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Lyapunov stable homoclinic classes for smooth vector fields

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Abstract: In this paper, we show that for generic $C^1$, if a flow $X$ has the shadowing property on a bi-Lyapunov stable homoclinic class, then it does not contain any singularity and it is hyperbolic.

Keywords: homoclinic class; Lyapunov stable; shadowing; generic; hyperbolic

MSC: 37C50; 37C10; 37C20; 37C29; 37D05

1 Introduction

Let $M$ be a compact smooth Riemannian manifold, and let $\text{Diff}(M)$ be the space of diffeomorphisms of $M$ endowed with the $C^1$ topology. Hyperbolicity and stability have been important topics in differentiable dynamical systems since they were introduced by Smale [1]. For instance, a diffeomorphism $f : M \to M$ is structurally stable if and only if it satisfies Axiom A and the strong transversality condition. A diffeomorphism $f : M \to M$ satisfies Axiom A if the nonwandering set $\Omega(f)$ is $\overline{P(f)}$ and is hyperbolic, where $P(f)$ is the set of all periodic points of $f$. A set of diffeomorphisms is generic (or residual) if it contains a countable intersection of dense open sets of $\text{Diff}(M)$. Abraham and Smale [2] showed that the set of diffeomorphisms $f : M \to M$ satisfying Axiom A and the no-cycle condition is not dense in the space of $\text{Diff}(M)$.

If a diffeomorphism $f : M \to M$ satisfies Axiom A, then from the work of Smale [1], the nonwandering set $\Omega(f) = \bigcup_{i=1}^{n} A_i$, where each $A_i$ is a basic set. If a basic set contains a hyperbolic periodic point, then it is a homoclinic class. In general, a homoclinic class is not hyperbolic even in a generic sense. For a $C^1$ generic diffeomorphism $f : M \to M$, several extra conditions are imposed to obtain hyperbolicity of the homoclinic classes.

Let us give a short review of related results. Ahn et al. [3] proved that for generic $C^1$, if a diffeomorphism $f$ has the shadowing property on a locally maximal homoclinic class, then it is hyperbolic. Lee [4] proved that for generic $C^1$, if a diffeomorphism $f$ has the limit shadowing property on a locally maximal homoclinic class, then it is hyperbolic. Note that local maximality is quite a restrictive condition. Arbieto et al. [5] proved that for generic $C^1$, if a bi-Lyapunov stable homoclinic class is homogeneous and has the shadowing property, then it is hyperbolic. See [3, 4, 6–15] for related results.

We want to extend some of the above results for flows, that is, for a $C^1$ generic vector field $X \in \mathcal{X}(M)$, a condition under which we can obtain hyperbolicity of homoclinic classes. Unfortunately, we cannot use the same arguments as in the diffeomorphism case.

We say that a diffeomorphism $f$ satisfies the star condition if there is a $C^1$ neighborhood $\mathcal{U}(f) \subset \text{Diff}(M)$ such that for any $g \in \mathcal{U}(f)$, every periodic point of $g$ is hyperbolic. Aoki [16] and Hayashi [17] showed that if a diffeomorphism $f$ satisfies the star condition, then it is Axiom A and the no-cycle condition, that is, $\Omega$ stable.
We say that a flow $X^t$ satisfies the star condition if there is a $C^1$ neighborhood $U(X) \subset \mathcal{X}(M)$ such that for any $Y \in U(X)$, every critical point of $Y$ is hyperbolic. From the results of Guchenheimer [18], the Lorenz attractor satisfies the star condition, but it is not $\Omega$-stable because the attractor contains a hyperbolic singular point. However, if a flow does not contain singularities and satisfies the star condition, then it is $\Omega$ stable (see [19]).

### 2 Basic notions and main theorem

Let $M$ be a compact $n$-dimensional smooth Riemannian manifold, and let $d$ be the distance on $M$ induced from a Riemannian metric $\| \cdot \|$ on the tangent bundle $TM$, and denote by $\mathcal{X}(M)$ the set of $C^1$ vector fields on $M$ endowed with the $C^1$ topology. Then, every $X \in \mathcal{X}(M)$ generates a $C^1$ flow $X^t : M \times \mathbb{R} \to M$; that is, a $C^1$ map such that $X^t : M \to M$ is a diffeomorphism satisfying $X^0(x) = x$, and $X^{t+s}(x) = X^t(X^s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. The flow of $X$ will be denoted by $X^t, t \in \mathbb{R}$. For $X \in \mathcal{X}(M)$, a point $x \in M$ is singular of $X$ if $X(x) = 0$. Denote by $\text{Sing}(X)$ the set of all singular points of $X$. A point $x \in M$ is regular if $x \in M \setminus \text{Sing}(X)$. Denote by $R(M)$ the set of all regular points of $X$. A point $p \in M$ is periodic if there is $\pi(p) > 0$ such that $X^{\pi(p)}(p) = p$, where $\pi(p)$ is the prime period of $p$. Denote by $\text{Per}(X)$ the set of all closed orbits of $X$. Let $\text{Crit}(X) = \text{Sing}(X) \cup \text{Per}(X)$.

For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1, i \in \mathbb{Z}\}$ is a $\delta$-pseudo-orbit of $X$ if $d(X^{t_i}x_i, x_{i+1}) < \delta$ for any $i \in \mathbb{Z}$.

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a reparametrization of $\mathbb{R}$. Denote by $\text{Rep}(\mathbb{R})$ the set of reparametrizations of $\mathbb{R}$. Fix $\epsilon > 0$ and define $\text{Rep}(\epsilon)$ as follows:

$$\text{Rep}(\epsilon) = \{ h \in \text{Rep} : \frac{h(t)}{t} - 1 < \epsilon \}.$$ 

For a closed $X^t$-invariant set $\Lambda \subset M$, we say that $X$ has the shadowing property on $\Lambda$ if for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following property: given any $\delta$-pseudo-orbit $\xi = \{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\}$ with $x_i \in \Lambda$, there is a point $y \in M$ and an increasing homeomorphism $h \in \text{Rep}(\epsilon)$ such that $d(X^{h(t)}y, X^{t-i}(x_i)) < \epsilon$ for any $s_i < t < s_{i+1}$, where $s_i$ is defined as

$$s_i = \begin{cases} t_0 + t_1 + \cdots + t_{i-1}, & \text{if } i > 0 \\ 0, & \text{if } i = 0 \\ -t_{i-2} - \cdots - t_i, & \text{if } i < 0. \end{cases}$$

The point $y \in M$ is said to be a shadowing point of $\xi$.

Let $X^t$ be the flow of $X \in \mathcal{X}(M)$, and let $\Lambda$ be a $X^t$-invariant compact set. The set $\Lambda$ is called hyperbolic for $X^t$ if there are constants $C > 0, \lambda > 0$ and a splitting $T_XM = E^s_x \oplus \langle X(x) \rangle \oplus E^u_x$ such that the tangent flow $DX^t : TM \to TM$ leaves the continuous splitting invariant and

$$\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t} \quad \text{and} \quad \|DX^t|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$. We say that $X \in \mathcal{X}(M)$ is Anosov if $M$ is hyperbolic for $X^t$.

Let $\gamma$ be a hyperbolic closed orbit of a vector field $X \in \mathcal{X}(M)$, and we define the stable and unstable manifolds of $\gamma$ by

$$W^s(\gamma) = \{ y \in M : \omega(y) = \gamma \}$$

and

$$W^u(\gamma) = \{ y \in M : \alpha(y) = \gamma \}.$$ 

Let $X \in \mathcal{X}(M)$, and let $\gamma$ be a hyperbolic closed orbit of $X^t$. A point $x \in W^s(\gamma) \cap W^u(\gamma)$ is called a transversal homoclinic point of $X^t$ associated to $\gamma$. The closure of the transversal homoclinic points of $X^t$ associated to $\gamma$ is called the homoclinic class of $X^t$ associated to $\gamma$, and it is denoted by

$$H_X(\gamma) = \overline{W^s(\gamma) \cap W^u(\gamma)}.$$ 

It is clear that $H_X(\gamma)$ is a compact, transitive, and $X^t$-invariant set.
For two hyperbolic closed orbits $\gamma_1$ and $\gamma_2$ of $X'$, we say that $\gamma_1$ and $\gamma_2$ are **homoclinic related**, denoted by $\gamma_1 \sim \gamma_2$, if $W^s(\gamma_1) \cap W^u(\gamma_2) \neq \emptyset$ and $W^u(\gamma_1) \cap W^s(\gamma_2) \neq \emptyset$. It is clear that if $\gamma_1 \sim \gamma$, then $\text{index}(\gamma_1) = \text{index}(\gamma)$, where $\text{index}(\gamma) = \dim W^s(\gamma)$. Note that if $\gamma$ is a hyperbolic closed orbit of $X'$, then there exist a $C^1$ neighborhood $U(\gamma)$ of $\gamma$ such that for any $Y \in \text{int}(U(\gamma))$, there exists a unique hyperbolic closed orbit $\gamma_Y$ that equals $\bigcap_{t \in \mathbb{R}} Y'(U)$. The hyperbolic closed orbit $\gamma_Y$ is called the *continuation* of $\gamma$ with respect to $Y$, and $\text{index}(\gamma_Y) = \text{index}(\gamma)$.

A closed invariant set $\Lambda$ is **Lyapunov stable** if for any neighborhood $U$ of $\Lambda$, there is a neighborhood $V$ of $\Lambda$ such that $X'(V) \subset U$ for all $t > 0$. We say that $\Lambda$ is **bi-Lyapunov stable** if it is Lyapunov stable for $X$ and for $-X$.

We say that a subset $\mathcal{F} \subset \mathcal{X}(M)$ is **residual** if $\mathcal{F}$ contains the intersection of a countable family of open and dense subsets of $\mathcal{X}(M)$. In this case $\mathcal{F}$ is dense in $\mathcal{X}(M)$. A property “$P$” is said to be $C^1$-**generic** if “$P$” holds for all vector fields that belong to some residual subset of $\mathcal{X}(M)$. We write for $C^1$ **generic** $X \in \mathcal{X}(M)$ in the sense that there is a residual set $\mathcal{F} \subset \mathcal{X}(M)$ for any $X \in \mathcal{F}$. In this paper, we prove the following theorem, which is an extension of a result of Arbieto et al. [5] for flows.

**Theorem.** For $C^1$ **generic** $X \in \mathcal{X}(M)$, if a flow $X'$ has the shadowing property on a bi-Lyapunov stable homoclinic class $H_X(\gamma)$, then $H_X(\gamma) \cap \text{Sing}(X) = \emptyset$ and $H_X(\gamma)$ is hyperbolic.

### 3 Proof of the Theorem

Let $M$ be as previously, and let $X \in \mathcal{X}(M)$. We define the strong stable and unstable manifolds of a hyperbolic periodic point $p$ respectively as follows:

$$W^{ss}(p) = \{y \in M : d(X'(y), X'(p)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^s(\text{Orb}(p)) = \bigcup_{t \in \mathbb{R}} W^{ss}(X'(p)),$$

where $\text{Orb}(p)$ is the orbit of $p$. If $\epsilon > 0$, the local strong stable manifold is defined as

$$W^{ss}_{\epsilon}(p) = \{y \in M : d(X'(y), X'(p)) < \epsilon, \text{ if } \epsilon \geq 0\}.$$

By the stable manifold theorem, there is an $\epsilon = \epsilon(p) > 0$ such that

$$W^{ss}(p) = \bigcup_{t \in \mathbb{R}} X^{-t}(W^{ss}_{\epsilon(p)}(X'(p))).$$

We can define the unstable manifolds similarly. If $\sigma$ is a hyperbolic singularity of $X$, then there exists an $\epsilon = \epsilon(\sigma) > 0$ such that

$$W^u_{\epsilon}(\sigma) = \{x \in M : d(X'(x), \sigma) < \epsilon \text{ as } t \geq 0\}$$

and

$$W^u(\sigma) = \bigcup_{t \in \mathbb{R}} X^{t}(W^u_{\epsilon}(\sigma)).$$

Analogous definitions hold for unstable manifolds.

### 3.1 Transversal intersection and the absence of singularities

The following lemma states that there are transversal intersections between invariant manifolds of hyperbolic closed orbits and singularities.
Lemma 3.1. Let $\gamma$ be a hyperbolic closed orbit of $X$. If a flow $X^t$ has the shadowing property on $H_X(\gamma)$, then for every hyperbolic $\sigma \in H_X(\gamma) \cap \text{Crit}(X)$, we have

$$W^s(\gamma) \cap W^u(\sigma) \neq \emptyset \quad \text{and} \quad W^u(\gamma) \cap W^s(\sigma) \neq \emptyset.$$ 

Proof. First, we assume that $\eta \in H_X(\gamma) \cap \text{Per}(X)$. Let $p \in \gamma$ and $q \in \eta$. Take $\varepsilon = \min\{\varepsilon(p), \varepsilon(q)\}$ and let $0 < \delta \leq \varepsilon$ be given by the shadowing property according to $\varepsilon$. Since $H_X(\gamma)$ is transitive, there is $x \in H_X(\gamma)$ such that $\omega(x) = H_X(\gamma)$. Then, there are $t_1 > 0$ and $t_2 > 0$ such that $X^{t_1}(x) \in B_{\delta}(p)$ and $X^{t_2}(x) \in B_{\delta}(q)$. Assume that $t_2 = t_1 + k$ for some $k > 0$. Then, the sequence

$$\{p, X^{t_1}(x), X^{t_1+1}(x), \ldots, X^{t_1+k-1}(x), q\} \subset H_X(\gamma)$$

is a finite $\delta$-pseudo-orbit of $X$. We construct a $\delta$-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset H_X(\gamma)$ as follows:

(i) $X^{-i}(p) = x_{-i}$ for $i \geq 0$;

(ii) $X^{t_1+i}(x) = x_i$ for $i = 1, \ldots, k-1$; and

(iii) $X^{t}(p) = x_{k+i}$ for all $i \geq 0$.

Since $X^t$ has the shadowing property on $H_X(\gamma)$, there is $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$d(X^{h(t)}(y), X^{t-s}(x_i)) < \varepsilon \quad \text{and} \quad d(X^{h(t)}(y), X^{t-s}(x_i)) < \varepsilon,$$

where $s_i < t < s_{i+1}$ and $s_{-i} < t < s_{-i+1}$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$. Then $y \in W^u_\varepsilon(p)$ and there is $\tau > 0$ such that $X^\tau(y) \subset W^s_\varepsilon(q)$. Thus, we have

$$\text{Orb}(y) \cap W^u(\gamma) \cap W^s(\eta) \neq \emptyset.$$

The other case is similar.

Now, we assume that $\sigma \in H_X(\gamma) \cap \text{Sing}(X)$. Let $p \in \gamma$. Take $\varepsilon = \min\{\varepsilon(p), \varepsilon(\sigma)\}$ and let $0 < \delta \leq \varepsilon$ be given by the shadowing property according to $\varepsilon$. Since $H_X(\gamma)$ is transitive, there is $x \in H_X(\gamma)$ such that $\omega(x) = H_X(\gamma)$. Then, there are $t_1 > 0$ and $t_2 > 0$ such that $X^{t_1}(x) \in B_{\delta}(\sigma)$ and $X^{t_2}(x) \in B_{\delta}(p)$. Assume that $t_2 = t_1 + k$ for some $k > 0$. We can thus construct a $\delta$-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset H_X(\gamma)$ as follows:

(i) $\sigma = x_{-i}, t_{-i} = 1$ for $i \geq 0$;

(ii) $X^{t_1+i}(x) = x_i$ for $i = 1, \ldots, k-1$; and

(iii) $X^{t}(p) = x_{k+i}$ for all $i \geq 0$.

Since $X^t$ has the shadowing property on $H_X(\gamma)$, as in the proof of previous arguments, we have $W^u(\sigma) \cap W^s(\gamma) \neq \emptyset$. The other case is similar. \hfill $\square$

We say that $X$ is Kupka–Smale if every $\sigma \in \text{Crit}(X)$ is hyperbolic, and their invariant manifolds intersect transversally. Denote by $KS$ the set of all Kupka–Smale vector fields. It is known that $KS \subset X(M)$ is a residual subset (see [20]).

Lemma 3.2. There is a residual set $\mathcal{S} \subset X(M)$ such that for any $X \in \mathcal{S}$, if a flow $X^t$ has the shadowing property on $H_X(\gamma)$, then for all $\eta \in H_X(\gamma) \cap \text{Crit}(X)$, we have

$$W^s(\gamma) \pitchfork W^u(\eta) \neq \emptyset \quad \text{and} \quad W^u(\gamma) \pitchfork W^s(\eta) \neq \emptyset.$$ 

Proof. Let $X \in \mathcal{S}$ and let $\eta \in H_X(\gamma) \cap \text{Crit}(X)$. Since a flow $X^t$ has the shadowing property on $H_X(\gamma)$, by Lemma 3.1, $W^u(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^s(\gamma) \cap W^s(\eta) \neq \emptyset$. Since $X \in KS$, $W^u(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$. \hfill $\square$

Proposition 3.3. For any $X \in \mathcal{S}$, if a flow $X^t$ has the shadowing property on $H_X(\gamma)$, then we have

$$H_X(\gamma) \cap \text{Sing}(X) = \emptyset.$$ 

Proof. Let $X \in KS$ and let $\gamma$ be a hyperbolic periodic orbit of $X$ in $H_X(\gamma)$ with index $j$. Suppose that $X$ has a hyperbolic singularity $\sigma \in H_X(\gamma)$ with index $i$. If $j < i$, then $\dim W^u(\sigma) + \dim W^s(\gamma) \leq \dim M$. Since $X$ is a Kupka–Smale vector field, we have $\dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. By assumption, we can take $x \in W^u(\sigma) \cap W^s(\gamma)$. \hfill $\square$
Then $\text{Orb}(x) \in W^u(\sigma) \cap W^s(\gamma)$ and we can split
$$T_x(W^u(\sigma)) = T_x(\text{Orb}(x)) \oplus E^1 \quad \text{and} \quad T_x(W^s(\gamma)) = T_x(\text{Orb}(x)) \oplus E^2.$$ 

Thus, we know that
$$\dim(T_x(W^u(\sigma)) + T_x(W^s(\gamma))) < \dim W^u(\sigma) + \dim W^s(\gamma) = \dim M.$$ 

This is a contradiction, because $X$ is a Kupka–Smale vector field. If $j \geq i$, then
$$\dim W^s(\sigma) + \dim W^u(\gamma) \leq \dim M.$$ 

By the previous arguments, we have a contradiction. Thus, $H_\gamma(\gamma) \cap \text{Sing}(X) = \emptyset$. 

3.2 Chain recurrent class and homoclinic class

For any $x, y \in M$, we say that $x \rightarrow y$ if for any $\delta > 0$, there is a finite $\delta$-pseudo-orbit $\{(x_i, t_i) : 0 \leq i < n\}$ with $n > 1$ such that $x_0 = x$ and $d(X^{i+1}(x_{i+1}, t_i), y) < \delta$ and a $\delta$-pseudo-orbit $\{(z_i, s_i) : 0 \leq i < m\}$ with $m > 1$ such that $z_0 = y$ and $d(X^{i+1}(z_{i+1}, t_i), x) < \delta$. It is easy to see that $\rightarrow$ gives an equivalent relation on the chain recurrent set $\mathbb{C}(X)$. We denoted the equivalence class as
$$C_\gamma(\gamma) = \{x \in M : x \leadsto \gamma \text{ and } \gamma \rightarrow x\}$$

and called the chain recurrence class associated to $\gamma$. It is known that $H_\gamma(\gamma) \subseteq C_\gamma(\gamma)$, but the converse is not true in general. We now summarize some results about homoclinic classes and chain recurrence classes.

**Lemma 3.4.** There is a residual set $\mathcal{H}_2 \subset X(M) \text{ such that every } \gamma \in \mathcal{H}_2 \text{ satisfies:}\n
(a) the chain recurrence class $C_\gamma(\gamma) = H_\gamma(\gamma)$ (see [21]);
(b) if a closed orbit $\eta \in H_\gamma(\gamma)$, then $H_\gamma(\gamma) = H_\eta(\eta)$ (see [22]);
(c) $H_\gamma(\gamma) = W^s(\gamma) \cap W^u(\gamma)$ (see [22]);
(d) $W^s(\gamma)$ is Lyapunov stable for $-X$ and $W^u(\gamma)$ is Lyapunov stable for $X$ (see [22]);
(e) if $H_\gamma(\gamma)$ is Lyapunov stable for $X$, then there is a $C^1$ neighborhood $\mathcal{U}(\gamma)$ of every index in $[a, \beta]$; moreover, every closed orbit in $H_\gamma(\gamma)$ has its index in that interval (see [24]).

Let $X \in X(M)$ have no singularities and let $N \subset TM$ be the sub-bundle such that the fiber $N_x$ at $x \in M$ is the orthogonal linear subspace of $\langle X(x) \rangle$ in $T_x M$, that is, $N_x = \langle X(x) \rangle^\perp$. Here $\langle X(x) \rangle$ is the linear subspace spanned by $X(x)$ for $x \in M$. Let $\pi : TN \rightarrow N$ be the projection along $X$, and let
$$P_\gamma^X(v) = \pi(D_v X^\gamma(v)),$$

for $v \in N_x$ and $x \in M$. Let $\Lambda$ be a closed $X^\gamma$-invariant regular set. We say that $\Lambda$ is hyperbolic if the bundle $N_\Lambda$ has a $P_\gamma^X$-invariant splitting $\Delta^s \oplus \Delta^u$ and there exists an $l > 0$ such that
$$\left\|P_\gamma^X|_{\Delta^s}\right\| \leq \frac{1}{2} \quad \text{and} \quad \left\|P_\gamma^X|_{\Delta^u}\right\| \leq \frac{1}{2},$$

for all $x \in \Lambda$. Then, Doering [25] proved the following result, which is a method of proof for hyperbolicity.

**Proposition 3.5.** Let $\Lambda \subset M$ be a compact invariant set of $X^\gamma$. Then, $\Lambda$ is a hyperbolic set of $X^\gamma$ if and only if the linear Poincaré flow restriction on $\Lambda$ has a hyperbolic splitting $N_\Lambda = \Delta^s \oplus \Delta^u$, where $N = \bigcup_{x \in M^\gamma} N_x$. 

3.3 Weak hyperbolic periodic points

An exponential map \( \exp_p : T_p M(1) \to M \) is well defined for all \( p \in M \), where \( T_p M(\delta) \) denotes the ball \( \{ v \in T_p M : \| v \| \leq \delta \} \). For every regular point \( x \in X(\tau) \), let \( N_x = (X(x))^{\perp} \subset T_x M \), and \( N_x(\delta) \) be the \( \delta \)-ball in \( N_x \). Let \( N_{x,r} = \exp_r(N_x(r)) \). Given any point \( x \in R(M) \) and \( t \in \mathbb{R} \), there are \( r > 0 \) and a \( C^1 \) map \( \tau : N_{x,t} \to \mathbb{R} \) with \( \tau(x) = t \) such that \( X^{\tau(y)}(y) \in N_{X^{\tau(x)},1} \) for any \( y \in N_{x,r} \). We define the Poincaré map as

\[
fx,t : N_{x,r} \to N_{X^{\tau(x)},1} \\
y \mapsto fx,t(y) = X^{\tau(y)}(y).
\]

Let \( X \in X(M) \), and suppose \( p \in \gamma \in \text{Per}(X) \) and let \( f : N_{p,r} \to N_p(r > 0) \) be the Poincaré map of \( X \). For any \( \delta > 0 \), if the eigenvalue \( \lambda \) of \( D_p f \) is \( 1 < \lambda < 1 + \delta \), then there is \( g \) that is \( C^1 \) close to \( f \) such that \( D_p g \) has an eigenvalue \( \mu \) with \( \mu \leq 1 - \delta \), where \( g \) is the Poincaré map associated to \( Y \).

**Proof.** Let \( A_p \) be the eigenspace corresponding to \( \lambda \) with index \( p = i \), and let \( N_p = A_p \oplus A_p^{\perp} \). For the splitting, we have

\[
Df(p) = \begin{pmatrix}
Df(p)|_{A_p} A_1(f) \\
O \\
A^2(f)
\end{pmatrix}.
\]

Applying Gourmelon’s result [26] (see also [5, Theorem 2.5]), we define the map \( T : [0,1] \to G_1 \) as follows

\[
T(t) = \begin{pmatrix}
(1-t)Df(p)|_{A_p} + t\frac{1-\delta}{1+\delta}Df(p)|_{A_p} A_1(f) \\
O \\
A^2(f)
\end{pmatrix},
\]

for \( t \in [0,1] \). Then, we have

\[
D_pf = T(0) = \begin{pmatrix}
Df(p)|_{A_p} A_1(f) \\
O \\
A^2(f)
\end{pmatrix}
\]

and

\[
D_pg = T(1) = \begin{pmatrix}
\frac{1-\delta}{1+\delta}Df(p)|_{A_p} A_1(f) \\
O \\
A^2(f)
\end{pmatrix}.
\]

Thus, one can see that

\[
\frac{1-\delta}{1+\delta}Df(p)|_{A_p} \leq \frac{1-\delta}{1+\delta}(1+\delta) = (1-\delta).
\]

The proof is complete. \( \square \)

For any \( \delta > 0 \), we say that a point \( p \in \gamma \in \text{Per}(X) \) is \( \delta \)-weak hyperbolic periodic if there is an eigenvalue \( \lambda \) of \( D_p f \) such that \( (1-\delta) < |\lambda| < (1+\delta) \), where \( f : N_{p,r} \to N_p \) is the Poincaré map associated to \( X \).

Let \( X \in G_2 \) and let \( H_X(\gamma) \) be bi-Lyapunov stable with \( \text{index}(\gamma) = i(0 < i < \dim M - 1) \). Then, there is \( u(X) \) of \( X \) such that for any \( Y \in u(X) \), \( H_Y(\gamma_Y) \) is bi-Lyapunov stable and every closed orbit in \( H_Y(\gamma_Y) \) has the same index \( i \). From this fact, we have the following result.

**Lemma 3.7.** There is a residual set \( G_3 \subset X(M) \) such that for any \( X \in G_3 \), if a homoclinic class \( H_X(\gamma) \) is bi-Lyapunov stable, then \( H_X(\gamma) \) does not contain a \( \delta \)-weak hyperbolic periodic point.

**Proof.** Let \( X \in G_3 = G_1 \cap G_2 \) have the shadowing property on \( H_X(\gamma) \). Since \( X \) has the shadowing property on \( H_X(\gamma) \), by Lemmas 3.1 and 3.2, we have \( \eta \sim \gamma \) for every \( \eta \in H_X(\gamma) \cap \text{Per}(X) \). Suppose, by contradiction, that
for any $\delta > 0$ there is $p \in \eta \in H_X(\gamma) \cap Per(X)$ such that $p$ is a $\delta$ weak hyperbolic periodic point. Then, there is an eigenvalue $\lambda$ of $D_pf$ such that

$$1 - \delta < |\lambda| < 1 + \delta,$$

where $f : N_{p,r} \to N_p$ is the Poincaré map corresponding to the flow $X^t$. Assume that $1 < \lambda < 1 + \delta$ (the other case is similar). Let $p \in \gamma$ and $q \in \eta \in H_X(\gamma) \cap Per(X)$. Take $x \in W^S(p) \cap W^{uu}(q)$ and choose a neighborhood $U$ of $q$ such that:

(i) $U \cap \{\gamma\} = \emptyset$;
(ii) $U \cap Orb^*(x) = \emptyset$; and
(iii) $\text{Orb}^-(x) \subset U$.

Then, by [5, Theorem 2.5] and Lemma 3.6, there is $g C^1$ close to $f$ such that:

(i) $\text{index}(\eta_Y) > \text{index}(\gamma_Y)$;
(ii) it preserves the $i$ strong stable manifold of $q_\delta \in \eta_Y$ outside $U$; and
(iii) $W^{uu}(p_\delta) \cap W^{ss}(q_\delta) \neq \emptyset$;

where $\gamma_Y$ is the continuation of $\gamma$, $\eta_Y$ is the continuation of $\eta$, $Y^i$ is the flow corresponding to $g$, and $p_\delta \in \gamma_Y$. Using the $\lambda$-lemma, we have $q_\delta \in W^{uu}(p_\delta)$. Since $H_Y(\gamma_Y)$ is Lyapunov stable for $Y$, we have $W^{uu}(\gamma_Y) \subset H_Y(\gamma_Y)$, and so $q_\delta \in \eta_Y \subset H_Y(\gamma_Y)$. This is a contradiction. Since $X \in S_3$, if every $\eta \in H_X(\gamma) \cap Per(X)$ has index $i$, then by Lemma 3.4, every $\eta_Y \in H_Y(\gamma_Y) \cap Per(Y)$ has index $i$.

By Proposition 3.3, $H_X(\gamma) \cap Sing(X) = \emptyset$. Then, we have the following lemma, which is a flow version of the result proved by Wang [27].

**Lemma 3.8.** There is a residual set $S_4 \subset \mathcal{X}(M)$ such that for any $X \in S_4$, if a homoclinic class $H_X(\gamma)$ is not Hyperbolic, then for any $\delta > 0$, there is a periodic point $q \in \eta \subset H_X(\gamma) \cap Per(X)$ such that $\eta \sim \gamma$ and $q$ is a $\delta$-weak hyperbolic periodic point.

**Proof of the Theorem.** Let $X \in S_3 \cap S_4$ have the shadowing property on $H_X(\gamma)$. Suppose, by contradiction, that $H_X(\gamma)$ is not hyperbolic. Since $X$ has the shadowing property on $H_X(\gamma)$, by Lemma 3.2 we have $\eta \sim \gamma$, for all $\eta \in H_X(\gamma) \cap Per(X)$. Then, by Lemma 3.8, for any $\delta > 0$ there is $q \in \eta \in H_X(\gamma) \cap Per(X)$ such that $q$ is a weak hyperbolic periodic point. This is a contradiction by Lemma 3.7.

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