On the $\alpha$-Amenability of Hypergroups

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Abstract

Let $UC(K)$ denote the Banach space of all bounded uniformly continuous functions on a hypergroup $K$. The main results of this article concern on the $\alpha$-amenability of $UC(K)$ and quotients and products of hypergroups. It is also shown that a Sturm-Liouville hypergroup with a positive index is $\alpha$-amenable if and only if $\alpha = 1$.

Keywords. Hypergroups: Sturm-Liouville, Chébli-Trimèche, Bessel-Kingman. $\alpha$-Amenable Hypergroups.

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1 Introduction

In [20] M. Skantharajah systematically studied the amenability of hypergroups. Among other things, he obtained various equivalent statements on the amenability of hypergroups. Let $K$ be a locally compact hypergroup. Let $L^1(K)$ and $UC(K)$ denote the hypergroup algebra and the Banach space of all bounded uniformly continuous functions on $K$, respectively. He showed that $K$ is amenable if and only if there exists an invariant mean on $UC(K)$. Observe that, contrary to the group case, $UC(K)$ may fail to be an algebra in general. Commutative or compact hypergroups are amenable, and the amenability of $L^1(K)$ implies the amenability of $K$; however, the converse is not valid any longer even if $K$ is commutative (see also [2, 7]).

Recently the notion of $\alpha$-amenable hypergroups was introduced and studied in [7]. Suppose that $K$ is commutative, let $\alpha \in \widehat{K}$, and denote by $I(\alpha)$ the maximal ideal in $L^1(K)$ generated by $\alpha$. As shown in [7], $K$ is $\alpha$-amenable if and only if either $I(\alpha)$ has a bounded approximate identity or $K$ satisfies the modified Reiter condition of $P_1$-type in $\alpha$. The latter condition together with the recursion formulas for orthogonal polynomials yields a sufficient condition for the $\alpha$-amenability of a polynomial hypergroup. However, this condition is not available for well known hypergroups on the non-negative real axis.

The purpose of this article is to generalize the notion of $\alpha$-amenability for $K$ to the Banach space $UC(K)$. It then turns out that the $\alpha$-amenability of $K$ is equivalent to the $\alpha$-amenability of $UC(K)$,
and a $\alpha$-mean on $UC(K)$ is unique if and only if $\alpha$ belongs to $L^1(K) \cap L^2(K)$. Furthermore, some results are obtained on the $\alpha$-amenability of quotients and products of hypergroups. Given a Sturm-Liouville hypergroup $K$ with a positive index, it is also shown that there exist non-zero point derivations on $L^1(K)$. Therefore, $L^1(K)$ is not weakly amenable, $\{\alpha\} (\alpha \neq 1)$ is not a spectral set, and $K$ is not $\alpha$-amenable if $\alpha \neq 1$. However, an example (consisting of a certain Bessel-Kingman hypergroup) shows that in general $K$ is not necessarily $\alpha$-amenable if $\{\alpha\}$ is a spectral set.

This article is organized as follows: Section 2 collects pertinent concepts concerning on hypergroups. Section 3 considers the $\alpha$-amenability of $UC(K)$. Section 4 contains the $\alpha$-amenability of quotients and products of hypergroups, and Section 5 is considered on the question of $\alpha$-amenability of Sturm-Liouville hypergroups.

## 2 Preliminaries

Let $(K, \omega, \sim)$ be a locally compact hypergroup, where $\omega : K \times K \to M^1(K)$ defined by $(x, y) \mapsto \omega(x, y)$, and $\sim : K \to K$ defined by $x \mapsto \hat{x}$, denote the convolution and involution on $K$, where $M^1(K)$ stands for all probability measures on $K$. $K$ is called commutative if $\omega(x, y) = \omega(y, x)$, for every $x, y \in K$.

Throughout the article $K$ is a commutative hypergroup. Let $C_c(K), C_0(K),$ and $C^b(K)$ be the spaces of all continuous functions, those which have compact support, vanishing at infinity, and bounded on $K$ respectively; both $C^b(K)$ and $C_0(K)$ will be topologized by the uniform norm $\|\cdot\|_{\infty}$. The space of complex regular Radon measures on $K$ will be denoted by $M(K)$, which coincides with the dual space of $C_c(K)$ [11, Riesz’s Theorem (20.45)]. The translation of $f \in C_c(K)$ at the point $x \in K$, $T_x f$, is defined by $T_x f(y) = \int_K f(t) d\omega(x, y)(t)$, for every $y \in K$. Being $K$ commutative ensures the existence of a Haar measure on $K$ which is unique up to a multiplicative constant [21]. Thus, according to the translation $T$, let $m$ be the Haar measure on $K$, and let $(L^1(K), \|\cdot\|_1)$ denote the usual Banach space of all integrable functions on $K$ [12, 6.2]. For $f, g \in L^1(K)$ the convolution and involution are defined by $f \ast g(x) := \int_K f(y) T_y g(x) dm(y) \ (m$-a.e. on $K)$ and $f^*(x) = \overline{f(\hat{x})}$, respectively, that $(L^1(K), \|\cdot\|_1)$ becomes a commutative Banach $\ast$-algebra. If $K$ is discrete, then $L^1(K)$ has an identity; otherwise $L^1(K)$ has a b.a.i. (bounded approximate identity), i.e. there exists a net $\{e_i\}_i$ of functions in $L^1(K)$ with $\|e_i\|_1 \leq M$, for some $M > 0$, such that $\|f \ast e_i - f\|_1 \to 0$ as $i \to \infty$ [3]. The dual space $L^1(K)^*$ can be identified with the space $L^\infty(K)$ of essentially bounded Borel measurable complex-valued functions on $K$. The bounded multiplicative linear functionals on $L^1(K)$ can be identified with

$$\mathcal{X}^b(K) := \{\alpha \in C^b(K) : \alpha \neq 0, \ \omega(x, y)(\alpha) = \alpha(x)\alpha(y), \ \forall x, y \in K\},$$

where $\mathcal{X}^b(K)$ is a locally compact Hausdorff space with the compact-open topology. $\mathcal{X}^b(K)$ with its subset

$$\hat{K} := \{\alpha \in \mathcal{X}^b(K) : \alpha(\hat{x}) = \alpha(x), \ \forall x \in K\}$$
are considered as the character spaces of $K$.

The Fourier-Stieltjes transform of $\mu \in M(K)$, $\widehat{\mu} \in C_b(\hat{K})$, is defined as $\widehat{\mu}(\alpha) := \int_K \overline{\alpha(x)}d\mu(x)$, which by restriction on $L^1(K)$ it is called Fourier transform and $\widehat{f} \in C_b(\hat{K})$, for every $f \in L^1(K)$.

There exists a unique regular positive Borel measure $\pi$ on $\hat{K}$ with the support $\mathcal{S}$ such that

$$\int_K |f(x)|^2 dm(x) = \int_{\mathcal{S}} |\widehat{f}(\alpha)|^2 d\pi(\alpha),$$

for all $f \in L^1(K) \cap L^2(K)$. $\pi$ is called Plancherel measure and its support, $\mathcal{S}$, is a nonvoid closed subset of $\hat{K}$. Observe that the constant function 1 is in general not contained in $\mathcal{S}$. We have $\mathcal{S} \subseteq \hat{K} \subseteq \mathcal{X}^b(K)$, where proper inclusions are possible; see [12, 9.5].

### 3 $\alpha$-Amenability of $UC(K)$

**Definition 3.1.** Let $K$ be a commutative hypergroup and $\alpha \in \hat{K}$. Let $X$ be a subspace of $L^\infty(K)$ with $\alpha \in X$ which is closed under complex conjugation and is translation invariant. Then $X$ is called $\alpha$-amenable if there exists a $m_\alpha \in X^*$ with the following properties:

(i) $m_\alpha(\alpha) = 1$,

(ii) $m_\alpha(T_x f) = \alpha(x) m_\alpha(f)$, for every $f \in X$ and $x \in K$.

The hypergroup $K$ is called $\alpha$-amenable if $X = L^\infty(K)$ is $\alpha$-amenable; in the case $\alpha = 1$, $K$ respectively $L^\infty(K)$ is called amenable. As shown in [7], $K$ is $\alpha$-amenable if and only if either $I(\alpha)$ has a b.a.i. or $K$ has the modified Reiter’s condition of $P_1$ type in the character $\alpha$. The latter is also equivalent to the $\alpha$-left amenability of $L^1(K)$, if $\alpha$ is real-valued [2]. For instance, commutative hypergroups are amenable, and compact hypergroups are $\alpha$-amenable for every character $\alpha$.

If $K$ is a locally compact group, then the amenability of $K$ is equivalent to the amenability of diverse subalgebras of $L^\infty(K)$, e.g. $UC(K)$ the algebra of bounded uniformly continuous functions on $K$ [17]. The same is true for hypergroups although $UC(K)$ fails to be an algebra in general [20]. We now prove this fact in terms of $\alpha$-amenability.

Let $UC(K) := \{ f \in C^b(K) : x \mapsto T_x f \text{ is continuous from } K \text{ to } (C^b(K), \|\cdot\|_\infty) \}$. The function space $UC(K)$ is a norm closed, conjugate closed, translation invariant subspace of $C^b(K)$ containing the constants and the continuous functions vanishing at infinity [20, Lemma 2.2]. Moreover, $\mathcal{X}^b(K) \subset UC(K)$ and $UC(K) = L^1(K) \ast L^\infty(K)$. Let $B$ be a subspace of $L^\infty(K)$ such that $UC(K) \subseteq B$. $K$ is amenable if and only if $B$ is amenable [20, Theorem 3.2].

The following theorem provides a further equivalent statement to the $\alpha$-amenability of $K$.

**Theorem 3.2.** Let $\alpha \in \hat{K}$. Then $UC(K)$ is $\alpha$-amenable if and only if $K$ is $\alpha$-amenable.

**Proof:** Let $UC(K)$ be $\alpha$-amenable. There exists a $m^{ac}_\alpha \in UC(K)^*$ such that $m^{ac}_\alpha(\alpha) = 1$ and $m^{ac}_\alpha(T_x f) = \alpha(x) m^{ac}_\alpha(f)$ for all $f \in UC(K)$ and $x \in K$. Let $g \in L^1(K)$ such that $\widehat{g}(\alpha) = 1$. Define
that \( m_\alpha|_{UC(K)} = m_\alpha^{uc} \). Since \( \varphi * g \in UC(K) \), \( m_\alpha \) is a well-defined bounded linear functional on \( L^\infty(K) \), \( m_\alpha(\alpha) = 1 \), and for all \( x \in K \) we have

\[
m_\alpha(T_x \varphi) = m_\alpha^{uc}((T_x \varphi) * g) \\
= m_\alpha^{uc}((\delta_x * \varphi) * g)) \\
= m_\alpha^{uc}(\delta_x * (\varphi * g)) \\
= m_\alpha^{uc}(T_x(\varphi * g)) \\
= \alpha(x)m_\alpha^{uc}(\varphi * g) \\
= \alpha(x)m_\alpha(\varphi).
\]

The latter shows that every \( \alpha \)-mean on \( UC(K) \) extends on \( L^\infty(K) \). Plainly the restriction of any \( \alpha \)-mean of \( K \) on \( UC(K) \) is a \( \alpha \)-mean on \( UC(K) \). Therefore, the statement is valid.

**Corollary 3.3.** Let \( \alpha \in \widehat{K} \) and \( UC(K) \subseteq B \subseteq L^\infty(K) \). Then \( K \) is \( \alpha \)-amenable if and only if \( B \) is \( \alpha \)-amenable.

The Banach space \( L^\infty(K)^* \) with the Arens product defined as follows is a Banach algebra:

\[
\langle m \cdot m', f \rangle = \langle m, m' \cdot f \rangle, \text{ in which } \langle m' \cdot f, g \rangle = \langle m', f \cdot g \rangle,
\]

and \( \langle f \cdot g, h \rangle = \langle f, g \ast h \rangle \) for all \( m, m' \in L^\infty(K)^*, f \in L^\infty(K) \) and \( g, h \in L^1(K) \) where \( \langle f, g \rangle := f(g) \).

The Banach space \( UC(K)^* \) with the restriction of the Arens product is a Banach algebra, and it can be identified with a closed right ideal of the Banach algebra \( L^1(K)^{**} \) [14].

If \( m, m' \in UC(K)^*, f \in UC(K) \), and \( x \in K \), then \( m' \cdot f \in UC(K) \) and we may have

\[
\langle m \cdot m', f \rangle = \langle m, m' \cdot f \rangle, \langle m' \cdot f, x \rangle = \langle m', T_x f \rangle.
\]

If \( y \in K \), then \( \int_K T_t f d\omega(x, y)(t) = T_y(T_x f) \) which implies that

\[
T_x(m \cdot f) = m \cdot T_x f,
\]

as

\[
T_x(m \cdot f)(y) = \int_K m \cdot f(t)d\omega(x, y)(t) \\
= \int_K \langle m, T_t f \rangle d\omega(x, y)(t) \\
= \langle m, \int_K T_t f d\omega(x, y)(t) \rangle.
\]
Let $X = UC(K)$, $f \in X$ and $g \in C_c(K)$ ($g \geq 0$) with $\|g\|_1 = 1$. Since the mapping $x \to T_x f$ is continuous from $K$ to $(C^b(K), \|\cdot\|_\infty)$ and the point evaluation functionals in $X^*$ separate points of $X$, we have

$$g \ast f = \int_K g(x)T_x f \, dm(x).$$

Therefore, for every 1-mean $m$ on $X$ we have $m(f) = m(g \ast f)$. Hence, two 1-means $m$ and $m'$ on $L^\infty(K)$ are equal if they are equal on $UC(K)$, as

$$m(f) = m(g \ast f) = m'(g \ast f) = m'(f) \quad (f \in L^\infty(K))$$

and $g$ is assumed as above. The latter together with [20, Theorem 3.2] show a bijection between means on $UC(K)$ and $L^\infty(K)$. So, if $UC(K)$ is amenable with a unique mean, then its extension on $L^\infty(K)$ is also unique which implies that $K$ is compact [13], therefore the identity character is isolated in $S$, the support of the Plancherel measure. We have the following theorem in general:

**Theorem 3.4.** If $UC(K)$ is $\alpha$-amenable with the unique $\alpha$-mean $m_{\alpha}^{uc}$, then $m_{\alpha}^{uc} \in L^1(K) \cap L^2(K)$, $\{\alpha\}$ is isolated in $S$ and $m_{\alpha}^{uc} = \frac{\pi(\alpha)}{\|\alpha\|_2^2}$, where $\pi : L^1(K) \to L^1(K)^{**}$ is the canonical embedding. If $\alpha$ is positive, then $K$ is compact.

**Proof:** Let $m_{\alpha}^{uc}$ be the unique $\alpha$-mean on $UC(K)$, $n \in UC(K)^*$ and $\{n_i\}$ be a net converging to $n$ in the $w^*$-topology. Let $x \in K$ and $f \in UC(K)$. Then

$$\langle m_{\alpha}^{uc} \cdot n_i, T_x f \rangle = \langle m_{\alpha}^{uc}, n_i \cdot (T_x f) \rangle$$

$$= \langle m_{\alpha}^{uc}, T_x(n_i \cdot f) \rangle$$

$$= \alpha(x) \langle m_{\alpha}^{uc}, n_i \cdot f \rangle$$

$$= \alpha(x) \langle m_{\alpha}^{uc} \cdot n_i, f \rangle.$$

For $\lambda_i = \langle n_i, \alpha \rangle \neq 0$, since the associated functional to the character $\alpha$ on $UC(K)^*$ is multiplicative [27], $m_{\alpha}^{uc} \cdot n_i / \lambda_i$ is a $\alpha$-mean on $UC(K)$ which is equal to $m_{\alpha}^{uc}$. Then the mapping $n \to m_{\alpha}^{uc} \cdot n$ defined on $UC(K)^*$ is $w^*-w^*$ continuous, hence $m_{\alpha}^{uc}$ is in the topological centre of $UC(K)^*$, i.e. $M(K)$; see [13, Theorem 3.4.3]. Since $m_{\alpha}^{uc}(\beta) = \delta_\alpha(\beta)$ and $m_{\alpha}^{uc} \in C^b(\hat{K})$, $\{\alpha\}$ is an open-closed subset of $\hat{K}$. The algebra $L^1(K)$ is a two-sided closed ideal in $M(K)$ and the Fourier transform is injective, so $m_{\alpha}^{uc}$ and $\alpha$ belong to $L^1(K) \cap L^2(K)$. The inversion theorem, [3, Theorem 2.2.36], indicates $\alpha = \hat{\alpha}$, accordingly $\alpha \in S$. Let $m_\alpha := \pi(\alpha)/\|\alpha\|_2^2$. Obviously $m_{\alpha}$ is a $\alpha$-mean on $L^\infty(K)$, and the restriction of $m_{\alpha}$ on $UC(K)$ yields the desired unique $\alpha$-mean.

If $\alpha$ is positive, then

$$\alpha(x) \int_K \alpha(t) dm(t) = \int_K T_x \alpha(t) dm(t) = \int_K \alpha(t) dm(t)$$

which implies that $\alpha = 1$, hence $K$ is compact.

\[\square\]
Observe that in contrast to the case of locally compact groups, there exist noncompact hypergroups with unique $\alpha$-means. For example, for little $q$-Legendre polynomial hypergroups, we have $\hat{K}\setminus\{1\} \subset L^1(K) \cap L^2(K)$; see [8]. Therefore, by Theorem 3.4, $UC(K)$ and $K$ are $\alpha$-amenable with the unique $\alpha$-mean $m_\alpha$, whereas $K$ has infinitely many 1-means [20].

Let $\Sigma_\alpha(X)$ be the set of all $\alpha$-means on $X = L^\infty(K)$ or $UC(K)$. If $\alpha = 1$, then $\Sigma_1(X)$ is nonempty (as $K$ is commutative) weak*-compact convex set in $X^*$ [20]. If $\alpha \neq 1$ and $X$ is $\alpha$-amenable, then the same is true for $\Sigma_\alpha(X)$.

**Theorem 3.5.** Let $X$ be $\alpha$-amenable ($\alpha \neq 1$). Then $\Sigma_\alpha(X)$ is a nonempty weak*-compact convex subset of $X^*$. Moreover, $\Sigma_\alpha(X) \cdot M/\langle M, \alpha \rangle \subseteq \Sigma_\alpha(X)$, for all $M \in X^*$ with $\langle M, \alpha \rangle \neq 0$. Furthermore, if $m_\alpha \in \Sigma_\alpha(X)$, then $m_\alpha^n = m_\alpha$ for all $n \in \mathbb{N}$.

**Proof:** Let $0 \leq \lambda \leq 1$ and $m_\alpha, m'_\alpha \in \Sigma_\alpha(X)$. If $m''_\alpha := \lambda m_\alpha + (1 - \lambda)m'_\alpha$, then $m''_\alpha(\alpha) = 1$ and $m''_\alpha(T_x f) = \alpha(x)m''_\alpha(f)$, for every $f \in X$ and $x \in K$; hence $m''_\alpha \in \Sigma_\alpha(X)$.

If $\{m_i\} \subseteq \Sigma_\alpha(X)$ such that $m_i \xrightarrow{w^*} m$, then $m \in \Sigma_\alpha(X)$. We have $m(\alpha) = 1$ and

$$m(T_x f) = \lim_{i \to \infty} m_i(T_x f) = \alpha(x) \lim_{i \to \infty} m_i(f) = \alpha(x)m(f),$$

for all $f \in X$ and $x \in K$. Moreover, $\Sigma_\alpha(X)$ is $w^*$-compact by Alaoglu’s theorem [6, p.424]. Let $M \in X^*$ with $\lambda = \langle M, \alpha \rangle \neq 0$. Then $M' := m_\alpha \cdot M/\lambda$ is a $\alpha$-mean on $X$, as

$$\langle m_\alpha \cdot M, T_x f \rangle = \langle m_\alpha, M \cdot T_x f \rangle = \langle m_\alpha, T_x (M \cdot f) \rangle = \alpha(x) \langle m_\alpha, M \cdot f \rangle = \alpha(x) \langle m_\alpha \cdot M, f \rangle.$$

Since $g \cdot m_\alpha = m_\alpha \cdot g = \hat{g}^*(\alpha)m_\alpha$ for all $g \in L^1(K)$, the continuity of the Arens product in the first variable on $X$ together with Goldstein’s theorem yield $m^2_\alpha = m_\alpha$; hence, $m^n_\alpha = m_\alpha$ for all $n \in \mathbb{N}$. □

**Remark 3.6.** Observe that if $K$ is $\alpha$ and $\beta$-amenable, then $m_\alpha \cdot m_\beta = m_\beta \cdot m_\alpha$ if and only if $m_\alpha \cdot m_\beta = \delta_\alpha(\beta)m_\alpha$, as

$$\alpha(x)\langle m_\alpha \cdot m_\beta, f \rangle = \alpha(x)\langle m_\alpha, m_\beta \cdot f \rangle = \langle m_\alpha, T_x (m_\beta \cdot f) \rangle = \langle m_\alpha, m_\beta \cdot T_x f \rangle = \langle m_\alpha \cdot m_\beta, T_x f \rangle = \langle m_\beta \cdot m_\alpha, T_x f \rangle = \langle m_\beta, m_\alpha \cdot T_x f \rangle = \langle m_\beta, T_x (m_\alpha \cdot f) \rangle = \beta(x)\langle m_\beta, m_\alpha \cdot f \rangle = \beta(x)\langle m_\beta \cdot m_\alpha, f \rangle \quad (f \in X, x \in K).$$
A closed nonempty subset $H$ of $K$ is called a subhypergroup if $H \cdot H = H$ and $\tilde{H} = H$, where $\tilde{H} := \{ \tilde{x} : x \in H \}$. Let $H$ be a subhypergroup of $K$. Then $K/H := \{ xH : x \in K \}$ is a locally compact space with respect to the quotient topology. If $H$ is a subgroup or a compact subhypergroup of $K$, then

$$\omega(xH,yH) := \int_K \delta_{xH} d\omega(x,y)(z) \quad (x,y \in K)$$

defines a hypergroup structure on $K/H$, which agrees with the double coset hypergroup $K//H$; see [12]. The properties and duals of subhypergroups and quotient of commutative hypergroups have been intensively studied by M. Voit [23, 24].

**Theorem 4.1.** Let $H$ be a subgroup or a compact subhypergroup of $K$. Suppose $p : K \to K/H$ is the canonical projection, and $\tilde{p} : \tilde{K}/\tilde{H} \to \tilde{K}$ is defined by $\gamma \mapsto \gamma op$. Then $K/H$ is $\gamma$-amenable if and only if $K$ is $\gamma op$-amenable.

**Proof:** Let $K/H$ be $\gamma$-amenable. Then there exists a $M_\gamma : C^b(K/H) \to \mathbb{C}$ such that $M_\gamma(\gamma) = 1$, and $M_\gamma(T_{xH}f) = \gamma(xH) M_\gamma(f)$.

Since $H$ is amenable [20], let $m_1$ be a mean on $C^b(H)$. For $f \in C^b(K)$, define

$$f^1 : K \to \mathbb{C} \text{ by } f^1(x) := \langle m_1, T_x f \rangle_{H}.$$ 

The function $f^1$ is continuous, bounded, and since $m_1$ is a mean for $H$, we have

$$T_h f^1(x) = \int_K f^1(t) d\omega(h,x)(t)$$

$$= \int_K \langle m_1, T_t f \rangle_H d\omega(h,x)(t)$$

$$= \langle m_1, \int_K T_t f \rangle_H d\omega(h,x)(t)$$

$$= \langle m_1, T_h [T_x f]_H \rangle = \langle m_1, T_x f \rangle = f^1(x),$$

for all $h \in H$. Then according to the assumptions on $H$, [24, Lemma 1.5] implies that $f^1$ is constant on the cosets of $H$ in $K$. We may write $f^1 = F \circ f$, $F \in C^b(K/H)$. Define

$$m : C^b(K) \to \mathbb{C} \text{ by } m(f) = \langle M_\gamma, F \rangle.$$ 

We have

$$xH F(yH) = T_x f^1(y) = \int_K \langle m_1, T_u f \rangle_H d\omega(x,y)(u) = \langle m_1, T_y (T_x f \rangle_H),$$

as $u \to T_u f \rangle_H$ is continuous from $K$ to $(C^b(H), \| \cdot \|_\infty)$ and the point evaluation functionals in $C^b(H)^*$ separates the points of $C^b(H)$; hence $T_x H F \circ op = (T_x f)^1$. Therefore,

$$m(T_x f) = \langle M_\gamma, T_x H F \rangle = \gamma(xH) \langle M_\gamma, F \rangle = \alpha(x)m(f).$$
Moreover, \( \langle m, \alpha \rangle = \langle M_\gamma, \gamma \rangle = 1 \), where \( \gamma \circ p = \alpha \). Then \( m(T_x f) = \alpha(x)m(f) \) for all \( f \in C_b(K) \) and \( x \in K \).

To show the converse, let \( m_\alpha \) be a \( \alpha \)-mean on \( C_b(K) \), and define
\[
M : C_b(K/H) \to \mathbb{C} \quad \text{by} \quad \langle M, f \rangle = \langle m_\alpha, f \circ p \rangle.
\]
Since
\[
T_{xH} f(yH) = T_{xH} f \circ p(y) = \int_{K/H} f d\omega(xH, yH) = \int_K f \circ p d\omega(x, y) = T_x f \circ p(y),
\]
so \( M(T_{xH} f) = \langle m_\alpha, T_x f \circ p \rangle = \alpha(x) \langle m_\alpha, f \circ p \rangle = \alpha(x) \langle M, f \rangle \). Since \( \hat{p} \) is an isomorphism, \([23, \text{Theorem 2.5}]\), and \( \gamma \circ p = \alpha \), we have
\[
\langle M, \gamma \rangle = \langle m_\alpha, \gamma \circ p \rangle = \langle m_\alpha, \alpha \rangle = 1.
\]
Therefore, \( M \) is the desired \( \gamma \)-mean on \( C_b(K/H) \).

4.2. Example: Let \( H \) be compact and \( H' \) be discrete commutative hypergroups. Let \( K := H \vee H' \) denotes the joint hypergroup that \( H \) is a subhypergroup of \( K \) and \( K/H \cong H' \) \([25]\). The hypergroups \( K, H \) and \( H' \) are amenable, and \( H \) is \( \beta \)-amenable for every \( \beta \in \hat{H} \) \([7]\). By Theorem 4.1, \( K \) is \( \alpha \)-amenable if and only if \( H' \) is \( \gamma \)-amenable (\( \alpha = \gamma \circ p \)).

Remark 4.3. Let \( G \) be a \([FIA]B\)-group \([16]\). Then the space \( G_B, B\)-orbits in \( G \), forms a hypergroup \([12, 8.3]\). If \( G \) is an amenable \([FIA]B\)-group, then the hypergroup \( G_B \) is amenable \([20, \text{Corollary 3.11}]\). However, \( G_B \) may not be \( \alpha \)-amenable for \( \alpha \in \widehat{G_B} \setminus \{1\} \). For example, let \( G := \mathbb{R}^n \) and \( B \) be the group of rotations which acts on \( G \). Then the hypergroup \( K := G_B \) can be identified with the Bessel-Kingman hypergroup \( \mathbb{R}_0 := [0, \infty) \) of order \( \nu = \frac{n-2}{2} \). Theorem 5.5 will show that if \( n \geq 2 \) then \( \mathbb{R}_0 \) is \( \alpha \)-amenable if and only if \( \alpha = 1 \). Observe that \( L_1(\mathbb{R}_0) \) is an amenable Banach algebra for \( n = 1 \) \([26]\); hence every maximal ideal of \( L_1(\mathbb{R}_0) \) has a b.a.i. \([4]\), consequently \( G_B \) is \( \alpha \)-amenable for every \( \alpha \in \widehat{G_B} \) \([7]\).

Let \( K \) and \( H \) be hypergroups with left Haar measures. Then it is straightforward to show that \( K \times H \) is a hypergroup with a left Haar measure. If \( K \) and \( H \) are commutative hypergroups, then \( K \times H \) is a commutative hypergroup with a Haar measure. As in the case of locally compact groups \([5]\), we have the following isomorphism
\[
\phi : L^\infty(K) \times L^\infty(H) \to L^\infty(K \times H) \quad \text{by} \quad (f, g) \to \phi(f, g),
\]
where \( \phi(f, g)(x, y) = f(x)g(y) \) for all \((x, y) \in K \times H\). Let \((x', y') \in K \times H\) and \((f, g) \in L^\infty(K) \times L^\infty(H)\). Then
\[
T_{(x', y')}(\phi(f, g))(x, y) = \int_{K \times H} \phi(f, g)(t, t') d\omega(x', x) \times \omega(y', y)(t, t')
\]
\[
= \int_{K} \int_{H} f(t)g(t') d\omega(x', x)(t)d\omega(y', y)(t')
\]
\[
= T_{x'} f(x)T_{y'} g(y)
\]
\[
= \phi(T_{x'} f, T_{y'} g)(x, y).
\]
Theorem 4.4. Let $K$ and $H$ be commutative hypergroups. Then

(i) the map $\phi$ defined in (3) is a homeomorphism between $\hat{K} \times \hat{H}$ and $\hat{K} \times \hat{H}$, where the dual spaces bear the compact-open topologies.

(ii) $K \times H$ is $\phi(\alpha,\beta)$-amenable if and only if $K$ and $H$ are $\alpha$ and $\beta$-amenable respectively.

Proof: (i) It is the special case of [4, Proposition 19].

(ii) As in the case of locally compact groups [5], we have $L^1(K \times H) \cong L^1(K) \otimes_p L^1(H)$, where $\otimes_p$ denotes the projection tensor product of two Banach algebras. If $K$ is $\alpha$-amenable and $H$ is $\beta$-amenable, then $I(\alpha)$ and $I(\beta)$, the maximal ideals of $L^1(K)$ and $L^1(H)$ respectively, have b.a.i. [7]. Since $L^1(K)$ and $L^1(H)$ have b.a.i., $L^1(K) \otimes_p I(\beta) + I(\alpha) \otimes_p L^1(H)$, the maximal ideal in $L^1(K \times H)$ associated to the character $\phi(\alpha,\beta)$, has a b.a.i. [5, Proposition 2.9.21], that equivalently $K \times H$ is $\phi(\alpha,\beta)$-amenable.

To prove the converse, suppose $m(\alpha,\beta)$ is a $\phi(\alpha,\beta)$-mean on $L^\infty(K \times H)$, define

$$m_\alpha : L^\infty(K) \longrightarrow \mathbb{C} \text{ by } \langle m_\alpha, f \rangle := \langle m(\alpha,\beta), \phi(f,\beta) \rangle.$$ 

We have

$$\langle m_\alpha, T_x f \rangle = \langle m(\alpha,\beta), \phi(T_x f,\beta) \rangle$$

$$= \langle m(\alpha,\beta), T_x \phi(f,\beta) \rangle$$

$$= \alpha(x) \beta(e) \langle m(\alpha,\beta), \phi(f,\beta) \rangle$$

$$= \alpha(x) \langle m(\alpha,\beta), \phi(f,\beta) \rangle$$

$$= \alpha(x) \langle m_\alpha, f \rangle,$$

for all $f \in L^\infty(K)$ and $x \in K$. Since $\langle m_\alpha, \alpha \rangle = \langle m(\alpha,\beta), \phi(\alpha,\beta) \rangle = 1$, $K$ is $\alpha$-amenable. Similarly $m_\beta : L^\infty(H) \longrightarrow \mathbb{C}$ defined by $m_\beta(g) := \langle m(\alpha,\beta), \phi(\alpha,g) \rangle$ is a $\beta$-mean on $L^\infty(H)$, hence $H$ is $\beta$-amenable. \hfill \Box

Remark 4.5. The proof of the previous theorem may also follow from Theorem 3.2 with a modifying [20, Proposition 3.8].

5 $\alpha$-Amenability of Sturm-Liouville Hypergroups

Suppose $A : \mathbb{R}_0 \rightarrow \mathbb{R}$ is continuous, positive, and continuously differentiable on $\mathbb{R}_0 \setminus \{0\}$. Moreover, assume that

$$\frac{A'(x)}{A(x)} = \frac{\gamma_0(x)}{x} + \gamma_1(x), \quad (4)$$

for all $x$ in a neighbourhood of 0, with $\gamma_0 \geq 0$ such that

SL1 one of the following additional conditions holds.
The function \( A \) is called Chébli-Trimèche if \( A \) is a Sturm-Liouville function of type SL1.1 satisfying the assumptions that the quotient \( \frac{A'}{A} \geq 0 \) is decreasing and that \( A \) is increasing with \( \lim_{x \to \infty} A(x) = \infty \). In this case SL2 is fulfilled with \( \alpha = \frac{1}{2} \). Let \( A \) be a Sturm-Liouville function satisfying (4) and SL2. Then there exists always a unique commutative hypergroup structure on \( \mathbb{R}_0 \) such that \( A(x)dx \) is the Haar measure. A hypergroup established by this way is called a Sturm-Liouville hypergroup and it will be denoted by \( (\mathbb{R}_0, A(x)dx) \). If \( A \) is a Chébli-Trimèche function, then the hypergroup \( (\mathbb{R}_0, A(x)dx) \) is called Chébli-Trimèche hypergroup.

The characters of \( \mathbb{R}_0, A(x)dx \) can be considered as solution \( \varphi_\lambda \) of the differential equation

\[
\left( \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)} \frac{d}{dx} \right) \varphi_\lambda = - (\lambda^2 + \rho^2) \varphi_\lambda, \quad \varphi_\lambda(0) = 1, \quad \varphi'_\lambda(0) = 0,
\]

where \( \rho := \lim_{x \to \infty} \frac{A'(x)}{2A(x)}, \) and \( \lambda \in \mathbb{R}_\rho := \mathbb{R}_0 \cup i[0, \rho] \); see [3, Proposition 3.5.49]. As shown in [3, Sec.3.5], \( \varphi_0 \) is a strictly positive character, and \( \varphi_\lambda \) has the following integral representation

\[
\varphi_\lambda(x) = \varphi_0(x) \int_{-x}^{x} e^{-i\lambda t} \, d\mu_x(t)
\]

where \( \mu_x \in M^1([-x, x]) \) for every \( x \in \mathbb{R}_0 \) and all \( \lambda \in \mathbb{C} \). In the particular case \( \lambda := i\rho \), the equality (5) yields \( |\varphi_0(x)| \leq e^{-\rho x} \), as \( \varphi_{i\rho} = 1 \).

**Proposition 5.1.** Let \( \varphi_\lambda \) be as above. Then

\[
\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq x^n e^{(|\text{Im}\lambda| - \rho)x}
\]

for all \( x \geq 0, \lambda \in \mathbb{C} \) and \( n \in \mathbb{N} \).

Proof: For all \( \lambda \in \mathbb{C} \setminus \{0\} \) and \( x > 0 \), applying the Lebesgue dominated convergence theorem [11] yields

\[
\frac{d^n}{d\lambda^n} \varphi_\lambda(x) = \varphi_0(x) \int_{-x}^{x} (-it)^n e^{-i\lambda t} \, d\mu_x(t),
\]

hence

\[
\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq \varphi_0(x) x^n \int_{-x}^{x} \left| e^{-i\lambda t} \right| \, d\mu_x(t)
\]

which implies that

\[
\left| \frac{d^n}{d\lambda^n} \varphi_\lambda(x) \right| \leq x^n e^{(|\text{Im}\lambda| - \rho)x}.
\]

To study the \( \alpha \)-amenability of Sturm-Liouville hypergroups, we may require the following fact in general. The functional \( D \in L^1(K)^* \) is called a \( \alpha \)-derivation \( (\alpha \in \hat{K}) \) on \( L^1(K) \) if

\[
D(f \ast g) = \widehat{f}(\alpha) D_\alpha(g) + \widehat{g}(\alpha) D_\alpha(f) \quad (f, g \in L^1(K)).
\]

Observe that if the maximal ideal \( I(\alpha) \) has an approximate identity, then \( D|_{I(\alpha)} = 0 \).
Lemma 5.2. Let $\alpha \in \widehat{K}$. If $K$ is $\alpha$-amenable, then every $\alpha$-derivation on $L^1(K)$ is zero.

Proof: Let $D \in L^1(K)^*$ be a $\alpha$-derivation on $L^1(K)$. Since $I(\alpha)$ has a b.a.i. [7], $D|_{I(\alpha)} = 0$. Assume that $g \in L^1(K)$ with $\hat{g}(\alpha) = 1$. Consequently, $g * g - g \in I(\alpha)$ which implies that $D(g) = 0$. \hfill $\square$

Theorem 5.3. Let $K$ be the Sturm-Liouville hypergroup with $\rho > 0$ and $\varphi_{\lambda} \in \widehat{K}$. Then $K$ is $\varphi_{\lambda}$-amenable if and only if $\lambda = i\rho$.

Proof: By Proposition 5.1, the mapping

$$D_{\lambda_0} : L^1(K) \to \mathbb{C}, \quad D_{\lambda_0}(f) = \frac{d}{d\lambda} \hat{f}(\lambda)|_{\lambda = \lambda_0} (\lambda_0 \neq i\rho),$$

is a well-defined bounded nonzero $\varphi_{\lambda_0}$-derivation. In that $R_0$ is amenable [20], applying Lemma 5.2 will indicate that $K$ is $\varphi_{\lambda}$-amenable if and only if $\lambda = i\rho$. \hfill $\square$

Remark 5.4. (i) Let $K$ be the Sturm-Liouville hypergroup with $\rho > 0$. By the previous theorem, $L^1(K)$ is not weakly amenable as well as $\{\varphi_{\lambda}\}, \lambda \neq i\rho$, is not a spectral set. Theorem 5.5 will show that the maximal ideals associated to the spectral sets do not have b.a.i. necessarily.

(ii) A Sturm-Liouville hypergroup is of exponential growth if and only if $\rho > 0$ [3, Proposition 3.5.55]. Then by Theorem 5.3 such hypergroups are $\varphi_{\lambda}$-amenable if and only if $\lambda = i\rho$. However, for $\rho = 0$ we do not have a certain assertion.

We now study special cases of Sturm-Liouville hypergroups in more details.

(i) **Bessel-Kingman hypergroup**

The Bessel-Kingman hypergroup is a Chébli-Trimèche hypergroup on $\mathbb{R}_0$ with $A(x) = x^{2\nu+1}$ when $\nu \geq -\frac{1}{2}$. The characters are given by

$$\alpha_{\lambda}^\nu(x) := 2^{\nu} \Gamma(\nu + 1) J_{\nu}(\lambda x)(\lambda x)^{-\nu},$$

where $J_{\nu}(x)$ is the Bessel function of order $\nu$, and $\lambda \in \mathbb{R}_0$ represents the characters. The dual space $\mathbb{R}_0$ has also a hypergroup structure and the bidual space coincides with the hypergroup $\mathbb{R}_0$ [3]. As shown in [26], the $L^1$-algebra of $(\mathbb{R}_0, dx)$, the Bessel-Kingman hypergroup of order $-\frac{1}{2}$, is amenable; as a result, $(\mathbb{R}_0, dx)$ is $\alpha_{\lambda}^\nu$-amenable for every $\lambda \in \mathbb{R}_0$.

Suppose $L^1_{\text{rad}}(\mathbb{R}^n)$ is the subspace of $L^1(\mathbb{R}^n)$ of radial functions and $\nu = \frac{n+2}{2}$. It is a closed self adjoint subalgebra of $L^1(\mathbb{R}^n)$ which is isometrically $*$-isomorphic to the hypergroup algebra $L^1(\mathbb{R}_0, dm_n)$, where $dm_n(r) = 2^{n/2} \Gamma(n/2) r^{n-1} dr$.

**Theorem 5.5.** Let $K$ be the Bessel-Kingman hypergroup of order $\nu \geq 0$. If $\nu = 0$ or $\nu \geq \frac{1}{2}$, then $K$ is $\alpha_{\lambda}^\nu$-amenable if and only if $\alpha_{\lambda}^\nu = 1$.

---

1This subsection is from parts of the author’s Ph.D. thesis at the Technical University of Munich.
Proof: (i) Let \( \nu = 0 \) and \( \alpha^0_\lambda \in \hat{K} \). Since \( K \) is commutative, \( K \) is \((1-)\)amenable [20]. Suppose now \( \alpha^0_\lambda \neq 1 \) and \( K \) is \( \alpha^0_\lambda \)-amenable, so \( I(\alpha^0_\lambda) \) has a b.a.i. If \( \ell_\nu(\alpha^0_\lambda) \) is the corresponding ideal to \( I(\alpha^0_\lambda) \) in \( L^1_{rad}(\mathbb{R}^2) \), then \( \ell_\nu(\alpha^0_\lambda) \) has a b.a.i., say \( \{e'_i\} \). Let \( I := [\ell_\nu(\alpha^0_\lambda) * L^1(\mathbb{R}^2)]^{cl} \). The group \( \mathbb{R}^2 \) is amenable [17], so let \( \{e_i\} \) be a b.a.i for \( L^1(\mathbb{R}^2) \). For every \( f \in \ell_\nu(\alpha^0_\lambda) \) and \( g \in L^1(\mathbb{R}^2) \), we have

\[
\|f * g - (f * g) * (e'_i * e_i)\|_1 \leq \|g\|_1 \|f * e'_i - f\|_1 + \|f * e'_i\|_1 \|g - g * e_i\|_1.
\]

The latter shows that \( \{e'_i * e_i\} \) is a b.a.i for the closed ideal \( I \). In [18, Theorem 17.2], it is shown that \( Co(I) \), cospectrum of \( I \), is a finite union of lines and points in \( \mathbb{R}^2 \). But this contradicts the fact that \( Co(I) \) is a circle with radius \( \lambda \) in \( \mathbb{R}^2 \); hence, \( K \) is \( \alpha^0_\lambda \)-amenable if and only if \( \alpha^0_\lambda = 1 \).

(ii) Following [19], \( \hat{f}(\lambda) = \int_0^\infty f(x)\alpha^\nu_\lambda(x)dm(x) \) is differentiable, for all \( f \in L^1(K), \nu \geq \frac{1}{2}, \) and \( \lambda \neq 0 \). Since \( \frac{d}{dx} (x^{-\nu} J_\nu(x)) = -x^\nu J_{\nu+1}(x) \) and \( J_\nu(x) = O(x^{-1/2}) \) as \( x \to \infty \) [1], there exists a constant \( A_\nu(\lambda_0) > 0 \) such that \( \left| \frac{d}{d\lambda} \hat{f}(\lambda) \right|_{\lambda=\lambda_0} \leq A_\nu(\lambda_0) \|f\|_1 \) (\( \lambda_0 \neq 0 \)). Hence, the mapping

\[
D_{\lambda_0} : L^1(\mathbb{R}^n_0, x^{2\nu+1}dx) \longrightarrow \mathbb{C}, \quad D_{\lambda_0}(f) = \frac{d}{d\lambda} \hat{f}(\lambda) \bigg|_{\lambda=\lambda_0},
\]

is a well-defined bounded nonzero \( \alpha_\nu \)-derivation. Hence, Lemma 5.2 implies that \( K \) is \( \alpha_\nu \)-amenable if and only if \( \alpha_\nu = 1 \).

\[\square\]

(ii) Jacobi hypergroup of noncompact type

The Jacobi hypergroup of noncompact type is a Chébli-Trimèche hypergroup with

\[
A^{(\alpha, \beta)}(x) := 2^{\nu} \sinh^{2\alpha+1}(x) \cosh^{2\beta+1}(x),
\]

where \( \rho = \alpha + \beta + 1 \) and \( \alpha \geq \beta \geq -\frac{1}{2} \). The characters are given by Jacobi functions of order \( (\alpha, \beta) \), \( \varphi^{(\alpha, \beta)}(\lambda) := 2 \binom{\frac{1}{2} (\rho + i\lambda)}{\frac{1}{2} (\rho - i\lambda)} \binom{1}{2} (\nu - i\lambda); \alpha + 1; -\sinh^2 t) \), where \( _2F_1 \) denotes the Gaussian hypergeometric function, \( \alpha \geq \beta \geq -\frac{1}{2}, t \in \mathbb{R}_0 \) and \( \lambda \) is the parameter of character \( \varphi^{(\alpha, \beta)}(\lambda) \) which varies on \( \mathbb{R}_0 \cup [0, \rho] \). It is straightforward to show that \( \rho = \alpha + \beta + 1 \). As we have seen, if \( \rho > 0 \), then \( \mathbb{R}_0 \) is \( \varphi^{(\alpha, \beta)} \)-amenable if and only if \( \varphi^{(\alpha, \beta)}(1) = 1 \). If \( \rho = 0 \), then \( \alpha = \beta = -\frac{1}{2} \), hence \( \varphi^{(-\frac{1}{2}, -\frac{1}{2})}(t) = \cos(\lambda t) \) so that turns \( \mathbb{R}_0 \) to the Bessel-Kingman hypergroup of order \( \nu = -\frac{1}{2} \), which is \( \varphi^{(\alpha, \beta)} \)-amenable for every \( \lambda \).

Let \( A(\alpha, \beta) := \{ \varphi : \varphi \in L^1(\mathbb{R}_0, \pi) \} \). By the inverse theorem, [3, Theorem 2.2.32], we have \( A(\alpha, \beta) \subseteq C_0([0, \infty)) \). There exists a convolution structure on \( \mathbb{R}_0 \) such that \( \pi \) is the Haar measure on \( \mathbb{R}_0 \) and \( A(\alpha, \beta) \) is a Banach algebra of functions on \( [0, \infty) \); see [9]. The following estimation with Lemma 5.2 will indicate that for \( \alpha \geq \frac{1}{2} \) and \( \alpha \geq \beta \geq -\frac{1}{2} \), the maximal ideals of \( A(\alpha, \beta) \) related to the points in \( (0, \infty) \) do not have b.a.i.

Theorem 5.6. [15] For \( \alpha \geq \frac{1}{2}, \alpha \geq \beta \geq -\frac{1}{2} \) and \( \varepsilon > 0 \) there is a constant \( k > 0 \) such that if \( f \in A(\alpha, \beta) \) then \( f|_{[\varepsilon, \infty)} \in C^{[\alpha+1/2]}([\varepsilon, \infty)) \) and

\[
\sup_{t \geq \varepsilon} |f^{(j)}(t)| \leq k \|f\|_{(\alpha, \beta)}, \quad 0 \leq j \leq [\alpha + 1/2].
\]
Remark 5.7. Here are the special cases of Jacobi hypergroups of noncompact type which are 1-amenable only:

(i) Hyperbolic hypergroups, if $\beta = -\frac{1}{2}$ and $\rho = \alpha + \frac{1}{2} > 0$.

(ii) Naimark hypergroup, if $\beta = -\frac{1}{2}$ and $\alpha = \frac{1}{2}$.

(iii) **Square hypergroup**

The Square hypergroup is a Sturm-Liouville hypergroup on $\mathbb{R}_0$ with $A(x) = (1 + x)^2$ for all $x \in \mathbb{R}_0$. Obviously we have $\rho = 0$ and the characters are given by

$$
\varphi_\lambda(x) := \begin{cases} 
\frac{1}{1+x} \left( \cos(\lambda x) + \frac{1}{\lambda} \sin(\lambda x) \right) & \text{if } \lambda \neq 0 \\
1 & \text{if } \lambda = 0.
\end{cases}
$$

If $\lambda \neq 0$, then

$$
\frac{d}{d\lambda} \varphi_\lambda(x) = \frac{x}{1+x} \left( -\sin(\lambda x) + \frac{\cos(\lambda x)}{\lambda} - \frac{1}{x\lambda^2} \sin(\lambda x) \right),
$$

which is bounded as $x$ varies. Therefore, the mapping

$$
D_{\lambda_0} : L^1(\mathbb{R}_0, Adx) \longrightarrow \mathbb{C}, \quad D_{\lambda_0}(f) := \frac{d}{d\lambda} \hat{f}(\lambda) \bigg|_{\lambda = \lambda_0},
$$

is a well-defined bounded nonzero $\varphi_\lambda$-derivation on $L^1(\mathbb{R}_0, Adx)$. Applying Lemma 5.2 results that $\mathbb{R}_0$ is 1-amenable only.

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