Modified Euler approximation scheme for stochastic differential equations driven by fractional Brownian motions

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Abstract

For a stochastic differential equation driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ it is known that the classical Euler scheme has the rate of convergence $2H - 1$. In this paper we introduce a new numerical scheme which is closer to the classical Euler scheme for diffusion processes, in the sense that it has the rate of convergence $2H - \frac{1}{2}$. In particular, the rate of convergence becomes $\frac{1}{2}$ when $H$ is formally set to $\frac{1}{2}$ (the rate of Euler scheme for classical Brownian motion). The rate of weak convergence is also deduced for this scheme. The main tools are fractional calculus and Malliavin calculus. We also apply our approach to the classical Euler scheme.

Keywords. Fractional Brownian motion, Euler scheme, fractional calculus, Malliavin calculus, stochastic differential equations, strong convergence, weak convergence, rate of convergence.

1 Introduction

Consider the following stochastic differential equation on $\mathbb{R}^d$

$$X_t = X_0 + \int_0^t b(X_{\eta(s)})ds + \sum_{j=1}^m \int_0^t \sigma_j(X_s)dB^j_s,$$

where $B = (B^1, \ldots, B^m)$ is an $m$-dimensional fractional Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$ and $b, \sigma^1, \ldots, \sigma^m : \mathbb{R}^d \to \mathbb{R}^d$ are continuous mappings. The above stochastic integrals are of path-wise Riemann-Stieltjes type. If $\sigma^1, \ldots, \sigma^m$ are continuously differentiable and their partial derivatives are bounded and locally Hölder continuous of order $\delta > \frac{1}{H} - 1$ and $b$ is Lipschitz, then the above equation (1.1) has a unique solution and the solution is Hölder continuous of order $\gamma > 0$ for any $\gamma < H$. This result was proved first by Lyons [7] using Young integrals (see [15]) and $p$-variation estimates, and later by Nualart and Rascànu [13] using fractional calculus (see [16]).

We are interested in the numerical approximations for the solution to Equation (1.1). For simplicity of the presentation we consider uniform partitions of the interval $[0, T]$, $t_i = i\frac{T}{n} = ih$, $i = 0, \ldots, n$. For every positive integer $n$, we define $\eta(t) = t_i$ when $t_i \leq t < t_i + h$, and $\epsilon(t) = t_i + h$ if $t_i < t \leq t_i + h$. The following Euler numerical approximation scheme has been previously studied

$$X^n_t = X_0 + \int_0^t b(X^n_{\eta(s)})ds + \int_0^t \sigma(X^n_{\eta(s)})dB_s, \quad t \in [0, T].$$

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This scheme can also be written as

\[ X_t^n = X^n_{t_k} + b(X^n_{t_k})(t - t_k) + \sum_{j=1}^{m} \sigma^j(X^n_{t_k})(B^j_{t_k} - B^j_t), \quad t_k \leq t \leq t_{k+1}, k = 0, 1, \ldots, n - 1. \]

It was proved by Mishura [9] that for any \( \epsilon > 0 \) there exist a random variable \( C_\epsilon \) such that almost surely,

\[ \sup_{0 \leq t \leq T} |X^n_t - X_t| \leq C_\epsilon n^{1-2H+\epsilon}. \]

This means that this approximation scheme has the rate of convergence \( 2H - 1 \). Moreover, the convergence rate \( n^{1-2H} \) is also sharp for this scheme, in the sense that \( n^{2H-1}[X^n_t - X_t] \) converges almost surely to a finite and nonzero limit. This has been proved in the one-dimensional case by Nourdin and Neuenkirch [10] by using the Doss representation of the solution (see also Theorem 6.1 below). Notice that if \( H = \frac{1}{2} \), then \( 2H - 1 = 0 \), which means that \( X^n_t \) does not converge to \( X_t \). This is not surprising. In fact, if \( H \) is formally set to be \( \frac{1}{2} \) (standard Brownian motion case), then it is well-known from the classical results of numerical approximations (see [3], [5]) that \( X^n_t \) converges to \( \tilde{X}_t \) which is the solution to the following Itô stochastic differential equation

\[ \tilde{X}_t = X_0 + \int_0^t b(\tilde{X}_s)ds + \frac{1}{2} \int_0^t \sum_{j=1}^{m} (\nabla \sigma^j)(\tilde{X}_s)ds. \tag{1.3} \]

In the above and throughout this paper, \( d \) denotes the Stratonovich integral and \( \delta \) denotes the Itô integral. Moreover, \( \nabla \sigma^j \) denotes the \( d \times d \) matrix \( \left( \frac{\partial \sigma^j}{\partial x_k} \right)_{1 \leq i,k \leq d} \). However, in the case where \( H = \frac{1}{2} \), Equation (1.1) becomes the Stratonovich equation \( X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s \) driven by a standard Brownian motion \( W \), whose solution will then not be the limit of \( X^n_t \).

From the above we see that the numerical scheme (1.2) has a completely different rate of convergence than the Euler-Maruyama scheme (see [3], [5]) for classical Brownian motion. It is then natural to ask the question: how to construct a numerical scheme analogous to the Euler-Maruyama scheme? In particular, we need the convergence rate of this new scheme to be \( \frac{1}{2} \) when \( H \) is formally set to be \( \frac{1}{2} \). This paper will answer this question. We shall introduce and study a new approximation scheme, which can be viewed as an authentical modified version of the Euler-Maruyama scheme (1.2).

To obtain the new numerical scheme, we rewrite the equation (1.1) on the interval \( t_k \leq t \leq t_{k+1} \) as

\[ X_t = X_{t_k} + \sum_{j=1}^{m} \int_{t_k}^{t} \sigma^j(X_s)dB^j_s, \quad t_k \leq t \leq t_{k+1}, \]

where we assume \( b = 0 \) for simplicity. Then, the chain rule for the Young’s integral yields for any \( s \geq t_k \)

\[ \sigma^j(X_s) = \sigma^j(X_{t_k}) + \sum_{l=1}^{m} \int_{t_k}^{s} \nabla \sigma^j(X_u)\sigma^l(X_u)dB^l_u. \]
Hence,

\[ X_t = X_{t_k} + \sum_{j=1}^{m} \int_{t_k}^{t} \sigma^{j}(X_{t_k})dB_{s}^{j} + \sum_{j,l=1}^{m} \int_{t_k}^{t} \int_{t_k}^{s} \nabla \sigma^{j}(X_u)\sigma^{l}(X_u)dB_{u}^{j}dB_{s}^{l} \]

\[ = X_{t_k} + \sum_{j=1}^{m} \int_{t_k}^{t} \sigma^{j}(X_{t_k})dB_{s}^{j} + \sum_{j,l=1}^{m} \int_{t_k}^{t} \int_{t_k}^{s} \nabla \sigma^{j}(X_{t_k})\sigma^{l}(X_{t_k})dB_{u}^{j}dB_{s}^{l} \]

\[ + \sum_{j=1}^{m} \int_{t_k}^{t} \int_{t_k}^{s} [\nabla \sigma^{j}(X_u)\sigma^{j}(X_u) - \nabla \sigma^{j}(X_{t_k})\sigma^{j}(X_{t_k})] dB_{u}^{j}dB_{s}^{j} \]

\[ + \sum_{j \neq l}^{m} \int_{t_k}^{t} \int_{t_k}^{s} \nabla \sigma^{j}(X_{t_k})\sigma^{l}(X_{t_k})dB_{u}^{j}dB_{s}^{l} \]

\[ = X_{t_k} + \sum_{j=1}^{m} \sigma(X_{t_k})(B_{t}^{j} - B_{t_k}^{j}) + \frac{1}{2} \sum_{j=1}^{m} \nabla \sigma^{j}(X_{t_k})\sigma^{j}(X_{t_k})(t - t_k)^{2H} \]

\[ + R_{k,n}(t_k, t), \quad t_k \leq t \leq t_{k+1}, \tag{1.4} \]

where

\[ R_{k,n}(t_k, t) = \sum_{j \neq l}^{m} \int_{t_k}^{t} \int_{t_k}^{s} \nabla \sigma^{j}(X_{t_k})\sigma^{l}(X_{t_k})dB_{u}^{j}dB_{s}^{l} \]

\[ + \sum_{j=1}^{m} \int_{t_k}^{t} \int_{t_k}^{s} [\nabla \sigma^{j}(X_u)\sigma^{j}(X_u) - \nabla \sigma^{j}(X_{t_k})\sigma^{j}(X_{t_k})] dB_{u}^{j}dB_{s}^{j} \]

\[ + \frac{1}{2} \sum_{j=1}^{m} \nabla \sigma^{j}(X_{t_k})\sigma^{j}(X_{t_k}) \left[ (B_{t}^{j} - B_{t_k}^{j})^2 - (t - t_k)^{2H} \right] . \]

Now we obtain our new numerical scheme by throwing away the higher order stochastic term \( R_{k,n}(t_k, t) \) in (1.4). More precisely, the new approximation scheme (including the term \( b \)) is defined by

\[ X_{t}^n = X_{0} + \int_{0}^{t} b(X_{\eta(s)})ds + \int_{0}^{t} \sigma(X_{\eta(s)})dB_{s} + H \sum_{j=1}^{m} \int_{0}^{t} \nabla \sigma^{j}(X_{\eta(s)})\sigma^{j}(X_{\eta(s)})(s - \eta(s))^{2H-1}ds, \tag{1.5} \]

or

\[ X_{t}^n = X_{t_k} + b(X_{t_k})(t - t_k) + \sum_{j=1}^{m} \sigma(X_{t_k})(B_{t}^{j} - B_{t_k}^{j}) + \frac{1}{2} \sum_{j=1}^{m} \nabla \sigma^{j}(X_{t_k})\sigma^{j}(X_{t_k})(t - t_k)^{2H} \]

for any \( t \in [t_k, t_{k+1}] \). Notice that, taking into account (1.3), if we take \( H = \frac{1}{2} \) and replace \( B \) by a standard Brownian motion \( W \), this is the classical Euler scheme for the Stratonovich stochastic differential equation

\[ X_t = X_0 + \int_{0}^{t} b(X_s)ds + \int_{0}^{t} \sigma(X_s)dW_s. \]

For this numerical scheme we shall prove

\[ \sup_{0 \leq t \leq T} \mathbb{E}|X_t - X_t^n|^2 \leq \begin{cases} Cn^{-(2H-\frac{3}{2})} & \text{if } \frac{1}{2} < H < \frac{3}{4}, \\ Cn^{-1} \sqrt{\log n} & \text{if } H = \frac{3}{4}, \\ Cn^{-1} & \text{if } \frac{3}{4} < H < 1. \end{cases} \]
The proof of this result combines the techniques of Malliavin calculus with classical fractional calculus. The main idea is to express the path-wise Riemann Stieltjes integral appearing in (1.1) and in (1.3) as the sum of a Skorohod integral plus a correction term which involves the trace of the Malliavin derivative. A key ingredient in the study of the numerical schemes is the asymptotic behavior of weighted quadratic variations. We refer to [11] for a discussion on the asymptotic behavior of general Hermite weighted variations. On the other hand, we make use of uniform estimates for the moments of all orders of the processes \(X_n\) and their first and second order Malliavin derivatives, which can be obtained using techniques of fractional calculus, following the approach used, for instance, by Hu and Nualart in [4].

We also obtain a weak approximation result for our new numerical scheme. In this case, the rate is of the order \(n^{-1}\) for all values of \(H\). More precisely, we are able to show that \(n \left( E(f(X_t)) - E(f(X^n_t)) \right) \) converges to a finite non zero limit which can be explicitly computed. Finally, let us mention the fact that the techniques of Malliavin calculus also allows us to provide an alternative and simpler proof of the fact that the rate of convergence of the numerical scheme (1.2) is of the order \(n^{1-2H}\) and this rate is optimal, extending to the multidimensional case the result by Neuenkirch and Nourdin [10]. The Malliavin calculus technique has first been introduced to the study of weak approximation by Kohatsu-Higa in [9].

In the case \(\frac{1}{4} < H < \frac{1}{2}\) the stochastic differential equation (1.1) can be solved using the theory of rough paths introduced by Lyons (see [8]). There are also a number of results on the rate of convergence of Euler-type numerical schemes in this case (see, for instance, the paper by Deya, Neuenkirch and Tindel [1] for a Milstein-type scheme without Lévy area in the case \(\frac{1}{3} < H < \frac{1}{2}\), and the monograph by Friz and Victoir [2]).

The paper is organized as follows. The next section contains some basic material on fractional calculus and Malliavin calculus that will be used along the paper. In Section 3 we derive the necessary estimates for the uniform and Hölder norms of the processes \(X, X^n\) and their first and second Malliavin derivatives. In Section 4 we prove our main result on the rate of convergence in \(L^2\) for the numerical scheme (1.3). The weak approximation result is discussed in Section 5. In Section 6 we deal with the numerical scheme (1.2). In the last section, we prove some auxiliary results. To simplify the presentation, we shall set \(b(x) = 0\), and our results can be easily extended to the case where the equation includes a drift term.

2 Preliminaries

Let \(B = \{(B^1_t, \ldots, B^m_t), t \in [0, T]\}\) be an \(m\)-dimensional fractional Brownian motion (fBm) with Hurst parameter \(H \in (\frac{1}{2}, 1)\), defined on a complete probability space \((\Omega, \mathcal{F}, P)\). Namely, \(B\) is a mean zero Gaussian process with covariance

\[
\mathbb{E}(B^i_t B^j_s) = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}) \delta_{ij},
\]

for all \(s, t \in [0, T]\), where \(\delta\) is the Kronecker delta function.

2.1 Elements of fractional calculus

In this subsection we introduce the definitions of the fractional integral and derivative operators that will be used to estimate path-wise Riemann Stieltjes integrals.

Let \(a, b \in [0, T]\) with \(a < b\). Let \(\beta \in (0, 1)\). We denote by \(C^0(a, b)\) the space of \(\beta\)-Hölder continuous functions on the interval \([a, b]\). For a function \(x : [0, T] \to \mathbb{R}\), \(\|x\|_{a, b, \beta}\) denotes the \(\beta\)-Hölder norm of \(x\) on \([a, b]\), that is,

\[
\|x\|_{a, b, \beta} = \sup \left\{ \frac{|x_u - x_v|}{|u - v|^{\beta}} ; a \leq u < v \leq b \right\}.
\]
We recall that for each \( n \geq 1 \) and \( i = 0, \ldots, n \), we set \( t_i = \frac{iT}{n} = ih \) and we define \( \eta(t) = t_i \) when \( t_i \leq t < t_i + h \). We will also make use of the following seminorm:

\[
\|x\|_{a,b,\beta,n} = \sup \left\{ \frac{|x_u - x_v|}{|u - v|^{\beta}}, a \leq u < v \leq b, \eta(u) = u \right\}.
\]

We will denote the supnorm of \( x \) on the interval \([a, b]\) as \( \|x\|_{a,b,\infty} \). When \( a = 0 \) and \( b = T \), we will simply write \( \|x\|_{\infty} \) and \( \|x\|_{\beta} \).

Let \( f \in L^1([a, b]) \) and \( \alpha > 0 \). The left-sided and right-sided fractional Riemann-Liouville integrals of \( f \) of order \( \alpha \) are defined for almost all \( t \in (a, b) \) by

\[
I_{a+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha - 1} f(s)ds
\]

and

\[
I_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_t^b (s - t)^{\alpha - 1} f(s)ds,
\]

respectively, where \((-1)^{\alpha} = e^{-i\alpha\pi}\) and \( \Gamma(\alpha) = \int_0^{\infty} r^{\alpha-1} e^{-r} dr \) is the Gamma function. Let \( I_{a+}^\alpha(L^p) \) (resp. \( I_{b-}^\alpha(L^p) \)) be the image of \( L^p([a, b]) \) by the operator \( I_{a+}^\alpha \) (resp. \( I_{b-}^\alpha \)). If \( f \in I_{a+}^\alpha(L^p) \) (resp. \( f \in I_{b-}^\alpha(L^p) \)) and \( 0 < \alpha < 1 \) then the fractional Weyl derivatives are defined as

\[
D_{a+}^\alpha f(t) = \frac{1}{\Gamma(1 - \alpha)} \left( \frac{f(t)}{(t-a)^\alpha} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right) \tag{2.1}
\]

and

\[
D_{b-}^\alpha f(t) = \frac{(-1)^{-\alpha}}{\Gamma(1 - \alpha)} \left( \frac{f(t)}{(b-t)^\alpha} + \alpha \int_t^b \frac{f(t) - f(s)}{(s-t)^{\alpha+1}} ds \right), \tag{2.2}
\]

where \( a \leq t \leq b \).

Suppose that \( f \in C^\lambda(a,b) \) and \( g \in C^\mu(a,b) \) with \( \lambda + \mu > 1 \). Then, according to [15], the Riemann-Stieljes integral \( \int_a^b f dg \) exists. The following proposition can be regarded as a fractional integration by parts formula, and provides an explicit expression for the integral \( \int_a^b f dg \) in terms of fractional derivatives. We refer to [16] for additional details.

**Proposition 2.1** Suppose that \( f \in C^\lambda(a,b) \) and \( g \in C^\mu(a,b) \) with \( \lambda + \mu > 1 \). Let \( \lambda > \alpha \) and \( \mu > 1 - \alpha \). Then the Riemann-Stieltjes integral \( \int_a^b f dg \) exists and it can be expressed as

\[
\int_a^b f dg = (-1)^\alpha \int_a^b D_{a+}^\alpha f(t) D_{b-}^{1-\alpha} g_{b-}(t) dt,
\]

where \( g_{b-}(t) = g(t) - g(b) \).

For any integer \( k \geq 1 \), we denote by \( C_b^k(\mathbb{R}^d; \mathbb{R}^M) \) the space of \( k \) times continuously differentiable functions \( f : \mathbb{R}^d \to \mathbb{R}^M \) which are bounded together with their first \( k \) partial derivatives. Also \( C_b^\infty(\mathbb{R}^d; \mathbb{R}^M) \) is the space of infinitely differentiable functions which are bounded together with all their partial derivatives. If \( M = 1 \) we simply write \( C_b^k(\mathbb{R}^d) \) and \( C_b^\infty(\mathbb{R}^d) \).

### 2.2 Elements of Malliavin calculus

Let us introduce some basic notation and results on the Malliavin calculus of variations with respect to the \( m \)-dimensional fBm \( B \). We refer to Nualart [12] for a complete presentation of these notions.
Let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on $[0, T]$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \frac{1}{2} (t^{2H} + s^{2H} - |t-s|^{2H}).$$

The space $L^F([0, T])$ is continuously embedded into $\mathcal{H}$, and for $\phi, \psi \in L^F([0, T])$ the scalar product in $\mathcal{H}$ can be expressed as

$$\langle \phi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T \phi(s) \psi(t) |t-s|^{2H-2} ds dt,$$

where $\alpha_H = H(2H - 1)$.

Let $S$ be the set of smooth and cylindrical random variables of the form

$$F = f(B_{s_1}, \ldots, B_{s_N}),$$

where $N \geq 1$ and $f \in C^\infty_0(\mathbb{R}^{m \times N})$. The derivative operator is defined in $S$ as the $\mathcal{H}^m$-valued random variable such that for $j = 1, \ldots, m$ and for $t \in [0, T]$

$$D^j_t F = \sum_{i=1}^N \frac{\partial f}{\partial x_{i,j}} (B_{s_1}, \ldots, B_{s_N}) 1_{[0,s_i]}(t).$$

We can iterate this expression and get higher order derivatives $D^{j_1} \cdots D^{j_k}_t$. For any $p \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $S$ with respect to the seminorm

$$\|F\|_{k,p}^p = \mathbb{E} [\|F\|^p] + \mathbb{E} \left[ \sum_{l=1}^k \left( \sum_{j_1, \ldots, j_l=1}^m \|D^{j_1} \cdots D^{j_l} F\|_{\mathcal{H}^{\otimes l}}^2 \right)^{\frac{p}{2}} \right].$$

We can also fix $j = 1, \ldots, m$ and introduce the Sobolev space $\mathbb{D}^{j,k,p}$ of random variables which are $k$ times differentiable with respect to the one-dimensional fBm $B^j$. That is, $\mathbb{D}^{j,k,p}$ is the completion of $S$ with respect to the seminorm

$$\|F\|_{j,k,p}^p = \mathbb{E} [\|F\|^p] + \mathbb{E} \left[ \sum_{l=1}^k \|D^j \cdots D^j F\|_{\mathcal{H}^{\otimes l}}^p \right].$$

Clearly $\cap_{j=1}^m \mathbb{D}^{j,k,p} = \mathbb{D}^{k,p}$.

For any $j = 1, \ldots, m$ we denote by $\delta^j$ the adjoint of the derivative operator $D^j$. That is, the domain of $\delta^j$ in $L^2$ is a subspace of $L^2(\Omega; \mathcal{H})$ and for any $u \in \text{Dom}\delta^j$ and $F \in \mathbb{D}^{j,1,2}$ the following duality relationship holds

$$\mathbb{E} (\langle u, D^j F \rangle_{\mathcal{H}}) = \mathbb{E} (\delta^j(u) F).$$

Then, $\delta^j(u)$ is also called the Skorohod integral of $u$ with respect to the fBm $B^j$ and we use the notation $\delta^j(u) = \int_0^T u_t^j \delta B^j_t$.

Suppose that $u = \{u_t, t \in [0, T]\}$ is a stochastic process whose trajectories are Hölder continuous of order $\gamma > 1 - H$. Then, for any $j = 1, \ldots, m$, the path-wise Riemann-Stieltjes integral $\int_0^T u_t dB^j_t$ exists by the results of Young [15]. On the other hand, if $u \in \mathbb{D}^{j,1,2}(\mathcal{H})$ and the derivative $D^j u_t$ satisfies almost surely

$$\int_0^T \int_0^T |D^j u_t||t-s|^{2H-2} ds dt < \infty,$$
and \( \mathbb{E} \left( \left\| Du \right\|^2_{L^2([0,T]^2)} \right) < \infty \), then (see Proposition 5.2.3 in [12]) we can write
\[
\int_0^T u_t dB_t^j = \int_0^T u_t \delta B_t^j + \alpha_H \int_0^T \int_0^T D_s^j u_t |t - s|^{2H-2} ds dt. \tag{2.3}
\]

Let \( p > 1 \), and \( u \in \mathbb{D}^{1,p}(\mathcal{H}) \). The following inequality gives an estimate of \( L^p \) norm of the Skorohod integral of \( u \) with respect to the BM \( B^j \) (see Proposition 1.5.8 in [12])
\[
\mathbb{E} \left( \left\| \int_0^T u_t dB_t \right\|^p \right) \leq C_p \left[ \mathbb{E} \left( \left\| u_t \right\|^p_{L^2([0,T]^2)} \right) + \mathbb{E} \left( \left\| D_s^j u_t \right\|^p_{L^2([0,T]^2)} \right) \right]. \tag{2.4}
\]

For simplicity, in the remaining part of this section we assume \( m = 1 \). For every \( n \geq 1 \), let \( \mathcal{H}_n \) be the \( n \)th Wiener chaos of \( B \), that is, the closed linear subspace of \( L^2(\Omega, \mathcal{F}, \mathbb{P}) \) generated by the random variables \( \{ H_n(B(h)), h \in \mathcal{H}, \| h \|_{\mathcal{H}} = 1 \} \), where for an integer \( n \geq 2 \), we denote by \( H_n \) the Hermite polynomial with degree \( n \) defined by
\[
H_n(x) = \frac{(-1)^n}{n!} e^x \frac{d^n}{dx^n} (e^{-x}).
\]
The mapping \( I_n(h^{\otimes n}) = n! H_n(B(h)) \) provides a linear isometry between the symmetric tensor product \( \mathcal{H}^{\otimes n} \) (equipped with the modified norm \( \| \cdot \|_{\mathcal{H}^{\otimes n}} = \frac{1}{\sqrt{n!}} \cdot \| \cdot \|_{\mathcal{H}^{\otimes n}} \)) and \( \mathcal{H}_n \). The following duality formula holds
\[
\mathbb{E}(FI_n(h)) = \mathbb{E}(\langle D^n F, h \rangle_{\mathcal{H}^{\otimes n}}), \tag{2.5}
\]
for any element \( h \in \mathcal{H}^{\otimes n} \) and any random variable \( F \in \mathbb{D}^{n,2} \), and where \( D^n F \) denotes the \( n \)th iteration of the derivative operator.

Let \( \{ e_k, k \geq 1 \} \) be a complete orthonormal system in \( \mathcal{H} \). Given \( f \in \mathcal{H}^{\otimes n} \) and \( g \in \mathcal{H}^{\otimes m} \), for every \( r = 0, \ldots, n \wedge m \), the contraction of \( f \) and \( g \) of order \( r \) is the element of \( \mathcal{H}^{\otimes(n+m-2r)} \) defined by
\[
f \otimes_r g = \sum_{k_1, \ldots, k_r=1}^{\infty} \langle f, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle_{\mathcal{H}^{\otimes r}} \otimes \langle g, e_{k_1} \otimes \cdots \otimes e_{k_r} \rangle_{\mathcal{H}^{\otimes r}}.
\]
We denote the symmetrization of \( f \otimes_r g \) by \( f \bar{\otimes}_r g \in \mathcal{H}^{\otimes(n+m-2r)} \). We have the following product formula for multiple stochastic integrals. If \( f \in \mathcal{H}^{\otimes n} \) and \( g \in \mathcal{H}^{\otimes m} \), then
\[
I_n(f)I_m(g) = \sum_{r=0}^{n \wedge m} r! \binom{n}{r} \binom{m}{r} I_{n+m-2r}(f \bar{\otimes}_r g). \tag{2.6}
\]

Throughout the paper, we will assume that \( \beta \) and \( \alpha \) satisfy \( \frac{1}{2} < \beta < H \) and \( \frac{1}{2} > \alpha > 1 - \beta \). Also, \( C \) and \( k \) will represent constants that are independent of the \( n \) and whose value may change from line to line.

### 3 Estimates of the processes \( X^n \), \( X \) and their Malliavin derivatives

In this section, we will consider the case when \( m = 1 \) in order to simplify the notation. All results developed here can be generalized to general case \( m > 1 \). In the case \( m = 1 \), \( \sigma : \mathbb{R}^d \rightarrow \mathbb{R}^d \). Recall that \( \alpha \) and \( \beta \) are two numbers such that \( \frac{1}{2} < \beta < H \) and \( \frac{1}{2} > \alpha > 1 - \beta \).
3.1 Uniform and Hölder estimates for $X^n$ and $X$

Our first result are uniform and Hölder bounds for the process $X^n$ defined in (1.3).

**Proposition 3.1** Let $X^n$ be the process defined in (1.3). Assume $\sigma \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$. Then there exist positive constants $k$ and $k'$ depending on $\alpha$, $\beta$, $H$, $T$, $\|\sigma\|_\infty$ and $\|\nabla\sigma\|_\infty$, such that, almost surely,

$$\|X^n\|_\infty \leq |X_0| + k \left( \|B\|_{\beta}^3 + 1 \right), \quad (3.1)$$

and

$$\|X^n\|_{\beta} \leq k' \left( \|B\|_{\beta}^3 + 1 \right). \quad (3.2)$$

**Proof:** We first prove (3.1). Fix $s, t \in [0, T]$ such that $s = \eta(s)$ and $s \leq t$. By the definition of $X^n$, we can write

$$\left| X^n_t - X^n_s \right| \leq \left| \int_s^t \sigma(X^n_{\eta(r)}) dB_r \right| + H \left| \int_s^t (\nabla \sigma \sigma)(X^n_{\eta(r)})(r - \eta(r))^{2H-1} dr \right|.$$ 

From Lemma 7.1 and using $r - \eta(r) \leq r - s$, we obtain

$$\left| X^n_t - X^n_s \right| \leq k_1 \|B\|_{\beta} \|\sigma\|_\infty (t-s)^\beta + k_3 \|B\|_{\beta} \|\nabla \sigma\|_\infty \|X^n\|_{s,t,\beta,n} (t-s)^{2\beta} + \frac{1}{2} \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H},$$

where $k_1$ and $k_3$ are the constants appearing in Lemma 7.1. Dividing both sides by $(t-s)^\beta$ we get

$$\left| \frac{X^n_t - X^n_s}{(t-s)^\beta} \right| \leq k_1 \|B\|_{\beta} \|\sigma\|_\infty + k_3 \|B\|_{\beta} \|\nabla \sigma\|_\infty \|X^n\|_{s,t,\beta,n} (t-s)^\beta + \frac{1}{2} \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H-\beta}. \quad (3.3)$$

Notice that the above equality is still true if we replace $s, t$ by $s', t'$, where $s \leq s' \leq t', t' = \eta(s')$. Therefore, we can write

$$\|X^n\|_{s,t,\beta,n} \leq k_1 \|B\|_{\beta} \|\sigma\|_\infty + k_3 \|B\|_{\beta} \|\nabla \sigma\|_\infty \|X^n\|_{s,t,\beta,n} (t-s)^\beta + \frac{1}{2} \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H-\beta}. \quad (3.4)$$

We can assume that $\|\nabla \sigma\|_\infty \neq 0$ otherwise the inequality (3.1) is straightforward. Let us define

$$\Delta = 1 \wedge (2k_3 \|B\|_{\beta} \|\nabla \sigma\|_\infty)^{-\frac{1}{\beta}}. \quad (3.3)$$

Then we have $k_3 \|B\|_{\beta} \|\nabla \sigma\|_\infty \Delta^\beta \leq \frac{1}{2}$ and, as a consequence, if $s, t \in [0, T]$ satisfy $s \leq t$, $s = \eta(s)$ and $t - s \leq \Delta$, we obtain

$$\|X^n\|_{s,t,\beta,n} \leq k_1 \|B\|_{\beta} \|\sigma\|_\infty + \frac{1}{2} \|X^n\|_{s,t,\beta,n} + \frac{1}{2} \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H-\beta},$$

and thus

$$\|X^n\|_{s,t,\beta,n} \leq 2k_1 \|B\|_{\beta} \|\sigma\|_\infty + \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H-\beta}. \quad (3.4)$$
Furthermore, we have
\[
\|X^n\|_{s,t,\infty} \leq |X^n_s| + \|X^n\|_{s,t,\beta,n}(t-s)\beta
\leq |X^n_s| + \|\sigma\|_{\infty}(2k_1\|B\|_{\beta} + \|\nabla\sigma\|_{\infty}\Delta^{2H-\beta})\Delta^\beta
\leq |X^n_s| + k_5,
\]
where \(k_5 = \frac{4}{\beta}\|\nabla\sigma\|_{1,\infty}\|\sigma\|_{\infty} + \|\nabla\sigma\|_{\infty}\|\sigma\|_{\infty}\) and the last inequality is obtained by observing that 
\(\Delta^\beta \leq (2k_3\|B\|_{\beta}\|\nabla\sigma\|_{\infty})^{-1}\) and \(\Delta \leq 1\).

If \(\Delta > \frac{2T}{n}\), we divide the interval \([0, m\frac{2T}{n}] \subset [0, T]\) into \(m = \lfloor \frac{2T}{n} \rfloor\) intervals of length \(\frac{T}{n}\). Here \(\lfloor \frac{2T}{n} \rfloor\) denotes the integer part of \(\frac{2T}{n}\). Since the length of each of these subintervals is larger than \(\frac{T}{n}\), we are able to choose \(m\) points \(s_1, s_2, \ldots, s_m\) from each of these intervals such that \(\eta(s_i) = s_i, i = 1, 2, \ldots, m\). On the other hand, we have \(s_{i+1} - s_i \leq \Delta\) for all \(0 = 1, \ldots, m - 1\). Using (3.5) repeatedly, we obtain
\[
\sup_{0 \leq t \leq T} |X^n_t| \leq |X_0| + (m + 1)k_5 \leq |X_0| + k_5 \left(2T \vee 2T (\frac{2k_3}{n}\|B\|_{\beta}\|\nabla\sigma\|_{\infty})^{\frac{1}{\beta}} + 1\right),
\]
which implies (3.1) with \(k = k_5 \max \left(1 + 2T, 2T \left(\frac{2k_3}{n}\|B\|_{\beta}\|\nabla\sigma\|_{\infty}^{\frac{1}{\beta}}\right)\right)\).

If \(\Delta \leq \frac{2T}{n}\), that is, \(n \leq 2T \left\{1 + (2k_3\|B\|_{\beta}\|\nabla\sigma\|_{\infty}^{\frac{1}{\beta}})\right\}\), then
\[
\sup_{0 \leq t \leq T} |X^n_t| \leq |X_0| + \sup_{t \in [0, T]} \left|\int_0^t \sigma(X^n_{\eta(s)})dB_s\right|
\leq |X_0| + \sup_{t \in [0, T]} \left|\sum_{i=0}^{\lfloor \frac{T}{n} \rfloor - 1} \sigma(X^n_{\frac{iT}{n}})(B_{\frac{(i+1)T}{n}} - B_{\frac{iT}{n}}) + \sigma(X^n_{\frac{T}{n}})(B_t - B_{\frac{T}{n}})\right|
\leq |X_0| + n\|\sigma\|_{\infty}\|B\|_{\beta}\left(\frac{T}{n}\right)^\beta
\leq |X_0| + 2T^{1+\beta}\|\sigma\|_{\infty}\|B\|_{\beta}\left[1 + (2k_3\|B\|_{\beta}\|\nabla\sigma\|_{\infty})^{\frac{1-\beta}{\beta}}\right],
\]
which implies (3.1) with \(k = 2T^{1+\beta}\|\sigma\|_{\infty}\left(1 + (2k_3\|B\|_{\beta}\|\nabla\sigma\|_{\infty})^{\frac{1}{\beta}}\right)\). Therefore, the inequalities (3.7) and (3.6) allow us to complete the proof of (3.1).

In order to show (3.2), let \(s, t\) be such that \(0 \leq t - s \leq \Delta\). Using (3.1) and the definition of \(X^n\), we can write
\[
\frac{|X^n_t - X^n_s|}{|t - s|^{\beta}} \leq \frac{|X^n_t - X^n_{\eta(s)+\frac{T}{n}}|}{|t - s|^{\beta}} + \frac{|X^n_{\eta(s)+\frac{T}{n}} - X^n_s|}{|t - s|^{\beta}}
\leq \frac{|X^n_t - X^n_{\eta(s)+\frac{T}{n}}|}{|t - (\eta(s)+\frac{T}{n})|^{\beta}} + \frac{|X^n_{\eta(s)+\frac{T}{n}} - X^n_s|}{|t - s|^{\beta}}
\leq \|X^n\|_{s,t,\beta,\eta} + \left|\sigma(X^n_{\eta(s)})(B_{\eta(s)+\frac{T}{n}} - B_s) + H(\nabla\sigma\sigma)(X^n_{\eta(s)}) \int_{\eta(s)+\frac{T}{n}}^\eta s(r - \eta(r))^{2H-1} dr\right|
\leq 2k_1\|B\|_{\beta}\|\sigma\|_{\infty} + \|\nabla\sigma\|_{\infty}\|\sigma\|_{\infty}(t-s)^{2H-\beta} + \|B\|_{\beta}\|\sigma\|_{\infty}
+ H\|\nabla\sigma\|_{\infty}\|\sigma\|_{\infty}\eta(s) + \frac{T}{n} - s|^{1-\beta}
\leq (2k_1 + 1)\|B\|_{\beta}\|\sigma\|_{\infty} + \|\nabla\sigma\|_{\infty}\|\sigma\|_{\infty}(\Delta^{2H-\beta} + T^{1-\beta}),
\]
which implies
\[
\|X^n\|_{s,t,\beta} \leq k_6 (\|B\|_{\beta} + 1),
\]
with a constant \( k_6 = \max \left( (2k_1 + 1)\|\sigma\|_\infty, \|\nabla\sigma\|_\infty \right) \).

On the other hand, if \( t - s \geq \Delta \), we can write
\[
\frac{|X^n_t - X^n_s|}{|t - s|^\beta} \leq \frac{|X^n_{s+\Delta} - X^n_s| + |X^n_{s+2\Delta} - X^n_{s+\Delta}| + \cdots + |X^n_t - X^n_{\frac{t}{s} \Delta}|}{|t - s|^\beta},
\]
where \( \left\lfloor \frac{t}{s} \right\rfloor \) denotes the integer part of \( \frac{t}{s} \). Then, using (3.8) we can write
\[
\frac{|X^n_t - X^n_s|}{|t - s|^\beta} \leq k_6 \left( \left\lfloor \frac{t - s}{\Delta} \right\rfloor + 1 \right) \Delta^\beta \left( \frac{|B|_\beta + 1}{|t - s|^\beta} \right) \leq k_6 \left( T^{1-\beta} \left( 1 + (2k_3\|B\|_\beta\|\nabla\sigma\|_\infty)^{\frac{1}{\beta}} \right) \right) (\|B\|_\beta + 1),
\]
which implies the estimate (3.2). ■

The following result has been obtained in [4]. Here we give a concise proof for the sake of completeness.

**Proposition 3.2** Let \( X \) be the process defined in (1.1). Assume \( \sigma \in C^1_b(\mathbb{R}^d; \mathbb{R}^d) \). Then there exists positive constant \( k \) depending on \( \alpha, \beta, H, T, \|\sigma\|_\infty \) and \( \|\nabla\sigma\|_\infty \), such that almost surely
\[
\|X\|_\infty \leq |X_0| + k \left( \|B\|_\beta^\frac{1}{\beta} + 1 \right),
\]
and
\[
\|X\|_\beta \leq k \left( \|B\|_\beta^\frac{1}{\beta} + 1 \right).
\]

**Proof:** We first show (3.9). Let \( s, t \in [0, T] \) be such that \( s \leq t \). Using Lemma 7.1 we have
\[
|X_t - X_s| \leq k_1\|B\|_\beta \|\sigma\|_\infty (t - s)^\beta + k_2\|B\|_\beta \|\nabla\sigma\|_\infty \|X\|_{s,t,\beta} (t - s)^2^\beta,
\]
where \( k_1 \) and \( k_2 \) are the constants in Lemma 7.1. Hence
\[
\|X\|_{s,t,\beta} \leq k_1\|B\|_\beta \|\sigma\|_\infty + k_2\|B\|_\beta \|\nabla\sigma\|_\infty \|X\|_{s,t,\beta} (t - s)^\beta.
\]
Let \( \Delta_1 \) be defined by
\[
\Delta_1 = 1 \wedge (2k_2\|B\|_\beta \|\nabla\sigma\|_\infty)^{-\frac{1}{\beta}}.
\]
If we assume \( 0 \leq t - s \leq \Delta_1 \), we obtain
\[
\|X\|_{s,t,\beta} \leq 2k_1\|B\|_\beta \|\sigma\|_\infty.
\]
Therefore,
\[
\|X\|_{s,t,\infty} \leq |X_s| + \|X\|_{s,t,\beta} (t - s)^\beta \leq |X_s| + 2k_1\|\sigma\|_\infty \|B\|_\beta \Delta_1^\beta.
\]
By (3.13) and the definition of \( \Delta_1 \) we have \( \|X\|_{s,t,\infty} \leq |X_s| + \frac{k_1}{k_2}\|\sigma\|_\infty \|\nabla\sigma\|_\infty^{-1} \). Then, we divide the interval \([0, T]\) into \( m = \left\lfloor \frac{T}{\Delta_1} \right\rfloor + 1 \leq T \vee (2Tk_2\|B\|_\beta \|\nabla\sigma\|_\infty)^{\frac{1}{\beta}} + 1 \) subintervals, and we choose \( s_1, \ldots, s_m \) from each subinterval. In this way we obtain
\[
\sup_{0 \leq t \leq T} |X_t| \leq |X_0| + \frac{k_1}{k_2}\|\sigma\|_\infty \|\nabla\sigma\|_\infty^{-1}
\]
\[
\leq |X_0| + \left( T \vee (2Tk_2\|B\|_\beta \|\nabla\sigma\|_\infty)^{\frac{1}{\beta}} + 1 \right) \frac{k_1}{k_2}\|\sigma\|_\infty \|\nabla\sigma\|_\infty^{-1},
\]
which completes the proof of (3.9).

In order to show (3.10), we can estimate \( \|X\|_{s,t,\beta} \) by (3.12) if \( t - s \leq \Delta_1 \), and if \( t - s > \Delta_1 \), we use the same method as in the proof of (3.2) in Proposition 3.1. ■
3.2 Estimates for solutions of two SDE’s driven by fBm

The following results in this section are tailored for our use in the next section.

**Lemma 3.3** Let $X$ and $X^n$ be the processes defined in (1.1) and (1.2), respectively. Assume $\sigma \in C^1_b(\mathbb{R}^d; \mathbb{R}^d)$. Fix $\tau \in [0, T]$ and consider a $d$-dimensional process $Q = \{Q_t, t \in [\tau, T]\}$ with $\beta$-Hölder continuous trajectories. Assume that $V = \{V_t, t \in [\tau, T]\}$ is a $d$-dimensional process satisfying

$$V_t = Q_t + \int_{\tau}^{t} f(c_1X_u + c_2X^n_u) V_u dB_u,$$

where $f : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is of class $C^1_b$ and $c_1, c_2 \in [0, 1]$. Then, there exists a positive constant $k$ depending on $\alpha, \beta, H, T, \|\sigma\|_\infty, \|\nabla \sigma\|_\infty, \|f\|_\infty$ and $\|\nabla f\|_\infty$ such that

$$\|V\|_{\tau, T, \beta} \leq k e^{\int_{\tau}^{T} |Q_t| + \|Q_t\|_\beta} \tag{3.14}$$

and

$$\|V\|_{\tau, T, \beta} \leq k e^{\int_{\tau}^{T} |Q_t| + \|Q_t\|_\beta} \tag{3.15}$$

**Proof:** First we show (3.14). Let $\tau \leq s \leq t \leq T$. By the definition of $V$,

$$|V_t - V_s| \leq |Q_t - Q_s| + \left| \int_s^t f(c_1X_u + c_2X^n_u) V_u dB_u \right|. \tag{3.16}$$

Lemma 7.1(ii) applied to the vector valued function $(x, v, y) \to f(c_1x + c_2y)v$ yields

$$\left| \int_s^t f(c_1X_u + c_2X^n_u) V_u dB_u \right| \leq k_1 \|f\|_\infty \|V\|_{s, t, \infty} \|B\|_\beta (t-s)^\beta + k_2 \|f\|_\infty \|V\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta} \tag{3.17}$$

$$+ k_3 \|\nabla f\|_\infty \|V\|_{s, t, \infty} \|X\|_{s, t, \beta} + \|X^n\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta}.$$  

Now (3.16) and (3.17) imply

$$|V_t - V_s| \leq |Q_t|_{\beta}(t-s)^{\beta} + k_1 \|f\|_\infty \|V\|_{s, t, \infty} \|B\|_\beta (t-s)^{\beta}$$

$$+ k_2 \|f\|_\infty \|V\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta}$$

$$+ k_3 \|\nabla f\|_\infty \|V\|_{s, t, \infty} \|X\|_{s, t, \beta} + \|X^n\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta}.$$  

Hence

$$\|V\|_{s, t, \beta} \leq |Q|_{\beta} + k_1 \|f\|_\infty \|V\|_{s, t, \infty} \|B\|_\beta (t-s)^{\beta}$$

$$+ k_2 \|f\|_\infty \|V\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta}$$

$$+ k_3 \|\nabla f\|_\infty \|V\|_{s, t, \infty} \|X\|_{s, t, \beta} + \|X^n\|_{s, t, \beta} \|B\|_\beta (t-s)^{2\beta}.$$  

Choose $\Delta_2$ such that

$$\Delta_2 = \Delta \wedge \Delta_1 \wedge (2k_2 \|f\|_\infty \|B\|_\beta)^{-\frac{1}{\beta}}, \tag{3.18}$$

where $\Delta_1$ is defined in (3.11) and $\Delta$ is defined in (3.3). Let $s, t$ be such that $0 \leq t-s \leq \Delta_2$. Then we have

$$\|V\|_{s, t, \beta} \leq 2|Q|_{\beta} + k_1 \|f\|_\infty \|V\|_{s, t, \infty} \|B\|_\beta$$

$$+ 2k_2 \|f\|_\infty \|V\|_{s, t, \beta} \|B\|_\beta (t-s)^{\beta}$$

$$+ 2k_3 \|\nabla f\|_\infty \|V\|_{s, t, \infty} \|X\|_{s, t, \beta} + \|X^n\|_{s, t, \beta} \|B\|_\beta \Delta_2.$$
From (3.8) and (3.12), we have
\[ \|V\|_{s,t,\beta} \leq 2\|Q\|_\beta + 2k_1\|f\|_\infty\|V\|_{s,t,\infty}\|B\|_\beta \\
+ 2k_3\|\nabla f\|_\infty\|V\|_{s,t,\infty}[2k_1\|\sigma\|_\infty\|B\|_\beta + k_6(\|\|B\|_\beta + 1)]\|\|B\|_\beta\Delta_2^\beta \\
\leq 2\|Q\|_\beta + \|V\|_{s,t,\infty}\|B\|_\beta \left\{k_7 + k_8\|\|B\|_\beta\Delta_2^\beta \right\}, \] (3.19)
where
\[ k_7 = 2k_1\|f\|_\infty + 2k_3k_6\|\nabla f\|_\infty, \]
\[ k_8 = 2k_3\|\nabla f\|_\infty[2k_1\|\sigma\|_\infty + k_6]. \]

Hence
\[ \|V\|_{s,t,\infty} \leq \|V\|_{s,\infty} + 2\|Q\|_\beta\Delta_2^\beta + \|V\|_{s,t,\infty}\|B\|_\beta\Delta_2^\beta \left\{k_7 + k_8\|\|B\|_\beta\Delta_2^\beta \right\}. \]

If we further assume \( \Delta_2 \) satisfying
\[ \|B\|_\beta\Delta_2^\beta \left\{k_7 + k_8\|\|B\|_\beta\Delta_2^\beta \right\} \leq \frac{1}{2}, \] (3.20)
then we obtain, taking into account that \( \Delta_2 \leq 1 \),
\[ \|V\|_{s,t,\infty} \leq 2(\|V\|_{s,\infty} + 2\|Q\|_\beta). \]

If we divide the interval \([\tau,T]\) into \( m = \left\lfloor \frac{T-\tau}{\Delta_2} \right\rfloor + 1 \) subintervals and choose \( s_1, s_2, \ldots, s_m \) from each of these intervals, we have
\[ \|V\|_{\tau,T,\infty} \leq 2^{m+1}(\|V\|_{\tau} + 4\|Q\|_\beta). \] (3.21)

Notice that for (3.20) to hold it suffices that
\[ \Delta_2\|B\|_\beta \leq \frac{\sqrt{k_7^2 + 2k_8} - k_7}{2k_8} = K_1. \] (3.22)

Thus by (3.18) and (3.22),
\[ m \leq 1 + T \left(1 \lor (K_1^{-1}\|B\|_\beta) \lor (2(k_3 \lor k_2)(\|\nabla \sigma\|_\infty \lor \|f\|_\infty)\|\|B\|_\beta)^{\frac{1}{\beta}} \right). \] (3.23)
Finally, from equations (3.18) and (3.23) we obtain the estimate (3.14).

In order to show (3.15), we notice that if \( 0 \leq t - s \leq \Delta_2 \), from (3.19) and (3.14), we deduce
\[ \|V\|_{s,t,\beta} \leq 2\|Q\|_\beta + k (\|B\|_\beta^2 + 1) e^{k\|\|B\|_\beta^2} (\|Q\|_\beta + 4\|Q\|_\beta), \]
for some constant \( k \), which provides the desired estimate. On the other hand, if \( t - s > \Delta_2 \), we use the same method as in the proof of (3.2) in Proposition 3.1. The proof of the lemma is now complete. ■

Consider now a second type of stochastic differential equation driven by a fractional Brownian motion.
Lemma 3.4 Let $X$ and $X^n$ be the processes defined by (1.1) and (1.3), respectively. Assume $\sigma \in C^1_b(\mathbb{R}^d; \mathbb{R}^d)$. Fix $\tau \in [0, T]$ and consider a $d$-dimensional process $Q = \{Q_t, t \in [\tau, T]\}$ with Hölder continuous paths of order $\beta$. Let $U = \{U_t, \tau \leq t \leq T\}$ be a $d$-dimensional process that satisfies

$$U_t = Q_t + \int_{\epsilon(\tau)}^t g(X^n_{\eta(u)}(u)) dU(u) + \int_{\epsilon(\tau)}^t h(X^n_{\eta(u)}(u)) (u - \eta(u))^{2H - 1} du,$$

(3.24)

for any $t$ such that $\epsilon(\tau) \leq t \leq T$, and $U_t = Q_t$ when $\tau \leq t \leq \epsilon(\tau)$. Here $g, h \in C^1_b(\mathbb{R}^d; \mathbb{R}^{d \times d})$, and $\epsilon(\tau) = t_i + h$ if $t_i < \tau \leq t_i + h$ for some $i = 0, \ldots, n$.

Then, there exists a positive constant $k$ depending on $\alpha, \beta, H, T, \|\sigma\|_\infty, \|\nabla\sigma\|_\infty, \|g\|_\infty, \|\nabla g\|_\infty, \|h\|_\infty$ and $\|\nabla h\|_\infty$ such that

$$\|U\|_{\tau, T, \infty} \leq k(|Q_\tau| + \|Q\|_\beta) e^{k\|B\|_\beta^\frac{1}{\beta}},$$

(3.25)

and

$$\|U\|_{\tau, T, \beta} \leq k(|Q_\tau| + \|Q\|_\beta) e^{k\|B\|_\beta^\frac{1}{\beta}}.$$  

(3.26)

Proof: We first prove (3.25). Let $s, t \in [0, T]$ be such that $\tau \leq s \leq t$ and $s = \eta(s)$. This implies $s \geq \epsilon(\tau)$. Then we can write

$$|U_t - U_s| \leq |Q_t - Q_s| + \int_s^t g(X^n_{\eta(u)}(u)) dU(u) + \int_s^t h(X^n_{\eta(u)}(u)) (u - \eta(u))^{2H - 1} du.$$

By Lemma 7.1(i) applied to the vector-valued function $(u, x) \to g(x)u$ we obtain

$$\left| \int_s^t g(X^n_{\eta(u)}(u)) dU(u) \right| \leq k_1 |g|_\infty \|U\|_{s, t, \infty} \|B\|_\beta (t - s)^\beta + k_3 |g|_\infty \|U\|_{s, t, \beta, n} \|B\|_\beta$$

$$+ k_2 \|\nabla g\|_\infty \|U\|_{s, t, \infty} \|X^n\|_{s, t, \beta, n} \|B\|_\beta (t - s)^{2\beta}.$$ 

Therefore,

$$\|U\|_{s, t, \beta, n} \leq |Q|_\beta + k_1 |g|_\infty \|U\|_{s, t, \infty} \|B\|_\beta (t - s)^\beta + k_3 |g|_\infty \|U\|_{s, t, \beta, n} \|B\|_\beta (t - s)^\beta$$

$$+ k_2 \|\nabla g\|_\infty \|U\|_{s, t, \infty} \|X^n\|_{s, t, \beta, n} \|B\|_\beta (t - s)^{2\beta} + \frac{1}{2H} \|h\|_\infty \|U\|_{s, t, \infty} (t - s)^{2H - \beta}.$$

Choose $\Delta_3$ such that

$$\Delta_3 = \Delta \wedge (2k_3 |g|_\infty \|B\|_\beta)^{-\frac{1}{\beta}},$$

(3.27)

where $\Delta$ is defined in (3.3). Then, when $|t - s| \leq \Delta_3$, we can write

$$\|U\|_{s, t, \beta, n} \leq 2|Q|_\beta + 2k_1 |g|_\infty \|U\|_{s, t, \infty} \|B\|_\beta$$

$$+ 2k_2 \|\nabla g\|_\infty \|U\|_{s, t, \infty} \|X^n\|_{s, t, \beta, n} \|B\|_\beta (t - s)^\beta$$

$$+ \frac{1}{H} \|h\|_\infty \|U\|_{s, t, \infty} (t - s)^{2H - \beta}.$$
Using the estimate (3.4) yields
\[
\|U\|_{s,t,n} \leq 2\|Q\|_\beta + \|U\|_{s,t,\infty} \left\{ 2k_1\|g\|_\infty \|B\|_\beta \\
+ 2k_2\|\nabla g\|_\infty \|\nabla \sigma\|_\infty \|\sigma\|_\infty (t-s)^{2H-\beta}\|B\|_\beta (t-s)\beta \\
+ 4k_1k_2\|\nabla g\|_\infty \|B\|_\beta^2\|\sigma\|_\infty (t-s)^{\beta} + \frac{1}{H}\|h\|_\infty (t-s)^{2H-\beta} \right\} \\
\leq 2\|Q\|_\beta + \|U\|_{s,t,\infty} \left\{ (k_9 + k_{10}\|B\|_\beta \Delta_3^\beta)\|B\|_\beta + \frac{1}{H}\|h\|_\infty \Delta_3^{2H-\beta} \right\},
\]
where
\[
k_9 = 2k_1\|g\|_\infty + 2k_2\|\nabla g\|_\infty |\nabla \sigma|_\infty \|\sigma\|_\infty; \\
k_{10} = 4k_1k_2\|\nabla g\|_\infty \|\sigma\|_\infty.
\]
As a consequence,
\[
\|U\|_{s,t,\infty} \leq |U_s| + 2\|Q\|_\beta \Delta_3^\beta \\
+ \|U\|_{s,t,\infty} \left\{ (k_9 + k_{10}\|B\|_\beta \Delta_3^\beta)\|B\|_\beta \Delta_3^\beta + \frac{1}{H}\|h\|_\infty \Delta_3^{2H} \right\},
\]
If we further assume that \(\Delta_3\) satisfies
\[
\frac{1}{H}\Delta_3^{2H}\|h\|_\infty \leq \frac{1}{4} \quad \text{and} \quad (k_9 + k_{10}\|B\|_\beta \Delta_3^\beta)\|B\|_\beta \Delta_3^\beta \leq \frac{1}{4},
\]
then
\[
\|U\|_{s,t,\infty} \leq 2|U_s| + 4\|Q\|_\beta.
\]
If \(\Delta_3 > \frac{2T}{n}\), we divide the interval \([0,m\frac{\Delta_3}{2}] \subset [0,T]\) into \(m = \lfloor \frac{2T}{\Delta_3} \rfloor\) intervals of length \(\frac{\Delta_3}{2}\).
Since the length of each of these subintervals is larger than \(\frac{T}{n}\), we are able to choose \(m\) points \(s_1, s_2, \ldots, s_m\) from each of these intervals such that \(\eta(s_i) = s_i, i = 1, 2, \ldots, m\). On the other hand, we have \(s_{i+1} - s_i \leq \Delta_3\) for all \(0 = 1, \ldots, m - 1\). Using (3.30) repeatedly, we obtain
\[
\|U\|_{T,T,\infty} \leq 2^{m+1}(|U_T| + 4\|Q\|_\beta).
\]
Notice that for (3.29) to hold, it suffices that
\[
\Delta_3^\beta\|B\|_\beta \leq \frac{\sqrt{k_9^2 + k_{10} - k_9}}{2k_{10}} = K_2 \quad \text{and} \quad \Delta_3^\beta \leq \frac{H}{4}\|h\|_\infty^{-1}.
\]
By (3.27), (3.29), (3.31) and the definition of \(\Delta_3\), we have
\[
m \leq \frac{2T}{\Delta_3} \leq k + \|B\|_\beta^{\frac{1}{\beta}},
\]
for some constant \(k\) not depending on the partition. Therefore
\[
\|U\|_{T,T,\infty} \leq k2^k\|B\|_\beta^{\frac{1}{\beta}} \left(|U_T| + 4\|Q\|_\beta\right).
\]
When \(\Delta_2 \leq \frac{2T}{n}\), that is, when \(n \leq \frac{2T}{\Delta_2} \leq m \leq k + k\|B\|_\beta^{\frac{1}{\beta}}\), by the definition of \(U\) in (3.21), we can write for any \(t \in [\tau,T]\)
\[
|U_t| \leq |U_{\eta(t)}| + |Q_t - Q_{\eta(t)}| + \|g\|_\infty |U_{\eta(t)}||B\|_\beta (T/n)^\beta + \frac{1}{2H}\|h\|_\infty |U_{\eta(t)}|(T/n)^{2H} \\
= |Q_t - Q_{\eta(t)}| + \|U_{\eta(t)}| \left[ 1 + \|g\|_\infty \|B\|_\beta (T/n)^\beta + \frac{1}{2H}\|h\|_\infty (T/n)^{2H} \right] \\
\]
Iterating this estimate, we obtain
\[
\sup_{\tau \leq t \leq T} |U_t| \leq Q_\tau \left[ 1 + \|g\|_\infty \|B\|_\beta(T/n)^\beta + \frac{1}{2} \|h\|_\infty (T/n)^{2H} \right]^n
\]
\[
+ \|Q\|_\beta(T/n)^\beta n \left[ 1 + \|g\|_\infty \|B\|_\beta(T/n)^\beta + \frac{1}{2} \|h\|_\infty (T/n)^{2H} \right]^{n-1}
\]
\[
\leq k(|Q_\tau| + \|Q\|_\beta) e^{k\|B\|_\beta^{1/\beta}},
\]
for some constant \(k\) independent of \(n\), where we have used the inequalities
\[
1 + \|g\|_\infty \|B\|_\beta(T/n)^\beta + \frac{1}{2} \|h\|_\infty (T/n)^{2H} \leq 1 + k(1 + \|B\|_\beta)n^{-\beta},
\]
for some positive constant \(k\) and
\[
\exp \left( n \log(1 + k(1 + \|B\|_\beta)n^{-\beta}) \right) \leq e^{k(1 + \|B\|_\beta)n^{1-\beta}}.
\]
This completes the proof of (3.25).

In order to show (3.26), we notice that if \(0 \leq t - s \leq \Delta_3\), from (3.28) and (3.25), we deduce
\[
\|U\|_{s,t,\beta,n} \leq k(1 + \|B\|_\beta^2)(|Q_\tau| + \|Q\|_\beta)e^{k\|B\|_\beta^{1/\beta}}.
\]
for some constant \(k\). Then, we can finish the proof using the same approach as in the proof of Proposition 3.1. The proof of the lemma is now complete. □

3.3 Estimates for the Malliavin derivatives of \(X\) and \(X^n\)

We are now ready to derive upper bounds for the processes that we will need in the proof of the main result, including the Malliavin derivative of the solution of (1.1) and the modified Euler scheme process (1.5). We refer the reader to Nualart and Saussereau [14] for results on Malliavin regularity of the solution of Equation (1.1). In particular, if \(\sigma\) belongs to \(C^2_b(\mathbb{R}^d;\mathbb{R}^d)\), then \(X^i_t\) belongs to \(\mathbb{D}^{2,p}\) for all \(p \geq 1\), \(i = 1, \ldots, d\) and \(t \in [0,T]\).

**Proposition 3.5** Let \(X\) and \(X^n\) be the processes defined in (1.1) and (1.5), respectively. Suppose that \(\sigma \in C^2_b(\mathbb{R}^d;\mathbb{R}^d)\). Then, there exists a positive constant \(k\) such that for all \(s, r \in [0,T]\) and for all \(n\)
\[
\max (\|D_sX\|_\infty, \|D_rX\|_\beta, \|D_sD_rX\|_\infty, \|D_sD_rX\|_\beta) \leq ke^{k\|B\|_\beta^{1/\beta}},
\]
and
\[
\max (\|D_sX^n\|_\infty, \|D_rX^n\|_\beta, \|D_sD_rX^n\|_\infty, \|D_sD_rX^n\|_\beta) \leq ke^{k\|B\|_\beta^{1/\beta}}.
\]
**Proof:** Let us first prove part (3.33). For any \(0 \leq r \leq t \leq T\) we can write
\[
D_rX_t = \sigma(X_r) + \int_r^t \nabla \sigma(X_u) D_r X_u dB_u.
\]
On the other hand, \(D_rX_t = 0\) if \(r > t\). Applying Lemma 3.3 with \(\tau = r\), \(V_t = D_rX_t\), \(Q_t = \sigma(X_r)\), \(f = \nabla \sigma\), \(c_1 = 1\) and \(c_2 = 0\) we deduce the estimates
\[
\|D_rX\|_\infty \leq k\|\sigma\|_\infty e^{k\|B\|_\beta^{1/\beta}}.
\]
\[\|D_r X\|_\beta \leq k \|\sigma\|_{\infty} e^{k\|B\|_\beta^{\frac{1}{2}}}. \tag{3.36}\]

Taking the second derivative yields for \(0 \leq s \leq r \leq t\)

\[D_s D_r X_t = \nabla \sigma(X_r) D_s X_r + \int_r^t D_s [\nabla \sigma(X_u)] D_r X_u dB_u + \int_r^t \nabla \sigma(X_u) D_s D_r X_u dB_u.\]

Set

\[Q_t = \nabla \sigma(X_r) D_s X_r + \int_r^t D_s [\nabla \sigma(X_u)] D_r X_u dB_u. \tag{3.37}\]

Applying (3.35) yields

\[|Q_r| = |\nabla \sigma(X_r) D_s X_r| \leq k \|\nabla \sigma\|_{\infty} \|\sigma\|_{\infty} e^{k\|B\|_\beta^{\frac{1}{2}}}.\]

On the other hand, Lemma 7.1(ii) leads to the estimate

\[\|Q\|_\beta \leq k \left( \|D_s X\|_{\infty} \|D_r X\|_{\infty} \|B\|_\beta + \|D_s X\|_{\infty} \|D_r X\|_{\infty} \|X\|_\beta \right) + \|D_s X\|_\beta \|D_r X\|_{\infty} + \|D_s X\|_{\infty} \|D_r X\|_\beta,\]

where the constant \(k\) depends on the uniform bounds of the partial derivatives of \(\sigma\) up to the third order. Then, (3.10), (3.35) and (3.36) imply

\[\|Q\|_\beta \leq k \left( 1 + \|B\|_\beta^{\frac{1}{2}} \right) e^{k\|B\|_\beta^{\frac{1}{2}}}.\]

Finally, applying Lemma 3.3 to \(V_t = D_s D_r X_t\), with \(\tau = r\), \(f = \nabla \sigma\), \(c_1 = 1\) and \(c_2 = 0\) and \(Q\) given by (3.37), we obtain

\[\|D_s D_r X\|_{\infty} \leq k e^{k\|B\|_\beta^{\frac{1}{2}}}, \tag{3.38}\]

and

\[\|D_s D_r X\|_\beta \leq k e^{k\|B\|_\beta^{\frac{1}{2}}}. \tag{3.39}\]

Then, part (3.34) follows from (3.35), (3.36), (3.38) and (3.39).

In order to prove part (3.34), let \(r, t\) be such that \(0 \leq r \leq t \leq T\). We have

\[D_r X_t^n = \sigma(X_{u(r)}^n) + \int_{\epsilon(r)}^t \nabla \sigma(X_{u(u)}^n) D_r X_{u(u)}^n dB_u \]
\[+ H \int_{\epsilon(r)}^t [\nabla(\nabla \sigma)](X_{u(u)}^n) D_r X_{u(u)}^n (u - \eta(u))^{2H-1} du. \tag{3.40}\]

Applying Lemma 3.4 with \(\tau = r\), \(U_t = D_r X_t^n\), \(Q_t = \sigma(X_{u(u)}^n)\), \(g = \nabla \sigma\) and \(h = [\nabla(\nabla \sigma)]\) we deduce the estimates

\[\|D_r X^n\|_\infty \leq k \|\sigma\|_{\infty} e^{k\|B\|_\beta^{\frac{1}{2}}}, \tag{3.41}\]
\[ \| D_r X^n \|_\beta \leq k \| \sigma \|_\infty e^{k \| B \|_\beta^\frac{1}{\beta}}, \]  
(3.42)

which implies the desired result for \( \| D_r X^n \|_\infty \) and \( \| D_r X^n \|_\beta \).

Now we show the same type of estimate for the second derivative \( D^2 X^n \). Let \( 0 \leq s \leq r \leq t \). Differentiating (3.40), we obtain

\[
D_s D_r X^n_t = \nabla \sigma (X^n_{\eta(r)}) D_s X^n_{\eta(r)} + \int_{\epsilon(r)}^t D_s [\nabla \sigma (X^n_{\eta(u)})] D_s D_r X^n_{\eta(u)} dB_u \\
+ \int_{\epsilon(r)}^t D_s [\nabla \sigma (X^n_{\eta(u)})] D_r X^n_{\eta(u)} dB_u \\
+ H \int_{\epsilon(r)}^t [\nabla (\nabla \sigma \sigma)] (X^n_{\eta(u)}) D_s D_r X^n_{\eta(u)} (u - \eta(u))^{2H-1} du \\
+ H \int_{\epsilon(r)}^t D_s [\nabla (\nabla \sigma \sigma)] (X^n_{\eta(u)}) D_r X^n_{\eta(u)} (u - \eta(u))^{2H-1} du.
\]

Denote

\[
Q_t = \nabla \sigma (X^n_{\eta(r)}) D_s X^n_{\eta(r)} + \int_{\epsilon(r)}^t D_s [\nabla \sigma (X^n_{\eta(u)})] D_r X^n_{\eta(u)} dB_u \\
+ H \int_{\epsilon(r)}^t D_s [\nabla (\nabla \sigma \sigma)] (X^n_{\eta(u)}) D_r X^n_{\eta(u)} (u - \eta(u))^{2H-1} du,
\]

for \( t \geq \epsilon(r) \) and \( Q_t = \nabla \sigma (X^n_{\eta(r)}) D_s X^n_{\eta(r)} \) if \( r \leq t < \epsilon(r) \). Then, by (3.41),

\[ |Q_t| \leq \| \nabla \sigma \|_\infty \| D_s X^n \|_\infty \leq k \| \nabla \sigma \|_\infty \| \sigma \|_\infty e^{k \| B \|_\beta^\frac{1}{\beta}}. \]

(3.43)

Lemma 7.1(i) and the estimates (3.41) and (3.42) yield

\[ \| Q \|_\beta \leq k e^{k \| B \|_\beta^\frac{1}{\beta}}. \]

(3.44)

Finally, we obtain the desired bound for \( \| D_s D_r X^n \|_\infty \) and \( \| D_s D_r X^n \|_\beta \) applying Lemma 3.4 and using the estimates (3.43) and (3.44). This completes the proof of the proposition. ■

We also need the following result on estimates for the first and second derivatives of solutions to linear equations.

**Proposition 3.6** Let \( X \) and \( X^n \) be the processes defined in (1.1) and (1.5), respectively. Assume that \( \sigma \in C^3_b (\mathbb{R}^d; \mathbb{R}^d) \). Let \( V \) be a \( d \)-dimensional process satisfying the equation

\[ V_t = V_0 + \int_0^t f(c_1 X_u + c_2 X^n_u) V_u dB_u, \]

where \( f \in C^2_b (\mathbb{R}^d; \mathbb{R}^{d \times d}) \) and \( c_1, c_2 \in [0, 1] \). Then, there exists a positive constant \( k \) such that (which does not depend on \( c_1 \) and \( c_2 \)) such that for all \( s, r \in [0, T] \) and for all \( n \)

\[ \max (\| V \|_\infty, \| V \|_\beta, \| D_r V \|_\infty, \| D_r V \|_\beta, \| D_s D_r V \|_\infty, \| D_s D_r V \|_\beta) \leq k e^{k \| B \|_\beta^\frac{1}{\beta}}. \]

(3.45)
Proof: We only need to show that for all \(0 \leq s \leq r \leq T\), the processes \(\{V_t, t \in [0, T]\}\), \(\{D_rV_t, t \in [r, T]\}\) and \(\{D_sD_rV_t, t \in [r, T]\}\) satisfy the conditions of Lemma 3.3 for a suitable process \(Q\) such that

\[
|Q_r| + \|Q\|_\infty \leq ke^{k\|B\|_\beta^2}. \tag{3.46}
\]

For the process \(V\) we take \(\tau = 0\) and \(Q_t = V_0\) and (3.46) is obvious. For the derivative \(DV_t\), we have, if \(0 \leq r < t \leq T\)

\[
D_rV_t = f(c_1X_r + c_2X^n_r)V_r + \int_r^t D_r\left[f(c_1X_u + c_2X^n_u)\right]V_udB_u
\]

In that case,

\[
Q_t = f(c_1X_r + c_2X^n_r)V_r + \int_r^t D_r\left[f(c_1X_u + c_2X^n_u)\right]V_udB_u.
\]

Therefore,

\[
|Q_r| \leq \|f\|_\infty \|V\|_\infty
\]

and by Lemma 3.1(ii) we obtain that \(\|Q\|_\beta\) is bounded by a polynomial in the variables \(\|B\|_\infty\), \(\|V\|_\infty\), \(\|V\|_\beta\), \(\|X\|_\beta\), \(\|X^n\|_\beta\), \(\|D_rX\|_\infty\), \(\|D_rX^n\|_\infty\), \(\|D_rX\|_\beta\) and \(\|D_rX^n\|_\beta\). Then, again this process \(Q\) satisfies (3.46), by Lemma 3.3 applied to \(V\) and the estimates (3.2), (3.10), (3.35), (3.36), (3.41) and (3.42).

Finally, let \(0 \leq s \leq r \leq t \leq T\). We have

\[
D_sD_rV_t = D_s\left[f(c_1X_r + c_2X^n_r)V_r\right] + \int_r^t D_sD_r\left[f(c_1X_u + c_2X^n_u)\right]V_udB_u
\]

\[
+ \int_r^t D_r\left[f(c_1X_u + c_2X^n_u)\right]D_sV_udB_u + \int_r^t D_s\left[f(c_1X_u + c_2X^n_u)\right]D_rV_udB_u
\]

Denote for \(r \leq t \leq T\)

\[
Q_t = D_s\left[f(c_1X_r + c_2X^n_r)V_r\right] + \int_r^t D_sD_r\left[f(c_1X_u + c_2X^n_u)\right]V_udB_u
\]

\[
+ \int_r^t D_r\left[f(c_1X_u + c_2X^n_u)\right]D_sV_udB_u + \int_r^t D_s\left[f(c_1X_u + c_2X^n_u)\right]D_rV_udB_u.
\]

Then

\[
|Q_r| \leq k\|DV\|_\infty(1 + \|D_rX\|_\infty + \|D_rX^n\|_\infty),
\]

and by Lemma 3.1(ii) we obtain that \(\|Q\|_\beta\) is bounded by a polynomial in the variables \(\|B\|_\infty\) and supremum and \(\beta\)-Hölder norms of \(V\), \(D_rV\), \(D_sV\), \(X\), \(D_rX\), \(D_sX\), \(X^n\), \(D_rX^n\), \(D_sX^n\) and \(D_sD_rX^n\). It is then easy to check that this process \(Q\) satisfies (3.46). \(\blacksquare\)

**Remark 3.7** All the results obtained in this section hold true when the approximation process \(X^n_t\) is replaced by the one defined by the recursive scheme (1.2). In this case we would need one less derivative of the coefficient \(\sigma\). We omit the details of the proof. We use \(X^n_t\) to represent both the solutions computed by (1.2) and (1.3). This will not cause confusion since in Sections 4 and 5 deal with the scheme (1.3) and in Section 6 we consider the classical Euler scheme (1.2).
4 Rate of convergence for the modified Euler scheme

The main result of this section is the convergence in $L^2$ of the scheme defined in (1.5) to the solution of the SDE (1.1).

**Theorem 4.1** Let $X$ and $X^n$ be solutions to equations (1.1) and (1.5), respectively. We assume $\sigma \in C^6_b(\mathbb{R}^d;\mathbb{R}^d)$. Then there exists a constant $C$ independent of $n$ such that

$$
\sup_{0 \leq t \leq T} (\mathbb{E}(|X^n_t - X_t|^2))^\frac{1}{2} \leq \begin{cases} 
\mathcal{O}(n^{-1}), & \frac{3}{4} < H < 1, \\
\mathcal{O}(n^{-1} \sqrt{\ln n}), & H = \frac{3}{4}, \\
\mathcal{O}(n^{\frac{1}{2} - 2H}), & \frac{1}{2} < H < \frac{3}{4}.
\end{cases}
$$

**Proof:** We split the proof into five steps.

**Step 1.** Let $Y_t = X_t - X^n_t$, $t \in [0, T]$. By the definition of the processes $X$ and $X^n$, we can write

$$
Y_t = \int_0^t \left[ \sigma(X_s) - \sigma(X^n_{\eta(s)}) \right] dB_s - H \sum_{j=1}^m \int_0^t (\nabla \sigma^j)(X^n_{\eta(s)})(s - \eta(s))^{2H-1} ds
$$

$$
= \int_0^t \left[ \sigma(X_s) - \sigma(X^n_s) + \sigma(X^n_s) - \sigma(X^n_{\eta(s)}) \right] dB_s
$$

$$
- H \sum_{j=1}^m \int_0^t (\nabla \sigma^j)(X^n_{\eta(s)})(s - \eta(s))^{2H-1} ds.
$$

Since

$$
\sigma^j(X_s) - \sigma^j(X^n_s) = \int_0^1 \nabla \sigma^j(\theta X_s + (1 - \theta)X^n_s) d\theta,
$$

and

$$
\sigma^j(X^n_s) - \sigma^j(X^n_{\eta(s)}) = \int_0^1 \nabla \sigma^j(\theta X^n_s + (1 - \theta)X^n_{\eta(s)}) \left[ \sigma(X^n_{\eta(s)})(B_s - B_{\eta(s)}) + \frac{1}{2} \sum_{l=1}^m (\nabla \sigma^l \sigma^j)(X^n_{\eta(s)})(s - \eta(s))^{2H} \right] d\theta,
$$

we have

$$
Y_t = \sum_{j=1}^m \int_0^t \sigma^j_1(s) Y_s dB^j_s + \sum_{j=1}^m \int_0^t \sigma^j_2(s)(B_s - B_{\eta(s)}) dB^j_s
$$

$$
+ \sum_{j=1}^m \int_0^t \sigma^j_3(s)(s - \eta(s))^{2H} dB^j_s - H \sum_{j=1}^m \int_0^t \sigma^j_4(s)(s - \eta(s))^{2H-1} ds,
$$

where, for $j = 1, 2, \ldots, m$ we set

$$
\sigma^j_1(s) = \int_0^1 \nabla \sigma^j(\theta X_s + (1 - \theta)X^n_s) d\theta,
$$

$$
\sigma^j_2(s) = \int_0^1 \nabla \sigma^j(\theta X^n_s + (1 - \theta)X^n_{\eta(s)}) \sigma(X^n_{\eta(s)}) d\theta,
$$

$$
\sigma^j_3(s) = \frac{1}{2} \int_0^1 \nabla \sigma^j(\theta X^n_s + (1 - \theta)X^n_{\eta(s)}) \left[ \sum_{l=1}^m (\nabla \sigma^l \sigma^j)(X^n_{\eta(s)}) \right] d\theta,
$$

$$
\sigma^j_4(s) = \nabla \sigma^j \sigma^j(X^n_{\eta(s)}).
$$
Let \( \Lambda^n = \{ \Lambda^n_t, t \in [0,T] \} \) be the \( d \times d \) matrix-valued solution of the following linear stochastic differential equation

\[
\Lambda^n_t = I + \sum_{j=1}^{m} \int_0^t \sigma^j_t(s) \Lambda^n_{s} dB^j_s,
\]

where \( I \) denotes the identity matrix. By applying the chain rule for the Young’s integral to \( P_t \Lambda^n_t \), where \( P = \{ P_t, t \in [0,T] \} \) is the unique solution of the equation

\[
P_t = I - \sum_{j=1}^{m} \int_0^t P_s \sigma^j_t(s) dB^j_s,
\]

we see that \( P_t \Lambda^n_t = I \) for all \( t \in [0,T] \). Therefore, \( (\Lambda^n_t)^{-1} \) exists and satisfies for all \( t \in [0,T] \)

\[
(\Lambda^n_t)^{-1} = I - \sum_{j=1}^{m} \int_0^t (\Lambda^n_s)^{-1} \sigma^j_t(s) dB^j_s.
\]

With this process \( \Lambda^n \), we can express the process \( Y_t \) explicitly as

\[
Y_t = \Lambda^n_t \left\{ \sum_{j=1}^{m} \int_0^t (\Lambda^n_s)^{-1} \sigma^j_2(s) (B_{s} - B_{\eta(s)}) dB^j_s + \sum_{j=1}^{m} \int_0^t (\Lambda^n_s)^{-1} \sigma^j_3(s) (s - \eta(s))^{2H} dB^j_s \right\}
- H \sum_{j=1}^{m} \int_0^t (\Lambda^n_s)^{-1} \sigma^j_3(s) (s - \eta(s))^{2H-1} ds.
\]

(4.1)

**Remark 4.2** Proposition 3.4 and Fernique’s theorem imply that the norms \( \| \Lambda^n \|_\infty, \| \Lambda^n \|_\beta, \| D_r \Lambda^n \|_\infty, \| D_r \Lambda^n \|_\beta, \| D_s \Lambda^n \|_\infty \) and \( \| D_r \Lambda^n \|_\beta \) are bounded uniformly in \( s \leq r \leq t \) and \( n \) by random variables that have moments of all orders. The same property holds \( (\Lambda^n)^{-1} \).

**Step 2.** We consider the first sum in (4.1). Without loss of generality, we only consider the \( j \)th term of this sum. Using (2.3), we obtain

\[
\Lambda^n_t \int_0^t (\Lambda^n_s)^{-1} \sigma^j_2(s) (B_{s} - B_{\eta(s)}) dB^j_s = \int_0^t \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j_2(s) (B_{s} - B_{\eta(s)}) \delta B^j_s
+ \alpha_H \int_0^t \int_0^t D^j_r \left[ \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j_2(s) (B_{s} - B_{\eta(s)}) \right] |r - s|^{2H-2} ds dr
= \int_0^t \Lambda^n_t \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] (B_{s} - B_{\eta(s)}) \delta B^j_s
+ \alpha_H \int_0^t \int_0^t D^j_r \left[ \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j_2(s) \right] (B_{s} - B_{\eta(s)}) |r - s|^{2H-2} ds dr
+ H \int_0^t \Lambda^n_t (\Lambda^n_s)^{-1} [\sigma^j_2(s)]_j (s - \eta(s))^{2H-1} ds
= A^1_n + A^2_n + A^3_n + A^4_n,
\]

where \( [\sigma^j_2(s)]_j \) is the \( j \)th column of the matrix \( \sigma^j_2(s) \), that is, for \( j = 1, \ldots, m \) we have

\[
[\sigma^j_2(s)]_j = \int_0^1 \nabla \sigma^j(\theta X^n_s) + (1 - \theta) X^n_{\eta(s)} \sigma^j(X^n_{\eta(s)}) d\theta.
\]

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We claim that the following estimate holds for the \( L^2 \) norm of the first term in (4.2)
\[
\mathbb{E} (|A_n|) \leq C n^{-2H-2\beta},
\] (4.3)
for any \( \beta \) such that \( \frac{1}{2} < \beta < H \). In order to prove (4.3) we first apply (2.4) and we obtain
\[
\mathbb{E} \left( \left| \int_0^t \Lambda_t^n \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] (B_s - B_{\eta(s)}) \delta B^j_s \right|^2 \right) \\
\leq C \left( \mathbb{E} \left( \left| \int_0^t \Lambda_t^n \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] (B_s - B_{\eta(s)}) \right|^2 \right) \right)^{\frac{1}{2}} \\
+ C \mathbb{E} \left( \left\| \sum_{i=1}^m D^i_s \left[ \Lambda_t^n \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] (B_s - B_{\eta(s)}) \right] \right\|^2 \right)^{\frac{1}{2}}.
\] (4.4)

By Proposition 3.6 (see also Remark 3.7), the estimate (3.2) for the \( \beta \)-Hölder norm of \( X^n \) and Fernique’s theorem, for any \( \beta \) such that \( \frac{1}{2} < \beta < H \),
\[
\left| (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right| \leq n^{-\beta} F_\beta,
\]
for some nonnegative random variable \( F_\beta \), which has finite moments of all orders.

Therefore,
\[
\mathbb{E} \left( \int_0^t \Lambda_t^n \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] (B_s - B_{\eta(s)}) \right) \\
\leq \mathbb{E} \left( \left| \int_0^t \Lambda_t^n \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) - (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] \right|^2 \right)^{\frac{1}{2}} \mathbb{E} \mathbb{E}[|B_s - B_{\eta(s)}|^2]^{\frac{1}{2}} \\
\leq C \beta n^{-H-\beta}.
\]

So the first term in (4.4) is less than or equal to \( C \beta n^{-2H-2\beta} \), where \( C \beta \) is a constant independent of \( n \).

By Proposition 3.6 (see also Remark 3.7), Proposition 3.5, the estimate (3.2) for the \( \beta \)-Hölder norm of \( X^n \) and Fernique’s theorem, we also have
\[
\left| D^j_s \left[ \Lambda_t^n (\Lambda^n_s)^{-1} \sigma^j_2(s) \right] - D^j_s \left[ \Lambda_t^n (\Lambda^n_{\eta(s)})^{-1} \sigma^j_2(\eta(s)) \right] \right| \leq n^{-\beta} F_\beta,
\]
where \( F_\beta \) is a nonnegative random variable having finite moments of all orders, and \( \beta \) is such that \( \frac{1}{2} \leq \beta < H \). Using this inequality, we see that the second term in (4.3) is less than or equal to \( C n^{-2H-2\beta} \). This completes the proof of the estimate (4.3).

**Step 3.** Consider the second term in (4.2). We can write, using Equation (2.3)
\[
A_{n}^2 = \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \Lambda_t^n (\Lambda^n_{r})^{-1} \sigma^j_2(t_l) (B_s - B_{t_l}) \delta B^j_s \\
= \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \Lambda_t^n (\Lambda^n_{r})^{-1} \sigma^j_2(t_l) (B_s - B_{t_l}) dB^j_s \\
- \alpha_H \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \int_0^t D^j_r \left\{ \Lambda_t^n (\Lambda^n_{r})^{-1} \sigma^j_2(t_l) (B_s - B_{t_l}) \right\} |r - s|^{2H-2} dr ds \\
= B_{n}^1 - B_{n}^2.
\] (4.5)
The term $B_n^1$ can be expressed as follows

$$
B_n^1 = \sum_{l=0}^{n-1} \sum_{i \neq j} \int_{t_l}^{t_{l+1}} \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_i (B_s^i - B_{t_l}^i) dB_s^j
$$

$$
+ \frac{1}{2} \sum_{l=0}^{n-1} \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j (B_{t_{l+1}}^j - B_{t_l}^j)^2.
$$

(4.6)

As before, $[\sigma_2^j(t_l)]_i$, $i = 1, \ldots, m$, denotes the $i$th column of the matrix $\sigma_2^j(t_l)$.

For the term $B_n^2$ in equation (4.5), we can write

$$
B_n^2 = \alpha_H \sum_{l=0}^{n-1} \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \int_{t_l}^{t_{l+1}} \int_0^t D_r^j(B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds
$$

$$
+ \alpha_H \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds
$$

$$
= \alpha_H \sum_{l=0}^{n-1} \int_{t_l}^{t_{l+1}} \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds
$$

$$
+ \alpha_H \int_0^t \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds.
$$

(4.7)

Substituting (4.6) and (4.7) into (4.5), we have

$$
A_n^2 = \sum_{l=0}^{n-1} \sum_{i \neq j} \int_{t_l}^{t_{l+1}} \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_i (B_s^i - B_{t_l}^i) dB_s^j
$$

$$
+ \frac{1}{2} \sum_{l=0}^{n-1} \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \left[ (B_{t_{l+1}}^j - B_{t_l}^j)^2 - (t_{l+1} - t_l)^{2H} \right]
$$

$$
- \alpha_H \int_0^t \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(t_l)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds
$$

$$
= A_n^{2,1} + A_n^{2,2} + A_n^{2,3}.
$$

Proposition 7.4 and Proposition 7.5 in Section 7.4 provide the desired estimates for the $L^2$ norm of the terms $A_n^{2,1}$ and $A_n^{2,2}$, respectively. On the other hand, we can combine the above $A_n^{2,3}$ with the term $A_n^3$ in (4.2) to obtain

$$
A_n^{2,3} + A_n^3 = \alpha_H \left[ \int_0^t \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(s)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} ds dr
$$

$$
- \int_0^t \int_0^t D_r^j \left\{ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(s)]_j \right\} (B_s^j - B_{t_l}^j)|r - s|^{2H-2} dr ds \right].
$$

By Proposition 3.3 (see also Remark 4.2), Proposition 3.5, the estimate (3.2) for the $\beta$-Hölder norm of $X^n$ and Fernioque’s theorem, we have

$$
\left| D_r^j \left[ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(s)]_j \right] - D_r^j \left[ \Lambda_i^n(\Lambda_j^n)^{-1} [\sigma_2^j(s)]_j \right] \right| \leq n^{-\beta} F_{\beta},
$$

22
where $\beta$ is any number such that $\frac{1}{2} < \beta < H$ and $F_\beta$ is a nonnegative random variable having finite moments of all orders and independent of $n$. As a consequence, we obtain

$$\mathbb{E} \left( |A_{n}^{23} + A_{n}^{3} |^2 \right) \leq C n^{-2H-2\beta}.$$ 

**Step 4.** The sum of the last term in (4.1) and the forth term $A_{n}^{4}$ in (4.2) is

$$H \sum_{j=1}^{m} \int_{0}^{t} \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{2}^{j}(s) |s - \eta(s)|^{2H-1} ds - H \sum_{j=1}^{m} \int_{0}^{t} \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{4}^{j}(s) |s - \eta(s)|^{2H-1} ds$$

$$= H \sum_{j=1}^{m} \int_{0}^{t} \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \left[ |\sigma_{2}^{j}(s) - \sigma_{4}^{j}(s)| |s - \eta(s)||^{2H-1} ds.$$ 

We can easily verify that the $L^2$ norm of the above expression is less than or equal to a constant times $n^{1-3H}$. 

**Step 5.** We consider the second sum in (4.1). It suffices to consider the $j$th term of the sum. By (2.3) we write

$$\Lambda_{n}^{j} \int_{0}^{t} \frac{d}{ds} \left[ \Lambda_{s}^{n}(\Lambda_{s}^{n})^{-1} \sigma_{3}^{j}(s) |s - \eta(s)|^{2H} \right] ds = \int_{0}^{t} \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \left[ \sigma_{3}^{j}(s) |s - \eta(s)|^{2H} \delta B_{s}^{j} + \int_{0}^{t} D_{s}^{j} \left[ \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{3}^{j}(s) \right] |s - \eta(s)|^{2H} (r-s)^{2H-2} ds dr \right].$$

Taking into account that $\sup_{r,s,t} \left| D_{s}^{j} \left[ \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{3}^{j}(s) \right] \right|$ has bounded moments of all orders by Remark 4.2, we obtain

$$\mathbb{E} \left\{ \left| \int_{0}^{t} D_{s}^{j} \left[ \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{3}^{j}(s) \right] |s - \eta(s)|^{2H} (r-s)^{2H-2} ds dr \right|^{2} \right\} \leq C n^{-4H}.$$ 

By (2.4), we also have

$$\mathbb{E} \left\{ \left| \int_{0}^{t} \Lambda_{n}^{j}(\Lambda_{s}^{n})^{-1} \sigma_{3}^{j}(s) |s - \eta(s)|^{2H} \delta B_{s}^{j} \right|^{2} \right\} \leq C n^{-4H}.$$ 

This completes the proof of the theorem. ■

5  Weak approximation

The next result provides the rate of the convergence of the weak approximation associated with the scheme (1.5).

**Theorem 5.1** Let $X$ and $X^{n}$ be the solution to the equations (1.1) and (1.3), respectively. Suppose that $\sigma \in C_{b}^{4}(\mathbb{R}^{d}; \mathbb{R}^{d \times m})$ and consider a function $f \in C_{b}^{3}(\mathbb{R}^{d})$. Then

$$n \left\{ \mathbb{E} \left[ f(X_{t}) \right] - \mathbb{E} \left[ f(X_{t}^{n}) \right] \right\}$$

$$- \frac{\alpha_{n}^{2} T}{2} \sum_{j=1}^{m} \sum_{l=1}^{m} \int_{0}^{t} \int_{0}^{t} \mathbb{E} \left( D_{u}^{l} D_{v}^{j} \left[ \nabla f(X_{t}) \Lambda_{t} \Lambda_{s}^{-1} (\nabla \sigma^{j} \cdot \sigma^{l})(X_{s}) \right] \right)$$

$$\times |u - s|^{2H-2} |s - r|^{2H-2} du dv,$$ 

(5.1)
as \( n \) tends to infinity, where \( \Lambda \) is the solution of the equation
\[
\Lambda_t = I + \sum_{j=1}^{m} \int_{0}^{t} \nabla \sigma^j(X_s) \Lambda_s dB_s^j. \tag{5.2}
\]

In particular, there exists a constant \( C \) independent of \( n \) such that
\[
\sup_{0 \leq t \leq T} \left| \mathbb{E}[f(X_t)] - \mathbb{E}[f(X^n_t)] \right| \leq \frac{C}{n}. \tag{5.3}
\]

**Proof:** Recall that \( Y_t = X_t - X^n_t, \ t \in [0, T] \). Given a function \( f \in C^3_b(\mathbb{R}^d) \), we can write
\[
\mathbb{E}[f(X_t)] - \mathbb{E}[f(X^n_t)] = \int_{0}^{1} \mathbb{E} \left[ \nabla f \left( \theta X_t + (1 - \theta)X^n_t \right) Y_t \right] d\theta = \int_{0}^{1} \mathbb{E} \left[ \nabla f(Z_t^\theta) Y_t \right] d\theta,
\]
where we denote \( Z_t^\theta = \theta X_t + (1 - \theta)X^n_t, \ 0 \leq t \leq T \). By (4.1), we have
\[
\mathbb{E} \left[ \nabla f(Z_t^\theta) Y_t \right] = \mathbb{E} \left[ \nabla f(Z_t^\theta) \Lambda^n_t \left\{ \sum_{j=1}^{m} \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j(s)(B_s - B_{\eta(s)}) dB_s^j \right. 
+ \sum_{j=1}^{m} \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j(s)(s - \eta(s))^{2H} dB_s^j 
- H \sum_{j=1}^{m} \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j(s)(s - \eta(s))^{2H-1} ds \right\} 
\right.
+ \sum_{j=1}^{m} \left( I_1^j + I_2^j - I_3^j \right).
\]

**Step 1.** We consider first the term \( I_1^j \) in (5.4). Applying (2.3) yields
\[
I_1^j = \mathbb{E} \left[ \nabla f(Z_t^\theta) \Lambda^n_t \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j(s)(B_s - B_{\eta(s)}) dB_s^j \right ]
= \alpha_H \mathbb{E} \int_{0}^{t} \int_{0}^{r} \mathbb{E} \left\{ D_r^j \left[ \nabla f(Z_t^\theta) \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j(s)(B_s - B_{\eta(s)}) \right] \right| r - s|^{2H-2} dr ds \]
\[
= \alpha_H \int_{0}^{t} \int_{0}^{r} \mathbb{E} \left\{ D_r^j \left[ \nabla f(Z_t^\theta) \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j(s)(B_s - B_{\eta(s)}) \right] \right| r - s|^{2H-2} dr ds \]
\[
+ \alpha_H \int_{0}^{t} \int_{0}^{r} \mathbb{E} \left\{ \nabla f(Z_t^\theta) \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j(s) D_r^j(B_s - B_{\eta(s)}) \right| r - s|^{2H-2} dr ds \]
\[
= \alpha_H \sum_{l=1}^{m} \int_{0}^{t} \int_{0}^{r} \int_{0}^{s} \mathbb{E} \left\{ D_u^l D_r^j \left[ \nabla f(Z_t^\theta) \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j(s) \right] \right| r - s|^{2H-2} du ds \]
\[
+ \alpha_H \int_{0}^{t} \mathbb{E} \left\{ \nabla f(Z_t^\theta) \Lambda^n_t (\Lambda^n_s)^{-1} \sigma^j(s) \right| s - \eta(s)|^{2H-1} ds
= J_1^j + J_2^j.
\]

Taking into account Proposition 3.6 (see also Remark 4.2), Proposition 3.5 and the estimates (3.1) and (3.9) on the uniform norm of \( X^n \) and \( X \), we can easily deduce that \( |J_1^j| \leq Cn^{-1} \). The term \( J_2^j \) will be treated in Step 3.
Step 2. Consider the term \( I^j_2 \) in (5.4). Applying again (2.3) yields
\[
|I^j_2| = \left| \mathbb{E} \left[ \nabla f(Z_t^n)\Lambda_t^n \int_0^t (\Lambda_s^n)^{-1}\sigma^j_3(s)(s - \eta(s))^{2H} dB_s^j \right] \right|
\]
\[
= \alpha_H \left| \mathbb{E} \left[ \int_0^t \int_0^t D_t^j D^j_0 \left[ \nabla f(Z_t^n)\Lambda_t^n (\Lambda_s^n)^{-1}\sigma^j_3(s)(s - \eta(s))^{2H} \right] [r - s]^{2H-2} ds dr \right| 
\]
\[
\leq C n^{-2H}.
\]

Step 3. Finally, the difference between \( J^j_2 \) and the term \( I^j_3 \) in (5.4) is given by
\[
J^j_2 - I^j_3 = H \int_0^t \mathbb{E} \left[ \nabla f(Z_t^n)\Lambda_t^n (\Lambda_s^n)^{-1} ([\sigma^j_2(s)] - [\sigma^j_4(s)]) (s - \eta(s))^{2H-1} ds \right]
\]
\[
= H \int_0^t \mathbb{E} \left[ \nabla f(Z_t^n)\Lambda_t^n (\Lambda_s^n)^{-1} \int_0^t \left[ \nabla \sigma^j \left( \theta X^n_s + (1 - \theta) X^n_{\eta(s)} \right) - \nabla \sigma^j \left( X^n_s \right) \right] d\theta \right]
\]
\[
\times \sigma^j \left( X^n_{\eta(s)}(s - \eta(s))^{2H-1} ds. 
\]

We can easily verify that \( |J^j_2 - I^j_3| \leq C n^{-2H} \). In summary, we have proved the estimate (5.3).

Step 4. From the above estimates, to prove (5.1) we only need to show that for any \( j = 1, \ldots, m \),
\[
n J^j_1 \rightarrow \frac{\alpha_H^2 T}{2} \sum_{l=1}^m \int_0^t \int_0^t \int_0^t \mathbb{E} \left[ \int_0^t D_t^l D^l_0 \left[ \nabla f(X_t)\Lambda_t \Lambda_s^{-1}(\nabla \sigma^l \sigma^l)(X_s) \right] \right] ds dr,
\]
as \( n \) tends to infinity. We show this in two steps. Denote
\[
\Phi_{u,s} = \sum_{l=1}^m \int_0^t \mathbb{E} \left[ D_t^l D^l_0 \left[ \nabla f(X_t)\Lambda_t \Lambda_s^{-1}(\nabla \sigma^l \sigma^l)(X_s) \right] \right] |s - r|^{2H-2} dr.
\]
Notice that \( \Phi_{u,s} \) is uniformly bounded by Proposition 3.6 (see also Remark 4.2), Proposition 3.5 and the estimate (3.9). Then, applying Lemma 7.3 the following convergence holds
\[
\lim_{n \to \infty} n \alpha_H^2 T \int_0^t \int_0^t \int_0^t \mathbb{E} \left[ 1_{[\eta(s),s]}(v) \right] |u - v|^{2H-2} dudsdr
\]
\[
= \frac{\alpha_H^2 T}{2} \int_0^t \int_0^t \mathbb{E} \left[ \Phi_{u,s} |u - s|^{2H-2} ds dr. 
\]

The second step is to show that
\[
\lim_{n \to \infty} n \left[ J^j_1 - \alpha_H^2 \int_0^t \int_0^t \int_0^t \mathbb{E} \left[ 1_{[\eta(s),s]}(v) \right] |u - v|^{2H-2} dudsdr \right] = 0. \tag{5.5}
\]

We can write
\[
J^j_1 - \alpha_H^2 \int_0^t \int_0^t \int_0^t \mathbb{E} \left[ 1_{[\eta(s),s]}(v) \right] |u - v|^{2H-2} dudsdr
\]
\[
= \alpha_H^2 \sum_{l=1}^m \int_0^t \int_0^t \int_0^t \mathbb{E} \left[ D_t^l D^l_0 \left[ \nabla f(Z_t^n)\Lambda_t^n (\Lambda_s^n)^{-1}[\sigma^j_2(s)]_l \right.ight.
\]
\[
\left. - \nabla f(X_t)\Lambda_t \Lambda_s^{-1}(\nabla \sigma^l \sigma^l)(X_s) \right] \right] 1_{[\eta(s),s]}(v) |u - v|^{2H-2} dudsdr.
\]
Then, \((5.5)\) follows by the dominated convergence theorem, taking into account that
\[
\lim_{n \to \infty} \mathbb{E}\left\{ D_u^i D_r^j \left[ \nabla f(Z_t^\theta) \Lambda_t^n(\Lambda_s^n)^{-1}[\sigma^j_2(s)]_t - \nabla f(X_t) \Lambda_t \Lambda_t^{-1}(\nabla \sigma^j \sigma^j)(X_s) \right] \right\} = 0
\]
The proof of Theorem 5.1 is then completed. ■

6 Rate of convergence for the Euler scheme

In this section the approach based on Malliavin calculus to study the numerical schemes developed in Section 4 shall be applied to study the rate of convergence of the classical Euler scheme defined in (1.2).

Our first result in this section is the strong convergence of the classical Euler scheme. The proof is significantly shorter comparing to that of the modified Euler scheme which provides a finer approximation of the solution of the stochastic differential equation, where weighted quadratic variation terms are involved. As we will see, the rate of weak convergence and the rate of strong convergence are the same for the Euler scheme.

**Theorem 6.1** Let \(X\) and \(X^n\) be the processes defined in (1.1) and (1.2), respectively. Suppose that \(\sigma \in C^3_b(\mathbb{R}^d; \mathbb{R}^{d \times m})\). Then as \(n\) tends to infinity,
\[
n^{2H-1}(X_t - X^n_t) \rightarrow \frac{T^{2H-1}}{2} \sum_{j=1}^m \int_0^T \Lambda_t \Lambda_t^{-1}(\nabla \sigma^j \sigma^j)(X_s) ds,
\]
where \(\Lambda\) is the solution to the linear equation (5.2), and the convergence holds both almost surely and in \(L^p\) for all \(p \geq 1\).

**Proof:** Let \(Y_t = X_t - X^n_t, t \in [0, T]\). By the definition of \(X_t\) and \(X^n_t\)
\[
Y_t = \int_0^t [\sigma(X_s) - \sigma(X^n_{\eta(s)})] dB_s
\]
\[
= \int_0^t [\sigma(X_s) - \sigma(X^n_s) + \sigma(X^n_s) - \sigma(X^n_{\eta(s)})] dB_s.
\]
Since
\[
\sigma^i(X_s) - \sigma^i(X^n_s) = \int_0^1 \nabla \sigma^i(\theta X_s + (1 - \theta)X^n_s) Y_s d\theta,
\]
and
\[
\sigma^i(X^n_s) - \sigma^i(X^n_{\eta(s)}) = \int_0^1 \nabla \sigma^i(\theta X^n_s + (1 - \theta)X^n_{\eta(s)}) \sigma(X^n_{\eta(s)})(B_s - B_{\eta(s)}) d\theta,
\]
we have
\[
Y_t = \sum_{j=1}^n \int_0^t \sigma^i_1(s) Y_s dB_s^j + \sum_{j=1}^m \int_0^t \sigma^i_2(s)(B_s - B_{\eta(s)}) dB_s^j,
\]
where
\[
\sigma^i_1(s) = \int_0^1 \nabla \sigma^i(\theta X_s + (1 - \theta)X^n_s) d\theta,
\]
\[
\sigma^i_2(s) = \int_0^1 \nabla \sigma^i(\theta X^n_s + (1 - \theta)X^n_{\eta(s)}) \sigma(X^n_{\eta(s)}) d\theta.
\]
The process \( Y_t \) can be expressed explicitly as follows

\[
Y_t = \Lambda_t^n \sum_{j=1}^{m} \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j_2(s)(B_s - B_{\eta(s)})dB^j_s, \tag{6.1}
\]

where \( \Lambda^n \) is the solution of the equation

\[
\Lambda_t^n = I + \sum_{j=1}^{m} \int_{0}^{t} \sigma^j_2(s) \Lambda^n_s dB^j_s.
\]

Furthermore, we have

\[
\Lambda_t^n \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j_2(s)(B_s - B_{\eta(s)})dB^j_s = \Lambda_t^n \int_{0}^{t} (\Lambda^n_s)^{-1} \sigma^j_2(s)(B_s - B_{\eta(s)})d\delta B^j_s
\]
\[
+ \alpha_H \int_{0}^{t} \int_{0}^{t} D^j_r \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s)(B_s - B_{\eta(s)}) \right] |r - s|^{2H-2} dsdr
\]
\[
= \int_{0}^{t} \Lambda_t^n (\Lambda^n_s)^{-1} \sigma^j_2(s)(B_s - B_{\eta(s)})d\delta B^j_s
\]
\[
+ \alpha_H \int_{0}^{t} \int_{0}^{t} D^j_r \left[ (\Lambda^n_s)^{-1} \sigma^j_2(s) \right] (B_s - B_{\eta(s)}) |r - s|^{2H-2} dsdr
\]
\[
+ \alpha_H \int_{0}^{t} \int_{0}^{t} \Lambda_t^n (\Lambda^n_s)^{-1} \left[ \sigma^j_2(s) \right] 1_{[r,s]}(r) |r - s|^{2H-2} dsdr
\]
\[
= A_t^1 + A_t^2 + A_t^3.
\]

Recall that \( [\sigma^j_2(s)] \) denotes the \( j \)-th column of the matrix \( \sigma^j_2(s) \).

As in the proof of Theorem 5.1 and taking into account Remark 3.7, we can show that the \( L^p \) norms of the first term \( A_t^1 \) and second term \( A_t^2 \) are bounded by a constant times \( n^{-H} \). Therefore,

\[
n^{2H-1} \left( \mathbb{E}(|A_t^1 + A_t^2|^p) \right)^{\frac{1}{p}} \leq C_p n^{H-1}. \tag{6.2}
\]

This implies that \( n^{2H-1}(A_t^1 + A_t^2) \) converges to zero almost surely and in \( L^p \) for all \( p \geq 1 \).

Then, to prove the theorem we only need to show that for \( j = 1, 2, \ldots, m \),

\[
n^{2H-1} \alpha_H \int_{0}^{t} \int_{0}^{t} \Lambda_t^n (\Lambda^n_s)^{-1} \left[ \sigma^j_2(s) \right] 1_{[r,s]}(r) |r - s|^{2H-2} dsdr
\]
\[
\to \frac{T^{2H-1}}{2} \int_{0}^{t} \Lambda_t \Lambda^{-1}_t \nabla \sigma^j \sigma^j(X_s) ds
\]

almost surely and in \( L^p \) for all \( p \geq 1 \).

As in the proof of Theorem 5.1 we can show this in two steps. It is clear that

\[
n^{2H-1} \int_{0}^{t} \int_{0}^{t} \Lambda_t \Lambda^{-1}_s \nabla \sigma^j \sigma^j(X_s) 1_{[r,s]}(r) |r - s|^{2H-2} dsdr
\]
\[
= n^{2H-1} \int_{0}^{t} \Lambda_t \Lambda^{-1}_s \nabla \sigma^j \sigma^j(X_s) \frac{(s - \eta(s))^{2H-1}}{2H - 1} ds
\]
\[
\to \frac{T^{2H-1}}{2 \alpha_H} \int_{0}^{t} \Lambda_t \Lambda^{-1}_s \nabla \sigma^j \sigma^j(X_s) ds,
\]

almost surely and in \( L^p \) for all \( p \geq 1 \).
Set
\[
\Phi_n = \int_0^t \int_0^t \Lambda_t^n (\Lambda_s^n)^{-1} [\sigma_2^j(s)]_1 [\eta(s),s]_1 (r) |r - s|^{2H - 2} ds dr \\
- \int_0^t \int_0^t \Lambda_t A_s^{-1} \nabla \sigma^j (X_s) [1 [\eta(s),s]_1 (r) |r - s|^{2H - 2} ds dr.
\]

The next step is to show that \( n^{2H - 1} \Phi_n \) converges to zero as \( n \) tends to infinity almost surely and in all \( L^p \). We make the decomposition \( \Phi_n = \Phi_n^1 + \Phi_n^2 \), where

\[
\Phi_n^1 = \int_0^t \int_0^t \Lambda_t^n (\Lambda_s^n)^{-1} [\sigma_2^j(s)]_1 [\eta(s),s]_1 (r) |r - s|^{2H - 2} ds dr \\
- \int_0^t \int_0^t \Lambda_t A_s^{-1} \nabla \sigma^j (X_s^n) [1 [\eta(s),s]_1 (r) |r - s|^{2H - 2} ds dr,
\]

and

\[
\Phi_n^2 = \int_0^t \int_0^t \Lambda_t A_s^{-1} [\nabla \sigma^j (X_s^n) - \nabla \sigma^j (X_s)] [1 [\eta(s),s]_1 (r) |r - s|^{2H - 2} ds dr.
\]

With the help of (6.1), we can write for \( r < t, i = 1, \ldots, m, \)

\[
D_t^i X_t^n - D_t^i X_t = D_t^i X_t^n - D_t^i X_t + \sum_{j=1}^m \int_0^t (\Lambda_s^n)^{-1} \sigma_2^j (s) (B_s - B_{\eta(s)}) dB_s^j + \Lambda_t^n (\Lambda_s^n)^{-1} \sigma_2^j (r) (B_r - B_{\eta(r)}) \\
+ \Lambda_t^n \sum_{j=1}^m \int_0^t D_t^i ([\Lambda_s^n])^{-1} \sigma_2^j (s) (B_s - B_{\eta(s)}) dB_s^j \\
+ \Lambda_t^n \sum_{j=1}^m \int_0^t (\Lambda_s^n)^{-1} [\sigma_2^j (s)]_1 [\eta(s),s]_1 (r) dB_s^j.
\]

Applying (2.3) and (2.4) to the right-hand side of the above expression and taking into account Remark 4.2 adapted to the classical Euler scheme (see Remark 3.7), we can show that for any \( p > 1, \)

\[
n^{2H - 1} \sup_{t \in [b,T]} \mathbb{E}(|D_t X_t^n - D_t X_t|^p)^{1/p} \leq C. \tag{6.3}
\]

By the chain rule for the Young’s integral, we can verify the expression

\[
\Lambda_t^n - \Lambda_t = \Lambda_t \sum_{j=1}^m \int_0^t \Lambda_s^{-1} [\sigma_1^j (s) - \nabla \sigma^j (X_s)] \Lambda_s^n dB_s^j.
\]

Applying again (2.3), (2.4), (6.3), Proposition 3.6 (see also Remark 4.2) and the estimate (3.2) for the Hölder norm of \( X^n \), adapted to the classical Euler scheme (see Remark 3.7), we can show that

\[
n^{2H - 1} \sup_{t \in [b,T]} \mathbb{E}(|\Lambda_t^n - \Lambda_t|^p)^{1/p} \leq C. \tag{6.4}
\]

Now it follows from (6.4) that for all \( p \geq 1, \)

\[
(\mathbb{E}(|\Phi_n^1|^p))^{\frac{1}{p}} \leq C_p n^{2 - 4H}. \tag{6.5}
\]
On the other hand, we can also show that \((\mathbb{E}(|A_n^3|^p))^{\frac{1}{p}} \leq C_p n^{1-2H}\), which together with (6.2) implies that the \(L^p\) norm of \(Y_t\) is bounded uniformly in \(t\) by a constant times \(n^{1-2H}\). This leads to the inequality

\[
(\mathbb{E}(|\Phi_n^2|^p))^{\frac{1}{p}} \leq C_p n^{2-4H}.
\] (6.6)

Then (6.5) and (6.6) imply that \(\Phi_n\) converges to zero as \(n\) tends to infinity almost surely and in \(L^p\) for all \(p \geq 1\). The proof of the theorem is now complete. ■

As a consequence of the above theorem, we can deduce the following result.

**Corollary 6.2** Let \(X\) and \(X_n\) be the processes defined in (1.1) and (1.2), respectively. Suppose that \(\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^{d \times m})\) and \(f \in C_b^2(\mathbb{R}^d)\). Let \(\Lambda\) be the process defined in (5.2). Then as \(n\) tends to infinity,

\[
n^{2H-1} [f(X_n^t) - f(X_t)] \xrightarrow{\text{a.s., in } L^p} \frac{T^{2H-1}}{2} \sum_{j=1}^m \int_0^t \nabla f(X_t) \Lambda_j \sigma_j \sigma_j \sigma_j (X_s) ds,
\]

almost surely and in \(L^p\) for all \(p \geq 1\).

**Proof:** First, we can write

\[
n^{2H-1} [f(X_n^t) - f(X_t)] = n^{2H-1} \left( \int_0^1 \nabla f(Z_t^\theta) d\theta \right) (X_n^t - X_t),
\]

where Equation (6.1) has been applied, and we denote \(Z_t^\theta = \theta X_t + (1-\theta)X_n^t\), \(t \in [0, T]\). The result follows from Theorem 6.1, the convergence of \(X_n^t\) to \(X_t\) and the assumption on \(f\). ■

The above corollary implies the following weak approximation result

\[
\lim_{n \to \infty} n^{2H-1} [\mathbb{E}[f(X_t)] - \mathbb{E}[f(X_n^t)] = \frac{T^{2H-1}}{2} \sum_{j=1}^m \int_0^t \mathbb{E}[\nabla f(X_t) \Lambda_j \sigma_j \sigma_j \sigma_j (X_s)] ds.
\]

# 7 Appendix

## 7.1 Estimates of a stochastic integral driven by fBm

In this section, we give an estimate on a stochastic integral driven by a fractional Brownian motion with Hurst parameter \(H > \frac{1}{2}\) using fractional calculus.

**Lemma 7.1** Let \(B = \{B_t, t \in [0, T]\} \) be a one-dimensional fractional Brownian motion with Hurst parameter \(H \in (\frac{1}{2}, 1)\). Suppose that \(f : \mathbb{R}^{l+m} \to \mathbb{R}\) is continuously differentiable. Let \(\alpha\) and \(\beta\) be constants such that \(\frac{1}{2} < \beta < H\) and \(\alpha : \frac{1}{2} > \alpha > 1 - \beta\). We denote by \(\nabla_x f\) the \(l\)-dimensional vector with coordinates \(\frac{\partial f}{\partial x_i}\), \(i = 1, \ldots, l\), and by \(\nabla_y f\) the \(m\)-dimensional vector with coordinates \(\frac{\partial f}{\partial x_{l+i}}\), \(i = 1, \ldots, m\). Consider processes \(x = \{x_t, t \in [0, T]\}\) and \(y = \{y_t, t \in [0, T]\}\) with dimensions \(l\) and \(m\), respectively, such that \(\|x\|_{0,T,\beta}\) and \(\|y\|_{0,T,\beta,n}\) are finite for each \(n \geq 1\). Then, we have the following estimates:

(i) For any \(s, t \in [0, T]\) such that \(s \leq t\) and \(s = \eta(s)\) we have

\[
\left| \int_s^t f(x_r, y_{\eta(r)}) dB_r \right| \leq k_1 \sup_{r \in [0, T]} |f(x_r, y_{\eta(r)})| \|B\|_\beta (t-s)^\beta + k_2 \sup_{r \in [0, T]} |\nabla_x f(x_r, y_{\eta(r)})| \|x\|_{s,t,\beta} \|B\|_\beta (t-s)^{2\beta} + k_3 \sup_{r \in [0, T]} |\nabla_y f(x_r, y_{\eta(r)})| \|y\|_{s,t,\beta,n} \|B\|_\beta (t-s)^{2\beta},
\]

where

- \(k_1, k_2, k_3\) are constants depending on \(\alpha, \beta, \gamma\), and \(\eta\).
where the \( k_i, i = 1, 2, 3 \), are constants depending on \( \alpha \) and \( \beta \).

(ii) If the function \( f \) only depends on the first \( l \) variables, then the above estimate holds for all \( 0 \leq s \leq t \leq T \).

Proof: Fix \( s, t \in [0, T] \) such that \( s = \eta(s) \) and \( s \leq t \). We use the fractional integration by parts formula established in Proposition \([2.1]\) to obtain

\[
\left| \int_s^t f(x_r, y_{\eta(r)}) dB_r \right| \leq \int_s^t \left| D_{s+}^\alpha f(x_r, y_{\eta(r)}) \right| |D_{1-}^{1-\alpha}(B_r - B_t)| \, dr. \tag{7.1}
\]

By the definition of fractional differentiation in \([2.2]\), we can write

\[
\left| D_{1-}^{1-\alpha}(B_r - B_t) \right| \leq k_0 \|B\| (t - r)^{\alpha + \beta - 1}, \quad s \leq r \leq t,
\]

where \( k_0 = \frac{1}{\Gamma(\alpha)} \left( 1 + \frac{\alpha}{\beta + \alpha - 1} \right) \). On the other hand, using \([2.1]\) we obtain

\[
\left| D_{s+}^\alpha f(x_r, y_{\eta(r)}) \right| \leq \frac{1}{\Gamma(1 - \alpha)} \left[ \sup_{r \in [0, t]} |f(x_r, y_{\eta(r)})|(r - s)^{-\alpha} + \alpha \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \int_s^t (r - u)^{\beta - \alpha - 1} \, du \right]
\]

\[
\leq \frac{1}{\Gamma(1 - \alpha)} \left[ \sup_{r \in [0, t]} |f(x_r, y_{\eta(r)})|(r - s)^{-\alpha} + \alpha \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \int_s^r (r - u)^{\beta - \alpha - 1} \, du \right]
\]

\[
+ \alpha \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \int_s^r \left| \eta(r) - \eta(u) \right| \beta \, du \right]. \tag{7.3}
\]

The inequalities (7.1), (7.3) and (7.2) together imply

\[
\left| \int_s^t f(x_r, y_{\eta(r)}) dB_r \right|
\]

\[
\leq \frac{1}{\Gamma(1 - \alpha)} \int_s^t \left[ \sup_{r \in [0, t]} |f(x_r, y_{\eta(r)})|(r - s)^{-\alpha} + \alpha \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \int_s^r (r - u)^{\beta - \alpha - 1} \, du \right]
\]

\[
+ \alpha \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \int_s^r \left| \eta(r) - \eta(u) \right| \beta \, du \right]
\]

\[
\leq k_1 \sup_{r \in [0, T]} \|f(x_r, y_{\eta(r)})\| \|B\| (t - s)^{\beta} + k_2 \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \|B\| (t - s)^{2\beta}
\]

\[
+ k_3 \sup_{r \in [0, T]} \|x\|_{s, t, \beta} \|B\| (t - s)^{2\beta},
\]

where \( k_1 = k_0 \frac{\Gamma(\alpha + \beta)}{\Gamma(\beta + 1)} \) and \( k_2 = k_0 \frac{\alpha \Gamma(\alpha + \beta) \Gamma(\beta + 1)}{\Gamma(2\beta + 1)(\beta - \alpha)} \) and \( k_3 = k_0 k_4 \frac{\alpha}{\Gamma(1 - \alpha)} \), \( k_4 \) being the constant in Lemma \([7.2]\). This completes the proof. \( \blacksquare \)

**Lemma 7.2** Let \( \beta \) and \( \alpha \) satisfy \( \frac{1}{2} < \beta < H \) and \( \frac{1}{2} > \alpha > 1 - \beta \). Choose \( s, t \in [0, T] \) such that \( s < t \), \( s = \eta(s) \). Then there exists a constant \( k_4 \) depending on \( \alpha \), \( \beta \) and \( T \), such that

\[
\int_s^t (t - r)^{\alpha + \beta - 1} \int_s^r \frac{|\eta(r) - \eta(u)|^\beta}{(r - u)^{\alpha + 1}} \, du \, dr \leq k_4 (t - s)^{2\beta}.
\]

**Proof:** Without loss of generality, we let \( T = 1 \). Note that when \( \eta(s) = s < t \leq \eta(s) + \frac{1}{n} \), the double integral equals zero. In the following we will assume \( t > \eta(s) + \frac{1}{n} \). We can write

\[
\int_s^t (t - r)^{\alpha + \beta - 1} \int_s^r \frac{|\eta(r) - \eta(u)|^\beta}{(r - u)^{\alpha + 1}} \, du \, dr
\]

\[
= \int_{\eta(s) + \frac{1}{n}}^t (t - r)^{\alpha + \beta - 1} \int_s^{\eta(r)} \frac{|\eta(r) - \eta(u)|^\beta}{(r - u)^{\alpha + 1}} \, du \, dr.
\]

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In the case when $\eta(s) + \frac{1}{n} \leq t \leq \eta(s) + \frac{2}{n}$, noticing that in the above integral the inequalities $t - r \leq t - s$ and $\eta(r) - \eta(u) = \frac{1}{n}$ are always correct, we have

\[
\int_{\eta(s)+\frac{1}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \\
\leq \int_{\eta(s)+\frac{1}{n}}^{t} (t - s)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{(\frac{1}{n})^{\beta}}{(r - u)^{\alpha + 1}} du dr \\
= n^{-\beta} (t - s)^{\alpha + \beta - 1} \int_{\eta(s)+\frac{2}{n}}^{t} \int_{\eta(s)+\frac{1}{n}}^{\eta(r)} (r - u)^{-\alpha - 1} du dr \\
\leq C n^{-\beta} (t - s)^{\alpha + \beta - 1} n^{\alpha - 1} \leq C (t - s)^{2\beta},
\]

since in this case $t - s \leq \frac{2}{n}$ or $n^{-\beta + \alpha - 1} \leq (t - s)^{\beta - \alpha + 1}$.

Now we consider the case when $t > \eta(s) + \frac{2}{n}$. We can write

\[
\int_{\eta(s)+\frac{1}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \\
= \left(\int_{\eta(s)+\frac{2}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \right) + \int_{\eta(s)+\frac{1}{n}}^{\eta(s)+\frac{2}{n}} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr = I_1 + I_2.
\]

Following the same lines as in (7.4), we have $I_2 \leq (t - s)^{2\beta}$. Finally, we write

\[
I_1 = \int_{\eta(s)+\frac{1}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \\
= \int_{\eta(s)+\frac{2}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{\eta(s)+\frac{1}{n}}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \\
= I_{11} + I_{12}.
\]

Notice that in the term $I_{12}$, we always have $r - u > \frac{1}{n}$. Thus $\eta(r) - \eta(u) \leq r - u + \frac{1}{n} \leq 2(r - u)$. Therefore,

\[
I_{12} \leq \int_{\eta(s)+\frac{2}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{s}^{\eta(r)-\frac{1}{n}} \frac{2^\beta (r - u)^\beta}{(r - u)^{\alpha + 1}} du dr \leq k(t - s)^{2\beta}.
\]

On the other hand, we have

\[
I_{11} = \int_{\eta(s)+\frac{2}{n}}^{t} (t - r)^{\alpha + \beta - 1} \int_{\eta(r)-\frac{1}{n}}^{\eta(r)} \frac{\eta(r) - \eta(u)}{(r - u)^{\alpha + 1}} du dr \\
\leq k n^{-\beta} (t - s)^{\alpha + \beta - 1} \int_{\eta(s)+\frac{2}{n}}^{t} \left[ \frac{1}{(r - \eta(r))^\alpha} - \frac{1}{(r - \eta(r) + \frac{1}{n})^\alpha} \right] dr \\
\leq k n^{-\beta} (t - s)^{\alpha + \beta - 1} \int_{\eta(s)+\frac{2}{n}}^{t} \frac{1}{(r - \eta(r))^\alpha} dr \\
\leq k n^{-\beta} (t - s)^{\alpha + \beta - 1} (\eta(t) + \frac{1}{n}) - (\eta(s) + \frac{2}{n}) n^{\alpha - 1} \\
\leq k(t - s)^{2\beta}.
\]

The lemma is now proved. □
7.2 A technical lemma

Lemma 7.3 Suppose that $\Phi : [0,T]^2 \to \mathbb{R}$ is a bounded measurable function. Then, for any $t \in [0,T]$,

$$
\lim_{n \to \infty} n \int_0^t \int_0^t \Phi_{u,s} 1_{[\eta(s),s]}(v) |u - v|^{2H-2} dudv = \frac{T}{2} \int_0^t \int_0^t \Phi_{u,s} |u - s|^{2H-2} duds.
$$

Proof: The results is clearly true if $\Phi$ is continuous. On the other hand, we can approximate any bounded function on $[0,T]^2$ by a continuous function $\Phi^\epsilon$ in $L^p([0,T]^2)$, where $p > \frac{1}{2H-1}$. Then, the Hölder inequality yields the estimate

$$
n \int_0^t \int_0^t |\Phi_{u,s} - \Phi_{u,s}^\epsilon| 1_{[\eta(s),s]}(v) |u - v|^{2H-2} dudv \leq C \|\Phi - \Phi^\epsilon\|_{L^p([0,T]^2)}
$$

where the constant $C$ does not depend on $n$, and this allows us to complete the proof. ■

The above result holds true also for a bounded function $\Phi$ taking values in some separable Banach space.

7.3 Estimates of the $L^2$-norm of weighted quadratic variations

For notational convenience, we let $T = 1$. Let $B = \{B_t, t \in [0,1]\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. For any $n \geq 1$ and $k = 1, \ldots, n$ we define $\delta_{k/n} = 1_{[(k-1)/n,k/n]}$ and $\Delta B_k = B_{k/n} - B_{(k-1)/n}$. We need an estimate of the $L^2$ norm of the quadratic variation defined as

$$
V_n^{(2)}(F) = \sum_{k=1}^n F_k^n \left[ (\Delta B_k)^2 - n^{-2H} \right], \quad (7.5)
$$

where $F_k^n, k = 1, \ldots, n, n \geq 1$ is a family of random variables.

Proposition 7.4 Let $B = \{B_t, t \in [0,1]\}$ be a one-dimensional fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. Consider random variables $F_k^n, k = 1, \ldots, n, n \geq 1$ in $\mathbb{D}^{4,p}$ for all $p \geq 1$, such that for each $k = 1, \ldots, n$ and $j = 1, 2, 3, 4$ there exist a version of the derivative $D^j F_k^n$ satisfying

$$
\sup_{k,n \in \mathbb{N}, k \leq n} \sup_{t_1, \ldots, t_j \in [0,1]} \mathbb{E} \left( |D^j_{t_1, \ldots, t_j} F_k^n| \right) < \infty. \quad (7.6)
$$

Then the weighted quadratic variation defined in (7.5) satisfies

$$
\mathbb{E}(V_n^{(2)}(F))^2 = \begin{cases} 
O(n^{1-4H}) & H < 3/4, \\
O(n^{-2} \ln n) & H = 3/4, \\
O(n^{-2}) & H > 3/4.
\end{cases}
$$

Proof: Notice that $(\Delta B_k)^2 - n^{-2H} = I_2(\delta_{k/n}^\otimes)$, where $I_2$ denotes the double stochastic integral. In this way we obtain

$$
\mathbb{E}(V_n^{(2)}(F))^2 = \sum_{k,l=1}^n \mathbb{E} \left( F_k^n F_l^n I_2(\delta_{k/n}^\otimes) I_2(\delta_{l/n}^\otimes) \right).
$$
The product formula of multiple stochastic integrals (see (2.4)) yields
\[ I_2(\delta^\otimes_2)I_2(\delta^\otimes_2) = I_4(\delta^\otimes_2 \otimes \delta^\otimes_2) + 4I_2(\delta_k/n \otimes \delta_l/n) \langle \delta_k/n, \delta_l/n \rangle_H + 2\langle \delta_k/n, \delta_l/n \rangle_H^2. \]

Therefore, using the duality relationship between the iterated derivative and multiple stochastic integrals (see (2.6)), we can write
\[
\begin{align*}
\mathbb{E}(V_n^{(2)}(F)^2) &= \sum_{k,l=1}^{n} \mathbb{E} \left( F^n_k F^n_l I_4(\delta^\otimes_2 \otimes \delta^\otimes_2) \right) + 4 \sum_{k,l=1}^{n} \mathbb{E} \left( F^n_k F^n_l I_2(\delta_k/n \otimes \delta_l/n) \right) \langle \delta_k/n, \delta_l/n \rangle_H \\
& \quad + 2 \sum_{k,l=1}^{n} \mathbb{E} \left( F^n_k F^n_l \right) \langle \delta_k/n, \delta_l/n \rangle_H^2 \\
& = \sum_{k,l=1}^{n} \mathbb{E} \left( D^4(F^n_k F^n_l), \delta^\otimes_2 \otimes \delta^\otimes_2 \right) \langle \delta_k/n, \delta_l/n \rangle_H \\
& \quad + 4 \sum_{k,l=1}^{n} \mathbb{E} \left( D^2(F^n_k F^n_l), \delta_k/n \otimes \delta_l/n \right) \langle \delta_k/n, \delta_l/n \rangle_H \\
& \quad + 2 \sum_{k,l=1}^{n} \mathbb{E} \left( F^n_k F^n_l \right) \langle \delta_k/n, \delta_l/n \rangle_H^2 \\
& = I_1 + I_2 + I_3.
\end{align*}
\]

(7.7)

For the second term in the above expression, we have
\[
\int_0^1 \int_0^1 \int_0^1 \int_0^1 \left| \mathbb{E}(D_{t_1,t_2}(F^n_k F^n_l)) \right| \delta_{k/n}(s_1) \delta_{l/n}(s_2) \\
\times \left| s_1 - t_1 \right|^{2H-2} \left| s_2 - t_2 \right|^{2H-2} ds_1 dt_1 ds_2 dt_2 \leq Cn^{-2},
\]
where \( C \) is a constant independent of \( n \). Therefore,
\[
|I_2| \leq Cn^{-2} \sum_{k,l=1}^{n} \langle \delta_k/n, \delta_l/n \rangle_H = Cn^{-2}.
\]

(7.8)

Similarly, for the first term we have
\[
|I_1| \leq \sup_{t_1,t_2,t_3,t_4 \in [0,1]} \left| \mathbb{E}(D_{t_1,t_2,t_3,t_4}(F^n_k F^n_l)) \right| \left| \left< 1, \delta^\otimes_2 \otimes \delta^\otimes_2 \right>_{H\otimes 4} \right| \leq Cn^{-4},
\]
where \( C \) is a constant independent of \( n \). In addition, by the results in [11] (Lemma 5 and Lemma 6), we have
\[
\sum_{k,l=1}^{n} \left| \langle \delta_k/n, \delta_l/n \rangle_H \right|^2 = \left\{ \begin{array}{ll}
O(n^{1-4H}) & H < 3/4, \\
O(n^{-2} \ln n) & H = 3/4, \\
O(n^{-2}) & H > 3/4.
\end{array} \right.
\]

(7.10)

Now applying these estimates to the third term in (7.7) and using (7.8) and (7.9) we obtain the desired result.

The above result can be generalized to weighted Hermite variations of order \( q \geq 2 \) of the form
\[
\sum_{k=1}^{n} F^n_k H_q(n^H \Delta B_k),
\]
where \( H_q, q \geq 2, \) denotes the Hermite polynomial with degree \( q \).

In the particular case where \( F^n_k = f(B_k/n) \) for a suitable function \( f \), the asymptotic behavior of the weighted Hermite variations of order \( q \geq 2 \) has been analyzed in [11].
7.4 Estimates of the $L^2$-norm of the weighted covariation of two independent fractional Brownian motions

We also need an estimate of the weighted covariation of two independent fractional Brownian motions.

**Proposition 7.5** Let $B$ and $\tilde{B}$ be two independent one-dimensional fractional Brownian motions, with the same Hurst parameter $H > \frac{1}{2}$. Let $F^n_k, k = 1, \ldots, n, n \geq 1$ be random variables satisfying (7.6). Then, if $t_k = k/n$,

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} F^n_k (\tilde{B}_s - \tilde{B}_{t_{k-1}}) dB_s \right)^2 \right] = \begin{cases} 
O(n^{1-4H}) & H < 3/4, \\
O(n^{1-4H} \ln n) & H = 3/4, \\
O(n^{-2}) & H > 3/4.
\end{cases}
\]

**Proof:** Using the product formula for double stochastic integrals we can write

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} F^n_k (\tilde{B}_s - \tilde{B}_{t_{k-1}}) dB_s \right)^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{n} F^n_k F^n_{t_k} \int_{t_{k-1}}^{t_k} (\tilde{B}_s - \tilde{B}_{t_{k-1}}) dB_s \int_{t_{l-1}}^{t_l} (\tilde{B}_s - \tilde{B}_{t_{l-1}}) dB_s \right] = \mathbb{E} \left[ \sum_{k=1}^{n} F^n_k F^n_{t_k} I_2 \left( \left[ (\tilde{B}_s - \tilde{B}_{t_{k-1}}) \delta_{k/n} \right] \otimes \left[ (\tilde{B}_s - \tilde{B}_{t_{l-1}}) \delta_{l/n} \right] \right) + \mathbb{E} \left[ \sum_{k=1}^{n} F^n_k F^n_{t_k} \left( (\tilde{B}_s - \tilde{B}_{t_{k-1}}) \delta_{k/n}, (\tilde{B}_s - \tilde{B}_{t_{l-1}}) \delta_{l/n} \right) \right] \right].
\]

The duality formula for the double stochastic integral yields

\[
\mathbb{E} \left[ \left( \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} F^n_k (\tilde{B}_s - \tilde{B}_{t_{k-1}}) dB_s \right)^2 \right] = \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbb{E} \left[ \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} D_{t_1,t_2}(F^n_k F^n_{t_k}) \left[ (\tilde{B}_{s_1} - \tilde{B}_{t_{k-1}}) \delta_{k/n}(s_1) \right] \times \left[ (\tilde{B}_{s_2} - \tilde{B}_{t_{l-1}}) \delta_{l/n}(s_2) \right] |t_1 - s_1|^{2H-2} |t_2 - s_2|^{2H-2} dt_1 ds_1 dt_2 ds_2 \right] + \alpha \mathbb{E} \left[ \sum_{k=1}^{n} \sum_{l=1}^{n} \mathbb{E} \left[ F^n_k F^n_{t_k} \int_{0}^{1} \int_{0}^{1} (\tilde{B}_t - \tilde{B}_{t_{k-1}}) \delta_{k/n}(t) (\tilde{B}_s - \tilde{B}_{t_{l-1}}) \delta_{l/n}(s) |t-s|^{2H-2} ds \right] \right] = I_1 + I_2.
\]

Notice that

\[
\mathbb{E} \left[ D_{t_1,t_2}(F^n_k F^n_{t_k}) (\tilde{B}_{s_1} - \tilde{B}_{t_{k-1}}) (\tilde{B}_{s_2} - \tilde{B}_{t_{l-1}}) \right] \leq \mathbb{E} \left[ D_{t_1,t_2}(F^n_k F^n_{t_k}) I_2 \left( \mathbb{1}_{[t_{k-1},s_1]} \otimes \mathbb{1}_{[t_{l-1},s_2]} \right) \right] + \mathbb{E} \left[ D_{t_1,t_2}(F^n_k F^n_{t_k}) \right] \mathbb{E} \left[ \mathbb{1}_{[t_{k-1},s_1]} \otimes \mathbb{1}_{[t_{l-1},s_2]} \right] \mathbb{H}^{\otimes 2} + C \mathbb{E} \left( \mathbb{1}_{[t_{k-1},s_1]} \otimes \mathbb{1}_{[t_{l-1},s_2]} \right) \mathbb{H}^{\otimes 2} \leq \frac{C}{n^2} + C \mathbb{E} \left( \mathbb{1}_{[t_{k-1},s_1]} \otimes \mathbb{1}_{[t_{l-1},s_2]} \right) \mathbb{H}^{\otimes 2}.
\]

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Substituting (7.12) into the first term $I_1$ in (7.11), we get

$$|I_1| \leq \frac{C}{n^2} \sum_{k,l=1}^{n} \left[ \int_0^1 \int_0^1 \int_0^1 \delta_{k/n}(s_1)\delta_{l/n}(s_2)|t_1 - s_1|^{2H-2}|t_2 - s_2|^{2H-2}dt_1ds_1dt_2ds_2 \right]$$

$$+ C \sum_{k,l=1}^{n} \int_{t_{l-1}}^{t_l} \int_{t_{k-1}}^{t_k} \int_0^1 \left( 1_{[t_{k-1},s_1]}, 1_{[t_{l-1},s_2]} \right)_{H}|t_1 - s_1|^{2H-2}|t_2 - s_2|^{2H-2}dt_1ds_1dt_2ds_2$$

$$\leq Cn^{-2}.$$

Similarly,

$$\left| \mathbb{E} \left[ F^n_k F^n_l \left( \tilde{B}_t - \tilde{B}_{t_{k-1}} \right) \left( \tilde{B}_s - \tilde{B}_{t_{l-1}} \right) \right] \right|$$

$$\leq \mathbb{E} \left[ F^n_k F^n_l I_2 \left( 1_{[t_{k-1},t]} \otimes 1_{[t_{l-1},s]} \right) \right] + \mathbb{E} \left[ F^n_k F^n_l \langle 1_{[t_{k-1},t]}, 1_{[t_{l-1},s]} \rangle_H \right]$$

$$\leq \mathbb{E} \left[ D^2 \left[ F^n_k F^n_l \right], 1_{[t_{k-1},t]} \otimes 1_{[t_{l-1},s]} \right]_{H^{\otimes 2}} + C \left\langle 1_{[t_{k-1},t]}, 1_{[t_{l-1},s]} \right\rangle_H$$

$$\leq Cn^{-2} + C \left\langle 1_{[t_{k-1},t]}, 1_{[t_{l-1},s]} \right\rangle_H.$$ (7.13)

Substituting (7.13) into the second term $I_2$ of (7.11), we get

$$|I_2| \leq \frac{C}{n^2} \sum_{k,l=1}^{n} \left[ \int_0^1 \int_0^1 \delta_{k/n}(t)\delta_{l/n}(s)|t - s|^{2H-2}dt\right]$$

$$+ C \sum_{k,l=1}^{n} \int_{t_{l-1}}^{t_l} \int_{t_{k-1}}^{t_k} \int_0^1 \left( 1_{[t_{k-1},t]}, 1_{[t_{l-1},s]} \right)_{H}\delta_{k/n}(t)\delta_{l/n}(s)|t - s|^{2H-2}dt\right]$$

$$\leq Cn^{-2} + \sum_{k,l=1}^{n} \left| \left\langle \delta_{k/n}, \delta_{l/n} \right\rangle_H \right|^2.$$ (7.14)

The above estimates together with (7.10) implies the result. ■

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