GAP THEOREMS ON CRITICAL POINT EQUATION OF THE TOTAL SCALAR CURVATURE WITH DIVERGENCE-FREE BACH TENSOR

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Abstract. On a compact $n$-dimensional manifold, it is well known that a critical metric of the total scalar curvature, restricted to the space of metrics with unit volume is Einstein. It has been conjectured that a critical metric of the total scalar curvature, restricted to the space of metrics with constant scalar curvature of unit volume, will be Einstein. This conjecture, proposed in 1987 by Besse, has not been resolved except when $M$ has harmonic curvature or the metric is Bach flat. In this paper, we prove some gap properties under divergence-free Bach tensor condition for $n \geq 5$, and a similar condition for $n = 4$.

1. Introduction

Let $M$ be an $n$-dimensional compact manifold, and let $\mathcal{M}_1$ be the set of all smooth Riemannian structures of unit volume on $M$. The total scalar curvature $S$ on $\mathcal{M}_1$ is given by

$$S(g) = \int_M s_g \, dv_g,$$

where $s_g$ is the scalar curvature of $g \in \mathcal{M}_1$. Hilbert showed that critical points of $S$ on $\mathcal{M}_1$ are Einstein. In [5], Koiso introduced the space $\mathcal{C}$ of constant scalar curvature metrics of unit volume. The Euler-Lagrange equation of $S$ restricted to $\mathcal{C}$ may be written in the form of the following critical point equation

$$z_g = s_g^{rs}(f).$$

Here, $z_g$ is the traceless Ricci tensor corresponding to $g$, and the operator $s_g^{rs}$ is the $L^2$ adjoint of the linearization $s'_g$ of the scalar curvature, given by

$$s'_g(f) = D_g df - (\Delta_g f) g - fr_g,$$

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where $D_g d$ and $\Delta_g$ denote the Hessian and the (negative) Laplacian, respectively, and $r_g$ is the Ricci curvature of $g$. If $f = 0$ in (1.1), then $g$ is clearly Einstein. By taking the trace of (1.1), we obtain
\[ \Delta_g f = -\frac{s_g}{n-1} f. \]
Thus, if $s_g/(n-1)$ is not in the spectrum of $\Delta_g$, then the critical metric $g$ is again Einstein. For example, if $s_g \leq 0$, then $g$ is Einstein. Note that if a non-trivial solution $(g, f)$ of (1.1) is Einstein, then (1.1) is reduced to the Obata equation, and so $(M, g)$ should be isometric to a standard $n$-sphere (6).

We remark that the existence of a non-trivial solution is a strong condition. The only known case satisfying this is that of the standard sphere. It was conjectured in [2] that this is the only possible case.

**Besse Conjecture.** Let $(g, f)$ be a solution of (1.1) on an $n$-dimensional compact manifold $M$. Then, $(M, g)$ is Einstein.

There are some partial answers to this conjecture. For example, it was proved that the Besse conjecture holds if $M$ has harmonic curvature (see Theorem 1.2 of [10] and also [11]). A Riemannian manifold $(M, g)$ is said to have harmonic curvature if $\delta R = 0$, where $R$ is the full Riemann tensor, and $\delta$ is the negative divergence operator. In particular, a locally conformally flat non-trivial solution $(g, f)$ of (1.1) with $s_g > 0$ is clearly isometric to a standard sphere. Note that when $s_g$ is constant, $\delta R = 0$ if and only if $\delta W = 0$ (cf. (2.1) below), where $W$ is the Weyl tensor. Qing and Yuan showed in [8] that the Besse conjecture holds if $g$ is Bach-flat, i.e., $B = 0$, where $B$ is the $n$-dimensional Bach tensor (see Section 2 for its definition). In fact, they proved that Bach-flatness implies harmonic curvature. Thus, it is natural to consider the divergence-free Bach tensor condition in the critical point equation (1.1) as a way to generalize Bach-flat condition. It turns out that $\delta B$ vanishes automatically when $n = 4$, and $\delta B = 0$ if and only if $\langle i_X C, z_g \rangle = 0$ for any vector $X$ when $n \geq 5$ (see Proposition 5 below). Here, $C$ is the Cotton tensor defined by (2.2) below.

In this paper, we will prove some gap properties under the assumption $\delta B = 0$. For a non-trivial solution $(g, f)$ of (1.1), we define $\mu$ by
\[ \mu = \max \left\{ \min_M (1 + f), \max_M (1 + f) \right\}. \]
If $f$ satisfies $f \geq -1$, we can easily show that $(M, g)$ is Einstein. In fact, one can show from (1.1) that $\text{div}(z_g(\nabla f, \cdot)) = (1 + f)|z_g|^2$, and so rigidity follows...
from the divergence theorem. Note that $\mu \geq 1$, because we have

$$0 = \int_M \Delta f = -\frac{s}{n-1} \int_M f,$$

which implies that there exists a point $p \in M$ satisfying $f(p) = 0$. In fact, $\mu > 1$ unless $f$ is trivial.

Our first main result for gap property on the critical point equation is the following.

**Theorem 1.** Let $(g, f)$ be a non-trivial solution of (1.1) on an $n$-dimensional compact manifold $M$. Assume that $\langle i_X C, z_g \rangle = 0$ for any vector $X$ and $n \geq 4$. If $|z_g|^2 \leq \frac{s^2}{4n(n-1)}$, then $(M, g)$ is isometric to a standard $n$-sphere.

As mentioned above, when $n \geq 5$, the condition that the Bach tensor is divergence-free implies the first hypothesis in Theorem 1. Thus, for $n \geq 5$ we have the following result.

**Corollary 2.** Let $n \geq 5$ and $(g, f)$ be a non-trivial solution of (1.1) on an $n$-dimensional compact manifold $M$ having divergence-free Bach tensor. If $|z_g|^2 \leq \frac{n^2}{4(n-1)^2}$, then $(M, g)$ is isometric to a standard $n$-sphere.

In [10] and [11], we proved the Besse conjecture is true when $(M, g)$ has harmonic curvature. In this case, the traceless Ricci tensor $z_g$ can be decomposed into $\nabla f$-direction and its orthogonal complement. In other words, for a vector $X$ orthogonal to $\nabla f$, we have $z_g(\nabla f, X) = 0$, and so $z_g$ can be controlled by $z_g(N, \cdot) = i_N z_g$ with $N = \frac{\nabla f}{|\nabla f|}$ on each hypersurface given by a level set of $f$. Related to $i_N z_g$, we have the following gap property.

**Theorem 3.** Let $(g, f)$ be a non-trivial solution of (1.1) on an $n$-dimensional compact manifold $M$. Assume that $\langle i_X C, z_g \rangle = 0$ for any vector $X$ and $n \geq 4$. If

$$|z_g|^2 \leq \min \left\{ 2|i_N z_g|^2, \frac{s^2}{4n(n-1)} \right\},$$

then $(M, g)$ is isometric to a standard sphere.

It is comparable with Theorem 2 of [1], which states that a non-trivial solution $(g, f)$ of (1.1) has zero radial Weyl curvature with

$$|z_g|^2 \leq \frac{s^2}{n(n-1)},$$

then $(M, g)$ is isometric to a standard sphere. We say that $g$ has zero radial Weyl curvature if $\tilde{i}_{\nabla f} W = 0$, where $\tilde{i}_X$ is defined in (3.1).
This paper is organized as follows. In Section 2, we give some properties of Bach tensor and Cotton tensor with their divergences. In particular, we include the fact that the divergence of Bach tensor is given by the inner product of the Cotton tensor with traceless Ricci tensor (Proposition 5). In Section 3, we introduce a covariant 3-tensor and derive some properties of it to handle the critical point equation. In section 4 and 5, we prove our main results, Theorem 1 and 3.

2. Divergences of a Bach tensor

Let \((M, g)\) be an \(n\)-dimensional Riemannian manifold. For convenience, we denote \(s_g, r_g\) and \(z_g\) by \(s, r\) and \(z\), respectively, if there is no ambiguity. Throughout the paper, we will assume that the dimension \(n \geq 4\).

Let \(D\) be the Levi-Civita connection on \((M, g)\) and let us denote by \(\mathcal{C}_\infty(\mathsf{S}^2M)\) the space of sections of symmetric 2-tensors on \((M, g)\). Then, the differential operator \(dD\) from \(\mathcal{C}_\infty(\mathsf{S}^2M)\) to \(\mathcal{C}_\infty(\Lambda^2 M \otimes T^*M)\) is defined by

\[
dD\eta(X, Y, Z) = (DX\eta)(Y, Z) - (DY\eta)(X, Z)
\]

for \(\eta \in \mathcal{C}_\infty(\mathsf{S}^2M)\) and vectors \(X, Y,\) and \(Z\). In particular, the following result is well known (\[2\]): under the identification of \(\mathcal{C}_\infty(T^*M \otimes \Lambda^2M)\) with \(\mathcal{C}_\infty(\Lambda^2 M \otimes T^*M)\),

\[
\delta R = -dD r.
\]

(2.1)

For a function \(\varphi \in C_\infty(M)\) and \(\eta \in C_\infty(\mathsf{S}^2M)\), \(d\varphi \wedge \eta\) is defined by

\[
(d\varphi \wedge \eta)(X, Y, Z) = d\varphi(X)\eta(Y, Z) - d\varphi(Y)\eta(X, Z).
\]

Here, \(d\varphi\) denotes the usual total differential of \(\varphi\).

The Cotton tensor \(C \in \Gamma(\Lambda^2 M \otimes T^*M)\) is defined by

\[
C = dD r - \frac{1}{2(n-1)} ds \wedge g
\]

(2.2)

and the \(n\)-dimensional Bach tensor \(B\) is defined by

\[
B = \frac{1}{n-3} \delta D \delta W + \frac{1}{n-2} \hat{W} z.
\]

Here, \(\delta D\) is the \(L^2\) adjoint operator of \(dD\), and

\[
\hat{W} z(X, Y) = \sum_{i=1}^{n} z(W(X, E_i)Y, E_i)
\]

for an orthonormal frame \(\{E_i\}_{i=1}^{n}\). \(\delta r\) is defined similarly. From now on, we will omit the summation notation, as we employ the Einstein convention.
Because
\[ \delta W = -\frac{n-3}{n-2} d^D \left( r - \frac{s}{2(n-1)} g \right), \]
we have
\[ C = -\frac{n-2}{n-3} \delta W \quad \text{and} \quad \delta C = -\frac{n-2}{n-3} \delta^D \delta W. \]
(2.3)

As a consequence, we have
\[ (n-2) B = -\delta C + \dot{W} z. \]
(2.4)

**Proposition 4.** (Corollary 1.22 of [2]) For any tensor \( h \), we have
\[ D^2_{X,Y} h - D^2_{Y,X} h = -R(X,Y) h \]
and
\[ D^3_{X,Y,Z} h - D^3_{Y,X,Z} h = -R(X,Y) D_Z h + D_{R(X,Y)Z} h. \]

Recall that the Schouten tensor \( A \) is defined by
\[ A = r - \frac{s}{2(n-1)} g \]
so that \( C = d^D A \). The following is Lemma 5.1 of [3]. We include the proof for the sake of completeness.

**Proposition 5.** For any vector field \( X \) we have
\[ (n-2) \delta B(X) = -\frac{n-4}{n-2} \langle i_X C, z \rangle = -\frac{n-4}{n-2} \left( \frac{1}{2} X(|z|^2) + \delta(z \circ z)(X) \right). \]

Here,
\[ i_X C(Y, Z) = C(X, Y, Z), \]
\[ \langle i_X C, z \rangle = \sum_{i,j=1}^n i_X C(E_i, E_j) z(E_i, E_j), \]
and a 2-tensor \( z \circ z \) is defined by
\[ z \circ z(X, Y) = \sum_{i=1}^n z(X, E_i) z(E_i, Y) \]
for any orthonormal frame \( \{ E_i \}_{i=1}^n \) and vector fields \( X, Y, \) and \( Z \).

**Proof.** Let \( \{ E_i \}_{i=1}^n \) be a geodesic frame. Denoting \( z_{ij} = z(E_i, E_j) \) and \( r_{ij} = r(E_i, E_j) \), it follows from (2.3) and (2.4) that
\[ (n-2) \delta B(X) = -\delta \delta C(X) + \dot{W} z(X) \]
\[ = -\delta \delta C(X) - \frac{n-3}{n-2} C(X, E_i, E_j) z_{ij} + \frac{1}{2} W(E_i, E_j, E_k, X) C(E_i, E_j, E_k). \]
Note that
\[
\delta \delta C(X) = \delta \delta d^P A(X)
= D_{E_i} D_{E_k} (D_{E_k} A(E_i, X) - D_{E_i} A(E_k, X))
= (D_{E_i} D_{E_k} - D_{E_k} D_{E_i}) D_{E_k} A(E_i, X).
\]
Thus, by Proposition 4 we have
\[
\delta \delta C(X) = R(E_k, E_i) D_{E_k} A(E_i, X) - D_{R(E_k, E_i)} E_k A(E_i, X)
= -D_{E_k} A(R(E_k, E_i) E_i, X) - D_{E_k} A(E_i, R(E_k, E_i) X)
= r_{kj} D_{E_k} A(E_j, X) + (R(E_k, E_i) E_s, X) D_{E_k} A(E_i, E_s)
= 1/2 (R(E_k, E_i) E_s, X) C(E_k, E_i, E_s).
\]
Hence,
\[
(n-2) \delta B(X) = -n-3 \frac{n-3}{n-2} C(X, E_i, E_j) z_{ij} + \frac{1}{2} (W - R)(E_i, E_j, E_k, E_k) C(E_i, E_j, E_k).
\]
From the decomposition of Riemann tensor, it follows
\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} \left( g_{ik} A_{jl} + g_{jl} A_{ik} - g_{jk} A_{il} - g_{il} A_{jk} \right),
\]
and so
\[
(W - R)(E_i, E_j, E_k, E_i) C(E_i, E_j, E_k) = -\frac{2}{n-2} (A_{ij} C_{ik} + A_{ik} C_{ij}) = -\frac{2}{n-2} r_{ik} C_{ilk}.
\]
Here, \( C_{ijk} = C(E_i, E_j, E_k) \) and we have used the fact that \( \sum_i C(E_i, Y, E_i) = 0 \) for any \( Y \). By substituting these results, we obtain the desired equation:
\[
(n-2) \delta B(X) = -n-4 \frac{n-4}{n-2} C(X, E_j, E_k) z_{jk}.
\]
Finally, it is obvious from the definition of \( C \) that
\[
\langle i_X C, z \rangle = \frac{1}{2} d|z|^2(X) + \delta (z \circ z)(X).
\]
\[\square\]
By Proposition 5, it is clear that \( \delta B = 0 \) when \( n = 4 \), and \( \langle i_X C, z \rangle = 0 \) for any vector field \( X \) if and only if \( \delta B = 0 \) when \( n \geq 5 \). In particular, since
\[
0 = \langle i_X C, z \rangle = \frac{1}{2} d|z|^2(X) + \delta (z \circ z)(X),
\]
we have
\[
(2.5) \quad \frac{1}{2} \Delta |z|^2 = \delta \delta (z \circ z).
\]
The following result holds when the scalar curvature is constant.
Proposition 6. ((10) of [9]) Assume that $s_g$ is constant. Then,

$$\delta d^D r = D^* D z + \frac{n}{n-2} z \circ z + \frac{s}{n-1} z - \frac{1}{n-2} |z|^2 g - \tilde{W} z.$$

Proof. The proof follows from the identity (see 4.71 in [2])

$$\delta d^D r = D^* D r + \frac{1}{2} D d s + r \circ r - \tilde{R} r,$$

and the relation in [9]

$$\tilde{R} r = \tilde{W} z + \frac{1}{n-2} |z|^2 g + \frac{(n-2)s}{n(n-1)} z - \frac{2}{n-2} z \circ z + \frac{s^2}{n^2} g,$$

which comes from the decomposition of the Riemann tensor. □

3. Critical metrics

In this section, we turn our attention to a non-trivial solution $(g, f)$ of (1.1). To do this, we will introduce a covariant 3-tensor $T$ defined by

$$T = \frac{1}{n-2} df \wedge z + \frac{1}{(n-1)(n-2)} i_{\nabla f} z \wedge g,$$

Also we define the interior product $\tilde{i}$ to the final factor by

$$(3.1) \quad \tilde{i}_V \omega(X, Y, Z) = \omega(X, Y, Z, V)$$

for a $(4,0)$-tensor $\omega$ and a vector field $V$.

Now, from the critical metric equation (1.1) we have

$$(3.2) \quad (1 + f) z = D df + \frac{s f}{n(n-1)} g.$$

By applying $d^D$ to both sides of this equation and using the Ricci identity

$$d^D D df(X, Y, Z) = R(X, Y, Z, \nabla f)$$

for any vector fields $X, Y, Z$ on $M$, we obtain

$$(df \wedge z + (1 + f)d^D z)(X, Y, Z) = \tilde{i}_{\nabla f} R(X, Y, Z) + \frac{s}{n(n-1)} df \wedge g(X, Y, Z).$$

Since $C = d^D z$ when $s$ is constant, we obtain

$$(3.3) \quad (1 + f) C = \tilde{i}_{\nabla f} R - df \wedge z + \frac{s}{n(n-1)} df \wedge g = \tilde{i}_{\nabla f} \tilde{W} - (n-1) T.$$

Here, we used the fact that

$$\tilde{i}_{\nabla f} R = \tilde{i}_{\nabla f} \tilde{W} - \frac{1}{n-2} i_{\nabla f} r \wedge g - \frac{1}{n-2} df \wedge r + \frac{s}{(n-1)(n-2)} df \wedge g,$$
which follows from the curvature decomposition (c.f. [2], p.48])

\[
R(X, Y, Z, W) = \mathcal{W}(X, Y, Z, W) + \frac{1}{n-2}(g(X, Z)r(Y, W) + g(Y, W)r(X, Z) - g(Y, Z)r(X, W) - g(X, W)r(Y, Z)) - \frac{s}{(n-1)(n-2)}(g(X, Z)g(Y, W) - g(Y, Z)g(X, W)).
\]

From the definition of the Bach tensor (2.4) and (3.3), we have

\[
(\mathcal{B}, z) = -\delta C + \mathcal{W}z = -\delta \left( \frac{1}{1+f} \tilde{\nabla} f \mathcal{W} - (n-1) \frac{T}{1+f} \right) + \mathcal{W}z.
\]

Since

\[
\delta \tilde{\nabla} f \mathcal{W}(X, Y) = -\frac{n-3}{n-2} d^2 r(Y, \nabla f, X) + (1+f) \mathcal{W}z(X, Y),
\]

we have

\[
-\delta \left( \frac{1}{1+f} \tilde{\nabla} f \mathcal{W} \right)(X, Y) = -\frac{1}{(1+f)^2} \mathcal{W}(\nabla f, X, Y, \nabla f) - \frac{1}{1+f} \delta \tilde{\nabla} f \mathcal{W}(X, Y) = \frac{1}{(1+f)^2} \mathcal{W}(X, \nabla f, Y, \nabla f) + \frac{n-3}{n-2} \frac{1}{1+f} d^2 r(Y, \nabla f, X) - \mathcal{W}z(X, Y).
\]

Therefore, it follows from (3.3) and (3.4) that

\[
(n-2)(f)B = -\delta C + \mathcal{W}z = -\delta \left( \frac{1}{1+f} \tilde{\nabla} f \mathcal{W} - (n-1) \frac{T}{1+f} \right) + \mathcal{W}z.
\]

On the other hand, by taking the divergence of \( T \), we have

\[
(n-1)(n-2)\delta T(X, Y) = \frac{n-2}{n-1} s f z(X, Y) - (n-2) D_{\nabla f} z(X, Y) + C(Y, \nabla f, X) + n(1+f)z \circ z(X, Y) - (1+f)|z|^2 g(X, Y).
\]

By combining (3.5) and (3.6), we obtain the followings.

**Proposition 7.** On \( M \), we have

\[
(n-1)(\delta T, z) = (n-2)(1+f)(B, z) = \frac{s f}{n-1} |z|^2 - \frac{1}{2} \nabla f (|z|^2) + \frac{n}{n-2}(1+f)(z \circ z, z).
\]

Also we have

**Proposition 8.** On \( M \), we have

\[
(n-1)\delta T(X) = \frac{n-1}{n-2}(1+f)(i_X C, z) + (n-1)(i_X T, z).
\]
Proof. By taking the derivative of (3.5), we have
\[(n-2)B(\nabla f, X) = (n-2)(1+f)\delta B(X) - \delta C(\nabla f, X) + \frac{n-3}{n-2}(1+f)i_XC, z) - (n-1)\delta T(X).\]
Thus, by (2.4) and Proposition 5 we have
\[(n-1)\delta T(X) = -\hat{\nu}z(\nabla f, X) + \frac{1}{n-2}(1+f)i_XC, z) + (n-1)i_XT, z).\]
where the last equality comes from (3.3). \(\square\)

We also have the following.

Lemma 9. We have
\[|T|^2 = \frac{2}{n-2}(i_{\nabla f}T, z),\]
and
\[\frac{(n-2)^2}{2}|T|^2 = |z|^2|\nabla f|^2 - \frac{n}{n-1}z \circ z(\nabla f, \nabla f).\]

Proof. It is a straightforward computation. From the definition of $T$,
\[|T|^2 = \frac{1}{n-2} \sum_{i,j,k} T(E_i, E_j, E_k) \left( df \wedge z + \frac{1}{n-1} i_{\nabla f} z \wedge g(E_i, E_j, E_k) \right) = \frac{2}{n-2}(i_{\nabla f}T, z). \]
Also
\[(n-2)^2|T|^2 = \sum_{i,j,k} |df(E_i)z_{jk} - df(E_j)z_{ik} + \frac{1}{n-1} (z(\nabla f, E_i)g_{jk} - z(\nabla f, E_j)g_{ik})|^2 \]
\[= 2|\nabla f|^2|z|^2 - \frac{2n}{n-1}z \circ z(\nabla f, \nabla f). \]
\(\square\)

4. Proof of Theorem 1

In this section, we prove Theorem 1. Throughout the section and the next section, we assume that \(i_XC, z) = 0\) for any vector $X$ with $n \geq 4$. To prove Theorem 1, we first need the following.

Lemma 10. Let $(g, f)$ be a non-trivial solution of (1.1) on an $n$-dimensional compact manifold $M$, $n \geq 4$. Assume that $(i_XC, z) = 0$. Then
\[\int_M (1+f)(z \circ z, z) = \frac{(n-2)s}{2n(n-1)} \int_M |z|^2. \]
Proof. Note that
\[ \frac{1}{2} \int_M (1 + f) \Delta |z|^2 = \frac{1}{2} \int_M |z|^2 \Delta f = -\frac{s}{2(n-1)} \int_M f |z|^2. \]
Also, by (2.5) we have
\[ \frac{1}{2} \int_M (1 + f) \Delta |z|^2 = \int_M (1 + f) \delta (z \circ z) = \int_M \delta (z \circ z) (\nabla f) = \int_M (z \circ z, Dd f) \]
\[ = \int_M (1 + f) (z \circ z) - \frac{s}{n(n-1)} \int_M f |z|^2. \]
Thus,
\[ \int_M (1 + f) (z \circ z) = -\frac{(n-2)s}{2n(n-1)} \int_M f |z|^2. \]
However, by (1.1) it is easy to see that
\[ \delta (i \nabla f z) = -(1 + f) |z|^2, \]
which implies that
\[ (4.1) \quad \int_M f |z|^2 = -\int_M |z|^2. \]

The following is the Okumura inequality which can be found in Lemma 2.6 of [7] (c.f. see also Lemma 2.4 of [4]).

**Lemma 11.** For any real numbers \( a_1, \ldots, a_n \) with \( \sum_{i=1}^{n} a_i = 0 \), we have
\[ -\frac{n-2}{\sqrt{n(n-1)}} \left( \sum_{i=1}^{n} a_i^2 \right)^{3/2} \leq \sum_{i=1}^{n} a_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}} \left( \sum_{i=1}^{n} a_i^2 \right)^{3/2}, \]
and equality holds if and only if at least \( n-1 \) of the \( a_i \)'s are all equal.

Now, we are ready to prove Theorem 1

Let \( M_0 := \{ f \leq -1 \} \) and \( M^0 := \{ f > -1 \} \). Note that \( (z \circ z, z) = \text{tr}(z^3) \). By applying Lemma 11 to the traceless Ricci tensor \( z \), we have
\[ (1 + f) (z \circ z, z) \leq -\frac{n-2}{\sqrt{n(n-1)}} (1 + f) |z|^3 \]
on the set \( M_0 \), and
\[ (1 + f) (z \circ z, z) \leq \frac{n-2}{\sqrt{n(n-1)}} (1 + f) |z|^3 \]
on the set \( M^0 \).
Therefore, by Lemma 10 with $M = M_0 \cup M_0$, 
\[
\frac{(n-2)s}{2n(n-1)} \int_M |z|^2 = \int_{M_0} (1+f)(z \circ z, z) + \int_{M_0^0} (1+f)(z \circ z, z)
\leq -\frac{n-2}{\sqrt{n(n-1)}} \int_{M_0} (1+f)|z|^3 + \frac{n-2}{\sqrt{n(n-1)}} \int_{M_0^0} (1+f)|z|^3
\leq \frac{n-2}{\sqrt{n(n-1)}} \max_M (1+f) \int_{M_0} |z|^3 + \frac{n-2}{\sqrt{n(n-1)}} \min_M (1+f) \int_{M_0^0} |z|^3.
\]
Recall that $\mu = \max\{\min_M (1+f), \max_M (1+f)\}$. Consequently, we obtain 
\[
\frac{n-2}{\sqrt{n(n-1)}} \int_M \left( \frac{s}{2\sqrt{n(n-1)}} - \mu |z| \right) |z|^2 \leq 0.
\]
From the assumption, 
\[
\frac{s}{2\sqrt{n(n-1)}} - \mu |z| \geq 0.
\]
As a result, we have either $z = 0$, or 
\[
|z| = \frac{s}{2\mu \sqrt{n(n-1)}}.
\]
From (5.1) and the fact that $\int_M f = 0$, the second case should be excluded; otherwise 
\[
0 = \int_M (1+f)|z|^2 = \frac{s^2}{4n(n-1)\mu^2} \int_M (1+f) = \frac{s^2}{4n(n-1)\mu^2},
\]
which is a contradiction. \(\square\)

5. Proof of Theorem 3

In this section, we prove Theorem 3. To do this, we first show the following integral identity.

**Lemma 12.** We have 
\[
(5.1) \quad \frac{(n-1)(n-2)}{2} \int_M |T|^2 = \frac{s}{n} \int_M f^2 |z|^2 + \int_M z \circ z (\nabla f, \nabla f)
\leq + \frac{2(n-1)}{n-2} \int_M f (1+f)(z \circ z, z).
\]

**Proof.** It follows from Proposition 8 and Lemma 9 together with the assumption $\langle i_X C, z \rangle = 0$ for any vector $X$ that 
\[
\delta \delta T(\nabla f) = \langle i_{\nabla f} T, z \rangle = \frac{n-2}{2} |T|^2.
\]
Thus,
\[
\int_M f \delta \delta \delta T = \int_M \delta \delta T(\nabla f) = \frac{n-2}{2} \int_M |T|^2.
\]
By Proposition 8 again with \(\langle i_X C, z \rangle = 0\), we have
\[
\delta \delta T(X) = \langle i_X T, z \rangle,
\]
(5.3)
From the definition of \(T\),
\[
\langle T, C \rangle = \frac{n}{n-2} \langle df \wedge z, C \rangle = \frac{n}{n-2} \langle i_{\nabla f} C, z \rangle = 0,
\]
and so, by taking the divergence of \(\delta \delta T\), it follows from (5.3) that
\[
\delta \delta \delta T = \langle \delta T, z \rangle - \frac{1}{2} \langle T, C \rangle = \langle \delta T, z \rangle.
\]
Thus, by Proposition 7
\[
(n-1) \delta \delta T = \frac{s f}{n-1} |z|^2 - \frac{1}{2} \nabla f(|z|^2) + \frac{n}{n-2} (1+f) \langle z \circ z, z \rangle.
\]
From this together with (5.2), we have
\[
\int_M |T|^2 = \frac{s f}{n-1} \int_M |z|^2 - \frac{1}{2} \int_M f \nabla f(|z|^2)
+ \frac{n}{n-2} \int_M (1+f) \langle z \circ z, z \rangle.
\]
(5.4)
Next, by Proposition 5 with the assumption that \(\langle i_X C, z \rangle = 0\), we have
\[
\frac{1}{2} \int_M f \nabla f(|z|^2) = - \int_M \delta (z \circ z)(f \nabla f)
= - \int_M z \circ z(\nabla f \cdot f) - \int_M f \langle z \circ z, Dd f \rangle
= - \int_M z \circ z(\nabla f \cdot f) - \int_M f (1+f) \langle z \circ z, z \rangle + \frac{s}{n(n-1)} \int_M f^2 |z|^2.
\]
Here, the last equality comes from (3.2). Substituting this into (5.4), we obtain (5.1).

Now, we are ready to prove Theorem 3.

Proof. By Lemma 9
\[
\frac{(n-1)(n-2)}{2} \int_M |T|^2 = \frac{n-1}{n-2} \int_M |z|^2 |\nabla f|^2 - \frac{n}{n-2} \int_M z \circ z(\nabla f, \nabla f).
\]
Comparing this to (5.1), we have
\[
\frac{n-1}{n-2} \int_M |z|^2 |\nabla f|^2 - \frac{2(n-1)}{n-2} \int_M z \circ z(\nabla f, \nabla f)
= \frac{s}{n} \int_M f^2 |z|^2 + \frac{2(n-1)}{n-2} \int_M f (1+f) \langle z \circ z, z \rangle.
\]
From the assumption
\[
|z|^2 \leq \min \left\{ 2|Nz|^2, \frac{s^2}{4n(n-1)} \right\},
\]
we have
\[
|\nabla f|^2 |z|^2 \leq 2|\nabla f z|^2 = 2z \circ z (\nabla f, \nabla f),
\]
and so
\[
\frac{s}{n} \int_M f^2 |z|^2 + \frac{2(n-1)}{n-2} \int_M f (1 + f) (z \circ z, z) \leq 0.
\]
Note that, by Lemma 10
\[
\int_M f (1 + f) (z \circ z, z) = \int_M f^2 (z \circ z, z) + \int_M f (z \circ z, z)
\]
\[
= \int_M f^2 (z \circ z, z) + \frac{(n-2)s}{2n(n-1)} \int_M |z|^2 - \int_M (z \circ z, z).
\]
Therefore, applying Lemma 11
\[
\frac{s}{n} \int_M (1 + f^2) |z|^2 \leq \frac{2(n-1)}{n-2} \int_M (1 - f^2) (z \circ z, z) \leq \frac{2\sqrt{n-1}}{\sqrt{n}} \int_M (1 + f^2) |z|^3,
\]
which implies
\[
0 \leq \int_M (1 + f^2) |z|^2 \left( s \frac{s}{2\sqrt{n(n-1)}} - |z| \right) \leq 0,
\]
where the first inequality follows from the assumption
\[
|z| \leq \frac{s}{2\sqrt{n(n-1)}}
\]
Hence, we may conclude that \( z = 0 \) on all of \( M \). If the equality holds,
\[
|z| = \frac{s}{2\sqrt{n(n-1)}}
\]
then we reach a contradiction as in the proof of Theorem 11. □

References

[1] H. Baltazar, On critical point equation of compact manifolds with zero radial Weyl curvature, \url{arXiv:1709.09681}.
[2] A.L. Besse, Einstein Manifolds, New York: Springer-Verlag 1987
[3] H.D. Cao and Q. Chen, On Bach-flat gradient shrinking Ricci solitons, Duke Math. J. 162 (2013), no.6, 1149–1169.
[4] G. Huisken, Ricci deformation of the metric on a Riemannian manifold, J. Diffeom. Geom. 21 (1985), 47–62.
[5] N. Koiso, A decomposition of the space \( \mathcal{M} \) of Riemannian metrics on a manifold, Osaka J. Math. 16 (1979), 423–429.
[6] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan 14 (1962), no.3, 333–340.

[7] M. Okumura, *Hypersurfaces and a pinching problem on the second fundamental tensor*, Amer. J. Math. 96 (1974), 207–213.

[8] J. Qing, W. Yuan, *A note on static spaces and related problems*, J. of Geom. and Phys. 74 (2013), 13–27.

[9] G. Yun, J. Chang, S. Hwang, *On the structure of linearization of the scalar curvature*, Tohoku Math. J. 67 (2015), 281–295.

[10] G. Yun, J. Chang, S. Hwang, *Total scalar curvature and harmonic curvature*, Taiwanese J. Math. 18 (2014), no.5, 1439–1458.

[11] G. Yun, J. Chang, S. Hwang, *Erratum to: Total scalar curvature and harmonic curvature*, Taiwanese J. Math. 20 (2016), no.3, 699–703.

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