Bifurcations of a harmonically excited system impacting between two moving rigid bodies near a 1:4 strong resonance point

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Abstract. A harmonically excited system with rigid body impacts is considered. The periodic motions and Poincaré mapping of the vibro-impact system are derived analytically. A center manifold theorem technique is applied to reduce the Poincaré mapping to a two-dimensional one, and the normal form mapping associated with 1:4 strong resonance is obtained. Two-parameter bifurcations of the fixed points of 1:4 strong resonance in the vibro-impact system are analyzed. The simulation results illustrate some interesting dynamical features: in the vibro-impact system, there exist Neimark-Sacker bifurcations of periodic-impact motions and tangent and fold bifurcations of period-4 orbits near the bifurcation point of 1:4 strong resonance.

1. Introduction

The vibro-impact phenomena, such as the components of a vibrating system collide with rigid obstacles, exist in a variety of engineering applications, particularly in mechanisms and machines with clearances. It is urgent and important to investigate the dynamical behaviors of the vibro-impact systems. The physical process during impacts is strongly nonlinear and discontinuous, but it can be described theoretically and numerically by discontinuities in good agreement with reality. In the past several years, the dynamics of the vibro-impact systems have been studied in great detail in references, which include stability and bifurcations [1-4], grazing singularities [5-6], quasi-periodic impacts [7], period-doubling motion [8], bifurcations in strong resonance case [9], chaos control [10], etc. Based on the references [2-4, 9], the purpose of the present study is to focus attention on two-parameter bifurcation of the fixed points associated with 1:4 strong resonance. A harmonically excited system having repeated impacts between two moving rigid bodies is considered. Periodic motions and bifurcations of the vibro-impact system, associated with 1:4 strong resonance, are analyzed. Some

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complicated bifurcations, e.g., $T_{nm}$ types of tangent bifurcations of period-4 orbits, are found to exist near the bifurcation point of 1:4 strong resonance.

2. Mechanical model and Poincaré mapping

In the mechanical model shown in Figure 1, the masses $M_{L1}$, $M_{L2}$, ..., and $M_{Ln}$ are connected to linear springs with stiffnesses $K_{L1}$, $K_{L2}$, ..., and $K_{Ln}$, and linear viscous dashpots with damping constants $C_{L1}$, $C_{L2}$, ..., and $C_{Ln}$. The masses $M_{R1}$, $M_{R2}$, ..., and $M_{Rm}$ are connected to linear springs with stiffnesses $K_{R1}$, $K_{R2}$, ..., and $K_{Rm}$, and linear viscous dashpots with damping constants $C_{R1}$, $C_{R2}$, ..., and $C_{Rm}$. All the excitations on these masses are harmonic, and $F_{Li} = P_{Li} \sin(\Omega t + \tau)$, $F_{Rj} = P_{Rj} \sin(2\Omega T + \rho)$, $i = 1,2,\cdots n$, $j = 1,2,\cdots m$. All the masses move horizontally, and their displacements are represented by $X_{Li}$, $X_{L2}$, ..., $X_{Ln}$ and $X_{R1}$, $X_{R2}$, ..., $X_{Rm}$. The two masses $M_{L1}$ and $M_{R1}$ impact each other when the displacement $X_{R1} - X_{L1}$ equals the gap $\Delta$. The impact is described by a coefficient of restitution $R$. Damping is assumed as proportional damping of the Rayleigh type. Compared to the force period, the duration of the impact can be negligible.

Figure 1. Model of the harmonically excited system impacting between two moving rigid bodies.

Suppose $M_{L1} \neq 0$, $K_{L1} \neq 0$, and let $P_0 = |P_{L1}| + |P_{L2}| + \cdots + |P_{Ln}| + |P_{R1}| + |P_{R2}| + \cdots + |P_{Rm}|$, then the non-dimensional quantities are given by

$$m_{Li} = \frac{M_{L1}}{M_{L1}}$$
$$k_{Li} = \frac{K_{Li}}{K_{L1}}$$
$$f_{Li} = \frac{P_{Li}}{P_0}$$
$$\zeta_{Li} = \frac{C_{Li}}{2\sqrt{K_{Li}M_{Li}}}$$
$$x_{Li} = \frac{X_{Li}K_{Li}}{P_0}$$
$$\zeta_{Li} = \frac{r_{K_{Li}}}{i = 1,2,\cdots n}$$

$$m_{Rj} = \frac{M_{R1}}{M_{R1}}$$
$$k_{Rj} = \frac{K_{Rj}}{K_{L1}}$$
$$f_{Rj} = \frac{P_{Rj}}{P_0}$$
$$\zeta_{Rj} = \frac{C_{Rj}}{2\sqrt{K_{Rj}M_{Rj}}}$$
$$x_{Rj} = \frac{X_{Rj}K_{Rj}}{P_0}$$
$$\zeta_{Rj} = \frac{r_{K_{Rj}}}{j = 1,2,\cdots m}$$

$$\omega = \frac{\Omega M_{L1}}{K_{L1}}$$
$$t = \frac{\tau M_{L1}}{P_0}$$
$$\delta = \frac{\Delta K_{L1}}{P_0}$$

Between any two consecutive impacts, the time $T$ is set to zero directly at the instant when the former impact is over, and the phase angle $\tau$ is used to make a suitable choice for the origin of time in the calculation. The differential equations of the system can be given by

$$M_{L1}\ddot{x}_L + 2rK_L\dot{x}_L + K_Lx_L = F_L \sin(\omega t + \tau)$$
$$M_{R1}\ddot{x}_R + 2rK_R\dot{x}_R + K_Rx_R = F_R \sin(\omega t + \tau)$$

(1)
where \( M_L = \text{diag} \{ m_{L1}, \ldots, m_{Ln} \} \), \( M_R = \text{diag} \{ m_{R1}, \ldots, m_{Rm} \} \), \( x_L = [x_{L1}, x_{L2}, \ldots, x_{Ln}]^T \), \( x_R = [x_{R1}, x_{R2}, \ldots, x_{Rm}]^T \),

\[
K_L = \begin{bmatrix}
    k_{L1} & -k_{L1} & & & & \\
    -k_{L1} & k_{L1} + k_{L2} & \ddots & & & \\
    & \ddots & \ddots & \ddots & & \\
    & & -k_{L(n-1)} & k_{L(n-1)} + k_{Ln} & & \\
\end{bmatrix}, \quad K_R = \begin{bmatrix}
    k_{R1} & -k_{R1} & & & & \\
    -k_{R1} & k_{R1} + k_{R2} & \ddots & & & \\
    & \ddots & \ddots & \ddots & & \\
    & & -k_{R(m-1)} & k_{R(m-1)} + k_{Rm} & & \\
\end{bmatrix},
\]

\( F_L = [f_{L1}, f_{L2}, \ldots, f_{Ln}]^T \), \( F_R = [f_{R1}, f_{R2}, \ldots, f_{Rm}]^T \), the dot (\( \dot{} \)) denotes differentiation with respect to the non-dimensional time \( t \).

Let \( \dot{x}_{L1}, \dot{x}_{L2}, \ldots, \dot{x}_{Ln} \), and \( \dot{x}_{R1}, \ldots, \dot{x}_{Rm} \), represent the instant approach and departure velocities of the two masses \( M_L \) and \( M_R \) respectively whenever the impact occurs. According to the impact law, the impact equation of the two masses \( M_L \) and \( M_R \) can be given by

\[
\dot{x}_{Li+1} + m_{Li} \dot{x}_{Ri} = \dot{x}_{Li}, \quad \dot{x}_{Ri+1} - \dot{x}_{Ri} = -R(\dot{x}_{Li+1} - \dot{x}_{Ri+1}), \quad (x_{Ri} - x_{Li}) = \delta.
\]  

Under suitable system parameter conditions, the system can exhibit periodic impact behavior. The periodic motion of the system can be characterized by using the symbol \( p - q \), where \( p \) is the impact number and \( q \) is the forcing cycle number. The Poincaré section associated with the system’s state, just immediately after impact, is chosen, and period \( q \) single-impact motion and its disturbed mapping are derived analytically in Reference [3]. Let \( \theta = \omega t \), we choose the section \( \sigma = ((x_{L1}, \ldots, x_{Ln}, \dot{x}_{L1}, \ldots, \dot{x}_{Ln}, x_{R2}, \ldots, x_{Rm}, \dot{x}_{Ri}, \ldots, \dot{x}_{Rm}, \theta) \in \mathbb{R}^{2(n+m)} \times S, x_{Ri} - x_{Li} = \delta, \dot{x}_{L1} = \dot{x}_{L2}, \ldots, \dot{x}_{Ri} = \dot{x}_{Ri+1}) \) to establish the Poincaré mapping of periodic motion, which is given by the style \( AX' = \tilde{f}(v, X) - X^* = f(v, AX) \), where \( X \in \mathbb{R}^{2(n+m)} \), \( v \) is a varying parameter, and \( v \in \mathbb{R}^1 \) or \( \mathbb{R}^2 \); \( X \) is the fixed point, \( X^* = X^* + AX' \). \( X = (x_{L1}, \ldots, x_{Ln}, \dot{x}_{L1}, \ldots, \dot{x}_{Ln}, \dot{x}_{R1}, \ldots, \dot{x}_{Rm})^T \) is a fixed point, \( AX = (\Delta x_{L1}, \ldots, \Delta x_{Ln}, \Delta \dot{x}_{L1}, \ldots, \Delta \dot{x}_{Ln}, \Delta \dot{x}_{R1}, \ldots, \Delta \dot{x}_{Rm})^T \) and \( AX' = (\Delta x'_{L1}, \ldots, \Delta x'_{Ln}, \Delta \dot{x}_{L1}, \ldots, \Delta \dot{x}_{Ln}, \Delta \dot{x}_{R1}, \ldots, \Delta \dot{x}_{Rm})^T \) are the disturbed vectors of the fixed point \( X^* \).

Define function \( g(AX, \Delta \theta) = \tilde{x}_{Ri}(t_c) - \tilde{x}_{L1}(t_c) - \delta = 0 \) and suppose \( \partial g / \partial \Delta \theta \big|_{X^*} \neq 0 \), we can get the Poincaré mapping

\[
AX' = \tilde{f}(v, X) - X^* = f(v, AX) \quad \text{Def}.
\]  

Linearizing the Poincaré mapping at the fixed point \( X^* \) results in the matrix \( Df(v, X^*) = \frac{\partial f(v, AX)}{\partial AX} \big|_{v, X^*} \). The stability of 1-1 motion is determined by computing and analyzing the eigenvalues of Jacobian matrix \( Df(v, X^*) \). If one of the eigenvalues passes through the unit circle in the complex plane, the associated bifurcation will occur. In general, bifurcation occurs in various ways according to the numbers of the eigenvalues on the unit circle and their position on the unit circle. In this paper, two-parameter bifurcations of the fixed points in the vibro-impact system are considered, and dynamics of the system is studied with special attention to the bifurcations for 1:4 strong resonance, in which \( Df(v, X^*) \) satisfies two assumptions: H1. \( Df(v, X^*) \) has a complex-conjugate pair eigenvalues \( \lambda_{1,2}(v_c) \) on the unit circle \( (|\lambda_{1,2}(v_c)| = 1) \), the other eigenvalues \( \lambda_3(v_c), \ldots, \lambda_{2(m+n)}(v_c) \) stay inside the unit circle; H2. \( d|\lambda_i(v)| / dv \big|_{v=v_c} > 0 \) and \( \lambda_i'(v_c) = 1 \).
3. Center manifold and normal form mapping associated with 1:4 resonance case
By using the center manifold technique and normal form method of mapping, we can reduce the mapping (3) to the normal form mapping, which is expressed in the complex form by

\[ \phi_\mu(\zeta, \bar{\zeta}) = \lambda(\mu)\zeta + C(\mu)\zeta^2 + D(\mu)\zeta^3 + O(\zeta^4), \]  

(4)

The fourth iterate of the mapping (4) associated with 1:4 resonance case can be represented, for all sufficiently small \( \mu \) in the form

\[ F^4(\zeta) = \phi_\mu(\zeta) + O(\zeta^4), \]

where \( C(\mu) \) and \( D(\mu) \) are smooth complex-valued functions of \( \mu \), \( \omega(0) = 0 \), and \( C(0) = -4iC(0) \), \( D(0) = -4iD(0) \), as seen in References[11, 12].

If the complex number \( D(0) \neq 0 \), then we can scale the planar system (5) by taking \( \zeta = \gamma(\beta)\eta \). Let

\[ A(\beta) = C(\mu(\beta)) |D(\mu(\beta))|^2, \]

then the scaling results in \( \eta = (\beta_1 + i\beta_2)\eta + A(\beta)\eta^2 + \bar{\eta}^3 \).

The bifurcation analyses of the above system are very complicated and requires analytical and numerical techniques. For \( A = A(0) \), the \( A \)-plane is divided into eleven regions by different bifurcation diagrams. The bifurcation sequences and phase portraits for every region are studied and illustrated in References[11, 12]. Near 1:4 resonance point there exist very complicated bifurcation phenomena, such as Neimark-Sacker bifurcation of nontrivial equilibria, tangent bifurcations of \( T \) in \( T_{in}, T_{on} \) or \( T_{out} \) types, etc.

4. Numerical analyses of Neimark-Sacker and tangent bifurcations
For the convenience, choosing the parameters \( n = 1 \) and \( m = 1 \), we can obtain a two-degree-of-freedom vibro-impact system shown in Figure 2. We can apply the former analyses to this system so that we can demonstrate the analyses’ effectiveness. The existence and stability of 1-1 motion are analyzed explicitly and local bifurcations are considered.

![Figure 2. Model of the two-degree-of-freedom reciprocating impact system with harmonic excitation](image)

The vibro-impact system shown in Figure 2, with the non-dimensional parameters: \( m_{1L} = 1 \), \( m_{R1} = 6.084071 \), \( k_{L1} = 1 \), \( k_{R1} = 32.184736 \), \( \delta = 0.01 \), \( f_{L1} = f_{R1} = 0.5 \) and \( R = 0.8 \) have been chosen for analyses. The forcing frequency \( \omega \) and the ratio \( r \) are taken as the control parameters, i.e. \( v = (\omega, r) \). By gradually increasing \( \omega \) and \( r \) to change the control parameter \( v \), we found that a complex conjugate pair of eigenvalues locate at the points \( (0, \pm i) \) of the unit circle of the complex plane, and the other eigenvalues still stay inside the unit circle as \( v \) equals \( v_c = (4.255997, 0.002175) \). Here \( v_c \) is a bifurcation point associated with 1:4 resonance, and all eigenvalues of \( Df(v, X) \) are given as \( \lambda_{1,2}(v_c) = -0.00000024 \pm 1.00000100i, \lambda_{3,4}(v_c) = 0.37018050 \pm 0.43189620i. \)

We choose the different ratio \( r \) and change the forcing frequency \( \omega \) in numerical analyses. For choosing \( r_c = 0.002175 \), the simulation results are shown in Figure 3 in projected Poincaré sections.
Figure 3. Projected Poincaré sections \( r = r_c \): (a) the 4-4 fixed points starting from the unstable 1-1 fixed point, undergoing sub-harmonic bifurcation, \( \omega = 4.26 \); (b) the stable 4-4 fixed points, \( \omega = 4.39 \); (c) the quasi-periodic attractor (the “clover” heteroclinic cycle), starting from the initial condition near the unstable 1-1 fixed point, \( \omega = 4.392 \); (d) the quasi-periodic attractor (the unsmooth “clover” heteroclinic cycle), \( \omega = 4.485 \); (e) phase locking, \( \omega = 4.487 \); (f) chaos, \( \omega = 4.488 \).

For choosing \( r = r_c + \Delta r \) (here \( \Delta r = 0.000145 \)), the simulation results show that the system exhibits stable 1-1 motion with \( \omega \in [4, 4.185341] \). As \( \omega \) passes through \( \omega_{c,1} = 4.185341 \) increasingly, Neimark-Sacker bifurcation of 1-1 motion occurs. The quasi-periodic attractor associated with 1-1 fixed point is represented by an attracting invariant circle in projected Poincaré section as seen in Figures 4(a) and (b). With increasing \( \omega \), \( T_{on} \) type tangent bifurcation of 4-4 fixed points occurs so that it changes the quasi-periodic attractor to two families of 4-4 fixed points: one is unstable, the other is stable, as seen in Figures 4(c)-(f). A full understanding of Figures 4(b)-(f) can be obtained by making a comparison with Figure 9.19 of page 443 in Reference [12]. With increasing \( \omega \), 4-4 motion changes its stability, and Neimark-Sacker bifurcation of fixed points occurs so that the system exhibits the quasi-periodic attractor (the “clover” heteroclinic cycle), starting from the initial condition near the unstable 1-1 fixed point, as seen in Figure 4(g). With the further increasing \( \omega \), phase locking takes place, as seen in Figure 4(h), so that the quasi-periodic motion gets locked into a periodic attractor of higher (than period four) period, which subsequently becomes unstable and chaotic; as seen in Figure 4(i).
Figure 4. Projected Poincaré sections ($r = r_c + \Delta r$): (a) quasi-periodic attractor associated with 1-1 fixed point, $\omega = 4.19$; (b) quasi-periodic attractor, $\omega = 4.2169$; (c) 4-4 fixed points generated via $T_{on}$ type of tangent bifurcation of 4-4 fixed points, $\omega = 4.217$; (d) 4-4 fixed points (stable nodes), $\omega = 4.2171$; (e) 4-4 fixed points, $\omega = 4.35$; (f) 4-4 fixed points, $\omega = 4.355$; (g) quasi-periodic attractor (the “clover” heteroclinic cycle), $\omega = 4.36$; (h) chaos, $\omega = 4.39$; (i) chaos, $\omega = 4.465$.

For choosing $r = r_c - \Delta r$, the simulation results show that the system exhibits the similar bifurcation sequences and the routes to chaos as Figure 4, so not described here.

5. Conclusions
A harmonically excited system having repeated impacts between two moving rigid bodies is considered. The periodic motions and Poincaré mapping of the vibro-impact system are derived analytically. Two-parameter bifurcations of the fixed points in the vibro-impact system, associated with 1:4 strong resonance case, are analyzed. Sub-harmonic bifurcation in 1:4 strong resonance occurs, and near the bifurcation point of 1:4 strong resonance there exist quasi-periodic impact motion, stable and unstable 4-4 motions produced via $T_{on}$ type tangent (fold) bifurcation, etc. The simulation results show that the routes of 4-4 motions to chaos are multiple due to Neimark-Sacker bifurcation of 1-1 fixed points, the “clover” heteroclinic cycle formed by coinciding stable and unstable separatrices, etc.
The strict bifurcation conditions, associated with 1:4 strong resonance cases, are not easy to encounter in engineering application. However, there exists the possibility that nonlinear dynamical systems, with two or more varying parameters, work near the critical values of 1:4 strong resonance bifurcations by changing the parameters. It is necessary to study the bifurcations caused by changing the parameters and reveal nonlinear system dynamical behavior near the strong resonance points.

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