DIRICHLET SERIES WITH PERIODIC COEFFICIENTS, RIEMANN’S FUNCTIONAL EQUATION AND REAL ZEROS OF DIRICHLET L-FUNCTIONS

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Abstract. In this paper, we give Dirichlet series with periodic coefficients that have Riemann’s functional equation and real zeros of Dirichlet L-functions. The details are as follows. Let \( L(s, \chi) \) be the Dirichlet L-function and \( G(\chi) \) be the Gauss sum associate with a primitive Dirichlet character \( \chi \pmod{q} \). Put \( f(s, \chi) := q^s L(s, \chi) + i^{-\kappa(\chi)} G(\chi) L(s, \overline{\chi}) \), where \( \overline{\chi} \) is the complex conjugate of \( \chi \) and \( \kappa(\chi) := (1 - \chi(-1))/2 \). Then we prove that \( f(s, \chi) \) satisfies Riemann’s functional equation appearing in Hamburger’s theorem if \( \chi \) is even. In addition, we show that \( f(\sigma, \chi) \neq 0 \) all \( \sigma \geq 1 \). Moreover, we prove that \( f(\sigma, \chi) \neq 0 \) for all \( 1/2 \leq \sigma < 1 \) if and only if \( L(\sigma, \chi) \neq 0 \) for all \( 1/2 \leq \sigma < 1 \). When \( \chi \) is real, all zeros of \( f(s, \chi) \) with \( \Re(s) > 0 \) are on the line \( \sigma = 1/2 \) if and only if GRH for \( L(s, \chi) \) is true. However, \( f(s, \chi) \) has infinitely many zeros off the critical line \( \sigma = 1/2 \) if \( \chi \) is non-real.

1. Introduction

1.1. The Riemann zeta function and the Dirichlet L-function. Let \( s \) be a complex variable expressed as \( s = \sigma + it \), where \( \sigma, t \in \mathbb{R} \). Then, the Riemann zeta function is defined by the series

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \sigma > 1.
\]

It is widely known that \( \zeta(s) \) is continued meromorphically and has a simple pole at \( s = 1 \) with residue 1. The Dirichlet L-function is defined by the series

\[
L(s, \chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \sigma > 1,
\]

where \( \chi(n) \) is a Dirichlet character \( \pmod{q} \). The Dirichlet L-function \( L(s, \chi) \) can be analytically continued to the whole complex plane to a holomorphic function if \( B_0(\chi) := \sum_{r=0}^{q-1} \chi(r)/q = 0 \), otherwise to a meromorphic function with a simple pole, at \( s = 1 \), with residue \( B_0(\chi) \) (see for example [4, Corollary 10.2.3]). For simplicity, put

\[
\Gamma_\pi(s) := \frac{\Gamma(s)}{(2\pi)^s}, \quad \Gamma_{\cos}(s) := 2\Gamma_\pi(s) \cos\left( \frac{\pi s}{2} \right), \quad \Gamma_{\sin}(s) := 2\Gamma_\pi(s) \sin\left( \frac{\pi s}{2} \right).
\]

Then, the Riemann zeta function \( \zeta(s) \) satisfies Riemann’s functional equation

\[
\zeta(1 - s) = \Gamma_{\cos}(s) \zeta(s) \quad (1.1)
\]

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(see for example [11, (2.1.8)]). When \( \chi \) is a primitive character (mod \( q \)), the Dirichlet \( L \)-function \( L(s, \chi) \) satisfies the functional equation
\[
L(1-s, \chi) = q^{s-1} \Gamma(s) (e^{-\pi is/2} + \chi(-1) e^{\pi is/2}) G(\chi) L(s, \overline{\chi}),
\]
(1.2)
where \( \overline{\chi} \) is the complex conjugate of \( \chi \) and the Gauss sum \( G(\chi) \) is defined as \( G(\chi) := \sum_{r=1}^{\varphi(q)} \chi(r) e^{2\pi ir/q} \) (see for example [1, Theorem 12.11]).

Next, we review some of the standard conjectures and facts on zeros of \( \zeta(\chi) \). All real zeros of \( \zeta(s) \), so-called trivial zeros, are simple and only at the negative even integers. The famous Riemann hypothesis is a conjecture that the Riemann zeta function \( \zeta(s) \) has its non-real zeros only on the critical line \( \sigma = 1/2 \). When \( \chi \) is an even primitive character, then the only zeros of \( L(s, \chi) \) with \( \Re(s) < 0 \) are at the negative even integers. If \( \chi \) is an odd primitive character, then the only zeros of \( L(s, \chi) \) with \( \Re(s) < 0 \) are at the negative odd integers. Note that \( \zeta(s) \) and \( L(s, \chi) \) do not vanish when \( \sigma > 1 \) by their Euler products. The generalized Riemann hypothesis asserts that, for every Dirichlet character \( \chi \) and every complex number \( s \) with \( \Re(s) > 0 \) and \( L(s, \chi) = 0 \), then the real part of \( s \) is 1/2. Note that the existence of a real zero at \( s = 1/2 \) is not prohibited by the GRH.

1.2. **Dirichlet series with periodic coefficients.** For \( \sigma > 1 \), define two Dirichlet \( L \)-functions (mod 5) by
\[
L_1(s) := \frac{1}{1^s} + \frac{i}{2^s} - \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \cdots, \quad L_2(s) := \frac{1}{1^s} - \frac{i}{2^s} + \frac{i}{3^s} - \frac{1}{4^s} + \frac{1}{6^s} + \cdots.
\]
And we define real numbers 0 < \( \theta < \pi/2 \) and \( \xi > 0 \) by
\[
\tan \theta := \xi, \quad \xi := (5^{1/2} - 1)^{-1} \left( (10 - 2 \cdot 5^{1/2})^{1/2} - 2 \right).
\]
Then, the Davenport-Heilbronn function (see [11, Chapter 10.25]) is defined by
\[
f(s) := \frac{\sec \theta}{2} \left( e^{-i\theta} L_1(s) + e^{i\theta} L_2(s) \right).
\]
We have the following for the Davenport-Heilbronn function \( f(s) \) (see [2, Section 1], [3, Section 5] or [11, Chapter 10.25]). It should be emphasised that the gamma factor of the functional equation (1.3) below is similar to that of (1.1) but definitely different from that of (1.1). More precisely, the Riemann zeta function \( \zeta(s) \) has trivial zeros at the negative even integers but the Davenport-Heilbronn function \( f(s) \) has trivial zeros at the negative odd integers.

**Theorem A.** The function \( f(s) \) satisfies the functional equation
\[
f(1-s) = 5s^{-1/2} \Gamma_{\sin}(s) f(s).
\]
Furthermore, the function \( f(s) \) has an infinity of zeros in the half-plane \( \sigma > 1 \).

Recently, Vaughan [9] studied the following functions. Let
\[
\lambda(\chi) := \frac{G(\chi)}{i^{\kappa(\chi)} \sqrt{q}}, \quad \kappa(\chi) := \begin{cases} 1 & \text{if } \chi \text{ is odd,} \\ 0 & \text{if } \chi \text{ is even.} \end{cases}
\]
Then we define the functions \( V_+(s, \chi) \) and \( V_-(s, \chi) \) by
\[
V_+(s, \chi) := \frac{L(s, \chi) + \lambda(\chi) L(s, \overline{\chi})}{1 + \lambda(\chi)} \quad \text{and} \quad V_-(s, \chi) := \frac{L(s, \chi) - \lambda(\chi) L(s, \overline{\chi})}{1 - \lambda(\chi)}
\]
when \( \lambda(\chi) \neq -1 \) and \( \lambda(\chi) \neq 1 \), respectively. Then, he showed the following (see [9, Theorems 2.1, 2.2, 3.1 and 3.2]).
Theorem B. We have the functional equations
\[ V_+(1 - s, \chi) = q^{s-1/2} \Gamma_\cos(s - \kappa(\chi)) V_+(s, \chi), \]
\[ V_-(1 - s, \chi) = -q^{s-1/2} \Gamma_\cos(s - \kappa(\chi)) V_-(s, \chi). \]

Moreover, the function \( V_+(s, \chi) \) has \( \asymp T \) zeros in the region \( \{ s : \sigma > 1, |t| < T \} \). The same statement also holds for \( V_-(s, \chi) \).

The existence of complex zeros of \( f(s) \) and \( V_\pm(s, \chi) \) are proved by the theorem below. Let \( N_F(\sigma_1, \sigma_2, T) \) be the number of zeros of a function \( F(s) \) in the rectangle \( \sigma_1 < \Re(s) < \sigma_2, |\Im(s)| \leq T \), counted with their multiplicities. Then we have the following which is proved by Saias and Weingartner [10, Theorem].

Theorem C. Let \( \{a(n)\} \) be a periodic sequence of complex numbers such that \( F(s) := \sum_{n=1}^{\infty} a(n)n^{-s} \) is not of the form \( P(s)L(s, \chi) \), where \( P(s) \) is a Dirichlet polynomial and \( L(s, \chi) \) is a Dirichlet L-function. Then there exists a positive number \( \eta \) such that, for all real numbers \( \sigma_1 \) and \( \sigma_2 \) with \( 1/2 < \sigma_1 < \sigma_2 < 1 + \eta \), there exist positive numbers \( c_1, c_2 \), and \( T_0 \) such that for all \( T \geq T_0 \) we have
\[ c_1 T \leq N_F(\sigma_1, \sigma_2, T) \leq c_2 T. \]

2. The main theorem and remarks

2.1. Main theorem. As an analogue or improvement of the Davenport-Heilbronn function \( f(s) \) or Vaughan functions \( V_\pm(s, \chi) \), we define the function
\[ f(s, \chi) := q^s L(s, \chi) + i^{-\kappa(\chi)} G(\chi) L(s, \chi), \]
where \( \chi \) is a primitive Dirichlet characters (mod \( q \)) and \( \kappa(\chi) := (1 - \chi(-1))/2 \). Note that all the functions \( f(s), V_\pm(s, \chi) \) and \( f(s, \chi) \) are expressed as a sum of two Dirichlet L-functions. In the present paper, we show the following which implies that \( f(s, \chi) \) have Riemann’s functional equation (if \( \chi \) is even) and real zeros of Dirichlet L-functions.

Theorem 2.1. We have the four statements below;
(i). The function \( f(s, \chi) \) satisfies Riemann’s functional equation
\[ f(1 - s, \chi) = \Gamma_\cos(s)f(s, \chi) \tag{2.1} \]
if \( \chi(-1) = 1. \) When \( \chi(-1) = -1, \) the function \( f(s, \chi) \) fulfills
\[ f(1 - s, \chi) = \Gamma_\sin(s)f(s, \chi). \tag{2.2} \]

(ii). One has \( f(\sigma, \chi) \neq 0 \) for all \( \sigma \geq 1 \) and primitive Dirichlet character \( \chi. \) Moreover, we have \( f(\sigma, \chi) \neq 0 \) for all \( 1/2 \leq \sigma < 1 \) if and only if \( L(\sigma, \chi) \neq 0 \) for all \( 1/2 \leq \sigma < 1 \).

(iii). Fix a real Dirichlet character \( \chi. \) Then all complex zeros of \( f(s, \chi) \) with \( \Re(s) > 0 \) are on the vertical line \( \sigma = 1/2 \) if and only if GRH for \( L(s, \chi) \) is true.

(iv). Fix a non-real Dirichlet character \( \chi. \) Then, there exists a positive number \( \eta \) such that, for all real numbers \( \sigma_1 \) and \( \sigma_2 \) with \( 1/2 < \sigma_1 < \sigma_2 < 1 + \eta, \) there exist positive numbers \( C_1, C_2, \) and \( T_0 \) such that for all \( T \geq T_0 \) we have
\[ C_1 T \leq N_{f(s, \chi)}(\sigma_1, \sigma_2, T) \leq C_2 T, \]
where \( N_{f(s, \chi)}(\sigma_1, \sigma_2, T) \) is the number of zeros of the function \( f(s, \chi) \) in the rectangle \( \sigma_1 < \Re(s) < \sigma_2, |\Im(s)| \leq T, \) counted with their multiplicities.
We remark that the gamma factor of the functional equation (2.1) completely coincides with that of Riemann’s functional equation (1.1) or (H3) given below. Moreover, the statement (ii) of Theorem 2.1 implies that the real zeros of \( f(s, \chi) \) completely coincide with those of \( L(s, \chi) \) (there are no theorem on real zeros of \( f(s) \) or \( V_\pm(s, \chi) \) in [2, Section 1], [3, Section 5], [9, Section 3] or [11, Chapter 10.25]). However, \( f(s, \chi) \) with non-real character has zeros in the half-plane \( \Re(s) > 1/2 \) from (iv) of Theorem 2.1. Therefore, for any non-real even character \( \chi \), the function \( f(s, \chi) \) or \( g(s, \chi) \) has Riemann’s functional equation, real zeros of Dirichlet \( L \)-functions and non-real zeros of Dirichlet series with periodic coefficients not of the form \( P(s)L(s, \chi) \) as the title of this paper says.

In the next subsection, we give some remarks for Riemann’s functional equation (1.1). We give proofs of Theorem 2.1 and Corollary 2.2 and some remarks on them in Section 3.

2.2. Remarks on Riemann’s functional equation. The following converse theorem on \( \zeta(s) \) is widely-known (see also [11, Chapter 2.13]).

**Theorem D** (Hamburger [5, Satz 1]). Suppose that \( F(s) \) satisfies

\[
F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \text{ where } a(n) \in \mathbb{C}, \text{ converges absolutely for } \sigma > 1, \tag{H1}
\]

\[
P(s)F(s) \text{ is an entire function of finite order for some polynomial } P(s), \tag{H2}
\]

\[
\xi_F(1-s) = \xi_F(s), \text{ where } \xi_F(s) := \pi^{-s/2}\Gamma(s/2)F(s). \tag{H3}
\]

Then, one has \( F(s) = C\zeta(s) \), where \( C \) is a constant.

Clearly, the functions \( \zeta(s) \) and \( f(s, \chi) \) with \( \chi(-1) = 1 \) satisfy the condition (H2) and Riemann’s functional equation (H3) from the definitions of them and equalities (1.1) and (2.1). However, the functions \( L(s, \chi) \) with primitive characters, \( f(s) \) and \( V_\pm(s, \chi) \) satisfy (H1) and (H2) but they do not fulfill (H3). It should be mentioned that the functional equation (2.1) does not contradict Hamburger’s Theorem because \( f(s, \chi) \) satisfies (H1). In [7, Theorem 1], Knopp showed that there are infinitely many linearly independent solutions which satisfy (H2), (H3) and

\[
F(s) = \sum_{n=1}^{\infty} a(n)n^{-s}, \text{ where } a(n) \in \mathbb{C}, \text{ converges absolutely in some half-plane}. \tag{K}
\]

Knopp gives no explicit representation for the coefficients of \( a(n) \) of the Dirichlet series satisfy the condition (K). Using \( f(s, \chi) \), we give explicitly some functions satisfy (H1) and (H3) as an analogue or improvement of Knopp’s theorem.

**Corollary 2.2.** Put \( H(s, q) := (q^s + q^{1-s})^{-1} \) and \( g(s, \chi) := H(s, q)f(s, \chi) \). Then the function \( g(s, \chi) \) with \( \chi(-1) = 1 \) can be expressed as an ordinary Dirichlet series and satisfies Riemann’s functional equation

\[
g(1-s, \chi) = \Gamma(s)g(s, \chi). \tag{2.3}
\]

Namely, the function \( g(s, \chi) \) with \( \chi(-1) = 1 \) fulfills (H1), (H3) and (D).

Let \( 0 < r < q \) be relatively prime integers and put

\[
Q(s, r/q) := \frac{1}{2\varphi(q)} \sum_{\chi \mod q} (1 + \chi(-1)) (\chi(r)q^s + G(r, \chi))L(s, \chi),
\]

where \( G(r, \chi) \) is the generalized Gauss sum defined by \( G(r, \chi) := \sum_{n=1}^{q} \chi(n)e^{2\pi i rn/q} \). In [8, Corollary 1.5], it is proved that \( H(s, q)Q(s, r/q) \) satisfies (H1), (H3) and (D). It should be emphasised that \( g(s, \chi) \) is simpler than \( H(s, q)Q(s, r/q) \) because the function \( f(s, \chi) \) is expressed as a sum of two Dirichlet \( L \)-functions.
3. Proofs

3.1. Proofs of (i) and (iii) of Theorem 2.1.

Proofs of (2.1) and (2.2). Supposes \( \chi(-1) = 1 \). Then we obtain
\[
\Gamma_{\cos}(s)G(\chi)L(s, \overline{\chi}) = q^{1-s}L(1 - s, \chi)
\]
from (1.2). By replacing \( s \) by \( 1 - s \) in (1.2) we have
\[
q^sL(s, \chi) = \Gamma_{\cos}(1 - s)G(\chi)L(1 - s, \overline{\chi})
\]
By multiplying both sides of the equality above by \( \Gamma_{\cos}(s) \), we obtain
\[
\Gamma_{\cos}(s)q^sL(s, \chi) = \Gamma_{\cos}(s)\Gamma_{\cos}(1 - s)G(\chi)L(1 - s, \overline{\chi}) = G(\chi)L(1 - s, \overline{\chi})
\]
(3.2) since it holds that
\[
\Gamma_{\cos}(s)\Gamma_{\cos}(1 - s) = \frac{2\Gamma(s)}{(2\pi)^s} \cos \left( \frac{\pi s}{2} \right) \cdot \frac{2\Gamma(1 - s)}{(2\pi)^{1-s}} \sin \left( \frac{\pi s}{2} \right) = \frac{\Gamma(s)\Gamma(1 - s)}{2\pi} \cdot 2\sin \pi s = 1
\]
according to the well-known formulas
\[
\cos \left( \frac{\pi(1 - s)}{2} \right) = \sin \left( \frac{\pi s}{2} \right), \quad \Gamma(s)\Gamma(1 - s) = \frac{\pi}{\sin \pi s}.
\]
Therefore, from (3.1) and (3.2), it holds that
\[
\Gamma_{\cos}(s)f(s, \chi) = \Gamma_{\cos}(s)\left(q^sL(s, \chi) + G(\chi)L(s, \overline{\chi})\right)
\]
\[
= G(\chi)L(1 - s, \overline{\chi}) + q^{1-s}L(1 - s, \chi) = f(1 - s, \chi)
\]
which implies (2.1). Similarly, we can prove (2.2), namely, the case \( \chi(-1) = -1 \). \( \square \)

Proof of (iii) of Theorem 2.1. Assume that \( \chi \) is a real character \((\text{mod } q)\). Then we have
\[
f(s, \chi) = (q^s + i^{-\kappa(\chi)}G(\chi))L(s, \chi).
\]
(3.3)
For any real primitive Dirichlet character, all zeros of the factor \( q^s + i^{-\kappa(\chi)}G(\chi) \) are on the critical line \( \sigma = 1/2 \) according to the well-known formula
\[
|G(\chi)| = \sqrt{q}.
\]
(3.4)
It should be emphasized that this phenomenon is distinctly rare since Dirichlet polynomials \( a_0 + a_1q^{-s} \) have zeros off the vertical line \( \sigma = 1/2 \) for almost all \( a_0, a_1 \in \mathbb{C} \). Thus, we obtain that all complex zeros of \( f(s, \chi) \) with \( \Re(s) > 0 \) are on the vertical line \( \sigma = 1/2 \) if and only if GRH for \( L(s, \chi) \) is true. Hence, we have (iii) of Theorem 2.1. \( \square \)

Remark. For \( c \in \mathbb{C} \), consider the function
\[
f_c(s, \chi) := cq^sL(s, \chi) + i^{-\kappa(\chi)}G(\chi)L(s, \overline{\chi})
\]
(3.5)
as a generalization of \( f(s, \chi) = f_t(s, \chi) \). Clearly, we have
\[
f_c(s, \chi) = (cq^s + i^{-\kappa(\chi)}G(\chi))L(s, \chi)
\]
when \( \chi \) is a real character. By modifying the proof above, we can see that the factor \( cq^s + i^{-\kappa(\chi)}G(\chi) \) with \( |c| \neq 1 \) vanishes for some \( \sigma \neq 1/2 \) and \( t \in \mathbb{R} \). Hence, the function \( f_c(s, \chi) \) with a real character \( \chi \) and \( |c| \neq 1 \) has complex zeros off the critical line \( \sigma = 1/2 \) even if GRH for \( L(s, \chi) \) is true. In this case, we have,
\[
f_c(1 - s, \chi) \neq \Gamma_{\cos}(s)f_c(s, \chi) \quad \text{or} \quad f_c(1 - s, \chi) \neq \Gamma_{\sin}(s)f_c(s, \chi), \quad |c| \neq 1.
\]
Hence, when \( \chi \) is real, the functional equations (2.1) and (2.2) prohibit the existence of complex zeros of \( f(s, \chi) \) off the critical line \( \sigma = 1/2 \) under GRH.
3.2. Proofs of (ii) and (iv) of Theorem 2.1 and Corollary 2.2. As usual, we say that two Dirichlet characters \( \chi_1, \chi_2 \) are non-equivalent if the primitive characters \( \chi_1^*, \chi_2^* \) inducing \( \chi_1, \chi_2 \) are distinct. In [6, Lemma 8.1], Kaczorowski and Perelli proved the following.

**Theorem E.** For \( l = 1, \ldots, m \), let \( P_l(s) \) be Dirichlet polynomials and \( \chi_l \) be non-equivalent Dirichlet characters such that
\[
\sum_{l=1}^{m} P_l(s, \chi_l) = 0 \quad \text{identically.}
\]
Then \( P_l(s) = 0 \) identically for \( l = 1, \ldots, m \).

By using this theorem, we show the statements on non-real zeros of \( f(s, \chi) \).

**Proof of (iv) of Theorem 2.1.** Let \( \chi \) be a non-real Dirichlet character. Then, from Theorem E, the function \( q^{-s}f(s, \chi) \) can not be expressed as \( P_0(s)L(s, \chi_0) \), where \( P_0(s) \) is a Dirichlet polynomial and \( \chi_0 \) is a Dirichlet character. Clearly, the Dirichlet series of \( L(s, \chi) \) and \( q^{-s}G(\chi)L(s, \chi) \) have periodic coefficients of period \( q \) and \( q^2 \), respectively. Hence, \( q^{-s}f(s, \chi) \) is a Dirichlet series with periodic coefficients of period \( q^2 \) and not of the form \( P_0(s)L(s, \chi_0) \). Therefore, by Theorem C, we have (iv) of Theorem 2.1. \( \square \)

We are now in a position to prove the statements on real zeros of \( f(s, \chi) \).

**Proof of (ii) of Theorem 2.1.** Assume that \( L(\sigma_0, \chi) = 0 \) for some \( \sigma_0 > 1/2 \). Then, clearly one has \( L(\sigma_0, \overline{\chi}) = 0 \) and
\[
f(\sigma_0, \chi) = q^sL(\sigma_0, \chi) + i^{-\kappa(\chi)}G(\chi)L(\sigma_0, \overline{\chi}) = 0.
\]

Conversely, suppose that \( L(\sigma, \chi) \neq 0 \) for all \( \sigma > 1/2 \). Then, for \( \sigma > 1/2 \), the statement \( f(\sigma, \chi) = 0 \) is equivalent to
\[
\frac{L(\sigma, \overline{\chi})}{L(\sigma, \chi)} = \frac{i^{-\kappa(\chi)}q^\sigma}{G(\chi)}.
\]
(3.6)
The absolute value of the left hand side of (3.6) is 1 from
\[
|L(\sigma, \chi)| = |L(\sigma, \overline{\chi})| = |L(\sigma, \overline{\chi})|.
\]
(3.7)
On the contrary, the absolute value of the right hand side of (3.6) is greater than 1 by (3.4) if \( \sigma > 1/2 \). Hence, there are no \( \sigma > 1/2 \) which satisfies (3.6). Next suppose that \( \sigma = 1/2 \). Then we have
\[
q^{1/2}L(1/2, \chi) = i^{-\kappa(\chi)}G(\chi)L(1/2, \overline{\chi})
\]
according to the functional equation (1.2). Hence, we obtain
\[
f(1/2, \chi) = q^{1/2}L(1/2, \chi) + i^{-\kappa(\chi)}G(\chi)L(1/2, \overline{\chi})
\]
\[
= 2q^{1/2}L(1/2, \chi) = 2i^{-\kappa(\chi)}G(\chi)L(1/2, \overline{\chi}).
\]
Therefore, we have \( f(\sigma, \chi) \neq 0 \) for all \( \sigma \geq 1/2 \) if \( L(\sigma, \chi) \) does not vanish for all \( \sigma \geq 1/2 \).

Furthermore, we can easily see that \( f(\sigma, \chi) \neq 0 \) for all \( \sigma \geq 1 \) and primitive Dirichlet character \( \chi \) from (3.3), (3.4), (3.6), (3.7) and the fact that \( L(\sigma, \chi) \neq 0 \) for all \( \sigma \geq 1 \) which is proved by the existence of the Euler product of \( L(s, \chi) \) and the well-known fact \( L(1, \chi) \neq 0 \) (see [4, Theorem 10.5.29]). Thus, we obtain (ii) of Theorem 2.1. \( \square \)
**Remark.** Recall that \( f_c(s, \chi) \) is defined by (3.5) and suppose that \( L(\sigma, \chi) \neq 0 \) for all \( \chi \) and \( \sigma \geq 1/2 \). By modifying the proof above, for any fixed \( \chi_0 \) and \( \sigma_0 \geq 1/2 \), we can find \( c_0 \neq 1 \) such that \( f_{c_0}(\sigma_0, \chi_0) = 0 \), namely, the function \( f_{c_0}(s, \chi_0) \) vanishes on the real line. In this case, we have,

\[
f_c(1-s, \chi) \neq \Gamma_{\cos}(s)f_c(s, \chi) \quad \text{or} \quad f_c(1-s, \chi) \neq \Gamma_{\sin}(s)f_c(s, \chi), \quad c \neq 1.
\]

Therefore, there are no real zeros of \( f(s, \chi) \) with \( \sigma \geq 1/2 \) owing to the functional equations (2.1) and (2.2). However, due to (iv) of Theorem 2.1, the function \( f(s, \chi) \) with a non-real character \( \chi \) has non-real zeros in the half-plane \( \sigma > 1/2 \) despite the existence of the functional equations (2.1) and (2.2). In other words, the functional equations (2.1) and (2.2) forbid the existence of real zeros of \( f(s, \chi) \) but allow the existence of non-real zeros of \( f(s, \chi) \) when \( \chi \) is non-real.

**Proof of Corollary 2.2.** Clearly, the function \( q^{-s}f(s, \chi) \) can be expressed as an ordinary Dirichlet series, namely, satisfies (H1). By the definition of \( H(s, q) \), we have \( H(1-s, q) = H(s, q) \) and all poles of \( H(s, q) \) are on the vertical line \( \sigma = 1/2 \). Moreover, one has

\[
q^s H(s, q) = \frac{q^s}{q^s + q^{1-s}} = \frac{1}{1 + q^{1-2s}} = \sum_{k=0}^{\infty} (-q)^k q^{2ks}
\]

if \( \sigma > 1/2 \). Therefore, the function \( g(s, \chi) = q^{-s}f(s, \chi) \cdot q^s H(s, q) \) can be also expressed as an ordinary Dirichlet series. Hence we have Corollary 2.2. \( \Box \)

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