Invariant Subspace Method and Fractional Modified Kuramoto-Sivashinsky Equation

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Abstract

In this paper, the invariant subspace method is applied to the time fractional modified Kuramoto-Sivashinsky partial differential equation. The obtained reduced system of nonlinear ordinary fractional equations is solved by the Laplace transform method and with using of some useful properties of Mittag-Leffler function. Then, some exact solutions of the time fractional nonlinear studied equation are found.

Keywords

Invariance subspace method, Caputo fractional derivative, fractional modified Kuramoto-Sivashinsky equation, Mittag-Leffler function.

1 Introduction

In the last decade, fractional calculus attracted a great interest of many researchers. The idea of fractional order derivative was started with half-order derivative as discussed in the literature by Leibniz and L'Hôpital. Next, it was extended to an arbitrary order derivative by Liouville, Riemann, Grünwald, Letnikov, Caputo etc. In addition, different approaches to define fractional derivatives are known [1, 2, 3, 4]. The study of fractional differential equations becomes of great interest, since for their widely applications including fluid flow, dynamical processes in self-similar and porous structures, electromagnetic waves, probability and statistics, viscoelasticity, signal processing, and so on [1, 4, 5].

The construction of particular exact solutions of fractional differential equations is not an easy task and it remains a relevant problem. This is the reason
why a powerful methods for solving those fractional equations were recently developed in the literature, including Adomian decomposition method [6], first integral method [7], homotopy perturbation method [8], Lie group theory method [9, 10, 11, 12, 13] and so on. Most recently, according to invariance principles, the invariant subspace method developed by V.A. Galaktionov and S.R. Svirshchevski [14] to study partial differential equations was extended by R.K. Gazizov and A.A. Kasatkin [15] to construct some particular exact solutions for time fractional differential equations.

The invariant subspace method used in the present paper, yields us with an exact solutions of the time fractional modified Kuramoto-Sivashinsky equation in terms of the well known Mittag-Leffler function. In the paper [15], the invariant subspace method and Lie group analysis are joined to solve the reduced fractional ordinary differential system and the original studied equation. In our case, resolution of the reduced system is done by the Laplace transform method and by using of some remarkable properties of the well known Mittag-Leffler function [16, 17, 18].

This paper is organized as follows: In section 2, we recall some main results of fractional derivatives and integrals. Section 3, is devoted to describe the invariance subspace method. While in section 4, we use the described method to construct exact solution admitted by the time fractional modified Kuramoto-Sivashinsky equation. Finally, a conclusion is given.

2 SOME BASIC RESULTS ON FRACTIONAL CALCULUS

This section is devoted to recall briefly some definitions and basic results on fractional calculus. For more details and proofs of the results, we refer to [1, 2, 3, 4].

The Riemann-Liouville fractional integral is defined by:
\[
J^\gamma_0 f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \tau)^{\gamma-1} f(\tau) \, d\tau, \tag{2.1}
\]
where \( \gamma \in \mathbb{R}^+ \), and
\[
\Gamma(\gamma) = \int_0^{+\infty} x^{\gamma-1} e^{-x} \, dx, \tag{2.2}
\]
is the Euler Gamma function.

By definition \( J^\gamma_0 f(t) = f(t) \) and it satisfies the property \( J^\alpha_0 J^\gamma_0 f(t) = J^{\alpha+\gamma}_0 f(t) \).

First recall that there are various contributions [1, 2, 3, 4] to define fractional derivatives. In this paper, we adopt the fractional derivative in the sense of Caputo [11, 2, 3, 4]. The Caputo definition is used not only because it makes
easy the consideration of initial conditions but also because the derivative of a constant is equal to zero. In what follows, we recall some important results and properties of fractional derivatives and integrals. For more details see for example [4]. Before going on, let us denote by $AC^n([0, t]), n \in \mathbb{N}$ the class of functions $f(x)$ which are continuously differentiable in $[0, t]$ up to order $(n - 1)$ and with $f^{(n-1)} \in AC([0, t])$.

**Theorem 1** Let $n - 1 < \alpha < n$, with $n \in \mathbb{N}$. If $f(x) \in AC^n([0, t])$, then the Caputo fractional derivative exists almost everywhere on $[0, t]$ and it is represented in the form:

$$D^\alpha_t f(t) = J^{\alpha}_{n-m} D^n_t f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d\tau, \quad \alpha \neq n. \quad (2.3)$$

The Caputo derivative (2.3) and the Riemann-Liouville integral (2.1) satisfy the following properties [4]:

$$D^n_t J^\alpha_t f(t) = f(t), \quad \alpha > 0, \quad (2.4)$$

$$J^\alpha_t D^n_t f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k, \quad \alpha > 0, \quad t > 0, \quad (2.5)$$

$$J^\alpha_t t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}, \quad \alpha > 0, \quad \gamma > -1, \quad t > 0, \quad (2.6)$$

$$D^n_t t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \alpha > 0, \gamma \in ]-1, 0[\cup]0, +\infty[, \quad t > 0. \quad (2.7)$$

Here, it is important to mention that the studied equation is a time fractional partial differential equation of order $0 < \alpha < 1$, so the integer $n$ appearing in the relation (2.3) is equal to one. Consequently, the formulae (2.5) becomes:

$$J^\alpha_t D^n_t f(t) = f(t) - f(0), \quad t > 0. \quad (2.8)$$

### 3 DESCRIPTION OF THE INVARIANT SUBSPACE METHOD

The aim of this section is to collect and to present some necessary and essentials results from invariant subspace theory. The invariant subspace method [14] was firstly used to construct particular exact solutions of evolutionary partial differential equations of the form:

$$\frac{\partial u}{\partial t} = F(u, u_{1x}, u_{2x}, \ldots, u_{kx}), \quad k \in \mathbb{N}, \quad (3.1)$$
where \( u = u(t, x) \), \( u_{ix} = \frac{\partial^i u}{\partial x^i} \) is the \( i \)-th order derivative of \( u \) with respect to the space variable \( x \) and \( F \) is a nonlinear differential operator.

Recently, Gazizov and Kasatkin [15] showed that the invariant subspace method can be applied also to equations with time fractional derivative:

In fact, consider the time fractional partial differential equation of the form:

\[
D_\alpha^\alpha u(t, x) = F[u],
\]

(3.2)

where \( F[u] = F(u, u_1^x, u_2^x, \ldots, u_k^x) \) and \( D_\alpha^\alpha \) is the time fractional derivative in the sense of Caputo. The invariant subspace method is based on the following basic definitions and results [14, 15].

Let \( f_1(x), \ldots, f_n(x) \) be an \( n \) linearly independent functions and \( W_n \) is the \( n \)-dimensional linear space namely \( W_n = \langle f_1(x), \ldots, f_n(x) \rangle \). \( W_n \) is said to be invariant under the given operator \( F[u] \) if \( F[u] \in W_n \) whenever \( u \in W_n \).

**Proposition 3.1** Let \( W_n \) be an invariant subspace of \( F[u] \). A function \( u(t, x) = \sum_{i=1}^{n} f_i(x)u_i(t) \) is a solution of equation (3.2) if and only if the expansion coefficients \( u_i(t) \) satisfy the following system of fractional ordinary differential equations:

\[
\begin{align*}
    D_\alpha^\alpha u_1 &= F_1(u_1, \ldots, u_n), \\
    D_\alpha^\alpha u_2 &= F_2(u_1, \ldots, u_n), \\
    \vdots & \vdots \\
    D_\alpha^\alpha u_n &= F_n(u_1, \ldots, u_n),
\end{align*}
\]

where \( F_1, \ldots, F_n \) are given by:

\[
F(c_1f_1(x) + \cdots + c_nf_n(x)) = F_1(c_1, \ldots, c_n)f_1(x) + \cdots + F_n(c_1, \ldots, c_n)f_n(x).
\]

(3.3)

**Remark 1** A crucial question in the theory of invariant subspace method was how to get the corresponding invariant subspace of a given differential operator. This question is solved by the following proposition and for more details see [14].

**Proposition 3.2** Let \( f_1(x), \ldots, f_n(x) \) form the fundamental set of solutions of a linear \( n \)-th order ordinary differential equation

\[
L[u] = y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0,
\]

(3.4)

and \( F[y] = F(x, y, y', \ldots, y^{(k)}) \) a given differential operator of order \( k \leq n - 1 \), then the subspace \( W_n = \langle f_1(x), \ldots, f_n(x) \rangle \) is invariant with respect to \( F \) if and only if:

\[
L[F[y]] = 0,
\]

(3.5)

whenever \( y \) satisfies equation (3.4).
Remark 2 Condition of invariance appearing in the above proposition is the invariance criterion for equation (3.4) with respect to the Lie-Bäcklund generator $V = F[y] \frac{\partial}{\partial y}$. This criterion shows us how the invariant subspace method is related to the techniques used in Lie symmetry analysis, see for more details [12, 20, 21, 22].

4 EXACT SOLUTION OF THE FRACTIONAL MKS EQUATION

In this section, we use the invariant subspace method to construct some exact solutions of the time fractional modified Kuramoto-Sivashinsky equation (mKS) which is given by:

$$D_\alpha^t (u) = -u_{4x} - u_{2x} + (1 - \lambda)(u_x)^2 + \lambda (u_{xx})^2, \quad 0 < \alpha < 1, \ t > 0,$$

where $u = u(t, x)$ and $\lambda \in [0; 1]$. In the case $\alpha = 1$, the (mKS) equation (4.1) is a model for the dynamics of a hyper-cooled melt [23, 24]. A more general class of such models was introduced and discussed in [25].

Proposition 4.1 For any $\lambda \in [0; 1]$ the nonlinear operator $F[u]$ given by:

$$F[u] = -u_{4x} - u_{2x} + (1 - \lambda)(u_x)^2 + \lambda (u_{xx})^2,$$

admits $W_3 = \langle 1, \cos \gamma x, \sin \gamma x \rangle$ with $\gamma = \sqrt{\frac{1 - \lambda}{\lambda}}$ as an invariant subspace.

Proof. For any function

$$h(t, x) = C_1 + C_2 \cos \gamma x + C_3 \sin \gamma x,$$

with $C_i = C_i(t)$ arbitrary functions, we get:

$$F[h] = -\gamma^4 C_2 \cos \gamma x - \gamma^4 C_3 \sin \gamma x + \gamma^2 C_2 \cos \gamma x + \gamma^2 C_3 \sin \gamma x$$

$$+ (1 - \lambda)(\gamma^2 C_2^2 \cos^2 \gamma x + \gamma^2 C_3^2 \sin^2 \gamma x - 2\gamma^2 C_2 C_3 \cos \gamma x \sin \gamma x)$$

$$+ \lambda(\gamma^4 C_2^2 \cos^2 \gamma x + \gamma^4 C_3^2 \sin^2 \gamma x + 2\gamma^4 C_2 C_3 \cos \gamma x \sin \gamma x)$$

$$= (1 - \lambda)(\gamma^2 C_2^2 + C_3^2) + (\gamma^2 - \gamma^4) C_2 \cos \gamma x + (\gamma^2 - \gamma^4) C_3 \sin \gamma x \in W_3.$$ 

Now, we search an exact solution admitted by the time fractional (mKS) equation (4.1) of the form:

$$u(t, x) = C_1 + C_2 \cos \gamma x + C_3 \sin \gamma x.$$ 

Consequently, a function $u(t, x)$ of the form (4.4) is a solution of the time fractional (mKS) equation if the expansion coefficients $C_i(t)$ satisfy the following system of ordinary fractional differential equations:

$$\begin{cases}
D_\alpha^t C_1 = (1 - \lambda)\gamma^2 C_2^2 + (1 - \lambda)\gamma^2 C_3^2, \\
D_\alpha^t C_2 = \gamma^2(1 - \gamma^2) C_2, \\
D_\alpha^t C_3 = \gamma^2(1 - \gamma^2) C_3.
\end{cases}$$

(4.5)
To get a non trivial solution needs to assume the condition $C_2(0)C_3(0) \neq 0$ and for convenience we suppose $C_2(0) = C_3(0) = 1$. This last condition will be clear when the Laplace transform will be used. We start to construct solution of the third equation in the above reduced system of ordinary fractional differential equations. We mention that, with the Laplace transform it is frequently possible to avoid working with equations of different differential orders by translating the problem into an easy one.

Recalling some useful properties of the Laplace transform [1]:

$$\mathcal{L}\{D_t^{\alpha} f(t)\} = s^{\alpha} \tilde{f}(s) - s^{\alpha - 1} f(0), \quad 0 < \alpha < 1, \quad (4.6)$$

where

$$\mathcal{L}\{f(t)\} = \tilde{f}(s) = \int_{0}^{\infty} e^{-st} f(t) dt, \quad (4.7)$$

By putting $\theta = \gamma^2 (1 - \gamma^2)$ and applying the Laplace transform on both sides of the third equation appearing in the fractional ordinary differential system, we obtain:

$$s^\alpha \mathcal{L}\{C_3(t)\} - s^{\alpha - 1} C_3(0) = \theta \mathcal{L}\{C_3(t)\}, \quad (4.8)$$

it yields:

$$\mathcal{L}\{C_3(t)\}(s) = \frac{s^{\alpha - 1}}{s^{\alpha} - \theta}, \quad (4.9)$$

then, with the inverse Laplace transform, it gives:

$$C_3(t) = E_{\alpha,1}(\theta t^\alpha), \quad (4.10)$$

where $E_{\alpha,\beta}(\theta t^\alpha)$ is the Mittag-Leffler function given by:

$$E_{\alpha,\beta}(\theta t^\alpha) = \sum_{i=0}^{\infty} \frac{\theta^{i\alpha}}{\Gamma(\alpha i + \beta)}. \quad (4.11)$$

Not that when $\beta = 1$, $E_{\alpha,1} \equiv E_{\alpha}$.

Two last equations in the fractional ordinary differential system [1.5] are the same, hence,

$$C_2(t) = C_3(t) = E_{\alpha,1}(\theta t^\alpha). \quad (4.12)$$

Substituting the obtained expressions of $C_2$ and $C_3$ in the first equation of the system [1.5], it leads to:

$$D_t^{\alpha} C_1 = 2(1 - \lambda) \gamma^2 (E_{\alpha,1}(\theta t^\alpha))^2. \quad (4.13)$$

The Mittag-Leffler function does not satisfy the composition property:

$$E_{\alpha}(x)E_{\alpha}(y) \neq E_{\alpha}(x + y), \quad (4.14)$$
but it can be observed that the function \( E_\alpha(x^\alpha) \):

\[
E_\alpha(x^\alpha) = \sum_{i=0}^{\infty} \frac{x^{\alpha i}}{\Gamma(\alpha i + 1)}
\]

does satisfy the composition property:

\[
E_\alpha(x^\alpha)E_\alpha(y^\alpha) = E_\alpha((x + y)^\alpha), \quad \alpha > 0,
\]

Using of the above relation (4.16), so the equation (4.13) becomes:

\[
D^\alpha C_1 = 2(1 - \lambda)\gamma^2 E_{\alpha,1}(\theta(2t)^\alpha).
\]

Applying \( J^\alpha \) on both sides of equation (4.17), and using integration of the Mittag-Leffler function relation \[1\] (p. 25), we obtain:

\[
\frac{1}{\Gamma(\nu)} \int_0^t (t - \tau)^{\nu-1} E_{\alpha,\beta}(\eta t^\alpha)\tau^{\beta-1} d\tau = t^{\beta+\nu-1} E_{\alpha,\beta+\nu}(\eta t^\alpha),
\]

where \( \alpha > 0, \beta > 0 \) and \( \nu > 0 \), it leads by taking \( \alpha = \nu, \eta = \theta^2 \) and \( \beta = 1 \) to:

\[
J^\alpha E_{\alpha,1}(\theta(2t)^\alpha) = t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha).
\]

According to the following relation, it yields:

\[
J^\alpha D^\alpha C_1(t) = C_1(t) - C_1(0), \quad 0 < \alpha < 1,
\]

we obtain

\[
C_1(t) = 2(1 - \lambda)\gamma^2 t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha) + C_1(0).
\]

We assume \( C_1(0) = 0 \). Hence, the obtained solution of fractional ordinary differential system (4.13) yields the following exact solution of the nonlinear time fractional modified Kuramoto-Sivashinsky equation (4.1):

\[
u(t, x) = 2(1 - \lambda)\gamma^2 t^\alpha E_{\alpha,\alpha+1}(\theta(2t)^\alpha) + E_{\alpha,1}(\theta t^\alpha)(\cos \gamma x + \sin \gamma x),
\]

where \( \gamma = \sqrt{1 - \lambda} \) and \( \theta = \gamma^2 - \gamma^4 \).

5 SOME PARTICULAR CASES

In this section, we extract some particular cases, precisely exact solutions corresponding to \( \lambda = \frac{1}{m} \), with \( m \in \mathbb{N}^* - \{1\} \).

Case 1 \( \lambda = \frac{1}{2} \)

This particular value of \( \lambda \) leads to \( \gamma = 1 \) and \( \theta = 0 \). Consequently, an exact solution of the studied fractional equation (4.1) is given by:

\[
u(t, x) = \frac{t^\alpha}{\Gamma(\alpha + 1)} + \cos x + \sin x, \quad 0 < \alpha \leq 1.
\]
Case 2 $\lambda = \frac{1}{m}$, $m > 2$

In this case we obtain that $\gamma = \sqrt{m-1}$ and $\theta = (m-1)(2-m)$. The constructed exact solution takes the form:

$$u_{\alpha,m}(t, x) = 2^{1-\alpha} \frac{(m-1)}{m(2-m)} \{E_{\alpha,1} [(m-1)(2-m)2^\alpha t^\alpha] - 1\}$$
$$+ E_{\alpha,1} [(m-1)(2-m)t^\alpha] \cos \sqrt{m-1}x$$
$$+ E_{\alpha,1} [(m-1)(2-m)t^\alpha] \sin \sqrt{m-1}x.$$  \hspace{1cm} (5.2)

Now we look for solutions of nonlinear time fractional equation (4.1) corresponding to $\alpha = 1$ and $\alpha = \frac{1}{2}$.

**Subcase 2.1** $\lambda = \frac{1}{m}$, $m > 2$, $\alpha = 1$.

According to the relation

$$E_{1,1}(z) = E_1(z) = e^z,$$  \hspace{1cm} (5.3)

the corresponding exact solution of equation (4.1) is obtained to be of the following form:

$$u_{1,m}(t, x) = \frac{m-1}{m(2-m)} \{e^{2(m-1)(2-m)t} - 1\} + e^{(m-1)(2-m)t} \cos \sqrt{m-1}x$$
$$+ e^{(m-1)(2-m)t} \sin \sqrt{m-1}x.$$  \hspace{1cm} (5.4)

**Subcase 2.2** $\lambda = \frac{1}{m}$, $m > 2$, $\alpha = \frac{1}{2}$.

According to the relation:

$$E_{\frac{1}{2},1}(z) = E_{\frac{1}{2}}(z) = e^{z^2} \left(1 + erf(z) \sqrt{\frac{2}{\pi}} \int_0^z e^{-y^2} dy\right),$$  \hspace{1cm} (5.5)

where

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-y^2} dy,$$  \hspace{1cm} (5.6)

the corresponding exact solution of equation (4.1) is obtained in this subcase to be of the form:

$$u_{\frac{1}{2},m}(t, x) = \sqrt{2} \frac{(m-1)}{m(2-m)} \{E_{\frac{1}{2}} \left(\sqrt{2}(m-1)(2-m)\sqrt{t}\right) - 1\}$$
$$+ e^{(m-1)(2-m)t} (\cos \sqrt{m-1}x + \sin \sqrt{m-1}x).$$  \hspace{1cm} (5.7)

We end this section by giving some corresponding graphs of some found particular solutions.
Figure 1: Solution with $\alpha = 1$ and $m = 3$

Figure 2: Solution with $\alpha = 1$ and $m = 4$

Figure 3: Solution with $\alpha = \frac{1}{2}$ and $m = 3$

Figure 4: Solution with $\alpha = \frac{1}{2}$ and $m = 4$

Figure 5: Solution with $\alpha = 1$ and $\lambda = \frac{1}{2}$

Figure 6: Solution with $\alpha = \frac{1}{2}$ and $\lambda = \frac{1}{2}$
6 Conclusion

Here, by using the Laplace transform method and some basic properties of the Mittag-Leffler function, we succeed to solve the obtained reduced system of ordinary fractional equations. Consequently, the invariant subspace method was appropriate to construct some exact solutions of the time fractional nonlinear modified Kuramoto-Sivashinsky equation. Finally, we note that the method used in this paper can be extended to obtain exact solutions of other nonlinear time fractional differential equations.

References

[1] I. Podlubny, Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications, New York: Academic Press, 1999.

[2] K.B. Oldham, J. Spanier, The fractional calculus.-Academic Press, 234p., 1974.

[3] K. Miller and B. Ross, An introduction to the fractional calculus and fractional differential Equations, John Wiley, Sons Inc., New York, 1993.

[4] A. Kilbas, H. Srivastava, and J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, Elsevier, 2006.

[5] L. Debnath, Recent applications of fractional calculus to science and engineering, International Journal of Mathematics and Mathematical Sciences, Vol. 54, 3413-3442, 2003.

[6] S. Momani, Z. Odibat, Analytical solution of a time-fractional Navier-Stokes equation by Adomian decomposition method, Comput. Math. Appl. 488-494, 2006.

[7] M. Eslami, B. Fathi Vajargah, M. Mirzazadeh and Anjan Biswas, Application of first integral method to fractional partial differential equations, Indian Journal of Physics. Volume 88, Issue 2, 177-184, 2014.

[8] H. Jafari, A. Golbabai, S. Seifi, K. Sayyand, Homotopy analysis method for solving multi-term linear and nonlinear diffusion-wave equations of fractional order, Comput. Math. Appl. 59, 1337-1344, 2010.

[9] R.K. Gazizov, A.A. Kasatkin, S.Yu. Lukashchuk, Continuous transformation groups of fractional differential equations, Vestnik, USATU 9, 125-135, 2007.

[10] R.K. Gazizov, A.A. Kasatkin, S.Yu. Lukashchuk, Symmetry properties of fractional diffusion equations, Phys. Scr. T 136, 014-016, 2009.
11 E. Buckwar, Y. Luchko, Invariance of a partial differential equation of fractional order under the Lie group of scaling transformations, J. Math. Anal. Appl. 227, 81-97, 1998.

12 A. Ouhadan and E. H. EL Kinani, Exact solution of time fractional Kolmogorov equation by using Lie symmetry analysis. Journal of Fractional Calculus and Applications, Vol. 5(1) Jan., 97-104, 2014.

13 A. Ouhadan and E. H. EL Kinani, Lie symmetry analysis of some time fractional partial differential equations. International Journal of Modern Physics: Conference Series, In press, 2014.

14 V. Galaktionov and S. Svirshchevskii, Exact solutions and invariant subspaces of nonlinear partial differential equations in mechanics and physics. Chapman and Hall/CRC applied mathematics and nonlinear science series, 2007.

15 R. Gazizov and A. Kasatkin, Construction of exact solutions for fractional order differential equations by the invariant subspace method, Computers and Mathematics with Applications, vol. 66, no. 5, 576-584, 2013.

16 G. Jumarie, Table of some basic fractional calculus formulae derived from a modified Riemann- Liouville derivative for non-differentiable functions, Appl. Math. Lett. 22 no. 3, 378-385, 2009.

17 G. Jumarie, Laplaces transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative, Appl. Math. Letter, 22, 11, 1659-1664, 2009.

18 J. C. Prajapati, Certain properties of Mittag-Leffler function with argument $x^\alpha$, $\alpha > 0$, Italian Journal of Pure and Applied Mathematics n. 30, 411-416, 2013.

19 P.J. Olver, Application of Lie groups to Differential Equation. Springer, New York, 1986.

20 G. Bluman, S. Kumei, Symmetries and Differential Equations. Applied Mathematical Sciences Series. 81 (Second ed.). New York, Springer-Verlag, 1989.

21 G.W. Bluman and J. D. Cole, The general similarity solutions of the heat equation. Journal of Mathematics and Mechanics, vol. 18, 1025-1042, 1969.

22 L.V. Ovsiannikov, Group Analysis of Differential Equations. Academic Press, New York, NY, USA, 1982.

23 D.C. Sarocka and A.J. Bernoff, An intrinsic equation of interfacial motion for the solidification of a pure hypercooled melt, Phys. D 85, 348-374, 1995.

24 G.I. Sivashinsky, On cellular instability in the solidification of a dilute binary alloy, Phys. D 8, 243-248, 1983.
[25] T. Hocherman and P. Rosenau, On KS-type equations describing the evolution and rupture of a liquid interface, Phys. D, 67, 113-125, 1993.