text{Var}_i \) for some \( i \geq 0 \), then write \( \sigma-a \) for the substitution that maps \( a \) to \( a \), and maps every other \( b \) to \( \sigma(b) \).
5. Suppose \( \sigma \) is a substitution. Let \( t(\sigma) \) be the valuation (Definition 2.8(1)) taking \( a \) to \( t(\sigma(a)) \in T_{lev(a)} \), where \( t(\sigma(a)) \) is from Figure 7. In symbols:

\[ t(\sigma) : a \rightarrow t(\sigma(a)) \quad \text{Fig. 7} \quad \{t(b) \mid U \models b \in \sigma(a)\}. \]

6. In the case that \( \sigma = \text{id} \), this simplifies to:

\[ t(\text{id}) : a \rightarrow t(a) = \{t(b) \mid U \models b \in a\}. \]

**Lemma 3.12.** Suppose \( \phi \) is a (possibly open) PNF predicate and \( \sigma \) is a substitution. Write

\[ \varsigma = t(\sigma) \]

(Definition 3.11(5)). Then

\[ [\phi]_\varsigma \iff U \models \phi \sigma. \]

**Proof.** By induction on syntax:

— **The cases of \( \bot \) and \( T \).** \( [\bot]_\varsigma \neq T \) and \( [T]_\varsigma = T \) by Figure 2, and \( \bot \notin U \) and \( T \in U \) by Figure 6.

—— **The case of \( \neg \phi \).** We reason as follows:

\[ [\neg \phi]_\varsigma \iff \neg [\phi]_\varsigma \quad \text{Figure 2} \]

\[ \iff U \not\models \phi \sigma \quad \text{Ind. hyp. on} \ \phi \ \text{a subterm of} \ \neg \phi \]

\[ \iff U \models \neg(\phi \sigma) \quad \text{Figure 6} \]

\[ \iff U \models (\neg \phi) \sigma \quad \text{Fact of substitution} \]

— **The cases of \( \phi \land \phi' \) and \( \phi \lor \phi' \).** Routine induction as for \( \neg \), using how the cases match up for \( \land \) and \( \lor \) in Figures 2 and 6.

—— **The case of \( b \in a \).** We reason as follows:

\[ [b \in a]_\varsigma \iff [\varsigma(b)]_{\varsigma(a)} \quad \text{Figure 2} \]

\[ \iff U \models (b \in a) \sigma \quad \text{Corollary 3.8} \]

\[ \iff U \models (b \in a) \sigma \quad \text{Fact of substitution} \]
The cases of \( \forall a. \phi \) and \( \exists a. \phi \). We reason as follows:

\[
\begin{align*}
\forall a. \phi &\iff \forall a' \in \text{Var}_{\text{lev}(a)}. [\phi]_{\varsigma[a:=a']} \\
&\iff \forall a' \in \text{Var}_{\text{lev}(a)}. U \models \phi((\sigma-a)[a:=a']) \\
&\iff U \models \forall a. (\phi(\sigma-a)) \\
&\iff U \models (\forall a. \phi)\sigma
\end{align*}
\]

Remark 3.10
Ind. hyp. on \( \phi \) subterm of \( \forall a. \phi \)
Figure 6
Fact of syntax

The case for \( \exists a. \phi \) is just as for \( \forall a. \phi \), but with \( \exists \).

The ECT sets model \( \mathcal{T} \) from Figure 7 is sound and complete with respect to \( U \):

**Proposition 3.13.** Suppose \( \phi \) is a predicate in prenex normal form. Let \( \varsigma = t(id) \), meaning that \( \varsigma(a) = t(a) \). Then

\[
[\phi]_{\varsigma} \quad \text{if and only if} \quad U \models \phi.
\]

**Proof.** From Lemma 3.12, taking \( \sigma = id \).

From the above we have:

**Corollary 3.14(1).** Suppose an ECT theory \( U \) exists (Definition 3.3(4)).

Then the sets model \( \mathcal{T} \) from Definitions 3.6 and 3.9 which we build out of \( U \), forms a sets model of TST+.

(2) As a corollary, if there exists an ECT theory then TST+ is consistent.

**Proof.** We must show that the notion of validity \( [-] \) obtained from the sets model \( \mathcal{T} \) satisfies the TST+ axioms from Figure 1. But this is now routine:

(1) The ECT model is a sets model and so extensional by construction.
(2) Fix a valuation \( \varsigma \) (it does not matter which). By our assumption that \( U \) is comprehensive (Definition 3.3(2)), and by Proposition 3.13, \( [\text{pnf}((\text{Cmp}_{b, \phi}))]_{\varsigma} \) holds. It follows from Lemma 2.18 that \( [(\text{Cmp}_{b, \phi})]_{\varsigma} \).
(3) Similarly for axiom \( (\text{TA}_{\phi}) \), we can conclude that \( [\phi \leftrightarrow \phi^+]_{\varsigma} \) for every closed \( \phi \).

We conclude with another simple corollary of Proposition 3.13:

**Lemma 3.15 (Size of the model).** The ECT sets model is countably infinite (Definition 2.10).

**Proof.** \( T_0 \) from Figure 7 is countably infinite by construction, since \( T_0 = \text{Var}_0 \) is a countably infinite set (specifically, of variable symbols at level 0).

An upper size limit is imposed by the cardinality of syntax, which is countable, so that the cardinality of \( T_i \) in Figure 7 is at most countable for each \( i \geq 1 \).

Furthermore it is easy to check that for every \( i \geq 1 \), if \( T_i \) from Figure 7 has finite cardinality \( n \) then using Proposition 3.13 and by a simple diagonalisation argument using Comprehension (Definition 3.3(2)), \( T_{i+1} \) has finite cardinality \( 2^n \). We note that \( n \neq 2^n \), but this contradicts Typical Ambiguity (Definition 3.3(2)), since the predicates “There are precisely \( n \) elements” and “There are precisely \( 2^n \) elements” are both easily expressible using closed PNF predicates. It follows that the cardinality of \( T_i \) is also at least countable.

**Remark 3.16.** Note that \( U \) assumes in Definition 3.3(3) precisely that Typical Ambiguity axiom which makes proving consistency of TST+ hard in the first place, so that this Section can be read as

*If we had a theory that satisfies Extensionality, Comprehension, and Typical Ambiguity, then we could build a countably infinite and extensional sets model that satisfies Comprehension and Typical Ambiguity.*
This needed to be proved of course, but it leaves open the question of where we intend to find this ECT theory. Constructing one occupies the rest of the paper.

4. THE POWERSETS MODEL OF TST SYNTAX, AND SOME NOTIONS OF VALIDITY

4.1. The basic definitions

Definition 4.1. Fix a sets model (Definition 2.8) \( \mathcal{P} = (P_i \mid i \geq 0) \) as follows:

\[
P_0 = \mathbb{N} \quad P_{i+1} = pow(P_i).
\]

We will call \( \mathcal{P} \) the (full) powersets model.

\( \mathcal{P} \) is arguably the canonical TST model. The rest of the constructions in this paper will be with respect to \( \mathcal{P} \) unless stated otherwise. So when the reader sees \([\phi]\_\_\_\_\_\_, henceforth, this means validity from Figure 2 as determined over the full powersets model \( \mathcal{P} \) defined here.

Definition 4.2. Let a context \( F, G \) be a finite set of predicates in prenex normal form (Definition 2.13).

Definition 4.3. Suppose \( F \) is a context. We write \( Orb(F) \) for the (shift-)orbit of \( F \), which is the least set of predicates such that:

1. \( F \subseteq Orb(F) \).
2. If \( \phi \) is closed then \( \phi \in Orb(F) \) implies \( \phi^+ \in Orb(F) \).
3. If \( \phi \) is closed then \( \phi^+ \in Orb(F) \) implies \( \phi \in Orb(F) \).

Example 4.4. Suppose \( b \in \text{Var}_0 \) and \( a \in \text{Var}_1 \). Then:

1. \( Orb(\{\bot\}) = \{\bot\} \).
2. \( Orb(\{\forall b. b \in a\}) = \{\forall b. b \in a\} \).
3. This is a closed predicate and has a non-singleton orbit.
4. \( Orb(\{\forall b. b \in a\} \cup \{\forall b'. b. b' \in a' \mid \text{lev}(a') = \text{lev}(b') + 1\}) \).

Definition 4.5. Suppose \( F \) is a context (Definition 4.2).

1. Possible context. Define \( \models F \) by

\[
\models F \quad \text{when} \quad \exists \varsigma, \forall \phi \in Orb(F). [\phi]_\varsigma.
\]

When \( \models F \) holds, we call \( F \) possible. In words, this means that for some valuation, the whole of the shift-orbit of \( F \) is valid.

2. Necessary context. Suppose \( F \) is a context. Define \( \models^\circ F \) by

\[
\models^\circ F \quad \text{when} \quad \forall \varsigma, \forall \phi \in Orb(F). [\phi]_\varsigma.
\]

When \( \models^\circ F \) holds, we call \( F \) necessary. In words, this means that for every valuation, the whole of the shift-orbit of \( F \) is valid.

We will only care about the case when \( F = \{\phi\} \), so we may call \( \phi \) necessary when \( \models \phi \) holds.
Examples of necessary predicates are $\top$, and $\exists a.\forall b. b \in a$ where $\text{lev}(a) = \text{lev}(b)+1$. It does not matter to what level we shift $a$ and $b$, or what the evaluation context is: they are always valid. See Lemma 4.16 for more examples.

(3) **Impossible context.** Write $F \models F'$ for the assertion $\neg(\models F)$. Unpacking this, we obtain that $F \models F'$ when $\forall \varsigma. \exists \phi \in \text{Orb}(F). \neg[\phi]_{\varsigma}$.

When $F \models F'$ holds, we call $F$ **impossible**. In words, this means that for every valuation $\varsigma$, it is impossible for the whole of the shift-orbit of $F$ to be valid (i.e. there is some shift $k$ and some predicate $\phi \in F$ such that $\phi^{+k}$ is not valid).

Examples of impossible contexts are $\{\bot\}$, $\{b \in a, \neg(b \in a)\}$, and $\{\forall b. \forall a. b \in a\}$.

**Remark 4.6.** $\models$ and $\models F$ are related as notation and terminology suggest: if $F$ is necessary then it is possible; if $\models F$ then $\models F'$. The converse implication does not hold. For instance $\models \exists b. b \in a$ but not $\models \exists b. b \in a$. In words: the context $\{\exists b. b \in a\}$ is possible (because we could set $\varsigma(a) = \{\varsigma(b)\}$), but not necessary (because we could set $\varsigma(a) = \emptyset$).

**4.2. (Soundness) properties of the definitions**

This subsection collects some lemmas which will be useful for the Soundness results in the next Section (see in particular Theorem 5.4), and elsewhere.

**Lemma 4.7.**

1. $[\bot]_{\varsigma} = \bot$.
2. As a corollary, if $F$ is a context then $F, \bot \models$.

*Proof.*

1. A fact of Figure 2.
2. From Definition 4.5(1).

**Lemma 4.8.** Suppose $\phi$ is a TST predicate and $b \in \text{Var}_i$ and $a \in \text{Var}_{i+1}$. Then:

1. At least one of $[b \in a]_{\varsigma}$ and $[\neg(b \notin a)]_{\varsigma}$ must be false.
2. As a corollary, if $F$ is a context then $F, b \in a, \neg(b \in a) \models$.

*Proof.*

1. A fact of Figure 2.
2. From Definition 4.5(1).

**Lemma 4.9.** Suppose $\phi$ and $\phi'$ are TST predicates and $\varsigma$ is a valuation. Then:

1. If $[\phi \land \phi']_{\varsigma}$ then $[\phi \land \phi', \phi, \phi']_{\varsigma}$.
2. As a corollary, if $F$ is a context and $\phi$ and $\phi'$ are quantifier-free PNF predicates then $F, \phi, \phi' \models$ implies $F, \phi \land \phi' \models$.

*Proof.*

1. From the fact that $[\phi \land \phi']_{\varsigma}$ if and only if $[\phi]_{\varsigma}$ and $[\phi']_{\varsigma}$.
2. If $\phi \land \phi'$ is quantifier-free then it is a fact of Definition 4.3 that $\text{Orb}(\phi \land \phi') = \{\phi \land \phi'\}$. We use part 1.

**Lemma 4.10.** Suppose $\phi$ and $\phi'$ are TST predicates and $\varsigma$ is a valuation.

1. $[\phi \lor \phi']_{\varsigma}$ implies $[\phi]_{\varsigma}$ or $[\phi']_{\varsigma}$.
(2) As a corollary, if $F$ is a context and $\phi$ and $\phi'$ are quantifier-free PNF predicates then $F, \phi \vDash$ and $F, \phi' \vDash$ implies $F, \phi \vee \phi' \vDash$.

Proof.  
(1) From the fact that $[\phi \vee \phi']_\varsigma$ if and only if $[\phi]_\varsigma$ or $[\phi']_\varsigma$.  
(2) If $\phi \vee \phi'$ is quantifier-free then it is a fact of Definition 4.3 that $\text{Orb}(\phi \vee \phi') = \{\phi \vee \phi'\}$. We use part 1.

Lemma 4.11. Suppose $\forall a. \phi$ is a PNF predicate (Definition 2.13). Then:

(1) $[\forall a. \phi]_\varsigma$ implies $[\phi]_\varsigma[a:=x]$ for every $x \in P_{lev(a)}$.
(2) As a corollary, if $F$ is a context and $F, \phi[a:=a'] \vDash$, for some $a' \in \text{Var}_{lev(a)}$, then $F, \forall a. \phi \vDash$.

Proof.  
(1) From the fact of Figure 2 that $[\forall a. \phi]_\varsigma$ if and only if $[\phi]_\varsigma[a:=x]$ for every $x \in P_{lev(a)}$.
(2) We work with the contrapositive. Suppose $\not\vDash F, \forall a. \phi$, meaning by Definition 4.5(1) that there exists a valuation $\varsigma$ such that $[\psi]_\varsigma$ for every $\psi \in \text{Orb}(F, \forall a. \phi)$. There are now three cases:
   (a) Suppose $\forall a. \phi$ is open.  
       Then $\text{Orb}(\forall a. \phi) = \{\forall a. \phi\}$ and $\text{Orb}(\phi[a:=a']) = \{\phi[a:=a']\}$. By assumption we have $[\forall a. \phi]_\varsigma$, so using part 1 of this result it must be the case that $[\phi[a:=a']]_\varsigma$ (regardless of the value of $\varsigma(a')$). We conclude that $\vDash \phi[a:=a']$ as required.
   (b) Suppose $\forall a. \phi$ is closed and $\phi[a:=a']$ is open.  
       Then $\forall a. \phi \in \text{Orb}(\forall a. \phi)$ so $[\forall a. \phi]_\varsigma$ and we reason as for the previous case.
   (c) The case that the binding is vacuous — so $\forall a. \phi$ is closed, and $\phi[a:=a']$ is also closed because $a$ is not free in $\phi$ — is excluded by condition 3 in Definition 2.13. This excludes the edge case where $\text{Orb}(\forall a. \phi)$ and $\text{Orb}(\phi)$ differ because of the level of some vacuous binding by $a$.

Lemma 4.12. Suppose $\exists a. \phi$ is a PNF predicate. Then:

(1) $[\exists a. \phi]_\varsigma$ implies $[\phi]_\varsigma[a:=x]$ for some $x \in P_{lev(a)}$.
(2) As a corollary, if $F$ is a context and $F, \phi \vDash$ and $a$ is fresh for $F$, then $F, \exists a. \phi \vDash$.

Proof.  
(1) From the fact of Figure 2 that $[\exists a. \phi]_\varsigma$ if and only if $[\phi]_\varsigma[a:=x]$ for some $x \in P_{lev(a)}$.
(2) We work with the contrapositive. Suppose $\not\vDash F, \exists a. \phi$, meaning by Definition 4.5(1) that there exists a valuation $\varsigma$ such that $[\psi]_\varsigma$ for every $\psi \in \text{Orb}(F, \exists a. \phi)$. There are now three cases:
   (a) Suppose $\exists a. \phi$ is open.  
       Then $\text{Orb}(\exists a. \phi) = \{\exists a. \phi\}$ and $\text{Orb}(\phi) = \{\phi\}$. By assumption we have $[\exists a. \phi]_\varsigma$, so using part 1 of this result it must be the case that $[\phi]_\varsigma[a:=x]$ for some $x \in P_{lev(a)}$. We can take $\varsigma[a:=x]$ as our witnessing valuation for $\vDash F, \exists a. \phi$, because $a$ is fresh for $F$ and $\exists a. \phi$.
   (b) Suppose $\exists a. \phi$ is closed and $\phi$ is open.  
       Then $\exists a. \phi \in \text{Orb}(\exists a. \phi)$ so $[\exists a. \phi]_\varsigma$ and we reason as for the previous case.
   (c) The case that the binding is vacuous — so $\exists a. \phi$ is closed, and $\phi$ is also closed because $a$ is not free in $\phi$ — is excluded by condition 3 in Definition 2.13.

Lemma 4.13. Suppose $\phi$ is a TST predicate. Then:

\footnote{The key lemma being used is the (standard) substitution lemma of sets models, that $[\phi[a:=a']]_\varsigma = [\phi]_\varsigma[a:\varsigma(a')]$.}
(1) \( [\neg \phi]_\varsigma = [\sim \phi]_\varsigma \).
(2) As a corollary, if \( F \) is a context and \( \phi \) is a PNF predicate then \( F, \sim \phi \models \) implies \( F, \neg \phi \models \).

Proof. (1) From Lemma 2.20.
(2) From part 1 of this result and Definition 4.5(1).

Lemma 4.14. Suppose \( \phi \) is a closed predicate and \( \models \mathcal{O} \phi \). Then \( [\phi]_\varsigma = \top \).

As a corollary, if \( F \) is a context and \( \phi \) is a closed PNF predicate such that \( \models \mathcal{O} \phi \), then \( F, \phi \models \) implies \( F \models \).

Proof. Just from unpacking Definition 4.5(1&2).

Lemma 4.15. Suppose \( F \) is a context and \( \phi \) is a closed PNF predicate. Then
\[ F, \phi \models \text{ if and only if } F, \phi^+ \models . \]

Proof. By construction in Definition 4.3 \( \text{Orb}(\phi) = \text{Orb}(\phi^+) \). The result is then a fact of the definition of \( \models \) in Definition 4.5(1).

We will use part 2 of Lemma 4.16 in Corollary 5.5, and parts 3 and 4 in Proposition 7.5:

Lemma 4.16. Recall the axioms (Ext) and (Cmp\(_{b,\phi}\)) from Figure 1. Then:

(1) \( \models \emptyset \).
(2) \( \emptyset \not\models \).
(3) \( \models \mathcal{O} \text{(Ext)} \).
(4) \( \models \mathcal{O} \text{(Cmp\(_{b,\phi}\))} \).

Proof. Take any valuation \( \varsigma \) — the predicates above are all closed, so the arguments below work for any such choice. Then:

(1) Unpacking Definition 4.5(1) it is trivially the case that \( [\phi]_\varsigma \) for every \( \phi \in \text{Orb}(\emptyset) = \emptyset \).
(2) From part 1 noting from Definition 4.5(3) that \( \emptyset \models \) means \( \neg (\models \emptyset) \).\(^7\)
(3) Sets are extensional, so as we noted already in Lemma 2.11, \( [\text{(Ext)}]_\varsigma \) is just a fact of sets.
(4) The powersets model sets \( P_\varsigma = \text{pow}^t(\mathbb{N}) \) in Definition 4.1 makes full powersets available, and \( [\text{(Cmp\(_{b,\phi}\))}]_\varsigma \) is just a fact of powersets.

Lemma 4.17. Suppose \( \phi \) is a closed predicate. Then
\[ \models \mathcal{O} \phi^+ \text{ if and only if } \models \phi \text{.} \]

Proof. From the definition of \( \models \mathcal{O} \) in Definition 4.5(2) using the fact that \( \text{Orb}(F) \) from Definition 4.3 is closed under \( \phi \mapsto \phi^+ \) if \( \phi \) is closed.

5. A SOUND DERIVATION SYSTEM

5.1. Derivability

Recall from Definition 4.2 that a context is a finite set of predicates in prenex normal form (Definition 2.13).

Definition 5.1(1) Call a set of contexts deductively closed when it is closed under the inference rules in Figure 8. (Recall that \( \models \mathcal{O} \) is from Definition 4.5(2) and \( \phi^+ \) is from Notation 2.4(3).)

\(^7\)A direct argument is just that it is trivially impossible that \( \neg [\phi]_\varsigma \) for some \( \phi \in \text{Orb}(\emptyset) = \emptyset \).
(2) Write \( \vdash \) for the least deductively closed set of contexts. As is standard, we may write \( F \vdash \) for “\( F \in \vdash \)” and \( F \not\vdash \) for “\( F \not\in \vdash \)”.

(3) If \( F \vdash \) holds then call the context \( F \) inconsistent.

(4) If \( \neg(F \vdash) \) holds then call \( F \) consistent and write \( F \not\vdash \).

(5) Suppose \( X \) is a (possibly infinite) set of predicates in prenex normal form. Then:

\[ \begin{align*}
\text{— Write } X \vdash \text{ and call } X \text{ inconsistent when there exists a context } F \subseteq_{\text{fin}} X \text{ such that } F \vdash. \\
\text{— Write } X \not\vdash \text{ and call } X \text{ consistent when } F \not\vdash \text{ for every context } F \subseteq_{\text{fin}} X.
\end{align*} \]

**Remark 5.2.** Some general observations on the derivation system in Figure 8:

(1) Our derivation system only has left-intro rules, though to be fair the de Morgan dual \( \neg\phi \) from Figure 4 recovers some of the power of the ‘missing’ right-intro rules. This is a feature, not a bug: we use the derivation system to help us build a maximal consistent set, so any additional deductive power can only complicate the proofs.

(2) Here is a simple but illustrative example derivation of \( \neg(T \land T) \vdash \), showing how these rules work:

\[ \begin{align*}
\bot \vdash (\bot) & \quad (\bot) \vdash (\bot) \\
(\bot \lor \bot) \vdash (\lor) & \quad (\neg) \vdash (\neg)
\end{align*} \]

Above, we use the fact that \( \neg(T \land T) = \bot \lor \bot \), as can be easily calculated from Figure 4.

(3) The restrictions on \( (\land) \), \( (\lor) \), and \( (\neg) \) reflect that predicates in contexts are required to be in prenex normal form (a quantifier prefix scoping a quantifier-free purely propositional part) — so these constraints are ‘soft’ in the sense that, while accurate, they are more a reminder for us of something that is guaranteed to hold anyway. This will give us Lemma 6.4, and it also simplifies away a corner case; see Remark 6.6.

(4) \( (\Box) \) is a style of rule often called a deletion rule. It lets us introduce into the context any closed PNF predicate \( \phi \) that is necessarily valid in the powersets model, in the sense of \( \vdash \Box \phi \) from Definition 4.5(2).
We care about (☐) because, noting Lemma 4.16(3&4), it inserts axioms into our derivation system for Extensionality and (prenex normal forms of) Comprehensions. Looking forward to Corollary 5.5 and Theorem 5.4, (☐) is sound because it is valid in the powersets model.8

(5) (☐) is not computable: it is in general hard to decide whether \([\phi^k]_\xi\) is valid in the powersets model for every \(\xi\) and every \(k\). Thus it is in general not computable whether \(F \vdash\).

This is not a problem: we do not need to compute these relations, we just need them to be mathematically well-defined (just like “the set of terminating programs” is not computable, but is mathematically well-defined).

(6) In rule (Shift) the bottom-left context is bracketed as \(F, (\phi^+)\) — that is: the shift applies to an individual predicate, not to the whole context.

Intuitively, (Shift) ensures that derivability is typically ambiguous, in the sense of Definition 3.3(3).

Further discussions will follow as the proofs develop.

The generalisation of (Ax) to non-atomic predicates is admissible:

**Lemma 5.3.** Suppose \(\phi\) is a predicate in prenex normal form, and \(F\) is a context. Then:

1. \(F, \phi, \neg \phi \vdash\).
2. \(F, \phi, \neg \phi \vdash\).

**Proof.** Part 2 follows from part 1 using (¬L). Part 1 is by a routine induction on \(\phi\) for every \(F\), using the rules in Figure 8 to break \(\phi\) down until we reach the base case (Ax) or (⊥L):

\[
\begin{align*}
F, \phi, \neg \phi \vdash \\
F, \forall a, \phi, \neg \phi \vdash \quad \text{(a fresh for } F) \\
\quad F, \forall a, \phi, \exists a, \neg \phi \vdash \\
F, \phi', \neg \phi' \vdash \quad \text{(\(\wedge\)L)} \\
F, \phi \wedge \phi', \neg \phi' \vdash \\
F, \phi \wedge \phi', \neg \phi' \vdash \quad \text{(\(\forall\)L)} \\
F, \phi \wedge \phi', \neg \phi' \vdash \\
\end{align*}
\]

\[F, \bot, T \vdash (\neg L)\]

\[\square\]

### 5.2. Soundness and consistency

We now relate derivability \(F \vdash\) from Figure 8 to the denotation \(F \models\) from Definition 4.5(3) (which means \(\not\models F\)):

**Theorem 5.4 (Soundness).** The rules in Figure 8 are sound with respect to the notion of validity from Definition 4.5. In symbols:

\[F \vdash \quad \text{implies } F \models\]

**Proof.** By a routine induction on the structure of derivations in Figure 8:

— *Soundness of (⊥L).* By Lemma 4.7.
— *Soundness of (Ax).* By Lemma 4.8.
— *Soundness of (\(\wedge\)L).* By Lemma 4.9.
— *Soundness of (\(\forall\)L).* By Lemma 4.10.
— *Soundness of (\(\forall\)L).* By Lemma 4.11.

---

8Our consistency argument in Corollary 5.5 will be semantic, not structural, so having a deletion rule here is fine. We will not argue “we can eliminate Cut so the system is consistent”; such an argument is false in the presence of (☐) and we will not make it. We will eliminate Cut, or to be more precise we will eliminate some but not all Cuts (see Remark 6.15), but this will be applied in an ultrafilter completion argument that follows the semantic consistency argument and depends on it.
Soundness of ($\exists L$). By Lemma 4.12.

Soundness of ($\neg L$). By Lemma 4.13.

Soundness of ($\Box$). By Lemma 4.14.

Soundness of (Shift). By Lemma 4.15.

\[\square\]

**Corollary 5.5 (Consistency).** $\emptyset \not\vdash$, or more succinctly:
\[\nvdash\]

**Proof.** Suppose it is the case that $\emptyset \vdash$, so that there exists a concrete derivation $\Gamma$ with conclusion $\emptyset \vdash$.

This suggestion is not *structurally* absurd because the rule ($\Box$) does allow us to remove (certain) predicates from the context — however, we shall see in the next paragraph that it is *semantically* absurd.

By Soundness (Theorem 5.4) each of the rules in Figure 8 is sound with respect to the notion of validity $\models$ from Definition 4.5, and so it must be the case that $\emptyset \models$. By Lemma 4.16(2) this is semantically impossible, a contradiction. $\square$

**5.3. Weakening and strengthening**

Lemmas 5.6 and 5.7 will be helpful:

**Lemma 5.6.** Suppose $F$ is a context and $\Gamma$ is a derivation of $F \vdash$ and $G$ is any context.

Then $Wk(\Gamma, G)$ the tree obtained by adding $G$ to every context in $\Gamma$, is a derivation of $F, G \vdash$, and has the same height as $\Gamma$.

**Proof.** We check the rules in Figure 8 and note that they all remain valid under expanding the context. $\square$

**Lemma 5.7.** Suppose $F$ is a context and $\Gamma$ is a derivation of $F, T \vdash$. Let $St(\Gamma)$ be the tree obtained by removing $T$ from every context in $\Gamma$ if $T \in F$, and let $St(\Gamma) = \Gamma$ otherwise.

Then $St(\Gamma)$ is a derivation of $F \vdash$ and has no greater height than $\Gamma$.

**Proof.** We check the rules in Figure 8 and note that they all remain valid under removing $T$ from the context.

There is, admittedly, a little bookwork for the cases of a contracting instance of ($\Box$) where $\phi = T$, or an instance of (Shift) where $\phi = T$. But such instances can trivially just be deleted, so this can only reduce derivation height. $\square$

**6. Admissibility of Cut**

Cut-elimination proofs tend to be long, but the structure of the arguments below is as close as possible to a standard cut-elimination argument for first-order logic.

Note that Cut-elimination on its own does not prove consistency for the logic in Figure 8, because of the deletion rule ($\Box$). We prove consistency by semantic methods in Corollary 5.5 instead; but we still need Cut-elimination because consistency alone is not enough.

We need a set that is consistent and *also* interacts appropriately with predicate structure: e.g. if $\phi \vee \phi'$ is in the set then at least one of $\phi$ and $\phi'$ must be in the set. What Cut-elimination — along with the structural nature of the rules — is useful for, is ultimately to give us Lemma 7.3, which is what permits us to construct a consistent set that is also an ECT theory.

---

9 An instance of ($3L$) inside $\Gamma$ might need to choose a ‘fresher’ fresh variable, to avoid name-chash with $G$. This is standard.
6.1. Interactions of \((\text{Shift})\) with other derivation rules

Before we start to treat Cut, it will be useful to show that the \((\text{Shift})\) rule from Figure 8 can always be eliminated or pushed down to the end of a derivation. Intuitively, this will mean that our proofs of cut-admissibility will only need to deal with derivations in shift-normal form, as we now define:

**Notation 6.1.**

1. Call a derivation shift-free when it contains no instances of rule \((\text{Shift})\) from Figure 8.
2. Say that a derivation is in shift-normal form when (if read bottom-up from the conclusion towards the leaves of the derivation-tree) it consists of finitely many (possibly zero) instances of \((\text{Shift})\), followed by a shift-free subderivation.

Intuitively, a derivation in shift-normal form has all of its instances of \((\text{Shift})\) shifted (pun intended) to the end. In this Subsection we set about proving Proposition 6.7, that every derivable sequent has a derivation in shift-normal form.

**Lemma 6.2.** \((\text{Shift})\) commutes down past instances of \((\land L), (\lor L), (\forall L), (\exists L), (\neg L), (\square),\) and \((\text{Shift})\) that are focussed on on other predicates.

**Proof.** We consider the case of \((\lor L)\):

\[
\frac{F, \psi^+, \phi \vdash \text{(Shift)}}{F, \psi, \phi \vdash \text{(Shift)}} \quad \frac{F, \psi, \phi' \vdash}{F, \psi, \phi \lor \phi' \vdash \text{(\lor L)}} \quad \frac{F, \psi^+, \psi, \phi' \vdash \text{(Shift)}}{F, \psi, \phi \vdash}
\]

Above we use Weakening (Lemma 5.6) to weaken the right-hand branch with \(\psi^+\) and the left-hand branch with \(\psi\) (this does not increase derivation-depth).

The other rules are similar but simpler. We give one example:

\[
\frac{F, \psi^+, \phi \vdash \text{(Shift)}}{F, \psi, \phi \vdash} \quad \frac{F, \psi, \exists a. \phi \vdash \text{(Shift)}}{F, \psi^+, \exists a. \phi \vdash \text{(\exists L)}}
\]

Lemma 6.3 will be helpful for Lemma 6.4:

**Lemma 6.3.** Suppose \(\phi\) is a predicate that is closed and quantifier-free. Then

1. \(\phi^+ = \phi\)
2. \(\phi^+ = \phi\)

**Proof.** Closed syntax in prenex normal form is is constructed only from \(\bot, \top, \neg, \land,\) and \(\lor\) (no \(\forall\) or \(\exists\) and no \(b \in a\)). It is a fact that this mentions no variables, and on such syntax, \(^+\) acts as the identity, because there are no variables to shift.

Our restriction to prenex normal forms in the derivation system gives us the following simple but important elimination reduction:

**Lemma 6.4.** An instance of \((\text{Shift})\) followed by an instance of \((\land L), (\lor L),\) or \((\neg L)\) acting on the same formula, may be eliminated.

**Proof.** This is because we restrict \((\text{Shift})\) in Figure 8 to closed predicates, and we restrict \((\land L), (\lor L),\) and \((\neg L)\) to quantifier-free predicates. We use Lemma 6.3:
20

\[
\frac{F, \kappa \vdash G, \sim \kappa^+ k \vdash}{F, G \vdash} \text{(Cut)}
\]

Fig. 9: The shift offset Cut rule (Definition 6.8)

\[\square\]

**Lemma 6.5.** An instance of (**Shift**) followed by an instance of (\(\square\)) acting on the same formula, may be eliminated.

**Proof.** Suppose \(F\) is a context and \(\phi\) is a closed predicate and \(\vdash \square \phi\) (Definition 4.5(2)). Note from Lemma 4.17 that also \(\vdash \square^+ \phi\). Then we can reduce as follows:

\[
\frac{F, \phi^+ \vdash}{F, \phi \vdash} \text{(**Shift**)} \quad \Rightarrow \quad \frac{F, \phi \vdash}{F \vdash} \text{(\(\square\))}
\]

\[\square\]

Remark 6.6 continues Remark 5.2(3):

**Remark 6.6.** There is no case where (**Shift**) is followed by an instance of (\(\forall L\)) or (\(\exists L\)) focussed on the same predicate, because

— (**Shift**) in Figure 8 requires its principal formula \(\phi\) to be closed, and
— prenex normal form requires bindings to be non-trivial (Definition 2.13(3)) so that \(\forall a. \phi\) or \(\exists a. \phi\) would not be in prenex normal form.

**Proposition 6.7.** If \(F \vdash\) is derivable, then it is derivable with a derivation in shift-normal form (Notation 6.1(2)).

**Proof.** By Lemma 6.2 we can commute an instance of (**Shift**) down past other rules if they are focussed on other predicates. By Lemmas 6.4 and 6.5, if (**Shift**) is followed by (\(\land L\)), (\(\lor L\)), (\(\neg L\)), or (\(\square\)) then it can be eliminated. By Remark 6.6 there is no case that (**Shift**) is followed by (\(\forall L\)) or (\(\exists L\)) focussed on the same predicate. \[\square\]

### 6.2. The shift-offset Cut rule

**Definition 6.8.** Let the **Cut rule** be as in Figure 9, where:

1. \(F\) and \(G\) are contexts (Definition 4.2).
2. \(k \in \mathbb{Z}\).
3. \(\kappa^+ k\) extends Notation 2.4(3) in the natural way to denote \(k\) applications of \(^+\). If \(k\) is negative, then \(\kappa^+ k\) is that predicate \(\kappa'\) such that \((\kappa')^{+ - k} = \kappa\), if this exists ((\(\kappa')^{+ - k}\) denotes a shift **down** by \(k\), which since \(k\) is assumed negative means a shift **up** by \(-k\)).
4. \(\kappa\) and \(\kappa^+ k\) exist and are TST predicates in prenex normal form.
5. If \(\kappa\) is either open or propositional, then \(k = 0\) — so if we have a nonzero shift offset \(k \neq 0\) in a Cut then this requires \(\kappa\) to be a closed prenex form with a nonempty quantifier prefix.\(^{10}\)

We call the predicate \(\kappa\) above, the **cut formula** or **focus** of the Cut, and we call \(k\) the **shift offset** of the Cut.

**Remark 6.9(1)** Note that contexts are sets, so that it may be that \(F, G = F\) and it may be that \(\kappa \in F\) or \(\sim \kappa^+ k \in G\).

\(^{10}\)Our notion of prenex normal form also requires quantifiers to be nonvacuous (Definition 2.13(3)), so having a nonempty quantifier prefix means that there are variables and they do get bound.
(2) The reader familiar with cut-elimination proofs might read the line “\( \kappa \) is either open or propositional, then \( k = 0 \)” above with extra care. What about a Cut-elimination step between \( \kappa = \forall a.\kappa' \) and \( (\sim \kappa)^{+k} = (\exists a.\sim \kappa')^{+k} \) for \( k \neq 0 \)? Typically, elimination of this Cut would lead us to a smaller cut between \( \kappa' [a := a'] \) and \( (\sim \kappa' [a := a'])^{+k} \) for \( k \neq 0 \) — but this would be illegal because \( k \neq 0 \) and \( \kappa [a := a'] \) is open.

Indeed, it is precisely here that the Cut-elimination procedure does something unusual. See the concluding reductions in Subsection 6.5 (search for “Second reduction, for when the shift offset is strictly positive (\( k > 0 \))”)

The rest of this Section is devoted to showing that (Cut) is admissible. The proof is by proving commutations and essential cases and by an induction on the size of the cut formula, and cut-height (defined below), lexicographically ordered.

**Definition 6.10.**

1. Define the **pair ordering** \( \preceq \) on pairs of ordinals such that

\[
(\alpha, \beta) \preceq (\alpha', \beta') \quad \text{when} \quad \alpha \leq \alpha' \land \beta \leq \beta'.
\]

Note that \( \preceq \) is a well-ordering.

2. Let the **cut-height** of a Cut between derivations \( \Gamma \) and \( \Delta \) be the pair \((\text{height}(\Gamma), \text{height}(\Delta))\). We order cut-height with the pair ordering \( \preceq \) from Definition 6.10; \( \preceq \) is the relation in use whenever we induct on cut-height in what follows.

**6.3. Commutations of (Cut) with other rules**

In this Subsection we will show how the rules in Figure 8 interact with (Cut), where it is not the focus of the cut.

The commutation rules are illustrated below. We show the commutations from the left, but the behaviour on the right is no harder.

**Commutation for \( \land \).**

\[
\begin{array}{c}
\vdash \Gamma \\
F, \phi, \phi', \kappa \vdash (\land) \\
F, \phi \land \phi', \kappa \vdash \\
F, \phi \land \phi', G \vdash \\
\end{array}
\begin{array}{c}
\vdash \Delta \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\Rightarrow
\begin{array}{c}
\vdash \Gamma \\
F, \phi, \phi', \kappa \vdash \\
F, \phi \land \phi', G \vdash \\
(\land)
\end{array}
\begin{array}{c}
\vdash \Delta \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\]

**Commutation for \( \lor \).**

\[
\begin{array}{c}
\vdash \Gamma \\
F, \phi, \kappa \vdash \\
F, \phi', \kappa \vdash \\
F, \phi \lor \phi', \kappa \vdash \\
F, \phi \lor \phi', G \vdash \\
\end{array}
\begin{array}{c}
\vdash \Gamma' \\
F, \phi', \kappa \vdash \\
(\lor) \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\Rightarrow
\begin{array}{c}
\vdash \Delta \\
F, \phi \lor \phi', G \vdash \\
(\text{Cut})
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma \\
F, \phi, \kappa \vdash \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\begin{array}{c}
\vdash \Delta \\
F, \phi', \kappa \vdash \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\Rightarrow
\begin{array}{c}
\vdash \Delta \\
F, \phi', G \vdash \\
(\lor)
\end{array}
\begin{array}{c}
F, (\phi \lor \phi'), G \vdash \\
(\text{Cut})
\end{array}
\]

\[
\begin{array}{c}
\vdash \Gamma \\
F, \phi, \kappa \vdash \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\begin{array}{c}
\vdash \Delta \\
F, \phi', \kappa \vdash \\
G, \sim \kappa^{+k} \vdash \\
(\text{Cut})
\end{array}
\Rightarrow
\begin{array}{c}
\vdash \Delta \\
F, \phi', G \vdash \\
(\lor)
\end{array}
\begin{array}{c}
F, (\phi \lor \phi'), G \vdash \\
(\text{Cut})
\end{array}
\]
Commutation for $\neg$.

$$\frac{\Gamma \vdash F, \neg \phi, \kappa \quad \Delta \vdash G, \neg K^k \kappa}{\Gamma \vdash F, \neg \phi, G} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \neg \phi, \kappa \quad \Delta \vdash G, \neg K^k \kappa}{\Gamma \vdash F, \neg \phi, G} \quad \text{(Cut)}$$

Commutation for $\forall$.

$$\frac{\Gamma \vdash F, \phi[a:=a'], \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \forall a \phi, G} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \phi[a:=a'], \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \forall a \phi, G} \quad \text{(Cut)}$$

Commutation for $\exists$. Renaming if necessary as standard, we make sure that the fresh $a$ in the left-hand ($\exists$) branch is fresh also for $G$.

$$\frac{\Gamma \vdash F, \phi, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \exists a \phi, G} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \phi, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \exists a \phi, G} \quad \text{(Cut)}$$

Commutation for ($\Box$).

$$\frac{\Gamma \vdash F, \phi, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \phi, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+} \quad \text{(Cut)}$$

Commutation for (Shift).

$$\frac{\Gamma \vdash F, \phi^+, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \phi, G} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \phi^+, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \phi^+, G} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \phi^+, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \phi, G} \quad \text{(Shift)}$$

$$\frac{\Gamma \vdash F, \phi^+, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \phi^+, G} \quad \text{(Shift)}$$

Lemma 6.11. The derivation reductions above strictly reduce cut-height, and do not increase the size of the cut formula.

Proof. One of the branches always get strictly shallower, and the formula $\kappa$ is unchanged. $\square$

6.4. Reductions for (Shift)

$$\frac{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+} \quad \text{(Cut)}$$

$$\frac{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+}{\Gamma \vdash F, \kappa \quad \Delta \vdash G, \neg \kappa \kappa^+} \quad \text{(Cut)}$$
Lemma 6.12. (Cut) can absorb instances of (Shift) in either branch acting on the principal formula.

Proof. From the derivation reductions above.

Lemma 6.12 is more powerful than it may look, because in combination with Proposition 6.7 this tells us that all shifts on the principal formula of a (Cut) can be absorbed into the Cut itself. Furthermore, we saw at the end of Subsection 6.3 how shifts not on the principal formula can be commuted down through the Cut.

Thus, in the context of the results we have seen, what Lemma 6.12 really means is that a (Cut) always can pass up through a (Shift) or absorb it.

6.5. The essential cases for rules introducing logical connectives

We now illustrate essential cases for rules other than (Shift). In the non-quantifier essential cases (i.e. all but the final essential case for (∀L) and (∃L) below) the conditions on (Cut) in Definition 6.8 ensure that the shift offset is zero, so we do not write it in. We spell this out just for the first case, for (Ax):

The essential case of any rule with (Ax). We reduce as follows:

\[
\frac{\vdash \Gamma \quad \vdash \Delta}{\vdash \Gamma, \sim \Phi \vdash} \quad \text{(Shift)}
\]

\[
\frac{\vdash \Gamma, \sim \Phi \vdash}{\vdash \Gamma} \quad \text{(Cut)}
\]

\[
\frac{\vdash \Gamma}{\vdash \Gamma, \sim \Phi \vdash} \quad \text{(Cut)}
\]

Wk(Γ) above is from Lemma 5.6.

We do not need to worry about the shift offset \(k\) in the Cut (see Definition 6.8): this must be zero because \(b \in a\) is definitely not closed with a nontrivial quantifier prefix.

The corresponding Cut of \(F, \sim (b \in a) \vdash\) with \(G, b \in a, \sim (b \in a) \vdash\) is precisely similar (recall from Figure 4 that \(\sim (\sim (b \in a)) = (b \in a)\)).

The essential cases of any rule with (∀L).

\[
\frac{\vdash \Gamma}{\vdash \Gamma, \sim \Phi \vdash} \quad \text{(∀L)}
\]

\[
\frac{\vdash \Gamma, \sim \Phi \vdash}{\vdash \Gamma} \quad \text{(Cut)}
\]

\[
\frac{\vdash \Gamma}{\vdash \Gamma, \sim \Phi \vdash} \quad \text{(Cut)}
\]

Wk above is from Lemma 5.6, and St is from Lemma 5.7.

The essential case between (∃L) and (∀L), where \(\phi\) and \(\phi'\) are quantifier-free. There are two cases, depending on whether the cut formula is contracted or not. First, the non-contracting case:
Here we have replaced a Cut on \( \phi \land \phi' \) with two cuts on strictly smaller predicates \( \phi \) and \( \phi' \). The contracting case is similar, just a bit longer:

Here we have replaced a Cut on \( \phi \land \phi' \) with three cuts: two on strictly smaller predicates \( \phi \) and \( \phi' \), and one on \( \phi \land \phi' \) but with lesser cut-height (Definition 6.10).

*The essential case of any rule with \((\neg \land)\).* If \( \phi = (b \in a) \) then \( \neg \phi = \neg \phi \) by Figure 4 and we reduce as follows:

Here size is unchanged and cut-height is decreased. If \( \phi \) is not atomic then there are two cases, depending on whether the cut formula is contracted or not. Note below that \( \neg \neg \neg \phi = \phi \). First, the non-contracting case:
Above, we replace a Cut on $\neg\phi$ with a Cut on the (since $\phi$ is not atomic) strictly smaller $\phi$.

The contracting case is just a bit longer:

\[
\begin{array}{c}
\vdots \Gamma \\
F, \neg\phi, \sim\phi \vdash (\neg L) \\
\vdots \Delta \\
G, \phi \vdash (\text{Cut}) \\
\hline \\
F, G \vdash
\end{array}
\quad 
\Rightarrow 
\begin{array}{c}
\vdots \Gamma \\
\vdots \Delta \\
F, \neg\phi, \sim\phi \vdash G, \phi \vdash (\text{Cut}) \\
\hline \\
F, G \vdash
\end{array}
\]

Above we replace a Cut on $\neg\phi$ with two Cuts, both on the strictly smaller predicate $\phi$.

The essential case between $(\forall L)$ and $(\exists L)$. There is a contracting essential case, and a non-contracting essential case. We give just the non-contracting case; the contracting case is just like those above.

There is then a further case split:

1. A case where the principal formula is open\(^{11}\) and the shift offset $k$ is zero (see condition 5 of Definition 6.8), and
2. A case where the principal formula is closed and the shift offset $k$ may be nonzero.

We consider each case in turn.

\textit{First reduction, for when the shift offset is zero} ($k = 0$).

\[
\begin{array}{c}
\vdots \Gamma \\
F, \phi[a:=a'] \vdash (\forall L) \\
\vdots \Delta \\
G, \sim\phi \vdash \text{fresh } a \\
\hline \\
F, \forall a.\phi \vdash (\exists L) \\
\vdots \Delta[a:=a'] \\
F, \phi[a:=a'] \vdash G, \sim\phi[a:=a'] \vdash (\text{Cut}) \\
\hline \\
F, G \vdash
\end{array}
\]

Above, $\Delta[a:=a']$ denotes a copy of $\Delta$ in which the substitution $[a:=a']$ has been applied throughout (in a capture-avoiding manner of course).

This reduction replaces a Cut between $\forall a.\phi$ and $\exists a.\sim\phi$ with one between $\phi[a:=a']$ and $\sim\phi[a:=a']$, which is a strictly smaller Cut.

\textit{Second reduction, for when the shift offset is strictly positive} ($k > 0$). (Note that a strictly negative offset from left-to-right, is also a strictly positive offset from right-to-left, so there is no real loss of generality taking $k > 0$. What really matters is that it is nonzero.)

We now impose an additional \textbf{closure restriction}, that $F$ and $G$ are contexts consisting entirely of closed predicates. This anticipates Notation 6.14, and the closure restriction will be exploited in Corollary 6.18. We make no claim that Cut can be eliminated for the more general case between $\forall a.\kappa$ and $(\exists a.\sim\kappa)^{+k}$ where $k \neq 0$ and where the predicates in $F$ and $G$ are not necessarily all closed — but our proof will not require this.

\(^{11}\)It can’t be propositional because this is an essential case for $\forall$ and $\exists$. \vspace{1em}
Then:

\[
\begin{array}{c}
\Gamma \\
\vdash F, \phi[a:=a'] \\
\vdash G, \neg \phi^{+k} \\
\vdash (\forall L) \\
\vdash \Delta \\
\vdash \exists L \quad \text{fresh } a^{+k} \\
\vdash (\exists L) \\
\hline
\vdash F \land G \\
\hline
\hline
\vdash F, G \vdash \\
\hline
\hline
\vdash \Gamma^{+k} \\
\vdash \Delta[a^{+k}:=a'^{+k}] \\
\vdash F^{+k}, \phi[a:=a']^{+k} \\
\vdash G, \neg \phi[a:=a']^{+k} \\
\vdash \exists L \quad \text{fresh } a^{+k} \\
\vdash (\exists L) \\
\hline
\vdash F^{+k} \\
\vdash G, \neg \phi[a:=a']^{+k} \\
\vdash (\exists L) \\
\hline
\vdash F^{+k}, G \\
\vdash (\text{Shift})^{*} \\
\end{array}
\]

Above:

— The reduction replaces a Cut between \( \forall a. \phi \) and \((\exists a. \phi)^{+k}\) with one between \( \phi[a:=a']^{+k} \) and \( \neg \phi[a:=a']^{+k} \), which is a strictly smaller Cut (and a valid one according to the conditions in Definition 6.8, since the offset between the branches is zero).

— \((\text{Shift})^{*}\) denotes multiple instances of \((\text{Shift})^{*}\); one for each of the (closed) predicates in \( F \). This is why we had to assume that \( F \) contains only closed predicates.

— \( \Gamma^{+k} \) denotes a copy of \( \Gamma \) in which every variable has been shifted up by \( k \), and similarly elsewhere, e.g. \( \Delta^{+k} \). It suffices here to pick fixed but arbitrary bijections \( \text{Var}_i \cong \text{Var}_{i+1} \) for each \( i \geq 0 \), and apply these \( k \) times throughout.

— In particular, \( \phi[a:=a']^{+k} \) denotes a copy of \( \phi[a:=a'] \) in which we shift all variables up by \( k \) using the bijections we have just chosen. Note that \( \phi[a:=a']^{+k} = \phi^{+k}[a^{+k}:=a'^{+k}] \).

— \( \Delta[a^{+k}:=a'^{+k}] \) denotes a copy of \( \Delta \) in which the substitution \( [a^{+k}:=a'^{+k}] \) has been applied throughout (in a capture-avoiding manner of course).

— Choosing a fresh name for the bound variable \( a \) and substituting it for \( a' \) is somewhat bogus, because \( F, G, \text{and } \forall a. \phi \) are assumed closed. Thus for this particular instance, the presentation could be simplified by just omitting \( a' \) and the associated substitution. However, this might confuse, so for clarity we follow the form of the rule \((\forall L)\) in Figure 8.

We illustrate a case with a contraction on the left (the case with a contraction on the right is no harder, and the case with contractions on both sides is just by combining the two cases):

\[
\begin{array}{c}
\Gamma \\
\vdash F, \forall a. \phi, \phi[a:=a'] \\
\vdash G, \neg \phi^{+k} \\
\vdash (\forall L) \\
\vdash \Delta \\
\vdash \exists L \quad \text{fresh } a^{+k} \\
\vdash (\exists L) \\
\hline
\vdash F \land G \\
\hline
\hline
\vdash F^{+k}, \phi[a:=a']^{+k}, \forall a. \phi^{+k} \\
\vdash G, \exists a. \neg \phi^{+k} \\
\vdash (\exists L) \\
\hline
\vdash \Delta[a:=a'] \\
\vdash (\exists L) \\
\hline
\vdash F^{+k}, G \\
\vdash (\text{Shift})^{*} \\
\end{array}
\]

**Lemma 6.13.** All of the essential cases above strictly reduce the size of the cut formula, or leave size unchanged and reduce the cut-height.
Proof. This is just a fact, as discussed above for each case.

6.6. Admissibility of Cut

Notation 6.14. Recall that Definition 4.2 defines a context to be a finite set of PNF predicates. It is convenient to extend Notation 2.4(2):

(1) Call a context closed when it contains only closed predicates.
(2) Call a context $F$ possibly open when it may contain predicates that are not closed. Contexts are possibly open anyway in Definition 4.2, so a possibly open context is just a context, and to call it ‘possibly open’ is simply to emphasise that it need not be closed.

Remark 6.15 (An important non-theorem). It may be useful to pause for a moment to take stock.

We are about to prove some Cut-admissibility results below — Theorem 6.17, Corollary 6.18, and Theorem 6.19 — but what is arguably most interesting, and worth pointing out explicitly here, is a non-theorem of these three results in this Subsection: that if $F$ and $G$ are possibly open contexts and $\phi$ is a closed predicate, and $F, \phi \vdash$ and $G, \neg\neg\neg\phi \vdash$, then it is not necessarily the case that $F, G \vdash$.

The reason for this is the closure condition on contexts in the second reduction case for $(\forall L)$ and $(\exists L)$ with a nonzero shift offset, above. Note there the context $F^+$, followed by instances of $(\text{Shift})^*$ to convert it back to $F$. This only works when $F$ is closed.

So what the Cut-admissibility results below are really saying is that for every situation that is not this non-theorem, Cut can be eliminated. This non-theorem affects the ultrafilter completion construction which follows below, in the sense that we must take care to proceed in two stages:

(1) saturate closed predicates using Corollary 6.18 (to get $C$ in Theorem 7.1) and then
(2) saturate open predicates using Theorem 6.19 (to get $Q$ in Theorem 7.4).

This two-stage construction is there precisely to avoid a situation of having to decide whether to add $\phi$ or $\neg\neg\neg\phi$ to our nascent ultrafilter set, when $\phi$ is closed but the set contains open predicates. We now proceed with the proofs:

Remark 6.16. Recall that a PNF predicate is one that is in prenex normal form (Definition 2.13), and that a context is a finite set of PNF predicates (Definition 4.2). The Cut rule is in Figure 9.

Theorem 6.17. Suppose that

— $F$ and $G$ are contexts (finite sets of PNF predicates) and
— $\phi$ is a PNF predicate.

Then:

— if $F, \phi \vdash$ and $G, \neg\phi \vdash$ are derivable by shift-free derivations (Notation 6.1(1)),
— then $F, G \vdash$ is derivable by a shift-free derivation.

Proof. By a routine induction on the size of the cut formula and cut-height, lexicographically ordered:

— By Lemmas 6.12 and 6.13, essential cases either eliminate the Cut, or they strictly reduce the size of the cut formula, or (for one $\neg\neg\neg$ case) the essential case does not change the cut formula and strictly reduces the cut-height.
— By Lemma 6.11, commutations do not increase the size of the cut formula, and they strictly reduce cut-height.

Thus we can apply commutations and essential cases to a Cut — note that the Cut is ‘ordinary’ in the sense that it has zero shift offset, since we are cutting $\phi$ against $\neg\neg\neg\phi$ — and thus eliminate it in the standard way.

Corollary 6.18. Suppose that

— $F$ and $G$ are closed contexts (finite sets of closed PNF predicates) and
— φ is a closed PNF predicate.

Then

— if \( F, \phi \vdash \) and \( G, \neg \phi \vdash \) are derivable, then
— so is \( F, G \vdash \).

Proof. By Proposition 6.7 we may assume without loss of generality that the derivations of \( F, \phi \vdash \) and \( G, \neg \phi \vdash \) are in shift-normal form, i.e. have all their instances of \( \text{Shift} \) at the end (Notation 6.1(2)).

We then note:

— Reductions in Subsection 6.4 can absorb the shifts into the \( \text{Cut} \), if they act on the principal formula.
— The final commutation in Subsection 6.3 can commute shifts down through the Cut, if they do not act on the principal formula.
— If the Cut is on a closed quantified predicate with nonzero shift-offset then we use the final essential case (between (\( \forall L \)) and (\( \exists L \)) in Subsection 6.5 to reduce to a Cut on an open predicate with zero shift-offset.

What remains is now guaranteed to be a Cut with zero shift-offset and on shift-free derivations, so we can use Theorem 6.17 to eliminate it.

Theorem 6.19. Suppose that

— \( F \) and \( G \) are (possibly open) contexts and
— \( \phi \) is an open PNF predicate (so \( \phi \) is not closed; it has at least one free variable).

Then

— if \( F, \phi \vdash \) and \( G, \neg \phi \vdash \) are derivable, then
— so is \( F, G \vdash \).

Proof. By Proposition 6.7 we may assume without loss of generality that the derivations of \( F, \phi \vdash \) and \( G, \neg \phi \vdash \) are in shift-normal form. By the closure condition in \( \text{Shift} \) in Figure 8, none of the instances of shift that conclude the derivations can be focused on \( \phi \) or \( \neg \phi \), because we carefully assumed that \( \phi \) is not closed.

We may therefore use the reductions in Subsection 6.4 to eliminate the Cut or move it up until it reaches a pair of shift-free subderivations. Then we eliminate the Cut on this pair of shift-free subderivations, using Theorem 6.17.

Corollary 6.20(1) Suppose \( \mathcal{F} \) is a possibly infinite set of closed PNF predicates, and \( \phi \) is a closed PNF predicate. Then

\[
\mathcal{F} \not\vdash \implies \mathcal{F}, \phi \not\vdash \text{ or } \mathcal{F}, \neg \phi \not\vdash .
\]

(2) Suppose \( \mathcal{F} \) is a possibly infinite set of possibly open PNF predicates, and \( \phi \) is an open PNF predicate. Then

\[
\mathcal{F} \not\vdash \implies \mathcal{F}, \phi \not\vdash \text{ or } \mathcal{F}, \neg \phi \not\vdash .
\]

Proof. (1) Suppose \( \mathcal{F} \not\vdash \) and suppose we have contexts

— \( F \subseteq_{\text{fin}} \mathcal{F} \) such that \( F, \phi \vdash \), and
— \( F' \subseteq_{\text{fin}} \mathcal{F} \) such that \( F', \neg \phi \vdash \).

By Lemma 5.6 therefore \( F, F', \phi \vdash \) and \( F, F', \neg \phi \vdash \). By Corollary 6.18 \( F, F' \vdash \), contradicting our assumption that \( \mathcal{F} \not\vdash \).

(2) As for part 1 of this result, but using Theorem 6.19 for the open predicate \( \phi \).
7. THE ULTRAFILTER CONSTRUCTION, AND BUILDING AN ECT THEORY

So given a consistent set, Corollary 6.20 allows us to extend it, and as standard this invites us to saturate to a maximally consistent set. Also as standard, we will do this such that disjunctions are witnessed — if \( \phi \lor \phi' \) then \( \phi \) or \( \phi' \) — and existentials are witnessed — if \( \exists a. \phi \) then \( \phi[a:=a'] \) for some witness \( a' \). The result will be an ECT theory:

7.1. Saturating closed predicates

Theorem 7.1. We can construct a set of closed PNF predicates \( C \) such that:

1. For every closed \( \phi \), at least one of \( \phi \in C \) or \( \neg \phi \in C \) holds.
2. \( C \) is consistent. In symbols: \( C \not\vdash \).

We call such a \( C \) a closed PNF theory.

Proof. Note first from Corollary 5.5 that \( \emptyset \not\vdash \). Now we enumerate closed predicates in prenex normal form as \( \{ \phi_i \mid i \geq 0 \} \) and iterate through this list. For each \( \phi_i \) in turn, we use Corollary 6.20(1) to consistently add either \( \phi \) or \( \neg \phi \). The result is a \( C \) with the required properties. \( \square \)

Notation 7.2. Using Theorem 7.1 above, make a fixed but arbitrary choice of closed PNF theory \( C \).

7.2. Saturating open predicates, to obtain an ECT theory

We now set about extending \( C \) above (a theory of closed PNF predicates) to \( Q \) below (a theory of possibly open PNF predicates). We will do this by adding predicates one by one, using Lemma 7.3:

Lemma 7.3. Suppose \( F \) is a set of predicates and suppose \( \bigcup \{ \text{fv}(\phi) \mid \phi \in F \} \) is finite. Then:

1. Suppose \( F, \exists a. \phi \not\vdash \), and let \( a \) be not free in any predicate in \( F \) — some such \( a \) exists, by our assumption that \( \bigcup \{ \text{fv}(\phi) \mid \phi \in F \} \) is finite. Then \( F, \exists a. \phi \not\vdash \).
2. Suppose \( F, \forall a. \phi \not\vdash \). Then \( F, \forall a. \phi, \phi \not\vdash \).
3. Suppose \( F, \phi \land \phi' \not\vdash \). Then at least one of \( F, \phi \lor \phi' \) or \( F, \phi \lor \phi' \) must hold.
4. Suppose \( F, \phi \lor \phi' \not\vdash \). Then \( F, \phi \land \phi', \phi, \phi' \not\vdash \).
5. Suppose \( F, \neg \phi \not\vdash \). Then \( F, \neg \phi, \neg \phi \not\vdash \).

Proof. For each case we prove a contrapositive:

1. Suppose \( F \subseteq_{\text{fin}} F, \exists a. \phi, \phi \) and \( F \not\vdash \). Then, using Lemma 5.6 (weakening) if required, we have that \( F, \exists a. \phi, \phi \not\vdash \) and using \( (\exists \mathbf{L}) \) \( F, \exists a. \phi \not\vdash \) as required.
2. Suppose \( F \subseteq_{\text{fin}} F, \forall a. \phi, \phi[a:=a'] \) and \( F \not\vdash \). Then using Lemma 5.6 and \( (\forall \mathbf{L}) \) \( F, \forall a. \phi \not\vdash \).
3. Suppose we have
   
   \[ F \subseteq_{\text{fin}} F \] such that \( F, \phi \lor \phi' \not\vdash \), and
   
   \[ F' \subseteq_{\text{fin}} F \] such that \( F', \phi \lor \phi' \not\vdash \).

   Then \( F \cup F' \subseteq_{\text{fin}} F \) and using Lemma 5.6 \( F, F', \phi \lor \phi' \not\vdash \), so using \( (\forall \mathbf{L}) \) we conclude that \( F, F', \phi \lor \phi' \not\vdash \).
4. Suppose \( F \subseteq_{\text{fin}} F, \phi \land \phi', \phi, \phi' \) and \( F \not\vdash \). Then using Lemma 5.6 and \( (\land \mathbf{L}) \) we conclude that \( F, \phi \land \phi' \not\vdash \).
5. Suppose \( F \subseteq_{\text{fin}} F, \neg \phi, \neg \phi \) and \( F \not\vdash \). Then using Lemma 5.6 and \( (\neg \mathbf{L}) \) we conclude that \( F, \neg \phi \not\vdash \).

\( \square \)

Theorem 7.4. The closed PNF theory \( C \) from Notation 7.2 can be extended to an ECT theory \( Q \) (Definition 3.3(4)).

Proof. Set \( Q_0 = C \). By construction in Theorem 7.1 \( Q_0 \) is consistent; in symbols:

\[ Q_0 \not\vdash \]
Enumerate open PNF predicates as \( (\phi_i \mid i \geq 0) \). We will use this enumeration to build an ascending chain of consistent sets \( (Q_i \mid i \geq 0) \), starting from \( Q_0 \) as just defined.

To define \( Q_{i+1} \) given \( Q_i \) we reason by cases on \( \phi_i \) using Lemma 7.3, as follows:

- If \( \phi_i = \exists a.\phi \) and \( Q_i, \exists a.\phi \not\vdash \) then we set
  \[ Q_{i+1} = Q_i \cup \{\exists a.\phi, \phi\}. \]
  where we choose \( a \) fresh for all the elements in \( Q_i \). By Lemma 7.3(1) this is consistent.

- If \( \phi_i = \exists a.\phi \) and \( Q_i, \forall a.\neg \phi \not\vdash \) then by Corollary 6.20(2) \( Q_i, \forall a.\neg \phi \not\vdash \). We set
  \[ Q_{i+1} = Q_i \cup \{\forall a.\neg \phi\}. \]

- If \( \phi_i = \forall a.\phi \) and \( Q_i, \forall a.\phi \not\vdash \) then set
  \[ Q_{i+1} = Q_i \cup \{\exists a.\phi\}. \]
  where we choose \( a \) fresh for all the elements in \( Q_i \).

- If \( \phi_i = \phi \lor \phi' \) and \( Q_i, \phi \lor \phi' \not\vdash \) then by Lemma 7.3(3) \( Q_i, \phi \lor \phi' \not\vdash \) or \( Q_i, \phi \lor \phi', \phi' \not\vdash \). We set
  \[ Q_{i+1} = Q_i \cup \{\phi \lor \phi', \phi\} \quad \text{or} \quad Q_{i+1} = Q_i \cup \{\phi \lor \phi', \phi'\}. \]
as appropriate.

- If \( \phi_i = \phi \land \phi' \) and \( Q_i, \phi \land \phi' \not\vdash \) then by Corollary 6.20(2) \( Q_i, \phi \land \phi' \not\vdash \) and by Lemma 7.3(4) \( Q_i, \neg \phi \lor \neg \phi' \not\vdash \). We set
  \[ Q_{i+1} = Q_i \cup \{\neg \phi \lor \neg \phi', \neg \phi, \neg \phi'\}. \]

- The cases that \( \phi_i = \phi \land \phi' \) are symmetric with those for \( \phi \lor \phi' \).

- If \( \phi_i = \neg \phi \) and \( Q_i, \neg \phi \not\vdash \) then by Lemma 7.3(5) \( Q_i, \neg \phi, \neg \phi \not\vdash \) and we set\(^{12}\)
  \[ Q_{i+1} = Q_i \cup \{\neg \phi, \neg \phi\}. \]

- If \( \phi_i = \neg \phi \) and \( Q_i, \neg \phi \not\vdash \) then by Corollary 6.20(2) \( Q_i, \phi \not\vdash \) (note from Figure 4 that \( \neg \neg \phi = \phi \)) and we set
  \[ Q_{i+1} = Q_i \cup \{\phi\}. \]

Finally, we set
\[ Q = \bigcup_i Q_i. \]

\[ \square \]

**Proposition 7.5.** \( Q \) from Theorem 7.4 is an ECT theory in the sense of Definition 3.3.

**Proof.** We check that the properties of an ECT theory as outlined in Definition 3.3 do indeed hold. There are two things to check:

1. The PNF theory properties in Figure 6 hold.
   - This is a fact of the constructions above: we take care to preserve consistency; we make sure to witness existentials and disjunctions; we always add either \( \neg \phi \) or \( \phi \) (and by Lemma 5.3 and consistency, we do not add both).

\(^{12}\)Strictly speaking we do not need to add \( \neg \phi \), because this would get added anyway by whatever clause handles its top-level predicate-former. But we do so for consistency with all the other clauses.
(2) The ECT properties from Definition 3.3 hold.

Typical Ambiguity is built in from having the derivation rule (Shift) in Figure 8. Extensionality and Comprehension are built in from having the derivation rule (□), and from Lemma 4.16(3&4).

Corollary 7.6. There exists a countably infinite sets model of TST+. As a corollary, NF is consistent.

Proof. We combine Proposition 7.5 with Corollary 3.14 and Lemma 3.15. That is: we take Q above and feed it into the argument in Section 3 to build a concrete countably infinite sets model T of TST+. It is known [Spe62] that this implies the consistency of NF (we reproduce the relevant model construction in Appendix A for the reader’s convenience).

Remark 7.7 (Consistency strength). The argument in this paper requires natural numbers N and an ability to iterate powersets. We do make choices (e.g. in Theorems 7.1 and 7.4) but these choices are over predicates, which can be well-ordered so we can just pick the least one in the well-ordering, and we do not require any Axiom of Choice to do this. This locates the consistency strength of the argument as being equal to that of higher-order arithmetic (i.e. numbers and sets of numbers and sets of sets of numbers and so forth).

Remark 7.8. We briefly compare the proof above with the known proof of consistency of NFU (NF with urelemente) using Ehrenfeucht-Mostowski (EM) models.

The Ehrenfeucht-Mostowski theorem is a remarkable result which allows us, under extremely general conditions, to build models with lots of automorphisms. A clear presentation is in [Gai67, Theorem 11] and a detailed recent discussion is in [Hod93, Theorem 11.2.8]. A detailed discussion is out of scope for this paper but what matters is that the EM models of NFU work using symmetry: intuitively, we get Typical Ambiguity (TA) by building a model in levels such that every level is isomorphic (symmetric) with every other level. These symmetries cannot be represented within the model, but they are easily observable from outside it.

The NF/TST+ model in this paper has a different flavour from these NFU models: it is highly asymmetric. Specifically, asymmetry is introduced in the ultrafilter completion of Theorem 7.4.

We retain TA over closed predicates because of the (Shift) rule in Figure 8, which is all we really need. Away from closed predicates — in the world of open predicates as handled by Theorem 7.4 — the model is allowed to become asymmetric.

Thus the model in this paper seems to be in a distinct spirit than an EM-style model. It has to lose design freedom by excluding urelemente (so that it yields a model of NF rather than NFU), but it regains some design freedom by not imposing the kind of global symmetry that typifies EM-style models: we trade the freedom of having urelemente for the freedom to be asymmetric.

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A. SKETCH OF TRANSFORMATION BETWEEN (COUNTABLY INFINITE) MODELS OF TST+ AND QUINE’S NF

In this Appendix we will re-express in a slightly more modern notation the simple model-theoretic argument described towards the end of the third page of [Spe62]. First, a preliminary definition which is clearly related to Definition 2.10:

**Definition A.1.** A countably infinite model of Quine’s NF consists of a countably infinite set \( V \) and a membership relation \( \varepsilon \subseteq V \times V \), such that extensionality and (stratifiable) comprehension are valid over the structure \( (V, \varepsilon) \).

**Theorem A.2.** There exists a countably infinite model of TST+ if and only if there exists a countably infinite model of Quine’s NF.

**Proof.** We outline the manipulation on models from [Spe62] to convert between the models:

Suppose we have a countably infinite sets model of TST+, so we have

\[
(V_0, V_1 \subseteq \text{pow}(V_0), V_2 \subseteq \text{pow}(V_1), \ldots)
\]

and each \( V_i \) is countably infinite.

To build a model of Quine’s NF, we choose\(^{13}\) isomorphisms \( f_i : V_i \cong V_{i+1} \) and set

\[
V = V_0 \quad \text{and} \quad y \in x \text{ when } y \in f_i(x).
\]

In this way, \( f_i \) assigns a \( V_i \)-extension to each element of \( V_i \). It is now routine to check that \( (V, \varepsilon) \) is an extensional model (because the sets model is) with stratifiable comprehension (because the sets model has stratified comprehension) and so is a model of Quine’s NF.

Conversely, given a countable model \( (V, \varepsilon) \) of Quine’s NF we construct a countable model of TST+ by setting \( V_0 = V \) and \( V_1 = \{ \{ v' \in V \mid v' \in v \} \mid v \in V \} \) and repeatedly unrolling the sets extension to obtain a sequence of countably infinite sets \( V_i \subseteq \text{pow}^i(V) \).

\[\square\]

B. GENERALISING RUSSELL’S PARADOX: CYCLIC STRATIFIED SYNTAX DOES NOT NORMALISE

The comprehension rewrite rule on TST syntax augmented with stratified comprehension terms \( \{a \mid \phi\} \) — a.k.a. \( \beta \)-reduction

\[
t \in \{a \mid \phi\} \rightarrow \phi[a := t]
\]

\(^{13}\)In fact we can assume the sets are delivered to us with an enumeration, since we build them in this paper; so it suffices for \( f_i \) to biject the first element of \( V_i \) with the first element of \( V_{i+1} \), and so forth.
— is normalising [Gab18]. What if we stratify, but using some cyclic graph like integers modulo some $n$, instead of $\mathbb{N}$? We take a few moments to illustrate that we do indeed lose normalisation:

**Definition B.1.** Define a naïve sets term $r_1$ by

$$r_1 = \{ a \mid \neg (a \in a) \}.$$

**Remark B.2.** *Russell's paradox* corresponds to the following rewrite:

$$r_1 \not\in r_1 \rightarrow \neg (r_1 \not\in r_1).$$

This rewrite is non-normalising, in the sense that the term we started with is a subterm of the term we finish with — we have a loop.

**Notation B.3.** Given a naïve sets term $t$ and $i \geq 0$, define $i^t$ by:

$$i^0 t = t \quad \text{and} \quad i^{i+1} t = i(i^t).$$

**Remark B.4.** We note the following rewrites:

$$s \not\in i t \rightarrow t \not\in s \quad \text{and} \quad t \not\in i s \rightarrow s \not\in i t \rightarrow t \not\in s.$$

**Definition B.5.** We define

$$r_2 = \{ a \mid \neg (i a \in a) \}.$$

**Remark B.6.** If we try to normalise $i r_2 \not\in r_2$, we obtain:

$$i r_2 \not\in r_2 \rightarrow \neg (i^2 r_2 \not\in r_2) \rightarrow^* \neg (i r_2 \not\in r_2).$$

($\rightarrow^*$ above denotes multiple rewrites.)

**Definition B.7.** Generalising the above, if $n \geq 1$ then we write

$$r_n = \{ a \mid \neg (i^{n-1} a \in a) \}.$$

**Remark B.8.** If we try to normalise $i^{n-1} r_n \not\in r_n$, we obtain

$$i^{n-1} r_n \not\in r_n \rightarrow \neg (i^{2n-2} r_n \not\in r_n) \rightarrow^* \neg (i^{n-1} r_n \not\in r_n).$$

So if we stratify naïve sets with finitely many levels $\mathbb{Z}_{\text{mod}(n)}$ for any finite $n$, then there exist predicates with no normal form.