On the clique number of the square of a line graph and its relation to Ore-degree

Maxime Faron ∗ Luke Postle †

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Abstract

In 1985, Erdős and Nešetřil conjectured that the square of the line graph of a graph $G$, that is $L(G)^2$, can be colored with $\frac{5}{2}\Delta(G)^2$ colors. This conjecture implies the weaker conjecture that the clique number of such a graph, that is $\omega(L(G)^2)$, is at most $\frac{5}{2}\Delta(G)^2$. In 2015, Śleszyńska-Nowak proved that $\omega(L(G)^2) \leq \frac{3}{2}\Delta(G)^2$. In this paper, we prove that $\omega(L(G)^2) \leq \frac{4}{3}\Delta(G)^2$. This theorem follows from our stronger result that $\omega(L(G)^2) \leq \sigma(G)^2$ where $\sigma(G) := \max_{u,v \in E(G)} d(u) + d(v)$, is the Ore-degree of the graph $G$.

1 Introduction

The strong chromatic index, $\chi'_s(G)$, of a graph $G$ is defined as the least integer $k$ for which there exists a $k$-coloring of $E(G)$ such that edges at distance at most 2 receive different colors. Equivalently, $\chi'_s(G) = \chi(L(G)^2)$, where $L(G)^2$ denotes the square of the line graph of $G$. Since $\Delta(L(G)^2) < 2\Delta(G)^2$, the trivial upper bound on the chromatic number gives that $\chi'_s(G) \leq 2\Delta(G)^2$. However Erdős and Nešetřil (see [4, 5]) conjectured a much stronger upper bound as follows.

Conjecture 1.1. If $G$ is a graph, then $\chi'_s(G) \leq 1.25\Delta(G)^2$.

Note that if Conjecture 1.1 is true, then the bound would be tight as the following example shows. Indeed, if $G_k$ denotes the graph obtained from a 5-cycle by blowing up each vertex into an independent set of $k$ vertices, then $\Delta(G_k) = 2k$ and $L(G_k)^2$ is a clique with $5k^2 = 1.25\Delta(G_k)^2$ vertices. Figure 1 depicts the graph $G_3$.

In 1997, Molloy and Reed [7] made the first step towards Conjecture 1.1. They showed that for all graphs $G$, the graph $L(G)^2$ is 1/36-sparse. Thus the naive coloring procedure guarantees that such a graph can be colored with $(1 - \varepsilon)(\Delta(L(G)^2) + 1)$ colors for some $\varepsilon > 0$. With their naive coloring procedure, the value of $\varepsilon$ that Molloy and Reed obtain is approximately $0.0238 - \frac{1}{36} \approx 0.0007$. Bruhn and Joos [3] improved the bound on the neighborhood sparsity by showing that $L(G)^2$ is 1/4-sparse. Combining this with an improved coloring procedure, they deduce that $\varepsilon \approx 0.0347$ suffices. Bonamy, Perrett and the second author [2] improved the bound even further by using

∗Ecole Normale Superieure de Lyon, Lyon, France. Email: maxime.faron@ens-lyon.org.
†Department of Combinatorics and Optimization, University of Waterloo. Email: lpostle@uwaterloo.ca. Canada Research Chair in Graph Theory. Partially supported by NSERC under Discovery Grant No. 2014-06162, the Ontario Early Researcher Awards program and the Canada Research Chairs program.
an iterative version of the improved coloring procedure as well as an improved sparsity bound for critical graphs, proving $\chi'(G) \leq 1.835\Delta^2$.

Conjecture [11] implies a bound on the clique number of $L(G)^2$. Since the tight example mentioned above is also a clique in $L(G)^2$, we have the following weaker conjecture by Faudree, Gyárfás, Schelp, and Tuza [6] from 1990 (see also [1]).

**Conjecture 1.1.** If $G$ is a graph, then $\omega(L(G)^2) \leq 1.835\Delta^2$.

Some progress on Conjecture [12] has already been made. First, the case of bipartite graphs has been solved in a stronger form by Faugree et. al. [6] since 1990 as follows.

**Theorem 1.3.** If $G$ is a bipartite graph, then $\omega(L(G)^2) \leq \Delta(G)^2$.

Indeed, this bound is tight since the set of edges of a complete bipartite graph $K_{\Delta,\Delta}$ is a clique in $L(K_{\Delta,\Delta})^2$ of size $\Delta^2$. In a recent article, Śleszyńska-Nowak [9] gave a new proof of theorem 1.3. Furthermore, she improved the bound for the general case as follows.

**Theorem 1.4.** If $G$ is a graph, then $\omega(L(G)^2) \leq 1.5\Delta(G)^2$.

### 1.1 Main Results

In [9], Theorem 1.4 is proved by counting edges from a vertex $v$ of maximum degree. The crucial part of the argument is counting the number of edges which are not incident with any neighbor of $v$. This set is counted by means of vertex covers; however, the proof does not make use of the fact that this set of edges must also induce a clique in $L(G)^2$. If only induction could be applied to this set, then the bound might be improved; indeed, an unsophisticated inductive application of the result could be used to improve the bound to about $1.49\Delta(G)^2$. The trouble is that in this new graph the maximum degree might not decrease. Yet for any edge in the clique in $L(G)^2$, the sum of the degrees of its ends does decrease dramatically. This motivates then the use of Ore-degree, defined as follows, to more directly apply induction on said subgraph.

**Definition 1.5 (Ore-degree of a subgraph).** Let $G$ be a graph and $H$ be a subgraph of $G$. We define the Ore-degree of $H$ in $G$ as $\sigma_G(H) = \max_{xy \in E(H)}(d_G(x) + d_G(y))$. Moreover, the Ore-degree of $G$, which we denote by $\sigma(G)$, is defined as $\sigma_G(G)$.

As will be seen later, Ore-degree is flexible enough to be used inductively in these kinds of proofs. Moreover, Ore-degree is perhaps the more natural parameter to bound $\omega(L(G)^2)$ since if $G$ is simple, $\sigma(G) = \Delta(L(G)) + 2$. In addition, there is also the natural question of whether
analyses of the conjectures and theorems stated above hold for Ore-degree. For example, it is
not immediate that the analogue of the trivial upper bound, that is \(\omega(L(G)) \leq \frac{5}{4}\sigma(G)^2\), is true; however, we found a short proof of this fact which we omit since this is implied by our stronger
result Corollary 1.10.

Our first main result relates the clique number in the square of the line graph of a bipartite
graph (even a multigraph) to Ore-degree as follows.

**Theorem 1.6.** If \(G\) is a bipartite multigraph, then \(\omega(L(G)^2) \leq \frac{1}{4}\sigma(G)^2\).

Theorem 1.6 implies Theorem 1.3 since \(\sigma(G) \leq 2\Delta(G)\) and indeed Theorem 1.6 is tight for the
complete bipartite graph.

In fact, we prove a stronger result wherein we use only the Ore-degree of the clique instead of
the whole graph as follows.

**Theorem 1.7.** If \(G\) is a bipartite multigraph and \(H\) is a subgraph of \(G\) such that \(E(H)\) is a clique
in \(L(G)^2\), then \(|E(H)| \leq \Delta(H)(\sigma_G(H) - \Delta(H)) \leq \frac{1}{4}\sigma_G(H)^2\).

Theorem 1.7 is more useful for inductive purposes since Theorem 1.6 since we may only be able
to control the Ore-degree of edges in \(H\). Moreover, we conjecture that the same result holds when
only \(H\) is bipartite as follows.

**Conjecture 1.8.** If \(G\) is a graph, and \(H\) is a bipartite subgraph of \(G\) such that \(E(H)\) is a clique
in \(L(G)^2\), then \(|E(H)| \leq \frac{1}{4}\sigma_G(H)^2\).

While we cannot prove Conjecture 1.8 we can prove that it implies Conjecture 1.2 as follows.

**Theorem 1.9.** Let \(G\) be a graph, \(H\) a subgraph of \(G\) such that \(E(H)\) is a clique in \(L(G)^2\), and \(a \in \left[\frac{1}{3}, \frac{1}{2}\right]\).
If the following assumption holds:

**Assumption 1.** For all bipartite subgraphs \(H'\) of \(H\) such that \(|E(H')| < |E(H)|\), we have
\(|E(H')| \leq a \cdot \sigma_{G[V(H')]}(H')^2\).

Then \(|E(H)| \leq \left(\frac{1+a}{4}\right)\sigma_G(H)^2\).

Note than that if Assumption 1 holds with \(a = \frac{1}{3}\), then by Theorem 1.9 \(\omega(L(G)^2) \leq \frac{5}{16}\sigma_G(H)^2 \leq \frac{5}{4}\Delta(G)^2\), and hence Conjecture 1.8 implies Conjecture 1.2. Moreover, Assumption 1 is itself true
inductively for \(a = \frac{1}{3}\) as the following corollary demonstrates.

**Corollary 1.10.** If \(G\) is a graph and \(H\) is a subgraph of \(G\) such that \(E(H)\) is a clique in \(L(G)^2\),
then \(|E(H)| \leq \frac{1}{2}\sigma_G(H)^2 \leq \frac{1}{3}\sigma(G)^2\).

**Proof.** We proceed by induction on \(|E(H)|\). If \(|E(H)| \leq 1\), then the the result follows trivially. So
we may assume \(|E(H)| \geq 2\). Let \(H'\) be a bipartite subgraph of \(H\) such that \(|E(H')| < |E(H)|\).
Now \(E(H')\) is also a clique in \(L(G)^2\) since \(E(H)\) is. But then \(E(H')\) is a clique in \(L(G[V(H')])^2\).
So by induction, \(|E(H')| \leq \frac{5}{16}\sigma_{G[V(H')]}(H')^2\). Thus Assumption 1 of Theorem 1.9 holds for \(H\) with
\(a = \frac{1}{3}\). But then by Theorem 1.9 with \(a = \frac{1}{3}\), \(|E(H)| \leq \left(\frac{1+\frac{1}{3}}{4}\right)\sigma_G(H)^2 = \frac{1}{3}\sigma_G(H)^2\), as desired. □
Since, $\sigma(G) \leq 2\Delta(G)$, we have the following progress towards Conjecture 1.2.

**Corollary 1.11.** If $G$ is a graph, then $\omega(L(G)^2) \leq \frac{4}{3}\Delta(G)^2$.

In addition, we should mention that Reed [8] proved that $\chi_f(G) \leq \left\lfloor \frac{\Delta(G)+1+\omega(G)}{2}\right\rfloor$ where $\chi_f(G)$ is the fractional chromatic number. Hence Corollary 1.11 implies that $\chi_f(L(G)^2) \leq \frac{5}{3}\Delta(G)^2$ which is progress toward the fractional chromatic version of Conjecture 1.1.

Lastly, one may wonder if the results are tight. Corollary 1.11 is not if Conjecture 1.2 is true. Yet when $G$ is bipartite, Theorem 1.3 is tight for the complete bipartite graph. Is this the only extremal example? Essentially yes as we prove the following stability version of Theorem 1.3.

**Theorem 1.12.** For all $\varepsilon \in [0,1]$, if $G$ is a bipartite graph such that $\omega(L(G)^2) \geq (1-\varepsilon)\Delta(G)^2$, then $G$ contains a subgraph isomorphic to $K_{r,r}$ where $r = (1-\sqrt{5}\varepsilon^{1/4})\Delta(G)$.

The paper is organized as follows. In Section 2, we prove Theorem 1.7. In Section 3, we prove Theorem 1.9. Finally in Section 4, we prove Theorem 1.12.

# 2 Proof of the Bipartite Result

**Proof of Theorem 1.7**. We may assume $E(H) \neq \emptyset$ as otherwise the result follows trivially. For ease of reading, let us denote $\Delta(H)$ by $\Delta_H$ and $\sigma_G(H)$ by $\sigma$. Let $v$ be a vertex of maximum degree in $H$, that is $d_H(v) = \Delta_H$.

We may assume without loss of generality that $V(G) = V(H)$ since $E(H)$ is a clique in $L(G[V(H)])^2$. Moreover, since $E(H)$ is a clique in $L(G)^2$, all edges of $H$ must be at distance at most 2 in $G$ of each edge in $H$ incident to $v$. Thus each vertex in $H$ must be at distance at most 3 in $G$ of $v$. Since $V(G) = V(H)$, we have that every vertex in $G$ is at distance at most 3 in $G$ of $v$.

Let $A = N_H(v)$, $C = N_G(v) \setminus A$ and finally let $S = \{u \in V(H) : d_G(u,v) = 2, \exists vw \in E(H), d_G(v,w) = 3\}$. Let $E_C$ be the set of edges of $H$ incident with a vertex of $C$, $E_S$ be the set of edges of $H$ incident with a vertex of $S$ and finally let $E_A$ be the set of edges of $H$ incident with a vertex of $A$ but not a vertex in $S$ (see Figure 2 for an illustration).

Note that every edge $e$ of $H$ that is not incident with a vertex in $N_G(v)$ must be incident with a vertex $u$ such that $d_G(u,v) = 2$ since $E(H)$ is a clique in $L(G)^2$. Since $G$ is bipartite, the other end, call it $w$, of such an edge must be odd distance from $v$ and hence $d_G(v,w) = 3$. But then $u \in S$ by definition. That is, every edge of $H$ is incident with a vertex in $A, C$ or $S$. Thus $E(H) = E_C \cup E_S \cup E_A$.

![Figure 2: Diagram of notations from the proof of theorem 1.7](image)

We claim that every vertex $u \in S$ is adjacent to all vertices in $A$. To see this recall that by definition of $S$, there exists $uw \in E(H), d_G(v,w) = 3$. But for all $x \in A$, $d_G(vx, uw) = 2$ and hence
\[ xu \in E(G) \text{ as claimed since none of the other edges } vu, vw, xw \text{ exist in } G \text{ given the distances of } x, u, \text{ and } w \text{ to } v \text{ in } G. \]

Now \(|E_C| \leq |C| \Delta_H = (d_G(v) - |A|) \Delta_H \) and \(|E_S| \leq |S| \Delta_H\). Meanwhile, for every \( x \in A \), there are at most \( d_G(x) - |S| \) edges in \( E_A \) that are incident to \( x \) since every vertex in \( A \) is adjacent to all vertices in \( S \) and such edges are not in \( E_A \). Thus \(|E_A| \leq \sum_{x \in A} (d_G(x) - |S|)\). Yet for all \( x \in A \), \( d_G(x) + d_G(v) \leq \sigma \). Hence \(|E_A| \leq |A| (\sigma - d_G(v) - |S|)\).

So we have
\[
|E(H)| \leq |E_C| + |E_A| + |E_S| \\
\leq (d_G(v) - |A|) \Delta_H + |A| (\sigma - d_G(v) - |S|) + |S| \Delta_H.
\]

However, \(|A| = \Delta_H \) and hence
\[
|E(H)| \leq \Delta_H (\sigma - \Delta_H) \leq \left( \frac{\sigma}{2} \right)^2.
\]

3 Proof of the General Result

To prove Theorem \[ \text{1.9} \], we decompose the graph \( H \) into several sets of edges and count such sets in two different ways, one involving Assumption \[ \text{1.1} \] and the other involving trivial bounds.

Proof of Theorem \[ \text{1.9} \]. We may assume without loss of generality that \( V(G) = V(H) \) since \( E(H) \) is also a clique in \( L(G[V(H)])^2 \). Furthermore, we may assume \( E(H) \neq \emptyset \) as otherwise the result follows trivially. For ease of reading, we let \( d(v) \) denote the degree in \( H \) of a vertex \( v \), that is \( d_H(v) \), and we also let \( \sigma \) denote \( \sigma_G(H) \).

Let \( x \) be a vertex of maximum degree in \( H \) and let \( y \) be a neighbor in \( H \) of \( x \) such that \( d_H(y) \) is maximized.

Note also that since \( xy \in E(H) \), every vertex of \( H \) (and hence also of \( G \)) is at distance at most two in \( G \) from at least one of \( x \) or \( y \). Thus we let \( A = N_G(x) \cup N_G(y) \setminus \{x, y\} \), that is \( A = \{z \in V(H) : d_G(z, \{x, y\}) = 1\} \) and we let \( B = \{z \in V(H) : d_G(z, \{x, y\}) = 2\} \). So \( V(G) = \{x, y\} \cup A \cup B \).

Let us further partition \( A \) as follows. Let \( A_1 = N_G(x) \setminus N_G(y) \), \( A_2 = N_G(x) \cap N_G(y) \) and \( A_3 = N_G(y) \setminus N_G(x) \), see Figure \[ \text{3} \] for an illustration.

![Diagram](https://via.placeholder.com/150)

Figure 3: Diagram of notations from the proof of Theorem \[ \text{1.9} \]
Finally it will be convenient to consider the neighbors in $H$ of $x$, so let $C = N_H(x) \setminus \{y\}$. Note that $C \subseteq A_1 \cup A_2$. Moreover, for every vertex $v \in C$, $d(v) \leq \sigma - d(x)$ and yet also $d(v) \leq d(y)$ since $d(y)$ is maximized over $N_H(x)$.

Let $E_A$ be the set of edges with both ends in $A$. Note that every edge of $H$ other than $xy$ is incident with a vertex in $A$. Hence $|E(H)| = 1 + \sum_{v \in A} d(v) - |E_A|$. Yet $\sum_{v \in C} d(v) \leq |C| \cdot d(y)$ while $\sum_{v \in A \setminus C} d(v) \leq |A \setminus C| \cdot d(x)$. Thus we have the following trivial bound on $|E(H)|$:

$$|E(H)| \leq 1 + |C| \cdot d(y) + |A \setminus C| \cdot d(x) - |E_A|.$$ 

Yet $|C| = d(x)$ and $|A| \leq \sigma - |A_2| - 2$. Thus we have:

$$|E(H)| \leq 1 + d(x) \cdot d(y) + (\sigma - |A_2| - 2 - d(x))d(x) - |E_A|$$

$$\leq 1 - 2d(x) + d(x)\sigma + d(x)d(y) - d(x)^2 - |A_2|d(x) - |E_A|.$$ 

We deduce the following simpler bound from the bound above:

$$|E(H)| \leq d(x)(\sigma - d(x) + d(y)).$$

Now let us define two bipartite subgraphs of $H$ as follows: let $H_1$ be the graph such that $V(H_1) = A_1 \cup B$ and $E(H_1) = \{uv \in E(H) : u \in A_1, v \in B\}$; let $H_2$ be the graph such that $V(H_2) = A_3 \cup B$ and $E(H_2) = \{uv \in E(H) : u \in A_3, v \in B\}$. Since $E(H)$ is a clique in $L(G)^2$, so are $E(H_1)$ and $E(H_2)$. Furthermore, $E(H_1)$ is a clique in $L(G_1)^2$ and $E(H_2)$ is a clique in $L(G_2)^2$, where we let $G_1 = G[V(H_1)]$ and $G_2 = G[V(H_2)]$.

In addition, for every edge $uv \in E(H_1)$ and $v \in N_H(y)$, either $u$ or $v$ must be adjacent to $w$. This implies that if $uv \in E(H_1)$, then $d_{G_1}(u) + d_{G_1}(v) \leq \sigma - d(y)$. So $\sigma_{G_1}(H_1) \leq \sigma - d(y)$. Thus by Assumption II applied to $H_1$, we find that

$$|E(H_1)| \leq a(\sigma - d(y))^2.$$ 

Similarly $\sigma_{G_2}(H_2) \leq \sigma - d(x)$, and by Assumption II applied to $H_2$, we have

$$|E(H_2)| \leq a(\sigma - d(x))^2.$$ 

Now every edge of $H$ is either incident to one of $x$, $y$ or a vertex of $A_2$, or is in one of $E(H_1)$, $E(H_2)$ or $E_A$. Thus we get a new bound as follows:

$$|E(H)| \leq d(x) + d(y) - 1 + |E_A| + |E(H_1)| + |E(H_2)| + |A_2| \cdot d(x).$$ 

Substituting the bounds for $|E(H_1)|$ and $|E(H_2)|$ from above now gives:

$$|E(H)| \leq d(x) + d(y) - 1 + |E_A| + a(\sigma - d(y))^2 + a(\sigma - d(x))^2 + |A_2| \cdot d(x).$$ 

Then, the sum of the bound above and our first trivial bound is also a bound as follows:

$$2|E(H)| \leq d(y) - d(x) + (1 - 2a)d(x)\sigma - 2ad(y)\sigma - (1 - a)d(x)^2 + d(x)d(y) + 2a\sigma^2 + ad(y)^2.$$ 

Factoring out $d(x)$ and recalling that $d(y) \leq d(x)$ gives

$$|E(H)| \leq \frac{1}{2} (d(x)((1 - 2a)\sigma + d(y) - (1 - a)d(x)) - 2ad(y)\sigma + 2a\sigma^2 + ad(y)^2).$$
Recall that \(a \in \left[\frac{1}{4}, \frac{1}{3}\right]\). Now we have two bounds, a simple one and an average one as follows:

\[
|E(H)| \leq d(x)(\sigma - d(x) + d(y)). \tag{1}
\]

\[
|E(H)| \leq \frac{1}{2} \left( (1 - 2a)\sigma + d(y) - (1 - a)d(x) \right) - 2a \cdot d(y)\sigma + 2a\sigma^2 + ad(y)^2. \tag{2}
\]

Next we set \(s = \sqrt{1 + a} - 1\). We now distinguish two cases, depending on whether \(\frac{d(y)}{\sigma}\) is more or less than \(s\).

**Case 1:** \(d(y) \leq s\sigma\).

Then, by (1), \(|E(H)| \leq d(x)(\sigma - d(x) + d(y))\). Since \(d(y) \leq s\sigma\), we have that

\[
|E(H)| \leq d(x)((1 + s)\sigma - d(x)),
\]

which is at most

\[
\left( \frac{1 + s}{2}\sigma \right)^2 = \frac{1 + a}{4}\sigma,
\]

as desired.

**Case 2:** \(d(y) \geq s\sigma\).

By (2),

\[
|E(H)| \leq \frac{1}{2} \left( (1 - 2a)\sigma + d(y) - (1 - a)d(x) \right) - 2a \cdot d(y)\sigma + 2a\sigma^2 + ad(y)^2
\]

\[
\leq \frac{1}{2} \left( (1 - a)d(x) \left( \frac{(1 - 2a)\sigma + d(y)}{1 - a} - d(x) \right) - 2a \cdot d(y)\sigma + 2a\sigma^2 + ad(y)^2 \right).
\]

We would like to say that the right side of the inequality is maximized when \(d(x) = \frac{(1 - 2a)\sigma + d(y)}{2(1 - a)}\) but we should first distinguish whether or not \(\frac{(1 - 2a)\sigma + d(y)}{2(1 - a)}\) is greater than \(\sigma - d(y)\) which is an upper bound on \(d(x)\).

**Case 2.1:** \(\frac{(1 - 2a)\sigma + d(y)}{2(1 - a)} \leq \sigma - d(y)\), that is \(d(y) \leq \frac{\sigma}{3 - 2a}\).

Then (2) is maximized when \(d(x) = \frac{(1 - 2a)\sigma + d(y)}{2(1 - a)}\), whence we get

\[
|E(H)| \leq \frac{1}{2} \left( (1 - a) \left( \frac{(1 - 2a)\sigma + d(y)}{2(1 - a)} \right)^2 - 2a \cdot d(y)\sigma + 2a\sigma^2 + ad(y)^2 \right)
\]

\[
\leq \frac{1}{8(1 - a)} \left( (1 + 4a - 4a^2)\sigma^2 + (2 - 12a + 8a^2)d(y)\sigma + (1 + 4a - 4a^2)d(y)^2 \right).
\]

Let

\[
f(t) = \frac{1}{8(1 - a)} \left( (1 + 4a - 4a^2)\sigma^2 + (2 - 12a + 8a^2)t\sigma + (1 + 4a - 4a^2)t^2 \right)
\]

\[
= \frac{1}{8(1 - a)} \left( (1 + 4a - 4a^2)(\sigma + t)^2 + 4(4a^2 - 5a)t\sigma \right).
\]

Then \(f\) is a second-degree polynomial, whose leading coefficient is \(\frac{1 + 4a - 4a^2}{8(1 - a)}\). But, as \(0 < a \leq 1\), \(\frac{1 + 4a - 4a^2}{8(1 - a)} > 0\). So \(f\) is a convex function. Hence
\[
\max_{t \in [s \sigma, \sigma \frac{3 - 2a}{3 - 2a}]} (f(t)) = \max \left( f(s \sigma), f \left( \frac{\sigma}{3 - 2a} \right) \right).
\]

**Claim 3.1.** \(f(s \sigma) \leq \frac{1 + a}{4} \sigma^2\).

**Proof.** Note that \(4a^2 - 5a \leq 0\) since \(a \leq 5/4\). Meanwhile \(s = \sqrt{a + 1} - 1 \geq a/2 - a^2/8\) since \(a + 1 \geq (1 + a/2 - a^2/8)^2 = 1 + a - a^3/8 + a^4/64\) since \(0 \leq a \leq 8\). Thus

\[
f(s \sigma) = \frac{1}{8(1 - a)} ((1 + 4a - 4a^2)(1 + s)^2 \sigma^2 + 4(4a^2 - 5a)s \sigma^2)
\leq \frac{\sigma^2}{8(1 - a)} ((1 + 4a - 4a^2)(a + 1) + 4(4a^2 - 5a)(a/2 - a^2/8))
= \frac{\sigma^2}{8(1 - a)} (1 + 5a - 10a^2 + 6.5a^3 - 2a^4).
\]

However,

\[
\frac{1 + a}{4} \leq \frac{1 + 5a - 10a^2 + 6.5a^3 - 2a^4}{8(1 - a)}
= \frac{2 - 2a^2 - (1 + 5a - 10a^2 + 6.5a^3 - 2a^4)}{8(1 - a)}
= \frac{1 - 5a + 8a^2 - 6.5a^3 + 2a^4}{8(1 - a)}
= \frac{(1 - 3a)(1 - 2a + 2a^2 - 5a^3) + .5a^4}{8(1 - a)}.
\]

Yet \(1 - 3a \geq 0\) as \(a \leq 1/3\), \(1 - a \geq 0\) as \(a \leq 1\), and \(1 - 2a + 2a^2 - 5a^3 = 1 - 2a(1 - a) - .5a^3 \geq 1 - 2(.5)^2 - .5(1)^3 = 0\) as \(0 \leq a \leq 1\). Thus the difference in the equation above is at least 0 and hence \(\frac{1 + a}{4} \sigma^2\) is at least \(f(s \sigma)\) as desired. \(\square\)

**Claim 3.2.** \(f(\frac{\sigma}{3 - 2a}) \leq \frac{1 + a}{4} \sigma^2\).

**Proof.** Note that

\[
f \left( \frac{\sigma}{3 - 2a} \right) = \frac{-16a^4 + 48a^3 - 36a^2 - 12a + 16}{8(1 - a)(3 - 2a)^2} \sigma^2.
\]

But \(-16a^4 + 48a^3 - 36a^2 - 12a + 16 = 4(1 - a)(4a^3 - 8a^2 + a + 4)\sigma^2\) and hence

\[
f \left( \frac{\sigma}{3 - 2a} \right) = \frac{4a^3 - 8a^2 + a + 4}{2(3 - 2a)^2} \sigma^2.
\]

However,
\[
\begin{align*}
1 + a - \frac{4a^3 - 8a^2 + a + 4}{2(3 - 2a)^2} &= \frac{(1 + a)(3 - 2a)^2 - 2(4a^3 - 8a^2 + a + 4)}{4(3 - 2a)^2} \\
&= \frac{(9 - 3a - 8a^2 + 4a^3) - (8a^3 - 16a^2 + 2a + 8)}{4(3 - 2a)^2} \\
&= \frac{1 - 5a + 8a^2 - 4a^3}{4(3 - 2a)^2} \\
&= \frac{(1 - a)(2a - 1)^2}{4(3 - 2a)^2}.
\end{align*}
\]

Thus since \(a \leq 1\), this is at least 0. Hence \(\frac{1 + a}{4}\) is at least \(f\left(\frac{\sigma}{3 - 2a}\right)\), as desired.

Thus \(|E(H)| \leq \max \left( f(s\sigma), f\left(\frac{\sigma}{3 - 2a}\right)\right)\), which by Claims 3.1 and 3.2 is at most \(\frac{1 + a}{4}\) as desired.

**Case 2.2:** \(\frac{(1 - 2a)\sigma + d(y)}{2(1 - a)} > \sigma - d(y)\), that is \(d(y) \geq \frac{\sigma}{3 - 2a}\).

By (2),

\[
|E(H)| \leq \frac{1}{2} \left( (1 - a)d(x) \left( \frac{(1 - 2a)\sigma + d(y)}{1 - a} - d(x) \right) - 2ad(y)\sigma + 2a\sigma^2 + ad(y)^2 \right).
\]

Let

\[
g(t) = \left( \frac{(1 - 2a)\sigma + d(y)}{1 - a} - t \right) t = \frac{(1 - 2a)\sigma + d(y)}{1 - a} - t^2.
\]

So \(g\) is a second-degree polynomial whose leading coefficient is \(-1 \leq 0\) and hence is concave. Yet \(g\) is maximized when \(t = \frac{(1 - 2a)\sigma + d(y)}{2(1 - a)}\) which is greater than \(\sigma - d(y)\). So \(g\) is an increasing function on \([0, \sigma - d(y)]\) and hence

\[
\max_{t \in [0, \sigma - d(y)]} g(t) = g(\sigma - d(y)).
\]

Thus

\[
|E(H)| \leq \frac{1}{2} \left( (1 - a)(\sigma - d(y)) \left( \frac{(1 - 2a)\sigma + d(y)}{1 - a} - (\sigma - d(y)) \right) - 2ad(y)\sigma + 2a\sigma^2 + ad(y)^2 \right)
\]

\[
\leq \frac{1}{2} \left( a\sigma^2 - 2(1 - a)d(y)^2 + 2(1 - a)d(y)\sigma \right)
\]

\[
\leq \frac{1}{2} \left( a\sigma^2 + 2(1 - a)d(y)(\sigma - d(y)) \right)
\]

\[
\leq \frac{1}{2} \left( a\sigma^2 + 2(1 - a)\frac{\sigma^2}{4} \right)
\]

\[
\leq \frac{1}{8} \left( (4a + 2 - 2a)\sigma^2 \right)
\]

\[
\leq \frac{1 + a}{4} \sigma^2.
\]

\[\square\]
4 Proof of the Stability Result

We prove Theorem 4.2 in two parts. First we prove that if there is a large clique in \(L(G)^2\), then there are two sets of size at most \(\Delta\) with many edges between them as follows.

**Lemma 4.1.** Let \(\varepsilon \in [0, 1]\). If \(G = (A, B)\) is a bipartite graph and \(H\) is a subgraph of \(G\) such that \(E(H)\) is a clique in \(L(G)^2\) and \(|E(H)| \geq (1 - \varepsilon)\Delta(G)^2\), then there exists \(A' \subseteq A, B' \subseteq B\) such that \(|A'|, |B'| \leq \Delta(G)\) and \(|E(H) \cap E(A', B')| \geq (1 - 2\varepsilon - 2\sqrt{\varepsilon})\Delta(G)^2\).

**Proof.** Let \(a\) be a vertex of \(A\) such that \(d_H(a)\) is maximum over all vertices in \(A\). Let \(S_a = \{u \in A : N_H(u) \setminus N_G(a) \neq \emptyset\}\). Note that every vertex of \(S_a\) is adjacent in \(G\) to every vertex of \(N_H(a)\) since otherwise \(E(H)\) is not a clique in \(L(G)^2\). Let \(E_a\) be the set of edges of \(H\) not incident with a vertex in \(N_G(a)\). Note that \(|E_a| \leq |S_a|(\Delta(G) - |N_H(a)|)\). Yet \(|S_a| \leq \Delta(G)\) since \(S_a\) is contained in the neighborhood of any vertex in \(N_H(a)\). Thus \(|E_a| \leq \Delta(G)(\Delta(G) - |N_H(a)|)\).

Similarly let \(b\) be a vertex of \(B\) such that \(d_H(b)\) is maximum over all vertices in \(B\). Let \(S_b = \{v \in B : N_H(v) \setminus N_G(b) \neq \emptyset\}\) and let \(E_b\) be the set of edges of \(H\) not incident with a vertex in \(N_G(b)\). A symmetric argument to the one above shows that \(|E_b| \leq \Delta(G)(\Delta(G) - |N_H(b)|)\).

Let \(A' = N_G(a)\) and \(B' = N_G(b)\). Note that \(|A'|, |B'| \leq \Delta(G)\). Moreover, \(|E(H) \cap E(A', B')| \geq |E(H)| - |E_a| - |E_b| \geq |E(H)| - \Delta(G)(2\Delta(G) - |N_H(a)| - |N_H(b)|)\).

Now we may assume without loss of generality that \(d_H(a) \geq d_H(b)\). Thus \(d_H(a) = \Delta(G)\). By Theorem 4.7, \(|E(H)| \leq \Delta(H)(2\Delta(G) - \Delta(H))\). Thus \(\Delta(H) \geq (1 - \varepsilon - \sqrt{\varepsilon})\Delta(G)\), for otherwise \(|E(H)| < (1 - \sqrt{\varepsilon})(1 + \sqrt{\varepsilon})\Delta(G)^2 = (1 - \varepsilon)^2\Delta(G)^2\), contrary to our assumption.

But then \(|E_a| \leq \sqrt{\varepsilon}\Delta(G)^2\). So the number of edges of \(H\) with one end in \(B'\) is at least \((1 - \varepsilon - \sqrt{\varepsilon})\Delta(G)^2\). Thus there must exist a vertex \(v\) in \(B'\) such that \(|N_H(v)| \geq (1 - \varepsilon - \sqrt{\varepsilon})\Delta(G)(1 - \varepsilon - \sqrt{\varepsilon})\Delta(G)^2\) and hence \(|E_b| \geq \sqrt{\varepsilon}\Delta(G)^2\). Thus \(|E(H) \cap E(A', B')| \geq |E(H)| - |E_a| - |E_b| \geq (1 - 2\varepsilon - 2\sqrt{\varepsilon})\Delta(G)^2\), as desired. □

Yet we can also prove that two such sets with many edges between them must contain a large complete bipartite subgraph as follows.

**Lemma 4.2.** Let \(\alpha \in [0, 1]\). If \(G = (A, B)\) is a bipartite graph with \(|A|, |B| \leq n\) and \(H\) is a subgraph of \(G\) such that \(|E(H)| \geq (1 - \alpha)n^2\) and \(E(H)\) is a clique in \(L(G)^2\), then there exists a subgraph \(J\) of \(G\) isomorphic to \(K_{r,r}\) where \(r = (1 - \sqrt{2\alpha})n\).

**Proof.** We may assume without loss of generality that \(|A| = |B| = n\) for otherwise we may add isolated vertices to \(A\) and \(B\) while maintaining the hypotheses of the lemma. Now let \(G'\) be the graph such that \(V(G') = (A, B)\) and \(E(G') = \{ab \in E(G) : a \in A, b \in B\}\). Let \(C\) be a minimum vertex cover of \(G'\) and \(M\) be a maximum matching of \(G'\), say of size \(m\). By König’s theorem, \(|V(C)| = m|\).

If \(e_1 = a_1b_1, e_2 = a_2b_2\) are two distinct edges of \(M\), then at least one of \(a_1b_2, a_2b_1\) must not be an edge in \(H\) as otherwise \(E(H)\) is not a clique in \(L(G)^2\) since the distance between \(a_1b_2\) and \(a_2b_1\) is at least 3 in \(G\). But there are at least \(m(m - 1)/2\) such pairs. So \(|E(H)| \leq |A||B| - m - \frac{m(m - 1)}{2} \leq n^2 - m^2/2\). Since \(|E(H)| \geq (1 - \alpha)n^2\) we find that \(m \leq \sqrt{2\alpha}n\). Yet \(|V(C)| = m|\).
Now $G[V(G) - V(C)]$ is a complete bipartite graph since every edge of $G'$ is incident with a vertex of $C$. Let $A' \subseteq A - V(C)$ of size $r$ and $B' \subseteq B - V(C)$ of size $r$, which is possible since $m \leq n - r$. Let $J = G[A' \cup B']$. Now $J$ is isomorphic to $K_{r,r}$ as desired. 

We are now ready to prove Theorem 1.12 using the two lemmas above.

Proof of Theorem 1.12. Let $G = (A, B)$ be a bipartition of $G$. Let $H$ be a subgraph of $G$ such that $E(H)$ is a clique in $L(G)^2$ and $|E(H)| \geq (1 - \varepsilon)\Delta(G)^2$. By Lemma 4.1, there exist $A' \subseteq A, B' \subseteq B$ such that $|A'|, |B'| \leq \Delta(G)$ and $|E(H) \cap E(A', B')| \geq (1 - 2\varepsilon - 2\sqrt{\varepsilon})\Delta(G)^2$.

Let $G' = G[A' \cup B']$ and let $H' = H[A' \cup B']$. Now $H'$ is a subgraph of $G'$. Moreover, $E(H')$ is a clique in $L(G)^2$ since $E(H)$ is a clique in $L(G)^2$. But then $E(H')$ is a clique in $L(G[V(H')])^2 \subseteq L(G')^2$.

Apply Lemma 4.2 to $G' = (A', B')$ and $H'$ with $n = \Delta(G)$ and $\alpha = 2\varepsilon + 2\sqrt{\varepsilon} \leq 4\sqrt{\varepsilon}$. Thus $G'$ contains a subgraph isomorphic to $K_{r,r}$ where $r = (1 - \sqrt{2\alpha})n \geq (1 - \sqrt{8\varepsilon})\Delta(G) = (1 - \sqrt{8\varepsilon^{1/4}})\Delta(G)$ as desired.

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