A Weak Galerkin Finite Element Scheme
for the Biharmonic Equations by Using Polynomials of Reduced Order

Ran Zhang · Qilong Zhai

Received: 2 May 2014 / Revised: 25 October 2014 / Accepted: 1 November 2014 / Published online: 3 December 2014 © Springer Science+Business Media New York 2014

Abstract A new weak Galerkin (WG) finite element method for solving the biharmonic equation in two or three dimensional spaces by using polynomials of reduced order is introduced and analyzed. The WG method is on the use of weak functions and their weak derivatives defined as distributions. Weak functions and weak derivatives can be approximated by polynomials with various degrees. Different combination of polynomial spaces leads to different WG finite element methods, which makes WG methods highly flexible and efficient in practical computation. This paper explores the possibility of optimal combination of polynomial spaces that minimize the number of unknowns in the numerical scheme, yet without compromising the accuracy of the numerical approximation. Error estimates of optimal order are established for the corresponding WG approximations in both a discrete $H^2$ norm and the standard $L^2$ norm. In addition, the paper also presents some numerical experiments to demonstrate the power of the WG method. The numerical results show a great promise of the robustness, reliability, flexibility and accuracy of the WG method.

Keywords Weak Galerkin finite element methods · Weak Laplacian · Biharmonic equation · Polyhedral meshes

Mathematics Subject Classification Primary: 65N30, 65N15, 65N12, 74N20 · Secondary: 35B45, 35J50, 35J35
1 Introduction

This paper will concern with approximating the solution $u$ of the biharmonic equation

$$\Delta^2 u = f, \quad \text{in } \Omega,$$

with clamped boundary conditions

$$\begin{align*}
\Delta^2 u &= f, \quad \text{in } \Omega, \\
u &= g, \quad \text{on } \partial\Omega, \\
\frac{\partial u}{\partial n} &= \phi, \quad \text{on } \partial\Omega,
\end{align*}$$

where $\Delta$ is the Laplacian operator, $\Omega$ is a bounded polygonal or polyhedral domain in $\mathbb{R}^d$ for $d = 2, 3$ and $n$ denotes the outward unit normal vector along $\partial\Omega$. We assume that $f, g, \phi$ are given, sufficiently smooth functions.

This problem mainly arises in fluid dynamics where the stream functions $u$ of incompressible flows are sought and elasticity theory, in which the deflection of a thin plate of the clamped plate bending problem is sought [26,34,36].

Due to the significance of the biharmonic problem, a large number of methods for discretizing (1.1)–(1.3) have been proposed. These methods include dealing with the biharmonic operator directly, such as discretizing (1.1)–(1.3) on a uniform grid using a 13-point or 25-point direct approximation of the fourth order differential operator [9,24]; mixed methods, that is, splitting the biharmonic equation into two coupled Poisson equations [1,4–7,12,15,17–20,25,27]. Also there are some other approaches to the biharmonic problems, like the conformal mapping methods [11,35], integral equations [29], orthogonal spline collocation method [8] and the fast multipole methods [23], etc.

Among these methods, finite element methods are one of the most widely used technique, which is based on variational formulations of the equations considered. In fact, the biharmonic equation is also one of the most important applicable problems of the finite element methods, cf. [2,13,14,16,22,41]. The Galerkin methods, discretizing the corresponding variational form of (1.1) is given by seeking $u \in H^2(\Omega)$ satisfying

$$u|_{\partial\Omega} = g, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = \phi$$

such that

$$\langle \Delta u, \Delta v \rangle = \langle f, v \rangle, \quad \forall v \in H^2_0(\Omega),$$

where $H^2_0(\Omega)$ is the subspace of $H^2(\Omega)$ consisting of functions with vanishing value and normal derivative on $\partial\Omega$.

Standard finite element methods for solving (1.1)–(1.3) based on the variational form (1.4) with conforming finite element require rather sophisticated finite elements such as the 21-degrees-of-freedom of Argyris (see [3]) or nonconforming elements of Hermite type. Since the complexity in the construction for the finite element with high continuous elements, $H^2$ conforming element are seldom used in practice for the biharmonic problem. To avoid using of $C^1$-elements, besides the mixed methods, an alternative approach, nonconforming and discontinuous Galerkin finite element methods have been developed for solving the biharmonic equation over the last several decades. Morley element [28] is a well known nonconforming element for the biharmonic equation for its simplicity. A $C^0$ interior penalty method was developed in [10,21]. In [30], a hp-version interior penalty discontinuous Galerkin method was presented for the biharmonic equation.
Recently a new class of finite element methods, called weak Galerkin (WG) finite element methods were developed for the biharmonic equation for its highly flexible and robust properties. The WG method refers to a numerical scheme for partial differential equations in which differential operators are approximated by weak forms as distributions over a set of generalized functions. This thought was first proposed in [38] for a model second order elliptic problem, and this method was further developed in [31, 39, 40]. In [32], a weak Galerkin method for the biharmonic equation was derived by using discontinuous functions of piecewise polynomials on general partitions of polygons or polyhedra of arbitrary shape. After that, in order to reduce the number of unknowns, a $C^0$ WG method [33] was proposed and analyzed. However, due to the continuity limitation, the $C^0$ WG scheme only works for the traditional finite partitions, while not arbitrary polygonal or polyhedral grids as allowed in [32].

In order to realize the aim that reducing the unknown numbers and suit for general partitions of polygons or polyhedra of arbitrary shape at the same time, in this paper we construct a reduction WG scheme based on the use of a discrete weak Laplacian plus a new stabilization that is also parameter free. The goal of this paper is to specify all the details for the reduction WG method for the biharmonic equations and present the numerical analysis by presenting a mathematical convergence theory.

An outline of the paper is as follows. In the remainder of the introduction we shall introduce some preliminaries and notations for Sobolev spaces. In Sect. 2 is devoted to the definitions of weak functions and weak derivatives. The WG finite element schemes for the biharmonic Eqs. (1.1)–(1.3) are presented in Sect. 3. In Sect. 4, we establish an optimal order error estimates for the WG finite element approximation in an $H^2$ equivalent discrete norm. In Sect. 5, we shall drive an error estimate for the WG finite element method in the standard $L^2$ norm. Section 6 contains the numerical results of the WG method. The theoretical results are illustrated by these numerical examples. Finally, we present some technical estimates for quantities related to the local $L^2$ projections into various finite element spaces and some approximation properties which are useful in the convergence analysis in “Appendix”.

Now let us define some notations. Let $D$ be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^d$, $d = 2, 3$. We use the standard definition for the Sobolev space $H^s(D)$ and their associated inner products $(\cdot, \cdot)_{s,D}$, norms $\| \cdot \|_{s,D}$, and seminorms $| \cdot |_{s,D}$ for any $s \geq 0$.

The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\| \cdot \|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript $D$ in the norm and in the inner product notation.

The space $H(\text{div}; D)$ is defined as the set of vector-valued functions on $D$ which, together with their divergence, are square integrable; i.e.,

$$H(\text{div}; D) = \left\{ \mathbf{v} : \mathbf{v} \in [L^2(D)]^d, \nabla \cdot \mathbf{v} \in L^2(D) \right\}.$$  

The norm in $H(\text{div}; D)$ is defined by

$$\| \mathbf{v} \|_{H(\text{div}; D)} = \left( \| \mathbf{v} \|_D^2 + \| \nabla \cdot \mathbf{v} \|_D^2 \right)^{\frac{1}{2}}.$$  

2 Weak Laplacian and Discrete Weak Laplacian

For the biharmonic equation (1.1), the underlying differential operator is the Laplacian $\Delta$. Thus, we shall first introduce a weak version for the Laplacian operator defined on a class of discontinuous functions as distributions [32].
Let $K$ be any polygonal or polyhedral domain with boundary $\partial K$. A weak function on the region $K$ refers to a function $v = \{v_0, v_b, v_g\}$ such that $v_0 \in L^2(K), v_b \in L^2(\partial K)$, and $v_g \cdot n \in L^2(\partial K)$, where $n$ is the outward unit normal vector along $\partial K$. Denote by $W(K)$ the space of all weak functions on $K$, that is,

$$W(K) = \{v = \{v_0, v_b, v_g\} : v_0 \in L^2(K), v_b, v_g \cdot n \in L^2(\partial K)\}. \quad (2.1)$$

Recall that, for any $v \in W(K)$, the weak Laplacian of $v = \{v_0, v_b, v_g\}$ is defined as a linear functional $\Delta_w v$ in the dual space of $H^2(K)$ whose action on each $\phi \in H^2(K)$ is given by

$$(\Delta_w v, \phi)_K = (v_0, \Delta \phi)_K - \langle v_b, \nabla \phi \cdot n \rangle_{\partial K} + \langle v_g \cdot n, \phi \rangle_{\partial K}, \quad (2.2)$$

where $(\cdot, \cdot)_K$ stands for the $L^2$-inner product in $L^2(K)$ and $(\cdot, \cdot)_{\partial K}$ is the inner product in $L^2(\partial K)$.

The Sobolev space $H^2(K)$ can be embedded into the space $W(K)$ by an inclusion map $i_W : H^2(K) \to W(K)$ defined as follows

$$i_W(\phi) = \{\phi|_K, \phi|_{\partial K}, (\nabla \phi \cdot n)|_{\partial K}\}, \quad \phi \in H^2(K).$$

With the help of the inclusion map $i_W$, the Sobolev space $H^2(K)$ can be viewed as a subspace of $W(K)$ by identifying each $\phi \in H^2(K)$ with $i_W(\phi)$.

Analogously, a weak function $v = \{v_0, v_b, v_g\} \in W(K)$ is said to be in $H^2(K)$ if it can be identified with a function $\phi \in H^2(K)$ through the above inclusion map. Here the first components $v_0$ can be seen as the value of $v$ in the interior and the second component $v_b$ represents the value of $v$ on $\partial K$. Denote $\nabla v \cdot n$ by $v_n$, then the third component $v_g$ represents $(\nabla v \cdot n)|_{\partial K} = v_g \cdot n$. Obviously, $v_g \cdot n = \nabla v \cdot n$. Note that if $v \notin H^2(K)$, then $v_b$ and $v_g$ may not necessarily be related to the trace of $v_0$ and $(\nabla v_0 \cdot n)n$ on $\partial K$, respectively.

For $v \in H^2(K)$, from integration by parts we have

$$(\Delta_w v, \phi)_K = (v, \Delta \phi)_K - \langle v_b, \nabla \phi \cdot n \rangle_{\partial K} + \langle \nabla v \cdot n, \phi \rangle_{\partial K}$$

$$= (v_0, \Delta \phi)_K - \langle v_b, \nabla \phi \cdot n \rangle_{\partial K} + \langle v_g \cdot n, \phi \rangle_{\partial K}.$$ 

Thus the weak Laplacian is identical with the strong Laplacian, i.e.,

$$\Delta_w i_W(v) = \Delta v$$

for smooth functions in $H^2(K)$.

For numerical implementation purpose, we define a discrete version of the weak Laplacian operator by approximating $\Delta_w$ in polynomial subspaces of the dual of $H^2(K)$. To this end, for any non-negative integer $r \geq 0$, let $P_r(K)$ be the set of polynomials on $K$ with degree no more than $r$.

**Definition 2.1** ([32]) A discrete weak Laplacian operator, denoted by $\Delta_{w,r,K}$, is defined as the unique polynomial $\Delta_{w,r,K} v \in P_r(K)$ satisfying

$$(\Delta_{w,r,K} v, \phi)_K = (v_0, \Delta \phi)_K - \langle v_b, \nabla \phi \cdot n \rangle_{\partial K} + \langle v_g \cdot n, \phi \rangle_{\partial K}, \quad \forall \phi \in P_r(K). \quad (2.3)$$

From the integration by parts, we have

$$(v_0, \Delta \phi)_K = (\Delta v_0, \phi)_K + \langle v_0, \nabla \phi \cdot n \rangle_{\partial K} - \langle \nabla v_0 \cdot n, \phi \rangle_{\partial K}.$$ 

Substituting the above identity into (2.3) yields

$$(\Delta_{w,r,K} v, \phi)_K - (\Delta v_0, \phi)_K = \langle v_0 - v_b, \nabla \phi \cdot n \rangle_{\partial K} - \langle (\nabla v_0 - v_g) \cdot n, \phi \rangle_{\partial K}, \quad (2.4)$$

for all $\phi \in P_r(K)$. 

\[ Springer\]
3 Weak Galerkin Finite Element Scheme

Let $\mathcal{T}_h$ be a partition of the domain $\Omega$ into polygons in 2D or polyhedra in 3D. Assume that $\mathcal{T}_h$ is shape regular in the sense as defined in [39]. Denote by $\mathcal{E}_h$ the set of all edges or flat faces in $\mathcal{T}_h$, and let $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega$ be the set of all interior edges or flat faces.

Since $v_n$ represents $\nabla v \cdot n$, then $v_n$ is naturally dependent on $n$. To ensure a single valued function $v_n$ on $e \in \mathcal{E}_h$, we introduce a set of normal directions on $\mathcal{E}_h$ as follows

$$\mathcal{N}_h = \{ n_e : n_e \text{ is unit and normal to } e, \ e \in \mathcal{E}_h \}. \quad (3.1)$$

For any given integer $k \geq 2$, $T \in \mathcal{T}_h$, denote by $\mathcal{W}_k(T)$ the discrete weak function space given by

$$\mathcal{W}_k(T) = \{ (v_0, v_b, v_n n_e) : v_0 \in P_k(T), v_b, v_n \in P_{k-1}(e), e \subset \partial T \}. \quad (3.2)$$

By patching $\mathcal{W}_k(T)$ over all the elements $T \in \mathcal{T}_h$ through a common value on the interface $\mathcal{E}_h^0$, we arrive at a weak finite element space $V_h$ given by

$$V_h = \{ (v_0, v_b, v_n n_e) : (v_0, v_b, v_n n_e) \mid_{T} \in \mathcal{W}_k(T), \ \forall T \in \mathcal{T}_h \}.$$

Denote by $V_h^0$ the subspace of $V_h$ constituting discrete weak functions with vanishing traces; i.e.,

$$V_h^0 = \{ (v_0, v_b, v_n n_e) : (v_0, v_b, v_n n_e) \in V_h, v_b \mid_{e} = 0, v_n \mid_{e} = 0, \ e \in \partial T \cap \partial \Omega \}.$$

Denote by $\Lambda_h$ the trace of $V_h$ on $\partial \Omega$ from the component $v_b$. It is obvious that $\Lambda_h$ consists of piecewise polynomials of degree $k - 1$. Similarly, denote by $\Upsilon_h$ the trace of $V_h$ from the component of $v_n$ as piecewise polynomials of degree $k - 1$. Denote by $\Delta_{w,k-2}$ the discrete weak Laplacian operator on the finite element space $V_h$ computed by using (2.3) on each element $T$ for $k \geq 2$, that is,

$$\left( \Delta_{w,k-2} v \right) \mid_{T} = \Delta_{w,k-2,T}(v \mid_{T}) \ \forall v \in V_h. \quad (3.3)$$

For simplicity, we shall drop the subscript $k - 2$ in the notation $\Delta_{w,k-2}$ for the discrete weak Laplacian operator. We also introduce the following notation

$$\left( \Delta_w v, \Delta_w w \right)_h = \sum_{T \in \mathcal{T}_h} \left( \Delta_w v, \Delta_w w \right)_T.$$

For each element $T \in \mathcal{T}_h$, denote by $Q_0$ the $L^2$ projection onto $P_k(T)$, $k \geq 2$. For each edge/face $e \subset \partial T$, denote by $Q_b$ the $L^2$ projection onto $P_{k-1}(e)$. Now for any $u \in H^2(\Omega)$, we shall combine two projections together to define a projection into the finite element space $V_h$ such that on the element $T$

$$Q_hu = \{ Q_0u, Q_b u, (Q_b(\nabla u \cdot n_e))n_e \}.$$

**Theorem 3.1** Let $Q_h$ be the local $L^2$ projection onto $P_{k-2}$. Then the following commutative diagram holds true on each element $T \in \mathcal{T}_h$:

$$\Delta_w Q_h u = Q_h \Delta u, \quad \forall u \in H^2(T). \quad (3.4)$$

**Proof** For any $\phi \in P_{k-2}(T)$, from the definition of the discrete weak Laplacian and the $L^2$ projection.
\[(\Delta u_h^b u, \phi)^T = (Q_0 u, \Delta \phi)^T - (Q_b u, \nabla \phi \cdot n)_{\partial T} + (Q_b (\nabla u \cdot n)_{e}, n \cdot \phi)_{\partial T} \]
\[= (u, \Delta \phi)^T - (u, \nabla \phi \cdot n)_{\partial T} + (\nabla u \cdot n, \phi)_{\partial T} \]
\[= (\Delta u, \phi)^T = (Q_h \Delta u, \phi), \]

which implies (3.4).

The commutative property (3.4) indicates that the discrete weak Laplacian of the \(L^2\) projection of \(u\) is a good approximation of the Laplacian of \(u\) in the classical sense. This is a good property of the discrete weak Laplacian in application to algorithm and analysis.

For any \(u_h = \{u_0, u_b, u_n n_e\}\) and \(v = \{v_0, v_b, v_n n_e\}\) in \(V_h\), we introduce a bilinear form as follows

\[s(u_h, v) = \sum_{T \in T_h} h_T^{-1} (\nabla u_0 \cdot n_e - u_n, \nabla v_0 \cdot n_e - v_n)_{\partial T} + \sum_{T \in T_h} h_T^{-3} (Q_b u_0 - u_b, Q_b v_0 - v_b)_{\partial T}.\]

**Weak Galerkin Algorithm 1** Find \(u_h = \{u_0, u_b, u_n n_e\} \in V_h\) satisfying \(u_b = Q_b g\) and \(u_n = Q_b \phi\) on \(\partial \Omega\) and the following equation:

\[(\Delta w u_h, \Delta w v)_h + s(u_h, v) = (f, v_0), \quad \forall v = \{v_0, v_b, v_n n_e\} \in V_h^0. \quad (3.5)\]

**Lemma 3.2** For any \(v \in V_h^0\), let \(\|v\|\) be given by

\[\|v\|^2 = (\Delta w v, \Delta w v)_h + s(v, v). \quad (3.6)\]

Then, \(\|\cdot\|\) defines a norm in the linear space \(V_h^0\).

**Proof** For simplicity, we shall only prove the positivity property for \(\|\cdot\|\). Assume that \(\|v\| = 0\) for some \(v \in V_h^0\). It follows that \(\Delta w v = 0\) on each element \(T\), \(Q_b v_0 = v_b\) and \(\nabla v_0 \cdot n_e = v_n\) on each edge \(\partial T\). We claim that \(\Delta v_0 = 0\) holds true locally on each element \(T\). To this end, for any \(\varphi \in P_{k-2}(T)\) we use \(\Delta w v = 0\) and the identify (2.4) to obtain

\[0 = (\Delta w v, \varphi)^T = (\Delta v_0, \varphi)^T + (Q_b v_0 - v_b, \nabla \varphi \cdot n)_{\partial T} + (v_n n_e \cdot n - \nabla v_0 \cdot n, \varphi)_{\partial T} \]
\[= (\Delta v_0, \varphi)^T, \]

where we have used the fact that \(Q_b v_0 - v_b = 0\) and

\[v_n n_e \cdot n - \nabla v_0 \cdot n = \pm (v_n - \nabla v_0 \cdot n_e) = 0 \]

in the last equality. The identity (3.7) implies that \(\Delta v_0 = 0\) holds true locally on each element \(T\).

Next, we claim that \(\nabla v_0 = 0\) also holds true locally on each element \(T\). For this purpose, for any \(\phi \in P_k(T)\), we utilize the Gauss formula to obtain

\[(\nabla v_0, \nabla \phi)^T = - (\Delta v_0, \phi)^T + (\nabla v_0 \cdot n, \phi)_{\partial T} = (\nabla v_0 \cdot n, \phi)_{\partial T}. \quad (3.8)\]

By letting \(\phi = v_0\) on each element \(T\) and summing over all \(T\) we obtain

\[\sum_{T \in T_h} (\nabla v_0, \nabla v_0)^T = \sum_{T \in T_h} (\nabla v_0 \cdot n, v_0)_{\partial T}. \quad (3.9)\]

\(\Box\) Springer
For two elements \( T_1, T_2 \in \mathcal{T}_h \), which share an edge \( e \in \mathcal{E}_h \setminus \partial \Omega \) as a common edge, denote \( v^1_0, v^2_0 \) the values of \( v \) in the interior of \( T_1, T_2 \), respectively. It follows from \( Q_b v^1_0 = Q_b v^2_0 = v_b \) on edge \( e \) and the fact \( \nabla v_0 \cdot n_e = v_n \in P_{k-1}(e) \) that
\[
\langle \nabla v^1_0 \cdot n_{T_1}, v^1_0 \rangle_e + \langle \nabla v^2_0 \cdot n_{T_2}, v^2_0 \rangle_e = \pm \langle v_n, v^1_0 - v^2_0 \rangle_e = \pm \langle v_n, Q_b v^1_0 - Q_b v^2_0 \rangle_e = 0,
\]
where \( n_{T_1}, n_{T_2} \) denote the outward unit normal vectors on \( e \) according to elements \( T_1, T_2 \), respectively. This, together with \( \nabla v_0 \cdot n = v_n = 0 \) on the boundary edge \( e \in \mathcal{E}_h \cap \partial \Omega \) implies
\[
\sum_{T \in \mathcal{T}_h} (\nabla v_0 \cdot n, v_0)_{\partial T} = 0.
\]
It follows from Eq. (3.9) that \( \| \nabla v_0 \|_T = 0 \) on each element \( T \). Thus, \( v_0 = \text{const} \) locally on each element and is then continuous across each interior edge \( e \) as \( v_0|_e = \tilde{Q}_b v_0 = v_b \). The boundary condition of \( v_b = 0 \) then implies that \( v \equiv 0 \) on \( \Omega \), which completes the proof.

**Lemma 3.3** The weak Galerkin finite element scheme (3.5) has a unique solution.

**Proof** Assume \( u^{(1)}_h \) and \( u^{(2)}_h \) are two solutions of the WG finite element scheme (3.5). It is obvious that the difference \( \rho_h = u^{(1)}_h - u^{(2)}_h \) is a finite element function in \( V^0_h \) satisfying
\[
(\Delta_w \rho_h, \Delta_w v)_h + s(\rho_h, v) = 0, \quad \forall v \in V^0_h.
\]
By letting \( v = \rho_h \) in above Eq. (3.10) we obtain the following identity
\[
(\Delta_w \rho_h, \Delta_w \rho_h)_h + s(\rho_h, \rho_h) = 0.
\]
It follows from Lemma 3.2 that \( \rho_h \equiv 0 \), which shows that \( u^{(1)}_h = u^{(2)}_h \). This completes the proof.

**4 An Error Estimate**

The goal of this section is to establish an error estimate for the WG-FEM solution \( u_h \) arising from (3.5).

First of all, let us derive an error equation for the WG finite element solution obtained from (3.5). This error equation is critical in convergence analysis.

**Lemma 4.1** Let \( u \) and \( u_h \in V_h \) be the solution of (1.1)–(1.3) and (3.5), respectively. Denote by
\[
e_h = Q_h u - u_h
\]
the error function between the \( L^2 \) projection of \( u \) and its weak Galerkin finite element solution. Then the error function \( e_h \) satisfies the following equation
\[
(\Delta_w e_h, \Delta_w v)_h + s(e_h, v) = \ell_u(v)
\]
(4.1)
for all $v \in V^0_h$. Here
\[
\ell_u(v) = \sum_{T \in T_h} (\Delta u - Q_h \Delta u, \nabla v_0 \cdot n - v_n n_e \cdot n)_{\partial T} \tag{4.2}
- \sum_{T \in T_h} (\nabla (\Delta u - Q_h \Delta u) \cdot n, v_0 - v_b)_{\partial T} + s(Q_h u, v).
\]

Proof Using (2.4) with $\varphi = \Delta u Q_h u = Q_h \Delta u$ we obtain
\[
\begin{align*}
(\Delta u, \Delta v_0)_T &= (\Delta u, \Delta v)_T + (v_0 - v_b, \nabla (Q_h \Delta u) \cdot n)_{\partial T} - ((\nabla v_0 - v_n n_e) \cdot n, Q_h \Delta u)_{\partial T} \\
&= (\Delta u, \Delta v_0)_T + (v_0 - v_b, \nabla (Q_h \Delta u) \cdot n)_{\partial T} - ((\nabla v_0 - v_n n_e) \cdot n, Q_h \Delta u)_{\partial T},
\end{align*}
\]
which implies that
\[
(\Delta u, \Delta v_0)_T = (\Delta u Q_h u, \Delta u v)_T - (v_0 - v_b, \nabla (Q_h \Delta u) \cdot n)_{\partial T} + ((\nabla v_0 - v_n n_e) \cdot n, Q_h \Delta u)_{\partial T}. \tag{4.3}
\]

Next, it follows from the integration by parts that
\[
(\Delta u, \Delta v_0)_T = (\Delta^2 u, v_0)_T + (\Delta u, \nabla v_0 \cdot n)_{\partial T} - (\nabla (\Delta u) \cdot n, v_0)_{\partial T}.
\]

By summing over all $T$ and then using the identity $(\Delta^2 u, v_0) = (f, v_0)$ we arrive at
\[
\sum_{T \in T_h} (\Delta u, \Delta v_0)_T = (f, v_0) + \sum_{T \in T_h} (\Delta u \nabla v_0 \cdot n - v_n n_e \cdot n)_{\partial T} - \sum_{T \in T_h} (\nabla (\Delta u) \cdot n, v_0 - v_b)_{\partial T},
\]
where we have used the fact that $v_n$ and $v_b$ vanish on the boundary of the domain. Combining the above equation with (4.3) yields
\[
(\Delta u Q_h u, \Delta u v)_h = (f, v_0) + \sum_{T \in T_h} (\Delta u - Q_h \Delta u, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T} - \sum_{T \in T_h} (\nabla (\Delta u - Q_h \Delta u) \cdot n, v_0 - v_b)_{\partial T}.
\]

Adding $s(Q_h u, v)$ to both sides of the above equation gives
\[
(\Delta u Q_h u, \Delta u v)_h + s(Q_h u, v)
= (f, v_0) + \sum_{T \in T_h} (\Delta u - Q_h \Delta u, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T} - \sum_{T \in T_h} (\nabla (\Delta u - Q_h \Delta u) \cdot n, v_0 - v_b)_{\partial T} + s(Q_h u, v). \tag{4.4}
\]

Subtracting (3.5) from (4.4) leads to the following error equation
\[
(\Delta u e_h, \Delta u v)_h + s(e_h, v) = \sum_{T \in T_h} (\Delta u - Q_h \Delta u, (\nabla v_0 - v_n n_e) \cdot n)_{\partial T} - \sum_{T \in T_h} (\nabla (\Delta u - Q_h \Delta u) \cdot n, v_0 - v_b)_{\partial T} + s(Q_h u, v)
\]
for all $v \in V^0_h$. This completes the derivation of (4.1). \qed
The following Theorem presents an optimal order error estimate for the error function $e_h$ in the trip-bar norm. We believe this trip-bar norm provides a discrete analogue of the usual $H^2$-norm.

**Theorem 4.2** Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \geq 2$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\Omega)$. Then, there exists a constant $C$ such that

$$\|u_h - Q_h u\| \leq C h^{k-1} \|u\|_{k+2}. \quad (4.5)$$

The above estimate is of optimal order in terms of the meshsize $h$, but not in the regularity assumption on the exact solution of the biharmonic equation.

**Proof** By letting $v = e_h$ in the error Eq. (4.1), we have

$$\|e_h\|^2 = \ell(e_h). \quad (4.6)$$

where

$$\ell(e_h) = \sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla e_0 - e_n n_e) \cdot n \rangle_{\partial T}$$

$$- \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot n, e_0 - e_b \rangle_{\partial T}$$

$$+ \sum_{T \in T_h} h_T^{-1} \langle \nabla Q_0 u \cdot n_e - Q_b (\nabla u \cdot n_e), \nabla e_0 \cdot n_e - e_n \rangle_{\partial T}$$

$$+ \sum_{T \in T_h} h_T^{-3} \langle Q_b Q_0 u - Q_b u, Q_b e_0 - e_b \rangle_{\partial T}. \quad (4.7)$$

The rest of the proof shall estimate each of the terms on the right-hand side of (4.7). For the first term, we use the Cauchy–Schwarz inequality and the estimates (7.5) and (7.6) in Lemma 7.4 (see “Appendix”) with $m = k$ to obtain

$$\left| \sum_{T \in T_h} \langle \Delta u - Q_h \Delta u, (\nabla e_0 - e_n n_e) \cdot n \rangle_{\partial T} \right|$$

$$\leq \left( \sum_{T \in T_h} h_T \|\Delta u - Q_h \Delta u\|^2_{\partial T} \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \|\nabla e_0 \cdot n_e - e_n\|^2_{\partial T} \right)^{1/2}$$

$$\leq C h^{k-1} \|u\|_{k+1} \|e_h\|. \quad (4.8)$$

For the second term, using Lemmas 7.4, 7.6 and 7.9 we obtain

$$\left| \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot n, e_0 - e_b \rangle_{\partial T} \right|$$

$$\leq \left| \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot n, Q_b e_0 - e_b \rangle_{\partial T} \right|$$

$$+ \left| \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot n, e_0 - Q_b e_0 \rangle_{\partial T} \right|$$

$$\leq C h^{k-1} \|u\|_{k+1} \|e_h\|.$$
\[
\begin{align*}
&\left| \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot n, Q_b e_0 - e_h \rangle_{\partial T} \right| \\
&\quad + \left| \sum_{T \in T_h} \langle (\nabla (\Delta u) - Q_b (\nabla (\Delta u))) \cdot n, e_0 - Q_b e_0 \rangle_{\partial T} \right| \\
&\quad \leq \left( \sum_{T \in T_h} h_T^3 \| \nabla (\Delta u - Q_h \Delta u) \|^2_{\partial T} \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} h_T^{-3} \| Q_b e_0 - e_b \|^2_{\partial T} \right)^{\frac{1}{2}} \\
&\quad + \left( \sum_{T \in T_h} \| \nabla (\Delta u) - Q_b (\nabla (\Delta u)) \|^2_{\partial T} \right)^{\frac{1}{2}} \cdot \left( \sum_{T \in T_h} \| e_0 - Q_b e_0 \|^2_{\partial T} \right)^{\frac{1}{2}} \\
&\quad \leq C h^{k-1} \| u \|_{k+2} \| e_h \|. \tag{4.9}
\end{align*}
\]

where the $H^{k+2}$-norm of $u$ is used because the estimate in Lemma 7.9 is not optimal in terms of the mesh parameter $h$.

The third and fourth terms can be estimated by using the Cauchy–Schwarz inequality and the estimates (7.7) and (7.8) in Lemma 7.4 as follows

\[
\left| \sum_{T \in T_h} h_T^{-1} \langle \nabla Q_0 u \cdot n_e - Q_b (\nabla u \cdot n_e), \nabla e_0 \cdot n_e - e_h \rangle_{\partial T} \right| \leq C h^{k-1} \| u \|_{k+1} \| e_h \| \tag{4.10}
\]

and

\[
\left| \sum_{T \in T_h} h_T^{-3} \langle Q_b Q_0 u - Q_b u, Q_b e_0 - e_b \rangle_{\partial T} \right| \leq C h^{k-1} \| u \|_{k+1} \| e_h \|. \tag{4.11}
\]

Substituting (4.8)–(4.11) into (4.6) gives

\[
\| e_h \|^2 \leq C h^{k-1} \| u \|_{k+2} \| e_h \|^2,
\]

which implies (4.5) and hence completes the proof. \qed

\section{5 Error Estimates in $L^2$}

In this section, we shall establish some error estimates for all three components of the error function $e_h$ in the standard $L^2$ norm.

First of all, let us derive an error estimate for the first component of the error function $e_h$ by applying the usual duality argument in the finite element analysis. To this end, we consider the problem of seeking $\varphi$ such that

\[
\begin{align*}
\Delta^2 \varphi &= e_0, & \text{in } \Omega, \\
\varphi &= 0, & \text{on } \partial \Omega, \\
\frac{\partial \varphi}{\partial n} &= 0, & \text{on } \partial \Omega.
\end{align*}
\]
Assume that the dual problem has the $H^4$ regularity property in the sense that the solution function $\varphi \in H^4$ and there exists a constant $C$ such that
\[ \|\varphi\|_4 \leq C \|e_0\|. \]  
(5.2)

**Theorem 5.1** Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \geq 2$. Let $k_0 = \min\{3, k\}$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\Omega)$ and the dual problem (5.1) has the $H^4$ regularity. Then, there exists a constant $C$ such that
\[ \|u_0 - Q_0 u\| \leq Ch^{k+k_0-2} \|u\|_{k+1}, \]  
(5.3)
which means we have a sub-optimal order of convergence for $k = 2$ and optimal order of convergence for $k \geq 3$.

**Proof** Testing (5.1) by error function $e_0$ and then using the integration by parts gives
\[
\|e_0\|^2 = (\Delta^2 \varphi, e_0) = \sum_{T \in T_h} (\Delta \varphi, \Delta e_0)_T + \sum_{T \in T'_h} \langle \nabla (\Delta \varphi) \cdot n, e_0 \rangle_{\partial T} - \sum_{T \in T'_h} (\Delta \varphi, \nabla e_0 \cdot n)_{\partial T} \\
= \sum_{T \in T_h} (\Delta \varphi, \Delta e_0)_T + \sum_{T \in T'_h} \langle \nabla (\Delta \varphi) \cdot n, e_0 - e_b \rangle_{\partial T} \\
- \sum_{T \in T'_h} (\Delta \varphi - Q_h \Delta \varphi, \nabla e_0 - e_n n_e) \cdot n)_{\partial T}.
\]
where we have used the fact that $e_n$ and $e_b$ vanishes on the boundary of the domain $\Omega$. By letting $u = \varphi$ and $v_0 = e_h$ in (4.3), we can rewrite the above equation as follows
\[
\|e_0\|^2 = (\Delta_w Q_h \varphi, \Delta_w e_h)_h + \sum_{T \in T_h} \langle (\nabla (\Delta \varphi) - \nabla (Q_h \Delta \varphi)) \cdot n, e_0 - e_b \rangle_{\partial T} \\
- \sum_{T \in T_h} (\Delta \varphi - Q_h \Delta \varphi, (\nabla e_0 - e_n n_e) \cdot n)_{\partial T}.
\]
Next, by letting $v = Q_h \varphi$, from the error equation (4.1), we have
\[
(\Delta_w Q_h \varphi, \Delta_w e_h)_h = \sum_{T \in T_h} \langle (\Delta u - Q_h \Delta u, (\nabla Q_0 \varphi) \cdot n - Q_b (\nabla \varphi \cdot n_e) n_e \cdot n)_{\partial T} \\
- \sum_{T \in T_h} (\nabla (\Delta u - Q_h \Delta u) \cdot n, Q_0 \varphi - Q_b \varphi)_{\partial T} \\
- s(e_h, Q_h \varphi) + s(Q_h u, Q_h \varphi).
\]
Combining the two equations above gives
\[
\|e_0\|^2 = \sum_{T \in T_h} \langle (\nabla (\Delta \varphi) - \nabla (Q_h \Delta \varphi)) \cdot n, e_0 - e_b \rangle_{\partial T} \\
- \sum_{T \in T_h} (\Delta \varphi - Q_h \Delta \varphi, (\nabla e_0 \cdot n_e - e_n) \cdot n)_{\partial T}.
\]
\[ + \sum_{T \in T_h} \langle (\Delta u - Q_h \Delta u, (\nabla Q_0 \varphi) \cdot \mathbf{n} - Q_b (\nabla \varphi \cdot \mathbf{n}_e) \mathbf{n} \cdot \mathbf{n} \rangle_{\partial T} \]
\[ - \sum_{T \in T_h} \langle \nabla (\Delta u - Q_h \Delta u) \cdot \mathbf{n}, Q_0 \varphi - Q_b \varphi \rangle_{\partial T} \]
\[ - s(e_h, Q_h \varphi) + s(Q_h u, Q_h \varphi). \quad (5.4) \]

From the Cauchy–Schwarz inequality and Lemma 7.4, we can estimate the six terms on the right-hand side of the identity above as follows.

For the first term, it follows from Lemmas 7.4, 7.9 and the fact \( k_0 = \min\{k, 3\} \leq 3 \) that

\[ \left| \sum_{T \in T_h} \langle (\nabla (\Delta \varphi) - \nabla (Q_h \Delta \varphi)) \cdot \mathbf{n}, e_0 - e_b \rangle_{\partial T} \right| \]
\[ \leq \left( \sum_{T \in T_h} h_T^3 \| \nabla (\Delta \varphi) - \nabla (Q_h \Delta \varphi) \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-3} \| Q_b e_0 - e_b \|_{\partial T}^2 \right)^{\frac{1}{2}} \]
\[ + \left( \sum_{T \in T_h} \| \nabla (\Delta \varphi) - Q_b \nabla (\Delta \varphi) \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} \| e_0 - Q_b e_0 \|_{\partial T}^2 \right)^{\frac{1}{2}} \]
\[ \leq \left( \sum_{T \in T_h} h_T^3 \| \nabla (\Delta \varphi) - \nabla (Q_h \Delta \varphi) \|_{\partial T}^2 \right)^{\frac{1}{2}} \| e_h \| \]
\[ + C \lambda \left( \sum_{T \in T_h} \| \nabla (\Delta \varphi) - Q_b \nabla (\Delta \varphi) \|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot h^{-\frac{1}{2}} \| e_0 \| \]
\[ + C \left( \sum_{T \in T_h} \| \nabla (\Delta \varphi) - Q_b \nabla (\Delta \varphi) \|_{\partial T}^2 \right)^{\frac{1}{2}} \cdot h^\frac{3}{2} \| e_h \| \]
\[ \leq Ch^{k_0-1}(\| \varphi \|_{k_0+1} + h \delta_{k_0,2} \| \varphi \|_4) \| e_h \| + C \lambda h^\frac{1}{2} \| \varphi \|_4 \cdot h^{-\frac{1}{2}} \| e_0 \| \]
\[ + Ch^{k_0-\frac{5}{2}} (\| \varphi \|_{k_0+1} + h \delta_{k_0,2} \| \varphi \|_4) \cdot h^\frac{3}{2} \| e_h \| \]
\[ \leq Ch^{k_0-1} \| \varphi \|_4 \| e_h \| + C \lambda \| \varphi \|_4 \| e_0 \|. \quad (5.5) \]

For the second term, it follows from (7.3) with \( m = k_0 \) that

\[ \left| \sum_{T \in T_h} (\Delta \varphi - Q_h \Delta \varphi, (\nabla e_0 \cdot \mathbf{n}_e - e_n) \cdot \mathbf{n})_{\partial T} \right| \]
\[ \leq \left( \sum_{T \in T_h} h_T \| \Delta \varphi - Q_h \Delta \varphi \|_{\partial T}^2 \right)^{\frac{1}{2}} \left( \sum_{T \in T_h} h_T^{-1} \| \nabla e_0 \cdot \mathbf{n}_e - e_n \|_{\partial T}^2 \right)^{\frac{1}{2}} \]
\[ \leq Ch^{k_0-1} \| \varphi \|_{k_0+1} \| e_h \| \leq Ch^{k_0-1} \| \varphi \|_4 \| e_h \|. \quad (5.6) \]
As to the third term, it follows from Cauchy–Schwarz inequality and Lemma 7.4 that

\[
\sum_{T \in T_h} \langle \Delta u - \Omega_h \Delta u, (\nabla Q_0 \varphi) \cdot \mathbf{n} - Q_b (\nabla \varphi \cdot \mathbf{n}_e) \mathbf{n}_e \cdot \mathbf{n} \rangle_{\partial T} \\
\leq \left( \sum_{T \in T_h} h_T \| \Delta u - \Omega_h \Delta u \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-1} \| (\nabla Q_0 \varphi) \cdot \mathbf{n} - Q_b (\nabla \varphi \cdot \mathbf{n}_e) \|_{\partial T}^2 \right)^{1/2} \\
\leq C h^{k-1} \| u \|_{k+1} h^{k_0-1} \| \varphi \|_{k_0+1} \leq C h^{k+k_0-2} \| u \|_{k+1} \| \varphi \|_4.
\] (5.7)

For the fourth term, by using Lemma 7.3, we have

\[
\sum_{T \in T_h} \langle \nabla (\Delta u - \Omega_h \Delta u) \cdot \mathbf{n}, Q_0 \varphi - Q_b \varphi \rangle_{\partial T} \\
\leq \left( \sum_{T \in T_h} h_T^3 \| \nabla (\Delta u - \Omega_h \Delta u) \|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in T_h} h_T^{-3} \| Q_0 \varphi - \varphi \|_{\partial T}^2 \right)^{1/2} \\
\leq C h^{k-1} (\| u \|_{k+1} + h \delta_{k,2} \| u \|_4) h^{k_0-1} \| \varphi \|_{k_0+1} \\
\leq C h^{k-1} (\| u \|_{k+k_0-2} + h \delta_{k,2} \| u \|_4) \| \varphi \|_4.
\] (5.8)

As to the fifth term, we also use the Cauchy–Schwarz inequality and Lemma 7.4 to obtain

\[
| s(e_h, Q_k \varphi) | \leq \left| \sum_{T \in T_h} h_T^{-1} \langle \nabla e_0 \cdot \mathbf{n}_e - e_n, \nabla Q_0 \varphi \cdot \mathbf{n}_e - Q_b (\nabla \varphi \cdot \mathbf{n}_e) \rangle_{\partial T} \right| \\
+ \left| \sum_{T \in T_h} h_T^{-3} \langle Q_b e_0 - e_b, Q_b Q_0 \varphi - Q_b \varphi \rangle_{\partial T} \right| \\
\leq C h^{k_0-1} \| \varphi \|_4 \| e_h \|.
\] (5.9)

The last term can be estimated as follows

\[
| s(Q_k u, Q_k \varphi) | \leq \left| \sum_{T \in T_h} h_T^{-1} \langle (\nabla Q_0 u \cdot \mathbf{n}_e - Q_b (\nabla u \cdot \mathbf{n}_e), (\nabla Q_0 \varphi \cdot \mathbf{n}_e - Q_b (\nabla \varphi \cdot \mathbf{n}_e) \rangle_{\partial T} \right| \\
+ \left| \sum_{T \in T_h} h_T^{-3} \langle Q_b Q_0 u - Q_b u, Q_b Q_0 \varphi - Q_b \varphi \rangle_{\partial T} \right| \\
\leq C h^{k-1} \| u \|_{k+1} h^{k_0-1} \| \varphi \|_{k_0+1} \\
\leq C h^{k+k_0-2} \| u \|_{k+1} \| \varphi \|_4.
\] (5.10)

Substituting all the six estimates into (5.4) we obtain

\[
\| e_0 \|^2 \leq C h^{k+k_0-2} (\| u \|_{k+1} + h \delta_{k,2} \| u \|_4) \| \varphi \|_4 \\
+ C h^{k_0-1} \| \varphi \|_4 \| e_h \| + C \lambda \| \varphi \|_4 \| e_0 \|.
\]
Using the regularity estimate (5.2) and choosing constant $\lambda$ such that $C\lambda\|\varphi\|_4 < \frac{1}{2}\|e_0\|$, we arrive at
\[
\|e_0\| \leq Ch^{k_0-1}\|e_h\| + Ch^{k+k_0-2}(\|u\|_{k+1} + h\delta_{k,2}\|u\|_4) \\
\leq Ch^{k+k_0-2}\|u\|_{k+2}.
\]
Together with the $H^2$ error estimate (4.5) we have the desired $L^2$ error estimate (5.3). □

In order to study the error estimates on edges, we shall introduce the edge-based $L^2$ norm here. To keep the consistency of order, the edge-based $L^2$ norm is different from the standard $L^2$ norm.

**Definition 5.2** For any function $v$ defined on the edges $\mathcal{E}_h$,
\[
\|v\|^2_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} h_e\|v\|^2_{L^2(e)},
\]
where $h_e$ is the measure of edge $e \in \mathcal{E}_h$.

Next, we shall derive the estimates for the second and third components of the error function $e_h$.

**Theorem 5.3** Let $u_h \in V_h$ be the weak Galerkin finite element solution arising from (3.5) with finite element functions of order $k \geq 2$. Let $k_0 = \min\{k, 3\}$. Assume that the exact solution of (1.1)–(1.3) is sufficiently regular such that $u \in H^{k+2}(\omega)$ and the dual problem (5.1) has the $H^4$ regularity property. Then, there exists a constant $C$ such that
\[
\|u_b - Q_b u\|_{\mathcal{E}_h} \leq C h^{k+k_0-2}\|u\|_{k+2},
\]
\[
\|u_n - Q_b (\nabla u_0 \cdot n_e)\|_{\mathcal{E}_h} \leq C h^{k+k_0-3}\|u\|_{k+2}.
\]

**Proof** It is obvious that
\[
\|e_b\|^2_{L^2(e)} \leq 2(\|Q_b e_0\|^2_{L^2(e)} + \|Q_b e_0 - e_b\|^2_{L^2(e)}).
\]
Summing over all edges, we have
\[
\|u_b - Q_b u\|^2_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} h_e\|u_b - Q_b u\|^2_{L^2(e)} \\
\leq 2 \left( \sum_{e \in \mathcal{E}_h} h_e\|Q_b e_0\|^2_{L^2(e)} + \sum_{e \in \mathcal{E}_h} h_e\|Q_b e_0 - e_b\|^2_{L^2(e)} \right) \\
\leq C \left( \sum_{T \in T_h} h_T\|Q_b e_0\|^2_{L^2(\partial T)} + \sum_{T \in T_h} h_T\|Q_b e_0 - e_b\|^2_{L^2(\partial T)} \right). \tag{5.13}
\]

We shall discuss the two terms separately. For the first part, by applying the trace inequality (7.1), the inverse inequality (7.2) and the error estimate for $e_0$ in Theorem 5.1, we have
\[
\sum_{T \in T_h} h_T\|Q_b e_0\|^2_{L^2(\partial T)} \leq \sum_{T \in T_h} h_T\|e_0\|^2_{L^2(\partial T)} \\
\leq C \sum_{T \in T_h} \left( \|e_0\|^2_{L^2(T)} + h_T^2\|\nabla e_0\|^2_{L^2(T)} \right)
\]
\[ \leq C \sum_{T \in T_h} \| e_0 \|^2_{L^2(T)} \leq C h^{2k+2k_0-4} \| u \|_{k+2}^2. \]  

(5.14)

For the second part, we use the trip-bar norm to handle the second part.

\[ \sum_{T \in T_h} h_T \| Q_b e_0 - e_b \|^2_{L^2(\partial T)} \leq h^4 \sum_{T \in T_h} h_T^{-3} \| Q_b e_0 - e_b \|^2_{L^2(\partial T)} \leq h^4 \| e_b \|^2 \]  

(5.15)

Combining the above two estimates gives the desired error estimate (5.11).

Similarly, we establish the error estimates for \( e_n \).

\[ \| e_n \|^2_{E_h} = \sum_{e \in E_h} h_e \| e_n \|^2_{L^2(e)} \leq C \left( \sum_{T \in T_h} h_T \| \nabla e_0 \cdot n_e \|_{\partial T} + \sum_{T \in T_h} h_T \| \nabla e_0 \cdot n_e - e_n \|_{\partial T} \right) \leq C \left( \sum_{T \in T_h} \| \nabla e_0 \|_T + h^2 \| e_h \| \right) \leq C \left( \sum_{T \in T_h} h_T^{-2} \| e_0 \|_T + h^2 \| e_h \| \right) \leq C \left( h^{2k+2k_0-6} + h^2 \right) \| u \|_{k+2}^2. \]  

(5.16)

Thus, we have

\[ \| e_n \|_{E_h} \leq C h^{k+k_0-3} \| u \|_{k+2}, \]

which completes the proof.

\[ \square \]

6 Numerical Results

In this section, we would like to report some numerical results for the weak Galerkin finite element method proposed and analyzed in previous sections. Here we use the following finite element space

\[ \tilde{V}_h = \{ v = \{ v_0, v_b, v_n n_e \}, v_0 \in P_2(T), v_b, v_n \in P_1(e), T \in T_h, e \subset E_h \}. \]

For any given \( v = \{ v_0, v_b, v_n n_e \} \in \tilde{V}_h \) and \( \varphi \in P_0(T) \), we compute the discrete weak Laplacian \( \Delta_w v \) on each element \( T \) as a function in \( P_0(T) \) as follows

\[ (\Delta_w v, \varphi)_T = (v_0, \Delta \varphi)_T - (v_b, \nabla \varphi \cdot n)_{\partial T} + (v_n n_e \cdot n, \varphi)_{\partial T}, \]

which could be simplified as

\[ (\Delta_w v, \varphi)_T = (v_n n_e \cdot n, \varphi)_{\partial T}. \]
Table 1  Errors and orders of Example 6.1 in $H^2$ and $L^2$ with $k = 2$

| $h$         | $\| u_h - Q_h u \|$ | Order | $\| u_0 - Q_0 u \|$ | Order |
|-------------|----------------------|-------|----------------------|-------|
| 3.74355e−01 | 3.69061e−01          |       | 4.29897e−02          |       |
| 1.91955e−01 | 1.89785e−01          | 9.59493e−01 | 1.11418e−02          | 1.94801 |
| 9.56362e−02 | 1.01110e−01          | 9.08440e−01 | 2.97175e−03          | 1.90660 |
| 4.78382e−02 | 5.57946e−02          | 8.57728e−01 | 8.08649e−04          | 1.87773 |
| 2.20971e−02 | 3.00721e−02          | 8.91700e−01 | 2.14457e−04          | 1.91483 |
| 1.10485e−02 | 1.55286e−02          | 9.53498e−01 | 5.49264e−05          | 1.96512 |

Table 2  Errors and orders of Example 6.1 in $L^2$ and $L^\infty$ for $\epsilon_b$ with $k = 2$

| $h$         | $\| Q_b u - u_b \|_{\xi_b}$ | Order | $\| Q_b u - u_b \|_{\infty}$ | Order |
|-------------|-----------------------------|-------|-------------------------------|-------|
| 3.74355e−01 | 1.21967e−01                 |       | 1.18101e−01                   |       |
| 1.91955e−01 | 3.12884e−02                 | 1.91858 | 3.27686e−02                   | 1.84964 |
| 9.56362e−02 | 8.39049e−03                 | 1.89880 | 8.84728e−03                   | 1.88901 |
| 4.78382e−02 | 2.28623e−03                 | 1.87578 | 2.39957e−03                   | 1.88246 |
| 2.20971e−02 | 6.06514e−04                 | 1.91436 | 6.33868e−04                   | 1.92052 |
| 1.10485e−02 | 1.55351e−04                 | 1.96501 | 1.62044e−04                   | 1.96780 |

The error for the weak Galerkin solution is measured in six norms defined as follows:

$$
\| e_h \|_2^2 = \sum_{T \in T_h} \left( \int_T |\Delta u_h|v_h|^2 \,dT + h_T^{-1} \int_{\partial T} (\nabla v_0) \cdot n_e - v_n |^2 \,ds \right)
+ h_T^{-3} \int_{\partial T} (Q_b v_0 - v_b)^2 \,ds \tag{A discrete $H^2$ norm}$$

$$
\| Q_0 v - v_0 \|_2^2 = \sum_{T \in T_h} \int_T |Q_0 v - v_0|^2 \,dT \tag{Element based $L^2$ norm}$$

$$
\| Q_b v - v_b \|_{\xi_b}^2 = \sum_{e \in \xi_b} h_e \int_e |Q_b v - v_b|^2 \,ds \tag{Edge based $L^2$ norm for $v_b$}$$

$$
\| Q_b v - v_n \|_{\xi_b}^2 = \sum_{e \in \xi_b} h_e \int_e |Q_b v - v_n|^2 \,ds \tag{Edge based $L^2$ norm for $v_n$}$$

$$
\| Q_b v - v_b \|_{\infty} = \max_{e \in \xi_b} |Q_b v - v_b| \tag{Edge based $L^\infty$ norm for $v_b$}$$

$$
\| Q_b v - v_n \|_{\infty} = \max_{e \in \xi_b} [Q_b (\nabla u_0 \cdot n_e) - v_n] \tag{Edge based $L^\infty$ norm for $v_n$}$$

Example 6.1  Consider the biharmonic problem (1.1)–(1.3) in the square domain $\Omega = (0, 1)^2$. It has the analytic solution $u(x) = x^2(1 - x)^2y(1 - y)^2$, and the right hand side function $f$ in (1.1) is computed to match the exact solution. The mesh size is denoted by $h = 1/n$. Table 1 shows that the convergence rates for the WG-FEM solution in the $H^2$ and $L^2$ norms are of order $O(h)$ and $O(h^2)$ when $k = 2$, respectively.

Table 2 shows that the errors and orders of Example 6.1 in $L^2$ and $L^\infty$ for $\epsilon_b$. The numerical results are in consistency with theory for these two cases.

Table 3 shows that the errors and orders of Example 6.1 in $L^2$ and $L^\infty$ for $\epsilon_n$. The numerical results are in consistency with theory for these two cases.
### Table 3 Errors and orders of Example 6.1 in $L^2$ and $L^\infty$ for $e_n$ with $k = 2$

| $h$          | $\|Q_h(\nabla u \cdot n_e) - u_n\|_{E_h}$ | Order | $\|Q_h(\nabla u \cdot n_e) - u_n\|_{L^\infty}$ | Order |
|--------------|------------------------------------------|-------|-----------------------------------------------|-------|
| 3.74355e−01 | 1.18286e−01                             |       | 5.28497e−02                                   |       |
| 1.91858e+00 | 3.12884e−02                             | 1.1858| 1.51029e−02                                   | 1.80707|
| 9.56362e−02 | 8.39049e−03                             | 1.89880| 7.33970e−03                                   | 1.04103|
| 4.78382e−02 | 2.28623e−03                             | 1.87578| 3.41617e−03                                   | 1.10334|
| 2.20971e−02 | 6.06514e−04                             | 1.91436| 1.18287e−03                                   | 1.53009|
| 1.10485e−02 | 1.55351e−04                             | 1.96501| 3.30602e−04                                   | 1.83912|

### Table 4 Errors and orders of example 6.1 in $H^2$ and $L^2$ with $k = 3$

| $h$          | $\|u_h - Q_hu\|$ | Order | $\|u_0 - Q_0u\|$ | Order |
|--------------|------------------|-------|-------------------|-------|
| 3.74355e−01 | 1.17819e−01      |       | 4.56114e−03       |       |
| 1.91955e−01 | 3.56257e−02      | 1.72558| 4.16403e−04       | 3.45334|
| 9.56362e−02 | 1.00915e−02      | 1.81977| 5.55158e−05       | 3.55145|
| 4.78382e−02 | 2.56977e−03      | 1.97343| 2.30985e−06       | 3.94259|
| 2.20971e−02 | 6.44317e−04      | 1.99580| 1.44990e−07       | 3.99378|
| 1.10485e−02 | 1.61222e−04      | 1.99873| 9.07702e−09       | 3.99759|

### Table 5 Errors and orders of example 6.1 in $L^2$ and $L^\infty$ for $e_b$ with $k = 3$

| $h$          | $\|Q_hu - u_b\|_{E_h}$ | Order | $\|Q_hu - u_b\|_{L^\infty}$ | Order |
|--------------|-------------------------|-------|-------------------------------|-------|
| 3.74355e−01 | 8.34847e−03             |       | 1.15414e−02                   |       |
| 1.91955e−01 | 8.06272e−04             | 3.37217| 1.08014e−03                   | 3.41753|
| 9.56362e−02 | 7.89345e−05             | 3.35254| 9.02080e−05                   | 3.58181|
| 4.78382e−02 | 5.19889e−06             | 3.92438| 5.93961e−06                   | 3.92481|
| 2.20971e−02 | 3.26604e−07             | 3.99259| 3.72799e−07                   | 3.99390|
| 1.10485e−02 | 2.04554e−08             | 3.99699| 2.33003e−08                   | 3.99998|

### Table 6 Errors and orders of example 6.1 in $L^2$ and $L^\infty$ for $e_n$ with $k = 3$

| $h$          | $\|Q_h(\nabla u \cdot n_e) - u_n\|_{E_h}$ | Order | $\|Q_h(\nabla u \cdot n_e) - u_n\|_{L^\infty}$ | Order |
|--------------|------------------------------------------|-------|-----------------------------------------------|-------|
| 3.74355e−01 | 5.23031e−02                             |       | 1.15371e−01                                   |       |
| 1.91858e+00 | 8.83906e−03                             | 2.56493| 1.96390e−02                                   | 2.55449|
| 9.56362e−02 | 1.50030e−03                             | 2.55865| 3.59916e−03                                   | 2.44799|
| 4.78382e−02 | 1.89000e−04                             | 2.98878| 4.60320e−04                                   | 2.96695|
| 2.20971e−02 | 2.33468e−05                             | 3.01709| 5.56932e−05                                   | 3.04707|
| 1.10485e−02 | 2.89988e−06                             | 3.00916| 6.86324e−06                                   | 3.02054|
Table 7  Errors and orders of Example 6.2 in $H^2$ and $L^2$ with $k = 2$

| $h$          | $\|u_h - Q_h u\|$ | Order | $\|u_0 - Q_0 u\|$ | Order |
|--------------|---------------------|-------|---------------------|-------|
| 3.74355e−01 | 3.51847e+01         |       | 4.18608e+00         |       |
| 1.91955e−01 | 1.79831e+01         | 9.68306e−01 | 1.06553e+00         | 1.97403 |
| 9.56362e−02 | 9.36621e+00         | 9.41104e−01 | 2.74735e−01         | 1.95546 |
| 4.78382e−02 | 4.90899e+00         | 9.32039e−01 | 7.07013e−02         | 1.95823 |
| 2.20971e−02 | 2.51557e+00         | 9.64541e−01 | 1.79112e−02         | 1.98087 |
| 1.10485e−02 | 1.26858e+00         | 9.87671e−01 | 4.9750e−03          | 1.99367 |

Table 8  Errors and orders of Example 6.2 in $L^2$ and $L^\infty$ for $e_b$ with $k = 2$

| $h$          | $\|Q_b u - u_b\|_{E_h}$ | Order | $\|Q_b u - u_b\|_{\infty}$ | Order |
|--------------|--------------------------|-------|---------------------------|-------|
| 3.74355e−01 | 1.15398e+01              |       | 1.10028e+00               |       |
| 1.91955e−01 | 2.99335e+00              | 1.94679 | 2.97577e+00               | 1.88654 |
| 9.56362e−02 | 7.75705e−01              | 1.94818 | 7.77671e−01               | 1.93603 |
| 4.78382e−02 | 1.99884e−01              | 1.95635 | 2.00300e−01               | 1.95700 |
| 2.20971e−02 | 5.06547e−02              | 1.98039 | 5.07058e−02               | 1.98194 |
| 1.10485e−02 | 1.27205e−02              | 1.99354 | 1.27268e−02               | 1.99428 |

Table 9  Errors and orders of Example 6.2 in $L^2$ and $L^\infty$ for $e_n$ with $k = 2$

| $h$          | $\|Q_b (\nabla u \cdot n_e) - u_n\|_{E_h}$ | Order | $\|Q_b (\nabla u \cdot n_e) - u_n\|_{\infty}$ | Order |
|--------------|-----------------------------------------|-------|-----------------------------------------|-------|
| 3.74355e−01 | 1.15398e+01                            |       | 4.02986e+00                            |       |
| 1.91955e−01 | 2.99335e+00                            | 1.94679 | 1.26437e+00                            | 1.67231 |
| 9.56362e−02 | 7.75705e−01                            | 1.94818 | 4.40635e−01                            | 1.52076 |
| 4.78382e−02 | 1.99884e−01                            | 1.95635 | 1.74400e−01                            | 1.33718 |
| 2.20971e−02 | 5.06547e−02                            | 1.98039 | 5.22660e−02                            | 1.73846 |
| 1.10485e−02 | 1.27205e−02                            | 1.99354 | 1.37655e−02                            | 1.92482 |

Table 10  Errors and orders of example 6.2 in $H^2$ and $L^2$ with $k = 3$

| $h$          | $\|u_h - Q_h u\|$ | Order | $\|u_0 - Q_0 u\|$ | Order |
|--------------|---------------------|-------|---------------------|-------|
| 3.74355e−01 | 9.17084e+00         |       | 3.37369e+01         |       |
| 1.91955e−01 | 2.46720e+00         | 1.89418 | 2.77383e+02         | 3.60438 |
| 9.56362e−02 | 6.52418e−01         | 1.91900 | 2.14578e+03         | 3.69231 |
| 4.78382e−02 | 1.65736e−01         | 1.97691 | 1.36946e+04         | 3.96982 |
| 2.20971e−02 | 4.16442e−02         | 1.99270 | 8.50154e+06         | 4.00974 |
| 1.10485e−02 | 1.04302e−02         | 1.99734 | 5.29568e−07         | 4.00484 |
In Tables 4, 5 and 6 we investigate the same problem for \( k = 3 \). Table 4 shows that the convergence rates for the WG-FEM solution in the \( H^2 \) and \( L^2 \) norms are of order \( O(h^2) \) and \( O(h^4) \). Tables 5 and 6 show the errors and orders in \( L^2 \) and \( L^\infty \) for \( e_b \) and \( e_n \), which are also consistent with theoretical conclusions.

Example 6.2 Consider the biharmonic problem (1.1)–(1.3) in the square domain \( \Omega = (0, 1)^2 \). It has the analytic solution \( u(x) = \sin(\pi x) \sin(\pi y) \), and the right hand side function \( f \) in (1.1) is computed accordingly.

The numerical results are presented in Tables 7, 8, 9, 10, 11 and 12 which confirm the theory developed in previous sections.

Acknowledgments We gratefully acknowledge Professor Junping Wang for presenting this problem and giving us many valuable suggestions. The authors also thank the anonymous referees and editor for their careful reading of the manuscript and their valuable comments to improvement the work.

7 Appendix: \( L^2 \) Projection and Some Technical Results

In this section, we shall present some technical results for the \( L^2 \) projection operators with respect to the finite element space \( V_h \). These results are useful for the error estimates of the WG finite element method.

Lemma 7.1 ([39]) (Trace Inequality) Let \( T_h \) be a partition of the domain \( \Omega \) into polygons in 2D or polyhedra in 3D. Assume that the partition \( T_h \) satisfies the assumptions (A1), (A2), and (A3) as specified in [39]. Then, there exists a constant \( C \) such that for any \( T \in T_h \) and edge/face \( e \in \partial T \), we have
\[ \| \theta \|^p \leq Ch^{-1}(\| \theta \|^p + h^p T \| \nabla \theta \|^p_T), \]  

(7.1)

where \( \theta \in H^1(T) \) is any function.

**Lemma 7.2** ([39]) (Inverse Inequality) Let \( T_h \) be a partition of the domain \( \Omega \) into polygons or polyhedra. Assume that \( T_h \) satisfies all the assumptions (A1)–(A4) as specified in [39]. Then, there exists a constant \( C(n) \) such that

\[ \| \nabla \varphi \|_T \leq C(n) h^{-1} \| \varphi \|_T, \quad \forall T \in T_h \]  

(7.2)

for any piecewise polynomial \( \varphi \) of degree \( n \) on \( T_h \).

### 7.1 Approximation Properties

The following lemma provides some approximation properties for the projection operators \( Q_h \) and \( Q_b \).

**Lemma 7.3** ([32]) Let \( T_h \) be a finite element partition of \( \Omega \) satisfying the shape regularity assumptions. Then, for any \( 0 \leq s \leq 2 \) and \( 2 \leq m \leq k \) we have

\[ \sum_{T \in T_h} h^2_T \| \Delta u - Q_h \Delta u \|_{s,T}^2 \leq Ch^{2(m+1)} \| u \|_{m+1}^2, \]  

(7.3)

\[ \sum_{T \in T_h} h^2_T \| \Delta u - Q_h \Delta u \|_{s,T}^2 \leq Ch^{2(m-1)} \| u \|_{m+1}^2, \]  

(7.4)

**Lemma 7.4** Let \( 2 \leq m \leq k, \omega \in H^{m+2}(\Omega) \). There exists a constant \( C \) such that the following estimates hold true:

\[ \bigg( \sum_{T \in T_h} h_T \| \Delta \omega - Q_h \Delta \omega \|_{\partial T}^2 \bigg)^{1/2} \leq Ch^{m-1} \| \omega \|_{m+1}, \]  

(7.5)

\[ \bigg( \sum_{T \in T_h} h^3_T \| \nabla (\Delta \omega - Q_h \Delta \omega) \|_{\partial T}^2 \bigg)^{1/2} \leq Ch^{m-1} (\| \omega \|_{m+1} + h \delta_{m,2} \| \omega \|_4), \]  

(7.6)

\[ \bigg( \sum_{T \in T_h} h^{-1}_T \| \nabla (Q_0 \omega) \cdot n_e - Q_b (\nabla \omega \cdot n_e) \|_{\partial T}^2 \bigg)^{1/2} \leq Ch^{m-1} \| \omega \|_{m+1}, \]  

(7.7)

\[ \bigg( \sum_{T \in T_h} h^3_T \| Q_0 Q_0 \omega - Q_b \omega \|_{\partial T}^2 \bigg)^{1/2} \leq Ch^{m-1} \| \omega \|_{m+1}, \]  

(7.8)

\[ \bigg( \sum_{T \in T_h} \| \nabla (\Delta \omega) - Q_b (\nabla (\Delta \omega)) \|_{\partial T}^2 \bigg)^{1/2} \leq Ch^{m-3} \| \omega \|_{m+2}. \]  

(7.9)

Here \( \delta_{i,j} \) is the usual Kronecker’s delta with value 1 when \( i = j \) and value 0 otherwise.

**Proof** To derive (7.5), we use the trace inequality (7.1) and the estimate (7.4) to obtain...
\[
\sum_{T \in \mathcal{T}_h} h_T \| \Delta \omega - Q_h \Delta \omega \|_{\partial T}^2 \\
\leq C \sum_{T \in \mathcal{T}_h} \left( \| \Delta \omega - Q_h \Delta \omega \|_{T}^2 + h_T^2 \| \nabla (\Delta \omega - Q_h \Delta \omega) \|_{T}^2 \right) \\
\leq C h^{2m-2} \| \omega \|_{m+1}^2.
\]

As to (7.6), we use the trace inequality (7.1) and the estimate (7.4) to obtain
\[
\sum_{T \in \mathcal{T}_h} h_T^3 \| \nabla (\Delta \omega - Q_h \Delta \omega) \|_{\partial T}^2 \\
\leq C \sum_{T \in \mathcal{T}_h} \left( h_T^2 \| \nabla (\Delta \omega - Q_h \Delta \omega) \|_{T}^2 + h_T^4 \| \nabla^2 (\Delta \omega - Q_h \Delta \omega) \|_{T}^2 \right) \\
\leq C h^{2m-2} \left( \| \omega \|_{m+1}^2 + h^2 \delta_{m,2} \| \omega \|_4^2 \right).
\]

As to (7.7), we have from the definition of \( Q_b \), the trace inequality (7.1), and the estimate (7.3) that
\[
\sum_{T \in \mathcal{T}_h} h_T^{-1} \| \nabla Q_0 \omega \cdot n_e - Q_b (\nabla \omega \cdot n_e) \|_{\partial T}^2 \\
\leq \sum_{T \in \mathcal{T}_h} h_T^{-1} \| \nabla Q_0 \omega - \nabla \omega \|_{\partial T}^2 \\
\leq C \sum_{T \in \mathcal{T}_h} \left( h_T^{-2} \| \nabla Q_0 \omega - \nabla \omega \|_{T}^2 + \| \nabla Q_0 \omega - \nabla \omega \|_{1,T}^2 \right) \\
\leq C h^{2m-2} \| \omega \|_{m+1}^2.
\]

Notice that \( Q_b \) is a linear bounded operator, we use the definition of \( Q_b \) and the trace inequality (7.1) to obtain
\[
\sum_{T \in \mathcal{T}_h} h_T^{-3} \| Q_b Q_0 \omega - Q_b \omega \|_{\partial T}^2 \\
\leq \sum_{T \in \mathcal{T}_h} h_T^{-4} \| Q_0 \omega - \omega \|_{T}^2 + h_T^{-2} \| \nabla (Q_0 \omega - \omega) \|_{T}^2 \\
\leq C h^{2m-2} \| \omega \|_{m+1}^2.
\]

To derive (7.9), we use the trace inequality (7.1) and the estimate (7.4) to obtain
\[
\sum_{T \in \mathcal{T}_h} \| \nabla (\Delta \omega) - Q_b (\nabla (\Delta \omega)) \|_{\partial T}^2 \\
\leq C \sum_{T \in \mathcal{T}_h} \left( h_T^{-1} \| \nabla (\Delta \omega) - Q_b (\nabla (\Delta \omega)) \|_{T}^2 + h_T \| \nabla (\nabla (\Delta \omega) - Q_b (\nabla (\Delta \omega))) \|_{T}^2 \right) \\
\leq C h^{2m-3} \| \omega \|_{m+2}^2.
\]

This completes the proof of (7.9), and hence the lemma.  \( \Box \)

### 7.2 Technical Inequalities

The goal here is to present some technical estimates useful for deriving error estimates for the WG finite element scheme (3.5).
Lemma 7.5 There exists a constant $C$ such that, for any $v = \{v_0, v_b, v_n e\} \in V_h$, the following holds true

$$\sum_{T \in \mathcal{T}_h} \|\Delta v_0\|_{T}^2 \leq C\|v\|^2. \quad (7.10)$$

Proof From the identity (2.4) with $\phi = \Delta v_0$ we have

$$\|\Delta v_0\|_{T}^2 = (\Delta_w v, \Delta v_0)_T - (Q_b v_0 - v_b, \nabla (\Delta v_0) \cdot n)_{\partial T} + (\nabla v_0 - v_n n_e) \cdot n, \Delta v_0)_{\partial T}. \quad (7.12)$$

Thus, using the Cauchy–Schwarz inequality, trace inequality, and the inverse inequality we obtain

$$\|\Delta v_0\|_{T}^2 \leq \|\Delta_w v\|_T \|\Delta v_0\|_T + \|Q_b v_0 - v_b\|_{\partial T} \|\nabla (\Delta v_0) \cdot n\|_{\partial T}$$

$$+ \|(\nabla v_0 - v_n n_e) \cdot n\|_{\partial T} \|\Delta v_0\|_T \leq C(\|\Delta_w v\|_T \|\Delta v_0\|_T + h_T^{-1} \|Q_b v_0 - v_b\|_{\partial T} \|\nabla (\Delta v_0) \cdot n\|_T$$

$$+ h_T^{-1} \|(\nabla v_0 - v_n n_e) \cdot n\|_{\partial T} \|\Delta v_0\|_T) \leq C(\|\Delta_w v\|_T \|\Delta v_0\|_T + h_T^{-3} \|Q_b v_0 - v_b\|_{\partial T}^2 + h_T^{-1} \|(\nabla v_0 - v_n n_e) \cdot n\|^2_{\partial T}).$$

Hence,

$$\|\Delta v_0\|_{T}^2 \leq C(\|\Delta_w v\|_T^2 + h_T^{-3} \|Q_b v_0 - v_b\|_{\partial T}^2 + h_T^{-1} \|(\nabla v_0 - v_n n_e) \cdot n\|^2_{\partial T}),$$

which verifies the inequality (7.10).

Lemma 7.6 ([37], Lemma 10.4) There exists a constant $C$ such that, for any $v \in V_h^0$, we have the following Poincaré inequality:

$$\|v_0\|^2 \leq C \left(\sum_{T \in \mathcal{T}_h} \|\nabla v_0\|_{T}^2 + h^{-1} \sum_{T \in \mathcal{T}_h} \|Q_b v_0 - v_b\|_{\partial T}^2\right). \quad (7.11)$$

The following lemma provides an estimate for the term $\sum_{T \in \mathcal{T}_h} \|\nabla v_0\|_{T}^2$. Note that $v_0$ is a piecewise polynomial of degree $k \geq 2$. Thus, Lemma 7.7 is concerned only with piecewise polynomials; no boundary condition is necessary.

Lemma 7.7 Let $\varphi$ be any piecewise polynomial of degree $k \geq 2$ on each element $T$. Denote by $\nabla h \varphi$ and $\Delta h \varphi$ the gradient and Laplacian of $\varphi$ taken on each element. Then, for any $\varepsilon > 0$, there exists a constant $C$ such that

$$\|\nabla h \varphi\|^2 \leq \varepsilon \|\varphi\|^2 + C \varepsilon^{-1} \|\Delta h \varphi\|^2$$

$$+ C \varepsilon^{-1} h^{-1} \left(\sum_{e \in \mathcal{E}_h} \int_e \left(\frac{\partial \varphi_L}{\partial n_L} + \frac{\partial \varphi_R}{\partial n_R}\right)^2 ds\right) \quad (7.12)$$

$$+ Ch^{-1} \left(\sum_{e \in \mathcal{E}_h} \int_e (Q_b \varphi_R - Q_b \varphi_L)^2 ds\right).$$

Here $\varphi_L$ is the trace of $\varphi$ on $e$ as seen from the “left” or the opposite direction of $n_e$. If $e$ is a boundary edge, then the trace from the outside of $\Omega$ is defined as zero.
Proof. On each element $T$, we have

$$\int_T |\nabla \varphi|^2 dT = - \int_T \varphi \Delta \varphi dT + \int_{\partial T} \frac{\partial \varphi}{\partial n} \varphi ds$$

$$= - \int_T \varphi \Delta \varphi dT + \int_{\partial T} \frac{\partial \varphi}{\partial n} Q_b \varphi ds.$$  

Summing over all $T \in T_h$, we have

$$\|\nabla h \varphi\|^2 = - \int_{\Omega} \varphi \Delta h \varphi dT + \sum_{T \in T_h} \int_{\partial T} \frac{\partial \varphi}{\partial n} Q_b \varphi ds. \quad (7.13)$$

Using the identity $a_L b_L + a_R b_R = (a_L + a_R) b_L + a_R(b_R - b_L)$ we obtain

$$\sum_{T \in T_h} \int_{\partial T} \frac{\partial \varphi}{\partial n} Q_b \varphi ds = \sum_{e \in E_h} \int_e \left( \frac{\partial \varphi_L}{\partial n_L} Q_b \varphi_L + \frac{\partial \varphi_R}{\partial n_R} Q_b \varphi_R \right) ds$$

$$= \sum_{e \in E_h} \int_e \left( \frac{\partial \varphi_L}{\partial n_L} + \frac{\partial \varphi_R}{\partial n_R} \right) Q_b \varphi_L ds$$

$$+ \sum_{e \in E_h} \int_e \frac{\partial \varphi_R}{\partial n_R} (Q_b \varphi_R - Q_b \varphi_L) ds.$$  

Thus, from the Cauchy–Schwarz inequality we have

$$\left| \sum_{T \in T_h} \int_{\partial T} \frac{\partial \varphi}{\partial n} Q_b \varphi ds \right| \leq \left( \sum_{e \in E_h} \int_e \left( \frac{\partial \varphi_L}{\partial n_L} + \frac{\partial \varphi_R}{\partial n_R} \right)^2 ds \right)^{1/2} \left( \sum_{e \in E_h} \int_e |Q_b \varphi_L|^2 ds \right)^{1/2}$$

$$+ \left( \sum_{e \in E_h} \int_e \left( \frac{\partial \varphi_R}{\partial n_R} \right)^2 ds \right)^{1/2} \left( \sum_{e \in E_h} \int_e (Q_b \varphi_R - Q_b \varphi_L)^2 ds \right)^{1/2}. \quad (7.14)$$

Next, we use the trace inequality (7.1) and the inverse inequality (7.2) to obtain

$$\int_e |Q_b \varphi_L|^2 ds \leq \int_e |\varphi_L|^2 ds$$

$$\leq C \left[ h^{-1} \int_T \varphi^2 dT + h \int_T |\nabla \varphi|^2 dT \right]$$

$$\leq Ch^{-1} \int_T \varphi^2 dT, \quad (7.15)$$

and

$$\int_e \left| \frac{\partial \varphi_R}{\partial n_R} \right|^2 ds \leq C \left[ h^{-1} \int_T |\nabla \varphi|^2 dT + h \int_T |\nabla^2 \varphi|^2 dT \right]$$

$$\leq Ch^{-1} \int_T |\nabla \varphi|^2 dT. \quad (7.16)$$
Substituting (7.15) and (7.16) into (7.14) yields
\[
\left| \sum_{T \in T_h} \int_{\partial T} \frac{\partial \varphi}{\partial n} Q_b \varphi \, ds \right| \leq \text{Ch}^{-\frac{1}{2}} \| \varphi \| \left( \sum_{e \in \mathcal{E}_h} \int_{e} \left( \frac{\partial \varphi_L}{\partial n_L} + \frac{\partial \varphi_R}{\partial n_R} \right)^2 \, ds \right)^{\frac{1}{2}} \\
+ \text{Ch}^{-\frac{1}{2}} \| \nabla_h \varphi \| \left( \sum_{e \in \mathcal{E}_h} \int_{e} (Q_b \varphi_R - Q_b \varphi_L)^2 \, ds \right)^{\frac{1}{2}} .
\]
(7.17)

Substituting (7.17) into (7.13) gives
\[
\| \nabla_h \varphi \|^2 \leq \| \Delta_h \varphi \| \| \varphi \| + \text{Ch}^{-\frac{1}{2}} \| \varphi \| \left( \sum_{e \in \mathcal{E}_h} \int_{e} \left( \frac{\partial \varphi_L}{\partial n_L} + \frac{\partial \varphi_R}{\partial n_R} \right)^2 \, ds \right)^{\frac{1}{2}} \\
+ \text{Ch}^{-\frac{1}{2}} \| \nabla_h \varphi \| \left( \sum_{e \in \mathcal{E}_h} \int_{e} (Q_b \varphi_R - Q_b \varphi_L)^2 \, ds \right)^{\frac{1}{2}} ,
\]
which, through an use of Young’s inequality, implies the desired estimate (7.12). This completes the proof. □

**Lemma 7.8** There exists a constant C such that for any \( v = \{ v_0, v_b, v_n \} \in \mathcal{V}_h^0 \) the following Poincaré type inequality holds true
\[
\| \nabla_h v_0 \| \leq C \| v \|. \quad (7.18)
\]
In addition, we have the following estimate
\[
\| \nabla_h v_0 \| \leq \lambda h^{-1} \| v \| + \text{Ch} \| v \| , \quad (7.19)
\]
where \( \lambda \) is a positive constant.

**Proof** The first component \( v_0 \) is a piecewise polynomial of degree \( k \geq 2 \). Using the estimate (7.12) in Lemma 7.7 we have
\[
\| \nabla_h v_0 \|^2 \leq \varepsilon \| v \|^2 + C \varepsilon^{-1} \| \Delta_h v_0 \|^2 \\
+ \text{Ch}^{-1} \left( \sum_{e \in \mathcal{E}_h} \int_{e} \left( \frac{\partial v_0 L}{\partial n_L} + \frac{\partial v_0 R}{\partial n_R} \right)^2 \, ds \right) \\
+ \text{Ch}^{-1} \left( \sum_{e \in \mathcal{E}_h} \int_{e} (Q_b v_{0R} - Q_b v_{0L})^2 \, ds \right) . \quad (7.20)
\]
By inserting \( v_n n_e \cdot n \) in each integrand we obtain
\[
\sum_{e \in \mathcal{E}_h} \int_{e} \left( \frac{\partial v_0 L}{\partial n_L} + \frac{\partial v_0 R}{\partial n_R} \right)^2 \, ds \leq C \sum_{T \in T_h} \| \nabla v_0 \cdot n_e - v_n \|^2_{\partial T} .
\]
Similarly, by inserting \( v_b \)
\[
\sum_{e \in \mathcal{E}_h} \int_{e} (Q_b v_{0R} - Q_b v_{0L})^2 \, ds \leq C \sum_{T \in T_h} \| Q_b v_0 - v_b \|^2_{\partial T} .
\]
Substituting the above two inequalities into (7.20) yields
\[
\|\nabla h v_0\|^2 \leq \varepsilon \|v\|^2 + C \varepsilon^{-1} \|\Delta_h v_0\|^2 + Ch^{-1} \sum_{T \in T_h} \|Q_b v_0 - v_b\|^2_{\partial T} \\
+ C \varepsilon^{-1} h^{-1} \sum_{T \in T_h} \|\nabla v_0 \cdot n_e - v_n\|^2_{\partial T}.
\] (7.21)

Using the Poincaré inequality (7.11) and the estimate (7.10) we arrive at
\[
\|\nabla h v_0\|^2 \leq \varepsilon C \|\nabla h v\|^2 + C \varepsilon^{-1} \|v\|^2,
\]
which leads to the inequality (7.18) for sufficiently small \(\varepsilon\).

Finally, by setting \(\varepsilon = \lambda h^{-2}\) in (7.21) we arrive at
\[
\|\nabla h v_0\|^2 \leq \lambda h^{-2} \|v\|^2 + Ch^2 \|v\|^2,
\]
where \(\lambda\) is a positive constant. This verifies the inequality (7.19), and hence completes the proof of the lemma.

**Lemma 7.9** There exists a constant \(C\) such that for any \(v = \{v_0, v_b, v_n\} \in V_h^0\) one has
\[
\sum_{T \in T_h} \int_{\partial T} (v_0 - Q_b v_0)^2 ds \leq Ch \|v\|^2
\] (7.22)
and
\[
\sum_{T \in T_h} \int_{\partial T} (v_0 - Q_b v_0)^2 ds \leq C \lambda h^{-1} \|v\|^2 + Ch^3 \|v\|^2.
\] (7.23)

**Proof** From the trace inequality (7.1) and the inverse inequality (7.2), we have
\[
\int_{\partial T} (v_0 - Q_b v_0)^2 ds \leq Ch \int_T |\nabla v_0|^2 dT.
\]
Summing over all \(T \in T_h\) yields
\[
\sum_{T \in T_h} \int_{\partial T} (v_0 - Q_b v_0)^2 ds \leq Ch \sum_{T \in T_h} \int_T |\nabla v_0|^2 dT,
\] (7.24)
which, combined with (7.18) and (7.19), completes the proof of the lemma.

**Remark 7.1** The estimate (7.22) in Lemma 7.9 is sufficient for us to derive an optimal order error estimate for the WG finite element solution arising from (3.5). But the estimate (7.22) is sub-optimal in terms of the mesh parameter \(h\). We conjecture that the following inequality holds true
\[
\sum_{T \in T_h} \int_{\partial T} (v_0 - Q_b v_0)^2 ds \leq Ch^3 \|v\|^2.
\] (7.25)
However, with the current mathematical approach, we are unable to verify the validity of (7.25). This estimate is then left to interested readers or researchers as an open problem.
References

1. Arad, M., Yakhot, A., Ben-Dor, G.: A highly accurate numerical solution of a biharmonic equation. Numer. Methods Partial Differ. Equ. 13, 375C391 (1998)
2. Argyris, J.H., Fried, I., Scharpf, D.W.: The TUBA family of plate elements for the matrix displacement method. Aeronaut. J. R. Aeronaut. Soc. 72, 514C517 (1968)
3. Argyris, J.H., Dunne, P.C.: The finite element method applied to fluid dynamics. In: Hewitt, B.L., Illingworth, C.R., Lock, R.C., Mangler, K.W., McDonnel, J.H., Richards, C., Walkden, F. (eds.) Computational Methods and Problems in Aeronautical Fluid Dynamics, pp. 158–197. Academic Press, London (1976)
4. Arnold, D.N., Brezzi, F.: Mixed and nonconforming finite element methods: implementation, postprocessing and error estimates. RAIRO Modél. Math. Anal. Numer. 19(1), 7–32 (1985)
5. Behrens, E.M., Guzmán, J.: A mixed method for the biharmonic problem based on a system of first-order equations. SIAM J. Numer. Anal. 49, 789–817 (2011)
6. Bialecki, B., Karageorghis, A.: A Legendre spectral Galerkin method for the biharmonic Dirichlet problem. SIAM J. Sci. Comput. 22(5), 1549–1569 (2000)
7. Bialecki, B., Karageorghis, A.: Spectral Chebyshev collocation for the Poisson and biharmonic equations. SIAM J. Sci. Comput. 32(5), 2995–3019 (2010)
8. Bialecki, B.: A fast solver for the orthogonal spline collocation solution of the biharmonic Dirichlet problem on rectangles. J. Comput. Phys. 191, 601–621 (2003)
9. Bjorstad, P.: Fast numerical solution of the biharmonic dirichlet problem on rectangles. SIAM J. Numer. Anal. 20, 59–71 (1983)
10. Brenner, S., Sung, L.: C0 interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. J. Sci. Comput. 22/23, 83–118 (2005)
11. Chan, R.H., DeLillo, T.K., Horn, M.A.: The numerical solution of the biharmonic equation by conformal mapping. SIAM J. Sci. Comput. 18, 1571–1582 (1997)
12. Chen, G., Li, Z., Lin, P.: A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible Stokes flow. Adv. Comput. Math. 29, 113–133 (2008)
13. Chen, H.R., Chen, S.C., Qiao, Z.H.: C0-nonconforming tetrahedral and cuboid elements for the three-dimensional fourth order elliptic problem. Numer. Math. 124(1), 99–119 (2013)
14. Chen, H.R., Chen, S.C., Qiao, Z.H.: C0-nonconforming triangular prism elements for the three-dimensional fourth order elliptic problem. J. Sci. Comput. 55(3), 645–658 (2013)
15. Ciarlet, P.A., Raviart, P.G.: A mixed finite element method for the biharmonic equation. In: Mathematical Aspects of Finite Elements in Partial Differential Equations. Academic Press, New York, pp. 125–145 (1974)
16. Clough, R.W., Tocher, J.L.: Finite element stiffness matrices for analysis of plates in bending. In: Proceedings of the Conference on Matrix Methods in Structural Mechanics. Wright Patterson A.F.B., Ohio (1965)
17. Cockburn, B., Dong, B., Guzmán, J.: A hybridizable and superconvergent discontinuous Galerkin method for biharmonic problems. J. Sci. Comput. 40, 141C187 (2009)
18. Davini, C., Pitacco, I.: An unconstrained mixed method for the biharmonic problem. SIAM J. Numer. Anal. 38, 820C836 (2001)
19. Dean, E.J., Glowinski, R., Pironneau, O.: Iterative solution of the stream functionvorticity formulation of the Stokes problem, applications to the numerical simulation of incompressible viscous flow. Comput. Methods Appl. Mech. Eng. 87, 117–155 (1991)
20. Ehrlich, L.W., Gupta, M.M.: Some difference schemes for the biharmonic equation. SIAM J. Numer. Anal. 12, 773–790 (1975)
21. Engel, G., Garikipati, K., Hughes, T., Larson, M.G., Mazzei, L., Taylor, R.: Continuous/discontinuous finite element approximations of fourth order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. Comput. Meth. Appl. Mech. Eng. 191, 3669–3750 (2002)
22. Fréneis de Veubeke, B.: A conforming finite element for plate bending. In: Zienkiewicz, O.C., Holister, G.S. (eds.) Stress Analysis, p. 145C197. Wiley, New York (1965)
23. Greenbaum, A., Greengard, L., Mayo, A.: On the numerical solution of the biharmonic equation in the plane. Phys. D 60, 216–225 (1992)
24. Gupta, M.M., Manohar, R.P.: Direct solution of biharmonic equation using noncoupled approach. J. Comput. Phys. 33, 236–248 (1979)
25. Heinrichs, W.: A stabilized treatment of the biharmonic operator with spectral methods. SIAM J. Sci. Stat. Comput. 12, 1162–1172 (1991)
26. Jaswon, M.A., Symm, G.T.: Integral Equation Methods in Potential Theory and Elastostatics, pp. 99–126. Academic Press, New York (1977)
27. Linden, J.: A Multigrid Method for Solving the Biharmonic Equation on Rectangular Domains, Notes Numer. Fluid Mech. 11, Vieweg, Braunschweig (1985)
28. Morley, L.S.D.: The triangular equilibrium element in the solution of plate bending problems. Aero. Quart. 19, 149–169 (1968)
29. Mayo, A.: The fast solution of Poissons and the biharmonic equations on irregular regions. SIAM J. Numer. Anal. 21, 285–299 (1984)
30. Mozolevski, I., Sli, E., Bsing, P.R.: hp-Version a priori error analysis of interior penalty discontinuous Galerkin finite element approximations to the biharmonic equation. J. Sci. Comput. 30, 465–491 (2007)
31. Mu, L., Wang, J., Ye, X.: Weak Galerkin finite element methods on polytopal meshes. arXiv:1204.3655v2
32. Mu, L., Wang, J., Ye, X.: Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes, preprint
33. Mu, L., Wang, J., Ye, X., Zhang, S.: A $C^0$-weak Galerkin finite element method for the biharmonic equation, preprint
34. Muskhelishvili, N.I.: Some Basic Problems in the Mathematical Theory of Elasticity. Noordhoff Groningen, The Netherlands (1953)
35. Pandit, S.K.: On the use of compact streamfunction-velocity formulation of steady Navier–Stokes equations on geometries beyond rectangular. J. Sci. Comput. 36, 219–242 (2008)
36. Roache, P.: Computational Fluid Dynamics. Hermosa, Albuquerque (1972)
37. Wang, C., Wang, J.: An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes. arXiv:1303.0927
38. Wang, J., Ye, X.: A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math. 241, 103–115 (2013)
39. Wang, J., Ye, X.: A weak Galerkin mixed finite element method for second-order elliptic problems. Math. Comput. 83(289), 2101–2126 (2014)
40. Wang, J., Ye, X.: A weak Galerkin finite element method for the Stokes equations. arXiv:1302.2707v1
41. Zlámal, M.: On the finite element method. Numer. Math. 12, 394–409 (1968)