Gap statistics close to the quantile of a random walk

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Abstract
We consider a random walk of \(n\) steps starting at \(x_0 = 0\) with a double exponential (Laplace) jump distribution. We compute exactly the distribution \(p_{k,n}(\Delta)\) of the gap \(d_{k,n}\) between the \(k\)th and \((k + 1)\)th maxima in the limit of large \(n\) and large \(k\), with \(\alpha = k/n\) fixed. We show that the typical fluctuations of the gaps, which are of order \(O(n^{-1/2})\), are described by a universal \(\alpha\)-dependent distribution, which we compute explicitly. Interestingly, this distribution has an inverse cubic tail, which implies a non-trivial \(n\)-dependence of the moments of the gaps. We also argue, based on numerical simulations, that this distribution is universal, i.e. it holds for more general jump distributions (not only the Laplace distribution), which are continuous, symmetric with a well-defined second moment. Finally, we also compute the large deviation form of the gap distribution \(p_{\alpha,n}(\Delta)\) for \(\Delta = O(1)\), which turns out to be non-universal.

Keywords: random walks, extreme statistics, Brownian motion

(Some figures may appear in colour only in the online journal)

1. Introduction

During recent decades, extreme value statistics (EVS) have found a lot of applications in statistical physics, ranging from disordered systems [1–4], directed polymers and stochastic growth processes in the Kardar–Parisi–Zhang universality class [5–12] to fluctuating interfaces [13–16], random matrices [17, 18], random walks and Brownian motions [19–21] all the way to cold atoms [22]. The basic question concerns the distribution of the maximum \(x_{\text{max}}\) (or equivalently the minimum \(x_{\text{min}}\)) among a collection of \(N\) random variables \(x_1, x_2, \ldots, x_N\), and in particular in the limit \(N \to \infty\), i.e. in the thermodynamic limit. This problem is fully
understood in the case of independent and identically distributed (i.i.d.) random variables $x_i$, for which it is well known that there exist three distinct universality classes (Gumbel, Fréchet and Weibull) depending only on the tail of the parent distribution of the variables $\{x_i\}$ [23]. However, in many situations in statistical physics, it turns out that, often, one has to deal with strongly correlated variables [24]. In fact, there exist at present very few exact results for the EVS in strongly correlated systems and it is thus crucial to identify physically relevant models for which the EVS can be computed exactly. A prototypical example of such models is the discrete-time random walk (RW), which constitutes a useful laboratory to test the effects of strong correlations on EVS [25]. Here we are interested in the statistics of the gaps between the consecutive maxima of a discrete-time RW.

Indeed, the distribution of the global maximum $x_{\text{max}}$ (or the minimum $x_{\text{min}}$) is certainly interesting but it gives only a partial information on the system—it concerns one single variable out of $n \gg 1$—and in some cases it is useful to consider the more general question of order statistics which concern the joint statistics of the $k$th maxima $M_{k,n}$, such that $x_{\text{max}} = M_{1,n} > M_{2,n} > \cdots > M_{n+1,n} = x_{\text{min}}$. Natural observables are then the gaps between successive maxima, $d_{k,n} = M_{k,n} - M_{k+1,n}$, which are useful, for instance, to quantify the phenomenon of ‘crowding’ near the extremes [26–29]. In physics, the statistics of the gaps were studied in the context of branching Brownian motions [30, 31], as well as for noisy signals with power spectrum in $1/f^\alpha$ [32], with applications in cosmology [33]. For RWs, such questions related to the $k$th maximum belong to the general realm of ‘fluctuation theory’ [34] and their statistics have been computed using probabilistic methods [35–41]. However, much less is known about the gaps from fluctuation theory (see however [42]).

Rather recently, two of us developed an independent method, based on backward equations (see below), which allowed us to solve exactly the gap statistics for random walks with symmetric exponential jumps [20]. From this exact result, the distribution of the gaps near $x_{\text{max}}$ (i.e. at the ‘edge’), for long random walks, was obtained and it was shown to exhibit a very rich behaviour, which was conjectured to be, to a large extent, universal, i.e. independent of the details of the jump distribution provided it has a well defined second moment. This conjecture at the ‘edge’ was partly confirmed by a more recent work where it was shown that, for long RWs, the same distribution describes the typical fluctuations of the gaps of RWs with symmetric gamma-distributed jumps [43].

The goal of this paper is to reconsider this problem of the gaps of random walks and study their statistics in the ‘bulk’, i.e. far away from $x_{\text{max}}$ near a quantile of the RW, for instance near the median which is at half-way between $x_{\text{min}}$ and $x_{\text{max}}$. We find that the gaps in the bulk also display a very rich behaviour which is quite different from the one found at the edge. We conjecture that this behaviour is also universal.

2. Model and main results

Let us thus consider a one-dimensional RW in continuous space and discrete time defined by

$$x_i = x_{i-1} + \eta_i \quad \text{for } i = 1, \cdots, n \quad \text{starting from } x_0 = 0,$$  \hspace{1cm} (1)

where the $\eta_i$’s are i.i.d. random variables with a double exponential (or Laplace) probability distribution function (PDF)

$$f(\eta) = \frac{e^{-\sqrt{2|\eta|}}}{\sqrt{2\pi} \sigma},$$  \hspace{1cm} (2)

Note that the terms ‘bulk’ and ‘edge’ are borrowed from the terminology used in random matrix theory [44, 45].
where $\sigma^2 = \int_{-\infty}^{\infty} \eta^2 f(\eta) \, d\eta$ is the variance of the jump distribution. Hence in the large $n$ limit the RW in (1) converges to the Brownian motion. Note that for a walk of $n$ steps, there are $n+1$ positions $\{x_0, x_1, \ldots, x_n\}$. We order these positions and define the random variables $M_{k,n}$ as the $k$th maximum among the positions $\{x_i\}$ of the RW (see figure 1), such that

$$M_{1,n} = x_{\text{max}} \geq M_{2,n} \geq \cdots \geq M_{n,n} \geq M_{n+1,n} = x_{\text{min}}.$$  

(3)

Since the jump distribution is symmetric, i.e. $f(\eta) = f(-\eta)$, $x_{\text{max}} = M_{1,n}$ has the same distribution as $-x_{\text{min}} = -M_{n+1,n}$. Similarly $M_{2,n}$ has the same distribution as $-M_{n+2-k,n}$ and more generally $M_{k,n}$ has the same distribution as $-M_{n+2-k,n}$. Therefore the distribution $P_{k,n}(x)$ of $M_{k,n}$ satisfies the relation

$$P_{k,n}(x) = P_{n+2-k,n}(-x).$$  

(4)

2.1. Distribution of the $k$th maximum

We first compute the PDF $P_{k,n}(x)$ of $M_{k,n}$ for a double exponential jump PDF, using the method introduced in [20]. In the limit of large $n$, with $\alpha = k/n$ fixed, one finds that $P_{k=\alpha,n}(x)$ takes the scaling form

$$P_{k=\alpha,n}(x) \approx \frac{1}{\sqrt{n\sigma}} P_{\alpha} \left( \frac{x}{\sqrt{n\sigma}} \right)$$

with $P_{\alpha}(z) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-z^2} \text{erfc} \left( z \sqrt{\frac{\alpha}{2(1-\alpha)}} \right), & z \geq 0 \\ \sqrt{\frac{2}{\pi}} e^{-z^2} \text{erfc} \left( |z| \sqrt{\frac{\alpha}{2(1-\alpha)}} \right), & z < 0. \end{cases}$

(5)

Note that the limiting distribution satisfies the relation $P_{\alpha}(z) = P_{1-\alpha}(-z)$ which reflects the symmetry between maxima and minima noted in equation (4). In figure 2, we plot the PDF $P_{\alpha}(x)$ for $\alpha = 0.1$. We see clearly that it is quite asymmetric, and from the exact expression (5), it is easy to check that $P_{\alpha}(z) \approx 2 \sqrt{\frac{1-\alpha}{\alpha}} e^{-z^2/(2(1-\alpha))}$ for $z \to +\infty$ while

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Figure 1. Left: RW $x_i$ of $n = 30$ steps starting from $x_0 = 0$ with a global maximum $x_{16} = M_{1,30}$ and a global minimum $x_8 = M_{31,30}$. For this realisation, one has for instance $M_{3,30} = x_5$ and $M_{5,30} = x_28$. Right: values of $M_{k,n}$ as a function of $k$ for this particular realisation.

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3 Note that we use the notation $f(x) \approx g(x)$ as $x \to x_0$ (respectively $f_n \approx g_n$ as $n \to \infty$) to indicate that $f(x)/g(x) = 1 + o(1)$ (respectively $f_n/g_n = 1 + o(1)$) in this limit.
\[ P_\alpha(z) \approx \frac{2}{\pi |z|} \sqrt{\frac{\alpha}{1 - \alpha}} e^{-z^2/(2\alpha)} \text{ for } z \to -\infty. \]

Close to \( z = 0 \), where \( P_\alpha(0) = \sqrt{\frac{2}{\pi}} \), the distribution \( P_\alpha(z) \) has a cusp. From this result (5), one can easily compute the first moment \( \langle M_{k,n} \rangle \) in the limit of large \( n \) and \( k \) with \( k/n = \alpha \) fixed

\[ \langle M_{k,n} \rangle \approx \sigma \sqrt{n} \mathcal{M} \left( \alpha = \frac{k}{n} \right), \quad \mathcal{M}(\alpha) = \sqrt{\frac{2}{\pi}} \left( \sqrt{1 - \alpha} - \sqrt{\alpha} \right). \quad (6) \]

Note that \( \mathcal{M}(1 - \alpha) = -\mathcal{M}(\alpha) \), in agreement with the property in (4). In the limit \( \alpha \to 0 \), one recovers that \( \langle x_{\text{max}} \rangle = \langle M_{1,n} \rangle \approx \sigma \sqrt{n} \mathcal{M}(0) = \sigma \sqrt{2n/\pi} \), as expected from the Brownian motion result.

Incidentally, in this scaling limit where both \( n \) and \( k \) are large with \( k/n = \alpha \) fixed, \( M_{k,n} \) corresponds exactly to what is called, in the probability literature, the \((1 - \alpha)\)-quantile for Brownian motion [37]. Roughly speaking, \( M_{k=n,n} \) is such that a fraction \((1 - \alpha)\) of the points of the trajectory of the random walk are below \( M_{k=n,n} \), while a fraction \( \alpha \) of the points are above it. Since in the large \( n \) limit the RW converges to Brownian motion, \( M_{k=n,n}/(\sigma \sqrt{n}) \), converges, in the scaling limit where both \( n \) and \( k \) are large with \( k/n = \alpha \) fixed, to the \((1 - \alpha)\)-quantile \( q_{1-\alpha} \) of the Brownian motion, i.e.

\[ \frac{M_{k=n,n}}{\sigma \sqrt{n}} \xrightarrow{n \to \infty} q_{1-\alpha} = \inf \{ x : \int_0^1 \Theta(x(\tau) - x) \, d\tau \geq 1 - \alpha \}, \quad (7) \]

where \( x(\tau) \) is a standard Brownian motion (with diffusion constant \( D = 1/2 \)) starting from 0 on the time interval \([0, 1]\) and \( \Theta(x) \) is the Heaviside theta function. In fact, one can check that the formula for \( P_\alpha(z) \) in equation (5), obtained here using a backward-equation formalism, coincides with the result obtained previously in the mathematics literature using quite different probabilistic methods for Brownian motion [37–39].
2.2. Distribution of the kth gap

Our main new results concern the gaps
\[ d_{k,n} = M_{k,n} - M_{k+1,n} \geq 0, \quad k = 1, \ldots, n, \] (8)
between two consecutive maxima. For a double exponential jump distribution (2), the mean value of the gap \( \langle d_{k,n} \rangle = \langle M_{k,n} \rangle - \langle M_{k+1,n} \rangle \) can be computed for any finite \( n \) and \( k \), yielding [20]
\[ \frac{\langle d_{k,n} \rangle}{\sigma} = \frac{\Gamma (k + 1.5)}{\sqrt{2\pi k!}} + \frac{\Gamma (n - k + 1.5)}{\sqrt{2\pi (n - k + 1)!}}. \] (9)

It is straightforward to extract the large \( n \) behaviour of this exact expression (9) in the two scaling regimes corresponding to \( k = O(1) \) and \( k = O(n) \) as
\[ \frac{\langle d_{k,n} \rangle}{\sigma} \approx \begin{cases} \frac{\Gamma (k + 1.5)}{\sqrt{2\pi k!}} & \text{, } n \to \infty, \quad k = O(1) \\ \frac{\mu(\alpha)}{\sqrt{n}} & \text{, } n \to \infty, \quad \alpha = \frac{k}{n} = O(1), \end{cases} \] (10)
where the scaling function \( \mu(\alpha) \) reads
\[ \mu(\alpha) = \frac{1}{2\pi n} \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{1 - \alpha}} \right). \] (11)

Note that one can check that \( \mu(\alpha) = -M'(\alpha) \), where \( M(\alpha) \) is given in equation (6), as expected from equations (8) and (6). This result (10) clearly shows that, for large \( n \), there are two different scales for the gaps \( d_{k,n} \) depending on \( k = O(1) \) or \( k = O(n) \). It is useful to think about the values of the \( k \)th maxima of the random walks after step \( n \) as a point process on the line, as illustrated in figure 3. Near the edges, i.e. near the maximum \( x_{\max} = M_{1,n} \) and the minimum \( x_{\min} = M_{n+1,n} \), the gaps are of order \( O(1) \) (see the first line of equation (10)) while they are of order \( O(n^{-1/2}) \) (see the second line of equation (10)) in the bulk, i.e. far from \( x_{\max} \) and \( x_{\min} \). Note that by taking the large \( k \) limit in the first line of equation (10) one obtains \( \langle d_{k,n} \rangle \approx \sigma / \sqrt{2\pi k} \), while by taking the small \( \alpha = k/n \) limit in the second line of equation (10) one also obtains \( \langle d_{k,n} \rangle \approx \sigma / \sqrt{2\pi \alpha n} = \sigma / \sqrt{2\pi k} \); this shows that there is a smooth matching between the edge and the bulk at the level of the first moment \( \langle d_{k,n} \rangle \).

What about the full PDF \( p_{k,n}(\Delta) \) of the gaps \( d_{k,n} \) in the large \( n \) limit? Near the edge, this PDF was computed for jumps with a double exponential jump distribution in [20] and subsequently for symmetric gamma-distributed jumps in [43]. In particular, it was shown in [20] that for a double exponential jump distribution, the PDF \( p_{k,n}(\Delta) \) becomes independent of \( n \) in the large \( n \) limit, i.e. \( \lim_{n \to \infty} p_{k,n}(\Delta) = p_{k,\infty}(\Delta) \), consistent with the first line of equation (10). In the large \( k \) limit, it turns out [20] that the limiting distribution \( p_{k,\infty}(\Delta) \) has two different scaling behaviours, depending on \( \Delta \): (i) a regime of typical fluctuations for \( \Delta = O(1/\sqrt{k}) \) and (ii) a large deviation regime for \( \Delta = O(1) \). The most interesting result obtained in [20] concerns the typical fluctuations where \( p_{k,n}(\Delta) \) takes the scaling form [20]
\[ p_{k,n}(\Delta) \approx \frac{\sqrt{k}}{\sigma} P \left( \frac{\sqrt{k} \Delta}{\sigma} \right) \] (12)
where the scaling function \( P(\hat{\delta}) \) is given by [20]
Based on numerical simulations, it was conjectured in [20] that the typical distribution $P(\delta)$ is universal, i.e. it does not depend on the jump distribution $f(\eta)$ as long as it is symmetric and has a finite variance $\sigma^2 < \infty$. The validity of this conjecture was then reinforced by an exact analytical computation for gamma distributed jump distribution $f(\eta) = |\eta|^p e^{-|\eta|}$ with $p \in \mathbb{N}$ [43]. From this expression (13), it is easy to obtain the asymptotic behaviours of $P(\delta)$ for small and large $\delta$

$$P(\delta) \approx \begin{cases} 
4 \sqrt{\frac{2}{\pi}} (1 + 2\delta^2) - \delta(4\delta^2 + 3)e^{2\delta^2} \text{erfc}(\sqrt{2}\delta), & \delta \to 0, \\
\frac{3}{\sqrt{2\pi}} \frac{1}{\delta}, & \delta \to \infty.
\end{cases}$$

(14)

In particular, it exhibits an interesting power law tail $P(\delta) \propto \delta^{-4}$ for large $\delta$. The large deviation regime of $p_{k,n}(\Delta)$, for $\Delta = O(1)$, can also be computed explicitly for the double exponential jump distribution (2) but, unlike $P(\delta)$ in (14), it turns out to be non-universal, i.e. it depends explicitly on the jump distribution [20, 43] and, for this reason, it is somewhat less interesting than the typical fluctuation regime.

In this paper, we derive the full PDF $p_{k,n}(\Delta)$ of the gap $d_{k,n}$ in the bulk, i.e. for large $n$ and large $k$ but keeping the ratio $\alpha = k/n$ fixed. We show that the behaviour in the bulk is rather different from the one found at the edge in [20, 43] recalled above in equations (13) and (14), which corresponds instead to the limit $\alpha \to 0$ (i.e. $1 \ll k \ll n$). We find that this PDF $p_{k,n}(\Delta)$ again exhibits two different scaling regimes depending on $\Delta$: a typical regime for $\Delta = O(n^{-1/2})$, consistent with the second line of equation (10), and a large deviation regime for $\Delta = O(1)$. Our most interesting results concern the typical regime, for $\Delta = O(n^{-1/2})$, where $p_{k,n}(\Delta)$ takes the scaling form

$$p_{k=n,\alpha}(\Delta) \approx \frac{\sqrt{\pi}}{\sigma} \mathcal{P}_\alpha\left(\frac{\sqrt{n}\Delta}{\sigma}\right),$$

(15)

where the scaling function $\mathcal{P}_\alpha(\delta)$ depends continuously on the parameter $\alpha$ and is given explicitly by

$\mathcal{P}_\alpha(\delta) = \int\frac{\sqrt{\pi \delta^2 - \sqrt{2\pi \delta^2}}}{\delta^2} e^{-\frac{\delta^2}{2\delta^2}} d\delta$.

In this context, the PDF $p_{k,n}(\Delta)$ can be thought of as describing the typical fluctuations of the process $M_{k,n}$ in the bulk, while the PDF $p_{k,n}(\Delta)$ at the edge can be thought of as describing the typical fluctuations of the process $M_{k,n}$ at the edge.

We note that the typical fluctuation regime $p_{k,n}(\Delta)$ in the bulk is anisotropic, i.e. it depends on the ratio $\alpha = k/n$, and is given by

$$p_{k,n}(\Delta) \approx \frac{\sqrt{\pi \delta^2 - \sqrt{2\pi \delta^2}}}{\delta^2} e^{-\frac{\delta^2}{2\delta^2}} d\delta,$$

where $\mathcal{P}_\alpha(\delta)$ depends continuously on the parameter $\alpha$ and is given explicitly by

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$\mathcal{P}_\alpha(\delta) = \int\frac{\sqrt{\pi \delta^2 - \sqrt{2\pi \delta^2}}}{\delta^2} e^{-\frac{\delta^2}{2\delta^2}} d\delta$.
\[
\mathcal{P}_\alpha(\delta) = \int_0^\infty \frac{y e^{-y}}{\pi \sqrt{\alpha(1-\alpha)}} \left[ y e^{-\frac{y^2}{4\sqrt{\alpha(1-\alpha)}}} \text{erfc} \left( \frac{y}{2\sqrt{2\alpha}} \right) + \frac{y e^{-\frac{y^2}{2}}}{4\sqrt{2\pi\alpha^2}} \text{erfc} \left( \frac{y}{2\sqrt{2(1-\alpha)}} \right) \right] \, dy.
\]  

(16)

For generic \( \alpha \) this integral over \( y \) can be evaluated in terms of hypergeometric functions of two variables (namely Humbert series, see equation (A.6a) in appendix A). For the special case \( \alpha = 1/2 \), which describes the gaps near the median, \( \mathcal{P}_{1/2}(\delta) \) can be expressed in terms of elementary functions (see equation (A.7)). One can also show that in the limit \( \alpha \to 0 \), our result in equations (15) and (16) yields back the edge result in equations (13) and (14)—that limit is however a bit subtle and is studied in detail below. Under this form (16), we notice the limit is however a bit subtle and is studied in detail below. Under this form (16), we notice the

Interestingly, we see that the tail \( \mathcal{P}_\alpha(\delta) \propto \delta^{-3} \), for finite \( \alpha \) and in the bulk, is different from the tail \( \propto \delta^{-4} \) obtained at the edge [see equation (14)]. From this inverse cubic tail one would naively conclude that the moments of the gaps (beyond the first one given in equations (10) and (11)) do not exist. However, this power law behaviour of the gap distribution \( p_{k,n}(\Delta) \) is cut-off for \( \Delta \gg n^{-1/2} \) and the higher moments are actually dominated by the large deviation regime of the PDF \( p_{k,n}(\Delta) \) for \( \Delta = O(1) \gg n^{-1/2} \) which we can also compute exactly (see below). The latter turn out to be non-universal. Consequently, the moments of the gaps beyond the first one, that we also study below, are to a large extent non-universal.

The paper is organised as follows. Section 3 is dedicated to the distribution of the \( k \)th maximum \( M_{k,n} \). In section 4 we derive the results for the gap \( d_{k,n} \), which constitute our main results, before we conclude in section 5. Some details of the computations have been relegated in appendices A–C.

3. Distribution of the \( k \)th maximum

We first expose a method which allows us to obtain the distribution of \( M_{k,n} \) that we will generalise in the next section to obtain the distribution of the gaps \( d_{k,n} \). It relies on the identity for the cumulative distribution function (CDF) of the \( k \)th maximum \( M_{k,n} \).
where $N_x$ is the counting process for the number of steps where the RW takes values above $x$. Indeed, there are exactly $k$ positions among the $n + 1$ positions $x'_i$ of the walk such that $x_i \leq M_{k,n}$ (see figure 1 for an example). Note that this identity remains valid for any discrete time stochastic process. We introduce the probability $q_{k,n}(x)$ that a RW of $n$ steps starting at $x_0 = x$ has exactly $k$ points on the negative axis between step 1 and step $n$, i.e. $N_0 = n - k$ for the walk starting from $x_0 = x$. The CDF $F_{k,n}(x)$ of $M_{k,n}$ can then be expressed from an elementary path transformation (see figure 5) as

$$F_{k,n}(x) = \begin{cases} 
\sum_{l=0}^{k-1} q_{l,n}(x) & , x \geq 0 \\
\sum_{l=0}^{k-2} q_{n-l,n}(-x) & , x < 0.
\end{cases}$$

The probability $q_{k,n}(x)$ can be constructed recursively, using the equation [20]

$$q_{k,n}(x) = \int_{0}^{\infty} dx' f(x' - x) q_{k,n-1}(x') + \int_{-\infty}^{0} dx' f(x' - x) q_{n-k,n-1}(-x'),$$

(together with the initial condition $q_{0,0}(x) = 1$ and $q_{k,0}(x) = 0$ for $k > n$. The first term of this equation describes the case where the walk has an additional initial jump from $x > 0$ to $x' > 0$, while the second term describes a jump from $x > 0$ to $x' < 0$. To solve this equation (20) it is useful [20] to introduce the auxiliary function $r_{k,n}(x) = q_{n-k,n}(x)$ which is the probability that a RW starting from $x_0 = x$ has $k$ points above 0 between step 1 and step $n$. One can then write two coupled equations for $q_{k,n}(x)$ and $r_{k,n}(x)$ [20].
Finally, the solutions read

\[ q_{k,n}(x) = \int_0^\infty dx' f(x' - x) q_{k,n-1}(x') + \int_0^\infty dx' f(x + x') r_{k-1,n-1}(x') \quad (21) \]

\[ r_{k,n}(x) = \int_0^\infty dx' f(x' - x) r_{k-1,n-1}(x') + \int_0^\infty dx' f(x + x') q_{k,n-1}(x'). \quad (22) \]

These integral equations (21) can be solved by generating function techniques. We introduce

\[ \tilde{q}(z,s;x) = \sum_{n=0}^\infty \sum_{k=0}^n s^n z^k q_{k,n}(x) \quad \text{and} \quad \tilde{r}(z,s;x) = \sum_{n=0}^\infty \sum_{k=0}^n s^n z^k r_{k,n}(x), \quad (23) \]

and obtain from (21) the set of coupled integral equations

\[ \tilde{q}(z,s;x) = 1 + s \int_0^\infty dx' f(x' - x) \tilde{q}(z,s,x') + zs \int_0^\infty dx' f(x' + x) \tilde{r}(z,s,x') \quad (24) \]

\[ \tilde{r}(z,s;x) = 1 + zs \int_0^\infty dx' f(x' - x) \tilde{r}(z,s,x') + s \int_0^\infty dx' f(x' + x) \tilde{q}(z,s,x'). \quad (25) \]

These equations, valid for any distribution of jumps \( f(\eta) \), turn out to be very difficult to solve in general. However, for a double exponential jump distribution (2) they can be solved exactly using the identity \( f''(\eta) = \frac{2}{\lambda} [f(\eta) - \delta(\eta)] \). Differentiating twice equations (24) and (25) with respect to \( x \), we obtain two decoupled differential equations

\[ \frac{\sigma^2}{2} \partial_x^2 \tilde{q}(z,s;x) = (1 - s) \tilde{q}(z,s;x) - 1 \quad (26) \]

\[ \frac{\sigma^2}{2} \partial_x^2 \tilde{r}(z,s;x) = (1 - zs) \tilde{r}(z,s;x) - 1. \quad (27) \]

Discarding the diverging solution for \( x \to +\infty \), we obtain

\[ \tilde{q}(z,s;x) = a(z,s)e^{-\sqrt{2(1-s)} z} + \frac{1}{1-s}, \quad \tilde{r}(z,s;x) = b(z,s)e^{-\sqrt{2(1-s)} z} + \frac{1}{1-zs} \quad (28) \]

The values of \( a(z,s) \) and \( b(z,s) \) are obtained by substituting back these forms in equations (26) and (27). Finally, the solutions read

\textbf{Figure 5.} Left: RW \( x_i \) of \( n = 20 \) steps starting from \( x_0 = 0 \) with jump distribution \( f(\eta) \).

For this walk, the 4th maximum \( M_{4,20} = x_{10} \geq 0 \). Right: affine transformation of the RW \( y_i = x_{10} - x_i \) starting from \( y_0 = x_{10} \). This walk has the same jump distribution \( f(\eta) \) and has exactly 4 points below \( y = 0 \).
\[ \tilde{q}(z, s; x) = \left( \frac{1}{\sqrt{1 - zs}} - \frac{1}{\sqrt{1 - s}} \right) \frac{e^{-\sqrt{2(1 - s)}z}}{\sqrt{1 - s}} + \frac{1}{1 - s}, \quad (29a) \]

\[ \tilde{r}(z, s; x) = \left( \frac{1}{\sqrt{1 - s}} - \frac{1}{\sqrt{1 - zs}} \right) \frac{e^{-\sqrt{2(1 - zs)}z}}{\sqrt{1 - zs}} + \frac{1}{1 - zs}. \quad (29b) \]

The generating function \( \tilde{P}(z, s; x) \) of the PDF \( P_{kn}(x) \) can be worked out explicitly in terms of \( \tilde{q}(z, s; x) \) and \( \tilde{r}(z, s; x) \) using equation (19)

\[ \tilde{P}(z, s; x) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} s^{k} z^{n-k} P_{kn}(x) = \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{\sigma} \left( \frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\beta\tau}} \right) \frac{e^{-\sqrt{2(\tau - s)}z}}{\sqrt{2\pi \tau}} & , \quad x \geq 0 \\
\frac{\sqrt{2}}{\sigma} \left( \frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\beta\tau}} \right) \frac{e^{-\sqrt{2(\tau - s)}z}}{\sqrt{2\pi \tau}} & , \quad x < 0.
\end{array} \right. \quad (30) \]

To obtain the large \( n \) behaviour of \( P_{kn}(x) \), we perform a change of variables in the generating function \( \tilde{P}(z, s; x) \) and set \( s = \exp(-p) \) and \( z = \exp(-q) \) with \( p, q \to 0 \) but keeping \( p/q = O(1) \) fixed. In this limit, the discrete sums over \( n \) and \( k \) can be replaced by integrals. The generating function \( \tilde{P}(z, s; x) \) therefore converges towards the double Laplace transform of the PDF \( P_{kn}(x) \) with respect to \( n \) and \( k \),

\[ \tilde{P}(z = e^{-p}, s = e^{-q}; x) \approx \tilde{\Pi}(q, p; x) = \int_{0}^{\infty} dn \int_{0}^{\infty} dk e^{-px - qk} P_{kn}(x). \quad (31) \]

where the function \( \tilde{\Pi}(q, p; x) \) can be computed from equations (29) and (30). It reads, at leading order for \( p, q \to 0 \) with \( p/q = O(1) \) fixed,

\[ \tilde{\Pi}(q, p; x) \approx \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{\sigma} \left( \frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\beta\tau}} \right) \frac{e^{-\sqrt{2(1 - s)}z}}{s} & , \quad x \geq 0 \\
\frac{\sqrt{2}}{\sigma} \left( \frac{1}{\sqrt{\beta}} - \frac{1}{\sqrt{\beta\tau}} \right) \frac{e^{-\sqrt{2(1 - s)}z}}{s} & , \quad x < 0.
\end{array} \right. \quad (32) \]

Using the inverse Laplace transforms

\[ \mathcal{L}_{u \to \tau}^{-1} \left( e^{-\sqrt{2(\tau + v)}z} \right) = \frac{x e^{-\sqrt{2(1 + v)}z}}{\sqrt{2\pi (\tau + v)}}, \quad \mathcal{L}_{u \to \tau}^{-1} \left( \frac{1}{\sqrt{u + v}} \right) = \frac{e^{-\sqrt{\tau + v}}}{\sqrt{\tau + v}}, \quad (33) \]

we invert the Laplace transform from \( p \) to \( n \), yielding

\[ \mathcal{L}_{p \to n}^{-1} \left( \tilde{\Pi}(q, p; x) \right) \approx \left\{ \begin{array}{ll}
\frac{1}{\sigma} \int_{0}^{\infty} s \frac{e^{-\sqrt{2(1 - s)}z}}{\sqrt{2\pi \tau}} \frac{1 - e^{-\sqrt{2(1 - s)}z}}{q} & , \quad x \geq 0 \\
\frac{1}{\sigma} \int_{0}^{\infty} s \frac{e^{-\sqrt{2(1 - s)}z}}{\sqrt{2\pi \tau}} \frac{1 - e^{-\sqrt{2(1 - s)}z}}{q} & , \quad x < 0.
\end{array} \right. \quad (34) \]

Finally, using the identity

\[ \mathcal{L}_{v \to t}^{-1} \left( \frac{e^{-\sqrt{\tau + v}}}{\sqrt{\tau + v}} \right) = \Theta(t - \tau), \quad (35) \]

we invert the Laplace transform from \( q \) to \( k \), yielding
Our starting point is the joint probability of

\[ P_{x,n}(x) \approx \begin{cases} \frac{1}{\sigma} \int_0^{n-k} d\tau \frac{x e^{-\frac{x^2}{\sigma^2}}}{\pi \sqrt{n-\tau+1/2}}, & x \geq 0 \\ \frac{1}{\sigma} \int_0^{k} d\tau \frac{|x| e^{-|x|^2}}{\pi \sqrt{n-\tau+1/2}}, & x < 0. \end{cases} \]  

Changing the variable \( \tau \to n \tau \) in this expression, it takes the scaling form described in the first line of equation (5), where the scaling function \( P_\alpha(z) \) is given as

\[ P_\alpha(z) \approx \begin{cases} f_0^{1-\alpha} \int_0 z e^{\frac{x^2}{\sigma^2}} \frac{x}{\sqrt{2\pi(1-t)^{3/2}}}, & z \geq 0 \\ f_0^{\alpha} \int_0 z e^{\frac{x^2}{\sigma^2}} \frac{x}{\sqrt{2\pi(1-t)^{3/2}}}, & z < 0. \end{cases} \]  

The integral in equation (37) can be computed exactly using the identity

\[ \frac{\partial}{\partial t} \text{erfc} \left( \frac{z}{\sqrt{2/2t}} \right) = \frac{z e^{-(1-t)z^2/2t}}{\sqrt{2\pi(1-t)^{3/2}}}, \]  

where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2} du \) is the complementary error function, leading to the final expression of \( P_\alpha(z) \) in the second line of equation (5).

We conclude this section by mentioning that there is an alternative method to obtain the PDF of \( M_{k,n} \) for a discrete time RW. It was obtained in the limit of large \( n \) and for \( \alpha = k/n = O(1) \) in [40], making use of the identity in law derived in [35] and extended in [36] (see also the more recent work [41])

\[ M_{k,n} \equiv \max_{0 \leq i \leq n+1-k} x_i + \min_{0 \leq j \leq k} x'_j, \]

where \( \{x_i\} \) and \( \{x'_j\} \) are two independent random walks with same jump distribution \( f(\eta) \) starting at \( x_0 = x'_0 = 0 \). This alternative method can be exploited for any jump distribution \( f(\eta) \), even if \( \sigma^2 = \infty \). The distribution of \( M_{k,n} \) was shown to take universal scaling forms, depending on the behaviour of \( \hat{f}(k) = \int_0^\infty e^{-ik\eta} f(\eta) \) for small \( k \). However, to our knowledge, there is no direct extension of this method to compute the distribution of the gap \( d_{k,n} \). In the next section, we will show how to extend the method presented in this section to obtain exact results for the PDF \( p_{k,n}(\Delta) \) of the gap \( d_{k,n} \) for a double exponential jump distribution.

4. Distribution of the kth gap

Our starting point is the joint probability of \( M_{k,n} \) and \( M_{k+1,n} \),

\[ S_{k,n}(x,y) = \text{Prob.} \{ M_{k,n} \geq y, M_{k+1,n} \leq x \}, \]

from which the PDF of the gap \( d_{k,n} \) can be obtained as

\[ p_{k,n}(\Delta) = - \int dx \int dy \partial^2_{xy} S_{k,n}(x,y) \Theta(y-x) \delta(\Delta - y + x). \]  

To obtain the joint probability \( S_{k,n}(x, y) \), we introduce the probability \( Q_{k,n}(x, \Delta) \) that a RW of \( n \) steps starting at \( x_0 = x \) has exactly \( k \) points below \( 0 \) and no point in the interval \([-\Delta, 0]\) between step 1 and step \( n \). Using a simple path transformation, one obtains the relation [20]
The probability $Q_{k,n}(x, \Delta)$ can be obtained recursively using the relation

$$Q_{k,n}(x, \Delta) = \int_0^\infty dx' f(x-x')Q_{k,n-1}(x', \Delta) + \int_{-\infty}^0 dx' f(x-x' + \Delta)Q_{n-k,n-1}(-x', \Delta).$$

This is a similar recursion relation as for $q_{k,n}(x)$ in equation (20). Therefore, introducing the generating functions

$$\widetilde{Q}(z, s; x, \Delta) = \sum_{n=0}^\infty \sum_{k=0}^n s^knz^kQ_{k,n}(x, \Delta), \quad \widetilde{R}(z, s; x, \Delta) = \sum_{n=0}^\infty \sum_{k=0}^n s^knz^kQ_{n-k,n}(x, \Delta),$$

they satisfy a set of coupled integral equations very similar to equations (24) and (25) [20]

$$\widetilde{Q}(z, s; x, \Delta) = 1 + s\int_0^\infty dx' f(x-x')\widetilde{Q}(z, s; x', \Delta) + zs\int_0^\infty dx' f(x+x' + \Delta)\widetilde{R}(z, s; x', \Delta),$$

$$\widetilde{R}(z, s; x, \Delta) = 1 + zs\int_0^\infty dx' f(x-x')\widetilde{R}(z, s; x', \Delta) + s\int_0^\infty dx' f(x+x' + \Delta)\widetilde{Q}(z, s; x', \Delta).$$

For a double exponential jump distribution $f(\eta)$ as in equation (2), these integral equations can be recast as differential equations [20, 43]: indeed one can easily show that the generating functions follow the set of differential equations (26) and (27), with the substitutions $\tilde{q} \rightarrow \tilde{Q}$ and $\tilde{r} \rightarrow \tilde{R}$. Solving these equations, we obtain

$$\tilde{Q}(z, s; x, \Delta) = A_1(z, s; \Delta)e^{-\sqrt{\frac{2}{1-s}}x} + \frac{1}{1-s},$$

$$\tilde{R}(z, s; x, \Delta) = B_1(z, s; \Delta)e^{-\sqrt{\frac{2}{1-s}}x} + \frac{1}{1-zs}.$$  

The coefficients $A_1$ and $B_1$ are then determined by inserting these solutions back in equations (45) and (46). This yields

$$A_1(z, s; \Delta) = \frac{\sqrt{\frac{2}{1-s}} - \sqrt{\frac{2}{1-s}}\sqrt{1-\zeta} + \sqrt{\frac{2}{1-s}}\sqrt{1-\zeta} + \sqrt{\frac{2}{1-s}} + \sqrt{\frac{2}{1-s}}\sqrt{1-\zeta}}{(\sqrt{1-\zeta})(1-s) + 1}\left[\sqrt{1-\zeta}\cosh\left(\frac{\sqrt{\frac{2}{1-s}}\Delta}{\sigma}\right) + \sinh\left(\frac{\sqrt{\frac{2}{1-s}}\Delta}{\sigma}\right)\right],$$

and $B_1(z, s; \Delta) = A_1(z^{-1}, zs; \Delta)$. From equations (41) and (42), we can express the generating function $p(z, s; \Delta) = \sum_{k=0}^\infty z^k\tilde{p}_{k,n}(\Delta)$, in terms of the coefficients $A_1$ and $B_1$ (see appendix A of [43] for more details). This yields
\[ \tilde{p}(z, s; \Delta) = \partial_{\Delta} A_1(z, s; \Delta) + \frac{\sigma}{\sqrt{2(1-s)}} \partial_{\Delta}^2 A_1(z, s; \Delta) \]
\[ + \frac{ze^{\sqrt{2(1-z)\Delta}}}{2(1-z)} \left( \partial_{\Delta} B_1(z, s; \Delta) + \frac{\sigma}{\sqrt{2(1-z)}} \partial_{\Delta}^2 B_1(z, s; \Delta) \right). \] (50)

As in the case of the PDF of \( M_{k,n} \), we are interested in the limit \( n \to \infty \) and \( k \to \infty \), which is conveniently obtained by performing the changes of variables \( z = e^{-q} \) and \( s = e^{-p} \) and by taking the limit \( p, q \to 0 \). In this limit, the discrete sums over \( k \) and \( n \) can then be replaced by integrals, yielding
\[ \tilde{p}(z = e^{-q}, s = e^{-p}; \Delta) \approx \tilde{\pi}(p + q, p; \Delta) = \int_0^\infty d(n-k) \int_0^\infty dk e^{-(n-k)-(p+q)} \tilde{p}_{k,n}(\Delta), \] (51)

where \( \tilde{\pi}(p + q, p; \Delta) \) is the double Laplace transform of the PDF \( p_{k,n}(\Delta) \) with respect to \( k \) and \( n - k \). To simplify the notations, we denote from now on \( r = p + q \). Note that we anticipate that \( \tilde{\pi}(r, p + q; \Delta) \) is symmetric in \( p \) and \( q \), since \( p_{k,n}(\Delta) = p_{n-k,n}(\Delta) \). We will now analyse the PDF in the large \( n \) limit and treat separately the typical fluctuations for \( \Delta = O(n^{-1/2}) \) and the atypically large fluctuations for \( \Delta = O(1) \).

4.1. Typical regime of fluctuation

To analyse the typical regime, we need to obtain the behaviour of \( A_1(z = e^{-q}, s = e^{-p}; \Delta) \) (resp. \( B_1 \)) in the regime \( p, q, \Delta \to 0 \) but keeping \( \Delta^2/p = O(1) \) and \( p/q = O(1) \) both fixed. In this regime, the coefficients take the scaling form
\[ A_1(z = e^{-q}, s = e^{-p}; \Delta) \approx a_1(r = p + q, p; \Delta) = -\frac{r - p + \frac{\sqrt{2}\Delta}{\sigma}}{p \sqrt{r}} \left( \frac{\sqrt{p}}{\sqrt{p} + \sqrt{r} + \frac{\sqrt{2}\Delta}{\sigma}} \right), \] (52)
\[ B_1(z = e^{-q}, s = e^{-p}; \Delta) \approx b_1(r = p + q, p; \Delta) = -\frac{p - r + \frac{\sqrt{2}\Delta}{\sigma}}{r \sqrt{p}} \left( \frac{\sqrt{r}}{\sqrt{p} + \sqrt{r} + \frac{\sqrt{2}\Delta}{\sigma}} \right). \] (53)

Note that \( b_1(r, p; \Delta) = a_1(p, r; \Delta) \). Inserting these expressions in equation (50), we realise that the leading terms are the second derivatives with respect to \( \Delta \), as \( \Delta \) and \( p \approx 1 - s \) are small in this limit. The scaling function \( \tilde{\pi}(r = p + q, p; \Delta) \) thus reads in this limit
\[ \tilde{\pi}(r, p; \Delta) \approx \frac{\sigma}{\sqrt{2}} \partial_{\Delta}^2 \left( a_1(r, p; \Delta) \sqrt{p} + b_1(p, r; \Delta) \sqrt{r} \right) \] (54)
\[ = \frac{2\sqrt{2}}{\sigma \left( \sqrt{r} + \sqrt{p} + \frac{\sqrt{2}\Delta}{\sigma} \right)^3} \left( \frac{1}{\sqrt{r}} + \frac{1}{\sqrt{p}} \right)^2. \] (55)

It is symmetric in \( p \) and \( r = p + q \), reflecting the symmetry of the PDF \( p_{k,n}(\Delta) \) in \( k \) and \( n - k \). To invert the Laplace transforms with respect to \( r \) and \( s \), we first use the identity
\[ \frac{2}{(x + p)^3} = \int_0^\infty y^2 e^{-y(x+p)} dy, \] (56)
to obtain
\[
\tilde{\pi}(r, p; \Delta) = \frac{1}{2\sigma} \int_0^\infty y^2 e^{-\psi} \left( \frac{\Psi + \sqrt{\eta^2 + \sqrt{\eta^2}}}{\sqrt{\eta^2 + \sqrt{\eta^2}}} \right)^2 \, dy.
\]  
Finally, using the Laplace inversion formulæ
\[
\mathcal{L}_{u \to \tau}^{-1} \left( \frac{e^{-\sqrt{u}}}{\sqrt{u}} \right) = \frac{xe^{-\frac{\pi}{4}x^2}}{2\sqrt{\pi}x^{3/2}}, \quad \mathcal{L}_{u \to \tau}^{-1} \left( \frac{e^{-\sqrt{u}}}{\sqrt{u}} \right) = \frac{e^{-\frac{\pi}{4}u^2}}{\sqrt{\pi^3/2}}.
\]  
we obtain the PDF
\[
p_{\alpha,n}(\Delta) \approx \int_0^\infty y^2 e^{-\psi} \left[ \frac{e^{-\frac{\pi}{4}y^2}}{\pi \sqrt{k(n-k)}} + \frac{ye^{-\frac{\pi}{4}(n-k)^2}}{4\sqrt{\pi}k(n-k)^2} \text{erfc} \left( \frac{y}{2\sqrt{2k}} \right) \right. \\
+ \left. \frac{ye^{-\frac{\pi}{4}(n-k)^2}}{4\sqrt{2\pi k^3}} \frac{y}{2\sqrt{2(n-k)}} \right] \, dy.
\]  
Performing the change of variable \( y \to y/\sqrt{n} \) in equation (60), we eventually obtain that \( p_{\alpha,n}(\Delta) \) takes the scaling form announced in equation (15) with the scaling function given in equation (16) that we reproduce here
\[
P_\alpha(\delta) = \int_0^\infty y^2 e^{-\psi} \left[ \frac{e^{-\frac{\pi}{4}y^2}}{\pi \sqrt{\alpha(1-\alpha)}} + \frac{ye^{-\frac{\pi}{4}(1-\alpha)^2}}{4\sqrt{\pi}(1-\alpha)^2} \text{erfc} \left( \frac{y}{2\sqrt{2\alpha}} \right) \right. \\
+ \left. \frac{ye^{-\frac{\pi}{4}(1-\alpha)^2}}{4\sqrt{2\pi \alpha^3}} \frac{y}{2\sqrt{2(1-\alpha)}} \right] \, dy.
\]  
For generic \( \alpha \) this integral has a rather complicated expression in terms of hypergeometric functions of two variables (namely Humbert series, see equation (A.6a) in appendix A). However, for \( \alpha = 1/2 \) it has an explicit expression in terms of elementary functions given in (A.7).

From this expression (61), we can compute the mean of the distribution, recovering the result of equation (11) (see appendix B for details). However, this distribution has a heavy tail as seen in equation (17) and its moments of order \( p \geq 2 \) are infinite. In figure 4, we compare the scaling function \( P_\alpha(\delta) \) to numerical results obtained for \( 10^6 \) simulations of random walks of \( n = 10^5 \) steps with exponential, Gaussian and uniform distribution of jump \( f(\eta) \), suggesting the universality of the result.

**The limit \( \alpha \to 0 \).** We first check that in the limit \( \alpha \to 0 \), this distribution \( P_\alpha(\delta) \) yields back the result at the edge obtained in [20], given in equations (12) and (13). To recover this edge result, we need to take simultaneously the limit \( \alpha = k/n \to 0 \) and \( \delta = \sqrt{n} \Delta \to \infty \) but keeping \( \sqrt{\alpha} \Delta = \sqrt{K} \Delta \) fixed (see equation (13)). In this scaling limit, we show that \( P_\alpha(\delta) \) in equation (16) takes the scaling form
\[
P_\alpha(\delta) \approx \sqrt{\alpha} P(\sqrt{\alpha} \delta),
\]
where \( P(\delta) \) is given in equation (13). To show this result (62), we demonstrate equivalently, setting \( \delta = \delta'/\sqrt{\alpha} \) with \( \delta' \) fixed, that
\[
\lim_{\alpha \to 0} \frac{1}{\sqrt{\alpha}} P_{\alpha} \left( \frac{\delta'}{\sqrt{\alpha}} \right) = P(\delta').
\]  
(63)
To show (63) we write \( P_{\alpha}(\delta'/\sqrt{\alpha}) \) starting from equation (61) and perform the change of variable \( y \to z = y/\sqrt{\alpha} \) to obtain
\[
\frac{1}{\sqrt{\alpha}} P_{\alpha} \left( \frac{\delta'}{\sqrt{\alpha}} \right) = \int_{0}^{\infty} z e^{-\delta z} \left[ \frac{\sqrt{\alpha} e^{-\frac{z^2}{2\alpha}}}{\pi \sqrt{1-\alpha}} + \frac{z \alpha^{3/2} e^{-\frac{z^2}{2\alpha^2}}}{4\sqrt{2\pi}(1-\alpha)^2} \text{erfc} \left( \frac{z}{2\sqrt{2\alpha}} \right) + \frac{z e^{-\frac{z^2}{2}} \text{erfc} \left( \frac{z \sqrt{\alpha}}{2\sqrt{2(1-\alpha)}} \right) }{4\sqrt{2\pi}} \right] dz \approx \int_{0}^{\infty} z^3 e^{-\frac{z^2}{2} - \delta z} + \frac{z e^{-\frac{z^2}{2}} \text{erfc} \left( \frac{z \sqrt{\alpha}}{2\sqrt{2(1-\alpha)}} \right) }{4\sqrt{2\pi}} dz, \quad \alpha \to 0.
\]  
(64)
since only the last term (in the integrand) survives in the limit \( \alpha \to 0 \). Finally, evaluating explicitly the remaining integral over \( z \) in equation (64) yields back the scaling function \( P(\delta') \) given in equation (13). This shows the scaling form in equation (62).

The limit \( \alpha \to 0 \) deserves yet another remark. Indeed, in the limit \( \delta \to \infty \), the scaling function \( P_{\alpha}(\delta) \), in the bulk, has an inverse cubic tail (see the second line of equation (17)), i.e. \( P_{\alpha}(\delta) \propto \delta^{-3} \). On the other hand, at the edge (corresponding to the limit \( \alpha \to 0 \)), the PDF of the gap decays as \( P(\delta) \propto \delta^{-2} \) (see equation (14)). This indicates that the two limits \( \alpha \to 0 \) and \( \delta \to \infty \) do not commute. In fact, from the full expression in (61) it is rather straightforward to show that there exists a scaling regime corresponding to \( \alpha \to 0 \) and \( \delta \to \infty \) but keeping \( \xi = \alpha \delta = O(1) \) fixed, which smoothly interpolates between these two different tail behaviours. In this scaling regime, we find that \( P_{\alpha}(\delta) \) takes the scaling form
\[
P_{\alpha}(\delta) \approx \alpha^{5/2} F(\alpha \delta) \quad \text{with} \quad F(\xi) = \frac{2}{\pi} \frac{1}{\xi^3} + \frac{3}{\sqrt{8\pi}} \frac{1}{\xi^4}.
\]  
(65)
A plot of this scaling function \( F(\xi) \) together with a comparison of \( \alpha^{-5/2} P_{\alpha}(\delta) \) (evaluated from the exact formula in equation (61)) is provided in the left panel of figure 6. This scaling form (65) indicates that, for large \( \delta \gg 1 \) and small \( \alpha \ll 1 \), the function \( P_{\alpha}(\delta) \) exhibits two different tail behaviours: if \( 1 \ll \delta \ll \alpha^{-1} \) then \( P_{\alpha}(\delta) \propto \delta^{-4} \) while if \( \delta \gg \alpha^{-1} \) then \( P_{\alpha}(\delta) \propto \delta^{-3} \). This is summarised in the right panel of figure 6.

4.2. Large deviation regime

Since the PDF governing the typical fluctuations of the gaps has a power law tail \( P_{\alpha}(\delta) \propto \delta^{-3} \), higher order moments of the gaps are dominated by the large deviation regime of \( p_{\alpha}(\Delta) \) for \( \Delta = O(1) \). To study this regime, we compute the behaviour of the coefficients \( A_1(\zeta = e^{-q}, s = e^{-p}; \Delta) \) for \( p, q \to 0 \) with \( p/q = O(1) \) fixed but with \( \Delta = O(1) \) fixed. This yields
\[
A_1(\zeta = e^{-q}, s = e^{-p}; \Delta) \approx -\frac{1}{p} + a_2 \left( p + q, p; \frac{\Delta}{\sigma} \right) + a_3 \left( p + q, p; \frac{\Delta}{\sigma} \right)
\]  
(66)
\[
B_1(\zeta = e^{-q}, s = e^{-p}; \Delta) \approx -\frac{1}{p + q} + a_2 \left( p, p + q; \frac{\Delta}{\sigma} \right) + a_3 \left( p, p + q; \frac{\Delta}{\sigma} \right).
\]  
(67)
where the functions $a_2(r = p + q; \tilde{\Delta})$ and $a_3(r = p + q; \tilde{\Delta})$ read

$$a_2(r; p; \tilde{\Delta}) = \frac{\coth(\sqrt{2\tilde{\Delta}})}{\sqrt{p}} + \frac{1}{\sqrt{r}\sinh(\sqrt{2\tilde{\Delta}})}, \quad (68)$$

$$a_3(r; p; \tilde{\Delta}) = -\sqrt{r} \frac{1}{p \sinh^2(\sqrt{2\tilde{\Delta}})} - \coth^2(\sqrt{2\tilde{\Delta}}) - \left(1 + \sqrt{\frac{p}{r}}\right) \frac{\cosh(\sqrt{2\tilde{\Delta}})}{\sinh^2(\sqrt{2\tilde{\Delta}})} \quad (69)$$

In the limit $p, q \to 0$ with $p/q = O(1)$ fixed, the leading contribution in the general formula (50) is again given by the terms involving the second derivatives, yielding

$$\tilde{\pi}(r; p; \Delta) \approx \frac{\sigma}{\sqrt{2}} \sqrt{\Delta} \left( a_2(r; p; \frac{\sqrt{\Delta}}{\sigma}) + a_3(r; p; \frac{\sqrt{\Delta}}{\sigma}) + a_2(p; r; \frac{\sqrt{\Delta}}{\sigma}) + a_3(p; r; \frac{\sqrt{\Delta}}{\sigma}) \right). \quad (70)$$

In this expression, we will discard the terms that depend only on $p$ or $r = p + q$ as they will lead, after Laplace inversion (i.e. $p \to n$ and $q \to k$), to contributions which are respectively proportional to $\delta(\alpha)$ or $\delta(1 - \alpha)$. Indeed, as we are interested here in the bulk regime $0 < \alpha < 1$, these contributions can be ignored. The remaining terms read

$$\tilde{\pi}(r = p + q; p; \Delta) \approx \frac{\sqrt{2}}{\sigma} \left[ \frac{1}{\sqrt{p}} \frac{3 + \cosh\left(\frac{\sqrt{\Delta}}{\sigma}\right)}{\sinh^2\left(\frac{\sqrt{\Delta}}{\sigma}\right)} - 2 \left(\frac{\sqrt{r}}{p} + \frac{\sqrt{p}}{r}\right) \frac{1 + \cosh\left(\frac{\sqrt{\Delta}}{\sigma}\right)}{\sinh^2\left(\frac{\sqrt{\Delta}}{\sigma}\right)} \right]. \quad (71)$$

For this large deviation form, the Laplace transforms are simple to invert, using that

$$u^{-a} = \int_0^\infty e^{-xu} x^{a-1} \frac{1}{\Gamma(a)} \, dx, \quad a > 0. \quad (72)$$

For large $n$ and large $k$ we obtain finally for $\Delta = O(1)$
\[ p_{\lambda,n}(\Delta) \approx \frac{1}{n\sigma} \sqrt{\frac{2}{\pi}} \frac{\Psi\left(\frac{\Delta}{\sigma}\right)}{\sqrt{\alpha(1-\alpha)}} + \frac{1}{\sigma n^{3/2}} \varphi_0\left(\frac{\Delta}{\sigma}\right), \quad (73) \]

where \( \Psi(\tilde{\Delta}) \) and \( \varphi_0(\tilde{\Delta}) \) are given by

\[ \Psi(\tilde{\Delta}) = \frac{\sqrt{2}}{\pi} \left[ 3 + \cosh(2\sqrt{2}\tilde{\Delta}) \right], \quad \varphi_0(\tilde{\Delta}) = \frac{\sqrt{2}}{\pi} \frac{2 + \cosh(2\sqrt{2}\tilde{\Delta})}{\sinh^2\left(\sqrt{2}\tilde{\Delta}\right)}. \quad (74) \]

The leading behaviour of this large deviation form (73) will be different in the large limit, depending on whether \( k = O(1) \) remains fixed or \( k = O(n) \). Indeed, taking first the limit \( n \to \infty \) with \( k \) fixed (i.e. \( \alpha \to 0 \), corresponding to the edge), the dominant contribution is the term \( \alpha^{-3/2} \) in equation (74), leading to

\[ p_{\lambda,n}(\Delta) \approx \frac{1}{\sigma k^{3/2}} \varphi_0\left(\frac{\Delta}{\sigma}\right), \quad \Delta = O(1) \quad \text{and} \quad k = O(1), \quad (75) \]

recovering the result of [20].

On the other hand, in the bulk, for \( n, k \to \infty \) with \( \alpha = k/n \) fixed, the leading contribution is given by the term of order \( O(1/n) \) in (73). Therefore, in the bulk, the large deviation form of the PDF of the gap reads

\[ p_{\lambda,n}(\Delta) \approx \frac{1}{n\sigma} \varphi_0\left(\frac{\Delta}{\sigma}\right), \quad \Delta = O(1) \quad \text{and} \quad k = O(n). \quad (76) \]

Note that in the limit \( \tilde{\Delta} = \Delta/\sigma \to 0 \), the large deviation function \( \Psi(\tilde{\Delta}) \) behaves as \( \Psi(\tilde{\Delta}) \approx (2/\pi)\tilde{\Delta}^{-3} \), which matches smoothly with the tail behaviour of the typical regime (see the second line of equation (17)).

Finally, we end up this section on the large deviations by noting that all the terms in equation (73) become of the same order in the intermediate regime where \( k = O(\sqrt{n}) \). Indeed, setting \( \lambda = k/\sqrt{n} = O(1) \) one obtains, from (73), that \( p_{\lambda,n}(\Delta) \) takes the scaling form

\[ p_{\lambda,n}(\Delta) \approx \frac{1}{\sigma k^{3/2}} \mathcal{G}\left(\frac{k}{\sqrt{n}}, \frac{\Delta}{\sigma}\right), \quad \mathcal{G}(\lambda, \tilde{\Delta}) = \lambda \Psi(\tilde{\Delta}) + \varphi_0(\tilde{\Delta}), \quad (77) \]

which smoothly interpolates between (75) in the limit \( \lambda \to 0 \) and (76) in the limit \( \lambda \to \infty \).

Let us now investigate the consequences of this behaviour (73) on the moments of the gaps.

### 4.3. Computation of the moments

Since the scaling function \( p_\alpha(\delta) \) that describes the typical gap fluctuations behaves as \( p_\alpha(\delta) \propto \delta^{-3} \) for large \( \delta \), the first moment of the gap is indeed completely dominated by the typical region [see equation (B.5)]. This is however not the case for higher order moments. Indeed, the two non-trivial contributions to the large deviation form in equation (73) will both contribute to the moments of order \( p \geq 2 \). In the case \( p > 3 \), we obtain

\[ \frac{\langle d_{\lambda,n}^p \rangle}{\sigma^p} \approx \frac{M_p}{\sqrt{k(n-k)}} + \left( \frac{1}{k^{3/2}} + \frac{1}{(n-k)^{3/2}} \right) m_p, \quad p > 3, \quad (78) \]

where the values of \( M_p \) and \( m_p \) can be computed explicitly (see appendix C),

\[ M_p = \int_0^\infty dx x^p \Psi(x) = \frac{2^{\frac{p+1}{2}}}{\pi} p! (1 - 2^{1-p}) \zeta(p-1), \quad (79) \]
\[ m_p = \int_0^\infty dx x^p \varphi_0(x) = \frac{2^{2-p}}{\sqrt{\pi}} p! \zeta(p-2). \]  

(80)

where \( \zeta(s) = \sum_{k=1}^\infty k^{-s} \) is the Riemann Zeta function. The cases \( p = 2 \) and \( p = 3 \) are particular since there are logarithmic corrections. For \( p = 2 \) we obtain

\[ \frac{\langle d_{\Delta,n}^2 \rangle}{\sigma^2} \approx \frac{\ln n}{\sqrt{\pi k(n-k)}} + \frac{1}{2} \left( \frac{1}{k^{3/2}} + \frac{1}{(n-k)^{3/2}} \right), \]

(81)

while, for \( p = 3 \) we have

\[ \frac{\langle d_{\Delta,n}^3 \rangle}{\sigma^3} \approx \frac{3\pi}{2\sqrt{2k(n-k)}} + \frac{3}{4\sqrt{2\pi}} \left( \frac{\ln k}{k^{3/2}} + \frac{\ln(n-k)}{(n-k)^{3/2}} \right). \]

(82)

Finally, these behaviours can be summarised as follows

\[ \frac{\langle d_{\Delta,n}^p \rangle}{\sigma^p} \approx \begin{cases} \frac{1}{\sqrt{2\pi k}} + \frac{1}{\sqrt{2\pi(n-k)}} & , \ p = 1 \\ \frac{\ln n}{\pi \sqrt{k(n-k)}} + \frac{1}{2} \left( \frac{1}{k^{3/2}} + \frac{1}{(n-k)^{3/2}} \right) & , \ p = 2 \\ \frac{3\pi}{2\sqrt{2\pi(n-k)}} + \frac{3}{4\sqrt{2\pi}} \left( \frac{\ln k}{k^{3/2}} + \frac{\ln(n-k)}{(n-k)^{3/2}} \right) & , \ p = 3 \\ \frac{M_p}{\sqrt{\delta(n-k)}} + \left( \frac{1}{k^{3/2}} + \frac{1}{(n-k)^{3/2}} \right) m_p & , \ p > 3, \end{cases} \]

(83)

In the regime \( \alpha = k/n = O(1) \), the first term in the last three lines of equation (83) gives the leading contribution to the moments. On the other hand, in the regime \( k = O(1) \), it is the term in \( k^{-3/2} \) that is dominant. Finally, in the intermediate regime mentioned above \( k = O(\sqrt{n}) \), both of these terms are of the same order for \( p > 3 \) while the term with the logarithmic correction is dominant for \( p = 2, 3 \).

5. Conclusion

In this article, we have computed exactly the PDF \( p_{\Delta,n}(\Delta) \) of the gap between two successive maxima \( M_{\Delta,n} \) and \( M_{\Delta+1,n} \) for a RW with double exponential (Laplace) jump distribution. The main focus of the present paper has been the limiting distribution of the gap \( d_{\Delta,n} \) in the scaling limit where both \( n \) and \( k \) are large, keeping the ratio \( \alpha = k/n \) fixed. This allowed us to study the gaps in the bulk, i.e. far from the global maximum \( x_{\text{max}} \) of the RW after \( n \) steps (see figure 3). Our main result is an explicit expression for the distribution \( P_\alpha(\delta) \) (see equation (16)) which governs the typical fluctuations of \( d_{\Delta,n} \) in this scaling limit, namely for \( d_{\Delta,n} = O(n^{-1/2}) \). We conjecture that this distribution \( P_\alpha(\delta) \) is universal for all random walks with a jump distribution \( f(\eta) \) which is continuous, symmetric and possesses a finite second moment. What happens for heavy tailed jump distributions, i.e. the case of Lévy flights, remains a challenging open question, in particular because we do not know how to solve the backward integral equations (45) and (46) in this case. We hope that the results obtained here will motivate further works to develop alternative methods to study the gap statistics of Lévy flights.
We found rather useful to think about the different positions of the random walker after \( n \) steps as a point process on a line, as illustrated in figure 3. By analogy with random matrices, we naturally identified edge regions, close to the extremal positions of the RW, as well as the bulk region, far from the maximum and the minimum. In particular, we have shown that the gaps behave quite differently in these two regions. Pursuing this analogy with random matrices, one may wonder whether one can define a ‘density’ associated to this point process that would capture the existence of these edge regions, and would be the equivalent of the Wigner semi-circle in random matrix theory. This is left for future investigations [46].

**Appendix A. Explicit expression for \( P_\alpha(\delta) \)**

The integrals in the expression (16) for the PDF \( P_\alpha(\delta) \) can be expressed in terms of the following integrals

\[
I_2(\delta, b) = \int_0^\infty y^3 e^{-\delta y - b y^2} = \frac{\sqrt{\pi}}{8 b^{3/2}} e^{\frac{\delta^2}{4 b^2}} (2 b + \delta^2) \text{erfc} \left( \frac{\delta}{2 \sqrt{b}} \right) - \frac{\delta}{4 b^2}
\]

(A.1)

\[
I_3(\delta, b) = \int_0^\infty y^3 e^{-\delta y - b y^2} = \frac{4 b + \delta^2}{8 b^3} - 4 e^{\frac{\delta^2}{2 b}} \text{erfc} \left( \frac{\delta}{2 \sqrt{b}} \right)
\]

(A.2)

as well as (see formula 4 p 178 of [47])

\[
J(\delta, b, c) = \int_0^\infty e^{-\delta y} e^{-b y^2} \text{erf}(cy) = \frac{3 c}{4 b^{3/2}} \Psi_1 \left( \frac{5}{2}; \frac{3}{2}; 1; \frac{3}{2}; -\frac{c^2}{b} \frac{\delta^2}{4 b} \right)
\]

(A.3)

\[
\frac{-2 c \delta}{\sqrt{\pi} b^3} \Psi_1 \left( \frac{3}{2}; \frac{1}{2}; \frac{3}{2}; \frac{3}{2} - \frac{c^2}{b} \frac{\delta^2}{4 b} \right)
\]

(A.4)

where \( \Psi_1 \) is a confluent hypergeometric series of two variables (sometimes called Humbert series [48]) defined as

\[
\Psi_1(a, b; c, x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m (b)_n}{(c)_m (c)_n} \frac{x^m y^n}{m! n!}
\]

(A.5)

where \( (a)_n = \Gamma(a + n)/\Gamma(a) \) is the Pochhammer symbol. In terms of \( I_2(\delta, b) \) and \( I_3(\delta, b) \) in (A.1) and \( J(\delta, b, c) \) in (A.3), the gap distribution \( P_\alpha(\delta) \) in (16) reads

\[
P_\alpha(\delta) = \frac{1}{\pi \sqrt{\alpha(1 - \alpha)}} I_2 \left( \delta, \frac{1}{8 \alpha(1 - \alpha)} \right)
\]

(A.6a)

\[
+ \frac{1}{4 \sqrt{2 \pi}} \left( \frac{1}{(1 - \alpha)^{3/2}} I_3 \left( \delta, \frac{1}{8 (1 - \alpha)} \right) + \frac{1}{\alpha} I_3 \left( \delta, \frac{1}{8 \alpha} \right) \right)
\]

(A.6b)

\[
- \frac{1}{4 \sqrt{2 \pi}} \left( \frac{1}{(1 - \alpha)^{3/2}} J \left( \delta, \frac{1}{8 (1 - \alpha)}, \frac{1}{2 \sqrt{2 \alpha}} \right) + \frac{1}{\alpha} J \left( \delta, \frac{1}{8 \alpha}, \frac{1}{2 \sqrt{2 (1 - \alpha)}} \right) \right)
\]

(A.6c)

Note that this expression (A.6a) is explicitly symmetric under the change \( \alpha \rightarrow 1 - \alpha \), as it should, i.e. \( P_\alpha(\delta) = P_{1-\alpha}(\delta) \).
In the special case $\alpha = 1/2$ (which corresponds to the vicinity of the median), the integrals in equation (A.6a) can be performed in terms of elementary functions (using in particular formula 2 p 175 of [47]). This yields

$$
\mathcal{P}_{\alpha=1/2}(\delta) = 2e^{\delta^2} (2\delta^2 + 3) \delta \left( \text{erfc} \left( \frac{\delta}{\sqrt{2}} \right)^2 - 2\text{erfc}(\delta) \right) - 2 \sqrt{\frac{2}{\pi}} e^{\delta^2} (3\delta^2 + 2) \text{erfc} \left( \frac{\delta}{\sqrt{2}} \right) + \frac{8}{\sqrt{\pi}} (\delta^2 + 1) + \frac{4}{\pi} \delta.
$$

(A.7)

which is clearly different from the scaling function found at the edge [20] (corresponding to the limit $\alpha \to 0$), given in equation (13). In particular, its asymptotic behaviours are given by

$$
\mathcal{P}_{\alpha=1/2}(\delta) \approx \begin{cases} 
\frac{4(2-\sqrt{2})}{\sqrt{\pi}}, & \delta \to 0 \\
\frac{4}{\pi} \delta^{-3}, & \delta \to \infty.
\end{cases}
$$

(A.8)

which is fully consistent with the behaviours given in equation (17) specified for $\alpha = 1/2$.

**Appendix B. Computation of $\langle d_{k,n} \rangle$ from $\mathcal{P}_\alpha(\delta)$**

From equation (61), the mean value of the gap is obtained as

$$
\frac{\sqrt{n}\langle d_{k,n} \rangle}{\sigma} \approx \int_0^\infty \delta \mathcal{P}_\alpha(\delta) \, d\delta.
$$

(B.1)

Using that $\int_0^\infty y^2 e^{-y^2} \, dy = 1$, we obtain the expression

$$
\frac{\sqrt{n}\langle d_{k,n} \rangle}{\sigma} \approx \int_0^\infty dy \frac{e^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{\pi \sqrt{\pi(1-\alpha)^2}} + \int_0^\infty dy \frac{ye^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{4\sqrt{2\pi(1-\alpha)^2}} \text{erfc} \left( \frac{y}{2\sqrt{2}\alpha} \right) + \int_0^\infty dy \frac{ye^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{4\sqrt{2\pi\alpha^2}} \text{erfc} \left( \frac{y}{2\sqrt{2}(1-\alpha)} \right).
$$

(B.2)

The second integral is identical to the third under $\alpha \to 1 - \alpha$. It can be computed using an integration by part,

$$
\int_0^\infty dy \frac{ye^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{4\sqrt{2\pi(1-\alpha)^2}} \text{erfc} \left( \frac{y}{2\sqrt{2}\alpha} \right) = \left[ -\frac{e^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{\sqrt{2\pi(1-\alpha)^2}} \text{erfc} \left( \frac{y}{2\sqrt{2}\alpha} \right) \right]_0^\infty - \int_0^\infty \frac{e^{-\frac{y^2}{4\pi(1-\alpha)^2}}}{2\pi\sqrt{\alpha(1-\alpha)}} dy.
$$

(B.3)

where we used that $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dz e^{-z^2}$. Note that the last term of this equation allows to simplify half of the first term of equation (B.2). As there are two such terms (coming from the integrals with $\alpha$ and $1 - \alpha$), the final result reads

$$
\frac{\sqrt{n}\langle d_{k,n} \rangle}{\sigma} \approx \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{1-\alpha}} \right) = \mu(\alpha),
$$

(B.5)

where we used that $\text{erfc}(0) = 1$ and $\text{erfc}(x \to \infty) = 0$. 

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Appendix C. Computations of $M_p$ and $m_p$

To obtain the value of $M_p$, we compute the moment of order $p$ of the large deviation scaling function of the PDF $\Psi(\tilde{\Delta})$. Using that $\Psi(x) = (\sqrt{2}/\pi)\partial^2_x \sinh^{-1}(\sqrt{2}x)$, $M_p$ reads after integration by part

$$M_p = \int_0^\infty dx x^p \Psi(x) = \frac{\sqrt{2}}{\pi} p(p-1) \int_0^\infty dx \frac{x^{p-2}}{\sinh(\sqrt{2}x)}. \quad (C.1)$$

Changing variable from $x \to z = \sqrt{2}x$, we obtain

$$M_p = \frac{p(p-1)2^{p-2}}{\pi} \int_0^\infty dz \frac{z^{p-2}}{\sinh(z)}. \quad (C.2)$$

Finally, using the integral representation of the Riemann Zeta function [49],

$$\zeta(s) = \frac{1}{2(1-2^{-s})\Gamma(s+1)} \int_0^\infty dx \frac{x^{s-1}}{\sinh(s)}, \quad (C.3)$$

we obtain the final result in equation (79).

To obtain the value of $m_p$, we proceed similarly, computing the moment of order $p$ of the large deviation scaling function of the PDF $\varphi_0(\tilde{\Delta})$. Using that $\varphi_0(x) = (8\pi)^{-1/2}\partial^2_x \sinh^{-2}(x)$, $m_p$ reads after integration by part

$$m_p = \int_0^\infty dx x^p \varphi_0(x) = \frac{p(p-1)}{8\pi} \int_0^\infty dx \frac{x^{p-2}}{\sinh^2(\sqrt{2}x)}. \quad (C.4)$$

Changing variable $x \to z = \sqrt{2}x$, we obtain

$$m_p = \frac{p(p-1)}{2^{p-2} \sqrt{\pi}} \int_0^\infty dz \frac{z^{p-2}}{\sinh^2(z)}. \quad (C.5)$$

Finally, using the integral representation [50] of the Riemann Zeta function,

$$\zeta(s) = \frac{2^{s-1}}{\Gamma(s+1)} \int_0^\infty dx \frac{x^s}{\sinh^2(s)}, \quad (C.6)$$

we obtain the final expression in equation (80).

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References

[1] Derrida B 1981 Random-energy model: an exactly solvable model of disordered systems Phys. Rev. B 24 2613
[2] Bouchaud J P and Mézard M 1997 Universality classes for extreme-value statistics J. Phys. A: Math. Gen. 30 7997
[3] Dean D S and Majumdar S N 2001 Extreme-value statistics of hierarchically correlated variables deviation from Gumbel statistics and anomalous persistence Phys. Rev. E 64 046121
[4] Le Doussal P and Monthus C 2003 Exact solutions for the statistics of extrema of some random 1D landscapes, application to the equilibrium and the dynamics of the toy model Physica A 317 140
[5] Baik J, Deift P and Johansson K 1999 J. Am. Math. Soc. 12 1119
[6] Johansson K 2000 Commun. Math. Phys. 209 437
[7] Prâhofer M and Spohn H 2000 Phys. Rev. Lett. 84 4882
[8] Sasamoto T and Spohn H 2010 Phys. Rev. Lett. 104 230602
[9] Calabrese P, Le Doussal P and Rosso A 2010 Europhys. Lett. 90 20002
[10] Dotsenko V 2010 Europhys. Lett. 90 20003
[11] Amir G, Corwin I and Quastel J 2011 Commun. Pure Appl. Math. 64 466
[12] Baik J, Liechty K and Schehr G 2012 On the joint distribution of the maximum and its position of the Airy2 process minus a parabola J. Math. Phys. 53 083303
[13] Raychaudhuri S, Cranston M, Przybala C and Shapir Y 2001 Maximal height scaling of kinetically growing surfaces Phys. Rev. Lett. 87 136101
[14] Györgyi G, Holdsworth P C, Portelli B and Racz Z 2003 Statistics of extremal intensities for Gaussian interfaces Phys. Rev. E 68 056116
[15] Majumdar S N and Comtet A 2004 Exact maximal height distribution of fluctuating interfaces Phys. Rev. Lett. 92 225501
[16] Majumdar S N and Comtet A 2005 Airy distribution function: from the area under a Brownian excursion to the maximal height of fluctuating interfaces J. Stat. Phys. 119 777
[17] Tracy C A and Widom H 1994 Commun. Math. Phys. 159 151
[18] For a short review, see Majumdar S N and Schehr G 2014 Top eigenvalue of a random matrix: large deviations and third order phase transition J. Stat. Mech. P01012 (and references therein)
[19] Comtet A and Majumdar S N 2005 Precise asymptotics for a random walker’s maximum J. Stat. Mech. P06013
[20] Schehr G and Majumdar S N 2012 Universal order statistics of random walks Phys. Rev. Lett. 108 040601
[21] Majumdar S N, Mounaix Ph and Schehr G 2013 Exact statistics of the gap and time interval between the first two maxima of random walks and Lévy flights Phys. Rev. Lett. 111 070601
[22] Dean D S, Le Doussal P, Majumdar S N and Schehr G 2017 Statistics of the maximal distance and momentum in a trapped Fermi gas at low temperature J. Stat. Mech. 063301
[23] Gumbel E J 1958 Statistics of Extremes (New York: Columbia University Press)
[24] Majumdar S N and Pal A 2014 Extreme value statistics of correlated random variables (arXiv:1406.6768)
[25] Majumdar S N 2010 Universal first-passage properties of discrete-time random walks and Levy flights on a line: statistics of the global maximum and records Physica A 389 4299
[26] Sabhapandit S and Majumdar S N 2007 Density of near-extreme events Phys. Rev. Lett. 98 14021
[27] Perret A, Comtet A, Majumdar S N and Schehr G 2013 Near-extreme statistics of Brownian motion Phys. Rev. Lett. 111 240601
[28] Ramola K, Majumdar S N and Schehr G 2014 Universal order and gap statistics of critical branching Brownian motion Phys. Rev. Lett. 112 210602
[29] Ramola K, Majumdar S N and Schehr G 2015 Branching Brownian motion conditioned on particle numbers Chaos Soliton Fract. 74 79
[30] Brunet E and Derrida B 2009 Statistics at the tip of a branching random walk and the delay of traveling waves Europhys. Lett. 87 60010
[31] Brunet E and Derrida B 2011 A branching random walk seen from the tip J. Stat. Phys. 143 420
[32] Moloney N R, Ogzoy K and Racz Z 2011 Order statistics of $|f^\alpha|$ signals Phys. Rev. E 84 061101
[33] Tremaine S and Richstone D 1977 A test of a statistical model for the luminosities of bright cluster galaxies Astrophys. J. 212 311
[34] Feller W 1968 An Introduction to Probability Theory and its Applications vol I and II, 3rd edn (New York: Wiley)
[35] Wendel J G 1960 Order statistics of partial sums Ann. Math. Stat. 31 1034
[36] Port S C 1963 An elementary probability approach to fluctuation theory J. Math. Anal. Appl. 6 109–51
[37] Yor M 1995 The distribution of Brownian quantiles J. App. Probab. 32 405
[38] Dassios A 1995 The distribution of the quantiles of a Brownian motion with drift and the pricing of related path-dependent options Ann. Appl. Probab. 5 389
[39] Embrechts P, Rogers L C G and Yor M 1995 A proof of dassios’ representation of the $|\alpha|$-quantile of Brownian motion with drift Ann. Appl. Probab. 5 757
[40] Dassios A 1996 Sample quantiles of stochastic processes with stationary and independent
increments Ann. Appl. Probab. 6 1041
[41] Chaumont L 1999 A path transformation and its applications to fluctuation theory J. London Math.
Soc. 59 729
[42] Pitman J 2019 Limit laws and Bessel processes in the extreme order statistics of random walks (in
preparation)
[43] Battilana M, Majumdar S N and Schehr G 2018 Gap statistics for random walks with gamma
distributed jumps Markov Process. Relat. (arXiv:1711.08744) accepted
[44] Mehta M L 1991 Random Matrices (Boston, MA: Academic)
[45] Forrester P J 2010 Log-Gases and Random Matrices (Princeton, NJ: Princeton University Press)
[46] Lacroix-A-Chez-Toine B, Majumdar S N and Schehr G in preparation
[47] Prudnikov A P, Brychkov Y A and Marichev O I 1992 Integrals and Series. vol. 4: Direct Laplace
Transforms (New York: Gordon and Breach)
[48] See Humbert series https://en.wikipedia.org/wiki/Humbert_series
[49] NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/25.5.E8
[50] NIST Digital Library of Mathematical Functions http://dlmf.nist.gov/25.5.E9