Burling graphs revisited, part II: Structure

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Abstract

The Burling sequence is a sequence of triangle-free graphs of increasing chromatic number. Any induced subgraph of a graph in this sequence is called a Burling graph. These graphs have attracted some attention because on one hand they have geometric representations, and on the other hand they provide counter-examples to several conjectures about bounding the chromatic number in classes of graphs.

Using an equivalent definition from the first part of this work (called derived graphs), we study several structural properties of Burling graphs. In particular, we give decomposition theorems for the class using in-star cutsets, study holes and their interactions in Burling graphs, and analyze the effect of subdividing some arcs of a Burling graph. Using mentioned results, we introduce new techniques for providing new triangle-free graph that are not Burling graphs.

Among other applications, we prove that wheels are not Burling graphs. This answers an open problem of the second author and disproves a conjecture of Scott and Seymour.

1 Introduction

Graphs in this paper have neither loops nor multiple edges. In this introduction they are non-oriented, but oriented graphs will sometimes be considered in the rest of the paper. A class of graphs is hereditary if it is

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closed under taking induced subgraphs. A triangle in a graph is a set of three pairwise adjacent vertices, and a graph is triangle-free if it contains no triangle.

In 1965, Burling [2] proved that triangle-free intersection graphs of axis-parallel boxes in $\mathbb{R}^3$ have unbounded chromatic number. The work of Burling uses geometric terminology. However, one may rephrase it into a more modern graph theoretic setting by defining in a combinatorial way a sequence of triangle-free graphs, called the Burling sequence, with increasing chromatic number, and proving that each of them is isomorphic to the intersection graph of a set of axis-parallel boxes in $\mathbb{R}^3$. It was later proved that every graph in the Burling sequence is isomorphic to the intersection graph of various geometrical objects, see [3, 8, 9, 11].

The Burling sequence also attracted attention lately because it is a good source of non-$\chi$-bounded classes of graphs. Let us explain this. A hereditary class of graphs is $\chi$-bounded if there exists a function $f$ such that for every graph $G$ in the class, $\chi(G) \leq f(\omega(G))$, where $\chi(G)$ denotes the chromatic number of $G$ and $\omega(G)$ the maximum number of pairwise adjacent vertices in $G$. In [8], it is shown that there exist graphs $H$ such that graphs in the Burling sequence (and therefore some triangle-free graphs with arbitrary large chromatic number) do not contain any subdivision of $H$ as an induced subgraph. This provides counter-examples to a well-studied conjecture of Scott (Conjecture 8 in [13]), saying that for every $H$, the class of graphs that do not contain any subdivision of $H$ as an induced subgraphs are $\chi$-bounded. A counter-example to Scott conjecture is called a non-weakly pervasive graph. Despite the fact that Scott’s conjecture is disproved in [8], classifying graphs into weakly pervasive and non-weakly pervasive remains of interest.

This paper is the second part in a series of three articles, so let us explain briefly the content of the other parts. In all the series, we call Burling graph any graph that is isomorphic to an induced subgraph of some graph in the Burling sequence. In the first part of this work [11], we give several equivalent definitions of Burling graphs, including geometrical characterizations. In the third part of this work [12], our goal is to use results from the previous parts to make progress toward understanding weakly pervasive graphs by giving new examples of graphs that are not weakly pervasive.

In this second part, we use one of the new equivalent definitions from the first part [11] (namely derived graphs) to study the structure of Burling graphs. Among other results, we give a decomposition theorem for oriented Burling graphs, characterize the subdivisions of $K_4$ that are Burling graphs,
and give a new characterization of Burling graphs (called $k$-sequential graphs). We also answer some open questions in $\chi$-boundedness: an open problem of the second author in \[16\] and a conjecture of Scott and Seymour in \[14\] (both concerning wheel-free graphs). Moreover, most of the results in this part have applications in the third part \[12\] as well. We postpone the explanations for these applications to part III, and in the following brief outline of the paper, we explain how the structural results of this contribution are motivated in their own rights.

In section 2, we describe the notation that we use.

In section 3, we recall the definition of derived graphs (first defined in Part I of this work \[11\] where they are shown to be equivalent to Burling graphs) which are oriented graphs obtained from a tree by some precise rules. We also show how their arcs can be subdivided, providing generic constructions of Burling graphs. This study of subdivisions seems important to us, because the class of Burling graphs is so far the only source of non-weakly pervasive graphs.

In section 4, we define the notion of nobility of a Burling graph (roughly, it is the maximum number of vertices of the graph that lie in a branch of a tree from which the graph can be derived). We then show how Burling graphs of a given nobility can be constructed from graphs of smaller nobility. The section is mostly motivated by the need of the so-called 2-Burling graphs that serve as basic graphs in the decomposition theorem of the next section. We also present $k$-Burling graphs that enables us to provide another equivalent definition of Burling graphs.

In Section 5, we give a decomposition theorem for Burling graphs using star cutsets. This theorem was already proved by Chalopin, Esperet, Li and Ossona de Mendez in \[3\]. Our contribution is to give a stronger result for oriented Burling graphs that we need in this part (Section 6) and in part 3. Interestingly, star cutsets have been proved very useful in the proof of the strong perfect graph theorem. They were invented by Chvátal \[6\] who generalized them to skew partitions, and who proved that in some sense, star cutsets preserve being perfect. It is not the place here to give too much detail about that, but briefly speaking, many classes of perfect graphs, including the class of all perfect graphs, have decomposition theorems \[5\] (a graph in the class is basic or has a decomposition), and star cutsets or one of their variants (skew partitions, double star cutsets, etc.) are often one of the decompositions. It is therefore striking to see a class of triangle-free graphs of high chromatic number that can still be decomposed into very simple graphs by star cutsets. Moreover, we observe here for the first time that the decomposition theorem of Burling graphs in \[3\] disproves an open question
from [4] about star cutset preserving in some sense bounds on the chromatic number. We believe that this explains why the decomposition method that was so efficient to prove the strong perfect graph theorem did not provide many results in the more general field of $\chi$-boundedness.

In section 6, we describe properties of holes in Burling graphs, where a hole is a chordless cycle of length at least 4. Results from this section provide essential tools for the proofs in Section 7 and for the results in part 3 of this work. In particular, the orientation of a hole in a Burling graph is used intensively in the sequel. Also, how a graph is organized regarding its holes is very often of interest in studying the structure of many classes of graphs, so we believe that our study might help future works about the structure of Burling graphs as well.

In section 7, using the results of our structural studies, we give several examples of graphs that are not Burling. Among these examples are wheels (graphs built from a hole and a vertex with at least three neighbors in that hole). This results proves that the class of wheel-free graphs is not $\chi$-bounded, which answers negatively a question of the second author in [16], and disproves a conjecture of Scott and Seymour in [15]. This result is independently proved by Davies [7] (with a different technique) and has been claimed by Scott and Seymour [14] in a personal communication. However, our proof, appearing first in the master’s thesis of the first author [10] is the first written proof of this theorem.

To sum up, besides answering some open problems, we study here many structural aspects of Burling graphs, most notably, their closure under subdivision, decomposition theorem, structure and holes. We believe that these studies, along with the examples we provide, might help solving the following questions.

**Question 1.** What is the list of minimal forbidden induced subgraphs that defines Burling graphs?

**Question 2.** What is the complexity of deciding whether a graph is Burling or not?

## 2 Notation

We denote by $N(v)$ the neighborhood of $v$, and denote by $N[v]$ the closed neighborhood of $v$, that is $N(v) \cup \{v\}$. If the graph is oriented, $N^-(v)$ and $N^+(v)$ denote the set of in-neighbors and out-neighbors of $v$ respectively, and $N^-[v]$ and $N^+[v]$ are the closed versions of them.
There is a difficulty regarding notations in this paper. The graphs we are interested in will be defined from trees. More specifically, a tree $T$ is considered and a graph $G$ is derived from it by following some rules that are defined in the next section. We have $V(G) = V(T)$ but $E(G)$ and $E(T)$ are different (disjoint, in fact). Also, even if we are originally motivated by non-oriented graphs, it turns out that $G$ has a natural orientation, and considering it is essential in many of our proofs.

So, in many situations we have to deal simultaneously with the tree, the oriented graph derived from it and the underlying graph of this oriented graph. A last difficulty is that since we are interested in hereditary classes, we allow removing vertices from $G$. But we have to keep all vertices of $T$ to study $G$ because the so-called shadow vertices, the ones from $T$ that are not in $G$, capture essential structural properties of $G$. All this will be clearer in the next section. For now, it explains why we need to be very careful about the notation. For any classical notion that we do not define, refer to [1].

**Notation for trees**

A tree is a graph $T$ such that for every pair of vertices $u, v \in V(T)$, there exists a unique path from $u$ to $v$. A rooted tree is a pair $(T, r)$ such that $T$ is a tree and $r \in V(T)$. The vertex $r$ is called the root of $(T, r)$. By abuse of notation, we often refer to the rooted tree $(T, r)$ as $T$, in particular when $r$ is clear from the context.

In a rooted tree, each vertex $v$ except the root has a unique parent which is the neighbor of $v$ in the unique path from the root to $v$. If $u$ is the parent of $v$, then $v$ is a child of $u$. A leaf of a rooted tree is a vertex that has no child. Note that every tree has at least one leaf.

A branch in a rooted tree is a path $v_1v_2\ldots v_k$ such that for each $1 \leq i < k$, $v_i$ is the parent of $v_{i+1}$. This branch starts at $v_1$ and ends at $v_k$. A branch that starts at the root and ends at a leaf is a principal branch. Note that every rooted tree has at least one principal branch.

If $T$ is a rooted tree, the descendants of a vertex $v$ are all the vertices that are on a branch starting at $v$. The ancestors of $v$ are the vertices on the unique path from $v$ to the root of $T$. Notice that a vertex is a descendant and an ancestor of itself. We call a strict descendant (resp. a strict ancestor) of a vertex any descendant (resp. ancestor) other than the vertex itself.

We will no more use words such as neighbors, adjacent, path, etc for trees. Only parent, child, branch, descendant and ancestor will be used.
**Notation for graphs and oriented graphs**

Graphs in this article have no loops and no multiple edges. From now on, we use the term *oriented graph* to refer to a graph whose edges (called *arcs*) are all oriented and no arc is oriented in both directions, and we use the term *non-oriented graph* or simply *graph* only to refer to a graph with none of its edges oriented.

When \( G \) is a graph or an oriented graph, we denote by \( V(G) \) its vertex set. We denote by \( E(G) \) the set of edges of a graph \( G \) and by \( A(G) \) the set of arcs of an oriented graph. When \( u \) and \( v \) are vertices, we use the same notation \( uv \) to denote an edge and an arc. Observe the arc \( uv \) is different from the arc \( vu \), while the edge \( uv \) is equal to the edge \( vu \).

When \( G \) is an oriented graph, its *underlying graph* is the graph \( H \) such that \( V(H) = V(G) \) and for all \( u, v \in V(H) \), \( uv \in E(H) \) if and only if \( uv \in A(G) \) or \( vu \in A(G) \). We then also say that \( G \) is an *orientation of* \( H \). When there is no risk of confusion, we often use the same letter to denote an oriented graph and its underlying graph.

In the context of oriented graphs, we use the words *in-neighbor*, *out-neighbor*, *in-degree*, *out-degree*, *sink* and *source* with their classical meaning. Terms from the non-oriented realm, such as *degree*, *neighbor*, *isolated vertex* or *connected component*, when applied to an oriented graph, implicitly apply to its underlying graph.

**Notation for paths and cycles**

For \( k \geq 0 \), a *path* of length \( k \) is a graph \( P \) with vertex-set \( \{p_0, \ldots, p_k\} \) and edge-set \( \{p_0p_1, \ldots, p_{k-1}p_k\} \). We denote it by \( P_k \). For \( k \geq 3 \), a *cycle* of length \( k \) is a graph \( C \) with vertex-set \( \{c_1, \ldots, c_k\} \) and edge-set \( \{c_1c_2, \ldots, c_{k-1}c_k, c_kc_1\} \). We denote it by \( C_k \). Since we deal mostly with the “induced subgraph” containment relation, when we say that \( X \) is a path (resp. a cycle) in some graph \( G \), we mean that \( X \) is an induced subgraph of \( G \) that is isomorphic to a path (resp. a cycle).

We define directed paths and cycles similarly (the arc-sets are \( \{p_0p_1, \ldots, p_{k-1}p_k\} \) and \( \{c_1c_2, \ldots, c_{k-1}c_k, c_kc_1\} \) respectively).

A path (resp. a cycle) in an oriented graph is a path (resp. a cycle) of its underlying graph. Possibly, it is not a directed path (resp. cycle), it can have any orientation. Notations \( P_k \) and \( C_k \) will be used only in the context of non-oriented graphs. We use no specific notation for directed paths and cycles. A *hole* in a graph \( G \) is a cycle of length at least 4 that is an induced subgraph of \( G \). A *hole* in an oriented graph means a hole of its underlying
3 Derived graphs

In this section, we recall the definition and main properties of derived graphs that were introduced in [11]. A Burling tree is a 4-tuple \((T, r, \ell, c)\) in which:

(i) \(T\) is a rooted tree and \(r\) is its root,

(ii) \(\ell\) is a function associating to each vertex \(v\) of \(T\) which is not a leaf, one child of \(v\) which is called the last-born of \(v\),

(iii) \(c\) is a function defined on the vertices of \(T\). If \(v\) is a non-last-born vertex of \(T\) other than the root, then \(c\) associates to \(v\) the vertex-set of a (possibly empty) branch in \(T\) starting at the last-born of \(p(v)\). If \(v\) is a last-born or the root of \(T\), then we define \(c(v) = \emptyset\). We call \(c\) the choose function of \(T\).

By abuse of notation, we often use \(T\) to denote the 4-tuple.

The oriented graph \(G\) fully derived from the Burling tree \(T\) is the oriented graph whose vertex-set is \(V(T)\) and such that \(uv \in A(G)\) if and only if \(v\) is a vertex in \(c(u)\). A non-oriented graph \(G\) is fully derived from \(T\) if it is the underlying graph of the oriented graph fully derived from \(T\).

A graph (resp. oriented graph) \(G\) is derived from a Burling tree \(T\) if it is an induced subgraph of a graph (resp. oriented graph) fully derived from \(T\). The graph (resp. oriented graph) \(G\) is called a derived graph if there exists a Burling tree \(T\) such that \(G\) is derived from \(T\).

Observe that if the root of \(T\) is in \(V(G)\), then it is an isolated vertex of \(G\). Observe that a last-born vertex of \(T\) that is in \(G\) is a sink of \(G\). This does not mean that every oriented derived graph has a sink, because it could be that no last-born of \(T\) is in \(V(G)\).

In 1965, Burling [2] introduced a sequence \(\{G_k\}_{k \geq 1}\) of graphs such that \(\chi(G_k) = k\) and where each \(G_k\) is a triangle-free intersection graph of some boxes (i.e. axis parallel cuboids) in \(\mathbb{R}^3\) (see [2] or [11] for the definition). The class of Burling graphs is the class of all induced subgraphs of the graphs in the sequence \(\{G_k\}\). It was proved in [11] that a graph is a Burling graph if and only if it can be derived from a Burling tree (Theorem 4.9 of [9]). So in what follows, we use “Burling graph” or “graph derived from a tree” according to what is most convenient.
Figure 1: Complete bipartite graphs derived from trees.

Figure 2: Cycle of length 6 derived from a tree.

Now, let us give some examples of derived graphs. We use the convention in all figures in this paper that the tree $T$ is represented with black edges while the arcs of $G$ are represented in red. The last-born of a vertex of $T$ is its rightmost child. Moreover, denoting a vertex of $T$ in white means that this vertex in not in $G$.

In the first graph represented in Figure 1, we have $c(u) = c(v) = \{x, y\}$. It shows that there exists an orientation of $C_4$ which is an oriented derived graph, so $C_4$ is a derived graph. The second graph shows that $K_{3,3}$ is a derived graph, and it is easy to generalize this construction to $K_{n,m}$ for all integers $n, m \geq 1$. In both graphs, the vertex $r$ of $T$ is not a vertex of $G$. Figure 2 is a presentation of $C_6$ as a derived graph. Notice that in this presentation, the vertex $v$ is not in $G$.

Notice that if a graph $G$ is derived from $T$, the branches of $T$, restricted to the vertices of $G$, are stable sets of $G$. In particular, no edge of $T$ is an edge of $G$. 

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Lemma 3.1. Suppose that $G$ is an oriented graph derived from a Burling tree $T$. If $uv \in A(G)$, then $p(u)$ is an ancestor of $p(v)$.

Proof. Follows directly from the definition of derived graphs. □

Lemma 3.2. An oriented Burling graph has no directed cycles.

Proof. Let $G$ be a Burling graph. It is therefore a derived graph (by Theorem 4.9 of [11]). Assume that $G$ is derived from a Burling tree $T$. If a directed cycle $v_1 \ldots v_k v_1$ exists in $G$, then by Lemma 3.1, for every $1 \leq i \leq k$, $p(v_i)$ is an ancestor of $p(v_{i+1})$ (with addition modulo $k$). This is possible only if $p(v_1) = \cdots = p(v_k)$. So, all $v_i$’s have the same parent $p$, so we may assume up to symmetry that $v_1$ and $v_2$ are not last-borns of $p$. This is a contradiction since $v_1v_2 \in A(G)$. □

Lemma 3.3. Every oriented Burling graph contains a source.

Proof. Follows from Lemma 3.2. □

Lemma 3.4. A Burling graph contains no triangle.

Proof. Suppose, for the sake of contradiction, that this is not true. Let $G$ be an oriented graph derived from a Burling tree $T$ whose underlying graph contains a triangle. Up to symmetry, a triangle has only two possible orientations. By Lemma 3.2, the triangle in $G$ is not a directed cycle. Therefore, it is transitively oriented, i.e. it has an arc $uw$ and two arcs $uw$ and $wv$. So, $v$ and $w$ are both out-neighbors of $u$, and thus are included in the a common branch of $T$. Therefore, there is no arc between $v$ and $w$, a contradiction. □

The next lemma shows that all oriented Burling graphs can be derived from Burling trees with specific properties. This will reduce complication in some proofs. Let $G$ be an oriented graph derived from a Burling tree $T$. An arc $uv$ of $G$ is a top-arc with respect to $T$ if $v$ is the out-neighbor of $u$ that is closest (in $T$) to the root of $T$. An arc $uv$ of $G$ is a bottom-arc with respect to $T$ if $v$ is the out-neighbor of $u$ that is furthest (in $T$) from the root of $T$. Note that it is possible that a top (resp. bottom) arc with respect to $T$ is no longer a top-arc (resp. bottom) when $G$ is derived from another tree $T'$. This is why we add “with respect to $T$” but we omit it when $T$ is clear from the context. Informally, top-arcs and bottom-arcs are arcs that can be subdivided while preserving being Burling, as we will show in Lemma 3.8.
Lemma 3.5. If $G$ is an oriented Burling graph derived from a Burling tree $(T, r, \ell, c)$, then $G$ can be derived from a Burling tree $(T', r', \ell', c')$ such that: $v \in V(G)$ if and only if $v$ is neither the root nor a last-born of $T'$. Moreover, for all arcs $uv$ of $G$, $uv$ is a top (resp. bottom) arc of $G$ with respect to $T$ if and only if it is a top (resp. bottom) arc of $G$ with respect to $T'$.

Proof. We apply several transformations to $(T, r, \ell, c)$ in order to obtain a Burling tree $(T', r', \ell', c')$ from which $G$ can be derived and that satisfies the conclusion. We start with $(T', r', \ell', c') = (T, r, \ell, c)$.

If $r \in V(G)$, then add a vertex $r'$ adjacent to $r$, and set $\ell'(r) = r$.

If some vertex $v$ of $G$ is a last-born (possibly $r$ after the first step), then let $u$ be its parent. Delete the edge $uv$ from $T'$ and add a vertex $w$ adjacent to $u$ and $v$. Set $\ell'(u) = w$. Add a vertex $w'$ adjacent to $w$, and set $w' = \ell(w)$. For every vertex $x$ such that $v \in c(x)$, replace $c'(x)$ by $c'(x) \cup \{w\}$. See Figure 3. Note that all the new vertices added to $T'$ at this step are last-borns and are not in $G$.

Finally, suppose that some non-root vertex $v$ of the tree is not a last-born of the tree and is not in $V(G)$ neither. In that case, let $p$ be the parent of $v$ (which exists, because $v$ is not the root) and let $u$ be the last-born of $p$. So, $u \neq v$. Let $v_1 = v$ and for $i \geq 1$, if $v_i$ is not a leaf, let $v_{i+1} = \ell(v_i)$. Let $v_k$ be a leaf. Define a new Burling tree $(T', r, \ell, c')$ as follows: remove the edge $pu$ and add the edge $v_ku$. This defines the tree $T'$. Now, define $\ell'(p) = v_1$, $\ell'(v_k) = u$, and for any other vertex $w$, define $\ell'(w) = \ell(w)$. Moreover,
define $c'(v) = \emptyset$, for every vertex $w \in V(T') \setminus \{v\}$ with $u \in c(w)$, define $c'(w) = c(w) \cup \{v_1, \ldots, v_k\}$, and for any other vertex $w'$, define $c'(w') = c(w)$. In the new Burling tree, $v$ is a last-born. Notice that because of the previous transformations, none of the vertices $v_1, \ldots, v_k$ are in $V(G)$, and therefore the adjacencies in $G$ do not change. Moreover, this transformation does not cancel the effect of the previous ones.

Applying these transformations in the presented order leads to a Burling tree that satisfies the conclusions of the lemma and for every vertex $x$, we have $c'(x) \cap V(G) = c(x) \cap V(G)$ for $x \in V(G)$, so that $G$ can be derived from $T'$. The orders in which the vertices of $G$ appear along branches of $T$ is the same as along the branches of $T'$, so top and bottom vertices are the same for the two representations.

Subdivisions

We here study how to obtain new oriented Burling graphs by subdividing arcs. Before starting the proofs, it is worth observing that in a Burling tree $(T, r, \ell, c)$, when $v$ is the last-born of some vertex $u$, one may obtain another Burling tree $(T', r, \ell', c')$ by setting $(T', r, \ell', c') = (T, r, \ell, c)$ and then applying the following transformations to $T'$. First, delete the edge $uv$ from $T'$ and add a new vertex $w$ adjacent to $u$ and $v$. Then set $\ell'(u) = w$ and $\ell'(w) = v$. To define $c'$, add $w$ to all sets $c(x)$ that contain $v$. A fact that we do not state formally but is easy to check and is implicit in some of the proofs below is that this new Burling tree $T'$ is equivalent to $T$ in the sense that every graph that can be derived from $T$ can be derived from $T'$. This is simply because $V(G) \subseteq V(T) \subseteq V(T')$ and because for all $x \in V(G)$, $c(x) \cap V(G) = c'(x) \cap V(G)$.

Subdividing an arc $uv$ into $uwv$ in an oriented graph means removing the arc $uv$ and adding instead a directed path $uwv$ where $w$ is a new vertex.

**Lemma 3.6.** Let $G$ be an oriented graph derived from a Burling tree $T$ and $uv$ be a bottom-arc of $G$. The graph $G'$ obtained from $G$ by subdividing $uv$ into $uwv$ can be derived from a Burling tree $T'$ in such a way that:

- $uw$ is a bottom-arc of $G'$,
- $uw$ is both a bottom-arc and a top-arc of $G'$,
- every top-arc of $G$ with respect to $T$ (except $uv$) is a top-arc of $G'$ with respect to $T'$,
- every bottom-arc of $G$ with respect to $T$ (except $uv$) is a bottom-arc of $G'$ with respect to $T'$.
Figure 5: Subdividing a bottom-arc.

Proof. By Lemma 3.5, we may assume that $v$ is not a last-born. Let $x$ be the parent of $v$, and let $t$ be the last-born of $x$. See Figure 5.

Build from $T$ a Burling tree $(T', r', \ell', c')$ by removing the edge $xt$ from $T$. Then add to $T'$ a path $xx't$, and set $\ell'(x) = x'$, and $\ell'(x') = t$. Add to $T'$ a new vertex $w$ adjacent to $x$. Set $c'(u) = \{w\} \cup c(u) \setminus \{v\}$. Set $c'(w) = \{x', v\}$.

We see that the oriented graph $G'$ obtained from $G$ by subdividing arc $uv$ into $uwv$ can be derived from $T'$.

Top-subdividing an arc $uv$ into $wu$ and $wv$ means removing $uv$ and add instead two arcs $wv$ and $wu$ where $w$ is a new vertex.

Lemma 3.7. Let $G$ be an oriented graph derived from a Burling tree $T$ and $uv$ be a top-arc of $G$ such that $u$ is a source of $G$. The graph $G'$ obtained from $G$ by top-subdividing $uv$ can be derived from a Burling tree $T'$ in such a way that:

- $wv$ is a top-arc,
- $wu$ is a bottom-arc,
- every top-arc of $G$ with respect to $T$ (except $uv$) is a top-arc of $G'$ with respect to $T'$,
- every bottom-arc of $G$ with respect to $T$ (except $uv$) is a bottom-arc of $G'$ with respect to $T'$.

Proof. Note that $u$ is not a last-born since there exists an arc $uv$ in $G$. Let $x$ be the parent of $u$. Let $y$ be the last-born of $v$ (if $v$ is a leaf of $T$, just add $y$ to $T$). By Lemma 3.5, we may assume that $y$ is not in $G$. Let $v'$ be the child of $v$ such that $v' \in c(u)$ (it is possible that $v' = y$ if no child of $v$ is in $c(u)$, but in that case we just add $y$ to $c(u)$). See Figure 6 where the cases
Figure 6: Top-subdividing a top-arc.
and \( v' \neq y \) and \( v' = y \) are represented. Notice that the proof below applies to
the two cases at the same time.

Build from \( T \) a Burling tree \( (T', r', \ell', c') \) by removing the edges \( xu, vv' \) and \( vy \) from \( T \). Then add to \( T' \) the edge \( vu \), the path \( vy'y \) and the edge \( y'y' \), and set \( \ell'(v) = y' \) and \( \ell'(y') = y \). Add to \( T' \) a new vertex \( w \) adjacent to \( x \) (in \( T' \)). Set \( c'(u) = c(u) \setminus V(P) \) where \( P \) is the path of \( T' \) from \( x \) to \( v \). Set \( c'(w) = \{u\} \cup V(P) \setminus \{x\} \). Replace \( c'(z) \) by \( c'(z) \cup \{y'\} \) for all \( z \) such that \( y \in c(z) \) or \( v' \in c(z) \).

We see that the oriented graph \( G' \) obtained from \( G \) by top-subdividing \( uv \) into \( uw \) and \( wv \) can be derived from \( T' \) because \( v \notin c'(u) \) and \( u, v \in c'(w) \), and the rest of the arcs between the vertices of \( G \) have remained unchanged.

Observe that \( v \) and \( u \) are the only vertices of \( G \) in \( V(P) \setminus \{x\} \) since \( uv \) is a top-arc of \( G \). Observe that no vertex \( z \) of \( G \) has \( u \) in \( c(z) \) since \( u \) is a source of \( G \).

**Lemma 3.8.** Let \( G \) be an oriented Burling graph derived from a Burling tree \( T \). Any graph obtained from \( G \) after performing the following operations is an oriented Burling graph:

- Replacing some bottom-arcs \( uv \) by a path of length at least 1, directed from \( u \) to \( v \).
- Replacing some top-arcs \( uv \) such that \( u \) is a source of \( G \) by an arc \( wv \) and a path of length at least 1 from \( w \) to \( u \).

**Proof.** Clear by repeatedly applying Lemmas 3.6 and 3.7.

Let us now give applications of Lemma 3.8. In Figures 7 and 8, some oriented Burling graphs are represented. In both figures, dotted arcs represent top-arcs that have a source as one of their end-points, dashed arcs represent bottom-arc, and all the other arcs are represented by solid arcs. Therefore, by Lemma 3.8, by top-subdividing any of the dotted arcs and subdividing any of the dashed arcs, we obtain an oriented Burling graph. In Figure 7, the oriented graphs derived from the Burling trees from Figure 1 are represented. In Figure 8, a Burling tree, together with the oriented graph derived from it, are represented. By considering its underlying graph, we see how to obtain several subdivisions of \( K_4 \), namely any subdivision in which every edge except \( uy \) and \( wy \) is possibly subdivided. We will see later that this figure in fact provides all subdivisions of \( K_4 \) that are Burling graphs (see Lemma 7.4). As a consequence, all graphs arising from the three non-oriented graphs in Figure 9 by subdividing dashed edges are Burling graphs.
Figure 7: Some subdivisions of complete bipartite graphs as Burling graphs.

Figure 8: Some subdivisions of $K_4$ that are Burling graphs.

Figure 9: Some Burling graphs. Dashed edges can be subdivided.
Lemma 3.9. Let $G$ be an oriented derived graph, and let $uv$ be an arc such that $N^+(u) = \{v\}$ and $N^-(v) = \{u\}$. Then the graph $G'$ obtained by contracting $uv$ is also an oriented derived graph and the top-arcs (resp. bottom-arcs) of $G$ except $uv$ are the top-arcs (resp. bottom-arcs) of $G'$.

Proof. Suppose that $G$ is derived from the Burling tree $(T, r, \ell, c)$. Let $S$ be the vertex-set (possibly empty) of the path starting at the last-born of the parent of $u$ and ending at the parent of $v$ in $T$. Notice that $S \subseteq c(u)$, and since $c(u) \cap V(G) = \{v\}$, no vertex of $S$ and no descendant of $v$ in $c(u)$ is a vertex of $G$.

Define $c'(u) = S \cup c(v)$ and define $c'(w) = c(w)$ for any vertex $w$ of $T$ other than $u$. It is easy to see that $(T, r, \ell, c')$ is a Burling tree. The graph $G'$ is derived from this new Burling tree. Indeed, $G'$ is the subgraph of the graph fully derived from $(T, r, \ell, c')$ induced on $V(G) \setminus \{v\}$.

Finally, it is easy to see that no top-arcs or bottom-arcs are changed except for $uv$. 

4 $k$-Burling graphs

An oriented graph $G$ is a oriented $k$-Burling graph if it can be derived from a Burling tree $T$ such that on each branch of $T$, at most $k$ vertices belong to $G$. In such a case, we say that $G$ is derived from $T$ as a $k$-Burling graph. Note that the empty graph is the unique 0-Burling graph (in fact, the empty graph is $k$-Burling for all integers $k \geq 0$). In the next sections, 2-Burling graphs will be useful and we need to describe them more precisely. Since it is not much harder to describe $k$-Burling graphs in general, we do this here by the mean of the so-called $k$-sequential graphs.

The nobility of an oriented graph is the smallest integer $k$ such that $G$ is a $k$-Burling graph. The nobility of a non-oriented Burling graph $G$ is the smallest nobility of an oriented Burling graph $G'$ such that $G$ is the underlying graph of $G'$.

Let us see examples. On Figure 10, two oriented graphs $G_1$ and $G_2$ are represented. Since $G_1$ can be derived from $T_1$, we see that $G_1$ is a 2-Burling graph. In fact, the nobility of $G_1$ must be 2, because since $c$ is a source of degree 2, its two out-neighbors must be on the same branch. Similarly, $G_2$ is a 3-Burling graph and has nobility 3 (because of $y$ being a source of degree 3). Hence, $G$ is a 2-Burling graph (in fact it has nobility 2 because we will soon see that only forests have nobility 1). This shows that an oriented graph (for instance $G_2$) may have a nobility different from the nobility of its underlying graph.
Figure 10: Nobility of a graph.

Figure 11: $G$ can be derived from $T_1$ and $T_2$. 
In Figure 12, an oriented Burling graph $G$ with nobility 3 is represented together with two Burling graphs $T_1$ and $T_2$. Observe that $T_2$ has a branch that contains four vertices of $G$. So, $G$ is derived from $T_1$ as a 3-Burling graph, and is derived from $T_2$ as a 4-Burling graph.

It should be pointed out that the nobility of an oriented graph may be strictly greater than its maximum out-degree, as shown on Figure 12. The graph $G$ has three sources with out-neighborhood $\{1, 2, 3\}$, $\{2, 3, 4\}$ and $\{3, 4, 5\}$. Assume that $G$ is derived from a Burling tree $T$. At least four vertices among 1, 2, 3, 4 and 5 must lie on the same branch of $T$. We only sketch the proof: on the branch of $T$ that contains 1, 2, and 3, either 1 is the farthest vertex from the root or it is not. In the former case, if 4 is not a descendant of both 2 and 3, then 1, 2, 3, and 4 are all on a common branch, and if 4 is a descendant of both 2 and 3, then any branch containing 4, contains both 3, and 5. So, in particular, 2, 3, 4, and 5 are on a common branch. In the latter case (where 1 is not the farthest from root among 1, 2, and 3), any branch containing both 2 and 3 contains 1 as well. So, in particular 1, 2, 3, and 4 are on a common branch. So, the nobility of $G$ is 4.

1-Burling graphs

An in-tree is any oriented graph obtained from a rooted tree $(T, r)$ by orienting every edge towards the root. Formally, $e = uv$ is oriented from $u$ to $v$ if and only if $v$ is on the unique path of $T$ from $u$ to $r$. Notice that in an in-tree, every vertex but the unique sink, has a unique out-neighbor. An in-forest is an oriented forest whose connected components are in-trees.

**Lemma 4.1.** An oriented graph $G$ is an in-forest if and only if it is a 1-Burling oriented graph.

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Proof. Suppose that $G$ is derived from a Burling tree $T$ as a 1-Burling graph. By Lemma 3.3, every Burling graph contains a source and since the out-neighborhood of any vertex is included in a branch and $G$ is 1-Burling, this source has degree at most 1. So, every 1-Burling oriented graph has a source of degree at most 1. This implies by an easy induction that every 1-Burling oriented graph is an in-forest.

To prove the converse statement, it is enough to check that for every 1-Burling graph $G$ derived from a Burling tree $(T, r, \ell, c)$ and every vertex $v$ of $G$, adding an in-neighbor $u$ of $v$ with degree 1 yields a 1-Burling graph $G'$. Here is how to construct $G'$. Build a rooted tree $T'$ from $T$ by adding a new root $r'$ adjacent to $r$. Define for $V(T')$ the functions $\ell'$ and $c'$ as equal to $\ell$ and $c$ for vertices of $T$. Nominate $r$ as the last-born child of $r'$ and add $u$ as a non-last-born child of $r'$. Then consider a branch $B$ of $T$ that contains $v$ and set $c'(u) = B$. Note that by definition of 1-Burling graphs, $B \cap V(G) = \{v\}$. So $G'$ is indeed derived from $(T', r', \ell', c')$ and it is clearly a 1-Burling graph.

Top-sets

When $G$ is derived form a Burling tree $(T, r, \ell, c)$, we call the top-set of $G$ the set $S$ of all vertices $v$ of $G$ such that $v$ is the unique vertex of $G$ in the branch of $T$ from $r$ to $v$. See Figure 13. The top-set of the graph presented in Figure 13 is $\{a, b, c, d, e\}$.
Lemma 4.2. If $S$ is the top-set of the oriented graph $G$ derived from a Burling tree, then $G[S]$ is an in-forest. Moreover, if $G$ is a $k$-Burling graph ($k \geq 1$), then $G \setminus S$ is a $(k - 1)$-Burling graph.

Proof. The graph $G[S]$ is clearly 1-Burling from the definition of the top-sets and also $G \setminus S$ is a $(k - 1)$-Burling graph. So, $G[S]$ is an in-forest by Lemma 4.1.

When $G$ is derived from a Burling tree $T$, every vertex $u$ of $G$ has a unique ancestor in the top-set of $G$. This ancestor is called the top-ancestor of $u$.

Lemma 4.3. Let $G$ be derived from a Burling tree $(T, r, \ell, c)$. Let $u$ and $v$ be two vertices of $G$ with top-ancestors $u'$ and $v'$, respectively. If $uv$ is an arc of $G$ then either:

(i) $u' = v'$, $u' \neq u$, and $v' \neq v$, or

(ii) $u = u'$ and $uv' \in A(G)$.

Proof. Suppose $u = u'$. So, $v \in c(u)$. The branch from $r$ to $v$ therefore contains $p(u)$, and since $u$ is in the top-set, $v'$ must be in the branch from $p(u)$ to $v$ (and $v' \neq p(u)$). Hence, $v' \in c(u')$. So, $u = u'$ and $uv' \in A(G)$.

Suppose $u \neq u'$. So, $u'$ is ancestor of $p(u)$. By Lemma 3.1, $p(u)$ is an ancestor of $p(v)$, so $u'$ is an ancestor of $v$. Hence $u' = v'$. Also $v \neq v'$ because $u$ and $v'$ are in the same branch.

Sequential graphs

Top-sets suggest defining Burling graphs as the graphs obtained from the empty graph by repeatedly adding in-forests, with several precise rules about the arcs between them. A graph obtained after $k$ steps of such a construction will be called a $k$-sequential graph, and we will prove that $k$-sequential graphs are equivalent to $k$-Burling graphs. The advantage of $k$-sequential graphs is that they have no shadow vertices like in the definition of derived graphs. Also, they directly form a hereditary class, there is no need to say that we take all induced subgraphs of something previously defined by induction as in the definition of Burling graphs through the Burling sequence. The price to pay for that it that we have to maintain a set of stable sets in the inductive process.

Recall that in an in-forest, every vertex has at most one out-neighbor. Also every connected component of an in-forest is an in-tree, and therefore contains a unique sink.
The 0-sequential graph is the pair $(G, S)$ where $G$ is the empty graph (so $V(G) = \varnothing$) and $S = \{\varnothing\})$. For $k \geq 1$, a $k$-sequential graph is any pair $(G, S)$, where $G$ is an oriented graph and $S$ is a set of stable sets of $G$, obtained as follows:

(i) Pick a (possibly empty) in-forest $H$.

(ii) For every vertex $v$ of $H$, pick a $(k - 1)$-sequential graph $(H_v, R_v)$.

(iii) For every vertex $u$ of $H$ that is not a sink, consider the unique out-neighbor $v$ of $u$, choose a stable set $R$ in $R_v$ and add all possible arcs from $u$ to $R$.

(iv) The previous steps define all the vertices and arcs of $G$.

(v) Set $S = \{\varnothing\} \cup \{\{v\} \cup R : v \in V(H), R \in R_v\}$.

An oriented graph $G$ is $k$-sequential if, for some set $S$, $(G, S)$ is $k$-sequential. The in-forest $H$ in the definition above is called the base forest of the $k$-sequential graph. Observe that the graph on one vertex is $k$-sequential for all $k \geq 1$ and the empty graph is $k$-sequential for all $k \geq 0$. The graph $G$ in Figure 13 is a 2-sequential graph. The in-forest $H$ is the subgraph of $G$ induced by $\{a, b, c, d, e\}$. The graphs $H_a, H_b, and H_c$ are shown in the figure, and they are all 1-sequential graphs.

**Lemma 4.4.** For all $k \geq 0$, an oriented graph $G$ is a $k$-Burling graph with top-set $S$ if and only if it is a $k$-sequential graph with base forest $H$ and $H = G[S]$.

**Proof.** Let $G$ be a $k$-Burling graph derived from a Burling tree $(T, r, \ell, c)$ as a $k$-Burling graph, and let $S$ be the top-set of $G$. Our goal is to show that $G$ is $k$-sequential with base forest $H = G[S]$. Call any branch of $T$ that contains $r$ or is empty a top-branch of $T$. We prove by induction on $k$ that $(G, S)$ is a $k$-sequential graph where

$$S = \{\varnothing\} \cup \{S : S \text{ is the intersection of a top-branch of } T \text{ with } V(G)\}.$$  

For $k = 0$, this is clear. Suppose that $k \geq 1$ and that the statement holds for $k - 1$. Let us prove that $(G, S)$ is a $k$-sequential graph, by building it as in the definition above. Define $H = G[S]$ where $S$ is the top-set of $G$. By Lemma 4.2, $H$ is an in-forest.

For each vertex $v$ of $H$, consider the Burling tree $(T_v, v, \ell_v, c_v)$ where $T_v$ is induced by all descendants of $v$ in $T$ and $\ell_v, c_v$ are the restrictions to
$V(T_v)$ of $\ell_v$ and $c_v$ respectively. By the induction hypothesis, the subgraph of $G$ which is derived from $(T_v,v,\ell_v,c_v)$ is a $(k-1)$-sequential graph, and we denote it by $(H_v,\mathcal{R}_v)$ and

$$\mathcal{R}_v = \{\emptyset\} \cup \{S : S \text{ is the intersection of a top-branch of } T_v \text{ with } V(H_v)\}.$$  

By Lemma 4.3 all arcs of $G$ are either arcs of $H$, or arcs of $H_v$ for some $v \in V(H)$, or arcs of the form $uv$ where $w$ is a descendant of $v$ such $uv \in A(H)$. It follows that $G$ can be obtained from $H$ and the $H_v$'s by adding for every arc $uv$ of $H$ all arcs of the form $uw$ where $w \in c(u) \cap (V(G) \setminus \{v\})$. It follows that for every vertex $u$ of $H$ that is not a top-vertex and the unique out-neighbor $v$ of $u$, all possible arcs from $u$ to $R$ are added where $R = c(u) \cap (V(G) \setminus \{v\}) \in \mathcal{R}_v$. It follows that $(G,S)$ is a $k$-sequential graph.

Let us prove the converse statement. Consider a $k$-sequential graph $(G,S)$ obtained as in the definition from a base forest $H$ and $(k-1)$-sequential graphs $(H_v,\mathcal{R}_v)$ for each $v \in V(H)$. We have to prove that $G$ is a $k$-Burling graph and $H = G[S]$ where $S$ is the top-set of $G$. We prove by induction on $k$ that $G$ can be derived from a Burling tree $(T,r,\ell,c)$ as a $k$-Burling graph such that

$$S = \{S : S \text{ is the intersection of a top-branch of } T \text{ with } V(G)\}.$$  

For $k = 0$, this is clear, so suppose $k \geq 1$ and the statement is true for $k-1$. So, $G$ is obtained from $H$ as in the definition of $k$-sequential graphs. By Lemma 4.1 $H$ is a 1-Burling graph derived from a tree $(T_H,r_H,\ell_H,c_H)$. By the induction hypothesis, for every $v \in V(H)$, $H_v$ can be derived from a Burling tree $(T_v,r_v,\ell_v,c_v)$ and

$$\mathcal{R}_v = \{S : S \text{ is the intersection of a top-branch of } T_v \text{ with } V(H_v)\}.$$  

We now build a Burling tree $(T,r,\ell,c)$ from $T_H$ and the $T_v$'s. For every $v \in V(H)$, we add an edge from $v$ to $r_v$. This defines $T$. We set $r = r_H$. We define the last-borns in $T$ as inherited from the last-borns in $T_H$ and the $T_v$'s, and declare $r_v$ to be the last-born of $v$ (except if $v$ is not a leaf of $T_H$, in which case it keeps its last-born). For every vertex $u$ of $H$ that is not a sink, we consider the unique out-neighbor $v$ of $u$, and the chosen set $R$ in $\mathcal{R}_v$. We set $c(u) = \{v\} \cup R$. For every vertex $u$ of $H$ that is a sink, we set $c(u) = \emptyset$. For all other vertices, we define $c(u)$ as inherited from $c_H$ or $c_v$. We that $G$ can be derived from $(T,r,\ell,c)$ and

$$S = \{S : S \text{ is the intersection of a top-branch of } T \text{ with } V(G)\}.$$  

$\Box$
Pivots and antennas

Suppose that $G$ is derived from a Burling tree $(T, r, ℓ, c)$ with top-set $S$, we call a pivot of $G$ any sink of $G[S]$ and an antenna of $G$ any source of $G[S]$.

**Lemma 4.5.** If $G$ is derived from a Burling tree $(T, r, ℓ, c)$, then every pivot of $G$ is a sink of $G$ and every antenna of $G$ is a source of $G$.

*Proof.* Let $S$ be the top-set of $G$. By Lemma 4.4 there exists $k$ such that $G$ is a $k$-sequential graph with base forest $H = G[S]$. By Lemma 4.3, if $u$ is a pivot of $G$, there cannot be an arc $uv$ in $G$, so $u$ is a sink of $G$. Similarly, if $v$ is an antenna of $G$, an arc $wv$ would contradict Lemma 4.3 so $v$ is a source of $G$. 

**Lemma 4.6.** If a connected oriented graph $G$ is derived from a Burling tree $(T, r, ℓ, c)$ with top-set $S$, then $G[S]$ is an in-tree (in particular $G$ has a unique pivot). Moreover, no vertex of $G$ is a strict descendant of an antenna of $G$.

*Proof.* By Lemma 4.4 there exists an integer $k$ such that $G$ is a $k$-sequential graph with base forest $H = G[S]$. For the sake of contradiction, suppose that $H$ is disconnected. Let $X$ and $Y$ be two connected components of $H$. By the definition of $k$-sequential graphs, $X$ and $Y$ are in distinct components of $G$, a contradiction to $G$ being connected. So, $H$ is connected and thus is an in-tree.

Again, for the sake of contradiction, let $u$ be a strict descendant of an antenna of $G$. By the construction of $k$-sequential graphs, $u$ and the unique pivot of $G$ are in distinct connected components of $G$, a contradiction to $G$ being connected. Therefore, no vertex of $G$ is a strict descendant of an antenna of $G$. 

The following lemma gives more properties of the top-set under stronger connectivity assumptions.

An in-star is an in-tree whose unique sink is adjacent to all other vertices.

**Lemma 4.7.** Let $G$ be a connected oriented graph derived from a Burling tree $(T, r, ℓ, c)$. Suppose that $G$ has no cut-vertex and no vertex of degree at most 1. Then the following statements hold:

(i) The top-set of $G$ is an in-star $S$ with at least two leaves (so $G$ has a unique pivot and its in-neighbors are the antennas of $G$).
(ii) All vertices of $S \setminus \{v\}$ are sources of $G$ where $v$ is the unique sink of $S$.

(iii) The pivot of $G$ is an ancestor of all vertices of $V(G) \setminus S$.

Proof. Let $v$ be the sink of $G[S]$. If $G[S]$ is not an in-star, then there exists a directed path $yvx$ in $G[S]$. By the construction of $k$-sequential graphs, we see that $x$ is a cut-vertex of $G$ that separates $v$ from $y$. A contradiction. By Lemma 4.5 all vertices of $S \setminus \{v\}$ are sources of $G$ because they are the antennas of $G$. Since $G$ is connected, the antennas have no strict descendants, by Lemma 4.6 the pivot of $G$ is an ancestor of all vertices of $V(G) \setminus S$. 

2-Burling graphs

A leaf in an in-tree is a vertex with no in-neighbors (so the root is not a leaf unless the in-tree has only one vertex). An oriented chandelier is any oriented graph $G$ obtained from an in-tree $G'$ whose root is of degree at least 2 by adding a vertex $v$ and all arcs $uv$ where $u$ is a leaf of $G'$.

Lemma 4.8. An oriented graph is an oriented chandelier if and only if it is a connected 2-Burling graph with no cut-vertex and no vertex of degree at most 1.

Proof. First, suppose that $G$ is an oriented chandelier. So $G$ is clearly connected, has no cut-vertex, and has no vertex of degree at most 1. It remains to prove that it is a 2-Burling graph. Let $G'$ and $v$ be as in the definition of oriented chandelier. Let $u_1, \ldots, u_k$ be the leaves of $G'$, and for $i \in [k]$, let $v_i$ be the neighbor of $u_i$ in $G$ ($v_i$'s are not necessarily distinct). Set $G'' = G' \setminus \{u_1, \ldots, u_k\}$. Since in $G'$, the root has degree at least 2, $G''$ contains the root, and thus is a non-empty in-tree. By Lemma 4.1, $G''$ can be derived from a Burling tree $(T, r, \ell, c)$ as a 1-Burling graphs (i.e. on every branch of $T$, at most one vertex belongs to $V(G)$). Let us build a tree $T'$ from $T$. Add a new root $r'$ adjacent to $r$, and add $k$ new children $v_1, \ldots, v_k$ to $r'$. This defines the rooted tree $(T', r')$. Then, define $\ell'(r') = r$ and for any vertex $x \in V(T') \setminus \{r', v_1, \ldots, v_k\}$, we define $\ell'(x) = \ell(x)$. Notice that the vertices $v_1, \ldots, v_k$ are leaves in $T'$, thus $\ell'$ is not defined for them. Now, for every $i \in [k]$, let $B_i$ be the branch of $T$ starting at $r$ and ending at $u_i$. Define $c'(v_i) = B_i$ and $c'(r') = \emptyset$. For every vertex $x \in V(T') \setminus \{r', v_1, \ldots, v_k\}$, set $c'(x) = c(x)$. The tuple $(T', r', \ell', c')$ is a Burling tree. Renaming $r$ as $v$, we see that $G$ can be derived from $T'$. Indeed, $G$ is the subgraph of the oriented graph fully derived from $T'$ induced by $V(G'') \cup \{v, v_1, \ldots, v_k\}$. Moreover,
on each branch of $T'$, at most 2 vertices are in $V(G)$, thus $G$ is a 2-Burling graph.

Conversely, suppose that $G$ is a connected graph with no cut-vertex and no vertex of degree at most 1 that is derived as a 2-Burling graph from a Burling tree $T$. By Lemma 4.7, $G$ has a unique pivot $v$, all antennas of $G$ are in-neighbors of $v$, and the rest of the vertices of $G$ are all descendants of $v$ in $T$. In particular, considering $v$ as a shadow vertex of $T$, we see that $G \setminus v$ is a 1-Burling graph. Therefore, by Lemma 4.1, $G \setminus v$ is an in-forest. On the other hand, since $G$ has no vertex cut, $G \setminus v$ is connected and thus is an in-tree. Let $r$ be the root of this in-tree. Since $r$ cannot be of degree 1 in $G$, it has at least two in-neighbors. But $v$ is not an in-neighbor of $r$ (because it is among its ancestor). Therefore, in $G \setminus v$, the root $r$ has at least 2 children. Moreover, if a leaf $u$ of $G \setminus v$ is not adjacent to $v$ in $G$, then, $u$ has degree at most 1 in $G$, a contradiction. So, $v$ is adjacent to all leaves of $G'$. Hence $G$ is an oriented chandelier.

In the construction of oriented chandeliers in the proof above, $v$ is the pivot of $G$ and its neighbors are the antennas. The unique sink of $G \setminus v$ is called the bottom of $G$. Note that every source of $G$ is an antenna. The pivot and the bottom are the only sinks of $G$. Also, in the Burling tree $T$ from which $G$ is derived, every vertex of $G$ except the antennas are descendants of the pivot.

5 Star cutsets

In this section, we study star cutsets in derived graphs.

**Lemma 5.1.** Suppose that $G$ is an oriented graph derived from a Burling tree $T$. Let $v$ and $w$ be two vertices of $G$ such that $v$ is an ancestor of $w$ in $T$. Then every neighbor of $w$ in $G$ is either an in-neighbor of both $v$ and $w$ or a descendant of $v$.

**Proof.** Let $u$ be a neighbor of $w$ in $G$. If $u$ is an out-neighbor of $w$, then $p(w)$ is an ancestor of $u$. However, $p(w)$ is a descendant of $v$ (possibly $v$ itself). So $u$ is a descendant of $v$. If $u$ is an in-neighbor of $w$, then $p(u)$ is an ancestor of $w$, and therefore it is on the unique branch in $T$ between $w$ and the root. This branch includes $v$ as well. There are two cases: either $p(u)$ is a descendant of $v$ or $p(u)$ is an ancestor of $v$. In the former case, $u$ is a descendant of $v$. In the latter case $u$ is an in-neighbor of $v$ because $u$ is connected to every vertex in the path between $w$ and the last-born of $p(u)$, and $v$ is on this path. 


Lemma 5.2. Suppose that $G$ is an oriented graph derived from a Burling tree $T$. Let $u$, $v$, and $w$ be three vertices of $G$ such that $w$ is a descendant of $v$ and $u$ is not a descendant of $v$. Then every path (not necessarily directed) in $G$ between $u$ and $w$ contains an in-neighbor of $v$ in $G$.

Proof. Let $P$ be a path in $G$ from $u$ to $w$. Since $u$ is not a descendant of $v$ while $w$ is, $P$ must contain an edge $u'w'$ such that $u'$ is not a descendant of $v$ while $w'$ is. By Lemma 5.1 applied to $v$ and $w'$, $u'$ is a in-neighbor of $v$.

A full in-star cutset in an oriented graph $G$ is a set $S = N^-[v]$ for some vertex $v \in V(G)$ such that $G \setminus S$ is disconnected. A full star cutset in a graph $G$ (oriented or not) is a set $S = N[v]$ for some vertex $v \in V(G)$ such that $G \setminus S$ is disconnected. A star cutset in a graph $G$ (oriented or not) is a set $S$ such that for some vertex $v \in V(G)$, $\{v\} \subseteq S \subseteq N[v]$, and $G \setminus S$ is disconnected. In this case, we say that the star cutset $S$ is centered at $v$.

We say that in graph $G$, the star cutset $S$ separates two vertices $u$ and $v$ if $u$ and $v$ are in distinct connected components of $G \setminus S$.

Lemma 5.3. Suppose that $G$ is an oriented graph derived from a Burling tree $T$. Let $u$, $v$, and $w$ be three vertices of $G$ appearing in this order along a branch of $T$. Then every path (not necessarily directed) in $G$ from $u$ to $w$ goes through an in-neighbor of $v$ in $G$.

In particular, $N^-[v]$ is a full in-star cutset of $G$ and $N[v]$ is a full star cutset of $G$, which separated $u$ and $v$.

Proof. Follows from Lemma 5.2 and $u$ and $w$ are in distinct connected components of $G \setminus N^-[v]$.

Lemma 5.4. If a triangle-free oriented graph $G$ has a cut-vertex, then either $G$ has a full in-star cutset, or $G$ has a vertex of degree at most 1.

Proof. Let $v$ be a cut-vertex of $G$. Let $A$ and $B$ be two connected components of $G \setminus v$. If $|A| \leq 1$ or $|B| \leq 1$, then $G$ has a vertex of degree at most 1, so let us assume that $|A| \geq 2$ and $|B| \geq 2$. Since $G$ is triangle-free, $A$ (resp. $B$) contains a non-neighbor $a$ (resp. $b$) of $v$. It follows that $a$ and $b$ are in distinct connected components of $G \setminus N^-[v]$. So, $G$ has a full in-star cutset centered at $v$.

Theorem 5.5. If $G$ is an oriented Burling graph, then either $G$ has a full in-star cutset, or $G$ is an oriented chandelier, or $G$ contains a vertex of degree at most 1.
Figure 14: Some subdivisions of $K_4$ that are not Burling.

Proof. Suppose that $G$ has no vertex of degree at most 1. In particular, $|V(G)| \geq 3$. By Lemma 5.4 we may assume that $G$ has no cut-vertex (in particular $G$ is connected since $|V(G)| \geq 3$). We may assume that $G$ is a 2-Burling graph, for otherwise some branch of $T$ contains at least three vertices of $G$ and by Lemma 5.2 $G$ has a full in-star cutset. So, $G$ is a connected 2-Burling graph with no cut-vertex and no vertex of degree at most 1. Hence by Lemma 4.8 $G$ is an oriented chandelier.

Theorem 5.5 is best possible in the sense that every oriented chandelier has no full in-star cutset.

The non-oriented case

We may prove a theorem similar to Theorem 5.5 for non-oriented graphs with the same method. We do not present its proof, because it was already obtained in [3] for a superclass of Burling graphs, the so-called restricted frame graphs. (For definition of restricted frame graphs, see Definition 2.2 of [3].)

A non-oriented graph obtained from a tree $H$ by adding a vertex $v$ adjacent to every leaf of $H$ is called in [3] a chandelier. If the tree $G$ has the property that the neighbor of each leaf has degree two, then the chandelier is a luxury chandelier. Observe that if $G$ is an oriented chandelier, then $G$ has no full in-star cutset, but it may have a full star cutset. It can be proved that the luxury chandeliers are exactly the graphs with no full star cutset that are also underlying graphs of oriented chandeliers. The following theorem is from [3], Corollary 3.4.

Theorem 5.6 (Chalopin, Esperet, Li and Ossona de Mendez). If $G$ is a non-oriented connected Burling graph, then $G$ has a full star cutset, or $G$ is luxury chandelier, or $G$ is an induced subgraph of $P_4$.

As explained in [3], Theorem 5.6 conveniently gives graphs that are not Burling. For instance, the three graphs represented in Figure 14 are not
Burling because they have no full star cutset and are not luxury chandelier (because in a luxury chandelier, there exists a vertex that is contained in all cycles).

In [4], the following question is asked: is there a constant $c$ such that if a graph $G$ is triangle-free and all induced subgraphs of $G$ either are 3-colorable or admit a star cutset, then $G$ is $c$-colorable? It was seemingly never noticed that Theorem 5.6 answers the question in the negative, since luxury chandelier are 3-colorable and Burling graphs are triangle-free graphs of unbounded chromatic number.

6 Holes in Burling graphs

A hole in a graph $G$ is a chordless cycle of length at least 4. We call hole of an oriented graph any hole of its underlying graph. By Theorem 5.5, since a hole has no in-star cutset (whatever the orientation) and no vertex of degree 1, every hole in an oriented graph derived from a Burling tree $T$ is an oriented chandelier. In particular, the explanations given after the proof of Lemma 4.8 apply. Therefore, every hole $H$ has four special vertices that we describe here:

- two sources called the antennas,
- one common neighbor of the antennas that is also an ancestor in $T$ of all the vertices but the antennas, called the pivot,
- one sink distinct from the pivot, called the bottom.

Every other vertex of $H$ lies on a directed paths from an antenna to the bottom. We call subordinate vertex of a hole any vertex distinct from its pivot and antennas (in particular, the bottom is subordinate and is therefore a descendant of the pivot).

**Lemma 6.1.** Let $H$ be a hole in an oriented graph $G$ derived from a Burling tree $T$. Let $p$ be the pivot of $H$ and $C$ be the connected component of $G \setminus N^-[p]$ that contains $H \setminus N^-[p]$. Then every vertex of $C$ is a descendant of $p$.

**Proof.** Suppose for the sake of contradiction, that the statement does not hold. So, $C$ contains a vertex $u$ that is not a descendant of $p$. Since every vertex of $H \setminus N^-[p]$ is a descendant of $p$, there exists a descendant $v$ of $p$ in $C$. Let $P$ be a path from $u$ to $v$ in $C$. By Lemma 5.2, $P$ contains an in-neighbor of $p$. This contradicts the definition of $C$. 

A dumbbell is a graph made of path $P = x \ldots x'$ (possibly $x = x'$), a hole $H$ that goes through $x$ and a hole $H'$ that goes through $x'$. Moreover
$V(H) \cap V(P) = \{x\}$, $V(H) \cap V(P') = \{x'\}$, $V(H) \cap V(H') = \{x\} \cap \{x'\}$ and there are no other edges than the edges of the path and the edges of the holes.

**Lemma 6.2.** Suppose a dumbbell with holes $H$, $H'$ and path $P = x \ldots x'$ as in the definition is the underlying graph of some oriented graph $G$ derived from a Burling tree $T$. Then either $x$ is not a subordinate vertex of $H$ or $x'$ is not a subordinate vertex of $H'$.

**Proof.** Suppose for the sake of contradiction, that the statement does not hold. So, the pivot $p$ of $H$ is in the interior of the path $H \setminus x$ and the pivot $p'$ of $H'$ is in the interior of the path $H' \setminus x'$. By Lemma 6.1 applied to $H$, every vertex of $G \setminus N[p]$ is a descendant of $p$ in $T$ (notice that $N[p] = N^-[p]$ since the pivot is a sink). By Lemma 6.1 applied to $H'$, every vertex of $G \setminus N[p']$ is a descendant of $p'$ in $T$. It follows that $p$ and $p'$ are on the same branch of $T$, so up to symmetry, we may assume that $p$ is an ancestor of $p'$.

Let $q$ and $r$ be vertices of $H$ such that $p$, $q$ and $r$ are consecutive along $H$. So, $q$ is an antenna of $H$ (because it is adjacent to the pivot), and $r$ is a descendant of $p$, but also of $p'$. Thus $p'$ is between $p$ and $r$ in some branch of $T$. Now because $p$ and $r$ are both in $c(q)$, so is $p'$. Hence $q$ is adjacent to $p'$, a contradiction to the definition of dumbbells. 

A **domino** is a graph made of one edge $xy$ and two holes $H_1$ and $H_2$ that both go through $xy$. Moreover $V(H_1) \cap V(H_2) = \{x, y\}$ and there are no other edges than the edges of the holes.

**Lemma 6.3.** Suppose a domino with holes $H_1$, $H_2$ and edge $xy$ as in the definition is the underlying graph of some oriented graph $G$ derived from a Burling tree $T$. Then for some $z \in \{x, y\}$ and some $i \in \{1, 2\}$, $z$ is the pivot of $H_i$ and $z$ is a subordinate vertex of $H_{3-i}$.

**Proof.** Let us first prove that one of $x$ or $y$ is the pivot of one of $H_1$ or $H_2$. Otherwise, the pivot $p_1$ of $H_1$ is in the path $H_1 \setminus \{x, y\}$ and the pivot $p_2$ of $H_2$ is in the path $H_2 \setminus \{x, y\}$. Suppose up to symmetry that $yx$ is an arc of $G$. It follows that $x \in V(G) \setminus (N[p_1] \cup N[p_2])$. Because $x \neq p_1, p_2$ by assumption, and $x \notin N(p_1) \cup N(p_2)$ because in a hole, the neighbors of the pivot are sources. By Lemma 6.1 applied to $H_1$ and to $H_2$, $x$ is a descendant of both $p_1$ and $p_2$. It follows that up to symmetry, we may assume that $p_1$ is a descendant of $p_2$.

Let $a$ and $a'$ be the antennas of $H_2$. Note that $a, a' \neq x$. Up to symmetry, suppose that $x$, $a$, $p_2$ and $a'$ appear in this order along $H_2$. Let $x'$ be the neighbor of $a$ in $H \setminus p_2$ (possibly $x = x'$). Since $x$ and $x'$ are in the same
component of $G \setminus (N[p_1] \cup N[p_2])$, $x'$ is a descendant of both $p_1$ and $p_2$. And since $a x' \in A(G)$, we have $x' \in c(a)$, so $p_1 \in c(a)$ and $a p_1 \in A(G)$, a contradiction to the definition of dominos.

We proved one of $x$ or $y$ is the pivot of one of $H_1$ or $H_2$. Up to symmetry, suppose that $x$ is the pivot of $H_1$. It remains to prove that $x$ is a subordinate vertex of $H_2$. First, $x$ cannot be an antenna of $H_2$ because $y x \in A(G)$. Hence, we just have to prove that $x$ being the pivot of $H_2$ yields a contradiction. So, suppose that $x$ is the pivot of $H_2$. It follows that $y$ is an antenna of both $H_1$ of $H_2$, so it is a source of $G$. Let $y_1$ and $y_2$ be the neighbors of $y$ in $H_1 \setminus x$ and $H_2 \setminus x$ respectively. Vertices $x$, $y_1$ and $y_2$ are on the same branch $B$ of $T$ (because they are all in $c(y)$). So, by Lemma 5.2, $y_1$, $x$ and $y_2$ appear either in this order or in the reverse order along $B$, because $y_1$ and $y_2$ are not centers of star cutsets of $G$. If $y_2$ is the deepest vertex in $T$ among the three, then $y_1$ is an ancestor of $x$ while being in the hole $H_1$ for which $x$ is the pivot, a contradiction. On the other hand, if $y_1$ is the deepest, then $y_2$ is an ancestor of $x$ while being in the hole $H_2$ for which $x$ is the pivot, again a contradiction. \(\Box\)

A theta is a graph made of three internally vertex-disjoint paths of length at least 2, each linking two vertices $u$ and $v$ called the apexes of the theta (and such that there are no other edges than those of the paths). A long theta is a theta such that all the paths between the two apexes of the theta have length at least 3.

**Lemma 6.4.** Suppose a long theta with apexes $u$ and $v$ is the underlying graph of some oriented graph $G$ derived from a Burling tree $T$. Then exactly one of $u$ and $v$ is the pivot of every hole of $G$.

**Proof.** Consider the three paths between $u$ and $v$, and let $Q_1$, $Q_2$, and $Q_3$ denote the set of the internal vertices of these three paths respectively (so, $|Q_i| \geq 2$). For $i = 1, 2, 3$, let $H_i$ be the hole induced by $Q_i \cup Q_{i+1} \cup \{u, v\}$ (with subscript taken modulo 3) and let $p_i$, $a_i$, and $a'_i$ denote the pivot and the two antennas of $H_i$.

For the sake of contradiction, assume that there is a hole in $G$, say $H_1$, for which neither of $u$ and $v$ is a pivot. So, without loss of generality, assume that $p_1 \in Q_1$. Also, notice that $a_1$ and $a'_1$ are the two neighbors of $p_1$. Thus:

(i) neither of $a_1$ and $a'_1$ are in $Q_2$, and consequently, no vertex of $Q_2$ is a source in $G$,

(ii) because the underlying graph is a long theta, at least one of $a_1$ and $a'_1$, say $a_1$, is in $Q_1$. 30
Figure 15: Theta+ and an orientation of it.

Now consider the hole $H_2$. Since the theta is long, if $p_2$ is in $Q_2 \cup \{u, v\}$, then at least one antenna of $H_2$ must be in $Q_2$ which contradicts (i). Thus, $p_2 \in Q_3$. Therefore, with the same argument as before, at least one of $a_2$ and $a'_2$, say $a_2$ also should be in $Q_3$.

Finally, consider the hole $H_3$. Notice that $a_1, p_1 \in Q_1$ respectively form a source and a sink for $H_3$. On the other hand, $a_2, p_2 \in Q_3$ also, respectively form a source and a sink for $H_3$. So, these are the four extrema of $H_3$. But then at least three of these four vertices should be consecutive, which is impossible. This contradiction finishes the proof of the lemma.

7 New examples of non-Burling graphs

We are now able to describe non-Burling triangle-free graphs that do have full star cutsets, so going beyond the method following from [3], where all examples have no star cutsets.

We call the (non-oriented) graph represented in Figure 15 (left) Theta+.

**Theorem 7.1.** Theta+ is not a Burling graph.

*Proof.* For the sake of contradiction, suppose that $G$ is a Burling graph. So, some orientation of $G$ can be derived from a Burling tree. Hence, every $C_4$ of this orientation must contain a pivot, a bottom and two antennas. One can check that with this condition, up to symmetry, the orientation of $G$ is as $G'$ shown in Figure 15 (right). Note that $a, b$ and $c$ are out-neighbors of $x$, so they must be on the same branch of the Burling tree. Therefore, by Lemma 5.2, one of $a, b$ or $c$ must be the center of a full in-star cutset, a contradiction. \[\square\]
A flower is a graph $G$ made of a hole $H$ where every edge $e$ is part of a hole $H_e$. Moreover, $V(H) \cap V(H_e) = e$, for all edges $e, f$ of $H$, $V(H_e) \cap V(H_f) = e \cap f$, and the only edges and vertices of $G$ are those of the $H_e$’s. In Figure 16, two examples of flowers are represented.

**Theorem 7.2.** No flower is a Burling graph.

*Proof.* Suppose $G$ is a flower with a hole $H$ as in the definition. Let $v$ be the pivot of $H$, and $u, w$ the two neighbors of $v$ in $H$. So, $H_{uv}$ and $H$ form a domino, and by Lemma 6.3 one of the two vertices $u$ and $v$ should be the pivot of one of the two holes, and a subordinate vertex of the other. Notice that $u$ cannot be a pivot of any of the two holes because $uv$ is an arc. So, $v$ is a subordinate vertex of $H_{uv}$. Similarly, $v$ is a subordinate vertex of $H_{vw}$. Hence, $H_{vw}$ and $H_{uv}$ contradict Lemma 6.2 (since $H_{vw}$ and $H_{uv}$ form a dumbbell).

A wheel is a graph made of hole $H$ called the rim together with a vertex $c$ called the center that has at least three neighbors in $H$. Wheels are restricted frame graphs (see Theorem A.1. and Figure 7 in [3]). It was claimed by Scott and Seymour in private communication that wheels are not derived graphs, and independently, Davies also proved it recently. The first written proof of the theorem that we are aware of is in the master’s thesis of the first author, see [10].

**Theorem 7.3** (Scott and Seymour [14], Pournajafi [10], Davies [7]). *No wheel is a Burling graph.*
Proof. Suppose that a graph $G$ is wheel with rim $H$ and center $c$. Let $v$ be the pivot of $H$, $u$ and $u'$ its antennas, and $w$ its bottom. So, there is an edge-partition of $H$ into a directed path $P_u$ from $u$ to $w$, a directed path $P_{u'}$ from $u'$ to $w$ and the edges $uv$ and $u'v$.

We claim that $c$ has at most one neighbor in $P_u$. Otherwise, $c$ and a subpath $P_u$ form a hole $J$, and since $P_u$ is directed, this hole cannot contain two sources, a contradiction. Similarly, $P_{u'}$ contains at most one neighbor of $c$. Hence, the only possibility for $c$ to have at least three neighbors in $H$ is that $c$ is adjacent to $v$, to one internal vertex of $P_u$ and to one internal vertex of $P_{u'}$. Notice that $c$ cannot be adjacent to $u$ or $u'$ otherwise there will be a triangle in $G$.

Two holes $H_u$ and $H_{u'}$ of $G$, containing respectively $u$ and $u'$, go through the edge $vc$, forming a domino. Since $c$ is not adjacent to the sources of $u$ and $u'$, it can be the pivot of neither $H_u$ nor $H_{u'}$. Hence, by Lemma 6.3 $v$ must be the pivot of either $H_u$ or $H_{u'}$, say of $H_u$ up to symmetry. Let $x$ be the neighbor of $c$ in $P_u$. Since $v$ is the pivot of $H_u$, $cx$ is an arc of $G$. Since $x$ is not the pivot of $H_u$, by Lemma 6.3 $x$ is the pivot of $H_w$, that is the hole of $G$ containing $c$ and $w$. Hence, $x$ is a sink of $H_u$, a contradiction to $P_u$ being directed from $u$ to $w$.

As shown in Figures 17 and 18 graphs that are quite close to flowers or wheels can be Burling graphs.

In [16], Trotignon asked whether the class of wheel-free graphs is $\chi$-bounded (see Question 5.1 of [16]). In [14], Scott and Seymour made a closely related conjecture that the class of all graphs that for all $k$ that do not contain an induced cycle such that some vertex has at least $k$ neighbours on the cycle is $\chi$-bounded (see 12.16 in [14]). Theorem 7.3 answers in negative to the former and disproves the latter.

The next theorem fully characterizes subdivisions of $K_4$ that are Burling graphs.

**Theorem 7.4.** Let $G$ be a non-oriented graph obtained from $K_4$ by subdividing edges. Then $G$ is a Burling graph if and only if $G$ contains four vertices $a$, $b$, $c$ and $d$ of degree 3 such that $ab, ac \in E(G)$ and $ad, bc \notin E(G)$.

**Proof.** Suppose that $G$ is a Burling graph. Let $a$, $b$, $c$ and $d$ be the four vertices of degree 3 of $G$. If $G[a, b, c, d]$ contains no vertex of degree at least 2, then $G$ is isomorphic to one of the graphs represented in Figure 14 so $G$ has no star cutset, a contradiction to Theorem 5.6. So, up to symmetry, we may assume that $a$ has degree at least 2 in $G[a, b, c, d]$, so up to symmetry $ab, ac \in E(G)$. If $bc \in E(G)$, then $G$ contains a triangle, a
Figure 17: Burling graphs close to flowers.
Figure 18: Burling graphs close to wheels.
contradiction to Lemma 3.4. So, $bc \notin E(G)$. If $ad \in E(G)$, then $G$ is a wheel, a contradiction to Lemma 7.3. So, $ad \notin E(G)$. We proved that $ab, ac \in E(G)$ and $ad, bc \notin E(G)$.

Conversely, if we suppose that $ab, ac \in E(G)$ and $ad, bc \notin E(G)$, then $G$ is obtained by subdividing dashed edges of the graph represented in Figure 9. It is therefore a Burling graph as explained after the proof of Lemma 3.8.

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