Delta shocks in the relativistic full Euler equations for a Chaplygin gas

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Abstract

The relativistic full Euler equations for a Chaplygin gas are studied. The Riemann problem is solved constructively. There are two kinds of Riemann solutions, in which one consists of three contact discontinuities and the other involves a delta shock wave on which both state variables the rest mass density and the proper energy density simultaneously contain the Dirac delta functions. It is quite different from the previous ones on which only one state variable contains the Dirac delta function. The formation mechanism, generalized Rankine-Hugoniot relation and entropy condition are clarified for this type of delta shock wave. Under the generalized Rankine-Hugoniot relation and entropy condition, the existence and uniqueness of delta shock solutions are also established.

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1. Introduction

The relativistic Euler equations of the conservation laws of baryon numbers, momentum and energy reads (see [8, 27])

\[
\begin{align*}
\left(\frac{n}{\sqrt{1-v^2/c^2}}\right)_t + \left(\frac{n v}{\sqrt{1-v^2/c^2}}\right)_x &= 0, \\
\left(\frac{(p/c^2+p)u}{1-v^2/c^2}\right)_t + \left(\frac{(p/c^2+p)v^2}{1-v^2/c^2} + p\right)_x &= 0, \\
\left(\frac{(p/c^2+p)v^2/c^2}{1-v^2/c^2} + \rho\right)_t + \left(\frac{(p/c^2+p)v}{1-v^2/c^2}\right)_x &= 0,
\end{align*}
\]

where \(n, \rho, p\) and \(v\) represent the rest mass density, the proper energy density, the pressure and the particle speed, respectively, and the constant \(c\) is the speed of light.

In his fundamental work of 1948, Taub [42] derived system (1.1) and then obtained the Hugoniot curve of the relativistic shocks, and also showed that \(\gamma\), the ratio of specific heats, must be less than \(\frac{5}{3}\). He gave a more systematic description of relativistic hydrodynamics in his later work [43]. In 1986,
Thompson [44] established several relations on the relativistic shock curves. He observed that “the relativistic shock equations are much more complicated and do not lend themselves to expressions that are both simple and general”. Since the high complexity of the system itself, up to now, there are few results for this system in the literature. Chen [8] solved the Riemann problem to system (1.1) for the polytropic gas with the equations of state

\[ p = (\gamma - 1)c^2(\rho - n) \]  

and

\[ p = kSn^\gamma. \]  

Recently, its vanishing pressure limit problem was studied by Yin and Sheng [48]. Besides, Geng and Li [15] studied the non-relativistic global limits of the entropy solutions to the Cauchy problem of system (1.1) for the isothermal flow

\[ p = k\rho. \]  

Ding [14] proved the global stability of the strong rarefaction wave to 1-D piston problem of system (1.1) for the polytropic gas with the equations of state

\[ p = (\gamma - 1)c^2(\rho - n) \]  

and

\[ p = kSn^\gamma. \]  

Here, we concern with the equation of state is

\[ p = -\frac{1}{\rho}, \]  

which was introduced by Chaplygin [5], Tsien [45] and von Karman [19] as a suitable mathematical approximation for calculating the lifting force on a wing of an airplane in aerodynamics. A gas is called a Chaplygin gas if it satisfies the equation of state (1.2). The Chaplygin gas owns a negative pressure and occurs in certain theories of cosmology. Such a gas has been advertised as a possible model for dark energy [1, 16].

In recent years, astrophysicists have growing interests in the Chaplygin gas dynamics, which replaces the polytropic equation of state

\[ p(\rho) = \rho^\gamma (\gamma > 1) \]  

(e.g., see [6-7, 18, 37, 40]) with

\[ p(\rho) = -\rho^{-1}. \]  

The typical feature of the Chaplygin gas dynamics is that the \( \delta \)-shocks appear in non-zero pressure cases. The Riemann problem was solved for the nonrelativistic case by Brenier [2] and Serre [35], relativistic case by Cheng and Yang [10], followed by its vanishing pressure limit problem by Yin and Song [49].

In this paper, we are interested in the Riemann problem for (1.1) and (1.2) with Riemann initial data

\[ (n, \rho, v)(0, x) = \begin{cases} (n_-, \rho_-, v_-), & x < 0, \\ (n_+, \rho_+, v_+), & x > 0, \end{cases} \]  

where \( \rho_{\pm}, n_{\pm}, v_{\pm} \) are given constant states. The Riemann problem is a special initial value problem where the initial data consisting of two piecewise constant states are separated by a jump discontinuity at the origin for the one-dimensional hyperbolic systems of conservation laws. It is well known that the Riemann problem is the most fundamental problem in the field of nonlinear hyperbolic conservation laws. Theories of hyperbolic systems of conservation laws can be found in [3-4, 11, 25, 27, 36, 39] etc.

For the Chaplygin gas, the considered relativistic full Euler equations possess three linearly degenerate characteristic fields, thus the classical elementary waves only involves contact discontinuities. The rarefaction wave curves and the shock wave curves are actually coincided to the so-called contact discontinuities in the state space. Although the system is much more complicated and the results are much harder to obtain, with the help of the contact discontinuity curves, by the analysis on the physically relevant region and the method of characteristic analysis, we construct Riemann solutions only involving contact discontinuities when

\[ v_{+} + \frac{c_+}{\sqrt{1 + \frac{\rho_+}{\rho_{-}c_+^2}}} > v_{-} - \frac{c_-}{\sqrt{1 + \frac{\rho_-}{\rho_{+}c_-^2}}} \]  

However, for the case

\[ v_{+} + \frac{c_+}{\sqrt{1 + \frac{\rho_+}{\rho_{-}c_+^2}}} \leq v_{-} - \frac{c_-}{\sqrt{1 + \frac{\rho_-}{\rho_{+}c_-^2}}} \]

we find that the Riemann solution can not be constructed by these classical contact discontinuities and
delta shocks should occur. In this case, we rigorously analyze the formation of mechanism for delta shock wave with Dirac delta function in the state variables the rest mass density and the proper energy density. By the definition of the delta shock wave solution to (1.1) and (1.2) in the sense of distributions, we propose the generalized Rankine-Hugoniot relation and entropy condition for this type of delta shock wave. Thus both existence and uniqueness of delta shock wave solutions can be obtained by solving the generalized Rankine-Hugoniot relation under entropy condition.

In this work, it is proven that the delta shock wave with Dirac delta function in both state variables the rest mass density and the proper energy density develops in solutions of the relativistic full Euler equations for the Chaplygin gas. It is quite different from the previous ones on which only one state variable contains the Dirac delta function. To our knowledge, this type of delta shock wave has not been found in the previous studies on the relativistic Euler equations. For related researches of delta shock waves, we refer the readers to [6, 9-10, 12-13, 17, 20-24, 26, 28-35, 37-38, 41, 46-49] and the references cited therein for more details. For the theory of the delta shock wave with Dirac delta function in multiple state variables, interested readers may refer to [9, 30-33, 46-47] for further details. Besides, substantially different from the works [9, 30-31, 46-47], where the delta shock wave with Dirac delta function in multiple state variables has been found only in some non-strictly hyperbolic systems of conservation laws, we find this type of delta shock wave in a linearly degenerate and strictly hyperbolic systems of conservation laws.

The rest of this paper is organized as follows. In Sections 2, we first clarify the physically relevant region where we can present classical solutions and delta shock waves, and deduce the classical contact discontinuity curves, then construct Riemann solutions only involving the classical contact discontinuities. In Section 3, we analyze the formation of mechanism for delta shock wave with Dirac delta function in both the rest mass density and the proper energy density. We also propose the generalized Rankine-Hugoniot and entropy condition for this type of delta shock wave and then prove the existence and uniqueness of delta shock wave solutions under the generalized Rankine-Hugoniot relation and entropy condition.

2. Preliminaries and classical Riemann solutions

In this section, we present some preliminary knowledge for system (1.1) and construct classical Riemann solutions of (1.1)-(1.2) with initial data (1.3). The physically relevant region for solutions is

$$\Lambda = \left\{(n, \rho, v)|n > 0, \rho > \frac{1}{c}, |v| < c\right\}, \quad (2.1)$$

that is, the sonic speed $\sqrt{p'(\rho)}$ should be strictly less than the speed of light (see [8]).

For any smooth solution, system (1.1) with (1.2) can be written in matrix form

$$A \begin{pmatrix} n \\ \rho \\ v \end{pmatrix}_t + B \begin{pmatrix} n \\ \rho \\ v \end{pmatrix}_x = 0, \quad (2.2)$$
Then Riemann problem (1.1), (1.2) and (1.3) is reduced to the following boundary value problem of

where

\[ A = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & \frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} \\ 0 & \frac{(\frac{1}{\sqrt{1-v^2/c^2}} - \frac{1}{\sqrt{1-v^2/c^2}})}{1-v^2/c^2} & \left(\frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \\ 0 & \frac{1}{1-v^2/c^2} & \left(\frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \end{pmatrix}, \tag{2.3} \]

and

\[ B = \begin{pmatrix} \frac{v}{\sqrt{1-v^2/c^2}} & 0 & \frac{n}{(1-v^2/c^2)^{3/2}} \\ 0 & \frac{v^2 + \frac{1}{\rho}}{1-v^2/c^2} & \left(\frac{n}{(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \\ 0 & \frac{1}{1-v^2/c^2} & \left(\frac{n}{(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \end{pmatrix}. \tag{2.4} \]

It follows from (2.3) and (2.4) that

\[ A^{-1}B = \begin{pmatrix} v & \frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} & \frac{n}{(1-v^2/c^2)^{3/2}} \\ 0 & \frac{v^2 + \frac{1}{\rho}}{1-v^2/c^2} & \frac{n}{(1-v^2/c^2)^{3/2}} \\ 0 & \frac{1}{1-v^2/c^2} & \frac{n}{(1-v^2/c^2)^{3/2}} \end{pmatrix}. \tag{2.5} \]

By (2.5), it is not difficult to see that system (1.1) with (1.2) has three real and distinct eigenvalues

\[ \lambda_1 = \frac{v - \frac{1}{\rho}}{1 - \frac{v^2}{c^2}}, \quad \lambda_2 = \frac{v + \frac{1}{\rho}}{1 - \frac{v^2}{c^2}}, \quad \lambda_3 = \frac{1}{\rho}. \tag{2.6} \]

with the corresponding right eigenvectors

\[ \varphi_1 = \left(\frac{\rho - \frac{1}{\rho}}{\rho - \frac{1}{\rho}}, \frac{-n}{\rho - \frac{1}{\rho}}, \frac{1}{\rho - \frac{1}{\rho}}\right)^T, \quad \varphi_2 = (1, 0, 0)^T, \quad \varphi_3 = \left(\frac{n}{1 - v^2/c^2}, \frac{1}{1 - v^2/c^2}, \frac{1}{1 - \rho}\right)^T, \tag{2.7} \]

satisfying

\[ \nabla \lambda_i \cdot \varphi_i = 0 \quad (i = 1, 2, 3). \]

Therefore, system (1.1) with (1.2) is strictly hyperbolic and fully linearly degenerate, and the associated waves are contact discontinuities.

Since system (1.1) with (1.2) and the Riemann data (1.3) are invariant under stretching of coordinates: \((t, x) \rightarrow (\alpha t, \alpha x)\) \((\alpha \text{ is a constant})\), we seek the self-similar solution

\[ (n, \rho, v)(t, x) = (n, \rho, v)(\xi), \quad \xi = \frac{x}{t}. \]

Then Riemann problem (1.1), (1.2) and (1.3) is reduced to the following boundary value problem of ordinary differential equations:

\[ \begin{cases} -\xi \left(\frac{n}{\sqrt{1-v^2/c^2}}\right) \frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} + \left(\frac{\sqrt{n}}{c^2(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \xi = 0, \\
-\xi \left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) + \left(\frac{n}{(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \xi = 0, \\
-\xi \left(\frac{n}{(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) + \left(\frac{n}{(1-v^2/c^2)^{3/2}} + \frac{1}{\sqrt{1-v^2/c^2}}\right) \xi = 0, \end{cases} \tag{2.8} \]
with \((n, \rho, v)(\pm \infty) = (n_{\pm}, \rho_{\pm}, v_{\pm})\).

For any smooth solution, system (2.8) can be rewritten as

\[
\begin{pmatrix}
\frac{v-\xi}{\sqrt{1-v^2/c^2}} & 0 & \frac{nc^2-nv\xi}{c^2(1-v^2/c^2)^{3/2}} \\
0 & \frac{c^2(1-v^2/c^2)-\xi^2(c^2+v^2)}{c^2-c^2v^2} & \frac{(-\rho+c^2)(\rho^2-v^2-\xi^2)}{(c^2-v^2)^2} \\
0 & \frac{(c^2-v^2)c^2v-\xi^2(c^2+v^2)}{c^2-c^2v^2} & \frac{(-\rho+v^2)(c^2-v^2+2v\xi)}{(c^2-v^2)^2}
\end{pmatrix}
\begin{pmatrix}
 dn \\
d\rho \\
dv
\end{pmatrix} = 0. \tag{2.9}
\]

It provides either the general solution (constant state)

\[(n, \rho, v) = \text{Constant},\]

or the singular solutions

\[
\begin{align*}
\xi &= \lambda_1 = \frac{v-\frac{1}{\rho}}{\rho} = 0, \\
\frac{dn}{d\rho} &= \frac{\rho}{\rho_c^2 - 1}, \\
\xi &= \lambda_2 = v, \\
d\rho &= 0, \ dv = 0, \ dn \neq 0, \\
\xi &= \lambda_3 = \frac{v+\frac{1}{\rho}}{1+\rho_c^2}, \\
\frac{dn}{d\rho} &= \frac{\rho}{\rho_c^2 - 1}.
\end{align*} \tag{2.10}
\]

Integrating (2.10) from \((n_-, \rho_-, v_-)\) to \((n, \rho, v)\) yields that

\[
\xi = \lambda_1 = \frac{v-\frac{1}{\rho}}{1-\frac{v}{\rho_c^2}} = \frac{v_--\frac{1}{\rho_-}}{1-\frac{v}{\rho_-}c^2} \quad \text{and} \quad \frac{n}{n_-} = \sqrt{\frac{(\rho c - 1)(\rho c + 1)}{\rho_- c - 1}(\rho_- c + 1)}. \tag{2.13}
\]

Similarly, we have

\[
\xi = \lambda_2 = v = v_- \quad \rho = \rho_- \quad \text{and} \quad n \neq n_- , \tag{2.14}
\]

\[
\xi = \lambda_3 = \frac{v+\frac{1}{\rho}}{1+\frac{1}{\rho_c^2}} = \frac{v_--\frac{1}{\rho_-}}{1+\frac{v}{\rho_-}c^2} \quad \text{and} \quad \frac{n}{n_-} = \sqrt{\frac{(\rho c - 1)(\rho c + 1)}{\rho_- c - 1}(\rho_- c + 1)}. \tag{2.15}
\]

For a bounded discontinuity at \(\xi = \sigma\), the Rankine-Hugoniot relation holds:

\[
\begin{align*}
-\sigma \left[ \frac{n}{\sqrt{1-v^2/c^2}} \right] + \left[ \frac{nv}{\sqrt{1-v^2/c^2}} \right] &= 0, \\
-\sigma \left[ \frac{\rho}{\rho_c^2 + 1} \right] + \left[ \frac{\rho v}{\rho_c^2 + 1} \right] &= 0, \\
-\sigma \left[ \frac{1}{\rho_c^2 + 1} \right] + \left[ \frac{1}{\rho_c^2 + 1} \right] &= 0,
\end{align*} \tag{2.16}
\]

where \([q] = q_--q_-\) is the jump of \(q\) across the discontinuity and \(\sigma\) is the velocity of the discontinuity.

Eliminating \(\sigma\) in the second and third equations of (2.16), we have

\[
\left( -\frac{1}{\rho} + \frac{1}{\rho_-} \right) \left( \rho - \rho_- \right) \left( 1 - \frac{v^2}{c^2} \right) \left( 1 - \frac{v_-^2}{c^2} \right) = \left( -\frac{1}{\rho c} + \rho \right) \left( -\frac{1}{\rho_- c} + \rho_- \right) (v - v_-)^2. \tag{2.17}
\]
Then, from (2.17) it follows that
\[
\frac{(v - v_-)^2}{(1 - \frac{v^2}{c^2})(1 - \frac{v_-^2}{c^2})} = \frac{(\rho - \rho_-)^2}{(\rho^2 - \frac{1}{c^2})(\rho_-^2 - \frac{1}{c^2})},
\]  
(2.18)
and
\[
\left(\frac{v - v_-}{c^2 - 1}\right)^2 = \frac{\left(-\frac{1}{\rho} + \frac{1}{\rho_-}\right)(\rho - \rho_-)}{\left(-\frac{1}{\rho^2} + \rho_-\right)\left(-\frac{1}{\rho_-^2} + \rho\right)}.
\]  
(2.19)
By a simple calculation, it is easy to see that (2.19) is equivalent to
\[
\frac{v - v_-}{c^2 - 1} = \pm \left(\frac{\rho - \rho_-}{\rho_- - \frac{1}{c^2}}\right).
\]  
(2.20)
Thus, we have two cases, namely,

Case 1: \(\frac{v - v_-}{c^2 - 1} = \frac{\rho - \rho_-}{\rho_- - \frac{1}{c^2}}\), which gives a 1-shock
\[
S_1 : \frac{v - v_-}{c^2 - 1} = \frac{v_- - 1}{\rho_- - \frac{1}{c^2}} \text{, with } p > p_- , \rho > \rho_-, v < v_-.
\]  
(2.21)
Hence, from (2.21) and (2.18) it follows that
\[
\sqrt{1 - \frac{v^2}{c^2}} = \frac{\rho - \rho_-}{\sqrt{\rho^2 - \frac{1}{c^2}}(\rho_-^2 - \frac{1}{c^2})}
\]  
(2.22)
and
\[
v = \frac{v_0 - a}{1 - \frac{v_0^2}{c^2}}.
\]  
(2.23)
\[
\sqrt{1 - \frac{v^2}{c^2}} = \frac{\sqrt{c^2 - a^2}}{c} \sqrt{1 - \frac{v^2}{c^2}}
\]  
(2.24)
\[
\sqrt{c^2 - a^2} = \frac{c \sqrt{\rho_-(-\frac{1}{p_- c} + \rho)(\rho - \rho_-)} - \rho_-}{\rho_- - \frac{1}{c^2}},
\]  
(2.25)
where
\[
a = \frac{\rho - \rho_-}{\rho_- - \frac{1}{c^2}}.
\]
Eliminating \(\sigma\) in the first and second equations of (2.16), we have
\[
\frac{v - v_-}{\sqrt{1 - \frac{v^2}{c^2}}(1 - \frac{v_-^2}{c^2})} = \frac{n_-v_0(-\frac{1}{\rho_- c^2} + \rho)}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{n(-\frac{1}{\rho c^2} + \rho_0)}{\sqrt{1 - \frac{v_-^2}{c^2}}} = \left(-\frac{1}{\rho} + \frac{1}{\rho_-}\right)\left(\frac{n}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{n_-}{\sqrt{1 - \frac{v_-^2}{c^2}}}\right).
\]  
(2.26)
Substituting (2.22)-(2.25) into (2.26), we get, after a straightforward calculation that
\[
\left(\frac{n_-v_0\left(\rho_- - \frac{1}{\rho_- c^2}\right)}{\sqrt{\rho_-\left(-\frac{1}{\rho c^2} + \rho\right)\left(-\frac{1}{\rho_- c^2} + \rho\right)}} - \frac{n\left(\rho_- - \frac{1}{\rho c^2}\right)}{-\frac{1}{\rho_- c^2} + \rho}\right)\left(v_0 - \frac{1}{\rho}\right)(\rho - \rho_-) = 0.
\]  
(2.27)
When \(\rho \neq \rho_-\), the second part of the left side in the above expression will not be zero if \(v_- \neq \frac{1}{\rho_-}\), which means
\[
\frac{n_-}{n} = \sqrt{\frac{(p c - 1)(p c + 1)}{(\rho_- c - 1)(\rho_- c + 1)}} \quad \text{if } v_- \neq \frac{1}{\rho_-}.
\]  
(2.28)
Because we rearrange terms to get that

Substituting (2.22)-(2.25) into (2.29), and noting that which means

This defines the Hugoniot curve of the relativistic shock.

When \( \rho = \rho_\pm \), the second part of the left side in the above expression will not be zero if \( v_- \neq \rho_- c^2 \), which means

Because \( v_- = \rho_- c^2 \) contradicts with \( v_- = \frac{1}{\rho_-} \), the above expression (2.31) together with (2.28) yields that

This defines the Hugoniot curve of the relativistic shock.

Substituting (2.23)-(2.25) and (2.32) into the first equation of (2.16), we get, after a straightforward calculation that

When \( \rho \neq \rho_- \), from (2.33), it is easy to find that

When \( \rho = \rho_- \), the situation is simple. From (2.23) and (2.16), we can easily obtain that

Case 2: \( \frac{v_- v}{c^2} - 1 = -\frac{\rho_+ - \rho_\pm}{\rho_\pm c^2} \), which gives a 3-shock

where \( p < \rho_- \), \( \rho < \rho_- \), \( v < v_- \).
Then, from (2.36) and (2.18) it follows that

$$\frac{v - v_\pm}{\sqrt{(1 - \frac{v_\pm^2}{c^2})(1 - \frac{v^2}{c^2})}} = \frac{\rho - \rho_\pm}{\sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}}.$$  \quad (2.37)

$$v = v_\pm + \frac{\rho - \rho_\pm}{1 + \frac{\rho - \rho_\pm}{\sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}}}.$$  \quad (2.38)

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{\sqrt{c^2 - a^2}}{c(1 + \frac{\rho - \rho_\pm}{\sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}})} \sqrt{1 - \frac{v_\pm^2}{c^2}}.$$

$$\sqrt{c^2 - a^2} = c \frac{\sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}}{\rho \rho_\pm - \frac{\rho - \rho_\pm}{c^2}}.$$  \quad (2.40)

where

$$a = \frac{\rho - \rho_\pm}{\rho \rho_\pm - \frac{\rho - \rho_\pm}{c^2}}.$$

Substituting (2.37)-(2.40) into (2.26), we get, after a straightforward calculation that

$$\left(\frac{n(\rho_\pm - \frac{1}{c^2})}{\rho_\pm \sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}} - \frac{n - (\rho - \frac{1}{c^2})}{\rho(\rho^2 - \frac{1}{c^2})}\right)(\rho \rho_\pm - \frac{\rho - \rho_\pm}{c^2}) = 0.$$  \quad (2.41)

When $\rho \neq \rho_\pm$, the second part of the left side in the above expression will not be zero if $v_\pm \neq -\frac{1}{\rho_\pm}$, which means

$$\frac{n}{n_\pm} = \sqrt{\frac{(pc - 1)(pc + 1)}{(\rho_\pm c - 1)(\rho_\pm c + 1)}}$$

if $v_\pm \neq -\frac{1}{\rho_\pm}$.  \quad (2.42)

Substituting (2.37)-(2.40) into (2.29), and noting that $\frac{v - v_\pm}{c^2 - 1} = -\frac{\rho - \rho_\pm}{\rho \rho_\pm - \frac{\rho - \rho_\pm}{c^2}}$, we get, after a straightforward calculation that

$$\left(\frac{n(\rho_\pm - \frac{1}{c^2})}{\sqrt{(\rho^2 - \frac{1}{c^2})(\rho_\pm^2 - \frac{1}{c^2})}} - \frac{n - (\rho - \frac{1}{c^2})}{\rho(\rho^2 - \frac{1}{c^2})}\right)\left(\frac{v_\pm}{\rho \rho_\pm - \frac{\rho - \rho_\pm}{c^2}} + 1\right)(\rho - \rho_\pm) = 0.$$  \quad (2.43)

When $\rho \neq \rho_\pm$, the second part of the left side in the above expression will not be zero if $v_\pm \neq -\rho_\pm c^2$, which means

$$\frac{n}{n_\pm} = \sqrt{\frac{(pc - 1)(pc + 1)}{(\rho_\pm c - 1)(\rho_\pm c + 1)}}$$

if $v_\pm \neq -\rho_\pm c^2$.  \quad (2.44)

Because $v_\pm = -\rho_\pm c^2$ contradicts with $v_\pm = -\frac{1}{\rho_\pm}$, the above expression (2.44) together with (2.42) yields that

$$\frac{n}{n_\pm} = \sqrt{\frac{(pc - 1)(pc + 1)}{(\rho_\pm c - 1)(\rho_\pm c + 1)}}.$$  \quad (2.45)

This defines the Hugoniot curve of the relativistic shock.

Substituting (2.38)-(2.40) and (2.45) into the first equation of (2.16), we get, after a straightforward calculation that

$$\sigma = \frac{v_\pm \rho - \rho_\pm + a(\rho \rho_\pm - \frac{1}{c^2})}{\rho_\pm \rho - \rho_\pm + \frac{\rho - \rho_\pm}{c^2}(\rho \rho_\pm - \frac{1}{c^2})} = \frac{(\rho - \rho_\pm)(v_\pm \rho_\pm + 1)}{\rho_\pm \rho - \rho_\pm + \frac{\rho - \rho_\pm}{c^2}(\rho \rho_\pm + \frac{1}{c^2})}.$$  \quad (2.46)

When $\rho \neq \rho_\pm$, from (2.46), it is easy to find that

$$\sigma = \frac{v_\pm + \frac{1}{\rho_\pm}}{1 + \frac{\rho - \rho_\pm}{c^2}}.$$  \quad (2.47)
When $\rho = \rho_-$, the situation is simple. From (2.38) and (2.16), we can easily obtain that
\[
\sigma = v = v_-, \quad \rho = \rho_- \quad \text{and} \quad n \neq n_-.
\]

According to above discussions, there are two types of shock curves $S_i$ ($i = 1, 3$), which are given by
\[
S_i : \begin{cases}
\sigma = \frac{v - 1}{\rho_-} = \frac{v_- - 1}{1 - \frac{1}{\rho_-^2}} \\
n = \sqrt{\frac{(\rhoc - 1)(\rhoc + 1)}{(\rhoc - 1)(\rhoc + 1)}}
\end{cases}
\]
with
\[
p > p_-, \quad \rho > \rho_-, \quad v < v_-
\]
and
\[
S_3 : \begin{cases}
\sigma = \frac{v + 1}{\rho^2} = \frac{v_+ + 1}{1 + \frac{1}{\rho_+^2}} \\
n = \sqrt{\frac{(\rhoc - 1)(\rhoc + 1)}{(\rhoc - 1)(\rhoc + 1)}}
\end{cases}
\]
with
\[
p < p_-, \quad \rho < \rho_-, \quad v < v_-
\]

From (2.13), (2.15) and (2.49)-(2.50), we can find that the rarefaction waves and the shock waves are coincident in the state space, which correspond to contact discontinuities of the first and the third families:
\[
J_1 : \xi = \sigma = \frac{v - 1}{\rho_-} = \frac{v_- - 1}{1 - \frac{1}{\rho_-^2}} \quad \text{and} \quad n = \sqrt{\frac{(\rhoc - 1)(\rhoc + 1)}{(\rhoc - 1)(\rhoc + 1)}}
\]
\[
J_3 : \xi = \sigma = \frac{v + 1}{\rho^2} = \frac{v_+ + 1}{1 + \frac{1}{\rho_+^2}} \quad \text{and} \quad n = \sqrt{\frac{(\rhoc - 1)(\rhoc + 1)}{(\rhoc - 1)(\rhoc + 1)}}
\]

Remark 1. In [8], the author derived the Hugoniot curve of the relativistic shocks by considering such a coordinate system that the shock speed is zero. Different from that, in this paper, we obtain the result generally and ulteriorly give the analytical formula of relativistic shocks (also see [15, 48]).

In the state space, starting from a given state $(n_-, \rho_-, v_-)$, we draw the contact discontinuity curves (2.51) and (2.52) for $\rho > \frac{1}{c}$, the projections of which onto the $(\rho, v)$-plane are denoted by $J_1$ and $J_3$, respectively. So, $J_1$ has the asymptotic line $v = \frac{v_- - 1}{\rho_-}$ and the singularity point $(\rho, v) = (\frac{1}{c}, c)$, and $J_3$ has the asymptotic line $v = \frac{v_+ + 1}{1 + \frac{1}{\rho_+^2}}$ and the singularity point $(\rho, v) = (\frac{1}{c}, -c)$. Also, starting
from the point \( \left( n_-, \rho_-, \frac{\rho_- v_- c^2 - 2\rho_- v_- + \rho_-}{\rho_- c^2 - 2v_- + \rho_-} \right) \), in the state space, we draw the contact discontinuity curve (2.52), the projection of which onto the \((\rho, v)\)-plane is denoted by \( S_5 \). Then, \( S_5 \) has the asymptotic line \( v = \frac{v_- - \rho_-}{1 - \rho_- c^2} \) and the singularity point \((\rho, v) = (\frac{1}{c}, -c)\). The projections of these curves onto the \((\rho, v)\)-plane divide the \((\rho, v)\)-plane into five regions I, II, III, IV and V, as shown in Fig. 1.

Fig. 1. The projections of the curves \( J_1 \) and \( J_3 \) onto the \((\rho, v)\)-plane.

For any given right state \((n_+, \rho_+, v_+)\), according to Fig. 1, we can construct Riemann solutions of (1.1), (1.2) and (1.3). When the projection of the state \((n_+, \rho_+, v_+)\) onto the \((\rho, v)\)-plane lies in \( I \cup II \cup III \cup IV \), the Riemann problem can be solved in the following way. On the physically relevant region, we draw the contact discontinuity curves \( J_1(n_-, \rho_-, v_-) \) and \( J_3(n_+, \rho_+, v_+) \). The projections of these contact discontinuity curves onto the \((\rho, v)\)-plane have a unique intersection point \((\rho_*, v_*\)
Then, we draw the contact discontinuity curve determined by
\[
\begin{align*}
J & = \frac{v_s - \frac{1}{\rho - c^2}}{\frac{1}{\rho - c^2}} = \frac{v_s}{\frac{1}{\rho - c^2}}, \\
J_2 & = \frac{v_s + \frac{1}{\rho + c^2}}{\frac{1}{\rho + c^2}} = \frac{v_s + \frac{1}{\rho + c^2}}{\frac{1}{\rho + c^2}}.
\end{align*}
\]

Theorem 2.1. For Riemann problem (1.1), (1.2) and (1.3), on the physically relevant region, under the condition \( \frac{v_+ - \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}} > \frac{v_- - \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}} \), there exists a unique entropy solution, which can be expressed as
\[
(n, \rho, v)(t, x) = \begin{cases} 
(n_-, \rho_-, v_-), & -\infty < x/t < \frac{v_+ - \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}}, \\
(n_{1+}, \rho_{1+}, v_{1+}), & \frac{v_+ - \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}} \leq x/t \leq v_{1+}, \\
(n_{2+}, \rho_{2+}, v_{2+}), & v_{1+} < x/t \leq \frac{v_- + \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}}, \\
(n_+, \rho_+, v_+), & \frac{v_- + \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}} < x/t < +\infty,
\end{cases}
\]

where
\[
\begin{align*}
\rho_{1+} &= \rho_{2+} = \rho_*, \\
v_{1+} &= v_{2+} = v_*, \\
\frac{v_+ - \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}} &= \frac{v_- - \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}}, \\
\frac{v_+ + \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}} &= \frac{v_- + \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}}, \\
n_{1+} &= n_+ \sqrt{\frac{(\rho_- c - 1)(\rho_+ c + 1)}{(\rho_- c - 1)(\rho_+ c + 1)}} \\
n_{2+} &= n_+ \sqrt{\frac{(\rho_- c - 1)(\rho_+ c + 1)}{(\rho_- c - 1)(\rho_+ c + 1)}}.
\end{align*}
\]

Fig. 3. The characteristic lines from initial data for the case \( \frac{v_+ - \frac{1}{\rho_+ c^2}}{\frac{1}{\rho_+ c^2}} > \frac{v_- - \frac{1}{\rho_- c^2}}{\frac{1}{\rho_- c^2}} \).
3. Delta shock solutions

In this section, we construct the Riemann solutions of (1.1)-(1.2) with initial data (1.3) when the projection of the state \((n_+, \rho_+, v_+)\) onto the \((\rho, v)\)-plane lies in \(V\), namely,

\[
\frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n} \geq \frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}}. \tag{3.1}
\]

At this moment, the linearly degenerate characteristic lines from initial data will overlap in a domain \(\Omega = \{(t, x) | \frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}} \leq x \leq \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}}, 0 \leq t < +\infty\}\) shown in Fig. 3. So, the singularity must develop in \(\Omega\). It is easy to know that the singularity is impossible to be a jump with finite amplitudes because the Rankine-Hugoniot relation is not satisfied on the bounded jump.

To analyze the singularity, we first study the special case \(v_n - \frac{1}{\rho_n} = \frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}}\). Let us consider the limit of the solution \((n, \rho, v)(\xi)\) when \(n_-, \rho_-, v_-, n_+\) and \(\rho_+\) are fixed, \(\frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}} \to \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}} + 0\). When \(\frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}} > \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}}\), the solution is given by (2.55), where \(\rho_\ast\) and \(v_\ast\) satisfy

\[
\begin{align*}
\frac{c^2(1 + \frac{1}{\rho_\ast})}{c^2 - \frac{1}{\rho_\ast}} &= \frac{c^2(v_\ast - \frac{1}{\rho_\ast})}{c^2 - \frac{1}{\rho_\ast}}, \\
\frac{c^2(1 + \frac{1}{\rho_\ast})}{c^2 - \frac{1}{\rho_\ast}} &= \frac{c^2(v_\ast + \frac{1}{\rho_\ast})}{c^2 - \frac{1}{\rho_\ast}}. \tag{3.2}
\end{align*}
\]

We can employ (3.2) and calculate to obtain

\[
\rho_\ast = \frac{c^2 - ab + \sqrt{c^4 + a^2b^2 - c^2(a^2 + b^2)}}{c^2(b - a)}, \tag{3.3}
\]

where

\[
a = \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}} \quad \text{and} \quad b = \frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}}.
\]

Therefore, as \(\frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}} \to \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}} + 0\), namely, \(b \to a^+\), the combination of (3.2)-(3.3) and (2.56) yields

\[
\begin{align*}
\rho_{s1} &= \rho_{s2} = \rho_\ast \to +\infty, n_{s1} \to +\infty, n_{s2} \to +\infty, \\
v_{s1} &= v_{s2} = v_\ast \to \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}}. \tag{3.4}
\end{align*}
\]

These show that contact discontinuities \(J_1, J_2\) and \(J_3\) coincide to form a new type of nonlinear hyperbolic wave. Now let us calculate the total quantities of \(n, \rho\) and \(v\) between \(J_1\) and \(J_2\) as \(n_-, \rho_-, v_-\), \(n_+\) and \(\rho_+\) are fixed, \(\frac{v_+ + \frac{1}{\rho_+ c_+}}{1 + \frac{v_+}{\rho_+ c_+}} \to \frac{v_n - \frac{1}{\rho_n}}{1 - \frac{v_n}{\rho_n c_n}} + 0\),

\[
\lim_{b \to a^+} \int_a^b \rho(\xi) d\xi = \lim_{b \to a^+} \int_a^b \rho d\xi = \lim_{b \to a^+} \int_a^b \rho(-a) = \lim_{b \to a^+} \int_a^b v d\xi = \lim_{b \to a^+} \int_a^b v d\xi = \lim_{b \to a^+} v(b - a) = 0. \tag{3.5}
\]
and
\[
\lim_{b \to a^+} \int_a^b n(\xi) d\xi = \lim_{b \to a^+} \left( \int_a^b \frac{n}{\sqrt{c^2 \rho^2 - 1}} \sqrt{c^2 \rho^2 - 1} d\xi + \int_{b}^g \frac{n}{\sqrt{c^2 \rho^2 - 1}} d\xi \right) = \lim_{b \to a^+} \left( \frac{(c^2 - v_n a)n_{-}}{\rho_{c}c^2} \sqrt{c^2 \rho^2 - 1} + \frac{(c^2 - v_n b)n_{+}}{\rho_{c}c^2} \sqrt{c^2 \rho^2 - 1} \right) = \frac{c^2 - a^2}{c^2} \left( \frac{n_{-}}{\sqrt{\rho_{-}^2 - c^2}} + \frac{n_{+}}{\sqrt{\rho_{+}^2 - c^2}} \right) \neq 0.
\] (3.7)

Hence, (3.5) and (3.7) show that \( \rho(\xi) \) and \( n(\xi) \) have the same singularity as a weighted Dirac delta function at \( \xi = \frac{v_n}{1 - \rho_{c}c^2} \), while (3.6) implies that \( v(\xi) \) has a bounded variation. Thus, such a type of nonlinear hyperbolic wave of (1.1) and (1.2) is a delta shock wave with a weighted Dirac delta function in both \( n \) and \( \rho \), denoted by \( S_{\delta} \). It is quite different from the previous ones on which only one state variable contains the Dirac delta function. To our knowledge, this type of delta shock wave has not been found in the previous studies on the relativistic Euler equations. Moreover, for \( S_{\delta} \) in this case, the inequality
\[
\lambda_1(n_{+}, \rho_{+}, v_{+}) < \lambda_2(n_{+}, \rho_{+}, v_{+}) < \lambda_3(n_{+}, \rho_{+}, v_{+}) = \sigma = \lambda_1(n_{-}, \rho_{-}, v_{-}) < \lambda_2(n_{-}, \rho_{-}, v_{-}) < \lambda_3(n_{-}, \rho_{-}, v_{-})
\]
holds, where \( \sigma = \frac{v_n}{1 - \rho_{c}c^2} \) is the propagation speed of \( S_{\delta} \). It means that none of the six characteristic lines on both sides of \( S_{\delta} \) is outgoing with respect to \( S_{\delta} \).

By the above analysis, for the case \( \frac{v_n}{1 - \rho_{c}c^2} > \frac{v_{+} + \rho_{+}}{1 + \rho_{+}c^2} \), the delta shock wave solution containing a Dirac delta function in both \( n \) and \( \rho \) will be considered. Thus we first introduce three definitions as follows.

**Definition 3.1.** A triple \((n, \rho, v)\) constitutes a solution of (1.1) in the sense of distributions if it satisfies
\[
\int_{0}^{+\infty} \int_{-\infty}^{+\infty} \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial t} \phi_{t} + \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial x} \phi_{x} \right) dx dt = 0,
\]
\[
\int_{0}^{+\infty} \int_{-\infty}^{+\infty} \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial t} \phi_{t} + \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial x} \phi_{x} \right) dx dt = 0,
\]
\[
\int_{0}^{+\infty} \int_{-\infty}^{+\infty} \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial t} \phi_{t} + \left( \frac{\rho}{\sqrt{1 - \rho_{c}c^2}} \right) \frac{\partial}{\partial x} \phi_{x} \right) dx dt = 0,
\]
for all test functions \( \phi \in C_{0}^{\infty}(R^{+} \times R^{2}) \).

**Definition 3.2.** A two-dimensional weighted delta function \( w(s)\delta_{L} \) supported on a smooth curve \( L \) parameterized as \( t = t(s), x = x(s) \) \((c \leq s \leq d)\) is defined by
\[
\langle w(s)\delta_{L}, \phi \rangle = \int_{c}^{d} w(s)\phi(t(s), x(s)) ds,
\]
for all test functions \( \phi \in C_{0}^{\infty}(R^{2}) \).

**Definition 3.3.** A triple distribution \((n, \rho, v)\) is called a delta shock wave solution of (1.1) if it is
represented in the form

\[
(n, \rho, v)(t, x) = \begin{cases} 
(n_-, \rho_-, v_-), & x < x(t), \\
(h(t)\delta(x - x(t)), w(t)\delta(x - x(t)), v_0(t)), & x = x(t), \\
(n_+, \rho_+, v_+), & x > x(t),
\end{cases}
\] 

and satisfies Definition 3.1, where \((n, \rho, v)(t, x)\) and \((n_+, \rho_+, v_+)(t, x)\) are piecewise smooth bounded solutions of (1.1).

With Definitions 3.1-3.3, we seek a delta shock wave solution with the discontinuity \(x = x(t)\) of (1.1) with (1.2) in the form

\[
(n, \rho, v)(t, x) = \begin{cases} 
(n_-, \rho_-, v_-), & x < x(t), \\
(h(t)\delta(x - x(t)), w(t)\delta(x - x(t)), v_0(t)), & x = x(t), \\
(n_+, \rho_+, v_+), & x > x(t),
\end{cases}
\] 

where \(x(t), h(t), w(t) \in C^1[0, +\infty)\), \(\delta(\cdot)\) is the standard Dirac measure supported on the curve \(x = x(t)\), and \(h(t), w(t)\) are the weights of the delta shock wave on the state variables \(n, \rho\), respectively. Similar to [2, 10, 32], we define \(\rho^{-1}\) as follows

\[
\rho^{-1}(x) = \begin{cases} 
\rho_-, & x < x(t), \\
0, & x = x(t), \\
\rho_+, & x > x(t).
\end{cases}
\] 

We assert that (3.11) is a delta shock wave solution of (1.1) with (1.2) in the sense of distributions if it satisfies the following generalized Rankine-Hugoniot relation

\[
\begin{align*}
\frac{dx(t)}{dt} &= v_k(t), \\
\frac{d}{dt} \left( \frac{h(t)}{\sqrt{1 - v_k^2(t)/c^2}} \right) &= v_k(t) \left[ \frac{\rho}{\sqrt{1 - v_k^2(t)/c^2}} \right] - \left[ \frac{n v}{\sqrt{1 - v_k^2(t)/c^2}} \right], \\
\frac{d}{dt} \left( \frac{w(t)v_k(t)}{\sqrt{1 - v_k^2(t)/c^2}} \right) &= v_k(t) \left[ \frac{\rho}{\sqrt{1 - v_k^2(t)/c^2}} \right] - \left[ \frac{n v}{\sqrt{1 - v_k^2(t)/c^2}} \right] - \frac{1}{\rho}, \\
\frac{d}{dt} \left( \frac{w(t)}{\sqrt{1 - v_k^2(t)/c^2}} \right) &= v_k(t) \left[ \frac{\rho}{\sqrt{1 - v_k^2(t)/c^2}} \right] + \left[ \frac{n v}{\sqrt{1 - v_k^2(t)/c^2}} \right],
\end{align*}
\] 

where \([q] = q_+ - q_-\), etc. In fact, if the relation (3.13) holds, then for any test function \(\phi \in C_0^\infty([0, +\infty) \times R)\), by Green’s formulation, we have

\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} \left( \left( \frac{n}{\sqrt{1 - v^2/c^2}} \right) \phi_t + \left( \frac{n v}{\sqrt{1 - v^2/c^2}} \right) \phi_x \right) dx dt
\]

\[
= \int_0^{+\infty} \int_{-\infty}^{x(t)} \frac{n_+}{\sqrt{1 - v^2/c^2}} \phi_t + \frac{n_+ v_+}{\sqrt{1 - v^2/c^2}} \phi_x dx dt
\]

\[
+ \int_0^{+\infty} \int_{x(t)}^{+\infty} \frac{n_-}{\sqrt{1 - v^2/c^2}} \phi_t + \frac{n_- v_-}{\sqrt{1 - v^2/c^2}} \phi_x dx dt
\]

\[
+ \int_0^{+\infty} \frac{h(t)}{\sqrt{1 - v_k^2(t)/c^2}} \phi_k(t, x(t)) dx dt
\]

\[
+ \int_0^{+\infty} \frac{h(t)v_k(t)}{\sqrt{1 - v_k^2(t)/c^2}} \phi_k(t, x(t)) dx dt.
\]
Without loss of generality, we assume that $v_\delta(t) := \sigma_\delta$ is a constant and $\sigma_\delta > 0$. By exchanging the ordering of integral and using the change of variables, the first term on the right-hand side of (3.14) equals

\[
\int_0^{+\infty} \int_0^{\infty} n_- \frac{n_-}{\sqrt{1 - v_-^2/c^2}} \phi_1 dx dt + \int_0^{+\infty} \int_0^{x(t)} n_- \frac{n_-}{\sqrt{1 - v_-^2/c^2}} \phi_1 dx dt \\
+ \int_0^{+\infty} \int_{-\infty}^{x(t)} n_- v_- \frac{n_-}{\sqrt{1 - v_-^2/c^2}} \phi_3 dx dt
\]

\[
= \int_0^{+\infty} dx \int_0^{+\infty} n_- \phi_1 dx dt + \int_0^{+\infty} \frac{n_- v_-}{\sqrt{1 - v_-^2/c^2}} \phi(t, x(t)) dt
\]

\[
= - \int_0^{+\infty} \int_0^{t(x)} n_- \frac{n_-}{\sqrt{1 - v_-^2/c^2}} \phi(t(x), x) dx + \int_0^{+\infty} \frac{n_- v_-}{\sqrt{1 - v_-^2/c^2}} \phi(t, x(t)) dt
\]

\[
= \int_0^{+\infty} \left( \frac{n_- v_-}{\sqrt{1 - v_-^2/c^2}} - \frac{n_- v_\delta(t)}{\sqrt{1 - v_\delta^2/c^2}} \right) \phi(t, x(t)) dt. \tag{3.15}
\]

Similarly, the second term on the right-hand side of (3.14) equals

\[
\int_0^{+\infty} dx \int_0^{t(x)} \frac{n_+}{\sqrt{1 - v_+^2/c^2}} \phi_2 dx dt - \int_0^{+\infty} \frac{n_+ v_+}{\sqrt{1 - v_+^2/c^2}} \phi(t, x(t)) dt
\]

\[
= \int_0^{+\infty} \frac{n_+}{\sqrt{1 - v_+^2/c^2}} \phi(t(x), x) dx - \int_0^{+\infty} \frac{n_+ v_+}{\sqrt{1 - v_+^2/c^2}} \phi(t, x(t)) dt
\]

\[
= \int_0^{+\infty} \left( \frac{n_+ v_\delta(t)}{\sqrt{1 - v_\delta^2/c^2}} - \frac{n_+ v_+}{\sqrt{1 - v_+^2/c^2}} \right) \phi(t, x(t)) dt. \tag{3.16}
\]

By using integrating by parts, from (3.14)-(3.16), we obtain

\[
\int_0^{+\infty} \int_{-\infty}^{+\infty} \left( \left( \frac{n}{\sqrt{1 - v^2/c^2}} \right) \phi_t + \left( \frac{n v}{\sqrt{1 - v^2/c^2}} \right) \phi_x \right) dx dt
\]

\[
= \int_0^{+\infty} v_\delta(t) \left[ \frac{n}{\sqrt{1 - v^2/c^2}} \right] - \left[ \frac{n v}{\sqrt{1 - v^2/c^2}} \right] \frac{d}{dt} \left( \frac{h(t)}{\sqrt{1 - v_\delta^2(t)/c^2}} \right) \phi(t, x(t)) dt = 0, \tag{3.17}
\]

which yields the first equality of (3.8). Similarly, one can prove the second and third equalities of (3.8). Thus, the assertion is true.

**Remark 2.** The generalized Rankine-Hugoniot relation (3.13) reflects the exact relationship among the limit states on both sides of the delta shock wave and the location, propagation speed, weights and the reassignment of $v$ on the delta shock wave.

In addition, to guarantee uniqueness, we should propose the following entropy condition

\[
\frac{v_+ + 1}{1 + \frac{v_+}{\rho_+ c^2}} \leq v_\delta(t) \leq \frac{v_- - 1}{1 - \frac{v_-}{\rho_- c^2}}, \tag{3.18}
\]

which means that all characteristic lines on both sides of the delta shock wave are incoming. A discontinuity satisfying (3.13) and (3.18) will be called a delta shock wave to system (1.1).

At this moment, the Riemann problem is reduced to solve the ordinary differential equations (3.13) with initial data

\[
t = 0 : \quad x(0) = 0, \quad v_\delta(0) = \frac{v_0}{15}, \quad h(0) = 0, \quad w(0) = 0. \tag{3.19}
\]
where \( v_0 \) is an undetermined constant.

Integrating (3.13) from 0 to \( t \) with initial data (3.19), we have

\[
\left\{ \begin{array}{ll}
\frac{h(t)}{\sqrt{1-v_0^2(t)/c^2}} = \left[ \frac{n}{\sqrt{1-v^2/c^2}} \right] x(t) - \left[ \frac{n_0}{\sqrt{1-v_0^2/c^2}} \right] t, \\
\frac{w(t)}{v_0(t)} = Fx(t) - Gt, \\
\frac{w(t)}{1-v_0^2(t)/c^2} = Ex(t) - Ft,
\end{array} \right.
\]

(3.20)

where

\[
E = \left[ \frac{\left( \frac{1}{\rho v} + \rho \right) v^2/c^2} {1 - v^2/c^2} + \rho \right], \quad F = \left[ \frac{\left( \frac{1}{\rho v} + \rho \right) v} {1 - v^2/c^2} \right], \quad G = \left[ \frac{\left( \frac{1}{\rho v} + \rho \right) v^2} {1 - v^2/c^2} - \frac{1}{\rho} \right].
\]

In what follows, under the entropy condition (3.18), we can solve (3.20) to obtain

\[
\left\{ \begin{array}{l}
x(t) = \frac{F+\sqrt{F^2-EG}}{E} t, \\
v_0(t) = \frac{F+\sqrt{F^2-EG}}{E}, \\
w(t) = \sqrt{F^2-EG} \left( 1 - \left( \frac{F+\sqrt{F^2-EG}}{E} \right)^2 \right) t, \\
h(t) = \sqrt{1 - \left( \frac{F+\sqrt{F^2-EG}}{E} \right)^2} \left( \left[ \frac{n}{\sqrt{1-v^2/c^2}} \right] \frac{F+\sqrt{F^2-EG}}{E} - \left[ \frac{n_0}{\sqrt{1-v_0^2/c^2}} \right] \right) t,
\end{array} \right.
\]

(3.21)

for \( E \neq 0 \), and

\[
\left\{ \begin{array}{l}
x(t) = \frac{G}{EF} t, \\
v_0(t) = \frac{G}{EF}, \\
w(t) = -F \left( 1 - \left( \frac{G}{EF} \right)^2 \right) t, \\
h(t) = \sqrt{1 - \left( \frac{G}{EF} \right)^2} \left( \left[ \frac{n}{\sqrt{1-v^2/c^2}} \right] \frac{G}{EF} - \left[ \frac{n_0}{\sqrt{1-v_0^2/c^2}} \right] \right) t,
\end{array} \right.
\]

(3.22)

for \( E = 0 \). The proof is similar to that in [10], so we omit it.

Thus, we have proved the following result.

**Theorem 3.1.** On the physically relevant region, under the condition \( \frac{v_+ + \rho_+}{\rho_+ c^2} \leq \frac{v_- - \rho_-}{\rho_- c^2} \), Riemann problem (1.1), (1.2) and (1.3) admits a unique entropy solution in the sense of distributions of the form

\[
(n, \rho, v)(t, x) = \begin{cases} 
(n_-, \rho_, v_-), & x < x(t), \\
(h(t)\delta(x - x(t)), w(t)\delta(x - x(t)), v_0(t)), & x = x(t), \\
(n_+, \rho_+, v_+), & x > x(t),
\end{cases}
\]

(3.23)

where \( x(t), v_0(t), h(t) \) and \( w(t) \) are shown in (3.21) for \( E \neq 0 \) or (3.22) for \( E = 0 \).

At last, combining with the results in Section 2, we can conclude

**Theorem 3.2.** For Riemann problem (1.1), (1.2) and (1.3), on the physically relevant region, there exists a unique entropy solution, which consists of three contact discontinuities when

\[
\frac{v_+ + \rho_+}{\rho_+ c^2} > \frac{v_- - \rho_-}{\rho_- c^2}
\]

and a delta shock wave on which both \( \rho \) and \( n \) contain Dirac delta function simultaneously when

\[
\frac{v_+ + \rho_+}{\rho_+ c^2} \leq \frac{v_- - \rho_-}{\rho_- c^2}.
\]
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