Generalized solutions of the stochastic Burgers equation in multi-dimension

P. Catuogno*, J.F. Colombeau†, C. Olivera‡

Key words: Stochastic Burgers equation, Generalized functions, multiplication of distributions, generalized solutions.

MSC2000 subject classification: 60H15, 46F99.

Abstract

In this paper we introduce a new concept of weak solution for the conservative stochastic Burgers equation in multi-dimension. Our definition is based on weak solution concepts introduced by various authors to point out solutions when the equations do not have solutions in the sense of distributions. In one dimension our solution reduces to the classical distributional solution of the 1-D stochastic Burgers equation.

1 Introduction

The aim of this paper is to study the existence of solution to the multi-dimensional stochastic Burgers equation in $\mathbb{R}^d$ of the following form:

*Departamento de Matemática, Universidade Estadual de Campinas, Brazil. E-mail: pedrojc@ime.unicamp.br.
†Institut Fourier, Université de Grenoble., France. E-mail: jf.colombeau@wanadoo.fr.
‡Departamento de Matemática, Universidade Estadual de Campinas, Brazil. E-mail: colivera@ime.unicamp.br.
\[
\begin{aligned}
\left\{
\begin{array}{l}
\partial_t U(t,x) = \Delta_x U(t,x) + \nabla_x \|U(t,x)\|^2 + \nabla_x \dot{W}(t,x), \\
U(0,x) = \nabla_x f(x).
\end{array}
\right.
\end{aligned}
\]

where \(\dot{W}(t,x)\) is a space-time white noise.

During the past few decades, the stochastic Burgers equation has found applications in diverse fields ranging from statistical physics, cosmology, and fluid dynamics to engineering. In fact, the problem of Burgers turbulence, that is the study of the solutions to the Burgers equation with random initial conditions or random forcing is a central issue in the study of nonlinear systems out of equilibrium. See J. Bec, K. Khanin [3] and W. E [16].

The multidimensional conservative stochastic Burgers equation has been studied by several authors in the case \(\dot{W}(t,x)\) is a noise white in time and correlated in space. See A. Dermoune [15], Z. Brezniak, B. Goldys, M. Neklyudov [8] and R. Iturriaga, K. Khanin [20]. In the case that the nonlinear term is interpreted in the sense of Wick calculus the stochastic Burgers equation has been well studied, see S. Assing [2] and also H. Holden, T. Lindstrom, B. Oksendal, J. Uboe, T. Zhang [18] and [19]. They showed existence and uniqueness results for the solution regarded as a stochastic process with values in a Kondratiev space of stochastic distributions.

Up to our knowledge there is no proof of existence of solution in any sense for equation (1) for \(d \geq 2\). In the case that \(d = 1\), the problem of existence of solutions for stochastic Burgers equation is well understood, see G. da Prato, A. Debussche, R. Teman [14], L. Bertini, N. Cancrini, G. Jona-Lasinio [5] and P. Catuogno, C. Olivera [9].

A main difficulty with the multidimensional stochastic Burgers equation is
that the solutions take values in a distributional space. Therefore, it is necessary to give meaning to the non-linear term $\nabla_x \|U(t, x)\|^2$. In the 1-dimensional case the Cole-Hopf transformation is used to sort out this problem, but this does not make sense in the multidimensional case since the solution of the stochastic heat equation is not a standard stochastic process [31].

We remark that it is believed that the Cole-Hopf solution $U(t, x) := \nabla_x \log Z(t, x)$, where $Z(t, x)$ denotes the solution of the stochastic heat equation with multiplicative noise, is the correct physical solution of (1), see [6] and [11].

In order to give sense to this expected solution of the multidimensional stochastic Burgers equation we introduce a new concept of solution strongly inspired by previous works of numerous authors using generalized solutions for equations of mathematical physics in cases in which the expected or known (for instance from numerical investigations) solutions do not make sense within distribution theory: [7, 10, 11, 12, 17, 13, 22, 24, 28, 29]. In particular such generalized solutions have been considered for stochastic equations in Nedeljkov-Rajter [23] and Oberguggenberger et al [27, 25, 26].

We denote by $\mathcal{D}((0, T) \times \mathbb{R}^d; \mathbb{R}^d)$ the space of the infinitely differentiable functions with compact support in $(0, T) \times \mathbb{R}^d$ and values in $\mathbb{R}^d$.

Let us denote by $\mathcal{L}_B$ the Burgers operator,

$$\mathcal{L}_B \varphi = \partial_t \varphi - \Delta_x \varphi - \nabla_x \|\varphi\|^2$$

for $\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d; \mathbb{R}^d)$.

Now, we introduce our concept of weak solution for the stochastic Burgers equation.
Definition 1.1 We say that a sequence \((U_n)\) of smooth semimartingales is a weak generalized solution of the equation (1) if

1. For each \(\varphi \in \mathcal{D}((0, T) \times \mathbb{R}^d; \mathbb{R}^d)\),
\[
\lim_{n \to \infty} <\mathcal{L}_B U_n, \varphi > = \int_{[0,T] \times \mathbb{R}^d} \varphi(t, x) \cdot \nabla_x \dot{W}(t, x) \, dt \, dx.
\]

2. \(U_n(0, x) = \nabla f(x)\).

The above means that, after averaging on any smooth test function, the sequence \((U_n)\) tends to satisfy the equation when \(n \to \infty\). In general we ignore the nature of the possible limit of the sequence \((U_n)\): this limit can be a distribution which is not necessarily solution of the equation in the sense of distribution theory, such as in the case of shock waves for nonconservative systems [11, 1], in the case of delta waves in general and a fortiori delta-prime wave [29], or it can be an object which is not a distribution, such as objects put in evidence in general relativity in [31, 32, 33]. In one space dimension the limit of the sequence \((U_n)\) that we construct exists as a distribution which, further, is solution in the sense of distributions.

In this article we show that there exist a weak generalized solution for the multi-dimensional stochastic Burgers equation, which is a generalized type of Cole-Hopf solution. In the next section we review some facts on space time white noise. In section 3 we show the existence of such a weak solution for the stochastic Burgers equation (1). Then we recover the stronger result of existence of a distributional solution for the stochastic Burgers equation in dimension 1.
2 Preliminaries

We say that a random field \( \{S(t, x) : t \in [0, T], x \in \mathbb{R}^d\} \) is a spatially dependent semimartingale if for each \( x \in \mathbb{R}^d \), \( \{S(t, x) : t \in [0, T]\} \) is a \( \mathbb{R}^d \)-valued semimartingale in relation to the same filtration \( \{\mathcal{F}_t : t \in [0, T]\} \).

If \( S(t, x) \) is a \( C^\infty \)-function of \( x \) and continuous in \( t \) almost everywhere, it is called a smooth semimartingale. See [21] for a rigorous study of spatially depend semimartingales and applications to stochastic differential equations.

A distribution valued Gaussian process with mean zero \( \{\dot{W}(t, x) : t \in [0, T], x \in \mathbb{R}^d\} \) is a space-time white noise if

\[
E(\dot{W}(t, x)\dot{W}(s, y)) = \delta(t-s)\delta(x-y).
\]

More precisely, for any \( \xi \in L^2([0, T] \times \mathbb{R}^d) \) the random variables

\[
\int_{[0,T] \times \mathbb{R}^d} \xi(t, x)\dot{W}(t, x)dxdt
\]

are jointly Gaussian with mean zero and covariance

\[
E(\int_{[0,T] \times \mathbb{R}^d} \xi(t, x)\dot{W}(t, x)dxdt \int_{[0,T] \times \mathbb{R}^d} \rho(t, x)\dot{W}(t, x)dxdt) = \int_{[0,T] \times \mathbb{R}^d} \xi \cdot \rho(t, x)dxdt.
\]

It is not difficult to construct a space time white noise. In fact, let \( \{f_j : j \in \mathbb{N}\} \) be a complete orthonormal basis of \( L^2([0, T] \times \mathbb{R}^d) \) and \( \{Z_j : j \in \mathbb{N}\} \) be a family of independent Gaussian random variables with mean zero and variance one. Then

\[
\dot{W}(t, x) := \sum_{j=1}^{\infty} f_j(t, x)Z_j \tag{2}
\]

is a space time white noise.

The integral of \( \xi \in L^2([0, T] \times \mathbb{R}^d) \) with respect to the space-time white noise \( \dot{W} \) is given by the series

\[
\int_{[0,T] \times \mathbb{R}^d} \xi(t, x)\dot{W}(t, x)dxdt = \sum_{j=1}^{\infty} <\xi, f_j> Z_j.
\]
The cylindrical Wiener process \( \{ W_t : L^2(\mathbb{R}^d) \to L^2(\Omega) : t \in [0, T] \} \) associated to \( \hat{W} \) is given by

\[
W_t(\varphi) := \int_{[0,T] \times \mathbb{R}^d} 1_{[0,t]}(s)\varphi(x)\hat{W}(s, x)dx ds.
\]

It is clear that \( W_t(\varphi) \) is a Brownian motion with variance \( t\|\varphi\|^2 \) for each \( \varphi \in L^2(\mathbb{R}^d) \).

We say that a sequence of smooth semimartingales \( (W^n) \) is a weak approximation to the cylindrical Wiener \( W \) if for all \( \varphi \in L^2(\mathbb{R}^d) \),

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \varphi(x)W^n_t(x)dx = W_t(\varphi). \tag{3}
\]

A weak approximation \( (W^n) \) to the cylindrical Wiener process \( W \) is good if for each \( n \in \mathbb{N} \), \( W^n_t(x) \) is a Brownian motion with quadratic variation

\[
< W^n(x) >_t = C_n \cdot t,
\]

where \( C_n \) is a constant depending on \( n \).

A main ingredient in our approach for solving the Burgers equation is the use of regularization techniques for the white noise with respect to the space, we refer the reader to [6] and [24] for the background material.

Let \( \rho : \mathbb{R}^d \to [0, \infty) \) be an infinitely differentiable symmetric function with compact support such that \( \int_{\mathbb{R}^d} \rho(x) \, dx = 1 \). We will consider the mollifiers \( \rho_n(x) = n^d\rho(nx) \), with \( n \in \mathbb{N} \). The regularizations by \( \rho \) of the space-time white noise \( \hat{W} \), denoted by \( \hat{W}_{\rho_n} \), are defined to be

\[
\hat{W}_{\rho_n}(t, x) := \rho_n * \hat{W}(t, x).
\]

We observe that \( \hat{W}_{\rho_n} \) is white in time and colored in space, in fact we have that \( \hat{W}_{\rho_n}(t, x) \) is a distribution Gaussian valued process with mean zero and covariance,
\[
\mathbb{E}(\dot{W}_{\rho_n}(t,x)\dot{W}_{\rho_n}(s,y)) = \delta(t-s)h_n(x-y)
\]
where \(h_n : \mathbb{R}^d \to \mathbb{R}\) is given by \(h_n(z) = \int_{\mathbb{R}^d} \rho_n(u)\rho_n(u+z)du\).

In terms of the expansion (2) we have that
\[
\dot{W}_{\rho_n}(t,x) := \sum_{j=1}^{\infty} \rho_n * f_j(t,x)Z_j
\]
and the integral of \(\xi \in L^2([0,T] \times \mathbb{R}^d)\) with respect to \(\dot{W}_{\rho_n}\) is given by
\[
\int_{[0,T] \times \mathbb{R}^d} \xi(t,x)\dot{W}_{\rho_n}(t,x)dxdt = \sum_{j=1}^{\infty} \langle \xi * \rho_n, f_j \rangle Z_j.
\]

We observe that \(\dot{W}_{\rho_n}(t,x)\) converges weakly to the space-time white noise \(\dot{W}(t,x)\), this is,
\[
\lim_{n \to \infty} \int_{[0,T] \times \mathbb{R}^d} \xi(t,x)\dot{W}_{\rho_n}(t,x)dxdt = \int_{[0,T] \times \mathbb{R}^d} \xi(t,x)\dot{W}(t,x)dxdt \quad (4)
\]
for all \(\xi \in L^2([0,T] \times \mathbb{R}^d)\).

The mollified cylindrical Wiener process \(W_t^{\rho_n}(x)\) associated with the space-time white noise \(\dot{W}(t,x)\) is defined by
\[
W_t^{\rho_n}(x) := W_t(\rho_n(x-\cdot)). \quad (5)
\]

The distributional time derivative of \(W_t^{\rho_n}(x)\) is \(\dot{W}_{\rho_n}(t,x)\).

We have that \(W_t^{\rho_n}(x)\) is a Brownian motion with quadratic variation,
\[
< W_t^{\rho_n} >_t = \| \rho \|^2 n^d \cdot t. \quad (6)
\]

In particular we have proved the existence of good weak approximations to cylindrical Wiener processes.

**Proposition 2.1** \((W_t^{\rho_n})\) is a good weak approximation to the cylindrical Wiener process \(W\). In case that \(\varphi \in L^2(\mathbb{R}^d)\) has compact support the convergence in (5) is almost everywhere.
3 Existence of weak solution in any dimension

Let \((W^n)\) be a good weak approximation to the cylindrical Wiener \(W\). We will denote by \(H_n(t, x)\) the process \(\log Z_n(t, x)\), where \(Z_n\) is the solution of the regularized stochastic heat equation in the Itô sense

\[
\begin{aligned}
    dZ_n &= \Delta_x Z_n \, dt + Z_n \, dW^n, \\
    Z_n(0, x) &= \exp(f(x)).
\end{aligned}
\]  

(7)

Applying the Feymann-Kac formula we obtain the following representation of the solution to equation (7) with \(f \in C^\infty_b(\mathbb{R}^d)\),

\[
Z_n(t, x) = \mathbb{E}(\exp(f(x + B_t)) \exp(\int_0^t W^n(t - s, x + B_s) ds))
\]

(8)

where \(B_t\) is a Brownian motion independent of \(W^n\) such that \(B_0 = 0\) defined in an auxiliary probability space, see [15].

The sequence of smooth semimartingales \((U_n)\) given by \(U_n := \nabla_x H_n\) is called the Cole-Hopf sequence associated to \((W^n)\).

**Theorem 3.1** Let \(f \in C^\infty_b(\mathbb{R}^d)\). The Cole-Hopf sequence associated to \((W^n)\) is a weak generalized solution of the stochastic Burgers equation (1).

**Proof:** Let \((U_n)\) be the Cole-Hopf sequence given by \(U_n(t, x) := \nabla_x H_n(t, x) = \nabla_x \log Z_n(t, x)\). Since \(Z_n\) satisfies the equation (7), applying Itô formula we have that

\[
dH_n = \frac{dZ_n}{Z_n} - \frac{d < Z_n >}{2Z_n^2} = \frac{\Delta_x Z_n dt + Z_n dW^n}{Z_n} - \frac{Z_n^2 C_n dt}{2Z_n^2} = \frac{\Delta_x Z_n dt}{Z_n} + dW^n - \frac{C_n dt}{2}.
\]
It is easy to check that
\[ \frac{\Delta_x Z_n}{Z_n} = \Delta_x H_n + \|\nabla_x H_n\|^2. \]

Combining the above equations we obtain
\[ dH_n = (\Delta_x H_n + \|\nabla_x H_n\|^2)dt + dW_n - \frac{C_n dt}{2}, \] (9)

that is
\[ H_n(t, x) = f(x) + \int_0^t (\Delta_x H_n(s, x) + \|\nabla_x H_n(s, x)\|^2)ds + W^n_t(x) - \frac{C_n t}{2}. \]

Let \( \varphi = (\varphi_1, \cdots, \varphi_d) \in D((0, T) \times \mathbb{R}^d; \mathbb{R}^d) \). Multiplying (9) by \( \partial_t \partial_{x_i} \varphi(t, x) \) and integrating in \([0, T] \times \mathbb{R}^d\), we obtain that
\[
\int_{[0,T] \times \mathbb{R}^d} H_n \partial_t \partial_{x_i} \varphi dx dt = -\int_{[0,T] \times \mathbb{R}^d} (\Delta_x H_n + \|\nabla_x H_n(s, x)\|^2) \partial_{x_i} \varphi dx dt - \int_{[0,T] \times \mathbb{R}^d} \partial_{x_i} \varphi(t, x) dW^n_t(x) dx.
\]

Thus
\[
< \mathcal{L}_B U_n, \varphi > = -\int_{[0,T] \times \mathbb{R}^d} \nabla_x \cdot \varphi(t, x) dW^n_t(x) dx.
\] (10)

Finally, by (3) we have
\[
\lim_{n \to \infty} < \mathcal{L}_B U_n, \varphi > = \int_{[0,T] \times \mathbb{R}^d} \varphi(t, x) \cdot \nabla_x \tilde{W}(t, x) dx dt.
\] (11)

\( \square \)

**A classical distributional solution in the one dimensional case.** We observe that in the 1-dimensional case the sequence \((Z_n)\) of solutions of the regularized stochastic heat equations (7) converges uniformly on compacts of \((0, T) \times \mathbb{R}\) to \(Z\), where \(Z\) is the solution of the stochastic heat equation in the Itô sense
\[
\begin{aligned}
\frac{dZ}{Z} &= \Delta_x Z dt + Z_n dW, \\
Z(0, x) &= \exp(f(x)).
\end{aligned}
\] (12)
See L. Bertini and N. Cancrini [4], Theorem 2.2.

We will denote by $U(t, x)$ the gradient in the sense of the distributions of $\log Z(t, x)$, that is, $U(t, x) = \nabla_x \log Z(t, x)$.

**Corollary 3.2** $U$ is a distributional solution of the 1-dimensional stochastic Burgers equation [7]. That is, $U$ verifies

1. For all Cole-Hopf sequence $(U_n)$ associated to a good weak approximation to the cylindrical Wiener process $W$,

   $$U = \lim_{n \to \infty} U_n$$

   and there exists $\nabla_x \|U\|^2$ such that

   $$\nabla_x \|U\|^2 := \lim_{n \to \infty} \nabla_x \|U_n\|^2 \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}).$$

2. For all $\varphi \in \mathcal{D}((0, T) \times \mathbb{R}; \mathbb{R})$,

   $$< \mathcal{L}_B U, \varphi > = \int_{[0, T] \times \mathbb{R}^d} \varphi(t, x) \nabla_x \dot{W}(t, x) dx dt.$$ 

3. $\nabla_x f$ is the section of $U$ at $t = 0$ in the sense of S. Lojasiewicz, see [24].

**Proof:** It is clear that,

$$\lim_{n \to \infty} < U_n, \partial_t \varphi >= < U, \partial_t \varphi >$$ (13)

and

$$\lim_{n \to \infty} < U_n, \Delta_x \varphi >= < U, \Delta_x \varphi > .$$ (14)

We have that $\int_{[0, T] \times \mathbb{R}} \varphi(t, x) dW_t(x) dx$ defines a continuous linear functional from $\mathcal{D}((0, T) \times \mathbb{R}; \mathbb{R})$ to $\mathbb{R}$ (see K. Schaumloffel [30]) and

$$\lim_{n \to \infty} \int_{[0, T] \times \mathbb{R}} \nabla_x \varphi(t, x) dW_t^n(x) dx = \int_{[0, T] \times \mathbb{R}} \nabla_x \varphi(t, x) dW_t(x) dx.$$ (15)
From the equation (10) and the convergences (13), (14) and (15) we have that for all $\varphi \in \mathcal{D}((0,T) \times \mathbb{R}; \mathbb{R})$,
\[
\int_{[0,T] \times \mathbb{R}} \nabla_x \|U_n(t,x)\|^2 \varphi(t,x) \, dt \, dx
\]
converges and defines a linear functional. Thus the nonlinearity
\[
< \nabla_x \|U(t,x)\|^2, \varphi > := \lim_{n \to \infty} \int_{[0,T] \times \mathbb{R}} \nabla_x \|U_n(t,x)\|^2 \varphi(t,x) \, dt \, dx
\]
is well defined.
Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $\{\rho_\varepsilon : \varepsilon > 0\}$ be a strict delta net. From the continuity of $Z(t,x)$,
\[
\lim_{\varepsilon \to 0} \int_{[0,T] \times \mathbb{R}} \nabla_x \ln Z(t,x) \, \rho_\varepsilon(t) \, \varphi(x) \, dt \, dx = - \int_{\mathbb{R}} f(x) \, \nabla_x \varphi(x) \, dx = \int_{\mathbb{R}} \nabla_x f(x) \, \varphi(x) \, dx.
\]
Thus $\nabla_x f$ is the section of $U$ at $t = 0$. □

**Acknowledgements**

The authors P. Catuogno and C. Olivera are partially supported by CNPq through the grant 460713/2014-0. C. Olivera also by FAPESP through the grant 2012/18739-0. J.F. Colombeau is supported by FAPESP through the grant 2012/18940-7.

**References**

[1] J. Aragona; J.F. Colombeau and S.O. Juriaans. *Nonlinear Generalized functions and jump conditions for a standard liquid-gas model*. J. Math. Anal. Appl. 418,2,2014, p. 964-971.
[2] S. Assing. A pregenerator for Burgers equation forced by conservative noise. Commun. Math. Phys. 2002, 225, 611-632.

[3] J. Bec ; K. Khanin. Burgers turbulence. Phys. Rep. 2007, 447, 1-66.

[4] L. Bertini ; N. Cancrini. The stochastic heat equation: Feynman-Kac formula and intermittence. Journal of Statistical Physics. 1995, 78, 5-6, 1377-1401.

[5] L. Bertini ; N. Cancrini ; G. Jona-Lasinio. The stochastic Burgers equation. Comm. Math. Phys. 1994. 165, 2, 211-232.

[6] L. Bertini ; G. Giacomin. Stochastic Burgers and KPZ equations from particle systems. Comm. Math. Phys. 1997, 183, 571-607.

[7] H. A. Biagioni. A nonlinear theory of generalized functions. Springer Lecture Notes in Math. 1421, 1990.

[8] Z. Brzeźniak ; B. Goldys ; M. Neklyudov. Multidimensional stochastic Burgers equation. SIAM J. Math. Anal. 46, 2014, 1, 871-889.

[9] P. Catuogno ; C. Olivera. Strong solution of the stochastic Burgers equation. Appl. Anal. 93, 2014, 3, 646-652.

[10] J. Colombeau. Elementary introduction to new generalized functions. Math. Studies 113, North Holland, 1985

[11] J. Colombeau. Multiplication of distributions. Lecture Notes in Math. 1532, Springer Verlag, 1992

[12] V. G. Danilov, G.A. Omel’yanov, V.M. Shelkovich. Weak asymptotic method and interaction of nonlinear waves. AMS Translations 208, 2003, 33-164.
[13] V. G. Danilov; V. Yu. Rudnev, *Weak asymptotic solution of the phase-field system in the case of fusion of free boundaries in the Stefan-Gibbs-Thomson problem*, Journal of Mathematical Sciences, 151, 1, 2008, 2664-2676.

[14] G. Da Prato; A. Debussche; R. Temam. *Stochastic Burgers equation*. NoDEA Nonlinear Differential Equations Appl. 1994, 1, 4, 389-402.

[15] A. Dermoune. *Around the stochastic Burgers equation*. Stochastic Anal. Appl. 1997. 15, 3, 295-311.

[16] W. E. *Stochastic hydrodynamics*, Current developments in mathematics, 109-147, Int. Press, Somerville, 2000.

[17] M. Grosser; M. Kunzinger; M. Oberguggenberger and R. Steinbauer. *Geometric theory of generalized functions*. Kluwer, Dordrecht, 2001.

[18] H. Holden ; T. Lindstrom ; B. Oksendal ; J. Uboe ; T. Zhang. *The Burgers equation with a noisy force and the stochastic heat equation*. Comm. Partial Differential Equations. 1994, 19, 1-2, 119-141.

[19] H. Holden ; T. Lindstrom ; B. Oksendal ; J. Uboe ; T. Zhang. *The stochastic Wick type Burgers equation*. Stochastic Partial Differential Equations. 141-161, London Math. Soc. LNS, 216, Cambridge, 1995.

[20] R. Iturriaga ; K. Khanin. *Burgers turbulence and random Lagrangian systems*. Comm. Math. Phys. 2003, 232, 3, 377-428.

[21] H. Kunita. *Stochastic flows and stochastic differential equations*. Cambridge University Press, 1990.

[22] V. Maslov ; G. Omelyanov. *Geometric Asymptotics for Nonlinear PDE I*. Translations of Mathematical Monographs 202. AMS, 2001.
[23] M. Nedeljkov; D. Rajter. *Nonlinear stochastic wave equations with Colombeau generalized stochastic processes*. Math. Models Meth. applied Sciences. 12, 5, 2002, 665-688.

[24] M. Oberguggenberger. *Multiplication of distributions and applications to partial differential equations*. Pitman Research Notes in Math. Series 259. Ed. Longman Science and Technology, 1993.

[25] M. Oberguggenberger; F. Russo. *Nonlinear stochastic wave equations*. Integral Transform Special functions. 6, 1-4, 1998, 71-83.

[26] M. Oberguggenberger; F. Russo. *Nonlinear SPDEs: Colombeau solutions and pathwise limits*. Stochastic analysis and related topics, VI (Geilo, 1996), Progr. Probab., 42, Birkhauser Boston, Boston, MA, 1998.

[27] M. Oberguggenberger; F. Russo. *White noise driven stochastic partial differential equations: triviality and non-triviality*. Nonlinear theory of generalized functions (Vienna, 1997), 315-333, Chapman-Hall/CRC Res. Notes Math., 401, Chapman-Hall/CRC, 1999.

[28] G. Omelyanov ; I. Segundo-Caballero. *Asymptotic and numerical description of the kink/antikink interaction*. Electronic J. of Differential Equations, 2010, 150, 1-19.

[29] E. Panov ; V. Shelkovich. *δ-shock waves as a new type of solutions to systems of conservation laws*. J. Differential Equations 2006, 228, 49-86.

[30] K. Schaumloffel. *White noise in space and time as the time-derivative of a cylindrical Wiener process*. Stochastic Partial Differential Equations and Applications II, 225-229, LNM 1390, Springer, 1989.
[31] R. Steinbauer. *The ultra relativistic Reissner-Nordstrom field in the Colombeau algebra.* J. Math. Phys. 38, 3, 1977, 1614-1622.

[32] R. Steinbauer ; J A Vickers. *The use of generalized functions and distributions in general relativity.* Classical Quantum Gravity. 23, 10, R91-R114, 2006.

[33] J. A. Vickers. *Distributional geometry in general relativity.* J. Geom Phys. 62, 3, 2012, 692-705.

[34] J. Walsh. *An introduction to stochastic partial differential equations.* École d été de probabilités de Saint-Flour, XIV 1984, 265439, LNM 1180, Springer, 1986.