MULTIPLE SOLUTIONS FOR THE
VAN DER WAALS–ALLEN–CAHN–HILLIARD EQUATION
WITH A VOLUME CONSTRAINT

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ABSTRACT. We give multiplicity results for the solutions of a nonlinear elliptic equation, with an asymmetric double well potential of Van der Waals-Allen–Cahn–Hilliard type, satisfying a linear volume constraint, on a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$. The number of solutions is estimated in terms of topological and homological invariants of the underlying domain $\Omega$.

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1. Introduction

1.1. Formulation of the problem (P_{V,\varepsilon}). In this paper we are concerned with the existence of multiple solutions of the following nonlinear problem (P_{V,\varepsilon}): for fixed positive constants V and \varepsilon, find u \in H^{1}_0(\Omega), and \lambda \in \mathbb{R} such that

\begin{align}
-\varepsilon^2 \Delta u + W'(u) &= \lambda, \\
\int_{\Omega} u(x) \, dx &= V,
\end{align}

where \Omega is an open bounded Lipschitz domain in \mathbb{R}^N, and W: \mathbb{R} \to \mathbb{R} is a \text{C}^2 function which satisfies the following assumptions:

\begin{align}
(1.2) \quad &W(0) = W'(0) = 0; \quad W''(0) > 0, \\
(1.3) \quad &\exists s_0 \in ]0, +\infty[ \text{ s.t. } -m := W(s_0) = \min \{W(s) : s \in \mathbb{R}\} < 0, \\
(1.4) \quad &W'(s) > 0, \quad \forall s \in ]s_0, s_0 + \delta[, \text{ for some } \delta > 0.
\end{align}

In particular (1.4), is fulfilled for potentials W of class \text{C}^2 when s_0 is a minimum point and

\begin{align}
(1.5) \quad W''(s_0) > 0.
\end{align}

In what follow we denote by s_0 the minimum positive real number for which (1.3) is satisfied. These assumptions imply that the potential W is not an even function, as opposed to the standard Allen–Cahn potential, which has symmetric minima. Moreover, such a W takes different values at the two local minima. We will refer to this situation by saying that W is asymmetric. However, this entails no essential difference in the geometry of the solutions of the problem, see discussion in Section 1.4.

For the central result of the paper, we also need an asymptotic growth condition for W, given by assuming the existence of positive constants A, B such that

\begin{align}
(1.6) \quad |W'(s)| \leq A + B|s|^{p-1}, \quad p < \frac{2N}{N-2} \quad (p < \infty \text{ if } N = 1, 2).
\end{align}

The graph of a typical potential function W satisfying the above axioms is given in Figure 1.1.

Remark 1.1. A simple and instructive example of potentials satisfying (1.2), (1.3), (1.4) and is given by the non-symmetric Allen-Cahn-Hilliard potential:

\begin{align}
W(s) = s^2(s - s_1)(s - s_2),
\end{align}

where 0 < s_1 < s_0 < s_2. The usual Allen-Cahn potential is taken with s_1 = s_2. The reader should observe that, when N \geq 3, assumption (1.6) is not satisfied by (1.7). However, we will also discuss a result of multiplicity of solutions without assuming the growth condition.

Equations of type (1.1), such as the Allen-Cahn equation [1] and the Cahn-Hilliard equation [8] (see also the book [22]), appear naturally in many problems of mathematical physics and applied mathematics. In theoretical biology equations of
The linearized problem. Assume that \((u, \lambda) \in H^1_0(\Omega) \times \mathbb{R}\) is a solution of (1.1). Linearizing the problem along \(u\) gives the following:

\[
-\varepsilon^2 \Delta \vartheta + W''(u) \vartheta = \Lambda, \\
\int_{\Omega} \vartheta(x) \, dx = 0.
\]

Definition 1.2. A solution \((u, \lambda)\) of Problem \((P_{V,\varepsilon})\) is said to be degenerate if (1.8) admits a non-trivial solution \((\vartheta, \Lambda) \in H^1_0(\Omega) \times \mathbb{R}\), and nondegenerate otherwise.

It is not hard to see that \((u, \lambda)\) is a nondegenerate solution of \((P_{V,\varepsilon})\) when \(u\) is a nondegenerate critical point of the associated energy functional, see Section 1.5.

1.3. Statement of the existence results. The focus of this paper is on the existence of solutions which are not necessarily minima of the associated energy functional (see Section 1.5 below), and on their multiplicity. We recall that in the literature there are many results relative to the existence of multiple solutions which are critical points of the energy. However in these cases, usually it is exploited the fact
that $W(0)$ is not a minimum value of $W$ (see the book [23]). In other references, multiplicity results are obtained for even potentials $W$, in which case the topology of the real projective space plays a crucial role. In all these situations, the solutions found present many nodal regions.

In the present paper, we find multiple solutions exploiting the topology of the domain $\Omega$. A lower bound for the number of solutions will be given using Lusternik–Schnirelman theory and Morse theory.

For a topological space $X$, let us denote by $\text{cat}(X)$ the Lusternik–Schnirelman category of $X$, see Definition 2.1.

**Theorem 1.3.** Under assumptions (1.2), (1.3), (1.4), (1.6), for $V > 0$ sufficiently small, there exists $\varepsilon(V) > 0$ such that for all $\varepsilon \in [0, \varepsilon(V)]$, Problem $(P_{V, \varepsilon})$ admits:

- at least one solution if $\Omega$ is contractible;
- at least $\text{cat}(\Omega) + 1$ distinct solutions if $\Omega$ is non-contractible.

Moreover, if $\Omega$ is non-contractible and all solutions of Problem $(P_{V, \varepsilon})$ are non-degenerate (Definition 1.2), then there are at least $2P_1(\Omega) - 1$ distinct solutions, where $P_1(\Omega)$ is the sum of the Betti numbers of $\Omega$.

It is a natural conjecture that the nondegeneracy assumption in the last statement of Theorem 1.3 should hold for generic choices of the quadruple $(\Omega, W, V, \varepsilon)$. It is also interesting that the last claim of Theorem 1.3 holds without the nondegeneracy assumption, provided that the solutions are counted with a suitable notion of multiplicity, see Definition 2.8.

The method employed for the construction of the solutions of (1.1) also provides bounds for the energy and the Morse index, see Proposition 1.4 below.

### 1.4. A brief discussion on the assumptions.

In the proof of our results, we will use assumptions (1.2) to deduce that $W(s) > 0$ for $s < 0$ (needed in Lemma 4.5), that $W(s) \geq -ks$ for some $k > 0$ and for $s > 0$ small (needed in the proof of Theorem 3.11). Assumption (1.3), i.e., the fact that the absolute minimum of $W$ is negative, is used to deduce that the minimum of the functional (1.9) is negative, which plays a crucial role in the proof of Theorem 3.11. Namely, this fact will imply that the solution $U_\gamma$ of a certain auxiliary problem (see Section 3) has compact support. Studying the regularity of such a function $U_\gamma$ will require a rather involved analysis of a certain variational inequality, whose solutions are subject to an affine constraint, which is discussed in Sections 3.3, 3.4, and 3.5. It is important to remark, however, that the fact that $W$ takes different values at the two local minima, is irrelevant for the geometry of the solutions of the problem. Namely, given a potential $W$ as above, one can consider a linear perturbation of the form $\tilde{W}(s) = W(s) + A s$, with $A > 0$. When $A$ is suitably chosen, the new potential $\tilde{W}$ has two global minima at the zero level; clearly, a pair $(u, \lambda)$ is a solution of Problem $(P_{V, \varepsilon})$ with the potential $W$ if and only if $(u, \lambda - A)$ is a solution of Problem $(P_{V, \varepsilon})$ with potential $\tilde{W}$.

Finally assumption (1.4) is used to guarantee that solutions of the auxiliary minimization problem are bounded from above and (1.2) is used to show that solutions...
of the auxiliary minimization problem are bounded from below. The subcritical
growth condition imposed by (1.6) is needed for technical reason, as it makes the
corresponding variational problem well defined in the appropriate Sobolev setting.

In a forthcoming paper we will develop a theory that allows to obtain a mul-
tiplicity result that does not employ the subcritical growth condition (1.6). This
will be obtained by showing suitable a priori bounds for the low energy solutions,
including bounds on the corresponding Lagrange multiplier.

1.5. The variational framework. Under assumption (1.6), solutions of Problem
(P_{V,\varepsilon}) are characterized as critical points of the energy functional
\[ E_{\varepsilon} : H^1_0(\Omega) \rightarrow \mathbb{R} \]
defined by:
\[ E_{\varepsilon}(u) = \frac{\varepsilon^2}{2} \int_\Omega |\nabla u|^2 \, dx + \int_\Omega W(u(x)) \, dx, \]
under the constraint
\[ \int_\Omega u(x) \, dx = V. \]
Assumption (1.6) guarantees that \( E_{\varepsilon} \) is a well defined functional on \( H^1_0(\Omega) \) (see
for instance [23, Proposition B.10]) which is of class \( \mathcal{C}^2 \). The differential of the
functional \( E_{\varepsilon} \) is given by:
\[ E'_{\varepsilon}(u)v = \varepsilon^2 \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega W'(u) \cdot v \, dx, \quad u, v \in H^1_0(\Omega). \]
Moreover, assumption (1.3) implies that \( E_{\varepsilon} \) is bounded from below:
\[ E_{\varepsilon}(u) \geq \int_\Omega W(u(x)) \, dx \geq -m |\Omega|, \quad \forall u \in H^1_0(\Omega). \]

1.6. Bounds on the energy and the Morse index. In view to applications to the
constant mean curvature problem in Riemannian manifolds, which requires taking
limits to the singular case \( \varepsilon \rightarrow 0 \), one needs uniform estimates of the modulus of
\( \frac{E_{\varepsilon}}{\varepsilon} \) and the Morse index of the families of solutions \( (u_{\varepsilon})_\varepsilon \). The methods developed
in the paper allow to obtain the following result:

**Proposition 1.4.** Under the assumptions of Theorem 1.3, for \( V > 0 \) sufficiently
small and for all \( \varepsilon \in [0, \varepsilon_0(V)] \), at least \( \text{cat}(\Omega) \) solutions of Problem \( (P_{V,\varepsilon}) \) have
energy \( \frac{E_{\varepsilon}}{\varepsilon} \) which is uniformly bounded in \( \varepsilon \). Moreover, in the nondegenerate case,
at least \( P_1(\Omega) \) solutions of Problem \( (P_{V,\varepsilon}) \) have energy \( \frac{E_{\varepsilon}}{\varepsilon} \) and Morse index which
is uniformly bounded in \( \varepsilon \).

A proof of Proposition 1.4 will be given at the end of Section 4.
2. Notation and Preliminary Facts

In this section we present some known results related to the Lusternik–Schnirelmann theory and Morse theory which will be used in the sequel.

**Definition 2.1.** Let \((X, \tau)\) be a topological space and \(Y \subseteq X\) be a closed subset. The Lusternik-Schnirelmann category of \(Y\) in \(X\) is the number \(\text{cat}_X(Y) \in \mathbb{N} \cup \{+\infty\}\) defined as the minimum number \(k\) such that there exist \(U_1, \ldots, U_k\) open subsets of \(X\) contractible in \(X\) such that \(Y \subseteq \bigcup_i U_i\). Furthermore, we set \(\text{cat}(X) := \text{cat}_X(X)\).

Let us also recall the following

**Definition 2.2.** Let \(M\) be a \(C^1\)-Hilbert manifold, \(J : M \to \mathbb{R}\) a \(C^1\) functional, and \((u_n)\) a sequence in \(M\). We say that \(u_n\) is a Palais–Smale sequence (or a PS-sequence, for short) for \(J\) if

\[
\lim_{n \to \infty} J(u_n) = c \in \mathbb{R},
\]

and

\[
\lim_{n \to \infty} \|J'(u_n)\|_{T_{u_n}^* M} = 0,
\]

where \(T_{u_n}^* M\) denotes the (topological) dual of the tangent space \(T_{u_n} M\).

**Definition 2.3.** Let \(M\) be a \(C^2\)-Hilbert manifold, \(J : M \to \mathbb{R}\) a \(C^1\) functional. We say that \(J\) satisfies the Palais-Smale condition, if every Palais-Smale sequence has a convergent subsequence in the strong topology of \(M\).

2.1. Abstract Lusternik–Schnirelman and Morse theory. To prove our main results we need the following theorem.

**Theorem 2.4.** Let \(M\) be a \(C^2\)-Hilbert manifold and let \(J : M \to \mathbb{R}\) be a \(C^1\) functional. Assume that

(i) \(\inf_{u \in M} J(u) > -\infty\);

(ii) \(J\) satisfies the Palais–Smale condition;

(iii) there exists a topological space \(X\) and two continuous maps \(f : X \to J^c\), \(g : J^c \to X\) such that \(g \circ f\) is homotopic to the identity map of \(X\).

Then there are at least \(\text{cat}(X)\) critical points of \(J\) in \(J^c\). Furthermore, if \(M\) is contractible and \(\text{cat}(X) > 1\), or more generally if \(\text{cat}(X) > \text{cat}(M)\), there is at least one additional critical point \(u \notin J^c\).

**Proof.** Under assumption (iii), \(\text{cat}(X) \leq \text{cat}(J^c)\). The result follows by applying standard variational techniques, see [3] or [4] for details. \(\square\)

The above result can be improved in the nondegenerate case using Morse theory.

Let \(X\) be a topological space and denote by \(H^n(X)\) its \(n\)-th Alexander–Spanier cohomology group with coefficients in \(\mathbb{R}\); let \(\beta_n(X)\) denote the \(n\)-th Betti number of \(X\), i.e., the dimension of \(H^n(X)\). For an account in book form of the Alexander-Spanier cohomology we refer the interested reader to the classical text [17].
Definition 2.5 (Poincare’s Polynomial). The Poincare’s Polynomial \( P_t(X) \) of \( X \) is defined as the formal power series in the variable \( t \):

\[
P_t(X) := \sum_{n=0}^{+\infty} \beta_n \ t^n.
\]

Remark 2.6. If \( X \) is a compact manifold, we have that \( H^n(X) \) is a finite dimensional vector space and the formal series (2.3) is actually a polynomial.

In the following definition we give the notion of Morse index of a critical point, which is necessary in our treatment to establish a relation between the Poincare’s polynomial \( P_t(\Omega) \) and the number of solutions of the Euler–Lagrange equation associated to a given functional \( J \). For our purposes, it is necessary to employ an extension of Morse theory to functionals that are not necessarily of class \( C^2 \), which uses generalized notions of nondegeneracy and Morse index. We will follow here the approach to Morse theory developed in [2] which is suitable in problems arising from PDE’s.

Given a pair \( Y \subset X \) of topological spaces and \( k \geq 0 \), let \( H^k(X,Y) \) denote the \( k \)-th relative Alexander–Spanier cohomology group of the pair, and denote by \( \beta_k(X,Y) \) its dimension.

Definition 2.7 (Morse Index). Let \( M \) be a \( C^2 \)-Hilbert manifold, \( J: \mathcal{M} \to \mathbb{R} \) a \( C^1 \) functional and let \( u \in \mathcal{M} \) be an isolated critical point of \( J \) at level \( c \in \mathbb{R} \). We denote by \( i_t(u) \) the following formal power series in \( t \)

\[
i_t(u) := \sum_{k=0}^{+\infty} \beta_k(J^c \cap U, (J^c \setminus \{u\}) \cap U) \ t^k,
\]

where \( J^c = \{ v \in \mathcal{M} : J(v) \leq c \} \), and \( U \) is a neighborhood of \( u \) containing only \( u \) as a critical point. We call \( i_t(u) \) the polynomial Morse index of \( u \). The number \( i_1(u) \) is called the multiplicity of \( u \).

If \( J \) is of class \( C^2 \) in a neighborhood of \( u \) and \( J''[u] \) is not degenerate, we say that \( u \) is a nondegenerate critical point. In this case we have that

\[
i_t(u) = t^{\mu(u)},
\]

where \( \mu(u) \) is the Morse index of \( u \), i.e., the dimension of a maximal subspace on which the bilinear form \( J''[u](\cdot, \cdot) \) is negative-definite. This suggests the following definition.

Definition 2.8. Let \( \mathcal{M} \) be a \( C^2 \)-Hilbert manifold, \( J: \mathcal{M} \to \mathbb{R} \) be a \( C^1 \) functional and let \( u \in \mathcal{M} \) be an isolated critical point of \( J \) at level \( c \). We say that \( u \) is (topologically) nondegenerate, if \( i_t(u) = t^{\mu(u)} \), for some \( \mu(u) \in \mathbb{N} \).

\[\text{1This means that } J(u) = c, J'[u] = 0, \text{ and there exists a neighbourhood } U \text{ of } u \text{ in } \mathcal{M} \text{ such that } u \text{ is the only critical critical point of } J \text{ in } U.\]
Theorem 2.9. Let the assumptions (i), (ii), and (iii) of Theorem 2.4 hold, and assume additionally that all the critical points of $J^c$ are isolated. Then the following identity of formal power series holds:

$$(2.6) \sum_{u \in \text{Crit}(J)} i_t(u) = P_t(X) + t[P_t(X) - 1] + (1 + t)Q(t),$$

where $Q(t)$ is a polynomial with nonnegative integer coefficients, and $\text{Crit}(J)$ denotes the set of critical points of $J$ on $J^c$. Moreover, if all the critical points are nondegenerate, there are at least $P_1(X)$ critical points with energy less than or equal to $c$, and at least $P_1(X) - 1$ critical points with energy greater than $c$.

Proof. See [3] or [4] for details. \qed

Remark 2.10. In Theorem 2.9, each one of the $P_1(X)$ critical points with energy less than or equal to $c$ has Morse index that varies in the range $\{0, \ldots, k_*\}$, with $k_* = \max \{k \in \mathbb{N} : \beta_k(X) \neq 0\}$. In particular, when $X$ is a smooth manifold, then the Morse index of these critical points is less than or equal to $\dim(X)$.

2.2. Notations. We will use the following notations throughout the paper:

- Given $N \geq 1$ and a Borel subset $B \subset \mathbb{R}^N$, we will denote by $|B|$ the Lebesgue $N$-measure of $B$, and by $\chi_B$ the characteristic function of $B$;
- $\omega_N$ is the volume of the unit ball of $\mathbb{R}^N$, and $\alpha_N$ is the N-area of the unit sphere in $\mathbb{R}^{N+1}$;
- given a functional $\mathcal{F}$ on a set $S$, we will denote by $\text{arg} \min \{\mathcal{F}(x) : x \in S\}$ the (possibly empty) set of minimizers of $\mathcal{F}$ in $S$;
- given subset $A \subset B \subset \mathbb{R}^N$, we write $A \Subset B$ to mean that the closure of $A$ is compact and it is contained in $B$;
- for a function $u : X \to \mathbb{R}$, we denote by $u^+$ (resp., $u^-$) the positive (resp., negative) part of $u$, defined by $u^+(x) = \max(u(x), 0) = \frac{1}{2}(u(x) + |u(x)|)$ (resp., $u^-(x) = \max(-u(x), 0) = \frac{1}{2}(-u(x) + |u(x)|)$);
- for an integer $k \geq 0$ and $\alpha \in [0, 1]$, the Hölder spaces $C^{k,\alpha}(\Omega)$ and $C^{k,\alpha}_{\text{loc}}(\Omega)$, and their Banach space norms, are defined as in [13, § 4.1].
- Given any set $\mathcal{X}$ and real valued functions $f_1, f_2 : \mathcal{X} \to \mathbb{R}$, we define $f_1 \vee f_2, f_1 \wedge f_2 : \mathcal{X} \to \mathbb{R}$ by setting $f_1 \vee f_2(x) = \max \{f_1(x), f_2(x)\}$ and $f_1 \wedge f_2(x) = \min \{f_1(x), f_2(x)\}$.

3. The auxiliary problem

In order to prove Theorem 1.3, we will exploit the properties of a certain solution $U_\gamma$ of an auxiliary variational problem in $\mathbb{R}^n$. Such $U_\gamma$ is a radial function with compact support in $\mathbb{R}^n$, that will be used to define a homotopy inverse for the barycenter map, see Section 4. In order to study the properties of $U_\gamma$, we will employ some results from the classical theory of variational inequalities, for which a standard reference is [15].
3.1. The auxiliary problem \((P_\gamma)\). We consider the following minimization problem \((P_\gamma)\): for fixed \(\gamma \in ]0, +\infty[\), find \(u \in H^1(\mathbb{R}^N)\) minimizing

\[
E(u) = \int_{\mathbb{R}^N} \left[ \frac{1}{2} |\nabla u|^2 + W(u) \right] \, dx,
\]

over the convex set

\[
K_\gamma = \left\{ u \in H^1(\mathbb{R}^N) : u \geq 0, \int_{\mathbb{R}^N} u \, dx \leq \gamma \right\},
\]

where the potential \(W: \mathbb{R} \to \mathbb{R}\) is defined in the introduction, i.e., it is a map of class \(C^2\) satisfying \((1.2), (1.3)\) and \((1.6)\). Observe that, by Fatou’s Lemma, the set \(K_\gamma\) is weakly closed\(^2\) in \(H^1(\mathbb{R}^N)\). The problem \((P_\gamma)\) is translation invariant so, if a minimum exists, all its translates are minima too.

As it is well known, if a minimum \(U_\gamma\) for problem \((P_\gamma)\) exists, then there exists \(\lambda_\gamma = \lambda(U_\gamma) \in \mathbb{R}\) such that \(U_\gamma\) satisfies the associated variational inequality

\[
\int \left[ \langle \nabla U_\gamma, \nabla (v - U_\gamma) \rangle + W'(U_\gamma)(v - U_\gamma) \right] \, dx \geq \lambda_\gamma \int (v - U_\gamma) \, dx,
\]

for every \(v \in K_\gamma\) (see \([15, \text{Proposition 5.1, p. 15}]\)). On the other hand, by \([15, \text{Theorem 2.1, p. 24, Chapter II}]\) applied to the variational inequality \((3.2)\), for each fixed \(\lambda_\gamma\) and \(U_\gamma\) the following variational inequality

\[
\int \left[ \langle \nabla u, \nabla (v - u) \rangle + W'(U_\gamma)(v - u) \right] \, dx \geq \lambda_\gamma \int (v - U_\gamma) \, dx, \quad \forall v \in K_\gamma,
\]

admits at most one solution \(u\).

3.2. Analysis of the variational inequality.

**Definition 3.1.** (see \([15, \text{Definition 5.1, p. 35, Ch. II}]\)) Let \(\Omega \subset \mathbb{R}^N\) be an open subset, \(u \in W^{1,2}(\Omega)\) and \(E \subset \overline{\Omega}\). The function \(u\) is nonnegative on \(E\) in the sense of \(W^{1,2}(\Omega)\) if there exists a sequence \(u_n \in W^{1,\infty}(\Omega)\) such that

\[
u_n(x) \geq 0, \quad \forall x \in E, \quad \text{and} \quad u_n \rightharpoonup u \text{ in } W^{1,2}(\Omega).
\]

We say that \(u \succeq v\) on \(E\) in the sense of \(W^{1,2}(\Omega)\), if \(u - v \geq 0\) on \(E\) in the sense of \(W^{1,2}(\Omega)\).

**Definition 3.2.** (See \([15, \text{Definition 6.7, Ch. II, p. 45}]\)) Let \(\Omega \subset \mathbb{R}^N\) be an open set, \(x_0 \in \Omega\), and \(u \in W^{1,2}(\Omega)\). We say that \(\phi(x_0) > 0\) in the sense of \(W^{1,2}(\Omega)\), if there exists an open ball \(B_\rho(x_0)\) with \(\rho > 0\) and \(\phi \in W^{1,\infty}_0(B_\rho(x_0))\). \(\phi \geq 0\) and \(\phi(x_0) > 0\), such that \(u - \phi \rightharpoonup 0\) on \(B_\rho(x_0)\) in the sense of \(W^{1,2}(\Omega)\). For any \(\psi \in W^{1,2}(\Omega)\) we say that \(u(x_0) > \psi(x_0)\) in the sense of \(W^{1,2}(\Omega)\), if \(u(x_0) - \psi(x_0) > 0\) in the sense of \(W^{1,2}(\Omega)\).

\(^2\)This is the reason for using the constraint \(\int_{\mathbb{R}^N} u \, dx \leq \gamma\), rather than \(\int_{\mathbb{R}^N} u \, dx = \gamma\). In fact, we will later show that the two constraints define the same minimization problem when \(\gamma\) is sufficiently large (see Theorem 3.11, formula (3.26)).
It is easy to see that, for all \( u \in W^{1,2}(\Omega) \), the set:

\[
\left\{ x \in \Omega : u(x) > 0 \text{ in the sense of } W^{1,2}(\Omega) \right\}
\]

is open.

Carrying on our analysis of the variational inequality (3.2), it is not too hard to prove that, given a minimizer \( U_\gamma \) for Problem \((P_\gamma)\), on the open subset \( \Gamma \subset \mathbb{R}^N \):

\[
\Gamma = \Gamma(U_\gamma) = \left\{ x \in \mathbb{R}^N : U_\gamma(x) > 0 \text{ in the sense of } W^{1,2}(\mathbb{R}^N) \right\},
\]

we have that

\[
\int_\Gamma \left[ (\nabla U_\gamma, \nabla \varphi) + W'(U_\gamma) \varphi \right] \, dx = \lambda_\gamma \int_\Gamma \varphi \, dx, \quad \forall \varphi \in C_0^\infty(\Gamma).
\]

We also need the following notation

\[
H^1_{\text{rad}}(\mathbb{R}^N) = \{ u \in H^1(\mathbb{R}^N) : u \text{ is radially symmetric} \}.
\]

**Definition 3.3.** Let \( \Lambda \) be a measurable set of finite volume in \( \mathbb{R}^N \). Its symmetric rearrangement \( \Lambda^* \) is the open ball centered at the origin whose volume agrees with the volume of \( \Lambda \). Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a nonnegative measurable function that vanishes at infinity, in the sense that all its superlevel sets have finite measure, i.e., \( |\{ x : f(x) > t \}| < +\infty \), for all \( t > 0 \). The symmetric decreasing rearrangement of \( f \) is the radially symmetric function \( f^* \) whose superlevel sets are the symmetric rearrangements of the superlevel sets of \( f \). Thus:

\[
f^*(x) = \int_0^{+\infty} \chi_{\{y : f(y) > y\}}^*(t) \, dt.
\]

The symmetric decreasing rearrangement \( f^* \) of a measurable function \( f \) is lower semicontinuous (since its level sets are open), and it is uniquely determined by the distribution function \( \mu_f(t) := |\{ x : f(x) > t \}|. \) By construction, \( f^* \) is equimeasurable with \( f \), i.e., corresponding superlevel sets of \( f \) and of \( f^* \) have the same volume, \( \mu_f(t) = \mu_{f^*}(t) \) for all \( t > 0 \).

**Lemma 3.4.** \( E(u^*) \leq E(u) \), for every \( u \in W^{1,2}(\mathbb{R}^N) \), \( u \geq 0 \).

**Proof.** By the Polya-Szego inequality we have that \( \int |\nabla u^*|^2 \, dx \leq \int |\nabla u|^2 \, dx \). Using the Layer-Cake integral representation of a nonnegative function we have that \( \int W(u^*) \, dx = \int W(u) \, dx \) (compare with [25, Proposition 2.6]). The conclusion follows easily. \( \square \)

Let us recall the following result from [7].

**Theorem 3.5.** Let \( u : \mathbb{R}^N \to [0, +\infty[ \) be a function with \( \text{supp}(u) \subset \mathbb{R}^N \), and let \( \Lambda : [0, +\infty[ \to [0, +\infty[ \) be a function of class \( C^2 \), such that \( \Lambda(0) = 0 \), and with \( \Lambda^p \) convex for some \( 1 \leq p < +\infty \). Assume \( \int_{\mathbb{R}^N} \Lambda(|\nabla u|) \, dx < +\infty \). Then \( \nabla u^* \) is a measurable function and

\[
\int_{\mathbb{R}^N} \Lambda(|\nabla u^*|) \, dx \leq \int_{\mathbb{R}^N} \Lambda(|\nabla u|) \, dx.
\]

\( \text{see } [15, \text{page 43]} \) and use a partition of unity argument
Moreover, if $p > 1$, if
\[\left| \nabla u^* - \frac{1}{\partial u^*/\partial x} \right| = 0,\]
if $A$ is strictly increasing, and if equality holds in (3.5), then there exists $x_0 \in \mathbb{R}^N$ such that $u^*(x_0 + x) = u(x)$ a.e. in $\mathbb{R}^N$.

**Proof.** See [7, Theorem 1.1].

**Theorem 3.6.** Let $D$ be an open bounded domain of $\mathbb{R}^N$, $u \in L^1(D)$, $f \in L^p(D)$, $1 < p < \infty$, be such that $\Delta u = f$, in $D$ in the sense of distributions. Then $u \in W^{2,p}_{\text{loc}}(D)$ and there exists a constant $C = C(p, n, K, D) > 0$ such that:
\[\|u\|_{W^{2,p}(K)} \leq C \left\{ \|u\|_{L^1(D)} + \|f\|_{L^p(D)} \right\},\]
for any $K \subset D$. In particular, $\Delta u = f$, a.e. $\Omega$.

**Proof.** See [21, Theorem 1.1].

It will also be useful to keep in mind standard elliptic regularity results, such as [13, Theorem 9.19], that will play an important role in the proof of Theorem 3.11 below. Let us also recall a celebrated result of Gidas, Ni and Nirenberg concerning the symmetry of solutions of certain elliptic PDEs:

**Theorem 3.7.** Let $f$ be of class $C^1$ and let $u > 0$ be a positive solution in $C^2(\overline{\mathbb{R}^N}(0, R))$ of $\Delta u + f(u) = 0$ with $u = 0$ on $|x| = R$. Then $u$ is radially symmetric, and $\frac{\partial u}{\partial r} < 0$ for $0 < r < R$. In particular $\|u\|_\infty = u(0)$.

**Proof.** See [12, Theorem 1].

### 3.3. Regularity of the obstacle problem under the volume constraint.

We will need a regularity result for solution of variational problems with constraints. The following theorem is obtained with a slight modification of the arguments used in the proof of [10, Thm. 1, Thm. 2]. Compared with the results of [10], here we consider the case where an extra non-homogeneous term is present. For our purposes this extra term denoted by $f$ is in $L^\infty$, and it only depends on the unknown function $u$ (and not on its gradient $\nabla u$). For the sake of completeness, we will prove here a statement which is more general than the one we need in the proof of Theorem 3.13.

Let us consider a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$; given a constant $V \in \mathbb{R}$ and functions $\psi_1, \psi_2, \chi \in H^1(\Omega)$ with $\psi_1 \leq \chi \leq \psi_2$ and:
\[\int_\Omega \psi_1 \, dx < V < \int_\Omega \psi_2 \, dx,\]
let us denote by
\[K_{\psi_1, \psi_2, \chi, V, \Omega} = \left\{ v \in H^1(\Omega) : (v - \chi) \in H^1_0(\Omega), \int_\Omega v \, dx = V, \psi_1 \leq v \leq \psi_2 \right\}.\]

In our next result, we will consider only the case where the function $\psi_1$ is constant.
Theorem 3.8. If \( u \in K_{\psi_1,\psi_2,X,V,\Omega} \) is a solution of the variational inequality:

\[
(3.8) \quad \int_{\Omega} \nabla u \cdot \nabla (u - v) \, dx \leq \int_{\Omega} f(x)(u(x) - v(x)) \, dx,
\]

for all \( v \in K_{\psi_1,\psi_2,X,V,\Omega} \), where \( \psi_1 \) is a constant function, and

\[ f \in L^\infty(\Omega), \quad \psi_2 \in C^{1,\alpha}(\Omega), \]

then \( \nabla u \in C^{0,\alpha}_{loc}(\Omega) \), i.e., \( u \in C^{1,\alpha}(\Omega) \). Moreover, if \( \psi_2 \in C^{1,\alpha}(\overline{\Omega}) \), then \( u \in C^{1,\alpha}(\overline{\Omega}) \).

Remark 3.9. Theorem 3.8 is an essential regularity results, that will needed in the proof of Theorem 3.13 to establish that the radial solution of a certain auxiliary problem has vanishing normal derivative along the boundary of its support.

Proof of Theorem 3.8. For the sake of brevity, we will denote \( K := K_{\psi_1,\psi_2,u,V,\Omega} \).

For every subset \( A \subseteq \mathbb{R}^n \), \( |A| \) denotes the Lebesgue’s measure of \( A \). When

\[
\left\{ x \in \Omega : \psi_1 < u(x) < \psi_2(x) \right\} = 0,
\]

the statement of the theorem becomes trivial since it means that \( u = \psi_1 \), a.e., or \( u = \psi_2 \), a.e. Thus from now on we can assume that

\[
\left\{ x \in \Omega : \psi_1 < u(x) < \psi_2(x) \right\} > 0;
\]

thus, we get the existence of \( \varepsilon_0 = \varepsilon_0(u, \psi_1, \psi_2) > 0 \) such that

\[
\left\{ x \in \Omega : \psi_1 + \varepsilon_0 < u(x) < \psi_2(x) - \varepsilon_0 \right\} > 0.
\]

Now we construct a function \( \phi : \mathbb{R} \to [0, 1] \), \( \phi \in C^\infty(\mathbb{R}) \), such that \( \phi|_{[-\infty, \varepsilon_0]} = 0 \) and \( \phi|_{[2\varepsilon_0, +\infty]} = 1 \). We can find a small ball \( B \Subset \Omega \) and a function \( v \in W^{1,2}_0(B) \), satisfying \( 0 \leq v \leq 1 \), with the property that

\[
\varphi(x) := v(x) \cdot \phi(u(x) - \psi_1(x)) \cdot \psi_2(x) - u(x)
\]

does not vanish identically, and

\[
\int_B \varphi(x) \, dx > 0.
\]

From the construction of \( \varphi \) we have

\[
\psi_1 \leq u + t\varphi \leq \psi_2 \quad \text{a.e. in } \Omega, \quad \forall \, t \in [-\varepsilon_0, \varepsilon_0[.
\]

Now let \( B_R(x_0) \subset \Omega \), where \( R > 0 \) is arbitrary, and \( B_R(x_0) \cap B = \emptyset \).

Remark 3.10. In order to obtain the following parts of the proof for all \( B_R(x_0) \subset \Omega \) we need the existence of two disjoint small balls \( B^1, B^2 \), and two functions \( \varphi_1 \in H^1(B^1), \varphi_2 \in H^1(B^2) \) with the same properties as \( \varphi \), i.e., satisfying (3.9) and (3.10). The existence of such function can be shown by simply repeating the existence argument above, with \( B^1 \) small enough. Then, for \( R_0 > 0 \) sufficiently small, we have \( B_{R_0}(x_0) \subset \Omega \), and either \( B_{R_0}(x_0) \cap B^1 = \emptyset \) or \( B_{R_0}(x_0) \cap B^2 = \emptyset \).
For sake of simplicity we deal only with $B$ and $\phi$, for a more detailed treatment of this standard isoperimetric argument compare [16, Example 2.13, p. 279–280]. Choose $R_0$ small enough as prescribed by the preceding remark and set $B := B^i$ and $\phi := \phi_i$, where $i \in \{1, 2\}$ is such that $B_{R_0}(x_0) \cap B^i = \emptyset$. We want to show the desired regularity of $U$ inside the ball $B_R(x_0)$ for any $0 < R < R_0$ and $x_0 \in \Omega$. With this aim in mind let $U$ be the harmonic function on $B := B_R(x_0)$ with the boundary values of $u$, i.e.,

\[(3.11) \int_{B_R} \nabla U \nabla \phi \, dx = 0, \forall \phi \in W^{1,2}_0(B_R), \text{ and } U - u \in W^{1,2}_0(B_R).\]

By a standard argument (see [9, Lemma 7.I]) we have for $0 < \rho < R$

\[(3.12) \int_{B_{\rho}} |\nabla U|^2 \, dx \leq \left(\frac{\rho}{R}\right)^n \int_{B_R} |\nabla U|^2 \, dx.\]

Furthermore we introduce

$$\tilde{U} = \begin{cases} U, & \text{in } B_R; \\ u, & \text{in } \Omega \setminus B_R. \end{cases}$$

and we set $v := (\tilde{U} \lor \psi_1) \land \psi_2$. By construction we know that there is $i \in \{1, 2\}$, such that $B_i \cap B_R = \emptyset$. Set $\varphi := \varphi_i$, $B := B_i$, and let $\tilde{t} := \tilde{t}_{B_R} \in \mathbb{R}$ verifying

$$\int_{\Omega} (v + \tilde{t}\varphi) \, dx = V = \int_{\Omega \setminus B_R} u + \int_{B_R} u = \int_{\Omega \setminus B_R} v + \int_{B_R} v + \tilde{t} \int_{B} \varphi.$$ 

It follows that

$$\tilde{t} = \int_{B_R} (u - (\tilde{U} \lor \psi_1) \land \psi_2) \, dx \left(\int_{B} \varphi \, dx\right)^{-1}.$$ 

As it is immediate to check, inside the ball $B_R$ and also in the entire $\mathbb{R}^n$ we have

\[(3.13) |u - v| \leq |u - \tilde{U}|.\]

This simple observation allow us to estimate the value of $\tilde{t}$, i.e.,

\[(3.14) |\tilde{t}| \leq \int_{B_R} |u - \tilde{U}| \, dx \left(\int_{B} \varphi \, dx\right)^{-1}.\]

We claim that if we choose $0 < R_0$ small enough then for every $0 < R < R_0$ we have $|\tilde{t}_{B_R}| < \varepsilon_0$. By an application of Hölder inequality and Gagliardo-Nirenberg inequality (which is possible because $U - u \in W^{1,2}_0(B_R)$) we have that

\[(3.15) \int_{B_R} |u - U| \, dx \leq \left(\omega_n R^n \right)^{\frac{1}{2}} \left(\int_{B_R} |\nabla (u - U)|^2 \, dx\right)^{\frac{1}{2}}.\]

On the other hand

$$\int_{B_R} |\nabla U|^2 \, dx \leq \int_{B_R} |\nabla u|^2 \, dx,$$
since \( \int_{B_R} \nabla U \cdot \nabla u = \int_{B_R} |\nabla U|^2 \, dx \) and
\[
- \int_{B_R} \nabla U \cdot \nabla u + \int_{B_R} |\nabla u|^2 = \int_{B_R} |\nabla (U-u)|^2 \geq 0.
\]
Hence
\[
\int_{B_R} |u-U|^2 \, dx \leq \left( \omega_n R^n \right)^{\frac{1}{n} + \frac{1}{2}} \left( 2 \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \cdot
\]
From (3.14) and (3.16) we get easily
\[
|\tilde{t}| \leq \left( \omega_n r^n \right)^{\frac{1}{n} + \frac{1}{2}} \left( 2 \int_{\Omega} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{B} \phi \, dx \right)^{-1}.
\]
From the last inequality we see that for a suitable \( R_0 = R_0(\varphi, u) > 0 \) we have that for every \( 0 < R < R_0 \) we have \( v + \tilde{t} \varphi \in K \). Thus
\[
\int_{\Omega} \nabla u \nabla (u - v - \tilde{t} \varphi) \, dx \leq \int_{\Omega} f(x)(u - v)(x) \, dx + \int_{B} f \varphi \, dx.
\]
At first we reduce the problem to the case \( \psi_1 = 0 \).
In fact, \( u \in K \) is a solution of (3.8) with \( f_i = 0, \forall i \in \{1, \ldots, n\} \) if and only if \( \overline{U} = (u - \psi_1) \in \overline{K} \) solves
\[
\int_{\Omega} \nabla \overline{U} \nabla (\overline{U} - \overline{v}) \, dx \leq \int_{\Omega} f(u - v) \, dx,
\]
for all
\[
\overline{v} \in \overline{K} := \left\{ \tilde{v} \in W^{1,2}(\Omega) | \tilde{v} - \overline{U} \in W^{1,2}_0(\Omega), \int_{\Omega} \tilde{v} \, dx = V - \int_{\Omega} \psi_1 \, dx, \ 0 \leq \tilde{v} \leq \psi = \psi_2 - \psi_1 \right\}.
\]
Let \( \overline{U} \) be the harmonic function on \( B_R := B_R(x_0) \subset \Omega \) with boundary values \( u \), then for \( 0 < \rho < R \leq R_0 \)
\[
\int_{B_\rho} |\nabla \overline{U} - (\nabla \overline{U})_\rho|^2 \, dx \leq c_4 \left( \frac{\rho}{R} \right)^{n+2} \int_{B_R} |\nabla \overline{U} - (\nabla \overline{U})_R|^2 \, dx,
\]
where \( (\nabla \overline{U})_\rho := \frac{1}{|B_\rho(x_0)|} \int_{B_\rho(x_0)} \nabla \overline{U} \, dx \), and \( c_4 = c_4(n) > 0 \). We set
\[
\overline{v} = \begin{cases} \overline{U} & \text{on } \Omega \setminus B_R, \\ \overline{U} \wedge \psi & \text{on } B_R. \end{cases}
\]
With \( \overline{v} \) and \( \tilde{t} \) as before we have that \( \overline{v} + \tilde{t} \varphi \in K \) and for \( 0 < R \leq R_0 \),
\[
\int_{B_R} \nabla \overline{U} \cdot \nabla (\overline{U} \wedge \psi) \leq \tilde{t} \int_{B} \nabla \overline{U} \cdot \nabla \varphi \, dx
\]
Recall the following easy inequality:

\[(3.18) \quad \left| \int_{B_R} f(x)(u - v)(x) \, dx \right| \leq \|f\|_{\infty, \Omega} \int_{B_R} |u - \bar{U}| \, dx \]

\[\leq \|f\|_{\infty, \Omega} (\omega_n R^n)^{\frac{1}{n} + \frac{1}{2}} \left( \int_{B_R} |\nabla (u - \bar{U})|^2 \, dx \right)^{\frac{1}{2}}.\]

We write \( \bar{u} - \bar{U} \wedge \psi = \bar{u} - \bar{U} + \bar{U} - \bar{U} \wedge \psi \), we make use of

\[0 = \int_{B_R} -\gamma \nabla (\bar{u} - \bar{U} \wedge \psi) \, dx\]

which is true for all \( \gamma \in \mathbb{R}^n \) and observe that (3.11), (3.14), (3.15), and (3.18) still hold when applied to \( \bar{u}, v, U \). Then we get

\[
\int_{B_R} \nabla \bar{u} \nabla (\bar{u} - \bar{U} \wedge \psi) \leq \bar{t} \int_{B_R} \nabla \bar{u} \nabla \varphi \, dx \\
+ \int_{B_R} f(x)(\bar{u} - \bar{U} \wedge \psi)(x) + \bar{t} \int_{B_R} f \varphi \, dx.
\]

In fact we are able to prove by elementary meanings the following inequality just in the case \( \psi_1 = \text{const.} \in \mathbb{R} \) (which implies \( \bar{U} = \bar{U} \))

\[
\int_{B_R} |\nabla (\bar{u} - \bar{U})|^2 \, dx \leq \bar{t} \int_{B_R} \nabla \bar{u} \nabla \varphi \, dx \\
+ \int_{B_R} f(x)(\bar{u} - \bar{U} \wedge \psi)(x) + \bar{t} \int_{B_R} f \varphi \, dx \\
+ \int_{B_R} \nabla \bar{u} \nabla (\bar{U} \wedge \psi - \bar{U}).
\]

Hence

\[(3.19) \quad \int_{B_R} |\nabla (\bar{u} - \bar{U})|^2 \, dx \leq c_5 \left\{ \int_{B_R} |\nabla (\bar{U} - \bar{U} \wedge \psi)|^2 \, dx + \int_{B_R} |\nabla \psi_1 - \gamma|^2 \, dx + R^{n+2} \right\},\]

where \( c_5 = c_5(n, \varphi, f, u) > 0 \) depends on \( \bar{C} \) and \( n \). To estimate

\[
\int_{B_R} |\nabla (\bar{U} - \bar{U} \wedge \psi)|^2 \, dx,
\]

we observe that

\[
\int_{B_R} \nabla (\bar{U} - \bar{U} \wedge \psi) \nabla \varphi \, dx = -\int_{B_R} (\nabla (\bar{U} \wedge \psi) - \delta) \cdot \nabla \varphi \, dx
\]
being true for each $\delta \in \mathbb{R}^n$ and each $\phi \in W^{1,2}_0(B_R)$. We choose $\phi := (\mathcal{U} - \mathcal{U} \land \psi)$ and get

$$
\int_{B_R} |\nabla (\mathcal{U} - \mathcal{U} \land \psi)|^2 \, dx = \int_{\{\mathcal{U} > \psi\}} |\nabla (\mathcal{U} - \mathcal{U} \land \psi)|^2 \, dx \\
- \int_{\{\mathcal{U} > \psi\}} (\nabla (\mathcal{U} \land \psi) - \delta) \cdot \nabla (\mathcal{U} - \mathcal{U} \land \psi) \, dx \\
= \int_{\{\mathcal{U} > \psi\}} (\nabla (\psi) - \delta) \cdot \nabla (\mathcal{U} - \psi) \, dx \\
H^\text{Hölder} \leq \left\{ \int_{\{\mathcal{U} > \psi\}} |\nabla (\psi) - \delta|^2 \right\}^2 \left\{ \int_{\{\mathcal{U} > \psi\}} |\nabla (\mathcal{U} - \psi)|^2 \right\}^2.
$$

(3.20)

Dividing by $\left\{ \int_{\{\mathcal{U} > \psi\}} |\nabla (\mathcal{U} - \psi)|^2 \right\}^2$ yields to

$$
\int_{B_R} |\nabla (\mathcal{U} - \mathcal{U} \land \psi)|^2 \, dx = \int_{\{\mathcal{U} > \psi\}} |\nabla (\mathcal{U} - \psi)|^2 \, dx \\
\leq \int_{B_R} |\nabla \psi - \delta|^2 \, dx.
$$

Combining (3.19) and (3.20) we deduce

$$
\int_{B_R} |\nabla (\mathcal{U} - \mathcal{U})|^2 \, dx \leq c_5 \left\{ \int_{B_R} |\nabla \psi - \delta|^2 \, dx + \int_{B_R} |\gamma|^2 \, dx + R^{n+2} \right\}.
$$

(3.21)

As $\psi_1, \psi_2, \psi \in C^{1,\alpha}_\text{loc}(\Omega)$, it is obvious that choosing $\delta = (\nabla \psi)(x_0)$ and $\gamma = 0$ we obtain

$$
\int_{B_R} |\nabla (\mathcal{U} - \mathcal{U})|^2 \, dx \leq c_6 R^{n+2\alpha},
$$

(3.22)

for $0 < R \leq R_0 \leq 1$. From this, we conclude that

$$
\left| \int_{B_R} |\nabla \mathcal{U} - (\nabla \mathcal{U})_R|^2 \, dx - \int_{B_R} |\nabla \mathcal{U} - (\nabla \mathcal{U})_R|^2 \, dx \right| \\
\leq \tilde{c}_6 R^{n+2\alpha} + \int_{B_R} |(\nabla \mathcal{U})_\rho - (\nabla \mathcal{U})_\rho|^2 \, dx \\
J^\text{Jensen} \leq \tilde{c}_6 R^{n+2\alpha} + \int_{B_R} |\nabla (\mathcal{U} - \mathcal{U})|^2 \, dx,
$$

this last equation together with (3.17) for $0 < \rho \leq R$

$$
\int_{B_\rho} |\nabla \mathcal{U} - (\nabla \mathcal{U})_\rho|^2 \, dx \leq 2 \left( \frac{\rho}{R} \right)^{n+2} \int_{B_R} |\nabla \mathcal{U} - (\nabla \mathcal{U})_R|^2 \, dx \\
+ 2 \int_{B_\rho} |\nabla (\mathcal{U} - \mathcal{U})|^2 \, dx \\
+ 2 \int_{B_\rho} |(\nabla \mathcal{U})_\rho - (\nabla \mathcal{U})_\rho|^2 \, dx \\
\overset{(3.22)}{\leq} 2 \left( \frac{\rho}{R} \right)^{n+2} \int_{B_R} |\nabla \mathcal{U} - (\nabla \mathcal{U})_R|^2 \, dx + 2 c_6 R^{n+2\alpha}
$$
and a well-known result of Campanato (see also [11]) says that
\[ u \in C^{\gamma} \]
\[ C(3.27) \]
satisfying:
\[ the restriction of W \]
\[ The function U \]
\[ every \]
\[ Without loss of generality we can minimize \]
\[ Proof. \]
\[ below. \]
\[ growth condition (1.6). This observation will be useful later, see Remark 3.15 \]
\[ R \]
\[ and there exists \]
\[ Problem (3.12) \]
\[ Remark \]
\[ results of the paper, which gives the existence of a compactly supported radial \]
\[ 3.4. Existence of a radial solution. We are now ready to prove one of the central \]
\[ Theorem 3.11. Problem (P_\gamma) has at least one solution U_\gamma \in K_\gamma \cap H^{1,\alpha}_{\text{rad}}(\mathbb{R}^N) \]
\[ E(\gamma) = \gamma_0(\mathbb{R}, W|_{(0,s_0)}) \in ]0, +\infty[ \]
\[ U_{\gamma} \]
\[ and there exists R_\gamma = R(U_{\gamma}) > 0 \]
\[ (3.27) \]
\[ The function U_{\gamma} is of class C^{1,\alpha} in \mathbb{R}^N, for every \alpha \in ]0, 1[, and it is of class C^2 \]
\[ on its positivity set. \]
\[ Remark 3.12. Among other things, the above theorem says that \gamma_0 \]
\[ Remark \]
\[ Proof. Without loss of generality we can minimize E over K_{\gamma} \cap H^{1,\alpha}_{\text{rad}}(\mathbb{R}^N). In \]
\[ fact, by Lemma 3.4, E(u^*) \leq E(u), and \int_{\mathbb{R}^N} u^* dx = \int_{\mathbb{R}^N} u dx, where u^* \in \]
\[ H^{1,\alpha}_{\text{rad}}(\mathbb{R}^N) is the symmetric decreasing rearrangement of u (see Definition 3.3). \]
\[ We will prove that a minimizing sequence (u_j) in H^{1,\alpha}_{\text{rad}}(\mathbb{R}^N) is bounded in \]
\[ H^1(\mathbb{R}^N). By assumptions (1.2), (1.6) and (1.3) on the potential W, we have that \]
\[ W(s) \geq -ks \] for some k \in \mathbb{R}^+ and every s \geq 0. Then, there exists C \in \mathbb{R} \]
\[ satisfying: \]
\[ C \geq \int |\nabla u_j|^2 + \int W(u_j) \geq \int |\nabla u_j|^2 - k\gamma, \]
and so:

\[(3.29) \quad \int |\nabla u_j|^2 \leq k\gamma + C.\]

This says that $|\nabla u_n|$ is bounded in $L^2(\mathbb{R}^N)$, hence by Sobolev’s inequality $u_j$ is bounded in $L^2(\mathbb{R}^N)$. On the other hand, $\int u_j = \gamma$ and $u_j \geq 0$, so $u_j$ is bounded in $L^1(\mathbb{R}^N)$ too. Using interpolation, we have that $u_j$ is bounded in $L^2(\mathbb{R}^N)$, and so $u_j$ is bounded in $H^1(\mathbb{R}^N)$. Thus, up to subsequences, $u_j$ is weakly convergent to a function $U_\gamma \in H^1(\mathbb{R}^N)$. The theorem of Strauss [24] (see also [5, Appendix A.1, Theorem 142]), asserts that for any $N \geq 2$, one has a compact inclusion of $H^1_{rad}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, for every $2 < p < 2^* = \frac{2N}{N-2}$ ($2^* = +\infty$, when $N = 2$).

Then, by a standard application of Nemytskii’s theorem, we have that $u \mapsto W \circ u$ is a compact operator from $H^1_{rad}(\mathbb{R}^N)$ to its topological dual $[H^1_{rad}(\mathbb{R}^N)]'$. In particular, the functional $u \mapsto \int_{\mathbb{R}^N} W(u(x)) \, dx$ is weakly continuous. Moreover $u \mapsto \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$ is weakly lower-semi-continuous. The direct method of the calculus of variations ensures the existence of a minimum $U_\gamma \in K_\gamma$, since $K_\gamma$ is convex and strongly closed, so a fortiori it is also weakly closed. This proves the first assertion of the theorem.

In order to prove (3.25), let us consider a nonnegative function $\varphi \in H^1_{rad}(\mathbb{R}^N)$, such that $\int_{\mathbb{R}^N} \varphi = 1$, and such that

\[(3.30) \quad B_\varphi := \int_{\mathbb{R}^N} W(\varphi) \, dx < 0.\]

The existence of such a function $\varphi$ follows from assumption (1.3). Set $A_\varphi = \int_{\mathbb{R}^N} \frac{1}{2} |\nabla \varphi|^2 \, dx > 0$. Next, let us set $\varphi_\rho(x) = \varphi(x/\rho)$. Recalling the definition (3.1), it is easy to check that

\[(3.31) \quad \varphi_\rho \in H^1_{rad}(\mathbb{R}^N) \cap K_{\rho^N},\]

and that

\[(3.32) \quad E(\varphi_\rho) = A_\varphi \rho^{N-2} + B_\varphi \rho^N.\]

So, for every $\rho \geq \rho_0 = \rho_0(n, W) := \sqrt{-\frac{A_\varphi}{B_\varphi}} \to 0$, we have $E(\varphi_\rho) < 0$.

Set $\gamma_0 = \gamma_0(N, W) = \rho_0^N > 0$ (observe that actually with the right choice of $\varphi$ in (3.30) we have $\gamma_0 = \gamma_0(N, W_{1[0,s_0]})$); then for every $\gamma \geq \gamma_0$, the following inequalities hold

\[(3.33) \quad E(U_\gamma) \leq E(\varphi_\rho^N) < 0.\]

This proves (3.25).

In order to prove (3.26), we define $\Psi_\rho = U_\gamma(x/\rho)$ and we set

\[A_{U_\gamma} = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla U_\gamma|^2 \, dx > 0, \quad \text{and} \quad B_{U_\gamma} = \int_{\mathbb{R}^N} W(U_\gamma) \, dx.\]

Then we have

\[(3.34) \quad E(\Psi_\rho) = A_{U_\gamma} \rho^{N-2} + B_{U_\gamma} \rho^N.\]
Using (3.34) we obtain \( E(\psi_\rho) \big|_{\rho=1} = A_{U_\gamma} + B_{U_\gamma} = E(U_\gamma) < 0 \) and so (3.35) \( B_{U_\gamma} < 0. \)

Using (3.33) and (3.35), we then get:

\[
\frac{d}{dp} [E(\psi_\rho)]_{\rho=1} = \left[(N-2)\rho^{N-3}A_{U_\gamma} + NB_{U_\gamma}\right]_{\rho=1} = (N-2)A_{U_\gamma} + NB_{U_\gamma} = (N-2)(A_{U_\gamma} + B_{U_\gamma}) + 2B_{U_\gamma} = (N-2)E(U_\gamma) + 2B_{U_\gamma} < 0.
\]

From last inequality it follows that \( \int U_\gamma = \gamma \), otherwise for some \( \rho > 1 \) we would have \( \int \psi_\rho = \gamma \) and \( E(\psi_\rho) < E(U_\gamma) \), this facts being in contradiction with the fact that \( U_\gamma \) is an absolute minimum in \( K_\gamma \). Thus the proof of (3.26) is accomplished.

It remains to prove (3.27). First, let us observe that the support of \( U_\gamma \) is either \( \mathbb{R}^N \) or a ball of finite radius centered at the origin. Namely, \( U_\gamma^* = U_\gamma \) where \( U_\gamma^* \) is the Schwartz’s symmetrization, and therefore \( U_\gamma \) is radially symmetric. By our assumptions \( U_\gamma \) satisfies equation (3.4) in the weak sense of \( W^{1,2} \) in \( \Gamma \) (see [15, p. 43]), where \( \Gamma \) is the positivity set of \( U_\gamma \), see (3.3). By standard elliptic regularity results (e.g. [13, Theorem 9.19]), \( U_\gamma \) is of class \( C^2 \) on \( \Gamma \). Moreover, the stationarity condition for \( U_\gamma \) yields:

\[
E'(U_\gamma)(U_\gamma) = \int [(\nabla U_\gamma, \nabla U_\gamma)] + W'(U_\gamma)U_\gamma = \lambda_\gamma \int U_\gamma = \lambda_\gamma \gamma,
\]

for some \( \lambda_\gamma \in \mathbb{R} \). But \( E'(U_\gamma)(U_\gamma) = \frac{d}{dt} [E(tU_\gamma)]_{t=1} \leq 0 \), since \( U_\gamma \) is a minimum in \( K_\gamma \). Combining (3.36) with (3.37) we get readily (3.38)

\[
\lambda_\gamma \leq 0.
\]

It is easy to show that inequality (3.38) is strict, for otherwise, using (3.4) we would have \( E'(U_\gamma) = 0 \), which contradicts (3.36).

Using standard \( a \ priori \) estimates (see Proposition A.1), it is easy to show that \( U_\gamma \) is bounded from above, namely, \( 0 \leq U_\gamma \leq s_0 \). Therefore \( U_\gamma \in L^\infty(\Gamma) \), and the \( L^\infty \)-norm is bounded uniformly with respect to \( \gamma \).

By standard elliptic regularity (see for instance [13, Theorem 9.19]) \( U_\gamma \) is in \( C^2_{\text{loc}}(\Gamma) \) for every \( \alpha \). To deduce that \( U_\gamma \in C^{1,\alpha}(\mathbb{R}^N) \) we apply Theorem 3.8 with \( \Omega \) equal to \( B_{R^n}(0, R_\gamma + 1) \) (for instance) and \( f = -W'(U_\gamma) + \lambda_\gamma \), \( \psi_1 \equiv 0 \), \( \psi_2 = \|U_\gamma\|_{\infty, B_{R^n}(0, R_\gamma + 1)} \). So \( f \in L^\infty(\mathbb{R}^N) \) and for every \( 0 < \alpha < 1 \), \( \psi_1, \psi_2 \in C^{1,\alpha}(\overline{B_{R^n}(0, R_\gamma + 1)}) \) since \( \psi_1, \psi_2 \) are constants, then \( U_\gamma \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N) \), for all \( \alpha \in [0, 1] \). We recall from a result contained in [24] (see also [5, Lemma 141]) that, since \( U_\gamma \in H^{1}_{\text{rad}}(\mathbb{R}^N) \) the following estimate due to Strauss holds

\[
|U_\gamma(x)| \leq C \frac{\|U_\gamma\|_{H^{1}(\mathbb{R}^N)}}{|x|^{\frac{N+2}{2}}}, \quad \text{for a.e. } x \in \mathbb{R}^N,
\]

\(^4\)For this conclusion, it suffices to assume that \( W(s) > W(s_0) \) for \( s \) in a right neighborhood of \( s_0 \), see Remark A.2.
for some positive constant $C$. Now we argue indirectly and we assume that the support of $U_\gamma$ is $\mathbb{R}^N$, i.e., that $\Gamma = \mathbb{R}^N$, to obtain a contradiction. Fix $\alpha \in [0, +\infty[$ small enough so that $W'(s) \geq ks$ for every $s \in [0, a]$. This choice is always possible by (1.2). From (3.39) we deduce the existence of $r_0 > 0$ such that $U_\gamma(x) \leq \alpha$ if $|x| \geq r_0$, and then $W'(U_\gamma(x)) \geq 0$. Since $U_\gamma$ is radially symmetric, we can write $U_\gamma(x) = u_\gamma(|x|)$, where $u_\gamma : [0, +\infty[ \to \mathbb{R}$. Recalling that $U_\gamma \in C^{2,\alpha}(\Omega')$ for every $\Omega' \Subset \Gamma$ equation (3.40) below

\begin{equation}
- \Delta U_\gamma + W'(U_\gamma) = \lambda_\gamma,
\end{equation}

is satisfied in the classical sense, and it gives the following ordinary differential equation for $u_\gamma$:

\begin{equation}
\frac{d}{dr} \left[ r^{N-1} u_\gamma'(r) \right] = \left[ -\lambda_\gamma + W'(u_\gamma(r)) \right] r^{N-1}, \quad \forall \ r \in [0, +\infty[.
\end{equation}

Integrating (3.41) on the interval $[r_0, r]$ we get

\begin{align*}
r^{N-1} u_\gamma'(r) - r_0^{N-1} u_\gamma'(r_0) &= \frac{-\lambda_\gamma}{N} (r^N - r_0^N) + \int_{r_0}^{r} W'(u_\gamma(s)) s^{N-1} ds \\
&\geq \frac{-\lambda_\gamma}{N} (r^N - r_0^N) \geq -\frac{\lambda_\gamma}{N} r^N + c_0,
\end{align*}

where $c_0 \in \mathbb{R}$ is a constant independent of $r$. From the last inequality we see that

\begin{equation}
\left. u_\gamma'(r) \right|_{r_0} \geq -\frac{\lambda_\gamma}{N} r + c_1,
\end{equation}

where $c_1 \in \mathbb{R}$ is independent of $r$. Integrating again we get

\begin{equation}
\left. u_\gamma(r) \right|_{r_0} \geq c_2 - \frac{\lambda_\gamma}{2N} r^2,
\end{equation}

with $c_2 \in \mathbb{R}$ independent of $r$. Exploiting the fact that $\lambda_\gamma < 0$, the above equation contradicts the Strauss’s decay estimates (3.39). This contradictions shows that $\Gamma = B_{R_N}(0, R_\gamma)$, and this concludes the proof. \( \square \)

3.5. **Asymptotics for the radius $R_\gamma$.** We need to show that $R_\gamma \cong \gamma^{\frac{1}{N}}$ as $\gamma \to +\infty$. More precisely:

**Theorem 3.13.** In the notations of Theorem 3.11, there exist positive constants $\tilde{\gamma}_0 = \tilde{\gamma}_0 (N, \gamma_0, \|W\|_{[0,s_0]} , \|\|_\infty) \geq \gamma_0$, $C^- = C^- (W, N)$, and $C^+ = C^+ (W, N)$ such that the following inequalities hold:

\begin{equation}
C^- \gamma^{\frac{1}{N}} \leq R_\gamma < C^+ \gamma^{\frac{1}{N}},
\end{equation}

for all $\gamma > \tilde{\gamma}_0$.

**Remark 3.14.** The constants $C^+$ and $C^-$ in (3.44) can be estimated as follows:

\begin{equation}
C^+ = \frac{3}{2} \left( \frac{1}{s_1 \omega_N} \right)^{\frac{1}{N}},
\end{equation}

where $s_1 > 0$ is the first positive zero of $W'$:

\begin{equation}
s_1 = \min \{ s > 0 : W'(s) = 0 \},
\end{equation}

\begin{equation}
\left. u_\gamma(r) \right|_{r_0} \geq c_2 - \frac{\lambda_\gamma}{2N} r^2,
\end{equation}

where $c_2 \in \mathbb{R}$ independent of $r$. Exploiting the fact that $\lambda_\gamma < 0$, the above equation contradicts the Strauss’s decay estimates (3.39). This contradictions shows that $\Gamma = B_{R_N}(0, R_\gamma)$, and this concludes the proof. \( \square \)

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\end{equation}

where $s_1 > 0$ is the first positive zero of $W'$:

\begin{equation}
s_1 = \min \{ s > 0 : W'(s) = 0 \},
\end{equation}
and

\[(3.47) \quad C^-(W, N) = \left( \frac{1}{s_0 \omega_N} \right)^{\frac{1}{N}}. \]

By our assumptions, \(s_1 < s_0\), and therefore \(C^- < C^+\).

**Proof.** Since \(U_\gamma = U_\gamma^s\), from the definition of symmetric decreasing rearrangement that \(U_\gamma\) is nonincreasing. From this it follows that \(U_\gamma(0) = \|U_\gamma\|_\infty \leq s_0\) which implies

\[(3.48) \quad s_0 \omega_N R_\gamma^N \geq U_\gamma(0) \omega_N R_\gamma^N \geq \gamma, \]

from which the first inequality in \(3.44\) follows readily for every \(\gamma \geq \gamma_0\), with \(C^-\) given by \(3.47\).

Establishing the second inequality in \(3.44\) requires a much more involved argument, which will take the remainder of this section. Towards this goal, let us observe that, since \(0\) is a local maximum of \(U_\gamma\),

\[\Delta U_\gamma(0) = -\lambda_\gamma + W'(U_\gamma(0)) \quad \text{with} \quad -\lambda_\gamma > 0, \quad \text{and so} \quad W'(U_\gamma(0)) < 0, \]

which implies \(U_\gamma(0) > s_1 > 0\), where \(s_1\) is given in \(3.46\). Set

\[a = \sup \left\{ t \mid W'(s) \geq 0, \forall s \in [0, t] \right\} \]

and

\[\tilde{R}_\gamma = \inf \left\{ |x| : U_\gamma(x) \leq a \right\}; \]

clearly:

\[(3.49) \quad s_1 \omega_N \tilde{R}_\gamma^N < \gamma. \]

We now want to estimate the real number \(z = R_\gamma - \tilde{R}_\gamma\). Using elementary Taylor expansion we get:

\[a = u_\gamma(\tilde{R}_\gamma) = u_\gamma(R_\gamma) + u'_\gamma(R_\gamma) \left( \tilde{R}_\gamma - R_\gamma \right) + \frac{1}{2} u''_\gamma(\theta) \left( \tilde{R}_\gamma - R_\gamma \right)^2 \]

\[= u'_\gamma(R_\gamma) \left( \tilde{R}_\gamma - R_\gamma \right) + \frac{1}{2} u''_\gamma(\theta) \left( \tilde{R}_\gamma - R_\gamma \right)^2 \]

for some \(\theta \in [\tilde{R}_\gamma, R_\gamma[\). Our equation \(3.41\) becomes

\[\frac{d}{dr} \left[ r^{N-1} u'_\gamma(r) \right] = \left[ -\lambda_\gamma + W'(u_\gamma(r)) \right] r^{N-1}, \]

i.e.,

\[u''_\gamma(r) + \frac{N-1}{r} u'_\gamma(r) = -\lambda_\gamma + W'(u_\gamma(r)) > -\lambda_\gamma > 0, \]

whenever \(r \in [\tilde{R}_\gamma, R_\gamma[\). Since \(u'_\gamma(r) \leq 0\) for \(r \in [\tilde{R}_\gamma, R_\gamma[\) (this is a property of symmetric rearrangements), it follows that

\[(3.50) \quad U_\gamma''(r) > -\lambda_\gamma, \quad \forall r \in [\tilde{R}_\gamma, R_\gamma[. \]
We need to give an estimate for a positive lower bound \( \lambda_2 \geq \frac{1}{w^*_2} > 0 \). Towards this goal, we consider the following comparison function \( \hat{v}_\gamma(x) = \hat{v}_\gamma(|x|) \), where \( \hat{v}_\gamma : [0, +\infty[ \to \mathbb{R} \) is the piecewise affine function defined by:

\[
\hat{v}_\gamma(r) = \begin{cases} 
  s_0, & \text{if } r \in [0, t_0]; \\
  s_0 - s_0(r - t_0), & \text{for } r \in [t_0, t_0 + 1]; \\
  0, & \text{if } r > t_0 + 1,
\end{cases}
\]

with the constant \( t_0 > 0 \) suitably defined. It is easy to check that \( v_\gamma \in W^{1,2}_0(\mathbb{R}^N) \), that we can choose \( 0 < t_0 = t_0(\gamma) = c_1 \gamma \frac{N}{2} \), with \( c_1 = (\frac{4\omega_N s_0}{3})^{-\frac{1}{4}} \), so that

\[
\frac{1}{2}\gamma \leq \int_{\mathbb{R}^N} v_\gamma \, dx = \omega_N s_0 t_0^N \left( 1 + \frac{1}{2} \left[ (1 + \frac{1}{t_0})^N - 1 \right] \right) \leq \gamma,
\]

for large \( \gamma \) and \( v_\gamma \geq 0 \). Since \( U_\gamma \) is a minimizer for Problem (P\(_\gamma\)), we have \( E[U_\gamma] \leq E[v_\gamma] \). On the other hand, an explicit computation of \( E[v_\gamma] \) gives:

\[
\frac{1}{\alpha_{N-1}} E[v_\gamma] = \frac{1}{2} \int_{t_0}^{t_0 + 1} \hat{v}_\gamma'(r)^2 r^{N-1} \, dr + \int_{t_0}^0 W(\hat{v}_\gamma(r)) r^{N-1} \, dr
\]

\[
= \left( \frac{s_0^2}{2N} + \hat{w} \right) t_0^N \left[ (1 + \frac{1}{t_0})^N - 1 \right] - \frac{w}{N} t_0^N,
\]

where \( \hat{w} := \|W\|_{\infty, (0, s_0)} \) (recall that \( \alpha_{N-1} \) denotes the area of the unit ball in \( \mathbb{R}^N \)). We denote by \( \overline{E}_\gamma^* \) the last term of the above inequality, and set \( E_\gamma^* := \alpha_{N-1} \overline{E}_\gamma^* \), so that:

\[
E(U_\gamma) \leq E(v_\gamma) \leq E_\gamma^*.
\]

By (3.51), one has \( E_\gamma^* < 0 \) and, for \( \gamma \) sufficiently large, \( -E_\gamma^* \sim c_2 \gamma \) for some positive constant \( c_2 \).

We now use the classical Pohozaev identity in the bounded starshaped domain \( B_{\mathbb{R}^N}(0, R_\gamma) \) to obtain:

\[
\int_{B_{\mathbb{R}^N}(0, R_\gamma)} -\lambda_2 U_\gamma + W(U_\gamma)
\]

\[
= \left( \frac{1}{N} - \frac{1}{2} \right) \int_{B_{\mathbb{R}^N}(0, R_\gamma)} \|\nabla U_\gamma\|^2 - \frac{1}{2} \int_{\partial B_{\mathbb{R}^N}(0, R_\gamma)} (x \cdot \nu) \left( \frac{\partial U_\gamma}{\partial \nu} \right)^2,
\]

where \( \nu \) is the outward pointing normal unit field to the boundary of the ball. On the other hand by (3.33) and (3.53), we get

\[
E[U_\gamma] = -\int [W(U_\gamma)]^- + \int [W(U_\gamma)]^+ + \frac{1}{2} \int |\nabla U_\gamma|^2 < E_\gamma^* < 0.
\]
Combining the last two equations leads to

\[
-\lambda \gamma = \int [W(U_{\gamma})]^+ - \int [W(U_{\gamma})]^-
\]

\[
+ \left( \frac{1}{N} - \frac{1}{2} \right) \int_{B_{R_{\gamma}}(0, R_{\gamma})} \| \nabla U_{\gamma} \|^2 - \frac{1}{2} \alpha_{N-1} R_{\gamma}^N (u'_{\gamma}(R_{\gamma}))^2
\]

\[
\geq \frac{1}{N} \int_{B_{R_{\gamma}}(0, R_{\gamma})} \| \nabla U_{\gamma} \|^2 - E^*_\gamma - \frac{1}{2} \alpha_{N-1} R_{\gamma}^N (u'_{\gamma}(R_{\gamma}))^2
\]

\[
\geq -E^*_\gamma - \frac{1}{2} \alpha_{N-1} R_{\gamma}^N (u'_{\gamma}(R_{\gamma}))^2.
\]

(3.56)

Now, we claim that:

(3.57)

\[ U_{\gamma}'(R_{\gamma}) = 0, \]

for every \( \gamma \geq \gamma_0 \). This follows easily from the \( C^{1,\alpha} \)-regularity of \( U_{\gamma} \), keeping in mind that \( U_{\gamma}(r) = 0 \) for \( r > R_{\gamma} \). Combining (3.56) with (3.57), and taking \( \gamma \) large enough we get

(3.58)

\[ -\lambda \gamma \geq w^*_\gamma := -\frac{E^*_\gamma}{\gamma} > 0, \]

where \( w^* > 0 \) is a positive constant that could be chosen equal to \( \frac{m\omega_N}{2} = w^* > 0 \).

Further, since \( U_{\gamma}'(R_{\gamma}) \leq 0 \) and \( \left( \bar{R}_{\gamma} - R \right) < 0 \), we obtain

\[ a = U_{\gamma}(R_{\gamma}) + U_{\gamma}'(R) \left( \bar{R}_{\gamma} - R_{\gamma} \right) + \frac{1}{2} u''(0) \left( \bar{R}_{\gamma} - R_{\gamma} \right)^2 \]

\[ \geq \frac{1}{2} w^* \left( \bar{R}_{\gamma} - R_{\gamma} \right)^2, \]

from which we deduce

\[ R_{\gamma} - \bar{R}_{\gamma} \leq \sqrt{\frac{2a}{w^*_\gamma}}, \]

i.e.,

(3.59)

\[ R_{\gamma} \leq \bar{R}_{\gamma} + \sqrt{\frac{2a}{w^*_\gamma}} \leq k\gamma^{1/N} + \sqrt{\frac{2a}{w^*_\gamma}}. \]

The second inequality here follows easily by the estimate of \( \bar{R}_{\gamma} \) given below

\[ \gamma = \int_0^{R_{\gamma}} U_{\gamma}(r) \alpha_{N-1} r^{N-1} \, dr \geq \int_0^{\bar{R}_{\gamma}} U_{\gamma}(r) \alpha_{N-1} r^{N-1} \, dr \]

\[ \geq a \int_0^{\bar{R}_{\gamma}} \alpha_{N-1} r^{N-1} \, dr = a \omega_N \bar{R}_{\gamma}^N. \]

Thus

(3.60)

\[ \bar{R}_{\gamma} \leq \left( \frac{\gamma}{a\omega_N} \right)^\frac{1}{N} = k_{N,W}\gamma^{\frac{1}{N}}, \]
with this last equation we justify (3.59) and indeed we accomplish the proof of the theorem.

\[ \Box \]

Remark 3.15. Using the observation in Remark 3.12, and the inequalities (3.51), (3.52), it is easy to see that \( \tilde{\gamma}_0 \) depends only on \( W|_{[0,s_0]} \), which means that \( \tilde{\gamma}_0 \) can be defined also for potentials \( W \) that violate the subcritical growth condition (1.6). This is an important observation in view of a multiplicity result without the assumption of the subcritical growth condition (1.6).

4. PROOF OF THE MAIN RESULTS

Consider the open sets:

\[ \Omega^+_r = \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) < r \}, \]
\[ \Omega^-_r = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \}, \]

where dist is the usual Euclidean distance of \( \mathbb{R}^N \). Let \( r > 0 \) be small enough such that both \( \Omega^+_r \) and \( \Omega^-_r \) are homotopically equivalent to \( \Omega \) via some suitable maps \( f_0 : \Omega \rightarrow \Omega^+_r \) and \( g_0 : \Omega^-_r \rightarrow \Omega \).

The existence of such \( r > 0 \) and the homotopy equivalences \( f_0, g_0 \), follows from the assumption that \( \partial \Omega \) is Lipschitz.

A proof of our results is obtained by applying Theorems 2.4 and 2.9 to the following setup, recalling the constants \( \tilde{\gamma}_0 \) (Theorem 3.11), and \( C^+ \) (Theorem 3.13):

\( M = M_{V^c} \), where

\[ M_{V^c} = \left\{ u \in H^1_0(\Omega) : \int_{\Omega} u(x) \, dx = V \right\}, \]

with

\[ V \leq V_1 := \min \left\{ \left( \frac{r}{C^+} \right)^N, \left( \frac{r}{R\tilde{\gamma}_0} \right)^N \right\} = \left( \frac{r}{C^+} \right)^N, \]

\( J = E_{\epsilon|\Omega} \), where

\[ E_{\epsilon}(u) = \frac{\epsilon^2}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} W(u(x)) \, dx, \]

with

\[ \epsilon \leq \epsilon_1(V) := \left( \frac{V}{\tilde{\gamma}_0} \right)^\frac{1}{N}; \]

\( X = \Omega \); \( f = \Phi_{\epsilon,V} \circ f_0 : \Omega \rightarrow E^c_{\epsilon} \cap M_{V^c} \), where \( c = \epsilon^N E(\cup_{V/\epsilon^N}) \) (see (3.25)) and

\( \Phi_{\epsilon,V} : \Omega^-_r \rightarrow E^c_{\epsilon} \cap M_{V^c} \)

is the map \( \Omega^-_r \ni x_0 \mapsto \Phi_{\epsilon,V}^{x_0} \in E^c_{\epsilon} \cap M_{V^c} \) defined by

\[ \Phi_{\epsilon,V}^{x_0}(x) = \cup_{V/\epsilon^N} \left( \frac{x - x_0}{\epsilon} \right), \quad x \in \Omega. \]
\( g = g_0 \circ \beta : E_{\varepsilon} \cap M^V \rightarrow \Omega, \) where \( \beta : E_{\varepsilon} \cap M^V \rightarrow \Omega^+ \) is the map defined as follows

\[
\beta(u) := \int_{\Omega} \frac{|u(x)| \, \mathrm{d}x}{\int_{\Omega} |u(x)| \, \mathrm{d}x}.
\]

Note that:

\[
\beta \left( x_{\varepsilon, V} \right) = x_0, \quad \forall \varepsilon, V, x_0.
\]

Let us now show that all the above objects are well defined, and that, using this framework, the assumptions of Theorems 2.4 and 2.9 are satisfied.

**Lemma 4.1.** For every \( \epsilon > 0 \) and \( V > 0 \), the functional \( E_{\epsilon} \lvert_{M^V} \) satisfies the Palais-Smale condition.

**Proof.** Assume that \((u_n)\) is a Palais-Smale sequence at level \( c \). Observe that, writing equations (2.1) and (2.2) explicitly, we get:

\[
\frac{\epsilon^2}{2} \int_{\Omega} |\nabla u_n|^2 \, \mathrm{d}x + \int_{\Omega} W(u_n(x)) \, \mathrm{d}x \rightarrow c, \quad \text{as } n \rightarrow \infty,
\]

\[
-\epsilon^2 \Delta u_n + W'(u_n) = \lambda_n + T_n,
\]

where \( \lim_{n \rightarrow \infty} T_n = 0 \) strongly in \( H^{-1}(\Omega) \). Then by (4.7) and the assumptions (1.2), (1.6), we obtain

\[
c + 1 \geq \frac{\epsilon^2}{2} \int_{\Omega} |\nabla u_n|^2 \, \mathrm{d}x + \int_{\Omega} W(u_n(x)) \, \mathrm{d}x
\]

\[
\geq \frac{\epsilon^2}{2} \int_{\Omega} |\nabla u_n|^2 \, \mathrm{d}x - k \int_{\Omega} u \, \mathrm{d}x
\]

\[
= \frac{\epsilon^2}{2} \int_{\Omega} |\nabla u_n|^2 \, \mathrm{d}x - kV.
\]

Then \( |\nabla u_n| \) is bounded in \( L^2 \) and hence by the Poincaré inequality, \( u_n \) is bounded in \( H^1_0(\Omega) \), so there exists \( u \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u \). We have to show that \( u_n \rightarrow u \) strongly in \( M^V_{\epsilon,c} \). It is well known (Nemytskii’s theorem) that by (1.6), the map

\[
W' : H^1_0(\Omega) \ni u \mapsto W \circ u \in H^{-1}(\Omega)
\]

is a compact nonlinear operator. Thus \( W'(u_n) \rightarrow W'(u) \) strongly in \( H^{-1}(\Omega) \). Multiplying (4.8) by \( u_n \) and using the constraints \( \int u = V \) we get that \( \lambda_n \) is a bounded sequence. So, up to a subsequence, we can assume that \( \lambda_n \rightarrow \lambda \).

Now, recalling that \( \Delta^{-1} : H^{-1}(\Omega) \rightarrow H^1_0(\Omega) \) is an isomorphism, we obtain that

\[
u_n = \frac{1}{\varepsilon^2}(-\Delta^{-1}) \left[ \lambda_n - W'(u_n) + T_n \right]
\]

is a convergent sequence in \( M^V \). This concludes the proof. \( \square \)
For any open set $U \subseteq \mathbb{R}^N$, we denote by $\overline{U}$ its closure, and we define
\[
\mathcal{M}^V(U) = \left\{ u \in H^1(\mathbb{R}^N) : \text{supp}(u) \subseteq \overline{U}, \int u = V \right\}.
\]
Moreover, we set:
\[
\begin{align*}
m(\epsilon, \rho, V) &= \inf \left\{ E_\epsilon(u) : u \in \mathcal{M}^V(B_\rho(0)), u \geq 0 \right\}, \\
m^*(\epsilon, \rho, V) &= \inf \left\{ E_\epsilon(u) : u \in \mathcal{M}^V(\mathbb{R}^N \setminus B_\rho(0)), \beta(u) = 0, u \geq 0 \right\},
\end{align*}
\]
where $\beta(u)$ is defined in (4.5).

**Lemma 4.2.** For every $V \in ]0, V_1[$, $\epsilon \in ]0, \epsilon_1(V)[$, and for all $\rho > 0$, the following inequality holds:
\[
m^*(\epsilon, \rho, V) > m(\epsilon, \epsilon R_\gamma, V),
\]
where $\gamma = \frac{V}{\epsilon^N}$ and $R_\gamma$ is the radius of the closed ball that supports $U_\gamma$, see (3.27).

**Proof.** Let $\gamma_0 > 0$ be as in Theorem 3.11; by (3.27), for all $\rho > 0$ and for all $\gamma \geq \gamma_0$, the following holds:
\[
E(U_\gamma) = \min \left\{ E(u) : u \in \mathcal{M}^V(B_{R_\gamma}(0)), u \geq 0 \right\} \\
= \min \left\{ E(u) : u \in \mathcal{M}^V(\mathbb{R}^N), u \geq 0 \right\} \\
\leq \min \left\{ E(u) : u \in \mathcal{M}^V(\mathbb{R}^N \setminus B_\rho(0)), \beta(u) = 0, u \geq 0 \right\}.
\]
Next, we will show that this inequality is strict. We argue indirectly and we assume that $w$ is a minimizer of $E$ over the set
\[
\left\{ u \in \mathcal{M}^V(\mathbb{R}^N \setminus B_\rho(0)) : \beta(u) = 0, u \geq 0 \right\},
\]
and that
\[
E(U_\gamma) = E(w).
\]
Thus, $w$ is a map with barycenter at $0$, and with support contained in the exterior of a ball centered at $0$. Denote by $w^*$ the symmetric decreasing rearrangement of $w$ in $\mathbb{R}^N$, see Definition 3.3. Clearly, $\beta(w^*) = 0$, because $w^*$ is radially symmetric, and $w^* \neq w$, because the support of a decreasing rearrangement is always a ball centered at the origin. By Lemma 3.4, $E(w^*) \leq E(w)$. We cannot have $E(w^*) = E(w)$, because if such equality holds, then by Theorem 3.5 (whose application is allowed by the fact that a classical result of Gidas-Ni-Nirenberg, i.e., Theorem 3.7 ensures the validity of (3.6)) we would have $w^* = w(\cdot + x_0)$, and so $\beta(w^*) = \beta(w(\cdot + x_0)) = x_0 = 0$, which contradicts the fact that $w \neq w^*$. This implies $E(w^*) < E(w)$, which gives the following contradiction:
\[
E(U_\gamma) = E(w) > E(w^*) \geq E(U_\gamma),
\]
and therefore it shows that:
\[
m^*(1, \rho, \gamma) > m(1, R_\gamma, \gamma).
\]
Given \( u \in H^1(\mathbb{R}^N) \), set \( u_\varepsilon(x) := u(\varepsilon x) \). It is immediate to see that \( E_\varepsilon(u) = \varepsilon^N E(u_\varepsilon) \), which implies immediately:

\[
(4.13) \quad m(\varepsilon, r, V) = \varepsilon^N m\left(1, \frac{r}{\varepsilon}, \frac{V}{\varepsilon^N}\right) \quad \text{and} \quad m^*(\varepsilon, r, V) = \varepsilon^N m^*\left(1, \frac{r}{\varepsilon}, \frac{V}{\varepsilon^N}\right).
\]

Set \( \gamma = \frac{V}{\varepsilon^N} \); by our choice of \( \varepsilon_1 \), it is \( \gamma \geq \overline{\gamma}_0 \), thus inequality (4.11) follows readily from (4.12) and (4.13). Namely:

\[
m^*(\varepsilon, r, V) \overset{(4.13)}{=} \varepsilon^N m^*\left(1, \frac{r}{\varepsilon}, \frac{V}{\varepsilon^N}\right) \overset{(4.12)}{=} \varepsilon^N m\left(1, R_\gamma, \frac{V}{\varepsilon^N}\right) \overset{(4.13)}{=} m(\varepsilon, \varepsilon R_\gamma, V). \tag{□}
\]

**Remark 4.3.** From Theorem 3.11 it is easy to check that \( m(1, \rho, \gamma) = E(U_\gamma) \) for every \( \rho \geq R_\gamma \).

**Lemma 4.4.** Given \( \varepsilon \in ]0, \varepsilon_1[ \), \( V \in ]0, V_1[ \), and setting:

\[
(4.14) \quad c = c(\varepsilon, V, N, W) = m(\varepsilon, \varepsilon R_\gamma, V),
\]

where \( \gamma = \frac{V}{\varepsilon^N} \) and \( R_\gamma \) is as in (3.27), then \( E_\varepsilon \cap \mathcal{M}^V \) in nonempty, and the map \( f: \Omega \to E_\varepsilon \cap \mathcal{M}^V \) is well defined.

**Proof.** By (4.4), \( \gamma > \overline{\gamma}_0 \), and we obtain:

\[
\varepsilon \cdot R_\gamma \overset{(3.44)}{<} \varepsilon \cdot C^+ \cdot \gamma \overset{\mathcal{H}}{\leq} C^+ \cdot V_1^\gamma \overset{(4.3)}{<} r.
\]

From this inequality and the definition of \( \Phi_{\varepsilon, \gamma} \), it is immediate to deduce that

\[
\text{supp} \left( \Phi_{\varepsilon, \gamma}^{x_0, \gamma} \right) = \overline{B}_{\mathbb{R}^N}(x_0, \varepsilon R_\gamma) \subseteq \Omega
\]

for every \( x_0 \in \Omega^+ \). Now, using an elementary change of variables in the integrals we obtain:

\[
E_\varepsilon\left( \Phi_{\varepsilon, \gamma}^{x_0, \gamma} \right) = \frac{\varepsilon^2}{2} \int_{\Omega} \left| \nabla_x U_\gamma \left( \frac{x-x_0}{\varepsilon} \right) \right|^2 dx + \int_{\Omega} W\left(U_\gamma \left( \frac{x-x_0}{\varepsilon} \right) \right) dx
\]

\[
= \frac{\varepsilon^2}{2} \int_{B_{\mathbb{R}^N}(0, R_\gamma)} \left| \nabla_y U_\gamma(y) \right|^2 \varepsilon^{N-2} dy + \int_{B_{\mathbb{R}^N}(0, R_\gamma)} W(U_\gamma(y)) \varepsilon^N dy
\]

\[
= \varepsilon^N E[U_\gamma] = \varepsilon^N m(1, R_\gamma, \gamma) = m(\varepsilon, \varepsilon R_\gamma, V) = c,
\]

and

\[
(4.15) \quad \int_{\mathbb{R}^N} \Phi_{\varepsilon, \gamma}^{x_0, \gamma} dx = \varepsilon^N \int_{\mathbb{R}^N} U_\gamma dx = \varepsilon^N \gamma = V.
\]

Hence, \( \Phi_{\varepsilon, \gamma}^{x_0, \gamma} \in E_\varepsilon \cap \mathcal{M}^V \), (in particular \( E_\varepsilon \cap \mathcal{M}^V \neq \emptyset \)) and we are done. \( \tag{□} \)

**Lemma 4.5.** For \( V \in ]0, V_1[ \), \( \varepsilon \in ]0, \varepsilon_1(V)\), the function \( g: E_\varepsilon \cap \mathcal{M}^V \to \Omega \) is well defined, i.e., if \( u \in \mathcal{M}^V \), \( E_\varepsilon(u) \leq c = m(\varepsilon, \varepsilon R_\gamma, V) \) where \( \gamma = \frac{V}{\varepsilon^N} \), we have \( \beta(u) \in \Omega^+ \).
Proof. Let us argue by contradiction, assuming that there exists \( \mathfrak{u} \in E_\varepsilon \cap \mathcal{M}_V \) such that \( x := \beta(\mathfrak{u}) \notin \Omega_V^+ \). Then, \( \Omega \subset \mathbb{R}^N \setminus B_r(\bar{x}) \), and therefore \( m^*(\varepsilon, r, V) \leq E_\varepsilon(\bar{u}) \leq c = m(\varepsilon, \varepsilon R \gamma, V) \).

This contradicts (4.11), and concludes the proof. \( \square \)

We are now ready to finalize the proof of our main results.

Proof of Theorem 1.3. It is sufficient to verify assumptions (i), (ii), (iii) of Theorems 2.4 and 2.9 in our variational framework. For assumption (i) see (1.10). Assumption (ii) follows from Lemma 4.1. Assumptions (iii) follows from Lemmas 4.4 and 4.5. As to the last statement of Theorem 2.4, note that \( \mathcal{M}_V \) is contractible. Namely, it is an affine (closed) subspace of \( H_0^1(\Omega) \), see (4.2). \( \square \)

Proof of Proposition 1.4. For every \( V \in [0, V_1[ \) and all \( \varepsilon \in ]0, \varepsilon(V_1)[ \), the solution of problem (P\( _V,\varepsilon \)) found in the energy sublevel \( m(\varepsilon, \varepsilon R \gamma(\varepsilon, V), V) \), where \( \gamma(\varepsilon, V) = V/\varepsilon^N \), recall formula (4.10). Thus, a proof of Proposition 1.4 is obtained by showing that

\[
\limsup_{\varepsilon \to 0} m(\varepsilon, \varepsilon R \gamma(\varepsilon, V), V) < +\infty.
\]

This follows readily from the very definition of \( m(\varepsilon, \rho, V) \), see (4.10), observing that, by (3.44), the quantity \( \varepsilon R \gamma(\varepsilon, V) \) is bounded as \( \varepsilon \to 0 \). In the nondegenerate case, the statement about the boundedness of the Morse index of the low energy solutions follows readily from the observation in Remark 2.10. \( \square \)

Appendix A. Auxiliary results: a priori estimates

For the reader’s convenience, in this appendix we give the statement and a short proof of some a priori estimates for solutions of elliptic PDE’s, that were used in the paper.

Let us consider the elliptic PDE:

(A.1) \[-\Delta u + G'(u) = 0\]

on a bounded domain \( \Omega \subset \mathbb{R}^n \), with \( G: \mathbb{R} \to \mathbb{R} \) a function of class \( C^2 \) satisfying \( G(0) = 0 \) and

(A.2) \[ |G'(s)| \leq A + B|s|^{p-1}, \]

for some positive constants \( A, B \) and for some \( p < \frac{2n}{n+2} \).

A weak solution \( u \in H_0^1(\Omega) \) of (A.1) is a critical point\(^5\) of the functional \( E: H_0^1(\Omega) \to \mathbb{R} \) defined by:

(A.3) \[ E(u) = \int_\Omega \left[ \frac{1}{2}||\nabla u||^2 + G(u) \right] \, dx, \]

\(^5\)By standard elliptic regularity, such a weak solution \( u \) belongs to \( H^3(\Omega) \).
i.e., it satisfies:
(A.4)
\[ dE(u)v = \int_\Omega [\nabla u \cdot \nabla v + G'(u)v] \, dx = 0, \quad \forall v \in H^1_0(\Omega). \]
Assumption (A.2) implies that \( E \) is a well defined \( C^1 \)-functional in \( H^1_0(\Omega) \) and that \( dE(u) \) in (A.4) is a bounded linear operator on \( H^1_0(\Omega) \).

**Proposition A.1.** Let \( u \in H^1_0(\Omega) \) be a solution of (A.4). Assume that there exists \( s_- < 0 \)
(A.5)
\[ G'(s) < 0 \quad \text{for} \ s \leq s_- \]
Then,
\[ s_- \leq u \quad \text{a.e. in} \ \Omega. \]
Similarly, if there exists \( s_+ > 0 \) such that
(A.6)
\[ G'(s) > 0, \quad \text{for} \ s \geq s_+, \]
then
\[ u \leq s_+, \quad \text{a.e. in} \ \Omega. \]

**Proof.** For \( v: \Omega \to \mathbb{R} \), denote by \( v^+ \) and \( v^- \) the nonnegative functions defined by
\[ v^+(x) = \max \{ -v(x), 0 \}, \quad v^-(x) = \max \{ v(x), 0 \}. \]
Define:
\[ \Omega_- = \{ x \in \Omega : u \leq s_- \}, \quad \Omega_+ = \{ x \in \Omega : u \geq s_+ \}. \]
Set \( v = (u - s_-)^-; \) since \( u \in H^1_0(\Omega) \), then also \( v \in H^1_0(\Omega) \). Plugging such \( v \) in (A.4), we get:
\[ 0 = \int_\Omega \nabla u \cdot \nabla (u - s_-)^- + G'(u)(u - s_-)^- \, dx = \int_{\Omega_-} ||\nabla u||^2 + G'(u)(u - s_-) \, dx. \]
If \( u \leq s_- \) on a set of positive measure, then by (A.5) the last integral is strictly positive, giving a contradiction. Thus, \( u \geq s_- \) almost everywhere.

Similarly, now plug \( v = (u - s_+)^+ \) into (A.4):
\[ 0 = \int_\Omega \nabla u \cdot \nabla (u - s_+)^+ + G'(u)(u - s_+)^+ \, dx = \int_{\Omega_+} ||\nabla u||^2 + G'(u)(u - s_+) \, dx. \]
If \( u \geq s_+ \) on a set of positive measure, then by (A.6) the last integral is strictly positive, giving a contradiction. Thus, \( u \leq s_+ \) almost everywhere, which concludes the proof. \( \square \)

**Remark A.2.** The assumptions of Proposition A.1 can be somewhat weakened if one wants to obtain bounds only for solutions \( u \) of (A.4) that are *minima* of the corresponding energy functional \( E \) in (A.3). Namely, in order to conclude that \( s_- \leq u \), it is not necessary to assume (A.5). It suffices to assume that \( G(s) > G(s_-) \) for \( s_- \) in a left neighborhood of \( s_- \), for in this case, if \( u < s_- \) somewhere, then the function \( u^- \in H^1_0(\Omega) \) defined by \( u^-(x) = \max \{ u(x), s_- \} \) would satisfy \( E(u^-) < E(u) \), contradicting the minimality assumption for \( u \). Similarly, in order to conclude that \( u \leq s_+ \) it suffices to assume that \( G(s) > G(s_+) \) for \( s \) in a right neighborhood of \( s_+ \).
Let us now consider the eigenvalue equation on $\Omega$:

\[ -\varepsilon^2 \Delta u + W'(u) = \lambda, \]

for some $\lambda \in \mathbb{R}$.

**Proposition A.3.** Let $\lambda \in \mathbb{R}$ and $u \in H^1_0(\Omega)$ be a (weak) solution of (A.7), with $\int_\Omega u \, dx > 0$, and satisfying $u \leq s_+$ for some $s_+ > 0$. Then:

\[ \lambda \geq w^- := \min_{s \in [0, s_+]} W'(s). \]

**Proof.** The function $u$ satisfies:

\[ \varepsilon^2 \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega W'(u) v \, dx = \lambda v, \quad \forall v \in H^1_0(\Omega). \]

Denote by $\Omega^+ = \{ x \in \Omega : u(x) \geq 0 \}$ and observe that $|\Omega^+| > 0$, because $\int_\Omega u \, dx > 0$. Setting $v = u^+$ in (A.8) we get:

\[ \lambda \int_{\Omega^+} u \, dx = \varepsilon^2 \int_{\Omega^+} \| \nabla u \|^2 \, dx + \int_{\Omega^+} W'(u) u \, dx \geq \int_{\Omega^+} W'(u) u \, dx \geq w^- \int_{\Omega^+} u \, dx. \]

The conclusion follows readily. \qed

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