On the Continuum Limit of the Conformal Matrix Models

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Abstract

The double scaling limit of a new class of the multi-matrix models proposed in [1], which possess the $W$-symmetry at the discrete level, is investigated in details. These models are demonstrated to fall into the same universality class as the standard multi-matrix models. In particular, the transformation of the $W$-algebra at the discrete level into the continuum one of the paper [2] is proposed, the corresponding partition functions being compared. All calculations are demonstrated in full in the first non-trivial case of $W^{(3)}$-constraints.

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1 Introduction

The idea of the description of 2d theories of conformal matter coupled to gravity through the simpler models of the lattice type, which fall into the same universality classes, suggested in [3, 4] led to an explosion of the interest to different matrix models in 1989, when it was understood that the most convenient choice of the lattice models is related to the triangulations of random surfaces (in spirit of Regge approach [5]), which can be coded in terms of matrix models. More concretely, there was proposed a class of multi-matrix models whose (multi-)critical points correspond to minimal series of conformal matter coupled to gravity [3, 4, 8, 9, 10, 11, 12, 13]. The general description of the (double scaling) continuum limit of these models essentially involves the following two properties of the double scaled partition function: it should be the $\tau$-function of proper reduced KP hierarchy and it should satisfy the $W$-algebra [14, 2, 15]. The combination of these two properties leads to the string equation.

Indeed, it was the string equation that was derived firstly, and the general properties of the matrix models were guessed later from the properties of the string equation. Unfortunately, it is very difficult to prove them completely, as the the structure of the theory is very complicated when considering higher multi-critical points and/or multi-matrix models. From the other hand, let us point out that the determining properties of the multi-matrix model partition function are observed after taking the continuum limit. Therefore, it is reasonable to find out new multi-matrix models with these properties at the discrete level. In fact, at the discrete level one usually has a property of manifest distinguishing the integrability and the $W$-invariance simultaneously. Due to this fact, one can effectively investigate the continuum limit by observing only $W$-algebra (by modulo one subtlety, see below the subsect.5.2). As for the integrability, it is the standard thing that one can continue the model off critical point preserving this property. This continuation is rather natural and simultaneously gives rise to a proper regularization of the continuum theory (we also know this phenomenon in quantum integrable systems, where to regularize the theory one can put it on the lattice preserving the quantum integrability - see, for example, [16]).

All this was partially realized previously. Namely, there was proved that at the discrete
level the partition function of the Hermitian one-matrix model is the $\tau$-function of Toda chain hierarchy [17] and satisfies the Virasoro constraints [18, 19]. It was also shown [20] that these constraints turn into the continuum constraints of the paper [2]. Moreover, one can check [21] that in this continuum limit Toda chain hierarchy really turns into KdV hierarchy which corresponds to the double scaling limit of the Hermitian one-matrix model [14, 2, 15].

Unfortunately, the situation is not so clear in the multi-matrix model case. In the general description the standard multi-matrix models proposed in [11, 12, 13] have no any symmetry at the discrete level (though they still correspond to an integrable system, that is, to Toda lattice hierarchy [17, 22, 23]). It does the proof of their $W$-invariance in the continuum limit very difficult. There were also suggested other discrete multi-matrix models with the same integrability property and $W$-invariance [24, 25]. Their continuum limit was investigated [24], but the complete answer is not obtained up to now due to technical difficulties.

At last, in the paper [1] a rather natural multi-matrix model was introduced, with the property of $W$-invariance included by definition. This model has less trivial integrable structure and corresponds to multi-component KP hierarchy with a special reduction of AKNS type [20]. This model will be the main subject under consideration in the paper. As the corresponding partition function can be naturally presented in the form of the correlator in conformal theory, we call it ”conformal multi-matrix model ” (CMM).

We consider the paper as one of the series of two papers devoted to the investigation of CMM. Another one [26] mostly discusses the the integrability, while the continuum limit is described in less details. The present paper thoroughly deals with the continuum limit of CMM. But, for the sake of completeness, we describe general properties of the discrete model. More concretely, in the section 2 the definition of CMM is introduced, and some of its general properties are discussed. Here we are interested only in the continuum limit of $W$-algebra , the continuum limit of arbitrary multicomponent KP hierarchy of such type being out of the scope of this paper. In the sect.3 the general approach to taking the continuum limit is discussed. It is considered for the simplest case of the Virasoro constraints in the one-matrix model in rather invariant terms [20], and some natural generalizations are proposed. It turns out that this invariant formulation
naturally continues to the general case due to the very special form of the discrete $W$-algebra resembling the structure of the continuum $W$-algebra of the paper [2] (indeed, just the ”conformal” form of the partition function in this model is in charge of this). In the sect.4 the simplest non-trivial case of two-matrix model ($W^{(3)}$-constraints) is considered in full details, and the general case is discussed in the sect.5, some important but tedious calculations being shifted to the Appendix. The conclusion contains the list of the main results as well as some general discussion.

2 Conformal multi-matrix models

2.1 Formulation of the model

To begin with, we would like to introduce conformal multi-matrix models in accordance with the paper [1] (see also [26]). First, we show that the simplest example of discrete Hermitian one-matrix model can be easily reformulated in these terms.

Indeed, Hermitian one-matrix model ($p = 2$) can be defined as a solution to discrete Virasoro constraints:

$$L_n Z_{2,N}[t] = 0, \quad n \geq -1$$

$$L_n \equiv \sum_{k=0}^{\infty} k t_k \partial / \partial t_{k+n} + \sum_{a+b=n} \partial^2 / \partial t_a \partial t_b$$

$$\partial Z_{2,N} / \partial t_0 = -NZ_{2,N}$$

The Virasoro generators (1) have the well-known form of the Virasoro operators in the theory of one free scalar field. If we look for such solution in terms of holomorphic components of the scalar field

$$\phi(z) = \hat{q} + \hat{p} \log z + \sum_{k \neq 0} \frac{J_{-k}}{k} z^{-k}$$

$$[J_n, J_m] = n \delta_{n+m,0}, \quad [\hat{q}, \hat{p}] = 1$$

the procedure is as follows. Define vacuum states

$$J_k |0\rangle = 0, \quad \langle N | J_{-k} = 0, \quad k > 0$$

$$\hat{p} |0\rangle = 0, \quad \langle N | \hat{p} = N \langle N |, \quad (3)$$
the stress-tensor
\[ T(z) = \frac{1}{2}[(\partial \phi(z))^2] = \sum T_n z^{-n-2}, \quad T_n = \frac{1}{2} \sum_{k>0} J_{-k} J_{k+n} + \frac{1}{2} \sum_{a+b=n, a,b\geq 0} J_a J_b, \]  
(4)

\[ T_n |0\rangle = 0, \quad n \geq -1 \]  
(5)

and the Hamiltonian
\[ H(t) = \frac{1}{\sqrt{2}} \sum_{k>0} t_k J_k = \oint_{C_0} V(z) j(z) \]
\[ V(z) = \sum_{k>0} t_k z^k, \quad j(z) = \frac{1}{\sqrt{2}} \partial \phi(z). \]  
(6)

Now one can easily construct a “conformal field theory” solution to (1) in two steps. First,
\[ L_n \langle N | e^{H(t)} \ldots = \langle N | e^{H(t)} T_n \ldots \]  
(7)
can be checked explicitly. As an immediate consequence, any correlator of the form
\[ \langle N | e^{H(t)} G | 0 \rangle \]  
(8)
\((N\text{ counts the number of zero modes of } G)\) gives a solution to (1) provided
\[ [T_n, G] = 0, \quad n \geq -1. \]  
(9)

Second, the conformal solution to (9) (and therefore to (1)) comes from the properties of 2d conformal algebra. Indeed, any solution to
\[ [T(z), G] = 0 \]  
(10)
is a solution to (9), and it is well-known that the solution to (10) is a function of screening charges
\[ Q_{\pm} = \oint J_{\pm} = \oint e^{\pm \sqrt{2} \phi}. \]  
(11)

With a selection rule on zero mode it gives
\[ G = \exp Q_+ \rightarrow \frac{1}{N!} Q_N \]  
(12)

Of course, the general case must be \(G \sim Q_{N+M}^N Q_M^M\). Nevertheless, this choice of one of the two possible screening operators has a clear algebraic sense which we will discuss below.
in this section. It can be justified by the special prescription for integration contours, proposed in [1], which implies that the dependence of $M$ can be irrelevant and one can just put $M = 0$). In this case the solution

$$Z_{2,N}[t] = \langle N | e^{H(t)} \exp Q_+ | 0 \rangle$$

(13)

after computation of the free theory correlator gives well-known result

$$Z_{2,N} = (N!)^{-1} \prod_{i=1}^{N} dz_i \exp \left( - \sum t_k z_i^k \right) \Delta_N^2 (z) =$$

$$= (N! \text{Vol } U(N))^{-1} \int DM \exp \left( - \sum t_k M^k \right)$$

(14)

$$\Delta_N = \prod_{i<j} (z_i - z_j)$$

in the form of multiple integral over spectral parameters or integration over Hermitian matrices.

In the case of $p = 2$ (Virasoro) constraints this is just a useful reformulation of the Hermitian 1-matrix model. However, in what follows we are going to use this point of view as a constructive one. Indeed, instead of considering a special direct multi-matrix generalization of (14) [11, 12, 13] one can use powerful tools of conformal theories, where it is well known how to generalize almost all the steps of above construction: first, instead of looking for a solution to Virasoro constraints one can impose extended Virasoro or $W$-constraints on the partition function. In such case one would get Hamiltonians in terms of multi-scalar field theory, and the second step is generalized directly using screening charges for $W$-algebras. The general scheme looks as follows

(i) Consider Hamiltonian as a linear combination of the Cartan currents of a level one Kac-Moody algebra $G$

$$H(t^{(1)}, \ldots, t^{(\text{rank } G)}) = \sum_{\lambda,k>0} t_k^{(\lambda)} \mu_{\lambda} J_k,$$

(15)

where $\{\mu_i\}$ are basis vectors in Cartan hyperplane, which, say for $SL(p)$ case are chosen to satisfy

$$\mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p}, \quad \sum_{j=1}^{p} \mu_j = 0.$$

(ii) The action of differential operators $W_i^{(a)}$ with respect to times $\{t_k^{(\lambda)}\}$ can be now defined from the relation

$$W_i^{(a)} \langle N | e^{H(t)} \rangle \ldots = \langle N | e^{H(t_i)} W_i^{(a)} \rangle \ldots, \quad a = 2, \ldots, p; \quad i \geq 1 - a,$$

(16)
where

\[
W_i^{(a)} = \oint z^{a+i-1} W^{(a)}(z)
\]

\[
W^{(a)}(z) = \sum_{\lambda} \{\mu_{\lambda} \partial \phi(z)^a + \ldots
\]

(17)

are spin-\(a\) W-generators of \(W_p\)-algebra written in terms of rank \(\mathcal{G}\)-component scalar fields [27].

(iii) The conformal solution to \(W\)-constraints arises in the form

\[
Z_{p, N}^{CM}\{\{t\}\} = \langle N| e^{H(t)} G\{Q^{(a)}\} |0\rangle
\]

(18)

where \(G\) is an exponential function of screenings of level one Kac-Moody algebra

\[
Q^{(a)} = \oint J^{(a)} = \oint e^{\alpha \phi}
\]

(19)

\(\{\alpha\}\) being roots of finite-dimensional simply lanced Lie algebra \(\mathcal{G}\). (For the case of non-simply laced case see [28]. Below \(\mathcal{G} = SL(p)\) if not stated otherwise.) The correlator (18) is still a free-field correlator and the computation gives it again in a multiple integral form

\[
Z_{p, N}^{CM}\{\{t\}\} \sim \int \prod_{\alpha} \left[ N_{\alpha} \prod_{i=1}^{N_{\alpha}} dz_i^{(a)} \exp \left( - \sum_{\lambda, k > 0} i_k^{(\lambda)} (\mu_{\lambda} \alpha) (z_i^{(a)})^k \right) \right] \times \prod_{(\alpha, \beta)} N_{\alpha} \prod_{i=1}^{N_{\alpha}} \prod_{j=1}^{N_{\beta}} (z_i^{(\alpha)} - z_j^{(\beta)}) \alpha \beta
\]

(20)

The expression (20) is what we shall study in this paper: namely the solution to discrete \(W\)-constraints which can be written as multiple integral over spectral parameters \(\{z_i^{(a)}\}\) (this integral is sometimes called “eigenvalue model”). The difference with the one-matrix case (14) is that the expressions (20) have rather complicated representation in terms of multi-matrix integrals. Namely, the only non-trivial (Van-der-Monde) factor can be rewritten in the (invariant) matrix form:

\[
\prod_{i=1}^{N_{\alpha}} \prod_{j=1}^{N_{\beta}} (z_i^{(\alpha)} - z_j^{(\beta)}) \alpha \beta = \left[ \det \{ M^{(a)} \otimes I - I \otimes M^{(b)} \} \right]^{\alpha \beta},
\]

(21)

where \(I\) is the unit matrix. Still this is a model with a chain of matrices and with closest neighbour interactions only (in the case of \(SL(p)\)).

Now we would like to say some words on the general structure of this model. Let us point out that its partition function is nothing but the correlation function of objects
which have clear algebraic meaning. Indeed, the time dependent exponential is generated by Cartan currents of $SL(p)$, and the exponentials of the screening charges correspond to the exponentials of other (non-Cartan) generators of $SL(p)$. But we should stress that again it should not be an arbitrary combination of these generators, but only of any $p - 1$ of them. Indeed, the different choices of these $p - 1$ generators correspond to the different models but with the same properties of the integrability and the $W$-invariance.

In fact, it is very immediate thing to fermionize these expressions and merely write down the expression for proper $\tau$-function of reduced KP hierarchy of the generalized AKNS type \cite{29, 30} in the form of fermionic correlator \cite{26}. This generalized AKNS reduction has its origin in the reducing from the general case of $GL(p)$ algebra to its simple subalgebra $SL(p)$ \cite{29}, when considering proper $W^{(p)}$-algebra \cite{27}.

The purpose of this paper is to show that CMM defined by \eqref{20} as a solution to the $W$-constraints possesses a natural continuum limit. To pay for these advantages one should accept a slightly less elegant matrix integral with the entries like \eqref{21}.

\section{2.2 On the proper basis for CMM}

Now we would like to discuss briefly the manifest expressions for constraint algebras in terms of time variables.

The first non-trivial example (which we use as a demonstrating example in the section 4) is the $p = 3$ associated with Zamolodchikov’s $W_3$-algebra \cite{32} and serves as alternative to 2-matrix model. In this particular case one obtains

\begin{equation}
H(t, \bar{t}) = \frac{1}{\sqrt{2}} \sum_{k \geq 0} (t_k J_k + \bar{t}_k \bar{J}_k) \tag{22}
\end{equation}

\begin{equation}
W^{(2)}_n = L_n = \sum_{k=0}^{\infty} (kt_k \partial / \partial t_{k+n} + k \bar{t}_k \partial / \partial \bar{t}_{k+n} ) + \sum_{a+b=n} (\partial^2 / \partial t_a \partial t_b + \partial^2 / \partial \bar{t}_a \partial \bar{t}_b) \tag{23}
\end{equation}

\begin{equation}
W^{(3)}_n = \sum_{k,l>0} (kt_k l \partial / \partial t_{k+n+l} - k \bar{t}_k \bar{l} \partial / \partial \bar{t}_{k+n+l} - 2kt_k \bar{t}_l \partial / \partial \bar{t}_{k+n+l} + \tag{24}
\end{equation}

\begin{equation}
\text{\textsuperscript{1}Indeed, all this has an interpretation immediately in terms of proper Hamiltonian reduction - see also \cite{1, 31}.}
\end{equation}
\[\begin{align*}
+2 & \sum_{k>0} \left[ \sum_{a+b=n+k} (kt_k \partial^2/\partial t_a \partial t_b - kt_k \partial^2/\partial t_a \partial \bar{t}_b - 2k \bar{t}_k \partial^2/\partial t_a \partial \bar{t}_b) \right] + \\
+ \frac{4}{3} & \sum_{a+b+c=n} (\partial^3/\partial t_a \partial t_b \partial t_c - \partial^3/\partial t_a \partial \bar{t}_b \partial \bar{t}_c),
\end{align*}\] (24)

where times \( t_k \) and \( \bar{t}_k \) correspond to the two orthogonal directions in \( SL(3) \) Cartan plane. (We use the standard specification of the Cartan basis: \( e = \alpha_1/\sqrt{2}, \bar{e} = \sqrt{3} \nu_2/\sqrt{2} \).) In this case one has six screening charges \( Q^{(\pm \alpha_i)} \) \((i = 1, 2, 3)\) which commute with \( W^{(2)}(z) = T(z) = \frac{1}{2} [\partial \phi(z)]^2 \): (25)

and

\[W^{(3)}(z) = \sum_{\lambda=1}^{3} : (\mu_\lambda \partial \phi(z))^3 : \], (26)

where \( \mu_\lambda \) are vectors of one of the fundamental representations (3 or \( \bar{3} \)) of \( SL(3) \).

This basis was originally used in [1] as it just corresponds to the integrable flows, for the continuum limit we will use another basis in the Cartan plane connected with \( t \pm \bar{t} \). (In other words, this is the question what is the proper reduction, or what combinations of the “integrable” times should be eliminated.)

To begin with, we consider the simplest non-trivial case of \( p = 3 \). Then introducing the scalar fields

\[\partial \phi^{(1)}(z) = \sum_k kt_k^{(1)} z^{-k}, \quad \partial \phi^{(2)}(z) = \sum_k kt_k^{(2)} z^{-k},\] (27)

\[\partial \phi^{(1)}(z) = \sum_k \frac{\partial}{\partial t_k^{(2)}} z^{-k-1}, \quad \partial \phi^{(2)}(z) = \sum_k \frac{\partial}{\partial \bar{t}_k^{(1)}} z^{-k-1},\] (28)

with \( t_k^{(1)} = (i \bar{t}_k + t_k)/2\sqrt{2}, t_k^{(2)} = (i \bar{t}_k - t_k)/2\sqrt{2} \), one obtains the expressions:

\[W^{(2)}(z) = \frac{1}{2} : \partial \phi^{(1)}(z) \partial \phi^{(2)}(z): \], (29)

\[W^{(3)}(z) = \frac{1}{3\sqrt{3}} \sum_i : (\partial \phi^{(i)}(z))^3 :, \] (30)

instead of (23) and (24).

This choice of basis in the Cartan plane is adequate to the continuum limit of the system under consideration, as the latter one is described by completely analogous expressions [2]. Now let us describe this basis in more invariant terms and find the generalization to arbitrary \( p \).
Comparing (31) with (26), we can conclude that $\partial \phi^{(i)} \equiv \beta_i \partial \phi$ corresponds to the basis
\[ \beta_{1,2} = \frac{1}{2} (\sqrt{3} \mu_2 \pm i \alpha_2). \] (31)

This basis has the properties
\[ \beta_1 \cdot \beta_2 = 1, \quad \beta_1 \cdot \beta_1 = 0, \quad \beta_2 \cdot \beta_2 = 0. \] (32)

Now it is rather evident how this basis should look in the case of general $p$. Due to (2) we can guess what is the choice of the proper scalar fields:
\[ \partial \phi^{(i)}(z) = \sum_k k t_k^{(i)} z^{k-1} + \sum_k \partial \partial_k^{(p-i)} z^{k-1}. \] (33)

This choice certainly corresponds to the basis with defining property (it can be observed immediately from the relations (33) and (16)):
\[ \beta_i \cdot \beta_j = \delta_{p,i+j}, \] (34)

the proper choice of the Hamiltonians in (15) being
\[ H = \sum_{i,k} t_k^{(i)} \beta_i \cdot J_k, \] (35)

what determines new times adequate to the continuum limit.

Let us construct the basis (34) in a manifest way. To begin with, we define a set of vectors $\{\mu_i\}$ with the property:
\[ \mu_i \cdot \mu_j = \delta_{ij} - \frac{1}{p}, \quad \sum_i \mu_i = 0. \] (36)

The $W^{(n)}$-algebra can be written in this basis as follows [27]:
\[ W^{(n)} = (-)^{n+1} \sum_{1 \leq j_1 < \ldots < j_n \leq p} \prod_{m=1}^{n} : (\mu_{j_m} \cdot \partial \phi);, \quad n = 2, \ldots, p. \] (37)

Now the basis (34) can be constructed from (36) by diagonalization of the following cyclic permutation [2, 33]:
\[ \mu_i \rightarrow \mu_{i+1}, \quad \mu_p \rightarrow \mu_1 \quad i = 1, \ldots, p - 1. \] (38)

This transformation has $\{\beta_i\}$ as its eigenvectors, their manifest expressions being of the form:
\[ \beta_k = \frac{1}{\sqrt{p}} \sum_{j=1}^{p} \exp\left\{ \frac{2\pi i}{p} jk \right\} \mu_j, \quad k = 1, 2, \ldots, p - 1. \] (39)
It is trivial to check that the properties (34) are indeed satisfied. One can immediately rewrite the corresponding \(W\)-generators in the basis of \(\beta_i\)'s. After all, one obtain the expressions similar to the continuum \(W\)-generators \([2, 33]\), but with the scalar fields defined as in \([33]\) and without the “anomaly” corrections appearing in the continuum case due to the twisted boundary conditions. These corrections can be correctly reproduced by taking the \(p\)-th root of the partition function as well as simultaneously doing the reduction (see the sects.4-5).

Thus, the proposed procedure allows one to take the continuum limit immediately transforming the scalar fields as elementary building blocks. Nevertheless, before immediate doing of the continuum limit for any CMM let us consider this in the simplest case of the Hermitian one-matrix model \([20]\) and get some insight for the general case.

3 Double-scaling limit of CMM: preliminary comments

3.1 Results of [20] for the one-matrix model

To begin with let us briefly remind the main points of \([20]\).

It has been suggested in \([2]\) that the square root of the partition function of the continuum limit of one-matrix model is subjected to the Virasoro constraints

\[
\mathcal{L}_n^{\text{cont}} \sqrt{\mathcal{Z}} dz = 0, \quad n \geq -1, \quad (40)
\]

where

\[
\mathcal{L}_n^{\text{cont}} = \sum_{k=0}^n \left( k + \frac{1}{2} \right) T_{2k+1} \frac{\partial}{\partial T_{2(k+n)+1}} + G \sum_{0 \leq k \leq n-1} \frac{\partial^2}{\partial T_{2k+1} \partial T_{2(n-k-1)+1}} + \frac{\delta_{0,n}}{16} + \frac{\delta_{-1,n} T_{1}^2}{(16G)} \quad (41)
\]

are modes of the stress tensor

\[
\mathcal{T}(z) = \frac{1}{2} \left( \partial \Phi(z) \right)^2 - \frac{1}{16z^2} = \sum \frac{\mathcal{L}_n}{z^{n+2}} \quad (42)
\]

where

\[
\partial \Phi(z) = \sum_{n \geq 0} \left( \left( n + \frac{1}{2} \right) T_{2n+1} z^{n+\frac{1}{2}} + \frac{\partial}{\partial T_{2n+1}} z^{-n-\frac{3}{2}} \right). \quad (43)
\]
It was shown in [20] that these equations which reflect the $W^{(2)}$-invariance of the partition function of the continuum model can be deduced from analogous constraints in Hermitian one-matrix model by taking the double-scaling continuum limit. The procedure (generalized below to CMM) is as follows.

The partition function of Hermitian one-matrix model can be written in the form

$$Z\{t_k\} = \int \mathcal{D}M \exp \text{Tr} \sum_{k=0} t_k M^k$$

and satisfies [18, 19] the discrete Virasoro constraints (1)

$$L_n^H Z = 0, \quad n \geq 0$$

$$L_n^H = \sum_{k=0}^{n} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_k \partial t_{n-k}}.$$  \hspace{1cm} (45)

In order to obtain the above-mentioned relation between $W$-invariance of the discrete and continuum models one has to consider a reduction of model (14) to the pure even potential $t_{2k+1} = 0$.

Let us denote by the $\tau_{N}^{\text{red}}$ the partition function of the reduced matrix model

$$\tau_{N}^{\text{red}}\{t_{2k}\} = \int \mathcal{D}M \exp \text{Tr} \sum_{k=0}^{t_{2k}} M^{2k}$$

and consider the following change of the time variables

$$g_m = \sum_{n \geq m} (-)^{n-m} \frac{(n+\frac{3}{2}) a^{-n+\frac{1}{2}}}{(n-m)! \Gamma \left( m + \frac{1}{2} \right) T_{2n+1}} T_{2n+1},$$

where $g_m \equiv mt_{2m}$ and this expression can be used also for the zero discrete time $g_0 \equiv N$ that plays the role of the dimension of matrices in the one-matrix model. Derivatives with respect to $t_{2k}$ transform as

$$\frac{\partial}{\partial t_{2k}} = \sum_{n=0}^{k-1} \frac{\Gamma \left( k + \frac{1}{2} \right) a^{n+\frac{1}{2}}}{(k-n-1)! \Gamma \left( n + \frac{3}{2} \right) \partial \hat{T}_{2n+1}} \frac{\partial}{\partial \hat{T}_{2n+1}},$$

where the auxiliary continuum times $\hat{T}_{2n+1}$ are connected with “true” Kazakov continuum times $T_{2n+1}$ via

$$T_{2k+1} = \hat{T}_{2k+1} + a \frac{k}{k+1/2} \hat{T}_{2(k-1)+1},$$

and coincide with $T_{2n+1}$ in the double-scaling limit when $a \to 0$. 

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Let us rescale the partition function of the reduced one-matrix model by exponent of quadratic form of the auxiliary times \( \tilde{T}_{2n+1} \)

\[
\tilde{\tau} = \exp \left(-\frac{1}{2} \sum_{m,n \geq 0} A_{mn} \tilde{T}_{2m+1} \tilde{T}_{2n+1}\right) \tau_N^{\text{red}}
\]  

(50)

with

\[
A_{nm} = \frac{\Gamma \left(n + \frac{3}{2}\right) \Gamma \left(m + \frac{3}{2}\right)}{2\Gamma^2 \left(\frac{3}{2}\right)} \frac{(-)^{n+m}a^{-n-m-1}}{n!m!(n+m+1)(n+m+2)}.
\]

(51)

Then a direct though tedious calculation \(^{20}\) demonstrates that the relation

\[
\frac{\tilde{L}_n}{\tilde{\tau}} = a^{-n} \sum_{p=0}^{n+1} C_p \left(-1\right)^{n+1-p} \frac{L_{2p}^{\text{red}}}{{\tau}^{\text{red}}},
\]

(52)

is valid, where

\[
L_{2n}^{\text{red}} \equiv \sum_{k=0}^{k_{2k}} k t_{2k} \frac{\partial}{\partial t_{2(k+n)}} + \sum_{0 \leq k \leq n} \frac{\partial^2}{\partial t_{2k} \partial t_{2(n-k)}}
\]

(53)

and

\[
\tilde{L}_n = \sum_{k \geq 0} \left(k + \frac{1}{2}\right) T_{2k+1} \frac{\partial}{\partial T_{2(k+1)}} + \frac{T^2_{1}}{16},
\]

\[
\tilde{L}_0 = \sum_{k \geq 0} \left(k + \frac{1}{2}\right) T_{2k+1} \frac{\partial}{\partial T_{2k+1}},
\]

\[
\tilde{L}_n = \sum_{k \geq 0} \left(k + \frac{1}{2}\right) T_{2k+1} \frac{\partial}{\partial T_{2(k+n)}},
\]

\[
+ \sum_{0 \leq k \leq n-1} \frac{\partial}{\partial T_{2k+1} \partial T_{2(n-k)}} + \left(-\right)^{n} \frac{1}{16a^n}, \quad n \geq 1.
\]

(54)

Here \( C_p = \frac{n!}{p!(n-p)!} \) are binomial coefficients.

These Virasoro generators differ from the Virasoro generators \(^{11}\) \(^{,2,15}\) by terms which are singular in the limit \( a \rightarrow 0 \). At the same time \( L_{2p}^{\text{red}} \) at the r.h.s. of (52) do not need to vanish, since

\[
0 = L_{2p} \tau \bigg|_{t_{2k+1}=0} = L_{2p}^{\text{red}} \tau^{\text{red}} \bigg|_{t_{2k+1}=0} + \sum_{i} \frac{\partial^2 \tau}{\partial t_{2i+1} \partial t_{2(n-i-1)+1}} \bigg|_{t_{2k+1}=0}.
\]

(55)

It was shown in \(^{20}\) that these two origins of difference between (41) and (54) actually cancel each other, provided eq.(52) is rewritten in terms of the square root \( \sqrt{\tilde{\tau}} \) rather than \( \tilde{\tau} \) itself:

\[
\frac{\tilde{L}_n}{\sqrt{\tilde{\tau}}} = a^{-n} \sum_{p=0}^{n+1} C_p \left(-1\right)^{n+1-p} \frac{L_{2p} \tau}{\tau} \bigg|_{t_{2k+1}=0} \left(1 + O(a)\right).
\]

(56)
3.2 Generalization of the Kazakov variables

Now we would like to generalize the procedure proposed in the sect.3.1 to the case of the general multi-matrix model. However, before doing this, we shall describe a simpler way to guess proper Kazakov variables as well as tilded time variables and the connection between discrete and continuum $W$-algebra. It can be done already at the level of the "leading" (or "quasiclassical" in accordance with [18]) terms, i.e. those including only the first derivatives. It is possible to do due to the fact that the constraint algebra of CMM strongly resembles the structure of the continuum $W$-algebra [2] (see (29), (30)).

To begin with, we would like to consider the simplest multi-matrix case of two matrices in details. The general case which is completely analogous to the two-matrix one but requires considerably more tedious calculations will be considered in the Appendix.

The leading terms of the $W^{(3)}$ and $W^{(3)}$ generators are given by the formulas

$$
\sqrt{3}W^{(3)}_{n}\text{lead} = \sum_{i=1}^{2} \sum_{k,l>0} k^{(i)} t_{l(3-i)} \frac{\partial}{\partial t_{k+n+l}}, \quad g^{(i)} = t_{k}^{(i)},
$$

(57)

$$
\sqrt{3}W^{(3)}\text{lead} = \sum_{k,m \geq 0} \left( k + \frac{1}{3} \right) \left( m + \frac{1}{3} \right) T_{3k+1}T_{3m+1} \frac{\partial}{\partial T_{3(k+m+n)+1}}
+ \sum_{k,m \geq 0} \left( k + \frac{2}{3} \right) \left( m + \frac{2}{3} \right) T_{3k+2}T_{3m+2} \frac{\partial}{\partial T_{3(k+m+n-1)+2}}.
$$

(58)

Now, to connect these two algebras, we have to find out a change of time variables similar to (17), (18) which gives the relation between leading terms of the generators (57), (58). This problem can be solved in two steps. First, we make a reduction of the discrete times

$$
t_{3k+1}^{(i)} = t_{3k+2}^{(i)} = 0, \quad i = 1, 2, \quad k = 0, 1, \ldots.
$$

(59)

Second, we note that the changing of time variables

$$
g^{(i)} = \sum_{n \geq m} \frac{(-)^{n-m} \Gamma \left( n + 1 + \frac{i}{3} \right) a^{-n-\frac{1}{3}}}{(n-m)! \Gamma \left( m + \frac{i}{3} \right)} T_{3n+i}, \quad i = 1, 2,
$$

(60)

$$
\frac{\partial}{\partial t_{3k}^{(i)}} = \sum_{n=0}^{k-1} \frac{\Gamma \left( k + \frac{i}{3} \right) a^{n+\frac{1}{3}}}{(k-n-1)! \Gamma \left( n + 1 + \frac{i}{3} \right)} \frac{\partial}{\partial T_{3n+i}}, \quad i = 1, 2,
$$

(61)
where \( g_n^{(i)} \) is equal to \( mt_{3n}^{(i)} \), and the auxiliary continuum times \( \tilde{T}_{3n+i} \) are connected with the Kazakov continuum times \( T_{3n+i} \) through

\[
T_{3k+i} = \tilde{T}_{3k+i} + a \frac{k}{k+i/3} \tilde{T}_{3(k-1)+i}.
\] (62)

It gives rise to the following relation between \( W^{(3)\text{lead}} \) and \( \hat{W}^{(3)\text{lead}} \)

\[
a^{-n} \sum_{p=0}^{n+2} C_p^m (-)^{n-p} \frac{\partial}{\partial \tilde{T}_{3m+2}} = \hat{W}^{(3)\text{lead}}, \quad n \geq -2,
\] (63)

where \( \hat{W}^{(3)\text{lead}} \) is given by (58) with shifted limits of the summation

\[
\sqrt{3} \hat{W}_n^{(3)\text{lead}} = \sum_{k+m \geq -n} \left( k + \frac{1}{3} \right) \left( m + \frac{1}{3} \right) T_{3k+1} T_{3m+1} \frac{\partial}{\partial \tilde{T}_{3(k+m+n)+2}}
\]

\[
+ \sum_{k+m \geq -n-1} \left( k + \frac{2}{3} \right) \left( m + \frac{2}{3} \right) T_{3k+2} T_{3m+2} \frac{\partial}{\partial \tilde{T}_{3(k+m+n+1)+1}}.
\] (64)

The summation in (64) includes the terms with times \( T_m, \ m < 0 \) defined by trivial continuing the equation (60).

The proof of (63) is based on the formulas (60), (61) and the following identity for the \( \Gamma \)-functions

\[
\sum_{\gamma=0}^{a} (-)^{\gamma} \Gamma(\gamma + b) \frac{\Gamma(c - b + a)}{\Gamma(a - \gamma)! \Gamma(c + \gamma)} = \frac{\Gamma(b) \Gamma(c - b + a)}{\Gamma(a + 1) \Gamma(c + a) \Gamma(c - b)}.
\] (65)

To eliminate the incorrect negative mode terms in (64) we use the partition function rescaled by an exponential of a quadratic form of the auxiliary times \( \tilde{T}_{3n+i} \) (compare with (59))

\[
\tilde{\tau} = \exp \left( - \sum_{m,n \geq 0} A_{mn} \tilde{T}_{3m+1} \tilde{T}_{3n+2} \right) \tau_{N_1,N_2}^{\text{red}},
\] (66)

where \( \tau_{N_1,N_2}^{\text{red}} \) is the partition function of the conformal two-matrix model after the reduction (39) and

\[
A_{nm} = \frac{\Gamma \left( n + \frac{4}{3} \right) \Gamma \left( m + \frac{5}{3} \right)}{\Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{2}{3} \right)} (-)^{n+m} a^{-n-m-1} \frac{n!m!(n+m+1)(n+m+2)}{n!m!(n+m+1)(n+m+2)}.
\] (67)

\[\text{Similar to the case considered in the previous subsection we have to include properly into the set of the independent discrete variables the sizes of matrices } N_i. \text{ These are just new variables of zero indices } g_0^{(i)}. \text{ In the continuum limit they are related to the cosmological constant (generally there should be many different cosmological constants). All this can be treated completely along the line of [20], so we will not discuss it here.}\]
This rescaling of the partition function is equivalent to the following transformation of the generators $\hat{W}_n^{(3)}$

$$\tilde{\hat{W}}_n^{(3)} = \exp \left( - \sum_{m,n \geq 0} A_{mn} \tilde{T}_{3m+1} \tilde{T}_{3n+2} \right) \hat{W}_n^{(3)} \exp \left( \sum_{m,n \geq 0} A_{mn} \tilde{T}_{3m+1} \tilde{T}_{3n+2} \right),$$

(68)

where now not only leading terms should be taken into account in (68).

Then direct calculations similar to the calculations in [20] show that the changing of the times (60), (61) and the rescaling (66) establish the relation like (63)

$$\frac{1}{\tau} \tilde{\hat{W}}_p^{(3)} \tau = a^{-p} \sum_{n=0}^{p+2} C_{p+2}^n (-)^{p-n} W_{3n}^{(3)} \frac{\tau^{\text{red}}}{\tau^{\text{red}}}, \quad p \geq -2,$$

(69)

where the generators $W_{3n}^{(3)} \tau^{\text{red}}$ depend only on the times $t_{3n}^{(1,2)}$ and derivatives with respect to them, and the generators $\tilde{\hat{W}}_p^{(3)}$ are exactly the generators of the $W^{(3)}$-symmetry of the continuum model [2].

By definition, the generators $W_n^{(3)}$ annihilate the partition function (20) for $p = 3$ while $W_{3n}^{(3)} \tau^{\text{red}}$ does not need to vanish, since

$$0 = W_{3n}^{(3)} \tau \bigg|_{t_{3k+i}=0; \ i=1,2} = W_{3n}^{(3)} \tau^{\text{red}} + \ldots,$$

(70)

where $\ldots$ means terms similar to $\sum_{m,k} \partial^3 \log \tau / \partial t_{3m+1} \partial t_{3n-3(m+k)+1} \partial t_{3k-2}$ with all possible correct gradations which do not vanish under the reduction (59). In analogy with the results of [20], these additional terms provides the eq.(69) can be rewritten in the terms of the cubic root of the $\tau$-function $\sqrt[3]{\tau}$ (see the sect.5.2 for detailed explanations)

$$\frac{\mathcal{W}_n^{(3)} \text{ cont } \sqrt[3]{\tau}}{\sqrt[3]{\tau}} = a^{-n} \sum_{p=0}^{n+2} C_{n+2}^p (-)^{n-p} W_{3p}^{(3)} \frac{\tau^{\text{red}}}{\tau} \bigg|_{t_{3k+i}=0; \ i=1,2} \left(1 + O(a)\right).$$

(71)

We exclude here all simple but tedious calculations of general case, since for our consideration of CMM below we will use a more efficient way to deal with the changing of the time-variables $t \to T$ (which was also proposed in [20]), that is, a scalar field formalism. For example, in the case of the one-matrix model the Kazakov change of the time variables (47), (48) can be deduced from the following prescription. Let us consider the free scalar field with periodic boundary conditions (33) for $p = 2$

$$\partial \varphi(u) = \sum_{k \geq 0} g_k u^{2k-1} + \sum_{k \geq 1} \frac{\partial}{\partial t_{2k}} u^{-2k-1},$$

(72)
and analogous scalar field with antiperiodic boundary conditions (43):

\[
\partial \Phi(z) = \sum_{k \geq 0} \left( \left( k + \frac{1}{2} \right) T_{2k+1} z^{k+\frac{1}{2}} + \frac{\partial}{\partial T_{2k+1}} z^{-k-\frac{1}{2}} \right).
\]

(73)

Then the equation

\[
\frac{1}{\tau} \partial \Phi(z) \tilde{\tau} = a \frac{1}{\tau_{\text{red}}} \partial \varphi(u) \tau_{\text{red}}, \quad u^2 = 1 + az
\]

(74)
generates the correct transformation rules (47), (48) and gives rise to the expression (51) for \( A_{nm} \). Taking the square of both sides of the identity (74),

\[
\frac{1}{\tau} T(z) \tilde{\tau} = \frac{1}{\tau_{\text{red}}} T(u) \tau_{\text{red}},
\]

(75)
one can obtain after simple calculations that the relation (52) is valid.

4 The case of \( W(3) \) in scalar field formalism

4.1 Scalar field formalism

In this section we would like to consider the case of \( W(3) \)-algebra in full details in the framework of the scalar field formalism described at the end of the previous section. In this subsection we describe the formalism, and, in the following ones, we apply it to Virasoro and \( W(3) \)-algebra respectively.

Thus, let us consider the set of scalar fields for the discrete two-matrix model

\[
\partial \varphi^{(1)}(u) = \sum_{k \geq 0} g_k^{(1)} u^{3k-1} + \sum_{k \geq 1} \frac{\partial}{\partial T_{3k}} u^{-3k-1},
\]

\[
\partial \varphi^{(2)}(u) = \sum_{k \geq 0} g_k^{(2)} u^{3k-1} + \sum_{k \geq 1} \frac{\partial}{\partial T_{3k}} u^{-3k-1},
\]

(76)

and the scalar fields of the continuum model [2]

\[
\partial \Phi^{(1)}(z) = \sum_{k \geq 0} \left( \left( k + \frac{1}{3} \right) T_{3k+1} z^{k+\frac{2}{3}} + \frac{\partial}{\partial T_{3k+2}} z^{-k-\frac{1}{3}} \right),
\]

\[
\partial \Phi^{(2)}(z) = \sum_{k \geq 0} \left( \left( k + \frac{2}{3} \right) T_{3k+2} z^{k+\frac{2}{3}} + \frac{\partial}{\partial T_{3k+1}} z^{-k-\frac{2}{3}} \right).
\]

(77)

Then all the relations (60), (61) and (67) can be encoded in the equations

\[
\frac{1}{\tau} \partial \Phi^{(i)}(z) \tilde{\tau} = a_u u^{i-2} \frac{1}{\tau_{\text{red}}} \partial \varphi^{(i)}(u) \tau_{\text{red}}, \quad u^3 = 1 + az, \quad i = 1, 2.
\]

(78)
Let us prove (78) for the case of \( i = 2 \) only. The \( i = 1 \) case can be treated analogously. We have to check that the relation \( u^3 = 1 + az \), with using (60) and (61), gives rise to the following equation

\[
a \sum_{k \geq 0} g^{(2)}_k u^{3k-1} + a \sum_{k \geq 1} \frac{\partial \ln \tau}{\partial \tau_{3k}} u^{-3k-1} = \\
= \sum_{k \geq 0} \left( k + \frac{2}{3} \right) T_{3k+2} z^{k-\frac{1}{3}} + a \sum_{k \geq 0} \left( \frac{\partial \ln \tau}{\partial T_{3k+1}} z^{-k-\frac{4}{3}} - \sum_{m \geq 0} A_{mk} T_{3m+2} z^{-k-\frac{4}{3}} \right). \quad (79)
\]

Let us start our check with the derivative terms. Using the obvious relations

\[
u^3 = (1 + az)^\alpha = \sum_{m \geq 0} \frac{\Gamma(\alpha + 1)}{m! \Gamma(\alpha + 1 - m)} (az)^{\alpha - m} \quad (80)
\]

and (61), we obtain

\[
\sum_{k \geq 1} \frac{\partial}{\partial \tau_{3k}} u^{-3k-1} = \\
= \sum_{k \geq 1} ^{k-1} \sum_{f=0} a^f \sum_{m \geq 0} \frac{\Gamma(k + \frac{4}{3}) \Gamma(-k - \frac{2}{3}) a^{f-m-k}}{(k - f - 1)! \Gamma(f + \frac{4}{3}) m! \Gamma(-k - m + \frac{2}{3})} \frac{\partial}{\partial T_{3f+1}} z^{-m-k-\frac{4}{3}} = \\
= \sum_{f \geq 0} a^f \sum_{\gamma \geq 0} \Gamma(f + \frac{4}{3}) \Gamma(-\gamma - \frac{1}{3}) \frac{\partial}{\partial T_{3f+1}} z^{-\gamma-\frac{4}{3}} \delta_{f,\gamma} = a^{-1} \sum_{f \geq 0} \frac{\partial}{\partial T_{3f+1}} z^{-f-\frac{4}{3}}, \quad (81)
\]

where we replaced the variable of the summation \( k \to \gamma = k + m \) and used the identity

\[
\sum_{m \geq 0} \frac{(-m)^\gamma}{m!(\gamma - f - m)!} = \delta_{f,\gamma}. \quad (82)
\]

The first sum in the l.h.s. of (79), after the substitution \( u^3 = 1 + az \) and using (60), falls into two different sums

\[
\sum_{k \geq 0} kt_{3k}^{(2)} u^{3k-1} = \sum_{k \geq 0} \sum_{n \geq k} \sum_{m=0}^k \frac{\Gamma(\gamma + \frac{5}{3}) (\gamma) a^{k-m-n-1}}{\Gamma(k - m + \frac{2}{3}) (n - k)! m!} T_{3n+2} z^{k-m-\frac{1}{3}} \\
+ \sum_{k \geq 0} \sum_{n \geq k} \sum_{m \geq k+1} \frac{\Gamma(\gamma + \frac{5}{3}) (\gamma) a^{k-m-n-1}}{\Gamma(k - m + \frac{2}{3}) (n - k)! m!} T_{3n+2} z^{k-m-\frac{4}{3}}. \quad (83)
\]

The first sum in (83) can be rewritten in the form

\[
\sum_n \sum_{k=0}^n \sum_{m \geq k} \frac{\Gamma(\gamma + \frac{5}{3}) (\gamma) a^{k-m-n-1}}{\Gamma(k - m + \frac{2}{3}) (n - k)! m!} T_{3n+2} z^{k-m-\frac{1}{3}} = 
\]

18
\[
\sum_{n \geq 0} \sum_{\gamma=0}^{n} \sum_{k=0}^{n} T_{3n+2} \frac{\Gamma\left(n + \frac{5}{3}\right)}{\Gamma\left(\gamma + \frac{1}{3}\right)} \frac{(-)^{n} a^{\gamma-n-1}}{(n-k)! (k-\gamma)!} z^{\gamma-\frac{4}{3}} = \sum_{n \geq 0} T_{3n+2} \frac{\Gamma\left(n + \frac{5}{3}\right)}{\Gamma\left(\gamma + \frac{1}{3}\right)} a^{\gamma-n-1} z^{\gamma-\frac{4}{3}} \delta_{n,\gamma} = a^{-1} \sum_{n \geq 0} \left(n + \frac{2}{3}\right) T_{3n+2} z^{n-\frac{4}{3}},
\]

where we used the identity (82). After introducing the new variable of the summation \( \beta = m - k - 1 \), changing the order of the summation and using the identity

\[
\sum_{k=0}^{n} \frac{(-)^{k}}{(n-k)! (k+\beta+1)!} = \frac{1}{\beta! n!(\beta + n + 1)},
\]

the second sum in the r.h.s. of (83) can be rewritten in the form

\[
\frac{a^{-1}}{\beta \geq 0} \frac{\sum_{\alpha \geq 0} \left(\tilde{T}_{3n+2} + a^{-1} \frac{n+\frac{5}{3}}{\beta} \bar{T}_{3n+1}\right)}{\beta \geq 0 \frac{\sum_{n \geq 0} \tilde{T}_{3n+2} \frac{\Gamma\left(n + \frac{5}{3}\right)}{\Gamma\left(\beta - \frac{1}{3}\right)} (-)^{n} a^{-n-1}}{\beta \geq 0 \frac{\sum_{n \geq 0} A_{n,\beta} T_{3n+2} z^{\beta-\frac{4}{3}}}{\beta \geq 0 \frac{\sum_{n \geq 0} A_{n,\beta} T_{3n+2} z^{\beta-\frac{4}{3}}}{\beta \geq 0 \frac{\sum_{n \geq 0} A_{n,\beta} T_{3n+2} z^{\beta-\frac{4}{3}}}}}
\]

and from the last two lines of (84) the expression (77) for the matrix \( A_{nm} \) follows.

Performing the similar calculations for the case of \( i = 1 \) in (78), one can find the same expression for the matrix \( A_{nn} \). So, we may conclude that the changing of times (51), (61) results in the equations (78).

### 4.2 Virasoro constraints of the two-matrix model

It was proposed in [4] that the continuum “two-matrix” model possesses the \( \mathcal{W}^{(3)} \) and Virasoro symmetries, the Virasoro generators \( \mathcal{L}_n \) and the generators of the \( \mathcal{W}^{(3)} \) algebra being constructed from the scalar fields (77) in the following way

\[
\mathcal{T}(z) = \frac{1}{2} : \partial \Phi^{(1)}(z) \partial \Phi^{(2)}(z) : - \frac{1}{9z^2} = \sum_{n \geq 0} \frac{\mathcal{L}_n}{z^{n+2}},
\]

\[
\mathcal{W}^{(3)}(z) = \frac{1}{3\sqrt{3}} \left( : \partial \Phi^{(1)}(z) \partial \Phi^{(2)}(z) : \right)^3 = \sum_{n \geq 0} \frac{\mathcal{W}^{(3)}_n}{z^{n+3}}.
\]

We will prove below that the equations (78) yield the relation between the generators \( \tilde{\mathcal{L}}_n \) and the corresponding generators \( W^{(2)}_n \) associated with the reduction (59): \( \frac{1}{\tilde{T}} \tilde{\mathcal{L}}_n \tilde{T} = a^{-n} \sum_{p=0}^{n+1} C_{p+1}^{n+1} (-)^{n+1-p} \frac{L_{3p}^{\text{red}} \tau_{\text{red}}^{\text{red}}}{\tau_{\text{red}}}, \quad n \geq -1, \)
where Virasoro generators $L_{3n}^{\text{red}}$ are defined by the formula (29), where only $t^{(i)}_{3n}$ and corresponding derivatives are nonzero and the generators

$$
\tilde{L}_1 = \sum_{k \geq 1} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3k-2}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3k-1}} \right) + \frac{2}{9} T_1 T_2,
$$

$$
\tilde{L}_0 = \sum_{k \geq 0} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3k+1}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3k+2}} \right),
$$

$$
\tilde{L}_n = \sum_{k-m=-n} \left( \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3m+1}} + \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3m+2}} \right)
+ \sum_{m+k=n+1} \frac{\partial}{\partial T_{3k+2}} \frac{\partial}{\partial T_{3m+1}} + \frac{(-)^n}{9a^n}, \quad n \geq 1
$$

(90)

differ from the continuum generators $L_n$ of the ref. [2] by singular $c$-number terms. Indeed, again $L_{3n}^{\text{red}}$s do not annihilate $\tau^{\text{red}}$ exactly, but these two effects cancel each other, provided eq. (89) for $\tilde{\tau}$ is rewritten in terms of the cubic root of the $\tau$-function $\sqrt[3]{\tilde{\tau}}$. In other words, doing accurately the reduction procedure in the Virasoro constraints of the discrete two-matrix model one can rewrite (89) in the form

$$
\frac{1}{\sqrt[3]{\tilde{\tau}}} L_n^{\text{cont}} \sqrt[3]{\tilde{\tau}} = a^{-n} \sum_{p=0}^{n+1} C_p^{n+1} (-)^{n+1-p} \frac{L_{3p}^{\text{red}} \tau}{\tau} (1 + O(a)) \bigg|_{z_{3k+i}=0; \ i=1,2}, \quad n \geq -1.
$$

(91)

Thus, we conclude that, indeed, the continuum Virasoro constraints for the case of $p=3$ can be derived from the corresponding Virasoro constraints of the discrete conformal two-matrix model.

To prove (89) let us calculate

$$
\frac{\Phi^{(1)}(z)\tilde{\tau}}{\tilde{\tau}} \frac{\Phi^{(2)}(z)\tilde{\tau}}{\tilde{\tau}} = \sum_{n \geq 0} z^{-n-2} \left[ \sum_{k \geq 0} \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3(k+n)+1}} \right.
+ \left. \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3(k+n)+2}} \frac{n}{k=0} \frac{\partial}{\partial T_{3k+2}} \frac{\partial}{\partial T_{3(n-1-k)+2}} \right]
+ \frac{1}{z} \left[ \sum_{k \geq 1} \left( k + \frac{1}{3} \right) T_{3k+1} \frac{\partial}{\partial T_{3(k-1)+1}} \right. + \left. \left( k + \frac{2}{3} \right) T_{3k+2} \frac{\partial}{\partial T_{3(k-1)+2}} \frac{2}{9} T_1 T_2 \right],
$$

(92)

where in (82) only singular at $z \to \infty$ terms are taken into account.

Using at $n \geq 1$ the identity

$$
\sum_{k+m=n+1 \atop k,m \geq 0} A_{km} = \sum_{k+m=n+1 \atop k,m \geq 0} \frac{\Gamma \left( k + \frac{5}{3} \right) \Gamma \left( m + \frac{4}{3} \right) (-)^{n-1} a^{-n}}{k!m! \Gamma \left( \frac{2}{3} \right) \Gamma \left( \frac{1}{3} \right) n(n+1)} = \frac{(-)^{n-1}}{9a^n},
$$

(93)
the formula (92) can be rewritten in the compact form
\[
\Phi^{(1)}(z) \frac{\tau}{\tilde{\tau}} \Phi^{(2)}(z) \frac{\tau}{\tilde{\tau}} = \sum_{n \geq 1} \frac{\tilde{\mathcal{L}}_n \tau}{\tilde{\tau}} - \sum_{k+m=n-1} \frac{\partial^2 \ln \tau}{\partial T_{3k+2} \partial T_{3m+1}} .
\] (94)

Using the fact that the generators \(L_{3n}^\text{red}\) of the discrete Virasoro algebra can be obtained from the formula
\[
\sum_n u^{-3n-2} L_{3n} = \frac{1}{2} \partial \varphi^{(1)}(u) \partial \varphi^{(2)}(u) ,
\] (95)
one can easily show that
\[
a^2 \frac{\partial \varphi^{(1)}(u) \tau}{\partial \tau} \frac{\partial \varphi^{(2)}(u) \tau}{\partial \tau} = a^2 \sum_{n \geq 0} u^{-3(n+1)} \left[ \frac{L_{3n} \tau}{\tau} - \sum_{m+k=n, m,k \geq 1} \frac{\partial^2 \ln \tau}{\partial \mathcal{T}_{3k} \partial \mathcal{T}_{3m}} \right] ,
\] (96)
where again only the terms singular at \(u \to \infty\) are taken into account. Now using the equations (78) we can conclude that
\[
\sum_{n \geq 1} z^{-n-2} \left[ \frac{\tilde{\mathcal{L}}_n \tau}{\tilde{\tau}} - \sum_{k+m=n-1} \frac{\partial^2 \ln \tau}{\partial T_{3k+2} \partial T_{3m+1}} \right] = a^2 \sum_{n \geq 0} u^{-3(n+1)} \left[ \frac{L_{3n} \tau}{\tau} - \sum_{m+k=n, m,k \geq 1} \frac{\partial^2 \ln \tau}{\partial \mathcal{T}_{3k} \partial \mathcal{T}_{3m}} \right] .
\] (97)

It follows now from (80) that
\[
\frac{\tilde{\mathcal{L}}_n \tau}{\tilde{\tau}} - \sum_{k+m=n-1, k,m \geq 0} \frac{\partial^2 \ln \tau}{\partial T_{3k+2} \partial T_{3m+1}} = a^{-n} \sum_{p=0}^{n+1} C_{n+1}^p (-1)^{n+1-p} \left[ \frac{L_{3p} \tau}{\tau} - \sum_{m+k=n, m,k \geq 1} \frac{\partial^2 \ln \tau}{\partial \mathcal{T}_{3k} \partial \mathcal{T}_{3m}} \right] .
\] (98)

If we take into account that
\[
a^{-n} \sum_{p=0}^{n+1} (-1)^{n+1-p} C_{p}^{n+1} \sum_{m+k=p, m,k \geq 1} \frac{\partial^2}{\partial \mathcal{T}_{3k} \partial \mathcal{T}_{3m}} =
\]
\[
= a^{-n} (-1)^{n+1} \sum_{f \geq 0, g \geq 0} \frac{\partial^2}{\partial T_{3f+1} \partial T_{3g+2}} \frac{\Gamma \left( f + \frac{4}{3} \right) \Gamma \left( g + \frac{5}{3} \right)}{\Gamma \left( p + \frac{2}{3} \right) \Gamma \left( m + \frac{1}{3} \right)} \times \sum_{p=2}^{n+1} \sum_{m=1}^{p-1} (-1)^p C_{n+1}^p \frac{\Gamma \left( m + \frac{1}{3} \right) \Gamma \left( p - m + \frac{2}{3} \right)}{(m-f-1)!(p-m-g-1)!} = \sum_{f \geq 0, g \geq 0} \frac{\partial^2}{\partial T_{3f+1} \partial T_{3g+1}} ,
\] (99)
we may conclude that (89) is correct.
4.3 Connection between $\mathcal{W}_{n}^{(3)}$ and $W_{3n}^{(3)}$

Now we are going to repeat the procedure of the previous subsection for the $W^{(3)}$-algebra generators. The connection between the discrete and the continuum cases was formulated in the section 4.1 and based on the formula (69) which can be easily proved by means of the equations (78). Indeed, it is easy to see that the generators $W_{n}^{(3)\text{ red}}$ of the reduced discrete model can be rewritten using the scalar fields (76) as

$$\sum_{n} u^{-3n-3} W_{3n}^{(3)} = \frac{1}{3\sqrt{3}} \frac{1}{u^{3}} : (\partial \varphi^{(1)}(u))^{3} : + : (\partial \varphi^{(2)}(u))^{3} :$$  \hspace{1cm} (100)

Then the relation between the generators of the $W^{(3)}$ symmetry for the discrete model and the generators $\tilde{W}_{n}^{(3)}$ is

$$\frac{1}{\tau} \tilde{W}_{p}^{(3)} \tilde{\tau} = a^{-p} \sum_{n=0}^{p+2} C_{p+2}^{n} (-)^{p-n} \frac{W_{3n}^{(3)\text{ red}}}{\tau_{\text{red}}}, \quad p \geq -2$$  \hspace{1cm} (101)

and follows from the identity

$$\left( \frac{\partial \Phi^{(1)}(z) \tau}{\tilde{\tau}} \right)^{3} + \left( \frac{\partial \Phi^{(2)}(z) \tau}{\tilde{\tau}} \right)^{3} = a^{3} \left[ \frac{1}{u^{3}} \left( \frac{\partial \varphi^{(1)}(u)}{\tau_{\text{red}}} \right)^{3} + \left( \frac{\partial \varphi^{(2)}(u)}{\tau_{\text{red}}} \right)^{3} \right].$$  \hspace{1cm} (102)

The generators $\tilde{W}_{n}^{(3)}$ $n \geq -2$ are defined by the formulas (88) and the generators $\tilde{W}_{-2}$ and $\tilde{W}_{-1}$ (but not the $\tilde{W}_{0}^{(3)}$) have additional terms, cubic in times:

$$\tilde{W}_{-1}^{(3)} = \frac{1}{27} T_{1}^{3} + \cdots \quad \text{and} \quad \tilde{W}_{-2}^{(3)} = \frac{8}{27} T_{2}^{3} + \frac{4}{9} T_{1}^{2} T_{4} + \cdots.$$  \hspace{1cm} (103)

To obtain (101) from (102) one need the set of the combinatorial identities

$$a^{-n} \sum_{f=0}^{n+2} C_{n+2}^{f} (-)^{n-f} \sum_{k+m+p=f} \frac{\partial^{3}}{\partial T_{3k}^{(1)} \partial T_{3m}^{(1)} \partial T_{3p}^{(3)}} = \sum_{\alpha+\beta+\gamma=n-1} \frac{\partial^{3}}{\partial T_{3\alpha+i} \partial T_{3\beta+i} \partial T_{3\gamma+i}},$$

$$a^{-n} \sum_{f=0}^{n+2} C_{n+2}^{f} (-)^{n-f} \sum_{m,p \geq 1} \left( g_{m+p-n+1} + \frac{\partial \ln \tau}{\partial h_{n-1-m-p}^{(2)}} \right) \frac{\partial^{2}}{\partial h_{3m}^{(1)} \partial h_{3p}^{(1)}} =$$

$$= \sum_{\beta+\gamma+\alpha=n-1} \left( \alpha + \frac{2}{3} \right) T_{3\alpha+1} \frac{\partial^{2}}{\partial T_{3\beta+2} \partial T_{3\gamma+2}} + \sum_{\beta+\gamma+\alpha=n-2} \frac{\partial \ln \tilde{\tau}}{\partial T_{3\alpha+2} \partial T_{3\beta+2} \partial T_{3\gamma+2}},$$

$$a^{-n} \sum_{f=0}^{n+2} C_{n+2}^{f} (-)^{n-f} \sum_{m,p \geq 1} \left( g_{m+p-n+1} + \frac{\partial \ln \tau}{\partial h_{n-1-m-p}^{(2)}} \right) \frac{\partial^{2}}{\partial h_{3m}^{(1)} \partial h_{3p}^{(1)}} =$$

$$= \sum_{\beta+\gamma+\alpha=n} \left( \alpha + \frac{2}{3} \right) T_{3\alpha+2} \frac{\partial^{2}}{\partial T_{3\beta+1} \partial T_{3\gamma+1}} + \sum_{\beta+\gamma+\alpha=n-1} \frac{\partial \ln \tilde{\tau}}{\partial T_{3\alpha+1} \partial T_{3\beta+1} \partial T_{3\gamma+1}}.$$  

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which can be proved by the direct calculation using the formulas (60), (61) and identities for the Γ-functions similar to

\[
\sum_{m=0}^{d} \frac{\Gamma(m+b)\Gamma(c-m)}{m!(a-m)!} = \frac{\Gamma(c-d)\Gamma(b)\Gamma(b+c)}{d!\Gamma(b-c-d)}.
\]

(104)

5 General case

5.1 The continuum limit of \(W^{(n)}\)-algebra

It is easy to generalize the formalism of the scalar field described in the sect.4.2-4.3 to the general conformal multi-matrix models using the scalar fields with \(\mathbb{Z}_p\)-twisted boundary conditions. Let us introduce \(p-1\) sets of the discrete times \(t_{i}^{(i)}\), \(i = 1, 2, \ldots, p-1\) and \(k = 0, 1, \ldots\) for the discrete \((p-1)\)-matrix model and consider the reduction

\[i_{pk+j}^{(i)} = 0, \ i, j = 1, 2, \ldots, p-1, \ k = 0, 1, \ldots.
\]

(105)

One can choose the "discrete" and the "continuum" scalar fields in the form

\[
\partial \varphi^{(i)}(u) = \sum_{k \geq 0} g_{k}^{(i)} u^{pk-1} + \sum_{k \geq 0} \frac{1}{\partial t_{pk}^{(p)\nu}} u^{-pk-1}, \quad g_{k}^{(i)} = k^i_{pk}, \quad g_{0}^{(i)} = N_{i},
\]

(106)

\[
\partial \Phi^{(i)}(z) = \sum_{k \geq 0} \left\{ \left(k + \frac{i}{p}\right)^{k^i_{pk}+1} T_{pk+i} z^{-k^{-\frac{2\nu}{p}}} + \frac{1}{\partial T_{pk+p-i}^{(p)\nu}} z^{-k^{-\frac{2\nu}{p}}} \right\},
\]

(107)

\[
T_{pk+i} = T_{pk+i} \sum_{k \geq 0} \frac{1}{k + i/p} \tilde{T}_{pk+p-i}, \ i = 1, 2, \ldots, p-1.
\]

(108)

Then the equations

\[a u^{-p+1} \frac{1}{\tau_{\text{red}}} \partial \varphi^{(i)}(u) \tau_{\text{red}} = \frac{1}{\tau} \partial \Phi^{(i)}(z) \tau, \ u^{\nu} = 1 + az, \ i = 1, 2, \ldots, p-1,
\]

(109)

generate Kazakov-like change of the time variables

\[
g_{m}^{(i)} = \sum_{n \geq m} \frac{(-)^{n-m} \Gamma(n+1+\frac{i}{p}) a^{-\frac{n-\nu}{p}}}{(n-m)! \Gamma(m+\frac{i}{p})} T_{pn+i}, \ i = 1, 2, \ldots, p-1
\]

(110)

\[
\frac{\partial}{\partial t_{pk}^{(i)}} = \sum_{n=0}^{k-1} \frac{1}{(k-n-1)! \Gamma(n+1+\frac{i}{p})} \partial T_{pn+i}, \ i = 1, 2, \ldots, p-1.
\]

(111)
Then the consideration similar to those used in the previous subsections shows that the relation between the tilded continuum generators \( \tilde{W}_n^{(i)}, i = 2, \ldots, p \) of the \( \mathcal{W} \)-symmetry and the reduced generators of the discrete \( W \)-symmetry \( W_{\text{red}}^{(i)} \) is as follows:

\[
\frac{1}{\tau} \tilde{W}_n^{(i)} \tau = a^{-n} \sum_{s=0}^{n+i-1} C_{n+i-1}^s (-1)^{n+i-1-s} W_{\text{red}}^{(i)} \frac{\tau_{\text{red}}}{\tau}, \quad n \geq -i + 1, \tag{112}
\]

where the rescaled \( \tau \)-function is defined:

\[
\tilde{\tau} = \exp \left( -\frac{1}{2} \sum_{i=1}^{p-1} \sum_{m,n \geq 0} A_{nm}^{(i)} \tilde{T}_{pm+i} \tilde{T}_{pm+p-i} \right) \tau_{\text{red}} \{t_{pk}\} \tag{113}
\]

and the matrices \( A_{nm}^{(i)} \) are determined by:

\[
A_{nm}^{(i)} = \frac{\Gamma \left( n + \frac{p+i}{p} \right) \Gamma \left( m + \frac{2p-i}{p} \right)}{\Gamma \left( \frac{i}{p} \right) \Gamma \left( \frac{p-i}{p} \right)} \frac{(-1)^{n+m}a^{-n-m-1}}{n!m!(n+m+1)(n+m+2)}, \quad i = 1, 2, \ldots, p-1. \tag{114}
\]

The proof of the equivalence of (109) to (110), (111), (113) and (114) is the same as in the sect.4.2-4.3 and we omit it here. To prove the equations (112), we need the exact expressions how the generators \( \tilde{W}_n^{(i)} \) and \( W_{\text{red}}^{(i)} \) are connected with corresponding scalar fields (106), (107).

Performing the proper reduction procedure (105), which eliminates all time-variables excepting those of the form \( t_{pk}^{(i)} \) (i.e. leaves the \( 1/p \) fraction of the entire quantity of variables) we can obtain the relation:

\[
\frac{1}{\sqrt{\tau}} \mathcal{W}_n^{(i)} \sqrt{\tau} = a^{-n} \sum_{s=0}^{n+i-1} C_{n+i-1}^s (-1)^{n+i-1-s} W_{\text{red}}^{(i)} \tau \bigg|_{t_{pk}^{(i)} \neq 0 \text{ only}} (1 + O(a)), \quad n \geq -i + 1, \tag{115}
\]

where \( \mathcal{W}_n^{(i)} \)'s are the \( \mathcal{W} \)-generators of the paper \([2]\). Thus, we proved the \( \mathcal{W} \)-invariance of the partition function of the continuum \( p - 1 \)-matrix model and found the explicit connection of its partition function with the corresponding partition function of the discrete \( (p-1) \)-matrix model.

### 5.2 On the reduction of the partition function

To conclude this section we would like to discuss the problem of reduction (59) and (105) of the partition function in detail, with accuracy up to (non-leading) \( c \)-number contributions and only after the continuum limit is taken. More precisely, we reformulate the condition...
of a proper reduction in the continuum limit in order to reduce it to more explicit formulas which can be immediately checked. As a by-product of our consideration we obtain some restrictions on the integration contour in the partition function (20).

To get some insight, let us consider the simplest case of the Virasoro constrained Hermitian one-matrix model [20]. Before the reduction the Virasoro operators read as in (15). Then their action on \( \log \tau \) can be rewritten as

\[
\left[ \sum_k kt_k \frac{\partial \log \tau}{\partial t_{k+n}} + \sum_m \frac{\partial^2 \log \tau}{\partial t_m \partial t_{n-m}} \right] + \sum_m \left[ \frac{\partial \log \tau}{\partial t_m} \frac{\partial \log \tau}{\partial t_{n-m}} \right] = 0. \tag{116}
\]

After the reduction, we obtain

\[
\left[ \sum_k 2kt_{2k} \frac{\partial \log \tau_{\text{red}}}{\partial t_{2k+2m}} + \sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{2m} \partial t_{2n-2m}} + \sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{2m+1} \partial t_{2n-2m-1}} \right] + \sum_m \left[ \frac{\partial \log \tau_{\text{red}}}{\partial t_{2m}} \frac{\partial \log \tau_{\text{red}}}{\partial t_{2n-2m}} \right] = 0 \tag{117}
\]

under the condition

\[
\left. \frac{\partial \log \tau_{\text{red}}}{\partial t_{\text{odd}}} \right|_{t_{\text{odd}}=0} = 0. \tag{118}
\]

The last formula is a direct consequence of the “Schwinger-Dyson” equation induced by the transformation of the reflection \( M \rightarrow -M \) in (14). Indeed, due to the invariance of the integration measure under this transformation one can conclude that the partition function (14) depends only on a quadratic form of odd times.

Thus, the second derivatives of \( \log \tau \) over odd times do not vanish, and are conjectured to satisfy the relation

\[
\sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{2m} \partial t_{2n-2m}} \sim - \sum_m \frac{\partial^2 \log \tau_{\text{red}}}{\partial t_{2m+1} \partial t_{2n-2m-1}}, \tag{119}
\]

where the sign ”~” implies that this relation should be correct only after taking the continuum limit. In this case one obtains the final result (cf. (53))

\[
\left[ \sum_k kt_{2k} \frac{\partial \log \sqrt{\tau_{\text{red}}}}{\partial t_{2k+2n}} + \sum_m \frac{\partial^2 \log \sqrt{\tau_{\text{red}}}}{\partial t_{2m} \partial t_{2n-2m}} \right] = 0. \tag{120}
\]

Thus, it remains to check the correctness of the relation (119). To do this, one should use the manifest equations of integrable (Toda chain) hierarchy, and after direct but tedious calculations [20] one obtains the result different from the relation (119) by \( c \)-number terms which are singular in the limit \( a \rightarrow 0 \) and just cancell corresponding items in (14) (this is certainly correct only after taking the continuum limit).
All this (rather rough) consideration can be easily generalized to the \(p\)-matrix model case. In this case one should try to use all \(W^{(i)}\)-constraints with \(2 \leq i \leq p\). Thus, the second derivatives should be replaced by higher order derivatives, and one obtain a series of equations like (116). It is the matter of trivial calculation to check that these equations really give rise to the proper constraints satisfied by \(\sqrt[p]{\tau}\) (cf. (71) and (115)) provided by the two sets of the relations like (118) and (119).

Namely, the analog of the relation (118) in the \(p\)-matrix model case is the cancellation of all derivatives with incorrect gradation, i.e. with the gradation non-equal to zero by modulo \(p\). The other relation (119) should be replaced now by the conditions of the equality (in the continuum limit) of all possible terms with the same correct gradation. In the simplest case of \(p = 3\) these are

\[
\sum_m \frac{\partial^2 \log \tau^{\text{red}}}{\partial t_{3m+1} \partial t_{3n-3m-1}} \sim \sum_m \frac{\partial^2 \log \tau^{\text{red}}}{\partial t_{3m} \partial t_{3n-3m}},
\]

\[
\sum_{m,k} \frac{\partial^3 \log \tau^{\text{red}}}{\partial t_{3m+1} \partial t_{3n-3(m+k)+1} \partial t_{3k-2}} \sim \sum_{m,k} \frac{\partial^3 \log \tau^{\text{red}}}{\partial t_{3m+2} \partial t_{3n-3(m+k)+2} \partial t_{3k-4}} \sim \sum_{m,k} \frac{\partial^3 \log \tau^{\text{red}}}{\partial t_{3m} \partial t_{3n-3(m+k)} \partial t_{3k}}. \quad (121)
\]

Again, this second condition is correct modulo some singular in the limit of \(a \to 0\) terms, which appear only in the case of even \(p\). Unfortunately, we do not know the way to prove this statement without using the integrable equations, what is very hard to proceed in the case of higher \(p\).

On the other hand, the cancellation of derivatives with incorrect gradation can be trivially derived from the “Schwinger-Dyson” equations given rise by the transformations \(M \to \exp\left\{\frac{2\pi i}{p}\right\} M\) \((0 < k < p)\) of the integration variable in the corresponding matrix integral, the integration measure being assumed to be invariant. In its turn, it implies that the integration contour, instead of real line, should be chosen as a set of rays beginning in the origin of the co-ordinate system with the angles between them being integer times \(\frac{2\pi}{p}\).

This rather fancy choice of the integration contour is certainly necessary to preserve \(\mathbb{Z}_p\)-invariance of \(p\)-matrix model system.

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\(^3\)Simultaneously it means that the size of all matrices in (21) should be specially adjusted, say, to be times of \(p\).
6 Conclusion

To conclude, we have demonstrated that the partition function of a new class of multi-matrix models (CMM) proposed in [1] can be described by the same constraint algebras as in the case of the standard multi-matrix models [11, 12, 13]. It was conjectured in [2] that it defines the partition function in full. Therefore, one can conclude that the partition function of CMM lies in the same universality class (i.e. have the same (double scaling) continuum limit) as the standard models. Certainly, there is an essential difference in these models both in their formulation at the discrete level and in the procedure of taking the continuum limit. Indeed, in contrast to the standard multi-matrix models, CMM’s satisfy $W$-constraint algebra already at the discrete level. But as a price, it leads to very complicated (though still of the nearest neighbours) interaction of matrices in the matrix integral. From the other hand, the correct reduction procedure corresponds to taking the square root of the partition function in the standard case, in contrast to the root of the $p$-th degree, which should be involved for the proper reduction of the conformal $(p + 1)$-matrix model.

Now we would like to emphasize that the double scaling limit of the standard multi-matrix models was never investigated honestly in the whole space of possible coupling constants (i.e. not nearby a critical point). The problem certainly is that it is very difficult to describe any matrix model in the whole space without any explicit constraint algebra imposed on the partition function. Therefore, the continuum limit for multi-matrix models proposed in [2] should be considered rather as a definition than as the theorem.

In fact, we know another case when multi-matrix model possesses $W$-invariance. It is the partition function described in [25], whose continuum limit was investigated in [24]. The arguments mentioned above suggest that it should be possible to propose adequate procedure of taking the continuum limit for the $W$-constraints in this case as well, but it is not done up to now.

At last, we would like to stress some points which are to be understood better.

1) In the paper we have proved the correspondence between discrete and continuum constraint algebras at the level of leading terms (see the sect.3.2), but the whole proof
taking into account all corrections is still unknown. It is not a problem to do this in any concrete case, and we have demonstrated it in the case of $W^{(3)}$-constraint algebra, but it is very difficult to deal with the general case. One of the main reasons is that it is difficult to write down the general expression for the generators of $W^{(n)}$ algebra (see, for example, [33]).

Nevertheless, the main statement should be correct. We think that there should be more elegant and efficient way of doing the continuum limit, and that to find out this one should understand deeper the role which is played by the scalar fields with different boundary conditions.

2) We do not know how it is possible to prove the condition of proper reduction like (121). It was done explicitly only in the one-matrix model case (the relation (119)), where we used the equations of the integrable hierarchy (Toda chain hierarchy) in the manifest way and after rather tedious calculations proved the statement [20]. But even in the case of two-matrix model the integrable equations are very complicated (maybe, we do not understand them completely) and it is absolutely unclear how one should solve this problem in the general case. Again, we think it might be done in a simpler way.

The solution to all these problems seems to be important for a better understanding of the role of the double scaling continuum limit. We hope to return to all these problems elsewhere.

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8 Appendix

In this appendix we consider Kazakov-like change of the time variables in the general case of the conformal $p - 1$-matrix model as well as the relation between the discrete and the continuum $W$-algebras which can be easily extracted from the analyzes of the leading terms only (see the sect.3.2). We can use again the fact that the constraint algebra of CMM, by construction, strongly resembles the structure of the corresponding continuum
constraint algebra. This assertion means that the generators $W_{ps}^{(p)}$’s of the discrete algebra, after the reduction (105), depend on the scalar fields (106) through
\[
\sum_{s \in \mathbb{Z}} \frac{W_{ps}^{(p)}}{u^{|s+p|}} \sim \sum_{i=1}^{p-1} \left( \partial \varphi^{(i)}(u) \right)^p : u^{p(i-p+1)},
\]
(122)
where the sign “∼” means that we omit some numerical coefficient. Then it is obvious from (122) that the generators $W_{ps}^{(p)}$’s are given by the sum (with proper binomial coefficients) of the terms
\[
\prod_{a=1}^{\bar{j}} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \left| \sum_{m_b = \sum k_a + i - p + s + 1} \sum_{p-i} \sum_{a=1}^{p-j} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \right|,
\]
(123)
where $j = 0, 1, \ldots, p - 1, i = 1, 2, \ldots, p - 1$.

Now we are going to prove that the Kazakov change of the time variables (110), (111) results in the relation
\[
a^{-n} \sum_{s=0}^{n+p-1} C_{n+p-1}^{s} (-1)^{n+p-1-s} \prod_{a=1}^{\bar{j}} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \left| \sum_{m_b = \sum k_a + i - p + s + 1} \sum_{p-i} \sum_{a=1}^{p-j} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \right|,
\]
\[
= \prod_{a=1}^{\bar{j}} \prod_{b=1}^{j} \sum_{n_a \geq n-i-p-j} \sum_{f_b \geq 0} \left( n_a + \frac{i}{p} \right) T_{p_n+i} \frac{\partial}{\partial T_{p_f+p-i}} \left| \sum_{f_b = \sum n_a + n + i - p + j} \right| \sum_{f_b = \sum n_a + n + i - p + j}
\]
(124)
where the sums over $a$ and $b$ in (124) run in the same limits as in (123) and again in the sums over $n_a$ some incorrect shifts appear which should be removed by a proper rescaling of the reduced partition function of CMM. The expression (124) reproduces the connection (112).

The equation (124) is based on the following identity for the $\Gamma$-functions
\[
\sum_{s=0}^{n+p-1} C_{n+p-1}^{s} (-1)^{n+p-1-s} \prod_{a=1}^{\bar{j}} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \left| \sum_{m_b = \sum k_a + i - p + s + 1} \sum_{p-i} \sum_{a=1}^{p-j} \sum_{k_a \geq 0} g_{k_a}^{(i)} \prod_{b=1}^{j} \sum_{m_b \geq 1} \frac{\partial}{\partial \ell^{(p-i)}} \right| = \prod_{a=1}^{\bar{j}} \prod_{b=1}^{j} \sum_{n_a \geq n-i-p-j} \sum_{f_b \geq 0} \left( n_a + \frac{i}{p} \right) T_{p_n+i} \frac{\partial}{\partial T_{p_f+p-i}} \left| \sum_{f_b = \sum n_a + n + i - p + j} \right|
\]

\[
= \left\{ \begin{array}{ll}
1 & \text{if } \sum f_b = \sum n_a + n + i - p + j \\
0 & \text{otherwise},
\end{array} \right.
\]
(125)
which will be proved in several steps.
First, we note that the product of the sums over $b$ in the second line of (125) can be easily simplified using the identity
\[
\sum_{\gamma=0}^{a} \frac{1}{\gamma!(a-\gamma)!\Gamma(b-\gamma)\Gamma(c+\gamma)} = \frac{\Gamma(c+b+a-1)}{\Gamma(a+1)\Gamma(b)\Gamma(c+a)\Gamma(c+b-1)},
\]
and the product is equal to
\[
\frac{\Gamma \left( \sum k_a + i - 2p + s + 1 + j + (p-j)\frac{2p-i}{p} \right)}{(\sum k_a - \sum f_b + i - 2p + j + s + 1)!\Gamma \left( \sum f_b + (p-j)\frac{2p-i}{p} \right)}. \tag{127}
\]

Second, using the identity (65) we perform the sum over $s$ in (125) to obtain
\[
\prod_{a=1}^{j} \sum_{k_a=0}^{n_a} (-1)^{n_a-k_a} \frac{\Gamma \left( n_a + \frac{i}{p} \right)}{(n_a - k_a)!\Gamma \left( k_a + \frac{i}{p} \right)} \times \frac{\Gamma \left( \sum k_a + i - p + 1 + j + (p-j)\frac{2p-i}{p} \right)}{(\sum k_a - \sum f_b + p + j + i + n)!\Gamma \left( \sum f_b - n - p + 1 + (p-j)\frac{2p-i}{p} \right)}. \tag{128}
\]

At last, using the identities (126) $j-1$ times and (82), we conclude that (125) is correct.

References

[1] A.Marshakov, A.Mironov, and A.Morozov, Phys.Lett., B265 (1991) 99.

[2] M.Fukuma, H.Kawai, and R.Nakayama, Int.J.Mod.Phys., A6 (1991) 1385.

[3] V.Kazakov, Phys.Lett., B159 (1985) 303.

[4] F.David, Nucl.Phys., B257 (1985) 45.

[5] T.Regge, Nuovo Cim., 19 (1961) 558.

[6] V.Kazakov, Mod.Phys.Lett., A4 (1989) 2125.

[7] E.Brézin and V.Kazakov, Phys.Lett., B236 (1990) 144.

[8] M.Douglas and S.Shenker, Nucl.Phys., B335 (1990) 635.

[9] D.Gross and A.Migdal, Phys.Rev.Lett., 64 (1990) 127.

[10] D.Gross and A.Migdal, Nucl.Phys., B340 (1990) 333.
[11] E.Brezin, M.Douglas, V.Kazakov, and S.Shenker, Phys.Lett., 237B (1990) 43.

[12] Č.Crnković, P.Ginsparg, and G.Moore, Phys.Lett., 237B (1990) 196.

[13] D.Gross and A.Migdal, Phys.Rev.Lett., 64 (1990) 717.

[14] M.Douglas, Phys.Lett., B238 (1990) 179.

[15] R.Dijkgraaf, H.Verlinde, and E.Verlinde, Nucl.Phys., B348 (1991) 435.

[16] A.G.Izergin and V.E.Korepin, Nucl.Phys., B205 (1982) 401.

[17] A.Gerasimov et al., Nucl.Phys., B357 (1991) 565.

[18] A.Mironov and A.Morozov, Phys.Lett., 252B (1990) 47.

[19] J.Ambjørn, J.Jurkiewicz, and Yu.Makeenko, Phys.Lett., 251B (1990) 517.

[20] Yu.Makeenko et al., Nucl.Phys., B356 (1991) 574.

[21] A.Gerasimov et al., Mod.Phys.Lett., A6 (1991) 3079.

[22] S.Kharchev et al., Nucl.Phys., 366B (1991) 569.

[23] S.Kharchev et al., Generalized Kontsevich model versus Toda hierarchy and discrete matrix models, preprint FIAN/TD-03/92 – ITEP-M-3/92, hepth@xxx/9203043

[24] E.Gava and K.Narain, Phys.Lett., B263 (1991) 213.

[25] A.Marshakov, A.Mironov and A.Morozov, Mod.Phys.Lett., A7 (1992) 1345.

[26] S.Kharchev et al., Conformal matrix models as an alternative to conventional multi-matrix models, Preprint FIAN/TD-09/92 – ITEP-M-4/92, hepth@xxx/9208044.

[27] V.Fateev and S.Lukyanov, Int.J.Mod.Phys, A3 (1988) 507.

[28] B.Feigin and E.Frenkel, preprint MIT (1992).

[29] M.J.Ablowitz, D.J.Kaup, A.C.Newell, and H.Segur, Stud.Appl.Math., 53 (1974) 249.

[30] K.Ueno and K.Takasaki, Adv.Studies in Pure Math., 4 (1984) 1.
[31] See, for example, A. Morozov, *Nucl. Phys.*, **357B** (1991) 619.

[32] A. B. Zamolodchikov, *Teor. Mat. Fiz.*, **65** (1985) 347.

[33] M. Fukuma, H. Kawai, and R. Nakayama, *Comm. Math. Phys.*, **143** (1992) 371.