Abstract

The $A_{n-1}$ Gaudin model with integrable boundaries specified by non-diagonal $K$-matrices is studied. The commuting families of Gaudin operators are diagonalized by the algebraic Bethe ansatz method. The eigenvalues and the corresponding Bethe ansatz equations are obtained.

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1 Introduction

Gaudin type models constitute a particularly important class of one-dimensional many-body systems with long-range interactions. They have found applications in many branches of fields ranging from condensed matter physics to high energy physics. For example, Gaudin models have been used to establish the integrability of the reduced BCS theory of small metallic grains [1, 2, 3, 4] and the Seiberg-Witten supersymmetric Yang-Mills theory [5]. They have also provided a powerful tool for constructing the solutions to the Knizhnik-Zamolodchikov equation [6, 7, 8, 9, 10] of the Wess-Zumino-Novikov-Witten conformal field theory.

Recently Gaudin models with non-trivial boundaries have attracted much interest [9, 11, 12, 13, 14, 15]. So far, attention has largely been concentrated on Gaudin models with boundary conditions specified by diagonal K-matrices. In [12], the XXZ Gaudin model with boundaries given by the non-diagonal K-matrices in [16, 17] was constructed and solved by the algebraic Bethe ansatz method. In this paper we generalize the results in [12] and solve the $A_{n-1}$ Gaudin magnets with open boundary conditions corresponding to the non-diagonal K-matrices obtained in [18].

This paper is organized as follows. In section 2, we briefly review the inhomogeneous $A_{n-1}^{(1)}$ trigonometric vertex model with integrable boundaries, which also services as introducing our notation and some basic ingredients. In section 3, we construct the generalized Gaudin operators associated with non-diagonal K-matrices. The commutativity of these operators follows from applying the standard procedure [19, 20, 9, 11] to the inhomogeneous $A_{n-1}^{(1)}$ trigonometric vertex model with off-diagonal boundaries found in [18], thus ensuring the integrability of the Gaudin magnets. In section 4, we diagonalize the Gaudin operators simultaneously by means of the algebraic Bethe ansatz method. This constitutes the main new result of this paper. The diagonalization is achieved by means of the technique of the “vertex-face” transformation [21]. Section 5 is for conclusions. In the Apendix, we list the explicit matrix expressions of the K-matrices corresponding to the $n = 3, 4$ cases.
2 Preliminaries: inhomogeneous $A_{n-1}^{(1)}$ open chain

Let us fix a positive integer $n$ ($n \geq 2$) and a generic complex number $\eta$, and $R(u) \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R-matrix of the $A_{n-1}^{(1)}$ trigonometric vertex model given by

$$R(u) = \sum_{\alpha=1}^{n} R^{\alpha\alpha}_{\alpha\alpha}(u) E_{\alpha\alpha} \otimes E_{\alpha\alpha} + \sum_{\alpha \neq \beta} \left\{ R^{\alpha\beta}_{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\beta\beta} + R^{\beta\alpha}_{\alpha\beta}(u) E_{\beta\alpha} \otimes E_{\alpha\beta} \right\},$$  \hspace{1cm} (2.1)

where $E_{ij}$ is the matrix with elements $(E_{ij})_{lk} = \delta_{jk}\delta_{il}$. The coefficient functions are

$$R^{\alpha\beta}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(u)}{\sin(u+\eta)} e^{-i\eta}, & \alpha > \beta, \\ 1, & \alpha = \beta, \\ \frac{\sin(u)}{\sin(u+\eta)} e^{i\eta}, & \alpha < \beta, \end{cases} \hspace{1cm} (2.2)$$

$$R^{\beta\alpha}_{\alpha\beta}(u) = \begin{cases} \frac{\sin(\eta)}{\sin(u+\eta)} e^{iu}, & \alpha > \beta, \\ 1, & \alpha = \beta, \\ \frac{\sin(\eta)}{\sin(u+\eta)} e^{-iu}, & \alpha < \beta. \end{cases} \hspace{1cm} (2.3)$$

The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2),$$ \hspace{1cm} (2.4)

and the properties \cite{18}:

Unitarity : \hspace{0.5cm} $R_{12}(u)R_{21}(-u) = \text{id}$, \hspace{1cm} (2.5)

Crossing-unitarity : \hspace{0.5cm} $R_{12}^t(u)M_2^{-1}R_{21}^t(-u - n\eta)M_2 = \frac{\sin(u)\sin(u + n\eta)}{\sin(u + \eta)\sin(u + n\eta - \eta)} \text{id}$, \hspace{1cm} (2.6)

Quasi-classical property : \hspace{0.5cm} $R_{12}(u)\big|_{\eta \to 0} = \text{id}$. \hspace{1cm} (2.7)

Here $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with $P_{12}$ being the usual permutation operator and $t_i$ denotes the transposition in the $i$-th space, and $\eta$ is the so-called crossing parameter. The crossing matrix $M$ is a diagonal $n \times n$ matrix with elements

$$M_{\alpha\beta} = M_\alpha \delta_{\alpha\beta}, \hspace{0.5cm} M_\alpha = e^{-2i\alpha\eta}, \hspace{0.5cm} \alpha = 1, \ldots, n.$$ \hspace{1cm} (2.8)

Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbb{C}^n)$, $A_j$ is an embedding operator in the tensor space $\mathbb{C}^n \otimes \mathbb{C}^n \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the $i$-th and
The quasi-classical properties of the R-matrix enables one to introduce the corresponding classical r-matrix \( r(u) \)

\[
R(u) = \text{id} + \eta r(u) + O(\eta^2), \quad \text{when } \eta \to 0,
\]

\[
r(u) = \sum_{\alpha \neq \beta} \left\{ r_{\alpha\beta}^{\alpha\beta}(u) E_{\alpha\alpha} \otimes E_{\beta\beta} + r_{\alpha\beta}^{\beta\alpha}(u) E_{\beta\alpha} \otimes E_{\alpha\beta} \right\}.
\]

(2.9)

Here the coefficient functions in the expression of \( r(u) \) are given by

\[
r_{ij}^{kl}(u) = \left. \frac{\partial}{\partial \eta} \{ R_{kl}^{ij}(u) \} \right|_{\eta=0}.
\]

(2.10)

One introduces the “row-to-row” monodromy matrix \( T(u) \), which is an \( n \times n \) matrix with elements being operators acting on \( (\mathbb{C}^n)^{\otimes N} \)

\[
T(u) = R_{01}(u + z_1)R_{02}(u + z_2) \cdots R_{0N}(u + z_N).
\]

(2.11)

Here \( \{ z_i | i = 1, \ldots, N \} \) are arbitrary free complex parameters which are usually called inhomogeneous parameters. With the help of the QYBE (2.4), one can show that \( T(u) \) satisfies the so-called “RLL” relation

\[
R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v).
\]

(2.12)

Integrable open chains can be constructed as follows [25]. Let us introduce a pair of K-matrices \( K^-(u) \) and \( K^+(u) \). The former satisfies the reflection equation (RE)

\[
R_{12}(u_1 - u_2)K_1^-(u_1)R_{21}(u_1 + u_2)K_2^-(u_2) = K_2^-(u_2)R_{12}(u_1 + u_2)K_1^-(u_1)R_{21}(u_1 - u_2),
\]

(2.13)

and the latter satisfies the dual RE [25, 26]

\[
R_{12}(u_2 - u_1)K_1^+(u_1)M_1^{-1}R_{21}(-u_1 - u_2 - n\eta) M_1 K_2^+(u_2) = M_1 K_2^+(u_2)R_{12}(-u_1 - u_2 - n\eta) M_1^{-1} K_1^+(u_1)R_{21}(u_2 - u_1).
\]

(2.14)

For models with open boundaries, instead of the standard “row-to-row” monodromy matrix \( T(u) \) (2.11), one needs the “double-row” monodromy matrix \( T(u) \)

\[
T(u) = T(u)K^-(u)T^{-1}(-u).
\]

(2.15)
Using (2.12) and (2.13), one can prove that $T(u)$ satisfies

$$R_{12}(u_1 - u_2)T_1(u_1)R_{21}(u_1 + u_2)T_2(u_2) = T_2(u_2)R_{12}(u_1 + u_2)T_1(u_1)R_{21}(u_1 - u_2).$$  (2.16)

Then the double-row transfer matrix of the inhomogeneous $A_{n-1}^{(1)}$ trigonometric vertex model with open boundary is given by

$$\tau(u) = tr(K^+(u)T(u)).$$  (2.17)

The commutativity of the transfer matrices

$$[\tau(u), \tau(v)] = 0,$$  (2.18)

follows as a consequence of (2.4)-(2.6) and (2.13)-(2.14). This ensures the integrability of the inhomogeneous $A_{n-1}^{(1)}$ trigonometric vertex models with open boundaries specified by the K-matrices.

## 3 $A_{n-1}$ Gaudin model with boundaries

In this paper, we will consider the non-diagonal solutions $K^+(u)$ obtained in [18], which are respectively given by

$$K^-(u)_l^s = \sum_{j=1}^{n} k_j(u) \phi_j^s(u; \lambda) \tilde{\phi}_j^t(-u; \lambda),$$  (3.1)

$$K^+(u)_l^s = \sum_{j=1}^{n} \tilde{k}_j(u) \phi_j^s(-u; \lambda') \tilde{\phi}_j^t(u; \lambda').$$  (3.2)

There are $n$ different solutions, each parametrized by an integer $l$ ($1 \leq l \leq n$) and given by

$$k_j(u) = \begin{cases} 1, & 1 \leq j \leq l, \\ \frac{\sin(\xi - u)}{\sin(\xi + u)} e^{-2iu}, & l + 1 \leq j \leq n, \end{cases}$$  (3.3)

$$\tilde{k}_j(u) = \begin{cases} e^{-2i(jn)}, & 1 \leq j \leq l, \\ \frac{\sin(\xi + u + \frac{n}{2} \eta)}{\sin(\xi - u + \frac{n}{2} \eta)} e^{2i(u + \frac{n}{2} \eta)}, & l + 1 \leq j \leq n. \end{cases}$$  (3.4)

Hereafter we choose some particular $l$ without losing generality. In (3.1) and (3.2), $\phi$, $\tilde{\phi}$ and $\phi$ are intertwiners which are specified in section 4. Besides a discrete parameter $l$, the K-matrix $K^-(u)$ (resp. $K^+(u)$) depends on continuous parameters $\xi$, $\{\lambda_j\}$ (resp. $\tilde{\xi}$, $\{\lambda'_j\}$) and $\rho$ (whose dependence is through the intertwiner-matrix (4.2) below). Here and
throughout, associated with the boundary parameters \( \{ \lambda_j \} \) (resp. \( \{ \lambda'_j \} \)) we introduce a vector \( \lambda = \sum_{j=1}^{n} \lambda_j \epsilon_j \) (resp. \( \lambda' = \sum_{j=1}^{n} \lambda'_j \epsilon_j \)), where \( \{ \epsilon_j \mid j = 1, \ldots, n \} \) is the orthonormal basis of the vector space \( \mathbb{C}^n \) such that \( \langle \epsilon_j, \epsilon_k \rangle = \delta_{jk} \).

As can be seen below (e.g. (4.2), (4.11) and (4.12)), when \( m \) is specialized to \( \lambda \), \( K^-(u) \) does not depend on \( \eta \). Moreover, the dependence on \( \lambda_n \) disappears in the final expression of \( K^-(u) \) although it appears in the expression of \( \phi_i(u; \lambda) \). Without loss of generality in this paper we will assume \( \lambda_n = 0 \). Thus the K-matrix \( K^-(u) \) depends on \( n + 1 \) continuous free parameters \( \{ \lambda_j \mid j = 1, \ldots, n-1 \}; \rho, \xi \} \) for \( 1 \leq l \leq n-1 \), and \( n \) parameters \( \{ \lambda_j \mid j = 1, \ldots, n-1 \}; \rho \} \) for \( l = n \). Some explicit matrix forms of the K-matrices \( K^-(u) \) for the \( n = 3, 4 \) cases are given as (A.1)-(A.7) in the Appendix A.

Let us emphasize that a further restriction

\[
\lambda' + \eta \sum_{k=1}^{N} \epsilon_{jk} = \lambda, \tag{3.5}
\]

where \( \{ jk \mid k = 1, \ldots, N \} \) are positive integers such that \( 2 \leq j_k \leq n \), is necessary for the application of the algebraic Bethe ansatz method in section 4. Hereafter, we shall impose this constraint. We further restrict the complex parameters \( \xi \) and \( \bar{\xi} \) to be the same, i.e.,

\[
\bar{\xi} = \xi, \tag{3.6}
\]

so that (3.7) below is satisfied. Under these two constraints, the K-matrix \( K^+(u) \) depends on the same set of free parameters as the K-matrix \( K^-(u) \) (see above). Moreover, the K-matrices satisfy the following relation thanks to the restrictions (3.5) and (3.6)

\[
\lim_{\eta \to 0} \{ K^+(u) K^-(u) \} = \lim_{\eta \to 0} \{ K^+(u) \} K^-(u) = \text{id}. \tag{3.7}
\]

Let us now introduce the Gaudin operators \( \{ H_j \mid j = 1, 2, \ldots, N \} \) associated with the inhomogeneous \( A_{n-1}^{(1)} \) trigonometric vertex model with boundaries specified by the K-matrices \( (3.1) \) and \( (3.2) \):

\[
H_j = \Gamma_j(z_j) + \sum_{k \neq j}^{N} r_{kj}(z_j - z_k) + (K_j^-(z_j))^{-1} \left\{ \sum_{k \neq j}^{N} r_{jk}(z_j + z_k) \right\} K_j^-(z_j), \tag{3.8}
\]

where \( \Gamma_j(u) = \frac{\partial}{\partial \eta} \{ \bar{K}_j(u) \} \mid_{\eta = 0} K_j^-(u) \), \( j = 1, \ldots, N \), with \( \bar{K}_j(u) = tr_0 \{ K_0^+(u) R_{0j}(2u) P_{0j} \} \). Here \( \{ z_j \} \) are the inhomogeneous parameters of the inhomogeneous \( A_{n-1}^{(1)} \) trigonometric vertex model and \( r(u) \) is the classical r-matrix given by (2.9).
The generalized Gaudin operators (3.8) are obtained by expanding the double-row transfer matrix (2.17) at the point $u = z_j$ around $\eta = 0$:

$$\tau(z_j) = \tau(z_j)|_{\eta=0} + \eta H_j + O(\eta^2), \quad j = 1, \cdots, N,$$  \hspace{1cm} (3.9)

$$H_j = \frac{\partial}{\partial \eta} \tau(z_j)|_{\eta=0}. $$ \hspace{1cm} (3.10)

The relations (2.7) and (3.7) imply that the first term in the expansion (3.9) is equal to the identity,

$$\tau(z_j)|_{\eta=0} = \text{id}. $$ \hspace{1cm} (3.11)

Then the commutativity of the transfer matrices \{\tau(z_j)\} (2.18) implies

$$[H_j, H_k] = 0, \quad j, k = 1, \cdots, N. $$ \hspace{1cm} (3.12)

Thus the Gaudin system defined by (3.8) is integrable. Moreover, the fact that the Gaudin operators \{H_j\} (3.8) can be expressed in terms of the transfer matrix of the inhomogeneous $A_{n-1}^{(1)}$ trigonometric vertex model with open boundary enables us to exactly diagonalize the operators by the algebraic Bethe ansatz method with the help of the “vertex-face” correspondence technique, as will be shown in the next section. The aim of this paper is to diagonalize the generalized Gaudin operators $H_j, j = 1, \cdots, N$ (3.8), simultaneously.

4 Eigenvalues and Bethe ansatz equations

4.1 Intertwining vectors and face-vertex correspondence

For a vector $m \in \mathbb{C}^n$, set

$$m_j = \langle m, e_j \rangle, \quad |m| = \sum_{k=1}^{n} m_k, \quad j = 1, \ldots, n. $$ \hspace{1cm} (4.1)

We introduce \[18\] $n$ intertwining vectors (intertwiners) \{\phi_j(u; m)| j = 1, \ldots, n\}. Each $\phi_j(u; m)$ is an $n$-component column vector whose $\alpha$-th component is $\{\phi_{j}^{(\alpha)}(u; m)\}$. The $n$ intertwiners form an $n \times n$ the intertwiner-matrix (in which $j$ and $\alpha$ stand for the column
and the row indices respectively), with the non-vanishing matrix elements being

$$
\begin{pmatrix}
  e^{iF_1(m)} & e^{iF_2(m)} & \cdots & e^{iF_n(m) + \rho e^{2iu}} \\
  e^{iF_2(m)} & \ddots & & \vdots \\
  \vdots & & \ddots & \vdots \\
  e^{iF_n(m)} & \cdots & & e^{iF_{n-1}(m)}
\end{pmatrix}.
$$

(4.2)

Here $\rho$ is a complex constant, and $\{f_j(m) | j = 1, \ldots, n\}$ and $\{F_j(m) | j = 1, \ldots, n\}$ are linear functions of $m$:

$$f_j(m) = \sum_{k=1}^{j-1} m_k - m_j - \frac{1}{2}|m|, \quad j = 1, \ldots, n,$n

(4.3)

$$F_j(m) = \sum_{k=1}^{j} m_k - \frac{1}{2}|m|, \quad j = 1, \ldots, n - 1,$n

(4.4)

$$F_n(m) = -\frac{3}{2}|m|.

(4.5)

We remark that as is clear from (4.2), while $\phi_n(u; m)$ is a function of $u$, $\phi_j(u; m)$ does not depend on $u$ for $1 \leq j \leq n - 1$.

From the above intertwiner-matrix, one may derive the following face-vertex correspondence relation [18]:

$$R_{12}(u_1 - u_2)\phi_i(u_1; m) \otimes \phi_j(u_2; m - \eta \epsilon_i) = \sum_{k,l} W_{kl}^{ij}(u_1 - u_2)\phi_k(u_1; m - \eta \epsilon_i) \otimes \phi_l(u_2; m).$$

(4.6)

Here the non-vanishing elements of $\{W_{kl}^{ij}(u)\}$ are

$$W_{jj}^{ij}(u) = 1, \quad W_{jk}^{ij}(u) = \frac{\sin(u)}{\sin(u + \eta)}, \quad \text{for } j \neq k,$n

(4.7)

$$W_{kj}^{ij}(u) = \begin{cases} 
\frac{\sin(\eta)}{\sin(u + \eta)} e^{iu}, & j > k, \\
\frac{\sin(\eta)}{\sin(u + \eta)} e^{-iu}, & j < k,
\end{cases} \quad \text{for } j \neq k.

(4.8)

Associated with $\{W_{ij}^{kl}(u)\}$, one may introduce “face” type R-matrix $W(u)$

$$W(u) = \sum_{i,j,k,l} W_{ij}^{kl}(u) E_{ki} \otimes E_{lj}.

(4.9)
Note that the “face” type R-matrix $W(u)$ does not depend on the face type parameter $m$, in contrast to the $\mathbb{Z}_n$ elliptic case [27], and thus $W(u)$ and $R(u)$ satisfy the same QYBE, i.e. $W(u)$ obeys the usual (vertex type) QYBE rather than the dynamical one. Moreover the R-matrix $W(u)$ enjoys the quasi-classical property,

$$W(u)|_{\eta\to0} = \text{id}.$$ (4.10)

For a generic $\rho \in \mathbb{C}$, the determinant of the intertwiner matrix (4.2) is non-vanishing and thus the inverse of (4.2) exists [18]. This fact allows the introduction of other types of intertwiners $\tilde{\phi}$ and $\tilde{\phi}$ satisfying the following orthogonality conditions:

$$\sum_\alpha \tilde{\phi}_i^{(\alpha)}(u; m) \phi_j^{(\alpha)}(u; m) = \delta_{ij},$$ (4.11)

$$\sum_\alpha \tilde{\phi}_i^{(\alpha)}(u; m + \eta \epsilon_i) \phi_j^{(\alpha)}(u; m + \eta \epsilon_j) = \delta_{ij}.$$ (4.12)

From these conditions we derive the “completeness” relations:

$$\sum_k \tilde{\phi}_k^{(\alpha)}(u; m) \phi_k^{(\beta)}(u; m) = \delta_{\alpha\beta},$$ (4.13)

$$\sum_k \tilde{\phi}_k^{(\alpha)}(u; m + \eta \epsilon_k) \phi_k^{(\beta)}(u; m + \eta \epsilon_k) = \delta_{\alpha\beta}.$$ (4.14)

Corresponding to the vertex type K-matrices (3.1) and (3.2), one has the following face type K-matrices $K$ and $\tilde{K}$ [28]

$$K(\lambda|u)_i^j = \sum_{\alpha,\beta} \tilde{\phi}_j^{(\alpha)}(u; \lambda - \eta (\epsilon_i - \epsilon_j)) K_\beta^{(\alpha)}(u; \lambda) \phi_i^{(\beta)}(-u; \lambda),$$ (4.15)

$$\tilde{K}(\lambda'|u)_i^j = \sum_{\alpha,\beta} \tilde{\phi}_j^{(\alpha)}(-u; \lambda') K_\beta^{(\alpha)}(u; \lambda' - \eta (\epsilon_j - \epsilon_i)) \phi_i^{(\beta)}(u; \lambda').$$ (4.16)

Straightforward calculations show that the face type K-matrices are diagonal $^1$

$$K(\lambda|u)_i^j = \delta_i^j k_j(u; \xi), \quad \tilde{K}(\lambda'|u)_i^j = \delta_i^j \tilde{k}_j(u), \quad i, j = 1, \ldots, n,$$ (4.17)

where functions $\{k_i(u; \xi) = k_i(u)\}$ and $\{\tilde{k}_i(u)\}$ are respectively given by (3.3) and (3.4).

A remark is in order. Although the K-matrices $K^\pm(u)$ given by (3.1) and (3.2) are generally non-diagonal (in the vertex picture), after the face-vertex transformations (4.15) $^1$

$^1$The spectral parameter $u$ and the boundary parameter $\xi$ of the reduced double-row monodromy matrices constructed from $K(\lambda|u)$ will be shifted in each step of the nested Bethe ansatz procedure [29]. Therefore, it is convenient to specify the dependence of $K(\lambda|u)$ on the boundary parameter $\xi$ through $k_j(u; \xi)$.

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and (4.16), the face type counterparts $\mathcal{K}(\lambda|u)$ and $\tilde{\mathcal{K}}(\lambda'|u)$ become diagonal simultaneously. This fact enables ones to apply the generalized algebraic Bethe ansatz method to diagonalize the transfer matrix $\tau(u)$ \textsuperscript{30} \textsuperscript{12} \textsuperscript{21} (for $n = 2$ case, the corresponding transfer matrix was diagonalized alternatively by the fusion hierarchy of the transfer matrices with the anisotropy value being the roots of unity \textsuperscript{31} \textsuperscript{32}).

### 4.2 Algebraic Bethe ansatz

By means of (4.13), (4.14), (4.16) and (4.17), the transfer matrix $\tau(u)$ (2.17) can be recasted into the following face type form:

$$\tau(u) = \sum_{\mu, \nu} \tilde{\mathcal{K}}(\lambda'|u)_{\mu}^{\nu} T(\lambda'|u)_{\mu}^{\nu} = \sum_{\mu} \tilde{k}_{\mu}(u) T(\lambda'|u)_{\mu}^{\nu}. \quad (4.18)$$

Here we have introduced the face type double-row monodromy matrix $T(m|u)$,

$$T(\lambda'|u)_{\mu}^{\nu} = \left. T(m|u)_{\mu}^{\nu} \right|_{m=\lambda'} = \tilde{\phi}_{\nu}(u; m - \eta(\epsilon_{\mu} - \epsilon_{\nu})) T(u) \phi_{\mu}(-u; m) \right|_{m=\lambda'} \equiv \sum_{\alpha, \beta} \tilde{\phi}_{\nu}(u; \lambda' - \eta(\epsilon_{\mu} - \epsilon_{\nu})) T(u)^{\beta}_{\alpha} \phi_{\mu}(-u; \lambda'). \quad (4.19)$$

Moreover, (2.16), (4.16) and (4.17) imply the following exchange relations among $T(m|u)_{\mu}^{\nu}$:

$$\sum_{i_1, i_2 j_1, j_2} W_{i_1 j_1}^{i_2 j_2} (u_1 - u_2) T(m + \eta(\epsilon_{i_1} + \epsilon_{i_2})|u_1)_{i_2}^{i_1}$$

$$\times W_{j_2 i_3}^{j_1 i_3} (u_1 + u_2) T(m + \eta(\epsilon_{j_1} + \epsilon_{j_2})|u_2)_{j_3}^{j_2}$$

$$= \sum_{i_1, i_2 j_1, j_2} T(m + \eta(\epsilon_{i_1} + \epsilon_{i_2})|u_2)_{j_3}^{j_1} W_{j_2 i_3}^{i_1 j_1} (u_1 + u_2)$$

$$\times T(m + \eta(\epsilon_{j_2} + \epsilon_{j_3})|u_1)_{i_2}^{i_1} W_{j_3 i_3}^{j_2 i_2} (u_1 - u_2). \quad (4.20)$$

In the following we will use the standard notation,

$$\mathcal{A}(m|u) = T(m|u)_{1}^{1}, \quad \mathcal{B}_j(m|u) = T(m|u)_{1}^{j}, \quad \mathcal{C}_j(m|u) = T(m|u)_{j}^{1}, \quad j = 2, \ldots, n, \quad (4.21)$$

$$\mathcal{D}_i^j(m|u) = T(m|u)_{i}^{j} - \delta_i^j W_{1 j}^{1} (2u) \mathcal{A}(m|u), \quad i, j = 2, \ldots, n. \quad (4.22)$$

In order to apply the algebraic Bethe ansatz method, one needs to construct a pseudo-vacuum state (also called a reference state) which is the common eigenstate of the operators $\mathcal{A}$, $\mathcal{D}_i^j$ and is annihilated by the operators $\mathcal{C}_j$. In contrast to the models with diagonal $K^\pm(u)$, the usual highest-weight state

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix},$$
is no longer the pseudo-vacuum state. However, after the face-vertex transformations (4.15) and (4.16), the face type K-matrices $K(\lambda|u)$ and $\tilde{K}(\lambda|u)$ simultaneously become diagonal. This suggests that one can translate the $A^{(1)}_{n-1}$ trigonometric model with non-diagonal K-matrices into the corresponding SOS model with diagonal K-matrices $K(\lambda|u)$ and $\tilde{K}(\lambda|u)$ given by (4.15)-(4.16).

Consider the state [21],

$$|\Omega\rangle = \phi_1(-z_1; \lambda - (N - 1)\eta\varepsilon_1) \otimes \phi_1(-z_2; \lambda - (N - 2)\eta\varepsilon_1) \cdots \otimes \phi_1(-z_N; \lambda),$$

which depends on the boundary parameters \{\lambda_j\}, but not on the boundary parameter $\xi$. It is also independent of \{z_i\} because as mentioned before the vector $\phi_1(u;m)$ does not depend on $u$ regardless of $m$. The state (4.23) is the pseudo-vacuum state in the “face picture” since it can be shown to satisfy the following equations:

$$A(\lambda - N\eta\varepsilon_1|u)|\Omega\rangle = k_1(u;\xi)|\Omega\rangle,$$

$$D_j(\lambda - N\eta\varepsilon_1|u)|\Omega\rangle = \delta_j^0 \frac{\sin(2u)e^{i\eta}}{\sin(2u + \eta)} k_j(u + \frac{\eta}{2};\xi - \frac{\eta}{2})$$

$$\times \left\{ \prod_{k=1}^{N} \frac{\sin(u + z_k)\sin(u - z_k)}{\sin(u + z_k + \eta)\sin(u - z_k + \eta)} \right\} |\Omega\rangle,$$

$$C_j(\lambda - N\eta\varepsilon_1|u)|\Omega\rangle = 0, \quad j = 2, \ldots, n, \quad (4.25)$$

$$B_j(\lambda - N\eta\varepsilon_1|u)|\Omega\rangle \neq 0, \quad j = 2, \ldots, n, \quad (4.26)$$

as required.

For later convenience, we introduce a set of non-negative integers \{N_j|j = 1, \ldots, n - 1\} with $N_1 = N$ and complex parameters $\{v^{(j)}_k|k = 1, 2, \ldots, N_j+1, j = 0, 1, \ldots, n - 2\}$. As in [33, 34, 35, 21], the parameters $\{v^{(j)}_k\}$ will be used to specify the eigenvectors of the corresponding reduced transfer matrices (see below). We will also adopt the convention,

$$v_k = v^{(0)}_k, \quad k = 1, 2, \ldots, N. \quad (4.28)$$

We now apply the generalized algebraic Bethe ansatz method developed in [29] to diagonalize the transfer matrix (2.17) with the K-matrices $K^\pm(u)$ given by (3.1)-(3.5). We seek the common eigenvectors of the transfer matrix of the form

$$|v_1, \ldots, v_N\rangle = \sum_{i_1, \ldots, i_N=2}^n F_{i_1,i_2,\ldots,i_N} B_{i_1}(\lambda' + \eta\varepsilon_{i_1} - \eta\varepsilon_1|v_1) B_{i_2}(\lambda' + \eta(\varepsilon_{i_1} + \varepsilon_{i_2}) - 2\eta\varepsilon_1|v_2) \cdots$$

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\[ \times B_{iN-1}(\lambda' + \eta \sum_{k=1}^{N-1} \epsilon_{ik} - (N-1)\eta \epsilon_1 |v_{N-1}) \]
\[ \times B_{iN}(\lambda' + \eta \sum_{k=1}^{N} \epsilon_{ik} - N\eta \epsilon_1 |v_N) |\Omega \rangle. \] (4.29)

Here the summing indices should obey the constraint, \( \lambda' + \eta \sum_{k=1}^{N} \epsilon_{ik} = \lambda \), where \( \lambda' \) and \( \lambda \) are the boundary parameters which satisfy the restriction (3.5). Taking this constraint into account, (4.29) may be written as
\[ |v_1, \ldots, v_N\rangle = \sum_{i_1, \ldots, i_N=2}^{n} F^{i_1, i_2, \ldots, i_N} B_{i_1}(\lambda' + \eta \epsilon_{i_1} - \eta \epsilon_1 |v_1) B_{i_2}(\lambda' + \eta (\epsilon_{i_1} + \epsilon_{i_2}) - 2\eta \epsilon_1 |v_2) \cdots \]
\[ \times B_{i_{N-1}}(\lambda' + \eta \sum_{k=1}^{N-1} \epsilon_{ik} - (N-1)\eta \epsilon_1 |v_{N-1}) \]
\[ \times B_{i_N}(\lambda - N\eta \epsilon_1 |v_N) |\Omega \rangle. \] (4.30)

With the help of (4.18), (4.21) and (4.22) one may rewrite the transfer matrix (2.17) in terms of the operators \( \mathcal{A} \) and \( \mathcal{D}^j_i \)
\[ \tau(u) = \alpha^{(1)}(u) \mathcal{A}(\lambda'|u) + \sum_{j=2}^{n} \tilde{k}^{(1)}_j(u + \frac{\eta}{2}) \mathcal{D}(\lambda'|u) \] (4.31)

Here we have introduced the function \( \alpha^{(1)}(u) \),
\[ \alpha^{(1)}(u) = \sum_{j=1}^{n} \tilde{k}_j(u) W^{1}_{ij}(2u), \] (4.32)

and the reduced K-matrix \( \tilde{K}^{(1)}(\lambda'|u) \) with the elements given by
\[ \tilde{K}^{(1)}(\lambda'|u)_{ij} = \delta^{ij} \tilde{k}^{(1)}_j(u), \quad i, j = 2, \ldots, n, \] (4.33)
\[ \tilde{k}^{(1)}_j(u) = \tilde{k}_j(u - \frac{\eta}{2}), \quad j = 2, \ldots, n. \] (4.34)

Following [29, 13, 21], we introduce a set of reduced K-matrices \( \{ \tilde{K}^{(b)}(\lambda'|u)|b = 0, \ldots, n-1 \} \) which include the original one \( \tilde{K}(\lambda'|u) = \tilde{K}^{(0)}(\lambda'|u) \) and the ones in (4.33) and (4.34):
\[ \tilde{K}^{(b)}(\lambda'|u)_{ij} = \delta^{ij} \tilde{k}^{(b)}_j(u), \quad i, j = b+1, \ldots, n, \quad b = 0, \ldots, n-1, \] (4.35)
\[ \tilde{k}^{(b)}_j(u) = \tilde{k}_j(u - \frac{\eta}{2}), \quad j = b+1, \ldots, n, \quad b = 0, \ldots, n-1. \] (4.36)
Moreover we introduce a set of functions \( \{ \alpha^{(b)}(u) | b = 1, \ldots, n-1 \} \) (including the one in (4.32)) related to the reduced K-matrices \( \tilde{K}^{(b)}(\lambda|u) \)

\[
\alpha^{(b)}(u) = \sum_{j=b}^{n} W_{j}^{(b)}(2u) \tilde{K}_{j}^{(b-1)}(u), \quad b = 1, \ldots, n. \tag{4.37}
\]

Carrying out the nested Bethe ansatz \([21]\), one finds that, with the coefficients \( F^{i_1 i_2 \cdots i_N} \) properly chosen, the Bethe state \( |v_1, \ldots, v_N \rangle \) is the eigenstate of the transfer matrix \( 2.17 \),

\[
\tau(u) |v_1, \ldots, v_N \rangle = \Lambda(u; \xi, \{ v_k \}) |v_1, \ldots, v_N \rangle, \tag{4.38}
\]

with the eigenvalue given by

\[
\Lambda(u; \xi, \{ v_k \})
= \alpha^{(1)}(u) k_1(u; \xi) \prod_{k=1}^{N_1} \frac{\sin(u + v_k) \sin(u - v_k - \eta)}{\sin(u + v_k + \eta) \sin(u - v_k)}
+ \frac{\sin(2u) e^{i\eta}}{\sin(2u + \eta)} \left\{ \prod_{k=1}^{N_1} \frac{\sin(u - v_k + \eta) \sin(u + v_k + 2\eta)}{\sin(u - v_k) \sin(u + v_k + \eta)} \right. \\
\times \prod_{k=1}^{N} \frac{\sin(u + z_k) \sin(u - z_k)}{\sin(u + z_k + \eta) \sin(u - z_k + \eta)} \left. \right\} \Lambda^{(1)}(u + \frac{\eta}{2}; \xi - \frac{\eta}{2}, \{ v_k^{(1)} \}). \tag{4.39}
\]

The eigenvalues \( \{ \Lambda^{(j)}(u; \xi, \{ v_k^{(j)} \}) | j = 0, \ldots, n-1 \} \) (with \( \Lambda(u; \xi, \{ v_k \}) = \Lambda^{(0)}(u; \xi, \{ v_k^{(0)} \}) \)) of the reduced transfer matrices are given by the following recurrence relations:

\[
\Lambda^{(j)}(u; \xi^{(j)}, \{ v_k^{(j)} \})
= \alpha^{(j+1)}(u) k_{j+1}(u; \xi^{(j)}) \prod_{k=1}^{N_{j+1}} \frac{\sin(u + v_k^{(j)}) \sin(u - v_k^{(j)} - \eta)}{\sin(u + v_k^{(j)} + \eta) \sin(u - v_k^{(j)})}
+ \frac{\sin(2u) e^{i\eta}}{\sin(2u + \eta)} \left\{ \prod_{k=1}^{N_{j+1}} \frac{\sin(u - v_k^{(j)} + \eta) \sin(u + v_k^{(j)} + 2\eta)}{\sin(u - v_k^{(j)}) \sin(u + v_k^{(j)} + \eta)} \right. \\
\times \prod_{k=1}^{N_j} \frac{\sin(u + z_k^{(j)}) \sin(u - z_k^{(j)})}{\sin(u + z_k^{(j)} + \eta) \sin(u - z_k^{(j)} + \eta)} \left. \right\} \Lambda^{(j+1)}(u + \frac{\eta}{2}; \xi^{(j)} - \frac{\eta}{2}, \{ v_k^{(j+1)} \}).
\]
\[ \Lambda^{(n-1)}(u; \xi^{(n-1)}) = \tilde{k}_n^{(n-1)}(u) \Lambda_n(u; \xi^{(n-1)}). \]  

The reduced boundary parameters \( \{\xi^{(j)}\} \) and inhomogeneous parameters \( \{z^{(j)}_k\} \) are given by

\[ \xi^{(j+1)} = \xi^{(j)} - \frac{\eta}{2}, \quad z^{(j+1)}_k = v^{(j)}_k + \frac{\eta}{2}, \quad j = 0, \ldots, n - 2. \]  

Here we have adopted the convention: \( \xi = \xi^{(0)}, \ z^{(0)}_k = z_k \). The complex parameters \( \{v^{(j)}_k\} \) satisfy the following Bethe ansatz equations:

\[
\alpha^{(1)}(v_s) \Lambda_{1}^{(1)}(v_s; \xi) \frac{\sin(2v_s + \eta) e^{-i\eta}}{\sin(2v_s + 2\eta)} \times \prod_{k \neq s, k=1}^{N_1} \frac{\sin(v_s + v_k) \sin(v_s - v_k - \eta)}{\sin(v_s + v_k + 2\eta) \sin(v_s - v_k + \eta)} \\
= \prod_{k=1}^{N} \frac{\sin(v_s + z_k) \sin(v_s - z_k)}{\sin(v_s + z_k + \eta) \sin(v_s - z_k + \eta)} \times \Lambda^{(1)}(v_s + \frac{\eta}{2}; \xi - \frac{\eta}{2}, \{v^{(1)}_k\}),
\]

\[
\alpha^{(j+1)}(v_s) \Lambda_{j+1}^{(j+1)}(v_s; \xi) \frac{\sin(2v_s^{(j)} + \eta) e^{-i\eta}}{\sin(2v_s^{(j)} + 2\eta)} \times \prod_{k \neq s, k=1}^{N_{j+1}} \frac{\sin(v_s^{(j)} + v_k^{(j)}) \sin(v_s^{(j)} - v_k^{(j)} - \eta)}{\sin(v_s^{(j)} + v_k^{(j)} + 2\eta) \sin(v_s^{(j)} - v_k^{(j)} + \eta)} \\
= \prod_{k=1}^{N_j} \frac{\sin(v_s^{(j)} + z_k^{(j)}) \sin(v_s^{(j)} - z_k^{(j)})}{\sin(v_s^{(j)} + z_k^{(j)} + \eta) \sin(v_s^{(j)} - z_k^{(j)} + \eta)} \times \Lambda^{(j+1)}(v_s^{(j)} + \frac{\eta}{2}; \xi^{(j)} - \frac{\eta}{2}, \{v^{(j+1)}_k\}),
\]

\[ j = 1, \ldots, n - 2, \]  

\[ (4.40) \]

\[ (4.41) \]

4.3 Eigenstates and the corresponding eigenvalues

The relation (3.10) between \( \{H_j\} \) and \( \{\tau(z_j)\} \) and the fact that the first term on the r.h.s. of (3.39) is a c-number enable us to extract the eigenstates of the generalized Gaudin operators \( \{H_j\} \) and the corresponding eigenvalues from the results obtained in the previous subsection.

Introduce the functions \( \{\beta^{(j)}(u, \eta)\} \)

\[ \beta^{(j+1)}(u, \eta) \equiv \beta^{(j+1)}(u) = \alpha^{(j+1)}(u) k_{j+1}(u; \xi - \frac{j}{2}\eta), \quad j = 0, \ldots, n - 2. \]  

\[ (4.45) \]
Then by (4.40)-(4.42), the Bethe ansatz equations (4.43) and (4.44) become, respectively,

\[
\beta^{(j+1)}(v_s^{(j)}) \frac{\sin(2v_s^{(j)} + \eta)e^{-in\eta}}{\sin(2v_s^{(j)} + 2\eta)} \prod_{k \neq s, k=1}^{N_{j+1}} \frac{\sin(v_s^{(j)} + v_k^{(j)}) \sin(v_s^{(j)} - v_k^{(j)} - \eta)}{\sin(v_s^{(j)} + v_k^{(j)} + 2\eta) \sin(v_s^{(j)} - v_k^{(j)} + \eta)} = \beta^{(j+2)}(v_s^{(j)} + \frac{\eta}{2}) \prod_{k=1}^{N_j} \frac{\sin(v_s^{(j)} + v_k^{(j-1)} + \frac{\eta}{2}) \sin(v_s^{(j)} - v_k^{(j-1)} - \frac{\eta}{2})}{\sin(v_s^{(j)} + v_k^{(j-1)} + 3\frac{\eta}{2}) \sin(v_s^{(j)} - v_k^{(j-1)} + \frac{\eta}{2})},
\]

\[
\beta^{(n-1)}(v_s^{(n-2)}) \frac{\sin(2v_s^{(n-2)} + \eta)e^{-in\eta}}{\sin(2v_s^{(n-2)} + 2\eta)} \prod_{k \neq s, k=1}^{N_{n-1}} \frac{\sin(v_s^{(n-2)} + v_k^{(n-2)}) \sin(v_s^{(n-2)} - v_k^{(n-2)} - \eta)}{\sin(v_s^{(n-2)} + v_k^{(n-2)} + 2\eta) \sin(v_s^{(n-2)} - v_k^{(n-2)} + \eta)} = \tilde{k}_n^{(n-1)}(v_s^{(n-2)} + \frac{\eta}{2}) \tilde{k}_n(v_s^{(n-2)} + \frac{\eta}{2}; e^{(n-1)}) \prod_{k=1}^{N_{n-2}} \frac{\sin(v_s^{(n-2)} + v_k^{(n-3)} + \frac{\eta}{2}) \sin(v_s^{(n-2)} - v_k^{(n-3)} - \frac{\eta}{2})}{\sin(v_s^{(n-2)} + v_k^{(n-3)} + 3\frac{\eta}{2}) \sin(v_s^{(n-2)} - v_k^{(n-3)} + \frac{\eta}{2})}.
\]

Here we have used the convention: \(v_k^{(1)} = z_k, k = 1, \cdots, N\). The quasi-classical property (4.10) of the R-matrix \(W(u)\), (4.37) and (4.45) lead to the following relations

\[
\beta^{(j+1)}(u, 0) = 1, \quad \frac{\partial}{\partial u} \beta^{(j+1)}(u, 0) = 0, \quad j = 0, \cdots, n - 2.
\]

Then, one may introduce functions \(\{\gamma^{(j+1)}(u)\}\) associated with \(\{\beta^{(j+1)}(u, \eta)\}\)

\[
\gamma^{(j+1)}(u) = \frac{\partial}{\partial \eta} \beta^{(j+1)}(u, \eta) \Big|_{\eta=0}, \quad j = 0, \cdots, n - 2.
\]

Using (4.48), we can extract the eigenvalues \(h_j\) (resp. the corresponding Bethe ansatz equations) of the Gaudin operators \(H_j\) (4.3) from the expansion around \(\eta = 0\) for the first order of \(\eta\) of the eigenvalues (4.39) of the transfer matrix \(\tau(u = z_j)\) (resp. the Bethe ansatz equations (4.46) and (4.47)). Finally, the eigenvalues of the generalized Gaudin operators are

\[
h_j = \gamma^{(1)}(z_j) - \sum_{k=1}^{N_1} \{\cot(z_j + x_k) + \cot(z_j - x_k)\}.
\]

The parameters \(\{x_k^{(j)}\} k = 1, 2, \cdots, N_{j+1}, j = 0, 1, \cdots, n - 2\) (including \(x_k\) as \(x_k = x_k^{(0)}\), \(k = 1, \cdots, N_1\)) are determined by the following Bethe ansatz equations:

\[
\gamma^{(j+1)}(x_s^{(j)}) - \cot(2x_s^{(j)}) - i - 2 \sum_{k \neq s, k=1}^{N_{j+1}} \{\cot(x_s^{(j)} + x_k^{(j)}) + \cot(x_s^{(j)} - x_k^{(j)})\}
\]
\[ \gamma^{(j+2)}(x_k^{(j)}) - \sum_{k=1}^{N_j} \left\{ \cot(x_s^{(j)} + x_k^{(j-1)}) + \cot(x_s^{(j)} - x_k^{(j-1)}) \right\} - \sum_{k=1}^{N_{j+2}} \left\{ \cot(x_s^{(j)} + x_k^{(j+1)}) + \cot(x_s^{(j)} - x_k^{(j+1)}) \right\}, \]

\[ j = 0, \ldots, n - 3, \quad (4.51) \]

\[ \gamma^{(n-1)}(x_s^{(n-2)}) - \cot(2x_s^{(n-2)}) - i - 2 \sum_{k \neq s, k=1}^{N_{n-1}} \left\{ \cot(x_s^{(n-2)} + x_k^{(n-2)}) + \cot(x_s^{(n-2)} - x_k^{(n-2)}) \right\} \]

\[ = g(x_s^{(n-2)}) - \sum_{k=1}^{N_{n-2}} \left\{ \cot(x_s^{(n-2)} + x_k^{(n-3)}) + \cot(x_s^{(n-2)} - x_k^{(n-3)}) \right\}. \quad (4.52) \]

Here we have used the convention: \( x_k^{(-1)} = z_k, k = 1, \ldots, N \) in (4.51), and the function \( g(u) \) is given by

\[ g(u) = \frac{\partial}{\partial \eta} \left( k_n^{(n-1)}(u + \frac{\eta}{2}) k_n(u + \frac{\eta}{2}; \xi - \frac{n-1}{2} \eta) \right) \bigg|_{\eta=0}. \quad (4.53) \]

## 5 Conclusions

We have studied the \( A_{n-1} \) Gaudin model with boundaries specified by the non-diagonal K-matrices \( K^\pm(u) \), (3.1) and (3.2). In addition to the inhomogeneous parameters \( \{z_j\} \), the Gaudin operators \( \{H_j\} \), (3.8), have one discrete parameter \( l(1 \leq l \leq n) \) and \( n+1 \) continuous free parameters \( \{\lambda_j| j = 1, \ldots, n - 1; \rho, \xi \} \) for \( 1 \leq l \leq n - 1 \) (or \( n \) continuous free parameters \( \{\lambda_j| j = 1, \ldots, n - 1; \rho \} \) for \( l = n \)). As seen from section 4, although the “vertex type” K-matrices \( K^\pm(u) \) (3.1) and (3.2) are non-diagonal, the compositions, (4.15) and (4.16), lead to the diagonal “face-type” K-matrices after the face-vertex transformation. This has enabled us to successfully construct the corresponding pseudo-vacuum state \( |\Omega\rangle \) (4.23), diagonalize the generalized Gaudin operators \( \{H_j\} \) by means of the algebraic Bethe ansatz method, and derive the eigenvalues (4.50) as well as the Bethe ansatz equations (4.51) and (4.52) of the boundary Gaudin model.

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**Appendix: Explicit matrix forms of K-matrices**

In this appendix, we give the explicit matrix expressions of the K-matrices (3.1) for the cases \( n = 3, 4 \).

**The \( A_2^{(1)} \) case:**

There are three types of K-matrices \( K^{-}(u) \), which are labelled by the discrete parameters \( l \), for the trigonometric \( A_2^{(1)} \) model.

- For the case of \( l = 1 \), the 7 non-vanishing matrix elements \( K^{-}(u)^{k}_{j} \) are given by:

\[
\begin{align*}
K^{-}(u)_{1}^{1} &= \frac{e^{2iu}}{e^{2iu} + e^{\rho}} \left( 1 - e^{\rho} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \\
K^{-}(u)_{1}^{2} &= \frac{e^{-2i(\lambda_1 + \lambda_2) + \rho}}{e^{2iu} + e^{\rho}} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \\
K^{-}(u)_{1}^{3} &= -\frac{e^{-2i(\lambda_1 + \lambda_2) + \rho}}{e^{2iu} + e^{\rho}} \left( 1 + e^{2iu} \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \\
K^{-}(u)_{2}^{1} &= \frac{e^{2i\lambda_1 + i(u + \xi)}}{(e^{2iu} + e^{\rho}) \sin(u + \xi)}, \\
K^{-}(u)_{2}^{2} &= \frac{1}{e^{2iu} + e^{\rho}} \left( e^{\rho} - \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \\
K^{-}(u)_{2}^{3} &= -\frac{e^{-2i\lambda_2 - i(u - \xi) + \rho}}{(e^{2iu} + e^{\rho}) \sin(u + \xi)}, \quad K^{-}(u)_{3}^{3} = e^{-2iu} \frac{\sin(\xi - u)}{\sin(\xi + u)}. \quad (A.1)
\end{align*}
\]

- For the case of \( l = 2 \), the 7 non-vanishing matrix elements \( K^{-}(u)^{k}_{j} \) are given by:

\[
\begin{align*}
K^{-}(u)_{1}^{1}, \quad K^{-}(u)_{2}^{1}, \quad K^{-}(u)_{3}^{1} \ &= \text{the same as those in (A.1),} \\
K^{-}(u)_{2}^{2} &= 1, \quad K^{-}(u)_{3}^{3} = -\frac{e^{2i(\lambda_1 + \lambda_2) + i(u + \xi)}}{(e^{2iu} + e^{\rho}) \sin(u + \xi)}, \\
K^{-}(u)_{2}^{3} &= \frac{e^{2i\lambda_2 + i(u + \xi)}}{(e^{2iu} + e^{\rho}) \sin(u + \xi)}, \\
K^{-}(u)_{3}^{3} &= \frac{1}{e^{2iu} + e^{\rho}} \left( e^{\rho} - \frac{\sin(u - \xi)}{\sin(u + \xi)} \right). \quad (A.2)
\end{align*}
\]
The $A^{(1)}_3$ case:

There are four types of K-matrices $K^-(u)$, which are labelled by the discrete parameters $l$, for the trigonometric $A^{(1)}_3$ model.

- For the case of $l = 1$, the 10 non-vanishing matrix elements $K^-(u)^k_j$ are given by:

  $K^-(u)_1^1 = \frac{e^{2iu} + e^{4iu+\rho}}{e^{2iu} + e^{\rho}} \left(1 + e^\rho \frac{\sin (u - \xi)}{\sin (u + \xi)} \right)$,

  $K^-(u)_2^1 = -\frac{e^{-2i\lambda_1 + \rho}}{e^{2iu} - e^{\rho}} \left(1 + e^{2iu} \frac{\sin (u - \xi)}{\sin (u + \xi)} \right)$,

  $K^-(u)_3^1 = \frac{e^{-2i(\lambda_1 + \lambda_2) + \rho}}{e^{2iu} - e^{\rho}} \left(1 + e^{2iu} \frac{\sin (u - \xi)}{\sin (u + \xi)} \right)$,

  $K^-(u)_4^1 = -\frac{e^{-2i(\lambda_1 + \lambda_2 + \lambda_3) + \rho}}{e^{2iu} - e^{\rho}} \left(1 + e^{2iu} \frac{\sin (u - \xi)}{\sin (u + \xi)} \right)$,

  $K^-(u)_1^2 = \frac{e^{2i\lambda_1 + i(u+\xi)} \sin 2u}{(e^{2iu} - e^{\rho}) \sin (u + \xi)}$,

  $K^-(u)_2^2 = -\frac{1}{e^{2iu} - e^{\rho}} \left(e^{\rho} + \frac{\sin (u - \xi)}{\sin (u + \xi)} \right)$,

  $K^-(u)_3^2 = \frac{e^{-2i(\lambda_2 - i(u-\xi) + \rho} \sin 2u}{(e^{2iu} - e^{\rho}) \sin (u + \xi)}$,

  $K^-(u)_4^2 = -\frac{e^{-2i(\lambda_2 + \lambda_3 - i(u-\xi) + \rho}}{e^{2iu} - e^{\rho}} \sin (u + \xi)$,

  $K^-(u)_3^3 = K^-(u)_4^3 = e^{-2iu} \frac{\sin (\xi - u)}{\sin (\xi + u)}$.  \hspace{1cm} (A.4)

- For the case of $l = 2$, the 10 non-vanishing matrix elements $K^-(u)^k_j$ are given by:

  $K^-(u)_1^1, K^-(u)_2^1, K^-(u)_3^1, K^-(u)_4^1$ are the same as those in (A.4)

  $K^-(u)_2^2 = 1$,  $K^-(u)_1^3 = -\frac{e^{2i(\lambda_1 + \lambda_2 + i(u+\xi) \sin 2u}{(e^{2iu} - e^{\rho}) \sin (u + \xi)}$,

  $K^-(u)_2^3 = \frac{e^{2i(\lambda_2 + i(u+\xi) \sin 2u}{(e^{2iu} - e^{\rho}) \sin (u + \xi)}$,
\[ K^-(u)^3_3 = -\frac{1}{e^{2iu} - e^\rho} \left( e^\rho + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right), \]
\[ K^-(u)^3_4 = \frac{e^{-2\lambda_3 - i(u - \xi) + \rho} \sin 2u}{(e^{2iu} - e^\rho) \sin(u + \xi)}, \quad K^-(u)^4_4 = e^{-2iu} \frac{\sin(\xi - u)}{\sin(\xi + u)}. \] (A.5)

- For the case of \( l = 3 \), the 10 non-vanishing matrix elements \( K^-(u)^k_j \) are given by:
  \[ K^-(u)^1_1, K^-(u)^1_2, K^-(u)^3_3, K^-(u)^4_4 \] are the same as those in (A.4)
  \[ K^-(u)^2_2 = K(u)^3_3 = 1, \]
  \[ K^-(u)^4_1 = \frac{e^{2i(\lambda_1 + \lambda_2 + \lambda_3) + i(u + \xi)} \sin 2u}{(e^{2iu} - e^\rho) \sin(u + \xi)}, \]
  \[ K^-(u)^4_2 = \frac{e^{2i\lambda_3 + i(u + \xi)} \sin 2u}{(e^{2iu} - e^\rho) \sin(u + \xi)}, \]
  \[ K^-(u)^4_3 = \frac{e^{2i\lambda_3 + i(\xi + \xi)} \sin 2u}{(e^{2iu} - e^\rho) \sin(u + \xi)}, \]
  \[ K^-(u)^4_4 = -\frac{1}{e^{2iu} - e^\rho} \left( e^\rho + \frac{\sin(u - \xi)}{\sin(u + \xi)} \right). \] (A.6)

- For the case of \( l = 4 \), the 7 non-vanishing matrix elements \( K^-(u)^k_j \) are given by:
  \[ K^-(u)^1_1 = \frac{e^{2iu} - e^{4iu + \rho}}{e^{2iu} - e^\rho}, \quad K^-(u)^1_2 = \frac{e^{-2i\lambda_1 + \rho} (e^{4iu} - 1)}{e^{2iu} - e^\rho}, \]
  \[ K^-(u)^1_3 = -\frac{e^{-2i(\lambda_1 + \lambda_2) + \rho} (e^{4iu} - 1)}{e^{2iu} - e^\rho}, \]
  \[ K^-(u)^1_4 = \frac{e^{-2i(\lambda_1 + \lambda_2 + \lambda_3) + \rho} (e^{4iu} - 1)}{e^{2iu} - e^\rho}, \]
  \[ K^-(u)^2_2 = K^-(u)^3_3 = K^-(u)^4_4 = 1. \] (A.7)

The above explicit results confirm the following properties for the non-diagonal K-matrices of the \( A^{(1)}_{n-1} \) vertex model: for \( l = 1, \ldots, n - 1 \), \( K^-(u) \) depends on \( n + 1 \) continuous free parameters and has \( 3n - 2 \) non-vanishing matrix elements; for \( l = n \), \( K^-(u) \) depends on \( n \) continuous free parameters and has \( 2n - 1 \) non-vanishing matrix elements. So our K-matrices contain more boundary parameters and more non-vanishing matrix elements than those found in [36, 37, 38].

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