A locally symmetric Kähler Einstein structure on a tube in the nonzero cotangent bundle of a space form

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Abstract

We obtain a locally symmetric Kähler Einstein structure on a tube in the nonzero cotangent bundle of a Riemannian manifold of positive constant sectional curvature. The obtained Kähler Einstein structure cannot have constant holomorphic sectional curvature.

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Introduction

The differential geometry of the cotangent bundle $T^*M$ of a Riemannian manifold $(M,g)$ is almost similar to that of the tangent bundle $TM$. However, there are some differences because the lifts (vertical, complete, horizontal etc.) to $T^*M$ cannot be defined just like in the case of $TM$.

In [17] V. Oproiu and the present author have obtained a natural Kähler Einstein structure $(G, J)$ of diagonal type induced on $T^*M$ from the Riemannian metric $g$. The obtained Kähler structure on $T^*M$ depends on one essential parameter $u$, which is a smooth function depending on the energy density $t$ on $T^*M$. If the Kähler structure is Einstein they get a second order differential equation fulfilled by the parameter $u$. In the case of the general solution, they have obtained that $(T^*M, G, J)$ has constant holomorphic sectional curvature.

In this paper we study the singular case where the parameter $u = At$, $A \in \mathbb{R}$. The considered natural Riemannian metric $G$ of diagonal type on the nonzero cotangent bundle $T^*_0M$ is defined by using one parameter $\nu$ which is a
smooth function depending on the energy density $t$. The vertical distribution $VT_0^*M$ and the horizontal distribution $HT_0^*M$ are orthogonal to each other but the dot products induced on them from $G$ are not isomorphic (isometric).

Next, the natural almost complex structures $J$ on $T_0^*M$ that interchange the vertical and horizontal distributions depends of one essential parameter $v$.

After that, we obtain that $G$ is Hermitian with respect to $J$ and it follows that the fundamental 2-form $\phi$ associated to the almost Hermitian structure $(G, J)$ is the fundamental form defining the usual symplectic structure on $T_0^*M$, hence it is closed.

From the integrability condition for $J$ it follows that the base manifold $M$ must have constant sectional curvature $c$ and the parameter $v$ must be a rational function depending on energy density $t$.

If the constant sectional curvature $c$ is positive then we obtain a locally symmetric Kähler Einstein structure defined on a tube in $T_0^*M$. The Kähler Einstein manifold obtained cannot have constant holomorphic sectional curvature.

1. Some geometric properties of $T^*M$

Let $(M, g)$ be a smooth $n$-dimensional Riemannian manifold and denote its cotangent bundle by $\pi : T^*M \to M$. Recall that there is a structure of a $2n$-dimensional smooth manifold on $T^*M$, induced from the structure of smooth $n$-dimensional manifold of $M$. From every local chart $(U, \varphi) = (U, x^1, \ldots, x^n)$ on $M$, it is induced a local chart $(\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)$, on $T^*M$, as follows. For a cotangent vector $p \in \pi^{-1}(U) \subset T^*M$, the first $n$ local coordinates $q^1, \ldots, q^n$ are the local coordinates $x^1, \ldots, x^n$ of its base point $x = \pi(p)$ in the local chart $(U, \varphi)$ (in fact we have $q^i = \pi^*x^i = x^i \circ \pi$, $i = 1, \ldots, n$). The last $n$ local coordinates $p_1, \ldots, p_n$ of $p \in \pi^{-1}(U)$ are the vector space coordinates of $p$ with respect to the natural basis $(dx^1_{\pi(p)}, \ldots, dx^n_{\pi(p)})$, defined by the local chart $(U, \varphi)$, i.e. $p = p_idx^i_{\pi(p)}$.

An $M$-tensor field of type $(r, s)$ on $T^*M$ is defined by sets of $n^{r+s}$ components (functions depending on $q^i$ and $p_i$), with $r$ upper indices and $s$ lower
indices, assigned to induced local charts \((\pi^{-1}(U), \Phi)\) on \(T^*M\), such that the local coordinate change rule is that of the local coordinate components of a tensor field of type \((r, s)\) on the base manifold \(M\) (see [6] for further details in the case of the tangent bundle). An usual tensor field of type \((r, s)\) on \(M\) may be thought of as an \(M\)-tensor field of type \((r, s)\) on \(T^*M\). If the considered tensor field on \(M\) is covariant only, the corresponding \(M\)-tensor field on \(T^*M\) may be identified with the induced (pullback by \(\pi\)) tensor field on \(T^*M\).

Some useful \(M\)-tensor fields on \(T^*M\) may be obtained as follows. Let \(v, w : [0, \infty) \to \mathbb{R}\) be a smooth functions and let \(||p||^2 = g^{-1}_{\pi(p)}(p, p)\) be the square of the norm of the cotangent vector \(p \in \pi^{-1}(U)\) \((g^{-1} = \text{the tensor field of type } (2,0)\) having the components \((g_{kl}(x))\) which are the entries of the inverse of the matrix \((g_{ij}(x))\) defined by the components of \(g\) in the local chart \((U, \varphi)\). The components \(g_{ij}(\pi(p)), p_i, v(||p||^2)p_j, p_i, w(||p||^2)p_j\) define \(M\)-tensor fields of types \((0, 2)\), \((0, 1)\), \((0, 2)\) on \(T^*M\), respectively. Similarly, the components \(g^{kl}(\pi(p)), g^0_k = p_ng^{hi}, w(||p||^2)g^{0k}g^{0l}\) define \(M\)-tensor fields of type \((2, 0)\), \((1, 0)\), \((2, 0)\) on \(T^*M\), respectively. Of course, all the components considered above are in the induced local chart \((\pi^{-1}(U), \Phi)\).

The Levi Civita connection \(\nabla\) of \(g\) defines a direct sum decomposition

\[
TT^*M = VT^*M \oplus HT^*M.
\]

of the tangent bundle to \(T^*M\) into vertical distributions \(VT^*M = \text{Ker } \pi_*\) and the horizontal distribution \(HT^*M\).

If \((\pi^{-1}(U), \Phi) = (\pi^{-1}(U), q^1, \ldots, q^n, p_1, \ldots, p_n)\) is a local chart on \(T^*M\), induced from the local chart \((U, \varphi) = (U, x^1, \ldots, x^n)\), the local vector fields \(\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}\) on \(\pi^{-1}(U)\) define a local frame for \(VT^*M\) over \(\pi^{-1}(U)\) and the local vector fields \(\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}\) define a local frame for \(HT^*M\) over \(\pi^{-1}(U)\), where

\[
\frac{\delta}{\delta q^i} = \frac{\partial}{\partial q^i} + \Gamma^0_{ih} \frac{\partial}{\partial p_h}, \quad \Gamma^0_{ih} = p_k \Gamma^k_{ih}
\]

and \(\Gamma^k_{ih}(\pi(p))\) are the Christoffel symbols of \(g\).

The set of vector fields \((\frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n}, \frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n})\) defines a local frame on \(T^*M\), adapted to the direct sum decomposition (1).

We consider

\[
t = \frac{1}{2} ||p||^2 = \frac{1}{2} g^{-1}_{\pi(p)}(p, p) = \frac{1}{2} g^{ik}(x)p_ip_k, \quad p \in \pi^{-1}(U)
\]

the energy density defined by \(g\) in the cotangent vector \(p\). We have \(t \in [0, \infty)\)
for all \( p \in T^*M \).

From now on we shall work in a fixed local chart \((U, \varphi)\) on \(M\) and in the induced local chart \((\pi^{-1}(U), \Phi)\) on \(T^*M\).

## 2. An almost Kähler structure on the \( T^*_0M \)

The nonzero cotangent bundle \( T^*_0M \) of Riemannian manifold \((M, g)\) is defined by the formula: \( T^*M \) minus zero section. Consider a real valued smooth function \( v \) defined on \((0, \infty) \subset \mathbb{R}\) and a real constant \( A \). We define the following \( M \)-tensor field of type \((0, 2)\) on \( T^*_0M \) having the components

\[
G_{ij}(p) = Atg_{ij}(\pi(p)) + v(t)p_ip_j.
\]

(3)

It follows easily that the matrix \((G_{ij})\) is positive definite if and only if \( A > 0, A + 2v > 0 \). The inverse of this matrix has the entries

\[
H^{kl}(p) = \frac{1}{At}g^{kl}(\pi(p)) + w(t)g^{0k}g^{0l},
\]

(4)

where

\[
w = \frac{-v}{At^2(A + 2v)}.
\]

(5)

The components \( H^{kl} \) define an \( M \)-tensor field of type \((2, 0)\) on \( T^*_0M \).

**Remark.** If the matrix \((G_{ij})\) is positive definite, then its inverse \((H^{kl})\) is positive definite too.

Using the \( M \)-tensor fields defined by \( G_{ij}, H^{kl} \), the following Riemannian metric may be considered on \( T^*_0M \)

\[
G = G_{ij}dq^idq^j + H^{ij}Dp_idp_j,
\]

(6)

where \( Dp_i = dp_i - \Gamma^0_{ij}dq^j \) is the absolute (covariant) differential of \( p_i \) with respect to the Levi Civita connection \( \nabla \) of \( g \). Equivalently, we have

\[
G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = G_{ij}, \quad G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = H^{ij}, \quad G(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = G(\frac{\delta}{\delta q^j}, \frac{\partial}{\partial p_i}) = 0.
\]
Remark that $HT_0^*M$, $VT_0^*M$ are orthogonal to each other with respect to $G$, but the Riemannian metrics induced from $G$ on $HT_0^*M$, $VT_0^*M$ are not the same, so the considered metric $G$ on $T^*M$ is not a metric of Sasaki type. Remark also that the system of 1-forms $(dq^1, \ldots, dq^n, Dp_1, \ldots, Dp_n)$ defines a local frame on $T^*T^*M$, dual to the local frame $(\frac{\delta}{\delta q^i}, \ldots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n})$ adapted to the direct sum decomposition (1).

Next, an almost complex structure $J$ is defined on $T_0^*M$ by the same $M$-tensor fields $G_{ij}$, $H^{kl}$, expressed in the adapted local frame by

\[ J \frac{\delta}{\delta q^i} = G_{ik} \frac{\partial}{\partial p_k}, \quad J \frac{\partial}{\partial p_i} = -H^{ik} \frac{\delta}{\delta q^k} \]

From the property of the $M$-tensor field $H^{kl}$ to be defined by the inverse of the matrix defined by the components of the $M$-tensor field $G_{ij}$, it follows easily that $J$ is an almost complex structure on $T_0^*M$.

**Theorem 1.** $(T_0^*M, G, J)$ is an almost Kähler manifold.

**Proof.** Since the matrix $(H^{kl})$ is the inverse of the matrix $(G_{ij})$, it follows easily that

\[ G(J \frac{\delta}{\delta q^i}, J \frac{\delta}{\delta q^j}) = G(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}), \quad G(J \frac{\partial}{\partial p_i}, J \frac{\partial}{\partial p_j}) = G(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}), \]

\[ G(J \frac{\partial}{\partial p_i}, J \frac{\delta}{\delta q^j}) = G(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = 0. \]

Hence

\[ G(JX, JY) = G(X, Y), \quad \forall X, Y \in \Gamma(T_0^*M). \]

Thus $(T_0^*M, G, J)$ is an almost Hermitian manifold.

The fundamental 2-form associated with this almost Hermitian structure is $\phi$, defined by

\[ \phi(X, Y) = G(X, JY), \quad \forall X, Y \in \Gamma(T_0^*M). \]

By a straightforward computation we get

\[ \phi(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = 0, \quad \phi(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = 0, \quad \phi(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}) = \delta^j_i. \]

Hence

\[ \phi = Dp_i \wedge dq^i = dp_i \wedge dq^i, \]

(8)
due to the symmetry of $\Gamma^0_{ij} = p_h \Gamma^h_{ij}$. It follows that $\phi$ does coincide with the fundamental 2-form defining the usual symplectic structure on $T^*_0M$. Of course, we have $d\phi = 0$, i.e. $\phi$ is closed. Therefore $(T^*_0M, G, J)$ is an almost Kähler manifold.

3. A Kähler structure on a tube $T^*_{0A} M$

We shall study the integrability of the almost complex structure defined by $J$ on $T^*_0M$. To do this we need the following well known formulas for the brackets of the vector fields $\frac{\partial}{\partial p_i}, \frac{\partial}{\delta q^i}, i = 1, ..., n$

\[ \begin{align*}
[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}] &= 0; \\
[\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}] &= \Gamma^i_{jk} \frac{\partial}{\partial p_k}; \\
[\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}] &= R^0_{ij} \frac{\partial}{\partial p_k},
\end{align*} \]

where $R^h_{ij}((\pi(p))$ are the local coordinate components of the curvature tensor field of $\nabla$ on $M$ and $R^0_{ij}(p) = p_h R^h_{ij}$. Of course, the components $R^0_{ij}, R^h_{ij}$ define M-tensor fields of types $(0,3), (1,3)$ on $T^*M$, respectively.

**Theorem 2.** The Nijenhuis tensor field of the almost complex structure $J$ on $T^*_0M$ is given by

\[ N(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) = \begin{cases} 
\{At((v + A)(\delta^h_{ij} g_{jk} - \delta^h_{ij} g_{ik}) - R^h_{ij}\} p_h \frac{\partial}{\partial p_k}, \\
N(\frac{\delta}{\delta q^i}, \frac{\partial}{\partial p_j}) = H^k l H^j r \{At((v + A)(\delta^h_{ij} g_{rl} - \delta^h_{ij} g_{il}) - R^h_{ij}\} p_h \frac{\delta}{\partial q^r}, \\
N(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}) = H^k l H^j l \{At((v + A)(\delta^h_{ij} g_{rk} - \delta^h_{ij} g_{lk}) - R^h_{ij}\} p_h \frac{\partial}{\partial p_k}.
\end{cases} \]

**Proof.** Recall that the Nijenhuis tensor field $N$ defined by $J$ is given by

$N(X, Y) = [JX, JY] - J[X, Y] - J[JX, Y] - [X, JY]$, $\forall$ $X, Y \in \Gamma(T^*_0M)$.

Then, we have $\frac{\delta}{\delta q^i} t = 0$, $\frac{\partial}{\partial p_i} t = g^{0k}$ and $\hat{\nabla}_i G_{jk} = 0$, $\hat{\nabla}_i H^{jk} = 0$, where

$\hat{\nabla}_i G_{jk} = \frac{\delta}{\delta q^i} G_{jk} - \Gamma^l_{ij} G_{lk} - \Gamma^l_{ik} G_{lj}$

$\hat{\nabla}_i H^{jk} = \frac{\delta}{\delta q^i} H^{jk} + \Gamma^l_{ij} H^{lk} + \Gamma^l_{ik} H^{lj}$
The above expressions for the components of $N$ can be obtained by a quite long, straightforward computation.

**Theorem 3.** The almost complex structure $J$ on $T^*_0M$ is integrable if and only if the base manifold $M$ has constant sectional curvature $c$ and the function $v$ is given by

$$v = \frac{c - A^2 t}{At}. \quad (11)$$

**Proof.** From the condition $N = 0$, one obtains

$$\{At(v + A)(\delta^h_i g_{jk} - \delta^h_j g_{ik}) - R^h_{kij}\}p_h = 0.$$ 

Taking $p_l \neq 0$ and $p_h = 0 \forall h \neq l$, it follows that

$$R^l_{kij} = At(v + A)(\delta^l_i g_{jk} - \delta^l_j g_{ik}).$$

Thus the sectional curvature $c = At(v + A)$ of $(M, g)$ depends only on $q^l$. Using the Schur theorem (in the case where $M$ is connected and dim $M \geq 3$), we obtain that $c$ is constant. Then we obtain the expression (11) of $v$.

Conversely, if $(M, g)$ has constant sectional curvature $c$ and $v$ is given by (11), it follows in a straightforward way that $N = 0$.

**Remark.** The function $v$ must fulfill the condition

$$A + 2v = \frac{2c - A^2 t}{At} > 0, \quad A > 0. \quad (12)$$

If $c > 0$ then $(T^*_{0A}M, G, J)$ is a Kähler manifold, where $T^*_{0A}M$ is the tube in $T^*_0M$ defined by the condition $0 < \|p\|^2 < \frac{4c}{At}$.

The components of the Kähler metric $G$ on $T^*_{0A}M$ are

$$G_{ij} = Atg_{ij} + \frac{c - A^2 t}{At} p_ip_j; \quad (13)$$

$$H^{ij} = \frac{1}{At} g^{ij} - \frac{c - A^2 t}{At(2c - A^2 t)} g^{0i}g^{0j}. \quad (13)$$
4. A Kähler Einstein structure on $T_{0A}^*$

In this section we shall study the property of the Kähler manifold $(T_{0A}^*, G, J)$ to be Einstein.

The Levi Civita connection $\nabla$ of the Riemannian manifold $(T_{0A}^*, G)$ is determined by the conditions

$$\nabla G = 0, \quad T = 0,$$

where $T$ is its torsion tensor field. The explicit expression of this connection is obtained from the formula

$$2G(\nabla_X Y, Z) = X(G(Y, Z)) + Y(G(X, Z)) - Z(G(X, Y)) +$$

$$+ G([X, Y], Z) - G([X, Z], Y) - G([Y, Z], X); \quad \forall X, Y, Z \in \Gamma(T_{0A}^* M).$$

The final result can be stated as follows.

**Theorem 4.** The Levi Civita connection $\nabla$ of $G$ has the following expression in the local adapted frame $(\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}, \frac{\partial}{\partial p^1}, \ldots, \frac{\partial}{\partial p^n})$:

$$\begin{align*}
\nabla \frac{\partial}{\partial q^i} \frac{\partial}{\partial p^j} &= Q_{hi}^{ij} \frac{\partial}{\partial p^h}, \\
\nabla \frac{\delta}{\delta q^i} \frac{\partial}{\partial p^j} &= -\Gamma_{ih}^{ij} \frac{\partial}{\partial p^h} + P_{hi}^{ij} \frac{\delta}{\delta q^h}, \\
\nabla \frac{\partial}{\partial p^i} \frac{\delta}{\delta q^j} &= S_{hij}^{ij} \frac{\partial}{\partial p^h}, \\
\nabla \frac{\delta}{\delta q^i} \frac{\delta}{\delta q^j} &= \Gamma_{ij}^{h} \frac{\delta}{\delta q^h} + S_{hij}^{ij} \frac{\partial}{\partial p^h},
\end{align*}$$

where $Q_{hi}^{ij}, P_{hi}^{ij}, S_{hij}^{ij}$ are $M$-tensor fields on $T_{0A}^* M$, defined by

$$\begin{align*}
Q_{hi}^{ij} &= \frac{1}{2} G_{hk} \left( \frac{\partial}{\partial p^i} H^{jk} + \frac{\partial}{\partial p^j} H^{ik} - \frac{\partial}{\partial p^k} H^{ij} \right), \\
P_{hi}^{ij} &= \frac{1}{2} H^{hk} \left( \frac{\partial}{\partial p^i} G_{jk} - H^{dl} R_{ijkl}^0 \right), \\
S_{hij}^{ij} &= -\frac{1}{2} G_{hk} \frac{\partial}{\partial p^k} G^{ij} + \frac{1}{2} R_{hij}^0.
\end{align*}$$

After replacing of the expressions of the involved $M$-tensor fields , we obtain

$$\begin{align*}
Q_{hi}^{ij} &= \frac{1}{2t} \left\{ (g^{ij} + \frac{c}{t(2c-A^2t)} g^{0i} g^{0j}) p_h - (\delta^{i0} g_{h0}^{0j} + \delta_{h0}^{0j} g_{0h}^{0i}) \right\}, \\
P_{hi}^{ij} &= -Q_{ji}^{ih}, \\
S_{hij}^{ij} &= - \frac{2c-A^2t}{2} (g_{ij} p_h + g_{ih} p_j) + \frac{A^2t}{2} g_{hj} p_i + \frac{3c-2A^2t}{2} p_h p_i p_j.
\end{align*}$$
The curvature tensor field $K$ of the connection $\nabla$ is obtained from the well known formula

$$K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z, \quad \forall \ X, Y, Z \in \Gamma(T^*_{0,A}M).$$

The components of curvature tensor field $K$ with respect to the adapted local frame $(\frac{\delta}{\delta q^1}, \ldots, \frac{\delta}{\delta q^n}, \frac{\partial}{\partial p_1}, \ldots, \frac{\partial}{\partial p_n})$ are obtained easily:

$$
\begin{align*}
K\left(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} &= QQ Q^h_{ijk} \frac{\delta}{\delta q^l}, \\
K\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right)\frac{\delta}{\delta q^k} &= PP P^k_{ijh} \frac{\partial}{\partial p^h}, \\
K\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\delta}{\delta q^k} &= PP Q^j_{ikh} \frac{\partial}{\partial p^h}, \\
K\left(\frac{\partial}{\partial p_i}, \frac{\delta}{\delta q^j}\right)\frac{\partial}{\partial p^k} &= PP Q^j_{ik} \frac{\partial}{\partial p^k},
\end{align*}
$$

where

$$
\begin{align*}
QQ Q^h_{ijk} &= \frac{A^2 t}{2} (\delta^h g^i - \delta^h g^i) + \frac{A^2}{4} (g_{ik}p_j - g_{jk}p_i) g^{0h} - \\
&- \frac{A^2}{4} (\delta^h p_j - \delta^h p_i)p_k, \\
QQ P^k_{ijh} &= -QQ Q^k_{ijh}, \\
PP Q^j_{ikh} &= -\frac{1}{2t}(\delta^j g^i - \delta^j g^i) - \frac{1}{4t}(g^{ih}g^{0j} - g^{ih}g^{0j})p_k + \\
&+ \frac{1}{4t}(\delta^i g^{0j} - \delta^i g^{0j})g^{0h}, \\
PP P^h_{ijk} &= -PP Q^h_{ijk}, \\
PP Q^j_{ik} &= \frac{A^2}{2} \delta^j G_{hk} + \frac{2c - A^2 t}{4t}(\delta^j p_h + \delta^j p_k)p_j + \\
&+ \frac{A^2}{4} (g_{jh}p_k + g_{jk}p_h) g^{0i} - \frac{c}{2t} g^{0i} p_j p_h p_k, \\
PQQ^i_{hk} &= \frac{A^2}{2} \delta^i H^{hk} - \frac{1}{4t}(g^{ih}g^{0k} + g^{ik}g^{0h})p_j - \\
&- \frac{A^2}{4(2c - A^2 t)} (\delta^h g^{0k} + \delta^k g^{0h}) g^{0j} + \frac{c}{2t} (2c - A^2 t) g^{0h} g^{0h} g^{0k} p_j,
\end{align*}
$$

are M-tensor fields on $T^*_{0,A}M$. 

\[9\]
Remark. From the local coordinates expression of the curvature tensor field \( K \) we obtain that the Kähler manifold \((T^*M, G, J)\) cannot have constant holomorphic sectional curvature.

The Ricci tensor field \( \text{Ric} \) of \( \nabla \) is defined by the formula:

\[
\text{Ric}(Y, Z) = \text{trace}(X \rightarrow K(X, Y)Z), \quad \forall \ X, Y, Z \in \Gamma(T^*_0AM).
\]

It follows

\[
\begin{aligned}
\text{Ric}(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial p^j}) &= \frac{An}{2}H^{ij}, \\
\text{Ric}(\frac{\partial}{\partial p^i}, \frac{\partial}{\partial q^r}) &= \text{Ric}(\frac{\delta}{\delta q^r}, \frac{\partial}{\partial p^i}) = 0.
\end{aligned}
\]

Thus

\[
(19) \quad \text{Ric} = \frac{An}{2}G.
\]

By straightforward computation, using the relations (16),(18) and the package Ricci, the following formulas are obtained:

\[
\begin{aligned}
\frac{\delta}{\delta q^r}QQQ^h_{ijkl} &= -\Gamma^h_{ls}QQQ^s_{ijkl} + \Gamma^h_{li}QQQ^h_{skj} + \Gamma^h_{lj}QQQ^h_{isk} + \Gamma^h_{lk}QQQ^h_{ij}, \\
\frac{\delta}{\delta q^r}PPQ^i_{jk} &= \Gamma^i_{lk}PPQ^i_{sjk} - \Gamma^i_{ls}PPQ^i_{ijk} - \Gamma^i_{ij}PPQ^i_{skh} + \Gamma^i_{ih}PPQ^i_{jks}, \\
\frac{\delta}{\delta q^r}PPQ^i_{jkh} &= -\Gamma^i_{ls}PPQ^i_{skh} + \Gamma^i_{ij}PPQ^i_{skh} + \Gamma^i_{lk}PPQ^i_{jkh} + \Gamma^i_{lh}PPQ^i_{jks}, \\
\frac{\partial}{\partial p^i}QQQ^h_{ijk} &= -P^i_{hs}QQQ^h_{skj} + P^i_{si}QQQ^h_{skj} + P^i_{sj}QQQ^h_{skj} + P^i_{sh}QQQ^h_{skj}, \\
\frac{\partial}{\partial p^i}PPQ^i_{j} &= P^i_{ks}PPQ^i_{skh} - P^i_{si}PPQ^i_{skh} - P^i_{sj}PPQ^i_{skh} - P^i_{sh}PPQ^i_{skh}, \\
\frac{\partial}{\partial p^i}PPQ^i_{jkh} &= -P^i_{sk}PPQ^i_{skh} + P^i_{sl}PPQ^i_{skh} + P^i_{sj}PPQ^i_{skh} + P^i_{sh}PPQ^i_{skh}, \\
\frac{\partial}{\partial p^i}PPQ^i_{j} &= P^i_{sk}PPQ^i_{skh} - P^i_{sl}PPQ^i_{skh} - P^i_{sj}PPQ^i_{skh} - P^i_{sh}PPQ^i_{skh}.
\end{aligned}
\]
Due to the relations (14),(17), we have

\[
(\nabla_{\delta q^l} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) \frac{\delta}{\delta q^k} = \left( \frac{\delta}{\delta q^l} QQQ^h_{ijk} + \Gamma^h_{ls} QQ_Q^s_{ijk} - \Gamma^h_{ij} QQ_Q^h_{sk} - \Gamma^h_{lk} QQ_Q^h_{js} \right) \frac{\delta}{\delta q^h} + \\
+ (S_{hls} QQ_Q^s_{ijk} + S_{slk} QQ_Q^s_{ih} + S_{ssl} PQQ^s_{ikh} - S_{sli} PQQ^s_{jkh}) \frac{\partial}{\partial p^h}.
\]

The coefficient of \( \frac{\delta}{\delta q^h} \) is zero due to the relations (20). By straightforward computation, using the relations (16),(18) and the package Ricci, we obtain that the coefficient of \( \frac{\partial}{\partial p^h} \) is zero. Thus

\[
(\nabla_{\delta q^l} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) \frac{\delta}{\delta q^k} = 0.
\]

Similarly

\[
(\nabla_{\delta p^l} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) \frac{\delta}{\delta q^k} = \left( \frac{\partial}{\partial p^l} QQQ^h_{ijk} + P^h_{ls} QQ_Q^s_{ijk} - P^h_{ij} QQ_Q^h_{sk} - P^h_{lk} QQ_Q^h_{js} \right) \frac{\delta}{\delta q^h}.
\]

The coefficient of \( \frac{\delta}{\delta q^h} \) is zero due to the relations (20). Thus

\[
(\nabla_{\delta p^l} K)(\frac{\delta}{\delta q^i}, \frac{\delta}{\delta q^j}) \frac{\delta}{\delta q^k} = 0.
\]

Similarly, we have computed the covariant derivatives of curvature tensor field \( K \) in the local adapted frame \( (\delta_{q^i}, \delta_{p^j}) \) with respect to the connection \( \nabla \) and we obtained in all the cases that the result is zero. Therefore

\[
\nabla K = 0.
\]

Now we may state our main result.

**Theorem 5.** If the Riemannian manifold \((M, g)\) has positive constant sectional curvature \( c \), the conditions (12) are fulfilled and the components of the metric \( G \) are given by (13) then \((T^*_\theta A^\ast M, G, J)\) is a locally symmetric Kähler Einstein manifold.
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