Ultra slow-roll inflation demystified
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Abstract

Ultra-slow-roll (USR) inflation is a new mode of inflation which corresponds to the occasions when the inflaton field must traverse an extremely flat part of the scalar potential, when the usual slow-roll (SR) fails. We investigate USR and obtain an estimate for how long it lasts, given the initial kinetic density of the inflaton. We also find that, if the initial kinetic density is small enough, USR can be avoided and the usual SR treatment is valid. This has important implications for inflection-point inflation.

1 Introduction

Cosmic inflation is an organic component of the concordance model of cosmology. It is a period of exponential expansion in the early Universe, which determines the initial conditions for the subsequent Hot Big Bang cosmology. In particular, it makes the Universe spatially flat, large and uniform but also provides the necessary deviations from perfect uniformity in the form of the primordial curvature perturbation, which accounts for the eventual formation of the large scale structure. Typically, inflation is modelled through the inflationary paradigm, which suggests that the Universe undergoes inflation when dominated by the potential density of a scalar field (inflaton). This potential density remains roughly constant during inflation. As a result, the generated curvature perturbation is almost scale-invariant, as suggested by observations. In order to keep the potential density roughly constant, the variation of the field must be very small throughout inflation. Because the inflaton’s equation of motion is the same as a body rolling down a potential slope subject to friction, we need this roll to be slow for the inflaton, in field space, so as to keep the potential density roughly unchanged. Thus, in the inflationary paradigm, the inflaton undergoes slow-roll (SR) during inflation. Indeed, the latest CMB data favours single-field slow-roll inflation [1].

The SR solution is an attractor [2] as long as the potential is flat enough to support it. However, it was recently realised that SR may end not only when the potential becomes steep and curved, as is for the end of inflation, but also when it suddenly becomes extremely flat, too flat for the regular SR assumptions to apply. In this case, the system engages in so-called ultra slow-roll (USR) inflation. This new mode of inflation has been hitherto unknown. It can have a profound impact on inflationary observables, so it must be taken into account. However, even though diagnosed, USR has not been fully understood, with
most of its dynamics traced numerically. In this letter, we attempt to demystify USR and provide a conceptual understanding of its dynamics. Ignoring USR can lead to important miscalculations of inflationary observables.

USR arises when the potential becomes extremely flat, so much so that, SR would force the kinetic density of the field to reduce faster than it would if the field were in free-fall

\[ \rho_{\text{kin}} \equiv \frac{1}{2} \dot{\phi}^2 \propto a^{-6} , \]

which of course cannot happen. Thus, the system departs from SR and the field engages in USR, during which the kinetic density decreases as in free-fall, until the system can get back to SR, when the decreasing \(|\ddot{\phi}|\) catches up with the slope of the potential \(|V'|\), or until inflation ends, e.g. by a phase transition. Note that, even though the slope is very small, we still have potential domination \(V > \rho_{\text{kin}}\) so inflation continues. USR was first investigated in Ref. [3], which was followed by Refs. [4, 5] and recently by Ref. [6]. In Refs. [3] and [5] a constant potential is assumed, which cannot exhibit SR. In Ref. [4] it was shown that USR is not an attractor solution and the system departs from it as soon as the conditions which enforce USR allow it. But which conditions are these?

In this letter we explore this question. To obtain an insight of the dynamics of USR, we study USR in linear inflation and then generalise our findings for an arbitrary inflation model. We particularly consider inflection-point inflation because it can lead to USR. It is fair to say that the community seems little aware of USR, so the hope is that our treatment may be revealing of USR’s nature. This is a particularly acute problem in models of inflection-point inflation, where a region of USR exists around the inflection point. In USR this region is traversed in a moderate number of e-folds. However, were SR assumed, this number would grow substantially. As inflationary observables are determined by the correct number of e-folds, this can have profound implications on inflationary predictions and on the viability of inflection-point models.

We use natural units, where \(c = \hbar = 1\) and \(8\pi G = m_P^{-2}\), with \(m_P = 2.43 \times 10^{18}\) GeV being the reduced Planck mass.

## 2 Ultra-slow roll inflation

To explore USR inflation, we will look closely at the Klein-Gordon equation of motion of the canonical homogeneous inflaton field \(\phi\):

\[ \ddot{\phi} + 3H\dot{\phi} + V' = 0 , \quad (1) \]

where \(H \equiv \dot{a}/a\) is the Hubble parameter (with \(a\) being the scale factor) \(V\) is the scalar potential and the dot \{(prime\} denotes derivative with respect to the cosmic time \{(the inflaton field\}. We name each term of the above as the acceleration, the friction and the slope term respectively. We also employ the flat Friedman equation during inflation, when the Universe is dominated by the inflaton field:

\[ 3H^2m_P^2 = \frac{1}{2} \dot{\phi}^2 + V . \quad (2) \]

We define two slow-roll parameters

\[ \epsilon \equiv -\dot{H}/H^2 \quad (3) \]
and
\[ \epsilon_2 \equiv \frac{\dot{\epsilon}}{\epsilon H} = -6 - \frac{2V'}{H\dot{\phi}} + 2\epsilon = \frac{2\ddot{\phi}}{H\dot{\phi}} + 2\epsilon, \quad (4) \]
where we have employed Eqs. (1) and (2). It is easy to show that
\[ \epsilon = \frac{3}{2}(1 + w), \quad (5) \]
where \( w \) is the barotropic parameter of the homogeneous inflaton field, given by
\[ w = \frac{\rho_{\text{kin}} - V}{\rho_{\text{kin}} + V}, \quad (6) \]
where \( \rho_{\text{kin}} \equiv \frac{1}{2}\dot{\phi}^2 \). For inflation we need \( w < -\frac{1}{3} \), which means \( V > 2\rho_{\text{kin}} \). From Eq. (5), we see that inflation (accelerated expansion) occurs when \( \epsilon < 1 \).

Now, in the usual SR, the acceleration term in Eq. (1) is negligible, so the latter becomes
\[ 3H\dot{\phi} \simeq -V, \quad (7) \]
which shows that the friction term is locked to the slope term. In this case, Eq. (4) becomes
\[ \epsilon_2 = -2\eta + 4\epsilon, \quad (8) \]
where the usual SR parameters are
\[ \epsilon \simeq \epsilon_{\text{SR}} \equiv \frac{1}{2}m_p^2 \left( \frac{V'}{V} \right)^2 \quad \text{and} \quad \eta \equiv m_p^2 \frac{V''}{V}. \quad (9) \]
During SR, \( \epsilon, |\eta| \ll 1 \), which means that \( |\epsilon_2| \ll 1 \).

However, if the potential suddenly becomes extremely flat, the slope term in Eq. (1) may reduce drastically, which means that it virtually disappears. The equation is then rendered
\[ \ddot{\phi} + 3H\dot{\phi} \simeq 0, \quad (10) \]
which shows that the friction term is now locked with the acceleration term. In this case, Eq. (4) becomes
\[ \epsilon_2 = -6 + 2\epsilon, \quad (11) \]
During inflation \( \epsilon < 1 \), which means \( |\epsilon_2| \approx 6 \). Thus, if during inflation, the potential becomes suddenly very flat, \( |\epsilon_2| \), which is initially small grows to larger than unity, SR is applicable no-more and a period of USR begins.

Intuitively, one can understand this as follows. If we are in SR but the slope \( |V'| \) reduces drastically, it initially drags with it the friction term, by virtue of Eq. (7). This decreases the value of \( |\dot{\phi}| \), i.e. the kinetic density \( \rho_{\text{kin}} = \frac{1}{2}\dot{\phi}^2 \), but this value cannot decrease arbitrarily quickly. The fastest it can decrease is \( \rho_{\text{kin}} \propto a^{-6} \), which we call free-fall because it corresponds to a field with no potential density \( V = 0 \), such that its equation of motion
is Eq. (10). Therefore, if the kinetic density of SR is forced (by the decreasing slope) to reduce faster than free-fall then the system breaks away from SR. In SR the acceleration term is negligible, because it is very small, compared to the friction and slope terms, which are locked together as shown in Eq. (7). However, if the slope reduces drastically and drags the friction term with it, they both become small too and eventually comparable to the acceleration term. So all three terms in Eq. (1) are comparable. When this happens, the friction term changes allegiances and becomes locked with the acceleration term, resulting in USR.

Now, once in USR, the field becomes oblivious of the potential, as demonstrated by Eq. (10). This is similar to the kination period of quintessential inflation models \[7, 8\] but there is a crucial difference. In kination, the Universe is dominated by \(\rho_{\text{kin}}\), while in USR inflation, we still have potential domination and \(V > \rho_{\text{kin}}\). Being oblivious to the potential, the inflaton field can even climb up an ultra-shallow \(V\) \[4\]. Indeed, when the system enters the USR regime, it “flies over” the flat patch of the potential, sliding on its decreasing kinetic density. In that sense, the term ultra-SR is actually a misnomer, because the field rolls faster than it would have done if SR were still applicable over the extremely flat region.

Indeed, if \(|V'|\) decreases to almost zero, so does \(\epsilon_{\text{SR}}\). In SR the number of elapsing e-folds is
\[
\Delta N = \frac{1}{m_P} \int_{\phi_1}^{\phi_2} \frac{d\phi}{\sqrt{2\epsilon_{\text{SR}}}},
\]  
which increases substantially if \(\epsilon_{\text{SR}}\) becomes extremely small. In contrast, in USR \(\epsilon\) does not decrease too much, so we have \(\epsilon_{\text{SR}} \ll \epsilon < 1\). The number of elapsing e-folds is given in general by
\[
\Delta N = -\int \frac{dH}{\epsilon H},
\]  
and in USR it can be much smaller compared to SR if \(\epsilon_{\text{SR}} \ll \epsilon\). Thus, when considering an inflation model that results in periods of USR, but only SR is assumed, there is a danger of overestimating the number of e-folds it takes for the field to roll down.

It is evident that USR depends on having substantial kinetic density, which cannot decrease faster than free-fall. However, if one begins inflation at the extremely flat region with very small kinetic density, then SR may be attained, quickly, even immediately. Now, the initial conditions for inflation are shrouded by the no-hair theorem, which renders them academic, because all memory is lost once the inflationary attractor is reached. Thus, provided inflation begins comfortably before the cosmological scales exit the horizon, the initial conditions of the inflaton field can be taken to correspond to kinetic density small enough to avoid USR despite an extremely flat scalar potential. This can rescue inflation models such as inflection-point inflation, which may have problems with USR. To quantify how small the initial kinetic density needs to be, we first investigate linear inflation.
3 Ultra-slow-roll in linear inflation and beyond

We consider the inflation model:

\[ V = V_0 + M^3 \phi, \]

where \( V_0 \) is a constant density scale and \( M \) is a mass scale. Then the Klein-Gordon Eq. (1) becomes:

\[ \ddot{\phi} + 3H_0 \dot{\phi} + M^3 = 0, \]

where \( H_0^2 \equiv V_0/3m_B^2 \) and we assumed \( V_0 \gg M^3 \phi \). The above has the general solution

\[ \dot{\phi} = Ce^{-3\Delta N} - \frac{M^3}{3H_0}, \]

where \( \Delta N = H_0 \Delta t \) is the elapsing e-folds and \( C \) is a constant. We also find

\[ \ddot{\phi} = -3H_0 C e^{-3\Delta N}. \]

If initially (\( \Delta N = 0 \)) the velocity of the field is \( \dot{\phi}_0 = 0 \), then \( C = M^3/3H_0 \) and the Klein-Gordon suggests that \( \ddot{\phi}_0 = -M^3 \). Then, as time continues, the above suggest that the Klein-Gordon becomes

\[ -M^3 e^{-3\Delta N} + M^3(e^{-3\Delta N} - 1) + M^3 = 0. \]

Notice that, even though the friction term begins as zero it soon (in a single e-fold) dominates over the acceleration term and the slow-roll (SR) condition is recovered, where \( \ddot{\phi} \) is negligible and \( V' \) is balanced by \( 3H \dot{\phi} \). Thus if we start with zero velocity, we have SR immediately afterwards.

Now suppose that, originally \( \dot{\phi} \neq 0 \). If \( |C| \ll M^3/3H_0 \) then \( \dot{\phi} \simeq \dot{\phi}_0 \simeq -M^3/3H_0 \) (cf. Eq. (16)), which means that the friction term is \( 3H \dot{\phi} \simeq -M^3 = V' \) and we have SR. Thus we always obtain immediately SR if \( |C| \leq M^3/3H_0 \). If \( |C| \gg M^3/3H_0 \) then \( 3H_0 |\dot{\phi}_0| \simeq 3H_0 |C| > M^3 \), which means that the friction term initially dominates over the slope term and is balanced by the acceleration term, \( |\ddot{\phi}_0| = 3H_0 |C| \) according to Eq. (17). Thus, the Klein-Gordon is \( \dot{\phi} + 3H \dot{\phi} \simeq 0 \) (cf. Eq. (10)), which gives rise to USR. USR continues until \( 3H_0 |C| e^{-3\Delta N} = M^3 \), when all three terms in the Klein-Gordon become comparable. Afterwards, the friction term becomes \( 3H \dot{\phi} \simeq -M^3 \), which counterbalances the slope term, while the acceleration term becomes negligible. Thus, we recover SR.

Therefore, USR lasts

\[ \Delta N_{\text{USR}} = \frac{1}{3} \ln \left( \frac{3H_0 |C|}{M^3} \right) = \frac{1}{3} \ln \left( \frac{3H_0 \sqrt{2\rho_{\text{kin}}^0}}{M^3} \right), \]

where \( \rho_{\text{kin}}^0 \equiv \frac{1}{2} \dot{\phi}_0^2 \) is the initial kinetic density, which is \( \rho_{\text{kin}}^0 \simeq \frac{1}{2} C^2 \) for large \( |C| \).
All in all, we find that, to obtain a sizeable period of USR, we need

\[ |C| \gg M^3/3H_0 \iff \rho_{\text{kin}}^0 > \frac{M^6}{18H_0^2}. \tag{20} \]

Otherwise, we have SR only. Note that, if \( M = 0 \) and the potential is exactly flat, SR is never recovered [3].

We may generalise the above for an arbitrary potential, as follows. At extremely flat region of the potential we set \( M^3 \equiv V'(\phi_f) \) and \( H_0^2 = V(\phi_f)/3m_P^2 \) and enforce the bound in Eq. (20), where \( \phi_f \) corresponds to the flattest part of the potential. Thus, to avoid USR we need

\[ \rho_{\text{kin}}(\phi_f) \equiv \frac{1}{2} \dot{\phi}_f^2 \leq \frac{(V')^2m_P^2}{6V} \bigg|_{\phi_f} = \frac{1}{3} \epsilon_{\text{SR}}(\phi_f)V(\phi_f). \tag{21} \]

The above makes sense, because the kinetic density in SR is

\[ \rho_{\text{kin}}^{\text{SR}} = \frac{1}{2} \left( \frac{V'}{3H} \right)^2 = \frac{1}{3} \epsilon_{\text{SR}} V, \tag{22} \]

where we used Eqs. (7) and (9). Thus, the bound in Eq. (21) really requests that the kinetic density in the flat patch be at most the one corresponding to SR: \( \rho_{\text{kin}}(\phi_f) \leq \rho_{\text{kin}}^{\text{SR}} \). This makes sense because if one has kinetic density in excess of \( \rho_{\text{kin}}^{\text{SR}} \), the friction term in Eq. (1) cannot be balanced by the slope term and we have USR.

In view of the above, we can also recast Eq. (19) as

\[ \Delta N_{\text{USR}} = \frac{1}{6} \ln \left( \frac{3\rho_{\text{kin}}^0}{\epsilon_{\text{SR}} V} \right), \tag{23} \]

where we used the potentially dominated Friedman equation.

### 4 Ultra-slow-roll in inflection-point inflation

We now focus on inflection-point inflation, which may feature USR. Inflection point inflation corresponds to the case of a flat step on the otherwise steep potential wall. This step is formed because of opposing terms in the potential which almost cancel each other. There are many model realisations, most notably A-term inflation [9], MSSM inflation [10] and many others [11]. However, in the vast majority of these works the USR phase has not been considered, which may cast doubt on some of their findings.

To avoid the USR period, one only needs to assume that the initial kinetic density is small enough according to the bound in Eq. (21), where \( \phi_f \) now corresponds to the inflection point, which is the flattest part of the potential plateau. This can be understood as follows. The potential for inflection-point inflation can be crudely approximated by three consecutive segments of linear potential. Inflation only takes place along the flattest segment, and it is similar to linear inflation.
While rolling from large values of $\phi$ to small, when the field reaches the flat segment then there is an abrupt reduction in $|V'|$. Because the friction term in the Klein-Gordon was at least as large as the slope term before reaching the flat segment (i.e. we had SR or free-fall), afterwards, the friction term cannot be balanced by the (substantially reduced) slope term. Thus, the acceleration term rushes to balance it and we have USR.

Now, during USR, we have $\rho_{\text{kin}} \propto a^{-6}$ so that
\[
\dot{\phi} \ddot{\phi} = \dot{\rho}_{\text{kin}} = -6H \rho_{\text{kin}} \propto a^{-6},
\]
where we took $H \simeq$ constant. Because $|\dot{\phi}| = \sqrt{2\rho_{\text{kin}}} \propto a^{-3}$, the above suggests that $|\ddot{\phi}| \propto a^{-3}$. After crossing the inflection point, though, the slope of the potential begins to increase, while the acceleration decreases, as we have seen. At some point, they meet each other and then the friction term changes allegiances and becomes locked with the slope term, so that SR is recovered.

But what if the evolution of the field had already begun at the flat patch? Then, provided the kinetic density is small enough, one can immediately have SR inflation \cite{12}. The bound in Eq. (21) is a conservative estimate on the maximum kinetic density because the slope at the inflection point is smaller than at the rest of the plateau.

5 Quantum diffusion

Now, we briefly discuss quantum diffusion. If the potential is extremely flat, quantum fluctuations of the field may dominate its variation. The quantum variation of the field per Hubble time $\delta t = H^{-1}$ is typically given by the Hawking temperature in de Sitter space $\delta \phi = H/2\pi$. Thus, the kinetic density of quantum fluctuations is
\[
\rho_{\text{kin}}^{\text{diff}} = \frac{1}{2} \left( \frac{\delta \phi}{\delta t} \right)^2 = \frac{H^4}{8\pi^4}.
\]
This should be interpreted as the lowest value the kinetic density can have. If $|V'| < \frac{3}{2\pi}H^3$ then quantum diffusion overwhelms SR\textsuperscript{1}, so that the number of USR e-folds is
\[
\Delta N_{\text{USR}} = \frac{1}{6} \ln \left( \frac{\rho_{\text{kin}}^0}{\rho_{\text{kin}}^{\text{diff}}} \right) = \frac{1}{3} \ln \left( 6\pi \sqrt{\frac{2\rho_{\text{kin}}^0 m_P^2}{V}} \right),
\]
where we used $\rho_{\text{kin}} \propto a^{-6}$ in USR and the potential dominated Friedman equation.

6 Perturbations

We now comment briefly on the curvature perturbation during USR inflation. This has been studied extensively in Ref. \cite{5}. Here we note that, during USR there is a spike in the

\begin{footnote}
\textsuperscript{1}because $|\ddot{\phi}| = |V'|/3H < H^2/2\pi = \frac{\delta \phi}{\delta t}$
\end{footnote}
curvature perturbation, which may potentially lead to the copious production of primordial black holes, that can substantially contribute to the dark matter in the Universe [6, 13]. This can be understood as follows.

For the spectrum of the curvature perturbation we have

$$\sqrt{P} = \frac{H^2}{2\pi \dot{\phi}} \Rightarrow P = \frac{H^2}{8\pi^2 m_p^2 \epsilon},$$

where we used that $2m_p^2 \dot{H} = -\dot{\phi}^2$ and Eq. (3).

In SR inflation the variation of $\epsilon = \epsilon_{SR}$ is very small, so $P$ remains roughly constant, which corresponds to an almost scale-invariant spectrum of perturbations. Indeed, the variation of $\epsilon$ is traced by $\epsilon_2 \equiv \ddot{\epsilon}/\epsilon H$ (c.f. Eq. (4)). In SR, Eq. (8) suggests that $|\epsilon_2| = |4\epsilon - 2\eta| \ll 1$.

Things are different during USR, though. Because $\epsilon = \frac{3}{2} \dot{\phi}^2/V$, where $V \simeq 3m_p^2 H^2$ and $\dot{\phi}^2 = 2\rho_{\text{kin}} \propto a^{-6}$ during USR inflation, we find $\epsilon \propto a^{-6} \propto e^{-6\Delta N}$, where $\Delta N$ is the elapsing USR e-folds. Thus, we obtain that $P \propto e^{6N}$ and the curvature perturbation grows exponentially during USR inflation.\(^2\)

Note, though, that, were USR not considered, according to Eq. (27), the usual SR would lead to an even more dramatic increase in $P$ because $\epsilon_{SR} \ll \epsilon$ when traversing an extremely flat patch of the potential. This, however, does not happen as the field “overshoots” the flat patch [6] “surfing” on its decreasing kinetic density.

7 Conclusions

In conclusion, we have investigated ultra-slow-roll (USR) inflation, which may take place when the inflationary potential becomes extremely flat. We have showed that this is a temporary phase of inflation, not an attractor, and obtained an estimate of how many e-folds it lasts, depending on the initial kinetic density of the inflaton field. We have discussed how the field can depart from the usual slow-roll (SR) when crossing an extremely flat patch in the scalar potential. SR would force the field to spend a lot of time traversing the flat patch. Instead, the field “glosses over” the flat patch in a moderate number of e-folds. Because the number of e-folds is of paramount importance when calculating inflationary observables, we argued that USR has to be taken into account, when necessary. In particular, we looked into inflection-point inflation, which exhibits a flat patch near the inflection point in the potential, that may give rise to USR. Models which do not take this into account are in danger of miscalculating the values of inflationary observables. However, this danger can be averted if one assumes that the field begins its evolution already on the flat patch (e.g. near the inflection point) with small initial kinetic density. We obtained a conservative bound on the initial kinetic density of the field, which manages to avoid USR inflation and render the SR treatment valid.

\(^2\)Note that $P$ must be evaluated at the end of USR and not at horizon exit [3, 5].
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References

[1] P. A. R. Ade et al. [Planck Collaboration], Astron. Astrophys. 594 (2016) A20.
[2] D. H. Lyth and A. R. Liddle, Cambridge, UK: Cambridge Univ. Pr. (2009) 497 pp.
[3] W. H. Kinney, Phys. Rev. D 72 (2005) 023515.
[4] J. Martin, H. Motohashi and T. Suyama, Phys. Rev. D 87 (2013) no.2, 023514.
[5] M. H. Namjoo, H. Firouzjahi and M. Sasaki, Europhys. Lett. 101 (2013) 39001; S. Mooij and G. A. Palma, JCAP 1511 (2015) no.11, 025; A. E. Romano, S. Mooij and M. Sasaki, Phys. Lett. B 761 (2016) 119.
[6] C. Germani and T. Prokopec, arXiv:1706.04226 [astro-ph.CO].
[7] B. Spokoiny, Phys. Lett. B 315 (1993) 40; M. Joyce and T. Prokopec, Phys. Rev. D 57 (1998) 6022; C. Pallis, JCAP 0510 (2005) 015; Nucl. Phys. B 751 (2006) 129; M. E. Gomez, S. Lola, C. Pallis and J. Rodriguez-Quintero, JCAP 0901 (2009) 027.
[8] K. Dimopoulos, Nucl. Phys. Proc. Suppl. 95 (2001) 70; K. Dimopoulos, Phys. Rev. D 68 (2003) 123506; K. Dimopoulos and C. Owen, JCAP 1706 (2017) no.06, 027.
[9] J. C. Bueno Sanchez, K. Dimopoulos and D. H. Lyth, JCAP 0701 (2007) 015; R. Allahverdi, A. Kusenko and A. Mazumdar, JCAP 0707 (2007) 018; J. Garcia-Bellido, AIP Conf. Proc. 878 (2006) 277; C. M. Lin and K. Cheung, Mod. Phys. Lett. A 25 (2010) 1425.
[10] R. Allahverdi, K. Enqvist, J. Garcia-Bellido and A. Mazumdar, Phys. Rev. Lett. 97 (2006) 191304; D. H. Lyth, JCAP 0704 (2007) 006; R. Allahverdi, K. Enqvist, J. Garcia-Bellido, A. Kusenko and A. Mazumdar, JCAP 0706 (2007) 019; T. Matsuda, JCAP 0706 (2007) 029; K. Enqvist, L. Mether and S. Nurmi, JCAP 0711 (2007) 014; Z. Lalak and K. Turzynski, Phys. Lett. B 659 (2008) 669; S. Nurmi, JCAP 0801 (2008) 016; T. Matsuda, Nucl. Phys. B 822 (2009) 88; K. Kamada and J. Yokoyama, Prog. Theor. Phys. 122 (2010) 969; R. Allahverdi, A. Ferrantelli, J. Garcia-Bellido and A. Mazumdar, Phys. Rev. D 83 (2011) 123507; S. Choudhury, A. Mazumdar and S. Pal, JCAP 1307 (2013) 041.
[11] N. Itzhaki and E. D. Kovetz, JHEP 0710 (2007) 054; R. Allahverdi, B. Dutta and A. Mazumdar, Phys. Rev. D 78 (2008) 063507; M. Badziak and M. Olechowski, JCAP 0902 (2009) 010; K. Enqvist, A. Mazumdar and P. Stephens, JCAP 1006 (2010) 020;
S. M. Choi and H. M. Lee, Eur. Phys. J. C 76 (2016) no.6, 303; N. Okada and D. Raut, Phys. Rev. D 95 (2017) no.3, 035035; N. Okada, S. Okada and D. Raut, Phys. Rev. D 95 (2017) no.5, 055030.

[12] K. Dimopoulos, C. Owen and A. Racioppi, arXiv:1706.09735 [hep-ph].

[13] G. Ballesteros and M. Taoso, arXiv:1709.05565 [hep-ph]; Y. Gong, arXiv:1707.09578 [astro-ph.CO]; K. Kannike, L. Marzola, M. Raidal and H. Veerme, JCAP 1709 (2017) no.09, 020; J. M. Ezquiaga, J. Garcia-Bellido and E. Ruiz Morales, arXiv:1705.04861 [astro-ph.CO]; J. Garcia-Bellido and E. Ruiz Morales, Phys. Dark Univ. 18 (2017) 47; H. Motohashi and W. Hu, Phys. Rev. D 96 (2017) no.6, 063503.