ANALYTICAL MODELS FOR VALENCE FERMIONS IN ISOTROPIC TRAPS

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For isotropic confining Ioffe-Pritchard or TOP potentials, a valence fermion trapped with a closed core of other fermions can be described by an analytical effective one-particle model with a physical eigenspectrum. Related constructions exist for Paul and Penning traps. The analytical models arise from quantum-mechanical supersymmetry.

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Electromagnetic traps may be divided into two categories: those that trap neutral particles or atoms, and those that trap charged particles or ions. The trapping mechanism for the former usually exploits the force experienced by a magnetic dipole in a nonuniform magnetic field. Typical configurations consist of a magnetic quadrupole field supplemented with a mechanism to reduce trapping losses near the field zero. Two well-established examples are the Ioffe-Pritchard trap \( [1, 2] \), which has end coils ensuring a nonzero-field minimum, and the time-averaged orbiting-potential (TOP) trap \( [3] \), which employs a rotating magnetic field. Discussions of various neutral-particle magnetostatic traps are given in Ref. \([4]\). Charged particles and ions can be trapped with the Paul trap \( [5] \), which involves an oscillating electric potential and can simultaneously trap particles of both polarities. An alternative is the Penning trap \([6, 7]\), which combines electrostatic and magnetic fields and exists in many forms \([8, 9, 10]\). A general discussion of ion traps is given in Ref. \([11]\).

The quantum behavior of many particles confined in a trap has been the subject of much recent experimental and theoretical work \([12, 13]\). Most investigations have centered on systems involving large numbers of bosons. In the present paper, we address instead some quantum-physics issues for certain traps containing one to a few hundred fermions, with the system in the ground state. We seek to describe the physics of an additional valence fermion added to such a system, perhaps in a highly excited state, using a relatively simple analytical model. Note that the focus of the present work is primarily theoretical issues involving quantum physics. In particular, we disregard experimentally important issues such as thermal effects \([14, 13]\).

Part of our interest in situations of this type stems from the analogy with Rydberg atoms, which have played a central role in the development of tools for understanding multi-electron atoms \([15]\). Certain related developments could emerge in the context of multiparticle traps of the type we consider. Rydberg states of alkali-metal atoms are widely used for experimental and theoretical investigations, in part because the associated electronic core is relatively simple, forming a closed shell. For related reasons, in the present work we primarily consider systems of trapped fermions with the valence fermion lying outside a closed fermionic shell. The analogy with alkali-metal atoms is best for traps with a confining potential that is purely isotropic in
three dimensions. For certain values of the applied fields, the Ioffe-Pritchard and
TOP traps provide examples that closely approximate this situation.

Both the Ioffe-Pritchard and TOP traps capture magnetic dipoles $\vec{\mu}$ aligned op-
positely to an applied magnetic field $\vec{B}$ by drawing them into a region of minimum $|\vec{B}|$. The potential energy is $U(x, y, z) = \mu |\vec{B}|$. Dipoles aligned with the field are
expelled from the trap.

The Ioffe-Pritchard trap consists of two coils with aligned symmetry axes in, say,
the $\hat{e}_z$ direction and four conducting bars equidistant from and parallel to the $z$ axis.
The coils carry parallel currents, while the bars carry alternately oriented currents.
In cylindrical coordinates, the magnetic field to third order in $\rho$ and $z$ is

$$B_{IP}(\rho, \phi, z) = B_c \hat{e}_z + B'_t \rho [\cos 2\phi \hat{e}_\rho - \sin 2\phi \hat{e}_\phi] + \frac{B_c a_c}{l_c^4} \left[(z^2 - \rho^2/2)\hat{e}_z - z\rho \hat{e}_\rho \right].$$

In this expression, the coils are taken to have radius $R_c$, to carry currents $I_c \hat{e}_\phi$,
and to be positioned at $z = \pm A_c$. The bars are assumed to be located at $\rho = S_l$
with $\phi = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, and to carry currents of magnitude $I_l$. The other
quantities are a characteristic length, area, field, and field gradient: $l_c = (A_c^2 + R_c^2)^{1/2}$,
$a_c = 3(4A_c^2 - R_c^2)/2, B_c = \mu_0I_cR_c^2/l_c^3, B'_t = 2\mu_0I_l/\pi S_l^2$. Trapped magnetic dipoles have
potential energy

$$U_{IP}(\rho, \phi, z) = \mu B_c \left[1 + \frac{a_c}{l_c^4} z^2 + \frac{1}{2} \left(\frac{B'_t^2}{B_c^2} - \frac{a_c}{l_c^4} \right) \rho^2 \right].$$

If the condition $B'_t^2/B_c^2 = 3a_c/l_c^4$ is satisfied, as is readily feasible in practical situations, then the potential Eq. [2] is isotropic:

$$U_{IP}(r) = \mu B_c \left(1 + \frac{r^2}{r_c^2} \right),$$

where $r_c^2 = l_c^4/a_c$. Note that the orientation of the dipoles is anisotropic, for example,
lying roughly along the $-\hat{e}_z$ direction near the $z$ axis.

The TOP trap involves two parallel coils as above, but carrying oppositely oriented
currents, and two further pairs of coils providing an additional rotating bias field.
Let the quadrupole coils have radius $R_q$, be located at $z = \pm A_q$, and carry currents $\mp I_q \hat{e}_\phi$, respectively. Let the pairs of bias coils of radius $R_b$ be located at $x = \pm A_b$
on the $x$-axis and at $y = \pm A_b$ on the $y$-axis, and let the corresponding currents be $I_x(t) = I_b \cos \omega_b t$ and $I_y(t) = I_b \sin \omega_b t$. The time dependence creates a magnetic field vector that to lowest order lies parallel to the $z = 0$ plane and rotates with frequency $\omega_b$. To second order in $x$, $y$, and $z$, the magnetic field is

$$ B_{\text{TOP}}(x, y, z, t) = \left( B_b \cos \omega_b t + B'_q x + \frac{a_b B_b}{2l_b^4} \left[ (3x^2 - r^2) \cos \omega_b t - 2xy \sin \omega_b t \right] \right) \hat{e}_x + \left( B_b \sin \omega_b t + B'_q y + \frac{a_b B_b}{2l_b^4} \left[ (3y^2 - r^2) \sin \omega_b t - 2xy \cos \omega_b t \right] \right) \hat{e}_y - \left( 2B'_q z + \frac{B_b a_b}{l_b^4} [x \cos \omega_b t + y \sin \omega_b t] z \right) \hat{e}_z , \quad (4) $$

where the characteristic length, area, field, and field gradient are

$$ l_b = \left( A_b^2 + R_b^2 \right)^{1/2}, \quad a_b = \frac{3(4A_b^2 - R_b^2)}{2}, \quad B_b = \mu_0 I_b R_b^2 / l_b^3, \quad B'_q = 3\mu_0 I_q R_q^2 A_q / 2(A_q^2 + R_q^2)^{5/2}. $$

The potential energy is obtained by expanding $\mu |B_{\text{TOP}}|$ for small $\rho$, $z$ and time averaging, assuming that the frequency is high enough so the dipoles do not move appreciably over one cycle. The time averaging removes linear and cubic terms, producing the trapping potential

$$ U_{\text{TOP}}(\rho, z) = \mu B_b \left[ 1 + \frac{1}{4} \left( \frac{B'_q^2}{B_b^2} + \frac{a_b}{l_b^4} \right) \rho^2 + \frac{2B'_q^2}{B_b^2} - \frac{a_b}{2l_b^4} \right] z^2 \quad (5) $$

to third order. If the physical parameters satisfy $B'_q^2 / B_b^2 = 3a_b / 7l_b^4$, a readily attainable condition, then the potential becomes isotropic:

$$ U_{\text{TOP}}(r) = \mu B_b \left( 1 + \frac{r^2}{r_b^2} \right) , \quad (6) $$

where $r_b^2 = 14l_b^4 / 5a_b$. Note that the dipole orientation is anisotropic, being opposite to the time-averaged field Eq. (4) and pointing away from the origin along the $z$ axis and radially towards the origin in the $z = 0$ plane.

The Schrödinger equation for a particle in either isotropic trap is

$$ -\frac{\hbar^2}{2m} \nabla^2 + \mu B_0 + \mu B_0 \left( \frac{r^2}{r_s^2} \right) \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) , \quad (7) $$

where $m$ is the dipole mass, $B_0 = B_b$ or $B_0$, and $r_s = r_c$ or $r_b$. Separating in spherical polar coordinates, we write $\psi(r, \theta, \phi) = (r_0 / r) W(r) Y(\theta, \phi)$, where $Y(\theta, \phi)$
are spherical harmonics, \( r_0 = (\hbar/m\omega_0)^{1/2} \), and \( \omega_0 = (2\mu B_0/mr_s^2)^{1/2} \). This choice puts the radial equation for \( W(r) \) into a convenient form:

\[
\left(-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{L(L + 1)}{r^2} + \mu B_0 + \frac{1}{2} m\omega_0^2 r^2 \right) W(r) = E_N W(r) ,
\]

where the angular momentum quantum number \( L \) takes values \( L = 0, 1, 2, \ldots \) and the energy eigenvalues are \( E_N = \mu B_0 + \hbar \omega_0(N + 3/2) \), with principal quantum number \( N = L, L+2, L+3, \ldots \). The wave functions are given in terms of generalized Laguerre polynomials:

\[
W_{N,L}(r) = C_{N,L}(r/r_0)^{L+1} \exp(-r^2/2r_0^2)L^{(L+1/2)}_{N/2-L/2} \left( r^2/r_0^2 \right) ,
\]

where the quantities \( C_{N,L} \) are normalization constants.

In what follows, it is useful to consider situations where a cloud of fermions forms a closed shell of energy levels in an isotropic trap. The number of fermions forming a closed shell below a particular value of \( N \) is determined by the degeneracies of states in the single-particle system. Examining the full wave functions and recalling that the allowed values of \( N \) increase in steps of two units, the degeneracy for a given \( N \) is found to be \((N+1)(N+2)/2\). Note that the doubling in atomic systems due to the two spin orientations has no analogue here because only dipoles oriented against the magnetic field are trapped. The number of particles completely filling levels less than or equal to \( N \) is \((N+1)(N+2)(N+3)/6\).

In the present work, we seek to construct relatively simple analytical one-particle models for the valence fermion in cases with more than one trapped particle. In addition to their intrinsic interest, such models could be used to make analytical predictions of physical properties or could provide a favorable starting point for perturbative and other calculations. The methodology applied here to generate effective models for various trap systems is related to that adopted in the development of a relatively simple analytical model for the valence electron in Rydberg atoms [16]. This model has been used in a variety of contexts [17, 18], including recently the prediction of certain experimentally observable features of long-term revivals in Rydberg atoms.

Consider a system of fermions caught in a trap, with one particle excited relative to the others. This valence fermion can be regarded as moving in an effective potential
created by a combination of the trapping fields and interactions with the other trapped particles. We include among these interactions the quantum effects from the Pauli principle, which prevents the valence fermion from occupying filled levels, and also interparticle forces that act to modulate the trapping fields. In constructing analytical models for the valence fermion, we address first the issue of incorporating effects from the Pauli principle and subsequently examine an analytical extension that could describe other interparticle forces.

To illustrate the idea, consider the particular tower of states $|N, L = 0, M = 0\rangle$ accessible to a trapped valence fermion with angular quantum number $L = 0$ and azimuthal quantum number $M = 0$. If only one particle is trapped, then $N \geq 0$ with $N$ even. If there is also a core of four fermions filling the energy levels below $N = 2$, then a valence fermion with $L = 0$ is restricted by the Pauli principle to levels with even $N \geq 2$. Neglecting for the moment other interactions among the trapped particles, the valence fermion in each of the two situations can access states with identical eigenenergies, except that in the five-particle case the $N = 0$ level is inaccessible. The issue of constructing an analytical effective one-particle theory to describe the $L = 0$ states of the valence fermion in the five-particle case can therefore be rephrased as the problem of finding an analytically solvable effective trapping potential with $L = 0$ energy eigenstates identical to those of the one-particle case but with the lowest one-particle level missing.

Isospectral problems of this type can be treated in several ways. One approach might be the inverse method [19]. In the present work, we choose instead an alternative method with a definite physical interpretation that is both elegant and relatively simple, based on supersymmetric quantum mechanics with the superalgebra sqm(2) [16, 17]. This supersymmetric technique takes as input a Schrödinger hamiltonian $H^+$ for which the ground-state eigenenergy vanishes, and it determines via supersymmetry a complementary hamiltonian $H^-$. By construction, the eigenstates of $H^-$ are degenerate with those of the original potential except for the ground state, which is absent. The eigenstates of the two systems $H^+$ and $H^-$ are related by an explicit
map. The two associated Schrödinger equations can be written as

$$H^\pm \psi^\pm = \hbar^2 / 2m \left( -\frac{d^2}{dr^2} + \left( \frac{dU}{dr} \right)^2 \pm \frac{d^2 U}{dr^2} \right) \psi^\pm = E^\pm \psi^\pm ,$$  \hspace{1cm} (10)$$

where $\psi^\pm$ are the eigenstates of $H^\pm$ with eigenvalues $E^\pm$. The combinations of derivatives of $U(r)$ generate associated potentials $V^\pm(r)$. Since by assumption $H^+$ and hence $V^+(r)$ are given, the function $U(r)$ can be found by solving a differential equation. The form of $V^-(r)$ and $H^-$ can then be deduced. The existence of the degeneracy and explicit map between excited states in the two eigenspectra is a direct consequence of the supersymmetry.

This procedure can be applied to Eq. (8). Subtracting the ground-state energy $E_L = \mu B_0 + \hbar \omega_0 (L + 3/2)$ makes the lowest eigenvalue of the radial Hamiltonian vanish, as required by the method. This subtraction depends on the angular momentum $L$, which therefore must be fixed. The input potential becomes

$$V^+_L(r) = \hbar^2 / 2m \frac{L(L + 1)}{r^2} + \frac{1}{2} \frac{m \omega_0^2 r^2}{2} - \frac{\hbar \omega_0}{2} \frac{2L + 3}{2} ,$$

with eigenenergies $E^+_N,L = \hbar \omega_0 (N - L)$ and eigenfunctions $W^+_N,L(r) = W_{N,L}(r)$. The solution for the function $U(r)$ is

$$U(r) = \frac{1}{2} \left( \frac{r}{r_0} \right)^2 - (L + 1) \ln \left( \frac{r}{r_0} \right) ,$$

from which we obtain the form of $H^-$:

$$\left( -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{(L + 1)(L + 2)}{r^2} + \frac{1}{2} \frac{m \omega_0^2 r^2}{2} - \frac{\hbar \omega_0}{2} \frac{2L + 1}{2} \right) W^-_{N_s,L} = E^-_{N_s,L} W^-_{N_s,L} .$$

The energy eigenvalues are $E^-_{N_s,L} = \hbar \omega_0 (N_s - L)$, where $N_s = L + 2, L + 4, \ldots$, and the eigenfunctions are $W^-_{N_s,L}(r) = W_{N_s-2,L+2}(r)$.

The derivation shows that for the $L = 0$ case discussed above the eigenfunctions $(r_0/r)W^-_{N_s,L=0}(r)Y_{L=0,M=0}(\theta, \phi) \equiv (r_0/r)W_{N_s-2,L=2}(r)Y_{L=0,M=0}(\theta, \phi)$ are effective one-particle eigenfunctions for the valence particle in a trap containing a total of five fermions. We emphasize that these are not conventional oscillator eigenfunctions, as can be seen from the index structure.
It might be tempting instead to model the valence fermion directly using conventional oscillator wave functions, based on a shell-model approach where the Pauli principle is incorporated by hand. However, the supersymmetric eigenfunctions have several advantages. Unlike the conventional oscillator case, for which the lowest state must be excluded by hand, the supersymmetric states form a complete set. Moreover, the lowest-state radial eigenfunction $W_{N_s=2,L=0}^{-}(r)$ in our effective model has degree zero and hence zero nodes, as expected for the ground state of the five-fermion system. A conventional oscillator wave function would have one node instead.

The potentials $V_0^-$ and $V_0^+$ differ by an inverse-square repulsive term:

$$V_0^-(r) - V_0^+(r) = \frac{\hbar^2}{m r^2} + \hbar \omega_0.$$  \hspace{1cm} (14)

In the present context, the additional repulsion in $V_0^-$ plays the role of the Pauli principle by preventing the valence fermion from occupying the filled lower levels. Note that a change in the angular-momentum barrier would produce a similar effect on the potential but would not connect states with the same value of $L$.

The above arguments for the case $L = 0$ can also be applied to other values of $L$. For example, in the case where the valence fermion has $L = 1$ the method produces a relation between two effective one-particle models, the first involving two trapped fermions with one in the ground state and the second involving eleven trapped fermions with ten filling the levels below $N = 3$.

Iterations of the method produce further relations. When the valence fermion has $L = 0$, for instance, the effective one-particle model for the system with five trapped fermions described by Eq. (13) can in turn be related to another model for a system with 21 trapped fermions, 20 of which fill the levels below $N = 4$. Implementing this mathematically requires another shift of the energy zero so that the ground state of the five-particle system has zero energy. Repeating the procedure produces a series of interrelations between effective one-particle models for the $L = 0$ towers of states of systems with $n = 1, 5, 21, 57, \ldots$ trapped fermions. A similar iteration for $L = 1$ generates connections between systems with $n = 2, 11, 36, 85, \ldots$ trapped fermions. The two sequences of numbers are generated by the formula $n = 1 + (N+1)(N+2)(N+3)/6$, with $N = -1, 1, 3, \ldots$ or $N = 0, 2, 4, \ldots$.
The preceding discussion has largely disregarded effects of interactions between the trapped particles. If particle interactions are entirely neglected, the degeneracy of each fixed-$N$ level means that the same effective one-particle model applies to the valence fermion in traps containing $n_d = d + (N + 1)(N + 2)(N + 3)/6$ particles, where $d = 1, 2, \ldots, (N + 1)(N + 2)/2$ is the number of fermions lying outside a closed shell. Although there may be special situations where particle interactions are relatively small, the possibility of additional interactions among the $d$ valence fermions suggests that the best effective models would typically have $d = 1$. Similarly, the best effective models should also be ones describing small numbers of trapped particles and a relatively highly excited valence fermion. These considerations favor, for example, the model for the five-particle case with one highly excited fermion.

Issues involving particle interactions could be addressed using a variety of standard methods, such as mean-field theory, perturbation theory, and Monte-Carlo methods. Given our present focus, we pursue here instead the possibility of incorporating interactions via analytical modifications to the effective models.

Interactions typically shift the energy eigenvalues of the valence fermion. We treat this as a shift $\Delta = \Delta(N, L)$ in the principal quantum number. Since the effective model produces apparent integer shifts in $L$, we define for convenience an integer $I = I(L)$ and introduce the effective principal quantum number $N^* = N + I - \Delta$. It is also convenient to define $N_s = N + 2I$, a quantum number analogous to the spectroscopic principal quantum number in atomic systems. We therefore have $N^* = N_s - I - \Delta$, with new energy eigenvalues $E_{N^*} = \mu B_0 + \hbar \omega_0 \left( N^* + 3/2 \right)$.

The problem is to find a modification of the radial potential in the above effective one-particle models such that the eigensolutions of the corresponding radial equation remain analytical but are associated with the modified eigenenergies $E_{N^*}$. This minor miracle can be accomplished by adding the effective potential

$$V_{\text{EFF}}(r) = \frac{\hbar^2}{2m} \frac{L^*(L^* + 1) - L(L + 1)}{r^2} + \hbar \omega_0 (N - N^*)$$

(15)

to the operator on the left-hand side of Eq. (8), where $L^* = L + I - \Delta$. The resulting
differential equation,
\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2}{2m} \frac{L^* (L^* + 1)}{r^2} + \mu B_0 + \frac{1}{2} m \omega_0^2 r^2 \right) W(r) = E_{N^*} W(r)
\]
has eigenvalues \( E_{N^*} \) with \( N^* = L^*, L^* + 2, L^* + 4, \ldots \). In terms of the functional form given in Eq. (9), the eigensolutions are \( W_{N^*, L^*}(r) \). Note that the special case \( \Delta = 0 \) reproduces all the complementary Hamiltonians \( H^\pm \) discussed above when \( \mu B_0 + \hbar \omega_0 \left( L^* + 3/2 - 2S \right) \) is subtracted from both sides of Eq. (16). Thus, the radial equation for \( H^+ \) is recovered by selecting \( S = 0 \) and \( I = 0 \), while that for \( H^- \) is recovered by selecting \( S = 1 \) and \( I = 1 \). The \( I_0 \)th iteration of \( H^+ \) is obtained with \( S = 0 \) and \( I = I_0 \), while the corresponding \( H^- \) has \( S = 1 \) and \( I = I_0 + 1 \).

The analysis given here has some similarities to the derivation of a relatively simple analytical model for the valence electron in Rydberg atoms discussed in Refs. [16, 17, 18]. For example, our treatment of the \( L = 0 \) states of isotropic trap systems with 1, 5, 21, 57, \ldots fermions resembles that of the s-orbital states of alkali-metal atoms in the Rydberg case. Note that, since in the present trap context the valence fermion is typically neutral, its interactions involve a dipole moment rather than a monopole charge. This suggests that the effect of interactions may be significantly smaller than in the Rydberg case and the effective models correspondingly better. Note also that the form of the effective potential (15) is reminiscent of the special analytical model investigated in Ref. [20].

In the atomic case, orthogonality of the model eigenfunctions arises because the quantum defects in, for example, alkali-metal atoms are asymptotically independent of the principal quantum number. Similarly, in the present context the eigenfunctions \( W_{N^*, L^*}(r) \) form an orthonormalizable basis provided \( \Delta \) is independent of \( N \) with \( \Delta < L + I + 3/2 \). The \( N \)-independence implies that for each fixed \( L \) only one parameter is needed, corresponding to a simultaneous shift of the principal quantum numbers for that tower of states. Experience gained in the atomic case suggests that rapid asymptotic \( N \)-independence is likely to suffice for applicability of the models here. However, it is unclear \textit{a priori} which trap systems have this feature, as it depends on details of the many-body dynamics. Experimental investigations establishing energy spectra for the systems considered here would be of interest. The eigenfunctions
$W_{N^*,L^*}(r)$ could then be used to predict other quantities such as transition rates.

In contrast to the Ioffe-Pritchard or TOP traps, which can be isotropic in three dimensions, the Paul and Penning traps are generically isotropic in two dimensions. Effective one-particle models can nonetheless be obtained via a similar approach. For brevity, we restrict ourselves to outlining the treatment of the case where interactions of the valence fermion are neglected. Effective one-particle models analogous to Eq. (16) that allow for level shifts can also be constructed for the Paul and Penning traps.

The Paul trap consists of a time-dependent potential given in cylindrical coordinates by 
\[ \phi(\rho, \phi, z, t) = \phi(\rho, \phi, z) \cos \tilde{\Omega}_p t, \]
where $\phi$ is
\[ \phi(\rho, \phi, z) = \frac{V_p}{2d_p^2}(z^2 - \rho^2/2) \]  
with characteristic voltage $V_p$ and length $d_p$. A quantum solution for the one-particle case exists [21] and can be investigated experimentally [22], but for simplicity we consider here an alternative approach for large $\tilde{\Omega}_p \gg \Omega_p \equiv (\sqrt{2} |qV_p|/md_p^2)^{1/2}$ in which $q\phi$ is approximated by a time-independent effective potential [23]
\[ \nabla \equiv \frac{q^2}{4m\Omega_p^2} \nabla \phi \cdot \nabla \phi = \frac{1}{2}m\omega_p^2(\rho^2 + 4z^2), \]  
where $m$ and $q$ are the mass and charge of the trapped fermion and $\omega_p = \Omega_p^2/4\tilde{\Omega}_p^2$. The associated quantum problem separates with a wave function of the form $\Psi(\rho, \phi, z) = (\rho_p/\rho)^{1/2}X(\rho)Y(\phi, z)$, where $\rho_p = (\hbar/m\omega_p)^{1/2}$. Introducing the quantum numbers $K = 0, 1, 2, \ldots$ and $M = 0, \pm1, \pm2, \ldots$ gives
\[ Y_{M,K}(\phi, z) = A_K \exp(iM\phi) \exp(-z^2/\rho_p^2)H_K(\sqrt{2} z/\rho_p), \]  
where the $H_K$ are Hermite polynomials and the $A_K$ are normalization coefficients. The two-dimensional radial equation is
\[ \left( -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \frac{\hbar^2}{2m} \frac{M^2 - 1}{\rho^2} + \frac{1}{2}m\omega_p^2\rho^2 + \hbar\omega_p(2K + 1) \right) X(\rho) = EX(\rho), \]  
\[ ^1 \text{In a particular rotating frame, a single particle in a Penning trap with a special ratio of applied fields can experience an isotropic potential in three dimensions. The incorporation of rotation effects, including those on any fermion core, could then allow a treatment similar to that for the Ioffe-Pritchard and TOP traps.} \]
The associated energy eigenvalues are \( E_{N,K} = \hbar \omega_p (N + 2K + 2) \), where \( N = |M|, |M| + 2, |M| + 4, \ldots \), and the eigenfunctions are

\[
X_{N,|M|}(\rho) = C_{N,|M|}(\rho/\rho_p)^{|M|+1/2} \exp(-\rho^2/2\rho_p^2) L^{|M|}_{N/2-|M|/2}(\rho^2/\rho_p^2),
\]

where the \( C_{N,|M|} \) are normalization coefficients.

The Penning trap involves an electrostatic field of the form \( |17| \), along with a uniform magnetic field \( B = B_p \hat{e}_z \). Defining the axial frequency \( \omega_z = (|qV_p|/me_p^2)^{1/2} \), the cyclotron frequency \( \omega_c = |qB_p|/m \), and \( \Omega = (\omega_c^2 - 2\omega_z^2)^{1/2} \), the one-particle hamiltonian for \( q > 0 \) is

\[
H = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{8}m\Omega^2 \rho^2 + \frac{1}{2}m\omega_z^2 z^2 + \frac{1}{2}\hbar\omega_c i\partial_\phi.
\]

The equation separates via \( \Psi(\rho, \phi, z) = (\rho_0/\rho)^{1/2} X(\rho) \Theta(\theta, z) \), where \( \rho_0 = (\hbar/m\omega_c)^{1/2} \). The \( \rho \) equation is

\[
\left( -\frac{\hbar^2}{2m} \frac{d^2}{d\rho^2} + \frac{\hbar^2}{2m} M^2 - \frac{1}{4} \rho^2 + \frac{1}{8}m\Omega^2 \rho^2 + (K + \frac{1}{2})\hbar\omega_c - \frac{1}{2}M\hbar\omega_c \right) X(\rho) = EX(\rho),
\]

where \( M = 0, \pm 1, \pm 2, \ldots \) and \( K = 0, 1, 2, \ldots \). The energy eigenvalues are \( E_{N,K,M} = \hbar(\Omega N + 2\omega_z K - \omega_c M + \Omega + \omega_z)/2 \), where \( N = |M|, |M| + 2, |M| + 4, \ldots \). The full eigensolutions involve generalized Laguerre and Hermite polynomials:

\[
\Psi_{N,K,M}(\rho, \phi, z) = C_{N,K,|M|}(\rho/\rho_0)^{|M|} \exp \left[ -\frac{k}{4} \left( \frac{\rho}{\rho_0} \right)^2 - \frac{1}{2} \left( \frac{z}{z_0} \right)^2 + iM\phi \right] 
\times L^{|M|}_{N/2-|M|/2} \left( k\rho^2/2\rho_0^2 \right) H_K(z/z_0),
\]

where \( k = \Omega/\omega_c \), \( z_0 = (\hbar/m\omega_z)^{1/2} \), and \( C_{N,K,|M|} \) are normalization coefficients.

Equations (20) and (23) are both radial equations for a two-dimensional oscillator, and the same approach to effective one-particle models applies to each. For brevity, we treat primarily the Paul case in what follows. Subtracting the ground-state energy, the equation analogous to (11) is

\[
V_{|M|}^+ = \frac{\hbar^2}{2m} \frac{|M|^2 - 1/4}{\rho^2} + \frac{1}{2}m\omega_c^2 \rho^2 - \hbar\omega_p(|M| + 1).
\]

The partner potential is \( V_{|M|}^- = V_{|M|+1}^+ + 2\hbar\omega_p \). The difference \( V_{|M|}^- - V_{|M|}^+ \) again acts as an additional repulsion that can be regarded as preventing the valence fermion
from accessing a filled lower level. The corresponding partial eigenfunctions are 
\[ X_N^+ |M\rangle = X_{N+1,|M|}(\rho) \]
and
\[ X_{N-1,|M|}(\rho) = X_{N-|M|+1}(\rho), \]
where \( N_s = |M| + 2, |M| + 4, \ldots \). The eigenspectra
\[ E_N^+ |M\rangle = \hbar \omega_p (N - |M|) \]
and
\[ E_{Ns,|M|}^- = \hbar \omega_p (N_s - |M|) \]
degenerate, except for the ground state. Similar expressions arise for the Penning trap.

For both types of trap, the potential \( V_{|M|}^+ \) depends on \( M \) and the subtracted ground-state energy depends on \( K \) (see Eqs. (20) and (23)). The effective one-particle models therefore describe towers of states \( |N, M, K\rangle \) with fixed \( K \) and \( M \). For the Paul trap, the number of spin-1/2 states with energy less than or equal to \( E_{N,K} \) is

\[
n(\tilde{E}) = \begin{cases} 
\tilde{E}(\tilde{E} + 2)(2\tilde{E} - 1)/12, & \tilde{E} \text{ even}, \\
(\tilde{E}^2 - 1)(2\tilde{E} + 3)/12, & \tilde{E} \text{ odd},
\end{cases}
\]

(26)

where \( \tilde{E} = E_{N,K}/\hbar \omega_p \). The Paul-trap systems related by effective one-particle models are therefore those with 1, 3, 7, 15, 27, 45, 69, \ldots trapped fermions.

Similar considerations apply for the Penning trap. Note, however, that in this case the number of spin-1/2 states with energy less than or equal to \( E_{N,M,K} \) depends on the frequency tuning. The magnetron motion is also unstable, so the corresponding quantum-number combination \((N + M)/2\) cannot be too large.

In closing, we remark that the radial equations for all the traps considered in this paper can be mapped into various radial equations for Coulomb-type potentials [24], among which is the usual radial Coulomb equation in three dimensions. This suggests that under suitable circumstances the analogies drawn above between atoms and trapped systems might be more than heuristic. It would be amusing, for example, to identify a trap system for which the oscillator energy shifts correspond via a direct map to known quantum defects for an alkali-metal atom.

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