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Distributional Properties of Fluid Queues Busy Period and First Passage Times

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Abstract: In this paper, I analyze the distributional properties of the busy period in an on-off fluid queue and the first passage time in a fluid queue driven by a finite state Markov process. In particular, I show that the first passage time has an IFR distribution and the busy period in the Anick–Mitra–Sondhi model has a DFR distribution.

Keywords: fluid queue; busy period; IFR and DFR distributions; Laplace transform

MSC: 60G51; 60G50; 60K25

JEL Classification: C60

1. Introduction

Fluid queues are used to represent systems where some quantity accumulates or is gradually depleted over time, subject to some random (usually Markov) environment. The stochastic fluid queues offer powerful modeling ability for a wide range of real-life systems of significance. They are highly used in performance evaluation of telecommunication systems, modeling dams, health systems, environmental sciences, biology, physics, etc.; see [1] for overview and references therein. Existing theoretical frameworks of analysing this type of queues come mainly from stochastic processes theory and so-called analytic methods. In this paper, we consider two kinds of fluid queues:

The first fluid queue analyzed in this paper is the so-called on-off fluid queue. In this queue, we have $N$ sources. By an on-off $i$th flow ($i = 1, 2, \ldots, N$), we mean a $0 \rightarrow 1$ process $\xi_i(t)$, in which consecutive off periods alternate with on periods. Random on and off times are independent of each other. In this paper, we assume that the silence periods of $i$th source ($i = 1, 2, \ldots, N$) are exponentially distributed with parameter $\lambda_i$, whereas the activity periods are generally distributed with distribution function $A_i$. The inflow rate $r_i$ of each source $i$, when active, is assumed to be at least equal to the outflow rate of the buffer—which is assumed to be 1. Then, the input rate to the buffer at time $t$ equals $Z(t) = \sum_{i=1}^{N} r_i \xi_i(t)$ and the constant output rate 1, wherein the buffer content process $Q(t)$ is governed by the equation

$$\frac{dQ(t)}{dt} = \begin{cases} 
Z(t) - 1, & \text{for } Q(t) > 0 \\
(Z(t) - 1)_+, & \text{for } Q(t) = 0.
\end{cases}$$

The seminal model is the Anick–Mitra–Sondhi model, where $r_i = r$ and $A_i$ have exponential distribution; see [2] for details.

In the second fluid model, called the Markov fluid queue, the process $\{(Q(t), f(t)), t \geq 0\}$ consists of a continuous level process $Q(t) \geq 0$, a phase variable $f(t) \in \mathcal{I} := \{1, 2, \ldots, N\}$ for some fixed $N \in \mathbb{N}$ and real-valued rates $r_i$, $i \in \mathcal{S}$, such that

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the phase process \( \{ J(t), t \geq 0 \} \) is an irreducible, continuous-time Markov chain with finite state space \( I \);

- when \( J(t) = i \) and \( Q(t) > 0 \), the rate of change of \( Q(t) \) at time \( t \) is given by \( r_i \), that is, \( \frac{d Q(t)}{dt} = r_i \);

- when \( J(t) = i \) and \( Q(t) = 0 \), the rate of change of \( Q(t) \) at time \( t \) is given by \( \max\{0, r_i\} \).

The main goal of this article is to analyze the distributional properties of the busy periods and the first passage times in the fluid queue theory.

First, in the next section, we prove that in the first fluid model, the busy periods starting with an activity period of fixed source—say, \( i \)—have a completely monotone density function, hence they have a decreasing failure rate (a so-called DFR distribution). We recall that the function \( f \) satisfying
\[
(-1)^k f^{(k)}(x) \geq 0, \quad k = 0, 1, 2, \ldots,
\]
is called completely monotonic, and for a distribution function \( F \), its failure rate is defined via
\[
r_F(x) = \frac{F'(x)}{1 - F(x)}.
\]

Hence, in particular, the busy period in the Anick-Mitra-Sondhi model has a DFR distribution.

Later, we prove that for the second fluid queue, the first passage time over fixed level of the buffer content process \( Q(t) \) has an increasing failure rate (a so-called IFR distribution).

Another goal of this work is transferring old techniques from classical queueing theory to more modern queueing systems.

2. Main Results

2.1. Busy Period

Let us consider the classical on-off model, where we have \( N \) on-off sources with silence times of \( i \)-th source which are exponentially distributed with intensity \( \lambda_i \) and activity periods of \( i \)-th source have a completely monotone generic distribution \( A_i \) with the finite mean and the Laplace transform
\[
\alpha_i(\theta) = \int_0^\infty e^{-\theta x} A_i(dx).
\]

Source \( i \) constantly transmits at rate \( r_i \geq 1 \) when active and the total output rate from the buffer equals 1. Note that silence periods are exponentially distributed with the intensity \( \lambda = \sum_{i=1}^N \lambda_i \).

The busy period, denoted by \( P_i \), starts from activity of the \( i \)-th source, and let \( B_i \) be its distribution function with Laplace transform \( \pi_i(\cdot) \).

Our first main result is the following theorem.

**Theorem 1.** \( P_i \) (\( i = 1, \ldots, N \)) has a completely monotone distribution.

By [3] (Prop. 5.9A, p. 75) we can conclude the following crucial fact.

**Corollary 1.** \( P_i \) has a DFR distribution.

**Proof.** Since the speed at which the buffer size increases at some time \( t \), which is completely determined by which sources are active at \( t \), some specific sources may be given priority for transmission by the queue without affecting the busy periods. Isolating some source \( i \), we may assume that sources \( j, j \neq i \) are given preemptive priority over source \( i \). Hence, using the work conservation arguments, we can conclude that, as far as busy periods are concerned, everything works as if all the sources \( j \neq i \) were replaced by a single on-off source \( i' \) (with preemptive priority over \( i \)). In other words, sources \( j \neq i \) are treated as if source \( i \) were absent, hence they generate their own busy periods, namely the busy periods of the model without source \( i \). Moreover, the source \( i \) is served only during the corresponding idle periods. This argument produce the following key equation for the Laplace transforms of the busy periods.
\[ \pi_i(\theta) = a_i \left[ \lambda_i (r_i - 1) (1 - \pi_i(\theta)) + \sum_{j \neq i} \lambda_j r_j (1 - \pi_j(\theta)) \right]. \] (1)

We refer to [4] (Thm. 5.2) for detailed proof of this fact.

We will now mimic the idea presented in [5]. First, assume that some probability distribution functions \( K \) and \( L_i \) \((i = 1, \ldots, N)\) are completely monotone with the Laplace transforms \( k(\cdot) \) and \( l_i(\cdot) \), respectively. Then random variable with the Laplace transform:

\[ \beta(\theta) = k \left[ \theta + \sum_{i=1}^{N} c_i (1 - l_i(\theta)) \right], \]

where \( c_i \geq 0 \), has a completely monotone distribution function. Indeed, denoting \( c = \sum_{j=1}^{N} c_j \)

\[ \beta(\theta) = \sum_{m=0}^{\infty} \int_{0}^{\infty} e^{-ax} e^{-cx} \left( x \sum_{j=1}^{N} c_j l_j(\theta) \right)^m / m! \ K(dx) \]

and it is the transform of sums of convolutions of positive measures with purely positive support. Moreover, \( \beta(0) = k(0) = 1 \). Hence, \( \beta(\cdot) \) is the transform of a probability measure with positive support and it is the Laplace transform of some random variable. Furthermore, since \( K \) is completely monotone, it is the limit in distribution of some sequence of finite mixtures of exponentials. It is then clear that we need to prove the above statement only when \( K \) is an exponential distribution with parameter, say, \( s \). In this case,

\[ \beta(\theta) = s / \left\{ \theta + s + c - \sum_{j=1}^{N} \sum_{i=1}^{K_i} p_{ij} \gamma_{ji} / (\theta + \gamma_{ji}) \right\}, \]

where \( K_{ji} \) is finite and \( p_{ji}, \gamma_{ji} > 0 \) with \( \sum_{j=1}^{K_i} p_{ji} = 1 \). The denominator is an increasing function of \( \theta \) between its simple poles at \( -\gamma_{ji} \). It is then clear that this denominator has a simple zero between each pair of adjacent poles and a zero in the intervals \((-\infty, -\max_{j,i} \gamma_{ji})\) and \((-\min_{j,i} \gamma_{ji}, 0)\). Hence, there are \( \sum_{i=1}^{N} K_i + 1 \) zeros on the negative real \( \theta \) axis. The observation that residues are all positive completes the proof of above statement.

Consider now the sequence of distribution functions with positive support whose Laplace transforms are generated recursively by:

\[ \pi_{i,0}(\theta) = 1, \]

\[ \pi_{i,m+1}(\theta) = a_i \left[ \lambda_i (r_i - 1)(1 - \pi_{i,m}(\theta)) + \sum_{j \neq i} \lambda_j r_j (1 - \pi_{j,m}(\theta)) \right]. \]

Because of the positive support and finite first moment of \( A_i \), the Laplace transform \( a_i \) decreases with \( \theta \) and \( a'_i(\theta) < 0 \) on \((0, \infty)\). Hence,

\[ \pi_{i,m+1}(\theta) - \pi_{i,m}(\theta) = \int_{0}^{\pi_{i,m+1}(\theta) - \pi_{i,m}(\theta)} \frac{r_i \theta + \lambda_i (r_i - 1)(1 - \pi_{i,m}(\theta)) + \sum_{j \neq i} \lambda_j r_j (1 - \pi_{j,m}(\theta))}{r_i \theta + \lambda_i (r_i - 1)(1 - \pi_{i,m-1}(\theta)) + \sum_{j \neq i} \lambda_j r_j (1 - \pi_{j,m-1}(\theta))} a'_i(u) \ du < 0. \]

Since \( \pi_{i,m}(\theta) \geq a_i(\theta + \lambda) \), the sequence \( \pi_{i,m} \) decreases to \( \pi_i \) solving (1). The previous observation completes the proof. \( \square \)

2.2. First Passage Times

We will consider now the second fluid queue model with random environment process \( J(t) \) on the finite state space \( \mathcal{I} = \{1, \ldots, N\} \). Recall that \( r_i \) is a netput intensity when \( J(t) = i \). We order set \( \mathcal{I} \) in such a way that \( r_i \leq r_j \) as \( i < j \). We will discretize the time and consider the Markov chain
X_n = (Q(\frac{n}{m}), I(\frac{n}{m})) where Q(t) is a buffer content process at time t. We will consider partial ordering on R × I with (x, i) ≤ (y, j) if x ≤ y or x = y and i ≤ j.

**Theorem 2.** The first passage time \( \tau(x) = \inf\{t \geq 0 : Q(t) > x\} \) has an increasing failure rate (IFR).

**Proof.** In the first step, we prove that \( \tau^m(x) = \min\{n \in N : X_n \geq (x, 1)\} \) is IFR; then, we approximate \( \tau(x) \) by \( \tau^m(x) \) as \( m \to \infty \). To prove the first statement by [6] (Theorem 3.1), it suffices to prove that \( P(X_1 \leq z|X_0 = w) \) is \( TP_2 \) in \( z \) and \( w \) belonging to \( R \times I \), which means that for \( x < y, a < b \) and \( i_3 < i_4 \), if \( a = b \) or \( i_1 < i_2 \) if \( x = y \), we have

\[
P(X_1 \leq (y, i_2)|X_0 = (a, i_3)) \leq P(X_1 \leq (x, i_1)|X_0 = (a, i_3))
\]

or that

\[
P(X_1 \leq (x, i_1)|X_1 \leq (y, i_2), X_0 = (a, i_3)) \geq P(X_1 \leq (x, i_1)|X_1 \leq (y, i_2), X_0 = (b, i_4)).
\]

Now, observe that the LHS of Equation (2) equals 1 if \( x > a + r_{i_3} \). Similarly, RHS equals 0 if \( x < b + r_{i_4} \). For the same reasons, \( y \geq b + r_{i_4} > a + r_{i_3} \). Hence, the only possible left case is when \( a = b \) and \( a + r_{i_4} < x < a + r_{i_3} \), which is impossible by our assumptions. To conclude, \( \tau^m(x) \) is IFR, which is equivalent to the statement that \( \log P(\tau^m(x) > t) \) is concave.

It suffices thus to prove now that \( P(\tau(x) > t) = \lim_{m \to \infty} P(\tau^m(x) > t) \) or that \( P(\sup_{s \leq t} Q(s) \leq x) = \lim_{m \to \infty} P(\sup_{k/m \leq t} X_k \leq x) \). Note that \( \sup_{s \leq t} Q(s) - \sup_{k/m \leq t} X_k \leq r_{1/m} \to 0 \) as \( m \to \infty \). Moreover, clearly, \( \sup_{s \leq t} Q(s) \) has a well-defined density function. This completes the proof. \( \square \)

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