A thermodynamic motivation for dark energy

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Abstract

It is argued that the discovery of cosmic acceleration could have been anticipated on thermodynamic grounds, namely, the generalized second law and the approach to equilibrium at large scale factor. Therefore, the existence of dark energy—or equivalently, some modified gravity theory—should have been expected. In general, cosmological models that satisfy the above criteria show compatibility with observational data.

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I. INTRODUCTION

The standard cold dark matter (SCDM) model \[1\] was in good health until around the last decade of the previous century when it became apparent that the fractional density of matter falls well below the Einstein-de Sitter value, $\Omega_m = 1$—see e.g. \[2, 3\]. The death blow came at the close of the century with the discovery of the current cosmic acceleration \[4\], something the said model cannot accommodate by any means. However, to account for the acceleration in homogeneous and isotropic models one must either introduce some exotic energy component with a huge negative pressure (dubbed dark energy) or, more drastically, devise some theory of gravity more general than Einstein relativity \[5\]. Thus, both solutions appear somewhat forced and not very aesthetical. Here we argue that dark energy (or something equivalent) is demanded on thermodynamic grounds. In other words, we provide what we believe is a sound thermodynamic motivation for the existence of dark energy.

Our argument is based on that the natural tendency of systems to evolve toward thermodynamical equilibrium is characterized by two properties of its entropy function, $S(x)$, namely, it never decreases, $dS(x)/dx \geq 0$, and is convex, $d^2S(x)/dx^2 < 0$ \[6\]. In the context of an ever expanding Friedmann-Roberson-Walker (FRW) cosmology this translates in that the entropy of the apparent horizon plus that of matter and fields enclosed by it must fulfill $S'(a) \geq 0$ at any $a$—the generalized second law (GSL)—as well as $S''(a) \leq 0$ as $a \to \infty$, where $a$ is the scale factor of the FRW metric and the prime means $d/da$. The apparent horizon in FRW universes always exists (which is not generally true for the particle horizon and the future event horizon) and is known to posses not only an entropy proportional to its area \[7, 8\] but also a temperature \[9\]. Besides, it appears to be the appropriate thermodynamic boundary \[10\].

Before we proceed, it is fair to recall the (at least theoretical) existence of systems lacking any global maximum entropy state, such as Antonov’s sphere \[11, 12\]. The latter system consists in a sphere enclosing a number of particles that share some total energy. If the sphere radius happens to increase beyond some critical value, the system becomes unstable and the entropy function ceases to have a global maximum. We shall apply our argument under the assumption that the Universe tends to a state of maximum entropy irrespective of whether it may eventually reach it or not. It would be odd and frustrating that such a universal principle as the second law of thermodynamics could not be applied to the Universe as a
whole, especially given the close connection between thermodynamics and gravity \[13, 14\].

Section II illustrates why dark energy (or some or other modified gravity model) is required on thermodynamic basis and study some dark energy models to see whether they fulfill the thermodynamic criteria. Section III applies the said criteria to some representative modified gravity models. Finally, section IV summarizes and discusses our findings. As is customary, a naught subscript stands for the present value of the corresponding quantity.

II. WHY DARK ENERGY WAS TO BE EXPECTED

Let us consider a FRW universe. Its entropy is contributed by two terms: the entropy of the apparent horizon which is proportional to its area, \( \mathcal{A} = 4\pi \tilde{r}_A^2 \), and the entropy of the fluids enclosed by the horizon. Here \( \tilde{r}_A = \left( \sqrt{H^2 + (k/a^2)} \right)^{-1} \) denotes the radius of the horizon and \( H \) the Hubble factor of the FRW metric \([7]\). As is well-known, \( S_A \equiv \frac{k_B}{\ell_{Pl}} \frac{\mathcal{A}}{4} \), where \( \ell_{Pl} \) and \( k_B \) stand for Planck’s length and Boltzmann’s constant, respectively.

In virtue of the first Friedmann equation

\[
3H^2 + \frac{k}{a^2} = 8\pi G \rho ,
\]

where \( \rho \) is the total energy density, we can write

\[
\mathcal{A} = 4\pi \tilde{r}_A^2 = \frac{3}{2G} \frac{1}{\rho} ,
\]

as well as

\[
\mathcal{A}' = \frac{9}{2G} \frac{1 + w}{a \rho} ,
\]

where we have used the conservation equation \( \rho' = -3(1 + w)\rho/a \), where \( w = p/\rho \) denotes the overall equation of state parameter, i.e., not just of dark energy. From (3) the area will augment in expanding universes if \( 1 + w > 0 \) and decrease otherwise.

A further derivation with \( w = \) constant yields

\[
\mathcal{A}'' = -\frac{9}{2G} \frac{1 + w}{(a \rho)^2} (a \rho' + \rho) = \frac{9}{2G a^2 \rho} (1 + w)(2 + 3w) .
\]

Accordingly, \( \mathcal{A}'' \leq 0 \) for \(-1 \leq w \leq -2/3\), and \( \mathcal{A}'' > 0 \) otherwise. Thus, the above criterion disfavor the dominance of fluids at late times such that the overall equation of state is either of phantom type or larger than \(-2/3\).
We note in passing that when \( w \neq \text{constant} \) last equation generalizes to

\[
\mathcal{A}'' = \frac{9}{2G\alpha^3} \left[ (1 + w) \left( \frac{2 + 3w}{a} \right) \right].
\]

(5)

Let us consider the entropy associated to the fluid enclosed by the apparent horizon. If the fluid is just dust (pressureless matter, subscript \( m \)), i.e. \( \rho = \rho_m \) and \( p = p_m = 0 \), the unphysical result that the temperature vanishes follows whence Euler’s equation, \( Ts = \rho + p \), cannot be used to determine the entropy of cold matter (i.e., dust). We proceed instead as follows. Every dust particle contributes a given bit -say \( k_B \)- to the fluid entropy. So, within the apparent horizon we will have \( S_m = k_B N \) being \( N = (4\pi/3)\tilde{r}_A^3 n \) the number of particles there, and \( n = n_0 a^{-3/2} \) the (conserved) number density of dust particles. Then,

\[
S_m = k_B \frac{4\pi}{3} \tilde{r}_A^3 n_0 a^{-3} \propto a^{3/2}.
\]

(6)

Hence \( S_m'' \propto \frac{3}{4\sqrt{a}} > 0 \). We note parenthetically that the entropy augments in the volume enclosed by the horizon because the latter encompasses more and more particles as the universe expands (i.e., as \( H \) diminishes). Thus, \( S_m'' + S_A'' > 0 \).

We may now readily understand why sooner or later the Universe should accelerate, i.e., why it must be endowed with dark energy (subscript \( x \), equation of state \(-1 \leq w_x \leq -2/3\)), or something dynamically equivalent at the background level (as a suitably modified gravity). We note parenthetically that, at late times, dark energy is to dominate over all other energy components whereby \( w \simeq w_x \) as \( a \to \infty \). Were the Universe dominated by radiation and/or matter for ever, \( S_A'' \) could never become negative. And if the horizon entropy dominated the total entropy, then the Universe would never tend to a state of maximum entropy (compatible with the constraints of the system).

Figure 1 sketches the evolution of the horizon area for a spatially flat FRW universe dominated by a mixture of cold matter and dark energy, i.e., \( \mathcal{A} \propto [\Omega_m a^{-3} + \Omega_x a^{-3(1+w_x)}]^{-1} \) for \( w_x = -5/6 \) (solid line), \( w_x = -1 \) (dashed), and \( w_x = -1.2 \) (dot-dashed) -in the three cases we have assumed \( \Omega_m = 0.3 \). It is readily seen that for \(-1 \leq w_x < -2/3 \) (solid and dashed lines) the curvature evolves from positive to negative values and stays negative forever. By contrast, when the dark energy component is phantom, \( w_x < -1 \) (dot-dashed line) the curvature goes from positive to positive values with an intermediate, transient, phase of negative values about the maximum. Clearly, while in the two first instances the Universe tends to thermodynamic equilibrium at late times in the case of phantom it does
not. Needless to say, in the absence of dark energy (not shown) we would have $\mathcal{A}'' > 0$ for whatever value of the scale factor.

Figure 1. Schematic evolution of the area of the apparent horizon in a FRW universe dominated by cold matter and dark energy with $w_x = -5/6$ (solid line), $w_x = -1$ (dashed), and $w_x = -1.2$ (dot-dashed). In plotting the graphs we have taken $\Omega_{m0} = 0.3$.

For completeness, Fig. 2 illustrates the influence of the spatial curvature on the evolution of $\mathcal{A}$ when $a \gg 1$ for three $\Lambda$CDM models ($w_x = -1$). As can be seen, the curvature of the graph for positively curved models remains positive (recall that $\Omega_k \equiv -k/(a^2 H^2)$).

We believe that the GSL alongside the criterion that the total entropy $S$ must fulfill $S'' < 0$ at late times may serve to discard some cosmological models and to set limits on the evolution of $w_x$.

It will prove useful to express $\mathcal{A}$ and its derivatives in terms of the deceleration parameter, $q \equiv -\ddot{a}/(a H^2)$. The latter can be written as

$$q = -\left(1 + \frac{a H'}{H}\right).$$  \hfill (7)
Figure 2. Schematic evolution of the area of the apparent horizon in a FRW universe dominated by cold matter and cosmological constant \((w_x = -1)\). From top to bottom, \(\Omega_{k0} = -0.02, 0,\) and \(0.009,\) respectively, -see [19]. For \(a\) values of the order unity and below (not shown) the graphs practically overlap each other. In plotting the graphs we have chosen \(\Omega_{\Lambda0} = 0.7.\)

In spatially flat FRW universes \(A = 4\pi H^{-2},\) whence \(A' = 8\pi(1 + q)/H^2\) and

\[
A'' = 2A \left[ (1 + q)(1 + 2q) + \frac{q'}{a} \right].
\]  

(8)

It follows that for \(q \geq -1/2\) the second derivative of the horizon area, \(A'',\) cannot go from positive to negative values unless \(q' < 0.\) This excludes evolutions of \(q\) as in Fig. 3. The latter corresponds to cosmological models in which the current accelerated expansion is just transitory -after a period of dark energy dominance, pressureless matter is assumed to take over [13, 16].

At this stage it seems fitting to ask ourselves whether the second derivative of the entropy of the energy components -that for simplicity we will call fluids, subscript \(f\)- enclosed within the apparent horizon will be positive enough so that the sum \(S''_f + S''_A\) be positive. However, before proceeding we wish to remark that if the energy component is a scalar field in a pure quantum state, or the cosmological constant, it will have no entropy at all. Nevertheless, we wish to consider the possibility of the scalar field being in a mixture state and accordingly entitled to have entropy (obviously, this excludes the case \(w_f = -1\)).

For simplicity we will take \(k = 0\) and a single fluid component with \(w_f = \) constant. The entropy of the fluid filling the volume enclosed by the apparent horizon can be estimated by
Figure 3. Schematic of the evolution of the deceleration parameter, \( q(a) \), given by Eq. (11), in a FRW universe where the present acceleration stage is transitory and reverts for ever to matter domination.

Virtue of Gibbs equation,

\[ T_f dS_f = d \left( \rho_f \frac{4\pi}{3} H^{-3} \right) + w_f \rho_f d \left( \frac{4\pi}{3} H^{-3} \right) . \]  

(9)

With help of (1) and noting that \( \rho = \rho_f \)

\[ 2GT_f \frac{dS_f}{da} = -\frac{1}{H^2} (1 + 3w_f) \frac{dH}{da} . \]

Using the second Friedmann equation,

\[ \frac{dH}{da} = -4\pi G (1 + w_f) \frac{\rho_f}{aH} , \]  

(10)

and Eq. (11) once more, we get

\[ 2GT_f \frac{dS_f}{da} = \frac{3}{2} (1 + w_f) (1 + 3w_f) \frac{1}{aH} . \]  

(11)

Thus, \( dS_f/da \) will be negative or nil for \(-1 \leq w_f \leq -1/3\) and positive otherwise.

From last equation we can write,

\[ 2G \frac{d^2S_f}{da^2} = \frac{3}{2} (1 + w_f) (1 + 3w_f) \frac{d}{da} (aH T_f)^{-1} . \]

The evolution of the temperature of a perfect fluid, \( d \ln T_f / d \ln a = -3w_f \), readily follows from Gibbs’ equation and the condition for \( dS_f \) to be a differential expression -see e.g.
Recalling that \( w_f = \text{constant} \), it leads to \( T_f = T_{f0} a^{-3w_f} \). Using now \( \rho_f = \rho_{f0} a^{-3(1+w_f)} \) we arrive to
\[
2G \frac{da^2}{da^2} S_f = \frac{3}{4T_{f0} H_0} (1 + w_f) (1 + 3w_f) (1 + 9w_f) a^{(9w_f-1)/2}.
\] (12)

For \(-1 < w_f < -1/3\) (dark energy) as well as for \( w_f > -1/9 \) one follows that \( S_f'' > 0 \). However, \( S_f'' \) decreases with expansion for \( w_f < 1/9 \). The question arises whether the sum \( S_f'' + S_A'' \) will be positive or negative in the long run. To answer it we inspect the ratio between the right hand sides of Eqs. (12) and (4) and note that \( w = w_f \),
\[
\frac{S_f''}{S_A''} \propto a^{3(w_f-1)/2}.
\] (13)

This expression vanishes for \( w_f < 1 \) as \( a \to \infty \), whereby \( S_f'' + S_A'' < 0 \) in the long run provided \(-1 < w_f < -2/3\), i.e., if the fluid is dark energy.

It remains to be seen whether the GSL, \( S_f' + S_A' \geq 0 \), is fulfilled. Note that in virtue of Eqs. (3) and (11), with \( T_f \propto a^{-3w_f} \), the ratio
\[
\frac{S_f'}{S_A'} \propto \frac{1/(aT_f H)}{1/a \rho_f} \propto \rho_f^{1/2} a^{3w_f} \propto a^{3(w_f-1)/2}
\] (14)

vanishes in the long run provided \( w_f < 1 \). Thus albeit \( S_f' \) is negative for \(-1 < w_f < -1/3\), the GSL is satisfied for \( w_f > -1 \) when \( a \to \infty \).

Altogether, dark energy with constant equation of state in the interval \((-1, -2/3)\) respects the GSL as well as the criterion that \( S_f'' + S_A'' < 0 \) when \( a \to \infty \). This result is consistent with the tightest observational constraints available - cfr. Table in IV in [19]. In our view, cosmological models that meet both criteria should be preferred to those failing any of the two.

A. Parameterized \( w_x \) models

In this subsection we consider the model of Barboza and Alcaniz in which the dark energy equation of state is parameterized in terms of redshift as [20]
\[
w_x(z) = w_0 + \frac{w_1 z (1 + z)}{1 + z^2}.
\] (15)

It has the advantage over other parameterizations of not diverging at any time. We recall, parenthetically, that the redshift is related to the scale factor by \( 1 + z = 1/a \).
The pair of free, constants parameters \(w_0\) and \(w_1\) in (15) ought to be restricted by physical requirements and observation. A first constraint \(w_0 + w_1 < 0\) follows from demanding that dark energy be subdominant at early times (when \(z \gg 1\)), otherwise cosmic structure would never had formed. Next, we will use the GSL together with the condition that the sum \(S_m'' + S_x'' + S_A''\) be negative or nil to set further constraints on the said pair.

We begin by writing the Hubble function of a spatially-flat FRW universe dominated by cold matter and dark energy, with equation of state (15),

\[
H^2 = H_0^2 \left[ \Omega_m a^{-3} + \Omega_x a^{-3(1+w_0+w_1)} (2a^2 - 2a + 1)^{3w_1/2} \right].
\]

(16)

We next consider the entropy of the apparent horizon, \(S_A \propto A\). Two separate cases arise:

(i) \(w_1 < 2/3\) and \(-2/3 < w_0 < -w_1\) as well as (ii) \(w_0 < -2/3\) and \(w_0 + w_1 < 0\). In the first case, \(S_A'\) grows without bound and we do not consider it any further. In the second one \(S_A'\) approaches zero asymptotically from above. As \(a \to \infty\), we see that \(w_x(a) \to w_0\), \(\Omega_x \to 1\), \(aw_x' \to 0\) (\(\sim a^{-1}\)) and \(H^2 \sim a^{-3(w_0+1)}\) whence all the terms but the last one, that decays as \(a^{-1}\), in the second derivative of the horizon entropy,

\[
S_A'' \propto \mathcal{A}'' \propto \frac{1}{a^2 H^2} \left[ 3(w_x \Omega_x)^2 + \frac{5}{2} w_x \Omega_x - \frac{3}{2} w_x^2 \Omega_x + 1 + \frac{1}{2} a \Omega_x w_x' \right],
\]

(17)

present a like behavior. In consequence, the quantity in the square brackets tends to a constant, and \(S_A'' \sim a^{3w_0+1}\) as \(a \to \infty\). For \(S_A''\) to be negative that constant must be negative as well, then \(3w_0^2 + 5w_0 + 2 < 0\).

Finally, this constraint and the previous one set the accessible range to

\[
\begin{cases}
  w_1 < 2/3 \\
  -1 < w_0 < -2/3
\end{cases}
\bigcup
\begin{cases}
  2/3 \leq w_1 < 1 \\
  -1 < w_0 < -w_1,
\end{cases}
\]

(18)

as shown in Fig. (4).

Obviously, when the entropy of matter and dark energy are taken into account the previous result may vary. With the help of the equation for the temperature of dark energy

\[
T_x(a) = T_0 a^{-3(w_0+w_1)} (2a^2 - 2a + 1)^{3w_1/2},
\]

(19)

it can be checked that

\[
\frac{S_x'}{S_m'} \sim C_1 \quad \text{and} \quad \frac{S_x''}{S_m''} \sim C_2,
\]

where \(C_1\) and \(C_2\) are constants. Hence we focus on matter contribution with respect to the horizon.
Figure 4. The available region in the \((w_0, w_1)\) plane for the parameterized-\(w_x\) model of Barboza and Alcaniz [20].

When \(w_0 > 0\) it follows that
\[
\frac{S'_m}{S'_A} \sim \frac{S''_m}{S''_A} \sim a^{-3/2},
\]
whereby the horizon contribution prevails in the long run. However, we have already discarded this solution because \(S'_A\) diverges. In a sense, this is a consistency check on our criteria because observationally \(w_0\) is known to be negative.

In the opposite case, \(w_0 < 0\), one obtains
\[
\frac{S'_m}{S'_A} \sim \frac{S''_m}{S''_A} \sim a^{3(w_0-1)/2}.
\]
Again, the horizon contribution prevails, and the result (18) stands.

The FRW universe model dominated by cold matter plus dark energy, with equation of state parameterized according to Eq. (15), has been observationally constrained in [20] by using data from supernovae type Ia, baryon acoustic oscillations, and cosmic background radiation. The best fit values of the parameters at 1\(\sigma\) confidence level were found to lie in the ranges
\[
-1.35 \leq w_0 \leq -0.86, \quad -0.33 \leq w_1 \leq 0.91, \quad (20)
\]
in full consistency with the above result, Eq. (18).
B. Chaplygin gas model

The original Chaplygin gas model unifies matter and dark energy in the sense that they are no longer two separate components but a unique entity that mimics cold matter at early times and a cosmological constant at late times \[21\]. Its equation of state \( p = -A/\rho \), with \( A \) a positive constant and \( \rho \) the energy density of the gas (i.e., the total energy density), is obtainable from the Nambu-Goto action for a d-brane moving in the \( d + 1 \) dimensional bulk \[22\]. In a FRW universe, the dependence of the energy density on the scale factor reads

\[
\rho = \sqrt{A + (B/a^6)},
\]

where \( B \) is nonnegative integration constant; or equivalently,

\[
\rho = \rho_0 \left[ 1 - \Omega_* + \Omega_* a^{-6} \right]^{1/2}
\]  

with \( \Omega_* = B/(A + B) \).

The evolution of the horizon area \((A \propto 1/\rho)\) is akin to the one depicted by the solid line in Fig. that is to say, it grows monotonously and presents negative curvature at sufficiently large scale factor.

By contrast, since the entropy of the Chaplygin gas obeys \( S_{Ch} \propto (1 + w)/(HT) \), where

\[
w = \frac{p}{\rho} = - \left[ 1 + \frac{\Omega_*}{1 - \Omega_*} a^{-6} \right]^{-1},
\]

refers to the equation of state parameter of the gas (which evolves from zero to \(-1\)), and the temperature is governed by \( T'/T = -3w/a \), one follows that \( S'_{Ch} < 0 \) and \( S''_{Ch} > 0 \) for large scale factor. However, as it can be checked, \( S'_{Ch}/S'_{A} \to 0 \) and \( S''_{Ch}/S''_{A} \to 0 \) as \( a \to \infty \). Accordingly, the total entropy, \( S = S_{Ch} + S_{A} \), fulfills \( S' > 0 \) and \( S'' < 0 \) in the same limit. In other words, the GSL is respected in that limit and the Universe tends to thermodynamic equilibrium in the long run.

C. Holographic, interacting models

Here we first consider a spatially-flat, holographic, interacting model dominated by pressureless matter and dark energy. The latter component is assumed holographic in the sense that its energy density varies as the area of the apparent horizon and interacts with matter at a given, non-constant, rate. As a result \( w_x \) decreases with expansion and the fractional densities of both components remain constant. This much alleviates the cosmic coincidence problem \[23\] and the model shows compatibility with observation \[24\].
Inspection of the Hubble factor, given by Eq. (2.4) of [24],
\[ H(a) = H_0 \left[ \frac{\Gamma}{3H_0 r} + \left( 1 - \frac{\Gamma}{3H_0 r} \right) a^{-3/2} \right], \tag{23} \]
readily reveals that the graph of the evolution of the area of the apparent horizon, \( A \propto 1/H^2 \), has positive curvature at the beginning of the expansion and gently evolves to negative curvature values remaining thus for ever, similarly to the solid line in Fig. 1, for the best fit observational values, \( \Gamma/H_0 = 0.563 \) and \( r = 0.452 \), obtained in [24].

As can be readily checked, the second derivative of the entropy of the dark energy within the horizon, \( S_x \), is negative as well. Effectively,
\[ S_x = \frac{4\pi}{3} \tilde{r}_A^3 = \frac{4\pi}{3} \frac{1}{H^3} \frac{1 + w_x}{T_x} \rho_x \propto \frac{1 + w_x}{H T_x}, \tag{24} \]
where \( r = \rho_m/\rho_x \) and \( T_x \propto a^{-3w_x} \) when \( a \to \infty \). For \( a \gg 1 \) one has that \( w = \text{constant} \), and one obtains \( S''_x \propto 3(1 + w_x) w_x (3w_x - 1) < 0 \) since, in the model at hand, \( 1 + w_x < 0 \).

As for the entropy of pressureless matter
\[ S_m = k_B \frac{4\pi}{3} \tilde{r}_A^3 n = k_B \frac{4\pi}{3} \frac{\rho_m}{m H^3} = \frac{4\pi}{3} \frac{1}{m H^3} \frac{r}{1 + r} (\rho_m + \rho_x) \propto \frac{1}{H}, \tag{25} \]
where \( m \) stands for the mass of the particles. (Recall that in this model \( r = \text{constant} \)). We then have, \( S''_m \propto -H'/H^2 \) and \( S''_r \propto 2(\dot{H}/H^3) - (H''/H^2) \). Using (23) we find that \( S''_m > 0 \) and
\[ S''_m \propto \frac{\gamma \beta^2}{a^3 H^3}, \]
where \( \beta \) is a short-hand for the dimensionless combination \( \Gamma/(3H_0 r) \) and \( \gamma = (1 - \beta)/\beta \); both quantities lie in the range \((0,1)\). For \( a \gg 1 \), the second term is the leading one whence, at sufficiently large scale factor, \( S''_m \) is also negative. In summary, in this model, \( S''_A + S''_m + S''_x < 0 \) for \( a \to \infty \).

It would be misleading, however, to believe that for all holographic models \( A'' < 0 \) when \( a \gg 1 \). For instance, in the holographic model of Gao et al. [25], which uses the Ricci’s length as infrared cutoff, the Hubble equation reads
\[ H = H_0 \sqrt{\Omega_{k0} a^{-2} + \Omega_{m0} a^{-3} + \Omega_{r0} a^{-4} + \frac{\alpha}{2 - \alpha} \Omega_{m0} a^{-3} + f_0 a^{(2/\alpha)-4}}, \tag{26} \]
where the subscript \( r \) stands for radiation; \( \alpha \simeq 0.46 \) and \( f_0 \simeq 0.65 \) are dimensionless parameters. As Fig. 5 shows, the area of the apparent horizon grows to a maximum to monotonously decrease afterwards for ever. Consequently, \( A'' > 0 \) as \( a \to \infty \).
Figure 5. Schematic evolution of the area of the apparent horizon of the holographic, non-interacting model of Gao et al. [25]. The curvature is negative about the maximum and becomes larger than zero later on and remains thus for ever. In drawing the graph we have adopted the values in the said reference: \( \Omega_{k0} = 0, \Omega_{r0} = 8.1 \times 10^{-5}, \Omega_{m0} = 0.27, \Omega_{x0} = 0.73, \alpha = 0.46, \) and \( f_0 = 0.65. \)

On the other hand, the entropies of the fluid components (radiation, matter, and dark energy) decrease with expansion and their second derivatives are positive for large scale factor. Thus, this model violates the GSL and does not approach thermodynamic equilibrium at late times.

### III. MODIFIED GRAVITY MODELS

Models that depart from Einstein gravity may lead to a late acceleration era without the help of any exotic component of negative pressure. Here we examine some of them, namely: the one based in the brane-induced gravity model of Dvali et al. [26] (see [27, 28]), the original Cardassian model proposed by Freese and Lewis [29], and the torsion model of Bengochea and Ferraro [30], to check whether they fulfill these criteria. Before we proceed,
it is fair to say that, as far as we know, it has not been rigorously proven that the entropy associated to the apparent horizon of the two latter models simply obeys $S_A \propto A$, but it seems to us a very reasonable assumption and we shall adopt it.

\section*{A. Dvali-Gabadazze-Porrati’s model}

This model considers our 4-dimensional Universe as a brane embedded in a 5-dimensional bulk with flat Minkowski metric. As a consequence of the brane-induced term, the conventional Friedmann’s equation modifies to

$$H^2 + \frac{k}{a^2} = \left( \sqrt{\frac{\rho}{3M_{Pl}^2}} + \frac{1}{4r_c^2} + \frac{1}{2r_c} \right)^2,$$

where $\rho$ and $r_c$ stand for the total energy density (matter plus radiation in this model) and the crossover scale below which gravity appears as four dimensional, respectively.

The entropy of the apparent horizon, as computed by Sheykhi et al. \cite{31},

$$S_A = k_B \frac{3\pi \bar{r}_A^2}{\ell_{Pl}^2} \left[ 1 + \frac{\bar{r}_A}{r_c} \right],$$

is no longer proportional to the horizon area but contributed by two terms. The first one corresponds to the Bekenstein-Hawking entropy and accounts for gravity on the brane. The second one is related to the bulk. Its evolution in terms of the scale factor is sketched in the left panel of Fig. 6. From a certain point on it increases monotonously with negative curvature. As before, the entropy of dust and radiation are proportional to $(aH)^{-3}$. Both of them decrease monotonously with positive curvature. It is found that

$$\frac{S_m'}{S_A'} \sim \frac{S_r'}{S_A'} \rightarrow -\frac{1}{3\Omega_m r_c H_0},$$

(subscript $r$ for radiation), bear in mind that $r_c$ has physical dimensions of time. Thus, in the long run it leads to a negative constant -though the latter does not exactly coincides in the radiation and matter cases.

Once again the question arises whether the total entropy, $S = S_m + S_r + S_A$, obeys the GSL and presents negative curvature when $a \gg 1$. To answer this consider the limiting expressions when $a \rightarrow \infty$,

$$S' = S_A' \left[ 1 + \frac{S_m'}{S_A'} + \frac{S_r'}{S_A'} \right] \rightarrow 0,$$

$$S'' = S_A'' \left[ 1 + \frac{S_m''}{S_A''} + \frac{S_r''}{S_A''} \right] \rightarrow 0.$$
Figure 6. Left panel: Schematic evolution of the entropy of the horizon of the Dvali-Gabadadze-Porrati model. Right panel: The single graph depicts the qualitative evolution of the entropy of radiation as well as of cold matter inside the horizon. In drawing the graphs we have used the best-fit values of the parameters of the model, $r_c H_0 = 1.21$ and $\Omega_{m0} = 0.18$, obtained in [32].

The first one tends to zero from above, the second one from below (following the healthy behavior of the apparent horizon entropy). Both relations when taken together imply the inequality

$$1 - \frac{4\ell_p^2 c}{27 k_B \Omega_{m0} r_c H_0^2} \left[ k_B n_0 + \frac{4 \rho_{r0} c^2}{T_{r0}} \right] < 0,$$

which translates into an upper bound on the current number density of dust particles,

$$n_0 < \frac{1}{k_B} \left[ \frac{27 k_B \Omega_{m0} r_c H_0^2}{4 \ell_p^2 c} \right] \sim 10^{38} \text{ cm}^{-3}.$$  

Since it is fulfilled by a huge margin the GSL is satisfied and $S''$ results negative in the long run.

**B. Cardassian model**

In this spatially-flat FRW model the first Friedmann equation acquires an extra term that accounts for acceleration at sufficiently high redshifts,

$$H^2 = \frac{8 \pi G}{3} \rho + B \rho^\alpha.$$  

Here two new non-negative constants, $B$ and $\alpha$, appear while $\rho$ stands for the energy density of cold matter - the only energy component. For $\alpha < 2/3$ the universe features a transition
from \( q > 0 \) to \( q < 0 \) at redshift

\[
z_{tr} = \left[ \frac{(1 - \frac{3\alpha}{2}) B}{(4\pi G/3) \rho_0^{1-\alpha}} \right]^{1/(1-\alpha)},
\]

without need whatsoever of dark energy.

As can be checked, \( A = 1/H^2 \) evolves in such a way that while \( A' > 0 \) at all redshifts, \( A'' \) changes from positive to negative values at some point and stays thus for ever. This was to be expected since the extra term on the right hand side of (34) dynamically amounts to the presence of some dark energy component. Thus, the overall behavior of \( A \) should be qualitatively similar to that of a model dominated by a mixture of pressureless matter and dark energy. To be more specific, comparison of the right hand sides of (34) and the first Friedmann equation for a mixture of cold dark matter and dark energy with constant \( w_x \) and \( k = 0 \), namely, \( H^2 = (8\pi G/3)[\rho_{m0} a^{-3} + \rho_{x0} a^{-3(1+w_x)}] \), shows that, at the background level, every Cardassian model can be mapped to a spatially-flat dark energy one satisfying \( \alpha = 1 + w_x \) and \( B = (8\pi G/3) (\rho_{x0}/\rho_{m0}^\alpha) \). As a consequence \( A'' < 0 \) for sensible \( \alpha \) values (i.e., \( 0 < \alpha < 2/3 \)).

Although the entropy of the matter enclosed by the apparent horizon varies as \( (a H)^{-3} \), whence \( S_m' < 0 \) and \( S_m'' > 0 \) for \( a \gg 1 \), the ratios \( S_m'/A' \propto a^{3(w_x-1)/2} \) and \( S_m''/A'' \propto a^{3(w_x-1)/2} \) tend to zero in the same limit. It implies that \( S_m' + S_A' > 0 \) and \( S_m'' + S_A'' < 0 \) as \( a \to \infty \).

### C. Torsion model

The torsion model of Bengochea and Ferraro \cite{30} falls into a class of models that describes gravitation in terms of the torsion scalar \( \tau \) rather than curvature, \( R \). Thus, the action takes the form

\[
I = \frac{1}{16\pi G} \int d^4x \sqrt{-\tilde{g}} (\tau + f(\tau)) + I_{\text{matter}},
\]

where \( f(\tau) \) is a free function to be constrained by observation and experiments. An important advantage of this set of theories is that the field equations are second order as opposed to the four order equations of \( f(R) \) gravity. However, up to now, specific models have been proposed in the cosmological context only. Works investigating the symmetries and dynamics of the theory show that it can exhibit extra degrees of freedom since the theory is not local Lorentz invariant \cite{33}.
Bearing in mind that for the FRW metric $\tau = -6H^2$, from (35) the generalized Friedmann equations

\[ H^2 = \frac{8\pi G}{3}\rho - \frac{f}{6} - 2H^2 f' , \]  

\[ \frac{dH^2}{d\ln a} = \frac{16\pi G p + 6H^2 + f + 12H^2 f'}{24H^2 f' - 2 - 2f''} , \]  

follow. Here, $\rho$ and $p$ denote the energy density and pressure of the fluid component—which, we will assume, just cold matter.

In the model of Ref. [30]

\[ f(\tau) = -\alpha(-\tau)^{-n} , \]  

where $\alpha = (1 - \Omega_{m0})(6H^2_0)^{1-n}/(2n - 1)$.

We evaluate the entropy of the apparent horizon and matter entropy using the best fit values of the parameters, $n = -0.1$ and $\Omega_{m0} = 0.27$ -cfr. Ref. [30]. Left panel of Fig. 7 shows the evolution of area of the apparent horizon in terms of the scale factor. The plot starts with positive curvature and is ever increasing. Later on, the effect of the “dark torsion” is felt and the sign of the curvature changes to remain negative for ever. The right panel qualitatively depicts the evolution of the entropy of cold matter (proportional to $(aH)^{-3}$). It begins increasing to decrease after the maximum. The curvature is negative about the maximum only. Thus, the question arises whether for $a \to \infty$ the GSL will be respected and the sum $S_m'' + S_A''$ will be negative. A numerical study shows that the ratios $|S_m'|/S_A'$ and $S_m''/|S_A''|$ increase with the scale factor when $a \gg 1$. Therefore the answer to both questions appears to be no.

Before closing this subsection it is fair to recall that it is still unknown whether $f(\tau)$ models admit black holes solutions. If they don’t, it will become unclear that some connection entropy-horizon area really holds in these theories.

IV. DISCUSSION

As we have argued, neither a radiation nor a cold matter dominated universe can tend to thermodynamic equilibrium in the long run. By contrast, dark energy dominated universes may; this holds true irrespective of whether the dark energy component has entropy or not. Accordingly, dark energy (or something dynamically equivalent at the background level, such as a suitably modified gravity theory) appears thermodynamically motivated. In other
words, any of these two kind of ingredients was to be expected on thermodynamic grounds. We, therefore, should not wonder that the Universe is accelerating. However, it does not mean that every accelerating universe is thermodynamically motivated; that is the case, for instance, of any phantom dominated expansion with $w_x = \text{constant}$, and some modified gravity models.

One may object that if our reasoning were valid, the Universe would have never ceased to inflate as it would mean a transition from acceleration, $\mathcal{A}'' < 0$, to deceleration $\mathcal{A}'' > 0$. Clearly if the primordial inflation would have lasted for ever, big bang nucleo-synthesis would never have occurred, galaxies couldn’t have come into existence, and so on -something in stark contrast with observation. However, this reasoning is rather incomplete as it leaves aside the huge entropy generated during the reheating process at the end of inflation. In this explosive and quasi-instantaneous event the inflaton field relinquishes all its energy in the form of matter and radiation and, as a consequence, the Universe sees its temperature enormously increased [34]. Since this huge amount of matter and radiation thermalizes (a necessary condition for primordial nucleosynthesis) both second derivatives, $S_m''$ and $S_r''$, are negative and, as a result, we may well have that $S_r'' + S_m'' + S_A'' < 0$. Obviously, this will depend on the specific inflationary model and the particular reheating process involved, but we are not aware of any general argument against the fulfillment of this inequality. Specific calculations in this connection will be the subject of a future research.
Interestingly enough, cosmological models complying both with the GSL and the thermodynamic criterion that, at late times, the Universe should approach thermodynamic equilibrium appear to be compatible with observational data. Nevertheless, some models that do not comply with the said criteria (as some phantom models and some modified gravity models) seem also consistent with the said data.

Our argument could be falsified if it were discovered that in the past (but after the radiation-dominated period set in) the Universe experienced one (or more) transitions from deceleration to acceleration and back. Cosmologies of the type have been proposed to explain the seemingly periodic distribution of galaxies with redshift -see, e.g. [35], [36] - and as an expedient to solve the cosmic coincidence problem [37]. However, recent studies on the impact of hypothetical transient periods of acceleration-deceleration on the matter growth [38] and on the radiation power spectrum [39] from the decoupling era, $a \simeq 10^{-5}$, to $a = 0.5$ practically discard these periods in the redshift intervals considered.

If eventually it gets confirmed that the present phase of acceleration is to last forever, it may be seen as an indication that the Universe as a whole obeys the laws of thermodynamics (with the reservation that not all models that accelerate at late times comply with them). If, on the contrary, the Universe resumed a decelerated stage one should either call into question the validity of applying the said laws in a cosmological setting or wait for a later, and definitive, accelerating era.

Altogether, it is for the reader to decide which possibility looks the less queer: dark energy (or modified gravity) or a universe that will never approach thermodynamic equilibrium. The remaining possibility, either dark energy or modified gravity combined with an increasing departure from equilibrium, involves two oddities rather than one.

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