On evolution of multiphase nonlinear modulated waves

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Abstract

We present a fundamental solution to an initial value problem for the KdV-Whitham system in an explicit integral form. Monotonically decreasing initial data with finite number of breaking points are considered. Generating function for the commuting flows of the averaged KdV hierarchy producing the analytical solutions to the KdV-Whitham system is constructed.

1 Introduction

It is known [1,2] that evolution of smooth initial distribution

\[ u(x, 0) = u_0(x) \]  (1)

according to the Korteweg-de Vries (KdV) equation

\[ \partial_t u + 6u \partial_x u + \varepsilon^2 \partial_{xxx} u = 0 \]  (2)

is described locally as \( \varepsilon \to 0 \) by \( g \)-phase \( (g \)-gap) solution which is expressed in terms of theta-function of the hyperelliptic Riemann surface of genus \( g \) (see Refs. [3,4])

\[ \Gamma : \quad y^2 = \prod_{j=1}^{2g+1} (\lambda - r_j) \equiv R_{2g+1}(r, \lambda), \]  (3)

\[ r_1 \leq r_2 \leq ... \leq r_{2g+1}. \]

This solution has the form

\[ u(x, t) = -2 \frac{d^2}{dx^2} \ln \theta \left( \frac{kx - \omega t}{\varepsilon} \mid r_1, ..., r_{2g+1} \right) + c(r). \]  (4)

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Here the components of the wave number and the frequency vectors
\( k(r) = (k^{(1)}, \ldots, k^{(g)}) \),
\( \omega(r) = (\omega^{(1)}, \ldots, \omega^{(g)}) \) are expressed in terms of the branch points \( r_j \) by well-known "finite-gap" formulas
\[
\begin{align*}
    k^{(m)} &= \alpha_1^{(m)}(r), \\
    \omega^{(m)} &= \alpha_1^{(m)}(r) \sum_{j=1}^{2g+1} r_j + 4\alpha_2^{(m)}, \quad m = 1, \ldots, g .
\end{align*}
\]  
(5)

where the functions \( \alpha_i^{(m)}(r) \) are defined uniquely by the normalization conditions
\[
\int_{\alpha_k} d\Omega^{(m)} = \delta_{km}, \quad k, m = 1, \ldots, g
\]  
(6)

for the basis holomorphic differentials
\[
d\Omega^{(m)} = \sum_{k=1}^{g} \alpha_k^{(m)}(r) \frac{\lambda^{g-k}}{\sqrt{R_{2g+1}(r, \lambda)}} d\lambda ,
\]  
(7)

where the sign is given by \( \sqrt{1} = 1 \).

We consider the Riemann surface (3) with cuts along the intervals \((-\infty, r_1], [r_2, r_3], \ldots, [r_{2g}, r_{2g+1}]\). The canonical basis of cycles \( \alpha_1, \ldots, \alpha_g; \beta_1, \ldots, \beta_g \) is chosen so that the cycles \( \alpha_1, \ldots, \alpha_g \) lie on \( \Gamma \) over the permitted zones \([r_1, r_2], \ldots, [r_{2g-1}, r_{2g}]\). The genus \( g \) of the Riemann surface depends on the initial data \( u_0(x) \) (globally) and on the position on the \((x, t)\) – plane (locally). In the general case the \((x, t)\) – plane is split into a finite set of regions in every of which the solution is described by the formula (4) with certain \( g \) [5,6] (Fig. 1). The slow \( x, t \) – evolution of the branch points \( r_j \) parametrizing the "uniform" solution (4) is governed by the Whitham modulation equations [7,8,1]
\[
\begin{align*}
    \partial_t r_i + V_i(r) \partial_x r_i &= 0, \quad i = 1, 2, \ldots, 2g + 1
\end{align*}
\]  
(8)

with real and different characteristic speeds (pure hyperbolicity [9]) which will be specified below. The existence of global solutions to (5) corresponding to the monotonically decreasing analytic initial data (4) for the KdV equation was proven recently by Tian in Ref. [10] where a set of analytic solutions in the form of infinite series were constructed with the aid of the generalized hodograph transform [11] and Krichever algebraic – geometrical procedure [12].

Unlike that of Ref. [10] we propose the direct way to construct the general (nonanalytic) global solutions of (5) in an explicit integral form. This problem has been solved to date only in the simplest single-phase case [13-16]. Construction of the global solutions for the multiphase case is vital both from purely theoretical point of view to complete the theory of the multi-gap averaged integrable systems and for the physical applications where the complex multiphase nonlinear dissipationless wave behaviour is not uncommon (Alfvenic turbulence in the solar wind [17], the Earth’s bowshock [18] and oth.)
2 Generating function for the averaged KdV hierarchy

We start from the potential representation for the characteristic speeds $V_j$:

$$V_j(r) = \frac{\partial_j \omega^{(m)}}{\partial_j k^{(m)}}; \quad \partial_j \equiv \partial/\partial r_j$$  \hspace{1cm} (9)

which immediately follows from the "conservation of waves"

$$\partial_t k(r) + \partial_x \omega(r) = 0,$$  \hspace{1cm} (10)

considered as a consequence of (8) (for the single-phase case see Refs.[14,15,19]). We notice that the representation (9) is valid for any chosen phase number since the characteristic speeds $V_j$ do not depend on $m$.

Looking for the solution of (8) in the characteristic form

$$x - V_i(r)t = W_i(r)$$  \hspace{1cm} (11)

one arrives at the linear overdetermined compatible equation system for $W_i$'s [11,20].

$$\frac{\partial_i W_j}{W_i - W_j} = \frac{\partial_i V_j}{V_i - V_j}; \quad i \neq j.$$  \hspace{1cm} (12)

Construction (11), (12) is known as the generalized hodograph transformation and provides all the hydrodynamic symmetries of (8), i.e. equations of the form

$$\partial_t r_i + W_i(r) \partial_x r_i = 0,$$  \hspace{1cm} (13)

which commute with (8), i.e. $\partial_{t\tau} r_i = \partial_{\tau t} r_i$. Here $\tau$ is "new" independent time. Some of the solutions of (12) (the uniform ones) correspond to the averaged hierarchy of the KdV equation [8,12,20,21]. The system (12) can be locally parametrized by $2g + 1$ functions of one variable [20]. We remark that this corresponds to the number of arbitrary functions $r_j(x,0)$ in the initial value problem for the Whitham system (8).

Now we shall construct a generating function $W_j(r; \lambda)$ for the uniform commuting flows to (8) which will give the characteristic speeds of the averaged KdV hierarchy as the coefficients of the expansion of $W_j(r; \lambda)$ in powers of $\lambda^{-1}$ as $\lambda \to \infty$. The two first terms of this series are obvious and correspond to the zeroth (linear equation of translation) and the first (system (8)) terms of the averaged KdV hierarchy:

$$W_j^0 = 1, \quad W_j^1 = V_j.$$  \hspace{1cm} (14)

So we shall seek $W_j(r; \lambda)$ in the form predicted by (9), (12):

$$W_j(r; \lambda) = \frac{\partial_j \omega^{(m)}(r; \lambda)}{\partial_j k^{(m)}(r)}, \quad m = 1, ..., g,$$  \hspace{1cm} (15)
where
\[ \tilde{\omega}^{(m)}(\mathbf{r}; \lambda) = k^{(m)} + \frac{\text{const}}{\lambda} \omega^{(m)} + \ldots \]  \hspace{1cm} (16)

Such a generating function for the single–phase case was constructed by Pavlov in [22] and has the form
\[ \tilde{\omega}_{s.p.}(\mathbf{r}; \lambda) = \frac{k \lambda^{3/2}}{\sqrt{R_3(\mathbf{r}, \lambda)}}. \]  \hspace{1cm} (17)

With (15), (16), (17) in mind we propose the following project:
\[ \tilde{\omega}^{(m)}(\mathbf{r}; \lambda) = \lambda^{3/2} \sum_{k=1}^{g} a_k^{(m)} \frac{\lambda^{g-k}}{\sqrt{R_{2g+1}(\mathbf{r}, \lambda)}} = \lambda^{3/2} \frac{d^{(m)}(\lambda)}{d\lambda}, \]  \hspace{1cm} (18)

which has two first terms of expansion as \( \lambda \to \infty \) coinciding with (16) and turns into (17) in the case \( g = 1 \).

Let us check that commuting flows (15) with the frequencies (18) identically satisfy the linear equation system (12).

With this aim in view we consider the following combinations occurring in the left part of (12):
\[ \lambda^{-3/2} \partial_t \left( \frac{\partial_j \tilde{\omega}^{(m)}}{\partial_j k^{(m)}} \right) = p_{ij}^{(m)}(\mathbf{r}) \frac{\lambda^g + a_{g-1}^{(m)} \lambda^{g-1} + \ldots + a_0^{(m)}}{(\lambda - r_j)(\lambda - r_i) \sqrt{R_{2g+1}(\mathbf{r}, \lambda)}}, \]  \hspace{1cm} (19)

and
\[ \lambda^{-3/2} \left( \frac{\partial_i \tilde{\omega}^{(m)}}{\partial_j k^{(m)}} - \frac{\partial_j \tilde{\omega}^{(m)}}{\partial_j k^{(m)}} \right) = q_{ij}^{(m)}(\mathbf{r}) \frac{\lambda^g + b_{g-1}^{(m)} \lambda^{g-1} + \ldots + b_0^{(m)}}{(\lambda - r_j)(\lambda - r_i) \sqrt{R_{2g+1}(\mathbf{r}, \lambda)}}. \]  \hspace{1cm} (20)

Here \( p_{ij}^{(m)}(\mathbf{r}), q_{ij}^{(m)}(\mathbf{r}) \) are some certain (analytic) functions. The coefficients \( a_k^{(m)}(\mathbf{r}), b_k^{(m)}(\mathbf{r}) \) can be uniquely defined by the normalization conditions, namely, integrating (19), (20) over the \( \alpha \)–cycles we arrive taking into account (5), (18) at two sets of algebraic equations
\[ \int_{\alpha_k} \frac{\lambda^g + a_{g-1}^{(m)} \lambda^{g-1} + \ldots + a_0^{(m)}}{(\lambda - r_j)(\lambda - r_i) \sqrt{R_{2g+1}(\mathbf{r}, \lambda)}} d\lambda = 0, \]  \hspace{1cm} and
\[ \int_{\alpha_k} \frac{\lambda^g + b_{g-1}^{(m)} \lambda^{g-1} + \ldots + b_0^{(m)}}{(\lambda - r_j)(\lambda - r_i) \sqrt{R_{2g+1}(\mathbf{r}, \lambda)}} d\lambda = 0, \quad k = 1, \ldots, g, \]  \hspace{1cm} (21)

which are identical and imply \( a_j^{(m)}(\mathbf{r}) = b_j^{(m)}(\mathbf{r}), \quad j = 0, 1, \ldots, g - 1 \). Therefore the ratio
\[ S_{ij}(\mathbf{r}, \lambda) = \frac{\partial_t \left( \frac{\partial_j \tilde{\omega}^{(m)}}{\partial_j k^{(m)}} \right)}{\partial_j \tilde{\omega}^{(m)} - \partial_j k^{(m)}} = \frac{p_{ij}^{(m)}(\mathbf{r})}{q_{ij}^{(m)}(\mathbf{r})}; \]  \hspace{1cm} (22)
does not depend on \( \lambda \) and to check the fulfilment of the equality (12) for the functions \( W_j(\lambda; r) \) defined by (15), (18) it is suffice to check it for any of terms of the expansion (16). The equation is obviously satisfied by the second term which gives the simplest nontrivial solution \( W_j = V_j \). Hence the function (18) really is the generating function for the frequences of the KdV hierarchy. As a consequence, the generating equation for the phase wave number conservation laws of the KdV hierarchy has the form

\[
\partial_r k^{(m)} + \partial_x \left( \frac{d\Omega^{(m)}}{d\lambda} \lambda^{3/2} \right) = 0.
\]  

(23)

3 General solution of the hodograph equations

Now we are ready to construct the general solution to the hodograph equations (12). Remind that it has to contain \( 2g + 1 \) arbitrary functions of one variable. So using the generating function \( W_j(r, \lambda) \) as a kernel of convolution we arrive at the general solution in the form

\[
W_j(r) = \sum_{k=1}^{2g+1} \oint_{A_k} W_j(r; \lambda) \phi_k(\lambda) d\lambda,
\]  

(24)

where \( \phi_k(\lambda) \) and \( A_k \) are arbitrary functions and contours respectively on the Riemann surface \( \Gamma \). The important restriction that has to be imposed upon \( \phi_k \)’s and \( A_k \)’s follows from the pure hyperbolicity of the modulational system \( (8) \) \([9]\) and the form of the hodograph solution (11), namely, all \( W_j \)’s must be real. We remark that this requirement is not contained in the system (12) itself. Moreover, the contours \( A_k \) have to lie off the branch points. Finally, using the ”potential” representation (15), (19) we arrive at the solution

\[
W_j(r) = \frac{\partial_j \tilde{\omega}^{(m)}(r)}{\partial_j k^{(m)}(r)}, \quad j = 1, 2, ..., 2g + 1, \quad m = 1, ..., g,
\]  

(25)

where

\[
\tilde{\omega}^{(m)}(r) = \sum_{k=1}^{2g+1} \int_{a_k}^{r_k} \phi_k(\lambda) d\Omega^{(m)}.
\]  

(26)

Here ”new” arbitrary \( \phi_k \)’s correspond to \( \lambda^{3/2} \phi_k(\lambda) \) from (24). Arbitrary constants \( a_k \) are real therefore the integration is accomplished along the axes in real \( r \)-space (this corresponds to a special choice of \( A_k \)’s which can be tightened to a real axis on the Riemann surface). Choosing \( \phi_k(\lambda), a_k \) one can obtain all the commuting flows described in [21,23].

Now we shall discuss briefly connection between the solution (25) and the Euler–Darboux–Poisson type systems which were investigated much earlier by Eisenhart [24] for the case of three independent variables \( (g = 1) \) and have arisen recently when studying the finite–gap averaged KdV equation [25,14,15,16,10]. In fact, the solution (25) can be represented in the form

\[
\tilde{\omega}^{(m)}(r) = \sum_{i=1}^{g} a_i^{(m)}(r) f_i(r)
\]  

(27)
where $\omega_i(r)$ are the normalization coefficients defined by (29) and

$$f_i(r) = \sum_{k=1}^{2g+1} \lambda^{g-i} \phi_k(\lambda) \frac{d\lambda}{\sqrt{R_{2g+1}(r, \lambda)}}, \quad i = 1, \ldots, g$$

are the Eisenhart type solutions of the overdetermined compatible system of the Euler–Darboux–Poisson equations

$$2(r_i - r_j) \partial^2_{ij} f = \partial_i f - \partial_j f, \quad i \neq j, \quad i, j = 1, \ldots, 2g + 1.$$  

The solution (28) generally is unlimited when any two of neighboring invariants $r_j$ coalescing. For the further consideration we shall need the solution which is limited in all the domain of definition $r_1 \leq \ldots \leq r_{2g+1}$. The requirement of boundedness reduces the number of arbitrary functions in the general solution. We first demonstrate the way to construct the limited solution for the simplest single–phase case (see Refs. [14,15]). We start from the representation (26) which takes in the single–phase case the form

$$\tilde{\omega}(r) = \sum_{k=1}^{3} \int_{a_k}^{r_k} \phi_k(\lambda) d\Omega,$$

$$d\Omega = \frac{k}{\sqrt{R_3(r, \lambda)}} d\lambda, \quad \int_{r_1}^{r_3} d\Omega = 1.$$  

To provide boundedness of (30) at the singular surfaces $r_2 = r_1$ and $r_2 = r_3$ one splits the domain of definition $D: r_1 \leq r_2 \leq r_3$ into two parts $D_1 : \xi < r_2 \leq r_3$ and $D_2 : r_1 \leq r_2 < \xi$ with the deleted point (surface) $r_2 = \xi$, where $r_1 \leq \xi \leq r_3$ is some real parameter on the Riemann surface. Each of these subdomains contains only the one singularity. To obtain the limited solution one should put $a_1 = a_2 = a_3 = \xi$ and

$$\phi_2 = -\phi_3 \quad \text{for} \quad D_1,$$

$$\phi_2 = -\phi_1 \quad \text{for} \quad D_2.$$  

By this means the resulting solution is characterized by two (instead of initial three) arbitrary functions $\phi_1(\lambda)$ and $\phi_3(\lambda)$. Introducing new (real) arbitrary functions $\psi_1(\lambda) = -\phi_1(\lambda), \psi_2(\lambda) = \sqrt{-1} \phi_3(\lambda)$ we present the general bounded continuous solution in the form

$$\tilde{\omega}(r) =$$

$$\begin{cases} 
\int_{r_1}^{r_2} \psi_1(\lambda) d\Omega + \sqrt{-1} \int_{\xi}^{r_3} \psi_2(\lambda) d\Omega, & \text{for} \ r_1 \leq r_2 \leq \xi \\
\int_{r_1}^{\xi} \psi_1(\lambda) d\Omega + \sqrt{-1} \int_{r_2}^{r_3} \psi_2(\lambda) d\Omega, & \text{for} \ \xi \leq r_2 \leq r_3
\end{cases}$$

6
or more compactly
\[
\tilde{\omega}(r) = \min(r_2, \xi) \int_{r_1}^{r_3} \psi_1(\lambda) d\Omega + \sqrt{-1} \max(r_2, \xi) \int_{r_1}^{r_3} \psi_2(\lambda) d\Omega .
\] (33)

The obtained solution can be singular only at two points which are outside the domains \(D_1\) and \(D_2\), namely, \(r_2 = r_1 = \xi\) and \(r_2 = r_3 = \xi\). These singularities are removed by the additional requirement
\[
\psi_1(\xi) = \psi_2(\xi) = 0 .
\] (34)

We remark that the solution (33), (34) generally is nonanalytic at the plane \(r_2 = \xi\). This, however, does not contradict to a hyperbolic nature of the problem under consideration. Such a weak singularity was first revealed by Gurevich and Pitaevskii [13] in the numerical solution for the problem of cubic-like breaking (the exact solution of this problem, however, is analytic and does not contain any singularities [26,15,27]). The solution (33), (34) can be easily generalized to the multiphase case. The result is:
\[
\tilde{\omega}(m)(r) = \sum_{k=1}^{g} \tilde{\omega}_k^{(m)}(r) , \ m = 1, ..., g ,
\] (35)

where
\[
\tilde{\omega}_k^{(m)}(r) = \min(r_{2k}, \xi_k) \int_{r_{2k-1}}^{r_{2k+1}} \psi_{2k-1}(\lambda) d\Omega^{(m)} + \sqrt{-1} \max(r_{2k}, \xi_k) \int_{r_{2k-1}}^{r_{2k+1}} \psi_{2k}(\lambda) d\Omega^{(m)} ,
\]
\[
\psi_{2k-1}(\xi_k) = \psi_{2k}(\xi_k) = 0 ,
\]
\[
r_{2k-1} \leq \xi_k \leq r_{2k+1} .
\] (36)

One can see that this solution is continuous and limited in all the domain of definition and generally contains \(g\) weak singular points \(r_{2k} = \xi_k , \ k = 1, ..., g\). We remark in addition that formula (35) represents in some sense a decomposition of the \(g\)-phase solution into \(g\) single-phase ones in the hodograph space.

The analytic limited solutions which were obtained in [10] in the form of infinite series can be picked out from the general representation (35) by imposing the additional restriction
\[
\psi_{2k-1}(\lambda) = \psi_{2k}(\lambda) \equiv \psi(\lambda) , \ k = 1, ..., g ,
\] (37)

where \(\psi(\lambda)\) is a real analytic function with \(g\) real zeros \(\xi_k\).

4 Boundary value problem solution

Now we formulate the boundary problem to the system (12) corresponding to the initial value problem (1), (2). To avoid geometrical complications we shall consider
only monotonically decreasing initial data \([1]\). Evolution of such a curve with the only
breaking point \((u''(x_{br}) = 0)\) is described completely (in the weak limit sense \([1, 2, 6]\))
by the single–phase averaged system \([8]\) with the boundary conditions formulated
first by Gurevich and Pitaevskii \([13]\). The more recent investigations \([1, 2, 16, 28]\) have
shown that these boundary conditions arise naturally in the zero dispersion limit of
the KdV equation and are in the continuity of the Riemann invariants \(r_j\) on the phase
transition boundaries where \(r_{2k} = r_{2k-1}\) \((j \neq 2k; 2k - 1)\) or \(r_{2k} = r_{2k+1}\)
\((j \neq 2k; 2k + 1)\).

Earlier this problem was investigated by Avilov and Novikov \([29]\) numerically in the
framework of the theory of system of the hydrodynamic type. The effective analytic
solutions of the Gurevich – Pitaevskii problem for all single–phase evolving (including
nonanalytical and hump–like ) data were obtained in Refs. \([15, 30, 31]\). It is clear that
every additional breaking introduces a new phase (gap) into consideration and therefore
the genus of the problem \(g\) coincides with the number of breaking points on the initial
curve \([6]\). By this means the number of dynamical variables in the problem is \(2g + 1\).

At some moment \(t_{overlap}\) the evolving oscillatory phases will begin to overlap forming
the multiphase solution region. To be specific we consider for the beginning the case
when the initial curve \(u_0(x)\) has two distinct breaking points \(x_1\) and \(x_2\) (Fig. 2a):

\[
\begin{align*}
  u_0'(x) &< 0, \quad u_0''(x_1) = 0, \quad u_0''(x_2) = 0, \\
  u_0'''(x_1) &> 0, \quad u_0'''(x_2) > 0, \\
  u_0(x_1) &= c_1, \quad u_0(x_2) = c_2.
\end{align*}
\]

We assume in addition that the initial data are such that the breakings appear distinctly
in space (i.e. that there are no points like \(A\) in Fig. 1). This assumption is made only
for convenience of consideration and will be omitted afterwards. To obtain boundary
conditions to the system \([12]\) corresponding to the initial data \([1], \ [38]\) we apply
to the one–phase stage of the evolution when two single-phase modes do not overlap
yet (Fig. 2b). At this stage the global solution of \([12]\), which is generally defined
in five \((2g + 1) \) - dimensional \(r\)-space, lies on two \((g)\) separated three–dimensional
sections \((r_1, r_2, r_3), ..., (r_{2g-1}, r_{2g}, r_{2g+1})\) (Fig. 3) (see Ref.[15]) with common diagonal
surfaces \(r_2 = r_3 = r_4, ..., r_{2g-2} = r_{2g-1} = r_{2g}\). These diagonal surfaces are the domains
of definition of the zero-phase solution which separates the one-phase domains at the
stage considered. Due to hyperbolicity of the hodograph equations (which is inherited
from the hyperbolicity of the KdV-Whitham system) these 3-D domains do not interact
and can be considered independently. We remark that the initial data and their smooth
evolution until the first breaking are defined, as it will be shown below, on the pure
diagonal \(r_1 = r_2 = ... = r_{2g+1}\) in hodograph space \([16, 10]\).

Now we recall the important general fact proven in \([10]\), which provides existence
of the global solution to the initial value problem for the KdV-Whitham system: the
uniform commuting flows \(W_j(r)\) (including \(V_j(r)\), of course) do not depend on multiple
invariants \(r_{2k} = r_{2k+1}\), or \(r_{2k} = r_{2k-1}\) if \(j \neq 2k; 2k + 1\) or \(2k - 1\), and, moreover, they
turn into their analogs for the \( g - 1 \) genus case:

\[
W_j \begin{array}{c}
\left( r_1, \ldots, r_{2k-1}, \beta, \bar{\beta}, r_{2k+2}, \ldots, r_{2g+1} \right) \\
g
\end{array} = (39)
\]

and the analogous assertion is true for \( r_{2k} = r_{2k-1}, \ j \neq 2k, 2k-1 \). As the obvious consequence of the equations (12) these relationships take place for all (not only uniform) \( W_j \)'s. This fact allows us to formulate boundary conditions at the phase transition surfaces \( r_{2k} = r_{2k-1} \) and \( r_{2k} = r_{2k+1}, \ k = 1, \ldots, g \) which are single-phase – zero-phase transitions at the considered stage of evolution. Comparison of the hodograph solution (11) with the solution of the Hopf equation (zero-phase averaged KdV)

\[
x - 6rt = W(r),
\]

where \( W(r) \) is the inverse function to the initial curve \( u_0(x) \) (see Fig. 2a) shows that odd branches of the multivalued curve \( r_j(x, r), \ j = 1, \ldots, 5; \ r_1 \leq \ldots \leq r_5 \), have to be matched with the appropriate branches of the inverse function \( W(r) \) (see Fig. 2b). We shall demonstrate in details how to accomplish this matching at the boundaries where \( r_2 = r_3 \) and \( r_2 = r_1 \) (the right-hand single-phase wave in Fig. 2b). The rest matching can be accomplished in an analogous way.

As a first step we make a passage to the single-phase regime in the two-phase desired solution (12) with certain \( W_j(r_1, \ldots, r_5) \). This corresponds to passage to times \( t_{br} < t < t_{overlap} \).

Using (39) we have the transition

\[
W_j \begin{array}{c}
\left( r_1, r_2, r_3, \zeta, \zeta \right) \\
g = 2
\end{array} = W_j \begin{array}{c}
\left( r_1, r_2, r_3 \right) \\
g = 1
\end{array}, \ j = 1, 2, 3, (41)
\]

where \( \zeta \) is an arbitrary real parameter.

Now the boundary conditions for the function \( W_j(r_1, r_2, r_3) \) can be obtained by means of the usual Gurevich–Pitaevskii matching (see Refs. [13-15]) for the single-phase case, namely, at the edges where \( r_2 = r_3 \) and \( r_2 = r_1 \) we have, taking into account (39), the single-phase – zero-phase transitions which drive us to the solution of the Hopf equation (40) with the initial data (1), (38) that is:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{when } r_2 = r_3 \\
W_1 \end{array}
\end{array} \\
g = 1
\end{array} = W_1 \begin{array}{c}
\left( r_1, c_1, c_1 \right) \\
g = 1
\end{array} = W_1 \begin{array}{c}
\left( r_1 \right) \\
g = 0
\end{array} = W(r_1) (42)
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{when } r_2 = r_1 \\
W_3 \end{array}
\end{array} \\
g = 1
\end{array} = W_3 \begin{array}{c}
\left( c_1, c_1, r_3 \right) \\
g = 1
\end{array} = W_3 \begin{array}{c}
\left( r_3 \right) \\
g = 0
\end{array} = W(r_3)
\]

9
Here following the ideology of [14,15] we have placed the coalesced invariants into the breaking point $c_1$ which in a natural way marks the domain of definition of the sought solution with respect to the variables $r_1$ and $r_3$, namely (see Fig. 2b)

$$r_1 \leq c_1 \leq r_3$$

(43)

Using arbitrarity of the parameter $\zeta$ in (41) we present the boundary conditions for the desired $W_j$’s in the final form

$$W_1(r_1, c_1, c_1, c_1, c_1) = W(r_1) ,$$

$$W_3(c_1, c_1, r_3, r_3, r_3) = W(r_3) ,$$

(44)

An analogous consideration for the left-hand single-phase wave gives two more conditions

$$W_3(r_3, r_3, r_3, c_2, c_2) = W(r_3) ,$$

$$W_5(c_2, c_2, c_2, c_2, r_5) = W(r_5) ,$$

and, obviously,

$$r_3 \leq c_2 \leq r_5 .$$

(45)

(46)

So we have four ($2g$) boundary conditions of the Goursat type for the system (12) which is, as it was mentioned above, specified by five ($2g+1$) functions of one variable. The rest condition is the boundedness of the $W_j$’s in all the domain of definition which provides the boundedness of the hodograph solution (12) at finite $x,t$. The limited solution is given by formulas (25), (35) and is parametrized precisely by $2g$ arbitrary functions of one variable. Substituting this solution into the boundary conditions (44), (45) one arrives after some calculations at four linear Abel equations resolving of which gives a simple representation for the unknown functions $\psi_j(\lambda)$ in (35).

$$\psi_{2k-1}(\lambda) = \frac{(-1)^k}{2\pi} \int_{c_k}^{c_k} \frac{W(x)}{\sqrt{x-\lambda}} dx ,$$

$$\psi_{2k}(\lambda) = \frac{(-1)^k}{2\pi} \int_{c_k}^{c_k} \frac{W(x)}{\sqrt{\lambda-x}} dx , \quad k = 1, 2.$$  

(47)

It should be noted that the arbitrary parameters $\xi_k$ in (33) have been chosen as follows (compare (36) with (43), (46)):

$$\xi_1 = c_1 , \quad \xi_2 = c_2$$

(48)

to satisfy the conditions (36).
Examination of the general case of evolution of the initial perturbation with $g$ breaking points is completely analogous and does not involve any additional difficulties. The result is: the representation (47) is valid for any $g$. Thus, the arbitrary functions $\psi_k(\lambda)$ in the general limited solution (35) are the Abel transformations of the parts of the inverse to the initial perturbation function $W(r)$ which are divided by the breaking points. We emphasize that this result is valid only for monotonic initial data which have the one-valued inverse function. Evolution of the hump-like or large-scale oscillating initial perturbation requires an additional consideration which will be accomplished in the following papers. Appropriate results for the single-phase case can be found in Refs.[30,31].

So we have found the solution describing the motion of the multivalued curve $r_j(x, t)$ proceeding from the single-phase matching. Let us examine the initial value problem for the Whitham system (8) corresponding to the solved boundary problem. Making the consequent phase transitions like (41) in the hodograph solution (11) we arrive putting the arbitrary parameter $\zeta$ equal to $r_j$ at the system

$$x - V_j(r_j, ..., r_j)t = W_j(r_j, ..., r_j)$$

which is the system of the implicit solutions of $2g + 1$ Hopf equations (zero-phase averaged KdV) with the same initial data

$$x - 6r_jt = W_j(r_j) \quad j = 1, ..., 2g + 1.$$  \hspace{1cm} (50)

Therefore, both initial data and their smooth evolution are given on the pure diagonal $r_1 = r_2 = ... = r_{2g+1}$ in the real $r$-space. By this means the desired initial value problem for the Whitham-KdV system has the form

$$r_1(x, 0) = r_2(x, 0) = ... = r_{2g+1}(x, 0) = u_0(x).$$  \hspace{1cm} (51)

It should be noted that our consideration really did not require any assumptions of the distinct breakings so the latter can be omitted.

Now we determine the motion of the phase transition boundaries which are the multiple caustics for the obtained solution. Really, it follows from the potential representation (11) that $V_{2k}(r)$ coalesces with either $V_{2k-1}(r)$ or $V_{2k+1}(r)$ when $r_{2k}$ coalesces with either $r_{2k-1}$ or $r_{2k+1}$ respectively. By this means the lines $x_j(t)$ of the phase transition in the $x, t$-plane have the multiple characteristic direction in every point and are given by the ordinary equations

$$\frac{dx_j}{dt} = V_j^{\text{multiple}}(r)$$  \hspace{1cm} (52)

which should be considered together with the hodograph solution (11), (23), (35), (47).

Let us discuss briefly the evolution of the Riemann surface topology on the obtained family of solutions. The Riemann surface of genus $g$ is topologically equivalent to a sphere with $g$ handles. These handles evolve both in space and in time and some of
them degenerate into poles on the phase transition boundaries in \((x, t)\)-plane. Tending time to zero we arrive at the initial Riemann surface which represents a sphere with \(g\) singularities (poles). By this means we conclude that the topology of the evolving Riemann surface from the very beginning is given not only by the KdV equation itself but by the whole Cauchy problem formulation.

5 The fundamental solution to the initial value problem.

We give another representation for the solution to the initial value problem for the Whitham-KdV system. Substituting the Abel transformations (47) into the solution (35) and interchanging the order of integration we find

\[
\tilde{\omega}^{(m)} = \int_{r_1}^{r_{2g+1}} W(x)K^{(m)}(r, x)dx
\]

(53)

where \(K^{(m)}(r, x)\) is the fundamental solution (Riemann kernel function [32]) to the initial value problem (8), (51):

\[
2\pi K^{(m)}(r, x) = \begin{cases} 
(-1)^k \int_{r_{2k-1}}^{\min(r_{2k}, x)} \frac{d\Omega^{(m)}}{\sqrt{x - \lambda}} & \text{for } r_{2k-1} \leq x \leq c_k , \\
(-1)^k \int_{\max(r_{2k}, x)}^{r_{2k+1}} \frac{d\Omega^{(m)}}{\sqrt{x - \lambda}} & \text{for } c_k \leq x \leq r_{2k+1}.
\end{cases}
\]

(54)

6 Summary and Conclusions

The generating function of the uniform commuting flows for the multiphase averaged KdV equation (Whitham-KdV hierarchy) has been obtained in a direct way as depending on an additional parameter \(\lambda\) solution of the linear generalized hodograph equations. It has a potential form and the potential (generalized frequency) turns out to be proportional to the coefficient of the holomorphic basis differential on the Riemann surface of genus \(g\), where \(g\) coincides with the number of phases. The general solution in hodograph space is constructed as a superposition of \(2g + 1\) contour integrals of this generating function multiplied by arbitrary functions of one variable. The physical solutions corresponding to the Cauchy problem with monotonically decreasing initial data with \(g\) breaking points for the initial KdV equation are distinguished by the requirement of boundedness and depend on \(2g\) arbitrary functions. Resolving of the appropriate boundary problem to the Whitham equations in hodograph space shows that these functions are the linear Abel transformations of the parts of the initial
curve divided by breaking points. Evolution of the Riemann surface topology on the obtained family of solutions is discussed and it is shown that "initial" Riemann surface represents a sphere with $g$ singularities (poles). The initial value problem for the Whitham-KdV equations has been formulated and its fundamental solution (Riemann kernel function) has been constructed.

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Figure Captions

- 1. Splitting of the \((x,t)\)-plane in the zero dispersion limit.
- 2. Riemann invariants behaviour for the case \(g = 2\) at \(a) t = 0\) \(b) t_{br} < t < t_{overlap}\)
- 3. Domain of definition of the single-phase mode: \(r_{2k-1} \leq r_{2k} \leq r_{2k+1}\)