Some sufficient conditions for path-factor uniform graphs

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Abstract. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph $H$ of $G$ is called an $\mathcal{H}$-factor of $G$ if each component of $H$ is isomorphic to an element of $\mathcal{H}$. A graph $G$ is called an $\mathcal{H}$-factor uniform graph if for any two edges $e_1$ and $e_2$ of $G$, $G$ has an $\mathcal{H}$-factor covering $e_1$ and excluding $e_2$. Let each component in $\mathcal{H}$ be a graph with at least $d$ vertices, where $d \geq 2$ is an integer. Then an $\mathcal{H}$-factor and an $\mathcal{H}$-factor uniform graph are called a $P_{\geq d}$-factor and a $P_{\geq d}$-factor uniform graph, respectively. In this article, we verify that (i) a 2-edge-connected graph $G$ is a $P_{\geq 3}$-factor uniform graph if $\delta(G) > \frac{\alpha(G)+4}{2}$; (ii) a $(k+2)$-connected graph $G$ of order $n$ with $n \geq 5k+3 - \frac{3}{\gamma-1}$ is a $P_{\geq 3}$-factor uniform graph if $|N_G(A)| > \gamma(n-3k-2)+k+2$ for any independent set $A$ of $G$ with $|A| = \lfloor \gamma(2k+1) \rfloor$, where $k$ is a positive integer and $\gamma$ is a real number with $\frac{1}{3} \leq \gamma \leq 1$.

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1. Introduction

The graphs considered here are finite, undirected and simple. Let $G$ be a graph with edge set $E(G)$ and vertex set $V(G)$. We use $i(G)$, $\omega(G)$, $\alpha(G)$ and $\delta(G)$ to denote the number of isolated vertices, the number of connected components, the independence number and the minimum degree of $G$, respectively. Let $N_G(x)$ denote the set of neighbours of a vertex $x$ in $G$. By $d_G(x)$ we mean $|N_G(x)|$ and we call it the degree of a vertex $x$ in $G$. For any $X \subseteq V(G)$ or $X \subseteq E(G)$ the symbol $G[X]$ denotes the subgraph of $G$ induced by $X$. We write $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $G - X = G[V(G)\setminus X]$ for $X \subseteq V(G)$, and denote by $G - X$ the subgraph derived from $G$ by deleting edges of $X$ for $X \subseteq E(G)$. The edge joining vertices $x$ and $y$ is denoted by $xy$. A vertex subset $X$ of $G$ is called an independent set if $X \cap N_G(X) = \emptyset$. Let $P_n$ and $K_n$ denote the path and the complete graph with $n$ vertices, respectively. We
denote by $K_{m,n}$ the complete bipartite graph with the bipartition $(X,Y)$, where $|X| = m$ and $|Y| = n$. Let $G_1$ and $G_2$ be two graphs. By $G_1 \cup G_2$ we mean a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. By $G_1 \vee G_2$ we mean a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{e = xy : x \in V(G_1), y \in V(G_2)\}$. Recall that $\lfloor r \rfloor$ is the greatest integer with $|r| \leq r$, where $r$ is a real number.

A subgraph of $G$ is spanning if the subgraph includes all the vertices of $G$. For a set $\mathcal{H}$ of connected graphs, a spanning subgraph $H$ of $G$ is called an $\mathcal{H}$-factor of $G$ if each component of $H$ is isomorphic to an element of $\mathcal{H}$. A graph $G$ is called an $\mathcal{H}$-factor covered graph if $G$ admits an $\mathcal{H}$-factor covering $e$ for any $e \in E(G)$. A graph $G$ is called an $\mathcal{H}$-factor uniform graph if $G - e$ is an $\mathcal{H}$-factor covered graph for any $e \in E(G)$. Let each component in $\mathcal{H}$ be a path with at least $d$ vertices, where $d \geq 2$ is an integer. Then an $\mathcal{H}$-factor, an $\mathcal{H}$-factor covered graph and an $\mathcal{H}$-factor uniform graph are called a $P_{\geq d}$-factor, a $P_{\geq d}$-factor covered graph and a $P_{\geq d}$-factor uniform graph, respectively.

Amahashi and Kano [1] derived a characterization for a graph with a $\{K_{1,l} : 1 \leq l \leq m\}$-factor. Kano and Saito [11] posed a sufficient condition for the existence of $\{K_{1,l} : m \leq l \leq 2m\}$-factors in graphs. Kano, Lu and Yu [10] investigated the existence of $\{K_{1,2}, K_{1,3}, K_{5}\}$-factors and $P_{\geq 3}$-factors in graphs depending on the number of isolated vertices. Bazgan et al. [2] put forward a toughness condition for a graph to have a $P_{\geq 3}$-factor. Zhou, Bian and Pan [22], Zhou, Wu and Bian [28], Zhou, Wu and Xu [30], Wang and Zhang [13], Zhou [20] obtained some results on $P_{\geq 3}$-factors in graphs with given properties. Johansson [7] presented a sufficient condition for a graph to have a path-factor. Gao, Chen and Wang [4] showed an isolated toughness condition for the existence of $P_{\geq 3}$-factors in graphs with given properties. Kano, Lee and Suzuki [9] verified that each connected cubic bipartite graph with at least eight vertices admits a $P_{\geq 8}$-factor. Wang and Zhang [14], Zhou and Liu [23] presented some degree conditions for the existence of graph factors. Zhou, Wu and Liu [29], Zhou [21], Yuan and Hao [16] established some relationships between independence numbers and graph factors. Enomoto, Plummer and Saito [3], Zhou, Liu and Xu [25], Zhou [18,19], Zhou and Sun [26] derived some neighborhood conditions for the existence of graph factors. Some other results on graph factors can be found in Wang and Zhang [15], Zhou and Liu [24].

A graph $H$ is factor-critical if $H - x$ has a perfect matching for each $x \in V(H)$. To characterize a graph with a $P_{\geq 3}$-factor, Kaneko [8] introduced the concept of a sun. A sun is a graph formed from a factor-critical graph $H$ by adding $n$ new vertices $x_1, x_2, \ldots, x_n$ and $n$ new edges $y_1x_1, y_2x_2, \ldots, y_nx_n$, where $V(H) = \{y_1, y_2, \ldots, y_n\}$. According to Kaneko, $K_1$ and $K_2$ are also suns. A sun with at least six vertices is called a big sun. A component of $G$ is called a sun component if it is isomorphic to a sun. Let $sun(G)$ denote the
number of sun components of \( G \). Kaneko \[8\] put forward a criterion for a graph with a \( P_{\geq 3} \)-factor.

**Theorem 1.1.** \[8\] A graph \( G \) admits a \( P_{\geq 3} \)-factor if and only if

\[
\text{sun}(G - X) \leq 2|X|
\]

for all \( X \subseteq V(G) \).

Later, Zhou and Zhang \[17\] improved Theorem 1.1 and acquired a criterion for a \( P_{\geq 3} \)-factor covered graph.

**Theorem 1.2.** \[17\] Let \( G \) be a connected graph. Then \( G \) is a \( P_{\geq 3} \)-factor covered graph if and only if

\[
\text{sun}(G - X) \leq 2|X| - \varepsilon(X)
\]

for any vertex subset \( X \) of \( G \), where \( \varepsilon(X) \) is defined by

\[
\varepsilon(X) = \begin{cases} 
2, & \text{if } X \text{ is not an independent set;} \\
1, & \text{if } X \text{ is a nonempty independent set and } G - X \text{ has a non-sun component;} \\
0, & \text{otherwise.}
\end{cases}
\]

Zhou and Sun \[27\] got a binding number condition for the existence of \( P_{\geq 3} \)-factor uniform graphs. Gao and Wang \[5\], Liu \[12\] improved the above result on \( P_{\geq 3} \)-factor uniform graphs. Hua \[6\] investigated the relationship between isolated toughness and \( P_{\geq 3} \)-factor uniform graphs. It is natural and interesting to put forward some new sufficient conditions to guarantee that a graph is a \( P_{\geq 3} \)-factor uniform graph. In this article, we proceed to study \( P_{\geq 3} \)-factor uniform graphs and pose some new graphic parameter conditions for the existence of \( P_{\geq 3} \)-factor uniform graphs, which are shown in the following.

**Theorem 1.3.** Let \( G \) be a 2-edge-connected graph. If \( G \) satisfies

\[
\delta(G) > \frac{\alpha(G) + 4}{2},
\]

then \( G \) is a \( P_{\geq 3} \)-factor uniform graph.

**Theorem 1.4.** Let \( k \) be a positive integer and \( \gamma \) be a real number with \( \frac{1}{3} \leq \gamma \leq 1 \), and let \( G \) be a \((k + 2)\)-connected graph of order \( n \) with \( n \geq 5k + 3 - \frac{3}{\gamma - 1} \). If

\[
|N_G(A)| > \gamma(n - 3k - 2) + k + 2
\]

for any independent set \( A \) of \( G \) with \( |A| = \lfloor \gamma(2k + 1) \rfloor \), then \( G \) is a \( P_{\geq 3} \)-factor uniform graph.

The proofs of Theorems 1.3 and 1.4 will be given in Sections 2 and 3.
2. The proof of Theorem 1.3

Proof of Theorem 1.3. For any $e = xy \in E(G)$, let $G' = G - e$. To verify Theorem 1.3, we only need to prove that $G'$ is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that $G'$ is not a $P_{\geq 3}$-factor covered graph. Then it follows from Theorem 1.2 that

$$\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \quad (2.1)$$

for some vertex subset $X$ of $G'$.

**Claim 1.** $X \neq \emptyset$.

**Proof.** Assume that $X = \emptyset$. Then from (2.1) and $\varepsilon(X) = 0$ we have $\text{sun}(G') \geq 1$. On the other hand, since $G$ is 2-edge-connected, $G'$ is connected, which implies that $\omega(G') = 1$. Thus, we derive that $1 \leq \text{sun}(G') \leq \omega(G') = 1$, that is, $\text{sun}(G') = 1$. Note that $|V(G')| = |V(G)| \geq 3$ by $G$ being a 2-edge-connected graph. Hence, $G'$ is a big sun, which implies that there exist at least three vertices $x_1, x_2, x_3$ with $d_{G'}(x_i) = 1$, $i = 1, 2, 3$. Thus, there exists at least one vertex with degree 1 in $G$, which contradicts that $G$ is 2-edge-connected. Claim 1 is proved. □

**Claim 2.** $|X| \geq 2$.

**Proof.** Let $|X| \leq 1$. Combining this with Claim 1, we get $|X| = 1$.

If $G' - X$ admits a non-sun component, then $\varepsilon(X) = 1$ by the definition of $\varepsilon(X)$. According to (2.1) and $\varepsilon(X) = 1$, we obtain

$$\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 = 2|X| = 2. \quad (2.2)$$

Note that $G' - X$ includes a non-sun component. Combining this with (2.2), we get

$$\alpha(G') \geq \text{sun}(G' - X) + 1. \quad (2.3)$$

Since $G' = G - e$, we deduce $\alpha(G) \geq \alpha(G') - 2$. Then using (2.2) and (2.3), we infer

$$\alpha(G) \geq \alpha(G') - 2 \geq \text{sun}(G' - X) - 1 \geq 2 - 1 = 1. \quad (2.4)$$

By virtue of (2.2), $G' - X$ has at least two sun components, which implies that $G - X$ admits one vertex $v$ with $d_{G - X}(v) = 1$. Thus, we derive

$$\delta(G) \leq d_G(v) \leq d_{G - X}(v) + |X| = |X| + 1 = 2. \quad (2.5)$$

It follows from (2.4), (2.5) and $\delta(G) > \frac{\alpha(G) + 4}{2}$ that

$$2 \geq \delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{5}{2},$$

which is a contradiction.
If $G' - X$ does not admit a non-sun component, then $\varepsilon(X) = 0$ by the definition of $\varepsilon(X)$. By means of (2.1), $|X| = 1$ and $\varepsilon(X) = 0$, we get
\[ \alpha(G') \geq \text{sun}(G' - X) \geq 2|X| + 1 = 3. \tag{2.6} \]
From (2.6), we have
\[ \alpha(G) \geq \alpha(G') - 2 \geq 3 - 2 = 1. \tag{2.7} \]
Note that $\text{sun}(G - X) \geq \text{sun}(G' - X) - 2 \geq 3 - 2 = 1$ by (2.6), which implies that $G - X$ has at least one vertex $v$ with $d_{G-X}(v) \leq 1$. Thus, we infer
\[ \delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| \leq |X| + 1 = 2. \tag{2.8} \]
In terms of (2.7), (2.8) and $\delta(G) > \frac{\alpha(G)+4}{2}$, we derive
\[ 2 \geq \delta(G) > \frac{\alpha(G)+4}{2} \geq \frac{5}{2}, \]
which is a contradiction. This completes the proof of Claim 2. \hfill \Box

Suppose that there exist $a$ isolated vertices, $b$ $K_2$’s and $c$ big sun components $H_1, H_2, \ldots, H_c$, where $|V(H_i)| \geq 6$, in $G' - X$, and so
\[ \text{sun}(G' - X) = a + b + c. \tag{2.9} \]
It follows from (2.1), (2.9), $\varepsilon(X) \leq 2$ and Claim 2 that
\[ a + b + c = \text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1 \geq 3. \tag{2.10} \]
Claim 3. $\delta(G) \leq |X| + 1$.

Proof. If $a \neq 0$, then $d_{G'-X}(v) = 0$ for any $v \in V(aK_1)$. Note that $G' = G - e$. Thus, we infer $d_{G-X}(v) \leq 1$ for any $v \in V(aK_1)$, and so
\[ \delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| \leq |X| + 1. \]
If $a = 0$, then $b + c \neq 0$, which implies that $G' - X$ admits at least two vertices with degree 1, and so $G - X$ has at least one vertex $v$ with $d_{G-X}(v) = 1$. Thus, we obtain
\[ \delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| = |X| + 1. \]
This completes the proof of Claim 3. \hfill \Box

Next, we consider two cases by the value of $a + c$.

Case 1. $a + c = 0$.
In this case, $b \geq 3$ by (2.10).

Claim 4. $\alpha(G) \geq b$.

Proof. If $x \notin V(bK_2)$ or $y \notin V(bK_2)$, then we easily see that $\alpha(G) \geq b$. If $x \in V(bK_2)$ and $y \in V(bK_2)$, then $G - X$ has $(b-2)$ $K_2$’s and a $P_4$ component, and so we easily see that $\alpha(G) \geq (b-2) + 2 = b$. We have finished the proof of Claim 4. \hfill \Box
According to (2.10), \( a + c = 0 \), Claim 4 and \( \delta(G) > \frac{\alpha(G)+1}{2} \), we deduce
\[
\delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{b + 4}{2} = \frac{a + b + c + 4}{2} \\
\geq \frac{2|X| - 1 + 4}{2} = \frac{2|X| + 3}{2} > |X| + 1,
\]
which contradicts Claim 3.

**Case 2.** \( a + c \neq 0 \).

**Subcase 2.1.** \( a \neq 0 \).

If \( x \notin V(aK_1) \) and \( y \notin V(aK_1) \), then \( d_{G-X}(v) = 0 \) for any \( v \in V(aK_1) \). Thus, we derive
\[
\delta(G) \leq d_G(v) \leq d_{G-X}(v) + |X| = |X|.
\]
(2.11)

It follows from (2.10), (2.11), \( \delta(G) > \frac{\alpha(G)+4}{2} \) and \( \alpha(G) \geq \text{sun}(G - X) \geq \text{sun}(G' - X) - 2 \) that
\[
|X| \geq \delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{\text{sun}(G' - X) - 2 + 4}{2} = \frac{\text{sun}(G' - X) + 2}{2}
\]
\[
\geq \frac{2|X| - 1 + 2}{2} = \frac{|X| + 1}{2},
\]
which is a contradiction. In what follows, we discuss the case with \( x \in V(aK_1) \) or \( y \in V(aK_1) \). Without loss of generality, let \( x \in V(aK_1) \). We write \( Y = V(H_1) \cup \cdots \cup V(H_c) \).

**Subcase 2.1.1.** \( y \in V(bK_2) \cup Y \).

In this subcase, we deduce \( \alpha(G) \geq a + b + c \). Combining this with (2.10) and \( \delta(G) > \frac{\alpha(G)+4}{2} \), we infer
\[
\delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{a + b + c + 4}{2} \geq \frac{2|X| - 1 + 4}{2} = \frac{2|X| + 3}{2} > |X| + 1,
\]
which contradicts Claim 3.

**Subcase 2.1.2.** \( y \in V(G) \setminus (V(bK_2) \cup Y) \).

In this subcase, we have \( \text{sun}(G - X) \geq \text{sun}(G' - X) - 1 \). Combining this with (2.10), \( \alpha(G) \geq \text{sun}(G - X) \) and \( \delta(G) > \frac{\alpha(G)+4}{2} \), we derive
\[
\delta(G) > \frac{\alpha(G) + 4}{2} \geq \frac{\text{sun}(G - X) + 4}{2} \geq \frac{\text{sun}(G' - X) - 1 + 4}{2}
\]
\[
= \frac{\text{sun}(G' - X) + 3}{2} \geq \frac{2|X| - 1 + 3}{2} = |X| + 1,
\]
which contradicts Claim 3.

**Subcase 2.2.** \( c \neq 0 \).

Obviously, \( \alpha(G') \geq a+b+\sum_{i=1}^{c} \frac{|V(H_i)|}{2} \geq a+b+3c \) by \( |V(H_i)| \geq 6 \). Combining this with (2.10), \( c \neq 0 \) and \( \alpha(G) \geq \alpha(G') - 2 \), we obtain
\[
\alpha(G) \geq \alpha(G') - 2 \geq a + b + 3c - 2 \geq a + b + c \geq 2|X| - 1. \quad (2.12)
\]
By virtue of (2.12), Claim 3 and $\delta(G) > \frac{\alpha(G)+4}{2}$, we deduce
$$|X| + 1 \geq \delta(G) > \frac{\alpha(G)+4}{2} \geq \frac{2|X| - 1 + 4}{2} = |X| + \frac{3}{2},$$
which is a contradiction. This completes the proof of Theorem 1.3. $\square$

3. The proof of Theorem 1.4

Proof of Theorem 1.4. For any $e \in E(G)$, we write $G' = G - e$. To prove Theorem 1.4, we only need to justify that $G'$ is a $P_{\geq 3}$-factor covered graph. Suppose, to the contrary, that $G'$ is not a $P_{\geq 3}$-factor covered graph. Then by Theorem 1.2, we have
$$\text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \quad (3.1)$$
for some vertex subset $X$ of $G'$. We write $a = i(G - X)$ and $b = \lceil \gamma(2k + 1) \rceil$.

Claim 1. $b \geq a + 1$.

Proof. Let $b \leq a$. We may choose $b$ isolated vertices $x_1, x_2, \ldots, x_b$ in $G - X$. Write $A = \{x_1, x_2, \ldots, x_b\}$. Then $A$ is an independent set of $G$. Thus, we infer
$$\gamma(n - 3k - 2) + k + 2 < |N_G(A)| \leq |X|. \quad (3.2)$$

It follows from (3.1), (3.2) and $\varepsilon(X) \leq 2$, $\frac{1}{3} \leq \gamma \leq 1$ and $n \geq 5k + 3 - \frac{3}{5\gamma - 1}$ that
$$0 \geq |X| + \text{sun}(G' - X) - n \geq |X| + 2|X| - \varepsilon(X) + 1 - n$$
$$\geq 3|X| - n - 1 > 3(\gamma(n - 3k - 2) + k + 2) - n - 1$$
$$= (3\gamma - 1)n - 3\gamma(3k + 2) + 3k + 5$$
$$\geq (3\gamma - 1) \left(5k + 3 - \frac{3}{5\gamma - 1}\right) - 3\gamma(3k + 2) + 3k + 5$$
$$= (3\gamma - 1)(2k + 1) - \frac{3(3\gamma - 1)}{5\gamma - 1} + 3$$
$$\geq 3 - \frac{3(3\gamma - 1)}{5\gamma - 1} > 3 - 3 = 0,$$
which is a contradiction. We have finished the proof of Claim 1. $\square$

In what follows, we consider four cases by the value of $|X|$ and derive a contradiction in each case.

Case 1. $|X| = 0$.

Note that $G' = G - e$ and $G$ is $(k + 2)$-connected. Hence, $G'$ is $(k + 1)$-connected and $\omega(G') = 1$. Combining this with (3.1) and $\varepsilon(X) = 0$, we obtain
$$1 = \omega(G') \geq \text{sun}(G') \geq 1.$$ Thus, we have $\text{sun}(G') = \omega(G') = 1$. Then using $n \geq 5k + 3 - \frac{3}{5\gamma - 1} \geq 8 - \frac{3}{5x_{\frac{1}{2}}} = \frac{7}{2} > 3$, we see that $G'$ is a big sun, and
so \( G' \) has at least three vertices with degree 1, which contradicts that \( G' \) is a \((k + 1)\)-connected graph.

**Case 2.** \( 1 \leq |X| \leq k \).

Note that \( 1 \leq |X| \leq k \) and \( G' \) is \((k + 1)\)-connected. We derive \( \omega(G' - X) = 1 \). According to (3.1) and \( \varepsilon(X) \leq |X| \), we get

\[
1 = \omega(G' - X) \geq \text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq |X| + 1 \geq 2,
\]

which is a contradiction.

**Case 3.** \( |X| = k + 1 \).

Since \( G \) is \((k + 2)\)-connected, \( G - X \) is connected, and so \( \omega(G - X) = 1 \). Note that \( G' = G - e \). Thus, we deduce

\[
\omega(G' - X) \leq \omega(G - X) + 1 = 2.
\]

By virtue of (3.1), (3.3), \( k \geq 1 \) and \( \varepsilon(X) \leq 2 \), we infer

\[
2 \geq \omega(G' - X) \geq \text{sun}(G' - X) \geq 2|X| - \varepsilon(X) + 1 \geq 2|X| - 1
\]

\[
= 2(k + 1) - 1 = 2k + 1 \geq 3,
\]

which is a contradiction.

**Case 4.** \( |X| \geq k + 2 \).

In light of (3.1), \( \varepsilon(X) \leq 2 \) and \( \frac{1}{3} \leq \gamma \leq 1 \), we derive

\[
\text{sun}(G - X) \geq \text{sun}(G' - X) - 2 \geq 2|X| - \varepsilon(X) + 1 - 2 \geq 2|X| - 3
\]

\[
\geq 2(k + 2) - 3 = 2k + 1 \geq \gamma(2k + 1) \geq \lfloor \gamma(2k + 1) \rfloor = b,
\]

which implies that \( G - X \) admits an independent set of order at least \( b \). Then using Claim 1, we may choose \( a \) isolated vertices \( x_1, x_2, \ldots, x_a \) and \( (b - a) \) nonadjacent vertices \( x_{a+1}, \ldots, x_b \) with \( d_{G - X}(x_i) = 1 \) for \( a + 1 \leq i \leq b \), in \( G - X \). Set \( A = \{x_1, x_2, \ldots, x_a, x_{a+1}, \ldots, x_b\} \). Then \( A \) is an independent set of \( G \). Thus, we deduce

\[
\gamma(n - 3k - 2) + k + 2 < |N_G(A)| \leq |X| + b - a,
\]

that is,

\[
|X| > \gamma(n - 3k - 2) + k + 2 - b + a.
\]

It follows from (3.1), (3.4), \( \varepsilon(X) \leq 2 \) and \( n \geq 5k + 3 - \frac{3}{5\gamma - 1} \) that

\[
0 \geq |X| + 2\text{sun}(G' - X) - i(G' - X) - n
\]
\[
\geq |X| + 2(2|X| - \varepsilon(X) + 1) - (i(G - X) + 2) - n
\]
\[
\geq |X| + 2(2|X| - 1) - (a + 2) - n
\]
\[
= 5|X| - a - 4 - n
\]
\[
> 5(\gamma(n - 3k - 2) + k + 2 - b + a) - a - 4 - n
\]
which is a contradiction. This completes the proof of Theorem 1.4. □

4. Remarks

Remark 1. Next, we show that the condition \( \delta(G) > \frac{\alpha(G)+4}{2} \) in Theorem 1.3 cannot be replaced by \( \delta(G) \geq \frac{\alpha(G)+4}{2} \). We construct a graph \( G = K_{3+t} \cup (4+2t)K_2 \), where \( t \) is a nonnegative integer. Then \( G \) is \((3+t)\)-connected, \( \delta(G) = 4 + t \) and \( \alpha(G) = 4 + 2t \). Thus, we have \( \delta(G) = \frac{\alpha(G)+4}{2} \). For any \( e \in E((4+2t)K_2) \), let \( G' = G - e = K_{3+t} \cup ((3+2t)K_2 \cup (2K_1)) \). Select \( X = V(K_{3+t}) \subseteq V(G') \). Then \( |X| = 3 + t \) and \( \varepsilon(X) = 2 \). Thus, we derive

\[
\text{sun}(G' - X) = 5 + 2t > 4 + 2t = 2(3 + t) - 2 = 2|X| - \varepsilon(X).
\]

By Theorem 1.2, \( G' \) is not a \( P_{\geq 3}\)-factor covered graph, and so \( G \) is not a \( P_{\geq 3}\)-factor uniform graph.

Remark 2. The conditions with a \((k+2)\)-connected graph and \( |N_G(A)| > \gamma(n-3k-2)+k+2 \) in Theorem 1.4 cannot be replaced by a \((k+1)\)-connected graph and \( |N_G(A)| \geq \gamma(n-3k-2)+k+1 \). Let \( \gamma \) be a rational number such that \( \frac{1}{3} \leq \gamma \leq 1 \). Then we can write \( \gamma = \frac{b}{2k+1} \) for nonnegative integers \( b \) and \( k \). Let \( G = K_{k+1} \cup ((2k+1)K_2) \), where \( k \) is a positive integer. Then \( G \) is \((k+1)\)-connected and \( n = |V(G)| = 5k + 3 > 5k + 3 - \frac{3}{5\gamma-1} \). If \( A \) is an independent set of order \( b = \gamma(2k+1) \), then

\[
\gamma(n-3k-2)+k+2 > |N_G(A)| = \gamma(2k+1) + k + 1 = \gamma(n-3k-2)+k+1.
\]

For any \( e \in E((2k+1)K_2) \), let \( G' = G - e = K_{k+1} \cup ((2k)K_2 \cup (2K_1)) \). Select \( X = V(K_{k+1}) \subseteq V(G') \). Then \( |X| = k + 1 \) and \( \varepsilon(X) = 2 \). Thus, we infer

\[
\text{sun}(G' - X) = 2k + 2 > 2k = 2(k+1) - 2 = 2|X| - \varepsilon(X).
\]

According to Theorem 1.2, \( G' \) is not a \( P_{\geq 3}\)-factor covered graph, and so \( G \) is not a \( P_{\geq 3}\)-factor uniform graph.

5. Conclusion

The concept of path-factor uniform graph was first presented by Zhou and Sun [27], and they showed a binding number condition for the existence of
$P_{\geq 3}$-factor uniform graphs. Gao and Wang [5], Liu [12] improved Zhou and Sun’s above result. Hua [6] gave toughness and isolated toughness conditions for graphs to be $P_{\geq 3}$-factor uniform graphs. In our article, we study the relationships between some graphic parameters (for instance, minimum degree, independence number and neighborhood, and so on) and the existence of $P_{\geq 3}$-factor uniform graphs. The theorems derived in this article belong to existence theorems, that is, under what kind of conditions the path-factor uniform graph exists. However, in a specific computer network, it needs to use a certain algorithm to determine the values of some graphic parameters of the fix network graph and show the eligible path-factor uniform graph from the algorithm point of view. The problems of such algorithms are worthy of consideration in future research.

So far, results on the existence of path-factor uniform graphs are very few. There are many problems on graphs which can be considered for path-factor uniform graphs. For example, we can consider the structures and properties of path-factor uniform graphs. In what follows, we put forward open problems as the end of our article.

**Problem 1.** Find the necessary and sufficient conditions for a graph to be a path-factor uniform graph.

**Problem 2.** Find relationships between other graphic parameters and path-factor uniform graphs.

**Problem 3.** What are the structures and properties in path-factor uniform graphs?

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