Maximizing the Number of Spanning Trees in a Connected Graph

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Abstract

We study the problem of maximizing the number of spanning trees in a connected graph by adding at most \(k\) edges from a given candidate edge set, a problem that has applications in domains including robotics, network science, and cooperative control. By Kirchhoff’s matrix-tree theorem, this problem is equivalent to maximizing the determinant of an SDDM matrix. We give both algorithmic and hardness results for this problem:

- We give a greedy algorithm that, using submodularity, obtains an approximation ratio of \((1 - 1/e - \epsilon)\) in the exponent of the number of spanning trees for any \(\epsilon > 0\) in time \(O(ne^{-1} + (n + q)e^{-3})\), where \(n\) is the number of vertices, and \(m\) and \(q\) are the number of edges in the original graph and the candidate edge set, respectively. Our running time is optimal with respect to the input size, up to logarithmic factors, and substantially improves upon the \(O(n^3)\) running time of the previous proposed greedy algorithm, which has an approximation ratio \((1 - 1/e)\) in the exponent. Notably, the independence of our running time of \(k\) is novel, compared to conventional top-\(k\) selections on graphs that usually run in \(\Omega(mk)\) time. A key ingredient of our greedy algorithm is a routine for maintaining effective resistances under edge additions that is a hybrid of online and offline processing techniques; this routine may be of independent interest in areas including dynamic algorithms and data streams.

- We show the exponential inapproximability of this problem by proving that there exists a constant \(c > 0\) such that it is \(\text{NP}\)-hard to approximate the optimum number of spanning trees in the exponent within \((1 - c)\). By our reduction, the inapproximability of this problem can also be stated as there exists a constant \(d > 0\) such that it is \(\text{NP}\)-hard to approximate the optimum number of spanning trees within \((1 + d)^{-n}\). Our inapproximability result follows from a reduction from the minimum path cover in undirected graphs, whose hardness again follows from the constant inapproximability of the Traveling Salesman Problem (TSP) with distances 1 and 2. Thus, the approximation ratio of our algorithm is also optimal up to a constant factor in the exponent. To our knowledge, this is the first hardness of approximation result for maximizing the number of spanning trees in a graph, or equivalently, maximizing the determinant of an SDDM matrix.

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\*The notation \(\tilde{O}\) hides \(\log \log(n)\) factors.
1 Introduction

We study the problem of maximizing the number of spanning trees in a weighted connected graph $G$ by adding at most $k$ edges from a given candidate edge set. By Kirchhoff’s matrix-tree theorem [Kir47], the number of spanning trees in $G$ is equivalent to the determinant of a minor of the graph Laplacian $L$. Thus, an equivalent problem is to maximize the determinant of a minor of $L$, or, more generally, to maximize the determinant of an SDDM matrix. The problem of maximizing the number of spanning trees, and the related problem of maximizing the determinant of an SDDM matrix, have applications in a wide variety of problem domains. We briefly review some of these applications below.

In robotics, the problem of maximizing the number of spanning trees has been applied in graph-based Simultaneous Localization and Mapping (SLAM). In graph-based SLAM [TM06], each vertex corresponds to a robot’s pose or position, and edges correspond to relative measurements between poses. The graph is used to estimate the most likely pose configurations. Since measurements can be noisy, a larger number of measurements results in a more accurate estimate. The problem of selecting which $k$ measurements to add to a SLAM pose graph to most improve the estimate has been recast as a problem of selecting the $k$ edges to add to the graph that maximize the number of spanning trees [KHD15, KHD16, KSHD16a, KSHD16b]. We note that the complexity of the estimation problem increases with the number of measurements, and so sparse, well-connected pose graphs are desirable [DK06]. Thus, one expects $k$ to be moderately sized with respect to the number of vertices.

In network science, the number of spanning trees has been studied as a measure of reliability in communication networks, where reliability is defined as the probability that every pair of vertices can communicate [Myr96]. Thus, network reliability can be improved by adding edges that most increase the number of spanning trees [FL01]. The number of spanning trees has also been used as a predictor of the spread of information in social networks [BAE11], with a larger number of spanning trees corresponding to better information propagation.

In the field of cooperative control, the log-number of spanning trees has been shown to capture the robustness of linear consensus algorithms. Specifically, the log-number of spanning trees quantifies the network entropy, a measure of how well the agents in the network maintain agreement when subject to external stochastic disturbances [SM14, dBCM15, ZEP11]. Thus, the problem of selecting which edges to add to the network graph to optimize robustness is equivalent to the log-number of spanning trees maximization problem [SM18, ZSA13]. Finally, the log-determinant of an SDDM matrix has also been used directly as a measure of controllability in more general linear dynamical systems [SCL16]. In this paper, we provide an approximation algorithm to maximize the log-number of spanning trees of a connected graph by adding edges.

1.1 Our Results

Let $G = (V, E)$ denote an undirected graph with $n$ vertices and $m$ edges, and let $w : E \to \mathbb{R}^+$ denote the edge weight function. For another graph $H$ with edges supported on a subset of $V$, we write “$G$ plus $H$” or $G + H$ to denote the graph obtained by adding all edges in $H$ to $G$.

Let $L$ denote the Laplacian matrix of a graph $G$. The effective resistance $R_{\text{eff}}(u, v)$ between two vertices $u$ and $v$ is given by

$$R_{\text{eff}}(u, v) \overset{\text{def}}{=} (e_u - e_v)^T L^\dagger (e_u - e_v),$$

where $L^\dagger$ is the Moore-Penrose pseudoinverse of $L$.
where \( e_u \) denotes the \( u \)th standard basis vector and \( L^\dagger \) denotes the Moore-Penrose inverse of \( L \).

The weight of a spanning tree \( T \) in \( G \) is defined as
\[
    w(T) \equiv \prod_{e \in T} w(e),
\]
and the weighted number of spanning trees in \( G \) is defined as the sum of the weights of all spanning trees, denoted by \( \mathcal{T}(G) \equiv \sum_T w(T) \). By Kirchhoff’s matrix-tree theorem [Kir47], the weighted number of spanning trees equals the determinant of a minor of the graph Laplacian:
\[
    \mathcal{T}(G) = \det (L_{1:n-1,1:n-1}).
\]

In this paper, we study the problem of maximizing the weighted number of spanning trees in a connected graph by adding at most \( k \) edges from a given candidate edge set. We give a formal description of this problem below.

**Problem 1** (Number of Spanning Trees Maximization (NSTM)). Given a connected undirected graph \( G = (V,E) \), an edge set \( Q \) of \( q \) edges, an edge weight function \( w : (E \cup Q) \to \mathbb{R}^+ \), and an integer \( 1 \leq k \leq q \), add at most \( k \) edges from \( Q \) to \( G \) so that the weighted number of spanning trees in \( G \) is maximized. Namely, the goal is to find a set \( P \subseteq Q \) of at most \( k \) edges such that
\[
    P \in \arg\max_{S \subseteq Q, |S| \leq k} \mathcal{T}(G + S).
\]

**Algorithmic Results.** Our main algorithmic result is solving Problem 1 with an approximation factor of \((1 - \frac{1}{e} - \epsilon)\) in the exponent of \( \mathcal{T}(G) \) in nearly-linear time, which can be described by the following theorem:

**Theorem 1.1.** There is an algorithm \( P = \text{NSTMaximize}(G,Q,w,\epsilon,k) \), which takes a connected graph \( G = (V,E) \) with \( n \) vertices and \( m \) edges, an edge set \( Q \) of \( q \) edges, an edge weight function \( w : (E \cup Q) \to \mathbb{R}^+ \), a real number \( 0 < \epsilon \leq 1/2 \), and an integer \( 1 \leq k \leq q \), and returns an edge set \( P \subseteq Q \) of at most \( k \) edges in time \( \tilde{O}((m + (n + q)e^{-2})e^{-1}) \). With high probability, the following statement holds:
\[
    \log \frac{\mathcal{T}(G + P)}{\mathcal{T}(G)} \geq \left(1 - \frac{1}{e} - \epsilon \right) \log \frac{\mathcal{T}(G + O)}{\mathcal{T}(G)},
\]
where \( O \equiv \arg\max_{S \subseteq Q, |S| \leq k} \mathcal{T}(G + S) \) denotes an optimum solution.

The running time of \( \text{NSTMaximize} \) is independent of the number \( k \) of edges to add to \( G \), and it is optimal with respect to the input size up to logarithmic factors. This running time substantially improves upon the previous greedy algorithm’s \( O(n^3) \) running time [KSHD16b], or the \( \tilde{O}((m + (n + q)e^{-2})k) \) running time of its direct acceleration via fast effective resistance approximation [SS11, DKP+17], where the latter becomes quadratic when \( k \) is \( \Omega(n) \). Moreover, the independence of our running time of \( k \) is novel, comparing to conventional top-\( k \) selections on graphs that usually run in \( \Omega(mk) \) time, such as the ones in [BBCL14, Yos14, MTU16, LPS+18]. We briefly introduce these top-\( k \) selections in Section 1.3.
A key ingredient of the algorithm NSTMaximize is a routine AddAbove that, given a sequence of edges and a threshold, sequentially adds to the graph any edge whose effective resistance (up to a $1 \pm \epsilon$ error) is above the threshold at the time the edge is processed. The routine AddAbove runs in nearly-linear time in the total number of edges in the graph and the edge sequence. The performance of AddAbove is characterized in the following lemma:

**Lemma 1.2.** There is a routine $P = \text{AddAbove}(G, (u_i, v_i)_{i=1}^{q}, w, th, \epsilon, k)$, which takes a connected graph $G = (V, E)$ with $n$ vertices and $m$ edges, an edge sequence $(u_i, v_i)_{i=1}^{q}$, an edge weight function $w : (E \cup (u_i, v_i)_{i=1}^{q}) \rightarrow \mathbb{R}^{+}$, real numbers $th$ and $0 < \epsilon \leq 1/2$, and an integer $k$, and performs a sequential edges additions to $G$ and returns the set $P$ of edges that have been added with $|P| \leq k$. The routine AddAbove runs in time $\tilde{O}(m + (n + q)\epsilon^{-2})$. With high probability, there exist $(\hat{r}_i)_{i=1}^{q}$ such that AddAbove has the same return value as the following procedure, in which

$$(1 - 2\epsilon)R_{\text{eff}}^{G(i)}(u_i, v_i) \leq \hat{r}_i \leq (1 + 2\epsilon)R_{\text{eff}}^{G(i)}(u_i, v_i)$$

holds for all $i = 1, 2, \ldots, q$:

```
G^{(1)} \leftarrow G
for i = 1 to q do
    if $w(u_i, v_i) \cdot \hat{r}_i \geq th$ and $k > 0$ then
        $G^{(i+1)} \leftarrow G^{(i)} + (u_i, v_i)$, $k \leftarrow k - 1$
    else
        $G^{(i+1)} \leftarrow G^{(i)}$
Return the set of edges in $G^{(q+1)}$ but not in $G$.
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The routine AddAbove can be seen as a hybrid of online and offline processing techniques. The routine is provided a specific edge sequence as input, as is typical in offline graph algorithms. However, the routine does not know what operation should be performed on an edge (i.e., whether the edge should be added to the graph) until the edge is processed, in an online fashion. The routine thus has to alternately update the graph and query effective resistance. This routine may be of independent interest in areas including dynamic algorithms and data streams.

**Hardness Results.** To further show that the approximation ratio of the algorithm NSTMaximize is also nearly optimal, we prove the following theorem, which indicates that Problem 1 is exponentially inapproximable:

**Theorem 1.3.** There is a constant $c > 0$ such that given an instance of Problem 1, it is NP-hard to find an edge set $P \subseteq Q$ with $|P| \leq k$ satisfying

$$\log \frac{T(G + P)}{T(G)} > (1 - c) \cdot \log \frac{T(G + O)}{T(G)},$$

where $O$ is an optimum solution defined in Theorem 1.1.

The proof of Theorem 1.3 follows by Lemma 1.5. By the same lemma, we can also state the inapproximability of Problem 1 using the following theorem:
Theorem 1.4. There is a constant $d > 0$ such that given an instance of Problem 1, it is NP-hard to find an edge set $P \subseteq Q$ with $|P| \leq k$ satisfying
\[
\mathcal{T}(G + P) > \frac{1}{(1 + d)^n} \cdot \mathcal{T}(G + O),
\]
where $O$ is an optimum solution defined in Theorem 1.1.

Theorem 1.3 implies that the approximation ratio of NSTMaximize is optimal up to a constant factor in the exponent. To our knowledge, this is the first hardness of approximation result for maximizing the number of spanning trees in a graph, or equivalently, maximizing the determinant of an SDDM matrix (a graph Laplacian minor).

In proving Theorem 1.3, we give a reduction from the minimum path cover in undirected graphs, whose hardness follows from the constant inapproximability of the traveling salesman problem (TSP) with distances 1 and 2. The idea behind our reduction is to consider a special family of graphs, each graph from which equals a star graph plus an arbitrary graph supported on its leaves. Let $H = (V, E)$ be a graph equal to a star $S_n$ plus its subgraph $H[V'] = (V', E')$ supported on $S_n$’s leaves. We can construct an instance of Problem 1 from $H$ by letting the original graph, the candidate edge set, and the number of edges to add be, respectively
\[
G \leftarrow S_n, \quad Q \leftarrow E', \quad k \leftarrow |V'| - 1.
\]
We give an example of such an instance in Figure 1.

We then show in the following lemma that for two such instances whose $H[V']$s have respective path cover number 1 and $\Omega(|V'|)$, the optimum numbers of spanning trees differ by a constant factor in the exponent.

Lemma 1.5. Let $H = (V, E)$ be an unweighted graph equal to a star $S_n$ plus $H$’s subgraph $H[V'] = (V', E')$ supported on $S_n$’s leaves. For any constant $0 < \delta < 1$, there exists an absolute constant $c > 0$ such that, if $H[V'] = (V', E')$ does not have any path cover $P$ with $|P| < \delta n$, then
\[
\log \mathcal{T}(S_n + P) \leq (1 - c) \cdot \log \mathcal{T}(F_n)
\]
holds for any $P \subseteq E'$ with $|P| \leq n - 1$. Here $F_n$ is a fan graph with $n - 1$ triangles (i.e., a star $S_n$ plus a path supported on its leaves).

We remark that our reduction uses only simple graphs with all edge weights being 1. Thus, Problem 1 is exponentially inapproximable even for unweighted graphs without self-loops and multi-edges.

1.2 Ideas and Techniques

Algorithms. By the matrix determinant lemma [Har97], the weighted number of spanning trees multiplies by
\[
1 + w(u, v) R_{\text{eff}}(u, v)
\]
upon the addition of edge $(u, v)$. Then, the submodularity of $\log \mathcal{T}(G)$ follows immediately by Rayleigh’s monotonicity law [Sto87]. This indicates that one can use a simple greedy algorithm [NWF78] that picks the edge with highest effective resistance iteratively for $k$ times to
Figure 1: An instance of Problem 1 constructed from $H$, which equals a star graph plus $H[V']$ supported on its leaves. Here, $r$ is the central vertex of the star. All red edges and green edges belong to the candidate edge set, where red edges denotes a possible selection with size $|V'|-1$.

achieve a $(1 - \frac{1}{e})$-approximation. By computing effective resistances in nearly-linear time [SS11, DKP+17], one can implement this greedy algorithm in $\tilde{O}((m + (n + q)e^{-2})k)$ time and obtain a $(1 - \frac{1}{e} - \varepsilon)$-approximation. To avoid searching for the edge with maximum effective resistance, one can invoke another greedy algorithm proposed in [BV14], which maintains a geometrically decreasing threshold and sequentially picks any edge with effective resistance above the threshold. However, since the latter part of this greedy algorithm requires the recomputation of effective resistances after each edge addition, it still needs $\tilde{O}((m + q)k)$ running time. Thus, our task reduces to performing the sequential updates faster.

We note that for a specific threshold, the ordering in which we perform the sequential updates does not affect our overall approximation. Thus, by picking an arbitrary ordering of the edges in candidate set $Q$, we can transform this seemingly online task of processing edges sequentially into an online-offline hybrid setting. While we do not know whether to add an edge until the time we process it, we do know the order in which the edges will be processed. We perform divide-and-conquer on the edge sequence, while alternately querying effective resistance and updating the graph. The idea is that if we are dealing with a short interval of the edge sequence, instead of working with the entire graph, we can work with a graph with size proportional to the length of the interval that preserves the effective resistances of the edges in the sequence. As we are querying effective resistances for candidate edges, the equivalent graph for an interval can be obtained by taking the Schur complement onto endpoints of the edges in it. And, this can be done in nearly-linear time in the graph size using the approximate Schur complement routine in [DKP+17].
Specifically, in the first step of divide-and-conquer, we split the edge sequence \((f_i)_{i=1}^q\) into two halves
\[
f^{(1)} \overset{\text{def}}{=} f_1, \ldots, f_{\lfloor q/2 \rfloor} \quad \text{and} \quad f^{(2)} \overset{\text{def}}{=} f_{\lfloor q/2 \rfloor + 1}, \ldots, f_q.
\]
We note the following:
1. Edge additions in \(f^{(2)}\) do not affect effective resistance queries in \(f^{(1)}\).
2. An effective resistance query in \(f^{(2)}\) is affected by
   (a) edge additions in \(f^{(1)}\), and
   (b) edge additions in \(f^{(2)}\) which are performed before the specific query.

Since \(f^{(1)}\) is completely independent of \(f^{(2)}\), we can handle queries and updates in \(f^{(1)}\) by performing recursion to its Schur complement. We then note that edge additions in \(f^{(1)}\) are performed entirely before queries in \(f^{(2)}\), and thus can be seen as offline modifications to \(f^{(2)}\). Moreover, all queries in \(f^{(2)}\) are affected by the same set of modifications in \(f^{(1)}\). We thus address the total contribution of \(f^{(1)}\) to \(f^{(2)}\) by computing Schur complement onto \(f^{(2)}\) in the graph updated by edge additions in \(f^{(1)}\). In doing so, we have addressed \((2a)\) for all queries in \(f^{(2)}\), and thus have made \(f^{(2)}\) independent of \(f^{(1)}\). This indicates that we can process \(f^{(2)}\) by also performing recursion to its Schur complement. We keep recursing until the interval only contains one edge, where we directly query the edge’s effective resistance and decide whether to add it to the graph. Essentially, our algorithm computes the effective resistance of each edge with an elimination of the entire rest of the graph, while heavily re-using previous eliminations. This gives a nearly-linear time routine for performing sequential updates. Details for this routine can be found in Section 3.1.

**Hardness.** A key step in our reduction is to show the connection between the minimum path cover and Problem 1. To this end, we consider an instance of Problem 1 in which \(G\) is a star graph \(S_n\) with \(n\) leaves, the candidate edge set \(Q\) forms an underlying graph supported on \(S_n\)’s leaves, and the number of edges to add equals \(k = n - 1\). We show that for two instances whose underlying graphs have respective path cover number 1 and \(\Omega(n)\), their optimum numbers of spanning trees differ exponentially.

Consider any set \(P\) that consists of \(n-1\) edges from \(Q\), and any path cover \(\mathcal{P} = \{p_1, p_2, \ldots, p_t\}\) of the underlying graph using only edges in \(P\). Clearly \(t\) is greater than or equal to the minimum path cover number of the underlying graph. If \(P\) forms a Hamiltonian path \(\mathcal{P}^*\) in the underlying graph, \(T(S_n + P)\) can be explicitly calculated \([\text{MEM14}]\) and equals
\[
T(S_n + P) = T(S_n + \mathcal{P}^*) = \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^n - \left( \frac{3 - \sqrt{5}}{2} \right)^n \right).
\]

When the path cover number of the underlying graph is more than 1, \(T(S_n + P)\) can be expressed by a product of a sequence of effective resistances and \(T(G + \mathcal{P})\). Specifically, for an arbitrary ordering \((u_i, v_i)_{i=1}^{t-1}\) of edges in \(P\) but not in the path cover \(\mathcal{P}\), we define a graph sequence \(G^{(1)}, \ldots, G^{(t)}\) by
\[
G^{(1)} \overset{\text{def}}{=} G + \mathcal{P}, \\
G^{(i+1)} \overset{\text{def}}{=} G^{(i)} + (u_i, v_i) \quad \text{for} \quad i = 1, \ldots, t - 1.
\]
By the matrix determinant lemma, we can write the number of spanning trees in $G^{(t)}$ as

$$T(G^{(t)}) = T(G^{(1)}) \cdot \prod_{i=1}^{t-1} \left( 1 + R_{\text{eff}}^{G^{(i)}}(u_i, v_i) \right).$$  \hspace{1cm} (2)

Note that we omit edge weights here since we are dealing with unweighted graphs.

Let $l_i \overset{\text{def}}{=} |p_i|$ be the number of edges in path $p_i$. Since all paths $p_i \in \mathcal{P}$ are disjoint, $T(G^{(1)})$ can be expressed as

$$T(G^{(1)}) = \prod_{i=1}^{t} T(S_{l_i+1} + p_i) = \prod_{i=1}^{t} \left( \frac{1}{\sqrt{5}} \left( \frac{3 + \sqrt{5}}{2} \right)^{l_i+1} - \left( \frac{3 - \sqrt{5}}{2} \right)^{l_i+1} \right),$$  \hspace{1cm} (3)

where $S_{l_i+1}$ denotes the star graph with $l_i + 1$ leaves, and the second equality follows from (1).

When the path cover number of the underlying graph is at least $\Omega(n)$, we show that the number of spanning trees in $S_n + P$ is exponentially smaller than $T(S_n + P^*)$. Let $\mathcal{P}_1$ denote the set of paths in $\mathcal{P}$ with $O(1)$ lengths. Let $t_1$ denote the number of paths in $\mathcal{P}_1$. Then, $T(S_n + P)$ is exponentially smaller due to the following reasons. First, by (1) and (3), $T(G^{(1)})$ is less than $T(S_n + P^*)$ by at least a multiplicative factor of $(\sqrt{5})^{t-1} (1 + \theta)^{t_1}$ for some constant $\theta > 0$. Second, the effective resistances between endpoints of the $t - 1$ edges in $P$ but not in the path cover $\mathcal{P}$ are less than $\sqrt{5} - 1$ and hence cannot compensate for the $(\sqrt{5})^{t-1}$ factor. Third, $t_1$ is at least $\Omega(n)$ by Markov’s inequality, which ensures that the factor $(1 + \theta)^{t_1}$ is exponential. This leads to the exponential drop of $T(S_n + P)$. We defer our proof details to Section 4.

1.3 Related Work

Maximizing the Number of Spanning Trees. There has been limited previous algorithmic study of the problem of maximizing the number of spanning trees in a graph.

Problem 1 was also studied in [KSHD16b]. This work proposed a greedy algorithm which, by computing effective resistances exactly, achieves an approximation factor of $(1 - \frac{1}{e})$ in the exponent of the number of spanning trees in $O(n^3)$ time. As far as we are aware, the hardness of Problem 1 has not been studied in any previous work.

A related problem of, given $n$ and $m$, identifying graphs with $n$ vertices and $m$ edges that have the maximum number of spanning trees has been studied. However, most solutions found to this problem are for either sparse graphs with $m = O(n)$ edges [BLS91, Wan94], or dense graphs with $m = n^2 - O(n)$ edges [Shi74, Kel96, GM97, PBS98, PR02]. Of particular note, a regular complete multipartite graph has been shown to have the maximum number of spanning trees from among all simple graphs with the same number of vertices and edges [Che81].

Maximizing Determinants. Maximizing the determinant of a positive semidefinite matrix (PSD) is a problem that has been extensively studied in the theory of computation. For selecting a principle minor of a PSD with the maximum determinant under certain constraints, [Kha95, CM09, SEFM15, NS16, ESV17] gave algorithms for approximating the optimum solution. [SEFM15, Nik15] also studied another related problem of finding a $k$-dimensional simplex
of maximum volume inside a given convex hull, which can be reduced to the former problem under cardinality constraint. For finding the principal $k \times k$ submatrix of a positive semidefinite matrix with the largest determinant, [Nik15] gave an algorithm that obtains an approximation of $e^{- (k + o(k))}$. On the hardness side, all these problems have been showed to be exponentially inapproximable [Kou06, CM13, SEFM15].

The problem studied in [Nik15] can also be stated as the following:

Given $m$ vectors $x_1, \ldots, x_m \in \mathbb{R}^n$ and an integer $k$, find a subset $S \subseteq [m]$ of cardinality $k$ so that the product of the largest $l$ eigenvalues of the matrix $\Sigma_S \overset{\text{def}}{=} \sum_{i \in S} x_i x_i^T$ is maximized where $l \overset{\text{def}}{=} \min\{k, n\}$. ($\star$)

[Nik15] gave a polynomial-time algorithm that obtains an $e^{-(k + o(k))}$-approximation when $k \leq n$. When $k \geq n$, Problem ($\star$) is equivalent to maximizing the determinant of $\Sigma_S$ by selecting $k$ vectors. [SX18] showed that one can obtain an $e^{-n}$-approximation for $k \geq n$. Moreover, they showed that given $k = \Omega(n/\epsilon + \log(1/\epsilon)/\epsilon^2)$, one can obtain a $(1 + \epsilon)^{-n}$-approximation. Using the algorithms in [Nik15, SX18], we can obtain an $e^{-n}$-approximation to a problem of independent interest but different from Problem 1: Select at most $k$ edges from a candidate edge set to add to an empty graph so that the number of spanning trees is maximized. In contrast, in Problem 1, we are seeking to add $k$ edges to a graph that is already connected. Thus, their algorithms cannot directly apply to Problem 1.

In [ALSW17a, ALSW17b], the authors also studied Problem ($\star$). They gave an algorithm that, when $k = \Omega(n/\epsilon^2)$, gives a $(1 + \epsilon)^{-n}$-approximation. Their algorithm first computes a fractional solution using convex optimization and then rounds the fractional solution to integers using spectral sparsification. Since spectral approximation is preserved under edge additions, their algorithm can apply to Problem 1 obtaining a $(1 + \epsilon)^{-n}$-approximation. However, their algorithm needs $k$ to be $\Omega(n/\epsilon^2)$, given that the candidate edge set is supported on $O(n)$ vertices (which is natural in real-world datasets [KGS+11, KHD16]). In contrast, $k$ could be arbitrarily smaller than $n$ in our setting.

We remark that both our setting of adding edges to a connected graph and the scenario that $k$ could be arbitrarily smaller than $n$ have been used in previous works solving graph optimization problems such as maximizing the algebraic connectivity of a graph [KMST10, NXC+10, GB06]. We also remark that the algorithms in [Nik15, SX18, ALSW17a, ALSW17b] all need to solve a convex optimization for their continuous relaxation, which runs in polynomial time in contrast to our nearly-linear running time, while the efficiency is crucial in applications [KHD16, KSHD16a, SCL16, SM18].

We also note that [DPPR17] gave an algorithm that computes the determinant of an SDDM matrix to a $(1 + \epsilon)$-error in $\tilde{O}(n^2 \epsilon^{-2})$ time. In our algorithm, we are able to maximize the determinant in nearly-linear time without computing it.

**Fast Computation of Effective Resistances.** Fast computation of effective resistances has various applications in sparsification [SS11, ADK+16, LS17], sampling random spanning trees [DKP+17, DPPR17, Sch18], and solving linear systems [KLP16]. [SS11, KLP16] gave approximation routines that, using Fast Laplacian Solvers [ST14, CKM+14], compute effective resistances for all edges to $(1 + \epsilon)$-errors in $\tilde{O}(mc^{-2})$ time. [CGP+18] presents an algorithm that computes the effective resistances of all edges to $(1 + \epsilon)$-errors in $O(m^{1+o(1)} \epsilon^{-1.5})$ time.
For computing effective resistances for a given set of vertex pairs, [DKP+17] gave a routine that, using divide-and-conquer based on Schur complements approximation [KS16], computes the effective resistances between $q$ pairs of vertices to $(1 \pm \epsilon)$-errors in $\tilde{O}(m + (n + q)\epsilon^{-2})$ time. [DKP+17] also used a divide-and-conquer to sample random spanning trees in dense graphs faster. For maintaining $(1 + \epsilon)$-approximations to all-pair effective resistances of a fully-dynamic graph, [DGGP18] gave a data-structure with $\tilde{O}(m^{4/5} \epsilon^{-4})$ expected amortized update and query time. In [DPPR17], the authors combined the divide-and-conquer idea and their determinant-preserving sparsification to further accelerate random spanning tree sampling in dense graphs. A subset of the authors of this paper (Li and Zhang) recently [LZ18] used a divide-and-conquer approach to compute, for every edge $e$, the sum of effective resistances between all vertex pairs in the graph in which $e$ is deleted. Our routine for performing fast sequential updates in Section 3.1 is motivated by these divide-and-conquer methods and is able to cope with an online-offline hybrid setting.

Top-$k$ Selections on Graphs. Conventional top-$k$ selections on graphs that rely on submodularity usually run in $\Omega(mk)$ time, where $m$ is the number of edges. Here, we give a few examples of them.

In [BBCCL14], the authors studied the problem of maximizing the spread of influence through a social network. Specifically, they studied the problem of finding a set of $k$ initial seed vertices in a network so that, under the independent cascade model [KKT03] of network diffusion, the expected number of vertices reachable from the seeds is maximized. Using hypergraph sampling, the authors gave a greedy algorithm that achieves a $(1 - \frac{1}{\epsilon} - \epsilon)$-approximation in $O((m + n)k\epsilon^{-2}\log n)$ time.

[Yos14, MTU16] studied the problem of finding a vertex set $S$ with maximum betweenness centrality subject to the constraint $|S| \leq k$. Both algorithms in [Yos14, MTU16] are based on sampling shortest paths. To obtain a $(1 - \frac{1}{\epsilon} - \epsilon)$-approximation, their algorithms need at least $\Omega(mn\epsilon^{-2})$ running time according to Theorem 2 of [MTU16]. Given the assumption that the maximum betweenness centrality among all sets of $k$ vertices is $\Theta(n^2)$, the algorithm in [MTU16] is able to obtain a solution with the same approximation ratio in $O((m + n)k\epsilon^{-2}\log n)$ time.

A subset of the authors of this paper (Li, Yi, and Zhang) and Peng and Shan recently [LPS+18] studied the problem of finding a set $S$ of $k$ vertices so that the quantity

$$\sum_{u \in (V \setminus S)} R_{\text{eff}}(u, S)$$

is minimized. Here $R_{\text{eff}}(u, S)$ equals, in the graph in which $S$ is identified as a new vertex, the effective resistance between $u$ and the new vertex. By computing marginal gains for all vertices in a way similar to the effective resistance estimation routine in [SS11], the authors achieved a $(1 - \frac{k}{k-1} - \frac{1}{\epsilon} - \epsilon)$-approximation in $\tilde{O}(mk\epsilon^{-2})$ time.

We remark that there are algorithms for maximizing submodular functions that use only nearly-linear evaluations of the objective function [BV14, EN17]. However, in many practical scenarios, evaluating the objective function or the marginal gain is expensive. Thus, directly applying those algorithms usually requires superlinear or even quadratic running time.
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2 Preliminaries

2.1 Graphs, Laplacians, and Effective Resistances
Let $G = (V, E)$ be a positively weighted undirected graph. $V$ and $E$ is respectively the vertex set and the edge set of the graph, and $w : E \to \mathbb{R}^+$ is the weight function. Let $|V| = n$ and $|E| = m$. The Laplacian matrix $L$ of $G$ is given by

$$L_{[u,v]} = \begin{cases} -w(u,v) & \text{if } u \sim v, \\ \deg(u) & \text{if } u = v, \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(u) \triangleq \sum_{u \sim v} w(u,v)$, and we write $u \sim v$ iff $(u,v) \in E$. We will use $L$ and $L^G$ interchangeably when the context is clear.

If we assign an arbitrary orientation to each edge of $G$, we obtain a signed edge-vertex incident matrix $B_{m \times n}$ of graph $G$ defined as

$$B_{[e,u]} = \begin{cases} 1 & \text{if } u \text{ is } e \text{'s head,} \\ -1 & \text{if } u \text{ is } e \text{'s tail,} \\ 0 & \text{otherwise.} \end{cases}$$

Let $W$ be an $m \times m$ diagonal matrix in which $W_{[e,e]} = w(e)$. Then we can express $L$ as $L = B^T W B$. It follows that a quadratic form of $L$ can be written as

$$x^T L x = \sum_{u \sim v} w(u,v) (x_{[u]} - x_{[v]})^2.$$ 

It is then observed that $L$ is positive semidefinite, and $L$ only has one zero eigenvalue if $G$ is a connected graph. If we let $b_e$ be the $e$th column of $B^T$, we can then write $L$ in a sum of rank-1 matrices as $L = \sum_{e \in E} w(e) b_e b_e^T$.

The Laplacian matrix is related to the number of spanning trees $T(G)$ by Kirchhoff’s matrix-tree theorem [Kir47], which expresses $T(G)$ using any $(n-1) \times (n-1)$ principle minors of $L$. We denote by $L_{-u}$ the principle submatrix derived from $L$ by removing the row and column corresponding to vertex $u$. Since the removal of any matrix leads to the same result, we will usually remove the vertex with index $n$. Thus, we write the Kirchhoff’s matrix-tree theorem as

$$T(G) = \det(L_{-n}^G).$$  

(4)

The effective resistance between any pair of vertices can be defined by a quadratic form of the Moore-Penrose inverse $L^\dagger$ of the Laplacian matrix [KR93].

**Definition 2.1.** Given a connected graph $G = (V, E, w)$ with Laplacian matrix $L$, the effective resistance any two vertices $u$ and $v$ is defined as $R_{\text{eff}}(u,v) = (e_u - e_v)^T L^\dagger (e_u - e_v)$.

For two matrices $A$ and $B$, we write $A \preceq B$ to denote $x^T A x \leq x^T B x$ for all vectors $x$. If for two connected graph $G$ and $H$ their Laplacians satisfy $L^G \preceq L^H$, then $(L^H)^\dagger \preceq (L^G)^\dagger$. 

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2.2 Submodular Functions

We next give the definitions for monotone and submodular set functions. For conciseness we use $S + u$ to denote $S \cup \{u\}$.

**Definition 2.2 (Monotonicity).** A set function $f : 2^V \to \mathbb{R}$ is monotone if $f(S) \leq f(T)$ holds for all $S \subseteq T \subseteq V$.

**Definition 2.3 (Submodularity).** A set function $f : 2^V \to \mathbb{R}$ is submodular if $f(S + u) - f(S) \geq f(T + u) - f(T)$ holds for all $S \subseteq T \subseteq V$ and $u \in V \setminus T$.

2.3 Schur Complements

Let $V_1$ and $V_2$ be a partition of vertex set $V$, which means $V_2 = V \setminus V_1$. Then, we decompose the Laplacian into blocks using $V_1$ and $V_2$ as the block indices:

$$L = \begin{pmatrix} L_{[V_1,V_1]} & L_{[V_1,V_2]} \\ L_{[V_2,V_1]} & L_{[V_2,V_2]} \end{pmatrix}.$$

The Schur complement of $G$, or $L$, onto $V_1$ is defined as:

$$SC(G,V_1) = SC(L^G,V_1) \overset{\text{def}}{=} L_{[V_1,V_1]}^G - L_{[V_1,V_2]}^G \left( L_{[V_2,V_2]}^G \right)^{-1} L_{[V_2,V_1]}^G,$$

and we will use $SC(G,V_1)$ and $SC(L^G,V_1)$ interchangeably.

The Schur complement preserves the effective resistance between vertices $u, v \in V_1$.

**Fact 2.4.** Let $V_1$ be a subset of vertices of a graph $G$. Then for any vertices $u, v \in V_1$, we have:

$$R_{\text{eff}}^G(u,v) = R_{\text{eff}}^{SC(G,V_1)}(u,v).$$

3 Nearly-Linear Time Approximation Algorithm

By the matrix determinant lemma [Har97], we have

$$\det \left( (L + w(u,v)b_{u,v}b_{u,v}^T)_{-n} \right) = \left( 1 + w(u,v)b_{u,v}^T L_{-n}^T b_{u,v} \right) \det(L_{-n}).$$

Thus, by Kirchhoff’s matrix tree theorem, we can write the increase of $\log T(G)$ upon the addition of edge $(u, v)$ as

$$\log T(G + (u,v)) - \log T(G) = \log \left( 1 + w(u,v)R_{\text{eff}}^G(u,v) \right),$$

which immediately implies $\log T(G)$’s submodularity by Rayleigh’s monotonicity law [Sto87].

**Lemma 3.1.** $\log T(G + P)$ is a monotone submodular function.

Thus, one can obtain a $(1 - \frac{1}{e})$-approximation for Problem 1 by a simple greedy algorithm that, in each of $k$ iterations, selects the edge that results in the largest effective resistance times edge weight [NWF78].
Our algorithm is based on another greedy algorithm for maximizing a submodular function, which is proposed in [BV14]. Instead of selecting the edge with highest effective resistance in each iteration, the algorithm maintains a geometrically decreasing threshold and sequentially selects any edge with effective resistance above the threshold. The idea behind this greedy algorithm is that one can always pick an edge with highest effective resistance up to a \((1 \pm \epsilon)\)-error. In doing so, the algorithm is able to obtain a \((1 - \frac{1}{e} - \epsilon)\)-approximation using nearly-linear marginal value evaluations. We give this algorithm in Algorithm 1.

Algorithm 1: \(P = \text{GreedyTh}(G, Q, w, \epsilon, k)\)

**Input**: 
- \(G = (V, E)\): A connected graph.
- \(Q\): A candidate edge set with \(|Q| = q\).
- \(w: (E \cup Q) \to \mathbb{R}^+\): An edge weight function.
- \(\epsilon\): An error parameter.
- \(k\): Number of edges to add.

**Output**: 
- \(P\): A subset of \(Q\) with at most \(k\) edges.

1. \(P \leftarrow \emptyset\)
2. \(er_{max} \leftarrow \max_{(u,v) \in Q} w(u,v) \cdot R^G_{eff}(u,v)\)
3. \(th \leftarrow er_{max}\)
4. **while** \(th \geq \frac{e}{q}er_{max}\) **do**
5. **forall** \((u,v) \in Q \setminus P\) **do**
6. **if** \(|P| < k\) and \(w(u,v) \cdot R^G_{eff}(u,v) \geq th\) **then**
7. \(G \leftarrow G + (u,v)\)
8. \(P \leftarrow P \cup \{(u,v)\}\)
9. \(th \leftarrow (1 - \epsilon)th\)
10. **return** \(P\)

The performance of algorithm \(\text{GreedyTh}\) is characterized in the following theorem.

**Theorem 3.2.** The algorithm \(P = \text{GreedyTh}(G, Q, w, \epsilon, k)\) takes a connected graph \(G = (V, E)\) with \(n\) vertices and \(m\) edges, an edge set \(Q\) of \(q\) edges, an edge weight function \(w: (E \cup Q) \to \mathbb{R}^+\), a real number \(0 < \epsilon \leq 1/2\), and an integer \(1 \leq k \leq q\), and returns an edge set \(P \subseteq Q\) of at most \(k\) edges. The algorithm computes effective resistances for \(O(\frac{q}{\epsilon} \log \frac{q}{\epsilon})\) pairs of vertices, and uses \(O(\frac{q}{\epsilon} \log \frac{q}{\epsilon})\) arithmetic operations. The \(P\) returned satisfies the following statement:

\[
\log \frac{T(G + P)}{T(G)} \geq \left(1 - \frac{1}{e} - \epsilon\right) \log \frac{T(G + O)}{T(G)},
\]

where \(O \overset{\text{def}}{=} \arg \max_{S \subseteq Q, |S| \leq k} T(G + S)\) denotes an optimum solution.

A natural idea to accelerate the algorithm \(\text{GreedyTh}\) is to compute effective resistances approximately, instead of exactly, using the routines in [SS11, DKP+17]. To this end, we develop the following lemma, which shows that to obtain a multiplicative approximation of

\[
\log \left(1 + w(u,v)R^G_{eff}(u,v)\right),
\]
it suffices to compute a multiplicative approximation of \( R^{G'}_{\text{eff}}(u, v) \). We note that if given that \( a \) and \( b \) are within a factor of \( (1 + \epsilon) \) of each other, then it follows that \( \log (1 + a) \) and \( \log (1 + b) \) are within \( (1 + \epsilon) \) as well, as the function \( \log \log(1 + e^x) \) is a 1-Lipschitz function. Since we are using \( (1 \pm \epsilon)\)-approximation, we give an alternative proof.

**Lemma 3.3.** For any non-negative scalars \( a, b \) and \( 0 < \epsilon \leq 1/2 \) such that

\[
(1 - \epsilon)a \leq b \leq (1 + \epsilon)a,
\]

the following statement holds:

\[
(1 - 2\epsilon) \log(1 + a) \leq \log(1 + b) \leq (1 + 2\epsilon) \log(1 + a).
\]

**Proof.** Since \( \log(1 + a) = \ln(1 + a) / \ln 2 \) and \( \log(1 + b) = \ln(1 + b) / \ln 2 \), we only need to prove the \((1 + 2\epsilon)\)-approximation between \( \ln(1 + a) \) and \( \ln(1 + b) \).

Let \( x_2 \approx 2.51 \) be the positive root of the equation

\[
\ln(1 + x) = x/2.
\]

Then, we have \( \ln(1 + x) \geq x/2 \) for \( 0 \leq x \leq x_2 \) and \( \ln(1 + x) < x/2 \) for \( x > x_2 \).

For \( a \leq x_2 \), we have

\[
\frac{|\ln(1 + a) - \ln(1 + b)|}{|a - b|} \leq \frac{d \ln(1 + x)}{dx} \bigg|_{x = \min\{a, b\}} \leq \frac{d \ln(1 + x)}{dx} \bigg|_{x = 0} = 1
\]

\[
\Rightarrow \quad |\ln(1 + a) - \ln(1 + b)| \leq |a - b| \leq e\epsilon \leq 2\epsilon \ln(1 + a) \quad \text{by } \ln(1 + a) \geq a/2
\]

\[
\Rightarrow \quad (1 - 2\epsilon) \ln(1 + a) \leq \ln(1 + b) \leq (1 + 2\epsilon) \ln(1 + a).
\]

For \( a > x_2 \), we have

\[
\frac{|\ln(1 + a) - \ln(1 + b)|}{|a - b|} \leq \frac{d \ln(1 + x)}{dx} \bigg|_{x = \min\{a, b\}} \leq \frac{1}{1 + (1 - \epsilon)a}
\]

\[
\Rightarrow \quad |\ln(1 + a) - \ln(1 + b)| \leq \frac{|a - b|}{1 + (1 - \epsilon)a} \leq \frac{e\epsilon}{1 - \epsilon}a \leq \frac{e\epsilon}{2a} \quad \text{by } 0 < \epsilon \leq 1/2
\]

\[
= 2\epsilon \leq 2\epsilon \ln(1 + a) \quad \text{by } a > x_2 \approx 2.51
\]

\[
\Rightarrow \quad (1 - 2\epsilon) \ln(1 + a) \leq \ln(1 + b) \leq (1 + 2\epsilon) \ln(1 + a).
\]

\( \square \)

By using the effective resistance approximation routine in [DKP+17], one can pick an edge with effective resistance above the threshold up to a \( 1 \pm \epsilon \) error. Therefore, by an analysis similar to that of Algorithm 1 of [BV14], one can obtain a \((1 - \frac{1}{e} - \epsilon)\)-approximation in time \( \tilde{O}(m + (n + q)e^{-2}k) \). The reason that the running time has a factor \( k \) is that one has to recompute the effective resistances whenever an edge is added to the graph. To make the running time independent of \( k \), we will need a faster algorithm for performing the sequential updates, i.e., Lines 5-8 of Algorithm 1.
3.1 Routine for Faster Sequential Edge Additions

We now use the idea we stated in Section 1.2 to perform the sequential updates at Lines 5-8 of Algorithm 1 in nearly-linear time. We use a routine from [DKP+17] to compute the approximate Schur complement:

**Lemma 3.4.** There is a routine $S = \text{ApproxSchur}(L, C, \epsilon, \delta)$ that takes a Laplacian $L$ corresponding to graph $G = (V, E)$ with $n$ vertices and $m$ edges, a vertex set $C \subseteq V$, and real numbers $0 < \epsilon \leq 1/2$ and $0 < \delta < 1$, and returns a graph Laplacian $S$ with $O(|C| \epsilon^{-2} \log n)$ nonzero entries supported on $C$. With probability at least $1 - \delta$, $S$ satisfies

$$(1 - \epsilon)SC(L, C) \preceq S \preceq (1 + \epsilon)SC(L, C).$$

The routine runs in $\tilde{O}(m \log^2 (n/\delta) + n \epsilon^{-2} \log^4 (n/\delta))$ time.

We give the routine for performing fast sequential updates in Algorithm 2.
Algorithm 2: $P = \text{AddAbove}(G, (u_i, v_i)_{i=1}^q, w, th, \epsilon, k)$

**Input**: $G = (V, E)$: A connected graph.
- $(u_i, v_i)_{i=1}^q$: An edge sequence of $q$ edges.
- $w : (E \cup (u_i, v_i)_{i=1}^q) \rightarrow \mathbb{R}^+$: An edge weight function.
- $th$: A threshold.
- $\epsilon$: An error parameter.
- $k$: Number of edges to add.

**Output**: $P$: A subset of $Q$ with at most $k$ edges.

1. Let $\epsilon_1 = \frac{2}{3} \cdot \epsilon / \log q$ and $\epsilon_2 = (1 - 1 / \log q) \cdot \epsilon$.
2. Let $L$ be the Laplacian matrix of $G$.
3. if $q = 1$ then
   4. $S \leftarrow \text{ApproxSchur}(L, \{u_1, v_1\}, \epsilon, \frac{1}{100(m+q)})$
   5. Compute $S^\dagger$ by inverting $S$ in $O(1)$ time.
   6. if $w(u_1, v_1) \cdot b_{u_1,v_1}^T S^\dagger b_{u_1,v_1} \geq th$ and $k > 0$ then
      7. return $\{(u_1, v_1)\}$
   8. else
      9. return $\emptyset$
10. else
   11. Divide $(u_i, v_i)_{i=1}^q$ into two intervals
       
       $f^{(1)} \overset{\text{def}}{=} (u_1, v_1), \ldots, (u_{\lfloor q/2 \rfloor}, v_{\lfloor q/2 \rfloor})$ and
       $f^{(2)} \overset{\text{def}}{=} (u_{\lfloor q/2 \rfloor+1}, v_{\lfloor q/2 \rfloor+1}), \ldots, (u_q, v_q)$,
       
       and let $V^{(1)}$ and $V^{(2)}$ be the respective set of endpoints of edges in $f^{(1)}$ and $f^{(2)}$.
   12. $S^{(1)} \leftarrow \text{ApproxSchur}(L, V^{(1)}, \epsilon_1, \frac{1}{100(m+q)})$
   13. $P^{(1)} \leftarrow \text{AddAbove}(S^{(1)}, f^{(1)}, w, th, \epsilon_2, k)$
   14. Update the graph Laplacian by:
       
       $L \leftarrow L + \sum_{(u,v) \in P^{(1)}} w(u,v) b_{u,v} b_{u,v}^T$.
   15. $S^{(2)} \leftarrow \text{ApproxSchur}(L, V^{(2)}, \epsilon_1, \frac{1}{100(m+q)})$
   16. $P^{(2)} \leftarrow \text{AddAbove}(S^{(2)}, f^{(2)}, w, th, \epsilon_2, k - |P^{(1)}|)$
   17. return $P^{(1)} \cup P^{(2)}$

The performance of AddAbove is characterized in Lemma 1.2.

**Proof of Lemma 1.2.** We first prove the correctness of this lemma by induction on $q$.

When $q = 1$, the routine goes to lines 4-9. Lemma 3.4 guarantees that $S$ satisfies

$$(1 - \epsilon) SC(L, \{u_1, v_1\}) \preceq S \preceq (1 + \epsilon) SC(L, \{u_1, v_1\}),$$

which implies

$$(1 - 2\epsilon) b_{u_1,v_1}^T L^\dagger b_{u_1,v_1} \leq b_{u_1,v_1}^T S^\dagger b_{u_1,v_1} \leq (1 + 2\epsilon) b_{u_1,v_1}^T L^\dagger b_{u_1,v_1}.$$
Thus, the correctness holds for $q = 1$.

Suppose the correctness holds for all $1 \leq q \leq t$ where $t \geq 1$. We now prove that it also holds for $q = t + 1$. Since $q > 1$, the routine goes to Line 11-17. Again by Lemma 3.4, we have

$$(1 - \frac{2}{3} \cdot \epsilon / \log q)SC(L, V^{(1)}) \preceq S^{(1)} \preceq (1 + \frac{2}{3} \cdot \epsilon / \log q)SC(L, V^{(1)}).$$

By the inductive hypothesis, any effective resistance query in $f^{(1)}$ is answered with an error within

$$1 \pm (1 - 1/\log q) \cdot \epsilon.$$

Combining this with (5) gives the correctness for $f^{(1)}$. Then, by a similar analysis, we can obtain the correctness for $f^{(2)}$. By induction, the correctness holds for all $q$.

For the success probability, note that every time we invoke the routine APPROXSCHUR, we set the failure probability to $\frac{1}{10 n (m + q)}$. Thus, we get high probability by a union bound.

We next analyzie the running time.

Let $T(q, \epsilon)$ denote the running time of ADDABOVE($G, (u_i, v_i)_{i=1}^q, w, th, \epsilon, k$) when the number of edges in $G$ is $O(q \epsilon^2 \log n)$. It immediately follows that when $G$ contains $m$ edges, where $m$ is an arbitrary number, the total running time of ADDABOVE is at most

$$2 \cdot T(q/2, (1 - 1/\log q) \cdot \epsilon) + \tilde{O}(m + n \epsilon^2 \log^2 q),$$

since by Lemma 3.4, in the first step of divide-and-conquer, the routine will divide the graph into two Schur complements each with $O(q \epsilon^2 \log n)$ edges. Also by Lemma 3.4, we can write $T(q, \epsilon)$ in the following recurrence form:

$$T(q, \epsilon) = 2 \cdot T(q/2, (1 - 1/\log q) \cdot \epsilon) + \tilde{O}(q \epsilon^2 \log^2 q),$$

which gives $T(q, \epsilon) = \tilde{O}(q \epsilon^{-2})$. Combining this with (6) gives the total nearly-linear running time $\tilde{O}(m + (n + q) \epsilon^{-2})$.

3.2 Incorporating Fast Sequential Edge Additions into the Greedy Algorithm

We now incorporate ADDABOVE into Algorithm 1 to obtain our nearly-linear time greedy algorithm. To estimate the maximum effective resistance at Line 2 of Algorithm 1, we will also need the effective resistance estimation routine from [DKP+17]:

**Lemma 3.5.** There is a routine $(\hat{r}_{u,v})_{(u,v) \in Q} = \text{ERESt}({G = (V, E)}, w, Q, \epsilon)$ which takes a graph $G = (V, E)$ with $n$ vertices and $m$ edges, an edge weight function $w : E \rightarrow \mathbb{R}^+$, a set $Q$ of $q$ vertex pairs, and a real number $0 < \epsilon \leq 1/2$, and returns $q$ real numbers $(\hat{r}_{u,v})_{(u,v) \in Q}$ in $\tilde{O}(m + (n + q) \epsilon^{-2})$ time. With high probability, the following statement holds for all $(u, v) \in Q$:

$$(1 - \epsilon) R_{\text{eff}}(u, v) \leq \hat{r}_{u,v} \leq (1 + \epsilon) R_{\text{eff}}(u, v).$$

We give our greedy algorithm in Algorithm 3.
Algorithm 3: \( P = \text{NSTMaximize}(G, Q, w, \epsilon, k) \)

**Input:**  
\( G = (V, E) \): A connected graph.  
\( Q \): A candidate edge set with \( |Q| = q \).  
\( w : (E \cup Q) \rightarrow \mathbb{R}^+ \): An edge weight function.  
\( \epsilon \): An error parameter.  
\( k \): Number of edges to add.  

**Output:**  
\( P \): A subset of \( Q \) with at most \( k \) edges.

1. \( P \leftarrow \emptyset \)  
2. \( \hat{r}_{u,v}(u,v) \in Q \leftarrow \text{ERest}(G = (V, E), w, Q, \epsilon) \)  
3. \( er_{max} \leftarrow \frac{1 + \epsilon}{1 - \epsilon} \cdot \max_{(u,v) \in Q} w(u,v) \cdot \hat{r}_{u,v} \)  
4. \( th_0 \leftarrow \log(1 + er_{max}) \)  
5. \( th \leftarrow th_0 \)  
6. **while** \( th \geq \frac{\epsilon}{2q} th_0 \) **do**  
7. Pick an arbitrary ordering \((u_i, v_i)_{i=1}^{q-|P|}\) of edges in \( Q \setminus P \).  
8. \( P' \leftarrow \text{AddAbove}(G, (u_i, v_i)_{i=1}^{q-|P|}, w, 2^{th} - 1, \epsilon/12, k - |P|) \)  
9. Update the graph by \( G \leftarrow G + P' \).  
10. \( P \leftarrow P \cup P' \)  
11. \( th \leftarrow (1 - \epsilon/6) th \)  
12. **return** \( P \)

The performance of algorithm NSTMaximize is characterized in Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 1.2, the running time equals
\[
O(\epsilon^{-1} \log \frac{q}{\epsilon}) \cdot \tilde{O}(m + (n + q)\epsilon^{-2}) = \tilde{O}((m + (n + q)\epsilon^{-2})\epsilon^{-1}).
\]

We next prove the correctness of the approximation ratio.

We first consider the case in which the algorithm selects exactly \( k \) edges from \( Q \). When the algorithm selects an edge \((u_i, v_i)\) at threshold \( th \), the effective resistance between any \( u_j, v_j \) where \( i < j \leq q \) can be upper bounded by
\[
\log(1 + R_{\text{eff}}(u_j, v_j)) \leq \frac{1}{1 - \epsilon/6} \cdot \frac{th}{1 - \epsilon/6} \\
\leq \frac{(1 + \epsilon/6) \log(1 + R_{\text{eff}}(u_i, v_i))}{(1 - \epsilon/6)^2} \\
\leq \frac{\log(1 + R_{\text{eff}}(u_i, v_i))}{1 - \epsilon/2}, \tag{8}
\]
and the effective resistance between any \( u_j, v_j \) where \( 1 \leq j < i \) can be upper bounded by

\[
\log(1 + R_{\text{eff}}(u_j, v_j)) \leq \frac{1}{1 - \epsilon/6} \cdot \log(1 + R_{\text{eff}}(u_i, v_i)) \leq \frac{(1 + \epsilon/6) \log(1 + R_{\text{eff}}(u_i, v_i))}{1 - \epsilon/6} \leq \log(1 + R_{\text{eff}}(u_i, v_i)).
\]

Thus, the algorithm always picks an edge with at least \( 1 - \epsilon/2 \) times the maximum marginal gain. Let \( G^{(i)} \) be the graph after the first \( i \) edges are added, and let \( O \) be the optimum solution. By submodularity we have, for any \( i \geq 0 \)

\[
\log \frac{T(G^{(i+1)})}{T(G^{(i)})} \geq \frac{1 - \epsilon/2}{k} \cdot \log \frac{T(G + O)}{T(G^{(i)})}.
\]

Then, we have

\[
\log \frac{T(G^{(k)})}{T(G)} \geq \left(1 - \left(1 - \frac{1 - \epsilon/2}{k}\right)^k\right) \cdot \log \frac{T(G + O)}{T(G)} \geq \left(1 - \frac{1}{e^{1-\epsilon/2}}\right) \cdot \log \frac{T(G + O)}{T(G)} \geq \left(1 - \frac{1}{e} - \epsilon/2\right) \cdot \log \frac{T(G + O)}{T(G)}
\]

where the second inequality follows from \((1 - 1/x)^x \leq 1/e\).

When the algorithm selects fewer than \( k \) edges, we know by the condition of the while loop that not selecting the remaining edges only causes a loss of \((1 - \epsilon/2)\). Thus, we have

\[
\log \frac{T(G^{(k)})}{T(G)} \geq (1 - \epsilon/2) \left(1 - \frac{1}{e} - \epsilon/2\right) \cdot \log \frac{T(G + O)}{T(G)} \geq \left(1 - \frac{1}{e} - \epsilon\right) \cdot \log \frac{T(G + O)}{T(G)}
\]

which completes the proof. \(\square\)

4 Exponential Inapproximability

In this section, we will make use of properties of some special classes of graphs. Amongst these are: the star graph \( S_n \), which is an \((n+1)\)-vertex tree in which \( n \) leaves are directly connected to a central vertex; an \( n \)-vertex path graph \( P_n \); an \( n \)-vertex cycle \( C_n \); a fan graph \( F_n \), defined by \( S_n \) plus a \( P_n \) supported on its leaves; and a wheel graph \( W_n \), defined by \( S_n \) plus a \( C_n \) supported on its leaves. We define the length of a path or a cycle by the sum of edge weights in the path/cycle. Specifically, the lengths of \( P_n \) and \( C_n \) with all weights equal to 1 are \( n - 1 \) and \( n \), respectively. We also write \( K_n \) to denote the \( n \)-vertex complete graph.
4.1 Hardness of Approximation for Minimum Path Cover

We begin by introducing the definition of the minimum path cover problem.

**Problem 2** (Minimum Path Cover). Given an undirected graph $G = (V, E)$, a path cover is a set of disjoint paths such that every vertex $v \in V$ belongs to exactly one path. Note that a path cover may include paths of length 0 (a single vertex). The minimum path cover problem is to find a path cover of $G$ having the least number of paths.

We recall a known hardness result of TSP with distance 1 and 2 ($(1,2)$-TSP).

**Lemma 4.1** ([PY93, EK01]). There is a constant $\sigma > 0$, such that it is NP-hard to distinguish between the instances of $(1,2)$-TSP ($K_n$ with edge weights 1 and 2) having shortest Hamiltonian cycle length $n$ and shortest Hamiltonian cycle length at least $(1 + \sigma n)$.

Next, we reduce the $(1,2)$-TSP problem to a TSP-Path problem with distance 1 and 2 ($(1,2)$-TSP-Path).

**Lemma 4.2.** There is a constant $\delta > 0$, such that it is NP-hard to distinguish between the instances of $(1,2)$-TSP-Path ($K_n$ with edge weights 1 and 2) having shortest Hamiltonian path length $(n - 1)$ and shortest Hamiltonian path length at least $(1 + \delta n)$.

**Proof.** Completeness: If a $(1,2)$-weighted complete graph $K_n$ has a Hamiltonian cycle of length $n$, then it has a Hamiltonian path of length $n - 1$.

Soundness: Given a $(1,2)$-weighted complete graph $K_n$, let $Q$ be a shortest Hamiltonian cycle in $K_n$, and suppose its length satisfies $|Q| \geq (1 + \sigma)n$. Then the shortest Hamiltonian path in $K_n$ has length at least $(1 + \sigma)n - 2$. Let $\delta = \frac{\sigma}{2}$. Then for $n > \frac{4}{\delta}$, $|H| \geq (1 + \delta)n$. \qed

**Lemma 4.3.** There is a constant $\delta$ such that it is NP-hard to distinguish between graphs with minimum path cover number 1 and minimum path cover number at least $\delta n$.

**Proof.** Completeness: In a $(1,2)$-weighted complete graph $K_n$, if $H^*$ is a shortest Hamiltonian path in $K_n$, and its length satisfies $|H^*| = n - 1$, then $H^*$ is a Hamiltonian path of the subgraph consists of all edges in $K_n$ that have weights equal to 1.

Soundness: In a $(1,2)$-weighted complete graph $K_n$, in which any Hamiltonian path $H$ satisfies $|H| \geq (1 + \delta)n$, the minimum path cover number in the subgraph consists of all edges in $K_n$ with weight 1 is at least $\delta n + 2$. \qed

4.2 Exponential Inapproximability for NSTM

We now consider an instance of NSTM we described in Section 1.1. Recall that we obtain the number of spanning trees in $G + P$ by iteratively using the matrix determinant lemma:

$$T(G^{(i+1)}) = \left(1 + R_{\text{eff}}^{(i)}(u_i, v_i)\right)T(G^{(i)}).$$

We are interested in the effective resistance between endpoints of an edge $e \in P$ that is not contained in any path in a minimum path cover $P$ of the graph $H[V']$ using only edges in $P$. 

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Lemma 4.4. Let $H = (V, E)$ be an unweighted graph equal to a star $S_n$ plus $H$’s subgraph $H[V'] = (V', E')$ supported on $S_n$’s leaves (see Figure 1). Let $P$ be an arbitrary subset of $E'$ with $|P| = n - 1$, and $\mathcal{P}$ be the minimum path cover of $H[V']$ using only edges in $P$. Let $e = (u_i, v_i)$ be an edge that is contained in $P$ but not in the path cover $\mathcal{P}$, then:

1. If $u_i$ is not an endpoint of a path in the path cover $\mathcal{P}$, and $v_i$ is not an isolated vertex in $\mathcal{P}$, then
   
   $$R_{\text{eff}}(u_i, v_i) \leq \frac{7}{6}.$$

2. If $u_i$ is not an endpoint of a path in the path cover $\mathcal{P}$, and $v_i$ is an isolated vertex in $\mathcal{P}$, then
   
   $$R_{\text{eff}}(u_i, v_i) \leq \frac{3}{2}.$$

3. If $u_i$ and $v_i$ are endpoints of the same path in the path cover $\mathcal{P}$, then
   
   $$R_{\text{eff}}(u_i, v_i) < \sqrt{5} - 1.$$

Proof. From Rayleigh’s Monotonicity law, we know that for the first case

$$R_{\text{eff}}(u_i, v_i) \leq \frac{1}{2} + \frac{2}{3},$$

Similar analysis shows that

$$R_{\text{eff}}(u_i, v_i) \leq \frac{1}{2} + 1$$

holds for the second case in the lemma. Figure 2 shows how we obtain these bounds.

The last case follows directly from the fact that

$$\mathcal{T}(W_n) < \sqrt{5} \cdot \mathcal{T}(F_n),$$

which was shown in [MEM14].
We first prove the completeness of the reduction.

**Lemma 4.5.** Let $H = (V, E)$ be an unweighted graph equal to a star $S_n$ plus its subgraph $H[V'] = (V', E')$ supported on $S_n$’s leaves. In the NSTM instance where $G = S_n$, $Q = E'$, and $k = n - 1$, there exists $P \subseteq Q$, $|P| = k$ which satisfies

$$T(G + P) = \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^n - \left( \frac{3 - \sqrt{5}}{2} \right)^n \right)$$

if $H[V']$ has a Hamiltonian path.

**Proof.** Suppose there is a Hamiltonian path $P^*$ in graph $H[V']$, then $S_n + P^*$ is a fan $F_n$. It’s number of spanning trees is given in [MEM14] as

$$T(S_n + P^*) = T(F_n) = \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^n - \left( \frac{3 - \sqrt{5}}{2} \right)^n \right).$$ (10)

Before proving the soundness of the reduction, we warm up by proving the following lemma:

**Lemma 4.6.** Let $H = (V, E)$ be an unweighted graph equal to a star $S_n$ plus its subgraph $H[V'] = (V', E')$ supported on $S_n$’s leaves. In the NSTM instance where $G = S_n$, $Q = E'$, and $k = n - 1$, for any edge set $P \subseteq Q$ with $k$ edges that does not constitute a Hamiltonian path,

$$T(S_n + P) < T(S_n + P^*).$$

**Proof.** Suppose $P' = \{p'_1, p'_2, \ldots, p'_{t'}\}$ is a minimum path cover of $G'' = (V', P)$, with $|p'_i| = l'_i$ and $l'_1 \leq l'_2 \leq \cdots \leq l'_{t'}$. Suppose $l'_1 = \cdots = l'_{t'} = 0 < l'_{t'+1}$, then

$$T(S_n + P') = \prod_{i=1}^{t'} \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{l'_i+1} - \left( \frac{3 - \sqrt{5}}{2} \right)^{l'_i+1} \right),$$ (11)

Since $7/6 < \sqrt{5} - 1 < 3/2$

$$T(S_n + P) \leq \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{l'_{t'+1}} - \left( \frac{3 - \sqrt{5}}{2} \right)^{l'_{t'+1}} \right) \prod_{i=\kappa+1}^{t'} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{l'_i+1} - \left( \frac{3 - \sqrt{5}}{2} \right)^{l'_i+1} \right)$$ (12)

$$< \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^{n-\kappa} - \left( \frac{3 - \sqrt{5}}{2} \right)^{n-\kappa} \right)$$ (13)

$$< \frac{1}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^n - \left( \frac{3 - \sqrt{5}}{2} \right)^n \right)$$

$$= T(S_{n+1} + P^*).$$

where (13) follows from (12) by the fact that $\sum_{i=\kappa+1}^{t'} (l'_i + 1) = n - \kappa$.  

\[ \square \]
Next, we prove the soundness of the reduction.

**Proof of Lemma 1.5.** Suppose the minimum path cover number of \( H[V'] = (V', E') \) is at least \( \delta n \). Then any path cover \( \mathcal{P}' = \{ p_1', p_2', \ldots, p_{t'} \} \) of \( G'' = (V', P) \) must satisfy \( t' \geq t \). We let \(|p_i'| = l_i'\) for \( i \in \{1, \ldots, t'\} \) and \( l_1' \leq l_2' \leq \cdots \leq l_{t'}' \). Suppose \( l_1' = \cdots = l_{\kappa}' = 0 < l_{\kappa+1}' \); then according to (12), we obtain

\[
\mathcal{T}(S_n + P) \leq \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \prod_{t_i' \geq 1} \left( \frac{3 + \sqrt{5}}{2} l_i'^{t_i' + 1} - \frac{3 - \sqrt{5}}{2} l_i'^{t_i' + 1} \right)
\]

Since the average length of paths in \( \mathcal{P}' \) satisfies

\[
\mathbb{E}_i[l_i'] \leq \frac{n - \delta n}{\delta n} = \frac{1 - \delta}{\delta},
\]

by Markov's inequality,

\[
\mathbb{P}_i\left[l_i' \geq \frac{2(1 - \delta)}{\delta} \right] \leq \frac{1}{2}.
\]

Therefore

\[
\mathcal{T}(S_n + P) \leq \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \prod_{1 \leq t_i' < \frac{2(1 - \delta)}{\delta}} \left( \frac{3 + \sqrt{5}}{2} l_i'^{t_i' + 1} - \frac{3 - \sqrt{5}}{2} l_i'^{t_i' + 1} \right) \cdot \left( \prod_{t_i' \geq \frac{2(1 - \delta)}{\delta}} \left( \frac{3 + \sqrt{5}}{2} l_i'^{t_i' + 1} - \frac{3 - \sqrt{5}}{2} l_i'^{t_i' + 1} \right) \right)
\]

\[
< \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \left( \prod_{1 \leq t_i' < \frac{2(1 - \delta)}{\delta}} \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{2t_i' + 2} \right) \right) \left( \frac{3 + \sqrt{5}}{2} \right)^{n - \kappa}
\]

\[
= \left( \frac{5}{2} \right)^\kappa \cdot \frac{1}{\sqrt{5}} \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{\delta n - \kappa} \right) \left( \frac{3 + \sqrt{5}}{2} \right)^{n - \kappa}
\]

\[
= \frac{1}{\sqrt{5}} \left( \frac{5}{2} \right)^\kappa \left( \frac{3 + \sqrt{5}}{2} - \left( \frac{3 - \sqrt{5}}{2} \right)^{\delta n - \kappa} \right) \left( \frac{3 + \sqrt{5}}{2} \right)^{\delta n - \kappa} \left( \frac{3 + \sqrt{5}}{2} \right)^{\delta n - \kappa} \left( \frac{3 + \sqrt{5}}{2} \right)^{n - \delta n - \kappa}
\]

The third inequality follows by Markov's inequality. Let

\[
\alpha \overset{\text{def}}{=} \max \left\{ \left( \frac{15 - 5\sqrt{5}}{4} \right)^{\delta n - \kappa}, \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{\delta n - \kappa} \right) \right\}.
\]
Since $\delta$ is a positive constant, $\alpha$ is a constant that satisfies $0 < \alpha < 1$. Then

$$T(S_n + P) < \frac{1}{\sqrt{5}} \left( \alpha \cdot \frac{3 + \sqrt{5}}{2} \right)^n.$$ 

Therefore

$$\frac{\log T(S_n + P)}{\log T(F_n)} < \frac{-\frac{1}{2} \log 5 + n \cdot \log \left( \alpha \cdot \frac{3 + \sqrt{5}}{2} \right)}{-\frac{1}{2} \log 5 + n \cdot \log \left( \frac{3 + \sqrt{5}}{2} \right) + \log \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{2n} \right)} < \frac{n \log \left( \alpha \cdot \frac{3 + \sqrt{5}}{2} \right)}{n \log \left( \frac{3 + \sqrt{5}}{2} \right) + \log \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{2n} \right)}.$$ 

If $n$ satisfies

$$n > \max \left\{ \frac{1}{2 \log \frac{3 + \sqrt{5}}{2}}, \frac{2}{\log \frac{1}{\alpha}} \right\},$$

then

$$\frac{\log T(S_n + P)}{\log T(F_n)} < \frac{n \cdot \log (\alpha) + n \cdot \log \left( \frac{3 + \sqrt{5}}{2} \right)}{n \cdot \log \left( \frac{3 + \sqrt{5}}{2} \right) + \log \left( 1 - \left( \frac{3 - \sqrt{5}}{2} \right)^{2n} \right)} < \frac{\frac{1}{2} \log (\alpha) + \log \left( \frac{3 + \sqrt{5}}{2} \right)}{\log \left( \frac{3 + \sqrt{5}}{2} \right)} = 1 - \frac{\log (1/\alpha)}{2 \log \left( \frac{3 + \sqrt{5}}{2} \right)}.$$ 

Thus, we obtain the constant $c$ stated in Lemma 1.5:

$$c = \frac{\log (1/\alpha)}{2 \log \left( \frac{3 + \sqrt{5}}{2} \right)}.$$ (15)
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