Vanishing viscosity for linear-quadratic mean-field control problems

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Abstract—We consider a mean-field control problem with linear dynamics and quadratic control. We prove the existence of an optimal control by applying the vanishing viscosity method: we add a (regularizing) heat diffusion with a small viscosity coefficient and let such coefficient go to zero. The main result is that, in this case, the limit optimal control is exactly the optimal control of the original problem.

I. INTRODUCTION

Linear-Quadratic problems (LQ from now on) for finite-dimensional systems are the easiest non-trivial examples in optimal control, see [19]. Their use in control theory is ubiquitous, in particular as the simplest stabilizers around a nominal trajectory. Thus, any optimal control theory for a new class of systems should confront itself with LQ problems. This article aims to define and solve LQ problems for deterministic mean-field control systems.

Mean-field equations are the natural limit of a large number \(N\) of interacting particles when \(N\) tends to infinity. The state of the system is then a density or, more in general, a measure. We can apply a control to steer the system to a desired configuration or, as in our case, to optimize some cost. The resulting dynamics is called a mean-field control problem. See a general treatment of these problems in [2], [3].

Mean-field control problems are intimately related to mean-field games, as clearly explained in [2]. We recall that mean-field games (first introduced in [10], [14]) describe the limit of a large number of interacting particles in which each particle optimizes a personal cost, in the spirit of Nash differential games. Instead, in mean-field control an (external) controller aims to minimize a global cost for the whole population. In this sense, our contribution aims to provide one more theoretical tool (the vanishing viscosity method) to the study of mean-field control problems. We will focus on LQ models: for mean-field games, they were first studied in [11], where they are called LQG systems (G stands for Gaussian noise).

In this article, we study two deeply connected mean-field control problems. The first corresponds to the mean-field of a deterministic ordinary differential equation, that is a continuity equation for the measure. Its expression is

\[
\partial_t \mu_t + \text{div}(b(t,x,\mu_t,u)\mu_t) = 0,
\]

where \(\mu_t\) is a time-dependent measure and \(b(t,x,\mu_t,u)\) is a vector field, depending on time, space, the measure itself, and the control \(u\). The second mean-field control equation is the mean-field of a stochastic ordinary differential equation with additive Brownian motion \(\sqrt{2\varepsilon} W_t\), that is an advection-diffusion equation. Its explicit expression is

\[
\partial_t \mu_t + \text{div}(b(t,x,\mu_t,u)\mu_t) = \varepsilon \Delta \mu_t,
\]

where \(\varepsilon \Delta \mu_t\) represents the heat diffusion. It is natural to ask if solutions of (2) converge to (1) when \(\varepsilon \to 0\), i.e. when passing from the probabilistic to the deterministic mean-field problem. Such limit is known as the vanishing viscosity method, and it can be traced back to [12], [13], [17].

Such more theoretical aspect also has an applied interest: indeed, regularity of the viscous problem implies that standard numerical methods to solve it can be applied, then the deterministic problem can be solved by convergence. Moreover, the introduction of inherent noises is standard in real life models, like for example [8].

We now couple the (deterministic or probabilistic) dynamics for the measure \(\mu_t\) with a (given) running and final cost

\[
J(\mu, u) = \int_0^T \int_{\mathbb{R}^d} f(t,x,\mu_t,u)\,d\mu_t\,dt + \int_{\mathbb{R}^d} g(x,\mu_T)\,d\mu_T.
\]

We then have two optimal controls: on one side, the minimizer \(u\) of the deterministic optimal control problem coupling (1) and (3); on the other side, the family of minimizers \(u^\varepsilon\) of the probabilistic optimal controls coupling (2) and (3), indexed by \(\varepsilon > 0\). In this case, one can again ask the following questions related to vanishing viscosity:

- Do we have convergence of optimal controls \(u^\varepsilon \to u\)?
- Do we have convergence of optimal trajectories \(\mu^\varepsilon_t \to \mu_t\)?
- Do we have convergence of costs \(J(\mu^\varepsilon, u^\varepsilon) \to J(\mu,u)\)?

Such questions do not have a general answer. Our main result (Theorem 1) states that, if the dynamics is linear and the cost is quadratic, all answers are positive. Up to our knowledge, this work is the first application of the vanishing viscosity in mean-field control.

The structure of the article is the following. We first fix notation in Section I-A. In Section II we define the deterministic and probabilistic problems \((\mathcal{P}_t), (\mathcal{P}_\varepsilon)\) and state our main result. In Section III we collect some preliminary tools (Wasserstein distances, derivatives of functions of measures) and we briefly recall well-posedness for solutions of (1) and (2). We then study the probabilistic problem \((\mathcal{P}_\varepsilon)\) in Section IV, where we explicitly build the associated optimal control. The tools and the results of the sections III and IV
are then used in Section V to prove our main result, that is convergence of the solution of \((P_\varepsilon)\) to the one of \((P)\). We draw some conclusions in Section VI.

A. Notation

We denote with \(x', M'\) the transpose of the vector \(x\) and matrix \(M\). We write that a matrix \(Q\) satisfies \(Q > 0, Q \geq 0\) when it is positive definite (resp. semi-definite).

We will work on the Euclidean space \(\mathbb{R}^d\) and we denote by \(\mathcal{P}(\mathbb{R}^d)\) the space of probability measures on \(\mathbb{R}^d\). We use the notation \(\mathcal{L}^d\) for the standard Lebesgue measure on \(\mathbb{R}^d\).

The set \(\mathcal{P}_c(\mathbb{R}^d)\) is the subset of the probability measures with finite second moment, that is of measures \(\mu\) satisfying

\[
\int_{\mathbb{R}^d} |x|^2 \, d\mu < \infty.
\]

We will mostly focus on the subspace of absolutely continuous measures with compact support \(\mathcal{P}_c^a(\mathbb{R}^d)\).

We denote with \(\bar{\mu}\) the barycenter of the measure \(\mu\), that is defined as

\[
\bar{\mu} := \int_{\mathbb{R}^d} \xi \, d\mu(\xi).
\]

We also denote with \(\mu_n \rightharpoonup \mu\) the standard weak-* convergence of measures. We recall that it means the following:

\[
\forall \varphi \in C_c^\infty(\mathbb{R}^d) \quad \text{it holds} \quad \int_{\mathbb{R}^d} \varphi \, d\mu_n \to \int_{\mathbb{R}^d} \varphi \, d\mu.
\]

Finally, we denote the space of Lipschitz functions as

\[
\text{Lip}(\mathbb{R}^d) := \{ f : \mathbb{R}^d \to \mathbb{R}^d \mid |L| > 0 \text{ s.t. } \forall x, y \in \mathbb{R}^d \mid |f(x) - f(y)| \leq L|x - y| \}.
\]

II. Problem Statement

In this article, we consider two LQ mean-field control problems. This means that:

- the dynamics is linear, i.e. the vector field is of the form
  \[
  b(t, x, \mu, u) = A(t)x + B(t)u + \bar{A}(t)\bar{\mu}_t,
  \]
- the cost is quadratic, i.e the cost \(J(\mu, u)\) defined in (3) is of the form
  \[
  J(\mu, u) = \frac{1}{2} [x'Q(t)x + u'R(t)u + (x - S(t)\bar{\mu}_t)'\bar{Q}(t)(x - S(t)\bar{\mu}_t)],
  \]
  \[
  g(x, \mu) = \frac{1}{2} [x'Q_Tx + (x - S_T\bar{\mu}_T)'\bar{Q}_T(x - S_T\bar{\mu}_T)].
  \]

Here, all operators \(A(t), B(t), \bar{A}(t), Q(t), R(t), S(t), \bar{Q}(t), Q_T, S_T, \bar{Q}_T\) are linear operators \(\mathbb{R}^d \to \mathbb{R}^d\), i.e. square matrices. The first 7 of them are continuous functions of time, defined for all \(t \in [0, T]\). We will often omit the dependence of matrices on time, for simplicity of notation.

Remark 1: It is interesting to observe that both the dynamics \(b\) and the cost \(J(\mu, u)\) contains terms depending on the barycenter \(\bar{\mu}_t\) of the state, following [2]. Indeed, in mean-field control problems, it is interesting to steer the state measure either towards its barycenter or far from it (i.e. concentrate or dissipate the population).

We will assume standard symmetry and positive-definiteness of matrices from now on. We also need to add two conditions related to the barycenter terms. They are summarized here:

\[
\text{(M)}
\]

- Running cost: for all \(t \in [0, T]\), the matrices \(Q(t), \bar{Q}(t), R(t)\) are symmetric with \(Q(t), \bar{Q}(t) \geq 0\) and \(R(t) > 0\).
- Barycenter in the running cost: for all \(t \in [0, T]\) it holds
  \[
  Q(t) + (I - S(t))'\bar{Q}(t)(I - S(t)) \geq 0.
  \]
- Final cost: it holds \(Q_T, \bar{Q}_T\) symmetric, both satisfying \(Q_T, \bar{Q}_T \geq 0\).
- Barycenter in the final cost: it holds
  \[
  Q_T + (I - S_T)'\bar{Q}_T(I - S_T) \geq 0.
  \]

We are now ready to define the deterministic problem \((P)\):

\[
\text{PROBLEM } (P)
\]

Find

\[
\min_{u \in \mathcal{U}} J(\mu_t, u),
\]

such that \(\mu_t \in C([0, T]; \mathcal{P}_c(\mathbb{R}^d))\) is a solution of

\[
\begin{aligned}
\partial_t \mu_t + \text{div}(b(t, x, \mu_t, u)) & = 0, \\
\mu_t|_{t=0} & = \mu_0.
\end{aligned}
\]

The cost \(J(\mu, u)\) is quadratic, given by (3), (5), (6) and the matrices satisfy (M).

The dynamics is linear, given by (4). The initial state satisfies \(\mu_0 \in \mathcal{P}_c^a(\mathbb{R}^d)\).

The set of admissible controls is

\[
\mathcal{U} := L^1((0, T); \text{Lip}(\mathbb{R}^d)).
\]

The probabilistic (or viscous) problem is very similar:

\[
\text{PROBLEM } (P_\varepsilon)
\]

Take \((P)\) and replace the dynamics (9) with

\[
\begin{aligned}
\partial_t \mu_t + \text{div}(b(t, x, \mu_t, u)) & = \varepsilon \Delta \mu_t, \\
\mu_t|_{t=0} & = \mu_0.
\end{aligned}
\]

As already stated above, the crucial difference between \((P)\) and \((P_\varepsilon)\) is given by the presence of a viscosity term \(\varepsilon \Delta \mu_t\). This corresponds to the fact that the underlying dynamics in \((P)\) is deterministic, while in \((P_\varepsilon)\) a Brownian motion \(\sqrt{2\varepsilon W_t}\) is added. From the mathematical point of
view, the dynamics of \((P_\varepsilon)\) is easier to study than \((P)\), as the viscosity term (that is a heat diffusion) ensures stronger regularity of the solution. For this reason, it is classical to study the regular viscous problem \((P_\varepsilon)\) and hope to pass to the limit to infer something on the original deterministic problem \((P)\). Our aim is then to apply the vanishing viscosity method to our problem: we study the convergence of solutions of \((P_\varepsilon)\) to solutions of \((P)\) for \(\varepsilon \to 0\). In the LQ setting, we have convergence, as stated in our main result.

**Theorem 1:** Let \((\hat{\mu}_\varepsilon, \hat{v}_\varepsilon)\) be solutions of \((P_\varepsilon)\).

Then exists a solution \((\hat{\mu}, \hat{v})\) of \((P)\) such that, for \(\varepsilon \to 0\), it holds:

(i) \(\hat{v}_\varepsilon \to \hat{v}\) in \(U = L^1((0, T); \text{Lip}(\mathbb{R}^d))\);

(ii) \(\hat{\mu}_\varepsilon \to \hat{\mu}\) in \(C([0, T], \mathcal{P}_2(\mathbb{R}^d))\);

(iii) \(J(\hat{\mu}_\varepsilon, \hat{v}_\varepsilon) \to J(\hat{\mu}, \hat{v})\).

**Remark 2:** One usually looks for admissible controls \(u \in L^1((0, T); L^1(\mathbb{R}^d; \text{d} \mu_\varepsilon))\) in \((P)\) and \(L^1((0, T); L^1(\mathbb{R}^d; \text{d} \mu_\varepsilon'))\) in \((P_\varepsilon)\). This is indeed the minimum requirement in order to give a (weak) meaning to equations \((9)\) and \((10)\).

However, for our purposes it is not restrictive to search for a Lipschitz optimal control for the following reasons:

- in \((P)\), conditions \((7)-(8)\) imply that the coercivity condition of [4] is satisfied; this ensures that there exists at least one Lipschitz optimal control;
- we know a priori that even if in \((P_\varepsilon)\) we look for controls in the larger class \(L^1((0, T); L^1(\mathbb{R}^d; \text{d} \mu_\varepsilon'))\), the optimum is Lipschitz. See [6, Lemma 6.18].

Therefore, by restricting ourselves to Lipschitz controls, we have the advantage that equation \((9)\) is well-posed, see Theorem 2 below. Moreover, the functional spaces do not depend on the solutions \(\mu_t, \mu'_t\) anymore.

**III. MEAN-FIELD EQUATIONS**

In this section, we recall the main tools that allow to study mean-field equations. The most relevant examples of such equations are the ones in which the vector field is given by a convolution, i.e. of the form

\[
\partial_t \mu_t + \text{div}(V(\mu_t \ast H) \mu_t) = 0.
\]

Convolutions indeed describe long-range interaction between particles. The mean-field limit of several key agent-based models have been described and studied, such as the limits of bounded confidence models [5] or Cucker-Smale [9].

More in general, we study the following Cauchy problem:

\[
\begin{cases}
\partial_t \mu_t + \text{div}(v[\mu_t](t, \cdot) \mu_t) = 0, \\
\mu_t|_{t=0} = \mu_0,
\end{cases}
\]

where the vector field \(v[\mu] : (0, T) \times \mathbb{R}^d \to \mathbb{R}^d\) and the initial probability measure \(\mu_0 \in \mathcal{P}(\mathbb{R}^d)\) are given.

Observe that the vector field depends on the whole measure \(\mu\) itself. It represents the fact that the dynamics in a point of the density actually depends on the density elsewhere, due to long-range interactions. The presence of such phenomenon, known as non-locality, requires to develop a specific theory to ensure well-posedness of \((12)\).

**A. Well-posedness of the deterministic Cauchy problem**

In this section, we study well-posedness of \((12)\). Such theory is based on Wasserstein distance, that we recall here.

**Definition 3:** Let \(\mu, \nu \in \mathcal{P}^2(\mathbb{R}^d)\). We say that \(\gamma \in \mathcal{P}(\mathbb{R}^{2d})\) is a transport plan between \(\mu\) and \(\nu\), denoted by \(\gamma \in \Pi(\mu, \nu)\), when \(\gamma(A \times \mathbb{R}^d) = \mu(A)\) and \(\gamma(\mathbb{R}^d \times B) = \nu(B)\) for any pair of Borel sets \(A, B \subset \mathbb{R}^d\).

Given two measures \(\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)\), the Wasserstein distance between \(\mu\) and \(\nu\) is given by

\[
W_2^2(\mu, \nu) = \min_{\gamma \in \Pi(\mu, \nu)} \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \gamma(\text{d}x, \text{d}y) \right\}.
\]

We will consider the space \(\mathcal{P}_2(\mathbb{R}^d)\) endowed with the Wasserstein distance \(W_2\) from now on. It is remarkable to observe that the Wasserstein distance metrizes the weak-* topology of probability measures. More precisely, it holds:

\[
W_2(\mu, \mu_n) \to 0 \iff \mu_n \rightharpoonup^* \mu,
\]

where the vector field \(v\) be uniformly Lipschitz with respect to the Wasserstein distance on \(\mathcal{P}_2(\mathbb{R}^d)\) and the Euclidean distance in \(\mathbb{R}^d\), i.e. there exists a constant \(L\) such that

\[
|v[\mu](t, x) - v[\nu](t, y)| \leq L (W_2(\mu, \nu) + |x - y|),
\]

for all \(t \in \mathbb{R}, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)\), and \(x, y \in \mathbb{R}^d\). Then, for each \(\mu_0 \in \mathcal{P}_c(\mathbb{R}^d)\) there exists a unique solution to \((12)\).

The key consequence of this theorem for the study of \((P)\) is that the exponential map, associating to each Lipschitz control \(u\) the corresponding solution of \((9)\), is well defined.

**B. Derivative with respect to a measure**

In this section, we provide some background material on differential calculus in the space of probability measures. The definitions and the proofs of the results of this section are taken from [6].

Since we deal with minimization of a functional \(J\), it is useful to have a first-order condition with respect to perturbations of the state \(\mu\). We recall that there are several different concepts of derivatives with respect to measures, see e.g. [6]. For our problem, we need the so-called L-derivative. Let \((\Omega, F, \nu)\) be an atomless probability space. Given a map \(h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}\) we define the lifting

\[
\hat{h}(X) = h(\mathcal{L}(X)), \quad \forall X \in L^2(\Omega; \mathbb{R}^d),
\]

where \(\mathcal{L}(X) := X \# \nu\). Note that \(\mathcal{L}(X) \in \mathcal{P}_2(\mathbb{R}^d)\), since \(X \in L^2(\Omega; \mathbb{R}^d)\).
Definition 5: A function $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be $L$-differentiable at $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ if there exists a random variable $\hat{h}$ at $X$ such that $h$ is Fréchet differentiable at $X$, and $X$ has a Fréchet differentiable at $X$, and $(X, D\hat{h}(X))$ has the same law as $(X', D\hat{h}(X'))$.

Proposition 6: Let $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ be a $L$-differentiable function. Then for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ there exists a measurable function $\xi : \mathbb{R}^d \to \mathbb{R}^d$ such that for all $X \in L^2(\Omega; \mathbb{R}^d)$ with law $\mu$, it holds that $Dh(X) = \xi(X)$ $\mu$-almost surely.

With this notation, the equivalence class of $\xi \in L^2(\mathbb{R}^d; \mathbb{R}^d)$ is uniquely defined and we denote by $\partial_\mu h(\mu_0)$. We call $L$-derivative of $h$ at $\mu_0$ the function

$$\partial_\mu h(\mu_0) : x \in \mathbb{R}^d \mapsto \partial_\mu h(\mu_0)(x).$$

Finally, a function $h : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is said to be $L$-convex if it is $L$-differentiable and

$$h(\mu) - h(\mu') - \mathbb{E}[\partial_\mu h(\mu_0)(X) : (X - X')] \geq 0,$$

whenever $X, X' \in L^2(\Omega; \mathbb{R}^d)$ with law, respectively, $\mu, \mu'$.

IV. The Viscous LQ Problem ($\mathcal{P}_\varepsilon$)

In this section, we solve the viscous problem ($\mathcal{P}_\varepsilon$). We prove existence and uniqueness of the optimal control and provide an explicit expression by a cascade of two Riccati equations. This section is mostly based on [1], [2], [6].

The main idea to solve ($\mathcal{P}_\varepsilon$) is to use stochastic control to find the optimal pair $(\mu^*_\varepsilon, u^*_\varepsilon)$ for it. Indeed, we may think of $\mu^*_\varepsilon$ as the law of a stochastic process $\mathcal{X}^\varepsilon$ which solves the stochastic differential equation

$$dX^\varepsilon_t = b(X^\varepsilon_t, \mu^*_\varepsilon_t,\alpha^*_\varepsilon_t) dt + \sqrt{2\varepsilon} dW_t, \quad X^\varepsilon_0 = X_0. \quad (14)$$

Here $W_t$ is a standard Brownian motion, $\alpha$ is the control, $X_0$ is a random variable independent from $W_t$ with law $\mu_0$, and the equation has to be understood in the Itô sense. With these notations the cost functional can be rewritten as

$$J(\alpha) = \mathbb{E} \left[ \int_0^T f(t, X^\varepsilon_t, \mu^*_\varepsilon_t,\alpha^*_\varepsilon_t) dt + g(X^\varepsilon_T, \mu^*_\varepsilon_T) \right],$$

where the control $\alpha_t$ is a measurable process with values in $\mathbb{R}^d$, which satisfies $\mathbb{E} \left[ \int_0^T |\alpha_t|^2 dt \right] < \infty$. We remark that this latter conditions is required in order to have well-posedness of the equation (14), see [6].

When the data of the optimization problem are smooth enough, solving ($\mathcal{P}_\varepsilon$) and applying the stochastic approach just described are equivalent. This is the case of the present LQ setting, see [6], [15] for the precise description of the two approaches and the connection between them. We will use the second approach from now on.

A. The adjoint variable and stochastic maximum principle

In this section we define the adjoint process of a controlled state $X_t$ and we recall the Stochastic Pontryagin Maximum Principle (SPMP) for optimality. Since we will give general definitions, in this subsection we drop the superscript $\varepsilon$ for simplicity. We will use the notation $(\Omega, \mathcal{F}, \mathbb{P})$ for a copy of $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{E}$ for the expectation under $\mathbb{P}$.

First, we define the Hamiltonian $H$ as

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha), \quad (15)$$

for $(t, x, \mu, y, \alpha) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d$.

Definition 8: We call an adjoint processes of $X_t$ any couple $(Y_t, Z_t)$ satisfying the equation

$$\left\{ \begin{array}{l}
- dY_t = (\nabla_\alpha H(t, X_t, \mu_t, Y_t, Z_t, \alpha_t)) dt - Z_t dW_t, \\
Y_T = g(X_T, \mu_T) + \mathbb{E}[\partial_\mu g(Y_T, \mu_T) / \mu_T].
\end{array} \right. \quad (16)$$

where $(\tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{\alpha}_t)$ is an independent copy of $(X_t, Y_t, Z_t, \alpha_t)$ defined on the space $(\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

We have the following sufficient condition for optimality associated to the SPMP, see [6].

Theorem 3: Let $b, f, g$ be Linear-Quadratic, i.e. (4)-(5)-(6) hold. Let $\alpha_0$ be an admissible control, $X_t$ the corresponding controlled state process, and $(Y_t, Z_t)$ the corresponding adjoint processes. Assume that:

• $(x, \mu) \mapsto g(x, \mu)$ is convex,
• $(x, \mu, \alpha) \mapsto H(t, x, \mu, Y_t, Z_t, \alpha)$ is convex.

If it holds $L^1 \otimes \mathbb{P}$-a.e. that

$$H(t, X_t, \mu_t, Y_t, Z_t, \alpha^*_\varepsilon) = \inf_{\alpha} H(t, X_t, \mu_t, Y_t, Z_t, \alpha),$$

then $\alpha^*_\varepsilon$ is an optimal control. Moreover, if $H$ is strictly convex in $\alpha$, then the optimal control is also unique.

The convexity in the measure variable in the Theorem above is intended as L-convexity. By assuming $H$ and $g$ defined as (15) and (6), the hypothesis of Theorem 3 are satisfied and we can compute explicitly the unique optimal control. This is the content of the next section.

B. Computation of the optimal control

In this section, we explicitly find the optimal control $\hat{u}^\varepsilon$ of ($\mathcal{P}_\varepsilon$), showing moreover that it does not depend on the parameter $\varepsilon$. With this goal, we first solve the forward-backward system of equations given by the SPMP. We then compute $\hat{u}^\varepsilon$.

By Theorem 3, the optimal control is given by the minimizer of the Hamiltonian, i.e.

$$\hat{\alpha} = \hat{\alpha}(t, x, \mu, y) = -R^{-1}B'y,$$

while the L-derivative of $H$ is

$$\partial_\mu H(t, x, \mu, y)(x') = \bar{A}'y - S'\bar{Q}(x - S\bar{\mu}).$$
Then, by plugging $\hat{\alpha}$ in (14) and (16), and defining $\bar{x}_\varepsilon^t := \mathbb{E}[X_\varepsilon^t], \bar{y}_\varepsilon^t := \mathbb{E}[Y_\varepsilon^t]$, we obtain the following forward-backward system:

$$
\left\{
\begin{aligned}
\dot{X}_\varepsilon^t &= (AX_\varepsilon^t - BR^{-1}B'Y_\varepsilon^t + \hat{\alpha}) dt + \sqrt{\varepsilon} dW_t, \\
X_\varepsilon^0 &= x_0, \\
-\dot{Y}_\varepsilon^t &= (AY_\varepsilon^t + (Q + \hat{\alpha}^2)X_\varepsilon^t - QS\bar{x}_\varepsilon^t) dt \\
Y_\varepsilon^0 &= (Q_T + \hat{Q}_T)X_\varepsilon^T + (S_T Q_T S_T - S_T' Q_T - \hat{Q}_T S_T)\bar{x}_\varepsilon^T.
\end{aligned}
\right.
$$

By taking the expectations in (17), we have that $\bar{x}_\varepsilon^t, \bar{y}_\varepsilon^t$ satisfies the system

$$
\left\{
\begin{aligned}
\dot{x}_\varepsilon^t &= (A + \hat{A})\bar{x}_\varepsilon^t - BR^{-1}B'\bar{y}_\varepsilon^t, \\
-\dot{y}_\varepsilon^t &= (Q + (I - S')\hat{Q}(I - S))\bar{x}_\varepsilon^t + (A + \hat{A})\bar{y}_\varepsilon^t, \\
x_0^\varepsilon = \mu_0, \quad \bar{y}_0^\varepsilon = (Q_T + (I - S_T')\hat{Q}_T(I - S_T))\bar{x}_\varepsilon^T.
\end{aligned}
\right.
$$

To solve the finite-dimensional control problem (18) we proceed as follows: consider the following linear advection-diffusion equation.

$$
\left\{
\begin{aligned}
\dot{x}_\varepsilon^t &= (A + \hat{A})x_\varepsilon^t - BR^{-1}B'p_\varepsilon^t, \\
-\dot{y}_\varepsilon^t &= (Q + (I - S')\hat{Q}(I - S))x_\varepsilon^t + (A + \hat{A})y_\varepsilon^t, \\
x_0^\varepsilon = \mu_0, \quad y_0^\varepsilon = (Q_T + (I - S_T')\hat{Q}_T(I - S_T))x_\varepsilon^T.
\end{aligned}
\right.
$$

where $\bar{y}_\varepsilon^t$ is the unique solution of the Riccati equation (see for example [19])

$$
\left\{
\begin{aligned}
\dot{\Sigma} + \Sigma (A + \hat{A}) + (A + \hat{A})' \Sigma - \Sigma BR^{-1}B'\Sigma + Q + (I - S')\hat{Q}(I - S) = 0, \\
\Sigma(T) = Q_T + (I - S_T')\hat{Q}_T(I - S_T),
\end{aligned}
\right.
$$

which is symmetric and positive definite. Then, the system (18) can be reinterpreted as the system given by the Pontryagin’s Maximum principle applied to the above finite dimensional problem, where $\bar{x}_\varepsilon^T$ is the optimal trajectory and $\bar{y}_\varepsilon^T$ the associated co-state. Then it holds $\bar{y}_\varepsilon^t = \Sigma \bar{x}_\varepsilon^t$, and substituting in (18) we get

$$
\left\{
\begin{aligned}
\dot{x}_\varepsilon^t &= (A + \hat{A} - BR^{-1}B'\Sigma)x_\varepsilon^t, \\
\dot{y}_\varepsilon^t &= \mu_0.
\end{aligned}
\right.
$$

Since (19) is a linear ODE with smooth coefficients, it admits a unique solution $\bar{x}_t$, which therefore does not depend on $\varepsilon$. It immediately follows that $\bar{y}_\varepsilon^t = \bar{y}_\varepsilon^T$. Then, once $\mathbb{E}[X_\varepsilon^t]$ and $\mathbb{E}[Y_\varepsilon^t]$ are replaced with $\bar{x}_t, \bar{y}_t$, the system (17) can be associated to a standard stochastic control problem with strictly convex Hamiltonian, hence it admits a unique solution, see [6]. The affine structure of (17) suggests to write $Y_\varepsilon^t = P^\varepsilon x_\varepsilon^t + \mu_\varepsilon^t$, where

$$
\left\{
\begin{aligned}
\dot{\mu}_\varepsilon^t &= A^\varepsilon \mu_\varepsilon^t + P^\varepsilon A - P^\varepsilon BR^{-1}B'P^\varepsilon + Q + \bar{Q}, \\
P^\varepsilon(T) &= Q_T + \bar{Q}_T, \\
\dot{\mu}_\varepsilon^t &= (A^\varepsilon - P^\varepsilon BR^{-1}B')\mu_\varepsilon^t + (A\Sigma + P^\varepsilon A + S^T\bar{Q}S - S^T\bar{Q} - \bar{Q})\bar{x}_t, \\
p^\varepsilon(t) &= (S_T^t \bar{Q}_T S_T - S_T^t \bar{Q}_T - \bar{Q}_T S_T)\bar{x}_t, \\
Z^\varepsilon_t &= \sqrt{\varepsilon} P^\varepsilon_t.
\end{aligned}
\right.
$$

Arguing as before, the first equation in (20) is of Riccati-type, then it admits a unique solution $P^\varepsilon$. Hence, uniqueness of the solution implies that the matrix $P^\varepsilon$ does not depend on $\varepsilon$. Therefore, once we have the matrix $P$, the equation for $p^\varepsilon$ is a linear ODE and again by uniqueness it admits a unique solution, which then does not depend on $\varepsilon$. On balance, $Z^\varepsilon_t$ is the only term which depends on the viscosity.

We can now write the optimal control $\hat{\alpha}$ as

$$
\hat{\alpha}_t = \hat{u}(t, x_\varepsilon^t) = -R^{-1}B'P x_\varepsilon^t - R^{-1}B'p_\varepsilon^t,
$$

which is in feedback form. Finally, by the connection with (12) described above, we know that the optimal control $\hat{u}$ is then given by

$$
\hat{u}(t, x) = -R^{-1}B'P x - R^{-1}B'p.
$$

The equation (21) is a linear advection-diffusion equation. It is still non-local, since the dependence on the barycenter is contained in the term $b$.

V. PROOF OF THE MAIN RESULT

In this section we prove our main result, that is Theorem 1. We show that the unique optimal pair $(\hat{\mu}_\varepsilon^t, \hat{u}_\varepsilon(t))$ of $(P_\varepsilon)$ converges to an optimal pair $(\mu_\varepsilon, \hat{u})$ for $(P)$.

A. Preliminary lemmas

In this subsection we prove a stability lemma for the continuity equation; we then apply it to our optimal control problem. It is worth to note that we assume that the control $u$ is fixed in the following Lemma.

**Lemma 9:** Let $b : (\mathbb{R}^d)^2 \mapsto \text{Lip}((0, T) \times \mathbb{R}^d)$ be a vector field satisfying (13), such that there exists $C$ for which

$$
|b(t, x, \mu)| \leq C \left( 1 + |x| + \int_{\mathbb{R}^d} |y| \mu(dy) \right).
$$

Let $\mu_\varepsilon, \mu_\varepsilon^t$ be the unique solution of the deterministic (9) and the viscous equation (10) with vector field $b$, respectively. It then holds

$$
\lim_{\varepsilon \to 0} \sup_{t \in [0, T]} W_2(\mu_\varepsilon^t, \mu_t) = 0.
$$

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Moreover, the barycenter and costs converge too:
\[ \bar{\mu}_t^\varepsilon \to \bar{\mu}_t, \quad \text{as } \varepsilon \to 0, \text{ uniformly in } [0, T], \]
\[ J(\mu_t^\varepsilon, u) \to J(\mu_t, u), \quad \text{as } \varepsilon \to 0. \]

**Proof:** The core of the proof is to prove (22). Since \( \mu_t, \mu_t^\varepsilon \in \mathcal{P}_2(\mathbb{R}^d) \), one can use \(|x|^2\) as a test function in the dynamics (9), (10). By a Gronwall type argument, thanks to the growth assumptions on \( b \), it holds
\[ \sup_{t \in [0, T]} \int_{\mathbb{R}^d} |x|^2 \mu_t^\varepsilon (dx) < C, \]
where \( C > 0 \) is a constant independent on \( \varepsilon \). Then, by properly choosing the test function in (10) and, again by the growth assumptions on \( b \), it also holds
\[ \mu_t^\varepsilon (\mathbb{R}^d \setminus B_R) \leq CT/R, \quad (23) \]
where \( C \) is a constant which depends only on the parameters of the problem and
\[ \sup_{t \in [0, T]} \int_{B_R^c} |x|^2 \mu_t^\varepsilon (dx) \to 0, \quad \text{uniformly as } R \to \infty. \quad (24) \]
The bound (23) implies tightness of \( \mu_t^\varepsilon \). Then, up to sub-sequences, there exists a measure \( \lambda_t \in L^\infty((0, T); \mathcal{P}_2(\mathbb{R}^d)) \) such that
\[ \lim_{\varepsilon \to 0} \sup_{t \in [0, T]} W_2(\mu_t^\varepsilon, \lambda_t) = 0. \quad (25) \]
Because of the uniform Lipschitz assumption on \( b \), the convergence in (25) is enough to pass to the limit in (10) and hence uniqueness for (9) implies \( \mu_t = \lambda_t \). In particular, the whole original sequence \( \mu_t^\varepsilon \) converges to \( \mu_t \). Finally, convergence of the barycenter and of the costs follows from (22), by a direct computation.

**B. Proof of the main result**

We are now able to prove our main result.

**Proof of Theorem 1.** First of all, by the analysis given in Section IV, the convergence of controls \( \hat{u}^\varepsilon \to \hat{u} \) is a direct consequence of the fact that the optimal control \( \hat{u} \) does not depend on \( \varepsilon \). Then, we consider \( \hat{\mu}_t \), the unique solution of (9) with control \( \hat{u} \). Since the control does not depend on the viscosity parameter, the convergence of optimal trajectory \( \hat{\mu}_t^\varepsilon \to \hat{\mu} \) and of the cost \( J(\hat{\mu}_t^\varepsilon, \hat{u}) \to J(\hat{\mu}, \hat{u}) \) is a direct consequence of Lemma 9.

To conclude, we must show that \((\hat{\mu}_t, \hat{u})\) is actually an optimal pair for \((\mathcal{P})\). Let \( u \neq \hat{u} \) be a Lipschitz control and \( \mu_t \) the corresponding trajectory. We define \( \mu_t^\varepsilon \) to be the unique solution of (10) with control \( u \) and, since \((\mu_t^\varepsilon, u)\) is an admissible pair for \((\mathcal{P}_2)\) and \((\hat{\mu}_t^\varepsilon, \hat{u})\) is optimal, it holds
\[ J(\hat{\mu}_t^\varepsilon, \hat{u}) < J(\mu_t^\varepsilon, u). \quad (26) \]
By \((iii)\) and Lemma 9, it holds
\[ J(\hat{\mu}_t, \hat{u}) = \lim_{\varepsilon \to 0} J(\hat{\mu}_t^\varepsilon, \hat{u}), \quad J(\mu_t, u) = \lim_{\varepsilon \to 0} J(\mu_t^\varepsilon, u), \quad (27) \]
and combining (26) and (27) we get that
\[ J(\hat{\mu}_t, \hat{u}) \leq J(\mu_t, u), \]
for any admissible pair \((\mu_t, u)\). Then, \((\hat{\mu}_t, \hat{u})\) is an optimal pair for \((\mathcal{P})\) and the proof is complete.

**VI. CONCLUSIONS AND FUTURE PERSPECTIVES**

In this article, we proved that the vanishing viscosity method works properly when applied to the deterministic LQ mean-field control problem \((\mathcal{P})\): one can add a small noise of amplitude \( \varepsilon \), solve the optimal control problem, and is ensured that for \( \varepsilon \to 0 \) the limit is the solution of the deterministic LQ mean-field problem.

Our goal now is to study more general mean-field optimal control problems in [7], in particular with coercive costs as in [4]. Moreover, it will be very interesting to study coupled mean-field controlled systems and mean-field games, i.e. with two levels of optimization.

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