Elliptic umbilic representations
connected with the caustic

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Abstract
We investigate the elliptic umbilic canonical integral with an approach based on a series expansion of its initial distribution shifted to the caustic points. An absolutely convergent integral representation for the elliptic umbilic is obtained. Using it, we find the elliptic umbilic particular values in terms of 2F2 hypergeometric functions. We also derive an integral over the product of Gaussian and two Airy functions in terms of Bessel functions of fractional orders. Some other corollaries including 3F2 hypergeometric function special values and the Airy polynomials relations are also discussed.

Keywords: elliptic umbilic integral, Airy functions, hypergeometric functions, Bessel functions

1 Introduction
This paper is inspired by the relation connecting the elliptic umbilic integral [1] with Bessel functions of orders \( \pm \frac{1}{6} \), which was found by Berry and Howls in [2]:

\[
E(0, 0, z) = \sqrt{\frac{\pi |z|}{27}} \exp \left( -\frac{2i z^3}{27} \right) \left\{ J_{-1/6} \left( \frac{2|z|^3}{27} \right) - i \text{sgn} \ z \cdot J_{1/6} \left( \frac{2|z|^3}{27} \right) \right\}
\]

(1.1)

\[
= \frac{\sqrt{\pi}}{3} \exp \left( -\frac{2i z^3}{27} \right) \left\{ \frac{1}{\Gamma \left( \frac{5}{6} \right)} \cdot {}_0 F_1 \left( \frac{5}{6}, \frac{-z^6}{3^6} \right) - \frac{iz}{3\Gamma \left( \frac{7}{6} \right)} \cdot {}_0 F_1 \left( \frac{7}{6}, \frac{-z^6}{3^6} \right) \right\}.
\]

(1.2)

(We prefer the hypergeometric description of the result since it demonstrates that \( E(0, 0, z) \) is an entire function of \( z \). Besides, it leads to the value \( E(0, 0, 0) \) immediately. On the other hand, the expression in terms of Bessel functions is useful for large values of \( z \) due to well-known asymptotic of \( J_\nu(z) \).) The elliptic umbilic integral is defined by the formula

\[
E(x, y, z) = \mathcal{F} \left[ \exp \left\{ -iz(\xi^2 + \eta^2) + i(\xi^3 - 3\xi \eta^2) \right\} \right](x, y),
\]

(1.3)

where \( \mathcal{F}[f(\xi, \eta)](x, y) \) is the 2D Fourier transform,

\[
\mathcal{F}[f(\xi, \eta)](x, y) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \exp \{ -i(x \xi + y \eta) \} f(\xi, \eta) \, d\xi \, d\eta,
\]

(1.4)
Fig. 1: The caustic, amplitude and phase distributions of the elliptic umbilic \(E(x, y, z)\). The points \((z^2, 0)\) and \((-z^2/3, 0)\) marked by black discs correspond to the parameter values \(t = 0\) and \(t = \pi\), respectively. The distributions are shown for \(z = 4.5\) in the square \(-25 \leq x, y \leq 25\).

and \(x, y, z\) are real variables. (We use the notation proposed in [1] instead of its later modification \(\Psi_E(x, y, z) = 2\pi E(-x, -y, -z)\), see [2] and [3, Chap. 36].)

It was quite unexpected for us that \(E(0, 0, z)\) requires the use of \(J_{\pm 1/6}\) functions, especially since \(E(x, y, 0)\) is described in terms of Airy functions, i.e., in terms of \(J_{\pm 1/3}\) functions [1, Eq. (3.4)]:

\[
E(x, y, 0) = \gamma \pi \text{Re} \left\{ \text{Ai} \left( -\frac{x + iy}{3\gamma} \right) \text{Bi} \left( -\frac{x - iy}{3\gamma} \right) \right\},
\]

where \(\gamma = (2/3)^{2/3}\) is an auxiliary constant used for brevity.

The function \(E(x, y, z)\) has been thoroughly investigated by Berry et al [12]. We recall some properties of \(E(x, y, z)\) proven in [1] and used below. First, the function \(E(x, y, z)\) satisfies the symmetry relations:

\[
E(x, y, -z) = E^\ast(x, y, z),
\]

\[
E(x, -y, z) = E(x, y, z),
\]

\[
E \left( x \cos \frac{2\pi n}{3} - y \sin \frac{2\pi n}{3}, x \sin \frac{2\pi n}{3} + y \cos \frac{2\pi n}{3}, z \right) = E(x, y, z),
\]

where an asterisk means complex conjugation, \(n\) is an integer, and the last expression demonstrates the invariance of \(E(x, y, z)\) under rotation by \(2\pi n/3\) in the \((x, y)\) plane.

Second, the equation of the caustic in parametric form is

\[
x + iy = \frac{z^2}{3} e^{-\frac{2\pi t}{3}} (2 + e^{-3it}), \quad t \in [0, 2\pi).
\]

And third, \(E(x, y, z)\) may be expanded into a series in powers of \(z\):

\[
E(x, y, z) = \gamma \pi \sum_{n \geq 0} \frac{(i\gamma z)^n}{n!} \text{Re} \left\{ \text{Ai}^{(n)} \left( -\frac{x + iy}{3\gamma} \right) \text{Bi}^{(n)} \left( -\frac{x - iy}{3\gamma} \right) \right\},
\]

which provides a simple description of the function behaviour for small \(z\) (see figure 1).
Since both functions $\text{Ai}(x)$ and $\text{Bi}(x)$ satisfy the same differential equation $y'' = xy$, the $n$-th derivative of each function can be written as a linear combination of the function itself and its first derivative:

$$
\begin{align*}
\text{Ai}^{(n)}(x) &= P_n(x)\text{Ai}(x) + Q_n(x)\text{Ai}'(x), \\
\text{Bi}^{(n)}(x) &= P_n(x)\text{Bi}(x) + Q_n(x)\text{Bi}'(x).
\end{align*}
$$

Here $P_n(x)$ and $Q_n(x)$ are some polynomials (the index $n$ corresponds to the derivative order but not the polynomials degree). In particular,

$$
\begin{align*}
P_0(x) &= 1, & P_1(x) &= 0, & P_2(x) &= x, & P_6(x) &= x^3 + 4, & P_{12}(x) &= x^6 + 260x^3 + 280, \\
Q_0(x) &= 0, & Q_1(x) &= 1, & Q_2(x) &= 0, & Q_6(x) &= 6x, & Q_{12}(x) &= 30x^4 + 600x.
\end{align*}
$$

The polynomials can be expressed in terms of particular values of Gegenbauer polynomials \[4\], however, corresponding formulae are quite unusual to apply them for evaluation of the series \(1.10\) easily.

Nevertheless, an attempt to transform and reorder the terms in the series expansion \(1.10\) is one of possible ways to find an expression of $E(x, y, z)$, which can be helpful for numerical simulations. Of course, there are various methods for evaluating infinite range oscillatory integrals. In particular, the Weniger transform demonstrates the effectiveness being applied to the elliptic umbilic \[5,6\]. However, this method is mainly due to divergent integrals and series, whereas the right side of \(1.3\) is an ordinary integral. In a sense, the situation with the Airy function, $\text{Ai}(x)$, is a good example. Initially, $\text{Ai}(x)$ is defined by a conditionally convergent integral for real values of $x$ only. Then, by contour integration, it reduces to an absolutely convergent integral which is valid for any complex $x$ (see Eqs. (9.5.1) and (9.5.4) in \[3, Chap.9\]). Here we are interested in similar procedure for the elliptic umbilic.

The paper is organized as follows. In Section 2 we find various series expansions for the elliptic umbilic depending on the derivatives of the Airy functions product. This section also gives the main result of the paper, namely an absolutely convergent integral representation of $E(x, y, z)$ closely connecting with the caustic \(1.9\). In Sections 3 and 4 some corollaries of this result are presented. First, we analyze the series expansions and obtain the elliptic umbilic values in the caustic points $(z^2, 0, z)$ and $(-z^2/3, 0, z)$ in terms of $2F_2$ hypergeometric functions. Then, an improper integral of the product of Gaussian and two Airy functions is evaluated in terms of $0F_1$ hypergeometric functions (the Bessel functions of fractional orders). The paper ends with concluding remarks and perspectives for future research. Some auxiliary integrals and series containing the Airy functions are presented in Appendix A.
2 Elliptic umbilic and absolutely convergent integral representation

Let us return to the elliptic umbilic definition (1.3). By changing variables \( \xi \to \xi + a \) and \( \eta \to \eta + b \), where \( a, b \) are real parameters, we obtain

\[
E(x, y, z) = \frac{1}{2\pi} \exp \left( -iax - iby - iz[a^2 + b^2] + i[a^3 - 3ab^2] \right) \\
\times \int_{\mathbb{R}^2} \exp \left( -i[x + 2az - 3(a^2 - b^2)]\xi - i[y + 2bz + 6ab]\eta \right) \\
- i[z - 3a]\xi^2 - 6ib\xi\eta - i[z + 3a]\eta^2 + i[\xi^3 - 3\xi\eta^2] \, d\xi \, d\eta. \tag{2.1}
\]

Expanding the factor \( \exp(-i[z - 3a]\xi^2 - 6ib\xi\eta - i[z + 3a]\eta^2) \) as a power series and replacing monomials in \( \xi, \eta \) by derivatives in \( x, y \) leads to an integral which is known due to (1.5):

\[
E(x, y, z) = \exp \left( -iax - iby - iz[a^2 + b^2] + i[a^3 - 3ab^2] \right) \\
\times \sum_{n \geq 0} \frac{i^n}{n!} \left\{ [z - 3a]\partial_x^2 + 6b\partial_y \partial_x + [z + 3a]\partial_y^2 \right\}^n E(x + 2az - 3[a^2 - b^2], y + 2bz + 6ab, 0) \tag{2.2}
\]

(cf. [7], Eq. (27)).

Choosing various values of \( a, b \) in (2.2) gives various series expansions. For example, if the elliptic umbilic in the right side of (2.2) is \( E(x, y, 0) \), then we have

\[
\begin{align*}
2az - 3[a^2 - b^2] &= 0 \\
2bz + 6ab &= 0
\end{align*}
\Rightarrow (a, b) = \left\{ (0, 0), \left( -\frac{z}{3}, \frac{z}{\sqrt{3}} \right), \left( -\frac{z}{3}, -\frac{z}{\sqrt{3}} \right), \left( \frac{2z}{3}, 0 \right) \right\}.
\]

For the case \( a = b = 0 \), the expansion (2.2) reduces to (1.10):

\[
E(x, y, z) = \sum_{n \geq 0} \frac{(iz)^n}{n!} \left\{ \partial_x^2 + \partial_y^2 \right\}^n E(x, y, 0) \\
= \sum_{n \geq 0} \frac{(iz)^n}{n!} \left\{ \gamma \partial_x \partial_y \right\}^n E(x, y, 0) \\
= \gamma \pi \sum_{n \geq 0} \frac{(i\gamma z)^n}{n!} \Re \{ A_i^{(n)}(v)B_i^{(n)}(v^*) \}, \tag{2.3}
\]

where \( v = -(x + iy)/3\gamma \). For other three cases, the expansions are

\[
E(x, y, z) = \exp \left( \frac{iz}{3} (x - y\sqrt{3}) - \frac{4iz^3}{27} \right) \sum_{n \geq 0} \frac{(2iz)^n}{n!} \left\{ \partial_x^2 + \sqrt{3}\partial_x \partial_y \right\}^n E(x, y, 0),
\]

\[
= \exp \left( \frac{iz}{3} (x + y\sqrt{3}) - \frac{4iz^3}{27} \right) \sum_{n \geq 0} \frac{(2iz)^n}{n!} \left\{ \partial_x^2 - \sqrt{3}\partial_x \partial_y \right\}^n E(x, y, 0),
\]

\[
= \exp \left( -\frac{2ixz}{3} - \frac{4iz^3}{27} \right) \sum_{n \geq 0} \frac{(iz)^n}{n!} \left\{ 3\partial_y^2 - \partial_x^2 \right\}^n E(x, y, 0). \tag{2.4}
\]
It is evident that all of them are various versions of one and the same expansion due to the symmetry relation (1.8).

We can simplify the differential operator in the right side of (2.2) by removing two terms of three:

\[ E(x, y, z) = \exp \left( \frac{ixz}{3} - \frac{4iz^3}{27} \right) \sum_{n \geq 0} \frac{(2iz)^n}{n!} \partial_x^{2n} E(x - z^2, y, 0), \quad (2.5) \]

\[ = \exp \left( -\frac{ixz}{3} + \frac{2iz^3}{27} \right) \sum_{n \geq 0} \frac{(2iz)^n}{n!} \partial_y^{2n} E \left( x + \frac{z^2}{3}, y, 0 \right). \quad (2.6) \]

Here \((a, b) = (-z/3, 0)\) and \((a, b) = (z/3, 0)\), respectively. Moreover, expanding \(E(x, y, 0)\) as a power series in \(x, y\) (see Appendix A), it is easy to find the derivatives in both expressions.

And finally, substituting \(z = 0\) into (2.2) one can find a differential difference relation for \(E(x, y, 0)\):

\[ E(x, y, 0) = \exp \left( -i [ax + by] + i [a^3 - 3ab^2] \right) \]
\[ \times \sum_{n \geq 0} \frac{(3i)^n}{n!} \left\{ a [\partial_y^2 - \partial_x^2] + 2ab \partial_x \partial_y \right\}^n E(x - 3[a^2 - b^2], y + 6ab, 0). \quad (2.7) \]

The formulae above are useful in certain analytic descriptions (special values of the hypergeometric \(3F_2\) function, Airy polynomials \(\Pi_n, \text{etc.}\) which we discuss later. However, all the expansions of \(E(x, y, z)\) did not look perfect because their invariance under a rotation by \(2\pi/3\) in \((x, y)\) plane cannot be read immediately. The expansion (2.3) looks a little better than others since its invariance follows from the known properties of the Airy functions:

\[ \text{Ai}(\omega v) = \frac{1}{2}e^{\pi i/3} [\text{Ai}(v) - i \cdot \text{Bi}(v)], \quad \text{Bi}(\omega v) = \frac{1}{2}e^{-\pi i/6} [3\text{Ai}(v) + i \cdot \text{Bi}(v)], \quad (2.8) \]

where \(\omega = e^{2\pi i/3}\) is the cubic root of unity.

We can try to construct the desired expansion, noting that the elliptic umbilic shifts in the formulae (2.5) and (2.6) are coordinates of the caustic points placed on the \(x\) axis (see figure 1). Namely, we select parameters \(a, b\) in (2.2) such that the elliptic umbilic in the right side has been shifted to a point of the caustic, \(E(x - x_0, y - y_0, 0)\), where

\[ x_0 = \frac{z^2}{3} (2 \cos t_0 + \cos 2t_0), \quad y_0 = \frac{z^2}{3} (2 \sin t_0 - \sin 2t_0) \quad (2.9) \]

for some \(t_0 \in [0, 2\pi)\). Then

\[ 2az - 3[a^2 - b^2] = -x_0, \quad \frac{2bz + 6ab}{6} = -y_0 \]

\[ \Rightarrow \quad a = -\frac{z}{3} \cos t_0, \quad b = -\frac{z}{3} \sin t_0, \quad (2.10) \]

\[ \left\{ [z - 3a] \partial_x^2 + 6bd \partial_x \partial_y + [z + 3a] \partial_y^2 \right\}^n = (2z)^n \left\{ \cos \frac{t_0}{2} \partial_x - \sin \frac{t_0}{2} \partial_y \right\}^{2n}, \]

and the expansion (2.2) has the form

\[ E(x, y, z) = \exp \left( \frac{iz}{3} \left[ x \cos t_0 + y \sin t_0 \right] - \frac{iz^3}{27} \left[ 3 + \cos 3t_0 \right] \right) \]
\[ \times \sum_{n \geq 0} \frac{(2iz)^n}{n!} \left\{ \cos \frac{t_0}{2} \partial_x - \sin \frac{t_0}{2} \partial_y \right\}^{2n} E(x - x_0, y - y_0, 0). \quad (2.11) \]
Now, we transform the right side of (2.11) to an integral with Airy functions. Let \(X = x - x_0, Y = y - y_0\) and \(V = -(X + iY)/3\gamma\). Then \(E(X, Y, 0) = \gamma \pi \text{Re} \{\text{Ai}(V)\text{Bi}(V^*)\}\),

\[
\cos \frac{t_0}{2} \partial_X - \sin \frac{t_0}{2} \partial_Y = -\frac{1}{3\gamma} \left\{ e^{-it_0/2} \partial_V + e^{it_0/2} \partial_{V^*} \right\}
\]

and

\[
\begin{align*}
\left\{ \cos \frac{t_0}{2} \partial_X - \sin \frac{t_0}{2} \partial_Y \right\}^{2n} E(X, Y, 0) &= \\
&= \frac{\gamma \pi}{2} \left( -\frac{1}{3\gamma} \right)^{2n} \sum_{k=0}^{2n} \binom{2n}{k} \left( e^{-it_0/2} \partial_V \right)^k \left( e^{it_0/2} \partial_{V^*} \right)^{2n-k} \{\text{Ai}(V)\text{Bi}(V^*) + \text{Ai}(V^*)\text{Bi}(V)\} = \\
&= \frac{\gamma \pi}{2} \left( \frac{\gamma}{4} \right)^n \sum_{k=0}^{2n} \binom{2n}{k} \left[ e^{it_0(n-k)} \text{Ai}^{(k)}(V)\text{Bi}^{(2n-k)}(V^*) + e^{-it_0(n-k)} \text{Ai}^{(k)}(V^*)\text{Bi}^{(2n-k)}(V) \right].
\end{align*}
\]

(2.12)

Denoting by \(S(n, k)\) the expression within the square brackets and substituting the right side of (2.12) into the series in (2.11), we have

\[
\sum_n \frac{(2iz)^n}{n!} \left\{ \cos \frac{t_0}{2} \partial_X - \sin \frac{t_0}{2} \partial_Y \right\}^{2n} E(X, Y, 0) = \\
\frac{\gamma \pi}{2} \sum_{n \geq 0} \frac{1}{n!} \left( \frac{i\gamma z}{2} \right)^n \sum_{k=0}^{2n} \binom{2n}{k} S(n, k) = \frac{\gamma \pi}{2} \sum_{k \geq 0} \sum_{n \geq \frac{k}{2}} \binom{2n}{k} \frac{1}{n!} \left( \frac{i\gamma z}{2} \right)^n S(n, k) = \\
\frac{\gamma \pi}{2} \sum_{\epsilon=0, 1} \sum_{k \geq 0} \sum_{n \geq k+\epsilon} \binom{2n+2k+2\epsilon}{n+k+\epsilon} ! \left( \frac{i\gamma z}{2} \right)^{n+k+\epsilon} S(n+k+\epsilon, 2k+\epsilon) = \\
\frac{\gamma \pi}{2} \sum_{\epsilon=0, 1} \sum_{k \geq 0} \sum_{n \geq 0} \binom{2n+2k+2\epsilon}{n+k+\epsilon} ! \left( \frac{i\gamma z}{2} \right)^{n+k+\epsilon} S(n+k+\epsilon, 2k+\epsilon) (2n+\epsilon)!. 
\]

(2.13)

Since

\[
\frac{(2N)!}{N!} = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-t^2} (4t^2)^N \, dt,
\]

then

\[
\sum_n = \frac{\gamma \sqrt{\pi}}{2} \sum_{\epsilon=0, 1} \int_{\mathbb{R}} e^{-t^2} \sum_{k \geq 0} \sum_{n \geq 0} \frac{(2i\gamma z t^2)^{n+k+\epsilon}}{(2k+\epsilon)!(2n+\epsilon)!} \\
\times \left\{ e^{it_0(n-k)} \text{Ai}^{(2k+\epsilon)}(V)\text{Bi}^{(2n+\epsilon)}(V^*) + e^{-it_0(n-k)} \text{Ai}^{(2k+\epsilon)}(V^*)\text{Bi}^{(2n+\epsilon)}(V) \right\} \, dt.
\]

(2.14)

Returning to (2.11) and using the relations

\[
\begin{align*}
\sum_{k \geq 0} F^{(2k)}(x) \frac{a^{2k}}{(2k)!} &= \sum_{k \geq 0} \frac{1}{2} \left( \frac{1 - (-1)^k}{2} \right) F^{(k)}(x) \frac{a^k}{k!} = \frac{1}{2} \left\{ F(x + a) + F(x - a) \right\}, \\
\sum_{k \geq 0} F^{(2k+1)}(x) \frac{a^{2k+1}}{(2k+1)!} &= \sum_{k \geq 0} \frac{1}{2} \left( \frac{1 - (-1)^k}{2} \right) F^{(k)}(x) \frac{a^k}{k!} = \frac{1}{2} \left\{ F(x + a) - F(x - a) \right\},
\end{align*}
\]
we obtain an absolutely convergent integral because both Airy functions are of order $\frac{3}{2}$:

$$E(x, y, z) = \frac{\gamma \sqrt{\pi}}{2} \exp \left( \frac{iz}{3} \left[ x \cos t_0 + y \sin t_0 \right] - \frac{i z^3}{27} [3 + \cos 3 t_0] \right)$$

$$\times \int_{\mathbb{R}} e^{-t^2} \left\{ \text{Ai}(V + t \sqrt{2ie^{-it_0} \gamma z}) \text{Bi}(V^* + t \sqrt{2ie^{it_0} \gamma z}) + \right.$$ 

$$\left. + \text{Ai}(V^* + t \sqrt{2ie^{it_0} \gamma z}) \text{Bi}(V + t \sqrt{2ie^{-it_0} \gamma z}) \right\} dt. \quad (2.15)$$

Here, $V = -\{ (x - x_0) + i(y - y_0) \}/3\gamma$ and $(x_0, y_0)$ is a point of the caustic (2.9). In particular, if $z = 0$, then $x_0 = y_0 = 0$ and (1.5) follows immediately. As is seen from (2.14), the expression (2.15) does not depend on a square root branch selection.

3 **Elliptic umbilic and hypergeometric functions**

In what follows, we will restrict ourselves to the case $z \geq 0$ for simplicity (see Eq. (1.6)), considering $z$ as a parameter. To evaluate the integral (2.15), a caustic point $(x_0, y_0)$ is required. Of course, this point may be chosen arbitrary. However, for evaluation of $E(x, y, z)$ it seems natural to choose such $(x_0, y_0)$ that $x = cx_0$, $y = cy_0$, where $c \geq 0$. In particular, if $(x, y)$ is placed on the caustic, the choice $x_0 = x$, $y_0 = y$ is the best since then $V = 0$ and the oscillating influence of the Airy functions on the integrand is minimal. Nevertheless, we will not exclude the case $c < 0$ hoping to obtain a nice formula.

Substituting $x = cx_0$ and $y = cy_0$ into (2.15), one gets

$$E(cx_0, cy_0, z) = \frac{\gamma \sqrt{\pi}}{2} \exp \left( \frac{ic z^3}{9} [2 + \cos 3 t_0] - \frac{i z^3}{27} [3 + \cos 3 t_0] \right)$$

$$\times \int_{\mathbb{R}} e^{-t^2} \left\{ \text{Ai}(X + iY) \text{Bi}(X - iY) + \text{Ai}(X - iY) \text{Bi}(X + iY) \right\} dt, \quad (3.1)$$

where

$$X = \frac{V + V^*}{2} + t \sqrt{2i \gamma z} \frac{e^{-it_0/2} + e^{it_0/2}}{2} = \frac{(1 - c)^2}{9\gamma} (2 \cos t_0 + \cos 2 t_0) + t \sqrt{2i \gamma z} \cos \frac{t_0}{2},$$

$$Y = \frac{V - V^*}{2i} + t \sqrt{2i \gamma z} \frac{e^{-it_0/2} - e^{it_0/2}}{2i} = \frac{(1 - c)^2}{9\gamma} (2 \sin t_0 - \sin 2 t_0) - t \sqrt{2i \gamma z} \sin \frac{t_0}{2}. \quad (3.2)$$

Applying the results of Appendix A, we can expand the expression within the figure brackets into a Taylor series:

$$\text{Ai}(X + iY) \text{Bi}(X - iY) + \text{Ai}(X - iY) \text{Bi}(X + iY) =$$

$$= \frac{8}{\sqrt{\pi}} \sum_{m,n \geq 0} \frac{(-1)^m \text{Im} \omega^{m+1} X^m Y^{2n}}{12(5+2n-2m)/6 \Gamma(\frac{5+2n-2m}{6}) m! n!}. \quad (3.3)$$

Then substituting (3.2) into (3.3), expansion by Newton’s binomial formula and integrat-
ing term by term converts the integral in (3.1) into a multiple series:

\[
E(cx_0, cy_0, z) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \exp\left(\frac{[6c - 3] + [3c - 1] \cos 3t_0}{27}\right) \sum_{m,n \geq 0} \frac{\{\text{Im } \omega^{n-m+1}\}}{\Gamma\left(\frac{3+2n-2m}{6}\right)m!n!} \times \sum_{0 \leq 2\ell \leq m+2n} T(\ell, m, n) \left\{ \frac{1}{2} \right\}^\ell \frac{4^\ell (c - 1)^{m+2n-2\ell}}{3^{m+3n-2\ell}} (iz)^{2(m+2n)-3\ell},
\]  

where

\[
T(\ell, m, n) = \sum_{k \geq 0} \frac{1}{2^k} \binom{m}{2\ell - k} \binom{2n}{k} \\
\times (1 - 2 \cos t_0 - 2 \cos^2 t_0)^{m-k} (1 + \cos t_0)^{n-\ell-k} (1 - \cos t_0)^{3n-k}. \tag{3.5}
\]

The series \(T(\ell, m, n)\) is naturally terminating, so we do not indicate its limits here. (In fact, \(\max(0, 2\ell - m) \leq k \leq \min(2\ell, 2n)\) and \(T(\ell, m, n)\) is a terminating \(2F_1\) hypergeometric series.)

The right side of (3.4), after some algebra, may be written in the manner of (1.2),

\[
E(cx_0, cy_0, z) = \frac{\sqrt{\pi}}{3} \exp\left(\frac{[6c - 3] + [3c - 1] \cos 3t_0}{27}\right) \times \left\{ \frac{1}{\Gamma\left(\frac{5}{6}\right)} \sum_{\nu \geq 0} W_{0, 3\nu} \left(\frac{iz}{3}\right)^{3\nu} - \frac{iz}{3\Gamma\left(\frac{2}{6}\right)} \sum_{\nu \geq 0} W_{1, 3\nu+1} \left(\frac{iz}{3}\right)^{3\nu} \right\}. \tag{3.6}
\]

where

\[
W_{\delta, N} = \sum_{0 \leq 2\nu \leq N} 4^{N-2\nu} \binom{1}{2}^{N-2\nu} (c - 1)^{\nu} \\
\times \sum_{0 \leq 2\nu \leq N} 3^{N-2\nu} \binom{5+2\nu}{6} \frac{1}{\Gamma\left(\frac{5+2\nu}{6}\right)} \frac{1}{\Gamma\left(\frac{5-4\nu}{6}+n+k\right)(2N-3k-2n)!(n+2k)!} T(N - 2\nu, 2N - 3k - 2m, n). \tag{3.7}
\]

In particular, \(W_{0, 0} = W_{1, 1} = 1\), \(W_{0, 3} = (6c - 1) + (3c - 1) \cos 3t_0\), \(W_{1, 4} = (\frac{15}{4} c^2 + 6c - 1) + (3c^2 + 3c - 1) \cos 3t_0\). Unfortunately, we could not rewrite \(W_{\delta, N}\) as polynomials in \(\cos 3t_0\), in general.

Now, we consider two cases of (3.4), when its quadruple series can be shortened drastically. These are \(t_0 = 0\) and \(t_0 = \pi\) that correspond to caustic points placed on the \(x\) axis:

\[
E(cz^2, 0, z) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \exp\left(\frac{[9c - 4]iz^3}{27}\right) \sum_{m,n \geq 0} \left\{ \text{Im } \omega^{n-m+1}\right\} \frac{([c - 1]z^2)^m(2iz)^n}{\Gamma\left(\frac{5-4n}{6}+n+k\right)(2N-3k-2n)!(n+2k)!m!n!}. \tag{3.8}
\]

\[
E\left(-\frac{cz^2}{3}, 0, z\right) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \exp\left(\frac{[3c - 2]iz^3}{27}\right) \sum_{m,n \geq 0} \left\{ \text{Im } \omega^{n-m+1}\right\} \frac{\left(\frac{1}{3}\right)_n \left(\frac{1}{3}(1 - c)z^2\right)^m(\frac{2}{3}iz)^n}{\Gamma\left(\frac{5-4n}{6}+n+k\right)(2N-3k-2n)!(n+2k)!m!n!}. \tag{3.9}
\]
Since $\omega^{n-m+1} = \omega^{1+2m+n}$, it is reasonable to collect terms with $2m + n = k$ together. Then we have

$$E(cz^2, 0, z) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \exp\left(\frac{[9c - 4]i^3z^3}{27}\right) \times \sum_{k=0} \{\text{Im} \omega^{k+1}\} \cdot \frac{(2iz)^k}{\Gamma\left(\frac{5-4k}{6}\right)k!} \cdot _2F_1\left(-\frac{k}{2}, \frac{1-k}{2}; \frac{5-4k}{6} \mid 1 - c\right),$$

$$(3.10)$$

$$E\left(-\frac{c^2}{3}, 0, z\right) = \frac{2\sqrt{\pi}}{3\sqrt{3}} \exp\left(\frac{[3c - 2]i^3z^3}{27}\right) \times \sum_{k=0} \{\text{Im} \omega^{k+1}\} \frac{\left(\frac{1}{3}\right)_k}{\Gamma\left(\frac{5+4k}{6}\right)k!} \cdot _3F_2\left(-\frac{k}{2}, \frac{1-k}{2}, \frac{1-2k}{4}; \frac{3}{4} \mid 1 - c\right),$$

$$(3.11)$$

where the $_2F_1$ functions may be reduced to Gegebauer polynomials.

Of course, the case $c = 1$ is the simplest for both formulae:

$$E(z^2, 0, z) = \frac{\sqrt{\pi}}{3} \exp\left(\frac{5iz^3}{27}\right) \times \left\{\frac{1}{\Gamma\left(\frac{5}{6}\right)} \cdot _2F_2\left(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}, \frac{2}{3}; \frac{32}{27}iz^3\right) - \frac{iz}{3\Gamma\left(\frac{7}{6}\right)} \cdot _2F_2\left(\frac{5}{12}, \frac{7}{12}; \frac{5}{3}, \frac{5}{3}; \frac{32}{27}iz^3\right)\right\},$$

$$(3.12)$$

$$E\left(-\frac{z^2}{3}, 0, z\right) = \frac{\sqrt{\pi}}{3} \exp\left(\frac{iz^3}{27}\right) \times \left\{\frac{1}{\Gamma\left(\frac{5}{6}\right)} \cdot _2F_2\left(\frac{\frac{1}{3}, \frac{2}{3}}{\frac{2}{3}, \frac{2}{3}}, \frac{8}{27}iz^3\right) - \frac{iz}{3\Gamma\left(\frac{7}{6}\right)} \cdot _2F_2\left(\frac{\frac{5}{12}, \frac{5}{12}}{\frac{5}{3}, \frac{5}{3}}, \frac{8}{27}iz^3\right)\right\}. $$

$$(3.13)$$

If $c = 0$, then [3.10] leads to [1.2] due to Gauss formula for a $_2F_1$ of unit argument [3 Eq. (15.4.20)] and the relation between Kummer and Bessel functions, $1F_1(a; 2a\mid 4z) = e^{2z} \cdot _0F_1(a + \frac{1}{2}\mid z^2)$.

The asymptotic expansion of the $_2F_2$ function as $z \gg 1$ is well known [3 Sect. 16.11]. As result,

$$E(z^2, 0, z) \approx \frac{1}{2\sqrt{6\pi}} \exp\left(\frac{5iz^3}{27}\right) \left\{e^{-\pi i/8} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{(2z)^{1/4}} - e^{\pi i/8} \left(\frac{\Gamma\left(\frac{7}{4}\right)}{(2z)^{7/4}}\right)\right) + \frac{\exp(-iz^3)}{4\sqrt{2}z}\right\},$$

$$(3.14)$$

$$E\left(-\frac{z^2}{3}, 0, z\right) \approx \frac{\sqrt{6}}{2\sqrt{\pi}} \exp\left(\frac{iz^3}{27}\right) \left\{e^{-\pi i/4} \left(\frac{\Gamma\left(\frac{1}{4}\right)}{(4z)^{1/4}} - e^{\pi i/4} \left(\frac{\Gamma\left(\frac{7}{4}\right)}{(4z)^{7/4}}\right)\right) + \frac{1}{2z} \exp\left(-\frac{7iz^3}{27}\right)\right\}. $$

$$(3.15)$$

(Each of the terms should be multiplied by $\{1 + O(z^{-3})\}$ to see the remainder term order.)

Following [2], we compare the asymptotic expansions (3.14) and (3.15) with the exact solutions (3.12) and (3.13) in figure 2.

4 Integrals containing the products of Airy functions

Let us return to the integral representation (2.15) and consider the cases, for which the value of $E(x, y, z)$ is already known, Eqs. (1.2), (3.12) and (3.13). Then the integral in
the right side of (2.15) may be expressed in hypergeometric terms. Moreover, since both
sides of (2.15) are entire functions, it holds for the complex values of \( z \) and \( t_0 \) also.

We start with the case \( E(0,0,z) \). Introducing parameters \( 2a = \sqrt{2ie^{-it_0z}} \) and \( 2b = \sqrt{2ie^{it_0z}} \), we get the relation

\[
\int_{\mathbb{R}} e^{-t^2} \left\{ \text{Ai}(2at - a[a^3 + 2b^3]) \text{Bi}(2bt - b[2a^3 + b^3]) \right. \\
\left. + \text{Ai}(2bt - b[2a^3 + b^3]) \text{Bi}(2at - a[a^3 + 2b^3]) \right\} dt \\
= \frac{2}{12^{1/3}} \exp(-\frac{(a^3 + b^3)^2}{3}) \left\{ \frac{1}{\Gamma\left(\frac{3}{6}\right)} \cdot \, _0F_1\left(\frac{5}{6} \mid \frac{4}{9}(ab)^6\right) - \frac{2ab}{12^{1/3} \Gamma\left(\frac{2}{6}\right)} \cdot \, _0F_1\left(\frac{7}{6} \mid \frac{4}{9}(ab)^6\right) \right\},
\]

which is valid for all \( a, b \in \mathbb{C} \).

In particular, if \( b = -a \), then the integrand is the even function and

\[
\int_{\mathbb{R}} e^{-t^2} \text{Ai}(2at + a^4) \text{Bi}(-2at + a^4) dt \\
= \frac{1}{12^{1/3}} \left\{ \frac{1}{\Gamma\left(\frac{3}{6}\right)} \cdot \, _0F_1\left(\frac{5}{6} \mid \frac{4}{9}a^{12}\right) + \frac{2a^2}{12^{1/3} \Gamma\left(\frac{2}{6}\right)} \cdot \, _0F_1\left(\frac{7}{6} \mid \frac{4}{9}a^{12}\right) \right\}.
\]

In the same way as in (A17), we can use Airy atoms and separate the relation (4.2) into three parts:

\[
\int_{\mathbb{R}} e^{-t^2} f(2at + a^4)f(-2at + a^4) dt = \sqrt{\pi} \cdot \, _0F_1\left(\frac{5}{6} \mid \frac{4}{9}a^{12}\right),
\]

\[
\int_{\mathbb{R}} e^{-t^2} g(2at + a^4)g(-2at + a^4) dt = -2\sqrt{\pi} a^2 \cdot \, _0F_1\left(\frac{7}{6} \mid \frac{4}{9}a^{12}\right),
\]

\[
\int_{\mathbb{R}} e^{-t^2} f(2at + a^4)g(-2at + a^4) dt = \int_{\mathbb{R}} e^{-t^2} g(2at + a^4)f(-2at + a^4) dt = 0.
\]

Combining them properly, it is easy to find similar integrals for other products of Airy functions:

\[
\int_{\mathbb{R}} e^{-t^2} \text{Bi}(2at + a^4)\text{Bi}(-2at + a^4) dt = 3 \int_{\mathbb{R}} e^{-t^2} \text{Ai}(2at + a^4)\text{Ai}(-2at + a^4) dt \\
= \frac{\sqrt{3}}{12^{1/3}} \left\{ \frac{1}{\Gamma\left(\frac{3}{6}\right)} \cdot \, _0F_1\left(\frac{5}{6} \mid \frac{4}{9}a^{12}\right) - \frac{2a^2}{12^{1/3} \Gamma\left(\frac{2}{6}\right)} \cdot \, _0F_1\left(\frac{7}{6} \mid \frac{4}{9}a^{12}\right) \right\}.
\]
For other two cases, $E(z^2,0,z)$ and $E(-z^2/3,0,z)$, the relations are as follows:

$$
\int_{\mathbb{R}} e^{-t^2} \left\{ \text{Ai}(t_1) \text{Bi}(t_2) + \text{Ai}(t_2) \text{Bi}(t_1) \right\} dt = \frac{2}{12^{1/3}} \exp \left( -\frac{1}{3} (a-b)^4 (a^2 + 4ab + b^2) \right) 
\times \left\{ \frac{1}{\Gamma \left( \frac{5}{6} \right)} \cdot _2F_2 \left( \frac{1}{12}; \frac{7}{3}; 1; \frac{64}{3} \right) \right\} - \frac{2ab}{12^{1/3} \Gamma \left( \frac{5}{6} \right)} \cdot _2F_2 \left( \frac{5}{12}; \frac{11}{4}; 1; \frac{64}{3} \right),
$$

(4.5)

where $t_1 = 2at - a(a-b)^2(a+2b)$ and $t_2 = 2bt - b(a-b)^2(2a+b)$,

$$
\int_{\mathbb{R}} e^{-t^2} \left\{ \text{Ai}(t_3) \text{Bi}(t_4) + \text{Ai}(t_4) \text{Bi}(t_3) \right\} dt = \frac{2}{12^{1/3}} \exp \left( -\frac{1}{3} \left( (a^3 + b^3)^2 + 3a^2b^2 \right) \right) 
\times \left\{ \frac{1}{\Gamma \left( \frac{5}{6} \right)} \cdot _2F_2 \left( \frac{1}{6}; \frac{3}{2}; 1; \frac{16}{3} \right) \right\} - \frac{2ab}{12^{1/3} \Gamma \left( \frac{5}{6} \right)} \cdot _2F_2 \left( \frac{5}{6}; \frac{7}{4}; 1; \frac{16}{3} \right),
$$

(4.6)

where $t_3 = 2at - a(a+b)(a^2 - ab + 2b^2)$ and $t_4 = 2bt - b(a+b)(2a^2 - ab + b^2)$. The simplest corollaries of both formulae (when $b = a$ and $b = -a$, respectively) may be proven directly, without using the theory above.

5 Concluding remarks

In the paper, we obtain the absolutely convergent integral representation for the elliptic umbilic and derive some of its corollaries. The formulae presented above may be useful for establishing possibly new hypergeometric identities and relations containing Airy polynomials. We discuss them briefly.

Since $\omega^{n-m+1} = \omega^{1-m-2n}$, we can transform (3.8) and (3.9) collecting terms with $m + 2n = k$. It leads to the elliptic umbilic expansions which are valid for all $c$. They contain $_2F_2$ hypergeometric functions but have a more complicated structure than (3.12) and (3.13).

The hypergeometric functions appear in other expansions of $E(x, y, z)$, presented in Section 2. For example, equating the coefficients of $z^n$ in (1.2) and (2.5), where we put $x = y = 0$, gives a closed-form expression for $F_1(\frac{1}{2}; a + \frac{2}{3}; a + \frac{2}{3}; a + \frac{1}{2})$, which is known. (More exactly, we first obtain the value of $F_1$ function for discrete $n$ and then replace $n$ by a continuous parameter, $-n/3 \to a$. The last step can be justified by Carlson’s theorem [9] § 5.8.1; see also [4, Appendix1].)

Applying this approach to (1.2) and (2.6), we find that

$$3F_2 \left( \frac{a}{a + \frac{1}{2}, \frac{4a+1}{6}; 1; 1; 1; 3; 4} \right) = \frac{\Gamma \left( \frac{5-4a}{6} \right) \Gamma \left( \frac{17+2a}{6} \right) \Gamma \left( 1 + \frac{a}{3} \right)}{2^{2-2a} \sqrt{\pi} \Gamma \left( \frac{1}{2} - 2a \right) \Gamma \left( 1 + 2a \right) \cos \frac{2\pi a}{3} \cos \frac{\pi (1+2a)}{3}}.
$$

(5.1)

Next, equating the coefficients of $x^\ell y^m z^n$ in (2.5) and (2.6) leads to the value of $3F_2 \left( \frac{a}{2}; \frac{a}{2}; \frac{a}{2}; \frac{a}{2}; \frac{a}{2}; \frac{a}{2}; \frac{a}{2} \right)$ depending on three parameters, which is too cumbersome. Besides, replacing discrete parameters $\ell, m, n$ by continuous ones, we could not check the conditions of Carlson’s theorem.

Finally, Eq. (2.7) provides a way to connect the polynomials $P_n(x)$, $Q_n(x)$ in (1.11) with Airy polynomials [10]. The last ones are a particular case of the Kampé de Fériet
polynomials [11] and may be defined by the generating function

$$\sum_{n \geq 0} \Pi_n(x) \frac{t^n}{n!} = \exp\left( xt - \frac{t^3}{3} \right). \quad (5.2)$$

Substituting $a = ic\sqrt{\gamma}$ and $b = 0$ into (2.7), after some algebra we have

$$\sum_{n \geq 0} \text{Ai}^{(2n)}(x) \frac{e^n}{n!} = \exp\left( cx + \frac{2c^3}{3} \right) \text{Ai}(x + c^2). \quad (5.3)$$

Then

$$\text{Ai}^{(2n)}(x) = n! \cdot [e^n] \left\{ \sum_{\ell \geq 0} \Pi_{\ell} \left( -\frac{x}{2^{1/3}} \right) \frac{(-2^{1/3}e)^{\ell}}{\ell!} \sum_{k \geq 0} \text{Ai}^{(k)}(x) \frac{c^{2k}}{k!} \right\}, \quad (5.4)$$

where we use the notation proposed in [12]. Namely, if $A(z)$ is any power series $\sum_k a_k z^k$, then $[z^k]A(z)$ denotes the coefficient of $z^k$ in $A(z)$. In our view, this notation is more convenient to manipulate power series than usual analytic description, $[z^k]A(z) = A^{(k)}(0)/k!$.

Separating the components of (5.4), one gets the relations

$$P_{2n}(x) = \sum_{0 \leq 2k \leq n} n! \frac{(-2^{1/3})^{n-2k}}{(n-2k)!k!} \Pi_{n-2k} \left( -\frac{x}{2^{1/3}} \right) P_k(x),$$

$$Q_{2n}(x) = \sum_{0 \leq 2k \leq n} n! \frac{(-2^{1/3})^{n-2k}}{(n-2k)!k!} \Pi_{n-2k} \left( -\frac{x}{2^{1/3}} \right) Q_k(x). \quad (5.5)$$

\section{A The power series for $\text{Re}\{\text{Ai}(x + iy)\text{Bi}(x - iy)\}$}

Let us find the expansion of $\text{Re}\{\text{Ai}(x + iy)\text{Bi}(x - iy)\}$ in power series. We deduce it from the expansion for $\text{Ai}(x + y)\text{Ai}(x - y)$. There are at least two ways to prove that

$$\text{Ai}(x + y)\text{Ai}(x - y) = \frac{2}{\sqrt{\pi}} \sum_{m,n \geq 0} \frac{(-1)^{m+n} x^m y^{2n}}{12^{(5+2n-2m)/6} \Gamma\left( \frac{5+2n-2m}{6} \right) m! n!}. \quad (A1)$$

Here, we use the approach based on the Mellin transform,

$$\mathfrak{M}[f(x)](\alpha) = \int_0^\infty x^{\alpha-1} f(x) \, dx, \quad \alpha > 0, \quad (A2)$$

applying to the shifted Airy functions.

We begin with the well-known formula [13, Eq. (2.21)]

$$\text{Ai}^2(x) + \text{Bi}^2(x) = \frac{1}{\pi^{3/2}} \int_0^\infty \exp\left( xt - \frac{t^3}{12} \right) \frac{dt}{\sqrt{t}}, \quad (A3)$$

replacing $x$ by $\omega x$ and using (2.8):

$$\text{Ai}^2(\omega x) + \text{Bi}^2(\omega x) = 2e^{-\pi i/3} \left\{ \text{Ai}^2(x) + i \cdot \text{Ai}(x)\text{Bi}(x) \right\}, \quad (A4)$$

$$\text{Ai}^2(x) + i \cdot \text{Ai}(x)\text{Bi}(x) = \frac{e^{\pi i/3}}{2\pi^{3/2}} \int_0^\infty \exp\left( \omega xt - \frac{t^3}{12} \right) \frac{dt}{\sqrt{t}}. \quad (A5)$$
Then

\[
\int_0^\infty x^{\alpha-1} \{ \text{Ai}^2 + i \cdot \text{AiBi} \} (x + c) \, dx
\]

\[
= \frac{e^{\pi i/3}}{2\pi^{3/2}} \int_0^\infty \exp \left( \frac{\omega ct - t^3}{12} \right) \frac{dt}{\sqrt{t}} \int_0^\infty \exp \left( -e^{-\pi i/3}tx \right) x^{\alpha-1} \, dx
\]

\[
= \frac{e^{\pi i/3}}{2\pi^{3/2}} \int_0^\infty \exp \left( \frac{\omega ct - t^3}{12} \right) \frac{\Gamma(\alpha)}{(e^{-\pi i/3}t)^\alpha} \frac{dt}{\sqrt{t}}
\]

\[
= \frac{\Gamma(\alpha)}{2\pi^{3/2}} e^{\pi i(\alpha+1)/3} \sum_{n \geq 0} \frac{(\omega c)^n}{n!} \int_0^\infty \exp \left( -\frac{t^3}{12} \right) t^{n-\alpha-1/2} \, dt
\]

\[
= \frac{\Gamma(\alpha)}{6\pi^{3/2}} e^{\pi i(\alpha+1)/3} \sum_{n \geq 0} \frac{\Gamma \left( \frac{2n+1-2\alpha}{6} \right)}{\Gamma \left( \frac{5+2\alpha-2n}{6} \right)} \frac{12^{(2n+1-2\alpha)/6}}{n!} (\omega c)^n
\]

\[
= \frac{2\Gamma(\alpha)}{\sqrt{\pi}} \sum_{n \geq 0} \frac{1 - i \cdot \cot \left( \frac{\pi}{6} \frac{5+2\alpha-2n}{6} \right)}{12^{(5+2\alpha-2n)/6} \Gamma \left( \frac{5+2\alpha-2n}{6} \right)} \frac{(-c)^n}{n!}
\]

(A6)

The last equality follows after using the reflection formula: \( \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \).

To insure the convergence of the integral we restrict our study to the case of small \( \alpha \)'s, namely, \( \alpha \in (0, \frac{1}{2}) \). Of course, this restriction is due to the imaginary part only. As for the real part, the resulting expression

\[
\Re [\text{Ai}^2(x + c)](\alpha) = \frac{2\Gamma(\alpha)}{\sqrt{\pi}} \sum_{n \geq 0} \frac{12^{(2n+2\alpha-5)/6}}{\Gamma \left( \frac{5+2\alpha-2n}{6} \right)} \frac{(-c)^n}{n!}
\]

(A7)

is valid for all \( \alpha > 0 \) and may be established by analytic continuation (see [14][15] for details).

Next, we use the formula [11, Eq. (B18)]

\[
\text{Ai} \left( \frac{a + b}{2^{2/3}} \right) \text{Ai} \left( \frac{a - b}{2^{2/3}} \right) = \frac{1}{2^{2/3} \pi} \int_{\mathbb{R}} e^{ibt} \text{Ai}(a + t^2) \, dt
\]

(A8)
to get the relation between the Mellin transforms of the shifted $\text{Ai}$ and $\text{Ai}^2$ functions:

$$
\int_0^\infty x^{\alpha-1} \text{Ai} \left( \frac{x + c + b}{2^{2/3}} \right) \text{Ai} \left( \frac{x + c - b}{2^{2/3}} \right) \, dx
= \frac{1}{2^{2/3} \pi} \int_0^\infty x^{\alpha-1} \, dx \int_\mathbb{R} e^{ibt} \text{Ai}(x + c + t^2) \, dt
= \frac{1}{2^{2/3} \pi} \int_0^\infty \left( e^{ibt} + e^{-ibt} \right) \, dt \int_0^\infty x^{\alpha-1} \text{Ai}(x + t^2 + c) \, dx
= \frac{4}{2^{2/3} \pi} \int_0^\infty \int_0^\infty \cos(bt) s^{2\alpha-1} \text{Ai}(s^2 + t^2 + c) \, ds \, dt
= \frac{4}{2^{2/3} \pi} \int_0^\infty r^{2\alpha} \text{Ai}(r^2 + c) \, dr \int_0^{\pi/2} \cos(br \sin \phi) \cos^{2\alpha-1} \phi \, d\phi
= \frac{4}{2^{2/3} \pi} \int_0^\infty r^{2\alpha} \text{Ai}(r^2 + c) \sum_{n \geq 0} \left( -b^2 r^2 \right)^n \frac{(2n)!}{(2n)!} \cdot \frac{1}{2} B \left( n + \frac{1}{2}, \alpha \right) \, dr
= \frac{2 \Gamma(\alpha)}{2^{2/3} \pi} \sum_{n \geq 0} \frac{\Gamma(n + \frac{1}{2}) (-b^2)^n}{\Gamma(n + \alpha + \frac{1}{2}) (2n)!} \int_0^\infty r^{2n+2\alpha} \text{Ai}(r^2 + c) \, dr
= \frac{2 \Gamma(\alpha)}{2^{2/3} \pi} \sum_{n \geq 0} \frac{\left( -\frac{1}{2} b^2 \right)^n}{\Gamma(n + \alpha + \frac{1}{2}) n!} \int_0^\infty r^{2n+2\alpha} \text{Ai}(r^2 + c) \, dr
= \frac{2 \Gamma(\alpha)}{2^{2/3} \pi \Gamma(\alpha + \frac{1}{2})} \int_0^\infty {}_0F_1 \left( \alpha + \frac{1}{2}, -\frac{b^2 r^2}{4} \right) r^{2\alpha} \text{Ai}(r^2 + c) \, dr
= \frac{\Gamma(\alpha)}{2^{2/3} \sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \int_0^\infty {}_0F_1 \left( \alpha + \frac{1}{2}, -\frac{b^2 x^2}{4} \right) x^{\alpha-1/2} \text{Ai}(x + c) \, dx. \quad (A9)
$$

Neglecting quite interesting integral relations between Airy and Bessel functions (see also [10]), we substitute $b = 0$ and obtain

$$
2^{2\alpha/3} \mathfrak{M} \left[ \text{Ai}^2 \left( x + \frac{c}{2^{2/3}} \right) \right] (\alpha) = \frac{\Gamma(\alpha)}{2^{2/3} \sqrt{\pi} \Gamma(\alpha + \frac{1}{2})} \mathfrak{M} \left[ \text{Ai}(x + c) \right] \left( \alpha + \frac{1}{2} \right). \quad (A10)
$$

As a corollary,

$$
\mathfrak{M} \left[ \text{Ai}(x + c) \right] (\alpha) = \frac{2^{(2\alpha+1)/3} \sqrt{\pi} \Gamma(\alpha)}{\Gamma(\alpha - \frac{1}{2})} \mathfrak{M} \left[ \text{Ai}^2 \left( x + \frac{c}{2^{2/3}} \right) \right] \left( \alpha - \frac{1}{2} \right)
\equiv \frac{\Gamma(\alpha)}{\Gamma(\alpha - \frac{1}{2})} \sum_{n \geq 0} \frac{3^{(n-\alpha-2)/3}}{n!} (-c)^n. \quad (A11)
$$
Now, one can find the power series for the product of two Airy functions:

\[
\text{Ai}\left(\frac{a+b}{2^{2/3}}\right)\text{Ai}\left(\frac{a-b}{2^{2/3}}\right) = \frac{1}{2^{2/3}\pi} \int_{\mathbb{R}} e^{ibx} \text{Ai}(a + x^2) \, dx
\]

\[
= \frac{1}{2^{2/3}\pi} \sum_{n \geq 0} \frac{(ib)^n}{n!} \int_{\mathbb{R}} x^n \text{Ai}(a + x^2) \, dx = \frac{1}{2^{2/3}\pi} \sum_{n \geq 0} \frac{(-b^2)^n}{(2n)!} \int_0^\infty t^{n-1/2} \text{Ai}(a + t) \, dt
\]

\[
= \frac{1}{\sqrt{\pi}} \sum_{m \geq 0} \sum_{n \geq 0} \frac{3^{(m-5/2-n)/3}(-a)^m(-b^2)^n}{2^{2n+2/3} \Gamma\left(\frac{5+2n-2m}{6}\right) m! n!}
\]

which is analogous to the formula (A11).

Because of the symmetry relations (2.8), the following formulae hold:

\[
4 \text{Re}\{\omega \text{Ai}(\omega[x + y])\text{Ai}(\omega[x - y])\} = \text{Ai}(x + y)\text{Ai}(x - y) - \text{Bi}(x + y)\text{Bi}(x - y),
\]

\[
4 \text{Im}\{\omega \text{Ai}(\omega[x + y])\text{Ai}(\omega[x - y])\} = \text{Ai}(x + y)\text{Bi}(x - y) + \text{Ai}(x - y)\text{Bi}(x + y).
\]

Then it is easy to get the power series for other products of Airy functions:

\[
\text{Ai}(x + y)\text{Ai}(x - y) = \sum_{m,n \geq 0} c_{m,n} \Rightarrow
\]

\[
\text{Bi}(x + y)\text{Bi}(x - y) = \sum_{m,n \geq 0} \{1 - 4 \text{Re}\omega^{n-m+1}\} c_{m,n},
\]

\[
\text{Ai}(x + y)\text{Bi}(x - y) + \text{Ai}(x - y)\text{Bi}(x + y) = 4 \sum_{m,n \geq 0} \text{Im}\{\omega^{n-m+1}\} c_{m,n},
\]

where we rewrite (A11) in short-hand. The last expression helps to find the desired result:

\[
\text{Re}\{\text{Ai}(x + iy)\text{Bi}(x - iy)\} = \frac{4}{\sqrt{\pi}} \sum_{m,n \geq 0} \frac{\text{Im}\{\omega^{n-m+1}\}(-x)^m y^{2n}}{12^{(5+2n-2m)/6} \Gamma\left(\frac{5+2n-2m}{6}\right) m! n!}.
\]

An alternative way is to use Airy functions \(f(x)\) and \(g(x)\) which are connected with Airy functions by the relations [17]:

\[
\text{Ai}(x) = c_1 f(x) - c_2 g(x), \quad \frac{\text{Bi}(x)}{\sqrt{3}} = c_1 f(x) + c_2 g(x),
\]

where \(c_1 = 3^{-2/3}\Gamma\left(\frac{2}{3}\right)\) and \(c_2 = 3^{-1/3}\Gamma\left(\frac{1}{3}\right)\). Since the coefficients of power series for both Airy atoms are rational,

\[
f(x) = \sum_{k \geq 0} \left(\frac{1}{3}\right) \frac{3^k x^{3k}}{(3k)!} = {}_0F_1\left(\frac{2}{3} \left| \frac{x^3}{9}\right.\right),
\]

\[
g(x) = \sum_{k \geq 0} \left(\frac{2}{3}\right) \frac{3^k x^{3k+1}}{(3k+1)!} = x \cdot {}_0F_1\left(\frac{4}{3} \left| \frac{x^3}{9}\right.\right),
\]
we can separate (A11) into three parts:

\[
\text{Ai}(x+y)\text{Ai}(x-y) = \left\{ c_1 f(x+y) - c_2 g(x+y) \right\}\left\{ c_1 f(x-y) - c_2 g(x-y) \right\}
\]

\[
= \frac{f(x+y)f(x-y)}{12^{1/3}\sqrt{3}\pi \Gamma\left(\frac{5}{6}\right)} - \frac{f(x-y)g(x+y) + f(x+y)g(x-y)}{2\pi \sqrt{3}} + \frac{g(x+y)g(x-y)}{12^{1/6}\sqrt{\pi} \Gamma\left(\frac{4}{6}\right)},
\]

\[
\frac{f(x+y)f(x-y)}{12^{1/3}\sqrt{3}\pi \Gamma\left(\frac{5}{6}\right)} = \sum_{m,n \geq 0} c_{m,n},
\]

\[
- \frac{f(x-y)g(x+y) + f(x+y)g(x-y)}{2\pi \sqrt{3}} = \sum_{m,n \geq 0} c_{m,n}, \quad (A17)
\]

\[
\frac{g(x+y)g(x-y)}{12^{1/6}\sqrt{\pi} \Gamma\left(\frac{4}{6}\right)} = \sum_{m,n \geq 0} c_{m,n}.
\]

Applying these expansions to \(\text{Re}\{\text{Ai}(x+iy)\text{Bi}(x-iy)\}\), we get a double series

\[
\text{Re}\{\text{Ai}(x+iy)\text{Bi}(x-iy)\} = \sqrt{3}c_1^2 \left| f(x+iy) \right|^2 - \sqrt{3}c_2^2 \left| g(x+iy) \right|^2
\]

\[
= \frac{2\sqrt{3}}{\sqrt{\pi}} \left\{ \sum_{m,n \geq 0} \frac{(-x)^m y^{2n}}{12^{(5+2n-2m)/6} \Gamma\left(\frac{5+2n-2m}{6}\right) m! n!} - \sum_{m,n \geq 0} \frac{(-x)^m y^{2n}}{12^{(5+2n-2m)/6} \Gamma\left(\frac{5+2n-2m}{6}\right) m! n!} \right\}, \quad (A18)
\]

which evidently coincides with (A14).

As a final remark, we provide the counterpart of (A14) for completeness,

\[
\text{Im}\{\text{Ai}(x+iy)\text{Bi}(x-iy)\} = -\frac{1}{\pi} \sum_{m,n \geq 0} \frac{(-x)^m y^{6n+2m+1}}{12^n \left(\frac{5}{2}\right)_3 n! m! n!}
\]

\[
= -\frac{1}{\pi} \sum_{k \geq 0} \frac{(-4)^k y^{2k+1}}{(2k+1)!} \cdot \Pi_k(2^{3/2}x)^{(2k+1)/(2k+3)}, \quad (A19)
\]

and mention that another way to prove (A11) is based on a slightly modified version of Moyer’s formula [18] (see also [19]),

\[
\text{Ai}(x+y)\{\text{Ai}(x-y) + i \cdot \text{Bi}(x-y)\} = \frac{1}{2\pi^{3/2}} \int_0^\infty \exp\left(i\left[\frac{t^3}{12} + xt - \frac{y^2}{t} + \frac{\pi}{4}\right]\right) \frac{dt}{\sqrt{t}}. \quad (A20)
\]

namely,

\[
|\text{Ai}(x+iy) + i \cdot \text{Bi}(x+iy)|^2 = \frac{1}{\pi^{3/2}} \int_0^\infty \exp\left(-\frac{t^3}{12} + xt - \frac{y^2}{t}\right) \frac{dt}{\sqrt{t}}, \quad (A21)
\]

where \(x \in \mathbb{R}\) and \(y \geq 0\).

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