LOWER ORDER TERMS IN SZEGÖ TYPE LIMIT THEOREMS
ON ZOLL MANIFOLDS

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Abstract. We compute the third order term in a generalization of the Strong Szegö Limit Theorem for a zeroth order pseudodifferential operator (PsDO) on a Zoll manifold of an arbitrary dimension. In [GO2], the second order term was computed by V. Guillemin and K. Okikiolu. In the present paper, an important role is played by a certain combinatorial identity which we call the generalized Hunt–Dyson formula [Gi3]. This identity is a different form of the renowned Bohnenblust–Spitzer combinatorial theorem which is related to the maximum of a random walk with i.i.d. steps on the real line. A corollary of our main result is a fourth order Szegö type asymptotics for a zeroth order PsDO on the unit circle, which in matrix terms gives a fourth order asymptotic formula for the determinant of the truncated sum $P_n(T_1 + T_2D)P_n$ of a Toeplitz matrix $T_1$ with the product of another Toeplitz matrix $T_2$ and a diagonal matrix $D$ of the form $\text{diag}(0, 1, \frac{1}{2}, \frac{1}{3}, \cdots)$. Here $P_n = \text{diag}(1, \cdots, 1, 0, \cdots)$, $n$ ones.

1. Introduction

The main motivation for this work was to find an explicit formula for a “Szegö–regularized” determinant of a zeroth order pseudodifferential operator (PsDO) on a Zoll manifold introduced in [GO1, after (3)] and [O2], see Remark 1.7. Our main result, Theorem 1.3, is valid for any dimension $d \in \mathbb{N}$. In the case $d = 2$, Theorem 1.3 gives such a formula.

1.1. Notations and main results. Let $M = S^1$ be the unit circle $\mathbb{R}/2\pi\mathbb{Z}$. Denote by $P_n$, $n \in \mathbb{N}$, the orthogonal projection from $L^2(S^1)$ to the subspace spanned by $\{e^{ikx}\}_{|k| \leq n}$. For a function $f \in L^1(S^1)$ denote its $k$th Fourier coefficient by $\hat{f}_k := \int_0^{2\pi} f(x) e^{-ikx} \frac{dx}{2\pi}$, $k \in \mathbb{Z}$. Let $b(x)$ be a positive function on $S^1$ such that $\sum_{k \in \mathbb{Z}} |k| (|\log b|)_k^2 < \infty$. Denote by $B$ the operator of multiplication by $b$ acting in $L^2(S^1)$. The matrix representation of the operator $B$ in the basis $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is the Toeplitz matrix $(\hat{b}_{j-k})_{j,k \in \mathbb{Z}}$. The classical Strong Szegö Limit Theorem (SSLT) [Sz2] states that

$$\text{Tr} \log P_n BP_n = \text{Tr} P_n (\log B) P_n + \sum_{k=1}^{\infty} k (\log b)_k (\log b)_{-k} + o(1), \quad n \to \infty.$$ 

Here $\text{Tr} \log P_n BP_n = \log \det P_n BP_n$ and $\text{Tr} P_n (\log B) P_n = (2n + 1) \int_0^{2\pi} \log b(x) \frac{dx}{2\pi}$. It has been shown by H. Widom that the remainder is $O(n^{-\infty})$ if $b(x) \in C^\infty(S^1)$, see [W1].

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The main result of this paper is Theorem 1.3, in which we find a third order generalization of the SSLT for a zeroth order pseudodifferential operator (PsDO) \( B \) on a Zoll manifold \( M \) of an arbitrary dimension \( d \in \mathbb{N} \).

Recall that \( M \) is called a Zoll manifold \([GO1]\) if it is compact, closed and such that the geodesic flow on \( M \) is simply periodic with period \( 2\pi \). The unit circle and the standard sphere of any dimension are Zoll manifolds. A second order generalization of the SSLT for a Zoll manifold \( M \) of any dimension has been obtained by V. Guillemin and K. Okikiolu \([GO1, GO2]\), see also an important preceding work \([O1]\) by K. Okikiolu for \( M = S^2 \) and \( S^3 \). The proofs in \([O1, GO1, GO2]\) use a combinatorial identity due to G. A. Hunt and F. J. Dyson and proceed in the spirit of the combinatorial proof of the classical SSLT by M. Kac \([K]\). See also \([GO3, O2]\) where the combinatorial approach and the Hunt–Dyson formula (HD) are used in a different setting to obtain a second order generalization of the SSLT for a manifold with the set of closed geodesics of measure zero in the unit cotangent bundle.

In the proof of Theorem 1.3 we use the method of \([GO2]\). A central role in our proof is played by a certain combinatorial identity which generalizes the Hunt–Dyson formula mentioned above to an arbitrary natural power. We call this identity the generalized Hunt–Dyson formula (gHD), see Theorem 8.2 and \([Gi3]\). After having discovered and proved the gHD we realized that it is related to another combinatorial theorem, which has a long history. This theorem is a result due to H. F. Bohnenblust that appeared in an article by F. Spitzer on random walks \([S]\), and is now commonly known as the Bohnenblust–Spitzer theorem (BSt). A major application of, and motivation for the BSt, is the computation of the characteristic function of the maximum of a random walk with independent identically distributed (i.i.d.) steps carried out in \([S]\). Note that the expectation of such a maximum was computed earlier in \([K]\) with the help of the usual HD.

Let \( M \) be a Zoll manifold of dimension \( d \in \mathbb{N} \). Let \( \Psi^m(M) \), \( m \in \mathbb{Z} \), denote the space of classical PsDO’s of order \( m \) on \( M \). Recall that for a given \( G \in \Psi^m(M) \), its principal symbol \( \sigma_m(G) \) and subprincipal symbol \( \text{sub}(G) \) are well-defined on \( T^*M \). Let \( \Delta \) denote the Laplace–Beltrami operator on \( M \). It is known \([DO]\) that there exists a constant \( \alpha \in \mathbb{R} \) such that the spectrum of \( \sqrt{-\Delta} \) lies in bands around the points \( k + \frac{\alpha}{4} \), \( k \in \mathbb{N} \). Moreover, it has been shown in \([CDV]\) that there exists \( A_{-1} \in \Psi^{-1}(M) \) such that \( [\Delta, A_{-1}] = 0 \) and the spectrum of the operator

\[
A := \sqrt{-\Delta} - \frac{\alpha}{4} - A_{-1}
\]

is \( \mathbb{N} \). Let \( P_n \), \( n \in \mathbb{N} \), denote the projection from \( L^2(M) \) onto the subspace spanned by the eigenfunctions of \( A \) corresponding to the eigenvalues \( 1, 2, \cdots, n \). Let \( dxd\xi \) be the standard measure on \( S^*M := \{(x, \xi) : \sigma_1(A)(x, \xi) = 1\} \) divided by \((2\pi)^d\).

Following \([GO2]\) we will assume that \( \sigma_1(A)(x, \xi) = \sigma_1(A)(x, -\xi) \) for all \((x, \xi) \in T^*M \). In \([Gi1]\) Chapter 1 this is not assumed which leads to more complicated expressions. Let \( \Theta^t(x, \xi) \) denote the shift of the point \((x, \xi) \in S^*M \) by \( t \) units along geodesic flow. For any function \( f \in C^\infty(S^*M) \) introduce the \( k \)th Fourier coefficient of the closed geodesic of length \( 2\pi \) starting at a given point \((x, \xi)\)

\[
\hat{f}_k(x, \xi) := \int_0^{2\pi} e^{-ikt} f(\Theta^t(x, \xi)) \frac{dt}{2\pi}, \quad k \in \mathbb{Z}.
\]

The simplest form of our result is for the case of \( M = S^1 \) with \( f(z) = \log z \). Note that \( f(z) \) is analytic in a disk of radius 1 about the point \( z = 1 \). In the proofs
in Section 2, we require that the function $f(z)$ is analytic on a disk the radius of which depends on a certain norm of the operator $B \in \Psi^0(M)$. For our purposes the following norm is convenient

$$
|||B|||_d := ||B|| + ||[A, [A, B]]|| + \|\nabla^{\max(d+2,6)}\sigma_0(B)\|_{\infty} + \|\nabla^3 \text{sub}(B)\|_{\infty} + \int_{S^*M} |\nabla^3 \sigma_d(A^{2-d}B)| \, dx d\xi,
$$

where $d = \dim M$, $\| \cdot \|$ is the operator norm in $L^2(M)$, $\nabla$ includes both $x$- and $\xi$-derivatives in local coordinates on $T^*M$, and $\|g\|_{\infty} := \max_{(x,\xi) \in S^*M} |g(x,\xi)|$. (The integral in (1.3) is well-defined being a Guillemin–Wodzicki residue, see (2.8) below.)

**Theorem 1.1.** Let $M = \mathbb{S}^1$ and $P_n$ be the projection on the linear span of $\{e^{ikx}\}_{k \leq n}$. Let $B \in \Psi^0(M)$ and assume that $\sigma_0(B)$ is strictly positive, and that the symbolic norm $|||I - B|||$ is sufficiently small. Then $\log B \in \Psi^0(M)$ and the following holds as $n \to \infty$,

$$
\text{Tr} \log P_n BP_n = \text{Tr} P_n (\log B) P_n + \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\sigma_0(\log B))_k (\sigma_0(\log B) - k) \, dx d\xi
$$

$$
+ \frac{1}{n} \cdot \frac{1}{2} \int_{S^*M} \sum_{k=1}^{\infty} k (\sigma_0(\log B))_k (\text{sub}(\log B) - k) \, dx d\xi + O\left(\frac{1}{n^2}\right).
$$

In (1.3) the argument $(x, \xi) \in S^*M$ is omitted for brevity and for each $(x, \xi) \in S^*M$ the Fourier coefficient is understood in the sense of (1.2).

We need to fix more notation to formulate the result for $M = \mathbb{S}^1$ and an arbitrary analytic $f(z)$. Let $\mathcal{A}_1$ denote the set of all analytic functions on $\mathbb{C}$ with no constant term

$$
\mathcal{A}_1 := \{ f(z) : f(z) = \sum_{m=1}^{\infty} c_m z^m, \, z \in \mathbb{C} \}.
$$

In [LRS] the authors introduce a linear map $W_2$ from $\mathcal{A}_1$ to the space of continuous functions from $\mathbb{C}^2$ to $\mathbb{C}$, defined by

$$
W_2[f](x_1, x_2) := \frac{1}{2} \int_{0}^{x_1} \int_{0}^{x_2} f'_{\xi_1} - f'_{\xi_2} \, d\xi_1 d\xi_2.
$$

Let $j \in \mathbb{N}$. We will call a linear map $V$ from $\mathcal{A}_1$ to the space of continuous functions $\mathbb{C}^j \to \mathbb{C}$ a linear $j$-map. The linearity means that for arbitrary $f, g \in \mathcal{A}_1$, $\alpha, \beta \in \mathbb{C}$, $x_1, \ldots, x_j \in \mathbb{C}$

$$
V[\alpha f + \beta g](x_1, \ldots, x_j) = \alpha V[f](x_1, \ldots, x_j) + \beta V[g](x_1, \ldots, x_j).
$$

A 2-map $U$, which is equivalent to $W_2$, was earlier constructed by H. Widom [W0, W7]. We will need also a 2-map $W_2$ whose action on an arbitrary $f \in \mathcal{A}_1$ is prescribed by

$$
\tilde{W}_2[f](x_1, x_2) := \frac{1}{2} \int_{0}^{x_1} \int_{0}^{x_2} f'_{\xi_1} - f'_{\xi_2} \, d\xi_1 d\xi_2.
$$

For an arbitrary $B \in \Psi^0(M)$, let us write $b_0 := \sigma_0(B)$ and $b_{\text{sub}} := \text{sub}(B)$, $b'_0(x, \xi) := b_0(\Theta'(x, \xi))$, $b'_{\text{sub}}(x, \xi) := b_{\text{sub}}(\Theta'(x, \xi))$, and omit the argument $(x, \xi) \in \mathbb{S}^1$.
and \( \Phi \) notation of the Poisson brackets see Remark 1.2. Secondly, we introduce notations needed to describe the contributions of Theorem 1.2. Let \( B \to \infty \) and also that for \( 0 < j \leq m \), see Section 7 and (7.2) for an expression for \( \Phi \) evaluated at the point \((\xi_1, \cdots, \xi_j, 0, \cdots)\), see (7.3). Now for \( x, y \in \mathbb{R} \)

\[
S^*M. \text{ It is convenient to introduce the notations }
\]

\[
\Upsilon_2[f](B) := \int_{S^*M} dx d\xi \sum_{k=1}^{\infty} k \int_0^{2\pi} \int_0^{2\pi} e^{ik(t_1-t_2)} W_2[f](b^1_0, b^2_0) \frac{dt_1 dt_2}{2\pi 2\pi}
\]

and

\[
\Upsilon_{3,\text{sub}}[f](B) := \int_{S^*M} dx d\xi \sum_{k=1}^{\infty} k \int_0^{2\pi} \int_0^{2\pi} e^{ik(t_1-t_2)} \tilde{W}_2[f](b^1_0, b^2_0) b^2_{\text{sub}} \frac{dt_1 dt_2}{2\pi 2\pi}.
\]

**Theorem 1.2.** Let \( M = \mathbb{S}^1 \) and \( P_n \) be the projection on the linear span of \( \{e^{ikx}\}_{k \leq n} \). Let \( B \in \Psi^0(M) \) and \( f \in A_1 \). Then \( f(B) \in \Psi^0(M) \) and the following holds as \( n \to \infty \),

\[
\text{Tr } f(P_n BP_n) = \text{Tr } P_n f(B) P_n + \Upsilon_2[f](B) + \frac{1}{n} \cdot \Upsilon_{3,\text{sub}}[f](B) + O\left(\frac{1}{n^2}\right).
\]

Observe that Theorem 1.1 follows from Theorem 1.2 setting \( f(t) = \log z \) and noting following H. Widom and [LRS, O2] that

\[
(1.7) \quad W_2[\log](x_1, x_2) = -\frac{1}{2} \log x_1 \log x_2,
\]

and also that for \( 0 \leq t_1, t_2 \leq 2\pi \)

\[
\tilde{W}_2[\log](b^1_0, b^2_0 b^2_{\text{sub}}) = (\log b^1_0)(b^2_{\text{sub}}/b^1_0)^t_1 (\text{sub}(\log B))^t_2.
\]

In the higher dimensional case two additional contributions to the third Szegö term now arise. First, define a 3-map \( W_3 \) such that for any \( f \in A_1 \)

\[
W_3[f](x_1, x_2, x_3) := x_3 \int_0^{x_1} \int_0^{x_2} \left( \frac{f(\xi_1)}{\xi_1(\xi_1-x_3)(\xi_1-\xi_2)} - \frac{f(\xi_2)}{\xi_2(\xi_2-x_3)(\xi_2-\xi_2)} + \frac{f(x_3)}{x_3(\xi_1-x_3)(\xi_2-x_3)} \right) d\xi_1 d\xi_2,
\]

and introduce the notation

\[
\Upsilon_{3,0}[f](B) := (d-1) \int_{S^*M} dx d\xi \times \left[ \sum_{k=1}^{\infty} k \left( k^2 + (1 + \alpha/2)k \right) \int_0^{2\pi} \int_0^{2\pi} e^{ik(t_1-t_2)} W_2[f](b^1_0, b^2_0) \frac{dt_1 dt_2}{2\pi 2\pi} \right. \\
+ \sum_{k,l=1}^{\infty} kl \int_0^{2\pi} \int_0^{2\pi} e^{ik(t_1-t_2) + il(t_2-t_3)} W_3[f](b^1_0, b^2_0, b^3_0) \frac{dt_1 dt_2 dt_3}{2\pi 2\pi 2\pi},
\]

see Remark 1.2. Secondly, we introduce notations needed to describe the contribution of the Poisson brackets \( \{b^0_0, b^0_0\} \), \( 0 \leq t, s \leq 2\pi \), to the third Szegö term. Define for each \( j \in \mathbb{N} \) a linear \( j \)-map \( \Phi_j \) such that for any \( m \in \mathbb{N}, m \geq j \),

\[
(1.9) \quad \Phi_j[z^m](x_1, \cdots, x_j) := \sum_{t_1, \cdots, t_j \geq 1 \atop t_1 + \cdots + t_j = m} \frac{x_1^{t_1}}{t_1} \cdots \frac{x_j^{t_j}}{t_j},
\]

and \( \Phi_j[z^m] = 0 \) for \( m = 1, \cdots, j - 1 \). This together with the linearity defines \( \Phi_j \) uniquely on \( A_1 \), see Section 3 and (7.2) for an expression for \( \Phi_j \) acting on an arbitrary \( f \in A_1 \). We can write (1.9) in terms of the complete symmetric function of degree \( m-j \) evaluated at the point \((\xi_1, \cdots, \xi_j, 0, \cdots)\), see (7.3). Now for \( x, y \in \mathbb{R} \)
denote \(-x_- := \min(0, x)\) and \(M_2(x, y) := \min(0, x, x+y)\). For any \(\varkappa_1, \varkappa_2, \mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_k, \rho_1, \ldots, \rho_l \in \mathbb{Z}\) set

\[
\Omega_j^{(1)}(x_1, x_2, \mu_1, \ldots, \mu_j) := M_2(x_1, x_2) - (x_1 + x_2 - M_2(x_1, x_2) - (\mu_1)_- - \cdots - (\mu_j)_-)
\]

(1.10)

\[-(x_1)_- - \left( (x_1)_+ - (\mu_1)_- - \cdots - (\mu_j)_- \right)
- (\mu_1)_+ + \cdots + (\mu_j)_+ + x_2 - (\nu_1)_- - \cdots - (\nu_k)_-)
- (\mu_1)_- - \cdots - (\mu_j)_- - (\mu_1)_+ + \cdots + (\mu_j)_+ + x_1 - x_2
- (x_2)_+ - (\nu_1)_- - \cdots - (\nu_k)_-
- (\mu_1)_- - \cdots - (\mu_j)_- - (\nu_1)_+ + \cdots + (\nu_k)_+ + x_1
- (\nu_1)_- - \cdots - (\nu_k)_- - (\nu_1)_+ + \cdots + (\nu_k)_+ + x_2_-.
\]

(1.11)

and

\[
\Omega_j^{(2)}(x_1, x_2, \mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_k)
:= -(\mu_1)_- - \cdots - (\mu_j)_- - (\mu_1)_+ + \cdots + (\mu_j)_+
\]

(1.12)

\[+ x_1 - (\nu_1)_- - \cdots - (\nu_k)_- - (\nu_1)_+ + \cdots + (\nu_k)_+
+ x_2 - (\rho_1)_- - \cdots - (\rho_l)_-.
\]

For any \(f \in \mathcal{A}_1\) denote by \(T_j[f](z)\) its Taylor polynomial of degree \(j \in \mathbb{N}\) about the point \(t = 0\). For any \(f \in \mathcal{A}_1\) and \(B \in \Psi^0(M)\) introduce the notation

\[
\Upsilon_{3, \text{Poi}}[f](B) := \int_{S^*M} \left( \Lambda^{(1)}[f](b_0) + \Lambda^{(2)}[f](b_0) + \Lambda^{(3)}[f](b_0) \right) dx d\xi
\]

where

\[
\Lambda^{(1)}[f](b_0) := \frac{1}{2\pi} \sum_{j=1}^{\infty} \frac{1}{j!} \Omega_j^{(1)}(x_1, x_2, \mu_1, \ldots, \mu_j)
\]

(1.13)

\[
\times \int_0^{2\pi} \int_0^{2\pi} e^{i(x_1 u_1 + x_2 u_2)} \left\{ b_0^{u_1}, b_0^{u_2} \right\} \frac{du_1}{2\pi} \frac{du_2}{2\pi}
\]

\[
\times \cdots e^{i(\mu_1 r_1 + \cdots + \mu_j r_j)}
\]

\[
\times \Phi_j \left[ -z^{-2}(f(z) - T_2[f](z)) \right] (b_0^{r_1}, \ldots, b_0^{r_j}) \frac{dr_1}{2\pi} \cdots \frac{dr_j}{2\pi},
\]
and

\[
\Lambda^{(2)}[f](b_0) := \frac{1}{2^l} \sum_{j,k,l=1}^{\infty} \frac{1}{j! k!} \times \sum_{\kappa_1+\kappa_2+\ldots+\mu_j+\ldots+\mu_k} \Omega_{j,k}^{(2)}(\kappa_1, \kappa_2, \mu_1, \ldots, \mu_j, \nu_1, \ldots, \mu_k) \\
\times \int_0^{2\pi} \int_0^{2\pi} e^{i(\kappa_1 u_1 + \kappa_2 u_2)} \left\{ b_{u_1}^{b_{u_2}}, b_{0}^{b_{0}} \right\} du_1 du_2 \frac{2\pi}{2\pi} \\
\times \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left\{ (\mu_1 r_1 + \ldots + \mu_j r_j + \nu_1 s_1 + \ldots + \nu_k s_k) dr_1 \ldots dr_j ds_1 \ldots ds_k \frac{2\pi}{2\pi} \frac{2\pi}{2\pi} \\
\times \Phi_{j+k+1} \left[ z^{-2} (f(z) - T_3[f](z)) \right] \left( b_{r_1}^{b_{r_j}}, b_{0}^{b_{0}}, b_{s_1}^{b_{s_k}}, b_{0}^{b_{0}}, \ldots, b_{0}^{b_{0}} \right) \right)
\]

(1.14)

and

\[
\Lambda^{(3)}[f](b_0) := \frac{1}{2^l} \sum_{j,k,l=1}^{\infty} \frac{1}{j! k! l!} \times \sum_{\kappa_1+\kappa_2+\ldots+\mu_j+\ldots+\mu_k+\rho_1+\ldots+\rho_j} \Omega_{j,k,l}^{(3)}(\kappa_1, \kappa_2, \mu_1, \ldots, \mu_j, \nu_1, \ldots, \nu_k, \rho_1, \ldots, \rho_l) \\
\times \int_0^{2\pi} \int_0^{2\pi} e^{i(\kappa_1 u_1 + \kappa_2 u_2)} \left\{ b_{u_1}^{b_{u_2}}, b_{0}^{b_{0}} \right\} du_1 du_2 \frac{2\pi}{2\pi} \\
\times \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \left\{ (\mu_1 r_1 + \ldots + \mu_j r_j + \nu_1 s_1 + \ldots + \nu_k s_k + \rho_1 t_1 + \ldots + \rho_l t_l) dr_1 \ldots dr_j ds_1 \ldots ds_k dt_1 \ldots dt_l \frac{2\pi}{2\pi} \frac{2\pi}{2\pi} \\
\times \Phi_{j+k+l+1} \left[ z^{-2} (f(z) - T_3[f](z)) \right] \left( b_{r_1}^{b_{r_j}}, b_{0}^{b_{0}}, b_{s_1}^{b_{s_k}}, b_{0}^{b_{0}}, \ldots, b_{0}^{b_{0}} \right) \right)
\]

(1.15)

Now the result for any dimension and an arbitrary \(f \in A_1\). Denote

\( \Upsilon_3[f](B) := \Upsilon_{3,0}[f](B) + \Upsilon_{3,\text{sub}}[f](B) + \Upsilon_{3,\text{Poi}}[f](B) \).

**Theorem 1.3.** Let \( M \) be a Zoll manifold of dimension \( d \in \mathbb{N} \). Let \( A \) be defined by (1.1). Assume that \( \sigma_1(A)(x, \xi) = \sigma_1(A)(x, -\xi) \) for all \((x, \xi) \in T^* M\). Let \( B \in \Psi^0(M) \) and \( f \in A_1 \). Then \( f(B) \in \Psi^0(M) \) and the following holds as \( n \to \infty \),

\[
\text{Tr} f(P_n B P_n) = \text{Tr} P_n f(B) P_n + n^{d-1} \cdot \Upsilon_2[f](B) + n^{d-2} \cdot \Upsilon_3[f](B) + O(n^{d-3})
\]

(1.16)

Most of the paper is devoted to the proof of Theorem 1.3. Theorem 1.2 follows from Theorem 1.3 in view of the following. For \( d = 1 \), \( \Upsilon_{3,0} \) vanishes, and also \( \Upsilon_{3,\text{Poi}} \) vanishes, because all the Poisson brackets vanish in this case (for each of the two cotangent directions the angle does not change and \( \sigma_0(B) \) is homogeneous of degree 0 in \( \xi \)). Note that for the function \( f(z) = \log z \), the terms involving \( W_2 \) and \( W_3 \) in Theorem 1.3 for any \( d \in \mathbb{N} \) take a simpler form as in Theorem 1.1. We make several remarks, and then state a corollary to Theorem 1.1 and 1.3 which gives an explicit formula for \( \log \det P_n B P_n \), as \( n \to \infty \). For \( d = 1, 2 \), that formula gives an expression for a possible generalized determinant of the operator \( B \in \Psi^0(M) \), see Remark 1.7 below.
Remark 1.1. The existence of a full expansion of the type \((1.16)\) for \(f(z) = z^m, m \in \mathbb{N},\) \(f(z) = \log z,\) has been proven in \([GO1]\). Explicit expressions for the first two coefficients were given in \([GO1, GO2]\) in the case when \(f(z) = z^m, m \in \mathbb{N},\) \(f(z) = \log z,\) and for general \(f \in \mathcal{A}_1,\) in \([LRS].\) See Remark 2.2 in subsection 2.3 for a discussion of the formulas for further coefficients.

Remark 1.2. The formula \((L8)\) for \(W_3[f]\) has a structure similar to the coefficient in the third asymptotic term in a Szegö type expansion for convolution operators obtained by R. Roccaforte in \([RR].\) A certain combinatorial identity known as Spitzer’s formula (see \([DyMcK, Section 3.7]\)), which is a version of the Bohnenblust–Spitzer combinatorial theorem, is also used in the proof in \([RR]\). (See also \([Bax, Ro]\) and \([Gi3, Remark 1.6]\)). A second order Szegö type expansion for convolution operators was established by H. Widom in \([W9]\) with the help of the usual Hunt–Dyson combinatorial formula \((8.4)\). A full asymptotic expansion for convolution operators is obtained in \([W4]\).

Remark 1.3. If \(f \in \mathcal{A}_1\) is a polynomial then each of the sums over \(j, k, l\) in \(\mathcal{T}_{3, \text{Poi}}[f](B)\) reduces to a finite sum for any Zoll manifold \(M\) and any given \(B \in \Psi^0(M),\) because \(\Phi_m[z^n] = 0\) identically for \(m > n.\) The term \(\mathcal{Y}_{3, \text{Poi}}[f](B)\) vanishes for any \(f \in \mathcal{A}_1\) in the case of \(d = 1,\) and also in the following case: Assume \(f(z) = \sum_{k=1}^{\infty} c_{2k-1} z^{2k-1},\) let \(M = S^d, d \in \mathbb{N}, d \geq 2,\) and let \(B \in \Psi^0(M)\) have a principal symbol \(b_0 \in C^\infty(S^d)\) which is independent of \(\xi,\) and in addition is an odd function on \(S^d.\) Thus \(b_0(x) = -b_0(x_a)\) for all pairs of antipodal points \(x, x_a \in S^d.\) Then the Poisson brackets in \(\mathcal{T}_{3, \text{Poi}}[f](B)\) are odd with respect to \(\xi,\) and vanish after the integration over \(S^*M.\) (We refer to \([W3]\) for an auxiliary calculation of the needed Poisson bracket on \(S^2,\) which can be easily modified for any \(S^d, d \geq 3.\)) Under these assumptions the coefficient of the first asymptotic term in \((1.16)\) of order \(n^d,\) which is a part of \(\text{Tr} P_n f(B) P_n,\) also vanishes, but not the coefficients of the second and the third asymptotic terms of orders \(n^{d-1}\) and \(n^{d-2},\) respectively.

Remark 1.4. In order to define the third Szegö asymptotic term in \((1.16)\) the condition \(f \in C^4(I)\) is necessary, where \(I\) is the closed set of values of \(\sigma_0(B).\) If the term \(\mathcal{T}_{3, \text{Poi}}[f](B)\) is absent (see Remark 1.3) then the condition \(f \in C^3(I)\) is necessary. It was shown in \([LRS]\) that if \(B\) is self-adjoint and \(f'' \in L^\infty(I)\) then the second order Szegö formula holds, see also \([L8].\) It would be interesting to know if the condition of essential boundedness of the fourth derivative of \(f\) is sufficient for Szegö asymptotic formula with three terms to hold. Another question concerns the best possible norm in place of \((1.3).\)

Remark 1.5. We have discovered and proved the generalized Hunt–Dyson combinatorial formula (Theorem 8.2) being unaware of the Bohnenblust–Spitzer theorem (Theorem 8.1). A derivation of the gHD from the BST and vice versa can be found in \([Gi3, Chapter 2]\) for an independent proof of the gHD. Also the importance of the BST in the theory of the maximum of a random walk with real i.i.d. steps and related results are discussed in \([Gi3, Remark 1.5 and 1.6].\)

1.2. Explicit asymptotic formulas for \(\log \text{det} P_n BP_n,\) as \(n \to \infty.\) Let \(f(z) = \log z.\) Theorem 1.1 and 1.3 give an expression for \(\log \text{det} P_n BP_n = \text{Tr} \log P_n BP_n\) as a sum of \(\text{Tr} P_n[\log B] P_n\) and two lower order corrections, as \(n \to \infty.\) Proposition 1.4 below gives, for an arbitrary \(G \in \Psi^0(M),\) an auxiliary asymptotic expansion for \(\text{Tr} P_n GP_n,\) as \(n \to \infty.\) For dimension \(d = 1, 2,\) we need the constant
coefficient in this expansion, which is more complicated than the other ones, see the proof of Proposition 1.4 in Section 3 for details. Let \( R_l(G) \), \( l = 0, 1, 2, \ldots \), be the Guillemin–Wodzicki residues as given in (2.8) below. For \( d = 1, 2 \) make an additional assumption

\[
(1.17) \quad \sum_{l=0}^{\infty} |R_l(G)| < \infty
\]

under which the following sums are absolutely convergent

\[
(1.18) \quad C(G) := \sum_{k=1}^{\infty} \left( \text{Tr} (\pi_k G) - \sum_{l=0}^{+\infty} k^{d-1-l} R_l(G) \right),
\]

see Section 3 for the proof.

Let \( \gamma \) denote the Euler constant and \( \zeta \) the Riemann zeta function.

**Proposition 1.4.** Let \( M \) be a Zoll manifold of dimension \( d \in \mathbb{N} \). Let \( P_n \) be as above and assume that \( G \in \Psi^0(M) \). For \( d = 1, 2 \), assume in addition that (1.17) holds, and let \( C(G) \) be defined by (1.18). Then the following holds as \( n \to \infty \),

(i) for \( d = 1 \),

\[
\text{Tr} \ P_n G P_n = n \cdot R_0(G) + \log n \cdot R_1(G) + \left( C(G) + \gamma R_1(G) + \sum_{l=2}^{\infty} \zeta(l) R_l(G) \right) + O \left( \frac{1}{n^2} \right),
\]

(ii) for \( d = 2 \),

\[
\text{Tr} \ P_n G P_n = n^2 \cdot \frac{1}{2} R_0(G) + n \cdot \left( \frac{1}{2} R_0(G) + R_1(G) \right) + \log n \cdot R_2(G) + \left( C(G) + \gamma R_2(G) + \sum_{l=2}^{\infty} \zeta(l) R_{l+1}(G) \right) + O \left( \frac{1}{n^2} \right),
\]

(iii) for \( d \geq 3 \),

\[
\text{Tr} \ P_n G P_n = n^d \cdot \frac{1}{d} R_0(G) + n^{d-1} \cdot \left( \frac{1}{2} R_0(G) + \frac{1}{d-1} R_1(G) \right) + n^{d-2} \cdot \left( \frac{d-1}{12} R_0(G) + \frac{1}{2} R_1(G) + \frac{1}{d-2} R_2(G) \right) + \log n \cdot R_d(G) + O(n^{d-3}).
\]

**Remark 1.6.** The coefficients of \( n^d \), \( n^{d-1} \) and \( \log n \) for \( d \geq 2 \) can be found in (GO2 after Lemma 0.2]. From Proposition 1.4 with \( G = \log B \) we see that \( \text{Tr} \ P_n (\log B) P_n \) in Theorem 1.3 and 1.4 contributes to the leading asymptotic term of order \( n^d \), and also to lower order terms of order \( n^j \), \( j = d - 1, \ldots, 1, 0, -1, \ldots \), and to the logarithmic term \( \log n \), as \( n \to \infty \). In the classical SSLT the situation is much simpler: \( \log B \) is just the Toeplitz matrix of the operator of multiplication by \( \log b \), and so \( \text{Tr} \ P_n (\log B) P_n = (2n + 1)(\log b)_0 \).

Now we are ready to state two corollaries.

**Corollary 1.5.** Let \( B \in \Psi^0(S^1) \) have a strictly positive principal symbol and with a sufficiently small symbolic norm \( |||I - B|||_1 \). Assume also that (1.17) holds. Then the following holds as \( n \to \infty \),

\[
\log \det P_n B P_n = c_1 \cdot n + c_{\log} \cdot \log n + c_0 + c_{-1} \cdot \frac{1}{n} + O \left( \frac{1}{n^2} \right),
\]
where the coefficients are the sums of the corresponding coefficients from Theorem 1.3 and Proposition 1.4(i) for $G = \log B$.

Assume further that $\sigma_0(B)$ and $\text{sub}(B)$ do not depend on the direction of $\xi$, that is $\sigma_0(B)(x, \xi) = b_0(x)$ and $\text{sub}(B)(x, \xi) = b_{\text{sub}}(x) |\xi|^{-1}$, for $(x, \xi) \in S^*\mathbb{S}^1$. Assume also that $b_2 = 0$. Then the following holds as $n \to \infty$,

\begin{equation}
\log \det P_n B P_n = n \cdot 2 \int_0^{2\pi} \log b_0(x) \frac{dx}{2\pi} + \log n \cdot 2 \int_0^{2\pi} \frac{b_{\text{sub}}(x)}{b_0(x)} \frac{dx}{2\pi} + \left( \sum_{k=1}^{\infty} k (\log b_0)_k (\log b_0)_{-k} + C(\log B) + \gamma R_1(\log B) + \sum_{l=2}^{\infty} \zeta(l) R_l(\log B) \right) + \frac{1}{n} \left( \sum_{k=1}^{\infty} k (\log b_0)_k (b_{\text{sub}}/b_0)_{-k} + \int_0^{2\pi} \left[ \frac{b_{\text{sub}}(x)}{b_0(x)} + \left( \frac{b_{\text{sub}}(x)}{b_0(x)} \right)^2 \right] \frac{dx}{2\pi} \right) + O\left( \frac{1}{n^2} \right),
\end{equation}

where $C(\log B)$ is given by (1.18).

The proof of (1.13) is an exercise in the calculus of PsDO’s together with Proposition 1.4(i), and is left to the reader. In some simple cases, for instance for $b_{\text{sub}}(x) = \pm \frac{1}{2} b_0(x)$, the left-hand side in (1.19) can be computed explicitly. The coefficients of $n, \log n, \frac{1}{n}$ on the right in (1.19) in these cases are as expected, see also Remark 1.3 below.

**Corollary 1.6.** Let $M$ be a Zoll manifold of dimension $d \geq 2$. Assume that $P_n$ and $A$ are as in Theorem 1.3. Let $B \in \Psi^0(M)$ have a strictly positive principal symbol and a sufficiently small symbolic norm $||I - B||_1$. For $d = 2$, assume in addition (1.17). Then the following holds, as $n \to \infty$,

\begin{equation}
\log \det P_n B P_n = C^{(d)}_d \cdot n^d + C^{(d)}_{d-1} \cdot n^{d-1} + C^{(d)}_{d-2} \cdot n^{d-2} + C^{(d)}_{\log} \cdot \log n + O\left( n^{d-3} \right),
\end{equation}

where the coefficients are the sums of the corresponding coefficients from Theorem 1.3 for $f(z) = \log z$ and Proposition 1.4(ii) or (iii) for $G = \log B$. If one counts the logarithmic term, this expansion is fourth order for $d = 2, 3$ and third order for $d \geq 4$.

**Remark 1.7.** The coefficients $C^{(d)}_d$ and $C^{(d)}_{d-1}$, $d \in \mathbb{N}$, have been found in [3O1, GO2]. The most interesting coefficient in (1.20) is the constant one, since one can think of $\exp C^{(d)}_0$ as a regularized determinant of $B \in \Psi^0(M)$, see [3O1, after (3)] and [O2]. The sum

\[ \gamma R_d(\log B) + \sum_{l=2}^{\infty} \zeta(l) R_{l+d-1}(\log B) \]

will for all $d \in \mathbb{N}$ be a part of $C^{(d)}_0$. For $d = 1$, Corollary 1.3 gives a full expression for $C^{(1)}_0$. For $d = 2$, Corollary 1.6 gives a full expression for $C^{(2)}_0$, which is quite lengthy.
Remark 1.8. Let us compare the result of Corollary 1.3 with a generalization of SSLT to the case of $B$ being an operator of multiplication by a function $b(x)$ having discontinuities which is due to H. Widom and E. Basor in [W8]. In this case $\log b(x)$ also has discontinuities, and so the series $\sum_{k \in \mathbb{Z}} |k| |(\log b)_k|^2$ diverges logarithmically. The following third order asymptotic formula holds for the operator of multiplication by a piecewise $C^2$ function $b(x)$

\begin{equation}
\log \det P_n BP_n = a_1 \cdot n + a_2 \cdot \log n + a_3 + o(1), \quad n \to \infty,
\end{equation}

where $a_1$ as in (1.19), the coefficient $a_2$ has been computed by H. Widom in [W6], and the constant term $a_3$ has been found by E. Basor in [B]. Note that the matrix $B$ in (1.21) is still Toeplitz, the logarithmic order of the subleading term being due to a slower decay of the Fourier coefficients of $b(x)$. In our case the matrix of the operator $B \in \Psi^0(S^1)$ is not Toeplitz (see Remark 1.9 below), and the $\log n$ term comes from the contribution of sub($B$).

It would be interesting to find a compact formula for the constant term in (1.19). We mention that the constant $a_3$ in (1.21) found in [B] has a form similar to the one in (1.19). It contains a “finite” term and an infinite series of certain integrals multiplied by the values of the Riemann zeta function at the points 3, 5, · · · . Interestingly, an “invariant” form of that series has been found in [W8]. It is written as a single integral involving the function

$$
\Psi(x) := \frac{d}{dx} \log \Gamma(x).
$$

This gives the hope that a similar formula can be found for the constant in (1.19).

Remark 1.9. The matrix interpretation of Corollary 1.3 is as follows. Assume for simplicity that $B \in \Psi^0(S^1)$ is as in the second part of Corollary 1.3, that is $\sigma_0(B)(x, \xi) = b_0(x)$ and sub($B$)$^{-1}(x, \xi) = b_{sub}(x)|\xi|^{-1}$, for all $(x, \xi) \in S^*S^1$. Assume also that $b_{-2} = b_{-3} = \cdots = 0$. Let $B_0$ and $B_{sub}$ be the operators of multiplication by $b_0$ and $b_{sub}$, respectively. Let $D$ be the linear operator in $L^2(S^1)$ such that

$$
D e^{ikx} = \begin{cases} 
\frac{1}{|k|} e^{ikx}, & |k| \geq 1 \\
0, & k = 0.
\end{cases}
$$

Note that this is not a differential, but rather a smoothing operator of order $-1$. There is known a correspondence between the classical PsDO’s on the circle and their discrete counterparts, see [K] for details. By that correspondence, the zeroth order PsDO $B$ we started with equals $B_0 + B_{sub}D$. Introduce two Toeplitz matrices, $\hat{B}_0 := \{\{b_0\}_{j, k} \}_{j, k \in \mathbb{Z}}$ and $\hat{B}_{sub} := \{\{b_{sub}\}_{j, k} \}_{j, k \in \mathbb{Z}}$. Set also $\hat{D} := \text{diag}(\cdots, \frac{1}{2}, 1, 0, 1, \frac{1}{2}, \cdots)$. Then the matrix representation of $B_0 + B_{sub}D$ is $\hat{B}_0 + \hat{B}_{sub} \cdot \hat{D}$. Finally, set $\tilde{P}_n = \text{diag}(\cdots, 0, 1, \cdots, 1, 0, \cdots)$ ($2n + 1$ ones). Then Corollary 1.3 gives a fourth order asymptotics for the determinant of the truncated matrix $\tilde{P}_n \cdot (\hat{B}_0 + \hat{B}_{sub} \cdot \hat{D}) \cdot \tilde{P}_n$.

Now we can reformulate the question of finding the constant term in (1.19) in purely matrix terms. Drop the hats and the dots for brevity. Let $C_1$ be a Toeplitz matrix that corresponds to the operator of multiplication by $b_{sub}/b_0$, and let the matrix $D$ be as above. Clearly, the matrices $C_1$ and $D$ do not commute. Assume that the matrix $\log(I - C_1D)$ is well-defined. The question is to compute the constant coefficient in $\text{Tr} P_n \log(I - C_1D) P_n$, or which is the same, the constant
coefficient in
\[ \text{Tr } P_n \log(I - D^{1/2}C_1D^{1/2})P_n, \quad n \to \infty. \]
As we have noticed in Remark 1.4, this question is trivial for a Toeplitz matrix \( T \) in place of \( D^{1/2}C_1D^{1/2} \).

1.3. A related result for the maximum of a random walk. Let us explain how we can use the \( j \)-maps \( \Phi_j \), \( j \in \mathbb{N} \), to write the bivariate characteristic function of the maximum of a random walk and its position at a smaller time. There is a lack of symmetricity in this problem, and this case is not considered in [3]. Let \( X_1, X_2, \cdots \) be independent real valued random variables which assume real values and have the same distribution density \( \phi(X) \). Assume for simplicity that \( \phi \) is Schwartz class. (The result below holds for much more general \( \phi \), e.g., for discrete random variables, if understood in the sense of distributions.) In our case the characteristic function
\[ \mathbb{E}\{e^{itX}\} := \int_{-\infty}^{\infty} e^{itX} \phi(X) \, dX \]
is well-defined and invertible as a Fourier transform. Let \( S, \) \( \Theta \), interval \( 0 \) which is the length of the maximal excursion to the right of 0 during the time interval \( 0, \cdots, p \). Then for \( q \in \mathbb{N} \) introduce the random variable
\[ T_{p+q} := \max(0, S_1, \cdots, S_p, S_{p+1}, \cdots, S_{p+q}), \]
which is the length of the maximal excursion to the right of 0 during the time interval \( 0, \cdots, p+q \). Note that the time \( p+q \) is strictly larger than \( p \).

For arbitrary \( j, k \in \mathbb{N} \) and \( y_1, \cdots, y_j, z_1, \cdots, z_k \in \mathbb{R} \) we set
\[ \Omega_{j,k}(y_1, \cdots, y_j, z_1, \cdots, z_k) := (y_1)_+ + \cdots + (y_j)_+ + \cdots + (z_k)_+. \]
Denote
\[ \hat{\phi}(\eta) := \int_{-\infty}^{\infty} e^{-i\eta y} \phi(y) \, dy, \quad \mathcal{F}^{-1}_{\eta \to y}[f(\eta)] := (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\eta y} f(\eta) \, d\eta. \]

**Theorem 1.7.** Let \( \phi, X_p, S_p, \) and \( T_{p+q}, p, q \in \mathbb{N} \), be as above. Let \( \Phi_j, j \in \mathbb{N} \), be as in \([1, 4]\). Then for \( |a|, |b| < 1 \) the following holds
\[ \sum_{p,q=1} a^p b^q \mathbb{E}\left\{ e^{iaS_p + ibT_{p+q}} \right\} = \sum_{j,k=1}^{\infty} \frac{1}{j!} \frac{1}{k!} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{ia(y_1 + \cdots + y_j) + ib\Omega_{j,k}(y_1, \cdots, y_j, z_1, \cdots, z_k)} \times \mathcal{F}^{-1}_{\eta_1 \to y_1, \cdots, \eta_j \to y_j}[\Phi_j \left[ \frac{as}{1 - as} \right] \hat{\phi}(\eta_1), \cdots, \hat{\phi}(\eta_j)] \times \mathcal{F}_{\zeta_1 \to z_1, \cdots, \zeta_k \to z_k}[\Phi_k \left[ \frac{bt}{1 - bt} \right] \hat{\phi}(\zeta_1), \cdots, \hat{\phi}(\zeta_k)] \, dy_1 \cdots dy_j \, dz_1 \cdots dz_k. \]
In this formula, the \( j \)-maps \( \Phi_j \) and \( \Phi_k \) act on the function of \( s \) and \( t \), respectively. The arguments of these actions are the values of \( \hat{\phi} \) at the corresponding points.

The proof for fixed \( p, q \in \mathbb{N} \) is carried out analogously to the computations in Section 5, see [2, Section 1.10] for details.
1.4. Organization of the article. The paper is organized as follows. In Section 2, we recall the method of \([\text{GO2}]\), estimate the remainder after the third Szegö term, and justify the passage from the set of polynomials to an analytic function \(f\). After that we deal with an arbitrary monomial \(f(z) = z^m, m \in \mathbb{N}\). An operator 

\[ B_{\kappa_1} \cdots B_{\kappa_m} \in \Psi^0(M), \kappa_1, \cdots, \kappa_m \in \mathbb{Z}, \]

arises in that computation (see (2.3) below for the definition of the Fourier coefficient \(B_{\kappa}, \kappa \in \mathbb{Z}\)). In Section 3 we compute the contribution of \(\sigma_0(B_{\kappa_1} \cdots B_{\kappa_m}), m \geq 2\), to higher order Szegö terms. The resulting expression involves only \(\sigma_0(B)\). In Section 4, we calculate the contribution of the symmetric part of \(\text{sub}(B_{\kappa_1} \cdots B_{\kappa_m}), m \geq 2\), to the third Szegö term. It depends on \(\sigma_0(B)\) and \(\text{sub}(B)\). In Section 5, we compute the contribution of the non-symmetric part of \(\text{sub}(B_{\kappa_1} \cdots B_{\kappa_m}), m \geq 2\), to the third Szegö term, which involves Poisson brackets of the principal symbol of \(B\) shifted along by geodesic flow. This contribution depends only on \(\sigma_0(B)\). In Section 6, a proof of Proposition 1.4 is given. In Section 7, we find an expression for \(\Phi_0(j), j \in \mathbb{N}\), and also for \(W_3[j]\), for an arbitrary \(f \in A_1\). In Section 8, the auxiliary combinatorial background is given. The results of this paper (Theorem 1.1, 1.3 and 8.2, Proposition 1.4 and Corollary 1.5 and 1.6) were announced in \([\text{Gi2}]\), where we also gave an outline of the proofs.

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2. The full third Szegö term and the remainder estimate

2.1. The abstract scheme from \([\text{GO2}]\). We start by expanding the analytic function \(f(z)\) in a power series about 0 and proving (1.10) for \(f(z) = z^m\) for an arbitrary \(m \in \mathbb{N}\). After that we justify the passage from the set of polynomials to analytic functions. Let us recall the method of \([\text{GO2}]\). Let \(\pi_k, k \in \mathbb{N}\), be the projection on the \(k\)th eigenspace of the operator \(A\) and set \(\pi_k := 0\) for \(k \leq 0\). Then 

\[ P_n = \sum_{k=1}^{n} \pi_k \quad \text{for} \quad n \in \mathbb{N}, \quad \text{and we set} \quad P_n := 0, \quad n \leq 0. \]

For an arbitrary \(B \in \Psi^0(M)\) and \(t \in \mathbb{R}\) introduce the operator 

\[ B^t := e^{-itA} Be^{itA}. \]

By Egorov’s theorem, \(B^t \in \Psi^0(M)\), and also 

\[ \sigma_0(B^t)(x, \xi) = \sigma_0(B)(\Theta^t(x, \xi)), \]

where \(\Theta^t\) stands for the shift by \(t\) units along the geodesic flow. Note that because \(\text{sub}(A) = \text{const}\) the following also holds \([\text{Gi}]\) 

\[ \text{sub}(B^t)(x, \xi) = \text{sub}(B)(\Theta^t(x, \xi)). \]

Because \(\text{spec}(A) = \mathbb{N}\), the operator \(B^t\) is periodic in \(t\) with period \(2\pi\). Therefore we can introduce the Fourier expansion \(B = \sum_{\kappa \in \mathbb{Z}} B_{\kappa}\) where \(B_{\kappa} \in \Psi^0(M), \kappa \in \mathbb{Z}\),
is defined by
\begin{equation}
B_\kappa = \sum_{k=1}^{\infty} \pi_{k+\kappa} B \pi_k = \frac{1}{2\pi} \int_0^{2\pi} e^{i\kappa t} e^{-itA} Be^{itA} dt.
\end{equation}

This together with (2.1) and (2.2) implies

**Lemma 2.1.** For any \( B \in \Psi^0(M) \) and \( \kappa \in \mathbb{Z} \)
\[
\sigma_0(B_\kappa)(x, \xi) = \int_0^{2\pi} e^{i\kappa t} \sigma_0(B)(\Theta^t(x, \xi)) \frac{dt}{2\pi}
\]

and
\[
\operatorname{sub}(B_\kappa)(x, \xi) = \int_0^{2\pi} e^{i\kappa t} \operatorname{sub}(B)(\Theta^t(x, \xi)) \frac{dt}{2\pi}.
\]

For \( m \in \mathbb{N} \) and \( \kappa_1, \cdots, \kappa_m \in \mathbb{Z} \) introduce the notation
\begin{equation}
M_m(\kappa) := \min(0, \kappa_1, \kappa_1 + \kappa_2, \cdots, \kappa_1 + \cdots + \kappa_m).
\end{equation}

Now using the remarkable commutation relation
\begin{equation}
B_\kappa P_n = P_{n+\kappa} B_\kappa, \quad n \in \mathbb{N}, \ \kappa \in \mathbb{Z},
\end{equation}
we move all the projectors to the left in the expression
\[
(P_n B P_n)^m = \sum_{\kappa_1, \cdots, \kappa_m} P_n B_{\kappa_1} P_n B_{\kappa_2} P_n \cdots P_n B_{\kappa_m} P_n
\]
obtaining \( P_n B^m P_n \) plus another term. This implies for all \( n \in \mathbb{N} \)
\[
\text{Tr}(P_n B P_n)^m - \text{Tr} P_n B^m P_n
\]
\begin{equation}
= - \sum_{\kappa_1 + \cdots + \kappa_m = 0} \text{Tr} \left( (P_n - P_{n+\kappa_1} \cdots P_{n+\kappa_1 + \cdots + \kappa_m}) B_{\kappa_1} \cdots B_{\kappa_m} \right)
\end{equation}
\begin{equation}
= - \sum_{\kappa_1 + \cdots + \kappa_m = 0} \sum_{j = M_m(\kappa) + 1}^0 \text{Tr}(\pi_{n+k} B_{\kappa_1} \cdots B_{\kappa_m}).
\end{equation}

**Remark 2.1.** The relation (2.3) follows readily after writing both sides using the definition (2.3). We note that the Fourier coefficients \( B_\kappa \) are PsDO’s even if \( B \) is a multiplication operator. Also even in the simplest case \( M = S^1, B = b(x), B_\kappa \neq \hat{b}_\kappa e^{i\kappa x} \). This explains the non-symmetricity of (2.4) with respect to positive and negative \( \kappa \).

Next, for any \( G \in \Psi^0(M) \), \( M \) being a Zoll manifold, there exists a full asymptotic expansion for \( \text{Tr}(\pi_k G) \), as \( k \to \infty \), see Lemma 2.2 below. This result is due to Y. Colin de Verdière [CdV]. The coefficients in that expansion are certain Guillemin–Wodzicki residues. Recall that for any compact closed manifold \( M \) of dimension \( d \in \mathbb{N} \) the Guillemin–Wodzicki residue of a pseudodifferential operator \( G \in \Psi^m(M) \) of order \( m \in \mathbb{Z} \) is defined by
\begin{equation}
\text{Res}(G) := \int_{S^*M} \sigma_{-d}(G)(x, \xi) \, dx d\xi
\end{equation}
(recall that we have included \((2\pi)^{-d}\) in the notation \( dx d\xi \)). For an arbitrary \( G \in \Psi^0(M) \) denote
\begin{equation}
R_l(G) := \text{Res}(A^{-d+\ell} G), \quad l = 0, 1, 2, \cdots.
\end{equation}
Lemma 2.2. Let $M$ be a Zoll manifold of dimension $d \in \mathbb{N}$. Assume $G \in \Psi^0(M)$. Then for any $N = 0, 1, 2, \cdots$, there exists $C_N < \infty$ such that

$$\left| \text{Tr}(\pi_k G) - \sum_{l=0}^{N} k^{d-1-l} R_l(G) \right| \leq C_N k^{d-2-N}, \ k \in \mathbb{N}. \tag{2.9}$$

See [CdV] and [GO2, Appendix] for the proof of (2.9). We need the expressions for the first two residues.

Lemma 2.3. Let $A$ be defined by (1.1). For any $G \in \Psi^0(M)$

$$R_0(G) = \int_{S^* M} \sigma_0(G) \, dx \, d\xi \tag{2.10}$$

and

$$R_1(G) = \int_{S^* M} \left( (d-1)(\alpha/4) \sigma_0(G) + \text{sub}(G) \right) \, dx \, d\xi. \tag{2.11}$$

Proof. The equality (2.10) follows easily from the computation rules for PsDO’s and the fact that $\sigma_1(A) = 1$ on $S^* M$. Next, the equality

$$R_1(G) = \int_{S^* M} \text{sub}(A^{-d+1} G) \, dx \, d\xi. \tag{2.12}$$

follows from the definition of the subprincipal symbol (see, e.g., [DG]), and the fact that the integral over $S^* M$ of a Poisson bracket equals zero. (We could also refer to Proposition 29.1.2 in [Ho4], differentiate with respect to the spectral parameter, make the same remark concerning the integral involving the Poisson bracket.) Now (2.11) follows from (2.12) by the computation rules concerning the subprincipal symbol: by [DG, (1.4)]

$$\text{sub}(A^{-d+1} G) = \text{sub}(A^{-d+1}) \sigma_0(G) + \sigma_{-d+1}(A^{-d+1}) \text{sub}(G)$$

$$+ (2i)^{-1} \{ \sigma_{-d+1}(A^{-d+1}), \sigma_0(G) \},$$

where $\{ , \}$ denotes the Poisson bracket. Now by [DG, (1.3)]

$$\text{sub}(A^{-d+1}) = (-d + 1) (\sigma_1(A))^{-d} \text{sub}(A),$$

and we use the definition (1.1), and that $\text{sub}(-\Delta) = 0$ to complete the proof. \qed

2.2. Beginning of the proof of Theorem 1.3. We do not assume that the Fourier expansion $B = \sum_{\kappa \in \mathbb{Z}} B_{\kappa}$ has only a finite number of terms. Denote

$$\mathcal{B} := (x_1, \cdots, x_m), \quad B_{\mathcal{B}} := B_{x_1} \cdots B_{x_m}. \tag{1.3}$$

Below $c(B)$ and $C(B)$ will denote various constants which depend only on $B$ (and not on $m$). We will indicate which seminorm of $B$ enters a certain $C(B)$ when necessary. For any $M > 0$ denote

$$B^{(M)} := \sum_{|\kappa| \leq M} B_{\kappa}. \tag{2.13}$$

Introduce for $m \in \mathbb{N}$, $M > 0$ the set of indices

$$Q^m(M) := \{(x_1, \cdots, x_m) : |x_l| \leq M, l = 1, \cdots, m\}. \tag{1.4}$$
We split the right-hand side of (2.6) into the following three sums. The first sum is

\[ - \sum_{\varpi \in Q^n \cap \{ n/(2m) \}} \sum_{j = M_m(\varpi) + 1}^{0} \left( (n + j)^{d-1} \Res(A^{-d}B_{\varpi}) + (n + j)^{d-2} \Res(A^{1-d}B_{\varpi}) \right), \]  

the second

\[ - \sum_{\varpi \in Q^n \cap \{ n/(2m) \}} \sum_{j = M_m(\varpi) + 1}^{0} \left[ \Tr(\pi_{n+j} B_{\varpi}) - (n + j)^{d-1} \Res(A^{-d}B_{\varpi}) - (n + j)^{d-2} \Res(A^{1-d}B_{\varpi}) \right], \]

and the third

\[ - \sum_{\varpi \in Q^n \cap \{ n/(2m) \}} \sum_{j = M_m(\varpi) + 1}^{0} \Tr(\pi_{n+j} B_{\varpi}). \]

The contributions to the second and third Szegő term come from (2.14), the expressions (2.15) and (2.16) having a lower order, as \( n \to \infty \). In the rest of this section we estimate the remainder after the third Szegő term. It is very important that the symbol of the operator \( B \) of any order is smooth. We will repeatedly refer to subsection 2.7 where the most technical part of the computation is carried out.

2.3. Computation of the sum (2.14). Let us single out the terms involving \( n^{d-1} \) and \( n^{d-2} \) in (2.14). Then the latter splits into the sum of

\[ - \sum_{\varpi \in Q^n \cap \{ n/(2m) \}} \sum_{j = M_m(\varpi) + 1}^{0} \left[ n^{d-1} \Res(A^{-d}B_{\varpi}) + n^{d-2} \left( (d - 1) \cdot j \cdot \Res(A^{-d}B_{\varpi}) + \Res(A^{1-d}B_{\varpi}) \right) \right], \]

and the remainder

\[ - \sum_{\varpi \in Q^n \cap \{ n/(2m) \}} \sum_{j = M_m(\varpi) + 1}^{0} \left[ \sum_{k=0}^{d-3} n^k \binom{d-1}{k} j^{d-1-k} \Res(A^{-d}B_{\varpi}) + \sum_{k=0}^{d-3} n^k \binom{d-2}{k} j^{d-2-k} \Res(A^{1-d}B_{\varpi}) \right]. \]

Taking into account Lemma 2.3 for \( G = B_{\varpi} \) and the formulas

\[ \sum_{j = M_m(\varpi) + 1}^{0} 1 = -M_m(\varpi), \quad \sum_{j = M_m(\varpi) + 1}^{0} j = -\left( \frac{M_m(\varpi)}{2} \right)^2 + M_m(\varpi) \]
we conclude that $\mathcal{P}$ equals

$$
\sum_{\pi \in \mathcal{Q}^m(\pi, n/2m)} n^{d-1} \cdot M_m(\pi) \int_{S^*M} \sigma_0(B_{\pi}) \, dx d\xi
$$

(2.19)

$$
+ n^{d-2} \left[ \frac{d-1}{2} \left( (M_m(\pi))^2 + (1 + \frac{\alpha}{2}) M_m(\pi) \right) \int_{S^*M} \sigma_0(B_{\pi}) \, dx d\xi
+ M_m(\pi) \int_{S^*M} \operatorname{sub}(B_{\pi}) \, dx d\xi \right].
$$

In Section 3, the expression

$$
\sum_{\kappa_1, \ldots, \kappa_m = 0} (M_m(\pi))^n \int_{S^*M} \sigma_0(G_{\kappa_1} \cdots G_{\kappa_m}) \, dx d\xi
$$

(2.20)

for $n \in \mathbb{N}$ and any $G \in \Psi^0(M)$ is computed (we set $G := B^{(n/2m)}$). The important point here is that both the domain of summation and the second factor in (2.20) are symmetric, and so we can symmetrize the first factor. The integral in (2.20) is symmetric with respect to $\kappa_1, \ldots, \kappa_m$ because already the integrand is symmetric

$$
\sigma_0(G_{\kappa_1} \cdots G_{\kappa_m}) = \sigma_0(G_{\kappa_1}) \cdots \sigma_0(G_{\kappa_m}).
$$

The gHD for $n = 1, 2$ (Theorem 8.2) is needed in this computation.

In Section 4 and 5, the expression

$$
\sum_{\kappa_1, \ldots, \kappa_m = 0} M_m(\pi) \int_{S^*M} \operatorname{sub}(G_{\kappa_1} \cdots G_{\kappa_m}) \, dx d\xi
$$

for any $G \in \Psi^0(M)$ is computed (we set $G := B^{(n/2m)}$). Note that

$$
\operatorname{sub}(G_{\kappa_1} \cdots G_{\kappa_m}) = \sum_{k=1}^m \operatorname{sub}(G_{\kappa_k}) \prod_{\substack{\pi = 1 \atop \pi \neq k}}^m \sigma_0(B_{\pi})
$$

(2.21)

$$
+ \frac{1}{2i} \sum_{1 \leq k < l \leq m} \{ \sigma_0(G_{\kappa_k}), \sigma_0(G_{\kappa_l}) \} \prod_{\substack{\pi = 1 \atop \pi \neq k, \pi \neq l}}^m \sigma_0(B_{\pi})
$$

where the first sum is symmetric with respect to $\kappa_1, \ldots, \kappa_m$, and the second one is generally speaking not (not even after the integration over $S^*M$). Because of that circumstance the original method of symmetrization $\mathfrak{S}$ fails, and we have to modify it. It turns out that each of the $m(m-1)/2$ terms in the second sum in (2.21) possesses a partial symmetry. For instance, if $1 \leq r$, $r + 2 \leq s$ and $s + 2 \leq m$ then in the expression

$$
\sigma_0(G_{\kappa_1} \cdots G_{\kappa_r}) \{ \sigma_0(G_{\kappa_{r+1}}), \sigma_0(G_{\kappa_{s+1}}) \} \sigma_0(G_{\kappa_{s+2}} \cdots G_{\kappa_m})
$$

the indices $\kappa_1, \ldots, \kappa_r$ can be permuted without changing the resulting expression, and the same holds for the groups of indices $\kappa_{s+2}, \ldots, \kappa_s$ and $\kappa_{s+2}, \ldots, \kappa_m$ (and we are even allowed to interchange the indices within the three groups). However we can neither interchange any index from any of the three groups with $\kappa_{r+1}, \kappa_{s+1}$, nor interchange the latter two. Here the original form of the BSt (Theorem 8.1) is needed.
Remark 2.2. In view of (2.6) and Lemma 2.2, the computation of all terms in the Szegő asymptotics is reduced to the evaluation of the following expression for different \(n \in \mathbb{N}\) and \(l = 0, 1, 2, \cdots\),

\[
\sum_{\kappa_1 + \cdots + \kappa_m = 0} (M_m(\kappa_1, \cdots, \kappa_m))^n \text{Res}(A^{-d+l}B_{\kappa_1} \cdots B_{\kappa_m}).
\]

The gHD (Theorem 8.2) makes the computation possible for any \(n \in \mathbb{N}\), provided that the second factor in (2.22) is symmetric with respect to \(\kappa_1, \cdots, \kappa_m\). The problem is that this is the case generally speaking only for \(l = 0\). The case of \(l = 1\) and \(n = 1\) is dealt with in Section 4 and 5.

We would like to mention that in the work [RR] by R. Roccaforte an interesting way to rewrite expressions of the type (2.22) with non-symmetric (matrix-valued) second factor has been suggested. The idea is to consider all the cases when the minimum is attained on the sum

\[
\kappa_1 + \cdots + \kappa_j, \quad j = 1, \cdots, m - 1,
\]

and make certain changes of summation indices. More precisely, for an operator \(B \in \Psi^0(M)\) written as a Fourier series \(B = \sum_{\kappa \in \mathbb{Z}} B_\kappa\) following [RR] we set

\[
\begin{align*}
B_- &:= \sum_{\kappa < 0} B_\kappa, & \quad B_{j+}^1 &:= (B \ast B_j^1)_-, & \quad j &\in \mathbb{N}, \\
B_+ &:= \sum_{\kappa \geq 0} B_\kappa, & \quad B_{j+}^1 &:= (B_j^1 \ast B)_+, & \quad j &\in \mathbb{N},
\end{align*}
\]

where \(\ast\) denotes the discrete convolution on the Fourier series side. The order of the operators being convolved is important. Then (2.22) can be rewritten as

\[
\sum_{\kappa} (-\kappa)_-^n \text{Res}\left[A^{-d+l} \sum_{j=1}^{m-1} \left(B_j^1\right)_- (B_{j+}^m - B)ight].
\]

Note that the summation in (2.23) is over a single variable \(\kappa\). This reminds the expression which appears in the usual second Szegő term [K, GO2]. Also the formulas from [W2] have a similar structure. However the formula (2.23) for \(f(z) = z^m\) is not very explicit, and we could not write a reasonable formula for an arbitrary analytic \(f(z)\) starting from (2.23) (nor was it done in [RR]).

Bringing together the results of Section 3, 4, and 5, we conclude that (2.19) equals

\[
\sum_{\kappa} (\kappa)_-^{n-1} \cdot \Upsilon_2[z^m](B^{(n/(2m))}) + \sum_{\kappa} (\kappa)^{n-2} \cdot \Upsilon_3[z^m](B^{(n/(2m))}).
\]

This in turn equals

\[
n^{d-1} \cdot \Upsilon_2[z^m](B) + n^{d-2} \cdot \Upsilon_3[z^m](B)
\]

plus an error whose absolute value does not exceed

\[
n^{d-1} \left| \Upsilon_2[z^m](B^{(n/(2m))}) - \Upsilon_2[z^m](B) \right| + n^{d-2} \left| \Upsilon_3[z^m](B^{(n/(2m))}) - \Upsilon_3[z^m](B) \right|, \quad n \in \mathbb{N}.
\]

We need the following auxiliary statement which is proved in subsection 2.7.
Lemma 2.4. For \( l \in \{2,3\} \), an arbitrary \( B \in \Psi^0(M) \) and any \( L \in \mathbb{N} \) there exist two constants \( c_{l,L}(B) \) and \( C_l(B) \) such that for \( m \in \mathbb{N}, m \geq 2 \)
\[
\left| \mathcal{Y}_l[z^m](B^{(M)}) - \mathcal{Y}_l[z^m](B) \right| \leq c_{l,L}(B) \cdot m^2 \cdot (C_l(B))^m \cdot M^{-L}, \quad M \geq 1.
\]

It follows from the proof of Lemma 2.4 that \( C_l(B) \), \( l = 2,3 \), involves \( \|\nabla^6 \sigma_0(B)\|_\infty \) and \( \|\nabla^2 \text{sub}(B)\|_\infty \), but does not depend on \( L \). Applying Lemma 2.4 with \( M = n/(2m) \) and \( l = 2, L = 2 \), respectively, \( l = 3, L = 1 \) to the first, respectively, second term in (2.25), we estimate the latter by
\[
n^{d-3} \cdot c \cdot n^2 \cdot (C_l(B))^m, \quad n \in \mathbb{N},
\]
where \( C_l(B), l = 2,3 \), depends on \( \|\nabla^6 \sigma_0(B)\|_\infty \) and \( \|\nabla^2 \text{sub}(B)\|_\infty \).

We finish this subsection by showing that the absolute value of the remainder (2.18) can be estimated, for \( n \in \mathbb{N} \), by
\[
n^{d-3} \cdot c(B) \cdot m \cdot (C_l(B))^m.
\]
For instance, the absolute value of each of the \( d-2 \) terms in the first sum in the square bracket in (2.18) does not exceed
\[
n^{d-3} c_d \sum_{|\kappa_1|, \ldots, |\kappa_m| \in \mathbb{Z}^m} |M_m(\kappa)|^d \cdot |\text{Res}(A^{-d}B_{\kappa_1} \cdots B_{\kappa_m})| \leq n^{d-3} c_d \sum_{|\kappa_1|, \ldots, |\kappa_m| \in \mathbb{Z}^m} (1 + |\kappa_1|)^d \cdots (1 + |\kappa_m|)^d
\]
\[
\times \int_{S^* M} d\xi |\sigma_0(B_{\kappa_1})| \cdots |\sigma_0(B_{\kappa_m})| \leq n^{d-3} c(B) \cdot m \cdot (C_l(B))^m, \quad n \in \mathbb{N}.
\]
The last inequality is due to the fast decay of the maximum over \( S^* M \) of \( |\sigma_0(B_{\kappa})| \), as \( |\kappa| \to \infty \) (smoothness of \( \sigma_0(B) \) and Egorov’s theorem), see subsection 2.7 for details. We remark also that in (2.26) \( C_l(B) \) depends on \( \|\nabla^{d+2} \sigma_0(B)\|_\infty \). In the same way one shows that the absolute value of each of the \( d-2 \) terms in the second sum in the square bracket in (2.18) is estimated by the right-hand side of (2.26).

2.4. Computation of the sum (2.16). For any \( \kappa \in \mathbb{Z}^m \setminus Q_m(n/(2m)) \) the absolute value of at least one of the components of \( \kappa \) is \( > n/(2m) \). Redenote it by \( \kappa_1 \).

We also note that \( j \geq -n \) in (2.16) (otherwise \( \pi_{n+j} = 0 \)). Then the absolute value of the sum (2.16) does not exceed
\[
m \sum_{j=-n}^{0} \left| \sum_{\kappa \in \mathbb{Z}^m \mid |\kappa_1| > n/(2m)} \sum_{\kappa_1 + \cdots + \kappa_m = 0} \text{Tr}(\pi_{n+j} B_{\kappa_1} \cdots B_{\kappa_m}) \right|
\]
\[
\leq n^d \sum_{j=-n}^{0} \sum_{\kappa \in \mathbb{Z}^m \mid |\kappa_1| > n/(2m)} c_d(n+j)^{d-1} \|B_{\kappa_1}\| \cdots \|B_{\kappa_m}\|
\]
\[
\leq n^d \cdot m \cdot c_d \sum_{|\kappa_1| > n/(2m)} \|B_{\kappa_1}\| \left( \sum_{\kappa \in \mathbb{Z}^m, \kappa_1 + \cdots + \kappa_m = 0} \|B_{\kappa}\| \right)^{m-1},
\]
where the factor $c_d(n + j)^{d-1}$ is an estimate of the multiplicity of the eigenvalue $n + j$ and $\| \cdot \|$ is the operator norm in $L^2(M)$. By (2.39) below, there is a constant $c(B)$ such that
\[
\sum_{|x_1| > n/(2m)} \| B_{x_1} \| \leq c(B) \left( \frac{n}{2m} \right)^{-3}, \quad n \in \mathbb{N}.
\]
Set $C(B) := \sum_{\mu \in \mathbb{Z}} \| B_{\mu} \|$. By (2.41) below, $C(B) \leq \| B \| + (\pi^2/3) \|[A, [A, B]]\|$.
It follows now from (2.27) that the absolute value of the sum (2.16) is estimated, for $n \in \mathbb{N}$, by
\[
n^{-3} \cdot c_d(c(B) \cdot m^4 \cdot (C(B))^m).
\]

2.5. Computation of the sum (2.15). For an arbitrary $G \in \Psi^0(M)$ and any $\mu \in \mathbb{N}$ denote
\[
T_\mu G := \mu^{-d+3} \left[ \text{Tr}(\pi_\mu G) - \mu^{-d-1} \text{Res}(A^{-d}G) - \mu^{-d-2} \text{Res}(A^{1-d}G) \right].
\]
We wish to prove that there exist two constants $c(B)$ and $C(B)$ so that the limit, as $n \to \infty$, of the absolute value of (2.15) divided by $n^{-3}$ is $\leq c(B) \cdot m \cdot (C(B))^m$.
Let us estimate the absolute value of (2.15) divided by $n^{-3}$ by
\[
\sum_{\pi \in Q^m(n/(2m))} \sum_{j=M(n/2)}^0 \left( \frac{n + j}{n} \right)^{d-3} \left| T_{n+j}B_{\pi} \right|
\leq \sum_{\pi \in Q^m(n/(2m))} \sum_{j=M(n/2)+1}^0 \left| T_{n+j}B_{\pi} \right|
\]
where $\pi \in Q^m(n/(2m))$. We need to interchange the $\lim_{n \to \infty}$ to the right-hand side in (2.29) and the sums over infinitely growing sets. To justify that we find a summable over $\pi$ and $j$ majorant and then refer to the Lebesgue dominated convergence. We use the principle of uniform boundedness to present a summable majorant.

Introduce a Banach space $X = \Psi^0(M)/\Psi^{-3}(M)$ with a norm $\| \cdot \|_X$ given by
\[
\| G \|_X := \| G \| + \int_{S^* M} \left( |\sigma_{-d}(A^{-d}G)| + |\sigma_{-d}(A^{1-d}G)| \\
+ |\sigma_{-d}(A^{2-d}G)| \right) dx d\xi, \quad G \in X,
\]
where the integrands are well-defined by the definition of the Guillemin–Wodzicki residue (2.7). Then it is clear from the definition (2.28) that each $T_\mu : X \to \mathbb{C}$ is a linear and bounded functional, because the multiplicity of each eigenvalue $\mu$ is finite and
\[
|\text{Res}(A^{1-d}G)| \leq \int_{S^* M} |\sigma_{-d}(A^{1-d}G)| \, dx d\xi, \quad l = 0, 1, 2.
\]
Also by Lemma 2.2, for all $G \in X$ there exists the limit
\[
\lim_{\mu \to \infty} T_\mu G = \text{Res}(A^{2-d}G).
\]
Therefore for all $G \in X$ one has $\sup_{\mu \in \mathbb{N}} |T_\mu G| < \infty$. By the principle of uniform boundedness, there exists a constant $N < \infty$ such that
\[
\sup_{\mu \in \mathbb{N}} |T_\mu G| \leq N \| G \|_X, \quad G \in X.
\]
We finish the construction of the majorant by showing that it is summable. The right-hand side in (2.29) is estimated by

\[
\sum_{m=0}^{\infty} \left| T_{n+j} (B^{(n/2m)} \Psi) \right| = \sum_{m=0}^{\infty} \left| \int_{S^* M} \sigma_{-d} (A^{2^{-d} B_{\Psi}}) \right| dx d\xi 
\]

where the notation (2.13) has been employed. Due to the infinite smoothness of the full symbol \(a(B)\) and Egorov’s theorem, and also by (2.30)

\[
\lim_{n \to \infty} \left| T_{n+j} (B^{(n/2m)} \Psi) \right| = \left| \int_{S^* M} \sigma_{-d} (A^{2^{-d} B_{\Psi}}) \right| dx d\xi. 
\]

Therefore the right-hand side of (2.31) does not exceed

\[
\sum_{m=0}^{\infty} \left| \int_{S^* M} \sigma_{-d} (A^{2^{-d} B_{\Psi}}) \right| dx d\xi \leq c(B) \cdot m \cdot (C(B))^m, 
\]

again due to the smoothness of the symbol of any order of \(B\), see subsection 2.7.

We notice that \(c(B)\) depends on \(\int_{S^* M} \left| \nabla^3 \sigma_{-d} (A^{2^{-d} B}) \right| dx d\xi, \|\nabla^3 \sigma_0(B)\|_\infty\) and \(\|\nabla^3 \text{sub}(B)\|_\infty\).

2.6. End of the proof of Theorem 1.3. We have shown that there exist two constants \(c(B)\) and \(C(B)\) (the latter constant is small together with the norm (1.3)) such that for all \(m \in \mathbb{N}, m \geq 2\), one has

\[
\left| \text{Tr}((P_n B^n P_n)^m) - \text{Tr}(P_n B^n P_n) - n^{d-1} \cdot \Upsilon_2[z^m](B) - n^{d-2} \cdot \Upsilon_3[z^m](B) \right| 
\]

\[
\leq n^{d-3} \cdot c(B) \cdot m^4 \cdot (C(B))^m, \quad n \to \infty. 
\]

Take now an arbitrary function \(f(z) = \sum_{m=1}^{\infty} c_m z^m \in A_1\) which is analytic on a neighborhood of \(\{z : |z| < C(B)\}\). Because the trace is a linear operation and the functionals \(\Upsilon_l, l = 2, 3\), are linear in the first argument, we can write

\[
\left| \text{Tr}(f(P_n B^n P_n)) - \text{Tr}(P_n f(B) P_n) - n^{d-1} \cdot \Upsilon_2[f(z)](B) - n^{d-2} \cdot \Upsilon_3[f(z)](B) \right| 
\]

\[
= \left| \sum_{m=1}^{\infty} c_m \left( \text{Tr}((P_n B^n P_n)^m) - \text{Tr}(P_n B^n P_n) - n^{d-1} \cdot \Upsilon_2[z^m](B) - n^{d-2} \cdot \Upsilon_3[z^m](B) \right) \right|. 
\]
Now we estimate the absolute value of the latter sum by the sum of the absolute values of its terms and employ (2.32). We obtain then
\[
\left| \text{Tr}(f(P_n BP_n)) - \text{Tr}(P_n f(B)P_n) - n^{d-1} \cdot \Upsilon_2[f](B) - n^{d-2} \cdot \Upsilon_3[f](B) \right| \\
\leq n^{d-3} \cdot c(B) \sum_{m=1}^{\infty} |c_m| m^4 \cdot (C(B))^m, \quad n \to \infty,
\]
the series on the right is convergent due to the analyticity of \( f \) on a neighborhood of \( \{z : |z| < C(B)\} \).

### 2.7. Proof of Lemma 2.4

We have to prove that for an arbitrary \( B \in \Psi^0(M) \), \( l = 2, 3 \), and any \( L \in \mathbb{N} \) there exist two constants \( c_{l,L}(B) \) and \( C_l(B) \) so that for \( m \in \mathbb{N} \), \( m \geq 2 \)
\[
\left| \Upsilon_l[z^m](B(M)) - \Upsilon_l[z^m](B) \right| \\
\leq c_{l,L}(B) \cdot m^2 \cdot (C_l(B))^m \cdot M^{-L}, \quad M \to \infty.
\]

We consider the case \( l = 2 \) first. By the definition of \( \Upsilon_2 \)
\[
\left| \Upsilon_2[z^m](B(M)) - \Upsilon_2[z^m](B) \right| \\
= \left| \sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\Xi) \int_{S^*M} \sigma_0(B(B(M) - B \Xi)) \, dx d\xi \right|
\]
We have
\[
B(x_1) \cdots B(x_m) - B(x_1) \cdots B(x_m) = (B(x_1) - B(x_1)) B(x_2) \cdots B(x_m) \\
+ B(x_1) (B(x_2) - B(x_2)) B(x_3) \cdots B(x_m) + \cdots + B(x_1) \cdots B(x_{m-1}) (B(x_m) - B(x_m)).
\]
Let us estimate the integral of the first summand, the other \( m - 1 \) are estimated analogously. Note that \( \left| M_m(\Xi) \right| \leq (1 + |\kappa_1|) \cdots (1 + |\kappa_m|) \). Also because the sum of the indices in (2.33) is zero we have
\[
|\kappa_1| = | - \kappa_2 - \cdots - \kappa_m| \leq |\kappa_2| + \cdots + |\kappa_m|
\]
and hence
\[
\left| M_m(\Xi) \right| \leq (1 + |\kappa_2|)^2 \cdots (1 + |\kappa_m|)^2.
\]
(We do not want \( \kappa_1 \) to appear on the right-hand side in (2.33) below.) Introduce the notation for \( G \in \Psi^0(M) \)
\[
\|\sigma_0(G)\|_{\infty} := \max_{(x, \xi) \in S^*M} |\sigma_0(G)(x, \xi)|.
\]
Hence
\[
\left| \sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\Xi) \int_{S^*M} \sigma_0(B(B(M) - B(x_1)) B(x_2) \cdots B(x_m)) \, dx d\xi \right|
\leq |S^*M| \sum_{\kappa_2, \cdots, \kappa_m \in \mathbb{Z}} (1 + |\kappa_2|)^2 \cdots (1 + |\kappa_m|)^2 \\
\times \|\sigma_0(B^{(M)}_{\kappa_2} \cdots \kappa_m) - B_{\kappa_2} \cdots \kappa_m\|_{\infty} \|\sigma_0(B^{(M)}_{\kappa_2})\|_{\infty} \cdots \|\sigma_0(B^{(M)}_{\kappa_m})\|_{\infty}.
\]
For any \( G \in \Psi^0(M) \) we write for brevity \( g_0 := \sigma_0(G) \), \( g_{\text{sub}} := \text{sub}(G) \), introduce the notation
\[
(\text{Ad} A)(G) := [A, G] \in \Psi^0(M)
\]
and state the following important fact (see [GO2, Lemma 1.3]).

Lemma 2.5. For any $G \in \Psi^0(M)$ and any $\nu \in \mathbb{Z}$

$$\nu G_{\nu} = [A, G]_{\nu},$$

(2.37)

where the right-hand side is the $\nu$th Fourier coefficient of $[A, G]$. Also for any $L \in \mathbb{N}$ there exist finite $c_L(G)$ and $C_L(G)$ such that

$$\|\sigma_0(G_{\nu})\|_\infty \leq C_L \nu^{-L}, \quad |\nu| \to \infty,$$

and

$$\sum_{|\lambda| > M} \|\sigma_0(G_{\lambda})\|_\infty \leq C_L M^{-L}, \quad M \to \infty,$$

and

$$\|\text{sub}(G_{\nu})\| \leq C_L \nu^{-L}, \quad |\nu| \to \infty,$$

and

$$\sum_{|\lambda| > M} \|\text{sub}(G_{\lambda})\|_\infty \leq C_L M^{-L}, \quad M \to \infty,$$

and for the operator norm in $L^2(M)$

$$\sum_{|\lambda| > M} \|G_{\lambda}\| \leq C_L M^{-L}, \quad M \to \infty.$$  

(2.39)

Proof. The equation (2.37) follows readily from the definition (2.36) and the fact that $\pi_\mu$ is the projector on the corresponding to the eigenvalue $\mu \in \mathbb{N}$ eigenspace of the operator $A$. Due to the infinite smoothness of $\sigma_0(G)$ we can repeat the operation (2.37) any finite number of times, and consequently the norm $\|\sigma(G_{\nu})\|_\infty$ decays rapidly, as $|\nu| \to \infty$. By the fast decay of $\|\sigma_0(G_{\lambda})\|_\infty$, as $|\lambda| \to \infty$, for any $L \in \mathbb{N}$ there is a constant $C_L(G)$ such that for all $(x, \xi) \in S^*M$ and $0 \leq t \leq 2\pi$

$$\left|\frac{(\vartheta^{(M)}_t)^{\nu}}{\vartheta^{(M)}_0} - \frac{\vartheta^t}{\vartheta^0}\right| \leq \sum_{|\lambda| > M} \|\sigma_0(G_{\lambda})\|_\infty \leq C_L(G) \cdot M^{-L}, \quad M \to \infty,$$

which together with Lemma 2.1 proves (2.38). The result for the subprincipal symbol also follows from its infinite smoothness and Lemma 2.1. For (2.39), see [GO2, Lemma 1.3].

For any $G \in \Psi^0(M)$ we now define inductively $(\text{Ad } A)^N(G)$, $N \in \mathbb{N}$, and note that

$$(\text{Ad } A)^N(G_{\nu}) = ((\text{Ad } A)^N(G))_{\nu}, \quad \nu \in \mathbb{Z}, \quad N \in \mathbb{N}.$$  

For $N = 1$ this follows readily from (2.36), for $N \geq 2$ we proceed by induction.

Lemma 2.6. For any $H \in \Psi^0(M)$ and $\nu \in \mathbb{Z}$

$$\|\sigma_0(H_{\nu})\|_\infty \leq \|\sigma_0(H)\|_\infty$$

and

$$\|\text{sub}(H_{\nu})\|_\infty \leq \|\text{sub}(H)\|_\infty$$

Proof. Follows from Egorov’s theorem (2.1) and (2.2).
Lemma 2.7. For any \( G \in \Psi^0(M) \)

\[
\sum_{\nu \in \mathbb{Z}} \| \sigma_0(G_\nu) \|_\infty \leq \| \sigma_0(G_0) \|_\infty + \frac{\pi^2}{3} \| \sigma_0((\text{Ad} A)^2(G)) \|_\infty
\]

and

\[
\sum_{\nu \in \mathbb{Z}} \| \text{sub}(G_\nu) \|_\infty \leq \| \text{sub}(G_0) \|_\infty + \frac{\pi^2}{3} \| \text{sub}((\text{Ad} A)^2(G)) \|_\infty
\]

and for the operator norm in \( L^2(M) \)

\[
\sum_{\nu \in \mathbb{Z}} \| G_\nu \| \leq \| G_0 \| + \frac{\pi^2}{3} \| (\text{Ad} A)^2(G) \|.
\]

Proof. Applying (2.37) twice we get

\[ G_\nu = \nu^{-2} [A, [A, G]]_\nu, \quad \nu \neq 0, \]

which implies (2.40) because \( \sum_{\nu \neq 0} \nu^{-2} = \pi^2 / 3 \). For (2.41), see [GO2, Lemma 1.3].

Now we estimate (2.35) by

\[
\sum_{\kappa_1 \in \mathbb{Z}} \| \sigma_0((B(M) - B)_{\kappa_1}) \|_\infty \\
\times \left( \sum_{\kappa_2 \in \mathbb{Z}} (1 + |\kappa_2|)^2 \| \sigma_0(B^{(M)}_{\kappa_2}) \|_\infty \right) \cdots \left( \sum_{\kappa_m \in \mathbb{Z}} (1 + |\kappa_m|)^2 \| \sigma_0(B^{(M)}_{\kappa_m}) \|_\infty \right).
\]

By (2.38) for any \( L \in \mathbb{N} \) there is \( c_{2, L}(B) \) such that, as \( M \to \infty \),

\[
\sum_{\kappa_1 \in \mathbb{Z}} \| \sigma_0((B^{(M)} - B)_{\kappa_1}) \|_\infty = \sum_{|\kappa_1| > M} \| \sigma_0(B_{\kappa_1}) \|_\infty \leq c_{2, L}(B) M^{-L}.
\]

Also

\[
\sum_{\kappa_2 \in \mathbb{Z}} (1 + |\kappa_2|)^2 \| \sigma_0(B^{(M)}_{\kappa_2}) \|_\infty = \sum_{\kappa_2 \in \mathbb{Z}} (1 + 2|\kappa_2| + |\kappa_2|^2) \| \sigma_0(B^{(M)}_{\kappa_2}) \|_\infty
\]

and for example

\[
\sum_{\kappa_3 \in \mathbb{Z}} |\kappa_3|^2 \| \sigma_0(B^{(M)}_{\kappa_3}) \|_\infty = \sum_{|\kappa_3| \leq M} \| \sigma_0(B^2_{\kappa_3}) \|_\infty
\]

\[
= \sum_{|\kappa_3| \leq M} \| \sigma_0((\text{Ad} A)^2(B))_{\kappa_3} \|_\infty
\]

\[
\leq \sum_{\kappa_3 \in \mathbb{Z}} \| \sigma_0((\text{Ad} A)^2(B))_{\kappa_3} \|_\infty =: C(B),
\]

where \( C(B) \) is estimated by \( \| \nabla^4 \sigma_0(B) \|_\infty \), in view of Lemma 2.4. We have proved that

\[
| \Upsilon_2[z^m](B^{(M)}) - \Upsilon_2[z^m](B) | \leq c_{2, L}(B) \cdot m \cdot (C_2(B))^m \cdot M^{-L}, \quad M \geq 1.
\]

where \( C_2(B) \) depends on \( \| \nabla^4 \sigma_0(B) \|_\infty \).
We use (2.34) and estimate for instance the integral of the first of the three parts of (2.33), and is estimated in the same way as above. The second part is
\[
\left| \sum_{\kappa_1 + \cdots + \kappa_m = 0} (M_m(\mathcal{F}))^2 \int_{s^*M} \sigma_0(B^{(M)}_{\mathcal{F}} - B_{\mathcal{F}}) \, dx \, d\xi \right|
\]
and the only difference with the above argument is that here we have \((1 + |\kappa_l|)^4, l = 2, \ldots, m\) in the estimates, and consequently
\[
C(B) := \sum_{\kappa \in \mathbb{Z}} \|\sigma_0((Ad A)^4(B))\|_\infty
\]
can be estimated in terms of \(\|\nabla^6 \sigma_0(B)\|_\infty\), by Lemma 2.7. The absolute value of the third part of \(\mathcal{Y}_3[z^m](B^{(M)}) - \mathcal{Y}_3[z^m](B)\) is estimated by
\[
\left| \sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\mathcal{F}) \int_{s^*M} \text{sub}(B^{(M)}_{\mathcal{F}} - B_{\mathcal{F}}) \, dx \, d\xi \right|
\]
We use (2.34) and estimate for instance the integral of the first of the \(m\) summands
\[
\text{sub} \left( (B^{(M)}_{\kappa_1} - B_{\kappa_1})B^{(M)}_{\kappa_2} \cdots B^{(M)}_{\kappa_m} \right) = \text{sub}(B^{(M)}_{\kappa_1} - B_{\kappa_1})\sigma_0(B^{(M)}_{\kappa_2} \cdots B^{(M)}_{\kappa_m})
\]
\[
+ \sigma_0(B^{(M)}_{\kappa_1} - B_{\kappa_1}) \text{sub}(B^{(M)}_{\kappa_2} \cdots B^{(M)}_{\kappa_m})
\]
\[
+ \frac{1}{2^m} \left\{ \sigma_0(B^{(M)}_{\kappa_1} - B_{\kappa_1}), \sigma_0(B^{(M)}_{\kappa_2} \cdots B^{(M)}_{\kappa_m}) \right\}
\]
Denote \(G := (Ad A)^2(B)\). By Lemma 2.7 for any \(L \in \mathbb{N}\) we obtain
\[
\left| \sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\mathcal{F}) \int_{s^*M} \text{sub}(B^{(M)}_{\kappa_1} - B_{\kappa_1})\sigma_0(B^{(M)}_{\kappa_2} \cdots B^{(M)}_{\kappa_m}) \, dx \, d\xi \right|
\]
\[
\leq c_L \cdot m \cdot (C(G))^m \cdot M^{-L}, \quad M \geq 1.
\]
We note that \(C(G(B))\) can be estimated in terms of \(\|\nabla^2 \text{sub}(B)\|_\infty\) and \(\|\nabla^4 \sigma_0(B)\|_\infty\). Now for the Poisson bracket term in (2.43),

**Lemma 2.8.** For any \(G \in \Psi^0(M)\) and \(L \in \mathbb{N}\) there exist \(c_L(G)\) and \(C_L(G)\) such that
\[
\|\partial_\nu \sigma_0(G_\nu)\|_\infty \leq C_L \nu^{-L}, \quad |\nu| \to \infty
\]
and
\[
\|\partial_\nu \sigma_0(G_\nu)\|_\infty \leq C_L \nu^{-L}, \quad |\nu| \to \infty
\]
and
\[
C_1(G) := \sum_{\kappa \in \mathbb{Z}} \|\partial_\kappa \sigma_0(G_\kappa)\|_\infty < \infty
\]
and
\[
C_2(G) := \sum_{\kappa \in \mathbb{Z}} \|\partial_\kappa \sigma_0(G_\kappa)\|_\infty < \infty
\]

Now we recall (2.24), estimate the absolute value of the Poisson brackets by the products of the supremum norms of the derivatives, employ Lemma 2.8 and conclude that the contribution of the Poisson brackets involves \((C(B))^m\), where \(C(B)\) depends on \(\|\nabla^5 \sigma_0(B)\|_\infty\). Thus for a certain \(C_3(B) < \infty\) and any \(L \in \mathbb{N}\)
\[
\left| \mathcal{Y}_3[z^m](B^{(M)}) - \mathcal{Y}_3[z^m](B) \right| \leq c_3L(B) \cdot m^2 \cdot (C_3(B))^m \cdot M^{-L}, \quad M \geq 1,
\]
where $C_3(B)$ depends on $\|\nabla^6 \sigma_0(B)\|_{\infty}$ and $\|\nabla^2 \text{sub}(B)\|_{\infty}$. The factor $m^2$ is due to the fact that there are $m$ terms in the expansion of the Poisson bracket in (2.43).

3. Contribution of $\sigma_0(B_{x_1} \cdots B_{x_m})$

Let $B \in \Psi^0(M)$ (we redenote $G$ by $B$). Our goal here is to calculate

$$\sum_{x_1 + \cdots + x_m = 0} \frac{1}{m!} \sum_{\tau \in S_m} (M_m(\overline{x}))^n \int_{S^* M} \sigma_0(B_{\overline{x}}) \, d\xi$$

for $n = 1, 2$. Assume for the moment that $n$ is any natural number. Recall that $S_m$ denotes the set of all permutations $\tau$ of the numbers $1, \ldots, m$. Denote $\overline{x}_\tau := (x_{\tau_1}, \ldots, x_{\tau_m})$. Note that both the second factor in (3.1) and the domain of summation are symmetric in $\overline{x}$. Therefore

$$\sum_{x_1 + \cdots + x_m = 0} \frac{1}{m!} \sum_{\tau \in S_m} (M_m(\overline{x}_\tau))^n \int_{S^* M} \sigma_0(B_{\overline{x}}) \, d\xi$$

$$= \sum_{x_1 + \cdots + x_m = 0} \frac{1}{m!} \sum_{\tau \in S_m} \sum_{p=1}^{m-1} \left[ (M_p(\overline{x}_\tau))^n - (M_{p-1}(\overline{x}_\tau))^n \right] \int_{S^* M} \sigma_0(B_{\overline{x}}) \, d\xi.$$

Now an application of the gHD (Theorem 8.2) gives

$$\sum_{x_1 + \cdots + x_m = 0} \frac{1}{m!} \sum_{\tau \in S_m} \sum_{p=1}^{m-1} \frac{1}{j!} \sum_{k_1, \ldots, k_j \geq 1} \sum_{l_1, \ldots, l_j \geq 1} \left( \begin{array}{c} n \\ l_1, \ldots, l_j \end{array} \right) \int_{S^* M} \sigma_0(B_{\overline{x}}) \, d\xi,$$

where the notation (3.1) has been used. We make the sum $\frac{1}{m!} \sum_{\tau \in S_m}$ the outmost one and then drop it, because it is the number of $\overline{x}$'s that matters, not their indices (symmetry of the last factor and of the domain of summation again). We get

$$\sum_{x_1 + \cdots + x_m = 0} \sum_{p=1}^{m-1} \frac{1}{j!} \sum_{k_1, \ldots, k_j \geq 1} \sum_{l_1, \ldots, l_j \geq 1} \left( \begin{array}{c} n \\ l_1, \ldots, l_j \end{array} \right) \int_{S^* M} \sigma_0(B_{\overline{x}}) \, d\xi.$$

We have $m - 1$ independent summation variables $\overline{x}$. Let us make the change of variables

$$\nu_1 := k_1(\overline{x}) = x_1 + \cdots + x_{k_1},$$

$$\nu_2 := k_2(\overline{x}) = x_{k_1+1} + \cdots + x_{k_1+k_2},$$

$$\cdots$$

$$\nu_j := k_j(\overline{x}) = x_{k_1+\cdots+k_{j-1}+1} + \cdots + x_{k_1+\cdots+k_{j-1}+k_j}$$

for $j = 1, \ldots, \min(p, n)$. We also interchange the summations over $p$ and $j$, introduce the new variable $k_{j+1} := m - p$, and perform the $m - j - 1$ unconditional summations in (3.2). We need the following important fact (\text{3.2}) after Lemma 1.2 which states that the convolution on the Fourier transform side corresponds to raising $B$ to a power on the original side.
Lemma 3.1. For any $B \in \Psi^0(M)$ and arbitrary $j \in \mathbb{N}$ and $\nu \in \mathbb{Z}$ one has

$$\sum_{\kappa_1, \ldots, \kappa_j=\nu} B_{\kappa_1} \cdots B_{\kappa_j} = (B^j)^\nu.$$ 

Notice also that by Egorov’s theorem (2.3) and because $\sigma_0(B^k) = (\sigma_0(B))^k$,

$$\sigma_0((B^k)^\nu)(x, \xi) = \int_0^{2\pi} (b_0^k(x, \xi))^k e^{i\nu} \frac{dt}{2\pi} = \int_0^{2\pi} (b_0^k(x, \xi))^t e^{i\nu} \frac{dt}{2\pi}.$$ 

Then (3.2) becomes

$$\sum_{j=1}^{\min(n,m-1)} \frac{1}{j!} \sum_{\nu_1, \ldots, \nu_j} \sum_{l_1+\cdots+l_j = n} \left( \begin{array}{c} n \\ l_1, \ldots, l_j \end{array} \right) \cdot \rho \cdot (-\nu_1)_- \cdots (-\nu_j)_-^j$$

(3.3) becomes

$$\times \int_{S^*M} dx d\xi \int_0^{2\pi} \cdots \int_0^{2\pi} e^{i(\nu_1 t_1 + \cdots + \nu_j t_j - (\nu_1 + \cdots + \nu_j) t_{j+1})} \frac{dt_1}{2\pi} \cdots \frac{dt_j}{2\pi} \frac{dt_{j+1}}{2\pi}$$

$$\times F_j[\sigma^m](b_0^{t_1}, \ldots, b_0^{t_j}, b_0^{t_{j+1}})$$

where we have introduced a linear $(j+1)$-map $F_{j+1}$, whose action on $f(z) = z^m$, $m \geq j + 1$, is prescribed by

$$F_{j+1}[\sigma^m](x_1, \ldots, x_j, x_{j+1}) := \sum_{k_1 \cdots k_j \geq 1 \atop k_1 + \cdots + k_j + k_{j+1} = m} \frac{x_1^{k_1}}{k_1} \cdots \frac{x_j^{k_j}}{k_j} x_{j+1}^{k_{j+1}}$$

and $F_{j+1}[\sigma^m] := 0$ for $m = 1, \cdots, j$.

Remark 3.1. It is important that for any fixed $n$ the sum over $j$ in (3.3) terminates at $n$, no matter how large $m$ is.

We need the following fact [GO2, after (1.3)].

Lemma 3.2. Let $M$ be a Zoll manifold and $A$ defined by (1.4). Assume $\sigma_1(A)(x, \xi) = \sigma_1(A)(x, -\xi)$ for all $(x, \xi) \in T^*M$. Let $f, g \in C^\infty(S^*M)$, $\nu \in \mathbb{Z}$, and recall the notation (1.2). Then

$$\int_{S^*M} \widehat{f}_\nu \cdot \widehat{g}_\nu \, dx d\xi = \int_{S^*M} \widehat{f}_\nu \cdot \widehat{g}_\nu \, dx d\xi.$$ 

We need to compute (3.3) only for $n = 1, 2$. The computation for $n = 1$ has been carried out in [GO2]. We repeat it here for the sake of completeness. For $n = 1$ (3.3) becomes

$$\sum_{\nu_1} (-\nu_1)_- \int_{S^*M} dx d\xi \int_0^{2\pi} \int_0^{2\pi} e^{i\nu_1 t_1 - t_2} \sum_{k_1, k_2 \geq 1 \atop k_1 + k_2 = m} \frac{(b_0^k)^{t_1}}{k_1} \frac{(b_0^k)^{t_2}}{k_2} \frac{dt_1}{2\pi} \frac{dt_2}{2\pi}$$

(3.5)

$$= \sum_{\nu_1} (-\nu_1)_- \int_{S^*M} dx d\xi \sum_{k_1, k_2 \geq 1 \atop k_1 + k_2 = m} \left( \frac{b_0^{k_1}}{k_1} \right)_{\nu_1} \left( \frac{b_0^{k_2}}{k_2} \right)_{\nu_1}$$

This can be rewritten as

$$\sum_{\nu} (-\nu)_- \int_{S^*M} dx d\xi \sum_{j=1}^{m-1} \left( \frac{b_0^j}{j} \right)_{\nu} \left( \frac{b_0^{m-j}}{m-j} \right)_{\nu}$$

(3.6)
Now we sum (3.6) and (3.7) and use \( \frac{1}{\nu} \) where the last equality is obtained after the change of summation index \( j \to m - j \). Now we sum (3.4) and (3.7) and use \( \frac{1}{j} + \frac{1}{m-j} = \frac{m}{j(m-j)} \) and Lemma 3.2 to conclude that (3.5) equals
\[
\sum_{\nu} \left( \frac{b_{0j}^{m}}{m-j} \right)^{\nu} \nu - \nu
\]
where the last equality is obtained after the change of summation index \( j \to m - j \).

For (3.8) first case we repeat the above argument, the corresponding contribution becomes
\[
W_{\nu} = \max(0, \nu)\sum_{\nu} \left( \frac{b_{0j}^{m}}{m-j} \right)^{\nu} \nu - \nu
\]
where following [LRS] we have noticed that (3.7) equals
\[
W_{2}[z^{m}](x_{1}, x_{2}) = \frac{m}{2} \sum_{\nu} \frac{x_{j}^{m-j}}{j - j}
\]
In view of (3.10) we rewrite (3.8) as
\[
\int_{S^{*}M} f \hat{g} \hat{h} \nu dx d\xi = \int_{S^{*}M} f \nu \hat{g} \hat{h} \nu dx d\xi.
\]

Now for the case when \( n = 2 \) in (3.3). In this case \( j \) can be either 1 or 2. In the first case we repeat the above argument, the corresponding contribution becomes
\[
\sum_{k=1}^{\infty} k^{2} \int_{S^{*}M} dx d\xi \int_{0}^{2\pi} e^{ik(t_{1} - t_{2})} W_{2}[z^{m}](b_{0}^{l}, b_{0}^{l}) \frac{dt_{1}}{2\pi} \frac{dt_{2}}{2\pi}.
\]
For \( j = 2 \), (3.3) becomes
\[
\sum_{\nu_{1}, \nu_{2}} \left( \frac{b_{0j}^{m}}{m-j} \right)^{\nu_{1} - \nu_{2}} \nu_{1} - \nu_{2}
\]
where by (3.4)
\[
F_{3}[z^{m}](x_{1}, x_{2}, x_{3}) := \sum_{k_{1} + k_{2} + k_{3} = m} x_{1}^{k_{1}} x_{2}^{k_{2}} x_{3}^{k_{3}}, \quad m \geq 3,
\]
and \( F_{3}[z^{m}] := 0 \) for \( m = 1, 2 \). Let us redenote \( W_{3} := F_{3} \). The formula (1.8) for \( W_{3}[f], f \in A_{1} \), is justified in Section 7. Now the last detail, the following can be proved in the same way as Lemma 3.2. For any \( f, g, h \in C^{\infty}(S^{*}M) \) and \( \lambda, \mu, \nu \in \mathbb{Z} \)
\[
\int_{S^{*}M} \hat{f} \hat{g} \hat{h} \nu dx d\xi = \int_{S^{*}M} \hat{f} \nu \hat{g} \nu \hat{h} \nu dx d\xi.
\]
4. Contribution of the symmetric part of \(_{B_{\kappa_1, \cdots, \kappa_m}}\)

Let us choose any \(B \in \Psi^0(M)\). The goal of this section is to calculate for an arbitrary \(m \in \mathbb{N}, m \geq 2\), the sum

\[
\sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\kappa) \int_{S^* M} dx d\xi \\
\times \sum_{r=1}^m \sigma_0(B_{\kappa_1} \cdots B_{\kappa_{r-1}}) \sigma_0(B_{\kappa_r}) \sigma_0(B_{\kappa_{r+1}} \cdots B_{\kappa_m}).
\]

(4.1)

As in Section 3, the key observation is that both the second factor in (4.1) and the domain of summation are symmetric in \(\kappa\). We permute all the \(\kappa\)'s in the first factor in (4.1), make use of Theorem 8.2 for the power \(n = 1\), that is the classic HD,

\[
\sum_{\tau \in S_m} M_m(\kappa_\tau) = \sum_{\tau \in S_m} \sum_{j=1}^m \frac{-(\kappa_r + \cdots + \kappa_j)}{j},
\]

and drop the summation over \(S_m\), because, as in Section 3, it is the number of \(\kappa\)'s in a particular sum that matters, not their indices (again, the domain of summation and the second factor are still symmetric). After that we get

\[
\int_{S^* M} dx d\xi \sum_{j=1}^{m-1} \sum_{\kappa_1 + \cdots + \kappa_m = 0} \frac{-(\kappa_1 + \cdots + \kappa_j)}{j}
\]

\[
\times \sigma_0(B_{\kappa_1} \cdots B_{\kappa_{r-1}}) \sigma_0(B_{\kappa_r}) \sigma_0(B_{\kappa_{r+1}} \cdots B_{\kappa_m}).
\]

We split the above sum into two: for \(r = 1, \cdots, j\) and for \(r = j + 1, \cdots, m\). In the first case we set

\[
\nu := \kappa_1 + \cdots + \kappa_j, \\
-\mu + \nu := \kappa_r.
\]

Then

\[
\kappa_1 + \cdots + \kappa_{r-1} + \kappa_{r+1} + \cdots + \kappa_j = \mu, \\
\kappa_{j+1} + \cdots + \kappa_m = -\nu.
\]

We carry out the independent summations and recall Egorov's theorem (2.1) and Lemma 3.1 to write the sum over \(r = 1, \cdots, j\) as

\[
\int_{S^* M} dx d\xi \sum_{j=1}^{m-1} \frac{1}{j} \sum_{\nu, \mu} (-\nu) \sigma_0((B^j)_{\nu-\mu}) \sigma_0((B^{m-j})_{-\nu})
\]

(4.2)

Now let us write \(b^t_0 := \sigma_0(\Theta^t(x, \xi))\) and \(b^t_{\text{sub}} := \text{sub}(H)(\Theta^t(x, \xi)), H \in \Psi^0(M)\). We have

\[
\sum_{\mu} \sigma_0((B^j)_{\nu-\mu}) \sigma_0((B^{m-j})_{-\nu})
\]

\[
= \frac{1}{j} \int_0^{2\pi} e^{i\nu s} (b^t_0)_{s-j} b^t_{\text{sub}} ds 2\pi = \frac{1}{j} \text{sub}((B^j)_\nu).
\]

Then (4.2) becomes

\[
\int_{S^* M} dx d\xi \sum_{j=1}^{m-1} \frac{1}{j} \sum_{\nu} (-\nu) \sigma_0((B^j)_\nu) \sigma_0((B^{m-j})_{-\nu})
\]

(4.3)
Analogously, in the case \( r = j + 1, \cdots, m \) we set
\[
\nu := \kappa_1 + \cdots + \kappa_j, \quad -\mu - \nu := \kappa_r.
\]
Then
\[
\kappa_{j+1} + \cdots + \kappa_{r-1} + \kappa_{r+1} + \cdots + \kappa_j = \mu, \quad \kappa_{j+1} + \cdots + \kappa_m = -\nu.
\]
Then the sum for \( r = j + 1, \cdots, m \) becomes
\[
\int_{S^*M} dxd\xi \sum_{j=1}^{m-1} \frac{1}{j} \sum_{\nu+r+1}^{m} (-\nu) \sigma_0((B^j)_{\nu}) \text{sub}(B_{-\nu-\mu}) \sigma_0((B^{m-j})_{\mu})
\]
\[
= \int_{S^*M} dxd\xi \sum_{j=1}^{m-1} \frac{1}{j} \sum_{\nu} (-\nu) \sigma_0((B^j)_{\nu}) \text{sub}(B^{m-j})_{-\nu})
\]
which in view of Lemma 3.2 equals
\[
(4.4) \quad \int_{S^*M} dxd\xi \sum_{\nu} (-\nu) \sum_{j=1}^{m} \frac{1}{j} \text{sub}(B^{m-j}) \sigma_0((B^j)_{-\nu})
\]
We make now in (4.4) a change of index \( j \to m - j \) and get
\[
(4.5) \quad \int_{S^*M} dxd\xi \sum_{\nu} (-\nu) \sum_{j=1}^{m} \frac{1}{m-j} \text{sub}(B^j) \sigma_0((B^{m-j})_{-\nu})
\]
We sum (4.5) with (4.3), use \( \frac{1}{j} + \frac{1}{m-j} = \frac{m}{j(m-j)} \), refer to Lemma 3.2 and replace \( j \to m - j \) again to conclude that (4.1) equals
\[
(4.6) \quad -m \int_{S^*M} \sum_{\nu} \sum_{j=1}^{m-1} \sigma_0((B^j)_{\nu}) \text{sub}(B^{m-j})_{-\nu}) dxd\xi.
\]
It remains to notice that (4.4) in view of (2.2) equals
\[
\int_{S^*M} dxd\xi \sum_{k=1}^{\infty} k \int_{0}^{2\pi} \int_{0}^{2\pi} e^{i(k_1 t_1 + k_2 t_2)} W_2(z^m)(b_0^{i_1}, b_0^{i_2}) b_{\text{sub}}^{j_1} b_{\text{sub}}^{j_2} \frac{dt_1}{2\pi} \frac{dt_2}{2\pi}.
\]

5. Contribution of the non-symmetric part of \( \text{sub}(B_{\varphi_1} \cdots B_{\varphi_m}) \)

Let us choose any \( B \in \Psi^0(M) \). The goal of this section is to calculate for an arbitrary \( m \in \mathbb{N}, m \geq 2 \), the expression
\[
(5.1) \quad \sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\varphi) \int_{S^*M} dx d\xi \sum_{1 \leq k < \ell \leq m} \{ \sigma_0(B_{\varphi_k}), \sigma_0(B_{\varphi_\ell}) \} \prod_{p=1}^{m} \sigma_0(B_{\varphi_p}).
\]
The domain of summation is symmetric in \( \varphi \). However the integrated over \( S^*M \) sum is generally speaking not symmetric. Each of the terms in the sum in (5.1) possesses a partial symmetry.

Depending on \( k \) and \( l \) in (5.1) there are three possible cases. In the first case all the indices except for those in the Poisson bracket form one continuous block. In this case it is convenient to rewrite \( M_m(\varphi) \) as follows. Note that for \( a, b \in \mathbb{R} \)
\[
(5.2) \quad \min(a, b) = a - (b - a)_,
\]
let $p \in \mathbb{N}$, $p \geq 2$, $\mu_1, \ldots, \mu_p \in \mathbb{Z}$ and set $s_j := \mu_1 + \cdots + \mu_j$, $j = 1, \ldots, p$. Then for any $j = 1, \ldots, p - 1$

\[
\min(0, s_1, \ldots, s_j, s_{j+1}, \ldots, s_p) = \min \left( \min(0, s_1, \ldots, s_j), \min(s_{j+1}, \ldots, s_p) \right) = \min \left( \min(0, s_1, \ldots, s_j), s_j + \min(0, s_{j+1} - s_j, \ldots, s_p - s_j) \right).
\]

Therefore it follows from (5.2)

\[
M_p(\mu_1, \ldots, \mu_p) = M_j(\mu_1, \ldots, \mu_j)
\]

(5.3)

\[
- \left( \mu_1 + \cdots + \mu_j - M_j(\mu_1, \ldots, \mu_j) + M_{p-j}(\mu_{j+1}, \ldots, \mu_p) \right) _-.
\]

The three subcases of the first case are

$k = 1, l = 2$; \hspace{1em} $k = 1, l = m$; \hspace{1em} $k = m - 1, l = m$.

In these three subcases using (5.3) we rewrite $M_m(\varpi_1, \ldots, \varpi_m)$, respectively, as

\[
M_2(\varpi_1, \varpi_2) - \left( \varpi_1 + \varpi_2 - M_2(\varpi_1, \varpi_2) + M_{m-2}(\varpi_3, \ldots, \varpi_m) \right) _-,
\]

and

\[
- (\varpi_1) _- \left( \varpi_1, \ldots, \varpi_{m-1} \right)
\]

\[
- (\varpi_2 + \cdots + \varpi_{m-1} - M_{m-2}(\varpi_2, \ldots, \varpi_{m-1}) + \varpi_m) _-,
\]

and

\[
M_{m-2}(\varpi_1, \ldots, \varpi_{m-2}) - \left( \varpi_1 + \cdots + \varpi_{m-2} - M_{m-2}(\varpi_1, \varpi_2) + M_{m-2}(\varpi_{m-1}, \varpi_m) \right) _-.
\]

In the second case all the indices but the two from the Poisson bracket form two continuous blocks. In this case we rewrite $M_m(\varpi)$ as follows. Note that for $p \in \mathbb{N}$, $p \geq 3$, $\mu_1, \ldots, \mu_p$ and $s_1, \ldots, s_p$ as above, and any $j = 1, \ldots, p - 2$

\[
\min(0, s_1, \ldots, s_j, s_{j+1}, \ldots, s_p) = \min \left( \min(0, s_1, \ldots, s_j), \min(s_{j+1}, \ldots, s_p) \right) = \min \left( \min(0, s_1, \ldots, s_j), s_j + \min(0, s_{j+1} - s_j, \ldots, s_p - s_j) \right)
\]

which together with (5.2) implies

\[
M_p(\mu_1, \ldots, \mu_p) = M_j(\mu_1, \ldots, \mu_j)
\]

(5.4)

\[
- \left( \mu_1 + \cdots + \mu_j - M_j(\mu_1, \ldots, \mu_j) + M_{p-j-1}(\mu_{j+2}, \ldots, \mu_p) \right) _-.
\]

There are three subcases: first $k = 1, l = r + 1$, $2 \leq r \leq m - 2$, in which $M_m(\varpi)$ after first using (5.3) and then (5.4) becomes

\[
- (\varpi_1) _- \left( \varpi_1, \varpi_2, \ldots, \varpi_r \right)
\]

\[
- M_{r-1}(\varpi_2, \ldots, \varpi_r) + \varpi_{r+1} + M_{m-r-1}(\varpi_{r+2}, \ldots, \varpi_m) _-,
\]

second: $k = r + 1, l = r + 2$, $1 \leq r \leq m - 3$, in which $M_m(\varpi)$ after first using (5.4) and then (5.3) becomes

\[
M_r(\varpi_1, \ldots, \varpi_r) - \left( \varpi_1 + \cdots + \varpi_r - M_r(\varpi_1, \varpi_2, \ldots, \varpi_r) + \varpi_{r+1} \right)
\]

\[
- (\varpi_{r+2}) _- \left( (\varpi_{r+2})_+ + M_{m-r-2}(\varpi_{r+3}, \ldots, \varpi_m) \right) _-,
\]
third: \( k = r + 1, l = m, 1 \leq r \leq m - 3 \), in which \( M_m(\mathcal{F}) \) after first using first (5.4) and then (5.3) becomes

\[
M_r(\kappa_1, \ldots, \kappa_r) - \left( \kappa_1 + \cdots + \kappa_r - M_r(\kappa_1, \ldots, \kappa_r) + \kappa_{r+1} + M_{m-r-2}(\kappa_{r+2}, \ldots, \kappa_{m-1}) - (\kappa_{r+2} + \cdots + \kappa_{m-1}) - M_{m-r-2}(\kappa_{r+2}, \ldots, \kappa_{m-1}) + \kappa_m \right)_-.
\]

Finally in the third case the indices from the Poisson bracket are taken to be \( \kappa_{r+1} \) and \( \kappa_{s+1} \), where

\[
1 \leq r, \quad r + 2 \leq s, \quad s + 2 \leq m,
\]

and there are three continuous blocks \( 1, \ldots, r \) and \( r + 2, \ldots, s \) and \( s + 2, \ldots, m \).

In that case \( M_m(\mathcal{F}) \) after using (5.4) twice becomes

\[
M_r(\kappa_1, \ldots, \kappa_r) - \left( \kappa_1 + \cdots + \kappa_r - M_r(\kappa_1, \ldots, \kappa_r) + \kappa_{r+1} + M_{s-r-1}(\kappa_{r+2}, \ldots, \kappa_s) - (\kappa_{r+2} + \cdots + \kappa_s) - M_{s-r-1}(\kappa_{r+2}, \ldots, \kappa_s) + \kappa_{s+1} + M_{m-s-1}(\kappa_{s+2}, \ldots, \kappa_m) \right)_-.
\]

We make the computation for the third case, the first and the second are treated in the same way. We use the a convenient reformulation of the original form of the BSt (Theorem 8.1), and one property of the \( j \)-maps \( \Phi_j, j \in \mathbb{N} \), see Lemma 5.1 below. We rewrite the corresponding to the third case part of (5.1) as

\[
\sum_{\kappa_1 + \cdots + \kappa_m = 0} M_m(\mathcal{F}) \int_{S^* M} dx d\xi \sum_{r=1}^{m-3} \sum_{s=r+2}^{m-2} \{ \sigma_0(B_{\kappa_{r+1}}), \sigma_0(B_{\kappa_{s+1}}) \} \\
\times \sigma_0(B_{\kappa_1} \cdots B_{\kappa_r}) \sigma_0(B_{\kappa_{r+2}} \cdots B_{\kappa_s}) \sigma_0(B_{\kappa_{s+2}} \cdots B_{\kappa_m}),
\]

where \( k = r + 1, l = s + 1 \), and each of the three products under the \( \sigma_0 \) sign contains at least one factor (therefore this expression is non-zero only for \( m \geq 5 \)). Now we observe that both domain of summation and each of the three products are symmetric if we interchange the indices \( \kappa_1, \ldots, \kappa_r, \kappa_{r+2}, \ldots, \kappa_s \) and \( \kappa_{s+2}, \ldots, \kappa_m \) separately, that is within each of these three sets. Then their sums do not change, and we can therefore consider the representation (5.5) as a function of

\[
M_r(\kappa_1, \ldots, \kappa_r), \quad M_{s-r-1}(\kappa_{r+2}, \ldots, \kappa_s), \quad M_{m-s-1}(\kappa_{s+2}, \ldots, \kappa_m)
\]

only. We interchange the indices within the three groups, use Theorem 8.1 and take in account Remark 8.2. After that because of the symmetricity of domain of summation, and of the principal symbols of the three products we may, and will, drop the coefficients \( 1/(r!1), 1/(s-r-1)! \) and \( 1/(m-s-1)! \) and the summations over the permutations over the three groups of indices (just as in Section 3). We
conclude that \((5.6)\) becomes

\[
(5.7) \sum_{\kappa_1 + \cdots + \kappa_m = 0} \sum_{r=1}^{m-3} \sum_{s=r+2}^{m-2} \sum_{a=1}^{r} \frac{1}{a!} \sum_{j_1, \ldots, j_a \geq 1 \atop j_1 + \cdots + j_a = r} \frac{1}{j_1 \cdots j_a} \sum_{s-r=1}^{\gamma} \frac{1}{\beta!} \sum_{k_1, \ldots, k_\beta \geq 1 \atop k_1 + \cdots + k_\beta = s-r-1} \frac{1}{k_1 \cdots k_\beta} 
\]

\[
\times \sum_{\gamma=1}^{m-s-1} \frac{1}{\gamma!} \sum_{l_1, \ldots, l_\gamma \geq 1 \atop l_1 + \cdots + l_\gamma = m-s-1} \frac{1}{l_1 \cdots l_\gamma} \int_{S^M} dx \, d\xi \{ \sigma_0(B_{\kappa_1 + \cdots + \kappa_m}), \sigma_0(B_{\kappa_{r+1} + \cdots + \kappa_m}) \}
\]

\[
\times \left[ - (j_1(\kappa))_+ \cdots - (j_\alpha(\kappa))_+ \left( (j_1(\kappa))_+ \cdots + (j_\alpha(\kappa))_+ + \kappa_{r+1} \right) - (k_1(\kappa))_+ \cdots - (k_\beta(\kappa))_+ \left( (k_1(\kappa))_+ \cdots + (k_\beta(\kappa))_+ + \kappa_{s+1} \right) - (l_1(\kappa))_+ \cdots - (l_\gamma(\kappa))_+ \right] 
\]

\[
\times \sigma_0(B_{\kappa_1} \cdots B_{\kappa_m}) \sigma_0(B_{\kappa_{r+2} + \cdots + \kappa_m}) \sigma_0(B_{\kappa_{s+2} + \cdots + \kappa_m}).
\]

Here for all possible values of the indices each of the summation variables \(\kappa_1, \cdots, \kappa_m\) is involved in some of the \(j_1(\kappa), \cdots, \) but only once. We rewrite now the sum over \(r\) and \(s\) as a sum over three summation variables \(a, b, c \geq 1\), which are the lengths of the three continuous blocks, with the condition \(a + b + c = m - 2\). Note also that it does not matter anymore on which place \(\kappa_{r+1}\) and \(\kappa_{s+1}\) stands, it is only important that the same letter is used in the non-positive valued function coming from the square bracket in \((5.7)\). Let us make two changes of variables

\[
\kappa_1 := \kappa_{r+1}, \quad \kappa_2 := \kappa_{s+1}
\]

and for all \(a, b, c \geq 1\) with \(a + b + c = m - 2\)

\[
\mu_1 := j_1(\kappa), \cdots, \mu_\alpha := j_\alpha(\kappa), \quad 1 \leq \alpha \leq a \\
\nu_1 := k_1(\kappa), \cdots, \nu_\beta := k_\beta(\kappa), \quad 1 \leq \beta \leq b \\
\rho_1 := l_1(\kappa), \cdots, \rho_\gamma := l_\gamma(\kappa), \quad 1 \leq \gamma \leq c.
\]

Then the square bracket in \((5.7)\) becomes exactly the defined by \((1.12)\) non-positive valued function \(O^{(3)}_{\alpha, \beta, \gamma}\). Now we carry out the \(((a - \alpha) + (b - \beta) + (c - \gamma))\) free summations over the rest of \(\kappa\)'s, just as in Section 3. We get

\[
(5.8) \int_{S^M} dx \, d\xi \sum_{a, b, c \geq 1} \sum_{a + b + c = m} \sum_{a=1}^{a} \sum_{b=1}^{b} \sum_{c=1}^{c} \frac{1}{a!} \frac{1}{b!} \frac{1}{c!} 
\]

\[
\times \sum_{\kappa_1 + \kappa_2 + \cdots + \kappa_m = 0} O^{(3)}_{\alpha, \beta, \gamma}(\kappa_1, \kappa_2, \mu_1, \cdots, \mu_\alpha, \nu_1, \cdots, \nu_\beta, \rho_1, \cdots, \rho_\gamma) 
\]

\[
\times \{ \sigma_0(B_{\kappa_1}), \sigma_0(B_{\kappa_2}) \} \sigma_0 \left( \sum_{j_1, \ldots, j_a \geq 1 \atop j_1 + \cdots + j_a = a} \frac{(B_{j_1})_{\mu_1} \cdots (B_{j_a})_{\mu_\alpha}}{j_1 \cdots j_a} \right)
\]
\[
\times \sigma_0 \left( \sum_{k_1, \ldots, k_j \geq 1} \frac{(B^{k_1})_{l_1} \cdots (B^{k_j})_{l_j}}{k_1 \cdots k_j} \right) \sigma_0 \left( \sum_{l_1, \ldots, l_j \geq 1} \frac{(B^{l_1})_{l_1} \cdots (B^{l_j})_{l_j}}{l_1 \cdots l_j} \right).
\]

Now by the definition of the Fourier coefficient, by Egorov’s theorem and the fact that \(\sigma_0(B^j) = (\sigma_0(B))^j\), \(j \in \mathbb{N}\), we obtain, just as in Section 3:

\[
\int_{S^* M} dxd\xi \sum_{\alpha, \beta, \gamma} \sum_{\alpha=1}^a \sum_{\beta=1}^b \sum_{\gamma=1}^c \frac{1}{\alpha! \beta! \gamma!} \times \sum_{x_1 + x_2 + \cdots + x_a + \cdots + x_a} \frac{1}{\alpha! \beta! \gamma!} \Omega_{\alpha, \beta, \gamma}^j (x_1, \ldots, x_a, \mu_1, \cdots, \mu_p, \nu_1, \cdots, \nu_p, \rho_1, \cdots, \rho_p)
\]

(5.9)

\[
\times \{\sigma_0(B^{x_1}), \sigma_0(B^{x_2})\}
\]

\[
\times \left( \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} dt_1 \cdots dt_{p-2} \sum_{p=2}^{2p} dt_1 \right)
\]

\[
\times e^{i(m_1 t_1 + \cdots + m_p t_p + \cdots + s_1 t_1 + \cdots + s_p t_p + \cdots + r_1 t_p)}
\]

\[
\times \Phi_{\alpha}[x^a](b_0^a, \cdots, b_0^a) \Phi_{\beta}[y^b](b_0^b, \cdots, b_0^b) \Phi_{\gamma}[z^c](b_0^c, \cdots, b_0^c)
\]

We use the fact that \(\Phi_{\alpha}[z^a] = 0\) for \(\alpha > a\) to extend the summation over each of the variables \(\alpha, \beta, \gamma\) to the whole of \(\mathbb{N}\), and make these summations the outer ones. Finally we use the following important property of the \(j\)-maps \(\Phi_j\), \(j \in \mathbb{N}\), whose proof immediately follows from the definition of \(\Phi_j\) and is omitted.

**Lemma 5.1.** Let choose an arbitrary \(p \in \mathbb{N}\). For any fixed real or complex \(x_j, y_k, z_l, j, k, l = 1, \cdots, p - 2\) the following identity holds

\[
\sum_{\alpha, \beta, \gamma} \sum_{\alpha=1}^a \sum_{\beta=1}^b \sum_{\gamma=1}^c \frac{1}{\alpha! \beta! \gamma!} \times \Phi_{\alpha}[x^a](x_1, \cdots, x_\alpha) \Phi_{\beta}[y^b](y_1, \cdots, y_\beta) \Phi_{\gamma}[z^c](z_1, \cdots, z_\gamma)
\]

(5.10)

\[
= \sum_{\alpha=1}^\infty \sum_{\beta=1}^\infty \sum_{\gamma=1}^\infty \frac{1}{\alpha! \beta! \gamma!} \times \Phi_{\alpha+\beta+\gamma}[z^p](x_1, \cdots, x_\alpha, y_1, \cdots, y_\beta, z_1, \cdots, z_\gamma).
\]

Note that the sums terminate for \(\alpha + \beta + \gamma > p\), and \(\alpha, \beta, \gamma \geq 1\). Therefore only \(p\) of the variables \(x_1, \cdots, z_{p-2}\) are present at the each term on the right-hand side in (5.10).

The formula (1.13) now follows from (5.9) and Lemma 5.1 with \(p = m - 2\). It is important that we have reduced the number of j-maps from three to one, and that the \(j\)-map \(\Phi_j[f]\) is linear in \(f\) for all \(j \in \mathbb{N}\). Therefore (1.13) holds for any \(f \in A_1\).

We divide by \(z^2\), because for a monomial \(f(z) = z^m\), \(m \geq 5\), the \(j\)-map should be evaluated at \(z^{m-2}\). We subtract the fourth degree Taylor polynomial \(T_4[f]\) of \(f\) about \(t = 0\) because the term under consideration is absent for all polynomials \(f\) of degree \(\leq 4\).

The formulas (1.13) and (1.14) are proved analogously. Because the corresponding expressions appear for the monomials of degree at least 3 and 4, respectively, we subtract \(T_2[f]\) and \(T_3[f]\), respectively.
We will need the following statement (see for instance [GrRy]).

**Lemma 6.1.** Let $n, m \in \mathbb{N}$. Then the following holds, as $n \to \infty$,

\[
\sum_{k=1}^{n} k^m = \frac{n^{m+1}}{m+1} + \frac{1}{2} n^m + \frac{m}{12} n^{m-1} + O(n^{m-2})
\]

\[
\sum_{k=1}^{n} k^{-1} = \log n + \gamma + \frac{1}{2} n^{-1} + O(n^{-2})
\]

\[
\sum_{k=1}^{n} k^{-2} = \zeta(2) - \frac{1}{n} + O(n^{-2})
\]

\[
\sum_{k=1}^{n} k^{-m} = \zeta(m) + O(n^{-m+1}), \quad m \geq 3.
\]

Note that (6.1)

\[
\text{Tr}(P_n G) = \sum_{k=1}^{n} \text{Tr}(\pi_k G), \quad n \in \mathbb{N}.
\]

Assume first that $d \geq 4$. Then by (2.9)

\[
\left| \text{Tr}(\pi_k G) - k^{d-1} R_0(G) - k^{d-2} R_1(G) - k^{d-3} R_2(G) \right| \leq c_1(G) k^{d-4}, \quad k \in \mathbb{N},
\]

and so in view of (6.1), for any $n \in \mathbb{N},$

\[
\left| \text{Tr}(P_n G) - n^d \cdot \frac{1}{d} R_0(G) - n^{d-1} \cdot \left( \frac{1}{2} R_0(G) + \frac{1}{d-1} R_1(G) \right) 
- n^{d-2} \cdot \left( \frac{d-1}{12} R_0(G) + \frac{1}{d-2} R_1(G) + \frac{1}{d-2} R_2(G) \right) 
+ c_3(R_0(G), R_1(G), R_2(G), d) \cdot n^{d-3} \right| \leq c_2(G,d) n^{d-3}.
\]

For $d = 3$ there also appears a term with $\log n$ in the left-hand side of (6.2). This proves part (iii).

For $d = 1, 2$ when we sum over $k = 1, \ldots, n$ in (2.9) there is a subtle point, namely, the constant coefficient in (6.1), as $n \to \infty$. The terms of all orders in (2.9), and also the possible rapidly decaying term, will contribute to it. Assume (1.17). Then the following series is absolutely concergent

\[
\epsilon_k(G) := \text{Tr}(\pi_k G) - \sum_{l=0}^{+\infty} k^{d-1-l} R_l(G), \quad k \in \mathbb{N}.
\]

Furthermore, for any $N \in \mathbb{N}$ and all $k \in \mathbb{N}$ by (2.3)

\[
|\epsilon_k(G)| \leq \left| \text{Tr}(\pi_k G) - \sum_{l=0}^{N+d-2} k^{d-1-l} R_l(G) \right| + \left| \sum_{l=N+d-1}^{\infty} k^{d-1-l} R_l(G) \right| 
\leq c_N(G) k^{-N} + k^{-N} \sum_{l=0}^{\infty} |R_l(G)| \leq \hat{c}_N(G) \cdot k^{-N}.
\]
in view of (1.17). Note that \( C(G) \) defined by (1.18) equals
\[
(6.5) \quad C(G) = \sum_{k=1}^{\infty} \epsilon_k(G),
\]
the series being absolutely convergent. Now summing over \( k = 1, \cdots, n \) in (6.3) we obtain
\[
(6.6) \quad \text{Tr}(P_n G) = \sum_{l=0}^{n} R_l(G) \sum_{k=1}^{n} k^{d-1-l} + \sum_{k=1}^{n} \epsilon_k(G).
\]
Because \( \epsilon_k(G) \) decays rapidly (6.4), as \( k \to \infty \), and by (6.3), \( \sum_{k=1}^{n} \epsilon_k(G) \) converges rapidly to \( C(G) \). From this and (6.6) we can obtain an asymptotics of \( \text{Tr}(P_n G) \) up to any negative order. In particular, taking into account Lemma 6.1 we prove (i) and (ii).

7. A formula for \( \Phi_j[f], f \in \mathcal{A}_1, j \in \mathbb{N} \)

We find a formula for \( \Phi_j[f], f \in \mathcal{A}_1 \), in terms of an auxiliary linear \( j \)-map \( \tilde{\Phi}_j \), \( j \in \mathbb{N} \), see (7.2). The latter acts on monomials as follows
\[
(7.1) \quad \tilde{\Phi}_j[z^m](\xi_1, \cdots, \xi_j) := \sum_{i_1+\cdots+i_j = m} \xi_1^{i_1} \cdots \xi_j^{i_j}, \quad m = 0, 1, 2, \cdots,
\]
which is the complete symmetric function of degree \( m \) evaluated at the point \((\xi_1, \cdots, \xi_j, 0, \cdots)\). To write a formula for \( \Phi_j[f], f \in \mathcal{A}_1 \), we use an idea suggested by Kurt Johansson. It uses a Cauchy integral representation of (7.1) via the generating function. Namely, by the identity \( (1.2.5) \) for \((\xi_1, \cdots, \xi_j, 0, \cdots)\)
\[
\sum_{n=0}^{\infty} \Phi_j[z^n](\xi_1, \cdots, \xi_j) \cdot \frac{1}{\zeta^n} = \prod_{k=1}^{j} \frac{\zeta}{\zeta - \xi_k}.
\]
After a multiplication by \( \zeta^{m-1} \) and an integration over a contour \( \gamma \subset \mathbb{C} \) which circumferences the points \( 0, \xi_1, \cdots, \xi_j \) we single out \( \tilde{\Phi}_j[z^m] \) and obtain
\[
\tilde{\Phi}_j[z^m](\xi_1, \cdots, \xi_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{\zeta^{j-1}}{(\zeta - \xi_1) \cdots (\zeta - \xi_j)} \zeta^m d\zeta.
\]
Therefore for \( f \in \mathcal{A}_0 \) (an analytic on \( \mathbb{C} \) function which might have a constant term) we can define
\[
\tilde{\Phi}_j[f](\xi_1, \cdots, \xi_j) = \frac{1}{2\pi i} \int_{\gamma} \frac{\zeta^{j-1}}{(\zeta - \xi_1) \cdots (\zeta - \xi_j)} f(\zeta) d\zeta.
\]
The \( j \)-map \( \Phi_j \) is now defined for any \( f \in \mathcal{A}_1 \) by
\[
(7.2) \quad \Phi_j[f](x_1, \cdots, x_j) = \int_{0}^{x_1} \cdots \int_{0}^{x_j} \tilde{\Phi}_j[z^j f(z)](\xi_1, \cdots, \xi_j) d\xi_1 \cdots d\xi_j.
\]
Now we take (7.2) as a definition of \( \Phi_j \), then (1.9) with \( m \geq j \) holds, and it only remains to prove that \( \Phi_j \) vanishes on the set of polynomials of degree \( j - 1 \) with no constant term. Note that the integrand in (7.2) equals
\[
(7.3) \quad \frac{1}{2\pi i} \int_{\gamma} \frac{\zeta^{-1} f(\zeta)}{(\zeta - \xi_1) \cdots (\zeta - \xi_j)} d\zeta.
\]
The integral does not depend on the contour of integration, if only all \( \xi_1, \cdots, \xi_j \) are inside it. Let \( \gamma = \{ \xi : |\xi| = R \} \), \( R \to \infty \). If \( f(z) = z^k \), \( k = 1, \cdots, j - 1 \), then the absolute value of (7.3) is estimated by \( (2\pi)^{-1}R^{j-2} \cdot 2\pi R/R^j = R^{-1} \to 0 \), as \( R \to \infty \). Therefore (7.3) must be 0.

If \( f \in \mathcal{A}_1 \) then \( z^{-1}f(z) \) is analytic. Therefore (7.3) equals the sum of \( j \) residues at the points \( \xi_1, \cdots, \xi_j \). For instance, for \( j = 2 \) and any \( f \in \mathcal{A}_1 \)

\[
\Phi_2[f](x_1, x_2) = \int_0^{x_1} \int_0^{x_2} \frac{\xi_1^{-1}f(\xi_1) - \xi_2^{-1}f(\xi_2)}{\xi_1 - \xi_2} \, d\xi_1 d\xi_2.
\]

**Remark 7.1.** Another way to construct \( \tilde{\Phi}_j \), \( j \in \mathbb{N} \), is by induction on \( j \). In that case one uses a simple formula

\[
(7.4) \quad \frac{u^{r+1} - v^{r+1}}{u - v} = \sum_{p, q \geq 0, p + q = r} u^p v^q, \quad u, v \in \mathbb{C},
\]

for \( r = 0, 1, 2, \cdots \). This derivation is however longer than the above argument. The formula (7.4) was used in [LRS] for the computation of the 2-map \( W_2 \) defined in (1.5). In that case the induction is not needed.

Finally we find a formula for \( W_3[f], f \in \mathcal{A}_1 \), where the action \( W_3[z^m], m \in \mathbb{N} \), is given by (3.9). In view of (7.1) using the integration as in (7.2) and moving out the parts has length at least one. For an arbitrary partition of \( m \) into \( j \) parts

\[
k_1 \geq 1, \cdots, k_j \geq 1, \quad k_1 + \cdots + k_j = m,
\]

we introduce the notation

\[
k_1(\tau) := \kappa_{r_1} + \cdots + \kappa_{r_{k_1}}
\]
\[
k_2(\tau) := \kappa_{r_{k_1}+1} + \cdots + \kappa_{r_{k_1+k_2}}
\]
\[\vdots\]
\[
k_j(\tau) := \kappa_{r_{k_1}+\cdots+k_{j-1}+1} + \cdots + \kappa_{r_{k_1}+\cdots+k_{j-1}+k_j}.
\]

Each of \( k_l(\tau), l = 1, \cdots, j \), is a sum of \( k_l \) permuted variables out of \( \kappa_{r_1}, \cdots, \kappa_m \) so that each of the permuted variables enters exactly one sum. Note also that because \( k_1 + \cdots + k_j = m \) one has

\[
k_1(\tau) + \cdots + k_j(\tau) = \kappa_{r_1} + \cdots + \kappa_m
\]
\[
= \kappa_1 + \cdots + \kappa_m,
\]
For $a \in \mathbb{R}$, denote $-(a)_- := \min(0,a)$ and $(a)_+ := \max(0,a)$. We state now a combinatorial formula, called CF, which is an equivalent form of the BSt [2, Theorem 2.2], and does not involve any advanced combinatorial coefficients. This formula is very suitable for the calculation of sums which arise in Section 3 and in computations of the joint distributions for random walks as in Theorem 1.7. Recall the notation (2.4).

**Theorem 8.1** (CF: an equivalent version of the BSt). For any $m \in \mathbb{N}$, arbitrary $\kappa_1, \ldots, \kappa_m \in \mathbb{R}$, and any real- or complex-valued function $f$ defined on the left half-axis, the following holds

$$
\sum_{\tau \in S_m} f(M_m(\tau)) = \sum_{\tau \in S_m} \frac{1}{j!} \sum_{k_1, \ldots, k_j \geq 1 \atop k_1 + \cdots + k_j = m} \frac{f(- (k_1(\tau)_-) - \cdots - (k_j(\tau)_-))}{k_1 \cdots k_j}
$$

This holds because the sets of the values of the arguments of $f$ on the left- and on the right-hand side in (8.2) contain the same numbers with the same multiplicities, by the BSt, the rest being just an account of the number of conjugacy classes in $S_m$, see Section 3 and especially the proof of Lemma 3.3 in [Gi3].

For a monomial $f(z) = z^n$, $n \in \mathbb{N}$, a further calculation in CF can be carried out, of one subtracts the value of $f$ at the “previous” maximum. The right-hand side of the resulting formula has a multiplicative, and not additive, as in the CF, form, which is important for the calculation of the sums of convolution type in Section 3.

**Theorem 8.2** (Generalized Hunt–Dyson formula). For any power $n \in \mathbb{N}$, an arbitrary number of variables $m \in \mathbb{N}$ and for arbitrary fixed values of the real variables $\kappa_1, \ldots, \kappa_m$ one has

$$
\sum_{\tau \in S_m} [(M_m(\tau))^n - (M_{m-1}(\tau))^n] = \sum_{\tau \in S_m} \min(m,n) \sum_{j=1}^{m-1} \frac{1}{j!} \sum_{k_1, \ldots, k_j \geq 1 \atop k_1 + \cdots + k_j = m \atop l_1 + \cdots + l_j = n} \binom{n}{l_1, \ldots, l_j} \times \frac{(-(k_1(\tau)_-))^{l_1}}{k_1} \cdots \frac{(-(k_j(\tau)_-))^{l_j}}{k_j}.
$$

**Remark 8.1.** In the case $n = 1$ the identity (8.3) becomes the usual Hunt–Dyson formula [3, after (4.8)]

$$
\sum_{\tau \in S_m} [M_m(\tau) - M_{m-1}(\tau)] = \sum_{\tau \in S_m} \frac{- (\kappa_{\tau_1} + \cdots + \kappa_{\tau_m})}{m} = (m-1)! (- (\kappa_1 + \cdots + \kappa_m)_-).
$$

**Remark 8.2.** It is important that Theorem 8.1 and 8.2 holds if the symmetric group $S_m$ is replaced with a larger symmetric group of indices. One does however need the summation over the whole group $S_m$. The usual formula HD holds for the sums over the cyclic subgroup $C_m$, as well.
Remark 8.3. The usual formula HD of course can be viewed as a corollary of the Bohnenblust–Spitzer identity. However one needs the formula (8.3) for all $n \in \mathbb{N}$ to reprove the BST as is done in [Gi3, Section 4].

Remark 8.4. The BST, the gHD, and the HD hold if the minima are replaced with the maxima, and the negative parts $-(\cdot)_-$ are replaced with $(\cdot)_+$.

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