Hamiltonian analysis of SO(4, 1)-constrained BF theory

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Abstract

In this paper we discuss the canonical analysis of SO(4, 1)-constrained BF theory. The action of this theory contains topological terms appended by a term that breaks the gauge symmetry down to the Lorentz subgroup SO(3, 1). The equations of motion of this theory turn out to be the vacuum Einstein equations. By solving the B field equations one finds that the action of this theory contains not only the standard Einstein–Cartan term but also the Holst term proportional to the inverse of the Immirzi parameter, as well as a combination of topological invariants. We show that the structure of the constraints of an SO(4, 1)-constrained BF theory is exactly that of gravity in the Holst formulation. We also briefly discuss the quantization of the theory.

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1. Introduction

One of the most remarkable developments in general relativity of the last decades was Ashtekar’s discovery that the phase space of gravity can be described with the help of a background-independent theory of self-dual SU(2) connection [1]. This discovery became a foundation of the research program of loop quantum gravity [2, 3]. The original Ashtekar’s formulation was generalized few years later by Barbero to the case of real connections [4], parametrized by a single real number γ, called the Immirzi parameter [5]. It turns out that this parameter is in fact an additional dimensionless coupling constant of the gravitational action, which takes the symbolic form [6]

\[ S_{\text{grav}} = \frac{1}{G} \int e^\alpha e^\beta R_{\rho\gamma\delta} \left( e_{\alpha\beta}^{\gamma\delta} + \frac{1}{\gamma} \delta^{\gamma\delta}_{\alpha\beta} \right) \epsilon^{\mu\nu\rho\sigma} - \frac{\Lambda}{3G} \epsilon_{\mu\nu\rho\sigma} e^\alpha_{\mu} e^\beta_{\nu} e^\gamma_{\rho} e^\delta_{\sigma} . \]  

In the action above \( e^\alpha \) is the tetrad one-form and \( R_{\alpha\beta} \) is the curvature two-form of the Lorentz connection \( \omega_{\alpha\beta} \), where the Lorentz algebra indices \( \alpha, \beta, \ldots \) run from 0 to 3. The term proportional to the inverse of \( \gamma \), called the Holst term, regardless of not being a total derivative, does not affect field equations because its contribution vanishes on shell (for zero torsion) by
virtue of the Bianchi identity. In spite of this, its presence is not completely innocent: it affects the canonical structure of the classical theory, and quantum theories for different $\gamma$ lead to different physical predictions (for example the expression for black hole entropy calculated in this framework depends on $\gamma$ [7]).

It has been noted in [8] that from Wilsonian perspective it would be quite unnatural not to append the action (1) with all possible terms that are compatible with the field content ($\epsilon$ and $\omega$) and (local Lorentz and diffeomorphism) symmetries of the theory. It turns out that there are only three such terms corresponding to three topological invariants (Pontryagin, Euler and Nieh–Yan classes, see (49)–(51)). Again, the presence of these terms does not influence the classical field equation when the constant time slices of the spacetime are compact without boundaries. However, they may play an important role in quantum theory and/or in the case when boundaries are present. In the formulation of [8] all these terms come with a priori independent coupling constants and one wonders if it would be possible to find a formulation of the theory so as to organize them in a unified way.

Such a formulation is known for quite some time and is dubbed constrained BF theory. The idea that gravity can be formulated as a constrained topological BF theory has its roots in works of MacDowell and Mansouri [9] and of Plebanski [10]. The starting point of the present work is the following action, proposed and discussed in [11] (see also [12]):

$$S = \int d^4x \epsilon^{\mu\nu\lambda\rho} \left( B_{\mu\nu IJ} F_{I J}^{\mu\nu} - \frac{B}{2} B_{\mu I J} B_{I J}^{\mu\nu} - \frac{a}{4} \epsilon^{IJKL} B_{\mu I}^{J} B_{\nu J}^{K L} \right).$$ (2)

In this action,

$$F_{\mu I J} = \partial_\mu A_{I J} - \partial_I A_{\mu J} + A_{\mu K} A^{K J} - A_{\mu} A^{K J}$$

is the field strength of the $SO(4, 1)$ (or $SO(3, 2)$) connection $A_{\mu I J}$, while $B_{\mu\nu IJ}$ is a two-form field valued in the algebra of the same gauge group. The capital Latin indices $I, J, K, \ldots$ are the algebra ones and run from 0 to 4, when the Lorentz subalgebra of the gauge algebra is labeled by Greek indices from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots$ running from 0 to 3. We decompose them into timelike 0 and spacelike $a, b, c, \ldots$. Below, in the course of Hamiltonian analysis, we also decompose the spacetime indices $\mu, \nu$ into time and space denoting the space indices by letters from the middle of the Latin alphabet $i, j, k, \ldots$.

As we show in the next section, the theory defined by the action (2) is equivalent to the Einstein–Cartan theory with an action accompanied with the Holst term and the topological terms described above. The six coupling constants of [8] are then replaced by two dimensionless couplings $a$ and $\beta$ of (2) and one dimensionful scale $\ell$.

The plan of the paper is as follows. In the next section we show that the theory defined by (2) is (classically) equivalent to general relativity with cosmological constant. In section 3 we discuss the canonical formulation of this theory, while in sections 4 and 5 we show how these constraints can be simplified and recast into the form proposed by Holst. In the final section we make some comments concerning the perturbative quantization of the theory around the Kodama state.

2. Gravity as a constrained BF theory

In this section we recall some properties of the action (2). It has been shown in [11] that this action is equivalent to the standard action of Einstein–Cartan gravity. To see this one first decomposes the connection $A_{\mu I J}$ into the tetrad and Lorentz connection

2
\[ A^\alpha_{\mu} = \frac{1}{\ell} e^\alpha_\mu, \quad A^\alpha_{\mu} = \omega^\alpha_{\mu}, \]  

(3)

with \( \ell \) being a length scale, necessary for dimensional reasons since the connection on the left-hand side has the dimension of inverse length, while tetrad is dimensionless\(^1\) associated with the cosmological constant

\[ \frac{1}{\ell^2} = \frac{\Lambda}{3}. \]

Then one solves the equations of motion for \( B \) and substitutes the result back into the action.

As a result one finds Einstein action appended with a number of topological invariants. To find its canonical form one has to associate the dimensionless coupling constants \( \alpha \) and \( \beta \) of (2) with the physical ones: Newton’s constant \( G \), the cosmological constant \( \Lambda \) and the Immirzi parameter \( \gamma \):

\[ \alpha = \frac{G \Lambda}{3 (1 + \gamma^2)}, \quad \beta = \frac{G \Lambda}{3 (1 + \gamma^2)}, \quad \gamma = \frac{\beta}{\alpha}. \]  

(4)

Instead of repeating this derivation here, let us show that field equations resulting from the action (2) are the standard vacuum Einstein equations. The field equations read

\[ \epsilon^{\mu
\nu\rho\sigma} \left( D^A_{\mu} B_{\nu\rho} \right)^{IJ} = 0, \]  

(5)

\[ \epsilon^{\mu
\nu\rho\sigma} \left( F_{\mu\nu}^{IJ} - \beta B_{\mu\nu}^{IJ} - \frac{\alpha}{2} \epsilon^{IJKL} B_{\mu\nu KL} \right) = 0. \]  

(6)

In (5) \( D^A_{\mu} \) is the covariant derivative defined by connection \( A \), so that

\[ \left( D^A_{\mu} B_{\nu\rho} \right)^{IJ} = \partial_{\mu} B_{\nu\rho}^{IJ} + A^K_{\mu \nu} B_{\rho}^{JK} + A^K_{\mu \rho} B_{\nu}^{JK}. \]

The theory defined by (2) for non-zero \( \alpha \) breaks the original de Sitter \( \text{SO}(4, 1) \) gauge symmetry down to Lorentz \( \text{SO}(3, 1) \). It is, therefore, convenient to decompose the covariant derivative \( D^A_{\mu} \) into Lorentz \( \text{so}(3, 1) \) and translational parts, and to use the Lorentz covariant derivative defined by the Lorentz connection \( \omega \), to wit

\[ \left( D^A_{\mu} B_{\nu\rho} \right)^{\alpha\beta} = \partial_{\mu} B_{\nu\rho}^{\alpha\beta} + \omega^K_{\mu \nu} B_{\rho}^{K\beta} + \omega^K_{\mu \rho} B_{\nu}^{K\alpha}, \]  

(7)

\[ \left( D^A_{\mu} B_{\nu\rho} \right)^{\alpha4} = \left( D^\omega_{\mu} B_{\nu\rho} \right)^{\alpha4} - \frac{1}{\ell} e^{\mu \beta} B_{\nu\rho}^{\alpha\beta}, \]  

(8)

where

\[ \left( D^\omega_{\mu} B_{\nu\rho} \right)^{\alpha\beta} = \partial_{\mu} B_{\nu\rho}^{\alpha\beta} + \omega^K_{\mu \nu} B_{\rho}^{K\beta} + \omega^K_{\mu \rho} B_{\nu}^{K\alpha}, \]  

(9)

with an obvious generalization for another Lorentz tensors. Using this decomposition, we rewrite the field equations (6) and (7) as

\[ \epsilon^{\mu
\nu\rho\sigma} \left( D^\omega_{\mu} B_{\nu\rho}^{\alpha\beta} - \frac{1}{\ell} e^{\mu \alpha} B_{\nu\rho}^{\beta4} + \frac{1}{\ell} e^{\mu \beta} B_{\nu\rho}^{\alpha4} \right) = 0, \]  

(10)

\[ \epsilon^{\mu
\nu\rho\sigma} \left( D^\omega_{\mu} B_{\nu\rho}^{\alpha4} - \frac{1}{\ell} e^{\mu \beta} B_{\nu\rho}^{\alpha\beta} \right) = 0, \]  

(11)

\(^1\) In our approach all generators of the gauge algebra are dimensionless. Alternatively, one can use dimensionful generators of the translational part of the algebra (as it is usually done when one wants eventually to make the algebra contraction). Then momentum generators have the canonical dimension of inverse length and \( \ell \) shows up in the algebra as well.
\[
\begin{align*}
F_{\mu\nu}^{\alpha\beta} - \beta B_{\mu\nu}^{\alpha\beta} - \frac{\alpha}{2} \epsilon^{\alpha\beta\gamma\delta} B_{\mu\nu}^{\gamma\delta} &= 0, \\
F_{\mu\nu}^{\alpha4} - \beta B_{\mu\nu}^{\alpha4} &= 0.
\end{align*}
\] (12)

Note that the curvature in (12) is the sum of the Riemann tensor of \( \omega \) and the cosmological curvature
\[
F_{\mu\nu}^{\alpha\beta} = R_{\mu\nu}^{\alpha\beta} - \frac{1}{\ell^2} (\epsilon_{\mu}^{\alpha} e_{\nu}^{\beta} - \epsilon_{\nu}^{\alpha} e_{\mu}^{\beta})
\] (14),
while that in (13) is just the torsion
\[
F_{\mu\nu}^{\alpha4} = \frac{1}{\ell} (D_{\mu}^{\alpha} e_{\nu}^{\beta} - D_{\nu}^{\alpha} e_{\mu}^{\beta}) = \frac{1}{\ell} T_{\mu\nu}^{\alpha4}.
\] (15)

Solving (12) and (13) for \( B \) we find
\[
B_{\mu\nu}^{\alpha4} = \frac{1}{\beta} F_{\mu\nu}^{\alpha4}, \quad B_{\mu\nu}^{\alpha\beta} = \frac{1}{2} M_{\alpha\beta}^{\gamma\delta} F_{\mu\nu}^{\gamma\delta},
\] (16)

where
\[
M_{\alpha\beta}^{\gamma\delta} = \frac{1}{(\alpha^2 + \beta^2)} (\delta_{\gamma\delta}^{\alpha\beta} - \alpha \epsilon^{\alpha\beta\gamma\delta}),
\] (17)

with \( \delta^{\alpha\beta}_{\gamma\delta} \equiv \delta^{\alpha\beta}_{\gamma\delta} - \delta^{\alpha\beta}_{\delta\gamma} \). The tensor \( M \) is a sum of the Lorentz invariant tensors and, therefore, its covariant derivative \( D_{\mu}^{\alpha} \) vanishes.

Substituting (16) into (10) and using the Bianchi identity for the Riemann curvature one can check that the resulting equation forces the torsion \( T_{\mu\nu}^{\alpha4} = \ell F_{\mu\nu}^{\alpha4} \) to vanish\(^2\). Using this it is easy to see that (11) is equivalent to the Einstein equations with the cosmological constant \( \Lambda = 3/\ell^2 \). This completes the proof that field equations following from the action (2) reproduce the standard Einstein equations.

It should be noted that when the coupling constant \( \alpha = 0 \) the theory becomes topological, so that the last term in the action (2) that explicitly breaks the gauge symmetry from the topological \( \text{SO}(4, 1) \) down to physical \( \text{SO}(3, 1) \) carries all the information about dynamical local degrees of freedom of gravity. As we will see below this fact is clearly reflected in the structure of constraints algebra.

3. Canonical analysis

In the first step of canonical analysis of the constrained BF theory defined by (2), let us decompose the curvature \( F_{\mu\nu}^{IJ} \) into electric and magnetic parts
\[
F_{\mu\nu}^{IJ} \rightarrow \left( F_{\mu\nu}^{\hat{I}\hat{J}}, F_{\mu\nu}^{ij} \right),
\] (18)

with
\[
F_{\hat{0}i}^{\hat{I}\hat{J}} = \dot{A}_i^{\hat{I}\hat{J}} - \partial_i A_{0}^{\hat{I}\hat{J}} + A_{0}^{\hat{I}} A_{0}^{\hat{J}} - A_{0}^{\hat{I}} A_{0}^{\hat{J}} = \dot{A}_i^{\hat{I}\hat{J}} - \mathcal{D}_i A_0^{\hat{I}\hat{J}},
\] (19)

where the dot denotes the time derivative, \( \mathcal{D}_i \) is the covariant derivative for the connection \( A_i^{\hat{I}\hat{J}} \) and
\[
F_{ij}^{\hat{I}\hat{J}} = \partial_i A_j^{\hat{I}\hat{J}} + A_i^{\hat{I}} A_j^{\hat{J}} - i \leftrightarrow j.
\] (20)

As usual the zero component of the connection becomes a Lagrange multiplier for the Gauss law. Further we decompose the \( B \) field into
\[
B_{\mu\nu}^{IJ} \rightarrow \left( B_{\mu\nu}^{I\hat{J}} \equiv 2 B_{0\hat{I}}^{I\hat{J}}, \mathcal{T}^{I\hat{J}} \equiv 2 \epsilon^{I\hat{J}k} B_{\hat{J}k}^{I\hat{J}} \right).
\] (21)

\(^2\) To prove this one has to assume the invertibility of the tetrad.
As we will see shortly, $P^{iJ}$ turn out to be momenta associated with spatial components of the gauge field $A$, while the remaining components of $B$ play a role of Lagrange multipliers.

Using these definitions and integrating by parts we can rewrite the action as follows:

$$ S = \int dt L, \quad (22) $$

$$ L = \int d^4x \left( \mathcal{P}^{iJ}_A \dot{A}^{iJ} + B^{iJ}_I \Pi^{IJ}_I + A^{iJ}_0 \Pi^{IJ}_I \right). \quad (23) $$

It is clear that $B^{iJ}_I$ and $A^{iJ}_0$ are the Lagrange multipliers enforcing the constraints $\Pi^{IJ}_I$ and $\Pi^{IJ}_I$, which explicitly read

$$ \Pi^{IJ}_I(x) = (\mathcal{D}_i \mathcal{P}^{iJ})_I(x) = (\partial_i \mathcal{P}^{iJ} + A^{iK}_J \mathcal{P}^{iJ}_K + A^{iJ}_K \mathcal{P}^{iJ}_K)(x), \quad (24) $$

which is the Gauss law for the $SO(4,1)$ invariance (see below), and

$$ \Pi^{IJ}_I(x) = \left( 2\epsilon^{ijk} F_{jkIJ} - \beta \mathcal{P}^{iJ} - \frac{\alpha}{2} \epsilon^{ijkl} \mathcal{P}^{ijkl} \right)(x). \quad (25) $$

The Poisson bracket of the theory is

$$ \{ A^{iJ}_I(x), \mathcal{P}^{iJ}_K(y) \} = \frac{1}{2} \delta(x - y) \delta^{ij} \delta_{KL}. \quad (26) $$

(The factor 1/2 results from the fact that the canonical momentum associated with $A$ defined as $\delta L/\delta \dot{A}$ is $2\mathcal{P}$, not $\mathcal{P}$.) The Lagrangian (23) contains just the standard $(p\dot{q})$ kinetic term appended with a combination of constraints, reflecting the manifest diffeomorphism invariance of the action (2) that we have started with. It is worth noting that prior to taking care of the constraints the dimension of phase space of the system is $2 \times 3 \times 10 = 60$ at each space point. As we see the dimension of the physical phase space is going to be 4, as it should be.

The Poisson brackets of the constraints can be straightforwardly computed and read

$$ \{ \Pi^{IJ}_I(x), \Pi^{KL}_K(y) \} = \delta(x - y)(\eta_{IL} \Pi^{JK}_J - \eta_{JL} \Pi^{IK}_I + \eta_{IK} \Pi^{JL}_L)(x) \approx 0, \quad (27) $$

which means that $\Pi^{IJ}_I$ form a representation of the gauge group $SO(4,1)$ of the unconstrained theory ($\alpha = 0$), as expected. Further

$$ \{ \Pi^{IJ}_I(x), \Pi^{iJ}_K(y) \} = 2\epsilon^{ijk} \delta(x - y) \left( \epsilon_{KLIP} A^P_I(x) - \epsilon_{KLPi} A^P_I(x) \right) \quad (28) $$

and

$$ \{ \Pi^{IJ}_I(x), \Pi^{iJ}_K(y) \} = -\frac{\alpha}{2} \delta(x - y) \left( \epsilon_{KLPi} \mathcal{P}^{iP} J(x) - \epsilon_{KLip} \mathcal{P}^{iP} J(x) \right) $$

$$ + \frac{\alpha}{4} \delta(x - y)(\eta_{IL} \epsilon_{JMKN} - \eta_{JL} \epsilon_{IMKN} - \eta_{IK} \epsilon_{LMKN} + \eta_{JK} \epsilon_{ILMN}) \mathcal{P}^{iMN}(x) $$

$$ + \frac{1}{2} \delta(x - y) (\eta_{IL} \Pi^{iJ}_I(x) - \eta_{JL} \Pi^{iJ}_I(x) - \eta_{IK} \Pi^{iJ}_I(x) + \eta_{JK} \Pi^{iJ}_I(x)). \quad (29) $$

It is worth noting that in the topological limit $\alpha = 0$ all the constraints are first class. This observation leads to the following, apparent puzzle. Namely, as we said above the kinematical phase space is 60 dimensional. On the other hand, for $\alpha = 0$ we have 10 + 30 first-class constraints that remove from this phase space 80 degrees of freedom. How is this possible? To answer this let us note that not all the constraints are independent. Indeed taking the covariant divergence of the $\Pi^{IJ}_I$ constraint and making use of the Bianchi identity we see that

$$ (\mathcal{D}_i \Pi^{ij})_I = -\beta \Pi^{ij}, \quad (30) $$

As a result the constraints $\Pi^{ij}_I$ are not independent.
and thus the set of constraints is reducible. It follows that we have only 30 independent first-class constraints \( \Pi_{iJ} \), which remove exactly 60 dimensions from the phase space, as it should be, since the theory with \( \alpha = 0 \) is topological.

Returning to the case \( \alpha \neq 0 \) we note that the action (2) is invariant under local gauge transformations that belong to the Lorentz subgroup \( \text{SO}(3, 1) \) of the initial de Sitter group \( \text{SO}(4, 1) \). It follows that it is natural to expect that one can simplify the algebra of constraints (27)–(29) if one decomposes the constraints into that belonging to the Lorentz and the translational parts of the algebra. From (24) we get

\[
\Pi_{a4}(x) \equiv \Pi_a(x) = (D_i^\mu P_{\mu}^i)_{a4}(x) - \frac{1}{\ell} e^i_\alpha(x) P_{\mu}^i a\beta(x) \approx 0
\]

(31)

\[
\Pi_{a\beta}(x) = (D_i^\mu P_{\mu}^i)_{a\beta}(x) - \frac{1}{\ell} e^i_\alpha(x) P_{\mu}^i a\beta(x) + \frac{1}{\ell} e^i_\beta(x) P_{\mu}^i a\beta(x) \approx 0,
\]

(32)

while from (25)

\[
\Pi_{i4}(x) \equiv \Phi^i_\alpha(x) = 2 \epsilon^{ijk} F_{jk} a\beta(x) - \beta P_{i4}(x) \approx 0
\]

(33)

\[
\Pi_{i\beta}(x) \equiv \Phi^i_\alpha(x) = 2 \epsilon^{ijk} F_{jk} a\beta(x) - \beta P_{i\beta}(x) - \frac{\alpha}{2} \epsilon_{\alpha\beta\gamma\delta} P_{\mu\nu\sigma}(x) \approx 0.
\]

(34)

One then finds that the algebra of constraints (28) and (29) simplifies a lot, and the only brackets that do not vanish weakly are

\[
\{ \Pi_a(x), \Phi^i_{\gamma\delta}(y) \} \approx -\frac{\alpha}{2} \delta(x - y) \epsilon_{\gamma\kappa\lambda\sigma} P_{i\lambda\sigma}(x)
\]

(35)

\[
\{ \Pi_a(x), \Phi^{i\gamma}(y) \} \approx -\frac{\alpha}{4} \delta(x - y) \epsilon_{\gamma\kappa\lambda\sigma} P_{ij\lambda\sigma}(x)
\]

(36)

\[
\{ \Phi^i_a(x), \Phi^{i\gamma}(y) \} \approx 2 \alpha \epsilon^{ijk}(x - y) \epsilon_{\alpha\beta\gamma\delta} A_{k\delta}(x) = \frac{2\alpha}{\ell} \epsilon^{ijk}(x - y) \epsilon_{\alpha\beta\gamma\delta} \delta_{ijk}(x)
\]

(37)

Now we can turn to the next step of canonical analysis, i.e. to checking if there are any tertiary constraints. The Hamiltonian, being a combination of constraints, reads

\[
H = -2A^\alpha \Pi_a - A^{a\beta} \Pi_{a\beta} - 2B_{i\alpha} \Phi^i_a - B_{i\alpha\beta} \Phi^{i\beta}(x).
\]

(38)

It follows from (35)–(37) that we have to satisfy the following conditions to ensure that the constraints are preserved by time evolution, generated by the Hamiltonian (38):

\[
\dot{\Pi}_a = \frac{\alpha}{2} (B_{i\beta} p^{i\gamma\delta} + B_{i\gamma} p^{i\beta\delta}) \epsilon_{\alpha\beta\gamma\delta} \approx 0
\]

(39)

\[
\dot{\Phi}^i_a = -\alpha \left( \frac{2}{\ell} \epsilon^{ijk} B_j \gamma_{\epsilon} \delta - \frac{1}{2} A^{\beta} p^{i\gamma\delta} \right) \epsilon_{\alpha\beta\gamma\delta} \approx 0
\]

(40)

\[
\dot{\Phi}^{i\beta} = -\alpha \left( \frac{4}{\ell} \epsilon^{ijk} B_j \gamma_{\epsilon} \delta + A^{\gamma} p^{4\delta} \right) \epsilon_{\alpha\beta\gamma\delta} \approx 0.
\]

(41)

These equations can be solved for Lagrange multipliers (we have 34 equations for 34 unknowns \( B_{i\beta}, B_{i\gamma}, A^{\gamma} \) with arbitrary coefficients), and thus there are no tertiary constraints.

Note however that there is an ambiguity in the Dirac procedure in the case of diff-invariant systems, i.e. such that the Hamiltonian is a combination of constraints. The usual approach

3 In what follows we restrict ourselves to the positive cosmological constant case; the negative cosmological constant and the anti-de Sitter group \( \text{SO}(3, 2) \) can be analyzed analogously.
is to check if one can solve the vanishing of the time derivative of the constraints condition for Lagrange multipliers, as we did above. But this is, clearly, not a general solution of these conditions. In general, one may look for the solutions with the arbitrary values of the Lagrange multipliers but instead restricting the phase space (for example if we impose the condition that all the Lagrange multipliers in (39)–(41) are arbitrary, there would be additional constraints saying that the components of tetrad and momenta are to be equal to zero). Note that this problem does not arise in the case of the Hamiltonian not being weakly zero, because then the resulting equations pertaining to the time invariance of the constraints are non-homogeneous. Thus the procedure that is usually employed does not seem to provide a complete characterization of the phase space, but we adopt it here, leaving the discussion of this subtle point to future work.

Let us finish this section with a simple, but important remark. Constraint (24) is a Gauss constraint for SO(4, 1) gauge symmetry and, as is well known and can be checked by straightforward calculation, they generate the infinitesimal SO(4, 1) gauge transformation through the Poisson bracket: if

$$G = \int \xi^{IJ} \Gamma_{IJ},$$

then

$$\delta A_{IJ}^i = \{A_{IJ}^i, G\} = -(D_i \xi)^{IJ}, \quad \delta \mathcal{P}_{IJ} = \{\mathcal{P}_{IJ}, G\} = [\xi, \mathcal{P}_{IJ}].$$

It should be noted however that when $\alpha \neq 0$, as the analysis presented above shows, only the Lorentz part of these constraints $\Pi_{\alpha \beta}$ remains first class. This result is an obvious consequence of the fact that for $\alpha \neq 0$ the SO(4, 1) invariance of the topological action is broken down to the local Lorentz invariance.

4. Simplifying the constraints

The aim of this section is to rewrite the system of constraints (27)–(29) in a form that makes contact with the constraints structure of general relativity with the Holst term, discussed in [6]. In what follows we borrow some ideas from the paper of Perez and Rezende [8]. (Similar ideas, albeit in more restricted setting, were discussed, e.g., in [13] and [14].)

In the first step let us rearrange constraints (27)–(29) so as to write them in the following form:

$$\Phi^i_\alpha = \mathcal{P}^i_\alpha - \frac{4}{\ell \beta} \epsilon^{ijk} D_j e_k^\alpha \approx 0 \quad (42)$$

$$\Phi^i_{\alpha \beta} = \mathcal{P}^i_{\alpha \beta} - M_{\alpha \beta} \gamma^\delta F_{jk \gamma \delta} \epsilon^{ijk} \approx 0 \quad (43)$$

$$\Pi_{\alpha \beta} = \frac{2}{\ell^2} \epsilon^{ijk} D_j (K_{\alpha \beta} \gamma^\delta e_j^\alpha e_k^\delta) \approx 0 \quad (44)$$

$$\Pi_\alpha = \frac{1}{\ell} \epsilon^{ijk} K_{\alpha \beta} \gamma^\delta e_i^\beta R_{jk \gamma \delta} - \frac{2\alpha}{(\alpha^2 + \beta^2)\ell^3} \epsilon^{ijk} e_{\alpha \gamma \delta} e_j^\delta e_k^\gamma e_i^\alpha \approx 0 \quad (45)$$

Recall that the coupling constants $\alpha$ and $\beta$ satisfy the identity $\alpha/(\alpha^2 + \beta^2) = \ell^3/G$, while the operators $M$ and $K$ are defined to be

$$M^{\alpha \beta}_{\gamma \delta} = \frac{\alpha}{(\alpha^2 + \beta^2)} (\gamma \delta^{\alpha \beta} - \epsilon^{\alpha \beta}_{\gamma \delta}), \quad (46)$$
\[ K^{ab}_{\gamma\delta} \equiv \alpha \frac{1}{(\alpha^2 + \beta^2)} \left( \frac{1}{\gamma} y^{ab}_{\gamma\delta} + e^{ab}_{\gamma\delta} \right). \]  

(47)

Also recall that the action (2), after solving for \( B \) and expressing the resulting action in terms of the SO(3, 1) connection \( \omega \) and tetrad \( e \), has the form [11]

\[
S = \frac{1}{G} \int e^{ab} \left( R_{\mu\nu}^{ab} e\rho_{\mu\nu}^{\rho} e\sigma_{\mu\nu}^{\sigma} - \frac{\Lambda}{3} e_{\mu\nu} a e_{\rho\sigma} e_{\mu\nu}^{\rho\sigma} \right) \epsilon^{\mu\nu\rho\sigma} \\
+ \frac{2}{G \gamma} \int R_{\mu\nu}^{ab} e_{\rho}^{\alpha} e_{\gamma}^{\beta} e^{\mu\nu\rho\sigma} \gamma^{2} + \frac{1}{2} N_{\gamma} + \frac{3 \gamma}{2 G \Lambda} P_{\gamma} - \frac{3}{4 G \Lambda} E_{\gamma}.
\]  

(48)

One immediately recognizes here the standard gravitational action in the first line, and the Holst term, whose strength is governed by the Immirzi parameter \( \gamma = \beta / \alpha \) in the second. The last three terms are proportional to topological invariants (Nieh–Yan, Pontryagin and Euler):

\[
N_{\gamma} = \int \left( T_{\mu\nu}^{a} T_{\rho\sigma}^{\alpha} - 2 R_{\mu\nu}^{ab} e_{\rho}^{\alpha} e_{\gamma}^{\beta} \right) \epsilon^{\mu\nu\rho\sigma},
\]

(49)

\[
P_{\gamma} = \int R_{\mu\nu}^{ab} R_{\rho\sigma}^{ab} e^{\mu\nu\rho\sigma},
\]

(50)

\[
E_{\gamma} = \int R_{\mu\nu}^{ab} R_{\rho\sigma}^{ab} e^{\mu\nu\rho\sigma}.
\]

(51)

As we show, in the case when the constant time surface is without boundaries \( \partial \Sigma = 0 \), the topological terms play the role of the generating functional for canonical transformations, which simplify the constraints considerably [8]. The key observation is that Pontryagin and Nieh–Yan invariants can be expressed as total derivatives

\[
N_{\gamma} = 4 \int \partial_{\mu} \left( e_{\gamma}^{a} \partial_{\rho} e_{\gamma}^{a} \partial_{\rho} \right) \epsilon^{\mu\nu\rho\sigma},
\]

(52)

\[
P_{\gamma} = 4 \int \partial_{\mu} \left( \omega_{\rho a} b \partial_{\rho} \omega_{\rho a} b + \frac{2}{3} \omega_{\rho a} b \omega_{\rho c} b \partial_{\rho} \omega_{\rho c} b \right) \epsilon^{\mu\nu\rho\sigma}.
\]

(53)

The same holds for the Euler class. However, in this case one has to make use of self- and anti-self-dual combinations of the Lorentz connection

\[
\pm \omega_{i}^{\rho ab} = \frac{1}{2} \left( \omega_{i}^{\rho ab} \pm \frac{i}{2} e_{\gamma}^{a} \omega_{i}^{\rho b} \right), \quad \pm \omega_{i}^{\gamma b} \epsilon_{\gamma}^{a} = \pm i \omega_{i}^{ab}
\]

(54)

and curvature (see e.g. [15])

\[
\pm R_{\mu\nu}^{ab} = \frac{1}{2} \left( R_{\mu\nu}^{ab} \pm \frac{i}{2} e_{\gamma}^{a} R_{\mu\nu}^{b} \right).
\]

(55)

It can be checked that both Pontryagin and Euler class can be rewritten with the help of \( \pm R_{\mu\nu}^{ab} \) as follows:

\[
P_{\gamma} = \int \epsilon^{\mu\nu\rho\sigma} \left( \pm R_{\mu\nu}^{ab} \partial_{\rho} \omega_{\gamma}^{a} b + \frac{2}{3} \omega_{\rho} a b \omega_{\rho c} b \partial_{\rho} \omega_{\rho c} b \right) \epsilon^{\mu\nu\rho\sigma}.
\]

(56)

\[
E_{\gamma} = 2i \int \epsilon^{\mu\nu\rho\sigma} \left( \pm R_{\mu\nu}^{ab} \partial_{\rho} \omega_{\gamma}^{a} b - \frac{2}{3} \omega_{\rho} a b \omega_{\rho c} b \partial_{\rho} \omega_{\rho c} b \right) \epsilon^{\mu\nu\rho\sigma}.
\]

(57)

Introducing the Chern–Simons four-vector

\[
C_{\mu}^{a} (\omega) = \left( \omega_{\rho a b} \partial_{\rho} \omega_{\gamma}^{a} b + \frac{2}{3} \omega_{\rho a b} \omega_{\rho c} b \omega_{\rho c} b \right) \epsilon^{\mu\nu\rho\sigma},
\]

(58)
we write the Pontryagin and Euler classes as total derivatives,
\[ P_4 = 4 \int (\partial_{\mu} C^\mu(\gamma, \omega) + \partial_{\mu} C^\mu(-\omega)) \]
(59)
\[ E_4 = 8i \int (\partial_{\mu} C^\mu(\gamma, \omega) - \partial_{\mu} C^\mu(-\omega)). \]
(60)

Therefore, the topological part of action (48) takes the form
\[ S_T = \frac{2\alpha}{(\alpha^2 + \beta^2)} \int \partial_{\mu} (C^\mu(\gamma, \omega) + C^\mu(-\omega)) - \frac{2\alpha}{(\alpha^2 + \beta^2)} \int \partial_{\mu} (C^\mu(\gamma, \omega) - C^\mu(-\omega)) \]
\[ + \frac{4}{\beta \ell^2} \int \partial_{\mu} (e_{\alpha a} D_{\mu}^a e_\beta) \epsilon^{\mu \nu \rho \sigma}. \]
(61)

It is worth noting that in spite of the presence of the imaginary i here, the action \( S_T \) is real (for real \( \gamma \)).

For constant time surfaces being a manifold without boundary (\( \partial \Sigma = 0 \)), all total spacial derivatives terms drop out and only the ones with total time derivative survive
\[ S_T = \int \partial_0 W(e, \omega), \]
where \( W(\omega, e) \) is a functional of torsion and self- and anti-self-dual Chern–Simons forms \( L_{CS} \equiv C^0 \)
\[ W(e, \omega) = \frac{2\alpha}{(\alpha^2 + \beta^2)} \int_\Sigma ((\gamma - i) L_{CS}^+(\gamma, \omega) + (\gamma + i) L_{CS}^-(\gamma, \omega)) + \frac{4}{\beta \ell^2} \int_\Sigma \epsilon^{ijk}(e_{\alpha a} D_{\mu}^a e_{\beta}) \epsilon^{\mu \nu \rho \sigma}. \]
(62)

Having the functional \( W \) we can make canonical transformation, which defines the new momenta \( P_i^\alpha \) and \( P_{ij}^{\alpha \beta} \) of the tetrad \( e \) and the connection \( \omega \), respectively:
\[ \mathcal{P}_i^\alpha = P_i^\alpha + \{ P_i^\alpha, W(\omega, e) \}, \quad \mathcal{P}_{ij}^{\alpha \beta} = P_{ij}^{\alpha \beta} + \{ P_{ij}^{\alpha \beta}, W(\omega, e) \}, \]
(63)
with
\[ \{ e_{i0}^\gamma, \mathcal{P}_j^\rho \} = \frac{1}{2} \delta_{i}^{\gamma} \delta_{j}^{\rho} \quad \text{and} \quad \{ \omega_{ij}^{\alpha \beta}, \mathcal{P}_i^{\gamma} \} = \frac{1}{2} \delta_{i}^{\gamma} \delta_{i}^{\beta}. \]
(64)

Since the variations of the functional \( W(\omega, e) \) are
\[ \frac{1}{2} \frac{\delta W}{\delta \omega_{ij}^{\alpha \beta}} = M_{ij}^{\gamma \delta} R_{jky}^{\gamma \delta} e_{i}^{jk} - \frac{4}{\beta \ell^2} e_{j}^{\epsilon} e_{k}^{\delta} e_{i}^{\epsilon} e_{j}^{\epsilon} e_{k}^{\delta}, \]
(65)
\[ \frac{1}{2} \frac{\delta W}{\delta e_{i}^{\alpha}} = \frac{4}{\beta \ell^2} e_{j}^{\epsilon} D_{i}^{\alpha \beta} e_{k}^{\delta}, \]
(66)
we find that the resulting constraints, expressed in terms of new momenta (63), take the form
\[ \Phi_i^{\gamma} = \mathcal{P}_i^{\gamma} \approx 0, \]
(67)
\[ \Phi_{ij}^{\alpha \beta} = \mathcal{P}_{ij}^{\alpha \beta} - \frac{2}{\ell^2} K_{ij}^{\gamma \delta} e_{j}^{\epsilon} e_{k}^{\delta} e_{l}^{\epsilon} \approx 0 \]
(68)
\[ \Pi_{ij}^{\alpha \beta} = \frac{2}{\ell^2} e_{j}^{\epsilon} K_{ij}^{\gamma \delta} D_{i}^{\epsilon \alpha \beta} (e_{j}^{\epsilon}, e_{k}^{\delta}) \approx 0 \]
(69)
\[ \Pi_i^{\alpha} = \frac{1}{\ell^2} e_{j}^{\epsilon} K_{ij}^{\gamma \delta} e_{j}^{\epsilon} F_{j}^{\gamma \delta} \approx 0. \]
(70)
This form of constraints is our starting point in checking equivalence with the ones proposed by Holst [6], which prove, in turn, that they describe general relativity, as expected. To establish this equivalence we have to fix the time gauge. We turn to this problem in the next section.

Before closing this section let us make an important remark. The considerable simplification of the constraints relies heavily on the fact that the constant time surfaces are manifolds without boundary. In the case when boundaries are present the analysis of the constraints becomes much more involved. We will address this issue in the forthcoming paper.

5. Time gauge

In order to make contact with the Hamiltonian analysis of Holst, we have to fix the gauge so as to remove the time component of the tetrad and then to relate momenta associated with the Lorentz connection to an appropriate combination of the remaining tetrad components.

To this end, let us introduce the gauge fixing condition which must be added to the list of constraints

\[ e^0_i \approx 0. \] (71)

In this gauge the constraints \( \mathcal{P}^i_0 \approx 0 \) can be removed by turning to the Dirac bracket so the remaining constraints (67) and (68) take the form

\[ \mathcal{P}^i_a \approx 0 \] (72)

\[ \mathcal{P}^i_{0a} + \frac{2\alpha}{\ell^2(\alpha^2 + \beta^2)} \epsilon^{ijk} \epsilon_{abc} e^b_j e^c_k \approx 0 \] (73)

\[ \mathcal{P}^i_{ab} - \frac{2\alpha}{\ell^2(\alpha^2 + \beta^2)} \frac{1}{\gamma} \epsilon^{ijk} \frac{\gamma}{\ell^2} \epsilon_{abc} e^b_j e^c_k \approx 0, \] (74)

where we have used the convention \( \epsilon^{0abc} = \epsilon^{abc} \) and \( \epsilon_{0abc} = -\epsilon_{abc} \).

Combining the last two equations we find constraints for generalized self- and anti-self-dual parts of \( \mathcal{P}^i \)

\[ +P^i_a \approx 0 \] (75)

\[ -P^i_a + \frac{4\alpha}{(\alpha^2 + \beta^2)\ell^2} \epsilon^{ijk} \epsilon_{abc} e^b_j e^c_k \approx 0, \] (76)

where we define

\[ \pm P^i_a = \mathcal{P}^i_{0a} \pm \frac{\gamma}{2} \epsilon_{abc} \mathcal{P}^i_{bc}. \] (77)

It can be easily checked that \( \pm P \) are the momenta associated with generalized (anti-) self-dual combinations of the Lorentz connection (which for \( \gamma = \pm i \) become usual self- and anti-self-dual ones)

\[ \pm w^a_i = \omega^a_{0i} \pm \frac{1}{2\gamma} \epsilon^{abc} \omega_{bc}, \] (78)

with the Poisson brackets being

\[ \{ \pm w^a_i, \pm P^j_a \} = 0, \quad \{ \pm w^a_i, \pm P^j_d \} = \delta^j_i \delta^a_d. \] (79)

---

4 This condition fixes the gauge symmetry generated by the boost part of the Gauss constraint (23), which, from the analysis of section 3, we know to be first class. However, as is well known, there is a subtlety here because one more constraint (94) arises later in the formalism. For detailed discussion of the time gauge see, e.g., [25] and [3].
Let us now turn to the constraint $\Pi_{\alpha\beta}$ (69). Decomposing it into components we find

$$\Pi_{ab} = \frac{2\alpha}{(\alpha^2 + \beta^2)\ell^2} e^{ijk} \left( \frac{2}{\gamma} \left( \partial_i (e_j a e_k b) + \omega_{i\alpha} e_j e_{k\beta} - \omega_{i\beta} e_j e_{k\alpha} \right) - 2\epsilon_{abc} \omega^d_{ij} e_{jd} e^e_k \right)$$  \hspace{1cm} (80)

$$\Pi_{0\nu} = -\frac{2\alpha}{(\alpha^2 + \beta^2)\ell^2} e^{ijk} \left( \epsilon_{abc} \left( \partial_i (e_j^b e_k^c) + 2\omega_{i\alpha} b e_j d e^e_k + \frac{2}{\gamma} \omega_{i\beta} b e_j a e_k \right) \right).$$  \hspace{1cm} (81)

Taking the combination $\Pi_{0\nu} \pm \frac{\gamma}{2} \epsilon_{abc} \Pi^{bc}$ we get

$$\frac{4\alpha}{(\alpha^2 + \beta^2)\ell^2} e^{ijk} \left( \left( \frac{1 + \gamma^2}{\gamma} \right) \omega^b_{i\alpha} e_j a e_k b - \epsilon_{abc} \left( \partial_i (e_j^b e_k^c) + 2\omega_{i\alpha} b e_j d e^e_k \right) \right) \approx 0.$$  \hspace{1cm} (82)

From these two equations it follows that

$$\omega^b_{i\alpha} e_j a e_k b \approx 0,$$  \hspace{1cm} (83)

and

$$\left( \partial_i (e_j^b e_k^c) + 2\omega_{i\alpha} b e_j d e^e_k \right) \epsilon^{ijk} \epsilon_{abc} \approx 0.$$  \hspace{1cm} (84)

which expressed in terms of the new variables

$$\omega^b_{i\alpha} = \frac{1}{2} \left( \nu^b_i - \nu^a_i \right), \quad \omega^b_{i\beta} = \frac{\gamma}{2} \epsilon^{abc} \left( \nu^b_i - \nu^a_i \right)$$  \hspace{1cm} (86)

take the form of the Gauss and the boost constraints

$$G_a \equiv \left( (\nu^b_i - \nu^a_i) e_j a e_k b \right) \epsilon^{ijk} \approx 0,$$  \hspace{1cm} (87)

$$B_a \equiv \left( \partial_i (e_j^b e_k^c) + \gamma \left( \nu^b_i - \nu^a_i \right) e_j a e_k b \right) \epsilon^{ijk} \approx 0.$$  \hspace{1cm} (88)

We can handle the scalar part of (70)

$$S = \frac{\alpha}{(\alpha^2 + \beta^2)\ell^2} \left( \frac{2}{\gamma} e_{i\alpha} R^{0b}_{jk} - \epsilon_{abc} e^a_j F^{bc}_{jk} \right) e^{ijk} \approx 0,$$  \hspace{1cm} (89)

similarly, obtaining as a result the expression

$$\left[ \left( \frac{1 + \gamma^2}{\gamma} \right) e^{dbc} \partial_j (e_k d b) + \left( 1 - \frac{\gamma^2}{2} \right) e^{dabc} \partial_j (e_k d b) + \left( \frac{3 - \gamma^2}{2} \right) e^b_j e^c_k - \frac{2}{\ell^2} e^b_j e^c_k \right] e_{abc} e^{dijk} \approx 0.$$  \hspace{1cm} (90)

As for the vector part of (70)

$$V_a = \frac{\alpha}{(\alpha^2 + \beta^2)\ell^2} \left( \frac{2}{\gamma} e^a_j R_{jk} a b - 2\epsilon_{abc} e^a_j \omega^b_{jk} \right) e^{ijk} \approx 0,$$  \hspace{1cm} (90)

we find

$$\left[ 2\epsilon_{abc} \partial_j (e_k^b e_l^c b) - \left( \frac{1 + \gamma^2}{2\gamma} \right) e^b_j e^c_k + \left( \frac{3\gamma^2 - 1}{2\gamma} \right) e^c_k - e^b_k \right] e_{abc} e^{dijk} \approx 0.$$  \hspace{1cm} (90)

This is obvious for $\gamma^2 \neq -1$. For $\gamma^2 = -1$ equation (82) is identically satisfied, but then, since the connection is real, the real and imaginary parts of (83) lead to (84) and (85).
It should be noted that in the case $\gamma^2 = -1$ all terms containing $^+w_{i\mu}$ cancel.

Constraints (87), (89) and (90), expressed in terms of $\pm w_i \epsilon_i$, are exactly the ones found by Holst [6] with the only modification being that in the scalar constraint we have the cosmological constant term. This extra term does not affect the resulting algebra, neither do topological terms shifting the definitions of the momenta, which do not change relations between the constraints, and leave the total number of physical degrees of freedom unaffected.

In the next step we try to get rid of the constraint $^+P_{i\mu} \approx 0$ and eliminate the dependence on $^+w_{j\mu}$ of all the remaining constraints. In order to be able to do that we must first simplify the form of the boost constraint (88). To this end, we multiply constraint (75) by tetrad and decompose the resulting constraint $C_{ab}$ into symmetric and antisymmetric parts

$$C_{(ab)} = ^+P_{i\mu}e_{i\mu}^b, \quad C_{[ab]} = ^+P_{i\mu}e_{i\mu}^b.$$

Let us now calculate the Poisson bracket of $C_{ab}$ with the scalar constraint

$$\{C_{ab}, S\} = \frac{(1 + \gamma^2)\ell}{\gamma G} e_{ijk} e_{j\mu} (2\partial_k e_{i\mu} - \gamma (^+w_k - w_k) e_{i\mu} e_{\alpha\gamma\delta}).$$

The bracket of the antisymmetric part $C_{[ab]}$ gives exactly the boost constraint (88). However, the bracket of the symmetric part $C_{(ab)}$ leads to the secondary constraint

$$B_{ab} \equiv 2e_{ijk} (e_{j\mu} \partial_k e_{i\mu} + e_{j\mu} \partial_k e_{i\mu} - \gamma (^+w_k - w_k) e_{i\mu} (e_j b e_{\alpha\gamma\delta} + e_j a e_{\alpha\gamma\delta}) \approx 0.$$ (93)

Clearly, this constraint would arise if we impose the requirement that all the constraints are to be preserved in the time evolution. Therefore, one has to add $B_{ab}$ to the set of constraints of the theory. But then the road suddenly becomes sunny. It suffices to note that the boost constraints $B_{a}$ and the newly derived constraints $B_{ab}$ are just the antisymmetric and symmetric parts of the simple constraint

$$(2\Gamma_i - \gamma (^+w_i - w_i)) \approx 0,$$ (94)

where $\Gamma_i$ is a (unique) solution of the Cartan first structural equation

$$\partial_i e_{j\mu} + \epsilon_{\alpha\gamma\delta} \Gamma_{[i} e_{j\mu]} = 0.$$ (95)

Calculating the Poisson bracket between constraints (94) and (75) shows that they are both second class. Therefore, we can solve them strongly and replace $^+w_{j\mu}$ with the solution in all the remaining constraints. Similarly using (72) and (76) we can identify the momentum of $^+w_{j\mu}$ with

$$-\frac{4\alpha}{(a^2 + \beta^2)\ell} \epsilon_{ijk} e_{i\mu} e_{j\mu} e_k = -\frac{8}{G} e_{i\mu} = -\frac{8}{G} E_i^\mu.$$ (98)

What remains are therefore three Gauss, three vector and one scalar constraints, all of them first class, constraining the 18-dimensional phase space of $^+w_{j\mu}$ and its momenta. Explicitly, our final phase space is defined by the variable $^+w_{j\mu}(x)$, and $E_i^\mu = e_i e_{i\mu}$ with the bracket

$$\{-^+w_{j\mu}(x), E_i^\mu(y)\} = -\frac{1}{8} G \delta_i^j \delta_{\mu\nu} \delta(x - y).$$ (96)

subject to the constraints

$$\left(\frac{1 + \gamma^2}{\gamma^2}\right)e_{ijk} \partial_j \Gamma_k - G \delta_{\alpha\beta} e_{ijk} + \frac{1}{\gamma} e_{ijk} \partial_i \Gamma_j + \frac{1}{\gamma} \Gamma_{[i} e_{j\mu} [^+w_k - w_k] e_{i\mu} e_{j\mu} e_k - \frac{2}{\ell^2} e \approx 0.$$ (97)

$$G_{ab} \equiv \left(\frac{2}{\gamma} \Gamma_i + 2w_i\right) e_{i\mu} e_{k\mu} e_{j\mu} e_{j\mu} e_k e_{i\mu} e_{j\mu} e_{j\mu} e_k e_{i\mu} e_{j\mu} e_{j\mu} e_k e_{i\mu} e_{j\mu} e_{j\mu} e_k \approx 0.$$ (98)

6 Instead of $C_{[ab]} = ^+P_{i\mu}e_{i\mu}^b$ just take the expression $C_{[ab]} = ^+P_{i\mu}e_{i\mu}^b$; so $\{C_{[ab]}$, $S\} = \frac{1 + \gamma^2}{\gamma^2} B^\mu$. 12
\[
V_b \equiv 2(e_{abc} \partial^- w^a_k + \gamma^- w_j b^c k) e^i e^{ijk} \approx 0. \tag{99}
\]

It is now a matter of standard calculation [2] to show that the system of these constraints is first class; thus, the dimension of physical phase space is \(18 - 14 = 4\) as it should. Of course, the final set of constraints we have obtained has exactly the form of the constraints describing gravity, cf [6]. This completes our analysis of the canonical structure of SO\((4, 1)\)-constrained BF theory.

6. Comments on quantization

Let us conclude this paper with some comments concerning quantization. Clearly, one can take the first-class Gauss vector and scalar constraints as a starting point in the construction of the quantum theory, as it is done in loop quantum gravity [2, 3]. However, the structure of constraints of the original theory opens another possibility of devising a perturbative expansion in the parameter \(\alpha\) around topological vacuum. Here, we describe briefly this perturbative theory leaving details to a separate publication.

Our starting point is the set of constraints (42)–(45). Consider now the canonical transformation (63). Its quantum counterpart can be easily found. To see how, take the mechanical model in which one makes the transformation (see [16])

\[
p_i \to p^\prime_i = p_i + \{p_i, f(q)\},
\]

so that quantum mechanically we have

\[
\hat{p}_i \to \hat{p}^\prime_i = \hat{p}_i + i [\hat{p}_i, f(\hat{q})].
\]

If we represent \(\hat{p}_i = i \partial / \partial q^i\), then \(\hat{p}^\prime_i = i \partial / \partial q^i - \partial f(q) / \partial q^i\). Therefore, if we decompose the wavefunction \(\psi(q) = \exp(-i f(q)) \psi(q)\), then

\[
\hat{p}^\prime \psi(q) = \exp(-i f(q)) \hat{p}^\prime \psi(q),
\]

which means that we just have to multiply the wavefunction with the phase \(\exp(-i f(q))\) and then use the standard representation of the new momenta \(\hat{P}\) as the derivatives over positions.

In the case at hand (63), it is therefore sufficient to multiply the wavefunction by the prefactor \(\exp(-i W(e, \omega))\) where \(W(e, \omega)\) is given by (62), and replace all the momenta \(P\) with the new ones \(\hat{P}\). Then we can just use constraints (67)–(70).

When \(\alpha = 0\) these constraints reduce to the first-class set

\[
\mathcal{P}^l_\alpha \approx 0, \quad \mathcal{P}^l_{\alpha\beta} \approx 0.
\]

The wavefunction annihilated by them is just a constant, and thus the full physical wavefunction is a phase \(\exp(-i W(e, \omega))\). Clearly, and not surprisingly, in this case the wavefunction is the Kodama state [17] (strictly speaking this is the Kodama state for SO\((4, 1)\) multiplied by the phase proportional to the Euler class of a constant time manifold). Note that here this state is delta function normalizable because all our constraints are real (cf [18]). The simplicity of the zeroth-order (in \(\alpha\)) solution reflects the fact that to this order the theory is topological.

Let us now turn to devising the \(\alpha\)-perturbative theory. Constraints (67)–(70) have the form \(\Phi = \Phi^{(0)} + \alpha \Phi^{(1)}\) (for the last two \(\Phi^{(0)} = 0\)). We also expand the wavefunction in the series in \(\alpha\), to wit

\[
\Psi = \Psi^{(0)} + \alpha \Psi^{(1)} + \cdots. \tag{100}
\]

The problem we are facing now is that for non-zero \(\alpha\) the constraints are no longer first class and therefore we need a nonstandard procedure to handle them. One possibility would be the Gupta–Bleuler quantization [19], but the required procedure of splitting the constraints into
holomorphic and anti-holomorphic parts is technically complex and, presumably, leads to the explicit breaking of Lorentz covariance (see [20] for discussion in a similar context). Another possibility would be to make use of the master constraint program [21–23], and [20], but this is again technically involved.

Instead we adopt the definition of the physical wavefunction $\Psi$ such that the matrix elements of all the constraints are zero

$$\langle \Psi | \Phi | \Psi \rangle = 0,$$

which is a weakened version of the Gupta–Bleuler scheme. It should be stressed that expression (101) is formal because to make the precise sense of it we must specify the inner product in the Hilbert space of states.

Now we use (101) to define the perturbative theory in $\alpha$. In the zeroth order we have

$$\langle \Psi^{(0)} | \Phi^{(0)} | \Psi^{(0)} \rangle = 0,$$

while in the first order in $\alpha$ we find

$$\langle \Psi^{(1)} | \Phi^{(0)} | \Psi^{(0)} \rangle + \langle \Psi^{(0)} | \Phi^{(1)} | \Psi^{(0)} \rangle + \langle \Psi^{(0)} | \Phi^{(0)} | \Psi^{(1)} \rangle = 0.$$  

Inspecting (67)–(70) we find that it follows from (102) and (103) that the zeroth-order wavefunction has to satisfy the following four conditions:

$$0 = \langle \Psi^{(0)} | \frac{\delta}{\delta \epsilon_k Y_k} | \Psi^{(0)} \rangle$$

$$0 = \langle \Psi^{(0)} | \frac{\delta}{\delta \omega_k} | \Psi^{(0)} \rangle$$

$$0 = \langle \Psi^{(0)} | \epsilon^{ijk} K_{\alpha\beta} \gamma^j D^\alpha_i (e_{jk} e_{kl}) | \Psi^{(0)} \rangle$$

$$0 = \langle \Psi^{(0)} | \epsilon^{ijk} K_{\alpha\beta} \gamma^k F_{jk} \gamma^\alpha | \Psi^{(0)} \rangle.$$  

Knowing $\Psi^{(0)}$ one can turn to the remaining first-order equation, resulting from (68), along with some of the second-order ones, to find $\Psi^{(1)}$, and then go to the next-order analysis. We stop the discussion at this point leaving the details to another paper.

7. Conclusions

In this paper we have performed the canonical analysis of the constrained $\text{SO}(4,1)$ BF theory. This analysis, although quite involved, seems to be significantly simpler than the analogous one of the Plebanski theory reported in [24], leading however to the slightly more general effective description of the dynamical degrees of freedom provided by Holst constraints that include the Immirzi parameter. This suggests that it might not be only simpler, but also more natural to consider the spin foam model associated with this particular formulation of gravity. Unfortunately, not much work has been done till now on the $\text{SO}(4,1)$ spin foam models, which would require to handle somehow not only the quadratic $B$ field term, but also the representation theory of $\text{SO}(4,1)$ group, which is more complicated than the one of the $\text{SU}(2)$ group, usually used in the spin foam context.

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