Time-Varying Optimization: Algorithms and Engineering Applications

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Abstract

This is the write-up of the talk I gave at the 23rd International Symposium on Mathematical Programming (ISMP) in Bordeaux, France, July 6th, 2018. The talk was a general overview of the state of the art of time-varying, mainly convex, optimization, with special emphasis on discrete-time algorithms and applications in energy and transportation. This write-up is mathematically correct, while its style is somewhat less formal than a standard paper.

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1 Introduction

With time-varying optimization, we mean the task of finding the minimum of an optimization problem that changes continuously in time. Let $f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ be a convex function parametrized over time, i.e., $f(x; t)$, where $x \in \mathbb{R}^n$ is the decision variable and $t \geq 0$ is time. Let $X(t) \subseteq \mathbb{R}^n$ be a convex set, also changing in time. Then the problem at hand can be formulated as finding

$$\min_{x \in X(t)} f(x; t), \quad \text{for all } t \geq 0.$$  \hfill (1)

That is, we want to find the minimum at each point in time. These types of problems appear naturally in many applications, for example energy, robotics, transportation, as we will see.

In this talk, we will sample Problem (1) at defined sampling times $t_k$, with $k = 0, 1, \ldots$ and sampling period $h = t_{k+1} - t_k$, and arrive at a sequence of time-invariant problems

$$\min_{x \in X_k} f(x; t_k).$$  \hfill (2)

When one can sample Problem (1) at the desired sampling frequency and solve the resulting time-invariant problems (2) at the desired accuracy within the sampling period, we are in a batch solution mode. This batch approach is hardly viable, except for low dimensional problems that can be sampled with sufficient long sampling periods (i.e., when the problem changes sufficiently slowly). We won't follow this approach, instead we will pursue an on-line approach, which will find approximate solutions of each of the time-invariant problems (2) and eventually will get close to the minimum trajectory.

To generate approximate solutions we will use running (or correction-only, or catching-up) algorithms and prediction-correction algorithms. We will touch upon primal and dual algorithms.

1.1 Background

Time-varying optimization has been around for quite some time, e.g., [1].
Continuous-time platforms have been discussed, e.g., in [2–7].
Running methods on discrete-time platforms can be traced back to Moreau [8], and subsequently have appeared in many contexts [2, 9–20].
A recent and fairly complete treatment is in [21] (which forms also the basis for part of the results I will present here).
Prediction-correction methods arise from non-stationary optimization [1, 9], parametric programming [12, 22–25], and continuation methods in numerical mathematics [26]. It also resembles evolutionary variational inequalities [27, 28] and path-following methods in interior point solvers [29].
Part of the work that I present here is in [30–33].

2 Formulation

Our main starting point is the time-invariant problem (2). Unless otherwise said, we will assume that the function is nicely behaving, that for us means that

**Assumption 1** Function \( f(x; t) \) is \( m \) strongly convex \( (m > 0) \) and \( L \) strongly smooth \( (L > 0) \) over \( x \in \mathbb{R}^n \), uniformly in time.

Assumption 1 guarantees that the solution (i.e., the minimizer of (2)) exists and its unique at every time \( t_k \) (of course assuming that \( X_k \) is non-empty). This implies also that the solution trajectory \( x^*(t) \) of (1) is well-defined.
To see situations for which this is not true see [34].

3 Running algorithms

Running algorithms or, as we said, correction-only/catching-up algorithms have always the same prototype structure. Here I present the work of [21] (so theorems and results are properly defined there); see the original paper for references to previous work.
Running algorithms start with a approximate solution \( x_0 \) and generate a sequence \( \{x_k\} \) by acquiring a new function at time \( t_{k+1} \) and performing \( C \) iterations of the selected method. For example, for the case of the running projected gradient, one does the following

- **Time** \( t_0 \), guess \( x_0 \)
- **Time** \( t_{k+1} \)
  1. Acquire a new function \( f(\cdot; t_{k+1}) \) and the constraint set \( X_{k+1} \)
  2. Set \( y_0 = x_k \)
  3. Perform \( C \) times:

\[
y_{i+1} = \Pi_{X_{k+1}}[y_i - \alpha \nabla_x f(y_i; t_{k+1})]
\]

4. Set \( x_{k+1} = y_C \)

In [17], \( \alpha > 0 \) is the stepsize, while \( \Pi_X \) is the projection onto the convex set \( X \).

3.1 Theoretical results

Typical theoretical results of running algorithms go as follow. Assume that the optimizer trajectory is well-behaved, that is that
**Assumption 2** The change in the optimizers of \( (2) \) is upper bounded as
\[
\|x^*(t_{k+1}) - x^*(t_k)\| \leq K, \quad \forall k \geq 0.
\]

Assumption 2 guarantees that the optimizers are indeed trackable. Note that \( K \) can be big, so we are not limited to small variations; on the other hand, the bounds will be big too if \( K \) is big.

Then, we have

**Theorem 1** (Informal) If your favorite method \( \mathcal{M} \) converges Q-linearly to the optimizer of a time-invariant problem as
\[
\|x_k - x^*(t_k)\| \leq \varrho C \|x_{k-1} - x^*(t_{k-1})\|, \quad \varrho < 1,
\]
then the same method \( \mathcal{M} \) converges Q-linearly to the optimizer trajectory of a time-varying problem up to an error bound as
\[
\|x_k - x^*(t_k)\| \leq \varrho^C (\|x_{k-1} - x^*(t_{k-1})\| + K),
\]
and
\[
\limsup_{k \to \infty} \|x_k - x^*(t_k)\| = \varrho^C O(K).
\]

The theorem is fairly general and its based on the triangle inequality. What it says is that the sequence \( \{x_k\} \) will track the solution trajectory up to a ball of size \( \varrho^C O(K) \). If \( C \to \infty \), we solve the time-invariant problem exactly and we are back to the time-invariant/batch mode (and the error is 0).

Based on this theorem, one can derive a corollary for the projected gradient method

**Corollary 1** For \( \alpha < 2/L \), the projected gradient method applied in a running mode generate a sequence \( \{x_k\} \) that converges to the error bound \( \varrho^C O(K) \) Q-linearly, with rate \( \varrho = \max\{1 - \alpha m, 1 - \alpha L\} \).

Equation (17) can be substituted with other methods, and Theorem 1 is true for a variety of methods \( \mathcal{M} \), such as
- Proximal point method, for \( f(\cdot; t_k) \) strongly convex, \( X_k \subseteq \mathbb{R}^n \);
- Forward-backward splitting (minimizing \( f(x; t) + g(x; t) \)), for \( f(\cdot; t_k) \) strongly smooth and strongly convex, \( g \) CCP, \( X_k \subseteq \mathbb{R}^n \);
- Dual ascent (for the problem \( \min_{x \in \mathbb{R}^n} f(x; t) \) subject to \( Ax = b \)), for \( f(\cdot; t_k) \) strongly smooth and strongly convex;
- D-R splitting, ADMM, doubly-regularized saddle-points, \ldots, for similar assumptions.

### 3.2 Beyond strong convexity/strong smoothness

We briefly touch here (and in this subsection alone) the more general case of relaxing the Assumption 1 to generic convex problems. In particular, the previous tracking results can be extended also in case of more general \( f(x; t) \), by using fixed-point theory in compact sets \( X(t) \).

E.g., for the projected gradient, if the function is only strongly smooth and \( \alpha < 2/L \), one can arrive at results of the form of

- Average fixed-point residual tracking:
\[
\frac{1}{T} \sum_{k=1}^{T} \|P_{X_k}(x_k - \alpha \nabla_x f(x_k; t_k)) - x_k\|^2 \leq O(1/T) + O(\hat{K} D + \hat{K}^2) \quad (4)
\]

where \( \hat{K} \) is a bound on a sequence of optimizers \( \{x^*(t_k)\} \), i.e., \( \|x^*(t_{k+1}) - x^*(t_{k})\| \leq \hat{K} \), (note that the optimizers need not be unique now); and \( D = \max_{t} \text{diam } X(t) \).
• Dynamic regret (aka objective function tracking):

\[ \frac{1}{T} \sum_{k=1}^{T} f(x_k; t_k) - f(x^*_k; t_k) \leq O(1/T) + O(\hat{K}D + \hat{K}^2) + O(K') \]

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where \( \hat{K}, D \) are as before, and \( K' \) is a bound on the functional variations as \( |f(x; t_{k+1}) - f(x; t_k)| \leq K', \forall x \in X(t) \).

Similar results hold for other methods.

4 Interlude: functions Lipschitz in time

An interesting and useful result can be derived when the dependence of the cost function \( f(x; t) \) over time is bounded in some sense. In particular, assume that \( f \) has a well-defined gradient and the time derivative of the gradient is bounded, i.e.

**Assumption 3** The time derivative of the gradient of \( f(x; t) \) in bounded uniformly in time,

\[ \|\nabla_{tx} f(x; t)\| \leq C_0, \quad \forall x \in \mathbb{R}^n, t \geq 0. \]

Assumption 3 is more restrictive than Assumption 1 and it implies it

**Assumption 3** \( \implies \) **Assumption 1**

with \( K = h C_0/m \), where we remind that \( h \) is the sampling period. (One can see this in e.g., [23]).

Assumption 3 is a sort of Lipschitz condition in time, and when it is valid implies that all the running methods yield an asymptotical error of the order of \( O(h) \) (i.e., linear in the sampling period).

5 Prediction-correction algorithms

Prediction-correction are methods that attempt at reducing the asymptotical error below \( O(h) \). We look here at the results presented in [30,33].

To get better bounds, one needs stronger assumptions. In addition to Assumption 1 here we will assume

**Assumption 4** Higher derivatives of the cost function are bounded as

\[ \|\nabla_{xxx} f(x; t)\| \leq C_1, \quad \|\nabla_{txx} f(x; t)\| \leq C_2, \quad \|\nabla_{tx} f(x; t)\| \leq C_3 \]

uniformly in time and for all \( x \in \mathbb{R}^n \).

Assumption 4 is an extension of Newton’s assumptions (for Newton’s method one requires \( C_1 \)) that also requires the time variations of the Hessian and gradient to be bounded.

Prediction-correction methods attempt at inferring how the optimizers are changing in time, by applying a pertinent Taylor’s expansion of the optimality conditions.

For example, if we were to solve the unconstrained problem

\[ \min_{x \in \mathbb{R}^n} f(x; t), \]

and we wanted to predict the optimizer at time \( t_{k+1} \), only from data available at time \( t_k \), one could start from the optimality condition at time \( t_{k+1} \)

\[ \nabla_x f(x^*(t_{k+1}); t_{k+1}) = 0 \]
(which we can’t solve) and Taylor expand as
\[ \nabla_x f(x^*(t_{k+1}); t_{k+1}) \approx \nabla_x f(x^*(t_k); t_k) + h \nabla_{tx} f(x^*(t_k); t_k) + \nabla_{xx} f(x^*(t_k); t_k) \delta x = 0 \] (8)

Since we don’t have \( x^*(t_k) \), we can substitute \( x_k \) and obtain a class of prediction schemes:
\[ \nabla_x f(x_k; t_k) + h \nabla_{tx} f(x_k; t_k) + \nabla_{xx} f(x_k; t_k) \delta x = \gamma \nabla_x f(x_k; t_k), \] (9)
and the prediction \( x_{k+1|k} \) is given by
\[ x_{k+1|k} = x_k + \delta x = x_k - [\nabla_{xx} f(x_k; t_k)]^{-1}(h \nabla_{tx} f(x_k; t_k) + (1 - \gamma) \nabla_x f(x_k; t_k)). \] (10)

Here \( \gamma \in [0, 1] \) is an extra tuning parameter. For \( \gamma = 1 \), we have a tangential update: we are moving along the “iso-suboptimal manifold”. When \( \gamma = 0 \), we have a Newton-like update, so that on top of predicting we are also going towards the optimizer. See a nice figure in [35].

If we were to solve the constrained problem
\[ \min_{x \in X} f(x; t), \] (11)
then we would do prediction over the generalized inequality
\[ \nabla_x f(x^*(t_{k+1}); t_{k+1}) + N_X(x^*(t_{k+1})) \ni 0, \] (12)
where \( N_X \) is the normal cone operator, which leads to
\[ \nabla_x f(x_k; t_k) + h \nabla_{tx} f(x_k; t_k) + \nabla_{xx} f(x_k; t_k) (x_{k+1|k} - x_k) + N_X(x_{k+1|k}) \ni 0 \] (13)
or equivalently, calling \( Q_k = \nabla_x f(x_k; t_k), c_k = h \nabla_{tx} f(x_k; t_k) \),
\[ x_{k+1|k} = \arg\min_{y \in X} \{1/2 y^T Q_k y + c_k^T y\}. \] (14)

Now, since we don’t want to solve an optimization problem with another optimization problem (however easy), we can set up an approximate scheme for (14) as
\[ y_{i+1} = \Pi_X[y_i - \beta(Q_k y_i + c_k)] \] (15)
that is a projected gradient method that has to run for \( P \) prediction steps and with stepsize \( \beta > 0 \).

If we were to solve a linearly constrained problem, a similar construct would apply for both primal and dual variable in a dual ascent setting.

5.1 Prototypical algorithm

As for the running methods, we report here a prototypical prediction-correction algorithm, here focussed on the projected gradient (but similar for gradient and dual ascent)

- Time \( t_0 \), guess \( x_0 \)
- Time \( t_k \)
  1. Set \( Q_k = \nabla_{xx} f(x_k; t_k), c_k = h \nabla_{tx} f(x_k; t_k) \)
  2. Set \( y_0 = x_k \)
  3. Perform \( P \) prediction steps:
     \[ y_{i+1} = \Pi_X[y_i - \beta(Q_k y_i + c_k)] \] (16)
  4. Set \( \hat{x}_{k+1|k} = y_P \) (approximate prediction)
- Time \( t_{k+1} \)
  1. Acquire a new function \( f(\cdot; t_{k+1}) \)
2. Set $y_0 = \hat{x}_{k+1|k}$

3. Perform $C$ correction steps:
   \[ y_{i+1} = \Pi_X[y_i - \alpha \nabla_x f(y_i; t_{k+1})] \]  
   (17)

4. Set $x_{k+1} = y_C$

Note that updates (16) are computationally cheap to carry out, once $Q_k$ and $c_k$ have been computed once, while updates (17) may be more expensive.

5.2 Theoretical results

Typical theoretical results goes as follows. Under Assumptions \[ 4 \] and a proper selection of stepsizes, number of prediction and correction steps, and sampling period, one is expect to track the solution trajectory up to a bound that depends on the problem properties and the sampling period. Depending on the method and on $P$ and $C$ we can have asymptotical errors that range from $O(h)$ to $O(h^4)$.

We report the result for projected gradient in prediction-correction mode as defined in the previous subsection.

**Theorem 2** (Informal) The projected gradient method in prediction-correction mode generates a sequence $\{x_k\}$ as follows.

Choose $\alpha, \beta < 2/L$.

Under Assumptions \[ 4 \] there exists a minimal number of prediction and correction steps $P, C$ for which globally

\[ \limsup_{k \to \infty} \|x_k - x^*(t_k)\| = O(\varrho_C h) \]

In addition, under Assumptions \[ 4 \] then locally (and for small $h$), there exists a minimal number of prediction and correction steps $P, C$ so that

\[ \limsup_{k \to \infty} \|x_k - x^*(t_k)\| = O(\varrho_C^1 h^2) + O(\varrho_C^2 h) \]

where $\varrho_1, \varrho_2 < 1$, and $\varrho_1, \varrho_2$ are the contraction rates for $\alpha$ and $\beta$, respectively.

Convergence is $O$-linear in both cases.

Theorem 2 says that tracking is not worse than correction-only method in the worst case. If the function has extra properties and we are interested in a local result, then a better asymptotical error can be achieved, provided some (stricter) conditions on the number of prediction and correction steps are verified.

The asymptotical error is composed of two terms; one which is labeled as approximation error, which is due to the early termination of the prediction step (if $P \to \infty$ and prediction is exact, this term goes to 0). The other, named prediction gain is the gain coming from using a prediction step, which brings the error down to a $O(h^2)$ dependence on the sampling period.

If $C$ grows, then the error reduces, as expected.

Theorem 2 can be modified for gradient methods and dual ascent methods.

6 A summary

We give now a short comparison between running and prediction-correction methods.
Table 1: Comparison between correction-only and prediction-correction methods

|                     | Correction-only | Prediction-correction |
|---------------------|-----------------|-----------------------|
| Assumptions         | Weak (mainly standard) | Stronger              |
| Complexity          | Low             | Higher                |
| Error               | $O(h)$          | $O(h) - O(h^2)$       |
| Methods             | Many, see [21]  | A few, see [30–33]    |

As one can see, correction-only method can tackle a larger class of problems up to a limited accuracy. Prediction-correction methods can achieve a better asymptotical error at the price of stronger assumptions and computational complexity.

We note that, in some cases, even keeping the computational time fixed, prediction-correction may achieve better errors than correction-only methods. This is because prediction steps are computationally easier than correction steps, and one can trade-off a few correction steps for many prediction ones. So, prediction-correction are very relevant even in practice.

7 Applications

Many applications entail some degree of time-varying optimization. We report below a collection of tested applications (either in correction-only mode or prediction-correction).

- Energy: e.g., time-varying optimal power flow and related, see [36–44]
- Transportation: [45, 46]
- Robotics: e.g., dynamic consensus [6, 7, 13, 15, 47, 48]
- Control: e.g., model predictive control [49–53]
- Signal processing: e.g., estimation in data streams, [20, 54–59]
- Others: economics [12], computational history [28]

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