Latent Signal Analysis and the Analytic Signal

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Abstract—In this paper, we present the latent signal analysis problem as a recasting of the complex extension problem. Almost universally, the approach has been to use the Hilbert Transform (HT) to construct Gabor's analytic signal. This approach depends on harmonic correspondence and forces the use of simple harmonic components in the analysis, which may lead to incorrect Instantaneous Amplitude (IA) and Instantaneous Frequency (IF) parameters. We show that by relaxing the harmonic correspondence condition, the resulting complex extension can still be an analytic function and we can arrive at alternate IA/IF parameterizations which may be more accurate at describing the latent signal. However, in relaxing the harmonic correspondence condition there is no longer a unique rule for the complex extension. In addition, we discuss the problems associated with the use of the HT as an AM–FM demodulator for the latent signal. Finally we provide solutions to an example latent signal analysis problem.

Index Terms—Hilbert space, Signal analysis.

I. INTRODUCTION

MANY physical phenomena are characterized by a complex signal

\[ z(t) = x(t) + jy(t) = \rho(t)e^{j\Theta(t)} \]  

(1)

where \( \rho(t) \) is the signal’s Instantaneous Amplitude (IA), \( \Theta(t) \) the signal’s phase, and \( \Omega(t) = \frac{d}{dt}\Theta(t) \) the signal’s Instantaneous Frequency (IF). We will call \( z(t) \) the latent signal because only the real part, \( x(t) \) is observed and the imaginary part, \( y(t) \) is hidden, i.e. the act of observation corresponds to the real operator

\[ x(t) = \Re\{z(t)\}. \]  

(2)

It is often desirable to analyze the latent signal—which is known to completely parameterize the physical phenomenon. Thus the complex extension problem becomes that of determining \( z(t) \) from the observation \( x(t) \), i.e. recovering the quadrature \( y(t) \). In the classical approach, we seek a unique rule \( \mathcal{L}\{\cdot\} \) such that the estimate of \( y(t) \), \( \hat{y}(t) = \mathcal{L}\{x(t)\} \). These relations and the complex extension problem are illustrated in Figure 1.

In the context of time-frequency signal analysis problems, we desire the instantaneous parameters, \( \hat{\rho}(t) \) and \( \hat{\Omega}(t) \). Once we have determined a rule for \( \hat{y}(t) \), the instantaneous estimates are given by

\[ \hat{\rho}(t) = \pm |\hat{z}(t)| = \pm \sqrt{x^2(t) + y^2(t)} \]  

(3)

\[ \hat{\Omega}(t) = \frac{d}{dt}\arctan\left(\frac{\hat{y}(t)}{x(t)}\right). \]  

(4)

The very definition of IA and IF hinges on \( \hat{y}(t) \) and hence \( \hat{\Omega} \).

In 1937, Carson and Fry formally defined the IF based on the phase derivative of a complex FM signal [1]-[5]. In Gabor’s seminal 1946 paper [6], he introduced the Quadrature Method (QM) as a practical approach for obtaining the complex extension of a real signal and showed its equivalence to the Hilbert transform (HT). In this context, the rule is given by

\[ \hat{y}(t) = \mathcal{H}\{x(t)\} \]  

(5)

where \( \mathcal{H} \) is the HT operator and

\[ \hat{z}(t) = x(t) + j\mathcal{H}\{x(t)\} \]  

(6)

term is termed the Analytic Signal (AS).

In 1948, Ville defined the IF of a real signal by using Gabor’s complex extension and Carson’s definition of IF [7],[8]. By defining the IF as the derivative of the phase of Gabor’s analytic signal, Ville was able to show the average spectral frequency is equal to the time average of the IF. He then formulated the Wigner-Ville Distribution (WVD) and showed that the first moment of the WVD with respect to frequency yields the IF [8]. Using Gabor’s AS to extend a real signal results in a number of very convenient relations. As a result, the AS is almost universally viewed as the correct complex extension and subsequently the correct way to define IA, IF, and phase for real signals [9].

However shortly afterward, Shekel pointed out the ambiguity problem in Ville’s work defining the instantaneous parameters of a real signal [8],[10],[11]. As an example, consider

\[ x(t) = \mathbb{R}\{a_0(t)e^{j[\omega_0(t)t + \phi_0]}\}. \]  

(7)

There are an infinite set of pairs \( a_0(t) \) and \( \omega_0(t) \) for which a given \( x(t) \) may be equivalently described and hence an infinite set of IA/IF parameterizations. Shekel pointed this...
ambiguity out “with the hope of banishing it [IF] forever from the dictionary of the communication engineer.” Others such as Hupert, suggested that despite the ambiguity, the concept of instantaneous parametrization of real signals was still useful and could be applicable [12], [13].

In order to constrain the ambiguity problem so that a unique complex extension can be justified for a real signal, Vakman proposed three conditions tied to physical reality [13–17]: 1) amplitude continuity, 2) phase independence of scaling and homogeneity, and 3) harmonic correspondence. With these conditions, Vakman showed that the unique complex extension is given by the rule $\hat{y}(t) = H\{x(t)\}$.

More recently, other authors have proposed other conditions to constrain the ambiguity, such as bounded amplitude and bounded IF variation, leading to alternate IA/IF solutions [17], [18]. Finally, we point out that a number of other methods have also been introduced, without explicitly stating the conditions. Vakman has shown that all these methods violate one or more of the conditions that he proposed and almost all retain the harmonic correspondence condition [19], [20]. Like many authors, we believe Vakman’s conditions justifying Gabor’s use of the HT are reasonable and sensible for several special cases of $z(t)$. However, Vakman’s conditions can be too restrictive for more general $z(t)$ and in the worst case, may lead to incorrect results and interpretations.

The contributions of this letter are the following. We present latent signal analysis as an alternate interpretation of the classic complex extension problem where our goal is not a unique complex extension of $x(t)$ but to find the latent signal, $z(t)$. We provide arguments against the restrictive assumption of harmonic correspondence and hence the use of the HT. Without harmonic correspondence, we prove that one can still maintain desirable mathematical properties. Although the HT can be used to recover the quadrature from the observation in special cases where the latent signal is well constrained, e.g. in communications, we point out that in general signal analysis problems, this is inappropriate. By recasting the complex extension problem, we provide alternate latent signal models, that in turn correspond to different IA/IF parameterizations. Although the existence of other IA/IF parameterizations is not new, Vakman argued that the AS is the only physically-justifiable complex extension. We argue that by modifying the differential equation for simple harmonic motion, our parameterizations are also physically justified [21], [22].

The rest of this paper is organized as follows. In Section II, we review the HT and its motivations and the AS. In Section III, we give a proof showing we can still maintain analyticity without harmonic correspondence and also give a proof that no universal rule for the quadrature exists. In Section IV, we provide an example of a latent signal analysis problem and several solutions. Finally, in Section V we conclude the article.

II. THE HILBERT TRANSFORM AND ANALYTIC SIGNAL

Although there exist several methods for estimating the instantaneous parameters, use of the HT still dominates science and engineering. The three main motivations are: 1) Vakman’s physical conditions, 2) analyticity of the resulting complex extension, and 3) ease of computation via Gabor’s QM. In this section, we review the motivations. For an historical reference on the HT, we recommend Titchmarsh [2].

A. Vakman’s Physical Conditions

Vakman proposed conditions in order to constrain the ambiguity in choosing the complex extension [14–16], [20], [23].

**Condition 1 Amplitude Continuity:** Simply stated, amplitude continuity requires that the IA, $\rho(t)$ is a continuous function. This implies that the rule, $\hat{y}(t) = L\{x(t)\}$ must be continuous, i.e.

$$L\{x(t) + \epsilon w(t)\} \rightarrow L\{x(t)\} \text{ for } ||\epsilon w(t)|| \rightarrow 0.$$  \hfill (8)

**Condition 2 Phase Independence of Scaling and Homogeneity:** Let $x(t)$ have a complex extension, $z(t) = \rho(t) exp[j\theta(t)]$. Then for a real constant $c > 0$, $cx(t)$ has associated complex extension $z_1(t) = [c\rho(t)] exp[j\theta(t)]$, i.e. only the IA of the complex representation is affected and $\theta(t)$ and $\Omega(t)$ remain unchanged. This implies that the rule for performing the complex extension is scalable

$$L\{cx(t)\} = cL\{x(t)\}.$$  \hfill (9)

**Condition 3 Harmonic Correspondence:** Let $x(t) = a_0 \cos(\omega_0 t + \phi_0)$, then harmonic correspondence forces the complex extension,

$$\hat{z}(t) = a_0 e^{j(\omega_0 t + \phi_0)},$$  \hfill (10)

where we note the IA and IF must be constant. This implies that

$$L\{a_0 \cos(\omega_0 t + \phi_0)\} = a_0 \sin(\omega_0 t + \phi_0)$$  \hfill (11)

and $\hat{z}(t)$ is a simple harmonic component (SHC).

As Vakman showed [14], the HT is the only operator that satisfies the above conditions and as a result, the HT (or Gabor’s practical QM implementation) is viewed as the correct way to complex extend a signal. With his work, Vakman was able to refute most objections as to whether use of the HT is physically justified.

**Condition 4 Phase Continuity:** In addition to real signals having an ambiguity in instantaneous parameterization, complex signals also have an ambiguity: although we construct $\hat{z}(t)$ using a rule for $y(t)$, we want the IA/IF pair $\rho(t)$, $\Omega(t)$ for signal analysis. However, there can be an ambiguity in the coordinate transformation [24]. There exist at least two choices for resolving this ambiguity:

- **Condition 4a Positive IA:** $\rho(t) \geq 0 \ \forall \ t$.
- **Condition 4b Phase continuity:** the phase $\theta(t)$ is a continuous function.

Although 4a is the traditional choice, we advocate condition 4b to ensure the IF is well defined.

B. Gabor’s Quadrature Method

The HT is an ideal operator that in practice is not physically realizable. Gabor’s QM provides a frequency-domain approach that under certain conditions is equivalent to the HT. The QM steps are:
1) decompose $x(t)$ into SHCs (i.e. compute the Fourier spectrum)
2) double the magnitude of the non-negative frequency components
3) negate the negative frequency components.
The QM results in a complex signal formulated in terms of spectral frequencies,
\[ \hat{\dot{z}}(t) = \sum_{k=0}^{K-1} a_k \exp \{ j [\omega_k t + \phi_k] \} \quad (12) \]
i.e. we have decomposed the signal into SHCs each having constant IA, $a_k$ and constant IF, $\omega_k$. By comparing the expressions for $\hat{\dot{z}}(t)$ in (12) and (1), we can effectively collapse $K$ SHCs into a single AM-FM component and then obtain an IA/IF pair. Implementing the QM using a FT is very convenient and is one reason for the method’s popularity.

C. Analyticity of the Analytic Signal
Unfortunately, the word “analytic” has two distinct meanings when used in signal processing and in addition, another meaning in mathematics. A signal is said to be analytic if it consists only of non-negative frequency components \[9\], \[25\]. The analytic signal refers to the complex extension of a real signal using the HT, i.e. Gabor’s method \[7\], \[9\], \[25\], \[26\].

If $z(t)$, where $t \equiv t + j \tau$, is an analytic function then the real and imaginary parts of the complex function,
\[ z(t) = u(t, \tau) + j v(t, \tau) \quad (13) \]
satisfy the Cauchy-Riemann (CR) conditions
\[ \frac{\partial}{\partial t} u(t, \tau) = \frac{\partial}{\partial \tau} v(t, \tau) \quad \text{and} \quad \frac{\partial}{\partial \tau} u(t, \tau) = - \frac{\partial}{\partial t} v(t, \tau). \quad (14) \]
For the AS $\hat{\dot{z}}(t) = x(t) + j H\{x(t)\}$, such that $t \leftarrow \tau \equiv t + j \tau$, the complex function $\hat{\dot{z}}(t) = \hat{u}(t, \tau) + j \hat{v}(t, \tau)$ is an analytic function \[27\]. Hence the reason for calling a HT-extended signal the analytic signal \[9\].

III. RELAXING THE CONDITION OF HARMONIC CORRESPONDENCE
A. On Harmonic Correspondence
We can understand Vakman’s motivation for tying harmonic correspondence to physical reality by considering the differential equation
\[ \frac{d^2}{dt^2} z(t) + \omega_0^2 z(t) = 0 \quad (15) \]
which describes many ideal systems e.g. an LC circuit or mass/spring models. The solution to (15) is the SHC in (10).

Any deviation from this ideal case, requires modification of the differential equation. For example, when the differential equation describes a circuit and resistance is included or describes motion and damping is included, the equation becomes
\[ \frac{d^2}{dt^2} z(t) + c \frac{dz}{dt} z(t) + \omega_0^2 z(t) = 0 \quad (16) \]
where $c$ is a constant. In this case, the solution includes an Amplitude Modulation (AM) term and has the form
\[ z(t) = a_0 e^{-\eta t} \exp(j[\omega_0 t + \phi_0]) \quad (17) \]
which is not a SHC \[22\]. As another example, when the differential equation is further modified to include time-varying coefficients, non-linearity, or partial derivatives with respect to $\tau$, the resulting solution may include both AM and Frequency Modulation (FM) terms, which is also not a SHC \[7\], \[20\].

While most authors believe harmonic correspondence is a reasonable condition to describe physical phenomena, we advocate that this condition is overly constraining. For many analysis problems, this can lead to incorrect interpretations because real physical systems always deviate from the ideal case. By not assuming harmonic correspondence, we gain a degree of freedom in our analysis that allows us to construct other complex extensions that may be better suited to describing the underlying physical phenomena associated with the signal. We believe that any attempt to find a unique rule to infer $z(t)$ of the form in (1) from $x(t)$ in the form of (1) is fundamentally flawed and that no such universal rule can exist.

B. On Analyticity of the Complex Extension
Choosing $\hat{\dot{y}}(t) = H\{x(t)\}$ to obtain the analytic signal, $\hat{\dot{z}}(t)$ ensures $z(t)$ is an analytic function. However, there are other choices for $\hat{\dot{y}}(t) \neq H\{x(t)\}$ such that $z(t)$ is an analytic function. We believe that the use of the term “analytic” has in most cases become too restrictive in signal processing. To wit, one may falsely believe that the HT is the only way to complex-extend a real signal in order to result in an analytic function. Other complex extensions of real signals can be constructed that result in analytic functions.

**Theorem** If we do not assume harmonic correspondence, there exists at least one choice for the quadrature, $y(t) \neq H\{x(t)\}$ that results in $z(t)$ being an analytic function.

**Proof:** Let $x(t) = a_0 \cos(\omega_0 t)$ and choose the quadrature such that
\[ z(t) = a_0 \cos(\omega_0 t) + j \alpha a_0 \sin(\beta \omega_0 t) \quad (18) \]
with real $\alpha$. The complex function is given by
\[ z(t) = a_0 \cos(\omega_0 [t + j \tau]) + j \alpha a_0 \sin(\beta \omega_0 [t + j \tau]) \quad (19) \]
where
\[ u(t, \tau) = a_0 \cos(\omega_0 t) \cosh(\beta \omega_0 \tau) - \alpha a_0 \cos(\beta \omega_0 t) \sinh(\beta \omega_0 \tau) \quad (20) \]
and
\[ v(t, \tau) = \alpha a_0 \sin(\beta \omega_0 t) \cosh(\beta \omega_0 \tau) - a_0 \sin(\omega_0 t) \sinh(\omega_0 \tau). \quad (21) \]
It can easily be shown that this choice leads to $z(t)$ satisfying the CR conditions and hence is an analytic function. Any choice of $\alpha \neq 1$ or $\beta \neq 1$ does not imply harmonic correspondence.

**Corollary** No unique rule for the quadrature, $\hat{\dot{y}}(t) = \mathcal{L}\{x(t)\}$ exists to obtain the latent signal, $z(t)$ from the observation, $x(t) = \Re\{z(t)\}$ for all $z(t)$.

**Proof:** Consider the latent signal, $z(t)$ of the form in (13). Assume a unique rule, $\hat{\dot{y}}(t) = \mathcal{L}\{x(t)\} = \alpha_0 a_0 \sin(\beta \omega_0 t)$ exists to obtain $z(t)$. This implies a unique $\hat{\dot{z}}(t) = a_0 \cos(\omega_0 t) + j \alpha_0 a_0 \sin(\beta \omega_0 t)$. If $\alpha_0 \neq \alpha$ or $\beta_0 \neq \beta$ then $\hat{\dot{z}}(t) \neq \hat{\dot{z}}(t)$ and $\mathcal{L}\{\cdot\}$ is not unique.
C. On Harmonic Conjugate Functions

If \( z(t) = u(t, \tau) + jv(t, \tau) \) is an analytic function, then \( u(t, \tau) \) and \( v(t, \tau) \) are unique and are harmonic conjugates [27]. Even though \( x(t) = u(t, 0) \) and \( y(t) = v(t, 0) \) in (1) this does not imply that \( u(t, \tau) = x(t) \) and \( v(t, \tau) = y(t) \), but rather \( u(t, \tau) \) and \( v(t, \tau) \) each contain terms from both \( x(t) \) and \( y(t) \):

\[
\begin{align*}
\hat{z}(t) &= x(t) + jy(t) \\
&= [x_R(t, \tau) + j x_I(t, \tau)] + j [y_R(t, \tau) + j y_I(t, \tau)] \\
&= [x_R(t, \tau) - y_I(t, \tau)] + j [x_I(t, \tau) + y_R(t, \tau)] \\
&= u(t, \tau) + jv(t, \tau) \\
\end{align*}
\]  

(22)

where \( R, I \) denote the real and imaginary parts, respectively. This is easily seen in (20), where \( \alpha \) and \( \beta \) are present in \( u(t, \tau) \) despite \( \alpha \) and \( \beta \) appearing only in \( y(t) \) in (18). Thus the problem of finding \( y(t) \) cannot be solved by finding a harmonic conjugate of \( u(t, \tau) \) because \( y(t) \) must be known to obtain \( u(t, \tau) \).

D. On AM–FM Demodulation

The HT is widely used as a demodulation algorithm for AM–FM signals. This is typically justified as a valid approach because of Vakman and also Bedrosian’s theorem [1]. The HT can be used to determine the IA/IF for a small subset of latent signals, i.e. those with harmonic correspondence. The HT can also be used to closely approximate the IA/IF for latent signals whenever \( H\{x(t)\} \approx y(t) \) [11]. However, the HT cannot be used in general to obtain the IA/IF of all latent signals. This is illustrated in Figure [1] and will be demonstrated in the next section.

IV. Example

Suppose we have three ideal systems: a Frequency Modulator, an Amplitude Modulator, and a LTI system in the steady state. We observe a triangle waveform \( x(t) \) that could have come from any of the three ideal systems. What is the corresponding latent signal \( z(t) \) or more strictly the instantaneous parameters, \( \rho(t) \) and \( \Omega(t) \)?

Let the observation be the periodic, even-symmetric triangle waveform where one period is given by

\[
x(t) = \begin{cases} 
-2A\omega_0 t/\pi + A, & 0 \leq t \leq T/2 \\
2A\omega_0 t/\pi + A, & -T/2 \leq t \leq 0 
\end{cases} 
\]

with amplitude \( A \), period \( T \), and fundamental frequency \( \omega_0 \).

A. Solution Assuming Harmonic Conjugate

If we assume harmonic correspondence then

\[
\begin{align*}
\hat{z}_1(t) &= x(t) + j\mathcal{H}\{x(t)\} \\
&= \sum_{k=0}^{\infty} \frac{8A}{\pi^2(2k+1)^2} \cos[(2k+1)\omega_0 t] \\
&\quad + j \sum_{k=0}^{\infty} \frac{8A}{\pi^2(2k+1)^2} \sin[(2k+1)\omega_0 t]
\end{align*}
\]  

(24a)

(24b)

where the IA is

\[
\dot{\rho}(t) = \frac{8A}{\pi^2} a_0(t)
\]

(25)

and the IF is

\[
\dot{\Omega}(t) = \omega_0 + \frac{d}{dt} M_0(t)
\]

(26)

where \( a_0(t) \) and \( M_0(t) \) are related through

\[
a_0(t)e^{jM_0(t)} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} e^{2k\omega_0 t}.
\]

(27)

B. Solution Assuming Constant IA

If we assume constant IA \( \dot{\rho}(t) = \rho_0 \), then

\[
\hat{z}_2(t) = \dot{\rho}(t) \cos(\omega_0 t) + j \dot{\rho}(t) \sin(\omega_0 t)
\]

with the IA given by

\[
\dot{\rho}(t) = x(t)/\cos(\omega_0 t).
\]

(28)

(29)

In this solution, \( \hat{y}(t) = \dot{\rho}(t) \sin(\omega_0 t) \neq \mathcal{H}\{\dot{\rho}(t) \cos(\omega_0 t)\} \), because Bedrosian’s theorem cannot be applied.

C. Solution Assuming Constant IA

If we assume constant IA \( \dot{\rho}(t) = A \), then

\[
\hat{z}_3(t) = A \cos(\omega_0 t + M_1(t)) + jA \sin(\omega_0 t + M_1(t))
\]

(30)

with IF given by

\[
\dot{\Omega}(t) = \omega_0 + \frac{d}{dt} M_1(t)
\]

(31)

where

\[
M_1(t) = \arccos[x(t)/A] - \omega_0 t.
\]

(32)

As in the solution for constant IF above,

\[
\hat{y}(t) = A \sin[\omega_0 t + M_1(t)] \neq \mathcal{H}\{A \cos[\omega_0 t + M_1(t)]\}.
\]

V. Conclusions

We have presented the latent signal analysis problem and reviewed the motivation for the use of the HT and AS in signal analysis. We proved the harmonic correspondence condition is not necessary for analyticity and that no unique rule for the complex extension exists to obtain the latent signal. It was argued that by relaxing the harmonic correspondence condition, which forces the use of the HT, we gain alternate choices for the latent signal. Although the HT is widely used for demodulation of AM–FM signals, it cannot be used to obtain the IA/IF of all latent signals. In a strict sense, the HT can only be used when the latent signal is a superposition of SHCs and can approximate the latent signal whenever \( H\{x(t)\} \approx y(t) \), e.g. communication signals where Bedrosian’s theorem is approximately satisfied. Finally, we provided an example illustrating various solutions to a latent signal analysis problem.

ACKNOWLEDGMENTS

The authors wish to thank Prof. Joe Lakey of the Dept. of Mathematical Sciences at New Mexico State University and Prof. Antonia Papandreou-Suppappola of the School of Electrical, Computer and Energy Engineering at Arizona State University for reviewing an early draft of this manuscript.
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