Continuous actions on wrapped Fukaya categories

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Abstract

We establish the continuous functoriality of wrapped Fukaya categories, yielding a way to probe the homotopy type of the automorphism group of a Liouville sector. In the case of a cotangent bundle, we show that the Abouzaid equivalence between the wrapped category and the \( \infty \)-category of local systems intertwines our action with the action of diffeomorphisms of the zero section; this shows that if homotopy groups of the diffeomorphism group survive in the string topology algebra, so do homotopy groups in the Liouville automorphism group. In particular, our methods yield a typically non-trivial map from the rational homotopy groups of Liouville automorphisms to the rational string topology algebra of the zero section.

Our methods employ several tools from the theory of \( \infty \)-categories, especially as they relate to \( A_\infty \)-categories and their localizations. Having found a dearth of written resources for this toolkit, we give brief expositions for the reader’s convenience.

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1 Introduction

Liouville manifolds, and Liouville sectors more generally, are fruitful objects of study in symplectic geometry. They give rise to interesting geometric questions whose solutions are amenable to flexible, and often topological, techniques. However, there are fundamental questions about Liouville sectors whose answers are wanting in the literature.

One question concerns the study of automorphisms of Liouville sectors. For compact symplectic manifolds the automorphism groups $\text{Symp}$ and $\text{Ham}$ enjoy enticing formal properties—$\text{Ham}$ is simple, for example, and its group isomorphism class is a complete invariant of a compact symplectic manifold. Our knowledge of these groups is quite sparse in most examples. The same is true concerning automorphisms of Liouville sectors.

A second question concerns what continuous dependence is enjoyed by wrapped Fukaya categories of Liouville sectors. While it is known that stop removal and open inclusions of Liouville sectors induce functors on their wrapped Fukaya categories [GPS17, GPS18b], it has remained unclear whether wrapped categories display a dependence on families of embeddings—that is, on isotopies. We initiate the study of such continuous dependencies. (They exist.)

These questions are related, of course; one can use continuous functoriality to probe spaces of automorphisms via Floer-theoretic invariants. The present work is the first, as far as we know, that produces non-trivial elements in the homotopy groups of Liouville automorphism spaces by precisely constructing such a toolkit.

Here is a sample application of our methods. Let $Q$ be a compact, oriented manifold, and let $\text{hAut}(Q)$ denote the space of those continuous maps $Q \to Q$ that happen to be homotopy equivalences. We note that the inclusion

$$\text{Diff}(Q) \to \text{hAut}(Q)$$

of orientation-preserving diffeomorphisms is continuous. Likewise, the inclusion

$$\text{Diff}(Q) \to \text{Aut}^o(T^*Q)$$

to the space of Liouville automorphisms is continuous.$^1$ Thus both maps induce maps of homotopy groups, and in particular, of rational homotopy groups. We prove the following at the end of Section 1.1:

**Theorem 1.0.1.** If $Q$ is simply connected, any element of the rational homotopy groups of $\text{Diff}(Q)$ that survives in $\text{hAut}(Q)$ detects a non-trivial element in the rational homotopy groups of $\text{Aut}^o(T^*Q)$.

More precisely, if an element $\alpha \in \pi_{\geq 2} \text{Diff}(Q) \otimes \mathbb{Q}$ has non-zero image in $\pi_{\geq 2} \text{hAut}(Q) \otimes \mathbb{Q}$, then it also has non-zero image in $\pi_{\geq 2} \text{Aut}^o(T^*Q) \otimes \mathbb{Q}$.

The map $\text{Diff} \to \text{hAut}$ has long been a subject of interest in smooth topology; as a result of the above theorem, computations in smooth topology yield non-trivial elements in the rational homotopy groups of $\text{Aut}^o(T^*Q)$. (See Remark 1.2.4.)

1.1 Main results

Let $M$ be a Liouville sector, and let $\text{Aut}^o(M)$ denote the topological group of Liouville automorphisms (Definition 3.5.2). The most basic theorem of this work is that $\text{Aut}^o(M)$ acts coherently on the wrapped Fukaya category of $M$:

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$^1$See Convention 1.1.2 for an explanation of the superscript in $\text{Aut}^o$. 

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**Theorem 1.1.1.** For any Liouville sector $M$, there exists an $A_\infty$ homomorphism

$$\text{Aut}^o(M) \to \text{Aut}(W(M))$$

from the space of Liouville automorphisms of $M$ to the space of automorphisms of the wrapped Fukaya category of $M$.

We refer the reader to Theorem 1.3.2 for a more precise statement. For example, it would be unnatural to treat the space of automorphisms of a Fukaya category as a topological group with emphasis on its homeomorphism class. We rather treat $\text{Aut}(W(M))$ as a group-like $A_\infty$-algebra in the $\infty$-category of spaces.

(Regardless, for the purposes of this introduction, the reader will not lose too much intuition by imagining that Theorem 1.1.1 produces a continuous group homomorphism between two topological groups.)

Before we go on, we should be explicit about what decorations we put on $M$—this affects what kind of category $W(M)$ is. For example, Theorem 1.1.1 as stated is only true for $W(M)$ being two-periodically graded and linear over $\mathbb{Z}/2\mathbb{Z}$. This is because a Liouville automorphism $\phi \in \text{Aut}^o(M)$ does not “know” how to respect any choice of grading $gr$, nor of background class $b \in H^2(M; \mathbb{Z}/2\mathbb{Z})$—and such data would be required to form a $\mathbb{Z}$-graded and $\mathbb{Z}$-linear Fukaya category. Moreover, given any class of decorations one imposes on $M$, there is a natural automorphism group consisting of Liouville automorphisms equipped with data respecting these decorations. (See Section 3.6 for details.)

**Convention 1.1.2.** Because the notation will become cumbersome, and because none of our methods depend in any essential way on decorations, we will often suppress decoration choices from the notation unless it is necessary (as it sometimes will be) to highlight the choices. We will write

$$\text{Aut}^o(M), \qquad \text{Aut}^{gr,b}(M), \qquad \text{Aut}(M)$$

to respectively denote the group of Liouville automorphisms, of Liouville automorphisms with data respecting a chosen $gr$ and $b$, and of Liouville automorphisms with data respecting some unspecified decorative choices.

In general, the reader should assume that any statement about the non-superscripted Aut is true for all possible decorations, while a statement about $\text{Aut}^{gr,b}$ is only true for the wrapped category of a Liouville sector $M$ equipped with a grading and background class, and a statement about $\text{Aut}^o(M)$ is true for the undecorated case.

We will write $W(M)$ for the wrapped category, suppressing dependence on decorations.

Then our methods also yield:

**Theorem 1.1.3.** For any Liouville sector $M$ equipped with decorations, there exists an $A_\infty$ homomorphism

$$\text{Aut}(M) \to \text{Aut}(W(M))$$

from the space of Liouville automorphisms of $M$ (equipped with data respecting chosen decorations) to the space of automorphisms of the wrapped Fukaya category of $M$.

In Theorem 1.1.3, the wrapped category on the right depends on the decorations, but we suppress this dependence from the notation (following Convention 1.1.2).

**Remark 1.1.4.** It turns out that in the examples we care about, this decorated automorphism group is not too homotopically different from $\text{Aut}^o(M)$ itself—for example, when the decorations are a grading $gr$ and a background class $b \in H^2(M; \mathbb{Z}/2\mathbb{Z})$, then the forgetful map $\text{Aut}^{gr,b}(M) \to \text{Aut}^o(M)$ induces an isomorphism on $\pi_{\geq 3}$. (See Proposition 3.6.8.)
By taking based loop spaces and applying Dunn’s additivity theorem (or, if one likes, May’s recognition principle), we obtain:

**Corollary 1.1.5.** There exists a map of $E_2$-algebras

$$\Omega \text{Aut}(M) \to \Omega \text{Aut}(W(M)).$$

where the base points of both are taken at the identity elements.

**Remark 1.1.6.** By generalities concerning $A_\infty$-categories, one can identify the endomorphisms of the identity functor of any $A_\infty$-category $\mathcal{C}$ with the (topological space associated to the) non-positive truncation of the Hochschild cochains of $\mathcal{C}$. The based loop space at the identity functor $\text{id}_C$ is precisely the space of those invertible endomorphisms—i.e., natural equivalences of the identity functor. Put another way, the based loop space $\Omega \text{Aut}(W(M))$ is the space of units of the Hochschild cochain algebra. The $E_2$-algebra structure of $\Omega \text{Aut}(W(M))$ coincides with the $E_2$-algebra structure inherited from the Hochschild cochains of $W(M)$.

By taking homotopy groups of domain and target, we have the following:

**Corollary 1.1.7.** The above maps induce group homomorphisms

$$\pi_{k+1} \text{Aut}(M) \to \begin{cases} HH^0(W(M))^\times & k = 0 \\ HH^{-k}(W(M)) & k \geq 1. \end{cases}$$

Here, $HH^0(W(M))^\times$ indicates the units of the multiplication of degree 0 Hochschild cohomology elements, and the homomorphism is with respect to this multiplication in degree 0. For $k \geq 1$, the group structure on the target is additive.

To put the above corollary in context, recall that the Hochschild cohomology of the Fukaya category of $M$ is isomorphic to the quantum cohomology ring of $M$ in various contexts: when $M$ is monotone [She16], and when $M$ is toric [AFO+], for example. So one can view the above results as an analogue of the Seidel homomorphism $\pi_1 \text{Ham}(M) \to QH(M)^\times$ [Sei97], but generalized to higher homotopy groups of the appropriate analogue of Ham in the Liouville setting. Indeed, one of our main geometric constructions is inspired by work of Savelyev who generalized the Seidel homomorphism to higher homotopy groups of Ham in the monotone setting [Sav13].

We now apply the above results to the special case when $M = T^*Q$ is a cotangent bundle. We fix an orientation on $Q$ and we choose $b \in H^2(M; \mathbb{Z}/2\mathbb{Z})$ to be pulled back from the second Stiefel-Whitney class of $Q$. We also fix the canonical grading on $T^*Q$.

Then a result of Abouzaid [Abo12] states that the resulting wrapped Fukaya category $W(T^*Q)$ is equivalent to the $A_\infty$-category $\text{Loc}(Q)$ of local systems on the zero section $Q$; in turn, $\text{Loc}(Q)$ is equivalent to $\text{Mod}(C_*(\Omega Q))$, modules over the $A_\infty$-algebra of chains on the based loop space. We thus have a quasi-isomorphism of chain complexes from the Hochschild chain complex to chains on the free loop space:

$$C_*^{\text{Hoch}}(W(T^*Q)) \simeq C_*^{\text{Hoch}}(\text{Mod}(C_*(\Omega Q))) \simeq C_*(\mathcal{L}Q)$$

where the second equivalence is due to Goodwillie [Goo85] and Burghelea-Fiedorowicz [BF86]. Finally, by a generalization of Poincaré duality, a choice of fundamental class on $Q$ results in an equivalence.

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2Given any cochain complex $C$, the non-positive truncation $\tau^{\leq 0} C$ is the cochain complex whose $i$th group is given by $C^i$ if $i < 0$, by 0 if $i > 0$, and by $\ker d^0$ if $i = 0$. The space associated to a non-positive cochain complex is constructed via the Dold-Kan correspondence.
isomorphism of graded commutative algebras\(^3\)

\[ HH^*(C_*QQ) \simeq H_{*+\dim Q}(C_*LQ) \]

from the Hochschild cohomology groups of the \(A_\infty\)-algebra \(C_*\Omega Q\) to the shifted homology of the free loop space. We thus have the following:

**Corollary 1.1.8.** Let \(Q\) be compact and orientable, and choose an orientation of \(Q\). The above maps induce group homomorphisms

\[ \pi_{k+1} \operatorname{Aut}^{gr,b}(T^*Q) \to \begin{cases} H_{\dim Q}(LQ)^\times & k = 0 \\ H_{k+\dim Q}(LQ) & k \geq 1 \end{cases} \quad (1.1) \]

That is, the homotopy groups of \(\operatorname{Aut}^{gr,b}(T^*Q)\) map to the homology of the free loop space of the zero section.

(As before, when \(k = 0\), the group structure on the target is multiplicative with respect to the string multiplication on \(C_*LQ\), and when \(k \geq 1\), the group structure is the additive one on homology.)

Note that because \(\operatorname{Aut}^{gr,b} \to \operatorname{Aut}^o\) induces an isomorphism on homotopy groups in degrees \(\geq 3\) (Proposition 3.6.8), the corollary also allows us to map these higher homotopy groups of \(\operatorname{Aut}^o\) to the homology of the free loop space of \(Q\).

At this point, the reader may be concerned that the results above are trivially true: One could always construct a trivial action, and induce null homomorphisms. To address this, note that there is a natural action of the diffeomorphism group \(\operatorname{Diff}(Q)\) on \(\operatorname{Loc}(Q)\).\(^4\) On the other hand, any diffeomorphism \(\phi\) induces a Liouville automorphism by pushing forward the effect of \(\phi\) on cotangent vectors. This induced map can be lifted to naturally respect the choices of \(b\) and \(gr\), so we have a map \(\operatorname{Diff}(Q) \to \operatorname{Aut}^{gr,b}(T^*Q)\).

**Theorem 1.1.9.** Let \(Q\) be closed and oriented. The diagram

\[
\begin{array}{ccc}
\operatorname{Diff}(Q) & \longrightarrow & \operatorname{Aut}^{gr,b}(T^*Q) \\
\downarrow & & \downarrow \text{Thm 1.1.1} \\
\operatorname{Aut}(\operatorname{Loc}(Q)) & \xrightarrow{\sim} & \operatorname{Aut}(\mathcal{W}(T^*Q))
\end{array}
\]

commutes up to homotopy. Here, the bottom arrow is induced by the Abouzaid equivalence [Abo12], the left vertical arrow is induced by the action of \(\operatorname{Diff}(Q)\) on \(\operatorname{Loc}(Q)\), and the top horizontal arrow takes a diffeomorphism to an automorphism of the cotangent bundles.

Informally, Theorem 1.1.9 states that a version of the Abouzaid equivalence \(\mathcal{W}(T^*Q) \simeq \operatorname{Loc}(Q)\) can be made equivariant with respect to the natural \(\operatorname{Diff}(Q)\) action on both categories.

**Corollary 1.1.10.** Let \([\alpha] \in \pi_{k+1} \operatorname{Diff}(Q)\) and suppose the image of \([\alpha]\) is non-zero under the map \(\pi_{k+1} \operatorname{Diff}(Q) \to HH^{-k}(C_*\Omega Q)\) from (1.1). Then the image of \([\alpha]\) is non-zero in \(\pi_{k+1} \operatorname{Aut}^{gr,b}(T^*Q)\). In particular, such \([\alpha]\) detect non-trivial elements in the homotopy groups of \(\operatorname{Aut}^{gr,b}(T^*Q)\).

\(^3\)See for example Malm [Mal11]; this is in fact an isomorphism of BV algebras with the string topology operations on the right-hand side, and the natural BV structure on Hochschild cohomology on the left-hand side.

\(^4\)Indeed, this factors through the homotopy automorphisms of \(Q\).
Now let \( \text{hAut}(Q) \) denote the space of those continuous maps \( f: Q \to Q \) that happen to be homotopy equivalences. By taking based loops, and noting that the action of Diff on \( \text{Loc} \) factors through \( \text{hAut} \) (as in Footnote 4), we have the following:

**Corollary 1.1.11.** Let us assume \( Q \) is compact and oriented. The following diagram commutes:

\[
\begin{array}{ccc}
\pi_{k+1} \text{Diff}(Q) & \longrightarrow & \pi_{k+1} \text{Aut}^{gr,b}(T^*Q) \\
\downarrow & & \downarrow \\
\pi_{k+1} \text{hAut}(Q) & \longrightarrow & H_{\dim Q + k}(LQ)
\end{array}
\]

where the maps for \( k = 0 \) are group homomorphisms when the lower-right corner is taken to be the multiplicative units of \( H_{\dim Q + k}(LQ) \).

**Example 1.1.12.** Let \( Q = S^1 \) and take \( R = \mathbb{Z} \). We must contemplate the composite

\[
\mathbb{Z} \cong \pi_0 \Omega \text{Diff}(S^1) \simeq \pi_0 \Omega \text{hAut}(S^1) \to (HH^0(C_\ast \Omega S^1))^\times \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.
\]

To understand the last map, we note the equivalence \( C_\ast \Omega S^1 \simeq \mathbb{Z}[x^\pm 1] \) as an \( A_\infty \) algebra. The 0th Hochschild cohomology is (the center of) this ring, and its units are the monomials of the form \( \pm x^i \) for \( i \in \mathbb{Z} \); the homomorphism \( \mathbb{Z} \cong \pi_1 \text{hAut}(S^1) \to (HH^0)^\times \) is given by \( i \mapsto x^i \).

In particular, Theorem 1.1.9 shows that \( \pi_1 \text{Diff}(S^1) \to \pi_1 \text{Aut}^{gr,b}(T^*S^1) \) is an injection. While less is known about the space of diffeomorphisms of tori \( T^n \), we regardless produce non-trivial elements in \( \pi_1 \text{Aut}^{gr,b}(T^*T^n) \) straightforwardly using the same method.

The above results imply Theorem 1.0.1:

**Proof of Theorem 1.0.1.** A result of Felix-Thomas [FT04] states that the map

\[
\pi_{k+1} \text{hAut}(Q) \otimes \mathbb{Q} \to H_{\dim Q + k}(LQ; \mathbb{Q})
\]

is an injection. (See Remark 1.2.5.) Moreover, the map \( \text{Aut}^{gr,b}(T^*Q) \to \text{Aut}^o(T^*Q) \) induces an injection on \( \pi_k \) for \( k \geq 2 \). (See Proposition 3.6.8.) Now combine Corollary 1.1.10 and Corollary 1.1.11.

\[\square\]

**1.2 Future directions**

We enumerate natural avenues of pursuit through a sequence of remarks:

**Remark 1.2.1** (Other notions of automorphisms). While we have taken \( \text{Aut}(M) \) and \( \text{Aut}^o(M) \) to be natural notions of automorphism group of a Liouville sector \( M \), it is not clear that these are the only such natural choice (even up to homotopy equivalence). In another direction, one could contemplate the automorphism group of the skeleton of a Liouville manifold; the correct notion of automorphism presumably depends both on the stratified homotopy type of the skeleton, and on some tubular or infinitesimal differential-geometric data attached to the skeleton (to be able to recover the equivalence type of its wrapped Fukaya category, for example).
Remark 1.2.2 (Filtered enhancements). When \( M = T^*Q \), all our computations “factor” through the homotopy type of \( M \cong Q \); this is unsurprising given that the wrapped Fukaya category of \( T^*Q \) depends only on the homotopy type of \( Q \) and our techniques incurably pass through wrapped Floer constructions. However, we suspect that by filtering automorphism groups in a way compatible with the action filtration on Floer complexes, one can glean far richer symplectic data.

Remark 1.2.3 (Is symplectic geometry helping differential topology, or vice versa?). Even for the case of \( M = T^*Q \) with \( Q \) compact and oriented, we note that the factorization \( \pi_* \text{Diff}(Q) \to \pi_* \text{hAut}(Q) \to H_{s+\dim Q}(LQ) \) from Corollary 1.1.11 is difficult to study using tools of homotopy theory. It is not yet clear in which direction information will naturally flow in the future: Whether the higher homotopy groups of \( \text{Aut} \) are amenable to Floer-theoretic techniques (and hence yield information about the relation between \( \text{Diff} \) and \( \text{hAut} \)), or whether the homotopy theory will yield information about Liouville automorphisms.

Remark 1.2.4. At present, the relation between \( \text{Diff} \) and \( \text{hAut} \) is best understood rationally, and the best-understood examples are the even-dimensional “genus \( g \)” manifolds \( (S^n \times S^n)^g \) obtained by taking the connect sum of \( S^n \times S^n \) \( g \) times. Even better understood are the manifolds obtained by removing a small open disk from \( (S^n \times S^n)^g \). The obvious low-hanging fruit is to detect rational homotopy groups of \( \text{Aut} \) of the cotangent bundles of these manifolds (with boundary). Indeed, computations by Berglund and Madsen [BM14] already produce non-trivial elements in light of Theorem 1.0.1.

Remark 1.2.5 (The present invariant factors through linearization). Again when \( Q \) is compact and oriented, the map \( \Omega \text{hAut}(Q) \to C_*(LQ)[-n] \) (with homological shifting conventions) has another description. At the level of homotopy/homology groups, it is equal to the composition

\[
\pi_* \Omega \text{hAut}(Q) \to H_* \Omega \text{hAut}(Q) \xrightarrow{[Q]} H_* \Omega \text{hAut}(Q) \otimes H_n(Q) \xrightarrow{H_* \text{ev}} H_{s+n}(LQ).
\]

The first arrow is the Hurewicz map, the next is tensoring with a choice of fundamental class, and the last is induced by the evaluation map \( \Omega \text{hAut}(Q) \times Q \to LQ \). The verification of this presentation is independent of any Floer theory. (We also remark that this composition was considered by Felix-Thomas [FT04] and, rationally, was shown to induce an isomorphism between the rational homotopy groups of \( \Omega \text{hAut}(Q) \) and the first filtered piece of the Hodge filtration on Hochschild (co)homology.)

In other words, outside of \( \pi_0 \), the information our actions obtains about \( \text{Aut} \) factors through the Hurewicz map of \( \text{hAut} \), and in fact of \( \text{Aut} \) as well. This is not satisfying, and at the very least, one would hope for a construction that factors through the stable homotopy type (which sees more about the topology of a space than its homology). This is another good reason to seek situations when we can define wrapped Fukaya categories over the sphere spectrum.

To explain: The fundamental reason that all maps factor through homology is because of an adjunction between spaces and chain complexes, whose unit is given by the Hurewicz map. (The adjunction has left adjoint given by the singular chains functor and right adjoint the Dold-Kan construction.) The chain complexes here arise because the wrapped category is an \( A_\infty \) category over a classical ring.

If rather we had a version of the wrapped category defined over the sphere spectrum, the unit of the adjunction would be the stabilization map \( X \to QX = \Omega^\infty \Sigma^\infty X \), which means all our higher homotopy invariants would factor through the stable homotopy groups of \( \text{Aut} \) rather than its homology.

Finally, we note it is inevitable that the higher homotopy invariants factor through the stable homotopy type using Fukaya-categorical methods, as any invariant obtained by acting on a linear
object—like a stable ∞-category—will always factor through the linearization of the groups. A
less tautological construction (for example, by transferring to a group whose stable homotopy type
detects more about the unstable homotopy type of Aut) would get around this.

Remark 1.2.6. Let us state two further reasons we take an interest in the continuous functoriality
of Fukaya categories. First, and independently of any symplectic considerations, many useful al-
gebraic invariants now have concrete geometric interpretations. The study of symplectic geometry
has benefitted from the congruence of the geometry of symplectic phenomena with the geometry
of algebraic phenomena. For example, the circle action on framed $E_2$-algebras, known as the BV
operator in characteristic 0, is visible in the Fukaya categories having Calabi-Yau like properties
simply by geometric rotation of Reeb orbits. Another example is the ability to encode the para-
cyclic structure of the s-dot construction of Fukaya categories using configurations of points on the
boundary of a disk [Tan19]. Our Corollary 1.1.5 follows this storyline.

Second, the moduli space of embeddings $M \to M'$ also encodes a part of the moduli space of
Lagrangian correspondences, or more generally of bimodules between Fukaya categories. Via mirror
symmetry, understanding this moduli space on the A model is expected to yield insights about the
moduli space of bimodules between derived categories of sheaves on the B model.

1.3 Outline of proofs

As mentioned in the introduction, decorating $M$ with a grading or a background class $b \in H^2(M; \mathbb{Z}/2\mathbb{Z})$
forces us to change our automorphism group. Regardless, we follow Convention 1.1.2, and when a
result below holds regardless of which decorations we demand of $M$, we will simply write Aut for
the automorphism group, with decorational choices suppressed from the notation.

To describe our proofs, let us first state a precise form of Theorem 1.1.1 and Theorem 1.1.3.

We let $\mathbb{B} \text{Aut}(M)$ denote the classifying ∞-category of the $A_\infty$-space Aut($M$). It has a single
object with endomorphism space Aut($M$).

Choice 1.3.1. We also fix a base ring $R$ and we let $\text{Cat}_{A_\infty}$ denote the ∞-category of $A_\infty$-categories
over $R$. By choosing appropriate structures on $M$ and appropriate brane structures on our La-
grangian, we assume $W(M)$ is $R$-linear. (For example, with the usual relative Pin structures on
our branes, one can take $R$ to be $\mathbb{Z}$. When gradings are chosen, we can take $\text{Cat}_{A_\infty}$ to consist of
usual $A_\infty$-categories, while when we have no gradings, we must demand that our $A_\infty$-categories
and functors are 2-periodic.)

We let $\text{Aut}(W(M))$ denote the space of $R$-linear automorphisms of $W(M)$.

Theorem 1.3.2. There exists a functor of ∞-categories

$$\mathbb{B} \text{Aut}(M) \to \text{Cat}_{A_\infty}$$

sending a distinguished base point of the domain to the wrapped Fukaya category $W(M)$. In
particular (by taking based loops) one has a map

$$\text{Aut}(M) \to \text{Aut}(W(M))$$

of group-like $A_\infty$-spaces.

Remark 1.3.3. “Group-like $A_\infty$-spaces” may seem a mouthful. The intuition is that the map
$\text{Aut}(M) \to \text{Aut}(W(M))$ is morally a continuous group homomorphism; that is, $\text{Aut}(M)$ acts con-
tinuously on the wrapped Fukaya category.
To construct the functor (1.2) we use an infinite-categorical version of the following tautology: The barycentric subdivision of a space is homotopy equivalent to the original space. While one imagines one could get away without such a trick, this saves us an enormous amount of analytic legwork.

To explain this, let us denote by $B \text{Aut}(M)$ the classifying space. Let $\text{Simp}(B \text{Aut}(M))$ denote the following category (in the classical sense): An object is a continuous map $j : |\Delta^n| \to B \text{Aut}(M)$ from an $n$-simplex to the classifying space. A morphism from $j$ to $j'$ is the data of an injective poset map $[n] \to [n']$ such that the induced composition $|\Delta^n| \to |\Delta^n| \xrightarrow{j'} B \text{Aut}(M)$ is equal to $j$. To see why we call this a model for a barycentric subdivision, we encourage the reader to fix some $j$ and enumerate all the objects admitting a map to $j$. We note that the only invertible morphisms are the identity morphisms.

Then the infinite-categorical version of the above tautology is as follows: If we localize this strict category along all morphisms, one obtains the infinite-category $\mathbb{B} \text{Aut}(M)$. (See Proposition 4.5.3.)

Thus, by the universal property of localizations, to construct a functor from $\mathbb{B} \text{Aut}(M)$ to $\text{Cat}_{A_{\infty}}$ as in Theorem 1.3.2, we need only construct a functor $W : \text{Simp}(B \text{Aut}(M)) \to \text{Cat}_{A_{\infty}}$ for which every morphism of the domain is sent to an equivalence of $A_{\infty}$-categories.

Warning 1.3.4. While this is all well and good, our geometric constructions require the composition $|\Delta^n| \to B \text{Aut}(M) \to B\text{Aut}^o(M)$ to be smooth, and there are several non-trivial ways to articulate smoothness of a map to an infinite-dimensional entity such as $B\text{Aut}^o(M)$. We choose to utilize the technology of diffeological spaces, and prove a smooth approximation theorem in the form of Theorem 4.4.4; informally, the theorem states that the barycentric subdivision intuition is not betrayed when we restrict ourselves to considering only smooth simplices. For technical reasons, we will also be led to mainly consider smooth maps from the extended simplices $|\Delta^n_e|$ to $B\text{Aut}^o(M)$. We will ignore this point in the introduction, but we do point out that all of Section 4 is devoted to this warning.

So how do we construct $W$? Given a smooth map $j : |\Delta^n| \to B \text{Aut}(M)$, one can informally describe the wrapped Fukaya category $W_j$ as follows:

We note that there is a tautological $M$-bundle over $B \text{Aut}(M)$. Consider the $M$-bundle over $|\Delta^n|$ obtained by pulling back along $j$. $W_j$ has objects given roughly by branes living in the fibers above the vertices of $|\Delta^n|$. Its morphism complexes are obtained by localizing a simple-to-define Fukaya category $O_j$ along non-negative continuation maps (and parallel transports thereof); this localization idea follows Abouzaid-Seidel’s unpublished work, also utilized in [GPS17]. There are some subtleties here we would like to point out:

- That one can define a “family” wrapped Fukaya category over $|\Delta^n|$ for $n \neq 0$ relies on Savelyev’s observation [Sav13] that one can operadically map the tautological family of holomorphic disks to $|\Delta^n|$. We count holomorphic sections (from the disks in this family to the pulled back bundle) to define $A_{\infty}$ structure maps.

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5 Note that font distinguishes the infinite-category $\mathbb{B} \text{Aut}(M)$ from the space $B \text{Aut}(M)$. 

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• As usual, one needs to choose auxiliary data (connections and almost-complex structures) to ensure a sensible count of holomorphic curves. Our definition of $O_j$ is inductive on the dimension of the domain $|\Delta^n|$ of $j$; the inductive step requires us to take data defined on the boundary of $|\Delta^n|$ and extend to the interior. This is the step at which one must invoke that $j$ is a map to the decorated version $\text{Aut}$ of the automorphism group (so that $j$ encodes already the homotopical information needed to extend auxiliary data to the interior). (See Choice 6.2.4.)

• Independent of any familial issues, we localize $O$ along continuation maps that are defined by counting holomorphic disks with one boundary puncture (not by counting strips). This is because in $O$, there is no immediate geometric interpretation of the identity morphism of an object, so the usual Yoneda Lemma trick to relate a natural continuation map $CF^*(-, L_0) \to CF^*(-, L_1)$ (defined by counting strips) to a morphism in $CF^*(L_0, L_1)$ is unavailable. The compatibility between the strip construction and the disk-with-one-puncture construction of continuation maps is verified in Proposition 7.3.1. This result itself is well-known to experts but detailed proofs are not easily found in the literature. Because of this, we give a full proof for the reader’s benefit. See Section 7.3 for the details.

• After the above technicalities are set up, one must then verify that inclusions of simplices define $A_\infty$-functors $O_j \to O_{j'}$, and that the induced map on localizations $W_j \to W_{j'}$ are equivalences of $A_\infty$-categories. This requires a way to compute morphisms in the wrapped categories. To this end, we show (using methods analogous to [GPS17]) that a “sequential colimit” definition of wrapped Floer cochain complexes recovers the morphism complexes in $W_j$ (Lemma 9.2.2).

This concludes the summary of the construction of the functor $W$ in (1.3), and hence of the functor in Theorem 1.3.2.

Remark 1.3.5. Note that the construction involves two distinct localizations—one to pass from a family of non-wrapped “directed” Fukaya categories $O$ to a family of wrapped Fukaya categories $W$, and the other to pass from a category $\text{Simp}(B\text{Aut}(M))$ to the classifying $\infty$-category $\mathbb{B}(\text{Aut}(M))$. Both localizations are used to avoid difficult or tedious analytical constructions concerning wrapped Floer theory.

Now we explain the proof of Theorem 1.1.9 which, in the example of $M = T^*Q$, verifies that the above construction yields non-trivial actions.

Fix a smooth manifold $Q$. Note that $B\text{Diff}(Q)$ has a tautological $Q$-bundle over it, and we can pull the $Q$-bundle back along any simplex $j : |\Delta^n| \to B\text{Diff}(Q)$. By assigning to each $j$ the $A_\infty$-category of local systems on the pulled back bundle, we obtain a functor $6$

$$\text{TwC}_*\mathcal{P} : \text{Simp}(B\text{Diff}(Q)) \to \mathcal{C}_{A_\infty}.\$$

On the other hand, there is a natural inclusion $\text{Diff}(Q) \to \text{Aut}^{gr,b}(T^*Q)$, and this induces a map of their classifying spaces (and hence of their categories of simplices). Call this induced map $\mathbb{D}$, and consider the composite

$$\text{Simp}(B\text{Diff}(Q)) \overset{\mathbb{D}}{\to} \text{Simp}(B\text{Aut}^{gr,b}(T^*Q)) \to \mathcal{C}_{A_\infty}.\$$

---

$6$The notation $\text{TwC}_*\mathcal{P}$ is an artifact of a particular presentation of the $A_\infty$-category of local systems we utilize in Section 10.1. This presentation turns out to play well with the construction from [Abo12], so we will lug around the notation for this pay-off.
We thus have two functors $\mathcal{O} \circ \mathcal{D}$ and $\text{Tw}_{C_*}\mathcal{P}$ from $\text{Simp}(B\text{Diff}(Q))$ to $\text{Cat}_{A^\infty}$. In Section 10.2, we construct a natural transformation from the latter to the former. This natural transformation is a non-wrapped, family-friendly version of the construction Abouzaid utilized in [Abo12] to exhibit a local system on a Lagrangian from any brane in the Fukaya category.

The next step is to show that the natural transformation $\mathcal{O} \circ \mathcal{D} \rightarrow \text{Tw}_{C_*}\mathcal{P}$ factors through the localization $W \circ \mathcal{D}$. This requires us to prove the following geometric theorem, which we state in greater generality (beyond the case of the zero section of the cotangent bundle case):

**Theorem 1.3.6** (Theorem 10.3.1). Let $M$ be a Liouville sector, and $c : L \rightarrow L'$ a continuation element associated to a non-negative isotopy. We also fix a compact test brane $X \subset M$. Then the map of twisted complexes

$$c_* : (L \cap X, D) \rightarrow (L' \cap X, D')$$

—induced by the Abouzaid functor from $\mathcal{O}(M)$ to $\text{Tw}_{C_*}\mathcal{P}(X)$—is an equivalence.

By the universal property of localization, we conclude that the natural transformation $\mathcal{O} \circ \mathcal{D} \rightarrow \text{Tw}_{C_*}\mathcal{P}$ induces a natural transformation $W \circ \mathcal{D} \rightarrow \text{Tw}_{C_*}\mathcal{P}$. (1.4)

The remaining key step is to prove that this natural transformation is in fact a natural equivalence. This is a family version of the Abouzaid equivalence, and we prove it in Theorem 11.0.1. We utilize Abouzaid’s original result in our proof, and in particular, we must relate the quadratic definition of the wrapped complex to the localization definition. This is accomplished through a combination of Proposition 11.1.9 and Lemma 2.4.34.

As a consequence we have a (homotopy) commutative diagram of $\infty$-categories

$$
\begin{array}{ccc}
\mathcal{O} \circ \mathcal{D} & \rightarrow & \text{Tw}_{C_*}\mathcal{P} \\
\uparrow & & \downarrow W \\
\mathcal{B} \text{Diff}(Q) & \rightarrow & \mathcal{B} \text{Aut}^{gr,b}(T^*Q) \\
\downarrow \text{Tw}_{C_*}\mathcal{P} & & \downarrow \text{Cat}_{A^\infty} \\
\end{array}
$$

The commutativity of the diagram is exhibited by the natural equivalence (1.4). Note that the left-hand map sends a distinguished vertex of $\mathcal{B} \text{Diff}(Q)$ to $\text{Tw}_{C_*}\mathcal{P}$.

The final step in proving Theorem 1.1.9 is taken in Proposition 12.1.13. There, we show that this left-hand functor is naturally equivalent to the functor exhibiting the natural $\text{Diff}(Q)$ action on $\text{Loc}(Q) \simeq \text{Tw}_{C_*}\mathcal{P}$. Then, because the right-hand functor sends a vertex to $W(Q)$, Theorem 1.1.9 follows by taking based loops of each $\infty$-category in the above commutative triangle.

**Remark 1.3.7.** Let us comment on the proof of Theorem 11.0.1, which shows that our family of wrapped Fukaya categories is equivalent to a family of local system categories. A non-trivial aspect of proving this equivalence is that Abouzaid’s construction in [Abo12] utilized quadratic Hamiltonians to define wrappings; this is in contrast to the definition of $W$ in the present work (following [GPS17]), which is a result of localizing with respect to non-negative, linear Hamiltonian continuation maps. In particular, morphisms of $W$ are not so tractable using pure geometry.

Put another way, we must confront the fact that there are multiple definitions of the wrapped Fukaya category in the literature.

While possible, it is non-trivial to write down the analysis (and in particular, compactness arguments) to see that one has an $A^\infty$ algebra map between the different versions of wrapped endomorphisms, and we do not do this. Instead, we formally conclude that such an algebra map exists.
by the universal property of localizations—i.e., by making use of category theory. The analytical
legwork, via this strategy, is reduced to checking that the underlying map of endomorphism com-
plexes is a quasi-isomorphism, which one can do by straightforward arguments invoking the action
filtration and relating a cofinal sequence of linear wrappings to a single quadratic wrapping.

We refer the reader also to Section 2.2 of [Syl19] for a separate approach.

Remark 1.3.8. A consequence of Theorem 11.0.1 is that the localization-style definition of the
wrapped Fukaya category of a cotangent bundle is equivalent to Abouzaid’s quadratic definition
in [Abo12], but this equivalence is not proven by writing an explicit, analytically defined functor
from one to the other. In the present work, it is rather obtained by comparing both $A_\infty$-categories
to the $A_\infty$-category of local systems, and even relies on the (independent) generation results of
Abouzaid and of Ganatra-Pardon-Shende [GPS18b]. Our proof that $\mathcal{W} \simeq \text{TwC}_* \mathcal{P}$ differs from that
of [GPS18a] in that we compute the equivalence of endomorphism algebras in a way that passes
through and relies on Abouzaid’s construction, while [GPS18a] does not.

Of course, in principle one need not pass through the local system category: it is possible to
write down a comparison functor from the cofinally wrapped category to the quadratically wrapped
category directly, but we do not do this here.

Remark 1.3.9. As mentioned in the abstract, we make heavy use of $\infty$-categorical machinery. We
give a very brief review of some of these techniques in Section 2.

Remark 1.3.10. Let us also highlight some of the notable results we utilize in technical portions
of the work.

1. In [GPS17], the term localization is used to refer to a particular construction of an $A_\infty$
category, but the literature is not explicit about whether this construction satisfies the ho-
motopical universal properties that a localization ought to satisfy. We record that it does
in Theorem 2.4.32. This relies on various results of Faonte, Lyubashenko, Lyubashenko-
Ovisienko, Lyubashenko-Manzyuk, Tabuada, and Toën [Fao13, Lyu03, LM08, LO06, Tab05,
Tab10, Toën07], and we do indeed need to assimilate these varied results to demonstrate the
compatibility with our $\infty$-categorical applications.

2. We also prove a smooth approximation result for diffeological automorphism groups—that
is, the simplicial set of smooth simplices of $\text{Aut}^o$ is homotopy equivalent to the simplicial set
of continuous simplices. (This is necessary because our geometric constructions for Fukaya-
categorical techniques require smoothness, while our interest is in the homotopy theory of
$\text{Aut}^o$, which is a priori understood through continuous simplices.) Our technique is quite
general and applies to any diffeological space of automorphisms whose behavior outside com-
pact sets is controlled—for example, to the space of compactly supported automorphisms, or
to the Liouville automorphisms of our paper. Indeed, to prove smooth approximation, the
reader will note that the only time we specifically invoke that our automorphism group is
$\text{Aut}^o$ is in Lemma 4.6.5.

3. We also utilize something we imagine will be useful for other users of $\infty$-categories: Given
a fibration $p : E \to B$ of simplicial sets, we replace it by a functor $p^{-1} : \text{subdiv}(B) \to \text{Cat}_\infty$
with the following properties:

(a) $\text{subdiv}(B)$ is the nerve of a strict category; informally, it is the barycentric subdivision
of $B$.

(b) There is a natural localization map $\text{subdiv}(B) \to B$. 

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(c) $p^{-1}$ is the functor assigning any simplex of $B$ the fiber of $E$ above that simplex. Note $p^{-1}$ induces a functor $B \to \text{Cat}_\infty$ by the universal property of localization, and we have:

(d) The induced functor $B \to \text{Cat}_\infty$ is equivalent to the functor classified by $p$.

This is a useful technique for us because $\text{subdiv}(B)$ is (the nerve of) a strict category, and we happen to precisely use this strict category to organize the families of (un)wrapped Fukaya categories living over $B$. We refer the reader to Lemma 12.1.3 and its surrounding discussions; there we only prove our claims for when $B$ is a Kan complex, but the same techniques go through when $B$ is an $\infty$-category in general.

### 1.4 Conventions

Here we make explicit our Fukaya-categorical conventions.

**Remark 1.4.1.** We follow the conventions of almost all the literature concerning wrapped categories, with one notable exception. The work [GPS17] reads boundary branes clockwise with respect to the boundary of a disk, while we use the more standard counterclockwise reading (Convention 1.4.4(3)).

This does result in some (very minor) differences in the algebra, which we enumerate here for the reader’s convenience:

1. In our work, a key filtered colimit will be of the form $\text{CF}^*(X,Y_0) \to \text{CF}^*(X,Y_1) \to \ldots$, induced by a sequence of morphisms $Y_0 \to Y_1 \to \ldots$ (see Lemma 9.2.2, where we relate the morphism complex of a localization to a filtered definition of wrapped Floer cohomology). In [GPS17], however, the analogous colimit is of the form $\text{CF}^*(X_0,Y) \to \text{CF}^*(X_1,Y) \to \ldots$, induced by maps $\ldots \to X_1 \to X_0$.

2. When we define our directed $A_\infty$-category $\mathcal{O}$, we will define $\text{hom}_{\mathcal{O}}((i,L,w),(i',L',w'))$ to be zero if $w > w'$; this is opposite the convention utilized in [GPS17].

**Convention 1.4.2** (Strip coordinates and positive/negative punctures). First, we will often denote an element of the infinite strip $\mathbb{R} \times [0,1]$ by $(\tau,t)$.

Every boundary-punctured holomorphic disk $S$ will be equipped with strip-like ends—i.e., holomorphic embeddings

$$
\epsilon : [0,\infty) \times [0,1] \to S, \quad \epsilon : (-\infty,0] \times [0,1] \to S,
$$

(called positive and negative, respectively) that converge as $\tau \to \pm \infty$ to the boundary punctures of $S$. We will call these boundary punctures positive and negative accordingly.

In all our applications, there will exactly one negative boundary puncture of $S$, while all other boundary punctures will be positive.

**Convention 1.4.4** (Morphisms and $\mu^k$). The operations $\mu^1, \mu^2, \ldots$ of a Fukaya category will be defined as usual by counting (pseudo)holomorphic maps

$$u : S \to M$$

satisfying certain boundary conditions.
1. (Generating chords.) The generators of the Floer cochain complex $CF^*(L_0, L_1)$ are in bijection with certain chords $[0, 1] \to M$ from $L_0$ to $L_1$. In particular, in strip-like end coordinates, the limiting chords

$$\lim_{\tau \to \pm \infty} u \circ \epsilon(\tau, -)$$

will be read as a morphism from the brane labeled at time $t = 0 \in [0, 1]$ to the brane labeled at time $t = 1 \in [0, 1]$.

Our chords will almost always be constant in our applications, but it will be healthy to conjure this convention as a rule of thumb.

2. (Inputs and outputs.) The negative puncture defines an output, while the positive punctures are inputs. More precisely, a strip-like end of a negative puncture will be decorated by boundary conditions giving rise to an output of the $\mu^k$ operations, and the positive strip-like ends will be decorated by boundary conditions encoding the inputs of the $\mu^k$ operations.

3. (Ordering the boundary arcs.) The branes decorating the boundary arcs of $S$ will be read in an order given by the boundary orientation of $\partial S$ induced from the standard holomorphic orientation of $S$—in particular, when we draw a picture of $S$, we read the brane labels counterclockwise.

4. (Orienting the boundary arcs.) Likewise, when we equip a boundary arc of $S$ with a moving boundary condition, the “positivity” of the moving boundary isotopy of a brane will be with respect to the boundary orientation of the arc. (See for example Definition 8.3.4.)

5. (Composition order.) Counting holomorphic disks with boundary conditions $L_0, \ldots, L_k$ (read counter-clockwise from the negative puncture) gives rise to the operation

$$\mu^k : CF(L_{k-1}, L_k) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_k).$$

(1.5)

For a summary of these conventions, see Figure 1.4.3.

**Example 1.4.5.** Fix a brane $L \subset M$ and fix an isotopy from $L$ to $L'$. If the isotopy is non-negative (Definition 3.11.4), one obtains an element of $CF(L, L')$ by counting holomorphic disks with a single boundary puncture, and with moving boundary condition dictated by the isotopy (Construction 7.1.8). See also Figure 1.4.6.
Figure 1.4.6. A holomorphic disk with one boundary puncture and with a moving boundary condition given by a non-negative isotopy $\mathcal{L}$. The count of such disks gives rise to an element of $CF(L, L')$.

Example 1.4.7. Fix a brane $L \subset M$ and fix an isotopy $\mathcal{L}$ from $L$ to $L'$. Fix also a brane $K \subset M$. If the isotopy is non-negative (Definition 3.11.4), one obtains a chain map $CF(K, L) \to CF(K, L')$ by counting holomorphic strips with moving boundary condition at $t = 1$ (Construction 7.2.5). See also Figure 1.4.8.

![Holomorphic Strip](https://via.placeholder.com/150)

Figure 1.4.8. A holomorphic strip with moving boundary condition $\mathcal{L}$ at $t = 1$ and fixed boundary condition $K$ at $t = 0$, counts of which define a continuation map $CF(K, L) \to CF(K, L')$. Note that the moving boundary condition places $L$ near $\tau = \infty$, and places $L'$ near $\tau = -\infty$. (In particular, the isotopy evolves in the $-\partial/\partial \tau$ direction.)

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2 \(\infty\)-categories and \(A_\infty\)-categories

We collect some basic definitions and useful results concerning \(\infty\)-categories and \(A_\infty\)-categories.

Fix a base ring \(R\). The most fundamental \(\infty\)-category we will utilize is the \(\infty\)-category \(\mathsf{Cat}_{A_\infty}\) of \(A_\infty\)-categories. Morphisms are \(R\)-linear \(A_\infty\)-categories with \(R\)-linear functors. Informally, the higher morphisms are given by natural equivalences, homotopies of natural equivalences, and higher homotopies thereof.

Because it has not seemed easy to find references in the literature, we collect useful results here. None of the individual results seems original and is most likely known to experts; we hope only that the synthesis is convenient for our readers so that people other than experts can access this knowledge. On top of just defining \(\mathsf{Cat}_{A_\infty}\), we also explain what we mean by localizations of \(\infty\)-categories and \(A_\infty\)-categories.

Let us mention some features of \(\mathsf{Cat}_{A_\infty}\) that may motivate the reader:

1. The higher homotopies of \(\mathsf{Cat}_{A_\infty}\) are a useful receptacle; for example, our main theorem proves that \(\mathsf{Cat}_{A_\infty}\) houses enough homotopical information to detect higher homotopy groups of \(B\ Aut(M)\).

2. In the usual one-category of \(A_\infty\)-categories, quasi-equivalences are not necessarily invertible (especially when the base ring \(R\) is not a field). In \(\mathsf{Cat}_{A_\infty}\), any quasi-equivalence of \(A_\infty\)-categories is an equivalence (i.e., admits an inverse up to homotopy).

2.1 Simplicial sets

We will frequently pass between the notion of a topological space, and of a simplicial set. Let us briefly make explicit this passage, which hereafter we will not invoke explicitly until danger arises.

**Notation 2.1.1 (Simplices).** We let \(\Delta\) denote the category whose objects are finite, non-empty, linearly ordered sets, and whose morphisms are weakly order-preserving maps.

Every object of \(\Delta\) is uniquely isomorphic to the linear poset \([n] := \{0 < 1 < \ldots < n\}\) of \(n + 1\) elements.

There exist exactly \(n + 1\) surjections from \([n + 1]\) to \([n]\). These are called *degeneracy maps*.

There exist exactly \(n + 1\) injections from \([n - 1]\) to \([n]\). These are called *face maps*.

We note that every morphism in \(\Delta\) can be factored as a succession of degeneracy maps followed by face maps.

**Definition 2.1.2.** A simplicial set is a contravariant functor from \(\Delta\) to the category of sets:

\[X : \Delta^{\text{op}} \to \text{Sets}.\]

We will let \(X_n\) denote the set assigned by \(X\) to \([n]\). We call \(X_n\) the *set of \(n\)-simplices of \(X\).*

A map of simplicial sets is a natural transformation.

**Example 2.1.3 (Simplices).** We let \(\Delta^n\) be the simplicial set represented by the object \([n]\). That is, \((\Delta^n)_k = \text{hom}_{\Delta}([k], [n])\).

By the Yoneda Lemma, if \(X\) is a simplicial set, the set of maps \(\Delta^n \to X\) is in natural bijection with \(X_n\).

**Example 2.1.4 (Nerves).** Let \(\mathcal{C}\) be a small category (in the usual sense). Then there is a functor \(N(\mathcal{C}) : \Delta^{\text{op}} \to \text{Sets}\) called the *nerve of \(\mathcal{C}\).* The set \(N(\mathcal{C})_0\) is the set of objects of \(\mathcal{C}\), the set \(N(\mathcal{C})_1\)
is the set of all morphisms in $C$, and the set $N(C)_k$ is the set of all commutative diagrams in $C$ in the shape of a $k$-simplex.

The two natural injections $[0] \to [1]$ are sent to two functions $N(C)_1 \to N(C)_0$ sending a morphism to its source, and to its target. Other face maps forget faces of a commutative diagram. Degeneracy maps insert identity morphisms into the commutative diagram.

**Example 2.1.5** (Simplicial nerves). More generally, if $C$ is a category enriched in topological spaces, one can construct its simplicial nerve $N(C)$, which is another $\infty$-category. Informally (and vaguely), a $k$-simplex in $N(C)$ encodes the data of $(k + 1)$ objects, sequences of composable morphisms between them, homotopies and higher homotopies among their compositions. More details may be found in Section 1.1.5 of [Lur09].

**Example 2.1.6** (Sing). Let $A$ be a topological space. Then there is a simplicial set $\text{Sing}(A)$ called the singular complex of $A$. The set $\text{Sing}(A)_k$ is the set of continuous maps from the topological $k$-simplex $|\Delta^k|$ to $A$.

**Example 2.1.7** (Horns). Fix $n \geq 0$ and $0 \leq i \leq n$. The $i$th $n$-horn is the subsimplicial set $\Lambda^n_i \subset \Delta^n$ assigning $[k]$ to the set of functions $[k] \to [n]$ that do not surject onto the set $[n] \setminus \{i\}$.

**Example 2.1.8** (Direct products). If $X$ and $Y$ are simplicial sets, one can define a new simplicial set $X \times Y$ by declaring $(X \times Y)_n = X_n \times Y_n$ (with the obvious effect on face and degeneracy maps).

Let us record the following construction here, which we utilize crucially in the main text.

**Construction 2.1.9** ($\text{subdiv}(B)$). Let $B$ be a simplicial set. Associated to it, there is a slice category $(\Delta\text{inj})/B$, where

1. An object is the choice of a pair $(a, j)$ where $a \geq 0$ is an integer and $j$ is a simplex $j : \Delta^a \to B$.

2. A morphism from $(a, j)$ to $(a', j')$ is a choice of an order-preserving inclusion $[a] \hookrightarrow [a']$ such that the composite $\Delta^a \to \Delta^a' \xrightarrow{j'} B$ is equal to $j$.

We let $\text{subdiv}(B)$ denote the simplicial set given by the nerve of this slice category:

$$\text{subdiv}(B) := N((\Delta\text{inj})/B).$$

**Example 2.1.10.** Thus there is exactly one 0-simplex in $\text{subdiv}(B)$ for every simplex $j$ (of any dimension) in $B$. There is an edge from a 0-simplex $j_0$ to another 0-simplex $j_1$ if and only if $j_0$ is “contained” in $j_1$ as simplices of $B$.

**Remark 2.1.11.** This construction is also called the subdivision of $B$ in Section III.4 of [GJ09].

**Remark 2.1.12.** Any simplicial set $B : \Delta^{op} \to \text{Set}$ determines a functor $\Delta^{op}_{\text{inj}} \to \text{Set}$ given by restricting to the injective morphisms in $\Delta$. This in turn classifies a Cartesian fibration over $\Delta_{\text{inj}}$ by the un/straightening construction (Section 2.2.6 below); $\text{subdiv}(B)$ may be identified with the domain of this Cartesian fibration.

### 2.2 $\infty$-categories

We give a rapid review of the facts we will be using about $\infty$-categories. For further details we recommend [Lur09] and the Appendix of [NT11]. Other applications of $\infty$-categories in symplectic geometry can be found in [Par16, Tan16, Tan, Tan18].

To motivate the next definition, we observe that every category gives rise to an $\infty$-category (Example 2.2.4), as does every topological space (Example 2.2.3). Thus the notion of $\infty$-category easily allows us to fuse category theory with homotopy theory.
Remark 2.2.1. Another highly successful framework for speaking of homotopically rich categories is the language of model categories due to Quillen. We will make reference to model categories but we will not give detailed accounts of techniques from model categories. Let us say that all model categories can be reasonably transported into the language of simplicially enriched categories using Dwyer-Kan localization (Notation 2.2.10), and all simplicially enriched categories may be rendered an \(\infty\)-category by the nerve construction (Example 2.1.5). Moreover, there exist plenty of \(\infty\)-categories that do not arise from model categories.

On the other hand, the language of model categories is still incredibly useful in concrete computations and in proving the equivalence of various homotopical categories.

Definition 2.2.2. An \(\infty\)-category is a simplicial set \(\mathcal{C}\) satisfying the following condition: For every \(n \geq 2\) and \(0 < i < n\), and for every map \(\Lambda^n_i \to \mathcal{C}\), there exists a map (indicated by a dashed arrow below) making the following diagram commute:

\[
\begin{array}{ccc}
\Lambda^n_i & \longrightarrow & \mathcal{C} \\
\downarrow & \searrow & \\
\Delta^n & \longrightarrow & \\
\end{array}
\]

If the horn-fillers exist for \(i = 0\) and \(i = n\) as well, we call \(\mathcal{C}\) an \(\infty\)-groupoid.

A functor is a map of simplicial sets.

A natural transformation between functors \(F_0, F_1 : \mathcal{C} \to \mathcal{D}\) is a map of simplicial sets \(\mathcal{C} \times \Delta^1 \to \mathcal{D}\) whose restrictions to \(\mathcal{C} \times \{0\}\) and \(\mathcal{C} \times \{1\}\) agree with \(F_0\) and \(F_1\), respectively.

Example 2.2.3. If \(\mathcal{C} = N(\mathcal{C})\) is the nerve of a small category, \(\mathcal{C}\) is an \(\infty\)-category. In fact, the horn-filling map \(\Delta^n \to \mathcal{C}\) exists uniquely. Note that if the horn-filling is satisfied for \(i = 0\) and \(i = n\) as well, one can conclude that \(\mathcal{C}\) is a groupoid.

Example 2.2.4. If \(\mathcal{C} = \text{Sing}(A)\) is the singular complex of a topological space, \(\mathcal{C}\) is an \(\infty\)-category, and in fact, an \(\infty\)-groupoid. The horn-fillers are almost never unique.

Remark 2.2.5. Let \(\mathcal{C}\) be an \(\infty\)-category. One should think of an \(n\)-simplex in \(\mathcal{C}\) as a homotopy commutative diagram in the space of an \(n\)-simplex, with specified homotopies rendering the diagram homotopy commutative. This fuses the notion that an \(n\)-simplex of \(N(\mathcal{C})\) is a commutative diagram in the shape of an \(n\)-simplex, with the notion that an \(n\)-simplex of \(\text{Sing}(A)\) is the data of homotopies between points (called the edges), homotopies between these edges (called triangles), and so forth until the data of the \(n\)-simplex parametrizes the homotopies dictated by its faces.

Now let us state some facts making only references to their proofs.

2.2.1 Spaces and Kan complexes

Given a topological space \(X\), one can form a simplicial set \(\text{Sing}(X)\) whose set of \(k\)-simplices consists of those continuous maps \(|\Delta^k| \to X\) from the standard \(k\)-simplex to \(X\). \(\text{Sing}(X)\) defines not only an \(\infty\)-category, but a stronger horn-filling property: In Definition 2.2.2, we may take \(0 \leq i \leq n\). Such a simplicial set is called a Kan complex. (This is synonymous to an \(\infty\)-groupoid.)

Given a simplicial set \(A\), one can form its geometric realization \(|A|\); this is a colimit rendering a topological space glued out of standard simplices.

This passage defines an equivalence between the \(\infty\)-category of topological spaces and of Kan complexes; for example, the natural maps

\[|\text{Sing}(X)| \to X, \quad A \to \text{Sing}(|A|)\]

are weak homotopy equivalences.
Remark 2.2.6. We will often refer to the “space” of maps, or the “space” of functors, when these “spaces” are actually most conveniently constructed as Kan complexes. This passage is only useful when we care about spaces up to weak homotopy equivalence; we certainly lose track of the homeomorphism type of spaces.

2.2.2 Limits and colimits

For any functor \( I \to \mathcal{C} \) of \( \infty \)-categories, one can define the notion of a limit and colimit; these (when they exist) satisfy a universal property analogous to the classical one. Indeed, when \( \mathcal{C} = N(C) \) is the nerve of a category (in the usual sense), the \( \infty \)-categorical definition of (co)limit agrees with the classical one.

Warning 2.2.7. There is no notion of a “strict (co)limit” in the setting of \( \infty \)-categories, as \( \infty \)-categories have a priori no composition law; only a space of ways to fill in horns. Some will say that “every (co)limit in an \( \infty \)-category is a homotopy (co)limit;” for example, if \( \mathcal{C} \) arises from a model category, the (co)limits of \( \mathcal{C} \) are indeed homotopy (co)limits in that model category.

Regardless, in everyday conversation, there could be some ambiguity about whether or not one is considering a category (in the usual sense) or an \( \infty \)-category. For example, the collection of simplicial sets may be organized into either sort of category. When such ambiguities may arise, we may use the term “homotopy (co)limit” rather than “(co)limit” to be explicit.

2.2.3 \( \infty \)-category of \( \infty \)-categories

There exists an \( \infty \)-category of \( \infty \)-categories. That is, there exists an \( \infty \)-category

\[
\mathcal{C}_{\infty}
\]

where each vertex is an \( \infty \)-category, every edge is a functor of \( \infty \)-categories, and higher simplices contain the data of natural equivalences and homotopies between these.

In particular, one can talk of limits and colimits of \( \infty \)-categories.

For details, we refer the reader to Section 1.2.13 and Chapter 4 of [Lur09].

Remark 2.2.8. We caution the reader that to deal with size issues, one needs at least three choices of Grothendieck universes. Equivalently, one needs to choose three strongly inaccessible cardinals. We do not elaborate on size issues here, but we merely remark that one needs to assume the existence of strongly inaccessible cardinals; this is an axiom independent of ZFC.

2.2.4 Kan completions and localizations

Let \( \mathcal{Gpd}_{\infty} \subset \mathcal{C}_{\infty} \) be the full subcategory consisting of \( \infty \)-groupoids. The inclusion \( \mathcal{Gpd}_{\infty} \to \mathcal{C}_{\infty} \) admits both a left and a right adjoint.

We will be interested in the left adjoint, which sends an \( \infty \)-category \( \mathcal{C} \) to the “smallest” \( \infty \)-groupoid \( |\mathcal{C}| \) containing it; we will call this the Kan completion of \( \mathcal{C} \).

More generally, let \( \mathcal{C} \) be an \( \infty \)-category, and \( W \subset \mathcal{C} \) a sub-\( \infty \)-category. (A subsimplicial set that also happens to be an \( \infty \)-category.) Then one can form the homotopy pushout of the following:

\[
\begin{array}{ccc}
W & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow \\
|W| & \longrightarrow & \\
\end{array}
\]
and the result is an \( \infty \)-category that we will call the localization of \( \mathcal{C} \) along \( W \), and we denote it by \( \mathcal{C}[W^{-1}] \). By the definition of homotopy pushout and of the Kan completion \(|W|\), the localization satisfies a universal property: If \( \mathcal{C} \rightarrow \mathcal{D} \) is any functor sending \( W \) to equivalences in \( \mathcal{D} \), then there exists a unique (up to contractible choice) factorization \( \mathcal{C} \rightarrow \mathcal{C}[W^{-1}] \rightarrow \mathcal{D} \). Here, by a factorization we mean the data of a 2-simplex

\[
\begin{array}{ccc}
\mathcal{C} & \rightarrow & \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{C}[W^{-1}] & \nearrow & 
\end{array}
\]

in \( \text{Cat}_\infty \).

**Remark 2.2.9.** We place emphasis on the fact that the localization of a category is rarely a category, but usually an \( \infty \)-category. Informally, this is because as we freely attach new simplices to render certain edges invertible, we necessarily attach higher simplices, and hence begin to see higher homotopical data.

### 2.2.5 Dwyer-Kan localizations

Let \( \mathcal{C} \) be a category and \( W \subset \mathcal{C} \) a subcategory. Out of this data one can construct a category enriched in simplicial sets:

**Notation 2.2.10.** We let \( L(\mathcal{C}, W) \) denote the Dwyer-Kan localization of \( \mathcal{C} \) with respect to \( W \).

This was defined in [DK80b], where it is referred to as a simplicial localization.

We will soon use some facts about relating mapping spaces of model categories to localizations:

**Lemma 2.2.11.** Let \( \mathcal{C} \) be a category and \( W \subset \mathcal{C} \) a subcategory. Let \( L(\mathcal{C}, W) \) be the Dwyer-Kan localization, and \( N(\mathcal{C})[N(W)^{-1}] \) the localization of \( \infty \)-categories. Then the natural map

\[
N(\mathcal{C})[N(W)^{-1}] \rightarrow N(L(\mathcal{C}, W))
\]

is an equivalence of \( \infty \)-categories.

**Proof.** The nerve functor defines a Quillen equivalence from the model category of simplicially enriched categories to the Joyal model category of \( \infty \)-categories. (See Theorem 2.2.5.1 of [Lur09].) It is standard to show that (i) the equivalence between simplicially enriched categories and \( \infty \)-categories respects \( \infty \)-groupoidification (by characterizing both as left adjoints) and (ii) that Dwyer-Kan localization computes a homotopy pushout. The Quillen equivalence preserves homotopy pushouts, so we are finished.

**Lemma 2.2.12.** Let \( \mathcal{C} \) be a model category with \( W \) its class of weak equivalences. For any two objects \( X, Y \in \text{Ob} \mathcal{C} \), let \( \text{Map}(X, Y) \) be a simplicial set of morphisms induced by the model structure.\(^7\) Let \( \text{hom}_{L(\mathcal{C}, W)}(X, Y) \) be the simplicial set of morphisms induced by Dwyer-Kan localization. Then the simplicial sets

\[
\text{Map}(X, Y) \quad \text{and} \quad \text{hom}_{L(\mathcal{C}, W)}(X, Y)
\]

are weakly homotopy equivalent.

**Proof.** See Proposition 4.4 of [DK80a]. \( \Box \)

\(^7\)For example, by taking a simplicial resolution of \( Y \) and a cosimplicial resolution of \( X \); or, when \( \mathcal{C} \) is a closed simplicial model category, by the usual function complex.

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2.2.6 Colimits via localization of unstraightenings

We will invoke one useful fact about computing colimits of $\infty$-categories. Recall that in classical category theory, a functor $F : \mathcal{C} \to \text{Cat}$ to the category of categories is the same thing as a Grothendieck (op)fibration $E_F \to \mathcal{C}$. The construction of this (op)fibration is called the Grothendieck construction of $F$.

Likewise, given a functor $F : \mathcal{C} \to \text{Cat}_\infty$, one can construct the unstraightening, which is the data of an $\infty$-category $\mathcal{E}_F$ and a functor $\mathcal{E}_F \to \mathcal{C}$ which is a coCartesian fibration. Likewise, one can construct a Cartesian fibration $\mathcal{D}_F \to \mathcal{C}^{\text{op}}$. The passage between functors $F$ and their associated fibrations is an equivalence of $\infty$-categories. (See Sections 2.2.1 and 3.2 of [Lur09] for details.)

We then have the following:

**Theorem 2.2.13** (Corollary 3.3.4.3 of [Lur09].) Let $\mathcal{C}$ be a small $\infty$-category. For any functor $F : \mathcal{C} \to \text{Cat}_\infty$, and any coCartesian fibration $p : \mathcal{E}_F \to \mathcal{C}$ modeling $F$, let $W \subset \mathcal{E}_F$ denote the collection of $p$-coCartesian edges. Then there is a natural equivalence

$$\mathcal{E}_F[W^{-1}] \simeq \text{colim} \ F.$$  

That is, colimits of $\infty$-categories can be computed as localizations of ($\infty$-categorical analogues of) Grothendieck constructions. A brief exposition may also be found in Section 4.3 of [Tan19].

2.3 The $\infty$-category of $A_\infty$-categories

2.3.1 dg and $A_\infty$ are equivalent theories

In what follows, all categories are assumed unital; if the reader seeks further generality, they may assume all categories are cohomologically unital (or c-unital; see (2a) of [Sei08] for notions of units in $A_\infty$-categories).

**Notation 2.3.1** ($A_\infty \text{Cat}$ and $dg \text{Cat}$.) We have fixed a base ring $R$. We do not recall the notion of $R$-linear $A_\infty$-categories and we do not choose a sign convention, of which one may find three in [Sei08, Kel06, Lyu03]; but regardless of the reader’s preference, we let

$$A_\infty \text{Cat}$$

denote the category of $R$-linear $A_\infty$-categories. Its objects are $A_\infty$-categories over $R$, and its morphisms are $R$-linear functors of $A_\infty$-categories. We note that this is a category in the usual sense: Composition is strict, and we have no notion of higher morphisms (e.g., natural transformations) that we incorporate.

Likewise, we let

$$dg \text{Cat}$$

denote the category of $R$-linear dg-categories.

**Definition 2.3.2.** A functor between $A_\infty$- or dg-categories is called an equivalence if it is essentially surjective on objects, and if it induces a quasi-isomorphism on all morphism complexes.

We let $W_{dg} \subset dg \text{Cat}$ and $W_{A_\infty} \subset A_\infty \text{Cat}$ denote the subcategories consisting of equivalences.

**Remark 2.3.3.** Some refer to what we call an equivalence of $A_\infty$-categories as a “quasi-equivalence” of $A_\infty$-categories.
Notation 2.3.4. We let
\[ \mathcal{C}_{A}\infty := N(A_{\infty}Cat)[N(W_{A_{\infty}})^{-1}] \]
\[ \mathcal{C}_{dg} := N(dgCat)[N(W_{dg})^{-1}] \]
be the localizations of \( N(A_{\infty}Cat) \) and \( N(dgCat) \) along the subcategory of equivalences (see Section 2.2.4). We refer to these as the \( \infty \)-category of \( A_{\infty} \)-categories, and of dg-categories, respectively.

Localizations are often difficult to handle without some serious machinery. Let us first make the following simplifying observation:

**Proposition 2.3.5.** The inclusion of \( dgCat \) into \( A_{\infty}Cat \) induces an equivalence of \( \infty \)-categories \( \mathcal{C}_{dg} \simeq \mathcal{C}_{A_{\infty}} \).

**Proof of Proposition 2.3.5.** For any \( A_{\infty} \)-category \( A \), let \( Y(A) \) denote the image of its Yoneda embedding. That is, this is the full subcategory of \( Fun_{A_{\infty}}(A^{op}, \text{Chain}_{R}) \) consisting of \( A_{\infty} \)-functors represented by objects of \( A \). We note two properties:

- \( Y(A) \) is a dg-category, and moreover, if \( f : A \to B \) is a c-unital \( A_{\infty} \) functor, then the induced map \( Y(A) \to Y(B) \) is a dg functor. Hence \( Y \) defines a functor of usual categories \( Y : A_{\infty}Cat \to dgCat. \)

We let \( i : dgCat \to A_{\infty}Cat \) denote the inclusion functor.

Clearly, \( i(W_{dg}) \subset W_{A_{\infty}} \) and \( Y(W_{A_{\infty}}) \subset W_{dg} \). Thus we have induced functors of \( \infty \)-categories

\[
\begin{array}{c}
N(dgCat) \xrightarrow{N(i)} N(A_{\infty}Cat) \xrightarrow{N(Y)} N(dgCat) \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{C}_{dg} \xrightarrow{i'} \mathcal{C}_{A_{\infty}} \xrightarrow{Y'} \mathcal{C}_{dg} .
\end{array}
\]

We note that there are natural transformations

\[ \text{id}_{dgCat} \to Y \circ i, \quad \text{id}_{A_{\infty}Cat} \to i \circ Y. \]

whose arrows are contained in \( W_{dg} \) and \( W_{A_{\infty}} \), respectively. This induces functors

\[ \mathcal{C}_{dg} \times \Delta^1 \to \mathcal{C}_{dg}, \quad \mathcal{C}_{A_{\infty}} \times \Delta^1 \to \mathcal{C}_{A_{\infty}} \]

such that \( \{X\} \times \Delta^1 \) is sent to an equivalence for every \( X \in \mathcal{C}_{dg} \) and every \( X \in \mathcal{C}_{A_{\infty}} \); that is, the natural transformations induce homotopies \( \text{id}_{\mathcal{C}_{dg}} \simeq Y' \circ i' \) and \( \text{id}_{\mathcal{C}_{A_{\infty}}} \simeq i' \circ Y' \). This shows \( i' \) and \( Y' \) are equivalences, and in fact inverse to each other up to homotopy. \( \square \)

**Warning 2.3.6.** The collection of functors from \( A \) to \( B \) can be made into an \( A_{\infty} \)-category, but \( \mathcal{C}_{dg} \) and \( \mathcal{C}_{A_{\infty}} \) do not see this enrichment. Informally, these \( \infty \)-categories only detect natural equivalences, and not arbitrary natural transformations.

Now let us relate these \( \infty \)-categories to other models. We begin with dg-categories. We recall that Tabuada put a model structure on dg-categories, and Tabuada’s weak equivalences are precisely given by \( W_{dg} \) [Tab05].

By Lemmas 2.2.11 and 2.2.12, we have:
Proposition 2.3.7. For any two dg-categories $A$ and $B$,

$$\text{Map}_{\text{Tabuada}}(A, B) \simeq \text{hom}_{\text{Cat}_{dg}}(A, B).$$

That is, the simplicial set of dg-functors from $A$ to $B$ as defined through the model structure of Tabuada is weakly equivalent to the hom-space resulting from the $\infty$-categorical localization.

Warning 2.3.8. The above weak equivalence is as simplicial sets, i.e., a homotopy equivalence on the space of functors from $A$ to $B$. But this does not underly an equivalence of functor categories. That is, the dg-category of functors from $A$ to $B$ is not equivalent to an enrichment that we will articulate later on. Confusingly, all enrichments we encounter will yield equivalent mapping spaces just as in Proposition 2.3.7, but only two of them will have equivalent functor categories.

2.3.2 Functors

Notation 2.3.9. To a pair of $A_\infty$-categories $A, B$ one can associate a functor $A_\infty$-category

$$\text{Fun}_{A_\infty}(A, B).$$

Objects are functors, closed degree zero morphisms are natural transformations, and higher operations encode homotopies between natural transformations. We refer the reader to [Sei08, Lyu03] for details.

Notation 2.3.10. Given an $A_\infty$-category $A$, one can construct an $\infty$-category $N(A)$ called the nerve, or the $A_\infty$ nerve of $A$. This construction is due independently to Faonte [Fao13] and to the second author [Tan16]. It is a functor from $A_\infty\text{Cat}$ to the category of simplicial sets.

Theorem 2.3.11 (Faonte [Fao13].) Fix two $A_\infty$-categories $A$ and $B$. There exist natural weak homotopy equivalences

$$\text{hom}_{\text{Cat}_{A_\infty}}(A, B) \rightarrow N(\text{Fun}_{A_\infty}(A, B))^\sim.$$

Here, the domain is the space of morphisms from $A$ to $B$ in $\text{Cat}_{A_\infty}$, while the target is the largest $\infty$-groupoid contained in the nerve $N(\text{Fun}_{A_\infty}(A, B))$.

Moreover, this equivalence is natural in both $A$ and $B$ variables.

Proof. This is Theorem 0.4 of Faonte [Fao13]. There, the author works over a field of characteristic 0, but as far as we can tell, none of the results rely on this assumption. (One need only have a commutative base ring.)

2.3.3 Hochschild cochains

Let $A$ be any small $A_\infty$-category. Then one can define the Hochschild cochain complex $CH^*$. Let us extract this invariant from the abstractions above.

While there is a way to recover the whole complex by enriching the $\infty$-category of dg-categories (or of $A_\infty$-categories) over itself, because we have no need for this in the present work, we will be content with the following observation:

Proposition 2.3.12 (Corollary 8.3 of [Toë07].). For any small $A_\infty$-category $A$, there exist natural isomorphisms

$$\pi_1(\text{Aut}(A)) \cong HH^0(A)^\times, \quad \pi_{i+1}(\text{Aut}(A)) \cong HH^{-i}(A).$$

Here, $HH^0(A)^\times$ denotes the units (under multiplication) of 0th degree Hochschild cohomology.
**Proof.** Clearly, the homotopy type of Aut(\(\mathcal{A}\)) is unchanged under equivalences of \(A_{\infty}\)-categories. Likewise, it is well-known that Hochschild cohomology is unchanged under equivalences of \(A_{\infty}\)-categories (in fact, it is even Morita invariant). So we may as well assume \(\mathcal{A}\) is a dg-category, and Toën’s cited work showed the result for dg-categories using mapping spaces given by the Tabuada model structure; the result follows from Proposition 2.3.7. \(\square\)

### 2.4 Localizations of \(A_{\infty}\)-categories

Fix a base commutative ring \(R\). All \(A_{\infty}\)-categories and functors in this section will be assumed \(R\)-linear.

In [GPS17], given an \(A_{\infty}\)-category \(\mathcal{A}\) with morphisms satisfying a “cofibrancy” condition\(^8\), and given a collection of morphisms \(W \subseteq \mathcal{A}\), a new \(A_{\infty}\)-category is constructed which we will denote by \(\mathcal{A}[W^{-1}]\). This \(A_{\infty}\)-category is called a localization in loc.cit.. The goal of this section is to prove the universal property of \(\mathcal{A}[W^{-1}]\) to justify this nomenclature.

**Remark 2.4.1.** Proving the universal property is by no means groundbreaking, as the ingredients are already in the literature; as mentioned in our abstract, our intent in presenting a proof is to collect scattered results into a single source for the reader’s convenience.

#### 2.4.1 Exactness and linearity

**Definition 2.4.2.** A chain complex is called *acyclic* if all its cohomology groups vanish.

Fix an \(A_{\infty}\)-category \(\mathcal{A}\). An object \(Z\) of \(\mathcal{A}\) is called a zero object if \(\text{hom}(Z, X)\) and \(\text{hom}(X, Z)\) are acyclic for any object \(X \in \mathcal{A}\).

We say that \(\mathcal{A}\) is *stable* or *pretriangulated* if it has a zero object, and if its image under the Yoneda embedding \(\mathcal{A} \to \mathcal{A}\text{Mod}\) is closed under mapping cones and direct sums.

**Remark 2.4.3** (Mapping cones). Let us elaborate on the notion of mapping cones. Consider a homotopy coherent diagram in the \(A_{\infty}\)-category of modules

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C.
\end{array}
\]

This is the data of not only the indicated arrows, but also of a homotopy rendering the square homotopy-commutative. We say that the above is a mapping cone sequence if, for every object \(X \in \mathcal{A}\), the induced homotopy-coherent diagram of \(R\)-linear chain complexes

\[
\begin{array}{ccc}
A(X) & \longrightarrow & B(X) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & C(X)
\end{array}
\]

is a homotopy pushout diagram. (That is, up to quasi-isomorphism, the above exhibits \(C(X)\) as a mapping cone of the map \(A(X) \to B(X)\).)

Note that if \(Z\) is a zero object of \(\mathcal{A}\), then the module represented by \(Z\) is a zero object in \(\mathcal{A}\text{Mod}\).

If the image of \(\mathcal{A}\) under the Yoneda embedding is closed under mapping cones, this implies that for any \(X \in \mathcal{A}\), there is a corresponding object \(X[1]\)—the cone of the zero map \(X \to 0\)— whose representing module represents a shift of the module represented by \(X\).

---

\(^8\)The only use of this condition for the present work is that morphism complexes are homotopically flat—i.e., tensor product with a morphism complex preserves acyclicity.
Notation 2.4.4. We let $\mathcal{C}at^Ex_{A_\infty}$ denote the full subcategory consisting of stable $A_\infty$-categories. (The superscript stands for “exact.”)

Recall there is a functor $\mathcal{C}at_{A_\infty} \rightarrow \mathcal{C}at_{A_\infty}$ taking any $A$ to its stable closure. A specific model for this functor is classically given by the twisted complex construction$^9$.

Notation 2.4.5 (Twisted complexes). Let $\mathcal{Tw}_A \subset \mathcal{A}Mod$ denote the smallest full, stable subcategory containing the 0 object of $\mathcal{A}Mod$ and containing the image of the Yoneda embedding $A \rightarrow \mathcal{A}Mod$.

We call $\mathcal{Tw}_A$ the $A_\infty$-category of twisted complexes of $A$, or the stable closure of $A$.

Remark 2.4.6. The stable closure can be modeled at the level of ordinary categories $A_\infty Cat \rightarrow A_\infty Cat$. It sends equivalences to equivalences$^{10}$, so induces a functor of $\infty$-categories

$$\mathcal{Tw} : \mathcal{C}at_{A_\infty} \rightarrow \mathcal{C}at_{A_\infty}.$$ 

(Note we refer to the induced functor of $\infty$-categories by the same notation.) The non-infinity categorical version $\mathcal{Tw} : A_\infty Cat \rightarrow A_\infty Cat$ admits a natural equivalence

$$\mathcal{Tw} \sim \mathcal{Tw} \circ \mathcal{Tw}.$$ 

Accordingly, $\mathcal{Tw}$ induces an idempotent functor of $\infty$-categories.

Remark 2.4.7. Because $\mathcal{Tw}$ is an idempotent functor of $\infty$-categories, it follows that it is left adjoint to a fully faithful right adjoint (given by the full inclusion of the essential image of $\mathcal{Tw}$)$^{11}$. The fully faithful right adjoint is precisely the inclusion $\mathcal{C}at^Ex_{A_\infty} \rightarrow \mathcal{C}at_{A_\infty}$.

Definition 2.4.8. Let $A$ and $B$ be stable. An $R$-linear functor $A \rightarrow B$ is called exact if it respects 0 objects, and if it sends mapping cone sequences to mapping cone sequences.

Proposition 2.4.9 (Lemma 3.30 of [Sei08]). Let $f : A \rightarrow B$ be an $R$-linear functor between arbitrary $R$-linear $A_\infty$-categories. Then the induced functor $\mathcal{Tw}f : \mathcal{Tw}A \rightarrow \mathcal{Tw}B$ is exact.

Because $\mathcal{Tw}$ is idempotent, we conclude:

Corollary 2.4.10. Any $R$-linear functor between stable $R$-linear $A_\infty$-categories is automatically exact.

Remark 2.4.11. The above corollary shows the deep connection between the notion of stability and the notion of linearity. (Though one can be $R$-linear without being stable.) For this reason, some authors who work only in the stable setting will define an $R$-linear stable presentable $\infty$-category to be an $\infty$-category equipped with an action of the $\infty$-category $RMod$ (where the action $RMod \times \mathcal{C} \rightarrow \mathcal{C}$ preserves colimits in each variable). The reader may verify that such an action endows the morphisms of $\mathcal{C}$ with the structure of $R$-linear chain complexes. (This is most easily verified if one uses that $R$-linear chain complexes are the same thing as $HR$-linear spectra.)

Remark 2.4.12. The corollary also explains the notation $\mathcal{C}at^Ex_{A_\infty}$ from Notation 2.4.4. $\mathcal{C}at^Ex_{A_\infty}$ is the full subcategory of stable $A_\infty$-categories with exact $R$-linear functors; i.e., with all $R$-linear functors.

---

$^9$See Bondal-Kapranov [BK90] for the dg construction, or Seidel’s book [Sei08] for the $A_\infty$ construction.

$^{10}$Lemma 3.25 of [Sei08].

$^{11}$Proposition 5.2.7.4 of [Lur09].
2.4.2 Quotients via universal property

**Definition 2.4.13** (Quotients). Fix an $A_\infty$-category $A$. Let $\mathcal{B} \subset A$ be a full subcategory, and let $0$ be a zero category. (For example, a category with a single object and only the 0 morphism.) Then the quotient $A_\infty$-category $A/\mathcal{B}$ is defined to be the pushout

\[
\begin{array}{ccc}
\mathcal{B} & \longrightarrow & A \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A/\mathcal{B}
\end{array}
\]

in $\text{Cat}_{A_\infty}$.

**Remark 2.4.14** (Quotients exist). One can conclude that quotients exist in at least two ways. First, Tabuada put a model structure on dg-categories where weak equivalences are equivalences of dg-categories. Since $\text{Cat}_{dg} \simeq \text{Cat}_{A_\infty}$ arises from this model category, and model categories have all homotopy colimits, we conclude that pushouts exist in $\text{Cat}_{dg}$, hence in $\text{Cat}_{A_\infty}$.

One could also use the model of enriched $\infty$-categories as defined by Gepner-Haugseng. Since $R\text{Mod}$ with the derived tensor product $\otimes^L_R$ is presentably monoidal, the $\infty$-category of $R\text{Mod}$-enriched $\infty$-categories is presentable\(^{12}\); hence it has all colimits. It is proven by Haugseng [Hau15] that the $\infty$-category of $R\text{Mod}$-enriched $\infty$-categories is equivalent to $\text{Cat}_{dg}$.

Since $\text{Tw}$ is a left adjoint, it preserves colimits (and in particular, pushouts). Since $\text{Tw}$ is an idempotent, it follows that $\text{Tw}A/\text{Tw}\mathcal{B}$ is stable. Noting that $\text{Tw}0 \simeq 0$, we conclude:

**Proposition 2.4.15.** The natural map $\text{Tw}(A/\mathcal{B}) \rightarrow (\text{Tw}A)/(\text{Tw}\mathcal{B})$ is an equivalence.

Finally, let us conclude by characterizing quotients another way.

**Proposition 2.4.16.** Let $\mathcal{A}$ be stable and fix $\mathcal{B} \subset \mathcal{A}$ a full subcategory. Fix further a stable $A_\infty$-category $\mathcal{Q}$ along with a functor $\mathcal{A} \rightarrow \mathcal{Q}$. Then the following are equivalent:

1. The natural map $\text{hom}_{\text{cat}_{A_\infty}}(\mathcal{Q}, -) \rightarrow \text{hom}_{\text{cat}_{A_\infty}}(\mathcal{A}, -)$ (induced by restriction along $\mathcal{A} \rightarrow \mathcal{Q}$) is an injection on $\pi_0$ and a homotopy equivalence for each component, and identifies (for any stable test category $\mathcal{D}$) the connected components of the domain as spanned by those functors $\mathcal{A} \rightarrow \mathcal{D}$ sending objects of $\mathcal{B}$ to zero objects in $\mathcal{D}$.

2. One can complete $\mathcal{A} \rightarrow \mathcal{Q}$ to a pushout diagram

\[
\begin{array}{ccc}
\mathcal{B} & \longrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{Q}
\end{array}
\]

where the top horizontal arrow is the inclusion $\mathcal{B} \subset \mathcal{A}$.

**Proof.** Suppose (1) holds. Taking $\mathcal{D} = \mathcal{Q}$, the identity functor of $\mathcal{Q}$ is send to the functor $\mathcal{A} \rightarrow \mathcal{Q}$ under restriction; the assumption implies that all objects of $\mathcal{B}$ are sent to zero objects in $\mathcal{Q}$. Now we simply note that if a functor $f : \mathcal{A} \rightarrow \mathcal{D}$ completes to any homotopy-commutative diagram

\[
\begin{array}{ccc}
\mathcal{B} & \longrightarrow & \mathcal{A} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{D}
\end{array}
\]

\(^{12}\)Corollary 5.4.5 of [GH15].
then the space of such homotopy-commutative diagrams\(^{13}\) is contractible given \(f\). This is obvious because 0 is a zero object of \(\mathfrak{Cat}_{A_{\infty}}^{\text{exact}}\), and because the space of natural transformations from a 0 functor \(B \to D\) to itself is contractible. Put another way, sending \(B\) to zero objects is homotopically a property, rather than data.

We use the same fact for the converse. The space of homotopy commutative diagrams involving \(D\) as above is naturally equivalent to the space of functors \(A \to D\) sending \(B\) to zero objects, which in turn is naturally equivalent to \(\text{hom}_{\mathfrak{Cat}_{A_{\infty}}}(Q, D)\) under restriction. Again because the condition of sending \(B\) to zero objects is property rather than data, \(\text{hom}_{\mathfrak{Cat}_{A_{\infty}}}(Q, D)\) is homotopy equivalent to the space of maps between the pushout diagram containing \(Q\) and square diagrams exhibiting \(A \to D\) as nullifying \(B\).

\[\square\]

2.4.3 Localizations via quotients

**Assumption 2.4.17** (\(W\) is small.). In what follows, we will always assume a collection denoted by the letter \(W\) to be small. This is for the simplicity of avoiding size issues and discussions of Grothendieck universes.

**Definition 2.4.18.** Let \(A\) be an \(A_{\infty}\)-category, and let \(W \subset H^0 \text{hom}_A\) be a collection of (cohomology classes of) morphisms in \(A\). A localization of \(A\) along \(W\) is the data of an \(A_{\infty}\)-category \(A[[W^{-1}]]\), equipped with a functor

\[\iota: A \to A[[W^{-1}]]\]

such that the following holds: For any \(A_{\infty}\)-category \(D\), the induced map of \(\infty\)-groupoids

\[\iota^*: \text{hom}_{\mathfrak{Cat}_{A_{\infty}}}(A[[W^{-1}]], D) \to \text{hom}_{\mathfrak{Cat}_{A_{\infty}}}(A, D)\]

is fully faithful, and identifies the essential image of \(\iota^*\) with those functors \(A \to D\) sending \(W\) to (cohomologically) invertible morphisms in \(D\).

Now let us construct localizations from the existence of quotients.

**Remark 2.4.19.** Note that Toën proceeds in the reverse direction; once one has a model structure on \(dg\)-categories, one can construct localizations, then conclude the existence of quotients \([\text{To"e07}]\).

**Notation 2.4.20** \((B_W)\). Let \(A\) be an \(R\)-linear \(A_{\infty}\)-category and fix a collection \(W \subset H^0(\text{hom}_A)\). We let \(B_W \subset \text{Tw}A\) denote the full subcategory of those subjects arising as mapping cones of elements of \(W\).

**Definition 2.4.21** \((L_W)\). Now consider the quotient \(\text{Tw}A/\text{Tw}B_W\), where \(B_W\) is as in Notation 2.4.20. The quotient receives a natural functor from \(A\), given by the composite \(A \to \text{Tw}A \to \text{Tw}A/\text{Tw}B\).

We let \(L_W \subset \text{Tw}A/\text{Tw}B\) be the full subcategory consisting of objects in the essential image of \(A \to \text{Tw}A/\text{Tw}B\).

**Remark 2.4.22.** The natural map \(\text{Tw}L_W \to \text{Tw}A/\text{Tw}B\) is an equivalence.

To see this, note the commutative diagram

\[
\begin{array}{ccc}
A & \to & L_W \\
\downarrow & & \downarrow \\
\text{Tw}A & \to & \text{Tw}L_W \\
& & \sim \\
& & \text{Tw}(\text{Tw}A/\text{Tw}B).
\end{array}
\]

\[^{14}\text{i.e., the space of natural equivalences from } B \to A \to D \text{ to } B \to 0 \to D\]
The composite map $\text{Tw} A \to \text{Tw} A / \text{Tw} B$ is essentially surjective (because it is the pushout along an essentially surjective map $\text{Tw} B \to 0$), as is the bottom-left horizontal arrow. Thus the map $\text{Tw} \mathcal{L}_W \to \text{Tw} A / \text{Tw} B$ is essentially surjective. Moreover, the bottom-right horizontal arrow is fully faithful because all other arrows in the right-hand side square are fully faithful.

Our goal is to prove the following:

**Proposition 2.4.23.** The functor $\mathcal{A} \to \mathcal{L}_W$ exhibits $\mathcal{L}_W$ as a localization of $\mathcal{A}$ along $\mathcal{L}_W$.

We first begin with a stable analogue:

**Proposition 2.4.24.** The quotient $\text{Tw} A / \text{Tw} B W$ is the localization of $\text{Tw} A$ along $W$ in the $\infty$-category of stable $A_\infty$-categories. That is, for any $D \in \mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}$, restriction along $\text{Tw} A / \text{Tw} B W$ induces an inclusion of connected components

$$\text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}} (\text{Tw} A / \text{Tw} B W, D) \to \text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}} (\text{Tw} A, D)$$

whose essential image consists of those functors $\text{Tw} A \to D$ sending any morphism in $W$ to an equivalence in $D$.

**Proof.** Let $F : \text{Tw} A \to D$ be a functor. If $f \in W$ is sent to an equivalence by $F$, then $\text{Cone}(F(f)) \simeq 0$ in $D$ because the cone of an equivalence is a zero object. On the other hand, because $F$ is exact, we have that $F(\text{Cone}(f)) \simeq \text{Cone}(F(f))$, meaning $F$ sends any object in $B_W$ to a zero object in $D$. If $F$ sends all object of $B_W$ to a zero object, then $F(f)$ must be an equivalence because its mapping cone is a zero object.

This shows that a functor $\text{Tw} A \to D$ sends an object of $\text{Tw} B W$ to a zero object if and only if it sends morphisms of $W$ to equivalences in $D$.

On the other hand, the full subspace of $\text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}} (\text{Tw} A, D)$ consisting of functors $A \to D$ factoring sending $\text{Tw} B W$ to zero is equivalent (via restriction) to the space of functors $\text{Tw} A / \text{Tw} B \to D$. This completes the proof. \qed

**Proof of Proposition 2.4.23.** For any pair of $A_\infty$-categories $\mathcal{A}$ and $\mathcal{D}$, one can naturally identify $\text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\mathcal{A}, \mathcal{D})$ as the full subspace of $\text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\text{Tw} A, \text{Tw} D)$ that factors $\mathcal{A}$ through the essential image of $D \subset \text{Tw} D$.

Let us now consider the iterated pullback squares

$$\begin{array}{ccc}
Q & \to & \text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\mathcal{A}, \mathcal{D}) \\
\downarrow & & \downarrow \\
Q' & \to & \text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\mathcal{A}, \text{Tw} \mathcal{D}) \\
\simeq & & \simeq \\
\text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\text{Tw} A / \text{Tw} B, \text{Tw} D) & \to & \text{hom}_{\mathcal{C}_{\text{cat}^{\text{Ex}}_{A_\infty}}}(\text{Tw} A, \text{Tw} D).
\end{array}$$

The lower-right vertical arrow is an equivalence is an equivalence because $\text{Tw}$ is an idempotent; the lower-left vertical arrow is an equivalence being the pullback of an equivalence. Note that the two other vertical arrows are fully faithful (because the top-right vertical arrow is).

By considering the outermost rectangle, we see that $Q$ is identified as the space of functors $\mathcal{A} \to \mathcal{D}$ sending morphisms in $W$ to equivalences (by Proposition 2.4.24).
Now we examine the two inner pullback squares. One can identify $Q'$ with hom\textsubscript{Cat\_A\infty}(L_W, Tw\mathcal{D})$, while the bottom-left vertical arrow is induced by the composite

\[ L_W \to TwL_W \simeq TwA/TwB. \]

(see Remark 2.4.22). Finally, the horizontal arrow from $Q'$ can further be identified with restriction along $A \to L_W$. This gives a second identification of $Q$: The space of functors from $L_W$ to $Tw\mathcal{D}$ factoring through $\mathcal{D}$; that is, functors from $L_W$ to $\mathcal{D}$.

This completes the proof.

\[ \square \]

2.4.4 Naturality of localizations

This is a straightforward consequence of the universal property of localizations, and we record it here for use in other sections.

**Proposition 2.4.25.** Let $\mathcal{C}$ be an ordinary category, and fix a functor $F : \mathcal{C} \to A\text{\_}\infty Cat$. Moreover suppose that for every $x \in \mathcal{C}$, we have chosen a collection $W_x \subset H^0\text{hom}_F(x)$, and that for every morphism $x \to y$ in $\mathcal{C}$, we have that $W_x$ has image contained in $W_y$.

Then localization induces a functor of $\infty$-categories

\[ N(\mathcal{C}) \to \text{Cat\_A}\infty \]

sending $x$ to $F(x)[W_x^{-1}]$, and sending a morphism $x \to y$ to the induced functor $F(x)[W_x^{-1}] \to F(y)[W_y^{-1}]$.

**Proof.** $F$ defines a diagram $\mathcal{C} \times \Delta^1 \to \text{Cat\_A}\infty$ whose value at $\{x\} \times \Delta^1$ is given by

\[ TwB_{W_x} \to TwF(x). \]

By the naturality of colimits (in particular, quotients) and functoriality of $Tw$, we then have an induced diagram $\mathcal{C} \times \Delta^1 \to \text{Cat\_A}\infty$ whose value on $\{x\} \times \Delta^1$ is

\[ a_x : F(x) \to TwF(x)/TwB_{W_x}. \]

Thus we have an induced functor $\mathcal{C} \to \text{Cat\_A}\infty$ by taking the induced arrows among the essential images of $a_x$. \[ \square \]

2.4.5 Models for quotients and localizations

Now that we see quotients and localizations exist, let us review concrete models of them. In our work, we will use the model $\mathcal{D}$ below (whose morphism complexes are described in (2.1)). This particular model allows us to infer a useful lemma computing morphism complexes of a localization as sequential colimits (Lemma 2.4.34).

**Remark 2.4.26.** This specific model is most likely unnecessary to prove the properties we need of the wrapped Fukaya category; for example, an $R$-linear versions of arguments used in I.3 of [NS+18] seem to suffice to prove Lemma 2.4.34.

**Remark 2.4.27.** In principle, the equivalence $\text{Cat\_A}\infty \simeq \text{Cat\_dg}$ means we can model quotients of $A\infty$-categories using their Yoneda embeddings, then resorting to a dg quotient. For example, the construction of Drinfeld [Dri04] together with the universal property verified by Tabuada [Tab10]
means we may construct quotients by considering the image \( A \subset \mathcal{A}\text{Mod} \), finding an appropriate replacement of this image, and constructing a quotient.

However, our goal is to compare this to a specific model of localization used in [GPS17], which is the model introduced by Lyubashenko-Ovisienko [LO06]. This model is particularly useful when the base ring \( R \) is a field, but it is not straightforward to prove its universal property directly. For example, an inconvenience presents itself when trying to carry out the previous paragraph, which is that arbitrary direct products of homotopically flat complexes need not be homotopically flat. So instead we will also rely on a model introduced by Lyubashenko-Manzyuk [LM08], which is equivalent to the previous model of Lyubashenko-Ovsienko regardless of base ring, and for which the relevant universal property may be deduced from known results (see Theorem 2.4.30).

Notation 2.4.28 (D and Q). Fix an \( A_\infty \)-category \( A \) and a full subcategory \( B \). There are various constructions of quotients in the literature, and we hone in on two. The first we will denote by \( D(A|B) \) following Lyubashenko-Ovsienko [LO06], and the second by \( Q(A|B) \) following Lyubashenko-Manzyuk [LM08]. We not recall their full definitions here and refer the reader to the papers just cited. We do, however, recall the morphism complexes for \( D \) in (2.1).

Remark 2.4.29. The localization \( \mathcal{O}[C^{-1}] \) used in [GPS17] is modeled on \( D \).

Let us also recall:

Theorem 2.4.30 ([LM08]). For any \( A_\infty \)-category \( A \) and any full subcategory \( B \subset A \), there is a functor \( \eta : A \to Q(A|B) \) satisfying the following property: For any \( A_\infty \)-category \( \mathcal{D} \), restriction along \( \eta \) induces a fully faithful embedding of \( A_\infty \)-categories

\[
\text{Fun}_{A_\infty}(Q(A|B), \mathcal{D}) \xrightarrow{\eta^*} \text{Fun}_{A_\infty}(A, \mathcal{D})
\]

whose essential image consists of those functors sending objects of \( B \) to zero objects in \( \mathcal{D} \).

Theorem 2.4.31 ([LM08]). There is a functor \( A \to D(A|B) \) sending all objects of \( B \) to zero objects. The functor \( \eta : Q(A|B) \to D(A|B) \) induced by Theorem 2.4.30 above is an equivalence.

Then we have

Theorem 2.4.32. When \( A \) is stable, \( D(A|B) \) is a quotient of \( A \) along \( B \) in the sense of Definition 2.4.13.

Proof. By Theorem 2.4.31, we need only prove that \( Q(A|B) \) is a quotient.

Let us first begin with a digression on how to compute mapping spaces. Let \( \text{Cat}_{A_\infty} \) be the \( A_\infty \)-category of \( A_\infty \)-categories. By the equivalence \( \text{Cat}_{dg} \simeq \text{Cat}_{A_\infty} \), the mapping spaces \( \text{hom}_{\text{Cat}_{A_\infty}}(\mathcal{E}, \mathcal{E}) \) may be computed by the mapping spaces of their corresponding Yoneda embeddings, \( \text{hom}_{\text{Cat}_{dg}}(Y(\mathcal{E}), Y(\mathcal{E})) \). This mapping space, in turn, can be computed by first taking the \( A_\infty \)-nerve of \( \text{Fun}_{A_\infty}(Y(\mathcal{E}), Y(\mathcal{E})) \), and then its underlying \( \infty \)-groupoid (i.e., the largest Kan complex contained therein). (This is Theorem 2.3.11.) But because \( \mathcal{E} \simeq Y(\mathcal{E}) \) and \( \mathcal{E} \simeq Y(\mathcal{E}) \) as \( A_\infty \)-categories, we have an equivalence of \( A_\infty \)-categories

\[
\text{Fun}_{A_\infty}(Y(\mathcal{E}), Y(\mathcal{E})) \simeq \text{Fun}_{A_\infty}(\mathcal{E}, \mathcal{E}).
\]

Thus the nerves of each are equivalent, as are their underlying Kan complexes.

---

14We call an \( A_\infty \)-category homotopically flat if for any pair of objects \( X, Y \) and any acyclic complex \( A \), \( \text{hom}(X,Y) \otimes A \) is also acyclic. (Note that Spaltenstein [Spa88] refers to this property as \( X \)-flatness, while Drinfeld [Dri04] refers to it as homotopical flatness.) The issue is that arbitrary direct products of homotopically flat complexes need not be homotopically flat.
Now consider the functor \( \mathcal{A} \to \mathcal{Q}(\mathcal{A}|\mathcal{B}) \). We have an induced natural transformation
\[
\hom_{\text{cat}_{\mathcal{A},\infty}}(\mathcal{Q}(\mathcal{A}|\mathcal{B}), -) \to \hom_{\text{cat}_{\mathcal{A},\infty}}(\mathcal{A}, -)
\]
which, on each test object \( \mathcal{D} \), is induced by the restriction functor from Theorem 2.4.30. By (taking the nerve of) that theorem, this restriction identifies \( \hom_{\text{cat}_{\mathcal{A},\infty}}(\mathcal{Q}(\mathcal{A}|\mathcal{B}), \mathcal{D}) \) with the space of those functors \( \mathcal{A} \to \mathcal{D} \) sending \( \mathcal{B} \) to a zero object. By Proposition 2.4.16, this proves that the map \( \mathcal{A} \to \mathcal{Q}(\mathcal{A}|\mathcal{B}) \) exhibits \( \mathcal{Q}(\mathcal{A}|\mathcal{B}) \) as a quotient.

**Corollary 2.4.33.** Let \( \mathcal{A} \) be an arbitrary \( R \)-linear \( A_{\infty} \)-category and \( W \subset H^0 \hom_{\mathcal{A}} \) a collection of (cohomology classes of) morphisms. Then the localization \( \mathcal{A}[W^{-1}] \) (Definition 2.4.18) is equivalent to the full subcategory of \( \mathcal{D}(\text{Tw}\mathcal{A}|\mathcal{B}_W) \) spanned by the essential image of the composite \( \mathcal{A} \to \text{Tw}\mathcal{A} \to \mathcal{D}(\text{Tw}\mathcal{A}|\mathcal{B}_W) \).

**Proof.** Combine Theorem 2.4.32 with Proposition 2.4.23.

### 2.4.6 Morphisms in a localization

Fix an \( A_{\infty} \)-category \( \mathcal{A} \) and a collection of (cohomology classes of) morphisms \( W \subset H^0 \hom_{\mathcal{A}} \).

Fix two objects \( X, Y \in \mathcal{A} \). By Corollary 2.4.33, the morphism complex \( \hom_{\mathcal{A}[W^{-1}]}(X, Y) \) in the localized category may be computed as the morphism complex
\[
\hom_{\mathcal{D}(\text{Tw}\mathcal{A}|\mathcal{B}_W)}(X, Y).
\]

We recall here that, by definition [LM08, GPS17], this complex is given by a bar-type construction:

\[
\begin{array}{c}
\vdots \\
\oplus_{Z_1, Z_2 \in \mathcal{B}_W} \hom_{\text{Tw}\mathcal{A}}(X, Z_1) \otimes \hom_{\text{Tw}\mathcal{A}}(Z_1, Z_2) \otimes \hom_{\text{Tw}\mathcal{A}}(Z_2, Y) \\
\downarrow \\
\oplus_{Z_1 \in \mathcal{B}_W} \hom_{\text{Tw}\mathcal{A}}(X, Z_1) \otimes \hom_{\text{Tw}\mathcal{A}}(Z_1, Y) \\
\downarrow \\
\hom_{\mathcal{A}}(X, Y)
\end{array}
\]

(2.1)

We will use the following lemma to compute morphism complexes in our wrapped Fukaya categories. A dual assertion is made in Lemmas 3.11 and 3.14 of [GPS17].

**Lemma 2.4.34.** Fix an \( A_{\infty} \)-category \( \mathcal{A} \) such that for any pair of objects \( A, A' \in \mathcal{A} \), we have that \( \hom_{\mathcal{A}}(A, A') \) is \( K \)-flat.\(^{15}\) Fix also a sequence of objects \( Y_0 \to Y_1 \to \ldots \) in \( \mathcal{A} \). (That is, a collection of objects \( Y_i \) equipped with morphisms \( Y_i \to Y_{i+1} \).) Suppose moreover that for any morphism \( Q \to Q' \) in \( W \), the induced map
\[
\text{hocolim}_i \hom_{\mathcal{A}}(Q', Y_i) \to \text{hocolim}_i \hom_{\mathcal{A}}(Q, Y_i)
\]
(2.2)
is a quasi-isomorphism. Then for any \( X \in \mathcal{A} \), the induced map
\[
\text{hocolim}_i \hom_{\mathcal{A}}(X, Y_i) \to \text{hocolim}_i \hom_{\mathcal{A}[W^{-1}]}(X, Y_i)
\]
is a quasi-isomorphism.

\(^{15}\)This is satisfied for \( \mathcal{O}_j \) (Definition 6.3.10), as each morphism complex is bounded in degree and degreewise free.
Remark 2.4.35. For the sake of coherence (and to define the homotopy colimit), one should extend
the collection of morphisms $Y_i \to Y_{i+1}$ to a functor $Z_{\geq 0} \to N(A)$, where $N(A)$ is the $A_\infty$-nerve and
$Z_{\geq 0}$ is the partially ordered set of non-negative integers. However, there is a contractible choice of
such extensions, so we ignore this detail.

Proof. A sequential homotopy colimit may be computed using a mapping telescope construction.
The telescope construction commutes with direct sums and tensor products, so for any $i$,

$$\text{hocolim}_i \text{hom}_{\mathcal{A}[W^{-1}]}(X, Y_i)$$

may be computed by a chain complex built from a simplicial object (just as (2.1) is) where the
$k$-simplices are as follows:

$$\bigoplus_{Z_1, \ldots, Z_k} \text{hom}_{\mathcal{A}}(X, Z_1) \otimes \text{hom}_{\mathcal{A}}(Z_1, Z_2) \otimes \cdots \otimes \text{hom}_{\mathcal{A}}(Z_{k-1}, Z_k) \otimes \text{hocolim}_i \text{hom}_{\mathcal{A}}(Z_k, Y_i)$$

We now claim that for every $k$, $\text{hocolim}_i \text{hom}_{\mathcal{A}}(Z_k, Y_i)$ is acyclic because $Z_k$ arises as a cone of a
morphism $Q \to Q'$ in $W$. That is, the mapping cone sequence $Q \to Q' \to Z_k$ induces a mapping
cone sequence

$$\text{hom}_{\mathcal{A}}(Z_k, Y_i) \to \text{hom}_{\mathcal{A}}(Q', Y_i) \to \text{hom}_{\mathcal{A}}(Q, Y_i)$$

of cochain complexes; since the mapping telescope construction commutes with mapping cones, the
diagram

$$\begin{array}{ccc}
\text{hocolim}_i \text{hom}_{\mathcal{A}}(Z_k, Y_i) & \longrightarrow & \text{hocolim}_i \text{hom}_{\mathcal{A}}(Q', Y_i) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \text{hocolim}_i \text{hom}_{\mathcal{A}}(Q, Y_i)
\end{array}$$

is still a homotopy pushout; in particular, the right vertical map being a quasi-isomorphism by
assumption (2.2), the top-left cochain complex is acyclic.

Finally, we note that if each morphism complex in $\mathcal{A}$ is $K$-flat, the same holds for $\mathcal{T}_{\mathcal{A}}$ (as
morphism complexes are defined as iterated mapping cones and shifts of the morphism complexes
of $\mathcal{A}$). This means that all the terms in the bar complex are acyclic except for the 0-simplex
term, which is $\text{hocolim}_i \text{hom}_{\mathcal{A}}(X, Y_i)$ to begin with. Now a standard argument, for example using
the length filtration and seeing that the associated graded are all acyclic, gives the statement we
desire. \qed
3 Liouville geometry and bundles

3.1 Liouville domains

The notion of Liouville domain will not make a frequent appearance in our work; but it is a convenient stepping stone to the notion of Liouville manifold.

Definition 3.1.1. Fix a symplectic manifold \((M, \omega)\). A vector field \(Z\) on \(M\) is said to be a Liouville vector field if the Lie derivative of \(\omega\) along \(Z\) is \(\omega\) itself:

\[
\mathcal{L}_Z \omega = \omega \tag{3.1}
\]

Given a Liouville vector field \(Z\), its flow will be called the Liouville flow.

Definition 3.1.2. Given a Liouville vector field \(Z\), let \(\lambda\) be the 1-form defined by the equation

\[
\lambda = \omega(Z, \cdot) \tag{3.2}
\]

We call \(\lambda\) the Liouville form.

Remark 3.1.3. Fix a vector field \(Z\) and its dual \(\lambda\) as in Equation (3.2). Then Equation (3.1) is equivalent to the condition that \(\lambda\) is an anti-derivative of \(\omega\):

\[
\omega = d\lambda.
\]

In particular, any symplectic manifold equipped with a Liouville vector field is an exact symplectic manifold. Conversely, given a 1-form \(\lambda\) satisfying \(d\lambda = \omega\), one sees that the dual vector field defined by (3.2) is automatically a Liouville vector field.

Definition 3.1.4. A Liouville domain is a compact symplectic manifold \(W\) with boundary, equipped with a Liouville vector field \(Z\) which points strictly outward along the boundary.

Remark 3.1.5. By the exactness witnessed in Remark 3.1.3, any Liouville domain \(W\) must have non-empty boundary unless \(W\) is 0-dimensional.

Notation 3.1.6 (The boundary of a Liouville domain). Fix a Liouville domain \(W\) (Definition 3.1.4). We will let \(\partial_{\infty}W\) be the boundary manifold.

Remark 3.1.7. Let \(W\) be a Liouville domain. It follows that \(\xi = \ker \lambda|_{\partial_{\infty}W}\) is a contact structure on \(\partial_{\infty}W\).

Remark 3.1.8 (Co-orientation). Recall that a co-orientation on a contact manifold is a choice of 1-form whose kernel is equal to the contact distribution. We see that the boundary \(\partial_{\infty}W\) of any Liouville domain is a contact manifold co-oriented by \(\lambda|_{\partial_{\infty}W}\).

3.2 Symplectizations

Notation 3.2.1 (Symplectization \(SY\) of a contact manifold). Given a co-oriented contact manifold \((Y, \alpha)\), its symplectization \(SY\) is the manifold

\[
SY = \mathbb{R} \times Y = \{(s, y)\}.
\]

We equip \(SY\) with the Liouville form (Definition 3.1.2)

\[
e^s \pi^* \alpha
\]

where \(\pi : SY \to Y\) is the projection map.

Notation 3.2.2 \((r)\). We will often use the change of coordinates

\[
r = e^s.
\]
3.3 Liouville manifolds

We now pass from the setting of a Liouville domain to a “manifold-with-conical-end” setting:

**Notation 3.3.1.** Let $M$ be a smooth manifold. We define an equivalence relation on the set of smooth 1-forms on $M$ as follows: We say $\theta \sim \theta'$ if and only if there exists a smooth, compactly supported function $f : M \to \mathbb{R}$ for which

$$\theta = \theta' + df.$$ 

We let $[\theta]_{\text{Liou}}$ denote the equivalence class of $\theta$.

**Definition 3.3.2 (Liouville manifold).** Fix the data of a pair $(M, [\theta]_{\text{Liou}})$, where $[\theta]_{\text{Liou}}$ is as in Notation 3.3.1. We say this pair is a Liouville manifold if for some (and hence any) choice $\theta \in [\theta]$, the pair $(M, \theta)$ is a completion of a Liouville domain.

**Remark 3.3.3.** Let us explain what we mean by a completion. We mean there exists a compact, co-oriented contact manifold $(Y, \alpha)$, and a map from the ‘positive half’ of $SY$

$$\iota : \mathbb{R}_{s \geq 0} \times Y \to M$$  \hspace{1cm} (3.3)

such that

1. $\iota$ respects Liouville forms, i.e., $\iota^*(\theta) = e^{s\pi^*\alpha}$,
2. $\iota$ is a diffeomorphism (of manifolds with boundary) onto its image, and
3. The complement $M \setminus \iota(\mathbb{R}_{s > 0} \times Y)$ is a Liouville domain when equipped with (the restriction of) $\theta$.

**Remark 3.3.4.** We have reserved $M$ to denote possibly-non-compact exact symplectic manifolds, while $W$ always denotes (compact) Liouville domains.

**Remark 3.3.5.** It is common to define a Liouville manifold as equipped with a choice of $\theta$, rather than just of $[\theta]_{\text{Liou}}$. While we do utilize particular choices of $\theta$ to perform certain geometric constructions, we find that a particular choice is distracting from the appropriate notion of automorphism. (See Definition 3.5.1.)

**Remark 3.3.6.** One may pass freely between a Liouville domain to a Liouville manifold (by completion), and vice versa (by choosing an $\iota$ as in (3.3)). However, the notion of Liouville manifold will be more canonical—e.g., less choice-dependent—in our applications.

3.4 Liouville sectors

**Remark 3.4.1.** The notion of Liouville sector is due to [GPS17], and extends the notion of Liouville manifold to the setting with boundary.

Just as Liouville manifolds are naturally presented as completions of exact symplectic manifolds with boundary, a Liouville sector is naturally the completion of an exact symplectic manifold $W$ with corners. As we will see shortly, the corners separate the boundary of $W$ into two components: A contact boundary (which will define the conical end of the completion) and another boundary (which will extend conically, but must further satisfy a convexity constraint—see (LDCB3) below.)
**Definition 3.4.2** (Liouville domain with convex boundary). Fix a compact exact symplectic manifold \((W, \theta)\) with corners. We let \(DW\) denote the entire boundary of \(W\)—i.e., the union of all faces and corners of \(M\).

We say the pair \((W, \theta)\) is a **Liouville domain with convex boundary** if the following are satisfied:

**LDCB1** (There are two kinds of boundary.) \(DW\) admits two smooth, codimension zero submanifolds-with-boundary \(\partial W\) and \(\partial_{\infty} W\) such that

\[
\partial_{\infty} W \cap \partial W
\]

is precisely the locus of corners of \(W\), and

\[
DW = \partial_{\infty} W \bigcup_{\partial_{\infty} W \cap \partial W} \partial W.
\]

**LDCB2** (\(\partial_{\infty} W\) is contact.) We demand that the Liouville vector field \(Z\) is strictly outward-pointing with respect to \(\partial_{\infty} W\). In particular, \(\theta|_{\partial_{\infty} W}\) renders \(\partial_{\infty} W\) a (co-oriented) contact manifold with boundary.

**LDCB3** (\(\partial W\) is convex.) We demand that there exists a smooth function \(I: W \to \mathbb{R}\) satisfying

\[
ZI = \alpha I
\]

for some \(\alpha > 0\) whose Hamiltonian flow along \(\partial W\) is strictly outward pointing. (See \[GPS17, Definition 2.4\].)

**LDCB4** (The \(\partial W\) boundary can be extended along \(Z\).) For simplicity, we will further assume that in some neighborhood of \(\partial_{\infty} W\), \(Z\) is contained in \(T(\partial W)\). (So near \(\partial_{\infty} W\), \(Z\) is tangent to \(\partial W\).

One can always deform \(\theta\) so that this is the case).

**Remark 3.4.3.** Let \((W, \theta)\) be a Liouville domain with convex boundary (Definition 3.4.2). Using the notation from (LDCB1), one may informally think of \(\partial W\) as the wall of \(W\), while one may think of \(\partial_{\infty} W\) as the ceiling. (There are no floors.)

Based on (LDCB4), the reader may imagine that the Liouville flow thus allows one to push the ceiling higher toward the sky, in a way such that the walls may similarly be extended upward. The Liouville flow may push on the walls inwards or outwards, but it only does so away from a neighborhood of the ceiling.

The reader should compare the following to the definition of Liouville domain (Definition 3.3.2). It is equivalent to Definition 2.4 of \[GPS17\].

**Definition 3.4.4.** Fix a pair \((M, [\theta]_{\text{Liou}})\) where \(M\) is a smooth manifold with boundary. We say that \((M, [\theta]_{\text{Liou}})\) is a **Liouville sector** if, for some (and hence all) \(\theta \in [\theta]_{\text{Liou}}\), the pair \((M, \theta)\) is the completion of a Liouville domain with convex boundary.

**Remark 3.4.5.** By a completion, we mean the data of a co-oriented contact manifold \((Y, \alpha)\) with boundary, and a map \(\iota: \mathbb{R}_{s \geq 0} \times Y \to M\) such that the appropriate analogues of Remark 3.3.3 are satisfied. In particular, \(\iota\) is a diffeomorphism of smooth manifolds with corners, and the restriction of \(\theta\) to the complement \(M \setminus \iota(\mathbb{R}_{s \geq 0} \times Y)\) exhibits the complement as a Liouville domain with convex boundary.

**Remark 3.4.6.** Let \((M, [\theta]_{\text{Liou}})\) be a Liouville sector. Then \(M\) only has one “type” of boundary, \(\partial M\), which one may informally think of as the conical extension of the wall \(\partial W\) by the Liouville flow.

**Remark 3.4.7.** Henceforth, we only use the term Liouville sector with the understanding that if a Liouville sector has empty boundary, then it is in particular a Liouville manifold.
### 3.5 Liouville automorphisms

**Definition 3.5.1** (Liouville automorphisms). Let $M_i$, $i = 0, 1$, be Liouville sectors. A *Liouville isomorphism* from $M_0$ to $M_1$ is a diffeomorphism $\phi : M_0 \to M_1$ satisfying

$$\phi^* [\theta_1]_{\text{Liou}} = [\theta_0]_{\text{Liou}}.$$ 

(See Notation 3.3.1.) If $M_0 = M_1$, we call $\phi$ a Liouville *automorphism*.

**Definition 3.5.2.** Let $M$ be a Liouville manifold, or a Liouville sector. We let

$$\text{Aut}^0(M)$$

denote the topological group of Liouville automorphisms of $M$. It is topologized as a subspace of $C^\infty(M, M)$ with the strong Whitney topology.

**Warning 3.5.3.** Note that the choice of $[\theta]_{\text{Liou}}$ not explicit in the notation $\text{Aut}^0(M)$.

**Remark 3.5.4.** We also treat $\text{Aut}^0(M)$ as a diffeological group; i.e., a set equipped with a notion of what it means to receive a “smooth” map from a smooth manifold $A$. See Example 4.1.14.

### 3.6 Liouville automorphisms for decorations

We now describe automorphisms groups of Liouville manifolds that “respect” particular decorations that are extrinsic to the data of $[\theta]$, focusing on the examples of gradings and background $b \in H^2(M : \mathbb{Z}/2\mathbb{Z})$. The methods here carry over to other decorations we anticipate will be of use in Floer theory, especially when one must trivialize more than $\det^2(TM)$.

#### 3.6.1 For gradings

A grading on $M$ is the data of a trivialization $\det^2(TM) \cong \mathbb{C} \times M$ as a complex line bundle. One may equivalently encode this data in the following homotopy-coherent diagram:

$$\begin{array}{ccc}
E^U(1) & \cong & * \\
\downarrow & & \\
M & \xrightarrow{\det^2(TM)} & BU(1) \cong K(\mathbb{Z}, 2).
\end{array}$$

Explicitly, the line bundle $\det^2(TM)$ is classified by a map to $BU(1) \cong K(\mathbb{Z}, 2)$, and a trivialization is given by the data of a null-homotopy of this map. We emphasize that when we say the above diagram is homotopy coherent, we are not merely asserting the existence of a null-homotopy; the diagram represents the data of the null-homotopy.

Thus, a Liouville automorphism equipped with data respecting this null-homotopy is not merely the data of a Liouville automorphism $\phi : M \to M$; it is also the data of a higher homotopy cohering the following diagram:

$$\begin{array}{ccc}
* & \to & * \\
\downarrow & & \\
M & \xrightarrow{\phi} & M
\end{array}$$

The space of such data is encoded as a homotopy fiber product:
Definition 3.6.1 (Aut\textsuperscript{gr}). We let Aut\textsuperscript{gr}(M) denote the space of Liouville automorphisms of M respecting gradings. It is defined to be the homotopy pullback:

\[ \begin{array}{ccc} \text{Aut}^{\text{gr}}(M) & \rightarrow & \text{hom}_{\text{Top}/(M,M)} \simeq \text{hom}_{\text{Top}}(M,M) \\ \downarrow & & \downarrow \\ \text{Aut}^{\alpha}(M) & \rightarrow & \text{hom}_{\text{Top}/BU(1)}(M,M) \end{array} \tag{3.4} \]

Remark 3.6.2. Let us explain the maps in (3.4).

Top\textsubscript{X} is the slice ∞-category of topological spaces equipped with a map to X; we have utilized this notation for X = * and X = BU(1). hom\textsubscript{Top}\textsubscript{X} is the morphism space in this ∞-category. Let us also explain the bottom horizontal arrow. This is most efficiently encoded by first observing a homotopy equivalence \( \text{Aut}^{\alpha}(M) \sim \text{Aut}^{\text{comp}} \) from the space of those Liouville automorphisms of M equipped with a homotopy between choices of \( \omega \)-compatible almost complex structures. The forgetful map is a homotopy equivalence because the space of almost-complex structures compatible with \( \omega \) is contractible. On the other hand, \( \text{Aut}^{\text{comp}}(M) \) has a natural map to hom\textsubscript{Top}/BU(1)(M,M) where the choice of almost-complex structure \( J \) on \( TM \) defines a map from \( M \to BU(1) \) (classifying det\textsuperscript{2}(TM)), and the homotopy between \( J \) and \( \phi^*J \) determines the homotopy between \( M \to BU(1) \) and the composite \( M \xrightarrow{\phi} M \to BU(1) \).

In summary, the bottom horizontal arrow is determined by considering the composite
\[ \text{Aut}^{\alpha}(M) \xleftarrow{\sim} \text{Aut}^{\text{comp}} \to \text{hom}_{\text{Top}/BU(1)}(M,M) \]
and choosing a homotopy inverse (together with a homotopy exhibiting the homotopy inverseness—this is a choice in a contractible space of choices) to the left-hand arrow.

Remark 3.6.3. All the spaces in (3.4) are endomorphism spaces in particular ∞-categories. For example, \( \text{Aut}^{\alpha}(M) \) is the endomorphism space of \( M \), considered as an object of the topologically enriched category of Liouville sectors with morphisms being Liouville isomorphisms. In particular, \( \text{Aut}^{\text{gr}}(M) \) is the endomorphism space of a corresponding fiber product category; thus it is an \( A_\infty \)-space (and in fact, group-like). Because the forgetful map \( \text{Aut}^{\text{gr}}(M) \to \text{Aut}^{\alpha}(M) \) is now seen to arise from a functor, it gives rise to a map of \( A_\infty \)-algebras (in the ∞-category of spaces).

We are interested in studying \( \text{Aut}^{\alpha}(M) \), so it will be useful to know how far away the homotopy type of \( \text{Aut}^{\text{gr}}(M) \) is from that of \( \text{Aut}^{\alpha}(M) \). So let us study the fibers of the natural forgetful map \( \text{Aut}^{\text{gr}}(M) \to \text{Aut}^{\alpha}(M) \) in (3.4).

Because the forgetful map is a homomorphism (see Remark 3.6.3), it suffices to compute the fiber over the identity map \( \phi = \text{id}_M : M \to M \). This in turn means it suffices to compute the homotopy fiber of the map \( \text{hom}_{\text{Top}}(M,M) \to \text{hom}_{\text{Top}/BU(1)}(M,M) \) over the identity morphism. This homotopy fiber is straightforwardly seen to be homotopy equivalent to
\[ \text{hom}(M,\Omega BU(1)) \simeq \text{hom}(M,S^1). \]

That is, we have a fiber sequence
\[ \text{hom}(M,U(1)) \to \text{Aut}^{\text{gr}}(M) \to \text{Aut}^{\alpha}(M). \]

The fiber is clearly 1-connected (i.e., has no homotopy groups in degrees \( \geq 2 \)), so from the Serre long exact sequence of homotopy groups, we conclude:

Proposition 3.6.4. The forgetful map \( \text{Aut}^{\text{gr}}(M) \to \text{Aut}^{\alpha}(M) \) induces an isomorphism on homotopy groups \( \pi_k \) for \( k \geq 3 \). The induced map on \( \pi_2 \) is an injection.
3.6.2 For a background class b

Likewise, fix an element \( b \in H^2(M; \mathbb{Z}/2\mathbb{Z}) \). This is classified by a map \( \tilde{b} : M \to K(\mathbb{Z}/2\mathbb{Z}, 2) \), well-defined up to homotopy. We fix \( \tilde{b} \) once and for all.

**Definition 3.6.5** (*Aut\(^b\)(M)). We let \( \text{Aut}^b(M) \) denote the space of Liouville automorphisms of \( M \) equipped with data respecting \( b \). It is defined via the following homotopy pullback square:

\[
\begin{array}{ccc}
\text{Aut}^b(M) & \longrightarrow & \text{hom}_{\text{Top}/K(\mathbb{Z}/2\mathbb{Z}, 2)}((M, \tilde{b}), (M, \tilde{b})) \\
\downarrow & & \downarrow \\
\text{Aut}^\alpha(M) & \longrightarrow & \text{hom}_{\text{Top}}(M, M)
\end{array}
\]

Informally, an element of \( \text{Aut}^b(M) \) is the data of a Liouville automorphism \( \phi : M \to M \), together with a homotopy from \( \tilde{b} \circ \phi \) to \( \tilde{b} \). The same observation as in Remark 3.6.3 shows \( \text{Aut}^b(M) \) is a group-like \( A_\infty \)-algebra in spaces, and \( \text{Aut}^b(M) \to \text{Aut}^\alpha(M) \) lifts to an \( A_\infty \) map.

Moreover, the fiber of this map can be computed as the fiber of the map \( \text{hom}_{\text{Top}/K(\mathbb{Z}/2\mathbb{Z}, 2)}(M, M) \to \text{hom}_{\text{Top}}(M, M) \), which is given by

\[ \text{hom}_{\text{Top}}(M, \Omega K(\mathbb{Z}/2\mathbb{Z}, 2)) \simeq \text{hom}_{\text{Top}}(M, \mathbb{R} P^\infty). \]

As before, we conclude:

**Proposition 3.6.6.** The forgetful map \( \text{Aut}^b(M) \to \text{Aut}^\alpha(M) \) induces an isomorphism on homotopy groups \( \pi_k \) for \( k \geq 3 \). The induced map on \( \pi_2 \) is an injection.

3.6.3 For both

**Definition 3.6.7** (*Aut\(^{gr,b}\)(M)). We define \( \text{Aut}^{gr,b}(M) \) as the homotopy pullback

\[
\begin{array}{ccc}
\text{Aut}^{gr,b}(M) & \longrightarrow & \text{Aut}^{gr}(M) \\
\downarrow & & \downarrow \\
\text{Aut}^b(M) & \longrightarrow & \text{Aut}^\alpha(M).
\end{array}
\]

**Proposition 3.6.8.** The forgetful map \( \text{Aut}^{gr,b}(M) \to \text{Aut}^\alpha(M) \) induces an isomorphism on homotopy groups \( \pi_k \) for \( k \geq 3 \). The induced map on \( \pi_2 \) is an injection.

**Proof.** Combine Proposition 3.6.4 and 3.6.6. \( \square \)

3.7 Liouville bundles

**Definition 3.7.1** (Liouville bundle). Fix a Liouville sector \( M \). A Liouville bundle with fiber \( M \) is the choice of a smooth \( M \)-bundle \( p : E \to B \), together with a smooth reduction of the structure group from \( \text{Diff}(M) \) to \( \text{Aut}^\alpha(M) \).

**Remark 3.7.2.** Definition 3.7.1 applies when \( p : E \to B \) is a smooth map of diffeological spaces (see Definition 4.1.2), or smooth manifolds with corners. By a smooth reduction of structure group, we mean that for an open cover, the specified transition maps \( U_{\alpha\beta} \to \text{Aut}^\alpha(M) \) must be smooth (in the sense of the diffeology on \( \text{Aut}^\alpha(M) \) (Example 4.1.14) and the diffeology of \( B \)).
**Notation 3.7.3** ($\partial E$). Let $E \to B$ be a Liouville bundle whose fibers are Liouville sectors, and suppose these fibers are all isomorphic to some Liouville sector $M$. We denote by

$$\partial E \to B$$

the induced fiber bundle whose fibers are diffeomorphic to $\partial M$. Note that we use the symbol $\partial E$ regardless of whether the base $B$ has boundary, corners, et cetera.

**Remark 3.7.4** (Θ). Let $E \to B$ be a Liouville bundle and suppose $E$ and $B$ are both smooth manifolds, possibly with corners. First let $B$ be smoothly contractible. Then there exists a global choice of 1-form

$$\Theta \in \Omega^1(E; \mathbb{R})$$

such that:

(Θ1) for every $b \in B$, the fiberwise restriction $\Theta|_{E_b}$ defines a 1-form on the fiber $E_b$ exhibiting $E_b$ as a Liouville completion.

By the paracompactness of $B$ and a partition of unity argument, we thus have a global 1-form $\Theta$ on $E \to B$ satisfying property (Θ1) for arbitrary base manifolds $B$.

**Remark 3.7.5.** Fix a Liouville bundle $E \to B$ where $B$ (and hence $E$) is a smooth manifold, possibly with corners. Then the space of $\Theta$ satisfying (Θ1) is convex, and in particular, smoothly contractible.

**Example 3.7.6.** Fix a Liouville sector $M$. There will be two classes of Liouville bundles of interest associated to $M$.

The first is the *universal* Liouville bundle. (See Section 4.3.) This is constructed from the universal principle bundle

$$E \overset{\mathrm{Aut}^\partial(M)}{\to} B \overset{\mathrm{Aut}^\partial(M)}{\to}$$

by taking the induced principle $M$-bundle

$$E \to B \overset{\mathrm{Aut}^\partial(M)}{\to}$$

(whose structure group is canonically smoothly reduced to $\mathrm{Aut}^\partial(M)$). The reader may appreciate why we carefully inserted the phrase “if $B$ is a manifold” in Remark 3.7.4—in the example of $B = B \overset{\mathrm{Aut}^\partial(M)}{\to} M$, $B$ is not a manifold.

The second main example is given by taking a smooth map from an extended smooth simplex (see Definition 4.2.1)

$$j : |\Delta^n_\varepsilon| \to B$$

and pulling back to obtain a smooth Liouville bundle $j^* E \to |\Delta^n_\varepsilon|$; in particular, one obtains another smooth Liouville bundle by restricting further to the standard $n$-simplex $|\Delta^n| \subset |\Delta^n_\varepsilon|$.

### 3.8 Connections on bundles

**Definition 3.8.1** (Connection). Let $\pi : E \to B$ be a smooth fiber bundle. Recall that a (Ehresmann) connection is a choice of splitting

$$TE \cong HTE \oplus VTE$$

(3.5)

where $VTE = \ker(d\pi)$. As usual we will call $HTE$ the horizontal distribution (associated to the connection).
Fix a Liouville bundle $\pi : E \to B$ over a smooth manifold $B$, and equip $E$ with a choice of global 1-form $\Theta \in \Omega^1(E)$ as in Remark 3.7.4. Then one has a natural connection on $\pi : E \to B$, defined as follows:

**Definition 3.8.2.** The **connection associated to** $\Theta$ is the subbundle of $TE$ consisting of those tangent vectors $x$ for which

$$VTE \subset \ker (d\Theta(-,x)).$$

That is, any vertical tangent vector is annihilated when paired with $x$ using $d\Theta$.

In particular, any Liouville bundle equipped with a $\Theta$ as in Remark 3.7.4 can choose Liouville form has a well-defined notion of parallel transport along smooth curves.

### 3.9 Almost complex structures

**Definition 3.9.1.** Let $E \to B$ be a Liouville bundle. Let $J$ be a smooth choice of fiber-wise almost complex structures on $E$.

We say that $J$ is **conical near infinity** if for some (hence any) choice of $\Theta$ (as in Remark 3.7.4), there exists a subset $K \subset E$, proper over $B$, such that the following holds:

1. For each $b \in B$, $K \cap E_b$ is a Liouville domain (exhibiting the fiber $E_b$ as the Liouville completion of $K \cap E_b$), and

2. Writing $E_b$ as the completion of $K \cap E_b$ with conical coordinate $s$ (3.3), we have that

$$\Theta|_{E_b} \circ |_{E_b} = d(e^s).$$

**Remark 3.9.2.** If $J$ is conical near infinity (Definition 3.9.1), it follows that along each fiber of $E \to B$, the Lie derivative of $J|_{E_b}$ with respect to the Liouville flow vanishes outside some compact subset (for example, outside of $K \cap E_b$).

**Example 3.9.3.** If $B$ is a point, then a choice of $J$ as in Definition 3.9.1 is a choice of conical-near-infinity almost-complex structure $J$ on the fiber Liouville sector, in the usual sense.

Later, we will have occasion to consider Liouville bundles $\pi : E \to B$ where the base $B = \overline{S}^{d+1}$ is the universal family of possibly-broken, boundary-marked holomorphic disks (with nodal points removed). (See Notation 5.1.1.)

**Notation 3.9.4 ($\mathcal{J}$).** For the time being, for every $b \in B$, let $\mathcal{S}_b \subset B$ denote the Riemann surface containing $b$. Then over $E$ there is a natural bundle

$$\mathcal{J} \to E$$

whose fibers above $x \in E$ consist of almost-complex structures on the vector bundle

$$(d\pi)^{-1}(T_b\mathcal{S}_b) \subset T_xE.$$  \hspace{1cm} (3.6)

**Remark 3.9.5.** Here is another description of $\mathcal{J}$. Let $B = \overline{S}_{d+1}^{\circ} \to \overline{R}_{d+1}$ denote the projection map for the universal family of curves, and let $\mathcal{H} \subset TB$ denote the vertical tangent bundle of this projection. Fix further a Liouville form $\Theta$ on $\pi : E \to B$, so that we have an induced splitting $TE \cong HTE \oplus VTE$ as in (3.5). By the identification $HTE \cong \pi^*TB$, we have an induced subbundle $\pi^*\mathcal{H} \oplus VTE \subset TE$. $\mathcal{J}$ is the bundle whose global sections are choices of almost-complex structures on $\pi^*\mathcal{H} \oplus VTE$. 

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**Definition 3.9.6** (\(\mathcal{J}\) Suitable for counting sections). Let \(B = \overline{S}_{d+1}^\circ\) and fix a Liouville bundle \(\pi : E \to B\). Let \(\mathcal{J}\) be the bundle from Notation 3.9.4. We say that a global section \(\mathcal{J}\) of \(\mathcal{J}\) is *suitable for counting sections* when the following are satisfied:

1. For every member of the universal family \(S_r \subset \overline{S}_{d+1}^\circ\), let \(E_r \to S_r\) denote the pulled back Liouville bundle. We demand that the projection map is holomorphic—that is, \(d\pi \circ \mathcal{J}|_{E_r} = j_r \circ d\pi\).

(Here, \(j_r\) is the complex structure on \(S_r\).)

2. \(\mathcal{J}\) preserves the vertical tangent space \(VTE\), and \(\mathcal{J}|_{VTE}\) is a conical-near-infinity almost-complex structure for the bundle \(E \to B\) as in Definition 3.9.1.

3. Finally, we demand that for some (and hence any) choice of global Liouville form \(\Theta\) on \(E\) as in Remark 3.7.4, there exists a subset \(K \subset E\) (independent of \(r\)), proper over \(B\), such that \(\mathcal{J}(HTE_r) = HTE_r\). (Here, \(HTE_r\) is the horizontal tangent space induced by pulling back the connection on \(E\) to a connection on \(E_r\).)

By abuse of notation, we will refer to \(\mathcal{J}\) also as a choice of almost-complex structure. (Even though, strictly speaking, \(\mathcal{J}\) only defines almost-complex structures on each \(E_r\), and not on all of \(E\).)

**Remark 3.9.7.** Let \(\mathcal{J}\) be an almost-complex structure suitable for counting sections (Definition 3.9.6). Choose a splitting \(T_xE_r \cong VTE \oplus T_{\pi(x)}S_r\), condition 1. says that \(\mathcal{J}_x\) may be written as a block triangular matrix. Condition 3. says that, outside controlled, fiber-wise compact subset, \(\mathcal{J}_x\) is block diagonal. In particular the space of \(\mathcal{J}\) is seen to be contractible.

### 3.10 Defining functions and barriers on families

Let us recall from [GPS17] that if \(M\) is a Liouville sector, there exists a smooth map

\[
\pi : \text{Nbd}(\partial M) \to \mathbb{C}_{\mathbb{R} \geq 0}
\]

from a neighborhood of \(\partial M\) to the complex numbers with positive real coordinate. This (possibly non-surjective) map satisfies the following:

1. The imaginary coordinate of \(\pi\) defines a smooth, linear-near-infinity function \(I\) whose Hamiltonian vector field is outward pointing at \(\partial M\). (This is called a “defining function” in [GPS17].)

2. Moreover, there is a contractible space of almost-complex structures \(J\) on \(M\), compatible with the Liouville structure of \(M\), such that \(\pi\) is \(J\)-holomorphic.

**Remark 3.10.1.** This allows one to use a standard “barrier” type argument using the open mapping theorem to conclude the following: If a holomorphic curve \(u : S \to M\) has boundary Lagrangians supported away from \(\partial M\), then the image of \(u\) must be bounded away from \(\partial M\) as well. (See 2.10.1 of [GPS17].) This is the main utility of the definition of Liouville sector, and in particular of the defining function \(I\). (Informally, while \(I\) defines the imaginary coordinate of \(\pi\), its negative Hamiltonian flow-time away from \(\partial M\) defines the real coordinate—see the proof of Proposition 2.24 of [GPS17].)
Remark 3.10.2. Now if $E \to B$ is a Liouville bundle of Liouville sectors over a smooth manifold, a partition of unity argument defines a global function $\pi : \text{Nbhd}(\partial E) \to \mathbb{C}_{\geq 0}$ whose imaginary part restricts on each fiber to a defining function $I$. (Note $\partial E$ is defined in Notation 3.7.3.) By contractibility of the space of almost-complex structures, one concludes that one can choose a smooth family of almost complex-structures $\mathcal{J}$ on $E$ such that

$$D\pi|_{\text{VTE}} \circ \mathcal{J} = j_{\mathcal{C}_{\mathbb{R}_{\geq 0}}} \circ D\pi|_{\text{VTE}}$$
onumber

on $\text{Nbhd}(\partial E)$. (Here $\text{VTE} = \ker D\pi$ is the vertical tangent bundle of $E$.) In particular, given any map $u : S \to E$ which is holomorphic with respect to $J$, the composite $\pi \circ u : S \to \mathbb{C}_{\mathbb{R}_{\geq 0}}$ is holomorphic, and the same barrier argument as in Remark 3.10.1 shows that the image of $u$ must be bounded away from the boundary of each fiber Liouville sector.

In particular, if one has a prior $C^0$ bounds on the strip-like ends of $S$, then one has an a priori $C^0$ bound on $u$ given the boundary conditions.

3.11 Branes and cofinal sequences of wrappings

Let $M$ be a Liouville sector.

Definition 3.11.1. A subset $A \subset M$ is called conical near infinity if for some (and hence all) $\theta \in \mathcal{C}_{\text{Liou}}$, and for some compact subset $K$, the complement $A \setminus K$ is closed under the positive Liouville flow.

There are standard decorations one should put on Liouville sectors and their Lagrangians to obtain a $\mathbb{Z}$-graded, $\mathbb{Z}$-linear Fukaya category—for example, gradings and Pin structures. We assume these structures to be chosen throughout. To that end:

Definition 3.11.2. Let $M$ be a Liouville sector. A brane is a conical-near-infinity Lagrangian $L \subset M$ equipped with the relevant decorations.

Example 3.11.3. So for example, if $L$ is a compact Lagrangian, then $L$ (when equipped with the appropriate decorations) is a compact brane. Note also that our branes have no boundary—by definition of Lagrangian (submanifold), $L$ is locally diffeomorphic to Euclidean space, and hence boundaryless.

Definition 3.11.4 (Non-negative isotopy). Now fix an exact Lagrangian isotopy $j : L \times [0, 1] \to M$ through conical-near-infinity Lagrangians. (In particular, this induces an isotopy of Legendrians inside $\partial_{\infty}M$.) We say this is a non-negative wrapping\(^{14}\), of a non-negative isotopy (of the Lagrangians) if for some (and hence any) choice of Liouville form $\theta$ on $M$, we have the following outside a compact subset of $L$:

$$\theta(Dj(\partial_0)) \geq 0.$$

Put another way, the flow of $L$ in $\partial_{\infty}M$ is non-negative with respect to the contact form induced by $\theta$.

Definition 3.11.5 (Cofinal sequence of non-negative wrappings). Now suppose one has chosen a sequence of conical-near-infinity branes

$$L^{(0)}, L^{(1)}, \ldots,$$

together with a non-negative wrapping from $L^{(i)}$ to $L^{(i+1)}$ for every $i$. We say this is a cofinal sequence of non-negative wrappings if the following holds: For any non-negative wrapping of $L^{(0)}$ to another conical-near-infinity brane $L'$, there exists

\(^{14}\)In [GPS17], this notion is called a positive wrapping (see Definition 3.20 of loc. cit.).
1. \( w \in \mathbb{Z} \) and
2. a non-negative wrapping from \( L' \) to \( L^{(w)} \)

such that the composite isotopy

\[
L^{(0)} \rightarrow L' \rightarrow L^{(w)}
\]

is homotopic to the composite isotopy

\[
L^{(0)} \rightarrow L^{(1)} \rightarrow \ldots \rightarrow L^{(w)}
\]

through non-negative isotopies.

**Remark 3.11.6.** The non-negativity of the wrapping allows us to define so-called continuation elements (see Definition 7.1.10); these yield in particular cohomology classes

\[
c \in HF^*(L^{(i-1)}, L^{(i)}).
\]

For any brane \( K \) transversal to the \( L^{(i)} \), we will hence be able to define a sequence of cohomology groups

\[
\ldots \rightarrow HF^*(K, L^{(i-1)}) \xrightarrow{c_i} HF^*(K, L^{(i)}) \rightarrow \ldots
\]

by using the (cohomology-level) \( \mu^2 \) operation. The directed limit (i.e., colimit) of this sequence will be isomorphic to the cohomology of the morphism complexes in our family wrapped categories (Lemma 9.2.2).
4 Smooth approximation for $B\text{Aut}^o(M)$

Following Convention 1.1.2, we let $\text{Aut}^o$ denote the undecorated automorphism space; that is, this is the usual topological space of diffeomorphisms $\phi : M \to M$ that happen to be Liouville automorphisms. In Section 4.8, we will describe what results here generalize to the decorated case, and how we utilize these results in the rest of our text.

Let us motivate some of the ingredients of the section.

Let $G$ be a topological group. Then the classifying space $BG$ is well-understood—we can compute its homotopy groups in terms of those of $G$, we have useful models for its homotopy type, and we also know that (homotopy classes of) maps into $BG$ classify (isomorphism classes of) principle $G$-bundles.

When $G = \text{Aut}^o(M)$ for a Liouville sector $M$, we in particular want to be able to understand a smooth object classifying smooth Liouville bundles (so we can pull back bundles along smooth maps to $B\text{Aut}^o(M)$), and to be able to make sense of its homotopy type smoothly (i.e., we want a smooth approximation theorem so we can prove that the complex of smooth maps to $BG$ is homotopy equivalent to its complex of all continuous maps to $BG$).

We accomplish this in the form of Theorem 4.4.4, the main goal of this section. It shows that the weak homotopy type of $B\text{Aut}^o(M)$ can be understood by localizing a certain category of smooth simplices in $B\text{Aut}^o(M)$. (If smoothness issues could be ignored, this would be a straightforward result about the category of barycentric simplices.)

Notation 4.0.1 (Smooth players wear hats.). In this paper, when due diligence is needed, we will refer to an object with smooth structure by $\hat{B}$ (i.e., by making the symbol wear a hat). $B$ will often denote an underlying set, or space, associated to $\hat{B}$. For example, $B\text{Aut}^o(M)$ is a diffeological space, while $B\text{Aut}^o(M)$ as the Milnor classifying space associated to the topological group $\text{Aut}^o(M)$.

(These have the same underlying set.)

Notation 4.0.2. We will use $|\Delta^n| \subset \mathbb{R}^{n+1}_{\geq 0}$ to denote the standard topological $n$-simplex, while $\Delta^n$ denotes the simplicial set given by the $n$-simplex (Example 2.1.3).

4.1 Reminders on diffeological spaces

We collect various results, many of which are due to the papers of Christensen-Sinnamon-Wu, Christensen-Wu, and Magnot-Watts [CSW14, CW14, CW17, MW17].

Notation 4.1.1 (Mfld and Euc.). Let Mfld denote the category of smooth manifolds—its objects are smooth manifolds, and morphisms are smooth maps. We let Euc $\subset$ Mfld denote the full subcategory of those manifolds that are diffeomorphic to an open subset of $\mathbb{R}^n$ for some $n$.

When defining a geometric object, one can take a Lawvere-type approach to define functions on that object, or one can take a functor-of-points approach to define functions into that object. A diffeological space is defined by the latter approach: We will often define a diffeological space by beginning with the data of a set $X$, and then for all $U \in \text{Ob Euc}$, specifying which functions $U \to X$ are “smooth.” This defines a functor $\hat{X} : \text{Euc}^{op} \to \text{Sets}$ as in the following definition:

Definition 4.1.2 (Diffeological space). Fix a functor $\hat{X} : \text{Euc}^{op} \to \text{Sets}$ (i.e., a presheaf on Euc). We say that $\hat{X}$ is a diffeological space if the following two conditions hold:
1. For any \( U \in \text{Euc} \), the function

\[
\hat{X}(U) \to \hom_{\text{Sets}}(\hom_{\text{Euc}}(\mathbb{R}^0, U), \hat{X}(\mathbb{R}^0))
\]

is an injection. (That is, functions are determined by their values on points of \( U \).

2. \( \hat{X} \) is a sheaf (with the usual notion of open cover on smooth manifolds).

A map of diffeological spaces—also known as a smooth map of diffeological spaces—is a map of presheaves.

**Remark 4.1.3.** The map in 1. is induced by the structure map for presheaves

\[
\hat{X}(U) \times \hom_{\text{Euc}}(\mathbb{R}^0, U) \to \hat{X}(\mathbb{R}^0).
\]

**Notation 4.1.4** (Underlying set \( X \)). Let \( \hat{X} \) be a diffeological space. Then we say that \( \hat{X}(\mathbb{R}^0) \) is the underlying set of \( \hat{X} \), and we denote it \( X \). Note that by 1., every element of \( \hat{X}(U) \) determines a function \( f : U \to X \). If \( f \) is in the image of the map in 1., we say that \( f \) is a smooth map from \( U \) to \( X \). (In the literature, this is also called a plot.)

Unwinding the definitions, we thus see that a diffeological space is equivalent to the data of a set \( X \), and for every \( U \in \text{Ob Euc} \), a subset \( \hat{X}(U) \subset \hom_{\text{Sets}}(U, X) \). These must satisfy the following properties:

- \( \hat{X}(U) \) contains all the constant maps.
- If \( U \to U' \) is smooth, then the composite \( U \to U' \to X \) is in \( \hat{X}(U) \) whenever the function \( U' \to X \) is in \( \hat{X}(U') \).
- If there is an open cover \( \{U_i\} \) of \( U \) such that the function \( U \to X \) factors as \( U_i \to X \), and each \( U_i \to X \) is in \( \hat{X}(U_i) \), then the function \( U \to X \) is in \( \hat{X}(U) \).

**Definition 4.1.5** (Smoothness of maps). Let \( \hat{X} \) and \( \hat{Y} \) be diffeological spaces. A function \( X \to Y \) of underlying sets is called smooth if it is induced by a map of diffeological spaces.

**Remark 4.1.6** (\( D \)-topology). Let \( \hat{X} \) be a diffeological space and \( X \) its underlying set. One can endow \( X \) with the finest topology for which every smooth function \( f : U \to X \) determined by \( \hat{X} \) is continuous. This is called the \( D \)-topology in the literature.

**Warning 4.1.7.** However, we will almost never make use of the \( D \)-topology, and in fact our main example \( X = \text{Aut}^\circ(M) \) will not be endowed with the \( D \)-topology. So the reader should not assume that the underlying set \( X \) of a diffeological space \( \hat{X} \) is endowed with the \( D \)-topology.

**Example 4.1.8** (Smooth manifolds). Let \( X \) be a smooth manifold. Then one can define a diffeological space by declaring \( \hat{X}(U) = \hom_{\text{Mfd}}(U, X) \); note that \( X \) is indeed the underlying set of \( \hat{X} \) as implied by our notation, and the \( D \)-topology coincides with the usual one. This construction gives a fully faithful embedding of the category of smooth manifolds into the category of diffeological spaces.

**Example 4.1.9** (Subspaces). Let \( \hat{X} \) be a diffeological space, and let \( A \subset X \) be a subset. Then \( A \) determines a subsheaf \( \hat{A} \subset \hat{X} \) where \( \hat{A}(U) \) consists of all those elements \( f : U \to X \) whose image lies in \( A \). We call this the subspace diffeology on \( A \). Note this is an example where there is ambiguity in the topology of \( A \)—one could give it the subspace topology with respect to the \( D \)-topology on \( X \), or give it the \( D \)-topology induced by the diffeological structure \( \hat{A} \). These topologies do not always coincide.
Example 4.1.10 (Function spaces). Let $X$ and $Y$ be smooth manifolds, and let $C^\infty(X,Y)$ be the set of smooth functions. We can endow this set with a diffeological space structure by declaring a function $U \to C^\infty(X,Y)$ to be smooth if and only if the adjoint map $U \times X \to Y$ is smooth. In general, the $D$-topology of this set is finer than the compact-open topology, finer than the weak Whitney topology, but coarser than the strong Whitney topology.

Remark 4.1.11. The category of diffeological spaces has all limits and colimits; in fact, the functor sending a diffeological space to its underlying set has both left and right adjoints, so the limit and colimit can be understood as sets in the usual way. Moreover, the $D$-topology of the resulting space can be understood as spaces in the usual way. We also have an explicit description of the colimit diffeology: A function $U \to \operatorname{colim} f$ is smooth if and only if there is an open cover $\{U_i\}$ of $U$, and an object $f(j)$ in the diagram given by $f$, such that the function factors $U_i \to f(j) \to \operatorname{colim} f$ with $U_i \to f(j)$ being smooth.

One can also show that the category of diffeological spaces is Cartesian closed. The hom-objects are precisely the function spaces with the diffeological space structure of Example 4.1.10.

We also remark that these observations follow straightforwardly from the fact that the category of diffeological spaces is equivalent to the category of so-called “concrete” sheaves on a site.

(Concreteness is precisely equivalent to condition 1 of Definition 4.1.2.)

Definition 4.1.12. A diffeological group is a group object in the category of diffeological spaces. Concretely, this is the data of a diffeological space $\hat{G}$, together with a group structure whose inverse and multiplication operations are smooth.

Example 4.1.13. Combining Examples 4.1.9 and 4.1.10, we see that for any smooth manifold $X$, the diffeomorphism group $\operatorname{Diff}(X)$ is a diffeological space. It is in fact a diffeological group (Definition 4.1.12).

It follows that any subgroup (closed as a subspace or otherwise) is also a diffeological group.

Example 4.1.14. In particular, for any Liouville sector $M$, the group $\operatorname{Aut}(M) \subset \operatorname{Diff}(M)$ is a diffeological group.

4.2 Reminders on some homotopy theory of diffeological spaces

One of the most useful tools in homotopy theory is the ability to convert any topological space into a simplicial set. We recall the analogue of this for diffeological spaces.

Definition 4.2.1 ($|\Delta^k|$). Let $|\Delta^k| \subset \mathbb{R}^{k+1}$ denote the affine hyperplane defined by the equation $\sum_{i=0}^k t_i = 1$. We refer to $|\Delta^k|$ as the extended $k$-simplex, and consider it a smooth manifold in the obvious way. (It is diffeomorphic to the standard Euclidean space $\mathbb{R}^k$.)

We will refer to a map $|\Delta^k| \to |\Delta^{k'}|$ as simplicial if it is the restriction of the linear map $\mathbb{R}^k \to \mathbb{R}^{k'}$ induced by some map of sets $[k] \to [k']$. (This map need not respect order.)

Remark 4.2.2. Since any smooth manifold is a diffeological space (Example 4.1.8), the assignment $[k] \mapsto |\Delta^k|$ defines a cosimplicial object in the category of diffeological spaces.

Notation 4.2.3. Let $\hat{X}$ be a diffeological space. We let $\operatorname{Sing}^{C^\infty}(\hat{X})$ denote the simplicial set

$$\operatorname{Sing}^{C^\infty}(\hat{X}) : \Delta^\text{op} \to \text{Sets}, \quad [k] \mapsto C^\infty(|\Delta^k|, \hat{X})$$

of smooth maps from extended simplices to $\hat{X}$.
Warning 4.2.4. In [CW14], the notation $S^D(X)$ is used to denote what we write as $\text{Sing}^{C\infty}(\hat{X})$. Also in loc. cit., the $\hat{X}$ notation is not used to distinguish a diffeological space from its underlying set.

Example 4.2.5. Let $X$ be a smooth manifold. We let $\hat{X}$ denote the associated diffeological space (Example 4.1.8), and let $\text{Sing}(X)$ denote the usual simplicial set of continuous simplices $|\Delta^n| \to X$. The natural map $\text{Sing}^{C\infty}(\hat{X}) \to \text{Sing}(X)$ is a weak homotopy equivalence by the smooth approximation theorem (sometimes called the Whitney approximation theorem in this generality).

Warning 4.2.6. The simplicial set $\text{Sing}^{C\infty}(\hat{X})$ need not be a Kan complex. (In contrast: For a space $X$, $\text{Sing}(X)$ is always a Kan complex.)

In general, it seems that a “homogeneity” property is needed to conclude that $\text{Sing}^{C\infty}(\hat{X})$ is a Kan complex. For example, when $\hat{X}$ is the diffeological space associated to a smooth manifold, $\text{Sing}^{C\infty}(\hat{X})$ is a Kan complex (Corollary 4.36 of [CW14]). But when $\hat{X}$ is associated to a smooth manifold with non-empty boundary, this is no longer true (Corollary 4.47 of ibid.).

Following the theme of “homogeneity implies Kan,” we have the following.

Proposition 4.2.7 (Proposition 4.30 of [CW14]). For any diffeological group $\hat{G}$ (see Definition 4.1.12.), $\text{Sing}^{C\infty}(\hat{G})$ is a Kan complex.

Finally, in usual homotopy theory, we can compare two different notions of homotopy groups. The usual notion $\pi_n$ is defined by homotopy classes of continuous maps $S^n \to X$, and the combinatorial definition is defined by classes of maps $\partial \Delta^n \to \text{Sing}(X)$.

In the diffeological space setting, we make the following definitions:

Definition 4.2.8. The nth smooth homotopy groups $\pi_n^{C\infty}(\hat{X},x_0)$ for $x_0 \in X$ is defined to be smooth homotopy classes of maps $f : |\Delta^n_e| \to \hat{X}$ satisfying the condition that $f(y) = x_0$ for any $y \in |\Delta^n_e|$ for which $y$ has some coordinate equal to zero.

Remark 4.2.9. There are many different but equivalent definitions of the smooth homotopy groups of $\hat{X}$; for example, one can take smooth, based maps $S^n \to \hat{X}$ up to smooth homotopy. But we prefer the above model for the following construction:

We as usual have the combinatorial homotopy groups defined for the simplicial set $\text{Sing}^{C\infty}(\hat{X})$; the homotopy groups are defined as equivalence classes of maps $\Delta^n \to \text{Sing}^{C\infty}(\hat{X})$ that are constant along $\partial \Delta^n$. One can check that the homotopy relation defining $\pi_n^{C\infty}$ is respected by the relation defining the simplicial homotopy groups, so we have a natural map

$$\pi_n^{C\infty}(\hat{X},x_0) \to \pi_n(\text{Sing}^{C\infty}(\hat{X}),x_0). \quad (4.1)$$

Moreover, if $\hat{X}$ is a diffeological space for which $\text{Sing}^{C\infty}(\hat{X})$ is a Kan complex, then the map on $\pi_0$ is a bijection, and for any point $x_0 \in X$, the maps (4.1) are group isomorphisms (Theorem 4.11 of [CW14]). In particular, this holds for diffeological groups by Proposition 4.2.7.

4.3 Reminders on $\hat{B}G$

Now let us recall some constructions of classifying spaces for diffeological groups. We follow [MW17] and [CW17].

Let $G$ be a diffeological group. Then in [MW17] and [CW17], the authors construct two diffeological spaces $\hat{E}G$ and $\hat{B}G$, together with a smooth map $\hat{E}G \to \hat{B}G$. We employ the diffeology of [CW17], as the resulting statements are a bit more general.
Remark 4.3.1. The constructions of $\hat{E}G$ and $\hat{B}G$ below are modeled on Milnor’s join construction [Mil56]. One obvious reason to prefer Milnor’s construction as opposed to the usual simplicial space construction is that the map $EG \to BG$ need not be locally trivial for the latter; this was pointed out as early as [Seg68].

Construction 4.3.2 ($\hat{E}G$.). Let $|\Delta^\omega|$ denote the infinite-dimensional simplex. That is, it is the set of those $(t_i), i \in \mathbb{Z}_{\geq 0} \oplus \omega \mathbb{R}$ for which only finitely many $t_i$ are non-zero, and $\sum t_i = 1$. As a diffeological space, we have that $|\Delta^\omega| \cong \text{colim}_{i \geq 0} |\Delta^i|$ where $i$ is the standard $i$-dimensional simplex, given the subspace diffeology from $\mathbb{R}^{i+1}$.

Then $|\Delta^\omega| \times \prod G$ can be given the product diffeology, and we define $\hat{E}G$ to be the quotient by identifying $(t_i, g_i) \sim (t'_i, g'_i)$ when the following two conditions hold:

1. $t_i = t'_i$ for all $i$, and
2. If $t_i = t'_i \neq 0$, then $g_i = g'_i$.

Obviously $\hat{E}G$ retains the projection map to $|\Delta^\omega|$; its fibers above an element $(t_i)$ can be identified with the product $\prod_{i \text{ s.t. } t_i \neq 0} G$.

Of course, $\hat{E}G$ is a diffeological space by virtue of the category of diffeological spaces having all limits and colimits (Remark 4.1.11). Likewise, the following is a diffeological space:

Construction 4.3.3 ($\hat{B}G$.). We let $\hat{B}G$ be the quotient by the natural action $\hat{E}G \times G \to \hat{E}G, [(t_i, g_i)] \cdot g = [(t_i, g_ig)]$.

These satisfy a series of properties that we state as a single theorem.

Theorem 4.3.4. The following are true:

1. The underlying set of $\hat{B}G$ is the Milnor construction of the classifying space $BG$ [Mil56].
2. $\hat{E}G$ is smoothly contractible. (Corollary 5.5 of [CW17].)
3. The map $\hat{E}G \to \hat{B}G$ is a diffeological principle $\hat{G}$-bundle. (Theorem 5.3 of [CW17].)
4. For any diffeological space $\hat{X}$, pull-back induces a bijection between smooth homotopy classes of maps $\hat{X} \to \hat{B}G$, and principle $\hat{G}$-bundles over $\hat{X}$. (Theorem 5.10 of [CW17].)

Remark 4.3.5. That $\hat{E}G$ is smoothly contractible means that we can find a smooth map $|\Delta^1| \times \hat{E}G \to \hat{E}G$ which interpolates between a constant map and the identity map. By Lemma 4.10 of [CW14], this induces a simplicial homotopy between the identity map of $\text{Sing}^{C\infty} (\hat{E}G)$ and a constant map—that is, $\text{Sing}^{C\infty} (\hat{E}G)$ is weakly contractible as a simplicial set, hence weakly homotopy equivalent to a point.

Remark 4.3.6. Let $\hat{E} \to \hat{B}$ be a diffeological bundle, and assume that the fibers are diffeological spaces for whom $\text{Sing}^{C\infty}$ is a Kan complex. Then the induced map $\text{Sing}^{C\infty} (\hat{E}) \to \text{Sing}^{C\infty} (\hat{B})$ is a Kan fibration. (See Proposition 4.28 of [CW14].)

As a result, Theorem 4.3.4(3) implies that we have a Kan fibration sequence of simplicial sets

$\text{Sing}^{C\infty} (\hat{G}) \to \text{Sing}^{C\infty} (\hat{E}G) \to \text{Sing}^{C\infty} (\hat{B}G)$. 

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4.4 Set-up and statement of Theorem 4.4.4.

Notation 4.4.1 \(\text{Simp}(\hat{B})\). Let \(\hat{B}\) be any diffeological space. We let \(\text{Simp}(\hat{B})\) denote the category of smooth, extended simplices of \(\hat{B}\). That is, an object of \(\text{Simp}(\hat{B})\) is the data of a smooth map \(j : |\Delta^k_e| \rightarrow \hat{B}\). (See Definition 4.2.1 for \(|\Delta^k_e|\).) A morphism is a commutative diagram

\[
\begin{array}{ccc}
|\Delta^k_e| & \rightarrow & |\Delta^k_e'| \\
\downarrow^j & & \downarrow^{j'} \\
\hat{B} & \rightarrow & \hat{B}
\end{array}
\]

where the map \(|\Delta^k_e| \rightarrow |\Delta^k_e'|\) is (induced by) a simplicial map.

Remark 4.4.2. As usual, if \(\mathcal{C}\) is a category, we let \(N(\mathcal{C})\) denote its nerve. (Example 2.1.4.) We have also introduced the subdivision of a simplicial set (Notation 2.1.9). In fact, we have a natural isomorphism of simplicial sets

\[N(\text{Simp}(\hat{B})) \cong \text{subdiv}(\text{Sing}^{C^\infty}(\hat{B})).\]

Throughout this section, we fix a Liouville sector \(M\). As before, we let \(\text{Aut}^o(M)\) denote the topological group of Liouville automorphisms of \(M\) (Definition 3.5.2), and \(\hat{\text{Aut}}^o(M)\) the diffeological group (Example 4.1.14).

Notation 4.4.3. Following Construction 4.3.3 for \(\hat{G} = \hat{\text{Aut}}^o(M)\), we denote the classifying diffeological space by \(B\text{Aut}^o(M)\).

The main theorem of this section is as follows:

Theorem 4.4.4. Consider the \(\infty\)-category \(N(\text{Simp}(B\hat{\text{Aut}}^o(M)))\). The Kan completion is homotopy equivalent to the Kan complex \(\text{Sing}(B\hat{\text{Aut}}^o(M)))\).

Remark 4.4.5. Note that \(\text{Simp}\) relies on the diffeological structure on \(B\hat{\text{Aut}}^o(M)\), while \(\text{Sing}\) relies only on the homotopy type of the space \(B\text{Aut}^o(M)\).

Suppose that \(Q\) is a smooth, compact, finite-dimensional manifold, possibly with corners. Then the exact same proofs laid out in this section show that Theorem 4.4.4 follows by replacing \(B\hat{\text{Aut}}^o(M)\) by \(B\text{Diff}(Q)\):

Theorem 4.4.6. Let \(Q\) be a smooth, compact, finite-dimensional manifold, possibly with corners. Consider the \(\infty\)-category \(N(\text{Simp}(B\hat{\text{Diff}}(Q)))\). The Kan completion is homotopy equivalent to the Kan complex \(\text{Sing}(B\text{Diff}(Q)))\).

4.5 Realizations of subdivisions

The following lemma illustrates one power of localization: It turns (the nerve of) a strict category into a homotopically rich object.

Lemma 4.5.1. The Kan completion of \(N(\text{Simp}(B\hat{\text{Aut}}^o(M)))\) is weakly homotopy equivalent to \(\text{Sing}^{C^\infty}(B\hat{\text{Aut}}^o(M)))\).

We give two proofs for the reader’s edification. The second proof has the advantage that one sees an explicit map leading to the homotopy equivalence.
Proof of Lemma 4.5.1 using coCartesian fibrations. \(N(\widehat{\text{Simp}}(B\widehat{\text{Aut}}^\circ(M)))\) is the total space of a Cartesian fibration over \(N(\Delta_{\text{inj}})\) with discrete fibers; in particular, the opposite category is a coCartesian fibration over \(\Delta_{\text{inj}}^\text{op}\). (See Remark 2.1.12.) This coCartesian fibration classifies the functor
\[
\Delta_{\text{inj}}^\text{op} \to \text{Sets} \subset \mathcal{G}_{\text{pd}}^\infty \subset \mathcal{C}_{\text{at}}^\infty,
\]
otherwise known as \(\text{Sing}^{\infty}\). Recall that the colimit of a diagram of \(\infty\)-categories is computed by localizing the total space of the corresponding coCartesian fibration along coCartesian edges. Thus we have an equivalence of \(\infty\)-categories
\[
\text{colim}_{\Delta_{\text{inj}}^\text{op}} \text{Sing}^{\infty}(B\widehat{\text{Aut}}^\circ(M)) \to N(\widehat{\text{Simp}}(B\widehat{\text{Aut}}^\circ(M)))[C^{-1}]
\]
where \(C\) is the collection of coCartesian edges. Because the inclusion \(\mathcal{G}_{\text{pd}}^\infty \subset \mathcal{C}_{\text{at}}^\infty\) admits a right adjoint, the colimit of this functor into \(\mathcal{C}_{\text{at}}^\infty\) may be computed via the colimit in \(\mathcal{G}_{\text{pd}}^\infty\), but this is of course the usual geometric realization because all of the relevant \(\infty\)-groupoids in the simplicial diagram are discrete. So the domain of the equivalence is the geometric realization (or Kan completion) of \(\text{Sing}^{\infty}(B\widehat{\text{Aut}}^\circ(M))\).

On the other hand, the localization is precisely the Kan completion of \(N(\widehat{\text{Simp}}(B\widehat{\text{Aut}}^\circ(M)))^\text{op}\) (because every edge is coCartesian), and hence the Kan completion of the non-opposite category. □

Here is another proof.

Notation 4.5.2. Fix a simplicial set \(S\). As usual we let \(\text{subdiv}(S)\) be the (nerve of) the category of simplices of \(S\). (See Construction 2.1.9.) We denote by
\[
\text{max} : \text{subdiv}(S) \to S
\]
the map of simplicial sets given by evaluating \(j\) at the maximal vertex of \(\Delta^n\).

Proposition 4.5.3. If \(S\) is a Kan complex, \(\text{max}\) exhibits \(S\) as the Kan completion of \(\text{subdiv}(S)\). More generally, \(\text{max}\) is a weak homotopy equivalence, so the Kan completion of \(S\) is homotopy equivalent to the Kan completion of \(\text{subdiv}(S)\).

Proof. It suffices to show that the induced map of geometric realizations \(|\text{max}| : |\text{subdiv}(S)| \to |S|\) is a homotopy equivalence; in fact, it is a homeomorphism. This is a classical result, as \(\text{subdiv}(S)\) is nothing more than the barycentric subdivision of \(S\). See for example III.4 of [GJ09]. □

By the universal property of localization, and because \(\text{max}\) is natural in the \(S\) variable, we have

Corollary 4.5.4. \(\text{max}\) induces a natural homotopy equivalence
\[
\text{hom}_{\mathcal{C}_{\text{at}}^\infty}(\text{subdiv}(S), \mathcal{C}_{\text{at}}^\infty) \simeq \text{hom}_{\mathcal{C}_{\text{at}}^\infty}(S, \mathcal{C}_{\text{at}}^\infty)
\]

Remark 4.5.5. If \(B\) is any topological space, one can analogously define the strict category \(\text{Simp}(B)\) of continuous simplices in \(B\). The proof of Lemma 4.5.1 adapts straightforwardly to show that the Kan completion of \(N(\text{Simp}(B))\) is homotopy equivalent to \(\text{Sing}(B)\).

Proof of Lemma 4.5.1 using \(\text{max}\) map. First note that if \(S = \text{Sing}^{\infty}(\hat{B})\), then \(N(\text{Simp}(\hat{B})) = \text{subdiv}(S)\) (Remark 4.4.2). Now use Proposition 4.5.3. □
4.6 Smooth approximation

The next few lemmas relate smoothness in the diffeological sense to continuity in the usual sense.

**Lemma 4.6.1.** Let $S$ be compact, and fix a function $f : S \to \text{Aut}^o(M)$ such that the adjoint map $S \times M \to M$ is smooth. Then $f$ is continuous in the strong Whitney topology.

**Proof.** For this proof, we fix a 1-form $\theta \in [\theta]_{\text{Liu}}$ on $M$ realizing $M$ as a Liouville completion. (See Notation 3.3.1.)

Let $g : S \times M \to M$ be the adjoint map, and for every $s \in S$, let $g_s : M \to M$ be the obvious function. Because $S$ is compact and because $f$ has image in $\text{Aut}^o(M)$, we know that there is some compact subspace $M_0 \subset M$ such that for all $s \in S$, we have $g_s(M_0) \subset M_0$, and that $g_s|_{M\setminus M_0}$ respects $\theta$. Note that there is a uniform bound on the variation in the derivatives of $g_s|_{M\setminus M_0}$ by the compactness of $M_0$ and of $S$. This shows that the smoothness of $g$ guarantees that $f$ is a continuous map in the strong Whitney topology. \hfill $\square$

**Lemma 4.6.2.** Consider the restriction map

$$(f : \Delta^k \to \hat{X}) \mapsto (f|_{\Delta^k} : |\Delta^k| \to X).$$

For $\hat{X} = \text{Aut}^o(M), E\text{Aut}^o(M), B\text{Aut}^o(M)$, the restriction map induces a commutative diagram of simplicial sets

$$
\begin{array}{ccc}
\text{Sing}^{C_{\infty}}(\text{Aut}^o(M)) & \longrightarrow & \text{Sing}^{C_{\infty}}(E\text{Aut}^o(M)) \\
\downarrow & & \downarrow \\
\text{Sing}(\text{Aut}^o(M)) & \longrightarrow & \text{Sing}(E\text{Aut}^o(M))
\end{array}
\begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \\
& & \text{Sing}(B\text{Aut}^o(M)).
\end{array}
$$

**Remark 4.6.3.** On the bottom row, we have topologized $\text{Aut}^o(M)$ with the strong Whitney topology, and we have used the induced topology on $E\text{Aut}^o(M), B\text{Aut}^o(M)$ (these are induced by Milnor’s join constructions for these spaces–see Constructions 4.3.2 and 4.3.3).

**Proof of Lemma 4.6.2.** For brevity, let us set $G = \text{Aut}^o(M)$.

It is obvious that the diagram commutes, so one need only check that the restriction of $f$ to $|\Delta^k|$ is continuous for every choice of $\hat{X}$.

For $\hat{X} = \hat{G}$, this is the content of Lemma 4.6.1.

For $\hat{X} = \hat{E}G$, the definition of the colimit diffeology guarantees that there is some cover $|\Delta^k| = \bigcup_i \hat{U}_i$ and smooth maps $f_i : \hat{U}_i \to |\Delta^{n_i}| \times G^{n_i}$ such that $f$ factors as

$$
\begin{array}{ccc}
U & \overset{f}{\longrightarrow} & \hat{E}G \\
\Bigg\downarrow & & \downarrow \\
\bigcup_i \hat{U}_i & \overset{\bigcup_i f_i}{\longrightarrow} & \bigcup_i |\Delta^{n_i}| \times G^{n_i} \\
\Bigg\uparrow & & \uparrow \\
\bigcup_i U_i & \overset{\bigcup_i f_i}{\longrightarrow} & \bigcup_i |\Delta^{n_i}| \times G^{n_i} \\
\end{array}
$$

where the bottom-right horizontal arrow is the (union of the) obvious inclusion map. (The inclusion map identifies $G^{n_i}$ with the subspace of $\prod_\omega G$ consisting of those $(g_j)$ whose components for $j > n_i$ are all equal to the identity element of $G$.)

Now replacing $U_i$ by (a possibly greater quantity of) smaller disks and taking their closures, we may assume each $U_i$ is compact. Then the same argument as in Lemma 4.6.1 shows that each $f_i$
may be assumed continuous. In particular, since $|\Delta^n| \times G^n \to |\Delta^n| \times \prod G$ is obviously continuous, as is the projection map to $E G$, we conclude that $f$ is continuous. So the middle vertical arrow in the statement of the lemma indeed lands in Sing($E G$).

A similar argument shows that the right vertical arrow also lands in Sing($B G$).

We next prove Lemma 4.6.5; informally, the lemma states that smooth approximation holds when $\text{Aut}^o$ is given the strong Whitney topology.

Remark 4.6.4. In general, smooth approximation for diffeological spaces must be dealt with carefully; for example, it is not usually true that Simp($\hat{X}$) is homotopy equivalent to Sing($X$) when $X$ is given the $D$-topology. (See Remark 3.13 of [CW14].)

However, when $\hat{X}$ is a smooth manifold, then usual smooth approximation shows that the map Sing$C^\infty (\hat{X}) \to$ Sing$(X)$—where Sing is defined by giving $X$ the usual topology as a manifold—is a homotopy equivalence.

Lemma 4.6.5. The map

$$\text{Sing}C^\infty (\hat{\text{Aut}}^o(M)) \to \text{Sing}(\text{Aut}^o(M))$$

 appearing as the left vertical map in Lemma 4.6.2) is a weak homotopy equivalence.

Proof. Suppose we have a continuous map $f : |\Delta^k| \to \text{Aut}^o(M)$ such that $\partial|\Delta^k|$ is sent to the identity. We will prove that $f$ is continuously homotopic to the restriction of a smooth map $|\Delta^k| \to \text{Aut}^o(M)$ through maps constant along $\partial|\Delta^k|$.

Given $t \in |\Delta^k|$, consider the composite diffeomorphism $f_t \circ f_t^{-1}$. By continuity of $f$, we know there is a small neighborhood $U_t \subset |\Delta^k|$ so that for all $t' \in U_t$, the element $f_{t'} \circ f_t^{-1}$ is a Liouville automorphism associated to the Hamiltonian flow of some very small function $H_{t'} : M \to M$. The function $H_{\bullet} : U_t \to C^\infty(M)$ may not be smooth, but by usual smooth approximation, we can find an arbitrarily small continuous homotopy to smooth $G_{\bullet} : U_t \to C^\infty(M)$.

Doing this for every $t \in |\Delta^k|$ and patching together by a partition of unity, we have constructed a homotopy (through Hamiltonians) from $f$ to a smooth map $F : |\Delta^k| \times M \to M$.

Finally, we can shrink a small neighborhood of $\partial|\Delta^k| \subset |\Delta^k|$ (through a smooth homotopy to the identity) to render $F$ smoothly collared near $\partial|\Delta^k|$; in particular, $F$ extends to a smooth map $|\Delta^k| \times M \to M$. Thus, our original map $f$ is homotopic to a map arising from a simplex of Sing$C^\infty (\hat{\text{Aut}}^o(M))$. So the induced map on $\pi_k$ is a surjection.

The same argument as above applied to maps $|\Delta^k| \times |\Delta^1| \to \text{Aut}^o(M)$ shows that the map on $\pi_k$ is also an injection. \qed

4.7 Proof of Theorem 4.4.4

Proof of Theorem 4.4.4. Consider the fiber sequences in Lemma 4.6.2. The top row is a Kan fibration (Remark 4.3.6), and the bottom row is as well, as fiber bundles of spaces are sent to Kan fibrations of simplicial sets under the Sing functor. So one has an induced map between long exact sequences of homotopy groups.

We note that $\hat{B}G$ and $B G$ are always path-connected, so the map on $\pi_0$ is a bijection.

By Lemma 4.6.5, the left vertical arrow of Lemma 4.6.2 is a weak homotopy equivalence. By Theorem 4.3.4(2), so is the middle vertical arrow. By the five lemma, we conclude that the right vertical arrow is also a weak homotopy equivalence. We conclude by applying Lemma 4.5.1. \qed
4.8 The case of Liouville automorphisms with decorations

Now suppose we want to decorate $M$ with some choices—such as a grading and a background class $b \in H^2(M; \mathbb{Z}/2\mathbb{Z})$. As before (Convention 1.1.2), we let $\text{Aut}(M)$ denote the automorphism space of $M$; informally, an element of $\text{Aut}(M)$ is the data of a Liouville automorphism $\phi : M \to M$ together with homotopies exhibiting a compatibility between the decorations on $M$ and the decorations pulled back along $\phi$. See Section 3.6 for details and examples.

We have a map of $A_{\infty}$-spaces

$$\text{Aut}(M) \to \text{Aut}^o(M)$$

and in turn a Kan fibration of simplicial sets

$$\text{Sing}(B\text{Aut}(M)) \to \text{Sing}(B\text{Aut}^o(M)) .$$

Now, we have not (and will not) put a diffeological space structure on $\text{Aut}(M)$; regardless, we will use the following notation:

**Notation 4.8.1.** The simplicial set $\text{Sing}^{C^\infty}(B\text{Aut}(M))$ is defined as the following pullback of simplicial sets:

$$\begin{array}{ccc}
\text{Sing}^{C^\infty}(B\text{Aut}(M)) & \longrightarrow & \text{Sing}(B\text{Aut}(M)) \\
\downarrow & & \downarrow \\
\text{Sing}^{C^\infty}(B\text{Aut}^o(M)) & \longrightarrow & \text{Sing}(B\text{Aut}^o(M)).
\end{array} \tag{4.2}$$

Thus a simplex of $\text{Sing}^{C^\infty}(B\text{Aut}(M))$ is given by a pair $(j, \eta)$ where $j : |\Delta^n| \to \text{Aut}^o(M)$ is a smooth map, and $\eta$ is a continuous map from $|\Delta^p| \subset |\Delta^n|$ to $B\text{Aut}(M)$.

**Remark 4.8.2.** We note that the bottom horizontal arrow in (4.2) is a weak homotopy equivalence by smooth approximation (Lemma 4.6.5). Because weak homotopy equivalences are preserved under pullback, we conclude that the top horizontal arrow $\text{Sing}^{C^\infty}(B\text{Aut}(M)) \to \text{Sing}(B\text{Aut}(M))$ in (4.2) is also a weak homotopy equivalence.

**Notation 4.8.3 (Simp$(B\text{Aut}(M))$).** We let

$$\text{Simp}(B\text{Aut}(M))$$

denote the category of simplices of the simplicial set $\text{Sing}^{C^\infty}(B\text{Aut}(M))$. So for example, we have that

$$N(\text{Simp}(B\text{Aut}(M))) = \text{subdiv}(\text{Sing}^{C^\infty}(B\text{Aut}(M))).$$

**Remark 4.8.4.** We thus have a commutative diagram of simplicial sets

$$\begin{array}{ccc}
N(\text{Simp}(B\text{Aut}(M))) & \xrightarrow{\text{max}} & \text{Sing}^{C^\infty}(B\text{Aut}(M)) \\
\downarrow & & \downarrow \\
N(\text{Simp}(B\text{Aut}^o(M))) & \xrightarrow{\text{max}} & \text{Sing}^{C^\infty}(B\text{Aut}^o(M))
\end{array}$$

where the horizontal arrows are weak homotopy equivalences (Proposition 4.5.3).

We have the following generalization of Theorem 4.4.4:
Theorem 4.8.5. The map

\[ N(\text{Simp}(B \text{ Aut}(M))) \to \text{Sing}(B \text{ Aut}(M)) \]

exhibits \( \text{Sing}(B \text{ Aut}(M)) \) as the Kan completion of \( N(\text{Simp}(B \text{ Aut}(M))) \). In particular, \( \text{Sing}(B \text{ Aut}(M)) \) is the localization of the \( \text{Simp}(B \text{ Aut}(M)) \) along all its morphism.

Proof. The morphism in question factors as

\[
\begin{align*}
N(\text{Simp}(B \text{ Aut}(M))) &\xrightarrow{\text{max}} |\text{Simp}(B \text{ Aut}(M))| \\
&\xrightarrow{\max} |\text{Sing}^{\infty}(B \text{ Aut}(M))| \\
&\to \text{Sing}(B \text{ Aut}(M))
\end{align*}
\]

where the first arrow is the localization (Kan completion), the max morphism is known to be a weak equivalence upon passage to Kan completion (Proposition 4.5.3), and the last arrow is a homotopy equivalence by Remark 4.8.2.

\[\square\]
5 Simplices, and families of disks

**Notation 5.0.1** (Standard simplices). Fix an integer \(d \geq 0\). We let \(\Delta^d\) denote the standard topological \(d\)-dimensional simplex, given by the subset of those \((t_0, \ldots, t_d) \in \mathbb{R}^{d+1}\) satisfying \(t_i \geq 0\) and \(\sum t_i = 1\). More generally, given any linear order \(A\), we let \(\Delta^A\) denote the subset of \(\mathbb{R}^A\) given by those \((t_a)_{a \in A}\) satisfying \(t_a \geq 0\) and \(\sum_{a \in A} t_a = 1\).

We will sometimes refer to \(\Delta^A\) as the geometric realization of \(A\).

**Definition 5.0.2** (ith vertex). Let \(\Delta^d\) be a standard simplex. Given \(i \in \{0, \ldots, d\}\), the ith vertex of \(\Delta^d\) is the unique point whose ith coordinate is equal to 1.

Likewise, if \(\Delta^n_e\) is the extended simplex, the ith vertex is the same point (with ith coordinate 1 and other coordinate 0).

**Remark 5.0.3** (Standard and extended simplices). We will often be passing between the extended simplices \(\Delta^n_e\) (Definition 4.2.1) and the standard simplices \(\Delta^n\) (Notation 5.0.1).

Of course, because the natural inclusion \(\Delta^n \to \Delta^n_e\) is a smooth map (from a manifold with corners), it will make sense to pullback smooth objects living over an extended simplex to a standard simplex.

Finally, we will do our best to use the letter \(h\) to denote maps from a standard simplex, and the letter \(j\) to denote maps from the extended simplex:

\[ h : \Delta^n \to B, \quad j : \Delta^n_e \to B. \]

We review Savelyev’s observation from [Sav13] that two fundamental objects of our fields—(i) universal families of holomorphic disks with \(k+1\) boundary punctures, and (ii) standard simplices—have compatible operadic structures.

Let us explain how we use this observation. Our first goal (accomplished in Section 6) is to associate, for every smooth map \(j : \Delta^n_e \to B\) and every Liouville bundle \(E \to B\), an non-wrapped Fukaya category \(\mathcal{O}_j\). This means that given \(j\) and a Liouville bundle, we must associate a \(d\)-ary \(A_\infty\) operation for every \((d+1)\)-tuple of objects.

In a way we make explicit later, this is done by taking a map

\[ \Delta^d \to \Delta^n \]

induced by the \((d+1)\)-tuple of objects, and noticing that the \(d\)-simplex \(\Delta^d\) itself can be (modulo a neighborhood of the boundary) identified with the total space of the universal family

\[ S_{d+1} \to \mathbb{R}_{d+1} \]

of \((d+1)\)-ary holomorphic disks (see Notation 5.1.1 and Remark 5.3.7). At the very end of the present section, we will choose such an identification \(\Delta^d \approx S_{d+1}\) once and for all. (Here, we use \(\approx\) rather than \(\cong\) to indicate that this is an identification modulo boundary.)

Roughly speaking, we will then define the \(d\)-ary operation \(m_d\) to be given by counts of holomorphic sections \(u\) (with Lagrangian boundary conditions) from fibers of \(S_{d+1}\) to the bundle obtained by pulling back \(E\) along the composite \(S_{d+1} \approx \Delta^d \to \Delta^n \to \Delta^n_e \to B\):

\[ E|_{S_{d+1}} \xrightarrow{\text{pullback}} E \]

\[ S_{d+1} \to \Delta^d \to \Delta^n \to \Delta^n_e \to B. \]
(Here, \(S_r \subset S_{d+1}\) is a holomorphic disk with \(d+1\) boundary marked points; it is the fiber above \(r \in \mathcal{R}_{d+1}\).

The reader may now appreciate that for such counts to satisfy the \(A_\infty\)-relations, one must impose some compatibilities on the structures chosen to define these operations—and especially the identifications \(S_{d+1} \approx |\Delta|^d\)—as one approaches the boundary moduli of nodal disks. To articulate these compatibilities, we will also be forced to choose maps

\[
\nu_\beta : \mathbb{S}^d_{d+1} \to |\Delta^n|
\]

for each simplicial map \(\beta : |\Delta^d| \to |\Delta^n|\). (See Notation 5.1.1 for the notation \(\mathbb{S}^0_{d+1}\).) Moreover, we would later like these non-wrapped Fukaya categories to be functorial in the choice of \(j\), meaning that if we have a simplicial inclusion \(|\Delta^n| \subset |\Delta'_n|\), the composite map \(j' : |\Delta'_n| \to |\Delta_n| \xrightarrow{\nu_j} B\), induces a functor \(\mathcal{O}_{j'} \to \mathcal{O}_j\) of non-wrapped Fukaya categories. This imposes further compatibilities on our choices.

The main purpose of this section is to define what these compatibilities are in terms of the maps \(\nu_\beta\), which for the special case of \(\beta = \text{id}\) recovers the identifications \(S_{d+1} \approx |\Delta^d|\). This is given in Definition 5.3.4. We record the existence of such choices in Proposition 5.3.5.

### 5.1 Universal families of curves and gluing along strip-like ends

**Notation 5.1.1 \((\mathcal{R}, \mathbb{S}, \mathbb{S}^0)\).** Let \(\mathcal{R}_{d+1}\) denote the compactified moduli space of holomorphic disks with \(d+1\) boundary punctures; we demand that one of these boundary punctures is distinguished, and we refer to it as the outgoing marked point, or the 0th marked point. Using the boundary orientation of a holomorphic disk, any other marked point may uniquely be labeled as the \(i\)th marked point for some \(1 \leq i \leq d\).

We let \(\mathbb{S}_{d+1} \to \mathcal{R}_{d+1}\) denote the universal family of (possibly nodal) disks living over \(\mathcal{R}_{d+1}\). Note that a fiber is never compact; every disk—nodal or not—has boundary punctures.

Finally, we let \(\mathbb{S}^0_{d+1} \subset \mathbb{S}_{d+1}\) denote the open subspace obtained by removing the nodal points of each fiber.

For any \(r \in \mathcal{R}_{d+1}\), we let \(S_r \subset \mathbb{S}^0_{d+1}\) denote the fiber above \(r\).

**Example 5.1.2.** If \(d = 2\), then \(\mathcal{R}_{2+1}\) is homeomorphic to a single point. \(\mathbb{S}_{2+1}\) is homeomorphic to a closed disk with three boundary points missing, as is \(\mathbb{S}^0_{2+1}\).

If \(d = 3\), then \(\mathcal{R}_{3+1}\) may be identified a closed unit interval \([0, 1]\). The universal family \(\mathbb{S}_{3+1} \to [0, 1]\) has the property that the fiber over any element of the open interval \((0, 1)\) is homeomorphic to a closed disk minus four boundary points. Over either endpoint—0 or 1—the fiber is a wedge sum of two disks with two boundary points missing on each disk; in each fiber, the wedge point is the nodal point. Finally, the space \(\mathbb{S}^0_{3+1}\) is obtained by removing exactly two points (the nodal points—one nodal point from each boundary element of \([0, 1]\)) from \(\mathbb{S}_{3+1}\).

More generally, \(\mathbb{S}^0_{d+1}\) is obtained from \(\mathbb{S}_{d+1}\) by removing \(i\) wedge points (i.e., \(i\) nodal points) from every fiber living over a codimension \(i\) stratum of \(\mathcal{R}_{d+1}\).

**Choice 5.1.3** (Strip-like ends \(\epsilon\)). We assume we have chosen strip-like ends near the nodes and boundary marked points of each fiber of \(\mathbb{S}^0_{d+1} \to \mathcal{R}_{d+1}\). See Sections (8d), (9a), and (9c) of [Sei08].

We denote these strip like ends \(\epsilon\) when necessary.

We assume we have also chosen diffeomorphisms \(|\Delta^1| \cong [0, 1]\) once and for all, so that the strip like ends are biholomorphic embeddings

\[
\epsilon : [0, \infty) \times |\Delta^1| \to S_r \quad \text{or} \quad \epsilon : (-\infty, 0] \times |\Delta^1| \to S_r.
\]

We denote by \(\epsilon_i\) the strip-like end at the \(i\)th puncture.
Notation 5.1.4 ($\circ_i$). Recall that every codimension one stratum of $\mathcal{K}_{d+1}$ can be written as a direct product $\mathcal{K}_{d+1} \times \mathcal{K}_{d_1+1}$; indeed, for a given $d_1$, and for any $1 \leq i \leq d_1$, there is an $i$th wedging map

$$\circ_i : \mathcal{K}_{d+1} \times \mathcal{K}_{d_1+1} \to \mathcal{K}_{d_1+1}, \quad d_2 + d_1 - 1 = d$$

(5.1)

which glues the 0th boundary vertex of a disk with $d_2 + 1$ marked points to the $i$th boundary vertex of a disk with $d_1 + 1$ marked points. For $r_1 \in \mathcal{K}_{d_1+1}$ and $r_2 \in \mathcal{K}_{d_2+1}$, we let $r_2 \circ_i r_1$ denote the image.

We will also write $S_{r_2} \circ_i S_{r_1}$ for the corresponding nodal disk.

Finally, because of our choice of strip-like ends, we can parametrize the corners of $\mathcal{K}_{d+1}$; for instance, in codimension one, the maps from (5.1) extend to maps

$$\mathcal{K}_{d+1} \times \mathcal{K}_{d_1+1} \times [0, \epsilon) \to \mathcal{K}_{d_1+1}, \quad d_2 + d_1 - 1 = d$$

(5.2)

(the dependence on $1 \leq i \leq d_1$ is suppressed in the above notation). See also Sections (9e) and (9f) of [Sei08].

Our main interest is in a lift of (5.2):

Notation 5.1.5 ($\sharp_i$ and $\sharp_i, r$). Let $\mathcal{K}_{d_2,d_1} \to \mathcal{K}_{d+1} \times \mathcal{K}_{d_1+1} \times [0, \epsilon)$ denote the map obtained by pulling back $\mathcal{K}_{d_1+1}$ and $\mathcal{K}_{d_2+1}$ along the projections to $\mathcal{K}_{d_1+1}$ and $\mathcal{K}_{d_2+1}$, then taking the coproduct of these two pullbacks. Concretely, a fiber of $\mathcal{K}_{d_2,d_1}$ over $(r_2, r_1, \tau)$ is the disjoint union $S_{r_2} \coprod S_{r_1}$. Then the gluing operation induced by the strip-like ends defines a map

$$\sharp_i : \mathcal{K}_{d_2,d_1} \times [0, \epsilon) \to \mathcal{K}_{d_1+1}, \quad d_2 + d_1 - 1 = d$$

(5.3)

where the restriction of $\sharp_i$ to time $\tau \in [0, \epsilon)$ will be denoted by $\sharp_i, \tau$.

Remark 5.1.6. Note the font $\mathcal{K}_{d_2,d_1}$ rather than $\mathcal{K}_{d_2,d_1}$; we use the former because Savelyev uses the latter font to indicate a different entity in [Sav13].

Notation 5.1.7 ($\sharp_i, r$). Let us describe $\sharp_i, \tau$ for the sake of establishing further notation. Fix $\tau \in [0, \epsilon)$ and elements $r_1 \in \mathcal{K}_{d_1+1}, r_2 \in \mathcal{K}_{d_2+1}$. Having fixed our strip-like ends long ago, the $i$th gluing operation identifies two open subsets of $S_{r_1}$ and $S_{r_2}$ to obtain a new disk $S_r$. The strip-like ends endow $S_r$ with a thick-thin decomposition, where we can holomorphically identify the “thin” region of $S_r$ with $(-\tau, \tau) \times |\Delta|$, and these thin regions are precisely the regions where the gluing operation has non-singleton fibers (i.e., this is the region over which $S_{r_1}$ and $S_{r_2}$ are glued). The gluing maps

$$S_{r_2} \coprod S_{r_1} \times \{\tau\} \to S_r$$

(where $r$ depends on $\tau$) define the maps $\sharp_i, \tau$. When $\tau = 0$, we have a map

$$\sharp_i, 0 : S_{r_2} \coprod S_{r_1} \to S_{r_2} \circ_i S_{r_1}.$$  

(5.4)

5.2 Simplices and inserting posets

Now let us consider the simplicial analogue of the previous section’s constructions.

Notation 5.2.1. Fix an integer $d \geq 0$. We let $[d]$ denote the linear poset given by

$$[d] = \{0 < 1 < \ldots < d\}.$$  

It is the unique linear order with $d + 1$ elements (up to unique choice of order-preserving isomorphism).
Notation 5.2.2 \((A_2 \circ_i A_1)\). Let \(A_2\) and \(A_1\) be finite, non-empty, linearly ordered posets. We let \(d_1 = \#A_1 - 1\). Then for any \(1 \leq i \leq d_1\), we can construct a new linear poset

\[A_2 \circ_i A_1\]

by gluing \(A_2\) into \(A_1\) as follows: Identify \(\min A_2\) with the \((i-1)\)st element of \(A_1\), and identify \(\max A_2\) with the \(i\)th element of \(A_1\). We see that \(A_2 \circ_i A_1 \cong [d_2 + d_1 - 1]\) as posets.

We have a natural gluing map of posets

\[\sharp_i : A_2 \coprod A_1 \to A_2 \circ_i A_1.\] (5.5)

By taking the geometric realization, we obtain a continuous map of topological spaces

\[|\Delta^{A_2}| \coprod |\Delta^{A_1}| \to |\Delta^{A_2 \circ_i A_1}|\]

which, by (canonically) identifying \(A_2 \cong [d_2]\) and \(A_1 \cong [d_1]\) as linear posets, is equivalent to a continuous map

\[\sharp_i : |\Delta^{d_2}| \coprod |\Delta^{d_1}| \to |\Delta^{d_2 + d_1 - 1}|.\] (5.6)

Concretely, \(\sharp_i\) simplicially includes \(|\Delta^{d_2}|\) and \(|\Delta^{d_1}|\) as subsimplices of \(|\Delta^{d_2 + d_1 - 1}|\), and these inclusions overlap along the edge between the \(i\)th and \((d_2 + i)\)th vertices of \(|\Delta^{d_2 + d_1 - 1}|\).

Remark 5.2.3. (5.6) should be compared with the map (5.4) from Section 5.1.

5.3 Operadic compatibility

Now we wish to relate the two operadic structures (of universal families of curves, and of simplices).

Notation 5.3.1 (The indices \(n\) and \(d\)). Fix a Liouville bundle \(P \to X\). We eventually want to define an \(A_\infty\)-category \(O_j\) associated to any smooth map \(j : |\Delta^n| \to X\); in particular, we must define the \(A_\infty\)-operations \(\mu^d\) for the \(A_\infty\)-category \(O_j\). In this section, the integers \(n\) and \(d\) will be used precisely for these purposes.

Suppose we are given a simplicial map \(\beta : |\Delta^d| \to |\Delta^n|\). (This is induced by a function \([d] \to [n]\), but this function need not be order-preserving.)\(^{17}\) We seek smooth maps

\[\nu_\beta : \bar{\Delta}^n_{d+1} \to |\Delta^n|\]

satisfying the following properties. (See the first row of Figure 5.3.2.)

(NS1) Fix any \(0 \leq k \leq d\). For any \(r \in \bar{\Delta}^d_{d+1}\), consider the fiber \(S_r \subset \bar{\Delta}^n_{d+1}\). Then for the edge from \(k - 1\) to \(k\) in \(|\Delta^d|\), the diagram

\[
\begin{array}{ccc}
[0, \infty) \times |\Delta^1| & \xrightarrow{\epsilon_k} & S_r \subset \bar{\Delta}^n_{d+1} \\
\downarrow \beta_{k-1,k} & & \downarrow \nu_\beta \\
|\Delta^1| & \xrightarrow{\nu_\beta} & |\Delta^n| \\
\end{array}
\]

\(^{17}\) Savelyev uses the notation \(u(m_1, \ldots, m_d, n)\), but the data of \(m_1, \ldots, m_d, n\) is equivalent to the data of a single simplicial map \(\beta : |\Delta^d| \to |\Delta^n|\), or equivalently, a map of sets \(\beta : [d] \to [n]\).
commutes. (Here, \( \beta_{k-1,k} \) is the simplicial inclusion of the edge from the \((k-1)\)st vertex to the \(k\)th.) In plain English, this means that \( \nu_\beta \) is compatible with the strip-like end parametrization by \(|\Delta^1|\) near the \(k\)th puncture of \( S_r \).

Note we are using Choice 5.1.3; also, when \( k = 0 \), the domain of \( \epsilon_k \) should be \((-\infty, 0] \times |\Delta^1|\) as opposed to \([0, \infty) \times |\Delta^1|\).

**(NS2)** Now consider the boundary of \( S_r \), and remove the images of the strip-like ends from this boundary. This results in \( d + 1 \) disconnected open intervals, and we enumerate them so that the \((k-1)\)st interval is contained in the boundary arc of \( S_r \) beginning at the \((k-1)\)st marked point and ending at the \(k\)th marked point.

We demand that \( \nu_\beta \) sends all of the \(k\)th interval to the vertex \( \beta(k) \).

At this point, we see that as \( r \) approaches a boundary stratum of \( \mathfrak{R}_{d+1} \) (i.e., as disks degenerate), we would like the restrictions \( \nu_\beta|_S \) to behave in a way compatible with the boundary faces of the \(n\)-simplex. See Figure 5.3.2. We make this compatibility (which Savelyev refers to as “natural” in [Sav13]) precise:

**Notation 5.3.3** \((\star_i)\). Fix \( d_2, d_1 \geq 2 \) such that \( d = d_2 + d_1 - 1 \). Choose \( 1 \leq i \leq d_1 \).

Then our fixed map \( \beta : |\Delta^d| \to |\Delta^n| \) induces maps \( \beta_1 : |\Delta^{d_1}| \to |\Delta^n| \) and \( \beta_2 : |\Delta^{d_2}| \to |\Delta^n| \) so that the diagram

\[
\begin{array}{ccc}
|\Delta^{d_2}| \bigcap |\Delta^{d_1}| & \xrightarrow{z_i} & |\Delta^d| \\
\beta_2 \bigcap \beta_1 & \xrightarrow{\beta} & |\Delta^n| \\
\end{array}
\]

commutes. (Here, \( z_i \) is the map from (5.6).)

On the other hand, if we are given maps \( \nu_{\beta_1} : \mathfrak{S}_{d_1+1}^\circ \to |\Delta^n| \) and \( \nu_{\beta_2} : \mathfrak{S}_{d_2+1}^\circ \to |\Delta^n| \), the conditions (NS1) and (NS2) guarantee the existence of a (unique) map making the following diagram commute:

\[
\begin{array}{ccc}
\mathfrak{S}_{d_1+1}^\circ \bigcap \mathfrak{S}_{d_1+1}^\circ & \xrightarrow{\nu_{\beta_1}, \tau=0} & \mathfrak{S}_{d_1+1}^\circ \bigcap \mathfrak{S}_{d_1+1}^\circ \\
\nu_{\beta_2} \bigcap \nu_{\beta_1} & \xrightarrow{\exists} & |\Delta^n| \\
\end{array}
\]

Here, the notation \( \mathfrak{S}_{d+1}^\circ \bigcap \mathfrak{S}_{d+1}^\circ \) denotes the family \( \mathfrak{S}_{d+1}^\circ \) restricted to the image of the map \( \circ_i \) from (5.1).

Extending the gluing parameter \( \tau \) from 0 to an element of \([0, \epsilon]\), there is a neighborhood \( U_{d_2,d_1,i} \supset \mathfrak{S}_{d_2+1}^\circ \circ_i \mathfrak{S}_{d_1+1} \) such that there is a unique extension \( \nu_{\beta_2} \circ_i \nu_{\beta_1} \) making the diagram below commute:

\[
\begin{array}{ccc}
\mathfrak{S}_{d_2+1}^\circ \bigcap \mathfrak{S}_{d_1+1}^\circ \times [0, \epsilon] & \xrightarrow{\tilde{z}_i} & \mathfrak{S}_{d_2+1}^\circ \bigcap U_{d_2,d_1,i} \\
\nu_{\beta_2} \bigcap \nu_{\beta_1} & \xrightarrow{\nu_{\beta_2} \circ_i \nu_{\beta_1}} & |\Delta^n| \\
\end{array}
\]

Explicitly, on the thin strips \(|\Delta^1| \times [-\tau, \tau]\), we declare \( \nu_{\beta_2} \circ_i \nu_{\beta_1} \) to equal the composition of the projection to \(|\Delta^1|\) with the simplicial inclusion of the edge from the \((i-1)\)st vertex to the \(i\)th vertex.
Figure 5.3.2. An image of $\nu_\beta$, restricted to $S_r$ for various $r \in \mathcal{R}_{3+1}$, when $\beta$ is the identity $\beta : |\Delta^3| \to |\Delta^3|$. The blue arcs on the left-hand disk are images of the 1-simplices $\{t\} \times |\Delta^1| \subset (0, \infty) \times |\Delta^1|$ under strip-like parametrizations; they are sent to the blue edges indicated on the right-hand 3-simplices. The red thick edges on the disks are the “open intervals” referred to in the main text (in practice, these thick edges are labeled by Lagrangians $L_i$); these red edges are collapsed to the vertices labeled in red on the 3-simplices. (The edge labeled by $L_i$ is sent to the $i$th vertex.) In yellow is a drawing of the image of $\nu_\beta$ in the 3-simplex. In the bottom most image, the two components of a nodal disk are labeled by orange and yellow, and these are sent to two faces of the 3-simplex as indicated. Note the new green strip-like ends, and the newly highlighted green edge of the 3-simplex.
Hence it is natural to demand the following:

(NS3) For all $d_1, d_2, i$ as above, we demand that $\nu_{\beta}$ agrees with $\nu_{\beta_2} \star_i \nu_{\beta_1}$ on some neighborhood of $\mathbb{R}_{d_2+1} \circ_i \mathbb{R}_{d_1+1}$. That is,

$$\nu_{\beta} = \nu_{\beta_2} \star_i \nu_{\beta_1} \quad \text{on} \quad \mathbb{S}_{d+1}^\circ |U_{d_2,d_1,i}|$$

(possibly after replacing $U_{d_2,d_1,i} \supset \mathbb{R}_{d_2+1} \circ_i \mathbb{R}_{d_1+1}$ with some other neighborhood containing $\mathbb{R}_{d_2+1} \circ_i \mathbb{R}_{d_1+1}$, if necessary).

Finally, while we have fixed $n$ up until now, we demand that $\nu_{\beta}$ is functorial as the codomain of $\beta$ varies:

(NS4) Let $\alpha : [n] \rightarrow [n']$ be a map of posets; by abuse of notation, we also denote the induced simplicial map $\alpha : |\Delta^n| \rightarrow |\Delta^{n'}|$. Then for any $\beta : |\Delta^d| \rightarrow |\Delta^n|$, we demand

$$\alpha \circ \nu_{\beta} = \nu_{\alpha \circ \beta}.$$

**Definition 5.3.4** (Natural system). For every $n, d \geq 0$ and every simplicial map $\beta : |\Delta^d| \rightarrow |\Delta^n|$, choose a smooth map

$$\nu_{\beta} : \mathbb{S}_{d+1}^\circ \rightarrow |\Delta^n|.$$ 

The collection $\{\nu_{\beta}\}$ is called a natural system if (NS1), (NS2), (NS3), and (NS4) above are satisfied.

A standard inductive argument shows the following:

**Proposition 5.3.5** (Proposition 3.4 of [Sav13]). Natural systems exist.

**Choice 5.3.6.** We will choose a natural system $\{\nu_{\beta}\}$ once and for all. (Note this is independent of any symplectic geometry or of any choice of Liouville bundle $E \rightarrow B$.)

**Remark 5.3.7.** One can prove that, given a natural system, each of the maps $\nu_{\beta} : \mathbb{S}_{n+1}^\circ \rightarrow |\Delta^n|$ (when $\beta = \text{id}$) is a degree one map on the interior; one can roughly think of $\nu_{\beta}$, then, as “homeomorphisms on the interior.” The naturality of the system says that these topological equivalences can be chosen in such a way that the gluing operations of disks are compatible with the insertion operation of simplices.

### 5.4 Collars on boundaries of simplices

We conclude this section with a final choice, made once and for all for every standard simplex $|\Delta^d| \subset \mathbb{R}^{d+1}$.

**Choice 5.4.1** (Collars of simplices). For every closed, codimension one face $F \subset |\Delta^d|$, we choose a small open neighborhood $U_F \subset |\Delta^d|$ together with a smooth retraction $\pi_F : U_F \rightarrow F$, which one thinks of as a projection map.

We choose the data of $(U_F, \pi_F)$ such that the following holds:

1. (The neighborhoods are mutually small.) If $A \subset |\Delta^d|$ is a closed subsimplex, let

$$U_A := \bigcap_{A \subset F} U_F.$$ 

If $A, A' \subset |\Delta^d|$ are two closed subsimplices such that $A \cap A' = \emptyset$, we demand that

$$U_A \cap U_{A'} = \emptyset. \quad (5.7)$$

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2. (The neighborhood do not “increase” with dimension.) Let $F$ and $F'$ be two codimension one faces with intersection $G = F \cap F'$. Then

$$U_F \cap F' \subset U_G.$$  

(5.8)

**Remark 5.4.2.** Let us motivate the conditions. First, the collars are chosen so that we may trivialize certain data over the collars. (See for example (Θ3) of Choice 6.2.4.)

The intersection property (5.7) guarantees that these local trivializations do not enforce a global trivialization.

The size constraint (5.8) enables choices that are made inductively by dimension. For example, if we have already chosen data on lower-dimensional face $F'$ for which a trivialization does not extend beyond $U_G$, $U_F$ must be sufficiently small for us to be able to trivialize on $U_F$.

**Remark 5.4.3.** Note that these collars are independent of our choice of natural systems. The collar choices are likewise independent of any symplectic geometry.
6 The non-wrapped Fukaya categories

Our definition of all relevant Fukaya categories will proceed in two steps: First, we will define a “non-wrapped” $A_\infty$ category $O$. Second, we will localize $O$ with respect to continuation maps of non-negative wrappings. (The intuition is that in any wrapped Fukaya category, a non-negative finite wrapping should induce an equivalence.) We denote the resulting $A_\infty$-category by $W$. This follows an idea originally due to Abouzaid and Seidel, which we understand is contained in forthcoming work; see also [GPS17], where we learned of the idea, and whose notation we largely follow.

Notation 6.0.1 ($E \to B$). Throughout this section, we fix a Liouville bundle $E \to B$ (Definition 3.7.1). While we will apply our constructions to the universal example of $B = \hat{\text{Aut}}^0(M)$ (Example 3.7.6), we state our constructions with the generality of an arbitrary base $B$.

Remark 6.0.2. We remind the reader that we have chosen a natural system once and for all (Choice 5.3.6), along with collaring data on simplices (Choice 5.4.1).

6.1 Choice of objects

As usual we have fixed a Liouville bundle $E \to B$.

Choice 6.1.1 ($\mathcal{L}_b$ and cofinal wrapping sequences.). For every point $b \in B$, we choose a countable collection $\mathcal{L}_b$ of eventually conical branes in the fiber $E_b$ such that the following holds: For every eventually conical brane $L' \subset E_b$, there exists an element $L \in \mathcal{L}_b$ such that $L$ admits a non-negative wrapping to $L'$, or $L'$ admits a non-negative wrapping to $L$.

Then, for every $L \in \mathcal{L}_b$, we choose a cofinal wrapping sequence $L = L^{(0)} \to L^{(1)} \to \ldots$ Finally, because this totality of choices is a countable collection, we may assume that if $L^{(w)}$ and $L'^{(w')}$ are in the same fiber, then they are either transverse, or $L = L'$ and $w = w'$.

Notation 6.1.2 (The wrapping index $w$). In the cofinal wrapping sequence of Choice 6.1.1, we will often denote the superscript index by $(w)$. The $w$ stands for “wrapping index.”

Warning 6.1.3. We warn the reader that $w$ does not correspond in any precise way to any “time” index for which we wrap; it is simply a bookkeeping device.

6.2 Choices of Floer data

Remark 6.2.1. Because we have already chosen a favorite collection of branes (Choice 6.1.1), any time we discuss a Lagrangian brane here, we assume that it equals $L^{(w)}$ for some $w \in \mathbb{Z}_{\geq 0}$ and some $L \in \mathcal{L}_b$ (for some $b \in B$). In particular, if two Lagrangians are in the same fiber $E_b$, they are either equal or transverse. This is not strictly necessary, but it will make certain things easier. See also Warning 6.2.6.

Notation 6.2.2 ($\vec{L}$). Recall we have fixed a Liouville bundle $E \to B$ (Notation 6.0.1). Fix $d \geq 0$ and a smooth map $h : |\Delta^d| \to B$. We denote by $\vec{L} = (L_0, \ldots, L_d)$ an ordered $(d + 1)$-tuple of branes with $L_i \subset h^*E$ contained above the $i$th vertex (Definition 5.0.2) of $|\Delta^d|$.

---

18 Definition 3.11.4.
19 Definition 3.11.5
Remark 6.2.3. In later notation, \( h \) will play the role of the composite \( \beta \circ j \). (See Definition 6.3.8 and Remark 6.3.9.)

Consider an ordered \((d+1)\)-tuple \( \tilde{L} \) (Notation 6.2.2).

Because \(|\Delta^d|\) is a smooth manifold with corners, one can construct a global 1-form \( \Theta_{\tilde{L}} \in \Omega^+(h^*E; \mathbb{R}) \) realizing a Liouville structure on each fiber of \( h^*E \). (Remark 3.7.4.) Using the natural system maps \( \nu_{id}: \mathbb{R}^{d+1} \to |\Delta^d| \), we may also choose almost-complex structures \( J \) on \( h^*E \) suitable for counting sections. (Definition 3.9.6.)

We now specify the choices we make to guarantee that the moduli spaces of holomorphic sections \( \mathcal{S}_r \subset \mathbb{R}^{d+1} \to |\Delta^d| \to h^*E \) are well-behaved moduli spaces.

Choice 6.2.4 (Liouville forms and almost complex structures suitable for counting sections). We begin with \( d = 0 \). Note that \( \Theta_{\tilde{L}} \) is simply a choice of Liouville structure on the fiber of \( E \to B \) determined by \( h \), and likewise for \( J_{\tilde{L}} \). Given this choice of \( \Theta_{\tilde{L}} \), we may assume that each brane \( L_i \) admits a primitive \( f_i: L_i \to \mathbb{R} \) such that \( df_i = \Theta_{\tilde{L}}|_{L_i} \) and such that \( f_i \) has compact support for all \( i = 0, \ldots, d \). (In other words, we require that \( f^*(\Theta) = 0 \) in \( H^1(L; \mathbb{R}_c) \). (Any brane admits a deformation so that this holds.)

We proceed inductively on \( d \). Assume that for all \( d' < d \), for all \( h' : |\Delta^{d'}| \to B \), and for all \((d'+1)\)-tuples \( \tilde{L}' = (L_0', \ldots, L_{d'}) \), we have chosen \( (\Theta_{\tilde{L}'}, J_{\tilde{L}'}) \) on \( (h')^*E \).

Fix an ordered \((d+1)\)-tuple \( \tilde{L} \). We choose \( (\Theta_{\tilde{L}}, J_{\tilde{L}}) \) on \( h^*E \) subject to the following conditions:

\( (\Theta 1) \) (Constancy implies constancy.) If \( h \) is constant, then so is \( (\Theta_{\tilde{L}}, J_{\tilde{L}}) \). More concretely, if \( h : |\Delta^d| \to B \) is constant, then for a constant map \( p : |\Delta^d| \to v \) to some (hence every) point of \(|\Delta^d|\), we have that \( \Theta_{\tilde{L}} = p^*(\Theta_{\tilde{L}}|_{h^*E}) \), and \( J_{\tilde{L}} = p^*(J_{\tilde{L}}|_{(h^*E)v}) \).

\( (\Theta 2) \) (Inductive step.) If \( \tilde{L}' \subset \tilde{L} \) is an order-preserving inclusion, consider the induced map \(|\Delta^{d'}| \to |\Delta^d| \). Then the data \( (\Theta_{\tilde{L}'}, J_{\tilde{L}'}) \) is equal to the pullback of \( (\Theta_{\tilde{L}}, J_{\tilde{L}}) \) along this induced map.

\( (\Theta 3) \) (Smooth collaring.) Recall the collaring choices from Choice 5.4.1. Suppose that \( F \subset |\Delta^d| \) is a codimension one face. \( F \) in particular determines an ordered \( d \)-tuple \( \tilde{L}' \subset \tilde{L} \). We demand

\[
\pi_F^*\Theta_{\tilde{L}'} = \Theta_{\tilde{L}}|_{U_F}, \quad \pi_F^*J_{\tilde{L}'} = (J_{\tilde{L}})|_{U_F}
\]

where \( \pi_F \) and \( U_F \) are as in Choice 5.4.1.

\( (\Theta 4) \) (Transversality.) For any \( 0 \leq i < j \leq d \), let \( \Pi_{ij} \) be the parallel transport along the simplicial edge from the \( i \)th vertex of \(|\Delta^d|\) to the \( j \)th. (See Definition 3.8.2.) We demand that \( \Pi_{ij}(L_i) \) and \( L_j \) are transverse.

\( (\Theta 5) \) (Regularity.) Further, we demand that the associated linearized del-bar operators are regular, so that the holomorphic disk moduli spaces (see Definition 6.3.8) are smooth manifolds.

\( (\Theta 6) \) (Coherent barriers) Finally, we demand that there exists some neighborhood of \( \partial h^*E \) such that, with respect to some global function \( \pi : \text{Nbhd}(\partial h^*E) \to \mathbb{C}_{\text{Re} \geq 0} \) as in Remark 3.10.2, \( \pi \) is \((J_{\tilde{L}})|_{VT_E}\)-holomorphic.

Remark 6.2.5. We may now further motivate the collaring choices made for simplices in Choice 5.4.1. If one chooses the above \( \Theta_{\tilde{L}} \) without collaring conditions, there is no guarantee that the \( \Theta_{\tilde{L}} \) glue smoothly along faces of a simplex.

\(^{20}\)When \( d = 0 \), a choice of \( J_{\tilde{L}} \) suitable for counting sections is simply a choice of almost-complex structure \( J \) on \( E_b = E \) for which \( J \) is compatible with the symplectic form and eventually cylindrical (Definition 3.9.1).

\(^{21}\)See also Warning 6.2.6 regarding compatibility with \( (\Theta 1) \).
**Warning 6.2.6.** The reader may be irked by an apparent incompatibility between (Θ1) and (Θ4). As stated, it is impossible to satisfy both conditions unless the branes $L_0, \ldots, L_d$ are a priori assumed transversal. This is the reason for Choice 6.1.1; see also Remark 6.2.1.

This a priori transversal assumption is not strictly necessary, as one could make choices ignoring (Θ1) altogether; but we later want an easy verification that a Fukaya category of a fixed fiber of $E$ is equivalent to the Fukaya category defined in [GPS17] (Example 9.1.5).

**Remark 6.2.7.** Recall we have fixed a natural system (Choice 5.3.6). We may pull back our choices $(\Theta_{\vec{L}}, J_{\vec{L}})$ along the map $S_r \subset S^5_{d+1} \stackrel{\nu_2}{\longrightarrow} |\Delta^d|$. Then by (NS1), along the strip-like ends, all our choices are translation-invariant. (Here, translation is by $[0, \infty)$ or by $(-\infty, 0]$ as parametrized by the strip-like end.)

**Remark 6.2.8.** We have used the notion of pulling back $J_{\vec{L}}$—a choice of almost-complex structure suitable for counting sections (Definition 3.9.6)—in articulating the conditions of Choice 6.2.4. We note that this pullback is defined by utilizing the natural systems from Choice 5.3.6; we leave the details to the reader.

**Proposition 6.2.9.** There exists choices $\{(\Theta_{\vec{L}}, J_{\vec{L}})\}_{\vec{L}}$ satisfying all the conditions laid out in Choice 6.2.4.

**Proof.** This follows from a standard argument using induction; see for example Lemma 3.8 and Section 3.3 of [Sav13]. Perhaps the main point to note in our present work is how to choose the $\Theta_{\vec{L}}$ compatibly. Given the bundle $h^* E \to |\Delta^d|$, the space of 1-forms $\Theta$ on $h^* E$ for which the fiberwise restrictions are Liouville forms is a smoothly contractible space (for example, it is easy to see that the space is convex). This contractibility is a necessary ingredient in the inductive step, as one must extend $\Theta$ from the boundary of an $n$-simplex to its interior.

---

**6.3 A non-wrapped Fukaya category over a simplex**

Fix a smooth map $j : |\Delta^n| \to B$. (Note $|\Delta^n|$ is an extended simplex as in Definition 4.2.1.)

We define in this section the non-wrapped Fukaya category $\mathcal{O}_j$ associated to $j$. The definition is inductive on $n$—we first define $\mathcal{O}_j$ for all $j$ having domain of dimension $\leq n$, then for those $j$ with domain having dimension $n + 1$.

**Remark 6.3.1.** The reader will note that $\mathcal{O}_j$ only depends on the restriction of $j$ to the standard simplex $|\Delta^n| \subset |\Delta^d|$. The reason we insist on the domain of $j$ being the extended simplex $|\Delta^n|$ is to make use of homotopy-theoretic results concerning diffeological spaces (for example, Remark 4.2.9).

**Notation 6.3.2 ($b_i$ and $L_{b_i}$).** For every $0 \leq i \leq n$, let $b_i$ be the image of the $i$th vertex (Definition 5.0.2) of $|\Delta^n|$ under $j$.

Recall we have chosen a countable collection of branes and a cofinal wrapping sequence of these (Choice 6.1.1). In particular, $L_{b_i}$ denote the countable collection of branes associated to $b_i$.

**Definition 6.3.3 (Objects).** An object of $\mathcal{O}_j$ is a triplet $(i, L, w)$ where

- $i \in \{0, \ldots, n\}$,
- $L \in L_{b_i}$, and
- $w \in \mathbb{Z}_{\geq 0}$.
Notation 6.3.4 \((L^w)\). One can informally think of the triplet \((i, L, w)\) as the brane \(L^w\) inside \(E_h\). For this reason, we will soon denote an object simply by \(L^w\), omitting \(i\). (See for example Definition 6.3.10.)

Notation 6.3.5 (Parallel transport \(\Pi\)). Fix a pair of objects \((L_0, i_0, w_0)\) and \((L_1, i_1, w_1)\). The integers \(i_0\) and \(i_1\) define a simplicial map \(\beta : |\Delta^1| \to |\Delta^n| \subset |\Delta^n|\) sending the initial vertex of \(|\Delta^1|\) to \(i_0\) and the final vertex to \(i_1\).

We let \(h = j \circ \beta\). One also has an underlying ordered pair of branes \(\bar{L} = (L_0, L_1)\). (The notation here is to be consistent with Notation 6.2.2.)

Because we have chosen \(\Theta_{\bar{L}}\) for \(h^*E\) (Choice 6.2.4), we have a parallel transport taking the initial fiber of \(h^*E\) (i.e., the fiber above the initial vertex of \(|\Delta^1|\)) to the final fiber of \(h^*E\).

We let \(\Pi_{i_0, i_1}\) denote this parallel transport.

We will render \(\Theta\) to be directed in the \(w\) index; this means that the morphism complex from \((i, L, w)\) to \((i', L', w')\) will be zero unless \(w < w'\), or \((i, L, w) = (i', L', w')\) (in which case the morphism complex is just the ground ring \(R\) in degree 0). Concretely:

Definition 6.3.6 (Morphisms). Fix two objects \((i_0, L_0, w_0)\) and \((i_1, L_1, w_1)\) of \(\Theta_j\).

We define the graded abelian group

\[ \text{hom}_{\Theta_j}((i_0, L_0, w_0), (i_1, L_1, w_1)) \]

to be

\[ \begin{aligned}
    \bigoplus_{x \in \Pi_{i_0, i_1}(L_0^{(w_0)}) \cap L_1^{(w_1)}} \mathbf{O}_x [-|x|], & \quad w_0 < w_1 \\
    R & \quad (i_0, L_0, w_0) = (i_1, L_1, w_1) \\
    0 & \quad \text{otherwise}.
\end{aligned} \]

Here, \(\Pi_{i_0, i_1}\) is the parallel transport map (Notation 6.3.5). We also note that \(\mathbf{O}_x\) is the orientation \(R\)-module of rank one associated to the intersection point \(x\), and \(|x|\) is the Maslov index associated to the brane data.

Remark 6.3.7. The set \(x \in \Pi_{i_0, i_1}(L_0) \cap L_1\) is also in bijection with the set of flat sections of \(h^*E \to |\Delta^1|\) (with respect to \(\Theta_{(L_0, L_1)}\)) beginning at \(L_0^{(w_0)}\) and ending at \(L_1^{(w_1)}\). (See Notation 6.3.5.)

Now we define the operation \(\mu^d\) for \(d \geq 1\).

Definition 6.3.8 (\(\mu^d\) for the non-wrapped categories). As usual, fix a smooth map \(j : |\Delta^d| \to B\). For \(d \geq 1\), fix a collection

\[ \bar{L} = \{(i_0, L_0, w_0), \ldots, (i_d, L_d, w_d)\}. \]

We may assume \(w_0 < \ldots < w_d\) by Definition 6.3.6 (otherwise \(\mu^d\) is forced to be zero).

Note that the integers \(i_0, \ldots, i_d\) induce a simplicial map \(\beta : |\Delta^d| \to |\Delta^n| \subset |\Delta^n|\) by sending the \(a\)th vertex of \(|\Delta^d|\) to the \(i_a\)th vertex of \(|\Delta^n|\). (This assignment, of course, need not be order-preserving.) Recall the map \(\nu_{\beta} : S^d_{d+1} \to |\Delta^n|\) as in Choice 5.3.6.

For a given collection of intersection points

\[ x_a \in \Pi_{i_{a-1}, i_a}(L_{a-1}^{(w_{a-1})}) \cap L_a^{(w_a)} \quad (a = 1, \ldots, d) \]

and

\[ x_0 \in \Pi_{i_0, i_d}(L_0^{(w_0)}) \cap L_d^{(w_d)}, \]

\[ x_0 \in \Pi_{i_0, i_d}(L_0^{(w_0)}) \cap L_d^{(w_d)}, \]

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we define

\[ M(x_d, \ldots, x_1; x_0) \]

to be the moduli space of holomorphic sections \( u \)

\[
\begin{array}{cccc}
S_r & \subset & S_{d+1} & \overset{\nu_\beta}{\longrightarrow} |\Delta^d| & \overset{\beta}{\longrightarrow} |\Delta^n| \subset |\Delta^u_e| \longrightarrow B
\end{array}
\]

satisfying the following boundary conditions:

1. Along the strip-line end near the \( a \)th puncture of \( S \), \( u \) converges to the parallel transport chord from \( L^{(wa-1)} \) to \( L^{(wa)} \) determined by \( x_a \).
2. Along the \( a \)th boundary arc of \( S \), but outside the strip-like ends, \( u \) is contained in the Lagrangian \( L^{(wa)} \subset E_{|\text{arc}} \). Note this makes sense due to the canonical trivialization of \( E|_{\text{arc}} \cong E|_{|\text{arc} \times \text{arc}} \); this is a consequence of (NS2).

As usual, the brane structures on the \( L^{(w)} \) allow us to orient these moduli spaces, and predict their dimension based on the degrees of the \( x_a \). We define

\[ \mu^d(x_d, \ldots, x_1) = \sum_{x_0} \sharp M(x_d, \ldots, x_1; x_0) x_0 \]

where the number \( \sharp M \) is counted with sign. In case our branes are not \( \mathbb{Z} \)-graded, we as usual we declare the \( x_0 \) coefficient of \( \mu^d \) to be zero when there is no zero-dimensional component of \( M(x_d, \ldots, x_1) \).

**Remark 6.3.9.** Given an ordered \((d+1)\)-tuple of objects in \( \mathcal{O}_j \) with underlying branes \( \vec{L} \), consider the induced map \( \beta : |\Delta^d| \to |\Delta^n| \subset |\Delta^u_e| \). The \( A_\infty \)-operations are defined by moduli spaces depending only on \( h = j \circ \beta \). (This follows from Definition 6.3.8 and (Θ1), (Θ2). Note that \( h \) is the same \( h \) as in Notation 6.2.2.)

**Definition 6.3.10.** Fix \( j : |\Delta^u_e| \to B \). We let \( \mathcal{O}_j \) denote the \( A_\infty \)-category where

- an object is the data of a brane \( L^{(w)} \) in one of the vertex-fibers (as in Definition 6.3.3),
- \( \text{hom}_{\mathcal{O}_j}(L_0^{(w_0)}, L_1^{(w_1)}) \) is as in Definition 6.3.6,
- The operations \( \mu^d \) are as in Definition 6.3.8.

**Remark 6.3.11.** When a \( \mu^d \) operation involves an element of an endomorphism hom-complex \( \text{hom}_{\mathcal{O}_j}(L, L) = R \), the operation is fully determined by demanding that the generator of the ring \( R \) be a strict unit.

**Remark 6.3.12.** One has an equivalence between two Floer complexes: One defined by a parallel-transport-induced boundary condition, and another defined by pushing forward along a parallel transport, then constructing a Floer complex where all the geometry is confined to a single fiber. See Lemma 7.4.1.

**Proposition 6.3.13.** \( \mathcal{O}_j \) is indeed an \( A_\infty \)-category.

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Proof. This is a standard consequence of regularity and Gromov compactness for holomorphic sections.

Conditions (Θ4) and (Θ5) guarantee that our moduli are manifolds.

We refer the reader to Section 8 for both $C^0$ estimates (so that holomorphic curves remain in an a compact subset determined by the boundary conditions) and energy estimates (so that the usual finite energy assumptions may be applied to encode how nodal curves may develop).

We note that condition (Θ2), together with (NS3) and (NS4), allow us to compactify the $d$-ary moduli space using products of $d'$-ary moduli for $d'<d$. That is, these conditions guarantee that the $A_\infty$ relations hold.

Example 6.3.14. Suppose $j : |\Delta_0^n| = |\Delta_0^n| \to B$ is the data of a point of $b \in B$. Then $\mathcal{O}_j$ is equivalent to the non-wrapped category $\mathcal{O}$ that [GPS17] associates to the fiber $E_b$ above $b$. This is because the base case of $n=0$ in Choice 6.2.4 implies that the boundary conditions for the holomorphic sections $u$ reduce to strip-like ends converging to intersection points $L_a-1 \cap L_a$ in $M$. (When the pull-back bundle is canonically trivialized as $E_b \times |\Delta^d|$, sections are equivalent to maps $u : S \to E_b$.)

6.4 Non-wrapped functoriality

Construction 6.4.1 (Functors $\mathcal{O}_j \to \mathcal{O}_{j'}$). Now suppose we have a map $\alpha : [n] \to [n']$, a smooth map $j' : |\Delta^n_{e'}| \to B$, and consider the induced diagram

\[
\begin{array}{ccc}
|\Delta^n_e| & \xrightarrow{\alpha} & |\Delta^n_{e'}| \\
j \downarrow & & \downarrow j' \\
B. & & \\
\end{array}
\]

(Equivalently, consider a morphism in the category $\text{Simp}(B)$.) Then there are induced assignments as follows:

1. An object $(i, L, w)$ of $\mathcal{O}_j$ is sent to the object $(\alpha(i), L, w)$ inside $\mathcal{O}_{j'}$. Here, we are identifying the fiber of $(j')^*E$ above $\alpha(i)$ with the fiber of $j^*E$ above $i$ in the obvious way.

2. If $\alpha$ is an injection, then condition (Θ2) of Choice 6.2.4 gives an isomorphism of graded abelian groups

\[
\text{hom}_{\mathcal{O}_j}((i_0, L_0, w_0), (i_1, L_1, w_1)) \to \text{hom}_{\mathcal{O}_{j'}}((\alpha(i_0), L_0, w_0), (\alpha(i_1), L_1, w_1)).
\]

Otherwise, factoring $\alpha$ as a surjection followed by an injection, condition (Θ1) gives the same isomorphism by identifying the fiber above a vertex $i' \in |\Delta^n_{e'}|$ with a fiber above any point in the preimage $\alpha^{-1}(i')$.

We call this assignment $\alpha_*$.

Proposition 6.4.2. The assignment $\alpha_*$ from Construction 6.4.1 is an $A_\infty$-functor $\mathcal{O}_{j'} \to \mathcal{O}_j$.

In fact, the assignments $j \mapsto \mathcal{O}_j$ and $\alpha \mapsto \alpha_*$ define a functor

\[
\text{Simp}(B \text{Aut})(M)) \to A_\infty \text{Cat}
\]

to the (strict) category of $R$-linear $A_\infty$-categories and $R$-linear functors between them, with the usual (strictly associative) composition of functors.
Proof. Remark 6.3.9 shows that the $\mu^d$ operations are respected on the nose, as one is counting sections over two bundles over $|\Delta^d|$ that admit an isomorphism respecting all boundary conditions and choice of $J$ and $A$. This shows $\alpha_*$ is a functor of $A_\infty$-categories, simply by defining the functor to have higher homotopies equaling zero.

Now suppose we have a commutative diagram of smooth maps

$$
\begin{array}{ccc}
|\Delta^n| & \xrightarrow{\alpha} & |\Delta^m| \\
\downarrow{j} & & \downarrow{j'} \\
B & \xrightarrow{\alpha'} & |\Delta^n'| \\
\downarrow{j''} & & \downarrow{j'''} \\
|\Delta^{n''}| & \xrightarrow{\alpha''} & |\Delta^{n}|
\end{array}
$$

We must show that $(\alpha' \circ \alpha)_* = \alpha'_* \circ \alpha_*$. This is straightforward from Construction 6.4.1. \qed

Remark 6.4.3. It follows immediately from the isomorphism observed in Construction 6.4.1 itself that when $\alpha : [n] \to [n']$ is a surjection, $\alpha_*$ is an equivalence of $A_\infty$-categories. (i.e., $\alpha_*$ is essentially surjective and induces a quasi-isomorphism of hom-complexes).
7 Continuation maps

If $L_0$ is a compact brane (Example 3.11.3), any Hamiltonian isotopy from $L_0$ to $L_1$ induces an element in the Floer cohomology $HF^*(L_0, L_1)$; this element is usually referred to as the continuation map, or sometimes the continuation element, associated to the isotopy.

Suppose $L_0$ is now a brane in a Liouville sector. If $L_0$ is not compact and the Hamiltonian isotopy is not compactly supported, one must further impose the restriction that the isotopy be non-negative to construct the continuation map (Definition 3.11.4). Non-negativity yields the necessary $C^0$ and energy bounds to achieve Gromov compactness for moduli of disks (and continuation maps are constructed by counting holomorphic disks); see Theorem 8.4.1.

In this section we review two constructions of a continuation element.

The first is by counting holomorphic disks with one boundary puncture, and with moving boundary conditions along the unique boundary arc of the disk. This will yield an element of $HF^*(L_0, L_1)$.

The second usually follows a two-step process, hence arguably more tedious: For any test brane $K$, one first counts holomorphic strips between $K \cap L_0$ and $K \cap L_1$, with one boundary arc of the strip given moving boundary condition from $L_0$ to $L_1$ (see Figure 1.4.8). In the second step, one proves the naturality of this construction at the level of cohomology—this allows one to invoke the Yoneda Lemma to infer an element of $H^*\hom(L_0, L_1) = HF^*(L_0, L_1)$. This method, however, is a bit unnatural for our purposes: To invoke the Yoneda Lemma, we must have a geometric interpretation of the unit element of $\hom(L_0, L_0)$. In our eventual construction of the non-wrapped Fukaya category $\mathcal{O}_j$, we will formally declare the endomorphisms of $L_0$ to be the base ring $R$; this forces us to lose the geometric interpretation of the unit unless we do a bit more work. Thus, in the present work, we do not take on the second step, and this version of the continuation element will be relative to a test brane $K$—i.e., it will define a degree zero map $HF^*(K, L_0) \to HF^*(K, L_1)$.

We will also prove that the two constructions yield equivalent elements in cohomology after applying the $\mu^2$ operation (Proposition 7.3.1). This result will later be employed in proving Theorem 10.3.1, which in turn allows us to conclude that the Abouzaid construction from the directed, non-wrapped $\mathcal{O}$ category to the category of local systems descends to a functor from the wrapped $\mathcal{W}$ category to the category of local systems.

7.1 Construction of continuation element

Throughout, we fix a Liouville sector $M$ along with an exact Lagrangian isotopy

$$\mathcal{L} : [0, 1] \times L_0 \to M$$

in $M$. We assume that

1. this isotopy is non-negative (Definition 3.11.4),

2. For each $s$, the image of the time $s$ embedding $L_s$ is an (eventually conical) brane (Definition 3.11.2), and

3. $L_0$ is transverse to $L_1$.

**Choice 7.1.1** (Choices for defining continuation map). Choose a marked point $z_0 \in \partial D^2$ and consider the Riemann surface with boundary $D^2 \setminus \{z_0\}$. We equip $D^2 \setminus \{z_0\}$ with a strip-like end near $z_0$. Further, we choose a function

$$\chi : \partial D^2 \to [0, 1]$$
such that $\chi$ is weakly increasing (with respect to the boundary orientation on $\partial D^2$), is locally constant outside a compact set, and is onto.

**Notation 7.1.2.** Given a non-negative Lagrangian isotopy $\mathcal{L}$ (7.1) and a function $\chi$ as in Choice 7.1.1, we let

$$ M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi) $$

be the set consisting of those maps $v : D^2 \setminus \{z_0\} \to M$ satisfying the following conditions:

$$ \begin{cases} \bar{\partial} Jv = 0, \\ \int_{D^2 \setminus \{z_0\}} |dv|^2 < \infty, \\ v(z) \in L^\chi(z) \text{ for } z \in \partial D^2 \setminus \{z_0\} \end{cases} \quad (7.2) $$

See Figure 1.4.6.

**Remark 7.1.3.** The non-negativity of the isotopy $\mathcal{L}$ guarantees the usual $C^0$-estimates. Thus the finite energy condition implies that as $z \to z_0$, the map $u$ converges exponentially (with respect to the strip-like coordinates near $z_0$) to a constant path supported at an intersection point $x \in L_0 \cap L_1$.

**Notation 7.1.4.** Remark 7.1.3 enables us to define the evaluation map

$$ ev_{z_0} : M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi) \to L_0 \cap L_1. $$

We denote

$$ M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x) := ev_{z_0}^{-1}(x) \quad (7.3) $$

for each $x \in L_0 \cap L_1$.

**Proposition 7.1.5** (Theorem C.3.1 [Oh15b]). For a generic choice of isotopy $\mathcal{L}$, $M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x)$ is a smooth manifold of dimension given by

$$ \dim M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, x) = \frac{n}{2} - \mu_{\mathcal{L}}(x) $$

where $\mu_{\mathcal{L}}(x)$ is the Maslov index of $x$ relative to $\mathcal{L}$.

**Remark 7.1.6.** The dimension count of Proposition 7.1.5 is compatible with the grading on $CF^*(L_0, L_1)$, in the sense that

$$ |x| = \dim M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi) = \frac{n}{2} - \mu_{\mathcal{L}}(x). \quad (7.4) $$

**Remark 7.1.7.** This definition of the cohomological degree is adopted because we put the output at $-\infty$ in the definition of the Floer moduli space. Another choice would be to take $|x|$ to be the codimension instead of the dimension of the relevant moduli space if the output were put at $\infty$.

**Construction 7.1.8.** We define a Floer cochain

$$ c^\chi(\mathcal{L}) := \sum_{x \in L_0 \cap L_1, |x| = 0} n^\chi_2(x) \langle x \rangle \quad (7.5) $$

where $n^\chi_2(x) = \#(M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi))$ (counted with sign as usual using orientations).

---

22That is, the images of all the $v$ are contained in an a-priori-determined compact subset of $M$. See Section 8.
Proposition 7.1.9. The cochain $c^\chi(L)$ is a cocycle. Moreover, its Floer cohomology class $[c^\chi(L)] \in HF^0(L_0, L_1)$ is independent of the choice of $\chi$.

Proof of Proposition 7.1.9. We compute the matrix coefficient of the Floer coboundary $\mu^1(c^\chi(L))$.

For each given $y \in L_0 \cap L_1$ with $|y| = 1$, we compute its coefficient in the linear expression of $\mu^1(c^\chi(L))$

$$\langle \mu^1(c^\chi(L)), y \rangle = \sum_{x \in L_0 \cap L_1} n^\chi_L(x)(\mu^1((x)), \langle y \rangle)$$

$$= \sum_{x \in L_0 \cap L_1} \#M(y, x) n^\chi_L(x).$$

Here—by the standard compactness-and-gluing theorems—the last sum is nothing but the count of the boundary elements of compact one-dimensional manifold $M(D^2 \setminus \{z_0\}; L, y)$; it hence vanishes. This proves $\langle \mu^1(c^\chi(L)), y \rangle = 0$ for all $y$ with $|y| = 1$ and so $\mu^1(c^\chi(L)) = 0$. Therefore $c^\chi(L)$ defines a Floer cohomology class in $HF^0(L_0, L_1)$.

Now the standard compactness-cobordism argument proves the second statement noting that the space of elongation functions $\chi$ is contractible (and in particular, connected).

Definition 7.1.10 (The continuation element). Let $L : [0,1] \times L_0 \to M$ be a non-negative, exact Lagrangian isotopy from $L_0$ to $L_1$. We denote by

$$c(L) \in HF^0(L_0, L_1)$$

the cohomology class associated to the cochain in Construction 7.1.8. We call it the continuation element associated to the isotopy.

7.2 Floer continuation using strips

Choice 7.2.1 ($\rho$). We fix an elongation function $\rho : \mathbb{R} \to [0,1]$ given by

$$\rho(\tau) = \begin{cases} 
1 & \text{for } \tau \geq 1 \\
0 & \text{for } \tau \leq 0 
\end{cases}$$

$$\rho'(\tau) > 0 \text{ for } 0 < \tau < 1. \quad (7.6)$$

Remark 7.2.2. For our purposes, any weakly monotone $\rho$ with value 0 near $-\infty$ and 1 near $\infty$ will suffice; we note that the space of such $\rho$ is contractible.

Notation 7.2.3. Given an exact Lagrangian isotopy $L$ and an elongation function $\rho$ as in Choice 7.2.1, we denote by

$$L^\rho : \tau \mapsto L_{\rho(\tau)}.$$

the induced $\mathbb{R}$-parametrized isotopy.

Choice 7.2.4. We also choose a smooth, 2-parameter family of eventually conical almost-complex structures on $M$ (Definition 3.9.1)

$$[0,1] \times [0,1] \to \{\text{Eventually conical } J\}, \quad (s,t) \mapsto J_{(s,t)}.$$
Construction 7.2.5 (Construction using strips). Fix an exact Lagrangian isotopy $\mathcal{L}$ and a brane $K$ such that $K \pitchfork L_i$ for $i = 0, 1$. The Floer continuation map

$$h^\rho_{\mathcal{L}} : CF(K, L_0) \to CF(K, L_1)$$

is defined by counting isolated solutions of the following system:

$$\begin{cases}
\frac{\partial u}{\partial \tau} + J_{\rho(\tau), t} \frac{\partial u}{\partial t} = 0 \\
u(\tau, 0) \in K, \quad u(\tau, 1) \in L_{\rho(1-\tau)}.
\end{cases} \tag{7.7}$$

See Figure 1.4.8.

A similar argument to the proof Proposition 7.1.9 shows the following:

**Proposition 7.2.6.** $h^\rho_{\mathcal{L}}$ is a chain map. Moreover, the map on cohomology

$$[h_{\mathcal{L}}] : HF(K, L_0) \to HF(K, L_1)$$

is independent of the choice of elongation function $\rho$.

### 7.3 Compatibility between the strip and disk definitions of continuation

Now let

$$[\mu^2] : HF(L_0, L_1) \otimes HF(K, L_0) \to HF(K, L_1)$$

denote the (cohomology-level) operation defined by counting holomorphic maps from a disk with three boundary punctures (with the usual boundary conditions). The continuation maps constructed from counting strips (Proposition 7.2.6), and constructed from counting disks with one puncture (Proposition 7.1.9), are compatible in the following sense:

**Proposition 7.3.1.** We have

$$[h_{\mathcal{L}}] = [\mu^2(c(\mathcal{L}), -)].$$

(The left-hand side is defined in Proposition 7.2.6.)

**Remark 7.3.2.** Recall that in our constructions of the continuation elements and the Floer continuation map in the previous Subsections, we have used two different kind of elongation functions, $\chi$ and $\rho$ (Choices 7.1.1 and 7.2.1) where the domain of $\chi$ is $\partial D^2 \setminus \{z_0\}$ and the domain of $\rho$ is $\mathbb{R}$. Different choices of such functions define the same map in cohomology. However, when we study compatibility between the strip and disk definitions of continuation maps, which requires us to examine a family of moduli spaces, we will require $\chi$ and $\rho$ to be compatible. This compatibility also has some freedom in our choices; see (7.16).

#### 7.3.1 Recollections on $\overline{M}_4$

We start our proof with considering the configuration space $\mathcal{M}_4$ of four boundary marked points of the unit disc modulo the action of $PSL(2, \mathbb{R})$. We denote an element thereof by an equivalence class $[z]$ of the tuple

$$z = (z_0, z_1, z_2, z_3).$$

It is easy to see that $\mathcal{M}_4$ is diffeomorphic to the open unit interval and its canonical compactification by stable curves, denoted by $\overline{\mathcal{M}}_4$, is obtained by adding two points on the boundary of the open interval. Each of these two points represents a singular disc with two irreducible components.
More specifically, modulo the action of $PSL(2;\mathbb{R})$, we may assume $z_0 = -1 \in \partial D^2$. Then we consider a diffeomorphism $\tau : \mathcal{M}_4 \to \mathbb{R}_{>0}$ given by the (real) cross ratio,

$$\tau([1, z_1, z_2, z_3]) = \frac{w_2 - w_3}{w_1 - w_2}, \quad w_i = \log z_i$$  \hspace{1cm} (7.8)

where $w_i \in \partial \mathbb{H} \subset \mathbb{C}$. (Here we take the logarithm $w_i = \log z_i$ with respect to the branch cut along the positive real axis.)

**Notation 7.3.3 ($\varphi_r$).** For later use, for each $r \in (0, \infty)$ we denote by

$$\varphi_r : D^2 \setminus \{z_0, z_1\} \to \mathbb{R} \times [0, 1]$$

the unique conformal map satisfying

$$\begin{cases}
    \varphi_r(z_0) = -\infty, \\
    \varphi_r(z_1) = \infty, \\
    \varphi_r(z_2) = (r, 1), \\
    \varphi_r(z_3) = (-r, 1)
\end{cases}$$ \hspace{1cm} (7.9)

for $r = \tau([z_0, z_1, z_2, z_3])$.

This realization on $\mathbb{R} \times [0, 1]$ (of the unit disk’s boundary points as prescribed by elements of $\mathcal{M}_4$) will be important in the study of the continuation equation and its relationship with the continuation element.

The two boundary points of $\overline{\mathcal{M}}_4$ represent singular curves of the types

$$\tau^{-1}(0) = (D^2, (z_0, z_1, \zeta)) \# (D^2, (\zeta, z_2, z_3)), \hspace{1cm} \tau^{-1}(\infty) = (D^2, (z_0, \zeta, z_3)) \# (D^2, (\zeta, z_1, z_2))).$$ \hspace{1cm} (7.10)

Under the above diffeomorphism $\tau : \mathcal{M}_4 \to [0, \infty]$, the unique conformal map $\varphi_r : D^2 \setminus \{z_0, z_1\} \to \mathbb{R} \times [0, 1]$ respects degenerations of the domain and of the target and we can express the limit of the sequence $[z_0, z_1, z_2, z_3]$ in $\overline{\mathcal{M}}_4$ at $r = \infty$ as a join of the morphisms between two stable curves

$$\psi : (D^2, (z_0, z_1, \zeta)) \to (\mathbb{R} \times [0, 1], \{(0, 1)\}),$$

$$\varphi : (D^2, \zeta) \to \Theta_-$$

where $\psi$ and $\varphi$ are uniquely defined by the symmetry (7.9) condition imposed on $\varphi_r$. We describe $\Theta_-$ in Notation 7.3.4 below.

For the later purpose of studying the degeneration of Floer moduli spaces that enter in the analysis of the image of the continuation element $c(\mathcal{L})$ under the Abouzaid functor $F$, it is useful to realize the above degeneration of curves through a Riemann surface $\Theta_-$ that is not only conformally equivalent to $D^2 \setminus \{z_0\}$ but also respects the symmetry of the kind (7.9).

**Notation 7.3.4 ($\Theta_-$).** We denote by $\Theta_-$ the domain (equipped with the strip-like coordinates) of the relevant moduli spaces, and denote by $\Theta_- \# Z$ the nodal curve obtained by the obvious grafting. (See Figure 10.3.6 for the image of the grafted domain.) We mention that we have conformal equivalences $\Theta_- \cong D^2 \setminus \{z_0\}$ and $Z \cong D^2 \setminus \{z_0, z_1, z_2\}$. We take the following explicit model for $\Theta_-$. Consider the domain

$$\{z \in \mathbb{C} \mid |z| \leq 1, \text{Im} z \geq 0 \} \cup \{z \in \mathbb{C} \mid |\text{Re} z| \leq 1, \text{Im} z \leq 0 \}$$

and take its smoothing around $\text{Im} z = 0$ that keeps the reflection symmetry about the $y$-axis of the domain. Then we take

$$Z = \{z \in \mathbb{C} \mid 0 \leq \text{Im} z \leq 1\} \setminus \{(0, 1)\}.$$ \hspace{1cm} (7.13)

Again we equip $Z$ with a strip-like coordinate at $z = (0, 1)$ that keeps the reflection symmetry.
7.3.2 Proof of Proposition 7.3.1

In this subsection we give the proof of Proposition 7.3.1. We start with Proposition 7.2.6. Using the independence of the map \([h^k_r]\) on \(\rho\), we deform \(\rho\) through a suitably chosen one-parameter family \(\{\rho_r\}_{0 < r < 1}\) starting with \(\rho_1 = \rho\) whose construction is now in order. We would like to degenerate the moduli space \(\mathcal{M}(K, \mathcal{L}^\rho; x_-, x_+)\) to the one associated to the right-hand side of Proposition 7.3.1 as \(r \to 0\).

Applying \(\varphi_1\) (see Notation 7.3.3), let us study the collection of maps
\[
v : D^2 \setminus \{z_0, z_1\} \to M \quad \text{such that } u = v \circ \varphi_1^{-1} \text{ satisfies (7.7) (7.14)}
\]
satisfying the finite energy condition. Any such solution converges to \(x_\pm\) with \(x_+ \in K \cap L_0\) and \(x_- \in K \cap L_1\) as \(r \to \pm \infty\) respectively in the coordinate \((\tau, t) \in \mathbb{R} \times [0, 1]\). Given points \(x_-\) and \(x_+\), we denote by
\[
\mathcal{M}(K, \mathcal{L}^\chi; x_-, x_+).
\]
the moduli space of (equivalence classes of) pairs \((v; z)\), where \(v\) is a solution of (7.14) converging to \(x_-\) and \(x_+\), modulo the biholomorphisms of \(D^2\). This is naturally isomorphic to \(\mathcal{M}(K, \mathcal{L}^\rho; x_-, x_+)\) if we set \(\chi = \rho \circ \varphi\) by definition. Therefore to make the following discussion consistent with the compactification of \(\mathcal{M}_4\), we will degenerate the moduli space (7.15) instead by deforming \(\chi\) used in Choice 7.1.1 through a one-parameter family of \(\chi\)'s parameterized by \(\mathcal{M}_4 \equiv (0, \infty)\) via the diffeomorphism \(\tau : \mathcal{M}_4 \to \mathbb{R}_{\geq 0}\) as follows.

We first fix an elongation function \(\rho : \mathbb{R} \to [0, 1]\) and define
\[
\chi_r = \rho \circ \varphi_r
\]
for \(r > 0\) and denote
\[
\rho_r = \rho \circ (\varphi_r \circ \varphi_1^{-1}) = \chi_r \circ \varphi_1^{-1}.
\]
Note that \(\rho_1 = \rho\). Then we introduce a parameterized moduli space of \((v; z)\) with \(z = (z_0, z_1, z_2, z_3)\) by adding two more marked points \(z_2, z_3\) to \((u; z_0, z_1)\). We have the natural fibration
\[
\mathfrak{f} : \mathcal{M}_4(K, \mathcal{L}^\chi; x_-, x_+) \to \mathcal{M}_4
\]
whose fiber is given by \(\mathcal{M}_4(K, \mathcal{L}^\chi; x_-, x_+)\) with \(\chi_r := \rho \circ \varphi_r\) for \(r = \tau([z_0, z_1, z_2, z_3])\): Each element of \(\mathcal{M}_4(K, \mathcal{L}^\chi; x_-, x_+)\) is a pair
\[
(v; z_0, z_1, z_2, z_3) \quad \text{satisfying } \tau([z_0, z_1, z_2, z_3]) = r
\]
where \(v\) is defined on \(D^2 \setminus \{z_0, z_1\}\). Since adding two additional (free) marked points \(z_2, z_3\) increases the dimension by 2, we need to cut it down by putting a codimension 1 constraint on the location of each of \(v(z_2)\) and \(v(z_3)\). We do this by taking local codimension 1 slices transversal to the image of \(v\) at \(v(z_2)\) and \(v(z_3)\), respectively. For this purpose, we use the following lemma: Recall that in the situation of Proposition 7.3.1 the moduli space \(\mathcal{M}_4(K, \mathcal{L}^\chi; x_-, x_+)\) is zero dimensional and compact for \(\chi = \chi_1\). In particular it consists of finitely many elements. We enumerate them by
\[
\mathcal{M}_4(K, \mathcal{L}^\chi; x_-, x_+) = \{v^{(1)}, \ldots, v^{(N)}\}.
\]

Lemma 7.3.5. Suppose the moduli space \(\mathcal{M}(K, \mathcal{L}^\chi; x_-, x_+)\) is nonempty and regular for \(\chi = \chi_1\). Then we can choose marked points \(z_\ell^\ell = (z_{0\ell}, z_{1\ell}, z_{2\ell}, z_{3\ell})\) so that
1. \(\tau([z_{0\ell}, z_{1\ell}, z_{2\ell}, z_{3\ell}]) = 1\),
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2. \( v^{(\ell)} \) is immersed at \( z_2^{\ell} \) and \( z_3^{\ell} \) for all \( \ell = 1, \ldots, N \).

3. there exists some \( \delta_0 > 0 \) such that

\[
d \left( v^{(\ell)}(z_i^{\ell}), v^{(\ell')}(z_j^{\ell'}) \right) \geq \delta_0
\]

for any pair \((\ell, i) \neq (\ell', j)\) with \( j = 2, 3 \) and \( 1 \leq \ell \leq N \).

The proof of this lemma—a simple application of the unique continuation of the image of pseudoholomorphic curves [FHS95, Oh97] and the implicit function theorem—is omitted.

We then pick a local transversal slice \( S_i^{\ell} \subset M \) for each \( i = 2, 3 \) and \( 1 \leq \ell \leq N \) such that

- \( \{S_i^{\ell}\} \) are pairwise disjoint for all \( \ell \) and \( i = 2, 3 \),
- \( v^{(\ell)}(z_i^{\ell}) \in S_i^{\ell} \cap L_{s_i^{\ell}} \) for all \( \ell \) for each \( i = 2, 3 \),
- Both \( S_i^{\ell} \cap v^{(\ell)} \) in \( M \) and \( (S_i^{\ell} \cap L_{s_i^{\ell}}) \cap \partial v^{(\ell)} \) hold at \( z_i^{0} \) for \( i = 2, 3 \) where \( s_i^{\ell} = \chi_{r^{\ell}}(z_i^{\ell}) \) with \( r^{\ell} = r([z_0^{\ell}, z_1^{\ell}, z_2^{\ell}, z_3^{\ell}]) \).

It follows from Condition 3 of Lemma 7.3.5 that by taking \( S_i^{\ell} \) sufficiently small, we may also assume

\[
d(S_2^{\ell}, S_3^{\ell}) > \frac{\delta_0}{2}
\]  
(7.17)

for all \( \ell = 1, \ldots, N \). We denote these collection of \( S_i^{\ell} \) by

\[
S_i = \{S_i^1, \ldots, S_i^N\}, \ i = 2, 3.
\]

**Proposition 7.3.6.** Suppose the moduli space \( M(K, \mathcal{L}^\chi; x_-, x_+) \) is nonempty and transversal. Let \( S_i^{\ell} \) for \( i = 2, 3 \) be the collection of the local slices for the \( M(K, \mathcal{L}^\chi; x_-, x_+) \) chosen in Lemma 7.3.5 above. We define a subset of \( M_4(K, \mathcal{L}^\chi; x_-, x_+) \) by

\[
\set{S}{S_2, S_3} = \mathcal{M}_4^{S_2, S_3}(K, \mathcal{L}^\chi; x_-, x_+)
\]

\[
\set{S}{S_2, S_3} = \set{(r, [(v; z)]) \in M_4(K, \mathcal{L}^\chi; x_-, x_+) \mid v(z_i) \in \cup_{\ell=1}^N S_i^{\ell}, \ i = 2, 3} \bigcap \mathcal{T}^{-1}((0, 1])
\]

where \( z = (z_0, z_1, z_2, z_3) \). Then provided the isotopy \( \{L_s\} \) is sufficiently small in fine \( C^\infty \) topology, this moduli space is a smooth submanifold of \( M_4(K, \mathcal{L}^\chi; x_-, x_+) \to (0, \infty) \) of codimension 2 and so of one dimension. Moreover the same property also persists to the compactification

\[
\overline{\mathcal{M}}_4^{S_2, S_3}(K, \mathcal{L}^\chi; x_-, x_+) \subset \overline{\mathcal{M}}_4(K, \mathcal{L}^\chi; x_-, x_+) \cap \mathcal{T}^{-1}((0, 1])
\]

1. \( v \) is immersed at \( z_2 \) and \( z_3 \) for all \( (v; z) \in \overline{\mathcal{M}}_4(K, \mathcal{L}^\chi; x_-, x_+) \),

2. \( v(z_i) \in \cup_{\ell=1}^N S_i^{\ell}, \ i = 2, 3 \).

Both \( S_i^{\ell} \cap v \) in \( M \) and \( (S_i^{\ell} \cap L_{s_i}) \cap \partial v \) hold at \( z_i \) for \( i = 2, 3 \) where \( s_i = \chi_{r}(z_i) \) with \( r = r([z_0, z_1, z_2, z_3]) \) for all \( \mathcal{T}^{-1}((0, 1]) \).

**Proof.** This is a standard local normal slice theorem; we refer readers to [FOOO09, p.424] for a proof and for the details of such a construction at the interior marked points. The current case of boundary marked points is the same except the following differences

- Here our slice \( S_i^{\ell} \) is of codimension 1 in \( M \) instead of codimension 2 in \( M \),
• We want this slice theorem for the whole family of the moduli spaces $\overline{M}_4(K, \mathcal{L}^r; x_-, x_+) \to r^{-1}([0, 1]) \subset \overline{M}^4$.

The $C^\infty$ smallness of the isotopy is required to ensure these properties for the whole family.

The $C^\infty$ smallness required in the proposition can be always achieved by breaking the given isotopy into a concatenation of smaller isotopies by choosing times

$$0 = t_0 < t_1 < t_2 < \cdots < t_N = 1.$$ 

We also remark that for the proof of Proposition 7.3.1 will follow if we prove it for the portion on each interval $[t_i, t_{i+1}]$ of the given isotopy. With these being said, we will assume from now on that the isotopy is sufficiently $C^\infty$ small so that the global slices $S^4_2, S^4_3$ exist independently of $r \in [0, 1]$.

Then by construction we have the following obvious one-one correspondence

$$\mathcal{M}^{S^4_2, S^3_3}(K, \mathcal{L}^r; x_-, x_+) \cong \mathcal{M}(K, \mathcal{L}^r; x_-, x_+) \cap (r)^{-1}((0, 1])$$

where we denote

$$\mathcal{M}^{S^4_2, S^3_3}(K, \mathcal{L}^r; x_-, x_+) := \mathcal{M}^{S^4_2, S^3_3}(K, \mathcal{L}^r; x_-, x_+) \cap (r)^{-1}(r).$$

Therefore we observe that there is a canonical diffeomorphism

$$\mathcal{M}^{S^4_2, S^3_3}(K, \mathcal{L}^r; x_-, x_+) \cong \mathcal{M}(K, \mathcal{L}^r; x_-, x_+)$$

induced by $v \mapsto v \circ \varphi_r^{-1}$ for the mapping part. We introduce a parameterized Floer continuation moduli space

$$\mathcal{M}^{\text{para}}(\mathcal{L}, L; x_-, x_+) = \coprod_{r \in (0, 1]} \mathcal{M}(K, \mathcal{L}^r; x_-, x_+)$$

defined on the domain $\mathbb{R} \times [0, 1]$. We have a natural fiberwise isomorphism

$$\tilde{\tau} : \mathcal{M}^{S^4_2, S^3_3}_4(K, \mathcal{L}^r; x_-, x_+) \to \mathcal{M}^{\text{para}}(K, \mathcal{L}; x_-, x_+)$$

given by

$$\tilde{\tau}(v; z_0, z_1, z_2, z_3) = (v \circ \varphi_r^{-1}, \tau([z_0, z_1, z_2, z_3]))$$

which makes the following commutative diagram commute,

$$\begin{array}{ccc}
\mathcal{M}^{S^4_2, S^3_3}_4(K, \mathcal{L}^r; x_-, x_+) & \xrightarrow{\tilde{\tau}} & \mathcal{M}^{\text{para}}(K, \mathcal{L}; x_-, x_+) \\
\downarrow r & & \downarrow r \\
(r \circ \text{ft})^{-1}((0, 1]) & \xrightarrow{\text{ft}} & (0, 1]
\end{array}$$

where $\text{ft}$ is the restriction of the forgetful map

$$\text{ft} : \mathcal{M}_4(K, \mathcal{L}^r; x_-, x_+) \to \overline{M}^4.$$ 

We note that the condition $v(z_i) \in S^4_i \cap L_{s_i}$ for $i = 2, 3$ in Proposition 7.3.6 and (7.17) imply the separating condition $d(v(z_2), v(z_3)) \geq \frac{\delta_0}{2} > 0$ for all $v \in \overline{M}^{S^4_2, S^3_3}_4(K, \mathcal{L}^r; x_-, x_+)$. Then it follows from the standard Gromov-Floer compactification and one-jet transversality that the compactified moduli space

$$\overline{M}^{S^4_2, S^3_3}_4(K, \mathcal{L}^r; x_-, x_+)$$

carries its boundary consisting of the types
1. $M_4^{S_2, S_3}(K, L^{x_1}; x_-, x_+)|_{r=0} \cong M_3^{S_2, S_3}(L^x; y) \# M_3(K, L_0, L_1; y, x_-, x_+)$ with $y \in L_0 \cap L_1$, 

2. $M_4^{S_2, S_3}(K, L^{x_1}; x_-, x_+)|_{r=1}$,

3. $M_1^{S_3}(K, L_1; x_-, x') \# M_3^{S_2}(K, L^x; x', x_+)$ for some $x' \in K \cap L_1$ with $|x_-| = |x'| + 1$,

4. $M_3^{S_3}(K, L^{x_1}; x_-, x) \# M_2^{S_2}(K, L_0; x, x_+)$ for some $x \in K \cap L_0$ with $|x| = |x_+| + 1$.

Let us explain the notation.

- $M_3^{S_2, S_3}(L^x; x)$ is the moduli space of equivalence classes of pairs $(w; \zeta, z_2, z_3)$ where $w$ is a function $D^2 \setminus \{\zeta\} \to (M, L)$ satisfying
  
  $$w(z_2) \in \bigcup_{\ell=1}^N S^\ell_2, \quad w(z_3) \in \bigcup_{\ell=1}^N S^\ell_3$$

  and
  
  $$\begin{cases}
  \partial w = 0, \\
  \lim_{z \to \zeta} w(z) = x, \\
  w(z) \in L_\chi(z) \quad \text{for} \quad z \in \partial D^2 \setminus \{\zeta\}
  \end{cases}$$

  for $\chi := \rho \circ \varphi|_{\partial D^2 \setminus \{\zeta\}}$ where $\varphi$ is the map given in (7.12). The forgetful map $(w; \zeta, z_2, z_3) \mapsto w$ induces an isomorphism between $M_3^{S_2, S_3}(L^x; y)$ and the moduli space $M(D^2 \setminus \{\zeta\}; L^x, y)$ studied in Section 7.1. Therefore the count of $M_3^{S_2, S_3}(L^x; y)$ gives rise to the operation $c(L)$.

- $M_3(K, L_0, L_1; y, x_-, x_+)$ is the moduli space whose count encodes the obvious coefficients in the usual $\mu^2$ operation.

- $M_3^{S_3}(K, L_1; x_-, x') \cong M(K, L_1; x_-, x')$ and $M_3^{S_2}(K, L_0; x, x_+) \cong M(K, L_0; x, x_+)$ are the similarly defined moduli of strips (with non-moving boundary conditions) whose counts encode the obvious coefficients in the usual $\mu^1$ operation. We note that one-jet transversality is used to establish that the bubbling of such type cannot occur at the interior parameter $0 < r < \infty$ by dimension counting.

Now we define the map $\mathcal{H}: CF^*(K, L_0) \to CF^{*-1}(K, L_1)$ of degree $-1$ by

$$\mathcal{H}(z) = \sum_{|x| = |z| - 1} \# \left(M_4^{S_2, S_3}(K, L^x; z, x)\right) \langle x \rangle.$$ 

We mention that $\dim M_4^{S_2, S_3}(K, L^x; z, x) = 0$ since $|z| = |x| - 1$. By summing up the sign counts of all the points in $\partial M_4^{S_2, S_3}(K, L^x; x_-, x_+)$ and utilizing

- The isomorphism $M_4^{S_2, S_3}(K, L^x; z, x_+) \mid_{r=1} \cong M(K, L^x; x_-, x_+)$ since $\rho_1 = \rho$,

- The definitions of $\mu^1$ and $\mu^2$ (which are standard),

- The definition of $c^\chi(L)$ (Construction 7.1.8),

- The definition of the continuation map $h^\rho_L$ (Construction 7.2.5), and

- The definition of $\mathcal{H}$ just given,
we have proven the identity
\[ h^\rhoL - \mu^2(c^\chi(L), *) = \mu^1H + \mathcal{H}^\mu. \]

This proves that the two maps \( h^\rhoL, \mu^2(c^\chi(L), *) \) are chain-homotopic. By taking cohomology, we have finished the proof.

**Remark 7.3.7.** The scheme we use in the proof is in the same spirit as in the definition of the stable map topology given in [FO99]. There, convergence for a sequence of maps with *unstable* domain is defined by first stabilizing the domain by adding additional marked points, taking the limit, then finally forgetting the added extra marked points. (See also [Oh15b, Section 9.5.2] for an amplification of this trichotomy.) In the above proof, we take a choice of the minimal (and so optimal) number of additional marked points by taking suitable transversal normal slices for the convergence proof: This is guided by our goal to prove the identity spelled out in Proposition 7.3.1. All these steps are a part of a standard process in the study of moduli space of pseudoholomorphic curves in general—for example, in the construction of the Kuranishi structure and abstract perturbation of the moduli space of stable maps.

### 7.4 Continuation maps and parallel transport

Let \( E \to |\Delta^1| \) be a Liouville bundle, and fix a countable collection \( \{X_i\} \) of branes in the fiber \( E_0 \) (assumed to be in general position), a brane \( Y \) in \( E_1 \), and a Liouville connection such that the parallel transports \( \Pi_{0,1}(X_i) \subset E_1 \) are all in general position with respect to \( Y \). (For example, such choices have already been made for \( E \to B \) in Choice 6.1.1 and Choice 6.2.4, so we may pull back these choices along a smooth map \( |\Delta^1| \to B \).)

Then for any choice of \( X \in \{X_i\} \), there are two natural Floer complexes to associate:

1. The Floer complex whose \( \mu^1 \) term is defined by counting holomorphic sections of the Liouville bundle \( E \). This is \( \text{hom}_O(X, Y) \).

2. The Floer complex obtained by first parallel transporting \( X \) to \( E_1 \) along the edge \( |\Delta^1| \), then computing the usual Floer complex \( CF^*(\Pi_{0,1}X, Y) \) in \( E_1 \).

There are obvious maps
\[ \text{hom}_O(X, Y) \cong CF^*(\Pi_{0,1}X, Y) \tag{7.18} \]
and (for any \( X, X' \in E_0 \))
\[ \text{hom}_O(X, X') \cong CF^*(\Pi_{0,1}X, \Pi_{0,1}X'). \tag{7.19} \]

We leave the proof of the following to the reader, as it is standard:

**Lemma 7.4.1.** The maps (7.18) and (7.19) are quasi-isomorphisms. Moreover, for any pair \( X, X' \in \{X_i\} \), the diagram

\[
\begin{array}{ccc}
H^* \text{hom}_O(X', X) \otimes H^* \text{hom}_O(X, Y) & \xrightarrow{\mu^2} & H^* \text{hom}_O(X', Y) \\
\downarrow{(7.19) \otimes (7.18)} & & \downarrow{(7.18)} \\
HF^*(\Pi_{0,1}X', \Pi_{0,1}X) \otimes HF^*(\Pi_{0,1}X, Y) & \xrightarrow{\mu^2} & HF^*(\Pi_{0,1}X', Y)
\end{array}
\]

commutes.
7.5 Continuation maps in Liouville bundles

Lemma 7.4.1 allows us to generalize the notion of continuation maps between branes to those in a fibration with a given connection and a given path:

**Definition 7.5.1** (Continuation elements for branes relative to a path). Fix a Liouville fibration $E \to [0, 1]$, along with a 1-form $\Theta \in \Omega^1(E)$. Fix a brane $X$ in the fiber above 0, and a brane $Y$ in the fiber above 1, and suppose that the parallel transport $\Pi_{0,1}(X)$ is transverse to $Y$. Note that this defines a cochain complex\footnote{The subscript “fib” stands for “fibration.”}

$$CF^*_\text{fib}(X,Y)$$

whose generators are flat chords from $X$ to $Y$, and whose differential is given by counting holomorphic sections of the $E$ pulled back along the projection $\mathbb{R} \times [0, 1] \to [0, 1]$. (See also Remark 7.5.2 below.)

Suppose further that there exists a non-negative Hamiltonian isotopy from $\Pi_{0,1}(X)$ to $Y$, and let $c \in HF^*(\Pi_{0,1}(X), Y)$ be the associated continuation element.

The associated **continuation element in** $HF^*_\text{fib}(X,Y)$ is the element corresponding to $c$ under the isomorphism of Lemma 7.4.1.

More generally, if $X$ and $Y$ are branes in two fibers of a Liouville fibration $E \to B$, and if $\gamma : [0, 1] \to B$ is a smooth map from $p(X)$ to $p(Y)$, a **continuation element relative to** $\gamma$ is an element of $HF^*_\text{fib}(X,Y)$ that arises from some continuation element (for some choice of non-negative Hamiltonian isotopy from $\Pi_{0,1}(X)$ to $Y$).

**Remark 7.5.2.** Given a Liouville fibration $E \to B$, fix a smooth map $j : |\Delta^n| \to B$ and consider the non-wrapped Fukaya category $\mathcal{O}_j$ (Definition 6.3.10). Suppose $X = L_0^{(w_0)}$ and $Y = L_1^{(w_1)}$ are branes underlying objects $(L_0, i_0, w_0)$ and $(L_1, i_1, w_1)$ of $\mathcal{O}_j$, respectively. We let $\beta : |\Delta^1| \to |\Delta^n|$ be the simplicial map sending the initial vertex of $|\Delta^1|$ to $i_0$ and the terminal vertex to $i_1$. We also assume that $w_0 < w_1$.

Then, setting $\gamma$ to be the composite $[0, 1] \cong |\Delta^1| \xrightarrow{\beta} |\Delta^n| \xrightarrow{j} B$, we have an isomorphism

$$\text{hom}_{\mathcal{O}_j}((L_0, i_0, w_0), (L_1, i_1, w_1)) \cong CF^*_\text{fib}(X,Y).$$

Thus, it makes sense to speak of continuation elements between objects of $\mathcal{O}_j$:

**Definition 7.5.3.** Given a Liouville fibration $E \to B$, fix a smooth map $j : |\Delta^n| \to B$ and consider the non-wrapped Fukaya category $\mathcal{O}_j$ (Definition 6.3.10). Fix two objects $(L_0, i_0, w_0)$ and $(L_1, i_1, w_1)$.

A **continuation element** from $(L_0, i_0, w_0)$ to $(L_1, i_1, w_1)$ is any element of

$$H^* \text{hom}_{\mathcal{O}_j}((L_0, i_0, w_0), (L_1, i_1, w_1))$$

arising from a continuation element of $HF^*_\text{fib}(L_0, L_1)$ (Definition 7.5.1) under the isomorphism

$$CF^*_\text{fib}(L_0, L_1) \cong \text{hom}_{\mathcal{O}_j}((L_0, i_0, w_0), (L_1, i_1, w_1))$$

(Remark 7.5.2).
8 Estimates

In this section, we study compactness properties of the moduli space of holomorphic sections of a Liouville bundle \( \pi : P \to \Sigma \) over a surface \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \). There are two fundamental analytical ingredients to establish: \( C^0 \)-estimates and energy estimates. These estimates imply Gromov compactness as usual, and establish both the \( A_\infty \) relations for \( \mathcal{O}_j \), and the existence and properties of continuation maps (which we study in the next section).

Remark 8.0.1. We reiterate that we are using cohomological conventions for our Floer complexes (Convention 1.4.4). So for example, given a morphism \( q \) from a brane \( L \) to another brane \( L' \), the differential \( \mu^1 q \) is computed by counting solutions to the equation

\[
\begin{align*}
\bar{\partial}_J u &= 0 \\
u(-\infty) &= p, \ u(\infty) &= q \\
u(\tau, 0) &\in L, \ u(\tau, 1) \in L'.
\end{align*}
\]

(8.1)

This is the same convention as in [Sei03].

8.1 Curvature

Notation 8.1.1 (\( \Omega \)). Fix a Liouville bundle \( E \to B \) with base \( B \), and fix a Liouville form \( \Theta \in \Omega^1(E) \) as in Remark 3.7.4; we have the associated two-form \( \Omega = d\Theta \) on \( E \).

Suppose that \( B = \Sigma \) is a Riemann surface. Then the curvature of the connection associated to \( \Theta \) (Definition 3.8.2) can be considered a 2-form on \( \Sigma \) with values in smooth functions of the fibers. More precisely:

Proposition 8.1.2. For any orientation-respecting volume form \( \omega_\Sigma \) on \( \Sigma \), we have

\[
\Omega|_{HTE} = f^*\omega_\Sigma|_{HTE}
\]

(8.2)

where \( f : E \to \mathbb{R} \) is a smooth function.

For proofs, see Theorem 4.2.9 of [Oh15a] or (1.4) of [Sei03].

Definition 8.1.3 (Compare with p.1007 [Sei03]). We say that the curvature of \( \Theta \) is non-negative if \( \Omega|_{HTE} \) is non-negative for the orientation on \( HTE \) (induced by that of \( \Sigma \)). More generally, we say \( \Theta \) is \((−C)\)-pinched (from below) if we have that

\[
\inf_{x \in E} f(x) \geq -C
\]

for some \( C \geq 0 \). (Here, the function \( f \) is as in (8.2).)

Example 8.1.4. \( \Theta \) has nonnegative curvature if and only if it is 0-pinned.

Remark 8.1.5. In all our choices of \( \Theta \), we can arrange for \( \Theta \) to be \((−C)\)-pinched. This is for two reasons: First, \( \Theta \) has behavior controlled outside a set \( K \subset E \) that is proper over \( \Sigma \) (by assumption, \( K \) is a set whose complement is a fiberwise cylindrical region). Second, on a strip-like end of \( \Sigma \), \( \Theta \) is trivialized in the \( \tau \) direction (e.g., in the \((-\infty, 0] \) direction of the strip \((\infty, 0] \times [0, 1])\)), so one may extend \( K \) over a compactification of \( \Sigma \) (e.g., by filling in the punctures of \( \Sigma \) with chords).
8.2 On the interior of disks

We now establish that holomorphic curves with certain Lagrangian boundary conditions must be contained in some compact region of a Liouville bundle. A key result is Proposition 8.2.5, which shows that—if a holomorphic curve \( u : \Sigma \to E \) intersects the cylindrical region of \( E \)—the natural cylindrical coordinate \( r = e^s \) is a subharmonic function on \( \Sigma \).

**Remark 8.2.1.** Recall that subharmonic functions behave like “convex up” functions, in that non-constant maxima are attained only along the boundary of the domain. Thus, knowing that the cylindrical coordinate of \( u \) is constrained along the interior of \( \Sigma \), by imposing appropriate boundary conditions on our Lagrangian family (to obtain constraints on the behavior of \( u \) along the boundary of \( \Sigma \)), we obtain the desired \( C^0 \) estimate in Theorem 8.4.1.

**Lemma 8.2.2.** Fix an conical-near-infinity conical almost-complex structure on \( E \to B \) (Definition 3.9.1). Let \( v : \Sigma \to E \) be any \((j,J)\)-holomorphic section. Then there exists some subset \( K \subset E \), proper over \( \Sigma \), outside of which we have

\[
\Delta (r \circ v)\omega_\Sigma = v^*\Omega.
\]  

**Proof.** We take \( K \) to be the same set as in the definition of conical-near-infinity almost-complex structure (Definition 3.9.1). We may then choose a global \( r \) coordinate. Using \( \Theta \circ J = dr \) and \( J \circ dv = dv \circ j \), we have

\[
d(r \circ v) = v^*dr = v^*(\Theta \circ J) = \Theta(J \circ dv) = \Theta(dv \circ j) = v^*\Theta \circ j.
\]

Therefore we have

\[
\Delta (r \circ v)\omega_\Sigma = -d(d(r \circ v) \circ j) = d(v^*\Theta) = v^*\Omega.
\]

This finishes the proof. \( \square \)

**Notation 8.2.3 \((l \text{ and } \omega_\beta)\).** Now let \( \beta \) be a positive 2-form on \( \Sigma \)—this means that

\[
\beta = l \omega_\Sigma
\]

for some positive function \( l : \Sigma \to \mathbb{R} \). We let

\[
\Omega_\beta = \Omega + \pi^*\beta.
\]

**Remark 8.2.4.** If \( \ell : \Sigma \to \mathbb{R} \) is sufficiently positive, then \( \Omega_\beta \) is a symplectic form. Moreover, given a conical-near-infinity choice of almost complex structure \( J \), choosing \( \ell \) sufficiently positive makes the projection \( E \to B \) a \((J,j)\)-holomorphic map (Lemma 2.1 of [Sei03]).

Consider the compatible metric

\[
g(X,Y) := \frac{1}{2}(\Omega_\beta(X,JY) + \Omega_\beta(Y,JX)).
\]

As usual, we have the identity

\[
|dv|^2 = |\partial_J v|^2 + |\partial_J v|^2,
\]

\[
2v^*\Omega_\beta = (|\partial_J v|^2 - |\partial_J v|^2) \omega_\Sigma
\]

for any smooth section \( v \). We have more when \( v \) is holomorphic:

---

24For example, by constructing an appropriate bundle of Liouville domains \( M_\Sigma \to \Sigma \), embedding \( M_\Sigma \hookrightarrow E \) over \( \Sigma \), then defining \( r \) by the Liouville flow time of the fiberwise contact boundaries.
Proposition 8.2.5. Suppose \( v : \Sigma \to E \) is a \((j,J)\)-holomorphic section. Then

\[
\Delta(r \circ v) = \frac{1}{2} |(dv)^v|^2 + f(v)\ell
\]

where \( f \) is the same function from (8.2) and \( \ell \) is from Notation 8.2.3.

In particular, if the pull-back connection of \( v^*E \) has nonnegative curvature, i.e., if \( f(v) \geq 0 \), and if \( \ell \) is large enough, then \( r \circ v \) is a subharmonic function on \( v^{-1}(E \setminus K) \). (Here, \( K \) is the same set as in Lemma 8.2.2.)

Proof. If \( \bar{\partial}_J v = 0 \), then we obtain

\[
v^*\Omega_\beta = \frac{1}{2} |dv|^2 \omega_\Sigma. \tag{8.4}
\]

Splitting \( dv = (dv)^v + (dv)^h \) into horizontal and vertical components of \( dv \), we compute

\[
\frac{1}{2} |(dv)^h|^2 \omega_\Sigma = (f(v) + 1)\beta
\]

where \( f \) is the function given by (8.2). Rewriting (8.4) as

\[
v^*\Omega + v^*\pi^*\beta = \frac{1}{2} |(dv)^v|^2 \omega_\Sigma + (f(v) + 1)\beta \tag{8.5}
\]

and using \( \pi \circ v = \text{id}_\Sigma \), we obtain

\[
v^*\Omega = \frac{1}{2} |(dv)^v|^2 \omega_\Sigma + (f(v) + 1)\beta - (\pi \circ v)^*\beta = \frac{1}{2} |(dv)^v|^2 \omega_\Sigma + \ell f(v)\omega_\Sigma. \tag{8.6}
\]

Now combine Lemma 8.2.2 with (8.6). \( \square \)

8.3 Boundary conditions

The definitions below simply give names to the conditions we can guarantee in our set-up.

Definition 8.3.1. Let \( E \to \Sigma \) be a Liouville bundle, and fix strip-like ends of \( \Sigma \). We say that \( E \) is translation-invariant over the strip-like ends if, over the strip-like ends, \( E \) is equipped with an isomorphism to a pullback bundle

\[
p^* E_{[0,1]} \to E_{[0,1]}
\]

where \( R \) is either \((-\infty,0]\) (for incoming strips) or \([0,\infty)\) (for outgoing strips).

The following is a variation on Section 2.1 of Seidel’s work [Sei03].

Definition 8.3.2 (Lagrangian boundary conditions for bundles). Fix a Liouville bundle \( E \to \Sigma \) and strip-like ends on \( \Sigma \). Assume \( E \) is translation-invariant over the strip-like ends (Definition 8.3.1). A Lagrangian boundary condition suitable for our purposes is an \((n+1)\)-dimensional submanifold \( \mathcal{L} \subset E|_{\partial \Sigma} \) equipped with the data of a smooth function \( K_{\mathcal{L}} : \mathcal{L} \to \mathbb{R} \) such that

1. \( \mathcal{L} \) is fiberwise conical near infinity, 
2. \( \pi|_{\mathcal{L}} : \mathcal{L} \to \partial \Sigma \) is a submersion,
3. \( \Theta|_{\mathcal{L}} = dK_{\mathcal{L}} \), and
4. \( \mathcal{L} \) is translation-invariant on the strip-like ends. (For example, for a strip-like end modeled on \( (-\infty, 0] \times [0, 1] \rightarrow [0, 1] \).)

**Remark 8.3.3.** The conditions of Definition 8.3.2 imply that for every \( z \in \partial \Sigma \), the fiber \( \mathcal{L}_z \) is an exact Lagrangian submanifold of \( E_z \), and that \( K_{\mathcal{L}|_{\mathcal{L}_z}} \) is a potential function thereof.

The following is the fibration version of non-negative wrapping.

**Definition 8.3.4.** Let \( \mathcal{L} \) be a boundary condition as in Definition 8.3.2. We call \( \mathcal{L} \) nonnegative (resp. nonpositive) relative to \( \partial \Sigma \) if \( \Theta(\xi) \geq 0 \) (resp. \( \Theta(\xi) \leq 0 \)) for all \( \xi \in T_x \mathcal{L} \) whose projection to \( T \partial \Sigma \) compatible with the orientation of \( \partial \Sigma \).

Finally, in our setting, one can arrange for the following:

**Definition 8.3.5.** Let \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \), and let \( \mathcal{L} \subset E \rightarrow \Sigma \) be a Lagrangian boundary condition as in Definition 8.3.2. Choose also a strip-like end for every \( z_i \), with \( z_0 \) incoming and others outgoing.

A connection induced by a Liouville form \( \Theta \) (as in Remark 3.7.4) will be called our kind of connection if it is trivial on \( \partial \Sigma \) outside the strip-like ends, and if the connection is invariant under the translation on the strip-like ends.

**Example 8.3.6.** For example, the conditions of Definition 8.3.5 hold on a strip-like end modeled after \( (-\infty, 0] \times [0, 1] \rightarrow [0, 1] \).)

**Remark 8.3.7.** In all our examples, \( E \rightarrow \Sigma \) is pulled back along a map \( \Sigma \rightarrow |\Delta^n| \) which collapses strip-like ends to a single edge of \( |\Delta^n| \), and collapses the rest of \( \partial \Sigma \) to vertices of \( |\Delta^n| \); as such, the connections pulled back from a bundle \( E' \rightarrow |\Delta^n| \) satisfy Definition 8.3.5.

**Remark 8.3.8.** If \( \Theta \) induces our kind of connection (Definition 8.3.5), and if for every connected component \( c_i \subset \partial \Sigma \), we choose some \( x_i \in c_i \) and a brane \( L_i \subset E_{x_i} \), one obtains a Lagrangian boundary condition \( \mathcal{L} \) by parallel transport along \( \partial \Sigma \). \( \mathcal{L} \) is non-negative in the sense of Definition 8.3.4.

**8.4 The \( C^0 \) estimate**

**Theorem 8.4.1.** Let \( (\Sigma, j) \) be a Riemann surface \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \) where \( \{z_0, \ldots, z_k\} \subset \partial D^2 \), and let \( \pi : E \rightarrow \Sigma \) be a Liouville bundle with translation invariance (Definition 8.3.1). Further choose:

- A non-negative boundary condition \( \mathcal{L} \) as in Definitions 8.3.2 and 8.3.4.
- A conical-near-infinity almost-complex structure \( J \) on \( E \) for which \( E \rightarrow \Sigma \) is \( (J, j) \)-holomorphic,
- A Liouville form \( \Theta \in \Omega^1(E) \) inducing our kind of connection \( \Theta \) (Definition 8.3.5) which is \( (−C) \)-pinched (Definition 8.1.3), and
- For each \( i \in 1, \ldots, k \), a parallel transport chord \( x_i \) from the \( (i - 1) \)st boundary brane to the \( i \)th boundary brane, along with a parallel transport chord from \( L_0 \) to \( L_k \).
Suppose that $v : \Sigma \to E$ is a $(j, J)$-holomorphic section such that

- $v(\partial \Sigma) \subset \mathcal{L}$, and
- $v$ converges to the parallel transport chords $x_i$ along the strip-like ends.

Then there exists some subset $A \subset E$, proper over $\Sigma$, and depending only on the $x_i$, such that $\text{image } v \subset A$.

**Proof.** By Proposition 8.2.5 we have

$$\Delta (r \circ v) \omega_\Sigma = \left( \frac{1}{2} |(dv)^v|^2 + f(v) \right) \beta$$

wherever the image of $v$ is contained in the cylindrical region of $E$. (That is, when $v$ has image outside of the set $K$ of the Proposition.)

We first consider the case of nonnegative curvature, i.e., $f \geq 0$. By the nonnegativity hypothesis, $r \circ v$ is a subharmonic function on $\Sigma$. Therefore it cannot have any local maximum at an interior point and we have only to check its boundary behavior.

We compute the radial derivative $\frac{\partial (r \circ v)}{\partial \nu}$—i.e., the derivative along an outward pointing boundary vector:

$$\frac{\partial (r \circ v)}{\partial \nu} = dr \left( \frac{\partial v}{\partial \nu} \right) = \Theta \circ J \left( \frac{\partial v}{\partial \nu} \right).$$

Since $v$ is $(j, J)$ holomorphic, we have

$$J \left( \frac{\partial v}{\partial \nu} \right) = J \circ dv \left( \frac{\partial}{\partial \nu} \right) = dv \left( j \frac{\partial}{\partial \nu} \right).$$

Therefore if $\partial / \partial \tau$ is any positively oriented tangent vector along $\partial \Sigma$, we have

$$\frac{\partial (r \circ v)}{\partial \nu} = -\Theta \left( \frac{\partial v}{\partial \tau} \right) \leq 0$$

by the nonnegativity assumption of $\mathcal{L}$.

Therefore the strong maximum principle implies that $r \circ u$ cannot have boundary local maximum anywhere on $\partial \Sigma$.

For the $(-C)$-pinched case, we have $\inf f \ell \geq -C$ and so the function $r \circ v$ satisfies the differential inequality

$$\Delta (r \circ v) \geq -C, \quad \frac{\partial (r \circ v)}{\partial \nu} \leq 0.$$

At this stage, we can apply the standard elliptic estimates (see for example [GT70, Theorem 3.7]). Another more explicit way of proceeding is to consider the function $g = (r \circ v) - \frac{C}{2} \ell^2$ where $t : \Sigma \to \mathbb{R}$ is the pull-back function of the standard coordinates $(\tau, t)$ of $\mathbb{R} \times [0, w]$ for some $w > 0$ via the slit domain representation of the conformal structure of $\Sigma = D^2 \setminus \{z_0, \cdots, z_k\}$. (See [BKO19, Section 3.2], for example.)

Then $g$ is a subharmonic function. We can apply the strong maximum principle to the function $g$ to conclude

$$\sup_{z \in \Sigma_{\text{end}}} g(z) \leq R_0$$

satisfying $\frac{\partial (r \circ v)}{\partial \nu} \leq 0$ since $\frac{\partial t}{\partial \nu} = 0$ along the boundary $\partial \Sigma$. Therefore we conclude

$$\sup_{z \in \Sigma_{\text{end}}} r \circ v(z) \leq R_0 + \frac{C}{2}.$$

This finishes the proof. \qed
8.5 The energy estimate

Fix a non-negative Lagrangian boundary condition \( \mathcal{L} \) (Definition 8.3.2 and 8.3.4). We have that

\[
\Theta|_\mathcal{L} = dK_\mathcal{L} + \pi^*(\kappa_\mathcal{L})
\]  \hspace{1cm} (8.7)

for a one-form \( \kappa_\mathcal{L} \in \Omega^1(\partial\Sigma) \) which vanishes on the strip-like ends. (In our case, \( \kappa_\mathcal{L} = 0 \) on all of \( \partial\Sigma \), but we include \( \kappa_\mathcal{L} \) in what follows for the interested reader.)

The action functional on the path space

\[ \mathcal{P}(L_0, L_1) = \{ \gamma \in C^\infty([0, 1], M) \mid \gamma(0) \in L_0, \gamma(1) \in L_1 \} \]

is given by

\[
A_{L_0, L_1}(\gamma) = -\int \gamma^* \theta + K_{L_1}(\gamma(1)) - K_{L_0}(\gamma(1)).
\]

Using (8.7), we also obtain

\[
\int v^* \Omega = \sum_{e \in I^-} A_{L_0, L_1}(x_e) - \sum_{e \in I^+} A_{L_0, L_1}(x_e) + \int_{\partial\Sigma} \kappa_\mathcal{L}. \hspace{1cm} (8.8)
\]

On the other hand, we derive from (8.5)

\[
\frac{1}{2} \left| (dv)^2 \right| = v^* \Omega - f(v)\beta
\]

and hence

\[
\frac{1}{2} \int_{\Sigma} \left| (dv)^2 \right| = \sum_{e \in I^-} A_{L_0, L_1}(x_e) - \sum_{e \in I^+} A_{L_0, L_1}(x_e) + \int_{\partial\Sigma} \kappa_\mathcal{L} - \int_{\Sigma} f(v)\beta.
\]

Here we would like to mention that both the integrals \( \int_{\partial\Sigma} \kappa_Q \) and \( \int_{\Sigma} f(v)\beta \) are finite since \( \kappa_Q = 0 \) and \( f(v) = 0 \) on the strip-like region of \( \Sigma \). We also have

\[
\frac{1}{2} \int_{W} \left| (dv)^2 \right| = \int_{W} (f(v) + 1)\beta = \int_{W} f(v)\beta + \int_{W} \beta
\]

for any compact domain \( W \subset \Sigma \).

We summarize the above discussion into the following uniform upper bound for the energy on any compact domain \( W \subset \Sigma \).

**Proposition 8.5.1.** Let \((\pi : E \to \Sigma, \Omega)\) and \( \mathcal{L} \) be as above and \( W \subset \Sigma \) be any given compact subdomain of \( \Sigma \). Then

\[
\frac{1}{2} \int_{W} |dv|^2 = \int_{\Sigma} v^*\Omega + \int_{W} \beta \hspace{1cm} (8.9)
\]

for any \((j, J)\)-holomorphic section \( v : \Sigma \to E \).

**Remark 8.5.2.** The reason why we restrict to compact domain \( W \subset \Sigma \) is that the form \( \beta \) may not be integrable, unlike \( f(v)\beta \). Moreover, the integrals above depend on the section \( v \) and may not be uniformly bounded, mainly because the strip-like regions of \( \Sigma = D^2 \setminus \{z_0, \ldots, z_k\} \) vary depending on the configuration of \( \{z_0, \ldots, z_k\} \).

**Remark 8.5.3.** By requiring translation invariance of \( \omega_\Sigma \) on the strip-like ends of \( \Sigma \), we conclude that the full integral \( \int_{\Sigma} \omega_\Sigma \) is infinite whenever there is at least one puncture on \( \Sigma \). In choosing the 2-form \( \beta = \ell \omega_\Sigma \), there are two competing interests:
• One one hand, we need the form $\Omega + \pi^* \beta$ to be nondegenerate,

• On the other hand, we wish to make the form $\beta$ have finite integral over $\Sigma$.

In general we cannot achieve both wishes simultaneously. This is the reason we need to consider the horizontal energy on compact domains $W$ e.g., on $W = \Sigma \setminus \Sigma^{\text{end}}$.

**Remark 8.5.4.** However, because we are given a connection that is translation-invariant on the strip-like ends, when we restrict $v$ along a strip-like end, we may write

$$v(\tau, t) = (\tau, t, u(\tau, t))$$

where $u$ is a function satisfying

$$\frac{\partial u}{\partial \tau} + J \left( \frac{\partial u}{\partial t} - X_H(\tau, t, u) \right) = 0.$$ 

This equation can be studied in the standard way of classical Floer theory.

Therefore with the uniform $C^0$-estimates at our disposal, we can apply the Gromov-Floer type of compactness arguments to the moduli space of pseudoholomorphic sections. (See [Sei03, Section 2.4] for some relevant details.)
9 The wrapped Fukaya categories

We fix as usual a Liouville fibration $E \rightarrow B$. The first agenda of this section is to define the wrapped Fukaya category $\mathcal{W}_j$ associated to a smooth simplex $j : \Delta^n_e \rightarrow B$ (Definition 9.1.4). We do this by localizing the non-wrapped categories $O_j$ from Section 6 along non-negative continuation maps.

We then prove that the assignment $j \mapsto \mathcal{W}_j$ is locally constant (i.e., forms a local system of wrapped Fukaya categories), in that any inclusion of simplices $j \subset j'$ induces an equivalence of $A_\infty$-categories (Proposition 9.3.3). This allows us to prove Theorem 1.3.2.

9.1 Equivalent choices of localizing morphisms

Fix a smooth map $j : \Delta^n_e \rightarrow B$ and consider the $A_\infty$-category $O_j$ from Definition 6.3.10. There are two families of localizing morphisms one can consider:

Definition 9.1.1 $(C$ and $C_{\Pi})$. We let $C_{\Pi}$ denote the collection of morphisms in $O_j$ arising as continuation maps $(i, L, w) \rightarrow (i', L', w')$ for $w' > w$ (see Definition 7.5.3).

On the other hand, we let $C \subset C_{\Pi}$ denote the collection of continuation maps with $i = i'$ and $L = L'$.

It turns out we can localize with respect to either $C$ or $C_{\Pi}$ from Definition 9.1.1, and end up with the same localization.

Lemma 9.1.2. The natural map $O_j[C^{-1}] \rightarrow O_j[C_{\Pi}^{-1}]$ between localizations (Definition 2.4.18) is an equivalence of $A_\infty$-categories.

Proof. Let $\mathcal{A}$ be any $A_\infty$-category, and let $W$ be any class of morphisms. Consider the localization functor $\mathcal{A} \rightarrow \mathcal{A}[W^{-1}]$, and let $\overline{W} \subset \mathcal{A}$ denote the subcategory of morphisms that are mapped to equivalences under the localization functor. It is immediate that $\mathcal{A}[W^{-1}] \simeq \mathcal{A}[\overline{W}^{-1}]$.

So it suffices to show that $\overline{C} \supset C_{\Pi}$. Suppose

$$[c_{01}] \in H^0_{\text{hom}}(O_j, (i_0, L_0, w_0), (i_1, L_1, w_1))$$

is a non-negative continuation element. Then choose a large enough wrapping index $w'_0$ of $(i_0, L_0, w_0)$ so that one can find a non-negative continuation element $c_{10'}$ from $(i_1, L_1, w_1)$ to $(i_0, L_0, w'_0)$. Then choose a positive enough wrapping index $w'_1$ so that one can find a non-negative continuation element $c_{01'}$ from $(i_0, L_0, w'_0)$ to $(i_1, L_1, w'_1)$. (Note that our assumption that the $(L, w)$ form a cofinal sequence allows for this.) Then one has a commutative diagram
in the cohomology category of \( O \), hence of \( O[C^{-1}] \) and of \( O[C^{-1}] \). In \( O[C^{-1}] \), the cohomology category admits a unique inverse to \([c_{00}]\) and \([C_{11}]\); then it follows that all the diagonal maps (and in particular, \([c_{01}]\)) are equivalences in \( O[C^{-1}] \). In particular, any \( c_{01} \in C_{11} \) is contained in \( \overline{C} \).

\[ \textbf{Remark 9.1.3.} \] We will use the definition of localizing along \( C \) (as opposed to \( C_{11} \)) in Lemma 9.2.2, as this choice simplifies computations of interest to us.

\[ \textbf{Definition 9.1.4} (W_j). \] Fix \( j : |\Delta^n| \to B \) a smooth map. We let \( W_j \) denote the localization \( O_j[C^{-1}] \), and we call it the partially wrapped Fukaya category associated to \( j \).

\[ \textbf{Example 9.1.5.} \] If \( n = 0 \) and \( j : |\Delta^0| \to B \) simply chooses a point \( b \in B \), then \( O_j[C^{-1}] \) is equivalent to the partially wrapped Fukaya category associated to the fiber \( E_b \) (as in [GPS17]).

### 9.2 Wrapped complexes as filtered colimits

Now we compute hom-complexes \( \text{hom}_{W_j}(X,Y) \) for two branes \( X \) and \( Y \) in the same fiber.

\[ \textbf{Remark 9.2.1.} \] We note that this also computes hom-complexes when the branes are in two different fibers of \( E \to B \): Simply find \( Y' \) in the same fiber as \( X \) and consider a parallel-transport continuation \( Y \to Y' \). In \( W_j \), this map induces an equivalence \( Y \simeq Y' \) by Lemma 9.1.2, so we have \( \text{hom}_{W_j}(X,Y) \simeq \text{hom}_{W_j}(X,Y') \).

\[ \textbf{Lemma 9.2.2.} \] The hom-complex \( \text{hom}_{W_j}(X,Y) \) has cohomology given by the wrapped Floer cohomology (computed in the fiber containing \( X \) and \( Y \)) as defined in [AS10, Section 3.7].

\[ \textbf{Proof.} \] This is essentially the same proof as in Lemma 3.37 of [GPS17] (which in turn is due to Abouzaid-Seidel’s unpublished work). The main difference is that—because of our counterclockwise orientation of the boundary of disks—our continuation maps behave covariantly, as opposed to contravariantly.

Let \( Y^{(0)} \to \ldots \) denote a cofinal sequence (Definition 3.11.5). We first claim that if \( c : Q \to Q' \) is a map in \( C \) (in particular, \( Q \) and \( Q' \) are in the same fiber), then the map

\[ \text{hocolim}_i \text{hom}_{O_j}(Q',Y^{(i)}) \to \text{hocolim}_i \text{hom}_{O_j}(Q,Y^{(i)}) \]  

is a quasi-isomorphism. To verify this claim, note that Lemma 7.4.1 shows that \( \mu^2 \) is respected at the level of cohomology, so we have induced maps of cohomology groups

\[ H^*\text{hocolim}_i \text{hom}_{O_j}(Q',Y^{(i)}) \cong \text{colim}_i H^*\text{hom}_{O_j}(Q',Y^{(i)}) \]

\[ \to \text{colim}_i H^*\text{hom}_{O_j}(Q,Y^{(i)}) \]

\[ \cong H^*\text{hocolim}_i \text{hom}_{O_j}(Q,Y^{(i)}) \].

There are isomorphisms in the lines above—this is because a filtered colimit of cohomology is naturally isomorphic to the cohomology of a filtered homotopy colimit. Now we note that the colimit of the cohomology groups is a definition for wrapped Floer cohomology (in the fiber containing \( Q \) and \( Y \)) employed in Abouzaid-Seidel’s paper [AS10, Section 3.7]. Moreover, in this definition of the wrapped Fukaya category, \( Q \) and \( Q' \) are exhibited as equivalent objects by a continuation map (this is proven in [BKO19]). So the arrow (9.2) is an isomorphism of groups. This shows that (9.1) is a quasi-isomorphism.

Because (9.1) is a quasi-isomorphism for any choice of \( Y^{(i)} \) and any choice of \( c : Q \to Q' \) in \( C \), a general fact about localizations (Lemma 2.4.34) shows that the natural map

\[ \text{hocolim}_i \text{hom}_{O_j}(X,Y^{(i)}) \to \text{hocolim}_i \text{hom}_{O_j|C^{-1}}(X,Y^{(i)}) \]
is an equivalence for any $X$. On the other hand, the latter homotopy colimit is indexed by a sequence of quasi-isomorphisms, because the maps $Y(i) \rightarrow Y(i+1)$ are already in $C$, being non-negative continuation maps. Thus we have

$$\text{colim}_i H^* \text{hom}_{\mathcal{O}_j}(X, Y(i)) \simeq \text{colim}_i H^* \text{hom}_{\mathcal{O}_j[D^{-1}]}(X, Y(i))$$

while the left-hand side is the colimit definition of wrapped Floer cohomology. This completes the proof.

9.3 Wrapped Functoriality

We first note that $\alpha^*$ (Construction 6.4.1) respects $C$ (Definition 9.1.1). That is, consider a diagram

$$\Delta^n \xrightarrow{\alpha} \Delta^{n'} \xrightarrow{j'} \Delta^n \xleftarrow{j} B,$$

where $\alpha$ is induced by an order-preserving injection $[n] \rightarrow [n']$. By definition, the continuation maps $C$ of $\mathcal{O}_j$ are sent to (some of the) continuation maps $C'$ of $\mathcal{O}_{j'}$, so we have an induced functor on the localizations

$$\alpha^*: W_j \rightarrow W_{j'}.$$ (9.3)

(Note that we have abused notation by using $\alpha^*$ again.)

Moreover, because $\mathcal{O}: \text{Simp}(B) \rightarrow A\infty\text{Cat}$ is a functor respecting each $C$, the naturality of localizations (Proposition 2.4.25) implies the following:

**Proposition 9.3.1.** The assignment $j \mapsto W_j$ and $\alpha \mapsto \alpha^*$ induces a functor of $\infty$-categories

$$W: N(\text{Simp}(B)) \rightarrow \text{Cat}_{A\infty}.$$  

**Remark 9.3.2.** Note that the target of $W$ is the $\infty$-category $\text{Cat}_{A\infty}$, rather than the (strict) category $A\infty\text{Cat}$. Indeed, the former is where we can articulate the universal property of localizations; see Section 2.

**Proposition 9.3.3** (Local triviality). The map $\alpha^*: W_j \rightarrow W_{j'}$ in (9.3) is an equivalence of $A\infty$-categories.

**Proof.** We note that it suffices to prove the proposition when $j: |\Delta^0| \rightarrow B$ is a 0-simplex in $B$.

It follows that $\alpha^*$ is essentially surjective from Lemma 9.1.2; in fact the proof shows directly that in $W_j$, a brane contained in a fiber above the $i$th vertex of $|\Delta^n|$ is—for any choice of $k \in \{0, \ldots, n\}$—equivalent to a brane above the $k$th vertex. Now note that $\alpha^*: W_j \rightarrow W_{j'}$ is obviously essentially surjective on branes in the fibers above vertices in the image of $[n] \rightarrow [n']$.

Now we must prove that the map is fully faithful (i.e., induces a quasi-isomorphism on morphism complexes). This follows from Lemma 9.2.2, as $\alpha$ induces the obvious identity map on wrapped Floer cohomology groups.

9.4 Proof of Theorem 1.3.2

Now we are ready to complete the proof of Theorem 1.3.2.
Proof of Theorem 1.3.2. By Proposition 9.3.1, we have a functor 

\[ \text{Simp}(B \text{Aut}(M)) \to A_\infty \text{Cat}. \]

Moreover, by Proposition 9.3.3, every edge in the domain is sent to an equivalence of \(A_\infty\)-categories; hence by the universal property of localization of \(\infty\)-categories, we have an induced diagram \(\Delta^2 \to \text{Cat}_\infty\) as follows:

\[
\begin{array}{ccc}
N(\text{Simp}(B \text{Aut}(M))) & \xrightarrow{W} & \text{Cat}_{A_\infty} \\
\downarrow & & \downarrow \\
\text{Sing}(B \text{Aut}(M)) & & \\
\end{array}
\]

(Informally, the above is a homotopy-commutative diagram of functors between \(\infty\)-categories.) Here, the left vertical map is the localization map along every edge of \(N(\text{Simp}(B \text{Aut}(M)))\). (We are using Theorem 4.8.5 to identify the localization with \(\text{Sing}(B \text{Aut}(M))\).) The dashed arrow is the induced map on localizations, and the map we seek, because of the obvious equivalence 

\[ B \text{Aut}(M) \simeq \text{Sing}(B \text{Aut}(M)). \]

This completes the proof. \(\square\)
10 Local systems and a bundle version of the Abouzaid map

Fix a Liouville domain $M$ and a compact brane $Q \subset M$. In [Abo12], Abouzaid constructed a functor from the quadratically wrapped category of $M$ to the category of local systems on $Q$. In this section we construct a bundle version of this functor for the non-wrapped Fukaya categories $\mathcal{O}$.

**Remark 10.0.1.** For concreteness, we construct this functor for the case of $M = T^*Q$; but one can do this for any Liouville bundle whose structure group has been reduced in such a way as to preserve the set $Q \subset M$.

To state the bundle version of the Abouzaid map, we first fix a smooth fiber bundle $E' \to B'$ with fiber given by a smooth, compact manifold $Q$. (The fiber may be a manifold with boundary). To this data we assign a collection of local system categories—specifically, to each $j : |\Delta^n| \to B'$, we also associate a dg-category $P_j$ whose twisted complex category is quasi-equivalent to a category of local systems on $j^*E'$. In particular, the assignment $j \mapsto TwP_j$ is a local system of local system categories. Our main interest in this construction is the universal example when $B' = B\text{Diff}(Q)$.

Setting $M = T^*Q$, we again utilize the result that the localization of a subdivision recovers the original homotopy type. (See Theorems 4.4.4 and 4.4.6.) This implies the existence of a functor of $\infty$-categories

$$TwP : BDiff(Q) \to CatA_{\infty}.$$ 

**Remark 10.0.2.** Note that here, the target is $CatA_{\infty}$, as opposed to $A_{\infty}Cat$ from Proposition 6.4.2. This is because the local constancy conditions above allow us to map into an $\infty$-category in which weak equivalences have been inverted; this is precisely the role of $CatA_{\infty}$. We refer the reader to Section 2.3 for more on $CatA_{\infty}$.

Then, the main result of the present section is to show that one can make the diagram

$$\begin{array}{ccc}
BDiff(Q) & \xrightarrow{Twp} & B\text{Aut}^{gr,b}(T^*Q) \\
\downarrow & & \downarrow W \\
CatA_{\infty} & \xleftarrow{\text{W}} & BDiff(Q)
\end{array}$$

commute up to a natural transformation from $W$ to $Twp$ (Corollary 10.3.3). That is, we have a $Diff(Q)$-equivariant functor from $W(T^*Q)$ to $Twp(Q)$.

To accomplish this goal, we first construct a natural transformation from $\mathcal{O}$ to $TwP$ (Proposition 10.2.7). Then, the main aim is to prove that the non-negative continuation maps in $\mathcal{O}$ are sent to equivalences in the category $TwC_*P$ of local systems (Theorem 10.3.1)—this is the most geometrically involved component of our arguments. By the universal property of the localization $W$, we conclude that the bundle version of the Abouzaid functor descends to the wrapped categories (Corollary 10.3.3).

We will prove in the next section that this natural transformation is a natural equivalence—i.e., that this is a $Diff(Q)$-equivariant equivalence (Theorem 11.0.1).

10.1 $C_*P$ (families of local system categories)

**Remark 10.1.1.** As in Section 4, we can endow the topological space $B\text{Diff}(Q)$ with a diffeological space structure. Because $Q$ is compact, the same techniques there allow us to see that $B\text{Diff}(Q)$ is a Kan complex, and that it satisfies smooth approximation. So, for example, if $Sing(B\text{Diff}(Q))$ denotes the usual singular complex of continuous simplices $|\Delta^n| \to B\text{Diff}(Q)$, and
if \( \text{Sing}^{C_{\infty}}(B\text{Diff}(Q)) \) denote the simplicial set whose \( n \)-simplices are smooth maps \( j : |\Delta^n_e| \to B\text{Diff}(Q) \) from extended simplices, then the inclusion of \( \text{Sing}^{C_{\infty}}(B\text{Diff}(Q)) \) to \( \text{Sing}(B\text{Diff}(Q)) \) is a homotopy equivalence of simplicial sets (Theorem 4.4.6).

**Notation 10.1.2** \((E_Q)\). Note that \( B\text{Diff}(Q) \) carries a principle \( \text{Diff}(Q) \) bundle—the universal one—and hence an associated fiber bundle with fibers \( Q \). We call this fiber bundle \( E_Q \to B\text{Diff}(Q) \).

**Notation 10.1.3** \((Q_a)\). Let \( j : |\Delta^n| \to B\text{Diff}(Q) \) be a smooth map. For any \( a \in [n] \), we let \( Q_a \) denote the fiber of \( j^*E_Q \) above the \( a \)-th vertex of \( |\Delta^n| \subset |\Delta^{|n|}_e| \).

Now we give some notation to the Moore path space category modeling the \( \infty \)-groupoid \( \text{Sing}(j^*E_Q) \).

**Construction 10.1.4** \((P_j)\). Let \( j : |\Delta^n| \to B\text{Diff} \) be a smooth map. We let \( P_j \) denote the topologically enriched category defined as follows:

We declare the object set to be the disjoint union of fibers

\[
P_j := \bigsqcup_{a \in [n]} Q_a.
\]

Given \( q_a \in Q_a, q_b \in Q_b \), we declare \( \text{hom}_{P_j}(q_a, q_b) \) to be the topological space of continuous maps \( \gamma : [0, \infty] \to j^*E_Q \) such that \( \gamma \) is compactly supported (i.e., constant beyond some finite time \( t \in [0, \infty] \)) and such that \( \gamma(0) = q_a \) and \( \gamma(\infty) = q_b \). Composition is defined in the obvious way: If \( t_\gamma \) is the smallest time for which \( \gamma \) is constant, \( \gamma' \circ \gamma \) is defined by setting

\[
(\gamma' \circ \gamma)(t) = \begin{cases} 
\gamma(t) & t \leq t_0 \\
\gamma'(t - t_0) & t \geq t_0.
\end{cases}
\]

**Notation 10.1.5.** Construction 10.1.4 defines a functor

\[
P : \text{Simp}(B\text{Diff}(Q)) \to \text{Cat}^{\text{Top}}, \quad j \mapsto P_j
\]

which we denote (as indicated) by \( P \). (It has the obvious effect on morphisms.) Here, \( \text{Cat}^{\text{Top}} \) is the category of categories enriched in topological spaces. For the notation \( \text{Simp}(B\text{Diff}(Q)) \), see Notation 4.4.1.

**Remark 10.1.6.** Given any map \( |\Delta^n_e| \to |\Delta^{|n|}_e| \to B\text{Diff}(Q) \), the induced map \( P_j \to P_{j'} \) is an equivalence\(^{25}\) of topologically enriched categories; this follows by noting that the inclusion \( j^*E_Q \to (j')^*E_Q \) is a homotopy equivalence.

**Notation 10.1.7** \((\text{T}wC_sP)\). Recall that the functor \( C_s \) sending a topological space \( P \) to its singular chain complex \( C_sP \) is lax monoidal. As a result, applying \( C_s \) to the morphism spaces to a topologically enriched category \( D \), we obtain a dg-category \( C_sD \).

We let \( C_sP_j \) denote the dg-category associated to \( P_j \). We denote the composite functor

\[
C_sP : \text{Simp}(B\text{Diff}(Q)) \xrightarrow{P} \text{Cat}^{\text{Top}} \xrightarrow{C_s} \text{dgCat}.
\]

We also denote by \( \text{T}wC_sP \) the composite of \( C_sP \) with the \( \text{T}w \) functor (Notation 2.4.5).

\(^{25}\)By an equivalence of topologically enriched categories, we mean an essentially surjective functors whose maps on morphism spaces are weak homotopy equivalences.
10.2 From Fukaya categories to local systems (the bundled Abouzaid functor)

Given any diffeomorphism \( \phi : Q \to Q \), one has an induced exact symplectomorphism \( D\phi : T^*Q \to T^*Q \) by pushing forward the effect of \( \phi \) on cotangent vectors. Because this clearly respects the diffeological smooth structures, and there exist natural lifts of \( D \) respecting the choices of \( gr \) and \( b = w_2(Q) \), we have an induced functor

\[
D : \text{Simp}(B\text{Diff}(Q)) \to \text{Simp}(B\text{Aut}_{gr,b}^{gr}(T^*Q)).
\]

Our present goal is to construct a natural transformation from \( O \circ D \) to \( \text{Tw}_{C^*P} \) (Proposition 10.2.7). We do so by utilizing an non-wrapped, family-friendly version of a construction we learned from Abouzaid’s paper [Abo12].

Remark 10.2.1. In [Abo12], the author constructs a functor whose domain is a quadratically wrapped Fukaya category—i.e., one whose morphisms are defined by using quadratic Hamiltonians. Here, we instead use our non-wrapped Fukaya categories \( O_j \) as the domain. Aside from this detail, the main ideas remain unchanged—in particular, the analytic input for the functor utilized in our present work is somewhat simpler.

Remark 10.2.2. The construction from [Abo12, Section 4.1] begins with a Liouville manifold \( M \) and a compact brane \( Q \subset M \) to define a functor from the quadratically wrapped Fukaya category of \( M \) to the category of local systems on \( Q \). The same construction carries through successfully when \( M \) is a Liouville sector (not necessarily a Liouville manifold). Below, we adapt the construction for our purposes. For the interested reader, let us remark that our bundle-version of the construction carries over to a setting in which the structure group of Aut(\( M \)) is reduced in such a way that every fiber admits a distinguished brane \( Q \); such is the case we are in, as Diff(\( Q \)) preserves the zero section of \( T^*Q \).

Construction 10.2.3 (The Abouzaid functor on objects). Fix a smooth map \( j : |\Delta^n| \to B\text{Diff}(Q) \) and consider the associated fibration \( P_j = j^*B\text{Diff}(Q) \to |\Delta^n| \). For an integer \( a \in \{0, \ldots, n\} \), let \( M_a \) be the fiber above the \( a \)th vertex of \( |\Delta^n| \), and let \( L_a \subset M_a \) be a brane. We assume \( L_a \) intersects the zero section \( Q_a \subset M_a \) transversally. To \( L_a \) we associate the following object of \( \text{Tw}_{C^*P}(Q_a) \):

\[
\left( \bigoplus_{x_a \in L_a \cap Q_a} x_a[-|x_a|], D \right).
\]

Let us explain the differential \( D \).

Given two intersection points \( x_a, x'_a \in L_a \cap Q_a \), we let

\[
\overline{\mathcal{H}}(x_a, x'_a)
\]

be the compactified moduli space of (possibly broken) holomorphic sections

\[
u : \mathbb{R} \times [0, 1] \to \mathbb{R} \times [0, 1] \times M_a
\]

with boundary on \( L_a \) and \( Q_a \), converging to \( x_a \) and \( x'_a \). Given a map \( u \), consider the restriction of \( u \) to the boundary line \( \mathbb{R} \times \{0\} \subset \mathbb{R} \times [0, 1] \) mapping to \( Q_a \). The resulting map \( \mathbb{R} \to M_a \) admits an arc-length parametrization (we use the Riemannian metric induced by the choice of almost-complex structure on \( M_a \)), and in particular, any \( u \) determines an arc-length parametrized path in \( Q_a \). This

\[\text{The functor } O : \text{Simp}(B\text{Aut}_{gr,b}^{gr}(T^*Q)) \to A_\infty\text{Cat} \text{ is from Proposition 6.4.2.}\]
induces a map from \( \mathcal{F}(x_a, x'_a) \) to the space of paths in \( Q_a \) from \( x_a \) to \( x'_a \), and in particular a map of chain complexes
\[
C_*\mathcal{F}(x_a, x'_a) \xrightarrow{\partial} \text{hom}_{\mathcal{F}}(x_a, x'_a).
\]
Then, one chooses fundamental chain classes for \( \mathcal{F} \), and pushes them forward. These elements of the hom-complexes of \( C_*\mathcal{F} \) determine the differential \( D \). We refer the reader to [Abo12, (2.27)] for details.

**Construction 10.2.4** (The rest of the Abouzaid functor). We now explain how the above assignment on objects extends to an \( A_\infty \)-functor \( \mathcal{O}_j \to \text{Tw} C_*\mathcal{F}_j \).

Fix an integer \( d \geq 1 \). For each \( i \in \{0, 1, \ldots, d\} \), choose also an integer \( a_i \in \{0, \ldots, n\} \) and a brane \( L_{a_i} \) above the \( a_i \)th vertex. The choices of \( a_i \) determine a unique simplicial map from the \( d \)-simplex to the \( n \)-simplex (by sending the \( i \)th vertex to the \( a_i \)th vertex). Let us take a cone on this map—specifically, the map
\[
|\Delta^{d+1}| \to |\Delta^n|, \quad \begin{cases} i \mapsto a_i & i \in \{0, \ldots, d\}, \\ i \mapsto a_d & i = d + 1. \end{cases}
\]
Given generators \( y_{i,i+1} \in \text{hom}_{\mathcal{O}_j}(L_{a_i}, L_{a_{i+1}}) \), we let
\[
\mathcal{H}(y_{0,1}, \ldots, y_{(d-1),d}; Q_{a_d})
\]
denote the moduli space of holomorphic sections \( u \)
\[
j^*E|_{\mathcal{S}_r} \xrightarrow{u} j^*E
\]
\[
\downarrow
\]
\[
\mathcal{S}_r \subset \mathcal{S}_{d+1+1} \xrightarrow{|\Delta^{d+1}|} |\Delta^d| \xrightarrow{|\Delta^n|}
\]
satisfying the following boundary conditions: On the strip like ends from the \( i \)th arc to the \((i+1)\)st arc, \( u \) converges to the parallel transport arc given by \( y_{i,i+1} \), while the \((d+1)\)st boundary arc—from the \((d+1)\)st puncture to the 0th puncture—is constrained by the parallel transport boundary condition from \( Q_{a_d} \) to \( Q_{a_0} \).

Given such a \( u \), one can measure the arc length of the restriction of \( u \) to the \((d+1)\)st boundary arc. (For example, by taking its *vertical* velocity; i.e., by using the fiberwise Riemannian metrics.) By the assumption of general position, an intersection \( x_0 \in L_{a_0} \cap Q_{a_0} \) and \( x_d \in L_{a_d} \cap Q_{a_d} \) are not related by parallel transport—we do not have triple intersections. So this guarantees that the vertically measured arc length is non-zero. (We point this out, as if the vertical velocity were zero the arc length parameterization would not be defined, and hence we would not have a map to \( \text{hom}_{\mathcal{F}} \).)

As before we have a map from \( \mathcal{H} \) to the space of paths in \( j^*E \) from \( Q_{a_0} \) to \( Q_{a_d} \), and this extends continuously to the compactification \( \mathcal{F} \). Pushing forward fundamental chains, we obtain the desired \( A_\infty \) functor maps.

**Remark 10.2.5.** We note one subtlety in the construction; it is somewhat unnatural to utilize the cone map \( |\Delta^{d+1}| \to |\Delta^n| \), as then verifying the \( A_\infty \) functor relations forces us to use (for example) the fact that for \( k \leq d \), the moduli space
\[
\mathcal{H}(y_{0,1}, \ldots, y_{(k-1),k}; Q_{a_d})
\]
is cobordant to the moduli space
\[
\mathcal{H}(y_{0,1}, \ldots, y_{(k-1),k}; Q_{a_k})
\]
so that the usual compactification of the moduli spaces yield the desired \( A_\infty \) functor relations.
Notation 10.2.6 (The Abouzaid functor \(\mathcal{F}\)). We will denote by 
\[ \mathcal{F} \]
the \(A_\infty\) functor defined in Constructions 10.2.3 and 10.2.4. We may write \(\mathcal{F}_j\) to denote the dependence on the simplex \(j\), but will largely leave this dependence implicit.

This construction clearly respects inclusions \(\Delta^n \hookrightarrow \Delta^n'\). As such, we have:

**Proposition 10.2.7.** The non-wrapped Abouzaid construction \(\mathcal{F}\) induces a natural transformation 
\[
\text{Simp}(B\text{Diff}(Q)) \xrightarrow{\mathcal{O} \circ \mathcal{D}} \text{Tw}_C^* \mathcal{P} \xrightarrow{\mathcal{W} \circ \mathcal{D}} A_\infty\text{Cat}.
\]

10.3 The Abouzaid functor descends to the wrapped category

Proposition 10.2.7 constructs a map from the non-wrapped, directed family of Fukaya categories to families of local system categories. To descend our construction to the wrapped setting, the main geometric result we must verify is the following.

**Theorem 10.3.1.** Let \(M\) be a Liouville sector, and \(c : L \rightarrow L'\) a continuation element associated to a positive isotopy. We also fix a compact test brane \(X \subset M\). Then the map on twisted complexes

\[ c_* : (X \cap L, D) \rightarrow (X \cap L', D') \]

—induced by the Abouzaid functor from \(\mathcal{O}(M)\) to \(\text{Tw}_C^* \mathcal{P}(X)\)—is an equivalence.

The theorem immediately implies:

**Corollary 10.3.2.** Let \(M = T^*Q\) and \(X = Q\). Then the map on twisted complexes

\[ c_* : (Q \cap L, D) \rightarrow (Q \cap L', D') \]

—induced by the Abouzaid functor from \(\mathcal{O}(M)\) to \(\text{Tw}_C^* \mathcal{P}(X)\)

By the universal property of localization and Lemma 9.1.2, we also conclude:

**Corollary 10.3.3.** The natural transformation from Proposition 10.2.7 induces a natural transformation from \(\mathcal{W} \circ \mathcal{D}\) to \(\text{Tw}_C^* \mathcal{P}\):

\[
\text{Simp}(B\text{Diff}(Q)) \xrightarrow{\mathcal{W} \circ \mathcal{D}} \text{Cat}_{A_\infty}
\]

By the universal property of \(\text{Tw}\), this in turn induces a natural transformation \(\text{Tw}\mathcal{W} \circ \mathcal{D} \rightarrow \text{Tw}_C^* \mathcal{P}\).

**Remark 10.3.4.** The functors \(\mathcal{W}\) and \(C_* \mathcal{P}\) both send morphisms of \(\text{Simp}(B\text{Diff}(Q))\) to equivalences in \(\text{Cat}_{A_\infty}\). Thus they both induce a functor

\[ \text{Sing}(B\text{Diff}(Q)) \simeq \mathbb{B}\text{Diff}(Q) \rightarrow \text{Cat}_{A_\infty} \]

by Theorem 4.4.6. In particular, these exhibit \(\text{Diff}(Q)\) actions on both \(\mathcal{W}(T^*Q)\) and \(C_* \mathcal{P}(Q)\). Corollary 10.3.3 says that the map \(\text{Tw}\mathcal{W}(T^*Q) \rightarrow \text{Tw}_C^* \mathcal{P}(Q)\) is \(\text{Diff}(Q)\)-equivariant.
10.3.1 Toward a proof of Theorem 10.3.1

Because Theorem 10.3.1 is a statement contained in single fiber of a Liouville bundle (and in particular, is a statement about a single Liouville sector), we now work in a Liouville sector $M$. We will also fix a compact test brane $X \subset M$; this $X$ plays the role of the zero section $Q$ above. For simplicity of exposition (and without a loss of generality), we will assume that $X$ is connected.

We fix a non-negative Hamiltonian isotopy

$$L(t) := \phi^t_F(L), \quad L(0) = L, \quad L(1) = L'$$

so that $L$ and $L'$ are transverse to $X$. By definition, $X_F = \beta Z$ for some constant $\beta \geq 0$ (Definition 3.11.4).

Let $c : L \to L'$ be a continuation map induced by a Hamiltonian isotopy generated by a positive linear Hamiltonian $F$ such that $F(r, y) = \lambda r$ on the cylindrical region of $L, L'$. We can construct $c_* : (X \cap L, D) \to (X \cap L', D')$ by considering the $A_\infty$-functor $\mathcal{F}$ whose linear term $\mathcal{F}^1 : CF^*(L, L') \to TwC_*\mathcal{P}(X)$ is given by

$$\mathcal{F}^1([y]) = \bigoplus_{x, x'} (-1)^{|y|+|x|+1} x_1, x_0 (|y|)|x_0| x_1, x_0$$

for $y \in X(L, L')$ and $x, x' \in X \cap L, x' \in X \cap L'$. (This is Construction 10.2.4; see [Abo12, (2.27)] for more details on this formula.) (Here, $[\mathcal{F}(x, y, x')]$ is a choice of fundamental class for $\mathcal{F}(x, y, x')$.)

As before, we denote by $c^\chi(L) \in CF(L, L')$ the continuation element from Construction 7.1.8. We must prove that $\mathcal{F}^1(c^\chi(L))$ is an equivalence in $TwC_*\mathcal{P}(X)$. By the definition (7.5) of $c^\chi(L)$, we have

$$\mathcal{F}^1(c^\chi(L)) = \mathcal{F}^1 \left( \sum_{y \in L_0 \cap L_1; |y| = 0} n_2^\chi(y)\langle y \rangle \right)$$

$$= \sum_{y \in L_0 \cap L_1; |y| = 0} x, x' \bigoplus (-1)^{|y|+|x|+1} n_2^\chi(y) \mathcal{F}_x^\chi([F(x, y, x')])$$

$$= \bigoplus_{x, x'} (-1)^{|y|+|x|+1} x_0 (|y|)|x_0| x_1, x_0 \mathcal{F}_x^\chi \left( \sum_{y \in L_0 \cap L_1; |y| = 0} n_2^\chi(y) [F(x, y, x')] \right) .$$

On the other hand, we have

$$\mathcal{F}_x^\chi \left( \sum_{y \in L_0 \cap L_1; |y| = 0} n_2^\chi(y) [F(x, y, x')] \right) = \mathcal{F}_x^\chi \left( \sum_{y \in L_0 \cap L_1; |y| = 0} [M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, y) \# [F(x, y, x')] \right) .$$

(See Figure 10.3.6.)

Let us explain the notation in the above lines. We defined $M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, y)$ in (7.3). We let

$$M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, y) \# \mathcal{F}(x, y, x') \equiv M(D^2 \setminus \{z_0\}; \mathcal{L}^\chi, y)_{ev_{z_0}} \times ev_{z_0} \mathcal{F}(x, y, x').$$

Notation 10.3.5 ($\#$ and $\#_+$. We denote by $\Theta_-$ and $Z$ the domains (equipped with the strip-like coordinates) of the two relevant moduli spaces, and denote by $\Theta_+ \# Z$ the nodal curve obtained by the obvious grafting.
Figure 10.3.6. The $\mathcal{F}^1$ term of the Abouzaid functor applied to a continuation element.

Note we have conformal equivalences $\Theta_\sim D^2 \{z_0\}$ and $Z \sim D^2 \{z_0, z_1, z_2\}$. Using a given strip-like coordinate around $\{(0, 1)\}$ in $Z \{\{0, 1\}\}$, we construct one-parameter family of the glued domains which we denote by $\Theta_{\sim} #_r Z$

for all sufficiently large $r > 0$.

**Notation 10.3.7.** We denote:

- The lower semi-disc by
  $$D^2_{\sim} = \{z \in D^2 \mid \text{Im } z \leq 0\}$$

- by
  $$\partial_{+} D^2_{\sim} \cong [-1, 1]$$

  the part of the boundary of $\partial D^2_{\sim}$ with $\text{Im } z = 0$, and

- we write
  $$\partial_{-} D^2_{\sim} = \partial D^2_{\sim} \setminus \text{Int } \partial_{+} D^2_{\sim} = \{|z| = 1 \& \text{Im } z \leq 0\}$$

  where $\text{Int } \partial_{+} D^2_{\sim} \cong (-1, 1)$ under the above identification.

By gluing the defining equations of the two moduli spaces on the glued domain $\Theta_{\sim} #_r Z$ for all sufficiently large $r > 0$ and then deforming the domain to the lower semi-disc $D^2_{\sim}$ and adjusting the positively moving boundary condition accordingly, we consider the equation

$$\begin{cases}
\bar{\partial} J_u = 0, \\
\int_{D^2_\sim} |du|^2 < \infty
\end{cases}$$

$$u(z) \in X \text{ for } z \in \partial_{-} D^2_{\sim}$$

$$u(z) \in L_{\chi(z)} \text{ for } z \in \partial_{+} D^2_{\sim}$$

where $\chi : \partial_{+} D^2_{\sim} \rightarrow [0, 1]$ is a monotone function such that $\chi(z) = 1$ and $\chi(z) = 0$ near $-1, 1 \in \partial_{+} D^2_{\sim}$ respectively.

Then by the standard gluing-deformation and compactness argument, we prove that the chain $\mathcal{F}^1(c^\chi(\mathcal{L}))$ is homotopic to

$$\text{ev}_* ([M(D^2 \{z_0\}; \mathcal{L}^\chi)])$$

where $M(D^2 \{z_0\}; \mathcal{L}^\chi, y)$ is the zero-dimensional moduli space of the equation (10.2). Therefore it remains to examine the moduli space of solutions to (10.2) for the given isotopy $\mathcal{L}$. 

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Remark 10.3.8. Note that the moduli space of solutions to (10.2) is equivalent to the moduli space defining the continuation map \( h_L \) (Construction 7.2.5). This is because under the conformal equivalence \( D^2 \setminus \{1, -1\} \cong \mathbb{R} \times [0, 1] \), the equation (10.2) is equivalent to the equation (7.7) after a suitable choice of nondecreasing elongation function \( \rho : \mathbb{R} \to [0, 1] \) satisfying

\[
\rho(\tau) = \begin{cases} 
1 & \text{for } \tau \geq a \\
0 & \text{for } \tau \leq 0 
\end{cases}
\]  

for some \( a \), just as in (7.6). (See also Remark 7.3.2 for the relationship between \( \chi \) and \( \rho \).)

By assumption on the isotopy, the isotopy is non-negative outside a compact set \( K \). By compactness of \( X \), we may enlarge \( K \) so that \( X \subset \text{Int}K \) without loss of generality. This allows us to apply the strong maximum principle to derive

\[
\text{image } u \subset K. \tag{10.4}
\]

(See [BKO19, section 13] for similar application of strong maximum principle.) Next we need to establish a uniform energy bound. Let \( h_X : X \to \mathbb{R} \) and \( h_s : L_s \to \mathbb{R} \) be Liouville primitives of \( X \) and \( L_s \) respectively. (So for example, \( dh_X = \lambda |_X \).) For a given exact Lagrangian isotopy generated by a Hamiltonian \( H \), a smooth family of Liouville primitives of \( L_s \) is given by

\[
h_s = h_0 + \int_0^s (\langle \lambda, X_H \rangle - H) \circ \phi^t_H \ dt. \tag{10.5}
\]

(See [Oh15a, Proposition 3.4.8], for example.) Then we have the following energy identity.

Lemma 10.3.9. Let \( \mathcal{L} \) be a nonnegative isotopy (near infinity) and \( H \) be a Hamiltonian generating the isotopy \( \mathcal{L} \). Then for any finite energy solution \( u \) of (7.7) with \( u(\pm \infty) = x_{\pm} \), we have

\[
E_J(u) = -(h_X(x_+) - h_X(x_-)) + (h_0(x_+) - h_1(x_-)) - \int_{-\infty}^{\infty} \rho'(\tau) H(u(1 - \tau, 1)) d\tau. \tag{10.6}
\]

where \( x_- \in X \cap L_1 \) and \( x_+ \in X \cap L_0 \). In particular, we have

\[
E_J(u) \leq C(X, \mathcal{L}, H; K) \tag{10.7}
\]

where \( C(X, \mathcal{L}, H; K) \) is some constant depending only on \( X, \mathcal{L}, H \) and \( K \).

Proof. While what follows mostly parallels the proof of [AOOdS18, Lemma 7.3], there are two differences:

- The current case treats the general case of Lagrangian branes in general Liouville sectors, while [AOOdS18, Lemma 7.3] treats the case of the zero section \( X = Q \) and the fiber \( L = 0 = T^*_qQ \) for which we can take \( h_X = 0 = h_0 \). Because of this, the statement of the current lemma is more general including [AOOdS18, Lemma 7.3] as a special case.

- There are differences in the details of the proof due to differences in conventions.

Because of these, we include a complete proof here for the reader’s convenience.

Consider the following family of the action functional

\[
A_{X, L_s}(\gamma) = -\int_{[0, 1]} \gamma^* \lambda + h_X(\gamma(1)) - h_s(\gamma(0)), \tag{10.8}
\]
where $\gamma \in \mathcal{P}(X, L_s)$. Obviously, we have

$$\mathcal{A}_{X, L_0}(x_+) - \mathcal{A}_{X, L_1}(x_-) = \int_{-\infty}^{\infty} \frac{d}{d\tau} \left( \mathcal{A}_{X, L_{\rho(1-\tau)}}(u(\tau)) \right) d\tau.$$  

We derive

$$\frac{d}{d\tau}(\mathcal{A}_{X, L_{\rho(1-\tau)}}(u(\tau))) = \frac{d}{d\tau} \left( \int_{[0,1]} -(u(\tau))^{*}\lambda - h_{X}(u(\tau, 0)) \right) - \frac{d}{d\tau} \left( h_{\rho(1-\tau)}(u(\tau, 1)) \right)$$

using the first variation of the action functional with free boundary condition (see [Oh15b, Equation (12.1.1)]), together with the fact that $u$ is a solution of (10.2). Therefore by combining the above calculations and integrating over $-\infty < \tau < \infty$, we have obtained

$$\mathcal{A}_{X, L_0}(x_+) - \mathcal{A}_{X, L_1}(x_-) = \mathcal{E}_J(u) - \int_{-\infty}^{\infty} \left\langle \lambda, \frac{\partial u}{\partial \tau}(\tau, 1) \right\rangle d\tau - (h_1(x_-) - h_0(x_+)).$$

Noticing that the boundary condition $u(\tau, 1) \in L_{\rho(1-\tau)}$ implies that $u(\tau, 1) = \phi_{H}^{\rho(1-\tau)}(v(\tau))$ for some curve $v(\tau) \in L_0$, we can write

$$h_{\rho(1-\tau)}(u(\tau, 1)) = \tilde{h}_{\rho(1-\tau)}(v(\tau))$$

for $\tilde{h}_{\rho(1-\tau)} = h_{\rho(1-\tau)} \circ \phi_{H}^{\rho(1-\tau)} : L_0 \to \mathbb{R}$. Then we compute

$$\frac{d}{d\tau}(h_{\rho(1-\tau)}(u(\tau, 1))) = d\tilde{h}_{\rho(1-\tau)} \left( \frac{dv}{d\tau} \right) - \rho'(1-\tau) \left. \frac{d\tilde{h}_s}{ds} \right|_{s=\rho(1-\tau)}(v(\tau))$$

It follows from the definitions that

$$\frac{\partial u}{\partial \tau}(\tau, 1) = d\phi_{H}^{\rho(1-\tau)} \left( \frac{dv}{d\tau} \right) - \rho'(1-\tau)X_H(u(\tau, 1)).$$

Plugging this in the previous equation and using the definition of Liouville primitive we obtain

$$\frac{d}{d\tau}(h_{\rho(1-\tau)}(u(\tau, 1))) = \left\langle \lambda, \frac{\partial u}{\partial \tau}(\tau, 1) \right\rangle + \rho'(1-\tau)H(u(\tau, 1)).$$

Substituting this into (10.9), we obtain

$$\frac{d}{d\tau}(\mathcal{A}_{H, \rho(1-\tau)}(u(\tau))) = \int_{[0,1]} \left| \frac{\partial u}{\partial t} \right|^2 dt - \rho'(1-\tau)H(u(\tau, 1))$$

which is equivalent to

$$\int_{[0,1]} \left| \frac{\partial u}{\partial t} \right|^2 dt = \frac{d}{d\tau}(\mathcal{A}_{H, \rho(1-\tau)}(u(\tau))) + \rho'(1-\tau)H(u(\tau, 1)).$$
By integrating this over $\tau \in \mathbb{R}$, we obtain
\[
E_J(u) = A_{X,L_1}(x_-) - A_{X,L_0}(x_+) + \int_{-\infty}^{\infty} \rho'(1 - \tau)H(u(\tau, 1)) \, d\tau
\]
\[
= A_{X,L_1}(x_-) - A_{X,L_0}(x_+) - \int_{-\infty}^{\infty} \rho'(\tau)H(u(1 - \tau, 1)) \, d\tau.
\]
Then we evaluate
\[
A_{X,L_1}(x_-) - A_{X,L_0}(x_+) = - (h_1(x_+) - h_X(x_+)) + (h_0(x_-) - h_X(x_-))
\]
\[
= (h_X(x_+) - h_X(x_-)) - (h_0(x_+) - h_1(x_-)).
\]
Combining these with the energy identity (10.6), we derive
\[
\int_{-\infty}^{\infty} \rho'(\tau)H(u(1 - \tau, 1)) \, d\tau \leq \int_{-\infty}^{\infty} \rho'(\tau)|H(u(1 - \tau, 1))| \, d\tau \leq \|H\|_K.
\]
Combining these with the energy identity (10.6), we derive
\[
E_J(u) \leq \max_K |\langle \lambda, X_H \rangle| + 2\|H\|_K + (\max h_X - \min h_X) =: C(X,L,H;K) \tag{10.11}
\]

10.3.2 Proof of Theorem 10.3.1

This section will by occupied by the proof of Theorem 10.3.1. Using the condition $X \subset K$, the support property image $u \subset K$ (10.4) and the above uniform energy bound (10.7) depending only on $K$, $\|H\|_K$ and $(\max h_X - \min h_X)$, we can safely deform the isotopy $L$ to $L' = \{L'(t)\}$ such that
\[
L'(t) \cap r^{-1}((\infty, R]) = L(t) \cap r^{-1}((\infty, R])
\]
where $R > 0$ is sufficiently large so that $K \subset r^{-1}((\infty, R/2])$, and we have
\[
L'(t) \cap M \setminus r^{-1}((\infty, 2R]) = L_0 \cap (M \setminus r^{-1}((\infty, 2R])
\]
which we suitably interpolate on the region $r^{-1}([R, 2R])$. In particular $L'(t) \cap X = L(t) \cap X$ for all $t \in [0,1]$ and
\[
L'(1) \cap M \setminus r^{-1}((\infty, 2R]) = L_0 \cap M \setminus r^{-1}((\infty, 2R]).
\]
Furthermore $L'(t) \equiv L(t)$ on the region $r^{-1}([R/2, R])$ and so the strong maximum principle can be applied which prevents any trajectory associated to $L'$ continued from $M(X,L)$ from penetrating into $r^{-1}([R/2, R])$. Furthermore we have
\[
X \cap L(t) = X \cap L'(t)
\]
and we may assume this intersection to be contained in the compact region $K$. 

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Define twisted complexes

\[ T = (X \cap L, D), \quad D = \{ ev_s([\mathcal{H}(X, L; x, x')]) \}_{x,y \in X \cap L}, \]

\[ T' = (X \cap L', D'), \quad D' = \{ ev_s([\mathcal{H}(X, L'; x', y')]) \}_{x',y' \in X \cap L'}. \]

and a morphism between them by

\[ S = \{ ev_s([\mathcal{M}(X, L; x, y)]) \}_{x \in X \cap L_0, y \in L_1}, \quad S' = \{ ev_s([\mathcal{M}(X, L'; x', y')]) \}_{x \in X \cap L_0, y \in L_1}. \]

**Proposition 10.3.10.**

\[ [S'] = [S] \]

in \( H_1^\mu(\mathcal{C}, \mathcal{D}) \) cohomology.

Assuming this proposition for the moment, we proceed with:

**Proof of Theorem 10.3.1.** We deform \( L'(1) \) to \( L_0 \) via a compactly supported isotopy \( L'' \). Since a compactly supported isotopy can be composed with its inverse so that their composition (through an isotopy of compactly supported isotopies) is isotopic to the constant isotopy \( \hat{L}_0 \), we obtain

\[
\bigoplus_{x,x'} ev_s([\mathcal{M}(X, L'\#L''; x, x')]) \sim \bigoplus_{x \in X \cap L_0} ev_s([\mathcal{M}(X, \hat{L}_0; x]) = id_{\mathcal{P}(X)}.
\]

But we have

\[
\bigoplus_{x,x'} ev_s([\mathcal{M}(X, L'\#L''; x, x')) \sim \bigoplus_{x,z} \bigoplus_{y} ev_s([\mathcal{M}(X, L'; x, y)]) \cdot ev_s([\mathcal{M}(X, L''; y, z)])
\]

\[
= \left( \bigoplus_{x,x'} ev_s([\mathcal{M}(X, L'; x, x')]) \right) \cdot \left( \bigoplus_{z,z'} ev_s([\mathcal{M}(X, L''; z, z')]) \right)
\]

Combining the above, we have proved

\[
\mathcal{F}(c^\chi(\mathcal{L})) = \bigoplus_{x_-, x_+} ev_s([\mathcal{M}(X, \mathcal{L}; x_-, x_+)]
\]

is an equivalence. This then finishes the proof of Theorem 10.3.1. \( \square \)

**Proof of Proposition 10.3.10.** We consider one-parameter family \( \mathcal{L}_s \) (i.e. a homotopy of isotopies) with \( 0 \leq s \leq 1 \) which are fixed on \( K \) but deforms outside \( r^{-1}([-\infty, R/2]) \) and consider the parameterized moduli space

\[
\mathcal{M}_{\text{para}}^{\mathcal{L}_s} (X, \{ \mathcal{L}_s \}; x_-, x_+) = \coprod_{s \in [0,1]} \{ s \} \times \mathcal{M}(X, \mathcal{L}_s; x_-, x_+)
\]

for the pairs \((x_-, x_+)\) with \(|x_-| = |x_+|\) with \( L_0 = \mathcal{L} \) and \( L_1 = L' \). Here we introduce the moduli spaces

\[
\mathcal{M}_{\text{para}}^{\mathcal{L}_s} (X, \{ \mathcal{L}_s \}; x_-, x_+)
\]

in general where the integer \( k \) appearing in the subindex \( (k) \) of the moduli space stands for the degree of the relevant operators which is the same as \( |x_-| - |x_+| \). This is one smaller than the dimension of the relevant parameterized moduli space.

We know that:
• the relevant Hamiltonians defining $L_s$ are only non-linear in a compact region contained in $r^{-1}((-\infty, 2R])$,
• there exists a uniform energy bound, and
• there is no bubbling,

so the boundary of $M^{para}_{(0)}(X, \{L_s\}; x^-, x^+)$ consists of the types

\[
\begin{align*}
M^{para}_{(-1)}(X, \{L_s\}; x^-, y) &\# \mathcal{H}(X, L_0; y, x^+), \\
\mathcal{H}(X, L_1; x^-, z) &\# M^{para}_{(-1)}(X, \{L_s\}; z, x^+), \\
M^{para}(X, \{L_s\}; x^-, x^+) &|_{s=0}, \\
M^{para}(X, \{L_s\}; x^-, x^+) &|_{s=1}.
\end{align*}
\]

Therefore we have

\[
e^{v_s}([M(X, L'; x^-, x^+)]) - e^{v_s}([M(X, L; x^-, x^+)]) = \bigoplus_{y \in X \cap L_0; |y| = |x^+|+1} e^{v_s}(M^{para}_{(0)}(X, \{L_s\}; x^-, y)) \# \mathcal{H}(X, L_0; y, x^+)) \\
+ \bigoplus_{z \in X \cap L_1; |z| = |x^-|-1} e^{v_s}(\mathcal{H}(X, L_1; x^-, z)) \# M^{para}_{(0)}(X, \{L_s\}; z, x^+)) .
\]

(10.12)

Now we introduce a collection of moduli spaces

\[
M^{para}_{(-1)}(X, \{L_x\}) = \{M^{para}(X, \{L_s\}; x, x')\}_{s \in [0, 1], x, x' ; |x'| = |x|-1}
\]

and define a 2-cochain

\[
\delta \mathfrak{H} := \left\{ e^{v_s}(M^{para}_{(-1)}(X, \{L_s\}; x, x')) \right\}_{s \in [0, 1], x, x' ; |x'| = |x|-1}.
\]

Then (10.12) gives rise to

\[
e^{v_s}([M_{(0)}(X, L')]) - e^{v_s}([M_{(0)}(X, L)]) = e^{v_s}(M^{para}_{(-1)}(X, \{L_s\})) \cdot e^{v_s}(\mathcal{H}(X, L_0)) \\
- e^{v_s}(\mathcal{H}(X, L_1)) \cdot e^{v_s}(M^{para}_{(-1)}(X, \{L_s\})).
\]

(10.13)

which can be rewritten into

\[
[S'] - [S] = \mu^1_{Tw(p)}([\delta \mathfrak{H}]) .
\]

By the definition of the $\mu^1$ on the twisted complex, we have

\[
\mu^1_{Tw(p)}(\delta \mathfrak{H}) = \mu^1_p([\delta \mathfrak{H}]) + \mu^2_p([\delta \mathfrak{H}], D_0) + \mu^2_p(D_1, [\delta \mathfrak{H}]).
\]

In the current case, we have $\mu^2_p([\delta \mathfrak{H}]) = \partial([\delta \mathfrak{H}]) = 0$ since $\delta \mathfrak{H}$ is a finite sum of zero-dimensional chains. This finishes the proof.

\[
\textbf{105}
\]
11 The equivalence $\text{Tw}W \simeq C_*P$

The main result of this section is

**Theorem 11.0.1.** The natural transformation $\text{Tw}W \circ D \rightarrow \text{Tw}C_*P$ from Corollary 10.3.3 is a natural equivalence. That is, for every smooth $j : |\Delta^n| \rightarrow B\text{Diff}(Q)$, the map $\text{Tw}W_{D,j} \rightarrow \text{Tw}C_*P_j$ is an equivalence of $A_\infty$-categories.

The proof requires some preliminary results that verify:

1. An isomorphism between two definitions of wrapped Floer cohomology—one using a colimit indexed over a non-negative sequence of wrappings (as in [AS10]) and the other using a Hamiltonian quadratic near infinity (as in [Abo12]). This is Proposition 11.1.9.

2. That quadratically wrapped Floer cohomology does not change under continuation maps of linear-near-infinity non-negative Hamiltonians. This is Lemma 11.2.2.

3. A compatibility between the non-wrapped Abouzaid map (Proposition 10.2.7) and the quadratically wrapped Abouzaid map from [Abo12] when mediated by the isomorphism from the just-mentioned Proposition 11.1.9. This compatibility is expressed in Corollary 11.3.2.

These ingredients will be mixed in Section 11.4 to give a proof of Theorem 11.0.1.

As it turns out, the above ingredients can be proven in large generality, so that is what we will do; moreover, we only need to verify these ingredients in a single fiber of a Liouville bundle, so in what follows, we will fix some Liouville sector $M$.

When we do apply the general results for our purposes, we will state this application as a corollary, and we will apply our general results in the following setting:

**Choice 11.0.2** (Choice for proving Theorem 11.0.1.). For any simplex $j : |\Delta^n| \rightarrow B\text{Diff}(Q)$ and for any $0 \leq a \leq n$, we set $M = T^*Q_a$ to be the fiber above the $a$th vertex of $|\Delta^n|$.

We also choose a point $q_a \in Q_a$ in the zero section above the $a$th vertex of $|\Delta^n|$. Choose also a cofinal sequence for the cotangent fiber $L = L^{(0)} = T^*_{q_a}Q_a$ (Definition 3.11.5).

We can arrange so that a cotangent fiber $T^*_{q_a}Q_a$ and all its cofinal wrappings are transverse to $Q_a$ and have only a single intersection point with $Q_a$, so that the natural transformation induced by the Abouzaid map sends $T^*_{q_a}Q_a$ to the object $q_a \in P_j$. We will assume so.

**Lemma 11.0.3.** Fix a Liouville sector $M$ and a cofinal sequence of wrappings for a brane $L = L^{(0)}$ (Definition 3.11.5). Then the continuation maps induce a functor of $\infty$-categories $Z_{\geq 0} \rightarrow N_{A_\infty}(\mathcal{O}(M))$ as follows:

$L^{(0)} \rightarrow L^{(1)} \rightarrow \ldots$

*Proof of Lemma 11.0.3.* The continuation maps determine, for every $i \in \mathbb{Z}_{\geq 0}$, a morphism in $\mathcal{O}(M)$ from the brane $L^{(i)}$ to the brane $L^{(i+1)}$; in particular, for each $i$ we have an edge in $N_{A_\infty}(\mathcal{O}_j)$. By the weak Kan property of $\infty$-categories, and because $Z_{\geq 0}$ is a poset, this sequence of edges lifts to a unique (up to contractible choice) functor from the $\infty$-category $Z_{\geq 0}$ to $N_{A_\infty}(\mathcal{O}(M))$. □

11.1 Comparing quadratic wrappings with cofinal wrappings

We have already shown that the colimit of a cofinal sequence of Floer cohomologies computes the hom-complex of the wrapped category $W$ (Lemma 9.2.2). In this section, we show that this colimit also computes the quadratically wrapped Floer cohomology (Proposition 11.1.9).

We first set some notation.
Notation 11.1.1 \((CF^*(L, L'; H), HF^*(L, L'; H))\). Let \(M\) be a Liouville sector and let \(L, L' \subset M\) be branes. Fix a smooth function \(H : M \to \mathbb{R}\). We denote the set of Hamiltonian chords of \(H\) from \(L\) to \(L'\)
\[
\text{Chord}(L, L'; H) = \{ \gamma \in \mathcal{P}(L, L') \mid \gamma(0) \in L, \gamma(1) \in L' \}
\]
and the associated the Floer cochain complex by
\[
CF^*(L, L'; H)
\]
whose differential is defined by considering the perturbed equation \(\overline{\partial}_{J,H}(u) = 0\), i.e.,
\[
\begin{cases}
\frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - X_H(t, u)) = 0 \\
u(\tau, 0) \in L, u(\tau, 1) \in L'.
\end{cases}
\]
(11.1)
The cohomology of this complex (i.e., the Floer cohomology) will be denoted
\[
HF^*(L, L'; H).
\]

Remark 11.1.2. We recall that there exists a natural chain isomorphism
\[
CF(L, L'; H) \to CF(L, \phi_H^1(L'); 0)
\]
induced by the correspondence
\[
\gamma \in \mathcal{P}(L, L') \mapsto \tilde{\gamma} \in \mathcal{P}(L, \phi_H^1(L')), \quad \tilde{\gamma}(t) := \phi_H^t(\gamma(t)),
\]
and \(J_t \mapsto \tilde{J}_t\) defined by
\[
\tilde{J}_t = (\phi_H^1)^*J_t
\]
which transforms the equation \(\overline{\partial}_{J,H}(u) = 0\) to
\[
\overline{\partial}_{\tilde{J}_0, 0}(v) = 0, \quad v(\tau, t) = \phi_H^1(u(\tau, t)).
\]

11.1.1 Hamiltonians

Definition 11.1.3 (Quadratic near infinity). Let \(M\) be a Liouville domain. A smooth function \(H : M \to \mathbb{R}\) is called quadratic near infinity if
\[
H = \frac{1}{2}r^2
\]
outside a compact subset of \(M\). (Here, we are using the coordinate \(r\) from Notation 3.2.2.)

Remark 11.1.4. Our wish is to compare \(CF(L, L'; H)\) with \(H\) quadratic near infinity—to a colimit of Floer complexes constructed from a cofinal sequence \(L^{(0)} \to L^{(1)} \to \ldots\).

Because each \(L^{(w)}\) in this sequence is conical near infinity, we may assume that each is the result of a Hamiltonian isotopy by a linear-near-infinity Hamiltonian \(F^{(w)} : M \to \mathbb{R}\). In what follows, we will choose a particular model for such a sequence of Hamiltonians—namely, we will set \(F^{(w)}\) to be obtained by increasing the slopes of a standard linear-near-infinity Hamiltonian (see (11.2)). The reader may make the necessary adjustments to the following proofs for a more general cofinal sequence of linear-near-infinity Hamiltonians. (For example, by altering (11.8) to interpolate \(H\) with a given \(F^{(w)}\), rather than with the sequence of \(F\)s we choose in (11.2).)
Notation 11.1.5 \((H \text{ and } F)\). In this section, we will use the symbol \(H\) to denote an autonomous Hamiltonian that is quadratic near infinity (Definition 11.1.3).

We will use the symbol \(F\) to denote a Hamiltonian that is autonomous and is linear near infinity, i.e., \(F = ar + b\) outside a compact subset for some constant \(a, b\) with \(a > 0\). Note that
\[
\phi_{vF}^1(L) = \phi_F^v(L)
\]
is still linear near infinity, and in particular outside \(\{r \leq R_K\} \supset K\) for any \(R_K\) large enough.

Notation 11.1.6 \((HF^*_{\text{quad}})\). Given a quadratic-near-infinity Hamiltonian \(H\), we define the notation
\[
HF^*_{\text{quad}}(L, L') = HF^*(L, \phi_H^1(L'); 0).
\]
Note that the dependence on \(H\) is suppressed on the left-hand side. (See also Notation 11.1.1 for the right-hand side.)

Lemma 11.1.7. Fix a Liouville embedding \(\iota : [0, \infty) \times \partial_{\infty}M \rightarrow M\) as in (3.3), and set \(r = e^s\) as in Notation 3.2.2. Let \(H : M \rightarrow \mathbb{R}\) be quadratic near infinity (Definition 11.1.3). Then there exists some constant \(C = C(\iota, H) > 0\) such that
\[
H - \lambda(X_H) \geq -C.
\]

Proof. Outside a large compact subset of \(M\), we have
\[
\lambda(X_H) = \lambda(rX_r) = rdr(-JX_r) = rdr\left(\frac{\partial}{\partial r}\right) = r
\]
so \(H - \lambda(X_H) = \frac{r^2}{2} - r > 0\) as long as \(r > 2\). The lemma follows because the (closure of) the complement of the image of \(\iota\) is compact. \(\square\)

Notation 11.1.8 (The chain maps \(\phi\)). Consider the interpolating homotopy
\[
(1 - s)vF + H, \quad s \in [0, 1]
\]
through Hamiltonian that induce non-negative wrappings near infinity. By the \(C^0\) and energy estimates established in Section 8, we have induced chain maps
\[
\phi_{vv'} : CF(L, L'; vF) \rightarrow CF(L, L'; v'F)
\]
for all \(v < v'\) and
\[
\phi_v : CF(L, L'; vF) \rightarrow CF(L, L'; H)
\]
for any \(v \in \mathbb{Z}_+\). (See Notation 11.1.1.)

Passing to cohomology, the maps \(\phi\) from Notation 11.1.8 induce a commutative diagram
\[
\begin{array}{cccccc}
HF(L, L'; F) & \rightarrow & HF(L, L'; 2F) & \rightarrow & \cdots & \rightarrow & HF(L, L'; vF) & \rightarrow & \cdots \\
\downarrow & & \downarrow & & & & \downarrow & & \\
HF(L, L'; H) & \rightarrow & HF(L, L'; H) & \rightarrow & \cdots & \rightarrow & HF(L, L'; H) & \rightarrow & \cdots
\end{array}
\]
and hence a homomorphism
\[
\varphi_\infty : \colim_{v \rightarrow \infty} HF^*(L, L'; vF) \rightarrow HF^*(L, L'; H) \cong HF^*_{\text{quad}}(L, L').
\]
(The last isomorphism is Remark 11.1.2.)

The main result we seek to prove now is:
Proposition 11.1.9. \( \varphi_{\infty} \) is an isomorphism.

Before proving Proposition 11.1.9, let us state the following consequence, which one obtains by setting \( L = L' \subset T^*Q \) to be a cotangent fiber to a point \( a \in Q \).

Corollary 11.1.10. In the setting of Choice 11.0.2, the induced map

\[
\operatorname{colim}_w H^* \operatorname{hom}_{\partial_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) \to H^{F^*}_{\operatorname{quad}}(T^*_q Q_a, T^*_q Q_a)
\]

is an isomorphism.

Proof. Observe that \( \operatorname{hom}_{\partial_j}(L, (L')^{(w)}) \) is naturally isomorphic to \( CF^*(L, L'; wF) \) when one chooses the cofinal sequence to be given by \( wF \). (See Remark 11.1.2 and Remark 11.1.4.) One then observes that given two cofinal sequences of wrappings \( (L')^{(w)} \) and \( (L')^{(v)} \), there is a natural category whose object set is given by the union \( \{ L^{(w)} \}_w \cup \{ L^{(v)} \}_v \), and a morphism is given by an isotopy class of a non-negative Hamiltonian isotopy between two objects. Each of the original cofinal sequences is cofinal in this category, so the colimits of the Floer cohomology groups agree.

11.1.2 Action filtration

Before proving Proposition 11.1.9, we set some notation.

Notation 11.1.11 (Action and energy). The holomorphic strip equation (11.1) may be interpreted as a gradient flow equation of the action functional

\[
\mathcal{A}(\gamma) = \mathcal{A}_{L, L'}(\gamma) = - \int \gamma^* \theta + \int H(\gamma(t)) \, dt + h_{L'}(\gamma(1)) - h_L(\gamma(0))
\]

where \( h_L : L \to \mathbb{R} \) is a function satisfying \( \lambda|_L = dh_L \). We have the basic energy identity

\[
E_{(J, H)}(u) = \mathcal{A}(u(\infty)) - \mathcal{A}(u(-\infty)).
\]

Notation 11.1.12. Each chain of \( CF(L', L; H) \) is a linear combination

\[
\alpha = \sum a_z \langle z \rangle \in CF(L, L'; H), \quad z \in \operatorname{Chord}(L, L'; H)
\]

where each \( z \) is a Hamiltonian chord from \( L_1 \) to \( L_0 \). We denote

\[
\operatorname{supp} \alpha = \{ z \in \operatorname{Chord}(L, L'; H) \mid a_z \neq 0 \text{ in (11.6)} \}
\]

and define the action \( \ell(\alpha) \) of the cycle by

\[
\ell(\alpha) = \max \{ \mathcal{A}(z) \mid z \in \operatorname{supp} \alpha \}.
\]

Notation 11.1.13 (Action filtration). In terms of this action, we have an increasing filtration

\[
CF_{\leq c}(L, L'; H) := \{ \alpha \in CF(L, L'; H) \mid \ell(\alpha) \leq c \}
\]

with \( CF_{\leq c}(L, L'; H) \subset CF_{\leq c'}(L, L'; H) \) for \( c < c' \). Each \( CF_{\leq c}(L, L'; H) \) is a subcomplex of \( CF(L, L'; H) \). \(^{28}\) By definition, we have

\[
CF(L, L'; H) = \operatorname{colim}_c CF_{\leq c}(L, L'; H).
\]

\(^{28}\) This is because we put the output of the differential at \( \tau = -\infty \), not at \( \tau = \infty \).
11.1.3 Proof of Proposition 11.1.9

We start with surjectivity. Let \( a \in HF(L, L'; H) \) and choose a cycle \( \alpha \) representing it, i.e., \( \mu_1(\alpha) = 0 \). Denote \( c = \ell(\alpha) \) (Notation 11.1.12). Now we consider the following linear adjustment of \( H \):

\[
H_v(x) = \begin{cases} 
H(x) & \text{if } r(x) \leq v \\
(v+1)r(x) - \frac{1}{2}(v+1)^2 & \text{if } r(x) \geq v+1
\end{cases}
\]

with suitable smooth interpolation in between. We fix a sequence of integers \( \{v_k\}_{k \in \mathbb{N}} \) diverging to \( \infty \) and consider the sequence \( H_v \) of linear near infinity Hamiltonians. We note that \( \{H_v\}_{k \in \mathbb{N}} \) is a monotone sequence of Hamiltonians converging to \( H \) uniformly on \( M^{cpt} \cup \{r \leq R\} \) for any large \( R > 0 \).

The following lemma will conclude surjectivity of \( \varphi_\infty \).

Lemma 11.1.14. There exists \( b \in HF(L, L'; H(v_k)) \) such that \( a = \varphi_\infty(b) \).

Proof. Let \( \alpha = \sum_{i=1}^k a_i(z_i) \) be a cycle representing \( a \) with \( z_i \in \text{Chord}(L, L'; H) \).

Clearly \( z_i \in \text{Chord}(L, L'; H(v_k)) \) as long as \( v_k \ell(\alpha) \) so large that

\[
\text{supp } \alpha \subset H^{-1}((\infty, v_k])
\]

where we abuse the notation \( \text{supp } \alpha \) by also denoting it as \( \cup_{z \in \text{supp } \alpha} \text{image } z \subset M \). Therefore \( \alpha \) can be regarded as a chain in \( CF(L, L'; H(v_k)) \). Denote the resulting chain of \( H(v_k) \) by \( \beta_k \). We now prove \( \beta_k \) is a cycle of \( H(v_k) \), if we choose \( k \) even larger if necessary.

To avoid confusion, we denote by \( \mu_1^H \) and \( \mu_1^{H(v_k)} \) be the \( \mu_1 \)-map for \( H \) and \( H(v_k) \) respectively. Then standing hypothesis is \( \mu_1^H(\alpha) = 0 \) and we want to prove

\[
\mu_1^{H(v_k)}(\beta_k) = 0
\]

by choosing a larger \( k \) if necessary. By definition, we have

\[
\mu_1^H(\alpha) = \sum_{i=1}^k a_i \mu_1^H(\langle z_i \rangle)
\]

where

\[
\mu_1^H(\langle z_i \rangle) = \sum_{y \in \text{Chord}(L, L'; H)} n_{(J, H)}(z_i, y) \langle y \rangle
\]

with \( n_{(J, H)}(z_i, y) = \#\mathcal{M}(z_i, y; J, H) \) where \( \mathcal{M}(z_i, y; J, H) \) is the moduli space of solutions \( u \) of (11.1) satisfying \( u(-\infty) = z_i, u(\infty) = y \). We rearrange the sum into

\[
\mu_1^H(\alpha) = \sum_{y \in \text{Chord}(L, L'; H)} \left( \sum_{i=1}^k a_i n_{(J, H)}(z_i, y) \right) \langle y \rangle.
\]

Therefore \( \mu_1^H(\alpha) = 0 \) is equivalent to

\[
\sum_{i=1}^k a_i n_{(J, H)}(z_i, y) = 0
\]
for all $y \in \text{Chord}(L, L'; H)$.

The same formula with $H$ replaced by $H(v_k)$ holds and so

$$
\mu_1^{H(v_k)}(\beta_k) = \sum_{y \in \text{Chord}(L, L'; H(v_k))} \left( \sum_{i=1}^{k} a_i n_i(J, H(v_k))(z_i, y) \right) \langle y \rangle.
$$

Therefore it remains to prove

$$
\sum_{i=1}^{k} a_i n_i(J, H(v_k))(z_i, y) = 0 \quad (11.8)
$$

for all $y \in \text{Chord}(L, L'; H(v_k))$ by choosing $k$ sufficiently large. This will follow if we establish

$$
\mathcal{M}(z_i, y; J, H(v_k)) = \begin{cases} 
\mathcal{M}(z_i, y; J, H) & \text{if } \mathcal{M}(z_i, y; J, H) \neq \emptyset \\
\emptyset & \text{if } \mathcal{M}(z_i, y; J, H) = \emptyset 
\end{cases} \quad (11.9)
$$

for all $i$ and $y$.

**Sublemma 11.1.15.** There are finitely many $y \in CF(L, L'; H)$ such that $\mathcal{M}(z_i, y; J, H) \neq \emptyset$ for some $i = 1, \ldots, k$.

**Proof.** By the energy identity, we have

$$
\mathcal{A}(y) \leq \ell(\alpha).
$$

On the other hand, for any Hamiltonian chord $y$ of $H$, we derive

$$
\mathcal{A}(y) = -\int y^* \theta + \int_0^1 H(y(t)) \, dt = -C
$$

by Lemma 11.1.7. Therefore under the given hypothesis, we have

$$
-C < \mathcal{A}(y) \leq \ell(\alpha).
$$

By the nondegeneracy assumption on $H$, this finishes the proof.

Set

$$
R_0 = \max_y \{ r(y) \mid y \text{ is as in the above sublemma} \}.
$$

Then it follows from Theorem 8.4.1 that there exists a sufficiently large $k$ such that

$$
\max r \circ u \leq R_0 + C'
$$

for $u \in \mathcal{M}(z_i, y; J, H)$ where $C'$ depends only on $\inf H > -\infty$.

The same discussion still applies to $H(v_k)$ since we still have

$$
H(v_k) - \theta \left( X_{H(v_k)} \right) \geq -C
$$

and

$$
\max r \circ u \leq R_0 + C'
$$
for \( u \in \mathcal{M}(z, y; J, H_{(v_k)}) \) for the same constant \( C, C' \) above respectively. Combining the two, we have proved (11.9) and so \( \mu_1^{H_{(v_k)}}(\beta_k) = 0 \).

Next we would like to prove

\[
[\phi_{v_k}(\beta_k)] = [\alpha] = a.
\]

For this we have only to know that

\[
(1 - s)H_{(v_k)} + sH \equiv H
\]
on \( r^{-1}(-\infty, R_0 + C') \) and the same \( C^0 \)-estimate as 8.4.1 applies for the continuation equation for \( H = \{H^s = (1 - s)H_{(v_k)} + sH\} \). This implies that any solution \( u \) of continuation equation satisfies (11.1) provided we choose \( v_k \) sufficiently large so that \( H^s \equiv H \) for all \( s \in [0, 1] \) on \( M \setminus \ell([R_0 + C', \infty) \).

This in fact implies

\[
\phi_{v_k}(\beta_k) = \alpha
\]
in chain level and hence proves \( [\phi_{v_k}(\beta_k)] = [\alpha] \).

This finishes the proof of surjectivity.

For the proof of injectivity, let \( \beta_k \) be a sequence of \( H_{(v_k)} \)-cycle such that \( [\beta_{k+1}] = [\phi_{k(k+1)}(\beta_k)] \)

and \( [\phi_{v_k}(\beta_k)] = 0 \) in \( HF(L, L'; H) \) for all \( k \geq k_1 \) with \( k_1 \) sufficiently large. Then

\[
\phi_{v_k}(\beta_k) = \mu_1^H(\alpha'\ell)
\]
for some \( H \)-chain \( \alpha'\ell \) or each \( k \geq k_1 \). Denote \( \lambda_2 = \ell(\alpha'\ell) \). Under this hypothesis, by the similar argument given in the surjectivity proof, we can find a sufficiently large \( \ell = \ell(k_1, \lambda_2) > k_1 \) such that

\[
\phi_{v_{k_1}\ell}(\beta_{k_1}) = \mu_1^{H_{(v_{k_1})}}(\alpha'\ell).
\]

(11.10)

Now we consider a conformally symplectic dilation \( f : M \to M \) defined by the Liouville flow for time \( \log(\rho) \) with \( \rho = \frac{v_{k_1}}{v_{k_2}} \) which becomes

\[
f(x) = (\rho r, y)
\]
for \( x = (r, y) \in M^\text{end} \). The isotopy \( t \mapsto f \circ \phi_{H_{(v_{k_1})}} \circ f^{-1} \) is still a Hamiltonian isotopy generated by the Hamiltonian

\[
\frac{v_{k_1}^2}{v_{k_2}^2}H_{(v_{k_1})} \circ f = : G_{k_1\ell}
\]

We note that \( \frac{v_{k_1}}{v_{k_2}}H_{(v_{k_1})} \circ f(r, y) = v_{k_1}r \) for any \( x = (r, y) \) such that \( H_{(v_{k_1})}(r, y) = v_{k_1}r \). Therefore we can find a chain isomorphism

\[
\eta_* : CF(L, L'; G_{k_1\ell}) \to CF(L, L'; H_{(v_{k_1})})
\]
associated to the isotopy \( \eta : s \mapsto \phi_{G_{k_1\ell}}^s \circ \phi_{H_{(v_{k_1})}}^{1-s} \) which is compactly supported.

We then define a map

\[
\psi_{k_1\ell} : CF(L, L'; H_{(v_{k_1})}) \to CF(L, L'; H_{(v_{k_1})})
\]
as the composition \( \psi_{k_1\ell} = (\eta)_* \circ f_* \) which is a quasi-isomorphism. Furthermore we also have
Sublemma 11.1.16. The map
\[ \psi_{k_1 \ell} \circ \phi_{v_{k_1}v_{\ell}} = (\eta)_s \circ f_s \circ \phi_{v_{k_1}v_{\ell}} \]
is chain homotopic to id on \( CF(L, L'; H(v_{k_1})) \).

Proof. We have only to notice that the isotopy
\[ g_s := \eta^{1-s} \circ f^{1-s} \circ (\phi_{H(v_{k_1})}^s \phi_{H(v_{\ell})}^s) : M \to M \]
and \( g_0 = \eta \circ f \circ \phi_{H(v_{k_1})}^1 \) and \( g_1 = \text{id} \). In particular, we obtain
\[ \psi_{k_1 \ell} \circ \phi_{v_{k_1}v_{\ell}} - \text{id} : CF(L, L'; H(v_{k_1})) \to CF(L, L'; H(v_{k_1})) \]
is chain homotopic to 0. This finishes the proof.

Now we apply the map \( \psi_{k_1 \ell} \) to (11.10) and get
\[ \psi_{k_1 \ell} \circ \phi_{v_{k_1}v_{\ell}}(\beta_{k_1}) = \psi_{k_1 \ell} \circ \mu_1^{H(v_{\ell})}(\alpha'_{k_1}) \]
The left-hand side can be written as
\[ \psi_{k_1 \ell} \circ \phi_{v_{k_1}v_{\ell}}(\beta_{k_1}) = \beta_{k_1} + \mu_1^{H(v_{k_1})}(\gamma) \]
for some chain \( \gamma \) of \( H(v_{k_1}) \) and the right-hand side coincides with
\[ \mu_1^{H(v_{k_1})} \circ \psi_{k_1 \ell}(\alpha'_{k_1}) \]
by the chain property of \( \psi_{k_1 \ell} \). Combining the two, we have derived
\[ \beta_{k_1} = \mu_1^{H(v_{k_1})} \circ \psi_{k_1 \ell}(\alpha'_{k_1}) + \mu_1^{H(v_{k_1})}(\gamma) = \mu_1^{H(v_{k_1})}(\psi_{k_1 \ell}(\alpha'_{k_1}) - \gamma) \]
This proves \( [\beta_{k_1}] = 0 \). By the compatibility of the sequence \( [\beta_k] \), this proves \( \lim_k [\beta_k] = 0 \) and hence the injectivity of the map \( \varphi_\infty \). This finishes the proof of the proposition.

11.2 Positive wrappings leave \( HF_{\text{quad}} \) unchanged

Notation 11.2.1 \( (H \# F) \). Given two (possibly time-dependent) Hamiltonians \( H \) and \( F \), we define a time-dependent Hamiltonian \( H \# F : \mathbb{R} \times M \to \mathbb{R} \) by
\[ H \# F(t, x) = H(t, x) + wF(\phi^t_H(x)) \]

Lemma 11.2.2. Suppose that \( (L')^{(w)} \) is obtained from \( L' \) by a non-negative Hamiltonian isotopy, and let \( H \) be a quadratic-near-infinity Hamiltonian. Then the continuation map
\[ HF(L, L' : H) \to HF(L, (L')^{(w)} ; H) \]
is an isomorphism.
Proof. Let $F$ be a linear-near-infinity Hamiltonian inducing a cofinal sequence of nonnegative isotopies $(L')^{(v)}$. Choose a linear-near-infinity Hamiltonian $G$ whose Hamiltonian flow realizes the isotopy from $L'$ to $(L')^{(w)}$. We have a commutative diagram of Floer cohomology groups

\[
\begin{array}{c}
HF(L, L'; vF) \rightarrow \ldots \rightarrow HF(L, L'; v'F) \rightarrow \ldots \rightarrow HF(L, L'; H) \\
\downarrow \downarrow \downarrow \downarrow \downarrow \\
HF(L, L'; vF\#G) \rightarrow \ldots \rightarrow HF(L, L'; v'F\#G) \rightarrow \ldots \rightarrow HF(L, L'; H\#G)
\end{array}
\]

(11.11)

where arrows are given by continuation maps, and $v < v'$. Moreover, the sequences $\{vF\}_v$ and $\{vF\#G\}_v$ are both cofinal in the spliced diagram of Hamiltonians $\{vF\}_v \cup \{vF\#G\}_v$.

Thus the colimits

\[
\colim_{v \to \infty} HF(L, L'; vF), \quad \colim_{v \to \infty} HF(L, L'; vF\#G)
\]

are equivalent. On the other hand, the top row and the bottom row of (11.11) are colimit diagrams by Proposition 11.1.9. Thus the rightmost vertical arrow of (11.11) is an isomorphism.

On the other hand, we have the obvious isomorphism

\[
HF(L, (L')^{(w)}; H) \cong HF(L, L'; H\#G).
\]

This completes the proof. \qed

Now suppose that $L = L'$ is a cotangent fiber of $T^*Q$ at a point $a \in Q$, and choose a cofinal sequence for $L$:

**Corollary 11.2.3.** In the setting of Choice 11.0.2, we have

\[
\colim_w HF_{quad}^*(T^*_q Q_a, T^*_q Q_a^{(w)}) \cong HF_{quad}^*(T^*_q Q_a, T^*_q Q_a).
\]

For later reference, we record explicitly the following fact, which follows from the commutativity of (11.11):

**Lemma 11.2.4.** In the setting of Choice 11.0.2, the diagram of cohomology groups

\[
\begin{array}{c}
H^* \hom_{O_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) \rightarrow HF_{quad}^*(T^*_q Q_a, T^*_q Q_a^{(w)}) \\
\downarrow \downarrow \\
H^* \hom_{O_j}(T^*_q Q_a, T^*_q Q_a^{(w+1)}) \rightarrow HF_{quad}^*(T^*_q Q_a, T^*_q Q_a^{(w+1)})
\end{array}
\]

is commutative. Here, all arrows are induced by continuation maps.

### 11.3 Comparing the non-wrapped Abouzaid map to the quadratically wrapped Abouzaid map

For this section, we let $|\Delta^n| = |\Delta^0|$, so that the Liouville bundle $E \rightarrow |\Delta^0|$ is simply a choice of Liouville sector $M$. In the following lemma, we thus drop the $j$ variable:
Lemma 11.3.1. Let $M$ be a Liouville sector and $Q \subset M$ a compact exact brane. Fix objects $X, L^{(w)} \in \text{Ob} \mathcal{O}(M)$. The Abouzaid functor defines an object of $\text{TwC}_* \mathcal{P}(Q)$ by (10.1) from any object $L$ of $\mathcal{O}(M)$, and we abbreviate this object as $L \cap Q$.

Then the diagram

$$
\begin{array}{ccc}
H^* \text{hom}_{\text{TwC}_* \mathcal{P}(Q)}(Q \cap X, Q \cap L^{(w)}) & \xrightarrow{\text{10.2.7}} & H^* \text{hom}_Q(X, L^{(w)}) \\
\uparrow_{\text{Abo12}} & & \downarrow_{\text{Abo12}} \\
H^* \text{hom}_Q(X, L^{(w)}) & \xrightarrow{\text{11.1.10}} & HF^*_{\text{quad}}(X, L^{(w)})
\end{array}
$$

commutes. Here, the horizontal map is a continuation map, while the two other maps are the non-wrapped (Proposition 10.2.7) and wrapped (Section 4 of [Abo12]) versions of the Abouzaid map.

Moreover, at the level of cohomology, these diagrams are compatible with the filtered diagram from Lemma 11.0.3.

Proof of Lemma 11.3.1. Because we are working at the level of cohomology, we may model the isotopy from $L$ to $L^{(w)}$ by some linear-near-infinity Hamiltonian $F$, and choose the quadratic Hamiltonian $H$ in such a way that $H = F$ on the region where $L$ and $L^{(w)}$ intersect.

Choosing a non-negative interpolating isotopy from $F$ to $H$ identifies

$$H^* \text{hom}_Q(L, (L')^{(w)}) = HF^*(L, L'; F) \to HF^*_{\text{quad}}(L, (L')^{(w)})$$

as a subcomplex of chords with action bounded by some $A$.

Both the non-wrapped and wrapped maps count holomorphic triangles one of whose vertices limit to Hamiltonian chords, and we find that the non-wrapped map counts precisely such triangles restricted to the subcomplex of chords with action less than or equal to $A$. This shows that the diagram commutes.

That the diagram is natural in $w$ follows straightforwardly by choosing the quadratic Hamiltonian to be equal to the Hamiltonian defining $L^{(w)}$ and $L^{(w')}$. (Note that while these choices certainly affect the chain-level maps, we may choose these Hamiltonians to have no effect at the level of cohomology.)

Now we set $M = T^* Q = T^* Q_a$ and set $L = L'$ to be a cotangent fiber at some point $q_a \subset Q_a$.

Corollary 11.3.2. In the setting of Choice 11.0.2, for every $w$, the diagram

$$
\begin{array}{ccc}
H^* \text{hom}_{\mathcal{P}}(q_a, q_a^{(w)}) & \xrightarrow{\text{10.2.7}} & H^* \text{hom}_Q(T_q^* Q_a, T_q^* Q_a^{(w)}) \\
\uparrow_{\text{Abo12}} & & \downarrow_{\text{Abo12}} \\
H^* \text{hom}_Q(T_q^* Q_a, T_q^* Q_a^{(w)}) & \xrightarrow{\text{11.1.10}} & HF^*_{\text{quad}}(T^* Q_a, T^* Q_a^{(w)})
\end{array}
$$

commutes. Here, the horizontal map is a continuation map, while the two other maps are the non-wrapped (Proposition 10.2.7) and wrapped [Abo12] versions of the Abouzaid map. Finally, $q_a^{(w)}$ is the intersection point of $T^* Q_a^{(w)}$ with the zero section. (We have chosen our cofinal wrapping so that there is only one intersection point.)

Moreover, at the level of cohomology, these diagrams are compatible with the filtered diagram from Lemma 11.0.3.
11.4 Proof of Theorem 11.0.1

Proof of Theorem 11.0.1. Because the cotangent fiber $T^*_q Q_a$ and the object $q_a \in Q_a$ generate\(^{29}\) the $A_\infty$-categories in which they reside, all that remains is to show that the map on endomorphism complexes of these objects—induced by Corollary 10.3.3—is a quasi-isomorphism.

We first claim the diagram below commutes:

\[
\begin{array}{c}
\text{colim}_w H^* \text{hom}_{O_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) \xrightarrow{\text{Cor 10.3.3}} H^* \text{hom}_{C^*_j P_j}(q_a, q_a) \\
\text{colim}_w H^* \text{hom}_{W_j}(T^*_q Q_a, T^*_q Q_a) \xrightarrow{\text{Prop 10.2.7}} H^* \text{hom}_{C^*_j P_j}(q_a, q_a) \\
\text{Prop 10.2.7} \cong \text{Cor 11.2.7} \cong |\text{Abo12}|
\end{array}
\]

(11.12)

Let us explain the diagram.

We first note that, for a fixed $w$, there is a homotopy commutative diagram of chain complexes

\[
\begin{array}{ccc}
\text{hom}_{W_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) & - & - \text{hom}_{C^*_j P_j}(q_a, q_a^{(w)}) \\
\text{hom}_{O_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) & - & - \text{hom}_{C^*_j P_j}(q_a, q_a^{(w)})
\end{array}
\]

where the diagonal map is the Abouzaid construction for $\emptyset$ (Proposition 10.2.7), and the upper, dashed, horizontal map is induced by the universal property of localization (Corollary 10.3.3). Here, $q_a^{(w)}$ is the intersection point of $T^*_q Q_a^{(w)}$ with the zero section. (We have chosen our cofinal wrapping so that there is only one intersection point.)

Moreover, there is a homotopy coherent functor $\mathbb{Z}_{\geq 0} \to O_j$ by Lemma 11.0.3, so the lower left corner of the triangle coheres into a homotopy-coherent sequential diagram indexed by $w$. Since we have functors $O_j \to W_j$ and $O_j \to C^*_j P_j$, we have an induced homotopy-coherent diagram of the colimits:

\[
\begin{array}{ccc}
\text{colim}_w \text{hom}_{W_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) & \longrightarrow & \text{colim}_w \text{hom}_{C^*_j P_j}(q_a, q_a^{(w)}) \\
\text{colim}_w \text{hom}_{O_j}(T^*_q Q_a, T^*_q Q_a^{(w)}) & \longrightarrow & \text{colim}_w \text{hom}_{C^*_j P_j}(q_a, q_a^{(w)})
\end{array}
\]

We note that the two sequential colimits in the top horizontal line is a colimit of isomorphisms upon passage to cohomology—this is because continuation maps are sent to equivalences in $W_j$ (by definition of localization) and in $C^*_j P_j$ (by Theorem 10.3.1). Thus, for both items in the top horizontal line, the cohomology of the colimit is isomorphic to the cohomology of the $w = 0$ term. This explains the upper-left triangle in (11.12).

The lower-right triangle of (11.12) is obtained by applying the $w$-indexed colimit (at the level of cohomology) to the triangle in Corollary 11.3.2. We observe that the filtered colimits on the right vertical edge consists of maps that are all equivalences, so in this way we may identify

\[
\text{colim}_w H^* \text{hom}_{C^*_j P_j}(q_a, q_a^{(w)}) \cong H^* \text{hom}_{C^*_j P_j}(q_a, q_a),
\]

and the lower-right corner of the triangle arises by using Corollary 11.2.3. This completes the explanation of the commutative diagram (11.12).

\[^{29}\text{For } TW_j C^*_j P_j \text{ this is obvious, while for } TW_W_j, \text{ this follows from Abouzaid’s Theorem and Proposition 9.3.3.}\]
Referring again to (11.12), note that the left-hand vertical arrow was verified to be an isomor-
phism at the level of cohomology in Lemma 9.2.2, the bottom horizontal arrow was verified to be an
isomorphism in Corollary 11.1.10, and the right-hand vertical arrow is an isomorphism by [Abo11].
Because these $\cong$-labeled arrows are isomorphisms, it follows that the top horizontal arrow is also
an isomorphism, which is what we sought to prove.
12 The diffeomorphism action on Loc

Notation 12.0.1 (Diff). Fix \( Q \) an oriented, compact manifold. We let \( \text{Diff}(Q) \) denote the topological group of orientation-preserving diffeomorphisms of \( Q \).

The goal of this section is to prove that the natural action of \( \text{Diff}(Q) \) on \( \text{Loc}(Q) \) is compatible with the action of \( \text{Aut}^{gr,b}(T^*Q) \) on \( W(T^*Q) \) from Theorem 1.1.1.

Given the results of our previous section, the only thing left to do is to verify that the diffeomorphism action on \( \text{Tw} C_\ast \mathcal{P} \) is the standard action of the diffeomorphism group on the \( A_\infty \)-category of local systems. This is proven in Proposition 12.1.13.

12.1 \( C_\ast \mathcal{P} \) is compatible with the diffeomorphism action

Our eventual goal is to prove that the (orientation-preserving) diffeomorphism group action on the \( \infty \)-category of local systems is compatible with its action on the wrapped Fukaya category of a cotangent bundle. So first let us show that our construction \( C_\ast \mathcal{P} \) encodes the usual action on the \( \infty \)-category of local systems.

For this, recall that the \( \infty \)-category of local systems on a space \( B \) with values in an \( \infty \)-category \( \mathcal{D} \) is equivalent to the \( \infty \)-category \( \text{Fun}(\text{Sing}(B), \mathcal{D}) \) of functors from \( \text{Sing}(B) \) to \( \mathcal{D} \). The evident action of \( \text{hAut}(B) \) on \( B \)—and hence on \( \text{Sing}(B) \)—exhibits the action of \( \text{hAut}(B) \) on the \( \infty \)-category of local systems.

On the other hand, our construction of \( C_\ast \mathcal{P} \) passes through a combinatorial trick that replaces \( B \text{Diff}(Q) \) by a category of simplices in \( B \text{Diff}(Q) \), which one can informally think of as the category encoding the barycentric subdivision of \( \text{Sing}(B \text{Diff}(Q)) \). We must show that this combinatorial trick allows us to recover the natural action of \( \text{Diff}(Q) \) on \( Q \); this is the content of Corollary 12.1.4 below.

Construction 12.1.1. Let \( p : E \to B \) be a Kan fibration of simplicial sets. We let \( \text{subdiv}(B) \) denote the subdivision simplicial set associated to \( B \) (see Construction 2.1.9).

We have an induced functor

\[
p^{-1} : \text{subdiv}(B) \to \text{Kan}, \quad (j : \Delta^k \to B) \mapsto p^{-1}(j)
\]

to the \( \infty \)-category of Kan complexes. Indeed, realizing \( \text{subdiv}(B) \) to be the nerve of a category, the above is induced by an actual functor to the category of simplicial sets, sending an object \( j \) to the simplicial set \( j^*E \).

Remark 12.1.2. Moreover, because \( p \) is a Kan fibration, every edge in \( \text{subdiv}(B) \) is sent to an equivalence in \( \text{Kan} \); thus the above functor factors through the localization of \( \text{subdiv}(B) \). Moreover, we know the localization to be equivalent as an \( \infty \)-category to the Kan complex \( B \) by Proposition 4.5.3. We draw this factorization as follows:

\[
\begin{array}{ccc}
\text{subdiv}(B) & \xrightarrow{p^{-1}} & \text{Kan} \\
\downarrow \text{Prop 4.5.3} & & \downarrow F \\
B & & \text{Kan}
\end{array}
\]

That is, \( p^{-1} \) induces some functor \( F : B \to \text{Kan} \) of \( \infty \)-categories.
On the other hand, the Kan fibration $p : E \to B$ classifies a functor of $\infty$-categories from $B$ to $\mathcal{K}an$ by the straightening/unstraightening correspondence (see Section 2.2.6). Our main goal is to prove:

**Lemma 12.1.3.** The functor classified by $p : E \to B$ admits a natural equivalence to the functor $F$ (induced by $p^{-1}$ in Construction 12.1.1).

Given the lemma, we have

**Corollary 12.1.4.** Let $EQ$ be the tautological $Q$ bundle over $B \text{Diff}(Q)$. We then have a Kan fibration $p : \text{Sing}(EQ) \to \text{Sing}(B \text{Diff}(Q))$, and the induced functor

$$N(\text{Simp}(B \text{Diff}(Q))) \to \mathcal{K}an, \quad (j : |\Delta^n| \to B) \mapsto \text{Sing}(j^* EQ).$$

Consider the induced functor $F$ from Remark 12.1.2:

$$\begin{array}{c}
N(\text{Simp}(B \text{Diff}(Q))) \\
\downarrow \text{Prop 4.5.3} \\
\text{Sing}(B \text{Diff}(Q)) \text{ } F \\
\downarrow \text{Sing}(\text{Diff}(Q)) \\
\mathcal{K}an
\end{array}$$

Then $F$ is naturally equivalent to the functor sending a distinguished vertex of $\text{Sing}(B \text{Diff}(Q))$ to $\text{Sing}(Q)$, and exhibiting the action of $\text{Sing}(\text{Diff}(Q))$ on $\text{Sing}(Q)$.

**Proof of Corollary 12.1.4.** The fibration $EQ \to B \text{Diff}(Q)$ classifies the functor $\text{Sing}(B \text{Diff}(Q)) \to \mathcal{K}an$ exhibiting the $\text{Diff}(Q)$ action on $Q$. Now apply Lemma 12.1.3. \hfill \Box

We need to recall a few tools before proving the lemma.

**Recollection 12.1.5 (Relative nerve).** Fix a functor $f : \mathcal{C} \to \text{sSet}$ from a category $\mathcal{C}$ to the category of simplicial sets. Then one can construct a coCartesian fibration $N_f(\mathcal{C}) \to N(\mathcal{C})$ called the relative nerve of $f$. (See Section 3.2.5 of [Lur09].)

$N_f(\mathcal{C})$ is a simplicial set defined as follows: For any finite, non-empty linear order $I$, an element of $N_f(\mathcal{C})(I)$ is given by the data of:

- A simplex $\phi : \Delta^I \to \mathcal{C}$ (where $\Delta^I \cong \Delta^n$ for $n = |I| - 1$), and
- For every subset $I' \subset I$, setting $i' = \max I'$, a simplex

$$\tau_{i'} : \Delta^{i'} \to f(\phi(i'))$$

These data must satisfy the following condition:

- For any $I' \subset I''$, the diagram of simplicial sets

$$\begin{array}{c}
\Delta^{i'} \xrightarrow{\tau_{i'}} f(\phi(i')) \\
\downarrow \\
\Delta^{i''} \xrightarrow{\tau_{i''}} f(\phi(i''))
\end{array}$$

must commute.
Example 12.1.6. Let us parse what the relative nerve \( N_{p^{-1}}(\text{subdiv}(B)) \) is, where \( p^{-1} \) is the functor from Construction 12.1.1. An \( n \)-simplex in \( N_{p^{-1}}(\text{subdiv}(B)) \) is the data of

- A collection of inclusions of simplices
  \[ \Delta^0 \hookrightarrow \Delta^1 \hookrightarrow \ldots \hookrightarrow \Delta^n \]
  together with a map \( j : \Delta^a \to B \), and
- A map \( \tau : \Delta^n \to j^*E \), such that
- For every \( i \in \{0, \ldots, n\} \), the \( i \)th vertex of \( \tau_{[i]} \) must be a vertex in \( j^*E|_{\Delta^a} \).

Because \( p : E \to B \) is assumed to be a Kan fibration, it follows that the forgetful map \( N_f(\text{subdiv}(B)) \to \text{subdiv}(B) \) is a coCartesian fibration (in fact, a left fibration)—see Proposition 3.2.5.21 of [Lur09].

Remark 12.1.7. Moreover, it is proven in [Lur09] that the fibration \( N_f(C) \), when straightened, classifies a functor naturally equivalent to \( f \). See again Proposition 3.2.5.21 of [Lur09].

Notation 12.1.8. On the other hand, note that we have a natural map

\[ \max : \text{subdiv}(B) \to B \]

for any simplicial set \( B \). On vertices, it sends \( j : \Delta^a \to B \) to the vertex \( j(\max[a]) \in B_0 \). This induces the obvious map on higher simplices.

Thus we have, for any Kan fibration \( E \to B \), the pulled back Kan fibration \( \max^*E \to \text{subdiv}(B) \) as follows:

\[
\begin{array}{ccc}
\max^*E & \rightarrow & E \\
\downarrow & & \downarrow p \\
\text{subdiv}B & \rightarrow & B \\
\end{array}
\]

Proof of Lemma 12.1.3. We have the map of coCartesian fibrations

\[ \max^*E \to N_{p^{-1}}(\text{subdiv}(B)) \]

which, on the fiber above \( j : \Delta^a \to B \), includes the simplicial set of all maps whose \( \tau \) lands in the fiber above \( j(\max[a]) \). This is obviously a weak homotopy equivalence along the fibers because \( E \to B \) is a Kan fibration. Thus we have a diagram

\[
\begin{array}{ccc}
N_{p^{-1}}(\text{subdiv}(B)) & \leftarrow & \max^*E & \rightarrow & E \\
\downarrow & & \downarrow & & \downarrow p \\
\text{subdiv}(B) & = & \text{subdiv}(B) & \rightarrow & B \\
\end{array}
\]

where each horizontal arrow is a weak homotopy equivalence (i.e., an equivalence in the model structure for Kan complexes). Thus every square in this diagram—upon passage to Kan complexes, i.e., their localizations—exhibits an equivalence of Kan fibrations. Because the straightening/unstraightening construction sends equivalences of fibrations to natural equivalences of functors, the result follows.

Now we make use of the Quillen adjunction employing Lurie’s dg nerve construction. See also [BD19].

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Notation 12.1.9 (Nerve and its adjoint). Fix a base ring $R$. Consider the adjunction

$$R[−] : sSet \leftrightarrow dgCat : N_{dg}.$$ 

Here, $N_{dg}$ is Lurie’s dg nerve. (See Construction 1.3.1.6 of [Lur12].) It is a functor sending any dg category to an $\infty$-category, and any dg-functor to a map of simplicial sets. We denote its left adjoint by $R[−].$

Remark 12.1.10. The adjunction of Notation 12.1.9 can be promoted to a Quillen adjunction. The Quillen adjunction is with respect to the Joyal model structure for simplicial sets, and the Tabuada model structure for dg-categories.

Note that these two model structures have simplicial localizations equivalent to $\text{Cat}_\infty$ and $\text{Cat}_{A_\infty}$, respectively. For the fact that the model structure on dg-categories recovers the $\infty$-category of $A_\infty$-categories, see Propositions 2.3.5 and 2.3.7.

Remark 12.1.11. When $\mathcal{C}$ is an $\infty$-groupoid—for example, $\text{Sing}(B)$ for some topological space $B$—then $R[\mathcal{C}]$ is equivalent to the dg-category $C_{s}\mathcal{P}$.

Notation 12.1.12 ($RQ$). Now let $RQ$ denote the following composition:

$$N(\text{Simp}(B \text{Diff}(Q))) \xrightarrow{p^{-1}} \text{Kan} \leftrightarrow sSet \xrightarrow{R[-]} \text{Cat}_{A_\infty}.$$ 

Proposition 12.1.13. There exists a natural equivalence

$$N(\text{Simp}(B \text{Diff}(Q))) \xrightarrow{\mathcal{C},\mathcal{P}} \text{Cat}_{A_\infty}.$$ 

Here, $RQ$ is the functor from Notation 12.1.12 and $C_{s}\mathcal{P}$ is the functor from Notation 10.1.7.

Proof. For every $j : |\Delta^n| \to B \text{Diff}(Q)$, let $D(j) \subset p^{-1}(j)$ denote the full subcategory spanned by those 0-simplices of $j^*E$ that are contained in a fiber above one of the vertices of $|\Delta^n|$. Then the natural transformation induced by the inclusion $D(j) \to p^{-1}(j)$ is essentially surjective and obviously fully faithful. On the other hand, $D(j)$ is equivalent to the path category $\mathcal{P}_j$. Thus the composite natural equivalences

$$p^{-1}(j) \leftarrow D \to \mathcal{P}$$

exhibits the natural equivalence we seek by choosing an inverse to either equivalence. 

Remark 12.1.14. The above proposition accomplishes the goal of seeing that $C_{s}\mathcal{P}$ exhibits the action of $\text{Diff}(Q)$ on the $\infty$-category of local systems. To see this, consider the composite

$$N(\text{Simp}(B \text{Diff}(Q))) \xrightarrow{\text{max}} \text{Sing}^{C_{s}}(B \text{Diff}(Q)) \xrightarrow{\sim} \text{Sing}(B \text{Diff}(Q)) \xrightarrow{\sim} B\text{Sing}(\text{Diff}(Q)) \xrightarrow{\sim} \text{Kan} \xrightarrow{R[-]} \text{Cat}_{A_\infty}.$$ 

Here,
• max is the natural map from a subdivision to the underlying simplicial set; the next arrow is the natural map from smooth, extended simplices to continuous simplices.

(Though we will not use this fact, this is a weak homotopy equivalence by the same arguments as in Lemma 4.6.5 and the fiber sequence of Lemma 4.6.2, applied to \( \text{Diff}(Q) \) as opposed to \( \text{Aut}^{sr,b}(T^*Q) \); here we are using the fact that \( Q \) is compact.)

• The next arrow identifies the singular complex of the classifying space \( B \text{Diff}(Q) \) with the \( \infty \)-category with one object, whose endomorphism space is given by \( \text{Sing}(\text{Diff}(Q)) \). The arrow \( \iota \) is the natural inclusion of this subcategory into \( \mathcal{Kan} \)—the unique object of \( \mathcal{B}(\text{Sing}(\text{Diff}(Q))) \) is sent to the Kan complex \( \text{Sing}(Q) \), and we have the obvious map on morphisms. Finally, \( R[-] \) is the left adjoint to the dg-nerve from Notation 12.1.9.

By the universal property of localization, \( R_Q \) factors as in the below diagram:

\[
\begin{array}{ccc}
N(\text{Simp}(B \text{Diff}(Q))) & \xrightarrow{R_Q} & \mathcal{C}at_{A_\infty} \\
\downarrow & & \downarrow \\
\text{Sing}(B \text{Diff}(Q)) & \cong & B \text{Diff}(Q)
\end{array}
\]

We know from Lemma 12.1.3 that the dashed arrow in the diagram is equivalent to our composite (12.1). Thus, by the natural equivalence of Proposition 12.1.13, we conclude that the functor \( \text{Sing}(B \text{Diff}(Q)) \to \mathcal{C}at_{A_\infty} \) induced by \( C_*\mathcal{P} \) is also equivalent to the composite map of (12.1). This was our goal.

12.2 Proof of Theorem 1.1.9

Proof of Theorem 1.1.9. Among functors from \( \text{NSimp}(B \text{Diff}(Q)) \) to \( \mathcal{C}at_{A_\infty} \), we have the following natural equivalences:

\[ \text{Tw}R_Q \xrightarrow{\text{Prop 12.1.13}} \text{Tw}C_*\mathcal{P} \xrightarrow{\text{Thm 11.0.1}} \text{Tw}\mathcal{W} \circ \mathcal{D}. \]

Because all three of the above functors—\( \text{Tw}R_Q \), \( \text{Tw}C_*\mathcal{P} \), and \( \text{Tw}\mathcal{W} \circ \mathcal{D} \)—map morphisms in \( \text{NSimp}(B \text{Diff}(Q)) \) to equivalences in \( \mathcal{C}at_{A_\infty} \), each induces a functor from the Kan completion of \( \text{NSimp}(B \text{Diff}(Q)) \). This Kan completion is an \( \infty \)-groupoid equivalent to \( \text{Sing}(B \text{Diff}(Q)) \), and hence to \( \mathcal{B}\text{Diff}(Q) \), by Theorem 4.4.6.

By Remark 12.1.14, the functor induced by \( \text{Tw}R_Q \) classifies the \( \text{Diff}(Q) \) action on \( C_*\mathcal{P}(Q) \). On the other hand, \( \text{Tw}\mathcal{W} \circ \mathcal{D} \) by construction classifies the action of \( \text{Diff}(Q) \) on \( \mathcal{W}(M) \) induced by the action of \( \text{Aut} \) on \( \mathcal{W}(M) \). This completes the proof.

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