ON PROJECTIVE 4-FOLDS OF GENERAL TYPE WITH THE GEOMETRIC GENUS GREATER THAN 1

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Abstract. We show that, for nonsingular projective 4-folds $V$ of general type with the geometric genus $p_g \geq 2$, $\varphi_{33}$ is birational onto the image and the canonical volume $\text{Vol}(V)$ has the lower bound $\frac{1}{620}$.

1. Introduction

Studying the behavior of pluricanonical maps of projective varieties has been one of the fundamental tasks in birational geometry. For varieties of general type, an interesting and critical problem is to find a positive integer $m$ so that $\varphi_m$ is birational onto the image. A momentous theorem given by Hacon-McKernan [11], Takayama [17] and Tsuji [18] says that there is some constant $r_n$ (for any integer $n > 0$) such that the pluricanonical map $\varphi_m$ is birational onto its image for all $m \geq r_n$ and for all minimal projective $n$-folds of general type. By using the birationality principle (see, for example, Theorem 2.2), an explicit upper bound of $r_n$ is determined by that of $r_n$. Therefore, finding the explicit constant $r_n$ for smaller $n$ is the upcoming problem. However, $r_n$ is known only for $n \leq 3$, namely, $r_1 = 3$, $r_2 = 5$ by Bombieri [2] and $r_3 \leq 57$ by Chen-Chen [4, 5, 6] and the first author [9].

The first partial result concerning the explicit bound of $r_4$ might be [6, Theorem 1.11] by Chen and the first author that $\varphi_{35}$ is birational for all nonsingular projective 4-folds of general type with $p_g \geq 2$. It is mysterious whether the numerical bound “35” is optimal under the same assumption that $p_g \geq 2$.

In this paper, we go on studying this question and prove the following theorem:

**Theorem 1.1.** Let $V$ be a nonsingular projective 4-fold of general type with $p_g(V) \geq 2$. Then

1. $\text{Vol}(V) \geq \frac{1}{620}$;
2. $\Phi_{m,V}$ is birational for all $m \geq 33$.

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Remark 1.2. As being pointed out by Brown and Kasprzyk [3], the requirement on $p_g$ in Theorem 1.1(2) is sharp from the following list of canonical fourfolds:

1. $X_{78} \subset \mathbb{P}(39, 14, 9, 8, 6, 1), K^4 = 1/3024$;
2. $X_{78} \subset \mathbb{P}(39, 13, 10, 8, 6, 1), K^4 = 1/3120$;
3. $X_{72} \subset \mathbb{P}(36, 11, 9, 8, 6, 1), K^4 = 1/2376$;
4. $X_{70} \subset \mathbb{P}(35, 14, 10, 6, 3, 1), K^4 = 1/1260$;
5. $X_{70} \subset \mathbb{P}(35, 1, 4, 10, 5, 4, 1), K^4 = 1/1400$;
6. $X_{68} \subset \mathbb{P}(34, 12, 8, 7, 5, 1), K^4 = 1/1680$.

Besides, the canonical 4-fold $X_{36} \subset \mathbb{P}(12, 9, 7, 5, 1, 1)$ has invariants:

- $p_g = 2$ and $K^4 = 1/105$.

Throughout all varieties are defined over a field $k$ of characteristic 0.

We will frequently use the following symbols:

- ‘$\sim$’ denotes linear equivalence or $\mathbb{Q}$-linear equivalence;
- ‘$\cong$’ denotes numerical equivalence;
- ‘$|A| \gg |B|$’ or, equivalently, ‘$|B| \ll |A|$’ means $|A| \supseteq |B|$ + fixed effective divisors.

2. Preliminaries

Let $V$ be a nonsingular projective 4-fold of general type. By the minimal model program (see, for instance [1, 14, 15, 16]), we can find a minimal model $Y$ of $V$ with at worst $\mathbb{Q}$-factorial terminal singularities. Since the properties, which we study on $V$, are birationally invariant in the category of normal varieties with canonical singularities, we do focus our study on $Y$ instead. Clearly the canonical sheaf satisfies $\omega_Y \cong \mathcal{O}_Y(K_Y)$ for any canonical divisor $K_Y$.

2.1. Set up for the map $\varphi_{m_0, Y}$.

Assume that there is a positive integer $m_0$ such that $P_{m_0}(Y) = h^0(Y, \mathcal{O}_Y(m_0K_Y)) \geq 2$, which means the $m_0$-canonical map $\varphi_{m_0, Y} : Y \dashrightarrow \mathbb{P}^{P_{m_0}-1}$ is non-trivial. Fix an effective divisor $K_{m_0} \sim m_0K_Y$.

By Hironaka’s theorem, we may take a series of blow-ups along nonsingular centers to obtain the model $\pi : Y' \rightarrow Y$, $Y'$ satisfies the following conditions:

(i) $Y'$ is nonsingular and projective;
(ii) the moving part of $|m_0K_{Y'}|$ is base point free so that $g_{m_0} = \varphi_{m_0, Y} \circ \pi : Y' \rightarrow \mathbb{P}^{P_{m_0}-1}$ is a non-trivial morphism;
(iii) the union of $\pi^*(K_{m_0})$ and all those exceptional divisors of $\pi$ has simple normal crossing supports.

Take the Stein factorization of $g_{m_0}$. We get

$$Y' \xrightarrow{f_{m_0}} \Gamma \xrightarrow{\Phi_{m_0, Y}} \mathbb{P}^{P_{m_0}-1}.$$
and hence the following diagram commutes:

\[
\begin{array}{ccc}
Y' & \xrightarrow{f_{m_0}} & \Gamma \\
\downarrow{\pi} & & \downarrow{s} \\
Y & \xrightarrow{\Phi_{m_0,Y}} & \Phi_{m_0,Y}(Y)
\end{array}
\]

We may write \( K_{Y'} = \pi^*(K_Y) + E_\pi \) where \( E_\pi \) is a sum of distinct exceptional divisors with positive rational coefficients. Denote by \( |M_{m_0}| \) the moving part of \( |mK_{Y'}| \) for any positive integer \( m \). Since \( Y \) has at worst \( \mathbb{Q} \)-factorial terminal singularities, we may write \( m_0\pi^*(K_Y) \sim M_{m_0} + E_{m_0} \) where \( E_{m_0} \) is an effective \( \mathbb{Q} \)-divisor as well. Clearly, one has \( 1 \leq \dim(\Gamma) \leq 4 \).

If \( \dim(\Gamma) = 1 \), we say that \( |M_{m_0}| \) is composed of a pencil and we have \( M_{m_0} \sim \sum_{i=1}^b F_i \equiv bF \) where \( F_i \) and \( F \) are smooth fibers of \( f_{m_0} \) and \( b = \deg f_{m_0} \cdot \mathcal{O}_{Y'}(M_{m_0}) \geq P_{m_0}(Y) - 1 \).

If \( \dim(\Gamma) > 1 \), we say that \( |M_{m_0}| \) is not composed of a pencil and, by Bertini’s theorem, we know that general members \( T_i \in |M_{m_0}| \) are nonsingular and irreducible.

Denote by \( T' \) a general fiber of \( f_{m_0} \) if \( \dim(\Gamma) = 1 \) or, otherwise, to be a general member of \( |M_{m_0}| \). Set

\[ \theta_{m_0} = \theta_{m_0,|M_{m_0}|} = \begin{cases} b, & \text{if } \dim(\Gamma) = 1; \\ 1, & \text{if } \dim(\Gamma) \geq 2. \end{cases} \]

So we naturally get \( m_0\pi^*(K_Y) \equiv \theta_{m_0}T' + E_{m_0} \). We also set \( \mu = \mu(T') \) to be the number

\[ \sup\{ \mu' \in \mathbb{Q} | \pi^*(K_Y) \equiv \mu'T' + E_{\mu'} \text{ for some effective } \mathbb{Q}\text{-divisor } E_{\mu'} \}. \]

Clearly we have \( \mu(T') \geq \frac{\theta_{m_0}}{m_0} \).

2.2. Convention.

For an arbitrary linear system \( |D| \) of positive dimension on a normal projective variety \( Z \), we may define a generic irreducible element of \( |D| \) in the following way. As we have \( |D| = \text{Mov}|D| + \text{Fix}|D| \), where \( \text{Mov}|D| \) and \( \text{Fix}|D| \) denote the moving part and the fixed part respectively. Consider the rational map \( \Phi_{|D|} = \Phi_{\text{Mov}|D|} \). We say that \( |D| \) is composed of a pencil if \( \dim \Phi_{|D|}(Z) = 1 \); otherwise, \( |D| \) is not composed of a pencil. A generic irreducible element of \( |D| \) is defined to be an irreducible component of a general member in \( \text{Mov}|D| \) if \( |D| \) is composed of a pencil or, otherwise, a general member of \( \text{Mov}|D| \).

Keep the above settings. We say that \( |D| \) can distinguish different generic irreducible elements \( X_1 \) and \( X_2 \) of a linear system \( |M| \) if neither
$X_1$ nor $X_2$ is contained in $\text{Bs}(D)$, and if $\Phi_{|D|}(X_1) \not\subseteq \Phi_{|D|}(X_2)$, $\Phi_{|D|}(X_2) \not\subseteq \Phi_{|D|}(X_1)$.

A nonsingular projective surface $S$ of general type with $K^2_{S_0} = u$ and $p_g(S_0) = v$ is referred to as a $(u,v)$-surface, where $S_0$ is the minimal model of $S$ and $\sigma : S \to S_0$ the corresponding contraction map.

2.3. Fixed notations.

Pick up a generic irreducible element $T'$ of $|M_{m_0}|$. Modulo further blow-ups on $Y'$, which is still denoted as $Y'$ for simplicity, we may have a birational morphism $\pi_T = \pi|_{T'} : T' \to T$ onto a minimal model $T$ of $T'$. Let $t_1$ be the smallest positive integer such that $P_{t_1}(T) \geq 2$.

Set $|N| = \text{Mov}(t_1K_{T'})$, which is a base point free linear system on $T'$. Denote by $j$ the induced projective morphism with connected fibers from $\Phi_{|N|}$ by Stein factorization and $c = \deg j_*O_{T'}(N) \geq P_{t_1}(T) - 1$. Set

$$a_{t_1,T} = \begin{cases} c, & \text{if } |N| \text{ is composed of a pencil;} \\ 1, & \text{if } |N| \text{ is not composed of a pencil.} \end{cases}$$

Let $S$ be a generic irreducible element of $|N|$. Then we have $t_1\pi_T^*(K_T) \equiv a_{t_1,T}S + E_N$, where $E_N$ is an effective $\mathbb{Q}$-divisor.

Suppose that $|H|$ is a base point free linear system on $S$ and $C$ is a generic irreducible element of $|H|$. As $\pi_T^*(K_T)|_S$ is nef and big, by Kodaira’s lemma, there is a rational number $\tilde{\beta} > 0$ such that $\pi_T^*(K_T)|_S \geq \tilde{\beta} C$.

Set

$$\beta = \beta(t_1,|N|) = \sup\{\tilde{\beta} | \tilde{\beta} > 0 \text{ s.t. } \pi_T^*(K_T)|_S \geq \tilde{\beta} C\}$$

$$\xi = \xi(t_1,|N|) = (\pi_T^*(K_T) \cdot C)|_{T'}.$$ 

2.4. Technical preparation.

We will use the following theorem which is a special form of Kawamata’s extension theorem.

**Theorem 2.1.** (cf. [13] Theorem A) Let $Z$ be a nonsingular projective variety on which $D$ is a smooth divisor. Assume that $K_Z + D \sim A + B$ where $A$ is an ample $\mathbb{Q}$-divisor and $B$ is an effective $\mathbb{Q}$-divisor such that $D \not\subseteq \text{Supp}(B)$. Then the natural homomorphism

$$H^0(Z, m(K_Z + D)) \longrightarrow H^0(D, mK_D)$$

is surjective for any integer $m > 1$.

In particular, when $Z$ is of general type and $D$, as a generic irreducible element, moves in a base point free linear system, the conditions of Theorem 2.1 are automatically satisfied. Keep the settings as in 2.1 and 2.3. Take $Z = Y'$ and $D = T'$. Then for sufficiently large and divisible integer $n > 0$ and a rational number $\mu'$ infinitesimally near the number $\mu$, it holds that

$$|n(\mu' + 1)K_{Y'}| \succ |n\mu'(K_{Y'} + T')|$$
and the homomorphism
\[ H^0(Y', n\mu'(K_{Y'} + T')) \to H^0(T', n\mu'K_{T'}) \]
is surjective. Noting that \( n(\mu' + 1)\pi^*(K_Y) \geq M_{n(\mu' + 1)} \) and
\[ \text{Mov}|n\mu'K_{T'}| = |n\mu'\pi^*(K_T)|, \]
one has
\[ n(\mu' + 1)\pi^*(K_Y)|_{T'} \geq M_{n(\mu' + 1)}|_{T'} \geq n\mu'\pi^*(K_T). \]
So we get the canonical restriction inequality:
\[ \pi^*(K_Y)|_{T'} \geq \frac{\mu}{\mu + 1}\pi^*(K_T) \geq \frac{\theta_{m_0}}{m_0 + \theta_{m_0}}\pi^*(K_T). \quad (2.1) \]
Similarly, one has
\[ \pi^*(K_T)|_{S} \geq \frac{a_{t_1,T}}{l_1 + a_{t_1,T}}\sigma^*(K_{S_0}). \quad (2.2) \]

We will tacitly use the following type of birationality principle.

**Theorem 2.2.** (cf. [5, 2.7]) Let \( Z \) be a nonsingular projective variety, \( A \) and \( B \) be two divisors on \( Z \) with \( |A| \) being a base point free linear system. Take the Stein factorization of \( \Phi|_A: Z \xrightarrow{h} W \to \mathbb{P}^{h_0(Z,A)-1} \) where \( h \) is a fibration onto a normal variety \( W \). Then the rational map \( \Phi|_{B+A} \) is birational onto its image if one of the following conditions is satisfied:

(i) \( \dim \Phi|_A(Z) \geq 2, |B| \neq \emptyset \) and \( \Phi|_{B+A}|_D \) is birational for a general member \( D \) of \( |A| \).

(ii) \( \dim \Phi|_A(Z) = 1 \), \( \Phi|_{B+A} \) can distinguish different general fibers of \( h \) and \( \Phi|_{B+A}|_F \) is birational for a general fiber \( F \) of \( h \).

2.5. **Some useful lemmas.**

The following results on surfaces and 3-folds will be used in our proof.

**Lemma 2.3.** (see [6, Lemma 2.4, Lemma 2.5]) Let \( \sigma: S \to S_0 \) be a birational contraction from a nonsingular projective surface \( S \) of general type onto the minimal model \( S_0 \). Assume that \( S \) is not a \((1,2)\)-surface. Then one has

1. \( (\sigma^*(K_{S_0}) \cdot C) \geq 2 \) for any moving irreducible curve \( C \) on \( S \) (i.e. \( C \) moves in a linear system of positive dimension);
2. \( (\sigma^*(K_{S_0}) \cdot \tilde{C}) \geq 2 \) for any very general irreducible curve \( \tilde{C} \) on \( S \).

**Lemma 2.4.** ([5, Lemma 2.5]) Let \( S \) be a nonsingular projective surface. Let \( L \) be a nef and big \( \mathbb{Q} \)-divisor on \( S \) satisfying the following conditions:

1. \( L^2 > 8 \);
2. \( (L \cdot C_x) \geq 4 \) for all irreducible curves \( C_x \) passing through any very general point \( x \in S \).

Then \( |K_S + [L]| \) gives a birational map.
3. Proof of the main theorem

In this section, we always assume \( V \) to be a nonsingular projective 4-fold of general type with \( p_g(V) \geq 2 \) and \( Y \) a minimal model of \( V \). Keep the same settings as in 2.1 and 2.3. We have an induced fibration \( f_1 : Y' \to \Gamma \). Since \( p_g(Y) \geq 2 \), we have \( m_0 = 1 \) and \( \theta_1 \geq 1 \).

3.1. Separation properties of \( \varphi_{m,Y} \).

**Lemma 3.1.** Let \( Y \) be a minimal 4-fold of general type with \( p_g(Y) \geq 2 \). Keep the settings in 2.1. Then \( |mK_{Y'}| \) can distinguish different generic irreducible elements of \( |M_1| \) for all \( m \geq 3 \).

**Proof.** Suppose \( m \geq 3 \). When either \( |M_1| \) is not composed of a pencil or \( |M_1| \) is composed of a rational pencil, we have \( h^0(Y', mK_{Y'} - M_1) > 0 \) and thus \( |mK_{Y'}| \) can distinguish different generic irreducible elements of \( |M_1| \). When \( |M_1| \) is composed of an irrational pencil, then \( M_1 \sim \sum_{i=1}^b T_i \) where \( T_i \) are smooth fibers of \( f_1 \) and \( b \geq 2 \). Pick two different generic irreducible elements \( T_1, T_2 \) of \( |M_1| \). Then by Kawamata-Viehweg vanishing theorem (12, 19), one has

\[
H^1(K_{Y'} + [(m - 2)\pi^*(K_Y)] + M_1 - T_1 - T_2) = 0,
\]

and the surjective map

\[
H^0(Y', K_{Y'} + [(m - 2)\pi^*(K_Y)] + M_1) 
\longrightarrow H^0(T_1, (K_{Y'} + [(m - 2)\pi^*(K_Y)] + M_1)|_{T_1}) \tag{3.1}
\]

\[
\oplus H^0(T_2, (K_{Y'} + [(m - 2)\pi^*(K_Y)] + M_1)|_{T_2}). \tag{3.2}
\]

As \( T_i \) is moving and \( M_1|_{T_i} \sim 0 \), both groups in (3.1) and (3.2) are nonzero. Therefore, \( |mK_{Y'}| \) can distinguish different generic irreducible elements of \( |M_1| \). \( \square \)

**Lemma 3.2.** Let \( Y \) be a minimal 4-fold of general type with \( p_g(Y) \geq 2 \). Keep the settings in 2.1 and 2.3. Pick up a generic irreducible element \( T' \) of \( |M_1| \). Then \( |mK_{Y'}|_{T'} \) can distinguish different generic irreducible elements of \( |N| \) for all \( m \geq 2t_1 + 4 \).

**Proof.** Suppose \( m \geq 2t_1 + 4 \). We have \( K_{Y'} \geq \pi^*(K_Y) \geq T' \). Similar to the proof of Lemma 3.1, we consider the following two situations: (i) \( |N| \) is not composed of a pencil or \( |N| \) is composed of a rational pencil; (ii) \( |N| \) is composed of an irrational pencil.

For (i), one has

\[
|2(t_1 + 1)K_{Y'}| \succ |(t_1 + 1)(K_{Y'} + T')|.
\]

By Theorem 2.1, one has

\[
|(t_1 + 1)(K_{Y'} + T')|_{T'} \succ |(t_1 + 1)K_{T'}|.
\]
As \((t_1 + 1)K_{T'} \geq N, |mK_{Y'}||_{T'}\) can distinguish different generic irreducible elements of \(|N|\).

For (ii), it holds that
\[ |2(t_1 + 2)K_{Y'}| > |(t_1 + 2)(K_{Y'} + T')|. \]

Using Theorem 2.1 again, one gets
\[ |(t_1 + 2)(K_{Y'} + T')||_{T'} > |(t_1 + 2)K_{T'}| \]

where \(S\) is two different generic irreducible elements of \(|N|\).

The vanishing theorem implies the surjective map
\[ H^0(T', K_{T'} + [\pi_T^*(K_T)] + N) \]
\[ \rightarrow H^0(S_1, (K_{T'} + [\pi_T^*(K_T)])|_{S_1}) \]
\[ \oplus H^0(S_2, (K_{T'} + [\pi_T^*(K_T)])|_{S_2}) \]

where we note that \((N - S_i)|_{S_i}\) is linearly trivial for \(i = 1, 2\). Since \(p_g(T) > 0\), both groups in (3.3) and (3.4) are non-zero. So \(|mK_{Y'}||_{T'}\) can distinguish different generic irreducible elements of \(|N|\) for any \(m \geq 2t_1 + 4\). \(\Box\)

**Lemma 3.3.** Let \(Y\) be a minimal 4-fold of general type with \(p_g(Y) \geq 2\). Keep the settings in 2.1 and 2.3. Pick up a generic irreducible element \(T'\) of \(|M_1|\) and a generic irreducible element \(S\) of \(|N|\). Define
\[ |H| = \begin{cases} 
\text{Mov}[K_S], & (K^2_{S_0}, p_g(S)) = (1, 2) \text{ or } (2, 3); \\
\text{Mov}[2K_S], & \text{otherwise}. 
\end{cases} \]

Then \(|mK_{Y'}||_{S}\) can distinguish different generic irreducible elements of \(|H|\) for all \(m \geq 4(t_1 + 1)\).

**Proof.** Similar to the proof of Lemma 3.2 we have
\[ |4(t_1 + 1)K_{Y'}||_{T'} > |2(t_1 + 1)K_{T'}|. \]

Since \(t_1K_{T'} \geq S\), we have
\[ |4(t_1 + 1)K_{Y'}||_{S} > |2(t_1 + 1)K_{T'}||_{S} \]
\[ > |2(K_{T'} + S)||_{S} = |2K_{S}|. \]

Since \(p_g(S) > 0\) and \(|H|\) is not composed of an irrational pencil, the statement automatically follows. \(\Box\)

**3.2. Two useful propositions.**

**Proposition 3.4.** Let \(Y\) be a minimal 4-fold of general type with \(p_g(Y) \geq 2\). Keep the settings in 2.1 and 2.3. Pick up a generic irreducible element \(T'\) of \(|M_1|\) and a generic irreducible element \(S\) of \(|N|\). If \(S\) is not a \((1, 2)\)-surface, then \(\varphi_m, Y\) is birational for all \(m > (2\sqrt{2} + 1)\frac{t_1}{a_{1, T}} + 1 + \frac{1}{\theta_1}\).
Proof. Since
\[(m - 1)\pi^*(KY) - T - \frac{1}{\theta_1}E_1 \equiv (m - 1 - \frac{1}{\theta_1})\pi^*(KY)\]
is nef and big, and it has simple normal crossing support, Kawamata-Viehweg vanishing theorem implies
\[|mKY||_{T'} \geq |KY + [(m - 1)\pi^*(KY) - \frac{1}{\theta_1}E_1]|_{T'}|\]
\[= |K_{T'} + [Q_{m,T'}]|, \quad (3.5)\]
where \(Q_{m,T'} = ((m - 1)\pi^*(KY) - T' - \frac{1}{\theta_1}E_1)|_{T'} \equiv (m - 1 - \frac{1}{\theta_1})\pi^*(KY)|_{T'}\) is nef and big and has simple normal crossing support.

By the canonical restriction inequality (2.1), we may write
\[\pi^*(KY)|_{T'} \equiv \frac{\theta_1}{1 + \theta_1}\pi_T^*(KT) + E_{1,T},\]
where \(E_{1,T}\) is certain effective \(\mathbb{Q}\)-divisor. As \(t_1\pi_T^*(KT)|_S \equiv a_{t_1,T}S + E_N\) where \(E_N\) is an effective \(\mathbb{Q}\)-divisor, one may obtain
\[|mKY)|_S \geq |K_{T'} + [Q_{m,T'} - \frac{1}{a_{t_1,T}}E_N]|_S|\]
\[= |KS + [(Q_{m,T'} - S - \frac{1}{a_{t_1,T}}E_N)|_S|\]
\[= |KS + [U_{m,S}]|, \quad (3.6)\]
where
\[U_{m,S} = (Q_{m,T'} - S - \frac{1}{a_{t_1,T}}E_N - (m - 1 - \frac{1}{\theta_1})E_{1,T})|_S\]
\[\equiv ((m - 1 - \frac{1}{\theta_1}) \cdot \frac{\theta_1}{1 + \theta_1} - \frac{t_1}{a_{t_1,T}})\pi_T^*(KT)|_S.\]
As (2.2) also gives
\[\pi_T^*(KT)|_S \equiv \frac{a_{t_1,T}}{t_1 + a_{t_1,T}}\sigma^*(KS_0) + E_{t_1,S}\]
for some effective \(\mathbb{Q}\)-divisor \(E_{t_1,S}\) on \(S\), together with (3.6), one has
\[U_{m,S}^2 \geq (((m - \frac{\theta_1 + 1}{\theta_1}) \cdot \frac{\theta_1}{\theta_1 + 1} - \frac{t_1}{a_{t_1,T}}) \cdot \frac{a_{t_1,T}}{t_1 + a_{t_1,T}})^2 \cdot K_{S_0}^2 > 8\]
where \(U_{m,S}\) is nef and big. Hence the statement clearly follows from Lemma 2.4, Lemma 2.3(2), Lemma 3.1, Lemma 3.2 and Theorem 2.2. 

Proposition 3.5. Let \(Y\) be a minimal 4-fold of general type with \(p_g(Y) \geq 2\). Keep the settings in 2.1 and 2.3. Pick up a generic irreducible element \(T'\) of \(|M_1|\) and a generic irreducible element \(S\) of
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$|N|$. If $S$ is neither a $(1,2)$-surface nor a $(2,3)$-surface, then $\varphi_{m,Y}$ is birational for all $m \geq 6(t_1 + 1)$.

**Proof.** Suppose $m \geq 6(t_1 + 1)$. Following the same procedures as in the proof of Lemma 3.2 and Lemma 3.3, one has

$$|mK_Y'| \geq |3(t_1 + 1)(K_Y' + T')|$$

and $|mK_{Y'}|_{T'} \geq |3(t_1 + 1)K_{T'}|$. Furthermore, one has

$$|mK_Y'|_S \geq |3(t_1 + 1)K_{T'}|_S \geq |3(K_{T'} + S)|_S = |3K_S|.$$

By the birationality principle, Lemma 3.1 and Lemma 3.2, the statement holds. $\square$

3.3. The case of $\dim(\Gamma) \geq 2$.

**Theorem 3.6.** Let $Y$ be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Keep the same settings as in 2.1 and 2.3. Assume that $\dim(\Gamma) \geq 2$. Then $\Phi_{m,V}$ is birational for all $m \geq 15$.

**Proof.** By Theorem 2.2, we may just consider $\varphi_{m,Y'}|_{T'}$ for a general member $T' \in |M_1|$. As we have $\theta_1 = 1$, (2.1) gives

$$\pi^*(K_Y)|_{T'} \geq \frac{1}{2} \pi^*_T(K_T).$$

(3.7)

Modulo some birational modifications, we may assume that $|M_1|_{T'}$ is a base point free linear system. Pick up a generic irreducible element $S$ of $|M_1|_{T'}$. It follows that

$$\pi^*(K_Y)|_{T'} \geq M_1|_{T'} \geq S.$$

Modulo $\mathbb{Q}$-linear equivalence, we have

$$K_{T'} \geq (\pi^*(K_Y) + T')|_{T'} \geq 2S.$$  

(3.8)

Using Theorem 2.1 we get

$$\pi^*_T(K_T)|_S \geq \frac{2}{3} \sigma^*(K_{S_0}).$$

(3.9)

Thus, combining (3.7) and (3.9), one gets

$$\pi^*(K_Y)|_S \geq \frac{1}{3} \sigma^*(K_{S_0}).$$

By (3.5), we already have

$$|mK_Y'|_{T'} \geq |K_{T'} + [Q_{m,T'}]|$$

where $Q_{m,T'} = ((m - 1)\pi^*(K_Y) - T' - \frac{1}{\sigma^*}E_1)|_{T'} \equiv (m - 2)\pi^*(K_Y)|_{T'}$. As $\pi^*(K_Y)|_{T'} \equiv S + E_S$ for some effective $\mathbb{Q}$-divisor $E_S$ on $T'$ and $Q_{m,T'} - S - E_S \equiv (m - 3)\pi^*(K_Y)|_{T'}$.
is nef and big, Kawamata-Viehweg vanishing theorem implies
\[ |mK_Y'|_S \geq |K_T' + [Q_{m,T'} - E_S]|_S \]
\[ \geq |K_S + [R'_{m,S}]| \]
where
\[ R'_{m,S} = (Q_{m,T'} - S - E_S)|_S \]
\[ \equiv (m - 3)\pi^*(K_Y)|_S. \]

Since \( R'_{m,S} \equiv \frac{m-3}{3} \sigma^*(K_S) + E_{m,S}' \) where \( E_{m,S}' \) is an effective \( \mathbb{Q} \)-divisor on \( S \), by Lemma 2.4 \( |K_S + [R'_{m,S}]| \) gives a birational map whenever \( m \geq 15 \).

Since \( \text{Mov}(K_{T'}) \geq |M_1|_{T'} \), we may take \( t_1 = 1 \) and by the proof of Lemma 3.2 we know that \( |mK_Y'| \) distinguishes different generic irreducible elements of \( |M_1|_{T'} \) for \( m \geq 6 \). Therefore, \( \varphi_{m,Y} \) is birational for all \( m \geq 15 \) in this case. □

3.4. The case of \( \dim(\Gamma) = 1 \).

**Theorem 3.7.** Let \( Y \) be a minimal 4-fold of general type with \( p_g(Y) \geq 2 \). Keep the same settings as in 2.1 and 2.3. Assume that \( \dim(\Gamma) = 1 \). Then \( \varphi_{m,Y} \) is birational for all \( m \geq 33 \).

**Proof.** We have \( \theta_1 \geq 1 \) and \( p_g(T') > 0 \). By Lemma 3.1 \( |mK_Y'| \) distinguishes different generic irreducible elements of \( |M_1| \) for all \( m \geq 3 \).

As an overall discussion, we study the linear system \( |mK_Y'|_C \) for generic irreducible element \( C \) of \( |H| \). Recall that, by (3.5) and (3.6), we already have
\[ |mK_Y'|_S \geq |K_S + [U_{m,S}]| \]
where \( U_{m,S} \equiv ((m - 1 - \frac{1}{\theta_1})\frac{\theta_1}{\theta_1 + 1} - \frac{t_1}{a_{t_1,T}})\pi^*(K_T)|_S \) is a nef and big \( \mathbb{Q} \)-divisor on \( S \). As we have \( \pi^*(K_T)|_S \sim \beta C + E_H \) for some effective \( \mathbb{Q} \)-divisor on \( S \), applying Kawamata-Viehweg vanishing theorem, we may get
\[ |mK_Y'|_C \geq |K_S + [U_{m,S} - \frac{1}{\beta}E_H]|_C \]
\[ = |K_C + [U_{m,S} - C - \frac{1}{\beta}E_H]|_C | \]
\[ = |K_C + D_m|, \quad (3.10) \]
where \( D_m = [U_{m,S} - C - \frac{1}{\beta}E_H]|_C \) with
\[ \deg D_m \geq ((m - 1 - \frac{1}{\theta_1})\frac{\theta_1}{\theta_1 + 1} - \frac{t_1}{a_{t_1,T}} - \frac{1}{\beta})(\pi^*(K_T)|_S \cdot C). \]

Thus, whenever \( m > (\frac{2}{\xi} + \frac{t_1}{a_{t_1,T}} + \frac{1}{\beta} + 1)\frac{\theta_1 + 1}{\theta_1} \), \( |mK_Y'|_C \) gives a birational map.
Therefore, by Theorem 2.2, Lemma 3.2, and Lemma 3.3, \( \varphi_{m,Y} \) is birational provided that

\[
m \geq 4t_1 + 4 \quad \text{and} \quad m > \frac{4}{\xi} + \frac{2t_1}{a_{t_1,T}} + \frac{2}{\beta} + 2.
\]

Now we study this problem according to the value of \( p_g(T) \).

**Case 1.** \( p_g(T) \geq 2 \)

Clearly, we may take \( t_1 = 1 \) and so \( a_{t_1,T} = 1 \). From [7, Section 2, Section 3], we know that one of the cases occur:

1. \( \beta = 1, \xi \geq 1; \) (correspondingly, \( d \geq 2 \) or \( d = 1 \) and \( b > 0 \) in [7])
2. \( \beta = \frac{1}{2}, \xi \geq \frac{1}{2}; \) (correspondingly, \( d = 1 \) and (1, 1)-surface case in [7])
3. \( \beta = \frac{1}{2}, \xi \geq \frac{1}{2}; \) (correspondingly, \( d = 1 \) and (1, 2)-surface case in [7])
4. \( \beta = \frac{1}{2}, \xi \geq 1; \) (correspondingly, \( d = 1 \) and (2, 3)-surface case in [7])
5. \( \beta = \frac{1}{4}, \xi \geq 2. \) (correspondingly, \( d = 1 \) and other surface case in [7])

So \( \varphi_{m,Y} \) is birational for all \( m \geq 16 \).

**Case 2.** \( p_g(T) = 1 \)

According to [6, Corollary 4.10], \( T \) must be of either types: (i) \( P_4(T) = 1, P_5(T) \geq 3; \) (ii) \( P_4(T) \geq 2 \).

For Type (i), we have \( t_1 = 5 \) and set \( |N| = \text{Mov}[5K_T] \). When \( |N| \) is composed of a pencil, we have \( a_{t_1,T} \geq 2 \) and \( S \) is exactly the general fiber of the induced fibration from \( \Phi_{|N|} \). If \( S \) is a not a (1, 2)-surface, by Proposition 3.4, \( \varphi_{m,Y} \) is birational for all \( m \geq 27 \). If \( S \) is a (1, 2)-surface, by [10, Proposition 4.1, Case 2], we have \( \beta \geq \frac{1}{2} \) and \( \xi \geq \frac{1}{2} \), and hence \( \varphi_{m,Y} \) is birational for \( m \geq 28 \). When \( |N| \) is not composed of a pencil, we have \( a_{t_1,T} \geq 1 \). Refer to the case by case argument of [10, Proposition 4.2, Proposition 4.3], to give an exact list, \((\beta, \xi)\) must be among one of the situations: \((1/5, 3/7), (1/5, 2/3), (1/3, 1/3), (1/5, 5/13), (1/5, 1), (1/2, 1/3), (1/5, 1/2), (2/5, 1/3), (1/4, 1/3)\). Hence \( \varphi_{m,Y} \) is birational for all \( m \geq 33 \).

For Type (ii), we have \( t_1 = 4 \) and set \( |N| = \text{Mov}[4K_T] \). When \( |N| \) is composed of a pencil and the generic irreducible element \( S \) of \( |N| \) is neither a (1, 2)-surface nor a (2, 3)-surface, by Proposition 3.5, \( \varphi_{m,Y} \) is birational for all \( m \geq 30 \). When \( P_4(T) = h^0(T, O_T(4K_T)) = 2 \) and \( |N| \) is composed of a rational pencil of (1, 2)-surfaces, the case by case argument of [10, Proposition 4.6, Proposition 4.7] tells that \((\beta, \xi)\) must be among one of the situations: \((2/7, 2/7), (1/5, 2/5), (2/5, 2/7), (1/3, 2/7), (1/4, 2/7), (1/5, 5/13), (1/5, 2/3), (1/5, 5/12), (1/5, 1/3)\). Hence \( \varphi_{m,Y} \) is birational for all \( m \geq 33 \). Otherwise, the case by case argument of [10, Proposition 4.5] tells that \((\beta, \xi)\) must be among one
of the situations: $(1/4, 2/5), (1/5, 2/5), (1/3, 1/3)$. Hence $\varphi_{m,Y}$ is birational for all $m \geq 31$.

In conclusion, $\varphi_{m,Y}$ is birational for all $m \geq 33$. \hfill \Box

**Remark 3.8.** Similar to the deduction of (3.10), one gets from (3.3) that

$$|mK_Y^*|_C \cong |K_C + D'_m|,$$

where

$$D'_m \equiv \left[(m - 1 - \frac{1}{\theta_1} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{t_1}{a_{1,T}} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{1}{\beta})\pi^*(K_Y^*)\right]_C.$$ Whenever $\deg D'_m \geq 2$, we know that $|K_C + D'_m|$ is base point free by the curve theory.

Set $\eta = (\pi^*(K_Y^*) \cdot C)_{Y^*}$ and $\alpha(m) = (m - 1 - \frac{1}{\theta_1} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{t_1}{a_{1,T}} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{1}{\beta})\eta$.

When $\alpha(m) > 1$, it follows that

$$m\eta \geq \deg K_C + \lceil \alpha(m) \rceil.$$ Consequently,

$$\eta \geq \frac{\deg K_C + \lceil \alpha(m) \rceil}{m}.$$

In particular, whenever $S$ is a $(1, 1)$-surface, we take $|H| = |2\sigma^*(K_{S_0})|$. It is clear that the generic irreducible element $C$ of $|H|$ is an even divisor and $\deg K_C = 6$. Thus, whenever $m - 1 - \frac{1}{\theta_1} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{t_1}{a_{1,T}} - \frac{\theta_1 + 1}{\theta_1} \cdot \frac{1}{\beta} > 0$, $\deg D'_m \geq 2$, which means $\eta \geq \frac{8}{m}$.

### 3.5. The canonical volume of 4-folds.

**Theorem 3.9.** Let $Y$ be a minimal 4-fold of general type with $p_g(Y) \geq 2$. Keep the same settings as in 2.1 and 2.3. Then $\text{Vol}(Y) \geq \frac{1}{620}$.

**Proof.** We have $\text{Vol}(Y) = K_Y^4$.

Recall that $m_0 \pi^*(K_Y^*) \equiv \theta_m T' + E_m$. One has

$$K_Y^4 = (\pi^*(K_Y^*))^4 \geq \frac{\theta_m^4}{m_0} (\pi^{*}(K_Y^*))^3 \cdot T' = \frac{\theta_m^4}{m_0} (\pi^*(K_Y^*)|_{T'})^3.$$ As we also have (2.1) and $t_1 \pi^*_T(K_T) \equiv a_{1,T} S + E_N$, it follows that

$$K_Y^4 \geq \frac{\theta_m^4}{m_0} \cdot \left(\frac{\theta_m^4}{m_0 + \theta_m^4}\right)^3 (\pi^*_T(K_T))^3 \geq \frac{\theta_m^4}{m_0} \cdot \left(\frac{\theta_m^4}{m_0 + \theta_m^4}\right)^3 \cdot \frac{a_{1,T}}{t_1} \cdot (S \cdot (\pi^*_T(K_T))^2) = \frac{\theta_m^4}{m_0} \cdot \left(\frac{\theta_m^4}{m_0 + \theta_m^4}\right)^3 \cdot \frac{a_{1,T}}{t_1} \cdot (\pi^*_T(K_T)|_S)^2.$$ By (2.2) and $\pi^*_T(K_T)|_S \geq \beta C$, we may get

$$K_Y^4 \geq \frac{\theta_m^4}{m_0} \cdot \left(\frac{\theta_m^4}{m_0 + \theta_m^4}\right)^3 \cdot \frac{a_{1,T}}{t_1} \cdot \left(\frac{a_{1,T}}{t_1 + a_{1,T}}\right)^2 K_{S_0}^2$$
or
\[
K_Y^4 \geq \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^3 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta (\pi_T^*(K_T)|_S \cdot C)
\]
\[
= \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^3 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta \xi.
\]
Alternatively we have
\[
K_Y^4 \geq \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^2 \cdot ((\pi_T^*(K_T))^2 \cdot (\pi^*(K_T)|_{T'}))
\]
\[
\geq \frac{\theta_{m_0}}{m_0} \cdot \frac{\theta_{m_0}}{m_0 + \theta_{m_0}}^2 \cdot \frac{a_{t_1,T}}{t_1} \cdot (S \cdot \pi_T^*(K_T) \cdot (\pi^*(K_Y)|_{T'}))
\]
\[
= \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^2 \cdot \frac{a_{t_1,T}}{t_1} \cdot ((\pi_T^*(K_T)|_S \cdot (\pi^*(K_Y)|_S))
\]
\[
\geq \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^2 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta ((\pi^*(K_Y)|_S) \cdot C)
\]
\[
\geq \frac{\theta_{m_0}}{m_0} \cdot \left( \frac{\theta_{m_0}}{m_0 + \theta_{m_0}} \right)^2 \cdot \frac{a_{t_1,T}}{t_1} \cdot \beta \eta.
\]

Now estimate the canonical volume according to the same classification of \(T\) and \(S\) as in Subsection 3.3 and Subsection 3.4.

(I) The case of \(\dim(\Gamma) \geq 2\)
Remember that in this case, \(m_0 = 1, \theta_{m_0} = 1, t_1 = 1, a_{t_1,T} = 2\) (by (3.8)) and \(\pi_T^*(K_T)|_S \geq \frac{2}{3} \sigma^*(K_{S_0})\) (by (3.9)). So we have \(K_Y^4 \geq \frac{1}{9}\).

(II) The case of \(\dim(\Gamma) = 1\)
We have \(m_0 = 1, \theta_{m_0} \geq 1\).

Subcase (II.1). \(p_g(T) \geq 2\).
As in Theorem 3.7, Case 1, \(t_1 = 1, a_{t_1,T} = 1\), so we correspondingly have the estimation as follows:

1. \(\beta = 1, \xi \geq 1, \text{ then } K_Y^4 \geq \frac{1}{8}\).
2. \(\beta = \frac{1}{4}, \xi \geq \frac{2}{3}, \text{ then } K_Y^4 \geq \frac{5}{128}\).
3. \(\beta = \frac{1}{7}, \xi \geq \frac{2}{3}, \text{ then } K_Y^4 \geq \frac{1}{24}\).
4. \(\beta = \frac{1}{7}, \xi \geq 1, \text{ then } K_Y^4 \geq \frac{1}{16}\).
5. \(\beta = \frac{1}{7}, \xi \geq 2, \text{ then } K_Y^4 \geq \frac{1}{16}\).

Subcase (II.2). \(p_g(T) = 1\).
We follow the same classification of \(T\) as in Theorem 3.7, Case 2.
Recall that for Type (i), we have \(t_1 = 5\). When \(|N|\) is composed of a pencil and the general fiber \(S\) of the induced fibration from \(\Phi_{|N|}\) is not a \((1,2)\)-surface, we have \(a_{t_1,T} \geq 2, \beta \geq \frac{2}{7}, \xi \geq \left(\frac{2}{7} \sigma^*(K_{S_0}) \cdot C\right) \geq \frac{2}{7}\), and thus \(K_Y^4 \geq \frac{1}{215}\). When \(|N|\) is composed of a pencil and the general fiber \(S\) is a \((1,2)\)-surface, we have \(a_{t_1,T} \geq 2, \beta \geq \frac{2}{7}, \xi \geq \frac{2}{7}\), and hence
When $|N|$ is not composed of a pencil, we have $t_1,T \geq 1$. The corresponding lower bounds of $K_Y^4$ to those of $(\beta, \xi)$ are as follows:

$$\frac{3}{200}, \frac{3}{300}, \frac{3}{260}, \frac{3}{230}, \frac{3}{190}, \frac{1}{1200}, \frac{1}{180}, \frac{1}{240}.$$  

For Type (ii), we have $t_1=4$. When $|N|$ is not composed of a pencil, then $\beta \geq \frac{1}{4}, \xi \geq \frac{2}{5}$ and $K_Y^4 \geq \frac{1}{320}$. When $|N|$ is composed of an irrational pencil, then $\beta \geq \frac{1}{3}, \xi \geq \frac{1}{3}$ and $K_Y^4 \geq \frac{1}{288}$. When $|N|$ is composed of a rational pencil of surfaces with $K_S^2 \geq 2$, then $K_Y^4 \geq \frac{1}{100}$. When $|N|$ is composed of a rational pencil of $(1,1)$-surfaces, taking $|H| = |2\sigma^*(K_S)|$, we get $\beta \geq \frac{1}{10}$. By Remark 3.8, $\eta \geq \frac{5}{31}$ and so $K_Y^4 \geq \frac{1}{620}$. When $P_3(T) = h^0(T, \mathcal{O}_T(4K_T)) = 2$ and $|N|$ is composed of a rational pencil of $(1,2)$-surfaces, the corresponding lower bounds of $K_Y^4$ to those of $(\beta, \xi)$ are as follows: $\frac{3}{392}, \frac{1}{400}, \frac{1}{280}, \frac{1}{336}, \frac{1}{448}, \frac{1}{416}, \frac{1}{240}, \frac{1}{384}, \frac{1}{480}$.

So we have shown Vol($Y$) $\geq \frac{1}{620}$. □

3.6. Proof of Theorem 1.1

Proof. Theorem 3.6, Theorem 3.7 and Theorem 3.9 directly implies Theorem 1.1. □

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