Horizon Strings and Interior States of a Black Hole

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We provide an explicit construction of classical strings that have endpoints on the horizons of the 2D Lorentzian black hole. We argue that this is a dual description of geodesics that are localized around the horizon which are the Lorentzian counterparts of the winding strings of the Euclidean black hole (the cigar geometry). Identifying these with the states of the black hole, we can expect that issues of black hole information loss can be posed sharply in terms of a fully quantizable string theory.

Any resolution of the black hole information paradox requires some account of the interior of the black hole. Stating the question thus carries the implication that computations of the entropy of a black hole, which rely on a microcanonical ensemble description valid at $G_N = 0$ (such as the degeneracy counting based on D-brane models) will not suffice to address unitarity of quantum processes at the horizon since the horizon vanishes at $G_N = 0$. An approach using perturbative string theory could conceivably lead to novel insights, but this requires the knowledge of the complete string spectrum on a black hole background which has not yet made an appearance.

In [1], Susskind has suggested that the entropy of the black hole be modelled in terms of strings that end on the horizon. Such strings will be accessible to the external observer and to the interior observer alike and will presumably be implicated in a unitarity description of black hole evaporation etc. It will therefore be of much interest if such strings can be constructed explicitly and perhaps even quantized.

In this context, there are two exactly solvable sigma models whose target space interpretation is that of a black hole: the BTZ black hole in 3D [2] and the 2D black hole [3] (one can construct other exact sigma models by building on these CFTs). The presence of a trapped region bounded by a horizon enables one to formulate all the usual black hole puzzles. It is therefore interesting to explore the spectrum of these models to isolate string configurations that might be relevant in studies of the information paradox and black hole complementarity.

The AdS/CFT correspondence has been a significant guiding force in recent studies of these issues and suggests that the interior of the black hole must be assigned a Hilbert space of states. Several studies have attempted to identify correlators of the dual field theory that would involve these interior modes and hence provide insights into the quantum black hole. The recent ER=EPR idea [4] suggests that one should search for Einstein-Rosen bridge like configurations as being the source of the Bekenstein-Hawking entropy which is now to be thought of as entanglement entropy between the two asymptotically flat regions of the extended black hole geometry.

For both the 2D black hole and the BTZ geometry, some version of AdS holography presumably exists but has not been completely identified. Thus, a construction of the full quantum spectrum of string theory on these backgrounds can be expected to be a fertile playground for the exploration of issues relating to the information paradox and black hole complementarity.

In this letter, we shall focus on the 2D black hole and identify classical configurations which possess the various desiderata as discussed above, and are also quantized relatively easily.

The 2D black hole as a gauged sigma model

We shall present a quick summary of the sigma model describing strings propagating on the Lorentzian 2D black hole as a gauged conformal field theory [3]. This black hole has a dual description as a condensate of non-singlet modes of a matrix model [5] and was in fact, first constructed as a solution to the low energy effective action[6].

To begin with, we start with the WZNW model based on the group $SL(2, \mathbb{R})$ which is a noncompact conformal field theory. The quantum spectrum of (the universal cover) of this model was fully realized in the work of Maldacena and Ooguri [7] who showed that a modular invariant partition function maybe obtained upon including spectrally flowed representations of the current algebra.

The symmetry that is gauged corresponds to a hyperbolic subgroup of $SL(2, \mathbb{R})$ generated by the current $J^3 - \bar{J}^3$, which acts on

$$SL(2, \mathbb{R}) \ni g = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \rightarrow g' = e^{i\sigma_3} g e^{i\sigma_3} \quad (1)$$

where $\epsilon(\sigma, t)$ is the gauge parameter.

The target space interpretation of the coset theory maybe obtained by gauge fixing and integrating out the gauge fields. The resultant sigma model has a metric and a dilaton

$$ds^2 = -k \frac{dudv}{1 - uv} \quad \Phi = -\frac{1}{2} \log(1 - uv)$$
where the off-diagonal entries of the $SL(2, \mathbb{R})$ matrix descend as embedding coordinates. This metric is that of a black hole with horizon represented by the diagonal lines $uv = 0$ (see Fig. 1), while there is a curvature singularity at $w = 1$.

Early studies of this string theory [8, 9] showed that the spectrum consisted of a single massless scalar field, together with the possibility of massive states at discrete values of the momenta. In view of the necessity of spectrally flowed representations for the parent $SL(2, \mathbb{R})$ theory, it is of interest to revisit the spectrum of the black hole. In this context, we note that the Euclidean black hole which has the shape of a cigar (and is itself another gauged CFT) has winding strings in its spectrum which can be shown to arise from these spectrally flowed representations.

We can approach the spectrum by studying classical particle-like solutions to the sigma model (for earlier work see [10]). In this two dimensional case, particle-like modes of the string, in the classical limit, will propagate on geodesics of the black hole geometry which are easily constructed. As an example, a null geodesic of the black hole is

$$ u = -1 \quad v = (e^{2E\tau} - 1), $$

with $E$ the energy, which we will regard as a possible solution of the black hole sigma model. We can then uplift this to the WZNW model as an $SL(2, \mathbb{R})$ matrix

$$ g = \begin{pmatrix} e^{E(\tau - \sigma)} & -1 \\ 1 - e^{2E\tau} e^{E(\tau + \sigma)} & e^{E(\tau + \sigma)} \end{pmatrix}. $$

Because of the gauge symmetry, the diagonal entries are ambiguous and we have used this to introduce $\sigma$ dependence in the diagonal entries. It is now easy to see that this matrix can be factorized into a product of left and right movers

$$ g = e^{\frac{E}{2}\sigma_3} g_+ (\sigma^+) g_- (\sigma^-) e^{\frac{E}{2}\sigma_3} $$

and hence is a solution of the $SL(2, \mathbb{R})$ WZNW model. Further, it can be shown that the product $g_+ (\sigma^+) g_- (\sigma^-)$ represents a spacelike geodesic of $SL(2, \mathbb{R})$ and hence upon quantization, will be a state in the principal continuous series of $SL(2, \mathbb{R})$. The additional factors of $e^{\pm \frac{E}{2} \sigma_3}$ is the operation of spectral flow. Thus the null geodesic of the black hole is obtained from the spacelike geodesic of $SL(2, \mathbb{R})$ after spectral flow along the $\sigma_3$-direction in $SL(2, \mathbb{R})$. The $J^3$ current of $SL(2, \mathbb{R})$ and the worldsheet stress tensor undergo shifts

$$ J^3_\pm = \tilde{J}^3_\pm + \frac{k E^2}{2} \quad T_\pm = \tilde{T}_\pm + E \tilde{J}^3_\pm + k E^2. $$

under the spectral flow operation where $\tilde{T}_\pm, \tilde{J}^3_\pm$ refer to the currents of the $SL(2, \mathbb{R})$ CFT.

We may similarly uplift the timelike geodesics of the black hole geometry as $SL(2, \mathbb{R})$ matrices. The timelike geodesics come in three classes depending on the sign of $E^2 - m^2$. In the case $E^2 > m^2$, the geodesics of the black hole scatter out to asymptotic infinity and can be shown to be again obtained from spacelike geodesics of $SL(2, \mathbb{R})$ after the spectral flow operation. Thus, the null geodesics and timelike geodesics with $E^2 > m^2$ are both obtained from the principal continuous series of $SL(2, \mathbb{R})$, and as such can be mapped into each other by an $SL(2, \mathbb{R})$ transformation (before spectral flow). This is probably the origin of the duality observed in [9] which interchanges the massive and the massless states of the black hole theory.

We shall focus our attention on the geodesics with $E^2 < m^2$. These are all localized around the black hole horizon and never reach asymptotic infinity (the dashed magenta curve in the middle in Fig. 2). For e.g.,

$$ u = -\frac{e^{-E\tau}}{\sin \phi} \sin(\beta \tau + \phi) \quad v = \frac{e^{E\tau}}{\sin \phi} \sin(\beta \tau - \phi) $$

which is seen to satisfy $uv > -\cot^2 \phi i.e., \text{it never reaches the asymptotically flat region at } uv \rightarrow -\infty$. Here the parameters $\beta^2 = m^2 - E^2$ and $\tan^2 \phi = \frac{E^2}{m^2}$. When uplifted to $SL(2, \mathbb{R})$, this geodesic is represented by

$$ g = \frac{1}{\sin \phi} \begin{pmatrix} e^{-E\sigma} \sin \beta \tau & -e^{-E\sigma} \sin(\beta \tau + \phi) \\ e^{E\sigma} \sin(\beta \tau - \phi) & e^{E\sigma} \sin \beta \tau \end{pmatrix}. $$

Here, we will regard the parameters $\beta$ and $\phi$ as being independent of $E$, and we may expect that the level matching and Virasoro conditions of the string theory determine the relations between $\beta, E, \phi$. It can be shown that this matrix is obtained from the time-like geodesic of $SL(2, \mathbb{R})$ by applying the spectral flow operation, i.e., $g = e^{-E\sigma - \sigma_3} U e^{i\sigma \tau} V e^{E\sigma - \sigma_3} (U, V \text{ are constant } SL(2, \mathbb{R}) \text{ matrices})$. Hence, upon quantization, these geodesics will give states in the Discrete Series of $SL(2, \mathbb{R})$. It can now be shown that these geodesics, upon “Wick rotation”, map to the winding strings of the Euclidean black hole. Hence, we propose that these should be interpreted as the single particle internal energy levels of the black hole. Since the quantum wavefunctions corresponding to these trajectories will have vanishing norm.
at asymptotic infinity, this supports their interpretation as “bound states” of the geometry. This interpretation will be justified by a dual description below. We also note that the geodesics extend into both regions I and II through the interior (see Fig.1) of the black hole geometry which means that the boundary quantum states corresponding to these are entangled (across the horizon). Thus, a count of such quantum strings should account for the Bekenstein-Hawking entropy, now viewed as the entanglement entropy of the left and right black holes [4].

**T-duality and New states**

In the usual description of T-duality, which is a symmetry of string perturbation theory, one either performs a spacetime $R \to \frac{1}{R}$ transformation or a worldsheet $\tau \to \sigma$ transformation. Either way, we end up with the same string moving on a different target space (with radius $1/R$). If, however, we perform both transformations, then a string state $|n,w\rangle$ with momentum and winding quantum numbers $n,w$ is transformed into a different physical state with quantum numbers $|w,n\rangle$. A similar transformation can be performed for this black hole sigma model as well, and will give us new, T-dual extended string configurations corresponding to each geodesic of the previous section.

In this case, the analogue of T-duality is the axial-vector interchange of the symmetry that is being gauged. A special feature of the black hole sigma model is that it is self dual under this interchange, that is if the symmetry we gauge is vectorial $g \to g' = \exp(\epsilon \sigma_3) g \exp(-\epsilon \sigma_3)$, then the resultant coset sigma model turns out to have the same target space as in the former case Eq.1. In this case, the diagonal entries of the $SL(2,\mathbb{R})$ matrix are gauge invariant and form the coordinates for the extended black hole spacetime. Since $ab = 1 - uv$, the asymptotically flat region I ($uv < 0, ab > 1$) in front of the horizon is dual to the region V ($uv > 1, ab < 0$) on the other side of the singularity.

We can therefore consider one of the matrices that represent a geodesic of the black hole geometry, and now regard the entries $a(\tau, \sigma)$ and $b(\tau, \sigma)$ as the embedding coordinates of a string worldsheet in the dual region. This is equivalent to right multiplication by $i\sigma_2$, and hence will also give a classical solution of the $SL(2,\mathbb{R})$ theory. This will however result in flipping the right moving $J^3$ quantum number which will not be compatible with the level matching condition of the axially gauged theory (unless the quantum number is zero).

If, in addition to right multiplication by $i\sigma_2$, we perform the worldsheet $\tau \to \sigma$ transformation, the resultant worldsheet will be a classical solution of the original string theory. Now the $J^3$ changes sign twice and hence, if the original solution was physical, the dual will also be physical.

For instance, in the matrix (2) representing a null geodesic in region I, $a = \exp(E(\tau - \sigma))$ and $b = \exp(E(\tau + \sigma))$ from which, after interchanging $\tau \to \sigma$ and reinterpreting, we get $uv = e^{2E\sigma}$ $t = -E\tau$ which we expect to be a solution in region V (and III). If we start with a timelike geodesic in region V (the horizontal black dashed curve in Fig.2), upon dualising we get folded strings $uv = -\cosh^2\beta\sigma$ that reach out to the boundary at $uv \to -\infty$, but do not reach the horizon at $uv = 0$. The black curves ending at the right boundary in Fig.2 represent this folded string at a few instants of worldsheet time. We should, using a holographic interpretation, regard these as being dual to operators of the boundary theory living at $uv \to -\infty$. Thus, we obtain “boundary operators” from the particle trajectories of region V upon dualization.

**Horizon Strings**

However, the geodesics with $E^2 < m^2$ (belonging to the Discrete series) are the most interesting of them all. Firstly, considering the $a,b$ entries from Eq.4 as representing a solution of the vectorially gauged theory, we can show that these represent the Lorentzian version of the “winding modes” of the trumpet geometry (which is region V upon Wick rotation). This is as expected if the original modes were the winding modes of the cigar since the trumpet geometry is T-dual to the cigar. In [5], these are the modes that condense in the Sine-Liouville description to form the black hole.

By multiplying the matrix in Eq.4 by $i\sigma_2$ and interchanging $\tau, \sigma$, we get a new solution of the original theory

$$u = \frac{\sin \beta \sigma}{\sin \phi} e^{-E\tau} \quad v = \frac{\sin \beta \sigma}{\sin \phi} e^{E\tau}$$

which is always inside the horizon but extends across the singularity ($0 \leq uv \leq \cosh^2 \phi$). In Fig.2, the solid vertical curves in the middle in magenta and blue are two such representative configurations, drawn at fixed $\tau$. These worldsheets are “stuck” to the horizon at $\sigma = \ldots$
0 and folded over at $\sigma = 0.2\pi$ (since we might expect that $\beta \in \mathbb{Z}$), and thus meson-like (to use the holographic terminology). This is interesting because one might wish to regard the black hole as being a condensate of these “mesons” consistent with a holographic interpretation of the finite temperature state of the dual theory.

Thus, we might choose to regard the black hole as a condensate of “mesons” as above. Or else, regard the “bound” geodesics as the degrees of freedom of the black hole visible to the external observer. These are complementary to each other in the sense of vector-axial duality and also possibly complementary to each other in the sense of black hole complementarity because the interior modes are visible only to an observer who has crossed the horizon. Note that both the geodesic and the folded string description are simultaneously applicable only in the interior of the black hole geometry. Also, in region V (which is the Lorentzian version of the trumpet geometry), only the folded string interpretation exists, there are no geodesics with $E^2 < m^2$.

**Level Matching and Physical State Condition**

For a classical solution to represent a physical state of the string theory, we need to impose the Virasoro and level matching conditions. The latter, in the case of axial of the string theory, we need to impose the Virasoro and have constructed satisfy this condition.

Upon applying either the $\tau \rightarrow \sigma$ transformation or the $i\sigma_2$ multiplication, $J^3 = \bar{J}^3$. All the geodesic solutions that we have constructed satisfy this condition.

Upon quantization and following the analysis in [5, 6], we can obtain massive states for discrete values of the quantum numbers. For instance, in the case of null geodesics, we observe imposes conditions on the various quantum numbers. For instance, in the case of null geodesics, we obtain the dispersion relation $\beta^2 = E^2$. Classically, the massive geodesics all do not satisfy the Virasoro conditions. Upon quantization and following the analysis in [8, 9], we can obtain massive states for discrete values of the quantum numbers (the difference with that work being the inclusion of spectral flow).

**Discussion**

To summarise, we have argued that the string spectrum requires spectrally flowed current algebra representations based on the Discrete series as well as the Continuous series of $SL(2, \mathbb{R})$. This construction of the full Hilbert space of the string theory has some attractive features which we would like to highlight. Although the above discussion was entirely with reference to the 2D black hole, much of it carries over, mutatis mutandis, to the BTZ black hole (and is already present in the literature [11]). Thus, one can compare calculations (at least those that pertain to the information paradox) in these two systems.

Upon quantization, the states of the Continuous series can be shown to be doubled in a manner that reflects the presence of the two asymptotically flat regions. This doubling, which is required by the representation theory of $SL(2, \mathbb{R})$, leads to entanglement of regions I and II (Fig. 1) of the black hole. The operators of the Discrete and Continuous series are then represented as 2-d matrices (besides the vertex operator part) acting on both copies of the Hilbert space [13]. The curious identity remarked in [12] following earlier work, suggests that it is necessary to include the Discrete series in the spectrum to achieve modular invariance (we have not established necessity). It is of course important to check that ghosts are absent in the spectrum of the string theory, and establish modular invariance in the hyperbolic basis that is necessitated by the black hole interpretation. The localised geodesics which extend across the horizon appear to be the right configurations to create the entangled state represented by the black hole along the lines of the ER=EPR proposal of Maldacena and Susskind [4]. It will be fascinating to investigate this using the full string description of this black hole.

Among the many questions worth exploring further, a crucial one is to relate these horizon modes with the microscopic degrees of freedom of a microcanonical description. If such a correspondence can be constructed, perhaps then one can truly say that we have a complete account of black hole entropy (not unitarity). A discouraging observation in this context is that, at the level of the gravity effective action, a Hawking-Page type transition does not occur between the linear dilaton theory and the black hole. This however, does not prevent us from exploring the issues relating to the information paradox and black hole complementarity using the black hole sigma model.

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