Energy and Momentum Distributions of a (2 + 1)-dimensional black hole background

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Abstract

Using Einstein, Landau-Lifshitz, Papapetrou and Weinberg energy-momentum complexes we explicitly evaluate the energy and momentum distributions associated with a non-static and circularly symmetric three-dimensional spacetime. The gravitational background under study is an exact solution of the Einstein’s equations in the presence of a cosmological constant and a null fluid. It can be regarded as the three-dimensional analogue of the Vaidya metric and represents a non-static spinless (2 + 1)-dimensional black hole with an outflux of null radiation. All four above-mentioned prescriptions give exactly the same energy and momentum distributions for the specific black hole background. Therefore, the results obtained here provide evidence in support of the claim that for a given gravitational background, different energy-momentum complexes can give identical results in three dimensions. Furthermore, in the limit of zero cosmological constant the results presented here reproduce the results obtained by Virbhadra who utilized the Landau-Lifshitz energy-momentum complex for the same (2 + 1)-dimensional black hole background in the absence of a cosmological constant.

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Introduction

Energy-momentum localization has been one of the oldest, most interesting but also most controversial problems in gravitation. Many renowned physicists have been working on this problematic issue with Einstein to be first in the row. After Einstein’s seminal work on energy-momentum complexes a large number of expressions for the energy distribution were proposed. However, the idea of the energy-momentum complex was severely criticized for a number of reasons. Therefore, this approach was abandoned for a long time. In 1990 Virbhadra revived the interest in this approach and since then numerous works on evaluating the energy and momentum distributions of several gravitational backgrounds have been completed. Later attempts to deal with this problematic issue were made by proposers of quasi-local approach. The seminal work on quasilocal mass was that of Brown and York. However, it should be stressed that the determination as well as the computation of the quasilocal energy and quasilocal angular momentum of a (2 + 1)-dimensional gravitational background were first presented by Brown, Creighton and Mann. Many attempts since then have been performed to give new definitions of quasilocal energy in General Relativity. Considerable efforts have also been performed in constructing superenergy tensors. Motivated by the works of Bel and independently of Robinson, many investigations have been given in this field.

In this paper we are implementing the approach of energy momentum complexes. The gravitational background under investigation is a three-dimensional black hole spacetime. Our interest in three-dimensional gravitational backgrounds stems to the fact that (2 + 1) dimensions provide a simpler framework than (3 + 1) and a more realistic one than (1 + 1) to investigate various problems. We evaluate the energy and momentum distributions associated with the (2 + 1)-dimensional black hole using four different energy-momentum complexes, specifically we are implementing the prescriptions of Einstein, Landau-Lifshitz, Papapetrou and Weinberg. The specific (2 + 1)-dimensional black hole background is a non-static although spinless (c.f. spinless BTZ black hole solution), and circularly symmetric. It is an exact solution of the Einstein equations in the presence of a cosmological constant Λ and a null fluid. Therefore, it can be regarded as the three-dimensional analogue of the Vaidya metric and represents a non-static spinless (2 + 1)-dimensional black hole with an outflux of null radiation.

Recently, there was some interest in using the approach of the energy-momentum complexes in the framework of teleparallel equivalent of General Relativity, i.e. teleparallel gravity (for a recent review on teleparallel gravity see).
The remainder of the paper is organized as follows. In Section 1 we briefly present the 
(2 + 1)-dimensional black hole spacetime in which energy and momentum distributions 
by using four different prescriptions are to be calculated. In the subsequent four Sections 
using Einstein, Landau-Lifshitz, Papapetrou and Weinberg energy-momentum complexes, 
respectively, we explicitly evaluate the energy and momentum distributions contained in a 
“sphere” of fixed radius. The results extracted in the four different prescriptions associated 
with the same gravitational background are identical. Finally, Section 6 is devoted to a 
brief summary of results and concluding remarks.

1 The non-static (2 + 1)-dimensional black hole

In 1992 Banados, Teitelboim and Zanelli discovered a black hole solution (known as BTZ 
black hole) in (2 + 1) dimensions [58]. Till that time it was believed that no black hole 
solution exists in three-dimensional spacetimes [60]. Since this discovery, many investi-
gations have been performed in order to extend the BTZ black hole solution. Virbhadra 
found one of these extended solutions [59] by including a cosmological constant \( \Lambda \) and an 
energy-momentum tensor of a null fluid, in the Einstein’s field equations\(^3\), i.e.

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa T_{\mu\nu}^{\text{fluid}}
\]

where \( \kappa \) is the gravitational coupling constant.

The energy-momentum of the null fluid \( T_{\mu\nu}^{\text{fluid}} \) is given by

\[
T_{\mu\nu}^{\text{fluid}} = U_\mu U_\nu
\]

where \( U_\mu \) is the fluid current vector satisfying the equation

\[
U_\mu U^\mu = 0
\]

since the fluid is null.

An exact solution of the field equations (1) is given by the line element

\[
ds^2 = -\left( -m(u) - \Lambda r^2 \right) du^2 - 2du \, dr + r^2 d\phi^2
\]

where \( u \) is the retarded null coordinate \( u = t - r \). This solution\(^4\) is obviously non-static 
and circularly symmetric, and thus there is only one killing vector \( \partial / \partial \phi \) related to the 
rotational invariance.

\(^3\)A classification of solutions of the three-dimensional Einstein equations was given by Yamazaki and 
Ida in [62].

\(^4\)It is noteworthy that at the same period of time Chan, Chan and Mann [61], and Husain [62] trying 
to investigate the phenomenon of mass inflation discovered non-static (2 + 1) solutions describing black 
holes irradiated by an influx of null radiation.
Therefore, solution (4) can be regarded as the three-dimensional analogue of the Vaidya metric and represents a non-static (2 + 1)-dimensional black hole with an outflux of null radiation. In the case where the mass function \( m(u) \) is constant the black hole under consideration becomes the spinless BTZ black hole.

## 2 Einstein’s Prescription

The energy-momentum complex of Einstein [11] in a three-dimensional background is given as
\[
\theta^\mu_\nu = \frac{1}{2\kappa} h^{\mu\lambda}_{\nu\lambda}
\]
where the Einstein’s superpotential \( h^{\mu\lambda}_{\nu\lambda} \) is of the form
\[
h^{\mu\lambda}_{\nu\lambda} = \frac{1}{\sqrt{-g}} g_{\nu\sigma} \left[ -g \left( g^{\mu\sigma} g^{\lambda\kappa} - g^{\lambda\sigma} g^{\mu\kappa} \right) \right]_{,\kappa}
\]
with the antisymmetric property
\[
h^{\mu\lambda}_{\nu\lambda} = -h^{\lambda\mu}_{\nu\mu} .
\]
Thus, the energy and momentum in Einstein’s prescription for a three-dimensional background are given by
\[
P_\mu = \int \int \int \theta^\mu_\nu dx^1 dx^2
\]
and specifically the energy of the physical system in a three-dimensional background is
\[
E = \int \int \int \theta^t_t dx^1 dx^2 .
\]
It should be noted that the calculations have to be restricted to the use of quasi-Cartesian coordinates.

In order to evaluate the energy and momentum distributions in the non-static (2 + 1)-dimensional black hole background described by the line element (4), we firstly have to calculate the Einstein’s superpotentials. There are sixteen non-vanishing superpotentials in Einstein’s prescription for the gravitational background under study but the required ones are the following
\[
\begin{align*}
h^{tt}_t &= h^{tt}_x = h^{tt}_y = 0 , \\
h^{tx}_t &= -\frac{x}{r^2} \left( 1 + \Lambda r^2 + m \right) ,
\end{align*}
\]
\[5\]It should be noted that it was actually von Freud who showed that the Einstein energy-stress complex can be written as the divergence of an antisymmetric superpotential [11].
\begin{align}
  h_{tx} &= \frac{x^2 + \Lambda (x^4 + 3x^2y^2 + 2y^4) + x^2m + y^2r (\dot{m} + m')}{r^3}, \\
  h_{ty} &= \frac{-xy(-1 + \Lambda (x^2 + y^2) - m + r (\dot{m} + m'))}{r^3}, \\
  h_{ty} &= \frac{-xy(-1 + \Lambda (x^2 + y^2) - m + r (\dot{m} + m'))}{r^3}, \\
  h_{ty} &= \frac{y^2 + \Lambda (y^4 + 3x^2y^2 + 2x^4) + y^2m + x^2r (\dot{m} + m')}{r^3},
\end{align}

where the dot and prime denote the partial derivatives with respect to time \( t \) and radial coordinate \( r \), respectively. The radial coordinate in terms of the quasi-Cartesian coordinates is given as

\[ r = \sqrt{x^2 + y^2} \]

while the mass parameter \( m = m(u) \) is given by

\[ m(u) = m(t, \sqrt{x^2 + y^2}) \]

By substituting the Einstein’s superpotentials as given by equation (10) into equation (5), one gets the energy density distribution

\[ \theta^t = \frac{m'}{2\kappa r} + \frac{\Lambda}{\kappa} \]

while the momentum density distributions are given by

\[ \theta^t_x = -\frac{x\dot{m}}{2\kappa r^2} \]
\[ \theta^t_y = -\frac{y\dot{m}}{2\kappa r^2} \]

Therefore, if we substitute expressions (13) and (14) into equations (9) and (8), respectively, the energy and momentum distributions associated with the non-static (2 + 1)-dimensional black hole under study, which are contained in a “sphere”\(^6\) of radius \( r_0 \), are given by

\[ E(r_0) = \frac{\pi}{\kappa} (m + \Lambda r_0^2) \]
\[ P_x = 0 \]
\[ P_y = 0 \]

\(^6\)Since the spatial section of the (2 + 1) spacetime is two-dimensional the “sphere” is just a circle.
A couple of comments are in order. Firstly, a neutral test particle experiences at a finite
distance $r_0$ the gravitational field of the effective gravitational mass given by the first of
the expressions in (15). What is worth mentioning is the fact that if the cosmological
constant is negative, i.e.
\[
\Lambda = -\frac{1}{l^2}
\]
then it seems that it is possible, although in a gravitational field, the neutral particle to
move freely on the circle of radius $r_0$ whenever the condition
\[
m(t, r_0) = \frac{r_0}{l^2}
\]
is fulfilled. Secondly, it easily seen that the energy-momentum complex of Einstein as
formulated in the gravitational background under consideration satisfies the local conserva-
tion laws, i.e the conservation laws in the ordinary sense,
\[
\frac{\partial}{\partial x^\mu} \theta_\nu^\mu = 0
\]

3 Landau-Lifshitz’s Prescription

The energy-momentum complex of Landau-Lifshitz [2] in a three-dimensional background
is given by
\[
L^{\mu\nu} = \frac{1}{2\kappa} S^{\mu\nu\kappa\lambda} \]s,\kappa,\lambda
\]
where the Landau-Lifshitz’s superpotential $S^{\mu\nu\kappa\lambda}$ is of the form
\[
S^{\mu\nu\kappa\lambda} = -g (g^{\mu\nu} g^{\kappa\lambda} - g^{\mu\kappa} g^{\nu\lambda})
\]
The energy-momentum complex of Landau-Lifshitz is symmetric in its indices
\[
L^{\mu\nu} = L^{\nu\mu}
\]
The energy and momentum in Landau-Lifshitz’s prescription for a three-dimensional back-
ground are given by
\[
P^\mu = \int \int \int L^{t\mu} dx^1 dx^2
\]
and specifically the energy of the physical system in a three-dimensional background is
\[
E = \int \int \int L^{tt} dx^1 dx^2
\]
It should be noted that the calculations in the Landau-Lifshitz’s prescription have to be
restricted to the use of quasi-Cartesian coordinates as in the Einstein’s prescription.
Since we intend to evaluate the energy and momentum distributions in the non-static \((2+1)\)-dimensional black hole background described by the line element \((4)\), we firstly calculate the Landau-Lifshitz’s superpotentials. There are thirty six non-vanishing superpotentials in Landau-Lifshitz’s prescription for the specific gravitational background given by \((4)\) but the required ones are the following

\[
S^{tttt} = S^{tttx} = S^{ttty} = S^{txtt} = S^{yttt} = S^{tttt} = 0 ,
S^{sttx} = \frac{-x^2 + 2y^2 + \Lambda \left( x^2 y^2 + y^4 \right) + y^2 m}{r^2} ,
S^{stty} = \frac{x y \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r^2} ,
S^{stxt} = \frac{x y \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r^2} ,
S^{styt} = \frac{y^2 + 2x^2 + \Lambda \left( x^2 y^2 + x^4 \right) + x^2 m}{r^2} ,
S^{xxtx} = \frac{x^2 + 2y^2 + \Lambda \left( x^2 y^2 + y^4 \right) + y^2 m}{r^2} ,
S^{xxtt} = \frac{-x y \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r^2} ,
S^{xxtt} = S^{xxtx} = S^{xxtt} = S^{xytt} = S^{yttt} = S^{xxtt} = 0 ,
S^{yttx} = \frac{-x y \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r^2} ,
S^{yxtt} = \frac{x \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r} ,
S^{yxtx} = \frac{-x y \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r^2} ,
S^{yxty} = \frac{x \left( 1 + \Lambda \left( x^2 + y^2 \right) + m \right)}{r} ,
S^{yytt} = S^{yytx} = S^{yyty} = 0 .
\] (24)

The mass parameter \(m = m(u)\) in terms of the quasi-Cartesian coordinates is given again by

\[
m(u) = m(t, \sqrt{x^2 + y^2}) .
\] (26)

By substituting the Landau-Lifshitz’s superpotentials as given in equation \((24)\) into equa-
tion (19) one gets the energy density distribution

\[ L^\mu = \frac{m'}{2\kappa r} + \frac{\Lambda}{\kappa} \]  

(27)

while the momentum density distributions are given by

\[ L^{tx} = -\frac{x\dot{m}}{2\kappa r^2} \]
\[ L^{ty} = -\frac{y\dot{m}}{2\kappa r^2} . \]  

(28)

Therefore, if we substitute expressions (27) and (28) into equations (23) and (22), respectively, the energy and momentum distributions associated with the non-static (2 + 1)-dimensional black hole under study, which are contained in a “sphere” of radius \( r_0 \), are given by

\[ E(r_0) = \frac{\pi}{\kappa} \left( m + \Lambda r_0^2 \right) \]
\[ P_x = 0 \]
\[ P_y = 0 . \]  

(29)

It is clear that we have derived exactly the same energy and momentum distributions as in the case of the Einstein’s prescription (c.f. equations (13), (14) and (15)). The comment concerning the neutral test particle and the effective gravitational mass made in the previous section also holds here.

Furthermore, the energy-momentum complex of Landau-Lifshitz as formulated in the (2 + 1)-dimensional black hole under investigation, satisfies the local conservation laws

\[ \frac{\partial}{\partial x^\nu} L^{\mu\nu} = 0 . \]  

(30)

4 Papapetrou’s Prescription

The energy-momentum complex of Papapetrou [4] in a three-dimensional background is given by

\[ \Sigma^{\mu\nu} = \frac{1}{2\kappa} N^{\mu\nu\kappa\lambda} \kappa\lambda \]  

(31)

where the Papapetrou’s superpotential \( N^{\mu\nu\kappa\lambda} \) is given by

\[ N^{\mu\nu\kappa\lambda} = \sqrt{-g} \left( g^{\mu\kappa} g^{\nu\lambda} - g^{\mu\kappa} \eta^{\nu\lambda} + g^{\kappa\lambda} \eta^{\mu\nu} - g^{\nu\lambda} \eta^{\mu\kappa} \right) \]  

(32)
where $\eta^{\mu\nu}$ is the Minkowskian metric, i.e.

$$
(\eta^{\mu\nu}) = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix}.
$$

(33)

The energy-momentum complex of Papapetrou like the Landau-Lifshitz one, is symmetric in its indices

$$
\Sigma^{\mu\nu} = \Sigma^{\nu\mu}.
$$

(34)

The energy and momentum in Papapetrou’s prescription for a three-dimensional background are given by

$$
P^\nu = \int \int \int \Sigma^{t\nu} dx^1 dx^2 dx^3
$$

(35)

and specifically the energy of the physical system in a three-dimensional background is

$$
E = \int \int \int \Sigma^{tt} dx^1 dx^2 dx^3.
$$

(36)

It should be noted that the calculations in the Papapetrou’s prescription have to be restricted to the use of quasi-Cartesian coordinates as in the two afore-said prescriptions. Since our scope is to evaluate the energy and momentum distributions in the non-static (2 + 1)-dimensional black hole background described by the line element (4) we firstly calculate the Papapetrou’s superpotentials. There are fifty four non-vanishing superpotentials in Papapetrou’s prescription for the specific gravitational background given by (4) but the required ones are the following

\[
\begin{align*}
N^{tttt} & = 0, \\
N^{tttx} & = -x \left(1 + \Lambda \left(x^2 + y^2\right) + m\right) / r, \\
N^{ttty} & = -y \left(1 + \Lambda \left(x^2 + y^2\right) + m\right) / r, \\
N^{ttxt} & = x \left(1 + \Lambda \left(x^2 + y^2\right) + m\right) / r, \\
N^{ttxx} & = \frac{2x^2 + 3y^2 + \Lambda \left(x^2 y^2 + y^4\right) + y^2 m}{r^2}, \\
N^{ttxy} & = \frac{-xy \left(1 + \Lambda \left(x^2 + y^2\right) + m\right)}{r^2}, \\
N^{ttxt} & = \frac{y \left(1 + \Lambda \left(x^2 + y^2\right) + m\right)}{r^2}, \\
N^{ttx} & = \frac{-xy \left(1 + \Lambda \left(x^2 + y^2\right) + m\right)}{r^2}.
\end{align*}
\]
The mass parameter $m = m(u)$ in terms of the quasi-Cartesian coordinates is given as before by

$$m(u) = m(t, \sqrt{x^2 + y^2}) \ .$$  \hspace{1cm} (38)

By substituting the Papapetrou's superpotentials (37) into equation (31) one gets the energy density distribution

$$\Sigma_{tt} = \frac{m' + \Lambda}{2\kappa r} + \frac{\Lambda}{\kappa}$$  \hspace{1cm} (39)

while the momentum density distributions are given by

$$\Sigma_{tx} = -\frac{x \dot{m}}{2\kappa r^2}$$

$$\Sigma_{ty} = -\frac{y \dot{m}}{2\kappa r^2} .$$  \hspace{1cm} (40)

Therefore, if we substitute expressions (39) and (40) into equations (36) and (35), respectively, respectively, the energy and momentum distributions associated with the non-static
(2 + 1)-dimensional black hole under investigation, which are contained in a “sphere” of radius $r_0$, are given by

$$E(r_0) = \frac{\pi}{\kappa} (m + \Lambda r_0^2)$$
$$P_x = 0$$
$$P_y = 0 \, .$$

(41)

It is obvious that we have again derived exactly the same energy and momentum distributions as in the cases of the Einstein’s and Landau-Lifshitz’s prescriptions. The comment concerning the neutral test particle and the effective gravitational mass made in Section 1 also holds here.

Furthermore, the energy-momentum complex of Papapetrou as formulated in the (2 + 1)-dimensional black hole under study, satisfies the local conservation laws

$$\frac{\partial}{\partial x^\nu} \Sigma^\mu \nu = 0 \, .$$

(42)

5 Weinberg’s Prescription

The energy-momentum complex of Weinberg in a three-dimensional background is given by

$$\tau^{\nu\lambda} = \frac{1}{2\kappa} Q^{\rho\nu\lambda} \, .$$

(43)

where the Weinberg’s superpotential $Q^{\rho\nu\lambda}$ is given by

$$Q^{\rho\nu\lambda} = \frac{\partial h^\mu_{\nu}}{\partial x^\rho} \eta^{\rho\lambda} - \frac{\partial h^\mu_{\rho}}{\partial x^\nu} \eta^{\nu\lambda} + \frac{\partial h^{\mu\nu}}{\partial x^\rho} \eta^{\rho\lambda} + \frac{\partial h^{\nu\lambda}}{\partial x^\rho} - \frac{\partial h^{\rho\lambda}}{\partial x^\nu}$$

(44)

where the symmetric tensor $h^{\mu\nu}$ is

$$h^{\mu\nu} = g^{\mu\nu} - \eta^{\mu\nu}$$

(45)

and $\eta^{\mu\nu}$ is the Minkowskian metric given by. It should be pointed out that all indices on $h^{\mu\nu}$ and/or $\partial/\partial x^\mu$ are raised or lowered with the use of the Minkowskian metric. Additionally, the energy-momentum complex of Weinberg is symmetric in its indices

$$\tau^{\mu\nu} = \tau^{\nu\mu}$$

(46)

while the superpotential $Q^{\rho\nu\lambda}$ is antisymmetric in its first two indices

$$Q^{\rho\nu\lambda} = -Q^{\nu\rho\lambda} \, .$$

(47)
The energy and momentum in Weinberg’s prescription for a three-dimensional background are given by
\[ P^\nu = \int \int \int \tau^{\nu t} dx^1 dx^2 \]  
(48)
and specifically the energy of the physical system in a three-dimensional background is
\[ E = \int \int \int \tau^{tt} dx^1 dx^2 . \]  
(49)
It should be noted again that the calculations in the Weinberg’s prescription as in all three afore-mentioned prescriptions have to be restricted to the use of quasi-Cartesian coordinates.
Since our aim is to evaluate the energy and momentum distributions in the non-static \((2 + 1)\)-dimensional black hole background described by the line element (4), we firstly evaluate the Weinberg’s superpotentials. There are sixteen nonvanishing superpotentials in the Weinberg’s prescription for the specific gravitational background given by (4) but the required ones are the following
\[
\begin{align*}
Q^{ttt} &= 0, \\
Q^{xtt} &= \frac{x(1 + \Lambda(x^2 + y^2) + m)}{r^2}, \\
Q^{ytt} &= \frac{y(1 + \Lambda(x^2 + y^2) + m)}{r^2}, \\
Q^{txt} &= -\frac{x(1 + \Lambda(x^2 + y^2) + m)}{r^2}, \\
Q^{xxt} &= 0, \\
Q^{yxt} &= 0, \\
Q^{tyt} &= -\frac{y(1 + \Lambda(x^2 + y^2) + m)}{r^2}, \\
Q^{xyt} &= 0, \\
Q^{yyt} &= 0 .
\end{align*}
\]  
(50)
The mass parameter \( m = m(u) \) in terms of the quasi-Cartesian coordinates is of the form
\[ m(u) = m(t, \sqrt{x^2 + y^2}) . \]  
(51)
Substituting the Weinberg’s superpotentials (50) into equation (43), the energy density distribution takes the form
\[ \tau^{tt} = \frac{m'}{2\kappa r} + \frac{\Lambda}{\kappa} . \]  
(52)
while the momentum density distributions are given by

\[
\tau^{xt} = -\frac{x\dot{m}}{2\kappa r^2}, \\
\tau^{yt} = -\frac{y\dot{m}}{2\kappa r^2}.
\]  

(53)

Thus, we substitute expressions (52) and (53) into equations (49) and (48), respectively, and the energy and momentum distributions associated with the non-static (2 + 1)-dimensional black hole under study, which are contained in a “sphere” of radius \( r_0 \), are given by

\[
E(r_0) = \frac{\pi}{\kappa} (m + \Lambda r_0^2), \\
P_x = 0, \\
P_y = 0.
\]  

(54)

It is evident again that we have again derived exactly the same energy and momentum distributions associated with the the non-static (2 + 1)-dimensional black hole as in the cases of the Einstein’s, Landau-Lifshitz’s and Papapetrou’s prescriptions. The comment concerning the neutral test particle and the effective gravitational mass made in Section 1 also holds here.

Furthermore, the energy-momentum complex of Weinberg as formulated in the (2 + 1)-dimensional black hole under study, satisfies the local conservation laws

\[
\frac{\partial}{\partial x^\nu} \tau^{\mu\nu} = 0.
\]  

(55)

6 Conclusions

In this work we have explicitly evaluated the energy and momentum distributions contained in a “sphere” of fixed radius of a (2 + 1)-dimensional black hole. The gravitational background under consideration is an exact solution of Einstein’s field equations in the presence of a cosmological constant \( \Lambda \) and a null fluid. Thus, it is regarded as a (2 + 1)-dimensional black hole with an outflux null radiation. The energy and momentum distributions are evaluated using four different energy-momentum complexes, specifically these are the energy-momentum complexes of Einstein, Landau-Lifshitz, Papapetrou and Weinberg. All four prescriptions give exactly the same energy and momentum distributions for the specific gravitational background. Consequently, the results obtained here support the
claim that for a given gravitational background, different energy-momentum complexes can give exactly the same results in three dimensions as they do in four dimensions.

The (2 + 1)-dimensional black hole solution under study was found by Virbhadra. He utilized the Landau-Lifshitz energy-momentum complex in order to evaluate the energy and momentum distributions in the case where the cosmological constant is zero. The results presented here can be viewed as a generalization of the ones derived by Virbhadra since our results in the limit of zero cosmological constant reproduce those obtained by Virbhadra.

It should also be pointed out that the energy distribution derived here can be regarded as the effective gravitational mass experienced by a neutral test particle placed in the (2 + 1)-dimensional black hole background under consideration. We have shown that in the case of a negative cosmological constant it is possible for the neutral test particle although in a gravitational background to move freely when a condition is fulfilled.

It is also noteworthy to observe that when the mass function of the (2+1)-dimensional black hole under investigation is set constant, the resultant gravitational background is the spinless BTZ black hole. Accordingly, one can derive from our results presented here the energy and momentum distributions associated with the spinless (2 + 1)-dimensional BTZ black hole just by setting the mass function constant. Furthermore, it seems quite interesting to calculate the energy and momentum distributions associated with the BTZ black hole by utilizing the Møller’s energy-momentum complex and to compare these results with the corresponding results derived here. Additionally, it will be also of some interest to investigate the contributions, if any, of the angular momentum of the BTZ black hole to the energy and momentum distributions. We hope to return to these issues in a future work.

Finally, since the (2+1)-dimensional BTZ black hole is an asymptotically Anti-de-Sitter spacetime (AAdS), it would be an oversight not to mention that in the last years due to the AdS/CFT correspondence there has been much progress in obtaining finite stress energy tensors of AAdS spacetimes. The gravitational stress energy tensor is in general infinite due to the infinite volume of the spacetime. In order to find a meaningful definition of gravitational energy one should subtract the divergences. The proposed prescriptions so far were ad hoc in the sense that one has to embed the boundary in

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1Recently, it was proven by the author that this is not the case for the two-dimensional stringy black hole backgrounds. Particularly, it was shown that Einstein’s energy-momentum complex provides meaningful physical results, i.e. energy and momentum distributions, while Møller’s energy-momentum complex fails to do so.

2For a short review see.
some reference spacetime. The important drawback of this method is that it is not always possible to find the suitable reference spacetime. Skenderis and collaborators\(^9\) \( [67, 71, 72, 73] \), and also Balasubramanian and Kraus \( [74] \), described and implemented a new method which provides an intrinsic definition of the gravitational stress energy tensor. The computations are universal in the sense that apply to all AAdS spacetimes. Therefore, it is nowadays right to state that the issue of the gravitational stress energy tensor for any AAdS spacetime has been thoroughly understood.

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\(^9\) Right after the first work of Henningson and Skenderis \( [67] \), Nojiri and Odintsov \( [68] \) calculated a finite gravitational stress energy tensor for an AAdS space where the dual conformal field theory is dilaton coupled. Furthermore, Nojiri and Odintson \( [69] \), and Ogushi \( [70] \) found well-defined gravitational stress energy tensors for AAdS spacetimes in the framework of higher derivative gravity and of gauged supergravity with single dilaton respectively.
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