CLUSTER ALGEBRAS AND THEIR BASES

FAN QIN

ABSTRACT. We give a brief introduction to (upper) cluster algebras and their quantization using examples. Then we present several important families of bases for these algebras using topological models. We also discuss tropical properties of these bases and their relation to representation theory.

This article is an extended version of the talk given at the 19th International Conference on Representations of Algebras (ICRA 2020).

1. INTRODUCTION

1.1. Cluster algebras and the dual canonical basis. [FZ02] invented cluster algebras as an algebraic framework to study the dual canonical basis $B^*$ [Lus90, Lus91][Kas90] and total positivity [Lus94]. These are commutative algebras with distinguished generators called cluster variables, which are grouped into overlapping sets called clusters. The monomials of cluster variables from the same cluster are called the cluster monomials. See the first example in Section 2.1.

[FZ02] expected that the coordinate rings of many interesting varieties arising from a simply-connected semisimple group $G$ have a natural structure of a cluster algebra. Moreover, they should possess an analogue of the original dual canonical bases, and these bases contain all cluster monomials. Therefore, it is a natural and important question to construct well-behaved bases for cluster algebras which contain all cluster monomials.

[BZ05] introduced quantization for cluster algebras. As a main motivation of [FZ02], the expected relation between quantum cluster monomials and the original dual canonical basis can be formulated as below ([Kim12, Conjecture 1.1]).

Conjecture 1.1 (Quantization conjecture). Given any symmetrizable Kac-Moody algebra $\mathfrak{g}$ and any Weyl group element $w$, up to scalar multiples in $q^{\mathbb{Z}}$, the corresponding quantum unipotent subgroup $\mathbb{A}_q[N_-(w)]$ is a quantum cluster algebra and its dual canonical basis contains all quantum cluster monomials.

Here $N_-(w)$ is the unipotent subgroup and $\mathbb{A}_q[N_-(w)]$ the quantum analog of its ring of functions, see Sections 2.4 and 5.2 for more details.
Proof. The cluster structure on $\mathbb{A}_q[N_-(w)]$ was verified by [GLS11, GLS13] and [GY16, GY21]. The dual canonical basis part in Conjecture 1.1 was verified for acyclic quivers by [KQ14] (based on [HL10][Nak11]), for simply-laced semisimple Lie algebras and partially for symmetric Kac-Moody algebras by [Qin17], for all symmetric Kac-Moody algebras by [KKKO18], and for all cases by [Qin20b]. □

1.2. Well-behaved bases. The approach in [Qin17] [Qin20b] for Conjecture 1.1 is based on showing that the dual canonical basis provides the common triangular basis proposed in [Qin17], see Section 5 and Theorem 5.8. This approach heavily depends on the analysis of tropical properties of (upper) cluster algebras (Section 2.3 and 4).

More precisely, the geometric counterpart for a cluster algebra is called the cluster variety, which is constructed by gluing split tori corresponding to the clusters. Its ring of functions is called the upper cluster algebra. The cluster algebra is contained in the upper cluster algebra and they often agree. Any upper cluster algebra element can be expressed as a Laurent polynomial ring in the cluster variables from any chosen cluster, and its Laurent expansion depends on the choice of the cluster. By tropical properties, we mean how the Laurent degrees change when we change the cluster. See Section 4 for details.

It is expected by Fock and Goncharov that the upper cluster algebra possesses a basis with good tropical properties (i.e. parametrized by the tropical points, see Section 4.4), see Conjecture 4.14.

In the literature, for studying bases of cluster algebras, we usually need to add a condition called the full rank assumption (Assumption 1). Under this assumption, there are three families for an upper cluster algebra that are well-behaved, important and intensively-studied:

- The generic basis in the sense of [Dup11], see Section 6: For cluster algebras arising from quivers which satisfy the injective-reachable assumption (Assumption 3), the existence of the generic basis is proved in [Qin21]. By [GLS12, GLS11], this family of bases includes the dual semi-canonical basis for the coordinate ring $\mathbb{C}[N_-(w)]$ in the sense of Lusztig [Lus00]. [GLFS20a, GLFS20b] showed that this family includes the bangle basis for the Skein algebra of an unpunctured surface with marked points [MSW13], see Section 3.

- The common triangular basis in the sense of [Qin21]: This family of bases includes the dual canonical basis by [Qin20b] (Theorem 5.8). Moreover, it often includes the set of the isoclasses of simple modules when a monoidal categorification is given. See Section 5 for more details.

- The theta basis in the sense of [GHKK18]: Many cluster algebras possess this basis (for example, when the injective-reachable
assumption holds) [GHKK18]. It consists of the \textit{theta functions} constructed in the \textit{scattering diagrams}, which arise from the geometry of the cluster varieties. This family includes the greedy basis [LLZ14][CGM+17]. [MQ21] shows that it includes the bracelet basis for Skein algebras of surfaces with marked points [MSW13].

All three families of bases above are parametrized by the tropical points (i.e. meet the expectation of Fock and Goncharov). By [Qin21, Theorem 1.2.1], there are infinitely many bases with such properties, see Section 4.5. Nevertheless, sometimes as conjectures, all known good bases for cluster algebras in literature should belong to these three families. See Section 7 for a further discussion.

1.3. \textbf{Contents.} This article is an extended version of the author’s talk given at the 19th International Conference on Representations of Algebras (ICRA 2020). We try to give a concise introduction to cluster algebras and their bases, in particular for those arising from quantum groups [GLS11, GLS13][GY16, GY21] or from surfaces [FST08]. The constructions and results we present are mostly from [Qin17, Qin21, Qin20b][MSW13][Thu14].

There has been a wide range of literature on cluster algebras, which focuses on different problems and is based on different methods. We omit many important structures and results for cluster algebras. We refer the reader to [Kel] for a good survey covering many aspects of cluster algebras, to [HL10][KKKO18] for their monoidal categorification and to [GHK15][GHKK18] for cluster varieties and the theta basis.

In Section 2, we give a brief introduction to cluster algebras. We start by giving an example of a cluster algebra using unitriangular matrices in Section 2.1. This will be our first running example for cluster algebras with a Lie theoretic background. Then we give general definitions for cluster algebras, upper cluster algebras, and their quantization. In Section 2.4, we discuss the quantized version of the first example, which provides the first example for the quantum unipotent subgroup $A_q[N_{-}(w)]$, its localization $A_q[N_{w}]$, their dual canonical bases and their quantum cluster structure.

Section 3 is independent. By using topological models, it aims to provide intuition to readers unfamiliar with cluster algebras and their bases. We introduce Skein algebras for surfaces with marked points, which we assume to have no punctures for simplicity. These algebras are (often) cluster algebras. Following [Thu14], we present the three important families of bases for these algebras using curves on surfaces. We give our second running example on the annulus (Example 3.2).

In Section 4, we introduce the dominance orders for Laurent degrees of Laurent polynomials, define the tropical transformations, and give a simplified definition for the tropical points. Then we clarify what
we mean by good tropical properties (i.e. parametrized by the tropical points), and how these properties can be used to study bases.

In Section 5, we define the triangular basis and the common triangular basis. We relate these bases to the dual canonical basis for quantum groups and the simple modules in monoidal categories.

In Section 6, we give a brief introduction to the generic basis, which is constructed from quiver Grassmannians for quiver representations via the Caldero-Chapoton map. We discuss the coefficient-free version of our second running example on the annulus (Example 3.3).

In Section 7, we give some important and open questions concerning bases for cluster algebras.

2. A BRIEF INTRODUCTION TO CLUSTER ALGEBRAS

2.1. A first example for total positivity, cluster algebras and their bases. Choose the base ring to be $k = \mathbb{R}$. Let $G = SL_3(k)$ denote the group of the matrices with determinant 1. It has the following subgroup (unipotent radical):

$$N_- := \left\{ g = \begin{pmatrix} 1 & 0 & 0 \\ X_1 & 1 & 0 \\ X_2 & X'_1 & 1 \end{pmatrix}, \forall X_1, X'_1, X_2 \right\}$$

The ring of functions $k[N_-]$ is the polynomial ring $k[X_1, X'_1, X_2]$. The monomials in $X_1, X'_1, X_2$ form a $k$-basis for $k[N_-]$, which is the dual PBW basis for the corresponding quantum group evaluated at $q = 1$.

A matrix $g$ is said to be totally non-negative if all of its minors are non-negative. It has four interesting minors $X_1, X'_1, X_2, X_3 := \begin{vmatrix} X_1 & 1 \\ X_2 & X'_1 \end{vmatrix}$ subject to the following algebraic relation

$$X'_1 = X_1^{-1} \cdot (X_3 + X_2)$$

They provide examples of cluster variables and the corresponding exchange relation.

We group the four minors into two overlapping subsets called clusters $\{X_1, X_2, X_3\}, \{X'_1, X_2, X_3\}$. The monomials from each subset are called cluster monomials. In this example, the cluster monomials form a $k$-basis for $k[N_-]$, which is the dual canonical basis $B^\ast$ for the corresponding quantum group evaluated at $q = 1$.

We refer the reader to Fomin’s ICM talk [Fom10] for more details on total positivity and cluster algebras.

2.2. A general definition for cluster algebras. Choose a base ring $\mathbb{k}$, which will be $\mathbb{Z}$ for the classical case or $\mathbb{Z}[v^\pm]$ for the quantum case in this paper. Here, $v$ denotes a formal quantum parameter, and we denote $q = v^2$ and $v = q^{1/2}$. 


A seed. Let $I$ denote a finite set of vertices endowed with a partition into unfrozen vertices and frozen vertices $I = I_{uf} \cup I_f$. We further fix a collection of strictly positive integers $d_i$, $i \in I$. Denote the diagonal matrix $D = \text{diag}(d_i)_{i \in I_{uf}}$.

A seed $t$ consists of indeterminates $X_i$ for $i \in I$, and an $I \times I \mathbb{Z}$-matrix $(b_{ij})$. We further require that $(b_{ij})$ is skew-symmetric via $D$, i.e., $b_{ij}d_j = -b_{ji}d_i$, $\forall i, j \in I$.

Let $\tilde{B}$ denote the $I \times I_{uf}$-submatrix and $B$ the $I_{uf} \times I_{uf}$-submatrix. $(b_{ij})$ is called the $B$-matrix for $t$, $B$ its principal part, $X_i$ its cluster variables on the vertices $i$, and $\{X_i| i \in I\}$ its cluster. When $(b_{ij})$ is skew-symmetric, we also associate to $t$ a unique quiver $\tilde{Q}$ without loops or oriented 2-cycles (called an ice quiver), whose vertex set is $I$ and

$$b_{ij} = |\{\text{arrows from } i \text{ to } j\}| - |\{\text{arrows from } j \text{ to } i\}|.$$ 

Its full subquiver on $I_{uf}$ is denoted by $Q$, called the principal quiver.

We associate to $t$ a rank-$|I|$ lattice $M^\circ(t) = \mathbb{Z}^I$ with the unit vectors $f_i$, $i \in I$. Identify the corresponding group ring $k[M^\circ(t)] = \bigoplus_{m \in M^\circ(t)} kX^m$ with the Laurent polynomial ring $k[X_i^\pm | i \in I]$, such that $X_i^f = X_i$. We use $\cdot$ to denote the commutative product.

Throughout this paper, we make the following assumption unless otherwise specified.

**Assumption 1 (Full rank assumption).** We assume that the matrix $\tilde{B} = (b_{ij})_{i \in I, j \in I_{uf}}$ is of full-rank, i.e., its column vectors $\text{col}_k \tilde{B}$, $k \in I_{uf}$, are linearly independent.

Now consider the quantum case $k = \mathbb{Z}[v^\pm]$. Any skew-symmetric $\mathbb{Z}$-matrix $\Lambda = (\Lambda_{ij})_{i,j \in I}$ induces a skew-symmetric bilinear form $\Lambda$ on $\mathbb{Z}^I$ (and thus on $M^\circ(t)$) such that $\Lambda(m, m') := m^T\Lambda m'$, where $(\ )^T$ denote the matrix transpose. By [GSV03, GSV05], under the full rank assumption, we can choose $\Lambda$ such that it is compatible with $t$ in the following sense: for any $i \in I$, $k \in I_{uf}$,

$$\Lambda(f_i, \text{col}_k \tilde{B}) = -\delta_{ik}d_k'$$

for some strictly positive integers $d_k'$.

The seed $t$ endowed with a compatible bilinear form $\Lambda$ is often called a quantum seed.

For a quantum seed $t$, we further endow $k[M^\circ(t)]$ with the following twisted product $*$:

$$X^m * X^{m'} = v^{\Lambda(m, m')} X^{m+m'}.$$ 

Unless otherwise specified, we use the twisted product $*$ for the multiplication in algebras. The skew-field of fractions for $k[M^\circ(t)]$ is denoted by $\mathcal{F}(t)$. 


Mutations. We denote the function $[a]_+ = \max(a, 0)$ for any $a \in \mathbb{R}$, and $[(a_i)]_+ = ([a_i]_+)$ for any $\mathbb{R}$-vector $(a_i)$.

Let there be given a seed $t$. For any unfrozen vertex $k \in I_{uf}$, we can construct a new seed $t' = \mu_k t$ by an operation $\mu_k$ called the mutation in the direction $k$. More precisely, $t'$ consists of new cluster variables $X'_i$ and a matrix $(b'_{ij})$, such that

$$b'_{ij} = \begin{cases} -b_{ik} & i = k \text{ or } j = k; \\ b_{ij} + b_{ik} [b_{kj}]_+ + [-b_{ik}]_+ b_{kj} & \text{else} \end{cases}$$

and $X'_i$ are the following elements in the (skew-)field $\mathcal{F}(t)$:

$$X'_i = \begin{cases} X_i & i \neq k; \\ X_i^{-f_k + \sum_j [-b_{ik}]_+ f_j} + X_i^{-f_k + \sum_j b_{ik}} & i = k. \end{cases}$$

Equation (2.1) is usually called the exchange relation. Correspondingly, we identify two (skew-)fields $\mu_k : \mathcal{F}(t') \simeq \mathcal{F}(t)$, such that $\mu_k(X'_i)$ is given by the Laurent polynomial defined in (2.1).

For the quantum case, one can check that (see [BZ05]), $X'_i \ast X'_j = v^{2\Lambda'_{ij}} X'_i \ast X'_j$ for some $\mathbb{Z}$-matrix $\Lambda' := (\Lambda'_{ij})_{i, j \in I}$. Moreover, $t'$ becomes a quantum seed after equipping with the matrix $\Lambda'$.

As before, we associate to $t'$ a lattice $M^0(t') = \mathbb{Z}^I$ with unit vector $f'_i$. We further identify the group ring $\mathbb{K}[M^0(t')] = \bigoplus_{m' \in M^0(t')} \mathbb{K} \cdot (X')^{m'}$ with $\mathbb{K}[(X')^\pm | i \in I]$, such that $(X')^{f_i} = X'_i$. Note that $\Lambda'$ induces a skew-symmetric bilinear form $\Lambda'$ on $M^0(t')$ such that $\Lambda'(f'_i, f'_j) = \Lambda'_{ij}$.

One can check that $\mu_k(\mu_k t) = t$, see [FZ02][BZ05].

We use $\Delta^+_t$ to denote the set of seeds obtained from the initial seed $t$ by iterated mutations. Note that $\Delta^+_t = \Delta^+_{\mu_k t}$, which we will denote by $\Delta^+_t$ for simplicity.

Cluster algebras. Let there be given an initial seed $t_0 = ((X_i)_{i \in I}, (b_{ij}))$ and consider the set of seeds $\Delta^+ = \Delta^+_{t_0}$ obtained from $t_0$. The cluster algebra $\overline{A}$ is the $\mathbb{K}$-algebra generated by the cluster variables $X_i(t)$, $i \in I$, for every seed $t \in \Delta^+$.

The (localized) cluster algebra $\mathcal{A}$ is the localization of $\overline{A}$ at the frozen variables $X_j$, $j \in I_{uf}$. We denote the multiplicative group of frozen factors by $\mathcal{P} := \{X^m | m \in \mathbb{Z}^I\}$, called the coefficient ring.

**Theorem 2.1** (Laurent phenomenon [FZ02][BZ05]). For any $t \in \Delta^+$, $\mathcal{A}$ is contained in the Laurent polynomial ring $\mathbb{K}[M^0(t)]$.

**Example 2.2** ($\mathfrak{sl}_3$ example [Qin20b, Example 8.2.9]). Consider the first example in Section 2.1. The set of vertices is $I = \{1, 2, 3\}$ such that
\[ I_{uf} = \{1\} \text{ and } I_f = \{2, 3\}. \]

We choose the initial seed \( t \) which consists of the cluster \((X_1, X_2, X_3)\) and the matrix \((b_{ij}) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}\).

The mutation at the only unfrozen vertex 1 generates the new seed \( t' = \mu_1 t \), which consists of the cluster \((X'_1, X_2, X_3)\) and the matrix \((b'_{ij}) = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\). In this example, the set of all seeds is \( \Delta^+ = \{t, t'\} \).

For the classical case \( k = \mathbb{Z} \), it is straightforward to check that the exchange relation is \( X_1 \cdot X'_1 = X_3 + X_2 \). For the quantum case \( k = \mathbb{Z}[v^\pm] \), we can choose \( \Lambda = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\). Then the quantized exchange relation is \( X_1 \ast X'_1 = vX_3 + v^{-1}X_2 \).

2.3. **Upper cluster algebras and cluster varieties.** Let there be given any initial seed \( t_0 \) and let \( \Delta^+ \) denote the corresponding set of seeds. Identify the skew-fields \( \mathcal{F}(t) \) by \((2.1)\), \( t \in \Delta^+ \). The upper cluster algebra is defined as

\[
\mathcal{U} := \bigcap_{t \in \Delta^+} k[M^\circ(t)].
\]

By the Laurent phenomenon (Theorem 2.1), we have \( \mathcal{A} \subset \mathcal{U} \). It often happens that \( \mathcal{A} = \mathcal{U} \). Moreover, the upper cluster algebra has the following useful property.

**Theorem 2.3** (Starfish Theorem \cite{BFZ05}). Under the full rank assumption, for any \( t \in \Delta^+ \), we have

\[
\mathcal{U} = k[M^\circ(t)] \cap (\bigcap_{t \in \Delta^+} k[M^\circ(\mu_k t)]).
\]

Let us describe the geometric counterpart for the upper cluster algebra \( \mathcal{U} \). Take \( k = \mathbb{C} \). Then each Laurent polynomial ring \( k[M^\circ(t)] \) is the ring of functions of the split torus \( T(t) = (\mathbb{C}^*)^d \). The mutation map \( \mu_k^* : \mathcal{F}(\mu_k t) \simeq \mathcal{F}(t) \) induces a birational map \( \mu_k : T(t) \dashrightarrow T(\mu_k t) \). The cluster \( K_2 \)-variety\(^1\) \( A \) is defined as the union of tori \( \bigcup_{t \in \Delta^+} T(t) \) glued by the mutation maps, and the upper cluster algebra is ring of functions \( k[A] \) on \( A \).

For completeness, let us sketch the construction of the cluster Poisson variety \( \mathbb{X} \) to \( A \). For each lattice \( M^\circ(t) = \oplus \mathbb{Z}f_i \), we can construct a dual lattice \( N(t) = \oplus \mathbb{Z}e_i \) with the pairing \( \langle e_i, f_j \rangle = \frac{1}{d} \). The Laurent polynomial ring \( k[N(t)] = \oplus_{n \in N(0)} kY^n \) is the ring of function of a dual torus \( T^\vee(t) \). One can define the mutation map \( \mu_k : T^\vee(t) \dashrightarrow

\(^1\)Following \cite{FG16}, we call \( A \) the cluster \( K_2 \)-variety because it has a canonical \( K_2 \)-class, see \cite[Section 2.5]{FG09a}. \)
\( T^\vee(\mu_k t) \) using mutation of \( y \)-variables (defined in Section (4.1), see [FZ07] or [Qin20b, (2.3)]) for the mutation rule. The corresponding cluster Poisson variety \( X \) is the union of the dual tori \( \cup_{t \in \Delta^+} T^\vee(t) \) glued by the mutation maps.

We refer the reader to [GHK15] for more details on cluster varieties.

2.4. Cluster structure on quantum groups: an \( \mathfrak{sl}_3 \) example. The quantized coordinate ring associated to the first example in Section 2.1 is a quantum cluster algebra. We make a brief introduction to this algebra and its bases, following the convention in [Kim12].

Consider the Lie algebra \( \mathfrak{g} = \mathfrak{sl}_3 \) and its Weyl group element \( w = w_0 \), where \( w_0 \) denotes the longest element. We choose a reduced word \( \overrightarrow{w} = s_1 s_2 s_1 \) for \( w \).

The quantized enveloping algebra \( U_q(\mathfrak{n}_-) \) is the \( \mathbb{C}(q) \)-algebra generated by \( F_1, F_2 \) (called Chevalley generators) satisfying the \( q \)-Serre relation:

\[
F_1^2 F_2 - (q + q^{-1}) F_1 F_2 F_1 + F_1 F_2 = 0.
\]

One can construct the following PBW generators for \( U_q(\mathfrak{n}_-) \) (called root vectors, see [Lus93]):

\[
F_-(\beta_1) = F_1, \\
F_-(\beta_2) = F_1 F_2 - q F_2 F_1,
\]

where we view the vectors \( \beta_i \) as a formal symbol for simplicity.

For any \( \underline{c} \in \mathbb{N}^3 \) (called Lusztig’s parametrization), we construct the ordered product

\[
F_-(\underline{c}) := F_-(\sum c_i \beta_i) := F_-(\beta_3)^{c_3} F_-(\beta_2)^{c_2} F_-(\beta_1)^{c_1}
\]

where \( \sum c_i \beta_i \) should be viewed as a formal sum.

The set \( \{ F_-(\underline{c}) | \forall \underline{c} \} \) is a basis for \( U_q(\mathfrak{n}_-) \), called the PBW basis. There is a perfect pairing on \( U_q(\mathfrak{n}_-) \), called Lusztig’s bilinear form \( (\ , \)_L \) (alternatively, we can use Kashiwara’s bilinear form). Then we construct the dual PBW basis \( \{ F_+^{\text{up}}(\underline{c}) | \forall \underline{c} \} \) using the bilinear form \( (\ , \)_L \). Following [Kim12], explicitly, we have

\[
F_+^{\text{up}}(c_i \beta_i) = \prod_{s=1}^{c_i} (1 - q^{2s}) \cdot F_-(c_i \beta_i) \\
F_+^{\text{up}}(\underline{c}) = F_+^{\text{up}}(\sum c_3 \beta_3) F_+^{\text{up}}(\sum c_2 \beta_2) F_+^{\text{up}}(\sum c_1 \beta_1)
\]

**Definition 2.4.** The quantum unipotent subgroup \( A_q[N_-(w)] \) is the \( \mathbb{Q}(q) \)-algebra spanned by the dual PBW basis.
In this example, let us compute the dual PBW generators for $A_q[N-(w)]$:

\[
\begin{align*}
F_{-}^{up}(\beta_1) &= (1 - q^2)F_1 \\
F_{-}^{up}(\beta_2) &= (1 - q^2)F_2 \\
F^{up}(\beta_2) &= (1 - q^2)(F_1F_2 - qF_2F_1)
\end{align*}
\]

There is the dual bar involution $\sigma$ on $A_q[N-(w)]$, see [Kim12]. For example, we have

\[
\begin{align*}
\sigma(F_i) &= -q^2F_i, \\
\sigma(F_1F_2) &= q^{-1}\sigma(F_2)\sigma(F_1).
\end{align*}
\]

Let $<$ denote the lexicographical order on the set of all $\underline{c} \in \mathbb{N}^3$.

**Definition 2.5.** The dual canonical basis (or upper global basis) $B^{up} = \{B^{up}(\underline{c})|\forall \underline{c}\}$ for $A_q[N-(w)]$, also denoted by $B^*$, is the unique basis subject to the following conditions:

- $B^{up}(\underline{c})$ are $\sigma$-invariant;
- $B^{up}(\underline{c}) \in F^{up}(\underline{c}) + \sum_{\underline{c}' < \underline{c}} q\mathbb{Z}[q]F^{up}(\underline{c}')$.

For example, we can explicitly check that

\[
\begin{align*}
B^{up}_-(\beta_1) &= F^{up}_-(\beta_1), \\
B^{up}_-(\beta_1 + \beta_3) &= (1 - q^2)(F_2F_1 - qF_2F_1).
\end{align*}
\]

Let us define $X_1 = q^{-1}F^{up}_-(\beta_1)$, $X'_1 = q^{-1}F^{up}_-(\beta_3)$, $X_2 = q^{-\frac{3}{2}}F^{up}_-(\beta_2)$ and $X_3 = q^{-\frac{1}{2}}B^{up}_-(\beta_1 + \beta_3)$. It is straightforward to check that they verify the exchange relation among quantum cluster variables as in Example 2.2:

\[
X_1X'_1 = q^{\frac{1}{2}}X_3 + q^{-\frac{1}{2}}X_2
\]

Thus, we have seen that $A_q[N-(w)]$ is the quantum cluster algebra $\mathcal{A}$, because they are both generated by these cluster variables.

Let $A_q[N^w]$ denote the localization of $A_q[N-(w)]$ at $X_2$, $X_3$ (frozen variables). It is called the quantum unipotent cell. Then $A_q[N^w]$ equals the corresponding (localized) quantum cluster algebra $\mathcal{A}$.

### 3. Topological models for bases

In this section, following [Thu14], we briefly introduce topological models to provide intuition to readers unfamiliar with cluster algebras and their bases.
3.1. **Skein algebras and cluster algebras.** A marked surface $\Sigma = (S,M)$ consists of an oriented topological surface $S$ and a finite set $M$ of marked points in $S$. As a topological surface, $\Sigma$ is viewed as $S \setminus M$. We require that each connected component of the boundary $\partial S$ contains at least one marked point. The marked points contained in the interior of $S$ are called punctures.

For simplicity, we make the following assumption. Many results can be generalized without this assumption.

**Assumption 2.** We assume that $\Sigma$ has no punctures.

For technical reasons, we exclude surfaces with empty boundary or containing connected components which are discs with 1 or 2 marked points.

We consider curves $C_i$ in $\Sigma$ that either end at marked points or are closed loops. A multicurve (or a diagram) is a finite union of curves, denoted by $C = \cup C_i$. We will consider (multi)curves up to isotopy fixing the marked points and the crossings. The isotopy class of $C$ is denoted by $[C]$.

Choose the base ring $\mathbb{k} = \mathbb{Z}$. The Skein algebra $\text{Sk}(\Sigma)$ is the quotient of the free $\mathbb{k}$-module $\bigoplus C[\mathbb{k}C]$ by the Skein relations in Figure 3.1, see [Thu14]. The multiplication in the Skein algebra is given by the union

$$[C] \cdot [C'] = [C \cup C'].$$

Note that the empty set provides the multiplicative unit for $\text{Sk}(\Sigma)$.

A curve is said to be simple if it is non-contractible and has no self-crossing. An arc $\gamma$ is a simple curve ending at marked points. A triangulation $^2\Delta$ is a maximal collection of pairwise non-isotopic non-crossing arcs $\gamma_i$ in $\Sigma$.

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$^2$We should not confuse the symbol $\Delta$ for a triangulation and $\Delta^+$ for the set of seeds.
Table 1. Comparison: topology and cluster theory

| Topology       | Cluster theory                      |
|----------------|-------------------------------------|
| triangulation $\Delta$ | seed $t_\Delta$                     |
| arc            | cluster variable                    |
| boundary arc   | coefficient/frozen variable         |
| $\cup \gamma_i$ for $\gamma_i \in \Delta$ | cluster monomial                   |
| union          | multiplication                      |
| crossing resolution | algebra relation                  |

Following [FST08], we associate a seed $t_\Delta$ to $\Delta$ as follows. Associate to each $\gamma_i \in \Delta$ a vertex denoted by $i$. The $i$-th cluster variable of $t_\Delta$ is defined to be the Skein algebra element $[\gamma_i]$, which is frozen if $\gamma_i$ is a boundary arc. For each pair of arcs $\gamma_i, \gamma_j \in \Delta$, we define $b_{ij}$ to be the following sum over all triangles $T$ in $\Delta$:

$$b_{ij} := \sum_{T \subset \Delta} \begin{cases} 
0 & \text{if } \gamma_i \text{ and } \gamma_j \text{ are not both in } T; \\
1 & \text{if } \gamma_j \text{ is the arc immediately clockwise of } \gamma_i \text{ in } T; \\
-1 & \text{if } \gamma_j \text{ is the arc immediately counterclockwise of } \gamma_i \text{ in } T.
\end{cases}$$

(3.1)

See Examples 3.2.

In this way, we obtain a classical cluster algebra $\overline{A} = \overline{A}(t_\Delta)$, which is independent of the triangulation we choose. We skip the details and refer the reader to [FST08] for the precise construction. Such cluster structures are closely related to the Teichmüller theory for $\Sigma$, see [FT18][FG06].

For unpunctured $\Sigma$, [Mul16b] proposed a natural quantization of $\overline{A}(t_\Delta)$ and $\text{Sk}(\Sigma)$.

**Theorem 3.1** ([Mul16b]). We have natural inclusions of $\mathbb{k}$-algebras $\overline{A} \subset \text{Sk}(\Sigma) \subset \overline{U}$. When there are at least two marked points on each connected component of the boundary $\partial S$, we have $\overline{A} = \text{Sk}(\Sigma) = \overline{U}$.

Table 1 summarizes analogous notions and structures appearing in the topology of $\Sigma$ and the corresponding cluster algebra. The symbol $\sim$ means a correspondence or an analogy.

**Example 3.2** (Annulus). We consider an annulus with two marked points on its boundary, see Figure 3.2. Denote its boundary arcs by $b_1, b_2$. Choose the initial triangulation $\Delta = \{x_1, x_2\} \cup \{b_1, b_2\}$. By rotating boundary components, it is clear that this marked surface has infinitely many triangulations, i.e., infinitely many cluster variables.

---

$^3$Here, we view a triangle $T$ as an oriented circle via homotopy, whose orientation is induced by that of $S$, and we investigate the relative positions of the arcs $\gamma_i$ and $\gamma_j$ on the circle.
Let us associate vertices 1, 2, 3, 4 to the arcs $x_1, x_2, b_1, b_2$ respectively. By (3.1), the matrix $(b_{ij})$ for the initial seed $t_\Delta$ is given by

\[
(b_{ij}) = \begin{pmatrix}
0 & -2 & 1 & 1 \\
2 & 0 & -1 & -1 \\
-1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}.
\]

By evaluating the frozen variables to 1 in $\text{Sk}(\Sigma)$, we obtain the coefficient-free Skein algebra $\text{Sk}'(\Sigma)$.

**Example 3.3** (Kronecker type). We continue Example 3.2, but removing the frozen vertices and consider the coefficient-free Skein algebra $\text{Sk}'(\Sigma)$. Then $\text{Sk}'(\Sigma)$ is the following classical cluster algebra $\overline{\mathcal{A}}$ without frozen variables.

Denote the initial cluster variables by $X_i = [x_i], \; i = 1, 2$. We have the initial seed $t_0 = ((X_1, X_2), B)$ whose $B$-matrix is $B = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$.

Its quiver is the Kronecker quiver $Q : 2 \to 1$.

The cluster variables $X_n, n \in \mathbb{Z}$, are computed recursively via the exchange relations

\[
X_{n+2} = X_n^{-1}(1 + X_{n+1}^2).
\]

The seeds are $t_n = ((X_{n+1}, X_{n+2}), B)$ for $n \in 2\mathbb{Z}$, and $t_n = ((X_{n+2}, X_{n+1}), -B)$, for $n \in 2\mathbb{Z} + 1$.

In this example, we have $\overline{\mathcal{A}} = \mathcal{A} = \mathcal{U} = \mathbb{k}[X_n]_{n \in \mathbb{Z}}$. Its cluster monomials do NOT form a basis.

### 3.2. Bangle basis (generic basis)

By a bangle, we mean a multicurve $C$ without self-crossing or contractible components. $[C]$ is called a bangle element.

It follows from the Skein relations for $\text{Sk}(\Sigma)$ that any multicurve $[D]$ can be written as a finite sum of bangle elements $[C]$. We further have the following result.
Theorem 3.4 ([MSW13][Thu14]). The bangle elements form a basis for $\text{Sk}(\Sigma)$.

The basis is called the bangle basis.

[GLFS20a] showed that such bangle elements are generic basis elements for upper cluster algebras, see Section 6.

Example 3.5. We continue Example 3.2. Let $L$ denote the non-contractible simple loop in the annulus (unique up to isotopy). The bangle elements are either the cluster monomials, such as $[x_1]^{m_1}[x_2]^{m_2}[b_1]^{m_3}[b_2]^{m_4}$, or the elements $[L]^{m_1}[b_1]^{m_2}[b_2]^{m_3}$, $m_i \geq 0$. See Figure 3.3.

3.3. Bracelet basis (theta basis). Assume that $C$ is a bangle. We can denote $C = \sqcup w_i C_i$, $w_i \geq 0$, such that $C_i$ and $C_j$ are non-isotopic curves, and $w_i$ denote the multiplicity of $C_i$ appearing in $C$ (up to isotopy).

Then, for any closed loop $C_i$, we can replace $w_i C_i$ by a closed loop having winding number $w_i$ and minimal self-crossings ($(w_i - 1)$-many), denoted by $\text{Brac}^{w_i}(C_i)$, see Figures 3.4 and 3.5. The new multicurve $C'$
obtained from $C$ is called a bracelet, and $[C'] \in \text{Sk}(\Sigma)$ the bracelet element. The following result easily follows from the analogous statement for the bangle basis (Theorem 3.4).

**Theorem 3.6 ([MSW13][Thu14]).** The bracelet elements form a basis for $\text{Sk}(\Sigma)$.

The basis is called the bracelet basis.

**Example 3.7.** We continue Examples 3.2. The bangle elements are either the cluster monomials or the elements $[\text{Brac}^{m_1}(L)][b_1]^{m_2}[b_2]^{m_3}, m_i \geq 0$. See Figure 3.3.

By [MQ21], the (quantized) bracelet elements belong to the (quantized) theta basis for the corresponding upper cluster algebra.

### 3.4. Chebyshev polynomials and bases.

Let us compute the bracelet elements $\text{Brac}^k(L)$ in Example 3.7 explicitly.

**Definition 3.8.** The Chebyshev polynomials of the first kind are polynomials $T_k(z), k \in \mathbb{N}$, in an indeterminate $z$, subject to the following recursive definition:

$$
T_0(z) = 2,
T_1(z) = z,
T_{k+1}(z) = zT_k(z) - T_{k-1}(z).
$$

They have the following properties:

$$
T_k(z)T_l(z) = T_{k+l}(z) + T_{|k-l|}(z), \quad \forall k, l \in \mathbb{N}
$$

$$
T_k(e^x + e^{-x}) = e^{kx} + e^{-kx}, \quad \forall k \in \mathbb{N}.
$$

In particular, $\forall M \in SL_2$, its trace satisfies $\text{tr}(M^k) = T_k(\text{tr}M)$.

**Theorem 3.9 ([MSW13]).** For any $k \in \mathbb{N}$, we have $[\text{Brac}^k(L)] = T_k([L])$.

In particular, we see that $[\text{Brac}^2(L)] = [L]^2 - 2$ in Figure 3.4.

[Thu14] observed that one can associate the trivial $k$-element permutation to the bangle $kL$ and a non-trivial permutation for the bracelet $\text{Brac}^k(L)$, see Figure 3.5. Correspondingly, [Thu14] proposed considering the formal average of all multicurves corresponding to any $k$-element permutation. More precisely, $kL$ and $\text{Brac}^kL$ only differ by a link diagram contained in a small rectangular neighborhood (we project the link diagram on the surface and forget its orientation). And such a link diagram represents a $k$-element permutation. By changing the link diagrams, we obtain new multicurves, see [Thu14, Figure 1].
Let $\text{Band}^k(L)$ denote the formal average of the multicurves corresponding to all $k$-elements permutations and call it a band curve. Let us compute the band element $[\text{Band}^k(L)]$ explicitly.

**Definition 3.10.** The Chebyshev polynomials of the second kind are polynomials $U_k(z)$, $k \in \mathbb{N}$, in an indeterminate $z$, subject to the following recursive definition:

\[
U_0(z) = 1, \\
U_1(z) = z, \\
U_{k+1}(z) = zU_k(z) - U_{k-1}(z).
\]

They have the following properties:

\[
U_k(z)U_l(z) = U_{k+l}(z) + U_{k+l-2}(z) + \cdots + U_{|k-l|}(z), \quad \forall k,l \in \mathbb{N} \\
U_k(e^x + e^{-x}) = e^{kx} + e^{(k-2)x} + \cdots + e^{-kx}, \quad \forall k \in \mathbb{N}.
\]

**Theorem 3.11 ([Thu14]).** For any $k \in \mathbb{N}$, we have $[\text{Band}^k(L)] = U_k([L])$.

For example, we have $[\text{Band}^2(L)] = [L]^2 - 1$ in Figure 3.6.

### 3.5. Band basis (triangular basis/dual canonical basis)

Assume that $C$ is a bangle. We can denote $C = \cup w_iC_i$, $w_i \geq 0$, such that $C_i$ and $C_j$ are non-isotopic curves, and $w_i$ denotes the multiplicity of $C_i$ appearing in $C$ (up to isotopy). Then, for any closed loop $C_i$, we replace $w_iC_i$ by the corresponding band curve $\text{Band}^{w_i}(C_i)$, see [Thu14, ...]
Figure 3.6. Bands

\[ \text{Band}^2(L) = \frac{1}{2}(\text{Brac}^2(L) + L^2) \]

Figure 1, and obtain a formal average of multicurves \( C' \). The corresponding element \([C'] \in \mathbf{Sk}(\Sigma)\), which will be called a band element, is defined naturally. The following result easily follows from the analogous statement for the bangle basis (Theorem 3.4).

**Theorem 3.12** ([Thu14]). *The band elements form a basis for \( \mathbf{Sk}(\Sigma) \).*

The basis is called the band basis.

**Example 3.13.** We continue Examples 3.2 and 3.5. The band elements are either the cluster monomials or the elements \([\text{Band}^{m_1} (L)] [b_1]^{m_2} [b_2]^{m_3}\), \( m_i \geq 0 \). See Figure 3.3.

**Conjecture 3.14.** The (quantized) band basis coincides with the common triangular basis in the sense of [Qin17] (Section 5.1), after localization at the frozen variables.

**Remark 3.15.** The only known evidence for Conjecture 3.14 is the widely known Example 3.2, see [Thu14, Remark 4.22]. For that case, the common triangular basis is the dual canonical basis (after a change of frozen variables [Qin20b, Section 4][Qin14, Section 9]), and the Chebyshev recursion was found for the dual canonical basis [Lam11].

4. Tropical properties and bases

4.1. Cluster expansions and \( g \)-vectors. Let there be given any initial seed \( t = ((X_i), (b_{ij})) \). Let us define its Laurent monomials \( Y_k := X^{c_{ek}} \tilde{B} \) and \( Y^n := X^{\tilde{B}n} \) for any \( n \in \mathbb{N}^{I_{uf}} \). \( Y_k \) are called its \( y \)-variables.

**Theorem 4.1** ([FZ07][DWZ10][Tra11][GHKK18]). For any given seed \( t' \in \Delta^+ \), the (quantum) cluster monomial \( X(t')^m \) in \( t' \) for any \( m \in \mathbb{N}^t \) must have the following Laurent expansion in \( k[\mathcal{M}^\circ(t)] \):

\[
X(t')^m = X^g \cdot (\sum_{n \in \mathbb{N}^{I_{uf}}} c_n Y^n)
\]
for some \( c_n \in k \) and \( g \in M^\circ(t) \). Moreover, \( c_0 = 1 \).

Note that, under the full rank assumption, by (4.1), the cluster monomial \( X(t')^m \) uniquely determines the vector \( g \) (called its extended \( g \)-vector) and the polynomial \( F' = \sum_{n \in \mathbb{N}^I} c_n Y^n \) with constant term 1 (called its \( F \)-polynomial).

Let \( \text{pr}_{I,uf} : \mathbb{Z}^I \to \mathbb{Z}^I_{uf} \) denote the natural projection. \( \text{pr}_{I,uf} g \) is called the principal \( g \)-vector.

**Example 4.2.** In Example 2.2, we have

\[
X'_1 = X_1^{-1} \cdot X_3 \cdot (1 + X_2 \cdot X_3^{-1}) = X^{(-1,0,1)^T} \cdot (1 + Y_1),
\]

where \( \cdot \) denotes the commutative product.

The following conjecture was formulated for the classical case in [FZ02], and its quantum version is also expected.

**Conjecture 4.3** (Positivity conjecture). The coefficients \( c_n \) for cluster monomials are contained in \( \mathbb{N} \) (or in \( \mathbb{N}[v^\pm] \) for the quantum case).

**Remark 4.4.** First consider Conjecture 4.3 for the classical case \( k = \mathbb{Z} \). Partial results were due to [MSW11][HL10][Nak11]. The conjecture was verified for skew-symmetric seeds by [LS15]. It was verified for all skew-symmetrizable seeds by [GHKK18].

Next consider the quantum case \( k = \mathbb{Z}[v^\pm] \). Partial results were due to [Qin12] [KQ14]. The conjecture was verified for skew-symmetric seeds by [DMSS15][Dav18].

Conjecture 4.3 remains open for skew-symmetrizable quantum seeds.

### 4.2. Pointedness and dominance orders.

Let there be given any seed \( t = ((X_i),(b_{ij})) \) subject to the full rank assumption. We are interested in the following type of Laurent polynomials, which share a similar form with the cluster monomials in (4.1).

**Definition 4.5** (Pointedness, dominance order [Qin17]). Any element \( Z \in k[M^\circ(t)] \) contained in \( X^g \cdot (1 + \sum_{n>0} kY^n) \), for some \( g \in M^\circ(t) = \mathbb{Z}^I \), is said to be \( g \)-pointed. We define its degree by \( \text{deg} Z = g \).

A subset \( Z = \{Z_g | g \in M^\circ(t)\} \subset k[M^\circ(t)] \) with \( g \)-pointed elements \( Z_g \) is said to be \( M^\circ(t) \)-pointed.

In order to justify the definition of the degree for \( Z \), we need to introduce the following partial order on \( M^\circ(t) \) (viewed as the lattice of Laurent degrees).

**Definition 4.6.** The dominance order \( \prec_t \) on \( M^\circ(t) \) is defined such that we have \( \text{deg} X(t)^n Y(t)^m \prec_t \text{deg} X(t)^g \forall n > 0 \).

Equivalently, \( g' \prec_t g \) if and only if there exists some \( 0 \neq n \in \mathbb{N}^I_{uf} \) such that \( g' = g + Bn \).
Remark 4.7. The definitions of pointedness, degree, and dominance order are inspired by representation theory.

More precisely, inspired by the monoidal categorification of cluster algebras [HL10] (see Section 5.3), we hope that $Z_g$ can be compared with the module character $\chi S(w)$ of a highest weight module $S(w)$ for some algebra (such as a quantum affine algebra [HL10]). From this point of view, $g$ plays the role of the highest weight $w$ for $S(w)$. Therefore, the $g$-pointed element $Z_g$ should be parametrized by its degree $g$.

Our dominance order is inspired by the dominance order for stratification of graded quiver varieties in [Nak11].

4.3. Injective-reachability.

Definition 4.8. A seed $t$ is injective-reachable if there is another seed $t[1] \in \Delta^+$ and a permutation $\sigma$ of $I_{uf}$, such that its cluster variables $X_{\sigma k}(t[1]), k \in I_{uf}$, have the principal $g$-vector $-f_k$, where $f_k$ denote the $k$-th unit vector.

An (upper) cluster algebra is said to be injective-reachable if its seeds are. Unless otherwise specified, we make the following assumption from now on.

Assumption 3. The seeds are injective-reachable.

We denote the cluster variable $I_k(t) := X_{\sigma k}(t[1])$. For any $m \in \mathbb{N}^{I_{uf}}$, define the ordered product $I(t)^m := \nu^a \prod I_k(t)^{m_k}$ using the twisted product, where $a \in \mathbb{Z}$ is chosen so that this is a pointed element.

Remark 4.9. The injective-reachable assumption holds for (almost) all well-known cluster algebras arising from representation theory or higher Teichmüller theory (except for once-punctured closed surfaces), see [Qin21] for a list.

If a seed $t$ is injective-reachable, so are all seeds in $\Delta^+$ [Mul16a][Qin17, Lemma 5.1.1].

A seed $t$ is injective-reachable if and only if there exists a green to red sequence in the sense of [Kel11].

When $t$ is skew-symmetric, one can construct the (completed) Jacobian algebra $\widehat{J}$ from the corresponding quiver with potential [DWZ08, DWZ10]. Then $t$ is injective-reachable if and only if the indecomposable injective modules $I_k$ of $\widehat{J}$ can be constructed from the simple modules by finitely many mutations, whence the name injective-reachable.

Example 4.10. In Example 4.2, we have $\deg X'_1 = (-1, 0, 1)^T \equiv -f_1 \mod \mathbb{Z}f_2 \oplus \mathbb{Z}f_3$. Therefore, the injective-reachable assumption holds.

4.4. Tropical properties. Let us define the piecewise linear map $\phi_k : M^\circ(t) \simeq M^\circ(\mu_k t)$ such that, for any $g = (g_i) \in M^\circ(t) = \mathbb{Z}^I$, its image $g' = (g'_i) \in M^\circ(\mu_k t) = \mathbb{Z}^I$ is given by
They are called tropical transformations. Then we have the following result.

**Theorem 4.11** ([FZ07, Conjecture 6.10][DWZ10][GHKK18]). If the extended $g$-vector of a cluster monomial with respect to the initial seed $t$ is $g$, then $\phi_k g$ is its extended $g$-vector with respect to the initial seed $\mu_k t$.

Denote the tropical semifield $\mathbb{Z}^T = (\mathbb{Z}, \max, +)$. We refer the reader to [GHKK18, Section 2] for the basics of tropicalization of cluster varieties $A$ and $X$ on $\mathbb{Z}^T$. Let us give the following simplified definition, hiding all geometric details.

**Definition 4.12.** The tropical points are the equivalence classes in $\bigsqcup_{t \in \Delta^+} M^\circ(t)$, such that $g \in M^\circ(t)$ and $\phi_k g \in M^\circ(\mu_k t)$ are equivalent. We denote the set of tropical points by $M^\circ$.

Let $[g]$ denote the equivalence class of $g \in M^\circ(t)$ in $M^\circ$. It can be shown that $[g]$ has exactly one representative $g'$ in $M^\circ(t')$ for any $t' \in \Delta^+$. Recall that we have a natural identification $M^\circ(t) = \oplus_{i \in I} \mathbb{Z} f_i \simeq \mathbb{Z}^I$. Then we obtain $M^\circ \simeq M^\circ(t) \simeq \mathbb{Z}^I$ as sets. Note that, while $M^\circ$ does not depend on $t$, the isomorphism $M^\circ \simeq \mathbb{Z}^I$ does.

**Definition 4.13.** An element $Z \in \mathcal{U}$ is said to be $[g]$-pointed, or parametrized by $[g]$, if for any $t' \in \Delta^+$, $Z$ is $g'$-pointed in $k[M^\circ(t')]$, where $g'$ is the representative of $[g]$ in $M^\circ(t')$.

A set $Z = \{Z_{[g]}[g] \in M^\circ\}$ is said to be parametrized by the tropical points if its elements $Z_{[g]}$ are parametrized by the tropical points $[g]$.

**Conjecture 4.14** (Fock-Goncharov conjecture [FG06, FG09b]). The upper cluster algebra $\mathcal{U}$ possesses a basis such that it is parametrized by the tropical points.\footnote{These are the tropical points for the cluster Poisson variety $X$ of the Langlands dual seed [FG09b][GHK15].}

**Remark 4.15.** The theta basis $\Theta$ in [GHKK18] provides such a basis for many $\mathcal{U}$, but fails in the general case. It was suggested in [GHKK18] Conjecture 4.14 should be modified in general.

\footnote{[FG06, FG09b] further expected the basis to be positive, i.e., its structure constants are non-negative.}
4.5. **Bases with good tropical properties.** We have seen three families of bases via topological models in Section 3. Any of them, if it exists, is known to be parametrized by the tropical points (and thus verifies Conjecture 4.14). Therefore, the tropical property in Conjecture 4.14 cannot uniquely determine a basis.

In general, there exist infinitely many bases parametrized by the tropical points: see [Qin21, Theorem 1.2.1] for a description of the infinite space of all such bases. At this moment, the three families of bases that we have seen in Section 3 are the most well-known and interesting ones. See Section 7 for a further discussion.

By the following lemma, any basis parametrized by the tropical points contains all cluster monomials.

**Lemma 4.16** ([Qin21, Lemma 3.4.12]). Assume that the upper cluster algebra $U$ satisfies the injective-reachable assumption. If there is an element $Z$ in $U$ and a cluster monomial $M$, such that they are both parametrized by the same tropical point, then $Z = M$.

Moreover, the tropical property provides the following criterion for verifying a basis.

**Theorem 4.17** ([Qin21, Theorem 4.3.1]). Assume that the upper cluster algebra satisfies the injective-reachable assumption. Let $S = \{S_{[g]}|\forall [g]\}$ denote a subset of $U$ whose elements $S_{[g]}$ are parametrized by the tropical points $[g]$. Then $S$ is a basis of $U$.

Theorem 4.17 immediately implies the existence of the generic basis, see Section 6.

5. **THE TRIANGULAR BASES FROM QUANTUM GROUPS AND MONOIDAL CATEGORIES**

We assume the injective-reachable assumption throughout this section.

5.1. **Triangular basis.**

**Definition 5.1** (Triangular basis [Qin17]). Consider the quantum case $\mathbb{k} = \mathbb{Z}[v^\pm]$. For any seed $t \in \Delta^+$, a triangular basis $L^t$ of $U$ is a $\mathbb{k}$-basis such that

- $L^t$ contains the quantum cluster monomials in $t, t[1]$.
- (pointedness) $L^t = \{L^t_g|g \in M^\circ(t)\}$ such that $L^t_g$ are $g$-pointed.
- (bar-invariance) $L^t_g$ are invariant under the bar-involution $v^{-\alpha}X(t)^m = v^{\alpha}X(t)^m$.
- (degree triangularity) $\forall i \in I, \exists \alpha \in \mathbb{Z}$, such that
  $$v^{\alpha}X_i(t) \ast L^t_g \in L^t_{f_i+g} + \sum_{g' < t^g+f_i} v^{-1}\mathbb{Z}[v^{-1}]L^t_{g'}.$$
Definition 5.1 can be generalized for subalgebras of the upper cluster algebra, see [Qin20b, Section 6.2]. Here we only consider it for upper cluster algebras for simplicity.

Lemma 5.2 ([Qin20b, Lemma 6.1.4]). For any \( t \in \Delta^+ \), the triangular basis \( L_t \) is unique if it exists.

Proof. Although we do not have a dual PBW basis for \( \mathcal{U} \), we can construct an analogous set: for any \( g \in M^0(t) \), we can always construct a unique \( g \)-pointed element \( I_g \) of the form \( v^\alpha X^m \ast X(t)^{m'} \ast I(t)^{m''} \), for some \( \alpha \in \mathbb{Z} \), \( m \in \mathbb{Z}^h \), \( m' = [pr_{I_d} g]_+ \), and \( m'' = [-pr_{I_d} g]_+ \), where \( pr_{I_d} g \) is the principal part of \( g \).

The set \( \{ I_g \mid \forall g \} \) is NOT a basis for \( \mathcal{U} \). Nevertheless, with the help of this set and the dominance order, there is a unique solution \( S \) in the formal completion \( k[M^0(t)] = k[M^0(t)] \otimes_{k[Y_k|k \in I_d]} k[Y_k|k \in I_d] \) via Lusztig’s lemma (see [Nak04, Lemma 8.4] [Lus90, 7.10]), such that \( S \) satisfies the conditions in Definition 5.1. See [Qin20b, Theorem 6.1.3] for an elementary calculation. Therefore, if \( L_t \) exists, \( L_t = S \). \( \square \)

The \( g \)-pointed elements \( I_g \) in the proof of Lemma 5.2 will be called the distinguished functions. They form a topological basis for the ring of the formal Laurent series \( k[M^0(t)] \), see [Qin21, Section 4.1][DM21, Section 2.2.2].

Remark 5.3. Unfortunately, we cannot verify the existence of \( L_t \) by elementary calculation, because we only have a candidate in the ring of the formal Laurent series (see the proof of Lemma 5.1), and it is hard to tell if the candidate only consists of Laurent polynomials.

Remark 5.4. The notion of the triangular basis was first proposed by [BZ14] for acyclic seeds (i.e. seeds corresponding to valued quivers without oriented cycles). Their definition is very different from ours. It was shown in [Qin19, Qin20b] that their triangular basis equals ours, when the seed is acyclic.

Definition 5.5 (Common triangular basis). Take \( k = \mathbb{Z}[v^\pm] \). A \( k \)-basis \( L \) of \( \mathcal{U} \) is said to be the common triangular basis if \( L \) is the triangular basis \( L_t \) for all seeds \( t \in \Delta^+ \).

Note that the common triangular basis in Definition 5.5 contains all quantum cluster monomials. By [Qin20b, Proposition 6.4.3], it is parametrized by the tropical points.

Remark 5.6. Definition 5.5 is simpler than but equivalent to the original definition in [Qin17], which imposes the extra condition that \( L \) is parametrized by the tropical points.

5.2. From the dual canonical basis to the common triangular basis.

\(^{6}\)One might need to restrict to subalgebras of an upper cluster algebra when Conjecture 4.14 fails, see [GHKK18][Zho20].
Cluster structure on quantum groups. Following the convention of [Kim12][KO21], we generalize the \( \mathfrak{sl}_3 \) example in Section 2.4 to any Kac-Moody algebra \( \mathfrak{g} \) and any element \( w \) in its Weyl group \( W \).

As before, we choose any reduced word \( \overrightarrow{w} \). Then we can construct the quantum unipotent subgroup \( A_q[N_-(w)] \) using the dual PBW basis. It is a quantum analog of the ring of functions \( \mathbb{C}[N_-(w)] \) for the unipotent subgroup \( N_-(w) = N_- \cap wNW^- \). Note that, for \( \mathfrak{g} \) a semi-simple Lie algebra and \( w_0 \) the longest element, \( N_-(w) = N_- \).

The \( q \)-center of \( A_q[N_-(w)] \) is defined as
\[
Z_q = \{ P \in A_q[N_-(w)] | \forall x \in A_q[N_-(w)], \exists \alpha \in \mathbb{Z} \text{ such that } Px = q^\alpha xP \}.
\]
In fact, it corresponds to the monoid generated by the frozen variables of the corresponding quantum cluster algebra.

The quantum unipotent cell \( A_q[N^-_w] \) is defined as the localization of \( A_q[N_-(w)] \) at the \( q \)-center. It is a quantum analog of the ring of functions \( \mathbb{C}[N^-_w] \) for the unipotent cell \( N^-_w = N_- \cap BwB \).

It is known that \( A_q[N_-(w)] \) has the dual canonical basis \( B^* \). Then the dual canonical basis for \( A_q[N^-_w] \) is defined as the localization of \( B^* \) at the \( q \)-center:
\[
\{ q^\alpha bP^{-1} | b \in B^*, P \in Z_q \},
\]
where the factor \( q^\alpha \) is chosen such that \( q^\alpha bP^{-1} \) is invariant under the dual bar involution. See [Kim12][KO21] for more details.

Following [GLS11, GLS13][GY16, GY21], we have a natural quantum seed \( t_0 = t_0(\overrightarrow{w}) \) associated to \( \overrightarrow{w} \).

\textbf{Theorem 5.7} ([GLS11, GLS13][GY20, GY21]). Consider quantum cluster algebras for the quantum case \( k = \mathbb{Z}[w^\pm] = \mathbb{Z}[q^{\pm \frac{1}{2}}] \). We have natural isomorphisms between \( \mathbb{Q}(q^{\frac{1}{2}}) \)-algebras:
\[
A_q[N_-(w)] \otimes \mathbb{Q}(q^{\frac{1}{2}}) \simeq \mathcal{A}(t_0(\overrightarrow{w})) \otimes \mathbb{Q}(q^{\frac{1}{2}}),
\]
\[
A_q[N^-_w] \otimes \mathbb{Q}(q^{\frac{1}{2}}) \simeq \mathcal{A}(t_0(\overrightarrow{w})) \otimes \mathbb{Q}(q^{\frac{1}{2}}).
\]

We refer the reader to [Qin20b, Section 8] for a brief introduction to the above isomorphisms.

Triangular basis on quantum groups.

\textbf{Theorem 5.8} ([Qin20b]). The dual canonical basis for \( A_q[N^-_w] \) is the common triangular basis for \( \mathcal{A}(t_0(\overrightarrow{w})) \) up to scalar multiples in \( q^{\frac{1}{2}} \mathbb{Z} \).

In particular, all quantum cluster monomials \( X(t)^m \) belong to the dual canonical basis up to scalar multiples.

\textbf{Remark} 5.9 (Obstruction). It is well-known that the structure constants of \( B^* \) are positive for a symmetric Kac-Moody algebra \( \mathfrak{g} \), the proof of which can be based on geometric representation theory or monoidal categorification via quiver Hecke algebras. But this property is NOT true for symmetrizable \( \mathfrak{g} \).
Table 2. Comparison: quantum groups and cluster algebras

| Quantum groups                      | cluster theory          |
|-------------------------------------|-------------------------|
| reduced words $\vec{w}'$ for fixed $w$ | $\subset$ seeds $t \in \Delta^+$ |
| Lusztig parametrizations $c$        | $\sim$ $g$-vectors in $M^c(t_0)$ |
| lexicographical order               | dominance order         |
| dual PBW basis $F^\text{comp}_{-1}(c, \vec{w})$ | $\neq$ distinguished functions $I^t$ |
| dual canonical basis                | $=$ common triangular basis $L$ |
| crystal structure                   | $?$                      |
| representations                     |                          |

To prove Theorem 5.8, we rely on an analysis of tropical properties [Qin21] instead of the positivity (or geometric representation theory as in [Qin17], or categorification). See [Qin20b] for details.

A comparison between quantum groups and cluster theory. A comparison of analogous notions and structures in quantum groups and cluster theory is summarized in Table 2, where $\sim$ means a correspondence or an analogy. Here, to any reduced word $\vec{w}'$ of $w$, we can associate a seed $t(\vec{w}') \in \Delta^+ = \Delta^+_0(\vec{w})$. But the set $\Delta^+$ is usually infinite and, in particular, not every seed arises from $\vec{w}'$. A Lusztig parametrization $c$ can be translated into a vector $g \in M^c(t_0)$ via a linear map, and one can compare the (partial) orders on both sides, see [Qin20b, Section 9.1] and [Cas20][Cas19].

Remark 5.10 (Generalization of $B^*$). Note that the notion of the common triangular basis makes sense for all injective-reachable quantum cluster algebras, which do not necessarily arise from quantum groups. In view of Theorem 5.8, the common triangular basis $L$ is the generalization of the dual canonical basis in cluster theory.

5.3. The common triangular basis in monoidal categories.

Monoidal categorification of cluster algebras. [HL10] proposed the notion of monoidal categorification for classical cluster algebras. Roughly speaking, for a given cluster algebra $\overline{A}$, one wants to find a monoidal category $(C, \otimes)$, so that its Grothendieck ring $K_0(C)$ is isomorphic to the cluster algebra $\overline{A}$. Moreover, the cluster monomials should correspond to isoclasses of simple modules.

Remark 5.11. Our requirement for monoidal categorification is weaker than the original proposal by [HL10], which demanded that the cluster monomials are in bijection with all real simple modules. It still remains open if they are in bijection, see the reachability conjectures [Qin20a, Remark 5.9].
[HL10] considered the quantum affine algebra $U_q(\hat{g})$ for a simply-laced simple Lie algebra $g$ (Dynkin type $ADE$). Its finite dimensional module category $\text{mod} U_q(\hat{g})$ is a monoidal category. For any level $N \in \mathbb{N}$, [HL10] considered the level-$N$ monoidal subcategories $C_N$ of $\text{mod} U_q(\hat{g})$. They showed that $K_0(C_N)$ is a classical cluster algebra. They expected that $C_N$ provides the monoidal categorification for this cluster algebra, for which one needs to verify that the cluster monomials correspond to simple modules.

The $v$-deformation $K_v(C)$ of $K_0(C)$ can be constructed using Nakajima’s quiver varieties [Nak04][VV03], and it is isomorphic to the corresponding quantum cluster algebra.

**Theorem 5.12 ([Qin17, Theorem 1.2.1(II)])**. The set of (bar-invariant) elements in $K_v(C_N)$ corresponding to the simple modules provides the common triangular basis for the corresponding quantum cluster algebra (after localization at the frozen variables, see (5.1)).

In particular, the cluster monomials correspond to simple modules in $C_N$, thus verifying (a weaker form of) [HL10, Conjecture 13.2].

Recently, [KKOP20] showed that more monoidal categories consisting of modules of the quantum affine algebra $U_q(\hat{g})$ provide monoidal categorification for cluster algebras.

Quantum groups and monoidal categorification. For any symmetric Kac-Moody algebra $g$ and any $w \in W$, one can construct a monoidal category $C_w$ consisting of ($\mathbb{Z}$-graded) finite dimensional modules of the corresponding quiver Hecke algebras [KL09, KL10][Rou08][VV11], such that the quantum unipotent subgroup $A_q[N_{-}(w)]$ is isomorphic to the Grothendieck ring $K_0(C_w)$. See [GLS11][KKKO18].

By [VV11], under this isomorphism, the dual canonical basis $B^*$ is identified with the set of the isoclasses of the self-dual simple modules in $C_w$ (see [KKKO18]). Then Theorem 5.8 implies that the common triangular basis is the set of the isoclasses of the self-dual simple modules (up to scalar multiplication and localization at the $q$-center, see (5.1)).

[KKKO18] proposed the notion of quantum monoidal categorification and proved the following result.

**Theorem 5.13 ([KKKO18])**. The category $C_w$ provides the quantum monoidal categorification for the corresponding quantum cluster algebra $\mathcal{A}(t_0(w))$. In particular, all quantum cluster monomials correspond to self-dual simple modules in $C_w$ up to scalar multiples.

Monoidal categorification and cluster theory. In the literature, for (almost) all cluster algebras known to admit a (quantum) monoidal categorification, such as those appearing in [Qin17][KKKO18][CW19], the

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7It is usually called the $t$-deformed Grothendieck ring $K_t(C)$ in literature. We use the quantum parameter $v$ instead of $t$ in our convention.
Table 3. Comparison: monoidal categories and cluster theory

| Monoidal categories $\mathcal{C}$ | cluster theory |
|----------------------------------|----------------|
| $v$-deformed Grothendieck ring $\otimes$ | (quantum) cluster algebra $\mathcal{A}$ |
| $\oplus$ | $\ast$ |
| homomorphism $M_i \otimes M_j \not\cong M_j \otimes M_i$ | $X_i \ast X_j \neq X_j \ast X_i$ |
| short exact sequence of ($\mathbb{Z}$-graded) objects $vM_3 \rightarrow M_1 \otimes M_1' \rightarrow v^{-1}M_2$ | (quantized) exchange relations $X_1 \ast X_1' = v X_3 + v^{-1}X_2$ |
| Rigidity: $\mathcal{C}$ has left/right dual | $t$ is Injective-reachable: $\exists t[\pm 1]$ |
| $\{$(self-dual) simple objects$\}$ | dominance order |
| standard modules | $\neq$ distinguished functions $I_g'$ |
| $\{$(self-dual) simple objects$\}$ | common triangular basis $\mathcal{L}$ |
| $\hat{J}$ | $\mathcal{L}_g = X^g(1 + \sum c_n Y^n)$ |

set of the isoclasses of (self-dual) simple objects produces the common triangular basis (up to scalar multiplication and localization at the frozen variables). This is implied by [Qin20b, Proposition 6.4.3] or [Qin17, Theorem 6.5.7]. From this point of view, the existence of the common triangular basis $\mathcal{L}$ suggests that there might exist a monoidal categorification.

In Remark 4.7, we have mentioned the similarity between the character of a highest weight module $S(w)$ and the Laurent expansion of a $g$-pointed element. The dominance order in cluster theory has been studied in [KK19] from the point of view of $R$-matrices. Table 3 summarizes the analogous notions and structures in monoidal categories and cluster theory, when a monoidal categorification is known.

6. The generic basis from representation theory

Throughout this section, we assume that the principal $B$-matrix for the initial seed $t_0$ is skew-symmetric, so that we can associate a principal quiver $Q$ to it.

6.1. Generic cluster characters. Let $Q$ denote the principal quiver. A potential $W$ is a $\mathbb{C}$-linear combination of (possibly infinitely many) oriented cycles in $Q$. The set of potentials is an infinite dimensional space, and we take a generic one such that $W$ is non-degenerate (see [DWZ08]).

We define $\hat{J}$ to be the completed Jacobian algebra associated to the quiver with potential $(Q, W)$, see [DWZ08] for details. In particular,
\( \hat{J} \) is a quotient algebra of the completion of the path algebra \( kQ \). See Section 6.2 for an example.

In our convention, we will consider left modules for the opposite algebra \( \hat{J}^{op} \). Let \( I_k \) denote the \( k \)-th indecomposable injective module for \( \hat{J}^{op} \). Under the injective-reachable assumption, each \( I_k \) is finite dimensional. Note that the converse is not true, see the Markov quiver example [Pla13, Example 4.3].

For any \( \hat{J}^{op} \)-module \( V \), consider its minimal injective resolution

\[
0 \to V \to I^m \to I^{m'}
\]

where \( m, m' \in \mathbb{N}^{I_{uf}} \), and we denote \( I^m = \oplus I^{m_k}_k \). Define the principal \( g \)-vector for \( V \) by \( g_V = m' - m \in \mathbb{Z}^{I_{uf}} \).

Consider the classical case \( k = \mathbb{Z} \). We have the following Caldero-Chapoton map (CC-map for short) sending any \( \hat{J}^{op} \)-module \( V \) to a Laurent polynomial in \( k[M^{o}(t_0)] \).

**Definition 6.1** (Caldero-Chapoton map [CC06][DWZ10]). For any \( \hat{J}^{op} \)-module \( V \), we define \( CC(V) \) as the following element in \( k[M^{o}(t_0)] \):

\[
CC(V) = X^{g_V} \cdot \left( \sum_{n \in \mathbb{N}^{I_{uf}}} \chi(\text{Gr}_n V) Y^n \right),
\]

where \( \chi(\ ) \) denote the topological Euler characteristic, and \( \text{Gr}_n V \) denote the quiver Grassmannian consisting of the \( n \)-dimensional submodules of \( V \).

For any vector \( g \in \mathbb{Z}^{I_{uf}} \), we have the affine space \( \text{Hom}_{\hat{J}^{op}}(I^{-g}+, I^{g}+) \). Then it has an open dense subset on which \( CC(\ker f) \) is constant, where \( f \) belongs to \( \text{Hom}_{\hat{J}^{op}}(I^{-g}+, I^{g}+) \). This constant value is called the generic cluster character associated to \( g \) and is denoted by \( L_g \).

Embedding \( \mathbb{Z}^{I_{uf}} \) and \( \mathbb{Z}^{I_{f}} \) into \( \mathbb{Z}^{I_{uf}} \) by adding 0 entries, we define the generic cluster character \( L_{g+m} = X^m \cdot L_g \) for any \( g \in \mathbb{Z}^{I_{uf}} \) and \( m \in \mathbb{Z}^{I_{f}} \).

When the generic cluster characters form a basis for \( U \), the basis is called the generic basis. In this case, we say that the generic basis for \( U \) exists.

**Theorem 6.2** ([Qin21, Theorem 1.2.3]). Consider the classical case \( k = \mathbb{Z} \). For any skew-symmetric injective-reachable upper cluster algebra \( U \) under the full rank assumption, the generic basis exists.

**Proof.** [Pla13] showed that the generic cluster characters are parametrized by the tropical points. The desired claim then follows as a consequence of Theorem 4.17. \( \square \)

6.2. Calculating bases using representation theory: Kronecker quiver. Let us discuss the representation theory for the generic basis in Example 3.3.
In this case, the potential $W$ vanishes, and the completed Jacobian algebra $\hat{J}$ is the path algebra $\mathbb{C}Q$. Consider the opposite quiver $Q^{op} : 1 \implies 2$. Let $\text{Rep}(Q^{op}, d)$ denote the affine space consisting of the $d$-dimensional $Q^{op}$-representations over $\mathbb{C}$, which correspond to $\hat{J}^{op}$-modules, see [CB92].

Each rigid module $M$ for $\hat{J}^{op} = \mathbb{C}Q^{op}$ corresponds to an open dense subset in $\text{Rep}(Q^{op}, d)$. Moreover, $CC(M)$ is a cluster monomial. It is well-known that all cluster monomials not divisible by the initial cluster variables take this form (to restore the initial cluster variables, one needs the notion of decorated representations [DWZ10], $\tau$-rigid pairs [AIR14], or cluster categories [BBMR06] [CCS06]).

It is known that the points in an open dense subset of $\text{Rep}(Q^{op}, (1,1))$ correspond to the indecomposable modules of dimension $(1,1)$, which are usually parametrized as $V_L(\lambda)$, $\lambda \in \mathbb{C}P^1$. Choose any such module and denote it by $V_L$. Independent of the choice, we have

\[
CC(V_L) = X_1X_2^{-1}(1 + Y_2 + Y_1Y_2)
\]

It is straightforward to check that $CC(V_L)$ equals the element $[L]$ in the coefficient-free Skein algebra $Sk'(\Sigma)$ for the closed simple loop $L$.

Next, for any $k \in \mathbb{N}$, there is an open dense subset of $\text{Rep}(Q^{op}, (k,k))$, such that its points correspond to modules isomorphic to $\oplus_{i=1}^k V_L(\lambda_i)$, $\lambda_i \neq \lambda_j$. Choose any such module and denote it by $V_{L^k}$. Independent of the choice, we have

\[
CC(V_{L^k}) = CC(V_L)^k = [L^k] \in Sk'(\Sigma).
\]

$CC(V_{L^k})$ is a generic cluster character.

**Remark 6.3.** For any chosen $V_L$, we also have a $(k,k)$-dimensional indecomposable module $V_{\text{Band}^k(L)}$, unique up to isomorphism, which is obtained from $V_L$ by iterated self-extensions. It turns out that

\[
CC(V_{\text{Band}^k(L)}) = [\text{Band}^k(L)] \in Sk'(\Sigma).
\]

We conjecture that this result can be generalized to more surfaces.

**Remark 6.4.** In this example, we can compute the bracelets $[\text{Brac}^k(L)]$ using transverse quiver Grassmannians instead of quiver Grassmannians in the CC-map, see [Dup10, IDE13]. By [All19], this formula holds true for unpunctured surfaces.

### 7. Further topics

#### 7.1. The theta basis and representation theory.

At first sight, there seems to exist no representation theoretic interpretation for the
theta basis. To see this, let there be given a monoidal categorification for a cluster algebra. Then the common triangular basis elements should correspond to the simple modules, which are often viewed as the minimal building blocks in representation theory, see Section 5.3. However, as in the case of Example 3.3, one can expect that a common triangular basis element is sometimes a positive linear combination of theta basis elements, see Section 3. So the theta basis elements appearing cannot correspond to any genuine modules.

However, it seems that the theta basis might be understood by considering the affine Grassmannians and the Langlands dual. More precisely, the recent work [BKK19] defined the Mirković-Vilonen basis for the algebra $C[N_-]$ for a simple simply-connected algebraic group $G$. The basis is constructed from cycles on the affine Grassmannian for the Langlands dual group $G^\vee$. By [BKK19], calculation in small examples seems to suggest an affirmative answer for the following question.

**Question 7.1.** Does the Mirković-Vilonen basis coincide with the theta basis?

### 7.2. A new family of bases.
Recall that, by a positive basis, we mean a basis with positive structure constants.

Assume $g$ is a semi-simple Lie algebra for simplicity. When its Cartan matrix is symmetrizable, we cannot find a positive basis for the unipotent quantum subgroup $A_q[N_-]$ from the previous three families of bases. More precisely, it is known that the generic basis is often not positive and we do not know its quantization. The dual canonical basis (the common triangular basis) is not positive for the symmetrizable case. The theta basis is positive at the classical level, but not positive at the quantum level for the symmetrizable case.

Nevertheless, finite-dimensional self-dual simple modules of quiver Hecke algebras give rise to a positive basis for $A_q[N_-]$. Note that this basis is the same as the dual canonical basis if the Cartan matrix is symmetric, but different if not.

We hope that this basis is well-behaved for cluster theory. If so, it provides a new family of interesting bases for cluster algebras. In particular, we hope to have affirmative answers for the following questions.

**Question 7.2.** Does this basis contain all cluster monomials? Is it parametrized by the tropical points?

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8Scattering diagrams and some theta functions can be constructed using representation theory, see [Bri17] [BST17].
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Email address: qin.fan.math@gmail.com