Magnitude, Diversity, Capacities, and Dimensions of Metric Spaces

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Abstract Magnitude is a numerical invariant of metric spaces introduced by Leinster, motivated by considerations from category theory. This paper extends the original definition for finite spaces to compact spaces, in an equivalent but more natural and direct manner than in previous works by Leinster, Willerton, and the author. The new definition uncovers a previously unknown relationship between magnitude and capacities of sets. Exploiting this relationship, it is shown that for a compact subset of Euclidean space, the magnitude dimension considered by Leinster and Willerton is equal to the Minkowski dimension.

Keywords Magnitude of metric spaces · Capacity · Minkowski dimension

Mathematics Subject Classifications (2010) Primary 51F99 · Secondary 31B15 · 49Q15

1 Introduction

The magnitude of a metric space is a numerical isometric invariant introduced by Leinster in [9]. From the perspective of geometry, its definition was motivated in a rather unusual way. In [6], Leinster had defined the Euler characteristic of a finite category, which generalizes the Euler characteristic of a topological space or of a poset. This notion of Euler characteristic can be naturally generalized from categories to enriched categories, a family of algebraic structures which, as observed by Lawvere in [5], includes metric spaces; in this context the generalization of Euler characteristic is named “magnitude”. Specialized then to metric spaces, one obtains Leinster’s definition of the magnitude of a finite metric space, stated in Definition 2.1 below. Magnitude was extended to compact metric spaces in multiple ways in [9, 11, 19, 20], which were shown by the author in [13] to agree with each other for many spaces (specifically, for so-called positive definite spaces, which include all compact subsets of Euclidean space).
Given this exotic provenance, it may come as a surprise that magnitude turns out to be closely related to classical invariants of integral geometry; see [9, 11, 20] for a number of results along these lines. Conjectures in [9, 11], which are supported by partial results in those papers and by heuristics and numerical computations in [19], suggest that the relationship between magnitude and integral geometry runs deep enough that all the intrinsic volumes of convex bodies can be recovered from magnitude. A second surprise, in a completely different direction, is that the magnitude of a finite metric space has been introduced in the literature before, in connection with quantifying biodiversity in [16]. Although the theory of magnitude was not developed at all in [16], the relationship between magnitude and diversity has been investigated more fully in [7].

The present work grew out of the author’s search for a more satisfactory definition of magnitude for compact metric spaces. The approach of [13] was to introduce yet another definition, a measure-theoretic generalization of a variational formula for the magnitude of a finite positive definite space derived in [7, 9], and to prove that this new definition agrees with all the earlier ones. Here we instead take a more functional-analytic approach to generalize directly the original definition of magnitude for finite metric spaces. Besides the aesthetic appeal of a more direct approach, the resulting definition, which again agrees with the earlier ones, can be used to prove new properties of magnitude in Euclidean space. More interestingly, it uncovers previously unknown connections between magnitude and potential theory. In fact, another surprise is that the magnitude of a compact subset of Euclidean space has (almost) been introduced in an equivalent form in the literature before both [9] and [16], as a type of capacity. (It seems likely that magnitude is the unique notion to have arisen independently in potential theory, theoretical ecology, and category theory.)

The relationship between magnitude and capacity has important consequences. There is a notion of dimension associated to magnitude which was first investigated in [11], and which provided some of the first compelling evidence that magnitude encodes interesting geometric information. Using in part a deep result on relationships between different capacities, we will see that in Euclidean space, this magnitude dimension turns out to be the same as Minkowski dimension. In establishing this result, we find an apparently new formulation of Minkowski dimension in terms of capacity. In addition, the conjectures from [9, 11] mentioned earlier would, if true, indicate previously unknown connections between capacities and intrinsic volumes of convex bodies.

The rest of this paper is organized as follows. Section 2 presents Leinster’s original definition of the magnitude of a finite metric space and some related definitions we will need. Section 3 develops the functional-analytic generalization of the definition of magnitude for compact spaces. Section 4 presents a dual perspective on the definitions of Section 3, and discusses a quantity closely related to magnitude, the maximum diversity of a metric space. Section 5 specializes the constructions of the previous sections to subsets of Euclidean space, and uses them to prove new results about the behavior of magnitude in Euclidean space. Section 6 discusses the connections between magnitude, maximum diversity, and capacities. Section 7 proves a new characterization of Minkowski dimension in terms of maximum diversity, and uses a result from potential theory recalled in Section 6 to deduce that magnitude dimension and Minkowski dimension are equal in Euclidean space. Finally, in Section 8, we briefly investigate another, closely related instance of the magnitude of an enriched category: the case of ultrametric spaces, whose theory turns out to be much simpler but nevertheless intriguingly similar to that of metric spaces.