A PRIORI ESTIMATES FOR THE SCALAR CURVATURE EQUATION ON $S^3$

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ABSTRACT. We obtain a priori estimates for solutions to the prescribed scalar curvature equation on $S^3$. The usual non-degeneracy assumption on the curvature function is replaced by a new condition, which is necessary and sufficient for the existence of a priori estimates, when the curvature function is a positive Morse function.

1. Introduction

Let $N \geq 3$ and $S^N$ be the standard sphere with round metric $g_0$ induced by $S^N = \partial B_1(0) \subset \mathbb{R}^{N+1}$. We study the problem: Which functions $K$ on $S^N$ occur as scalar curvature of metrics $g$ conformally equivalent to $g_0$? Writing $g = \varphi^{4/(N-2)} g_0$ this is equivalent to solving (see [3])

$$-4(N-1) \Delta_{S^N} \varphi + N(N-1) \varphi = K(\varphi) \varphi^{N+2 \over N-2}, \varphi > 0 \text{ in } S^N. \quad (1.1)$$

In stereographic coordinates $S_{\theta}(\cdot)$ centered at some point $\theta \in S^N$ equation (1.1) is equivalent to

$$-\Delta u = {K \circ S_{\theta}(x) \over N(N-1)} u^{N+2 \over N-2}, \ u > 0 \text{ in } \mathbb{R}^N, \quad (1.2)$$

where

$$u(x) = R_{\theta}(\varphi)(x) := (N(N-2))^{N-2 \over 2} (1+|x|^2)^{-N-2 \over 2} \varphi \circ S_{\theta}(x). \quad (1.3)$$

Obviously, to solve (1.1) the function $K$ has to be positive somewhere. Moreover, there are the Kazdan-Warner obstructions [7, 16], if $\varphi$ solves (1.1) then

$$\int_{S^N} \nabla x_j \cdot \nabla K \varphi^{2N \over N-2} = 0 \text{ for } j = 1 \ldots N+1.$$

In particular, a monotone function of $x_1$ can not be realized as the scalar curvature of a metric conformal to $g_0$.

Numerous studies have been made on equation (1.1) and various sufficient conditions for its solvability have been found (see [2, 4, 6, 11, 12, 18, 19] and the reference therein), usually under a non-degeneracy assumption on $K$. On $S^3$ a positive function $K$ is non-degenerate, if

$$\Delta_{S^3} K(\theta) \neq 0 \text{ if } \nabla K(\theta) = 0. \quad (nd)$$

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For positive Morse functions $K$ on $S^3$ it is shown in [3, 10, 21] that (1.1) is solvable if $K$ satisfies (nd) and
\[
d := -\left(1 + \sum_{\nabla K(\theta) = 0, \Delta_{S^3} K(\theta) < 0} (-1)^{\text{ind}(\theta)}\right) \neq 0, \tag{1.4}\]
where $\text{ind}(\theta)$ is the Morse index of $K$ at $\theta$. We are interested in the case when $N = 3$ and the non-degeneracy assumption (nd) is not satisfied.

To obtain the existence result Bahri and Coron [5] use a detailed analysis of the gradient flow of (1.1) and Schoen and Zhang [21] approximate (1.1) by subcritical problems $p > \frac{N+2}{N-2}$, which are always solvable, and analyze the possible blow-up of solutions.

We follow the approach suggested in [10] and use a continuity method. We join the curvature function $K$ to the constant function $K_0 \equiv 6$ by a one parameter family $K_t(\theta) := 6(1 + tk(\theta))$, where $k(\theta) := \frac{1}{6}(K(\theta) - 6)$, and consider
\[
-8\Delta_{S^3} \varphi + 6\varphi = 6(1 + tk(\theta))\varphi^5, \quad \varphi > 0 \quad \text{in } S^3, \tag{1.5}
\]
or in stereographic coordinates using (1.3) and $k_\theta(x) := k \circ S_\theta(x)$
\[
-\Delta u = (1 + tk_\theta(x))u^5 \quad \text{in } \mathbb{R}^3, \quad u > 0. \tag{1.6}
\]

In general there are no a priori $L^\infty$-estimates for (1.5) or (1.1) due to the noncompact group of conformal transformations of $S^N$ acting on solutions: the solutions of (1.2) for $k \equiv 0$ form a noncompact manifold (see [3, 14])
\[
Z := \{z_{\mu,y}(x) := \mu^{-\frac{N+2}{2}}(N(N-2))^{\frac{N-2}{4}}\left(1 + \left|\frac{x-y}{\mu}\right|^2\right)^{-\frac{N-2}{2}} : y \in \mathbb{R}^N, \mu > 0\},
\]
where $z_{\mu,y}(y) \to \infty$ as $\mu \to 0$.

Chang, Gursky and Yang [10] show that if $K \in C^2(S^3)$ is positive and satisfies (nd) then for every $\delta > 0$ there is a constant $C = C(\delta, K) > 0$ such that for all $t \in [\delta, 1]$ and solutions $\varphi_t$ of (1.5) we have
\[
C^{-1} \leq \varphi_t(\theta) \leq C \quad \text{and } \|\varphi_t\|_{C^{2,\alpha}(S^3)} \leq C.
\]
Furthermore, they compute the Leray-Schauder degree for (1.5) for $t > 0$ small, and show that it equals $d$ in (1.4) if $K$ is a Morse function. The a priori estimate implies the invariance of the degree as the parameter $t$ moves to 1 and gives a solution to (1.5) if $d \neq 0$. Chen and Lin [11] show that if $K \in C^2(S^3)$ is a non-degenerate Morse function then $C$ may be chosen independently of $\delta > 0$.

Hence, if (nd) fails, we face two problems: Is the a priori bound still valid and how do critical points of $K$ with $\Delta_{S^3} K = 0$ occur in the index count condition (1.4). Here, we will mainly deal with the question about the a priori bound of solutions.

In the following, unless otherwise stated, we will always assume $N = 3$ and that the function $K \in C^5(S^3)$ is positive. To give our main results we need the following notation. For $k \in C^5(S^3)$ we write $k_\theta = k \circ S_\theta$ and for a critical
point $\theta$ of $k$ we let

$$a_0(\theta) := \oint_{\mathbb{R}^3} \left( k_\theta(x) - \sum_{\ell=0}^{2} \frac{1}{\ell!} D^\ell k_\theta(0)(x)^\ell \right) |x|^{-6},$$

$$a_1(\theta) := \Delta^2 k_\theta(0) + \nabla(\Delta k_\theta(0)) \cdot (D^2 k_\theta(0))^{-1} \nabla(\Delta k_\theta(0)), \quad (1.7)$$

$$a_2(\theta) := k_\theta(0) a_1(\theta) - \frac{15}{8\pi} \int_{\partial B_1(0)} |D^2 k_\theta(0)(x)|^2 |x|^2,$$

where all differentiations are done in $\mathbb{R}$ and $\phi$ and solutions $\theta$ to Theorem 1.1. Thus, $\mathcal{A}$ is discrete and hence finite. Denote by $M$ the finite set

$$M := \{ \theta \in \mathbb{S} : \theta \in \mathcal{A}, a_0(\theta) = 0, \text{ and } a_2(\theta) \neq 0 \}.$$ 

Then for every $\delta > 0$ there is a constant $C = C(k, \delta) > 0$ such that for all

$$t \in (0, 1] \setminus \bigcup_{\theta \in M \setminus B_\delta(0)} B_\delta \left( -\frac{a_1(\theta)}{a_2(\theta)} \right),$$

and solutions $\varphi_t$ of (1.5) we have

$$C^{-1} \leq \varphi_t(x) \leq C \quad \text{and} \quad ||\varphi_t(x)||_{C^{2,\alpha}(S^3)} \leq C.$$ 

Theorem 1.1 extends the known a priori estimates to the case when $M$ may fail. If $k$ satisfies (nd) the solutions are uniformly bounded with respect to $t \in (0, 1]$. Moreover, we get uniform estimates for $t \in (0, 1]$ if $M^* = \emptyset$, where

$$M^* := \{ \theta \in M : 0 \leq -a_1(\theta)/a_2(\theta) \leq 1 \}.$$ 

Our results are optimal since we construct for every $\theta \in M^*$ solutions $\varphi_t$ which blow up as $t \to -a_1(\theta)/a_2(\theta)$. We say that $(t_i, \varphi_{t_i})$ blow up at the blow-up point $\theta \in S^3$, if $\varphi_i$ solves (1.5) with $t = t_i$, the sequence $(t_i)$ is bounded, and there is $(\theta_i)$ converging to $\theta$ such that $\varphi_i(\theta_i) \to \infty$.

**Theorem 1.2.** Under the assumptions of Theorem 1.1 let

$$M^* := \{ \theta \in M : 0 < -a_1(\theta)/a_2(\theta) \leq 1 \}.$$ 

Then there is $\delta > 0$ such that for any $\theta \in M^*$ there exists a unique $C^1$-curve

$$\{0 < \mu < \delta\} \ni \mu \mapsto (t^\theta(\mu), \varphi^\theta(\mu, \cdot)) \in (\delta, 1 + \delta) \times C^{2,\alpha}(S^3),$$

such that as $\mu \to 0$

$$t^\theta(\mu) = \frac{a_1(\theta)}{a_2(\theta)} + O(\mu^{\frac{4}{3}}),$$

and $\varphi^\theta(\mu, \cdot)$ solves (1.5) for $t = t^\theta(\mu)$ and blows up like

$$||R_\theta(\varphi^\theta(\mu, x)) - (1 + t^\theta(\mu))k(\theta)||_{C^{2,\alpha}((B_1(0))} = O(\mu^2).$$
The curves are unique, in the sense that, if \((t_i, \varphi_i) \in (\delta, 1 + \delta) \times C^{2,\alpha}(S^3)\) blow up at some \(\theta \in S^3\) then \(\theta \in M^*_+\) and there is a sequence of positive numbers \((\mu_i)\) converging to zero such that \((t_i, \varphi_i) = (t^\theta(\mu_i), \varphi^\theta(\mu_i, \cdot))\) for all but finitely many \(i \in \mathbb{N}\).

Hence, for Morse functions we obtain

**Corollary 1.3.** Suppose \(1 + k \in C^5(S^3)\) is a positive Morse function. There exists \(\delta_0 > 0\), such that for any \(0 < \delta < \delta_0\) the solutions of \(1.5\) are uniformly bounded for \(t \in [\delta, 1 + \delta]\), if and only if \(M^*_+ = \emptyset\).

For \(\theta \in M\) with \(a_1(\theta) = 0\) there is always the trivial curve of solutions,

\[
\mu \mapsto \left(0, (\mathcal{R}_\theta)^{-1} z_{\mu,0}\right) \in \mathbb{R} \times C^{2,\alpha}(S^3),
\]

which blow up at \(\theta\) as \(\mu \to 0\). In order to find a nontrivial curve, i.e. \(t(\mu) \in \mathbb{R} \setminus \{0\}\), we need to consider

\[
a_3(\theta) := \frac{12}{\pi^2} \left( D^2 k_\theta(0) \right)^{-1} \nabla(\Delta k_\theta(0)) \cdot \int_{\mathbb{R}^3} \left( \nabla k_\theta(x) - T^2_{k_\theta,0}(x) \right) \frac{|x|^{-6}}{|x|} dx + \frac{48}{\pi} (D^2 k_\theta(0))^{-1} \nabla(\Delta k_\theta(0)) \cdot \int_{\mathbb{R}^3} \left( k_\theta(x) - T^3_{k_\theta,0}(x) \right) \frac{x_i}{|x|^8} \quad (1.8)
\]

where we abbreviate the \(m\)th Taylor polynomial of \(k\) in \(y\) by

\[
T^m_{k,y}(x) := \sum_{\ell=0}^m \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell.
\]

**Theorem 1.4.** Under the assumptions of Theorem 1.1 suppose \(k \in C^6(S^3)\) and let

\[
M^*_0 := \{ \theta \in M : a_1(\theta) = 0 \text{ and } a_3(\theta) \neq 0 \}.
\]

Then there is \(\delta > 0\) such that for any \(\theta \in M^*_0\) there exists a unique \(C^1\)-curve

\[
\{0 < \mu < \delta\} \ni \mu \mapsto (t(\mu), \varphi(\mu, \cdot)) \in ((-\delta, 1 + \delta) \setminus \{0\}) \times C^{2,\alpha}(S^3),
\]

such that as \(\mu \to 0\)

\[
t(\mu) = \frac{a_3(\theta)}{a_2(\theta)} \mu + O(\mu^{1+\frac{1}{4}}),
\]

and \(\varphi(\mu, \cdot)\) solves \(1.5\) for \(t = t(\mu)\) and blows up like

\[
\| \mathcal{R}_\theta(\varphi(\mu, x)) - z_{\mu,0}(x) \|_{D^{1,2}(\mathbb{R}^3) \cap C^2(B_1(0))} = O(\mu^{2}).
\]

The curve is unique, in the sense that, if \((t_i, \varphi_i) \in ((-\delta, 1 + \delta) \setminus \{0\}) \times C^{2,\alpha}(S^3)\) blow up at \(\theta \in M^*_0\) then there is a sequence of positive numbers \((\mu_i)\) converging to zero such that \((t_i, \varphi_i) = (t(\mu_i), \varphi(\mu_i, \cdot))\) for all but finitely many \(i \in \mathbb{N}\).

To illustrate our results we give an example. Suppose \(k_\theta\) is given by

\[
k_\theta(x) = 1 + \frac{3x^2 - 2x^2 - x^2}{(1 + |x|^2)^2} + \frac{|x|^4}{(1 + |x|^2)^3} \left( b - \frac{a}{(1 + |x|^2)} - \frac{1 - a}{(1 + |x|^2)^2} \right).
\]
Then $1 + tk_\theta(x)$ is strictly positive for all $t \geq 0$, if $b \geq 0$ and $a \leq 3$, and
\[ \Delta k_\theta(0) = 0 \text{ and } \nabla k_\theta(0) = \nabla \Delta k_\theta(0) = 0. \]
Furthermore,
\[ a_0(\theta) = -\frac{\pi^2}{64}(35 - 48b + 5a), \quad a_1(\theta) = 120(b - 1), \]
\[ a_2(\theta) = 120(b - 1) - 56, \quad a_3(\theta) = 75 \left( 6b + \frac{7}{8}a - \frac{63}{8} \right). \]

Our results show: $\theta$ is not a blow-up point, if $a_0(\theta) \neq 0$, that is $a \neq -7 + \frac{48}{N}b$, or $a_2(\theta) = 0$, that is $b = \frac{22}{15}$. Moreover, if $a_0(\theta) = 0$ and $a_2(\theta) \neq 0$ then there is a curve of solutions $(t(\mu), \varphi(\mu, \cdot))$ which blow up at $\theta$ such that
\[ t(\mu) = \frac{b - 1}{\frac{22}{15} - b} + O(\mu^{\frac{1}{2}}), \quad \text{if } b \neq 1 \]
\[ t(\mu) = \frac{30}{56} \mu + O(\mu^{1 + \frac{1}{2}}), \quad \text{if } b = 1. \]

We sketch the strategy of the proofs of our main results and outline the remaining part of the paper. The transformation in (1.3) gives rise to a Hilbert space isomorphism between $H^{1,2}(S^N)$ and $D^{1,2}(\mathbb{R}^N)$, where $D^{1,2}(\mathbb{R}^N)$ denotes the closure of $C^\infty_c(\mathbb{R}^N)$ with respect to
\[ \|u\|^2 := \int_{\mathbb{R}^N} |\nabla u|^2 = \langle u, u \rangle. \]
Due to elliptic regularity (see [8]) and Harnack’s inequality it is enough to find a weak nonnegative solution of (1.1) in $H^{1,2}(S^N)$, or of the equivalent equation. Although we take advantage of both formulations, we mainly consider (1.2). We use a finite dimensional reduction of Melnikov type developed in [1, 2] and find solutions of (1.2) as critical points of $f_t : D^{1,2}(\mathbb{R}^N) \to \mathbb{R}$, where
\[ f_t(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 - \frac{N - 2}{2N} \int_{\mathbb{R}^N} (1 + tk(x))|u|^{\frac{2N}{N - 2}}. \]
For $t = 0$ the functional $f_0$ possesses, as seen above, a $N + 1$ dimensional manifold of critical points $Z$. To setup the finite dimensional reduction we need to analyze $Z$ and the spectrum of $f''_0(z)$ in detail, which is done for all $N \geq 3$ in Section 2. For the rest of the paper we will only deal with the case $N = 3$. In Section 3 we recall without proof that if $N = 3$ a sequence of solutions to (1.5) can only blow-up in a single point (see [18, 21]) and fit this result into our framework. Section 4 contains the finite dimensional reduction of our problem. In contrast to [2], where the reduction is performed for small $t$, we show that a finite dimensional reduction of (1.6) for large $t$ is still possible. We end up with a function $\bar{\alpha} : U \to \mathbb{R}^4$, where $U \subset \mathbb{R} \times Z$, such that the zeros of $\bar{\alpha}(t, \cdot)$ correspond to solutions of (1.6) with large $L^\infty$ norm. We recall that $Z$ is parametrized by $\mu$ and $y$. Now, to construct or to rule out blow-up sequences it is enough to construct or exclude zeros of $\bar{\alpha}(t, \cdot)$ for small $\mu$. To this end we need to expand $\bar{\alpha}$ up to order 5 in $\mu$ and to compute derivatives of $\bar{\alpha}$, which is done in Sections 4 and 5. We see that $\theta$ can only be a blow-up point if $\nabla k(\theta) = 0$ and $\Delta k_\theta(0) = 0$. In Section 6 we finally obtain under the assumptions of Theorems 1.1-1.4 that there are $(t_i, \varphi_i)$ which blow up at $\theta$ if and only if
Moreover, we see that
\[ \nabla k(\theta) = 0, \ \Delta k_{\theta}(0) = 0, \] and there exist positive \((\mu_i)\) converging to 0 such that
\[ 0 = a_0(\theta) + \mu_i (a_1(\theta) + t_1 a_2(\theta)) + \mu_i^2 a_3(\theta) + O(t_1 \mu_i^{1+\frac{1}{4}} + \mu_i^{2+\frac{1}{4}}). \]
This gives our main results, which are stated and proved in Section 7.

In a subsequent paper [20] we use the above a priori estimates and compute under the assumptions of Theorem 1.1, the Leray-Schauder degree \(d\) of the problem (1.5). We show that if \(M^* = \emptyset\) and \(k\) is a Morse function then,
\[ d = -\left(1 - \sum_{\theta \in \text{Crit}(k)^-} (-1)^{\text{ind}(\theta)}\right), \] where
\[ \text{Crit}(k)^- := \{ \theta \in S^3 : \nabla k(\theta) = 0 \} \]
generalizing the existence result in [3, 11, 21].

Our approach yields information about blow-up sequences as precise as we want, that is of any order in \(\mu\) or \(y\). For instance it is possible to compute the term of order \(\mu^0\), which is of interest when \(a_3(\theta)\) is zero. But the necessary computations and terms, as may already be seen in the expansion of order 5, are getting rather bulky. In higher dimensions \(N \geq 4\) solutions may blow up in more than one point and our method, which still applies with minor changes to \(N \geq 4\), will only give information about “one bubble” blow-up.

2. Preliminaries

We define for \(\mu > 0\) and \(y \in \mathbb{R}^N\) the maps \(U_\mu, T_y : \mathcal{D}^{1,2}(\mathbb{R}^N) \rightarrow \mathcal{D}^{1,2}(\mathbb{R}^N)\) by
\[ U_\mu(u) := \mu^{-\frac{N}{2}} u\left(\frac{\cdot}{\mu}\right) \text{ and } T_y(u) := u(\cdot - y). \]
With this notation the critical manifold \(Z\) is given by
\[ Z = \{ z_{\mu,y} = T_y \circ U_\mu(z_{1,0}) : y \in \mathbb{R}^N, \mu > 0 \}. \]

It is easy to check that the dilation \(U_\mu\) and the translation \(T_y\) conserve the norms \(\| \cdot \|\) and the \(L^2\)-Norm \(\| \cdot \|_{2^*}\), where \(2^* := 2N/(N - 2)\). Thus for every \(\mu > 0\) and \(y \in \mathbb{R}^N\)
\[ (U_\mu)^{-1} = (U_\mu)^t = U_{\mu^{-1}}, \ (T_y)^{-1} = (T_y)^t = T_{-y}, \] and
\[ f_0 = f_0 \circ U_\mu = f_0 \circ T_y \] (2.1)
where \((\cdot)^t\) denotes the adjoint. Twice differentiating the identities for \(f_0\) in (2.1) yields
\[ f_0''(v) = (T_y \circ U_\mu)^{-1} \circ f_0''(U_\mu(v)) \circ (T_y \circ U_\mu) \quad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^N). \] (2.2)
Moreover, we see that \(U(\mu, y, z) := T_y \circ U_\mu(z)\) maps \((0, \infty) \times \mathbb{R}^N \times Z\) into \(Z\), hence
\[ \frac{\partial U}{\partial z}(\mu, y, z) = T_y \circ U_\mu : T_z Z \to T_{T_y \circ U_\mu(z)Z} Z \] and
\[ T_y \circ U_\mu : (T_z Z)^\perp \to (T_{T_y \circ U_\mu(z)Z})^\perp. \] (2.3)
The tangent space $T^\ast_{z_{\mu,y}}Z$ at a point $z_{\mu,y} \in Z$ is spanned by $N+1$ orthonormal functions $\hat{\xi}_{\mu,y}^i$,

$$T^\ast_{z_{\mu,y}}Z = \langle \hat{\xi}_{\mu,y}^i : i = 0 \ldots N \rangle,$$

where $\hat{\xi}_{\mu,y}^i$ denotes for $i = 0$ the normalized tangent vector $\frac{d}{dz}z_{\mu,y}$ and for $1 \leq i \leq N$ the normalized tangent vector $\frac{d}{dz}z_{\mu,y} = -\frac{\partial}{\partial x}z_{\mu,y}$. By (2.3) we obtain

$$\hat{\xi}_{\mu,y}^i = \mathcal{T}_y \circ \mathcal{U}_\mu(\hat{\xi}_{1,0}^i).$$

An explicit calculation gives for $1 \leq i \leq N$

$$(\hat{\xi}_{1,0})^i = \tau_1 (1 + |x|^2)^{-\frac{N+3}{2}} (1 - 2 \frac{2}{1 + |x|^2})^i, \quad \tau_1^2 := \frac{(N + 1)\Gamma(N)}{\pi^{N/2} \Gamma(2 + N/2)N}.$$ 

For $i = 0$ we find

$$(\hat{\xi}_{1,0})^0 = \tau_0 (1 + |x|^2)^{-\frac{N+3}{2}} (1 - 2 \frac{2}{1 + |x|^2}), \quad \tau_0^2 := \frac{(N + 1)\Gamma(N)}{\pi^{N/2} \Gamma(2 + N/2)N}.$$ 

Using the canonical identification of the Hilbert space $D^{1,2}(\mathbb{R}^N)$ with its dual induced by the scalar-product we shall consider $f''_0(u)$ as an element of $D^{1,2}(\mathbb{R}^N)$ and $f''_0(u)$ as one of $L(D^{1,2}(\mathbb{R}^N))$. With this identification $f''_0(u)$ is of the form identity $-$ compact (see [2]) and hence a Fredholm operator of index zero.

Since $f''_0(z_{\mu,y})$ is a self-adjoint, compact perturbation of the identity map in $D^{1,2}(\mathbb{R}^N)$, its spectrum $\sigma(f''_0(z_{\mu,y}))$ consists of point-spectrum, possibly accumulating at 1. We fix $\lambda \in \sigma(f''_0(z_{\mu,y}))$ and a corresponding eigenfunction $u$. Then $u$ solves

$$-\Delta u - \frac{N + 2}{N - 2}z_{\mu,y}^{-2}u = \lambda (-\Delta u). \quad (2.4)$$

We expand $u$ in spherical harmonics with center $y$

$$u(y + r\vartheta) = \sum_{i=0}^{\infty} \sum_{l=1}^{c_i} v_{i,l}(r) Y_{i,l}(\vartheta), \quad r \in \mathbb{R}^+, \quad \vartheta \in S^{N-1},$$

where

$$v_{i,l}(r) = \int_{S^{N-1}} u(y + r\vartheta) Y_{i,l}(\vartheta) d\vartheta, \quad c_i := \left( \frac{N - 1 + i}{N - 1} \right) - \left( \frac{N - 3 + i}{N - 1} \right),$$

and $\{Y_{i,l}\}$ denote a $L^2(S^{N-1})$-basis of (real valued) spherical harmonics satisfying for all $i \in \mathbb{N}_0$ and $1 \leq l \leq c_i$

$$-\Delta_{S^{N-1}} Y_{i,l} = i(N + i - 2)Y_{i,l}.$$ 

There is a freedom in choosing such a $L^2$-basis and because the cases $i = 1, 2$ will be of special interest in the sequel we fix the basis-vectors in these cases. We set for $i = 1$ and $1 \leq l \leq N$

$$Y_{1,l}(\frac{x}{|x|}) := \left( \frac{2\pi^{N/2}}{N\Gamma(N/2)} \right)^{1/2} \frac{x_l}{|x|}. \quad (2.5)$$
For $i = 2$ we introduce a more convenient notation and write
\[ Y_{2,(l_1,l_2)} \left( \frac{x}{|x|} \right) := \left( \frac{\pi^{N/2}}{\Gamma(2+N/2)} \right)^{-1/2} |x|^{-2} \]
\[ \times \begin{cases} \sqrt{2} x_{l_1}x_{l_2} & \text{if } 1 \leq l_1 < l_2 \leq N \\ \sqrt{\frac{l_1-1}{n}} \left( x_{l_1}^2 - \frac{1}{t_{l_1-1}} \sum_{m=1}^{l_1-1} x_m^2 \right) & \text{if } 2 \leq l_1 = l_2 \leq N. \end{cases} \]

Since $u$ solves (2.4) the functions $v_{i,l}$ satisfy for $i \in \mathbb{N}_0$ and $1 \leq l \leq c_i$
\[-v_{i,l}'' - \frac{N-1}{r} v_{i,l}' + \frac{i(N+i-2)}{r^2} v_{i,l} = \frac{N+2}{(N-2)(1-\lambda)} z^{2i-2} v_{i,l}.\]

Making the transformation
\[ v(r) = r \frac{\pi^{N-2}}{\Gamma(N)} \zeta (\ln \mu + \ln r), \]
we obtain the equation for $i \in \mathbb{N}_0$ and $1 \leq l \leq c_i$
\[-\zeta_{i,l}'' - \frac{N(N+2)}{4(1-\lambda)} \cosh^2(t) \zeta_{i,l} = \left( -\frac{(N-2)}{2} - i(N+i-2) \right) \zeta_{i,l}.\]

Using the results in [17, p. 74] or [15] as in [13] we find
\[ \lambda_{i,j} = 1 - \frac{N(N+2)}{((N+1)+2(i+j-1))^2 - 1}. \]

The corresponding eigenfunction is given by
\[ \psi_{i,j}(t) := (1 - \tanh(t)^2)^{-\frac{N-2}{4}} \mathcal{P}_j^{(\sigma_i,\sigma_i)}(\tanh(t)), \]
where $\mathcal{P}_j^{(\sigma,\sigma)}$ denotes the Jacobi polynomial defined in (A.1) and $\sigma_i$ is given by
\[ \sigma_i := \frac{N-2}{2} + i. \]

Consequently, $\sigma(f_0''(z_{\mu,y})) = \{ \lambda_{i,j} : i, j \in \mathbb{N}_0 \}$ and the eigenspace of the eigenvalue $\lambda_{i,j}$ has dimension $c_i$ and is spanned by, $(l = 1 \ldots c_i)$
\[ \Phi_{i,j}^{(l)}(x) := a_{i,j} \mathcal{U}_l \left( |x|^2 (1+|x|^2)^{-\frac{N-2}{2}} - i \right) \mathcal{P}_j^{(\sigma_i,\sigma_i)}(1 - 2(1+|x|^2)^{-1}) Y_{i,l}(\frac{x}{|x|}), \]
\[ a_{i,j} := \frac{2(N-1+2(i+j))j!\Gamma(N-1+2i+j)}{(N-2+2(i+j))(N+2(i+j))\Gamma(N/2+i+j)^2}, \]

where $a_{i,j}$ are given by
\[ a_{i,j}^2 := \frac{2(N-1+2(i+j))j!\Gamma(N-1+2i+j)}{(N-2+2(i+j))(N+2(i+j))\Gamma(N/2+i+j)^2}. \]

To assure that the $\Phi_{i,j}^{(l)}$ are orthonormal. Since $Z$ is a manifold of critical points of $f_0''$, the tangent space $T_z Z$ at a point $z \in Z$ is contained in the kernel $N(f_0''(z))$ of $f_0''(z)$. As $\lambda_{i,j} = 0$ if and only if $i+j = 1$, the dimension of $N(f_0''(z))$ is $N + 1$, which implies that
\[ T_z Z = N(f_0''(z)) \quad \text{for all } z \in Z. \]
More precisely, we have
\[ \Phi_{1,0,l} = \xi_l \text{ for } l = 1, \ldots, N \text{ and } \Phi_{0,1,1} = \xi_0. \]  
(2.9)

If (2.8) holds the critical manifold \( Z \) is called non-degenerate (see [1]) and the self-adjoint Fredholm operator \( f_0'(z) \) maps the space \( D^{1,2}(\mathbb{R}^N) \) into \( T_z Z \) and is invertible in \( L(T_z Z) \). From (2.2) and (2.3), we obtain in this case
\[ \| (f_0'(z(1,0))^{-1} \|_{L(T_z Z)} = \| (f_0'(z))^{-1} \|_{L(T_z Z)} \quad \forall z \in Z. \]  
(2.10)

3. Blow up analysis

Let \( (K_i) \in C^1(S^3) \) satisfy for some \( A_0 \)
\[ A_0^{-1} \leq K_i(x) \leq A_0 \text{ and } \| \nabla K_i \|_\infty \leq A_0. \]  
(3.1)

We have the following result (see [18, 21])

**Theorem 3.1.** Suppose \( (K_i) \in C^1(S^3) \) satisfies (3.1) with \( N = 3 \). Then after passing to a subsequence either \( (\varphi_i) \) is uniformly bounded in \( L^\infty(S^3) \) and hence in \( C^{2,\alpha}(S^3) \) by elliptic regularity or \( (\varphi_i) \) has precisely one isolated simple blow-up point \( \theta \), i.e. there exists a sequence \( (\theta_i) \) of maxima of \( \varphi_i \) converging to some \( \theta \in S^3 \) and \( C = C(A_0) \) such that \( \varphi_i(\theta_i) \to +\infty \) and in geodesic normal coordinates about \( \theta_i \) given by \( \exp_{\theta_i}(\cdot) \)
\[ i\varphi_i(\theta_i)^{-2} \to 0, \]
\[ \left\| \frac{\varphi_i(\exp_{\theta_i} \left( \frac{x}{\varphi_i(\theta_i)^2} \right)) - \left( 1 + \frac{K_i(\theta_i)}{24} |x|^2 \right)^{-\frac{1}{2}}}{\varphi_i(\theta_i)} \right\|_{C^{2,\alpha}(\{|x| \leq 3\})} \leq i^{-4}, \]
\[ \varphi_i(x) \leq C \varphi_i(\theta_i)^{-1} \text{dist}_{S^3}(x, \theta_i)^{-1} \quad \text{for \text{dist}_{S^3}(x, \theta_i) \geq i\varphi_i(\theta_i)^{-2}}. \]

We need a slightly different version of this result.

**Corollary 3.2.** Under the assumptions of Theorem 3.1 the sequence \( (\varphi_i) \in C^2(S^3) \) is, after passing to a subsequence, either uniformly bounded in \( C^{2,\alpha} \) or there exist \( \theta \in S^3 \) and sequences \( (\mu_i) \in (0, \infty), (y_i) \in \mathbb{R}^3 \) satisfying
\[ \lim_{i \to \infty} \mu_i = 0, \quad \lim_{i \to \infty} y_i = 0, \]
such that in stereographic coordinates \( S_\theta(\cdot) \) about \( \theta \) the function \( u_i \) defined by the transformation \( \text{exp}_{\theta_i}(\cdot) \) satisfies
\[ u_i - 6\frac{\mu_i}{\overline{\varphi_i}(\theta_i)} - \frac{\mu_i}{\overline{\varphi_i}(\theta_i)} z_{\mu_i, y_i} \text{ is orthogonal to } T_{z_{\mu_i, y_i}} Z, \]  
(3.2)
\[ \| u_i - 6\frac{\mu_i}{\overline{\varphi_i}(\theta_i)} - \frac{\mu_i}{\overline{\varphi_i}(\theta_i)} z_{\mu_i, y_i} \|_{D^{1,2}(\mathbb{R}^3)} = o_{A_0}(1). \]  
(3.3)

To prove the corollary we first need the following lemma, which is an easy consequence of Theorem 3.1.

**Lemma 3.3.** Under the assumptions of Theorem 3.1 the sequence \( (\varphi_i) \in C^2(S^3) \) is, after passing to a subsequence, either uniformly bounded in \( C^{2,\alpha} \) or there exists a sequence \( (\theta_i) \) of maxima of \( \varphi_i \) converging to some \( \theta \in S^3 \) and such that \( \varphi_i(\theta_i) \to +\infty \) and in stereographic coordinates \( S_\theta(\cdot) \) using the transformation \( \text{exp}_{\theta_i}(\cdot) \)
\[ \| u_i - 6\frac{\mu_i}{\overline{\varphi_i}(\theta_i)} - \frac{\mu_i}{\overline{\varphi_i}(\theta_i)} z_{\mu_i, y_i} \|_{D^{1,2}(\mathbb{R}^3)} = o_{A_0}(1), \]  
(3.3)
where $\mu_i \to 0$ and $y_i \to 0$ are given by

$$y_i := S^{-1}_\theta(\theta_i), \quad \mu_i := \left(6/K_1(\theta_i)\right)^{1/2} \varphi_1(\theta_i)^{-2}.$$  

**Proof of Corollary 3.2.** From Lemma 3.3 we infer that there are $\theta \in S^3$, $(\hat{\mu}_i)$, and $(\hat{y}_i)$ such that

$$(\mu_i + |y_i| + \|u_i - 6^{1/4} \left(K_i \circ S_\theta(\hat{y}_i)\right)^{-1/2} z_{\hat{\mu},\hat{y}}\|_{D^{1,2}(\mathbb{R}^3)} = o_{A_0}(1).$$

For $i$ fixed, consider

$$d_i := \inf_{\mu, y} \|u_i - 6^{1/4} \left(K_i \circ S_\theta(\hat{y}_i)\right)^{-1/2} z_{\hat{\mu},\hat{y}}\|_{D^{1,2}(\mathbb{R}^3)}^2.$$  

Clearly $d_i = o_{A_0}(1)$ and therefore $d_i$ is attained at $\mu_i$, $y_i$ and

$$u_i - 6^{1/4} \left(K_i \circ S_\theta(\hat{y}_i)\right)^{-1/2} z_{\hat{\mu}_i,\hat{y}_i}$$

is orthogonal to $T z_{\hat{\mu}_i,\hat{y}_i} Z$.  

Since $z_{\hat{\mu}_i,\hat{y}_i}$ is orthogonal to $T z_{\hat{\mu}_i,\hat{y}_i} Z$ relation 3.3 follows. To prove rest of the claim we need to estimate $|y_i - \hat{y}_i|$ and $|\mu_i - \hat{\mu}_i|$. To this end we observe that by construction

$$o_{A_0}(1) = \|z_{\hat{\mu}_i,\hat{y}_i} - z_{\hat{\mu}_i,\hat{y}_i} - z_{1,0}\|_{D^{1,2}(\mathbb{R}^3)}.$$  

Since

$$\lim_{\mu, \mu^{-1} + |y| \to \infty} \|z_{\mu, y} - z_{1,0}\|_{D^{1,2}(\mathbb{R}^3)}^2 = 2\|z_{1,0}\|_{D^{1,2}(\mathbb{R}^3)}^2,$$

we see that there is $R_1 = R_1(A_0) > 0$ such that

$$(R_1)^{-1} \leq \hat{\mu}_i/\mu_i \leq R_1 \text{ and } |y_i - \hat{y}_i| \leq R_1.$$  

Now by explicit calculations or elliptic regularity, we have

$$\max \left\{|y_i - \hat{y}_i|, \left|\frac{\hat{\mu}_i}{\mu_i} - 1\right|\right\} \leq \text{const}(A_0) \|z_{\hat{\mu}_i,\hat{y}_i} - z_{1,0}\|_{D^{1,2}(\mathbb{R}^3)} = o_{A_0}(1),$$

which gives the claim. \hfill \Box

### 4. Expansion of the perturbation terms $w$ and $\bar{\alpha}$

For the rest of the paper we will only treat the case $N = 3$. Unless otherwise indicated, integration extends over $\mathbb{R}^3$ and is done with respect to the variable $x$. Moreover, we will write $k$ instead of $k_\theta$ when there is no possibility of confusion to avoid cumbersome subindexing.

From the change of coordinates $x \mapsto \mu x + y$, H"older’s and Sobolev’s inequality we get

**Lemma 4.1.** Let $y \in \mathbb{R}^3$, $\tau > 0$ and $f, r : \mathbb{R}^3 \to \mathbb{R}$ measurable such that

$$|r(x)| \leq C_r |x - y|^\sigma \text{ in } B_1(y), \quad |r(x)| \leq C_r |x - y|^{\tilde{\sigma}} \text{ in } \mathbb{R}^3 \setminus B_1(y),$$

$$|f(x)| \leq C_f |x|^{-s} \text{ in } B_1(0), \quad |f(x)| \leq C_f |x|^{-m} \text{ in } \mathbb{R}^3 \setminus B_1(0),$$

for some $C_r, m, s > 0$ and $0 \leq \tilde{\sigma}, \sigma$. Then there is $C = C(\tau, C_r) > 0$ such that for $v, v_1, v_2 \in D^{1,2}(\mathbb{R}^3)$:

For $0 \leq \tilde{\sigma}, \sigma \leq m - 3 - \tau$ and $s + \tau \leq 3 + \sigma$ there holds

$$|\int r(x) \mu^{-3} f \left(\frac{x - y}{\mu}\right)| \leq C(\chi_{(0,1)}(\mu)\mu^\sigma + \chi_{(1,\infty)}(\mu)(\mu^{s-3} + \mu^{\tilde{\sigma}})),$$
if \(0 \leq \sigma < \mu \leq m - \frac{\mu}{2} - \tau\) and \(s + \tau < \sigma + \frac{\mu}{2}\) then
\[
\left\| \int \frac{r(x)}{\mu^2} f \left( \frac{x-y}{\mu} \right) \right\|_{D^{1,2}(\mathbb{R}^3)} \leq C \left( \chi_{(0,1)}(\mu) \mu^\sigma + \chi_{(1,\infty)}(\mu)(\mu^{s-\frac{\mu}{2}} + \mu^\sigma) \right),
\]
if \(0 \leq \sigma < m - 2 - \tau\) and \(s + \tau < \sigma + 2\) then
\[
\sup_{\|v\| \leq 1} \left\| \int r(x) \mu^{-2} f \left( \frac{x-y}{\mu} \right) v \cdot \right\|_{D^{1,2}(\mathbb{R}^3)} \leq C \left( \chi_{(0,1)}(\mu) \mu^\sigma + \chi_{(1,\infty)}(\mu)(\mu^{s-2} + \mu^\sigma) \right),
\]
if \(0 \leq \sigma < m - \frac{3}{2} - \tau\) and \(s + \tau < \sigma + \frac{3}{2}\) then we have
\[
\sup_{\|v_1\|,\|v_2\| \leq 1} \left\| \int r(x) \mu^{-\frac{3}{2}} f \left( \frac{x-y}{\mu} \right) v_1 v_2 \cdot \right\|_{D^{1,2}(\mathbb{R}^3)} \leq C \left( \chi_{(0,1)}(\mu) \mu^\sigma + \chi_{(1,\infty)}(\mu)(\mu^{s-\frac{3}{2}} + \mu^\sigma) \right).
\]

Using the above estimates we may prove the main ingredient for the finite dimensional reduction.

**Lemma 4.2.** Suppose \(k \in C^6(\mathbb{R}^3)\) and there are \(A_0, B_0, B_1 > 0\) such that
\[
\max \left( B_0, B_1, \sup_{|m| \leq 5} \| D^m k \|_{\infty} \right) \leq A_0 \text{ and } A_0^{-1} \leq 1 + tk(x) \forall (x,t) \in \mathbb{R}^3 \times [-B_0, B_1].
\]
Then there exist \(\rho_0 = \rho_0(A_0) > 0, t_0 = t_0(A_0) > 0,\) an upper continuous function \(\mu_0 : \Omega \to \mathbb{R}_+ \cup \{ \infty \},\) depending only on \(A_0,\) and two functions \(w : \Omega \to D^{1,2}(\mathbb{R}^3)\) and \(\tilde{\alpha} : \Omega \to \mathbb{R}^4,\) where
\[
\Omega := \{ (t,\mu,y) \in [-B_0,B_1] \times (0,\infty) \times \mathbb{R}^3 : 0 < \mu < \mu_0(t) \},
\]
\[
\mu_0(t) = +\infty \text{ if } |t| \leq t_0,
\]
such that for any \((t,\mu,y) \in \Omega\)
\[
w(t,\mu,y) \text{ is orthogonal to } T_{z_{\mu,y}} Z \quad (4.1)
\]
\[
f_t'(z_{\mu,y} + w(t,\mu,y)) = \tilde{\alpha}(t,\mu,y) \cdot \dot{z}_{\mu,y} \in T_{z_{\mu,y}} Z \quad (4.2)
\]
\[
\|w(t,\mu,y) - w_0(t,\mu,y)\| + \|\tilde{\alpha}(t,\mu,y)\| < \rho_0, \quad (4.3)
\]
where \(\{ \xi_i : i = 0 \ldots 3 \}\) denotes the orthonormal basis of \(T_{z_{\mu,y}} Z\) given in \(\mathbb{R}^4\) and
\[
w_0(t,\mu,y) := ((1 + tk(y))^{-\frac{1}{2}} - 1)z_{\mu,y}.
\]
The functions \(w\) and \(\tilde{\alpha}\) are of class \(C^2\) and unique in the sense that if \((v, \tilde{\beta})\) satisfies \(\|w, \tilde{\alpha}\| \leq \infty\) for some \((t,\mu,y) \in \Omega\) then \((v, \tilde{\beta})\) is given by \(w(t,\mu,y), \tilde{\alpha}(t,\mu,y)\)). Moreover, we have for \(1 \leq j \leq 3\)
\[
\|w(t,\mu,y) - \sum_{i=0}^2 w_i(t,\mu,y)\| + \|\tilde{\alpha}(t,\mu,y) - \sum_{i=1}^2 \tilde{\alpha}_i(t,\mu,y)\| \leq O_{A_0} \left( t(\|
abla k(y)\|^2 \min(1,\mu_0^2) + \min(1,\mu_0^2)) \right), \quad (4.4)
\]
\[
\|\tilde{\alpha}(t,\mu,y) - \sum_{i=1}^2 \tilde{\alpha}_i(t,\mu,y)\| \leq O_{A_0} \left( t \min(1,\mu_0^2) \right),
\]
where
\[ \tilde{\alpha}_1(t, \mu, y) := -t \min(1, \mu)(1 + tk(y))^{-\frac{3}{4}} \frac{\pi}{3^\frac{1}{4}} \sqrt{5} \left( 0 \right) \nabla k(y), \]
\[ \tilde{\alpha}_2(t, \mu, y) := -t \min(1, \mu^2)(1 + tk(y))^{-\frac{3}{4}} \frac{\pi}{3^\frac{1}{4}} \sqrt{5} \left( \Delta k(y) \right). \]

and
\[ w_1(t, \mu, y) := t \min(1, \mu)(1 + tk(y))^{-\frac{3}{4}} T_y \circ \mathcal{U}_\mu (\tilde{w}_1(y)), \]
\[ \tilde{w}_1(y) := \mathcal{F}_0^{-1} \left( \int \nabla k(y)x(z_{1,0})^5 \right), \]
\[ w_2(t, \mu, y) := t \min(1, \mu^2)(1 + tk(y))^{-\frac{3}{4}} T_y \circ \mathcal{U}_\mu (\tilde{w}_2(y)), \]
\[ \tilde{w}_2(y) := \mathcal{F}_0^{-1} \left( \frac{1}{2} \int D^2k(y)x^2(z_{1,0})^5 \right), \]
The operator \( \mathcal{F}_0^{-1} \in \mathcal{L}(\mathcal{D}^{1,2}(\mathbb{R}^3), T_{z_1,0} Z^\perp) \) is defined by
\[ \mathcal{F}_0^{-1} := (f''_0(z_{1,0})|_{T_{z_1,0} Z^\perp})^{-1} \circ \text{Proj}_{T_{z_1,0} Z^\perp}. \]

Proof. Define \( H : \mathbb{R} \times (0, \infty) \times \mathbb{R}^3 \times \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R}^4 \rightarrow \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathbb{R}^4 \)
\[ H(t, \mu, y, w, \tilde{\alpha}) := (f''_0(z_{\mu,y} + w) - \tilde{\alpha} \cdot \tilde{\xi}_{\mu,y}, (\langle w, (\tilde{\xi}_{\mu,y}) \rangle)), \]
If \( H(t, \mu, y, w, \tilde{\alpha}) = (0, 0) \) then \( w \) satisfies (2.10-2.12). We have
\[ \left( \frac{\partial H}{\partial (w, \tilde{\alpha})}(t, \mu, y, w, \tilde{\alpha}) \right) \begin{pmatrix} \varphi \\ \beta \end{pmatrix} = (f''_0(z_{\mu,y} + w)\varphi - \tilde{\beta} \cdot \tilde{\xi}_{\mu,y}, (\langle \varphi, (\tilde{\xi}_{\mu,y}) \rangle)), \]
\[ \text{(4.5)} \]
Note that
\[ \left\langle \left( \frac{\partial H}{\partial (w, \tilde{\alpha})}(0, \mu, y, 0, 0) \right) \begin{pmatrix} w \\ \beta \end{pmatrix}, (f''_0(z_{\mu,y})w - \tilde{\beta} \cdot \tilde{\xi}_{\mu,y}, (\langle w, (\tilde{\xi}_{\mu,y}) \rangle)) \right\rangle \]
\[ = \|f''_0(z_{\mu,y})w\|^2 + |\tilde{\beta}|^2 + |\langle w, (\tilde{\xi}_{\mu,y}) \rangle|^2. \]
\[ \text{(4.6)} \]
From (2.10) and (4.6) we infer that \( (\frac{\partial H}{\partial (w, \tilde{\alpha})}(0, \mu, y, 0, 0)) \) is an injective Fredholm operator of index zero, hence invertible and
\[ \left\| \left( \frac{\partial H}{\partial (w, \tilde{\alpha})}(0, \mu, y, 0, 0) \right)^{-1} \right\| \leq 1 + \|f''_0(z_{\mu,y})^{-1}\|_{\mathcal{L}(T_{z_1,0} Z^\perp)} := C_s. \]
\[ \text{(4.7)} \]
Clearly, \( H(t, \mu, y, w, \tilde{\alpha}) = (0, 0) \) if and only if \( (w, \tilde{\alpha}) = F_{t,\mu,y}(w, \tilde{\alpha}), \)
\[ F_{t,\mu,y}(w, \tilde{\alpha}) := -\left( \frac{\partial H}{\partial (w, \tilde{\alpha})}(0, \mu, y, 0, 0) \right)^{-1} H(t, \mu, y, w, \tilde{\alpha}) + (w, \tilde{\alpha}). \]

We will prove that \( F_{t,\mu,y}(w, \tilde{\alpha}) \) is a contraction in some ball
\[ B_\rho \left( \sum_{i=0}^2 w_i(t, \mu, y), \sum_{i=1}^2 \tilde{\alpha}_i(t, \mu, y) \right) \]
for any radius \( \rho \) such that
\[ O_{A_0} \left( t \left( |\nabla k(y)| \min(1, \mu^2) + \min(1, \mu^{2+\frac{1}{3}}) \right) \right) \leq \rho \leq \rho_0. \]
where \( \rho_0 = \rho_0(A_0) \) will be chosen later.
To this end we fix \( \rho > 0 \) and \( (w, \vec{\alpha}) \in B_\rho(0, 0) \). In the sequel we will suppress the dependence of \( \vec{\alpha}_i \) and \( w_i \) on \( t, \mu \) and \( y \). From \(^{(2.7)}\) and Sobolev's inequality

\[
\frac{1}{C_*} \| F_{t,\mu,y}(w + \sum_{i=0}^{2} w_i, \vec{\alpha} + \sum_{i=1}^{2} \vec{\alpha}_i) - (\sum_{i=0}^{2} w_i, \sum_{i=1}^{2} \vec{\alpha}_i) \| \\
\leq \| f_0'(z_{\mu,y} + w + \sum_{i=0}^{2} w_i) - \sum_{i=1}^{2} \vec{\alpha}_i : \dot{\xi}_{\mu,y} - f_0''(z_{\mu,y})w \| \\
\leq \| (z_{\mu,y} + \sum_{i=0}^{2} w_i - \sum_{i=1}^{2} \vec{\alpha}_i : \dot{\xi}_{\mu,y}) \cdot - 5 \int (z_{\mu,y})^4w. \\
- \int (1 + tk(x)) \left( \sum_{i=0}^{2} \left( \frac{5}{i} \right) (z_{\mu,y} + w_0)^{5-i}(w_1 + w_2 + w)^i \right) \| \\
+ O_{A_0}(\| w_1 + w_2 + w \|^3). \tag{4.8}
\]

Obviously, \((1 + tk(y))(z_{\mu,y} + w_0)^4 = (z_{\mu,y})^4 \) and

\[
(z_{\mu,y} + w_0, \varphi) = \int (1 + tk(y))(z_{\mu,y} + w_0)^5 \varphi. \tag{4.9}
\]

Inserting this in \(^{(4.8)}\) and using Lemma \(^{(4.1)}\) we get

\[
\frac{1}{C_*} \| F_{t,\mu,y}(w + \sum_{i=0}^{2} w_i, \vec{\alpha} + \sum_{i=1}^{2} \vec{\alpha}_i) - (\sum_{i=0}^{2} w_i, \sum_{i=1}^{2} \vec{\alpha}_i) \| \\
\leq \| f_0''(z_{\mu,y})(w_1 + w_2) - ((\vec{\alpha}_1 + \vec{\alpha}_2) : \dot{\xi}_{\mu,y}) \cdot \\
- t \int (k(x) - k(y))(z_{\mu,y} + w_0)^5 \cdot \| \\
+ O_{A_0}(t^2(\| \nabla k(y) \| \min(1, \mu^2) + \min(1, \mu^3)) \| \\
+ O_{A_0}(t \min(1, \mu)\| w \| + \| w \|^2). \tag{4.10}
\]

From \(^{(2.1)} \) and the definition of \( w_1 \) and \( w_2 \) we infer

\[
f_0''(z_{\mu,y})w_1 = t \min(1, \mu^{-1})\text{Proj}_{T_{z_{\mu,y}}}Z_\perp \left( \int \nabla k(y)(x - y)(z_{\mu,y} + w_0)^5 \cdot \right),
\]

\[
f_0''(z_{\mu,y})w_2 = t \min(1, \mu^{-2})\text{Proj}_{T_{z_{\mu,y}}}Z_\perp \left( \int \frac{1}{2} D^2 k(y)(x - y)^2(z_{\mu,y} + w_0)^5 \cdot \right).
\]

To find \( \vec{\alpha}_1 \) and \( \vec{\alpha}_2 \) we observe that since \( \dot{\xi}_{\mu,y} \) is even and \( \dot{\xi}_{\mu,y}_i \) is odd for \( 1 \leq i \leq 3 \) we have

\[
\int \nabla k(y)(x - y)(z_{\mu,y} + w_0)^5(\dot{\xi}_{\mu,y})_0 = 0,
\]

\[
\int \frac{1}{2} D^2 k(y)(x - y)^2(z_{\mu,y} + w_0)^5(\dot{\xi}_{\mu,y})_i = 0 \text{ for } 1 \leq i \leq 3.
\]
For $1 \leq i \leq 3$ we get from (A.4) - (A.7)

$$
\int \nabla k(y)(x - y)(z_{\mu, y} + w_0)^5(\xi_{\mu, y})_i = \mu \frac{\pi}{3^\frac{1}{2}} \frac{\pi}{\sqrt{3}} (1 + t k(y))^{\frac{2}{5}} \frac{\partial k}{\partial x_i}(y).
$$

For $i = 0$ we get

$$
\int \frac{1}{2} D^2 k(y)(x - y)^2(z_{\mu, y} + w_0)^5(\xi_{\mu, y})_i = \mu^2 \frac{\pi}{3^\frac{1}{2}} \frac{\pi}{\sqrt{3}} (1 + t k(y))^{\frac{2}{5}} \frac{\partial^2 k}{\partial x_i^2}(y).
$$

Finally, we obtain

$$
f''_0(z_{\mu, y})(w_1 + w_2) - (\alpha_1 + \alpha_2) \cdot \xi_{\mu, y}
= t \left( \int \sum_{\ell=1}^2 \frac{D^\ell k(y)}{\ell!}(x - y)^{\ell}(z_{\mu, y} + w_0)^5 \cdot \right) + O_{A_0}(t \chi_{(1, \infty)}(\mu)),
$$

(4.11)

which implies together with (4.10) and Lemma 4.1

$$
\frac{1}{C_*} \| F_{t, \mu, y}(w + \sum_{i=0}^2 w_i, \alpha + \sum_{i=1}^2 \alpha_i) - (\sum_{i=0}^2 w_i, \sum_{i=1}^2 \alpha_i) \|
\leq O_{A_0} \left( t \left( |\nabla k(y)| \min(1, \mu^2) + \min(1, \mu^{2+\frac{1}{4}}) \right) \right)
+ O_{A_0} \left( t \min(1, \mu) \|w\| + \|w\|^2 \right).
$$

(4.12)

Consequently, if we fix $0 < \rho_0 < 1/4$ we obtain functions $\mu_0$ and $t_0$ depending on $\rho_0$ and $A_0$ such that $F_{t, \mu, y}$ maps $B_{\rho}(\sum_{i=0}^2 w_i, \sum_{i=1}^2 \alpha_i)$ into itself for every $(t, \mu, y) \in \Omega$ and $\rho > 0$ satisfying

$$
\text{const}(A_0, \rho_0)(|\nabla k(y)| \min(1, \mu^2) + \min(1, \mu^{2+\frac{1}{4}})) < \rho \leq \rho_0.
$$

(4.13)

To show that $F_{t, \mu, y}$ is a contraction we fix $\rho > 0$ and two vectors $(v_1, \beta_1)$ and $(v_2, \beta_2)$ in $B_{\rho}(0, 0)$. Then using Lemma 4.1 and 4.7

$$
\frac{\| F_{t, \mu, y}(\sum_{i=0}^2 w_i + v_i, \sum_{i=1}^2 \alpha_i + \beta_1) - F_{t, \mu, y}(\sum_{i=0}^2 w_i + v_2, \sum_{i=1}^2 \alpha_i + \beta_2) \|}{C_* \|(v_1, \beta_1) - (v_2, \beta_2)\|}
\leq \int_0^1 \| f''_0(z_{\mu, y} + \sum_{i=0}^2 w_i + v_1 + s(v_2 - v_1)) - f''_0(z_{\mu, y}) \| \, ds
\leq \int_0^1 \| f''_0(z_{\mu, y} + w_0) - f''_0(z_{\mu, y}) \| \, ds + O_{A_0}(\rho + \|w_1 + w_2\|)
\leq O_{A_0}(\rho + t \min(1, \mu)).
$$

Thus we have shown there are $\rho_0 > 0$ and two function $\mu_0$ and $t_0$ depending only on $A_0$, as claimed above, such that $F_{t, \mu, y}$ is a contraction in $B_{\rho}(\sum_{i=0}^2 w_i, \sum_{i=1}^2 \alpha_i)$ for every $(t, \mu, y) \in \Omega$ and $\rho > 0$ satisfying (4.13). From Banach's fixed-point theorem we deduce the existence and uniqueness of the functions $w$ and $\alpha$. The usual inverse function theorem yields the $C^2$ dependence. The estimates in (4.2) hold due to the uniqueness of the fixed-point and because $F_{t, \mu, y}$ is a contraction for every $\rho > 0$ satisfying (4.13).
We need a precise expansion of $w$ and $\bar{\alpha}$ in critical points $y$ of $k$ in terms of $\mu$ and $t$. This will be done up to order 5 in $\mu$. We later see that we may assume $|\nabla k(y)|$ to be of order $O(\mu^2)$. First we compute $\bar{w}_2(y)$ in terms of the eigenfunctions of $f''_n(z_{1,0})$.

**Lemma 4.3.** Under the assumptions of Lemma 4.2 we have

$$
\bar{w}_2(y) = \frac{3^3\sqrt{\pi}}{8} \sum_{j=0}^{\infty} \frac{\Gamma(j + \frac{3}{2})(5 + 2j)a_{2,j}}{\Gamma(j + 6)} \times \left( \sum_{1 \leq l < m \leq 3} \psi_{l,m}(y)\Phi_{2,j,(l,m)}^{1,0} + \sum_{l=2}^{3} \psi_{l,l}(y)\Phi_{2,j,(l,l)}^{1,0} \right) - \frac{\pi 3^{3\frac{3}{2}}}{16} \Delta k(y)\Phi_{0,0,1}^{1,0} + \frac{3^3\pi}{4} \Delta k(y) \sum_{j=2}^{\infty} \frac{\Gamma(j + \frac{3}{2})(1 + 2j)a_{0,j}}{\Gamma(j + 2)(j + 3)(j - 1)} a_{0,j} \Phi_{0,j,1}^{1,0},
$$

where we use the basis defined in (2.6), (2.7) and

$$
\begin{align*}
\psi_{l,m}(y) &:= \frac{2\sqrt{\pi}}{\sqrt{15}} \frac{\partial^2 k(y)}{\partial x_l \partial x_m}, \\
\psi_{2,2}(y) &:= \sqrt{\pi} \frac{\partial^2 k(y)}{\partial x_2^2} - \frac{\partial^2 k(y)}{\partial x_1^2}, \\
\psi_{3,3}(y) &:= \sqrt{\pi} \frac{\partial^2 k(y)}{\partial x_3^2} - \frac{\Delta k(y)}{3}.
\end{align*}
$$

**Proof.** To prove the claim we observe that if

$$
\text{Proj}_{T_{z_{1,0}}} \left( \frac{1}{2} \int D^2 k(y)(x)^2(z_{1,0})^2 \right) = \sum_{i+j \neq 1} c_{i,j} \Phi_{i,j,i}^{1,0}
$$

then (2.7) implies

$$
\bar{w}_2(y) = \sum_{i+j \neq 1} \sum_{l=1}^{c_i} \beta_{i,j,l} \Phi_{i,j,l}^{1,0}.
$$

To this end we note that in the basis given in (2.6)

$$
\frac{1}{2} D^2 k(y)(x)^2 = \frac{\Delta k(y)}{6} ||x||^2 + \sum_{l=2}^{3} \psi_{l,l}(y)Y_{2,(l,l)}(x) + \sum_{1 \leq l < m \leq 3} \psi_{l,m}(y)Y_{2,(l,m)}(x).
$$

Consequently,

$$
\bar{w}_2(y) \in \langle \Phi_{2,j,l}^{1,0}, \Phi_{0,j,1}^{1,0} : j \in \mathbb{N}_0, 1 \leq l \leq c_2 \rangle.
$$

Using (A.1)-(A.7) we find for $n = (l,l)$ or $n = (l,m)$

$$
\begin{align*}
\int \psi_n Y_{2,n}(x/||x||)||x||^2(z_{1,0})^5 \Phi_{2,j,n}^{1,0} &= \psi_n a_{2,j,3}^{\frac{3}{2}} \xi_{2,j,n}^{\frac{3}{2}}(j + 1)\Gamma(3/2)\Gamma(j + 7/2) \Gamma(j + 5) \\
\int ||x||^2 \Phi_{0,j,0}^{1,0} &= a_{0,j} \xi_{0,j}^{\frac{3}{2}} 2\sqrt{\pi} \left( \frac{\Gamma(1/2)\Gamma(j + 3/2)}{2\Gamma(j + 2)} - \delta_{0,j} \frac{\Gamma(3/2)}{2\Gamma(3)} \right).
\end{align*}
$$

Now, the claim follows from (4.14) and (4.15).
Remark 4.4. Under the assumptions of Lemma 4.2, an explicit calculation together with (4.3) yields for \((t, \mu, y) \in \Omega\), where we suppress the dependence of \(w\) and \(\bar{\alpha}\) on \((t, \mu, y)\),

\[
f''_t(z_{\mu,y} + w)\varphi = f''_0(z_{\mu,y})\varphi - 5t \int (k(x) - k(y))(z_{\mu,y} + w_0)^4 \varphi - 20 \int (1 + tk(x))(z_{\mu,y} + w_0)^3(w - w_0)\varphi - 30 \int (1 + tk(x))(z_{\mu,y} + w_0)^2(w - w_0)^2 \varphi + t^5 O_{\alpha_0}(|\nabla k(y)|^3 \min(1, \mu^3) + \min(1, \mu^6)),
\]

which implies by Lemma 4.4

\[
\|f''_t(z_{\mu,y} + w) - f''_0(z_{\mu,y})\| \leq |t| O_{\alpha_0}(|\nabla k(y)| \min(1, \mu) + \min(1, \mu^5)).
\]

Consequently, from (4.5), after decreasing \(\mu_0\) and \(t_0\) if necessary,

\[
\left\| \frac{\partial H}{\partial (w, \bar{\alpha})}(t, \mu, y, w, \bar{\alpha}) - \frac{\partial H}{\partial (w, \bar{\alpha})}(0, \mu, y, 0) \right\| \leq \frac{1}{2} C_\ast
\]

Thus we may assume \(\frac{\partial H}{\partial (w, \bar{\alpha})}(t, \mu, y, w, \bar{\alpha})\) is invertible and its inverse is uniformly bounded with respect to \((t, \mu, y) \in \Omega\).

We begin the expansion of \(\bar{\alpha}\) by computing the third order term.

Lemma 4.5. Under the assumptions of Lemma 4.2 we have as \(\mu \to 0\)

\[
\|\bar{\alpha}(t, \mu, y) - \sum_{j=1}^3 \bar{\alpha}_j(t, \mu, y)\| = t O_{\alpha_0}(\mu^{3+\frac{1}{2}} + |\nabla k(y)|^2 \mu^2),
\]

where \(\bar{\alpha}_1, \bar{\alpha}_2\) are defined in Lemma 4.4 and \(\bar{\alpha}_3\) is given by

\[
\bar{\alpha}_3(t, \mu, y)_i := -t \mu^2 \frac{3\pi}{2\sqrt{15}} (1 + tk(y))^{-\frac{2}{3}} \frac{\partial}{\partial x_i} \Delta k(y),
\]

for \(i = 1 \ldots 3\) and

\[
\bar{\alpha}_3(t, \mu, y)_0 := -t \mu^3 (1 + tk(y))^{-\frac{2}{3}} \frac{3\pi}{\pi^{\frac{1}{2}}} \int (k(x + y) - T_{k(\cdot + y),0}^3(x)) \frac{1}{|x|^\nu}.
\]

Proof. In the sequel we will suppress the dependence of \(w\) and \(\bar{\alpha}\) on \(t, \mu\) and \(y\), when there is no possibility of confusion. Moreover, we always assume \(0 < \mu \leq 1\).

As in Lemma 4.2, we infer from Lemma 4.4 (4.9), and the definition of \(w_1\)

\[
f'_t(z_{\mu,y} + w) = f'_0(z_{\mu,y})(w - w_0) - t \int (k(x) - k(y))(z_{\mu,y} + w_0)^5.
\]

\[
- 5t \int (k(x) - T_{k,y}^3(x))(z_{\mu,y} + w_0)^4(w - w_0 - w_1).
\]

\[
- 10 \int (1 + tk(x))(z_{\mu,y} + w_0)^3(w - w_0 - w_1)^2.
\]

\[
+ t^2 O_{\alpha_0}(|\nabla k(y)|^2 \mu^2 + |\nabla k(y)|^3 + \mu^6). \quad (4.17)
\]
As \( f'(z_{\mu,y} + w) - \tilde{\alpha}_i \xi_{\mu,y} = 0 \) and \( (\xi_{\mu,y})_i \in N(f''(z_{\mu,y})) \) we obtain from (A.11) and testing (4.17) with \( (\xi_{\mu,y})_i \)

\[
(\tilde{\alpha} - \sum_{j=1}^{2} \tilde{\alpha}_j)_i = -t \int (k(x) - T_{2,\mu}^2(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_i \\
- 5t \int (k(x) - T_{1,\mu}^1(x)) (z_{\mu,y} + w_0)^4(w - w_0 - w_1)(\xi_{\mu,y})_i \\
- 10 \int (1 + tk(x))(z_{\mu,y} + w_0)^3(w - w_0 - w_1)^2(\xi_{\mu,y})_i \\
+ t^2 O_{A_0}(|\nabla k(y)|^2 \mu^2 + |\nabla k(y)| \mu^3 + \mu^6).
\]  

(4.18)

By Lemma 4.1 and 4.2 we have for \( 0 \leq i \leq 3 \)

\[
\int (k(x) - T_{1,\mu}^1(x)) (z_{\mu,y} + w_0)^4(w - w_0 - w_1)(\xi_{\mu,y})_i = O_{A_0}(t\mu^{3+\frac{1}{2}}),
\]

\[
\int (1 + tk(x))(z_{\mu,y} + w_0)^3(w - w_0 - w_1)^2(\xi_{\mu,y})_i = O_{A_0}(t\mu^4).
\]

If \( 1 \leq i \leq 3 \) then we obtain from Lemma 4.1

\[
\int (k(x) - T_{2,\mu}^2(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_i \\
= \int \frac{1}{6} D^3 k(y)(x - y)^3 (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_i + O_{A_0}(\mu^{3+\frac{1}{2}}),
\]

and from (A.9)

\[
\int \frac{1}{6} D^3 k(y)(x - y)^3 (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_i = \mu^3 \frac{3^4 \pi}{2\sqrt{15}} (1 + tk(y))^{-\frac{5}{4}} \frac{\partial}{\partial x_i} \Delta k(y).
\]

(4.20)

Hence, the assertion of the lemma for \( 1 \leq i \leq 3 \) follows from (4.18)-(4.20).

To treat the remaining case \( i = 0 \) we use the fact that \( D^3 k(y)(x)^3 \) is odd and get

\[
\int_{B_1(y)} (k(x) - T_{2,\mu}^2(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_0 \\
= \int_{B_1(y)} (k(x) - T_{1,\mu}^1(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_0
\]

(4.21)

Since, by Lemma 4.1

\[
\int_{\mathbb{R}^3} |k(x) - T_{3,\mu}^3(x)| \mu^{-3}(1 + \mu^{-2}|x - y|^2)^{-4} = O_{A_0}(\mu^4),
\]

there holds after a translation \( x \to x + y \)

\[
\int_{B_1(y)} (k(x) - T_{3,\mu}^3(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_0 \\
= \frac{3^4 4\mu^3}{\pi \sqrt{5}(1 + tk(y))^{\frac{5}{4}}} \int_{B_1(0)} \frac{(k(x + y) - T_{3,\mu}^3(x))}{(\mu^2 + |x|^2)^3} + O_{A_0}(\mu^4).
\]
From
\[(\mu^2 + |x|^2)^{-3} - |x|^{-6} = -(\mu^2 + |x|^2)^{-3}(3|x|^{-2}\mu^2 + 3|x|^{-4}\mu^4 + |x|^{-6}\mu^6)\]
and Lemma 4.1, we infer
\[
\int_{\mathbb{R}^3} |k(x + y) - T_{k,y}^3(x)| (\mu^2 + |x|^2)^{-3} - |x|^{-6} = O_{A_0}(\mu). \quad (4.22)
\]
Hence,
\[
\int (k(x) - T_{k,y}^2(x)) (z_{\mu,y} + w_0)^5(\xi_{\mu,y})_0
\]
\[
= \frac{3\frac{1}{4}\mu^3}{\pi\sqrt{5}(1 + tk(y))^{\frac{3}{2}}} \left( \int_{B_1(0)} (k(x + y) - T_{k,y}^3(x)) |x|^{-6} + \right.
\]
\[
\int_{\mathbb{R}^3 \setminus B_1(0)} (k(x + y) - T_{k,y}^2(x)) |x|^{-6} \bigg) + O_{A_0}(\mu^4)
\]
\[
= \frac{3\frac{1}{4}}{\pi\sqrt{5}} \mu^3 (1 + tk(y))^{-\frac{3}{2}} \int (k(x + y) - T_{k,\xi}^3(x)) \frac{1}{|x|^6} + O_{A_0}(\mu^4),
\]
which ends the proof. \(\square\)

**Remark 4.6.** From Lemmas 4.2, 4.17, and 4.18 we see that
\[\frac{1}{t\mu}\tilde{\alpha}(t, \mu, y)\]
is a well defined, continuous function for \((t, \mu, y) \in \Omega\).

**Lemma 4.7.** Under the assumptions of Lemma 4.2 we have as \(\mu \to 0\)
\[|\tilde{\alpha}(t, \mu, y) - \sum_{j=1}^{4} \tilde{\alpha}_j(t, \mu, y)| = tO_{A_0}(\mu^{4+\frac{1}{2}})\]
\[+ t^2O_{A_0}(\mu^2|\nabla k(y)|^2 + \mu^3|\nabla k(y)| + \mu^4|\Delta k(y)| + \mu^{4+\frac{1}{2}}),\]
where for \(1 \leq i \leq 3\)
\[\tilde{\alpha}_4(t, \mu, y)_i = -t\mu^4(1 + tk(y))^{-\frac{3}{2}} \frac{3\frac{1}{4}}{\pi\sqrt{5}} \int (k(x + y) - T_{k,y}^3(x)) \frac{x_i}{|x|^8}\]
and
\[\tilde{\alpha}_4(t, \mu, y)_0 = t\mu^4(1 + tk(y))^{-\frac{3}{2}} \frac{3\frac{1}{4} \pi \sqrt{5}}{30} \Delta^2 k(y)\]
\[\quad - t^2\mu^4(1 + tk(y))^{-\frac{9}{2}} \frac{3\frac{1}{4} \pi \sqrt{5}}{16} \left( \int |D^2 k(y)(x)|^2 \right)\]

**Proof.** We proceed as in Lemma 4.5 and suppress the dependence of \(w\) and \(\tilde{\alpha}\) on \(t, \mu,\) and \(y\). From Lemmas 4.1 and 4.2 we have for \(0 \leq i \leq 3\)
\[-5t \int (k(x) - T_{k,y}^3(x)) (z_{\mu,y} + w_0)^4(w - w_0 - w_1)(\xi_{\mu,y})_i\]
\[= -5t \int \frac{1}{2} D^2 k(y)(x - y)^2 (z_{\mu,y} + w_0)^4 w_2(\xi_{\mu,y})_i + t^2O_{A_0}(\mu^{4+\frac{1}{2}}).\]
\(4.23\)
Furthermore,

\[-10 \int (1 + tk(x))(z_{\mu,y} + w_0)^3(w - w_0 - w_1)^2(\xi_{\mu,y})_i \]

\[= -10 \int (1 + tk(y))(z_{\mu,y} + w_0)^3(w_2)^2(\xi_{\mu,y})_i + t^2O_{A_0}(\mu^{4+\frac{1}{2}}). \quad (4.24)\]

**Case** 1 \( \leq i \leq 3 \): Since \( w_2 \) is even and \( (\xi_{\mu,y})_i \) is odd for \( 1 \leq i \leq 3 \) the integrals in (4.23)-(4.24) vanish. Thus, from (4.18) and the definition of \( \tilde{\alpha}_3 \) we get

\[\left(\tilde{\alpha} - \sum_{j=1}^{3} \tilde{\alpha}_j\right)_i = -t\mu^4 \frac{3\pi^3}{(1 + tk(y))\pi \sqrt{5}} \int \frac{(k(x + y) - T^3_{k(\mu,y)_0}(x))x_i}{(\mu^2 + |x|^2)^4} + t^2O_{A_0}(|\nabla k(y)|^2\mu^2 + |\nabla k(y)|\mu^3 + \mu^{4+\frac{1}{2}}).\]

Since \( D^4k(y)(x)^4 \) is even we may proceed analogously as in (4.21) and prove the claim of the lemma if \( 1 \leq i \leq 3 \).

**Case** \( i = 0 \): From (4.18), the definition of \( \tilde{\alpha}_3 \), (4.22), (4.23) = (4.24), and the fact that \( D^3k(y)(x)^2 \) is odd, we arrive at

\[\left(\tilde{\alpha} - \tilde{\alpha}_1 - \tilde{\alpha}_2 - \tilde{\alpha}_3\right)_0 = \frac{3\pi^4}{4t(1 + tk(y))^\frac{1}{2}} \int \left(\begin{array}{c}
3\frac{x}{\mu} - 2 + 3\frac{x}{\mu} - 4 + \frac{x}{\mu} - 6 \\
\mu^3(1 + |x|^2)^3 + 2
\end{array}\right) + t^2O_{A_0}(|\nabla k(y)|^2\mu^2 + |\nabla k(y)|\mu^3 + \mu^{4+\frac{1}{2}}). \quad (4.25)\]

From Lemma (A.1), (A.6), and (A.10) we get

\[\int (k(x + y) - T^3_{k(\mu,y)_0}(x)) \left(\begin{array}{c}
3\frac{x}{\mu} - 2 + 3\frac{x}{\mu} - 4 + \frac{x}{\mu} - 6 \\
\mu^3(1 + |x|^2)^3 + 2
\end{array}\right) = \mu^4 \frac{1}{4!} D^4k(y)(x)^4 \frac{5 + 6|x|^2 + 4|x|-4|x|-6}{(1 + |x|^2)^4} + O_{A_0}(\mu^{4+\frac{1}{2}})\]

\[= \mu^4 \frac{\pi^2}{24} \Lambda^2 k(y) + O_{A_0}(\mu^{4+\frac{1}{2}}).\]

For the second term in (4.26) we obtain from Lemmas (4.2) and (4.3)

\[-5t \int \frac{1}{2} D^2k(y)(x - y)^2(z_{\mu,y} + w_0)^4w_2(\xi_{\mu,y})_0 \]

\[= -t^2\mu^4 \frac{\sqrt{15}}{\pi(1 + tk(y))^\frac{1}{4}} \int \frac{1}{2} D^2k(y)(x)^2 \tilde{w}_2(y)(1 + |x|^2)^{-\frac{3}{2}} \left(1 - \frac{2}{1 + |x|^2}\right).\]
Moreover, from Lemma 4.3, (4.15), and (A.12)
\[
\int \frac{1}{2} D^2 k(y)(x^2) \tilde{w}_2(y)(1 + |x|)^{-\frac{5}{2}}(1 - 2(1 + |x|)^{-1}) = O(A_0(|\Delta k(y)|^2) + \left( \sum_{1 \leq l < m \leq 3} \psi_{l,m}(y)^2 + \sum_{l=2}^3 \psi_{l,l}(y)^2 \right) \frac{3^\frac{7}{2} \sqrt{\pi}}{4} \times \sum_{j=0}^\infty \frac{(3 + j)j!}{\Gamma\left(\frac{7}{2} + j\right)} \int_0^\infty \frac{r^6}{(1 + r^2)^6} \left( 1 - \frac{2}{1 + r^2} \right) P_j^2(\frac{3}{2}) \left( 1 - \frac{2}{1 + r^2} \right)
\]
\[
\frac{3^\frac{7}{2} \pi}{64} \left( \int_{\partial B_1(0)} |D^2 k(y)(x^2)| \right) \frac{3^{-\frac{7}{2}} 4}{\pi \sqrt{15}} \int (1 + |x|)^{-2} \left( 1 - \frac{2}{1 + |x|} \right)(\tilde{w}_2(y))^2.
\]

By Lemma 4.3, (4.20), and we have with \( \beta_j := \frac{(3 + j)j!}{\Gamma\left(\frac{7}{2} + j\right)} \)
\[
\int (1 + |x|)^{-2} \left( 1 - \frac{2}{1 + |x|} \right)(\tilde{w}_2(y))^2 = O(A_0(|\Delta k(y)|) + \frac{1}{4} \left( \int_{\partial B_1(0)} |D^2 k(y)(x^2)| \right) \frac{3 \pi^2}{16} \times \sum_{j=0}^\infty \frac{2 + 6j + j^2}{(5 + j)(4 + j)(2 + j)(1 + j)} \sum_{j=0}^\infty \beta_j P_j^2(\frac{3}{2}) \left( 1 - \frac{2}{1 + r^2} \right)^2
\]
\[
= \frac{1}{4} \left( \int_{\partial B_1(0)} |D^2 k(y)(x^2)| \right) \frac{3 \pi^2}{16} + O(A_0(|\Delta k(y)|)).
\]

Combining the computations in (4.18), (4.24), and (4.27) ends the proof.

Lemma 4.8. Under the assumptions of Lemma 4.2 suppose \( k \in C^6(\mathbb{R}^3) \) and \( \|D^6 k\| \leq A_0 \). Then there holds
\[
\|(\tilde{\alpha}(t, \mu, y))_0 - \sum_{j=1}^5 (\tilde{\alpha}_j(t, \mu, y))_0\| = O(A_0(t\mu^6)
\]
\[
+ t^2 O(A_0(\mu^2|\nabla k(y)|^2 + \mu^3|\nabla k(y)| + \mu^4|\Delta k(y)| + \mu^{4 + \frac{1}{2}})).
\]
where \((\tilde{\alpha}_{5}(t, \mu, y))_{0}\) is given by
\[
(\tilde{\alpha}_{5}(t, \mu, y))_{0} := -\frac{t\mu^{5}}{(1 + tk(y))^{2\frac{3}{5}}} \frac{3^{\frac{3}{5}} 4\sqrt{5}}{\pi} \int \left( k(y + x) - T_{k(-y), 0}^{t}(x) \right) \frac{1}{|x|^{8}}.
\]

**Proof.** From the proof of Lemma 4.7 we infer
\[
(\tilde{\alpha}(t, \mu, y))_{0} - \sum_{j=1}^{4}(\tilde{\alpha}_{j}(t, \mu, y))_{0} = -t(1 + tk(y))^{-\frac{1}{2}} \frac{3^{\frac{3}{5}} 4}{\pi \sqrt{15}}
\int \left( k(y + x) - \sum_{\ell=0}^{4} \frac{1}{\ell!} D^{\ell}k(y)(x)^{\ell} \right) \frac{5 + 6|\frac{x}{\mu}|^{-2} + 4|\frac{x}{\mu}|^{-4} + |\frac{x}{\mu}|^{-6}}{\mu^{3} \left( 1 + \frac{|x|^{2}}{\mu^{2}} \right)^{4}}
\]
\[+ t^{2}O_{A_{0}}(\mu^{2} |\nabla_{k}(y)|^{2} + \mu^{3} |\nabla_{k}(y)| + \mu^{4} |\Delta_{k}(y)| + \mu^{4+\frac{1}{2}})\]
Since \(D^{5}k(y)(x)^{5}\) is odd, analogously as in Lemma 5.5 we find
\[
\int \left( k(y + x) - \sum_{\ell=0}^{4} \frac{1}{\ell!} D^{\ell}k(y)(x)^{\ell} \right) \frac{5 + 6|\frac{x}{\mu}|^{-2} + 4|\frac{x}{\mu}|^{-4} + |\frac{x}{\mu}|^{-6}}{\mu^{3} \left( 1 + \frac{|x|^{2}}{\mu^{2}} \right)^{4}}
\]
\[= 5\mu^{5} \oint \left( k(y + x) - \sum_{\ell=0}^{4} \frac{1}{\ell!} D^{\ell}k(y)(x)^{\ell} \right) \frac{1}{|x|^{8}} + O_{A_{0}}(\mu^{6}),
\]
which ends the proof. \(\square\)

**Lemma 5.1.** Under the assumptions of Lemma 4.2 we have for all \((t, \mu, y) \in \Omega\) with \(|\mu| \leq 1\) and \(1 \leq i, j \leq 3\)
\[
\left| \frac{1}{t\mu} \frac{\partial \alpha(t, \mu, y)}{\partial y_{j}} \right| + \frac{\pi}{3^{\frac{3}{5}} \sqrt{5}} (1 + tk(y))^{-\frac{1}{2}} \frac{\partial^{2}k(y)}{\partial x_{i} \partial x_{j}} \leq O_{A_{0}} \left( |\nabla_{k}(y)|^{2} + \mu^{1+\frac{1}{2}} \right),
\]
(5.1)
\[
\left| \frac{1}{t\mu^{2}} \frac{\partial \alpha(t, \mu, y)}{\partial y_{j}} \right| + \frac{\pi}{3^{\frac{3}{5}} \sqrt{5}} (1 + tk(y))^{-\frac{1}{2}} \frac{\partial}{\partial x_{j}} \Delta_{k}(y) \right| \leq O_{A_{0}} \left( |\nabla_{k}(y)|^{2} \mu^{-1} + \mu^{\frac{1}{2}} \right).
\]
(5.2)

**Proof.** In the sequel we will suppress the dependence of \(w\) and \(\tilde{\alpha}\) on \((t, \mu, y)\). Since \(H(t, \mu, y, w, \tilde{\alpha}) \equiv 0\) we have
\[
- \frac{\partial H}{\partial y} = \frac{\partial H}{\partial (w, \tilde{\alpha})} \left( \frac{\partial w}{\partial y} \frac{\partial \tilde{\alpha}}{\partial y} \right) = \frac{\partial H}{\partial w} \frac{\partial w}{\partial y} + \frac{\partial H}{\partial \tilde{\alpha}} \frac{\partial \tilde{\alpha}}{\partial y},
\]
(5.3)
where
\[
\frac{\partial H}{\partial y_{j}}(t, \mu, y, w, \tilde{\alpha}) = \left( f_{t}''(\zeta_{\mu, y} + w) \frac{\partial \zeta_{\mu, y}}{\partial y_{j}} - \tilde{\alpha} \cdot \frac{\partial \zeta_{\mu, y}}{\partial y_{j}} + \left( \langle w, \frac{\partial (\zeta_{\mu, y})_{t}}{\partial y_{j}} \rangle \right)_{t} \right).
\]
(5.4)
A direct calculation gives for \(0 \leq l \leq 3\)
\[
\left\| \frac{\partial (\zeta_{\mu, y})_{t}}{\partial y_{j}} \right\| + \left\| \frac{\partial \zeta_{\mu, y}}{\partial y_{j}} \right\| \leq \text{const } \mu^{-1}.
\]
(5.5)
Differentiating $\langle z_{\mu,y}, (\hat{\xi}_{\mu,y})_l \rangle \equiv 0$ leads to
\[
\langle z_{\mu,y}, \frac{\partial (\hat{\xi}_{\mu,y})}{\partial y_j} \rangle = \langle \frac{\partial z_{\mu,y}}{\partial y_j}, (\hat{\xi}_{\mu,y})_l \rangle,
\]
and with (4.4) and (5.5) we arrive at
\[
\left| \langle w, \frac{\partial (\hat{\xi}_{\mu,y})}{\partial y_j} \rangle \right| - \left| ((1 + tk(y))^{-\frac{1}{4}} - 1) \frac{\partial z_{\mu,y}}{\partial y_j}, (\hat{\xi}_{\mu,y})_l \right| = tO_A_0(|\nabla k(y)| + \mu).
\]

By (5.5), the expansion of $\bar{\alpha}$ in Lemma 4.2 and (4.16) we see
\[
\left\| f''_l (z_{\mu,y} + w) \frac{\partial z_{\mu,y}}{\partial y_j} \right\| + \left\| \bar{\alpha} \cdot \frac{\partial \hat{\xi}}{\partial y_j} \right\| \leq tO_A_0(|\nabla k(y)| + \mu).
\]

From (5.6) - (5.7) we get
\[
\left\| \frac{\partial H}{\partial (w, \bar{\alpha})} (t, \mu, y, w, \bar{\alpha}) \left( (1 + tk(y))^{-\frac{1}{4}} - 1 \right) \frac{\partial z_{\mu,y}}{\partial y_j} \right\| + \frac{\partial H}{\partial y_j} (t, \mu, y, w, \bar{\alpha}) \leq tO_A_0(|\nabla k(y)| + \mu)
\]
which implies due to the uniform bound of the inverse (see Remark 4.4)
\[
\left\| \frac{\partial w}{\partial y_j} - ((1 + tk(y))^{-\frac{1}{4}} - 1) \frac{\partial z_{\mu,y}}{\partial y_j} \right\| + \left\| \frac{\partial \hat{\alpha}}{\partial y_j} \right\| \leq tO_A_0(|\nabla k(y)| + \mu).
\]

From (5.6) and (5.9) + (5.5) we deduce after testing with $(\hat{\xi}_{\mu,y})_j$
\[
\frac{\partial \bar{\alpha}}{\partial y_i} = f''_l (z_{\mu,y} + w) (\frac{\partial z_{\mu,y}}{\partial y_i} + \frac{\partial w}{\partial y_i} (\hat{\xi}_{\mu,y})_j - \sum_{i=0}^{3} \bar{\alpha}(t, \mu, y)_l \langle \frac{\partial \hat{\xi}_{\mu,y}}{\partial y_i} , (\hat{\xi}_{\mu,y})_j \rangle.
\]

From Lemma 4.4 (11.10), (5.8), and the fact that $(\hat{\xi}_{\mu,y})_j \in N(f''_0 (z_{\mu,y}))$ we obtain
\[
f''_l (z_{\mu,y} + w) (\frac{\partial z_{\mu,y}}{\partial y_i} + \frac{\partial w}{\partial y_i} (\hat{\xi}_{\mu,y})_j = -\frac{5t}{(1 + tk(y))^\frac{3}{2}} \int \left( \sum_{\ell=1}^{3} \frac{1}{\ell!} D^\ell k(y)(x - y)^\ell \right) (z_{\mu,y})^{4} (\frac{\partial z_{\mu,y}}{\partial y_i}) (\hat{\xi}_{\mu,y})_j

- 20 \int (z_{\mu,y})^{3} (w - w_0) (\frac{\partial z_{\mu,y}}{\partial y_i}) (\hat{\xi}_{\mu,y})_j

+ O_A_0 (t^{2+\frac{3}{2}} + t^{2}(|\nabla k(y)|^{2} \mu + |\nabla k(y)|^{2} \mu^{2} + \mu^{2+\frac{3}{2}}))
\]

Differentiating the identity $f''_0 (z_{\mu,y})(\hat{\xi}_{\mu,y})_j = 0$ with respect to $y_i$ leads to
\[
0 = f''_l (z_{\mu,y}) (\frac{\partial z_{\mu,y}}{\partial y_i}) (\hat{\xi}_{\mu,y})_j + f''_0 (z_{\mu,y}) (\frac{\partial (\hat{\xi}_{\mu,y})_j}{\partial y_i}),
\]
and we get from Lemma 4.1, 4.2, and 4.7
\[-20 \int (z_{\mu,y})^3(w - w_0) \frac{\partial z_{\mu,y}}{\partial y_i}(\xi_{\mu,y})_j \]
\[= f''_0(z_{\mu,y})(w - w_0) \frac{\partial z_{\mu,y}}{\partial y_i}(\xi_{\mu,y})_j = - f''_0(z_{\mu,y}) \frac{\partial(\xi_{\mu,y})_j}{\partial y_i}(w - w_0) \]
\[= - \sum_{l=0}^3 \langle \alpha(t, \mu, y) \rangle \langle \xi_{\mu,y}, (\xi_{\mu,y})_t \rangle \frac{\partial(\xi_{\mu,y})_j}{\partial y_i} \]
\[- \frac{t}{(1 + tk(y))^\frac{3}{2}} \int \left( \sum_{\ell=1}^3 \frac{1}{\ell !} D^\ell k(y)(x - y)^\ell (z_{\mu,y})^5 \frac{\partial(\xi_{\mu,y})_j}{\partial y_i} \right) \]
\[+ tO_{A_0}(\mu |\nabla k(y)|^2 + \mu^{2+\frac{1}{4}}). \]

Differentiating \(\langle (\xi_{\mu,y})_t, (\xi_{\mu,y})_j \rangle \equiv \text{const} \) with respect to \(y_i \) we obtain
\[\langle \frac{\partial(\xi_{\mu,y})_t}{\partial y_i}, (\xi_{\mu,y})_j \rangle = - \langle (\xi_{\mu,y}), (\xi_{\mu,y})_j, \frac{\partial(\xi_{\mu,y})_t}{\partial y_i} \rangle. \]

Inserting the above computations in (5.10) and (5.9) leads to
\[\frac{\partial(\xi_{\mu,y})_t}{\partial y_i} = - \frac{t}{(1 + tk(y))^\frac{3}{4}} \int \left( \sum_{\ell=1}^3 \frac{1}{\ell !} D^\ell k(y)(x - y)^\ell (z_{\mu,y})^5 \frac{\partial(\xi_{\mu,y})_j}{\partial y_i} \right) \]
\[+ tO_{A_0}(\mu |\nabla k(y)|^2 + \mu^{2+\frac{1}{4}}). \]

As \(\frac{\partial}{\partial y_i} z_{\mu,y}(x) = - \frac{\partial}{\partial x_i} z_{\mu,y}(x)\) and \(\frac{\partial}{\partial y_i} (\xi_{\mu,y})_j(x) = - \frac{\partial}{\partial x_i} (\xi_{\mu,y})_j(x)\) we obtain by partial integration
\[\int \left( \sum_{\ell=1}^3 \frac{1}{\ell !} D^\ell k(y)(x - y)^\ell (z_{\mu,y})^5 (\xi_{\mu,y})_j \right) \]
\[= \int \left( \sum_{\ell=1}^3 \frac{1}{\ell !} D^\ell k(y)(x - y)^\ell (z_{\mu,y})^5 (\xi_{\mu,y})_j \right) \]
\[= \int \left( \sum_{\ell=0}^2 \frac{1}{\ell !} D^\ell \frac{\partial k}{\partial x_i} (x - y)^\ell (z_{\mu,y})^5 (\xi_{\mu,y})_j \right). \]

The latter integral may be evaluated as in Lemma 4.2 and yields the claim.

\[\square\]

**Lemma 5.2.** Under the assumptions of Lemma 5.1 we have
\[\left\| \frac{\partial w}{\partial t} - \sum_{i=0}^2 \frac{\partial w_i}{\partial t} \right\| + \left\| \frac{\partial(\alpha)}{\partial t} - \sum_{i=1}^2 \frac{\partial(\alpha_i)}{\partial t} \right\| = O_{A_0}(\mu |\nabla k(y)|^2 + \mu^{2+\frac{1}{4}}). \quad (5.11)\]

**Proof.** In the sequel we will suppress the dependence of \(w\) and \(\alpha\) on \((t, \mu, y)\). Since \(H(t, \mu, y, w, \alpha) \equiv 0\) we have
\[\left( \int k(x)(z_{\mu,y} + w)^5, \tilde{\alpha} \right) = - \frac{\partial H}{\partial t}(t, \mu, y, w, \alpha) = \frac{\partial H}{\partial w}(t, \mu, y, w, \alpha) \frac{\partial w}{\partial t}(t, \mu, y, w, \alpha) \frac{\partial(\alpha)}{\partial \alpha}(t, \mu, y, w, \alpha) \]
\[= \left( f''_0(z_{\mu,y} + w) \frac{\partial w}{\partial t} - \frac{\partial(\alpha)}{\partial \alpha} \cdot \xi_{\mu,y}, (\frac{\partial w}{\partial t}, (\xi_{\mu,y})_t) \right) \quad (5.12)\]
Differentiating the identities

\[ f'_0(z_{\mu,y} + w) = t \int k(y)(z_{\mu,y} + w)^5, \]
\[ f''_0(z_{\mu,y}) \sum_{i=1}^{2} w_i - \sum_{i=1}^{2} \bar{\alpha}_i \cdot \bar{\xi}_{\mu,y} = t \int \left( \sum_{\ell=1}^{2} \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell \right) (z_{\mu,y} + w_0)^5. \]

with respect to \( t \) leads to

\[ f''_0(z_{\mu,y}) \sum_{i=0}^{2} \frac{\partial w_i}{\partial t} - \sum_{i=1}^{2} \frac{\partial \bar{\alpha}_i}{\partial t} \cdot \bar{\xi}_{\mu,y} = t \int \left( \sum_{\ell=1}^{2} \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell \right) (z_{\mu,y} + w_0)^5. \]

Furthermore, we note that

\[ \frac{\partial w_0}{\partial t} = -\frac{k(y)}{4} (1 + tk(y))^{-1}(z_{\mu,y} + w_0). \]

For \( \frac{\partial w_0}{\partial t} = O_{A_0}(\mu^\ell) \) as \( \mu \to 0 \) we get from Lemma 5.3, 4.16, 5.13, and 5.14

\[ f'''_0(z_{\mu,y} + w) \sum_{i=0}^{2} \frac{\partial w_i}{\partial t} - \frac{\partial (\bar{\alpha}_1 + \bar{\alpha}_2)}{\partial t} \cdot \bar{\xi}_{\mu,y} \]

\[ = \int \left( \sum_{\ell=1}^{2} \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell \right) (z_{\mu,y} + w_0)^5. \]

\[ + 5 \int k(y)(z_{\mu,y} + w_0)^4(w - w_0) \cdot +O_{A_0}(|\nabla k(y)|^4 \mu^2 + \mu^4). \]

Moreover, by Lemmas 4.1 and 4.2

\[ \int k(x)(z_{\mu,y} + w)^5. \]

\[ = \int \left( \sum_{\ell=1}^{2} \frac{1}{\ell!} D^\ell k(y)(x-y)^\ell \right) (z_{\mu,y} + w_0)^5 \cdot O_{A_0}(\mu^\frac{9}{2}) \]

\[ + 5 \int k(y)(z_{\mu,y} + w_0)^4(w - w_0) \cdot +O_{A_0}(|\nabla k(y)|^2 \mu^2 + \mu^2). \]

Combining 5.15, 5.16 and the fact that \( \frac{\partial w_0}{\partial t} \) remains in \( T_{z_{\mu,y}} Z^\perp \) we get

\[ \frac{\partial H}{\partial (w, \bar{\alpha})} \left( \sum_{i=0}^{2} \frac{\partial w_i}{\partial t} + \sum_{i=1}^{2} \frac{\partial \bar{\alpha}_i}{\partial t} \right) = \frac{\partial H}{\partial t} + O_{A_0}(|\nabla k(y)|^2 \mu^2 + \mu^{2+\frac{1}{4}}). \]

and the claim of the lemma follows from Remark 4.4.

**Lemma 5.3.** Under the assumptions of Lemma 5.7 we have

\[ \frac{\partial \bar{\alpha}_0}{\partial t} = \frac{1}{t} \bar{\alpha}_0 + \sum_{j=2}^{4} \frac{\partial \bar{\alpha}_j}{\partial t} - \frac{1}{t} \bar{\alpha}_0 \]

\[ + tO_{A_0}(|\nabla k(y)|^2 \mu^2 + |\nabla k(y)|^3 \mu^2 + \mu^4 |\nabla k(y)|^2 + \mu^{4+\frac{1}{4}}). \]
and for $1 \leq i \leq 3$

$$\frac{\partial (\vec{a})_i}{\partial t} = \frac{1}{t} (\vec{a})_i + \sum_{j=1}^{3} \frac{\partial (\vec{a}_j)_i}{\partial t} - \frac{1}{t} (\vec{a}_j)_i + tO_{A_0}( (\nabla k(y))^2 \mu^2 + |\nabla k(y)| \mu^3 + \mu^{3 + \frac{1}{2}}). \quad (5.18)$$

**Proof.** By (5.12) we have for $0 \leq i \leq 3$

$$\frac{\partial (\vec{a})_i}{\partial t} = f''(z_{\mu,y} + w) \frac{\partial w}{\partial t} (\vec{e}_{\mu,y})_i - \int (x) (z_{\mu,y} + w)^5 (\vec{\xi}_{\mu,y})_i.$$

To prove the claim of the lemma we will proceed termwise. In the calculations below certain terms will vanish simply because we are integrating a product of an odd and an even function. Moreover, we often use Lemma 4.1 without mentioning it explicitly.

For $(\vec{\xi}_{\mu,y})_i \in N(f''(z_{\mu,y}))$ and by (1.16) we see

$$f''(z_{\mu,y} + w) \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i = -5t \int (x) - k(y) (z_{\mu,y} + w_0)^4 \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$- 20 \int (1 + tk(x)) (z_{\mu,y} + w_0)^3 (w - w_0) \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$- 30 \int (1 + tk(x)) (z_{\mu,y} + w_0)^2 (w - w_0)^2 \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$+ t^3 O_{A_0}( (\nabla k(y))^2 \mu^3 + \mu^6).$$

Due to Lemma 5.2 we may replace $\frac{\partial w}{\partial t}$ by $\sum_{i=0}^{2} \frac{\partial w}{\partial t}$. By (5.14) we obtain

$$-5t \int (x) - k(y) (z_{\mu,y} + w_0)^4 \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$= \frac{5k(y)}{4(1 + tk(y))} \int (x) - k(y) (z_{\mu,y} + w_0)^5 (\vec{\xi}_{\mu,y})_i$$

$$- 5t \int \frac{1}{2} D^2 k(y)(x - y)^2 (z_{\mu,y} + w_0)^4 \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$+ tO_{A_0}( \mu^2 |\nabla k(y)|^2 + \mu^3 |\nabla k(y)| + \mu^{4 + \frac{1}{2}}).$$

From (5.14) and as $(\vec{\xi}_{\mu,y})_i \in N(f''(z_{\mu,y}))$ and $w - w_0 \in T_{\mu,y} T^\perp$ we see

$$-20 \int (1 + tk(y)) (z_{\mu,y} + w_0)^3 (w - w_0) \frac{\partial w}{\partial t} (\vec{\xi}_{\mu,y})_i$$

$$= k(y) \langle (w - w_0), (\vec{\xi}_{\mu,y})_i \rangle$$

$$= 0.$$
Furthermore, we see
\[-30 \int (1 + tk(x))(z_{\mu,y} + w_0)^2(w - w_0)^2 \frac{\partial w}{\partial t}(\dot{\xi}_{\mu,y})_i \]
\[= -30 \int (1 + tk(y))(z_{\mu,y} + w_0)^2(w_2)^2 \frac{\partial w_0}{\partial t}(\dot{\xi}_{\mu,y})_i \]
\[+ t^2 O_{A_0}(\mu^2|\nabla k(y)|^2 + \mu^3|\nabla k(y)| + \mu^{4+\frac{1}{2}})\]
Since \(z_{\mu,y}\) and \(w\) are orthogonal to \((\dot{\xi}_{\mu,y})_i\) and \((\dot{\xi}_{\mu,y})_i \in N(f''_0(z_{\mu,y}))\), we may estimate using Lemma 4.2
\[-\int k(x)(z_{\mu,y} + w)^5(\dot{\xi}_{\mu,y})_i = \frac{1}{t} f'_t(z_{\mu,y} + w)(\dot{\xi}_{\mu,y})_i + \frac{1}{t} \int (z_{\mu,y} + w)^3(\dot{\xi}_{\mu,y})_i \]
\[\quad = \frac{1}{t}(\ddot{\alpha}) + \frac{10}{t} \int (z_{\mu,y} + w_0)^3(w_2)^2(\dot{\xi}_{\mu,y})_i \]
\[\quad + t^2 O_{A_0}(|\nabla k(y)|^2 \mu^2 + |\nabla k(y)| \mu^3 + \mu^{4+\frac{1}{2}}).\]
As
\[
\frac{\partial w_2}{\partial t} = \left( \frac{1}{t} - \frac{5k(y)}{4(1 + tk(y))} \right) w_2,
\]
we end up with integrals that are, up to a factor, computed in Section 4. Summing up the results will give the claim of the lemma. \(\Box\)

6. Solvability of \(\ddot{\alpha}(t, \mu, y) = 0\)

**Lemma 6.1.** Under the assumptions of Lemma 4.2 suppose \(y_0\) is a nondegenerate critical point of \(k\), i.e.
\[\nabla k(y_0) = 0\] and \(D^2 k(y_0)\) is invertible, with \(\|(D^2 k(y_0))^{-1}\| \leq A_0\).

Moreover, assume \(\Delta k(y_0) = 0\). Consider the function \(\ddot{\alpha}\), defined by
\[
\ddot{\alpha}(t, \mu, y) := \frac{3}{t \mu^\frac{5}{2}} (1 + tk(y_0)) \frac{5}{2} (\ddot{\alpha}(t, \mu, y)_1, \ldots, \ddot{\alpha}(t, \mu, y)_3)^T,
\]
which is well defined and continuous in \(\Omega\) (see Remark 4.6), analogously we define \(\ddot{\alpha}_j(t, \mu, y)\). Then there are \(\delta_1 = \delta_1(A_0) > 0\) and a \(C^2\)-function \(\beta\),
\[
\beta : \{(t, \mu) : t \in [-B_0, B_1], 0 < \mu < \delta_1\} \rightarrow \mathbb{R}^3,
\]
such that
\[
\ddot{\alpha}(t, \mu, \beta(t, \mu)) = 0 \text{ for all } t \in [-B_0, B_1], 0 < \mu < \delta_1,
\]
and
\[
\beta(t, \mu) = y_0 + (D^2 k(y_0))^{-1} \left( \sum_{j=3}^{4} \ddot{\alpha}_j(t, \mu, y_0) \right) + O_{A_0}(\mu^{3+\frac{1}{2}}).
\]
Moreover, \(\beta\) is unique in the sense that, if \(y \in B_{\delta_1}(y_0)\) satisfies \(\ddot{\alpha}(t, \mu, y) = 0\) for some \(t \in [-B_0, B_1]\) and \(0 < \mu < \delta_1\), then \(y = \beta(t, \mu)\).
Proof. In view of 5.1, we would like to apply the implicit function theorem to the function \((\bar{\alpha}(t, \mu, y))_{1 \leq i \leq 3}\) in the point \((t, 0, y_0)\), but unfortunately \(\bar{\alpha}\) may not be differentiable for \(\mu = 0\). Instead we mimic the proof of the implicit function theorem and apply Banach’s fixed-point theorem to the function

\[ F(t, \mu, y) := y + (D^2 k(y_0))^{-1} \bar{\alpha}(t, \mu, y) \]

in \(B_\delta(y_0)\), where \(\delta > 0\) will be chosen later. Fix \(y_1, y_2 \in B_\delta(y_0)\), then by Lemma 5.1

\[ |F(t, \mu, y_1) - F(t, \mu, y_2)| \]

\[ = \left| (y_1 - y_2) + \left(\int_0^1 (D^2 k(y_0))^{-1} D^2 k(y_2 + t(y_1 - y_2)) \, dt \right)(y_1 - y_2) \right| \]

\[ \leq |y_1 - y_2| - \left(\int_0^1 (D^2 k(y_0))^{-1} D^2 k(y_2 + t(y_1 - y_2)) \, dt \right)(y_1 - y_2) \]

\[ + O_{A_0} \left( \sup_{y \in B_\delta(y_0)} |\nabla k(y)| + \mu^{\frac{3}{2}} \right) |y_1 - y_2| \]

\[ \leq O_{A_0} \left( \delta + \mu^{\frac{3}{2}} \right) |y_1 - y_2|. \]

For \(y \in B_\delta(y_0)\) we estimate using Lemma 5.2

\[ |F(t, \mu, y) - y_0| = \left| y - y_0 + \left(\int_0^1 (D^2 k(y_0))^{-1} \bar{\alpha}(t, \mu, y) \right) \right| \]

\[ \leq |y - y_0 - \left(\int_0^1 (D^2 k(y_0))^{-1} \nabla k(y) + O_{A_0}(\mu^2) \right) | \]

\[ \leq O_{A_0}(\delta^2 + \mu^2). \]

Consequently, there is \(\delta_1 = \delta_1(A_0) > 0\) such that \(F(t, \mu, \cdot)\) is a contraction in \(B_{\delta_1}(y_0)\) for any \(0 < \mu < \delta_1\) and \(t \in [-B_0, B_1]\). From Banach’s fixed-point theorem we may define \(\beta(t, \mu)\) to be the unique fixed-point of \(F(t, \mu, \cdot)\) in \(B_{\delta_1}(y_0)\). After shrinking \(\delta_1\) if necessary we may apply Lemma 5.1 and the usual implicit function theorem to see that the function \(\beta\) is twice differentiable for \(\mu > 0\).

To deduce the expansion for small \(\mu\) we fix \(\rho > 0\) and

\[ y \in U_\rho := B_\rho \left( y_0 + \left(\int_0^1 (D^2 k(y_0))^{-1} \bar{\alpha}(t, \mu, y_0) \right) \right). \]

Then, by Lemmas 5.5 and 5.7

\[ \left| F(t, \mu, y) - y_0 - \left(\int_0^1 (D^2 k(y_0))^{-1} \bar{\alpha}(t, \mu, y_0) \right) \right| \]

\[ \leq \left| y - y_0 - \left(\int_0^1 (D^2 k(y_0))^{-1} \nabla k(y) + O_{A_0}(\mu \rho^2 + \mu^2 \rho + \mu^{3+\frac{1}{2}}) \right) \right| \]

\[ \leq O_{A_0}(\rho^2 + \mu^2 \rho + \mu^{3+\frac{1}{2}}). \]

Hence, we may choose for small \(\mu\) a radius \(0 < \rho = O_{A_0}(\mu^{3+\frac{1}{2}})\) such that \(F\) maps \(U_\rho\) into itself and \(U_\rho \subset B_{\delta_1}(y_0)\). Consequently, the unique fixed-point \(\beta(t, \mu)\) must lie in this ball. This ends the proof. \(\Box\)
Lemma 6.2. Under the assumptions of Lemma 6.1 if moreover \( k \in C^6(\mathbb{R}^3) \) and \( \|D^6k\|_\infty \leq A_0 \), then we have

\[
(\alpha(t, \mu, \beta(t, \mu)))_0 = -t\mu^3(1 + tk(y_0)) \frac{3^4}{\pi \sqrt{5}}a_0(y_0)
+ t\mu^4(1 + tk(y_0)) \frac{3^4}{30} \sqrt{5}(a_1(y_0) + ta_2(y_0))
+ t\mu^5(1 + tk(y_0)) \frac{3^4}{30} a_3(y_0)
+ O_{A_0}(t\mu^{5+\frac{1}{2}} + t^2\mu^{4+\frac{1}{2}}),
\]

where \( a_i(y_0) = a_i(\theta) \) given in (1.7) and (1.8) with \( k_0 = k(\cdot + y_0) \). If the assumption \( k \in C^6(\mathbb{R}^3) \) is dropped then the terms of order higher than 4 in \( \mu \) have to be replaced by \( O_{A_0}(t\mu^{4+\frac{1}{2}}) \).

Proof. In view of Lemma 6.1 and because \( \nabla k(y_0) = 0 \) we may estimate functions of \( y := \beta(t, \mu) \) and of \( k(y) = k(\beta(t, \mu)) \) as follows

\[
F(y) = F(y_0) + F'(y_0)(D^2k(y_0))^{-1}\left( \sum_{j=3}^4 \hat{\alpha}_j(t, \mu, y_0) \right) + O_{A_0}(\mu^{3+\frac{1}{4}}),
\]

(6.1)

\[
F(k(y)) = F(k(y_0)) + O_{A_0}(\mu^4).
\]

To prove the claim of the lemma we expand \( \alpha(t, \mu, \beta(t, \mu))_0 \) according to Lemma 4.8 and use (6.1). For instance we have

\[
(\hat{\alpha}_2(t, \mu, y))_0 = -t\mu^2(1 + tk(y_0)) \frac{\pi}{3^4 \sqrt{5}} \Delta k(y) + O_{A_0}(t\mu^6)
= t\mu^4(1 + tk(y_0)) \frac{\pi}{3^4 \sqrt{5}} \Delta k(y_0) \left( D^2k(y_0) \right)^{-1} \nabla(\Delta k(y_0))
+ \mu \frac{3^4}{\pi \sqrt{5}} \nabla \Delta k(y_0) \cdot (D^2k(y_0))^{-1} \oint \frac{k(x + y_0) - T^3_{k(\cdot + y_0), 0}(x)}{|x|^5} x_i
+ O_{A_0}(t\mu^{5+\frac{1}{4}}).
\]

If we continue expanding the remaining terms given in Lemma 4.8 the claim follows. \( \square \)

Lemma 6.3. Under the assumptions of Lemma 6.1 let \( \Delta k(y_0) = 0 = a_0(y_0) \) and define

\[
\gamma(t, \mu) := \frac{1}{t\mu^4}(1 + tk(y_0)) \frac{30}{\pi 3^4 \sqrt{5}} (\alpha(t, \mu, \beta(t, \mu)))_0.
\]

Then

\[
\frac{\partial \gamma(t, \mu)}{\partial t} = a_2(y_0) + O_{A_0}(\mu^{\frac{1}{2}}).
\]

Proof. We have

\[
\frac{d(\tilde{\alpha}(t, \mu, \beta(t, \mu)))_0}{dt} = \frac{\partial (\tilde{\alpha})_0}{\partial t} \bigg|_{(t, \mu, \beta(t, \mu))} + \frac{\partial (\tilde{\alpha})_0}{\partial y} \bigg|_{(t, \mu, \beta(t, \mu))} \frac{\partial \beta}{\partial t} \bigg|_{(t, \mu)}.
\]
The derivatives of \((\vec{\alpha})_0\) are computed in (5.2) and (5.17). In order to compute the derivative of \(\beta\) we use the fact that

\[
\vec{\alpha}(t, \mu, \beta) := ((\vec{\alpha}(t, \mu, \beta))_1, \ldots, (\vec{\alpha}(t, \mu, \beta))_3)^T = \vec{0}.
\]

By (6.1) and Lemmas 5.1 and 5.3 we have

\[
\frac{\partial \beta}{\partial t} \Big|_{(t, \mu)} = -\left( \frac{\partial \vec{\alpha}}{\partial y} \Big|_{(t, \mu, \beta(t, \mu))} \right)^{-1} \frac{\partial \vec{\alpha}}{\partial t} \Big|_{(t, \mu, \beta(t, \mu))}
\]

\[
= t^{-1} \mu^{-1} \frac{3}{\pi} (1 + t k(y_0)) \left( (D^2 k(y_0))^{-1} + O_{A_0}(\mu^{\frac{1}{2}}) \right)
\]

\[
\left[ \frac{1}{t} (\vec{\alpha}(t, \mu, \beta))_i - O_{A_0} \left( \sum_{j=1}^{3} (\vec{\alpha}_j(t, \mu, \beta))_i \right) + t O_{A_0}(\mu^{3+\frac{1}{2}}) \right]_{i=1\ldots3}
\]

\[
= O_{A_0}(\mu^{2+\frac{1}{2}}),
\]

where we used the fact that as \(\vec{\alpha}(t, \mu, \beta)_i \equiv 0\) for \(1 \leq i \leq 3\)

\[
\sum_{j=1}^{3} (\vec{\alpha}_j(t, \mu, \beta))_i = O_{A_0}(t \mu^{3+\frac{1}{2}}).
\]

From (5.2) we get

\[
\frac{\partial (\vec{\alpha})_0}{\partial y} \Big|_{(t, \mu, \beta(t, \mu))} \frac{\partial \beta}{\partial t} \Big|_{(t, \mu)} = O_{A_0}(t \mu^{4+\frac{1}{2}}).
\]

Furthermore, by Lemmas 5.1 and 5.3

\[
\frac{d\vec{\alpha}(t, \mu, \beta)_0}{dt} = \frac{1}{t} \vec{\alpha}(t, \mu, \beta)_0 + \sum_{j=1}^{4} \frac{\partial \vec{\alpha}_j(t, \mu, \beta)_0}{\partial t} - \frac{1}{t} \vec{\alpha}_j(t, \mu, \beta)_0
\]

\[
+ O_{A_0}(t \mu^{4+\frac{1}{2}}).
\]

The definition of \(\gamma\), (5.1), and Lemma 6.2 yield the claim. \(\square\)

**Lemma 6.4.** Under the assumptions of Lemma 6.2 suppose \(a_2(y_0) \neq 0\) and either \(a_1(y_0) \neq 0\) or \((a_1(y_0) = 0 \text{ and } a_3(y_0) \neq 0)\). Moreover let

\[
A_0 \geq \begin{cases} 
|a_1(y_0)|^{-1} + 2|a_1(y_0)||a_2(y_0)|^{-1} + |a_2(y_0)|^{-1} & \text{if } a_1(y_0) \neq 0, \\
|a_2(y_0)|^{-1} + |a_3(y_0)|^{-1} + |a_3(y_0)| & \text{if } a_1(y_0) = 0,
\end{cases}
\]

and \(-\frac{a_1(y_0)}{a_2(y_0)} \in (-B_0, B_1)\). Then there exist \(\delta_2 = \delta_2(A_0) > 0\) and a \(C^1\)-function \(\bar{t}\),

\[
\bar{t}: \{ \mu : 0 < \mu < \delta_2 \} \to (-B_0, B_1) \setminus \{0\},
\]

such that \((\alpha(\bar{t}(\mu), \mu, \beta(\bar{t}(\mu), \mu)))_0 \equiv 0\) for all \(0 < \mu < \delta_2\) and

\[
\bar{t}(\mu) = -\frac{1}{a_2(y_0)} \begin{cases} 
a_1(y_0) + O_{A_0}(\mu^{\frac{1}{2}}) & \text{if } a_1(y_0) \neq 0 \\
a_3(y_0) \mu + O_{A_0}(\mu^{1+\frac{1}{2}}) & \text{if } a_1(y_0) = 0.
\end{cases}
\]

Moreover \(\bar{t}\) is unique in the sense that, if \(t \in (-B_0, B_1)\) and \(0 < \mu < \delta_2\) satisfy \((\alpha(t, \mu, \beta(t, \mu)))_0 = 0\) then \(t = \bar{t}(\mu)\).
Proof. We only sketch the proof, which is similar to the proof of Lemma 6.1. We will apply Banach’s fixed-point theorem to the function

\[ F_\mu(t) = F(t, \mu) := t - a_2(y_0)^{-1}\gamma(t, \mu), \]

where \( \gamma \) is given in Lemma 6.3. To this end we show that for small \( \mu \) the map \( F_\mu \) is a contraction in some ball centered at \(-\frac{a_3(y_0)}{a_2(y_0)}\) if \( a_1(y_0) \neq 0 \) and in \( B_r(-\frac{a_3(y_0)}{a_2(y_0)}\mu) \), if \( a_1(y_0) = 0 \), where

\[ 0 < r \leq r_0 = r_0(\mu) := \frac{1}{2} \mu \frac{|a_3(y_0)|}{|a_2(y_0)|}. \]

To prove that \( F_\mu \) is a contraction we may proceed as in Lemma 6.1. We only need the derivative of \( \gamma \), which is given in Lemma 6.3. \( \square \)

7. A priori estimates

We combine the results of Sections 3-6 to prove the \( C^2 \)-a priori estimates announced in the introduction.

**Theorem 7.1.** Suppose there is \( A_0 > 2 \) such that \( k \in C^5(S^3) \) satisfies

\[ D^2k_\theta(0) \]

is invertible, if \( \theta \in A := \{ \theta \in S^3 : \nabla k(\theta) = 0 \text{ and } \Delta k(\theta) = 0 \}, \)

\[ (A_0)^{-1} \leq 1 + (1 + A_0^{-1})k(\theta) \leq A_0, \]

\[ \| k \|_{C^5(S^3)} \leq A_0, \]

and

\[ A_0 \geq \sup \{ \| (D^2k_\theta(0))^{-1} \| : \theta \in A \}. \]

Thus, \( A \) is discrete and there is \( r = r(A_0) > 0 \) such that

\[ \nabla k(\theta) \neq 0 \text{ for all } \theta \in \bigcup_{\theta_0 \in A} B_r(\theta_0) \setminus \{ \theta_0 \}. \]

Additionally, assume there is \( A_1 > 0 \) such that

\[ A_1 \geq \sup \{ |\Delta k_\theta(0)|^{-1} : |\nabla k(\theta)| \leq A_1^{-1} \text{ and } \theta \in S^3 \setminus \bigcup_{\theta_0 \in A} B_r(\theta_0) \}, \]

\[ A_1 \geq \sup \{ |a_0(\theta)|^{-1} : \theta \in A \text{ and } a_0(\theta) \neq 0 \}. \]

Denote by \( M \) the finite set

\[ M := \{ \theta \in S^3 : \theta \in A, a_0(\theta) = 0, \text{ and } a_2(\theta) \neq 0 \}. \]

Then for every \( \delta > 0 \) exists \( C = C(A, A_0, A_1, \delta) \) such that for all

\[ t \in (0, 1] \setminus \bigcup_{\theta \in M} B_\delta(-a_1(\theta)/a_2(\theta)), \]

solutions \( \varphi_t \) of (1.3) we have

\[ C^{-1} \leq \varphi_t(x) \leq C \text{ and } \| \varphi_t(x) \|_{C^2, \alpha(S^3)} \leq C. \]

**Proof.** Set \( I_{\delta,k} := (0, 1] \setminus \bigcup_{y \in M} B_\delta(-a_1(y)/a_2(y)). \) To obtain a contradiction, we assume that there are sequences \( (k_i) \in C^5(S^3) \), satisfying the assumptions of the theorem with \( (A, A_0, A_1, \delta) \) fixed, and \( (t_i, \varphi_{t_i}) \in I_{\delta,k_i} \times C^2(S^3) \) of solutions to (1.3) with \( k = k_i \) such that \( \| \varphi_{t_i} \|_{\infty} \to \infty \) as \( i \to \infty \). Passing to a subsequence we may assume \( t_i \to t_0 \) as \( i \to \infty \). By Corollary 3.2 there
are \( \theta \in S^3 \), \( \mu_i \to 0 \) and \( y_i \to 0 \) such that \( u_{t_i} \) defined by (1.3) in stereographic coordinates \( \mathcal{S}_\theta(\cdot) \) solves (1.6) and satisfies
\[
\tilde{w}_{t_i} := u_{t_i} - (1 + t_i(k_i)\theta(y_i))^{-\frac{1}{2}}z_{\mu_i,y_i} \text{ is orthogonal to } Tz_{\mu_i,y_i} Z,
\|	ilde{w}_{t_i}\|_{\mathcal{D}^1(\mathbb{R}^3)} = o(1).
\]
Using the notation of Lemma 4.2 we have with \( k = k_i \)
\[
0 = f'_t(u_{t_i}) = f'_t(z_{\mu_i,y_i} + w_0(t_i,\mu_i,y_i) + \tilde{w}_{t_i}).
\]
Consequently, for large \( i \), due to the uniqueness of \( \tilde{\alpha} \) and \( \tilde{w} \) in Lemma 4.2
\[
u_{t_i} = z_{\mu_i,y_i} + w(t_i,\mu_i,y_i,k_i) \quad \text{and} \quad \tilde{\alpha}(t_i,\mu_i,y_i,k_i) = 0,
\]
where we added the additional parameter \( k_i \) to express the dependence of \( \tilde{\alpha} \) and \( w \) on \( k_i \). From the expansion of \( \tilde{\alpha} \) in (4.4) we see
\[
\lim_{i \to \infty} \nabla(k_i)\theta(y_i) = 0 \quad \text{and} \quad \lim_{i \to \infty} \Delta(k_i)\theta(y_i) = 0.
\]
As \((\mathcal{A}, A_0, A_1, \delta)\) is fixed, the point \( \theta \) is in \( \mathcal{A} \), hence \( \theta \) is a nondegenerated critical point of each \( k_i \). We may apply Lemma 6.1 with \( k = k_i \) and get for large \( i \)
\[
y_i = \beta(t_i,\mu_i,k_i),
\]
where again the additional parameter \( k_i \) denotes the dependence on \( k_i \). From Lemma 6.2 we now get
\[
0 = \frac{1}{t_i\mu_i^3} \left( \tilde{\alpha}(t_i,\mu_i,\beta(t_i,\mu_i,k_i),k_i) \right)
= -(1 + t_i k_i(\theta))^{-\frac{3}{4}} \frac{3 \pi^6}{4^{\frac{3}{2}}} a_0(\theta,k_i)
+ \mu_i(1 + t_i k_i(\theta))^{-\frac{9}{4}} \frac{3 \pi^3 \sqrt{5}}{30} \left( a_1(\theta,k_i) + t_i a_2(\theta,k_i) \right)
+ O(\mu_i^{1+\frac{1}{4}}). \quad (7.1)
\]
Consequently, as \((\mathcal{A}, A_0, A_1, \delta)\) is fixed, \( a_0(\theta,k_i) = 0 \) for large \( i \).
We observe that \( |a_2(\theta,k_i)| \geq A_0^{-4} \) for all \( i \) large enough, if not then we get, up to a subsequence,
\[
|a_1(\theta,k_i)| \geq |k_i(\theta)|^{-1} \left( \frac{15}{8\pi} \int_{\partial B_{t_i}(0)} |D^2(k_i)\theta(0)(x)^2|^2 \right) - A_0^{-4}
\geq \text{const} A_0^{-3} (1 - A_0^{-2}),
\]
which yields a positive lower bound on \( |a_1(\theta,k_i) + t_i a_2(\theta,k_i)| \) contradicting the expansion in (7.1) for \( i \) large. Hence from (7.1) we infer
\[
|t_i + \frac{a_1(\theta,k_i)}{a_2(\theta,k_i)}| \leq |a_2(\theta,k_i)|^{-1} O(\mu_i^{\frac{1}{2}}) \leq O(\mu_i^{\frac{1}{2}}),
\]
which is impossible for \( \delta > 0 \). This shows that all solutions \( \varphi_t \) of (1.3) with \( t \in I_\delta \) are uniformly bounded. From Harnack’s inequality and standard elliptic estimates they are uniformly bounded below by a positive constant and uniformly bounded in \( C^{2,\alpha}(S^3) \), which ends the proof. \( \square \)
Proof of Theorems 1.2 and 1.4. If \( \theta \in M^*_+ \cup M^*_0 \) we may apply Lemmas 6.1 and 6.4 with \( k = k_0 \) and \( y_0 = 0 \). If we set \( y(\mu) := \beta(\tilde{t}(\mu), \mu) \) then we have \( \tilde{a}(\tilde{t}(\mu), \mu, y(\mu)) = 0 \) for all \( 0 < \mu < \min(\delta_1, \delta_2) \) and \( y(\mu) = O(\mu^2) \). From Lemma 1.2 we get that

\[
\psi(\mu) := z_{\mu, y(\mu)} + w(\tilde{t}(\mu), \mu, y(\mu))
\]

is a solution of (1.6) with \( t = \tilde{t}(\mu) \). As \( \nabla k_0(0) = 0 \) and \( y(\mu) = O(\mu^2) \) we may use (6.4) to obtain in \( D^{1,2}(\mathbb{R}^3) \)

\[
\psi(\mu) = (1 + \tilde{t}(\mu) k_0(0))^{-\frac{1}{4}} z_{\mu,0} + O(\mu^2).
\]

To show that \( \psi(\mu) \) is positive for small \( \mu \), we note that from Sobolev’s inequality \( \psi(\mu)^{-} \rightarrow 0 \) in \( L^6 \) as \( \mu \rightarrow 0 \), where \( \psi(\mu)^{-} := \min(\psi(\mu), 0) \). Testing \( f_k(\psi(\mu)) \) with \( \psi(\mu)^{-} \) and using Sobolev’s inequality we get for some \( c(k) > 0 \)

\[
\int |\nabla \psi(\mu)^{-}|^2 = \int (1 + \tilde{t}(\mu) k_0(x))(\psi(\mu)^{-})^6 \leq c(k) \left( \int |\nabla \psi(\mu)^{-}|^2 \right)^{\frac{3}{2}}.
\]

If \( \psi(\mu)^{-} \neq 0 \) for small \( \mu \) we obtain the contraction

\[
c(k)^{-\frac{3}{2}} \leq \int |\nabla \psi(\mu)^{-}|^2 = \int (1 + \tilde{t}(\mu) k_0(x))(\psi(\mu)^{-})^6 \xrightarrow{\mu \to 0} 0.
\]

The \( C^0 \)-estimate then follows from elliptic regularity (see [8]). Setting

\[
\phi^0(\mu, \cdot) := (R_\theta)^{-1}(\psi(\mu)) \text{ and } \tilde{t}(\mu) = \tilde{t}(\mu)
\]

yields the existence of the desired curve of solutions.

To prove uniqueness of the curves suppose \( (t_i, \varphi_i) \) blows up at \( \theta \in S^3 \). If \( t_i \in (\delta, 1 + \delta) \) then, as in the proof of Theorem 1.1 we get \( \theta \in M^*_+ \). Under the assumptions of Theorem 1.4 we already know that \( \theta \in M^*_0 \). If all but finitely many \( (t_i, \varphi_i) \) lie on the curve corresponding to \( \theta \in M^*_+ \cup M^*_0 \), we are done. Hence we may assume, going to a subsequence if necessary, that none of the \( (t_i, \varphi_i) \) lie on the curve. This is impossible since by Corollary 3.2 and Lemma 1.2 there are \( \mu_i, y_i \) converging to zero such that for \( i \) large

\[
R_\theta(\varphi_i) = z_{\mu_i, y_i} + w(t_i, \mu_i, y_i) \text{ and } \tilde{a}(t_i, \mu_i, y_i) = 0,
\]

and thus applying the uniqueness part in Lemmas 6.1 and 6.4 we see that \( y_i = \beta(t_i, \mu_i) \) and \( t_i = \tilde{t}(\mu_i) \) and the points \( (t_i, \varphi_i) \) have to lie on the curve. \( \square \)

**APPENDIX A. FORMULAS AND INTEGRALS**

The *Jacobi polynomial* \( P_j^{(\alpha,\beta)} \) is defined by

\[
P_j^{(\alpha,\beta)}(x) := \frac{(-1)^j}{2^j j!} (1 - x^2)^{-\frac{\alpha}{2}} \frac{d^j}{dx^j} ((1 - x^2)^{\alpha + j}) \quad (A.1)
\]

To compute integrals containing Jacobi-polynomials we will use

\[
\int_{-1}^1 (1 - \xi^2)^\sigma P_j^{(\sigma,\sigma)}(\xi) P_i^{(\sigma,\sigma)}(\xi) = \frac{2^{2\sigma + 1} \Gamma(j + \sigma + 1)^2}{(2j + 2\sigma + 1) j! \Gamma(j + 2\sigma + 1)} \delta_{i,j}, \quad (A.2)
\]
and the recurrence relation for $j \in \mathbb{N}_0$

$$
\xi P_j^{(\sigma,\sigma)}(\xi) = \frac{(j+1)(j+2\sigma+1)}{(2j+2\sigma+1)(j+\sigma+1)} P_{j+1}^{(\sigma,\sigma)}(\xi) + \frac{j+\sigma}{2j+2\sigma+1} P_{j+1}^{(\sigma,\sigma)}(\xi),
$$

$$
P_{-1}^{(\sigma,\sigma)}(\xi) = 0, \ P_0^{(\sigma,\sigma)}(\xi) := 1. \quad (A.3)
$$

For a detailed account on Jacobi polynomial we refer to [22]. To evaluate integrals of the form

$$
\int_0^\infty r^a(1+r^2)^{-b} P_j^{(\sigma,\sigma)} \left( 1 - \frac{2}{1+r^2} \right) dr,
$$

we use the following change of coordinates

$$
\xi = \frac{r - r^{-1}}{r + r^{-1}} = 1 - \frac{2}{1+r^2}, \quad (A.4)
$$

which gives

$$
r = \left( \frac{1+\xi}{1-\xi} \right)^{1/2} \text{ and } dr = \left( \frac{1+\xi}{1-\xi} \right)^{-1/2} (1-\xi)^{-2} d\xi,
$$

and leads to

$$
2^{-b} \int_{-1}^1 (1+\xi)^{\frac{a-1}{2}} (1-\xi)^{-2} \left( 1 - \frac{2}{1+r^2} \right) P_j^{(\sigma,\sigma)}(\xi) d\xi.
$$

Moreover, we note that for any $a,b > -1$

$$
\int_{-1}^1 (1+\xi)^a(1-\xi)^b d\xi = 2^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}. \quad (A.5)
$$

This gives for $a > -1$ and $2b - a > 1$

$$
\int_0^\infty r^a(1+r^2)^{-b} = \frac{\Gamma(1+(a-1)/2)\Gamma(b-1-(a-1)/2)}{2\Gamma(b)}. \quad (A.6)
$$

To compute integrals over $\mathbb{R}^N$ we use polar coordinates. To compute the resulting integrals over $\partial B_1(0)$ we use the following elementary results:

For $\vec{\beta} \in \mathbb{N}_0^N$ we have

$$
\int_{\partial B_1(0)} \prod_{i=1}^N x_i^{2\beta_i} = \frac{2 \prod_{i=1}^N \Gamma(\beta_i + \frac{1}{2})}{\Gamma(N^2 + \sum_{i=1}^N \beta_i)}. \quad (A.7)
$$

Let $m \geq 2$ and $P_m$ be a homogeneous polynomial of order $m$ in $x \in \mathbb{R}^N$. Then

$$
\int_{\partial B_1(0)} P_m(x) = \frac{1}{2N+(N+m)(m-2)} \int_{\partial B_1(0)} (\Delta P_m)(x), \quad (A.8)
$$

$$
\int_{\partial B_1(0)} P_m(x)x_i = \frac{1}{2(N+2)+(N+m+1)(m-3)} \int_{\partial B_1(0)} (\Delta P_m)(x)x_i. \quad (A.9)
$$
Lemma A.2. Let 

\[
\frac{\partial}{\partial x_i} D_\ell^k(y)(x)^\ell = \ell \frac{\partial}{\partial y_i} D_\ell^{k-1}(y)(x)^{\ell-1}
\]

we see

\[
\int \frac{D_\ell^k(y)(x)^{2\ell}}{(2\ell)!} = 2\pi N/2 \Delta_\ell k(y) \prod_{m=1}^{N} 2m(N + 2m - 2),
\]

(A.10)

\[
\int \frac{D_\ell^{k+1}(y)(x)^{2\ell+1} x_i}{(2\ell+1)!} = \frac{2\pi N/2 \frac{\partial}{\partial y_i} \Delta_\ell k(y)}{N! N/2 \prod_{m=1}^{N} 2m(N + 2m)}.
\]

(A.11)

Lemma A.1. Suppose \( j \in \mathbb{N}_0 \), then

\[
\int_0^\infty r^6(1 + r^2)^{-5} \left(1 - \frac{2}{1 + r^2}\right) P_j^{(\frac{5}{2}, \frac{5}{2})}(1 - \frac{2}{1 + r^2})
\]

\[
= \frac{\Gamma\left(\frac{5}{2}\right) \Gamma\left(\frac{j}{2} + j\right) (j^2 + 6j + 2)}{2 \Gamma(6 + j)}. \quad \text{(A.12)}
\]

Proof. We use the change of coordinates in \( \text{(A.4)} \) and obtain

\[
\int_0^\infty r^4(1 + r^2)^{-5} \left(1 - \frac{2}{1 + r^2}\right) P_j^{(\frac{7}{2}, \frac{7}{2})}(1 - \frac{2}{1 + r^2})
\]

\[
= 2^{-5} \int_{-1}^1 \left(1 - \xi^2\right)^{\frac{5}{2}} \frac{\xi}{(1 - \xi)^2} P_j^{(\frac{5}{2}, \frac{5}{2})}(\xi)
\]

\[
= \frac{(-1)^j}{2^{j+5} j!} \int_{-1}^1 \frac{1}{(1 - \xi)^2} - \frac{1}{1 - \xi} \frac{d^j}{d\xi^j} (1 - \xi^2)^{\frac{7}{2} + j}
\]

\[
= \frac{(-1)^j}{2^{j+5} j!} \int_{-1}^1 (-1)^j (j + 1)! - (-1)^j j! \frac{1}{(1 - \xi)^{2+j}} (1 - \xi^2)^{\frac{7}{2} + j}.
\]

Now, the claim follows from \( \text{(A.3)} \). \( \square \)

Lemma A.2. Let \( \beta_j := \frac{(3+j)!}{\Gamma(3+j)} \) for \( j \in \mathbb{N} \). Then

\[
\int_0^\infty r^6(1 + r^2)^{-7} \left(1 - \frac{2}{1 + r^2}\right) \left(\sum_{j=0}^{\infty} \beta_j P_j^{(\frac{7}{2}, \frac{7}{2})}(1 - \frac{2}{1 + r^2})\right)^2 = \frac{1}{288}.
\]

(A.13)

Proof. We use the change of variable given in \( \text{(A.4)} \) and the recurrence formula \( \text{(A.3)} \), applied to \( \xi P_j^{(\frac{7}{2}, \frac{7}{2})}(\xi) \), and get

\[
\int_0^\infty r^6(1 + r^2)^{-7} \left(1 - \frac{2}{1 + r^2}\right) \left(\sum_{j=0}^{\infty} \beta_j P_j^{(\frac{7}{2}, \frac{7}{2})}(1 - \frac{2}{1 + r^2})\right)^2
\]

\[
= 2^{-7} \int_{-1}^1 (1 - \xi^2)^{\frac{7}{2}} \xi \left(\sum_{j=0}^{\infty} \beta_j P_j^{(\frac{7}{2}, \frac{7}{2})}(\xi)\right)^2
\]

\[
= 2^{-7} \sum_{d=0}^{\infty} \sum_{j=0}^{d} \beta_j \beta_{d-j} \int_{-1}^1 (1 - \xi^2)^{\frac{7}{2}} \xi P_j^{(\frac{7}{2}, \frac{7}{2})}(\xi) P_{d-j}^{(\frac{7}{2}, \frac{7}{2})}(\xi)
\]

\[
= \sum_{l=0}^{\infty} \beta_l \beta_{l+1} \frac{\Gamma(l + \frac{7}{2})^2 (l + \frac{7}{2})}{\Gamma(l + 6) (2l + 8) (2l + 6)!} = \frac{1}{4} \sum_{l=0}^{\infty} \frac{(l + 1)!}{l (l + 5)!} = \frac{1}{288}.
\]

\( \square \)
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