Phantom Field with O(N) symmetry in Exponential Potential

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(Dated: October 30, 2018)

In this paper, we study the phase space of phantom model with O(N) symmetry in exponential potential. Different from the model without O(N) symmetry, the introduction of the symmetry leads to a lower bound \( w > -3 \) on the equation of state for the existence of stable phantom dominated attractor phase. The reconstruction relation between the potential of O(N) phantom system and red shift has been derived.

PACS numbers: 98.80.Cq

I. INTRODUCTION

Astronomical measurements from Supernovae and CMB anisotropy independently confirm that about two-thirds of the energy density in our Universe is dark energy whose effective equation of state may lie in the range \(-1.38 < w < -0.82\). A recent analysis for the new high red shift data as well as dark energy and dark matter halos also indicate that dark energy with \( w < -1 \) seems more favorable. While the equation of state of conventional quintessence models that based on a scalar field and positive kinetic energy cannot evolve to the regime of \( w < -3 \). Some authors investigated phantom models which have negative kinetic energy and can realize the \( w < -1 \) in its evolution. Although the introduction of a phantom field causes many theoretical problems such as the violation of some widely accepted energy condition and the rapid vacuum decay, it is still very interesting in term of fitting current observations. Comparing with the other approaches to realizing the \( w < -1 \), such as the modification of Friedmann equation, it seems economical.

On the other hand, the role of complex scalar field as quintessence has been studied and its generalization to the fields with O(N) symmetry was done in Ref. [11]. In this paper, we study the case that when phantom models have an O(N) symmetry constraint via a specific model in which the potential is exponentially dependent on the field. We show that the dynamical system admits the phantom dominated attractor phase and the introduction of O(N) symmetry leads to a lower bound on the equation of state as \( w > -3 \), which is absent in the singlet scalar field phantom models. It is worth noting that although the O(N) phantom and the O(N) quintessence lead to similar equations, but they have very different dynamical evolution and physical implications, especially they will lead to different evolution of the equation of state \( w \). We also derive the relation between the red-shift and the potential of the scalar fields, known as reconstruction relation. It turns out that the potential relates to the red-shift the same way as that in ordinary scalar fields theory while the variation of the fields does in a quite different manner.

II. O(N) PHANTOM

We start from the flat Robertson-Walker metric

\[
ds^2 = dt^2 - a^2(t)dx^2
\]

The Lagrangian density for the Phantom with O(N) symmetry is

\[
L_\Phi = -\frac{1}{2} g^{\mu\nu} (\partial_\mu \Phi^a)(\partial_\nu \Phi^a) - V(|\Phi^a|)
\]

where \( \Phi^a \) is the component of the scalar field, \( a = 1, 2, \cdots , N \). To make it possess a O(N) symmetry, we write it in the following form

\[
\Phi^1 = R(t) \cos \varphi_1(t)
\]
\[
\Phi^2 = R(t) \sin \varphi_1(t) \cos \varphi_2(t)
\]
\[
\Phi^3 = R(t) \sin \varphi_1(t) \sin \varphi_2(t) \cos \varphi_3(t)
\]
\[
\Phi^{N-1} = R(t) \sin \varphi_1(t) \cdots \sin \varphi_{N-2}(t) \cos \varphi_{N-1}(t)
\]
\[
\Phi^N = R(t) \sin \varphi_1(t) \cdots \sin \varphi_{N-2}(t) \sin \varphi_{N-1}(t)
\]

Therefore, we have \( |\Phi^a| = R \) and assume that the potential of the O(N) phantom depends only on \( R \).

The action for the universe is:

\[
S = \int d^4x \sqrt{-g} \left(-\frac{1}{16\pi G} R_s - p_\gamma + L_\Phi \right)
\]

where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \), \( G \) is the Newton’s constant, \( R_s \) is the Ricci scalar, and \( p_\gamma \) is the pressure of the baryotropic fluid whose equation of state is \( p_\gamma = (\gamma - 1)\rho_\gamma \). Varying the action, one can obtain the Einstein equations and the radial equation of scalar fields as

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\[ H^2 = \frac{8\pi G}{3} \rho \gamma + \rho \Phi \]  
\[ \frac{\ddot{a}}{a} = -\frac{8\pi G}{3} \left( \frac{3\gamma}{2} - 1 \right) \rho \gamma + 2\rho \Phi + V(R) \]  
\[ \dot{R} + 3H\dot{R} - \left( \frac{\Omega^2}{a^6R^3} - \frac{\partial V(R)}{\partial R} \right) = 0 \]

with \( \rho \) = \( \frac{1}{2} (\dot{R}^2 + \frac{\Omega^2}{a^6R^3}) + V(R) \) and \( \rho \Phi = -\frac{1}{2} (\dot{R}^2 + \frac{\Omega^2}{a^6R^3}) - V(R) \) are the energy density and pressure of the \( \Phi \) field respectively, and \( H \) is Hubble parameter. To obtain the above equations, we make use of the fact that the "angular components" of the system could be simplified by the first integrals and its net contribution to the dynamics could be manifested by the term containing \( \Omega \) in the radial equation. For details of this, one can refer to Ref. [11]. The equation of state for the O(\( N \)) phantom is

\[ w = \frac{p}{\rho} = \frac{\dot{R}^2 + \frac{\Omega^2}{a^6R^3} + 2V(R)}{\dot{R}^2 + \frac{\Omega^2}{a^6R^3} - 2V(R)} \]  

It is clear that the O(\( N \)) phantom could realize the equation of state \( w < -1 \) which is equivalent to

\[ 0 < \dot{R}^2 + \frac{\Omega^2}{a^6R^3} < 2V(R) \]  

where the term \( \frac{\Omega^2}{a^6R^3} \) comes from the "total angular motion".

**III. ATTRACTOR PROPERTY OF O(\( N \)) PHANTOM**

In this section, we firstly investigate the attractor property of the O(\( N \)) phantom field in an exponential potential. To do so, we rewrite the equations of motion as

\[ \dot{H} = -\frac{\kappa^2}{2} (\rho \gamma + p \gamma - \dot{R}^2 - \frac{\Omega^2}{a^6R^3}) \]  
\[ \dot{\rho} \gamma = -3H (\rho \gamma + p \gamma) \]  
\[ \dot{R} + 3H \dot{R} - \dot{\rho} \gamma \]  
\[ H^2 = \frac{\kappa^2}{3} [\rho \gamma - \frac{1}{2} (\dot{R}^2 + \frac{\Omega^2}{a^6R^3}) + V(R)] \]

The potential \( V(R) \) here is specified as \( V(R) = V_0 \exp(-\lambda R) \). Now, we will introduce the following variables to obtain the autonomous system for the above dynamical system. The variables could be defined as \( x = \frac{\sqrt{6\kappa}}{\sqrt{3H}} \dot{R} \), \( y = \frac{\kappa \sqrt{V(R)}}{\sqrt{3H}} \), \( z = \frac{1}{\sqrt{6\kappa}} \frac{\Omega}{\sqrt{3H}} \), \( \xi = \frac{1}{\kappa R} \), and \( N = \log a \). The equation systems become the following autonomous system:

\[ \frac{dx}{dN} = \frac{3}{2} \left[ \gamma (1 + x^2 - y^2 + z^2) - 2(x^2 + z^2) \right] \]
\[ - (3x - \sqrt{6}z^2 \xi + \sqrt{\frac{3}{2}} \lambda y^2) \]
\[ \frac{dy}{dN} = \frac{3}{2} \left[ y (1 + x^2 - y^2 + z^2) - 2(x^2 + z^2) \right] - \sqrt{\frac{3}{2}} \lambda x y \]
\[ \frac{dz}{dN} = -3z + \frac{3}{2} \left[ y (1 + x^2 - y^2 + z^2) - 2(x^2 + z^2) \right] - \sqrt{6} x z \xi \]
\[ \frac{d\xi}{dN} = -\sqrt{6} \xi^2 x \]

Also, we have a constraint equation

\[ \Omega_R + \frac{\kappa^2 \rho \gamma}{3H^2} = 1 \]  

where

\[ \Omega_R = \frac{\kappa^2 \rho \gamma}{3H^2} = y^2 - x^2 - z^2 \]  

The equation of state for the O(\( N \)) phantom could be expressed in term of the new variables as

\[ w = \frac{p}{\rho} = \frac{x^2 + y^2 + z^2}{x^2 - y^2 + z^2} \]  

The critical points of the above autonomous system are easily obtained by setting the right hand sides of the above equations to zero. We write the variables near the critical points \((x_c, y_c, z_c, \xi_c)\) in the form \( x = x_c + u, y = y_c + v, z = z_c + w \) and \( \xi = \xi_c + \chi, \) where \( u, v, w, \chi \) are perturbations of the variables near the critical points and form a column vector denoted as \( \mathbf{U} \). Substitute the above expression into the autonomous system \((12)\), one can obtain the equations for the perturbations up to the first order as:

\[ \mathbf{U}' = M \cdot \mathbf{U} \]  

where the prime denotes the derivative with respect to \( N \). The coefficients of the perturbation equations form a 4×4 matrix \( M \) whose eigenvalues determine the type and stability of the critical points. The only physically meaningful critical point corresponding to the autonomous system \((12)\) is \((x, y, z, \xi) = (\frac{-1}{\sqrt{3}}, \sqrt{1 + \frac{\lambda^2}{6}}, 0, 0)\), which corresponds to the eigenvalues \((0, -3 - \frac{\lambda^2}{2}, -3\gamma - \lambda^2, -3 + \frac{\lambda^2}{2})\). Therefore, it is a stable node of the autonomous system when \( \lambda^2 < 6 \). This corresponds to a late time.
FIG. 1: The phase diagram of the $O(N)$ Phantom system in terms of $x$, $y$, $z$ for different initial $x$, $y$, $z$ and $\xi$.

FIG. 2: The projection of the phase space in $x$ $y$ plane.

attractor solution which is a phantom dominated epoch $\Omega_R = 1$ and an equation of state $w = -\frac{\lambda^2}{3} - 1$. Unlike the singlet phantom field in exponential potential [23], the introduction of internal symmetry imposes a lower bound to the parameter $\lambda^2$ for attractor solution, which equivalent to the equation of state $w > -3$. This is a very interesting characteristic of the $O(N)$ phantom field. In Fig. 1 and Fig. 2, we show the numerical results of the phase space of the system. In this plots, we choose the parameter $\gamma = 1$ and $\lambda = 1.2$.

IV. RECONSTRUCTION AND DISCUSSION

Now, we will correlate the potential with red shift. To do so, following the earlier study in this field[42], we introduce the quantity

\[ r(z) = \int_{t(z)}^{t_0} \frac{du}{a(u)} = \int_0^z \frac{dx}{H(x)} \tag{17} \]

which is the Robertson-Walker coordinate distance to an object at red-shift $z$. Also, we denote

\[ \rho_M = \frac{3\Omega_M H_0^2 (1 + z)^3}{8\pi G} \tag{18} \]

where $H_0$ is the present Hubble constant, $\Omega_M$ is the fraction of non-relativistic matter to the critical density. We then readily have

\[ \left(\frac{\dot{a}}{a}\right)^2 = H(z)^2 = \frac{1}{(dr/dz)^2} \tag{19} \]

\[ \frac{\ddot{a}}{a} = \frac{1}{(dr/dz)^2} + (1 + z) \frac{d^2r/dz^2}{(dr/dz)^3} \]

\[ \frac{dz}{dt} = -(1 + z)H(z) = -(1 + z)\frac{dr}{dz} \]

From the above equations, it is not difficult to derive the reconstruction equations as

\[ V[R(z)] = \frac{1}{8\pi G} \left[ \frac{3}{(dr/dz)^2} + (1 + z) \frac{d^2r/dz^2}{(dr/dz)^3} \right] \tag{20} \]

\[ \left(\frac{dR}{dz}\right)^2 + \frac{\Omega^2}{R^2} \left[ (1 + z)^4 \left(\frac{dr}{dz}\right)^2 \right] \tag{21} \]

\[ = \frac{(dr/dz)^2}{4\pi G(1 + z)^2} \left[ (1 + z)\frac{d^2r/dz^2}{(dr/dz)^3} + \frac{3}{2}(1 + z)^3 \right] \]

Eq. (20) is the same as those ordinary quintessence while the Eq. (21) is different in that there is a sign difference: the right hand side of the Eq. (21) is positive while it is negative in conventional $O(N)$ quintessence model.
Up to now, the observation data do not tell us what should be the nature of dark energy. But the future observation will be helpful to determine whether the dark energy is phantom, quintessence, or cosmological constant. If the equation of state $w < -1$ is completely confirmed by observations, then its implications to fundamental physics would be astounding, since it cannot be achieved with substance with canonical Lagrangian. Phantom field could be a good candidate for such substance because of its negative kinetic energy and simplicity. In this paper, we study a class of new phantom models, in which the scalar field possesses a $O(N)$ internal symmetry. In a specific model, in which the potential of the phantom fields is an exponential potential, we show that there exists an attractor solution when $\lambda^2 < 6$ and the attractor solution corresponds to a phantom energy dominant phase and an equation of state $w = -\frac{\lambda^2}{3} - 1$. Most strikingly, the introduction of $O(N)$ symmetry imposes a restriction to the existence of the attractor solution $\lambda^2 < 6$, which accordingly puts a lower bound to the equation of state $w > -3$. On the other hand, the speed of sound of the dark energy is defined by $c_s^2 = \frac{\rho_X}{\rho_X} = \frac{L_\phi - 2XL_\phi}{L_\phi X - L_\phi X}$, where $p = L_\phi (\phi, X)$ and $p = 2XL_\phi - L_\phi (\phi, X)$ with $X = \frac{1}{2} (\partial_\phi \phi)^2$. So, the model in this paper will produce an acceptable sound of speed $c_s^2 = 1$.

Acknowledgments

This work was partially supported by NKBRSF under Grant No. 1999075406

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