Embedding Constructions of Tail-Biting Trellises for Linear Block Codes

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Abstract—In this paper, embedding construction of tail-biting trellises for linear block codes is presented. With the new approach of constructing tail-biting trellises, most of the study of tail-biting trellises can be converted into the study of conventional trellises. It is proved that any minimal tail-biting trellis can be constructed by the recursive process of embedding constructions from the well-known Bahl- Cockle-Jelinek-Raviv (BCJR) constructed conventional trellises. Furthermore, several properties of embedding constructions of tail-biting trellises are discussed. Finally, we give four sufficient conditions to reduce the maximum state-complexity of a trellis with one peak.

Keywords: Linear block code, conventional trellis, nonmergeable trellis, tail-biting trellis, embedding construction

Index Terms—Block code, linear trellis, nonmergeable trellis, tail-biting trellis, embedding construction

I. INTRODUCTION

To reduce decoding complexity of a linear block code, in the papers [1], [5], [8], [9], [10] and references therein, conventional trellis representations of a linear block code have been proposed and investigated extensively. With these representations, different efficient soft-decision decodings of codes can be applied to decode a linear block code, for example, the Viterbi algorithm.

To further reduce the complexity, just as indicated and studied in the papers [5], [6], [8], [9], [10], characterizing and constructing minimal trellises for conventional trellis representations have key of importance. Based on this consideration, tail-biting trellises for a linear block code have been appeared. Although much unknown for these trellis still remain, the papers [2], [17] have shown that the number of states in a tail-biting trellis for a linear code can be as low as the square root of the number of states which is used in the minimal conventional trellis. These results have greatly activated the interests and concerns of many researchers. In recent years, much advance has been made in this direction, for example, see [5], [6], [11], [12], [13], [14] and the references therein.

Differing to a conventional trellis representation of a linear block code, a tail-biting trellis representation may have several starting and ending status pairs, which helps to reduce the total status number and hence, reduce the decoding complexity, while there is only one starting and ending status pair in a conventional representation. Just because of this, there have more flexible designs of tail-biting trellis representations, and at the same time, it is more difficult to find out the optimal representation for any linear block code. Here, the optimality means that there is the smallest status in the trellis. In fact, a method to design the optimal trellis for any linear block code has not appeared until now. Fortunately, there are a lot of works on this direction. Koetter and Vardy, in the papers [5], [6], have made a detailed study of the structure of linear tail-biting trellises. In the paper [13], the authors followed the idea given in the papers [11], [12], presented new ways of describing and constructing linear tail-biting trellises for linear block codes. By following their consideration, the minimal tail-biting trellis computation problem may thus be formulated as the problem to find a suitable matrix. However, to find this suitable matrix still is a difficult task, moreover, the paper did not give any method to overcome this difficulty.

In this paper, we will demonstrate that an embedding construction of a tail-biting trellis can be converted into a construction of a conventional trellis. It turns out that many properties of a conventional trellis can be switched into ones of a tail-biting trellis. Thus, a tail-biting trellis can be obtained by using a corresponding conventional trellis for a given linear block code. Furthermore, we will prove that any minimal tail-biting trellis can be constructed by the recursive process of embedding constructions from the well-known BCJR constructed conventional trellises. Based on the conclusions above, moreover, several properties of embedding constructions of tail-biting trellises are discussed in this paper. Finally, we also will give four sufficient conditions to reduce the maximum state-complexity of a conventional or a tail-biting trellis.

The organization of this paper is as follows. In the next section, some preliminaries are given, and in the section III, the embedding method and main results are stated. Four sufficient conditions are presented in the section IV. At last, conclusions are given in the section V.

II. PRELIMINARIES

In this section, a number of definitions and concepts related to conventional and tail-biting trellises will be introduced. We will follow some notations and definitions in [6] and [13].

Firstly, we need a few of terminologies from graph theory. An edge-labeled directed graph is defined as a triple \((V, E, \Sigma)\), which consists of a set \(V\) of vertices, a finite set \(\Sigma\), and a set \(E\) of ordered triples \((u, a, v)\), with \(u, v \in V\) and \(a \in \Sigma\). Usually, \(\Sigma\) is called as the alphabet and \((u, a, v)\) is called as an edge.
Also an edge \((u, a, v) \in E\) means that it begins at \(u\), ends at \(v\), and has label \(a\).

The following definitions are also necessary for this paper.

**Definition 1:** A conventional trellis \(T = (V, E, \Sigma)\) of depth \(n\) is an edge-labeled directed graph, which satisfies the following property: the set \(V\) can be partitioned into \(n + 1\) vertex classes, denoted as

\[
V = V_0 \cup V_1 \cup \cdots \cup V_n,
\]

where \(|V_0| = |V_n| = 1\), such that every edge in \(E\) is labeled with a symbol from the alphabet \(\Sigma\), and begins at a vertex of \(V_i\) and ends at a vertex of \(V_{i+1}\), for some \(i \in \{0, 1, \ldots, n-1\}\).

The ordered index set \(I = \{0, 1, \ldots, n\}\) introduced by the partition of \(V\) in (1) is called the time indices for \(T\).

A conventional trellis \(T\) is reduced if every vertex in \(T\) lies on at least one path from a vertex in \(V_0\) to a vertex in \(V_n\).

**Definition 2:** A tail-biting trellis \(T = (V, E, \Sigma)\) of depth \(n\) is an edge-labeled directed graph, if it satisfies condition that the set \(V\) can be partitioned into \(n\) vertex classes

\[
V = V_0 \cup V_1 \cup \cdots \cup V_{n-1},
\]

such that every edge in \(T\) is labeled with a symbol from the alphabet \(\Sigma\), and begins at a vertex of \(V_i\) and ends at a vertex of \(V_{i+1} \mod n\), for some \(i \in \{0, 1, \ldots, n-1\}\).

Some remarks are required here. The first, from the definitions, it is obvious that a conventional trellis is a tail-biting trellis, but the inverse is not true. The second, in a conventional trellis, the sizes of \(V_0\) and \(V_n\) are all equal to 1. In contrast to this, there is no such requirement in a tail-biting trellis. Moreover, if the size of \(V_0\) is equal to 1, a tail-biting trellis is reduced to a conventional one. The third, if an edge begins at a vertex in \(V_{n-1}\), it will end at a vertex in \(V_0\) in a tail-biting trellis, on the contrast, it will end at a vertex in \(V_n\) in a conventional one.

We continue to define some terminologies. The indices in the set \(I = \{0, 1, \ldots, n-1\}\) for the partition in (2) are called as the time indices. Moreover, in this paper, the set \(I\) is identified with \(\mathbb{Z}_n\), the residue classes of integers modulo \(n\). And hence, an interval of indices \([i, j]\) means the sequence \(\{i, i+1, \ldots, j\}\) if \(i < j\), and the sequence \(\{i, i+1, \ldots, n-1, 0, \ldots, j\}\) if \(i > j\). Every cycle of length \(n\) in \(T\) starting at a vertex of \(V_0\) defines a vector \((a_0, a_1, \ldots, a_{n-1}) \in \Sigma^n\), which is an edge-label sequence. If every vertex in \(T\) lies on at least one cycle from a vertex in \(V_0\), the tail-biting trellis \(T\) is defined as reduced.

Secondly, some connections between a linear block code and an edge-labeled directed graph are needed. According the results given in papers [11, 13, 7, 10, 9, 18], any linear block code can be represented by using a conventional trellis or a tail-biting trellis. Let us make these representations more precisely.

Denote an \((n, k)\) linear block code over \(F_q\) as \((n, k)_q\). Assume that \(C = (n, k)_q\) is a linear block code. Thus, every codeword in \(C\) is a vector over \(F_q\) with size \(n\). Arranging all entries in this vector in the natural order becomes a sequences in \(F_q\) with length \(n\). If the set consisting of all these sequences is precisely the same as the one consisting of all edge-labeled sequences corresponding to those cycles in \(T\) that start at a vertex of \(V_0\), the conventional or tail-biting trellis \(T\) is said to represent a block code \(C\) of length \(n\) over \(\Sigma(=F_q)\).

Recall the facts that the number of states in a trellis code is an important factor in Viterbi decoding and it is directly related to decoding complexity. Hence, the quantity \(\log_{|\Sigma|} |V_i|\) is regarded as the state-complexity of the trellis, either conventional or tail-biting, at time index \(i\). At the same time, the sequence \(\{\log_{|\Sigma|} |V_i|, 0 \leq i < n\}\) gives the state-complexity profile (SCP) of the trellis. Therefore, a trellis \(T\) is said to be minimal if the maximum state-complexity over all time indices denoted by \(s_{\text{max}}(T)\) is minimized over all possible coordinate permutations of the code [10]. In the same paper, it is proved that the minimal conventional trellis for a linear block code is unique, and simultaneously, satisfies all definitions of minimality. Moreover, it is also improper (that is, any pair of edges directed towards a vertex has distinct labels, and so also any pair of edges leaving a vertex).

To help an understanding of the notations and concepts above, the trellis shown in Fig. 1 is the minimal conventional trellis for the (7, 4)2 Hamming code, which has a parity check matrix defined as follows:

\[
H = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}.
\]

![Fig.1 The minimal conventional trellis for a (7, 4)2 Hamming code.](image-url)
The path above represents a sequence \((1, 1, 1, 1, 1, 1)\), which corresponds to the codeword \([1, 1, 1, 1, 1, 1]\).

The trellis shown in Fig. 2 is a tail-biting trellis for the \((7, 4)_2\) Hamming code of Fig. 1.

Comparing to the figure 1, we can find that, in the figure 2, it has two starting and ending pairs. Moreover, a cycle \((\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array})\) from the left most vertex \(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\) to the right most vertex \(\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array}\) or from the left most vertex \(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\) to the right most vertex \(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\) corresponds a codeword in figure 2, while a path \(\begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \end{array}\) corresponds a codeword in the figure 1.

We also find that, in the figures, in addition to the labeling of edges, each vertex in the set \(V_i\) can be labeled by a sequence of length \(n - k\) of elements in \(\Sigma\), and all vertex labels at a given depth are distinct, just as shown in the figures 1 and 2. Thus, every path (or cycle) in this labeled conventional trellis (or tail-biting trellis) defines a sequence of length \(n(1 + n - k)\) over \(\Sigma\), consisting of alternating labels of vertices and edges in \(T\). The set of all label sequences in a labeled trellis is referred to as the label code represented by \(T\) and is denoted by \(S(T)\).

Fig. 2 illustrates a labeled tail-biting trellis, and Fig. 1 illustrates a labeled conventional trellis.

At last, we need two more definitions related to properties of a trellis.

**Definition 3:** A trellis \(T\) is said to be linear if there exists a vertex labeling of \(T\) such that \(S(T)\) is a vector space.

The notion of mergeability \([5], [15], [16]\) is also useful here.

**Definition 4:** A trellis is mergeable if there exist vertices in the same vertex class of \(T\) that can be replaced by a single vertex, while retaining the edges incident on the original vertices, without modifying \(C(T)\). If a trellis contains no vertices that can be merged, it is said to be nonmergeable.

Koetter and Vardy [5] have shown that if a linear trellis is nonmergeable, then it is also biproper. However, though the converse is true for conventional trellises, it is not true in general for tail-biting trellises. They show that for tail-biting trellises the following relation chain holds:

\[
\{\text{linear trellises}\} \cup \{\text{biproper linear trellises}\} \cup \{\text{nonmergeable linear trellises}\}
\]

In the discussion that follows, we restrict ourselves to trellises representing linear block codes over the alphabet \(\Sigma = \mathbb{F}_q\). We will occasionally refer to vertices in a trellis as “states”.

**III. BCJR Labeling and the Embedding Construction of Tail-Biting Trellis**

**A. The minimal BCJR labeling of a trellis**

The original BCJR algorithm [1] constructs the minimal and unique, up to isomorphism, conventional trellis for a linear block code. In the paper [13], the authors gave a simple method to describe this construction. Here, we only give two examples to illustrate this method. More details can be found in that paper.

**Example 1:** Consider a self dual \((4, 2)_2\) code with parity check matrix defined as follows:

\[
H = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 
\end{bmatrix}.
\]

We obtain a minimal BCJR labeling of the trellis for the \((4, 2)_2\) code as illustrated in Fig. 3.

**Example 2:** Similarly, consider the \((7, 4)_2\) Hamming code with parity check matrix defined as follows:

\[
H = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 1 
\end{bmatrix}.
\]

We obtain a minimal BCJR labeling of the trellis for the \((7, 4)_2\) Hamming code as illustrated in Fig. 1.
B. The embedding construction of tail-biting trellis

Now we can state our method to design a tail-biting trellis for a given linear block code. This method is demonstrated by following example.

Let us first consider the minimal conventional trellis $T$ for the $(4,2)_2$ code in Fig.3. Note that $\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V_2$.

Arranging this vector $\alpha$ to the first column and the last column in the parity check matrix $H$, we obtain

$$H' = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Thus, we can get a minimal BCJR labeling of the trellis for the parity check matrix $H'$ as illustrated in Fig. 4.

![Fig.4 The minimal conventional trellis for the parity check matrix $H'$](image)

In Fig.4, let $T_0$ be all paths from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V'_3$, and $C_0$ be the set consisting of all codewords corresponding to $T_0$; also let $T_1$ be all paths from $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V'_3$ to $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in V'_3$, and $C_1$ be the set consisting of all codewords corresponding to $T_1$. Comparing to Fig. 3, we can find out that both $C_0$ and $C_1$ are the $(4,2)_2$ codewords.

Now let us consider the set $T_0 \cap T_1$. This set can be divided into two parts, moreover, these two parts are isomorphic. In fact, the first part is consisted of the following vertexes: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and the four vertexes in the second part are $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Now we drop the four vertexes in the first part and the left most and the right most vertexes from the figure 4 and obtain the following trellis.

![Fig.4 The minimal conventional trellis for the parity check matrix $H'$](image)

It is easy to verify that the codewords corresponding to Fig.5 compose the linear block code $(4,2)_2$. In fact, let us consider the codewords in $C_0$ or $C_1$, passing only $V'_{3,0} = \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \}$. Suppose $c \in C_1$, represented by the path $p$ not passing $V'_{3,0}$. As $V'_{3,0}$ is a subspace of $V'_3$, the dimension of $V'_{3,0}$ is one less than that of $V'_3$, and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \notin V'_{3,0}$, thus by adding $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to each vertex label in $p$, we get the path $p'$, passing $V'_{3,0}$. It is clear that $p'$ represents a codeword $c \in C_0$. Similarly, suppose $c \in C_0$, represented by a path passing $V'_{3,0}$, then $c \in C_1$, represented by a path not passing $V'_{3,0}$. Thus, the codewords passing only $V'_{3,0}$ in $C_0$ or $C_1$ compose exactly the $(4,2)_2$ codewords.

Now we try to transform the operating steps above into a language of parity check matrix. To get Fig. 5, we deleted half paths in Fig. 4. In fact, it is equivalent to add a row to the parity check matrix. Let us go to more detail.

Let $C'$ be the codewords with the parity check matrix $H'$, and $C_1$ the codewords represented by all paths from $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in V'_6$, passing only $V'_{3,0}$. As $V'_{3,0}$ is a subspace of $V'_3$ and the dimension of $V'_{3,0}$ is one less than that of $V'_3$, thus the dimension of $C_1$ is one less than that of $C'$. Therefore, there exists a parity check matrix $H'^+$ for $C_1$, such that $H'^+$ is obtained by adding one more row to $H'$. In fact, it is enough to let

$$H'^+ = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$ 

Furthermore, the minimal BCJR labeling of the trellis for the parity check matrix $H'^+$ is illustrated in Fig. 6.
It is obvious that by deleting $V_0$ and $V_6$ and the corresponding edges, and deleting the first row of each vertex label in Fig.6, we get the Fig. 5, which turns out to be the labeled tail-biting trellis for the $(4,2)_2$ code.

Now we generalize the operating steps shown in the above example into a general operating method, and obtain the embedding construction of a tail-biting trellis as follows:

1. Let $C$ be an $(n,k)_q$ linear code with an $(n-k) \times n$ parity check matrix $H = (h_1, h_2, \ldots, h_n)$, and $T$ be its labeled BCJR trellis. Assume that $\alpha \in V_i (\alpha \neq 0)$. Let $s_i$ denote the dimension of $V_i$, $0 \leq i < n$. Since $V_i$ is a vector space, if $\alpha \in V_i$, then $2\alpha, 3\alpha, \ldots, (q-1)\alpha \in V_i$, there exists a linear subspace $V_{i,0}$ of dimension $s_i - 1$, such that $\alpha \notin V_{i,0}$. We now add $\alpha$ to $H$ before the first column and after the last column, respectively, and denote this new matrix as $H'$, that is, $H' = (\alpha, h_1, h_2, \ldots, h_n, \alpha)$. Construct a labeled BCJR trellis $T'$ for $H'$.

2. Let $C_i$ be the codewords represented by all paths from $i \alpha \in V_i'$ to $i \alpha \in V_{n+1}'$, $0 \leq i \leq q-1$. Then $C_i$ is the $(n,k)_q$ linear code $C$. Put $V_{i+1,0}' = V_{i,0}$. Then $V_{i+1,0}' \subset V_{i+1}'$, and the codewords only passing $V_{i+1,0}'$ in $C_0$ or $C_1$ or ... or $C_{q-1}$ compose exactly the $(n,k)_q$ linear code $C$. Let $C_i$ be the codewords represented by all paths passing only $V_{i+1,0}'$. Compute the parity check matrix $H^i$ for $C_i$. Obviously, $H^i$ has one more row than $H'$.

3. Let $T^i$ be the labeled BCJR trellis for parity check matrix $H^i$. By deleting $V_0'^i$ and $V_{n+2}'$ and relating edges, and deleting the first row of each vertex label in $T^i$, we get a labeled tail-biting trellis for the $(n,k)_q$ linear code $C$.

It is easy to show the validity of the embedding construction. Thus, with this new approach of constructing tail-biting trellises, most of the study of tail-biting trellises can be converted into that of conventional trellises.

Surprisingly, we can further process another embedding construction based on the obtained labeled BCJR trellis $T^i$.

For example, repeating the steps above on the parity check matrix $H^i$, which is corresponding to Fig. 6, we obtain a new parity check matrix

$$H^i = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$  

We can further get a BCJR trellis $T^i$ corresponding to $H^i$, and we find that the dimension of $V^i_4$ is 0.

Now some remarks are in order. The first one is that even if there exists an integer $q'$ for $\alpha \in V_i$, such that $0 < q' < q$, and $q'\alpha = 0$, $\alpha, 2\alpha, 3\alpha, \ldots, (q-1)\alpha$ are not distinct, but the embedding construction above can be similarly processed.

The second one is that $\alpha \notin V_{i,0}$ is a necessary condition. If $\alpha \in V_{i,0}$, then the codewords passing only $V_{i+1,0}' = V_{i,0}$ in $C_0$ or $C_1$ or ... or $C_{q-1}$ do not compose the $(n,k)_q$ linear code $C$.

The third one is that $\alpha$, in fact, specifies a coset decomposition $V_i/V_{i,0}$ of the vector space $V_i$, such that every coset is associated with a unique $ja, 0 \leq j < q$.

The fourth one is to notice that $V_{i,0}$ is not necessarily unique. For example, consider the trellis shown in Fig.3, let $i = 2, \alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then $V_{i,0} = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$ or $\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$. If $V_{i,0} = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$, $H^i$ will become

$$H^i = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix},$$

and we will get another labeled tail-biting trellis for the $(4,2)_2$ code as follows:

$$[0] [1] [0] [1] [1] [1] [0] [1] [1] [0] [0] [0] [1] [1] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0] [0]$$

Fig.7 The trellis constructed from Fig.3.

The fifth one is that even though $V_{i,0}$ are different, the corresponding tail-biting trellis is the same if $\alpha$ satisfies some conditions. Give an example as follows:

**Example 3:** Let $T$ be the labeled BCJR trellis in Fig.3. As $\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in V_1 \cap V_2 \cap V_3$, hence $V_{1,0} = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$, $V_{2,0} = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$, $V_{3,0} = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}$. Obviously, the embedding construction by $V_{1,0}$ or $V_{2,0}$ or $V_{3,0}$ gets the same tail-biting trellis.

To illustrate our method of construction, we demonstrate another example.
Example 4: Let \( T \) be the trellis for the \((7, 4)_2\) Hamming code in Fig.1, and \( \alpha = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \). Similarly, the embedding construction by \( V_{5,0} = \{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \} \)

gets a labeled tail-biting trellis as illustrated in Fig. 2.

Furthermore, if we repeat the construction on the trellis \( T^\dagger \) with \( H^\dagger \), a new tail-biting trellis can be obtained as follows:

Take \( \alpha^\dagger = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). Then \( V_{4,0} = \{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \} \) and get

\[
H^\dagger = \begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix},
\]

which generates another labeled tail-biting trellis, shown in the following figure. Notice that, in this trellis, the dimensions of both \( V_5^\dagger \) and \( V_6^\dagger \) are 1.

![Image of tail-biting trellis](image)

\[
\text{Table VIII for the parity check matrix } H^\dagger.
\]

C. Results on the embedding construction

For our construction above, some properties are important.

**Lemma 1**: Let \( T \) be a trellis for an \((n, k)_q\) linear code \( C \) with the parity check matrix \( H = [h_1, h_2, \ldots, h_n] \). Suppose \( \alpha \in V_i(\alpha \neq 0) \), \( V_{i,0} \) is a linear subspace of \( V_i \) of dimension \( s_i - 1 \), such that \( \alpha \notin V_{i,0} \). Let \( H' = (\alpha, h_1, h_2, \ldots, h_n, \alpha) \), and \( T' \) a labeled BCJR trellis for \( H' \). Let \( C_i \) be the codewords represented by all paths passing only \( V_{i+1,0} \). Suppose \( H^\dagger \) is an embedding construction by \( \alpha \) and \( V_{i,0} \), and \( H^\dagger \) has one more row \((x_1, x_2, \ldots, x_{n+2})\) than \( H' \). Then,

1. \((x_1, x_2, \ldots, x_{n+2})\) is not unique;
2. \( x_1 \) and \( x_{n+2} \) are distinct;
3. \((x_1, x_2, \ldots, x_{n+2})\) can be \((1, x_2, \ldots, x_{i+1}, 0, \ldots, 0)\), such that for each \((c_1, c_2, \ldots, c_{n+2}) \in C_i, c_1 + x_2c_2 + x_3c_3 + \ldots + x_{i+1}c_{i+1} = 0 \).

**Proof**: Let \( V_{i+1} = V_i(\alpha \neq 0) \), \( V_{i,0} \) a linear vector space generated by \( V_{i,0} \), and \( M^\dagger(\alpha) \) the map of \( M^\dagger \), where \( r > 1 \).

From Theorem 1, we prove the following lemma.

**Lemma 2**: Let \( V_i^\dagger, 0 < i < n \), denotes the state space of the trellis \( T^\dagger \) got by an embedding construction with \( \alpha \) and \( V_{i,0} \) from trellis \( T \). Let \( M(V_i), M(V_{i,0}) \) denote the map of \( V_{i,0} \), respectively. Then \( M(V_i) \) is a vector space. And, the space \( M^\dagger(V_i) = M^\dagger(V_{i,0}) \) and \( \alpha \in M^\dagger(V_{i,0}) \). Then \( V_{i+1}^\dagger = M^\dagger(V_i) \).

Case 2. \( M^\dagger(V_i) = M^\dagger(V_{i,0}) \) and \( \alpha \notin M^\dagger(V_{i,0}) \). Then \( V_{i+1}^\dagger \) is a vector space generated by \( M^\dagger(V_i) \) and \( \alpha \).

Case 3. \( M^\dagger(V_i) \neq M^\dagger(V_{i,0}) \) and all \( M^\dagger(\alpha) - \alpha \in M^\dagger(V_{i,0}) \). Then \( V_{i+1}^\dagger = M^\dagger(V_i) \).

Case 4. \( M^\dagger(V_i) \neq M^\dagger(V_{i,0}) \) and not all \( M^\dagger(\alpha) - \alpha \in M^\dagger(V_{i,0}) \). Then \( V_{i+1}^\dagger \) is a vector space generated by \( M^\dagger(V_{i,0}) \) and \( M^\dagger(\alpha) - \alpha \), we here select \( M^\dagger(\alpha) \) such that \( M^\dagger(\alpha) - \alpha \notin M^\dagger(V_{i,0}) \).
Proof: From Theorem 1, it is known that $V_i^\dagger$ is a vector space. We now show that $M(V_{i,0})$ is a vector space.

Let $a_i, b_i \in M(V_{i,0})$. Then there exist $x, y \in S(T)$, such that $x_i, y_i \in V_{i,0}$, and $a = x_{i+1}, b = y_{i+1}$, here $z_i$ denotes a state label of $z \in S(T)$ at time index $i$. From $x + y \in S(T)$ and $x_i + y_i \in V_{i,0}$, we have $a + b = x_{i+1} + y_{i+1} \in M(V_{i,0})$, hence $M(V_{i,0})$ is a vector space and so is $M^r(V_{i,0})$ for $r > 1$.

We only prove the Case 4. The others are similar.

Case 1. $M^r(V_i) = M^r(V_{i,0})$ and $\alpha \in M^r(V_{i,0})$. The trellis in Fig. 8 for $i = 4$ and $r = 1$ or 2 belongs to this case.

Case 2. $M^r(V_i) = M^r(V_{i,0})$ and $\alpha \notin M^r(V_{i,0})$. The trellis in Fig. 2 for $i = 4$ and $r = 2$ belongs to this case.

Case 3. $M^r(V_i) \neq M^r(V_{i,0})$ and all $M^r(\alpha) - \alpha \in M^r(V_{i,0})$. The trellis in Fig. 2 for $i = 3$ and $r = 1$ belongs to this case.

Case 4. $M^r(V_i) \neq M^r(V_{i,0})$ and not all $M^r(\alpha) - \alpha \in M^r(V_{i,0})$. The trellis in Fig. 2 for $i = 4$ and $r = 1$ belongs to this case.

Note that $(q - j)\alpha + \alpha M^r(\xi + \beta) = V_{i+r,0}, 0 \leq j < q, \beta \in V_{i,0}$. For any state $\beta \in V_{i,0}$, we know that

\[
M^r(\alpha + \beta) = M^r(\alpha) + M^r(\beta) = \beta M^r(\alpha) + M^r(\beta).
\]

This completes the proof.

In a similar way to the above discussion, one may discuss the case for $0 < j < i$.

Lemma 3: Let $T$ be a trellis for an $(n, k)$ linear code $C$. Suppose $\alpha \in V_i(\alpha \neq 0), V_i$ be a linear subspace of $V_i$ of dimension $s_i - 1$, such that $\alpha \notin V_{i,0}$. Then we can get a tail-biting trellis $T'$ with an embedding construction by $\alpha$ and $V_i,0$, such that the dimension of $V_i^\dagger$ is $s_i - 1$.

Proof: Let $H = (b_1, b_2, \ldots, b_n)$ be a parity check matrix for $T$, and let $H' = (a_1, b_1, b_2, \ldots, b_n, a)$. Construct a labeled conventional trellis $T'$ for $H'$.

Let $C_i$ be the codewords represented by all paths from $i \in V_i^\dagger$ to $\alpha \in V_{i+1}^\dagger, 0 \leq i \leq q - 1$. Then $C_i$ is the code for $T$.

Note that all paths from $0 \in V_1^\dagger$ to $0 \in V_{n+1}^\dagger$ compose exactly the trellis $T$, and adding $i_\alpha$ to each vertex label in all paths from $0 \in V_i^\dagger$ to $0 \in V_{n+1}^\dagger$ compose exactly all paths from $i \in V_i^\dagger$ to $\alpha \in V_{i+1}^\dagger, 0 \leq i < q - 1$. As $i \in V_i$, thus $V_i^\dagger = V_{i+1}^\dagger$.

By the process of embedding construction with $\alpha$ and $V_i,0$, it is clear that we get a tail-biting trellis $T'$, such that the dimension of $V_i^\dagger$ is $s_i - 1$.

An embedding construction has two key parameters: $\alpha$ and $V_i,0$. Therefore, to construct a minimal tail-biting trellis is to determine the sequence of $\alpha$ and $V_i,0$.

Now we can state one of the main results as a theorem.

Theorem 2: Any minimal tail-biting trellis for an $(n, k)$ linear code can be constructed by embedding constructions from a Bahl-Cocke-Jelinek-Raviv (BCJR) constructed conventional trellis.

Proof: Let $T$ be a minimal tail-biting trellis. Suppose $\alpha \in V_0$ but $\alpha \notin V_i$. From $T$, construct a new tail-biting $T'$ starting at time index $i$, i.e. $V_i = V_i', \alpha = V_0, V_{n-i} = V_0, V_{n-i} = V_0$.

From Lemma 3, the dimension of $V'_{n-i}$ can be reduced by 1, i.e. the dimension of $V_0$ can be reduced by 1.

Repeat the process above, we get a tail-biting trellis $T'$, such that $V_i = \{0\}$. As the Bahl-Cocke-Jelinek-Raviv (BCJR) constructed conventional trellis is unique, we know that $T'$ is a BCJR constructed conventional trellis.

Therefore, to construct a minimal tail-biting trellis, one just need to process conversely from $T'$.

IV. TO REDUCE THE MAXIMUM STATE-COMPLEXITY OF A TAIL-BITING TRELLIS WITH ONE PEAK

In this section, we restrict ourselves to trellises representing binary linear block codes.

Using embedding constructions, we discuss how to reduce the maximum state-complexity of a tail-biting (or conventional) trellis with one peak.

We first consider the following simplest case.

Proposition 1: Let $T$ be a trellis. Suppose $|V_p| > |V_{p-1}|$ and $|V_p| > |V_{p+1}|$, where $1 < p < n - 1$, and $|V_p| \geq 4$. We also assume that $|V_i| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 1 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.

Proof: We first show that $|V_{p-1} \cap V_p \cap V_{p+1}| > 1$.

Suppose $V_{p-1} = \{0, 1, \ldots, k-1\}$. Then $V_p = \{0, 1, \ldots, p-1\} \cup \{p, \ldots, n\}$, where $|V_{p-1}| \geq 4$. We also assume that $|V_{p-1}| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 1 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.

Proof: We first show that $|V_{p-1} \cap V_p \cap V_{p+1}| > 1$.

Suppose $V_{p-1} = \{0, 1, \ldots, k-1\}$. Then $V_p = \{0, 1, \ldots, p-1\}$, where $|V_{p-1}| \geq 4$. We also assume that $|V_{p-1}| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 1 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.

Proof: By the definition of the trellis for a linear code, every vertex of $V_i$ has at least out degree 1 or 2.

Proof: By the definition of the trellis for a linear code, every vertex of $V_i$ has at least out degree 1 or 2. If we note the following fact, then the proof is obvious.

For $\alpha \in V_i, \alpha \neq 0$, the out degree of 0 is 2. There exists a codeword $c = (0, \ldots, 0, 1, c_{i+2}, \ldots, c_n)$ if and only if the out degree of $\alpha$ is 2.

Proposition 2: Let $T$ be a trellis. Suppose $|V_p| > |V_{p-1}|$, $|V_p| = |V_{p+1}|$, and $|V_{p+1}| > |V_{p+2}|$, where $1 < p < n - 2$, and
Suppose that $|V_p| \geq 8$. We also assume that $|V_i| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 2 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.

Proof: Let $h_1, h_2, \ldots, h_p$ be the $n$ columns of $H$.

First consider the case that $h_{p+1} \in V_p$. Then $V_p = V_{p+1}$.

Now we show that $|V_{p-1} \cap V_p \cap V_{p+1} \cap V_{p+2}| > 3$.

Suppose $V_{p-1} = \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}\}$. Then $V_p = \{\alpha_0, \alpha_1, \ldots, \alpha_{k-1}, \alpha_0 + \beta, \alpha_1 + \beta, \ldots, \alpha_{k-1} + \beta\}$.

If $|V_{p-1}| \geq 8$, it is easy to see that there exist $\alpha_0, \alpha_1, \alpha_2, \alpha_3 \in V_{p+2}$, or $\alpha_0 + \beta, \alpha_1 + \beta, \alpha_2 + \beta, \alpha_3 + \beta \in V_{p+2}$, where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are distinct.

Let $\alpha = h_{p+1}$. Then a linear subspace $V_{p,0}$ of dimension $s_p - 1$ is existed, such that $V_{p,0} \subset V_p$ and $\alpha \notin V_{p,0}$. Then both $V_{p,0}$ and $M(V_{p,0})$ has the dimension $s_p - 1$, and $\alpha \notin M(V_{p,0})$.

Now we consider the trellis $T$ illustrated in Fig.1. Let $\alpha = (1)$.

Let $\beta, \alpha \in V_{p,0}$.

With an embedding construction by $\alpha, V_{p,0}$ and $H^1 = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1
\end{bmatrix}$, we obtain the trellis in Fig.9.

With a similar argument as Proposition 2, we have the following proposition.

Proposition 3: Let $T$ be a trellis. Suppose $|V_p| > |V_{p-1}|$, $V_p = V_{p+1} = V_{p+2}$ and $|V_{p+2}| > |V_{p+3}|$, where $1 < p < n - 3$, and $|V_{p-1}| \geq 8$. We also assume that $|V_i| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 3 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.

Proposition 4: Let $T$ be a trellis. Suppose $|V_p| > |V_{p-1}|$, $|V_p| = |V_{p+1}| = |V_{p+2}|$, $V_p \neq V_{p+1}$ or $V_{p+1} \neq V_{p+2}$ and $|V_{p+2}| > |V_{p+3}|$, where $1 < p < n - 3$, and $|V_{p-1}| \geq 8$. We also assume that $|V_i| < |V_{p-1}|$ for $0 \leq i < p - 1$ and $p + 3 < i < n$. Then the maximum state-complexity of $T$ can be reduced by 1 with an embedding construction.
Proof: We just show the case that $V_p \neq V_{p+1} = V_{p+2}$. The others are similar.

With a similar argument as Proposition 2, we may show that $|V_{p-1} \cap V_p \cap V_{p+1} \cap V_{p+2} \cap V_{p+3}| > 3$.

Suppose that $\alpha, \beta \in V_{p-1} \cap V_p \cap V_{p+1} \cap V_{p+2} \cap V_{p+3}, \alpha \neq \beta, \alpha \neq 0, \beta \neq 0$.

If $\beta = h_{p+2}$. Note that for $\alpha, \beta \in V_p$, $M(\alpha) = \alpha, M(\beta) = \beta$. Then a linear subspace $V_{p,0}$ of $V_p$ of dimension $s_p - 1$ is existed, such that $\alpha \notin V_{p,0}$. Then both $V_{p,0}$ and $M(V_{p,0})$ has the dimension $s_p - 1$, and $\alpha \notin M(V_{p,0}), \beta \in M(V_{p,0})$. Hence $M^2(V_{p,0}) = M(V_{p,0}), \alpha \notin M^2(V_{p,0})$. With an embedding construction by $\alpha$ and $V_{p,0}$, we have the proposition.

If $\alpha \neq h_{p+2}$ and $\beta \neq h_{p+2}$. Then $V_{p+2}$ has a linear subspace $V_{p+2,0}$ of dimension $s_{p+1}$, such that $\alpha \notin V_{p+2,0}, h_{p+2} \in V_{p+2,0}$. Then $M^{-1}(V_{p+2,0}) = V_{p+2,0}$, where $M^{-1}(V_{p+2,0})$ denotes the set $U \subset V_{p+1,1}$, such that $M(U) = V_{p+2,0}$. Hence $\alpha \notin M^{-2}(V_{p+2,0}), M^{-2}(V_{p+2,0}) \subset V_{p,1}$. With an embedding construction by $\alpha$ and $M^{-2}(V_{p+2,0})$, we have the proposition.

Similarly, we may further discuss how to reduce the maximum state-complexity of the trellis with one peak and $|V_p| = |V_{p+1}| = \cdots = |V_{p+j}|$ for $j > 2$.

V. CONCLUSION

We have presented a new approach of constructing tail-biting trellises for linear block codes, and have proved that any minimal tail-biting trellis can be constructed by the recursive process of embedding constructions from a BCJR constructed conventional trellis. We conclude this paper by observing that the minimal tail-biting trellis computation problem may thus be stated as follows:

Find the least embedding constructions, such that the minimal tail-biting trellis can be constructed from a BCJR constructed conventional trellis.

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