Dade’s ordinary conjecture implies the Alperin–McKay conjecture

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Abstract. We show that Dade’s ordinary conjecture implies the Alperin–McKay conjecture. We remark that some of the methods can be used to identify a canonical height zero character in a nilpotent block.

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Dade proved in [4] that his projective conjecture [4, 15.5] implies the Alperin–McKay conjecture. Navarro showed in [11, Theorem 9.27] that the group version of Dade’s ordinary conjecture implies the McKay conjecture. We show here that Dade’s ordinary conjecture [3, 6.3] implies the Alperin–McKay conjecture. Let $p$ be a prime number.

Theorem 1. If Dade’s ordinary conjecture holds for all $p$-blocks of finite groups, then the Alperin–McKay conjecture holds for all $p$-blocks of finite groups.

The proof combines arguments from Sambale [17] and formal properties of chains of subgroups in fusion systems from [7]. Let $(K, O, k)$ be a $p$-modular system. We assume that $k$ is algebraically closed, and let $\bar{K}$ be an algebraic closure of $K$. By a character of a finite group, we will mean a $\bar{K}$-valued character. For a finite group $G$ and a block $B$ of $OG$, let $\text{Irr}(B)$ denote the set of irreducible characters of $G$ in the block $B$, and let $\text{Irr}_0(B)$ denote the set of irreducible height zero characters of $G$ in $B$. For a central $p$-subgroup $Z$ of $G$ and a character $\eta$ of $Z$, let $\text{Irr}_0(B|\eta)$ denote the subset of $\text{Irr}_0(B)$ consisting of

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those height zero characters which cover the character $\eta$. The following lemma is implicit in [17].

**Lemma 2.** Let $P$ be a finite $p$-group, let $\mathcal{F}$ be a saturated fusion system on $P$, and let $Z \leq Z(\mathcal{F})$. Suppose that $\eta$ is a linear character of $P$. There exists a linear character $\hat{\eta}$ of $P$ such that $\hat{\eta}|_{Z} = \eta|_{Z}$ and $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$.

**Proof.** First consider the case that $\eta|_{Z}$ is faithful. Then $Z \cap \{P, P\} = 1$. Hence by [5, Lemma 4.3], $\text{foc}(\mathcal{F}) \cap Z = 1$. The result is now immediate. Now suppose $Z_0 = \text{Ker}(\eta|_{Z})$ and let $\mathcal{F} = \mathcal{F}/Z_0$. By the previous argument, applied to $P/Z_0$ and $\mathcal{F}$, there exists a character $\hat{\eta}$ of $P/Z_0$ such that $\hat{\eta}|_{Z/Z_0} = \eta|_{Z/Z_0}$ and $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$. Denote also by $\hat{\eta}$ the inflation of $\hat{\eta}$ to $P$. Then $\hat{\eta}$ has the required properties since $\text{foc}(\mathcal{F}) = \text{foc}(\mathcal{F})Z_0/Z_0$. □

The following result is a special case of a result due to Murai; we include a proof for convenience.

**Lemma 3** (cf. [9, Theorem 4.4]). Let $G$ be a finite group, $B$ be a block of $OG$, and $P$ a defect group of $B$. Let $Z$ be a central $p$-subgroup of $G$ and let $\eta$ be an irreducible character of $Z$ such that $\text{Irr}_0(B|\eta) \neq \emptyset$. Then $\eta$ extends to $P$.

**Proof.** By replacing $K$ by a suitable finite extension, we may assume that $K$ is a splitting field for all subgroups of $G$. Let $i \in B^P$ be a source idempotent of $B$ and let $V$ be a $KG$-module affording an element of $\text{Irr}_0(B|\eta)$. Then $n := \dim_K(iV)$ is prime to $p$ (see [13]). Since $i$ commutes with $P$, $iV$ is a $KP$-module via $x \cdot iv = ixv$, where $x \in P, v \in V$. Let $\rho : P \rightarrow \text{GL}_n(K)$ be a corresponding representation and let $\delta : P \rightarrow K^\times$ be the determinant character of $\rho$. Then $\delta|_{Z} = \eta^n$. The result follows since $n$ is prime to $p$. □

**Lemma 4.** Let $G$ be a finite group, let $B$ be a block of $OG$ with a defect group $P$, and let $Z$ be a central $p$-subgroup of $G$. Then $|\text{Irr}_0(B)|$ equals the product of $|\text{Irr}_0(B|1_Z)|$ with the number of distinct linear characters $\eta$ of $Z$ which extend to $P$.

**Proof.** Let $\mathcal{F} = \mathcal{F}_{(P,e_P)}(G,B)$ be the fusion system of $B$ with respect to a maximal $B$-Brauer pair $(P,e_P)$, and let $\eta$ be a linear character of $Z$ which extends to $P$. Since $Z \leq Z(\mathcal{F})$, by Lemma 2 there exists a linear character $\hat{\eta}$ of $P$ such that $\hat{\eta}|_{Z} = \eta$ and $\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})$. By the properties of the Broué-Puig $*$-construction [1,16] the map $\chi \mapsto \hat{\eta}^*\chi$ is a bijection between $\text{Irr}_0(B|1_Z)$ and $\text{Irr}_0(B|\eta)$. The result follows by Lemma 3. □

Slightly strengthening the terminology in [10], we say that a pair $(G,B)$ consisting of a finite group $G$ and a block $B$ of $OG$ is a minimal counterexample to the Alperin–McKay conjecture if $B$ is a counterexample to the Alperin–McKay conjecture and if $G$ is such that first $|G : Z(G)|$ is smallest possible and then $|G|$ is smallest possible.

**Proposition 5.** Let $(G,B)$ be a minimal counterexample to the Alperin–McKay conjecture. Then $O_p(G) = 1$. 

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Proof. By a result of Murai [10], we have that $Z := O_p(G)$ is central in $G$. Let $P$ be a defect group of $B$ and let $C$ be the block of $\mathcal{O}_N G(P)$ in Brauer correspondence with $B$. By Lemma 4, $|\text{Irr}_0(B)| = |\text{Irr}_0(C)|$ if and only if $|\text{Irr}_0(B)| = |\text{Irr}_0(C)|$, where $\bar{B}$ (resp. $\bar{C}$) is the block of $\mathcal{O}_N G(P)/Z$ dominated by $B$ (resp. $C$). The result follows since $\mathcal{N}_{G/Z}(P/Z) = \mathcal{N}_{G}(P)/Z$ and $\bar{B}$ and $\bar{C}$ are in Brauer correspondence. □

Let $\mathcal{F}$ be a saturated fusion system on a finite $p$-group $P$, and let $\mathcal{C}$ be a full subcategory of $\mathcal{F}$ which is upwardly closed; that is, if $Q$, $R$ are subgroups of $P$ such that $Q$ belongs to $\mathcal{C}$ and if $\text{Hom}_\mathcal{F}(Q, R)$ is nonempty, then also $R$ belongs to $\mathcal{C}$. Drawing upon notation and facts from [7, §5], $S_{<}(\mathcal{C})$ is the category having as objects nonempty chains $\sigma = Q_0 < Q_1 < \cdots < Q_m$ of subgroups $Q_i$ of $P$ belonging to $\mathcal{C}$ such that $m \geq 0$ and $Q_i$ is normal in $Q_m$, for $0 \leq i \leq m$. Morphisms in $S_{<}(\mathcal{C})$ are given by certain ‘obvious’ commutative diagrams of morphisms in $\mathcal{F}$; see [7, 2.1, 4.1] for details. With this notation, the length of a chain $\sigma$ in $S_{<}(\mathcal{C})$ is the integer $|\sigma| = m$. The chain $\sigma$ is called fully normalised if $Q_0$ is fully $\mathcal{F}$-normalised and if either $m = 0$ or the chain $\sigma_{\geq 1} = \sigma_1 < \sigma_2 < \cdots < \sigma_m$ is fully $N_\mathcal{F}(Q_0)$-normalised. Every chain in $S_{<}(\mathcal{C})$ is isomorphic (in the category $S_{<}(\mathcal{C})$) to a fully normalised chain. There is an involution $n$ on the set of fully normalised chains which fixes the chain of length zero $P$ and which sends any other fully normalised chain $\sigma$ to a fully normalised chain $n(\sigma)$ of length $|\sigma| \pm 1$. This involution is defined as follows. If $\sigma = P$, then set $n(\sigma) = \sigma$. If $\sigma = Q_0 < Q_1 < \cdots < Q_m$ is a fully normalised chain different from $P$ such that $Q_m = N_P(\sigma)$, then define $\sigma$ by removing the last term $Q_m$; if $Q_m < N_P(\sigma)$, then define $\sigma$ by adding $N_P(\sigma)$ as last term to the chain. Then $n(\sigma)$ is fully normalised, and $n(n(\sigma)) = \sigma$. Denote by $[S_{<}(\mathcal{C})]$ the partially ordered set of isomorphism classes of chains in $S_{<}(\mathcal{C})$, and for each chain $\sigma$ by $|\sigma|$ its isomorphism class. We have a partition

$$[S_{<}(\mathcal{C})] = \{[P]\} \cup \mathcal{B} \cup n(\mathcal{B}),$$

where $\mathcal{B}$ is the set of isomorphism classes of fully normalised chains $\sigma$ satisfying $|n(\sigma)| = |\sigma| + 1$. The following Lemma is a very special case of a functor cohomological statement [7, Theorem 5.11].

Lemma 6. With the notation above, let $f : [S_{<}(\mathcal{C})] \rightarrow \mathbb{Z}$ be a function on the set of isomorphism classes of chains in $S_{<}(\mathcal{C})$ satisfying $f([\sigma]) = f([n(\sigma)])$ for any fully normalised chain $\sigma$ in $S_{<}(\mathcal{C})$. Then

$$\sum_{[\sigma] \in [S_{<}(\mathcal{C})]} (-1)^{|\sigma|} f([\sigma]) = f([P]).$$

Proof. The hypothesis on $f$ implies that the contributions from chains in $\mathcal{B}$ cancel those from chains in $n(\mathcal{B})$, whence the result. □

Proposition 7. Let $G$ be a finite group such that $O_p(G) = 1$, and let $B$ be a block of $\mathcal{O}_G$ with nontrivial defect groups. Suppose that Dade’s ordinary conjecture holds for $B$ and that the Alperin–McKay conjecture holds for any block of any proper subgroup of $G$. Then the Alperin–McKay conjecture holds for the block $B$. 

Remark 9. Let \((P,e)\) be a maximal \(B\)-Brauer pair, and denote by \(\mathcal{F}\) the associated fusion system on \(P\). For \(d\) a positive integer, denote by \(k_d(G,B)\) the number of ordinary irreducible characters in \(B\) of defect \(d\). If \(p^d = |P|\), then \(k_d(G,B)\) is the number of height zero characters, and if \(p^d > |P|\), then \(k_d(G,B) = 0\).

Let \(\mathcal{C}\) be the full subcategory of \(\mathcal{F}\) consisting of all nontrivial subgroups of \(P\). We briefly describe the standard translation process between chains in a fusion system of a block and the associated chains of Brauer pairs. The map sending a chain \(\sigma = Q_0 < Q_1 < \cdots < Q_m\) in \(S_{\leq}(\mathcal{C})\) to the unique chain of nontrivial \(B\)-Brauer pairs \(\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)\) contained in \((P,e)\) induces a bijection between isomorphism classes of chains in \(S_{\leq}(\mathcal{C})\) and the set of \(G\)-conjugacy classes of normal chains of nontrivial \(B\)-Brauer pairs (cf. [7, 2.5]). If \(\sigma\) is fully normalised, then the corresponding chain of Brauer pairs \(\tau = (Q_0, e_0) < (Q_1, e_1) < \cdots < (Q_m, e_m)\) has the property that \(e_\tau = e_m\) remains a block of \(N_G(\tau)\), and by [7, 5.14], \(N_P(\sigma) = N_P(\tau)\) is a defect group of \(e_\tau\) as a block of \(N_G(\tau)\). Denote by \(n(\tau)\) the chain of Brauer pairs corresponding to \(n(\sigma)\).

Let \(d > 0\) such that \(p^d = |P|\). Define a function \(f\) on \(S_{\leq}(\mathcal{C})\) by setting

\[
f([\sigma]) = k_d(N_G(\tau), e_\tau)
\]

for any fully normalised chain \(\sigma\) and corresponding chain \(\tau\) of Brauer pairs. If \(N_P(\sigma)\) is a proper subgroup of \(P\), then \(f([\sigma]) = 0\), and if \(N_P(\sigma) = P\), then \(f([\sigma])\) is the number of height zero characters of the block \(e_\tau\) of \(N_G(\tau)\). Dade’s ordinary conjecture for \(B\), reformulated here in terms of chains of Brauer pairs, asserts that \(k_d(G,B)\) is equal to the alternating sum

\[
\sum_{[\sigma] \in S_{\leq}(\mathcal{C})} (-1)^{|[\sigma]|} f([\sigma]).
\]

The passage between formulations in terms of normalisers of chains of Brauer pairs rather than normalisers of chains of \(p\)-subgroups is well known; see e.g. [6, 4.5], [15].

If \(|n(\sigma)| = |\sigma| + 1\), then, setting \(H = N_G(\tau)\), we have \(N_G(n(\tau)) = N_H(N_P(\tau), e_{n(\tau)})\); that is, \((N_P(\tau), e_{n(\tau)})\) is a maximal \((H, e_\tau)\)-Brauer pair. By the assumptions, the Alperin–McKay conjecture holds for the block \(e_\tau\) of \(H\). This translates to the equality \(f([\sigma]) = f([n(\sigma)])\). That is, the function \(f\) satisfies the hypotheses of Lemma 6. Thus the above alternating sum is equal to \(f([P])\), which by definition is \(k_d(N_G(P,e), e)\), and thus the Alperin–McKay conjecture holds for \(B\).

Theorem 1 follows now immediately from combining Propositions 5 and 7.

Remark 8. By work of Dade [2] and Okuyama and Wajima [12], the Alperin–McKay conjecture holds for blocks of finite \(p\)-solvable groups. G. R. Robinson pointed out that Proposition 5 yields another short proof of this fact.

Remark 9. Let \(G\) be a finite group, \(B\) a block algebra of \(OG\), \((P, e_P)\) a maximal \((G, B)\)-Brauer pair with associated fusion system \(\mathcal{F}\) on \(P\), and let \(Z\) be a central \(p\)-subgroup of \(G\). Let \(\eta\) be a linear character of \(Z\), and suppose that \(\eta\) extends to a linear character \(\hat{\eta}\) of \(P\) satisfying \(\text{foc}(\mathcal{F}) \leq \text{Ker}(\hat{\eta})\). The proof of
Lemma 4 is based on the fact that the $*$-construction $\chi \mapsto \hat{\eta} \ast \chi$ yields a bijection $\text{Irr}(B|1) \to \text{Irr}(B|\eta)$. There is some slightly more structural background to this. For $\chi \in \text{Irr}(B)$, denote by $e(\chi)$ the corresponding central primitive idempotent in $K \otimes_O B$. Set

$$e_1 = \sum_{\chi \in \text{Irr}_0(B|1)} e(\chi), \quad e_\eta = \sum_{\chi \in \text{Irr}_0(B|\eta)} e(\chi).$$

Identify $B$ with its image in $K \otimes_O B$. Multiplying $B$ by the central idempotents $e_1$ and $e_\eta$ in $K \otimes_O B$ yields the two $O$-free $O$-algebra quotients $Be_1$ and $Be_\eta$ of $B$. By [8, Theorem 1.1], there is an $O$-algebra automorphism $\alpha$ of $B$ which induces the identity on $k \otimes_O B$ and which acts on $\text{Irr}(B)$ as the map $\chi \to \hat{\eta} \ast \chi$. Thus the extension of $\alpha$ to $K \otimes_O B$ sends $e_1$ to $e_\eta$, and hence induces an $O$-algebra isomorphism

$$Be_1 \cong Be_\eta.$$

We conclude this note with an observation regarding canonical height zero characters in nilpotent blocks, based in part on some of the above methods.

Let $G$ be a finite group, $B$ a block algebra of $OG$, $P$ a defect group of $B$, and $i \in B^P$ a source idempotent of $B$. Denote by $\mathcal{F}$ the fusion system of $B$ on $P$ determined by the choice of $i$. Suppose that $K$ is a splitting field for all subgroups of $G$. For $V$ a finitely generated $O$-free $B$-module, denote by

$$\Delta_{V,P,i} : P \to O^\times$$

the map sending $u \in P$ to the determinant of the $O$-linear automorphism of $iV$ induced by the action of $u$ on $V$ (this makes sense since all elements in $P$ commute with $i$). By standard properties of determinants, this map depends only on the $(B^P)^\times$-conjugacy class of $i$ and the isomorphism class of the $K \otimes_O B$-module $K \otimes_O V$. Thus if $V$ affords a character $\chi \in \text{Irr}(B)$, we write $\Delta_{\chi,P,i}$ instead of $\Delta_{V,P,i}$.

**Proposition 10.** With the notation above, let $\chi \in \text{Irr}(B)$ and $\eta \in \text{Irr}(P/\text{foc}(P))$. Regard $\eta$ as a linear character of $P$. We have

$$\Delta_{\eta \ast \chi,P,i} = \eta^{\chi(i)} \Delta_{\chi,P,i}.$$

**Proof.** The statement makes sense as the value of $\chi$ on an idempotent is a positive integer. Let $V$ be an $O$-free $OG$-module affording $\chi$. By [8, Theorem 1.1] there exists an $O$-algebra automorphism $\alpha$ of $B$ such that the module $V^\alpha$ (obtained from twisting $V$ by $\alpha$) affords $\eta \ast \chi$ and such that $\alpha(ui) = \eta(u)ui$ for all $u \in P$. Since in particular $\alpha(i) = i$, it follows that

$$\Delta_{V^\alpha,P,i}(u) = \Delta_{V,P,i}(\eta(u)u)$$

for all $u \in P$. The result follows as $\text{rank}_O(iV) = \chi(i)$. \qed

Denote by $\text{Irr}'(B)$ the set of all $\chi \in \text{Irr}(B)$ such that $\Delta_{\chi,P,i}$ is the trivial map (sending all elements in $P$ to 1). Set $\text{Irr}'_0(B) = \text{Irr}'(B) \cap \text{Irr}_0(B)$. The maximal local pointed groups on $B$ are $G$-conjugate. Thus if $P'$ is any other defect group of $B$ and $i' \in B^{P'}$ a source idempotent, then there exist $g \in G$ and $c \in (B^{P'})^\times$ such that $P' = gPg^{-1}$ and $i' = cgcg^{-1}c^{-1}$. Therefore the map
Δ_{V,P,i} is trivial if and only if the map Δ_{V,P',i'} is trivial, and hence the sets Irr′(B) and Irr′₀(B) are independent of the choice of P and i. The following is immediate.

**Proposition 11.** The sets Irr′(B) and Irr′₀(B) are invariant under any automorphism of G which stabilises B.

The next result shows that if B is nilpotent, then Irr′₀(B) consists of a single element.

**Proposition 12.** Suppose that B is nilpotent. Then |Irr′₀(B)| = 1. Moreover, if p is odd, then the unique element of Irr′₀(B) is the unique p-rational height zero character in B.

**Proof.** Let χ ∈ Irr₀(B). Since i is a source idempotent of B, χ(i) is prime to p. Hence if η, ζ are linear characters of P, then ηχ(i) = ζχ(i) implies that η = ζ. Since B is nilpotent, we have that fuc(F) = [P, P] and |Irr₀(B)| = |P : [P, P]|. Thus, by Proposition 10, the map χ → Δχ,P,i is a bijection from Irr₀(B) to Irr(P/[P, P]). This proves the first assertion.

Suppose that p is odd. Let χ₀ be the unique p-rational character in Irr₀(B). Let W(k) be the ring of Witt vectors in O. By the structure theory of nilpotent blocks (see [14]), there exists a W(k)G-module V affording χ₀. Since the source idempotent i can be chosen to be in W(k)G, we have that Δχ,P,i takes values in W(k). Since p is odd, it follows that the trivial character of P is the unique linear character of P which takes values in W(k).

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