Introduction

Let $p$ be a prime number. In this note, we combine the methods of Hida with the results of [HK1] to define a $p$-adic analytic function, the squares of whose special values are related to the values of triple product $L$-functions at their centers of symmetry. More precisely, let $f$, $g$, and $h$ be classical normalized cuspidal Hecke eigenforms of level 1 and (even) weights $k$, $\ell$, and $m$, respectively, with $k \geq \ell \geq m$; assume $k \geq \ell + m$. Let $L(s, f, g, h)$ be the triple product $L$-function [G1, G2, PSR]; its center of symmetry is the point $s = \frac{k + \ell + m - 2}{2}$. Let $\langle \cdot, \cdot \rangle_k$ be the normalized Petersson inner product for modular forms of weight $k$. Let $\mathbb{Q}\{f, g, h\}$ be the field generated over $\mathbb{Q}$ by the Fourier coefficients of $f$, $g$, and $h$. Using the integral representation for $L(s, f, g, h)$ [op. cit], Kudla and one of the authors have shown that the quotient

$$\frac{\pi^{2k} \langle f, f \rangle_k}{L(\frac{k + \ell + m - 2}{2}, f, g, h)}$$

is a square in $\mathbb{Q}\{f, g, h\}$. Here $C(k, \ell, m) \in \mathbb{Q}$ is a universal constant, depending only on $k$, $\ell$, and $m$. We construct $p$-adic measures which interpolate the square root of this quotient, as (the ordinary eigenform associated to) $f$ varies in a Hida family $f$. These are actually generalised measures, in the sense of [H1, II]: elements of the finite normal algebra extensions of the Iwasawa algebra which arise in Hida’s theory of the ordinary Hecke algebra. In the simplest case, we obtain the following formula (cf. Theorem 2.2.8):

$$\left( \frac{D_H(f, g, h)(k)}{H(k) \cdot K(k)} \right)^2 = \frac{L(\frac{w+1}{2}, f_k, g, h)}{\frac{1}{\pi^{2k}} \cdot \langle f_k, f_k \rangle_k \cdot C(k, \ell, m)}.$$

Here $f_k$ is (the primitive form associated to) the specialization in weight $k$ of $f$, $D_H(f, g, h)$ is the analytic function associated to our $p$-adic measure, $w = k + \ell + m - 3$, and $H(k)$ and $K(k)$ are normalizing factors depending on $f$.

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The construction of this measure is a modification of Hida’s approach to \( p \)-adic interpolation of Rankin products [H1,II]. Indeed, when the cusp form \( g \) is replaced by an appropriate Eisenstein series, Hida’s method defines a three-variable Rankin product: one variable for the value of \( s \) and the other two for the \( p \)-adic variation of \( f \) and \( h \). However, there is a subtle difference between our \( p \)-adic construction and that of Hida. Let \( Z = \mathbb{Z}_p^\times \) and let \( X \) be a \( p \)-adic manifold on which \( Z \) acts. Starting with measures \( dE \) on \( Z \) and \( \mu \) on \( X \), Hida constructed the convolution \( d(E \ast \mu) \) by the formula

\[
\int_{Z \times X} \phi(z,x) d(E \ast \mu) = \int_{Z \times X} \phi(z,z^{-1}x) dE(z) d\mu(x).
\]

The measure \( dE \) takes values in the space of \( p \)-adic modular forms. The fact that \( dE \) is supported on \( \mathbb{Z}_p^\times \) forces \( \int_{Z} \phi(z) dE(z) \) to be of level divisible by \( p \). Hence the \( p \)-adic symmetric square \( L \)-function constructed in [H3] comes with an Euler factor at \( p \) that gives a trivial zero at \( s = 1 \). In our present construction, we instead form the product of a measure and a fixed function. The calculation leading to (2.2.7) yields the correct Euler factor at \( p \), and thus to the \( p \)-adic interpolation formula (2.2.9). The same idea was used by one of the authors, together with R. Greenberg, to construct a modified symmetric square \( L \)-function, with the trivial zero removed, and thus to obtain a formula for the derivative at \( s = 1 \) of Hida’s symmetric square \( L \)-function at \( s = 1 \), with arithmetically interesting implications (see [GT], [HTU]).

The article [HK1] also considers the central critical values when \( k < \ell + m \). The \( p \)-adic interpolation of the critical values of such triple products, as opposed to the square roots, has recently been obtained by Böcherer and Panchishkin when each of the rank 2 motives \( M(f), M(g), \) and \( M(h) \) associated to \( f, g, \) and \( h \) is ordinary, cf. [P].

The notion that the square roots of central critical values of \( L \)-functions should have \( p \)-adic interpolations seems to have first arisen in connection with the thesis of A. Mori [M1, M2]. Mori showed that, if \( f \) is a holomorphic modular form and \( K \) is an imaginary quadratic field in which \( p \) splits, then the value of \( f \) at Heegner points associated to \( K \), suitably normalized, is naturally an Iwasawa function. When \( f \) is a new form, it should be possible to use Waldspurger’s results in [W] to show that this Iwasawa function \( p \)-adically interpolates the square roots of the central critical values of the \( L \)-functions \( L(s, f_K, \chi) \), where \( f_K \) is the base change of \( f \) to \( K \), as above, and \( \chi \) runs through a continuous family of algebraic Hecke characters for \( K \). Other examples have been studied by Stevens [St], Hida [H5], Sofer [So] and Villegas [V].

The first author would like to take this opportunity to thank his colleagues at the Université de Paris-Sud at Orsay, where most of this article was written in 1993, for providing a uniquely congenial working environment. Both authors would like to thank the referee, whose careful reading uncovered a significant error in the previous version, and whose suggestions led to a number of improvements. In particular, section 1.4 has been thoroughly rewritten on the basis of the referee’s suggestions.

1. \( p \)-adic measures associated to three modular forms

(1.1) Review of \( p \)-adic modular forms. Let \( \mathcal{O} \) be an algebra of finite-rank over \( \mathbb{Z}_p \), \( K = \mathcal{O} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). Let \( N \) be a positive integer prime to \( p \) and let \( Z_N = \lim_{\leftarrow} (\mathbb{Z}/Np^\alpha \mathbb{Z})^\times \);
\( \mathcal{M} = \mathcal{M}(N, \mathcal{O}) \) denotes the complete \( \mathcal{O} \)-algebra of \( \mathcal{O} \)-valued (\( p \)-adic) modular forms of level \( Np^\infty \) and any weight (i.e., \( p \)-adic modular forms in the sense of Katz; see [H1 I or II, §1]). For any integers \( k, \ell \geq 0 \), and any ring \( R \), we let \( M_\ell(Np^k, R) \) denote the module of modular forms of weight \( \ell \) for \( \Gamma_1(Np^k) \) – we say of level \( Np^k \), for short – whose \( q \)-expansion at the cusp at infinity lies in \( R[[q]] \). Let

\[ M_\ell(Np^\infty, R) = \bigcup_k M_\ell(Np^k, R). \]

Similarly, we define \( S_\ell(Np^k, R) \), \( S_\ell(Np^\infty, R) \), and \( \mathcal{S} = \mathcal{S}(N, \mathcal{O}) \) to be the corresponding modules of cusp forms. If \( g \in \mathcal{S} \), we write its \( q \)-expansion \( g = \sum_{n=1}^\infty a(n, g)q^n \), with \( a(n, g) \in \mathcal{O} \) for all \( n \). We use the same notation for modular forms with complex Fourier coefficients.

The following operators on \( M_\ell(Np^\infty, \mathcal{O}) \) are standard, preserve the submodules of cusp forms, and extend to the completion \( \mathcal{M} \) (cf. [H3, §1] and [Go] for details):

(1.1.1) For any prime number \( \lambda \), the Hecke operators \( T(\lambda) \) and, for \( \lambda \) relatively prime to \( N \), \( T(\lambda, \lambda) \), whose action on the \( q \)-expansion is given by \( [H2, (1.13 \ a)] \); more generally, for any \( \lambda \) relatively prime to \( N \), \( T(\lambda) \) and \( T(\lambda, \lambda) \) can be defined by the usual formulas.

(1.1.2) For any prime number \( \lambda \) relatively prime to \( Np \), the diamond operators \( < \lambda >_\ell \), whose action on \( M_\ell(Np^\infty, \mathcal{O}) \) is given by

\[ < \lambda >_\ell = \frac{T(\lambda)^2 - T(\lambda^2)}{\lambda^{\ell-1}} = \frac{T(\lambda, \lambda)}{\lambda^{\ell-1}}, \]

we let \( < \lambda > = < \lambda >_0 \).

(1.1.3) By continuity, the map \( \lambda \mapsto < \lambda > \) extends to an action of the group \( Z_N \) on \( M(Np^\infty, \mathcal{O}) \) given by \( < z > = z_p \cdot T(z, z) \). It coincides with that induced by \( Z_N \) acting on the \( p \)-adic moduli problem (cf. [H3,1] and [Go] for details).

Moreover

(1.1.4) The differential operator \( d = q \frac{dq}{q} \), operating on the \( q \)-expansion sends classical forms to \( p \)-adic modular forms.

(1.2) **Review of Hecke algebras.** We retain the notation from the previous section.

(1.2.1) **Definition.** The Hecke algebra

\[ h = h(Np^\infty, \mathcal{O}) \subset \text{End}_\mathcal{O}(\mathcal{S}(N, \mathcal{O})) \]

is the \( \mathcal{O} \)-subalgebra generated by the Hecke operators \( T(\lambda) \) for all primes \( \lambda \) and by \( T(\lambda, \lambda) \) for \( \lambda \) relatively prime to \( N \). Similarly,

\[ h^{\ell}(Np^\infty, \mathcal{O}) \subset \text{End}_\mathcal{O}(\bigoplus_{k \leq \ell} S_k(Np^\infty, \mathcal{K})) \cap \mathcal{O}[[q]]), \]

the Hecke algebra of weight \( \ell \) and level \( Np^\infty \), is the \( \mathcal{O} \)-subalgebra generated by the \( T(\lambda) \) for all primes \( \lambda \) and by the \( T(\lambda, \lambda) \) for \( \lambda \) relatively prime to \( N \).

There are canonical surjections of \( h(Np^\infty, \mathcal{O}) \) onto \( h^{\ell}(Np^\infty, \mathcal{O}) \) for all \( \ell, \alpha \), and (cf. [H2, pp 243 ff.])

(1.2.2) \[ h(Np^\infty, \mathcal{O}) = \lim_{\ell \to \infty} h^{\ell}(Np^\infty, \mathcal{O}). \]
Let \( h_\ell(Np^\alpha, \mathcal{O}) \) be the \( \mathcal{O} \)-algebra generated by the Hecke operators acting on \( S_\ell(Np^\alpha, \mathcal{O}) \) (note the difference between \( h_\ell(Np^\alpha, \mathcal{O}) \) and \( h^\ell(Np^\alpha, \mathcal{O}) \)). Let \( e \in h \) denote Hida’s ordinary idempotent \([H2, (1.17 \, b)]\), and let \( h_\ominus = eh \) be the universal ordinary \( p \)-adic Hecke algebra of level \( N \). We set \( S^\ominus_\ell(Np^\alpha, \mathcal{O}) = eS_\ell(Np^\alpha, \mathcal{O}) \), for \( \alpha \leq \ell \), and let \( h_\ominus^\ell(Np^\alpha, \mathcal{O}) = eh_\ell(Np^\alpha, \mathcal{O}) \).

Let \( \gamma = 1 + Np \in \mathbb{Z}_N \); then \( X = \gamma - 1 \) is a topologically nilpotent element in \( \Lambda_N = \varprojlim_n \mathcal{O}[[\mathbb{Z}/Np^n\mathbb{Z}]] \). Similarly, one lets \( \Lambda = \mathcal{O}[[\mathbb{Z}_p^\times]] \). Let \( \Lambda \) be the Iwasawa algebra \( \mathcal{O}[[X]] \subset \Lambda_N \). The map

\[
\lambda \mapsto \langle \lambda \rangle
\]

with the right hand side defined as in \((1.1.3)\), makes \( h \) and thus \( h_\ominus \) into continuous \( \Lambda \)-modules, and one of Hida’s main theorems is that

**Theorem.** The Hecke algebra \( h_\ominus \) is free of finite rank over \( \Lambda_\mathcal{O} \). Moreover, let \( \ell \geq 2, \ell \equiv 1 \pmod{p-1} \); let \( P_\ell \) denote the element \( (1 + X) - (1 + Np)^\ell \in \Lambda_\mathcal{O} \). Then the projection defines a canonical isomorphism

\[
h_\ominus^\ell/P_\ell h_\ominus^\ell \cong h_\ominus^\ell(Np, \mathcal{O}).
\]

**Remark.** Note that \( \Lambda_\mathcal{O}/P_\ell \Lambda_\mathcal{O} \) is canonically isomorphic to \( \mathcal{O} \) for every integer \( \ell \geq 1 \) and every \( \mathcal{O} \).

Let \( \mathfrak{M}^\ell = \mathfrak{M}^\ell(N, \mathcal{O}) = e \cdot \mathfrak{M} \subset \mathfrak{M}^\ell, \mathfrak{S}^\ell = \mathfrak{S}^\ell(N, \mathcal{O}) = e \cdot \mathfrak{S} \subset \mathfrak{S} \). We define an \( \mathcal{O} \)-bilinear pairing

\[
\langle \bullet, \bullet \rangle : \mathfrak{S}^\ell \times h_\ominus^\ell \to \mathcal{O}; \quad \langle g, T \rangle = a(1, g|T)
\]

where, as usual, we write \( g \mapsto g|T \) for the action of the Hecke operator \( T \). Then \((1.2.5)\) is a perfect pairing, with respect to which the action of \( h_\ominus \) is (tautologically) symmetric. For each \( \ell \geq 2 \), we obtain by restriction a perfect pairing

\[
\langle \bullet, \bullet \rangle : S^\ominus_\ell(Np, \mathcal{O}) \times h_\ominus^\ell(Np, \mathcal{O}) \to \mathcal{O}
\]

defined by the same formula as \((1.2.5)\). The pairing \((1.2.6)\) identifies \( S^\ominus_\ell(Np, \mathcal{O}) \) as the \( P_\ell \)-torsion submodule of \( \mathfrak{S}^\ell \).

**Congruence modules and \( \Lambda \)-adic forms.**

Henceforward we assume \( \mathcal{O} \) to be the ring of integers of a finite extension \( K \) of \( \mathbb{Q}_p \). We let \( \mathcal{L} \) denote the fraction field of \( \Lambda \), \( \mathcal{L}^\prime \) a finite extension of \( \mathcal{L} \), and let \( \mathfrak{I} \) be the integral closure of \( \Lambda \) in \( \mathcal{L}^\prime \). We denote by \( \mathcal{X}(\mathfrak{I}) \) the set of prime ideals of \( \mathfrak{I} \) of height 1, and let

\[
\mathcal{X}_k(\mathfrak{I}) = \{ P \in \mathcal{X}(\mathfrak{I}) | P \cap \Lambda_\mathcal{O} = P_k \}.
\]

For \( P \in \mathcal{X}(\mathfrak{I}) \) let \( \mathcal{O}_P = \mathfrak{I}/P \); we let \( k(P) = k \) if \( P \in \mathcal{X}_k(\mathfrak{I}) \).

Let \( \tau = \tau_f : h^\ell \to \mathfrak{I} \) be a homomorphism of \( \Lambda_\mathcal{O} \)-algebras. The subscript \( f \) refers implicitly to a family of \( p \)-adic modular forms, or to a \( \Lambda \)-adic modular form, in the sense of Wiles, cf. [H4].

**Definition.** We say \( \tau_f \), or \( f \), is \( N \)-primitive if, for some integer \( k > 1 \) (equivalently, for all \( k > 1 \)) and for some \( P \in \mathcal{X}_k(\mathfrak{I}) \), the homomorphism

\[
\tau_f(\text{mod} P_k) : h^\ell/P_k h^\ell \to \mathcal{O}_P = \mathfrak{I}/P
\]

(cf. Remark \((1.2.4)\)) is the homomorphism associated to a modular form of weight \( k \) of level dividing \( Np \) which is primitive at all primes dividing \( N \).

From now on, we consider such an \( N \)-primitive form \( f \).
(1.3.2) Proposition [H2, Cor. 3.7]. If $f$ is primitive, then the homomorphism $	au_f : \mathbb{h}^o \otimes_L \mathcal{L}' \to \mathcal{L}'$ is split over $\mathcal{L}'$.

Thus there are an idempotent $1 \in \mathbb{h}^o \otimes_L \mathcal{L}'$ and an isomorphism $1_f : \mathbb{h}^o \otimes_L \mathcal{L}' \sim \mathcal{L}'$ such that the homomorphism $\tau_f \otimes 1_{\mathcal{L}'}$ is given by multiplication by $1_f$. We write

\begin{equation}
\mathbb{h}^o \otimes_L \mathcal{L}' \sim \mathcal{L}' \oplus \mathcal{B};
\end{equation}

then $\tau_f$ corresponds to projection on the first factor.

Let $\mathbb{h}^o_f = \mathbb{h}^o \otimes_L \mathbb{I}$, $h(\mathcal{B}) = im(h^o_1) \subset \mathcal{B}$ with respect to the second projection in (1.3.3). It follows from Theorem (1.2.3) that (1.3.3) induces an injection $\mathbb{h}^o_1 \hookrightarrow \mathbb{I} \oplus h(\mathcal{B})$ of $\mathbb{I}$-modules. The congruence module

\begin{equation}
\mathcal{C} = (\mathbb{I} \oplus h(\mathcal{B}))/\mathbb{h}^o_1.
\end{equation}

is an $\mathbb{I}$-torsion module. On the other hand, if we take the natural embedding

\[ i : \mathbb{I} \hookrightarrow \mathbb{I} \oplus h(\mathcal{B}); \quad \lambda \mapsto (\lambda, 0) \]

then $\mathbb{H}_f = \mathbb{I} \cap i^{-1}(\mathbb{h}^o_1) \subset \mathbb{I}$ is an ideal in $\mathbb{I}$. Then $i$ induces an isomorphism $\mathcal{C} \xrightarrow{\sim} \mathbb{I}/\mathbb{H}_f$ of $\mathbb{I}$-modules. Thus

\begin{equation}
Denominator(1_f) = \{ a \in \mathbb{I} | 1_f \in \mathbb{h}^o_1 \} = \mathbb{H}_f.
\end{equation}

Once and for all we fix an element $H \in \mathbb{H}_f$ and define

\[ T_f = T_{f, H} = H \cdot 1_f \in \mathbb{h}^o_1. \]

Remark:

By assumption, $\mathbb{f}^\text{prim}_p$ is $N$-primitive. Let $\omega : (\mathbb{Z}/p\mathbb{Z})^\times \to \mathcal{O}$ be the Teichmüller character. Write

\[ \mathbb{h}^o = \Pi_{a \in \mathbb{Z}/(p-1)\mathbb{Z}} \mathbb{h}^o(\omega^a), \]

where $\mathbb{h}^o(\omega^a) \subset \mathbb{h}^o$ is the subalgebra on which $(\mathbb{Z}/p\mathbb{Z})^\times \subset \mathbb{Z}_N$ acts via the ath power $\omega^a$ of $\omega$; let $\mathbb{h}^o_1(\omega^a) = \mathbb{h}^o(\omega^a) \otimes_L \mathbb{I}$. Since $\mathbb{I}$ is assumed to be an integral domain, it follows that $\tau_f : \mathbb{h}^o_1 \to \mathbb{I}$ factors through the natural projection on $\mathbb{h}^o_1(\omega^a)$, for exactly one $a = a(f)$, say. For any $k \geq 2$ and any $a \in \mathbb{Z}/(p-1)\mathbb{Z}$ there is an isomorphism

\[ \mathbb{h}^o(\omega^a)/\mathbb{P}_k \mathbb{h}^o(\omega^a) \sim \mathbb{h}^o_1(\mathbb{N}_p, \omega^{a-k}, \mathcal{O}), \]

where $\mathbb{h}^o_1(\mathbb{N}_p, \omega^{a-k}, \mathcal{O}) \subset \mathbb{h}^o_1(\mathbb{N}_p, \mathcal{O})$ is the subalgebra on which the quotient $\mathbb{I}_{(1)}(\mathbb{O})/\mathbb{I}_{(n)}(\mathbb{O}) \sim (\mathbb{Z}/p\mathbb{Z})^\times$ acts via $\omega^{a-k}$ [H2, p.249]. If $P \in \mathbb{A}_k(\mathbb{I})$, $\mathbb{f}^\text{prim}_p$ is thus unramified at $p$ if and only if $a \equiv k(\text{mod } p-1)$, unless $k = 2$, in which case the $p$-component of the automorphic representation attached to $\mathbb{f}$ may be special; cf. [H1, II, p. 37].

(1.4) Arithmetic $p$-adic measures. Let $Z$ be a $p$-adic manifold of the form $\mathbb{Z}_p^r \times (\text{finite group})$. Let $\mathcal{O}$ be as in (1.1), and let $\mathcal{C}(Z, \mathcal{O})$ denote the space of continuous $\mathcal{O}$-valued functions on $Z$. For any subring $R \subset \mathcal{O}$ let $\mathcal{L}(Z, R)$ denote the space of locally constant $R$-valued functions on $Z$.

Let

\[ \text{Mes}(Z, \overline{\mathcal{M}}) = \text{Hom}_{\mathcal{O}}(\mathcal{C}(Z, \mathcal{O}), \overline{\mathcal{M}}) \]

be the set of $p$-adic measures on $Z$ with values in $\overline{\mathcal{M}}$. The measure $\mu \in \text{Mes}(Z, \overline{\mathcal{M}})$ is called ordinary (resp cuspidal) if it takes values in $\overline{\mathcal{M}}^\times$ (resp in $\overline{\mathcal{S}}$). As usual, we write $\int_Z \phi d\mu$ in place of $\mu(\phi)$. We assume $Z$ given with a continuous $Z_N$-action, denoted $(z, x) \mapsto z \cdot x$, for $z \in Z_N, x \in Z$. We let $z \mapsto z_p$ be the projection of $z \in Z_N$ on its $p$-adic part $z_p \in \mathbb{Z}_p^r$. 
(1.4.1) Definition. Let $\kappa$ be an integer and $\xi : \mathbb{Z}_N \to \mathcal{O}^\times$ a character of finite order. A $p$-adic measure on $\mathbb{Z}$ with values in $\overline{M}$ is arithmetic with character $(\kappa, \xi)$ (cf. [H1, II, (5.1)]) if
(a) for all $\phi \in \mathcal{L}(\mathbb{Z}, \mathcal{O} \cap \mathbb{Q})$
$$
\int_\mathbb{Z} \phi d\mu \in M_\kappa(Np^\infty, \overline{\mathbb{Q}});
$$
(b) For all $\phi$ as above
$$
\left( \int_\mathbb{Z} \phi d\mu \right) z = z_p^\kappa \xi(z) \int_\mathbb{Z} \phi |z d\mu
$$
where $\phi|z(x) = \phi(z \cdot x)$ for all $z \in \mathbb{Z}_N$.
(c) Let $d$ denote the operator of (1.1.4). There is a continuous function $\nu : \mathbb{Z} \to \mathcal{O}$ such that
$$
\nu |z = z_p^2 \nu
$$
for all $z \in \mathbb{Z}_N$ and such that, for any $\phi$ as above,
$$
d^r \left( \int_\mathbb{Z} \phi d\mu \right) = \int_\mathbb{Z} \nu^r \phi d\mu.
$$
Let $\mathcal{O}_{-\kappa, \xi^{-1}}$ be $\mathcal{O}$, viewed as $\mathcal{O}[[\mathbb{Z}_N]]$-module via the linear extension of the character $z \mapsto z_p^\kappa \xi^{-1}(z)$. Condition (b) can be rephrased:
(b′) The map
$$
\int_\mathbb{Z} \cdot d\mu : \mathcal{C}(\mathbb{Z}, \mathcal{O}) \to \overline{S}^0 \otimes \mathcal{O}_{-\kappa, \xi^{-1}}
$$
is $\mathfrak{A}_N$-linear, where $\mathfrak{A}_N$ acts on $\mathcal{C}(\mathbb{Z}, \mathcal{O})$ by $\mathcal{O}$-linear extension of the action $\phi \mapsto \phi|z$.

Comments: Let us give a useful reformulation of this definition when $\mathbb{Z} = \mathbb{Z}_p^\times$. The action of $\mathbb{Z}_N$ on $\overline{M}$ given by $z \mapsto <z>$ endows $\mathfrak{h}^0$ with a natural $\mathfrak{A}_N$-algebra structure. Let $\mathfrak{A} = \mathcal{O}[[\mathbb{Z}_p^\times]]$ and
$$
\psi : \mathbb{Z}_N \to \mathfrak{A}^\times, \quad z \mapsto z_p^\kappa \xi(z)[z_p]^2
$$
For any $\mathfrak{A}_N$-algebra $A$, let
$$
A(\psi) = A \otimes_{\mathfrak{A}_N} \mathfrak{A}
$$
where $\mathfrak{A}_N \to \mathfrak{A}$ is the natural extension of $\psi$. We view $A(\psi)$ as a $\mathfrak{A}_N$-algebra. Then, the group of ordinary cuspidal measures on $\mathbb{Z}_p^\times$ with character $(\kappa, \xi)$ can be identified with the $\psi$-isotypic $\mathfrak{A}$-submodule of $(\overline{S}^0)^\psi$ of $\overline{S}^0 \otimes \mathfrak{A}$ defined by
$$
(\overline{S}^0)^\psi = \bigcap_{z \in \mathbb{Z}_p^\times} (\overline{S}^0 \otimes \mathfrak{A})^{z \otimes 1 = 1 \otimes \psi(z)}
$$
As an example, we can take the following
(1.4.2) Example. Here $\mathbb{Z} = \mathbb{Z}_p^\times$ and $\nu$ in (c) is the tautological inclusion $\mathbb{Z} \hookrightarrow \mathcal{O}$.
The action of $\mathbb{Z}_N$ on $\mathbb{Z}$ is given by
$$
z \cdot x = z_p^2 x, \quad z \in \mathbb{Z}_N, x \in \mathbb{Z}$$
Let $R$ be the ring of algebraic integers in $\mathbb{C}$, and let $g \in S_{\ell}(N, R)$ for some $\ell$ and some $N$ relatively prime to $p$. We assume $g$ has nebentypus character $\xi$. Let $g_p$ be the twist of $g$ by the trivial character (mod $p$): If $g = \sum_{n=1}^{\infty} a(n, g) q^n$, we have $g_p = \sum_{(n,p)=1} a(n, g) q^n$. For any function $\phi \in L C(Z, R)$ set

\begin{equation}
\int_{Z} \phi \cdot d\mu_g = \sum_{(n,p)=1} \phi(n) a(n, g) q^n.
\end{equation}

Extend this by continuity to $C(Z, O)$ for varying $O$. Hida has verified (cf. [H1, I, Prop. 8.1]) that $d\mu_g$ is arithmetic with character $(\ell, \xi)$; its “moments” are given by

\begin{equation}
\int_{Z} x^r \cdot d\mu_g = d^r(g_p);
\end{equation}

here and in what follows we write $x^r = \nu(x)^r$.

**Example.** Here $Z, R, N,$ and $g$ are as in (1.4.2), and we let $h \in S_m(N, R)$ for some $m$. We let $\xi_g$ and $\xi_h$ denote the nebentypus characters of $g$ and $h$, respectively. The measure $h \cdot d\mu_g$ is defined by

\begin{equation}
\int_{Z} \phi \cdot h \cdot d\mu_g = h \cdot \sum_{n=1}^{\infty} \phi(n) a(n, g) q^n.
\end{equation}

It follows from (1.4.2) that $h \cdot d\mu_g$ is arithmetic with character $(\ell + m, \xi_g \cdot \xi_h)$; its “moments” are then given by

\begin{equation}
\int_{Z} x^r \cdot h \cdot d\mu_g = h \cdot d^r(g_p).
\end{equation}

**Constructions of arithmetic measures by Hida families.** Unless otherwise specified, we assume

\begin{enumerate}
\item $Z$ is a $p$-adic group containing $\mathbb{Z}_p^\infty$ as an open subgroup of finite index, and with an action of $(\mathbb{Z}/N\mathbb{Z})^\times$.
\end{enumerate}

Think for instance of $Z = \mathbb{Z}_p^\times$ with trivial action of $(\mathbb{Z}/N\mathbb{Z})^\times$, as we will assume in the next chapter.

The identification of the completed group algebra $O[[Z]]$ with the space of continuous $O$-valued distributions on $X$, as for example in [H4], yields an isomorphism

\[ O[[Z]] \cong Hom_O(C(Z, O), O). \]

By extension of scalars, we may thus identify

\begin{equation}
\{ \text{ordinary cuspidal measures on } Z \text{ with values in } \overline{M} \} \cong Hom_O(Hom_O(O[[Z]], O), \overline{S}).
\end{equation}

Let $Hom_O(Hom_O(O[[Z]], O), \overline{S})_{\kappa, \xi}$ be the $O$-submodule of the right-hand side of (1.4.4.1) corresponding to arithmetic measures with character $(\kappa, \xi)$.

Let $I$ and $\mathfrak{I}$ be as in §1.3 and let $(\kappa, \xi)$ be as in Definition 1.4.1. Let $Mes(Z, \overline{S})_{\kappa, \xi} = (\overline{S})^\psi$ be the set of arithmetic measures on $Z$ with character $(\kappa, \xi)$. Suppose
The pairing (1.2.5) induces by extending the scalars to \( \Lambda \), a pairing
\[
< \cdot, \cdot > : (S^0 \otimes \Lambda) \hat{\otimes}_\Lambda (h^0 \otimes \Lambda) \to \Lambda
\]
hence
\[
Mes(Z, S^0)_{h, \xi} \otimes_{\Lambda_N(\psi)} h^0(\psi) \to \Lambda_N(\psi)
\]
We base change it to \( h(\psi) \) and obtain
\[
(1.4.4.3) \quad Mes(Z, S^0)_{h, \xi} \otimes_{\Lambda_N(\psi)} h^0(\psi) \otimes_{\Lambda_N(\psi)} h^0(\psi) \to h^0(\psi)
\]
We now twist the Hida family \( \tau_f : h^0 \to I(\psi) \). We thus obtain an \( h^0(\psi) \)-algebra \( I(\psi) \). We use this algebra to base change (1.4.4.3). We get
\[
(1.4.4.4) \quad Mes(Z, S^0)_{h, \xi} \otimes_{\Lambda_N(\psi)} h^0(\psi) \otimes_{\Lambda_N(\psi)} I(\psi) \to I(\psi)
\]
(1.4.5) Definition. \( \ell_f : (S^0)^{\psi} \to I(\psi) \) be the \( \Lambda_N(\psi) \)-linear map given by
\[
\mu \mapsto < \mu, T_f \otimes 1 >.
\]
We call it the contraction against the Hida family \( f \).

Applying (1.4.5) to \( \mu = h \cdot d\mu_g \in Mes(Z, S^0)_{k, \xi} = (S^0)^{\psi} \) the desired element
\[
D_H(f, g, h) \in I(\psi).
\]
(1.5) Evaluation of \( D_H(f, g, h) \) at certain arithmetic points.

(1.5.1) Notation.

For any form \( h \in S_k(Np, \mathcal{O}) \), let \( h^\rho = \sum_{n=1}^{\infty} \overline{\alpha(n, h)} q^n \), where \( z \mapsto \overline{z} \) denotes complex conjugation; \( h^\rho \) is also an element of \( S_k(Np, \mathcal{Q}) \). We let
\[
(1.5.1.1) \quad \tilde{h} = h^\rho|_k \begin{pmatrix} 0 & 1 \\ -Np & 0 \end{pmatrix}.
\]

Let us denote by \( < \cdot, \cdot >_{p^m, k} \) the Petersson inner product for \( S_k(Np^m, \mathcal{O}) \), normalized to be linear in the first variable and anti-linear in the second. The formula for \( < \cdot, \cdot >_{p^m, k} \) is given as usual by:
\[
(1.5.1.2) \quad < f_1, f_2 >_{p^m, k} = \int_{\Gamma_0(p^m) \backslash \mathcal{H}} f_1(z) \overline{f_2(z)} y^{k-2} dx dy
\]
whenever \( f_1 \) and \( f_2 \) are modular forms of weight \( k \) for \( \Gamma_0(p^m) \) with same Nebentypus, and one of the two is a cusp form.

(1.5.2) Evaluation. In what follows, we let \( Z = Z^\phi_p \). We denote the set of height 1 prime ideals of \( I \) by \( \mathcal{X}(I) \). Any element of \( I(\psi) \) defines a function on \( \mathcal{X}(I) \).
Let \( f \) (or \( \tau_f \)) be as in 1.3, with \( \mathcal{O}' = \mathcal{O} \), and let \( g \in S_t(N, R) \) and \( h \in S_m(N, R) \) be as in (1.4.3), where \( R \) is the ring of integers in some number field, which we assume contained in \( \mathcal{O} \). The ordinary projection \( e(h \cdot d\mu_g) \) of the measure \( h \cdot d\mu_g \) is naturally an ordinary cuspidal measure on \( Z \) of character \((\kappa, \xi)\) for \( \xi = \xi_g \cdot \xi_h \) and \( \kappa = \ell + m \).

We shall compute special values at arithmetic points of \( D_H(f, g, h) \). For \( P \in \mathcal{X}_k(I) \), for some \( k \geq 2 \), let \( f_P \) be the \( e \)-eigenform associated to \( f_P^{prim} \), with \( q \)-expansion \( \sum_{n=1}^{\infty} a(n, f_P) q^n \), where

\[
a(np^r, f_P) = \alpha(f_P^{prim})^r \cdot a(n, f_P^{prim}) \quad \text{if} \quad (n, p) = 1,
\]

where \( \alpha(f_P^{prim}) \) is the \( p \)-adic unit root of the Hecke polynomial of \( f_P^{prim} \) at \( p \). If the nebentypus of \( f_P^{prim} \) is non-trivial then \( f_P^{prim} = f_P \). In particular, \( f_P \) is of level exactly \( Np \).

Let \( \mathcal{X}^{adm} = \{ P \in \mathcal{X}(I) \mid \exists k = k(P) \geq 2, k \equiv 1 \pmod{p-1}, P \in \mathcal{X}_k(I) \} \).

Then the set \( \mathcal{X}^{adm} \) is Zariski dense in \( \mathcal{X}(I) \). Therefore, the element \( D_H \in I(\psi) \) is determined by its values at points in \( \mathcal{X}^{adm} \). For any \( P \in \mathcal{X}^{adm} \), let \( H(P) \in \mathcal{O}_P \) denote the reduction of \( H \) modulo \( P \). Let \( P \in \mathcal{X}^{adm} \); let \( T_{f,P} = H(P) \cdot T_{f_P} \in h_N^r(\mathcal{O}, \mathcal{O}) \), where \( k = k(P) \). Let \( 2r = k - \ell - m \). Observe that by definition of \( \psi : [z] \mapsto z^{\ell+m} \cdot \xi_g \xi_h(z) \cdot [z_p^r] \), the image of \( P \) under the twisting map \( I \mapsto I(\psi) \) is above \( P_{k-\ell-m} = P \), \( \in \Lambda_{\mathcal{O}} \). Hence, by definition (1.4.5), we have

\[
D_H(f, g, h)(P) = \ell_{T_{f}}(e(h \cdot \int_Z x^r d\mu_g)) \\
(1.5.2.1)
= < e(h \cdot \int_Z x^r d\mu_g) , T_{f,P} > \\
= < e(h \cdot d^r g_p) , T_{f,P} > \\
= \ell_{T_{f}}(e(h \cdot d^r g_p))
\]

by compatibility of the pairings (1.2.5) and (1.2.6).

The \textit{Maass operators} \( \delta_{r}^{f} \), \( r = 1, 2, \ldots \), defined by Maass and Shimura, are the differential operators on the upper half plane given by the formula

\[
\delta_{\ell} = \frac{1}{2\pi i} \left( \frac{\ell}{2iy} + \frac{d}{dz} \right) ; \quad \delta_{r}^{f} = \delta_{\ell+2r-2} \circ \cdots \circ \delta_{\ell+2} \circ \delta_{\ell}.
\]

For any congruence subgroup \( \Gamma \), \( \delta_{r}^{f} \) takes holomorphic cusp forms of weight \( \ell \) for \( \Gamma \) to \( C^\infty \) functions on the upper half plane, rapidly decreasing at infinity and “nearly holomorphic” in Shimura’s sense [S], which transform under \( \Gamma \) like modular forms of weight \( \ell + 2r \). We refer to such functions as nearly holomorphic cusp forms. If \( f_1 \) are nearly holomorphic cusp forms of weights \( m_i \), \( i = 1, 2 \), then the product \( f_1 f_2 \) is a nearly holomorphic cusp form of weight \( m_1 + m_2 \).

If \( G \) is a nearly holomorphic cusp form of weight \( k \), then the holomorphic projection \( \mathcal{H}(G) \) is the unique holomorphic cusp form of weight \( k \) that satisfies

\[
< G, f >_{k} = < \mathcal{H}(G), f >_{k}
\]
for all holomorphic cusp forms $f$. Let $G = h \cdot \delta^r g$ and $G_p = h \cdot \delta^r g_p$. It follows from [H1, I, p. 185; II, Lemma 6.5, (iv)] that

$$e(h \cdot d^r g_p) = e(H(G_p)).$$

Thus, returning to formula (1.5.2.1), we find that

$$D_H(f, g, h)(Q) = \ell_P \circ e(H(G_p)).$$

Finally, appealing to [H1, I, prop. 4.5, II, 7.6], we find that

$$D_H(f, g, h)(Q) = H(P) \cdot \alpha(f_p^{prim})^{-1} \cdot p^{k-1} \cdot \frac{G_p \cdot \tilde{f}_P(pz) > p^2, k}{<f_P, f_P > p^2, k}.$$

### 2. Triple product $L$-functions

#### (2.1) A formula for the central critical value.

We retain the notation of the previous section. Let $f \in S_k(N,R), g \in S_\ell(N,R), h \in S_m(N,R)$ be three modular forms of level $N$, with $k \geq \ell \geq m$. We write their standard Hecke $L$-functions as follows:

$$L(s, ?) = \prod_{(q,N)=1} L_p(s, ?) \times \prod_{q \mid N} L_q(s, ?), \quad ? = f, g, h$$

where, for $(q,N) = 1$ the local Euler factors are of the form

$$L_q(s, f) = [(1 - \alpha_{1, q} q^{-s})(1 - \alpha_{2, q} q^{-s})]^{-1},$$

$$L_q(s, g) = [(1 - \beta_{1, q} q^{-s})(1 - \beta_{2, q} q^{-s})]^{-1},$$

$$L_q(s, h) = [(1 - \gamma_{1, q} q^{-s})(1 - \gamma_{2, q} q^{-s})]^{-1},$$

Here our $L$-functions are normalized so that $|\alpha_{i, q}| = q^{k-1}, |\beta_{i, q}| = q^{\ell-1}, |\gamma_{i, q}| = q^{m-1}, i = 1, 2$, for any archimedean absolute value. The triple product $L$-function is the convolution of these three:

$$L(s, f, g, h) = \prod_{(g.N)=1} \prod_{i, i', i''=1,2} (1 - \alpha_{i, q} \beta_{i', q} \gamma_{i'', q} q^{-s})^{-1} \times \prod_{q \mid N} L_q(s, f, g, h)$$

where the factors $L_q(s, f, g, h)$ for $q \mid N$ are the local Artin $L$-factors of the corresponding Weil-Deligne group representations, defined by reference to the local Langlands correspondence for $GL(2)$.

In what follows we restrict attention to the case $N = 1$, i.e., we assume our forms are all of level 1. This implies in particular that the weights $k, \ell, m$ are all even. We assume that $f, g,$ and $h$ correspond to cuspidal automorphic representations $\pi(f), \pi(g),$ and $\pi(h),$ respectively, of $GL(2)_{\mathbb{Q}}$, with trivial central characters $\xi(f), \xi(g), \xi(h)$, respectively. denote the respective central characters.

The analytic continuation of the triple product $L$-function has been proved by the method of Langlands-Shahidi [Sha] and by a variant of the Rankin method, due to Garrett [G1,G2] and generalized by Piatetski-Shapiro and Rallis [PSR]. It
is known to satisfy a functional equation of the usual type, relating the values at \( s \) and \( w + 1 - s \), where \( w = k + \ell + m - 3 \).

We assume henceforward that

\[
(2.1.3) \quad k \geq \ell + m.
\]

Under hypothesis (2.1.3), a formula is obtained in [HK1] – the Main Identity 9.2 – for the central critical value \( L\left(\frac{w+1}{2}, f, g, h\right) \) of the triple product \( L \)-function. The value is expressed as an integral of theta lifts of \( f, g, h \) to the orthogonal group attached to the split quaternion algebra \( M(2)_\mathbb{Q} \) over \( \mathbb{Q} \); i.e. to the split form of \( \text{O}(4) \).

The exact formula depends on several auxiliary choices. Let \( H \) denote the algebraic group \( (\text{GL}(2) \times \text{GL}(2))/d(\Gamma_m) \), where \( d \) is the diagonal embedding. Then \( H \) is naturally isomorphic to the identity component of the group of orthogonal similitudes of the split quaternion algebra. We let \( S \) be the space of Schwartz-Bruhat functions on \( M(2)(\mathbb{A}) \). To any \( \phi \in S \) that satisfies appropriate finiteness properties (\( K \)-finite for a maximal compact subgroup \( K \) of \( \text{GL}(2, \mathbb{R}) \times \text{O}(2, 2) \) with respect to the Weil representation) and any automorphic form \( F \) on \( \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A}) \), the theta correspondence associates an automorphic form \( \theta_\phi(F) \) on \( H(\mathbb{Q}) \backslash H(\mathbb{A}) \) (see [HK2], (5.1.12) for the precise formula, which also depends on the choice of a measure, specified in [HK1]).

Let \( r = \frac{k-\ell-m}{2} \), which by (2.1.3) is a positive integer. Let \( f^i \) be the normalized newform whose Hecke eigenvalues are the complex conjugates of those of \( f \). In fact \( f^i = f \), since \( N = 1 \), but we leave the notation \( f^i \) with a view to future generalizations. Then \( f^i \) is an antiholomorphic form with the same Hecke eigenvalues as \( f \); in other words, \( f^i \) lifts to an element \( f^i \) of \( \pi(f) \). Similarly, let \( g^i(i) \) and \( h^i \) be liftings of \( \delta_i^j(q) \), \( 0 \leq i \leq r \) and \( h \), respectively, to automorphic forms on \( \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A}) \); i.e., to elements of \( \pi(g) \) and \( \pi(h) \), respectively. Here we are using the fact that the Maass operators correspond to elements of \( \text{Lie}(\text{GL}(2)) \), cf. [HK1, Lemma 12.5] and the references cited there. It follows from [HK1, Theorem 7.2] that, for appropriate choices of \( \phi^i \in \mathcal{S}, i = 1, 2, 3 \), we can arrange that

\[
(2.1.4) \quad \theta_{\phi^i}(f^i) = f^i; \quad \theta_{\phi^i}(g^i(0)) = g^i(r); \quad \theta_{\phi^i}(h^i) = h^i
\]

We abbreviate \( \Phi = (\phi^1, \phi^2, \phi^3), F = (f^i, g^i(r), h^i) \). Let \( d\mu \) be the \( \text{GL}(2, \mathbb{A}) \)-invariant Haar measure on \( \mathbb{A}^\times \cdot \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A}) \) with total volume 1. Then we have the following formula:

**MAIN IDENTITY.** (\([HK1, 9.2]\)):

\[
(2.1.5) \quad Z_\infty(F, \Phi) \cdot L\left(\frac{w+1}{2}, f, g, h\right) = 2\zeta(2)^2 \cdot I(f^i, g^i(r), h^i)^2,
\]

where

\[
(2.1.6) \quad I(f^i, g^i(r), h^i) = \int_{\mathbb{A}^\times \cdot \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A})} f^i \cdot g^i(r) \cdot h^i d\mu
\]

Here \( \zeta(2) \) is the value at \( s = 2 \) of the Riemann zeta function, and \( Z_\infty(\bullet, \bullet) \) is the value at \( s = 0 \) of the normalized local zeta integrals, defined by Garrett and Piatetski-Shapiro-Rallis. The nature of \( Z_\infty(\bullet, \bullet) \) will be discussed in the next section; here we merely remark that the notation of [HK1] has been slightly simplified in the present account.
(2.2) $p$-adic interpolation of certain central critical values.

The Main Identity (2.1.6) can be rewritten

\[
(2.2.1) \quad \left( \frac{< h \cdot \delta_f^*(g), f >_k}{< f, f >_k} \right)^2 = \frac{Z_{\infty}(F, \Phi) L_{\frac{\omega+1}{2}, f, g, h}}{2\xi(2)^2} \left( \frac{< f, f >_k}{< f, f >_k} \right)^2.
\]

The left hand side of (2.2.1) has almost the same form as the square of a special value (1.5.2.2) of the $p$-adic measure constructed in 1.5. The only modification necessary is to replace $f$ by the value $f_P$ at a prime $P$ of an ordinary Hida family, and to incorporate the twist $f_P \mapsto \tilde{f}_P$.

Write $G = h \cdot \delta_f^*(g)$ and $G_P = h \cdot \delta_f^*(g_p)$, as in §1.5.2. Let $f, f_P$, and $f_P^{prim}$ be as in section 1.3. In what follows, we let $f = f_P^{prim}$. Recall that we have fixed the auxiliary level $N$ to be 1. Let $\alpha_1 = \alpha_1(f_P)$ be the $p$-adic unit root of the Hecke polynomial of $f_P^{prim}$ at $p$, and let $\alpha_2 = \alpha_2(f_P)$ denote its other root. Recall that $< \bullet, \bullet >_{p^m,k}$ ($m \geq 0$) has been defined in (1.3.11).

**Proposition 2.2.2.** With notations as above, the following formula is valid:

\[
< G_P, \tilde{f}_P >_{p^2,k} = \frac{E_p(f_P, g, h)}{p^{1-\frac{k}{2}}(1-\frac{\alpha_2}{\alpha_1})} \cdot < G_P^{prim}, f_P^{prim} >_{1,k},
\]

where

\[
E_p(f_P, g, h) = p^{-k}(p^2\alpha_1-\alpha_2a_p)-p^{2-\frac{\omega+1}{2}}-\alpha_1b_pc_p+p^{1-\frac{\omega+1}{2}}b_p^2+p^{1-m}c_p^2-\alpha_2p^{1-\frac{\omega+1}{2}}c_p-1
\]

**Proof.** The elements of this calculation are certainly well known to specialists. However, we were unable to find a complete comparison of the two sides in the literature, so we are including all details.

We extend the Petersson inner product $\langle \phi, \psi \rangle_{p^m,k}$ to $C^\infty$ forms of and weight $k$, level $p^m$ with trivial Nebentypus, one of them decreasing rapidly at cusps. Recall that $f_P = f - \alpha_1 \cdot f | [p]$ where:

- $a(p, f) = \alpha_1 + \alpha_2$, $\alpha_1\alpha_2 = p^{k-1}$ and $\alpha_1$ is a $p$-adic unit, and

- $\phi[m] = \phi(mz) = m^{-\frac{k}{2}} \cdot \phi \left( \begin{array}{c} m \\ 0 \\ 1 \end{array} \right)$

for any $\phi$ of weight $\ell$ and any $m \geq 1$.

From

\[
\tilde{f}_P = f_P \left( \begin{array}{cc} 0 & -1 \\ p & 0 \end{array} \right)
\]

using the equality \( \left( \begin{array}{cc} 0 & 1 \\ -p & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right) \), we find

\[
\tilde{f}_P(pz) = p^{-\frac{k}{2}} \cdot f \left( \begin{array}{cc} p^2 & 0 \\ 0 & 1 \end{array} \right) - \alpha_2 \cdot p^{-\frac{k}{2}} f \left( \begin{array}{cc} p & 0 \\ 0 & 1 \end{array} \right).
\]

Set $A(p^m) = \left( \begin{array}{cc} p^m & 0 \\ 0 & 1 \end{array} \right)$. We will repeatedly use the following
Lemma 2.2.3. The following formulas are valid:

1) Given any $C^\infty$ forms $\phi$ and $\psi$ of weight $w$, level 1, with $\phi$ eigen for $T_{p^m}$ of eigenvalue $\lambda_{p^m}$:

\begin{align*}
(i) & \quad <\phi, \psi|_w A(p^m) >_{p^m,w} = p^{m(1-\frac{w}{2})} \cdot \lambda_{p^m} < \phi, \psi >_{1,w}.
(ii) & \quad <\phi, \psi|_{p^m} = [\Gamma_0(p^m) : SL(2, \mathbb{Z})] < \phi, \psi >_{1,w}.
(iii) & \quad <\phi|_w A(p), \psi|_w A(p) >_{p^m,w} = (p+1) < \phi, \psi >_{1,w}.
\end{align*}

2) Similarly, if $\psi$ has level $p$ and $\phi$ level 1 and is eigen for $T_{p}$, one has:

\begin{align*}
(iv) & \quad <\phi, \psi|_w A(p) >_{p^2,w} = p^{1-\frac{w}{2}} \cdot \lambda_p < \phi, \psi >_{1,w} - < \phi|w A(p), \psi >_{p,w}.
\end{align*}

Proof (of Lemma 2.2.3). We proceed as in [PR,4.2 or H1, II, Lemma 5.3] by observing

$$SL(2, \mathbb{Z}) A(p^m) \Gamma_0(p^m) = SL(2, \mathbb{Z}) A(p^m)$$

and, if $\phi, \psi \in S_k(SL(2, \mathbb{Z}))$ and $\gamma \in GL(2, \mathbb{Q})$ has positive determinant,

$$< \phi, \psi|_{SL(2, \mathbb{Z}) \gamma \Gamma_0(p^m)} >_{p^m,w} = < \phi|_{SL(2, \mathbb{Z}) \gamma \Gamma_0(p^m) A(p)} , \psi >_{1,w}.$$ 

Here $\gamma \mapsto \gamma^t$ is the main involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Moreover, one checks that if

$$\Gamma_0(p^m) \begin{pmatrix} p^m \\ 0 \\ 0 \\ 1 \end{pmatrix} SL(2, \mathbb{Z}) = \prod_i \Gamma_0(p^m) \alpha_i$$

then

$$SL(2, \mathbb{Z}) \begin{pmatrix} p^m \\ 0 \\ 0 \\ 1 \end{pmatrix} SL(2, \mathbb{Z}) = \prod_i SL(2, \mathbb{Z}) \alpha_i$$

and since $\phi$ has level 1, one finds

$$p^m(\frac{w}{2} - 1) < \phi, \psi|_k A(p^m) >_{p^m,w} = < \phi|T_{p^m}, \psi >_{1,w} = \lambda_p < \phi, \psi >_{1,k},$$

which proves (i).

Next, assertion (ii) is obvious, and (iii) is similar, when $\Gamma_0(p^m)$ is replaced by $A(p^m)\Gamma_0(p^m)A(p^m)^{-1}$, which has the same index in $SL(2, \mathbb{Z})$.

For assertion (iv), one observes the equality of sets

$$\Gamma_0(p) A(p) = \Gamma_0(p) A(p) \Gamma_0(p^2)$$

then one uses the adjunction formula for

$$[\Gamma_0(p) A(p) \Gamma_0(p^2)]$$

together with the fact that $\Gamma_0(p^2) A(p) \Gamma_0(p) = U_p = \Gamma_0(p) A(p) \Gamma_0(p)$ admit a same set of representatives.

Step 1: Computation of $< G_p, \tilde{f}_p >_{p^2,k}$
We have
\[ <G_p, \tilde{f}_P >_{p^2,k} = p^{-k} \cdot (f|A(p^2)) - \alpha_2 \cdot p^{-\frac{k}{2}} f|A(p), h \cdot \delta_{\ell}(g|\tau_p)) >_{p^2,k} \]
where, if \( g = \sum_{n \geq 1} b_n q^n \), one has \( g|\tau_p = \sum_{(n,p)=1} b_n q^n \) and \( r = k - \ell - m \).
Let us observe that since \( g \) is a Hecke eigenform,
\[ g|\tau_p = g|(-T_p[p] + p^{\ell-1}[p^2]) \]
Therefore,
\[(2.2.4) \quad h \cdot \delta_{\ell}(g|\tau_p) = h \cdot \delta_{\ell} g - b_p p^{-\frac{k}{2}} h \cdot \delta_{\ell} \left(g\left(\begin{array}{c} p \\ 0 \\ 1 \end{array}\right)\right) + p^{-1} h \cdot \delta_{\ell} \left(g\left(\begin{array}{c} p^2 \\ 0 \\ 1 \end{array}\right)\right) \]
Recall by the way that \( \delta_{\ell}(g|\alpha) = (\delta_{\ell} g)|_{\ell+2,\alpha} \).
Now, by substituting (2.2.4) in the inner product
\[ \langle f|A(p^2) - \alpha_2 \cdot p^{-\frac{k}{2}} f|A(p), h \cdot \delta_{\ell}(g|\tau_p) >_{p^2,k} \]
one obtains a sum of six terms that we compute separately. Let \( G = h \cdot \delta_{\ell} g \).

\[ T_1 = \langle f|A(p^2), G >_{p^2,k} \]
Since \( G \) has level 1, we can apply Lemma 2.2.3 (i); one finds
\[ T_1 = p^{2-k} \cdot \langle f, G|T_{p^2} >_{1,k} = p^{2-k} (a_p^2 - p^{k-1}) \cdot \langle f, G >_{1,k} \]
\[ T_2 = -\alpha_2 p^{-\frac{k}{2}} \langle f|A(p), G >_{p^2,k} \]
Observe
\[ \langle f|A(p), G >_{p^2,k} = g \cdot \langle f\left(\begin{array}{c} p \\ 0 \\ 1 \end{array}\right), G >_{p,k} \]
Then, by the same reasoning as above, one has
\[ T_2 = -\alpha_2 p^{-\frac{k}{2}} a_p \cdot \langle f, G >_{1,k} \]
\[ T_3 = -b_p \cdot p^{-\frac{k}{2}} \cdot \langle f|A(p^2), h \cdot \delta_{\ell} g|A(p) >_{p^2,k} \]
One rewrites \( T_3 \) as
\[ -b_p \cdot p^{1-\frac{k+m}{2}} \cdot \langle f\left(\begin{array}{c} p \\ 0 \\ 1 \end{array}\right), \delta_{\ell} g |^{k-m} \rangle |G_0(p)\left(\begin{array}{c} p \\ 0 \\ 1 \end{array}\right) \Gamma_0(p^2), h >_{p^2,m} \]
By Lemma 2.2.3 (iv), one gets

\[ p^{\frac{m}{2} - 1} \langle \phi | A(p), h \rangle_{p^2, m} = \langle \phi, h | U_p | p, m \rangle \]

where \( h | U_p = h | T_p - p^{\frac{m}{2} - 1} \cdot h | A(p) \). Thus, one has

\[ T_3 = -p^{1 - \frac{k + m}{2}} b_p c_p \cdot \langle f | A(p) \cdot \overline{\delta g y^{k - m}}, h \rangle_{p, m} + p^{-\frac{k}{2}} b_p \langle f | A(p) \cdot \overline{\delta g y^{k - m}}, h | A(p) \rangle_{p, m} \]

and finally,

\[ T_3 = -p^{2 - \frac{k + m}{2}} a_p b_p c_p \cdot \langle f, G \rangle_{1, k} + p^{1 - \frac{k + m}{2}} b_p^2 \cdot \langle f, G \rangle_{1, k} \]

•

\[ T_4 = \alpha_2 b_p p^{\frac{k + m}{2}} \cdot \langle f | A(p), h \cdot \delta g | A(p) \rangle_{p^2, k} \]

We rewrite it as

\[ \alpha_2 b_p p^{\frac{k + m}{2}} \cdot \langle (f \delta g y^{k - m}) | A(p), h \rangle_{p^2, m}. \]

That is,

\[ T_4 = \alpha_2 b_p c_p p^{2 - \frac{k + m}{2}} \cdot \langle f, G \rangle_{1, k} \]

•

\[ T_5 = p^{-1} \cdot \langle f | A(p^2), h \cdot \delta g | A(p^2) \rangle_{p^2, k}. \]

By the same calculation, we get

\[ T_5 = p^{1 - m} (c_p^2 - p^{m - 1}) \cdot \langle f, G \rangle_{1, k} \]

•

\[ T_6 = -\alpha_2 p^{1 - \frac{k}{2}} \cdot \langle f | A(p), h \cdot \delta g \cdot A(p^2) \rangle_{p^2, k} \]

which is equal to

\[ -\alpha_2 p^{1 - \frac{k}{2}} p^{1 - \frac{m}{2}} \langle (f \delta g y^{k - m}) | A(p), h \rangle_{p^2, m}. \]

Hence by adjunction

\[ T_6 = -\alpha_2 p^{1 - \frac{k}{2}} [p^{1 - \frac{m}{2}} c_p \langle f \delta g y^{k - m}, h \rangle_{p, m} - \langle f \delta g y^{k - m}, h | A(p^2) \rangle_{p, m}] \]

Note that \( \langle f \delta g y^{k - m}, h \rangle_{p, m} = p \cdot \langle f \delta g y^{k - m}, h \rangle_{1, m} \) and

\[ \langle f \delta g | A(p) \cdot y^{k - m}, h | A(p) \rangle_{p, m} = \langle f, (\delta g \cdot h) | A(p) \rangle_{p, k} = p^{1 - \frac{k}{2} p} \cdot \langle f, G \rangle_{1, k} \]

Therefore,

\[ T_6 = -\alpha_2 p^{1 - \frac{k + m}{2}} c_p \cdot \langle f, G \rangle_{1, k} + \alpha_2 a_p p^{-k} \cdot \langle f, G \rangle_{1, k} \]
The sum of the terms $T_i (i = 1, \ldots , 6)$ is the product of $\langle f, G \rangle_{1,k}$ by
\[
p^{-k}(a_p^2 - p^{k-1}) - \alpha_2 p^{-k} a_p - p^{-2 - \frac{k+m}{2}} a_p b_p c_p + p^{1 - \frac{k+m}{2}} b_p^2 + \alpha_2 b_p c_p p^{2 - \frac{k+m}{2}} + \\
+ p^{1-m}(c_p^2 - p^{m-1}) - \alpha_2 p^{1 - \frac{k+m}{2}} c_p + \alpha_2 a_p p^{-k}
\]
That is,
\[
(2.2.5) \quad <G_p, \tilde{f}_P >_{p,k} = E_p(f_P, g, h)
\]

**Step 2: Computation of $< f_P, \tilde{f}_P >_{p,k}$**

\[
< f_P, \tilde{f}_P >_{p,k} = < f, f|_k A(p) >_{p,k} - \alpha_2 \cdot p^{\frac{k}{2}} < f|_k A(p), f|_k A(p) >_{p,k} \\
- \alpha_2 \cdot p^{\frac{k}{2}} < f, f >_{p,k} + \alpha_2^2 \cdot p^k < f|_k A(p), f >_{p,k}.
\]

It follows from Lemma 2.2.3 that
\[
< f_P, \tilde{f}_P >_{p,k} < f, f >_{1,k} = p^{-\frac{k}{2}}(p \cdot a_p - 2(p + 1)\alpha_2 + p^{1-k} \alpha_2 \cdot \pi_p).
\]

Since $f$ is of level 1, $\pi_p = a_p$ and $\alpha_1 \cdot \alpha_2 = p^{k-1}$. Thus,

\[
(2.2.6) \quad < f_P, \tilde{f}_P >_{p,k} < f, f >_{1,k} = p^{1-\frac{k}{2}} \alpha_1(1 - \frac{\alpha_2}{\alpha_1})(1 - \frac{\alpha_2}{p\alpha_1}).
\]

Proposition 2.2.2 now follows immediately by combining (2.2.5) and (2.2.6).

Recall we put
\[
E_p(f_P, g, h) = p^{-k}(p^2\alpha_1^2 - \alpha_2 a_p) - p^{2 - \frac{k+m}{2}} \alpha_1 b_p c_p + p^{1 - \frac{k+m}{2}} b_p^2 + p^{1-m}c_p^2 - \alpha_2 p^{1 - \frac{k+m}{2}} c_p - 1
\]

Let $S(P) = (1 - \frac{\alpha_2}{\alpha_1})(1 - \frac{\alpha_2}{p\alpha_1})$. It follows from Lemma 2.2.2 that the right-hand side of (1.5.2.2) equals

\[
(2.2.7) \quad H(P) \cdot \alpha_1^{-2} p^{k-2} \frac{E_p(f_P, g, h)}{S(P)} < G, f >_{1,k}.
\]

Let
\[
K(P) = \alpha_1^{-2} p^{k-2} \frac{E_p(f_P, g, h)}{S(P)}
\]

Combining (1.5.2.2) with (2.2.1), we then obtain our main result.
Theorem 2.2.8. Let $f$ be a $p$-adic family of ordinary cusp forms, in the sense of 1.3, unramified outside $p$. Let $g$ and $h$ be cusp forms of weights $\ell$ and $m$, respectively, of level 1. Let $H$ be an annihilator of the congruence module attached to $f$, and let $D_H(f, g, h)$ be the generalized $p$-adic measure constructed in section 1.4. For any integer $k \geq 2$, $k \equiv 1 \pmod{p-1}$ and for $P \in X_k(I)$, the value of $D_H(f, g, h)$ at $P$ is related to the central critical value $s = \frac{w+1}{2}$ of $L(s, f_P^{\text{prim}}, g, h)$ by the following formula:

\[
\left( \frac{D_H(f, g, h)(P)}{H(P) \cdot K(P)} \right)^2 = \frac{Z_{\infty}(F, \Phi)}{2\zeta(2)^2} \frac{L(1, f_P^{\text{prim}}, g, h)}{\langle f_P^{\text{prim}}, f_P^{\text{prim}} \rangle^2}.
\]

(2.3) Refinement of the main formula.

In order to compare the results described in the main formula to accepted conjectures on $p$-adic $L$-functions, or to formulate reasonable conjectures regarding the square roots of $p$-adic $L$-functions along “anti-cyclotomic” variables, it would be necessary to determine the $p$-adic behavior of the archimedean zeta integral $Z_{\infty}(F, \Phi)$ as the weight $k$ varies. The local nature of the calculations in [HK1] makes it clear that $Z_{\infty}(F, \Phi)$ depends only on the weights $k, \ell, m$. Our choices of archimedean data are dictated by the $p$-adic construction, and a full calculation of the archimedean integral would require summing $r = \frac{k-\ell-m}{2}$ separate terms for given $k$. Ikeda has recently computed archimedean triple product zeta integrals under very general hypotheses [I]. In the case $k \geq \ell + m$ his inputs are not quite the same as ours, but his techniques may be applicable to determine $Z_{\infty}(F, \Phi)$ explicitly. We note that $Z_{\infty}(F, \Phi)$ has been determined up to rational multiples in [HK1]. Bearing in mind the slightly different normalization used in [HK1], we find that

\[
\frac{Z_{\infty}(F, \Phi)}{\pi^{4-2k}} \in \mathbb{Q}^*.
\]

Since $\zeta(2) = \frac{\pi^2}{6}$, we recover the statement of the introduction.

We have restricted attention to forms $f, g,$ and $h$ of level 1. Allowing ramification at primes different from $p$ will modify the final formula. We may treat bad finite places $v$ as we have treated the infinite place, choosing local Schwartz-Bruhat functions $\phi_v$. Then nothing will change on the right-hand side of the Main Identity (2.1.6), but the left-hand side will include additional zeta integrals $Z_v(F, \Phi)$, reflecting these choices. Just as in the archimedean case, these local zeta integrals will depend only on the local components of the automorphic representations associated to $f, g,$ and $h$. The qualitative variation of these ramified components in a Hida family has been determined by Hida [H3, pp. 129-133] (the variation in a Hida family attached to Hecke characters of an imaginary quadratic field can be seen quite explicitly). In any case, for fixed conductor $N$, the number of distinct possible bad non-archimedean components is finite, so the possible denominators created by the local integrals $Z_v(F, \Phi)$ remain bounded.

Our restriction to level 1 is more serious at the prime $p$. Requiring that $f_P$ be unramified at $p$ for all $P$ amounts to restricting attention to a single branch of the Hida family $f$, namely to those $k$ congruent to $a(f)$ (mod $p-1$). In general, one wants to allow the conductor of $f_P$ to be divisible by $p$ but not by $p^2$. Removing the restriction that $f_P$ be unramified at $p$ only makes sense if we also allow $g$ and $h$
to have conductor divisible by $p$. In that case, the Main Identity will involve a zeta integral $Z_p(F, \Phi)$, which might introduce additional $p$-adic zeroes or poles. Explicit determination of such a local integral is also extremely difficult. The case of three special representations is considered in the article [GK] of Gross and Kudla; its explicit calculation is one of the most intricate in the theory of $L$-functions.

Anyone who successfully undertakes these calculations should find it easy, using the methods of this paper, to construct a $p$-adic measure in three variables (allowing $f$, $g$, $h$ and the nebentypus characters to vary, always subject to $\xi(f) \cdot \xi(g) \cdot \xi(h) = 1$), whose moments interpolate the square roots of normalized central critical values of triple product $L$-functions.

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