Electromagnetic field of a bunch intersecting a dielectric plate in a waveguide

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Abstract. The electromagnetic field (EMF) of a bunch moving uniformly and traversing a dielectric plate located in a waveguide is investigated. The main attention is focused on the case when Cherenkov radiation is generated in the plate. Analysis of the field components of the mode is performed with methods of the complex variable function theory. An algorithm of computation using the exact expressions for the EMF is also presented. Consideration of the EMF structure for different time moments is given. It is shown that Cherenkov-transition radiation (CTR) is generated in the vacuum area after the plate under certain conditions. Results obtained might be of interest for development of new methods of generation of electromagnetic radiation.

1. Introduction
Investigation of electromagnetic fields of charged particles in waveguides loaded with dielectrics is of interest for development of new methods of generation of electromagnetic radiation and acceleration of particles [1–3]. Radiation of a charged particle (or a small bunch) crossing a boundary between two homogenous media in a metal cylindrical waveguide has been analyzed in detail in our previous papers [4–6]. Such an effect as Cherenkov-transition radiation (CTR) at the dielectric-vacuum boundary [6] has been investigated. It has been shown that the CTR can be dominant in the vacuum area. It should be noted that, recently, the idea of self-amplified Cherenkov radiation (CR) in a waveguide filled with a periodic dielectric structure was discussed [7]. Therefore, it is of significance to consider an intermediate case when a beam intersects two boundaries in the waveguide. The case under investigation is one of the key problems that are important for discussion of main peculiarities of the CTR effect.

It should be noticed that the case of a charge intersecting a dielectric plate situated in a waveguide was analyzed earlier [8] (as well as the case when a charge intersects a waveguide with dielectric [9]). However, the main attention was paid to consideration of the energetic characteristics, and the EMF structure was not analyzed. Present paper is exactly devoted to a study of the total electromagnetic field for the problem under consideration.

Formulation of the problem is the following: a bunch of charge particles moves uniformly with a velocity \( V = c \beta \), in a circular waveguide of radius \( a \) along its axis through a dielectric plate with permittivity \( \varepsilon \) located in a vacuum. The length of the plate is \( d \). The bunch is characterized by a Gaussian distribution with the charge density \( \rho = q \delta(x) \delta(y) \exp\left(-\zeta^2/(2\sigma^2)\right) \) where \( \zeta = z - \beta ct \) and \( \sigma \) is a half-length of the beam. The middle of the bunch intersects the first boundary at the moment \( t = 0 \). We mark the vacuum areas before and after the plate as media 1 and 3 accordingly.
medium 2 (the dielectric plate) has a permittivity \( \varepsilon_2 = \varepsilon \). All of the media are nonmagnetic and have no dispersion.

The analytical solution to this problem is found traditionally as an expansion into a series of eigenfunctions of the transversal operator [8]. The electromagnetic field components in each media have two summands:

\[
E_j = E^f_j + E^b_j, \quad j = 1, 2, 3.
\]

The first one \( E^f_j \), called by V.L. Ginzburg [10], \( \varepsilon_2 \) is the free field connected with the influence of the boundary. It includes transition radiation (TR).

The forced field \( E^f_j \) has been well studied [11] and, mainly, analysis of the free field \( E^f_j \) presents some difficulties. Here we give expressions for longitudinal components of the free field in each of the media:

\[
E^f_j = \frac{2q j^2}{\pi a^4 \varepsilon_j} \sum_{n=1}^{\infty} \frac{Z_n^2 J_0(\chi_n r/a)}{J_2(\chi_n)} f^b_j, \quad j = 1, 2, 3,
\]

\[
I^b_j = \frac{i\beta(\varepsilon - 1)}{[1 - \beta^2]^{\frac{1}{2}} [1 - \beta^2 \varepsilon]} \int_{-\infty}^{\infty} \left( A_{nj} e^{ik_j z} + B_{nj} e^{-ik_j z} \right) e^{-\omega^2 \sigma^2 / \left(2 \beta^2 \varepsilon^2\right)} d\omega,
\]

where

\[
A_{n1} = 0, \quad B_{n1} = \frac{1}{q_n \Delta_n} \left( g^+ n^+ r^+ e^{-ik_{zd}} + g^- n^- r^- e^{ik_{zd}} - 2\sqrt{\omega^2 \varepsilon - \omega^2_n} p^+ n^+ e^{i\omega/\beta c} \right),
\]

\[
A_{n2} = \frac{1}{q_n \Delta_n} \left( -g^- n^+ r^+ e^{-ik_{zd}} - g^+ n^- r^- e^{ik_{zd}} \right), \quad B_{n2} = \frac{1}{q_n \Delta_n} \left( -g^- n^+ r^+ e^{-ik_{zd}} + g^+ n^- r^- e^{ik_{zd}} \right),
\]

\[
B_{n3} = 0, \quad A_{n3} = \frac{1}{q_n \Delta_n} \left( g^+ n^+ r^+ e^{ik_{zd}} + g^- n^- r^- e^{-ik_{zd}} \right), \quad g^\pm = \sqrt{\omega^2 \varepsilon - \omega^2_n} \pm \varepsilon \sqrt{\omega^2 \varepsilon - \omega^2_n},
\]

\[
\Delta_n = \left( g^+ n^+ \right)^2 e^{-ik_{zd}} - \left( g^- n^- \right)^2 e^{ik_{zd}}, \quad q_n = \left( \omega^2 + \beta^2 \omega^2_n \right) \left(1 - \beta^2 \varepsilon \right)^{-1}.
\]

\[
\alpha_n = \omega^2 \left(1 - \beta^2 - \beta^2 \varepsilon\right) + \beta^2 \omega^2_n, \quad \omega_n = \omega^2 \varepsilon - \omega^2_n.
\]

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\]

where

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A_{n1} = 0, \quad B_{n1} = \frac{1}{q_n \Delta_n} \left( g^+ n^+ r^+ e^{-ik_{zd}} + g^- n^- r^- e^{ik_{zd}} - 2\sqrt{\omega^2 \varepsilon - \omega^2_n} p^+ n^+ e^{i\omega/\beta c} \right),
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B_{n3} = 0, \quad A_{n3} = \frac{1}{q_n \Delta_n} \left( g^+ n^+ r^+ e^{ik_{zd}} + g^- n^- r^- e^{-ik_{zd}} \right), \quad g^\pm = \sqrt{\omega^2 \varepsilon - \omega^2_n} \pm \varepsilon \sqrt{\omega^2 \varepsilon - \omega^2_n},
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\Delta_n = \left( g^+ n^+ \right)^2 e^{-ik_{zd}} - \left( g^- n^- \right)^2 e^{ik_{zd}}, \quad q_n = \left( \omega^2 + \beta^2 \omega^2_n \right) \left(1 - \beta^2 \varepsilon \right)^{-1}.
\]

\[
\alpha_n = \omega^2 \left(1 - \beta^2 - \beta^2 \varepsilon\right) + \beta^2 \omega^2_n, \quad \omega_n = \omega^2 \varepsilon - \omega^2_n.
\]

We investigate the exact solution \( (2) \) – \( (8) \) both analytically and numerically. Analytical research is carried out with methods of the complex variable function theory. Computations are based on an algorithm using some separation of the integration path and the singularities of the integrands by taking into account the small absorption in the media. Next, we focus the main attention upon the case when the condition for CR in the dielectric plate is satisfied \( \beta^2 \varepsilon > 1 \).

### 2. Analytical research

In the analytical method, the singularities of integrands \( (3) \) are studied in a complex plane of \( \omega \). It can be seen, there are the following singularities:
two branch points $\pm \omega_0^{(1)} = \pm \omega_n - i\delta_1$, which are shown at figure 1 with corresponding branch cut defined by the equation $\text{Re}\left(\omega^2 - \omega_n^2\right) = 0$;

- two poles $\pm \omega_0^{(2)} = \pm i\beta \omega_n \left(1 - \beta^2\right)^{-1/2}$ which is always situated on an imaginary axis;

- two poles on the imaginary axis $\pm \omega_0^{(2)} = \pm i\beta \omega_n \left(\beta^2 - 1\right)^{-1/2} - i\delta_2$ if $\beta < e^{-1/2}$ or on the real axis $\pm \omega_0^{(2)} = \pm i\beta \omega_n \left(\beta^2 - 1\right)^{-1/2}$ if $\beta > e^{-1/2}$;

- a finite number of pairs of poles $\pm \omega_{ds} - i\delta_3$.

Here $\delta_1, \delta_2, \delta_3$ are positive infinitesimal quantities, as we take into account the very small losses in the media. Therefore, all of the singularities are located below a real axis (figure 1). The poles $\pm \omega_0^{(1)}$ and $\pm \omega_0^{(2)}$ are the zeros of function $q_n$ (8) in the denominators of the integrands. The zeros of function $\Delta_n$ (7) give a finite number of poles $\pm \omega_{ds}$. These poles are connected with the finite length of the dielectric plate. They can be found by solution of transcendental equation

$$\Delta_n = \text{tg}(x) - f(x, a) \bigg|_{\omega_{ds}} = 0,$$ (9)

where

$$f(x, a) = \frac{2\sqrt{\varepsilon} x \sqrt{\left(\lambda_n a\right)^2 (\varepsilon - 1) - x^2}}{x^2 (\varepsilon + 1) - \left(\lambda_n a\right)^2 (\varepsilon - 1)}.$$ $k\varepsilon_a d = d \sqrt{\omega_n^2 \varepsilon - \omega_{ds}^2} / \varepsilon \equiv x$, $\chi_n d / a \equiv \lambda_n d$.

This solution is represented graphically at figure 2. There is an upper branch of roots and also a lower branch of roots. Inequalities

$$\pi (k - 1) < \lambda_n d \sqrt{\varepsilon (\varepsilon - 1) / (\varepsilon + 1)} < \pi n,$$

$$\pi (21 - 1) / 2 < \lambda_n d \sqrt{\varepsilon (\varepsilon - 1) / (\varepsilon + 1)} < \pi (21 + 1) / 2,$$ (10)

indicate that there is always a finite number of such roots $s = k + 1$ which gives the finite number of pairs of poles $\pm \omega_{ds}$.

It can be obtained that all these poles are under inequality

$$\omega_n e^{-1/2} \leq \omega_{ds} / \omega_n, \quad j = 1, \ldots, s.$$ (11)

Thus the contributions from the poles $\pm \omega_{ds}$ to the free field give stationary waves inside the dielectric plate, and, outside the plate, they give the field exponentially decreasing with the distance from the boundaries.

If CR is generated in the dielectric plate, the contributions from the poles $\pm \omega_0^{(1)}$ to the free field give so-called Cherenkov-transition radiation (CTR) which is the reflected and transmitted waves of CR. They can be
evaluated by the residue theorem. It can be obtained that the CTR can exist at $\beta_C < \beta < \beta_{CT}$, where the lower threshold $\beta_C = \epsilon^{-1/2}$ is connected with the condition of the Cherenkov radiation generation and the upper threshold $\beta_{CT} = (\epsilon - 1)^{-1/2}$ is explained by total internal reflection of Cherenkov waves from the boundaries.

We obtain the expression for the CTR in medium 3 (after the plate) in the following form:

$$E_{CTR}^{3} = -\frac{4d}{a^2} \sum_{n=1}^{\infty} J_0 \left( \frac{2\epsilon}{\sigma_n} \right) \exp \left[ -\frac{\sigma_n^2}{2} \right] \Re \left[ \frac{(1 + \epsilon S) \exp \left( i \kappa_n \left( z - \frac{d}{\beta} \right) - \beta \epsilon \right)}{2 \epsilon S \cos (\kappa_n d) - \sin (\kappa_n d) \left( 1 - \epsilon^2 S^2 \right)} \right] \tag{12}$$

where $S = \sqrt{1 - \beta^2 (\epsilon - 1)}$, $\kappa_n = \lambda_n (\epsilon^2 - 1)^{-1/2} = \chi_n d^{-1} (\epsilon^2 - 1)^{-1/2}$. As distinct from the case of the dielectric-vacuum boundary [6], the CTR from the plate has the finite length which can be estimated with the results of our previous works. A rising edge of the CTR train can be described by formula:

$$z_1(t) = d + v_{g1} \left( t - d / \beta \right), \quad t > d / \beta,$$  \tag{13}

where $v_{g1} = c \sqrt{1 - \beta^2 (\epsilon - 1) \beta^{-1}}$ is a group velocity of the CTR in the vacuum area [6]. For a falling edge of the CTR we have

$$z_3(t) = d + v_{g1} \left( t - d / v_{g2} \right), \quad t > d / v_{g2},$$  \tag{14}

where $v_{g2} = c (\beta \epsilon)^{-1}$ is a group velocity of the CTR at the dielectric area [5]. So, the CTR train in area 3 has finite length that does not depend on time and the mode number but only on the parameters of the problem:

$$\Delta z_{CTR}^{3} = z_1(t) - z_3(t) = d \beta^{-2} \left( \beta^2 \epsilon - 1 \right) \sqrt{1 - \beta^2 (\epsilon - 1)}. \tag{15}$$

The CTR train has maximum length for some optimal parameters which are $\beta_0 = \left( \sqrt{1 + 8 \epsilon (\epsilon - 1) - 1} \right)^{1/2} (2 \epsilon)^{-1/2}$ if permittivity $\epsilon$ is fixed and $\epsilon_0 = \left( 3 + 2 \beta^2 \right) (3 \beta^2)^{-1}$ for the fixed velocity of the bunch motion. Therefore, for the case of the ultra-relativistic particle ($\beta \approx 1$) we find that $\epsilon_0 |_{\beta \to 1} \approx 1.67$. Note that even for these optimal parameters the maximum length of the CTR is less than the length of the dielectric plate: $\max \Delta z_{CTR}^{3} \approx 0.39 d$.

3. Numerical results and discussion

For calculations, we use the exact formulas (3). The integrands behave rather abruptly along the integration contour because it crosses the poles $\omega_{mn}$ and $\omega_{dn}$. To overcome these difficulties, an algorithm of computations is applied. As distinct from our previous works [4-6] we do not transform the initial integration contour (a real axis) but displace the singularities from the integration path (figure 1) by taking into account the small losses in the media. It should be underlined that the results can be obtained with a very good accuracy by optimizing the integration parameters. The exponential factor in (3) appearing because of the finite length $\sigma$ of the bunch results in decrease of significance of modes with large numbers.

The behavior of the first mode of the longitudinal component $E_z$ of the total field at different time moments $ct/\alpha$ for the plates with different lengths of the plate $d/\alpha$ is presented at figures 3 and 4 for the optimal parameters of the problem discussed above. The CTR determined by formulae (12)-(14) is also shown at pictures.
For the plate with length $d = 20a$ (figure 3) the situation in all the media is presented at different time moments. As one can see almost all the radiation is directed forward, and there is significant field not only inside the plate but also after it in the vacuum area. The moment when the CTR starts to form out of the plate is shown at figure 3,a. When the CTR train is entirely formed (figure 3,b), there is a coincidence between the exact solution and the analytical approximation in frequency but not in amplitude. At figure 3,c, the field has mainly a type of TR than the CTR.

The field in medium 3 after the plate is presented at figure 4 for relatively big plates. For the plates with length $d = 50a$ (figure 4, a, b, c) the CTR approximation is quite valid. At greater time moments (figure 4, b, c), the following wave trains are formed as well. These trains are connected with reflection and re-reflection of the CTR inside the plate. However, the approximation (12) describes only the first CTR train.

For the plates with bigger length $d = 100a$ (figure 4, d, e, f) the CTR train is longer, and the CTR approximation matches rather well with the exact solution but only in a certain limited area.

**Figure 3.** Dependence of the normalized longitudinal component $\vec{E}_x = E_x a^2 / q$ of the first mode of the total field (continuous line 1) and the CTR (dashed line 2) on distance $z/a$ for different dimensionless time moments $ct/a$ for the plate with length $d/a = 20$; $\beta = 0.99$, $\varepsilon = 1.67 + i 0.0001$, $\sigma/a = 1$, $\omega_0 = 2\pi \cdot 10$ GHz, $a = 5$ mm. The observation point is on the waveguide axis $r = 0$.

For the plate with length $d = 50a$ (figure 4, a, b, c) the CTR approximation is valid. At greater time moments (figure 4, b, c), the following wave trains are formed as well. These trains are connected with reflection and re-reflection of the CTR inside the plate. However, the approximation (12) describes only the first CTR train.

For the plates with bigger length $d = 100a$ (figure 4, d, e, f) the CTR train is longer, and the CTR approximation matches rather well with the exact solution but only in a certain limited area.

**Figure 4.** The same as in figure 3 for the field in medium 3 (after the plate) for different lengths of the plate $d/a$. 
4. Conclusion
The total electromagnetic field of a bunch moving through a dielectric plate located in a waveguide has been investigated numerically by using exact formulas. Analytical study has been made with methods of the complex variable function theory. The main attention has been paid to the case when the CR is generated in the dielectric plate.

Physical estimation shows that the CTR is generated in the vacuum area after the plate at $\beta_C < \beta < \beta_{CT}$ where the upper limit is explained by the total reflection of the CR from the boundary. The CTR train has finite length which does not depend on the time and the mode number but only on the parameters of the problem: the length of the plate $d$, permittivity $\varepsilon$ and velocity of the bunch motion $\beta$. The optimal parameters can be found in order that the length of the CTR train can be a maximum. It is shown that this length is always less than the length of the dielectric plate. So, for the CTR effect occurs, the length of the plate should be quite large in order that CR is formed inside the plate.

A complex structure of the field is revealed at the total field computations performed with using of the exact expressions. For thin plates, the CTR does not essentially signify. For thick enough layers, the CTR describes approximately the total field, but only in a certain limited spatial area (at fixed time moment). For sufficiently large time moments, the following CTR trains connected with reflection and re-reflection of radiation inside the plate are generated in addition to the first CTR train. Results obtained here might be of interest for development of new methods of generation of electromagnetic radiation.

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