FILLING THE GAP BETWEEN METRIC REGULARITY AND FIXED POINTS:
THE LINEAR OPENNESS OF COMPOSITIONS

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Abstract: This paper is devoted to the investigation of an important issue recently brought into attention by a recent paper of Arutyunov: the relation between openness of composition of set-valued maps and fixed point results. More precisely, we prove a general result concerning the openness of compositions and then we show that this result covers and implies most of the known openness results. In particular, we reobtain several recent results in this field, including a fixed point theorem of Dontchev and Frankowska.

Keywords: composition of set-valued mappings · linear openness · metric regularity · Lipschitz-like property · implicit multifunctions · fixed points

Mathematics Subject Classification (2010): 90C30 · 49J53 · 54C60

1 Introduction

The equivalent properties of metric regularity and openness at linear rate, which are nowadays mainly studied in the context of the set-valued maps, have an important history. Their origins are in the open mapping principle for linear operators obtained in the 1930s by Banach and Schauder. Subsequently, a nonlinear extension of open mapping principle was obtained by Lyusternik (1934) and Graves (1950) and, besides the importance on this result, another important contribution is its proof which became, along the years, a powerful instrument in the effort of getting generalizations, extensions and a better understanding of these seminal works. Another landmark was the extension of research to the case of set-valued maps with closed and convex graph and was done by Ursescu in 1975 and Robinson in 1976, respectively. The famous Robinson-Ursescu Theorem and a series of works of Milyutin in 1970s concerning the preservation of regularity (and openness) of set-valued maps under functional perturbation give a strong impetus to this direction of research and, therefore, since 1980s to our time many mathematicians have participated into a joint effort on the development and understanding of these problems. We mention here only a few, having major
contributions to the field: J. P. Aubin, A. Dontchev, H. Frankowska, A. Ioffe, B. S. Mordukhovich, J. P. Penot, R. T. Rockafellar, S. M. Robinson, C. Ursescu. We remark that in several works (of J. P. Penot and C. Ursescu, for instance) dealing with openness results, the techniques based on the original proof of Lyusternik-Graves Theorem (which is in fact an iteration procedure) have been replaced by the use of Ekeland Variational Principle. Detailed accounts on the historical facts, as well as on the evolution of terminology are given in the monographs of Rockafellar and Wets [23], Mordukhovich [18] and Dontchev and Rockafellar [8].

The present paper is motivated by some (very) recent developments, which suggest that the study of metric regularity can be done in relation with some fixed points results. This new approach is suggested in the monograph of Dontchev and Rockafellar [8], and it is more precisely emphasized in the work of Arutyunov [3]. Subsequently, the relation between fixed points results and metric regularity are further investigated in [6] and [16]. In [6], [16], some compositions of set-valued maps are involved and, in fact, the degrees of generality of considered compositions represent, in some extent, the main novelties in both these papers.

More precisely, Ioffe [16] investigates separately coincidence fixed point results and openness of compositions, while Dontchev and Frankowska in [6] present a fixed point theorem which generalized the previous result of Arutyunov, and, moreover, show that their result implies several well-known results concerning the metric regularity of sum of set-valued maps. Moreover, these papers emphasize the fact that the link between fixed points and metric regularity of the involved multifunctions has a purely metric behavior. As a consequence, their proofs make use of well-fitted adaptations of the initial iterative procedure of Lyusternik-Graves type, and work on general metric spaces, under local completeness assumptions.

In this paper we propose ourselves to bring into light a global approach and to fill several implications which are not previously given. Having this aim in mind, we follow the next procedure. Firstly, we prove some implicit multifunctions assertions which, besides their own importance are used several times as auxiliary results. Secondly, we prove a general openness result for compositions of set-valued maps. Our proof is a refinement of the method based on Ekeland Variational Principle. In particular, we prove that from our result one can get a nontrivial extension of the corresponding result of Ioffe. Moreover, the same result implies as well Dontchev-Frankowska fixed point theorem and this implication (i.e. from openness assertions to fixed points results) is not given elsewhere (up to our knowledge). At this point, it is clear that all the implications from Dontchev-Frankowska fixed point theorem apply, and therefore our openness result of compositions implies several very important results in this topic. At every step of our investigation we are trying to give an accurate account on the usual difficulties that arise at certain points of the proofs and to interrelate our results and arguments to the ones in literature. The setting we assume in this work is that of Banach spaces. Note that most of the results work equally well on normed vector spaces, with corresponding completeness assumptions around the reference points.

The paper is organized as follows. In the second section we introduce the basic notations, concepts and we recall some known results. The third section contains the main results of the paper. We present here some implicit multifunction theorems which are important as separate results, but their conclusions are in force in the proof of other results. Next, we formulate and prove the main result of the paper which is an openness theorem for compositions of set-valued maps. Then, on this basis, one result of Ioffe is significantly extended and completed. The last section is devoted to the proof of Dontchev-Frankowska fixed point theorem as a consequence of our main result and several interpretations and possibilities of generalizations are presented. The paper ends with a short conclusion section.
2 Preliminaries

This section contains some basic definitions and results used in the sequel. In what follows, we suppose that all the involved spaces are Banach. In this setting, $B(x, r)$ and $D(x, r)$ denote the open and the closed ball with center $x$ and radius $r$, respectively. Sometimes we write $D_X$ for the closed unit ball of $X$. If $x \in X$ and $A \subset X$, one defines the distance from $x$ to $A$ as $d(x, A) := \inf \{\|x - a\| \mid a \in A\}$. As usual, we use the convention $d(x, \emptyset) = \infty$. For a non-empty set $A \subset X$ we put $\text{cl} A$ for its topological closure. When we work on a product space, we consider the sum norm, unless otherwise stated.

Consider now a multifunction $F : X \rightrightarrows Y$. The domain and the graph of $F$ are denoted respectively by
\[ \text{Dom} F := \{x \in X \mid F(x) \neq \emptyset\} \]
and
\[ \text{Gr} F = \{(x, y) \in X \times Y \mid y \in F(x)\} \].

If $A \subset X$ then $F(A) := \bigcup_{x \in A} F(x)$. The inverse set-valued map of $F$ is $F^{-1} : Y \rightrightarrows X$ given by $F^{-1}(y) = \{x \in X \mid y \in F(x)\}$.

Recall that a multifunction $F$ is inner semicontinuous at $(x, y) \in \text{Gr} F$ if for every open set $D \subset Y$ with $y \in D$, there exists a neighborhood $U \in \mathcal{V}(x)$ such that for every $x' \in U$, $F(x') \cap D \neq \emptyset$ (where $\mathcal{V}(x)$ stands for the system of the neighborhoods of $x$).

We remind now the concepts of openness at linear rate, metric regularity and Lipschitz-likeness of a multifunction around the reference point.

**Definition 2.1** Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x}, \overline{y}) \in \text{Gr} F$.

(i) $F$ is said to be open at linear rate $L > 0$, or $L$-open around $(\overline{x}, \overline{y})$ if there exist a positive number $\varepsilon > 0$ and two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $\rho \in (0, \varepsilon)$ and every $(x, y) \in \text{Gr} F \cap [U \times V]$,
\[ B(y, \rho L) \subset F(B(x, \rho)). \]  

The supremum of $L > 0$ over all the combinations $(L, U, V, \varepsilon)$ for which (2.1) holds is denoted by $\text{lop} F(\overline{x}, \overline{y})$ and is called the exact linear openness bound, or the exact covering bound of $F$ around $(\overline{x}, \overline{y})$.

(ii) $F$ is said to be Lipschitz-like, or has Aubin property around $(\overline{x}, \overline{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$,
\[ F(x) \cap V \subset F(u) + L \|x - u\| D_Y. \]  

The infimum of $L > 0$ over all the combinations $(L, U, V)$ for which (2.2) holds is denoted by $\text{lip} F(\overline{x}, \overline{y})$ and is called the exact Lipschitz bound of $F$ around $(\overline{x}, \overline{y})$.

(iii) $F$ is said to be metrically regular around $(\overline{x}, \overline{y})$ with constant $L > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $(x, y) \in U \times V$,
\[ d(x, F^{-1}(y)) \leq L d(y, F(x)). \]  

The infimum of $L > 0$ over all the combinations $(L, U, V)$ for which (2.3) holds is denoted by $\text{reg} F(\overline{x}, \overline{y})$ and is called the exact regularity bound of $F$ around $(\overline{x}, \overline{y})$.

The next proposition contains the well-known links between the notions presented above. For more details about the proof, see [18, Theorems 1.49, 1.52].
Proposition 2.2 Let $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x}, \overline{y}) \in \text{Gr } F$. Then $F$ is open at linear rate around $(\overline{x}, \overline{y})$ iff $F^{-1}$ is Lipschitz-like around $(\overline{y}, \overline{x})$ iff $F$ is metrically regular around $(\overline{x}, \overline{y})$. Moreover, in every of the previous situations,

$$(\text{lop } F(\overline{x}, \overline{y}))^{-1} = \text{lip } F^{-1}(\overline{y}, \overline{x}) = \text{reg } F(\overline{x}, \overline{y}).$$

It is well known that the corresponding ”at point” properties are significantly different from the ”around point” ones. Let us introduce now some of these notions. For more related concepts we refer to [1].

Definition 2.3 Let $L > 0$, $F : X \rightrightarrows Y$ be a multifunction and $(\overline{x}, \overline{y}) \in \text{Gr } F$.

(i) $F$ is said to be open at linear rate $L$, or $L$–open at $(\overline{x}, \overline{y})$ if there exists a positive number $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B(\overline{y}, \rho L) \subset F(B(\overline{x}, \rho)).$$

The supremum of $L > 0$ over all the combinations $(L, \varepsilon)$ for which (2.4) holds is denoted by $\text{plop } F(\overline{x}, \overline{y})$ and is called the exact punctual linear openness bound of $F$ at $(\overline{x}, \overline{y})$.

(ii) $F$ is said to be pseudocalm with constant $L$, or $L$–pseudocalm at $(\overline{x}, \overline{y})$, if there exists a neighborhood $U \in \mathcal{V}(\overline{x})$ such that, for every $x \in U$,

$$d(\overline{y}, F(x)) \leq L \|x - \overline{x}\|.$$  

(2.5)

The infimum of $L > 0$ over all the combinations $(L, U)$ for which (2.5) holds is denoted by $\text{psdcm } F(\overline{x}, \overline{y})$ and is called the exact bound of pseudocalmness for $F$ at $(\overline{x}, \overline{y})$.

(iii) $F$ is said to be metrically hemiregular with constant $L$, or $L$–metrically hemiregular at $(\overline{x}, \overline{y})$ if there exists a neighborhood $V \in \mathcal{V}(\overline{y})$ such that, for every $y \in V$,

$$d(\overline{x}, F^{-1}(y)) \leq L \|y - \overline{y}\|.$$ 

(2.6)

The infimum of $L > 0$ over all the combinations $(L, V)$ for which (2.6) holds is denoted by $\text{hemreg } F(\overline{x}, \overline{y})$ and is called the exact hemiregularity bound of $F$ at $(\overline{x}, \overline{y})$.

For more details about these concepts, see [2], [12]. The next proposition presents the corresponding equivalences between the ”at point” notions introduced before.

Proposition 2.4 Let $L > 0$, $F : X \rightrightarrows Y$ and $(\overline{x}, \overline{y}) \in \text{Gr } F$. Then $F$ is $L$–open at $(\overline{x}, \overline{y})$ iff $F^{-1}$ is $L^{-1}$–pseudocalm at $(\overline{y}, \overline{x})$ iff $F$ is $L^{-1}$–metrically hemiregular at $(\overline{x}, \overline{y})$. Moreover, in every of the previous situations,

$$(\text{plop } F(\overline{x}, \overline{y}))^{-1} = \text{psdcm } F^{-1}(\overline{y}, \overline{x}) = \text{hemreg } F(\overline{x}, \overline{y}).$$

Recall that $\mathcal{L}(X, Y)$ denotes the normed vector space of linear bounded operators acting between $X$ and $Y$. If $A \in \mathcal{L}(X, Y)$, then the ”at” and ”around point” notions do coincide. In fact, $A$ is metrically regular around every $x \in X$ iff $A$ is metrically hemiregular at every $x \in X$ iff $A$ is open with linear rate around every $x \in X$ iff $A$ is open with linear rate at every $x \in X$ iff $A$ is surjective. Moreover, in every of these cases we have

$$\text{hemreg } A = \text{reg } A = (\text{lop } A)^{-1} = (\text{lop } A)^{-1} = \| (A^*)^{-1} \|,$$

$$\text{psdcm } A = (\text{psdcm } A)^{-1}.$$
where \( A^* \in L(Y^*, X^*) \) denotes the adjoint operator and hemreg \( A \), reg \( A \), plop \( A \) and lop \( A \) are common for all the points \( x \in X \) (see, for more details, [2 Proposition 5.2]).

Finally, we introduce the corresponding partial notions of linear openness, metric regularity and Lipschitz-like property around the reference point for a parametric set-valued map.

**Definition 2.5** Let \( L > 0 \), \( F : X \times P \rightrightarrows Y \) be a multifunction, \((\overline{x}, \overline{p}), \overline{y}) \in \text{Gr} F \) and for every \( p \in P \), denote \( F_p(\cdot) := F(\cdot, p) \).

(i) \( F \) is said to be open at linear rate \( L > 0 \), or \( L \)-open, with respect to \( x \) uniformly in \( p \) around \((\overline{x}, \overline{p}), \overline{y}) \) if there exist a positive number \( \varepsilon > 0 \) and some neighborhoods \( U \in \mathcal{V}(\overline{x}), V \in \mathcal{V}(\overline{p}), W \in \mathcal{V}(\overline{y}) \) such that, for every \( \rho \in (0, \varepsilon) \), every \( p \in V \) and every \((x, y) \in \text{Gr} F_p \cap [U \times W] \),

\[
B(y, \rho L) \subset F_p(B(x, \rho)), 
\]

(2.7)

The supremum of \( L > 0 \) over all the combinations \((L, U, V, W, \varepsilon) \) for which (2.7) holds is denoted by \( \operatorname{loup}_x F((\overline{x}, \overline{p}), \overline{y}) \) and is called the exact linear openness bound, or the exact covering bound of \( F \) in \( x \) around \((\overline{x}, \overline{p}), \overline{y}) \).

(ii) \( F \) is said to be Lipschitz-like, or has Aubin property, with respect to \( x \) uniformly in \( p \) around \((\overline{x}, \overline{p}), \overline{y}) \) with constant \( L > 0 \) if there exist some neighborhoods \( U \in \mathcal{V}(\overline{x}), V \in \mathcal{V}(\overline{p}), W \in \mathcal{V}(\overline{y}) \) such that, for every \( x, u \in U \) and every \( p \in V \),

\[
F_p(x) \cap W \subset F_p(u) + L \|x - u\| \mathbb{D}_Y. 
\]

(2.8)

The infimum of \( L > 0 \) over all the combinations \((L, U, V, W) \) for which (2.8) holds is denoted by \( \operatorname{lip}_x F((\overline{x}, \overline{p}), \overline{y}) \) and is called the exact Lipschitz bound of \( F \) in \( x \) around \((\overline{x}, \overline{p}), \overline{y}) \).

(iii) \( F \) is said to be metrically regular with respect to \( x \) uniformly in \( p \) around \((\overline{x}, \overline{p}), \overline{y}) \) with constant \( L > 0 \) if there exist some neighborhoods \( U \in \mathcal{V}(\overline{x}), V \in \mathcal{V}(\overline{p}), W \in \mathcal{V}(\overline{y}) \) such that, for every \((x, p, y) \in U \times V \times W \),

\[
d(x, F_p^{-1}(y)) \leq Ld(y, F_p(x)). 
\]

(2.9)

The infimum of \( L > 0 \) over all the combinations \((L, U, V, W) \) for which (2.9) holds is denoted by \( \operatorname{reg}_x F((\overline{x}, \overline{p}), \overline{y}) \) and is called the exact regularity bound of \( F \) in \( x \) around \((\overline{x}, \overline{p}), \overline{y}) \).

Similarly, one can define the notions of linear openness, metric regularity and Lipschitz-like property with respect to \( p \) uniformly in \( x \), and the corresponding exact bounds.

### 3 Linear openness of compositions

We start the main section of the paper with a refinement of a result previously given in [12]. Here, we present some conclusions in a slightly different form but without inner semicontinuity assumptions (see as well the comments after the proof).

**Theorem 3.1** Let \( X, P \) be metric spaces, \( Y \) be a normed vector space, \( H : X \times P \rightrightarrows Y \) be a set-valued map and \((\overline{x}, \overline{p}, 0) \in \text{Gr} \; H \). Denote by \( H_p(\cdot) := H(\cdot, p), H_x(\cdot) := H(x, \cdot) \).

(i) If \( H \) is open with linear rate \( c > 0 \) with respect to \( x \) uniformly in \( p \) around \((\overline{x}, \overline{p}, 0) \), then there exist \( \alpha, \beta, \gamma > 0 \) such that, for every \((x, p) \in B(\overline{x}, \alpha) \times B(\overline{p}, \beta) \),

\[
d(x, S(p)) \leq c^{-1}d(0, H(x, p) \cap B(0, \gamma)). 
\]

(3.1)
Suppose, in addition, that \( Y \) is a normed vector space and \( H \) is Lipschitz-like with respect to \( p \) uniformly in \( x \) around \((\overline{p}, 0)\). Then \( S \) is Lipschitz-like around \((\overline{p}, \overline{p})\) and

\[
\text{lips}(S) \leq c^{-1}\hat{\text{lips}}_{p}H((\overline{p}, \overline{p}), 0). \tag{3.2}
\]

(ii) If \( H \) is open with linear rate \( c > 0 \) with respect to \( p \) uniformly in \( x \) around \((\overline{p}, \overline{p}, 0)\), then there exist \( \alpha, \beta, \gamma > 0 \) such that, for every \((x, p) \in B(\overline{p}, \alpha) \times B(\overline{p}, \beta)\),

\[
d(p, S^{-1}(x)) \leq c^{-1}d(0, H(x, p) \cap B(0, \gamma)). \tag{3.3}
\]

If, moreover, \( H \) is Lipschitz-like with respect to \( x \) uniformly in \( p \) around \((\overline{p}, \overline{p}, 0)\), then \( S \) is metrically regular around \((\overline{p}, \overline{p})\) and

\[
\text{reg}\,S(\overline{p}, \overline{p}) \leq c^{-1}\hat{\text{reg}}_{p}H((\overline{p}, \overline{p}), 0). \tag{3.4}
\]

**Proof.** We will prove only the first item, because for the second one it suffices to observe that, defining the multifunction \( T := S^{-1} \), the proof is completely symmetrical, using \( T \) instead of \( S \). Moreover, using Proposition \( \ref{prop:lip} \) we know that \( \text{reg}\,S(\overline{p}, \overline{p}) = \text{lips}(\overline{p}, \overline{p}) \) and then \( (3.3) \) follows from \( (3.2) \).

For the (i) item, we know that there exist \( r, s, t, c, \epsilon > 0 \) such that, for every \( \rho \in (0, \epsilon) \), every \( p \in B(\overline{p}, t) \) and every \((x, y) \in \text{Gr}\,(\overline{p}, r) \cap [B(\overline{p}, r) \times B(0, s)]\),

\[
B(y, c\rho) \subset H_p(B(x, \rho)).
\]

Take now \( \rho \in (0, \min\{\epsilon, c^{-1}s\}) \). Set \( \alpha := r, \beta := t, \gamma := c\rho \) and fix arbitrary \((x, p) \in B(\overline{p}, \alpha) \times B(\overline{p}, \beta)\). If \( H(x, p) \cap B(0, c\rho) = \emptyset \), then \( d(0, H(x, p) \cap B(0, \gamma)) = +\infty \) and \( (3.1) \) trivially holds. Suppose next that \( H(x, p) \cap B(0, c\rho) \neq \emptyset \). If \( 0 \in H(x, p) \), then \( 0 \in H(x, p) \cap B(0, c\rho) \), and, again, \( (3.1) \) trivially holds. Suppose now that \( 0 \notin H(x, p) \cap B(0, c\rho) \). Then for every \( \xi > 0 \), there exists \( y_{\xi} \in H(x, p) \cap B(0, c\rho) \) such that

\[
\|y_{\xi}\| < d(0, H(x, p) \cap B(0, c\rho)) + \xi.
\]

Because \( d(0, H(x, p) \cap B(0, c\rho)) < c\rho \), we can choose \( \xi \) sufficiently small such that \( d(0, H(x, p) \cap B(0, c\rho)) + \xi < c\rho \). Consequently,

\[
0 \in B(y_{\xi}, d(0, H(x, p) \cap B(0, c\rho)) + \xi) \subset B(y_{\xi}, c\rho).
\]

Observe now that \( x \in B(\overline{p}, r) \), \( p \in B(\overline{p}, t) \), \( y_{\xi} \in B(0, d(0, H(x, p) \cap B(0, c\rho)) + \xi) \subset B(0, c\rho) \subset B(0, s) \), \( y_{\xi} \in H(x, p) \) and denote \( \rho_0 := c^{-1}(d(0, H(x, p) \cap B(0, c\rho)) + \xi) < \rho < \epsilon \).

But we know that

\[
B(y_{\xi}, c\rho_0) \subset H_p(B(x, \rho_0)),
\]

hence, using also \( (3.5) \), one obtains that there exists \( x_0 \in B(x, \rho_0) \) such that \( 0 \in H(x_0, p) \), which is equivalent to \( x_0 \in S(p) \). Then

\[
d(x, S(p)) \leq d(x, x_0) < \rho_0 = c^{-1}(d(0, H(x, p) \cap B(0, c\rho)) + \xi).
\]

Making \( \xi \to 0 \), we obtain \( (3.1) \).

Suppose now that \( H \) is Lipschitz-like with respect to \( p \) uniformly in \( x \) around \((\overline{p}, \overline{p}, 0)\). Then there exist \( l, a, b, \tau > 0 \) such that \( \tau < c\rho \) and for every \( x \in B(\overline{p}, a) \) and every \( p_1, p_2 \in B(\overline{p}, b) \),

\[
H(x, p_1) \cap D(0, \tau) \subset H(x, p_2) + ld(p_1, p_2)\mathbb{D}_Y.
\]

(3.6)
Take $\overline{\alpha} := \min\{a, \alpha\}$, $\overline{\beta} := \min\{b, \beta, (2l)^{-1}\tau\}$, $p_1, p_2 \in B(\overline{p}, \overline{\beta})$ and $x \in S(p_1) \cap B(\overline{x}, \overline{\alpha})$. Then $0 \in H(x, p_1) \cap D(0, \tau)$, whence, using (3.6), there exists $y'' \in D_Y$ such that $y' := l \cdot d(p_1, p_2)y'' \in H(x, p_2)$ with $\|y'\| \leq l \cdot [d(p_1, \overline{p}) + d(\overline{p}, p_2)] \leq \tau < c \rho$. Hence, $y' \in H(x, p_2) \cap B(0, \gamma)$, so using (3.1), we get that

$$d(x, S(p_2)) \leq c^{-1}d(0, H(x, p_2)) \cap B(0, \gamma) \leq c^{-1}\|y'\| \leq c^{-1}ld(p_1, p_2).$$

Consequently, because $l$ can be chosen arbitrarily close to $\widehat{\text{lip}}_PH((\overline{\alpha}, \overline{p}), 0)$, it follows that $S$ is Lipschitz-like around $(\overline{\alpha}, \overline{p})$ and $\text{lip} S(\overline{p}, \overline{x}) \leq c^{-1}\widehat{\text{lip}}_PH((\overline{\alpha}, \overline{p}), 0)$. The proof is now complete. \(\square\)

**Remark 3.2** If, in addition to the assumptions of Theorem 3.4, $H$ is inner semicontinuous at $(\overline{\alpha}, \overline{p}, 0)$, then the relation (3.1) becomes

$$d(x, S(p)) \leq c^{-1}d(0, H(x, p)).$$

Also, as one can see from the precedent proof, $P$ can be taken to be just topological space (see, for more details, [31, Theorem 3.2]).

We present the next result as a by-product of Theorem 3.1. Note that different versions of the second part of the following lemma are done in [2, Theorem 3.5] (with functions instead of multifunctions) and in [16, Lemma 2]. Here we also obtain some extra conclusions for a general situation. Taking into account that this lemma will be used in the sequel, we prefer to give all the details of its proof.

**Lemma 3.3** Let $Y, Z, W$ be normed vector spaces, $G : Y \times Z \Rightarrow W$ be a multifunction and $(\overline{y}, \overline{z}, \overline{w}) \in Y \times Z \times W$ be such that $\overline{w} \in G(\overline{y}, \overline{z})$. Consider next the implicit multifunction $\Gamma : Z \times W \Rightarrow Y$ defined by

$$\Gamma(z, w) := \{y \in Y \mid w \in G(y, z)\}.$$

Suppose that the following conditions are satisfied:
(i) $G$ is Lipschitz-like with respect to $z$ uniformly in $y$ around $(\overline{y}, \overline{z}, \overline{w})$ with constant $D \geq 0$;
(ii) $G$ is open at linear rate with respect to $y$ uniformly in $z$ around $(\overline{y}, \overline{z}, \overline{w})$ with constant $C > 0$.

Then the multifunction $H : Y \times Z \times W \Rightarrow W$, given by

$$H(y, (z, w)) := G(y, z) - w \text{ for every } (y, z, w) \in Y \times Z \times W,$$

is Lipschitz-like with respect to $(z, w)$ uniformly in $y$ around $(\overline{y}, (\overline{z}, \overline{w}), 0)$.

Moreover, there exists $\gamma > 0$ such that for every $(z, w), (z', w') \in D(\overline{z}, \gamma) \times D(\overline{w}, \gamma)$, and for every $\delta > 0$,

$$\Gamma(z, w) \cap D(\overline{z}, \gamma) \subset \Gamma(z', w') + \frac{1 + \delta}{C}(D\|z - z'\| + \|w - w'\|)D_Y. \tag{3.7}$$

**Proof.** Define $P := Z \times W$ and then $H : Y \times P \Rightarrow W$. Observe also that

$$\Gamma(z, w) = \{y \in Y \mid 0 \in H(y, (z, w))\},$$

and denote $H_{(\varepsilon, w)}(\cdot) := H(\cdot, (z, w))$. Because for every $z$ close to $\overline{z}$ we know from (ii) that $G_z$ is $C$--open at points from its graph around $(\overline{y}, \overline{w})$, we can conclude that there exists $\varepsilon > 0$ such that for every $\rho \in (0, \varepsilon)$, every $z \in B(\overline{z}, \varepsilon)$ and every $(y, w') \in \text{Gr} G_z \cap [B(\overline{y}, \varepsilon) \times B(\overline{w}, \varepsilon)]$, $B(w', C\rho) \subset G_z(B(y, \rho))$. 


Hence, for every \((z, w) \in B(\overline{z}, \varepsilon) \times B(\overline{w}, 2^{-1}\varepsilon)\) and every \((y, u) \in \text{Gr} H_{(z, w)} \cap [B(\overline{z}, \varepsilon) \times B(0, 2^{-1}\varepsilon)]\), setting \(w' := u + w \in B(\overline{w}, \varepsilon)\), we get that

\[
B(u, C^{-1}\rho) = B(w', C\rho) - w \subset G_z(B(y, \rho)) - w = H_{(z, w)}(B(y, \rho)).
\]

But this shows, applying Theorem 3.1, that there exist \(\beta > 0\) such that, for every \((y, z, w) \in B(\overline{y}, \beta) \times B(\overline{z}, \beta) \times B(\overline{w}, \beta)\),

\[
d(y, \Gamma(z, w)) \leq C^{-1}d(0, H(y, (z, w)) \cap B(0, \beta)). \tag{3.8}
\]

We want to prove that there exists \(c > 0\) such that \((D + 1)c < \beta\), for every \(y \in B(\overline{y}, c)\), \((z, w), (z', w') \in B(\overline{x}, c) \times B(\overline{w}, c)\),

\[
H(y, (z, w)) \cap D(0, c) \subset H(y, (z', w')) + (D \|z - z'\| + \|w - w'\|)D_Z. \tag{3.9}
\]

In particular, we will prove the first part of the conclusion.

Because of (i), we know that there exists \(a > 0\) such that for every \(y \in B(\overline{y}, a)\) and every \(z, z' \in B(\overline{z}, a)\),

\[
G(y, z) \cap D(\overline{w}, a) \subset G(y, z') + D \|z - z'\| D_Z. \tag{3.10}
\]

Choose now \(c \in (0, \min\{2^{-1}a, (D + 1)^{-1}\beta\})\) and take arbitrary \(y \in B(\overline{y}, c)\), \((z, w), (z', w') \in B(\overline{z}, c) \times B(\overline{w}, c)\). Furthermore, choose \(u' \in H(y, (z, w)) \cap D(0, c)\). Then \(\|u' + w - \overline{w}\| \leq 2c < a\), whence \(u' + w \in G(y, z) \cap D(\overline{w}, a)\) and because of (3.10), one obtains successively that

\[
\begin{align*}
  u' + w &\in G(y, z') + D \|z - z'\| D_Z, \\
  u' &\in G(y, z') - w' + (w - w') + D \|z - z'\| D_Z, \\
  u' &\in H(y, (z', w')) + (D \|z - z'\| + \|w - w'\|)D_Z.
\end{align*}
\]

Take now \(\gamma \in (0, \min\{\beta, 2^{-1}c\})\), \((z, w), (z', w') \in B(\overline{z}, \gamma) \times B(\overline{w}, \gamma)\) and \(y \in \Gamma(z, w) \cap D(\overline{y}, \gamma)\). Hence, \(0 \in H(y, (z, w)) \cap D(0, c)\). Then, using (3.9), we have that there exists \(\eta \in D_Z\) such that \((D \|z - z'\| + \|w - w'\|)\eta \in H(y, (z', w')) \cap B(0, \beta)\) (because \((D + 1)c < \beta\)). Finally, using (3.3), we get that

\[
d(y, \Gamma(z', w')) \leq C^{-1}d(0, H(y, (z', w')) \cap B(0, \beta)) \leq C^{-1}(D \|z - z'\| + \|w - w'\|),
\]

which completes the proof. \(\square\)

We are now in position to formulate and to prove our main result, an openness result for a fairly general set-valued composition.

**Theorem 3.4** Let \(X, Y, Z, W\) be Banach spaces, \(F_1 : X \rightrightarrows Y\), \(F_2 : X \rightrightarrows Z\) and \(G : Y \times Z \rightrightarrows W\) be three multifunctions and \((\overline{x}, \overline{y}, \overline{z}, \overline{w}) \in X \times Y \times Z \times W\) such that \((\overline{x}, \overline{y}) \in \text{Gr} F_1\), \((\overline{x}, \overline{z}) \in \text{Gr} F_2\) and \((\overline{y}, \overline{z}), \overline{w}) \in \text{Gr} G\). Let \(H : X \rightrightarrows W\) be given by

\[
H(x) := G(F_1(x), F_2(x)) \text{ for every } x \in X.
\]

and suppose that the following assumptions are satisfied:

(i) \(\text{Gr} F_1\), \(\text{Gr} F_2\) and \(\text{Gr} G\) are locally closed around \((\overline{x}, \overline{y}), (\overline{x}, \overline{z})\) and \(((\overline{y}, \overline{z}), \overline{w})\);

(ii) \(F_1\) is open at linear rate \(L > 0\) around \((\overline{x}, \overline{y})\);
(iii) $F_2$ is Lipschitz-like around $(\overline{\varphi}, \overline{\psi})$ with constant $M > 0$;
(iv) $G$ is open at linear rate with respect to $y$ uniformly in $z$ around $(\overline{\varphi}, \overline{\psi})$ with constant $C > 0$;
(v) $G$ is Lipschitz-like with respect to $z$ uniformly in $y$ around $(\overline{\varphi}, \overline{\psi})$ with constant $D \geq 0$;
(vi) $LC - MD > 0$.

Then there exists $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B(\overline{\psi}, (LC - MD)\rho) \subset H(B(\overline{\varphi}, \rho)).$$

Moreover, for every $\rho \in (0, 2^{-1}\varepsilon)$ and every $(x, y, z, w) \in B(\overline{\varphi}, 2^{-1}\varepsilon) \times B(\overline{\psi}, 2^{-1}\varepsilon) \times B(\overline{\varphi}, 2^{-1}\varepsilon)$ such that $(y, z) \in F_1(x) \times F_2(x)$ and $w \in G(y, z)$,

$$B(w, (LC - MD)\rho) \subset H(B(x, \rho)).$$

**Proof.** Using (i), one can find $\alpha > 0$ such that $\text{Gr } F_1 \cap \text{cl}[B(\overline{\varphi}, \alpha) \times B(\overline{\psi}, \alpha)]$, $\text{Gr } F_2 \cap \text{cl}[B(\overline{\varphi}, \alpha) \times B(\overline{\psi}, \alpha)]$ and $\text{Gr } G \cap [B(\overline{\varphi}, \alpha) \times B(\overline{\psi}, \alpha) \times B(\overline{\psi}, \alpha)]$ are closed. Also, using (ii) and (iii), there exist $\beta > 0$ such that, for every $(x', y') \in \text{Gr } F_1 \cap [B(\overline{\varphi}, \beta) \times B(\overline{\psi}, \beta)]$, $F_1$ is $L$-open at $(x', y')$ and for every $(v', u') \in \text{Gr } F_2^{-1} \cap [B(\overline{\varphi}, \beta) \times B(\overline{\psi}, \beta)]$, $F_2^{-1}$ is $M$-open at $(v', u')$. Finally, using (iv) and (v), we can find $\gamma > 0$ such that for every $(z, w), (z', w') \in B(\overline{\varphi}, \gamma) \times B(\overline{\psi}, \gamma)$ and every $\delta > 0$

$$\Gamma(z, w) \cap B(\overline{\psi}, \gamma) \subset \gamma \Gamma(z', w') + \frac{1}{C}(D \|z - z'\| + \|w - w'\|)\|Y\|,$$  \hspace{1cm} (3.11)

Without loosing the generality, using (v), we can suppose also that for every $y \in B(\overline{\psi}, \gamma)$ and every $z, z' \in B(\overline{\varphi}, \gamma)$,

$$G(y, z) \cap B(\overline{\psi}, \gamma) \subset G(y, z') + D \|z - z'\|\|Y\|.$$  \hspace{1cm} (3.12)

Fix $\varepsilon := \min\{\alpha, L^{-1}\alpha, M^{-1}\alpha, (LC + MD)^{-1}\alpha, \beta, 2^{-1}L^{-1}\beta, 2^{-1}M^{-1}\beta, \gamma, 2^{-1}L^{-1}\gamma, 2^{-1}M^{-1}\gamma, 2^{-1}(LC + MD)^{-1}\gamma\}$ and take $\rho \in (0, \varepsilon)$.

Define the multifunction $(F_1, F_2) : X \Rightarrow Y \times Z$ by $(F_1, F_2)(x) := F_1(x) \times F_2(x)$ and observe that $(\overline{\varphi}, \overline{\psi}, \overline{\psi}) \in \text{Gr } (F_1, F_2)$. Because of the choice of $\varepsilon$, we have that the set $\Omega \cap \text{cl } A$ is closed, where

$$A := B(\overline{\varphi}, \rho) \times B(\overline{\psi}, L\rho) \times B(\overline{\varphi}, M\rho) \times B(\overline{\psi}, (LC + MD)\rho),$$

$$\Omega := \{(x, y, z, w) \in X \times Y \times Z \times W \mid (y, z) \in (F_1, F_2)(x) \text{ and } w \in G(y, z)\}.$$  \hspace{1cm} (3.13)

(3.14)

Take $u \in B(\overline{\varphi}, (LC - MD)\rho)$. We must prove that $u \in H(B(\overline{\varphi}, \rho))$. There exists $\tau \in (0, 1)$ such that $\|u - \overline{\varphi}\| < \tau(LC - MD)\rho$. Endow the space $X \times Y \times Z \times W$ with the norm

$$\|(p, q, r, s)\|_0 := \tau(LC - MD) \max\{\|p\|, L^{-1}\|q\|, M^{-1}\|r\|, (LC + MD)^{-1}\|s\|\}$$

and apply the Ekeland variational principle to the function $\Omega \cap \text{cl } A \Rightarrow \mathbb{R}_+$,

$$h(p, q, r, s) := \|u - s\|.$$  \hspace{1cm} (3.15)

Then one can find a point $(a, b, c, d) \in \Omega \cap \text{cl } A$ such that

$$\|u - d\| \leq \|u - \overline{\varphi}\| - \|(a, b, c, d) - (\overline{\varphi}, \overline{\psi}, \overline{\psi})\|_0$$  \hspace{1cm} (3.16)

and

$$\|u - d\| \leq \|u - \overline{\varphi}\| + \|(a, b, c, d) - (p, q, r, s)\|_0, \forall (p, q, r, s) \in \Omega \cap \text{cl } A.$$
From (3.15) we have that
\[
\tau(LC - MD) \max \{\|a - \overline{x}\|, L^{-1} \|b - \overline{y}\|, M^{-1} \|c - \overline{z}\|, (LC + MD)^{-1} \|d - \overline{w}\|\}
\]
\[
= \|(a, b, c, d) - (\overline{x}, \overline{y}, \overline{z}, \overline{w})\|_0 \leq \|u - \overline{w}\| < \tau(LC - MD)\rho,
\]
hence \((a, b, c, d) \in A\), and, in particular, \(a \in B(\overline{x}, \rho)\).

If \(u = d\), then \(u \in H(a) \subset H(B(\overline{x}, \rho))\) and the desired assertion is proved.

We want to show that \(u = d\) is the sole possible situation. For this, suppose by means of contradiction that \(u \neq d\). Fix \(\omega > 0\) such that \(LC - \omega > 0\) and define next
\[
v := (LC - \omega) \|u - d\|^{-1} (u - d).
\]
Take arbitrary \(\zeta \in (0, 2^{-1}\rho)\). Remark that, from the choice of \(\epsilon, b \in B(\overline{y}, \gamma), c \in B(\overline{z}, \gamma)\) and \(d, d + \zeta v \in B(\overline{w}, \gamma)\). Hence, using (3.11) for \(\delta \in (0, (LC - \omega)^{-1}\omega)\), we get that
\[
b \in \Gamma(c, d) \cap B(\overline{y}, \gamma) \subset \Gamma(c, d + \zeta v) + (L - \epsilon')\zeta \mathbb{D}_Y,
\]
where \(\epsilon' := C^{-1}(\omega - \delta)(LC - \omega) \in (0, L)\). Therefore, there exists \(q \in \mathbb{D}_Y\) such that \(b + (L - \epsilon')\zeta q \in \Gamma(c, d + \zeta v)\), or, equivalently, \(d + \zeta v \in G(b + (L - \epsilon')\zeta q, c)\).

Now, from the \(L -\) openness of \(F_1\) at \((a, b) \in \text{Gr} F_1 \cap [B(\overline{x}, \beta) \times B(\overline{y}, \beta)]\), there exists \(\epsilon_0 < 2^{-1}\rho\) such that, for every \(\zeta \in (0, \epsilon_0)\),
\[
b + (L - \epsilon')\zeta q \in B(b, L\zeta) \subset F_1(B(a, \zeta)).
\]
(3.17)

Consequently, there exists \(p\) with \(\|p\| < 1\) such that \(b + (L - \epsilon')\zeta q \in F_1(a + \zeta p)\).

Also, using the \(M^{-1}\) - openness of \(F_2^{-1}\) at \((c, a) \in \text{Gr} F_2^{-1} \cap [B(\overline{z}, \beta) \times B(\overline{y}, \beta)]\), we can find \(\epsilon_1 < \epsilon_0\) such that for every \(\zeta \in (0, \epsilon_1)\),
\[
B(a, \zeta) \subset F_2^{-1}(B(c, M\zeta)).
\]
(3.18)

Hence, one can find \(e \in B(c, M\zeta)\) such that \(a + \zeta p \in F_2^{-1}(e)\) or, equivalently, \(e \in F_2(a + \zeta p)\).

Because we can write \(e = c + M\zeta r\) with \(\|r\| < 1\), we finally have that \((a + \zeta p, b + (L - \epsilon')\zeta q, c + M\zeta r) \in \text{Gr}(F_1, F_2)\).

From the choice of \(\epsilon\), we know that \(c, c + M\zeta r \in B(\overline{z}, \gamma)\) and \(b + (L - \epsilon')\zeta q \in B(\overline{y}, \gamma)\). Using now (3.12), we get that for every \(\zeta \in (0, \epsilon_1)\),
\[
d + \zeta v \in G(b + (L - \epsilon')\zeta q, c) \cap B(\overline{w}, \gamma)
\subset G(b + (L - \epsilon')\zeta q, c + M\zeta r) + MD\zeta \mathbb{D}_W,
\]
so there exist \(s \in \mathbb{D}_W\) such that \(d + \zeta v + MD\zeta s \in G(b + (L - \epsilon')\zeta q, c + M\zeta r)\).

In conclusion, for every \(\zeta \in (0, \epsilon_1)\), one can find \((p, q, r, s) \in (\mathbb{B}_X, \mathbb{D}_Y, \mathbb{B}_Z, \mathbb{D}_W)\) such that \((a + \zeta p, b + (L - \epsilon')\zeta q, c + M\zeta r, d + \zeta v + MD\zeta s) \in \Omega \cap A\). We use now (3.16) to obtain that
\[
\|u - d\| \leq \|u - (d + \zeta v + MD\zeta s)\| + \zeta \left\| (p, (L - \epsilon')q, Mr, v + MDs) \right\|_0
\leq \|u - d - \zeta v\| + MD\zeta + \zeta \left\| (p, (L - \epsilon')q, Mr, v + MDs) \right\|_0.
\]
(3.19)

But
\[
\|u - d - \zeta v\| = \|u - d\| - \zeta (LC - \omega).
\]
Set \( \varepsilon_2 := \min\{\varepsilon_1, (LC - \varepsilon)^{-1} ||u - d|| \} \). Then for every \( \zeta \in (0, \varepsilon_2) \), one obtains successively from (3.19) that
\[
||u - d|| \leq ||u - d|| - \zeta (LC - \omega) + MD\zeta + \zeta \left( ||p, (L - \varepsilon')q, Mr, v + MDs|| \right)_0,
\]
\[
LC - MD - \omega \leq \tau (LC - MD) \max\{||p||, L^{-1} ||(L - \varepsilon')q||, M^{-1} ||Mr||, (LC + MD)^{-1} ||v + MDs|| \},
\]
\[
LC - MD - \omega \leq \tau (LC - MD).
\]

Passing to the limit when \( \omega \to 0 \), we get that \( 1 \leq \tau \), which is a contradiction. The proof of the first part is now complete.

For the second part, the proof is similar. Suppose that the constants \( \alpha, \beta, \gamma, \varepsilon > 0 \) are chosen as above, take \( \rho \in (0, 2^{-1}\varepsilon) \) and \( (x, y, z, w) \in B(\bar{x}, 2^{-1}\varepsilon) \times B(\bar{y}, 2^{-1}\varepsilon) \times B(\bar{z}, 2^{-1}\varepsilon) \times B(\bar{w}, 2^{-1}\omega) \)
such that \( (y, z) \in F_1(x) \times F_2(x) \) and \( w \in G(y, z) \), and define \( A_1 := B(x, \rho) \times B(y, L\rho) \times B(z, M\rho) \times B(w, (LC + MD)\rho) \). Then \( \Omega \cap cl A_1 \) is again closed, because \( A_1 \subset B(\bar{x}, \omega) \times B(\bar{y}, \omega) \times B(\bar{z}, \omega) \times B(\bar{w}, \omega) \) and \( \omega < \alpha \). The rest of the proof is the same as before, observing that because \( (a, b, c, d) \in A_1 \), their small perturbations for \( \zeta > 0 \) sufficiently small remain in the desired balls, such that one can apply relations (3.11), (3.12), (3.17), (3.18).

As one can see, the first conclusion in Theorem 3.4 is an "at point" openness property for \( H \), while the second conclusion is not a genuine "around point" openness property, because we cannot guarantee that the property holds for any point \( (x, w) \in Gr H \) close to \( (\bar{x}, \bar{w}) \). We were able to prove that \( B(w, (LC - MD)\rho) \subset H(B(x, \rho)) \) exactly because the "intermediate" points \( y, z \) are also close to \( \bar{y} \) and \( \bar{z} \), respectively. In general, it is not possibly to take \( y, z \) with these properties and this is the same difficulty for getting local openness results for sum of multifunctions (see also Corollary 4.2 and the corresponding comments in [6]).

Next results, obtained on the basis of Theorem 3.4, improves and extends the conclusions of [16] Theorem 5.

**Theorem 3.5** Let \( X, Y, Z \) be Banach spaces, \( F : X \rightrightarrows Y \) and \( G : X \times Y \rightrightarrows Z \) be two multifunctions, and \( (\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Z \) such that \( (\bar{x}, \bar{y}) \in Gr F, (\bar{x}, \bar{y}, \bar{z}) \in Gr G \). Suppose that \( Gr F \) and \( Gr G \) are locally closed around \( (\bar{x}, \bar{y}) \) and \( (\bar{x}, \bar{y}, \bar{z}) \), respectively. Let \( \Phi : X \rightrightarrows Z \) be given by
\[
\Phi(x) := G(x, F(x)) \text{ for every } x \in X.
\]

(A) Moreover, suppose that:
(i) \( G \) is Lipschitz-like with respect to \( x \) uniformly in \( y \) around \( (\bar{x}, \bar{y}), \bar{z} \) with constant \( D \geq 0 \);
(ii) \( G \) is open at linear rate with respect to \( y \) uniformly in \( x \) around \( (\bar{x}, \bar{y}), \bar{z} \) with constant \( C > 0 \);
(iii) \( F \) is open at linear rate \( L > 0 \) around \( (\bar{x}, \bar{y}) \);
(iv) \( LC - D > 0 \).

Then there exists \( \varepsilon > 0 \) such that, for every \( \rho \in (0, \varepsilon) \) and every \( (x, y, z) \in B(\bar{x}, \varepsilon) \times B(\bar{y}, \varepsilon) \times B(\bar{z}, \varepsilon) \) such that \( y \in F(x) \) and \( z \in G(x, y) \),
\[
B(z, (LC - D)\rho) \subset \Phi(B(x, \rho)).
\]

Moreover, if \( G \) satisfies the condition that for every \( x \in B(\bar{x}, \varepsilon) \),
\[
G(x, y) \cap G(x, y') = \emptyset \text{ if } y \neq y',
\]
then the conclusion reads as follows: there exists $\epsilon' > 0$ such that, for every $\rho \in (0, \epsilon')$ and every $(x, z) \in \text{Gr } \Phi \cap [B(\overline{x}, \epsilon') \times B(\overline{y}, \epsilon')]$, the relation (3.20) holds.

(B) Moreover, suppose that:

(i) $G$ is open at linear rate with respect to $x$ uniformly in $y$ around $(\overline{x}, \overline{y}, z)$ with constant $C > 0$;
(ii) $G$ is Lipschitz-like with respect to $y$ uniformly in $x$ around $(\overline{x}, \overline{y}, z)$ with constant $D \geq 0$;
(iii) $F$ is Lipschitz-like around $(\overline{x}, \overline{y})$ with constant $M > 0$;
(iv) $C - MD > 0$.

Then there exists $\epsilon > 0$ such that, for every $\rho \in (0, \epsilon)$ and every $(x, y, z) \in B(\overline{x}, \epsilon) \times B(\overline{y}, \epsilon) \times B(\overline{z}, \epsilon)$ such that $y \in F(x)$ and $z \in G(x, y)$,
\[
B(z, (C - MD)\rho) \subset \Phi(B(x, \rho)).
\]
(3.22)

If, moreover, $F$ is Lipschitz around $(\overline{x}, \overline{y})$ with constant $M > 0$ and $F(\overline{x}) = \{\overline{y}\}$, then the conclusion reads as follows: there exists $\epsilon > 0$ such that, for every $\rho \in (0, \epsilon)$ and every $(x, z) \in \text{Gr } \Phi \cap [B(\overline{x}, \epsilon) \times B(\overline{z}, \epsilon)]$, the relation (3.22) holds.

**Proof.** The first parts of (A) and (B) easily follow from the last conclusion of Theorem 3.4 taking $F$ instead of $F_1$ or $F_2$, respectively, and setting the other multifunction as the identity mapping.

For the second part of (A), consider $\Gamma : X \times Z \rightrightarrows Y$ defined by $\Gamma(x, z) := \{y \in Y \mid z \in G(x, y)\}$. We can apply Proposition 3.3 to obtain $\gamma > 0$ such that for every $(x, z), (x', z') \in D(\overline{x}, \gamma) \times D(\overline{z}, \gamma)$, and every $\delta > 0$,
\[
\Gamma(x', z') \cap D(\overline{y}, \gamma) \subset \Gamma(x, z) + \frac{1 + \delta}{C}(D \|x - x'\| + \|z - z'\|)D_Y.
\]
(3.23)

Take $\delta > 0$ arbitrary small, set $\epsilon' \in (0, \min\{\gamma, \epsilon, C(D + 1)^{-1}(1 + \delta)^{-1}\})$, and pick $(x, z) \in \text{Gr } \Phi \cap [B(\overline{x}, \epsilon') \times B(\overline{z}, \epsilon')]$. Hence, there exists $y \in F(x)$ such that $z \in G(x, y)$. Apply next (3.23) for $(\overline{x}, \overline{z})$ instead of $(x', z')$ to obtain that there exist $y' \in \Gamma(x, z)$ such that $\|y' - \overline{y}\| < C^{-1}(1 + \delta)(D \|\overline{x} - x\| + \|\overline{z} - z\|) < C^{-1}(1 + \delta)(D + 1)\epsilon' < \epsilon$. Hence, $z \in G(x, y') \cap G(x, y)$, so $y = y' \in B(\overline{y}, \epsilon)$. The conclusion follows now from the first parts of (A).

For the second part of (B), set $\epsilon' \in (0, \min\{\epsilon, M^{-1}\})$, and pick $(x, z) \in \text{Gr } \Phi \cap [B(\overline{x}, \epsilon') \times B(\overline{z}, \epsilon')]$. Again, there exists $y \in F(x)$ such that $z \in G(x, y)$. Using now the additional assumptions,
\[
y \in \{\overline{y}\} + M \|\overline{x} - x\| D_Y,
\]
whence $\|y - \overline{y}\| \leq \epsilon$, and the first part applies in order to get the conclusion.

**Remark 3.6** The final part of the (i) item is [16, Theorem 5]. Note that the requirement of the fulfillment of (3.21) appears in [16]. We observe here that it is enough to have $x$ close to $\overline{x}$ and, moreover, $y'$ could be kept close to $\overline{y}$.

## 4 Filling the gap: metric regularity and fixed points

In this section we point out the relations between fixed point results and openness results. The direction that has been investigated up to now, starting with the Arutyunov’s work [3], and continued with Dontchev-Frankowska [6] and Ioffe’s [16] papers, is the possibility to get openness results from fixed point assertions. Here, we aim to emphasize that the opposite direction is also possible.

For the sake of completeness, we recall here the global version of the Lyusternik-Graves Theorem.
Theorem 4.1 Let \( F : X \rightrightarrows Y \) and \( G : Y \rightrightarrows X \) be two multifunctions such that \( \text{Gr} F \) and \( \text{Gr} G \) are locally closed. Suppose that \( \text{Dom}(F - G^{-1}) \) and \( \text{Dom}(G - F^{-1}) \) are nonempty and let \( L > 0 \) and \( M > 0 \) be such that \( LM > 1 \). If \( F \) is \( L \)--open at every point of its graph, and \( G \) is \( M \)--open at every point of its graph, then \( F - G^{-1} \) is \( (L - M^{-1}) \)--open at every point of its graph and \( G - F^{-1} \) is \( (M - L^{-1}) \)--open at every point of its graph.

This global and fully multivalued version of the Lyusternik-Graves Theorem appeared, for the first time, in [21, Theorem 1]. Meantime, different variants of this result were stated in [14], [4], [5], and the more general assumptions are that \( Y \) is a linear metric space with shift invariant metric, \( X \) is metric space, and the graphs of the involved multifunctions are complete. Another way to prove this theorem uses an iterative procedure of Lyusternik type. As we said before, was emphasized recently that it can also be obtained as a consequence of fixed point theorems, as in [3], [6], [16].

We apply now Theorem 3.4, for the special case where \( Y = Z = W \), \( G(y, z) := y - z \). Note that the following result was obtained in [12] as a consequence of the proof given there for global Lyusternik-Graves Theorem.

Corollary 4.2 Let \( F_1 : X \rightrightarrows Y \) and \( F_2 : X \rightrightarrows Y \) be two multifunctions and \( (\overline{x}, \overline{y}_1, \overline{y}_2) \in X \times Y \times Y \) such that \((\overline{x}, \overline{y}_1) \in \text{Gr} F_1 \) and \((\overline{x}, \overline{y}_2) \in \text{Gr} F_2 \). Suppose that the following assumptions are satisfied:

(i) \( \text{Gr} F_1 \) and is \( \text{Gr} F_2 \) are locally closed around \((\overline{x}, \overline{y}_1)\) and \((\overline{x}, \overline{y}_2)\), respectively;

(ii) \( F_1 \) is \( l \)--metrically regular around \((\overline{x}, \overline{y}_1)\);

(iii) \( F_2 \) is \( m \)--Lipschitz-like around \((\overline{x}, \overline{y}_2)\);

(iv) \( \text{Im} < 1 \).

Then there exists \( \varepsilon > 0 \) such that, for every \( \rho \in (0, \varepsilon) \), \((x, y, z) \in \text{Gr}(F_1, F_2) \cap [B(\overline{x}, \varepsilon) \times B(y_1, \varepsilon) \times B(y_2, \varepsilon)]\),

\[
B(y - z, (l^{-1} - m)\rho) \subset (F_1 - F_2)(B(x, \rho)).
\]

Remark 4.3 Let us observe that Theorem 3.4 opens several possibilities concerning the choices of the involved multifunctions. For instance, one can take \( G(y, z) := T_1(y) + T_2(z) \), where \( T_1 \in \mathcal{L}(Y, W) \) and \( T_2 \in \mathcal{L}(Z, W) \) with \( T_1 \) surjective, and suppose that \( L \parallel (T_1^{-1}) - M \parallel T_2 \parallel > 0 \). Then the conclusion of Theorem 3.4 applies in this special case.

After these comments we have the ingredients needed for closing the circle, showing that one has the following chain of implications:

- Theorem 3.4 \( \Rightarrow \) Corollary 4.2 \( \Rightarrow \) Dontchev-Frankowska Fixed Point Theorem \( \Rightarrow \) Arutyunov Fixed Point Theorem \( \Rightarrow \) global Lyusternik-Graves Theorem

- (Proof of) global Lyusternik-Graves Theorem \( \Rightarrow \) Corollary 4.2

Of course, on the first item, the first implication is given above, while the third one is part of [6]. As we have already said, it is shown in [12], that the proof of global Lyusternik-Graves Theorem based on the use of Ekeland variational principle can be used to obtain Corollary 4.2. The only implication to be proved is the second one on the first item. This is done next.
Consider the case \( \overline{y}_1 = \overline{y}_2 := \overline{y} \). Then 0 \( \in (F_1 - F_2)(\overline{y}) \). Denote
\[
S := (F_1 - F_2)^{-1}(0) = \{ x \in X \mid 0 \in (F_1 - F_2)(x) \}
\]
\[
= \{ x \in X \mid F_1(x) \cap F_2(x) \neq \emptyset \} = \text{Fix}(F_1^{-1}F_2).
\]
Or, \( S \) could be seen as the implicit multifunction associated to \( H : X \times P \Rightarrow Y \), \( H(x,p) := F_1(x) - F_2(x) \) for every \( (x,p) \in X \times P \), and in this case we identify the constant multifunction \( S \) with the set \( S \) given by (4.1).

**Theorem 4.4** Under the assumptions of Corollary \cite{4.2}, there exist \( \alpha, \beta > 0 \) such that for any \( x \in B(\overline{x}, \alpha) \) one has that
\[
d(x, S) \leq (l^{-1} - m)^{-1}d(F_1(x) \cap B(\overline{y}, \beta), F_2(x)).
\]

**Proof.** Under the notations of Theorem \cite{4.2} take \( \alpha, \beta > 0 \) such that \( \alpha < m, \alpha < \varepsilon, m\alpha < \beta, 3\beta < \varepsilon, 2(l^{-1} - m)^{-1}\beta < \varepsilon \). Suppose as well that \( \alpha \) is smaller than the radius of the neighborhood of \( \overline{y} \) involved in the Lipschitz-like property of \( F_2 \) at \( (\overline{x}, \overline{y}) \). Denote \( V := B(\overline{y}, \beta) \) and fix \( x \in B(\overline{x}, \alpha) \). If \( F_1(x) \cap B(\overline{y}, \beta) = \emptyset \) or 0 \( \in F_1(x) \cap B(\overline{y}, \beta) - F_2(x) \) the relation (4.2) automatically holds. On the contrary case, for every \( \mu > 0 \) there exists \( u_\mu \in F_1(x) \cap B(\overline{y}, \beta) - F_2(x) \) s.t.
\[
\| u_\mu \| \leq d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x)) + \mu.
\]
Since \( F_2 \) is Lipschitz-like around \( (\overline{x}, \overline{y}) \) and \( x \in B(\overline{x}, \alpha) \)
\[
\overline{y} \in F_2(\overline{x}) \cap V \subset F_2(x) + m \| x - \overline{x} \| \mathbb{D}_Y,
\]
whence there is \( u' \in F_2(x) \) with
\[
\| u' - \overline{y} \| \leq m \| x - \overline{x} \| \leq m\alpha < \beta.
\]
Therefore, \( F_2(x) \cap V \neq \emptyset \), whence
\[
\| u_\mu \| \leq d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x)) + \mu
\]
\[
\leq d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x) \cap B(\overline{y}, \beta)) + \mu
\]
\[
\leq 2\beta + \mu.
\]
On the other hand, there are \( u_1^1 \in F_1(x) \cap B(\overline{y}, \beta) \) and \( u_2^2 \in F_2(x) \) with \( u_\mu = u_1^1 - u_2^2 \). Accordingly,
\[
\| u_2^2 - \overline{y} \| \leq \| u_1^1 - \overline{y} \| + \| u_\mu \| \leq \beta + 2\beta + \mu = 3\beta + \mu.
\]
So, if we take \( \mu \) small enough one has \( \| u_2^2 - \overline{y} \| \leq \varepsilon \).
Take \( \rho_0 = (l^{-1} - m)^{-1}d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x)) + \mu < 2(l^{-1} - m)^{-1}\beta + \mu \) and hence, again for \( \mu \) small enough, \( \rho_0 < \varepsilon \). Since \( (x, u_1^1, u_2^2) \in \text{Gr}(F_1, F_2) \cap [B(\overline{x}, \varepsilon) \times B(\overline{y}, \varepsilon) \times B(\overline{y}, \varepsilon)] \),
\[
B(u_1^1 - u_2^2, (l^{-1} - m)\rho_0) \subset (F_1 - F_2)(B(x, \rho_0)).
\]
Since \( \| u_\mu \| \leq d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x)) + \mu \), one gets
\[
0 \in B(u_1^1 - u_2^2, (l^{-1} - m)\rho_0)
\]
whence there exists \( x' \in B(x, \rho_0) \) with \( 0 \in (F_1 - F_2)(x') \), whence
\[
d(x, S) \leq (l^{-1} - m)^{-1}d(0, F_1(x) \cap B(\overline{y}, \beta) - F_2(x)) + \mu.
\]
Taking \( \mu \to 0 \) one obtains the solution. □
5 Further remarks

Theorem 4.4 is the main result in [6], and is given there on metric spaces. Here we have obtained it on the base of Corollary 4.2, in order to close the circle between openness of compositions and fixed point theorems. Also, remark that a slightly different variant of this theorem can be stated as a consequence of Theorem 3.1:

Proposition 5.1 Under the assumptions of Corollary 4.2, there exist $\alpha, \beta > 0$ such that for any $x \in B(\pi, \alpha)$ one has that

$$d(x, S) \leq (l^{-1} - m)^{-1}d(0, (F_1 - F_2)(x) \cap B(0, \beta)).$$

Furthermore, the proof of this result, as the proof of the preceding theorem, does not use the full power of Theorem 3.1, which gives conclusions for the implicit multifunction $S$ depending on the parameter $p$. Supposing that $F_1 : X \times P \rightrightarrows Y$, $F_2 : X \rightrightarrows Y$, and taking $H : X \times P \rightrightarrows Y$ as $H(x, p) := F_1(x, p) - F_2(x)$, then the implicit multifunction $S : P \rightrightarrows X$ is given by

$$S(p) := (F_1(\cdot, p) - F_2)^{-1}(0) = \{x \in X \mid 0 \in F_1(x, p) - F_2(x)\} = \{x \in X \mid F_1(x, p) \cap F_2(x) \neq \emptyset\} = \text{Fix}(F_1(\cdot, p)^{-1}F_2).$$

In this way, with a very similar proof to the one of Theorem 4.4, one can also obtain [6] Theorem 7.

Of course, one can use directly Theorem 3.4 to obtain generalized relations with $G(F_1, F_2)$ instead of $F_1 - F_2$. Finally, observe that almost all the openness results given before can also be stated in parametric forms.

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