Asymptotic Dynamics of Stochastic $p$-Laplace Equations on Unbounded Domains

by

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ABSTRACT

This thesis is concerned with the asymptotic behavior of solutions of stochastic $p$-Laplace equations driven by non-autonomous forcing on $\mathbb{R}^n$. Two cases are studied, with additive and multiplicative noise respectively. Estimates on the tails of solutions are used to overcome the non-compactness of Sobolev embeddings on unbounded domains, and prove asymptotic compactness of solution operators in $L^2(\mathbb{R}^n)$. Using this result we prove the existence and uniqueness of random attractors in each case. Additionally, we show the upper semicontinuity of the attractor for the multiplicative noise case as the intensity of the noise approaches zero.

Keywords: Random Attractors; Upper Semicontinuity; $p$-Laplace Equation
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I release this document to the New Mexico Institute of Mining and Technology.

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CHAPTER 1

INTRODUCTION

Partial differential equations model a huge variety of phenomena in the physical, social, and life sciences. Unsurprisingly, solutions to these equations can often be just as varied and complex as the physical phenomenon being modeled. Therefore it becomes necessary to describe solutions to these equations, even when an analytic solution cannot be found. This qualitative endeavor has other benefits as well, such as being able to understand families of equations, and to predict behavior that numerical or approximate solutions would not readily find.

The subject of this thesis is the long-term behavior of solutions to two classes of stochastic degenerate parabolic equations with a $p$-Laplace term. Asymptotic behavior of solutions to these equations is investigated and used to establish the existence of random attractors. First suppose $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in \mathbb{R}})$ is a metric dynamical system where $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\theta_t\}_{t \in \mathbb{R}}$ is a measure-preserving transformation group on $\Omega$. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, consider the stochastic equation defined for $x \in \mathbb{R}^n$ and $t > \tau$,

$$\frac{\partial u}{\partial t} + \lambda u - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f(t, x, u) + g(t, x) + \alpha \eta(\theta_t \omega) u + \varepsilon h(x) \frac{dW}{dt},$$  (0.1)

with initial data,

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n.$$  (0.2)

The existence of attractors to this equation is studied in Chapter 3. We also investigate existence and upper semicontinuity of attractors for the equation with multiplicative noise,

$$\frac{\partial u}{\partial t} + \lambda u - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f(t, x, u) + g(t, x) + \alpha u \circ dW, \quad (0.3)$$

in Chapter 4. In both cases we assume that $p \geq 2$, $\alpha > 0$, $\lambda > 0$, $\varepsilon > 0$, $f$ is a time dependent nonlinearity, $g$ and $h$ are given functions, $\eta$ is a random variable and $W$ is a Wiener process on $(\Omega, \mathcal{F}, P)$.

Equations involving this type of nonlinear Laplacian operator have a rich mathematical theory, and play a role in some modern physical models. There are
some good references on the mathematical aspects of this type of nonlinearity, from existence, uniqueness, and regularity of equations, to its role in a nonlinear generalization of potential theory. The reader may consult Lindqvist for further details. The physical models involving this type of an operator are those with nonlinear diffusion, such as in non-Newtonian fluids or glacial flow. We refer to Lions [1969] for more details.

It should also be noted that these two types of noise model different phenomena. Additive noise occurs in physical systems where forcing occurs due to factors that are not included in the model. A simple example of this type of noise would be wind in an earthquake model of a building. In general, additive noise is small in magnitude, and is not proportional to the unknown function being studied. In contrast, something like uncertainties in air resistance calculations might lead one to introduce multiplicative noise, where the uncertainty is dependent on the velocity of the object. There are many technical details here, both from the modeling and the mathematical standpoint. The reader is advised to consult standard books on stochastic calculus for further information, such as Klebaner [2005].

An important observation in the study of dynamical systems is that analyzing long-term asymptotic behavior often reduces the possible dynamics of the system in question. This is frequently observed in dynamical systems which are dissipative in some sense. Many physical systems are modeled as dissipative dynamical systems due to friction and other thermodynamic losses. Studying these systems has given rise to various ways to model long-term asymptotic behavior. One particularly interesting approach is the idea of a global attractor which is an invariant subset of the phase space that attracts all trajectories.

The history of studying asymptotic dynamics is fairly old, and spans many areas of mathematics. Here we will restrict ourselves to looking at the recent developments, with particular emphasis on the theory of random attractors for partial differential equations. Several good references exist for the classical (deterministic autonomous) theory, such as Babin and Vishik [1992], Hale [1988], Ladyzhenskaya [1991], Robinson [2001], Temam [1988]. Likewise, the more modern theory treating random and nonautonomous systems is vast. Some of the references that are particularly useful are Carvalho et al. [2013], Cheban [2004], Chueshov [2001], Kloeden and Rasmussen [2011].

The definition of random attractor for autonomous stochastic systems was initially introduced in Crauel and Flandoli [1994], Flandoli and Schmalfuss [1996], Schmalfuss [1992]. Since then, such attractors for autonomous stochastic PDEs have been studied extensively, such as in Beyn et al. [2011], Caraballo et al. [2008a, 2010, 2011], Caraballo and Langa [2003], Caraballo et al. [2008b], Chueshov and Schutzow [2004], Chueshov [2001], Crauel et al. [1997], Crauel and Flandoli [1994], Flandoli and Schmalfuss [1996], Garrido-Atienza et al. [2011], Garrido-Atienza and Schmalfuss [2011], Garrido-Atienza et al. [2010], Gess et al. [2011], Gess [2013a,b], Huang and Shen [2009], Kloeden and Langa [2007], Schmalfuss [1992], Shen et al. [2010] in bounded domains, and in Bates et al. [2009], Wang [2009, 2011] in unbounded domains. For non-autonomous stochastic PDEs, the reader is referred to Adili and Wang [2013], Bates et al. [2013], Caraballo et al.
[2003], Duan and Schmalfuss [2003], Gess [2013a,b], Kloeden and Rasmussen [2011], Wang [2012, 2014b], Wang and Guo [2013] for examples of the existence of random attractors.

If $f$ and $g$ do not depend on time, then we call (0.1) an autonomous stochastic equation. In the autonomous case, the existence of random attractors of (0.1) has been established recently in Gess [2013a,b], Gess et al. [2011] by variational methods under the condition that the growth rate of the nonlinearity $f$ is not bigger than $p$. This result has been extended in Wang and Guo [2013] to the case where $f$ is non-autonomous and has a polynomial growth of any order. Note that in all papers mentioned above, the $p$-Laplace equation is defined in a bounded domain where compactness of Sobolev embeddings is available. Existence results on random attractors for the stochastic $p$-Laplace equation defined on unbounded domains have been studied in Krause and Wang [2014], Lewis et al. [2014], Li et al. [2014]. The goal of Chapter 3 is to overcome the non-compactness of Sobolev embeddings on $\mathbb{R}^n$ and prove the existence and uniqueness of random attractors for (0.1) in $L^2(\mathbb{R}^n)$. More precisely, we will show by a cut-off technique that the tails of solutions of (0.1) are uniformly small outside a bounded domain for large times. We then use this fact and the compactness of solutions in bounded domains to establish the asymptotic compactness of solutions in $L^2(\mathbb{R}^n)$. By the asymptotic compactness and absorbing sets of the equation, we can obtain the existence and uniqueness of random attractors. This random attractor is pathwise periodic if $f(t, x, u)$ and $g(t, x)$ are periodic in $t$.

Similar approaches also yield existence results for a random attractor for equation (0.3). Additionally, we demonstrate the convergence, in some sense, of the random attractor to the deterministic one as $\alpha \to 0$. These results can be found in Chapter 4, as well as in the journal article Lewis et al. [2014].
CHAPTER 2
RANDOM ATTRACTOR THEORY

The following notation will be used throughout this thesis: $\| \cdot \|$ for the norm of $L^2(\mathbb{R}^n)$ and $(\cdot, \cdot)$ for its inner product. The norm of $L^p(\mathbb{R}^n)$ is usually written as $\| \cdot \|_p$ and the norm of a Banach space $X$ is written as $\| \cdot \|_X$. The symbol $c$ or $c_i (i = 1, 2, \ldots)$ is used for a general positive number which may change from line to line.

Finally, we recall the following inequality which will be used to interpolate between some spaces:

$$\|u\|_p^p \leq \frac{q-p}{q-2} \|u\|^2 + \frac{p-2}{q-2} \|u\|_q^q,$$  \hspace{1cm} (0.1)

where $2 < p < q$ and $u \in L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$.

The following discussion will be the basic theory necessary to discuss the asymptotic dynamics in the following chapters. We will specialize all results to the phase space $L^2(\mathbb{R}^n)$, but it should be noted that it is not difficult to extend these definitions and theorems to more general Banach spaces.

The following definition of a cocycle was introduced in Wang [2012] in order to extend the notion of a cocycle for a random dynamical system to include systems that are also simultaneously driven by non-autonomous terms. Cocycles in general are extensions of the notion of semigroup or solution operator, in order to explicitly characterize the underlying dynamics of the probability space, and in this case the initial time.

**Definition 2.0.1.** A map $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is called a Cocycle if for all $\tau \in \mathbb{R}, \omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(\cdot, \tau, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(L^2(\mathbb{R}^n)), \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable.

(ii) $\Phi(0, \tau, \omega, \cdot)$ is the identity map on $L^2(\mathbb{R}^n)$.

(iii) $\Phi(t + s, \tau, \omega, \cdot) = \Phi(t, \tau, \cdot, \theta_s \omega, \cdot) \circ \Phi(s, \tau, \omega, \cdot)$.

(iv) $\Phi(t, \tau, \omega, \cdot) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is continuous.
Φ is said to be $T$-periodic if for every $t \in \mathbb{R}^+$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, 

$$
\Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot).
$$

In order to discuss the notion of an attractor, we have to first discuss the domain of attraction. The idea of global attractor is a set which attracts all bounded subsets of the phase space. We can also discuss more general domains of attraction by defining a collection of nonempty bounded subsets of the phase space, which become time and path dependent random sets. This extension is in fact very useful to discuss uniqueness and compactness of attractors in certain cases. The reader is referred to the book Carvalho et al. [2013] for further discussion of these details. We define such a collection as,

$$
D_\alpha = \{ D = \{ D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega \} \}.
$$

Elements of $D_\alpha$ must be bounded and nonempty, and the entire collection must also be inclusion-closed, which we define below.

**Definition 2.0.2.** A collection of sets $D_\alpha$ is called inclusion-closed if whenever $D \in D_\alpha$, and $D' \subseteq L^2(\mathbb{R}^n)$ is such that $D' \subseteq D$, then $D' \in D_\alpha$.

The set $D_\alpha$ is often referred to as a universe in the literature. We will discuss the particular choice of universe for the attractors discussed later in the thesis.

**Definition 2.0.3.** A set $A_\alpha = \{ A_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_\alpha$ is called a $D_\alpha$-random pullback attractor for $\Phi$ in $L^2(\mathbb{R}^n)$ if the following are satisfied:

(i) $A$ is measurable and $A(\tau, \omega)$ is compact for all $\tau \in \mathbb{R}$ and $\omega \in \Omega$.

(ii) $A$ is invariant, that is, for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\Phi(t, \tau, \omega, \cdot) = A(\tau + t, \theta_t \omega), \ \forall \ t \geq 0.
$$

(iii) For every $B = \{ B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D$ and for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$
\lim_{t \to \infty} \text{dist}_{L^2(\mathbb{R}^n)}(\Phi(t, \tau - t, \theta_{-t} \omega, B(\tau - t, \theta_{-t} \omega)), A(\tau, \omega)) = 0,
$$

where $\text{dist}_{L^2(\mathbb{R}^n)}$ is the Hausdorff semi-distance between two sets in $L^2(\mathbb{R}^n)$.

There are several important aspects of this definition. The $D_\alpha$-pullback random attractor is parameterized by both initial time, $\tau$, and the path in the probability space, $\omega$. There is also an important dependence on the universe $D_\alpha$ in which this object is attracting. We refer to the above cited literature for discussion and further motivation for why this particular object is interesting to study.

There are many different ways to prove the existence and uniqueness of an attractor, depending on the setting. Below we recall two definitions that will be used in this work, and are particularly important in the dynamics of PDEs, where compactness is a crucial technical concern.
Definition 2.0.4. Let \( K \in \mathcal{D}_\alpha \) be a family of nonempty closed subsets of \( L^2(\mathbb{R}^n) \). Then \( K \) is called a \( \mathcal{D}_\alpha \)-pullback absorbing set for \( \Phi \) if for all \( \tau \in \mathbb{R}, \omega \in \Omega \) and for every \( B \in \mathcal{D}_\alpha \), there exists \( T = T(B, \tau, \omega) > 0 \) such that

\[
\Phi(t\tau - t, \theta_{-\tau}\omega, B(t - t, \theta_{-\tau}\omega)) \subseteq K(\tau, \omega), \quad \text{for all } t \geq T.
\]

If, in addition, \( K \) is measurable with respect to \( \mathcal{F} \) in \( \Omega \), then \( K \) is called a closed measurable \( \mathcal{D}_\alpha \)-pullback absorbing set of \( \Phi \).

Definition 2.0.5. A Cocycle \( \Phi \) on \( L^2(\mathbb{R}^n) \) over \( \mathbb{R} \) and \( (\Omega, \mathcal{F}, \mathbb{P}, \theta) \) is called \( \mathcal{D}_\alpha \)-pullback asymptotically compact in \( L^2(\mathbb{R}^n) \) if for every \( \tau \in \mathbb{R}, \omega \in \Omega, D \in \mathcal{D}_\alpha, t_n \to \infty \) and \( u_{0,n} \in D(\tau - t_n, \theta_{-t_n}\omega) \), the sequence \( \Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, u_{0,n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \).

Finally, we recall the following proposition that will be used throughout this work to guarantee existence, uniqueness, and periodicity of \( \mathcal{D}_\alpha \)-pullback random attractors. It is an extension of classical existence results to the definition of cocycle in the non-autonomous stochastic setting. The proof and further discussion can be found in Wang [2012].

Proposition 2.0.6. Let \( \mathcal{D}_\alpha \) be the collection given above. If \( \Phi \) is \( \mathcal{D}_\alpha \)-pullback asymptotically compact in \( L^2(\mathbb{R}^n) \) and \( \Phi \) has a closed measurable \( \mathcal{D}_\alpha \)-pullback absorbing set \( K \) in \( \mathcal{D}_\alpha \), then \( \Phi \) has a unique \( \mathcal{D}_\alpha \)-pullback attractor \( \mathcal{A}_\alpha \) in \( L^2(\mathbb{R}^n) \) which is given by, for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),

\[
\mathcal{A}_\alpha(\tau, \omega) = \Omega(K, \tau, \omega) = \bigcup_{B \in \mathcal{D}} \Omega(B, \tau, \omega),
\]

where \( \Omega(K) = \{ \Omega(K, \tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is the \( \omega \)-limit set of \( K \).

If, in addition, there is \( T > 0 \) such that \( \Phi(t, \tau + T, \omega, \cdot) = \Phi(t, \tau, \omega, \cdot) \) and \( K(\tau + T, \omega) = K(\tau, \omega) \) for all \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), then the attractor \( \mathcal{A}_\alpha \) is pathwise \( T \)-periodic, i.e., \( \mathcal{A}_\alpha(\tau + T, \omega) = \mathcal{A}_\alpha(\tau, \omega) \).
CHAPTER 3

ADDITIVE NOISE

3.1 Cocycles Associated with Degenerate Equations

In this section, we first establish the well-posedness of equation (0.1) in $L^2(\mathbb{R}^n)$, and then define a continuous cocycle for the stochastic equation. This step is necessary for us to investigate the asymptotic behavior of solutions.

Let $(\Omega, \mathcal{F}, P)$ be the standard probability space where $\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}$, $\mathcal{F}$ is the Borel $\sigma$-algebra induced by the compact-open topology of $\Omega$ and $P$ is the Wiener measure on $(\Omega, \mathcal{F})$. Denote by $\{ \theta_t \}_{t \in \mathbb{R}}$ the family of shift operators given by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t) \quad \text{for all } \omega \in \Omega \text{ and } t \in \mathbb{R}. $$

From Arnold [1998] we know that $(\Omega, \mathcal{F}, P, \{ \theta_t \}_{t \in \mathbb{R}})$ is a metric dynamical system. Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, consider the following stochastic equation defined for $x \in \mathbb{R}^n$ and $t > \tau$,

$$\frac{\partial u}{\partial t} + \lambda u - \text{div}\left( |\nabla u|^{p-2} \nabla u \right) = f(t, x, u) + g(t, x) + \eta \eta(\theta_t \omega) u + \varepsilon h(x) \frac{dW}{dt} \quad (1.1)$$

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n, \quad (1.2)$$

where $p \geq 2$, $\alpha > 0$, $\lambda > 0$, $\varepsilon > 0$, $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, $h \in H^2(\mathbb{R}^n)$, $\eta$ is an integrable tempered random variable and $W$ is a two-sided real-valued Wiener process on $(\Omega, F, P)$. We assume the nonlinearity $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and satisfies, for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t, x, s) \leq -\gamma |s|^q + \psi_1(t, x), \quad (1.3)$$

$$|f(t, x, s)| \leq \psi_2(t, x)|s|^{q-1} + \psi_3(t, x), \quad (1.4)$$

$$\frac{\partial f}{\partial s}(t, x, s) \leq \psi_4(t, x), \quad (1.5)$$

where $\gamma > 0$ and $q \geq p$ are constants, $\psi_1 \in L^1_{\text{loc}}(\mathbb{R}, L^1(\mathbb{R}^n))$, $\psi_2, \psi_4 \in L^\infty_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{R}^n))$, and $\psi_3 \in L^{q_1}_{\text{loc}}(\mathbb{R}, L^{q_1}(\mathbb{R}^n))$. From now on, we always assume
\( h \in H^2(\mathbb{R}^n) \cap W^{2,q}(\mathbb{R}^n) \) and use \( p_1 \) and \( q_1 \) to denote the conjugate exponents of \( p \) and \( q \), respectively. Since \( h \in H^2(\mathbb{R}^n) \cap W^{2,q}(\mathbb{R}^n) \) and \( q \geq p \), by (0.1) we find \( h \in W^{2,p}(\mathbb{R}^n) \).

To define a random dynamical system for (1.1), we need to transfer the stochastic equation to a pathwise deterministic system. As usual, let \( z \) be the random variable given by:

\[
z(\omega) = -\lambda \int_{-\infty}^{0} e^{\lambda \tau} \omega(\tau) d\tau, \quad \omega \in \Omega.
\]

It follows from Arnold [1998] that there exists a \( \theta_t \)-invariant set \( \tilde{\Omega} \) of full measure such that \( z(\theta_t \omega) \) is continuous in \( t \) and \( \lim_{t \to \pm \infty} \frac{|z(\theta_t \omega)|}{|t|} = 0 \) for all \( \omega \in \tilde{\Omega} \). We also assume \( \eta(\theta_t \omega) \) is pathwise continuous for each fixed \( \omega \in \tilde{\Omega} \). For convenience, we will denote \( \tilde{\Omega} \) by \( \Omega \) in the sequel. Let \( u(t, \tau, \omega, u_\tau) \) be a solution of problem (1.7)-(1.8) with initial condition \( u_\tau \) at initial time \( \tau \), and define

\[
v(t, \tau, \omega, v_\tau) = u(t, \tau, \omega, u_\tau) - eh(x)z(\theta_t \omega) \quad \text{with} \quad v_\tau = u_\tau - ehz(\theta_\tau \omega). \quad (1.6)
\]

By (1.1) and (1.6), after simple calculations, we get

\[
\frac{\partial v}{\partial t} - \text{div} \left( |\nabla (v + eh(x)z(\theta_t \omega))|^{p-2} \nabla (v + eh(x)z(\theta_t \omega)) \right) + \lambda v \\
= f(t, x, v + eh(x)z(\theta_t \omega)) + g(t, x) + \alpha\eta(\theta_t \omega)v + \alpha\epsilon\eta(\theta_t \omega)z(\theta_t \omega)h, \quad (1.7)
\]

with initial condition

\[
v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n. \quad (1.8)
\]

In what follows, we first prove the well-posedness of problem (1.7)-(1.8) in \( L^2(\mathbb{R}^n) \), and then define a cocycle for (1.1)-(1.2).

**Definition 3.1.1.** Given \( \tau \in \mathbb{R}, \ \omega \in \Omega \) and \( v_\tau \in L^2(\mathbb{R}^n) \), let \( v(\cdot, \tau, \omega, v_\tau) : [\tau, \infty) \to L^2(\mathbb{R}^n) \) be a continuous function with \( v \in L^p_\text{loc}([\tau, \infty), W^{1,p}(\mathbb{R}^n)) \cap L^q_\text{loc}([\tau, \infty), L^q(\mathbb{R}^n)) \) and \( \frac{\partial v}{\partial t} \in L^p_\text{loc}([\tau, \infty), (W^{1,p})^*) + L^q_\text{loc}([\tau, \infty), L^q(\mathbb{R}^n)) \). We say \( v \) is a solution of (1.7)-(1.8) if \( v(\tau, \tau, \omega, v_\tau) = v_\tau \) and for every \( \zeta \in W^{1,p}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \),

\[
\frac{d}{dt}(v, \zeta) + \int_{\mathbb{R}^n} |\nabla (v + ehz(\theta_t \omega))|^{p-2} \nabla (v + ehz(\theta_t \omega)) \cdot \nabla \zeta dx + (\lambda - \alpha\eta(\theta_t \omega))(v, \zeta)
\]

\[
= \int_{\mathbb{R}^n} f(t, x, v + ehz(\theta_t \omega)) \zeta dx + (g(t, \cdot), \zeta) + \alpha\epsilon\eta(\theta_t \omega)z(\theta_t \omega)(h, \zeta)
\]

in the sense of distribution on \([\tau, \infty)\).
Next, we prove the existence and uniqueness of solutions of (1.7)-(1.8) in $L^2(\mathbb{R}^n)$. To this end, we set $O_k = \{ x \in \mathbb{R}^n : |x| < k \}$ for each $k \in \mathbb{N}$ and consider the following equation defined in $O_k$:

$$ \frac{dv_k}{dt} - \text{div} \left( |\nabla (v_k + \varepsilon h(x)z(\theta_t \omega))|^{p-2} \nabla (v_k + \varepsilon h(x)z(\theta_t \omega)) \right) + \lambda v_k $$

$$ = f(t, x, v_k + \varepsilon h(x)z(\theta_t \omega)) + g(t, x) + \alpha \eta(\theta_t \omega)v_k + \alpha \varepsilon \eta(\theta_t \omega)z(\theta_t \omega)h, \quad (1.9) $$

with boundary condition

$$ v_k(t, x) = 0 \quad \text{for all } t > \tau \text{ and } |x| = k \quad (1.10) $$

and initial condition

$$ v(\tau, x) = v_\tau(x) \quad \text{for all } x \in O_k. \quad (1.11) $$

By the arguments in Wang and Guo [2013], one can show that if (1.3)-(1.5) are fulfilled, then for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, system (1.9)-(1.11) has a unique solution $v_k(\cdot, \tau, \omega, v_\tau)$ in the sense of Definition 3.1.1 with $\mathbb{R}^n$ replaced by $O_k$. Moreover, $v_k(t, \tau, \omega, v_\tau)$ is $(\mathcal{F}, \mathcal{B}(L^2(O_k)))$-measurable with respect to $\omega \in \Omega$. We now investigate the limiting behavior of $v_k$ as $k \to \infty$. For convenience, we write $V_k = W_0^{1,p}(O_k)$ and $V = W^{1,p}(\mathbb{R}^n)$. Let $A: V_k \to V_k^*$ be the operator given by

$$ (A(v_1), v_2)_{V_k^*, V_k} = \int_{O_k} |\nabla v_1|^{p-2} \nabla v_1 \cdot \nabla v_2 \, dx, \quad \text{for all } v_1, v_2 \in V_k, \quad (1.12) $$

where $(\cdot, \cdot)_{V_k^*, V_k}$ is the duality pairing of $V_k^*$ and $V_k$. Note that $A$ is a monotone operator as in Showalter [1997] and $A: V \to V^*$ is also well defined by replacing $O_k$ by $\mathbb{R}^n$ in (1.12). The following uniform estimates on $v_k$ are useful.

**Lemma 3.1.2.** Suppose (1.3)-(1.5) hold. Then for every $T > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathbb{R}^n)$, the solution $v_k(t, \tau, \omega, v_\tau)$ of system (1.9)-(1.11) has the properties:

$$ \{v_k\}_{k=1}^\infty \text{ is bounded in } L^\infty(\tau, \tau + T; L^2(O_k)) \bigcap L^q(\tau, \tau + T; L^q(O_k)) \bigcap L^p(\tau, \tau + T; V_k), $$

$$ \{A(v_k + \varepsilon hz(\theta_t \omega))\}_{k=1}^\infty \text{ is bounded in } L^{p_1}(\tau, \tau + T; V_k^*) \text{ with } \frac{1}{p_1} + \frac{1}{p} = 1, $$

$$ \{f(t, x, v_k + \varepsilon hz(\theta_t \omega))\}_{k=1}^\infty \text{ is bounded in } L^{q_1}(\tau, \tau + T; L^{q_1}(O_k)), \quad \frac{1}{q_1} + \frac{1}{q} = 1, $$

and

$$ \left\{ \frac{dv_k}{dt} \right\} \text{ is bounded in } L^{p_1}(\tau, \tau + T; V_k^*) + L^2(\tau, \tau + T; L^2(O_k)) + L^{q_1}(\tau, \tau + T; L^{q_1}(O_k)). $$
Proof. By (1.9) we get

\[
\frac{1}{2} \frac{d}{dt} \|v_k\|^2 + \int_{O_k} |\nabla (v_k + ehz(\theta_t \omega))|^{p-2} \nabla (v_k + ehz(\theta_t \omega)) \cdot \nabla v_k \, dx + \lambda \|v_k\|^2
\]

\[
= \int_{O_k} f(t, x, v_k + ehz(\theta_t \omega))v_k \, dx + (g(t), v_k)
\]

\[
+ \alpha \eta(\theta_t \omega) \|v_k\|^2 + \alpha \eta(\theta_t \omega) z(\theta_t \omega)(h, v_k).
\]

(1.13)

For the second term on the left-hand side of (1.13), by Young's inequality we obtain

\[
\int_{O_k} |\nabla (v_k + ehz(\theta_t \omega))|^{p-2} \nabla (v_k + ehz(\theta_t \omega)) \cdot \nabla v_k \, dx
\]

\[
= \int_{O_k} |\nabla (v_k + ehz(\theta_t \omega))|^p \, dx
\]

\[
- \int_{O_k} |\nabla (v_k + ehz(\theta_t \omega))|^{p-2} \nabla (v_k + ehz(\theta_t \omega)) \cdot \nabla (ehz(\theta_t \omega)) \, dx
\]

\[
\geq \frac{1}{2} \int_{O_k} |\nabla (v_k + ehz(\theta_t \omega))|^p \, dx - c_1 |\epsilon z(\theta_t \omega)|^p \|\nabla h\|_p^p.
\]

(1.14)

For the first term on the right-hand side of (1.13), by (1.3) and (1.4) we get

\[
\int_{O_k} f(t, x, v_k + ehz(\theta_t \omega))v_k \, dx
\]

\[
= \int_{O_k} f(t, x, v_k + ehz(\theta_t \omega))(v_k + ehz(\theta_t \omega)) \, dx
\]

\[
- \epsilon z(\theta_t \omega) \int_{O_k} f(t, x, v_k + ehz(\theta_t \omega))h(x) \, dx
\]

\[
\leq - \gamma \int_{O_k} |v_k + ehz(\theta_t \omega)|^q \, dx + \int_{O_k} |\nabla \psi_1(t, x)| \, dx
\]

\[
+ \int_{O_k} |\nabla \psi_2(t, x)| |v_k + ehz(\theta_t \omega)| \, dx + \int_{O_k} |\nabla \psi_3(t, x)| |ehz(\theta_t \omega)| \, dx
\]

\[
\leq - \frac{\gamma}{2} \|v_k + ehz(\theta_t \omega)\|^q + \|\psi_1(t)\|_1 + \|\psi_3(t)\|_{q_1}^q + c_2 \int_{O_k} |ehz(\theta_t \omega)| \, dx.
\]

(1.15)

By Young's inequality we obtain

\[
\int_{O_k} g(t, x)v_k \, dx + \alpha \eta(\theta_t \omega) z(\theta_t \omega) \int_{O_k} h(x) v_k \, dx
\]

\[
\leq \frac{4}{\lambda} |\alpha \eta(\theta_t \omega) z(\theta_t \omega)|^2 \|h\|^2 + \frac{4}{\lambda} \|g(t)\|^2 + \frac{\lambda}{8} \|v_k\|^2.
\]

(1.16)
It follows from (1.13)-(1.16) that
\[
\frac{d}{dt} \|v_k\|^2 + \frac{7}{4} \lambda \|v_k\|^2 + \int_{\mathcal{O}_k} |\nabla (v_k + \varepsilon h z(\theta_1 \omega))|^p dx + \gamma \int_{\mathcal{O}_k} |v_k + \varepsilon h z(\theta_1 \omega)|^q dx \\
\leq 2 \alpha \eta(\theta_1 \omega) \|v_k\|^2 + c_3 \left( |\varepsilon z(\theta_1 \omega)|^p + |\varepsilon z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega) z(\theta_1 \omega)|^2 \right) \\
+ c_4 \left( \|g(t)\|^2 + \|\psi_1(t)\|_1 + \|\psi_3(t)\|_{q_1}^{q_1} \right). \tag{1.17}
\]
Multiplying (1.17) by \(e^{\frac{7}{4} \lambda t - 2 \alpha \int_0^t \eta(\theta_1 \omega) dr}\), and then integrating from \(\tau\) to \(t\), we get
\[
\|v_k(t, \tau, \omega, \nu_{\tau})\|^2 \\
+ \int_{\tau}^{t} e^{\frac{7}{4} \lambda (s-t) - 2 \alpha \int_s^t \eta(\theta_1 \omega) dr} \int_{\mathcal{O}_k} |\nabla (v_k(s, \tau, \omega, \nu_{\tau}) + \varepsilon h z(\theta_1 \omega)|^p dx ds \\
+ \gamma \int_{\tau}^{t} e^{\frac{7}{4} \lambda (s-t) - 2 \alpha \int_s^t \eta(\theta_1 \omega) dr} \int_{\mathcal{O}_k} |v_k(s, \tau, \omega, \nu_{\tau}) + \varepsilon h z(\theta_1 \omega)|^q dx ds \\
\leq c_3 \int_{\tau}^{t} e^{\frac{7}{4} \lambda (s-t) - 2 \alpha \int_s^t \eta(\theta_1 \omega) dr} \left( |\varepsilon z(\theta_1 \omega)|^p + |\varepsilon z(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega) z(\theta_1 \omega)|^2 \right) ds \\
+ c_4 \int_{\tau}^{t} e^{\frac{7}{4} \lambda (s-t) - 2 \alpha \int_s^t \eta(\theta_1 \omega) dr} \left( \|g(s)\|^2 + \|\psi_1(s)\|_1 + \|\psi_3(s)\|_{q_1}^{q_1} \right) ds \\
+ e^{\frac{7}{4} \lambda (t-\tau) - 2 \alpha \int_{\tau}^{t} \eta(\theta_1 \omega) dr} \|\nu_{\tau}\|^2. \tag{1.18}
\]
By (1.18) we get
\[
\{v_k\} \text{ is bounded in } L^\infty(\tau, \tau + T; L^2(\mathcal{O}_k)) \cap L^q(\tau, \tau + T; L^q(\mathcal{O}_k)) \cap L^p(\tau, \tau + T; V_k). \tag{1.19}
\]
By (1.4) and (1.19) we obtain
\[
\{f(t, x, v_k + \varepsilon h z(\theta_1 \omega))\}_{k=1}^{\infty} \text{ is bounded in } L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}_k)). \tag{1.20}
\]
By (1.12) and (1.19) we get
\[
\{A(v_k + \varepsilon h z(\theta_1 \omega))\}_{k=1}^{\infty} \text{ is bounded in } L^{p_1}(\tau, \tau + T; V^*_k). \tag{1.21}
\]
By (1.19)-(1.21) it follows from (1.9) that
\[
\left\{ \frac{dv_k}{dt} \right\} \text{ is bounded in } L^{p_1}(\tau, \tau + T; V^*_k) \\
+ L^2(\tau, \tau + T; L^2(\mathcal{O}_k)) + L^{q_1}(\tau, \tau + T; L^{q_1}(\mathcal{O}_k)),
\]
which completes the proof. \(\square\)
The next lemma is concerned with the well-posedness of (1.7)-(1.8) in $L^2(\mathbb{R}^n)$.

**Lemma 3.1.3.** Suppose (1.3)-(1.5) hold. Then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in L^2(\mathbb{R}^n)$, problem (1.7)-(1.8) has a unique solution $v(t, \tau, \omega, v_\tau)$ in the sense of Definition 3.1.1. In addition, $v(t, \tau, \omega, v_\tau)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega$ and continuous in $v_\tau$ in $L^2(\mathbb{R}^n)$ and satisfies

$$
\frac{d}{dt}\|v(t, \tau, \omega, v_\tau)\|^2 + 2(\lambda - \alpha \eta(\theta t \omega)) \|v\|^2 + 2\|\nabla(v + \varepsilon h(t, \omega, \theta))\|^p \nonumber
$$

$$
= 2\varepsilon \eta(\theta t \omega) \int_{\mathbb{R}^n} \left| \nabla(v + \varepsilon h(t, \omega, \theta)) \right|^{p-2} \nabla(v + \varepsilon h(t, \omega, \theta)) \cdot \nabla hdx
$$

$$
+ 2\int_{\mathbb{R}^n} f(t, x, v + \varepsilon h(t, \omega, \theta))vdx + 2(g(t), v) + 2\alpha \eta(\theta t \omega)z(\theta t \omega)(h, v) \quad (1.22)
$$

for almost all $t \geq \tau$.

**Proof.** Let $T > 0$, $t_0 \in [\tau, \tau + T]$ and $v_k(t, \tau, \omega, v_\tau)$ be the solution of system (1.9)-(1.11) defined in $O_k$. Extend $v_k$ to the entire space $\mathbb{R}^n$ by setting $v_k = 0$ on $\mathbb{R}^n \setminus O_k$ and denote this extension still by $v_k$. By Lemma 3.1.2 we find that there exist $\tilde{v} \in L^2(\mathbb{R}^n)$, $v \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathbb{R}^n))$, $\chi_1 \in L^{q_1}(\tau, \tau + T; L^{q_1}(\mathbb{R}^n))$, $\chi_2 \in L^{p_1}(\tau, \tau + T; V^*)$ such that, up to a subsequence,

$$
v_k \rightarrow v \text{ weak-star in } L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)), \quad (1.23)
$$

$$
v_k \rightarrow v \text{ weakly in } L^p(\tau, \tau + T; V) \text{ and } L^q(\tau, \tau + T; L^q(\mathbb{R}^n)), \quad (1.24)
$$

$$
A(v_k + \varepsilon h(t, \omega, \theta)) \rightarrow \chi_2 \text{ weakly in } L^{p_1}(\tau, \tau + T; V^*), \quad (1.25)
$$

$$
f(t, x, v_k + \varepsilon h(t, \omega, \theta)) \rightarrow \chi_1 \text{ weakly in } L^{q_1}(\tau, \tau + T; L^{q_1}(\mathbb{R}^n)), \quad (1.26)
$$

and

$$
v_k(t_0, \tau, \omega, v_\tau) \rightarrow \tilde{v} \text{ weakly in } L^2(\mathbb{R}^n). \quad (1.27)
$$

On the other hand, by the compactness of embedding $W^{1,p}(O_k) \hookrightarrow L^2(O_k)$ and Lemma 3.1.2, we can choose a further subsequence (not relabeled) by a diagonal process such that for each $k_0 \in \mathbb{N}$,

$$
v_k \rightarrow v \text{ strongly in } L^2(\tau, \tau + T; L^2(O_{k_0})). \quad (1.28)
$$

By (1.9) and (1.23)-(1.26) one can show that for every $\xi \in V \cap L^2(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$,

$$
\frac{d}{dt}(v, \xi) + (\chi_2, \xi)_{(V^*, V)} + (\lambda - \alpha \eta(\theta t \omega))(v, \xi) \nonumber
$$

$$
= (\chi_1, \xi)_{(L^{q_1}, L^q)} + (g(t), \xi) + \alpha \eta(\theta t \omega)z(\theta t \omega)(h, \xi) \quad (1.29)
$$
in the sense of distribution. By (1.29) we find

$$\frac{dv}{dt} = -\chi_2 + \chi_1 - (\lambda - \alpha \eta(\theta_1 \omega))v + g + \alpha \eta(\theta_1 \omega)z(\theta_1 \omega)h$$

(1.30)

in \(L^p(\tau, \tau + T; V^*) + L^q(\tau, \tau + T; L^2(\mathbb{R}^n)) + L^2(\tau, \tau + T; L^2(\mathbb{R}^n))\), which along with the fact \(v \in L^\infty(\tau, \tau + T; L^2(\mathbb{R}^n)) \cap L^p(\tau, \tau + T; V) \cap L^q(\tau, \tau + T; L^q(\mathbb{R}^n))\) implies (see, e.g., Lions [1969]) that \(v \in C([\tau, \tau + T], L^2(\mathbb{R}^n))\) and

$$\frac{1}{2} \frac{d}{dt} ||v||^2 = (\frac{dv}{dt}, v)_{V^* + L^q(\tau + T, \tau; V) \cap L^q(\tau + T, \tau; L^2)}$$

for almost all \(t \in (\tau, \tau + T)\). (1.31)

By (1.23)-(1.28), we can argue as in Wang and Guo [2013] to show that

$$\chi_2 = A(v + \epsilon hz(\theta_1 \omega)), \quad \chi_1 = f(t, x, v + \epsilon hz(\theta_1 \omega)), \quad v(\tau) = v_\tau \text{ and } v(t_0) = \tilde{v}.$$  

(1.32)

By (1.29) and (1.32) we find that \(v\) is a solution of problem (1.7)-(1.8) in the sense of Definition 3.1.1. On the other hand, by (1.30) and (1.32) we see that \(v\) satisfies energy equation (1.22).

We next prove the uniqueness of solutions. Let \(v_1\) and \(v_2\) be the solutions of (1.7) and \(\tilde{v} = v_1 - v_2\). Then we have

$$\frac{d}{dt} ||\tilde{v}||^2 + A(v_1 + \epsilon hz(\theta_1 \omega)) - A(v_2 + \epsilon hz(\theta_1 \omega)) + \lambda \tilde{v}$$

$$= \alpha \eta(\theta_1 \omega)\tilde{v} + f(t, x, v_1 + \epsilon hz(\theta_1 \omega)) - f(t, x, v_2 + \epsilon hz(\theta_1 \omega)),$$

which along with (1.5) and the monotonicity of \(A\) yields, for all \(t \in [\tau, \tau + T]\),

$$\frac{d}{dt} ||\tilde{v}||^2 \leq 2\alpha \eta(\theta_1 \omega) ||\tilde{v}||^2 + 2 \int_{\mathbb{R}^n} \psi_4(t, x)|\tilde{v}|^2 dx \leq c ||\tilde{v}||^2$$

for some positive constant \(c\) depending on \(\tau, T\) and \(\omega\). By Gronwall’s lemma we get, for all \(t \in [\tau, \tau + T]\),

$$||v_1(t, \tau, \omega, v_{1,\tau}) - v_2(t, \tau, \omega, v_{2,\tau})||^2 \leq e^{c(t-\tau)}||v_{1,\tau} - v_{2,\tau}||^2.$$  

(1.33)

So the uniqueness and continuity of solutions in initial data follow immediately.

Note that (1.27), (1.32) and the uniqueness of solutions imply that the entire sequence \(v_k(t_0, \tau, \omega, v_\tau) \rightarrow v(t_0, \tau, \omega, v_\tau)\) weakly in \(L^2(\mathbb{R}^n)\) for every fixed \(t_0 \in [\tau, \tau + T]\) and \(\omega \in \Omega\). By the measurability of \(v_k(t, \tau, \omega, v_\tau)\) in \(\omega\), we obtain the measurability of \(v(t, \tau, \omega, v_\tau)\) directly. \(\square\)

The following result is useful when proving the asymptotic compactness of solutions.
Lemma 3.1.4. Let (1.3)-(1.5) hold and \( \{v_n\}_{n=1}^\infty \) be a bounded sequence in \( L^2(\mathbb{R}^n) \). Then for every \( \tau \in \mathbb{R} \), \( t > \tau \) and \( \omega \in \Omega \), there exist \( v_0 \in L^2(\tau,t; L^2(\mathbb{R}^n)) \) and a subsequence \( \{v(\cdot, \tau, \omega, v_{n_m})\}_{m=1}^\infty \) of \( \{v(\cdot, \tau, \omega, v_n)\}_{n=1}^\infty \) such that \( v(s, \tau, \omega, v_{n_m}) \rightarrow v_0(s) \) in \( L^2(\mathcal{O}_k) \) as \( m \rightarrow \infty \) for every fixed \( k \in \mathbb{N} \) and for almost all \( s \in (\tau, t) \).

Proof. Let \( T \) be a sufficiently large number such that \( t \in (\tau, \tau + T) \). Following the proof of (1.28), we can show that there exists \( \tilde{v} \in L^2(\tau, \tau + T; L^2(\mathbb{R}^n)) \) such that, up to a subsequence,

\[
v(\cdot, \tau, \omega, v_n) \rightarrow \tilde{v} \text{ strongly in } L^2(\tau, \tau + T; L^2(\mathcal{O}_k)) \text{ for every } k \in \mathbb{N}.
\]

Thus, for \( k = 1 \), there exist a set \( I_1 \subseteq [\tau, \tau + T] \) of measure zero and a subsequence \( v(\cdot, \tau, \omega, v_{n_1}) \) such that

\[
v(s, \tau, \omega, v_{n_1}) \rightarrow \tilde{v}(s) \text{ in } L^2(\mathcal{O}_1) \text{ for all } s \in [\tau, \tau + T] \setminus I_1.
\]

Similarly, for \( k = 2 \), there exist a set \( I_2 \subseteq [\tau, \tau + T] \) of measure zero and a subsequence \( v(\cdot, \tau, \omega, v_{n_2}) \) of \( v(\cdot, \tau, \omega, v_{n_1}) \) such that

\[
v(s, \tau, \omega, v_{n_2}) \rightarrow \tilde{v}(s) \text{ in } L^2(\mathcal{O}_2) \text{ for all } s \in [\tau, \tau + T] \setminus I_2.
\]

Repeating this process we find that for each \( k \in \mathbb{N} \), there exist a set \( I_k \subseteq [\tau, \tau + T] \) of measure zero and a subsequence \( v(\cdot, \tau, \omega, v_{n_k}) \) of \( v(\cdot, \tau, \omega, v_{n_{k-1}}) \) such that

\[
v(s, \tau, \omega, v_{n_k}) \rightarrow \tilde{v}(s) \text{ in } L^2(\mathcal{O}_k) \text{ for all } s \in [\tau, \tau + T] \setminus I_k.
\]

Let \( I = \bigcup_{k=1}^\infty I_k \). Then by a diagonal process, we infer that there exists a subsequence (which is still denoted by \( v(\cdot, \tau, \omega, v_n) \)) such that

\[
v(s, \tau, \omega, v_n) \rightarrow \tilde{v}(s) \text{ in } L^2(\mathcal{O}_k) \text{ for all } s \in [\tau, \tau + T] \setminus I \text{ and } k \in \mathbb{N}. \tag{1.34}
\]

Note that \( I \) has measure zero and \( t \in (\tau, \tau + T) \), which along with (1.34) completes the proof. \( \square \)

Based on Lemma 3.1.3, we can define a continuous cocycle for problem (1.1)-(1.2) in \( L^2(\mathbb{R}^n) \). Let \( \Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n) \) be a mapping given by, for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( u_\tau \in L^2(\mathbb{R}^n) \),

\[
\Phi(t, \tau, \omega, u_\tau) = v(t + \tau, \tau, \theta_{t \tau} \omega, v_\tau) + eh(x)z(\theta_{t \omega}), \tag{1.35}
\]

where \( v \) is the solution of system (1.7)-(1.8) with initial condition \( v_\tau = u_\tau - eh(x)z(\omega) \) at initial time \( \tau \). Note that (1.6) and (1.35) imply

\[
\Phi(t, \tau, \omega, u_\tau) = u(t + \tau, \tau, \theta_{t \tau} \omega, u_\tau), \tag{1.36}
\]

where \( u \) is a solution of (1.1)-(1.2) in some sense. Since the solution \( v \) of (1.7)-(1.8) is \((\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))\)-measurable in \( \omega \) and continuous in initial data in \( L^2(\mathbb{R}^n) \),
we find that $\Phi(t, \tau, \omega, u_\tau)$ given by (1.35) is also $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^n)))$-measurable in $\omega$ and continuous in $u_\tau$ in $L^2(\mathbb{R}^n)$. In fact, one can verify that $\Phi$ is a continuous cocycle on $L^2(\mathbb{R}^n)$ over $(\Omega, \mathcal{F}, P, \{\theta\}_{t \in \mathbb{R}})$ in the sense of Definition (2.0.1). Note that the cocycle property (iii) of $\Phi$ can be easily proved by (1.35) and the properties of the solution $v$ of the pathwise deterministic equation (1.7)-(1.8). Our goal is to establish the existence of random attractors of $\Phi$ with an appropriate attraction domain. To specify such an attraction domain, we consider a family $D = \{D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of bounded nonempty sets such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{s \to -\infty} e^{\frac{5}{4} \lambda_5 s + 2\alpha \int_0^s \eta(\theta, \omega) dr} \|D(\tau + s, \theta_s \omega)\|^2 = 0, \quad (1.37)$$

where $\|S\| = \sup_{u \in S} \|u\|_{L^2(\mathbb{R}^n)}$ for a nonempty bounded subset $S$ of $L^2(\mathbb{R}^n)$. In the sequel, we will use $\mathcal{D}_{\alpha}$ to denote the collection of all families with property (1.37):

$$\mathcal{D}_{\alpha} = \{D = \{D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{satisfies (1.37)}\}. \quad (1.38)$$

We will construct a $\mathcal{D}_{\alpha}$-pullback attractor $A_{\alpha} = \{A_{\alpha}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathcal{D}_{\alpha}$ for $\Phi$ in $L^2(\mathbb{R}^n)$ in the sense of Definition (2.0.3).

We will apply the result of Proposition (2.0.6) from Wang [2012] to show the existence of $\mathcal{D}_{\alpha}$-pullback attractors for $\Phi$. Similar results on existence of random attractors can be found in Bates et al. [2006], Caraballo et al. [2003], Crauel and Flandoli [1994], Flandoli and Schmalfuss [1996], Gess [2013a], Schmalfuss [1992].

We remark that the $\mathcal{F}$-measurability of the attractor $A$ was given in Wang [2014b] and the measurability of $A$ with respect to the $P$-completion of $\mathcal{F}$ was given in Wang [2012]. For our purpose, we further assume the following condition on $g$, $\psi_1$ and $\psi_3$: for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^\tau e^{\lambda s} \left( \|g(s, \cdot)\|^2 + \|\psi_1(s, \cdot)\|_{L^1(\mathbb{R}^n)} + \|\psi_3(s, \cdot)\|_{L^{q_1}(\mathbb{R}^n)}^{q_1} \right) ds < \infty. \quad (1.39)$$

### 3.2 Uniform Estimates of Solutions

This section is devoted to uniform estimates of solutions of (1.1) and (1.7) which are needed for proving the existence of random attractors for $\Phi$. When deriving uniform estimates, the following positive number $\alpha_0$ is useful:

$$\alpha_0 = \frac{1}{8(1 + |E(\eta)|)} \lambda. \quad (2.1)$$
Lemma 3.2.1. Let $\alpha_0$ be the positive number given by (2.1). Suppose (1.3)-(1.5) and (1.39) hold. Then for every $\alpha \leq \alpha_0$, $\sigma \in \mathbb{R}$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}_a$, there exists $T = T(\tau, \omega, D, \sigma, \alpha) > 0$ such that for all $t \geq T$, the solution $v$ of problem (1.7)-(1.8) satisfies

$$
\left\| v(\sigma, \tau - t, \theta - \tau \omega, v_{\tau-t}) \right\|^2
+ \int_{\tau-t}^{\sigma} e^{\frac{5}{4} \lambda (s-\sigma) - 2 \alpha \int_{\tau-t}^{s} \eta(\theta, \omega) ds} \left\| \nabla \left( v(s, \tau - t, \theta - \tau \omega, v_{\tau-t}) + e \eta z(\theta - \tau \omega) \right) \right\|^p ds
+ c \int_{-\infty}^{\tau-t} e^{\frac{5}{4} \lambda (s-\sigma) - 2 \alpha \int_{\tau-t}^{s} \eta(\theta, \omega) ds} \left( \left| e \eta z(\theta - \tau \omega) \right|^p + \left| e \eta z(\theta - \tau \omega) \right|^q + |\alpha \eta z(\theta - \tau \omega) z(\theta - \tau \omega)|^2 \right) ds
\leq M,
$$
where $v_{\tau-t} \in D(\tau - t, \theta - \tau \omega)$ and $M = M(\tau, \omega, \sigma, \alpha, \epsilon)$ is given by

$$
M = c \int_{-\infty}^{\tau-t} e^{\frac{5}{4} \lambda (s-\sigma + \tau) - 2 \alpha \int_{\tau-t}^{s} \eta(\theta, \omega) ds} \left( \left| e \eta z(\theta - \tau \omega) \right|^p + \left| e \eta z(\theta - \tau \omega) \right|^q + |\alpha \eta z(\theta - \tau \omega) z(\theta - \tau \omega)|^2 \right) ds
+ c \int_{-\infty}^{\tau-t} e^{\frac{5}{4} \lambda (s-\sigma + \tau) - 2 \alpha \int_{\tau-t}^{s} \eta(\theta, \omega) ds} \left( \left\| \nabla g(s + \tau) \right\|^2 + \left\| \psi_1(s + \tau) \right\|_1 + \left\| \psi_3(s + \tau) \right\|_q \right) ds,
$$
with $c$ being a positive constant independent of $\tau$, $\omega$, $D$, $\alpha$ and $\epsilon$.

Proof. Using energy equation (1.22) and following the proof of (1.17), we obtain

$$
\frac{d}{dt} \left\| v \right\|^2 + \frac{7}{4} \lambda \left\| v \right\|^2 + \int_{\mathbb{R}^n} \left\| \nabla \left( v + e \eta z(\theta, \omega) \right) \right\|^p dx + \gamma \int_{\mathbb{R}^n} \left\| v + e \eta z(\theta, \omega) \right\|^q dx
\leq 2 \alpha \eta \left\| v \right\|^2 + c_3 \left( \left| e \eta z(\theta, \omega) \right|^p + \left| e \eta z(\theta, \omega) \right|^q + |\alpha \eta z(\theta, \omega) z(\theta, \omega)|^2 \right)
+ c_4 \left( \left\| \nabla g(t) \right\|^2 + \left\| \psi_1(t) \right\|_1 + \left\| \psi_3(t) \right\|_q \right). \quad (2.2)
$$

Multiplying (2.2) by $e^{\frac{5}{4} \lambda (\tau-t) - 2 \alpha \int_{\tau-t}^{\tau} \eta(\theta, \omega) ds}$, and then integrating from $\tau - t$ to $\sigma$ with $\sigma \geq \tau - t$, we get,

$$
\left\| v(\sigma, \tau - t, \omega, v_{\tau-t}) \right\|^2 + \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\frac{5}{4} \lambda (s-\sigma) - 2 \alpha \int_{\tau-t}^{s} \eta(\theta, \omega) ds} \left\| v(s, \tau - t, \omega, v_{\tau-t}) \right\|^2 ds
$$
Similarly, by the temperedness of \(\eta\)
\[
\begin{align*}
&+ \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| \nabla (v(s, \tau - t, \omega, v_{\tau-t}) + \epsilon h z(\theta, \omega)) \|_p^p ds \\
&+ \gamma \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v(s, \tau - t, \omega, v_{\tau-t}) + \epsilon h z(\theta, \omega) \|_q^q ds \\
&\leq e^{\frac{s}{4}} \lambda(\tau-t-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v_{\tau-t} \|^2 \\
+c_3 \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v(s, \tau - t, \theta, v_{\tau-t}) \|_1^2 ds \\
&+ c_4 \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v(s, \tau - t, \theta, v_{\tau-t}) \|_1^2 + \| \psi_3(s, \tau) \|_{q_1}^{q_1}) ds. \quad (2.3)
\end{align*}
\]
Replacing \(\omega\) with \(\theta - t\omega\) in (2.3), we get
\[
\begin{align*}
&\| v(\sigma, \tau - t, \theta - t\omega, v_{\tau-t}) \|^2 \\
&+ \frac{\lambda}{2} \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v(s, \tau - t, \theta - t\omega, v_{\tau-t}) \|^2 ds \\
&+ \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| \nabla (v(s, \tau - t, \theta - t\omega, v_{\tau-t}) + \epsilon h z(\theta - t\omega)) \|_p^p ds \\
&+ \gamma \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s-\sigma) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v(s, \tau - t, \theta - t\omega, v_{\tau-t}) + \epsilon h z(\theta - t\omega) \|_q^q ds \\
&\leq e^{\frac{s}{4}} \lambda(\tau-t-\sigma) + 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \| v_{\tau-t} \|^2 \\
+c_3 \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s+\tau-\omega) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \\
x (|\varepsilon z(\theta, \omega)|^p + |\varepsilon z(\theta, \omega)|^q + |\alpha \eta(\theta, \omega) z(\theta, \omega)|^2) ds \\
+c_4 \int_{\tau-t}^{\sigma} e^{\frac{s}{4}} \lambda(s+\tau-\omega) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \\
x (\| \psi_1(s, \tau) \|^2 + \| \psi_3(s, \tau) \|_{q_1}^{q_1}) ds. \quad (2.4)
\end{align*}
\]
By the ergodicity of \(\eta\), (2.1) and (1.39) one can verify that for all \(\alpha \leq \alpha_0\),
\[
\int_{-\infty}^{\sigma-t} e^{\frac{s}{4}} \lambda(s+\tau-\omega) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \\
x (\| g(s+\tau) \|^2 + \| \psi_1(s+\tau) \|_1 + \| \psi_3(s+\tau) \|_{q_1}^{q_1}) ds < \infty. \quad (2.5)
\]
Similarly, by the temperedness of \(\eta\) and \(z\), we can prove that for all \(\alpha \leq \alpha_0\),
\[
\int_{-\infty}^{\sigma-t} e^{\frac{s}{4}} \lambda(s+\tau-\omega) - 2\alpha \int_{\tau}^{s} \eta(\theta, \omega) d\theta \\
x (\| g(s+\tau) \|^2 + \| \psi_1(s+\tau) \|_1 + \| \psi_3(s+\tau) \|_{q_1}^{q_1}) ds < \infty.
\]
\begin{align*}
\times (|\varepsilon z(\theta s \omega)|^p + |\varepsilon z(\theta s \omega)|^q + |\varepsilon z(\theta s \omega)|^q + |\alpha z(\theta s \omega)z(\theta s \omega)|^2) ds < \infty. \quad (2.6)
\end{align*}

Since $v_{\tau-t} \in D(\tau - t, \theta_{-\tau} \omega)$ and $D \in D_{\alpha}$, by (1.37)-(1.38) we obtain
\begin{align*}
e^{-\frac{5\lambda}{4}(\tau-t) + 2\alpha \int_{0}^{\tau} \eta(\theta \omega) dr} |v_{\tau-t}|^2 \\
\leq e^{-\frac{5\lambda}{4}(\tau-t) + 2\alpha \int_{0}^{\tau} \eta(\theta \omega) dr} \left| D(\tau - t, \theta_{-\tau} \omega) \right|^2 \to 0,
\end{align*}
as $t \to \infty$. Therefore, there exists $T = T(\tau, \omega, D, \sigma, \alpha) > 0$ such that for all $t \geq T$,
\begin{align*}
e^{-\frac{5\lambda}{4}(\tau-t) + 2\alpha \int_{0}^{\tau} \eta(\theta \omega) dr} |v_{\tau-t}|^2 \\
\leq \int_{-\infty}^{\sigma - \tau} e^{-\frac{5\lambda}{4}(s+\tau) - 2\alpha \int_{s}^{\tau} \eta(\theta \omega) dr} \times \left( \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_1 + \|\psi_3(s + \tau)\|_{q_1} \right) ds,
\end{align*}
which along with (2.4)-(2.6) concludes the proof.

By Lemma 3.2.1, we obtain the following estimates.

**Lemma 3.2.2.** Suppose (1.3)-(1.5) and (1.39) hold. Then for every $\alpha \leq \alpha_0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_{\alpha}$, there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that for all $t \geq T$ and $\theta_{-\tau} \omega$, the solution $v$ of problem (1.7)-(1.8) satisfies
\begin{align*}
\|v(\tau, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 \\
+ \int_{\tau-t}^{\tau} e^{-\frac{5\lambda}{4}(s-\tau) - 2\alpha \int_{0}^{s} \eta(\theta \omega) dr} \|v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau-t})\|^2 ds \\
+ \int_{\tau-t}^{\tau} e^{-\frac{5\lambda}{4}(s-\tau) - 2\alpha \int_{0}^{s} \eta(\theta \omega) dr} \|\nabla(v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}) + \varepsilon h(x)z(\theta_{-\tau} \omega))\|_p^2 ds \\
+ \int_{\tau-t}^{\tau} e^{-\frac{5\lambda}{4}(s-\tau) - 2\alpha \int_{0}^{s} \eta(\theta \omega) dr} \|v(s, \tau - t, \theta_{-\tau} \omega, v_{\tau-t}) + \varepsilon h(x)z(\theta_{-\tau} \omega))\|_q^q ds \\
\leq R(\tau, \omega, \alpha, \varepsilon), \quad (2.7)
\end{align*}
where $v_{\tau-t} \in D(\tau - t, \theta_{-\tau} \omega)$ and $R(\tau, \omega, \alpha, \varepsilon)$ is given by
\begin{align*}
R(\tau, \omega, \alpha, \varepsilon) \\
= c \int_{-\infty}^{0} e^{-\frac{5\lambda}{4}s - 2\alpha \int_{0}^{s} \eta(\theta \omega) dr} \left( |\varepsilon z(\theta s \omega)|^p + |\varepsilon z(\theta s \omega)|^q + |\alpha z(\theta s \omega)z(\theta s \omega)|^2 \right) ds \\
+ c \int_{-\infty}^{0} e^{-\frac{5\lambda}{4}s - 2\alpha \int_{0}^{s} \eta(\theta \omega) dr} \left( \|g(s + \tau)\|^2 + \|\psi_1(s + \tau)\|_1 + \|\psi_3(s + \tau)\|_{q_1} \right) ds, \quad (2.8)
\end{align*}
with $c$ being a positive constant independent of $\tau$, $\omega$, $D$, $\alpha$ and $\varepsilon$. In addition, we have
\begin{align*}
\lim_{t \to \infty} e^{-\frac{5\lambda}{4}t + 2\alpha \int_{-\infty}^{0} \eta(\theta \omega) dr} R(\tau - t, \theta_{-\tau} \omega, \alpha, \varepsilon) = 0. \quad (2.9)
\end{align*}
Proof. (2.7) and (2.8) are special cases of Lemma 3.2.1 for \( \sigma = \tau \). We now prove (2.9). By (2.8) we have

\[
R(\tau - t, \theta_{-t}\omega, \alpha, \epsilon) = c \int_{-\infty}^{0} e^{\frac{5}{4} \lambda s - 2 \alpha} \int_{0}^{s} \eta(\theta_{-t}\omega) dr \times \left( |\varepsilon(\theta_{s-t}\omega)|^p + |\varepsilon(\theta_{s-t}\omega)|^q + |\alpha \varepsilon(\theta_{s-t}\omega)z(\theta_{s-t}\omega)|^2 \right) ds + c \int_{-\infty}^{0} e^{\frac{5}{4} \lambda s - 2 \alpha} \int_{0}^{s} \eta(\theta_{-t}\omega) dr \times \left( |\varepsilon(\theta_{s\omega})|^p + |\varepsilon(\theta_{s\omega})|^q + |\alpha \varepsilon(\theta_{s\omega})z(\theta_{s\omega})|^2 \right) ds + c \int_{-\infty}^{0} e^{\frac{5}{4} \lambda (t+s) - 2 \alpha} \int_{0}^{s} \eta(\theta_{t}\omega) dr \times \left( |\varepsilon(\theta_{\omega})|^p + |\varepsilon(\theta_{\omega})|^q + |\alpha \varepsilon(\theta_{\omega})z(\theta_{\omega})|^2 \right) ds.
\]

Therefore we get

\[
e^{-\frac{5}{4} \lambda t + 2 \alpha} \int_{-\infty}^{0} \eta(\theta_{t}\omega) dr R(\tau - t, \theta_{-t}\omega, \alpha, \epsilon)
\]

\[
= c \int_{-\infty}^{-t} e^{\frac{5}{4} \lambda s - 2 \alpha} \int_{0}^{s} \eta(\theta_{t}\omega) dr \times \left( |\varepsilon(\theta_{s\omega})|^p + |\varepsilon(\theta_{s\omega})|^q + |\alpha \varepsilon(\theta_{s\omega})z(\theta_{s\omega})|^2 \right) ds
\]

\[
+ c \int_{-\infty}^{-t} e^{\frac{5}{4} \lambda s - 2 \alpha} \int_{0}^{s} \eta(\theta_{s\omega}) dr \times \left( |\varepsilon(\theta_{s-t}\omega)|^p + |\varepsilon(\theta_{s-t}\omega)|^q + |\alpha \varepsilon(\theta_{s-t}\omega)z(\theta_{s-t}\omega)|^2 \right) ds
\]

\[
+ c \int_{-\infty}^{-t} e^{\frac{5}{4} \lambda (t+s) - 2 \alpha} \int_{0}^{s} \eta(\theta_{t}\omega) dr \times \left( |\varepsilon(\theta_{\omega})|^p + |\varepsilon(\theta_{\omega})|^q + |\alpha \varepsilon(\theta_{\omega})z(\theta_{\omega})|^2 \right) ds.
\]

Since the integrals in (2.8) are convergent, by (2.10) we obtain

\[
e^{-\frac{5}{4} \lambda t + 2 \alpha} \int_{-\infty}^{0} \eta(\theta_{t}\omega) dr R(\tau - t, \theta_{-t}\omega, \alpha, \epsilon) \to 0 \text{ as } t \to \infty.
\]

This completes the proof.

Next, we derive uniform estimates on the tails of solutions of (1.7)-(1.8) outside a bounded domain. These estimates are crucial for proving the asymptotic compactness of solutions on unbounded domains.

Lemma 3.2.3. Suppose (1.3)-(1.5) and (1.39) hold. Then for every \( \nu > 0, \alpha \leq \alpha_0, \epsilon > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( D \in D_{\alpha} \), there exists \( T = T(\tau, \omega, D, \alpha, \epsilon, \nu) > 0 \) and \( K = K(\tau, \omega, \alpha, \epsilon, \nu) \geq 1 \) such that for all \( t \geq T \) and \( \sigma \in [\tau - 1, \tau] \), the solution \( v \) of (1.7)-(1.8) satisfies

\[
\int_{|x| \geq K} |v(\sigma, \tau - t, \theta_{-t}\omega, v_{\tau-t})|^2 dx \leq \nu,
\]

where \( v_{\tau-t} \in D(\tau - t, \theta_{-t}\omega) \). In addition, \( T(\tau, \omega, D, \alpha, \epsilon, \nu) \) and \( K(\tau, \omega, D, \alpha, \epsilon, \nu) \) are uniform with respect to \( \epsilon \in (0,1) \).
Proof. Let $\rho$ be a smooth function defined on $\mathbb{R}^+$ such that $0 \leq \rho(s) \leq 1$ for all $s \in \mathbb{R}^+$, and
\[
\rho(s) = \begin{cases} 
0 & \text{for } 0 \leq s \leq 1; \\
1 & \text{for } s \geq 2.
\end{cases}
\]

Multiplying (1.7) by $\rho(\frac{|x|^2}{k^2})v$ and then integrating over $\mathbb{R}^n$ we get
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 \, dx
\]
\[
- \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) \text{div}(\nabla (v + \epsilon h\theta(t)\omega))|p^{-2}\nabla (v + \epsilon h\theta(t)\omega)) \, dx
\]
\[
= (\alpha\eta(\theta(t)) - \lambda) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 \, dx + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(t, x, v + \epsilon h\theta(t)\omega) \, dx
\]
\[\quad + \alpha\epsilon\eta(\theta(t))z(\theta(t)) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})h \omega \, dx + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g(t, x) \, dx. \tag{2.11}
\]

For the term involving the divergence we have
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2}) \text{div}(\nabla (v + \epsilon h\theta(t)\omega))|p^{-2}\nabla (v + \epsilon h\theta(t)\omega)) \, dx
\]
\[\quad = - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\nabla (v + \epsilon h\theta(t)\omega))|p \, dx
\]
\[\quad + \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\nabla (v + \epsilon h\theta(t)\omega))|p^{-2}\nabla (v + \epsilon h\theta(t)\omega)) \cdot \nabla (v + \epsilon h\theta(t)\omega) \, dx
\]
\[\quad - \int_{\mathbb{R}^n} \rho'(\frac{|x|^2}{k^2}) \frac{2x}{k^2} \cdot \nabla (v + \epsilon h\theta(t)\omega))|p^{-2}\nabla (v + \epsilon h\theta(t)\omega) \, dx
\]
\[\leq - \frac{1}{2} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\nabla (v + \epsilon h\theta(t)\omega))|p \, dx + c_1 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\nabla (v + \epsilon h\theta(t)\omega))|p \, dx
\]
\[\quad - \int_{|x| \leq 2k} \rho'(\frac{|x|^2}{k^2}) \frac{2x}{k^2} \cdot \nabla (v + \epsilon h\theta(t)\omega))|p^{-2}\nabla (v + \epsilon h\theta(t)\omega) \, dx
\]
\[\leq c_1 \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\nabla (v + \epsilon h\theta(t)\omega))|p \, dx + \frac{c_2}{k} (\|v\|_p^p + \|\nabla (v + \epsilon h\theta(t)\omega)\|_p^p). \tag{2.12}
\]

As in (1.15), for the nonlinearity $f$ we have
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(t, x, v + \epsilon h\theta(t)\omega) \, dx
\]
\[\leq \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(t, x, v + \epsilon h\theta(t)\omega) (v + \epsilon h\theta(t)\omega) \, dx
\]

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Since $K$ exists a $\gamma$ such that for all $\nu > 0$, there exists a $K_1 = K_1(\nu) \geq 1$ such that for all $k \geq K_1$,

\[
c_7 \int_{R^n} \rho(\frac{|x|^2}{k^2}) (|\nabla \text{ehz}(\theta_1 \omega)|^p + |\text{ehz}(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega) Z(\theta_1 \omega)| h^2) dx
\]

Note that

\[
\alpha \varepsilon \eta(\theta_1 \omega) Z(\theta_1 \omega) \int_{R^n} \rho(\frac{|x|^2}{k^2}) h v dx + \int_{R^n} \rho(\frac{|x|^2}{k^2}) g(t, x) v dx
\]

\[
\leq c_4 \int_{R^n} \rho(\frac{|x|^2}{k^2}) |\alpha \varepsilon \eta(\theta_1 \omega) z(\theta_1 \omega)| h^2 dx + c_5 \int_{R^n} \rho(\frac{|x|^2}{k^2}) |g(t, x)|^2 dx + \frac{3}{8} \lambda \int_{R^n} \rho(\frac{|x|^2}{k^2}) |v|^2 dx.
\]

It follows from (2.11)-(2.14) that

\[
\frac{d}{dt} \int R^n \rho(\frac{|x|^2}{k^2}) |v|^2 dx + \left( \frac{5}{4} \lambda - 2\alpha \varepsilon \eta(\theta_1 \omega) \right) \int R^n \rho(\frac{|x|^2}{k^2}) |v|^2 dx
\]

\[
\leq \frac{c_7}{k} (\|v\|_p^p + \|\nabla (v + \text{ehz}(\theta_1 \omega))\|_p^p)
\]

\[
+ c_7 \int_{R^n} \rho(\frac{|x|^2}{k^2}) (|g(t, x)|^2 + |\psi_1(t, x)| + |\psi_3(t, x)|^q_1) dx
\]

\[
+ c_7 \int_{R^n} \rho(\frac{|x|^2}{k^2}) (|\nabla \text{ehz}(\theta_1 \omega)|^p + |\text{ehz}(\theta_1 \omega)|^q + |\alpha \varepsilon \eta(\theta_1 \omega) Z(\theta_1 \omega)| h^2) dx.
\]
\[ \leq v \left( |\varepsilon z(\theta_t \omega)|^p + |\varepsilon z(\theta_t \omega)|^q + |\alpha \varepsilon \eta (\theta_t \omega) z(\theta_t \omega)|^2 \right). \]  \tag{2.16} 

By (2.15)-(2.16) we find that there exists \( K_2 = K_2(v) \geq K_1 \) such that for all \( k \geq K_2 \),

\[ \frac{d}{dt} \int \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 \, dx + \left( \frac{5}{4} \lambda - 2 \alpha \eta (\theta_t \omega) \right) \int \rho \left( \frac{|x|^2}{k^2} \right) |v|^2 \, dx \]

\[ \leq v\left( \|v\|_{L^p}^p + \|\nabla (v + \varepsilon h(z(\theta_t \omega)))\|_{L^p}^p \right) + c_7 \int_{|x| \geq k} (|g(t, x)|^2 + |\psi_1(t, x)| + |\psi_3(t, x)|^q) \, dx \]

\[ + v \left( |\varepsilon z(\theta_t \omega)|^p + |\varepsilon z(\theta_t \omega)|^q + |\alpha \varepsilon \eta (\theta_t \omega) z(\theta_t \omega)|^2 \right). \]  \tag{2.17} 

Multiplying (2.17) by \( e^{\frac{5}{4} \lambda t - 2 \alpha \int_0^t \eta(\theta_t \omega) \, dr} \), and integrating from \( \tau - t \) to \( \sigma \) with \( \sigma \geq \tau - t \), we get

\[ \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v(\sigma, \tau - t, \omega, v_{\tau - t})|^2 \, dx \]

\[ \leq e^{\frac{5}{4} \lambda (\tau - t - \sigma) - 2 \alpha \int_{\tau - t}^\sigma \eta(\theta_t \omega) \, dr} \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v_{\tau - t}|^2 \, dx \]

\[ + v \int_{\tau - t}^\sigma e^{\frac{5}{4} \lambda (s - \sigma) - 2 \alpha \int_{\tau - t}^s \eta(\theta_t \omega) \, dr} (\|v(s, \tau - t, \omega, v_{\tau - t})\|^p_{L^p} + \|\nabla (v + \varepsilon h(z(\theta_s \omega)))\|_{L^p}^p) \, ds \]

\[ + v \int_{\tau - t}^\sigma e^{\frac{5}{4} \lambda (s - \sigma) - 2 \alpha \int_{\tau - t}^s \eta(\theta_t \omega) \, dr} (|\varepsilon z(\theta_s \omega)|^p + |\varepsilon z(\theta_s \omega)|^q + |\alpha \varepsilon \eta (\theta_s \omega) z(\theta_s \omega)|^2) \, ds \]

\[ + c_7 \int_{\tau - t}^\sigma \int_{|x| \geq k} e^{\frac{5}{4} \lambda (s - \sigma) - 2 \alpha \int_{\tau - t}^s \eta(\theta_t \omega) \, dr} (|g(s, x)|^2 + |\psi_1(s, x)| + |\psi_3(s, x)|^q) \, dx \, ds. \]  \tag{2.18} 

Replacing \( \omega \) with \( \theta_{\tau - \omega} \) in (2.18), after simple calculations, we get for all \( k \geq K_2 \) and \( \sigma \in [\tau - 1, \tau] \),

\[ \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v(\sigma, \tau - t, \theta_{\tau - \omega}, v_{\tau - t})|^2 \, dx \leq e^{\frac{5}{4} \lambda (\tau - t - \sigma) + 2 \alpha \int_{\tau - t}^\sigma \eta(\theta_t \omega) \, dr} \|v_{\tau - t}\|^2 \]

\[ + v \int_{\tau - t}^\sigma e^{\frac{5}{4} \lambda (s - \sigma) - 2 \alpha \int_{\tau - t}^s \eta(\theta_t \omega) \, dr} \times (\|v(s, \tau - t, \theta_{\tau - \omega}, v_{\tau - t})\|^p_{L^p} + \|\nabla (v + \varepsilon h(z(\theta_s \omega)))\|_{L^p}^p) \, ds \]

\[ + v \int_{-\infty}^{\sigma - \tau} e^{\frac{5}{4} \lambda (s + \tau - \sigma) - 2 \alpha \int_{-\infty}^{s - \tau} \eta(\theta_t \omega) \, dr} \times (|\varepsilon z(\theta_s \omega)|^p + |\varepsilon z(\theta_s \omega)|^q + |\alpha \varepsilon \eta (\theta_s \omega) z(\theta_s \omega)|^2) \, ds \]

\[ + c_7 \int_{-\infty}^{\sigma - \tau} \int_{|x| \geq k} e^{\frac{5}{4} \lambda (s + \tau - \sigma) - 2 \alpha \int_{s - \tau}^s \eta(\theta_t \omega) \, dr} \times (|g(s, x)|^2 + |\psi_1(s, x)| + |\psi_3(s, x)|^q) \, dx \, ds. \]
\[\times \left( |g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^{q_1} \right) dx ds \leq e^{\frac{5}{4} + 2\alpha} f_{-1}^0 |\eta(\theta, \omega)| \, d\tau \, e^{-\frac{5}{4} \lambda + 2\alpha} f_{-t}^0 |\eta(\theta, \omega)| \, d\tau \left\| v_{\tau-t} \right\|^2 \]
\[+ \int_{-t}^T e^{\frac{5}{4} \lambda + 2\alpha} f_{-1}^0 |\eta(\theta, \omega)| \, d\tau \int_{-t}^T e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, ds \times (|v|^p + \| \nabla (v + \varepsilon h z(\theta_s - \tau \omega)) \|_p) \, ds \]
\[+ c_8 \int_{-\infty}^0 \int_{|x| \geq k} e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, ds \times (|g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^{q_1}) dx ds. \quad (2.19)\]
Since \( v_{\tau-t} \in D(\tau - t, \theta_s - \tau \omega) \), we see that for every \( v > 0, \tau \in \mathbb{R}, \omega \in \Omega \) and \( \alpha > 0 \), there exists a \( T_1(\tau, \omega, D, \alpha, v) > 0 \) such that for every \( t \geq T_1 \) and \( \sigma \in [\tau - 1, \tau] \),
\[e^{\frac{5}{4} \lambda + 2\alpha} f_{-1}^0 |\eta(\theta, \omega)| \, d\tau \, e^{-\frac{5}{4} \lambda t + 2\alpha} f_{-t}^0 |\eta(\theta, \omega)| \, d\tau \left\| v_{\tau-t} \right\|^2 \leq \frac{1}{v} \left\| D(\tau - t, \theta_s - \tau \omega) \right\|^2 \leq v. \quad (2.20)\]
Since \( \int_{-\infty}^0 \int_{\mathbb{R}^n} e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, ds \times (|g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^{q_1}) dx ds \) is convergent, we have
\[\int_{-\infty}^0 \int_{|x| \geq k} e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, ds \times (|g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^{q_1}) dx ds \to 0, \]
as \( k \to \infty \). Therefore, there exists \( K = K_3(\tau, \omega, \alpha, v) \geq K_2 \) such that for all \( k \geq K_3 \),
\[c_8 \int_{-\infty}^0 \int_{|x| \geq k} e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, ds \times (|g(s + \tau, x)|^2 + |\psi_1(s + \tau, x)| + |\psi_3(s + \tau, x)|^{q_1}) dx ds \leq v. \quad (2.21)\]
Note that
\[\left\| v(s, \tau - t, \theta_s - \tau \omega, v_{\tau-t}) \right\|_p^p \leq 2^p \left( \left\| v(s, \tau - t, \theta_s - \tau \omega, v_{\tau-t}) + \varepsilon h z(\theta_s - \tau \omega) \right\|_p^p + \left\| \varepsilon h z(\theta_s - \tau \omega) \right\|_p^p \right), \]
which along with (0.1) and Lemma 3.2.2 shows that there exists \( T_2 = T_2(\tau, \omega, D, \alpha, v) \geq T_1 \) such that for all \( t \geq T_2 \),
\[\int_{\tau-t}^T e^{\frac{5}{4} \lambda s - 2\alpha} f_{0}^s \eta(\theta, \omega) \, dw \left( \left\| v(s, \tau - t, \theta_s - \tau \omega, v_{\tau-t}) \right\|_p^p + \left\| \nabla (v + \varepsilon h z(\theta_s - \tau \omega)) \right\|_p^p \right) \, ds \]
that is, as $\subsequence$ (not relabeled) such that as

\begin{equation}
\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta + \tau \omega, v_{\tau - t})|^2 \, dx \leq 2v + \nu c_{11} R(\tau, \omega, \alpha, \epsilon) + \nu c_{12} \int_{-\infty}^0 e^{\frac{5}{2} \lambda s - 2\alpha} \int_0^s \eta(\theta, \omega) \, d\theta |\varepsilon_x(\theta, \omega)|^p \, ds,
\end{equation}

where $R(\tau, \omega, \alpha, \epsilon)$ is the number given by (2.8). It follows from (2.19)-(2.22) that for all $k \geq K_3$, $t \geq T_2$ and $\sigma \in [\tau - 1, \tau]$, we obtain

$$\int_{\mathbb{R}^n} \rho\left(\frac{|x|^2}{k^2}\right) |v(\sigma, \tau - t, \theta + \tau \omega, v_{\tau - t})|^2 \, dx \leq 2v + \nu c_{11} R(\tau, \omega, \alpha, \epsilon) + \nu c_{12} \int_{-\infty}^0 e^{\frac{5}{2} \lambda s - 2\alpha} \int_0^s \eta(\theta, \omega) \, d\theta |\varepsilon_x(\theta, \omega)|^p \, ds.$$

Note that $\rho\left(\frac{|x|^2}{k^2}\right) = 1$ when $|x|^2 \geq 2k^2$. This along with (2.23) concludes the proof. \hfill \Box

The asymptotic compactness of solutions of equation (1.7) is given below.

**Lemma 3.2.4.** Suppose (1.3)-(1.5) and (1.39) hold. Then for every $\alpha \leq \alpha_0$, $\epsilon > 0$, $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_\alpha$, the sequence $v(\tau, \tau - t_n, \theta + \tau \omega, v_{\tau - t_n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$ provided $t_n \to \infty$ and $v_{0,n} \in D(\tau - t_n, \theta, \tau \omega)$.

**Proof.** By Lemma 3.2.1 we find that there exists $N_1 = N_1(\tau, \omega, D, \alpha) > 0$ such that for all $n \geq N_1$,

$$\|v(\tau - 1, \tau - t_n, \theta + \tau \omega, v_{0,n})\| \leq c_1.$$  \hfill (2.24)

Applying Lemma 3.1.4 to the sequence $v(\tau, \tau - 1, \theta + \tau \omega, v(\tau - 1, \tau - t_n, \theta + \tau \omega, v_{0,n}))$, we find that there exist $s_0 \in (\tau - 1, \tau)$, $v_0 \in L^2(\mathbb{R}^n)$ and a subsequence (not relabeled) such that as $n \to \infty$,

$$v(s_0, \tau - 1, \theta + \tau \omega, v(\tau - 1, \tau - t_n, \theta + \tau \omega, v_{0,n})) \to v_0 \quad \text{in} \ L^2(\mathcal{O}_k) \text{ for every } k \in \mathbb{N},$$

that is, as $n \to \infty$,

$$v(s_0, \tau - t_n, \theta + \tau \omega, v_{0,n}) \to v_0 \quad \text{in} \ L^2(\mathcal{O}_k) \text{ for every } k \in \mathbb{N}. \quad (2.25)$$

By (1.33) we get

$$\|v(\tau, s_0, \theta + \tau \omega, v(s_0, \tau - t_n, \theta + \tau \omega, v_{0,n})) - v(\tau, s_0, \theta + \tau \omega, v_0)\|
\leq e^{c_1(\tau - s_0)} \|v(s_0, \tau - t_n, \theta + \tau \omega, v_{0,n}) - v_0\|.$$

Since $s_0 \in (\tau - 1, \tau)$, we obtain

$$\|v(\tau, s_0, \theta + \tau \omega, v(s_0, \tau - t_n, \theta + \tau \omega, v_{0,n})) - v(\tau, s_0, \theta + \tau \omega, v_0)\|^2
\leq e^{2c_1} \int_{|x| < k} |v(s_0, \tau - t_n, \theta + \tau \omega, v_{0,n}) - v_0|^2 \, dx.$$
\[ +e^{2c_1} \int_{|x| \geq k} |v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n}) - v_0|^2 \, dx \]
\[ \leq e^{2c_1} \int_{|x| < k} |v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n}) - v_0|^2 \, dx \]
\[ +2e^{2c_1} \int_{|x| \geq k} \left( |v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n})|^2 + |v_0|^2 \right) \, dx. \quad (2.26) \]

Since \( v_0 \in L^2(\mathbb{R}^n) \), given \( \nu > 0 \), there exists \( K_1 = K_1(\nu) \geq 1 \) such that for all \( k \geq K_1 \),
\[ 2e^{2c_1} \int_{|x| \geq k} |v_0|^2 \, ds \leq \nu. \quad (2.27) \]

On the other hand, by Lemma 3.2.3, there exist \( N_2 = N_2(\tau, \omega, D, \alpha, \varepsilon, \nu) \geq 1 \) and \( K_2 = K_2(\tau, \omega, \alpha, \varepsilon, \nu) \geq K_1 \) such that for all \( n \geq N_2 \) and \( k \geq K_2 \),
\[ 2e^{2c_1} \int_{|x| \geq k} |v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n})|^2 \, dx \leq \nu. \quad (2.28) \]

By (2.25) we find that there exists \( N_3 = N_3(\tau, \omega, D, \alpha, \varepsilon, \nu) \geq N_2 \) such that for all \( n \geq N_3 \),
\[ e^{2c_1} \int_{|x| < K_2} |v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n}) - v_0|^2 \, dx \leq \nu. \quad (2.29) \]

It follows from (2.26)-(2.29) that for all \( n \geq N_3 \),
\[ \|v(\tau, s_0, \theta - \tau \omega, v(s_0, \tau - t_n, \theta - \tau \omega, v_{0,n})) - v(\tau, s_0, \theta - \tau \omega, v_0)\|^2 \leq 3\nu, \]
that is, for all \( n \geq N_3 \),
\[ \|v(\tau - t_n, \theta - \tau \omega, v_{0,n}) - v(\tau, s_0, \theta - \tau \omega, v_0)\|^2 \leq 3\nu. \]

Therefore, \( v(\tau - t_n, \theta - \tau \omega, v_{0,n}) \) converges to \( v(\tau, s_0, \theta - \tau \omega, v_0) \) in \( L^2(\mathbb{R}^n) \). This completes the proof. \( \square \)

### 3.3 Random Attractors

In this section, we prove the existence of \( D_\alpha \)-pullback attractor for (1.1)-(1.2) in \( L^2(\mathbb{R}^n) \) by Proposition 2.0.6. To this end, we need to establish the existence of \( D_\alpha \)-pullback absorbing sets and the \( D_\alpha \)-pullback asymptotic compactness of \( \Phi \) in \( L^2(\mathbb{R}^n) \). The existence of absorbing sets of \( \Phi \) is given below.

**Lemma 3.3.1.** Suppose (1.3)-(1.5) and (1.39) hold. Then for every \( \alpha \leq \alpha_0 \) and \( \varepsilon > 0 \), the stochastic equation (1.1) with (1.2) has a closed measurable \( D_\alpha \)-pullback absorbing set \( K = \{ K(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in D_\alpha \) which is given by
\[ K(\tau, \omega) = \{ u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq 2\|\epsilon h(z(\omega))\|^2 + 2R(\tau, \omega, \alpha, \varepsilon) \}, \quad (3.1) \]
where \( R(\tau, \omega, \alpha, \varepsilon) \) is the number given by (2.8).
Proof. Let $D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_\alpha$. For every $\tau \in \mathbb{R}$ and $\omega \in \Omega$, denote by

$$\tilde{D}(\tau, \omega) = \{v \in L^2(\mathbb{R}^n) : v = u - \varepsilon h z(\omega) \text{ for some } u \in D(\tau, \omega)\}. \quad (3.2)$$

Since $z$ is tempered, we find that the family $\tilde{D} = \{\tilde{D}(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega\}$ belongs to $D_\alpha$ provided $D \in D_\alpha$. By (1.6) we have

$$u(\tau, \tau - t, \theta_{-t} \omega, u_{\tau - t}) = v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau - t}) + \varepsilon h z(\omega) \quad \text{with } v_{\tau - t} = u_{\tau - t} - \varepsilon h z(\theta_t \omega). \quad (3.3)$$

Thus, if $u_{\tau - t} \in D(\tau - t, \theta_{-t} \omega) \in D_\alpha$, then $v_{\tau - t} \in \tilde{D}(\tau - t, \theta_{-t} \omega) \in D_\alpha$. By Lemma 3.2.2 we find that there exists $T = T(\tau, \omega, D, \alpha, \varepsilon) > 0$ such that for all $t \geq T$,

$$\|v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau - t})\|^2 \leq R(\tau, \omega, \alpha, \varepsilon),$$

where $R(\tau, \omega, \alpha, \varepsilon)$ is as in (2.8). By (3.3) we get for all $t \geq T$,

$$\|u(\tau, \tau - t, \theta_{-t} \omega, u_{\tau - t})\|^2 \leq 2 \|\varepsilon h z(\omega)\|^2 + 2R(\tau, \omega, \alpha, \varepsilon).$$

This along with (1.36) and (3.1) shows that for all $t \geq T$,

$$\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K(\tau, \omega). \quad (3.4)$$

On the other hand, by (2.9) and the temperedness of $z$ we obtain

$$\lim_{t \to \infty} e^{-\frac{5}{2} \lambda t + 2 \lambda} \int_0^t \eta(\theta_r \omega) dr \|K(\tau - t, \theta_{-t} \omega)\| = 0. \quad (3.5)$$

By (3.4)-(3.5) we find that $K$ given by (3.1) is a closed $D_\alpha$-pullback absorbing set of $\Phi$ in $D_\alpha$. Note that the measurability of $K(\tau, \omega)$ in $\omega \in \Omega$ follows from that of $z(\omega)$ and $R(\tau, \omega, \alpha, \varepsilon)$ immediately. This completes the proof. \hfill \Box

The following is our main result regarding the existence of $D_\alpha$-pullback attractors of $\Phi$.

**Theorem 3.3.2.** Suppose (1.3)-(1.5) and (1.39) hold. Then for every $\alpha \leq \alpha_0$ and $\varepsilon > 0$, the stochastic equation (1.1) with (1.2) has a unique $D_\alpha$-pullback attractor $A_\alpha = \{A_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in D_\alpha$ in $L^2(\mathbb{R}^n)$. In addition, if there is $T > 0$ such that $f(t, x, s), g(t, x), \psi_1(t, x)$ and $\psi_3(t, x)$ are all $T$-periodic in $t$ for fixed $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$, then the attractor $A_\alpha$ is also $T$-periodic.

**Proof.** We first prove that $\Phi$ is $D_\alpha$-pullback asymptotically compact in $L^2(\mathbb{R}^n)$; that is, for every $\tau \in \mathbb{R}, \omega \in \Omega$, $D \in D_\alpha$, $t_n \to \infty$ and $u_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega)$, we want to show that the sequence $\Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0,n})$ has a convergent subsequence in $L^2(\mathbb{R}^n)$. Let $v_{0,n} = u_{0,n} - \varepsilon h z(\theta_{-t_n} \omega)$ and $D$ be the family given by (3.2). Since $u_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega)$, we find that $v_{0,n} \in \tilde{D}(\tau - t_n, \theta_{-t_n} \omega) \in D_\alpha$. Since $u_{0,n} \in D(\tau - t_n, \theta_{-t_n} \omega)$, we find that $v_{0,n} \in \tilde{D}(\tau - t_n, \theta_{-t_n} \omega) \in D_\alpha$. 26
Therefore, by (3.3) and Lemma 3.2.4 we find that \( u(\tau, \tau - t_n, \theta_{-\tau} \omega, u_{0, n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \). This together with (1.36) indicates that \( \Phi(t_n, \tau - t_n, \theta_{-t_n} \omega, u_{0, n}) \) has a convergent subsequence, and thus it is \( \mathcal{D}_\alpha \)-pullback asymptotically compact in \( L^2(\mathbb{R}^n) \). Since \( \Phi \) also has a closed measurable \( \mathcal{D}_\alpha \)-pullback absorbing set \( K \) given by (3.1), by Proposition 2.0.6 we get the existence and uniqueness of \( \mathcal{D}_\alpha \)-pullback attractor \( A_\alpha \in \mathcal{D}_\alpha \) of \( \Phi \) immediately.

Next, we discuss \( T \)-periodicity of \( A_\alpha \). Note that if \( f \) and \( g \) are \( T \)-periodic in their first arguments, then the cocycle \( \Phi \) is also \( T \)-periodic. Indeed, in this case, for every \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), by (1.36) we have

\[
\Phi(t, \tau + T, \omega, \cdot) = u(t + \tau + T, \tau + T, \theta_{-\tau - T} \omega, \cdot)
= u(t + \tau, \tau, \theta_{-\tau} \omega, \cdot) = \Phi(t, \tau, \omega, \cdot). \tag{3.6}
\]

In addition, if \( g(t, x), \psi_1(t, x) \) and \( \psi_3(t, x) \) are all \( T \)-periodic in \( t \), then by (2.8) and (3.1) we get \( K(\tau + T, \omega) = K(\tau, \omega) \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \). This along with (3.6) and Proposition 2.0.6 yields the \( T \)-periodicity of \( A_\alpha \). \( \square \)
CHAPTER 4

MULTIPlicative Noise

Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, consider the following stochastic equation defined for $x \in \mathbb{R}^n$ and $t > \tau$,

$$\frac{\partial u}{\partial t} + \lambda u - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f(t, x, u) + g(t, x) + \alpha u \circ dW$$  \hspace{1cm} (0.1)

with initial condition

$$u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (0.2)

where $p \geq 2$, $\alpha > 0$, $\lambda > 0$, $g \in L^2_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, and $W$ is a two-sided real-valued Wiener process on $(\Omega, \mathcal{F}, P)$. We assume the nonlinearity $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies, for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,

$$f(t, x, s) \leq -\gamma |s|^q + \psi_1(t, x),$$  \hspace{1cm} (0.3)

$$|f(t, x, s)| \leq \psi_2(t, x)|s|^{q-1} + \psi_3(t, x),$$  \hspace{1cm} (0.4)

$$\frac{\partial f}{\partial s}(t, x, s) \leq \psi_4(t, x),$$  \hspace{1cm} (0.5)

where $\gamma > 0$ and $q \geq p$ are constants, $\psi_1 \in L^1_{\text{loc}}(\mathbb{R}, L^1(\mathbb{R}^n))$, $\psi_2, \psi_4 \in L^\infty_{\text{loc}}(\mathbb{R}, L^\infty(\mathbb{R}^n))$, and $\psi_3 \in L^{q_1}_{\text{loc}}(\mathbb{R}, L^{q_1}(\mathbb{R}^n))$. From now on, we always use $p_1$ and $q_1$ to denote the conjugate exponents of $p$ and $q$, respectively.

To define a random dynamical system for (0.1), we need to transfer the stochastic equation to a pathwise deterministic system. As usual, let $z$ be the random variable given by:

$$z(\omega) = -\int_{-\infty}^{0} e^{\tau} \omega(\tau) d\tau, \quad \omega \in \Omega$$

Then $z$ solves the following stochastic equation:

$$\frac{d}{dt} z(\theta_t \omega) + z(\theta_t \omega) = \frac{dW}{dt}$$  \hspace{1cm} (0.6)

It follows from Arnold [1998] that there exists a $\theta_t$-invariant set $\tilde{\Omega}$ of full measure such that $z(\theta_t \omega)$ is continuous in $t$ and

$$\lim_{t \to \pm \infty} \frac{z(\theta_t \omega)}{t} = 0 \quad \text{and} \quad \lim_{t \to \pm \infty} \frac{1}{t} \int_0^t z(\theta_r \omega) dr = 0$$  \hspace{1cm} (0.7)
for all $\omega \in \tilde{\Omega}$. For convenience, we will denote $\tilde{\Omega}$ by $\Omega$ in the sequel. Let $u(t, \tau, \omega, u_\tau)$ be a solution of problem (0.1)-(0.2) with initial condition $u_\tau$ at initial time $\tau$, and define

$$v(t, \tau, \omega, v_\tau) = e^{-\alpha z(\theta t \omega)} u(t, \tau, \omega, u_\tau) \quad \text{with} \quad v_\tau = e^{-\alpha z(\theta t \omega)} u_\tau. \quad (0.8)$$

By (0.1) and (0.8), after simple calculations, we get

$$\frac{\partial v}{\partial t} - e^{\alpha(p-2)z(\theta t \omega)} \text{div} \left( |\nabla v|^{p-2} \nabla v \right) + \lambda v = \alpha z(\theta t \omega) v + e^{-\alpha z(\theta t \omega)} f(t, x, e^{\alpha z(\theta t \omega)} v) + e^{-\alpha z(\theta t \omega)} g(t, x), \quad (0.9)$$

with initial condition

$$v(\tau, x) = v_\tau(x), \quad x \in \mathbb{R}^n. \quad (0.10)$$

The well-posedness of the equations is investigated in Lewis et al. [2014], from which we also have the following result which is necessary to prove asymptotic compactness of the cocycle.

**Lemma 4.0.3.** Let (0.3)-(0.5) hold and $\{v_n\}_{n=1}^\infty$ be a bounded sequence in $L^2(\mathbb{R}^n)$. Then for every $\tau \in \mathbb{R}$, $t > \tau$ and $\omega \in \Omega$, there exist $v_0 \in L^2(\tau, t; L^2(\mathbb{R}^n))$ and a subsequence $\{v(n, \tau, \omega, v_{n,m})\}_{m=1}^\infty$ of $\{v(n, \tau, \omega, v_n)\}_{n=1}^\infty$ such that $v(s, \tau, \omega, v_{n,m}) \to v_0(s)$ in $L^2(\mathcal{O}_\tau)$ as $m \to \infty$ for every fixed $k \in \mathbb{N}$ and for almost all $s \in (\tau, t)$.

We must also specify the attraction domain for the multiplicative case. Here, we consider a family $D = \{D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\}$ of bounded nonempty sets such that for every $\tau \in \mathbb{R}$ and $\omega \in \Omega$,

$$\lim_{s \to -\infty} e^{\frac{3}{4} \lambda s - 2 \alpha \int_0^s z(\theta t \omega) dt} - 2 \alpha z(\theta t \omega) \|D(\tau + s, \theta t \omega)\|^2 = 0, \quad (0.11)$$

where $\|S\| = \sup_{u \in S}\|u\|_{L^2(\mathbb{R}^n)}$ for a nonempty bounded subset $S$ of $L^2(\mathbb{R}^n)$. In the sequel, we will use $\mathcal{D}_\alpha$ to denote the collection of all families with property (0.11):

$$\mathcal{D}_\alpha = \{D = \{D(\tau, \omega) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (0.11)}\}. \quad (0.12)$$

For our purpose, we further assume the following condition on $g, \psi_1$ and $\psi_3$: for every $\tau \in \mathbb{R}$,

$$\int_{-\infty}^{\tau} e^{\lambda s} \left( \|g(s, \cdot)\|^2 + \|\psi_1(s, \cdot)\|_{L^1(\mathbb{R}^n)} + \|\psi_3(s, \cdot)\|_{L^1(\mathbb{R}^n)}^{\frac{q_1}{2}} \right) ds < \infty. \quad (0.13)$$
4.1 Existence of Random Attractors

In this section, we prove the existence and uniqueness of $\mathcal{D}_\alpha$-pullback attractors for the stochastic equation (0.1). We also show the periodicity of the attractor when external terms are periodic in time. We start with the uniform estimates of solutions in $L^2(\mathbb{R}^n)$.

Lemma 4.1.1. If (0.3)-(0.5) and (0.13) hold, then for every $\alpha > 0$, $\tau \in \mathbb{R}$, $\sigma \in [\tau - 1, \tau]$, $\omega \in \Omega$, and $D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D}_\alpha$, there is $T = T(\tau, \omega, D, \alpha) > 0$ such that for all $t \geq T$, the solution $v$ of problem (0.9)-(0.10) satisfies

$$\|v(\sigma, \tau - t, \theta_{-\tau}\omega, v_{\tau_{-t}})\|^2 \leq M,$$

$$\int_{\tau_{-t}}^{\sigma} e^{\frac{5}{2}\lambda(s-\sigma)} - 2\alpha \int_{\tau_{-t}}^{s} z(\theta_{s-\tau}\omega) e^{-2\alpha z(\theta_{s-\tau}\omega)} \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau_{-t}})\|_{W^{1,p}}^p ds \leq M,$$

$$\int_{\tau_{-t}}^{\sigma} e^{\frac{5}{2}\lambda(s-\sigma)} - 2\alpha \int_{\tau_{-t}}^{s} z(\theta_{s-\tau}\omega) e^{-2\alpha z(\theta_{s-\tau}\omega)} \times \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau_{-t}})\|^2 + \|u(s, \tau - t, \theta_{-\tau}\omega, u_{\tau_{-t}})\|_{q}^q ds \leq M,$$

where $e^{-\alpha z(\theta_{-\tau}\omega)} v_{\tau_{-t}} \in D(\tau - t, \theta_{-\tau}\omega)$ and $M$ is given by

$$M = c + c \int_{-\infty}^{\sigma_{-t}} e^{\frac{5}{2}\lambda(s-\sigma)} - 2\alpha \int_{\tau_{-t}}^{s} z(\theta_{s-\tau}\omega) e^{-2\alpha z(\theta_{s-\tau}\omega)}$$

$$\times \left(\|\psi_1(s + \tau, \cdot)\|_1 + \|g(s + \tau, \cdot)\|^2\right) ds$$

for some positive constant $c$ depending only on $\lambda$, $q$, and $\gamma$.

Proof. Multiplying (0.9) through by $v$ and integrating over $\mathbb{R}^n$ we obtain

$$\frac{d}{dt} \|v\|^2 + 2e^{\alpha(p-2)z(\theta_{i}\omega)} \|\nabla v\|_p^p + \left(\frac{7}{4}\lambda - 2\alpha z(\theta_{i}\omega)\right) \|v\|^2 + 2\gamma e^{\alpha(q-2)z(\theta_{i}\omega)} \|v\|_{q}^q$$

$$\leq 2e^{-2\alpha z(\theta_{i}\omega)} \|\psi_1(t, \cdot)\|_1 + \frac{4}{\lambda} e^{-2\alpha z(\theta_{i}\omega)} \|g(t, \cdot)\|^2,$$

which along with (0.8) yields

$$\frac{d}{dt} \|v\|^2 + 2e^{-2\alpha z(\theta_{i}\omega)} \left(\frac{1}{2}\lambda \|u\|^2 + 2\gamma \|u\|_{q}^q + 2\|\nabla u\|_p^p\right) + \left(\frac{5}{4}\lambda - 2\alpha z(\theta_{i}\omega)\right) \|v\|^2$$

$$\leq e^{-2\alpha z(\theta_{i}\omega)} \left(2\|\psi_1(t, \cdot)\|_1 + \frac{4}{\lambda} \|g(t, \cdot)\|^2\right).$$
Multiply the above by $e^{\frac{5}{4}\lambda(t-s)}\int_0^s z(\theta, \omega) \, d\theta$ and integrate over $[\tau - t, \sigma]$, then replace $\omega$ by $\theta_{\tau-t}\omega$ in order to obtain

$$\|v(\sigma, \tau - t, \theta_{\tau-t}\omega, v_{\tau-t})\|^2$$

$$+ c_1 \int_{\tau-t}^\sigma e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha \int_0^s z(\theta_{\tau-t}\omega) \, d\theta_{\tau-t}\omega} \left( \|u\|^2 + \|u\|^q + \|\nabla u\|^p \right) \, ds$$

$$\leq e^{\frac{5}{4}\lambda(\tau-\sigma-t)-2\alpha \int_{\tau-t}^{\sigma-t} z(\theta, \omega) \, d\theta} e^{-2\alpha z(\theta_{\tau-t}\omega)} \|D(\tau - t, \theta_{\tau-t}\omega)\|^2$$

$$+ \int_{\tau-t}^\sigma e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha \int_0^s z(\theta_{\tau-t}\omega) \, d\theta_{\tau-t}\omega} \left( 2\|\psi_1(s, \cdot)\|_1 + \frac{4}{\lambda} \|g(s, \cdot)\|^2 \right) \, ds,$$

where $c_1 = \min\{\frac{1}{2}\lambda, 2\gamma, 2\}$. Since $e^{az(\theta_{\tau-t}\omega)}v_{\tau-t} \in D(\tau - t, \theta_{\tau-t}\omega)$ we further get

$$\|v(\sigma, \tau - t, \theta_{\tau-t}\omega, v_{\tau-t})\|^2$$

$$+ c_1 \int_{\tau-t}^\sigma e^{\frac{5}{4}\lambda(s-\sigma)-2\alpha \int_0^s z(\theta_{\tau-t}\omega) \, d\theta_{\tau-t}\omega} \left( \|u\|^2 + \|u\|^q + \|\nabla u\|^p \right) \, ds$$

$$\leq e^{\frac{5}{4}\lambda(\tau-\sigma-t)-2\alpha \int_{\tau-t}^{\sigma-t} z(\theta, \omega) \, d\theta} e^{-2\alpha z(\theta_{\tau-t}\omega)} \|D(\tau - t, \theta_{\tau-t}\omega)\|^2$$

$$+ c_2 \int_{-\infty}^{\sigma-t} e^{\frac{5}{4}\lambda(s-\sigma-t)+2\alpha \int_0^s z(\theta, \omega) \, d\theta} e^{-2\alpha z(\theta_{\tau-t}\omega)} \left( \|\psi_1(s + \tau, \cdot)\|_1 + \|g(s + \tau, \cdot)\|^2 \right) \, ds.$$

Note that the last integral in (1.3) exists because of (0.7) and (0.13). Since $D \in D_\alpha$ and $\sigma \in [\tau - 1, \tau]$, it follows from (1.37) that there exists $T = T(\tau, \omega, D, \alpha) > 0$ such that for all $t \geq T$,

$$e^{\frac{5}{4}\lambda(\tau-\sigma-t)-2\alpha \int_{\tau-t}^{\sigma-t} z(\theta, \omega) \, d\theta} e^{-2\alpha z(\theta_{\tau-t}\omega)} \|D(\tau - t, \theta_{\tau-t}\omega)\|^2 \leq 1.$$

which along with (1.3) and (0.1) completes the proof. □

As a special case of Lemma 4.1.1 we obtain the existence of $D_\alpha$-pullback absorbing sets for $\Phi$.

**Lemma 4.1.2.** If (0.3)-(0.5) and (0.13) hold, then for every $\alpha > 0$, the stochastic equation (0.1) with (0.2) has a closed measurable $D_\alpha$-pullback absorbing set $K_\alpha = \{K_\alpha(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \subset D_\alpha$ which is given by

$$K_\alpha(\tau, \omega) = \{u \in L^2(\mathbb{R}^n) : \|u\|^2 \leq e^{2az(\omega)} R(\alpha, \tau, \omega) \}, \quad (1.4)$$

where $R(\alpha, \tau, \omega)$ is given by

$$R(\alpha, \tau, \omega) = c + c \int_{-\infty}^0 e^{\frac{5}{4}\lambda s-2\alpha \int_0^s z(\theta, \omega) \, d\theta} e^{-2\alpha z(\theta, \omega)}$$

$$\times \left( \|\psi_1(s + \tau, \cdot)\|_1 + \|g(s + \tau, \cdot)\|^2 \right) \, ds \quad (1.5)$$

for some positive constant $c$ depending only on $\lambda$, $q$, and $\gamma$. 

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Proof. It follows from Lemma 4.1.1 with \( \sigma = \tau \) that there is \( T = T(\tau, \omega, D, \alpha) > 0 \) such that for all \( t \geq T \), the solution \( v \) of (0.9)-(0.10) satisfies
\[
\|v(\tau, \tau - t, \theta_{-t} \omega, v_{\tau-t})\|^2 \leq R(\alpha, \tau, \omega)
\]
for all \( e^{az(\theta-t)\omega}v_{\tau-t} \in D(\tau, \theta_{-t} \omega) \), which along with (0.8) indicates that for all \( t \geq T \) and \( u_{\tau-t} \in D(\tau, \theta_{-t} \omega) \),
\[
\|u(\tau, \tau - t, \theta_{-t} \omega, u_{\tau-t})\|^2 \leq e^{2az(\omega)}R(\alpha, \tau, \omega).
\]
Therefore, by (1.36) and (1.4) we get, for all \( t \geq T \),
\[
\Phi(t, \tau - t, \theta_{-t} \omega, D(\tau - t, \theta_{-t} \omega)) \subseteq K_\alpha(\tau, \omega). \tag{1.6}
\]
By (1.5) we have
\[
R(\alpha, \tau - t, \theta_{-t} \omega) = c + c \int_{-\infty}^{0} e^{\frac{5}{4}t - 2\alpha t s - 2\alpha t s^2} \left( \|\psi_1(s + \tau - t, \cdot)\|_1 + \|g(s + \tau - t, \cdot)\|_2 \right) ds
\]
\[
= c + c \int_{-\infty}^{-t} e^{\frac{5}{4}t(s+t) - 2\alpha t s} e^{-2\alpha t z(\theta, \omega)} \left( \|\psi_1(s + \tau, \cdot)\|_1 + \|g(s + \tau, \cdot)\|_2 \right) ds.
\]
This implies
\[
e^{-\frac{5}{4}t - 2\alpha t s} \int_{0}^{t} z(\theta, \omega) ds \|K_\alpha(\tau - t, \theta_{-t} \omega)\|^2
\]
\[
= e^{-\frac{5}{4}t - 2\alpha t s} \int_{0}^{t} z(\theta, \omega) ds R(\alpha, \tau - t, \theta_{-t} \omega) = c e^{-\frac{5}{4}t - 2\alpha t s} \int_{0}^{t} z(\theta, \omega) ds
\]
\[
+ c \int_{-\infty}^{-t} e^{\frac{5}{4}t s - 2\alpha t s} \left( \|\psi_1(s + \tau, \cdot)\|_1 + \|g(s + \tau, \cdot)\|_2 \right) ds\]
which along with (0.7) and the convergence of the integrals in (1.5) yields
\[
\lim_{t \to \infty} e^{-\frac{5}{4}t - 2\alpha t s} \int_{0}^{t} z(\theta, \omega) ds \|K_\alpha(\tau - t, \theta_{-t} \omega)\|^2 = 0. \tag{1.7}
\]
By (1.6)-(1.7) we find that \( K_\alpha \in D_\alpha \) is a \( D_\alpha \)-pullback absorbing set of \( \Phi \). \( \Box \)

We will need the following uniform estimates on the tails of solutions to (0.9) in order to establish the asymptotic compactness of solutions in \( L^2(\mathbb{R}^n) \).

Lemma 4.1.3. If (0.3)-(0.5) and (0.13) hold, then for every \( \eta > 0, \tau \in \mathbb{R}, \sigma \in [\tau - 1, \tau] \), \( \omega \in \Omega, \alpha > 0 \) and \( D \in D_\alpha \), there exists \( \bar{T} = T(\tau, \omega, D, \alpha, \eta) > 0 \) and \( \mathcal{K} = K(\tau, \omega, \alpha, \eta) \geq 1 \) such that for all \( t \geq \bar{T} \),
\[
\int_{|x| \geq \mathcal{K}} |v(\sigma, \tau - t, \theta_{-t} \omega, v_{\tau-t})|^2 \, dx \leq \eta
\]
where \( e^{az(\theta-t)\omega}v_{\tau-t} \in D(\tau - t, \theta_{-t} \omega) \).
Proof. Let \( \rho : \mathbb{R}^+ \to [0, 1] \) be a smooth function which satisfies
\[
\rho(s) = \begin{cases} 
0 & 0 \leq s \leq 1, \\
1 & s \geq 2.
\end{cases}
\]
Multiplying (0.9) through by \( \rho(\frac{|x|^2}{k^2})v \) and integrating over \( x \in \mathbb{R}^n \) we obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v^2 dx - e^{a(p-2)z(\theta_1, \omega)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v \text{ div}(|\nabla v|^{p-2}\nabla v) dx \\
+ (\lambda - az(\theta_1, \omega)) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v^2 dx \\
= e^{-2az(\theta_1, \omega)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})f(t, x, u)udx + e^{-az(\theta_1, \omega)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g(t, x)vdx. \quad (1.8)
\]
First, we have
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v \text{ div}(|\nabla v|^{p-2}\nabla v) dx \\
\leq - \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|\nabla v|^p dx - \frac{2}{k} \int_{k \leq |x| \leq \sqrt{k}} |v|^{p} \rho(\frac{|x|^2}{k^2})|\nabla v|^{p-2}|(x, \nabla v)| dx \\
\leq \frac{c}{k} \int_{k \leq |x| \leq \sqrt{k}} |v||\nabla v|^{p-1} dx \leq \frac{c_1}{k} (||v||^p_p + ||\nabla v||^p_p). \quad (1.9)
\]
Further, by Young’s inequality, we get
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})e^{-az(\theta_1, \omega)}|g\nabla v| dx \\
\leq \frac{3\lambda}{8} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v|^2 dx + \frac{2}{3\lambda} e^{-2az(\theta_1, \omega)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g^2(t, x) dx. \quad (1.10)
\]
Hence, using (0.3) and (1.9)-(1.10), we obtain from (1.8) that
\[
\frac{d}{dt} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v^2 dx + 2\gamma e^{-2az(\theta_1, \omega)} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|u|^q dx \\
+ \left( \frac{5}{4} \lambda - 2az(\theta_1, \omega) \right) \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})v^2 dx \leq \frac{2c_1}{k} e^{-2az(\theta_1, \omega)} (||u||^p_p + ||\nabla u||^p_p) \\
+ 2e^{-2az(\theta_1, \omega)} \left( \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})\psi_1(t, x) dx + \frac{2}{3\lambda} \int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})g^2(t, x) dx \right).
\]
Multiply the above by \( e^\frac{4}{5} \lambda t - 2a \int_{\tau}^{t} z(\theta, \omega) dr \) and integrate over \([\tau - t, \sigma]\), then replace \( \omega \) by \( \theta_{\tau, \omega} \) in order to obtain
\[
\int_{\mathbb{R}^n} \rho(\frac{|x|^2}{k^2})|v(\sigma, \tau - t, \theta_{\tau, \omega}, v_{\tau, t})|^2 dx
\]
Since the integral in (1.1) exists, we find that there exists \( K \) such that for all \( t \) provided \( t \in D \),

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v_{\tau-t}|^2 \, dx
\]

which along with Lemma 4.1.1 implies that there exists \( T_1 = T_1(\tau, \omega, D, \alpha) > 0 \) such that for all \( t \geq T_1 \),

\[
\int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v(\sigma, \tau - t, \theta - \tau \omega, v_{\tau-t})|^2 \, dx
\]

where \( M \) is the number given by (1.1). Since \( e^{\alpha z(\theta - t \omega)} v_{\tau-t} \in D(\tau - t, \theta - t \omega) \), by (1.37) there is \( T_2 = T_2(\tau, \omega, D, \alpha, \eta) \geq T_1 \) such that for all \( t \geq T_2 \),

\[
e^{-\frac{\alpha}{2} t - 2\alpha \int_{\tau-t}^\tau z(\theta, \omega) \, dr} e^{-2\alpha z(\theta, \omega)} \|
\]

Since the integral in (1.1) exists, we find that there exists \( K = K(\tau, \omega, \alpha, \eta) \geq 1 \) such that both \( \frac{2c_1}{k} M \) and the last integral in (1.11) are bounded by \( \eta \) for all \( k \geq K \). This together with (1.11) and (1.12) implies that for all \( t \geq T_2 \) and \( k \geq K \),

\[
\int_{|x| \geq \sqrt{2k}} |v(\sigma, \tau - t, \theta - \tau \omega, v_{\tau-t})|^2 \, dx \leq \int_{\mathbb{R}^n} \rho \left( \frac{|x|^2}{k^2} \right) |v(\sigma, \tau - t, \theta - \tau \omega, v_{\tau-t})|^2 \, dx < 3\eta,
\]

as desired. \( \square \)

We now prove the asymptotic compactness of solutions of equation (0.9) in \( L^2(\mathbb{R}^n) \).

**Lemma 4.1.4.** If (0.3)–(0.5) and (0.13) hold, then for every \( \alpha > 0 \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and \( D \in D_\alpha \), the sequence \( v(\tau, \tau - t_n, \theta - t_n \omega, v_{0,n}) \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \) provided \( t_n \to \infty \) and \( v_{0,n} e^{\alpha z(\theta - t_n \omega)} \in D(\tau - t_n, \theta - t_n \omega) \).
Proof. Let \( \tilde{v}_n = v(t - 1, r - t_n, \theta_\tau, v_{0, n}) \). It follows from Lemma 4.1.1 that \( \{\tilde{v}_n\}_{n=1}^\infty \) is bounded in \( L^2(\mathbb{R}^n) \). Therefore, by Corollary 3.1.4, there exist \( r \in (\tau - 1, \tau) \) and \( \xi \in L^2(\mathbb{R}^n) \) such that, up to a subsequence,

\[
v(r, \tau - t_n, \theta_\tau, v_{0, n}) = v(r, \tau - 1, \theta_\tau, \xi, v_n) \to \xi \quad \text{in} \quad L^2(\mathcal{O}_k) \quad \text{for all} \quad k \in \mathbb{N},
\]

which along with Lemma 4.1.3 shows that

\[
v(r, \tau - t_n, \theta_\tau, v_{0, n}) \to \xi \quad \text{in} \quad L^2(\mathbb{R}^n).
\]

(1.13)

On the other hand, by continuity in initial data and the fact \( r \in (\tau - 1, \tau) \) we get

\[
\|v(t, r, \theta_\tau, v(r, \tau - t_n, \theta_\tau, v_{0, n})) - v(t, r, \theta_\tau, \xi)\| \\
\leq c \|v(r, \tau - t_n, \theta_\tau, v_{0, n}) - \xi\|.
\]

(1.14)

Hence, by (1.13) and (1.14) we obtain

\[
v(t, r, \theta_\tau, v_{0, n}) \\
= v(t, r, \theta_\tau, v(r, \tau - t_n, \theta_\tau, v_{0, n})) \to v(t, r, \theta_\tau, \xi) \quad \text{in} \quad L^2(\mathbb{R}^n)
\]

which completes the proof. \( \square \)

We now present the existence and uniqueness of \( D_\alpha \)-pullback attractors of equation (0.1) in \( L^2(\mathbb{R}^n) \).

**Theorem 4.1.5.** If (0.3)-(0.5) and (0.13) hold, then for every \( \alpha > 0 \), system (0.1)-(0.2) possesses a unique \( D_\alpha \)-pullback attractor \( \mathcal{A}_\alpha = \{A_\alpha(t, \omega) : t \in \mathbb{R}, \omega \in \Omega\} \in D_\alpha \) in \( L^2(\mathbb{R}^n) \). Moreover, if there is \( T > 0 \) such that \( f(\cdot, x, s), g(\cdot, x) \) and \( \psi_1(\cdot, x) \) are periodic in their first argument with period \( T \) for every fixed \( x \in \mathbb{R}^n \) and \( s \in \mathbb{R} \), then the attractor is also \( T \)-periodic; that is, \( A_\alpha(t + T, \omega) = A_\alpha(t, \omega) \) for all \( t \in \mathbb{R} \) and \( \omega \in \Omega \).

**Proof.** Since \( \Phi \) has a closed measurable absorbing set \( K_\alpha \) given by (1.4) and is \( D_\alpha \)-pullback asymptotically compact by (0.8) and Lemma 4.1.4, the existence and uniqueness of \( D_\alpha \)-pullback attractor \( \mathcal{A}_\alpha \) follows from Wang [2012, 2014b] immediately. If \( f, g \) and \( \psi_1 \) are \( T \)-periodic in time, then both \( \Phi \) and \( K_\alpha \) are also \( T \)-periodic; that is, \( \Phi(t, \tau + T, \cdot) = \Phi(t, \tau, \cdot) \) and \( K_\alpha(t + T, \omega) = K_\alpha(t, \omega) \) for all \( t \in \mathbb{R}^+, \tau \in \mathbb{R} \) and \( \omega \in \Omega \). As a consequence, we obtain the periodicity of \( \mathcal{A}_\alpha \) from Wang [2012]. \( \square \)

### 4.2 Upper-Semicontinuity of Random Attractors

In this section, we prove convergence of random attractors of (0.1) as the intensity \( \alpha \) of noise approaches zero. From now on, we write the solution of (0.1)-(0.2) as \( u_\alpha \) and the corresponding cocycle as \( \phi_\alpha \). Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), let
\[ \tilde{R}(\tau, \omega) = c + c \int_{-\infty}^{0} e^{\frac{5}{2} \lambda s + 2 \int_{0}^{s} z(\theta_{t \omega}) dr} + 2|z(\theta_{t \omega})| (\| \psi_{1}(s + \tau, \cdot) \|_{1} + \| g(s + \tau, \cdot) \|^{2}) ds \]

and

\[ B(\tau, \omega) = \{ u \in L^{2}(\mathbb{R}^{n}) : \| u \|^{2} \leq e^{2|z(\theta_{t \omega})| \tilde{R}(\tau, \omega)} \} \]  

(2.16)

where the positive number \( c \) is as in (1.5). Then we have,

\[
\| B(\tau - t, \theta_{t \omega}) \|^{2} \leq e^{-2|z(\theta_{t \omega})|} + c e^{2|z(\theta_{t \omega})|} \int_{-\infty}^{0} e^{\frac{5}{2} \lambda s + 2 \int_{0}^{s} z(\theta_{t \omega}) dr} + 2|z(\theta_{t \omega})| \times (\| \psi_{1}(s + \tau - t, \cdot) \|_{1} + \| g(s + \tau - t, \cdot) \|^{2}) ds
\]

\[
\leq c e^{2|z(\theta_{t \omega})|} (1 + e^{\frac{5}{2} \lambda t} \int_{-\infty}^{t} e^{\frac{5}{2} \lambda s + 2 \int_{0}^{s} z(\theta_{t \omega}) dr} + 2|z(\theta_{t \omega})| \times (\| \psi_{1}(s + \tau) \|_{1} + \| g(s + \tau) \|^{2}) ds.
\]

Therefore, for all \( \alpha \in (0, 1] \) and \( t \geq 0 \), we obtain

\[
e^{-\frac{5}{4} \lambda t - 2\alpha} \int_{0}^{-t} z(\theta_{t \omega}) dr - 2\alpha z(\theta_{t \omega}) \| B(\tau - t, \theta_{t \omega}) \|^{2} \leq c e^{-\frac{5}{2} \lambda t + 2|\int_{0}^{0} z(\theta_{t \omega}) dr| + 4|z(\theta_{t \omega})|} + c e^{\frac{1}{2} \lambda t + 4|\int_{0}^{t} z(\theta_{t \omega}) dr| + 4|z(\theta_{t \omega})|} \int_{-\infty}^{-t} e^{\frac{5}{2} \lambda s + 2 \int_{0}^{s} z(\theta_{t \omega}) dr} + 2|z(\theta_{t \omega})| \times (\| \psi_{1}(s + \tau) \|_{1} + \| g(s + \tau) \|^{2}) ds.
\]

Note that the last integral exists by (0.13). Thus, for every \( \eta > 0, \tau \in \mathbb{R}, \) and \( \omega \in \Omega, \) there exists a \( T = T(\tau, \omega, \eta) \) such that for all \( t \geq T \) and for all \( \alpha \leq 1, \)

\[
e^{-\frac{5}{4} \lambda t - 2\alpha} \int_{0}^{-t} z(\theta_{t \omega}) dr - 2\alpha z(\theta_{t \omega}) \| B(\tau - t, \theta_{t \omega}) \|^{2} \leq \eta.
\]

Based on (2.17), we are able to prove the following uniform estimates on the tails of functions in random attractors.

**Lemma 4.2.1.** If (0.3)–(0.5) and (0.13) hold, then for every \( \eta > 0, \tau \in \mathbb{R} \) and \( \omega \in \Omega, \) there exists \( K = K(\tau, \omega, \eta) \geq 1 \) such that for all \( k \geq K, \)

\[
\int_{|x| \geq k} |\xi(x)|^{2} dx \leq \eta \text{ for all } \xi \in \bigcup_{0 < \alpha \leq 1} A_{\alpha \alpha}(\tau, \omega).
\]

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Proof. Following the proof of Lemmas 4.1.1 and 4.1.3, by (2.17) one can verify that for every \( \eta > 0 \), \( \tau \in \mathbb{R} \), and \( \omega \in \Omega \), there exists a \( T = T(\tau, \omega, \eta) \) and \( K = K(\tau, \omega, \eta) \geq 1 \), such that for all \( t \geq T \) and \( k \geq K \) and for all \( \alpha \in (0,1] \), the solution \( u_\alpha \) of (0.1)-(0.2) satisfies,

\[
\int_{|x| \geq k} |u_\alpha(\tau, \tau - t, \theta_{-\tau} \omega, u_{\alpha, \tau - t})|^2 dx \leq \eta \quad (2.18)
\]

where \( u_{\alpha, \tau - t} \in B(\tau - t, \theta_{-\tau} \omega) \) with \( B \) given by (2.16). By (1.5), (1.4) and (2.15)-(2.16) we have

\[
\bigcup_{0 < \alpha \leq 1} A_{\alpha}(\tau, \omega) \subseteq \bigcup_{0 < \alpha \leq 1} K_{\alpha}(\tau, \omega) \subseteq B(\tau, \omega). \quad (2.19)
\]

Let \( \xi \in A_{\alpha}(\tau, \omega) \) for some \( \alpha \in (0,1] \). By the invariance of \( A_{\alpha} \), there exists \( \zeta \in A_{\alpha}(\tau - T, \theta_{-\tau} \omega) \) such that \( \xi = u_\alpha(\tau, \tau - T, \theta_{-\tau} \omega, \zeta) \), which along with (2.18)-(2.19) implies that for all \( k \geq K \),

\[
\int_{|x| \geq k} |\xi(x)|^2 dx = \int_{|x| \geq k} |u_\alpha(\tau, \tau - T, \theta_{-\tau} \omega, \zeta)|^2 dx \leq \eta,
\]

as desired. \( \square \)

In the limiting case \( \alpha = 0 \), the stochastic equation (0.1) reduces to a deterministic one:

\[
\frac{\partial u}{\partial t} + \lambda u - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = f(t, x, u) + g(t, x) \quad \text{with} \quad u(\tau, x) = u_\tau(x) \quad (2.20)
\]

for \( x \in \mathbb{R}^n \). Denote the cocycle of (2.20) in \( L^2(\mathbb{R}^n) \) by \( \phi_0 \). As in (0.12), let \( D_{\alpha_0} \) be the collection of subsets of \( L^2(\mathbb{R}^n) \) given by,

\[
D_{\alpha} = \{ D = \{ D(\tau) \subseteq L^2(\mathbb{R}^n) : \tau \in \mathbb{R} \} : \lim_{s \to -\infty} e^{\frac{\lambda s}{\alpha}} \| D(\tau + s) \|^2 = 0, \forall \tau \in \mathbb{R} \}.
\]

Note that Theorem 4.1.5 is also valid when \( \alpha = 0 \); more precisely, \( \phi_0 \) has a unique \( D_{\alpha_0} \)-pullback attractor \( A_{\alpha_0} = \{ A_{\alpha_0}(\tau) : \tau \in \mathbb{R} \} \in D_{\alpha_0} \) and has a \( D_{\alpha_0} \)-pullback absorbing set \( K_0 = \{ K_0(\tau) : \tau \in \mathbb{R} \} \) where \( K_0(\tau) \) is defined by

\[
K_0(\tau) = \{ u \in L^2(\mathbb{R}^n) : \| u \|^2 \leq R_0(\tau) \} \quad (2.21)
\]

with

\[
R_0(\tau) = c + c \int_{-\infty}^0 e^{\frac{\lambda s}{\alpha}} (\| \phi_1(s + \tau) \|_1 + \| g(s + \tau) \|^2) ds. \quad (2.22)
\]

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The constant $c$ in (2.22) is the same as in (1.5). It follows from (1.5), (1.4) and (2.21)-(2.22) that
\[
\limsup_{\alpha \to 0} \|K_{\alpha}(\tau, \omega)\| = \|K_0(\tau)\|. \quad (2.23)
\]
In the sequel, we further assume there exists $\psi_5 \in L^\infty(\mathbb{R}, L^\infty(\mathbb{R}^n))$ such that for all $t, s \in \mathbb{R}$ and $x \in \mathbb{R}^n$,
\[
|\frac{\partial f}{\partial s}(t, x, s)| \leq \psi_5(t, x)(1 + |s|^{q-2}). \quad (2.24)
\]
Under condition (2.24), we have the following relations between solutions of (0.1) and (2.20).

**Lemma 4.2.2.** Suppose (0.3)-(0.5), (2.24) and (0.13) hold. If $u_\alpha$ and $u$ are the solutions of (0.1)-(0.2) and (2.20) with initial data $u_{\alpha, \tau}$ and $u, \tau$, respectively, then for every $\tau \in \mathbb{R}$, $\omega \in \Omega$, $T > 0$, and $\epsilon \in (0, 1)$, there exists $\alpha_0 = \alpha_0(\tau, \omega, T, \epsilon) \in (0, 1)$ such that for all $\alpha \leq \alpha_0$ and $t \in [\tau, \tau + T]$,
\[
\|u_\alpha(t, \tau, \omega, v_{\alpha, \tau}) - u(t, \tau, u_\tau)\|^2 \leq c\|v_{\alpha, \tau} - u_\tau\|^2 + c\epsilon(1 + \|v_{\alpha, \tau}\|^2 + \|u_\tau\|^2),
\]
where $c$ and $c_0$ are positive constants independent of $\alpha$ and $\epsilon$.

**Proof.** Let $\xi = v_\alpha - u$. Then by subtracting (0.9) from (2.20) and taking the inner product with $\xi$, we get
\[
\frac{1}{2} \frac{d}{dt} \|\xi\|^2 + \int_{\mathbb{R}^n} (e^\alpha(p-2)|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u) \cdot \nabla \xi \, dx + \lambda \|\xi\|^2 = \alpha z(\theta)|\xi|^2 + \alpha z(\theta)(u, \xi) + \int_{\mathbb{R}^n} (e^{-\alpha z(\theta)}f(t, x, e^{\alpha z(\theta)}v_\alpha) - f(t, x, u))\xi \, dx
\]
\[
+ (e^{-\alpha z(\theta)} - 1) \int_{\mathbb{R}^n} g(t, x)\xi \, dx.
\]
For the second term on the left hand side of (2.25) we have that,
\[
\int_{\mathbb{R}^n} (e^\alpha(p-2)|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u) \cdot \nabla \xi \, dx
\]
\[
= \int_{\mathbb{R}^n} e^\alpha(p-2)|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u) \cdot \nabla \xi \, dx \quad (2.26)
\]
\[
+ \int_{\mathbb{R}^n} (e^\alpha(p-2)(z(\theta)) - 1)|\nabla u|^{p-2}\nabla u) \cdot \nabla \xi \, dx
\]
By the monotonicity of the p-laplace operator, see e.g. DiBenedetto [1993], we have that there is a positive number $\beta$ such that
\[
(|\nabla v_\alpha|^{p-2}\nabla v_\alpha - |\nabla u|^{p-2}\nabla u) \cdot (\nabla v_\alpha - \nabla u) \geq \beta |\nabla v_\alpha - \nabla u|. \quad (2.27)
\]
By Young’s inequality, we find that for every $\tau \in \mathbb{R}, \omega \in \Omega$, $T > 0$, and $\epsilon \in [0, 1]$, there exists a $\alpha_1 = \alpha_1(\tau, \omega, T, \epsilon) > 0$ such that for all $\alpha \in [0, \alpha_1]$, 

$$\left| \int_{\mathbb{R}^n} (e^{\alpha(p-2)z(\theta_1 \omega)} - 1) |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{\zeta} dx \right|$$

(2.28)

$$\leq \frac{1}{2} \beta e^{\alpha(p-2)z(\theta_1 \omega)} \int_{\mathbb{R}^n} |\nabla \bar{\zeta}|^p dx + \epsilon \int_{\mathbb{R}^n} |\nabla u|^p dx.$$ 

It follows from (2.26)-(2.28) that

$$\int_{\mathbb{R}^n} (e^{\alpha(p-2)z(\theta_1 \omega)} |\nabla v_\alpha|^p - |\nabla u|^{p-2} \nabla u \cdot \nabla \bar{\zeta} dx)$$

(2.29)

$$\geq \frac{1}{2} \beta e^{\alpha(p-2)z(\theta_1 \omega)} \int_{\mathbb{R}^n} |\nabla \bar{\zeta}|^p dx - \epsilon \int_{\mathbb{R}^n} |\nabla u|^p dx.$$ 

For the third term on the right hand side of (2.25) by (0.4), (0.5), and (2.24) and Young’s inequality, we find that

$$\int_{\mathbb{R}^n} (e^{-\alpha z(\theta_1 \omega)} f(t,x,e^{\alpha z(\theta_1 \omega)}v_\alpha) - f(t,x,u)) \bar{\zeta} dx$$

$$= \int_{\mathbb{R}^n} e^{-\alpha z(\theta_1 \omega)} f(t,x,e^{\alpha z(\theta_1 \omega)}v_\alpha) - f(t,x,e^{\alpha z(\theta_1 \omega)}u)) \bar{\zeta} dx$$

$$+ \int_{\mathbb{R}^n} (e^{-\alpha z(\theta_1 \omega)} f(t,x,e^{\alpha z(\theta_1 \omega)}u) - f(t,x,e^{\alpha z(\theta_1 \omega)}u)) \bar{\zeta} dx$$

$$+ \int_{\mathbb{R}^n} (f(t,x,e^{\alpha z(\theta_1 \omega)}u) - f(t,x,u)) \bar{\zeta} dx = \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(t,x,s) \bar{\zeta}^2 dx$$

$$+(e^{-\alpha z(\theta_1 \omega)} - 1) \int_{\mathbb{R}^n} (f(t,x,e^{\alpha z(\theta_1 \omega)}u)) \bar{\zeta} dx + (e^{\alpha z(\theta_1 \omega)} - 1) \int_{\mathbb{R}^n} \frac{\partial f}{\partial s}(t,x,s) u \bar{\zeta} dx$$

$$\leq \int_{\mathbb{R}^n} \psi_2(t,x) (\bar{\zeta}^2 + |\bar{\zeta}|^q) dx + |e^{-\alpha z(\theta_1 \omega)} - 1| \int_{\mathbb{R}^n} \times (\psi_2(t,x)e^{\alpha(q-1)z(\theta_1 \omega)}|u|^{q-1} + \psi_3(t,x)) |\bar{\zeta}| dx$$

$$+ |e^{\alpha z(\theta_1 \omega)} - 1| \int_{\mathbb{R}^n} \psi_4(t,x)|\bar{\zeta}| dx \leq c_1|\bar{\zeta}|^2$$

+ c_1|e^{-\alpha z(\theta_1 \omega)} - 1| \int_{\mathbb{R}^n} (e^{\alpha(q-1)z(\theta_1 \omega)}(|u|^q + |v_\alpha|^q + \psi_3(t,x)|h_1|) dx$$

+ c_1|e^{\alpha z(\theta_1 \omega)} - 1||u||^2 dx. \quad (2.30)$$

So for every $\tau \in \mathbb{R}, \omega \in \Omega$, $T > 0$, and $\epsilon \in [0, 1]$, there exists a $\alpha_2 = \alpha_2(\tau, \omega, T, \epsilon) > 0$ such that for all $\alpha \in [0, \alpha_2]$ by (2.30) we have,

$$\int_{\mathbb{R}^n} (e^{-\alpha z(\theta_1 \omega)} f(t,x,e^{\alpha z(\theta_1 \omega)}v_\alpha) - f(t,x,u)) \bar{\zeta} dx$$
\[
\leq c_2\|\xi\|^2 + \varepsilon c_2 (1 + \|u\|^2 + \|v_\alpha\|^q + \|u\|^q).
\]  

(2.31)

Similarly, for every \( \tau \in \mathbb{R}, \omega \in \Omega, T > 0, \) and \( \varepsilon \in [0, 1], \) there exists a \( \alpha_3 = \alpha_3(\tau, \omega, T, \varepsilon) > 0 \) such that for all \( \alpha \in [0, \alpha_3] \) we have,

\[
(e^{-\alpha \omega(\eta, \omega)} - 1) \int_{\mathbb{R}^n} g(t, x)\xi dx \leq c_3\|\xi\|^2 + \varepsilon c_3\|g(t, \cdot)\|^2
\]  

(2.32)

So by (2.29)-(2.32), and Young’s inequality we have from (2.25) that

\[
\frac{d}{dt}\|\xi\|^2 \leq c_4\|\xi\|^2 + \varepsilon c_4 (1 + \|u\|^2 + \|v_\alpha\|^q + \|u\|^q + \|\nabla u\|^p + \|g(t, \cdot)\|^2).
\]

Solving this inequality we have,

\[
\|\xi\|^2 \leq e^{c_4(t-\tau)}\|\xi\|^2
\]

\[
+ \varepsilon c_4e^{c_4(t-\tau)} \int_\tau^t (1 + \|u(s, \tau, \omega, u_\tau)\|^2 + \|v_\alpha\|^q + \|u\|^q + \|\nabla u\|^p + \|g(t, \cdot)\|^2)ds.
\]

(2.33)

By (1.2) we have that for all \( \alpha \in [0, 1], \)

\[
\frac{d}{dt}\|v_\alpha\|^2 + 2e^{e/(p-2)z(\eta, \omega)} \|\nabla v_\alpha\|^p + c_5\|v_\alpha\|^2 + 2\gamma e^{e/(p-2)z(\eta, \omega)}\|v_\alpha\|^q
\]

\[
\leq c_6\|v_\alpha\|^2 + 2e^{-2\alpha z(\eta, \omega)}\|\psi_1(t, \cdot)\|^1 + \frac{4}{\lambda}e^{-2\alpha z(\eta, \omega)}\|g(t, \cdot)\|^2.
\]

(2.34)

By (2.34) we have that for all \( \alpha \in [0, 1] \) and \( t \in [\tau, \tau + T], \)

\[
\|v_\alpha\|^2 + \int_\tau^t \|\nabla v_\alpha\|^p + \|v_\alpha\|^2 + \|v_\alpha\|^q ds
\]

\[
\leq e^{c_6(t-\tau)}\|v_{\alpha, \tau}\|^2 + c_7e^{c_6(t-\tau)} \int_\tau^t (\|\psi_1(s, \cdot)\|^1 + \|g(s, \cdot)\|^2) ds.
\]

(2.35)

It is clear that (2.35) also holds for \( \alpha = 0, \) so we have by (2.33) and (2.35),

\[
\|v_\alpha(t, \tau, \omega, v_{\alpha, \tau}) - u(t, \tau, u_\tau)\|^2 \leq e^{c_4(t-\tau)}\|v_{\alpha, \tau} - u_\tau\|^2
\]

\[
+ \varepsilon c_\delta e^{c_\delta(t-\tau)} (1 + \|v_{\alpha, \tau}\|^2 + \|u_\tau\|^2 + \int_\tau^t (\|\psi_1(s, \cdot)\|^1 + \|g(s, \cdot)\|^2) ds.
\]

(2.36)

Since

\[
\|u_\alpha(t, \tau, \omega, v_{\alpha, \tau}) - v_\alpha(t, \tau, \omega, v_{\alpha, \tau})\|^2 = e^{\alpha z(\eta, \omega)} - 1 \|u_\alpha(t, \tau, \omega, v_{\alpha, \tau})\|
\]

we have the desired result from (2.35) and (2.36).
Lemma 4.2.3. Let (0.3)-(0.5), (2.24) and (0.13) hold. Given \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), if \( \alpha_n \to 0 \) and \( u_n \in A_{\alpha \alpha_n}(\tau, \omega) \), then \( \{u_n\} \) has a convergent subsequence in \( L^2(\mathbb{R}^n) \).

Proof. It follows from the invariance of \( A_{\alpha \alpha_n} \) and the fact \( u_n \in A_{\alpha \alpha_n}(\tau, \omega) \) that for each \( n \in \mathbb{N} \), there is \( \xi_n \in A_{\alpha \alpha_n}(\tau - 1, \theta - \omega) \) such that

\[
u_n = u_{\alpha_n}(\tau, \tau - 1, \theta - \omega, \xi_n).
\]

(2.37)

By (0.8), for the solution \( v_{\alpha_n} \) of (0.9)-(0.10) we get, for \( s \geq \tau - 1 \),

\[
u_{\alpha_n}(s, \tau - 1, \theta - \omega, \xi_n) = e^{-\alpha_n z_{\theta - \omega}} u_{\alpha_n}(s, \tau, \tau - 1, \theta - \omega, \xi_n) \quad \text{where} \quad \xi_n = e^{-\alpha_n z_{\theta - \omega}} \xi_n.
\]

(2.38)

By (2.19) and (2.38) we find that \( \{\xi_n\} \) is a bounded sequence in \( L^2(\mathbb{R}^n) \), and hence, as in Lemma 4.0.3, there exist \( r \in (\tau - 1, \tau) \) and \( \phi \in L^2(\mathbb{R}^n) \) such that, up to a subsequence,

\[
u_{\alpha_n}(r, \tau - 1, \theta - \omega, \xi_n) \to \phi \quad \text{in} \quad L^2(\mathcal{O}_k) \quad \text{for all} \quad k \in \mathbb{N},
\]

which along with (2.38) yields

\[u_{\alpha_n}(r, \tau - 1, \theta - \omega, \xi_n) \to \phi \quad \text{in} \quad L^2(\mathcal{O}_k) \quad \text{for all} \quad k \in \mathbb{N}.
\]

(2.39)

Since \( \xi_n \in A_{\alpha \alpha_n}(\tau - 1, \theta - \omega) \), by the invariance of attractor, we have \( u_{\alpha_n}(r, \tau - 1, \theta - \omega, \xi_n) \in A_{\alpha \alpha_n}(r, \theta - \tau) \). This together with (2.39) and Lemma 4.2.1 implies

\[u_{\alpha_n}(r, \tau - 1, \theta - \omega, \xi_n) \to \phi \quad \text{in} \quad L^2(\mathbb{R}^n).
\]

(2.40)

By (2.40) and Lemma 4.2.2 we get

\[u_{\alpha_n}(\tau, \tau - 1, \theta - \omega, \xi_n) = u_{\alpha_n}(\tau, \tau - 1, \theta - \omega, u_{\alpha_n}(r, \tau - 1, \theta - \omega, \xi_n)) \to u(\tau, r, \theta - \omega, \phi)
\]

in \( L^2(\mathbb{R}^n) \), which along with (2.37) completes the proof.

Lemma 4.2.4. If (0.3)-(0.5), (2.24) and (0.13) hold, then for every \( \tau \in \mathbb{R} \), and \( \omega \in \Omega \),

\[
\lim_{\alpha \to 0} \text{dist}_{L^2(\mathbb{R}^n)}(A_{\alpha \alpha}(\tau, \omega), A_{\alpha 0}(\tau, \omega)) = 0
\]

Proof. This follow from Theorem 3.7 in Wang [2014a] directly based on (2.23), and Lemmas 4.2.2 and 4.2.3.
CHAPTER 5

CONCLUSION

This thesis looked at two families of $p$-Laplace equations driven by stochastic and non-autonomous noise. Well-posedness of the equations was outlined briefly, allowing us to define a cocycle to model the dynamics. Estimates on the equations allowed us to define absorbing sets for these equations. Estimates on the tails of solutions to these equations were also used to show that these cocycles were pullback-asymptotically compact. These two properties immediately give us the existence and uniqueness of random pullback attractors, defined with respect to certain universes. The existence of these objects tells us that over long periods of time, the dynamics is confined to a compact subset of the phase space, and cannot grow arbitrarily largely.

In the multiplicative noise case, we also showed a structural property of the attractor. Specifically, we showed that the random pullback attractor for the stochastic equation approached that of the deterministic equation as $\alpha \to 0$ in an upper semicontinuous way. Roughly speaking, this means that the random attractor's dynamics was approximately contained within the deterministic one for very small $\alpha$. In other words, perturbing the equation with a small noise term does not cause the attractor to explode or bifurcate to a much larger attractor. Lower semicontinuity, that is, showing that the deterministic attractor is approximately contained in the random one for very small $\alpha$, is a much more difficult question in general, and requires more detailed knowledge of the asymptotic dynamics. The overall interpretation of having proven upper semicontinuity, but not lower semicontinuity, is that a small stochastic perturbation may cause the attractor to collapse or shrink in some sense, but it cannot become a much larger set.

There are many related open questions that were not addressed in this thesis. One may be able use the monotone properties of the $p$-Laplace operator to bound the attractors in some sense. It would also be interesting to look at other types of attracting sets, such as inertial manifolds or exponential attractors, which give some information about how quickly solutions will be attracted to them. It would also be interesting to explore other types of dynamics, such as forward attractors, although there are some difficulties in studying the forward dynamics of non-autonomous and stochastic equations. There are of course questions of regularity of attractors, and other detailed structural problems associated with them.
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Asymptotic Dynamics of Stochastic $p$-Laplace Equations on Unbounded Domains

by

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