Light Nonabelian Monopoles and Generalized $r$-Vacua in Supersymmetric Gauge Theories

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Abstract

We study a class of $\mathcal{N} = 1$ supersymmetric $U(N)$ gauge theories and find that there exist vacua in which the low-energy magnetic effective gauge group contains multiple nonabelian factors, $\prod_i SU(r_i)$, supported by light monopoles carrying the associated nonabelian charges. These nontrivially generalize the physics of the so-called $r$-vacua found in softly broken $\mathcal{N} = 2$ supersymmetric $SU(N)$ QCD, with an effective low-energy gauge group $SU(r) \times U(1)^{N-r}$. The matching between classical and quantum $(r_1, r_2, \ldots)$ vacua gives an interesting hint about the nonabelian duality.

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1. Introduction

The quantum behavior of nonabelian monopoles in spontaneously broken nonabelian gauge systems is of considerable interest. It could for instance be a key for understanding the confinement in QCD. In general, semiclassical “nonabelian monopoles” can either disappear, leaving only abelian monopoles detectable in low-energy theory, or survive as weakly coupled low-energy degrees of freedom, with genuine magnetic gauge interactions. Still another possibility is that the theory flows into a nontrivial conformal theory in which abelian or nonabelian monopole fields appear together with relatively nonlocal dyons and quarks. All of these possibilities are realized in various vacua of softly broken $\mathcal{N} = 2$ supersymmetric theories or in a large class of $\mathcal{N} = 1$ gauge theories. A point of fundamental importance \cite{1} is that nonabelian monopoles \footnote{For classical treatments and more recent developments on these, see \cite{2}}, in contrast to abelian ones, are essentially quantum mechanical. Often, a reasoning relying on the semiclassical approximation only gives an incorrect picture of the full quantum behavior. With the help of certain exact knowledge about the dynamical properties of supersymmetric gauge theories, we have now acquired considerable control on the quantum behavior of light nonabelian monopoles. Let us summarize the situation:

(i) In pure $\mathcal{N} = 2$ gauge theories, softly broken to $\mathcal{N} = 1$ by the adjoint scalar mass only, the low-energy theory is an effective magnetic abelian $U(1)^R$ gauge theory, where $R$ is the rank of the group; all monopoles are abelian and the theory effectively abelianizes \cite{3,4,5};

(ii) In $\mathcal{N} = 2$ gauge theories with $N_f$ flavors (fields in the fundamental representation of the gauge group), many vacua exist where the low-energy effective magnetic gauge symmetry is nonabelian, $SU(r) \times U(1)^{R-r}$, $R$ being the rank of the group, $r = 1, 2, \ldots, \frac{N_f}{2}$. The light degrees of freedom in these vacua are nonabelian monopoles transforming as $r$ of $SU(r)$ and carrying one of the $U(1)$ charges; abelian monopoles having charge in different $U(1)^{R-r-1}$ groups appear as well. Upon $\mathcal{N} = 1$ perturbation, these monopoles condense and give rise to confinement (nonabelian dual superconductor) \cite{6,7,8,9};

(iii) Abelian and nonabelian Argyres-Douglas vacua in $\mathcal{N} = 2$ theories \cite{10}: abelian or nonabelian monopoles and dyons together appear at a nontrivial infrared
fixed point theory. With a soft $\mathcal{N} = 1$ perturbation, such vacua confine;

**iv** Many $\mathcal{N} = 1$ conformal theories are known, having dual descriptions à la Seiberg, Kutasov, Kutasov-Schwimmer, etc. [11] [12] [13]

In this paper we continue our investigation and in particular, study systems in which at high energies the theory is a $U(N)$ gauge theory with an adjoint chiral multiplet $\Phi$ and a set of quark multiplets $Q, \tilde{Q}$ with some superpotential, while the low-energy magnetic gauge group contains more than one nonabelian factor, e.g., $SU(r_1) \times SU(r_2) \times SU(r_2) \ldots$, supported by light monopoles, so that they do not abelianize dynamically. The key ingredient turns out to be the superpotential,

$$W(\Phi, Q, \tilde{Q}) = W(\Phi) + \tilde{Q} m(\Phi) Q,$$

with a nontrivial structure in the function $m(\Phi)$.

**2. Bosonic $SU(N)$ theory with an adjoint scalar**

As a way of illustration let us consider first a bosonic $SU(N + 1)$ model

$$\mathcal{L} = \frac{1}{4g^2}(F_{\mu\nu}^A)^2 + \frac{1}{g^2}|(D_\mu \phi)^4|^2 - V(\phi),$$

where $\phi$ is a complex scalar field in the adjoint representation of $SU(N + 1)$. Suppose that the potential is such that at a minimum an adjoint scalar VEV takes the form

$$\langle \phi \rangle = \begin{pmatrix} v_1 \cdot 1_{r_1 \times r_1} \\ v_2 \cdot 1_{r_2 \times r_2} \\ \vdots \end{pmatrix}$$

where other diagonal elements, which we take all different, are represented by dots. Such a VEV breaks the gauge symmetry as

$$SU(N) \rightarrow \frac{SU(r_1) \times SU(r_2) \times U(1)^{N-r_1-r_2+1}}{\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2}}.$$ 

This system possesses several semiclassical monopoles. As is well known [2] [1] they lie in various broken $SU(2)$ subgroups,

i) in $(i, N + 1)$ subspaces, $i = 1, 2, \ldots, r_1$, giving rise to independent set of $r_1$ degenerate monopoles. These are the semiclassical candidates for the nonabelian monopoles in $\mathbb{R}^4$ of the dual $SU(r_1)$ group;
ii) in \((j, N + 1)\) subspaces, \(j = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2\), giving rise to \(r_2\) degenerate monopoles, possibly in \(r_2\) of the dual \(SU(r_2)\) group;

iii) in \((i, j)\) subspaces \(i = 1, 2, \ldots, r_1, j = r_1 + 1, r_1 + 2, \ldots, r_1 + r_2\), giving rise to \(r_1 r_2\) degenerate monopoles. These could be components of the nonabelian monopoles in \((r_1, r_2)\) representation of the dual \(SU(r_1) \times SU(r_2)\) group.

With appropriately chosen \(v_1, v_2\), it is easy to arrange things so that monopoles of the third type are the lightest of all. We shall find that such semiclassical reasoning however does not give a correct picture of the quantum theory, as \(v_1\) and \(v_2\) are taken small. Also, we would like to know whether there are vacua in which these monopoles appear as IR degrees of freedom and play the role of order parameters of confinement, and find out in which type of theories this occurs.

The question is highly nontrivial: for instance, in softly broken \(\mathcal{N} = 2\) SQCD with superpotential

\[
\mathcal{W} = \sqrt{2} \bar{Q}_i \Phi Q_i + \mu \text{Tr} \Phi^2 + m_i \bar{Q}_i Q_i, \quad m_i \rightarrow 0, \quad \mu \ll \Lambda, \quad (2.4)
\]

we know that possible quantum vacua in the limit \(m_i \rightarrow 0\) can be completely classified by an integer \(r\). The low-energy effective magnetic gauge symmetry is in general nonabelian and of type

\[
SU(r) \times U(1)^{N_c - r + 1}, \quad r = 0, 1, 2, \ldots, \frac{N_f}{2}. \quad (2.5)
\]

In other words, vacua with low-energy gauge symmetry of the type Eq.\((2.3)\) do not occur, even though this fact is not obvious from semiclassical reasoning. We are interested in knowing whether there exist systems in which an effective gauge symmetry with multiple nonabelian factors occur at low energies.

3. Generalized \(r\)-Vacua in \(\mathcal{N} = 1\) \(U(N)\) Theories: Semi-Classical Approximation

With the purpose of finding these new types of vacua, we enlarge the class of theories, and consider \(\mathcal{N} = 1\) supersymmetric \(U(N)\) gauge theories with a matter field \(\Phi\) in the adjoint representation and a set of quark superfields \(Q_i\) and \(\bar{Q}_i\) in the
fundamental (antifundamental) representation of $U(N)$, with a more general class of superpotentials:

$$W = W(\Phi) + \tilde{Q}^a_i m_i(\Phi)^b_a Q^i_b : (3.1)$$

$i = 1, 2, \ldots N_f$ is the flavor index; $a, b = 1, 2, \ldots N$ are the color indices. In the flavor-symmetric limit $m_i(\Phi) \rightarrow m(\Phi)$ (independent of $i$) the theory is invariant under a global $U(N_f)$ symmetry.

Apart from the simplest cases (up to cubic functions $W(\Phi)$ and up to linear function $m(\Phi)$) the models considered are not renormalizable. As explained in Ref. [13], however, these potentials represent “dangerously irrelevant” perturbations, and cannot be neglected in understanding the dynamical behavior of the theory in the infrared. We consider Eq. (3.1) as an effective Lagrangian at a given scale, and then explore the properties of the theory as the mass scale is reduced towards zero.

The semiclassical vacuum equations are

$$[\Phi, \Phi^\dagger] = 0 ; \quad (3.2)$$
$$0 = Q^i_a (Q^\dagger)^b_i - (\tilde{Q}^\dagger)^i_a \tilde{Q}^b_i ; \quad (3.3)$$
$$Q^i_a \frac{\delta m_i(\Phi)^b_a}{\delta \Phi^d_c} \tilde{Q}^b_i + \frac{\delta W(\Phi)}{\delta \Phi^d_c} = 0 ; \quad (3.4)$$
$$m_i(\Phi)^b_a Q^i_b = 0 \quad \text{(no sum over $i$)} ; \quad (3.5)$$
$$\tilde{Q}^b_i m_i(\Phi)^a_b = 0 \quad \text{(no sum over $i$)}. \quad (3.6)$$

As explained in [8], it is convenient first to consider flavor nonsymmetric cases, i.e., generic $m_i(\Phi)$ and nonvanishing $W$, so that the only vacuum degeneracy left is a discrete one, and to take the $U(N_f)$ limit $m_i(\Phi) \rightarrow m(\Phi)$ only after identifying each vacuum and computing the condensates in it. $m_i(\Phi)$ and $W(\Phi)$ are taken in the general form

$$W(\Phi) = \sum_k a_k \mathrm{Tr} (\Phi^k), \quad [m_i(\Phi)]_{ab} = \sum_k m_{i,k} \Phi^{k-1}_{ab} , \quad (3.7)$$

furthermore they are assumed to satisfy no special relations. We shall choose $m(\Phi)$ to be a polynomial of order quadratic or higher, and assume that the equation in the flavor symmetric limit

$$m(z) = 0 \quad (3.8)$$

has distinct roots

$$z = v^{(1)}, v^{(2)}, v^{(3)}, \ldots \quad (3.9)$$
Just as a reference, in the $\mathcal{N} = 2$ theory broken to $\mathcal{N} = 1$ only by the adjoint mass term, $W(\Phi) = \mu \text{Tr} \Phi^2$, $m_i(\Phi) = \sqrt{2}\Phi + m_i$, so that in that case the flavor symmetric equation $m(z) = 0$ would have a unique root, $-m/\sqrt{2}$.

We first use a gauge $U(N)$ rotation to bring the $\Phi$ VEV into a diagonal form,

$$\Phi = \delta_{ab} \phi_a = \text{diag} (\phi_1, \phi_2, \ldots, \phi_N),$$

which solves Eq. (3.2). $m(\Phi)$ is then also diagonal in color,

$$[m_i(\Phi)]_{ab} = \sum_k m_{i,k} \delta_{ab} \phi_{a}^{k-1} = \delta_{ab} m_i(\phi_a), \quad i = 1, 2, \ldots, N_f. \tag{3.11}$$

Eqs. (3.4), (3.5), (3.6) become

$$\sum_i m'_i(\phi_c) Q^i_c \tilde{Q}^c_i + W'(\phi_c) = 0; \tag{3.12}$$

$$m_i(\phi_c) Q^i_c = 0; \quad m_i(\phi_c) \tilde{Q}^c_i = 0; \tag{3.13}$$

(no sum over $c$), and

$$Q^i_c [\sum_k m_{i,k} \sum_{\ell} \phi_{c}^{\ell} \phi_{d}^{k-\ell-1}] \tilde{Q}^d_i = 0, \quad (c \neq d). \tag{3.14}$$

The diagonal elements $\phi_c$ (and the squark condensate) are of two different types. The first corresponds to one of the roots of

$$m_i(\phi^*_c) = 0. \tag{3.15}$$

For each $c$ this equation can be satisfied at most for one flavor, as we consider the generic, unequal functions $m_i(\Phi)$ first. Then there is one squark pair $Q^i_c, \tilde{Q}^c_i$ with nonvanishing VEVs and Eq. (3.12) yields their VEVs

$$Q^i_c = \tilde{Q}^c_i = \sqrt{-\frac{W'(\phi^*_c)}{m'_i(\phi^*_c)}} \neq 0. \tag{3.16}$$

Note that according to our assumption $m$ and $W$ satisfy no special relations so that $W'(\phi^*_c) \neq 0$.

The second group of $\langle \phi_c \rangle$ corresponds to:

$$W'(a_j) = 0, \quad j = 1, 2, \ldots. \tag{3.17}$$
As in general \( m_i(a_j) \neq 0 \), we have

\[
Q_i^c = \tilde{Q}_i^c = 0,
\]

for the corresponding color components of the squarks.

The classification of the vacua is somewhat subtle. The adjoint scalar VEV has the form

\[
\Phi = \text{diag} \left( v^{(1)}_1, v^{(1)}_2, \ldots, v^{(1)}_{r_1}, \ldots, v^{(p)}_1, \ldots, v^{(p)}_{r_p}, \ldots, a_1, \ldots, a_n \right),
\]

namely, there appear \( r_1 \) roots near \( v^{(1)}_1 \), \( r_2 \) roots near \( v^{(2)}_1 \), and so on.

The diagonal elements \( (a_1, \ldots, a_n) \) in Eq. (3.19) correspond to the roots of \( W'(z) = 0 \): these color components give rise to pure \( U(N_1) \times U(N_2) \times \ldots U(N_n) \) theory, \( \sum N_i = N - \sum r_j \), where \( N_j \) corresponds to the number of times \( a_j \) appears in Eq. (3.19).

No massless matter charged with respect to \( SU(N_j) \) group are there, \( \prod SU(N_j) \) interactions become strong in the infrared, and yields an abelian \( U(1)^n \) theory. The physics related to this part of the system has been recently discussed extensively, by use of a Matrix model conjecture proposed by Dijkgraaf-Vafa \[14\] as well as by a field theory approach initiated by Cachazo-Douglas-Seiberg-Witten \[15\]. We have nothing to add to them here.

Our focus of attention here is complementary: it concerns the first \( r_1 + r_2 + \ldots + r_p \) color components of \( \Phi \) in Eq. (3.19). This is interesting because in the flavor symmetric and \( W(\Phi) \to 0 \) limit, this sector describes a local

\[
U(r_1) \times U(r_2) \times \ldots U(r_p)
\]

(3.20)
geometry, supported by \( N_f \) massless quarks in the fundamental representation of \( SU(r_1) \), \( N_f \) massless quarks in the fundamental representation of \( SU(r_2) \), and so on. If \( r_i \leq \frac{N_f}{2} \), these interactions are non asymptotically free\(^2\) and they remain weakly coupled (or at most evolve to a superconformal theory) in the low energies. Furthermore, if all condensates are small or of order of \( \Lambda \), Eq. (3.20) describes a

\[\text{Note that the mass terms come from } \theta \Phi \text{ component of }\]

\[
\tilde{Q}_i m_i(\Phi) Q_b \sim (\tilde{Q} + \theta \tilde{\psi}_Q) (0 + \theta m'(v_1) \psi) (Q + \theta \psi_Q) + \ldots
\]

(3.21)

so in the limit \( Q_i \to 0 \) no mass terms arise and the beta function of \( N = 2 \) theory can be used even if interaction terms conserve the \( N = 1 \) supersymmetry only.
magnetic theory, the breaking of $U(r_1)$ by nonzero $W(\Phi)$ is a nonabelian Meissner effect (nonabelian confinement).

As for the structure of squark condensates, Eq.(3.12)-Eq.(3.14) appear to imply a color-flavor locked form of the squark VEVs of the form

$$Q^i_c \propto \delta^i_c, \quad \tilde{Q}^c_i \propto \delta^c_i.$$  

(3.22)

This however turns out to be true only if the groups of $r_1$ elements $v^{(1)}_i$, $r_2$ elements $v^{(2)}_i$, etc, correspond to zeros of mutually exclusive sets of flavor functions $m_i(z)$, which is not necessarily the case.

Let us explain this point better. Suppose that the two of the diagonal $\phi$ VEVs in Eq.(3.19), $x_1 \equiv v^{(1)}_1$ and $x_2 \equiv v^{(2)}_{r_1+1}$, correspond both to the first flavor. The VEVs of the first quark has the form

$$Q^1 = \begin{pmatrix} d_1 \\ 0 \\ \vdots \\ 0 \\ d'_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \tilde{Q}_1 = \begin{pmatrix} \tilde{d}_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$  

(3.23)

According to our assumption $x_1$ and $x_2$ are two distinct roots of $m_1(x) = 0$, that is,

$$\sum_k m_{1,k} x_i^{k-1} = 0, \quad i = 1, 2.$$  

(3.24)

Dividing the difference between these two equations by $x_1 - x_2$, one finds

$$\sum_k m_{1,k} \frac{1}{x_1^{k-1}} x_2^{k-\ell} = 0.$$  

(3.25)

This shows that the color nondiagonal vacuum equation Eq.(3.14) is indeed satisfied for $i = 1, c = 1, d = r_1 + 1$, by the quark VEVs Eq.(3.23).

The generalization for more nonabelian factors, and for more flavors getting VEVs in more than two color components, is straightforward.

For simplicity of notations and for definiteness, let us restrict ourselves in the following to the vacua with only two nonabelian factors. Their multiplicity is given
by the combinatorial factor
\[
\binom{N_f}{r_1} \times \binom{N_f}{r_2} \times \prod_{i=1}^{n} N_i
\] (3.26)
where the last factor corresponds to the Witten index of the pure $\prod SU(N_i) \subset U(N)$ theories.

Taking the above considerations into account, the classical VEVs in our theory take, in the flavor symmetric limit, the form
\[
\langle \phi \rangle = \begin{pmatrix} v_1 1_{r_1} & \cdots & v_2 1_{r_2} & a_1 1_{N_1} & \cdots & a_n 1_{N_n} \end{pmatrix},
\] (3.27)
where
\[
\sum_{j=1}^{n} N_j + r_1 + r_2 = N,
\] (3.28)
and
\[
Q = \begin{pmatrix} d_1 & \cdots & d_{r_1} & e_1 & \cdots & e_{r_2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} \tilde{d}_1 & \cdots & \tilde{d}_{r_1} & \tilde{e}_1 & \cdots & \tilde{e}_{r_2} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix},
\] (3.29)
where
\[
d_c = \tilde{d}_c = \sqrt{-\frac{W'(v^{(1)})}{m'(v^{(1)})}}, \quad e_c = \tilde{e}_c = \sqrt{-\frac{W'(v^{(2)})}{m'(v^{(2)})}}.
\] (3.30)
Note that classically, $0 \leq r_i \leq \min[N_f, N]$ apart from the obvious constraint $\sum r_i \leq N$.

As explained above, the condensates $d_a$ and $e_b$ can share the same flavor and so there are $\binom{N_f}{r_1} \binom{N_f}{r_2}$ ways to choose Eq.(3.29). We call $s$ the number of “superpositions”, that is, the number of flavors that are locked both to $v_1$ and $v_2$. The flavor
symmetry is broken then, in a \((r_1, r_2)\) vacuum with \(s\) superpositions, as

\[
U(N_f) \to U(r_1 - s) \times U(s) \times U(r_2 - s) \times U(N_f - r_1 - r_2 + s)
\]  

(3.31)

The meson condensates take the form

\[
\tilde{Q}Q = \begin{pmatrix}
-\frac{W'(v^{(1)})}{m'(v^{(1)})}1_{r_1-s} & - \left( \frac{W'(v^{(1)})}{m'(v^{(1)})} + \frac{W'(v^{(2)})}{m'(v^{(2)})} \right) 1_s & 0 \\
\frac{W'(v^{(2)})}{m'(v^{(2)})}1_{r_2-s} & 0 & \ddots \\
& & 0
\end{pmatrix},
\]

(3.32)

The vacuum counting becomes even simpler for a quadratic superpotential \(W\),

\[
W'(x) = g_0 + g_1 x.
\]  

(3.33)

In this case there is only one stationary point so the gauge group is broken by the condensates to \(U(N - r_1 - r_2)\). The \(SU\) part confines and gives a Witten index \(N - r_1 - r_2\), therefore the total number of classical vacua is

\[
\mathcal{N} = \sum_{r_1, r_2 = 0, \ldots, N_f, r_1 + r_2 \leq N} \left( N - r_1 - r_2 \right) \binom{N_f}{r_1} \binom{N_f}{r_2}.
\]  

(3.34)

In the particular case \(2N_f \leq N\) the second restriction on the sum over \(r_1\) and \(r_2\) is absent, and the summation can be performed to give a simple formula

\[
\mathcal{N} = (N - N_f) 2^{2N_f},
\]  

(3.35)

which is analogue of the formula for the softly broken \(\mathcal{N} = 2\) SQCD valid for \(N_f < N\) [8],

\[
\mathcal{N}_{SQCD} = (2 N - N_f) 2^{N_f-1},
\]  

(3.36)

and can be obtained from the latter by a formal replacement, \(N_f \to 2N_f\).

The semiclassical reasoning followed up to now is reliable when \(|v_i| \gg \Lambda, |Q| \gg \lambda\), and in this regime the massless matter multiplets simply correspond to the first \(r_1 + \ldots + r_p\) components of the original quark multiplets. When the parameters of the superpotential are such that the vacuum expectation values of \(\Phi\) and \(Q, \tilde{Q}\) are
of order of $\Lambda$ or smaller, we expect these massless multiplets to represent nonabelian magnetic monopoles. The vacuum with the symmetry breaking Eq. (3.20) is more appropriately seen as a vacuum in confinement phase, in which the order parameters of confinement are various magnetic monopoles carrying nonabelian charges

$$(r_1, 1, 1, \ldots), \quad (1, r_2, 1, \ldots), \quad (3.37)$$

etc.

4. Quantum $(r_1, r_2)$ Vacua

When $v_i$’s are small, of order of $\Lambda$ or less, the above semiclassical arguments are no longer reliable, but by varying continuously the parameters of $m_i(\Phi)$ from where the roots $v_i$’s are all very large to the region where they are small, we expect that $\prod SU(r_i)$ factors remain infrared free or superconformal, as long as

$$r_i \leq \frac{N_f}{2}, \quad i = 1, 2, \ldots \quad (4.1)$$

We conclude that in the theory Eq. (3.1) with $N_f$ quarks, in the limit

$$W(\Phi) \to 0; \quad m_i(\Phi) \to m(\Phi), \quad (4.2)$$

where $m(\Phi) = 0$ has at least two different roots, $\phi^* = v^{(1)}, v^{(2)}, v^{(3)}, \ldots$, there must be vacua with $\prod_i SU(r_i)$ effective gauge symmetry. If $|v_i| \leq \Lambda$ it must be a magnetic theory (the original $SU(N)$ interactions become strong).

We note that since the ultraviolet theory has no massless particles having multiple nonabelian charges, such as

$$(\mathcal{L}_1, \mathcal{L}_2, 1, \ldots), \quad (4.3)$$

we do not expect massless monopoles with such multiple charges to occur in the infrared either, in this theory.

The crucial information on the quantum system comes from the curve describing the Coulomb branch of our $\mathcal{N} = 1$ supersymmetric theory [17]:

$$y^2 = \prod_{i=1}^{N} (x - \phi_i)^2 - \Lambda^{2N - N_f} \det m(x) = \prod_{i=1}^{N} (x - \phi_i)^2 - \Lambda^{2N - N_f} \prod_i^{N_f} m_i(x), \quad (4.4)$$

10
where \( m_i(x) \) is the function appearing in the superpotential Eq. (3.1). The curve is valid for \( \ell N_f < N \) (\( \ell \) being the order of the polynomial \( m(\Phi) \)). For

\[
m_i(x) = C (x - v_i^{(1)}) (x - v_i^{(2)}),
\]

(4.5)

(so \( \ell = 2 \)) the curve is

\[
y^2 = \prod_{i=1}^{N} (x - \phi_i)^2 - C^{N_f} \Lambda^{2N-N_f} \prod_{i=1}^{N_f} (x - v_i^{(1)}) (x - v_i^{(2)})
\]

(4.6)

which is effectively equivalent to the curve of the \( \mathcal{N} = 2, SU(N) \) theory with \( 2N_f \) flavors.

The vacua with \( U(r_1) \times U(r_2) \times U(1)^{N-r_1-r_2} \) low-energy gauge symmetry arise at the point of QMS where the curve becomes singular

\[
y^2 = (x-\alpha)^{2r_1} (x-\beta)^{2r_2} \left[ \prod_{i=1}^{N-r_1-r_2} (x-\phi_i)^2 - C^{N_f} \Lambda^{2N-N_f} \prod_{i=1}^{N_f} (x-v_i^{(1)}) \prod_{i=1}^{N_f} (x-v_i^{(2)}) \right],
\]

(4.7)

with the factor in the square bracket factorized in maximum number of double factors. These clearly occurs only in the flavor symmetric limit \( v_i^{(1)} \to v^{(1)} \equiv \alpha, v_i^{(2)} \to v^{(2)} \equiv \beta \).

As \( r_1 \) of \( \phi_i \) can be equal to any \( r_1 \) of \( N_f v_i^{(1)} \)'s, and independently, \( r_2 \) of other \( \phi_i \)'s can be equal to any \( r_2 \) of \( N_f v_i^{(2)} \)'s, there is a multiplicity

\[
\binom{N_f}{r_1} \binom{N_f}{r_2}
\]

(4.8)

of vacua which converge to the \( (r_1, r_2) \) vacua in the flavor symmetric limit. Actually, the \( \mathcal{N} = 1 \) vacua exist at the maximally abelian singularity of the curve Eq. (4.7), which is, apart from the factor \( (x-v^{(1)})^{2r_1} (x-v^{(2)})^{2r_2} \), equivalent to that of \( \mathcal{N} = 2 \) \( SU(N-r_1-r_2) \) gauge theory with massive matter. Therefore the number of such singularities is equal to \( N-r_1-r_2 \). Collecting all the factors, we find that the total multiplicity

\[
\mathcal{N} = \binom{N_f}{r_1} \binom{N_f}{r_2} (N-r_1-r_2)
\]

(4.9)

which coincides with the semiclassical counting, Eq. (3.34), for corresponding values of \( r_1, r_2 \).

This last comment brings us to the subtle issue of correspondence between classical and quantum \( r \)-vacua. While classically, the values of \( r_i \) can reach \( \min(N, N_f) \), it is
evident from the above consideration that quantum vacua exist only for \( r_i \leq \frac{N_f}{2} \). This nicely confirms and generalizes the importance of quantum effects in deciding which nonabelian groups can survive in the infrared, emphasized repeatedly by us. At the same time, we find that when the UV theory contains an \( SU(r_i) \) factor supported by \( N_f \) massless quarks, \( r_i > N_f/2 \), so that the interactions become strong at low energies, such a sector is realized in the infrared by the dual, magnetic \( SU(N_f - r_i) \) theory. These nontrivial mappings between classical and quantum \( r \)-vacua in softly broken \( \mathcal{N} = 2 \) SQCD \[8\] and in more general models of the present paper appear to explain nicely the origin of Seiberg’s duality. We shall come back to this question in a separate publication.

5. Cachazo-Douglas-Seiberg-Witten Formulae

A further confirmation of our picture arises from the work by Cachazo et al. \[15\]. In particular they solved for the resolvents of the chiral operators

\[
M = \tilde{Q} \frac{1}{z - \Phi} Q; \quad R(z) = -\frac{1}{32\pi^2} \text{Tr} \frac{W_\alpha W^\alpha}{z - \Phi}.
\] (5.1)

The main result is a set of the generalized anomaly equations \(^3\)

\[
[W'(z) R(z)]_+ = R(z)^2, \\
[(M(z) m(z))]_+ = R(z) \delta^i_j; \quad [(m(z) M(z))]_+ = R(z) \delta^i_j.
\] (5.2)

\(\det m(z)\) has \( L = 2N_f \) zeros \( z_i \) (counted with their multiplicity), which for the choice Eq.(4.5) can be either \( v_1 \) or \( v_2 \) (in the \( SU(N_f) \) symmetric limit). In fact,

\[
m(z) = \text{diag} \left[ C \left( z - v_1^{(1)} \right) \left( z - v_1^{(2)} \right) \right],
\] (5.3)

\[
\frac{1}{m(z)} = \begin{pmatrix}
\frac{1}{C(z-v_1^{(1)})(z-v_1^{(2)})} \\
\cdot \\
\frac{1}{C(z-v_2^{(1)})(z-v_2^{(2)})}
\end{pmatrix}.
\] (5.4)

where \( v_1^{(1)} \rightarrow v^{(1)}, \ v_1^{(2)} \rightarrow v^{(2)} \) in the flavor symmetric limit.

\(^3\)The notation \([O(z)]_-\) stands for keeping only the negative powers in the Laurent expansion of \( O(z) \).
The solution of the anomaly equations for $R(z)$ is

$$2 R(z) = W'(z) - \sqrt{W'(z)^2 + f(z)},$$

(5.5)

where $f(z)$ is directly related to the gaugino condensates in the strong $\prod U(N_i)$ sectors [15]. Thus the zeros in $W'(z)$ (denoted by $a_i$) which appear classically as poles of $\frac{1}{z - \Phi}$ are replaced by cuts in a complex plane $z$ by the quantum effects. By defining

$$y = W'(z) - 2 R(z),$$

(5.6)

one has a $\mathcal{N} = 1$ curve

$$y^2 = W'(z)^2 + f(z)$$

(5.7)

on this Riemannian surface. The point of the construction of [15] lies in the fact that various chiral condensates are expressed elegantly in the form of integrals along cycles on this curve.

Taking the curve Eq.(5.7) as the double cover of a complex plane with appropriate branch cuts, the result of Eq.(5.5) refers to the “physical” (semiclassically visible) sheet; in the second sheet, the result is ($\tilde{z}$ lies in the second sheet at the same value as $z$)

$$W'(z) = W'(\tilde{z}); \quad 2 R(\tilde{z}) = W'(z) + \sqrt{W'(z)^2 + f(z)}.$$  

(5.8)

In the simplest model for $W$,

$$W(\Phi) = \mu \Phi^2, \quad W(z) = \mu z^2,$$

(5.9)

the only possible (classical) value for $a_i$ is zero; $f(z)$ is a constant, $f = -8\mu S$, where

$$S = -\frac{1}{32\pi^2} \langle \text{Tr} W_\alpha W^\alpha \rangle$$

is the VEV of the gaugino bilinear operator in the strong $SU(N - r_1 - r_2)$ super Yang-Mills theory. The solution for $R$ is explicitly,

$$2 R(z) = 2 \mu z - \sqrt{(2 \mu z)^2 + f}, \quad 2 R(\tilde{z}) = 2 \mu z + \sqrt{(2 \mu z)^2 + f}.$$  

(5.10)

The poles of $M$ in the classical theory, instead, remain poles in the full quantum theory [15]. We are interested in a vacuum in which $r_1 + r_2$ poles are in the physical
sheet: $r_1$ poles near $v^{(1)}$ and $r_2$ poles near $v^{(2)}$. The result of [15] for $M$ in this vacuum is

$$M(z) = R(z) \frac{1}{m(z)} - \sum_{i=1}^{r_1+r_2} \frac{R(\tilde{q}_i)}{z - z_i} \frac{1}{2\pi i} \oint_{z_i} \frac{1}{m(x)} \frac{dz}{x} - \sum_{j=1}^{2N_f-r_1-r_2} \frac{R(q_j)}{z - z_j} \frac{1}{2\pi i} \oint_{z_j} \frac{1}{m(x)} \frac{dz}{x}. \quad (5.11)$$

By definition the contour integrals must be done before the $SU(N_f)$ limit is taken, so

$$\oint_{z_i} \frac{1}{m(x)} \frac{dz}{x} = \begin{pmatrix} \ldots & 0 \\ \frac{1}{m'(z_i)} \\ 0 & \ldots \end{pmatrix}, \quad (5.12)$$

where $m'(v_i) = \pm C (v^{(1)}_i - v^{(2)}_i)$, the sign depending on whether the zero of $m(z)$ is the one near $v^{(1)}$ (+) or near $v^{(2)}$ (−). The $N_f \times N_f$ flavor structure is explicit in this formula: for instance the first term reads

$$[ R(z) \frac{1}{m(z)]_{ii} = \frac{R(z)}{C(z - v^{(1)}_i)(z - v^{(2)}_i)}. \quad (5.13)$$

In the $SU(N_f)$ symmetric limit, the first term is then $\propto 1_{N_f \times N_f}$; the second term of Eq. (5.11) consists of $r_1$ terms whose sum is invariant under $U(r_1) \times U(N_f - r_1)$ and $r_2$ terms which form an invariant under $U(r_2) \times U(N_f - r_2)$. The $r_1$ poles near $v^{(1)}$ can be related to any $r_1$ flavors out of $N_f$; analogously the $r_2$ poles near $v^{(2)}$ can be associated to any $r_2$ flavors out of $N_f$: there is no restriction between the two subsets of flavors. It follows that the global symmetry is broken to

$$U(N_f) \rightarrow U(r_1 - s) \times U(r_2 - s) \times U(s) \times U(N_f - r_1 - r_2 + s), \quad (5.14)$$

in a vacuum of this type, where $s$ is the number of “overlapping” flavors, $i.e.$, to which both roots $v^{(1)}_i$ and $v^{(2)}_i$ appear as poles in the first sheet. The result (5.14) is perfectly consistent with what was found semiclassically, Eq. (3.31).

Actually a more precise correspondence between the semiclassical and fully quantum mechanical results is possible. In order to compare with the semiclassical result for the meson condensate, Eq. (3.32), it suffices to evaluate the coefficient of $\frac{1}{z}$ in the large $z$ expansion of the quantum formula Eq. (5.11) (see Eq. 5.11). $f$ is a constant of
order of $\mu \Lambda^3$. We find

$$\tilde{Q}Q = \begin{pmatrix} A 1_{r_1-s} & B 1_s \\ C 1_{r_2-s} & D 1_{N_f-r_1-r_2+s} \end{pmatrix},$$

(5.15)

where

$$A = -\frac{R(\bar{v}^{(1)})}{m'(v^{(1)})} - \frac{R(v^{(2)})}{m'(v^{(2)})}; \quad B = -\frac{R(\bar{v}^{(1)})}{m'(v^{(1)})} - \frac{R(\bar{v}^{(2)})}{m'(v^{(2)})};$$

$$C = -\frac{R(v^{(1)})}{m'(v^{(1)})} - \frac{R(\bar{v}^{(2)})}{m'(v^{(2)})}; \quad D = -\frac{R(v^{(1)})}{m'(v^{(1)})} - \frac{R(v^{(2)})}{m'(v^{(2)})}. \quad (5.16)$$

In the classical limit, $f \to 0$, so

$$R(v^{(1)}), R(v^{(2)}) \to 0, \quad R(\bar{v}^{(1)}) \to W'(v^{(1)}), \quad R(\bar{v}^{(2)}) \to W'(v^{(2)}), \quad (5.17)$$

and the quantum expression for the meson condensates Eq.(5.15) correctly reduces to Eq.(3.32).

6. Classical vs Quantum $r$-Vacua: Illustration

Even though we found a nice corresponding between the semiclassical and fully quantum mechanical results, the precise (vacuum by vacuum) correspondence is slightly subtle, as the ranges of $r_i$ are different in the two regimes. In the quantum formulae, the vacua are parametrized by $r_1 = \min(N_f-r_1, r_1)$ and $r_2 = \min(N_f-r_2, r_2)$. The curve Eq.(4.4) factorizes as

$$y^2 = (x-v_1)^2 r_1 (x-v_2)^2 r_2 \ldots .$$

(6.1)

The low-energy degrees of freedom, carrying nontrivial nonabelian charges, are given in Table II.

Clearly, such a stricter condition for $r_i$ for the quantum theory reflects the renormalization effects due to which only for $r_i$ less than $N_f/2$ these nonabelian interactions remain non asymptotically free and can survive as low-energy gauge symmetries. We find the following correspondence between the classical $(r_1, r_2)$ vacua and the quantum
\[
\begin{array}{cccccc}
U(N_f) & SU(r_1) & U(1) & SU(r_2) & U(1) & U(1)^{N-r_1-r_2} \\
N_f & r_1 & 1 & 1 & 0 & 0 \\
N_f & 1 & 0 & r_2 & 1 & 0 \\
\end{array}
\]

Table 1: Massless dual-quarks in \(r_1, r_2\)-vacua.

\[(r_1, r_2) \text{ vacua:} \]
\[
\begin{array}{ccc}
  r_1 & r_2 \\
  r_1 & N_f - r_2 \\
N_f - r_1 & r_2 \\
N_f - r_1 & N_f - r_2 \\
\end{array}
\rightarrow r_1 \quad r_2
\]

(6.2)

and the total matching of the number of vacua in the two regimes must take into account of these rearrangements of \(r_1, r_2\). (This occurs also in the simpler case of the softly broken \(\mathcal{N} = 2\) SQCD of Carlino et.al.\cite{8})

In order to check the whole discussion and to illustrate some of the results found, we have performed a numerical study in the simplest nontrivial models. Figure 1 shows the situation for \(U(3)\) theory with nearly degenerate \(N_f = 4\) quark flavors and with a linear function \(m(\Phi)\). This corresponds basically to the seventeen vacua of the \(SU(3)\) theory studied earlier in the context of softly broken \(\mathcal{N} = 2\) SQCD \cite{8} (plus four vacua of \(r = 3\) due to the fact that here we consider \(U(3)\)). The second example, Figure 2, refers to a \(U(3)\) gauge theory with \(N_f = 2\), but with a quadratic function \(m(\Phi)\), illustrates well the situations studied in the present paper. We start from the curve Eq.(4.4) in the flavor-symmetric limit

\[
y^2 = G(x) = \prod_{i=1}^{N} (x-\phi_i)^2 - \Lambda^{2N-N_f} \det m(x) = \prod_{i=1}^{N} (x-\phi_i)^2 - \Lambda^{2N-N_f} m(x)^{N_f}, \quad (6.3)
\]

and apply the factorization equation of \cite{15,16}

\[
G(x) = F(x)H^2(x) \quad (6.4)
\]

\[
W'(x)^2 + f(z) = F(x)Q^2(x) \quad (6.5)
\]

In our model we take

\[
W(x) = \mu \ x^2, \quad (6.6)
\]

so \(\deg Q(x) \in \{0, 1\}\). In particular \(\deg Q = 0\) in a vacuum smoothly connected with a classical vacuum in which \(\Phi\) is completely higgsed, \(\langle \Phi \rangle = \text{diag}(z_1, \ldots, z_N)\), whereas
deg \( Q = 1 \) corresponds to a classical VEV for \( \Phi \) of the type \( \langle \Phi \rangle = \text{diag}(z_1, \ldots, z_r, 0, \ldots, 0) \), where some eigenvalues are zeros of the adjoint superpotential \( W(\Phi) \). In other words \( \deg Q = 0 \) or 1 if the matrix model curve Eq.\( (5.7) \) is degenerate or not respectively.

We must require the vanishing of the discriminant of the curve \( G(x) \), that is the resultant of \( G(x) \) and its first derivative:

\[
\mathcal{R} \left( G(x), \frac{dG(x)}{dx} \right) = 0. \tag{6.7}
\]

This guarantees the presence of a double zero in \( G(x) \).

**A.** \( U(3) \) \( N_f = 4 \) with linear \( m(\Phi) \) (see Fig.\( \PageIndex{4} \)).

\[y^2 = G(x) = 3 \prod_{i=1}^{3} (x - \phi_i)^2 - \Lambda^2 (x - m)^4 \tag{6.8}\]

As explained above quantum vacua are labeled by an integer \( r \) that ranges from 0 to \( N_f/2 = 2 \).

(i) \( r = 2 \).

After extracting a factor \( (x - m)^4 \) from \( G(x) \) we are left with a reduced curve

\[\bar{y}^2 = \tilde{G}(x) = (x - a)^2 - \Lambda^2 \tag{6.9}\]

Figure 1: Correspondence between the classical and quantum vacua in the \( U(3) \) theory with 4 flavors at degenerate mass.
Imposing Eqs. $(6.4),(6.5)$ we can fix $a = 0$ in order to get two opposite zeros for the reduced curve (the only possible choice is indeed $\deg Q = 0$). So there is only one solution, that is only one quantum vacuum with $r = 2$; its multiplicity is $N_f C_r = 6$, as we can check performing a quark mass perturbation on the curve. Then we have actually a sextet of $r = 2$ quantum vacua. These correspond exactly to the six classical vacua with $r = 2$.

(ii) $r = 1$.

We extract a factor $(x - m)^2$; the reduced curve is

$$
\tilde{y}^2 = \tilde{G}(x) = (x^2 - ax - b)^2 - \Lambda^2(x - m)^2
$$

(6.10)

The discriminant of $\tilde{G}(x)$ vanishes on particular (complex) 1-dimensional submanifolds of the moduli space parametrized by $(a, b)$. Clearly we must exclude those where $\tilde{G}(x) = (x - m)^2 \ldots$, because it belongs to the case $r = 2$ again. Afterwards we have two possibilities; looking at Eq. $(6.5)$ we can choose $\deg Q = 0$ and adjust the remaining free parameter to get one more double zero of $\tilde{G}(x)$ or $\deg Q = 1$ and let $\tilde{G}(x)$ have two opposite zeros. We recover one solution for the first choice (this is what is called baryonic root in $[8]$) and two solutions for the second. All these vacua have multiplicity $N_f C_r = 4$ and they correspond to classical vacua with $r = 1, 3$.

(iii) $r = 0$.

We work with the full curve Eq. $(6.8)$. Requiring the vanishing of the discriminant and avoiding the set of solutions that lead to the form $G(x) = (x - m)^2 \ldots$ (it would belong to the above cases) we find 3 solutions by imposing Eq. $(6.5)$ with $\deg Q = 0$. All these vacua have multiplicity $N_f C_r = 1$ and they correspond to classical $r = 0$ vacua.

B. $U(3) \quad N_f = 2$ with quadratic $m(\Phi)$ (see Fig.2).

$$
y^2 = G(x) = \prod_{i=1}^{3} (x - \phi_i)^2 - C^2 \Lambda^2 (x - v^{(1)})^2 (x - v^{(2)})^2
$$

(6.11)

Here we work with two “quantum” indices $r_1, r_2 \in \{0, 1\}$.

(i) $(r_1, r_2) = (1, 1)$.

The curve Eq. $(6.11)$ contains a factor $(x - v^{(1)})^2 (x - v^{(2)})^2$ so we are left
with
\[ \tilde{y}^2 = \tilde{G}(x) = (x - a)^2 - C^2 \Lambda^2 \quad (6.12) \]

There is only one solution for \( a \) just as in A.(i), but now the multiplicity of the vacuum is \( N_f C_{r_1} \times N_f C_{r_2} = 2 \times 2 = 4 \). This is what expected from the counting of classical \((r_1, r_2) = (1, 1)\).

(ii) \((r_1, r_2) = (0, 1)\).

The reduced curve is
\[ \tilde{y}^2 = \tilde{G}(x) = (x^2 - ax - b)^2 - C^2 \Lambda^2 (x - v^{(1)})^2. \quad (6.13) \]

As in A.(ii) we can have \( \deg Q = 0 \) or 1; we obtain respectively one and two solutions for the parameters \((a, b)\). The multiplicity of this three vacua is \( N_f C_{r_1} \times N_f C_{r_2} = 1 \times 2 = 2 \), so we recover the desired number coming from classical vacua with \((r_1, r_2) = (0, 1), (2, 1)\).

(iii) \((r_1, r_2) = (1, 0)\).

This case is exactly as in B.(ii) with \( v^{(1)} \leftrightarrow v^{(2)} \) and \( r_1 \leftrightarrow r_2 \).

(iv) \((r_1, r_2) = (0, 0)\).

Studying the full curve Eq. (6.11) and excluding solutions carrying a factor \((x - v^{(1)})^2 \) or \((x - v^{(2)})^2 \) in \( G(x) \), we find five solutions to the factorization

Figure 2: Correspondence between the classical and quantum vacua in the \( U(3) \) theory with 2 flavors and quadratic \( m(x) \) or equivalently 4 flavors with two different masses.
equations Eqs. (6.4), (6.5) (deg $Q = 0$). Their multiplicity is $N_r C_{r_1} \times N_r C_{r_2} = 1$. This allows us to recover the right number of vacua corresponding to classical $(r_1, r_2) = (0, 0)$ (multiplicity 3) and $(r_1, r_2) = (2, 0), (0, 2)$ (multiplicity 1).

7. Conclusion

In this paper we have shown that under some nontrivial conditions a $U(N)$ gauge theory with $\mathcal{N} = 1$ supersymmetry is realized dynamically at low energies as an effective multiply nonabelian gauge system

$$U(r_1) \times U(r_2) \times \ldots \times \prod U(1),$$

with massless particles having charges $(r_1, \ldots), (1, r_2, \ldots), \ldots$ In the fully quantum situation discussed in Sections 4, 5, 6, these refer to magnetic particles carrying nonabelian charges so the gauge symmetry breaking induced by the superpotential $W(\Phi)$ (dual Higgs mechanism) describes a nonabelian dual superconductor of a more general type than studied earlier.

We believe that the significance of our work lies not in a particular model considered or its possible applications, but in having given an existence proof of $U(N)$ systems which are realized at low energies as a magnetic gauge theory with multiple nonabelian gauge group factors. We found the conditions under which this type of vacua are realized. As a bonus, an intriguing correspondence between classical and quantum $(r_1, r_2, \ldots)$ vacua was found, generalizing an analogous phenomenon in the standard softly broken $\mathcal{N} = 2$ SQCD. We think that our findings constitute a small but useful step towards a more ambitious goal of achieving a complete classification of possible confining systems in 4D gauge theories, or finding the dynamical characterization of each of them.

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References

[1] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and H. Murayama, Nucl. Phys. B \textbf{701} (2004) 207 [arXiv:hep-th/0405070]; K. Konishi, in “Continuous Advances in QCD 2004”, Minneapolis, 2004, Edit. T. Gherghetta, World Scientific (2004), arXiv:hep-th/0407272.

[2] E. Lubkin, Ann. Phys. \textbf{23} (1963) 233; E. Corrigan, D.I. Olive, D.B. Farlie and J. Nuyts, Nucl. Phys. \textbf{B106} (1976) 475; C. Montonen and D. Olive, Phys. Lett. \textbf{72 B} (1977) 117; P. Goddard, J. Nuyts and D. Olive, Nucl. Phys. \textbf{B125} (1977) 1; F.A. Bais, Phys. Rev. \textbf{D18} (1978) 1206; E. J. Weinberg, Nucl. Phys. \textbf{B167} (1980) 500; Nucl. Phys. \textbf{B203} (1982) 445; N. Dorey, T.J. Hollowood and M.A.C. Kneipp, \texttt{hep-th/9512116} (1995); N. Dorey, C. Fraser, T.J. Hollowood and M.A.C. Kneipp, Phys. Lett. \textbf{B383} (1996) 422 [arXiv:hep-th/9605069]; C. J. Houghton, P. M. Sutcliffe, J.Math.Phys.\textbf{38} (1997) 5576 [arXiv:hep-th/9708006], B.J. Schroers and F.A. Bais, Nucl. Phys. \textbf{B535} (1998) 197 [arXiv:hep-th/9805163].

[3] N. Seiberg and E. Witten, Nucl. Phys. \textbf{B426} (1994) 19 (Erratum \textit{ibid.} \textbf{B430} (1994) 485) [arXiv:hep-th/9407087].

[4] P. Argyres and A. F. Faraggi, Phys. Rev. Lett \textbf{74} (1995) 3931 [arXiv:hep-th/9411047]; A. Klemm, W. Lerche, S. Theisen and S. Yankielowicz, Phys. Lett. \textbf{B344} (1995) 169 [arXiv:hep-th/9411048]; A. Klemm, W. Lerche and S. Theisen, Int. J. Mod. Phys. \textbf{A11} (1996) 1929 [arXiv:hep-th/9505150]; A. Hanany and Y. Oz, Nucl. Phys. \textbf{B452} (1995) 283-312 [arXiv:hep-th/9505075]; P. C. Argyres, M. R. Plesser and A. D. Shapere, Phys. Rev. Lett. \textbf{75} (1995) 1699 [arXiv:hep-th/9505100]; P. C. Argyres and A. D. Shapere, Nucl. Phys. \textbf{B461} (1996) 437 [arXiv:hep-th/9509175]; A. Hanany, Nucl. Phys. \textbf{B466} (1996) 85 [arXiv:hep-th/9509176].

[5] M. R. Douglas and S. H. Shenker, Nucl. Phys. \textbf{B447} (1995) 271 [arXiv:hep-th/9503163].

[6] N. Seiberg and E. Witten, Nucl. Phys. \textbf{B431} (1994) 484 [arXiv:hep-th/9408099].

[7] P. C. Argyres, M. R. Plesser and N. Seiberg, Nucl. Phys. \textbf{B471} (1996) 159 [arXiv:hep-th/9603042]; P.C. Argyres, M.R. Plesser, and A.D. Shapere, Nucl.
[8] G. Carlino, K. Konishi and H. Murayama, JHEP 0002 (2000) 004 [arXiv:hep-th/0001036]; Nucl. Phys. B590 (2000) 37 [arXiv:hep-th/0005076];

[9] S. Bolognesi and K. Konishi, Nucl. Phys. B645 (2002) 337-348 [arXiv:hep-th/0207161].

[10] P. C. Argyres and M. R. Douglas, Nucl. Phys. B448 (1995) 93 [arXiv:hep-th/9505062]; P. C. Argyres, M. R. Plesser, N. Seiberg and E. Witten, Nucl. Phys. B 461 (1996) 71 [arXiv:hep-th/9511154], T. Eguchi, K. Hori, K. Ito and S.-K. Yang, Nucl. Phys. B471 (1996) 430, [arXiv:hep-th/9603002];

[11] N. Seiberg, Nucl. Phys. B435 (1995) 129 [arXiv:hep-th/9411149].

[12] D. Kutasov, Phys. Lett. B 351, 230 (1995) [arXiv:hep-th/9503086]; D. Kutasov and A. Schwimmer, Phys. Lett. B 354, 315 (1995) [arXiv:hep-th/9505004];

[13] D. Kutasov, A. Schwimmer and N. Seiberg, Nucl. Phys. B 459 (1996) 455 [arXiv:hep-th/9510222].

[14] R. Dijkgraaf and C. Vafa, Nucl. Phys. B 644 (2002) 3 [arXiv:hep-th/0206255];

[15] F. Cachazo, M. R. Douglas, N. Seiberg and E. Witten, JHEP 0212 (2002) 071 [arXiv:hep-th/0211170]; F. Cachazo, N. Seiberg and E. Witten, JHEP 0302 (2003) 042 [arXiv:hep-th/0301006]; F. Cachazo, N. Seiberg and E. Witten, JHEP 0304 (2003) 018 [arXiv:hep-th/0303207]; for a review and further references, see: R. Argurio, G. Ferretti and R. Heise, Int. J. Mod. Phys. A19 (2004) 2015-2078 [arXiv:hep-th/0311066].
[16] Y. Ookouchi, JHEP 0401 (2004) 014 [arXiv:hep-th/0211287]; V. Balasubramanian, B. Feng, M. x. Huang and A. Naqvi, Annals Phys. 310, 375 (2004) [arXiv:hep-th/0303065].

[17] A. Kapustin, Phys. Lett. B 398, 104 (1997) [arXiv:hep-th/9611049].