A matricial view of the Karpelevič Theorem

Charles R. Johnson, Pietro Paparella

Department of Mathematics, College of William & Mary, Williamsburg, VA 23187-8795, USA
Division of Engineering and Mathematics, University of Washington Bothell, Bothell, WA 98011-8246, USA

Abstract

The question of the exact region in the complex plane of the possible single eigenvalues of all \(n\)-by-\(n\) stochastic matrices was raised by Kolmogorov in 1937 and settled by Karpelevič in 1951 after a partial result by Dmitriev and Dynkin in 1946. The Karpelevič result is unwieldy, but a simplification was given by Đoković in 1990 and Ito in 1997. The Karpelevič region is determined by a set of boundary arcs each connecting consecutive roots of unity of order less than \(n\). It is shown here that each of these arcs is realized by a single, somewhat simple, parameterized stochastic matrix. Other observations are made about the nature of the arcs and several further questions are raised. The doubly stochastic analog of the Karpelevič region remains open, but a conjecture about it is amplified.

Keywords: Stochastic matrix, Doubly stochastic matrix, Karpelevič arc, Karpelevič region, Ito polynomial, Realizing matrix

2010 MSC: 15A18, 15A29, 15B51

1. Introduction

In [11], Kolmogorov posed the problem of characterizing the subset of the complex plane, denoted by \(\Theta_n\), that consists of the individual eigenvalues of all \(n\)-by-\(n\) stochastic matrices.

One can easily verify that for each \(n \geq 2\), the region \(\Theta_n\) is closed, inscribed in the unit-disc, star-convex (with star-centers at zero and one), and symmetric with respect to the real-axis. Furthermore, it is clear that \(\Theta_n \subseteq \Theta_{n+1}, \forall n \in \mathbb{N}\). In view of these properties, \(\partial \Theta_n = \{\lambda \in \Theta_n : \alpha \lambda \notin \Theta_n, \forall \alpha > 1\}\), and each region is determined by its boundary.

Dmitriev and Dynkin [2] obtained a partial solution to Kolmogorov’s problem, and Karpelevič [10, Theorem B], expanding on the work of [2], resolved
it by showing that the boundary of $\Theta_n$ consists of curvilinear arcs (herein, Karpelevič arcs or K-arcs), whose points satisfy a polynomial equation that is determined by the endpoints of the arc (which are consecutive roots of unity). Đoković [13, Theorem 4.5] and Ito [7, Theorem 2] each provide a simplification of this result. However, notably absent in the Karpelevič Theorem (and the above-mentioned works) are realizing-matrices (i.e., a matrix whose spectrum contains a given point) for points on these arcs.

This problem has been addressed previously in the literature. Dmitriev and Dynkin [2, Basic Theorem] give a schematic description of such matrices for points on the boundary of $\Theta_n \setminus \Theta_{n-1}$ and Swift [16, §2.2.2] provides such matrices for $3 \leq n \leq 5$.

Our main result is providing, for every $n$ and for each arc, a single parametric matrix that realizes the entire $K$-arc as the parameter runs from 0 to 1. Aside from the theoretical importance – after all, the original problem posed by Kolmogorov is intrinsically matricial – possession of such matrices is instrumental in the study of nonreal Perron similarities in the longstanding nonnegative inverse eigenvalue problem [9], and provides a framework for resolving Conjecture 1 [12] vis-à-vis the results in [8].

In addition, we provide some partial results on the differentiability of the Karpelevič arcs. We demonstrate that some powers of certain realizing-matrices realize other arcs. Finally, we pose several problems that appeal to a wide variety of mathematical interests.

2. Notation & Background

The algebra of complex (real) $n$-by-$n$ matrices is denoted by $M_n(\mathbb{C})$ ($M_n(\mathbb{R})$). A real matrix is called nonnegative (positive) if it is an entrywise nonnegative (positive) matrix. If $A$ is nonnegative (positive), then we write $A \geq 0$ ($A > 0$).

An $n$-by-$n$ nonnegative matrix $A$ is called (row) stochastic if every row sums to unity; column stochastic if every column sums to unity; and doubly stochastic if it is row stochastic and column stochastic.

Given $n \in \mathbb{N}$, the set $F_n := \{p/q : 0 \leq p < q \leq n, \gcd(p, q) = 1\}$ is called the set of Farey fractions of order $n$. If $p/q, r/s$ are elements of $F_n$ such that $p/q < r/s$, then $(p/q, r/s)$ is called a Farey pair (of order $n$) if $x \notin F_n$ whenever $p/q < x < r/s$. The Farey fractions $p/q$ and $r/s$ are called Farey neighbors if $(p/q, r/s)$ or $(r/s, p/q)$ is a Farey pair.

The following is the celebrated Karpelevič Theorem in a form due to Ito [7].

**Theorem 2.1** (Karpelevič). The region $\Theta_n$ is symmetric with respect to the real axis, is included in the unit-disc $\{z \in \mathbb{C} : |z| \leq 1\}$, and intersects the unit-circle $\{z \in \mathbb{C} : |z| = 1\}$ at the points $\{e^{2\pi i p/q} : p/q \in F_n\}$. The boundary of $\Theta_n$ consists of these points and of curvilinear arcs connecting them in circular order.

Let the endpoints of an arc be $e^{2\pi i p/q}$ and $e^{2\pi i r/s}$ ($q < s$). Each of these arcs is given by the following parametric equation:

$$t^* (t^* - \beta)^{\lceil n/q \rceil} = \alpha^{\lfloor n/q \rfloor} t^{\lfloor n/q \rfloor}, \quad \alpha \in [0, 1], \quad \beta := 1 - \alpha.$$  

(2.1)
Figure 1 contains the regions $\Theta_3$, $\Theta_4$, and $\Theta_5$.

For $n \in \mathbb{N}$, we call the collection of such arcs the $K$-arcs (of order $n$) and we denote by $K(p/q, r/s) = K_n(p/q, r/s)$ the arc connecting $e^{i2\pi p/q}$ and $e^{i2\pi r/s}$, when $p/q$ and $r/s$ are Farey neighbors. Notice that the number of $K$-arcs equals $|F_n| = 1 + \sum_{k=1}^{n} \phi(k)$, where $\phi$ denotes Euler’s totient function.

For Farey neighbors $p/q$ and $r/s$, $q < s$, we call the collection of equations (2.1) the Ito equations (with respect to $\{p/q, r/s\}$) and the collection of polynomials

\[ f_\alpha(t) := t^q (t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor} t^{\lfloor n/q \rfloor}, \quad \alpha \in [0, 1] \]

the Ito polynomials (with respect to $\{p/q, r/s\}$).

A directed graph (or simply digraph) $\Gamma = (V, E)$ consists of a finite, nonempty set $V$ of vertices, together with a set $E \subseteq V \times V$ of arcs. For $A \in M_n(\mathbb{C})$, the directed graph (or simply digraph) of $A$, denoted by $\Gamma = \Gamma(A)$, has vertex set $V = \{1, \ldots, n\}$ and arc set $E = \{(i, j) \in V \times V : a_{ij} \neq 0\}$.

A digraph $\Gamma$ is called strongly connected if for any two distinct vertices $i$ and $j$ of $\Gamma$, there is a path in $\Gamma$ from $i$ to $j$. Following [1], we consider every vertex of $V$ as strongly connected to itself. A strong digraph is primitive if the greatest common divisor of all its cycle-lengths is one, otherwise it is imprimitive.

For $n \geq 2$, an $n$-by-$n$ matrix $A$ is called reducible if there exists a permutation matrix $P$ such that

\[ P^\top AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \]

where $A_{11}$ and $A_{22}$ are nonempty square matrices. If $A$ is not reducible, then $A$ is called irreducible. It is well-known that a matrix $A$ is irreducible if and only if $\Gamma(A)$ is strongly connected (see, e.g., [1] Theorem 3.2.1 or [5] Theorem 6.2.24).

An irreducible nonnegative matrix is called primitive if, in its digraph, the set of cycle-lengths is relatively prime; otherwise it is imprimitive.

For $n \in \mathbb{N}$, denote by $C_n$ the basic circulant, i.e.,

\[ C_n = \begin{bmatrix} 0 & I_{n-1} \\ 1 & 0 \end{bmatrix}. \]

Note that the digraph of $C_n$ is a cycle of length $n$. 

3
Given an \( n \times n \) matrix \( A \), the characteristic polynomial of \( A \), denoted by \( \chi_A \), is defined by \( \chi_A = \det (tI - A) \). The companion matrix \( C = C_f \) of a monic polynomial \( f(t) = t^n + \sum_{k=1}^{n} c_k t^{n-k} \) is the \( n \times n \) matrix defined by

\[
C = \begin{bmatrix}
0 & I_{n-1} \\
-c_n & -c
\end{bmatrix},
\]

where \( c = [c_{n-1} \cdots c_1] \). It is well-known that \( \chi_C = f \). Notice that \( C \) is irreducible if and only if \( c_n \neq 0 \).

3. Realizing-matrices

Lemma 3.1. Let \( A \in M_n(\mathbb{C}) \). If \( B = A + \alpha e_k e_k^\top \), then \( \det(B) = \det(A) + (-1)^{k+l} \alpha \det(A_{kl}) \).

Proof. Take either a Laplace-expansion along the \( k \)-th row or the \( k \)-th column of \( B \).

Theorem 3.2. For each \( K \)-arc \( K_n(p/q, r/s) \), there is a parametric, stochastic matrix \( M = M(\alpha), 0 \leq \alpha \leq 1 \), such that each point \( \lambda = \lambda(\alpha) \) of the arc is an eigenvalue of \( M \). Furthermore, if \( \alpha \in (0, 1) \), then \( M \) is primitive.

Proof. Let \( p/q \) and \( r/s \) be Farey neighbors, where \( q < s \). Note that \( s \neq q \lfloor n/q \rfloor \) since \( q \) and \( s \) are relatively prime.

First, we consider the case in which \( p/q = 0 \) and \( r/s = 1/n \) (which we call the Type 0 arc). Then (2.1) reduces to \( (t - \beta)^n - \alpha^n = 0 \). If

\[
M = M(\alpha) := \alpha C_n + \beta I = \begin{bmatrix}
\beta & \alpha & \alpha & \cdots & \alpha \\
\beta & \alpha & \cdots & \cdots & \beta \\
\alpha & \beta & \alpha & \cdots & \cdots \\
& \beta & \alpha & \cdots & \cdots \\
& & \beta & \alpha & \cdots
\end{bmatrix} \in M_R(n),
\]

then

\[
\chi_M(t) = \det (tI - (\alpha C_n + \beta I))
= \det ((t - \beta)I - \alpha C_n)
= \chi_{\alpha C_n}(t - \beta)
= (t - \beta)^n - \alpha^n.
\]

If \( \alpha \in (0, 1) \), then \( \Gamma(M) \) contains directed-cycles of length one and \( n \). Hence, \( M \) is irreducible and since the greatest common divisor of all cycle-lengths of \( \Gamma(M) \) is obviously one, \( M \) is primitive.
Next, we consider the case in which \( \lfloor n/q \rfloor = 1 \) (herein referred to as a Type I arc). Then (2.1) reduces to \( t^s - \beta t^{s-q} - \alpha = 0 \). If

\[
M = M(\alpha) := \begin{bmatrix} z & I \\ \alpha & \beta e_{s-q}^\top \end{bmatrix} \in M_s(\mathbb{R}),
\]

then \( M \geq 0 \) and \( \chi_M(t) = t^s - \beta t^{s-q} - \alpha \). If \( \alpha \in (0,1) \), then \( \Gamma(M) \) contains \( \Gamma(C_n) \). Hence, \( M \) is irreducible, and, since gcd \( (s-(s-q), s) = \gcd(q, s) = 1 \), it must be primitive.

Next, we consider the case in which \( \lfloor n/q \rfloor > 1 \) and \( s < q \lfloor n/q \rfloor \) (which we call a Type II arc). Then (2.1) reduces to

\[
(t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor} t^{q\lfloor n/q \rfloor - s} = 0.
\]

Consider the nonnegative matrix \( M = M(\alpha) := \alpha X + \beta Y \), where \( X \) is the nonnegative companion matrix of the polynomial \( t^q - t^{q\lfloor n/q \rfloor - s} \), and

\[
Y := \bigoplus_{k=1}^{\lfloor n/q \rfloor} C_q = \begin{bmatrix} C_q & \cdots & C_q \end{bmatrix} \in M_{q\lfloor n/q \rfloor}(\mathbb{R}).
\]

Since \( 1 < q \lfloor n/q \rfloor - s + 1 \leq n - s + 1 < q + 1 \), it follows that

\[
M = \begin{bmatrix}
1 & \alpha & \alpha & \cdots & \alpha \\
\beta & 1 & \alpha & \cdots & \alpha \\
\vdots & & \ddots & \ddots & \ddots \\
\alpha e_{q\lfloor n/q \rfloor-s+1}^\top & \cdots & \cdots & 1 & \beta \\
\alpha e_{q\lfloor n/q \rfloor-s+1}^\top & \cdots & \cdots & 1 & \beta \\
\end{bmatrix},
\]

where \( e_{q\lfloor n/q \rfloor-s+1} \in \mathbb{R}^q \). Because \( M - \alpha e_{q\lfloor n/q \rfloor} e_{q\lfloor n/q \rfloor-s+1}^\top \) is block upper-triangular, it follows from Lemma 3.1 that

\[
\chi_M(t) = (t^q - \beta)^{\lfloor n/q \rfloor} + (-1)^{q\lfloor n/q \rfloor-s+1}(\alpha)^{q\lfloor n/q \rfloor-s}(-\alpha)^{n/q-s}(\sigma)^{q\lfloor n/q \rfloor-1}(q^{n/q}-1)^{-1}(1^{q\lfloor n/q \rfloor-s} - (q^{n/q}-s)\sigma^{q\lfloor n/q \rfloor-s})
\]

\[
= (t^q - \beta)^{\lfloor n/q \rfloor} + (-1)^{q\lfloor n/q \rfloor-s+1}\alpha^{q\lfloor n/q \rfloor-s}
\]

\[
= (t^q - \beta)^{\lfloor n/q \rfloor} - \alpha t^{q\lfloor n/q \rfloor-s}.
\]

If \( \alpha \in (0,1) \), then the directed graph contains \( \lfloor n/q \rfloor \) strongly connected components and the graph on these components, determined whether off-diagonal
blocks are nonzero, is also strongly connected; hence, the entire graph is strongly connected, i.e., $M$ is irreducible. Furthermore, since $\Gamma(M)$ contains cycles of length $q$ and $q\lfloor n/q \rfloor -(q\lfloor n/q \rfloor -s+1)+1 = s$, it follows that $M$ is primitive.

Finally, we consider the case when $\lfloor n/q \rfloor > 1$ and $s > q\lfloor n/q \rfloor$ (herein referred to as a Type III arc). For convenience, let $d = s - q\lfloor n/q \rfloor$. Then (2.1) reduces to

$$t^d(t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor} = 0.$$  

Consider the nonnegative matrix $M = M(\alpha) := \alpha C_s + \beta Y$, where

$$Y = \begin{bmatrix} J_d(0) & C_q & \cdots & C_q \\ \vdots & \vdots & \ddots & \vdots \\ C_q & \cdots & C_q \end{bmatrix} + e_d e_d^\top \in M_s(\mathbb{R}).$$

Then

$$M = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 0 & 1 \\ 0 & 1 & \cdots & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 1 & \cdots & \cdots & 0 & 1 \\ \alpha & \cdots & \cdots & \cdots & \cdots & \alpha \\ \beta & \cdots & \cdots & \cdots & \cdots & \beta \\ 1 & \cdots & \cdots & \cdots & \cdots & 1 \end{bmatrix}.$$  

Since $M - \alpha e_s e_1^\top$ is block upper-triangular, following Lemma 3.1,

$$\chi_M(t) = t^d(t^q - \beta)^{\lfloor n/q \rfloor} + (-1)^{s+1}(-\alpha)(-\alpha)^{\lfloor n/q \rfloor-1}(-1)^{s-1-(\lfloor n/q \rfloor-1)}$$

$$= t^d(t^q - \beta)^{\lfloor n/q \rfloor} + (-1)^{2s+1} \alpha^{\lfloor n/q \rfloor}$$

$$= t^d(t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor}.$$  

If $\alpha \in (0,1)$, then $\Gamma(M)$ contains $\Gamma(C_n)$ as a subgraph. Hence, $M$ is irreducible, and since $\Gamma(M)$ clearly contains cycles of length $q$ and $s$, $M$ is primitive. \qed
Remark 3.3. Notice that the realizing matrices for arcs of Type I, II, and II all have trace zero.

Example 3.4. Table 1 contains realizing matrices illustrating each type of arc when $n = 9$ (the smallest order for which each arc-type appears).

\[
K\left(\frac{p}{q} : \frac{r}{s}\right) \quad \text{Type} \quad M(\alpha, \beta := 1 - \alpha)
\]

\[
K\left(\frac{1}{9} : \frac{1}{8}\right) \quad \text{I} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
K\left(\frac{2}{7} : \frac{1}{3}\right) \quad \text{II} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\beta & 0 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \beta & 0 & 0 & \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & \alpha & 0 & 0 & 0 & \beta & 0 & 0 \\
\end{bmatrix}
\]

\[
K\left(\frac{2}{9} : \frac{1}{4}\right) \quad \text{III} \\
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & \beta & 0 & 0 & 0 & \alpha & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\alpha & 0 & 0 & 0 & 0 & \beta & 0 & 0 & 0
\end{bmatrix}
\]

Table 1: Realizing matrices for arcs of Type I, II, and III when $n = 9$.

Let $\mathcal{M} := \{M(\alpha) : \alpha \in [0,1]\}$ be the set of realizing matrices for the arc $K(1/9, 1/8)$. For $d \in \mathbb{N}$, let $\mathcal{M}^d = \{M(\alpha)^d : M(\alpha) \in \mathcal{M}\}$. Theorem 5.4 shows that certain powers of the realizing matrices for the arc realize other arcs: in particular, $\mathcal{M}^2$, $\mathcal{M}^3$, and $\mathcal{M}^4$ form a set of realizing matrices for the arcs $K(2/9, 1/4)$, $K(1/3, 3/8)$, and $K(4/9, 1/2)$, respectively.
Differentiability of the Arcs

We investigate here the smoothness of the K-arcs, a natural question not previously addressed.

To that end, let \( f \) and \( g \) be monic polynomials of degree \( n \). For \( \alpha \in [0, 1] \), let \( c_\alpha := \alpha f + (1 - \alpha)g \). Since the roots of a polynomial vary continuously with respect to its coefficients, it follows that the locus \( L(f, g) := \{ t \in \mathbb{C} : c_\alpha(t) = 0, \, \alpha \in [0, 1] \} \) consists of \( n \) continuous paths (counting multiplicities), each of which connects a root of \( g \) to a root of \( f \), whose points depend continuously on the parameter \( \alpha \) (if \( f \) and \( g \) share a root, then there is a degenerate path at this root).

Denote by \( P(\mu, \lambda) \) the path that starts at the root \( \mu \) of \( g \) and terminates at the root \( \lambda \) of \( f \) (\( \mu \neq \lambda \)). If \( r = r(\alpha) \in P(\mu, \lambda), \alpha \in (0, 1) \), then

\[
0 = \alpha f(r) + (1 - \alpha)g(r).
\]

Differentiating with respect to \( \alpha \) yields

\[
0 = f(r) + \alpha g'(r)r' - g(r) + (1 - \alpha)g'(r)r' = f(r) - g(r) + r'c'_\alpha(r).
\]

If \( c'_\alpha(r) \neq 0 \) (i.e., if \( r \) is not a multiple root of \( c_\alpha \)), then

\[
r' = \frac{g(r) - f(r)}{c'_\alpha(r)}.
\]

Thus, the path \( P(\mu, \lambda) \) is differentiable at \( r \) if \( r \) is not a multiple root of \( c_\alpha \).

**Proposition 4.1.** For \( n \geq 4 \), let

\[
f_\alpha(t) := t^n - \beta t - \alpha, \quad \alpha \in [0, 1], \quad \beta := 1 - \alpha.
\]

(i) If \( n \) is even, then \( f_\alpha \) has \( n \) distinct roots.

(ii) If \( n \) is odd and \( \alpha \geq \beta \), then \( f_\alpha \) has \( n \) distinct roots.

(iii) If \( n \) is odd and \( \alpha < \beta \), then \( f_\alpha \) has a multiple root if and only if

\[
n^n\alpha^{n-1} - (n - 1)^n\beta \lambda - n^n\alpha^{n-1} + (n - 1)^n(\alpha - 1)^n = 0.
\]

**Proof.** Notice that \( f_\alpha(1) = 0 \), and, since \( C_f \) is primitive, if \( f_\alpha(\lambda) = 0, \lambda \neq 1 \), then

\[
|\lambda| < 1.
\]

It is well-known that a polynomial has a multiple root if and only if it shares a root with its formal derivative. Thus, \( f_\alpha \) has a multiple root \( \lambda \in \mathbb{C} \) if and only if \( f_\alpha(\lambda) = f'_\alpha(\lambda) = 0 \), i.e., if and only if

\[
\lambda^n - \beta \lambda - \alpha = 0 \quad \text{and} \quad n\lambda^{n-1} - \beta = 0.
\]
Solving for $\beta$ in (4.4) and substituting the result in (4.3) yields
\[ \lambda^n = -\frac{\alpha}{n-1}. \] (4.5)

Substituting for $\lambda_n$ in (4.3) yields
\[ \lambda = -\frac{\alpha n}{\beta(n-1)} < 0. \] (4.6)

We now consider each part separately:

(i) For contradiction, if $f_\alpha$ has a multiple root, then it must be negative (4.6); however, because $f_\alpha(-t) = t^n + \beta t - \alpha$, Descartes’ Rule of Signs ensures that $p$ has at least one negative root, a contradiction.

(ii) Suppose that $n$ is odd and $\alpha \geq \beta$. For contradiction, if $f_\alpha$ has a multiple root, then, following (4.6),
\[ |\lambda| = \frac{\alpha n}{\beta(n-1)} \geq \frac{n}{n-1} > 1, \]
contradicting (4.2).

(iii) It is well-known that a polynomial $f$ has a multiple root if and only if its resultant $R(f, f')$ vanishes. If
\[
S(f_\alpha, f'_\alpha) = \begin{bmatrix}
1 & \cdots & 1 & 1 & n & n+1 & 2n-1 \\
& & \ddots & \ddots & \ddots & \ddots & \\
& & & n \cdot & n-1 & -\beta & -\alpha \\
& & & & n \cdot & \ddots & -\beta \\
& & & & & \ddots & \ddots \\
& & & & & & 0 \cdot \\
& & & & & & 2n \cdot \\
& & & & & & 2n-1 \cdot
\end{bmatrix},
\]
then $R(f_\alpha, f'_\alpha) = |S(f_\alpha, f'_\alpha)| = |D - CB|$, where $B$, $C$, and $D$ denote the upper-right, lower-left, and lower-right blocks of $S(f_\alpha, f'_\alpha)$. Since
\[
D - CB = \begin{bmatrix}
(n-1)\beta & n\alpha \\
& \ddots & \ddots & \ddots \\
& & (n-1)\beta & n\alpha \\
& & & n \cdot \\
& & & & -\beta
\end{bmatrix},
\]
and $n$ is odd, it follows that $R(f_\alpha, f'_\alpha) = n^n\alpha^{n-1} - (n-1)^{n-1}\beta^n$ and the result is established.

Remark 4.2. If $n$ is odd and $f_\alpha$ has a multiple root $\lambda$ (which, following (4.6), must be negative), then Descartes’ Rule of Signs applied to $f_\alpha(-t) = -t^n + \beta t - \alpha$ forces the multiplicity of $\lambda$ as a root of $f_\alpha$ to be exactly two.
Remark 4.3. Under the hypotheses of part (iii) of Proposition 4.1, the resultant
\( R(f_\alpha, f'_\alpha) = \pi(\alpha) = n^n\alpha^{n-1} + (n-1)^{n-1}(\alpha - 1)^n \) for the polynomial \( f_\alpha \) defined in (4.1), is a univariate polynomial in \( \alpha \). Since \( \pi(0) = -(n-1)^{n-1} < 0 \) and \( \pi(1) = n^n > 0 \), it follows that \( \pi \) must have a root in \((0, 1)\). However, \( \pi'(\alpha) = n^n\alpha^{n-2} + (n-1)^{n-1}(\alpha - 1)^{n-1} \) and because \( n \) is odd, we have \( \pi(\alpha) \geq 0 \) for all \( \alpha \geq 0 \). Thus, \( \pi \) is strictly increasing on \((0, \infty)\) and hence has exactly one root in \((0, 1)\).

Corollary 4.4. Let \( n \geq 4 \) be a positive integer.

(i) If \( n \) is even and \( \lfloor n/2 \rfloor \leq m \leq n \), then the K-arc \( K_n(1/m, 1/m - 1) \) is
differentiable.

(ii) If \( n \) is odd and \( \lfloor n/2 \rfloor + 1 \leq m \leq n \), then the K-arc \( K_n(1/m, 1/m - 1) \) is
differentiable.

Proof. In view of Proposition 4.1, it suffices to consider the case when \( n \) is odd
and \( \alpha < \beta \), where \( f_\alpha \) is defined as in (4.1); however, this case is clear as well since
Remark 4.2 ensures that if \( f_\alpha \) has a multiple multiple root, then \( \lambda \) is real.

5. Powers of Realizing-matrices

For each of the arc types listed in the proof of Theorem 3.2, we refer to the
collection of polynomials

\[
\begin{align*}
  f_\alpha(t) &= (t - \beta)^n - \alpha^n & \text{(Type 0)} \\
  f_\alpha(t) &= t^s - \beta t^{s-q} - \alpha & \text{(Type I)} \\
  f_\alpha(t) &= (t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor} t^{\lfloor n/q \rfloor - s} & \text{(Type II)} \\
  f_\alpha(t) &= t^{s-q} (t^q - \beta)^{\lfloor n/q \rfloor} - \alpha^{\lfloor n/q \rfloor} & \text{(Type III)}
\end{align*}
\]

as the reduced Ito polynomials.

The following result is readily deduced from several well-known theorems
concerning Farey pairs (see, e.g., [3, pp. 28–29]).

Lemma 5.1. If \( p/q, r/s \) are elements of \( F_n \), then \( (p/q, r/s) \) is a Farey pair of
order \( n \) if and only if \( qr - ps = 1 \) and \( q + s > n \).

Lemma 5.2. If \( d \) is a positive integer such that \( 1 < d < n \), then \( (d/n, d/n - 1) \)
is a Farey pair of order \( n \) if and only if \( d \) divides \( n \) or \( d \) divides \( n - 1 \).

Proof. If there is a positive integer \( k \) such that \( n = dk \), then \( (d/n, d/n - 1) =
(1/k, d/n - 1) \). Since \( dk - (n - 1) = 1 \), it follows that \( d/n - 1 \notin F_n \). Because
\( k > 1 \), it follows that \( k + n - 1 > n \). Following Lemma 5.1, \( (1/k, d/n - 1) \) is a
Farey pair. A similar argument demonstrates that \( (d/n, 1/k) \) is a Farey pair if \( d \) divides \( n - 1 \).

Conversely, if \( d \) does not a divisor of either \( n \) or \( n - 1 \), then \( dn - d(n - 1) =
d \neq 1 \). The result now follows from Lemma 5.1.
Corollary 5.3. Let \( d, m, \) and \( n \) be positive integers such that \( d < m \leq n \), and suppose that \( (1/m, 1/m - 1) \) is a Farey pair of order \( n \).

(i) If \( d \) divides \( m \) and \( k := m/d \), then \( (1/k, d/m - 1) \) is a Farey pair of order \( n \) if and only if \( k + m - 1 > n \).

(ii) If \( d \) divides \( m - 1 \) and \( k := (m - 1)/d \), then \( (d/m, 1/k) \) is a Farey pair of order \( n \) if and only if \( m + k > n \).

Theorem 5.4. Let \( d, m, \) and \( n \) be positive integers such that \( 1 < d < m \leq n \). Suppose that \( (1/m, 1/m - 1) \) and \( (d/m, d/m - 1) \) are Farey pairs of order \( n \).

We distinguish the following cases:

(i) \( d \) divides \( m \): For \( f_\alpha(t) = t^m - \beta t - \alpha \), let \( M(\alpha) \) be defined as in \((3.1)\). If \( \mathcal{M} := \{M(\alpha) : \alpha \in [0,1]\} \), then \( M^d := \{M(\alpha)^d : \alpha \in [0,1]\} \) forms a set of realizing-matrices for \( K_n(1/k, d/m - 1) \), where \( k = m/d \).

(ii) \( d \) divides \( m - 1 \) and \( n > k \lfloor n/k \rfloor \), where \( k = m/d \): For \( f_\alpha(t) = t^m - \beta t - \alpha \), let \( M(\alpha) \) be defined as in \((3.1)\). If \( \mathcal{M} := \{M(\alpha) : \alpha \in [0,1]\} \), then \( M^d := \{M(\alpha)^d : \alpha \in [0,1]\} \) forms a set of realizing-matrices for \( K_n(d/m, 1/k) \), where \( k = m/d \).

Proof. Part (i): Since \( (1/k, d/m - 1) \) is a Farey pair, following Corollary 5.3, \( n < m + k - 1 \); consequently,
\[
d = \frac{m}{k} \leq \frac{n}{k} = \frac{m + k - 1}{k} = d + 1 - \frac{1}{k} < d + 1
\]
and hence \( \lfloor n/k \rfloor = d \). The Ito equations for \( (1/k, d/m - 1) \) are given by
\[
t^{m-1} (t^k - \beta)^{\lfloor n/k \rfloor} = \alpha^{\lfloor n/k \rfloor} t^{\lfloor n/k \rfloor}, \alpha \in [0,1], \beta := 1 - \alpha,
\]
and the reduced Ito polynomials for this arc are given by
\[
q_\alpha(t) = (t^k - \beta)^d - \alpha^d t, \alpha \in [0,1], \beta := 1 - \alpha.
\] (5.1)
Notice that \( \deg(q_\alpha) = m \), for every \( \alpha \in [0,1] \).

Let \( \lambda = \lambda(\alpha) \in K(1/k, d/m - 1) \). Consider the reduced Ito polynomial \( p_\beta(t) = t^m - \alpha t - \beta \) and its nonnegative companion matrix \( M = M(\beta) \). The Cayley-Hamilton theorem (see, e.g., [5] p. 109) ensures that \( M^m - \beta I = \alpha M \); hence
\[
q_\alpha(M^d) = (M^{dk} - \beta I)^d - \alpha^d M^d = (M^m - \beta I)^d - (\alpha M)^d = 0,
\]
i.e., \( q_\alpha \) is an annihilating polynomial for \( M^d \).

Denote by \( \psi_M \) the minimal polynomial of \( M \), i.e., \( \psi_M \) is the unique monic polynomial of minimum degree that annihilates \( M \) (see, e.g., [5] p. 192). Since \( M \) is a companion matrix, \( \psi_M = \chi_M \) (5 Theorem 3.3.14]). Hence, if \( J = S^{-1} MS \) is a Jordan canonical form of \( M \), then \( J \) is nonderogatory (5 Theorem 3.3.15]), i.e., \( J \) contains exactly one Jordan block corresponding to every distinct
eigenvalue. Since $M^d = SJ^dS^{-1}$, it follows that any Jordan canonical form of $J^d$ is nonderogatory – indeed, if $f(x) = x^d$, then $f'(x) = dx^{d-1}$ and $f'(x) = 0$ if and only if $x = 0$; since zero is not a repeated root (and hence not associated with a nontrivial Jordan block), the claim follows from [4, p. 424, Theorem 6.2.25] – thus, $M^d$ is nonderogatory and, following [5, Theorem 3.3.15], $\psi_{M^d} = \chi_{M^d}$ and $\deg(\psi_{M^d}) = m$. Since $\psi_{M^d}$ is the unique polynomial of minimum degree that annihilates $M$, and since $\deg(q_\alpha) = m$, it must be the case that $\chi_{M^d} = \psi_{M^d} = q_\alpha$. Hence, $M^d$ is a realizing-matrix for $\lambda$.

Part (ii): By hypothesis, $d = \frac{m-1}{k} < \frac{m}{k} \leq \frac{n}{k}$, hence $d \leq \lfloor \frac{n}{k} \rfloor$. Since $m > k\lfloor n/k \rfloor$, it follows that $m - k\lfloor n/k \rfloor \geq 1$ and $\lfloor n/k \rfloor \leq (m-1)/k = d$. Hence, $d = \lfloor n/k \rfloor$.

The Ito equations for $(d/m, 1/k)$ are given by

$$t^m (t^k - \beta)^d = \alpha^d t^{m-1}, \quad \alpha \in [0, 1], \quad \beta := 1 - \alpha,$$

and the reduced Ito polynomials for this arc are given by

$$q_\alpha(t) = t(t^k - \beta)^d - \alpha^d, \quad \alpha \in [0, 1], \quad \beta := 1 - \alpha.$$

Notice that $\deg(q_\alpha) = m$, for every $\alpha \in [0, 1]$.

Let $\lambda = \lambda(\alpha) \in K(d/m, 1/k)$. Consider the reduced Ito polynomial $f_\alpha(t) = t^m - \beta t - \alpha$ and its nonnegative companion matrix $M = M(\alpha)$. The Cayley-Hamilton theorem ensures that $M(M^{m-1} - \beta I) = M^m - \beta M = \alpha I$; hence

$$q_\alpha(M^d) = M^d(M^{m-1} - \beta I)^d - \alpha^d I = (M^m - \beta M)^d - (\alpha I)^d = 0,$$

i.e., $q_\alpha$ is an annihilating polynomial for $M^d$.

Using exactly the same argument as in part (i), it can be shown that $\chi_{M^d} = \psi_{M^d} = q_\alpha$. Hence, $M^d$ is a realizing-matrix for $\lambda$.

6. Additional Questions

In this section, we pose several problems and conjectures for further inquiry.

6.1. Karpelevič Arcs

Theorem [3,2] establishes the existence of parametric realizing-matrices for the K-arcs. Suppose that $M$ is a realizing-matrix for a given point on a given arc, and let $M_k$ be the irreducible component that realizes the arc. Clearly, $M_k^T$ and $P M_k P^T$ are also realizing-matrices. With the aforementioned in mind, we offer the following.

Problem 6.1. To what extent are the realizing-matrices unique?
Corollary 4.4 and Theorem 5.4 show that many, but not all arcs are differentiable. Given the empirical evidence, we pose the following.

**Conjecture 6.2.** All K-arcs of order n are differentiable for every n.

For $S \subseteq \mathbb{C}$, let $S^d := \{\lambda^d : \lambda \in S\}$. Theorem 5.4 demonstrates that $\sigma(M)^d = \sigma(M^d)$. Although the evidence is ample, a demonstration that the powered K-arc $K_n^d(1/m, 1/m - 1)$ corresponds to $K_n(1/k, d/m - 1)$ ($d$ divides $m$) or $K_n(d/m, 1/k)$ ($d$ divides $m - 1$ and $m > k\lfloor n/k \rfloor$) has proven elusive. Thus, we offer the following.

**Conjecture 6.3.** Let $d$, $m$, and $n$ be positive integers such that $1 < d < m \leq n$. Suppose that $(1/m, 1/m - 1)$ and $(d/m, d/m - 1)$ are Farey pairs of order $n$.

(i) If $d$ divides $m$, then $K_n^d(1/m, 1/m - 1) = K_n(1/k, d/m - 1)$, where $k = m/d$.

(ii) If $d$ divides $m - 1$ and $m > k\lfloor n/k \rfloor$, then $K_n^d(1/m, 1/m - 1) = K_n(d/m, 1/k)$, where $k = m/d$.

Let $K$ be a K-arc and let $d_K : [0, 1] \rightarrow \mathbb{R}^+$ be the function defined by $\alpha \mapsto |\lambda|$, where $\lambda = \lambda(\alpha)$ is the point on $K$ corresponding to $\alpha \in [0, 1]$. From Figure 1, we pose the following.

**Conjecture 6.4.** If $K$ is any K-arc, then the function $d_K$ is strictly convex.

### 6.2. The Levick-Pereira-Kribs Conjecture

For a natural number $n$, denote by $\Pi_n$ the convex-hull of the $n^{th}$ roots-of-unity, i.e.,

$$\Pi_n = \left\{ \sum_{k=0}^{n-1} \alpha_k \exp(2\pi ik/n) : \alpha_k \geq 0, \sum_{k=0}^{n-1} \alpha_k = 1 \right\}.$$

Denote by $\Omega_n$ the subset of the complex-plane containing all single eigenvalues of all $n$-by-$n$ doubly stochastic matrices. Perfect and Mirsky [15] conjectured that $\Omega_n = \bigcup_{k=1}^{n} \Pi_k$ and proved their conjecture when $1 \leq n \leq 3$. Levick et al. [12] proved Perfect-Mirsky when $n = 4$ but a counterexample when $n = 5$ was given by Mashreghi and Rivard [13]. Levick et al. conjectured that $\Omega_n = \Theta_{n-1} \cup \Pi_n$ ([12, Conjecture 1]).

In [8], necessary and sufficient conditions were found for a stochastic matrix to be similar to a doubly stochastic matrix. Thus, it is possible to investigate the Levick-Pereira-Kribs Conjecture via the realizing matrices given in Theorem 3.2 vis-à-vis the results in [8]. In particular, if $M$ is a realizing matrix for $\lambda$ on the boundary of $\Theta_n$ excluding the unit-circle (this case is clear), and $M \oplus 1$ is similar to a doubly stochastic matrix $D$, then $\Theta_{n-1} \cup \Pi_n \subseteq \Omega_n$. 


Acknowledgment

We would like to thank University of Washington Bothell undergraduate student Amber R. Thrall for proving that the polynomial $\pi$ is Remark 4.3 has only one root in $(0, 1)$.

References

[1] R. A. Brualdi and H. J. Ryser. *Combinatorial matrix theory*, volume 39 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1991.

[2] N. Dmitriev and E. Dynkin. On characteristic roots of stochastic matrices. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]*, 10:167–184, 1946.

[3] G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers*. Oxford University Press, Oxford, sixth edition, 2008.

[4] R. A. Horn and C. R. Johnson. *Topics in matrix analysis*. Cambridge University Press, Cambridge, 1994. Corrected reprint of the 1991 original.

[5] R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, second edition, 2013.

[6] R. B. Israel. Efficient computation of the trajectory of roots of a parameterized polynomial. Mathematics Stack Exchange. URL: http://math.stackexchange.com/q/93397 (version: 2011-12-22).

[7] H. Ito. A new statement about the theorem determining the region of eigenvalues of stochastic matrices. *Linear Algebra Appl.*, 267:241–246, 1997.

[8] C. R. Johnson. Row stochastic matrices similar to doubly stochastic matrices. *Linear and Multilinear Algebra*, 10(2):113–130, 1981.

[9] C. R. Johnson and P. Paparella. Perron similarities and the nonnegative inverse eigenvalue problem. In preparation.

[10] F. I. Karpelevič. On the characteristic roots of matrices with nonnegative elements. *Izvestiya Akad. Nauk SSSR. Ser. Mat.*, 15:361–383, 1951.

[11] A. N. Kolmogorov. Markov chains with a countable number of possible states. *Byull. Mosk. Gos. Univ., Mat. Mekh.*, 1(3):1–16, 1937.

[12] J. Levick, R. Pereira, and D. W. Kribs. The four-dimensional Perfect-Mirsky Conjecture. *Proc. Amer. Math. Soc.*, 143(5):1951–1956, 2015.

[13] J. Mashreghi and R. Rivard. On a conjecture about the eigenvalues of doubly stochastic matrices. *Linear Multilinear Algebra*, 55(5):491–498, 2007.
[14] D. Ž. Đoković. Cyclic polygons, roots of polynomials with decreasing non-negative coefficients, and eigenvalues of stochastic matrices. *Linear Algebra Appl.*, 142:173–193, 1990.

[15] H. Perfect and L. Mirsky. Spectral properties of doubly-stochastic matrices. *Monatsh. Math.*, 69:35–57, 1965.

[16] J. Swift. *The location of characteristic roots of stochastic matrices*. 1972. M. Sc. thesis, McGill University, Montréal.