1. Introduction

In 1979, Wintgen [21] proved a basic relation between the intrinsic Gauss curvature $K$, the extrinsic normal curvature $K_N$ and a squared mean curvature $\|H\|^2$ of any surface $M$ in Euclidean 4-space $\mathbb{E}^4$, namely

$$K + K_N \leq \|H\|^2$$

with the equality holding if and only if the curvature ellipse is a circle ([3], [11]). Following Verstraelen et. al. [7], [17], a surface $M$ in $\mathbb{E}^4$ is called Wintgen ideal if it satisfies the equality case of Wintgen inequality identically. Obviously Wintgen ideal surfaces in $\mathbb{E}^4$ are exactly superminimal [10] or superconformal [8] surfaces. In [2] B.Y. Chen completely classified Wintgen ideal surfaces in $\mathbb{E}^4$ with equal Gauss and normal curvatures.

Wintgen’s inequality is extended to surfaces in real space form by I.V. Guadalupe and L. Rodriguez [11]. However in [3] the authors make progress in the case of submanifolds in $(n + 2)$-dimensional real space form. In the same paper they conjectured that the above pointwise inequality is valid for higher dimensional cases (see also [9], [13] and [14] for some works on this pointwise inequality).

Recently, G. Jianguan and T. Zizhou give a proof of DDVV conjecture on a submanifold of a real space form. Furthermore they solved the problem of its equality case.

Let $M$ a submanifold of a $(n + d)$-dimensonal Euclidean space $\mathbb{E}^{n+d}$. Denote by $\overline{R}$ the curvature tensor of the Vander Waerden-Bortoletti connection $\overline{\nabla}$ of $M$ and $h$ is the second fundamental form of $M$ in $\mathbb{E}^{n+d}$. The submanifold $M$ is called semiparallel (or semi-symmetric [15]) if $\overline{R} \cdot h = 0$ [4]. This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which $R \cdot R = 0$ and a direct generalization of parallel submanifolds, i.e. submanifolds for which $\overline{\nabla}h = 0$. 

\textbf{Key words and phrases.} Normal curvature, Wintgen ideal surface, Superconformal surface, Semiparallel surface.

This paper is supported by Uludağ University Research found with Project No: KUAP(F)-2012/59.
In [4] J. Deprez showed the fact that the submanifold $M \subset \mathbb{E}^{n+d}$ is semi-parallel implies that $(M, g)$ is semi-symmetric. For references on semi-symmetric spaces, see [13]; for references on parallel immersions, see [9]. In [4] J. Deprez gave a local classification of semi-parallel hypersurfaces in Euclidean space. It is easily seen that all surfaces are semi-parallel.

In the present study we consider the Wintgen ideal surfaces in $n$-dimensional Euclidean space $\mathbb{E}^n$. We have shown that Wintgen ideal surfaces in $\mathbb{E}^n$ satisfying the semiparallelity condition $R(X, Y) \cdot h = 0$ are totally umbilical. Further, we obtain some results in $\mathbb{E}^4$.

2. Basic Concepts

Let $M$ be a smooth surface in $n$-dimensional Euclidean space $\mathbb{E}^n$ given with the surface patch $X(u, v) : (u, v) \in D \subset \mathbb{E}^2$. The tangent space to $M$ at an arbitrary point $p = X(u, v)$ of $M$ span $\{X_u, X_v\}$. In the chart $(u, v)$ the coefficients of the first fundamental form of $M$ are given by

$$E = \langle X_u, X_u \rangle, \ F = \langle X_u, X_v \rangle, \ G = \langle X_v, X_v \rangle,$$

where $\langle, \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u, v)$ is regular. For each $p \in M$, consider the decomposition $T_p \mathbb{E}^n = T_p M \oplus T^p_M$ where $T^p_M$ is the orthogonal component of the tangent plane $T_p M$ in $\mathbb{E}^n$, that is the normal space of $M$ at $p$.

Let $\chi(M)$ and $\chi^\perp(M)$ be the space of the smooth vector fields tangent and normal to $M$ respectively. Denote by $\nabla$ and $\nabla^\perp$ the Levi-Civita connections on $M$ and $\mathbb{E}^n$, respectively. Given any vector fields $X_i$ and $X_j$ tangent to $M$ consider the second fundamental map $h : \chi(M) \times \chi(M) \rightarrow \chi^\perp(M)$:

$$h(X_i, X_j) = \nabla^\perp X_i, X_j - \nabla X_i, X_j; \ 1 \leq i, j \leq 2,$$

where $\nabla^\perp$ is the induced. This map is well-defined, symmetric and bilinear.

For any normal vector field $N_\alpha, 1 \leq \alpha \leq n - 2$ of $M$, recall the shape operator $A : \chi^\perp(M) \times \chi(M) \rightarrow \chi^\perp(M)$:

$$A_{N_\alpha}X_i = -\nabla^\perp_{N_\alpha}X_i + D_{X_i}N_\alpha; \ 1 \leq i \leq 2,$$

where $D$ denotes the normal connection of $M$ in $\mathbb{E}^n$. This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_\alpha}X_i, X_j \rangle = \langle h(X_i, X_j), N_\alpha \rangle, \ 1 \leq i, j \leq 2.$$

The equation $2.2$ is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{\alpha=1}^{n-2} h^\alpha_{ij} N_\alpha, \ 1 \leq i, j \leq 2$$

where $h^\alpha_{ij}$ are the coefficients of the second fundamental form $h$. If $h = 0$ then $M$ is called totally geodesic. $M$ is totally umbilical if all shape operators are proportional to the identity map. $M$ is an isotropic surface if for each $p$ in $M$, $\|h(X, X)\|$ is independent of the choice of a unit vector $X$ in $T_p M$.

If we define a covariant differentiation $\nabla h$ of the second fundamental form $h$ on the direct sum of the tangent bundle and normal bundle $TM \oplus T^\perp M$ of $M$ by

$$\langle \nabla X_i h \rangle (X_j, X_k) = D_{X_i}h(X_j, X_k) - h(\nabla_{X_i}X_j, X_k) - h(X_j, \nabla_{X_i}X_k)$$
for any vector fields $X_i, X_j, X_k$ tangent to $M$. Then we have the Codazzi equation

$$ (\nabla_{X_i} h)(X_j, X_k) = (\nabla_{X_j} h)(X_i, X_k) $$

where $\nabla$ is called the Vanders Waerden-Bortoletti connection of $M$.

We denote $R$ and $\mathbf{R}$ the curvature tensors associated with $\nabla$ and $D$ respectively;

$$ R(X_i, X_j)X_k = \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k - \nabla_{[X_i, X_j]} X_k, $$

$$ R^+(X_i, X_j)N_\alpha = h(X_i, A_{N_\alpha} X_j) - h(X_j, A_{N_\alpha} X_i). $$

The equation of Gauss and Ricci are given respectively by

$$ \langle R(X_i, X_j)X_k, X_l \rangle = \langle h(X_i, X_l), h(X_j, X_k) \rangle - \langle h(X_i, X_k), h(X_j, X_l) \rangle, $$

$$ \langle R^+(X_i, X_j)N_\alpha, N_\beta \rangle = \langle [A_{N_\alpha}, A_{N_\beta}] X_i, X_j \rangle $$

for the vector fields $X_i, X_j, X_k$ tangent to $M$ and $N_\alpha, N_\beta$ normal to $M$.

Let us $X_i \wedge X_j$ denote the endomorphism $X_k \mapsto \langle X_j, X_k \rangle X_i - \langle X_i, X_k \rangle X_j$. Then the curvature tensor $R$ of $M$ is given by the equation

$$ R(X_i, X_j)X_k = \sum_{\alpha=1}^{n-2} (A_{N_\alpha} X_i \wedge A_{N_\alpha} X_j) X_k. $$

It is easy to show that

$$ R(X_i, X_j)X_k = K (X_i \wedge X_j) X_k, $$

where $K$ is the Gaussian curvature of $M$ defined by

$$ K = \langle h(X_1, X_1), h(X_2, X_2) \rangle - \|h(X_1, X_2)\|^2 $$

(see [1]).

The normal curvature $K_N$ of $M$ is defined by (see [3])

$$ K_N = \left\{ \sum_{1=\alpha<\beta}^{n-2} \langle R^+(X_1, X_2)N_\alpha, N_\beta \rangle \right\}^{1/2}. $$

We observe that the normal connection $D$ of $M$ is flat if and only if $K_N = 0$, and by a result of Cartan, this equivalent to the diagonalisability of all shape operators $A_{N_\alpha}$ of $M$, which means that $M$ is a totally umbilical surface in $\mathbb{E}^n$.

Further, the mean curvature vector $\mathbf{H}$ of $M$ is defined by

$$ \mathbf{H} = \frac{1}{2} \sum_{\alpha=1}^{n-2} tr(A_{N_\alpha}) N_\alpha. $$

3. Wintgen Ideal Surfaces

For the surface in Euclidean 3-space $\mathbb{E}^3$, the Euler inequality $K \leq \|\mathbf{H}\|^2$ holds. Obviously, $K = \|\mathbf{H}\|^2$ everywhere on $M$ if and only if the surface $M$ is totally umbilical in $\mathbb{E}^3$. So by theorem of Meusnier, $M$ is totally umbilical if and only if $M$ is part of a plane or a round sphere $S^2$ in $\mathbb{E}^3$. In 1979, Wintgen proved a basic relation between the Gauss curvature $K$, the normal curvature $K_N$ and a squared mean curvature $\|\mathbf{H}\|^2$ of any surface $M$ in Euclidean 4-space $\mathbb{E}^4$, namely

$$ K + K_N \leq \|\mathbf{H}\|^2 $$

(3.1)
with the equality holding if and only if the curvature ellipse of $M$ is a circle $[11]$. A surface $M$ in $\mathbb{E}^4$ is called Wintgen ideal if it satisfies the equality case of Wintgen inequality identically $[3,1]$ (see, $[7,17]$).

In $[2]$ B.Y. Chen gave the following result.

**Theorem 3.1.** $[2]$ Let $M$ be a smooth surface in Euclidean 4-space $\mathbb{E}^4$. Then the Wintgen inequality $[3,1]$ holds at every point in $M$. Moreover,

i) If $K_N \geq 0$ holds at a point $p \in M$, then the equality $[3,1]$ holds at $p$ if and only if, with respect to some suitable orthonormal frame $\{X_1, X_2, N_1, N_2\}$ at $p$, the shape operator at $p$ satisfies

$$A_{N_1} = \begin{pmatrix} \lambda_1 + \mu & 0 \\ 0 & \lambda_1 - \mu \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix},$$

ii) If $K_N < 0$ holds at a point $p \in M$, then the equality $[3,1]$ holds at $p$ if and only if, the shape operator at $p$ satisfies

$$A_{N_1} = \begin{pmatrix} \lambda_1 - 2\mu & 0 \\ 0 & \lambda_1 \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix}.$$  

Wintgen’s inequality is extended to surfaces in real space form by I.V. Guadalupe and L. Rodriguez $[13]$. However in $[3]$ the authors make progress in the case of submanifolds in $(n + 2)$-dimensional real space form.

In the same paper they conjectured that (DDVV conjecture) the pointwise inequality $[3,1]$ is valid for higher dimensional cases (see also $[6,13]$ and $[14]$).

Recently, G. Jianguan and T. Zizhou give a proof of DDVV conjecture on a submanifold of a real space form. Furthermore they solved the problem of its equality case;

**Theorem 3.2.** $[12]$ Let $M \subset \mathbb{E}^n$ a smooth surface given with the patch $X(u,v)$. Then the Wintgen inequality $[3,1]$ holds at every point in $M$. Moreover, the equalities holds at some point $p \in M$ if and only if there exits an orthonormal basis $\{X, Y\}$ of $T_p M$ and orthonormal basis $\{N_1, N_2, ..., N_{n-2}\}$ of $T_p^\perp M$, such that ($r > 3$);

$$A_{N_1} = \begin{pmatrix} h_{11} & 0 \\ 0 & h_{22} \end{pmatrix}, \quad A_{N_2} = \begin{pmatrix} \lambda_2 & \mu \\ \mu & \lambda_2 \end{pmatrix}, \quad A_{N_r} = \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_r \end{pmatrix}, \quad r \geq 3$$

where $h_{11} = \lambda_1 + \mu, h_{22} = \lambda_1 - \mu$ or $h_{11} = \lambda_1 - 2\mu, h_{22} = \lambda_1$. For the first case (resp. second case) $M$ is called a Wintgen ideal surface of first kind (resp. second kind).

4. Semi-parallel Surfaces

Let $M$ a smooth surface in $n$-dimensional Euclidean space $\mathbb{E}^n$. Let $\nabla$ be the connection of Vander Waerden-Bortoletti of $M$. Denote the tensors $\nabla$ by $\nabla$. Then the product tensor $\nabla \cdot h$ of the curvature tensor $\nabla$ with the second fundamental form $h$ is defined by

$$(\nabla(X_i, X_j) \cdot h)(X_k, X_l) = \nabla_{X_i} (\nabla_{X_j} h(X_k, X_l)) - \nabla_{X_j} (\nabla_{X_i} h(X_k, X_l)) - \nabla_{[X_i, X_j]} h(X_k, X_l)$$

for all $X_i, X_j, X_k, X_l$ tangent to $M$.  

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The surface \( M \) is said to be semi-parallel if \( \mathbf{R} \cdot h = 0 \), i.e. \( \mathbf{R}(X_i, X_j) \cdot h = 0 \) \([4], [5], [16]\). It is easy to see that the following equalities are hold:

\[
\begin{equation}
\mathbf{R}(X_i, X_j) \cdot h)(X_k, X_l) = R^+(X_i, X_j)h(X_k, X_l) - h(R(X_i, X_j)X_k, X_l) - h(X_k, R(X_i, X_j)X_l),
\end{equation}
\]

This notion is an extrinsic analogue for semi-symmetric spaces, i.e. Riemannian manifolds for which \( R \cdot R = 0 \) and a generalization of parallel surfaces, i.e. \( \nabla h = 0 \) \([9]\).

First, we proved the following result.

**Lemma 4.1.** Let \( M \subset \mathbb{E}^n \) a smooth surface given with the patch \( X(u, v) \). Then the following equalities are hold:

\[
\begin{align}
\mathbf{R}(X_1, X_2) \cdot h)(X_1, X_1) &= \left( \sum_{\alpha=1}^{n-2} h_1^{\alpha}(h_2^{\alpha} - h_1^{\alpha}) + 2K \right) h(X_1, X_2)  \\
&\quad + \sum_{\alpha=1}^{n-2} h_1^{\alpha}h_1^{\alpha}(h(X_1, X_1) - h(X_2, X_2)) \\
\mathbf{R}(X_1, X_2) \cdot h)(X_2, X_2) &= \left( \sum_{\alpha=1}^{n-2} h_2^{\alpha}(h_2^{\alpha} - h_1^{\alpha}) \right) h(X_2, X_2)  \\
&\quad + \sum_{\alpha=1}^{n-2} h_2^{\alpha}h_1^{\alpha}(h(X_1, X_1) - h(X_2, X_2))
\end{align}
\]

**Proof.** Substituting \([2.8] \) and \([2.4] \) into \([2.3] \) we get

\[
\begin{equation}
R^+(X_1, X_2)N_\alpha = h_2^{\alpha}(h(X_1, X_1) - h(X_2, X_2)) + (h_2^{\alpha} - h_1^{\alpha})h(X_1, X_2).
\end{equation}
\]

Further, by the use of \([2.13] \) we get

\[
\begin{align}
R(X_1, X_2)X_1 &= -KX_2  \\
R(X_1, X_2)X_2 &= KX_1.
\end{align}
\]

So, substituting \([1.3] \) and \([1.4] \) into \([1.1] \) we get the result. \( \Box \)

Semi-parallel surfaces in \( \mathbb{E}^n \) are classified by J. Deprez \([4]\):

**Theorem 4.1.** \([4]\) Let \( M \) a surface in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \). Then \( M \) is semi-parallel if and only if locally:

i) \( M \) is equivalent to a 2-sphere, or

ii) \( M \) has trivial normal connection, or

iii) \( M \) is an isotropic surface in \( \mathbb{E}^5 \subset \mathbb{E}^n \) satisfying \( \|H\|^2 = 3K \).

We get the following result.

**Theorem 4.2.** Let \( M \) a Wintgen ideal surface in \( \mathbb{E}^n \). If \( M \) is semi-parallel then it is a totally umbilical surface in \( \mathbb{E}^n \).
Proof. Let $M$ be a Wintgen ideal surface in $\mathbb{E}^n$ given with the patch $X(u,v)$. Then by Theorem 3.2 we get

$$
(4.5) \quad h(X_1, X_2) = \mu N_2, \quad h(X_1, X_1) - h(X_2, X_2) = (h_{11}^1 - h_{22}^1)N_1
$$

where $h_{11}^1 = \lambda_1 + \mu$, $h_{22}^1 = \lambda_1 - \mu$ or $h_{11}^1 = \lambda_1 - 2\mu$, $h_{22}^1 = \lambda_1$. Further, substituting (4.5) into (4.2) and using Lemma 4.1 one can get

$$
(4.6) \quad (\mathcal{R}(X_1, X_2) \cdot h)(X_1, X_1) = \left( \sum_{\alpha=1}^{n-2} \sum_{\alpha=1}^{d} h_{11}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha}) + 2K \right) \mu N_2
$$

$$
+ \left( \sum_{\alpha=1}^{n-2} h_{11}^{\alpha} h_{12}^{\alpha} \right) (h_{11}^1 - h_{22}^1)N_1
$$

$$
(4.7) \quad (\mathcal{R}(X_1, X_2) \cdot h)(X_1, X_2) = \left( \sum_{\alpha=1}^{n-2} h_{22}^{\alpha}(h_{22}^{\alpha} - h_{11}^{\alpha}) - 2K \right) \mu N_2
$$

$$
+ \sum_{\alpha=1}^{n-2} h_{22}^{\alpha} h_{12}^{\alpha}(h_{11}^1 - h_{22}^1)N_1
$$

Since $h_{12}^2 = \mu$, $h_{12}^2 = 0$, $\alpha \neq 2$, then the equation (4.6) becomes

$$
(\mathcal{R}(X_1, X_2) \cdot h)(X_1, X_1) = h_{11}^1 \mu(h_{11}^1 - h_{22}^1)N_1
$$

$$
+ [h_{11}^1(h_{22}^1 - h_{11}^1) + 2K] \mu N_2,
$$

$$
(\mathcal{R}(X_1, X_2) \cdot h)(X_1, X_2) = (\mu^2 - K)(h_{11}^1 - h_{22}^1)N_1
$$

$$
+ (h_{22}^1 - h_{11}^1) \mu^2 N_2,
$$

$$
(\mathcal{R}(X_1, X_2) \cdot h)(X_2, X_2) = h_{22}^1 \mu(h_{11}^1 - h_{22}^1)N_1
$$

$$
+ [h_{22}^1(h_{22}^1 - h_{11}^1) - 2K] \mu N_2.
$$

Suppose that, $M$ is semi-parallel then by definition $(\mathcal{R}(X_1, X_2) \cdot h)(X_i, X_j) = 0$, $(1 \leq i, j \leq 2)$. So, we get

$$
\begin{align*}
\mu^2(h_{11}^1 - h_{22}^1) &= 0, \\
\lambda_2 \mu(h_{11}^1 - h_{22}^1) &= 0, \\
(\mu^2 - K)(h_{11}^1 - h_{22}^1) &= 0, \\
\mu[h_{11}^1(h_{22}^1 - h_{11}^1) + 2K] &= 0, \\
\mu[h_{22}^1(h_{22}^1 - h_{11}^1) - 2K] &= 0.
\end{align*}
$$

where $h_{11}^1 = \lambda_1 + \mu$, $h_{22}^1 = \lambda_1 - \mu$ or $h_{11}^1 = \lambda_1 - 2\mu$, $h_{22}^1 = \lambda_1$. Now, we study the case $h_{11}^1 = h_{22}^1$. Then $\mu = 0$ by (4.8). This means that $R^\perp = 0$ by (4.3) and (4.5). This is equivalent to say that $M$ has vanishing normal curvature $K_N$, which means that $M$ is a totally umbilical surface in $\mathbb{E}^m$. □
5. Some Results in $E^4$

Rotation surfaces were studied in [19] by Vranceanu as surfaces in $E^4$ which are defined by the following parametrization:

\[ X(u, v) = (r(v) \cos v \cos u, r(v) \cos v \sin u, r(v) \sin v \cos u, r(v) \sin v \sin u) \]

where $r(v)$ is a real valued non-zero function.

We choose a moving frame $\{X_1, X_2, N_1, N_2\}$ such that $X_1, X_2$ are tangent to $M$ and $N_1, N_2$ are normal to $M$ as given the following (see [20]):

\[ X_1 = \frac{\partial}{r(v) \partial u} = (- \cos v \sin u, \cos v \cos u, - \sin v \sin u, \sin v \cos u), \]
\[ X_2 = \frac{\partial}{\partial v} = \frac{1}{A} (B(v) \cos u, B(v) \sin u, C(v) \cos u, C(v) \sin u), \]
\[ N_1 = \frac{1}{A} (-C(v) \cos u, -C(v) \sin u, B(v) \cos u, B(v) \sin u), \]
\[ N_2 = (- \sin v \sin u, \sin v \cos u, \cos v \sin u, - \cos v \cos u) \]

where

\[ A(v) = \sqrt{r^2(v) + (r')^2(v)}, \]
\[ B(v) = r'(v) \cos v - r(v) \sin v, \]
\[ C(v) = r'(v) \sin v + r(v) \cos v. \]

Furthermore, by covariant differentiation with respect to $X_1$ and $X_2$ a straightforward calculation gives:

\[ \nabla_{X_1} X_1 = -a(v)k(v)X_2 + a(v)N_1, \]
\[ \nabla_{X_1} X_2 = b(v)N_1, \]
\[ \nabla_{X_2} X_1 = -a(v)N_2, \]

where

\[ k(v) = \frac{r'(v)}{r(v)}, \]
\[ a(v) = \frac{1}{\sqrt{r^2(v) + (r')^2(v)}}, \]
\[ b(v) = \frac{2(r'(v))^2 - r(v)r''(v) + r^2(v)}{(r^2(v) + (r')^2(v))^{3/2}} \]
are differentiable functions.

Thus by the use of (2.9) together with (2.14) and (2.15) we get the following result.

**Proposition 5.1.** Let $M$ a Vranceanu surface given with the surface patch (5.1). Then the Gaussian curvature $K$ coincides with the normal curvature $K_N$ of $M$. That is:

\[ K = K_N = a(v)b(v) - a^2(v). \]
Theorem 5.1. Let $M$ a Vranceanu surface given with the surface patch (5.1). Then, $M$ is Wintgen ideal surface of first kind if and only if

\[(5.5) \quad r(v) = \pm \sqrt{c_1 \cos(2v) - c_2 \sin(2v)}, \quad c_1, c_2 \in \mathbb{R}\]

holds.

Proof. Suppose that the Vranceanu surface $M$ is given with the surface patch (5.1). If $M$ is a Wintgen ideal surface of first kind then by (3.4)

\[(5.6) \quad a(v) = \lambda_1 + \mu, b(v) = \lambda_1 - \mu, \mu = -a(v)\]

holds. Further, from (5.3) and (5.6) we get

\[(r'(v))^2 + r(v)r''(v) + 2r^2(v) = 0\]

which has a nontrivial solution (5.5). Conversely, if the equality (5.5) holds then the Vranceanu surface becomes Wintgen ideal surface of first kind.

\[\square\]

Theorem 5.2. Let $M$ be a Vranceanu surface given with the surface patch (5.1). Then, $M$ is Wintgen ideal surface of second kind if and only if $M$ is a minimal surface satisfying

\[(5.7) \quad r(v) = \pm \sqrt{1 \over \sqrt{c_1 \sin(2v) - c_2 \cos(2v)}}\]

where $c_1$ and $c_2$ are real constants.

Proof. Suppose that the Vranceanu surface $M$ is given with the surface patch (5.1) is Wintgen ideal surface of first kind then by the use of (3.4) the following equalities hold;

\[(5.8) \quad a(v) = \lambda_1 - 2\mu, b(v) = \lambda_1, \quad \mu = -a(v)\]

Further, from (5.3) and (5.8) we get

\[(5.9) \quad 3(r'(v))^2 - r(v)r''(v) + 2r^2(v) = 0\]

which has a nontrivial solution (5.7). Conversely, if the equality (5.7) holds then the Vranceanu surface becomes Wintgen ideal surface of second kind.

\[\square\]

Corollary 5.1. Let $M$ a Vranceanu surface given with the surface patch (5.1). If $M$ is semi-parallel then $M$ is a flat surface satisfying $r(v) = c_1 e^{c_2 v}$.

Proof. Suppose the Vranceanu surface $M$ is semi-parallel then by the use of (4.1) with (5.2) we get

\[(\vec{R}(X_1, X_2) \cdot h)(X_1, X_1) = (3a^2(v)(a(v) - b(v))) N_2\]

\[(\vec{R}(X_1, X_2) \cdot h)(X_1, X_2) = (a(v)(a(v) - b(v))(2a(v) - b(v))) N_1\]

\[(\vec{R}(X_1, X_2) \cdot h)(X_2, X_2) = a(v)(3a(v)b(v) - 2a(v)^2 - b(v)^2) N_2.\]

Suppose that, $M$ is semi-parallel then by (4.1) \((\vec{R}(X_1, X_2) \cdot h)(X_i, X_j) = 0, (1 \leq i, j \leq 2)\). Which implies that $a(v) - b(v) = 0$. So, by (5.4) $K = K_N = 0$. Further, from (5.3) we get the result.

\[\square\]
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