EXTREMAL CASES OF EXACTNESS CONSTANTS AND COMPLETELY BOUNDED PROJECTION CONSTANTS

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Abstract. We investigate some extremal cases of exactness constants and completely bounded projection constants. More precisely, for an $n$-dimensional operator space $E$ we prove that $\lambda_{cb}(E) = \sqrt{n}$ if and only if $\text{ex}(E) = \sqrt{n}$.

1. Introduction

Exactness constants and completely bounded (c.b.) projection constants are fundamental quantities in operator space theory.

For an operator space $E \subseteq B(H)$, the c.b. projection constant of $E$, $\lambda_{cb}(E)$, is defined by

$$\lambda_{cb}(E) = \inf \{ \| P \|_{cb} \mid P : B(H) \to E, \text{ projection onto } E \}.$$ 

Let $B = B(\ell_2)$ and $K$ be the ideal of all compact operators on $\ell_2$, and let

$$T_E : (B \otimes_{\min} E) / (K \otimes_{\min} E) \to (B/K) \otimes_{\min} E$$

be the map obtained from

$$q \otimes I_E : B \otimes_{\min} E \to (B/K) \otimes_{\min} E$$

by the taking quotient with respect to $K \otimes_{\min} E$, where $q : B \to K$ is the canonical quotient map. Then the exactness constant of $E$, $\text{ex}(E)$ is defined by

$$\text{ex}(E) = \| T_E^{-1} \|.$$ 

It is well known that the exactness constant is the same as $d_{SK}(E)$, where

$$d_{SK}(E) = \inf \{ d_{cb}(E, F) : F \subseteq K \},$$

when $E$ is finite dimensional ([9]).

The followings are well known facts about these quantities (Chapter 7 and 17 of [12] and Section 9 of [10]):

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**Fact 1.** For a finite dimensional operator space $E$ we have
\[ \text{ex}(E) = \text{dsk}(E) \leq \lambda_{cb}(E). \]

**Fact 2.** When $\dim(E) = n \in \mathbb{N}$, we have
\[ \lambda_{cb}(E) \leq \sqrt{n}. \]

Thus, for an $n$-dimensional operator space $E$, $\lambda_{cb}(E)$ and $\text{ex}(E)$ are both bounded by $\sqrt{n}$, and this upper bound is known to be asymptotically sharp. Indeed, we have $\text{ex}(\max \ell_n^1) \geq \frac{n}{2\sqrt{n-1}}$ for $n \geq 2$ ([9]). However, it is not yet known whether there is an $n$-dimensional operator space $E$ with $\lambda_{cb}(E) = \sqrt{n}$ or $\text{ex}(E) = \sqrt{n}$. In this paper we investigate the extremal cases $\lambda_{cb}(E) = \sqrt{n}$ and $\text{ex}(E) = \sqrt{n}$ and prove the following theorem.

**Theorem 1.** Let $n \geq 2$ and $E \subseteq B(H)$ be an $n$-dimensional operator space. Then we have $\lambda_{cb}(E) = \sqrt{n}$ if and only if $\text{ex}(E) = \sqrt{n}$. Equivalently, $\lambda_{cb}(E) < \sqrt{n}$ if and only if $\text{ex}(E) < \sqrt{n}$.

$\lambda_{cb}(E)$ is the operator space analogue of the projection constant $\lambda(X)$ of a Banach space $X$ given by $\lambda(X) = \sup \{ \lambda(X, Y) \mid X \subseteq Y \}$, where
\[ \lambda(X, Y) = \inf \{ \|P\| \mid P : Y \to Y \text{ projection onto } X \}. \]

See [4], [5], and [6] for more information on the Banach space case and [3] and [13] for the operator space case.

Throughout this paper, we assume that the reader is familiar with the standard results about operator spaces ([2], [10]), completely nuclear maps ([2]), and completely $p$-summing maps ([11]). For a linear map $T : E \to F$ between operator spaces and $1 \leq p < \infty$ we denote the completely nuclear norm and the completely $p$-summing norm of $T$ by $\nu^p(T)$ and $\pi^p_0(T)$, respectively.

For an index set $I$, $OH(I)$ denotes the operator Hilbert space on $\ell_2(I)$, which was introduced in [10]. When $I = \{ 1, \ldots, n \}$ for $n \in \mathbb{N}$, we simply write $OH_n$. For a family of operator spaces $(E_i)_{i \in I}$ and an ultrafilter $\mathcal{U}$ on $I$ we denote the ultraproduct of $(E_i)_{i \in I}$ with respect to $\mathcal{U}$ by $\prod_{\mathcal{U}} E_i$.

**2. Proof of the main result**

For the proof we need several lemmas. The first lemma is about the relationship between completely 1-summing maps and completely 2-summing maps.

**Lemma 2.** Let $v : E \to F$ be a completely 1-summing map. Then $v$ is completely 2-summing with $\pi^2_0(v) \leq \pi^1_0(v)$. 
Proof. Let $E \subseteq B(H)$ for some Hilbert space $H$. Then by Remark 5.7 of [11] we have an ultrafilter $\mathcal{U}$ over an index set $I$ and families of positive operators $(a_\alpha)_{\alpha \in I}, (b_\alpha)_{\alpha \in I}$, in the unit ball of $S_2(H)$ such that the following diagram commutes for some $u$ with $\|u\|_cb \leq \pi_1^u(v)$:

\[
\begin{array}{ccc}
E & \overset{u}{\longrightarrow} & F \\
\downarrow i & & \downarrow i \\
E_\infty & \overset{\mathcal{M}}{\longrightarrow} & E_1
\end{array}
\]

(2.1)

where $E_\infty = i(E)$ for the complete isometry

\[i : B(H) \hookrightarrow \prod_I B(H), x \mapsto (x)_{\alpha \in I},\]

$E_1 = \overline{Mi(E)}$ (the closure in $\prod_I S_1(H)$) for

\[M : \prod_I B(H) \to \prod_I S_1(H), (x_\alpha) \mapsto (a_\alpha x_\alpha b_\alpha),\]

and $\mathcal{M} = M|_{E_\infty}$.

Next, we split $M$ into $M = T_2 T_1$, where

\[T_1 : \prod_I B(H) \to \prod_I S_2(H), (x_\alpha) \mapsto (a_\alpha^{1/2} x_\alpha b_\alpha^{1/2})\]

and

\[T_2 : \prod_I S_2(H) \to \prod_I S_1(H), (x_\alpha) \mapsto (a_\alpha^{1/2} x_\alpha b_\alpha^{1/2}).\]

Note that

\[\|T_2\|_cb \leq \lim_{\mathcal{U}} \left\| M_\alpha : S_2(H) \to S_1(H), x \mapsto a_\alpha^{1/2} x b_\alpha^{1/2} \right\|_cb \leq 1,\]

since $M_\alpha^* = N_\alpha$ for

\[N_\alpha : B(H) \to S_2(H), x \mapsto a_\alpha^{1/2} x b_\alpha^{1/2}\]

and $\|N_\alpha\|_cb \leq 1$. Thus we have by Theorem 5.1 of [11] that

\[\|vx_{ij}\|_{M_n(F)} = \|(uT_2 T_1 i x_{ij})\|_{M_n(F)} \leq \pi_1^u(v) \| (T_2 T_1 i x_{ij})\|_{M_n(S_2(H))} \leq \pi_1^u(v) \| (a_\alpha^{1/2} x_{ij} b_\alpha^{1/2})\|_{M_n(S_2(H))} = \pi_2^u(v) \| (a_\alpha^{1/2} x_{ij} b_\alpha^{1/2})\|_{M_n(S_2(H))}
\]

for any $n \in \mathbb{N}$ and $(x_{ij}) \in M_n(F)$, which implies $\pi_2^u(v) \leq \pi_1^u(v)$. \qed

The second lemma is about the trace duality of completely 2-summing norms.

**Lemma 3.** Let $E$ and $F$ be operator spaces and $E$ be finite dimensional. Then for $v : F \to E$ we have

\[(\pi_2^u)^*(v) := \sup \{ |\text{tr}(vu)| \mid \pi_2^u(u : E \to F) \leq 1 \} = \pi_2^u(v).
\]
Proof. See Lemma 4.7 of [7]. □

The final lemma is about the relationship between the trace norm and the completely nuclear norm of a linear map on an operator space and the operator space approximation property.

**Lemma 4.** Let $E$ be an operator space with the operator space approximation property. Then for any completely nuclear map $u : E \to E$ we can define $\text{tr}(u)$, the trace of $u$, and we have

$$|\text{tr}(u)| \leq \nu^o(u).$$

**Proof.** Since $E$ has the operator space approximation property, the canonical mapping

$$\Phi : E \hat{\otimes} E^* \to E \otimes_{\min} E^*$$

is one-to-one by Theorem 11.2.5 of [2], where $\hat{\otimes}$ (resp. $\otimes_{\min}$) is the projective (resp. injective) tensor product in the category of operator spaces. Thus, $\mathcal{N}^o(E)$, the set of all completely nuclear maps on $E$, can be identified with $E \hat{\otimes} E^*$ with the same norm. Since we have the trace functional defined on $E \hat{\otimes} E^*$ (7.1.12 of [2]), we can translate it to $\mathcal{N}^o(E)$, so that we have

$$|\text{tr}(u)| \leq \|U\|_{E \hat{\otimes} E^*} = \nu^o(u),$$

where $U \in E \hat{\otimes} E^*$ is the element associated to $u \in \mathcal{N}^o(E)$. □

Let $E$ and $F$ be operator spaces. Then the $\Gamma_\infty$-norm and the $\gamma_\infty$-norm of a linear map $v : E \to F$ are defined by

$$\Gamma_\infty(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},$$

where the infimum is taken over all Hilbert spaces $H$ and all factorizations

$$i_Fv : E \overset{\alpha}{\to} B(H) \overset{\beta}{\to} F,$$

where $i_F : F \hookrightarrow F^{**}$ is the canonical embedding, and

$$\gamma_\infty(v) = \inf \|\alpha\|_{cb} \|\beta\|_{cb},$$

where the infimum is taken over all $m \in \mathbb{N}$ and all factorizations

$$v : E \overset{\alpha}{\to} M_m \overset{\beta}{\to} F.$$

See Section 4 of [3] or [1] for the details.

Now we are ready to prove our main result. The proof follows the classical idea of [4].

**Proof of Theorem 1.** By Fact 1 and Fact 2 it is enough to show that the condition $\lambda_{cb}(E) = \sqrt{n}$ is inconsistent with the condition $\text{ex}(E) = d_{SK}(E) < \sqrt{n}$. 

Step 1. $\pi_1^0(I_E) = \sqrt{n}$.

By trace duality and Lemma 4.1 and 4.2 of [3] (or see Theorem 7.6 of [1]) we have

$$\lambda_{cb}(E) = \Gamma_\infty(I_E) = \gamma_\infty(I_E) = \sup_{u \in \pi_1^0(E)} \frac{\|\text{tr}(u)\|}{\pi_1^0(u)}.$$

Since $E$ is finite dimensional, we can find $u \in CB(E)$ such that

$$\|\text{tr}(u)\| = \sqrt{n},$$

and by multiplying by a suitable constant we can also assume that $\pi_2^0(u) = \sqrt{n}$. Then, by Lemma 2, Lemma 3, and Theorem 6.13 of [11], we obtain

$$n = \sqrt{n} \pi_2^0(u) \leq \sqrt{n} \pi_1^0(u) = |\text{tr}(u)| \leq \pi_2^0(u) \pi_2^0(I_E) = n.$$

Thus, we get

$$\pi_1^0(u) = \sqrt{n} \text{ and } |\text{tr}(u)| = n.$$

Next, we show that $u$ is actually $I_E$. By Proposition 6.1 of [11] we have the factorization

$$u : E \xrightarrow{A} OH_n \xrightarrow{B} E \text{ with } \pi_2^0(A) \|B\|_{cb} \leq \sqrt{n}.$$

If we let $v : OH_n \to OH_n$ be defined by $v = AB$, we have $\text{tr}(v) = \text{tr}(v^*) = \text{tr}(u)$ and

$$\|I_{OH_n} - v\|_{HS}^2 = \text{tr}((I_{OH_n} - v)(I_{OH_n} - v)^*) = \text{tr}(I_{OH_n}) - 2 \text{tr}(u) + \text{tr}(v v^*) = n - 2n + \|v\|_{HS}^2 = (\pi_2^0(v))^2 - n \leq (\pi_2^0(A) \|B\|_{cb})^2 - n \leq 0,$$

which leads to the desired conclusion.

Step 2. Now we factorize $I_E$ as in the proof of Lemma 2. Then we have an ultrafilter $\mathcal{U}$, families of positive operators $(a_\alpha)_{\alpha \in I}$, $(b_\alpha)_{\alpha \in I}$, in the unit ball of $S_2(H)$, such that the diagram (2.1) commutes for some $u$ with

$$\|u\|_{cb} \leq \pi_1^0(I_E) = \sqrt{n}.$$

Then we can find a rank $n$ projection

$$w_1 : i(B(H)) \to i(B(H)) \text{ onto } E_\infty \text{ with } \pi_1^0(w_1) \leq \sqrt{n}.$$

Consider $i u : E_1 \to i(B(H))$. Since $i$ is a complete isometry, $i(B(H))$ is injective in the operator space sense, so that we can extend $i u$ to

$$\tilde{u} : \prod_{\mathcal{U}} S_1(H) \to i(B(H)) \text{ with } \|\tilde{u}\|_{cb} = \|i u\|_{cb}.$$

Now consider the same factorization $M = T_2 T_1$ as before. Note that

$$\pi_2^0(T_1) \leq 1 \text{ and } \|T_2\|_{cb} \leq 1.$$
by the same calculation as the proof for (5.8) of [11] and (2.2), respectively. Then for
\[ w := T_1 \tilde{u} T_2 : \prod_{i} S_2(H) \to \prod_{i} S_2(H) \]
we have
\[ \|w\|_{HS} = \pi_2^o(w) \leq \pi_2^o(T_1) \|\tilde{u}\|_{cb} \|T_2\|_{cb} \leq \pi_2^o(T_1) \|u\|_{cb} \leq \sqrt{n}. \]

Since \( T_1 i \) is 1-1, \( F := T_1 i(E) \) is n-dimensional. Furthermore, since
\[ w T_1 ix = T_1 \tilde{u} T_2 T_1 ix = T_1 i u M ix = T_1 ix \]
for all \( x \in E \), we have \( w|_F = I_F \), which means that \( |\lambda_k(w)| \geq 1 \) for \( 1 \leq k \leq n \),
where \( (\lambda_k(w))_{k \geq 1} \) is the sequence of eigenvalues of \( w \), in non-increasing order and counted according to multiplicity. By applying Weyl's inequality (Lemma 3.5.4 of [8]) and (2.3), we get
\[ n \leq \sum_{k=1}^{n} |\lambda_k(w)|^2 \leq \sum_{k=1}^{\infty} s_k(w)^2 = \|w\|_{HS}^2 \leq n, \]
where \( (s_k(w))_{k \geq 1} \) is the sequence of singular values of \( w \). Then we have
\[ |\lambda_k(w)| = \begin{cases} 1 & \text{if } 1 \leq k \leq n, \\ 0 & \text{if } k > n, \end{cases} \]
which implies that \( w \) has rank at most \( n \), as does
\[ w_1 := \tilde{u} M = \tilde{u} T_2 T_1 |_{i(B(H))} : i(B(H)) \to i(B(H)). \]
Actually, \( w_1 \) is our desired rank \( n \) projection. Indeed, we have
\[ w_1 ix = \tilde{u} M ix = i u M ix = ix \]
for all \( x \in E \), and since \( E_\infty \) is \( n \)-dimensional, \( w_1 \) maps onto \( E_\infty \). Moreover, we have
\[ \pi^o(w_1) \leq \|\tilde{u}\|_{cb} \pi^o_1(M) \leq \sqrt{n} \]
since \( \pi^o_1(M) \leq 1 \) \((5.7) \) of [11]).

**Step 3.** Since \( d_{SK}(E_\infty) = d_{SK}(E) < \sqrt{n} \), we have \( F \in K \) and an isomorphism
\[ T : E_\infty \to F \text{ with } \|T\|_{cb} \|T^{-1}\|_{cb} < \sqrt{n}. \]
By the fundamental extension theorem (Theorem 1.6 of [12]) we have extensions
\[ \overline{T} : i(B(H)) \to B(\ell_2) \text{ and } \overline{T}^{-1} : B(\ell_2) \to i(B(H)) \]
of \( T \) and \( T^{-1} \), respectively, with
\[ \|\overline{T}\|_{cb} = \|T\|_{cb} \text{ and } \|\overline{T}^{-1}\|_{cb} = \|T^{-1}\|_{cb}. \]
Let \( \tilde{w}_1 = \overline{T} w_1 \overline{T}^{-1} : B(\ell_2) \to B(\ell_2) \). Then clearly we have \( \text{ran}(\tilde{w}_1) \subseteq F \) and \( \tilde{w}_1|_F = I_F \), which means that \( \tilde{w}_1 \) is also a rank \( n \) projection from \( B(\ell_2) \) onto
Since $F \subseteq K$ and $K$ satisfies the operator space approximation property, we have by Lemma 4 and Corollary 15.5.4 of [2] that
\[ n = |\text{tr}(\tilde{w}_1|_K : K \to K)| \leq \nu_1(\tilde{w}_1|_K : K \to K) = \pi_1^q(\tilde{w}_1|_K : K \to K) \]
\[ = \pi_1^q(\tilde{w}_1|_K : K \to B(\ell_2)) \leq \|T\|_{cb} \|T^{-1}\|_{cb} \pi_1^q(w_1) \]
\[ \leq \|T\|_{cb} \|T^{-1}\|_{cb} \sqrt{n} < n, \]
This is a contradiction. \(\square\)

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