Foliated Lie and Courant Algebroids

by

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ABSTRACT. If $A$ is a Lie algebroid over a foliated manifold $(M, F)$, a foliation of $A$ is a Lie subalgebroid $B$ with anchor image $TF$ and such that $A/B$ is locally equivalent with Lie algebroids over the slice manifolds of $F$. We give several examples and, for foliated Lie algebroids, we discuss the following subjects: the dual Poisson structure and Vaintrob’s super-vector field, cohomology and deformations of the foliation, integration to a Lie groupoid. In the last section, we define a corresponding notion of a foliation of a Courant algebroid $A$ as a bracket-closed, isotropic subbundle $B$ with anchor image $TF$ and such that $B^\perp/B$ is locally equivalent with Courant algebroids over the slice manifolds of $F$. Examples that motivate the definition are given.

The main categories of interest for Differential Geometry are the $C^\infty$ category and the complex analytic category. However, there also is a significant interest in the category of foliated manifolds. On the other hand, in the last thirty years Lie algebroids became a central theme of differential-geometric research, usually within the framework of the $C^\infty$ category. Recently, a general study of holomorphic Lie algebroids has also been done (see [9, 10]). The aim of the present paper is to start a similar study of Lie algebroids in the $C^\infty$-foliated category.

Essentially, we will say that a Lie subalgebroid $B$ is a foliation of the Lie algebroid $A$ over the foliated manifold $(M, F)$ if the anchor image of $B$ is $TF$ and $A/B$ is locally equivalent with Lie algebroids over the slice manifolds of $F$. In Section 1, after giving the precise definitions and first properties, we discuss several examples of foliated Lie algebroids $(A, B)$: classical foliations, transitive foliations that correspond to foliated principal bundles, foliated Dirac structures, etc. In Section 2, we show that the tangent Lie algebroid of a foliated Lie algebroid is a foliated Lie algebroid. In Section 3, we establish the characteristic properties of the dual Poisson structure and of Vaintrob’s super-vector field of a foliated Lie algebroid. In Section 4, we define the cohomological spectral sequence of a foliated Lie algebroid and prove a Poincaré lemma in a particular case. Then, we define deformations of a foliation $B$ of $A$ and prove the existence of corresponding, cohomological, infinitesimal deformations. In

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Section 5, we show that, if $A$ is integrable to a Lie groupoid $G$ and $B$ is a foliation of $A$, $G$ has a foliation $\mathcal{G}$ such that the restriction of the construction of the Lie algebroid of $A$ to the leaves of $\mathcal{G}$ produces the Lie subalgebroid $B$. Finally, in Section 6, we give a corresponding definition of foliated Courant algebroids. With a notation similar to the above, in the Courant case we will ask $B$ to be a bracket-closed, isotropic subbundle of the Courant algebroid $A$, with anchor image $T F$, such that $B^\perp/B$ is locally equivalent with Courant algebroids over the slice manifolds of $F$. A usual foliation is a foliation of the Courant algebroid $TM \oplus T^*M$ in this sense.

In the whole paper, we will use the Einstein summation convention.

1 Definitions and examples

We begin by recalling some basic facts concerning foliations; the reader may find details in [13]. Consider an $m$-dimensional manifold $M$ endowed with a $C^\infty$-foliation $F$ with $n$-dimensional leaves and codimension $k = m - n$. Then, each point has local adapted coordinates $(x^a, y^u)$ $(a, b, ..., = 1, ..., k; u = 1, ..., n)$, with origin-centered, rectangular range, such that the local equations of $F$ are $x^a = \text{const.}$ and $y^u$ are leaf-wise coordinates. The foliation has the following, associated exact sequence of vector bundles

$$0 \rightarrow T F \overset{\iota}{\rightarrow} TM \overset{\psi}{\rightarrow} \nu F \rightarrow 0,$$

where $T F$ is the tangent bundle of the leaves, which will also be denoted by $F$, and $\nu F = TM/F$ is the transversal bundle of the foliation.

A mapping $\varphi : (M_1, F_1) \rightarrow (M_2, F_2)$ between two foliated manifolds is a foliated mapping if it sends leaves of $F_1$ into leaves of $F_2$. The local equations of such a map are of the form

$$x'^a_2 = x'^a_1(x'^{a_1}_1), \quad y'^u_2 = y'^u_2(x'^{a_1}_1, y'^{u_1}_1).$$

The mapping $\varphi$ is called a leaf-wise immersion, submersion, local diffeomorphism if the morphism induced by its differential $\varphi_*$ between the tangent bundles $T F_1, T F_2$ is an injection, surjection, isomorphism, respectively. There also exists a similar notion of a transversal immersion, submersion, local diffeomorphism that refers to the transversal bundles $\nu F_1, \nu F_2$.

Let $\pi_{P \rightarrow M} : P \rightarrow (M, F)$ (notice our notation for projections of bundles onto the base manifold; the index of $\pi$ will be omitted if no confusion is feared) be a principal $G$-bundle over the foliated manifold $(M, F)$. $P$ is a foliated bundle if it is endowed with a foliation $\mathcal{F}^P$ that satisfies the conditions:

(i) $T \mathcal{F}^P$ has a trivial intersection with the tangent spaces of the fibers (i.e., the intersection is equal to 0 and $T \mathcal{F}^P$ is “horizontal”),
(ii) $\mathcal{F}^P$ is invariant by $G$-right translations,
(iii) the projection $\pi$ is a foliated leaf-wise submersion.

In fact, because of (ii) $\pi$ induces a covering map between the leaves of $\mathcal{F}^P$ and the corresponding leaves of $F$. Equivalently, a foliated principal bundle
may be characterized by an atlas of local trivializations with \( G \)-valued, foliated transition functions where \( G \) is foliated by points. A foliated principal bundle \( P \) may be seen as the glue-up of pullbacks of \( G \)-principal bundles on the local transversal manifolds of the leaves of \( \mathcal{F} \).

The associated bundles of a foliated principal bundle are called foliated bundles too; they also have an induced foliation with leaves that cover the leaves of \( \mathcal{F} \). Accordingly, a vector bundle is foliated if the corresponding principal bundle of frames is foliated. It follows that a foliated structure on a vector bundle \( \pi: A \to M \) is a maximal local-trivialization atlas, defined on neighborhoods \( \{U\} \), where the local bases \( (b_i) \) are related by matrices of \( \mathcal{F} \)-foliated functions. Then, \( A \) has the \( \mathcal{F} \)-covering foliation \( \mathcal{F}^A \) and a cross section \( s \) of \( A \) is foliated if the corresponding mapping \( s: (M, \mathcal{F}) \to (A, \mathcal{F}^A) \) is foliated. Over a trivializing neighborhood \( U \) we have \( s = \alpha^i b_i \) where \( \alpha^i \) are \( \mathcal{F} \)-foliated functions. Furthermore, we have a sheaf \( \Phi_{\mathcal{F}}(A) \) of germs of foliated cross-sections, which is a locally free sheaf-module over the sheaf \( \Phi_{\mathcal{F}} \) of germs of foliated functions on \( M \) and which spans the sheaf \( \Phi(A) \) of germs of cross sections of \( A \) over \( C^\infty(M) \). Using the fact that \( \Phi(A) \) is a locally free sheaf it follows that every subsheaf \( \Phi_{\mathcal{F}}(A) \subseteq \Phi(A) \) that is a locally free sheaf-module over \( \Phi_{\mathcal{F}} \) and spans \( \Phi(A) \) yields a well defined foliated structure on \( A \). Indeed, let \( a_i \) be a basis of \( \mathcal{F} \)-foliated functions. Since \( \Phi_{\mathcal{F}}(A) \) spans \( \Phi(A) \), we have \( a_i = \sum \beta^i \alpha^j b_j \) with the germs of \( b_j \in \Phi_{\mathcal{F}}(A) \). The total number of elements \( b_j \) is finite and an independent subsystem may be used as a foliated local basis. Two such bases are related by foliated transition functions.

A foliated structure of \( A \) may be identified with a family of vector bundles \( A_{Q_u} \) over quotient spaces \( Q_u = U/\{u\_\mathcal{F}\} \) such that \( A|_U = \pi^{-1}_{u\_\mathcal{F}}(A_{Q_u}) \) (\( \{U\} \) is an open covering of \( M \) and \( \pi^{-1} \) denotes bundle pullback). Conversely, a family \( \{A_{Q_u}\} \) defines a foliated bundle \( A \) if \( \pi^{-1}_{u\_\mathcal{F}}(A_{Q_u}|_V) \) \( = \pi^{-1}_{v\_\mathcal{F}}(A_{Q_v}|_V) \). If \( \mathcal{F} \) consists of the fibers of a submersion \( M \to Q \), and if these fibers are assumed to be connected and simply connected, then a foliated bundle \( A \) over \( M \) is the pullback of a projected bundle \( A_Q \to Q \) (see Lemma 2.5 and Proposition 2.7 of [18]).

Finally, a foliated morphism \( \phi: E_1 \to E_2 \) between two foliated vector bundles over the same basis \( (M, \mathcal{F}) \) is a vector bundle morphism that sends foliated cross sections to foliated cross sections.

The most important example of a foliated vector bundle is the transversal bundle \( \nu\mathcal{F} \) of the foliation \( \mathcal{F} \). Notice that the foliated cross sections \( s \in \Gamma_{fol\nu\mathcal{F}} \) act on foliated functions \( f \in C^\infty_{\mathcal{F}_{fol}}(M, \mathcal{F}) \) (the index “\( \mathcal{F}_{fol} \)” is a shortcut for “\( \mathcal{F} \)-foliated”) by \( s(f) = Xf \in C^\infty_{\mathcal{F}_{fol}}(M, \mathcal{F}) \), for every \( X \) such that \( s = [X]_{mod\nu\mathcal{F}} \) (\( s(f) \) is well defined since \( X \) is a foliated vector field, i.e., a foliated cross section \( X: (M, \mathcal{F}) \to (TM, T\mathcal{F}) \)). Furthermore, on the space \( \Gamma_{fol\nu\mathcal{F}} \), there exists a well defined Lie algebra bracket

\[
[[X_1]_{mod\nu\mathcal{F}}, [X_2]_{mod\nu\mathcal{F}}]_{\nu\mathcal{F}} = [X_1, X_2]_{mod\nu\mathcal{F}}
\]

induced by the Lie bracket of the corresponding foliated vector fields.

Accordingly, the definition of a Lie algebroid suggests the following definition.
Definition 1.1. An $\mathcal{F}$-transversal-Lie algebroid over $(M, \mathcal{F})$ is a foliated bundle $E$ endowed with a Lie algebra bracket $[\cdot, \cdot]_E$ on $\Gamma_{fol}E$ and a foliated, transversal anchor morphism $\sharp_E : E \to \nu \mathcal{F}$ such that

1) $\sharp_E[e_1, e_2]_E = [\sharp_E e_1, \sharp_E e_2]_\nu \mathcal{F}$, $\forall e_1, e_2 \in \Gamma_{fol}E$,
2) $[e_1, fe_2]_E = f[e_1, e_2]_E + (\sharp_E e_1(f))e_2$, $\forall e_1, e_2 \in \Gamma_{fol}E, f \in C^\infty(M)$.

The symbols $\sharp, [\cdot, \cdot]$ will be used for any transversal-Lie and Lie algebroid while the index that denotes the bundle will be omitted if there is no risk of confusion. Notice also that we may not ask the morphism $\sharp_E$ to be foliated, a priori; this follows from condition 2), which implies $\sharp_e(f) \in C^\infty(M, \mathcal{F})$, $\forall f \in C^\infty(M, \mathcal{F})$. The following result is obvious.

Proposition 1.1. If $E$ is a transversal-Lie algebroid over $(M, \mathcal{F})$, $\nu \mathcal{F}$ projects to the tangent bundles of the local quotient spaces $Q_U$ of $\mathcal{F}$ and the local projected bundles $\{E_{Q_U}\}$ are Lie algebroids. In the case of a submersion $M \to Q$ with connected and simply connected fibers, $E \to M$ projects to a Lie algebroid over $Q$.

A transversal-Lie algebroid is not a Lie algebroid, and we shall define the notion of a foliated Lie algebroid as follows.

If $A$ is a Lie algebroid over $(M, \mathcal{F})$ and if $B$ is a Lie subalgebroid, a cross section $a \in \Gamma A$ will be called a $B$-foliated cross section if $[b, a]_A \in \Gamma B$, $\forall b \in \Gamma B$. The correctness of this definition follows from the fact that $[b, a]_A \in \Gamma B$ implies $[fb, a]_A = f[b, a]_A - (\sharp_A a(f))b$ $\in \Gamma B$. We shall denote by $\Gamma_B A$ the space of $B$-foliated cross sections of $A$. The Jacobi identity shows that the space $\Gamma_B A$ is closed by $A$-brackets. If $\sharp_A(B) \subseteq T \mathcal{F}$ then, $\forall a \in \Gamma_B A$ and $\forall f \in C^\infty(M, \mathcal{F})$, we have $[b, fa]_A \in \Gamma_B A$. Now, we give the following definition.

Definition 1.2. If $A$ is a Lie algebroid on $(M, \mathcal{F})$, a foliation of $A$ over $\mathcal{F}$ is a Lie subalgebroid $B \subseteq A$ that satisfies the following two conditions: 1) $\sharp_A(B) = F$, 2) locally, $\Gamma A$ is spanned by $\Gamma_B A$ over $C^\infty(M)$ (this is called the foliated-generation condition). A pair $(A, B)$ as above will be called a foliated Lie algebroid. If $\sharp_A : B \to F$ is a vector bundle isomorphism, $(A, B)$ is a minimally foliated Lie algebroid.

Remark 1.1. The condition that $B$ is closed by $A$-brackets may be replaced by the Frobenius condition: $\forall \alpha \in \text{ann } B$, the $A$-exterior differential $d_A \alpha$ belongs to the ideal of $A$-forms generated by $\text{ann } B$.

Lemma 1.1. If $B$ is a foliation of $A$ and $C$ is a complementary subbundle of $B$ in $A$ $(A = B \oplus C)$, $\Gamma C$ has local bases that consist of $B$-foliated cross sections of $A$.

Proof. Let $(c_h)$ be a local basis of $\Gamma C$ over an open neighborhood $U \subseteq M$. By the foliated-generation condition we have

$$c_h = \gamma_h^u a(h)_u = \gamma_h^u pr_C a(h)_u,$$
where $\gamma^u_h$ are differentiable functions and $a_{(h)u}$, therefore, $pr_C a_{(h)u}$ too, are local $B$-foliated cross sections of $A$. Since the set $\{pr_C a_{(h)u}\}$ is finite, after shrinking the neighborhood $U$ as necessary, there exists a subset of linearly independent, local cross sections $\{pr_C a_v\}$ such that $c_h = \lambda^v_h a_v pr_C a_v$. This subset obviously is the required local basis of $\Gamma C$.

**Remark 1.2.** The complementary subbundle $C$ may have elements with projection in $F$ since we did not ask $B = \sharp_A^{-1}(F)$. On the other hand, by condition 1) of Definition 1.2 there exist decompositions $B = ker \sharp_B \oplus P$ where $P$ is a subbundle of $B$.

**Proposition 1.2.** If $B$ is a foliation of the Lie algebroid $A$, then the vector bundle $E = A/B$ is a transversal-Lie algebroid on $(M,F)$.

**Proof.** Consider a decomposition $A = B \oplus C$ and take $B$-foliated local bases of $C$ over an open covering $\{U\}$ of $M$, which exist by Lemma 1.1. Let $\tilde{a}_w = \lambda^v_w a_w$ be a transition between two such bases and take the bracket by $b \in \Gamma B$. We get $(\sharp b) \lambda^v_w = 0$ and, since $\sharp(B) = F$, the functions $\lambda^v_w$ must be $F$-foliated. Thus, $C$ has an induced foliated structure, which transfers to $A/B \approx C$. We notice that $[a]_{mod.B} \in \Gamma_{fol}(A/B)$ if and only if $a \in \Gamma_B A$. Furthermore, the bracket of $B$-foliated cross sections of $A$, which is $B$-foliated too, descends to a well defined Lie bracket on $\Gamma_{fol} E$. Finally, the anchor $\sharp_A$ induces an anchor $\sharp_E : E \to \nu F$ as required by Definition 1.1. The fact that $\sharp_E$ is a foliated morphism may be justified as in the observation that follows Definition 1.1. Alternatively, we can see that, for all $a \in \Gamma_B A$, $\sharp_A a$ is a foliated vector field on $(M,F)$ as follows: condition 1) of Definition 1.2 shows that a vector field $Y \in \Gamma F$ is of the form $Y = \sharp_A b$, $b \in \Gamma B$, hence,

$$[Y,\sharp_A a]_A = [\sharp_A b, \sharp_A a]_A = \sharp_A [b, a] \in \Gamma F,$$

which is the required property. \qed

**Corollary 1.1.** A foliated Lie algebroid $(A,B)$ over $(M,F)$ produces a commutative diagram

$$
\begin{array}{ccc}
0 & \rightarrow & B \\
\sharp_B \downarrow & & \sharp_A \\
0 & \rightarrow & T\nu F
\end{array}
\quad
\begin{array}{ccc}
A & \rightarrow & E \\
\sharp_A \downarrow & & \downarrow \\
TM & \rightarrow & \nu F
\end{array}
\quad
0
$$

where the lines are exact sequences of vector bundles.

**Definition 1.3.** If the transversal-Lie algebroid $E$ is foliately equivalent with a quotient $A/B$ where $(A,B)$ is a foliated Lie algebroid, then $E$ may be placed in a diagram (1.4) and $A$ will be called an extension of $E$. An extension of $E$ such that $B \approx F$ will be called a minimal extension.

In what follows we give several examples of foliated Lie algebroids.

**Example 1.1.** For any foliation $F$ of a manifold $M$, $T\nu F$ is a foliation of the Lie algebroid $TM$. 

5
Example 1.2. If a Lie algebroid $A$ has a regular subalgebroid $B$, $M$ has the foliation $\mathcal{F}$ such that $F = \sharp(B)$. Then, $(A, B)$ is a foliated Lie algebroid if the foliated-generation condition is satisfied. For instance, if $A$ is regular, we may take $B = A$ and the foliated-generation condition is trivially satisfied. Another obvious example is that of a regular subalgebroid $B$ such that $B \subseteq \ker \sharp_A$. Then $B$ is a foliation of $A$ over the foliation of $M$ by points. For instance, let $J^1A$ be the first jet Lie algebroid of $A$ defined in [5]. It is well known that one has the exact sequence
\begin{equation}
0 \to \text{Hom}(TM, A) \xrightarrow{\pi} J^1A \xrightarrow{\sharp} A \to 0,
\end{equation}
where $\pi(j^1_x a) = a(x)$ ($x \in M, a \in \Gamma A$) is a Lie algebroid morphism, therefore, $\ker \pi = \text{Hom}(TM, A)$ is a regular Lie subalgebroid of $J^1A$. This subalgebroid is included in $\ker \sharp_{j^1A}$ because $j^1_x a \in \ker \pi_x$ implies $\sharp_{j^1A}(j^1_x a) = \sharp_A(a(x)) = 0$ (the equality $\sharp_{j^1A}(j^1_x a) = \sharp_A(a(x))$ is a part of the definition given in [5]). Thus, $\text{Hom}(TM, A)$ is a foliation of $J^1A$ over the points of $M$.

Example 1.3. The typical example of a transitive Lie algebroid is $A = TP/G$, where $\pi : P \to M$ is a principal bundle of structure group $G$ over $M$ and the quotient is the space of the orbits of $TP$ by the action of the differential of the right action of $G$ on $P$. The cross sections of $A$ may be identified with right-invariant vector fields on $P$, which allows to define the bracket, and the anchor is induced by $\pi_* : TP \to TM$ [15]. Now, assume that $M$ has the foliation $\mathcal{F}$ and $P$ is a foliated principal bundle with the covering foliation $\mathcal{F}^P$ of $\mathcal{F}$. Then, we get a Lie subalgebroid $B = T\mathcal{F}^P/G \subseteq A$ such that the restriction of the anchor of $A$ to $B$ has the image $T\mathcal{F}$. The foliated-generation condition of $A$ with respect to $B$ is satisfied because it is satisfied for $TP$ with respect to $T\mathcal{F}^P$ and $A/B = (TP/T\mathcal{F})/G$. Therefore, $(A, B)$ is a foliated Lie algebroid.

Example 1.4. Following [15], if $P$ is the bundle of frames of a vector bundle $\pi : E \to M$, Example 1.3 has the following equivalent form. Let $A$ be the transitive Lie algebroid $D(E)$ whose cross sections are the linear differential operators $\Gamma E \to \Gamma E$ of order $\leq 1$. The bracket is the commutant of the operators and the anchor is defined by the symbol of the operator. In the presence of a foliation $\mathcal{F}$ on $M$ and if we assume $E$ to be foliated, $D(E)$ has a Lie subalgebroid $B = \mathcal{D}_{\mathcal{F}}(E)$ defined by the operators whose symbol is a vector field tangent to $\mathcal{F}$ ($\mathcal{D}_{\mathcal{F}}(E)$ is commutant closed since $F = T\mathcal{F}$ is closed by Lie brackets), and the anchor sends $B$ onto the tangent bundle $F$: if $Y \in \Gamma F$, we get an operator $D$ of symbol $Y$ by putting $Ds = 0$ for $s \in \Gamma_{\text{fol}}(E)$ and
\begin{equation}
Dt = \sum (Y f_i) s_i, \quad \forall t = \sum f_i s_i, \quad s_i \in \Gamma_{\text{fol}}E, f_i \in C^\infty(M)
\end{equation}
(the definition is correct since if $s \in \Gamma_{\text{fol}}(E)$ and $f \in C^\infty_{\text{fol}}(M, \mathcal{F})$ [1.6] gives $D(fs) = 0$). The foliated-generation condition is satisfied too. Indeed, an operator of symbol $X$ is of the form $D = \nabla_X + \phi$, where $\nabla$ is a covariant derivative and $\phi \in \text{End}(E)$, therefore, $\phi \in \mathcal{D}_{\mathcal{F}}(E)$. The operator $D$ is foliated with respect to $\mathcal{D}_{\mathcal{F}}(E)$ if and only if $X$ is foliated with respect to $\mathcal{F}$ and it
is known that the vector fields on $M$ satisfy the foliated-generation condition. Thus, the pair $(A, B)$ considered above is a foliated Lie algebroid.

**Example 1.5.** Let $D$ be a Dirac structure on $(M, \mathcal{F})$ such that $F \subseteq D$ ($D$ is $\mathcal{F}$-projectable [23]). Then, $(D, F)$ is a foliated Lie algebroid. The fact that the foliated-generation condition is satisfied was proven in [25] (where, also, concrete examples were given). We notice that there are foliated Dirac structures where $F$ is strictly included in $\sharp_D^{-1}(F)$ ($\sharp_D = pr_{TM}$). For instance,

$$D = F \oplus \{([\sharp \alpha, \alpha]) / \alpha \in \text{ann } F\},$$

where $\alpha$ is a foliated bivector field on $M$ such that the Schouten-Nijenhuis bracket $[P, P]$ vanishes on $\text{ann } F$ ([24], Example 5.1).

**Example 1.6.** Let $D$ be a regular Dirac structure on $M$ with the (regular) characteristic foliation $\mathcal{E}$, $E = T\mathcal{E} = pr_{TM}D$ and let $\mathcal{F}$ be a subfoliation of $\mathcal{E}$. Put

$$D_\mathcal{F} = D \cap (F \oplus T^\ast M).$$

Notice that the mapping $\psi : D/D_\mathcal{F}(x) \rightarrow E/F(x)$ given by $[(X, \xi)]_{\text{mod. } D_\mathcal{F}} \mapsto [X]_{\text{mod. } F}$ is an isomorphism of vector spaces $\forall x \in M$. The existence of this isomorphism shows that $\text{dim } D_\mathcal{F}(x) = \text{const}$. Hence, $D_\mathcal{F}$ is a vector subbundle of $D$, which obviously is a Lie subalgebroid of $D$ with anchor image $F$. Moreover, since $\mathcal{F}$ is a subfoliation of $\mathcal{E}$, the vector bundle $E/F$ is $\mathcal{F}$-foliated and so is its isomorphic image $D/D_\mathcal{F}$. This shows that the pair $(D, D_\mathcal{F})$ satisfies the foliated-generation condition and is a foliated Lie algebroid.

**Proposition 1.3.** For any transversal-Lie algebroid $E$ over the foliated manifold $(M, \mathcal{F})$ and any lift $\rho : E \rightarrow TM$ of $\sharp_E$ ($\psi \circ \rho = \sharp_E$, $\psi = pr_{\nu E}$) there exists a canonical minimal extension $(A_0, \sharp_0, [\cdot, \cdot]_0)$ with $A_0 = F \oplus E$, $\sharp_0 = Id + \rho$ and

$$[Y_1, Y_2]_0 = [Y_1, Y_2], \ [Y, e]_0 = [Y, \rho e], \ [e, Y]_0 = [\rho e, Y]$$

$$[e_1, e_2]_0 = ([\rho e_1, \rho e_2] - \rho[e_1, e_2]) + [e_1, e_2],$$

$\forall Y, Y_1, Y_2 \in \Gamma F$, $\forall e, e_1, e_2 \in \Gamma_{\text{fol } E}$, where the non-indexed brackets are Lie brackets of vector fields. Conversely, for any minimal extension $A$ of $E$, there exists a lift $\rho$ such that $A$ is isomorphic to a twisted form of the canonical extension of $E$ by $F$.

**Proof.** The qualification “twisted” is similar to that used in the notion of a twisted Dirac structure and its exact meaning will appear at the end of the proof. We shall use the content and notation of diagram (1.4), which includes the mapping $\psi = pr_{\nu E}$. Notice that a lift $\rho$ has the property that $\rho e$ is a foliated vector field $\forall e \in \Gamma_{\text{fol } E}$. If we extend (1.7) by

$$[Y, f e]_0 = f[Y, \rho e] + (Y f)e, \ [e_1, f e_2]_0 = f[e_1, e_2]_0 + \rho e_1(f)e_2,$$

$\forall f \in C^\infty(M)$, we get a skew-symmetric bracket on $\Gamma A_0$. It is easy to see that the extension is compatible with (1.7) for $f \in C^\infty_{\text{fol }}(M, \mathcal{F})$. Also, straightforward calculations show that the axioms of a Lie algebroid, including the Jacobi
identity, are satisfied on arguments $Y \in \Gamma F, e \in \Gamma_{fol} E$. Since these types of cross sections locally generate $\Gamma A_0$, this proves the existence of the canonical extension. Notice that, for $E = \nu F$, $\rho$ is a splitting of the lower line of (1.4) and, if we identify $F \oplus \approx F \oplus \text{im} \rho = TM$, the previous construction yields the standard Lie algebroid structure of $TM$.

For the converse, we start with diagram (1.4), which yields $A = \phi(F) \oplus \tau(E)$ where $\phi : F \approx B, \phi^{-1} = \sharp_B$ (if exists because the extension is minimal) and $\tau : E \rightarrow A$ is a splitting of the upper exact sequence of the diagram ($\sigma \circ \tau = \text{Id}$). Then, we consider the lift $\rho = \sharp_A \circ \tau$, which implies $\sharp_A = \phi^{-1} + \rho \circ \sigma$. As for the brackets, we must have

$$\phi^{-1}[\phi Y_1, \phi Y_2]_A = \sharp_A[\phi Y_1, \phi Y_2]_A = [\sharp_A \phi Y_1, \sharp_A \phi Y_2] = [Y_1, Y_2],$$

i.e., $[\phi Y_1, \phi Y_2]_A = \phi[Y_1, Y_2]$. Then, since $\sigma[\phi Y, \tau e]_A = [\sigma \phi Y, \sigma \tau e]_E = 0, [\phi Y, \tau e]_A \in \Gamma B$ and

$$\phi^{-1}[\phi Y, \tau e]_A = \sharp_A[\phi Y, \tau e]_A = [Y, \sharp_A \tau e] \Leftrightarrow [\phi Y, \tau e]_A = \phi[Y, \rho e], \forall e \in \Gamma_{fol} E.$$

Finally, for $e_1, e_2 \in \Gamma_{fol} E$, we have

$$\sharp_A[\tau e_1, \tau e_2]_A - \rho[e_1, e_2]_E = [\rho e_1, \rho e_2] - \rho[e_1, e_2]_E \in \Gamma F$$

because, if we apply $\psi$ to the right hand side of the equality, we get

$$\psi[\rho e_1, \rho e_2] - \sharp_E[e_1, e_2] = [\psi \rho e_1, \psi \rho e_2]_E - \sharp_E[e_1, e_2] = 0$$

by the axioms of a transversal-Lie algebroid. Now, we notice that, if $a \in A$, $\sharp_A a \in \Gamma F$ if and only if $a = \phi Y + \tau e$ where $e \in \ker \sharp_E$; indeed, by the commutativity of diagram (1.4), $\psi \sharp_A \tau(e) = 0$ is equivalent to $\sharp_E e = 0$. This fact and the previous observation justify the formula

$$(1.8) \quad [\tau e_1, \tau e_2]_A = ([\tau e_1, \tau e_2]_A - \tau[e_1, e_2]_E) + \tau[e_1, e_2]_E$$

$$= [\phi([\rho e_1, \rho e_2] - \rho[e_1, e_2]_E) + \tau(\lambda(e_1, e_2))] + \tau[e_1, e_2]_E,$$

where $\lambda : \wedge^2 E \rightarrow E$ is a 2-form with values in the kernel of $\sharp_E$. Therefore, $A$ is isomorphic by $\phi + \tau$ to the canonical extension of $E$ by $F$ twisted by the addition of the form $\lambda$. The latter must also be asked to satisfy conditions that ensure the Jacobi identity.

**Remark 1.3.** In [3] the authors define a reduction of Lie algebroids that, essentially, is a projection along the fibers of a surjective submersion $\pi : M \rightarrow M'$. Namely, $A \rightarrow M$ reduces to $A' \rightarrow M'$ if there exists a Lie algebroid morphism $[15]$ $(\Pi : A \rightarrow A', \pi : M \rightarrow M')$. One can see that, if $B$ is a foliation of $A$ over the foliation of $M$ by the fibers of $\pi$ and if the fibers are connected and simply connected, then $A$ reduces to $A'$ defined by $\Pi/B = \pi^{-1}A'$.
2 The tangent Lie algebroid

In this section we will show that the tangent Lie algebroid of a foliated Lie algebroid \((M,B)\) over a foliated manifold \((M,F)\) has a natural structure of a foliated Lie algebroid over \((TM,F)\). For this purpose, we formulate the definition of the tangent Lie algebroid of a Lie algebroid \([16]\) in a suitable (but, not new) form.

Let \(\pi : A \to M\) be an arbitrary vector bundle. Take a neighborhood \(U \subseteq M\) with local coordinates \((x^i)_{i=1}^m\), with the local basis \((a_\alpha)_{\alpha=1}^r\) of \(\Gamma A (r=\text{rank} A)\) and the corresponding fiber coordinates \((\xi^\alpha)\). On the intersection of two such neighborhoods \(U \cap \tilde{U}\), these coordinates have transition functions of the following local form

\[
\tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{\xi}^\alpha = \Theta_{\tilde{\beta}}^\alpha(x^j)\xi^\beta \quad (\tilde{a}_\alpha = \Psi^\gamma_\alpha a_\gamma, \quad \Psi^\gamma_\alpha \Theta_{\tilde{\beta}}^\alpha = \delta^\beta_\gamma).
\]

Correspondingly, one has natural coordinates \((x^i,\dot{x}^i)\) on \(T_U M\) and \((x^i,\xi^\alpha,\dot{x}^i,\dot{\xi}^\alpha)\) on \(T_{\pi^{-1}U} A\), with the following transition functions:

\[
\tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{\xi}^\alpha = \Theta_{\tilde{\beta}}^\alpha(x^j)\xi^\beta, \quad \dot{\tilde{x}}^i = \frac{\partial \tilde{x}^i}{\partial x^j}\dot{x}^j, \quad \dot{\tilde{\xi}}^\alpha = \frac{\partial \Theta_{\tilde{\beta}}^\alpha}{\partial x^j}\xi^\beta + \Theta_{\tilde{\beta}}^\alpha \dot{\xi}^\beta.
\]

Formulas (2.2) show the existence of the double vector bundle

\[
\begin{array}{ccc}
TA & \xrightarrow{\pi} & TM \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi} & M
\end{array}
\]

where \(TA\) is a vector bundle over \(A\) with base coordinates \((x^i,\xi^\alpha)\) and fiber coordinates \((\dot{x}^i,\dot{\xi}^\alpha)\) and \(TA\) is a vector bundle over \(TM\) with base coordinates \((x^i,\dot{x}^i)\) and fiber coordinates \((\xi^\alpha,\dot{\xi}^\alpha)\). (Notice that the cross sections of \(\pi_*\) are not vector fields on \(A\))

If \(A = TM\), the indices \(\alpha, \beta, \ldots\) may be replaced by \(i, j, \ldots\) and \(\Theta_i^i = \partial \dot{x}^i/\partial x^i\), which shows the existence of the flip diffeomorphism \(\phi : TTM \to \tilde{TTM}\) defined by the local coordinate equations

\[
\phi(x^i,\xi^\alpha,\dot{x}^i,\dot{\xi}^\alpha) = (x^i,\dot{x}^i,\xi^\alpha,\dot{\xi}^\alpha).
\]

On the fibration \(TA \to A\) the fiber coordinates \((\dot{x}^i,\dot{\xi}^\alpha)\) are produced by the local bases of cross sections \((\partial/\partial x^i, \partial/\partial \xi^\alpha)\). On the fibration \(\pi_* : TA \to TM\) the fiber coordinates \((\xi^\alpha,\dot{\xi}^\alpha)\) are produced by local bases \((c_\alpha, \partial/\partial \xi^\alpha)\), where the vector \(c_\alpha\) is the image of \(a_\alpha\) under the natural identification of the tangent space of the fibers of \(A\) with the fibers themselves, which are vector spaces.

(The previous assertion is justified by checking the invariance of the expression \(\xi^\alpha c_\alpha + \dot{\xi}^\alpha (\partial/\partial \xi^\alpha)\) by the coordinate transformations (2.2).) It follows that for \(z \in TA\) the intersection space of the fibers of the two vector bundle structures of \(TA\) is

\[
P_z = \pi_{TA \to A}^{-1}(\pi_{TA \to A}(z)) \cap (\pi_{A \to M})_*^{-1}((\pi_{A \to M})_*(z)) = \text{span} \left\{ \frac{\partial}{\partial \xi^\alpha} \right\}.
\]
Therefore, $P = \cup_{z \in TA} P_z$ is a vector subbundle of $TA$, which we call the bi-vertical subbundle.

Furthermore, like for the tangent bundle $A = TM$, there are two lifting processes of cross sections of $\pi: A \to M$ to cross sections of $\pi_*: TA \to TM$.

One is the complete lift $a \mapsto a^C = a_\ast$, which sends the cross section $a$ to the differential of the mapping $a: M \to A$. The expression of $a_\ast$ by means of the local coordinates $(x^i, \dot{x}^i)$ on $TM$ and $(x^i, \dot{x}^i, \xi^\alpha, \dot{\xi}^\alpha)$ on $TA$ is

$$(x^i, \dot{x}^i) \mapsto (x^i, \dot{x}^i, \xi^\alpha(x^i), \dot{\xi}^\alpha \frac{\partial}{\partial x^i}),$$

where $a = \xi^\alpha(x^i) a_\alpha$, whence we get

$$a^C = \xi^\alpha c_\alpha + \dot{x}^i \frac{\partial}{\partial x^i}$$

The second lift, called the vertical lift and denoted by an upper index $V$, is the isomorphism between the pullback $\pi_*^{-1} TA \to A \to M$ of the vector bundle $A$ to $TM$ and the bi-vertical subbundle $P$ defined by

$$a = \eta^\alpha a_\alpha \mapsto a^V = \eta^\alpha \frac{\partial}{\partial \xi^\alpha}.$$

(Formulas 2.11 show that the bases $(a_\alpha)$ and $(\partial/\partial \xi^\alpha)$ have the same transition functions.)

Notice the following interpretation of the local basis $(c_\alpha, \partial/\partial \xi^\alpha)$ of $TA$ over $TM$

$$a_\alpha^C = c_\alpha, \quad a_\alpha^V = \frac{\partial}{\partial \xi^\alpha}.$$

Notice also the following properties

$$fa^C = fa^C + f^C a^V, \quad (fa)^V = fa^V \quad (f \in C^\infty(M), f^C = \dot{x}^i \frac{\partial f}{\partial x^i}).$$

Now, assume that $(A, \sharp_A, [\cdot, \cdot]_A)$ is a Lie algebroid. Define an anchor $\sharp_{TA}: TA \to TTM$ by putting

$$\sharp_{TA} c_\alpha = (\sharp_A a_\alpha)^C, \quad \sharp_{TA} \frac{\partial}{\partial \xi^\alpha} = (\sharp_A a_\alpha)^V,$$

where in the right hand side the lifts are those of the Lie algebroid $TM$. A simple calculation shows that $\sharp_{TA} = \phi \circ (\sharp_A)_*$ with $\phi$ defined by 2.4.

Furthermore, define a bracket of cross sections of $\pi_*$ by putting

$$[a_\alpha^V, a_\beta^V]_{TA} = 0, \quad [a_\alpha^C, a_\beta^V]_{TA} = [a_\alpha, a_\beta]_A, \quad [a_\alpha^C, a_\beta^C]_{TA} = [a_\alpha, a_\beta]_A,$$

which yields brackets of the elements of the basis of cross sections of $TA \to TM$, and by extending 2.11 to arbitrary cross sections via the axioms of a Lie
algebroid. One can check that the results of (2.11) hold for the lifts of arbitrary cross sections of $A$ and this justifies the independence of the bracket of the choice of the basis $a_\alpha$. Then, the axioms of a Lie algebroid hold for the basic cross sections $(e_\alpha, \partial/\partial \xi^\alpha)$, whence, the axioms also hold for any cross sections.

Thus, we have obtained a well defined structure of a Lie algebroid on the bundle $\pi_* : TA \to TM$. This Lie algebroid is called the tangent Lie algebroid of $A$ [10].

Now, we prove the announced result

**Proposition 2.1.** If the subalgebroid $B \subseteq A$ is a foliation of $A$ over $(M, \mathcal{F})$, then the tangent Lie algebroid $TB$ is a foliation of the Lie algebroid $TA$ over $(TM, T\mathcal{F})$.

**Proof.** We may use adapted local coordinates $(x^a, y^u)$ on $M$, such that $\mathcal{F}$ has the local equations $x^a = 0$, and local bases $(a_\alpha) \equiv (b_\alpha, q_\alpha)$ of $\Gamma A$ such that $(b_\alpha)$ is a local basis of $\Gamma B$ and $(q_\alpha)$ are $B$-foliated cross sections of $A$. If we use formulas (2.8), (2.10), (2.11), we see that the complete and vertical lifts of cross sections of $B \subseteq A$ to $TA$ and to $TB$ coincide, and that the bracket and anchor of $TB$ are the restrictions of the bracket and anchor of $TA$ to $TB \subseteq TA$. Therefore, $TB$ is a Lie subalgebroid of $TA$ over $TM$.

Furthermore, on $TM$ we have local coordinates $(x^a, y^u, \dot{x}^a, \dot{y}^u)$ and (2.2) shows that $x^a = \text{const.}, \dot{x}^a = \text{const.}$ are the equations of a foliation $T\mathcal{F}$ of the manifold $TM$ that consists of the tangent vectors of the leaves of $\mathcal{F}$. Since $\mathfrak{z}_{TB} = \phi \circ (\mathfrak{z}_B)_*$ where $\phi$ is the flip diffeomorphism and $\text{im} \mathfrak{z}_B = T\mathcal{F}$, we get $\text{im} \mathfrak{z}_{TB} = T\mathcal{F}$.

Finally, on one hand, $(b_\alpha^C, b_\alpha^V)$ is a local basis of cross sections of $TB$ and, on the other hand, using (2.11), it is easy to check that $(q_\alpha^C, q_\alpha^V)$ are $TB$-foliated, local cross sections of $TA \to TM$. Therefore, since $(b_\alpha^C, q_\alpha^C, b_\alpha^V, q_\alpha^V)$ is a local basis of cross sections of $TA \to TM$, the pair $(TA, TB)$ satisfies the foliated-generation condition. 

\[\square\]

### 3 The dual Poisson structure

It is well known that a Lie algebroid structure on the vector bundle $\pi : E \to M$ is equivalent with a specific Poisson structure on the total space of the dual bundle $E^*$, called the dual Poisson structure, defined by the following brackets of basic and fiber-linear functions:

\[
\Pi = \frac{1}{2} \sum_{h,k} \alpha^{jk}_{hk} \frac{\partial}{\partial \eta^h} \wedge \frac{\partial}{\partial \eta^k} + \alpha^a_k \frac{\partial}{\partial \eta^k} \wedge \frac{\partial}{\partial x^a},
\]

where $f, f_1, f_2 \in C^\infty(M)$, $s, s_1, s_2 \in \Gamma E$ and $l_s$ is the evaluation of the fiber of $E^*$ on $s$. If $x^a$ are local coordinates on $M$ and $\eta_k$ are the fiber-coordinates on $E^*$ with respect to the dual of a local basis $(e_h)$ of cross sections of $E$, then the corresponding Poisson bivector field is
where the coefficients \( \alpha \) are defined by the expressions

\[
\sharp_E(e_h) = \alpha^a_h \frac{\partial}{\partial x^a}, \quad [e_h, e_k]_E = \alpha^{i}_{hk} e_i.
\]

Conversely, formulas (3.2), (3.3) produce an anchor \( \sharp_E \) and a bracket \( [\cdot, \cdot]_E \) and the Poisson condition \( [\Pi, \Pi] = 0 \) implies the Lie algebroid axioms.

A Poisson structure of the form (3.2) will be called a fiber-linear structure, although this name is not totally appropriate since it does not describe the form of the second term of (3.2). The characteristic property of a bivector field of the form (3.2) is that the Poisson bracket of two fiber-polynomials of degrees \( h, k \) is a fiber-polynomial of degree \( h + k - 1 \).

In this section we establish properties of the dual Poisson structure of a foliated Lie algebroid.

**Proposition 3.1.** Let \( E \) be a foliated vector bundle over \( (M, \mathcal{F}) \) with the dual bundle \( E^* \), which has the covering foliation \( \mathcal{F}^{E^*} \) of \( \mathcal{F} \). A transversal-Lie algebroid structure on \( E \) is equivalent with a fiber-linear Poisson algebra structure on the space of \( \mathcal{F}^{E^*} \)-foliated functions on \( E^* \).

**Proof.** Let \((x^a, y^u)\) be \( \mathcal{F} \)-adapted local coordinates on \( M \) and \((e_h)\) be a local foliated basis of \( E \). Then, the space of foliated functions \( C^\infty_{\text{fol}}(E^*, \mathcal{F}^{E^*}) \) is locally spanned by \((x^a, \eta_h)\) and we get the Poisson algebra structure required by the proposition using formulas (3.1) for \( f, f_1, f_2 \in C^\infty_{\text{fol}}(M, \mathcal{F}) \), \( s, s_1, s_2 \in \Gamma_{\text{fol}}E \).

Conversely, the Poisson structure (3.1) on \( C^\infty_{\text{fol}}(E^*, \mathcal{F}^{E^*}) \) produces a transversal-Lie algebroid on \( E \) in the same way as in the case of a Lie algebroid. \( \square \)

**Proposition 3.2.** A foliated Lie algebroid \( (A, B) \) over \( (M, \mathcal{F}) \) is equivalent with a couple \((E^*, \Lambda)\), where \( E^* \) is a vector subbundle of \( A^* \) endowed with a foliated structure and the corresponding \( \mathcal{F} \)-covering foliation \( \mathcal{F}^{E^*} \), and \( \Lambda \) is a fiber-linear Poisson structure on the manifold \( A^* \) with the following properties:

1) \( \sharp_\Lambda (\text{ann}TE^*) = T\mathcal{F}^{E^*}, \)
2) \( \Lambda|_{E^*} \) induces a well defined Poisson algebra structure on \( C^\infty_{\text{fol}}(E^*, \mathcal{F}^{E^*}) \),
3) the total bundle manifold of \( A^*/E^* \) has a Poisson structure \( P \) such that the projection \((A^*, \Lambda) \to (A^*/E^*, P)\) is a Poisson mapping.

**Proof.** For the foliated Lie algebroid \((A, B)\), we take \( E^* = \text{ann} B \), which is foliated since its dual bundle is \( E = A/B \) and it is foliated. Furthermore, we take the dual Poisson structure \( \Lambda \) of the Lie algebroid \( A \).

In order to write down the Poisson bivector field \( \Lambda \) we define convenient local coordinates on the manifold \( A^* \) as follows. We take \( \mathcal{F} \)-adapted local coordinates \((x^a, y^u)\) on \( M \), we choose a splitting

\[
A = B \oplus C,
\]

and we take a local basis of \( \Gamma A \) that consists of the basis \( b_h \) of \( \Gamma B \) and the basis \( a_q \) of \( \Gamma C \) where \( a_q \) are foliated cross sections (see Lemma 1.1). For \( A^* \), we have the dual bases \((b^*h, a^*q)\) and corresponding fiber coordinates \((\eta_h, \zeta_q)\).
Since \((A, B)\) is a foliated Lie algebroid, the anchor and bracket have the following local expression
\[
\begin{align*}
\tilde{z}_A(b_h) &= \beta_h^a \frac{\partial}{\partial q^a} + \alpha_q^a \frac{\partial}{\partial y^a}, \\
\tilde{z}_A(a_q) &= \alpha_q^a \frac{\partial}{\partial q^a} + \alpha_q^a \frac{\partial}{\partial y^a} \quad (\text{rank}(\beta_h^a) = \dim \mathcal{F}),
\end{align*}
\]
where \(a_q^a, \alpha_q^a\) are foliated functions, i.e., locally, these are functions on the coordinates \(x^a\).

Then, the general formula \((3.6)\) implies
\[
\begin{align*}
\Lambda &= \frac{1}{2} \beta_{hk}^l \frac{\partial}{\partial q^h} \wedge \frac{\partial}{\partial q^k} + \gamma_{hl}^q \frac{\partial}{\partial y^q} \wedge \frac{\partial}{\partial \eta^l} + \frac{1}{2} (\alpha_{pq}^r + \alpha_{pq}^s \eta^r) \frac{\partial}{\partial \eta^p} \wedge \frac{\partial}{\partial \eta^q} \\
&\quad + \alpha_q^a \frac{\partial}{\partial \eta^q} \wedge \frac{\partial}{\partial x^a} + \beta_{hy}^a \frac{\partial}{\partial y^h} \wedge \frac{\partial}{\partial y^a} + \beta_{hy}^u \frac{\partial}{\partial \eta^h} \wedge \frac{\partial}{\partial y^u}.
\end{align*}
\]

The total space of the subbundle \(E^* \subseteq A^*\) has the local equations \(\eta_h = 0\), whence
\[
\tilde{z}_A(\text{ann } T E^*) = \text{span} \{i(d\eta_h)\Lambda|_{\eta^h=0}\} = \text{span} \{\beta_h^a \frac{\partial}{\partial y^a}\} = T \mathcal{F} E^*,
\]
which is property 1).

The space \(C^\infty_{\text{hol}}(E^*, \mathcal{F} E^*)\) is locally generated by \(x^a, \zeta_q \quad (\text{mod. } \eta_h = 0)\) and a function of local expression \(f(x^a, \zeta_q)\) extends to a function of local expression \(\tilde{f}(x^a, \zeta_q, \eta_h)\) in a neighborhood of \(E^*\) in \(A^*\). Formula \((3.14)\) restricted to \(\eta_h = 0\) shows that, for \(f_1, f_2 \in C^\infty_{\text{hol}}(E^*, \mathcal{F} E^*)\), \(\Lambda|_{\text{ann } T E^*}(df_1, df_2)\) depends only on \(f_1, f_2\) and that
\[
\{x^a, x^b\}_{\Lambda|_{\text{ann } T E^*}} = 0, \quad \{\zeta_q, x^a\}_{\Lambda|_{\text{ann } T E^*}} = \alpha_q^a(x^b), \quad \{\zeta_p, \zeta_q\}_{\Lambda|_{\text{ann } T E^*}} = \alpha_{pq}^s(x^b) \zeta_s.
\]

Therefore, \(\Lambda\) satisfies property 2).

Since \(A^*/E^* \approx B^*\), we may use as local coordinates on the manifold \(A^*/E^*\) the coordinates \((x^a, y^a, \eta_h)\). Then,
\[
P = \frac{1}{2} \beta_{hk}^l \frac{\partial}{\partial q^h} \wedge \frac{\partial}{\partial q^k} + \beta_{hy}^a \frac{\partial}{\partial y^h} \wedge \frac{\partial}{\partial y^a}
\]
is a bivector field on \(A^*/E^*\), which is Poisson because it has the same expression as the dual Poisson structure of the Lie algebroid \(B\). Property 3) is obviously satisfied.

Conversely, let \(A\) be a vector bundle over \((M, \mathcal{F})\) such that there exists a foliated subbundle \(E^*\) of the dual bundle \(A^*\) and a fiber-linear Poisson bivector field \(\Lambda\) of \(A^*\) with the properties 1), 2), 3). Define the subbundle \(B = \text{ann } E^* \subseteq A\) and take a decomposition \(A = B \oplus C\) and local bases of cross sections \(b_h \in B, a_q \in C\). Moreover, \(C \approx A/B \approx E = E^{**}\) is an \(\mathcal{F}\)-foliated bundle and \(a_q\) may be assumed to be \(\mathcal{F}\)-foliated cross sections. The corresponding local coordinates
\((x^a, y^a, \eta_h, \zeta_q)\) on \(A^*\) are similar to those used in (3.6) and, a priori, \(\Lambda\) is of the form
\[
\Lambda = \frac{1}{2} (\beta_{hk} \eta_l + \beta_{hk} \zeta_s) \frac{\partial}{\partial \eta_h} \wedge \frac{\partial}{\partial \eta_k} + (\gamma_{hq} \eta_l + \gamma_{hq} \zeta_s) \frac{\partial}{\partial \eta_h} \wedge \frac{\partial}{\partial \zeta_q} + \frac{1}{2} (\alpha_{pq} \eta_l + \alpha_{pq} \zeta_s) \frac{\partial}{\partial \zeta_p} \wedge \frac{\partial}{\partial \zeta_q} + \alpha_q \frac{\partial}{\partial \eta_h} \wedge \frac{\partial}{\partial \eta_q}.
\]

If we put \(\eta_h = 0\) and compute \(i(d\eta_h)\Lambda\) we see that property 1) implies \(\beta_{hk} = 0\), \(\gamma_{hq} = 0\), \(\lambda^a_h = 0\), \(\text{rank}(\beta_h^a) = \dim F\).

Furthermore, property 2) implies local expressions \(\alpha_q^a = \alpha_q^a(x^b)\), \(\alpha_{pq} = \alpha_{pq}^a(x^b)\).

Finally, the projection \((A^* \to A^*/E^*)\)-related bivector field \(P\) of property 3) is uniquely defined by \(\Lambda\); it must be given by (3.7) and it satisfies the Poisson condition. Therefore, \(\Lambda\) reduces to the form (3.6) and the corresponding Lie algebroid structure of \(A\) is given by formulas (3.5). Accordingly, the pair \((A, B)\) is a foliated Lie algebroid. (The foliated-generation condition is implied by the fact that \([b_h, a_q] \in B\).)

A. Vaintrob [21] gave an interpretation of the dual Poisson structure (3.2) as an odd, homological super-vector field, which led to important results on Lie algebroids seen as homological super-vector fields. We shall indicate the properties of this field in the foliated case.

The parity-changed vector bundle of \(A^*\) (in our case \(A^*\) is the dual of a Lie algebroid, but, the definition applies to any vector bundle) is a supermanifold \(\Pi A^*\), with local even coordinates \(x^i\) and local odd coordinates \(\bar{\eta}^\alpha\) associated to local, \(A^*\)-trivializing, coordinate neighborhoods \(U\) on \(M\), with the following local transition functions:
\[
\tilde{x}^i = \tilde{x}^i(x^j), \quad \tilde{\eta}^\alpha = b^\alpha_\beta \tilde{\eta}^\beta,
\]
where the first equations are the same as the transition functions of the local coordinates on \(M\) and the matrix \((b^\alpha_\beta)\) is defined by a change of local bases of cross sections of \(A\), \(a_\beta = b^\alpha_\beta \tilde{a}_\alpha\). The supertangent bundle \(T\Pi A^*\) has the local bases \((\partial/\partial x^i, \partial/\partial \tilde{\eta}^\alpha)\) with the local transition functions
\[
\frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}, \quad \frac{\partial}{\partial \tilde{\eta}^\beta} = b^\alpha_\beta \frac{\partial}{\partial \eta^\alpha}.
\]

These transition functions show that the correspondence
\[
x^i \mapsto x^i, \quad \frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial x^j}, \quad \eta_\alpha \mapsto \frac{\partial}{\partial \eta^\alpha}, \quad \frac{\partial}{\partial \eta_\alpha} \mapsto \tilde{\eta}^\alpha.
\]
produces a well defined mapping of the fiber-wise polynomial multivector fields on the manifold $A^*$ to super-multivector fields on $\Pi A^*$. In particular, the bivector field $\Pi$ defined by (3.2) is send to Vaintrob’s super-vector field

$$V = \frac{1}{2} \alpha^h h^k \frac{\partial}{\partial \eta^l} + \alpha^h \eta^k \frac{\partial}{\partial x^a}$$

and the Poisson condition $[\Pi, \Pi] = 0$ becomes the homology condition $[V, V] = 0$. To put terminology in agreement with the theory of Lie algebroids, the super-vector field $V$ will be called transitive if the $M$-rank $\rho$ of $V$ defined by $\rho = \text{rank}(\alpha^a_\eta)$ is equal to the dimension of $M$.

Furthermore, if $M$ is endowed with the foliation $\mathcal{F}$ and $A$ has a subbundle $B$, we have coordinates $(x^a, y^u)$ and bases $(b_h, a_q)$ like in the proof of Proposition 3.2 which yield coordinates $(x^a, y^u, \eta^h, \zeta^q)$ of $\Pi A^*$ with transition functions of the form

$$\bar{x}^a = \bar{x}^a(x^b), \quad \bar{y}^u = \bar{y}^u(x^a, y^v), \quad \bar{\eta}^h = b^h_k \eta^k + b^h_q \zeta^q, \quad \bar{\zeta}^q = b^q_y \zeta^p.$$

where the various matrices $b$ come from a change of the local basis. The transition functions (3.9) show that $\Pi A^*$ is endowed with the superfoliation

$$\bar{\mathcal{F}} = \text{span} \left\{ \frac{\partial}{\partial y^u}, \frac{\partial}{\partial \eta^h} \right\}.$$

Then, if we consider the vector bundle $E = A/B$, we have $E^* = \text{ann} B \subseteq A^*$ and $\Pi(A^*/E^*)$ may be seen as the submanifold of $\Pi A^*$ that has the local equations $\zeta^q = 0$. On the other hand, $\Pi E^*$ is a supermanifold with the local coordinates $(x^a, y^u, \zeta^q)$ and there exists a natural projection $\psi : \Pi A^* \rightarrow \Pi E^*$ locally defined by

$$(x^a, y^u, \zeta^q) \mapsto (x^a, y^u, \zeta^q).$$

In Vaintrob’s terminology a super-vector field on $\Pi A^*$ that corresponds to a bivector field on $A^*$ is said to be of degree 1. In our case, a super-vector field of degree 1 has the form

$$W = \frac{1}{2} \beta^h \eta^k \frac{\partial}{\partial \eta^l} + \frac{1}{2} \beta^h \eta^k \frac{\partial}{\partial \zeta^q} + \gamma^h \eta^k \zeta^q \frac{\partial}{\partial \eta^l} + \frac{1}{2} \alpha^k \eta^q \zeta^q \frac{\partial}{\partial \zeta^l} + \frac{1}{2} \alpha^k \eta^q \zeta^q \frac{\partial}{\partial \zeta^l} + \beta^h \eta^k \frac{\partial}{\partial y^u} + \alpha^k \eta^q \frac{\partial}{\partial y^u} + \lambda^h \eta^k \frac{\partial}{\partial x^a}.$$

**Proposition 3.3.** A foliated Lie algebroid $(A, B)$ over $(M, \mathcal{F})$ is equivalent with a couple $(E^*, W)$, where $E^*$ is an $\mathcal{F}$-foliated vector subbundle of $A^*$ and $W$ is a homological super-vector field of degree 1 on $\Pi A^*$ such that

i) $W$ is foliated (projectable) with respect to the super-foliation $\bar{\mathcal{F}}$ of $\Pi A^*$ associated to the pair $(\mathcal{F}, B)$,

ii) $W|_{\Pi(A^*/E^*)} \in T\bar{\mathcal{F}} \cap T\Pi(A^*/E^*)$ and is a homological super-vector field of degree 1 on $\Pi(A^*/E^*)$, transitive over the leaves of $\mathcal{F}$. 

15
and the bases \((b)\) which is why we only sketch it briefly here. Using the decomposition \((A,B)\) we get a bigrading of the algebras \(A\) and \(C\) with the corresponding cohomology homomorphism \(\#\) of \(\mathfrak{g}\)-algebras

\[
\gamma_{hq}^a = 0, \quad \alpha_{pq}^a = \alpha_{pq}(x^b), \quad \alpha_q^a = \alpha_q(x^b), \quad \lambda_h^a = 0, \quad \beta_{hk}^a = 0, \quad \text{rank} (\beta_h^a) = \dim \mathcal{F},
\]

and where the super-vector field

\[
W_{\Pi(A^*/E^*)} = \frac{1}{2} \beta_{hk}^h \eta^k \frac{\partial}{\partial y^l} + \beta_h^k \eta^h \frac{\partial}{\partial y^u} \in T \tilde{\mathcal{F}} \cap T \Pi(A^*/E^*)
\]

is a homological super-vector field on \(\Pi(A^*/E^*)\). The condition \(\text{rank} (\beta_h^a) = \dim \mathcal{F}\) is what we meant by the transitivity of \(W\). Conversely, if we start with the data \((A,E^*,W)\), \(A\), which has the Lie algebroid structure of Vaintrob super-vector field \(W\), has the subbundle \(B = ann E^*\) and \(IIA^*\) has the superfoliation \(\tilde{\mathcal{F}}\). Then, \(W\) may be represented under the form \((3.11)\). Condition i), which means that the terms of \((3.11)\) that contain \((\partial/\partial x^a, \partial/\partial \zeta^a)\) depend on the coordinates \((x^a, \zeta^a)\) only, implies the first four conditions \((3.11)\). Condition ii) ensures the fact that \(B\), with the Lie algebroid structure of Vaintrob super-vector field \(W_{\Pi(A^*/E^*)}\), is a foliation of \(A\).

\[ \square \]

4 Cohomology and deformations

The cohomology \(H^*(A)\) of a Lie algebroid \(A\) is that of the differential graded algebra \((\Omega(A) = \Gamma \wedge A^*, d_A)\) of \(A\)-forms where, if \(\lambda \in \Omega^k(A)\), then

\[
d_A \lambda (s_1,\ldots, s_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} d_A s_j(\lambda(s_1,\ldots, s_{j-1}, s_{j+1},\ldots, s_{k+1}))
\]

\[
+ \sum_{j<l} (-1)^{j+l} \lambda([s_j, s_l]_A, s_1,\ldots, s_{j-1}, s_{j+1},\ldots, s_{l-1}, s_{l+1},\ldots, s_{k+1})
\]

The anchor of the Lie algebroid produces a homomorphism of differential graded algebras \(\sharp_A : (\Omega(M), d) \rightarrow (\Omega(A), d_A)\) defined by

\[
(\sharp \theta)(s_1,\ldots, s_k) = \theta(\sharp s_1,\ldots, \sharp s_k), \quad \theta \in \Omega^k(M)
\]

with the corresponding cohomology homomorphism \(\sharp^*_A : H_{dr,Rham}^*(M) \rightarrow H^*(A)\). The general pattern of the cohomology of a foliated Lie algebroid \((A,B)\) over a foliated base manifold \((M,F)\) is the same like that of a foliated manifold \([23]\), which is why we only sketch it briefly here. Using the decomposition \(A = B \oplus C\) and the bases \((b_h, a_l)\), \((b^{bh}, a^{ast})\) where \(b_h \in \Gamma B, a_l \in \Gamma C\) and \(a_l\) are \(B\)-foliated, we get a bigrading of the \(A\)-forms such that a form of type \((bi\text{-degree}) (s,r)\) \((s\text{ is the }C\text{-grade and }r\text{ is the }B\text{-grade})\) has the local expression

\[
\lambda = \frac{1}{s!r!} \lambda_{l_1 \ldots l_r} b_{h_1} \ldots b_{h_r}(x^a, y^w) a^{s l_1} \wedge \ldots \wedge a^{s l_s} \wedge b^{s h_1} \wedge \ldots \wedge b^{s h_r}.
\]
We will denote by $\Omega^{s,r}(A)$ the space of $A$-forms of type $(s,r)$.

Since $B$ is bracket closed, one has $d_A a^i (b_h, b_k) = 0$ and we see that $d_A \lambda$ has only components of the type $(s + 1, r), (s, r + 1), (s + 2, r - 1)$. Hence there exists a decomposition

\begin{equation}
(4.3) \quad d_A = d_A' + d_A'' + \partial_A,
\end{equation}

where the terms are differential operators of types $(1, 0), (0, 1), (2, -1)$, respectively. The property $d_A^2 = 0$ is equivalent with

\begin{equation}
(4.4) \quad (d_A')^2 = 0, (d_A') d_A + d_A' (d_A') = 0, \quad (\partial_A)^2 = 0,
\end{equation}

$$d_A' \partial_A + \partial_A d_A' = 0, \quad (d_A'')^2 + d_A'' \partial_A + \partial_A d_A'' = 0.$$ 

The cohomology of $(\Omega^0(A), d_A')$ is the cohomology of the Lie algebroid $B$. On the other hand, an $A$-form $\lambda$ is said to be $B$-foliated if it is of type $(s, 0)$ and its local components $\lambda_{i_1, \ldots, i_s}$ are $\mathcal{F}$-foliated functions (this is an invariant condition because the bases $(a_l)$ are $B$-foliated and have $\mathcal{F}$-foliated transition functions). The operator $d_A'$ induces a cohomology of the $B$-foliated forms called the basic cohomology. Formulas (4.3) show that $(\Omega(A), d_A)$ is a semi-positive double cochain complex [23], hence, it produces a spectral sequence that connects among the previously mentioned cohomologies. Another component of the pattern is the truncated cohomology discussed in the Appendix of [26].

**Example 4.1.** In Example [12] we showed that for any Lie algebroid $A$ the pair $(J^1 A, \text{Hom}(TM, A))$ is a foliated Lie algebroid over the foliation of the basis $M$ by points, therefore, its cohomology has the pattern described above. On the other hand, the cohomology of $J^1 A$ may be interpreted as the 1-differentiable cohomology of $A$ in the sense of Lichnerowicz, who studied the 1-differentiable cohomology of many infinite dimensional Lie algebras on manifolds (e.g., see [12, 13] and the references therein; the case studied in [13] is that of the cotangent Lie algebroid of a Poisson manifold). This cohomology is defined by $k$-cochains $\lambda \in \text{Hom}_B(\Gamma(\wedge^k A), C^\infty(M))$, which are differential operators of order 1 in each argument, and by the differential $d$ defined by formula (4.1). Since a differential operator of order 1 is the composition of a $C^\infty(M)$-linear morphism by the first jet mapping $j^1$, the 1-differentiable cochains may be identified with usual cochains of the Lie algebroid $J^1 A$. Furthermore, since $[j^1 a_1, j^1 a_2]_{\mu, A} = j^1 [a_1, a_2]_A$ ($a_1, a_2 \in \Gamma A$) [5], $d$ for 1-differentiable cochains may be identified with $d_{J^1 A}$. Hence, $H^k_{1-\text{diff}}(A) \cong H^k(J^1 A)$ as claimed. A complementary subbundle $C$ of the foliation $\text{Hom}(TM, A)$ is given by the image of a splitting $\sigma : A \to J^1 A$ of the exact sequence [13] and the type decomposition (4.2) corresponds to the type decomposition used by Lichnerowicz.

Like in the case of a foliation [22], we get

**Proposition 4.1.** (The $d_A'$-Poincaré Lemma) Let $(A, B)$ be a minimally foliated Lie algebroid. For any local $A$-form $\lambda$ of type $(u, v)$ ($v > 0$) such that $d_A' \lambda = 0$, one has $\lambda = d_A'' \mu$ where $\mu$ is a local form of type $(u, v - 1)$.
Proof. We may proceed by induction on $k$, where $k$ is the number defined by the condition that the local expression of $\lambda$ does not contain the elements $a^{*k+1},...,a^q$ ($q = \text{rank } A - \text{rank } B$) of the basis used in (4.2).

Since we are in the case of a minimally foliated algebroid, $\sharp_B : B \to T\mathcal{F}$ is an isomorphism, which induces an isomorphism between the graded differential algebra $(\Omega^0(A), d_A^r)$ and the graded differential algebra of the $\mathcal{F}$-leaf-wise forms on $(M, \mathcal{F})$. Since the $d''$-Poincaré lemma holds for the latter [22, 23], it holds for the former as well, which justifies the conclusion for $k = 0$.

Now, we assume that the result holds for any $k < h - 1$ ($h = 1,...,q$) and take

$$\lambda = a^h \wedge \mu + \theta$$

where $\mu, \theta$ do not contain $a^h, ..., a^q$. Since $d''_A a^h = 0$, $d''_A \mu = 0$ if and only if $d''_A a^h = 0$. Then, the induction hypothesis yields the local expressions

$$\mu = d''_A \tau, \theta = d''_A \sigma,$$

therefore,

$$\lambda = d''_A (-a^h \wedge \tau + \sigma),$$

which means that the result also holds for $k = h - 1$ and we are done.

Obviously, the operator $d''_A$ makes sense for $V$-valued $A$-forms, where $V$ is any $\mathcal{F}$-foliated vector bundle on $M$, and the $d''$-Poincaré lemma (Proposition 4.1) holds for such forms as well.

The following corollary is a standard result.

Corollary 4.1. Let $(A, B)$ be a minimally foliated Lie algebroid and $V$ a foliated vector bundle over $(M, \mathcal{F})$. Let $\Phi^k$ be the sheaf of germs of $B$-foliated, $V$-valued $A$-forms. Then one has

$$H^l(M, \Phi^k) \approx \ker (d''_A : \Omega^{k,l}(A) \otimes V \to \Omega^{k,l+1}(A) \otimes V) / \im (d''_A : \Omega^{k,l-1}(A) \otimes V \to \Omega^{k,l}(A) \otimes V).$$

Remark 4.1. In the terminology of [26], Proposition 4.1 is a relative Poincaré lemma. Therefore the sheaf-interpretation of the truncated cohomology given in [26] holds for minimally foliated Lie algebroids.

There exists a general construction of characteristic $A$-cohomology classes of a Lie algebroid [4], which mimics the Chern-Weil theory and goes as follows. Take a vector bundle $V \to M$ of rank $r$. An $A$-connection (covariant derivative) on $V$ is an $\mathbb{R}$-bilinear operator $\nabla : \Gamma A \times \Gamma V \to \Gamma V$ that is $C^\infty(M)$-linear in the first argument and satisfies the condition

$$\nabla_a (fv) = f \nabla_a v + ((\sharp_A a) f)v, \quad (a \in \Gamma A, v \in \Gamma V, f \in C^\infty(M)).$$

The curvature operator of $\nabla$ is

$$R_\nabla (a_1, a_2) = \nabla_{a_1} \nabla_{a_2} - \nabla_{a_2} \nabla_{a_1} - \nabla_{[a_1, a_2], A}.$$

$R_\nabla$ is $C^\infty(M)$-bilinear and may be seen as a $V \otimes V^*$-valued $A$-form of degree 2. Then, $\forall \phi \in \mathbf{T}^k(\text{Gl}(r, \mathbb{R}))$, the space of real, ad-invariant, symmetric,
$k$-multilinear functions on the Lie algebra $gl(r, \mathbb{R})$, the $A$-cohomology classes $[\phi(R_V)] = [\phi(R_{V_1}, ..., R_{V_k})] \in H^{2k}(A)$ are the $A$-principal characteristic classes of $V$. The characteristic classes are spanned by the classes $[\epsilon_k(R_V)]$, where $\epsilon_k$ is the sum of the principal minors of order $k$ in $\text{det}(R_V - \lambda I)$. Indeed, if $\nabla^0, \nabla^1$ are two $A$-connections, Bott’s proof \[19\] gives

$$\phi(R_{V_1}) - \phi(R_{V^0}) = d_A \Delta(\nabla^0, \nabla^1) \phi,$$

where

$$\Delta(\nabla^0, \nabla^1) \phi = k \int_0^1 \phi(\nabla^1 - \nabla^0, R_{(t\nabla^1 + (1-t)\nabla^0)}, ..., R_{(t\nabla^1 + (1-t)\nabla^0)}) dt,$$

hence the cohomology classes defined by $\phi(R^0_V), \phi(R^1_V)$ coincide. In particular, the use of an orthogonal $A$-connection (such that $\nabla g = 0$ for a Euclidean metric $g$ on $A$) gives $[\epsilon_{2k-1}(R_V)] = 0$. The classes $[\epsilon_{2k}(R_V)]$ are the $A$-Pontryagin classes of $V$. For $V = A$ one gets the principal characteristic classes of the Lie algebroid $A$.

In the case of a foliated Lie algebroid $(A, B)$ over $(M, \mathcal{F})$ one can also mimic the construction of secondary characteristic classes. We define a Bott $A$-connection on a foliated vector bundle $V$ by asking it to satisfy the condition $\nabla_b v = 0$, $\forall b \in B$, $\forall v \in \Gamma_{\text{fol}} V$. Bott $A$-connections always exist: take an arbitrary $A$-connection $\nabla'$ on $V$ and a decomposition $A = B \oplus C$, then, define

$$\nabla_{b+c}(f v) = ((\nabla_b) f)v + \nabla'_{c}(f v) \ (f \in C^\infty(M), v \in \Gamma_{\text{fol}} V).$$

The curvature of a Bott $A$-connection satisfies the property

$$R_V(b_1, b_2)v = 0, \forall b_1, b_2 \in B, v \in V.$$

Indeed, due to $C^\infty(M)$-linearity with respect to $v$, it suffices to check the property for $v \in \Gamma_{\text{fol}} V$, which is trivial.

**Proposition 4.2.** (The Bott Vanishing Phenomenon) If $k > q = \text{rank } A - \text{rank } B$, $\forall \phi \in I^k(\text{Gl}(r, \mathbb{R}))$, the characteristic class defined by $\phi$ vanishes. Equivalently, $\text{Pont}_A^h(V) = 0$ for $h > 2q$, where $\text{Pont}_A^h(V)$ is the set of elements of total cohomological degree $h$ generated by the $A$-Pontryagin classes in the cohomology algebra $H^*(A)$.

**Proof.** Property \[17\] shows that $R_V$ belongs to the ideal generated by $\text{ann } B$, hence, if $k > q$, the $A$-form obtained by the evaluation of $\phi$ on $R_V$ is zero. \[\square\]

**Corollary 4.2.** Let $B$ be a Lie subalgebroid of the Lie algebroid $A$. If $\text{Pont}_A^h(A/B) \neq 0$ for some $h > 2(\text{rank } A - \text{rank } B)$, $B$ is not a foliation of $A$.

Now, if we take $\phi \in I^{2h-1}(\text{Gl}(r, \mathbb{R}))$ with $2h - 1 > q$, formula \[15\] yields $d_A(\Delta(\nabla^0, \nabla^1) \phi) = 0$ whenever $\nabla^0$ is an orthogonal $A$-connection and $\nabla^1$ is
A Bott $A$-connection on the $\mathcal{F}$-foliated vector bundle $V$. The $A$-cohomology classes $[\Delta(\nabla^0, \nabla^1)\phi] \in H^{4h-3}(A)$ are $A$-secondary characteristic classes of $V$. Like in [11] one can prove that these classes do not depend on the choice of $\nabla^0$ and remain unchanged if $\nabla^1$ is subject to a deformation via Bott connections. In the case $V = A/B$, the construction provides secondary $A$-characteristic classes of the foliated Lie algebroid $(A,B)$.

**Proposition 4.3.** The $A$-secondary characteristic classes of $V$ are anchor images of de Rham-secondary characteristic classes.

**Proof.** We may define the $A$-secondary characteristic classes by means of connections $\nabla^0, \nabla^1$ given by $\nabla^0_a = \nabla^0_{\sharp_a}$, $\nabla^1_a = \nabla^1_{\sharp_a}$, where $\nabla^0'$ is a usual orthogonal connection on $V \to M$ and $\nabla^1$ is a Bott connection with respect to the foliated Lie algebroid $(TM, T\mathcal{F})$. Then, the conclusion holds at the level of forms.

Deformations of foliations were studied by many authors [7, 8], etc., and one of the main results is that deformations produce infinitesimal deformations, which are cohomology classes of degree 1. Here, we extend this result to deformations of a foliation $B$ of a Lie algebroid $A$ over $(M, \mathcal{F})$.

**Definition 4.1.** A deformation of the foliation $B$ of $A$ is a differentiable family of foliations $B_t \subseteq A$ ($0 \leq t \leq 1$) where $B_0 = B$, each $B_t$ is regular and $F_t = \sharp(A(B_t))$ is a deformation of the foliation $F = \sharp_0$. By differentiability of $B_t$ we mean that each $x \in M$ has a neighborhood $U$ endowed with local cross sections $(b_h(x,t))_{h=1}^p$ of $A$ that are differentiable with respect to $(x,t)$ and define a basis of the local cross sections of the subbundle $B_t$ for all $t \in [0,1]$. In particular, all the subbundles $B_t$ have the same rank $p$.

The transition functions of the bases $(b_h(x,t))$ are of the form

$$\tilde{b}_h(x,t) = \beta_h^k(x,t)b_k(x,t). \quad (4.8)$$

If we apply the operator $\left(\frac{d}{dt}\right)_{t=0}$, we see that the correspondence

$$\eta^h b_h \mapsto [\eta^h \frac{\partial \tilde{b}_h(x,t)}{\partial t}]_{t=0} \quad (b_h = b_h(x,0))$$

provides a well defined $E = A/B$-valued $1$-form $\Xi$. The form $\Xi$ will be called the infinitesimal deformation form and it has the following invariant expression

$$\Xi(b) = \left[\frac{\partial \tilde{b}(x,t)}{\partial t}\right]_{t=0} \quad (4.9)$$

where $\tilde{b}(x,t) \in \Gamma B_t$ is an extension of $b \in B_x$; the result does not depend on the choice of the extension.

In what follows we identify the cochain complexes $(\Omega(B), d_B)$ and $(\Omega^p(A), d'_A)$ by using a complementary subbundle of $B$ in $A$; in particular, $\Xi$ is also seen as a $(0,1)$-form.

20
Proposition 4.4. The infinitesimal deformation form $\Xi$ is $d_A^*\mathcal{A}$-closed.

Proof. We begin by deriving a new expression of $\Xi$. Let us fix a subbundle $\mathcal{C} \subseteq \mathcal{A}$ such that $A = B_t \oplus \mathcal{C}$ holds $\forall t \in [0, \epsilon)$ (where $\epsilon$ is small enough) and consider a local basis $(b_h)$ of $B$ and a local basis $(c_s)$ of $\mathcal{C}$ consisting of $B$-foliated cross sections; this yields a foliated basis $e_s = [c_s]_{\text{mod.} B}$ of $A/B$. Then, we have expressions

\begin{equation}
\tag{4.10}
    b_h(x, t) = \lambda_h^k(x, t)b_k + \mu_h^s(x, t)c_s, \quad \text{rank}(\lambda_h^k) = p,
\end{equation}

therefore, $B_t$ also has a basis of the form

\begin{equation}
\tag{4.11}
    b'_h(x, t) = b_h + \theta_h^s(x, t)c_s, \quad \theta_h^s(x, 0) = 0.
\end{equation}

The dual cobases of the bases $(b'_h(x, t), c_s)$ are given by

\begin{equation}
\tag{4.12}
    b'_h(x, t) = b_h + \theta_h^s(x, t)c_s, \quad \theta_h^s(x, 0) = 0.
\end{equation}

The dual conditions

\begin{align*}
\langle b'_h(x, t), b'_k(x, t) \rangle &= \delta_h^k, \quad \langle c^*(x, t), b'_k(x, t) \rangle = 0, \\
\langle b'_h(x, t), c_s \rangle &= 0, \quad \langle c^*(x, t), c_u(x, t) \rangle = \delta_u^s.
\end{align*}

From (4.11) and (4.12) we get

\begin{equation}
\tag{4.13}
    \Xi = \frac{\partial \theta_h^s(x, t)}{\partial t} \bigg|_{t=0} \otimes c_s,
\end{equation}

if we denote by $\Xi^C$ the image of $\Xi$ by the vector bundle isomorphism $E \approx \mathcal{C}$, which following (4.13) means that

\begin{equation}
\tag{4.14}
    \Xi^C = -\frac{\partial c^*(x, t)}{\partial t} \bigg|_{t=0} \otimes c_s,
\end{equation}

the conclusion of the proposition is equivalent with

\begin{equation*}
    d_A \Xi^C(b_h, b_k) = 0,
\end{equation*}

which we see to hold by means of the following calculation. From (4.14) and using the definition of $d_A$ we have

\begin{align*}
    d_A \Xi^C(b_h, b_k) &= -\frac{\partial (d_A c^*(x, t)(b'_h(x, t), b'_k(x, t)))}{\partial t} \bigg|_{t=0} c_s \\
    &= \frac{\partial [c^*(x, t)(b'_h(x, t), b'_k(x, t))]}{\partial t} \bigg|_{t=0} c_s = \frac{\partial}{\partial t} \bigg|_{t=0} (pr^*_C [b'_h(x, t), b'_k(x, t)])_{A},
\end{align*}

where $pr^*_C$ is the projection defined by the decomposition $A = B_t \oplus \mathcal{C}$. Since $B_t$ is closed by $A$-brackets, the final term of the previous calculation is equal to zero. \qed
Continuing to follow the analogy with foliation theory, we give the following definition.

**Definition 4.2.** A trivial deformation of the foliation \( B \) of the Lie algebroid \( A \) is a deformation \( B_t \) such that \( B_t = \Psi_t(B) \), where \( (\Psi_t, \psi_t) (\psi_t : M \to M) \) is a family of automorphisms of \( A \) of the form \( \Psi_t = \exp(ta), \psi_t = \exp(t\sharp A\alpha) \), \( a \in \Gamma A \).

We recall that \( \Psi_t = \exp(ta) \) is a (local) 1-parameter group with respect to addition on \( t \), which is characterized by the property

\[
\left. \frac{d}{dt} \right|_{t=0} (\exp(ta))a' = [a, a']_A, \quad \forall a, a' \in \Gamma A.
\]

(4.15)

**Proposition 4.5.** The infinitesimal deformation form \( \Xi \) of a trivial deformation is \( d''_A \)-exact.

*Proof.* For a trivial deformation we may use the local bases

\[ b_h(x, t) = \exp(ta)(b_h(x, \exp(-t\sharp A\alpha)(x), 0), \]

which implies

\[
\left. \frac{\partial b_h(x, t)}{\partial t} \right|_{t=0} = [a, b_h]_A(x).
\]

Now, using a decomposition \( a = \alpha^h b_h + \beta^s c_s \), where \( (b_h, c_s) \) is the basis used in (4.10), the original definition of \( \Xi \) shows that the corresponding form \( \Xi^C \) is given by

\[
\Xi^C(\eta^h b_h) = pr_C(\eta^h[a, b_h]_A) = -\eta^h(\sharp A b_h(\beta^s))c_s = -d''_A(pr_Ca),
\]

which is equivalent with the required result.

**Definition 4.3.** Let \((A, B)\) be a foliated Lie algebroid. Two deformations \( B_t, \bar{B}_t \) of \( B \) are equivalent if there exists a cross section \( a \in \Gamma A \) such that \( \bar{B}_t = \exp(ta)(B_t) \), i.e., if they can be deduced from one another by composition with a trivial deformation.

Now, we get the following result:

**Proposition 4.6.** Two equivalent deformations have \( d''_A \)-cohomologous infinitesimal deformation forms.

*Proof.* For two equivalent deformations \( B_t, \bar{B}_t \) there are local bases related as follows

\[ b_h(x, t) = \exp(ta)(b_h(x, t)). \]

By applying the operator \( (\partial/\partial t)_{t=0} \) to this relation and using the computation (4.16) we get the following relation between the corresponding infinitesimal deformation forms

\[ \Xi^C = \Xi^C - d''_A(pr_Ca). \]

\[ \square \]
The cohomology class \( \Xi \in H^1(B; A/B) \) (the first cohomology space of \( A/B \)-valued \( B \)-forms) is the *infinitesimal deformation up to equivalence* of the foliation \( B \) of the Lie algebroid \( A \).

## 5 Integration of foliated Lie algebroids

In this section we briefly discuss the integrability of a foliated Lie algebroid.

**Definition 5.1.** Let \( l, r : G \rightrightarrows M \) be a Lie groupoid (\( l, r \) are the target (left) and source (right) projections, respectively), where the unit manifold \( M \) is endowed with a foliation \( F \). We call \( G \) an \( F \)-foliated Lie groupoid if it is endowed with a foliation \( G \) that has the following properties:

(i) the leaves of \( G \) are contained in the left fibers,
(ii) \( G \) is invariant by left translations,
(iii) the right projection \( r : (G, G) \to (M, F) \) is a foliated, leaf-wise submersion.

It is easy to get the following integrability result.

**Proposition 5.1.** Let \( l, r : (G, G) \rightrightarrows (M, F) \) be a foliated Lie groupoid. Then, the Lie algebroid \( A(G) \) has a natural structure of a foliated Lie algebroid. Conversely, if \((A, B)\) is a foliated Lie algebroid and \( A \) is integrable by the left-fiber connected Lie groupoid \( G \), then \( G \) has a natural foliation \( \mathcal{G} \) that makes \( G \) into a foliated Lie groupoid such that the restriction to the leaves of \( \mathcal{G} \) of the construction of \( A(G) \) produces a Lie subalgebroid that is isomorphic to \( B \).

**Proof.** If we start with the foliated groupoid \((G, \mathcal{G})\) and construct the Lie algebroid \( A(G) \) by means of the left-invariant vector fields, the invariant vector fields that are tangent to the leaves of \( \mathcal{G} \) produce a Lie subalgebroid \( B \subseteq A(G) \).

Furthermore, if we denote by \( T^lG \) the tangent bundle to the \( l \)-fibers we get the exact sequence

\[
0 \to T\mathcal{G} \to T^lG \to T^lG/T\mathcal{G} \to 0,
\]

where the quotient is a \( \mathcal{G} \)-foliated bundle. The foliated cross sections of \( T^lG \) are the infinitesimal automorphisms of \( T\mathcal{G} \) and (like for any foliation) span the bundle \( T^lG \). If the previous sequence is quotientized by left translations we get the exact sequence

\[
0 \to B \to A(G) \to A(G)/B \to 0
\]

and we see that the pair \((A(G), B)\) satisfies the foliated generation condition. Then, the anchor of \( A(G) \) is induced by the differential \( r_* \) of the right projection and its restriction to \( B \) is onto \( TF \) because of condition (iii) of Definition 5.1.

Therefore, \((A(G), B)\) is a foliated Lie algebroid over \((M, F)\).

Conversely, if we start with a foliated Lie algebroid \((A, B)\) over \((M, F)\) and \( A = A(G) \) for the Lie groupoid \( G \), we get the foliation \( \mathcal{G} \) by asking it to be tangent to the left invariant vector fields that produce cross sections of \( B \) (see Lemma 2.1 of [17]). Then, the conditions required in Definition 5.1 obviously...
hold. In particular, condition (iii) follows from the foliated-generation condition of \((A,B)\) together with the fact that \(\text{im} \; \sharp_B = TF\).

\[ \square \]

6 Foliated Courant algebroids

We do not intend to develop a theory of foliated Courant algebroid, but, we would like to clarify how such a notion should be defined. We refer to [14] for the general theory of Courant algebroids.

Definition 6.1. A transversal-Courant algebroid over the foliated manifold \((M,\mathcal{F})\) is a foliated vector bundle \(E \to M\) endowed with a symmetric, non degenerate, foliated, inner product \(g \in \Gamma_{\text{fol}} \odot^2 E^*\), with a foliated morphism \(\sharp_E : E \to \nu \mathcal{F}\) called the anchor and a skew-symmetric bracket \([\cdot, \cdot]_E : \Gamma_{\text{fol}} E \times \Gamma_{\text{fol}} E \to \Gamma_{\text{fol}} E\), such that the following conditions are satisfied:

1) \(\sharp_E [e_1, e_2]_E = [\sharp_E e_1, \sharp_E e_2]_\nu \mathcal{F}\),

2) \(\text{im}(\sharp_g \circ \sharp_E) \subseteq \ker \sharp_E\),

3) \(\sum_{\text{Cycl}} g([e_1, e_2]_E, e_3) = (1/3) \partial \sum_{\text{Cycl}} g([e_1, e_2]_E, e_3)\), \(\partial f = (1/2) \sharp_g (\sharp_E df)\),

4) \([e_1, f e_2]_E = f [e_1, e_2]_E + (\sharp_E e_1)f e_2 - g(e_1, e_2) \partial f\),

5) \((\sharp_E e)(g(e_1, e_2)) = g([e, e_1]_E + \partial g(e, e_1), e_2) + g(e_1, [e, e_2]_E + \partial g(e, e_2))\).

In these conditions, \(e, e_1, e_2, e_3 \in \Gamma_{\text{fol}} E\), \(f \in C^\infty_{\text{fol}}(M, \mathcal{F})\) and \(t\) denotes transposition.

Using local, foliated, \(g\)-canonical bases, it is easy to see that \(\sharp_E(\lambda) \in \Gamma_{\text{fol}} E^*\), \(\forall \lambda \in \Gamma_{\text{fol}} (\text{ann} \; TF)\). In particular, \(\forall f \in C^\infty_{\text{fol}}(M, \mathcal{F})\), \(\partial f \in \Gamma_{\text{fol}} E\).

Like for the Courant algebroids, which are obtained if \(\mathcal{F}\) is the foliation of \(M\) by points, we will say that the transversal-Courant algebroid \(E\) is transitive if \(\sharp_E\) is surjective and it is exact if it is transitive and \(\text{rank} \; E = 2 \text{rank} \; \nu \mathcal{F}\). The reason for this name is that \(E\) is exact if and only if the sequence of vector bundles

\[ (6.1) \quad 0 \longrightarrow \text{ann} \; TF \xrightarrow{\sharp_{(E,g)}} E \xrightarrow{\sharp_E} \nu \mathcal{F} \longrightarrow 0, \]

where \(\sharp_{(E,g)} = \sharp_g \circ \sharp_E\) (therefore, \(\partial f = \sharp_{(E,g)} df\)) is exact. (This follows like for Courant algebroids, e.g., Section 3 of [24].)

Remark 6.1. For an exact Courant algebroid any choice of a splitting of the exact sequence \((6.1)\) that has an isotropic image yields an isomorphism \(E \cong (TM \oplus T^* M)\) where the latter is endowed with a twisted Courant bracket [2]. The brackets of the transitive Courant algebroids were determined in [19, 24] and may be made explicit by using a splitting as above. These results extend to transversal Courant algebroids if we add the hypothesis that the exact sequence \((6.1)\) has a foliated splitting, which is no more an automatic fact.

Remark 6.2. If \(E\) is a transversal-Courant algebroid over \((M, \mathcal{F})\), the local projected bundles \(\{E_{Q_U}\}\) are Courant algebroids over the local transversal manifolds \(Q_U\) of \(\mathcal{F}\). In the case of a submersion \(M \to Q\) with connected and simply connected fibers, \(E \to M\) projects to a Courant algebroid over \(Q\).
Example 6.1. For any foliated manifold \((M,F)\) the vector bundle \(\nu F \oplus \nu^* F = \nu F \oplus \text{ann} TF\) has the structure of a transversal-Courant algebroid induced by the classical Courant structure of the local transversal manifolds \(Q_U\) of Remark 6.2.

Example 6.2. Let \(A\) be a Courant algebroid over the foliated manifold \((M,F)\) and let \(B\) be a subbundle of \(A\) that is a Courant algebroid with respect to the induced Courant bracket, anchor and metric. In particular, the metric induced in \(B\) must be non degenerate and \(A = B \oplus C\), \(C = B^{\perp_s}\). A cross section \(c \in \Gamma C\) will be called \(B\)-foliated if, \(\forall b \in \Gamma B\), \([b,c]_A \in \Gamma B\). By the axioms of Courant algebroids (see condition 4) of Definition 6.1 we see that, on one hand, if the previous condition holds for \(b\) it also holds for \(fb\) \((f \in C^\infty(M)\) and, on the other hand, if \(c\) is \(B\)-foliated and \(f \in C^\infty_fol(M,F)\), \(fc\) is foliated (use \(g(b,c) = 0\)). (The definition of a \(B\)-foliated cross section is not correct for an arbitrary \(a \in \Gamma A\).) Now, it makes sense to assume that the pair \((A,B)\) satisfies the following conditions (analogous to those in Definition 1.2): i) \(\sharp A(B) = TF\), ii) \(\Gamma C\) is locally spanned by the \(B\)-foliated cross sections. Like for Lie algebroids (Section 1), these conditions show that \(C\) has a natural structure of a foliated vector bundle and we may add condition iii) \(g|_C \in \Gamma_{fol} \otimes^2 C^*\). From i), we get \(\sharp A(\text{ann} TF) \subseteq \text{ann} B = C^*\), whence, \(\partial f \in \Gamma C\), \(\forall f \in C^\infty_fol(M,F)\). Furthermore, since \(A\) is a Courant algebroid we have (see condition 5), Definition 6.1

\[(6.2) \ (\sharp A c_1)(g(b,c_2)) = 0 = g([c_1,b]_A + \partial g(c_1,b), c_2) + g(b, [c_1, c_2]_A + \partial g(c_1, c_2)),\]

where \(b \in \Gamma B, c_1, c_2 \in \Gamma C\) and \(c_1, c_2\) are \(B\)-foliated. In view of condition iii), \(6.2\) becomes \(g(b, [c_1, c_2]_A) = 0\), therefore, \([c_1, c_2]_A \in \Gamma C\). Moreover, using axiom 3) for the Courant algebroid \(A\) and for the triple \((c_1, c_2, b)\) instead of \((e_1, e_2, e_3)\) we see that the bracket \([c_1, c_2]_A\) is \(B\)-foliated. Thus, conditions i), ii), iii) imply that \((C, \sharp A|_C, [\_ , \_]|_{\Gamma_{fol} C})\) is a transversal-Courant algebroid over \((M,F)\).

We have formulated the above example because, at the first sight, it could indicate the way to the notion of a foliated Courant algebroid. However, it seems that there are no corresponding, interesting, concrete examples and we shall propose another procedure below.

Assume that \(B\) is a \(g\)-isotropic subbundle of the Courant algebroid \(A\) of basis \((M,F)\) such that \(\Gamma B\) is closed by \(A\)-brackets and consider the \(g\)-coisotropic subbundle \(C = B^{\perp_s} \supseteq B\). By replacing \(c_1\) by \(b' \in \Gamma B\) in \(6.2\), we see that \(\Gamma B\) is closed by \(A\)-brackets if and only if

\[(6.3) \ [b, c]_A \in \Gamma C, \quad \forall b \in \Gamma B, \forall c \in \Gamma C.\]

Like in Example 6.2, a cross section \(c \in \Gamma C\) will be called \(B\)-foliated if, \(\forall b \in \Gamma B\), \([b, c]_A \in \Gamma B\) and this definition is correct. We will denote by \(\Gamma_B C\) the space of \(B\)-foliated cross sections of \(C\).
**Proposition 6.1.** If \((A, B)\) is a foliated Courant algebroid over \((M, \mathcal{F})\), the vector bundle \(E = C/B, C = B^{\perp_*}\), inherits a natural structure of an \(\mathcal{F}\)-transversal-Courant algebroid.

**Proof.** Since \(E\) is isotropic and \(g\) is non degenerate on the components \(B \oplus B'\) and \(S\), we fix such a decomposition and define a structure of transversal-Courant algebroid on \(S\).

Take
\[
g_S = g|_S, \quad \sharp_S = \sharp_A (\text{mod.} T\mathcal{F}), \quad [, ]_S = pr_S[, ,]_A,
\]
where the projection on \(S\) is defined by (6.4). Like in the proof of Proposition 1.2, properties ii) and iii) of Definition 6.2 yield a foliated structure of \(S\) such that \(s \in \Gamma S\) is foliated with respect to this structure if and only if \(s\) is \(B\)-foliated.

We check that, \(\forall s \in \Gamma B C, \sharp_S s\) is \(\mathcal{F}\)-foliated. Indeed, if \(Y \in \Gamma T\mathcal{F}\), property ii) allows us to write \(Y = \sharp_A b, b \in \Gamma B\) and we have
\[
[Y, \sharp_A s]_A = [\sharp_A b, \sharp_A s]_A = \sharp_A[b, s]_A \in \Gamma T\mathcal{F},
\]
therefore \(\sharp_A s\) is a foliated vector field on \((M, \mathcal{F})\). Thus, \(\sharp_S\) is a foliated morphism. Furthermore, property iv) implies that \(g_S\) is a foliated metric.

The decomposition (6.4) also gives \(A^* = B^* \oplus S^* \oplus B'^*\) and we have
\[
b_g(B) = B'^*, \quad b_g(S) = S^*, \quad b_g(B') = B^*.
\]
If \(\gamma \in \text{ann} T\mathcal{F}\), \(\sharp_A \gamma\) vanishes on \(B\), because of property ii), hence, \(\sharp_A \gamma \in S^* \oplus B'^*\) and \(\sharp_A \gamma \in C\). In particular, \(\forall f \in C^\infty_{\text{fol}}(M, \mathcal{F}), \partial f \in \Gamma C\). This fact allows us to use again (6.2), while taking \(s_1, s_2 \in \Gamma B C\) instead of \(c_1, c_2\), and we see that \([s_1, s_2]_A \in \Gamma C\). Now, if we write down the equality iii), Definition 6.2 for the Courant algebroid \(A\) and for \(c_1, c_2, e_3\) replaced by \(s_1, s_2 \in \Gamma B C, b \in \Gamma B\), the right hand side vanishes and we remain with \([s_1, s_2]_A, \partial A = 0\), which shows that \([, ,]_S\) takes foliated cross sections to foliated cross sections.

Thus, \(S\) is endowed with all the structures required by Definition 6.1 and it remains to check the conditions 1)-5). It is easy to check that \(\partial_S = \partial_A\).
on $C^\infty_f(M,F)$. Accordingly, 1)-5) are implied by the similar properties of the Courant algebroid $A$ (in particular, for 2) we may use the fact that this condition is equivalent with $g(\partial_s f_1, \partial_s f_2) = 0, \forall f_1, f_2 \in C^\infty_f(M,F)$ [14]). The conclusion of the proposition follows by transferring the structure obtained on $S$ to $E$ via the natural isomorphism $E \approx S$.

**Corollary 6.1.** If $(A,B)$ is a foliated Courant algebroid over $(M,F)$ and if $F$ consists of the fibers of a submersion $M \to Q$ with connected and simply connected fibers, the vector bundle $E = B^{\perp}/B$ projects to a Courant algebroid on $Q$.

**Remark 6.3.** If $B$ is a subbundle of the Courant algebroid $A$ that satisfies conditions i), ii) of Definition [6,2] and if we ask the quotient bundle $A/B$ to be a transversal-Courant algebroid such that its metric, anchor and bracket be induced by those of $A$, then, obviously, $B$ is a foliation of $A$.

The previous proposition and corollary show the interest of the notion of a foliated Courant algebroid. The following examples show that Definition [6,2] is reasonable. In these examples $T^{big}M = TM \oplus T^*M$ is the Courant algebroid defined in [3]: the anchor is the projection on $TM$ and the metric and bracket are defined by

\begin{equation}
(6.5) \quad g((X_1,\alpha_1),(X_2,\alpha_2)) = \frac{1}{2}(\alpha_1(X_2) + \alpha_2(X_1)),
\end{equation}

\begin{equation}
(6.6) \quad [(X_1,\alpha_1),(X_2,\alpha_2)] = ([X_1,X_2], L_{X_1}\alpha_2 - L_{X_2}\alpha_1 + \frac{1}{2}d(\alpha_1(X_2) - \alpha_2(X_1))).
\end{equation}

**Example 6.3.** A regular Dirac structure $D \subseteq T^{big}M$ is a foliation of the Courant algebroid $T^{big}M$ over $(M,E)$, where $E$ is the characteristic foliation of $D$. In this case the quotient bundle is $D^{\perp}/D = 0$.

**Example 6.4.** If $F$ is a foliation of $M$, $T^F$ is a foliation of the Courant algebroid $T^{big}M$. Indeed, using (6.5) we get $C = T^{\perp}\nu = TM \oplus \text{ann} T^F$. A pair $(X,\gamma) \in \nu$ is $T^F$-foliated if and only if $X$ is an $F$-foliated vector field and $\gamma$ is an $F$-foliated 1-form. The conditions of Definition [6,2] are obviously satisfied and the corresponding transversal-Courant algebroid is $E = C/T^F = \nu \oplus \text{ann} T^F$, already mentioned in Example [6,1]. Notice that $T^F$ remains a foliation of $T^{big}M$ if the Courant bracket is twisted by means of a closed 3-form $\Phi$ such that $i(Y)\Phi = 0, \forall Y \in T^F$ [20].

**Example 6.5.** Let $F$ be a foliation of $M$ and $\theta \in \Omega^2(M)$ be a closed 2-form. From [25], it is known that $E_\theta = \{(Y,i(Y)\theta) / Y \in T^F\}$ is an isotropic subbundle of $T^{big}M$, which is closed by the Courant bracket (6.6), and its orthogonal bracket is $E_\theta^\perp = \{(X,i(X)\theta + \gamma) / X \in TM, \gamma \in \text{ann} T^F\}$. A simple computation gives

\begin{align*}
[(Y,i(Y)\theta),(X,i(X)\theta + \gamma)] &= ([Y,X],i([Y,X])\theta + Ly\gamma).
\end{align*}
Accordingly, \((X, i(X)\theta + \gamma) \in \Gamma E'_\theta\) is \(E_\theta\)-foliated if and only if \(X \in \Gamma_{fol} TM, \gamma \in \Omega^1_{fol}(M)\) and, like in Example \(6.4\), the conditions of Definition \(6.2\) are satisfied and \(E_\theta\) is a foliation of \(T^{big} M\). Again, the corresponding quotient bundle is \(E'_\theta / E_\theta \approx \nu F \oplus \text{ann } TF\).

Below, we prove a result that has the flavor of a reduction and was inspired by \(27\). Let \(B\) be a foliation of the Courant algebroid \(A\) over the foliated manifold \((M, \mathcal{F})\). Let \(N\) be a submanifold of the base manifold \(M\), and consider the restricted vector bundles \(A_N, B_N, C_N = B_N^{\perp A}\).

**Proposition 6.2.** With the notation above, assume that:

(i) \(N\) is transversal to and has a clean intersection with the leaves of the foliation \(\mathcal{F}\).

(ii) (the reduction hypothesis) \(\sharp A(C_N) \subseteq TN\).

Then, the vector bundle \(E_N = C_N / B_N\) is a transversal-Courant algebroid over \((N, \mathcal{F}' = \mathcal{F} \cap N)\).

**Proof.** Since \(B\) is a foliation of \(A\), there exist local bases \((b_h \in \Gamma B, a_u \in \Gamma B C)\) of \(\Gamma C\) and the induced local bases of cross sections of \(E = C / B\) have local transition functions of the form \([\tilde{a}_u]_{\text{mod.} B} = \alpha^v_u [a_u]_{\text{mod.} B}\) such that \(\alpha^v_u\) are constant on the leaves of \(\mathcal{F}\). Then, \(\alpha^v_u\) are constant on the leaves of \(\mathcal{F}'\) too and the local bases \([a_u|N]_{\text{mod.} B_N}\) of \(\Gamma E_N\) define an \(\mathcal{F}'\)-foliated structure on \(E_N\).

Using these bases, we see that a cross section of \(E_N\) is foliated with respect to the \(\mathcal{F}'\)-foliated structure of \(E_N\) if and only if, locally, it is of the form \([c|N]_{\text{mod.} B_N}\) where \(c\) is \(B\)-foliated. Moreover, the metric induced on \(E_N\) by the metric \(g\) of \(A\) obviously is a foliated metric.

In view of the reduction hypothesis, the mapping

\[ [c]_{\text{mod. } B} \mapsto [\sharp A c]_{\text{mod. } \mathcal{F}} \quad (c \in \Gamma C) \]

produces a vector bundle morphism \(\sharp E_N : E_N \rightarrow \nu \mathcal{F}'\), which will be the anchor of the required Courant structure on \(E_N\). Like in the proof of Proposition \(6.1\), we can see that this anchor is a foliated morphism.

Now, we will show the existence of a bracket \([\cdot, \cdot]_{C_N} : \Gamma C_N \times \Gamma C_N \rightarrow \Gamma E_N\) induced by the Courant bracket of \(A\). The definition of this bracket is

\[ (6.7) \quad [c_1, c_2]_{C_N}(x) = [\tilde{c}_1, \tilde{c}_2]_A(x) \mod B_x, \quad x \in N, \]

where \(\tilde{c}_1, \tilde{c}_2\) are arbitrary extensions of \(c_1, c_2\) to \(C\). In order to prove that the result does not depend on the choice of the extensions it suffices to show that if \(\tilde{c}_2|_N = 0\) then \([\tilde{c}_1, \tilde{c}_2]|_A(x) \in B_x\). To see that, take a local basis \((b_h \in \Gamma B, a_u \in \Gamma B C)\) of \(\Gamma C\) and put \(\tilde{c}_2 = f^h b_h + k^u a_u\) where \(f^h, k^u\) vanish on \(N\). Using the axioms of a Courant algebroid (see \(4\)) of Definition \(6.1\), under the conditions above, we remain with

\[ [\tilde{c}_1, \tilde{c}_2]_A(x) = - \sum_u g_u(\tilde{c}_1(x), a_u(x)) \partial k^u(x). \]
Furthermore, for any $c \in \Gamma C$ we have
\[ g(c, \partial k^u) = \frac{1}{2}(\sharp A_c) k^u = 0 \]
(because $\sharp A_c \in TN$ and $k^u|_N = 0$), therefore, $\partial k^u(x) \in B_x$, which ends the proof of the correctness of the definition of the bracket (6.7).

The bracket (6.7) produces a bracket of foliated cross sections of $E_N$ as follows. If $\gamma \in \Gamma_{fol} E_N$, then, in a neighborhood of $x \in N$, we may write $\gamma = \sum_a \varphi^u(x^a)[a_u|_N]_{mod. B_N}$, which implies that $\gamma$ may be represented by a cross sections of $C_N$ that has the $B$-foliated, local extension $\tilde{\gamma} = \sum_a \varphi^u(x^a)a_u$. If we put
\[ [\gamma_1, \gamma_2]_{E_N}(x) = [\tilde{\gamma}_1|_N, \tilde{\gamma}_2|_N]_{C_N}(x) = [\tilde{\gamma}_1, \tilde{\gamma}_2]_{A}(x) \ (mod. B_x), \]
we get a well defined bracket on $\Gamma_{fol} E_N$.

Thus, on $E_N$ we have all the components required by the definition of a transversal-Courant algebroid over $(N, F')$ and, moreover, these components may be seen as restrictions to $N$ of the components of the transversal-Courant algebroid $E$ over $(M, F)$ given by Proposition 6.1. Hence, condition 1)-5) of Definition 6.1 are satisfied and we are done.

We finish by the observation that a definition similar to Definition 6.1 may be given for a notion of holomorphic Courant algebroid; we just have to replace the word “foliated” by “holomorphic” everywhere. Furthermore, if we state Definition 6.2 for complex vector bundles $A, B$ over a complex analytic manifold $M$, and with $TF$ replaced by the anti-holomorphic tangent bundle of $M$, we get the notion of a foliation of a complex Courant algebroid. The corresponding quotient $E$ will be a holomorphic Courant algebroid. Examples 6.4, 6.5 can be adapted to the holomorphic case.

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29
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