Exact Solutions for Matter-Enhanced Neutrino Oscillations

A. B. Balantekin

Department of Physics, University of Wisconsin
Madison, Wisconsin 53706 USA

and

Institute for Nuclear Theory, University of Washington, Box 351550
Seattle, WA 98195-1550 USA

and

Department of Astronomy, University of Washington, Box 351580
Seattle WA 98195-1580 USA

(November 26, 1997)

The analogy between supersymmetric quantum mechanics and matter-enhanced neutrino oscillations is exploited to obtain exact solutions for a class of electron density profiles. This integrability condition is analogous to the shape-invariance in supersymmetric quantum mechanics. This method seems to be the most direct way to obtain the exact survival probabilities for a number of density profiles of interest, such as linear and exponential density profiles. The resulting neutrino amplitudes can also be utilized as comparison amplitudes for the uniform semiclassical treatment of neutrino propagation in arbitrary electron density profiles.

14.60.Pq, 96.60.Jw, 26.65.+t

I. INTRODUCTION

Matter enhanced oscillations of neutrinos via the Mikheyev, Smirnov, Wolfenstein (MSW) mechanism \[1\] is of considerable current interest. MSW mechanism may account for the deficit of solar neutrinos \[2\]. Matter-enhanced neutrino oscillations may also play an important role in neutrino propagation through the core-collapse supernovae \[3\].

The equations of motion for the neutrinos in the MSW problem can be solved by direct numerical integration, which must be repeated many times when a broad range of mixing parameters are considered. This often is not very convenient; consequently various approximations are widely used. Exact or approximate analytic results allow a greater understanding of the effects of parameter changes. Exact solutions of the MSW equations for a number of density profiles are available in the literature. Landau-Zener method \[4\] is applicable to the level-crossing phenomenon inherent in matter-enhanced neutrino oscillations \[5,6\]. The standard Landau-Zener method is exact a linear density profile. Exact solutions for an exponential density profile \[7–9\] or for a profile in the form tanh \(x\) \[10\] are also given in the literature.

The analogy between supersymmetric quantum mechanics and matter-enhanced neutrino oscillations was pointed out in Ref. \[11\]. Indeed this analogy was used to obtain a semiclassical approximate expression for the hopping probability \[11,12\]. The aim of the current paper is to exploit this analogy further to obtain exact solutions for a class of electron density profiles. The integrability condition we investigate is analogous to the integrability condition called shape-invariance \[13\] in supersymmetric quantum mechanics.

For the purpose of establishing notation the MSW effect is outlined and salient formulas are given in Section II. In Section III the consequences of the shape-invariance of electron density profile are discussed, an exact expression for the electron neutrino amplitude for such density profiles is derived and its asymptotic limits are calculated. Several examples are given in Section IV and existing results in the literature are shown to follow from the formalism developed in the previous section. In Section V, the shape-invariance condition is shown to be a suitable first step for obtaining approximate expressions for the neutrino survival probability. Section VI includes a brief discussion of the results.

II. OUTLINE OF THE MSW EFFECT

The evolution of flavor eigenstates in matter is governed by the equation
\[ i \hbar \frac{\partial}{\partial x} \begin{bmatrix} \Psi_e(x) \\ \Psi_\mu(x) \end{bmatrix} = \begin{bmatrix} \varphi(x) & \sqrt{\Lambda} \\ \sqrt{\Lambda} & -\varphi(x) \end{bmatrix} \begin{bmatrix} \Psi_e(x) \\ \Psi_\mu(x) \end{bmatrix}, \tag{2.1} \]

where we defined
\[ \varphi(x) = \frac{1}{4E} \left(2\sqrt{2} G_F N_e(x) E - \delta m^2 \cos 2\theta_e \right) \tag{2.2} \]
and
\[ \sqrt{\Lambda} = \frac{\delta m^2}{4E} \sin 2\theta_e. \tag{2.3} \]

In these equations \( \delta m^2 \equiv m_2^2 - m_1^2 \) is the vacuum mass-squared splitting, \( \theta_e \) is the vacuum mixing angle, \( G_F \) is the Fermi constant and \( N_e(x) \) is the number density of electrons in the medium.

By making a change of basis
\[ \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix} = \begin{bmatrix} \cos \theta(x) & -\sin \theta(x) \\ \sin \theta(x) & \cos \theta(x) \end{bmatrix} \begin{bmatrix} \Psi_e(x) \\ \Psi_\mu(x) \end{bmatrix}, \tag{2.4} \]
the flavor-basis Hamiltonian of Eq. (2.1) can be simultaneously diagonalized. The matter mixing angle is defined via
\[ \sin 2\theta(x) = \frac{\sqrt{\Lambda}}{\sqrt{\Lambda + \varphi^2(x)}} \tag{2.5} \]
and
\[ \cos 2\theta(x) = \frac{-\varphi(x)}{\sqrt{\Lambda + \varphi^2(x)}}. \tag{2.6} \]

In the adiabatic basis the evolution equation takes the form
\[ i \hbar \frac{\partial}{\partial x} \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix} = \begin{bmatrix} -\sqrt{\Lambda + \varphi^2(x)} & -i\hbar \varphi'(x) \\ i\hbar \varphi'(x) & \sqrt{\Lambda + \varphi^2(x)} \end{bmatrix} \begin{bmatrix} \Psi_1(x) \\ \Psi_2(x) \end{bmatrix}, \tag{2.7} \]
where prime denotes derivative with respect to \( x \). Since the 2 × 2 “Hamiltonian” in Eq. (2.7) is an element of the \( SU(2) \) algebra, the resulting time-evolution operator is an element of the \( SU(2) \) group. Hence it can be written in the form
\[ U = \begin{bmatrix} \Psi_1(x) & -\Psi_2^*(x) \\ \Psi_2(x) & \Psi_1^*(x) \end{bmatrix}, \tag{2.8} \]
where \( \Psi_1(x) \) and \( \Psi_2(x) \) are solutions of Eq. (2.7) with the initial conditions \( \Psi_1(x_i) = 1 \) and \( \Psi_2(x_i) = 0 \).

To calculate the electron neutrino survival probability Eq. (2.8) needs to be solved with the initial conditions \( \Psi_e = 1 \) and \( \Psi_\mu = 0 \). Using Eq. (2.8) the general solution satisfying these initial conditions can be written as
\[ \Psi_e(x) = \cos \theta(x) [\cos \varphi_1^2(x) \sin \theta_1 \Psi_1^*(x) - \sin \theta_1 \Psi_1^*(x)] + \sin \theta(x) [\cos \varphi_2^2(x) \sin \theta_1 \Psi_1^*(x) + \sin \theta_1 \Psi_1^*(x)], \tag{2.9} \]
where \( \theta_1 \) is the initial matter angle. Once the neutrinos leave the dense matter (e.g. the Sun), the solutions of Eq. (2.7) are particularly simple. Inserting these into Eq. (2.9) we obtain the electron neutrino amplitude at a distance \( L \) from the solar surface to be
\[ \Psi_e(L) = \cos \theta_e [\cos \varphi_1^2(x) \sin \theta_1 \Psi_1^*(x) - \sin \theta_1 \Psi_1^*(x)] \exp \left( i \frac{\delta m^2}{4E} L \right) \]
\[ + \sin \theta_e [\cos \varphi_2^2(x) \sin \theta_1 \Psi_1^*(x) + \sin \theta_1 \Psi_1^*(x)] \exp \left( -i \frac{\delta m^2}{4E} L \right), \tag{2.10} \]
where \( \Psi_1(x) \) and \( \Psi_2(x) \) are the values of \( \Psi_1(x) \) and \( \Psi_2(x) \) on the solar surface. The electron neutrino survival probability averaged over the detector position, \( L \), is then given by
\[ P(\nu_e \rightarrow \nu_e) = \langle |\Psi_e(L)|^2 \rangle_L \]
\[ = \frac{1}{2} + \frac{1}{2} \cos 2\theta_e \cos 2\varphi_1 (1 - 2|\Psi_2(x)|^2) \]
\[ - \frac{1}{2} \cos 2\theta_e \sin 2\varphi_1 \left( \Psi_1^*(x) \Psi_2^*(x) + \Psi_1^*(x) \Psi_2^*(x) \right), \tag{2.11} \]
If the initial density is rather large, then \( \cos 2\theta_e \sim -1 \) and \( \sin 2\varphi_1 \sim 0 \) and the last term in Eq. (2.11) is very small. Different neutrinos arriving the detector carry different phases if they are produced over an extended source. Even if the initial matter density is not very large, averaging over the source position makes the last term very small as these phases average to zero. The completely averaged result for the electron neutrino survival probability is then given by
\[ P(\nu_e \rightarrow \nu_e) = \frac{1}{2} + \frac{1}{2} \cos 2\theta_e \cos 2\varphi_1 |\Psi_2(x)|^2 (1 - 2P_{\text{hop}}), \tag{2.12} \]
where the hopping probability is
\[ P_{\text{hop}} = |\Psi_2(x)|^2, \tag{2.13} \]
obtained by solving Eq. (2.7) with the initial conditions \( \Psi_1(x_i) = 1 \) and \( \Psi_2(x_i) = 0 \).

III. SHAPE-IN VaRIANCE OF THE ELECTRON DENSITY PROFILE

The coupled first-order equations for the flavor-basis wave functions can be decoupled to yield a second order equation for only the electron neutrino propagation
\[ -\hbar^2 \frac{\partial^2 \Psi_e(x)}{\partial x^2} - \left[ \Lambda + \varphi^2(x) + i\hbar \varphi'(x) \right] \Psi_e(x) = 0. \tag{3.1} \]
Introducing the operators
\[ \hat{A}_- = i\hbar \frac{\partial}{\partial x} - \varphi(x), \]
\[ \hat{A}_+ = i\hbar \frac{\partial}{\partial x} + \varphi(x), \] (3.2)
Eq. (2.1) takes the form
\[ \hat{A}_- \Psi_e(x) = \sqrt{\Lambda} \Psi_e(x), \]
\[ \hat{A}_+ \Psi_e(x) = \sqrt{\Lambda} \Psi_e(x), \] (3.3)
and Eq. (3.1) can be compactly expressed as
\[ \hat{A}_+ \hat{A}_- \Psi_e(x) = \Lambda \Psi_e(x). \] (3.4)
In these equations we take \( \varphi(x), \hat{A}_+, \) and \( \hat{A}_- \) a function of a set of parameters \( a \) that specify space-independent properties of the electron density.

The analogy between Eq. (3.1) and supersymmetric quantum mechanics was pointed out some time ago [11]. In a parallel development, in the context of supersymmetric quantum mechanics it was shown that a subset of the potentials for which the Schrödinger equations are exactly solvable share an integrability condition called shape invariance [13]. In this paper we consider a generalization of the shape-invariance condition to electron densities. We call a given electron density to be shape-invariant if a change of parameters satisfies the condition
\[ -\varphi^2(x; a_1) + i\hbar \varphi'(x; a_1) = -\varphi^2(x; a_2) - i\hbar \varphi'(x; a_2) + R(a_1), \] (3.5)
where \( a_2 \) is a function of \( a_1 \), and the remainder \( R(a_1) \) is independent of \( x \). As we illustrate in the forthcoming sections a number of electron density profiles satisfy this condition. Here we take \( a_1 \) to be the original set of parameters specifying the given electron density profile (hence real). The new set of parameters \( a_2 \) are, in general, complex. The shape invariance condition of Eq. (3.5) can be rewritten in terms of the operators defined in Eq. (3.2)
\[ \hat{A}_-(a_1) \hat{A}_+(a_1) = \hat{A}_+(a_2) \hat{A}_-(a_2) + R(a_1). \] (3.6)
We assume that replacing \( a_1 \) with \( a_2 \) in a given operator can be achieved with a similarity transformation:
\[ \hat{T}(a_1) \mathcal{O}(a_1) \hat{T}^{-1}(a_1) = \mathcal{O}(a_2). \] (3.7)
If the parameters \( a_1 \) and \( a_2 \) are related by a translation,
\[ a_2 = a_1 + \eta, \] (3.8)
then the operator \( \hat{T}(a_1) \) of Eq. (3.7) is simply given by
\[ \hat{T}(a_1) = \exp \left( \frac{\eta}{\partial a_1} \right). \] (3.9)
One should emphasize that the discussion in this article is valid for any relationship between \( a_1 \) and \( a_2 \) and is not limited to that given in Eq. (3.8). In the rest of this paper \( \hbar \) is set to one for typographical convenience, except when we discuss the semiclassical limit.

A. Exact Solutions

To solve the MSW equations we first introduce the operators [14]
\[ \hat{B}_+ = \hat{A}_+(a_1) \hat{T}(a_1) \]
\[ \hat{B}_- = \hat{T}^{-1}(a_1) \hat{A}_-(a_1). \] (3.10)
Using Eq. (3.4) one can easily prove the commutation relation:
\[ [\hat{B}_-, \hat{B}_+] = R(a_0), \] (3.11)
where we used the identity
\[ R(a_n) = \hat{T}(a_1) R(a_{n-1}) \hat{T}^{-1}(a_1), \] (3.12)
valid for any \( n \), to define \( a_0 \). One can also prove by induction two additional commutation relations
\[ [\hat{B}_+ \hat{B}_-, \hat{B}_+^n] = (R(a_1) + R(a_2) + \cdots + R(a_n)) \hat{B}_+^n, \] (3.13)
and
\[ [\hat{B}_+ \hat{B}_-, \hat{B}_-^{-n}] = (R(a_1) + R(a_2) + \cdots + R(a_n)) \hat{B}_-^{-n}. \] (3.14)

Using the operators introduced in Eq. (3.10), Eq. (3.4) can be rewritten as
\[ \hat{B}_+ \hat{B}_- \Psi_e(x) = \Lambda \Psi_e(x). \] (3.15)
Eqs. (3.13) and (3.14) imply that \( \hat{B}_+ \) and \( \hat{B}_- \) can be used as ladder operators to solve Eq. (3.15). To this end we introduce \( \Psi_e^{(0)} \) as the solution of the equation
\[ \hat{A}_-(a_1) \Psi_e^{(0)} = 0 = \hat{B}_- \Psi_e^{(0)}, \] (3.16)
which implies
\[ \Psi_e^{(0)} \sim \exp \left( -i \int \varphi(x; a_1) dx \right). \] (3.17)
If the function
\[ f(n) = \sum_{k=1}^{n} R(a_k) \] (3.18)
can be analytically continued so that the condition
\[ f(\mu) = \Lambda \] (3.19)
is satisfied for a particular (in general, complex) value of \( \mu \), then Eq. (3.13) implies that one solution of Eq. (3.15) is \( \hat{B}_+^n \Psi_e^{(0)} \). Similarly if \( \Psi_+ \) satisfies the equation
\[ \hat{B}_+ \Psi_+^{(0)} = 0, \] (3.20)
which implies that
The former limit the commutator 
\[ A \] traveling through the vacuum); and ii) \[ \psi \] vanishes. In the latter case this commutator can be ignored, that using Eq. (3.10) the two terms in Eq. (3.23) can be limits we can write Eqs. (3.24) and (3.25) as 
\[ B^\pm \psi^{(0)} \]

We consider two different limits: i) \( \varphi(x) \) is constant (i.e., either the electron density is constant, or the neutrino is traveling through the vacuum); and ii) \( \varphi(x) \to \pm \infty \). In the former limit the commutator 
\[ \frac{\partial}{\partial x} \varphi(x; a) = \varphi'(x; a) \]
vanishes. In the latter case this commutator can be ignored as 
\[ \varphi(x; a) \varphi(x; a) + i \varphi'(x; a) \]
\[ = \varphi(x; a) \varphi(x; a) \left( 1 + i \frac{\varphi'(x; a)}{\varphi(x; a) \varphi(x; a)} \right) \]
\[ \to \varphi(x; a) \varphi(x; a) \]
proved that \( \varphi'(x; a) / \varphi(x; a) \) remains finite (which is the case for all realistic density profiles). Hence it both limits we can write Eqs. (3.24) and (3.25) as 
\[ \hat{B}^n \psi^{(0)} = (\varphi(a_{n+1}) + \varphi(a_1))(\varphi(a_{n+1}) + \varphi(a_2)) \]
\[ \cdots (\varphi(a_{n+1}) + \varphi(a_n)) \exp \left( -i \int \varphi(x; a_{n+1}) dx \right) \]
and 
\[ \hat{B}^{-n} \psi^{(0)} = \left( -\varphi(a_n) - \varphi(a_1) \right)^{-1} \left( -\varphi(a_n) - \varphi(a_2) \right)^{-1} \]
\[ \cdots \times -\varphi(a_n) - \varphi(a_n) \right)^{-1} \exp \left( +i \int \varphi(x; a_1) dx \right) \]. (3.29)

In both of these equations the quantity \( \varphi(a_n) \) stands for \( \varphi(x; a_n) \).
If the density profile satisfies the condition 
\[ \varphi(x; a_n) = \varphi(x; a_1) + (n-1)\xi(x) \], (3.30)
then these asymptotic equations can be cast in a form suitable for analytic continuation:
\[ \hat{B}^n \psi^{(0)} = \zeta^n \frac{\Gamma(2n+2\varphi(a_1)/\xi)}{\Gamma(n+2\varphi(a_1)/\xi)} \exp \left( -i \int \varphi(x; a_{n+1}) dx \right) \], (3.31)
and 
\[ \hat{B}^{-n} \psi^{(0)} = (-\zeta)^n \frac{\Gamma(n-1+2\varphi(a_1)/\xi)}{\Gamma(2n-1+2\varphi(a_1)/\xi)} \]
\[ \times \exp \left( +i \int \varphi(x; a_n) dx \right) \]. (3.32)

The condition in Eq. (3.30) is satisfied for a number of density profiles as illustrated in the next section. If this condition is not satisfied, the analytic continuation may still be done, but will be more complicated. Whenever the condition in Eq. (3.30) is satisfied, in the asymptotic limits the electron neutrino amplitude can be written as 
\[ \psi_e = \beta \xi^n \frac{\Gamma(2\mu + 2\varphi(a_1)/\xi)}{\Gamma(\mu + 2\varphi(a_1)/\xi)} \exp \left( -i \int \varphi(x; a_{\mu+1}) dx \right) \]
\[ + \gamma (-\zeta)^{-\mu-1} \frac{\Gamma(\mu + 2\varphi(a_1)/\xi)}{\Gamma(2\mu + 1 + 2\varphi(a_1)/\xi)} \]
\[ \times \exp \left( +i \int \varphi(x; a_{\mu+1}) dx \right) \]. (3.33)

Also using the identity 
\[ \lim_{y \to \pm \infty} \frac{1}{\mu^y} \frac{\Gamma(2\mu + y)}{\Gamma(\mu + y)} = 1 \], (3.34)
the asymptotic value of Eq. (3.33) for \( \varphi \to \pm \infty \) is found to be 
\[ \psi_e = \beta (2\varphi(a_1))^\mu \exp \left( -i \int \varphi(x; a_{\mu+1}) dx \right) \]
\[ + \gamma (-2\varphi(a_1))^{-\mu-1} \exp \left( +i \int \varphi(x; a_{\mu+1}) dx \right) \]. (3.35)

The hopping probability can be easily obtained from these results as we illustrate in the next section.
IV. EXAMPLES

A. Linear density profile

For the linear density profile
\[ N_e(x) = N_0 - N_0'(x-x_R), \]
where \( N_0 \) is the resonant density:
\[ 2\sqrt{2} G_F N_0 E = \delta m^2 \cos 2\theta_v, \]  
(4.1)
we can write
\[ \varphi(x) = -\frac{\delta m^2}{4E} \cos 2\theta_v \frac{N'_0}{N_0}(x-x_R). \]  
(4.3)
Adopting the ansatz \( \varphi(x;a_n) = a_n(x-x_R) \), one finds that Eq. (3.33) is satisfied with
\[ a_1 = -\frac{\delta m^2}{4E} \cos 2\theta_v \frac{N'_0}{N_0} = a_2 = a_3 \cdots , \]  
(4.4)
and
\[ R(a_n) = 2i\alpha_1 = -i\frac{\delta m^2}{2E} \cos 2\theta_v \frac{N'_0}{N_0}, \]  
(4.5)
for all \( n \) (i.e., \( \zeta = 0 \) for a linear density profile in Eq. (3.30)). The function \( f(n) \) of Eq. (4.18) is then
\[ f(n) = -i \left( \frac{\delta m^2}{2E} \cos 2\theta_v \right) n, \]  
(4.6)
and hence the solution of Eq. (3.19) is given by
\[ \mu = \frac{i}{2} \Omega, \]  
(4.7)
where we defined
\[ \Omega = \frac{\delta m^2 \sin^2 2\theta_v N_0}{4E \cos 2\theta_v N'_0}. \]  
(4.8)
Linear density profile is special in the sense that both before and after the MSW resonance its absolute value gets to be very large. Hence Eq. (3.35) can be used to evaluate both asymptotic limits. Initially we take \( \varphi \to +\infty \) which gives
\[ \Psi_e(x) = \beta \left( 2\varphi(x) \right)^{i\Omega/2} \exp \left( -i \int_0^x \varphi(x_1) dx_1 \right). \]  
(4.9)
Imposing the initial condition \( \Psi(x_i) = 1 \) determines \( \beta \) and gives the final (\( \varphi \to -\infty \)) solution to be
\[ \Psi_e(x) = \left( \frac{\varphi(x)}{\varphi(x_i)} \right)^{i\Omega/2} \exp \left( -i \int_0^x \varphi(x_1) dx_1 \right). \]  
(4.10)
Using Eqs. (2.5) and (2.6) one finds that the asymptotic matter angles are \( \varphi \to +\infty, \sin \theta_i = 1, \cos \theta_i = 0, \) \( \varphi \to -\infty, \sin \theta_f = 0, \cos \theta_f = 1, \) \( \) and that
\[ \Psi_e(x) = -\Psi_e^*(x), \]  
(4.11)
Inserting Eq. (4.11) into Eq. (2.9) one finds that
\[ \Psi_e(x_f) = -\Psi_e^*(x_f). \]  
(4.12)
Comparing Eqs. (4.10) and (4.12) and using
\[ \lim_{x_i \to -\infty, x_f \to +\infty} \frac{\varphi(x_f)}{\varphi(x_i)} = -1 = e^{i\pi}, \]  
(4.13)
we find that the hopping probability is given by
\[ P_{\text{hop}} = |\Psi_e(x_f)|^2 = \exp (-\pi \Omega). \]  
(4.14)

B. Exponential Density Profile

In this case we have
\[ N_e(x) = N_0 e^{-\alpha(x-x_R)}, \]  
(4.15)
where \( N_0 \) is the resonant density given in Eq. (4.12); we can write
\[ \varphi(x) = \frac{\delta m^2}{4E} \cos 2\theta_v \left( e^{-\alpha(x-x_R)} - 1 \right). \]  
(4.16)
Adopting the ansatz
\[ \varphi(x; a_n) = \frac{\delta m^2}{4E} \cos 2\theta_v \left( e^{-\alpha(x-x_R)} - a_n \right), \]  
(4.17)
one can show that to satisfy Eq. (3.35) we should require
\[ a_n = a_1 - \frac{ie4E}{\delta m^2 \cos 2\theta_v} (n-1), \]  
(4.18)
and
\[ R(a_n) = -2i\alpha a_n \frac{\delta m^2}{4E} \cos 2\theta_v - \alpha^2. \]  
(4.19)
Hence
\[ \zeta = i\alpha \]  
(4.20)
In this case there are two solutions of Eq. (3.19) and the appropriate solution giving the correct limiting behavior (that \( \mu \) vanishes as \( \Lambda \) does) is
\[ \mu = \frac{i}{\alpha} \left( \frac{\delta m^2}{4E} \right) (1 - \cos 2\theta_v). \]  
(4.21)
Exponential density profile, like the linear one, can get to be very large (and of course positive) before the MSW resonance point. Hence initially the second term in Eq. (3.35) vanishes and the constant \( \beta \) can be fixed to be \( \beta = 2(\varphi(x_i))^{-\mu} \). However, unlike the linear density, much
after the MSW resonance point the exponential density vanishes and one should use the asymptotic form for the constant $\varphi$ derived in the previous section. In this limit we have

$$\varphi(x, a_1) = -\frac{\delta m^2}{4E} \cos 2\theta_v,$$

$$\varphi(x, a_{\mu+1}) = -\frac{\delta m^2}{4E}. \quad (4.22)$$

Inserting Eqs. (4.21) and (4.22) into Eq. (3.33) we find the final electron neutrino amplitude to be

$$\Psi_e(x_f) = \left(\frac{i\alpha}{2\varphi(x_f)}\right)^\mu \Gamma\left(\frac{i\delta m^2}{4E}\right) \Gamma\left(\frac{1}{2} + \frac{3m^2}{4E} (1 + \cos 2\theta_v)\right) \times \exp\left(-\frac{i\delta m^2}{4E} x_f\right)$$

$$+ \gamma(-ia)^{-\mu-1} \Gamma\left(\frac{i\delta m^2}{4E}(1 + \cos 2\theta_v)\right) \Gamma\left(1 + \frac{3m^2}{4E}\right) \times \exp\left(-\frac{i\delta m^2}{4E} x_f\right). \quad (4.23)$$

The asymptotic matter angles are now given by

$$\varphi \to +\infty, \sin \theta_i = 1, \cos \theta_i = 0,$$

$$\varphi \to 0, \sin \theta_f = \sin \theta_v, \cos \theta_f = \cos \theta_v. \quad (4.24)$$

Inserting Eq. (4.24) into Eq. (2.10) we get

$$\Psi_e(L) = -\cos \theta_v \Psi_{2,(s)}^* \exp\left(-\frac{i\delta m^2}{4E} L\right)$$

$$- \sin \theta_v \Psi_{1,(s)} \exp\left(-\frac{i\delta m^2}{4E} L\right). \quad (4.25)$$

Comparing Eqs. (4.23) and (4.25) one gets

$$\Psi_{2,(s)}^* = -\frac{1}{\cos \theta_v} \left(\frac{\alpha}{2\varphi(x_f)}\right)^\mu \Gamma\left(\frac{i\delta m^2}{4E}\right) \Gamma\left(\frac{1}{2} + \frac{3m^2}{4E} (1 + \cos 2\theta_v)\right) \times \exp\left(-\frac{\pi i\delta m^2}{8E} (1 - \cos 2\theta_v)\right), \quad (4.26)$$

which, upon squaring, gives the hopping probability to be

$$P_{\text{hop}} = e^{-\pi \delta(1-\cos 2\theta_v)} e^{-2\pi \delta} \frac{1}{1 - e^{-2\pi \delta}}, \quad (4.27)$$

where we defined

$$\delta = \frac{\delta m^2}{2E \alpha}. \quad (4.28)$$

Eq. (4.27) was previously obtained by direct solution of the differential equation [9]. The method presented here is not only more straightforward, but also explicitly demonstrates the role of the boundary conditions. Indeed using the infinite Landau-Zener method only provides the first term in the numerator of Eq. (4.26) [15]. To obtain the exact result, it is essential to take into account the fact that the electron density goes to zero (not to infinity) when the neutrinos move very far away from the MSW resonance region as illustrated above.

V. APPROXIMATE EXPRESSIONS

The method presented here is useful not only to write down exact expressions for the electron neutrino survival probabilities for shape-invariant electron densities, but can also be used as a starting point of several useful approximations. In particular, the transformation in Eq. (1.7) can be formally introduced without an explicit reference to the parameters $a_n$. Suppose, for example, we wish to satisfy Eq. (4.19) with $n = 1$ by introducing the function $\xi(x)$ in Eq. (3.3)

$$\xi^2(x) - \varphi^2(x) + i\hbar \varphi'(x) + i\hbar \xi'(x) = \Lambda. \quad (5.1)$$

In this section we write $\hbar$ explicitly in the equations to emphasize the semiclassical nature of the approximations we consider. By expanding the function $\xi(x)$ in powers of $\hbar$

$$\xi(x) = \xi_0(x) + \hbar \xi_1(x) + \hbar^2 \xi_2(x) + \cdots, \quad (5.2)$$

we find that Eq. (5.1) can be solved by matching the powers of $\hbar$:

$$\xi_0^2 = \varphi^2 + \Lambda,$$

$$2\xi_0 \xi_1 = -i(\varphi' + \xi_0'). \quad (5.3)$$

Using Eqs. (3.24), (3.24), and (3.25) the electron neutrino amplitude in this approximation can be written as

$$\Psi_e(x) = \beta \left(i\hbar \frac{\partial}{\partial x} + \varphi\right) \exp\left(-\frac{i}{\hbar} \int \xi(x) dx\right) + \gamma\left(i\hbar \frac{\partial}{\partial x} - \varphi\right)^{-1} \left(i\hbar \frac{\partial}{\partial x} - \xi\right)^{-1} \exp\left(+\frac{i}{\hbar} \int \xi(x) dx\right). \quad (5.4)$$

For a monotonically decreasing $\varphi$ we choose $\xi_0 = +\sqrt{\varphi^2 + \Lambda}$. To evaluate Eq. (5.4) in the lowest order in $\hbar$, we note that in the exponentials one needs to include $\xi_1$ (since $\hbar$’s cancel) whereas in the pre-exponential factors it is sufficient to retain only $\xi_0$. Inserting Eq. (5.3) into Eq. (5.4) and imposing the initial conditions we obtain to lowest order in $\hbar$

$$\Psi_e(x) = \frac{1}{2} T_-(0) T_-(x) \exp\left(+\frac{i}{\hbar} \int \sqrt{\varphi^2(x) + \Lambda} dx\right)$$

$$+ \frac{1}{2} T_+(0) T_+(x) \exp\left(-\frac{i}{\hbar} \int \sqrt{\varphi^2(x) + \Lambda} dx\right), \quad (5.5)$$

where we defined
\[ T_{\pm}(x) = \left( 1 \pm \frac{\varphi(x)}{\sqrt{\varphi'^2(x) + \Lambda}} \right)^{1/2}. \] (5.6)

Eq. (5.5) is the electron neutrino amplitude in the standard adiabatic approximation. The equivalence of the adiabatic and primitive semiclassical approximations was previously proved in Ref. [11]. Here we demonstrated that it can be obtained by solving a shape-invariance condition, valid for any electron density, to lowest order in \( h \).

Even though shape-invariant electron densities constitute a rather restrictive class, one can use these exact solutions as comparison solutions to obtain uniform semiclassical approximate solutions [6][11][17] for any given density that is not shape-invariant. To achieve this goal, one starts with a shape-invariant electron density profile for which the solution of the “mapping” equation

\[ \left( i \frac{\partial}{\partial S} + \xi(S) \right) \left( i \frac{\partial}{\partial S} - \xi(S) \right) \chi(S) = \Omega \chi(S) \] (5.7)

is known. In the next step, the variable \( S \) in Eq. (5.7) is taken to a function of \( x \) and the solutions of Eq. (5.4) are written as

\[ \Psi(x) = K(x) \chi(S(x)). \] (5.8)

By making the choice

\[ K(x) = \frac{1}{\sqrt{S'(x)}}, \] (5.9)

one can show that \( \Psi \) given in Eq. (5.8) satisfies Eq. (5.4) if the equality

\[ \hbar^2 \frac{K''}{K} - S'^2 \left( \Omega + \xi^2 + i\hbar \frac{\partial \xi}{\partial S} \right) \]

\[ + (\Lambda + \varphi^2 + i\hbar \varphi') = 0 \] (5.10)

is satisfied. Eq. (5.10) is usually solved by expanding \( S(x) \) in powers of \( \hbar \). Using a linear mapping density this mapping technique was used in Refs. [11] and [12] to obtain an approximate solution for the neutrino survival probability. It is possible to generalize this result by using not the linear density, but another shape-invariant mapping density, which most closely resembles the actual electron density that the neutrinos travel through.

\[ VI. \ CONCLUSIONS \]

We showed that the analogy between supersymmetric quantum mechanics and matter-enhanced neutrino oscillations can be used to obtain exact expressions for the neutrino amplitudes traveling in a class of electron density profiles. We explicitly worked out the neutrino survival probability for two shape-invariant electron density profiles that are most commonly used for studying matter-enhanced neutrino oscillations, namely linear and exponential densities. In addition to these two the tanh \( x \) and \( 1/(a + bx) \) density profiles are also shape-invariant and the MSW equations for these density profiles can be solved using the techniques described here. One should emphasize that although the neutrino propagation in these density profiles was studied before, the method described in this paper seems to be more direct and to have pedagogical value as it explicitly illustrates the role of the boundary conditions. The resulting neutrino amplitudes can also be utilized as comparison amplitudes for the uniform semiclassical treatment of neutrino propagation in arbitrary electron density profiles.

\[ ACKNOWLEDGMENTS \]

This work was supported in part by the U.S. National Science Foundation Grant No. PHY-9605140 at the University of Wisconsin, and in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation. I thank Institute for Nuclear Theory and Department of Astronomy at the University of Washington for their hospitality and Department of Energy for partial support during the completion of this work.

[1] S.P. Mikheyev and A. Yu. Smirnov, Sov. J. Nucl. Phys. 42, 913 (1985); Sov. Phys. JETP 64, 4 (1986); L. Wolfenstein, Phys. Rev. D 17, 2369 (1978); Phys. Rev. D 20, 2634 (1979).

[2] J.N. Bahcall, Neutrino Astrophysics (Cambridge, New York, 1989); W.C. Haxton, Ann. Rev. Astron. Astrophys. 33, 459 (1995); A.B. Balantekin, in Proceedings of the 1997 Jorge Andre Swieca Summer School, Sao Paulo, Brazil, C.A. Bertulani, et al., Eds. astro-ph/9706254.

[3] G.M. Fuller, R.W. Mayle, B.S. Mayer, and J.R. Wilson, Astrophys. J. 389, 517 (1992); S.E. Woosley and R. Hoffman, ibid. 395, 202 (1992); S.E. Woosley, G.J. Mathews, J.R. Wilson, R.D. Hoffman, and B.S. Meyer, ibid. 433, 229 (1994); G.M. Fuller, Phys. Rep. 277, 149 (1993); Y.-Z. Qian, G.M. Fuller, G.J. Matthews, R.W. Mayle, J.R. Wilson, and S.E. Woosley, Phys. Rev. Lett. 71, 1965 (1993); F.N. Loretii, Y.-Z. Qian, G.M. Fuller, and A.B. Balantekin, Phys. Rev. D 52, 6664 (1995).

[4] L. Landau, Phys. Z. Sowjetunion 2, 46 (1932); C. Zener, Proc. R. Soc. London, Ser. A 137, 696 (1932); E.C.G. Stueckelberg, Helv. Phys. Acta 5, 369 (1932).

[5] W.C. Haxton, Phys. Rev. Lett. 57, 1271 (1986); S.J. Parke, Phys. Rev. Lett. 57, 1275 (1986).

[6] W.C. Haxton, Phys. Rev. D 35, 2352 (1987).

[7] S.T. Petcov, Phys. Lett. B 200, 373 (1988).

[8] S. Toshev, Phys. Lett. B 196, 170 (1987).
[9] M. Bruggen, W.C. Haxton, and Y.-Z. Qian, Phys. Rev. D 51, 4028 (1995).
[10] D. Nötzold, Phys. Rev. D 36, 1625 (1987).
[11] A.B. Balantekin, S.H. Fricke, and P.J. Hatchell, Phys. Rev. D 38, 935 (1988).
[12] A.B. Balantekin and J.F. Beacom, Phys. Rev. D 54, 6323 (1996).
[13] L. Gendenshtein, JETP Lett. 38, 356 (1983).
[14] A.B. Balantekin, University of Wisconsin preprint MADNT-97-08 (quant-ph/9712018).
[15] P. Pizzochero, Phys. Rev. D 36, 2293 (1987).
[16] S.C. Miller, Jr. and R.H. Good, Phys. Rev. 91, 174 (1953); S.H. Fricke, A.B. Balantekin, P.J. Hatchell, and T. Uzer, Phys. Rev. A 37, 2797 (1988).
[17] J.F. Beacom, A.B. Balantekin, to be published in 'Springer Lecture Notes in Physics', H. Aratyn, et al., eds. [hep-th/9709117].