The \( q \)-Higgs and Askey–Wilson algebras

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Abstract

A \( q \)-analogue of the Higgs algebra, which describes the symmetry properties of the harmonic oscillator on the 2-sphere, is obtained as the commutant of the \( \mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \) subalgebra of \( \mathfrak{o}_{q^{1/2}}(4) \) in the \( q \)-oscillator representation of the quantized universal enveloping algebra \( U_q(\mathfrak{su}(4)) \). This \( q \)-Higgs algebra is also found as a specialization of the Askey–Wilson algebra embedded in the tensor product \( U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1)) \). The connection between these two approaches is established on the basis of the Howe duality of the pair \( (\mathfrak{o}_{q^{1/2}}(4), U_q(\mathfrak{su}(1,1))) \).

Introduction

The Higgs algebra was first obtained by Higgs \([1]\) as the algebra of the conserved quantities of the Coulomb problem and harmonic oscillator on the 2-sphere. Shown to be isomorphic to the Hahn algebra \([2]\), it was also identified as the symmetry algebra of the Hartmann potential \([3]\), of certain ring-shaped potentials \([4]\) and of the singular oscillator in two dimensions \([5, 6]\). The Higgs algebra stands between Lie algebras and quantized universal enveloping algebras, as it can be viewed both as a deformation of the \( \mathfrak{su}(2) \) Lie algebra \([7]\) and a truncation of the \( U_q(\mathfrak{sl}_2) \) quantum algebra \([8]\). It has been obtained as the quantum finite \( W \)-algebra \( W(\mathfrak{sp}(4), 2\mathfrak{sl}(2)) \) \([9, 10]\) and has also appeared in the context of Heisenberg quantization of identical particles \([11]\).

The Higgs algebra can be presented in the following form

\[
[D, A_\pm] = \pm 4A_\pm, \\
[A_+, A_-] = -D^3 + \alpha_1 D + \alpha_2,
\]

(0.1)
where $\alpha_1$, $\alpha_2$ are central elements.

We here aim to construct a $q$-deformation of (1.1) that preserves the general algebraic underpinnings of this structure. This will lead to an algebra that differs from the one in [12] where a certain $q$-extension of the Higgs algebra was defined by simply replacing the cubic expression in $D$ by one involving $q$-numbers (see (1.1)).

We propose to obtain a $q$-analogue of the Higgs algebra by following a commmutant approach similar to [13] (see also [14, 15]), where the ordinary Higgs algebra was obtained as the commutant of the $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ subalgebra of $\mathfrak{o}(4)$ in the oscillator representation of $U(\mathfrak{u}(4))$. This characterization was shown to be in duality in the sense of Howe [16–19] with the well-established embedding of the Hahn algebra in $U(\mathfrak{su}(1,1)) \otimes U(\mathfrak{su}(1,1))$ [20, 21]. While Howe duality, sometimes called “complementarity”, has not been thoroughly studied in the context of $q$-algebras (see for instance [22, 23]), the results in [24] will provide appropriate background for our purposes. The merit of the approach we propose is that the $q$-Higgs algebra obtained as a commutant also appears in a dual fashion as a specialization of the Askey–Wilson algebra [28, 30] in the tensor product $U_q(\mathfrak{su}(1,1))^{\otimes 2}$.

Let us now briefly present the contents of the paper. In Section 1, the algebra in $U_q(\mathfrak{su}(1,1))^{\otimes 2}$ will be presented in Section 3. As will be shown in Section 4, the $q$-Higgs algebra proves to be isomorphic to that specialization of the Askey–Wilson algebra [28, 30] in the tensor product $U_q(\mathfrak{su}(1,1))^{\otimes 2}$.

The same notation will be used for operators.

1 The $U_q(\mathfrak{su}(1,1))$, $\mathfrak{o}_q(n)$ algebras and their $q$-oscillator realizations

The duality connection that we shall invoke in our discussion involves the algebras $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_q(n)$. We shall thus begin by introducing these algebras and their $q$-oscillator realizations.

Let $q$ be a complex number such that $|q| < 1$. One defines for any number $x$ the following $q$-numbers:

$$ (x)_q := \frac{1 - q^x}{1 - q} \quad \text{and} \quad [x]_q := \frac{q^x - q^{-x}}{q - q^{-1}}. \quad (1.1) $$

The same notation will be used for operators.

1.1 The $U_q(\mathfrak{su}(1,1))$ and $\mathfrak{o}_q(n)$ quantum algebras

$U_q(\mathfrak{sl}_2)$ [31, 32] is the quantized universal enveloping algebra with three generators $j_0$ and $j_\pm$ subjected to the relations

$$ [j_0, j_\pm] = \pm j_\pm, \quad [j_+, j_-] = [2j_0]_q. \quad (1.2) $$

It is endowed with a Hopf structure with coproduct $\Delta : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2) \otimes U_q(\mathfrak{sl}_2)$

$$ \Delta(j_0) = j_0 \otimes 1 + 1 \otimes j_0, \quad \Delta(j_+) = j_+ \otimes q^{2j_0} + 1 \otimes j_+, \quad \Delta(j_-) = j_- \otimes 1 + q^{-2j_0} \otimes j_. \quad (1.3) $$

We shall denote by $U_q(\mathfrak{su}(1,1))$ the non-compact real form of $U_q(\mathfrak{sl}_2)$ that has the three generators $J_\pm$ and $J_0$ obeying

$$ [J_0, J_\pm] = \pm J_\pm, \quad J_- J_+ - q^2 J_+ J_- = q^{2J_0} \left[2J_0\right]_q. \quad (1.4) $$
The coproduct $\Delta : U_q(\mathfrak{su}(1,1)) \to U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$ will read

$$\Delta(J_0) = J_0 \otimes 1 + 1 \otimes J_0, \quad \Delta(J_\pm) = J_\pm \otimes q^{2J_0} + 1 \otimes J_\pm.$$  \hfill (1.5)

The Casimir operator $C$ of this algebra has the following expression

$$C = J_+ J_- q^{-2J_0+1} \frac{q}{(q^2-1)^2} (q^{2J_0-1} + q^{-2J_0+1}) + \frac{q^2+1}{(q^2-1)^2}.$$  \hfill (1.6)

The coproduct being an algebra morphism, the relations (1.5) define an embedding of $U_q(\mathfrak{su}(1,1))$ into $U_q(\mathfrak{su}(1,1)) \otimes U_q(\mathfrak{su}(1,1))$.

**Remark 1.1** In the limit $q \to 1$, one recovers the usual $\mathfrak{su}(1,1)$ Lie algebra with Casimir operator $C = J_+ J_- - J_0^2 + J_0$. Moreover, the standard presentation of $U_q(\mathfrak{su}(1,1))$ \cite{33} is recovered if one considers instead the generators $\tilde{J}_0 = J_0$, $\tilde{J}_+ = J_+ q^{-J_0}$ and $\tilde{J}_- = q^{-J_0} J_-$, which satisfy the commutation relations $[\tilde{J}_0, \tilde{J}_\pm] = \pm \tilde{J}_\pm$ and $[\tilde{J}_+, \tilde{J}_-] = [2\tilde{J}_0]_q$ and have co-commutative coproduct.

We introduce next the non-standard $q$-deformation $\mathfrak{o}_q(n)$ of $\mathfrak{o}(n)$ which is defined as the associative unital algebra with generators $L_{i,i+1}$ ($i = 1, \ldots, n-1$) and relations

$$L_{i-1,i} L_{i,i+1}^2 - (q + q^{-1}) L_{i,i+1} L_{i-1,i} + L_{i,i+1}^2 L_{i-1,i} = -L_{i-1,i},$$  \hfill (1.7a)

$$L_{i,i+1} L_{i-1,i}^2 - (q + q^{-1}) L_{i-1,i} L_{i,i+1} + L_{i-1,i}^2 L_{i,i+1} = -L_{i,i+1},$$  \hfill (1.7b)

$$[L_{i,i+1}, L_{j,j+1}] = 0 \text{ for } |i-j| > 1.$$  \hfill (1.7c)

In the litterature, this non-standard deformation is often denoted $U_q'(\mathfrak{so}_n)$, see for instance \cite{34, 37}. It has been shown in \cite{38} that $\mathfrak{o}_q(n)$ can be viewed as a $q$-analogue of the symmetric space based on the pair $(\mathfrak{gl}(n), \mathfrak{o}(n))$. Although it has no Hopf structure on its own, it is a coideal subalgebra of $U_q(\mathfrak{sl}(n))$ \cite{38} and appears in many areas of mathematical physics \cite{36}.

The two cases where $n = 3$ and $n = 4$ are especially of interest to us.

Let us first note that it is possible to consider a so-called “Cartesian” presentation \cite{39, 41} of $U_q(\mathfrak{sl}_2)$, in which the three generators play an “equitable” role, and which corresponds to the non-standard deformation $\mathfrak{o}_q(3)$ (equivalently $U_q'(\mathfrak{so}_3)$) in refs. \cite{40, 41} of the universal enveloping algebra $U(\mathfrak{so}(3))$, obtained by modifying the defining relations for the skew-symmetric generators of $\mathfrak{o}(3)$.

It goes like this. With $j_0$, $j_\pm$, the $U_q(\mathfrak{sl}_2)$ generators, form the following elements:

$$j_1 = ig\{q^{\pm j_0}, j_+ + j_-\}, \quad j_2 = gq^{\pm j_0} j_+ - j_-,$$

$$g = \frac{1}{(q^{\frac{1}{2}} + q^{-\frac{1}{2}})(q^{\frac{3}{2}} + q^{-\frac{3}{2}})}.$$  \hfill (1.8)

where $\{a, b\} = ab + ba$ is the anticommutator and $g$ is a normalization factor. Defining $j_3 \equiv [j_1, j_2]_q$, where $[a, b]_q := q^{\frac{1}{2}} ab - q^{-\frac{1}{2}} ba$ is the $q$-commutator, $j_1$, $j_2$ and $j_3$ then satisfy the “Cartesian” relations

$$[j_1, j_2]_q = j_3, \quad [j_2, j_3]_q = j_1, \quad [j_3, j_1]_q = j_2.$$  \hfill (1.9)

Upon identifying $L_{12} = j_1$, $L_{23} = j_2$, one finds that this corresponds precisely to the relations (1.7) for the algebra $\mathfrak{o}_q(3)$. Note that the relations (1.7c) do not exist in this case.

For what follows, it will also be useful to have the formulas for $\mathfrak{o}_q(4)$ in full. These relations read \cite{42}

$$L_{12} L_{23} - (q + q^{-1}) L_{23} L_{12} + L_{23}^2 L_{12} = -L_{12},$$  \hfill (1.10a)

$$L_{23}^2 L_{23} - (q + q^{-1}) L_{12} L_{23} L_{12} + L_{23}^2 L_{12} = -L_{23},$$  \hfill (1.10b)

$$L_{23} L_{34} - (q + q^{-1}) L_{34} L_{23} L_{23} + L_{34}^2 L_{23} = -L_{23},$$  \hfill (1.10c)

$$L_{34} L_{23} - (q + q^{-1}) L_{23} L_{34} L_{23} + L_{34}^2 L_{23} = -L_{34},$$  \hfill (1.10d)

$$[L_{12}, L_{34}] = 0.$$  \hfill (1.10e)
It is immediate to see that $L_{12}, L_{23}$ and $L_{23}, L_{34}$ respectively generate two $\mathfrak{o}(3)$ subalgebras of $\mathfrak{o}(4)$, however they do not appear within a direct sum, in contrast to what happens with $\mathfrak{o}(4)$.

If one introduces the following elements:

$$L_{13}^\pm = [L_{12}, L_{23}]_{q^\pm 1}, \quad L_{24}^\pm = [L_{23}, L_{34}]_{q^\pm 1}, \quad L_{14}^\pm = [L_{13}, L_{34}]_{q^\pm 1},$$  \hspace{1cm} (1.11)

where $[a, b]_q$ is defined as above and $[a, b]_{q^{-1}} := q^{-\frac{1}{2}} ab - q^{\frac{1}{2}} ba$, the two independent Casimir operators of the algebra $\mathfrak{o}(4)$ are then given by [27, 31, 41]

$$C_4 = q^{-2}L_{12}^2 + L_{23}^2 + q^2L_{34}^2 + q^{-1}L_{13}^2 L_{13} + qL_{24}^2 L_{24} + L_{14}^2 L_{14}, \hspace{1cm} (1.12a)$$

$$C_4' = q^{-1}L_{12} L_{34} - L_{13} L_{24} + qL_{23} L_{14}. \hspace{1cm} (1.12b)$$

1.2 The $q$-oscillator algebras, Schwinger and metaplectic realizations

Let us now recall the properties of the $q$-oscillator operators that will be used to realize the algebras presented above. The $q$-oscillator algebra $A_q(n)$ [43–45] is defined as the unital associative algebra over $\mathbb{C}$ generated by $n$ independent sets of $q$-oscillators $\{A_i^+, A_i^-\}$ verifying

$$[A_i^+, A_j^-] = \pm A_i^\pm, \quad [A_i^-, A_j^+] = q^{\frac{1}{2}0}, \quad A_i^- A_i^+ - qA_i^+ A_i^- = 1, \quad i = 1, \ldots, n, \hspace{1cm} (1.13)$$

and such that the commutators between elements with different indices $i$ are equal to zero. The last two relations allow one to express $N_i = A_i^+ A_i^-$ in terms of $A_i^0$:

$$N_i = A_i^+ A_i^- = \frac{1 - q^{a_i^0}}{1 - q} = (A_i^0)_q. \hspace{1cm} (1.14)$$

In the limit $q \to 1$, $A_i^0$ coincides with the usual number operator $N_i$.

The $q$-oscillator algebra has the following representation on the space spanned by the standard occupancy number states $|n_1, \ldots, n_n\rangle = |n_1\rangle \otimes \cdots \otimes |n_n\rangle$ ($n_i \in \mathbb{N}$):

$$A_i^0 |n_i\rangle = n_i |n_i\rangle, \quad A_i^+ |n_i\rangle = \sqrt{\frac{1 - q^{a_i^0}n_i + 1}{1 - q}} |n_i + 1\rangle, \quad A_i^- |n_i\rangle = \sqrt{\frac{1 - q^{a_i^0}n_i}{1 - q}} |n_i - 1\rangle. \hspace{1cm} (1.15)$$

These commuting $q$-oscillators can now be used to build realizations of the algebras considered above. Firstly, the algebra $\mathfrak{o}(3)$ can be realized à la Schwinger in terms of two $q$-oscillators. More precisely, using the homomorphism $\chi : U_q(\mathfrak{su}(2)) \to A_q(2)$ given by

$$\chi(j_0) = \frac{i}{2}(A_1^0 - A_2^0), \quad \chi(j_+) = q^{-\frac{1}{2}(A_1^0 + A_2^0 - 1)} A_1^+ A_2^-, \quad \chi(j_-) = q^{-\frac{1}{2}(A_1^0 + A_2^0 - 1)} A_1^- A_2^+, \hspace{1cm} (1.16)$$

and the identification [18], the following realization of $\mathfrak{o}(3)$ is obtained:

$$\chi(j_1) = \frac{ig^2}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} q^{-\frac{1}{2}A_2^0} (q^{\frac{1}{2}} A_1^- A_2^+ + q^{-\frac{1}{2}} A_1^+ A_2^-), \hspace{1cm} (1.17)$$

$$\chi(j_2) = \frac{g^2}{q^{\frac{1}{2}} + q^{-\frac{1}{2}}} q^{-\frac{1}{2}A_1^0} (q^{\frac{1}{2}} A_2^+ A_1^- - q^{-\frac{1}{2}} A_2^- A_1^+). \hspace{1cm} (1.18)$$

Another key ingredient is the metaplectic realization of $U_q(\mathfrak{su}(1, 1))$, which is given by the homomorphism $\mu : U_q(\mathfrak{su}(1, 1)) \to A_q(1)$:

$$\mu(J_0) = \mathcal{J}_0 = \frac{1}{2} \left( A_0^0 + \frac{1}{2} \right), \quad \mu(J_\pm) = \mathcal{J}_\pm = \frac{1}{2} i g^{1/2} (A_1^\pm)^2. \hspace{1cm} (1.19)$$
One sees immediately that it is a $q$-deformation of the usual metaplectic representation of $\mathfrak{su}(1,1)$.

Finally, we shall also use the realization of $\mathfrak{o}_{q^{1/2}}(4)$ in terms of $4$ $q$-oscillators which is provided by:

$$
\mathcal{L}_{i,i+1} = q^{-\frac{1}{q}A_0^+ + \frac{1}{q}(q^2 A_1^+ A_i^+ - A_i^+ A_1^+) - \frac{q}{q^2} A_i^- A_i^+}, \quad i = 1, 2, 3. 
$$

One checks that the $\mathcal{L}_{i,i+1}$ indeed verify relations of the form \(1.10\) but whose $q$'s have been replaced by $q^{1/2}$'s. Furthermore, $\mathcal{L}_{12}$, $\mathcal{L}_{34}$ commute and hence generate a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$.

2 The commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ in the $q$-oscillator realization of $U_q(\mathfrak{u}(4))$ and the $q$-Higgs algebra

It was shown in \[13\] that the Higgs algebra appears as the commutant of $\mathfrak{o}(2) \oplus \mathfrak{o}(2)$ in the universal enveloping algebra $U(\mathfrak{u}(4))$. This section aims to define the $q$-Higgs algebra through a $q$-analogue of this commutant picture.

We consider first the $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$ generated by $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$, and look for its commutant in $U_q(\mathfrak{u}(4))$.

Introduce the following three operators

$$
M^+ = \left( q^{A_0^+} + \frac{1}{q^2} (A_1^+)^2 + (A_2^+)^2 \right) \left( q^{A_0^-} + \frac{1}{q^2} (A_3^-)^2 + (A_4^-)^2 \right), 
$$

$$
M^- = \left( q^{A_0^+} + \frac{1}{q^2} (A_1^+)^2 + (A_2^+)^2 \right) \left( q^{A_0^-} + \frac{1}{q^2} (A_3^-)^2 + (A_4^-)^2 \right), 
$$

$$
L = (A_1^0 + A_0^0) - (A_0^0 + A_1^0), 
$$

which commute with the generators $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$ (in the limit $q \rightarrow 1$, $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$ correspond to rotations in the (1, 2) and (3, 4) planes).

One notes that each big parenthesis in the expression of the $M^\pm$ operators can actually be obtained by applying the coproduct of $U_q(\mathfrak{su}(1,1))$ to the $J_\pm$ generators. Recalling that the bilinears of the form $E_{ij} = A_i^+ A_j^-$, $i, j = 1, 2, 3, 4$ realize the $U_q(\mathfrak{u}(4))$ algebra \[40\], it can be observed that $M^\pm$, $L$ generate the non-trivial part of the commutant of $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ in the $q$-oscillator realization of $U_q(\mathfrak{u}(4))$.

It is immediate to see that $M^\pm$ and $L$ also commute with the central element

$$
H = \sum_{i=1}^{4} (A_i^0 + \frac{1}{q^2}).
$$

One could ask how were the expressions for $L$, $M^\pm$ obtained. First, the operator $L$ obviously commutes with $\mathcal{L}_{12}$ and $\mathcal{L}_{34}$. Second, instead of obtaining the factors in $M^\pm$ from the coproduct one can look for elements $T^\pm$ in $\mathfrak{A}_q(2)$ that commute with $\mathcal{L}_{12}$; this is most easily done “on-shell”, that is, by solving $[\mathcal{L}_{12}, T^\pm]|_{n_1, n_2} = 0$ for any two $q$-oscillator states. One thus arrives at

$$
T^\pm = q^{1/2}(A_0^0 + A_0^0) \left( q^{A_0^+ + \frac{1}{q^2}} (A_1^+)^2 + (A_2^+)^2 \right), \quad \alpha \in \mathbb{C}.
$$

Since $A_0^0 + A_0^0 = \frac{1}{2}(L + H)$, only the second factor of $T^\pm$ is relevant. The same is done with $\mathcal{L}_{34}$ on the direct product states $|n_3, n_4\rangle$. It is then clear that the only combinations of the operators $\mathcal{L}_{12}$ and their ($3, 4$) analogues that will belong to $U_q(\mathfrak{u}(4))$ are those occurring in $M^\pm$.

It now remains to determine the algebra formed by the three generators $M^\pm$ and $L$. 


Proposition 2.1 The operators $M^\pm$ and $L$ have the following commutators:

\[
[M^+, M^+] = -4M^\pm,
\]

\[
[M^+, M^-] = \left(\frac{1 + q}{q(1 - q)^3}\right) q^H \left((q + q^{-1})(q^L - q^{-L}) - 2(q^{\frac{1}{2}H} + q^{-\frac{1}{2}H})(q^{\frac{1}{2}L} - q^{-\frac{1}{2}L})\right) + \left(\frac{1 + q}{q^2(1 - q)}\right) q^H \left((q^{-\frac{1}{2}H}L_{12}^2 + q^{\frac{1}{2}H}L_{34}^2)q^{\frac{1}{2}L} - (q^{\frac{1}{2}H}L_{12}^2 + q^{-\frac{1}{2}H}L_{34}^2)q^{-\frac{1}{2}L}\right).
\] (2.4a)

The elements $L_{12}, L_{34}$ and $H$, which are central, play the role of structure constants. We shall take these relations to define abstractly the (universal) $q$-Higgs algebra.

Remark 2.1 Alternatively, if one considers the generator $q^{\frac{1}{2}L}$ instead of $L$, the first set of relations in (2.4a) becomes

\[
q^{\frac{1}{2}LM^\pm} = q^{-2}M^\pm q^{\frac{1}{2}L}.
\] (2.4b)

Proof: The first relations of (2.4a) are obvious. The last relation is obtained by a direct computation in the $q$-oscillator algebra. Starting with (2.1a)–(2.1b), and using the identity $[a_1^+, a_2^-, a_2^-] = [a_1^+, a_1^- a_2^-, a_2^-] = a_1^+, a_1^- a_2^-, a_2^-]$ for $a_1^+, a_1^- a_2^-, a_2^- = \left(q^{A_1^0 + \frac{1}{2}}(A_{21}^\pm)^2 + (A_{21}^\pm)^2\right)$, one gets

\[
[M^+, M^-] = \left[q^{2A_0^0+1}((A_1^+)^2, (A_1^-)^2) + [(A_2^+)^2, (A_2^-)^2] + q^{4A_0^0+\frac{1}{2}}(1 - q^2)((A_1^+)^2(A_2^-)^2 + q^{-2}(A_1^-)^2(A_2^+)^2]\right]
\times \left[q^{2A_0^0+1}(A_4^+)^2(A_4^-)^2 + (A_4^+)^2(A_4^-)^2 + q^{4A_0^0+\frac{1}{2}}((A_4^+)^2(A_4^-)^2 + q^{-2}(A_4^-)^2(A_4^+)^2)\right]
\times \left[q^{2A_0^0+1}((A_3^+)^2, (A_3^-)^2) + [(A_5^+)^2, (A_5^-)^2] + q^{4A_0^0+\frac{1}{2}}(1 - q^2)((A_5^+)^2(A_5^-)^2 + q^{-2}(A_5^-)^2(A_5^+)^2)\right]
\times \left[q^{2A_0^0+1}(A_6^+)^2(A_6^-)^2 + (A_6^+)^2(A_6^-)^2 + q^{4A_0^0+\frac{1}{2}}((A_6^+)^2(A_6^-)^2 + q^{-2}(A_6^-)^2(A_6^+)^2)\right].
\] (2.5)

Now, from the expression (1.20), one obtains

\[
L_{12}^2 = q^{-A_0^0 + \frac{1}{2}}(q(A_1^+)^2(A_2^-)^2 + q^{-1}(A_1^-)^2(A_2^+)^2 - q^{-\frac{1}{2}N_1 - q^{-\frac{1}{2}N_2 - q^{-\frac{1}{2}}q + \frac{1}{2}N_1 N_2}) (2.6)
\]

and a similar expression for $L_{34}^2$ with the replacement $A_1^0, A_2^0 \rightarrow A_3^0, A_4^0$.

Using the relations

\[
[(A_1^+)^2, (A_1^-)^2] = -(1 + q)q^{A_0^0}((q + q^{-1})N_1 + 1)
\] (2.7)

and

\[
q(A_1^+)^2(A_1^-)^2 = N_t^2 - N_i,
\] (2.8)

and after some algebra, one is left with the following equation

\[
[M^+, M^-] = \left[(1 + q)q^{A_0^0 + A_0^0 - 1}(1 - q)L_{12}^2 + \frac{q}{1 - q} q^{A_0^0 + A_0^0}(1 + q^2 - 2)\right][q^{A_0^0 + A_0^0 - 1}L_{34}^2 + [(A_0^0 + A_0^0)g]^2] - \left[(1 + q)q^{A_0^0 + A_0^0 - 1}(1 - q)L_{34}^2 + \frac{q}{1 - q} q^{A_0^0 + A_0^0}(1 + q^2 - 2)\right][q^{A_0^0 + A_0^0 - 1}L_{12}^2 + [(A_0^0 + A_0^0)g]^2].
\] (2.9)

Expressing the $A_0^0$ generators in terms of $L$ and $H$, one finally obtains the desired commutation relation.
Remark 2.2 In the limit $q \to 1$, noting that

$$\lim_{q \to 1} \mathcal{L}_{12} = 2i \mathcal{L}_{12}, \quad \lim_{q \to 1} \mathcal{L}_{34} = 2i \mathcal{L}_{34}, \quad \text{with} \quad \mathcal{L}_{jk} = -\frac{i}{2} \left( x_j \frac{\partial}{\partial x_k} - x_k \frac{\partial}{\partial x_j} \right),$$

(2.10)

one easily recovers from (2.4a) the commutation relations of the Higgs algebra (0.1) in the form:

$$[L, M^\pm] = \pm 4M^\pm,$$

$$[M^+, M^-] = -L^3 + \alpha_1 L + \alpha_2,$$

(2.11)

where $\alpha_1 = H^2 + 8(L_{12}^2 + L_{34}^2) - 4$, and $\alpha_2 = -8H(L_{12}^2 - L_{34}^2)$.

Hence, the relations (2.4) indeed define a $q$-deformation of the Higgs algebra.

3 The Askey–Wilson algebra and an embedding into $U_q(\mathfrak{su}(1, 1))^\otimes^2$

We now indicate how (a special case of) the Askey–Wilson algebra can be embedded in the tensor product $U_q(\mathfrak{su}(1, 1)) \otimes U_q(\mathfrak{su}(1, 1))$. With $\Delta$ the coproduct of $U_q(\mathfrak{su}(1, 1))$ given in (1.5), we can take

$$K_1 = \frac{1}{4} \left( 1 - q \frac{q_{J_0} \otimes q^{-J_0}}{1 - q} \right),$$

(3.1a)

$$K_2 = \frac{1}{2} \Delta(C) = \frac{1}{2} \left[ C \otimes q^{2J_0} + q^{-2J_0} \otimes C + J_+ q^{-2J_0 - 1} \otimes J_- + J_- q^{-2J_0 - 1} \otimes J_+ \right.$$

$$\left. + \frac{q^2 + 1}{(q^2 - 1)^2} \left( q^{-2J_0} \otimes q^{2J_0} - 1 \otimes q^{2J_0} - q^{2J_0} \otimes 1 + 1 \otimes 1 \right) \right],$$

(3.1b)

where $C$ denotes the Casimir operator given in (1.6).

Defining $K_3 = [K_1, K_2]$, a direct calculation gives

$$K_3 = \frac{1}{8} (1 + q^{-1})(J_+ \otimes J_- - J_- \otimes J_+)(q^{-J_0} \otimes q^{-J_0}).$$

(3.1c)

We now proceed to calculate the commutation relations of $K_1$, $K_2$, $K_3$. They are seen to take the form of the relations of the Askey-Wilson (AW) algebra which read

$$[K_1, K_2] = K_3,$$

(3.2a)

$$[K_2, K_3] = rK_2K_1K_2 + \xi_1 K_1 K_2 + \xi_2 K_2^2 + \xi_3 K_2 + \xi_4 K_1 + \xi_5,$$

(3.2b)

$$[K_3, K_1] = rK_1 K_2 K_1 + \xi_1 K_1^2 + \xi_2 K_1 K_2 + \xi_3 K_1 + \xi_6 K_2 + \xi_7,$$

(3.2c)

where $r$ is as in (3.3) below and $\xi_1, \ldots, \xi_7$ are arbitrary in the generic AW situation.

After a rather cumbersome calculation, using the expressions (3.1) for the $K_i$’s as well as the commutation relations (1.4), one finds that the $K_i$’s indeed obey the relations (3.2) with the following specific expressions for the parameters:

$$r = -(q - q^{-1})^2, \quad \xi_1 = \frac{1 + q^{-2}}{2}, \quad \xi_2 = \frac{(1 + q)^2(1 - q)}{4q^2}, \quad \xi_3 = 4(q - 1) \xi_7,$$

$$\xi_4 = 0, \quad \xi_5 = -\frac{(1 + q)(1 + q^2)(C^{(1)} - C^{(2)})[J_0^{(12)}]_q}{16q^4}, \quad \xi_6 = -\frac{(1 + q)^2}{16q^2},$$

$$\xi_7 = \frac{(1 + q)^2}{32q^2} \left( C^{(1)} q^{J_0^{(12)}} + C^{(2)} q^{-J_0^{(12)}} - (1 + q^{-2})[J_0^{(12)}]_q \right),$$

(3.3)
where $C^{(1)} = C \otimes 1$ and $C^{(2)} = 1 \otimes C$ are respectively the Casimir operators in the spaces 1 and 2 of the tensor product, and $J_{0}^{(12)} = \Delta(J_{0})$. These quantities $C^{(i)}$ and $J_{0}^{(12)}$ commute with $K_{1}$, $K_{2}$ and $K_{3}$ and we hence have a version of (3.2) that is centrally extended.

Since there are only three independant quantities entering the $\xi_{i}$'s (there are four in the general case), we conclude that the $K_{1}$, $K_{2}$, $K_{3}$ generate a specialization of the Askey–Wilson algebra. One checks that in the limit $q \to 1$, the parameters $r$, $\xi_{2}$, $\xi_{3}$ vanish and one recovers the Hahn algebra. The standard $q$-Hahn algebra is obtained from the Askey-Wilson algebra by setting for instance $\xi_{1} = 0$ in (3.2). The algebra satisfied by $K_{1}$, $K_{2}$ and $K_{3}$ is actually isomorphic to the $q$-Hahn algebra as the standard form of the latter [28, 47] is obtained by taking $K_{2} = K_{2} - \xi_{1}/r$. The limit $q \to 1$ is singular however if we adopt this presentation.

4 The $q$-Higgs algebra, the Askey–Wilson algebra, and the dual pair $(\mathfrak{o}_{q^{1/2}}(4), U_{q}(\mathfrak{su}(1,1)))$

We shall explain in this section how the $q$-Higgs algebra obtained as a commutant and the special Askey–Wilson algebra found from the embedding just described are connected through Howe duality and are in fact isomorphic.

Take 4 metaplectic representations defined as in (1.19). We will add them first pairwise using the $U_{q}(\mathfrak{su}(1,1))$ coproduct (1.10):

$$J_{0}^{(2i-1,2i)} = \frac{1}{2}(A_{2i-1}^{0} + A_{2i}^{0} + 1), \quad J_{\pm}^{(2i-1,2i)} = \frac{1}{2|q|^{1/2}} \left(q^{A_{2i}^{0} + \frac{1}{2}}(A_{2i+1}^{0})^{2} + (A_{2i}^{+})^{2}\right), \quad i = 1, 2. \quad (4.1)$$

and then using the coproduct once more will form

$$J_{0}^{(1234)} = J_{0}^{(12)} + J_{0}^{(34)} \quad \text{and} \quad J_{\pm}^{(1234)} = J_{\pm}^{(12)} q^{A_{21}^{0}} + J_{\pm}^{(34)}. \quad (4.2)$$

Mindful of Section 2, it is immediate to check that

$$[\mathcal{L}_{i,i+1}, J_{\pm}^{(1234)}] = 0, \quad i = 1, 2, 3, \quad (4.3)$$

where the $\mathcal{L}_{i,i+1}$ are defined as in (1.20). Let us stress that (4.3) makes the key statement that the algebras $U_{q}(\mathfrak{su}(1,1))$ and $\mathfrak{o}_{q^{1/2}}(4)$ are mutually commuting in the $q$-oscillator realization.

It has been shown [27] that $\mathfrak{o}_{q^{1/2}}(4)$ and $U_{q}(\mathfrak{su}(1,1))$ actually form a Howe dual pair. (They constitute precisely the quantum analogue of the classical pair $(\mathfrak{o}(4), \mathfrak{su}(1,1))$ which was used in the analysis of the Higgs and Hahn algebras [13].) This means that their representations can be connected through their Casimirs. We will now proceed to indicate explicitly how this is realized.

To that end, we first put the $J_{\bullet}^{(2i-1,2i)}$ in correspondance with the $J_{\bullet}$ from Section 3. Let us focus on the coproduct embeddings (1.1). As each pairing of $U_{q}(\mathfrak{su}(1,1))$ in the spaces $(1,2)$ and $(3,4)$ gives a copy of $U_{q}(\mathfrak{su}(1,1))$, we can embed the specialization of the Askey–Wilson algebra of Section 3 into these two copies of $U_{q}(\mathfrak{su}(1,1))$.

Indeed, upon substitution of (1.1) into equations (3.1) for $K_{1}$, $K_{2}$, $K_{3}$, we obtain the following $q$-
oscillator realization of the special Askey–Wilson algebra:

\[ \mathcal{K}_1 = \frac{1}{4} \frac{1 - q^2 \Delta}{1 - q}, \]  
\[ \mathcal{K}_2 = \frac{1}{2} \Delta^{(3)}(C) = \frac{1}{2} \left( (\mathcal{C}^{(1)} q^{H} + \mathcal{C}^{(2)} q^{-\frac{1}{2}H}) q^{-\frac{1}{2}L} + (1 + q^{-2}) q^{-\frac{1}{2}L} \right) \right), \]  
\[ \mathcal{K}_3 = \frac{1}{8(1 + q)} (M_+ - M_-) q^{-\frac{1}{2}H}. \]

where \( \Delta^{(n)}(x) = (i q^{(n-1)} \Delta)(x), \Delta^{(1)} = \Delta, \Delta^{(0)} = id; \) the generators \( M_\pm, L, H \) correspond to those given in (4.1) and (4.2) respectively and can alternatively be expressed as

\[ M_\pm = ([\mathcal{C}]_{q/2})^2 \mathcal{F}_\pm^{(12)} \mathcal{F}_\pm^{(34)}, \]  
\[ \frac{1}{2}L = \mathcal{F}_0^{(12)} - \mathcal{F}_0^{(34)}, \]  
\[ \frac{1}{2}H = \mathcal{F}_0^{(12)} + \mathcal{F}_0^{(34)} = \Delta^{(3)}(\mathcal{F}_0). \]

By construction the \( \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \) obey the relations of the special Askey–Wilson algebra with the parameters \( q/2 \).

Also note that the quantities \( \mathcal{C}^{(1)} \) and \( \mathcal{C}^{(2)} \) are the images of the Casimir operators \( C^{(1)} \) and \( C^{(2)} \) and that they are directly related to the \( \mathcal{L}_{12} \) and \( \mathcal{L}_{34} \) by

\[ \mathcal{C}^{(1)} = \frac{1}{(1 + q)^2} (\mathcal{L}_{12}^2 + 1), \quad \mathcal{C}^{(2)} = \frac{1}{(1 + q)^2} (\mathcal{L}_{34}^2 + 1). \]

In view of this, it is now evident that in the \( q \)-oscillator realization, the generators of the special Askey–Wilson algebra are expressible in terms of those of the \( q \)-Higgs algebra, and vice-versa. Hence these two algebras are isomorphic, as in the \( q \to 1 \) case.

To wrap things up, let us point out that the two Casimirs of \( \mathfrak{so}_{q/2}(4) \) given in (1.12) have a direct interpretation in this \( q \)-oscillator framework.

The first Casimir of \( \mathfrak{so}_{q/2}(4) \), denoted \( C_4 \), corresponds to the total Casimir of the quadruple tensor product of \( U_q(\mathfrak{su}(1, 1)) \):

\[ C_4 = \left( q^{-1} \mathcal{L}_{12}^2 + \mathcal{L}_{23}^2 + q \mathcal{L}_{34}^2 + q^{-\frac{1}{2}} \mathcal{L}_{13}^+ \mathcal{L}_{13}^- + q^{\frac{1}{2}} \mathcal{L}_{24}^+ \mathcal{L}_{24}^- + \mathcal{L}_{14}^+ \mathcal{L}_{14}^- \right) = \frac{(1 + q)^2}{2} \Delta^{(3)}(C). \]

This is precisely the pairing of the Casimirs of \( \mathfrak{so}_{q/2}(4) \) and \( U_q(\mathfrak{su}(1, 1)) \) that follows from the Howe duality.

The second Casimir of \( \mathfrak{so}_{q/2}(4) \), denoted \( C'_4 \), is identically zero in the \( q \)-oscillator realization:

\[ C'_4 = q^{-\frac{1}{2}} \mathcal{L}_{12} \mathcal{L}_{34} - \mathcal{L}_{13}^+ \mathcal{L}_{24}^- + q^{\frac{1}{2}} \mathcal{L}_{23} \mathcal{L}_{14}^+ = 0. \]

It can be seen as the \( q \)-analogue of the usual relation between the angular momenta, see for instance (4.1) in [13]: \( M_{12}M_{34} + M_{13}M_{24} + M_{14}M_{23} = 0 \).

Let us mention in closing this section that the \( q \to 1 \) limit of the above yields straightforwardly the duality presented in [13] between the Higgs or the Hahn algebras viewed as a commutant in \( U(\mathfrak{u}(4)) \) or embedded in \( U(\mathfrak{su}(1, 1)) \otimes U(\mathfrak{su}(1, 1)). \)
5 Conclusion

Summing up, we have introduced a $q$-analogue of the Higgs algebra by looking for the commutant of a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(4)$ in the $q$-oscillator representation of $U_q(u(4))$. This algebra was then seen to be isomorphic to a special case of the Askey–Wilson algebra (itself isomorphic to the standard $q$-deformation of the Hahn algebra) which has an embedding in $U_q(u(1,1)) \otimes U_q(u(1,1))$. The Howe dual pair $(\mathfrak{o}_{q^{1/2}}(4), U_q(u(1,1)))$ was then invoked as the reason behind this double picture.

The $q$-oscillator realization in which $\mathfrak{o}_{q^{1/2}}(n)$ and $U_q(u(1,1))$ commute can be generalized easily for $\mathfrak{o}_{q^{1/2}}(n)$ with $n$ arbitrary. It is known that $(\mathfrak{o}_{q^{1/2}}(n), U_q(u(1,1)))$ is a dual pair $^{[27]}$. This opens up the door to the study of the full Askey–Wilson algebra. We hypothesize that it should be possible to obtain this algebra as the commutant of a $\mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2) \oplus \mathfrak{o}_{q^{1/2}}(2)$ subalgebra of $\mathfrak{o}_{q^{1/2}}(6)$ in $U_q(u(6))$ in this $q$-oscillator representation. It would be also of high interest to see if the higher rank Askey–Wilson algebras $^{[49, 50]}$ could be obtained in a similar fashion.

It should moreover be mentioned that the dual pair $(\mathfrak{o}_{q^{1/2}}(n), U_q(u(1,1)))$ was analyzed in $^{[27]}$ in a $q$-commuting variable framework. It would be quite interesting to see if some sort of dimensional reduction in $q$-commuting variables could be performed to obtain a $q$-analogue of the superintegrable model on the $n$-sphere $^{[51]}$. We hope to address all these questions in the near future.

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