Abstract

The structure of $r$-fold tensor products of irreducible tame representations of $U(\infty) = \lim U(n)$ are described, versions of contragredient representations and invariants are realized, and methods of calculating multiplicities, Clebsch-Gordan and Racah coefficients are given using invariant theory on Bargmann-Segal-Fock spaces.

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I Introduction

Let $G_k$ and $G^C_k$ denote the unitary group and the general linear group, respectively. Then the inductive limits

$$G_\infty = \lim_{\rightarrow} G_k = \bigcup_{k=1}^\infty G_k$$
and

\[ G^c_\infty = \lim_{k \to \infty} G^c_k = \bigcup_{k=1}^{\infty} G^c_k \]

may be defined as follows:

\[ G^c_\infty = \{ g = (g_{ij})_{i,j \in \mathbb{N}} \mid g \text{ is invertible and all but a finite number of } g_{ij} - \delta_{ij} = 0 \} \]

and

\[ G_\infty = \{ u \in G^c_\infty \mid u^* = u^{-1} \}. \]

Representation theory of \( G_\infty \) and \( G^c_\infty \) was first studied by I. Segal in [1], then by A. Kirillov in [2], followed by S. Stratila and D. Voiculescu in [3], D. Pickerell in [4], G. Ol’shanskii in [5, 6, 7], I. Gelfand and M. Graev in [8], and V. Kac in [9]. This list is certainly not exhaustive, and the most complete list of references can be found in the comprehensive and important work of Ol’shanskii.

Following Ol’shanskii we call a unitary representation of \( G_\infty \) \textit{tame} if it is continuous in the group topology in which the descending chain of subgroups of the type \( \{ \begin{pmatrix} 1_k & 0 \\ 0 & * \end{pmatrix} \} \), \( k = 1, 2, 3, \ldots \) constitutes a fundamental system of neighborhoods of the identity \( 1_\infty \). Assume that for each \( k \) a unitary representation \( (R_k, H_k) \) of \( G_k \) is given and an isometric embedding (of Hilbert spaces) \( i^k_{k+1} : H_k \to H_{k+1} \) commuting with the action of \( G_k \) (i.e., \( i^k_{k+1} \circ R_k(u) = R_{k+1}(u) \circ i^k_{k+1} \)) is given. If \( H_\infty \) denotes the Hilbert space completion of \( \bigcup_{k=1}^{\infty} H_k \), then there exists uniquely a unitary representation
$R_\infty$ of $G_\infty$ on $H_\infty$ defined by

$$R_\infty(u)f = R_k(u)f \text{ if } u \in G_k \text{ and } f \in H_k.$$ 

The representation $(R_\infty, H_\infty)$ is called the inductive limit of the sequence $(R_k, H_k)$, and we have the following Theorem (see [5] for a proof).

**Theorem I.1.** If the representations $(R_k, H_k)$ are all irreducible then the inductive limit $(R_\infty, H_\infty)$ is also irreducible.

Let

$$\lambda_{G_k} = (m_1, \ldots, m_k), \quad m_1 \geq m_2 \geq \ldots \geq m_k \geq 0, \quad m_i \in \mathbb{N} \cup \{0\}, \quad i = 1 \ldots k.$$ 

then Ol’shanskii proved the following

**Theorem I.2.** All unitary irreducible tame representations of $G_\infty$ are the inductive limits of the sequences of the form $\{\rho_\lambda, V^{\lambda_{G_k}}\}$, where in each $(\lambda) = (m_1, m_2, \ldots)$ the $m_i$ are equal to 0 for sufficiently large $i$.

It follows from ‘Weyl’s unitarian trick’ that all irreducible tame representations of $G_\infty$ are inductive limits of sequences of the form $\{\rho_\lambda, V^{\lambda_{G_k}}\}$. Following Ol’shanskii a representation of $G_\infty$ is called holomorphic if it is a direct sum (of any number) of irreducible tame representations.

In this paper we consider the problem of decomposing an $r$-fold tensor product of unitary irreducible tame representations of $G_\infty$. Such a problem was investigated in [2] and [8] for the simplest type of tame irreducible representations, namely the fundamental (or principal) ones. In light of the
recent interest in Physics in the representation theory of $U_\infty$ it is natural to consider such an important problem in this theory.

The general problem can be stated as follows: Given $r$ tame irreducible $(G_\infty, V^{(\lambda_i)_\infty})$ modules, choose a basis $|\lambda_i, \xi_i\rangle$ for each $i$ (such a basis always exists, for example, the generalized Gelfand-Žetlin basis given in [8], but we do not limit ourselves only to this basis). Form the $r$-fold tensor product $(\lambda_1)_\infty \otimes \cdots \otimes (\lambda_r)_\infty$ and calculate the number of times the irreducible representation $(\lambda)_\infty$ occurs in the tensor product. The first method to compute this multiplicity is to observe that the spectral decomposition (or Clebsch-Gordan series) stabilizes for $k$ sufficiently large and then apply the Weyl determinant formula for $U(k)$ for sufficiently large $k$. This fact is proved rigorously as a theorem in Section III. In [10] it was shown that this multiplicity (for $SU(k)$) can be computed as solutions of Diophantine equations arising from the invariants of $SU(k)$. The first part of our program which is similar to the strategy given in [10] is as follows: instead of computing the multiplicity of $(\lambda)_\infty$ in the tensor product $(\lambda_1)_\infty \otimes \cdots \otimes (\lambda_r)_\infty$ we look at what is equivalent, the multiplicity of the identity representation in the augmented tensor product $(\lambda_1)_\infty \otimes \cdots \otimes (\lambda_r)_\infty \otimes (\lambda^\vee)_\infty$ where $(\lambda^\vee)_\infty$ is the contragredient representation of $(\lambda)_\infty$. But with this approach we are facing two major difficulties. The first one pertains to the contragredient representation $(\lambda^\vee)_\infty$: as it is well known (see e.g. [4]) an irreducible $(G_\infty, V^{(\lambda)_\infty})$-module can be realized as a subspace of a generalized Bargmann-Segal-Fock space in $n \times \infty$ complex variables (see Section IV for this realization), but it is not known
whether the irreducible \((G_\infty, V^{(\lambda^\vee)}_\infty)\)-module is realizable likewise. We prove in Section \[\text{Section}\] that by ‘twisting’ the action of the contragredient representation and by using an appropriate embedding of Bargmann-Segal-Fock spaces \(F_{n \times k} \subset F_{n \times (k+1)} \subset \cdots\) the \((G_\infty, V^{(\lambda^\vee)}_\infty)\)-module can also be realized as a submodule of a Bargmann-Segal-Fock space \(F_{n \times \infty}\). The notable difference is that the signature of \((\lambda^\vee)_\infty\) is characterized by the lowest weight instead of the highest weight and we will be dealing with lowest weight vectors instead of highest weight vectors as in the case \((G_\infty, V^{(\lambda)}_\infty)\). Another difficulty is that, realized as a submodule of the Bargmann-Segal-Fock space \(F_{n \times \infty}\), it is not clear that the tensor product \((\lambda_1)_\infty \otimes \cdots \otimes (\lambda_r)_\infty \otimes (\lambda^\vee)_\infty\) considered as a \(G_\infty\)-module is a holomorphic representation; in particular, the identity representation might not occur in this tensor product. Using a general reciprocity theorem for holomorphic representations of some infinite-dimensional groups (see \[\text{Reference}\]) we show that the tensor product \((\lambda_1)_\infty \otimes \cdots \otimes (\lambda_r)_\infty \otimes (\lambda^\vee)_\infty\) is indeed a holomorphic representation and that the multiplicity of the identity representation of \(G_\infty\) in this augmented tensor product is indeed equal to the multiplicity of \((\lambda)_\infty\) in the tensor product \((\lambda_1)_\infty \otimes \cdots (\lambda_r)_\infty\). Having overcome this difficulty first, we still have to deal with a second major difficulty; the generators of \(SU(k)\) used in \[\text{Reference}\] which are determinants of matrices of order \(k\) become unmanageable when \(k\) is large; furthermore, at the limit as \(k \to \infty\) these determinants are certainly not members of \(F_{n \times \infty}\). Both of these problems can be dealt with as follows: instead of using the determinant-invariants of \(SU(k)\) we use the classical invariants of \(U(k)\).
which are generated by a system of algebraically independent polynomials, but more importantly, the number of these polynomials depends only on the tensor product \((\lambda_1)_k \otimes \cdots \otimes (\lambda_r)_k \otimes (\lambda^\vee)_k\) and not on \(k\); in fact, the problems considered in [10] can be entirely solved using this new approach. Next, it can be shown (see [12]) that when \(k \to \infty\) these invariants tend to their inverse or projective limits which are infinite formal series of complex variables, but nevertheless remain algebraically independent and generate all \(G^C_\infty\) (or \(G_\infty\)) invariants. By analogy with the definition of rigged Hilbert Spaces (c.f. e.g. ”Generalized functions” by I. M. Gelfand and G. E. Shilov, Vol 4, P. 106) these infinite formal series may be thought of as differential operators. Thus if \(f \in \mathcal{F}_{n \times \infty}\) and \(p\) is a \(G_\infty\)-invariant then the inner product
\[
\langle p, f \rangle = p(D)f(\overline{Z})|_{Z=0}
\]
makes perfect sense since \(f \in \mathcal{F}_{n \times k}\) for some \(k\) and those terms in \(p(D)\) whose column indices are larger than \(k\) simply evaluate to zero. With this new interpretation of the \(G_\infty\)-invariants the method of computing Clebsch-Gordan and Racah coefficients in [10] can be adapted to the case of tensor products of \(G_\infty\); Actually, both the Diophantine equations and the computations of Clebsch-Gordan coefficients are much simpler, since the \(G_\infty\)-invariants are much simpler.
II  Preliminaries

Let $\mathbb{C}^{n \times k}$ denote the vector space of $n$ row by $k$ column matrices over $\mathbb{C}$, the field of complex numbers. If $Z = (Z_{ij})$ is an element of $\mathbb{C}^{n \times k}$, we let $\bar{Z}$ denote its complex conjugate, and write

$$Z = X_{ij} + \sqrt{-1} Y_{ij}; \quad 1 \leq i \leq n, \ 1 \leq j \leq k.$$  

If $dX_{ij}$ (resp. $dY_{ij}$) denotes the Lebesgue measure on $\mathbb{R}$ we let $dZ = \prod_{1 \leq i \leq n, \ 1 \leq j \leq k} dX_{ij} \cdot dY_{ij}$ denote the Lebesgue product measure on $\mathbb{R}^{nk}$. Define a Gaussian measure $d\mu$ on $\mathbb{C}^{n \times k}$ by

$$d\mu(Z) = \pi^{-nk} \exp[tr(Z\bar{Z}^t)]dZ$$

where $tr$ denotes the trace of a matrix. A map $f : \mathbb{C}^{n \times k} \to \mathbb{C}$ is said to be holomorphic square integrable if it is holomorphic on the entire domain $\mathbb{C}^{n \times k}$ and if

$$\int_{\mathbb{C}^{n \times k}} |f(Z)|^2 d\mu(Z) < \infty.$$  

The holomorphic square integrable functions form a Hilbert space with respect to the inner product

$$\langle f_1, f_2 \rangle = \int_{\mathbb{C}^{n \times k}} f_1(Z)\overline{f_2(Z)}d\mu(Z), \quad (1)$$

of which the polynomial functions form a dense subspace. The inner product (1) is equivalent to the inner product

$$\langle f_1, f_2 \rangle = f_1(D)\overline{f_2(Z)}|_{Z=0} \quad (2)$$
where $f(D)$ is the differential operator obtained by formally replacing $Z_{ij}$ by the partial derivative $\partial/\partial Z_{ij}$. We denote this Hilbert space by $\mathcal{F}_k = \mathcal{F}(\mathbb{C}^{n \times k})$. The natural embedding of $\mathbb{C}^{n \times k}$ into $\mathbb{C}^{n \times (k+1)}$ given by

$$Z \mapsto \begin{pmatrix} 0 & \cdots & 0 \\ Z & \cdots & Z \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{n \times (k+1)}$$

induces an isometric embedding

$$i_k^k : \mathcal{F}_k \longrightarrow \mathcal{F}_{k+1}$$

so that the collection $\{\mathcal{F}_k, i_k^k\}$ forms a directed system. We can then take the inductive limit $\mathcal{F}_\infty = \varinjlim_{k} \mathcal{F}_k$ (where the bar indicates closure with respect to the norm), with the natural inclusion

$$i_k : \mathcal{F}_k \longrightarrow \mathcal{F}_\infty.$$

Formally, elements of $\mathcal{F}_\infty$ are realized as equivalence classes $[f_\alpha]$, where

$$f_\alpha \sim f_\beta \text{ whenever } f_\beta = i_\beta^\alpha(f_\alpha) \text{ and } \alpha \leq \beta, \quad \alpha, \beta \in \mathbb{N}.$$ 

Since in our case we have $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, we can realize this space as

$$\mathcal{F}_\infty = \bigcup_{k=1}^{\infty} \mathcal{F}_k.$$

If $G_k = U(k)$ (or $U(k)^\mathbb{C} = GL(k, \mathbb{C})$) we also have the natural inclusion

$$j_k^k : G_k \longrightarrow G_{k+1}$$
given by

\[
g \mapsto \begin{pmatrix}
g & 0 \\
\vdots & \ddots & \ddots \\
0 & \cdots & 0 & 1
\end{pmatrix} \in G_{k+1}
\] (3)

We can then take the inductive limit \( G_\infty = \varinjlim G_k \), with the natural inclusion \( j_k : G_k \to G_\infty \). Again elements of \( G_\infty \) are formally defined as equivalence classes \([g_k]\), where we identify some \( g_k \in G_k \) with its inclusions into \( G_{k+1}, G_{k+2}, \) etc. If we let \( R_k \) denote the representation of \( G_k \) on \( \mathcal{F}_k \) given by right translation

\[
R_k(g)f(Z) = f(Zg), \quad Z \in \mathbb{C}^{m \times k}, \ g \in G_k.
\]

Then the following diagram commutes

\[
\begin{array}{ccc}
G_k \times \mathcal{F}_k & \xrightarrow{R_k} & \mathcal{F}_k \\
\downarrow j_{k+1} \times i_{k+1} & & \downarrow i_{k+1} \\
G_{k+1} \times \mathcal{F}_{k+1} & \xrightarrow{R_{k+1}} & \mathcal{F}_{k+1}
\end{array}
\] (4)

and so the representation \( R = \varinjlim R_k \) of \( G_\infty \) on \( \mathcal{F}_\infty \) is well defined by

\[
R([g_k])[f_k] = [R_k(g_k)f_k],
\] (5)

called the \textit{inductive limit} of the representations \( R_k \) on \( \mathcal{F}_\infty \), and we have commutativity of the diagram

\[
\begin{array}{ccc}
G_k \times \mathcal{F}_k & \xrightarrow{R_k} & \mathcal{F}_k \\
\downarrow j_k \times i_k & & \downarrow i_k \\
G_\infty \times \mathcal{F}_\infty & \xrightarrow{R} & \mathcal{F}_\infty
\end{array}
\]
Let $D_k \subset G_k$ be the diagonal subgroup. Let $\mathbf{Z}^+_k \subset G_k$ be the unipotent subgroup of upper triangular matrices with ones along the main diagonal and let $\mathbf{Z}^-_k$ be the analogous lower triangular subgroup. If $(M) = (M_1, \ldots, M_k)$ is any collection of integers, we define a holomorphic character

$$\pi^{(M)}(d) = d_1^{M_1}d_2^{M_2} \cdots d_k^{M_k} \quad d \in D_k.$$ 

In this context an element $f \in \mathcal{F}_k$ is said to be a weight vector of the representation $R_k$ with weight $(M)$ if

$$[R_k(d)f](Z) = f(Zd) = \pi^{(M)}(d)f(Z), \quad \forall \ d \in D_k.$$ 

If $f$ is a weight vector, and if

$$[R_k(\zeta)f](Z) = f(Z\zeta) = f(Z), \quad \forall \ \zeta \in \mathbf{Z}_k^+$$

then $f \in \mathcal{F}_k$ is a said to be a highest weight vector of the representation $R_k$. Similarly if $f$ is a weight vector, and if

$$[R_k(\zeta)f](Z) = f(Z\zeta) = f(Z), \quad \forall \ \zeta \in \mathbf{Z}_k^-$$

then $f \in \mathcal{F}_k$ is a said to be a lowest weight vector of the representation $R_k$. Since $G_k = U(k)$ or $GL(k, \mathbb{C})$, each irreducible representation of $G_k$ in $\mathcal{F}_k$ is finite dimensional and so admits a “unique” (up to multiplication by a nonzero scalar) highest weight vector with highest weight $(m) = (m_1, m_2, \ldots, m_k)$, and a unique lowest weight vector with lowest weight $(m_k, m_{k-1}, \ldots, m_1)$. This highest (or lowest) weight then characterizes each irreducible representation of $G_k$, and is called the signature of the representation. By the
Borel-Weil theorem a necessary and sufficient condition for \((m)\) to be the highest weight of an irreducible polynomial representation of \(G_k\) on \(\mathcal{F}_k\) is that \(m_1 \geq m_2 \geq \ldots \geq m_k \geq 0\). \(V^{(m)}\) is then cyclically generated as a \(G_k\)-module by the action of \(G_k\) on any one of its elements, in particular on its highest (or lowest) weight vector. Let \(B_k \subset GL(k,\mathbb{C})\) be the Borel subgroup of lower triangular matrices and for a \(k\)-tuple of non-negative integers \((m) = (m_1, m_2, \ldots, m_k)\) we define a holomorphic character

\[
\pi^{(m)}(b) = b_1^{m_1}b_2^{m_2} \ldots b_k^{m_k} b \in B_k. \tag{6}
\]

As a consequence of the Borel-Weil theorem (see for example [13]), any irreducible holomorphic representation of \(G_k\) with signature \((m) = (m_1, m_2, \ldots, m_n)\) can be explicitly realized as the representation \(R_k\) on the subspace of polynomial functions in \(\mathcal{F}_k = \mathcal{F}(\mathbb{C}^{n \times k})\) which satisfy the covariant condition

\[
f(bZ) = \pi^{(m)}(b)f(Z), \quad b \in B_n. \tag{7}
\]

We denote this subspace by \(V_k^{(m)}\), the restriction of \(R_k\) to this subspace by \(R_k^{(m)}\), and where necessary we explicitly designate this irreducible representation by the pair \((R_k^{(m)}, V_k^{(m)})\).

For each \(k = 1, 2, \ldots,\) let \(V_k\) be a subspace of \(\mathcal{F}_k\), on which the representation \(R_k\) of \(G_k\) is irreducible. Suppose also that the following diagram commutes

\[
\begin{array}{ccc}
G_k \times V_k & \xrightarrow{R_k} & V_k \\
\downarrow{j_k^{k+1} \times i_k^{k+1}} & & \downarrow{i_k^{k+1}} \\
G_{k+1} \times V_{k+1} & \xrightarrow{R_{k+1}} & V_{k+1}
\end{array} \tag{8}
\]
or equivalently, that the restriction of $R_{k+1}$ to $G_k$ contains a representation equivalent to $R_k$. In this case we write $V_k \preceq V_{k+1}$, and it is well documented in the literature that the representation $R = \lim \rightarrow R_k$ of $G_\infty$ on $V = \lim \rightarrow V_k$ is also irreducible. (For detailed expositions of inductive limit representations see [8] [2] [5].)

If $V_k$ is an irreducible representation of $G_k$ with signature

$$(m) = (m_1, m_2, \ldots, m_k) \text{ with } m_1 \geq m_2 \geq \ldots \geq m_k \geq 0,$$

and if $V_{k+1}$ is an irreducible representation of $G_{k+1}$ with signature $(h) = (h_1, h_2, \ldots, h_n, h_{n+1})$ it is also well known that $V_k \preceq V_{k+1}$ or equivalently written $(m) \preceq (h)$ if and only if

$$h_i \geq m_i \geq h_{i+1}, \quad i = 1, \ldots, k.$$ 

In particular, if $(m_1, m_2, \ldots, m_k)$ is the signature of an irreducible representation of $G_k$ and $(m_1, m_2, \ldots, m_k, 0)$ is the signature of an irreducible representation of $G_{k+1}$, then

$$(m_1, m_2, \ldots, m_k) \preceq (m_1, m_2, \ldots, m_k, 0)$$

and it is easy to see that if $f_{\max}$ is a highest weight vector for an irreducible representation of $G_k$ with highest weight $(m_1, m_2, \ldots, m_k)$ then $f_{\max}$ is also a highest weight vector of the irreducible representation of $G_{k+1}$ with highest weight $(m_1, m_2, \ldots, m_k, 0)$. We denote the inductive limit of the representations
(m_1, m_2, \ldots, m_k) \preceq (m_1, m_2, \ldots, m_k, 0) \preceq (m_1, m_2, \ldots, m_k, 0, 0) \preceq \cdots

by

(m_1, m_2, \ldots, m_k, 0, \ldots) = (m_1, m_2, \ldots, m_k, \vec{0}) = (m)^\infty

and realize this representation as the submodule of \( \mathcal{F}_\infty \) generated by the action of \( G_\infty \) on this highest weight vector. In the sequel we may also require more explicit notation: If \( (\underbrace{m_1, m_2, \ldots, m_l, 0, \ldots, 0}_k) \) is the signature of an irreducible representation of \( G_k \) call the integers \( m_1, m_2, \ldots, m_l, 0 \) the entries, we say \( l \) is the length of the signature (i.e., if the signature has at most \( l \) non-zero entries) and write

\[
\underbrace{(m_1, m_2, \ldots, m_l, 0, \ldots, 0)}_k = (m)^k_l
\]
or just \((m)^k\) if it is unnecessary to specify the length. With this notation we denote the signature of the inductive limit of the representations

\[
(m_1, m_2, \ldots, m_l) \preceq (m_1, m_2, \ldots, m_l, 0) \preceq (m_1, m_2, \ldots, m_l, 0, 0) \preceq \cdots
\]

\[
= (m)^k_l \preceq (m)^{k+1}_l \preceq (m)^{k+2}_l \preceq \cdots
\]

by

\[
(m)^\infty_l = (m_1, m_2, \ldots, m_l, 0, 0, 0, \ldots) = (m_1, m_2, \ldots, m_l, \vec{0}).
\]
III Stability of spectral decompositions

We motivate this section with the following example. It is readily computed using one of the standard formulae (for example \[13\]) that the tensor product of the irreducible representations of $G_2 = U(2)$ or $GL(2, \mathbb{C})$ decomposes as a direct sum

\[
(1, 0) \otimes (2, 0) \otimes (2, 0) \otimes (3, 0) = (8, 0) + 3(7, 1) + 5(6, 2) + 5(5, 3) + 2(4, 4) \quad (9)
\]

and the tensor product of irreducible representations of $G_4$

\[
(1, 0, 0, 0) \otimes (2, 0, 0, 0) \otimes (2, 0, 0, 0) \otimes (3, 0, 0, 0)
= (8, 0, 0, 0) + 3(7, 1, 0, 0) + 5(6, 2, 0, 0) + 5(5, 3, 0, 0) + 2(4, 4, 0, 0) \\
+ 3(6, 1, 1, 0) + 6(5, 2, 1, 0) + 5(4, 3, 1, 0) + 3(4, 2, 2, 0) + 2(3, 3, 2, 0) \\
+ (5, 1, 1, 1) + 2(4, 2, 1, 1) + (3, 3, 1, 1) + (3, 2, 2, 1) \quad (10)
\]

But notice that the first line of (10) is just (9), the spectrum of $G_2$ embedded in the spectrum of $G_4$. In this case we say that the spectrum of $G_4$ contains the spectrum of $G_2$, or that the spectrum of $G_2$ appears in the spectrum of $G_4$. Furthermore, it is routine to check that the spectral decomposition of
irreducible representations of $G_5$ is given by
\[(1, 0, 0, 0, 0) \otimes (2, 0, 0, 0, 0) \otimes (2, 0, 0, 0, 0) \otimes (2, 0, 0, 0, 0) \otimes (3, 0, 0, 0, 0) \]
\[= (8, 0, 0, 0, 0) + 3(7, 1, 0, 0, 0) + 5(6, 2, 0, 0, 0) + 5(5, 3, 0, 0, 0) + 2(4, 4, 0, 0, 0) \]
\[+ 3(6, 1, 1, 0, 0) + 6(5, 2, 1, 0, 0) + 5(4, 3, 1, 0, 0) + 3(4, 2, 2, 0, 0) \]
\[+ 2(3, 3, 2, 0, 0) + (5, 1, 1, 1, 0) + 2(4, 2, 1, 1, 0) + (3, 3, 1, 1, 0) \]
\[+ (3, 2, 2, 1, 0) \]  
(11)

and that the corresponding spectral decompositions of $G_6, G_7, \ldots$ are the same, i.e. composed entirely of the embedding of the spectrum of $G_4$. In this case we say the spectral decomposition stabilizes.

**Proposition III.1.** If $(\alpha)^k = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and $(\beta)^k = (\beta_1, \ldots, \beta_k)$ are the signatures of irreducible representations of $G_k$, and if $(\alpha)^{k+1} = (\alpha_1, \alpha_2, \ldots, \alpha_k, 0)$ and $(\beta)^{k+1} = (\beta_1, \ldots, \beta_k, 0)$ are the signatures of irreducible representations of $G_{k+1}$, then the spectrum of $(\alpha)^k \otimes (\beta)^k$ appears in the spectrum of $(\alpha)^{k+1} \otimes (\beta)^{k+1}$. Furthermore, the spectrum of $(\alpha)^K \otimes (\beta)^K$ stabilizes for $K$ sufficiently large.

**Proof.** We first note that, in the special case where $(\alpha)^k = (\alpha_1, 0, \ldots, 0)$ by [15] the spectral decomposition of $(\alpha)^k \otimes (\beta)^k$ is given by the ‘Weyl formula’, which is equivalent to applying the multiplier $\Gamma_{\alpha_1}$ to the signature $(\beta)^k = (\beta_1, \ldots, \beta_k)$ where

$$\Gamma_{\alpha_1}(\beta_1, \ldots, \beta_k) = \sum_{\nu_1+\ldots+\nu_k=\alpha_1 \atop 0 \leq \nu_i \leq s_i} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k)$$
Here, and in what follows of this proof, the Weyl formula also requires the condition that $0 \leq \nu_{i+1} \leq s_i$ where $s_i = m_i - m_{i+1}$, and we will refer to a multiplier of this type as a *simple multiplier*.

Now applying this simple multiplier to the signature $(\beta)^{k+1}_k$ we have

$$\Gamma_{\alpha_1}(\beta_1, \ldots, \beta_k, 0) = \sum_{\nu_1 + \cdots + \nu_k + \nu_{k+1} = \alpha_1 \atop 0 \leq \nu_{i+1} \leq s_i} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1})$$

$$= \sum_{\nu_1 + \cdots + \nu_k + \nu_{k+1} = \alpha_1 \atop \nu_{k+1} = 0} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1})$$

$$+ \sum_{\nu_1 + \cdots + \nu_k + \nu_{k+1} = \alpha_1 \atop \nu_{k+1} \neq 0} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1})$$

But the first sum, with $\nu_{k+1} = 0$, is just the spectrum of $(\alpha)^{k}_1 \otimes (\beta)^{k}_k$ contained in the spectrum of $(\alpha)^{k+1}_1 \otimes (\beta)^{k+1}_k$

We next note that a similar situation occurs when we apply a second simple multiplier $\Gamma_{\alpha_2}$ to the above spectral decomposition $(\alpha)^{k+1}_1 \otimes (\beta)^{k+1}_k$. That is, the sums are grouped into those terms whose last entry is zero, and those terms whose last entry is non-zero;

$$\Gamma_{\alpha_2}[(\alpha)^{k+1}_1 \otimes (\beta)^{k+1}_k] = \Gamma_{\alpha_2}(\Gamma_{\alpha_1}(\beta_1, \ldots, \beta_k, 0))$$
\[ \Gamma_{\alpha_2} \left[ \sum_{\nu_1 + \ldots + \nu_k + \nu_{k+1} = \alpha_1} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1}) \right. \\
+ \sum_{\nu_1 + \ldots + \nu_k + \nu_{k+1} = \alpha_1} (\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1}) \right] \\
= \sum_{\nu_1 + \ldots + \nu_k + \nu_{k+1} = \alpha_1} \Gamma_{\alpha_2}(\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0) \\
+ \sum_{\nu_1 + \ldots + \nu_k + \nu_{k+1} = \alpha_1} \Gamma_{\alpha_2}(\beta_1 + \nu_1, \ldots, \beta_k + \nu_k, 0 + \nu_{k+1}) \\
= \sum_{\nu_1 + \ldots + \nu_{k+1} = \alpha_1} \sum_{\mu_1 + \ldots + \mu_{k+1} = \alpha_2} (\beta_1 + \nu_1 + \mu_1, \ldots, \beta_k + \nu_k + \mu_k, 0) \\
+ \sum \text{(other terms involving signatures whose last entry is non-zero)} \\
\]

We then extend this idea to the general case where the spectral decomposition of \((\alpha)^k \otimes (\beta)^k\) is given by applying the multiplier \(\Gamma_{\alpha_k}\) to the signature \(\beta^k\) where \(\Gamma_{\alpha_k}\) is a compound multiplier computed as the ‘Weyl Determinant’:

\[ \Gamma_{\alpha_k} = \begin{vmatrix} \Gamma_{\alpha_1} & \Gamma_{\alpha_1+1} & \cdots & \Gamma_{\alpha_1+(k-1)} \\ \Gamma_{\alpha_2-1} & \Gamma_{\alpha_2} & \cdots & \Gamma_{\alpha_2+(k-2)} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma_{\alpha_k-(k-1)} & \Gamma_{\alpha_k-(k-2)} & \cdots & \Gamma_{\alpha_k} \end{vmatrix} \]
Here the simple multipliers $\Gamma_\alpha$ are regarded as permutable operators and the determinant is expanded in the usual way, with $\Gamma_0 = 1$ and $\Gamma_\alpha = 0$ for $\alpha < 0$. From this last statement it is obvious that the compound multiplier $\Gamma^{k+1}_\alpha$ is equal to $\Gamma^k_\alpha$ since the $k+1$st row used to compute the determinant corresponding to $\Gamma^{k+1}_\alpha$ is just $(0, \ldots, 0, 1)$.

Now for notational convenience we set the simple multiplier $\Gamma_{i,j} = \Gamma_{\alpha_i - (i-j)}$ and using the usual formula for determinant (summing over $S_k$, the symmetric group on $k$ symbols) we have

$$\Gamma^{k+1}_\alpha = \Gamma^k_\alpha = \sum_{\sigma \in S_k} sgn(\sigma) \Gamma_{\alpha_{\sigma(1)}} \cdots \Gamma_{\alpha_{\sigma(k)}}$$

So

$$\Gamma^{k+1}_\alpha (\beta^{k+1})$$

$$= \sum_{\sigma \in S_n} sgn(\sigma) \Gamma_{\alpha_{\sigma(1)}} \cdots \Gamma_{\alpha_{\sigma(k)}} (\beta_1, \ldots, \beta_k, 0)$$

$$= \sum_{\sigma \in S_k} sgn(\sigma) \sum_{\nu_1+\ldots+\nu_{k+1}=\alpha_1} \cdots \sum_{\mu_1+\ldots+\mu_{k+1}=\alpha_k} (\beta_1+\nu_1+\ldots+\mu_1, \ldots, 0+\nu_{k+1}+\ldots+\mu_{k+1})$$

$$= \sum_{\sigma \in S_k} sgn(\sigma) \sum \cdots \sum (\text{signatures whose last entry is zero})$$

$$+ \sum_{\sigma \in S_k} sgn(\sigma) \sum \cdots \sum (\text{signatures whose last entry is non-zero})$$

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But the sum $\sum_{13}$ is just the spectrum of $(\alpha)_k^k \otimes (\beta)_k^k$ appearing in the spectrum of $(\alpha)_k^{k+1} \otimes (\beta)_k^{k+1}$. Finally, the requirement that $0 \leq \nu_{i+1} \leq m_i - m_{i+1}$ guarantees that the application of a simple multiplier to a signature $(m_1, \ldots, m_l, 0, \ldots, 0)$ extends the length of the signature by at most one, since $0 \leq \nu_{l+1} \leq (m_{l+1} - m_{l+2}) = 0$. Thus, since there are only $k$ sums in $(12)$, application of a compound multiplier corresponding to a signature of length $k$ decomposes the tensor product into a spectrum of signatures of length at most $l + k$, proving that the spectrum stabilizes.

\[\square\]

IV A reciprocity theorem

According to [11] we have the following theorem regarding dual representations of Bargmann-Segal-Fock spaces.

**Theorem IV.1.** Let

\[G_1 \subset G_2 \subset \cdots \subset G_k \subset G_{k+1} \subset \cdots\]

be a chain of compact classical groups. Let $G_{\infty}$ denote the inductive limit of the $G_k$’s. Let $R_{G_{\infty}}$ and $R'_{G'}$ be given dual representations on $F_{n\times\infty}$. Let $H_{\infty}$ be the inductive limit of a chain of compact subgroups

\[H_1 \subset H_2 \subset \cdots \subset H_k \subset H_{k+1} \subset \cdots\]

with $H_k \subset G_k$, and let $R_{H_{\infty}}$ be the representation of $H_{\infty}$ on $F_{n\times\infty}$ obtained by restricting $R_{G_{\infty}}$ to $H_{\infty}$. If there exists a group $H' \supset G'$ and a representation
$R'_{H'}$ on $F_{n \times \infty}$ such that $R'_{H'}$ is the dual to $R_{H_{\infty}}$ and $R'_{G'}$ is the restriction of $R'_{H'}$ to the subgroup $G'$ of $H'$ then we have the following multiplicity free decompositions of $F_{n \times \infty}$ into isotypic components

$$F_{n \times \infty} = \bigoplus_{(\lambda)} \mathcal{I}^{(\lambda)}_{n \times \infty} = \bigoplus_{(\mu)} \mathcal{I}^{(\mu)}_{n \times \infty}$$

where $(\lambda)$ is a common irreducible signature of the pair $(G', G_{\infty})$ and $(\mu)$ is a common signature of the pair $(H', H_{\infty})$.

If $\lambda_{G_{\infty}}$ (resp. $\lambda'_{G'}$) denotes an irreducible unitary representation of class $(\lambda)$ and $\mu_{H_{\infty}}$ (resp. $\mu'_{H'}$) denotes an irreducible unitary representation of class $(\mu)$, then the multiplicity

$$\text{dim} \left[ \text{Hom}_{H_{\infty}} (\mu_{H_{\infty}} : \lambda_{G_{\infty}} |_{H_{\infty}}) \right]$$

of the irreducible representation $\mu_{H_{\infty}}$ in the restriction to $H_{\infty}$ of the representation $\lambda_{G_{\infty}}$ is equal to the multiplicity

$$\text{dim} \left[ \text{Hom}_{G'} (\lambda'_{G'} : \mu'_{H' | G'}) \right]$$

of the irreducible representation $\lambda'_{G'}$ in the restriction of the representation $\mu'_{H'}$.

Note that $G'$ and $H'$ are finite dimensional Lie groups and that we have a similar theorem for the pairs $(G', G_k)$ and $(H', H_k)$ where $G'$ and $H'$ remain fixed for all $k$. It follows that

$$\text{dim} \left[ \text{Hom}_{H_{\infty}} (\mu_{H_{\infty}} : \lambda_{G_{\infty}} |_{H_{\infty}}) \right]$$
remains constant and the spectral decomposition of $\lambda_{G_k \mid H_k}$ stabilizes for $k$ large. To apply this theorem to our problem we first let

$$G_\infty = U_\infty \times \cdots \times U_\infty$$

acting as exterior tensor product representations on

$$V^{(m_1)_\infty} \otimes \cdots \otimes V^{(m_r)_\infty} \subset F_{n \times \infty},$$

and $G' = U(p_1) \times \cdots U(p_r)$ acting on $F_{n \times \infty}$.

Then $H_\infty = U_\infty$ is the interior tensor product representation on

$$V^{(m_1)_\infty} \otimes \cdots \otimes V^{(m_r)_\infty},$$

and $H' = U(n)$, where recall that $n = p_1 + p_2 + \cdots + p_r$. This gives the multiplicity of the representation of $U_\infty$ with signature $(m)_\infty$ in $(m_1)_\infty \otimes \cdots \otimes (m_r)_\infty$ in terms of the multiplicity of the representation of the corresponding representations of $U(p_1) \times \cdots U(p_r)$ in the corresponding representation on $U(n)$. Next we let

$$G_\infty = U_\infty \times \cdots \times U_\infty$$

acting as exterior tensor product representations on

$$V^{(m_1)_\infty} \otimes \cdots \otimes V^{(m_r)_\infty} \otimes V^{(m')_\infty} \subset F_{n \times \infty},$$

and $G' = U(p_1) \times \cdots \times U(p_r) \times U(q)$ acting on $F_{n \times \infty}$. Then $H_\infty = U_\infty$ and $H' = U(p, q)$ where $p + q = n$, $p_1 + \cdots + p_r = p$. This gives the multiplicity of
the representation of $U_\infty$ with signature $(0, \ldots, 0, \ldots)^\infty$ in the tensor product $(m_1)^\infty \otimes \cdots \otimes (m_r)^\infty \otimes (m^\vee)^\infty$ in terms of the multiplicity of the representation with signature $(m_1)^n \otimes \cdots \otimes (m_r)^n \otimes (m^\vee)^n$ of $U(p_1) \times \cdots \times U(p_r) \times U(q)$ in the holomorphic discrete series of $U(p, q)$ with signature (lowest highest weight) \( (0, \ldots, 0, 0, \ldots, 0) \). Note that these two applications of this theorem can be used together to give another proof of Theorem VI.1 of Section VI.

V Realization of the contragredient representation

A representation $\rho$ of any group $G$ on a vector space $V$ induces in a natural way a representation $\rho^*$ (said to be contragredient to $\rho$) on its dual space $V^*$ by

$$\rho^*(g)\phi(v) = \phi(\rho(g^{-1})v) \quad \phi \in V^*, \ g \in G.$$ 

In this section, by making a formal change of variable, we are able to realize $R^*_{k}$ (the representation contragredient to $R_k$) as the representation $R^\vee_{k}$ on a subspace of polynomial functions of the Fock space $F^\vee_k$.

Let $\langle \cdot | \cdot \rangle$ be the inner product on the space $F_k$ given by (3) or the equivalent inner product (1). Then for any $f \in F_k$ and for each $k = 1, 2, 3, \ldots$ the mapping

$$\Phi : F_k \longrightarrow F_k^*$$
given by

\[ [\Phi f](h) = \langle h | f \rangle \quad h \in \mathcal{F}_k \]

is a conjugate linear isomorphism (or anti-isomorphism) from \( \mathcal{F}_k \) onto its dual space \( \mathcal{F}_k^\ast \), and it is routine to check (see \([10]\)) that \( \Phi \) intertwines the representations \( R_k \) and \( R_k^\ast \). It follows that if \( (R_k^{(m)}, V_k^{(m)}) \) is an irreducible representation of \( G_k \) and if \( V_k^{(m*)} = \Phi(V^{(m)}) \), then \( (R_k^{(m*)}, V_k^{(m*)}) \) is also an irreducible representation of \( G_k \). It is shown in appendix A of \([10]\) that the highest weight vector of \( (R_k^{(m)}, V_k^{(m)}) \) with highest weight \( (m_1, m_2, \ldots, m_k) \) is mapped to the lowest weight vector of \( (R_k^{(m*)}, V_k^{(m*)}) \) with weight \( (-m_1, -m_2, \ldots, -m_k) \), and the lowest weight vector of \( (R_k^{(m)}, V_k^{(m)}) \) with weight \( (m_k, \ldots, m_1) \) is mapped onto the highest weight vector of \( (R_k^{(m*)}, V_k^{(m*)}) \) with weight \( (-m_k, \ldots, -m_1) \).

We will realize \( (R_k^{(m*)}, V_k^{(m*)}) \) on a Fock space \( \mathcal{F}_k^\lor \) as the representation \( (R_k^{(m)}, V_k^{(m)}) \) constructed as follows.

Define \( (\mathbb{C}^{n \times k})^\lor \) as the vector space of complex \( n \times k \) matrices with the reverse ordering

\[
\begin{pmatrix}
w_{n,k} & \cdots & w_{n,1} \\
\vdots & \ddots & \vdots \\
w_{1,k} & \cdots & w_{1,1}
\end{pmatrix}
\in (\mathbb{C}^{n \times k})^\lor
\]

Let \( s = s_k \) be the \( k \times k \) matrix with ones along the off diagonal and zeros elsewhere
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & \ddots & & & 0 \\
\vdots & & \ddots & & \vdots \\
0 & & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} \in \mathbb{C}^{k \times k} \quad (14)
\]

For future reference one easily checks that if we set

\[
Z = \begin{pmatrix}
w_{11} & \cdots & w_{1k} \\
\vdots & & \vdots \\
w_{n1} & \cdots & w_{nk}
\end{pmatrix}
\]

that \( W = s_n Z s_k \) and that \( s = s^{-1} = s^T \). We then let \( \mathcal{F}_k^\vee \) be the Hilbert space of holomorphic square-integrable functions on \((\mathbb{C}^{n \times k})^\vee\) and define the action \( R_k^\vee \) of \( G_k \) on \( \mathcal{F}_k^\vee \) as

\[
[R_k^\vee(g)f](w) = f(w(sgs^\vee)) = f(w(sg^\vee s)).
\]

The embedding of \((\mathbb{C}^{n \times k})^\vee\) into \((\mathbb{C}^{n \times (k+1)})^\vee\) given by

\[
W \mapsto \begin{pmatrix}
0 & \vdots & W \\
0 & & \ddots & \vdots \\
& & \ddots & 0 \\
& & & 0
\end{pmatrix} \in \mathbb{C}^{n \times (k+1)}^\vee
\]

then induces an embedding \( i_{k+1}^\vee : \mathcal{F}_k^\vee \rightarrow \mathcal{F}_{k+1}^\vee \), we have commutativity of the diagram

\[
\begin{array}{ccc}
G_k \times \mathcal{F}_k^\vee & \xrightarrow{R_k^\vee} & \mathcal{F}_k^\vee \\
\downarrow{j_{k+1}^\vee \times i_{k+1}^\vee} & & \downarrow{i_{k+1}^\vee} \\
G_{k+1} \times \mathcal{F}_{k+1}^\vee & \xrightarrow{R_{k+1}^\vee} & \mathcal{F}_{k+1}^\vee
\end{array}
\]
as in (4), and the inductive limit representation $R^\vee = \lim_{\to} R^\vee_k$ of $G$ on $F^\vee_\infty = \lim_{\to} F^\vee_k$ is well defined as in (5).

We remark here that the space $F^\vee_\infty$ defined above and the space $F_\infty$ defined in Section [4] are certainly equal as sets, but are somewhat different as algebraic objects, being induced by different embeddings. In what follows, the arguments presented in developing properties of the various inductive limit representations are readily modified to any situation.

Let $f_{\text{max}}$ be the highest weight vector of $(R^m_k, V^m_k)$ with highest weight $(m_1, m_2, \ldots, m_k)$. Then by definition

$$R(d) f_{\text{max}}(Z) = d_1^{m_1} \cdots d_k^{m_k} f_{\text{max}}(Z) = \pi^{(m_1, m_2, \ldots, m_k)}(d) f_{\text{max}}(Z).$$

Define $f_{\text{min}}(W) := f_{\text{max}}(s_n W s_k) = f_{\text{max}}(Z)$, so that $f_{\text{max}}(Z) = f_{\text{min}}(s_n Z s_k)$. Then

$$R^\vee_k(d) f_{\text{min}}^\vee(W) = f_{\text{min}}^\vee(W d^\vee)$$

$$= f_{\text{max}}(s_n (W d^\vee) s), \text{ since } s_n W s = Z \text{ and } d^\vee = d^{-1}$$

$$= f_{\text{max}}(Z d^{-1})$$

$$= \pi^{(m_1, m_2, \ldots, m_k)}(d^{-1}) f_{\text{max}}(Z)$$

$$= \pi^{(-m_1, -m_2, \ldots, -m_k)}(d) f_{\text{min}}(s_n Z s_k)$$

$$= \pi^{(-m_1, -m_2, \ldots, -m_k)}(d) f_{\text{min}}^\vee(W)$$

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And for $\zeta \in \mathbb{Z}^{-}$

$$R^\vee_k(\zeta)f^\vee_{\text{min}}(W) = f^\vee_{\text{min}}(W s\zeta^\vee s)$$

$$= f_{\text{max}}(s_n(W s\zeta^\vee s)s)$$

$$= f_{\text{max}}(Z\zeta^\vee)$$

$$= f_{\text{max}}(Z)$$

$$= f^\vee_{\text{min}}(W)$$

since if $\zeta \in \mathbb{Z}^-$, then $\zeta^\vee \in \mathbb{Z}^+$. Thus $f^\vee_{\text{min}}$ is a lowest weight vector for the representation $R^\vee_k$, so if we let $V_k^{(m^\vee)}$ be the $G_k$-submodule generated by the action $R^\vee_k$ on $f^\vee_{\text{min}}$, then $(R^\vee_k, V_k^{(m^\vee)})$ is an irreducible representation of $G_k$ characterized by its lowest weight $(-m_1, -m_2, \ldots, -m_k)$, and it follows that this representation is equivalent to the contragredient representation on the dual space $(R_k^{(m^\ast)}, V_k^{(m^\ast)})$, with the same lowest weight $(-m_1, -m_2, \ldots, -m_k)$. Furthermore, if $b \in B_n$, and if we set $\tilde{b} = s_n b s_n$, then, using an argument similar to the one above, we see that

$$f^\vee_{\text{min}}(\tilde{b}W) = \pi^{(m)}(b)f^\vee_{\text{min}}(W)$$

and thus the space $(R^\vee_k, V_k^{(m^\vee)})$ can be characterized as the subspace of polynomial functions that transform covariantly with respect to the Borel subgroup, as in (6). We remark here that we refer to $f^\vee_{\text{min}}$ as a lowest weight vector because it is invariant under right translation by the subgroup $\mathbb{Z}^-$ which corresponds to the notion of lowest weight using the usual lexico-
graphic ordering. It is strictly a matter of choice whether or not to refer to it as a highest weight vector with respect to the reverse lexicographic ordering.

With this realization, if $(m) = (m_1, m_2, \ldots, m_k)$ is the signature of an irreducible representation of $G_k$, then its contragredient representation has signature $(m^\vee) = (-m_1, -m_2, \ldots, -m_k)$, and it is routine to check that

$$(-m_1, \ldots, -m_k) \prec (-m_1, \ldots, -m_k, 0) \prec (-m_1, \ldots, -m_k, 0, 0) \prec \cdots$$

i.e. the appropriate diagram (see (8)) commutes, and so the inductive limit of the irreducible representations (15) is an irreducible representation of $G_\infty$ with signature

$$(m^\vee)^\infty =: (-m_1, -m_2, \ldots, -m_l, \overrightarrow{0})$$

generated by the action $R^\vee$ on the vector $f^\vee_{\min}$. We will adopt the convention of referring to this as the representation contragredient to the irreducible representation with signature $(m)^\infty = (m_1, m_2, \ldots, m_k, \overrightarrow{0})$, although it is the inductive limit of contragredient representations. We summarize with

**Theorem V.1.** If the irreducible representation of $G_\infty$ with signature

$$(m_1, m_2, \ldots, m_l, \overrightarrow{0})$$

is the inductive limit of the representations

$$(m_1, m_2, \ldots, m_l) \preceq (m_1, m_2, \ldots, m_l, 0) \preceq (m_1, m_2, \ldots, m_l, 0, 0) \preceq \cdots$$

then the inductive limit of contragredient representations

$$(m_1, m_2, \ldots, m_l)^* \preceq (m_1, m_2, \ldots, m_l, 0)^* \preceq (m_1, m_2, \ldots, m_l, 0, 0)^* \preceq \cdots$$
is an also irreducible representation of $G_\infty$ with signature

$$(-m_1, -m_2, \ldots, -m_t, \underbrace{0, \ldots, 0}_{m}).$$

We illustrate this idea with the following example. For each $k = 1, 2, 3, \ldots$
consider $F_k = F(C^{1 \times k})$. If

$$f^{(m)}(Z) = z_1^m, \quad Z = (z_1, z_2, \ldots, z_k) \in C^{1 \times k}$$

it is easy to check that

$$f^{(m)}(Zd) = d^{m}a_1^m = \pi^{(m, 0, \ldots, 0)}(d)f^{(m)}(Z)$$

and

$$f^{(m)}(Z\zeta) = f^{(m)}(Z), \quad \zeta \in Z^+$$

so that $f^{(m)}(Z)$ is a highest weight vector of the representation, with highest weight $(m, 0, \ldots, 0)$. Right translation of $f^{(m)}$ by $G_k$ generates the finite dimensional vector space $P^{(m)}(C^{1 \times k})$, of homogeneous polynomials of degree $m$, so that $V^{(m, 0, \ldots, 0)} = P^{(m)}(C^{1 \times k})$ is an irreducible representation of $G_k$ with signature $(m, 0, \ldots, 0)$. Now $P^{(m)}(C^{1 \times k})$ embeds isometrically into $P^{(m)}(C^{1 \times (k+1)})$, which is also generated as a $G_{k+1}$-module by right translation of the highest weight vector $z_1^m$, which now has highest weight $(m, 0, \ldots, 0)$. Taking the inductive limit of the irreducible representations

$$(m) \varsubsetneq (m, 0) \varsubsetneq \ldots \varsubsetneq (m, 0, \ldots, 0) \varsubsetneq (m, 0, \ldots, 0) \varsubsetneq \ldots$$

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we obtain $V^{(m, \vec{0})}$, the irreducible representation of $G_\infty$ with signature

\[(m, \vec{0}) = (m, 0, 0, \ldots),\]

which is realized in $\mathcal{F}_\infty$ as the subspace of homogeneous polynomials of degree $m$, generated by the action $R$ of $G_\infty$ on the highest weight vector $f^{(m)}(Z) = z_1^m$.

\[V^{(m, \vec{0})} = P^{(m)}(z_1, z_2, \ldots), \quad Z \in \mathbb{C}^{1 \times \infty}\]

Now if $w = (w_k, \ldots, w_2, w_1) \in \mathbb{C}^{1 \times k \vee}$ set $f^{(-m)}(w) = w_1^m$.

If $d = \text{diagonal} \ (d_{11}, \ldots, d_{kk}) \in D_k$, then

\[\left[R_k^\vee(d)f^{(-m)} \right](w) = f^{(-m)}(w(sd^\vee s))\]

\[= (d_{11}^{-1}w_{11})^m = d_{11}^{-m}f^{(-m)}(w) = \pi^{(-m,0,\ldots,0)}(d)f^{(-m)}(Z)\]

and if $\zeta \in \mathbb{Z}^{-k}$ then

\[\left[R_k^\vee(\zeta)f^{(-m)} \right](w) = f^{(-m)}(w(s\zeta^\vee s))\]

\[= (\zeta_{11}w_{11})^m = f^{(-m)}(Z) \quad \text{since } \zeta_{11} = 1\]

Thus $f^{(-m)}$ is a lowest weight vector for the representation $R_k^\vee$ with lowest weight $(-m, \ldots, 0)$, and since this holds for all $k$, we denote the signature of the inductive limit of the representations

\[(-m) \prec (-m, 0) \prec \ldots \prec (-m, 0, \ldots, 0) \prec (-m, 0, \ldots, 0) \prec \ldots\]

by $(-m, 0, \ldots) = (-m, \vec{0})$, and the irreducible representation $V^{(-m, \vec{0})}$ is realized as the space of homogeneous polynomials of degree $m$ on $\mathbb{C}^{1 \times \infty \vee}$, generated by $f^{(-m)}$.

\[V^{(-m, \vec{0})} = P^{(m)}(\ldots, w_2, w_1), \quad w \in \mathbb{C}^{1 \times \infty \vee}\]

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VI Decomposing tensor products of irreducible representations

We now use this construction to realize the tensor product of inductive limits of irreducible representations. For

\[
Z^i = \begin{pmatrix}
    z_{11}^i & z_{12}^i & \cdots & z_{1k}^i \\
    \vdots & \vdots & \ddots & \vdots \\
    z_{p_1}^i & z_{p_2}^i & \cdots & z_{p_k}^i
\end{pmatrix} \in \mathbb{C}^{p_i \times k}
\]

set

\[
\left( \begin{array}{c}
    Z \\
    W
\end{array} \right) = \left( \begin{array}{c}
    Z^1 \\
    Z^2 \\
    \vdots \\
    Z^r
\end{array} \right) = \left( \begin{array}{c}
    z_{11}^1 & z_{12}^1 & \cdots & z_{1k}^1 \\
    \vdots & \vdots & \ddots & \vdots \\
    z_{p_1}^1 & z_{p_2}^1 & \cdots & z_{p_k}^1 \\
    w_{q_1} & w_{q_2} & \cdots & w_{q_1}
\end{array} \right) \in \mathbb{C}^{p \times k} \oplus \mathbb{C}^{q \times k^\vee}
\]

where \( p_1 + \cdots + p_r = p \) and \( p + q = n \). For economy of notation, we now let \( \mathcal{F}_k \) be the set of holomorphic square integrable functions on \( \mathbb{C}^{p \times k} \oplus \mathbb{C}^{q \times k^\vee} \) and define a representation of \( G_k \) on \( \mathcal{F}_k \) by

\[
[R_k \otimes R^\vee_k(g)f] \left( \begin{array}{c}
    Z \\
    W
\end{array} \right) = f \left( \begin{array}{c}
    Zg \\
    W(sg^\vee s)
\end{array} \right)
\]

(16)

We then obtain the inductive limit representation \( R \otimes R^\vee \) of the group \( G_\infty \) on \( \mathcal{F}_\infty = \lim_{k \to \infty} \mathcal{F}_k \) as the representation induced by the embedding of

\[
\mathbb{C}^{p \times k} \oplus \mathbb{C}^{q \times k^\vee} \longrightarrow \mathbb{C}^{p \times (k+1)} \oplus \mathbb{C}^{q \times (k+1)^\vee}
\]

given by
\[
\left(\begin{array}{c}
Z \\
W
\end{array}\right) \mapsto \left(\begin{array}{cccc}
z_{11} & z_{22} & \cdots & z_{1k} \\
\vdots & & & \vdots \\
z_{p1} & z_{p2} & \cdots & z_{pk} \\
0 & w_{qk} & \cdots & w_{q2} & w_{q1} \\
\vdots & & & \vdots \\
0 & w_{1k} & \cdots & w_{12} & w_{11}
\end{array}\right) \in \mathbb{C}^{p \times (k+1)} \oplus \mathbb{C}^{q \times (k+1)}^\vee
\]

and the embedding of \( G_k \to G_{k+1} \) given by (3).

If \((m_1, m_2, \ldots, m_{p_i}, 0, \ldots, 0)\) is the signature of an irreducible representation of \( G_k \), and if \((m)^\vee = (-m_1, -m_2, \ldots, -m_q, 0, \ldots, 0)\) is the signature of the representation contragredient to \((m) = (m_1, m_2, \ldots, m_q, 0, \ldots, 0)\) we form the \( n \)-tuple of positive integers

\[
\mu = (m_1^1, m_1^2, \ldots, m_{p_1}^1, m_1^2, \ldots, m_{p_2}^2, \ldots, m_r^1, \ldots, m_{p_r}^r, m_1, m_2, \ldots, m_q)
\]

If \( B_i, i = 1, \ldots, r \) is the Borel subgroup of lower triangular matrices of \( GL(p_i, \mathbb{C}) \) and if \( B_q \) is the Borel subgroup of \( GL(q, \mathbb{C}) \), for \( b \in B_q \) we first set \( \tilde{b} = sbs \), where \( s = s_q \) as in (14), and then set \( \tilde{B}_q = \{ \tilde{b} \mid b \in B_q \} \). The group \( B_1 \times B_2 \times \cdots \times B_r \times \tilde{B}_q \) can then be identified with the group of all lower triangular block matrices \( \beta \) of the form

\[
\beta = \left(\begin{array}{ccc}
b_1 & & 0 \\
b_2 & & \ddots \\
0 & b_r & \tilde{b}
\end{array}\right) \quad b_i \in B_i, \ b \in B_q
\]

\[31\]
where \( p_1 + \cdots + p_r = p \) and \( p + q = n \). It is a consequence of the Borel-Weil Theorem (see for example [10]) that for \( k \geq n \) the tensor product of irreducible \( G_k \) modules

\[
V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}
\]  

(20)

with the \( G_k \)-action given by (14), can be realized as the subspace of polynomial functions \( f \in \mathcal{F}_k \) which, using the terminology of this paper, satisfy the covariant condition

\[
f \left( \beta \begin{pmatrix} Z \\ W \end{pmatrix} \right) = \pi^{(\mu)}(\beta) f \left( \begin{pmatrix} Z \\ W \end{pmatrix} \right)
\]  

(21)

for \( \mu \) as in (13) and where \( \pi^{(\mu)}(\beta) = (b_1)^{m^1_1} \cdots (b_q)^{m^q_q} \), as in (3). Since this covariant condition holds for all \( k \geq n \), we realize the tensor product of irreducible \( G_\infty \)-modules

\[
V^{(m^1)^\infty} \otimes \cdots \otimes V^{(m^r)^\infty} \otimes V^{(m^\vee)^\infty}
\]

as the inductive limit of irreducible \( G_k \)-modules (20) induced by the embeddings (17) and (3), whose elements transform according to the covariant condition (21).

For each \( k \), let \( I_k \) denote the identity representation of \( G_k \) appearing in the tensor product

\[
V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}.
\]

then \( I_k \) has signature \( (0, \ldots, 0)_k \) and by definition there exists a non-zero element

\[
f_k \in V^{(m^1)^k} \otimes \cdots \otimes V^{(m^r)^k} \otimes V^{(m^\vee)^k}
\]
such that \([R_k \otimes R'_k](g)f_k = f_k\) for all \(g \in G_k\). This means that \(f_k\) is invariant under the action \(R_k \otimes R'_k\) of \(G_k\). Now it is well known from the theory of invariants (see for example [14]) that the algebra of polynomial invariants under this \(G_k\) action is generated by the \(pq\) algebraically independent polynomial functions

\[
P^k_{\alpha\beta}(Z,W) = (ZsW^T)_{\alpha\beta} = \sum_{t=1}^{k} Z_{\alpha,t}W_{\beta,t} \quad 1 \leq \alpha \leq p, \ 1 \leq \beta \leq q.
\]  

(22)

By our realization of the \(V^{(m^1)}k\) and \(V^{(m^\vee)}k\) as \(G_k\)-modules we obviously have the isometric embedding

\[
V^{(m^1)}k \otimes \cdots \otimes V^{(m^r)}k \otimes V^{(m^\vee)}k \subset V^{(m^1)}k+1 \otimes \cdots \otimes V^{(m^r)}k+1 \otimes V^{(m^\vee)}k+1
\]

of \(G_k\)-modules into \(G_{k+1}\)-modules. It is routine to check that the appropriate diagrams commute, and as in (4) and (8) we obtain the representation \(R \otimes R^\vee\) of \(G_\infty\) on \(V^{(m^1)}\infty \otimes \cdots \otimes V^{(m^r)}\infty \otimes V^{(m^\vee)}\infty\) as an inductive limit of representations of \(G_k, k = 1, 2, 3, \ldots\). But the case of the identity representation is entirely different. For each \(k\) let \(I^k\) denote the one-dimensional subspace of \(V^{(m^1)}k \otimes \cdots \otimes V^{(m^r)}k \otimes V^{(m^\vee)}k\) spanned by the invariant vector \(f_k\) mentioned above. Then we obviously can not define the inductive limit of \(I^k\). However we can define the inverse or projective limit of the family \(\{G_k, I^k, I^k\}\) as follows: For each pair of indices \(j, k\) with \(j \leq k\) define a continuous homomorphism \(\phi^k_j : I^k \rightarrow I^j\) such that

a) \(\phi^j_j\) is the identity map for all \(j\),

b) if \(i \leq j \leq k\), then \(\phi^k_i = \phi^j_j \circ \phi^i_j\).
Here we can take $\phi^k_j$ as the \textit{truncation homomorphism}, i.e. $\phi^k_j$ is defined on the generators $P^k_{\alpha\beta}$ by

$$\phi^k_j(P^k_{\alpha\beta}) = P^j_{\alpha\beta} \quad \text{for } j \leq k \quad (23)$$

The \textit{inverse limit} of the system $\{I^k, \phi^k_j\}$ is then formally defined by

$$I^\infty := \lim_{\leftarrow} I^k = \left\{ (f_k) \in \prod_k I^k \mid f_i = \phi^i_j(f_j) \quad \text{whenever } i \leq j \right\}$$

Concretely we can define the functions

$$P_{\alpha\beta} := P^\infty_{\alpha\beta} = \lim_{k \to \infty} P^k_{\alpha\beta} = \sum_{t=1}^\infty Z_{\alpha,t}W_{\beta,t} \quad 1 \leq \alpha \leq p, \ 1 \leq \beta \leq q \quad (24)$$

and make the following observations for each $\alpha, \beta$:

1) $P_{\alpha,\beta}$ is well defined on

$$\mathbb{C}^{p \times \infty} \oplus \mathbb{C}^{q \times \infty^\vee} = \bigcup_{k=1}^\infty \left( \mathbb{C}^{p \times k} \oplus \mathbb{C}^{q \times k^\vee} \right)$$

2) $P_{\alpha,\beta}$ is not an element of $\mathcal{F}_\infty$, but instead lies in $\lim_{\leftarrow} \mathcal{F}_k$, the \textit{projective limit} or \textit{inverse limit} of the of Bargmann-Segal-Fock spaces $\mathcal{F}_k$ (for details on the projective limit representations of $G_\infty$ see [13]).

It follows that any $f \in I^\infty$ has the form

$$f = \sum C_{IJK} \prod (P_{\alpha\beta})^\gamma \quad (25)$$

where the functions $P_{\alpha\beta}$ are as defined in (24) for $1 \leq \alpha \leq p, \ 1 \leq \beta \leq q$, the $\gamma$ are non-negative integers, the sums and products in (25) are finite, and the $C_{IJK}$ are constants with multi-indices $I, J$ and $K$. Let $\pi_k : I^\infty \to I^k$
denote the projection of $\mathcal{I}^\infty$ onto $\mathcal{I}^k$. Let $I^\infty$ denote the representation of $G_\infty$ on $\mathcal{I}^\infty$ given by the following equation

$$I^\infty(g)f = \sum C_{IJK} \prod \lim_{k \to \infty} \left[ \left( R \otimes R^\vee (g) P^k_{\alpha\beta} \right)^{\gamma} \right]$$

for $g \in G_\infty$ and $f \in \mathcal{I}^\infty$ (26)

Since $g \in G_\infty$ means that $g \in G_j$ for some $j$, and for $k \geq j$

$$\left[ R \otimes R^\vee (g) \right] P^k_{\alpha\beta} = P^k_{\alpha\beta}$$

equation (26) implies that $P_{\alpha\beta}$ are $G_\infty$-invariant, and hence $I^\infty(g)f = f$ for all $f \in \mathcal{I}^\infty$. It follows that $\pi_k(I^\infty(g)f) = \pi_k(f)$ for all $g \in G_\infty$ and $f \in \mathcal{I}^\infty$.

Recall that if $\mathcal{P}_k = \mathcal{P}(\mathbb{C}^{n \times k})$ denotes the subspace of all polynomial functions of $\mathbb{C}^{n \times k}$ then $\mathcal{P}_k$ is dense in $\mathcal{F}_k$. Let

$$\mathcal{P}_\infty = \bigcup_{k=1}^\infty \mathcal{P}_k$$

denote the inductive limit of $\mathcal{P}_k$, then clearly $\mathcal{P}_\infty$ is dense in $\mathcal{F}_\infty$. Let $\mathcal{P}_\infty^\ast$ (resp. $\mathcal{F}_\infty^\ast$) denote the dual or adjoint space of $\mathcal{P}_\infty$ (resp. $\mathcal{F}_\infty$). Then since $\mathcal{P}_\infty$ is dense in $\mathcal{F}_\infty$, $\mathcal{F}_\infty^\ast$ is dense in $\mathcal{P}_\infty^\ast$. By the Riesz representation theorem for Hilbert spaces, every element $f^\ast \in \mathcal{F}_\infty^\ast$ is of the form $\langle \cdot | f \rangle$ for some $f \in \mathcal{F}_\infty$, and the map $f^\ast \mapsto f$ is an anti-linear (or conjugate-linear) isomorphism. Thus we can identify $\mathcal{F}_\infty^\ast$ with $\mathcal{F}_\infty$ and obtain the rigged Hilbert space as the triple $\mathcal{P}_\infty \subset \mathcal{F}_\infty \subset \mathcal{P}_\infty^\ast$ (see [10] for the definition of rigged Hilbert spaces).

Typically and element $P_{\alpha\beta}$ defined by equation (24) belongs to $\mathcal{P}_\infty^\ast$, and if $f \in \mathcal{F}_\infty$ then $f \in \mathcal{F}_k$ for some $k$, so we can define the inner product

$$\langle P_{\alpha\beta}, f \rangle = \langle \pi_k(P_{\alpha\beta}), f \rangle = \langle P^k_{\alpha\beta}, f \rangle$$

(27)
in fact, in the calculation of
\[ P_{\alpha\beta}(D)f(Z) \bigg|_{Z=0} \]
the terms in \( P_{\alpha\beta} \) whose column indices are larger than \( k \) drop off.

**Theorem VI.1.** Let \( V^{(m_1)} \), \ldots, \( V^{(m_r)} \) and \( V^{(m_\infty)} \) be irreducible representations of \( G_\infty \). Using the convention of Section V, let \( V^{(m_\vee)} \) be the representation contragredient to \( V^{(m)} \). Let \( I^{\infty} \) be the identity representation defined by Equation [24]. Then the multiplicity of \( V^{(m)}^{\infty} \) in the tensor product
\[ V^{(m_1)}^{\infty} \otimes \ldots \otimes V^{(m_r)}^{\infty} \]
is equal to the multiplicity of \( I^{\infty} \) in the tensor product
\[ V^{(m_1)}^{\infty} \otimes \ldots \otimes V^{(m_p)}^{\infty} \otimes V^{(m_\vee)}^{\infty}. \]

**Proof.** From [10] we know that for sufficiently large \( k \) the multiplicity of \( V^{(m)}_k \) in
\[ V^{(m_1)}_k \otimes \ldots \otimes V^{(m_r)}_k \]
is equal to the multiplicity of the identity representation \( I^k \) in the augmented tensor product
\[ V^{(m_1)}_k \otimes \ldots \otimes V^{(m_r)}_k \otimes V^{(m_\vee)}_k. \]
For each \( k \) let \( h_k \) denote the homomorphism sending the irreducible representation of \( G_k \) with signature \((0, \ldots, 0)\) into the \( G_k \)-module
\[ V^{(m_1)}_k \otimes \ldots \otimes V^{(m_r)}_k \otimes V^{(m_\vee)}_k. \]
Then

\[ \phi_j^k \circ h_k = h_j \quad \text{for } j \leq k \]

where the homomorphisms \( \phi_j^k \) are defined as in (23). Let \((0, \ldots, 0)\) denote the signature of the representation of \( G_\infty \) as the inverse limit of irreducible representations of \( G_k \) with signature \((0, \ldots, 0)\) \( k \). Then we can define a homomorphism

\[ h : V^{(0, \ldots, 0)} \to V^{(m_1)} \otimes \cdots \otimes V^{(m_r)} \otimes V^{(m_\nu)} \]

by

\[ h(v) = \lim_{\leftarrow} h_k(\pi_k(v)) \quad (28) \]

where in Equation (28), \( \pi_k \) denotes the projection of \( V^{(0, \ldots, 0)} \) onto \( V^{(0, \ldots, 0)} \). Note that \( V^{(0, \ldots, 0)} \) or \( V^{(0, \ldots, 0)} \) are just the trivial \( G_k \) or \( G_\infty \) modules \( \mathbb{C} \), and that the \( G_\infty \)-module

\[ V^{(m_1)} \otimes \cdots \otimes V^{(m_r)} \otimes V^{(m_\nu)} \]

is considered as a \( G_\infty \)-submodule of the \( G_\infty \)-module \( P^* \). As remarked in Section III, the dimension of

\[ \text{Hom}_{G_k} \left( V^{(0, \ldots, 0)} \otimes V^{(m_1)} \otimes \cdots \otimes V^{(m_r)} \otimes V^{(m_\nu)} \right), \]

the space of all homomorphisms intertwining \( V^{(0, \ldots, 0)} \) and \( V^{(m_1)} \otimes \cdots \otimes V^{(m_r)} \otimes V^{(m_\nu)} \) stabilizes as \( k \) gets large. But this dimension is just the
The multiplicity of \( I^k \) in \( V^{(m^1)^k} \otimes \ldots \otimes V^{(m^r)^k} \otimes V^{(m^\nu)^k} \) which, in turn is equal to the multiplicity of \( V^{(m)^k} \) in \( V^{(m^1)^k}, \ldots, V^{(m^r)^k} \). It follows that at the (inductive) limit we have

\[
\dim \left[ \text{Hom}_{G^\infty} \left( V^{(0,\ldots,0)^\infty}, V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\nu)^\infty} \right) \right] = \dim \left[ \text{Hom}_{G^\infty} \left( V^{(m)^\infty}, V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \right) \right]
\]

or equivalently the multiplicity of \( V^{(m)^\infty} \) in the tensor product

\[
V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}
\]

is equal to the multiplicity of \( I^\infty \) in the tensor product

\[
V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty} \otimes V^{(m^\nu)^\infty}.
\]

\[
\square
\]

Let \( \{ f^{m^i}_{\xi_i} \} \), be a basis of state vectors for \( V^{(m^i)^\infty} \), \( i = 1 \ldots r \), let \( \{ f^m_{\xi} \} \) be a basis of state vectors for \( V^{(m)^\infty} \) and let \( \{ f^{m^\nu}_{\xi^*} \} \) be a basis of state vectors for \( V^{(m^\nu)^\infty} \). Then

\[
f^{m^1}_{\xi_1} \otimes f^{m^2}_{\xi_2} \otimes \ldots \otimes f^{m^r}_{\xi_r}
\]

is a natural basis for the tensor product of irreducible representations

\[
V^{(m^1)^\infty} \otimes \ldots \otimes V^{(m^r)^\infty}
\]

and

\[
f^{m^1}_{\xi_1} \otimes f^{m^2}_{\xi_2} \otimes \ldots \otimes f^{m^r}_{\xi_r} \otimes f^{m^\nu}_{\xi^*}
\]

is also a natural basis for the tensor product of irreducible representations...
is a natural basis for the tensor product of irreducible representations
\[ V^{(m_1)^\infty} \otimes \ldots \otimes V^{(m_r)^\infty} \otimes V^{(m^\nu)^\infty} \]

Let \( \{I_\eta\}_\eta \) be a basis for the \( G_\infty \)-invariant subspace which is ‘contained’ in
\[ V^{(m_1)^\infty} \otimes \ldots \otimes V^{(m_r)^\infty} \otimes V^{(m^\nu)^\infty} \]
in the sense described above. If we set
\[ I_\eta \left( \begin{pmatrix} Z \\ W \end{pmatrix} \right) = I_\eta(Z, W) \]
and consider \( I_\eta(Z, W) \) as a function of \( W \), and also note that any function \( f \in V^{(m^\nu)^\infty} \) is a function of \( W \) alone, then we can form the inner product, as defined in (27)
\[ \langle I_\eta | f \rangle_W = I_\eta(Z, D) \overline{f(W)} \big|_{W=0} \]
and thereby obtain a function of \( Z \).

Considering the remarks above, we adapt the statement and proof of Theorem 2.3 of [10], to our situation as follows

**Theorem VI.2.** Let
\[ \tilde{f}^{m,n}_\xi(Z) = \langle I_\eta(Z, W) | f^{m^\nu}_\xi(W) \rangle_W = I_\eta(Z, D) \overline{f^{m^\nu}_\xi(W)} \big|_{W=0} \]

Then \( \{\tilde{f}^{m,n}_\xi\}_\xi \) is an isomorphic image of \( \{f^m_\xi\}_\xi \) in \( V^{(m_1)^\infty} \otimes \ldots \otimes V^{(m_r)^\infty} \)
indexed by the multiplicity label \( \eta \) and we have the following relation of
Clebsch-Gordan coefficients

\[
\langle \tilde{f}^{m,n}_ξ | f^{m_1}_{ξ_1} f^{m_2}_{ξ_2} \cdots f^{m_r}_{ξ_r} \rangle = \langle I_η | f^{m_1}_{ξ_1} f^{m_2}_{ξ_2} \cdots f^{m_r}_{ξ_r} f^{m_∗}_{ξ_∗} \rangle
\]  

(30)

Proof. To first show that \( \tilde{f}^{m,n}_ξ (Z) \) in fact lies in \( V^{(m_1)}_∞ \otimes \cdots \otimes V^{(m_r)}_∞ \) it is sufficient to show (by the Borel-Weil theorem) that if \( b = (b_1, \ldots, b_r) \in B_1 \times \cdots \times B_r \) then, as in Equation (21)

\[
\tilde{f}^{m,n}_ξ (bZ) = \pi^{m_1}(b_1) \cdots \pi^{m_r}(b_r) \tilde{f}^{m,n}_ξ (Z)
\]

But since \( I_η \) 'lies' in

\[
V^{(m_1)}_∞ \otimes \cdots \otimes V^{(m_r)}_∞ \otimes V^{(m_*)}_∞
\]

it transforms covariantly with respect to the Borel subgroup defined in Equation (19) so we have

\[
\tilde{f}^{m,n}_ξ (bZ) = \langle I_η (bZ, W) | f^{m_∗}_{ξ_∗} (W) \rangle_W
\]

where \( W \) is the \( q \times q \) identity matrix

\[
= \langle \pi^{m}(β) I_η (Z, W) | f^{m_∗}_{ξ_∗} (W) \rangle_W \quad \text{where} \quad β = b \times Id
\]

\[
= \pi^{m}(β) \langle I_η (Z, W) | f^{m_∗}_{ξ_∗} (W) \rangle_W \quad \text{the inner product is linear in the first argument}
\]

\[
= \pi^{m}(b_1) \cdots \pi^{m_r}(b_r) \tilde{f}^{m,n}_ξ (Z) \quad \text{by Equation (21), as desired.}
\]

We next show that the \( \{ \tilde{f}^{m,n}_ξ \}_ξ \) transform under the representation \( R^{(m)} \) in the same manner as the \( \{ f^{m}_ξ \}_ξ \). Since \( I_η (Z, W) \) is invariant with respect to the action \( R \otimes R^\vee \) of \( G_∞ \) we have \( I_η (Zg, W) = I_η (Z, W s g^{-1} s) \) which can succinctly be written as \( R(g) I_η (Z, W) = R^\vee (g^{-1}) I_η (Z, W) \). We also have that

\[
R^{(m)}(g) f^{m}_ξ = \sum_{ξ'} D^{m}_{ξ'ξ}(g) f^{m}_{ξ'}
\]
where the $D^m_{\xi\xi'}$ are the $D$-functions for the representation $R^{(m)}$. Now for any $g \in G_\infty$ we can assume that $g \in U(k)$ for some $k$, so that $g^\vee = \bar{g}$. Hence $D^m_{\xi\xi'}(g^\vee) = \overline{D^m_{\xi\xi'}(g)}$, and it follows from the definitions of the symbols involved that

$$R^{(m)}(g)f^m_{\xi^*} = \sum_{\xi'} \overline{D^m_{\xi\xi'}(g)} f^m_{\xi'^*}$$

Thus we seek to show that

$$R^{(m)}(g)\tilde{f}^{m,\eta}_{\xi} = \sum_{\xi'} D^m_{\xi\xi'}(g)\tilde{f}^{m,\eta}_{\xi'}$$

By the preceding remarks and the definition of $\tilde{f}^{m,\eta}_{\xi}$ we then have

$$R^{(m)}(g)\tilde{f}^{m,\eta}_{\xi}(Z) = \langle I_\eta (Zg, W) \mid f^m_{\xi^*} (W) \rangle_w$$

$$= \langle R^{(m)}(g) I_\eta (Z, W) \mid f^m_{\xi^*} (W) \rangle_w$$

$$= \langle R^{(m)}(g^{-1}) I_\eta (Z, W) \mid f^m_{\xi^*} (W) \rangle_w$$

$$= \langle I_\eta (Z, W) \mid R^{(m)}(g)f^m_{\xi^*} (W) \rangle_w \text{ since the representation is unitary}$$

$$= \langle I_\eta (Z, W) \mid \sum_{\xi'} \overline{D^m_{\xi\xi'}(g)} f^m_{\xi'^*} (W) \rangle_w$$

$$= \sum_{\xi'} D^m_{\xi\xi'}(g) \langle I_\eta (Z, W) \mid f^m_{\xi'^*} (W) \rangle_w \text{ by conjugate linearity}$$

Finally we have

$$\langle I_\eta | f^m_{\xi_1} f^m_{\xi_2} \ldots f^m_{\xi_r} f^m_{\xi^*} \rangle = \langle I_\eta | f^m_{\xi^*} f^m_{\xi_1} f^m_{\xi_2} \ldots f^m_{\xi_r} \rangle$$

$$= \langle I_\eta (D, D) f^m_{\xi^*} (W) f^m_{\xi_1} (Z) \ldots f^m_{\xi_r} (Z) \rangle \mid (Z, W) = (0,0)$$

$$= \left[ \langle I_\eta (D, D) f^m_{\xi^*} (W) f^m_{\xi_1} (Z) \ldots f^m_{\xi_r} (Z) \rangle \rangle \right] \mid (Z, W) = (0,0)$$

$$= \tilde{f}^{m,\eta}_{\xi}(D) f^m_{\xi_1} (Z) \ldots f^m_{\xi_r} (Z) \right] \mid Z=0$$

$$= \langle \tilde{f}^{m,\eta}_{\xi} | f^m_{\xi_1} f^m_{\xi_2} \ldots f^m_{\xi_r} \rangle$$
which is Equation (30).

\[ \text{\textcopyright} \]

\section*{VII Example}

We illustrate the techniques described in this paper with the example

\[(7, 1, \vec{0}) \subset (1, \vec{0}) \otimes (2, \vec{0}) \otimes (2, \vec{0}) \otimes (3, \vec{0})\]

considered in (11) of Section III. By the results of Theorem VI.2 and Equation (21) we seek algebraically independent polynomials of the form

\[ P\left( \begin{pmatrix} Z \\ W \end{pmatrix} \right) = \sum C_{IJK} \prod (P_{\alpha\beta})^\gamma \alpha = 1, 2, 3, 4 \quad \beta = 1, 2 \quad (31) \]

that satisfy the covariant condition

\[ P\left( \beta \begin{pmatrix} Z \\ W \end{pmatrix} \right) = \pi^{(\mu)}(\beta) f \left( \begin{pmatrix} Z \\ W \end{pmatrix} \right) \quad (32) \]

where \( \mu = (1, 2, 2, 3, 7, 1) \) and

\[ \beta = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \\ b_5 \\ b_6 \end{pmatrix} \quad b_i, b* \in \mathbb{C}. \]

If \( D \) is the diagonal subgroup and \( \mathbb{Z}^+ \) is the upper triangular unipotent subgroup, then \( \beta \in D\mathbb{Z}^+ \) so we can first reduce the problem by solving (32) for the diagonal subgroup \( D \) which consists of elements of the form

\[ d = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \\ b_5 \\ b_6 \end{pmatrix} \quad b_i \in \mathbb{C}. \]
Hence we seek polynomials of the form

\[ P = P_{11}^{\ell_{11}} P_{12}^{\ell_{12}} P_{21}^{\ell_{21}} \cdots P_{41}^{\ell_{41}} P_{42}^{\ell_{42}} \]

that satisfy

\[ P \left( d \left( \begin{array}{c} Z \\ W \end{array} \right) \right) = \pi^{(d)}(d) f \left( \left( \begin{array}{c} Z \\ W \end{array} \right) \right), \forall d \in D. \]

This leads us to the system

\[
\begin{cases}
\ell_{11} + \ell_{12} = 1 \\
\ell_{21} + \ell_{22} = 2 \\
\ell_{31} + \ell_{32} = 2 \\
\ell_{41} + \ell_{42} = 3 \\
\ell_{11} + \ell_{21} + \ell_{31} + \ell_{41} = 7 \\
\ell_{12} + \ell_{22} + \ell_{32} + \ell_{42} = 1 
\end{cases}
\]

which gives us the following set of polynomials that transform covariantly with respect to the diagonal subgroup \( D \);

\[
P_1 = P_{11} P_{21} P_{22} P_{31}^2 P_{41}^3 \\
P_2 = P_{11} P_{21}^2 P_{31} P_{32} P_{41}^3 \\
P_3 = P_{11} P_{21}^2 P_{31}^2 P_{41}^2 P_{42} \\
P_4 = P_{12} P_{21}^2 P_{31}^3 P_{41}. 
\]

Next, from (31) and (32) we seek functions of the form

\[ P = C_1 P_1 + C_2 P_2 + C_3 P_3 + C_4 P_4 \] 

(33)

that transform covariantly with respect to the upper triangular unipotent
subgroup \( \mathbb{Z}^+ \) which consists of elements of the form

\[
\mathbb{Z}^+ = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 \\ 0 & 1 & b^* \\ 1 \end{bmatrix} \quad b^* \in \mathbb{C}.
\]

Checking this condition on \( P_1, P_2, P_3 \) and \( P_4 \) we see that

\[
P_1 \left( Z^+ \left( \frac{Z}{W} \right) \right) = P_1 + b^* P_{11} P_{21} P_{31} P_{41}^3
\]

\[
P_2 \left( Z^+ \left( \frac{Z}{W} \right) \right) = P_2 + b^* P_{11} P_{21}^2 P_{31}^2 P_{41}^3
\]

\[
P_3 \left( Z^+ \left( \frac{Z}{W} \right) \right) = P_3 + b^* P_{11} P_{21}^2 P_{31}^2 P_{41}^3
\]

\[
P_4 \left( Z^+ \left( \frac{Z}{W} \right) \right) = P_4 + b^* P_{11} P_{21}^2 P_{31}^2 P_{41}^3
\]

In order that

\[
P \left( Z^+ \left( \frac{Z}{W} \right) \right) = P \left( \left( \frac{Z}{W} \right) \right) \quad \forall Z^+ \in \mathbb{Z}^+
\]

we must have \( C_1 + C_2 + C_3 + C_4 = 0 \). Thus a convenient basis of \( G_{\infty} \)-invariants in this tensor product can be chosen as

\[
\mathcal{I}_1 = P_1 - P_2 = P_{11} P_{21} P_{22}^2 P_{31}^3 P_{41}^3 - P_{11} P_{21}^2 P_{31} P_{32} P_{41}^3
\]

\[
\mathcal{I}_2 = P_2 - P_3 = P_{11} P_{21}^2 P_{31} P_{32} P_{41}^3 - P_{11} P_{21}^2 P_{31} P_{41}^3 P_{42}
\]

\[
\mathcal{I}_3 = P_3 - P_4 = P_{11} P_{21}^2 P_{31}^2 P_{41}^2 P_{42} - P_{12} P_{21}^2 P_{31}^2 P_{41}^3
\]

Note that the space of invariants has dimension three, which is the multiplicity of \((7, 1, \overrightarrow{0})\) computed earlier.
Now a natural basis for the $G_\infty$-invariant subspace with signature $(1, \overrightarrow{0})$ contained in $F_\infty$ as described in Section VI is given by \( \{ Z_{1i} \}_{i=1}^\infty \). Similarly \( \{ Z_{2i} Z_{2j} \}_{i,j=1}^\infty \), \( \{ Z_{3i} Z_{3j} \}_{i,j=1}^\infty \) and \( \{ Z_{4i} Z_{4j} Z_{4k} \}_{i,j,k=1}^\infty \) are natural basis for the subspaces $(2, \overrightarrow{0})$, $(2, \overrightarrow{0})$ and $(3, \overrightarrow{0})$, respectively, and an element of $(7, 1, \overrightarrow{0})^\vee$ is its lowest weight vector $w_{11}^6 \det (w_{22} w_{21} w_{12})$. Thus an example of a basis element for the tensor product

\[(1, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (2, \overrightarrow{0}) \otimes (3, \overrightarrow{0}) \otimes (7, 1, \overrightarrow{0})^\vee\]

would be

\[Z_{11} Z_{21} Z_{31} Z_{41} (W_{11}^7 W_{22} - W_{11}^6 W_{21})\]

and to compute a Clebsch-Gordan coefficient we compute the inner product of (34) with, for example $I_1$.

\[
\langle I_1 \mid Z_{11} Z_{21} Z_{31} Z_{41} (W_{11}^7 W_{22} - W_{11}^6 W_{21}) \rangle = \left[ P_{11} P_{21} P_{22} P_{31} P_{41}^3 - P_{11} P_{21} P_{31} P_{32} P_{41}^3 \right] (D) Z_{11} Z_{21} Z_{31} Z_{41} (W_{11}^7 W_{22} - W_{11}^6 W_{21}) |_{(Z,W) = (0,0)}
\]

We remark that in the above computation, for example the product

\[P_{11} P_{21} P_{22} P_{31} P_{41}^3 (D) = \left( \sum_{t=1}^\infty Z_{1t} W_{1t} \right) \cdots \left( \sum_{t=1}^\infty Z_{4t} W_{1t} \right)^3 (D)\]

need only be evaluated up to $t = 2$ since those terms whose column indices are larger than two evaluate to zero in the above inner product. This is routinely accomplished using a computer algebra system, such as Maple.
VIII Conclusion

We have shown how the multiplicity problem the Clebsch-Gordan coefficients in the decomposition of r-fold tensor products of irreducible tame representations of $U(\infty)$ can be restated in terms of $U(\infty)$-invariants. Thus all the theorems for $U(k)$ treated in [10] can be generalized to $U(\infty)$. Actually the computational aspect of the problems are much simpler with this new approach and one can use computers to obtain invariant polynomials, and by differentiating these polynomials compute Clebsch-Gordan and Racah coefficients.

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