Quantum quench and thermalization to GGE in arbitrary dimensions and the odd-even effect

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Abstract

In many quantum quench experiments involving cold atom systems the post-quench system can be described by a quantum field theory of free scalars or fermions, typically in a box or in an external potential. We work with free scalars in arbitrary dimensions generalizing the techniques employed in our earlier work \cite{1} in 1+1 dimensions. In this paper, we generalize to $d$ spatial dimensions for arbitrary $d$. The system is considered in a box much larger than any other scale of interest. We start with the ground state, or a squeezed state, with a high mass and suddenly quench the system to zero mass ("critical quench"). We explicitly compute time-dependence of local correlators and show that at long times they are described by a generalized Gibbs ensemble (GGE), which, in special cases, reduce to a thermal (Gibbs) ensemble. The equilibration of local correlators can be regarded as ‘subsystem thermalization’ which we simply call 'thermalization' here (the notion of thermalization here also includes equilibration to GGE). The rate of approach to equilibrium is exponential or power law depending on whether $d$ is odd or even respectively. As in 1+1 dimensions, details of the quench protocol affect the long time behaviour; this underlines the importance of irrelevant operators at IR in non-equilibrium situations. We also discuss quenches from a high mass to a lower non-zero mass, and find that in this case the approach to equilibrium is given by a power law in time, for all spatial dimensions $d$, even or odd.
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1 Introduction and Summary

Thermalization in integrable systems\(^5\) has grown to be an important area of study in recent years [2–13]. Integrable systems possess an infinite number of conservation laws and are often described by free (quasi)particles; thus, to understand thermalization in integrable models, it is important to understand it in free field theories. Besides the theoretical motivation, it turns out that in many recent quantum quench experiments involving cold atom systems, the post-quench phase can be described by a quantum field theory of free scalars or fermions [14,15]. In a previous work [1] such systems were studied at length where the spatial dimension \(d\) of the system was \(d = 1\) and the post-quench hamiltonian was taken to be critical. Both bosonic and fermionic systems were studied with explicit quench protocols. Among other things, it was found that (i) the post-quench state can be represented by a generalized Calabrese-Cardy state, characterized by an infinite number of \(W_\infty\) charges, as postulated in earlier works [16,17], (ii) correlators of local operators as well as the reduced density matrix of finite subsystems asymptotically approached that of a generalized Gibbs ensemble (GGE), (iii) the relaxation was exponential, characterized by a rate given by conformal and \(W_\infty\) algebraic properties of the operators in question. Furthermore, it was found that (iv) it is possible to start with specially tailored squeezed states so that the post-quench state is a simple Calabrese-Cardy state, characterized by only one conserved quantity—namely the energy; in this case the equilibrium is described by the familiar Gibbs ensemble and the relaxation rate is a simple function of the energy and the conformal dimension of the operator.

It is natural to ask, both from an experimental as well as a theoretical point of view: (a) which of the above results, if any, generalize to \(d\) spatial dimensions for arbitrary \(d\), and, (b) in particular, if there is an asymptotic equilibrium, what is the rate of approach to the equilibrium? There are several reasons why the answer to these questions is not easy to guess. First of all, the proof of thermalization, for 1+1 dimensional conformal field theories in [16], relies on conformal maps which map a strip of the complex plane to an upper half plane. The quantitative aspects such as the relaxation rate, also arise from such conformal maps. In \(d + 1\) dimensions, there is no conformal map which maps the Euclidean time circle (or interval) times \(\mathbb{R}^d\) to (a part of) \(\mathbb{R}^{d+1}\) for \(d > 1\)\(^6\) Furthermore, [16] extensively used properties of \(W_\infty\) algebra which are specific to 1+1 dimensions, to derive thermalization to GGE. In addition, in holographic contexts, the GGE in 1+1 dimensions finds an interpretation in terms of 3D higher spin black holes which have an infinite dimensional algebra of large diffeomorphisms (to be precise, \(W_\infty\)). There is no obvious generalization of these to higher dimensions.

The explicit computations for free scalars and fermions, studied in [1], did not directly use the conformal map or the \(W_\infty\) algebra, but in terms of detail, used several features specific to 1+1 dimensions.

In this paper we consider quantum quench of the mass parameter of free scalar field theories in arbitrary dimensions (both to zero and non-zero final mass) and explicitly compute time-dependence of local correlators. For simplicity of the computation, we consider sudden quench for most part of the paper, although our method is applicable to more general quench protocols. A summary of our results is as follows:

1. General results for mass quench \((m_{in} \rightarrow m_{out})\)

   (a) If we start with the ground state of the initial Hamiltonian, the post-quench state turns out to be equivalent to a generalized Calabrese-Cardy (gCC) state, which is a boundary conformal state with a cut-off applied to each of an infinite family of commuting conserved charges. If we start with specially tailored squeezed states, the post-quench state turns out to be an ordinary Calabrese-Cardy (CC) state, namely a conformal boundary state with a cut-off only on the total energy of the system.

   (b) We show that at long times (a) the correlators, starting from a CC state, are described by a thermal (Gibbs) ensemble, and (b) the correlators, starting from a gCC state are described by

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\(^5\)We use the word “thermalization” here to mean asymptotic approach of a local correlator in a time-dependent pure state to its value in a generalized Gibbs ensemble (GGE). Sometimes the word “thermalization” is reserved only for cases where the asymptotia corresponds to a Gibbs ensemble; we will not make this distinction. Since we specifically refer to equilibration of local correlators, such equilibration to GGE may be called “subsystem equilibration” or, in our generalized sense, “subsystem thermalization”.

\(^6\)We thank Abhijit Gadde for a useful discussion in this regard.
a generalized Gibbs ensemble (GGE).

(c) The GGE is characterized by chemical potentials which are determined by the infinite number
of conserved charges of the initial gCC state. As found and remarked about in [1], this implies
not only a retention of detailed memory of the initial state, but also an apparent contradiction
of naive intuition that higher dimensional conserved charges (7) should not affect long time
behaviour. This naive Wilsonian intuition, which perfectly works at low energies, does not work
in situations where the initial state is a high energy state.

2. Quench to zero mass: \( m_{\text{in}} \to 0 \) (critical quench)

(a) The rate of approach to equilibrium is exponential or power law depending on whether the
number of spatial dimensions, \( d \), is odd or even, respectively. This is one of the main results of
the paper.

(b) In the case where the post-quench state is a CC state, there is a geometric understanding of
thermalization in terms of the method of images, which also elucidates the odd-even effect.

(c) Unlike in 1+1 dimensions, the decay exponents are not uniquely defined by the conformal di-
mension of the operator. For example, in a given odd spatial dimension \( d > 1 \), where \( \langle \phi \phi \rangle \) and
\( \langle \partial_i \phi \partial_i \phi \rangle \) both decay exponentially, with the same decay exponent.

(d) As in 1+1 dimensions, details of the quench protocol affect the long time behaviour. This
underlines the importance of irrelevant operators at long times in non-equilibrium situations
involving high energies.

3. Quench to small non-zero mass: In case the system is quenched from a high mass to a low non-zero
mass, exact time-dependence of correlators can again be computed. These generically approach a
GGE with an oscillatory power law behaviour, with a period of oscillation given by the final mass.
There is no odd-even effect unlike for the massless case mentioned above.

We summarize the main points a bit more precisely in a tabular form below.

| \( m_{\text{out}} = 0 \) | \( m_{\text{out}} \neq 0 \ll m_{\text{in}} \) |
|----------------|------------------|
| \( d = 1 \) | \( e^{-\gamma t} \cos(2m_{\text{out}}t+\delta) \) |
| \( d = 2 \) | \( \frac{1}{t^{\frac{1}{2}}} \cos(2m_{\text{out}}t+\delta) \) |
| odd \( d > 1 \) | \( e^{-\gamma t} \cos(2m_{\text{out}}t+\delta) \) |
| even \( d > 2 \) | \( \frac{1}{t^{\frac{d}{2}-1}} \) |

Table 1: Here \( d \) refers to the number of spatial dimensions. The table displays the structure of an equal
time two-point function \( G(t) \). For \( d > 2 \), \( G(t) \equiv \langle \phi(x,t)\phi(y,t) \rangle \); for \( d = 1, 2, m_{\text{out}} = 0 \) this quantity diverges
at long times, hence we display \( G(t) = \langle \partial_i \phi(x,t)\partial_i \phi(y,t) \rangle \) (in all cases, we drop the spatial dependence).
The decay coefficients \( \gamma \) are given in the text (see e.g. Table 4 which lists various values of \( \gamma \) for \( d = 3 \):
these values depend on the specific initial state); the exponential decay has a power law prefactor in case
of ground state quench. The phase factors \( \delta \) are given by \( d \pi/4 \). Note that the power laws in the massless
case (for even dimensions) differ from those in the massive case.

The paper is organized as follows:
In section 2 we discuss the main formalism of how to describe the post-quench state as a Bogoliubov transform of the out-vacuum, leading to its identification (with the help of Appendix A) as a generalized Calabrese-Cardy state (12). We discuss starting with ground states as well as squeezed states in the massive phase. Section 2.2.1 gives the formulae for the time-dependent two-point functions.

In section 3 we explicitly calculate the time-dependent part of the two-point function (additional material is provided in Appendices B and C). Tables 4, 5 and 6 show the decay of the time-dependent part for critical quench \( m_{\text{out}} = 0 \) for \( d = 1, 2, 3, 4 \); the results for general dimensions \( d \) are obtained by using a recursion relation. For non-critical quench \( m_{\text{out}} \neq 0 \), Tables 7, 8 and 9 present the time-dependent part of the two-point functions for \( d = 1, 2, 3, 4 \); the formula for the general dimension is given in (57). The overall behaviour is as indicated in Table 1 above.

The fact that the time-dependent parts of the two-point functions vanish at large times, already implies that these correlators (and hence all correlators, by the application of Wick’s theorem), asymptotically equilibrate. In section 4, using results from Appendix D, we explicitly show that the time-independent parts of the two-point functions, to which they equilibrate, are given by two-point functions in a generalized Gibbs ensemble (GGE). This constitutes a proof of thermalization of arbitrary local correlators (see Section 4.2).

In section 5 we look at the geometrical interpretation of two-point functions in the CC state, which allows up to have a better understanding of the odd-even effect.

In section 6 we calculate the GGE correlator for purely spatial separation for the simple case of the thermal ensemble and a critical quench. The calculation has an interpretation in terms of a Kaluza-Klein reduction along the thermal circle. For high enough temperature, or equivalently large enough spatial separation, only the Kaluza-Klein zero mode contribution survives, which has a power law behaviour. This result holds in any dimension, even or odd.

In section 7 we conclude with some comments and discussion of future directions.

## 2 Quantum quench in free scalar theories

![Figure 1: Quantum quench of mass of a scalar field in \( d + 1 \) dimensions with the protocol in eq. (2). This is shown in the left panel. For simplicity, we will consider a sudden quench for most of the paper, in which case, the mass profile looks like the figure on the right.](image)

Figure 1: Quantum quench of mass of a scalar field in \( d + 1 \) dimensions with the protocol in eq. (2). This is shown in the left panel. For simplicity, we will consider a sudden quench for most of the paper, in which case, the mass profile looks like the figure on the right.

The basic set-up is as follows. Consider a relativistic scalar free field theory in \( d \) spatial dimensions, with a time dependent mass (see fig. 1)

\[
\mathcal{L} = \frac{1}{2} \left( \partial_i \phi \partial_i \phi - \partial_0 \phi \partial_0 \phi - m^2(t) \phi \phi \right) ; \quad i = 1, \cdots, d
\]

In Fourier space,

\[
\phi(\vec{x}, t) = \int \frac{d^d k}{(2\pi)^d} e^{i \vec{k} \cdot \vec{x}} \phi(\vec{k}, t)
\]
the equation of motion for the Fourier mode $\phi(\vec{k}, t)$ (similar equation for $\phi^*(\vec{k}, t)$) is

$$-\partial_t^2 \phi(\vec{k}, t) - m^2(t) \phi(\vec{k}, t) = |\vec{k}|^2 \phi(\vec{k}, t)$$

For every $\vec{k}$, this can be identified with a Schrodinger problem on a line (coordinatized by $y$, say), with the identifications

$$t \rightarrow y, m^2(t) \rightarrow -V(y), |\vec{k}|^2 \rightarrow E, \phi(\vec{k}, t) \rightarrow \psi_E(y)$$

In the Schrodinger problem there are two equivalent bases of solutions: one which corresponds to particles coming in from the left: $\psi_E(y) = \text{linear combination of } u_{in}(E, y), u_{out}(E, y)$, where $u_{in}(E, y) \sim e^{i\omega_{in}y}$ as $y \rightarrow -\infty$, and another which corresponds to particles coming in from the right, $u_{out}(y) \sim e^{-i\omega_{out}y}$ as $y \rightarrow +\infty$. Taking cue from this, we have two sets of normal mode expansions:

$$\phi(\vec{k}, t) = a_{in}(\vec{k}, t)u_{in}(\vec{k}, t) + cc, \ u_{in} \sim e^{-i\omega_{in}t}, \ t \rightarrow -\infty,$$

$$\phi(\vec{k}, t) = a_{out}(\vec{k}, t)u_{out}(\vec{k}, t) + cc, \ u_{out} \sim e^{-i\omega_{out}t}, \ t \rightarrow \infty$$

$$\omega_{in} = \sqrt{|\vec{k}|^2 + m_{in}^2}, \ \omega_{out} = \sqrt{|\vec{k}|^2 + m_{out}^2} \tag{1}$$

Here. The two basis sets are of course linear combinations of each other

$$u_{in}(\vec{k}, t) = \alpha(\vec{k})u_{out}(\vec{k}, t) + \beta(\vec{k})u_{out}^*(-\vec{k}, t)$$

which implies

$$a_{in} = a^*(\vec{k})a_{out}(\vec{k}) - \beta^*(\vec{k})a_{out}^*(-\vec{k})$$

$$a_{out} = \alpha(\vec{k})a_{in}(\vec{k}) + \beta^*(\vec{k})a_{in}^*(-\vec{k})$$

Here $\alpha, \beta$ are the Bogoliubov coefficients. We take the mass profile to be (see Fig 1a)

$$m^2(t) = \frac{1}{2}(m_{in}^2 + m_{out}^2) - \frac{1}{2}(m_{in}^2 - m_{out}^2) \tan(\rho t) \tag{2}$$

Further we do all our calculations in the sudden limit $\rho \rightarrow \infty^7$ (see Fig 1b). For the quench protocol eq. (2), Bogoliubov coefficients are easy to compute explicitly [18]; in the sudden limit they become [1]

$$\alpha(k) = \frac{1}{2} \frac{\omega_{out} + \omega_{in}}{\sqrt{\omega_{out}\omega_{in}}}, \ \beta(k) = \frac{1}{2} \frac{\omega_{out} - \omega_{in}}{\sqrt{\omega_{out}\omega_{in}}} \tag{3}$$

In this limit the in- and out- waves also become especially simple

$$u_{in}(k, t) = \frac{e^{-i\omega_{in}t}}{\sqrt{2\omega_{in}}}, \ u_{out}(k, t) = \frac{e^{-i\omega_{out}t}}{\sqrt{2\omega_{out}}} \tag{4}$$

The final mass is arbitrary but taken to be less than the initial mass i.e. $m_{out} < m_{in}$. We also study the theory when the final mass is taken to be zero, when the final theory is critical. As we will see the more interesting results are obtained in the critical quench.

### 2.1 Quantum quench from the ground state

A natural pre-quench initial state (just before $t = 0$) is the ground state $|0_{in}\rangle$ of the initial Hamiltonian $H_{in}$, which is also the zero-particle state defined by the oscillators $a_{in}|0_{in}\rangle = 0$. Just after $t = 0$, the state remains $|\psi(0)\rangle = |0_{in}\rangle$ (remember we are working in the sudden quench limit, therefore the state has no time to change). We then define $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$ where $H := H_{out}$ is the final Hamiltonian.

The ‘in’ ground state can be written in terms of the out ground state through a Bogoliubov transformation $8$

$$|\psi(0)\rangle = |0_{in}\rangle = \exp \left[ \frac{1}{2} \sum_{\vec{k}} \gamma(\vec{k})a_{out}^\dagger(\vec{k})a_{out}^\dagger(-\vec{k}) \right] |0_{out}\rangle$$

---

7This limit has to be understood in the sense explained in detail in [1]

8This is easily checked by noting that the right hand side is annihilated by $a_{in} = a^*(\vec{k})a_{out}(\vec{k}) - \beta^*(\vec{k})a_{out}^*(-\vec{k})$
where $\gamma(k) = \beta^*(k)/\alpha^*(k)$ depends on the Bogoliubov coefficients and is only a function of $k := |\vec{k}|$. Further the ‘in’ ground state is also related to the Dirichlet boundary state [16] (see appendix A for more details)

$$|D\rangle = \exp \left[ -\frac{1}{2} \sum_{\vec{k}} a^\dagger_{\text{out}}(\vec{k})a^\dagger_{\text{out}}(-\vec{k}) \right] |0\rangle_{\text{out}}$$

through the relation

$$|0\rangle_{\text{in}} = \exp \left[ \frac{1}{2} \sum_{\vec{k}} \kappa(k)a^\dagger_{\text{out}}(\vec{k})a_{\text{out}}(\vec{k}) \right] |D\rangle$$

where

$$\kappa(k) = -\frac{1}{2} \log(-\gamma(k))$$

(5)

In general one can expand $\kappa(k)$ for small $k$:

$$\kappa(k) = \sum_{i=1}^\infty \kappa_i k^{i-1} = \kappa_1 + \kappa_2 k + \kappa_3 k^2 + ...$$

With this, the expression for $|0\rangle_{\text{in}}$ becomes

$$|\psi(0)\rangle = |0\rangle_{\text{in}} = \exp \left[ -\sum_{i=1}^\infty \kappa_i Q_i \right] |D\rangle$$

(6)

where

$$Q_i = \sum_{\vec{k}} |\vec{k}|^{i-1} a^\dagger_{\text{out}}(\vec{k})a_{\text{out}}(\vec{k})$$

(7)

are conserved charges (they obviously all commute with the ‘out’ Hamiltonian $H = \sum_{\vec{k}} \sqrt{|\vec{k}|^2 + m^2_{\text{out}}} a^\dagger_{\text{out}}(\vec{k})a_{\text{out}}(\vec{k})$). It is easy to explicitly compute the $\kappa_i$ coefficients by using the definition of $\kappa(k)$ in terms of the Bogoliubov coefficients which are given in (3):

$$\kappa(k) = \frac{1}{2} \log \left( \frac{\sqrt{k^2 + m^2 + \sqrt{k^2 + m^2_{\text{out}}}}}{\sqrt{k^2 + m^2 - \sqrt{k^2 + m^2_{\text{out}}}}} \right)$$

(8)

Note that henceforth we will call $m_{\text{in}} =: m$ for simplicity. It is assumed that $m > m_{\text{out}}$. From the above equation, the small $k$ expansion can be easily found.

For $m_{\text{out}} = 0$, the expansion contains only odd powers of $k$, thus the even $\kappa_i$’s are non-zero, e.g.

$$\kappa_2 = \frac{1}{m}, \quad \kappa_4 = -\frac{1}{6m^3}, ...$$

(9)

By contrast, for $m_{\text{out}} \neq 0$, only even powers of $k$ survive, leading to odd $\kappa_i$’s, e.g.

$$\kappa_1 = \frac{1}{2} \log \left( \frac{m + m_{\text{out}}}{m - m_{\text{out}}} \right), \quad \kappa_3 = \frac{1}{2m m_{\text{out}}}, ...$$

(10)

**Relation to a generalized Calabrese-Cardy (gCC) state** In case of critical quench ($m_{\text{out}} = 0$), the post-quench dynamics is conformal. For critical quenches leading to a generic (non-integrable) conformal field theory, Calabrese and Cardy [CC:2004-2005] postulated the following form for the post-quench wavefunction

$$|CC\rangle = \exp[-\kappa H]|Bd\rangle$$

(11)

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Footnotes:

9 Note that, because we focus on homogenous quench protocols in this paper, we have rotational invariance in the $d$ spatial dimensions. Hence the Bogoliubov coefficients are a function of $k := |\vec{k}|$.

10 For free scalars, the number operators $N(k) = a^\dagger_{\text{out}}(\vec{k})a_{\text{out}}(\vec{k})$ are themselves conserved, and provide an alternative basis of the algebra of conserved charges.

11 One assumes here that the quench takes a finite time to end, say $t = t_0$. In this paper, we are interested in a sudden quench which takes place at $t = 0$; thus $t_0 = 0$. 

5
where, \( \kappa \) is given by the inverse of the mass gap characterizing the initial state (before the quench). The state \( |Bd\rangle \) is a “boundary state” representing a conformal state subject to an appropriate boundary condition at imaginary time \( t = -i\kappa \).

In case the post-quench conformal theory is integrable, characterized by an infinite number of charges \( Q_i \), a more appropriate ansatz for the post-quench state is the gCC (generalized Cardy-Calabrese) state [1, 16]

\[
|\text{gCC}\rangle = \exp \left[ -\sum_{i=1}^{\infty} \kappa_i Q_i \right] |Bd\rangle
\]

(12)
The state (6) is clearly a special case of such a state, which was first found in [1] in 1+1 dimensions. Note that the Dirichlet state \( |D\rangle \) is a particular example of a boundary state. As mentioned above (9), here only the even charges \( Q_{2n}, n > 0 \) are non-zero; note that \( Q_2 = H \), as can be seen from the definition (7).

In case of a massive quench, the state (6) is of the form (12), where the conformal boundary state \( |Bd\rangle \) is replaced by the Dirichlet boundary state \( |D\rangle \). Although the post-quench theory is not conformal in this case, we will continue to use the name gCC for the state (6). In fact we will continue to use this nomenclature also for states like (20) obtained by noncritical quench from squeezed states.

### 2.1.1 Two-point functions

We will be interested in computing quantities like

\[
\langle \psi(t_1)|O(x_1)O(x_2)|\psi(t_2)\rangle = \langle \psi(0)|O(x_1', t_1)O(x_2', t_2)|\psi(0)\rangle
\]

(13)
The operators appearing on the RHS \( O(x,t) = e^{iH_{\text{out}}t}O(x)e^{-iH_{\text{out}}t} \) are in the Heisenberg picture defined with the Hamiltonian \( H_{\text{out}} \). This is applicable for \( t \geq 0 \). For time evolution to \( t < 0 \) we must use the Hamiltonian \( H_{\text{in}} \). With this understanding it is clear that the above definition comes with the prescription (9), here only the even charges \( Q_{2n}, n > 0 \) are non-zero; note that \( Q_2 = H \), as can be seen from the definition (7).

In our theory these are related to the ground-state two-point function

\[
G(x_1', t_1; x_2', t_2) \equiv \langle 0_{\text{in}}|\phi(x_1', t_1)\phi(x_2', t_2)|0_{\text{in}}\rangle = \int \frac{d^dk}{(2\pi)^d} u_{\text{in}}(k, t_1) u_{\text{in}}^*(k, t_2) e^{ik.(x_1'-x_2')}
\]

\[
= \int \frac{d^dk}{(2\pi)^d} \left[ \alpha(k)^2 u_{\text{out}}(k, t_1) u_{\text{out}}^*(k, t_2) + \alpha(k)^2 u_{\text{out}}(k, t_1) u_{\text{out}}^*(-k, t_2) + \alpha^*(k)^2 u_{\text{out}}^*(-k, t_1) u_{\text{out}}(-k, t_2) + \alpha(k)^2 u_{\text{out}}(-k, t_1) u_{\text{out}}(-k, t_2) \right] e^{ik.(x_1'-x_2')}
\]

Using the expressions (3) and (4), the ground-state two-point function becomes

\[
\langle 0_{\text{in}}|\phi(x_1', t_1)\phi(x_2', t_2)|0_{\text{in}}\rangle = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik.(x_1'-x_2')}}{4\sqrt{(k^2 + m^2)(k^2 + m_{\text{out}}^2)}} \left[ (2k^2 + m^2 + m_{\text{out}}^2) \cos \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 - t_2) \right) \right.
\]

\[
+ \left. (m_{\text{out}}^2 - m^2) \cos \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 + t_2) \right) - 2i \sqrt{(k^2 + m^2)(k^2 + m_{\text{out}}^2)} \sin \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 - t_2) \right) \right]
\]

(16)
As mentioned above this correlator is not time-ordered. The time-ordered correlator can be obtained from this by replacing \( t_1 - t_2 \) by \( |t_1 - t_2| \) but leaving \( t_1 + t_2 \) unaltered. As a consequence, the ETC’s considered here are already time-ordered. The expression above simplifies for \( t_1 = t_2 \) (ETC):

\[
\langle 0_{\text{in}}|\phi(x_1', t)\phi(x_2', t)|0_{\text{in}}\rangle = \int \frac{d^dk}{(2\pi)^d} \frac{e^{ik.(x_1'-x_2')}}{4\sqrt{k^2 + m^2}(k^2 + m_{\text{out}}^2)}
\]
\[
\left( m_{\text{out}}^2 - m^2 \right) \cos \left( 2t \sqrt{k^2 + m_{\text{out}}^2} \right) + 2k^2 + m^2 + m_{\text{out}}^2 \right)
\]

(17)

We also look at the 2-point function of the $\partial_\phi$ operator. Note that this correlator is directly obtainable from the unequal time two-point function $\langle \phi \phi \rangle$ (16) by applying the operator $\partial_t \partial_{\vec{x}_2} |_{t_1 = t_2 = t}$. Since the unequal time correlator is of the form $f(t_1 - t_2) + g(t_1 + t_2)$, it follows that $\langle 0_{in} | \partial_t \partial_{\vec{x}_1, t_1} \partial_{\vec{x}_2} \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) | 0_{in} \rangle$ is of the form $-f''(t_1 - t_2) + g''(t_1 + t_2)$. Hence the equal-time two-point function of $\partial_\phi \phi$ is of the form $-f''(0) + g''(2t)$; in particular, the time-dependent part of this correlator is $\partial_t^2$ applied to the time-dependent part of (17). By explicit calculation, one finds the expression

\[
\langle 0_{in} | \partial_t \phi(\vec{x}_1, t) \partial_t \phi(\vec{x}_2, t) | 0_{in} \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{4 \sqrt{k^2 + m^2}} \left[ - \left( m_{\text{out}}^2 - m^2 \right) \cos \left( 2t \sqrt{k^2 + m_{\text{out}}^2} \right) + 2k^2 + m^2 + m_{\text{out}}^2 \right]
\]

(18)

which indeed follows the general form $-f''(0) + g''(2t)$ mentioned above.

A primary motivation for considering two-point functions of the derivative operators $\langle \partial_t \phi \partial_t \phi \rangle$ is as follows. In low dimensions the time-dependent part $g(2t)$ of the equal time correlator $\langle \partial_t \phi \partial_t \phi \rangle$ (17) grows in time which masks the transients that signal thermalization (it grows linearly in time in $d = 1$, and logarithmically in $d = 2$). The extra time derivatives, leading to $g''(2t)$, get rid of these divergences.

For the same reason, we may also consider the correlators of the spatial derivatives $\vec{\partial} \phi$,

\[
\langle 0_{in} | \partial_\vec{t} \phi(\vec{x}_1, t_1) \partial_\vec{t} \phi(\vec{x}_2, t_2) | 0_{in} \rangle = \partial_{\vec{x}_1} \partial_{\vec{x}_2} \langle \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) \rangle
\]

The extra derivatives ensure decay with increasing $|\vec{x}|$ (in $d = 1, 2$ $\langle \phi \phi \rangle$ grows, respectively, linearly and logarithmically with $|\vec{x}|$). It is enough for this purpose to focus on

\[
\langle 0_{in} | \partial_\vec{t} \phi(\vec{x}_1, t) \partial_\vec{t} \phi(\vec{x}_2, t) | 0_{in} \rangle = -\partial_\vec{t} \partial_\vec{t} \langle \phi(\vec{x}_1, t_1) \phi(\vec{x}_2, t_2) \rangle, \quad \vec{x} = \vec{x}_1 - \vec{x}_2
\]

(19)

We will discuss more details of two-point functions of these derivative operators and their relations in Section 2.2.1 below.

### 2.2 More general quantum quench: from squeezed states

As we saw above, the post quench state (6) is built out of infinite number of chemical potentials acting on the Dirichlet boundary state. However certain special bosonic states can be obtained if the initial state are chosen to be specific squeezed states of the pre-quench Hamiltonian

\[
|\psi(0)\rangle = |f\rangle_{in} \equiv \exp \left[ \frac{1}{2} \sum_k f(\vec{k}) a^\dagger_{in}(\vec{k}) a_{in}(-\vec{k}) \right] |0_{in}\rangle
\]

This is just the Bogoliubov transformation of $|0_{in}\rangle$. As the $|0_{in}\rangle$ state can itself be written as a Bogoliubov transform of $|0_{out}\rangle$, the post quench state $|f_{in}\rangle$ is a composite Bogoliubov transform of $|0_{out}\rangle$:

\[
|f\rangle_{in} = \exp \left[ \frac{1}{2} \sum_k \gamma_{eff}(\vec{k}) a^\dagger_{in}(\vec{k}) a_{in}(-\vec{k}) \right] |0_{out}\rangle = \exp \left[ \frac{1}{2} \sum_k \gamma_{eff}(\vec{k}) a^\dagger_{out}(\vec{k}) a_{out}(\vec{k}) \right] |D\rangle
\]

(20)

where in the first line $\gamma_{eff}$ denotes the composite $\gamma$:

\[
\gamma_{eff}(\vec{k}) = \frac{\beta^*(\vec{k}) + f(\vec{k}) a(\vec{k})}{a^*(\vec{k}) + f(\vec{k}) \beta(\vec{k})}
\]

(21)
and in the second line we have the relation \( \kappa_{\text{eff}} = -\frac{1}{2} \log(-\gamma_{\text{eff}}) \), following similar arguments as for the ground state. Here \( \alpha, \beta \) are given by (3). The state (20) is of the form of a gCC state. Can we get any gCC (with a given \( \kappa(k) \)) starting from a suitably chosen squeezing function \( f \)?

The answer is clearly yes. By solving (21) for \( f \), we can clearly find an \( f \) for any \( \gamma_{\text{eff}} \). If we wish to generate a given \( \kappa_{\text{eff}}(k) \equiv \kappa(k) \) in (20), we must choose \( \gamma_{\text{eff}}(k) = -\exp[-2\kappa(k)] \). This gives us

\[
f(\vec{k}) = 1 - \frac{2\omega_{\text{out}}}{\omega_{in} \tanh(\kappa(k)) + \omega_{\text{out}}}
\]

This proves that we can prepare any gCC state, with a given \( \kappa(k) \), from squeezed states:

\[
|f\rangle_{in} = \exp \left[ \frac{1}{2} \sum_k \kappa(k) a_{\text{out}}^\dagger(\vec{k}) a_{\text{out}}(\vec{k}) \right] |D\rangle =: |gCC\rangle
\]  

(23)

**Critical quench from specific squeezed states** Let us consider the case of critical quench \( m_{\text{out}} = 0 \). We look at the following special states. From (22) it is clear that the choice \( f(\vec{k}) = f_4(k) = 1 - \frac{2\kappa}{\sqrt{k^2 + m^2 \tanh(\kappa_2 k) + k}} \) leads to a state with only two non-zero tunable parameters \( \kappa_2 \) and \( \kappa_4 \)

\[
|\psi(0)\rangle = |f_4\rangle = \exp[-\kappa_2 H - \kappa_4 Q_4]|D\rangle \equiv |gCC_4\rangle
\]

(24)

we call this the \( gCC_4 \) state because it is characterized by only two charges \( Q_2 \) and \( Q_4 \); we will find that the equilibrium state describing asymptotic correlators in \( |gCC_4\rangle \) is a grand canonical ensemble characterized by a temperature and one chemical potential. Further if \( \kappa_4 \) is also zero, i.e. \( f(\vec{k}) = f_2(k) = 1 - \frac{2\kappa}{\sqrt{k^2 + m^2 \tanh(\kappa_2 k) + k}} \) then we end up with the CC state

\[
|\psi(0)\rangle = |f_4\rangle = \exp[-\kappa_2 H]|D\rangle \equiv |CC\rangle
\]

(25)

Note that \( \kappa_2 \) here is not related to the mass parameter \( m \) unlike in (9) where \( \kappa_2 = 1/m \).

**2.1 The squeezed state 2-point function**

The 2-point function in the squeezed state is the same as in the ground state with \( \alpha \) and \( \beta \) replaced by \( \alpha_{\text{eff}} \) and \( \beta_{\text{eff}} \) above. The 2-point function in the general squeezed state (23) is given by

\[
\langle gCC|\phi(\vec{x}_1, t_1)\phi(\vec{x}_2, t_2)|gCC\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{2 \sqrt{k^2 + m_{\text{out}}^2}} \cos(2\kappa(k)) \left[ \cos \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 - t_2) + 2i\kappa(k) \right) - \cos \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 + t_2) \right) \right]
\]

(26)

When \( t_1 = t_2 = t \) this is

\[
\langle gCC|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|gCC\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{2 \sqrt{k^2 + m_{\text{out}}^2}} \left[ \coth(2\kappa(k)) - \cos \left( 2t \sqrt{k^2 + m_{\text{out}}^2} \right) \right] \cos \left( \sqrt{k^2 + m_{\text{out}}^2}(t_1 + t_2) \right)
\]

(27)

For the equal-time two-point function of \( \partial_\phi \), we apply to (26) the rule \(-f''(0) + g''(2t)\) mentioned above (18), which gives us

\[
\langle gCC|\partial_\phi \phi(\vec{x}_1, t)\partial_\phi \phi(\vec{x}_2, t)|gCC\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{2} \sqrt{k^2 + m_{\text{out}}^2} \left[ \coth \left( 2\kappa(k) \right) + \cos \left( 2t \sqrt{k^2 + m_{\text{out}}^2} \right) \right]
\]

(28)
For the two-point function of spatial derivatives, these derives act only on the exponential term, leading to
\[
\langle gCC | \partial_i \phi(\vec{x}_1, t) \partial_i \phi(\vec{x}_2, t) | gCC \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{k^2 e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{2\sqrt{k^2 + m_{\text{out}}^2}} \left[ \coth (2\kappa(k)) - \cos \left(2t\sqrt{k^2 + m_{\text{out}}^2} \right) \csch (2\kappa(k)) \right]
\]
(29)

Note that the time-dependent parts of the above correlators, calling them \((...)^{(td)}_{gCC}\), satisfy the relation
\[
\langle \partial_i \phi(\vec{x}_1, t) \partial_i \phi(\vec{x}_2, t) ...^{(td)}_{gCC} \rangle + \langle \partial_i \phi(\vec{x}_1, t) \partial_i \phi(\vec{x}_2, t) ...^{(td)}_{gCC} \rangle + m_{\text{out}}^2 \langle \phi(\vec{x}_1, t) \phi(\vec{x}_2, t) ...^{(td)}_{gCC} \rangle = 0
\]
(30)

### 2.3 gCC correlators encompass all cases

As is clear from the foregoing discussion, all the particular post-quench states \(|\psi(0)\rangle\) mentioned above, specifically \(|CC\rangle, |gCC_4\rangle\), and \(|0_{in}\rangle\), are examples of \(|gCC\rangle\) states, with specific \(\kappa(k)\) as in Table 2:

| post-quench state | \(|gCC]\) | \(|CC]\) \(_{m_{\text{out}}=0}\) | \(|gCC_4]\) \(_{m_{\text{out}}=0}\) | \(|0_{in}\) |
|------------------|-------|----------------|----------------|-------|
| form of \(\kappa(k)\) | \(\kappa(k)\) | \(\kappa_2 k\) | \(\kappa_2 k + \kappa_4 k^2\) | \(\frac{1}{2} \log \frac{\sqrt{k^2 + m_{\text{in}}^2} + \sqrt{k^2 + m_{\text{out}}^2}}{\sqrt{k^2 + m_{\text{in}}^2} - \sqrt{k^2 + m_{\text{out}}^2}}\) |

Table 2: Ground states and specific squeezed states as examples of gCC states. The \(\kappa(k)\) for the ground state (last column) is reproduced from (8). This table paves way for a uniform discussion all post-quench states discussed in this paper.

Consequently, the above formula (27), and the other related two-point functions, give the corresponding two-point functions in all these different cases. In particular, using the form of \(\kappa(k)\) (8) for ground state quench, one can recover all results of Section 2.1.1 from those of Section 2.2.1, and in particular recover (17) from (27) (after all, the ground state is a special squeezed state!). This implies, in principle, that if we derive some results on thermalization and relaxation for the general gCC states, using (27), we need not consider the above particular cases separately. However because each of these cases is significant on its own, we find it useful to often state the results separately.

### 3 Time-dependence of two-point Functions

Since we are dealing with free field theories, all correlators are related to the basic two point function \(\langle \psi(0) | \phi(x_1, t_1) \phi(x_2, t_2) | \psi(0) \rangle\). The primary goal of this paper is to study the long time behaviour of this quantity. We will find that it asymptotes ("thermalizes") to the corresponding observable in a GGE; we will find the characterization of the GGE and find the rate of approach to equilibrium. It is easy to generalize these results to multi-point functions, by Wick’s theorem. It is also straightforward to compute correlators of composite operators from the above two-point function by appropriate regularization procedures.

In this section we will analytically evaluate the various 2-point functions (17),(18), (27) and (28) (note that the first two are a special case of the last two). We will indicate the main steps and list the results, leaving some details to the various appendices.

#### 3.1 General remarks

Note that the various two-point functions mentioned in the previous paragraph are all of the form
\[
\int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{r}} \left[ F(k) + G(k) \cos(2\omega_k) \right], \quad \omega_k = \sqrt{k^2 + m_{\text{out}}^2}
\]
(31)
where $\vec{r} = \vec{x}_1 - \vec{x}_2$. For correlators of the $\langle \phi \phi \rangle$ type,

$$F(k) = \frac{1}{2\omega_k} \coth(2\kappa(k)), \quad G(k) = -\frac{1}{2\omega_k} \operatorname{csch}(2\kappa(k))$$

To proceed further, we write the vector $\vec{k}$ in polar coordinates $(k, \theta, \varphi_1, ..., \varphi_{d-2})$, oriented such that $\vec{k}.\vec{r} = kr \cos \theta$; the $\varphi_i$ parameterize a unit sphere $S^{d-2}$ (we assume $d > 1$ here). With this, the above integral becomes

$$\int_0^\infty dk \left[ F(k) + G(k) \cos \left( 2\sqrt{k^2 + m_{\text{out}}^2} \right) \right] \Omega_d(k, r)$$

$$A_d(k, r) := k^{d-1} \frac{\Omega_{d-2}}{(2\pi)^{d}} \int_0^\pi d\theta e^{ikr \cos \theta} (\sin \theta)^{d-2}$$

where $\Omega_{d-2} = \frac{2^{(d-3)/2}}{\Gamma((d-1)/2)}$ is the volume of $S^{d-2}$.

**Angular integrals**  The angular integration in (34) can be exactly done, which gives

$$A_d(k, r) = k^{d-1} 2^{1-d/2} \pi^{-\frac{d}{2}} \frac{1}{4\pi^2} \, \Phi_1 \left( \frac{d}{2}; -\frac{1}{4} k^2 r^2 \right)$$

The regularized Hypergeometric function $\Phi_1$ is some combination of trigonometric functions $\sin(kr)$ and $\cos(kr)$ in odd $d$ while it is some Bessel function in even $d$. The specific forms of $A_d(k, r)$ for various dimensions $d$ are tabulated below in Table 3:

| number of spatial dimensions $d$ | $A_d(k, r)$ |
|----------------------------------|----------------|
| 1                               | $\frac{1}{2} \cos(kr)$ |
| 2                               | $\frac{k J_0(kr)}{2\pi}$ |
| 3                               | $\frac{k \sin(kr)}{2\pi r}$ |
| 4                               | $\frac{k^2 J_1(kr)}{4\pi^2 r}$ |
| 5                               | $\frac{k (\sin(kr) - kr \cos(kr))}{4\pi^2 r^3}$ |

Table 3: The angular integral $A_d(k, r)$, (see eq.(35)) in the first few dimensions.

**Recursion relations for 2-point functions**  The structure of the integrals (31) occurring in the various correlators defined above makes it possible to connect them across dimensions through recursion relations(see Appendix B for details). From a correlator in $d + 1$ spacetime dimensions, one can obtain the correlator in $(d + 2) + 1$ dimensions by acting with a particular differential operator. This relation holds for both even and odd $d > 1$ (here $t_- = t_1 - t_2$ and $t_+ = t_1 + t_2$).

$$\langle \phi(x_1, t_1) \phi(x_2, t_2) \rangle^{(d+2)} = \frac{\Omega_{d-2}}{4\pi^2 \Omega_{d-4}} \left( -\partial_{t_-}^2 - \partial_{t_+}^2 + m_{\text{out}}^2 + \partial_x^2 \right) \langle \phi(x_1, t_1) \phi(x_2, t_2) \rangle^{(d)}, \quad d > 1$$

12We added $d = 1$ here for uniformity, using $\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} f(|k|) = \int_0^{\infty} dk |k| \frac{1}{2} \cos(|k||x|) f(|k|)$.

13Similar recursion relations appear in a somewhat different context in [19].
where $\langle...\rangle^{(d)}$ denotes an expectation value, in a $d+1$ dimensional theory, in a general post-quench state.

For the special case of equal time correlators which is our main focus, the recursion relation becomes

$$\langle \phi(x_1,t_1)\phi(x_2,t_2)\rangle^{(d+2)} = \frac{\Omega_{d-2}}{4\pi^2 \Omega_{d-4}} \left(-\partial^2_t + m^2_{\text{out}} + \partial^2_r\right) \langle \phi(x_1,t_1)\phi(x_2,t_2)\rangle^{(d)}, \ d > 1 \quad (37)$$

A similar relation holds for the equilibrium correlators. In an arbitrary GGE, we have the following recursion relation ($t$ here is same as $t_\mu$)

$$\langle \phi(\vec{r},t)\phi(0,0)\rangle^{(d)}_{\mu(k)} = \frac{\Omega_{d-2}}{4\pi^2 \Omega_{d-4}} \left(-\partial^2_t + m^2_{\text{out}} + \partial^2_r\right) \langle \phi(\vec{r},t)\phi(0,0)\rangle^{(d-2)}_{\mu(k)} \quad (38)$$

The 1+1 dimensional 2-pt function is not connected to its 3+1 dimensional counterpart in the above fashion, since in one dimensional space there is no angular coordinate which is crucial for this recursion relation to work (see Appendix B). Nevertheless, there is a different, specific relation connecting the two:

$$\langle \phi(x_1,t_1)\phi(x_2,t_2)\rangle^{(3+1)} = -\frac{1}{2\pi} \partial_r \langle \phi(x_1,t_1)\phi(x_2,t_2)\rangle^{(1+1)} \quad (39)$$

which follows straightforwardly, by inspection.

**Time-dependence** As remarked before, there is a time-dependent part and a time-independent part of the generic equal time two-point function (31). We will come back to an analysis of the time-independent part in Section 4 where we show that this part exactly matches the corresponding two-point function in a GGE (generalized Gibbs ensemble).

In the remainder of the section, we will therefore consider only the time-dependent, non-equilibrium, part. We will show that this part decays to zero at large times, which amounts to a proof of thermalization, and we will derive the form of the decay. It will also be convenient, for this purpose, to differentiate between massive quench $m_{\text{out}} \neq 0$ and critical quench $m_{\text{out}} = 0$. This is because in the latter case, the post-quench theory is conformal and we expect, and indeed verify, that this theory will have special properties as compared to the massive case. We will comment later (see Section 3.4) why we cannot obtain results for the critical quench by simply taking the $m_{\text{out}} \to 0$ limit of the power series expansion for $m_{\text{out}} \neq 0$.

### 3.2 Critical Quench Correlators

As mentioned above, we will mainly focus on the time-dependent parts of the 2-point functions, while the time-independent parts will be considered in Section 4.

#### 3.2.1 Time-dependent part of the $\langle \phi\phi \rangle$ correlator

Let us put $m_{\text{out}} = 0$. To keep the discussion general, let us consider the general gCC correlator (27) (as emphasized in Section 2.3, we can infer about the specific cases of ground states and the squeezed states from here). In the notation of (33), we now have

$$F(k) = \frac{1}{2k} \coth(2\kappa(k)), \ G(k) = -\frac{1}{2k} \cosh(2\kappa(k)), \ \cos\left(2t\sqrt{k^2 + m^2}\right) = \cos(2kt) \quad (40)$$

The functions $\kappa(k)$ are tabulated for the special cases of interest in Table 2, where for the ground state, we now have

$$\kappa(k) = \frac{1}{2} \log \left(\frac{\sqrt{k^2 + m^2} + k}{\sqrt{k^2 + m^2} - k}\right) \quad (41)$$

which has an expansion in odd powers of $k$, as mentioned above (9). Following this, we will focus, in the case of critical quench, only on gCC states with $\kappa(k)$ which have an expansion in odd powers of $k$. The $\kappa(k)$ for the CC and gCC$_4$ states, of course, satisfy this by definition.

With this assumption, the functions $F(k), G(k)$ are even in $k$. 

11
3.2.2 Large time behaviour for odd $d$

Note from the Table 3, $A_{d}k, r$ that for odd $d$ are even functions of $k$. Using the evenness of $F(k), G(k)$, we find that the integrand in (33) is even, which allows us to extend the integration contour to the entire real line (for even $d$, this step is not allowed, hence we will do something else). If one can close the contour on the upper or the lower half plane the integral can be evaluated by the method of residues. To see how it works, let us take the example of $d = 3$ (the procedure described below is similar to the case of $d = 1$ [1]; and in special cases, where exact results are known in $d = 1$, the 3-dimensional results can be derived by using (39)). For more details see Appendix C.2.

For the critical gCC correlator at $d = 3$, we get

$$
\langle \text{gCC}\phi(x_{1}, t)\phi(x_{2}, t)\rangle = \frac{1}{4\pi^{2}} \int_{-\infty}^{\infty} dk \sin(kr) \left[ \coth(2\kappa(k)) - \csch(2\kappa(k)) \cos(2kt) \right]
$$

$$
= \frac{1}{8\pi^{2}r} \int_{-\infty}^{\infty} dk e^{ikr} \left[ \coth(2\kappa(k)) - \csch(2\kappa(k)) \cos(2kt) \right]
$$

$$
= \frac{1}{8\pi^{2}r} \int_{-\infty}^{\infty} dk \left[ \frac{\cosh(2\kappa(k))e^{ikr}}{\sinh(2\kappa(k))} - \frac{e^{ik(r+2t)}}{2\sinh(2\kappa(k))} - \frac{e^{ik(r-2t)}}{2\sinh(2\kappa(k))} \right] \tag{42}
$$

Let us focus on the time-dependent terms $B$ and $C$ (the time-independent term $A$ is described later on, and in Appendix C.2). We will also assume large $t$ so that $2t > r$. Then from the asymptotic behaviour of functions $\kappa(k)$ in the Table 2, one can see that the functions $\kappa(k)$ grow as $k \to \infty$, in a way that ensures that the integrals are convergent, and for large $t$ (so that $2t > r$) it is possible to close the contour for the integral $B$ on the UHP and for $C$ on the LHP. At large times $t$, the slowest transient is given by the pole of the integrand singularity nearest to the origin (in the UHP for $B$ and LHP for $C$; the pole at the origin itself cancels between the terms $B$ and $C$; see Appendix C.2, especially Figure 6). It is easy to see that for a gCC with a finite number of chemical potentials, the nearest such singularity is a pole. This is given by the relevant zero of $\sinh(2\kappa(k))$, or, equivalently by the corresponding root of

$$
2\kappa(k) = \pm i\pi \tag{43}
$$

The root discussed above turns out to be of the form as $k = k_{1} \pm ik_{0}, k_{0} > 0$; here we choose the $+$ and $-$ signs respectively for the $B$ and $C$ terms; the value of $k_{1}$ may be zero, positive or negative. By applying residue calculus, we find that the slowest decay at large $t$ of the $\phi \phi$ correlator is given by

$$
b \exp[-k_{0}(t+2r)] + c \exp[-k_{0}(t-2r)] \tag{44}
$$

For non-zero $k_{1}$ this comes multiplied by an oscillatory term, of time period $2\pi/k_{1}$. In case of the CC state, the exact time-dependence can be calculated by summing over the poles (see Appendix C.2 for details). The exact two-point function for the CC state as well as for the ground state can also be derived from the their 1+1 dimensional counterparts found in [1] using the recursion relation (39) (see Appendix C.2). The net result of all this is that at large times the time-dependent part of the critical $\langle \phi \phi \rangle$ fall off exponentially, as in (44); these are tabulated in Table 4.

In case of a gCC with an infinite number of chemical potentials, such as when the gCC is the ground state, the poles can merge into a branch cut (see also a similar discussion in [1]). The structure (44) is retained even here, with additional power-law prefactors. Details of the above procedure for the CC, gCC, and the ground state, are worked out in Appendix C.2. Note that although for specificity we had taken the example of $d = 3$ here, the arguments go through for $d = 5$ or any higher odd space dimensions.

Since we are interested in the leading approach to equilibrium for simplicity, in the tables we only display the correlators in the limit $mt \gg 1 \gg m\tau$ or equivalently $\frac{t}{\kappa_{4}} \gg 1 \gg \frac{\tau}{\kappa_{4}}$. The ellipsis in the gCC, correlator represents $O(\kappa_{4}^{-2})$ terms.

12
\[ \langle \psi(0)|\phi(x_1, t)\phi(x_2, t)|\psi(0) \rangle \]

|\[ |\psi(0)\rangle\]|d = 3 | d = 4 |
|---|---|---|
|Ground state| \(-\frac{m^2}{16\pi^2}\sqrt{mt} e^{-2mt \left( 1 + O(mt)^2 \right)} \left( 1 + O(mt)^{-1} \right)\)| \( \frac{m}{128\pi^2 t^3} + O(\frac{1}{t^4}) \) |
|CC state| \(-\frac{1}{16\pi^2} e^{-\pi t/\kappa^2} \left( 1 + O(\frac{t}{\kappa^2})^2 \right) + O(e^{-2\pi t/\kappa^2})\)| \( \frac{1}{128\pi^2 \kappa^2} + O(\frac{1}{t}) \) |
|gCC state| \(-\frac{1}{16\pi^2} \left( 1 + \pi^2 \kappa_i + \cdots \right) e^{-\pi t/\kappa^2} \left( 1 + O(\frac{t}{\kappa^2})^2 \right) + O(e^{-2\pi t/\kappa^2})\)| \( \frac{1}{128\pi^2 \kappa^2} + \frac{3\pi^2 + 16\pi^2 + 24\kappa_i \kappa_i}{2048 \pi^2 \kappa^2} + O(\frac{1}{t}) \) |

Table 4: Time-dependent part of 2-point function \( \langle \psi(0)|\phi(x_1, t)\phi(x_2, t)|\psi(0) \rangle \) in critical quench for \( d = 3 \) (left column) and \( d = 4 \) (right column). Here \( r = |x_1 - x_2| \). Note that for even \( d \), e.g. \( d = 4 \), the leading large \( t \) dependence is the same in all three types of post-quench state \( |\psi(0)\rangle \), with the identification \( m = 1/\kappa^2 \) (see (9)). Further the gCC answer tells us that since the higher \( \kappa_i \)'s for \( i > 2 \) show up in subleading terms, the higher conserved charges do not affect the leading transient towards thermalization. However, as we will see later, the time-independent part of these two-point functions reflects an equilibrium given by a GGE which is characterized by all \( \kappa_i \) parameters.

**General odd \( d \) result:** From the recursion relation (36) it follows that for all odd \( d \geq 3 \), the slowest decay at large time of the time-dependent part of the \( \phi \phi \) correlator is given by replacing (44) with

\[
\begin{align*}
    &b_d \exp[-k_0(t + 2r)] + c_d \exp[-k_0(t - 2r)] \\
    &\text{where } k_0 \text{ has the same value discussed above and is independent of dimension } d. \text{ The constants } b_d, c_d \text{ depend on the dimension } d. \text{ The expression above may come with a possible oscillatory part as explained below (44).}
\end{align*}
\]

3.2.3 Large time behaviour for even \( d \)

Here the angular function, \( A_d(k, r) \) for even \( d \) are odd functions of \( k \). Therefore, unlike in odd \( d \), we cannot extend the integration contour to the entire real line and apply the method of residues. Fortunately there does exist a different approach in this case. Let us take the example of \( d = 4 \) to illustrate the method. As in the case of odd \( d \), we allow the most general \( \kappa(k) = \sum_{i=1}^{\infty} \kappa_i k^{2i-1} \) which grows at infinity such that the integral is convergent. The critical gCC correlator at \( d = 4 \) is

\[
(gCC)\phi(x_1, t)\phi(x_2, t)|gCC\rangle = \frac{1}{8\pi^2 r} \int_0^\infty dk J_1(kr) [\coth(2\kappa(k)) - \csch(2\kappa(k)) \cos(2kt)]
\]

In the notation of equation 33, \( G(k) = -\csch(2\kappa(k))/(2k) \). As before we are interested in large \( t \). Consider the following scaling

\[
k \to p = 2kt
\]

and rewrite the time-dependent part in terms of this new dimensionless momenta \( p \)

\[
\begin{align*}
    &- \frac{1}{8\pi^2 r} \int_0^\infty dk J_1(kr) \csch(2\kappa(k)) \cos(2kt) = - \frac{1}{32\pi^2 r} \frac{1}{t^2} \int_0^\infty dp \, p J_1 \left( \frac{pr}{2t} \right) \csch \left( 2\kappa \left( \frac{p}{2} \right) \right) \cos(p) \\
    &= - \frac{1}{64\pi^2 r} \frac{1}{t^2} \left\{ \int_0^{\infty(1 + i\epsilon)} dp \, p J_1 \left( \frac{pr}{2t} \right) \csch \left( 2\kappa \left( \frac{p}{2} \right) \right) e^{ip} + \int_0^{\infty(1 - i\epsilon)} dp \, p J_1 \left( \frac{pr}{2t} \right) \csch \left( 2\kappa \left( \frac{p}{2} \right) \right) e^{-ip} \right\}
\end{align*}
\]
\[
= -\frac{1}{256\pi^2\kappa_2} \left[ \frac{I(1)}{t^2} - \frac{(16\kappa_2^3 + 24\kappa_4 + 3\kappa_2 r^2)}{96\kappa_2 t^4} \right] I(3) + O(t^{-6}) \quad (47)
\]

In the second line we have rotated the contour anti-clockwise for \(e^{ip}\) and clockwise for \(e^{-ip}\). We can do this because the integrand \(J_1(kr) \cosh(2\kappa_2 k)\) vanishes on the arc at infinity due to the exponential damping in the \(\cosh(2\kappa_2 k)\) as \(k \to \infty\). In the last line we have Taylor expanded the integrand at large \(t\). We have also introduced the integrals \(I(n)\) defined by

\[
I(n) = \lim_{\epsilon \to 0} \int_0^{\infty} e^{ip} p^n dp + \lim_{\epsilon \to 0} \int_0^{\infty} e^{-ip} p^n dp
\]

\[
= \lim_{\epsilon \to 0} \int_0^{\infty} dp \left( e^{ip(i-\epsilon)} + e^{-ip(i+\epsilon)} \right) p^n + \mathcal{O}(\epsilon)
\]

\[
= \lim_{\epsilon \to 0} \int_0^{\infty} dp \left( e^{ip(i-\epsilon)} + e^{-ip(i+\epsilon)} \right) = \lim_{\epsilon \to 0} \left( \prod_{s=1}^{n} (\partial - \epsilon)^{s} \right)
\]

\[
I(n) = \begin{cases} 
(-1)^{(n+1)/2} \times \pi/2 & \text{n is odd} \\
0 & \text{n is even}
\end{cases}
\quad (48)
\]

where in the second line we have introduced \(p_\pm = p(1 \pm i\epsilon)\). Note that in spite of the growth of the \(I(n)\) with \(n\), because of the even more strongly convergent power series expansion for the product of the Bessel and cosech functions, the equation (47) has a convergent expansion in \(1/t^4\) for large enough \(t\).

Notice that \(\kappa_4\) shows up only from the \(O(t^{-4})\) onwards. Similarly \(\kappa_6\) starts appearing only from the \(O(t^{-6})\) term; explicitly that term is

\[
\frac{36 (14\kappa_2^3 - 30\kappa_3\kappa_4 - 45\kappa_2\kappa_6 + 45\kappa_4^2) + 15\kappa_2^2 r^4 + 120 r^2 (2\kappa_4^2 + 3\kappa_2 \kappa_4)}{46080\kappa_2^2 t^6} I(5)
\]

Having calculated \(I(n)\)'s, we find that the time-dependent part of the gCC 2-point function goes as

\[
-\frac{1}{8\pi^2 r} \int_0^{\infty} dk J_1(kr) \cosh(2\kappa_2 k) \cos(2kt) = \frac{1}{128\pi^2\kappa_2} \frac{1}{t^2} + \frac{3r^2 + 16\kappa_2^2 + 24\kappa_4}{2048\pi^2\kappa_2} \frac{1}{t^4} + O(t^{-6}) \quad (49)
\]

The effect of \(\kappa_4\) is suppressed by \(O(t^{-2})\) relative to the leading term and therefore it makes no difference to the leading power law fall off in the large \(t\) limit. The same is true for higher \(\kappa_n\)'s, as they are even more suppressed.

The equation (49) can be verified against explicit numerical integration of the momentum integral. In practical terms, the subleading term, of order \(1/t^4\), is possible to read off by first also calculate the time-dependent part in the gCC state numerically and see exact agreement in subleading \(O(t^{-4})\) behaviour after subtracting off the \(\kappa_2\)-dependent part (fig. 2).

\[
\langle \text{gCC} | \phi \phi | \text{gCC}\rangle - \langle \text{CC} | \phi \phi | \text{CC}\rangle
\]

Figure 2: We plot the gCC\textsubscript{4} equal-time correlator after subtracting off the CC part. This gives us the contribution depending solely on \(\kappa_4\). There is near perfect match between the analytic and numerical results. (Numerics done for values \(r = 0.1, \kappa_2 = 0.25\) and \(\kappa_4 = 0.025\))
General even \(d\) result: From the recursion relation (36) it follows that for all \(d \geq 4\), the time-dependent part of the \(\phi\phi\) correlator decays as (up to time-independent factors)

\[
\frac{1}{p^{d-2}}
\]  

(50)

The \(d = 2\) case is discussed separately below (see table 6) for a summary and also appendix C.1 for details).

One might wonder if the same reasoning would have given us the answer in odd \(d\), then we would not have to do the contour integrals there. In \(d = 3\) for example, the time-dependent part of the 2-point function has even powers of \(p\) (eq. 46)

\[
\begin{align*}
-\frac{1}{4\pi^2} & \int_0^\infty dk \sin(kr) \operatorname{csch}(2\kappa_2 k) \cos(2kt) \\
& = -\frac{1}{16\pi^2\kappa_2} \int_0^\infty dp \left[ \frac{1}{t} \left( \frac{4\kappa_2^3 + 6\kappa_4 + \kappa_2 r^2}{24\kappa_2 t^4} \right) + \mathcal{O}(t^{-5}) \right] \cos(p) \\
& = -\frac{1}{32\pi^2\kappa_2} \left[ \frac{I(0)}{t} - \frac{4\kappa_4 + \kappa_2 r^2}{24\kappa_2 t^4} I(2) \right] + \mathcal{O}(t^{-5}) \\
& = 0
\end{align*}
\]  

(51)

What this means is that either the answer is zero or something non-perturbative, i.e. a function with no Taylor expansion at \(t = \infty\). Since we already know the answer is an exponentially decaying function \(e^{-\pi t/\kappa_2}\), we know it is the latter. The above argument elucidates the odd-even difference in the approach to thermalization. We give another geometric understanding in the next section 5.

### 3.2.4 \(\partial_o \phi \partial_t \phi\) and \(\partial_o \phi \partial_t \phi\) Correlator

| \(\psi(0)\) | \(\partial_o \phi(\vec{x}_1, t) \partial_t \phi(\vec{x}_2, t)|\psi(0)\rangle | \(d = 3\) | \(d = 4\) |
|---|---|---|---|
| Ground state | \(\frac{-m^4}{16\pi^2 \sqrt{mt}} e^{-2mt} (1 + \mathcal{O}(mt)^2) (1 + \mathcal{O}(mt)^{-1})\) | \(\frac{3m}{256\pi^2 \frac{1}{t^4}} + \mathcal{O}(\frac{1}{t^5})\) |
| CC state | \(\frac{-\kappa_2^2}{64\kappa_2^2} e^{-\pi t/\kappa_2} (1 + \mathcal{O}(\frac{t}{\kappa_2})^2) + \mathcal{O}(e^{-2\pi t/\kappa_2})\) | \(\frac{3}{256\pi^2 \frac{1}{t^4}} + \mathcal{O}(\frac{1}{t^5})\) |
| gCC state | \(\frac{-\pi^2(1+2\kappa_4 + \cdots)}{64\kappa_2^2} e^{-\frac{\pi^2}{\kappa_2^2} (1+\frac{2\kappa_4 + \cdots}{\kappa_2})^2} (1+\mathcal{O}(\frac{t}{\kappa_2})^2) + \mathcal{O}(e^{-2\pi t/\kappa_2})\) | \(\frac{3}{256\pi^2 \frac{1}{t^4}} + \frac{15\kappa^2 + 80\kappa_4 + 120\kappa_2}{2048\pi^2 \kappa_2^2} + \mathcal{O}(\frac{1}{t^5})\) |

Table 5: Time-dependent part of 2-point function \(\langle \psi(0)|\partial_o \phi(\vec{x}_1, t) \partial_t \phi(\vec{x}_2, t)|\psi(0)\rangle\) in critical quench in \(d = 3\) in the left column and \(d = 4\) in the right column. Here \(r = |\vec{x}_1 - \vec{x}_2|\). \(|\psi(0)\rangle\) denotes the type of post-quench state.

As argued in section 2.1.1, the 2-point function of \(\partial \phi\) is not independent of the 2-point function of \(\phi\). As explained in that section, in \(d = 1\) and \(2\), our primary interest is in \(\langle \partial \phi \partial \phi \rangle\). For \(d = 1\) [1] this quantity for the general gCC state can be obtained by using the equations (28), (33) and (34):

\[
\langle \text{gCC}|\partial_o \phi(x_1, t) \partial_t \phi(x_2, t)|\text{gCC} \rangle = \frac{1}{4\pi} \int_{-\infty}^{\infty} dk e^{ikr} k \left[ \coth(2\kappa(k)) + \operatorname{csch}(2\kappa(k)) \cos(2kt) \right]
\]
approximation to perform the momentum integrals. In the notation of (31), the time dependent part of 2-point function in the gCC state (27) can be written

\[ \langle \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) \rangle_{td} = \frac{1}{4\pi} \int_0^\infty dk e^{ikr}k^2 J_0(kr) \cos(2kt) \cosh(2\kappa(k)) \]

This expression can be evaluated in the same manner as described in Section 3.2.3. We tabulate the results below.

| \(|\psi(0)\rangle|\) | d = 1 | d = 2 |
|---|---|---|
| Ground state | \[ \frac{m^2}{8\pi^{1/2}\sqrt{mt}} e^{-2mt} (1 + O(mt)^2) (1 + O(mt)^{-1}) \] | \[ -\frac{m}{8\pi^{1/2}} + O(\frac{1}{\kappa}) \] |
| CC state | \[ \frac{\kappa}{8\pi^2} e^{-\pi t/\kappa^2} (1 + O(\frac{\kappa}{\pi})^2) + O(e^{-2\pi t/\kappa^2}) \] | \[ -\frac{1}{8\pi^{1/2}} + O(\frac{1}{\kappa^4}) \] |
| gCC state | \[ \frac{\pi(1+\pi_4+\cdots)}{8\pi^2} e^{-2t} \left( \sqrt{\pi^2 + \frac{3\kappa_4+\cdots}{\kappa^2}} \right) (1 + O(\frac{\kappa}{\pi})^2) + O(e^{-2\pi t/\kappa^2}) \] | \[ -\frac{1}{8\pi^{1/2}} + \frac{8\pi^2+12\pi_4+3\pi_2^2}{12\pi^2\kappa^4} + O(\frac{1}{\kappa^4}) \] |

Table 6: Time-dependent part of 2-point function \( \langle \psi(0) | \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) | \psi(0) \rangle \) in critical quench in \( d = 1 \) in the left column and \( d = 2 \) in the right column. Here \( |\psi(0)\rangle \) denotes the type of post-quench state. \( r = |x_1 - x_2| \).

3.3 Massive Quench Correlators

In the massive or non-critical quench the final mass \( m_{\text{out}} \) is non-zero. Here we incorporate stationary phase approximation to perform the momentum integrals.

3.3.1 φφ Correlator

In the notation of (31), the time dependent part of 2-point function in the gCC state (27) can be written as

\[ \int_0^\infty dk G(k) \cos \left( 2t \sqrt{k^2 + m^2_{\text{out}}} \right) A_d(k, r) \]

\[ = \frac{1}{2} \int_0^\infty dk G(k) \left( e^{i2t\sqrt{k^2+m_{\text{out}}^2}} + e^{-i2t\sqrt{k^2+m_{\text{out}}^2}} \right) A_d(k, r) \]

\[ = \int_0^\infty dk f_d(k, r) e^{itw(k)} + \int_0^\infty dk f_d(k, r) e^{-itw(k)} \]

\[ = I_1 + I_2 \]

(52)

\( G(k) = -\frac{\cosh(2\kappa(k))}{2\sqrt{k^2+m_{\text{out}}^2}} \) for an arbitrary gCC state, while \( G(k) = \frac{(m_0^2-m^2)}{4\sqrt{k^2+m_{\text{out}}^2}} \) for the ground state.

\( A_d(k, r) = k^{d-1/2}r^{-d-\nu} F_1 \left( \frac{d-\nu}{2}; \frac{1}{2}k^2r^2 \right) \) is the angular function defined before (34) and \( f_d(k, r) = \frac{1}{2} G(k) A_d(k, r) \).
We consider the gCC states defined by
\[ \kappa(k) = \sum_{i=1}^{\infty} \kappa_{2i-1} k^{2(i-1)} = \kappa_1 + \kappa_3 k^2 + \kappa_5 k^4 + \cdots \] (53)

The reason for choosing only even powers of \( k \) is motivated by only even powers appearing in eq.(10). Let \( k = k_0 \) be a stationary point of the integral \( I_1 \) (for which \( w''(k_0) > 0 \)) having the general form

\[
I_1(t) = \left. m^{-d-1}_{\text{out}} \int df(k) e^{i tw(k)} \right|_{t \to \text{it}} = m^{-d-1}_{\text{out}} \int df(k) e^{-tw(k)}
\]

\[
= m^{-d-1}_{\text{out}} e^{\frac{-tw_0}{\sqrt{t}}} \int \frac{dy}{\sqrt{t}} \left( f_0 + f_0' \frac{y}{\sqrt{t}} + \frac{1}{2} f_0'' \frac{y^2}{t} + \frac{1}{3!} f_0''' \frac{y^3}{t^{3/2}} + \cdots \right) \times
\]

\[
\exp \left( \frac{w_0''}{2!} y^2 + \frac{w_0'''}{3!} y^3 + \frac{w_0''''}{4!} y^4 + \cdots \right)
\]

\[
= m^{-d-1}_{\text{out}} e^{\frac{-2t}{\sqrt{t}}} \int \frac{dy}{\sqrt{t}} \left( f_0 + f_0' \frac{y}{\sqrt{t}} + \frac{1}{2} f_0'' \frac{y^2}{t} + \frac{1}{3!} f_0''' \frac{y^3}{t^{3/2}} + \cdots \right) \exp \left( \frac{y^2}{4} - \frac{1}{4} y^4 + \cdots \right)
\]

\[
= m^{-d-1}_{\text{out}} e^{\frac{-2t}{\sqrt{t}}} \int \frac{dy}{\sqrt{t}} \exp \left( -\frac{1}{2} y^2 \right)
\]

where we Wick rotate \( t \to \text{it} \) in the first line (when \( w'(k_0) < 0 \) which is the case in \( d_2 \), we rotate in the opposite direction i.e. \( t \to -\text{it} \)). We expand both \( f \) and \( w \) around \( k = k_0 \) and introduce the variable \( y = \sqrt{t}(k-k_0) \) in the second line. We are using compact notation to write \( f_0 = f(k_0) \), \( f_0' = f'(k = k_0) \), etc. and similarly for \( w \). In the second line plug in \( w(k) = \sqrt{k^2 + 1} \) and finally in the last line we have brought down all terms in the exponent except for \( y^2 \). The factor \( m^{-d-1}_{\text{out}} \) is pulled out to make rest of the integral dimensionless. So every quantity inside the integral is multiplied by appropriate power of \( m_{\text{out}} \) to make it dimensionless.

\[
t \to m_{\text{out}} t; \quad r \to m_{\text{out}} r; \quad k \to k/m_{\text{out}}; \quad m \to m/m_{\text{out}}.
\]

\( G(k) \) is an even function of \( k \), while \( A_d(k, r) \) is an even function only in odd dimensions but an odd function in even dimensions. This means for even \( d \), the integration is neccesarily to be done for \( k \) belonging to \([0, \infty)\). This has important consequences for us because the stationary point which is the solution of the equation \( w'(k_0) = 0 \) is exactly \( k_0 = 0 \). So for even \( d \), the odd powers of \( y \) will also contribute. Since the most contribution comes near \( k = 0 \), we look at the behaviour of \( f(k) \) near the origin

\[
f(k) = -\frac{\pi^{-\frac{d}{2}}}{2^{d+1} \Gamma \left( \frac{d}{2} \right)} \text{csch}(2\kappa_1) k^{d-1} + \frac{\text{csch}(2\kappa_1) (d_2 + r^2 + 4d \kappa_3 \coth(2\kappa_1))}{2^{d+3} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} + 1 \right)} k^{d+1} + O(k^{d+3})
\]

(55)

Therefore the leading contribution comes from the \((d-1)\)th derivative i.e. \( f_0^{(d-1)} \frac{y^{d-1}}{\pi^{\frac{d-1}{2}}} \) which together with the \( e^{-2t}/\sqrt{t} \) goes as \( t^{-d/2} \). More explicitly, keeping track upto the sub-leading term in \( I_1 \)

\[
\frac{I_1(t)}{m^{-d-1}_{\text{out}}} \approx -\frac{\text{csch}(2\kappa_1)}{2^{d+1} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} + 1 \right)} \int_0^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2 - \frac{1}{4}} \text{csch}(2\kappa_1) \left( d + r^2 + 4d \kappa_3 \coth(2\kappa_1) \right) \frac{e^{-2t}}{t^{1+d/2}} \int_0^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2 - \frac{1}{4}} y^{d+3}
\]

\[
= -\frac{\text{csch}(2\kappa_1)}{2^{d+2} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} + 1 \right)} \frac{1}{4} \int_0^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2 - \frac{1}{4}} y^{d+3}
\]

\[
= -\frac{\text{csch}(2\kappa_1)}{2^{d+2} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} + 1 \right)} \frac{1}{4} \int_0^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2 - \frac{1}{4}} y^{d+3}
\]

\[
= -\frac{\text{csch}(2\kappa_1)}{2^{d+2} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} + 1 \right)} \frac{1}{4} \int_0^{\infty} \frac{dy}{\sqrt{y}} e^{-y^2 - \frac{1}{4}} y^{d+3}
\]

\[
= -\frac{\pi^{-\frac{d}{2}}}{2^{d+1} \Gamma \left( \frac{d}{2} \right)} \text{csch}(2\kappa_1) (d + r^2 + 4d \kappa_3 \coth(2\kappa_1)) \frac{e^{-2t}}{t^{1+d/2}} \text{csch}(2\kappa_1) \frac{e^{-2t}}{t^{1+d/2}}
\]

(56)

The third term in the first line of (56) comes from the product of the leading \( k^{d-1} \) term in expansion of \( f \) and \( w^{(d)} y^2/t \) brought down from the exponential. In the second line we have performed the integral using the general formula \( \int_0^{\infty} dy y^4 e^{-y^2} = \frac{\Gamma \left( \frac{d+1}{2} \right)}{2^{d/2}} \) and finally we Wick rotate back to \( t \to -\text{it} \). \( I_2 \) is calculated in a similar manner with the only difference being that in the final answer replace \( i \to -i \) (due
The above expression tells us that the leading transient only depends on \( \kappa(k) = \kappa_1 + \kappa_3 k^2 + \kappa_5 k^4 + \cdots \). Here \( m_{\text{out}} \) is the final mass and \( r = |\vec{x}_1 - \vec{x}_2| \).

### Table 7: Time-dependent part of 2-point function \( \langle gCC|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|gCC \rangle \) in the gCC state defined by \( \kappa(k) = \kappa_1 + \kappa_3 k^2 + \kappa_5 k^4 + \cdots \).

| Dimension \( d \) | Expression |
|-------------------|------------|
| 1                 | \[-\frac{\text{csch}(2m_{\text{out}})\cos(2m_{\text{out}}t + \frac{\pi}{2})}{4\sqrt{\pi}} + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^{3/2}}\right)\] |
| 2                 | \[-\frac{m_{\text{out}}\text{csch}(2m_{\text{out}})}{8\pi}\cos(2m_{\text{out}}t + \frac{\pi}{2}) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^2}\right)\] |
| 3                 | \[-\frac{m_{\text{out}}^2\text{csch}(2m_{\text{out}}\kappa_1)}{16\pi^2}\cos(2m_{\text{out}}t + 3\frac{\pi}{2}) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^{3/2}}\right)\] |
| 4                 | \[-\frac{m_{\text{out}}^3\text{csch}(2m_{\text{out}}\kappa_1)}{32\pi^3}\cos(2m_{\text{out}}t + \pi) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^3}\right)\] |

Table 8: Time-dependent part of 2-point function \( \langle 0_{\text{in}}|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|0_{\text{in}} \rangle \) in massive quench. \( r = |\vec{x}_1 - \vec{x}_2| \).

| Dimension \( d \) | Expression |
|-------------------|------------|
| 1                 | \[\frac{(m_{\text{out}}^2 - m_{\text{in}}^2)}{\pi^2 m_{\text{out}}} \cos(2m_{\text{out}}t + \frac{\pi}{2}) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^{3/2}}\right)\] |
| 2                 | \[\frac{(m_{\text{out}}^2 - m_{\text{in}}^2)}{16\pi^2 m_{\text{out}}} \cos(2m_{\text{out}}t + \frac{3\pi}{2}) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^3}\right)\] |
| 3                 | \[\frac{(m_{\text{out}}^2 - m_{\text{in}}^2)^2 m_{\text{out}}}{32\pi^3 m_{\text{out}}} \cos(2m_{\text{out}}t + \pi) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^{3/2}}\right)\] |
| 4                 | \[\frac{(m_{\text{out}}^2 - m_{\text{in}}^2)^2 m_{\text{out}}}{64\pi^4 m_{\text{out}}} \cos(2m_{\text{out}}t + \pi) + \mathcal{O}\left(\frac{1}{(m_{\text{out}}t)^3}\right)\] |

The above expression tells us that the leading transient only depends on \( \kappa_1 \). It neither cares about the the higher conserved charges nor the separation \( r \) which only show at subleading order. We tabulate the leading approach to equilibrium of (27) in Table 7. We also tabulate the 2-point function in the ground state state (17) in Table 8.

### 3.3.2 \( \partial_t \phi \partial_t \phi \) and \( \partial_t \phi \partial_t \phi \) Correlators

We also calculate the time-dependent piece of the correlator \( \langle \partial_t \phi(\vec{x}_1, t_1)\partial_t \phi(\vec{x}_2, t_2) \rangle |_{t_1 = t_2 = t} \) eq. (18).

It is a bit surprising that the leading power of \( t \) here is also same as before, but it is easy to understand why. For a moment let us rewrite \( 2t = t_1 + t_2 \). Then take derivatives on \( \langle \phi \phi \rangle \) correlator (in 1+1 dim. for
Table 9: Time-dependent part of 2-point function $\langle 0_{m}\mid\partial_{t}\phi(x_{1}, t)\partial_{t}\phi(x_{2}, t)\mid 0_{m}\rangle$ on the left and $\langle 0_{m}\mid\partial_{i}\phi(x_{1}, t)\partial_{i}\phi(x_{2}, t)\mid 0_{m}\rangle$ on the right, both for massive quench. Here $r = |x_{1} - x_{2}|$.

\[
\langle \partial_{t}\phi \partial_{t}\phi \rangle_{td} = \begin{cases} 
\frac{m_{\text{out}}(m_{\text{out}}^{2} - m^{2})}{8\pi^{2}m} \cos\left(2m_{\text{out}}t + \frac{\pi}{4}\right) \frac{1}{\sqrt{m_{\text{out}}}t^{3/2}} + O\left(\frac{1}{(m_{\text{out}}t)^{5/2}}\right) \\
\frac{m_{\text{out}}(m_{\text{out}}^{2} - m^{2})}{16\pi m} \cos\left(2m_{\text{out}}t + \frac{3\pi}{4}\right) \frac{1}{(m_{\text{out}}t)^{3/2}} + O\left(\frac{1}{(m_{\text{out}}t)^{5/2}}\right) \\
\frac{m_{\text{out}}(m_{\text{out}}^{2} - m^{2})}{32\pi^{3/2}m} \cos\left(2m_{\text{out}}t + \frac{5\pi}{4}\right) \frac{1}{(m_{\text{out}}t)^{5/2}} + O\left(\frac{1}{(m_{\text{out}}t)^{7/2}}\right) \\
\frac{m_{\text{out}}(m_{\text{out}}^{2} - m^{2})}{64\pi^{3}m} \cos\left(2m_{\text{out}}t + \frac{7\pi}{4}\right) \frac{1}{(m_{\text{out}}t)^{7/2}} + O\left(\frac{1}{(m_{\text{out}}t)^{9/2}}\right)
\end{cases}
\]

3.4 Some comments on 2-point functions in critical vs. massive quench

From the results of the previous subsections it seems that the critical correlators behave in a qualitatively different fashion from the those for the massive case, especially in odd spatial dimensions. This seems to indicate an apparent discontinuity as $m_{\text{out}} \to 0$. This is surprising since the master formula (40) for the critical correlator was obtained from that of the massive case (17) by putting $m_{\text{out}} = 0$!

Actually there is no true discontinuity at $m_{\text{out}} = 0$ in the exact time-dependent correlators. The apparent discontinuity emerges because the limits $m_{\text{out}} \to 0$ and $t \to \infty$ do not commute. This can be understood by considering the scales involved in the theory. For any given mass $m_{\text{out}}$, however small, at very large times we will have $t \gg 1/m_{\text{out}}$ (see Fig. 3 (a)). In other words, the dimensionless quantity $\hat{t} = m_{\text{out}}t \gg 1$.

![Figure 3: Relevant scales in the theory for the present consideration. Here $r/t$ is considered fixed as the large $t$ limit is taken. In (a) $m_{\text{out}}$ is a finite non-zero quantity. $t \to \infty$ limit means $m_{\text{out}}t \to \infty$. This is the case we discuss in the analysis of large time behaviour for quench to non-zero mass. In (b) the limit $m_{\text{out}} \to 0$ limit is taken first, so that $m_{\text{out}}t \to 0$; the large $t$ limit in this case can be defined as $m_{\text{out}}t \to \infty$. We do not consider this limit in the paper except when $m_{\text{out}} = 0$ (this limit is discussed a bit more at the end of the subsection).](image-url)
However when \( m_{\text{out}} = 0 \), \( 1/m_{\text{out}} = \infty \). No \( t \), however large, can exceed \( 1/m_{\text{out}} \). Indeed, the limit \( m_{\text{out}} \to 0 \) automatically implies \( \tilde{t} = m_{\text{out}}t \to 0 \) (see Fig. 3 (b)).

More mathematically speaking, the two-point functions involve scaling functions of the form \( F(mt,m_{\text{out}}t) \) (omitting the dimensionless variable \( r/t \)). For the massive two-point functions, late times imply that both scaled variables are taken to infinity; in particular \( m_{\text{out}}t \to \infty \). Indeed, the limit \( m_{\text{out}}t \to 0 \) (see Fig. 3 (b)).

The apparent discontinuity at \( m_{\text{out}} = 0 \) is only an artifact of the large time asymptotic expansion.

4 The Generalized Gibbs Ensemble (GGE)

We have seen above that the time-dependent part of the two-point functions (the part containing the cosine term in (31)) vanishes at large times. The remaining term, therefore, represents the asymptotic value of the two-point function. We will now show that the asymptotic value corresponds to that of the two-point function in a generalized Gibbs ensemble (GGE). Before that, we will first recall some relevant facts about a GGE.

4.1 GGE

The generalized Gibbs ensemble (GGE) can be regarded as a certain grand canonical ensemble with an infinite number of chemical potentials \([2, 3, 8]\) which are appropriate for the description of equilibrium for an integrable model. An integrable model has an infinite number of commutative conserved charges \( \hat{Q}_i \). A GGE is described by a density matrix

\[
\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left[ -\sum_i \mu_i \hat{Q}_i \right], \quad Z_{\text{GGE}} = \text{Tr} \exp \left[ -\sum_i \mu_i \hat{Q}_i \right]
\]

Let us consider a free scalar field \( \phi(x,t) \), described by the standard mode expansion

\[
\phi(\vec{k},t) = a(\vec{k},t)u(\vec{k},t) + cc, \quad u(k,t) = \frac{e^{-i\omega(k)t}}{\sqrt{2\omega(k)}}, \quad \omega(k) = \sqrt{k^2 + m_0^2}
\]

and a Hamiltonian \( H = \sum_k \omega(k)\hat{N}(k) \), where \( \hat{N}(k) = a^\dagger(k)a(k) \) are the occupation numbers.

This system is clearly integrable; a standard basis of the commuting conserved charges is (cf. (7)) \( \hat{Q}_i = \sum_k |k|^{-1}\hat{N}(k), \quad i = 1, 2, \ldots \) Another simple basis of charges are the set of occupation numbers \( \hat{N}(k) \) themselves. With this choice the GGE is described by the density matrix

\[
\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left[ -\sum_k \mu(k) \hat{N}(k) \right], \quad Z_{\text{GGE}} = \text{Tr} \exp \left[ -\sum_k \mu(k) \hat{N}(k) \right]
\]

The GGE partition is easy to evaluate (most calculations in this subsection are presented in greater detail in Appendix D):

\[
Z_{\text{GGE}} = \sum_{\{N(k)\},N(k)=0,1,2,\ldots} \exp \left( -\sum_k \mu(k)N(k) \right) = \prod_k (1 - \exp[-\mu(k)])^{-1}
\]

\[\text{Note that in } d = 1 \text{ and 2, there are infrared divergences associated with this limit for } \langle \phi\phi \rangle \text{ correlators but these problems are not there in correlators of the type } \langle \partial\phi\partial\phi \rangle.\]
It is also easy to evaluate the average value of the number operator
\[ \langle \hat{N}(k) \rangle_{\text{GGE}} = \text{Tr}(\rho_{\text{GGE}} \hat{N}(k)) = -\frac{\partial}{\partial \mu(k)} \ln Z_{\text{GGE}} = \frac{1}{e^\mu(k) - 1} \quad (62) \]

With the above ingredients, we can now compute the equal time two-point function in the GGE (for more details and also the case of unequal time correlator, see Appendix D):
\[
\langle \phi(\vec{x}_1, t)\phi(\vec{x}_2, t) \rangle_{\text{GGE}} = \text{Tr}[\rho_{\text{GGE}} \phi(\vec{x}_1, t)\phi(\vec{x}_2, t)] = \int \frac{d^d k}{(2\pi)^d} |u(k, t)|^2 e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} [2\langle N(k) \rangle_{\text{GGE}} + 1] = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2\sqrt{k^2 + m_0^2}} e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \coth(\mu(k)/2) \quad (63)
\]

Note that for massless scalars, \( H = \sum_k |k| a^\dagger(k)a(k) = Q_2 \). Also the total occupation number is always \( N = \sum_k a^\dagger(k)a(k) = Q_1 \). Hence, we can write (60) as
\[
\rho_{\text{GGE}} = \frac{1}{Z_{\text{GGE}}} \exp \left[ -\mu N + \beta H + \sum_{i=3}^{\infty} \mu_i \hat{Q}_i \right], \quad Z_{\text{GGE}} = \text{Tr} \exp \left[ -\mu N + \beta H + \sum_{i=3}^{\infty} \mu_i \hat{Q}_i \right] \quad (64)
\]
Thus, a standard thermal (Gibbs) ensemble is a GGE with all \( \mu_i \neq 2 = 0 \).

### 4.2 Equilibration to GGE

Note that the two-point function in the most general gCC state (27) is of the form
\[
\langle gCC|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|gCC \rangle = \int \frac{d^d k}{(2\pi)^d} e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \left[ \coth(2\kappa(k)) + \text{time dependent} \right] \quad (65)
\]
In the preceding sections we have shown that the time-dependent part decays at late times exponentially or by a power law. Hence the gCC two-point function (65) asymptotically approaches the GGE two-point function (63) provided we identify
\[
m_0 = m_{\text{out}}, \quad \mu(k) = 4\kappa(k), \quad \text{equivalently} \quad \mu_i = 4\kappa_i \quad (66)
\]
Indeed, these relations ensure
\[
\langle N(k) \rangle_{\text{GGE}} = \langle gCC|N(k)|gCC \rangle \quad (67)
\]
To prove this, note that by identifying \(|gCC \rangle \) with \(|f_{in} \rangle \) and using the top line of (20), we get that the right hand side of the above equation is given by \((1/\gamma_{\text{eff}}^2 - 1)^{-1}. \) By using \( \gamma_{\text{eff}}(k) = \exp[-2\kappa(k)] \) (see immediately above (22)), the RHS becomes \((e^{2\kappa(k)} - 1)^{-1} \) which agrees with equation ((62)) with the identification (66). Q.E.D.

Thus, we have established the following statement of equilibration:
\[
\langle gCC|\phi(\vec{x}_1, t)\phi(\vec{x}_2, t)|gCC \rangle = \langle \phi(\vec{x}_1, t)\phi(\vec{x}_2, t) \rangle_{\text{GGE}} + \left( \sim e^{-\gamma t} \text{ or } \sim 1/t^\beta \right) \quad (68)
\]
where the GGE is defined in terms of the gCC by the basic relation (67). Since for a free field theory, all correlation functions can be essentially built from the two-point function, the above statement of equilibration is true for all correlators. This constitutes a proof of the GGE version of quantum ergodic hypothesis in the quench models considered in this paper:
\[
\lim_{t \rightarrow \infty} \langle \psi(t)|O(x_1)O(x_2)O(x_3)\ldots|\psi(t) \rangle \quad (69)
\]
Eq. (66) gives us the important relation between the parameters of the equilibrium ensemble (GGE) with those of the initial state. In the massless case, in terms of the notation of (64), we can rewrite (66) as
\[
\beta = 4\kappa_2; \quad \mu_i = 4\kappa_i, i \neq 2 \quad (69)
\]
It is clear that the CC state (25) equilibrates to a standard Gibbs ensemble with \( \mu_i \neq 2 \neq 0 \).
5 Geometrical interpretation of the correlators in the CC state

Here we follow [20] to show how the two-point function in a CC state can be identified with an image sum in a slab geometry $R^d \times$ interval. This would help us better understand the origin of the difference in thermalization between odd and even dimensions (short, the odd-even effect). The slab propagator is defined as

$$G_{\text{slab}}(\tau_1, \vec{x}_1; \tau_2, \vec{x}_2) = \langle \phi(\tau_1, \vec{x}_1) \phi(\tau_2, \vec{x}_2) \rangle_{\text{slab}}$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} G_{\text{slab}}(\tau_1, \tau_2; \vec{k})$$

$$= \int \frac{d^d k}{(2\pi)^d} e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \left( S_{\text{th}}(\tau_1 - \tau_2) + S_{\text{diff}}(\tau_1 + \tau_2) \right)$$

where

$$S_{\text{th}}(\tau) = \sum_{n=0}^{\infty} e^{-|\vec{k}|(2nL+\tau)} + \sum_{n=1}^{\infty} e^{-|\vec{k}|(2nL-\tau)}$$

$$S_{\text{diff}}(\tau') = -\sum_{n=0}^{\infty} e^{-|\vec{k}|((2n+1)L-\tau')} - \sum_{n=1}^{\infty} e^{-|\vec{k}|((2n-1)L-\tau')}$$

The expression (70) is obtained by the method of images as shown in the diagram below (see figure 4). The geometry reflects the fact that a CC state of the form $e^{-\kappa_2 H} |Bd\rangle$ can be regarded as a Euclidean time evolution $\Delta \tau = \kappa_2$, here denoted by $L/2$. The suffices $\text{th}$ and $\text{diff}$ will be explained below.

![Diagram](image.png)

Figure 4: The two-point function between $\phi(\tau_1)$ and $\phi(\tau_2)$ can be regarded as the electrostatic potential between two positive charges of unit magnitude [20]. This follows from the fact that the massless Klein-Gordon equation in Euclidean space is simply the Laplace equation. Here we consider $\phi(\tau_1)$ as a source charge whose electrostatic potential in the slab geometry can be found out by the method of images by regarding the slab as two parallel mirrors. In the right panel we depict by green(red) circles the images obtained by an odd(even) number of reflections of the source. The electrostatic potential in the slab can now be regarded as a sum of that between the probe charge placed at $\tau_2$ and each of the images.

It is easy to verify that (70) reproduces the two-point function in a CC state. All the four terms in (70) can be easily summed to obtain

$$S_{\text{th}}(\tau_1 - \tau_2) + S_{\text{diff}}(\tau_1 + \tau_2) = \frac{1}{1 - e^{-2|\vec{k}|L}} \left[ e^{-|\vec{k}|\tau_1 - \tau_2} + e^{-|\vec{k}|(2L - \tau_1 - \tau_2)} - 2e^{-|\vec{k}|L} \cosh\left( |\vec{k}|(\tau_1 + \tau_2) \right) \right]$$

Now analytically continuing to real time by $\tau \to it$ and identifying $\kappa_2 = L/2$ we find the slab propagator is the same as (26) with $\kappa_2 = L/2$ and all other $\kappa_i = 0$

$$G_{\text{slab}}(t_1, \vec{x}_1; t_2, \vec{x}_2) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i \vec{k} \cdot (\vec{x}_1 - \vec{x}_2)}}{2|\vec{k}|} \csc\left( 2\kappa_2 |\vec{k}| \right) \left[ \cos\left( |\vec{k}|(2it_2 + t_1 - t_2) \right) - \cos\left( |\vec{k}|(t_1 + t_2) \right) \right]$$

This completes the explicit check that the slab propagator (70) indeed represents the two-point function in the CC state.
5.1 Comments on approach to thermalization and the odd-even effect

The understanding of thermalization from the above geometrical picture goes as follows. First, note that the terms in \( S_h \) correspond to Euclidean \( R^{d+1} \) 2-point function between the test charge with the ‘+’ images above and below it. These are precisely the terms which would appear in a cylindrical geometry, which represents a thermal 2-point function with Euclidean time period \( \beta = 2L_2 = 4\kappa_2 \); hence the suffix \( t_h \). The terms in \( S_{\text{diff}} \), on the other hand, correspond to \( R^{d+1} \) 2-point functions between the test charge with the ‘-’ images above and below it; below we will find that these terms vanish at long times, proving that the CC two-point function asymptotically approaches the thermal 2-point function \( S_h, \beta = 4\kappa_2 \), which verifies the relation (69).

It is clear from the above that \( S_{\text{diff}} \) represents the difference between CC correlator and the thermal correlator. For thermalization of the auto-correlator \( \langle \vec{x} = \vec{y} = \vec{r}, t_1 \neq 0 \rangle \), we need to consider the following difference

\[
G_{CC}(\tau_1, \vec{r}; \tau_2, \vec{r}) - G_{th}(\tau_1, \vec{r}; \tau_2, \vec{r}) = \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dk k^{d-2} S_{\text{diff}}(\tau_1 + \tau_2)
\]

\[
= -\frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dk k^{d-2} \left( \sum_{n=0}^\infty e^{-k((2n+1)L+\tau)} + \sum_{n=1}^\infty e^{-k((2n-1)L-\tau)} \right)
\]

\[
= -\frac{\Omega_{d-1}(d-2)!}{2(2\pi)^d} \left[ \frac{1}{L+\tau} \right]^{d-1} + \sum_{n=1}^\infty \left( \frac{1}{(2n+1)L+\tau} \right)^{d-1} + \sum_{n=1}^\infty \left( \frac{1}{(2n-1)L-\tau} \right)^{d-1} \right] \quad (73)
\]

Where \( \tau = \tau_1 + \tau_2 \). In the last line we have performed the \( k \) integral. Note that since \( \tau_1 \) and \( \tau_2 \) lie in the interval \([-L/2, L/2] \), allowed values of \( \tau \) are in the interval \([-L, L] \).

To see the difference between odd and even \( d \), let us choose particular values of \( d \).

\( d = 4 \): We have

\[
G_{CC}(\tau_1, \vec{r}; \tau_2, \vec{r}) - G_{th}(\tau_1, \vec{r}; \tau_2, \vec{r}) = -\frac{1}{8\pi^2} \left[ \frac{1}{L+\tau} \right]^{3} + \sum_{n=1}^\infty \left( \frac{1}{(2n+1)L+\tau} \right)^{3} + \sum_{n=1}^\infty \left( \frac{1}{(2n-1)L-\tau} \right)^{3} \right] \quad (74)
\]

Now we use the Euler-Maclaurin sum formula for an infinitely differentiable function

\[
\sum_{n=1}^\infty f(n) = \int_0^\infty dy f(y) + \frac{f(\infty) - f(0)}{2} + \sum_{k=1}^\infty B_{2k}/(2k)! \left( f^{(2k-1)}(\infty) - f^{(2k-1)}(0) \right)
\]

and apply it to the 2 sums above to get

\[
S_+ = \sum_{n=1}^\infty \left( \frac{1}{(2n+1)L+\tau} \right)^{3} = \frac{1}{4L(L+\tau)^3} - \frac{1}{2(L+\tau)^2} + \frac{L}{2(L+\tau)^4} + O((L+\tau)^{-6})
\]

\[
S_- = \sum_{n=1}^\infty \left( \frac{1}{(2n-1)L-\tau} \right)^{3} = \frac{1}{4L(L+\tau)^3} + \frac{1}{2(L+\tau)^2} + \frac{L}{2(L+\tau)^4} + O((L+\tau)^{-6})
\]

and therefore

\[
G_{CC}(\tau_1, \vec{r}; \tau_2, \vec{r}) - G_{th}(\tau_1, \vec{r}; \tau_2, \vec{r}) = -\frac{1}{8\pi^2} \left( \frac{1}{2L(L+\tau)^2} + \frac{1}{(L+\tau)^3} + \frac{L}{(L+\tau)^4} \right) + O((L+\tau)^{-6}) \quad (77)
\]

Analytically continuing to real time \( (\tau_1 \to it \text{ and } \tau_2 \to it) \), for \( t \to \infty^{15} \) we find (after putting \( L = 2\kappa_2 \))

\[
G_{CC}(t, \vec{r}; t, \vec{r}) - G_{th}(t, \vec{r}; t, \vec{r}) = \frac{1}{128\pi^2 \kappa_2 t^2} + \frac{\kappa_2}{128\pi^4 t^4} + O(t^{-6}) \quad (78)
\]

\( ^{15} \)Note that although \( |\tau| \leq L \), there is no such restriction on \( t \) which can be taken to infinity.
This gives us the power law decay in even dimensions. Also this is precisely the expression for the 2-point function of $\phi$ in the CC state ($\kappa_2n = 0, n > 1$) for $r = 0$. We match the Euler-Maclaurin result with the method used before to calculate eq. (49) to $O(t^{-20})$ and find exact agreement. Notice that there is a coincident singularity because of $t_1 = t_2$, but that would show up in ‘thermal’ part $G_{th}$, so we do not need to worry about it here. Now lets see what happens in odd dimensions.

$d = 3$: Here we have

$$G_{CC}(\tau_1, \vec{r}_1; \tau_2, \vec{r}) - G_{th}(\tau_1, \vec{r}_1; \tau_2, \vec{r}) = -\frac{1}{8\pi^2} \left( \frac{1}{L + \tau} + \sum_{n=1}^{\infty} \left( \frac{1}{(2n + 1)L + \tau} - \frac{1}{(2n - 1)L - \tau} \right)^2 \right)$$

The Euler-Maclaurin sum formula applied here results in

$$S_+ = \sum_{n=1}^{\infty} \left( \frac{1}{(2n + 1)L + \tau} \right)^2 - \frac{1}{2L(L + \tau)} - \frac{1}{2(L + \tau)} + \frac{L}{3(L + \tau)^3} + O((L + \tau)^{-5})$$

$$S_- = \sum_{n=1}^{\infty} \left( \frac{1}{(2n - 1)L - \tau} \right)^2 - \frac{1}{2L(L - \tau)} - \frac{1}{2(L - \tau)} - \frac{L}{3(L - \tau)^3} + O((L - \tau)^{-5})$$

Together with the $n = 0$ term

$$G_{CC}(\tau_1, \vec{r}_1; \tau_2, \vec{r}) - G_{th}(\tau_1, \vec{r}_1; \tau_2, \vec{r}) = -\frac{1}{8\pi^2} \left( \frac{1}{(L + \tau)^2} + S_+ + S_- \right) = 0$$

We expect the vanishing result found above to continue to all orders in the $1/t$ expansion (we have explicitly checked this fact till $O(t^{-20})$. Indeed this had to be the case because we know the answer goes as $e^{-\pi t/\kappa_2}$ (see (103)) which does not have a Taylor expansion in $1/t$ around 0.

It is not difficult to generalize the above results to arbitrary dimensions $d > 2$ for example by using the recursion relations (see Appendix B). The summary is that

$$G_{CC}(t_1, \vec{r}; t_2, \vec{r}) - G_{th}(t_1, \vec{r}; t_2, \vec{r}) \sim \begin{cases} t^{-(d-2)} & \text{d is even} \\ e^{-\pi t/\kappa_2} & \text{d is odd} \end{cases}$$

There is a pictorial way to understand the decay of the time-dependent part of the CC correlator. In the slab diagram imagine the analytic continuation of time ($\tau_j \rightarrow \tau_j + it_j$) with real time axis perpendicular to the plane of the paper (see figure 5). The spatial dimensions are all in the horizontal direction. Note that upon analytic continuation to real time, the ‘even’ and ‘odd’ images move in opposite directions. Also notice that $S_{th}$ involves the correlator between the probe (black) and the even images of the source (marked red); the way to see this is to note that such correlators all involve the time difference $t_1 - t_2$ as expected from (70). Similarly while $S_{diff}$ involves the correlator between the probe (black) and the odd images of the source (marked green); the way to see this is to note that such correlators all involve the time difference $t_1 + t_2$ again as expected from (70). When $t_1$ and $t_2$ are increased by the same amount, the Euclidean separations involved in $S_{th}$ remain the same whereas they go to infinity in case of $S_{diff}$. This is the reason why $S_{diff}$ decays in time in this limit and the slab correlator approaches the thermal value.\(^{16}\)

In the above we have considered separation between the source and the probe only in time but the entire argument goes through for fixed, finite spatial separation $r$ as we take $t \rightarrow \infty$.

### 5.2 The thermal auto-correlator

In the previous subsection we showed how the difference between the CC correlator and the thermal auto-correlator vanishes as $t_1, t_2 \rightarrow \infty$. In this section we will compute the thermal auto-correlator itself. We

\(^{16}\)We thank Abhijit Gadde for suggesting this.
Applying the Euler-Maclaurin formula
\[ d = 4 \]

Figure 5: In the above diagram we consider \( \phi(\tau_1) \) as the source and \( \phi(\tau_2) \) as a probe. As we analytically continue \( \tau \rightarrow \tau_1 + it \), and \( t_1 \) and \( t_2 \) are increased by the same amount, the red images (even reflections of the source) move in the same direction as the probe, while the green images (odd reflections) move in the opposite direction. For large \( t_1 \) and \( t_2 \) with \( t_1 - t_2 \) held fixed, the probe \( \phi(\tau_2 + it_2) \) sees the green images at progressively farther distances and hence their correlators decay. Thus in this limit the slab correlator approaches the thermal correlator.

will find that this too shows an odd-even difference between dimensions as one takes the limit of large \( t \equiv t_1 - t_2 \).

The quantity to compute is

\[
G_{th}(\tau_1 - \tau_2 = \tau; r = 0) = \frac{\Omega_{d-1}}{2(2\pi)^d} \int_0^\infty dk k^{d-2} \left[ \sum_{n=0}^\infty e^{-k(n\beta + \tau)} + \sum_{n=1}^\infty e^{-k(n\beta - \tau)} \right]
\]

\[
= \frac{\Omega_{d-1}(d-2)!}{2(2\pi)^d} \left[ \sum_{n=0}^\infty \left( \frac{1}{n\beta + \tau} \right)^{d-1} + \sum_{n=1}^\infty \left( \frac{1}{n\beta - \tau} \right)^{d-1} \right] 
\]

(83)

Note that here \( \tau = \tau_1 + \tau_2 \). As in the previous subsection lets analyze this in \( d = 4 \) and \( d = 3 \) separately.

\[ d = 4 \]

Here

\[
G_{th}(\tau_1 - \tau_2 = \tau; r = 0) = \frac{1}{8\pi^2} \left[ \frac{1}{\tau^3} + \sum_{n=1}^\infty \left( \frac{1}{n\beta + \tau} \right)^3 + \sum_{n=1}^\infty \left( \frac{1}{n\beta - \tau} \right)^3 \right] 
\]

(84)

Applying the Euler-Maclaurin formula

\[
S_+ = \sum_{n=1}^\infty \left( \frac{1}{n\beta + \tau} \right)^3 = \frac{1}{2\beta \tau^2} - \frac{1}{2\tau^3} + \frac{\beta}{4\tau^4} + O(\tau^{-6})
\]

(85)

\[
S_- = \sum_{n=1}^\infty \left( \frac{1}{n\beta - \tau} \right)^3 = \frac{1}{2\beta \tau^2} + \frac{1}{2\tau^3} + \frac{\beta}{4\tau^4} + O(\tau^{-6})
\]

After analytically continuing \( \tau = \tau_1 + \tau_2 \rightarrow it_1 + it_2 = it \), together with the \( n = 0 \) term, for large \( t \)

\[
G_{th}(\tau_1 - \tau_2 = \tau; r = 0) = \frac{i}{8\pi^2 t^3} - \frac{1}{8\pi^2 \beta t^3} + \frac{\beta}{16\pi^2 t^4} + O(t^{-6})
\]

(86)

This matches the precisely the answer (117), obtained from the integral formula (120).
\( d = 3 \): Here

\[
G_{th}(\tau_1 - \tau_2 = \tau; r = 0)_{\beta} = \frac{1}{8\pi^2} \left[ \frac{1}{\tau^2} + \sum_{n=1}^{\infty} \left( \frac{1}{n\beta + \tau} \right)^2 + \sum_{n=1}^{\infty} \left( \frac{1}{n\beta - \tau} \right)^2 \right]
\]

Applying the Euler-Maclaurin formula

\[
S_+ = \sum_{n=1}^{\infty} \left( \frac{1}{n\beta + \tau} \right)^2 = \frac{1}{\beta^2} - \frac{1}{2\tau^2} + \frac{\beta}{6\tau^3} + \mathcal{O}(\tau^{-5})
\]

\[
S_- = \sum_{n=1}^{\infty} \left( \frac{1}{n\beta - \tau} \right)^2 = -\frac{1}{\beta^2} - \frac{1}{2\tau^2} - \frac{\beta}{6\tau^3} + \mathcal{O}(\tau^{-5})
\]

After analytically continuing \( \tau = \tau_1 + \tau_2 \to it_1 + it_2 = it \), for large \( t \), together with the \( n = 0 \) term

\[
G_{th}(\tau_1 - \tau_2 = \tau; r = 0)_{\beta} = \frac{1}{8\pi^2} \left( \frac{1}{\tau^2} + S_+ + S_- \right) = 0 \text{ up to } \mathcal{O}(t^{-5})
\]

In fact, it turns out that the Taylor coefficients vanish to all orders. This is consistent with the behaviour at \( d = 3 \), \( G_{th} \sim e^{-\gamma t} \) which is independently obtained in Appendix C.2 see equation (111). Note that \( e^{-\gamma t} \) has an essential singularity at \( 1/t = 0 \) and hence does not have a Taylor expansion in terms of \( 1/t \).

### 6 Kaluza-Klein Interpretation of the thermal correlators

In the previous section we considered the time-dependence of thermal correlators, that is, thermal two-point functions with no spatial separation. In this section, we will look at thermal 2-point function at space-like separation. A convenient way to look at this is the Kaluza-Klein reduction. Demanding periodicity in Euclidean time, the normal mode expansion of a scalar field in \( R^d \times S^1 \) is given by

\[
\phi(\tau, \vec{r}) = \sum_n \phi_n(\vec{r}) e^{i 2\pi n \tau / \beta}
\]

In case the post-quench theory is critical, \( \phi \) obeys massless Klein-Gordon equation in \( d + 1 \) dimensions \( \partial^\mu \partial_\mu \phi(\tau, \vec{r}) = 0 \). In terms of the modes one gets

\[
(\partial^\mu \partial_\mu - \frac{4\pi^2 n^2}{\beta^2}) \phi_n(\vec{r}) = 0
\]

which is again a Klein-Gordon equation in one less dimension but now the fields \( \phi_n(\vec{r}) \) acquire a (thermal) mass \( m_n = \frac{4\pi^2 n^2}{\beta^2} \). The equal-time thermal 2-point function in terms of the modes is

\[
\langle \phi(0, \vec{r})\phi(0, \vec{0}) \rangle_{\beta} = \frac{1}{\beta} \sum_n \langle \phi_n(\vec{r})\phi_n(\vec{0}) \rangle_{\beta}
\]

\[
= \frac{1}{\beta} \sum_n \int \frac{d^dk}{(2\pi)^d} e^{ikr \cos \theta} e^{-\frac{m_n^2}{\beta}}
\]

This can be evaluated exactly

\[
\langle \phi(0, \vec{r})\phi(0, \vec{0}) \rangle_{\beta} = \frac{a_d}{r^{d-2}} + 2a_d \sum_{n=1}^{\infty} \frac{e^{-2\pi n\beta/\beta}}{r^{d-2}}
\]

where \( a_d \) is a dimension-dependent constant. In \( d = 2 \), \( r^{2-d} \) is replaced by \( \log r \). In all dimensions we get a Yukawa like exponential decay for the non-zero modes. But in the \( r \to \infty \) limit or equivalently the \( \beta \to 0 \) limit, only the power law term, corresponding to the \( n = 0 \) survives. This happens in all dimensions even or odd. This is expected because in the strict \( \beta \to 0 \), the circumference of the cylinder shrinks so that the insertions of the operators do not see the cylinder. Note that the surviving power law term corresponds to the Green's function for the dimensionally reduced Euclidean geometry \( \tilde{R}^d \).

Note the appearance of a power law behaviour at all dimensions in contrast with the thermal auto-correlator in the previous section, which has an exponential/power law behaviour depending on whether the spatial dimension \( d \) is odd/even.
7 Discussion

In this paper, we presented a detailed investigation of thermalization of local correlators in free scalar field theories. The main results, including a general proof of thermalization to GGE, and the difference in approach to thermalization depending on odd or even dimensions, are already summarised in the introduction. Here we will briefly mention some further questions and puzzles:

- In this paper we have considered mass quench of free scalar field theories. It should be straightforward to generalize to other quenches involving free scalars, such as a quench in an external confining potential, or in an external electromagnetic field. One expects that the post-quench state will be again given by a Bogoliubov transform of the out-vacuum, and hence will correspond to a generalized Calabrese-Cardy state. It would be interesting to see whether thermalization to GGE happens in all these cases, and whether the approach to thermalization follows similar patterns as in this paper. It would also be interesting to study theories of free fermions; once again the post-quench state should be describable in terms of a Bogoliubov transformation and the techniques of the present paper should be applicable.

- Along similar lines, it would be interesting to see if our results are true in more general integrable models. For transverse field Ising models, which can be reduced to free fermions (possibly with mass), our results are known to be true (see, e.g. [8]). It would be interesting to see if our results are true for integrable models without the explicit use of free field techniques.

- It is important to investigate whether the results obtained above generalize to interacting theories, away from integrability. As shown in [20], quantum quench in large $N$ $O(N)$ models can be reduced to a mass quench at long times in terms of an effective mass parameter. Using this result and following the techniques of the present paper, one can try to study thermalization in these models. Work along these lines is in progress [21].

- One of our results, that the leading transient towards thermalization is exponential or power law depending on, respectively, odd or even dimensionality of space, is strongly reminiscent of the behaviour of retarded propagators for massless particles, which have support only on the forward light cone in odd spatial dimensions, and inside the forward light cone in even spatial dimensions. The ‘leaking of the propagator’ inside the light cone in even space dimensions has been termed a violation of the strong Huygen’s principle (see, e.g. [22, 23] for some physical realization of this phenomenon), since Huygen’s construction of a propagating light front is based on propagation along the light cone. We found that there are mathematical similarities between this peculiarity of the retarded propagator and the odd-even effect described in this paper (both are due to difference in the integration over angles in odd vs even dimensions), although we were not able to derive one as a consequence of the other.\(^{17}\)

- The power law approach to thermalization does not appear to have a holographic counterpart; e.g. quasinormal decay of black hole perturbations has an exponential form. In the context of an $O(N)$ model in 2+1 dimensions, the above discussion would seem to suggest a power law decay even in the interacting model. It would be interesting to explore the holographic dual in this context.

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A Dirichlet boundary state and relation to post-quench state

A Dirichlet boundary state is defined by

\[ \phi(x)|D\rangle = 0 \]

In terms of the mode expansion (at \( t = 0 \))

\[ \phi(x)|D\rangle = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} \phi(\vec{k})|D\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{i\vec{k} \cdot \vec{x}}}{\sqrt{2\omega_k}} \left( a(\vec{k}) + a^\dagger(-\vec{k}) \right) |D\rangle \]

(91)

Clearly this is zero if \( \forall \vec{k} \) the following equation is satisfied

\[ \left[ a(\vec{k}) + a^\dagger(-\vec{k}) \right] |D\rangle = \left[ \frac{\partial}{\partial a^\dagger(\vec{k})} + a^\dagger(-\vec{k}) \right] |D\rangle = 0 \]

(92)

One can check that

\[ |D\rangle = \exp \left[ -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} a^\dagger(\vec{k})a^\dagger(-\vec{k}) \right] |0_{\text{out}}\rangle \]

(93)

easily satisfies the above equation. One can write the in ground state as (through a Bogoliubov transformation)

\[ |0_{\text{in}}\rangle = \exp \left[ \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \gamma(k) a^\dagger_{\text{out}}(\vec{k})a^\dagger_{\text{out}}(-\vec{k}) \right] |0_{\text{out}}\rangle \]

As we shown in [1] one can write

\[ |0_{\text{in}}\rangle = \exp \left[ -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \kappa(k) a^\dagger_{\text{out}}(\vec{k})a^\dagger_{\text{out}}(-\vec{k}) \right] |D\rangle \]

where \( \kappa(k) \) is related to \( \gamma(k) \) through equation (5).

B Recursion Relation

Here we derive the recursive differential operator for the for the most general gCC correlator. Starting from the 2 point function of \( \phi \) fields in \( d + 1 \) dimensions (\( \omega_{\text{out}} = \sqrt{k^2 + m_{\text{out}}^2} \))

\[ \langle \phi(x_1,t_1)\phi(x_2,t_2) \rangle^{(d)}_{\text{gCC}} = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot (\vec{x}_1 - \vec{x}_2)} \frac{\text{csch}(2\kappa(k)) \left[ \cos \left( \omega_{\text{out}}(t_1 - t_2) + 2i\kappa(k) \right) - \cos \left( \omega_{\text{out}}(t_1 + t_2) \right) \right]}{2\omega_{\text{out}}} \]

(94)

Here \( t_- = t_1 - t_2 \), \( t_+ = t_1 + t_2 \) and \( \kappa(k) = \sum_n \kappa_{2n} |\vec{k}|^{2n-1} \) represents the most general gCC state (see eq. (22)). We now connect this to the correlator in \( (d - 2) + 1 \) dimensions. One can write

\[ \langle \phi(x_1,t_1)\phi(x_2,t_2) \rangle^{(d)}_{\text{gCC}} = \frac{\Omega_{d-2}}{2(2\pi)^d} \int_0^\infty dk \frac{k^{d-3}}{\omega_{\text{out}}} \int_0^\pi d\theta e^{ikr \cos \theta} \left( \sin \theta \right)^{d-2} \left[ \cos \left( 2i\kappa(k) + \omega_{\text{out}} t_- \right) - \cos \left( \omega_{\text{out}} t_+ \right) \right] \]

(94)

\[ = \frac{\Omega_{d-2}}{2(2\pi)^d} \int_0^\infty \int_0^\pi d\theta e^{ikr \cos \theta} \left( \sin \theta \right)^{d-2} \left[ \cos \left( 2i\kappa(k) + \omega_{\text{out}} t_- \right) - \cos \left( \omega_{\text{out}} t_+ \right) \right] \times k^2 \sin^2 \theta \]

\[ = \frac{\Omega_{d-2}}{2(2\pi)^d} \int_0^\infty \int_0^\pi d\theta e^{ikr \cos \theta} \left( \sin \theta \right)^{d-4} \left[ \cos \left( 2i\kappa(k) + \omega_{\text{out}} t_- \right) - \cos \left( \omega_{\text{out}} t_+ \right) \right] \times \left( \omega_{\text{out}}^2 - m_{\text{out}}^2 - k^2 \cos^2 \theta \right) \]

\[ = \frac{\Omega_{d-2}}{2(2\pi)^d} \left( -\partial^2_{t_-} - \partial^2_{t_+} + m_{\text{out}}^2 + \partial^2_r \right) \int_0^\infty \int_0^\pi d\theta e^{ikr \cos \theta} \left( \sin \theta \right)^{d-4} \left[ \cos \left( 2i\kappa(k) + \omega_{\text{out}} t_- \right) - \cos \left( \omega_{\text{out}} t_+ \right) \right] \]
\[ \frac{\Omega_{d-2}}{4\pi^2\Omega_{d-4}} \left( -\partial^2_{t_+} - \partial^2_{t_-} + m^2_{out} + \partial^2_{r} \right) \frac{\Omega_{d-4}}{2(2\pi)^{d-2}} \int_0^\infty dk \frac{k^{d-3}}{\omega_{out}^2} \int_0\pi d\theta e^{ikr \cos \theta} (\sin \theta)^{d-4} \times [\cos(2i\kappa(k) + \omega_{out}t_-) - \cos(\omega_{out}t_+)] \]

\[ = \frac{\Omega_{d-2}}{4\pi^2\Omega_{d-4}} \left( -\partial^2_{t_-} - \partial^2_{t_+} + m^2_{out} + \partial^2_{r} \right) (\phi(x_1, t_1)\phi(x_2, t_2))^{(d-2)}_{\text{gCC}} \]  

\[ \Omega_{d-1} \] is the solid angle of a \( d - 1 \) dimensional spherical surface in \( d \) spatial dimensions. In the third line we have used \( k^2 = \omega_{out}^2 - m_{out}^2 \). The main point being used here is the fact that one can obtain the extra \( k^2 \sin(\theta)^2 \) by acting with the derivatives. Also notice that \( t_- \) and \( t_+ \) act as independent variables. This derivation also holds for the ground state quench as can be verified directly from equation (16).

## C Details of critical quench calculations

### C.1 2+1 dimensions

The details of calculation of correlators in \( d = 2 \) are presented here.

#### C.1.1 CC state

The time-dependent piece of the 2-point function of \( \partial_\phi \) in arbitrary gCC state is

\[ \langle \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) \rangle_{\text{td}} = \int_0^\infty \int_0^\infty \frac{d^dk}{2(2\pi)^d} e^{ik(x_1 - x_2)} \sqrt{k^2 + m^2_{out}} \cos \left( 2t \sqrt{k^2 + m^2_{out}} \right) \text{csch}(2\kappa(k)) \]  

which in 2+1 for critical quench is

\[ \langle \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) \rangle_{\text{td}} = \frac{1}{4\pi} \int_0^\infty dke^{ikr}J_0(kr) \cos(2kt) \text{csch}(2\kappa(k)) \]

\[ = \frac{1}{4\pi} \int_0^\pi d\phi \int_0^\infty dke^{ikr}k^2 \cos(kr \sin \phi) \cos(2kt) \text{csch}(2\kappa(k)) \]

\[ = \frac{1}{16\pi^2} \int_0^\pi d\phi \int_0^\infty dke^{ikr}k^2 \text{csch}(2\kappa(k)) \left( e^{ikap} + e^{-ikam} + e^{-ikap} + e^{-ikam} \right) \]

where we have exponentiated the cosine in terms of \( \kappa \) and \( \kappa_2 \), and \( \kappa(k) = \kappa_2k \). These integrals are easily performed in Mathematica in terms of the Polygamma function

\[ \int_0^\infty dke^{ikr}k^2 \text{csch}(2\kappa_2k)e^{ikap} = -\frac{\psi^{(2)}\left( \frac{1}{2} - \frac{iap}{4\kappa_2} \right)}{512\pi^2\kappa_2^2} \]

After performing the \( k \) integral, we expand the polygammas for large \( t \), and then perform the \( \phi \) integral to get

\[ \langle \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) \rangle_{\text{td}} = -\frac{1}{32\pi\kappa_2^2} \frac{8\kappa_2^2 + 3r^2}{256\pi\kappa_2^4} + \mathcal{O}(t^{-6}) \]  

#### C.1.2 Ground state

The time-dependent piece of 2-point function of \( \partial_\phi \) in the ground state is (18)

\[ \langle 0_{in} | \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) | 0_{in} \rangle_{\text{td}} = -\int \frac{d^dk}{(2\pi)^d} \frac{m^2_{out} - m^2}{4\sqrt{k^2 + m^2}} e^{ik(x_1 - x_2)} \cos \left( 2t \sqrt{k^2 + m^2_{out}} \right) \]
which in 2+1 d for critical quench is

\[
\langle 0_{in} | \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) | 0_{in} \rangle_{td} = \int_0^\infty dk \frac{k J_0(kr)}{8\pi \sqrt{k^2 + m^2}} m^2 \cos(2kt)
\]

\[
= \frac{m^2}{8\pi^2} \int_0^\pi d\phi \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} \cos(2kt) \cos(kr \sin \phi)
\]

\[
= \frac{m^2}{32\pi^2} \int_0^\pi d\phi \int_0^\infty dk \frac{k}{\sqrt{k^2 + m^2}} \left( e^{ikap} + e^{-ikap} + e^{ikam} + e^{-ikam} \right)
\]

(99)

Naively doing the integral, we find that \( \int_0^\infty \frac{m^2 k e^{ikr \phi}}{32\pi^2 \sqrt{k^2 + m^2}} \) diverges because as \( k \to \infty \), the integrand is just \( e^{ikr} \). We regulate this by \( k \to k(1 \pm i\epsilon) \), performing the \( k \) integral (alternatively Laplace transform with respect to \( \epsilon \)) and then taking \( \epsilon \to 0 \). The final result is

\[
\langle 0_{in} | \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) | 0_{in} \rangle_{td} = \int_0^\pi d\phi \frac{m^3}{32\pi} \left( -L_{-1}(amn) + I_1(amn) - L_{-1}(apn) + I_1(apn) \right)
\]

Now using the fact that modified Struve function has the asymptotic form

\[
L_\nu(x) - I_{-\nu}(x) \approx -\frac{1}{\sqrt{\pi}} \frac{(-1)^{j+1} \Gamma(j + 1/2)}{\Gamma(\nu + 1/2 - j)(x/2)^{2j - \nu + 1}}
\]

(100)

which to leading order is

\[
L_\nu(x) - I_{-\nu}(x) \approx -\frac{1}{\sqrt{\pi}} \frac{(-1)^{j+1} \Gamma(j + 1/2)}{\Gamma(\nu + 1/2 - j)(x/2)^{2j - \nu + 1}} \left( \frac{2}{x} \right)^{1-\nu}
\]

gives us

\[
\langle 0_{in} | \partial_t \phi(x_1, t) \partial_t \phi(x_2, t) | 0_{in} \rangle_{td} = -\int_0^\pi d\phi \frac{m}{8\pi^2} \frac{(r^2 \sin^2 \phi + 4t^2)}{(r^2 \sin^2 \phi - 4t^2)^{3/2}} = -\frac{mt}{4\pi (4t^2 - r^2)^{3/2}} = -\frac{m}{32\pi t^2} + O(t^{-4})
\]

C.2 3+1 dimensions

The details of calculation of correlators in \( d = 3 \) are presented here.

C.2.1 CC correlator

The integral in equation (27) for only \( \kappa_2 \) non-zero, can be evaluated directly in Mathematica. Even so we calculate it by hand as we would eventually need to do so in the gCC4 case. Performing the angular integral we land up with

\[
\langle f_2 | \phi(x_1, t) \phi(x_2, t) | f_2 \rangle = \frac{1}{4\pi^2 r} \int_0^\infty dk \sin(kr) \left[ \coth(2\kappa_2 k) - \text{csch}(2\kappa_2 k) \cos(2kt) \right]
\]

\[
= \frac{1}{8\pi^2 r} \int_{-\infty}^\infty dke^{ikr} \left[ \coth(2\kappa_2 k) - \text{csch}(2\kappa_2 k) \cos(2kt) \right]
\]

\[
= \frac{1}{8\pi^2 r} \int_{-\infty}^\infty dk \left[ \frac{\cosh(2\kappa_2 k)e^{ikr}}{\sinh(2\kappa_2 k)} - \frac{e^{ik(t+t_+)}}{2} - \frac{e^{ik(t-t_+)}}{2} \right]
\]

(101)

where we have used \( t_+ = 2t \). Now \( \sinh(2\kappa_2 k) \) has simple poles at \( 2\kappa_2 k = in\pi \) or \( k = in\pi/2\kappa_2 \) for any integer \( n \). We are interested in the \( t_+ > r \) limit, so we close the contour in the upper half plane for terms A and B while in the lower half for C. The residue at these poles is easily calculated

\[
\text{Res}_A(in\pi/2\kappa_2) = \frac{1}{2\kappa_2} e^{-n\pi r/2\kappa_2}
\]
\[ \text{Res}_B(-in\pi/2\kappa_2) = \left(\frac{-1}{4\kappa_2}\right)^n e^{-n\pi(t+\tau)/2\kappa_2} \]

\[ \text{Res}_C(-in\pi/2\kappa_2) = \left(\frac{-1}{4\kappa_2}\right)^n e^{-n\pi(t-\tau)/2\kappa_2} \]

We take half the contribution from the \( k = 0 \) pole. Thus we can write the integral as

\[
\langle f_2 | \phi(x_1,t) \phi(x_2,t) | f_2 \rangle = \frac{2\pi i}{8\pi^2 i r} \left[ \sum_{n=1}^{\infty} e^{-n\pi r/2\kappa_2} + \frac{1}{2} \left( \sum_{n=1}^{\infty} (-1)^n e^{-n\pi(t+\tau)/2\kappa_2} + \frac{1}{2} \right) \right]
\]

\[
= \frac{2\pi i}{16\pi^2 i r \kappa_2} \left[ \sum_{n=1}^{\infty} e^{-n\pi r/2\kappa_2} + \frac{1}{2} \left( \sum_{n=1}^{\infty} (-1)^n e^{-n\pi(t+\tau)/2\kappa_2} + \frac{1}{2} \right) \right]
\]

We have already done the summations

\[
\sum_{n=1}^{\infty} e^{-ns} + \frac{1}{2} = \frac{1}{e^s - 1} + \frac{1}{2} = \frac{1}{2} \coth(s/2)
\]

\[
\sum_{n=1}^{\infty} (-1)^n e^{-ns} + \frac{1}{2} = -\frac{1}{e^s + 1} + \frac{1}{2} = \frac{1}{2} \tanh(s/2)
\]

The \( \frac{1}{2} \)'s are coming from the \( n = 0 \) pole at the origin. The slowest decaying transient is \( (t >> \kappa_2 > r) \)

\[
-\frac{1}{16\kappa_2^2} e^{-\pi t/\kappa_2}
\]
C.2.2 gCC₄ correlator

The singularity structure of the correlator in the gCC₄ state (24) is similar to CC state.

\[ \langle f_4|\phi(0,t_1)\phi(\vec{r},t_2)|f_4 \rangle = \int \frac{d^4k}{(2\pi)^3} \frac{e^{i\vec{k} \cdot \vec{r}}}{2|k|} \left[ \coth(2\kappa_2|k| + 2\kappa_4|\vec{k}|^3) - \cos(2|k|t) \cosh(2\kappa_2|k| + 2\kappa_4|\vec{k}|^3) \right] \]

Doing this integral exactly is hard. But it is simpler, and more instructive, to figure out the behaviour of the integral by looking at the poles of the integrand in the complex k-plane. These are situated at the roots of \(2\kappa_2 k + 2\kappa_4 k^3 = i\pi\). Introducing a dimensionless parameter \(\kappa_4 = \kappa_4/\kappa_2\), in the small \(\kappa_4\) expansion, we find that roots of the above equation are

\[
\begin{align*}
    k_1(n) &= \frac{i\pi}{2\kappa_2} \left( n + \frac{\pi^2 n^3}{4} \kappa_4 + O(\kappa_4^2) \right) \\
    k_2(n) &= \frac{i}{\kappa_2} \left( \frac{1}{\sqrt{\kappa_4}} - \frac{3\pi^2 n^2}{32} \sqrt{\kappa_4} + O(\kappa_4) \right) \\
    k_3(n) &= -\frac{i}{\kappa_2} \left( \frac{1}{\sqrt{\kappa_4}} - \frac{3\pi^2 n^2}{32} \sqrt{\kappa_4} + O(\kappa_4) \right)
\end{align*}
\]

As we saw earlier (in particular, in the previous subsection), the leading large time behaviour is given by the slowest transient. So we only need to calculate the contribution of the poles nearest to the origin i.e. from \(k_1(n)\) with \(n = \pm 1\). Both \(k_2(n)\) and \(k_3(n)\) are very large due to the small \(\sqrt{\kappa_4}\) in the denominator.

Thus forgetting about all the other poles we do the integral. The residues are

\[
\begin{align*}
    \text{Res}_A(k_1(1)) &= \frac{1}{2\kappa_2} (1 + \frac{3}{4\pi^2} \kappa_4 + O(\kappa_4^2)) \exp \left[ -\frac{\pi r}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) \right] \\
    \text{Res}_B(k_1(1)) &= -\frac{1}{2\kappa_2} (1 + \frac{3}{4\pi^2} \kappa_4 + O(\kappa_4^2)) \exp \left[ -\frac{\pi (2t + r)}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) \right] \\
    \text{Res}_C(k_1(-1)) &= -\frac{1}{4\kappa_2} (1 + \frac{3}{4\pi^2} \kappa_4 + O(\kappa_4^2)) \exp \left[ -\frac{\pi (2t - r)}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) \right] \\
    \text{Res}_A(k_1(0)) &= \frac{1}{2\kappa_2}; \quad \text{Res}_B(k_1(0)) = \frac{1}{4\kappa_2}; \quad \text{Res}_C(k_1(0)) = \frac{1}{4\kappa_2}
\end{align*}
\]

As before we close the contour in the upper half plane for terms A and B while in the lower half for C. The result is

\[
\begin{align*}
    \langle f_4|\phi(0,t_1)\phi(\vec{r},t_2)|f_4 \rangle &= \frac{1}{4\pi \kappa_2} \left( \text{Res}_A(k_1(1)) + \frac{1}{2} \text{Res}_A(k_1(0)) - \text{Res}_B(k_1(1)) - \frac{1}{2} \text{Res}_B(k_1(0)) + \text{Res}_C(k_1(1)) + \frac{1}{2} \text{Res}_C(k_1(0)) \right) \\
    &= \frac{1}{16\pi \kappa_2} + \frac{1}{16\pi \kappa_2} (1 + \frac{3\pi^2}{4} \kappa_4 + O(\kappa_4^2)) \left\{ 2 \exp \left[ -\frac{\pi}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) r \right] + \right. \\
    &\left. \exp \left[ -\frac{\pi}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) (2t + r) \right] - \exp \left[ -\frac{\pi}{2\kappa_2} (1 + \frac{\pi^2}{4} \kappa_4 + O(\kappa_4^2)) (2t - r) \right] \right\} + O(\exp(-2\pi t/\kappa_2))
\end{align*}
\]
The exact value of \( k(1) \) is \( k(1) = i\gamma \) where
\[
\gamma = \frac{-2 6^{2/3} + \sqrt{6} (\sqrt{48 - 81\pi^2 K_4} + 9i\pi \sqrt{K_4})^{2/3}}{6\pi \sqrt{K_4} (\sqrt{48 - 81\pi^2 K_4} + 9i\pi \sqrt{K_4})}
\]
which implies that the exponentials appearing above are of the form
\[
e^{-\gamma r}, e^{-\gamma(2t - r)}
\]

### C.2.3 Ground state correlator

The above method of estimating the time-dependence at large times can also be applied to the ground state as it is just a particular squeezed state. But here we will use the special relation between correlators in \( 1 + 1 \) and \( 3 + 1 \) given by the action of the operation \( \frac{-1}{2\pi} \partial_r \) as already mentioned before in section 3.1. Let's see this in detail. The most general gCC correlator in \( 1 + 1 \) is given by
\[
(f|\phi(x_1, t_1)\phi(x_2, t_2)|f) = \int \frac{dk}{(2\pi)^2} e^{ikr} \frac{\sinh(2\kappa(k))}{2k} \left[ \cos(2\kappa(k) + kt_+) - \cos(kt_+) \right]
\]
where \( r = (x_1 - x_2) \). Acting with \( \frac{-1}{2\pi} \partial_r \) on this expression we obtain
\[
= \frac{1}{8\pi^2 r} \int_{-\infty}^{\infty} dk e^{ikr} \frac{\sinh(2\kappa(k))}{2} \left[ \cos(2\kappa(k) + kt_+) - \cos(kt_+) \right]
\]
This is precisely the expression for the most general gCC correlator in \( 3 + 1 \) after having done the angular integral. We are going to use this operation on the \( 1 + 1 \) ground state correlator to obtain the ground state correlator in \( 3 + 1 \). In \( 1 + 1 \) we are able to do the integral exactly in terms of Meijer G-functions.
\[
\langle 0_n|\phi(x_1, t)\phi(x_2, t)|0_m \rangle = -\frac{1}{16\pi^2 r^2} \left\{ -2G_{2,1}^{2,1} \left( \frac{r^2}{4} \right) \left| \frac{3}{2}, 1, \frac{1}{2} \right) + G_{1,3}^{2,1} \left( \frac{1}{4} m^2 (r + 2t)^2 \right| \left| \frac{3}{2}, 0, 0, \frac{1}{2} \right) 
\right. 
\]
\[
+ G_{1,3}^{2,1} \left( \frac{1}{4} m^2 (r - 2t)^2 \right| \left| \frac{3}{2}, 0, 1, \frac{1}{2} \right) - 8K_0(mr) \right\}
\]
Acting with \( \frac{-1}{2\pi} \partial_r \) on this we obtain the correlator in \( 3 + 1 \).
\[
\langle 0_n|\phi(x_1, t_1)\phi(x_2, t_2)|0_m \rangle = \frac{1}{32\pi^2 r^2 (r^2 - 4t^2)} \left\{ -2 \left( r^2 - 4t^2 \right) G_{1,3}^{2,1} \left( \frac{r^2}{4} \right) \left| \frac{3}{2}, 0, 1, \frac{1}{2} \right) 
\right. 
\]
\[
+ r(r + 2t)G_{1,3}^{2,1} \left( \frac{1}{4} m^2 (r - 2t)^2 \right| \left| \frac{3}{2}, 0, 1, \frac{1}{2} \right) + r(r - 2t)G_{1,3}^{2,1} \left( \frac{1}{4} m^2 (r + 2t)^2 \right| \left| \frac{3}{2}, 0, 1, \frac{1}{2} \right) 
\]
\[
+ 2m(r + 2t) \left( K_1(m(r + 2t)) - K_1(2mt - mr) + 2K_1(mr) \right) \right\}
\]

### C.2.4 The Thermal Correlator

The 2-point function in the thermal ensemble is (120)
\[
\langle \phi(\vec{x}_1, t_1)\phi(\vec{x}_2, t_2) \rangle_\beta = \frac{1}{2} \int \frac{d^3k}{(2\pi)^3} e^{ikr} \frac{\sinh(\omega_{out})}{\omega_{out}} \left[ \cos(\omega_{out}t) \coth \left( \frac{\mu(k)}{2} \right) - i \sin(\omega_{out}t) \right]
\]
\[
= \frac{1}{4\pi^2 r} \int_0^{\infty} dk \sin(kt) [\cos(\beta k/2) - i \sin(kt)]
\]

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Summing over all the poles we get the UHP while for the second term we close in the LHP. The contribution of the pole at the origin cancels. The last 2 terms give us Dirac-delta functions, while the first 2 terms are easily evaluated using contour integral with simple poles at the location $k = 2i\pi n/\beta$. Since $t > r$, for the first term we close the contour in the UHP while for the second term we close in the LHP. The contribution of the pole at the origin cancels. Summing over all the poles we get

$$\langle \phi(x_1, t_1)|\phi(x_2, t_2)\rangle_{\beta} = \frac{\coth\left(\frac{\pi (r-t)}{\beta}\right) + \coth\left(\frac{\pi (r+t)}{\beta}\right)}{8\pi \beta r} + i\delta(r-t) - i\delta(r+t)$$

The thermal autocorrelator is obtained by taking $r \to 0$

$$\langle \phi(x_1, t_1)|\phi(x_1, t_2)\rangle_{\beta} = -\frac{\operatorname{csch}^2\left(\frac{\pi t}{\beta}\right)}{4\beta^2}$$

\[C.3\] 4+1 dimensions

Here the details of calculation of particular correlators in $d = 4$ are provided, even though a general method of calculating the time-dependent part is presented in the text for an arbitrary gCC state.

\[C.3.1\] CC correlator

The time dependent part of eq. (27) with only $\kappa_2$ non-zero is

$$\langle f_2|\phi(x_1, t)|\phi(x_2, t)\rangle_{f_2} - \langle \phi(x_1, t)|\phi(x_2, t)\rangle_{\beta} = -\int \frac{d^4k}{(2\pi)^d} \frac{e^{i\vec{k}(x_1-x_2)}}{2|\vec{k}|} \cos\left(2|\vec{k}|t\right)$$

which in 4+1 simplifies to

$$=-\frac{1}{8\pi^2} \int_0^{\infty} dk \cdot k J_1(kr) \frac{\operatorname{csch}(2\kappa_2 k) \cos(2kt)}{\operatorname{csch}(2\kappa_2 k) \cos(2kt)}$$

$$=-\frac{1}{8\pi^4} \int_0^{\infty} dk \int_0^{\infty} d\phi \cos(kr \sin \phi - \phi) \frac{\operatorname{csch}(2\kappa_2 k) \cos(2kt)}{\operatorname{csch}(2\kappa_2 k) \cos(2kt)}$$

$$=-\frac{1}{16\pi^3} \int_0^{\pi} d\phi \int_0^{\infty} dk \frac{\kappa_2}{\kappa_2} \cos(k(2t + r \sin \phi - \phi) + \cos(k(2t - r \sin \phi + \phi))$$

$$=-\frac{1}{16\pi^3} \int_0^{\pi} d\phi \int_0^{\infty} dk \frac{\kappa_2}{\kappa_2} \cos(k(2t + r \sin \phi - \phi) + \cos(k(2t - r \sin \phi + \phi))$$

$$=-\frac{1}{256\pi^3 \kappa_2^2} \int_0^{\pi} d\phi \left[ e^{i\phi} \left( \psi^{(1)}\left( \frac{i(t_+ + r \sin \phi)}{4\kappa_2}\right) + \frac{1}{2} + \psi^{(1)}\left( \frac{t_+ - r \sin \phi)}{4\kappa_2}\right) \right) + e^{-i\phi} \left( \psi^{(1)}\left( \frac{i(t_+ - r \sin \phi)}{4\kappa_2}\right) + \frac{1}{2} + \psi^{(1)}\left( \frac{t_+ + r \sin \phi)}{4\kappa_2}\right) \right]\cos(1/\sqrt{r})$$

$$\approx \int d\phi \left[ \left. \frac{\sin^2(\phi)}{(16\pi^3 \kappa_2^2) t_+^2} + \frac{\sin^2(\phi)}{(16\pi^3 \kappa_2^2) t_+^2} \right] + O(\frac{1}{r^6})$$

$$=\frac{1}{128\pi^2 \kappa_2^2 t_+^2} + \frac{3\pi^2 + 16\kappa_2^2}{21\pi^4 \kappa_2^4 t_+^2} + O(t^{-6})$$

In the second line we used the integral representation $J_n(x) = \frac{1}{\pi} \int_0^\pi d\tau \cos(x \sin \tau - n\tau)$ of the Bessel function. Then we perform the $k$ integral to get Polygamma function. In the end we do the $\phi$ integral after series expanding Polygamma around $t = \infty$ to get leading large $t$ answer.
C.3.2 Ground state correlator

The time-dependent part of the equal-time ground state correlator here is

\[ C(\beta) \]

C.3.3 The Thermal Correlator

We have added and subtracted 1 from the coth to get to the final line. Now to calculate the first term and using the asymptotic form of these functions, eq. (100)

\[ L_\nu(x) - I_{-\nu}(x) \approx - \frac{1}{\sqrt{\pi}} \Gamma(\nu + 1/2) \left( \frac{2}{x} \right)^{1-\nu} \]

and knowing that \( K_0(x) \approx e^{-x}/\sqrt{x} \) is exponentially suppressed at large \( x \), one gets

\[ \int_0^\pi d\phi \frac{m \sin^2 \phi}{64\pi^3 t^2} = \frac{m}{128\pi^2 t^2} \] (114)

C.3.3 The Thermal Correlator

The 2-point function in the thermal ensemble is (120)

\[ \langle \phi(x_1, t_1) \phi(x_2, t_2) \rangle_\beta = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} e^{i k \cdot \vec{x}} \left[ \cos(\omega_{out} t) \coth \left( \frac{\mu(k)}{2} \right) - i \sin(\omega_{out} t) \right] \]

\[ = \frac{1}{8\pi^2} \int_0^\infty dk J_1(kr) k \left[ \cos(kt) \coth(\beta k/2) - i \sin(kt) \right] \]

\[ = \frac{1}{8\pi^2} \int_0^\infty dk J_1(kr) k \left[ \frac{2\cos(kt)}{e^{\beta k} - 1} + e^{-ikt} \right] \]

We have added and subtracted 1 from the coth to get to the final line. Now to calculate the first term and expand the integrand at large \( t \) (see eq. (47))

\[ \frac{1}{8\pi^2} \int_0^\infty dk J_1(kr) k \left[ \frac{2\cos(kt)}{e^{\beta k} - 1} + e^{-ikt} \right] = \frac{1}{4\pi^2 t^2} \int_0^\infty dp J_1(pr/t) p \cos(p) \frac{e^{\beta p/t} - 1}{e^{\beta p/t} - 1} \]

\[ = \frac{1}{16\pi^2 t^2 \beta} \left[ I(1) - \frac{\beta}{2t} I(2) + \frac{-3r^2 + 2\beta^2}{24t^2} I(3) + O(t^{-3}) \right] \]

\[ = - \frac{1}{8\pi^2 t^2 \beta} + \frac{-3r^2 + 2\beta^2}{32\pi^2 t^4} + O(t^{-6}) \] (115)
We have already rotated the contours appropriately and used the notation $I(n)$ of eq. (48). The $e^{-ikt}$ term is easily evaluated to give
\[
\frac{1}{8\pi^2r} \int_0^\infty dk J_1(kr)ke^{-ikt} = \frac{i}{8\pi^2r^3 (1 - \frac{r^2}{r^2})^{3/2}}
\]
This is just the UV singularity one expects in $d=4$. Combining we have
\[
\langle \phi(x_1', t_1)\phi(x_2', t_2) \rangle_\beta = \frac{i}{8\pi^2r^3 (1 - \frac{r^2}{r^2})^{3/2}} - \frac{1}{8\pi^2\beta t^2} + \frac{-3r^2 + 2\beta^2}{32\pi^2\beta t^4} + O(t^{-6})
\]
(116)
The thermal autocorrelator is obtained by taking $r \to 0$
\[
\langle \phi(x_1', t_1)\phi(x_2', t_2) \rangle_\beta = \frac{i}{8\pi^2r^3 (1 - \frac{r^2}{r^2})^{3/2}} - \frac{1}{8\pi^2\beta t^2} + \frac{\beta}{16\pi^2t^4} + O(t^{-6})
\]
(117)

D GGE Correlator

For the GGE “density matrix” $\rho = \frac{1}{Z} \exp\left(-\sum_k \mu(k) \hat{N}(k)\right)$, it’s easy to calculate the partition function
\[
Z = \text{tr} \exp\left(-\sum_k \mu(k) \hat{N}(k)\right)
= \sum_{\{N_k\}} \langle \{N_k\} | e^{-\sum_k \mu(k) \hat{N}(k)} | \{N_k\} \rangle = \sum_{\{N_k\}} \langle \{N_k\} | \prod_k e^{-\mu(k) \hat{N}(k)} | \{N_k\} \rangle
= \prod_k \sum_{N_k} \langle N_k | e^{-\mu(k) \hat{N}(k)} | N_k \rangle = \prod_k \sum_{N_k=1}^\infty e^{-\mu(k) N_k}
Z = \prod_k \left(1 - e^{-\mu(k)}\right)^{-1}
\]
(118)

Starting with the GGE 2-point function
\[
\langle \phi(x_1', t_1)\phi(x_2', t_2) \rangle_\rho = \frac{1}{Z} \text{tr} \left( e^{-\sum_k \mu(\vec{k}) \hat{N}(\vec{k})} \phi(x_1', t_1)\phi(x_2', t_2) \right)
= \frac{1}{Z} \sum_{\{N_\vec{k}\}} \langle \{N_\vec{k}\} | e^{-\sum_k \mu(\vec{k}) \hat{N}(\vec{k})} \phi(x_1', t_1)\phi(x_2', t_2) | \{N_\vec{k}\} \rangle
\]
(119)

In the second line we have used the occupation number basis. Using the partial Fourier transform for the field and the mode expansion
\[
\phi(x, t) = \int e^{i\vec{k} \cdot \vec{x}} \phi(\vec{k}, t) \frac{d^d k}{(2\pi)^d}
\]
where $\phi(\vec{k}, t) = a(\vec{k})a(\vec{k}, t) + a^\dagger(\vec{k})a^*(\vec{k}, t)$, gives
\[
\langle \phi(x_1', t_1)\phi(x_2', t_2) \rangle_\rho = \frac{1}{Z} \sum_{\{N_\vec{k}\}} \langle \{N_\vec{k}\} | e^{-\sum_k \mu(\vec{k}) \hat{N}(\vec{k})} \int \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}_1' + \vec{q} \cdot \vec{x}_2'} (a(\vec{k})a(\vec{q}, t_1) + a^\dagger(\vec{-k})a^*(\vec{-k}, t_1)) (a(\vec{q})a(\vec{q}, t_2) + a^\dagger(\vec{-q})a^*(\vec{-q}, t_2)) | \{N_\vec{k}\} \rangle
\]
Out of the resulting four terms only two terms give non-zero values.
\[
\langle \phi(x_1', t_1)\phi(x_2', t_2) \rangle_\rho = \frac{1}{Z} \int \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \sum_{\{N_\vec{k}\}} e^{-\sum_k \mu(\vec{k}) \hat{N}(\vec{k})} \langle \{N_\vec{k}\} | (a(\vec{k})a(\vec{k}, t_1)a^\dagger(\vec{-q})a^*(\vec{-q}, t_2) + a^\dagger(\vec{-k})a^*(\vec{-k}, t_1)a(\vec{q})a(\vec{q}, t_2)) | \{N_\vec{k}\} \rangle
\]
Using the commutation relation

\[ [a(\vec{k}), a^\dagger(-\vec{q})] = (2\pi)^d \delta^d(\vec{k} + \vec{q}) \]

and the form of the number operator

\[ a^\dagger(-\vec{q})a(\vec{k}) = N_{\vec{k}}(2\pi)^d \delta^d(\vec{k} + \vec{q}) \]

Therefore

\[ \langle \phi(x_1^t, t_1) \phi(x_2^t, t_2) \rangle = \frac{1}{Z} \int \frac{d^d k}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} e^{i(\vec{k} \cdot \vec{x}_1 + \vec{q} \cdot \vec{x}_2)} (2\pi)^d \delta^d(\vec{k} + \vec{q}) \sum_{\{N_{\vec{k}}\}} e^{-\sum_{\vec{k}} \mu(\vec{k}) N(\vec{k})} \]

\[ \sum_{\{N_{\vec{k}}\}} \left( \langle \{N_{\vec{k}}\} | (N_{\vec{k}} + 1) \{N_{\vec{k}}\} \rangle u(\vec{k}, t_1) u^\dagger(-\vec{q}, t_2) + \langle \{N_{\vec{k}}\} | N_{\vec{k}} \{N_{\vec{k}}\} \rangle u^\dagger(-\vec{k}, t_1) u(\vec{q}, t_2) \right) \]

Doing the q integral for the first term and k integral for the second and then writing it in terms of a single dummy variable:

\[ \langle \phi(x_1^t, t_1) \phi(x_2^t, t_2) \rangle = \frac{1}{Z} \int \frac{d^d k}{(2\pi)^d} \sum_{\{N_{\vec{k}}\}} e^{-\sum_{\vec{k}} \mu(\vec{k}) N(\vec{k})} \left[ \langle \{N_{\vec{k}}\} | (N_{\vec{k}} + 1) \{N_{\vec{k}}\} \rangle u(\vec{k}, t_1) u^\dagger(-\vec{k}, t_2) e^{i(\vec{k} \cdot \vec{x}_1 - \vec{k} \cdot \vec{x}_2)} + \langle \{N_{\vec{k}}\} | N_{\vec{k}} \{N_{\vec{k}}\} \rangle u^\dagger(\vec{k}, t_1) u(\vec{k}, t_2) e^{-i(\vec{k} \cdot \vec{x}_1 - \vec{k} \cdot \vec{x}_2)} \right] \]

Directly using \( N_{\vec{k}} = \frac{1}{2} \sum_{\{N_{\vec{k}}\}} \langle \{N_{\vec{k}}\} | e^{-\sum_{\vec{k}} \mu(\vec{k}) N(\vec{k})} N(\vec{k}) \{N_{\vec{k}}\} \rangle = N(k) = (e^{\mu(k)} - 1)^{-1} \), we get

\[ \langle \phi(x_1^t, t_1) \phi(x_2^t, t_2) \rangle = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} e^{i(\vec{k} \cdot \vec{x}_1 - \vec{k} \cdot \vec{x}_2)} \left[ \frac{u(\vec{k}, t_1) u^\dagger(\vec{k}, t_2)}{1 - e^{-\mu(k)}} + \frac{u^\dagger(\vec{k}, t_1) u(\vec{k}, t_2)}{e^{\mu(k)} - 1} \right] \]

Defining \( \vec{x} = \vec{x}_1 - \vec{x}_2 \), \( t = t_1 - t_2 \) and

\[ G_{\perp} = \frac{1}{\omega_{out}(e^\pm \mu(k) + 1)} \]

\[ \langle \phi(x_1^t, t_1) \phi(x_2^t, t_2) \rangle = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} e^{i(\vec{k} \cdot \vec{x})} \left( G_{-} e^{-i\omega_{out} t} + G_{+} e^{i\omega_{out} t} \right) \]

We can further simplify

\[ \frac{e^{-i\omega_{out} t}}{(1 - e^{-\mu(k)})} + \frac{e^{i\omega_{out} t}}{(e^{\mu(k)} - 1)} = \frac{e^{-i\omega_{out} t} e^{\mu(k)/2} + e^{i\omega_{out} t} e^{-\mu(k)/2}}{e^{\mu(k)/2} - e^{-\mu(k)/2}} \]

\[ = \frac{(\cos(\omega_{out} t) - i \sin(\omega_{out} t)) e^{\mu(k)/2} + (\cos(\omega_{out} t) + i \sin(\omega_{out} t)) e^{-\mu(k)/2}}{e^{\mu(k)/2} - e^{-\mu(k)/2}} \]

\[ = \cos(\omega_{out} t) \coth\left(\frac{\mu(k)}{2}\right) - i \sin(\omega_{out} t) \]

So

\[ \langle \phi(x_1^t, t_1) \phi(x_2^t, t_2) \rangle = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i(\vec{k} \cdot \vec{x})}}{\omega_{out} \omega_{out}} \left[ \cos(\omega_{out} t) \coth\left(\frac{\mu(k)}{2}\right) - i \sin(\omega_{out} t) \right] \] (120)

We would be particularly interested in the ETC when \( t_1 = t_2 \Rightarrow t = 0 \) then

\[ \langle \phi(x_1^t, t) \phi(x_2^t, t) \rangle = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{e^{i(\vec{k} \cdot \vec{x})}}{\omega_{out} \coth\left(\frac{\mu(k)}{2}\right)} \] (121)
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