A Fractional Lie Group Method For Anomalous Diffusion Equations

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Abstract
The Lie group method provides an efficient tool to solve nonlinear partial differential equations. This paper suggests a fractional partner for fractional partial differential equations. A space-time fractional diffusion equation is used as an example to illustrate the effectiveness of the Lie group method.

Keywords: Lie group method; Anonymous diffusion equation; Fractional characteristic method

1 Introduction
In the last three decades, researchers have found fractional differential equations (FDEs) useful in various fields: rheology, quantitative biology, electrochemistry, scattering theory, diffusion, transport theory, probability potential theory and elasticity [1], for details, see the monographs of Kilbas et al. [2], Kiriyaoka [3], Lakshmikantham and Vatsala [4], Miller and Ross [5], and Podlubny [6]. On the other hand, finding accurate and efficient methods for solving FDEs has been an active research undertaking.

Since Sophus Lie’s work on group analysis, more than 100 years ago, Lie group theory has become more and more pervasive in its influence on other mathematical disciplines [7, 8]. Then a question may naturally arise: is there a fractional Lie group method for fractional differential equations?

Up to now, only a few works can be found in the literature. For example, Buckwar and Luchko derived scaling transformations [9] for the fractional diffusion equation in Riemann-Liouville sense

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = D \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < \alpha, \quad 0 < x, \quad 0 < t, \quad 0 < D. \] (1)

Gazizov et al. find symmetry properties of fractional diffusion equations of Caputo derivative [10]

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k \frac{\partial (k(u)u_x(x,t))}{\partial x}, \quad 0 < \alpha, \quad 0 < x, \quad 0 < t, \quad 0 < k. \] (2)

Djordjevic and Atanackovic [11] obtained some similarity solutions for the time-fractional heat diffusion

\[ \frac{\partial^\alpha T(x,t)}{\partial t^\alpha} = k \frac{\partial^2(T(x,t))}{\partial x^2}, \quad 0 < \alpha, \quad 0 < x, \quad 0 < t. \] (3)

In this study, we investigate anonymous diffusion [12]

\[ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}}, \quad 0 < \alpha, \quad \beta \leq 1, \quad 0 < x, \quad 0 < t, \] (4)

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with a fractional Lie group method, and derive its classification of solutions. Here the fractional derivative is in the modified Riemann-Liouville sense [13] and is defined by \( \frac{\partial^\beta u(x,t)}{\partial x^\beta} \).

2 Characteristic Method for Fractional Differential Equations

Through this paper, we adopt the fractional derivative in modified Riemann-Liouville sense [13].

Firstly, we introduce some properties of the fractional calculus that we will use in this study.

(I) Integration with respect to \((dx)\alpha\) (Lemma 2.1 of [14])

\[
0 \mathcal{I}_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\xi)^{\alpha-1} f(\xi) d\xi = \frac{1}{\Gamma(\alpha+1)} \int_0^x f(\xi) (dx)\alpha, 0 < \alpha \leq 1.
\]

(II) Some other useful formulas

\[
f([x(t)])^{(\alpha)} = \frac{d}{dx} x^{(\alpha)}(t), \quad 0 D_x^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}.
\]

The properties of Jumarie’s derivative were summarized in [13]. The extension of Jumaire’s fractional derivative and integral to variational approach of several variables is done by Almeida et al. [15]. Fractional variational interactional method is proposed for fractional differential equations [16].

It is well known that the method of characteristics has played a very important role in mathematical physics. Preciously, the method of characteristics is used to solve the initial value problem for general first order. With the modified Riemann-Liouville derivative, Jumaire ever gave a Lagrange characteristic method [17]. We present a more generalized fractional method of characteristics and use it to solve linear fractional partial equations.

Consider the following first order equation,

\[
a(x,t) \frac{\partial u(x,t)}{\partial x} + b(x,t) \frac{\partial u(x,t)}{\partial t} = c(x,t).
\]

The goal of the method of characteristics is to change coordinates from \((x, t)\) to a new coordinate system \((x_0, s)\) in which the PDE becomes an ordinary differential equation along certain curves in the \(x-t\) plane. The curves are called the characteristic curves. More generally, we consider to extend this method to linear space-time fractional differential equations

\[
a(x,t) \frac{\partial^\beta u(x,t)}{\partial x^\beta} + b(x,t) \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = c(x,t), 0 < \alpha, \beta \leq 1.
\]

With the fractional Taylor’s series in two variables [13]

\[
du = \frac{\partial^\beta u(x,t)}{\Gamma(1+\beta)\partial x^\beta} (dx)^\beta + \frac{\partial^\alpha u(x,t)}{\Gamma(1+\alpha)\partial t^\alpha} (dt)^\alpha, 0 < \alpha, \beta \leq 1.
\]

Similarly, we derive the generalized characteristic curves

\[
\frac{du}{ds} = c(x,t),
\]

\[
\frac{(dx)^\beta}{\Gamma(1+\beta)ds} = a(x,t),
\]

\[
\frac{(dt)^\alpha}{\Gamma(1+\alpha)ds} = b(x,t).
\]

Eqs. (10)-(12) can be reduced to Jumarie’s result if \(\alpha = \beta\) in [17].
As an example, we consider the fractional equation
\[
\frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta u(x,t)}{\partial x^\beta} + \frac{2t^\alpha}{\Gamma(1+\alpha)} \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = 0, \quad 0 < \alpha, \beta \leq 1.
\]
(13)

We can have the fractional scaling transformation
\[
u = \nu(\frac{x^{2\beta}}{\Gamma^2(1+\beta)} + \frac{2t^\alpha}{\Gamma(1+\alpha)}).
\]
(14)

Note that when \(\alpha = \beta = 1\), as is well known, \(\frac{x^2}{\sqrt{t}}\) is one invariant of the line differential equation
\[
\frac{x}{\partial u(x,t)} + 2t \frac{\partial u(x,t)}{\partial t} = 0.
\]
(15)

3 Lie Group method for Fractional diffusion equation

With the proposed fractional method of characteristics, now we can consider a fractional Lie Group method for the fractional diffusion equation, which are the generalizations of the classical diffusion equations treating the super-diffusive flow processes. These equations arise in continuous-time random walks, modeling of anomalous diffusive and sub-diffusive systems, unification of diffusion and wave propagation phenomenon [18 - 23].

We assume the one-parameter Lie group of transformations in \((x, t, u)\) given by
\[
\begin{align*}
\frac{x^\beta}{\Gamma(1+\beta)} &= \frac{\hat{x}^\beta}{\Gamma(1+\beta)} + \varepsilon \xi(x, t, u) + O(\varepsilon), \\
\frac{t^\alpha}{\Gamma(1+\alpha)} &= \frac{\hat{t}^\alpha}{\Gamma(1+\alpha)} + \varepsilon \tau(x, t, u) + O(\varepsilon), \\
\hat{u} &= u + \varepsilon \phi(x, t, u) + O(\varepsilon),
\end{align*}
\]
(16)

where \(\varepsilon\) is the group parameter.

We start from the set of fractional vector fields instead of using the one of integer order [9 - 11]
\[
V = \xi(x, t, u) D_x^\beta + \tau(x, t, u) D_t^\alpha + \phi(x, t, u) D_u.
\]
(17)

The fractional second order prolongation \(\text{Pr}^{(2\beta)} V\) of the infinitesimal generators can be represented as
\[
\text{Pr}^{(2\beta)} V = V + \phi^{[t]} \frac{\partial \phi}{\partial D_t^\alpha u} + \phi^{[x]} \frac{\partial \phi}{\partial D_x^\beta u} + \phi^{[t]} \frac{\partial \phi}{\partial D_t^{2\alpha} u} + \phi^{[xx]} \frac{\partial \phi}{\partial D_x^{2\beta} u} + \phi^{[t]} \frac{\partial \phi}{\partial D_x^{2\beta} D_t^\alpha u}.
\]
(18)

As a result, we can have
\[
\text{Pr}^{(2\beta)} V(\Delta[u]) = 0,
\]
(19)

where \(\Delta[u] = \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} - \frac{\partial^\beta u(x,t)}{\partial x^{2\beta}}\).

Eq. (19) can be rewritten in the form
\[
(\phi^{[t]} - \phi^{[xx]}) \bigg|_{\Delta[u]=0} = 0.
\]
(20)

The generalized prolongation vector fields are defined as
\[
\begin{align*}
\phi^{[t]} &= D_t^\alpha \phi - (D_t^\alpha \xi) D_x^\beta u - (D_t^\alpha \tau) D_t^\alpha u, \\
\phi^{[x]} &= D_x^\beta \phi - (D_x^\beta \xi) D_x^\beta u - (D_x^\beta \tau) D_t^\alpha u, \\
\phi^{[xx]} &= D_x^{2\beta} \phi - 2(D_x^\beta \xi) D_x^{2\beta} u - (D_x^{2\beta} \xi) D_x^{2\beta} u - 2(D_x^{2\beta} \tau) D_x^{2\beta} D_t^\alpha u - (D_x^{2\beta} \tau) D_t^{2\alpha} u.
\end{align*}
\]
(21)-(23)

Substituting Eqs. (21)-(23) into Eq. (20) and setting the coefficients to zero, we can obtain some line fractional equations from which we can derive
\[\xi(x, t, u) = c_1 + c_4 \frac{x^\beta}{\Gamma(1+\beta)} + 2c_5 \frac{t^\alpha}{\Gamma(1+\alpha)} + 4c_6 \frac{x^\beta t^\alpha}{\Gamma(1+\beta)\Gamma(1+\alpha)},\]
\[\tau(x, t, u) = c_2 + 2c_4 \frac{x^\beta}{\Gamma(1+\beta)} + 4c_6 \frac{t^\alpha}{\Gamma(1+\alpha)},\]
\[\phi(x, t, u) = (c_3 - c_5 \frac{x^\beta}{\Gamma(1+\beta)} - 2c_6 \frac{t^\alpha}{\Gamma(1+\alpha)} - c_6 \frac{x^2 \beta}{\Gamma(1+2\beta)}u + a(x, t),\]
where \(c_i (i = 0...6)\) are real constants and the function \(a(x, t)\) satisfies
\[\frac{\partial^\alpha a(x, t)}{\partial a^{2\alpha}} = \frac{\partial^{2\beta} a(x, t)}{\partial a^{2\beta}}, \quad 0 < \alpha \leq 1, \quad 0 < \beta \leq 1.\] (24)

It is easy to check that the two vector fields \(\{V_1, V_2, V_3, V_4, V_5, V_6\}\) are closed under the Lie bracket. Thus, a basis for the Lie algebra is \(\{V_1, V_2, V_3, V_4, V_5\}\), which consists of the four-dimensional sub-algebra \(\{V_1, V_2, V_3, V_4\}\)

\[v_1 = \frac{\partial^\beta}{\partial x^\beta}, \quad v_2 = \frac{\partial^\alpha}{\partial t^\alpha}, \quad v_3 = \frac{\partial}{\partial x}, \quad v_4 = \frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta}{\partial x^\beta} + \frac{2t^\alpha}{\Gamma(1+\alpha)} \frac{\partial^\alpha}{\partial t^\alpha},\]
\[v_5 = \frac{2t^\alpha}{\Gamma(1+\alpha)} \frac{\partial}{\partial t^\alpha} - \frac{ux^\beta}{\Gamma(1+\beta)} \frac{\partial}{\partial x}, \quad v_6 = \frac{4\alpha}{\Gamma(1+\alpha)\Gamma(1+\beta)} \frac{x^\beta}{\Gamma(1+\beta)} \frac{\partial^\beta}{\partial x^\beta} + \frac{4(t^\alpha)^2}{\Gamma(1+2\alpha)} \frac{\partial^\alpha}{\partial t^\alpha} - \left(\frac{x^2 \beta}{\Gamma(1+2\beta)} + \frac{2t^{\alpha}}{\Gamma(1+\alpha)}\right)u \frac{\partial}{\partial a},\]
and one infinite-dimensional sub-algebra
\[v_7 = a(x, t) \frac{\partial}{\partial a}.\] (25)

Assume \(u = f\left(\frac{x^\beta}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}\right)\) is an exact solution of Eq. (4). Then with the proposed fractional method of characteristics, solving the above symmetry equations, we can derive
\[u(1) = f\left(\frac{x^\beta}{\Gamma(1+\alpha)} - \varepsilon \frac{t^\alpha}{\Gamma(1+\alpha)}\right),\]
\[u(2) = f\left(\frac{x^\beta}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} - \varepsilon\right),\]
\[u(3) = e^\varepsilon f\left(\frac{x^\beta}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} + \varepsilon\right),\]
\[u(4) = f\left(\frac{x^\beta}{\Gamma(1+\alpha)} e^{-\varepsilon} \frac{t^\alpha}{\Gamma(1+\alpha)} e^{-2\varepsilon}\right),\]
\[u(5) = e^\varepsilon \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{x^\beta}{\Gamma(1+\beta)} f\left(\frac{x^\beta}{\Gamma(1+\beta)} - 2\varepsilon \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\right),\]
\[u(6) = \frac{1}{\sqrt{1+4\varepsilon}} \frac{t^\alpha}{\Gamma(1+\alpha)} e^{\varepsilon^{2} \frac{t^\alpha}{\Gamma(1+\alpha)}} \frac{\Gamma(1+\alpha)x^\beta}{\Gamma(1+\beta)(1+\varepsilon)(1+\alpha)} \frac{\Gamma(1+\alpha)(1+\varepsilon)(1+\alpha)}{\Gamma(1+\beta)(1+\varepsilon)(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)},\]
\[u(7) = f\left(\frac{x^\beta}{\Gamma(1+\alpha)} + \frac{t^\alpha}{\Gamma(1+\alpha)} \right) + \varepsilon a(x, t),\]
which are all the classification of solutions of Eq. (4).

Take the solution \(u(6)\) as an example,
\[u(5) = e^{\varepsilon^{2} \frac{t^\alpha}{\Gamma(1+\alpha)}} \frac{x^\beta}{\Gamma(1+\beta)} f\left(\frac{x^\beta}{\Gamma(1+\beta)} - 2\varepsilon \frac{t^\alpha}{\Gamma(1+\alpha)} \frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}\right).\] (26)

Assume \(f\left(\frac{x^\beta}{\Gamma(1+\alpha)} - 2\varepsilon \frac{t^\alpha}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}\right) = c\), which can be set as the initial value of Eq. (4). Now we can check that \(u_1(5) = c e^{\varepsilon^{2} \frac{t^\alpha}{\Gamma(1+\alpha)}} \frac{x^\beta}{\Gamma(1+\beta)} \frac{t^\alpha}{\Gamma(1+\alpha)}\) is one of the exact solutions. If we make \(f\left(\frac{x^\beta}{\Gamma(1+\alpha)}, \frac{t^\alpha}{\Gamma(1+\alpha)}\right) = u_1(5) = c e^{\varepsilon^{2} \frac{t^\alpha}{\Gamma(1+\alpha)}} \frac{x^\beta}{\Gamma(1+\beta)} \frac{t^\alpha}{\Gamma(1+\alpha)}\), we can derive a new iteration solution \(u_2(5)\). As a result, by similar manipulations, we can give \(u_3(5) \ldots u_n(5)\) which are new exact solutions of Eq. (4).
4 Conclusions

Fractional differential equations have caught considerable attention due to their various applications in real physical problems. However, there is no systematic method to derive the exact solution. Now, the problem is partly solved in this paper.

Another problem may arise: can the Lie group method be extended to fractional differential equations of fractional order $0 \sim 2$? We will discuss such work in future.

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