The path to instability in multi-planetary systems

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Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT

The dynamical stability of tightly packed exoplanetary systems remains poorly understood. While for a two-planet system a sharp stability boundary exists, numerical simulations of three and more planet systems show that they can experience instability on timescales up to billions of years. Moreover, an exponential trend between the planet orbital separation measured in units of Hill radii and the survival time has been reported. While these findings have been observed in numerous numerical simulations, little is known of the actual mechanism leading to instability. Contrary to a constant diffusion process, planetary systems seem to remain dynamically quiescent for most of their lifetime before a very short unstable phase. In this work, we show how the slow chaotic diffusion due to the overlap of three-body resonances dominates the timescale leading to the instability for initially coplanar and circular orbits. While the last instability phase is related to scattering due to two-planet mean motion resonances (MMR), for circular orbits the two-planets MMR are too far separated to destabilize systems initially away from them. The studied mechanism reproduces very well the qualitative behaviour found in numerical simulations. We develop an analytical model to generalize the empirical trend obtained for equal mass and equally-spaced planets to general systems on initially circular orbits. We obtain an analytical estimate of the survival time consistent with numerical simulations over four orders of magnitude for the planet to star mass ratio \( \varepsilon \), and 6 to 8 orders of magnitude for the instability time. We also confirm that measuring the orbital spacing in terms of Hill radii is not adapted and that the right spacing unit scales as \( \varepsilon^{1/4} \). We predict that beyond a certain spacing, the three-planet resonances are not overlapped, which results in an increase of the survival time. We confirm these findings with the aid of numerical simulations of three-planet systems with different masses. We finally discuss the extension of our result to more general systems, containing more planets on initially non-circular orbits.

Key words. Celestial mechanics, Planets and satellites: dynamical evolution and stability

1. Introduction

One of the most astonishing results of the \textit{Kepler} mission has been the discovery of very compact Super-Earth multiplanetary systems (Borucki et al. 2011; Fabrycky et al. 2014). These systems, such as Kepler-11 (Lissauer et al. 2011a), can host more than six planets with masses between that of the Earth and Neptune, and with less than 100 days period. They have very low mutual inclinations and eccentricities (Johansen et al. 2012; Fung & Margot 2012; Xie et al. 2016) and for the majority, they are not in resonant chains (Lissauer et al. 2011b; Fabrycky et al. 2014). Understanding the orbital properties of these so-called Super-Earths/Mini-Neptunes is crucial, as it seems that at least 50\% of solar-type stars host a close-in planet with a radius that of Earth and Neptune (Mayor et al. 2011; Petigura et al. 2013; Fressin et al. 2013).

Studies of the \textit{Kepler} multiplanetary systems have shown that the architecture is most likely sculpted by dynamical stability (Johansen et al. 2012; Pu & Wu 2015). Indeed, it has been shown that the minimum spacing is mass dependent (Weiss et al. 2018), with a lower limit in observed Kepler systems of around 10 Hill radii. As a result, understanding the mechanism leading to the instability of more tightly packed systems is critical to our understanding of planet formation and architecture.

The question of the stability of exoplanetary systems is particularly challenging due to several factors. The observed close-in planets have most likely performed at least \( 10^9 \) to \( 10^{11} \) orbits since their formation, which make the numerical integration extremely costly if one wanted to integrate the system over its whole lifetime. The process is made even more costly because we do not know the exact orbital configuration, let alone the planet masses for systems detected by transits. But even if the present orbital configuration were known perfectly, planetary systems are extremely chaotic as it has been shown for our own Solar System (Laskar 1994; Laskar & Gastineau 2009). As a result, the only approach to a numerical stability analysis is to run several integrations with slight variations of the initial conditions to probe the outcome in a statistical manner. So, for each exoplanetary system, thousands of very costly numerical integrations would need to be run in order to obtain a satisfying understanding of its stability properties. The process could eventually be sped up thanks to the help of machine learning classification (Tamayo et al. 2016).
Another approach is to rely on analytical stability criteria. Under specific assumptions, it is possible to simplify the dynamics to obtain models describing the actual system behaviour. In particular, one can derive stability criteria that can delineate stable regions from unstable ones where systems will eventually experience close encounters and collisions. Among such analytical criteria, one can cite the Hill stability (Marchal & Bozis 1982; Gladman 1993; Petit et al. 2018) and the overlap of mean motion resonances (MMR, Wisdom 1980; Deck et al. 2013; Petit et al. 2017; Hadden & Lithwick 2018). For less compact, non-resonant systems, the dynamics are very well approximated by the secular model. In the secular approximation, one averages over the fast motion of the planets on their Keplerian orbits to only consider their long-term deformations. A well-known consequence of this averaging is the conservation of the planet semi-major axes, and thus of the Angular Momentum Deficit (AMD Laskar 1997, 2000). The AMD gives a dynamically motivated measure of the total eccentricities and mutual inclinations in a planetary system, and thus acts as a dynamical temperature. In particular if the AMD is low enough, there is no possible orbital rearrangement allowing for planetary close encounters. This concept has been defined as the AMD-stability (Laskar & Petit 2017), and it allows for a fast characterization of the stability of of planetary systems away from mean motion resonances, where the secular approximation is valid. Besides the AMD stability, the AMD has proven to be a versatile tool to understand planet dynamics (e.g., Volk & Malhotra 2020).

However, the transition from the secular regime to regions where the fast interactions between planets shall not be neglected is unclear. This is due to the influence of mean motion resonances (MMR) which forbid independent averaging over the planets fast angles. While theoretical studies in the two planet case have allowed to delimitate a sharp limit between the secular and non-secular region (Hadden & Lithwick 2018; Petit et al. 2018, and references therein), there are no complete studies for three and more planet systems. Numerical simulations (Chambers et al. 1996, and references in sec. 2) have shown a qualitative change of behaviour between 2 planet and 3+ planet systems: multi-planetary systems experience a long quiescent phase where the systems are almost secular before a very rapid transition to collisional dynamics. Preliminary analytical studies were proposed by Zhou et al. (2007) or Quillen (2011), but their models did not reproduce entirely the characteristics of the transition zone between long-lived systems, and systems where scattering occurs immediately.

This work attempts to study the mechanism leading to the instability of unstable tightly packed systems. Since the different stability regime between 2-planet and multi-planet systems starts at three planets, we focus on systems composed of three planets. Contrary to previous studies, we do not make any assumptions regarding the masses of the planets (providing that they remain small) and consider unevenly spaced planets. We however restrict ourselves to initially circular and coplanar systems. Indeed, due to interactions with the protoplanetary disk, compact, close-in systems most likely form in this state due to eccentricity and inclination damping (Lin & Papaloizou 1986). Note that we do not consider planets trapped into resonant chain here and refer to Pichierri & Morbidelli (2020) for an analytical study of stability of resonant chains. Besides, we are interested in systems that should be considered AMD-stable in the sense that no secular interactions can lead to their instability (Petit et al. 2018). Understanding the initially circular systems gives a lower bound for the eccentric ones. By analyzing individual simulations, we postulate, as Quillen (2011), that the instability is driven by the overlap of mean motion resonances between the three planets of the systems. Their prominent role comes from the presence of a dense subset of three-planet resonances that covers a large part of the phase space, even for circular orbits. Moreover, the system dynamics in presence of this subset are not secular, yet they preserve the total AMD, which is a characteristic observed in numerical simulations. This feature will be explained in sect. 4.2. Using estimates of the diffusion rate proposed by Chirikov (1979), we are able to compute an analytical expression for the survival time.

Our analytical approach allows us to determine features in numerical simulations that trace the particular mechanism we study, which leads us to conclude that we isolated the right mechanism for planetary instability. In particular, we confirm that the scaling in terms of Hill radius, widely used in numerical studies (Chambers et al. 1996; Smith & Lissauer 2009; Pu & Wu 2015; Obertas et al. 2017) is not appropriate. By comparing with numerical simulations, we show that our time estimate is valid over four orders of magnitude in mass and almost seven orders of magnitude in survival time.

In the context of exoplanet observations, three planet resonances are particularly significant as it is possible to assess their dynamical influence solely from transit data (Delisle 2017). They can also be a signpost of the disruption of MMR chains thanks to tidal dissipation (Charalambous et al. 2018; Pichierri et al. 2019). Yet, the interactions between such resonances has not been fully studied.

In section 2, we begin by a review of the works on the problem of tightly packed planetary systems and we do an in-depth qualitative analysis of the instability. In section 3, we introduce our framework to treat the problem of three planet MMR. Section 4 contains most of the technical details. We first describe the network of zeroth order three planet resonances, we then solve the dynamics for an isolated MMR to finally obtain a criterion delimitating the region where the MMRs overlap. Using the framework developed in section 4, we estimate in section 5 the survival time for a system of three planets, with arbitrary mass distribution and spacing (assuming that the planets are not too massive and tightly packed). We compare our analytical results to numerical simulations in section 6. Finally, we discuss possible extensions to more general systems than three planets on circular and coplanar orbits in section 7. While the analytical derivations make necessary to define auxiliary variables, we tried as much as possible to use only variables with a clear physical meaning in the figures to ease the comprehension of readers willing to skip the technical sections.

2. Qualitative description of the instability

The dynamics of tightly-packed systems is chaotic, and works on the subject have mainly focused on qualitative description of their behaviour due to the difficulty of the analytic approach. We review the qualitative description proposed by previous studies and highlight how the instability is triggered.
2.1. Stability in the two-planet case

While the three-body problem is not integrable in general, the problem of the stability of a two planet system is well understood. Most of the stability results come from the existence of a topological boundary in the three-body configuration space leading to the so-called Hill-stability (Marchal & Bozis 1982). In a Hill stable system, the two planets can never approach each other, which leads to a sharp difference of behaviour. The Hill-stability has been popularized by Gladman (1993) for circular orbits, as a minimal distance between orbits normalized by their Hill radius guaranteeing the system’s stability. This stability criterion can be written as

\[ \Delta = \frac{a_2 - a_1}{a_1} > 2 \sqrt[3]{\frac{R_H}{a_1}} = 3.46 \frac{R_H}{a_1}, \]  

(1)

where

\[ R_H = \frac{a_1 + a_2}{2} \left( \frac{m_1 + m_2}{3m_0} \right)^{1/3} \]  

(2)

is the mutual Hill radius with \( m_1, m_2 \) being the planet masses and \( m_0 \) the star mass. For inclined and eccentric orbits, there exists a critical AMD value depending only on semi-major axis and masses such that a system having a smaller AMD is Hill stable (Petit et al. 2018).

Another stability criterion for two planet system can be derived from the overlap of MMR (Wisdom 1980; Deck et al. 2013; Petit et al. 2017; Hadden & Lithwick 2018). While the unperturbed resonant problem is integrable, the interaction between neighbouring MMRs leads to the formation of a chaotic web such that the planets’ orbital elements wander in a random walk fashion. This behaviour is known as the Chirikov (1979) diffusion. For initially circular orbits, the overlap occurs at a distance scaling as \((m_1 + m_2)/m_0)^{2/7}\) (Wisdom 1980). The exponent \(2/7\) is close to \(1/3\) but it has been highlighted that there exists a regime where MMR overlap while the planets are Hill-stable, i.e., the system is long-lived while experiencing short-term chaos (Deck et al. 2013; Petit et al. 2018).

It results that a two planet system is either stable over timescale comparable with its star’s lifetime or unstable in a very short amount of time (less than \(10^6\) orbits). No such dichotomy is observed for multiplanetary systems. Indeed, a multiplanet system can seem stable if it is numerically integrated over a few million of orbits while becoming unstable in less than a billion years.

2.2. Survival time of tightly packed systems

The pioneering work on the stability of tightly packed multi-planetary systems was carried out by Chambers et al. (1996). They performed numerical simulations of systems with equal mass planets on initially equally spaced circular and coplanar orbits (hereafter called EMS systems). The constant orbit spacing is given by \( \Delta = (a_{k+1} - a_k)/a_k \). For various planetary masses and numbers of planets, they recorded the survival time of a system, defined as the integration time before the distance between two planet becomes smaller than a Hill radius. As shown by Rice et al. (2018), the time between such a close encounter and the proper collision is usually negligible. Chambers et al. (1996) observed that the survival time grows exponentially with the spacing \( \Delta \) rescaled by the Hill radius \( R_H \),

\[ \log_{10} \frac{T_{\text{surv}}}{P} = b \frac{\Delta}{R_H} + c \]  

(3)

where \( P \) is a typical orbital period and \( b \) and \( c \) are numerical factors\(^1\). \( b \) seems to have a small dependency in the mass ratio and the number of planets. \( c \) seems to also depend on the mass ratio. Analysing figure 4 from (Chambers et al. 1996), a more appropriate scaling seems to be

\[ \log_{10} \frac{T_{\text{surv}}}{P} = b' \Delta \left( \frac{m_p}{m_0} \right)^{-1/4} - c' - \log \frac{m_p}{m_0}, \]  

(4)

where \( b' \) and \( c' \) are positive numerical coefficients independent of the masses, \( m_p \) is the planet mass and \( m_0 \) the star mass. Note that such scaling was also chosen by Faber & Quillen (2007).

Subsequent numerical works on the stability of EMS have been carried out. As the computational capacities increased, Smith & Lissauer (2009) and then Obertas et al. (2017) obtained datasets with a much finer distribution of spacing and longer integration times showing systems going unstable after almost 10 Gyr. Beyond the trend already observed by Chambers et al. (1996), they showed that the survival time is reduced in the vicinity of low order two-planet MMR. Hussain & Tamayo (2020) show that the spreading of the 3+ planet case and the mechanism leading to instability have permitted to highlight the key features that the tightly packed system instability presents and that an analytical model should explain.

Following Chambers et al. (1996), most of the previously cited studies fit the survival time with curves similar to Eq. (3) because of the natural parallel with the two planet case. However, there is no generalization of the Hill stability in the 3+ planet case and the mechanism leading to instability has a priori no reason to be related to the Hill scaling \( R_H \). The discrepancy between the two proposed mass renormalizations Eqs. (3) and (4) is easily explained by the fact that most studies only considered a limited mass range and very small difference between the exponents. Zhou et al. (2007) estimated \( T_{\text{surv}} \) as a power-law in the spacing and using Nekhoroshev estimates, Yalinewich & Petrovich (2020) proposed a scaling similar to Eq. (4).

2.3. Phenomenology of the instability

These qualitative and quantitative studies on EMS systems have permitted to highlight the key features that the tightly packed system instability presents and that an analytical model should explain.

a. The survival time \( T_{\text{surv}} \) seems to have an exponential dependence in the orbital spacing, measured in units of \( \Delta \):
The survival time distribution suggests that the evolution is driven by a diffusion process (Hussain & Tamayo 2020). The dips close to first order two-planet MMRs indicate that they play a fundamental role in enabling the orbit crossing.

While the stability of EMS systems has been described extensively from numerical simulations, very few works have developed an analytical framework attempting to describe the observed behaviour. In the most elaborate model, Quillen (2011) proposed that the instability is driven by the overlap of zeroth-order three-planet MMRs. Resonances involving more than two planets emerge as the result of the first order averaging (e.g. Chapter 2, Morbidelli 2002, see also section 3) and are weaker than the two-planet MMR. Quillen (2011) shows that, despite of their smaller width, the three-planet MMR are more numerous and overlap at larger spacing and smaller eccentricities. The ansatz is that the planet semi-major axes evolve randomly through the rich network of these three-planet MMR, until a first order two planet MMR is encountered, leading to a rapid AMD increase and, shortly after, close encounters and collision. Moreover, the main resonances close to circular orbit preserve the total AMD (see section 4), which is consistent with simulations.

To illustrate the mechanism leading to instability, we perform the numerical integration of a typical EMS system. The planets have a mass $m_p = 10^{-3} \, M_\odot$ and orbit a solar-mass star. The inner orbit is at 1 au and the period ratios between adjacent planets is initially close$^2$ to $\nu_{k+1}/\nu_k = 1.175$. This particular value was chosen in order to observe the instability after roughly a few million inner planet’s orbits while being outside of a two planet MMR island. The orbits are initially circular and coplanar and the angles drawn randomly. As in previous studies, we run the simulations up to the first close encounter. In the considered case the integration lasts 3.33 Myr. The system is integrated with the hybrid integrator MERCURIUS (Rein et al. 2019) from the REBOUND code (Rein & Liu 2012) with a timestep of 0.01 yr.

We plot in Figure 1, the evolution of the semi-major axis of the three planets. The envelope around the curve corresponds to the extent of the orbits $i.e.$ the position of the periapses and apoapses, and is thus a measure of the orbits’ eccentricities. The curves are smoothed by performing a rolling averaging over the next 10 snapshots. The vertical axis is discontinuous to highlight the small variations during the large majority of the integration. As already described by previous authors, the system appears quiescent for the majority of its life time. Then after the pair 2-3 crosses the 7:6 resonance, the system becomes unstable in 127 kyr. This figure emphasizes the timescale difference between the lifetime of the system, and the proper unstable phase that is almost two orders of magnitude shorter. Explaining the lifetime of tightly packed systems should thus focus on the quiescent phase as the timescale to reach instability is dominated by this phase.

To show the rapid change of behaviour before the close encounter, we plot in Figure 2a, the evolution of the two adjacent period ratios $P_2/P_{k+1}$ as a function of the time to the close encounter (note the logarithmic scale). In Figure 2b, we plot the evolution of the system’s AMD $C$ (see eq. 7), rescaled by the total angular momentum $G$. The plotted quantity, $\sqrt{C/G}$ scales linearly with eccentricity for close to circular orbits. At the moment the pair 2-3 enters the 7:6 MMR region, the system enters the scattering phase. It is also the moment where the AMD starts to increase. Nevertheless, the initial phase is not secular, despite the almost conservation of the AMD, indeed we see that the period ratios are not constant but evolves over a long timescale.

The interaction and the position of the system with respect to the network of two planet MMR seems critical to the duration of the quiescent phase. However, Fig. 2 merely shows how the instability is triggered and not the mechanism leading to it. The slow evolution of the system is seen much more explicitly in the period ratio plane plotted in Figure 3. We plot the period ratio of the outer pair $P_2/P_3$ as a function of the inner pair period ratio $P_1/P_2$. In

\[^2\text{The initial period ratios are not rigorously equal in this specific example.}\]
First 3 Myr

This plane, the two planet MMR are vertical and horizontal dashed lines. We plot the neighbouring first order MMR i.e. the 7:6 and the 6:5 in black, indicating their approximate extent for circular orbit in grey. The second order resonance 13:11 is plotted in red, but it has a null width for circular orbits. We see the system starts outside of the two planet MMR. However, on top of the two-planet MMR network, there exist also the network of three-planet MMR. For circular orbits, the main three planet MMR are of zeroth order (see section 4.1). We plot the loci of the largest 3 planet MMR in the vicinity of the initial condition: this network is composed of a set of close to parallel lines which run transversally to the two-planet MMR lines. As predicted by Chirikov (1979) theory, the diffusion takes place perpendicularly to the network of the three-planet MMRs, up until the system reaches the two planet resonances where the trajectory wanders around rapidly.

This qualitative analysis seems to confirm Quillen’s hypothesis. The survival time is dominated by the diffusion along the three planet MMR network and the system becomes unstable once it reaches the two planet resonance where chaotic diffusion is faster and increases rapidly the total AMD. The survival time can be estimated by computing the diffusion rate according to Chirikov’s resonance overlap theory (see section 5.1). The scaling law for the survival time obtained by Quillen (2011) is a very steep power-law instead of the exponential behavior. In particular, the timescale is overestimated at short separations and underestimated for large ones. Quillen’s result and its difference with numerical simulations can be explained by some simplifications made in the computations, leading to an inexact determination of the effective diffusion rate as well as a limit of the three-planet MMR overlap.

In this study, we consider the general circular coplanar three planet problem. We remain in the framework of tightly packed systems but we relax the assumption on the initial equal spacing and equal masses. We show that it is possible to use Chirikov’s theory to explain the observed survival time scaling.

3. Problem considered and mean motion resonances (MMR)

We summarize most of the notations into table A.1. We consider a system of three planets of masses $m_1, m_2$ and $m_3$ orbiting a star of mass $m_0$. The canonical positions $r_j$ and momenta $\mathbf{p}_j$ are expressed in canonical heliocentric coordinates (Poincaré 1905; Laskar 1991). The initial orbits are assumed to be circular and coplanar. Let the semi-major axes $a_j$, the eccentricities $e_j$, the mean longitudes $\lambda_j$, and the periapses longitude $\sigma_j$ be the orbital elements defining the orbits. A set of canonical coordinates for the system is given by the modified Delaunay coordinates (e.g. Laskar 1991)

$$\Lambda_j = m_j \sqrt{\mu} a_j, \quad C_j = \Lambda_j \left(1 - \sqrt{1 - e_j^2}\right), \quad -\alpha_j,$$

where $\mu = \frac{g m_0}{G}$ and $G$ is the gravitational constant. Note that the gravitational parameter $\mu$ is the same for all three
planets as in Laskar & Petit (2017). This is possible if we consider the so-called democratic-heliocentric formulation of the planetary Hamiltonian (e.g. Morbidelli 2002). The couples of variables $(C_j, -σ_j)$ can also be replaced by their associated complex variables

$$x_j = \sqrt{C_j} e^{iσ_j},$$

with $i = \sqrt{-1}$ ($x_j$ are the canonical momenta and $-i\dot{x}_j$ the conjugated positions). For small eccentricities, we have

$$x_j = \sqrt{\Lambda_j/2ε} e^{iε}.\text{ The system total angular momentum } G \text{ and AMD } C \text{ are given by}$$

$$G = \sum_{j=1}^{3}(Λ_j - C_j) \quad \text{and} \quad C = \sum_{j=1}^{3} C_j. \quad (7)$$

The Hamiltonian $H$ describing the dynamics can be split into an integrable part

$$\mathcal{H}_0 = \frac{1}{2}\sum_{j=1}^{3} ||\vec{r}_j||^2 - \frac{\mu_m}{r_j} = -\sum_{j=1}^{3} \frac{\mu^2 m_j^3}{2Λ_j}, \quad (8)$$

describing the motion on unperturbed Keplerian orbits and a perturbation

$$\varepsilon \mathcal{H}_1 = \sum_{j=1}^{3} \sum_{j' \neq j} \frac{G m_j m_{j'}}{r_j - r_{j'}} + \frac{1}{2m_0} \left( \sum_{j=1}^{3} \vec{r}_j \right)^2 \quad (9)$$

describing the planet interactions. $ε$ is a dimensionless parameter of the order of the planet to star mass ratio to remind the scale difference between the two parts of the Hamiltonian. In terms of Poincaré coordinates, the perturbation part can be written as

$$\varepsilon \mathcal{H}_1 = \sum_{k,l} C_{k,l}(m_j, Λ_j) \left( \prod_{j=1}^{3} x_j^{j,l} \right) e^{iλ_j}, \quad (10)$$

where $k, l \in \mathbb{Z}^3, 1, l \in \mathbb{N}^3.$

Due to the conservation of angular momentum, the coefficient $C_{k,l}$ must vanish unless the indices $k,l, l \in \mathbb{N}^3.$

In the unperturbed case, the system is said to be in a mean motion resonance (MMR) if the mean motions

$$n_j = λ_j = \frac{\partial H_0}{\partial Λ_j} = \frac{μ^2 m_j^3}{Λ_j}, \quad (12)$$

verify an equation of the form

$$\sum_{j=1}^{3} k_j n_j = 0. \quad (13)$$

The sum $k = k_1 + k_2 + k_3$ is called the "order" of the resonance. The sum $K = |k_1| + |k_2| + |k_3|$ is called the "index" of the resonance. In the general case, the perturbation $\varepsilon \mathcal{H}_1$ also influences the resonant dynamics. The terms in eq. (10) contributing to the resonance are the ones that depend on the combination of mean longitudes $κ_1Λ_1 + κ_2Λ_2 + κ_3Λ_3.$ Because of d'Alembert rules, the leading order term in the perturbation is of order $k$ in eccentricities ($k$ being the resonance order).

Note that because each term in eq. (9) only contains contributions from two planets, this is also the case for (10). In particular, there are no terms in the non-averaged Hamiltonian $H = H_0 + ε \mathcal{H}_1$ that depend on angles of the form $κ \cdot Λ$ with $κ \neq 0$ for all $j.$ It results that there are no three planet resonances at the first order in $ε.$ There are instead of course $O(ε)$ two-planet MMR terms.

Three-planet MMR actually emerge in the perturbative Hamiltonian as $O(ε^3)$ terms which appear after applying a perturbation step which eliminates the fast angles $Λ$ to first order in $ε$ (this step is sometimes called "averaging", because to first order in $ε$ it is equivalent to averaging out the fast angles from the Hamiltonian). To do so, we assume that the system is "far enough" to the first order two planet MMR i.e. we assume that $κ_n = (k + 1)n_j$ is not too small with respect to $ε$ for some integer $k$ (Morbidelli 2002). This is for example the case for the system considered in the previous section. We sketch the main lines of these perturbative steps below, and we carry out the explicit calculation of the relevant terms in the next section.

Since we consider systems far enough from two planet resonances, we can perform one perturbation step and keep track of all terms up to order $O(ε^2)$ in the equations. We use the classical approach from the perturbations theory, the Lie series method (Deprit 1969). We refer to Morbidelli (2002), section 2.2 and references therein for a complete description of the method. Note that the method has already been applied to provide an analytical model of three body resonances when one of the body is a test particle (Nesvorný & Morbidelli 1998). The idea is to introduce a new set of variables (noted with a prime in the following equations), $ε$-close to the original ones, such that, in these new variables, the transformed Hamiltonian writes

$$\mathcal{H}' = H_0(Λ') + ε \mathcal{H}_1(Λ', Χ') + ε^2 \mathcal{H}_2(Λ', Χ', Χ) + O(ε^3), \quad (14)$$

where $ε \mathcal{H}_1$ is the average of $ε \mathcal{H}_1$ over the mean longitudes $λ_j$ and $ε^2 \mathcal{H}_2$ is the leading order term of a series in $ε.$ The transformation can be explicitly constructed as the flow between 0 and 1 of a generating Hamiltonian vector field $\text{exp}(ε\{\mathcal{H}_1, \cdot\})$ where $\{\cdot, \cdot\}$ is the Poisson bracket and $ε \mathcal{H}_1$ is the solution of the homological equation

$$[ε \mathcal{H}_1, H_0] + ε \mathcal{H}_1 = ε \mathcal{H}_1, \quad (15)$$

More precisely, if we note $h^{(k)} = h^{(k)}(Χ, Χ)$ the complex Fourier coefficients of $\mathcal{H}_1$ with respect to the mean longitudes, we can write

$$ε \mathcal{H}_1 = ε \sum_{k \neq 0} h^{(k)}(k \cdot n) e^{iλ j}. \quad (16)$$

We simplify the general case because the first order MMR are the only one with a non zero width for close to circular orbits. In general, we should require that $κ_n j_i + κ n_j j $ is small with respect to the largest coefficient corresponding to this particular order in the sum eq. (10). But because of their dependence in eccentricity, such a coefficient is negligible for higher order resonances.

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4 We use the convention $(f, g) = \sum_j \left( \frac{∂f}{∂p_j} \frac{∂g}{∂q_j} - \frac{∂f}{∂q_j} \frac{∂g}{∂p_j} \right)$ where $(p, q)$ is a set of conjugated coordinates.
Due to the expression of $\epsilon^2 H_1$ given in eq. (10), the denominators $k \cdot n$ are of the form $k_i n_j + k_j n_i$ and are "not too small" because we assumed the system to be far from two planet MMRs. Thus the formal series (16) is formally well defined; one can stop the summation at indices $k$ of sufficiently high order so that the remaining Fourier terms in $H_1$ have size smaller than $\epsilon^2$, which is ensured by the exponential decay of the Fourier coefficients. Thus, the solution (16) to the homological equation (15) is well defined. We can express the Hamiltonian $\epsilon^2 H_2$ as

$$
\epsilon^2 H_2 = \frac{1}{2} \{ \epsilon H_1 + \epsilon^2 H_1 \}. 
$$

The Poisson bracket in eq. (17) generates terms involving all three mean longitudes. In other words, three planets MMR that were not present in the initial Hamiltonian cannot be neglected at second order in averaging. The study of a particular three planet MMR can be done by a second averaging over the other fast angles, since all other terms will not contribute small divisors and can thus be eliminated by an additional perturbative step. In practice, this results in another change of coordinates, which are $\epsilon^2$ close to the first order averaged coordinates, and the new Hamiltonian is the average of $H'$ with respect to the fast (e.g. non resonant) angles (see next section). Because we do not need to change back to the initial coordinates, we will from now on drop the primes on the coordinates and Hamiltonian. We also drop the terms of order $\epsilon^3$ and more.

4. The three-planet, zeroth order, resonance network

In figure 3, we see that the diffusion mainly occurs perpendicularly to the zeroth order, three planet MMRs. This is expected for close to circular orbits since the resonant coefficients do not depend on eccentricity at the leading order in eccentricity. Besides, the structure of the network is easier to describe. We make the hypothesis that the zeroth order three planet MMRs are sufficient to explain most of the diffusion leading to the instability. This assumption is well supported in section 6, where we compare the analytical prediction of survival times calculated under this hypothesis with the results of numerical simulations. We analyze these MMRs and compute an overlap criterion in this section. We will consider the role of additional MMRs in section 7.

4.1. Network description

A zeroth order three planet MMR can be described by two integers $p$ and $q$. The resonance equation is

$$
pn_1 = (p + q)n_2 + qn_3 = 0. 
$$

Since such resonance does not depends on the longitude of the periapses, the AMD is unaffected by the resonant terms (see below). We thus restrict ourselves to the three\(^5\) degrees of freedom ($\lambda_1, \lambda_2, \lambda_3$). The resonance equation defines a plane in the frequency space ($n_1, n_2, n_3$). Because the gravitational interactions are scale invariant, we can restrict ourselves to

\(^5\) Each couple of conjugated variables counts for one degree of freedom.
as it lies between 0 and 1. One can remark that \( p/(p + q) \) can be extended as a continuous function in the period ratio plane. An adapted set of coordinates to describe the period ratio plane can be defined \(^6\) to take advantage of this property. We define the resonance locator

\[
\eta = \frac{1 - \nu_{23}}{\nu_{12} - \nu_{23}} = \frac{\nu_{12}(1 - \nu_{23})}{1 - \nu_{12}\nu_{23}} = \frac{p}{p + q}.
\]

(21)

The second equality is only valid on resonances. The position along the resonance can be defined by a generalized period ratio separation

\[
\nu = \frac{(1 - \nu_{23})(1 - \nu_{23})}{1 - \nu_{12}\nu_{23}} = \frac{p}{p + q} (\nu_{12} - 1) = p/\nu_{23} (1 - \nu_{23}).
\]

(22)

\( \eta \) is a constant on a specific resonance whereas \( \nu \) is a hyperbola along which the resonance strength is roughly comparable.

The variables \((\nu, \eta)\) are well adapted to describe the dynamics governed by the three planet MMRs. We can express the period ratios as a function of these variables

\[
\nu_{12} = \frac{\eta}{\eta + \nu}, \quad \text{and} \quad \nu_{23} = \frac{1 - \eta - \nu}{1 - \eta}.
\]

(23)

The levels of constant \( \nu \) are hyperbola with horizontal and vertical asymptotes \((1 + \nu)^{-1}\). We represent on Fig. 4, the curve \( \nu = 0.05 \) in orange.

### 4.2. Single zeroth order three-planet MMR Hamiltonian

Let us consider a specific resonance described by the integers \( p \) and \( q \). One can make a linear change of variables to use explicitly the resonant angle and average over the non-resonant ones. Let us define

\[
\theta_{\text{co}} = p\lambda_1 - (p + q)\lambda_2 + q\lambda_3,
\]

\[
\theta_{\Gamma} = \lambda_2 - \lambda_3,
\]

\[
\theta_\gamma = \lambda_3.
\]

(24)

The conjugated momenta are

\[
\Theta = \frac{1}{p}\Lambda_1,
\]

\[
\Gamma = \frac{p + q}{p}\Lambda_1 + \Lambda_2,
\]

\[
\mathbf{T} = \Lambda_1 + \Lambda_2 + \Lambda_3.
\]

(25)

We call \( \Gamma \) the scaling parameter by analogy with the two planet case (Michtchenko et al. 2008). Here \( \mathbf{T} \) is the circular and coplanar angular momentum and verifies \( \mathbf{T} = \mathbf{G} + \mathbf{C} \).

Using the method described in section 3, we can do a formal second order averaging over \( \theta_{\Gamma} \) and \( \theta_\gamma \) because these angles are not resonant. The Hamiltonian takes the form

\[
\mathcal{H} = \mathcal{H}_0(\Theta) + \varepsilon^2\mathcal{H}_1(\Theta, \theta_{\text{res}}) + \varepsilon^3\mathcal{H}_{2,\text{res}}(\Theta, \theta_{\text{res}}) + O(\varepsilon^3)
\]

(26)

where \( \varepsilon^2\mathcal{H}_{2,\text{res}}(\Theta, \theta_{\text{res}}) \) is the Hamiltonian of eq. (17) averaged over \( \theta_{\Gamma} \) and \( \theta_\gamma \), \( \Theta \) represents all the actions defined in (25). In eq. (26), we can neglect \( \varepsilon\mathcal{H}_1(\Theta) \) as it is small with respect to \( \mathcal{H}_0 \) and only accounts for a correction of the mean motions of order \( \varepsilon \). It should be noted that \( \Gamma \) and \( \mathbf{T} \) are integrals of motion of (26). As a result, the Hamiltonian only has one degree of freedom and is integrable. Another consequence of the conservation of \( \mathbf{T} \) is that the zeroth order three planet MMRs preserve the system AMD. In particular, if a system is only affected by these resonances, initially circular orbits will remain circular. As such behaviour is observed in numerical simulations before the late instability, this result confirms the decisive role of zeroth order three planet MMR in driving the instability.

We consider small variations of the actions around the resonance. Let us denote \( \Theta = \Theta_0 + \varepsilon \theta \) where \( \Theta_0 \) corresponds to the value of \( \Theta \) such that the system is on the resonance curve (20). Similarly, we have \( \Lambda_k = \Lambda_{k,0} + \varepsilon \Lambda_k \). We have

\[
d\Lambda_1 = p\,d\Theta,
\]

\[
d\Lambda_2 = -(p + q)d\Theta,
\]

\[
d\Lambda_3 = q\,d\Theta.
\]

(27)

So the inner and outer planet are moving on the same direction while the middle planet is moving in the opposite. At first order, we can express the change in the period ratio \( d\nu_{12} \) and \( d\nu_{23} \) as a function of \( d\Theta \).

\[
d\nu_{12} = 3\nu_{12,0} \left( \frac{p}{\Lambda_{1,0}} + \frac{p + q}{\Lambda_{2,0}} \right) d\Theta,
\]

\[
d\nu_{23} = -3\nu_{23,0} \left( \frac{p}{\Lambda_{2,0}} + \frac{q}{\Lambda_{3,0}} \right) d\Theta.
\]

(28)

We can take the ratio of the small variations \( d\nu_{23} \) and \( d\nu_{12} \) and using eq. (20) to replace \( p \) and \( q \), we obtain the differential equation

\[
\frac{d\nu_{23}}{d\nu_{12}} = \frac{\frac{\nu_{23}}{\nu_{12}} - \frac{m_1\nu_{12}^{-1/3}(1 - \nu_{12})}{m_2(1 - \nu_{23}) + m_3\nu_{12}^{-1/3}(1 - \nu_{12})}}{\nu_{12}}.
\]

(29)

that gives the direction of the change of period ratios anywhere in the plane \((\nu_{12}, \nu_{23})\) due to the neighbouring resonance. Note that the equation no longer depends explicitly on the integers \( p \) and \( q \). Indeed, while the strength of each resonance depends on the resonance index \( 2(p + q) \) (see the next section), the resonant motion direction can be extended as a continuous function of the period ratios using the resonance locus equation (20).

The solution of eq. (29) gives the direction of the Chirikov diffusion if the system dynamics were entirely governed by the zeroth order three planet MMRs. The differential equation can be integrated numerically given an initial condition, and the solution for the system studied in section 2 is displayed in orange on Fig. 3. We see that for the majority of the system’s lifetime, the system follows the diffusion direction. The system leaves it as soon as the dynamics are no longer driven by the three planet MMR network. Note that the problem has been reduced to study the diffusion along one dimensional curve.

### 4.3. Explicit size of the resonance width

From the previous section, we know that the dynamics of a single zeroth three planet resonance is integrable. Provided that these resonances overlap, we also have seen along which curve the motion should take place. However, there is no guarantee that the neighbouring three planet MMR interact. If the resonances are well isolated because their width

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\(^6\) These coordinates are not canonical, they are nevertheless convenient to describe the period ratio plane.

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is smaller than their separation, there is no possibility for large scale chaos. In this case, a system could be influenced by a single resonance and never jump to the other ones. The system will be almost secular and in principle could be considered as long term stable. Moreover, the diffusion timescale is linked to the resonance strength. It is also possible that while the resonances are overlapped, the diffusion along the network is so slow that it is meaningless for astrophysical applications.

One thus needs to study in detail the dynamics to evaluate the strength of the three planet resonances. We therefore carry out in this section a detailed and quantitative derivation of the perturbative steps described above, keeping track of all the relevant terms which contribute to three-planet MMRs. We limit ourselves to a leading order computation. As a result, we will only keep the terms that do not depend on the eccentricity in the final expression. We also neglect the indirect term of the perturbing Hamiltonian \( e_i H_1 \) as its value only affects the resonances when either \( p \) or \( q \) is equal to 1. It is instead necessary to keep terms to first order in eccentricity because they contribute to terms independent of the eccentricity at the second order in mass.

The terms of the perturbing Hamiltonian \( e_i H_1 \) that we consider are thus (Laskar & Robutel 1995; Murray & Dermott 1999)

\[
e_i H_1 = \sum_{12 < i \leq j < k} \sum_{12 \neq l} W_{ij}^l e^{i(l_i - l_j)}
\]

\[
+ \sum_{12 < i \leq j < k \geq 12 \neq l} (V_{ij, l} + V_{ij, l'}) e^{i(l_i - l_j) + c.c.},
\]

where \( c.c. \) designs the complex conjugate of the second sum, and

\[
W_{ij}^l = -\frac{m_i n_j A_i}{2 m_0} b_{ij}^{(l)}(\alpha_{ij}),
\]

\[
V_{ij, l}^q = \frac{m_i n_j A_j}{2 m_0} \frac{2}{\lambda_i} (l + 1 + \frac{a_{ij}}{2} \frac{\partial}{\partial \alpha_j}) b_{ij}^{(l)}(\alpha_{ij}),
\]

\[
V_{ij, l'}^q = \frac{m_i n_j A_i}{2 m_0} \frac{2}{\lambda_j} (l + 1 + \frac{a_{ij}}{2} \frac{\partial}{\partial \alpha_i}) b_{ij}^{(l)}(\alpha_{ij}),
\]

where \( a_{ij} = a_i / a_j \) and \( b_{ij}^{(l)}(\alpha) \) are the Laplace coefficients. We refer to appendix B for a definition and study of the Laplace coefficients and how to approximate them. Here we note that Quillen (2011) used a simplified approximation that presents the interest to be easy to manipulate

\[
b_{ij}^{(l)}(\alpha) \approx \frac{2 \alpha_i^{l_i}}{\pi \|l\| (1 - \alpha^2)}.
\]

In the limit of \( \alpha \) very close to 1, the asymptotic equivalent of the Laplace coefficients does not depend on \( l \) (Laskar & Robutel 1995). However, we find that for a fixed \( \alpha \), for large \( l \), the Laplace coefficient is asymptotic to

\[
b_{ij}^{(l)}(\alpha) \approx \frac{2 \alpha_i^{l_i}}{\pi \|l\| (1 - \alpha^2)}.
\]

In the close planet approximation, \( \alpha \to 1^- \), we get

\[
b_{ij}^{(l)}(\alpha) \approx \sqrt{\frac{2}{\pi \|l\| (1 - \alpha^2)}} e^{l (1 - \alpha)}.
\]

We refer to the appendix B for a detailed discussion.

The exponential dependency on \(-l\|l\| (1 - \alpha)\) of the Laplace coefficients has two consequences for \( W_{ij}^l \). The interactions between planet 1 and 3 are always negligible with respect to the interaction in adjacent pairs i.e. for a given \( l \), \( |W_{ij}^l| \ll |W_{ij}^{l+1}| \). Similarly, for a given resonance, higher order terms in the resonant angle such as \( e^{l N_k} \) for \( N > 1 \) are always negligible.

With this clarification, let us go back to the calculation of the perturbative steps, starting from the perturbative Hamiltonian (30). The solution \( \xi_1 \) to the corresponding homological equation has for expression

\[
\xi_1 = \sum_{1 \leq i < j < k \leq 3} \frac{W_{ij}^l}{i(l_i - n_j)} e^{i(l_i - l_j)}
\]

\[
+ \sum_{1 \leq i < j < k \geq 1 \leq l \geq 12} \left( V_{ij, l} + V_{ij, l'} \right) e^{i(l_i - l_j) + c.c.},
\]

where \( Z^* = Z \setminus \{0\} \). As explained schematically in the previous section, this perturbation step produces \( O(\varepsilon^2) \) terms, which we now calculate explicitly. In essence, we must only calculate the term \( e^{i \hat{H}^2} \) in (26). Since we would subsequently average over \( \theta_2 \) and \( \theta_1 \), the only terms that remain in \( e^{i \hat{H}^2} \) must depend on the angle \((p \lambda_1 - (p + q) \lambda_2 + q \lambda_3)\) or its opposite. Because of the form of \( \xi_1 \) and \( e_i H_1 \), the only terms contributing at zeroth order in eccentricity to the averaged Hamiltonian \( e \hat{H}^2 \), are of the form

\[
\left\{ \frac{W_{i2}^l}{i(p_1 - n_2)} e^{i(p_1 - l_2)}, \frac{W_{i2}^q}{i(p_1 - q_2)} e^{i(p_1 - q_2)} \right\}.
\]

or their complex conjugates. Note that in all the considered terms, the Poisson bracket can be reduced to the derivations with respect to the orbital elements \( A_2, A_3 \) and \( x_2, x_3 \) of planet 2 only, as they are the only ones to appear on both sides. For the two last terms, we only keep the terms where the eccentricity dependency is dropped due to the derivation operator \( \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x_3} = 0 \). It results that

\[
e^{i \hat{H}^2} = e^{i R_{pq} \cos(\theta_{pq})},
\]

where

\[
e^{i R_{pq}} = -\left( \frac{q}{p n_1 - n_2} + \frac{1}{n_1 - n_2} \right) \frac{\partial W_{i2}^p}{\partial A_2} \frac{1}{W_{i2}^q} + \left( \frac{1}{n_1 - n_2} + \frac{p}{q(n_2 - n_3)} \right) W_{i2}^p \frac{\partial W_{i2}^q}{\partial A_2} - \left( \frac{1}{n_1 - n_2} + \frac{p}{q(n_2 - n_3)} \right) W_{i2}^p \frac{\partial W_{i2}^q}{\partial A_2} + \left( \frac{1}{n_1 - n_2} + \frac{p}{q(n_2 - n_3)} \right) W_{i2}^p \frac{\partial W_{i2}^q}{\partial A_2}.
\]
Since $\varepsilon^2 \overline{\mathcal{H}}_2$ is small with respect to the Keplerian part, we evaluate the action at the nominal resonance value, that is, $m_1 - (p + q)m_2 + qn_3 = 0$. Using eqs. (18), (19), (31), (32), (33) as well as
\[
\frac{\partial \alpha_{12}}{\partial \Lambda_2} = -2 \frac{a_{12}}{\Lambda_2}, \quad \frac{\partial \alpha_{23}}{\partial \Lambda_2} = 2 \frac{a_{23}}{\Lambda_2} \quad \text{and} \quad \frac{\partial \alpha_2}{\partial \Lambda_2} = -3 \frac{n_3}{\Lambda_2},
\]
we can express $\varepsilon^2 R_{pq}$ as
\[
\varepsilon^2 R_{pq} = \frac{m_1 m_2 n_2 \Lambda_2}{m_0^2 v} \left[ \left( 1 - \eta \right) b_{1/2}^{(p)} \left( 1 + \alpha_{12} \frac{\partial}{\partial \alpha} \right) b_{1/2}^{(p)} + \eta \alpha_{23} b_{1/2}^{(q)} \frac{\partial b_{1/2}^{(q)}}{\partial \alpha} \right] + \frac{3 \eta (1 - \eta)}{2 v} \left[ b_{1/2}^{(p)} b_{1/2}^{(q)} \frac{\partial b_{1/2}^{(q)}}{\partial \alpha} \right] + \frac{m_1 m_2 n_2 \Lambda_2}{m_0^2 (v - (p + q)^{-1})} \left[ \left( 1 - \eta \right) b_{1/2}^{(p)} \left( 1 + \alpha_{12} \frac{\partial}{\partial \alpha} \right) b_{1/2}^{(p)} + \frac{(\eta + 1)}{4 (p + q)} \left( \alpha_{23} b_{1/2}^{(q)} \frac{\partial b_{1/2}^{(q)}}{\partial \alpha} \right) + \frac{\alpha_{12} \alpha_{23}}{4 (p + q)} \frac{\partial b_{1/2}^{(q)}}{\partial \alpha} \right] + \eta (1 - \eta) (p + q) b_{1/2}^{(p)} b_{1/2}^{(q)} \frac{\partial b_{1/2}^{(q)}}{\partial \alpha},
\]
the Laplace coefficients depending on $p$ (resp. $q$) are evaluated at $\alpha_{12}$ (resp. $\alpha_{23}$).

In this expression, the second prefactor can go to infinity for $v = 1/(p + q)$. For the resonance defined by $p$ and $q$, this happens at the intersection of the 2 planet MMRs $\nu_{12} = p/(p + 1)$ and $\nu_{23} = (q - 1)/q$. This result can be interpreted as the fact the second order averaging is not valid very close to a two planet MMR. Since we primarily focus on the regions outside of two planet MMR, we ignore this feature in the following developments. Moreover, for large $p + q$, the MMR intersections are within the region of two planet MMR overlap.

Under the assumptions made so far, the above expression is exact, and can be used for numerical explorations of the size and typical frequency of each three planet resonance without impediment (see below). To obtain further analytical insight, it is however quite cumbersome, and it does not clearly show which parameters of the planetary system and of the resonance play a role in determining the properties of the resonant motion. We thus aim at a simplification of the above expression, keeping always in mind that we are ultimately interested in the diffusion in period ratio space driven by these three planet resonances, and specifically in the timescale that is needed for large-scale diffusion. It is expected that the resonances with highest index dominate this timescale (see also below); in the remainder of this section we thus take the limit $1/(p + q) \to 0$ and expand around this value. We note moreover that for $1/(p + q) \to 0$, the second term in (42) blows up when $\nu \to 0$, in which case also the first term would go to infinity; however $\nu \to 0$ only happens when one of the period ratios $\nu_{i,j+1} \approx 1$: this limit is beyond the scope of the study, so we can exclude this case.

With these considerations in mind, the above expression can be considerably simplified. To this end, we make the close planet approximation: $1 - \alpha_{ij} \ll 1$. We define
\[
\delta_{ij} = 1 - \alpha_{ij} \approx \frac{2}{3} (1 - \nu_{ij}),
\]
which is an excellent estimate for period ratios from 0.5 to 1.

The product of the Laplace coefficients and their derivatives in $R_{pq}$ introduces an exponential factor of the form $e^{-\delta_{12}^2 - \delta_{23}^2}$ (cf. eq. 36) that sets the order of magnitude of the resonance term. We can thus simplify expression (42) by taking advantage of the resonance relationship. Indeed, for tightly packed systems, and in the vicinity of a resonance defined by $p$ and $q$, eq. (20) can be transformed into a relationship on the planet spacings
\[
\rho \delta_{12} \approx q \delta_{23},
\]
by analogy with the generalized period ratio separation $\nu$, we define a generalized orbital spacing that we note
\[
\delta = \frac{\delta_{12} \delta_{23}}{\delta_{12} + \delta_{23}} \approx \frac{p}{p + q} \delta_{12} \approx \frac{q}{p + q} \delta_{23},
\]
where the two last equalities are approximations using eq. (44). We have
\[
\nu \approx \frac{3}{2} \delta.
\]
Using the newly defined variable $\delta$ and $\eta$, the expression of the coefficient $\varepsilon^2 R_{pq}$ can be simplified to
\[
\varepsilon^2 R_{pq} = \frac{m_1 m_2 n_2 \Lambda_2 \eta (1 - \eta)}{3 m_0^2} \left( 17 + \frac{21}{(p + q) \delta} \right) e^{-2(p+q)\delta},
\]
where we only keep the terms up to the first order in $(p + q) \delta$. Since the exponential factor depends on $(p + q) \delta$, the resonance mainly matters in the region where $(p + q) \delta$ is of order unity, hence we choose to set $17 + 21/(p + q) \delta) = 38$, which is approximately the value taken for $(p + q) \delta \approx 1$ as it allows us to carry the computations analytically. This value also give a more accurate estimate for the resonances width (see below).

The expression obtained in (47) for the three-planet resonance perturbation Hamiltonian is remarkable. The resonance strength only depends on its index and not explicitly on $p$ and $q$. It results that all resonances with the same index can be compared very easily. In other words, the network of zeroth order three planet MMR can be partitioned into subnetworks consisting of resonances with the same index.

To fully describe the resonant dynamics, we now go back to equation (26) (we recall that we can safely drop the term $\varepsilon^2 \mathcal{H}_4$); we now expand the Keplerian part around the resonance center (Chirikov 1979; Petit et al. 2017). This is more easily done in the original Delaunay variables $\mathbf{A}$, and we have
\[
\mathcal{H}_0 = \frac{3}{2} \sum_{j=1}^3 \left( - \frac{m_3}{2 \Lambda j} + n_{j0} (\Lambda_j - \Lambda_{0,j}) - \frac{3 n_{j0}}{2 \Lambda_{j0}} (\Lambda_j - \Lambda_{0,j})^2 \right),
\]
where the constant terms can be safely dropped. Using eq. (18) and eq. (27), the first order term vanishes and the coefficient of the second order term has for expression
\[
-\frac{K_2}{2} = -\frac{3 m_2}{2 \Lambda_2} (p + q)^2 \left[ 1 + \frac{m_2}{m_1} \frac{\eta^2}{\alpha_{12}^2} + \frac{m_2}{m_3} (1 - \eta)^2 \right].
\]
Note that $K_2$ only depends on the index of the resonance and weakly on the planet masses. Passing finally to the resonant canonical variables (24) and (25), the resonant Hamiltonian has for expression
\[
\mathcal{H}_{\text{res}} = -\frac{K_2}{2}(\Theta - \Theta_0)^2 + \varepsilon^2 R_{pq} \cos \theta_{\text{res}}.
\]
This is the standard pendulum Hamiltonian. The width of the resonance in the action space is given by the expression (e.g. Ferraz-Mello 2007)
\[
\Delta \Theta = 2\varepsilon \sqrt{\frac{2R_{pq}}{K_2}}.
\]
The small oscillations frequency is given by
\[
\omega_{pq} = \varepsilon \sqrt{K_2R_{pq}} = n_2 \varepsilon M A \frac{\sqrt{\eta(1-\eta)}}{\delta} (p+q)e^{-(p+q)\delta}.
\]
where $A = \sqrt{\frac{3 \pi}{5}} = 3.47$ and
\[
\varepsilon M = \sqrt{\frac{m_1 m_2}{m_0^2}} \left( 1 + \frac{m_2}{m_1} \frac{\eta^2}{\sigma_{12}^2} + \frac{m_2}{m_5} \sigma_{23}^2 (1-\eta)^2 \right)
\]
is the relevant mass ratio for the studied problem. For equal mass and equal tight spacing we have $\varepsilon M = \sqrt{3 \pi}$. The resonances have a clearer geometrical interpretation in the period ratio space than in the action space, particularly when one needs to compare them. We thus compute the width of the resonances perpendicularly to the network i.e., the width in term of the variable $\eta$. Using, eqs. (21) and (28) and some algebraic manipulation, we have
\[
\frac{d\eta}{d\Theta} = \frac{K_2}{n_2 (p+q)} \frac{\eta(1-\eta)}{\nu}.
\]
It results that the width in terms of $\eta$ can be estimated as
\[
\Delta \eta_{pq} = 4 \sqrt{2} \eta(1-\eta) \omega_{pq} = 6.55 \varepsilon M (\eta(1-\eta))^{3/2} e^{-(p+q)\delta}.
\]
We have thus shown that the width of the resonances in the period ratio plane depends exponentially on the MMR index and the prefactor is a continuous function of the period ratios. In particular, it seems important to compare resonances with the same index because of their similar geometry.

### 4.4. Resonance overlap

We wish to determine the sections of the period ratio space ($v_{12}, v_{23}$) where resonances overlap. Because of the expression of resonance width $\Delta \eta_{pq}$, we see that the width of the resonances close to a given point $(v_{12}, v_{23})$ mainly depends on the index $p+q$. It is thus natural to consider the density of the resonances for a fixed value of $p+q$.

We denote $\rho_k(\delta, \eta)$, the local filling factor of the zeroth order three-planet MMRs of index $k = p+q$, $\rho_k$ corresponds to the proportion of the period ratio space occupied by this subnetwork. Let us also define $\rho_{\text{tot}}(\delta, \eta)$, the filling factor of all zeroth order three-planet MMRs. The filling factor measures the space locally\(^8\) occupied by all the nearby resonances of arbitrary index with respect to the available space in the period ratio plane. If $\rho_{\text{tot}}$ is larger than 1, then there are enough resonances to locally cover the period ratio plane.

Such a filling factor is introduced by Quillen (2011) for the same problem. They, however, only consider resonances such that $|p-q| \approx 1$, which led to neglecting the exponential dependency. We show here that taking into account all the resonances is critical to obtain an accurate diffusion rate and survival time. The idea to count all the resonances without taking care of their precise position in order to obtain Chirikov’s overlap estimate was also used with success for two-planet MMR of arbitrary order (Hadden & Lithwick 2018).

We have $\rho_{\text{tot}} \leq \sum_k \rho_k$ since some resonances are counted multiple times. Indeed, if $p$ and $q$ are not coprime, the resonance lies on top of a resonance of lower index\(^9\). Nevertheless, the contribution of a resonance defined by two integers of the form $N_p, N_q$ is negligible with respect to the contribution of the resonance $p, q$ because of the exponential decrease. As a result, we will consider that the overall resonance filling factor $\rho_{\text{tot}}$ is the sum of the subnetwork filling factors $\rho_{pq}$.

Let us consider the subnetwork of resonances with index\(^10\) $k = p+q$. The distance between two resonances in terms of $\eta$ is constant. Indeed let us consider the resonance defined by integers $p$ and $q$, its upper neighbour is defined by the integers $p+1$ and $q-1$, hence
\[
\eta_{p+1, q-1} - \eta_{pq} = \frac{1}{p+q} = \frac{1}{k}.
\]
The filling factor $\rho_k$ for the subnetwork of resonances with index $k$ can be determined by taking the ratio of the resonance width in terms of $\eta$ with the distance between two neighbouring resonances in $\eta$, i.e.
\[
\rho_k = k \Delta \eta_{pq} = \varepsilon M (\eta(1-\eta))^{3/2} \delta ke^{-(p+q)\delta}.
\]
The filling factor $\rho_k$ thus depends on the subnetwork index $k$, the generalized orbit spacing $\delta$, the masses, and the resonant locator $\eta$.

We approximate the total resonance filling factor $\rho_{\text{tot}}$ as the sum of the subnetwork ones. We thus have
\[
\rho_{\text{tot}} = 6.55 \varepsilon M (\eta(1-\eta))^{3/2} \frac{\int_0^{+\infty} ke^{-ks} \, dk}{\delta^2}.
\]
\(^8\) Here and later, by *locally*, we mean a region large enough to contain resonances of different indexes such that the exact resonance positions is not relevant, but small enough such that ($\delta$ and $\eta$) do not vary too much. A good example is a rectangle delimited by adjacent two-planet first order MMRs.

\(^9\) More precisely, the index of the largest resonance is $(p+q)/\text{gcd}(p, q)$.

\(^10\) Technically, we called $2(p+q)$ the index of the resonance; however the relevant quantities depend here on $p+q$ rather than $2(p+q)$. We thus call here $p+q$ the index of the resonance for simplicity.
where we have replaced the sum by an integral. The computations are also possible using the infinite sums but they result in more a complicated expression with a very limited gain in accuracy. This approximation is also done by Quillen (2011).

As Quillen (2011), we find that the filling factor depends linearly on the mass ratio and scales as $\delta^2$. However, our expression is valid for an arbitrary spacing and mass distribution, as long as the system is tightly packed. We confirm that the natural spacing rescaling for the problem is not the Hill radius (2), which scales as $\epsilon^{1/3}$, but rather a dependence in $\epsilon^{1/4}$. In particular, assuming that $M$ and $\eta$ are constant, we can define a critical spacing value $\delta_{ov}$ such that the zeroth order three planet MMR network fills the entire space. Taking $\rho_{\text{tot}} = 1$ and solving for $\delta$, one obtains

$$\delta_{ov} = \epsilon^{1/4} M^{1/4}(6.55)^{1/4} (\eta(1 - \eta))^{3/8}. \quad (59)$$

$\delta_{ov}$ is a function of the masses and $\eta$. We can rewrite the filling factor $\rho_{\text{tot}}$ as a function of $\delta_{ov}$ as a power law over $\delta$

$$\rho_{\text{tot}} = \left(\frac{\delta}{\delta_{ov}}\right)^{-4}. \quad (60)$$

In the case of equal mass and spacing systems, eq. (59) becomes

$$\delta_{ov,\text{eq}} = 1.16 \left(\frac{m_p}{m_0}\right)^{1/4}. \quad (61)$$

Note that $\delta_{ov,\text{eq}}$ corresponds to the generalized spacing defined in eq. (45). The actual orbit space is equal to $2\delta_{ov,\text{eq}}$ in this case. The overlap criterion obtained by Quillen (2011) is similar to ours since the exponent 1/4 make the numerical factors almost equal.

We plot in Figure 5, the number of resonances that overlap at a given point in the plane $(\nu_{12}, \nu_{23})$ for three equal mass planets. The mass of each planet is $10^{-3} M_{\odot}$. The image is computed by creating a square grid of 2000 equally spaced period ratios between 0.7 and 1. For each resonance index between 2 and 200, we compute for each point the closest resonance in term of $\eta$. The closest resonance indicator is defined by $\eta_{\text{res}}$. The closest point on the resonance $(\nu_{12,\text{res}}, \nu_{23,\text{res}})$, is found by gradient descent for the function $\eta - \eta_{\text{res}}$. We then compute the width in term of $\eta$ using eq. (55) at the point $(\nu_{12,\text{res}}, \nu_{23,\text{res}})$ and compare it to the distance to the resonance $\eta - \eta_{\text{res}}$. We use the exact expression for $R_{pq}$ (42). We count the resonances with multiplicities, i.e. even if $p$ and $q$ are not coprime. We see at the vicinity of the 2 planet MMRs that the three planet resonances width increase due to the second term in eq. (62).

The number of resonances is to first order a proxy for the filling factor $\rho_{\text{tot}}$ (eq. 58). We see in Figure 5 that the region where the overlap of the three planet MMR network takes place extends well beyond the Hill stability limits (eq. 1), particularly for comparable spacings between the two neighbouring planet pairs. However, for very unequal spacings (away from the main lower-left to upper-right diagonal) we see that the overlap of only three planet MMR is not enough to account for the instability and the two-planet interactions should be taken into account for the initial diffusion process.

To quantify how far the overlapping region extends, we consider systems of equally spaced planets with equal masses $m_a$ and plot in Figure 6 the minimal spacing given by the Hill stability limit (Gladman 1993; Petit et al. 2018), the Wisdom (1980) two-planet MMR overlap criterion, and our three planet MMR overlap criterion (eq. 61) as a function of the mutual Hill radius (eq. 2). We see that for small masses, the three planet MMR overlap region goes to orbital spacing of the order 10 Hill radius, comparable to what was observed in previous numerical simulations. It is also worth noting that we are only considering a restricted part of the resonance network. Higher order three planet res-
The diffusion direction is perpendicular to the resonance in the action space. It results that the diffusion is not isotropic and to study the trajectory, one would need to compute the diffusion tensor at each point, due to each of the contributing resonances. If the resonance lines do not intersect in the considered region, the diffusion will be well approximated by an unidimensional random walk perpendicular to the resonance network, with a negligible diffusion parallel to the resonances. We can therefore consider a scalar diffusion coefficient, given by the specific resonance that dominates the dynamics around the position in the phase space. In particular, the diffusion coefficient is not constant and depends on the closest resonance width. Such a diffusion corresponds to the behaviour observed in Fig. 3.

In section 4.4, we compute an overlap criterion taking into account every zeroth order three planet MMR. However, as the resonance index increases, the associated diffusion rate vanishes, such that in the limit where the diffusion is dominated by smaller and smaller resonances, the timescales effectively tend to infinity. However, for \( \delta < \delta_{ov} \), not all the resonances are necessary to cover the phase space. We therefore only need to consider the largest ones to compute the survival time.

### 5.2. Partial resonance overlap

We consider a small region around a point \((\delta, \eta)\) where the zeroth order three planet MMR network is locally overlapped. Since \(\rho_{bat} > 1\), not all the resonances are necessary to cover the phase space. The largest resonances lead to the fastest diffusion, hence, we need to only consider the subset of the widest resonances needed to cover the period ratio space in this given region. As the distance to reach a two-planet MMR is small, the main difference between the size of the resonances is governed by their index. We thus define an overlap index \(k_{ov}\) such that the subnetwork composed with the three planet MMR with index \(k\) smaller than \(k_{ov}\) locally cover the space. Using eqs. (57) and (60) we have

\[
\frac{\delta_{ov}^4}{\delta^2} \int_0^{k_{ov}} k e^{4\eta} dk = \left(\frac{\delta_{ov}}{\delta}\right)^4 \left[1 - (k_{ov} \delta + 1) e^{-k_{ov} \delta}\right] = 1. \tag{63}
\]

We thus have an implicit definition of \(k_{ov}\). We can also remark that the equation depends on \(k_{ov} \delta\) rather than \(k_{ov}\) alone. As a result, we define the variable

\[
\xi_{ov} = k_{ov} \delta, \tag{64}
\]

that is a function of \(\delta / \delta_{ov}\). There is no solution for eq. (63) in terms of elementary functions. However, an explicit solution can be obtained using the Lambert W function (Corless et al. 1996, the function is also called inverse product log)

\[
\xi_{ov} = -1 - W_{-1} \left(\frac{\delta^4 - \delta_{ov}^4}{e^{k_{ov} \delta}}\right), \tag{65}
\]

where \(e = \text{exp}(1)\) and \(W_{-1}\) is the real valued branch with values smaller than \(-e\) defined between \(-1\) and 0. The function \(W\) is the inverse function to \(z \rightarrow e^z\). Using bounds on \(W_{-1}\) by Chatzigergiou (2013), an excellent approximation of \(\xi_{ov}\) is

\[
\xi_{ov} \approx \sqrt{-2 \ln\left(1 - \frac{\delta^4}{\delta_{ov}^4}\right)} - \frac{2}{3} \ln\left(1 - \frac{\delta^4}{\delta_{ov}^4}\right). \tag{66}
\]

A more accurate expression can be derived by analysing the pertubations of Hamiltonian (50), we refer to Chirikov (1979); Cincotta (2002).
5.3. Instability timescale

Starting from a point \((\delta, \eta)\), in the period ratio space, we assume that the system wanders along the diffusion direction computed in section 4.2. While not exactly perpendicular to the resonant network in the period ratio space, the motion along the diffusion direction can be well parameterized by \(\eta\). We thus monitor the diffusion in terms of \(\eta\) since the resonance width and densities are easy to compute in terms of this variable. Moreover, as seen in figure 3, the systems are not too far from the first order MMRs, so the distances to cover are short and we can consider the period ratio as almost constant along the trajectory.

Since the diffusion coefficient \(D_{\eta\delta}\) mainly depends on \(p + q\), one can associate a diffusion rate to each of the equal resonance index subnetworks. We note \(\eta_{\delta}\), the distance in terms of \(\eta\) to the closest first order MMR. In order to describe the diffusion process, we adapt the framework developed by Morbidelli & Vergassola (1997) to compute escape rate of particles from the vicinity of invariant tori. Let us assume that the system starts initially at a position \(\eta_0\), and becomes unstable once \(\eta\) reaches \(\eta_0 + \Delta\). Furthermore, we assume that the dynamics behaves as a random walk. The position of the system along the resonance network is described by the equation

\[
d\eta\over dt = s(\eta) b(t),
\]

(67)

where \(s\) is related to the local diffusion coefficient and \(b(t)\) is a Gaussian white noise having zero average, verifying \((b(t)b(t')) = 2\delta(t - t')\).

For \(s = s_0\) constant, eq. (67) is the classical Langevin equation. The associated diffusion equation for the probability density \(p\) is

\[
\frac{\partial p}{\partial t} = s_0^2 \frac{\partial^2 p}{\partial \eta^2},
\]

(69)

where the diffusion coefficient is \(s_0^2\). By analogy with the case where \(s\) is constant, we define \(s(\eta) = \sqrt{D_{\eta\eta}\eta}\), where \(D\) and \(q\) define the largest MMR that contains the point \(\eta\). Since in the region considered, the resonances overlap, the interval \((\eta_0, \eta_0 + \Delta\)\) can be partitioned into the different subnetworks. Each value of \(\eta\) can be associated with an index \(k\) that corresponds to the widest resonance that contains it. The probability that a given point \(\eta\) is in a resonance of index \(k\) is given by

\[
P(\eta) = \text{3-pl. MMR of index } k = \begin{cases} 
\rho_k dk, & k \leq k_{\text{ov}}, \\
0, & k > k_{\text{ov}}.
\end{cases}
\]

(70)

Note that considered value of \(\eta\) could also be contained into a higher index resonance than \(k_{\text{ov}}\). However, the contribution of this higher order resonance to the diffusion is negligible.

Following Morbidelli & Vergassola (1997), eq. (67) can be solved by introducing the variable

\[
y(\eta) = \int_{\eta_0}^{\eta} \frac{d\eta'}{s(\eta')}.
\]

(71)

Indeed, computing the time derivative of \(y\) using eqs. (67) and (71), we have

\[
\frac{dy}{dt} = \frac{1}{s(\eta)} \frac{dy}{d\eta} = b(t),
\]

(72)

that is the Langevin equation with a unit diffusion coefficient. The evolution of \(y\) is therefore known and we obtain the evolution of \(\eta\) by inverting eq. (71). To do so, we need to determine the value of \(s(\eta)\).

Since we are interested in the overall diffusion speed and not the exact diffusion at a given point, we can attribute a probabilistic value to \(s\) using eq. (70). We thus compute the average value of \(y\) as a function of \(\eta\) over all the configurations of the resonance network. Noting \(\bar{y}\) and \(\overline{\delta}\) these average values, we have

\[
\bar{y}(\eta) = \int_0^{k_{\text{ov}}} \int_{\eta_0}^{\eta} y(\eta') P(\eta') d\eta' dk = \int_{\eta_0}^{\eta} d\eta' \int_0^{k_{\text{ov}}} \frac{d\delta}{\sqrt{D_k}} dk = (\eta - \eta_0) \int_0^{k_{\text{ov}}} \frac{\rho_k}{\sqrt{D_k}} dk.
\]

(73)

We see that the variation of \(\bar{y}\) is proportional to the variation of \(\eta\), the integral being almost constant\(^{12}\) around a given point of the period ratio space. In particular, we can define an effective diffusion coefficient, taking into account the contribution of all the resonances necessary to locally cover the phase space

\[
D_{\text{eff}} = \left( \int_0^{k_{\text{ov}}} \frac{\rho_k}{\sqrt{D_k}} dk \right)^2 = \left( \int_0^{k_{\text{ov}}} \frac{k}{\sqrt{\omega_k}} dk \right)^2.
\]

(74)

Indeed, deriving eq. (73) gives

\[
\frac{d\bar{y}}{dt} = \sqrt{D_{\text{eff}}} b(t).
\]

(75)

We refer to the Appendix C.1 for the exact expression. The exact expression involves several special functions to evaluate the integrals. It is nevertheless straightforward to evaluate them numerically using scipy (Virtanen et al. 2020) for instance. We also provide a jupyter notebook reproducing the figures of this article\(^{13}\).

For the rest of the discussion, a very good estimate\(^{14}\) is obtained with the expression

\[
D_{\text{eff}} \approx cMA n_2 \frac{9}{4\pi} \frac{\sqrt{1 - \frac{\delta}{\delta_{\text{ov}}}}}{\sqrt{2}} \frac{\delta_{\text{ov}}^3}{\delta^3} \left( 1 - \frac{\delta^2}{\delta_{\text{ov}}^2} \right) 10^{-\sqrt{\ln(1 - (\delta/\delta_{\text{ov}}))}}.
\]

(76)

This expression is surprisingly compact, mainly depends on \(\delta\) and \(\delta_{\text{ov}}\) and does not contain an exponential term. This expression is within an order of magnitude to the exact one over the majority of the considered range of \(\delta\). One can remark that the diffusion coefficient goes to 0 for \(\delta \to \delta_{\text{ov}}\).

We have reduced the initial problem to a simple continuous unidimensional random walk with constant diffusion coefficient. The system can wander up until it reaches a first order two planet MMR on the diffusion direction. The survival time distribution is thus given by the well studied hitting time probability of a brownian motion with two absorbing boundary conditions (Redner 2001; Borodin & Salminen 2002).

\(^{12}\) In the sense that it does not depend on the exact structure of the resonance network.

\(^{13}\) https://github.com/acpetit/PlanetSysSurvivalTime

\(^{14}\) The factor \(10^{-\sqrt{\ln(1 - (\delta/\delta_{\text{ov}}))}}\) was found by chance during exploratory work after having obtained the rest of the expression through power expansion for small \(\delta\). We have no clear explanation on its origin.
As shown in figure 7, the mean of \( \log_{10} T_{\text{surv}} / P_1 \) can be well approximated as

\[
\log_{10} \left( \frac{t_{\text{surv}}}{P_1} \right) = \log_{10} \left( \frac{T_0}{P_1} \right) + \log_{10} \left( \frac{3}{2} u_0^2 (1 - u_0)^2 \right),
\]

(78)

where we choose to normalize the survival time \( T_{\text{surv}} \) by the innermost planet period \( P_1 \) to include \( T_0 \) in the right hand side expression. Similarly, we approximate the standard deviation as a second order polynomial as

\[
\sigma \left( \log_{10} \left( \frac{T_{\text{surv}}}{P_1} \right) \right) = 0.9 - 2.25 u_0 (1 - u_0).
\]

(79)

Note that the standard deviation does not depend on the value of \( \log_{10} T_0 / P_1 \) itself. Thus the standard deviation does not depend on the order of magnitude of the instability time. This is a remarkable result as it has been shown in numerical simulations (Hussain & Tamayo 2020) that the standard deviation of the survival time of extremely close initial conditions has the same properties. Besides, Hussain & Tamayo (2020) measured the standard deviation to be 0.43 ± 0.16 dex which is consistent with the value we obtain for initial conditions not too close initially to two planets MMR as it can be seen in Figure 7.

Equation (78) gives the expression of the mean survival time for any initial condition and while it involves terms that are not easily tractable analytically, are easy to compute numerically given a specific system. In the remaining part of this section, we seek to obtain a simplified expression to show how the mean survival time depends on the spacing and planet masses.

Let us assume that the system initially starts in between the 2 planets MMR \( P:P+1 \) and \( P+1:P \) for the pair 1-2 and \( Q:Q+1 \) and \( Q+1:Q \) for the pair 2-3. In the example shown in section 2, \( P = Q = 6 \). By evaluating \( \eta \) at the edges of the square created by the resonances, one can find that the maximum variation of \( \eta \) without encountering a MMR is

\[
\Delta \eta_{\text{max}} = \frac{P}{P + Q} - \frac{P - 1}{P + Q} = \frac{1}{P + Q} = \nu,
\]

(80)

where the last equality is true for \( \nu \) evaluated at the intersections of the MMR \( P:P+1 \) and \( Q+1:Q \) or \( P+1:P \) and \( Q:Q+1 \). In practice, the variation of \( \nu \) in a given resonance rectangle is \( 2 \nu^2 \ll \nu \) so we neglect the variations of \( \nu \) in the rectangle. Thus, using \( \nu \) as the characteristic length for the diffusion interval of \( \eta \) is accurate. \( \eta_{\text{max}} \) describes the largest possible variation of \( \eta \), hence for any given point \( (v_{12}, v_{23}) \). For our simplified expression, we take \( \Delta \eta = \eta_{\text{max}} = 3/2 \delta \).

Furthermore, we assume \( u_0 = 0.5 \). Using eq. (76) to estimate the effective diffusion coefficient, we can get an order of magnitude for the survival time with the expression

\[
\log_{10} \left( \frac{T_{\text{surv}}}{P_1} \right) \approx - \log_{10} \left( 16 \sqrt{2} A_e M \sqrt{\eta_1 (1 - \eta_1)} / 3 \right) + \log_{10} \left( \frac{\delta^6}{\delta_{\text{av}}^4} \nu \right) + \sqrt{\log \left( 1 - \left( \frac{\delta}{\delta_{\text{av}}} \right)^4 \right)}
\]

(81)

This expression can be decomposed into a prefactor that mainly depends on the planet to star mass ratio and a function that only depends on \( \delta / \delta_{\text{av}} \). At first glance, this expression is not linear in \( \delta \) which seems to contradict the numerical results (e.g. Chambers et al. 1996; Obertas et al. 2017).

Salminen 2002; Wax & Cohen 2009). We note \( \eta_- \) and \( \eta_+ \) (\( \eta_+ > \eta_- \)), the value of \( \eta \) at the neighbouring first order two planet MMRs along the diffusion direction. The interval where the system can wander has for length \( \Delta \eta = \eta_+ - \eta_- \). The initial position on this interval can be measured by the quantity \( u_0 = (\eta_0 - \eta_-) / \Delta \eta \) that is between 0 and 1.

The time can be rescaled such that the considered segment has length unity and the diffusion coefficient is 1 if we take as time unit

\[
T_0 = \frac{\Delta \eta^2}{D_{\text{eff}}}.
\]

(77)

The distribution of \( \log_{10} T_{\text{surv}} / T_0 \) is plotted in figure 7 as well as the mean and standard deviation of \( \log_{10} T_{\text{surv}} / T_0 \). We give the expression of the distribution in appendix C.
which hinted at a linear dependency of the logarithm of the survival time with respect to the orbital spacing. However, the function has an inflection point for $\delta = 0.629 \delta_{\text{ov}}$ and can be well approximated by a linear function of $\delta / \delta_{\text{ov}}$. The linear approximation is correct in the regime of interest, i.e. for Hill stable planet pairs, not too close to the overlap limit. We have

$$\left( \log_{10} \frac{T_{\text{surv}}}{P_1} \right) \simeq - \log_{10} \left( \epsilon^2 M \sqrt{\eta (1 - \eta)} \right) - 6.72 + 6.08 \frac{\delta}{\delta_{\text{ov}}}.$$  \hfill (82)

To compare with previous numerical studies (Chambers et al. 1996; Faber & Quillen 2007), we compute the estimated survival time for equal mass and spacing planets

$$\left( \log_{10} \frac{T_{\text{surv}}}{P_1} \right) \simeq - \log_{10} \left( \frac{m_p}{m_0} \right) - 6.51 + 3.58 \left( \frac{m_0}{m_p} \right)^{1/4} \Delta.$$  \hfill (83)

The slope coefficient $3.56$ is very close to values obtained in previous works, Faber & Quillen (2007) estimated it at $3.7$ and Yalinewich & Petrovich (2020) at $3.4$. The prefactor proportional to $m_0/m_p$ is consistent with numerical simulations (Chambers et al. 1996; Faber & Quillen 2007) and the numerical constant is very close to the one obtained by Faber & Quillen (2007).

To summarize, we estimate the diffusion rate along the zeroth order three planet MMR network by only considering the widest resonances, up to the index $k = p + q$ such that they locally cover the period ratio plane. We show that the complex random walk along the resonance network can be represented by a diffusion process with an effective locally constant diffusion coefficient given by eq. (74). As observed in numerical simulations (Hussain & Tamayo 2020), we show that the survival time distribution is approximately lognormal and we recover the same standard deviation. Our estimation of the mean survival time scales as the planet separation in units of $\epsilon^{1/4}$ and not in units of Hill radii. In particular, we emphasize the importance of considering various mass system in such studies as it allows to discriminate between the physical mechanisms driving the dynamics. Moreover, while our estimate is not exponential in planet spacing as fitted in numerical simulations, we show that for the range of times of interest, it can be considered as such. As Quillen (2011), we predict that beyond the overlap limit, the survival time is likely much larger since the Chirikov diffusion is not an efficient process on its own.

6. Comparison with numerical simulations

Plenty of numerical studies has been performed recently on the problem of instability of tightly spaced planets (e.g., Obertas et al. 2017). However, the most recent study limiting itself to the minimal set-up: an equal mass and spacing (EMS) system with three planets on initially circular and coplanar orbit was performed by Faber & Quillen (2007). While they considered different mass ratios, the integration time was limited to $10^5$ inner planet orbits with a limited number of points.

In order to have a fine enough resolution and a longer integration time, we run our own suite of numerical simulations. We use REBOUND (Rein & Liu 2012) and the symplectic integrators WHFAST (Rein & Tamayo 2015) and MERCURUS (Rein et al. 2019). We initialize three planet systems, on initially circular and coplanar orbits. As previous studies (e.g. Chambers et al. 1996), the innermost planet semi-major axis is set to 1 au and the outer planet semi-major axis are chosen such that the two planet pairs have an equal semi-major axis ratio. The initial angles are randomly drawn. We do three suites of simulations with planet masses of $10^{-7} \, M_\odot$, $10^{-5} \, M_\odot$ and $10^{-3} \, M_\odot$. In each suite, the range of period ratios considered starts at 2 mutual Hill radii, i.e. in the region where even the planet pairs should be considered unstable, and extends beyond the predicted 3 planet MMR overlap criterion derived in section 4.4. Each system is integrated until two planets are closer than 1 Hill radius or until the inner planet has performed $10^9$ orbits. For each of these system, we report the final time as the survival time. A value of 1 Gyr should therefore be understood as a lower limit. We use WHFAST and a timestep of 1/20th of the inner orbit for the two smaller mass suites and MERCURUS with a timestep of 1/30th of the inner orbit for the largest mass planets. We stopped running simulations at larger separations when it became clear that the systems were stable for 1 Gyr, outside of the first order two planet MMRs. Each of the sets of simulations is composed of between 1,600 and 1,700 systems.

We respectively plot in Figures 8, 9 and 10, the survival times for the sets of simulations with planet masses of $10^{-7} \, M_\odot$, $10^{-5} \, M_\odot$, and $10^{-3} \, M_\odot$ as a function of the initial period ratio. In each of these figures, the survival time of a system corresponds to a blue dot and the predicted time (81) is plotted in red. Additionally, we add various other features that help understand the patterns that emerge in the survival time curves. The blue vertical line corresponds to the Hill stability limit (Petit et al. 2018), the orange vertical line to the two planet MMR overlap criterion (Wisdom 1980). The dashed black (resp. green) lines are the two planet first (resp. second) order MMR. The light orange rectangles show an estimate of the width of the two planet MMR (Petit et al. 2017).

![Fig. 8. Survival time for a three EMS planet system for planet masses of $10^{-7} \, M_\odot$ as a function of the initial period ratio.](image-url)
The first thing that should be noticed is that the survival time estimate (81) of the logarithm of the survival time is consistent with the numerical simulations in the range where the former is almost linear. The agreement is particularly striking for the intermediate case ($10^{-5} M_\odot$). We discuss explanations for the discrepancies for the small masses and Jupiter masses planets below. Moreover, the slope being correct in all three figures is another indication that the scaling in $\varepsilon^{1/4}$ is more appropriate than to renormalize the spacing by the Hill radius.

We focus on Figure 9 to describe more precisely the different features that should be pointed out. First, if one ignores the variations due to the proximity to the resonances, our estimate lies in the middle of the distribution of survival time from period ratios of 1.05 to the end of the region where we consider that the three planet MMR overlap, around 1.195. As a result, it means that our criterion slightly underestimate the diffusion time since in eq. 81, we assume a maximum value for the distance to the two planet MMR network. Nevertheless, Chirikov diffusion correctly predicts the slope as well as the right order of magnitude for the survival time.

We then notice, as previous studies have (Smith & Lithusauer 2009; Obertas et al. 2017), that the substructure on the curve is very well explained by the two planet first order MMR. By considering separately the regions outside of first order MMR and the regions inside, we can see that the survival time when the dynamics is dominated by the two planet MMR is roughly two to three order of magnitudes lower than outside of their influence. We can explain why the survival time is shorter for period ratios just below the Keplerian resonance, by noting that, due to the shape of the first order MMR, the unstable fixed point, where the first order MMR separatrices originate, is situated on the left of the resonance in the figures. Note that if we had introduced a fluctuation in the initial period ratio, the pattern would be much less clear. We can also note that the second order MMRs likely play a role in accelerating the diffusion. A similar effect is also clearly observed regarding the 2:1 MMR in Fig. 10 as well as in Fig. 8 where the larger apparent spread is due to the the very dense two planet MMR network.

Finally, we focus on the region close to where we predict that the three planet MMR stops overlapping (period ratios of about 1.19) in Fig. 9. We see that the linear trend followed in the range 1.05 to 1.16 no longer holds due to some systems surviving longer than expected, in particular beyond 1 Gyr. Moreover, the spread of the survival times increases instead of staying constant as shown by Hussain & Tamayo (2020). One can note that a similar feature is also visible in (Obertas et al. 2017) results, although the increase in the spread is less visible, most likely because they consider five planet systems instead of three. In particular, outside of the 6:5 and the 11:9 resonance, it appears that systems live much longer than one would have expected after extrapolating the linear trend fitted in previous numerical studies. The same behaviour is also observed in Fig. 10. However, the region where short lived and long lived systems coexist is much larger because of the larger size of high order, two planet MMR that are not taken into account in this analysis.

These two observations i.e. the longer survival times and the increased spreading around the overlap limit, are consequences of our analytical derivation. Indeed, beyond the overlap region, the Chirikov diffusion alone can no longer drive the instability over homogeneous regions of the phase space. This does not mean that systems beyond our approximate overlap limit will live indefinitely. However, the instability in these systems is driven by another mechanism than the Chirikov diffusion considered here. In particular, we have neglected the diffusion parallel to the resonances i.e. the Arnold diffusion (Cincotta 2002). However, these alternate pathways to instability are most likely much slower.

We now look more in detail at the apparent discrepancies between the estimated survival time and the two extreme cases. In Figure 8, we see that while the analyti-
The survival time at very close separations. We postulate that two planets with a large resolution in terms of period ratio. More importantly, the five and more planet overlap spacing is given by $K^{1/2} \delta_{ov}$. The survival time is also affected because, while $\rho_0$ has changed, the resonance width has not. As a result, the effective diffusion coefficient (eq. 74) becomes $K^{-2}D_{diff}$, where the change of $\delta_{ov}$ should be accounted for.

We can compare this expression to previous numerical results. We choose the simulations from Obertas et al. (2017) as they run systems composed of five Earth-mass planets with a large resolution in terms of period ratio. Moreover, their simulations have been run up to $10^{10}$ orbits. We plot in Figure 11, the survival time from Obertas et al. (2017) five Earth-mass planet EMS systems as a function of the initial period ratio as well as the three planet survival time estimate (81) and its extrapolation to a five planet system. As expected, the three planet survival time slightly overestimate the survival time, and more importantly, the MMR overlap criterion fails to account for the continuation of the trend beyond period ratios of 1.14. On the other hand, the extrapolation to five planets (with $K = 2$) goes almost to the region where the survival time starts increasing faster (around 1.165). We conclude that our approach can successfully account for the difference between three and more planet systems.

7. Beyond three planets on circular orbits

7.1. Systems with 4+ planets

As already noticed in previous numerical studies, increasing the number of planets beyond three does not change fundamentally the survival timescale. Chambers et al. (1996) show that while there is a slight change in the slope of the survival time between systems of five planet with systems of three planets, the survival time is mostly unchanged by the addition of other planets into the systems.\(^{15}\)

It is thus natural to try to extend our results to systems composed of more than three planets. Unfortunately, contrary to the three planet case, it is not possible to reduce the dynamics to an unidimensional Chirikov diffusion. Indeed, the resonance network cannot be projected into a two-dimensional plane as done in section 4. A solution is to consider triplets of adjacent planets, and assume that this triplet is perturbed by the additional planets. The influence of the other planets can be seen as a change in the period ratio $\nu_{12}$ and $\nu_{23}$ due to the resonances with the adjacent planets. Assuming the the planet spacings are comparable, the perturbation of the period ratio is of the same order of magnitude as the one induced by the three planet MMR from the considered triplet.

\(^{15}\)Note that Pichierri & Morbidelli (2020) show that adding more planets into a resonant chain changes its stability. Further studies on this topic are required.
We analyse the mechanism driving the instability time of closely packed planetary systems. Extending previous work by Quillen (2011), we use an integrable model to compute the dynamics of three zeroth order three planet MMR for systems on initially circular orbits, with arbitrary planet mass distribution and spacing. We then compute the region where these resonances overlap (eq. 59), as it is the region where large scale diffusion can occur (Chirikov 1979). We find that this region extends past the limit of overlap of two planet MMR and the spacing scales as $\varepsilon^{1/4}$, where $\varepsilon$ measures the planet to star mass ratio.

Inside the region of overlapping three planet MMR, the dynamics is not secular, despite an almost conservation of the AMD, and the period ratios can diffuse up until they reach one of the larger two planet MMR, then leading to a rapid instability. We derive an estimated diffusion coefficient by considering only the necessary resonances and as a result, estimate the survival time (eq. 81). Although in general, Chirikov diffusion leads to survival time following power-laws (Quillen 2011), our expression is well approximated by an exponential curve, as it is reported in numerical simulations. Moreover, we predict and observe on numerical simulations a change of behaviour in the region where three planet MMR are not overlapped. Beyond the overlap limit, the dynamics cannot be well represented by a relatively uniform diffusion mechanism, and while some other mechanism may drive some diffusion, it is expected to be much slower. The stability time therefore depends much more on the initial conditions since other mechanisms such as Arnold diffusion may be necessary to allow the planet to reach the instability.

We compare to numerical simulations and find an excellent agreement with our analytical estimate. Moreover, we discuss how apparent discrepancies can be explained. We also discuss how this result can be extended to systems containing more planets or on eccentric orbits. Moreover, we show that the classical fit where the instability time is an exponential function of the spacing measured in Hill radius fails to capture the physical mechanism at play. In particular, we see that for very small bodies, three body resonances can drive the instabilities over distances that are much larger than single two planet interactions. The tools necessary to compute the time estimates and reproduce the figures are made available at https://github.com/acpetit/PlanetSysSurvivalTime.

In this paper, we focused on systems initially outside of the influence of two planet MMR. A similar approach could be applied on the vicinity of a two planet MMR in order to track the system through the rapid final instability phase. Such works are necessary to understand the creation of AMD during scattering events, leading to planet collisions and ejections.

Acknowledgements. A.P. thanks A. Morbidelli and J. Laskar for useful discussions on the model and M. Pain for providing the references for the random walk exit time. We thank A. Obertas and D. Tamayo for allowing us to reproduce their data. This work by supported by the Royal Physiographic Society of Lund through the Fund of the Walter Gyllenberg Foundation (number 40730). M.D. and A.P. are supported by the project grant 2014.0017 ‘IMPACT’ from the Knut and Alice Wallenberg Foundation. A.J. and A.P. are supported by the European Research Council under ERC Consolidator Grant agreement 724687-PLANETESYS, the Swedish Research Council (grant 2018-04867), and the Knut and Alice Wallenberg Foundation (grants 2014.0017 and 2017.0287). GP thanks the European Research Council (ERC Starting Grant 757448-PAMDORA) for their financial support. This research was made possible by the open-source projects rebound (Rein & Liu 2012), Jupyter (Kluyver et al. 2016), iPython (Perez & Granger 2007), numpy (van der Walt et al. 2011), scipy (Virtanen et al. 2020), pandas (Wes McKinley 2010), and matplotlib (Hunter 2007).

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Fig. 11. Survival time of five Earth-mass planet EMS systems from (Obertas et al. 2017) as a function of the period ratio. The red curve corresponds to the survival time of a three Earth-mass planets system (eq. 81) and the dashed line is the extrapolation to a five planet system.

7.2. Eccentric planets

It is tempting to generalize our results to system with eccentric and inclined planets. As noted by Pu & Wu (2015), systems where planet are eccentric have similar survival time as systems with circular orbit if the spacing between the planet is measured by the distance between the apoapsis of the inner planet and the periapsis of the outer planet. In principle an overlap criterion could be computed by taking into account the full three planet MMR network with MMR of arbitrary order in the same way it was done by (Hadden & Lithwick 2018) for the two planet case. Another similarity with the two planet case is the fact that the dynamics of an isolated first order three planet MMR is integrable using the same strategy as in the two planet case (such a result will be presented in a separate article, Petit in prep.).

In practice, the structure of the full network is more complicated than the structure of zeroth order resonance network and the size of the resonances depends on individuals planet eccentricities. Moreover, in the case of eccentric orbits, the diffusion is not restricted to to a one dimensional direction. Indeed one has to take into account the diffusion along the eccentricity degrees of freedom contrarily to the diffusion for circular orbits because the zeroth order MMR conserve the total AMD. The extension to the eccentric case is the goal of future works.
Appendix A: Notations summary

We present in Table A.1 a summary of the notations used in this article.

Appendix B: Laplace coefficients

The Laplace coefficients appear naturally in the study of planetary systems through the development of the perturbation part in the three-body problem. The coefficient $b_l^j(\alpha)$ corresponds to the $l$-th Fourier coefficient of the function $(1 + \alpha^2 - 2\alpha \cos(\lambda))^{-1/2}$, i.e.

$$\frac{1}{2} b_l^j(\alpha) = \frac{\Gamma(l+1)}{\pi} \int_0^{\infty} \frac{\cos(\lambda)}{(1 + \alpha^2 - 2\alpha \cos(\lambda))^{l+1/2}} d\lambda.$$  \hspace{1cm} (B.1)

There are recurrence relations between them and we refer to Laskar & Robutel (1995) for a complete description.

One of the challenges of analytical studies of planet dynamics comes from the estimation of the Laplace coefficients. Indeed, due to Kepler thrid law, $a$ and $l$ are often tied to each other. For example, in the study of first order MMR, it is necessary to compute an approximation for large $l$ of the coefficient $b_l^j(\alpha)$ for $\alpha = (1 - 1/b)^{2/3}$ (Petit et al. 2017). In other words, the order in which the limits in terms of $a$ and $l$ are taken is relevant.

Laskar & Robutel (1995) give an alternative expression for the Laplace coefficients, in terms of hypergeometric functions

$$\frac{1}{2} b_l^j(\alpha) = \frac{\Gamma(s + \alpha d)}{\Gamma(s) \Gamma(l + 1 + \alpha^2)} F_1(s, s + l, l + 1; \alpha^2),$$  \hspace{1cm} (B.2)
Table A.1. Summary of the main notations used throughout the article. When possible, we give a short definition and/or refer to the equation where the quantity is defined.

| Name                     | Expression | Description                                      | Reference |
|--------------------------|------------|-------------------------------------------------|-----------|
| r_j                      |            | Heliocentric coordinate position                | Laskar (1991) |
| f_j                      |            | Heliocentric coordinate momentum                | Laskar (1991) |
| \Lambda_j                | m_j \sqrt{\mu a_j} | Circular angular momentum                       | (5)       |
| \Lambda_j                |            | Mean longitude                                  | (5)       |
| C_j                      | \Lambda_j\left(1 - \sqrt{1 - \epsilon_j^2}\right) | Planet j AMD                                    | (5)       |
| \omega_j                 |            | Longitude of the periapsis                      | (5)       |
| \epsilon_j               |            | Complex Poincaré coordinate                     | (6)       |
| C                         | C_1 + C_2 + C_3 | Total AMD                                       | (7)       |
| G                         | \Lambda_1 + \Lambda_2 + \Lambda_3 - C | Total angular momentum                          | (7)       |
| n_j                      | \frac{\epsilon_j}{\Lambda_j} | Mean motion                                      | (12)      |
| \epsilon                |            | Dimensionless parameter related to the planet to star mass ratio | (8)       |
| \mathcal{H}_0            |            | Keplerian part of the Hamiltonian               | (9)       |
| \epsilon \chi_i          |            | First order averaging generating Hamiltonian    | (16)      |
| \nu_i                    | \frac{P_i}{P_j} | Period ratio for planet i and j                | (19)      |
| \eta                      | \frac{\nu_i}{\nu_j} \left(\nu_i \nu_j (1 - \nu_i \nu_j)\right) | Resonance locator                               | (21)      |
| \nu                       |            | Generalized period ratio separation             |           |
| \theta_{res}             | p \lambda_l - (p + q) \lambda_2 + q \lambda_3 | Zeroth order three planet resonant angle         | (24)      |
| \Theta                    | \frac{\Lambda_l}{\nu_i} | Resonant action                                 | (25)      |
| \Gamma                    | \frac{\eta_i}{\nu_i} | Scaling parameter                               | (25)      |
| \Upsilon                 | \Lambda_1 + \Lambda_2 + \Lambda_3 | Circular angular momentum                       | (25)      |
| \alpha_{ij}              | \frac{a_i}{a_j} | Semi-major axis ratio                           | (49)      |
| b^{\delta}(\alpha)       |            | Laplace coefficients                            | (B.1)     |
| \epsilon^2 R_{pq}        |            | Resonant coefficient                            | (42), (47) |
| \delta_{ij}              | 1 - \alpha_{ij} | Planet orbital spacing                          | (43)      |
| \delta                    | \frac{a_{ij}}{a_{ij+1}} | Generalized orbital spacing                      | (45)      |
| \mathcal{K}_2            |            | Coefficient of the second order development of the Keplerian part | (49)      |
| \omega_{pq}              | \epsilon \sqrt{\mathcal{K}_2 R_{pq}} | Small oscillation frequency around the resonance | (52)      |
| A                         | 3.47       | Numerical factor appearing in \omega_{pq}      |           |
| \epsilon M               |            | Relevant mass ratio for the problem             | (53)      |
| \Delta \eta_{pq}         |            | Resonance width in period ratio space           | (55)      |
| \rho_{pq}                | (p + q) \Delta \eta_{pq} | Density of the subnetwork of resonances with index p + q | (57)      |
| \rho_{\text{tot}}        |            | Total density of the zeroth order three planet MMR network | (58)      |
| \delta_{ov}              |            | Generalized orbital spacing such that the MMR cover the full space | (59)      |
| \kappa_{ov}              |            | Minimum index such that the resonances with lower index locally cover the period ratio space | (63)      |
| \xi_{ov}                 | k_{ov} \delta |                                                |           |
| D_{pq}                    | \left(\Delta \eta_{pq}\right)^2 \omega_{pq}/(2\pi) | Diffusion coefficient linked to the resonance p, q | (62)      |

where \Gamma is the Gamma function and \(_2F_1\) is the Gaussian hypergeometric function. Laskar & Robutel (1995) use expression (B.3) to show that for \(\alpha \to 1\), the Laplace coefficients are independent of \(l\). However, we are interested in an estimate where we fix \(\alpha\) and make \(l\) take larger and larger values. We thus cannot use the equivalent they proposed.

In this article, we particularly focus on \(b_1^{\delta}(\alpha)\). For \(s = 1/2\), eq. (B.3) becomes

\[
\frac{1}{2} b_s^{\delta}(\alpha) = \frac{\alpha^s \Gamma\left(l + \frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(l + 1\right)} \left\{ \begin{array}{c} F_1\left(1, l + \frac{1}{2}, l + 1; ; \alpha^2\right) \end{array} \right. \tag{B.3}
\]

In the limit of large \(l\), the ratio of \(\Gamma\) functions is equivalent to \(l^{-1/2}\) (it’s worth noting that the estimate is already good for \(l = 1\)). We thus can focus on estimating the hypergeometric function. We find that for \(l \to \infty\), \(_2F_1\left(\frac{1}{2}, l + \frac{1}{2}; l + 1; \alpha^2\right)\) converges to a value depending on \(\alpha^2\) that we note \(f_{1/2}(\alpha^2)\).

Taking the limit \(l\) large into the differential equation verified by the hypergeometric function (Olver et al. 2010)\(^{16}\), we find that \(f_{1/2}\) is solution of

\[
(1 - x) f_{1/2}(x) - \frac{1}{2} f_{1/2}(x) = 0, \tag{B.4}
\]

\(^{16}\) see [https://dlmf.nist.gov/15.10](https://dlmf.nist.gov/15.10)
with \( f_{1/2}(0) = 1 \). As a result, we have

\[
 f_{1/2}(\alpha^2) = \frac{1}{\sqrt{1 - \alpha^2}}. 
\]  

(B.5)

\( f_{1/2}(\alpha^2) \) approximates extremely well the hypergeometric function as shown in Figure B.1 where we plot \( _2F_1\left(\frac{1}{2}, l + \frac{1}{2}; l + 1; \alpha^2\right) \) for different values of \( l \) as a function of \( \alpha \). Note that \( \alpha \) is plotted in logarithmic scale centered on 1 to show where the curve start to differ. We observe a fast convergence.

We therefore approximate the Laplace coefficients as

\[
 \frac{1}{2} b_{2n}^y(\alpha) = \frac{\alpha^l}{\sqrt{\pi l(1 - \alpha^2)}}. 
\]  

(B.6)

The approximation is very good (within 10%) for almost all values of \( l \). We plot in Figure B.2, the ratio of the exact Laplace coefficient and its estimate as a function of \( l \) for different values of \( \alpha \). In order to compare with Quillen (2011) estimate we plot with dashed line the ratio \( b_{2n}^y(\alpha)/(l(1 - \alpha^2))^{1/2} \). We use \( \alpha^l \) instead of \( e^{-l(1-\alpha)} \) in Quillen’s expression to avoid an unfair comparison since the difference in the exponential would dominate the difference in the prefactors. It is critical to properly estimate the prefactor. Indeed, because we integrate over the resonance index in section 4.4, the prefactor contributes significantly to the resonance density, and later to the estimate of the survival time. As a result, (Quillen 2011) estimate fails to fit the Laplace coefficient and as a result, overestimates the resonance width. In their work, this effect is partially compensated by an underestimation of the Laplace coefficient derivative.

Appendix C: Effective diffusion coefficient estimation

Appendix C.1: Exact diffusion coefficient

To compute the effective diffusion coefficient \( D_{\text{eff}} \) (eq. 62), one need to solve the integral

\[
 \left( \int_0^{\delta_{\text{ov}}} \frac{k}{\sqrt{\delta_{\text{ov}}}} \text{dk} \right)^{-2} = n_2 \varepsilon MA \sqrt{\eta(1 - \eta)} \left( \int_0^{\delta_{\text{ov}}} \varepsilon \text{d}k \right)^{-2}. 
\]  

(C.1)

The integral can be evaluated in term of the special Dawson function, which gives for \( D_{\text{eff}} \) the expression

\[
 D_{\text{eff}} = \varepsilon M_2 A \sqrt{\eta(1 - \eta)} \delta^2 \varepsilon^{\xi_{\text{ov}}} \left( 1 - \frac{2}{\xi_{\text{ov}}} D_{\text{s}} \left( \frac{\delta_{\text{ov}}}{2} \right) \right)^{-2}. 
\]  

(C.2)

where \( \varepsilon_{\text{ov}} = \varepsilon_{\text{ov}} \delta \) is given by eq. (65) and \( D_{\text{s}} \) is the Dawson function defined as

\[
 D_{\text{s}}(x) = e^{-x^2} \int_0^x e^{t^2} \text{dt} 
\]  

(C.3)

Using eq. (63), we can replace \( e^{-\xi_{\text{ov}}} \) to obtain

\[
 D_{\text{eff}} = \varepsilon M_2 A \sqrt{\eta(1 - \eta)} \delta^2 \varepsilon^{\xi_{\text{ov}}} \left( 1 - \delta^2 \varepsilon^{\xi_{\text{ov}}} \right)^{-2}. 
\]  

(C.4)

with

\[
 F(\xi_{\text{ov}}) = \left[ 1 - \frac{2}{\xi_{\text{ov}}} D_{\text{s}} \left( \frac{\sqrt{\xi_{\text{ov}}}}{2} \right) \right]^2 
\]  

(C.5)

As mentionned in the main text, we find that \( \xi_{\text{ov}}(\xi_{\text{ov}} + 1)F(\xi_{\text{ov}}) \) is extremely well approximated by

\[
 \xi_{\text{ov}}(\xi_{\text{ov}} + 1)F(\xi_{\text{ov}}) \approx \frac{2 \sqrt{2}}{9} \left( \frac{\delta_{\text{ov}}}{\varepsilon_{\text{ov}}} \right)^6 \left( 1 - \left( \frac{\delta_{\text{ov}}}{\varepsilon_{\text{ov}}} \right)^2 \right)^{1/2}. 
\]  

(C.6)

Indeed, the relative difference is below 1% for \( \delta < 0.96 \) and within a factor 2 overall. This expression was found
by chance after trying to improve an estimation based on a development around zero in terms of \( \delta/\delta_{ov} \). We plot in figure C.1 the exact expression of \( \xi_{ov}(\xi_{ov} + 1)F(\xi_{ov}) \) as a function of \( \delta/\delta_{ov} \) as well as its estimate. As it can be seen, the two curves lie on top of each other.

We also express the numerical factor as a function of \( \delta_{ov} \),

\[
es MA = \frac{3}{4} \sqrt{\frac{\delta_{ov}^4}{2(\eta(1-\eta))^{3/2}}} \tag{C.7}
\]

Combining these terms, we obtain an expression of \( D_{\text{eff}} \) that depends on \( \delta \) and \( \delta_{ov} \)

\[
D_{\text{eff}} \approx n^2 \frac{27}{16} \frac{\delta_{ov}^{10}}{\eta(1-\eta) \delta^4} \left( 1 - \frac{\delta^4}{\delta_{ov}^4} \right) 10^{-\frac{2}{3} \delta^4 \sqrt{-\ln(1-(\delta/\delta_{ov})^4)}} \tag{C.8}
\]

Appendix C.2: Exit time distribution

We provide here the distribution of the survival time. As in the main text, the interval where the system can wander has for length \( \Delta \eta = \eta_+ - \eta_- \). The initial position on this interval can be measured by the quantity \( u_0 = (\eta_0 - \eta_-)/\Delta \eta \) that is between 0 and 1. The distribution of the log of the survival time \( \log_{10} T_{\text{surv}}/T_0 \) is given by the expression (Borodin & Salminen 2002, eq. 3.0.2)

\[
\frac{dP_{\text{surv}}}{d \log_{10} \tau} = \sum_{k \in \mathbb{Z}} (-1)^k (1 - u_0 + k) \frac{\delta_{ov}^{10}}{\sqrt{4\pi \tau}} \exp \left( - \frac{(1 - u_0 + k)^2}{4\tau} \right), \tag{C.9}
\]

where \( \tau = T_{\text{surv}}/T_0 \) and \( T_0 = \frac{\Delta \eta^2}{D_{\text{eff}}} \).