NOETHERIANITY AND SPECHT PROBLEM
FOR VARIETIES OF BICOMMUTATIVE ALGEBRAS

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Abstract. Nonassociative algebras satisfying the polynomial identities
\[ x_1(x_2x_3) = x_2(x_1x_3) \text{ and } (x_1x_2)x_3 = (x_1x_3)x_2 \]
are called bicommutative. We prove the following results: (i) Finitely generated
bicommutative algebras are weakly noetherian, i.e., satisfy the ascending chain
condition for two-sided ideals. (ii) We give the positive solution to the Specht
problem (or the finite basis problem) for varieties of bicommutative algebras over
an arbitrary field of any characteristic.

1. Introduction

Let \( K \) be a field. A \( K \)-algebra \( R \) is called right-commutative if it satisfies the
polynomial identity
\[ (x_1x_2)x_3 = (x_1x_3)x_2, \]
i.e., \((r_1r_2)r_3 = (r_1r_3)r_2\) for all \( r_1, r_2, r_3 \in R \). Similarly one defines left-commutative
algebras as algebras which satisfy the identity
\[ x_1(x_2x_3) = x_2(x_1x_3). \]
Algebras which are both right- and left-commutative are called bicommutative. We
denote by \( \mathcal{B} \) the variety of all bicommutative algebras, i.e., the class of all algebras
satisfying the identities of right- and left-commutativity. The first example of a
one-sided commutative algebra is the right-symmetric Witt algebra \( W_1^{\text{rsym}} \) in one
variable which appeared already in the paper by Cayley \[ 6 \] in 1857,
\[ W_1^{\text{rsym}} = \left\{ f \frac{d}{dx} \mid f \in K[x] \right\} \]
 equipped with the multiplication
\[ \left( f_1 \frac{d}{dx} \right) \ast \left( f_2 \frac{d}{dx} \right) = \left( f_2 \frac{df_1}{dx} \right) \frac{d}{dx} \]
which is left-commutative. (Right-symmetric algebras satisfy the polynomial identity
\( (x_1, x_2, x_3) = (x_1x_2)(x_3), \) where \( (x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3) \) is the
associator.) Cayley also considered the realization of the right-symmetric Witt algebras
\( W_d^{\text{rsym}} \) in terms of rooted trees. The algebra \( W_1^{\text{rsym}} \) is an example of a Novikov
algebra (which is right-symmetric and left-commutative). (The algebras $W_{d}^{\text{rsym}}$ are not Novikov for $d > 1$ because are right-symmetric but are not left-commutative.) Novikov algebras and their opposite appeared in the 1970s and 1980s in the papers by Gel’fand and Dorfman [14] in their study of the Hamiltonian operator in finite-dimensional mechanics and by Balinskii and Novikov [9], see also [29], in relation with the equations of hydrodynamics. The basis of the free right-commutative algebras as a set of rooted trees was given by Dzhumadil’daev and Lőfwall [12]. Dzhumadil’daev, Ismailov, and Tulenbaev [11], see also the announcement [13], described the free bicommutative algebra $F(B)$ of countable rank and its main numerical invariants. In particular, they found a basis of $F(B)$ as a $K$-vector space, computed the Hilbert series of the $d$-generated free bicommutative algebra $F_d(B)$, the cocharacter and codimension sequences of the variety $B$.

It was shown in [13] that the square $F^2(B)$ of the algebra $F(B)$ is a commutative and associative algebra. Therefore one should expect that the algebra $F(B)$ itself has many properties typical for commutative and associative algebras. The simplest example of a finitely generated bicommutative algebra which is not noetherian is the one-generated free algebra $F_1(B)$ which has not finitely generated one-sided ideals. But we have established that any finitely generated bicommutative algebra is weakly noetherian, i.e., satisfies the ascending chain condition for two-sided ideals.

Let $K\{X\} = K\{x_1, x_2, \ldots \}$ be the absolutely free nonassociative algebra and let $f(r_1, \ldots, r_d) \in K\{X\}$. Recall that the $K$-algebra $R$ satisfies the polynomial identity $f = 0$ if $f(r_1, \ldots, r_d) = 0$ for all $r_1, \ldots, r_d \in R$. If $\{f_i \in K\{X\} \mid i \in I\}$ is a set of elements in $K\{X\}$, then the class $\mathfrak{V}$ of all algebras satisfying the polynomial identities $f_i = 0$, $i \in I$, is called the variety defined by the system of polynomial identities $\{f_i \mid i \in I\}$. The set $T(\mathfrak{V})$ of all polynomial identities satisfied by the variety $\mathfrak{V}$ is called the $T$-ideal or the verbal ideal of $\mathfrak{V}$. By definition, $T(\mathfrak{V})$ is generated as a $T$-ideal by any system of polynomials defining the variety $\mathfrak{V}$. One of the main problems in the theory of varieties of algebras is:

**Problem 1.1.** (The finite basis problem, or the Specht problem) Can any subvariety of a given variety of algebras be defined by a finite number of polynomial identities?

It follows from the description of the cocharacter sequence of the variety $\mathfrak{B}$ given in [11] that in characteristic 0 every $T$-ideal in $F(\mathfrak{B})$ is generated by its elements in two variables only. Hence the weak noetherian property for finitely generated bicommutative algebras immediately implies the positive solution to the Specht problem when $\text{char} K = 0$. In order to establish a similar result when the field $K$ is of positive characteristic we apply the classical method of Higman-Cohen [17] and [8]. Nowadays in many cases the Specht problem is solved into affirmative using the structure theory of $T$-ideals developed by Kemer for associative algebras in characteristic 0 (see his book [23] for an account), its further developments in positive characteristic (see, e.g., Belov-Kanel, Rowen, and Vishne [4]), and for other classes of algebras (e.g., Iltyakov [21] with detailed exposition in [22] for finite dimensional Lie algebras, Vajs and Zel’manov [32] for finitely generated Jordan algebras). But up to the 1970s, the Higman-Cohen method was one of the few methods to handle into affirmative the Specht problem:

- for groups: Cohen [8] for $\mathfrak{A}^2$ (below we use the standard notation for the varieties), Vaughan-Lee [33] for $\mathfrak{AN}_c \cap \mathfrak{N}_d \mathfrak{A}$;
• for Lie algebras: Vaughan-Lee [34] for \([\mathfrak{A}, \mathfrak{L}]\), \(\text{char} K \neq 2\), Bryant and Vaughan-Lee [5] for \(\mathfrak{N}_2\mathfrak{A}\), \(\text{char} K \neq 2\);
• for associative algebras in characteristic 0: Latyshev [25] and Genov [15, 16] for \(\mathfrak{N}_2\mathfrak{A}\), Latyshev [26] and Popov [30] for \(\mathfrak{N}_k\mathfrak{L}_2\);
• for associative algebras in positive characteristic: Chiripov and Siderov [7] for \(\mathfrak{N}_3\mathfrak{A}\), \(\text{char} K \neq 2\).

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2. Preliminaries

It the sequel we shall denote by \(F\) and \(F_d\) the free bicommutative algebras \(F(\mathfrak{B})\) and \(F_d(\mathfrak{B})\), respectively. It was established in [11] that the following monomials form a basis of the square \(F_d^2\) of the algebra \(F_d\) as a \(K\)-vector space:

\[
(1) \\
\quad u_{i,j} = x_i \cdot \cdots \cdot (x_{i,m-1} \cdot \cdots \cdot (x_{i,m-2} \cdot \cdots \cdot (x_{i,m-3} \cdot \cdots \cdot (x_{i,m-4}) \cdots )) \cdots )
\]

where \(m, n \geq 1\), \(1 \leq i_1 \leq \cdots \leq i_{m-1} \leq i_m \leq d\), \(1 \leq j_1 \leq j_2 \leq \cdots \leq j_n \leq d\). If \(L_{x_i}\) and \(R_{x_j}\) are the operators of left and right multiplication on \(F_d\), defined respectively by

\[
L_{x_i} : u \to x_i u \quad \text{and} \quad R_{x_j} : u \to ux_j, \quad u \in F_d,
\]

then \(1\) can be written as

\[
u_{i,j} = L_{x_{i_1}} \cdots L_{x_{i_{m-1}}} R_{x_{j_m}} \cdots R_{x_{j_n}} (x_{i_0}).
\]

For any permutations \(\sigma \in S_m\) and \(\tau \in S_n\) the element \(u_{i,j}\) from \(1\) satisfies the equality

\[
u_{i,j} = x_{i_{\sigma(1)}} \cdot \cdots \cdot (x_{i_{\sigma(m-1)}} \cdot \cdots \cdot (x_{i_{\sigma(m-2)}} \cdot \cdots \cdot (x_{i_{\sigma(m-3})} \cdots )) \cdots )) \cdots )
\]

i.e.,

\[
u_{i,j} = L_{x_{\sigma(1)}} \cdots L_{x_{\sigma(m-1)}} R_{x_{\tau(m)} \cdots R_{x_{\tau(2)}} \cdots R_{x_{\tau(1)}} (x_{i_{\sigma(m})})}
\]

By [11] the algebra \(F_d\) is isomorphic to the following algebra \(F(d)\). Let \(\mathbb{N}_d^d = \mathbb{N}_0 \times \cdots \times \mathbb{N}_0\) be the direct sum of \(d\) copies of \(\mathbb{N}_0\). Let \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}_0^d\), where all components except the \(i\)-th are equal to 0. The algebra \(F(d)\) has a basis

\[
(2) \quad \{x_i \mid i = 1, \ldots, d\} \cup \{u_{\alpha, \beta} \mid \alpha, \beta \in \mathbb{N}_0^d, \sum_{i=1}^d \alpha_i > 0, \sum_{i=1}^d \beta_i > 0\}
\]
and multiplication $\circ$ given by the following rules:

\begin{align}
    x_i \circ x_j &= v_{\epsilon_i, \epsilon_j}, \\
    x_i \circ v_{\alpha, \beta} &= v_{\alpha + \epsilon_i, \beta}, \\
    v_{\alpha, \beta} \circ x_j &= v_{\alpha, \beta + \epsilon_j}, \\
    v_{\alpha, \beta} \circ v_{\gamma, \delta} &= v_{\alpha + \gamma, \beta + \delta}.
\end{align}

The isomorphism $\pi : F_d \rightarrow F(d)$ is defined on the basis monomials of $F_d$ in the following way and then extended by linearity. We associate to any monomial $u_{i,j}$ in the $m$-tuple $X_d^{(i)} = (x_{i_1}, \ldots, x_{i_{m-1}}, x_{i_m})$ and the $n$-tuple $X_d^{(j)} = (x_{j_1}, x_{j_2}, \ldots, x_{j_n})$. If

$$
X_d^{(i)} = (x_{i_1}, \ldots, x_{i_\alpha_i}, \ldots, x_{i_m}), \quad X_d^{(j)} = (x_{j_1}, \ldots, x_{j_\beta_1}, \ldots, x_{j_n}),
$$

then $\pi(u_{i,j}) = v_{\alpha, \beta}$.

For our purposes it is more convenient to identify the element $v_{\alpha, \beta}$ with the monomial

$$
Y_d^\alpha Z_d^\beta = y_1^{\alpha_1} \cdots y_d^{\alpha_d} z_1^{\beta_1} \cdots z_d^{\beta_d}
$$

in the polynomial algebra $K[Y_d, Z_d] = K[y_1, \ldots, y_d, z_1, \ldots, z_d]$ in $2d$ commutative and associative variables. In this notation the algebra $F(d)$ is isomorphic to the algebra $G_d$ with basis

\begin{align}
\{ x_i \mid i = 1, \ldots, d \} \cup \{ Y_d^\alpha Z_d^\beta \mid \deg Y_d^\alpha, \deg Z_d^\beta > 0 \}
\end{align}

and multiplication

\begin{align}
    x_i \circ x_j &= y_i z_j, \\
    x_i \circ Y_d^\alpha Z_d^\beta &= y_i Y_d^\alpha Z_d^\beta, \\
    Y_d^\alpha Z_d^\beta \circ x_j &= Y_d^\alpha Z_d^\beta z_j, \\
    Y_d^\alpha Z_d^\beta \circ Y_d^\gamma Z_d^\delta &= Y_d^{\alpha+\gamma} Z_d^{\beta+\delta}.
\end{align}

The following lemma summarizes the properties of $G_d$ stated above.

**Lemma 2.1.** In the notation of (3) and (5):

(i) The algebra $F_d$ is isomorphic to the algebra $G_d$ generated by $X_d = \{ x_1, \ldots, x_d \}$. The square $G_d^2$ of $G_d$ has a basis $\{ Y_d^\alpha Z_d^\beta \mid \deg Y_d^\alpha, \deg Z_d^\beta > 0 \}$.

(ii) The left and the right multiplications by the elements of $X_d$ on $G_d^2$ define on it a natural structure of a $K[Y_d, Z_d]$-module.

As an immediate consequence of Lemma 2.1 we obtain the following description of the algebra $F$.

**Lemma 2.2.** The algebra $F$ is isomorphic to the algebra $G$ with basis

$$
X \cup \{ Y^\alpha Z^\beta \in K[Y, Z] = K[y_1, y_2, \ldots, z_1, z_2, \ldots] \mid \deg Y^\alpha, \deg Z^\beta > 0 \}.
$$

The algebra $G$ is generated by $X$ and the left and the right multiplications by the elements from $X$ on $G^2$ make it a $K[Y, Z]$-module.
3. Weak noetherianity

We start with an example showing that finitely generated bicommutative algebras are not necessarily noetherian.

**Proposition 3.1.** The free bicommutative algebra $F_1 = F_1(\mathcal{B})$ is not noetherian.

**Proof.** By Lemma 2.1 the algebras $F_1(G)$ and $G_1$ are isomorphic and we shall work in $G_1$ instead of in $F_1$. As a vector space $G_1$ has a basis

$$\{x_1\} \cup \{y_1^{\alpha_1}z_1^{\beta_1} \mid \alpha_1, \beta_1 > 0\}. $$

Consider the left ideal $I$ of $G_1$ generated by the monomials

$$y_1^\delta, \quad \delta = 1, 2, \ldots. $$

If $I$ is finitely generated, then it can be generated by a finite number of monomials $y_1z_1^\delta$, $1 \leq \delta \leq n$, from (6). Then, by (6), $I$ is spanned by the monomials

$$x_1 \circ \cdots \circ x_1 \circ y_1z_1^\delta = y_1^{k+1}z_1^\delta, \quad k = 0, 1, 2, \ldots, \quad 1 \leq \delta \leq n,$$

$$y_1^{\alpha_1}z_1^\delta \circ y_1z_1^\delta = y_1^{\alpha_1+1}z_1^{\beta_1+\delta}, \quad \alpha_1, \beta_1 \geq 1, \quad 1 \leq \delta \leq n.$$

Obviously, this list of monomials does not contain the monomials $y_1z_1^\delta$ from $I$ for $\delta > n$, i.e., the left ideal $I$ is not finitely generated. The considerations for not finitely generated right ideals of $G_1$ are similar. It is sufficient to consider the right ideal generated by $y_1^\gamma z_1$, $\gamma = 1, 2, \ldots$. □

The following theorem is the first main result of our paper.

**Theorem 3.2.** Finitely generated bicommutative algebras satisfy the ascending chain condition for two-sided ideals.

**Proof.** It is sufficient to work in the free algebra $F_d$, or, equivalently, in its isomorphic copy $G_d$. The factor algebra $G_d/G_d^2$ is finite dimensional and hence noetherian. Therefore the theorem will be established if we prove the weak noetherianity for the ideals in $G_d^2$. Every two-sided ideal $I$ of $G_d$ which is in $G_d^2$ is stable under the left and right multiplications by the generators $x_1, \ldots, x_d$ of $G_d$ and hence is a $K[Y_d, Z_d]$-submodule of $F_d^2$. As a $K[Y_d, Z_d]$-module $F_d^2$ is generated by the finite number of monomials $y_iz_j$, $i, j = 1, \ldots, d$. Hence the $K[Y_d, Z_d]$-submodule $I$ of $G_d^2$ is also finitely generated which implies that $I$ is finitely generated also as a two-sided ideal. □

**Remark 3.3.** By Theorem 3.2 every two-sided ideal $I$ of the free bicommutative algebra $F_d$ is finitely generated. It is an interesting problem how the number of the generators of $I$ depends on the rank $d$ of $F_d$. Since the square $F_d^2$ of $F_d$ is commutative and associative, we shall comment what happens for the ideals of the polynomial algebra $K[X_d]$. Seidenberg [31] formalized the problem in the following way. Given a function $f : \mathbb{N}_0 \to \mathbb{N}_0$, what is the maximal $k_0 \in \mathbb{N}$ with the property: There exists an ideal $I$ of $K[X_d]$ generated by the set $\{p_0(X_d), p_1(X_d), \ldots, p_{k_0}(X_d)\}$ such that deg$(p_k(X_d)) \leq f(k)$, $k = 0, 1, \ldots, k_0$, and the chain of ideals $I_0 \subset I_1 \subset \cdots \subset I_{k_0}$, where $I_k$ is generated by $\{p_0(X_d), p_1(X_d), \ldots, p_k(X_d)\}$, is strictly increasing. He showed that there exists a bound $q_f$ depending on $f$ and $d$ only which is recursive in $f$ for a fixed $d$. Moreno-Socías [28] found a simpler bound $q'_f$ which is primitive recursive in $f$ for all $d$ but there is no bound which is primitive recursive in $d$ in

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4. The Specht property

One of the most important numerical invariants of a given variety \( \mathcal{V} \) of algebras over a field of characteristic 0 is its cocharacter sequence

\[
\chi_n(\mathcal{V}) = \sum_{\lambda \vdash n} m(\lambda) \chi_\lambda, \quad n = 1, 2, \ldots.
\]

Here \( \chi_n(\mathcal{V}) \) is the \( S_n \)-character of the vector space \( P_n(\mathcal{V}) \) of the multilinear elements of degree \( n \) in the free algebra \( F_n(\mathcal{V}) \) of the variety \( \mathcal{V} \) under the natural left action of the symmetric group \( S_n \). We have denoted by \( \chi_\lambda \) the irreducible \( S_n \)-character indexed with the partition \( \lambda \) of \( n \) and \( m(\lambda) \in \mathbb{N}_0 \) is the multiplicity of \( \chi_\lambda \) in \( \chi_n(\mathcal{V}) \).

For the variety \( \mathcal{B} \) of bicommutative algebras it was shown in \cite{11} that

\[
\chi_n(\mathcal{B}) = \sum_{(\lambda_1, \lambda_2)} m(\lambda_1, \lambda_2) \chi_{(\lambda_1, \lambda_2)},
\]

where \( \lambda = (\lambda_1, \lambda_2) \) is a partition in two parts, with explicitly given values of \( m(\lambda_1, \lambda_2) \). This description, together with Theorem 4.2 easily implies the positive solution to the Specht problem in characteristic 0.

**Theorem 4.1.** Let \( \mathcal{V} \) be any variety of bicommutative algebras over a field \( K \) of characteristic 0. Then \( \mathcal{V} \) can be defined by a finite system of polynomial identities.

**Proof.** Let the base field \( K \) be of characteristic 0 and let \( \mathcal{V} \) be a variety of \( K \)-algebras. It is well known (see, e.g., \cite{9} Chapter 12) that if the nonzero multiplicities \( m(\lambda) \) in the cocharacter sequence \( \chi_n(\mathcal{V}) \) are only for partitions \( \lambda = (\lambda_1, \ldots, \lambda_d) \) in not more than \( d \) parts, then every subvariety \( \mathcal{W} \) of \( \mathcal{V} \) can be defined by polynomial identities \( f(x_1, \ldots, x_d) = 0 \), where \( f(x_1, \ldots, x_d) \in F_d(\mathcal{W}) \). In our case, if \( \mathcal{V} \) is a subvariety of \( \mathcal{B} \), it can be defined by its identities from \( F_2(\mathcal{B}) \). Hence the T-ideal \( T(\mathcal{V}) \subseteq F(\mathcal{B}) \) of the polynomial identities of \( \mathcal{V} \) is generated as a T-ideal by its elements in \( T(\mathcal{V}) \cap F_2(\mathcal{B}) \). The variety of all bicommutative algebras is defined by two identities. Hence to show that the variety \( \mathcal{V} \) has a finite basis of polynomial identities in the absolutely free algebra \( K\{X\} \) it is sufficient to show that \( T(\mathcal{V}) \cap F_2(\mathcal{B}) \) is finitely generated as a T-ideal in \( F_2(\mathcal{B}) \). Now Theorem 5.2 gives the much stronger result that \( T(\mathcal{V}) \cap F_2(\mathcal{B}) \) is finitely generated as an ordinary two-sided ideal. \( \square \)

**Remark 4.2.** As in Remark 5.3 we can ask how many polynomial identities we need to define a subvariety \( \mathcal{W} \) of \( \mathcal{B} \) in the case of characteristic 0. In the recent paper \cite{10} one of the authors has shown that if \( \mathcal{W} \) satisfies a polynomial identity of degree \( k \), then the number of the irreducible \( S_n \)-components in the \( S_n \)-module \( P_n(\mathcal{V}) \) of the multilinear elements in \( F(\mathcal{W}) \) is bounded by \( 3k^2 \). One can derive from here that a similar bound holds for the number of the generators of the T-ideal \( T(\mathcal{W}) \) in \( F(\mathcal{B}) \) if \( \mathcal{W} \) satisfies an identity of degree \( k \) and this bound does not depend on the degree of the other identities satisfied by \( \mathcal{W} \).
For the solution to the Specht problem over a field \( K \) of arbitrary characteristic we shall apply the Higman-Cohen method based on the technique of partially ordered sets.

**Definition 4.3.** The partially ordered set \((T, \preceq)\) is called *partially well-ordered* if for every subset \( S \) of \( T \) there is a finite subset \( S_0 \) of \( S \) with the property that for each \( s \in S \) there is an element \( s_0 \in S_0 \) such that \( s_0 \preceq s \).

Let \( \Phi \) be the set of all order preserving maps \( \varphi : \mathbb{N} \to \mathbb{N} \), i.e., if \( i < j \), \( i, j \in \mathbb{N} \), then \( \varphi(i) < \varphi(j) \). Let \((T, \preceq)\) be a partially ordered set and let \( T \) be the set of all finite sequences of elements in \( T \). Define the following partial order on \( T \). If \((a_1, \ldots, a_m)\) and \((b_1, \ldots, b_n)\) are two sequences, then

\[
(a_1, \ldots, a_m) \preceq (b_1, \ldots, b_n)
\]

if and only if there exists \( \varphi \in \Phi \) such that

\[
a_i \preceq b_{\varphi(i)} \quad \text{for all } i = 1, \ldots, m.
\]

One of the key ingredients of the Higman-Cohen method is the following result.

**Proposition 4.4.** [17, Theorem 4.3] If \((T, \preceq)\) is a partially well-ordered set, then the set \((T, \preceq)\) is also partially well-ordered.

For our purposes we shall need a restatement of a partial case of [5, Lemma 1]. We shall include the proof for completeness of the exposition.

**Proposition 4.5.** [5, Lemma 1] Let \((\{Y, Z\}, \preceq)\) be the set of all commutative and associative monomials in the variables \( Y = \{y_1, y_2, \ldots\} \) and \( Z = \{z_1, z_2, \ldots\} \) with the following partial order:

\[
Y^\alpha Z^\beta = y_1^{\alpha_1} \cdots y_m^{\alpha_m} z_1^{\beta_1} \cdots z_n^{\beta_n} \preceq y_1^{\gamma_1} \cdots y_p^{\gamma_p} z_1^{\delta_1} \cdots z_q^{\delta_q} = Y^\gamma Z^\delta
\]

if and only if there exists \( \varphi \in \Phi \) such that the monomial

\[
\varphi(Y^\alpha Z^\beta) = y_{\varphi(1)}^{\alpha_1} \cdots y_{\varphi(m)}^{\alpha_m} z_{\varphi(1)}^{\beta_1} \cdots z_{\varphi(n)}^{\beta_n}
\]

divides the monomial \( Y^\gamma Z^\delta \). Then the set \((\{Y, Z\}, \preceq)\) is partially well-ordered.

**Proof.** If \( m < n \) in the monomial \( Y^\alpha Z^\beta = y_1^{\alpha_1} \cdots y_m^{\alpha_m} z_1^{\beta_1} \cdots z_n^{\beta_n} \) we may multiply it by \( y_{m+1}^{\beta_{m+1}} \cdots y_n^{\beta_n} \) and similarly if \( n < m \). Hence we may assume that \( m = n \) in \( Y^\alpha Z^\beta \) and identify it with the sequence

\[
Y^\alpha Z^\beta = (y_1^{\alpha_1} z_1^{\beta_1}, \ldots, y_m^{\alpha_m} z_m^{\beta_m}).
\]

We partially order the set of monomials \([y, z] = \{y^a z^b \mid a, b \geq 0\}\) by divisibility:

\[
y^a z^b \preceq y^c z^d \quad \text{if and only if } y^a z^b \text{ divides } y^c z^d.
\]

The sets \(([y, z], \preceq)\) and \((\{Y, Z\}, \preceq)\) are isomorphic as partially ordered sets: If \( \varphi \in \Phi \) is an order preserving map then \( y_1^{\alpha_1} z_1^{\beta_1} \) divides \( y_{\varphi(i)}^{\alpha_{\varphi(i)}} z_{\varphi(i)}^{\beta_{\varphi(i)}} \) for all \( i = 1, \ldots, m \) if and only if the monomial \( \varphi(Y^\alpha Z^\beta) = y_{\varphi(1)}^{\alpha_1} \cdots y_{\varphi(m)}^{\alpha_m} z_{\varphi(1)}^{\beta_1} \cdots z_{\varphi(n)}^{\beta_n} \) divides the monomial \( Y^\gamma Z^\delta \). Clearly, the set \(([y, z], \preceq)\) is partially well-ordered. Applying Proposition 4.5 we derive that the set \(([y, z], \preceq)\) is also partially well-ordered. Hence the same holds for the set \((\{Y, Z\}, \preceq)\). \[\Box\]
In the sequel we shall consider the set $[Y, Z]$ equipped with the above partial order $\leq$. Now we shall equip $[Y, Z]$ with one more linear order $\leq$ which is a version of the reflected lexicographic order:

$$Y^\alpha Z^\beta = y_1^{\alpha_1} \cdots y_m^{\alpha_m} z_1^{\beta_1} \cdots z_m^{\beta_m} < y_1^{\gamma_1} \cdots y_m^{\gamma_m} z_1^{\delta_1} \cdots z_m^{\delta_m} = Y^\gamma Z^\delta$$

if and only if

- $\alpha_i < \gamma_i$ for some $i$ and $\alpha_j = \gamma_j$ for $j = i + 1, \ldots, m$,
- or $Y^\alpha = Y^\gamma$, $\beta_i < \delta_i$ for some $i$ and $\beta_j = \delta_j$ for $j = i + 1, \ldots, m$.

Obviously the set $([Y, Z], \leq)$ is well-ordered. If

$$f(Y, Z) = \sum_{i=1}^k \vartheta_{(i)} \alpha_{(i)} Y^{\alpha_{(i)}} Z^{\beta_{(i)}}, \quad 0 \neq \vartheta_{(i)} \alpha_{(i)} \in K,$$

and $Y^{\alpha_{(1)}} Z^{\beta_{(1)}} > \cdots > Y^{\alpha_{(k)}} Z^{\beta_{(k)}}$, then we call the monomial

$$\text{wt}(f(Y, Z)) = Y^{\alpha_{(1)}} Z^{\beta_{(1)}}$$

the weight of $f(Y, Z)$. It is easy to see, that if $f(Y, Z)$ belongs to the square $G^2$ of the algebra $G$, then

$$\text{wt}(x_i \circ f(Y, Z)) = y_i \text{wt}(f(Y, Z)), \quad \text{wt}(f(Y, Z) \circ x_i) = \text{wt}(f(Y, Z)) x_i,$$

(7) $\text{wt}(\varphi(f(Y, Z))) = \text{wt}(f(\varphi(1), \ldots, \varphi(m), \varphi(1), \ldots, \varphi(n))) = \varphi(\text{wt}(f(Y, Z)))$

for all $\varphi \in \Phi$.

The following lemma is the key step in the proof of the Specht property for varieties of bicommutative algebras over an arbitrary field $K$.

**Lemma 4.6.** Let $f$ and $g$ be two polynomials in the square $F^2$ of the free bicommutative algebra $F(\mathfrak{B})$ and let $f_1(Y, Z)$, $g_1(Y, Z)$ be their images in the algebra $G$. If $\text{wt}(f_1) \leq \text{wt}(g_1)$, then there is an element $h$ in the $T$-ideal of $F$ generated by $f$ with image $h_1$ in $G$ such that $\text{wt}(h_1) = \text{wt}(g_1)$.

**Proof.** Let

$$\text{wt}(f_1) = Y^\alpha Z^\beta, \quad \text{wt}(g_1) = Y^\gamma Z^\delta,$$

and let $\varphi \in \Phi$ be such that

$$\varphi(\text{wt}(f_1)) = \varphi(Y^\alpha Z^\beta) = y_1^{\alpha_{(1)}} \cdots y_m^{\alpha_{(m)}} z_1^{\beta_{(1)}} \cdots z_m^{\beta_{(n)}}$$

divides the monomial $Y^\gamma Z^\delta$. Hence there exists a monomial $Y^\xi Z^\eta \in [Y, Z]$ such that

$$Y^\xi Z^\eta \varphi(\text{wt}(f_1)) = \text{wt}(g_1).$$

Since $\varphi$ defines an endomorphism of the algebra $F$ and $T$-ideals are closed under endomorphisms, we obtain that the polynomial $\varphi(f)$ belongs to the $T$-ideal $(f)^T \subset F$ generated by $f$. Its image in $G^2$ is $\varphi(f_1)$. By \[ \text{wt}(\varphi(f_1)) = \varphi(\text{wt}(f_1)) \] the weight of $\text{wt}(g_1)$ can be obtained from $\varphi(\text{wt}(f_1))$ by consequent right- and left-multiplications by elements of $X$. The same multiplications produce a polynomial $h \in (f)^T$. By \[ Y^\xi Z^\eta \varphi(\text{wt}(f_1)) = \text{wt}(h_1) = \text{wt}(g_1). \]

The following theorem is the second main result of the paper.

**Theorem 4.7.** Any variety $\mathfrak{V}$ of bicommutative algebras over a field $K$ of arbitrary characteristic has a finite basis of its polynomial identities.


**Proof.** Again, since the variety of all bicommutative algebras is defined by two identities, it is sufficient to establish the finitely generation of the $T$-ideals in $F(\mathfrak{B})$. The algebra $F(\mathfrak{B})/F^2(\mathfrak{B})$ is the free algebra of the variety $\mathfrak{N}_2$ of all algebras with trivial multiplication defined by the polynomial identity $x_1x_2 = 0$. Clearly, $\mathfrak{N}_2$ has no proper subvarieties. Hence, for the proof of the theorem it is sufficient to consider varieties $\mathfrak{B}$ with $T$-ideals $I = T(\mathfrak{B})$ in $F^2(\mathfrak{B})$. Let $I_1$ be the image of $I$ in the square $G^2$ of the algebra $G$. The set $([Y, Z], \preceq)$ is partially well-ordered. Hence there is a finite set of polynomials $\{f^{(1)}, \ldots, f^{(k)}\}$ in $I$ with images $\{f^{(1)}_1, \ldots, f^{(k)}_1\}$ in $I_1$ with the property that for any $g \in I$ with image $g_1 \in I_1$ there exists an $f^{(i)}$ such that $\text{wt}(f^{(i)}_1) \preceq \text{wt}(g_1)$. Let us assume that the polynomials $f^{(1)}, \ldots, f^{(k)}$ do not generate the $T$-ideal $I$. Since the set $([Y, Z], \preceq)$ is well-ordered, there is a polynomial $g \in I \setminus \{f^{(1)}, \ldots, f^{(k)}\}^T$ such that the weight $\text{wt}(g_1)$ of its image in $I_1$ is minimal in the set $\text{wt}(I \setminus \{f^{(1)}, \ldots, f^{(k)}\})_1$. If $f^{(i)}$ is such that $\text{wt}(f^{(i)}_1) \preceq \text{wt}(g_1)$, then by Lemma 4.6 there exists an $h \in (f^{(i)})_1 \subseteq I$ such that its image $h_1 \in I_1$ satisfies $\text{wt}(h_1) = \text{wt}(g_1)$. If the coefficients of $\text{wt}(h_1)$ and $\text{wt}(g_1)$ in $h_1$ and $g_1$ are, respectively, $\mu$ and $\nu$, then the polynomial $u = g - \frac{\mu}{\nu}h$ belongs to $I$. If $u \neq 0$, then $u \in I \setminus (f^{(1)}, \ldots, f^{(k)})^T$. But this is impossible because $\text{wt}(u_1) < \text{wt}(g_1)$ for its image $u_1 \in G^2$ which contradicts to the minimality of $\text{wt}(g_1)$. □

**Remark 4.8.** Again, as in Remarks 3.3 and 4.2 we can ask about the number of generators of the $T$-ideals of $F(\mathfrak{B})$ when the base field is of positive characteristic. Since we work in the polynomial algebra $K[Y, Z]$ considered as a $K[X]$-bimodule we shall mention several results concerning the number of generators, theory of Gröbner bases and other algorithmic problems: Aschenbrenner, Hillar [1], Hillar, Windfeldt [20], Hillar, Sullivant [19], Krone [24], and Hillar, Kron, Leykin [13].

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