Research Article

A Surface Family with a Common Asymptotic Null Curve in Minkowski 3-Space \( E^3_1 \)

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This approach is on constructing a surface family with a common asymptotic null curve. It has provided the necessary and sufficient condition for the curve to be an asymptotic null curve and extended the study to ruled and developable surfaces. Subsequently, the study has examined the Bertrand offsets of a surface family with a common asymptotic null curve. Lastly, we support the results of this approach by some examples.

1. Introduction

In differential geometry of surfaces, an asymptotic direction is a curve with zero normal curvature. At a point on an asymptotic curve, we take a plane having both the curve's tangent and the normal of a surface. At such point, the trace curve of intersection of the surface and the plane has vanished curvature. Asymptotic directions exactly occur in case of negative (or zero) Gaussian curvature. Exactly speaking, there we will have two asymptotic directions bisected by the principal directions \([1–3]\).

Wang et al. \([4]\) addressed the constructions of family of surfaces sharing a given geodesic curve in Euclidean 3-space \( E^3 \). Inspired by the work of Wang et al. \([4]\), Bayram et al. \([5]\) considered surfaces with a shared asymptotic curve in \( E^3 \). They derived the parametric representation of a surface family by means of a given curve as an isoparametric and asymptotic curve. In \([6]\), Abdel-Baky and Al-Ghefari introduced some interesting developable and ruled surfaces as surface families in terms of given asymptotic curves. The study of Li et al. in \([7]\) was on forming a surface family by means of a given spatial curve and how to be a line of curvature on a surface. They provided three kinds of marching-scale functions and the necessary and sufficient conditions on them to meet both isoparametric and line of curvature requirements.

A lot of works dealing with family of surfaces having a common special curve in both Euclidean space and Minkowski space have been published (see, for instance, \([8–13]\)). The main interest of this work is to construct a surface family from a given asymptotic null curve. Hence, the sufficient and necessary conditions for the given curve to be the asymptotic null curve are given in details. As an application, some representative curves are selected to form their corresponding surfaces that have such curves as asymptotic null curves. We extend the study to ruled and developable surfaces. Finally, some related examples to these surfaces are illustrated.

2. Preliminaries

Consider the Minkowski 3-space \( E^3_1 \) as the ambient space. For our approach, we have utilized the relevant information from \([14–17]\). Let \( E^3_1 \) denote the Minkowski 3-space \( E^3_1 \), i.e., \( \mathbb{R}^3_1 \) equipped with the metric:

\[
\langle dx, dx \rangle = -dx_1^2 + dx_2^2 + dx_3^2,
\]  

(1)
in which \((x_1, x_2, x_3)\) is the canonical coordinates in \(\mathbb{R}^3\). A vector \(x\) in \(\mathbb{E}^3\) is referred to as light-like or null when \(\langle x, x \rangle = 0\) and \(x \neq 0\), space-like if \(\langle x, x \rangle > 0\), and time-like if \(\langle x, x \rangle < 0\). A light-like or time-like vector in \(\mathbb{E}^3\) is referred to as causal. We define the norm of \(x\) in \(\mathbb{E}^3\) as \(\|x\| = \sqrt{\langle x, x \rangle}\); then, \(x\) is a time-like unit vector if \(\langle x, x \rangle = -1\), and space-like unit if \(\langle x, x \rangle = 1\). Analogously, a regular curve \(\alpha: I \longrightarrow \mathbb{E}^3\) is time-like, space-like, or light-like if its tangent vector \(\alpha'\) is time-like, space-like, or light-like, respectively. Similarly, we say that a surface is time-like, space-like, or light-like if its tangent planes are time-like, space-like, or light-like, respectively.

\[\langle x, x \rangle = -x_1^2 + x_2^2 + x_3^2,\]  
\[x \cdot y = x_1 y_1 + x_2 y_2 + x_3 y_3,\]  
\[\langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3,\]

\[x \times y = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}.\]

Let \(\alpha = \alpha(s)\) be a null curve parameterized by its arc length; that is, the tangent vector \(\alpha' = A\) is null. Then, there exists a unique Cartan null frame \(\{A, B, C\}\) satisfying that \([16, 17]\)

\[\langle A, A \rangle = \langle B, B \rangle = \langle C, C \rangle = 0, \quad \langle A, B \rangle = \langle A, C \rangle = \langle B, C \rangle = 0, \quad A \times B = C, \quad A \times C = A, \quad B \times C = B.\]

The Frenet–Serret equations associated to such frame are as follows:

\[\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & 0 & \kappa \\ 0 & \omega & 0 \\ \omega & 0 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix},\]

where \(\kappa(s) \neq 0\) is a function on \(\alpha(s)\) and \(\omega\) is constant. Furthermore, if \(\omega = 0\), then \(\alpha = \alpha(s)\) is a generalized null cubic [13]. The vector fields \(B\) and \(C\) are referred to as the binormal vector field and principal normal vector field of \(\alpha(s)\), respectively.

\[\alpha' (s^*) = \alpha(s) \pm f C(s), \quad f = \mp \omega^{-1}\]

where \(f = \mp \omega^{-1}\) is the constant distance between their corresponding points. Note that, if \(\alpha'\) is a Bertrand mate of a curve \(\alpha\), then the converse is true. Therefore, the formulae link the Cartan frame of \(\alpha\) with that of its Bertrand mate \(\alpha'\) are \([17]\)

\[\begin{pmatrix} A' \\ B' \\ C' \end{pmatrix} = \begin{pmatrix} 0 & \mu & 0 \\ \mu & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix}, \quad \mu(s) = \pm \frac{\kappa}{\omega} \frac{ds}{ds}.\]

An isoparametric curve \(\alpha = \alpha(s)\) on a surface \(P(s, t)\) in \(\mathbb{E}^3\) is a curve in which there exists a parameter \(s_0\) or \(t_0\) such that \(\alpha(s) = P(s, t_0)\) or \(\alpha(t) = P(s_0, t)\). Given a parametric curve \(\alpha = \alpha(s)\), we call it an iso-asymptotic of the surface \(P(s, t)\) if it is both asymptotic and a parameter curve on \(P(s, t)\).

**Lemma 1.** A nonlinear regular curve \(\alpha(s)\) without inflections is an asymptotic on a surface \(P(u, v)\) if and only if it satisfies one of the following equivalent conditions:

1. The osculating plane at each noninflection point of \(\alpha(s)\) meets the surface tangential plane at \(\alpha(s) = P(u(s), v(s))\).
2. The binormal at each noninflection point of \(\alpha(s)\) is orthogonal to the surface tangential plane at \(\alpha(s) = P(u(s), v(s))\).

### 3. Surfaces with Common Asymptotic Null Curve

In this section, we present a new approach for constructing a surface family with a shared asymptotic null curve \(\alpha = \alpha(s)\), \(0 \leq s \leq L\), in which the curve osculating plane \(\{A, C\}\) is coincident with the surface tangential plane. The expression of the surface over \(\alpha(s)\) is given by

\[M: P(s, t) = \alpha(s) + u(s, t) A(s) + v(s, t) C(s), \quad 0 \leq t \leq T,\]

where \(u(s, t)\) and \(v(s, t)\) are \(C^1\) functions. If the parameter \(t\) is considered as the time, the functions \(u(s, t)\) and \(v(s, t)\) can be considered as directed marching distances of a point unit in the time \(t\) in the direction \(A\) and \(C\), respectively, and then, the position vector \(\alpha(s)\) is the initial location of the point. It is readily to check that the tangent vectors of \(M\) are given by

\[P_s = (1 + u_s(s, t) + \omega v(s, t)) A + \kappa v(s, t) B + (v_s(s, t) + \kappa u(s, t)) C, \quad P_t = u_s(s, t) A + v_t(s, t) C.\]

For convention, we use \(P_s = \partial P/\partial s\) and \(P_t = \partial P/\partial t\). Hence, the normal vector of \(M\) is

\[N(s, t) := P_s \times P_t = \eta_1(s, t) A(s) + \eta_2(s, t) B(s) + \eta_3(s, t) C(s),\]

where
\[
\begin{align*}
\eta_1 (s, t) &= -\left[1 + u_1 (s, t) + v (s, t) \omega \right] v_1 (s, t) - (v_1 (s, t) + \kappa u (s, t)) u_1 (s, t) \\
\eta_2 (s, t) &= -\kappa v (s, t) u_1 (s, t) \\
\eta_3 (s, t) &= -\kappa v (s, t) v_1 (s, t),
\end{align*}
\]

Additionally, since \(\alpha (s)\) is an isoparametric curve on \(M\), there is a parameter \(t = t_0 \in [0, T]\) in which \(P(s, t_0) = \alpha (s)\); that is,

\[
\begin{align*}
u (s, t_0) &= \nu_1 (s, t_0) = 0, \\
\frac{\partial u (s, t_0)}{\partial s} = \frac{\partial \nu (s, t_0)}{\partial s} = 0.
\end{align*}
\]

Therefore,

\[
N (s, t_0) = \eta_1 (s, t_0) A (s) + \eta_2 (s, t_0) B (s) + \eta_3 (s, t_0) C (s),
\]

where

\[
\begin{align*}
\eta_1 (s, t_0) &= -\nu_1 (s, t_0) \neq 0, \\
\eta_2 (s, t_0) &= \eta_3 (s, t_0) = 0.
\end{align*}
\]

Thus, when \(t = t_0\)—over \(\alpha (s)\)—, we obtain

\[
N (s, t_0) = -\nu_1 (s, t_0) A (s).
\]

This identifies the curve \(\alpha (s)\) as an asymptotic curve on the surface. Therefore, we provide the following theorem.

**Theorem 1.** The given spatial null curve \(\alpha (s)\) is iso-asymptotic on the surface \(P(s, t)\) if and only if

\[
\begin{align*}
u (s, t_0) &= \nu_1 (s, t_0) = 0, \\
\frac{\partial u (s, t_0)}{\partial s} = \frac{\partial \nu (s, t_0)}{\partial s} = 0,
\end{align*}
\]

The surfaces defined by equations (7) and (15) are referred to as the family of surfaces with a common asymptotic null curve. Any surface \(P(s, t)\) defined by equation (7) and satisfying equation (15) is a member of this family. As in [4], we also consider the case when the marching-scale functions \(u(s, t)\) and \(v(s, t)\) can be given into two factors as

\[
\begin{align*}
u (s, t) &= I (s) U (t), \\
v (s, t) &= M (s) V (t),
\end{align*}
\]

where \(I (s), M (s), U (t),\) and \(V (t)\) are \(C^1\) functions not identically zero. Therefore, we can derive the following corollary.

**Corollary 1.** The sufficient and necessary condition of the null curve \(\alpha (s)\) being an iso-asymptotic null curve on \(P(s, t)\) is

\[
U (t_0) = V (t_0) = 0, I (s) = \text{const}, m (s) = \text{const} \neq 0,
\]

\[
\frac{dV (t_0)}{dt} = \text{const} \neq 0, 0 \leq t_0 \leq T, 0 \leq s \leq L.
\]

It follows from Corollary 1 that, to attain a surface family, with a shared null asymptotic curve, we first pick the marching-scale functions as in equation (17) and then apply them to equation (7). For more convenience, although the marching-scale functions can be given in restricted forms, they can define a large class of surface family with a shared asymptotic null curve as follows:

(1) If we take

\[
\begin{align*}
u (s, t) &= \sum_{k=1}^{p} a_{1k} l (s)^k U (t)^k, \\
v (s, t) &= \sum_{k=1}^{p} a_{2k} m (s)^k V (t)^k.
\end{align*}
\]

We can write the sufficient condition for which the curve \(\alpha (s)\) is an iso-asymptotic null curve on the surface \(P(s, t)\) as follows:

\[
\begin{align*}
u (s, t) &= \sum_{k=1}^{p} a_{ik} l (s)^k U (t)^k, \\
v (s, t) &= \sum_{k=1}^{p} a_{2k} m (s)^k V (t)^k.
\end{align*}
\]

(2) If we pick

\[
\begin{align*}
u (s, t) &= f \left( \sum_{k=1}^{p} a_{1k} l (s)^k U (t)^k \right), \\
v (s, t) &= g \left( \sum_{k=1}^{p} a_{2k} m (s)^k V (t)^k \right).
\end{align*}
\]

We can rewrite condition (17), for which \(\alpha = \alpha (s)\) is iso-asymptotic null curve on the surface \(P(s, t)\), as follows:

\[
\begin{align*}
u (s, t) &= \sum_{k=1}^{p} a_{1k} l (s)^k U (t)^k, \\
v (s, t) &= \sum_{k=1}^{p} a_{2k} m (s)^k V (t)^k.
\end{align*}
\]

\[
\begin{align*}
u (s, t) &= f \left( \sum_{k=1}^{p} a_{1k} l (s)^k U (t)^k \right), \\
v (s, t) &= g \left( \sum_{k=1}^{p} a_{2k} m (s)^k V (t)^k \right).
\end{align*}
\]

in which \(l (s), m (s), U (t), V (t), f,\) and \(g\) are \(C^1\) functions. The factor-decomposition form is of
advantageous. Any set of functions $l(s)$ and $m(s)$ would meet (18) or (21). Therefore, the designer can adjust the shape of the surface by posing some conditions on the sets of $l(s)$ and $m(s)$ which guarantee resulting surface which belongs to the iso-asymptotic surface family with the null curve $\alpha(s)$ as the asymptotic.

**Example 1.** Given the null helix,

$$\alpha(s) = (s, \sin s, \cos s), 0 \leq s \leq 2\pi.$$  \hfill (22)

Therefore, we have the Cartan frame as follows:

$$A = (1, \cos s, -\sin s),$$

$$B = \frac{1}{2} (1, -\cos s, \sin s),$$

$$C = (0, -\sin s, -\cos s),$$

$$\kappa = 1,$$ and $$\omega = \frac{1}{2} \frac{1}{2}.$$

(23)

Then, we get a surface family with a common asymptotic null curve given by

$$P(s, t) = (s, \sin s, \cos s)$$

$$+ (u(s, t), 0, v(s, t)) \begin{pmatrix}
1 & \cos s & -\sin s \\
\frac{1}{2} & -\frac{1}{2} \cos s & \frac{1}{2} \sin s \\
0 & -\sin s & -\cos s
\end{pmatrix}.$$  \hfill (24)

Hence, we can get special members of the family as follows:

(1) By choosing $u(s, t) = t$ and $v(s, t) = -t, t_0 = 0$, where $\beta, \gamma \in \mathbb{R}, \gamma \neq 0$, and $-2 \leq t \leq 2$, then equation (17) is satisfied. Therefore, we attain a surface of the family (Figure 1)

$$P(s, t; \beta, \gamma) = (s, \sin s, \cos s)$$

$$+ t (\beta, 0, \gamma) \begin{pmatrix}
1 & \cos s & -\sin s \\
\frac{1}{2} & -\frac{1}{2} \cos s & \frac{1}{2} \sin s \\
0 & -\sin s & -\cos s
\end{pmatrix}.$$  \hfill (25)

(2) If we pick $u(s, t) = st^2$ and $v(s, t) = st$, where $t_0 = 0$ and $-6 \leq t \leq -6$, equation (19) is satisfied. Therefore, we get a surface of the family (Figure 2)

$$P(s, t, st^2, st) = (s, \sin s, \cos s)$$

$$+ ts(t, 0, 1) \begin{pmatrix}
1 & \cos s & -\sin s \\
\frac{1}{2} & \frac{1}{2} \cos s & \frac{1}{2} \sin s \\
0 & -\sin s & -\cos s
\end{pmatrix}.$$  \hfill (26)

(3) By choosing $u(s, t) = t^2$ and $v(s, t) = t$, where $t_0 = 0$ and $-1.7 \leq t \leq 1.7$, then equation (19) is satisfied. Therefore, we get a member of the family (Figure 3)

$$P(s, t, t^2, t) = (s, \sin s, \cos s)$$

$$+ t(t, 0, 1) \begin{pmatrix}
1 & \cos s & -\sin s \\
\frac{1}{2} & -\frac{1}{2} \cos s & \frac{1}{2} \sin s \\
0 & -\sin s & -\cos s
\end{pmatrix}.$$  \hfill (27)

(4) By choosing $u(s, t) = t \sin t$ and $v(s, t) = t$, where $t_0 = 0$ and $-2 \leq t \leq 2$, then equation (19) is satisfied. Therefore, we get a member of the family (Figure 4)

$$P(s, t, \sin t, \cos s) = (s, \sin s, \cos s)$$

$$+ (t \sin t, 0, t) \begin{pmatrix}
1 & \cos s & -\sin s \\
\frac{1}{2} & \frac{1}{2} \cos s & \frac{1}{2} \sin s \\
0 & -\sin s & -\cos s
\end{pmatrix}.$$  \hfill (28)

4. Ruled Surfaces with a Common Asymptotic Null Curve

Let $P(s, t)$ be a ruled surface with the directrix $\alpha(s)$ in which $\alpha(s)$ is an isoparametric null curve of $P(s, t)$; i.e., there is $t_0$ in which $P(s, t_0) = \alpha(s)$. Hence, the surface is defined as
Now, we utilize the conditions given in Theorem 1 to prove that \( \alpha(s) \) is asymptotic on \( P(s,t) \). Hence, these conditions become as
\[
\langle d, C \rangle \neq 0. \tag{32}
\]

Thus, the ruling direction \( d(s) \) is in the light-like plane \( Sp[A(s), C(s)] \) at any point on \( \alpha(s) \). Additionally, \( d(s) \) and \( A(s) \) must not be linearly dependent. Hence,
\[
d(s) = \beta(s)A(s) + \gamma(s)C(s), \quad 0 \leq s \leq L, \tag{33}
\]
for some real functions \( \beta(s) \) and \( \gamma(s) \neq 0 \). Utilizing it in the expressions in equation (30), we obtain:
\[
\begin{align*}
    u(s,t) &= t\beta(s), \\
    v(s,t) &= t\gamma(s), \quad \text{with } \gamma(s) \neq 0, \ 0 \leq s \leq L.
\end{align*}
\]

Therefore, the family of isoparametric-rulled surfaces with null asymptotic curve \( \alpha(s) \) would take the form
\[
M: P(s,t) = \alpha(s) + t \ d(s), \tag{35}
\]
where \( d(s) = \beta(s)A(s) + \gamma(s)C(s) \), for all \( 0 \leq s \leq L \), and the functions \( \beta(s) \) and \( \gamma(s) \neq 0 \) determine the shape of \( M \). We should point out that, in such a model, there are two asymptotic null curves moving through every point on \( \alpha(s) \), one of which is \( \alpha(s) \) itself and the other is a line in the null direction \( d(s) \) given by equation (33). Every member in the family of isoparametric-rulled surfaces with the shared null asymptotic \( \alpha(s) \) is determined by the two family parameters: \( \gamma(s) \) and \( \beta(s) \).

**Theorem 2.** The necessary and sufficient condition for \( P(s,t) \) to be a ruled surface with \( \alpha(s) \) as a common null asymptotic curve is that \( \exists t_0 \in [0,T] \) and functions \( \beta(s) \) and \( \gamma(s) \neq 0 \), in which \( P(s,t) \) can be constructed by equation (35).

Every member in the family isoparametric-rulled surfaces with the null asymptotic \( \alpha(s) \) is determined by \( \gamma(s) \) and \( \beta(s) \), that is, by the direction vector function \( d(s) \). In Example 1, for \( u(s,t) = t^2 \) and \( v(s,t) = t \), with \( -6 \leq t \leq -6 \), the corresponding ruled surface is shown in Figure 5, and for \( u(s,t) = 0 \) and \( v(s,t) = t \), with \( -2 \leq t \leq -2 \), Figure 6 shows the surface with \( u(s,t) = 0 \ v(s,t) = 0 \ -2 \leq t \leq 0.2 \).

### 4.1. Classification of Ruled Surfaces with a Common Asymptotic Null Curve

In this section, we classify the ruled surfaces with a common asymptotic null curve as time-like, light-like, and space-like surfaces. For this purpose, from equation (33), for all \( 0 \leq s \leq L \), we have
\[
\|d\|^2 = \gamma^2 > 0. \tag{36}
\]

By taking the partial derivatives regarding \( s \) and \( t \), respectively, we obtain
\[
\begin{align*}
    P_s &= A + td', \\
    P_t &= d, \tag{37}
\end{align*}
\]

Thus,
Theorem 5. Let \( M \) be a noncylindrical ruled surface \( M \) with parametrization (35) such that \( d' \) is a null vector. Then, \( M \) is a light-like surface in \( \mathbb{E}^1_3 \).

We now give the conditions for the ruled surface \( M \) defined by (35) to be cylindrical. To do that, we take the derivative of equation (33) regarding \( s \) and use equation (4); hence, we obtain

\[
d \times d' = [\beta (\gamma' + \beta \kappa) - \gamma (\beta' + \gamma \omega)] A + \gamma^2 \kappa B + \gamma \beta \kappa C.
\]

Therefore, \( P(s, t) \) is cylindrical if and only if

\[
\beta (\gamma' + \beta \kappa) - \gamma (\beta' + \gamma \omega) = 0, \quad \gamma^2 \kappa = 0, \quad \text{and} \quad \gamma \beta \kappa = 0.
\]

Since \( \gamma(s) \neq 0 \), for all \( 0 \leq s \leq L \), the above equation cannot equal zeros. Hence, there exists no cylindrical ruled surface given by (35).

Corollary 2. There exists no cylindrical and developable ruled surface represented in (35).

4.2. Bertrand Offsets for Surfaces with a Common Asymptotic Null Curve. Here, we examine the Bertrand offsets of a surface family with a shared asymptotic null curve. I\( \ddot{\text{u}} \)hen, given in

I\( \ddot{\text{u}} \)hen, analogous theory to the theory of Bertrand curves can be developed for these surfaces.

Definition 1. Let \( M \) and \( M^\ast \) be two surfaces based on iso-asymptotic null curves \( \alpha(s) \) and \( \alpha^\ast(s') \), respectively. \( M^\ast \) is the Bertrand offset of \( M \) if and only if \( \alpha(s) \) and \( \alpha^\ast(s') \) are the Bertrand mates.

Assume that \( M^\ast \) is a Bertrand offset of \( M \) with the Serret–Frenet frame \( (\alpha(s); A, B, C) \) given by equation (7). Then, \( M^\ast \) can take the form

\[
M^\ast; P^\ast(s', t) = \alpha^\ast(s') + u^\ast(s', t)A^\ast(s') + v(s', t)C^\ast(s'); \quad 0 \leq t \leq T,
\]

where \( u^\ast(s', t) \) and \( v(s', t) \) have the same meaning as in equation (7).

Now, we provide a representative example to illustrate such method and verify the correctness of the derived formulae.

Example 2. In Example 1, according to equation (5) and equation (6), we have \( f = \pm 2 \) and \( \mu = \pm 2(ds/ds^\ast) \). If we choose \( f = 2 \) and \( s^\ast = s \) and lower sign of \( \mu \), then

\[
\alpha^\ast(s) = (s, -\sin s, \cos s),
\]

with the Cartan frame:
Here, the surface family \( M^\ast \) can be written as \[ P^\ast(s, t) = (s, -\sin s, -\cos s) + (u^\ast(s, t), 0, v^\ast(s, t)) \] where

\[
\begin{align*}
A^\ast(s) &= (1, -\cos s, \sin s), \\
B^\ast(s) &= \frac{1}{2} (1, \cos s, -\sin s), \\
C^\ast(s) &= (0, \sin s, \cos s).
\end{align*}
\] (44)

Therefore, the surface family \( M^\ast \) can be written as
\[ P^\ast(s, t) = (s, -\sin s, -\cos s) + (u^\ast(s, t), 0, v^\ast(s, t)) \] where \((u^\ast(s, t), 0, v^\ast(s, t)) = (1, 1, 0 \cdot \cos s - \frac{1}{2} \sin s)\), \((0, \sin s, \cos s)\).

(45)

In view of equations (24) and (45), if we take \( u(s, t) = u^\ast(s, t) = 1 - \cos t \) and \( v(s, t) = v^\ast(s, t) = \sin t \), and \( t_0 = 0 \), where \( 0 \leq t \leq 2\pi \), then equation (15) is satisfied. The graphs of \( M \) and \( M^\ast \) are depicted in Figures 7 and 8.

Note that if we choose a different combination of characteristic curve, or even a number of curves, we would get and produce such series of surfaces.

5. Conclusion

In the Minkowski 3-space \( \mathbb{E}^3_1 \), there are many papers dealing with the problem of forming a family of surfaces from a given asymptotic curve [9, 10, 12, 13]. Here, we provide a different approach for building a surface family in which its members are sharing a given asymptotic null curve as isoparametric. Given a null space curve, we derive the characterization for the given curve to be asymptotic and for the resulting surface to be ruled. Finally, we analyzed the case that surfaces family having a Bertrand offset of a given curve as asymptotic null. Hopefully, the results of this study would be applicable for physicists and those of interest in general relativity theory.

Data Availability

All of the data are available within the paper.

Conflicts of Interest

The authors have no conflicts of interest.

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