Born-Infeld Type Extension of (Non-)Critical Gravity

Sang-Heon Yi

Center for Quantum Spacetime, Sogang University, Seoul 121-741, Korea

Abstract

We consider the Born-Infeld type extension of (non-)critical gravity which is higher-curvature gravity on Anti de-Sitter space with specific combinations of scalar curvature and Ricci tensor. This theory may also be viewed as a natural extension of three-dimensional Born-Infeld new massive gravity to arbitrary dimensions. We show that this extension is consistent with holographic $c$-theorem and scalar graviton modes are absent in this theory. After showing that ghost modes in the theory can be truncated consistently by appropriate boundary conditions, we argue that the theory is classically equivalent to Einstein gravity at the non-linear level. Black hole solutions are discussed in the view point of the full non-linear classical equivalence between the theory and Einstein gravity. Holographic entanglement entropy in the theory is also briefly commented on.

e-mail: shyi(at)sogang.ac.kr
1 Introduction

Holographic principle \cite{1} is expected to be deeply related to the fundamental nature of quantum gravity. Though its original insight is based on the semi-classical black hole physics, it is regarded as the fundamental principle of gravity. Based upon the spectacular progress in string theory, this principle is realized concretely as the Anti de-Sitter/Conformal Field Theory (AdS/CFT) correspondence \cite{2} which relates $d$-dimensional field theory to gravity or string theory on $(d+1)$-dimensional AdS space. According to this correspondence the large ’t Hooft coupling in field theory corresponds to small curvature scale in gravity. This opens up opportunities to study nonperturbative physics in field theory through gravity. In this regard, another interesting point in the correspondence is that the energy scale in field theory corresponds to the extra-dimension in gravity. This leads to the realization of renormalization group (RG) flow in the holographic setup, which is far to reach in the traditional perturbative approach. On physical ground it is natural that degrees of freedom are decreased along RG flow since RG flow is a coarse graining process in quantum field theory. However, it has been known to be very difficult to realize this intuitive picture concretely in continuum field theory except some special cases. One of these special cases is the famous Zamolodchikov $c$-theorem in two dimensional field theory \cite{3}, which says that the central charge decreases along RG flow. After the advent of the AdS/CFT correspondence there are various holographic studies about RG flow in higher dimensions, in particular about holographic $c$-theorem and its implication \cite{1}.

In recent years there are some interests in higher derivative gravity with several motivations. One motivation in the context of the AdS/CFT correspondence is to investigate the next order corrections in the large ’t Hooft coupling by higher derivative gravity. In this usual approach, higher derivative terms are treated as small corrections to the Einstein-Hilbert term. This is the usual strategy in string theory since stringy corrections are small at low energy. However, if one consider higher derivative terms as taking non-small contributions, these are plagued by ghosts which signal the deficiency of the theory. To avoid ghost problem in this case, one usually focus on two derivative higher curvature gravity like Gauss-Bonnet or Lovelock gravity. These higher curvature gravities or their related cousins of the effective two derivatives nature played important roles in seeing $c$-theorem holographically \cite{5}.

The situation is different in three-dimensional gravity since the usual Einstein gravity has no propagating degrees of freedom in this case. Through the study of higher derivative terms in three-dimensional topologically massive gravity \cite{6} on AdS space, it has been recognized that the fall-off boundary conditions are important to characterize higher derivative theories. Concretely speaking, it was found that by adding higher derivative terms, the central charge of dual CFT may vanish at the special value of the coefficient of higher derivative terms. In this case by taking different fall-off boundary conditions in bulk gravity, boundary dual CFTs are characterized differently. These specifications are now coined as chiral/log gravity conjectures \cite{7}.

Along the line of these developments four-dimensional critical gravity is proposed as the analogue of chiral/log gravity or critical new massive gravity \cite{8} on three-dimensional AdS space, which contains squared Weyl tensor as a higher derivative term. When the coefficient of
the Weyl tensor term is chosen appropriately, massless gravitons have zero excitation energy and mass of AdS-Schwarzschild black holes vanishes \cite{9}. Since the linearized equations of motion for gravitons in critical gravity takes the squared form of those in Einstein gravity, unwanted massive ghost modes are absent. Instead, there are ghost-like log modes which fall off more slowly than massless graviton modes but which may be unseparated with massless modes.

Another interesting recent development in higher curvature gravity is the claim that four-dimensional conformal gravity on AdS space with appropriate boundary conditions is equivalent to Einstein gravity on the same space at the classical but non-linear level \cite{10}. It was argued that by appropriate boundary conditions, ghosts can be truncated consistently or Einstein solutions are chosen consistently among all possible solutions. Subsequently, through linear analysis on graviton modes, non-critical gravity is introduced as the extension of critical gravity by relaxing the condition on the coupling of the Weyl tensor squared term \cite{11}. In this theory, when the coupling of higher derivative terms is chosen appropriately, massive ghost modes fall off more slowly than massless graviton modes while decoupled from massless modes. Then, massive ghost modes can be truncated consistently by boundary conditions.

Recently, non-critical gravity is shown to be equivalent to Einstein gravity by using auxiliary field formalism \cite{12}. To derive this equivalence in the framework of the AdS/CFT correspondence the Gibbons-Hawking boundary terms as well as the bulk Lagrangian are shown to be reduced to those in Einstein gravity by the consistent choice of subset among all possible solutions in non-critical gravity. More recently, four-dimensional non-critical gravity is extended to arbitrary dimensions and the equivalence between this theory and Einstein gravity on AdS space is verified in various setups at the non-linear level \cite{13}.

In this paper, we construct the Born-Infeld type (non-)critical gravity which is a natural extension of curvature quadratic (non-)critical gravity. This theory may also be viewed as a natural extension of three-dimensional Born-Infeld new massive gravity \cite{14} to arbitrary dimensions. After its introduction, we show that the theory is consistent with holographic c-theorem. This consistency with holographic c-theorem may be taken as a beneficial guideline or constraint for the extension of curvature quadratic (non-)critical gravity to even higher curvature gravity, as was shown in three-dimensional case \cite{15}. And then, through linear analysis we verify the absence of scalar graviton modes and find the range of the coupling of higher derivative terms to truncate ghosts consistently. Using these results, we argue that this theory is equivalent to Einstein gravity at the non-linear classical level. To see the full non-linear equivalence, black hole solutions and their some properties are also discussed. In the line of this, we briefly comment on the method to compute holographic entanglement entropy \cite{16} of the theory.

2 Born-Infeld Gravity

Non-critical gravity was introduced as a specific curvature quadratic gravity on AdS space after the truncation of ghosts by appropriate boundary conditions, which is shown to be
equivalent to Einstein gravity. More explicitly, curvature quadratic part consists only of scalar curvature square and Ricci tensor square terms with specific coefficients. It is natural question to ask there are even higher curvature (non-)critical gravity with similar properties. In the following, we consider the critical case simultaneously, in which ghost-like log-modes might be included.

The hint for the extension of (non-)critical gravity to even higher curvature gravity comes from three-dimensional new massive gravity(NMG) which was extended to even higher curvature gravity theory and named as $R^n$-NMG and BI-NMG $^{[14]}$ $^{[15]}$$^{[17]}$. As in the case of three-dimensional NMG, the extension to even higher curvature gravity may be controlled by requiring the existence of a simple central charge flow function. However, it is obvious that it becomes harder to restrict higher curvature couplings in higher dimensions by a single function. Nevertheless, the consistency with simple holographic $c$-theorem may be taken as a crucial guideline for the extension.

In this section we introduce the Born-Infeld type extension of $D$-dimensional (non-)critical gravity, which we call simply Born-Infeld(BI) gravity. After the introduction of BI gravity, it is shown to satisfy holographic $c$-theorem, which is the first consistency check for the theory. Now, let us consider the following $D$-dimensional action for BI gravity

$$S_{BI} = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-g} \mathcal{L}_{BI} = \frac{\kappa}{p} \frac{m^2}{(D-2)} \int d^Dx \sqrt{-g} \left[ \left( \frac{\det(A) - \frac{1}{m^2} G_{\mu}^{\mu}}{m^2} \right)^p - 1 + \frac{p(D-1)(D-2)}{2m^2\ell^2} \right]$$

where $\sigma$ takes $+1$ or $-1$ and $G_{\mu}^{\mu}$ denotes Einstein tensor, $G_{\mu}^{\mu} = R_{\mu}^{\mu} - \frac{1}{2} R \delta_{\mu}^{\mu}$. We take parameters $m^2$ and $\ell^2$ as real valued ones and their range is constrained only by allowance of AdS space. In the following the constant $p$ is taken as

$$p \equiv \frac{1}{D-1} = \frac{1}{d},$$

which turns out to be a natural choice consistent with simple holographic $c$-theorem. Furthermore, this choice of the value of $p$ leads, expanded in powers of $1/m^2$, to various known (non)-critical gravities as was shown in the following. When $D = 3$, the above action is identical with the one for BI-NMG up to some conventions.

Equations of motion expression, defined by $\mathcal{E}_{\mu\nu} \equiv (2\kappa^2/\sqrt{-g})\delta S_{BI}/\delta g^{\mu\nu}$, is given by

$$-\sigma(D-2)\mathcal{E}_{\mu\nu} = \frac{m^2}{p} \left[ \left( \frac{\det(A)^p - 1 + (D-2)^2}{2m^2\ell^2} \right) g_{\mu\nu} + 2B_{(\mu}^{\alpha}(\nabla_{\nu)\alpha} - BR_{\mu\nu}) - (\nabla_{\alpha}B^{\alpha\beta} - \nabla^2 B)g_{\mu\nu} + \nabla_{\mu}\nabla_{\nu} B + \nabla^2 B_{\mu\nu} - 2\nabla_{\alpha}\nabla_{(\mu}B^{\alpha\nu)} \right],$$

where we have defined

$$A_{\nu}^{\mu} \equiv \delta_{\nu}^{\mu} - \frac{1}{m^2} G_{\nu}^{\mu}, \quad B_{\nu}^{\mu} \equiv (\det(A)^p(A^{-1})_{\nu}^{\mu}, \quad B \equiv B_{\mu}^{\mu}. \quad$$

Our convention is such that the equations of motion are read as $\mathcal{E}_{\mu\nu} = T_{\mu\nu}$ when matters are coupled with the energy-momentum tensor convention $T_{\mu\nu} = -(2\kappa^2/\sqrt{-g})\delta S_{mat}/\delta g^{\mu\nu}$. 

3
Whenever the fractional power of the determinant expression is ill-defined, one should regard it as a formal expression which may have a concrete meaning after it is expanded in terms of $G^\mu_\nu/m^2$. Using the Taylor series expansion of the determinant expression, one can see that a few leading terms in the action are given by

$$S_{BI} = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g} \sigma \left[ R + \frac{(D-1)(D-2)}{\ell^2} + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{O}(R^4) \right], \quad (5)$$

$$\mathcal{L}_2 = -\frac{1}{m^2(D-2)} [R^\mu_\nu R^\nu_\mu - \frac{D}{4(D-1)} R^2],$$

$$\mathcal{L}_3 = -\frac{2}{3m^4(D-2)} \left[ R^\mu_\nu R^\nu_\rho R^\rho_\mu - \frac{3D}{4(D-1)} R R^\mu_\nu R^\nu_\mu + \frac{D^2 + 4D - 4}{16(D-1)^2} R^3 \right].$$

Note that the leading Lagrangian is nothing but the one in Einstein gravity and the Lagrangian up to $\mathcal{L}_2$ coincides with the one in noncritical gravity given in [13].

One can easily check that AdS space is allowed as the vacuum solution in BI gravity with $\bar{\mathcal{A}}^\mu_\nu = \frac{1}{L^2} (\delta^\mu_\nu - \bar{g}^\mu_\nu), \quad \bar{\mathcal{B}}^\mu_\nu = \frac{1}{L^2} \left( \delta^\mu_\nu \right)$. Clearly, the relation is the generalization of the three-dimensional BI-NMG case to the $D$-dimensional one. Expanding in powers of $1/m^2 L^2$, one obtains

$$\frac{1}{\ell^2} = \frac{L^2}{L^2} \left[ 1 - \frac{D-4}{4m^2 L^2} - \frac{(D-2)^2(D-6)}{24m^4 L^4} + \mathcal{O} \left( \frac{1}{m^6 L^6} \right) \right],$$

which reproduces the relation among parameters in the corresponding Lagrangian at each expansion order.

It is very convenient to introduce a useful quantity $a$, which appears repeatedly in the following, as

$$a \equiv 1 - \frac{(D-1)(D-2)}{2m^2 L^2}. \quad (8)$$

Then, $\mathcal{A}$ and $\mathcal{B}$ tensors on AdS space are given simply by

$$\bar{\mathcal{A}}^\mu_\nu = a \delta^\mu_\nu, \quad \bar{\mathcal{B}}^\mu_\nu = a^{D-1} \delta^\mu_\nu = a^p \delta^\mu_\nu.$$

Before verifying the consistency of BI gravity with holographic $c$-theorem, let us derive the so-called A-type trace anomaly, $a^*_d$ of even $d$-dimensional CFT dual to BI gravity. One of

\[1\]Interestingly, it was recently discovered that the so-called quasi-topological gravity [18] is effectively equivalent to the above Lagrangian up to $R^3$ order when the metric is confined to the conformal flat ones [19].
the easiest way to compute this quantity is the evaluation of the on-shell Lagrangian on AdS space [20,5], which is given in our convention as
\[
a_d^* = -\frac{\pi^{d/2} L^{d+1}}{d \Gamma\left(\frac{d}{2}\right)} \frac{1}{2 \kappa^2} \mathcal{L}_{BL}|_{\text{AdS}} = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{L^{d-1}}{\kappa^2} \sigma a^p .
\]
One may note that the final result is simply proportional to the one in Einstein gravity with factor $\sigma a^p$. This is the first indication of the non-linear equivalence of BI gravity and Einstein gravity with rescaled Newton’s constant, $\kappa^2/\sigma a^p$ and substituted cosmological constant, $L$.

In two derivative gravity, the holographic realization of RG flow between two conformal points in dual $d$-dimensional field theory is given by the AdS kink geometry [4,5]. Following this realization, let us consider the following geometry on $D = (d + 1)$ dimensions
\[
ds^2 = L^2 \left[ dr^2 + e^{2A(r)} \left( -dt^2 + dx_D^{2} \right) \right] \equiv L^2 \left[ dr^2 + g_{ij} dx^i dx^j \right].
\]
Its non-vanishing curvature tensor components are given by
\[
R^i_{\ \ ij} = -\frac{1}{L^2} (A'' + A^2) g_{ij}, \quad R^i_{\ \ jkl} = -\frac{A'^2}{L^2} (\delta^i_k g_{jl} - \delta^i_l g_{jk}),
\]
where the prime denotes the differentiation with respect to the radial coordinate $r$. Note that conformal point of dual field theory or corresponding AdS space is represented by $A(r) = r$. Non-trivial RG flow is realized by a certain function $A(r)$ connecting two AdS spaces, for which some additional matter action $S_{\text{mat}}$ needs to be introduced.

For computational purpose, introduce functions $C(r)$ and $C_0(r)$ as
\[
C(r) \equiv 1 - \frac{(D - 1)(D - 2)}{2m^2 L^2} A^2 ,
\]
\[
C_0(r) \equiv \left[ 1 - \frac{(D - 1)(D - 2)}{2m^2 L^2} A^2 - \frac{D - 2}{m^2 L^2} A'' \right],
\]
which become constants on AdS space as
\[
C(r)|_{\text{AdS}} = C_0(r)|_{\text{AdS}} = a .
\]
In terms of these functions, the values of $\mathcal{A}$ tensor, $\det \mathcal{A}$ and $\mathcal{B}$ tensor for the above metric are given by
\[
\mathcal{A}^r_{\ \ r} = C , \quad \mathcal{A}^i_{\ \ j} = C_0 \delta^i_j , \quad (\det \mathcal{A})^p = C^p C_0 ,
\]
\[
\mathcal{B}^r_{\ \ r} = C^{p-1} C_0 , \quad \mathcal{B}^i_{\ \ j} = C^p \delta^i_j.
\]
One may note that
\[
\mathcal{B}^i_{\ \ r} - \mathcal{B}^r_{\ \ i} = (C - C_0) C^{p-1} = \frac{D - 2}{m^2 L^2} C^{p-1} A'' .
\]
Now, let us introduce a central charge flow function as
\[
a_d(r) \equiv \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)} \frac{L^{d-1}}{\kappa^2} \sigma C^p ,
\]
which would represent the flow of central charge along RG trajectory of CFT dual to BI
gravity. The adequacy of this choice of the central charge function comes from two points.
First, this function on AdS space coincides with the holographic representation of the A-type
trace anomaly in even $d$-dimensional field theory (and its odd $d$-dimensional counterpart),
which is given in our convention as

$$a_d(r)|_{AdS} = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{L^{d-1}}{\kappa^2} \sigma a^p = a_d^\star. \quad (14)$$

Second, this function, when expanded in powers of $1/m^2 L^2$, reduces to the known form in
the case of noncritical gravity $[13]$

$$a_d(r) = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{L^{d-1}}{A^{d-1}} \sigma \left[ 1 - \frac{d-1}{2m^2 L^2} A'^2 + \mathcal{O}\left(\frac{1}{m^4 L^4}\right) \right]. \quad (15)$$

To address holographic $c$-theorem in this setup, one needs to evaluate $a'(r)_d$ and see its
monotonic property. By a straightforward calculation one can see that equations of motion
with matters gives us

$$T_{t}^t - T_r^r = E_t^t - E_r^r = \sigma (D - 2) \left[ 1 - \frac{D(D - 3)}{2m^2 L^2} A'^2 \right] C^{p-1} A'' \quad (16)$$

Some comments are in order. Since BI gravity is higher derivative theory, it is not guar-
anteed that $A''$ appears only as the overall factor in the above expression. Indeed, there
are various terms in the form of $A'A''$, $A'^2$, $A'^2 A''^2$, etc in each expression of $E_{\mu\nu}$. In
the above combination of $E_t^t$ and $E_r^r$, various such terms are completely canceled and the final
expression takes very similar form in Einstein gravity. It is interesting to note that even
the combination contains infinite number of derivatives as can be seen by expanding $C^{p-1}$ in
powers of $1/m^2 L^2$.

In the end, one can check that the monotonic property of this function comes from the null
energy condition on matters, $T_{t}^t - T_r^r \leq 0$:

$$a_d'(r) = -\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \frac{L^{d-1}}{A^d \kappa^2} \sigma (D - 2) \left[ 1 - \frac{D(D - 3)}{2m^2 L^2} A'^2 \right] C^{p-1} A'' \quad (17)$$

which shows us that BI gravity is consistent with the simple form of holographic $c$-theorem.
This is one of nice properties of BI gravity. (See $[21]$ for some related discussions in the
three-dimensional BI-NMG case).

One may note that the central charge flow function is proportional to the one in Einstein
gravity with simple factor $\sigma C^p$. Since RG flow or central charge function flow may be regarded
as non-linear characteristics, this simple proportionality at the level of the central charge flow
function between BI gravity and Einstein gravity is a strong indication for the equivalence of
two theories at the non-linear level.
3 Linear Analysis

In this section we perform linear analysis of BI gravity to show that potentially dangerous scalar graviton modes are absent in BI gravity [9], which was one of important ingredients to define critical gravity in four dimensions. And then, we show that one can truncate massive ghost modes consistently from massless graviton modes by boundary conditions just like non-critical gravity [12][13]. By supplementing this linear analysis with the consistency with holographic c-theorem in the previous section we argue that BI gravity is classically equivalent Einstein gravity at the non-linear level.

The absence of scalar graviton modes is one of minimum requirements for BI gravity to be equivalent to Einstein gravity at low energy. To see the absence of scalar graviton modes of BI gravity, let us expand metric around a background metric $\bar{g}_{\mu\nu}$ as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (18)$$

Then, the linearized Ricci tensor is given by

$$\delta R_{\mu\nu} = \nabla_\rho (\nabla_{(\mu} \bar{h}_{\nu)})\alpha - \frac{1}{2} \bar{\nabla}^2 h_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\mu \bar{\nabla}_\nu h,$$

where $h$ denotes the contraction with the background metric $\bar{g}_{\mu\nu}$ and $\bar{\nabla}$ is covariant derivative with respect to $\bar{g}_{\mu\nu}$. From now on, the background geometry is taken as AdS space.

Using the diffeomorphism invariance, one may choose the gauge $\nabla^\nu h_{\nu\mu} = \nabla_{\mu} h$. Under this gauge choice the linearized expression of $A$ tensor is given by

$$\delta A^\mu_{\nu} = -\frac{1}{m^2} \delta G^\mu_{\nu} = \frac{1}{2m^2} \left( \nabla^2 h^\mu_{\nu} - \nabla^\mu \nabla_\nu h \right) + \frac{1}{m^2 L^2} \left( h^\mu_{\nu} + \frac{D-3}{2} \delta^\mu_{\nu} h \right),$$

and the linearized expression of $B$ tensor is given by

$$\delta B^\mu_{\nu} = (\det \tilde{A})^{-p} \left[ p \tilde{B}^\beta_{\nu} \delta A^\alpha_{\beta} \tilde{B}^\alpha_{\nu} - \tilde{B}^\alpha_{\nu} \delta A^\alpha_{\beta} \tilde{B}^\beta_{\nu} \right]$$

$$= -\frac{\alpha^p(2-D)}{m^2} \left[ \frac{1}{2} \left( \nabla^2 h^\mu_{\nu} - \nabla^\mu \nabla_\nu h \right) + \frac{1}{L^2} \left( h^\mu_{\nu} - \frac{1}{2} \delta^\mu_{\nu} h \right) \right].$$

It is interesting to note that the linearized expression of $B$ tensor on AdS space contains only two derivatives on $h_{\mu\nu}$, which brings the linearized equations of motions into quartic differential equations. This contrasts with the fact that $B$ tensor generically contains infinite number of derivatives at the non-linear level.

After some calculation, one can obtain the linearized expression of contracted equations of motion as

$$0 = \delta \mathcal{E}^\mu_{\nu} = \frac{(D-1)(D-2)^2}{2L^2} \left[ 1 - \frac{D(D-3)}{2m^2 L^2} \right] h, \quad (19)$$

which shows us that the scalar graviton modes are decoupled and transverse traceless gauge can be taken.
Since the absence of scalar graviton modes is established, let us turn to ghost modes. In the transverse and traceless gauge, $\nabla^\mu h_{\mu\nu} = h = 0$, one can show that the linearized equations of motion, $\delta \mathcal{E}_{\mu\nu} = 0$ are given in the following quartic form

$$(\nabla^2 + \frac{2}{L^2})(\nabla^2 + \frac{2}{L^2} - M^2) h_{\mu\nu} = 0, \quad M^2 \equiv (D - 2)m^2 a.$$  \tag{20}$$

This quartic differential equation shows us that there exist massless and massive modes which are decoupled at the linear level. On general ground one can see that one of them should be ghosts. We take $\sigma$ appropriately such that massive modes become ghosts. Since the linearization and the expansion in powers of $1/m^2 L^2$ do not commute, the above linear wave equation, when expanded in power of $1/m^2 L^2$, does not reduce to the one in curvature quadratic (non-)critical gravity.

Now, we would like to specify BI gravity by restricting the parameter $m^2$. Critical BI gravity is defined by the condition $M^2 = 0$ or $a = 0$. In terms of parameters $m^2$ and $\ell^2$, the critical point is given by

$$m^2 = \frac{(D - 2)^2}{2\ell^2} = \frac{(D - 1)(D - 2)}{2L^2},$$

which belongs to the regime $m^2, \ell^2 > 0$. At this critical point the quartic differential equation becomes degenerate and then log modes appear instead of massive ghost modes. Furthermore, one can see that the A-type trace anomaly, $a^*_d$ of even $d$-dimensional dual CFT and its cousin in odd dimensions vanish at this point. All these are the analogues of curvature quadratic critical gravity and signal the null content of critical BI gravity when log modes, which might be mixed with massless modes, are truncated.

To define consistent non-critical BI gravity, we should truncate unwanted ghosts. As in curvature quadratic non-critical gravity, massive ghost modes can be truncated consistently by appropriate boundary conditions since massive ghost modes fall off more slowly than massless modes as far as $M^2 < 0$, while the classical instability by tachyons may be avoided when $M^2$ is above Breitenlohner-Freedman bound $[23]$, $M^2 > -(D - 1)^2/4L^2$. Thus, we take the range of coupling $m^2$ in the non-critical case as

$$\frac{(D - 1)(2D^2 - 9D + 9)}{4(D - 2)L^2} < m^2 < \frac{(D - 1)(D - 2)}{2L^2}.$$  

After ghost modes are truncated, non-critical BI gravity describes massless gravitons just like Einstein gravity. This linear equivalence supplemented by the consistency with holographic $c$-theorem strongly suggests that non-critical BI gravity is classically equivalent to Einstein gravity at the full non-linear level. In the next section we test this non-linear equivalence through black hole solutions.

---

2See also [22] for critical gravity on $D$-dimensions.
4 Black Hole Solutions and Their Properties

It is obvious that AdS-Schwarzschild black holes in Einstein gravity are also solutions in BI gravity since the scalar curvature and the Ricci tensor of those black holes are same with those of the vacuum AdS space. In the following, by computing some non-linear quantities in these black holes we check the non-linear equivalence between BI gravity and Einstein gravity. We also comment on holographic entanglement entropy in BI gravity.

In our conventions AdS-Schwarzschild black hole solutions are represented by

\[ ds^2 = L^2 \left[ \frac{d\rho^2}{f(\rho)} - f(\rho) dt^2 + \rho^2 d\Omega_{D-2}^2 \right], \quad f(\rho) \equiv \rho^2 + 1 - \frac{\omega}{\rho^{D-3}}, \quad (21) \]

where the constant \( \omega \) is related to the horizon radius \( \rho_H \) as \( \omega = \rho_H^{D-3}(\rho_H^2 + 1) \). The Hawking temperature of these black holes are given by

\[ T_H = \frac{1}{2\pi L} \left[ \rho_H + \frac{D-3}{2\rho_H}(\rho_H^2 + 1) \right]. \]

By Wald’s formula [24] it is straightforward to obtain the Beckenstein-Hawking-Wald entropy of these black holes as

\[ S_{BHW} = \frac{2\pi}{(D-2)\kappa^2} \int_H d^{D-2}x \sqrt{h} \sigma \left[ B_\mu^\mu - B_t^t - B_r^r \right]_{p=\rho_H} \]

\[ = \frac{2\pi A_H}{\kappa^2} \sigma a^p, \quad A_H = (\rho_H L)^{D-2} \Omega_{D-2}, \quad (22) \]

where \( A_H \) denotes the horizon area and \( \Omega_{D-2} \equiv 2\pi^{d/2}/\Gamma(d/2) \) is the volume of unit \( (D-2) \)-sphere. Note that the black hole entropy can be obtained by the corresponding one from Einstein gravity simply by rescaling the Newton’s constant \( \kappa^2 \rightarrow \kappa^2/\sigma a^p \) and substituting cosmological constant \( \ell \rightarrow L \). One can see that this entropy is related to the A-type trace anomaly \( a_d^* \) as

\[ S_{BHW} = 4\pi \rho_H^{d-1} a_d^*. \]

This shows us that the AdS-Schwarzschild black hole entropy vanishes whenever the central charge does in the critical case. It is also interesting to note that this relation is identical with the one in Einstein gravity. Though the generic formula for holographic entanglement entropy(HEE) in higher derivative gravity is unknown [25], it is natural to propose that the correct way to compute holographic entanglement entropy in BI gravity is just using Wald formula, which was the case in curvature quadratic non-critical gravity [26][13]. If we adopt this prescription for HEE, one can see that the equivalence of BI gravity and Einstein gravity persists in HEE, too.

Now, let us consider mass of these black holes. By assuming, in our case, the validity of the second law of black hole thermodynamics and then accordingly integrating the differential relation, \( dM = T_H dS_{BHW} \), one can see that mass of AdS-Schwarzschild black holes, \( M(BI) \), is given by

\[ M(BI) = \sigma a^p \frac{(D-2)\omega}{2\kappa^2} \Omega_{D-2} = \sigma a^p M(Einstein), \]
where $M(Einstein)$ denotes mass of AdS-Schwarzschild black holes in Einstein gravity.

One may try to check this result on mass of black holes by the Euclidean action formalism. First, one can see that the on-shell bulk Euclidean action on AdS-Schwarzschild black holes (or in fact on any Einstein space) is given by

$$I^b_{E}(BI)\bigg|_{\text{on-shell}} = -\frac{D-1}{2}\frac{m^2}{\kappa^2}\sigma \int d^Dx \sqrt{g_E} \left[ (\det{\mathcal{A}})^p - 1 + \frac{(D-2)^2}{2m^2\ell^2} \right] = \sigma a^p \frac{D-1}{\kappa^2 L^2} \int d^Dx \sqrt{g_E} ,$$

which shows us that the on-shell bulk Euclidean action is proportional to that of Einstein gravity. To compute mass of black holes in the Euclidean action formalism, one also needs to obtain the on-shell Gibbons-Hawking-York boundary term which is not yet known in BI gravity. Now, let us assume that the one-shell boundary action on AdS-Schwarzschild black holes is given by the corresponding one in Einstein gravity with the same factor, $\sigma a^p$, in the bulk action. Then, the on-shell Euclidean action on AdS-Schwarzschild black holes in BI gravity, after an appropriate subtraction by counter terms or background AdS space, is simply proportional to that in Einstein gravity: $I_E(BI) = \sigma a^p I_E(Einstein)$. As a result, one can obtain mass of AdS-Schwarzschild black holes in BI gravity as

$$M(BI) = T_H(I_E + S_{BHW})(BI) = \sigma a^p (I_E + S_{BHW})(Einstein) = \sigma a^p M(Einstein). \quad (23)$$

Conversely, this can be interpreted as the check of our assumption about the on-shell Gibbons-Hawking-York boundary term or the verification of the relation $I_E(BI) = \sigma a^p I_E(Einstein)$, if mass of black holes is regarded as already determined by black hole thermodynamics.

All the above results for AdS-Schwarzschild black holes are consistent with the claim on the non-linear equivalence between BI gravity and Einstein gravity at the classical level.

5 Conclusion

We have constructed BI gravity as the extension of (non-)critical gravity, which may also be viewed as the extension of BI-NMG. In the framework of the AdS/CFT correspondence, we showed that BI gravity is consistent with holographic $c$-theorem. Through linear analysis we also showed that scalar graviton modes are absent and massive ghost modes can be truncated consistently by boundary conditions. By combining various ingredients we argued that BI gravity at non-critical point is classically equivalent to Einstein gravity at the non-linear level. The way to compute HEE in BI gravity is also proposed consistently with this equivalence.

There are many ways to elaborate on BI gravity. First of all, there are some missing elements in BI gravity compared to curvature quadratic (non-)critical gravity. One interesting missing element is the Gibbons-Hawking-York boundary term of BI gravity, which is crucial for well-defined variational principle with boundary. In the context of the AdS/CFT correspondence this is also important for determining the stress tensor of dual CFT. Another missing point

---

3This assumption is not so groundless since this property was shown to be the case for curvature quadratic non-critical gravity which coincides with the expanded form of BI gravity up to relevant order.
is the absence of the auxiliary field method, which is very useful to justify the non-linear equivalence more convincingly.

Though we have showed that mass of AdS-Schwarzschild black holes in BI gravity is given by the corresponding one in Einstein gravity up to a factor, it would be very interesting to obtain mass of AdS-Schwarzschild black holes in BI gravity as conserved charge, for instance, by using Abbott-Deser-Tekin approach [27].

Another interesting question about BI gravity is to investigate its relation to counter terms in \((D + 1)\)-dimensional gravity as was the case in three-dimensional BI-NMG [28]. Though there are various undetermined parameters in counter term actions [28], these actions seem to be different from BI gravity action in this paper except the three dimensional case. In counter term actions, the absence of scalar graviton is not so clear and the consistency with holographic \(c\)-theorem is not guaranteed, either. However, since there are some open possibilities to determine counter term actions, it would be interesting to study further the relation between counter terms and BI gravity.

It is also interesting to verify or improve our proposal for HEE computation in BI gravity, since there is no generic prescription for HEE computation. The study on the gauge/fluid correspondence and Rényi entropy in BI gravity is also an interesting future direction. It is also interesting to find some solutions in BI gravity following BI-NMG case [29] [30].

In the long run, it would be very interesting to find some ways to confirm the classical equivalence between BI gravity and Einstein gravity and to understand better the meaning of the consistency of a certain gravity theory with holographic \(c\)-theorem in general.

Acknowledgments

This work is supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MEST) through the Center for Quantum Spacetime(CQUST) of Sogang University with grant number 2005-0049409. I would like to thank Seungjoon Hyun, Wooje Jang, Jaehoon Jeong, Yongjoon Kwon, Soonkeon Nam, and Jong-Dae Park for useful discussions.
References

[1] G. ’t Hooft, “Dimensional reduction in quantum gravity,” [arXiv:gr-qc/9310026] ;
L. Susskind, “The World As A Hologram,” J. Math. Phys. 36 (1995) 6377
[arXiv:hep-th/9409089].

[2] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,”
Adv. Theor. Math. Phys. 2 (1998) 231 ; Int. J. Theor. Phys. 38 (1999) 1113
[arXiv:hep-th/9711200 [hep-th]] ;
E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2 (1998)
253 [arXiv:9802150 [hep-th]] ;
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-
critical string theory,” Phys. Lett. B 428 (1998) 105 [arXiv:hep-th/9802109].

[3] A. B. Zamolodchikov, “Irreversibility of the Flux of the Renormalization Group in a 2D
Field Theory,” JETP Lett. 43 (1986) 730 ; Pisma Zh. Eksp. Teor. Fiz. 43 (1986) 565.

[4] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, “Renormalization group flows
from holography supersymmetry and a c theorem,” Adv. Theor. Math. Phys. 3 (1999)
363 [hep-th/9904017].

[5] R. C. Myers and A. Sinha, “Seeing a c-theorem with holography,” Phys. Rev. D 82
(2010) 046006 [arXiv:1006.1263 [hep-th]] ;
R. C. Myers and A. Sinha, “Holographic c-theorems in arbitrary dimensions,”
JHEP 1101 (2011) 125 [arXiv:1011.5819 [hep-th]].

[6] R. Jackiw, S. Templeton and S. Deser, “Three-Dimensional Massive Gauge Theories,”
Phys. Rev. Lett. 48 (1982) 975;
S. Deser, R. Jackiw and S. Templeton, “Topologically massive gauge theories,” Ann.
Phys. 140 (1982) 372 [Erratum-ibid. 1988 185 406 , 1988 281 409].

[7] W. Li, W. Song and A. Strominger, “Chiral Gravity in Three Dimensions,” JHEP 0804
(2008) 082 [arXiv:0801.4566 [hep-th]] ;
D. Grumiller and N. Johansson, “Instability in cosmological topologically massive gravity
at the chiral point,” JHEP 0807 (2008) 134 [arXiv:0805.2610 [hep-th]].

[8] E. A. Bergshoeff , O. Hohm and P. K. Townsend, “Massive Gravity in Three Dimen-
sions,” Phys. Rev. Lett. 102 (2009) 201301 [arXiv:0901.1766 [hep-th]].
E. A. Bergshoeff, O. Hohm, J. Rosseel, E. Sezgin and P. K. Townsend, “More on Massive
3D Supergravity,” Class. Quant. Grav. 28 (2011) 015002 [arXiv:1005.3952 [hep-th]].

[9] H. Lu and C. N. Pope, “Critical Gravity in Four Dimensions,” Phys. Rev. Lett. 106
(2011) 181302 [arXiv:1101.1971 [hep-th]].

[10] J. Maldacena, “Einstein Gravity from Conformal Gravity,” [arXiv:1105.5632 [hep-th]].
[11] H. Lu, Y. Pang and C. N. Pope, “Conformal Gravity and Extensions of Critical Gravity,” Phys. Rev. D 84, 064001 (2011) [arXiv:1106.4657 [hep-th]] ;
H. Lu, C. N. Pope, E. Sezgin and L. Wulff, “Critical and Non-Critical Einstein-Weyl Supergravity,” JHEP 1110, 131 (2011) [arXiv:1107.2480 [hep-th]].

[12] S. J. Hyun, W. J. Jang, J. H. Jeong and S. H. Yi, “Noncritical Einstein-Weyl Gravity and the AdS/CFT Correspondence,” JHEP 1201, 054 (2012) [arXiv:1111.1175 [hep-th]].

[13] S. Hyun, W. Jang, J. Jeong and S. -H. Yi, “On Classical Equivalence Between Non-critical and Einstein Gravity: The AdS/CFT Perspectives,” JHEP 1204 (2012) 030 [arXiv:1202.3924 [hep-th]].

[14] I. Gullu, T. C. Sisman and B. Tekin, “Born-Infeld extension of new massive gravity,” Class. Quant. Grav. 27 (2010) 162001 [arXiv:1003.3935 [hep-th]].

[15] A. Sinha, “On the new massive gravity and AdS/CFT,” JHEP 1006 (2010) 061 [arXiv:1003.0683 [hep-th]] ;
M. F. Paulos, “New Massive Gravity Extended With An Arbitrary Number Of Curvature Corrections,” Phys. Rev. D 82, 084042 (2010) [arXiv:1005.1646 [hep-th]].

[16] T. Takayanagi, “Holographic Dual of BCFT,” Phys. Rev. Lett. 107 (2011) 101602 [arXiv:1105.5165 [hep-th]] ; M. Fujita, T. Takayanagi and E. Tonni, “Aspects of AdS/BCFT,” JHEP 1111 (2011) 043 [arXiv:1108.5152 [hep-th]].

[17] S. Nam, J. D. Park and S. H. Yi, “AdS Black Hole Solutions in the Extended New Massive Gravity,” JHEP 1007 (2010) 058 [arXiv:1005.1619 [hep-th]].

[18] R. C. Myers and B. Robinson, “Black Holes in Quasi-topological Gravity,” JHEP 1008, 067 (2010) [arXiv:1003.5357 [gr-qc]] ;
J. Oliva and S. Ray, “A new cubic theory of gravity in five dimensions: Black hole, Birkhoff’s theorem and C-function,” Class. Quant. Grav. 27 (2010) 225002 [arXiv:1003.4773 [gr-qc]].

[19] U. C. dS, C. P. Constantinidis, A. L. A. Lima and G. M. Sotkov, “Domain Walls in Extended Lovelock Gravity,” [arXiv:1202.4682 [hep-th]] ;
U. d. Camara and G. M. Sotkov, “New Massive Gravity Domain Walls,” Phys. Lett. B 694 (2010) 94 [arXiv:1008.2553 [hep-th]].

[20] C. Imbimbo, A. Schwimmer, S. Theisen and S. Yankielowicz, “Diffeomorphisms and holographic anomalies,” Class. Quant. Grav. 17 (2000) 1129 [arXiv:hep-th/9910267] ;
A. Schwimmer and S. Theisen, “Entanglement Entropy, Trace Anomalies and Holography,” Nucl. Phys. B 801 (2008) 1 [arXiv:0802.1017 [hep-th]].

[21] I. Gullu, T. C. Sisman and B. Tekin, “c-functions in the Born-Infeld extended New Massive Gravity,” Phys. Rev. D 82, 024032 (2010) [arXiv:1005.3214 [hep-th]].

[22] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, “Critical Points of D-Dimensional Extended Gravities,” Phys. Rev. D 83 (2011) 061502 [arXiv:1101.4009 [hep-th]].
E. A. Bergshoeff, O. Hohm and P. K. Townsend, “On massive gravitons in 2+1 dimensions,” [arXiv:0912.2944 [hep-th]].

[23] P. Breitenlohner and D. Z. Freedman, “Stability In Gauged Extended Supergravity,” Annals Phys. 144, 249 (1982).

[24] R. M. Wald, “Black hole entropy is the Noether charge,” Phys. Rev. D 48 (1993) 3427 [arXiv:gr-qc/9307038];
T. Jacobson, G. Kang and R. C. Myers, “On Black Hole Entropy,” Phys. Rev. D 49 (1994) 6587 [arXiv:gr-qc/9312023];
T. Jacobson, G. Kang and R. Myers, “Black hole entropy in higher curvature gravity,” [arXiv:gr-qc/9502009].

[25] J. de Boer, M. Kulaxizi and A. Parnachev, “Holographic Entanglement Entropy in Lovelock Gravities,” JHEP 1107, 109 (2011) [arXiv:1101.5781 [hep-th]]; L. Y. Hung, R. C. Myers and M. Smolkin, “On Holographic Entanglement Entropy and Higher Curvature Gravity,” JHEP 1104 (2011) 025 [arXiv:1101.5813 [hep-th]]; N. Ogawa and T. Takayanagi, “Higher Derivative Corrections to Holographic Entanglement Entropy for AdS Solitons,” JHEP 1110, 147 (2011) [arXiv:1107.4363 [hep-th]].

[26] Y. Kwon, S. Nam, J. D. Park and S. H. Yi, “AdS/BCFT Correspondence for Higher Curvature Gravity: An Example,” [arXiv:1201.1988 [hep-th]].

[27] L. F. Abbott and S. Deser, “Stability of Gravity with a Cosmological Constant,” Nucl. Phys. B 195 (1982) 76.
S. Deser and B. Tekin, “Energy in generic higher curvature gravity theories,” Phys. Rev. D 67 (2003) 084009 [hep-th/0212292].

[28] D. P. Jatkar and A. Sinha, “New Massive Gravity and AdS4 counterterms,” Phys. Rev. Lett. 106 (2011) 171601 [arXiv:1101.4716 [hep-th]]; K. Sen, A. Sinha and N. V. Suryanarayana, “Counterterms, critical gravity and holography,” [arXiv:1201.1288 [hep-th]].

[29] M. Gurses, T. C. Sisman and B. Tekin, “Some exact solutions of all f(Ricci) theories in three dimensions,” [arXiv:1112.6346 [hep-th]]; H. Ahmedov and A. N. Aliev, “Type N Spacetimes as Solutions of Extended New Massive Gravity,” [arXiv:1201.5724 [hep-th]].

[30] A. Ghodsi and D. M. Yekta, “Black Holes in Born-Infeld Extended New Massive Gravity,” Phys. Rev. D 83, 104004 (2011) [arXiv:1010.2434 [hep-th]]; A. Ghodsi and D. M. Yekta, “On asymptotically AdS-like solutions of three dimensional massive [arXiv:1112.5402 [hep-th]].