A CLASS OF TOEPLITZ OPERATORS
WITH HYPERCYCLIC SUBSPACES

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ABSTRACT. We use a theorem by Gonzalez, Leon-Saavedra and Montes-Rodriguez to
construct a class of coanalytic Toeplitz operators which have an infinite-dimensional
closed subspace, where any non-zero vector is hypercyclic.

1. Introduction

Let \( X \) be a separable Banach space (or a Frechet space), and let \( T \) be a bounded
linear operator in \( X \). If there exists \( x \in X \) such that the set \( \{ T^n x, n \in \mathbb{N}_0 \} \) is dense
in \( X \), then \( T \) is said to be a hypercyclic operator and \( x \) is called its hypercyclic vector.

Here \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

Dynamics of linear operators and, as a special case, theory of hypercyclic operators
was actively developed for the last 20 years. A detailed review of the results up to the
end of 1990-s is given in the paper [7]. For a recent exposition of the theory see the
monographs [1] [8].

However, first examples of hypercyclic operators appeared much earlier. In 1929
Birkhoff has shown that the translation operator \( T_a : f(z) \mapsto f(z + a), a \in \mathbb{C}, a \neq 0, \)
is hypercyclic in the Frechet space of all entire functions \( \text{Hol}(\mathbb{C}) \) with topology of
uniform convergence on the compact sets. Later, McLane proved hypercyclicity of the
differentiation operator \( D : f \mapsto f' \) on \( \text{Hol}(\mathbb{C}) \). The first example of a hypercyclic
operator in the Banach setting was given in 1969 by Rolewicz [11] who showed that for
any \( \lambda \in \mathbb{C}, |\lambda| > 1 \), the operator \( \lambda S^* \) is hypercyclic on \( \ell^p(\mathbb{N}_0), 1 \leq p < \infty \), where \( S^* \) is
the backward shift on \( \ell^p(\mathbb{N}_0) \) which transforms a vector \( x = (x_0, x_1, \ldots, x_n, \ldots) \in \ell^p(\mathbb{N}_0) \)
to the vector \( (x_1, x_2, \ldots, x_{n+1}, \ldots) \).

Given a hypercyclic operator \( T \), what can be said about the set of its hypercyclic
vectors? Clearly, if \( x \) is a hypercyclic vector for the operator \( T \) then \( Tx, T^2x, T^3x, \ldots \)
are hypercyclic vectors for \( T \) as well. Hence, the set of hypercyclic vectors is dense
when it is non-empty.

The following result was proved by Bourdon [2] (a special class of operators commu-
ting with generalized backward shifts was previously considered by Godefroy and
Shapiro in [4]).

**Theorem (Bourdon, [2]).** Let \( T \) be a hypercyclic operator acting on a Hilbert space
\( H \). Then there exists a dense linear subspace, where any non-zero vector is hypercyclic
for \( T \).

**Definition.** Given a hypercyclic operator \( T \), an infinite-dimensional closed subspace
in which every non-zero vector is hypercyclic for \( T \) is called a hypercyclic subspace.

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Montes-Rodriguez [10] Theorem 3.4 proved that the operator $\lambda S^*$, $|\lambda| > 1$, on $\ell^2(N_0)$ has no hypercyclic subspaces. However, for some class of functions of the backward shift $S^*$ on $\ell^2(N)$ there exists a hypercyclic subspace, and it is the main result of the present paper. To state it, we need to introduce some notations. Let $D = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc and let $T = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Recall that the disc algebra $A(D)$ is the space of all functions which are continuous in the closed disc $D$ and analytic in $D$ (with the norm $\max_{z \in D} |\varphi(z)|$).

**Main Theorem.** For any function $\varphi \in A(D)$ such that $\varphi(T) \cap \mathbb{T} \not= \emptyset$ and $\varphi(D) \cap \mathbb{T} \not= \emptyset$ the operator $\varphi(S^*)$ on $\ell^2(N_0)$ has a hypercyclic subspace.

Note that the $\varphi(z) = \lambda z$, $|\lambda| > 1$, does not satisfy this condition.

The examples in the Main Theorem may be interpreted as certain Toeplitz operator on the Hardy space. The Hardy space $H^2 = H^2(D)$ is the space of all functions of the form $f(z) = \sum_{n \geq 0} c_n z^n$ with $\{c_n\} \in \ell^2(N_0)$, and thus is naturally identified with $\ell^2(N_0)$. Recall that for a function $\varphi \in L^\infty(T)$ the Toeplitz operator $T_\varphi$ with the symbol $\varphi$ is defined as $T_\varphi f = P_+(\varphi f)$, where $P_+$ stands for the orthogonal projection from $L^2(T)$ onto $H^2$. Then the backward shift on $S^*$ may be identified with the Toeplitz operator $T_\varphi$. It was shown in [3] that any coanalytic Toeplitz operator $T^*_\varphi$ (i.e., $\varphi$ is a bounded analytic function in $D$) is hypercyclic whenever $\varphi(D)$ intersects $T$. Our Main Theorem provides a class of coanalytic Toeplitz operators which have a hypercyclic subspace.

A general sufficient condition for the existence of a hypercyclic subspace was given by Gonzalez, Leon-Saavedra and Montes-Rodriguez in [6]. To state it we need the following stronger version of hypercyclicity:

**Definition.** Operator $T$ acting on a separable Banach space $B$ is said to be **hereditarily hypercyclic** if there exists a sequence of non-negative integers $\{n_k\}$ such that for each subsequence $\{n_{k_i}\}$ there exists a vector $x$ such that the sequence $\{T^{n_{k_i}}x\}$ is dense in $B$.

We also need to recall the notion of the essential spectrum.

**Definition.** Operator $U$ is called **Fredholm** if $\text{Ran} \ U$ is closed and has finite codimension and $\text{Ker} \ U$ is finite-dimensional. The **essential spectrum** of the operator $T$ is defined as

$$\sigma_e(T) = \{\lambda : T - \lambda I \text{ is non-Fredholm}\}.$$ 

**Theorem (Gonzalez, Leon-Saavedra, Montes-Rodriguez, [6] Theorem 3.2).** Let $T$ be a hereditary hypercyclic bounded linear operator on a separable Banach space $B$. Let the essential spectrum of $T$ intersect the closed unit disc. Then there exists a hypercyclic subspace for the operator $T$.

We intend to use this result in the proof of the Main Theorem.

Let us mention some other results on this topic. Shkarin in [12] proved that the differentiation operator on the standard Frechet space $Hol(C)$ has a hypercyclic subspace. Quentin Menet in [9] Corollary 5.5] generalized this result: he proved that for every non-constant polynomial $P$ the operator $P(D)$ has a hypercyclic subspace. He also obtained some results concerning weighted shifts on $\ell^p$. 

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2. Preliminaries on essential spectra of linear operators.

The following lemma is well known. We give its proof for the convenience of the reader.

**Lemma.** Essential spectrum of the operator $S^*$ is the unit circle.

**Proof:** Let us consider three cases:

Case 1: $|\lambda| > 1$. Then the operator $S^* - \lambda I = -\lambda(I - \frac{1}{\lambda}S^*)$ is invertible and, thus, it is Fredholm.

Case 2: $|\lambda| < 1$. We have $S^* - \lambda I = S^*(I - \lambda S)$. Since the operator $S^*$ is Fredholm (its kernel is one-dimensional, its image is the whole space $\ell^2$), and $I - \lambda S$ is invertible, their composition is also a Fredholm operator.

Case 3: $|\lambda| = 1$. Then the operator $S^* - \lambda I$ is not Fredholm, because its image has infinite codimension.

Indeed, the pre-image of the sequence $(\lambda y_1, \lambda^2 y_2, \lambda^3 y_3, \lambda^4 y_4, \ldots) \in \ell^2$ is given by $(a, \lambda(y_1 + a), \lambda^2(y_1 + y_2 + a), \ldots)$ and the equality $a = -\sum_{i=1}^{+\infty} y_i$ is necessary for the inclusion of this sequence into $\ell^2$.

Then the pre-image of the sequence

\[
\left(1, \frac{1}{2}, 0, \ldots, 0, \frac{1}{4}, 0, \ldots, 0, \ldots, \frac{1}{2^n}, 0, \ldots, 0, \ldots\right),
\]

multiplied componentwise by $(\lambda, \lambda^2, \lambda^3, \ldots)$, is given by

\[
\left(-2, -1, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{4}, -\frac{1}{4}, \ldots, -\frac{1}{2^n}, \ldots\right),
\]

multiplied componentwise by $(1, \lambda, \lambda^2, \ldots)$, but such sequences do not belong to $\ell^2$. All sequences of the form (1), as is easily seen, form an infinite-dimensional subspace in $\ell^2$. \qed

The following important theorem about the mapping of the essential spectra can be found, e.g., in [5, p. 107].

**Essential Spectrum Mapping Theorem.** For any bounded linear operator $T$ in a Hilbert space $H$ and for any polynomial $P$ one has $\sigma_e(P(T)) = P(\sigma_e(T))$.

3. Proof of the Main Theorem

In the proof of hereditary hypercyclicity of the operator $\varphi(S^*)$ we will use the following well-known criterion due to Godefroy and Shapiro [4] (for an explicit statement see, e.g., [8, Theorem 3.1]):

**Theorem (Godefroy–Shapiro criterion).** Let $T$ be a bounded linear operator in a separable Banach space. Suppose that the subspaces

\[
X_0 = \text{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{C}, \ |\lambda| < 1\},
\]

\[
Y_0 = \text{span}\{x \in X : Tx = \lambda x \text{ for some } \lambda \in \mathbb{C}, \ |\lambda| > 1\},
\]

and
are dense in \( X \). Then \( T \) is hereditarily hypercyclic.

**Proof of the Main Theorem** We should verify two conditions of the theorem of Gonzalez, Leon-Saavedra and Montes-Rodriguez.

Any function \( \varphi \) from disc-algebra can be approximated uniformly in \( \overline{D} \) by a sequence of polynomials \( P_n \). So \( P_n(S^*) \) tends to \( \varphi(S^*) \) in the operator norm.

We need to show that \( \sigma_S(\varphi(S^*)) \) intersects the closed unit disc. Since \( \varphi(\mathbb{T}) \cap \mathbb{T} \neq \emptyset \), there exist \( \lambda, \mu \in \mathbb{T} \) such that \( \varphi(\lambda) = \mu \). Then \( \mu_n = P_n(\lambda) \) tend to \( \mu \). By the Essential Spectrum Mapping Theorem for any polynomial \( P \) one has \( \sigma_S(P(S^*)) = P(\sigma_S(S^*)) = P(\mathbb{T}) \). In particular, \( \mu_n = P_n(\lambda) \in \sigma_S(P_n(S^*)) \) for any \( n \), and so \( P_n(S^*) - \mu_n I \) is not Fredholm.

Since the set of Fredholm operators is open in the operator norm (see, e.g., [3] Theorem 4.3.11), the set of non-Fredholm operators is closed, whence the limit of \( P_n(S^*) - \mu_n I \), which is equal to \( \varphi(S^*) - \mu I \), is not Fredholm, and \( \mu \) belongs to the essential spectrum of \( \varphi(S^*) \). The first condition of the theorem by Gonzalez, Leon-Saavedra and Montes-Rodriguez is verified.

It is well known that the condition \( \varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \) implies that \( \varphi(S^*) \) satisfies the Godefroy–Shapiro criterion. Let us briefly recall this argument.

Recall that the point spectrum of \( S^* \) equals \( \sigma_p(S^*) = \{ \lambda : |\lambda| < 1 \} \) and the eigenvector is given by \( (1, \lambda, \lambda^2, \cdots) \in \ell^2(\mathbb{N}_0) \), or, if we pass to the Hardy space \( H^2(\mathbb{D}) \) using the natural identification of \( H^2 \) with \( \ell^2(\mathbb{N}_0) \), by

\[
k_\lambda(z) = \frac{1}{1 - \lambda z} = \sum_{n=0}^{\infty} \lambda^n z^n.
\]

These are the Cauchy kernels, which are reproducing kernels of \( H^2 \). Clearly, \( k_\lambda, \lambda \in \mathbb{D} \), are also eigenvectors of \( \varphi(S^*) \) which correspond to eigenvalues \( \varphi(\lambda) \).

By the condition \( \varphi(\mathbb{D}) \cap \mathbb{T} \neq \emptyset \), we know that \( \varphi(\mathbb{D}) \) is an open set which intersects both \( \mathbb{D} \) and \( \mathbb{C} \setminus \overline{\mathbb{D}} \). Clearly, both of the sets \( \{ k_\lambda, \lambda \in \mathbb{D} : |\varphi(\lambda)| > 1 \} \) and \( \{ k_\lambda, \lambda \in \mathbb{D} : |\varphi(\lambda)| < 1 \} \) are dense in \( H^2 \). Indeed, \( f \in H^2 \) is orthogonal to \( k_\lambda \) if and only if \( f(\lambda) = 0 \) and both \( \{ \lambda \in \mathbb{D} : |\varphi(\lambda)| > 1 \} \) and \( \{ \lambda \in \mathbb{D} : |\varphi(\lambda)| < 1 \} \) are open sets. Thus the conditions of the Godefroy–Shapiro criterion are satisfied and the hereditarily hypercyclicity of the operator \( \varphi(S^*) \) follows.

Thus, by the theorem of Gonzalez, Leon-Saavedra and Montes-Rodriguez, the operator \( \varphi(S^*) \) has a hypercyclic subspace. \( \square \)

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**References**

[1] F. Bayart, E. Matheron, *Dynamics of Linear Operators*. Cambridge University Press, 2009.

[2] P. S. Bourdon, *Invariant manifolds of hypercyclic vectors*. Proceedings of the American Mathematical Society, (3) 118 (1993), pp. 845–847.

[3] E. B. Davies, *Linear Operators and Their Spectra*, Cambridge Studies in Advanced Mathematics, Vol. 106, Cambridge University Press, 2007.

[4] G. Godefroy, J. H. Shapiro, *Operators with dense, invariant, cyclic vector manifolds*. Journal of Functional Analysis, 98 (1991), pp. 229–269.

[5] S. Goldberg, *Unbounded Linear Operators*, McGraw-Hill, New York, 1966.

[6] M. Gonzalez, F. Leon-Saavedra, A. Montes-Rodriguez, *Semi-Fredholm Theory: Hypercyclic and supercyclic subspaces*. Proceedings of the London Mathematical Society, (3) 81 (2000), pp. 169–189.
[7] K.-G. Grosse-Erdmann, Universal families and hypercyclic operators. Bulletin of American Mathematical Society, (3) 36 (1999), pp. 345–381.
[8] K.-G. Grosse-Erdmann, A. Peris Manguillot, Linear Chaos, Springer, Berlin, 2011.
[9] Q. Menet, Hypercyclic subspaces and weighted shifts. arXiv:1208.4963v1 [math.FA], 24 Aug 2012.
[10] A. Montes-Rodriguez, Banach spaces of hypercyclic vectors. Michigan Mathematical Journal, 43 (1996), pp. 419–436.
[11] S. Rolewicz, On orbits of elements, Studia Math. 32 (1969), pp. 17–22.
[12] S. Shkarin, On the set of hypercyclic vectors for the differentiation operator. Israel Journal of Mathematics, 180 (2010), pp. 271–283.

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