Optical microresonator in form of hollow bottle for the measurement of hydrostatic pressure: A theoretical study of its sensitivity.

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Abstract. In this research, we report the theoretical study of an optical microresonator in form of hollow bottle for the measurement of hydrostatic pressure in microfluidics. The proposed device bases its operating principle on the excitation of the resonant modes WGMs that are confined to the interior of the optical cavity through the evanescent coupling of the light from the outside. The analytical study developed to determine the sensitivity of the device to changes in internal pressure in the hollow micro-bottle, is developed from a layer model with application of boundary conditions in the problem, where it is necessary to consider the excitation of the transverse modes WGMs and the elastic-optical effect produced by the internal pressurization of the resonant cavity. In the investigation we have concluded that the theoretical results present a good approximation with the experimental results using the layer model with which it is possible to optimize the design of this type of devices to improve the sensitivity for the measurement of the hydrostatic pressure in microfluidics.

1. Introduction
Optical micro-resonators are indispensable optical elements in integrated photonic systems because they allow the confinement of light in confined spaces and offer selectivity in specific optical wavelengths that make them ideal in WDM multiplexed wavelength systems. At present, there are different technologies for the manufacture of this type of devices, however, optical micro-resonators based on optical fibers have been studied, due to their relative ease of manufacture. This type of devices have been manufactured using different types of materials, such as silica [1] and some types of polymers [2], in the shape of rings [3], capillaries [4], hollow bottles [5], solid bottles [6], spheres [7], etc with applications in areas such as biology [8], medicine [9], physics [10,11], chemistry [7] and specifically in the area of sensors for temperature measurement [12], humidity [6], refractive index [13], and some other physical variables of interest. In the case of the micro-resonators in form of hollow bottle, their manufacture using silica has been reported through different techniques with a Q factor of the order of $10^8$ and small modal volumes, being ideal in experiments to demonstrate nonlinear optical effects. This type of devices have been studied analytically using the vector wave equation allowing to explore some of its properties, such as the distribution of field, the modal volume and the tunability of the resonance wavelengths [14] and also taking into account the complexity of its analytical solutions, the finite element method has been used to study some of its confinement properties [15].
The optical micro-resonators based on optical fibers are cavities designed with the objective that light experiences the phenomenon of total internal reflection in its interior at the interface between two materials with different refractive index. The confinement of the light inside the cavities can be studied from the excitation of the modes Whispering Gallery WGMs, which are supported by confining structures that morphologically present a rotational symmetry and whose spectra present a dependence on the shape and geometrical dimensions of the cavity. In this paper we report the theoretical study of a dielectric optical microresonator in form of a hollow bottle made from a slightly elastic material for its application in the measurement of hydrostatic pressure in microfluidics. The theoretical results allow analytically determine the sensitivity of the device in terms of the geometrical parameters of the microcavity using the Helmholtz scalar equation with the application of boundary conditions in each of the interface.

2. Theoretical description

An optical micro resonator in form of a hollow bottle is a microcavity in form of an elongated spheroid that can confine light in a 3D structure. The confinement of the light inside the structure is carried out in the equatorial zone of the cavity, allowing the excitation of the WGMs modes. In Figure 1, a general outline of the structure to be studied is observed. According to the theory of rays, these are coupled in the equatorial zone of the structure and only those wavelengths that meet the established resonance condition will be confined in the cavity. On the other hand, the modes in the cavity can generally be excited in spiral form on the confining layer of the structure between two axial return points \( \pm z_c \), these rays are commonly known as skew rays, however, in the present investigation we will consider only those modes that are excited along a same transverse plane on a point in the axial direction \( z \).

In figure 1, it is also observed that in our analytical model we will consider a resonant cavity with an approximately parabolic profile along the \( z \)-axis, which can be approximated by the expression

\[
R(z) = R_o \left[ 1 - \frac{1}{2}(\Delta k z)^2 \right] \tag{1}
\]

Where \( R_o \) represents the maximum radius measured along the equatorial axis and \( \Delta k \) represents the curvature profile. Figure 2 shows a cross-sectional view of the resonator on the equatorial axis, where \( a \) and \( b \) represent the internal and external radii of the cavity. The indices \( n_1, n_2 \) and \( n_3 \) represent the refractive indices of the media 1,2 and 3 respectively, the circles denoted by dashed lines with radii \( a' \).
and \( b' \) they represent the circumferential displacements of the internal and external radios of the cavity when these are slightly pressurized by an internal pressure \( p \). These circumferential displacements obey the model of plane-stress, which can be applied to thin-walled capillary structures and their mathematical expression will depend on the nature of the material.

For the concrete analytical study developed here, we will consider a dielectric microresonator in the form of a hollow bottle of approximately parabolic profile with cylindrical symmetry and we consider the excitation of the WGMs modes along a transverse plane of the cavity, that is, along a ring. In the cavity, the confinement of the field is given in the material with index of refraction \( n_2 \) which constitutes the wall of the capillary and to ensure the phenomenon of internal total reflection must be satisfied that \( n_2 > n_3 \) and \( n_1 > n_3 \) for the case of a microcavity in form of hollow bottle with thin wall thickness. To determine the way the field is distributed into the cavity, it is necessary to study the problem taking into account the polarization states of the WGMs modes confined within the microcavity, which are known as TE electric transverse modes and TM transverse magnetic modes. Here we will define the polarization of the field with respect to the plane of propagation of the WGMs modes, ie the TE resonances have the electric field perpendicular to the axis of the capillary, while the TM modes have the magnetic field perpendicular to the axis of the capillary. These polarization states are analyzed independently during the analysis. To study the propagation of the electric and magnetic field inside the micro-resonator, the electric and magnetic field functions must satisfy the respective Helmholtz equations in cylindrical coordinates, where \( r, \varphi, z, \) represent the radial, azimuthal and axial coordinates, respectively.

\[
\left( \nabla^2 + k^2 \right) \vec{H} = 0 \quad (2)
\]
\[
\left( \nabla^2 + k^2 \right) \vec{E} = 0 \quad (3)
\]

Where \( \nabla^2 \) is the operator Laplaciano, \( k = \omega m / c = 2 \pi n / \lambda_0 \) is the wave number with angular frequency \( \omega \) and wavelength of light in vacuum \( \lambda_0 \), propagating in a medium with a refractive index \( n \). Under the adiabatic approach applicable to microcavities with a smooth profile, as in the case of hollow microbottles with a parabolic profile, it must be fulfilled \( dR / dz << 1 \) and it is possible to establish a wave equation to determine the resonator's eigenfunctions. To do this, using the adiabatic approach along the \( z \) axis has a separable function for the fields

\[
E(r, \varphi, z) = \Phi(r, R(z)) Z(z) e^{i\omega t} \quad (4)
\]

Where \( m \) is a constant commonly called azimuthal order number. The expression (4) must satisfy the differential equation of Helmhotlz for the electric field and for the magnetic field respectively. In this way we can determine the differential equations for the radial and axial components

\[
\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \left( k^2 \varphi - \frac{m^2}{r^2} \right) \Phi = 0 \quad (5)
\]
\[
\frac{\partial^2 Z}{\partial z^2} + \left( k^2 - \frac{\mu_{mp}^2}{R_0^2} \right) Z = 0 \quad (6)
\]

Where \( k \) was previously defined, \( \mu_{mp} \) is the \( p \)-ezima root of the function of Bessel and \( p \) is an integer called a radial order number. The differential equation (6), is the differential equation of a simple harmonic oscillator, whose solution is

\[
Z_{mp}(z) = C_{mp} H_q \left( \frac{\Delta E_n}{2} \right) e^{\frac{\mu_{mp} z}{R_0}} \quad (7)
\]

Where \( H_q \) is the polynomial of Hermite of order \( q \) with constant normalization given by
\[ C_{mn} = \left[ \frac{\Delta E_m}{\pi 2^{2+m} (q!)} \right] \]  
\[ \Delta E_m = \frac{2\mu^2 \Delta k^2}{R_0^2} \]  

3. Theoretical results

In this research, we are interested in the analytical study of the sensitivity of an optical microresonator in form of a hollow bottle for the measurement of hydrostatic pressure in microfluidics. For this, it required to determine the displacements of the resonance wavelengths for a particular mode as a function of the pressure inside the cavity.

To determine the wavelengths of resonances, we applied the layer model for the determination of the eigenfunctions that contains the resonance wavelengths of the WGMs modes that are supported by the microcavity. In the model, we consider a structure that contains three layers of material with different refractive index, as shown in Figure 2. The medium with refractive index \( n_1 \), is a medium in the gaseous state that allows the pressurization of the microcavity, the medium with refractive index \( n_2 \), constitutes the material of the microcavity, which for simulation purposes is a polymer and the medium with index \( n_3 \), constitutes the external medium.

In order to determine the eigenvalue equation for TE and TM polarization states, it is required to solve the differential equation for the radial component. Taking into account that the differential equation (5) admits solutions in terms of the Bessel functions, we have proposed a solution taking into account that the field propagates mostly in the medium with refractive index \( n_2 \), in the form

\[
\Phi(r,R(z)) = \begin{cases} 
A_m J_\nu \left( \frac{k n_2 R}{R(z)} \right) & r \leq a(z) \\
B_m J_\nu \left( \frac{k n_2 R}{R(z)} \right) + C_m H^{(1)}_\nu \left( \frac{k n_2 R}{R(z)} \right) & a(z) < r \leq b(z) \\
H^{(2)}_\nu \left( \frac{k n_2 R}{R(z)} \right) & r > b(z) 
\end{cases}
\]  

Where \( A_m, B_m, C_m, D_m \) are complex constants, \( J_\nu(R) \) is the cylindrical Bessel functions of first order, \( H^{(1)}_\nu(R) \) is the first type Hankel function y \( H^{(2)}_\nu(R) = J_\nu(R) - iY_\nu(R) \) is the function of Hankel second type, \( R_c \) represents the caustic radius, which is obtained for a fixed resonance wavelength value. To determine the eigenfunctions that contains the information of the wavelengths of resonances, it is required to apply the boundary conditions in each one of the interfaces defined as

\[
(E_{n_2} - E_{n_1}) \times \hat{e}_n = 0; \quad (n_2^2 E_{n_2} - n_1^2 E_{n_1}) \cdot \hat{e}_n = 0; \quad (H_{n_2} - H_{n_1}) \times \hat{e}_n = 0; \quad (H_{n_2} - H_{n_1}) \cdot \hat{e}_n = 0
\]  

Where \( E_{n_1} \) and \( E_{n_2} \) are the fields in the media with index \( n_1 \) and \( n_2 \). The term \( \hat{e}_n \) it is a normal vector perpendicular to the interface. Analogously, it is required to apply the same conditions on the interface between the media with indexes \( n_2 \) and \( n_3 \). The application of boundary conditions in the problem allows complex constants to be determined \( A_m, B_m, C_m, D_m \) and at the same time the eigenvalue functions for the modes TE and TM. In the case of TE modes, the function of eigenvalues is described by equation (12).
\[
\left\{ a_n J_a \left( \frac{k_n R_a}{R(z)} \right) + b_n J_b \left( \frac{k_n R_b}{R(z)} \right) \right\} / \left\{ c_n J_a \left( \frac{k_n R_a}{R(z)} \right) + d_n J_b \left( \frac{k_n R_b}{R(z)} \right) \right\} = \frac{Y_a}{Y_b} \left( \frac{k_n R_a}{R(z)} \right) \quad (12)
\]

With
\[
a_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) J_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
b_n = \left[ n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) \right]
\]
\[
c_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) + n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
d_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) + n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) J_a \left( \frac{k_n R_a}{R(z)} \right)
\]

Where \( R(z) = R \), because the modes are supported in the central region of the curvature. In the case of TM modes, using the same analogous procedure for TE modes, the equation of eigenvalues is

\[
\left\{ e_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) + f_n J_a \left( \frac{k_n R_a}{R(z)} \right) + g_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) \right\} / \left\{ e_n J_a \left( \frac{k_n R_a}{R(z)} \right) + f_n J_a \left( \frac{k_n R_a}{R(z)} \right) + g_n Y_a \left( \frac{k_n R_a}{R(z)} \right) \right\} = \frac{J_a \left( \frac{k_n R_a}{R(z)} \right)}{J_a \left( \frac{k_n R_a}{R(z)} \right)} \quad (13)
\]

With
\[
e_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
f_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) J_a \left( \frac{k_n R_a}{R(z)} \right) + n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
g_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) J_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
h_n = n_n J_a \left( \frac{k_n R_a}{R(z)} \right) - n_n J_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
i_n = n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) + n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
o_n = n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]
\[
p_n = n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right) - n_n Y_a \left( \frac{k_n R_a}{R(z)} \right) Y_a \left( \frac{k_n R_a}{R(z)} \right)
\]

On the other hand, to determine the circumferential displacements of the internal and external radii, it is necessary to consider the elasto-optic effects experienced by the microcavity when it is internally pressurized by a gas, as shown in Figure 2. The circumferential displacement of the internal radius a is
denoted by \( a'(a, p) \) while the circumferential displacement of the external radio \( b \) is denoted by \( b'(b, p) \), where \( p \) represents the pressure. In Figure 2, the thickness of the wall of the hollow micro bottle is \( g=b-a \). It is possible to analytically determine the internal and external radii of the microcavity for a fixed pressure value, ie \( a(p) = a_s + a'(a, p) \) and \( b(p) = b_s + b'(b, p) \), where \( a_s \) and \( b_s \) represent the internal and external radios at zero pressure. To define the expression of the circumferential displacements in terms of pressure, it is necessary to define the material used for the manufacture of the microcavity, in this sense, we have chosen the polymer PMMA (Polymethyl-Methacrylate). For the PMMA, the circumferential displacement when the cavity is pressurized can be determined from the theory of elasticity assuming a linear behavior according to Hooke's law.

\[
a'(a, p) = a(r, p) \bigg|_{r=a} = \frac{2pb^2a}{E(a^2-b^2)} \quad (14)
\]

\[
b'(b, p) = b(r, p) \bigg|_{r=b} = \frac{pb}{E} \left(\frac{a^2 + b^2}{a^2 - b^2} + \nu\right) \quad (15)
\]

Where, \( r \) is the radial distance measured from the axial, \( E \) is the Young module and \( \nu \) is the Poisson coefficient.

4. Numerical simulations

To determine the sensitivity of the micro resonators in form of a hollow bottle, it is required to solve numerically the equations (12) and (13) with the help of equations (13) and (14) for the wavelengths of resonances for any value of \( p \) for TE and TM modes and different numbers of azimuthal and radial order. For the numerical determination of the sensitivities for the TE and TM modes, the following parameters were used: \( n_1=1.000299 \), \( n_2=1.49 \), \( n_3=1.0002924 \), \( m=2000 \), \( \nu=0.37 \) and \( E = 2.2 \times 10^8 \) Pa. In figure 3, the behavior of the microcavity sensitivity in the form of a hollow bottle for the measurement of the hydrostatic pressure for the TM modes is observed. The figure shows the sensitivity values for six different values of \( l=1,2,3,10,30,50 \). In the same way, in Figure 4 the sensitivity for TE modes and the same values of \( l \) is observed. For each one of the states of polarization, it is observed that the sensitivity values differ slightly, which is observed in the graph inside the circle of figures 3 and 4.

![Figure 3. Sensitivity of TM modes.](image)

![Figure 4. Sensitivity of TE modes.](image)

In the same way, in table 1, the sensitivity values are resolved in nm / bar for a hollow micro-bottle with external diameter 1520 µm, internal diameter 1430 µm and wall thickness 90 µm, which are recorded in graphs 3 and 4.
Table 1. Sensitivity values for the TE and TM modes.

| Number of order \( l \) | TE (nm/Bar) | TM (nm/Bar) |
|--------------------------|-------------|-------------|
| 1                        | 0.58548     | 0.58192     |
| 2                        | 0.61778     | 0.58249     |
| 3                        | 0.61425     | 0.58636     |
| 10                       | 0.61855     | 0.58055     |
| 30                       | 0.60995     | 0.58627     |
| 50                       | 0.61347     | 0.57019     |

On the other hand, we have developed a numerical simulation using the finite element method to determine the distribution of the field inside the cavity. In the simulation, we used the COMSOL Multiphysics 5.1 software to determine the regions of maximum intensity in the microcavity. For this, we have performed the simulation by developing an object with a paraboloid-shaped 2D geometry with a maximum external diameter of 1520 μm and a maximum internal diameter of 1430 μm. In the results, it is simple to observe that the WGMs modes confined in the structure are located on the surface of the cavity in opposite lateral zones.

![Figure 5](image)

**Figure 5.** Modes WGMs confined in the microcavity with external diameter of 1520 μm and internal diameter of 1430 μm.

5. References

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