Detecting the $c$-effectivity of motives, their weights, and dimension via Chow-weight (co)homology: a "mixed motivic decomposition of the diagonal"

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Abstract

We describe certain criteria for a motif $M$ to be $c$-effective, i.e., to belong to the $c$th Tate twist $\text{Obj} \, DM^eff_{gm,R}(c) = \text{Obj} \, DM^eff_{gm,R} \otimes L^c$ of effective Voevodsky motives (for $c \geq 1$; $R$ is the coefficient ring). In particular, $M$ is 1-effective if and only if a complex whose terms are certain Chow groups of zero-cycles is acyclic. The dual to this statement checks whether an effective motif $M$ belongs to the subcategory of $DM^eff_{gm,R}$ generated by motives of varieties of dimension $\leq c$. These criteria are formulated in terms of the Chow-weight (co)homology of $M$. This homology theory is introduced in the current paper and has several (other) remarkable properties: it yields a bound on the "weights" of $M$ (in the sense of the Chow weight structure defined by the first author) and detects the effectivity of "the lower weight pieces" of $M$. We also calculate the "connectivity" of $M$ (in the sense of Voevodsky’s homotopy $t$-structure) and prove that the exponents of the higher motivic homology groups (of an "integral" motif) are bounded whenever these groups are torsion. These motivic properties of $M$ have important consequences for its (co)homology.

Our results yield a vast generalization of the so-called "decomposition of the diagonal" statements.

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Introduction

The well-known technique of decomposition of the diagonal (cf. Remark 0.2 below) was introduced by Bloch and Srinivas in [BlS83]. Let us recall some easily formulated "motivic" results obtained via this method (and essentially established in [Via11]). For simplicity, we will state them for motives and Chow groups with rational coefficients over a universal domain $k$ (though certain generalizations of these results are also available).
Proposition 0.1. (i) Let $M$ be an effective Chow motif over $k$. Then $M$ is $c$-effective (i.e., it can be presented as $M' \otimes \mathbb{L}^\otimes c$ for some $c > 0$ and an effective $M'$) if and only if $\text{Chow}_j(M) = 0$ for $0 \leq j < c$ (see Remark 3.10 of [Via11]).

(ii) Let $h : N \to M$ be a morphism of effective Chow motives. Then $\text{Chow}_0(h)$ is surjective if and only if it "splits modulo 1-effective motives", i.e., if it corresponds to a representation of $M$ as a retract of $N \bigoplus Q \otimes \mathbb{L}$ for some effective motives $N$ and $Q$ (this is essentially Theorem 3.6 of ibid.; cf. Remark 0.2 below).

(iii) Let $h : N \to M$ be a morphism of effective Chow motives. Then the homomorphisms $\text{Chow}_j(h)$ are surjective for all $j \geq 0$ if and only if $h$ splits (this is Theorem 3.18 of ibid.).

Certainly, the Poincare duals to these results are also valid (cf. Remark 3.11 of ibid.). In statements of this sort one usually takes $M$ being the motif of a smooth projective $P/k$, whereas $N$ is obtained by resolving singularities of a closed subvariety $P'$ of $P$ (cf. Lemma 3 of [GoG13] and Theorem 3.6 of [Via11]). In this case, if $\text{Chow}_j(h)$ is surjective for all $j < c$ (this condition is called a cycle-theoretic connectivity one in [Par94]) then the diagonal cycle $P \subset P \times P$ is rationally equivalent to a sum of a cycle supported on $P' \times P$ and a one supported on $P \times W$ for some closed $W \subset P$ of codimension $\geq c$ (see Proposition 6.1 of [Par94]).

Remark 0.2. The latter formulation is an example of the "decomposition of the diagonal" statements in their "ordinary" form.

One can usually reformulate these cycle-theoretic statements using the following trivial observation: if $M$ is an object of an additive category $B$, $id_M = f_1 + f_2$ (for $f_1, f_2 \in B(M, M)$), and $f_i$ factor through some objects $M_i$ of $B$ (for $i = 1, 2$), then $M$ is a retract of $M_1 \bigoplus M_2$. In particular, if $B$ is Karoubian (see §1.1 below), then $M$ is direct summand of $M_1 \bigoplus M_2$.

One of the motivations for the results of this sort is that they reduce the study of various properties of $M$ to the study of "more simple motives" (i.e., of motives of "smaller dimension"); cf. Theorem 1 of [BIS83]. Certainly, these results also have nice cohomological consequences (that are called Hodge-theoretic connectivity ones in [Par94]); see Proposition 6.4 of ibid.

The goal of the current paper is to establish certain "mixed motivic" analogues of these results. Our main theorem includes a criterion for an effective Voevodsky motif to be $c$-effective (i.e., to belong to $\text{Obj} \ DM_{gm,R}^c(k) \otimes \mathbb{L}^\otimes c$; here $R$ is an arbitrary coefficient ring (containing $\frac{1}{p}$ if the characteristic of the base field $k$ is $p > 0$). A particular case of our results is the following statement: in the setting of Proposition 0.1(ii) the cone of $h$ is $c$-effective.
(i.e., the two-term complex $N \to M$ is homotopy equivalent to a cone of a morphism of $c$-effective Chow motives) if and only if $\text{Chow}_j(h)$ is bijective for all $j < c$ (for a general $k$ and $R$ one has to compute $\text{Chow}_j^R$ at arbitrary function fields over $k$ in this criterion). This is also equivalent to the existence of a morphism $h' : M \to N$ that is "inverse to $h$ modulo cycles supported in codimension $c$" (see Remark 3.2.5 below for more detail). We also establish a criterion for $\text{Chow}_j^R(h)$ to be bijective for $j < c_1$ and surjective for $c_1 \leq j < c_2$ (in Corollary 3.2.4). Even these very particular cases of our results seem to be new; they demonstrate that we really generalize the "decomposition of the diagonal" statements.

Our criteria for Voevodsky motives are formulated in terms of Chow-weight homology (of an object $M$ of $\text{DM}_{\text{gm},R}^{\text{eff}}$). These homology theories are new; they are defined as the (co)homology of the complexes obtained by applying $\text{Chow}_j^R$ (for $j \geq 0$) to the weight complex of $M$ (as defined in [Bon09] and [Bon10a]; the relation of the latter functor to the one of Gillet and Soulé is recalled in Remark 1.4.3(2) below). Chow-weight homology has several other remarkable properties; in particular, it "calculates weights" of motives (i.e., $M \in \text{DM}_{\text{gm},R}^{\text{eff}}$ in the notation of §2.2 below if and only if its Chow-weight homology vanishes in degrees $> n$; see Theorem 3.2.2(I.3)). It certainly follows that the higher degree Chow-weight homology of $M$ can be "detected" through the $E_2$-terms of (Chow-) weight spectral sequences for any (co)homology of $M$ (these weight spectral sequences generalize Deligne’s ones; see Remark 3.2.3(2) and Proposition 4.3.1 below for more detail). Moreover, one can "mix" the effectivity criteria with the "weight" ones; in particular, this yields a criterion for $M$ to be "$c - 1$-connective" (i.e., to belong to $\text{DM}_{\text{gm},R}^{\text{eff}}$ for $c \in \mathbb{Z}$, where $t_{\text{hom}}^R$ is the $R$-linear version of the homotopy $t$-structure of Voevodsky; see the end of §2.1 below). We also prove that the higher degree Chow-weight homology (resp. motivic homology) groups of $M$ are torsion if and only if their exponents are (finite and) uniformly bounded. One can also pass to the dual in our main results and obtain a bound on the "dimension" of $M$ (see Proposition 4.1.1 below). The facts described demonstrate the utility of Chow-weight (co)homology. Besides, the results of this paper (starting from §2.2) also illustrate that (Chow-) weight complexes and the Chow weight structure are very useful for the study of motives. They allow reducing several interesting questions on Voevodsky motives to the properties of Chow motives and birational motives (as defined in [KaS02]).

For the sake of the readers "scared of" Voevodsky motives, we note that our results can be applied to $K^*(\text{Chow}_R^{\text{eff}})$ instead of $\text{DM}_{\text{gm},R}^{\text{eff}}$ (see Remark 3.2.3(8) below). Yet even this "more elementary" version of the result is "quite triangulated", and its proof involves certain (triangulated) categories
of birational motives.

We also note that the vanishing of Chow-weight homology of $M$ in negative degrees does not yield the corresponding bound on the "weights" of $M$ (in contrast to Theorem 3.2.2(I.3)); see part 9 of the remark cited.

Now let us describe the contents of the paper; some more information of this sort can be found at the beginnings of sections.

In §1 we recall a large portion of the theory of weight structures (and prove two new lemmas on the subject).

In §2 we describe several properties of (various categories of) Voevodsky’s motives and of the Chow weight structures for them. They yield the computation of the intersection of $\text{DM}^\text{eff}_{\text{gm}, R}(c)$ with $d \leq n \text{DM}^\text{eff}_{\text{gm}, R}$ (this is a new result), and several other related statements. We also prove some auxiliary statements on the behaviour of complexes whose terms are certain (higher) Chow groups under morphisms of base fields; most of these results seem to be "well-known".

In §3 we define our main Chow-weight homology theories and study their properties. In particular we express the "weights" of a motif $M$ (defined in terms of the Chow weight structure) and its "effectivity" (i.e., whether it belongs to $\text{Obj} \text{DM}^\text{eff}_{\text{gm}, R} \otimes \mathbb{L}^c$ for a given $c > 0$) in terms of its Chow-weight homology. We describe several interesting consequences of our main theorem; they demonstrate that our results vastly generalize and extend the well-known "decomposition of the diagonal" ones. We also relate the vanishing of Chow-weight homology of $M$ with that of its motivic homology (and so, with its "homotopy $t$-structure connectivity")

In §4 we deduce some more corollaries from the results of the previous section. We dualize some of them (using the results of §2); so we calculate the dimensions of motives and bound their weights (from above) in terms of their Chow-weight cohomology. Next we prove the following: if certain (higher) Chow-weight homology (or motivic homology) groups of a motif $M$ are torsion, then their exponent is (uniformly) bounded. Lastly we note that the combination of two (more or less "standard") motivic conjectures yields the following: the vanishing (resp. the "$c$-effectivity") of the higher levels of the weight filtration on the singular homology of $M$ is equivalent to the corresponding vanishing of the Chow-weight homology of $M$ (and so, to the corresponding motivic "weight-effectivity" restrictions for it). We also discuss the seminal conservativity conjecture, and mention some of our future plans.

We conclude this introduction by a few more remarks.

In the current paper we take the down-to-earth point of view on Chow groups (and of related homology theories). Possibly, in one of the subsequent papers we will study certain Chow sheaves and related homology instead; for
motives over a base $S$ this should yield \textit{cycle modules over $S$} (see Remark 4.4.1(1) below for some more detail).

We will treat both the characteristic 0 and the positive characteristic case below; yet a reader may certainly assume that the characteristic of $k$ is 0 throughout the paper.

The current version of the text is somewhat preliminary. In particular, we only sketch some of the proofs (especially when we believe the corresponding arguments are well-known). Also, we will not say much on the perfectness issues in the case $p > 0$. Some more details and references will be added in the next versions of this paper. Besides, we are going to study the application of our results further (possibly, extending the current text or writing some new ones).

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1 Some preliminaries on weight structures

This section is dedicated to recalling the theory of weight structures in triangulated categories; yet Proposition 1.3.1(2) and Proposition 1.4.4 below are new.

In §1.1 we introduce some notation and conventions for (mostly, triangulated) categories; we also prove two simple lemmas.

In §1.2 we recall the definition and basic properties of weight structures.

In §1.3 we relate weight structures to localizations.

In §1.4 we recall several properties of weight complexes and weight spectral sequences, and prove a new lemma on "intersections of purely generated subcategories".

1.1 Some (categorical) notation and preliminaries

- For $a \leq b \in \mathbb{Z}$ we will denote by $[a, b]$ (resp. $[a, +\infty)$) the set \{ $i \in \mathbb{Z}$ : $a \leq i \leq b$ \} (resp. \{ $i \in \mathbb{Z}$ : $i \geq a$ \}; we will never consider real line segments in this paper).

- Given a category $C$ and $X, Y \in \text{Obj} \ C$, we denote by $C(X, Y)$ the set of morphisms from $X$ to $Y$ in $C$.

- For categories $C', C$ we write $C' \subset C$ if $C'$ is a full subcategory of $C$.
• Given a category $C$ and $X,Y \in \text{Obj}_C$, we say that $X$ is a retract of $Y$ if $\text{id}_X$ can be factored through $X$ (if $C$ is triangulated or abelian, then $X$ is a retract of $Y$ if and only if $X$ is its direct summand).

• An additive subcategory $H$ of additive category $C$ is called Karoubi-closed in $C$ if it contains all retracts of its objects in $C$. The full subcategory $\text{Kar}_C(H)$ of additive category $C$ whose objects are all retracts of objects of a subcategory $H$ (in $C$) will be called the Karoubi-closure of $H$ in $C$.

• The Karoubization $\text{Kar}(B)$ (no lower index) of an additive category $B$ is the category of “formal images” of idempotents in $B$. So, its objects are pairs $(A,p)$ for $A \in \text{Obj}_B$, $p \in B(A,A)$, $p^2 = p$, and the morphisms are given by the formula

$$\text{Kar}(B)((X,p),(X',p')) = \{ f \in B(X,X') : p' \circ f = f \circ p = f \}. \quad (1)$$

The correspondence $A \mapsto (A, \text{id}_A)$ (for $A \in \text{Obj}_B$) fully embeds $B$ into $\text{Kar}(B)$. Besides, $\text{Kar}(B)$ is Karoubian, i.e., any idempotent morphism yields a direct sum decomposition in $B$. Equivalently, $B$ is Karoubian if (and only if) the canonical embedding $B \to \text{Kar}(B)$ is an equivalence of categories. Recall also that $\text{Kar}(B)$ is triangulated if $B$ is (see [BaS01]).

• $C$ below will always denote some triangulated category; usually it will be endowed with a weight structure $w$.

• For any $A, B, C \in \text{Obj}_C$ we will say that $C$ is an extension of $B$ by $A$ if there exists a distinguished triangle $A \to C \to B \to A[1]$.

• A class $D \subset \text{Obj}_C$ is said to be extension-closed if it is closed with respect to extensions and contains $0$. We will call the smallest extension-closed subclass of objects of $C$ that contains a given class $B \subset \text{Obj}_C$ the extension-closure of $B$.

• Given a class $D$ of objects of $C$ we denote by $\langle D \rangle$ the smallest full Karoubi-closed triangulated subcategory of $C$ containing $D$. We will also call $\langle D \rangle$ the triangulated category generated by $D$ (yet note that in some previous papers of the authors a somewhat distinct definition was used).

• For $X,Y \in \text{Obj}_C$ we will write $X \perp Y$ if $C(X,Y) = \{0\}$. For $D, E \subset \text{Obj}_C$ we write $D \perp E$ if $X \perp Y$ for all $X \in D, Y \in E$. Given $D \subset \text{Obj}_C$ we denote by $D^\perp$ the class

$$\{ Y \in \text{Obj}_C : X \perp Y \ \forall X \in D \}. $$
Dually, \( \perp D \) is the class \( \{ Y \in \text{Obj}_C : Y \perp X \ \forall X \in D \} \).

- Given \( f \in C(X,Y) \), where \( X, Y \in \text{Obj}_C \), we will call the third vertex of (any) distinguished triangle \( X \xrightarrow{f} Y \rightarrow Z \) a cone of \( f \) (recall that different choices of cones are connected by non-unique isomorphisms).

- For an additive category \( B \) we denote by \( K(B) \) the homotopy category of (cohomological) complexes over \( B \). Its full subcategory of bounded complexes will be denoted by \( K^b(B) \). We will write \( X = (X^i) \) if \( X^i \) are the terms of the complex \( X \).

- Note yet that we will call any (covariant) homological functor a homology theory. So, for a complex \( A = (A^i,d^i) \) of abelian groups we call the quotient \( \text{Ker} d^i / \text{Im} d^{i-1} \) the \( i \)-th homology of \( A \).

In §4.2 below we will need the following simple piece of homological algebra.

**Definition 1.1.1.** We will say that \( M \in \text{Obj}_C \) is torsion (resp. \( n \)-torsion for an integer \( n \neq 0 \)) if there exists \( N_M > 0 \) (resp. \( r > 0 \)) such that the morphism \( N_M \text{id}_M \) is zero (resp. \( n^r \text{id}_M = 0 \)).

**Lemma 1.1.2.** For \( U, X \in \text{Obj}_C \) (\( C \) is a triangulated category) the following conditions are equivalent.

1. There exists a distinguished triangle \( U \rightarrow X \xrightarrow{f} Y \xrightarrow{d} U[1] \) in \( C \) such that \( Y \) is a torsion object (resp. an \( n \)-torsion object) of \( C \).

2. There exists a distinguished triangle \( Y' \rightarrow X \rightarrow U \rightarrow Y'[1] \) in \( C \) such that \( Y \) is a torsion object (resp. an \( n \)-torsion object) of \( C \).

**Proof.** It is sufficient to prove \( 1 \Rightarrow 2 \) because the implication \( 1 \Rightarrow 2 \) in the opposite category yields that \( 2 \Rightarrow 1 \) in \( C \). Assume that \( \lambda \text{id}_Y = 0 \) where \( \lambda > 0 \) is an integer (resp. a power of \( n \)).

We note that \( d \) factors through \( U/\lambda \), where \( U/\lambda \) denotes a cone of \( \lambda \text{id}_U \). Denote the corresponding morphism from \( U/\lambda \) to \( U[1] \) by \( r \), and denote the morphism from \( Y \) to \( U/\lambda \) by \( q \).

The octahedron axiom applied to the commutative triangle \( (q,r,r \circ q) \) (which we specify by its arrows) yields the existence of a distinguished triangle \( \text{Cone}(q)[-1] \rightarrow X \rightarrow U \). Note that \( \text{Cone}(q)[-1] \) is torsion (resp. \( n \)-torsion) since it is a cone of a morphism between torsion (resp. \( n \)-torsion) objects. So, we obtain the result.

**Lemma 1.1.3.** Let \( M \in \text{Obj}_C \) be an extension of \( D \in \text{Obj}_C \) by some \( T_1 \in \text{Obj}_C \), whereas \( D \) is an extension of \( N \in \text{Obj}_C \) by some \( T_2 \in \text{Obj}_C \). Then \( M \) is an extension of \( N \) by \( T \), where \( T \) is an extension of \( T_1 \) by \( T_2 \).
Proof. Denote the corresponding morphism from $M$ to $D$ by $q$ and denote the morphism from $D$ to $N$ by $q'$. The octahedron axiom applied to the commutative triangle $(q, q', q' \circ q)$ yields the result.

1.2 Weight structures: basics

Definition 1.2.1. I. A pair of subclasses $C_{w \leq 0}, C_{w \geq 0} \subset \text{Obj} \mathcal{C}$ will be said to define a weight structure $w$ for a triangulated category $\mathcal{C}$ if they satisfy the following conditions.

(i) $C_{w > 0}, C_{w \leq 0}$ are Karoubi-closed in $\mathcal{C}$ (i.e., contain all $\mathcal{C}$-retracts of their objects).

(ii) Semi-invariance with respect to translations.

$C_{w \leq 0} \subset C_{w \leq 0}[1], C_{w \geq 0} \subset C_{w \geq 0}$.

(iii) Orthogonality.

$C_{w \leq 0} \perp C_{w \geq 0}[1]$.

(iv) Weight decompositions.

For any $M \in \text{Obj} \mathcal{C}$ there exists a distinguished triangle

$$X \to M \to Y \to X[1]$$

such that $X \in C_{w \leq 0}, Y \in C_{w \geq 0}[1]$.

II. The category $\mathcal{H}_w \subset C$ whose objects are $C_{w=0} = C_{w \geq 0} \cap C_{w \leq 0}$ and morphisms are $\mathcal{H}_w(Z, T) = C(Z, T)$ for $Z, T \in C_{w=0}$, is called the heart of $w$.

III. $C_{w > i}$ (resp. $C_{w \leq i}$, resp. $C_{w=i}$) will denote $C_{w \geq 0}[i]$ (resp. $C_{w \leq 0}[i]$; resp. $C_{w=0}[i]$).

IV. We denote $C_{w \geq i} \cap C_{w \leq j}$ by $C_{i,j}$ (so it equals $\{0\}$ for $i > j$).

$C^b \subset C$ will be the category whose object class is $\bigcup_{i,j \in \mathbb{Z}} C_{i,j}$.

V. We will say that $(\mathcal{C}, w)$ is bounded if $C^b = C$ (i.e., if $\bigcup_{i \in \mathbb{Z}} C_{w \leq i} = \text{Obj} \mathcal{C} = \bigcup_{i \in \mathbb{Z}} C_{w \geq i}$).

VI. Let $\mathcal{C}$ and $\mathcal{C}'$ be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F : \mathcal{C} \to \mathcal{C}'$ be an exact functor.

$F$ is said to be weight-exact (with respect to $w, w'$) if it maps $C_{w \leq 0}$ into $C'_{w' \leq 0}$ and maps $C_{w > 0}$ into $C'_{w' > 0}$.

VII. Let $\mathcal{H}$ be a full subcategory of a triangulated category $\mathcal{C}$.

We will say that $\mathcal{H}$ is negative if $\text{Obj} \mathcal{H} \perp (\bigcup_{i > 0} \text{Obj}(\mathcal{H}[i]))$.

Remark 1.2.2. 1. A simple (and yet quite useful) example of a weight structure comes from the stupid filtration on $K^b(B)$ (or on $K(B)$) for an arbitrary additive category $B$. In this case $K^b(B)_{w \leq 0}$ (resp. $K^b(B)_{w > 0}$) will be the class of complexes that are homotopy equivalent to complexes concentrated in degrees $\geq 0$ (resp. $\leq 0$).
The heart of this weight structure is the Karoubi-closure of $B$ in $K^b(B)$.  
2. A weight decomposition (of any $M \in \text{Obj} \mathcal{C}$) is (almost) never canonical.

Yet for $m \in \mathbb{Z}$ we will often need some choice of a weight decomposition of $M[-m]$ shifted by $[m]$. So we obtain a distinguished triangle 
\[ w_{\leq m} M \rightarrow M \rightarrow w_{\geq m+1} M \] 
with some $w_{\geq m+1} M \in C_{w_{\geq m+1}}$, $w_{\leq m} M \in C_{w_{\leq m}}$.

We will often use this notation below (though $w_{\geq m+1} M$ and $w_{\leq m} M$ are not canonically determined by $M$).

3. In the current paper we use the “homological convention” for weight structures; it was previously used in [Wil09], [Bon13], and [Bon14], whereas in [Bon10a] and in [Bon10b] the “cohomological convention” was used. In the latter convention the roles of $C_{w_{\leq 0}}$ and $C_{w_{\geq 0}}$ are interchanged, i.e., one considers $C_{w_{\leq 0}} = C_{w_{\geq 0}}$ and $C_{w_{\geq 0}} = C_{w_{\leq 0}}$. So, a complex $X \in \text{Obj} K(A)$ whose only non-zero term is the fifth one (i.e., $X^5 \neq 0$) has weight $-5$ in the homological convention, and has weight 5 in the cohomological convention. Thus the conventions differ by “signs of weights”; $K(A)_{[i,j]}$ is the class of retracts of complexes concentrated in degrees $[-j,-i]$.

We also recall that D. Pauksztello has introduced weight structures independently in [Pau08]; he called them co-t-structures.

4. The orthogonality axiom in Definition 1.2.1(1) immediately yields that $Hw$ is negative in $\mathcal{C}$. We will formulate a certain converse to this statement below.

Let us recall some basic properties of weight structures. Starting from this moment we will assume that all the weight structures we consider are bounded (unless specified otherwise; this is quite sufficient for our purposes everywhere except in the proof of Proposition 3.1.2(6)).

**Proposition 1.2.3.** Let $\mathcal{C}$ be a triangulated category, $n \geq 0$; we will assume that $w$ is a fixed (bounded) weight structure on it everywhere except in assertion 7.

1. The axiomatics of weight structures is self-dual, i.e., for $\mathcal{D} = \mathcal{C}^{\text{op}}$ (so $\text{Obj} \mathcal{D} = \text{Obj} \mathcal{C}$) there exists the (opposite) weight structure $w'$ for which $\mathcal{D}_{w'_{\leq 0}} = C_{w_{\geq 0}}$ and $\mathcal{D}_{w'_{\geq 0}} = C_{w_{\leq 0}}$.

2. If $M \in C_{w_{\geq -n}}$ then $w_{\leq 0} M \in C_{[-n,0]}$.

3. $C_{w_{\leq 0}}$ is the extension-closure of $\cup_{i \leq 0} C_{w_{=i}}$ in $\mathcal{C}$; $C_{w_{\geq 0}}$ is the extension-closure of $\cup_{i \geq 0} C_{w_{=i}}$ in $\mathcal{C}$.
4. \( C_{w \geq 0} = (C_{w \leq -1})^\perp \) and \( C_{w \leq -1} = {}^\perp C_{w \geq 1} \).

5. Assume that \( w' \) is a weight structure for a triangulated category \( C' \). Then an exact functor \( F : C \to C' \) is weight-exact if and only if \( F(C_{w=0}) \subset C_{w'=0} \).

6. Let \( m \leq l \in \mathbb{Z} \), \( X, X' \in \text{Obj} \, C \); fix certain weight decompositions of \( X[-m] \) and \( X'[-l] \). Then any morphism \( g : X \to X' \) can be extended to a commutative diagram of the corresponding distinguished triangles (see Remark 1.2.2(2)):

\[
\begin{array}{ccc}
w_{\leq m}X & \longrightarrow & X \\
\downarrow & & \downarrow g \\
w_{\leq l}X' & \longrightarrow & X'
\end{array}
\]

Moreover, if \( m < l \) then this extension is unique (provided that the rows are fixed).

7. For a triangulated Karoubian category \( C \) let \( D \subset \text{Obj} \, C \) be a negative additive subcategory. Then there exists a unique weight structure \( w_T \) on \( T = \langle D \rangle \) in \( C \) such that \( D \subset T_{w_T=0} \). Its heart is equivalent to the Karoubization of \( D \).

Moreover, if there exists a weight structure \( w \) for \( C \) such that \( D \subset Hw \), then the embedding \( T \to C \) is strictly weight exact, i.e., \( T_{w_T \leq 0} = \text{Obj} \, T \cap C_{w \leq 0} \) and \( T_{w_T > 0} = \text{Obj} \, T \cap C_{w > 0} \).

8. Let \( M \in C_{w \geq 0} \), \( N \in \text{Obj} \, C \), \( f \in C(N, M) \). Then \( f \) factors through \( w_{\geq 0}N \).

9. Let \( D \) be a (full) triangulated subcategory of \( C \) such that \( w \) restricts to \( D \) (i.e., \( \text{Obj} \, D \cap C_{w \leq 0} \) and \( \text{Obj} \, D \cap C_{w > 0} \) give a weight structure for \( D \)); let \( M \in C_{w \leq 0} \), \( N \in C_{w \geq -n} \), and \( f \in C(M, N) \). Suppose that \( f \) factors through an object \( P \) of \( D \), i.e., there exist \( u_1 \in C(M, P) \) and \( u_2 \in C(P, N) \) such that \( f = u_2 \circ u_1 \). Then \( f \) factors through an element of \( D([-n, 0]) \).

**Proof.** All of these statements except the the two last ones and the second half of the preceding one can be found in [Bon10a] (pay attention to Remark 1.2.2(3)!). The "moreover" part of assertion 4 easily follows from assertion 6. Assertion 8 is an easy consequence of assertion 5.
Assertion \(8\) yields that \(u_2\) factors through \(w_{\geq -n}P\); so we can assume \(P \in D_{w_{\geq -n}}\). Next, the dual to assertion \(8\) (see assertion \(1\)) yields that \(u_1\) factors through \(w_{\leq 0}P\). It remains to note that we can choose \(w_{\leq 0}P \in D_{[-n,0]}\) (see assertion \(2\)).

\[\square\]

1.3 Weight structures in localizations

We call a category \(A\) a factor of an additive category \(A\) by its full additive subcategory \(B\) if \(\text{Obj}(A) = \text{Obj} B\) and \((A,B)(X,Y) = A(X,Y)/(\sum_{Z \in \text{Obj} B} A(Z,Y) \circ A(X,Z))\).

Proposition 1.3.1. Let \(D \subset C\) be a triangulated subcategory of \(C\); suppose that \(w\) induces a weight structure \(w_D\) on \(D\) (i.e., \(\text{Obj}(A) \cap \text{Obj} D_{w \leq 0}\) and \(\text{Obj}(A) \cap \text{Obj} D_{w \geq 0}\) give a weight structure for \(D\)). Denote by \(l\) the localization functor \(C \to C/D\) (the latter category is the Verdier quotient of \(C\) by \(D\)).

Then the following statements are valid.

1. \(w\) induces a weight structure on \(C/D\), i.e., the Karoubi-closures of \(l(\text{Obj} D_{w \leq 0})\) and \(l(\text{Obj} D_{w \geq 0})\) give a weight structure for \(C/D\).

2. For \(X \in \text{Obj} C\) assume that \(l(X) \in C/D_{w \leq 0}\). Then \(X\) is an extension of some element of \(C_{w \leq 0}\) by an element of \(D_{w \leq -1}\) (see §1.1).

3. The heart \(H(C/D)\) of the weight structure \(w_{C/D}\) obtained is the Karoubi-closure of (the natural image of) \(\underbrace{H_{w_{C/D}}}_{\text{H}}\) in \(C/D\).

4. If \((C,w)\) is bounded, then \(C/D\) also is.

Proof. Assertions 1, 3, and 4 were proved in §8.1 of [Bon10a].

Now we verify assertion 2. We use induction on the minimal \(n \in [1, +\infty)\) such that \(X\) belongs to \(C_{w \geq -n}\).

Suppose that \(X \in C_{w \geq -1}\). Consider a (shifted) weight decomposition of \(X\): \(w_{\leq -1}X \xrightarrow{i} X \xrightarrow{j} w_{\geq 0}X\). The orthogonality axiom for the weight structure \(w_{C/D}\) yields \(l(i) = 0\) (since we have assumed \(l(X) \in (C/D)_{w_{C/D} \geq 0}\)). Thus a well-known property of Verdier localizations yields the following: there exists a factorization of \(i\) through some object \(M\) of \(D\), i.e., there exist \(g \in C(w_{\leq -1}X, M)\) and \(f \in C(M, X)\) such that \(f \circ g = i\). By Proposition 1.2.3(9) we can assume that \(M \in D_{w = -1}\).

Now let us show that \(\text{Cone}(f) \in C_{w \geq 0}\); this would yield the assertion in our case. According to the octahedron axiom, there is a distinguished triangle \(\text{Cone}(g) \to w_{\leq -1}X \to \text{Cone}(f) \to \text{Cone}(g)[1]\). Since \(X'\) and \(M\) belong to \(C_{w = -1}\), \(\text{Cone}(g)[1]\) belongs to \(C_{w \geq 0}\). Thus \(\text{Cone}(f) \in C_{w \geq 0}\), since it is an extension of elements of \(C_{w \geq 0}\).
Now we describe the inductive step. If $X \in D_{w \geq -n}$, then one can apply the inductive assumption to $X[n-1]$. It yields a distinguished triangle of the form $M \rightarrow X \xrightarrow{f} X' \rightarrow M[1]$, where $M \in D_{w \leq -n}$ and $X' \in C_{w \geq -(n-1)}$. By the inductive assumption there also exists a distinguished triangle $M' \rightarrow X' \xrightarrow{g} X'' \rightarrow M'[1]$, where $M' \in D_{w \leq -1}$. Applying the octahedron axiom to the commutative triangle of morphisms $(p, q, q \circ p)$ we obtain a triangle of the type desired for $X$.

\[
\begin{array}{c}
\text{Remark 1.3.2.} \\
1. \text{Certainly, if the conditions of the proposition are fulfilled, the functor } l \text{ is weight-exact with respect to the corresponding weight structures.} \\
\text{Now assume that } l(M) \in C/D_{w \geq -n} \text{ (for some } n \in \mathbb{Z}). \text{ Then for any shifted weight decomposition } w_{-n-1}M \rightarrow M \rightarrow w_{-n}M \text{ the orthogonality axiom for } w_{C/D} \text{ yields that } l(g) = 0. \text{ Hence } l(h) \text{ splits.} \\
\text{Conversely, assume that there exists an element } N \in \text{C}_{w \leq -n} \text{ along with a } C \text{-morphism } h : M \rightarrow N \text{ such that } l(h) \text{ splits. Then } l(M) \text{ is a retract of } l(N), \text{ and so } l(M) \in C/D_{w \geq -n}.
\end{array}
\]

2. Part 2 of the proposition yields the existence of a distinguished triangle $d \rightarrow X \rightarrow c \rightarrow d[1]$ for some $c \in C_{w \geq 0}$, $d \in D_{w \leq -1}$. Certainly, this triangle is just a shifted weight decomposition of $X$. In particular, Proposition 1.2.3(3) (or part 2 of the proposition together with its dual) easily yields the following: if $X \in C_{[l,m]}$ for $l \leq 0 \leq m$, then $c \in C_{[0,m]}$ and $d \in C_{[l,-1]}$.

3. If $w$ is bounded then all weight structures compatible with it (for $D \subset C$) come from additive subcategories of $Hw$ (see Proposition 1.2.3(7)); it is actually not necessary to assume that $C$ is Karoubian here). Yet to ensure that there exists a weight structure for $C/D$ such that the localization functor is weight-exact it actually suffices to assume that $D$ is generated by some set of elements of $C_{[0,1]}$; see Theorem 4.2.2 of [BoS13].

1.4 On weight complexes and weight spectral sequences

We will need certain weight complex functors below. Applying the results of [Bon09], one can assume that all the weight complexes we need are given by "compatible" exact functors whose targets are the homotopy categories of complexes in the corresponding hearts; so that one may assume below that the weight complex of a $C$-morphism is well-defined up to homotopy. Yet (see §3 of [Bon10a]) one cannot construct canonical weight complex functors satisfying these properties without considering certain "enhancements" for their domains. The problem here that morphisms of objects a priori determine the morphisms of their weight complexes only up to a weak homotopy
(see §3.1 of ibid.). We don’t have to describe it here explicitly; so we will define weight complexes in the following way.

**Definition 1.4.1.** For an object $M$ of $C$ (where $C$ is endowed with a weight structure $w$) choose some $w_{\leq l} M$ (see Remark [1.2.2(2)]) for all $l \in \mathbb{Z}$. For all $l \in \mathbb{Z}$ connect $w_{\leq l-1} M$ with $w_{\leq l} M$ using Proposition [1.2.3(6)] (i.e., we consider those unique connecting morphisms that are compatible with $\text{id}_M$). Next, take the corresponding triangles

$$w_{\leq l-1} M \to w_{\leq l} M \to M^{-l} [l]$$

(4)

(so, we just introduce the notation for the corresponding cones). All of these triangles together with the corresponding morphisms $w_{\leq l} M \to M$ are called a choice of a *weight Postnikov tower* for $M$, whereas the objects $M^i$ together with the morphisms connecting them (obtained by composing the morphisms $M^{-l} \to (w_{\leq l-1} M)[1-l] \to M^{-l+1}$ that come from two consecutive triangles of the type (4)) will be denoted by $t(M)$ and is said to be a choice of a *weight complex* for $M$.

Respectively, for some $M, M' \in \text{Obj} C$, $f \in C(M, M')$, and some of their $t(M), t(M')$ we will say that a collection of arrows between the terms of these weight complexes is a choice of $t(f)$ if these arrows come from some morphisms of the weight Postnikov towers for $M, M'$ that is compatible with $f$.

Let us recall some basic properties of weight complexes (for a bounded $w$; this case is not much distinct from the general one).

**Proposition 1.4.2.** Let $M, M' \in \text{Obj} C$, $f \in C(M, M')$, where $C$ is endowed with a weight structure $w$.

Then the following statements are valid.

1. Any choice of $t(M) = (M^i)$ is a complex indeed (i.e., the square of the boundary is zero); all $M^i$ belong to $C_{w=0}$.
2. Any choice of $t(f)$ is a $C(Hw)$-morphism from the corresponding $t(M)$ to $t(M')$.
3. $t(M)$ is homotopy equivalent (i.e., $K(Hw)$-isomorphic) to a bounded complex.
4. $M$ determines its weight complex $t(M)$ up to a homotopy equivalence. In particular, if $M \in C_{w \geq 0}$, then any choice of $t(M)$ is $K(Hw)$-isomorphic to a complex with non-zero terms in non-positive degrees only.
5. If $t(M)$ is homotopy equivalent to 0, then $M = 0$.

6. Assume that $f[-m]$ yields a weight decomposition of $M'$ (so, one can take $w_{\leq m} M' = M$). Then for any choice of $t(f)$ its cone (see §7.7) is homotopy equivalent to a complex concentrated in degrees $\leq -m$.

7. Let $N \in \mathcal{C}_{w=0}$, $M \in \mathcal{C}_{w \geq 0}$; assume that an $f \in \mathcal{C}(N, M)$ factors through some $L \in \text{Obj} \mathcal{C}$. Then for any possible choice of $L^0$ (i.e., of the zeroth term of $t(L)$) $f$ can be factored through $L^0$.

8. Let $H_0 : Hw \to A$ ($A$ is an arbitrary abelian category) be an additive functor. Choose a weight complex $t(M) = (M^i)$ for each $M \in \text{Obj} \mathcal{C}$, and denote by $H(M)$ the zeroth homology of the complex $H_0(M^i)$. Then $H(\cdot)$ yields a homological functor from $\mathcal{C}$ to $A$ that does not depend on the choices of weight complexes for objects; we will call a functor of this type a $w$-pure one.

9. Let $\mathcal{C}'$ be a triangulated category endowed with a weight structure $w'$; let $F : \mathcal{C} \to \mathcal{C}'$ be a weight-exact functor. Then for any choice of $t(M)$ (resp. of $t(f)$) the complex $F(M^i)$ (resp. of $F_i(t(f))$) yields a weight complex of $F(M)$ (resp. a choice of $t(F(f))$) with respect to $w'$.

Proof. Assertions 1–6 follow immediately from Theorem 3.3.1 of [Bon10a]. In particular, assertion 6 is implied by parts I and IV of ibid.

Assertion 7 was essentially established in the course of proving of Proposition 1.2.3(9).

Assertion 8 is easy also; it is immediate from the dual statement given by Proposition 2.1.3(14) of [Bon13].

The last assertion is an immediate consequence of the definition of a weight complex (and of weight-exact functors). 

Remark 1.4.3. 1. Besides, Theorem 3.3.1(VI) easily yields that $t$ induces a bijection between the class of isomorphism classes of elements of $\mathcal{C}_{[0,1]}$ and the corresponding class for $K(Hw)$ (i.e., with the class of homotopy equivalence classes of complexes that have non-zero terms in degrees $-1$ and $0$ only).

2. The term "weight complex" originates from [GiS96], where a certain complex of Chow motives was constructed for a variety $X$ over a characteristic 0 field. The weight complex functor of Gillet and Soulé can be obtained via applying the "triangulated motivic" weight complex functor $DM^eff_{gm} \to K^b(\text{Chow}^{eff})$ (or $DM_{gm} \to K^b(\text{Chow})$; cf. Definition 3.1.1 below) to the motif with compact support of $X$ (see Proposition 6.6.2 of [Bon09]).

Certainly, our notion of weight complex is much more general.
These properties of weight complexes yield an interesting corollary.

Proposition 1.4.4. Let \( \mathcal{C} \) be a Karoubian triangulated category endowed with a weight structure \( w \); let \( C_1, C_2, C_3 \) be additive subcategories of \( \mathcal{H}_w \). Assume that all \( \mathcal{H}_w \)-morphisms from objects of \( C_1 \) to objects of \( C_2 \) can be factored through \( C_3 \). Then \( \text{Obj}(\langle C_1 \rangle_{\mathcal{C}}) \cap \text{Obj}(\langle C_2 \rangle_{\mathcal{C}}) \subset \text{Obj}(\langle C_3 \rangle_{\mathcal{C}}) \).

Proof. Let \( M \in \text{Obj}(\langle C_1 \rangle_{\mathcal{C}}) \cap \text{Obj}(\langle C_2 \rangle_{\mathcal{C}}) \). We can certainly assume that \( C_1 \) and \( C_2 \) are Karoubian. Then Proposition 1.4.2(9) yields that \( M \) possesses weight complexes \( t_1(M) \) and \( t_2(M) \) whose terms belong to \( C_1 \) and to \( C_2 \), respectively. Let \( u \) be the homotopy equivalence \( t_1(M) \Rightarrow t_2(M) \) (whose existence is guaranteed by part 4 of the proposition cited).

Now, consider the localization functor \( l : \mathcal{C} \to \mathcal{C}' = \mathcal{C}/\langle C_3 \rangle \). According to Proposition 1.3.1, \( w \) "induces" a weight structure on \( \mathcal{C}' \). Hence our assumption on \( C_3 \) yields that \( u \) becomes a zero morphism of complexes in \( K(\mathcal{H}_w \mathcal{C}') \). Thus Proposition 1.4.2(5) yields \( l(M) = 0 \). Certainly, the latter is equivalent to \( M \in \text{Obj}(\langle C_3 \rangle_{\mathcal{C}}) \).

Now recall some of the properties of weight spectral sequences established in §2 of [Bon10a].

Let \( A \) be an abelian category. In §2 of [Bon10a] for \( H : \mathcal{C} \to A \) that is either cohomological or homological (i.e., it is either covariant or contravariant, and converts distinguished triangles into long exact sequences) certain weight filtrations and weight spectral sequences (corresponding to \( w \) were introduced. Below we will be more interested in the homological functor case; certainly, one can pass to cohomology by a simple reversion of arrows (cf. §2.4 of ibid.).

Definition 1.4.5. Let \( H : \mathcal{C} \to A \) be a covariant functor, \( i \in \mathbb{Z} \).

1. We denote \( H \circ [i] : \mathcal{C} \to A \) by \( H_i \).

2. Choose some \( w_{\leq i}M \) and define the weight filtration for \( H \) by \( W_iH : M \mapsto \text{Im}(H(w_{\leq i}M) \to H(M)) \).

Recall that \( W_iH \) is functorial in \( M \) (in particular, it does not depend on the choice of \( w_{\leq i}M \)); see Proposition 2.1.2(1) of ibid.

Proposition 1.4.6. For a homological \( H : \mathcal{C} \to A \) and any \( M \in \text{Obj} \mathcal{C} \) there exists a spectral sequence \( T = T_\omega(H, M) \) with \( E_1^{pq}(T) = H_q(M^p) \), where \( M^i \) and the boundary morphisms of \( E_1(T) \) come from any choice of \( t(M) \), that converges to \( E_\infty^{p+q} = H_{p+q}(M) \) (at least) if \( M \) is bounded. \( T \) is \( \mathcal{C} \)-functorial in \( M \) and in \( H \) (with respect to composition of \( H \) with exact functors of abelian categories) starting from \( E_2 \).
Corollary 1.4.7. Let $M \in C_{w \geq 0}$, $N \in C_{w = 0}$.
Then the following statements are valid.

1. Choose some $t(M) = (M')$. Then $C(N, M)$ is isomorphic to the zeroth homology of $((H_w(N, M^*))$).

2. Let $D \subset C$ be a triangulated subcategory of $C$; suppose that $w$ induces a weight structure on $D$ (cf. Proposition 1.3.1). Assume that $f \in C(N, M)$ vanishes in the Verdier quotient $C/D$. Then $f$ factors through some object of $H_w D$.

Proof. 1. We may assume that $M^i = 0$ for $i > 0$ (see Proposition 1.2.3(2); note that this will not affect the homology of $(H_w(N, M^*))$, whereas certainly $N \perp M^i[−i]$ for all $i < 0$, $N \perp M^i[−i − 1]$ for all $i < −1$. Hence Proposition 1.4.6 yields the result.

2. The Verdier localization theory yields that $f$ factors through an object of $D$. Hence the assertion follows from Proposition 1.4.2(7).

Remark 1.4.8. Note that (for a fixed $H, q$) the functor $M \mapsto E_0^q(T_w(H, M))$ is a particular case of $w$-pure functors mentioned in Proposition 1.4.28. These functors (when $q$ varies) were called the virtual $t$-truncations of $H$; see §§2.3–2.4 of [Bon10b].

The virtual $t$-truncations of (Voevodsky's) motivic homology with respect to the Chow weight structure are the main subject of this paper. Some (other) functors of this type were considered in [KeS14]; cf. also Corollary 2.3.4 of [Bon13].

2 On motives, their "weights", and various (complexes of) Chow groups

In this section we study various motivic categories, Chow weight structures for them, and certain (complexes of) Chow groups.

In §2.1 we recall some basics on Voevodsky motives with coefficients in a $\mathbb{Z}^{[\frac{1}{p}]}$-algebra $R$. We also introduce some notation; the position of $R$ in it (as a lower or an upper index) is motivated by purely typographical reasons.

In §2.2 we introduce the Chow weight structures on various versions of $DM^R_{gm}$. Their properties allow the computation of the intersection of $DM^{eff}_{gm,R}(c)$ with $d_{\leq n}DM^{eff}_{gm,R}$ (this result is new). We also give a "pure motivic" characterization of morphisms that vanish when composed with open embeddings (that also seems to be quite new for $\mathbb{Z}^{[\frac{1}{p}]}$-motives if char $k =
$p > 0$), and deduce an important formula on morphisms in $DM_{gm}^R$ (that generalizes a formula from [KaS02]).

In §2.3 to extensions of $k$ and complexes of Chow motives we associate the homology of complexes consisting of their Chow groups (of fixed dimension and "highness"). We prove several properties of these homology theories (and of motivic homology). Most of these statements seem to be quite standard; yet to establish a relation between the corresponding homology over $K/k$ with that over its residue field, we invoke certain splitting results of [Bon10b].

2.1 Some notation and basics for Voevodsky motives

$k$ will be our perfect base field of characteristic $p$; we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ if $p = 0$. Denote the set of smooth varieties (resp. of smooth projective varieties) over $k$ by $SmVar$ (resp. by $SmPrVar$).

Recall that (as was shown in [MVW06]; cf. also [GiD14]), one can do the theory of motives with coefficients in an arbitrary commutative associative ring with a unit $R$. One should start with the naturally defined category of $R$-correspondences: $\text{Obj}(SmCor_R) = SmVar$; for $X, Y \in SmVar$ we set $SmCor_R(X, Y) = \bigoplus_U R$ for all integral closed $U \subset X \times Y$ that are finite over $X$ and also dominant over a connected component of $X$. Below we will always assume additionally that $R$ is an $\mathbb{Z}[\frac{1}{p}]$-algebra.

Proceeding as in [Voe00a] (i.e., taking the corresponding localization of $K^b(SmCor_R)$, and complexes of sheaves with transfers with homotopy invariant cohomology) one obtains the theory of motives (i.e., of the tensor triangulated category $DM_{gm}^{R_{eff}}$ together with its embeddings into $DM_{gm}^R$ and into $DM_{gm}^{eff}_R$; see below) that satisfies all the basic properties of the "usual" Voevodsky’s motives (i.e., of those with integral coefficients for $p = 0$). Being more precise, we recall that all of the results that were stated in [Voe00a] in this case are currently known for $\mathbb{Z}[\frac{1}{p}]$-motives (also if) $p > 0$; see [Kel13], [Deg08], and [Bon11]. So we will apply some of these properties of motives with $R$-coefficients without further mention. We will mostly be interested in the cases $R = \mathbb{Z}[\frac{1}{p}]$, $R = \mathbb{Q}$, and $R = \mathbb{Z}(l)$ for $l$ being a prime distinct from $p$ (so that $DM_{gm,R}^{eff} \subset DM_{gm}^R$ are the Karoubizations of the corresponding localizations of $DM_{gm}^{eff} \subset DM_{gm}$). Note also that the results of [Kel13] are proved in the case of an arbitrary $p$; so we will usually cite them without mentioning their well-known analogues from [Voe00a] established in the case $p = 0$ only. Still a reader can certainly restrict himself to the case $p = 0$; this would yield certain simplifications of the proofs (that include ignoring all the perfectness issues).

We note that the composition $SmCor_R \rightarrow K^b(SmCor_R) \rightarrow DM_{gm,R}^{eff}$
Certainly yields a functor $M^R_{gm}$ (of the $R$-motif) from the category of smooth $k$-varieties into $DM_{gm}^{eff}$. Actually, it extends to the category of all $k$-varieties (see [Voe00a] and [Kel13]); yet we will never need this extension.

$Chow_{gm,R}^{eff} \subset DM_{gm,R}^{eff}$ will denote the categories of effective homological Chow motives (resp. of Voevodsky effective geometric motives) over $k$ with coefficients in a $R$. For $c \geq 0$, $M \in \text{Obj} DM_{gm,R}^{eff}$, we denote by $M(c)$ the tensor product of $M$ by the $c$th tensor power of the Lefschetz motif $L$ (we will denote $L^\otimes c$ by $R(c)[2c]$ following the notation of [Voe00a]). Respectively, $M(c)$ will denote $M \otimes R(c) = M \otimes L^\otimes [-2c]$.

Next, recall that the functor $-(1)$ is a full embedding of $DM_{gm,R}^{eff}$ into itself that restricts to an embedding of $Chow_{gm,R}^{eff}$ into itself. It extends to an autoequivalence of the corresponding category $DM_{gm}^{R} = DM_{gm,R}^{eff}([-1])$ (i.e., we invert $- \otimes \mathbb{L}$); recall that this category contains $DM_{gm,R}^{eff}$ and $Chow_{gm,R} = Chow_{gm,R}^{eff}([-1])$. Moreover, $DM_{gm}^{R}$ is equipped with an exact Poincare duality functor $\sim : DM_{gm}^{R} \rightarrow DM_{gm}^{op}$ (constructed in [Voe00a] for $p = 0$; see Theorem 5.5.14 of [Kel13] or [Bon11] for the positive characteristic case) that sends $M_{gm}^{R}(P)$ into $M_{gm}^{R}(P)(-d)$ if $P$ is smooth projective everywhere of dimension $d$. It restricts to the "usual" Poincare duality for $Chow_{gm,R}$.

Both $DM_{gm,R}^{eff}$ and $DM_{gm}^{R}$ are Karoubian by definition.

An important property of motives is the Gysin distinguished triangle (see Proposition 4.3 of [Deg08] that establishes its existence in the case of an arbitrary $p$). For a closed embedding $Z \rightarrow X$ of smooth varieties, $Z$ is everywhere of codimension $c$ in $X$, it has the following form:

$$M_{gm}^{R}(X \setminus Z) \rightarrow M_{gm}^{R}(X) \rightarrow M_{gm}^{R}(Z)\langle c \rangle \rightarrow M_{gm}^{R}(X \setminus Z)[1].$$

(5)

Some of our formulations below will mention the homotopy $t$-structure for the Voevodsky motivic complexes. So we note that the methods of [Voe00a] yield an embedding $DM_{gm,R}^{eff}$ into a certain category $DM_{-R}^{eff}$, whereas the latter can be endowed with the so-called homotopy $t$-structure $t_{hom}^{R}$. Moreover, the arguments of [Deg11] yield an embedding of $DM_{-R}^{eff}$ into the triangulated category $DM_{gm}^{eff}$ of unbounded motivic complexes that is closed with respect to arbitrary coproducts. $t_{hom}^{R}$ can be extended to $DM_{gm}^{eff}$ (see Corollary 5.2 of ibid.) so that the class $DM_{gm}^{eff,t_{hom}^{R} \leq 0}$ coincides with $DM_{-R}^{eff,t_{hom}^{R} \leq 0}$ and equals the smallest extension-closed subclass of $\text{Obj} DM_{gm}^{eff}$ that is closed with respect to coproducts and contains $M_{gm}^{R}(X)$ for all smooth $X/k$. In particular, it follows that $DM_{gm}^{eff,t_{hom}^{R} \leq 0}$ is preserved by the functor $-(1)[1]$ and contains $\text{Obj} Chow_{gm,R}^{eff}(a)[a+b]$ for all $a, b \geq 0$ (since $R(1)[1]$ is a retract of $M_{gm}^{R}(G^m)$).
2.2 Chow weight structures for various $R$-motives and their applications

Now we note that the arguments used in the construction of the Chow weight structures in [Bon10a] and [Bon11] can be easily applied to $R$-motives (for any $\mathbb{Z}[\frac{1}{p}]$-algebra $R$).

**Proposition 2.2.1.** 1. There exists a bounded weight structure $w_{\text{Chow}}$ for $DM_{\text{gm}, R}^{\text{eff}}$ (resp. for $DM_{\text{gm}}^{R}$) whose heart consists of motives isomorphic to that coming from $\text{Chow}_{\text{gm}, R}^{\text{eff}}$ (resp. from $\text{Chow}_{R}$) via the corresponding full embedding. These weight structures for $DM_{\text{gm}, R}^{\text{eff}}$ and $DM_{\text{gm}}^{R}$ are compatible (i.e., the embedding $DM_{\text{gm}, R}^{\text{eff}} \rightarrow DM_{\text{gm}}^{R}$ is weight-exact).

Besides, $DM_{\text{gm}, R}^{\text{eff}}$ with $w_{\text{Chow}} \leq 0$ (resp. $DM_{\text{gm}}^{R}$ with $w_{\text{Chow}} \leq 0$) is the extension-closure of the set $\bigcup_{i \geq 0} \text{Obj} \text{Chow}_{\text{gm}, R}^{\text{eff}}[i]$ in $DM_{\text{gm}, R}^{\text{eff}}$ (resp. of the set $\bigcup_{i \geq 0} \text{Obj} \text{Chow}_{R}^{\text{eff}}[i]$ in $DM_{\text{gm}}^{R}$).

2. If $U \in \text{SmVar}$, $\dim U \leq m$, then $M_{R}(U) \in DM_{\text{gm}, R}^{\text{eff}}[−m, 0]$.

3. If $U \rightarrow V$ is an open dense embedding of smooth varieties, then $\text{Cone}(M_{R}(U) \rightarrow M_{R}(V)) \in DM_{\text{gm}, R}^{\text{eff}} w_{\text{Chow}} \leq 0$.

4. Let $k'$ be a perfect field extension of $k$. Then the extension of scalars functors $DM_{\text{gm}, R}^{\text{eff}}(k) \rightarrow DM_{\text{gm}, R}^{\text{eff}}(k')$ and $DM_{\text{gm}}^{R}(k) \rightarrow DM_{\text{gm}}^{R}(k')$ are weight-exact with respect to the corresponding Chow weight structures.

5. For any $n \in \mathbb{Z}$ the functor $−\langle n \rangle$ is weight-exact on $DM_{\text{gm}}^{R}$; the same is true for $DM_{\text{gm}, R}^{\text{eff}}$ if $n \geq 0$.

6. If $M \in \text{Obj} DM_{\text{gm}, R}^{\text{eff}}(n)$, $n \in \mathbb{Z}$, then there exists a choice of $t(M) = (M')$ (in $DM_{\text{gm}}^{R}$) with $M' \in \text{Chow}_{\text{gm}, R}^{\text{eff}}(n)$.

**Proof.** The first three assertions were stated in Theorem 2.2.1 of [Bon11] in the case $R = \mathbb{Z}[\frac{1}{p}]$; the proof carries over to the case of a general $R$ without any difficulty.

The remaining assertions are easy also. Assertions 4 and 5 are immediate from Proposition 1.2.3[5], whereas assertion 6 follows from the previous one by Proposition 1.3.2[9].

Now we deduce some simple corollaries from this proposition. Their formulation requires the following definition, that will be very important for us below.

**Definition 2.2.2.** 1. For $M \in \text{Obj} DM_{\text{gm}, R}^{\text{eff}}$, $c \geq 0$, we will say that $M$ is $c$-effective if it has the form $N\langle c \rangle$ for some $N \in \text{Obj} DM_{\text{gm}, R}^{\text{eff}}$. 

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2. We will say that the dimension of $M$ is not greater than an integer $m$ if $M$ belongs to $\langle M^R_m(P) : P \in SmPrVar, \dim P \leq m \rangle$.

We denote the (full) triangulated subcategory of $DM^eff_{gm,R}$ of motives of dimension $\leq m$ by $d_{\leq m}DM^eff_{gm,R}$ (cf. §3.4 of [Voe00a]; so, $d_{\leq m}DM^eff_{gm,R} = \{0\}$ if $m < 0$).

3. We denote by $DM^eff_R$ the Verdier quotient category $DM^eff_{gm,R} / DM^eff_{gm,R}(c+1)$; $^c$ will denote the corresponding localization functor.

**Remark 2.2.3.** If $p = 0$ then the arguments of [Voe00a] easily yield that $M^R_m(X) \in d_{\leq m}DM^eff_{gm,R}$ if $V$ is an arbitrary $k$-variety of dimension $\leq m$. For $p > 0$ this result can be easily established via the methods of §5.5 of [Kel13] (or of [Bon11] if $V$ is smooth; this case will be quite sufficient for us below).

**Corollary 2.2.4.** Let $c \geq 1$, $m \geq 0$.

1. The Chow weight structure "restricts" to a weight structure for $DM^eff_{gm,R}(c)$, i.e., there exists a weight structure $w^c$ for $DM^eff_{gm,R}(c) \subset DM^eff_{gm,R}$ such that $DM^eff_{gm,R}(c)_{w^c = 0} = DM^eff_{gm,Rw_{Chow} = 0)(\ObjDM^eff_{gm,R}(c))$, $DM^eff_{gm,R}(c)_{w^c > 0} = DM^eff_{gm,R,\WChow}(0)$ \ObjDM^eff_{gm,R}(c)$. Moreover, $DM^eff_{gm,R}(c)_{w^c = 0} = DM^eff_{gm,Rw_{Chow} = 0)(\ObjDM^eff_{gm,R}(c))$. More- 2. An object $M$ of $\Chow^eff_{gm,R}$ is $c$-effective (as an object of $DM^eff_{gm,R}$) if and only if it can be presented as $N(c)$ for $N \in \Obj\Chow^eff_{gm,R}$.

3. The Chow weight structure also restricts to a weight structure for $d_{\leq m}DM^eff_{gm,R}$ (which we will denote just by $w^c_{Chow}$). Its heart consists of those objects of this category that are isomorphic to Chow motives; the latter is equivalent to being a retract of $M^R_m(P)$ for some smooth projective $P/k$ of dimension $\leq m$.

4. If $U \to V$ is an open embedding of smooth varieties such that $V \setminus U$ is everywhere of codimension $c$ in $V$, $\dim V \leq m$, then $\Cone(M_{gm}(U) \to M^R_m(V)) \in (d_{\leq m-c}DM^eff_{gm,R})_{w_{Chow} = 0}(c)$.

**Proof.** 1. Note that $DM^eff_{gm,R}(c)$ is exactly the subcategory of $DM^eff_{gm,R}$ generated by $\Chow^eff_{gm,R}(c)$. Hence Proposition 1.2.3 yields the result immediately.

2. This is an immediate consequence of the "moreover" part of the previous assertion (since $-\langle c)$ yields an equivalence of $DM^eff_{gm,R}$ with $DM^eff_{gm,R}(c)$).

3. Immediate from Proposition 1.2.3 again.

4. There certainly exists a chain of open embeddings $U = U_0 \to U_1 \to U_2 \to \cdots \to U_m = V$ (for some $m \geq 1$) such that $U_i \setminus U_{i-1}$ are smooth for all $1 \leq i \leq m$. Hence the distinguished triangles together with Remark 2.2.3 imply (by induction on $m$) that $\Cone(M_{gm}(U) \to M^R_m(V)) \in$
Let us deduce some more lemmas that will be very important for us below.

**Proposition 2.2.5.** Let $m, j \geq 0$, $c \geq 1$.

1. Let $u : U \to V$ be an open embedding of smooth varieties such that $V \setminus U$ is everywhere of codimension $\geq c$ in $V$ and $\dim V \leq m$. Let $M \in DM_{gm,wChow}^R$, and assume that $g \in DM_{gm}^R(M_{gm}(V)\langle j \rangle, M)$ vanishes when composed with $M_{gm}^R(u)\langle j \rangle$. Then there exists a smooth projective $P/k$ of dimension $\leq m - c$ such that $g$ factors through $M_{gm}^R(P)\langle j + c \rangle$.

2. Any morphism $q : M_{gm}^R(Q) \to N\langle c \rangle$ for $N \in Obj Ch_{eff}^R$ and a smooth projective $R$ of dimension $\leq m$ can be factored through $M_{gm}^R(P)\langle c \rangle$ for some smooth projective $P/k$ of dimension $\leq m - c$.

3. $\text{Obj } d_{\leq m} DM_{gm,R}^{eff} \cap \text{Obj } DM_{gm,R}^{eff}\langle c \rangle = \text{Obj } (d_{\leq m} DM_{gm,R})\langle c \rangle$.

In particular, if $N \in \text{Obj } Ch_{eff}^R$ and $N\langle c \rangle$ is of dimension $\leq m$ (in $DM_{gm,R}^{eff}$), then $N$ is of dimension $\leq m - c$ (so, it is zero if $c > m$).

4. Let $M \in \text{Obj } Ch_{eff}^R$ and $g \in DM_{gm,R}^{eff}(M_{gm}^R(P)\langle j \rangle, M)$, where $P$ is a connected smooth projective variety (over $k$). Assume that the fibre of $g$ (considered as a rational equivalence class of cycles in the corresponding product of smooth projective varieties) over the generic point of $P$ vanishes. Then $g$ can be factored through an object of $Ch_{eff}^R(j + 1)$.

5. For $P$ and $M$ as in the previous assertion we have $DM_{gm}^{R,j}(M_{gm}^R(P)\langle j \rangle, M) \cong Ch_{j,R,k}(P)M$.

**Proof.** 1. Certainly, $g$ can be factored through $\text{Cone}(M_{gm}^R(u)\langle j \rangle)$. Next, Corollary 2.2.4(4) yields that $\text{Cone}(M_{gm}^R(u)\langle j \rangle) \in DM_{gm,R,wChow}^{eff}(j + c)$. Hence for $\text{Cone}(M_{gm}^R(u)) = M\langle c \rangle$ we can take $w_{\geq 0}(\text{Cone}(M_{gm}^R(u)\langle j \rangle)) = (w_{\geq 0}M')\langle j + c \rangle \in Ch_{eff}^R(j + c)$ (see Proposition 1.2.3(2)). Hence applying part 8 of the proposition cited we conclude the proof.

2. Let $Q = \sqcup Q_i$, where $Q_i$ are the connected components of $Q$ of some dimensions $m_i \leq m$; let $N$ be a retract of $M_{gm}^R(S)$ for some smooth projective $S$. By the "classical" theory of Chow motives, $q$ is given by a collection of algebraic cycles of dimensions $m_i - c$ in $Q_i \times S$. Hence there exists an open $U \subset Q$ such that $Q \setminus U$ is everywhere of codimension $\geq c$ in $Q$ and the "restriction" of $q$ onto $U$ vanishes. Thus the previous assertion yields that $q$ factors through some $M_{gm}^R(P)\langle c \rangle$ for a smooth projective $P/k$ of dimension $\leq m - c$. 22
3. The first part of the assertion is immediate from assertion 2 combined with Proposition 1.4.1.

In order to deduce the second part it suffices to note that all Chow motives in the heart of \( d_{m-c} \) are retracts of \( M_{gm,R}(P) \langle c \rangle \) for some smooth projective \( P/k \) of dimension \( \leq m - c \) (see Corollary 2.3.4(1,3)), and apply the Cancellation theorem.

4. The "continuity" of motivic cohomology groups (cf. also Proposition 2.3.2(1) and Lemma 3.4 of [Via11]) yields the existence of an open dense embedding \( u : U \to P \) such that \( g \) vanishes (i.e., it is rationally equivalent to zero if considered as an algebraic cycle) over \( U \) also. Hence assertion 1 yields the result.

5. Let \( \dim P = d \). Note (similarly to the proof of the previous assertion) that \( DM_{gm,R}(M_{gm}(P)(j), M) \cong Chow^R_{j+d}(M_{gm}(P) \otimes M) \). We obtain a natural surjective homomorphism \( DM_{gm,R}(M_{gm}(P)(j), M) \cong Chow^R_{j+d}(M_{gm}(P) \otimes M) \to Chow^{R,k}(P)(M) \). By Proposition 1.3.1(3), the natural homomorphism \( DM_{gm,R}(M_{gm}(P)(j), M) \to DM_{gm}^j(M_{gm}(P)(j), M) \) is surjective also. So, we should compare the kernels.

By the assertion cited, the latter kernel consists exactly of morphisms that can be factored through \( Chow^{eff}_{R}(j+1) \). Next, (the rational equivalence class of cycles representing) any morphism of the latter sort vanishes in \( Chow^{R,k}(P)(M) \) for simple dimension reasons (cf. Proposition 2.3.2(2) below). It remains to note that any element of \( \ker(DM_{gm,R}(M_{gm}(P)(j), M) \to Chow^{R,k}(P)(M)) \) can be factored through an object of \( Chow^{eff}_{R}(j+1) \) according to the previous assertion.

Remark 2.2.6. 1. The proof of (part 4) of the proposition uses an abstract version of the well-known decomposition of the diagonal arguments (cf. Proposition 1 of [BIS83]). The "usual" way to construct the factorization in question (see Theorem 3.6 of [Via11] and Lemma 3 of [GoG13]) is to resolve the singularities of \( P \setminus U \). Yet it seems difficult to apply this more explicit method if \( p > 0 \) (at least, for \( \mathbb{Z}[1/p] \)-coefficients). Besides, our reasoning is somewhat shorter than the one of loc. cit. (given the properties of Chow weight structures that are absolutely necessary for this paper anyway).

2. In the case \( R = \mathbb{Q} \) the "in particular" part of Proposition 2.2.5(3) was established in §3 of [Via11] (see Remark 3.11 of ibid.). The general case of the assertion is completely new.

3. The idea of studying \( DM_{gm}^j \) and the formulation of part 5 of the proposition was inspired by [KaS02] (where our assertion was established in the case \( j = 0 \)).
2.3 On higher Chow groups and motivic homology over various fields

We start with some simple definitions.

**Definition 2.3.1.** Let $K$ be a field.

1. We denote by $K^{perf}$ the perfect closure of $K$.
2. For $M$ being an object of $\text{Chow}_{\text{eff}}^R$ (or of $DM_{\text{eff}}^{R}$) and $j,l \in \mathbb{Z}$ we define $\text{Chow}_{j,l}^R(M)$ as $DM_{\text{gm}}^R(R(j)[2j+l], M)$; more generally, for an extension $K/k$ we set $\text{Chow}_{j,l}^{R,K}(M) = DM_{\text{gm}}^R(K^{perf})(R(j)[2j+l], M_K^{perf})$ (when we will use this notation for "general" $(l, M)$ we will usually take $j = 0$ in it).

Note that this definition can be naturally extended to $DM_{\text{eff}}^{-R}$.

3. We will say that $K$ is **essentially finitely generated** if it is the perfect closure of a field that is finitely generated over its prime subfield.

5. We call $K$ a universal domain if it is algebraically closed and of infinite transcendence degree over its prime subfield.

6. We will say that a field $K_0$ is a field of definition for an object $M$ of $DM_{\text{eff}}^{R}$ (resp. of $\text{Kb}(\text{Chow}^R)$) if it is equipped with a (fixed) perfect subfield $k_0$, an embedding $k_0 \to k$, and there is a fixed $M_0 \in \text{Obj } DM_{\text{eff}}^{R}(k_0)$ (resp. $M_0 \in \text{Obj } \text{Kb}(\text{Chow}^R(k_0))$) such that $M$ is isomorphic to $M$.

7. We call $K$ a rational extension of $k$ if $K \cong k(t_1, \ldots, t_n)$ for some $n \geq 0$.

8. We will say that $K$ is a function field over $k$ if $K$ is a finite separable extension of a rational extension $K'$ of $k$ (and so, it is the function field of some smooth connected variety $V/k$); we will call the transcendence degree of $K/k$ the dimension of $K$ over $k$.

Note that fields of definition for $M$ obviously form a category.

**Proposition 2.3.2.** Let $j,l \in \mathbb{Z}$, $d,r \geq 0$.

Then the following statements are valid.

1. Let $N \in \text{Obj } \text{Chow}^R$. Then $\text{Chow}_{j,l}^{R,K}(N) \cong DM_{\text{gm}}^R(K)(\hat{N}_K, R(-j)[-2j-l])$ for any perfect field $K/k$, where $\hat{N}$ is the Poincare dual of $N$ (in $\text{Chow}^R$ and in $DM_{\text{gm}}^R$).

2. For any $N \in \text{Obj } \text{Chow}_{\text{eff}}^R$ and any field $K/k$ we have $\text{Chow}_{j,l}^{R,K}(N(r)) = \{0\}$ if $j - r + l < 0$.

3. For an object $N$ of $DM_{\text{eff}}^{R}$ (or of $DM_{\text{eff}}^{-R}$) we have $N \in DM_{\text{eff}}^{R,R}_{\text{hom} \leq 0}$ (see the end of § 2.4) if and only if $\text{Chow}_{0,l}^{R,K}(N) = \{0\}$ for all $l < 0$ and all function fields $K/k$.

Moreover, for any $r \geq 0$ these conditions are equivalent to the vanishing of $\text{Chow}_{r,l+r}^{R,K}(N)$ for all $l < 0$ and all function fields $K/k$. 

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Proof. 1. This is an immediate consequence of Poincare duality for Voevodsky motives; see Theorem 5.23 of [Deg08].

2. Obviously, it suffices to establish the statement for $N = M^{R}_{gm}(P)$, where $P$ is as in the previous assertion; so we treat this particular case. Next, recall that motivic cohomology of smooth varieties can be computed as the (co)homology of certain (Suslin or Bloch) cycle complexes, see §4.2 of [Voe00a]. Therefore the group in question is a subquotient of a certain group of cycles of $K^{perf}$-dimension $j - r + l$. The result follows immediately.

3. The first part of the statement easily follows from Corollary 4.18 of [Voe00b]. The second half immediately follows from the following well-known fact: the functor $- (1)[1] : DM^{eff}_{R} \rightarrow DM^{eff}_{R}$ and its right adjoint $-1$ (in the notation of [Voe00b]) send $DM^{eff}_{R}^{hom} \leq 0$ into itself (see Theorem 5.3 of [Deg11] and §1.4 of [Bon10b]).

Now let us prove some facts relating (complexes of) higher Chow groups over various base fields. Our first statement is rather "classical" (cf. Lemma IA.3 of [Blo10] and §3 of [Via11]; one can also apply the more "advanced" formalism of [CiD14] to prove it), whereas the second one relies on the results of [Bon10b] (and [Bon13]) and seems to be new.

We will also study the ("ordinary") motivic homology of an $M \in \text{Obj } DM^{R}_{gm}$.

Proposition 2.3.3. Let $j, l \in \mathbb{Z}$.

Fix an object $(M^{i})$ of $K^{b}(\text{Chow}_{R})$; for a field of definition $K_{0}$ of $(M^{i})$ denote by $G(K_{0})$ the zeroth homology of the complex $\text{Chow}^{R,K_{0}}_{j,l}(M^{*})$ (certainly, $G$ is functorial with respect to morphisms of fields of definition for $(M^{i})$).

I. The following statements are valid.

1. Let $K_{0} \subset K_{0}'$ be fields of definitions for $M$. Then $G(K_{0}')$ is the (filtered) direct limit of $G(K_{0})$ if we take $K$ running through all finitely generated extensions of $K_{0}$ inside $K_{0}'$ such that the extension $K \cap K_{0}^{alg}/K_{0}$ is separable.

2. Let $K_{1}/K_{0}^{alg}$ and $K_{2}/K_{0}^{alg}$ be fields of definitions for $M$; let $s : K_{1} \rightarrow K_{2}$ be an embedding of fields such that for the corresponding motives obtained by extending scalars we have $(M^{i}_{K_{1}})_{K_{2}} \cong M^{i}_{K_{2}}$ (note that we do not require $s$ to be a morphisms of fields of definition over $k$). Then $s$ induces a homomorphism $G(K_{1}) \rightarrow G(K_{2})$ that is an isomorphism if $s(K_{1}) = K_{2}$, and is injective if $K_{1}$ is algebraically closed.

II. Let $R = \mathbb{Q}$. Then the following conditions are equivalent.

1. $G(K) = 0$ for any function field $K/k$.

2. $G(K_{0}) = 0$ for some universal domain of definition for $M$.

3. $G(K_{0}) = 0$ for any algebraically closed field of definition for $M$.
4. \( G(K_i) = 0 \) for some algebraically closed fields of definition of \( M \) such that the transcendence degrees of \( K_i \) over the corresponding prime field are not bounded above by any natural number.

5. \( G(K_0) = 0 \) for any field of definition for \( M \).

III. All the statements above remain valid if we define \( G(K) \) as \( \text{Chow}^{R,K}_{j,l}(M) \) for a fixed \( M \in DM^{R}_{gm} \).

Proof. We note (for convenience) that we can pass to the Poincare duals in all of these statements (see Proposition 2.3.2(1)). So, we can express \( G(K) \) in terms of motivic cohomology instead of motivic homology. We do not have to track the indices involved (since they coincide in all the assertions).

I Recall that the motivic cohomology of Chow motives can be (functorially) computed using certain complexes whose terms are expressed in terms of algebraic cycles. This fact easily yields all our assertions except the (very) last one (since any finitely generated extension of a field \( k \) is purely inseparable over some function field \( K/k \)).

In order to verify the remaining statement we note that for a (Voevodsky) motif \( N \) defined over a perfect field \( L \) the motivic cohomology of \( N_{L'} \) (for a perfect field extension \( L'/L \)) can be (functorially in \( N \)) expressed as the filtered direct limit of the corresponding cohomology of \( N \otimes M^{gm}_{R,L}(V_a) \) for certain smooth varieties \( V_a \) over \( L \). Next, if \( L \) is algebraically closed, then the \( DM^{R}_{gm,L} \)-morphism \( R \to M^{R}_{gm}(V_a) \) possesses a splitting given by any \( L \)-point of \( V_a \). Hence the homomorphism in question is injective since it can be presented as the direct limit of a system of (split) injections.

II Obviously, any \( M \) possesses an essentially finitely generated field of definition. Next, the existence of the trace maps for higher Chow groups (with respect to finite extensions of not necessarily perfect base fields; cf. Lemma 1.2 of [Via11]) yields the following: if \( K'_0/K_0 \) is an algebraic extension and \( G(K'_0) = \{0\} \), then \( G(K_0) = \{0\} \) also. Together with assertion I, these observations easily yield our claim.

III Note that the motivic (co)homology of any Voevodsky motif can also be computed in terms of certain complexes of algebraic cycles. The existence of these complexes is immediate from (the \( R \)-module analogue of) Theorem 3.1.1 of [Bon09] (note that this result is valid for \( p > 0 \) also; this is a consequence of Proposition 5.5.11(5) of [Kel13]). Hence the arguments above carry over to this setting without any difficulty.

Proposition 2.3.4. Again, let \( j,l \in \mathbb{Z}, (M^i) \in \text{Obj} \ K^b(\text{Chow}_R) \); let \( K_1 \) and \( K_2 \) be function fields over \( k \). For \( r > 0 \) assume that there exists a geometric \( k \)-valuation of rank \( r \) for \( K_2 \) such that the corresponding residue
field is isomorphic to $K_1$. Then there exists a split injection of the complex $\text{Chow}^{R,K^1}_{j,l}(M^*)$ into the complex $\text{Chow}^{R,K^2}_{j-r,1+r}(M^*)$.

Proof. Certainly, we can assume $j = l = 0$ in this statement. Once more, we apply the Poincare duality and obtain (see Proposition 2.3.2(1)) the following: for a complex $(N^i) \in \text{Obj} \text{Chow}_R$ we should construct a split injection of the complex $(DM^R_{gm}(N^i_{K_{perf}}; R))$ into $(DM^R_{gm}(N^i_{K_{perf}}; R)(r)[r])$. Note also that if Spec $K$ (for $i = 1, 2$) are the inverse (filtered) limit of some systems of smooth varieties $X^b/k$ and $O \in \text{Obj} \text{Chow}_R$, then $(DM^R_{gm}(O_{K_{perf}}; R)) \cong \text{lim}^R(\text{Chow}^{R}_{gm}(M^R_{gm}(X^b) \otimes O, R))$.

Hence the statement would be proved if we had a "motivic" category $\mathfrak{D} \supset DM^R_{gm}$, that contains certain homotopy limits $\text{lim}^R_{gm}(X^b_n)$ for $b = 1, 2$ (that can be denoted as $M^b_{gm}$(Spec $K$)), is equipped with a bi-additive tensor product bi-functor $DM^R_{gm} \times \mathfrak{D} \rightarrow \mathfrak{D}^R$ such that $\mathfrak{D}(\text{lim}^R_{gm}(X^b_n) \otimes O, R)$ are functorially isomorphic to $\text{lim}^R_{gm}(X^b_n) \otimes O, R$, and such that there exists a split $\mathfrak{D}^R$-morphism $\text{lim}^R_{gm}(X^b_n)(r)[r] \rightarrow \text{lim}^R_{gm}(X^b_n))$. 

Luckily, the results of previous papers yield the existence of $\mathfrak{D}$ having all these properties. Indeed, for $R = \mathbb{Z}$ a certain category of this sort was constructed in [Bon10b]. It suffered from two drawbacks: it only contained $DM^R_{eff}$ instead of $DM^R_{gm}$, and the splitting in question was established (see Corollary 4.2.2(2) of ibid.) only for $k$ being countable. Yet one can easily "correct" that category so that it would contain $DM^R_{gm}$, and in §6.4 of [Bon13] it was shown that the desired splitting exists for any perfect $k$.

\[ \square \]

Remark 2.3.5. 1. Since a function field of dimension $d$ is a finite separable extension of $k(t_1, \ldots, t_d)$, it is also a residue field for a (rank 1) geometric valuation of $k(t_1, t_2, \ldots, t_{d+1})$. Thus one may say that it "suffices to compute stalks at rational extensions of $k$" only!

2. One can prove the natural analogue of the previous proposition for the complex $\text{Chow}^{R,K}_{j,l}(M^*)$ replaced by the group $\text{Chow}^{R,K}_{j,l}(N)$, where $N$ is an object of $DM^R_{eff}$ (or of $DM^R_{eff}$; to this end it suffices to recall just a little more of the results of [Bon10b]).

Thus one obtains: $N \in DM^R_{eff}$ if and only if $\text{Chow}^{R,K}_{j,l}(N(1)[1]) = \{0\}$ for all rational fields $K/k$ (only!) and for all $l < 0$.

Possibly, the authors will dedicate a separate note to this observation.

3. One can also verify that $\text{Chow}^{R,K}_{j,-l,1+t}(M^*)$ contains (as a retract) the sum of any finite number of $\text{Chow}^{R,k_m}_{j,l}(M^*)$ for $k_m$ being residue fields for (distinct) geometric valuations of $K$ of rank $r$. Hence the homology groups of $\text{Chow}^{R,K}_{j,-l,1+t}(M^*)$ can be quite huge. So, we will not try to calculate them.
in general (at least, in the current paper); we will rather be interested in their vanishing (also, modulo torsion or after tensoring by \( \mathbb{Z}/l\mathbb{Z} \)) instead.

3 The main results

In this section we prove the central results of this paper.

In §3.1 we introduce the main homology theories of this paper (obtained via composing certain homology considered in the previous section with the weight complex functor) and prove several of their properties.

In §3.2 we establish the central theorem of this paper; it relates the Chow-weight homology with the \( c \)-effectivity of motives and their "weights". A very particular case of this result yields: a cone of a morphism \( h \) of Chow motives is \( c \)-effective if and only if \( h \) induces isomorphisms on Chow groups of dimensions less than \( c \). We obtain several other interesting consequences immediately: in particular, the non-vanishing of certain weight filtration pieces of (any) homology of \( M \) yields the non-vanishing of the corresponding Chow-weight homology for it. These results can be used to obtain vast generalizations of the well-known "decomposition of the diagonal" statements, and also of the relations between "cohomological and cycle-theoretic connectivity" (cf. the introduction and [Par94]). In §3.3 we prove that the properties of motives studied in the previous subsection can also be "detected" through the "higher" Chow-weight homology. As a consequence, we relate the vanishing of Chow-weight homology of \( M \) with that for its higher degree (zero-dimensional) motivic homology (and so, with its "\( t^R \)-hom-connectivity").

3.1 Chow-weight homology: definition and properties

Let us define the main homology theories of this paper.

Definition 3.1.1. 1. We denote by \( t_R(M) \) a choice of a weight complex for a motif \( M \in \text{Obj} \, DM^\text{eff}_{gm,R} \) (or of \( M \in \text{Obj} \, DM^R_{gm} \)) with respect to the Chow weight structure for \( DM^\text{eff}_{gm,R} \) (so, it is a \( \text{Chow}^\text{eff}_R \)-complex; as we have already said, one can assume that \( t_R \) is a functor \( DM^\text{eff}_{gm,R} \to K^b(\text{Chow}^\text{eff}_R) \)).

2. Let \( j,l,i \in \mathbb{Z} \); let \( K \) be a field extension of \( k \).

For an object \( M \) of \( DM^\text{eff}_{gm,R} \) (or of \( DM^R_{gm} \)), \((M^*)\) being a choice of \( t_R(M) \), we define \( \text{CWH}^j_{i,K}(M) \) (resp. \( \text{CWH}^j_{i,K}(M) \)) as the \( i \)-th homology of the complex \( \text{Chow}^R_{j,K}(M^*) \) (resp. of \( \text{Chow}^R_{j,K}(M^*) \)).

The reader should pay attention to the modification of the position of indices made in this notation for purely typographical reasons; besides, we omit \( R \) in it.
Let us prove some properties of $\mathrm{CWH}_{i,K}^r$.

**Proposition 3.1.2.** Let $l, i, K$ be as above, $r, j \geq 0$.

1. $\mathrm{CWH}_{i,K}^l(\_)$ yields a homological functor on $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}$ (that does not depend on any choices).

2. Assume that $r \geq j + 1$. Then $\mathrm{CWH}_{i,K}^j$ kills $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}(r + 1)$ (and so, induces a well-defined functor $\mathcal{D}_\mathrm{gm,R}^r \to \text{Ab}$; see Definition 2.2.3(3)).

3. Let $N \in \mathcal{D}_\mathrm{gm,R}^j$. Then for any smooth projective connective variety $P/k$ we have: $\mathcal{D}_\mathrm{gm,R}^j(P, \langle M_\mathrm{gm}(P) \rangle, N) \cong \mathcal{D}_{0,k}(N)$ (note that the latter group is well-defined according to the previous assertion).

4. Let $N \in \mathcal{D}_\mathrm{gm,R}^r$. Then $\mathrm{CWH}_{i,K}^j(N) = \{0\}$ for all $i > n, j \leq r - l$ (note that the corresponding $\mathrm{CWH}_{i,K}^j(N)$ are well-defined).

5. Assume that $0 \leq m \leq r$. Let $N \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}_{\mathrm{wChow} \geq i}$ (resp. $N \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}_{\mathrm{wChow} \geq -i}$) and assume that $\mathrm{CWH}_{i,K}^j(N) = \{0\}$ for all $0 \leq j \leq m$ and all function fields $K/k$. Then for any fixed choice of a (shifted) weight decomposition $w_{\leq i}N \to \to w_{\geq -i}N$ (see (3)) the morphism $g[i]$ can be factored through an object of $\mathrm{CWH}_{i,K}^{\text{eff}}_{\mathrm{wChow} \geq m + 1}$ (resp. through an image of such an object in $\mathcal{D}_\mathrm{gm,R}^r$).

6. If $N \in \text{Obj} \mathcal{D}_\mathrm{gm,R}^{\text{eff}} \cap \text{Obj} \mathcal{D}_\mathrm{gm,R}^{\text{eff}}_{\text{hom} \leq 0}$ (see the end of §2.1) and $i > j \geq 0$ then $\mathrm{CWH}_{i,K}^j(N) = \{0\}$.

**Proof.** 1. This is just a particular case of Proposition 1.4.2(3).

2. Recall that $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}(r)$ is generated by $\mathrm{CWH}_{i,K}^j(\_)$ (as a triangulated subcategory of $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}$). Hence the statement follows immediately from Proposition 2.3.2(2).

3. By Proposition 2.2.5(5), $\mathrm{CWH}_{i,0}(P, \langle M_\mathrm{gm}(P) \rangle, N) \cong H_0(\mathcal{D}_\mathrm{gm,R}^{\text{eff}}(P, \langle M_\mathrm{gm}(P) \rangle, N^*))$ (for $N^*$ being the terms of a weight complex for $N$). Hence it remains to apply Corollary 1.4.7(1).

4. We can certainly assume that the weight complex of $N$ is concentrated in degrees $\leq n$ (see Proposition 1.4.2(11)). Next, recall that any object of the heart of $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}$ is a retract of a one coming from $\mathrm{CWH}_{i,K}^{\text{eff}} \subset \text{Obj} \mathcal{D}_\mathrm{gm,R}^{\text{eff}}$. Hence the statement follows from Proposition 1.4.6.

5. Obviously, we can assume that $i = 0$.

We have $w_{\leq 0}N \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}_{\mathrm{wChow} = 0}$ (resp. $w_{\leq 0}N \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}_{\mathrm{wChow} = 0}$); so it is a retract of $M^{\text{eff}}_{\mathrm{gm}}(P)$ (resp. of $M^{\text{eff}}_{\mathrm{gm}}(P)$) for some $P \in \text{SmPrVar}$.

Hence it suffices to check the following for any $0 \leq j \leq m$ and $P, P_0 \in \text{SmPrVar}$: any $g_j \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}(M^{\text{eff}}_{\mathrm{gm}}(P) \langle j \rangle, N)$ (resp. $g_j \in \mathcal{D}_\mathrm{gm,R}^{\text{eff}}(M^{\text{eff}}_{\mathrm{gm}}(P) \langle j \rangle, N)$) can be factored through $M^{\text{eff}}_{\mathrm{gm}}(P^{j+1})(j + 1)$ (resp. through $\mathcal{D}_\mathrm{gm,R}^{\text{eff}}(P^{j+1})(j + 1)$) for some $P^{j+1} \in \text{SmPrVar}$. 29
By Corollary 1.4.7(2) applied to \( l^j \) (resp. to the localization functor \( l^j_r : DM^{eff}_{gm,R} \rightarrow DM^{eff}_{gm,R} \)), to achieve the goal it suffices to verify that the image of \( g^j \) in \( DM^{eff}_{gm} \) is 0. It remains to note that this image is an element of \( DM^{eff}_{gm}(l^j(M^R_{gm}(P^j)(j)), l^j(N)) \) (resp. of \( DM^{eff}_{gm}(l^j_r(M^R_{gm}(P^j)(j)), l^r(N)) \)), whereas the latter group is zero by assertion 3 and by our assumptions on CWH\(^k\)(P\(^j\))(N).

6. We certainly have \( \text{Obj} DM^{eff}_{gm,R} \cap DM^{eff}_{-R} \subseteq \text{Obj} DM^{eff}_{gm,R} \cap DM^{eff}_{-R} \subseteq 0 \) (see the end of §2.1). Note also that in the case \( p > 0 \) it suffices to establish the statement for \( R \) being a \( \mathbb{Z}(l) \)-algebra, where \( l \) is a prime distinct from \( p \). Indeed, the assertion for \( R \)-coefficients easily follows from all of its \( R \otimes \mathbb{Z} \mathbb{Z}(l) \)-analogues.

Next let us recall that in [Bon14] the following statement was proved (see Proposition 3.3.2 and Remark 3.3.3(1) of ibid.): \( DM^{eff}_{Q} \) is the smallest extension-closed subclass of \( \text{Obj} DM^{eff}_{Q} \) that is closed with respect to coproducts and contains \( \text{Obj} Chow^{eff}_{Q}(a)[a+b] \) for all \( a, b \geq 0 \). We note that the methods of loc. cit. easily yield the similar fact for \( DM^{eff}_{R} \) also (at least, if \( R \) is a \( \mathbb{Z}(l) \)-algebra, and for motives over a perfect field). Indeed, in the case \( p = 0 \) one should just replace the "ordinary" alterations argument used in the proof of Theorem 2.1.2(I.2) of ibid. by Hironaka’s resolution of singularities; in the case \( p > 0 \) one should use Gabber’s alterations of degree prime to \( l \) (see Theorem 3.2.12 of [Kel13] instead).

Next, we note that \( w_{Chow} \) can be extended (from \( DM^{eff}_{gm,R} \)) to \( DM^{eff}_{R} \) in a way that "respects coproducts" (by the categorical dual to Theorem 2.2.6 of [Bon13]; cf. also Theorem 4.5.2 and §7.1 of [Bon10a]). Hence Chow-weight homology (as well as any other \( w_{Chow} \)-pure homology theory whose target is an AB5 abelian category) can be extended to a homological functor \( DM^{eff}_{R} \rightarrow Ab \) that respects coproducts (here the unboundedness of the "non-geometric" version of \( w_{Chow} \) does not cause any problems).

Hence it suffices to verify the vanishing in question for \( N \in \text{Obj} Chow^{eff}_{R}(a)[a+b] \) (for some \( a, b \geq 0 \)). Thus it remains to apply Proposition 2.3.2(2).

3.2 Relating Chow-weight homology with \( \epsilon \)-effectivity: the central theorem

Now we can prove the main results of this paper. To formulate (the more general) assertion II.3 of the theorem below we will need the following technical definition.

**Definition 3.2.1.** Let \( I \) be a subset of \( \mathbb{Z} \times [0, +\infty) \) (see §1.1).
We will call it reasonable if for any \((i, j) \in \mathbb{Z} \times [0, +\infty), i' \geq i,\) and \(0 \leq j' \leq j,\) we have \((i', j') \in I.\)

For \(i \in \mathbb{Z}\) we denote by \(a_{I,i}\) the minimum of \(j \in \mathbb{Z}\) such that \((i, j) \notin I\) (note that \(a_{I,i}\) may be equal to \(+\infty\)). When there will be no risk of confusion, we will just write \(a_i.\)

**Theorem 3.2.2.** Let \(M \in \text{Obj } DM_{\text{gm,R}, c > 0, n \in \mathbb{Z}}.\)

1. For an arbitrary \(R\) the following statements are valid.
   - If \(M \in \text{Obj } DM_{\text{gm,R}\{c\}}\) if and only if \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all \(i \in \mathbb{Z}, 0 \leq j < c,\) and all function fields \(K/k.\)
   - More generally, \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all \(0 \leq j < c, i > n,\) and all function fields \(K/k\) if and only if \(M\) is an extension (see \((3.1)\)) of an element of \((DM_{\text{gm,R}}^c\{w\}_\text{Chow} \geq -n, c > 0\})\) by an element of \((DM_{\text{gm,R}}^c\{w\}_\text{Chow} \leq -n-1, c > 0\})\).
   - \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all \(j \geq 0, i > n,\) and all function fields \(K/k,\)
   if and only if \(M \in (DM_{\text{gm,R}}^c\{w\}_\text{Chow} \geq -n, c > 0\})\) for all function fields \(K/k\) and \((i, j) \in I\) is equivalent to the same vanishing for all field extensions \(K/k.\)

2. Let \(R = \mathbb{Q}.\) Then the vanishing of \(\text{CWH}_{i,K}^j(M)\) for all function fields \(K/k\) and \((i, j) \in I\) is also equivalent to \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all \((i, j) \in I\) and \(K\) being some fixed universal domain containing \(K.\)

3. Suppose \(I\) is reasonable. Then the following conditions are equivalent.
   - \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all function fields \(K/k\) and \((i, j) \in I.\)
   - \(i \leq n\) belongs to \((DM_{\text{gm,R}}^c\{w\}_\text{Chow} \geq -n-1, c > 0\})\) whenever \((i, j) \in I.\)
   - There exists a choice of \(w_{\leq -1} M\) (see \((3.3)\)) that is an object of \(DM_{\text{gm,R}}^c\{a_{I,i}\};\) here we set \(DM_{\text{gm,R}}^c\{+\infty\} = \{0\}.\)
   - There exists a choice of a weight complex for \(M\) such that its \(i\)-th term is \(j\)-effective whenever \((i, j) \in I.\)

Proof. 1.1. If \(M \in \text{Obj } DM_{\text{gm,R}}^c\{c\},\) then \(\text{CWH}_{i,K}^j(M) = \{0\}\) for all \(i, j,\) and \(K\) as in the assertion by Proposition \((3.1)\).2).

Conversely, assume that \(M\) satisfies the corresponding Chow-weight homology vanishing assumptions. Then is suffices to prove that \(l^{i-1}(M) \in DM_{\text{gm,R}}^c\{e\} \geq r\) for any \(r \in \mathbb{Z}\) (since the Chow weight structure for \(DM_{\text{gm}}^c\{e\}\) is bounded). Hence our assertion follows from the next one.

2. Assume that \(l^{i-1}(M) \in DM_{\text{gm,R}}^c\{e\} \geq -n.\) Then the vanishing of Chow-weight homology groups in question is immediate from part 4 of Proposition \((3.1)\).2.

Conversely, let the Chow-weight homology vanishing assumptions be fulfilled. Certainly, there exists an integer \(q\) such that \(l^{i-1}(M) \in DM_{\text{gm,R}}^c\{e\} \geq q.\)
By Proposition 1.3.1(2), it suffices to verify the following: if \( t^{-1}(M) \in \text{DM}_{\text{gm,\text{Chow}} \geq t}^R \) for some \( t < -n \), then \( t^{-1}(M) \) also belongs to \( \text{DM}_{\text{gm,\text{Chow}} \geq t+1}^R \).

Let us take a shifted weight decomposition \( w \leq t^{-1}(M) \to w_{\geq t+1}^{-1}(M) \). Proposition 3.1.2(5) yields that \( g = 0 \). Hence \( t^{-1}(M) \) is a retract of an element of \( \text{DM}_{\text{gm,\text{Chow}} \geq t+1}^R \); thus it belongs to \( \text{DM}_{\text{gm,\text{Chow}} \geq t+1}^R \).

3. If \( M \in \text{DM}_{\text{gm,\text{Chow}} \geq -n}^R \) then the previous assertion yields the vanishing of \( \text{CWH}_i^M(M) = \{0\} \) for all \( j \geq 0 \), \( i > n \), and all function fields \( K/k \).

Conversely, it suffices (similarly to the previous argument) to check the following: if \( M \in \text{DM}_{\text{gm,\text{Chow}} \geq t}^R \) for some \( t < -n \), then \( M \in \text{DM}_{\text{gm,\text{Chow}} \geq t+1}^R \). Again, we can fix a shifted weight decomposition \( w \leq M \to M \to w_{\geq t+1}M \) and check that \( g = 0 \). Assume that \( w \leq M[-l] \) is (a Chow motif) of dimension \( s \) for some \( s \geq 0 \). By Proposition 3.1.2(5), our Chow-weight homology assumptions yield that \( g[-l] \) can be factored through \( \text{Chow}^\text{eff}_R(s+1) \). Hence Proposition 2.2.5(2) implies that \( g = 0 \).

II. Assertions 1 and 2 follow from Proposition 2.3.3 immediately.

3. Note that \( I \) equals the union of the "strips" \( \bigcup_{(i_0,j_0) \in I} I_{i_0,j_0} \), where \( I_{i_0,j_0} = [i_0, +\infty) \times [0, j_0] \) (see §1.1). By assertion I.2 (cf. also its proof), the vanishing of \( \text{CWH}_i^M(M) \) for all function fields \( K/k \) and \( (i, j) \in I_{i_0,j_0} \) is equivalent to \( \text{I}^0(M) \in \text{DM}_{\text{gm,\text{Chow}} \geq -i_0+1}^R \). The combination of these equivalences for all \( (i_0,j_0) \in I \) yields the equivalence \( A \iff B \).

Next, condition B implies condition C for a fixed \( i \in \mathbb{Z} \) if \( a_{i+} < +\infty \) by assertion I.2 (since \( (i, a_i - 1) \in I \)). If \( a_{i+} = +\infty \) then one should apply assertion I.3 instead.

\( C \Rightarrow D \) by the definition of weight complexes (since \( j \)-effective motives form a triangulated subcategory in \( \text{DM}_{\text{gm,\text{eff}}}^R \)).

Finally, if we choose \( t_R(M) = (M^t) \) for \( M^t \) as in condition D, then Proposition 3.1.2(4) yields condition A (since the corresponding Chow-weight homology groups will be subfactors of certain vanishing Chow groups).

This finishes the proof.

\[ \square \]

Remark 3.2.3. One can easily make the following observations.

1. As a very particular case of the statement, we obtain the following fact: if \( h \) is a morphism of Chow motives then \( \text{Cone}(h) \) is \( c \)-effective (i.e., it is homotopy equivalent to a cone of a morphism of \( c \)-effective Chow motives; cf. part 3 of this remark) if and only \( h \) induces isomorphisms on the corresponding Chow groups of dimension less than \( c \). Certainly, here one should consider the Chow homology over all function fields over \( k \) for a general \( R \); for \( R = \mathbb{Q} \) a single universal domain \( K/k \) is
sufficient. Another equivalent condition is that "h possesses an inverse modulo cycles supported in codimension c" (see Corollary 3.2.4 and Remark 3.2.5 below for more detail).

We will also illustrate our theorem by a generalization of this result in Corollary 3.2.4.

Even for $R = \mathbb{Q}$ these particular cases of the theorem were not stated in the existing literature. The general case of the theorem appears to be completely new.

2. The Chow-weight homology vanishing conditions of the theorem are essentially formulated in terms of the weight complex of $M$. Hence Proposition 1.4.6 implies the following: if the vanishing conditions on $CWH^*_\varphi(M)$ described in part I.2 (or of its analogue mentioned in part II.2) of the theorem are fulfilled, then for any (co)homological functor $H : DM_{\text{gr},R} \to A$ and $p \geq n$ (resp. $p \leq -n$) the $E^2_{pq}$-terms of the Chow-weight spectral sequence $H_q(M^p) \Rightarrow H_{p+q}(M)$ (resp. $H_q(M^{-p}) \Rightarrow H_{p+q}(M)$) are subquotients of the (co)homology of some $c$-effective Chow motives.

Certainly, for any reasonable $I$ one can naturally extend this observation to any $M$ satisfying (condition A of) part II.3 of the theorem.

In particular, the study of the weight filtration on $H_*(M)$ for an arbitrary $H$ can yield the non-vanishing of certain Chow-weight and motivic homology groups (see §3.3 below); this is quite remarkable since the corresponding cycle class maps (for $H$ being a Weil cohomology theory) are very far from being surjective (in most cases). We will formulate the corresponding statement in more detail for $H$ being singular homology and also prove a certain converse to it in §4.3 below.

Besides, we note that these $E^2_{pq}$-terms are the corresponding factors of the Chow-weight filtration on $H_*(M)$ if this weight spectral sequence degenerates. In particular, this is the case if $H_*$ is the singular or étale (co)homology with rational coefficients (see Remark 2.4.3 of [Bon10a]); in this case these $E^2_{pq}$-terms can be functorially expressed in terms of $H_*(M)$ (using Deligne’s weights), and one also has a reasonable notion of $c$-effectivity for them.

3. Comparing $R$-motives for $R = \mathbb{Z}[1/p]$ with that for $R = \mathbb{Z}/l\mathbb{Z}$ (for $l$ being a prime number distinct from $p$) we obtain the following fact: $M$ does not become zero in the category of $\mathbb{Z}/l\mathbb{Z}$-motives if and only if there exists a function field $K/k$ such that some Chow-weight homology group of $M$ over $K$ is not uniquely $l$-divisible.

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One can also combine this observation with the previous part of this remark. Thus if a (co)homology theory \( H \) takes values in the category of finitely generated \( \mathbb{Z}[\frac{1}{p}] \)-modules (resp. of finitely generated \( \mathbb{Z}_l \)-modules), then the non-vanishing of the corresponding pieces of \( H_* \otimes \mathbb{Q} \) yields that the corresponding Chow-weight homology \( \mathbb{Z}[\frac{1}{p}] \)-modules are not infinitely divisible (resp. not infinitely \( l \)-divisible). In particular, one can take \( H \) being singular or étale (co)homology here.

4. The Chow-weight homology groups are rather difficult to calculate (and they tend to be huge, at least, over universal domains); still they are somewhat easier to treat than the "ordinary" motivic homology groups. In particular, \( CWH_* \) can be explicitly computed for any motif belonging to the subcategory of \( D\mathcal{M}_{gm,R}^{eff} \) generated by \( \cup_{j \geq 0}(d_{\leq 1}D\mathcal{M}_{gm,R}^{eff}(j)) \), whereas the motivic homology is very difficult to compute already for \( \mathbb{CP}^2 \). We will say more on the comparison of Chow-weight homology with the motivic one in §3.3 below (and especially in Remark 3.3.3(1)). Possibly, motives with "reasonable" Chow-weight homology (of small dimensions) will be the subject of one of the subsequent papers.

Note also that it is impossible to re-formulate the theorem and the previous parts of this remark in terms of motivic homology.

5. Now we describe (one more) setting when some of the Chow-weight homology groups are "known better".

Obviously, for a smooth \( X/k \) Poincare duality allows to express the motivic cohomology of \( X \) (that coincides with the corresponding cohomology of \( \mathcal{M}_{R}^{gm}(X) \)) in terms of the motivic homology of the dual of \( \mathcal{M}_{gm}^{R}(X) \); one can also twist the latter object by \( \langle \text{dim } X \rangle \) in order to obtain an effective motif. Now, in the case where \( X \) is equi-dimensional, one obtains the \( R \)-motif with compact support \( \mathcal{M}_{c,R}^{gm}(X) \) this way (see Theorem 5.5.14(3) of [Kel13]). We certainly have \( \mathcal{M}_{c,R}^{gm}(X) \in D\mathcal{M}_{gm,R}^{eff} \) (this is an easy consequence of Proposition 2.2.1(2); see the beginning of §4.1 below).

This yields (see Corollary 1.4.7(1)) the following: \( CWH^j_0(\mathcal{M}_{gm,R}^{c,R}(X)) \cong \text{Chow}_j^{R}(X) \) (for any \( j \geq 0 \)). In particular, our results can be applied to deducing the so-called "decomposition of the diagonal" statements and the "Hodge-theoretic connectivity" ones (see part 2 of this remark, as well as [Par94], [BIS83], [GoG12], [GoG13], [Via11], and several other related articles). We will probably say more on this matter in subsequent papers.
6. Obviously, all of the "motivic" conditions mentioned in Theorem 3.2.2 have "nice" (and easily formulated) tensor properties. These facts translate into the corresponding Chow-weight homology vanishing statements for tensor products of motives. This is quite remarkable since (certainly) there cannot exist any Kunneth-type formulas for Chow-weight homology.

7. Our theorem does not yield an explicit recipe for presenting a $c$-effective $M \in \text{Obj} \ DM_{\text{gm,R}}^{eff}$ as $N \langle c \rangle$. Yet certainly, if we have a morphism $N \langle c \rangle \to M$ that we suspect to be an isomorphism, we can check whether this is really the case using (part I.1 of) our theorem.

8. The reader can easily check that everywhere in the proof of Theorem 3.2.2 (and of the prerequisites to it) we could have replaced $DM_{\text{gm,R}}^{eff}$ by $K^b(\text{Chow}_{\text{eff}}(R))$ (certainly, then we would have to replace $DM_{\text{gm}}^R$ by the localization $K^b(\text{Chow}_{\text{eff}}(R))/K^b(\text{Chow}_{\text{eff}}(R))(j+1)$; the Chow weight structure for $K^b(\text{Chow}_{\text{eff}}(R))$ is just the stupid weight structure mentioned in Remark 1.2.2(1)). The corresponding statement is even somewhat more general than Theorem 3.2.2 itself since there can exist objects of $K^b(\text{Chow}_{\text{eff}}(R))$ that cannot be presented as weight complexes of motives. Besides, this result is easier to understand for persons that are not well-acquainted with Voevodsky motives. Its disadvantage is that is gives no possibility of controlling "substantially mixed" motivic phenomena; this includes motivic homology (cf. Corollary 3.3.2 below).

Now we apply the corresponding result for complexes of length 1. Note that we could have considered these complexes as objects of $DM_{\text{gm,R}}^{eff}$ instead (see Remark 1.4.3(1)); yet using $K^b(\text{Chow}_{\text{eff}}(R))$ makes our argument somewhat "more elementary".

9. $\text{Chow}_{R}^{eff}$-complexes of length 1 also yield an simple counterexample to the natural analogue of Theorem 3.2.2(I.3) for motives whose Chow-weight homology vanishes in degrees less than $n$ (and so, also to the corresponding analogues of parts I.2 and II.3 of the theorem). Let $R = \mathbb{Q}$, $k = \mathbb{C}$ (one can actually take any infinite field here), and consider a smooth projective $P/k$ (say, an elliptic curve) that possesses a 0-cycle $c_0$ that is numerically trivial but not rationally trivial (i.e., it is rationally non-torsion). We also denote by $c_0$ the corresponding morphism $\mathbb{Q} = M_{\text{gm}}^\mathbb{Q}(pt) \to M_{\text{gm}}^\mathbb{Q}(P)$; let $C$ be the cone of $c_0$ (i.e., $C = \ldots \to 0 \to \mathbb{Q} \xrightarrow{c_0} M_{\text{gm}}^\mathbb{Q}(P) \to 0 \ldots$, $M_{\text{gm}}^\mathbb{Q}(P)$ is in degree 0).

Since $c_0$ is rationally non-trivial (as a cycle with $\mathbb{Q}$-coefficients), $\text{Chow}_{R}^{eff}(c_0)$...
is an injection (and so, $Chow_{j,K}^Q(c_0)$ is injective for any $j \geq 0$ and $K/k$). On the other hand, $c_0$ does not split since it is numerically trivial as a cycle. Hence $C$ does not belong to $K^b(Chow_{eff}^c)$ (or to $DM_{gm,Q}^{d,c}$ if we "put it into" $DM_{gm,Q}^{d,c}$). So, the vanishing of Chow-weight homology of $M$ in negative degrees does not imply that the "weights of $M$ are non-negative".

More generally, it seems difficult to bound the weights of $M$ from below by any sort of homology. On the other hand, the Chow-weight cohomology considered in Proposition 3.2.1 below seems to be optimal for studying bounds of this sort.

**Corollary 3.2.4.** Let $h : N \to M$ be a morphism in $\text{Chow}_{eff}^c$, $0 \leq c_1 \leq c_2 \in \mathbb{Z}$. Then the following statements are equivalent.

1. $Chow_{j,K}^c(h)$ is a bijection for $j \in [0, c_1 - 1]$ and is a surjection for $j \in [c_1, c_2 - 1]$ for all function fields $K/k$.

2. The complex $N \to M$ is homotopy equivalent (i.e., $K^b(Chow_{eff}^c)$-isomorphic) to a complex $N'(c_1) \to M'(c_2)$ for some $N', M' \in \text{Obj} \text{Chow}_{eff}^c$.

3. There exists $h' \in Chow_{eff}^c(M, N)$ such that the morphism $id_M - h' \circ h$ factors through $Chow_{eff}^c(c_2)$, and $id_N - h' \circ h$ factors through $Chow_{eff}^c(c_1)$.

**Proof.** (1) $\iff$ (2). We take $C = \text{Cone} h \in \text{Obj} \text{Cone} K^b(Chow_{eff}^c)$ (or in $DM_{gm,R}^c$, we put $N$ in degree $-1$ and put $M$ in degree $0$), and consider the index set $I = [-1, +\infty) \times [0, c_1 - 1] \cup [0, +\infty) \times [c_1, c_2 - 1]$ (see §1.1).

We immediately obtain the equivalence of our condition 1 to the vanishing of $\text{CWH}^c_{i,K}(C)$ for $i \in I$. Combining the equivalence of conditions A and D in Theorem 3.2.2(II.3) with Remark 1.3.2(2) (see also Remark 3.2.3(5)), we obtain the result.

(2) $\implies$ (3). We have $t^{c-1}(C) \cong t^{c-1}(N'(c_1)[1])$. Next, this isomorphism certainly yields a similar isomorphism (of weight complexes) in $K^b(\text{Cone} \text{Cone} DM_{gm,R}^{d-1})$.

So, $C$ (considered as a $\text{Cone} \text{Cone} \text{Cone} DM_{gm,R}^{d-1}$-complex) is homotopy equivalent to $N'(c_1)[1]$; denote the corresponding morphisms $C \to N'(c_1)[1] \to C$ by $f$ and $g$, respectively. Since $id_C$ is $\text{Cone} \text{Cone} \text{Cone} DM_{gm,R}^{d-1}$-homotopic to $g \circ f$, there exists $h'' \in \text{Cone} \text{Cone} \text{Cone} DM_{gm,R}^{d-1}(M, N)$ such that $id_N - g \circ f = h'' \circ h$ and $h \circ h'' = id_M$.

Lifting $h''$ to $h' \in \text{Cone} \text{Cone} \text{Cone} DM_{eff}^c(M, N)$ (see Proposition 1.3.1(3)), we obtain the desired implication.

(3) $\implies$ (1). Arguing as above, we see that in $K^b(\text{Cone} \text{Cone} \text{Cone} DM_{gm,R}^{d-1})$ the morphism $id_C$ factors through an object of $\text{Cone} \text{Cone} \text{Cone} DM_{eff}^c(c_1)[1]$. The desired Chow-weight homology vanishing conditions follow immediately (cf. the proof of Theorem 3.2.2(II.2)).
Remark 3.2.5. 1. If $M = M_{gm}^R(P)$ and $N = M_{gm}^R(Q)$ for some $P, Q \in SmPrVar$, then condition 3 of the Corollary can be easily translated into the following assumption: $id_M - h \circ h'$ is rationally equivalent to a cycle supported on $P' \times P$, and $id_N - h' \circ h$ is rationally equivalent to a cycle supported on $Q' \times Q$, where $P' \subset P$ and $Q' \subset Q$ are some closed subvarieties of codimensions $c_2$ and $c_1$, respectively (see Proposition 2.2.5(1,2) and its proof).

2. Assume that $C \in d_{\leq m}K^b(\mathcal{Chow}^{\text{eff}}_R)$ (for some $m \geq 0$; this is certainly the case if $N$ and $M$ are of dimension at most $m$). Then $\text{CWH}_i^j(M) = \{0\}$ for $j$ greater than $m$ (and all $i \in \mathbb{Z}$). Thus if $c_2$ is greater than $m$ then our result yields that $h$ splits; if $c_1 > m$ then $h$ is an isomorphism. The first of these observations generalizes Theorem 3.18 of [Via11] (where the case $R = \mathbb{Q}$ was considered).

3.3 Criteria in terms of higher Chow-weight homology; an application to "the usual" motivic homology

Now we invoke Proposition 2.3.1.

**Proposition 3.3.1.** Let $I \subset \mathbb{Z} \times [0, +\infty)$ and $M \in \text{Obj } DM^{\text{eff}}_{gm,R}$ be fixed.

Consider the following conditions on $M$.

1. For some function $f_M : I \to [0, +\infty)$ we have $\text{CWH}_{i,K}^j(M) = \{0\}$ for all $(i, j) \in I$ and all function fields $K/k$.

2. $\text{CWH}_{i,K}^j(M) = \{0\}$ for all $(i, j) \in I$ and all function fields $K/k$.

3. For all rational extensions $K/k$ and $(i, j) \in I$ we have $\text{CWH}_{i,K}^j(M) = \{0\}$.

4. $\text{CWH}_{i,K}^0(M) = \{0\}$ for all $(i, j) \in I$ and all function fields $K/k$.

5. $\text{CWH}_{i,K}^a(M) = \{0\}$ for all $(i, j) \in I$, $a \in \mathbb{Z}$, and all field extensions $K/k$.

Then the following statements are valid.

1. Condition 2 implies conditions 3 and 4, either of the latter two conditions implies condition 1, whereas the first two conditions are equivalent.

2. Let $I$ be reasonable in the sense of Definition 3.2.1. Then our conditions 1-5 are equivalent.

3. Let $R = \mathbb{Q}$. Then our conditions are also equivalent to the vanishing of $\text{CWH}_{i,K}^j(M)$ for $K$ being a fixed universal domain containing $k$ and all $(i, j) \in I$.  

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Proof. 1. Certainly, condition 5 is the strongest of the five, whereas condition 1 follows from condition 2 and 4. The remaining implications are given by Proposition 2.3.4 (see also Remark 2.3.5(1)).

2. Since the first two conditions are equivalent, it suffices to verify that condition 2 implies condition 5.

By Theorem 3.2.2(II.3), M satisfies condition D of the theorem cited. Hence Proposition 3.1.2(4) yields the implication in question (cf. the proof of Theorem 3.2.2(II.3), D \( \implies \) A).

3. Similarly to the setting of Theorem 3.2.2(II.2), it suffices to combine assertion 2 with Proposition 2.3.3.

Now we describe an interesting particular case of the proposition.

**Corollary 3.3.2.** Let \( M \in \text{Obj } DM_{\text{gm},R}^{\text{eff}} \). Then the following conditions are equivalent.

1. \( M \in DM_{R}^{\text{eff},t_{\text{hom}} \leq 0} \).
2. \( \text{Chow}_{0,l}^{R,K}(M) = \{0\} \) for all \( l < 0 \) and all function fields \( K/k \).
3. Conditions 7,9 of the previous proposition for \( I = \{(i,j) : i > j \geq 0\} \) are fulfilled (note that it suffices to verify only one of these conditions).
4. \( M \) belongs to the extension-closure \( E \) of \( \bigcup_{a,b \geq 0} \text{Obj } \text{Chow}^{\text{eff}}_{R}(a)[a+b] \) (in \( \text{Obj } DM_{\text{gm},R}^{\text{eff}} \)).

Proof. The first assumption is equivalent to the second one by Proposition 2.3.3(3). They also imply the third assumption (i.e., all of the equivalent conditions from Proposition 3.3.1 by Proposition 3.1.2(6). Next, condition 2 from Proposition 3.3.1 yields our assumption 4 by Theorem 3.2.2(II.3) (the natural generalization of assertion 4 to the case of an arbitrary reasonable \( I \) was essentially established in the course of proving the implication \( C \implies D \) in the theorem cited).

Finally, our assumption 4 implies assumption 1 since \( \text{Obj } \text{Chow}^{\text{eff}}_{R}(a)[a+b] \subset DM_{R}^{\text{eff},t_{\text{hom}} \leq 0} \) for any \( a,b \geq 0 \) (see the end of §2.1).

\[\square\]

**Remark 3.3.3.** 1. Now consider the (Chow-) weight spectral sequences \( T(K) \) converging to the (zero-dimensional) motivic homology of \( M \) over \( K \): \( E_{1}^{p,q}(T(K)) = \text{Chow}^{R,K}_{0,-p}(M^{q}) \implies \text{Chow}^{R,K}_{0,-p-q}(M) \) (where \( t_{R}(M) = (M^{q}) \)). We certainly have \( E_{2}^{p,q}(T(K)) = \text{CWH}^{0,-p}_{q,K}(M) \). So (for any reasonable \( I \)) the equivalent conditions of Theorem 3.2.2(II.3) can be reformulated in terms of the vanishing of the corresponding \( E_{2} \)-terms of \( T(K) \) (for \( K \) running through function fields over \( k \)). In particular (by Corollary 3.3.2) the higher motivic homology of \( M \) (over any extension of \( k \)) vanishes whenever all the corresponding
$E_2^r(T(K))$ do. This is quite non-trivial since $T(K)$ do not have to degenerate at $E_2$!

Hence one may say that the usual motivic homology groups are "crude mixes" of the Chow-weight ones (via Chow-weight spectral sequences). Indeed, in contrast to the latter groups the motivic homology ones do not "detect" the $c$-effectivity and "weights" of motives (i.e., their vanishing in higher degrees does not yield any information of this sort).

2. Besides, $T(K)$ yield an alternative way of proving that condition 3 of our corollary implies condition 1.

3. For an (effective) Chow motif $N$ and $c \geq 0$ our corollary easily yields the following equivalence: $N \in DM_{\text{eff}}^{\text{et}, R_{\text{hom}}} \leq -c$ if and only if $N$ is $c$-effective. For $R = \mathbb{Q}$ one can also prove this statement by combining Proposition 2.3.4 with Lemma 3.9 of [Via11].

4 Supplements and applications

In this section we deduce some more implications from Theorem 3.2.2.

In §4.1 we dualize (some parts of) this theorem using the results of §2.2: this allows to bound the dimensions of motives and also their weights (from above) via calculating their Chow-weight cohomology. Hence in order to verify the vanishing of Chow-weight homology of $M$ (in higher degrees) over arbitrary extensions of $k$ it suffices to compute these groups over (rational) extensions of $k$ of bounded transcendence degrees (only).

In §4.2 we establish certain integral criteria for the rational vanishing of Chow-weight homology of a motif $M$. The purpose is to prove that if the Chow-weight homology (or motivic homology) groups of $M$ are torsion "in higher degrees", then their exponent is (uniformly) bounded.

In §4.3 we recall a result of [Bon09] and prove the following: a pair of (more or less) "standard" motivic conjectures imply the equivalence of the vanishing (resp. of the "$c$-effectivity") of certain levels of the weight filtration on the singular homology of $M$ to the corresponding vanishing conditions for its Chow-weight homology (and so, to the corresponding motivic "weight-effectivity" restrictions for it). We also discuss the conservativity conjecture.

In §4.4 we propose (briefly) a "sheaf-theoretic" approach to our results, and discuss their possible extensions to motives over a base and to certain "cobordism motives".

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4.1 Chow-weight cohomology and the dimension of motives

Now we dualise (parts I.1 and I.3 of) Theorem 3.2.2 together with some other properties of Chow-weight homology.

To this end we note that Proposition 2.2.1(1) yields the following: the Poincare duality for $DM_{gm}^R$ "respects" $w_{Chow}$, i.e., the dual of $DM_{gm,w_{Chow} \leq 0}^R$ is $DM_{gm,w_{Chow} \geq 0}^R$ (and also vice versa). Moreover, the categorical duality (cf. Proposition 1.2.3) essentially respects weight complexes (at least, for motives; this is explained in detail in Remark 1.5.9(1) of [Bon10a]). Thus one easily obtains the following results.

Proposition 4.1.1. For $M \in \text{Obj} \ DM_{gm}^R$, $j, l, i \in \mathbb{Z}$, $(M^*)$ being a choice of a weight complex for $M$, and a field extension $K/k$ we define $CWC_{i,K}^j(M)$ (resp. $CWC_{i,K}^j(M)$) as the $i$th homology of the complex $DM_{gm,K^{perf}}^R(M^*, R(j)[2j-l])$ (resp. of $DM_{gm,K^{perf}}^R(M^*, R(j)[2j])$).

The following statements are valid.

I The following statements are equivalent.

1. $CWC_{i,K}^j(M)$ yields a cohomological functor on $DM_{gm}^R$.

2. $CWC_{i,K}^j(M)$ vanishes on $d \leq n \ DM_{gm,K}^{eff}$ if $j - i > n$.

II. Assume that $M \in \text{Obj} \ d \leq n \ DM_{gm,K}^{eff}$ for some $n \geq 0$.

Then the following conditions are equivalent.

1. $M$ is also an object of $d \leq n - s \ DM_{gm,K}^{eff}$ for some $s \in [1, n]$.

2. $CWC_{i,K}^j(M) = \{0\}$ for all $i \in \mathbb{Z}$, $j \in [n - s + 1, n]$, and all function fields $K/k$.

3. $CWC_{i,K}^j(M) = \{0\}$ for all $i \in \mathbb{Z}$, $j \in [n - s + 1, n]$, and all rational extensions $K/k$.

4. $CWC_{i,K}^j(M) = \{0\}$ for all $i \in \mathbb{Z}$, $j \in [n - s + 1, n]$, $r \in \mathbb{Z}$, and all field extensions $K/k$.

III For $M$ as above and an integer $q$ also the following statements are equivalent.

1. $M \in DM_{gm,K}^{eff,w_{Chow} \leq q}$.

2. $CWC_{i,K}^j(M) = \{0\}$ for all $i > q$, $j \in [1, n]$, and all function fields $K/k$. 

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3. $\text{CWC}_{j+1,1}^{K}(M) = \{0\}$ for all $i > q$, $j \in [1, n]$, and all rational extensions $K/k$.

4. $\text{CWC}_{j+1,r}^{K}(M) = \{0\}$ for all $i > q$, $j \in [1, n]$, $r \in \mathbb{Z}$, and all field extensions $K/k$.

IV Now let $R = \mathbb{Q}$. Then it suffices to verify any of the assertions in part II and III of the proposition for $K$ being any (fixed) universal domain containing $k$.

Proof. We recall that the Poincare dual of $d_{\leq n}DM_{\text{gm},R}^{\text{eff}}$ is $d_{\leq n}DM_{\text{gm},R}^{\text{eff}}\langle -n \rangle$, and that the dual to $\text{Obj} d_{\leq n-4}DM_{\text{gm},R}^{\text{eff}}$ can (also) be described as $\text{Obj} d_{\leq n}DM_{\text{gm},R}^{\text{eff}}\langle s - n \rangle \cap \text{Obj} d_{\leq n}DM_{\text{gm},R}^{\text{eff}}\langle -n \rangle$ (see Proposition 2.2.5(3)). Together with the observations made prior to this proposition, this easily reduces our assertions to their duals that were proved in the previous section.

Remark 4.1.2. 1. One can certainly dualise part I.2 of Theorem 3.2.2, Remark 3.2.3(2), and the results of §3.3 in a similar way also.

In particular, it seems to be no problem to state and prove a vast "mixed motivic" generalization of Theorem 3.6 of [GoG12].

2. So, the Chow-weight cohomology theories described yield a mighty tool for computing the dimension of an effective motif. So it makes all the more sense to make the main "arithmetical" observation of this subsection (that seems to be more interesting either if $R \neq \mathbb{Q}$ or if we study motives over essentially finitely generated fields).

Let $M \in d_{\leq n}DM_{\text{gm},R}^{\text{eff}}$ (for some $n \geq 0$). We recall the proof of Theorem 3.2.2(1.2). There we have checked that $\gamma : w_{\leq t}l^{c-1}(M) \rightarrow l^{c-1}(M)$ is zero. By our assumption on $M$, we can assume that $w_{\leq t}l^{c-1}(M)$ is of dimension $\leq d$ (in $DM_{\text{gm},R}^{c-1}$). Hence the corresponding application of Proposition 3.1.2(5) reduces the verification of $g = 0$ to the vanishing of the corresponding $\text{CWH}_{j}^{\ast}_{k(P)}(M)$ for the dimension of $P_j$ being at most $n - j$.

So, we obtain the following statement.

Proposition 4.1.3. Let $M \in \text{Obj} d_{\leq n}DM_{\text{gm},R}^{\text{eff}}$ (for some $n \in \mathbb{Z}$). Then the following statements are valid.

1. To verify various assertions of Theorem 3.2.2(I) (resp. the condition $\mathcal{A}$ in the setting of Proposition 3.3.1(2), resp. the condition 2 of Corollary 3.3.2) it suffices to compute the corresponding $\text{CWH}_{j}^{\ast}_{K}(M)$ (resp. motivic homology groups over $K^{\text{perf}}$) for $K$ running through function fields of dimension $\leq d - j$ (resp. for $K/k$ of dimension $\leq d$) only.
2. In Proposition 3.3.1(2) it suffices to verify condition 3 for rational extensions $K/k$ of transcendence degree $\leq d - j + 1$.

3. For $R = \mathbb{Q}$, in the assertion mentioned in part 1 of this proposition it suffices to take $K$ being the algebraic closure of $k(t_1, \ldots, t_{d-j})$ (resp. of $k(t_1, \ldots, t_d)$) instead.

Remark 4.1.4. 1. Thus, if $M$ does not satisfy the (motivic) equivalence conditions of the statements mentioned in the previous proposition, there necessarily exists a function field $K/k$ of "small dimension" such that (at least) one of the corresponding Chow-weight homology (resp. motivic homology) groups does not vanish over $K$.

2. The question whether this dimension restriction is the best possible one seems to be quite difficult in general (especially if we consider geometric motives only; cf. Remark 4.4.1(2) below). Yet note that in the case $d = 1$, $R = \mathbb{Q}$, and a finite $k$ it is certainly not sufficient to compute Chow-weight homology over algebraic extensions of $k$ only.

4.2 Comparing integral and rational coefficients: bounding torsion of homology

Now, we deduce some consequences from our result by comparing $\mathbb{Z}[1/p]$-motives with $\mathbb{Q}$-ones. The general idea is the following one.

We have proved above several statements of the following sort: certain "weight-motivic" restrictions on $M \in \text{Obj} \, DM_{\text{eff}}^{\text{gm}, R}$ are equivalent to the vanishing of its Chow-weight homology (resp. of "ordinary" motivic homology, resp. of Chow-weight cohomology) in a certain range (and over arbitrary extensions of $k$). Now, let $M \in \text{Obj} \, DM_{\text{eff}}^{\text{gm}, \mathbb{Z}[1/p]}$, and assume that one of the "Chow-homological" vanishing conditions of the sort mentioned is fulfilled modulo torsion. This is certainly equivalent to the corresponding "motivic" vanishing restrictions for the image $M \otimes \mathbb{Q}$ of $M$ in $DM_{\text{gm}, \mathbb{Q}}^{\text{eff}}$ (via the corresponding extension of scalars). Next (as we will show) one can reformulate these conditions for $M \otimes \mathbb{Q}$ in terms of $DM_{\text{gm}, \mathbb{Z}[1/p]}^{\text{eff}}$; thus the corresponding motivic restrictions on $M$ are "valid up to a constant" $N_M > 0$. It follows immediately that the corresponding Chow-weight (co)homology groups of $M$ are killed by the multiplication by $N_M$. This is quite a remarkable fact, since a priori nothing prevents the Chow-weight homology (as well as motivic homology) from having really "weird" torsion.

So, assume $R$ is a $\mathbb{Z}[1/p]$-subalgebra of $\mathbb{Q}$ (of special interest are the cases $R = \mathbb{Z}[1/p, 1/n]$ and $R = \mathbb{Q}$). Denote by $\text{Chow}_{\mathbb{Z}[1/p], R}^{\text{eff}}$ (resp. $DM_{\text{gm}, \mathbb{Z}[1/p], R}^{\text{eff}}$) the full subcategory of $\text{Chow}_R^{\text{eff}}$ (resp. of $DM_{\text{gm}, R}^{\text{eff}}$) that comes from $\text{Chow}_{\mathbb{Z}[1/p]}^{\text{eff}}$.

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Lemma 4.2.1. Any object of $DM_{\text{eff}}^{\text{gm}, R}$ is a retract of some object of $DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}], R^\lambda}$.

Proof. By the main result of [BaS01], the Karoubization of $DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}], R}$ is triangulated; note that it is equivalent to the Karoubi-closure of $DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}], R}$ in $DM_{\text{eff}}^{\text{gm}, R}$. Since the Karoubi-closure mentioned contains $Chow_{\text{eff}}^{\text{gm}, R}$, the result follows from Proposition 2.2.1(1) (note the Chow weight structure for $DM_{\text{eff}}^{\text{gm}, R}$ is bounded!).

Remark 4.2.2. Since the methods used in this paper do not (currently) yield any "$p$-torsion information" on motives (if $p > 0$), multiplying $n$ by a power of $p$ does not affect the results of this section. So, till the end of this subsection we will assume that $p \nmid n$ (if $p > 0$; note also that we invert $p$ in $DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}]}$ anyway) just in order to write $\mathbb{Z}[1/n]$ instead of $\mathbb{Z}[1/p, 1/n]$.

Besides, we will set $R = \mathbb{Z}[\frac{1}{p}]$ when we compute Chow-weight homology.

Below we will also need the following technical lemma.

Lemma 4.2.3. Let $i, j, t \in \mathbb{Z}$; assume that $t \geq i$ and that for some $M \in DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{p}]的一切}M \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$ (resp. $CWH_{\nu}^j (M) \otimes \mathbb{Z}[1/n] = \{0\}$) for all $(i', j') \in [i, +\infty) \times [0, j - 1]$. Then for any choice of a shifted weight decomposition $w_{\leq -t} M \to w_{\leq -i+1} M \to w_{\leq -i} M[1]$ there exists a non-zero integer (resp., a power of $n$) $\lambda$ such that $\lambda v$ factors through an element of $DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{p}][-t-i, \nu]}(j)$.

Proof. Firstly, note that the image of the morphism $v$ in $DM_{\text{eff}}^{\text{gm}, \mathbb{Q}}(j)$ (resp. in $DM_{\text{eff}}^{\text{gm}, Z[1/n]}(j)$) is zero. Indeed, the Chow-weight homology groups of $M \otimes \mathbb{Q}$ (resp. of $M \otimes Z[1/n]$) are zero since they are equal to rational (resp. $Z[1/n]$-) hulls of certain torsion (resp. of $n$-torsion) groups. Thus the latter statement follows from Theorem 3.2.2(1.2–1.3) combined with the orthogonality axiom. Therefore (by the theorem cited), there exists an object $X'$ of $DM_{\text{eff}}^{\text{gm}, \mathbb{Q}}(j)$ (resp. of $DM_{\text{eff}}^{\text{gm}, Z[1/n]}(j)$) such that $v$ factors through it. By Lemma 4.2.1 we may take $X' = X \otimes \mathbb{Q}$ (resp. $X' = X \otimes Z[\frac{1}{n}]$) for some $X \in \text{Obj}(DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}])(j)$).

Since the functor $- \otimes \mathbb{Q} : DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}] \to DM_{\text{eff}}^{\text{gm}, Z[\frac{1}{n}] \to DM_{\text{eff}}^{\text{gm}, Z[1/n]}$ tensors morphism groups by $\mathbb{Q}$ (resp. by $Z[\frac{1}{n}]$), there exist
\( q \in DM_{gm, Z}^{eff} \left( w_{< -i} M, X \right) \) and \( q' \in DM_{gm, Z}^{eff} \left( X, M \right) \) such that \( q' \circ q = \lambda v \), where \( \lambda \) is a non-zero integer (resp. a power of \( n \)). Finally, by Proposition \[2.3.9\] we can assume that \( X \in DM_{gm, Z}^{eff} \left[ -t, -i \right] \).

Now let us prove the main result of this section.

**Theorem 4.2.4.** Let \( M \in \text{Obj} \ DM_{gm, Z}^{eff} \left[ 1/p \right]^I, I \subset \mathbb{Z} \times [0, +\infty) \).

1. The following conditions are equivalent.
   a. The group \( CWH^i_{i,K}(M) \) is torsion for any function field \( K/k \) and \( (i, j) \in I \).
   b. \( CWH^i_{i,K}(M) \) is torsion for any \( (i, j) \in I \) and \( K \) being some fixed universal domain containing \( K \).

   II Suppose also that \( I \) is reasonable (in the sense of Definition \[3.2.1\]) and \( n \) is a non-zero integer (that we assume to be divisible by \( p \) if \( p > 0 \)).

   Then the following conditions are equivalent.
   A. \( CHW^i_{i,K}(M) \otimes \mathbb{Q} = \{0\} \) (resp. \( CHW^i_{i,K}(M) \otimes \mathbb{Z}[1/n] = \{0\} \)) for all function fields \( K/k \) and \( (i, j) \in I \).
   B. For any \( i \in \mathbb{Z} \) there exists a distinguished triangle \( T_i \rightarrow M \rightarrow N_i \) such that \( N_i \) is an extension of an element of \( DM_{gm, Z}^{eff} \left[ 1/p \right]_{w_{\text{Chow}} \leq -i, j} \) by an element of \( DM_{gm, Z}^{eff} \left[ 1/p \right]_{w_{\text{Chow}} \leq -i} \) and \( T_i \) is a torsion motif (resp. is an \( n \)-torsion motif; see Definition \[1.1.1\]).
   C. There exists a distinguished triangle \( T \rightarrow M \rightarrow N \rightarrow T[1] \) such that \( CHW^i_{i,K}(N) = \{0\} \) for all \( (i, j) \in I \) and \( T \) is a torsion motif (resp. an \( n \)-torsion motif).
   D. \( N_M \cdot CHW^j_{i,K}(M) = \{0\} \), where \( N_M \) is a fixed non-zero integer (resp. a fixed power of \( n \)) for all function fields \( K/k \) and \( (i, j) \in I \).

**Proof.** I The group \( CWH^i_{i,K}(M) \) is torsion if and only if its rational hull is zero. So the result follows from Theorem \[3.2.2\] (II).

II First we prove A \( \Rightarrow \) B. To this end it suffices to verify the following claim for some fixed \( i \in \mathbb{Z} \): there exists a triangle of the form desired for any \( M \) such that \( CHW^j_{i', K}(M) \otimes \mathbb{Q} = \{0\} \) (resp. \( CHW^j_{i', K}(M) \otimes \mathbb{Z}[1/n] = \{0\} \)) for all \( (i', j') \in [i, +\infty) \times [0, a_i - 1] \).

Let \( w_{< -i} M \rightarrow M \rightarrow w_{< -i+1} M \) be a shifted weight decomposition of \( M \).

We start with constructing a triangle \( L \rightarrow M \rightarrow M' \rightarrow L[1] \) that satisfies the following properties:
1. \( M' \) is an extension of an element of \( DM_{gm, Z}^{eff} \left[ 1/p \right]_{w_{\text{Chow}} \geq -i+1} \) by a torsion or an \( n \)-torsion motif, respectively;
2. \( L \in DM_{gm, Z}^{eff} \left[ 1/p \right]_{w_{\text{Chow}} \leq -i(a_i)} \).
Since \( w_{\text{Chow}} \) is bounded, there exists an integer \( t \geq i \) such that \( M \) belongs to \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -t \). We prove the existence of a triangle of the form described by induction on \( t \).

If \( t = i \), by Lemma 1.2.3 there exist an \( X \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} = -i(\mathcal{q}) \) and morphisms \( \mathcal{q} \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} (w_{\leq-i}M, X) \), \( \mathcal{q}' \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} (X, M) \) such that \( \lambda \mathcal{v} = \mathcal{q}' \circ \mathcal{q} \), where \( \lambda \) is an integer (resp. a power of \( p \)). We certainly have \( \text{Cone}(\mathcal{q}) \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} [-i,-i+1] \). The octahedron axiom applied to the commutative triangle \((\text{Id}_{w_{\leq-i}M}, v, \lambda \mathcal{v})\) yields the existence of a distinguished triangle \( w_{\leq-i}M \to M \to E \to w_{\leq-i}M[1] \) such that \( E \) is an extension of an element \( E_{\text{pos}} = w_{\geq-i+1}M \) of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -i+1 \) by a torsion (resp. by an \( n \)-torsion) motif \( E_{\tau} \). Denote the corresponding morphism from \( E_{\tau} \) to \( E \) by \( h_1 \). The octahedron axiom applied to the commutative triangle \((\mathcal{q}, \mathcal{q}', \lambda \mathcal{v})\) yields that \( \text{Cone}(\mathcal{q}') \) is an extension of \( \text{Cone}(\mathcal{q})[1] \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -i+1 \) by \( E \). Lemma 1.1.3 applied to \( \text{Cone}(\mathcal{q}') \) certainly yields that \( \text{Cone}(\mathcal{q}') \) is an extension of an element of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -i+1 \) by a torsion (resp. by an \( n \)-torsion) motif. Hence, the triangle \( X \to M \to \text{Cone}(\mathcal{q}') \to X[1] \) yields the result for the base case of our induction.

Now we describe the inductive step. Obviously, we can apply the inductive assumption to \( M[t-i] \); hence there is a triangle of the form \( L_1 \to M \to M_1 \to L_1[1] \), where \( L_1 \in DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} = -t(\mathcal{a}_i) \) and \( M_1 \) is an extension of an element of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -t+1 \) by a torsion (resp. by an \( n \)-torsion) motif. By Lemma 1.1.2 \( M_1 \) is also an extension of a torsion (resp. by an \( n \)-torsion) motif \( T_1 \) by an element \( M_2 \) of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -t+1 \). Since \( \text{CWH}^{j'}_{i',K}(L_1) = \{0\} \), the corresponding long exact sequence for \( \text{CWH}^{j'}_{i',K} \otimes \mathbb{Q} \) (resp. for \( \text{CWH}^{j'}_{i',K} \otimes \mathbb{Z}[1/n] \)) yields that \( \text{CWH}^{j'}_{i',K}(M_1) \otimes \mathbb{Q} = \{0\} \) (resp. \( \text{CWH}^{j'}_{i',K}(M_1) \otimes \mathbb{Z}[1/n] = \{0\} \)) for \((i', j') \in [i, +\infty) \times [0, a_i - 1] \). Therefore, \( \text{CWH}^{j'}_{i',K}(M_2) \otimes \mathbb{Q} = \text{CWH}^{j'}_{i',K}(M_1) \otimes \mathbb{Q} = \{0\} \) (resp. \( \text{CWH}^{j'}_{i',K}(M_2) \otimes \mathbb{Z}[1/n] = \text{CWH}^{j'}_{i',K}(M_1) \otimes \mathbb{Z}[1/n] = \{0\} \)) for \((i', j') \in [i, +\infty) \times [0, a_i - 1] \) and we can apply the inductive assumption to \( M_2 \). It yields a triangle of the form \( L_2 \to M_2 \to Y \to L_2[1] \) such that \( Y \) is an extension of an element of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -t+1 \) by a torsion (resp. by an \( n \)-torsion) motif. By Lemma 1.1.2 \( Y \) is also an extension of a torsion (resp. an \( n \)-torsion) motif \( T_2 \) by an element \( Y_1 \) of \( DM_{\text{gm},\mathbb{Z}[\frac{1}{p}]}^{\text{eff}} w_{\text{Chow}} \geq -t+1 \). Denote the corresponding morphism from \( M_2 \) to \( M_1 \) by \( r \), the morphism from \( M_1 \) to \( Y_2 \) by \( v \). The octahedron axiom applied to the commutative triangle \((s, r, r \circ s)\) yields a distinguished
triangle $Y \xrightarrow{f} Y_2 \to T_2 \to Y[1]$. Applying the octahedron axiom to the commutative triangle $(u,v,v \circ u)$ we obtain that $\text{Cone}(v \circ u)[-1]$ belongs to $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\leq -i}(a_i)$. By Lemma 1.1.3, $Y_2$ is an extension of a torsion motif (resp. of an $n$-torsion motif) by an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\geq -i+1}$. Hence Lemma 1.1.2 yields that $Y_2$ is also an extension of an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\geq -i+1}$ by some torsion motif (resp. by some $n$-torsion motif).

Thus the triangle $\text{Cone}(v \circ u)[-1] \to M \to Y_2 \to \text{Cone}(v \circ u)$ yields the inductive step.

So we have a distinguished triangle $L \to M \xrightarrow{\alpha} M' \to L[1]$ where $L \in DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\leq -i}(a_i)$, and a distinguished triangle $M'_\text{tor} \to M' \xrightarrow{\beta} M'_\text{pos} \to M'_\text{tor}[1]$, where $M'_\text{pos} \in DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\geq -i+1}$ and $M'_\text{tor}$ is a torsion motif (resp. an $n$-torsion motif).

Applying the octahedron axiom to the commutative triangle $(\alpha, \beta, \beta \circ \alpha)$ we obtain that $P = \text{Cone}(\beta \circ \alpha)[-1]$ is an extension of a torsion motif (resp. of an $n$-torsion motif) by an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\leq -i}(a_i)$. By Lemma 1.1.2 $P$ is also an extension of an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\leq -i}(a_i)$ by a torsion motif (resp. by an $n$-torsion motif) $T$. Denote the corresponding morphism from $P$ to $M$ by $\phi$ and the morphism $T$ to $P$ by $\psi$. Applying the octahedron axiom to the commutative triangle $(\psi, \phi, \phi \circ \psi)$ we obtain a distinguished triangle of the form $T \to M \to N \to T[1]$, where $N$ is an extension of an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\geq -i+1}$ by an element of $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{\leq -i}(a_i)$. So we get condition B.

Now let us verify that $B \Rightarrow C$. Recall that there exist $t \geq s \in \mathbb{Z}$ such that $M \in DM_{\text{eff}}^{\text{tr}} \text{CHW}^{[-t-s]}$. Obviously, we can assume that $s = 0$ (since we can shift $M$ and $I$); so that $M \in DM_{\text{eff}}^{\text{tr}} \text{CHW}^{[-t,0]}$. In this case $\text{CHW}^{t}_{J,K}(M)$ is zero whenever $i \notin [0, t]$. Let $j'$ be the maximum integer such that $\text{CHW}^{j'}_{J,K}(M) \neq \{0\}$ for some $j$ such that $(t', j) \in I$. Let us use induction on $t' \geq 0$. If $t' = 0$ then the triangle $0 \to M^{id} \to M \to 0$ yields the result; this will be the base case of our induction.

Now we describe the inductive step. Suppose that triangles of the form desired exist for all motives in $DM_{\text{eff}}^{\text{tr}} \text{CHW}^{[-t-1,0]}$ that satisfy condition B. Since $\text{CHW}^{j}_{J,K}(N_t) = \{0\}$ for $0 \leq j < a_j$, we can apply the inductive assumption to $N_t$. This yields the existence of a distinguished triangle $T \to N_t \xrightarrow{\beta} N$ such that $T$ is a torsion motif (resp. an $n$-torsion motif) and $\text{CHW}^{j}_{J,K}(N) = 0$ for $(i, j) \in I$. Let $q$ be the corresponding morphism from $M$ to $N_t$. Applying

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the octahedron axiom to the commutative triangle \((q, r, r \circ q)\) we obtain a distinguished triangle \(T' \to M \overset{\text{eq}}{\to} N\), where \(T'\) is a torsion motif (resp. an \(n\)-torsion motif).

To prove \(C \Rightarrow D\) note that there exists an integer (resp. a power of \(n\)) \(N_T \neq 0\) such that \(N_T \cdot id_T = 0\) by the definition of a torsion motif (resp. of an \(n\)-torsion motif). Hence \(N_T \cdot CWH_{i,K}^j(T) = \{0\}\), where \(N_T\) is a non-zero integer (resp. a power of \(n\)) that does not depend on \((i, j)\). The corresponding long exact sequence for Chow-weight homology yields that \(CWH_{i,K}^j(M)\) is a quotient of the group \(CWH_{i,K}^j(T)\) for every \((i, j) \in I\); hence \(N_T \cdot CWH_{i,K}^j(M) = \{0\}\).

\(D \Rightarrow A\) is obvious.

\[\square\]

Now we combine the results proved with the ones of §3.3.

**Corollary 4.2.5.** Let \(M \in \text{Obj} DM^{\text{eff}}_{gm} \).

I. The "main" versions of the (equivalent) conditions A–D of Theorem 4.2.4 (i.e., we ignore the versions in brackets that mention \(n\)) are also equivalent to any of the following assertions (in the notation of the Theorem cited; so, \(I\) is a reasonable subset of \(\mathbb{Z} \times [0, +\infty)\)).

1. For all rational extensions \(K/k\) and \((i, j) \in I\) the groups \(CWH_{i,K}^{j-1,1}(M)\) are torsion.

2. The groups \(CWH_{i,K}^{a, j-a}(M)\) are torsion for all \((i, j) \in I\), \(a \in \mathbb{Z}\), and all field extensions \(K/k\).

3. The groups \(CWH_{i,K}^{i}(M)\) are torsion for \(K\) being a fixed universal domain containing \(k\) and \((i, j) \in I\).

4. There exists an integer \(N_M > 0\) such that \(N_M CWH_{i,K}^{a, j-a}(M) = \{0\}\) for all \((i, j) \in I\), \(a \in \mathbb{Z}\), and all field extensions \(K/k\).

II. The following conditions are equivalent.

1. \(M \otimes \mathbb{Q} \in DM^{\text{eff}}_{\mathbb{Q}} F_{\text{hom}} \leq 0\).

2. \(\text{Chow}_{0,l}^K(M) = \{0\}\) for all \(l < 0\) and all function fields \(K/k\).

3. The groups \(CWH_{i,K}^{a, j-a}(M)\) are torsion for all \(a \in \mathbb{Z}, i > j\), and all field extensions \(K/k\).

4. There exists an integer \(N_M > 0\) such that \(N_M CWH_{i,K}^{a, j-a}(M) = \{0\}\) for all \(a \in \mathbb{Z}, i > j\), and all field extensions \(K/k\).
5. There exists a distinguished triangle \( T \to M \to N \to T[1] \) such that
\[ N \in \text{Obj} \ DM_{gm, \mathbb{Z}[\frac{1}{p}]}^{\mathbb{Z}[\frac{1}{p}], \text{eff}} \bigcap DM_{\mathbb{Z}[\frac{1}{p}], \text{hom}}^{\mathbb{Z}[\frac{1}{p}], \text{eff}} \leq 0 \] and \( T \) is a torsion motif.

6. There exists \( N_M > 0 \) such that \( N_M \text{Chow}_{0,l}^\mathbb{Q,K} \) is \( \{0\} \) for all \( l < 0 \) and all field extensions \( K/k \).

7. There exists a universal domain \( K \) containing \( k \) such that \( \text{Chow}_{0,l}^\mathbb{Q,K} \) is \( \{0\} \) for all \( l < 0 \).

Proof. I. By Proposition 3.3.1 our conditions I.1-3 are equivalent to condition A of Theorem 4.2.4(II). It remains to note that the (method of the) proof of the implication \( D \implies A \) in the proposition cited also easily yields our condition I.4.

II. First we apply Corollary 3.3.2 for \( R = \mathbb{Q} \) (and with \( M \) replaced by \( M \otimes \mathbb{Q} \)). We immediately obtain that our conditions II.1, II.2, II.3, and II.7 are equivalent. Certainly, the latter condition is weaker than condition II.6.

Next, condition II.8 implies condition II.4 according to our assertion I (we take \( I = \{(i, j) : i > j \} \) in it). Besides, Theorem 4.2.4(II) yields the following for any \( M \) that fulfills one of these six conditions: there exists a distinguished triangle \( T \to M \to N \to T[1] \) such that \( T \) is a torsion motif and \( N \) belongs to the class \( E \) mentioned in condition 4 of Corollary 3.3.2 (for \( R = \mathbb{Z} \)). Hence \( N \in \text{Obj} \ DM_{gm, \mathbb{Z}[\frac{1}{p}]}^{\mathbb{Z}[\frac{1}{p}], \text{eff}} \bigcap DM_{\mathbb{Z}[\frac{1}{p}], \text{hom}}^{\mathbb{Z}[\frac{1}{p}], \text{eff}} \leq 0 \), so we obtain that our condition II.5 is the weakest one among the seven conditions of this assertion.

Thus it remains to verify that the latter condition implies condition II.6. We note that \( \text{CWH}_{i,K}^{a-l-a}(N) = \{0\} \) (see Corollary 3.3.2) and that the constant that kills \( T \) certainly kills all \( \text{CWH}_{i,K}^{a-l}(T) \). It remains to apply the long exact sequences that relate the Chow-weight homology of \( M \) with that of \( N \) and \( T \).

4.3 Conditional results: "detection" of c-effectivity and weights by singular homology

As we have already noted (see Remark 3.2.3(2)), for \( K = \mathbb{C} \) the equivalent conditions of Theorem 3.2.2(I.3) (resp. of (I.2)) yield the vanishing of certain weight factors of singular homology (resp. their c-effectivity). Now we explain this in somewhat more detail (for \( R = \mathbb{Q} \)) and prove (via the method used in §7.4 of [Bon10a]) that the converse implication is also valid if we assume that a pair of (more or less) standard motivic conjectures hold.
Proposition 4.3.1. Let $k = \mathbb{C}$, $M \in \text{Obj} \, DM_{gm,Q}^{\text{eff}}$, $s \in \mathbb{Z}$. Denote by $H : DM_{gm,Q}^{\text{eff}} \rightarrow \text{MHS}$ the singular homology functor with values in the category of mixed Hodge structures (with rational coefficients). Then the following statements are valid.

1. Assume that $M \in DM_{gm,Q}^{\text{eff}}_{w \text{Chow} \geq s}$. Then $(W_{s-1}H_i)(M) = 0$ for any $i \in \mathbb{Z}$.

2. Assume that $l^{c-1}(M) \in DM_{gm}^{c-1}_{w \text{Chow} \geq s}$. Then $(W_{s-1}H_i)(M)$ is $c$-effective for any $i \in \mathbb{Z}$ (here we define the $c$-effectivity for effective mixed Hodge structures similarly to Definition 2.2.3).

II Assume that the following conjectures hold.

A. The Hodge conjecture.

B. Any morphism of Chow motives (over $\mathbb{C}$) that induces an isomorphism on their singular homology is an isomorphism.

Then the converse implications are also valid.

Proof. I. As we have already said, both statements follow from Proposition 1.4.6 immediately (if we recall that $c$-effective mixed Hodge substructures form a Serre subcategory of the category of all effective mixed Hodge structures).

II. The converse implication for assertion I.1 (assuming conjectures A and B) is essentially the Poincare dual of Proposition 7.4.2 of [Bon09].

Now we explain that the method applied in loc. cit. yields the converse implication for assertion I.2 also. Indeed, $H$ certainly yields a homological functor $H^{(c)}$ from $DM_{gm}^{c-1}$ to the Serre localization of effective mixed Hodge structures by $c$-effective ones. Next, recall that the "ordinary" Hodge conjecture yields the generalized Hodge one; it follows that the restriction of $H^{(c)}$ to $H_{w \text{Chow}, DM_{gm}^{c-1}}$ satisfies the natural analogue of our Conjecture B. Hence the (dual to) the argument from loc. cit. can also be applied in this setting.

Remark 4.3.2. 1. Recall that the Chow-weight filtration on the singular homology of (the motif of) a complex variety differs from the Deligne’s weight filtration only by a certain shift; see Remark 2.4.3 of [Bon10a].

2. Certainly, our Conjecture B is a particular case of the famous conservativity conjecture (that predicts the following: if $H_*(M) = 0$ for $H_*$ being étale or singular (co)homology and $M \in \text{Obj} \, DM_{gm,Q}^{\text{eff}}$, then $M = 0$).

3. Applying Theorem 3.2.2 we can reformulate the assumptions described in part I of the proposition it terms of Chow-weight homology with rational coefficients (this was our reason to prove this statement here).

Besides, it is absolutely no problem to generalize our statements to a Hodge-theoretic characterization of the vanishing of $\text{CWH}_j^i(M)$ for $(i, j) \in I$,
where $I$ is reasonable (cf. Theorem 3.2.2(II.3)). In particular, this yields a conjectural description of $\text{Obj } \mathcal{D}_\text{eff}^\text{gm} \cap \mathcal{D}_\text{eff}^\text{gm} \cap \mathcal{Q} \cap \mathcal{D}_\text{eff}^\text{gm} - \mathcal{Q}$ in terms of singular homology.

We also hope to use the results of the current paper for attacking the conservativity conjecture. Note that (by Proposition 2.2.3(3)) it suffices to verify the following conjecture: if $H_*(M) = 0$ then $M$ is 1-effective.

We also make the following observation: one can construct "quite interesting" motives (with rational coefficients) with vanishing étale (and singular) (co)homology. We consider $A$ being a smooth (connected) affine variety of dimension $n > 0$; let $A'$ be its general hyperplane section (for a fixed embedding of $A$ into a projective space). Then the only non-zero étale homology group of $N = \text{Cone}(\mathcal{M}^\text{gm}(A') \to \mathcal{M}^\text{gm}(A))$ is in (the cohomological) degree $-n$ (see Lemma 3.3 of [Bei87]). Hence the $\mathbb{Q}_l$-étale homology (or singular homology with rational coefficients) of a high enough exterior power of $N[-n]$ is zero.

4.4 Some future plans: Chow sheaves, relative motives, and cycle modules

Let us indicate some possible modifications for Theorem 3.2.2: possibly they will be studied in consequent papers.

Remark 4.4.1. 1. In the current paper we treat Chow-weight homology (of a fixed $M \in \text{Obj } \mathcal{D}_\text{eff}^\text{gm}$) as functors that associate to field extensions of $k$ certain $R$-modules. Yet one can apply a "more structured" approach instead.

For any $U \in SmVar$ and $t_R(M) = (M^*_j)$, $j, l \in \mathbb{Z}$, one can consider the homology of the complex $\mathcal{D}_\text{eff}^\text{gm}(M^*_j(U)(j)[2j + l], M^*)$. Next the functors obtained can be sheafified with respect to $U$; this yields a sequence of certain Chow-weight homology sheaves (for any $(j, l)$). Moreover, if $j \geq 0$ then the sheafifications of $U \mapsto (M^*_j(U)(j)[2j], M^*)$ (that were called the Chow sheaves of $M^*$ in [Kal10]) are birational (in $U$, i.e., they convert open dense embeddings of smooth varieties into isomorphisms; see Remark 2.3 of [HuK06]). Hence the corresponding Chow-weight homology sheaves are birational also.

Possibly this sheaf-theoretic approach will be applied in a subsequent paper. In particular, it should yield a alternative proof of a (somewhat weaker) version of Proposition 2.3.4 that is sufficient for our purposes.

Moreover, the authors hope to extend these observations to the case of motives (with rational coefficients) over any "reasonable" base scheme.
S; one should study the corresponding perverse homotopy invariant Chow sheaves for $S$-motives (recall that those are conjecturally Rost’s cycle modules over $S$) and apply the results of [Bon14].

2. In the current paper (most of the time) we consider geometric motives only (this allows us to restrict ourselves to bounded weight structures). Yet it seems to be no problem to extend Proposition 3.1.2 to the case where $N$ is an object of $DM^{eff}_R$ or of $DM^{eff}_R$ (resp. of $DM^{eff}_R/DM^{eff}_R(r + 1)$ for an integer $r \geq 0$; see the proposition cited and its proof). This should yield a certain generalization of Theorem 3.2.2(I and II.3) to the setting of arbitrary "$w_{Chow}$-bounded below" effective motivic complexes.

This more general setting may shed some light on the computation of slices for motives (with respect to the effectivity filtration on $DM^{eff}_R$; note that slices are "usually non-geometric"). The authors are deeply grateful to prof. M. Levine for the suggestion to study this problem (and for mentioning Theorem 7.4.2 of [KaL10] as an interesting example of the calculation of slices). Yet the reader should be warned that our results do not yield the "naive" generalization of loc. cit. to arbitrary smooth projective varieties. The problem is that the (defined above) Chow sheaves of an arbitrary smooth projective $P$ may have non-zero Chow-weight homology (of dimension $j = 0$) in negative degrees (if we consider them as objects of $DM^{eff}_R$); so that they do not necessarily yield the slices of $M^{gm}_R(P)$.

3. Our arguments are rather formal and mostly rely on the existence of compatible Chow weight structures for various motivic categories. So the authors also hope to extend them to certain effective geometric cobordism motives (i.e., to the corresponding subcategory of the triangulated category of $MGL$-module spectra) at least if $p = 0$; cf. §6.3 of [Bon13].

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