The renormalization group and the effective potential in a curved spacetime with torsion

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Abstract

The renormalization group method is employed to study the effective potential in curved spacetime with torsion. The renormalization-group improved effective potential corresponding to a massless gauge theory in such a spacetime is found and in this way a generalization of Coleman-Weinberg’s approach corresponding to flat space is obtained. A method which works with the renormalization group equation for two-loop effective potential calculations in torsionful spacetime is developed. The effective potential for the conformal factor in the conformal dynamics of quantum gravity with torsion is thereby calculated explicitly. Finally, torsion-induced phase transitions are discussed.

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1 Introduction

It is a well-known fact that string theory contains an antisymmetric two-form whose only dynamical content is that of an axion [1]. The three-form gauge field strength corresponding to this antisymmetric two-form may be interpreted as the torsion (for a review of gravity with torsion see [2,3,15]). The $O(\alpha')$ string effective action [4] is equivalent to the usual Einstein-Cartan theory, and to first order in $\alpha'$ it is equivalent to higher-derivative gravity with torsion [5]. The equivalence between an axion in string theory and the presence of torsion has been discussed further in ref. [6], where it has been shown also that black holes might have axion hair [6,7].

On the other hand, the current interest in gravity with torsion [2,8] stems also from the search of the so-called fifth force (for a review, see [4]). Moreover, to be noticed is also the fact that cosmic strings can be naturally generated by torsion [9].

The present paper is devoted to the study of interacting quantum field theory in curved spacetime with torsion. More specifically, we apply the renormalization group approach to the calculation and analysis of the effective potential corresponding to several different theories.

The organization of the paper is as follows. In the next section we develop the procedure which yields a renormalization-group improvement of the effective potential corresponding to an arbitrary massless gauge theory in a curved spacetime with torsion. Some explicit examples are given. The phase structure of the RG improved effective potential for the $\lambda \varphi^4$ theory is discussed in detail. Sect. 3 is devoted to the calculation of the two-loop effective potential in a torsionful spacetime, in a situation in which all the $\beta$-functions are known. In sect. 4 the same procedure of sect. 3 is applied in order to find the effective potential corresponding to the effective theory of the conformal factor in quantum gravity with torsion. The phase transitions induced by torsion are discussed. Such phase transitions can actually remove the original singularity, a fact that might have very interesting consequences. The conclusions of the paper are to be found in sect. 5.
2 Renormalization-group improved effective potential in curved spacetime with torsion

In this section we discuss the renormalization-group improved effective potential for a theory in curved spacetime with torsion. This will be done by generalizing Coleman-Weinberg’s approach [10] (see also [12]). Let us consider a renormalizable, massless gauge theory which includes scalars \( \phi \), spinors \( \psi \) and gauge fields \( A_\mu \) in a curved spacetime with non-zero torsion. We shall denote by \( \tilde{g} \equiv (g, \lambda, h) \) the set of all coupling constants of the theory (\( g \) is the Yang-Mills, \( \lambda \) the scalar and \( h \) the Yukawa coupling), and \( \tilde{\xi} = (\xi, \zeta, \eta) \) are the scalar-gravitational couplings. The tree level potential has the following form

\[
V^{(0)} = a\lambda \phi^4 - b\xi R \phi^2 - d\zeta S_\mu S^\mu \phi^2,
\]

(1)

where \( a, b \) and \( d \) are some positive constants.

The renormalization group equation for the effective potential has the following form

\[
\left( \mu \frac{\partial}{\partial \mu} + \beta_g \frac{\partial}{\partial g} + \delta \frac{\partial}{\partial \alpha} + \beta_\xi \frac{\partial}{\partial \xi} - \gamma \phi \frac{\partial}{\partial \phi} \right) V = 0.
\]

(2)

We work in the Landau gauge (\( \alpha = 0 \)), in which \( \delta = 0 \) to one-loop order.

We adopt the approximation (linear on invariants of the gravitational field) in which \( \phi^2 \gg |R| \) and \( \phi^2 \gg |S_\mu S^\mu| \). In this approximation we split \( V \) in the following way

\[
V = V_1 + V_2 + V_3 = af_1(p, \phi, \mu)\phi^4 - bf_2(p, \phi, \mu)R\phi^2 - df_3(p, \phi, \mu)S_\mu S^\mu \phi^2,
\]

(3)

where \( p \equiv \{g, \xi, \alpha\} \) and \( f_1, f_2 \) and \( f_3 \) are some unknown functions. We also assume that each of the three \( V_1, V_2 \) and \( V_3 \) satisfy the renormalization group equation (2) (in this case, of course, \( V \) satisfies it too).

With all these considerations in mind, we can solve the renormalization group equation (2) as follows:

\[
V = a\lambda(t) f^4(t) \phi^4 - b\xi(t) f^2(t) R \phi^2 - d\zeta(t) f^2(t) S_\mu S^\mu \phi^2,
\]

(4)

where

\[
f(t) = \exp \left[ -\int_0^t dt' \tilde{\gamma} \left( \tilde{g}(t'), \tilde{\xi}(t'), \alpha(t') \right) \right], \quad t = \frac{1}{2} \ln \frac{\phi^2}{\mu^2}, \quad \tilde{g}(t) = \tilde{\beta}_g(t),
\]

3
\[
\dot{\xi}(t) = \bar{\beta}_{\xi}(t), \quad \dot{\alpha}(t) = \bar{\delta}(t), \quad g(0) = \bar{g}, \quad \xi(0) = \bar{\xi}, \quad \alpha(0) = \alpha
\]

and

\[
\begin{pmatrix}
\bar{\beta}_{\gamma}, 
\bar{\beta}_{\xi}, 
\bar{\gamma}, 
\bar{\delta}
\end{pmatrix} = \frac{1}{1 + \gamma} \begin{pmatrix}
\beta_{\gamma}, 
\beta_{\xi}, 
\gamma, 
\delta
\end{pmatrix}.
\]

The solution \([4]\) has been obtained using the following initial conditions

\[
V_1(t = 0) = a\lambda \varphi^4, \quad V_2(t = 0) = -b\xi R \varphi^2, \quad V_3(t = 0) = -d\zeta S_\mu S^\mu \varphi^2.
\] (5)

The initial conditions for \(V_1\) are slightly different from Coleman-Weinberg's [10], what will lead to some differences in the non-logarithmic terms; they are exactly the same as in ref. [13]. The renormalization-group improved effective potential \([4]\) in the absence of torsion (i.e., \(S_\mu = 0\)) has been obtained in ref. [11].

In the one-loop approximation —in which we shall actually work throughout this paper— the effective potential \([4]\) is formally the same, but now with

\[
f(t) = \exp \left[ -\int_0^t dt' \gamma \left( g(t'), \xi(t') \right) \right], \quad \dot{\bar{g}}(t) = \beta_{\bar{g}}(t), \quad \dot{\bar{\xi}}(t) = \beta_{\bar{\xi}}(t), \quad \bar{g}(0) = \bar{g}, \quad \bar{\xi}(0) = \bar{\xi}.
\] (6)

Expression \([4]\) with the functions of \(t\) being given by (6) can be applied to a variety of gauge theories. We shall now present a few examples.

(i) \(\lambda \varphi^4\)-theory. This is a quite simple example, which may however be interesting enough from a pedagogical point of view. Here it is not necessary to have \(\zeta \neq 0\) in order to obtain multiplicative renormalizability. One can always put \(\zeta = 0\) and then the theory does not interact with the torsion at all. Keeping, though, \(\zeta \neq 0\), and using the well-known values for the effective coupling constants, we get

\[
V = \frac{\lambda \varphi^4}{4! \left(1 - \frac{3\lambda}{(4\pi)^2}\right)} - \frac{1}{2} R \varphi^2 \left[ \frac{1}{6} + \left( \xi - \frac{1}{6} \right) \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1/3} \right] - \frac{1}{2} S_\mu S^\mu \varphi^2 \left[ \zeta \left( 1 - \frac{3\lambda t}{(4\pi)^2} \right)^{-1/3} \right],
\] (7)

where \(t = \frac{1}{2} \ln(\varphi^2/\mu^2)\). The potential \([\tilde{4}]\) is valid for the range of values of \(t\) which make it not to diverge (in particular, for any negative value of \(t\)). In flat space \((R = S_\mu = 0)\) the effective potential \([\tilde{4}]\) has been obtained in [10], and in curved space without torsion \((R \neq 0, S_\mu = 0)\), in [11].

(ii) Asymptotically free SU(2) theory. This is an example in which torsion appears in a much more natural way than in the previous case. The Lagrangian is given by (see
\[ L = -\frac{1}{4} G_{\mu\nu} G^{\alpha\beta} + \frac{1}{2} g^{\alpha\beta} (D_{\mu} \phi)^a (D_{\nu} \phi)^a + \frac{1}{2} \xi R \phi^a \phi^a + \frac{1}{2} \zeta S_{\mu\nu} \phi^a \phi^a \\
- \frac{1}{4!} \lambda (\phi^a \phi^a)^2 + i \bar{\psi}^a \left[ \gamma^\mu D^a_{\mu} + \eta \gamma^5 S_{\mu} \delta^a b \right] \psi^b - i e^{abc} \bar{\psi}^a \phi^c \psi^b. \]  

Here \((D_{\mu} \phi)^a = \partial_{\mu} \phi^a + g \epsilon_{abc} A^b_{\mu} \phi^c\) and \(D^a_{\mu} \psi^b = \nabla_{\mu} \psi^a + g \epsilon_{abc} A^b_{\mu} \phi^c\), the gauge group is \(\text{SU}(2)\), and \(\phi^a\) and \(\psi^a\) belong to the adjoint representation of the gauge group. It is known that the theory under discussion is asymptotically free for special solutions of the renormalization group \[14\].

It is not difficult to show that the theory minimally coupled to torsion and metric \((\eta = -1/8, \xi = \zeta = 0)\) is not multiplicatively renormalizable \[14\] (for a general discussion see \[15\]). In order to get multiplicative renormalizability, we must introduce the coupling parameters \(\xi, \zeta\) and \(\eta\) \[15\].

The renormalization group equations for the effective couplings have been obtained in refs. \[14\]:

\[ g^2(t) = \frac{g^2}{1 + l^2 t}, \quad l^2 = \frac{b^2 g^2}{(4\pi)^2}, \quad b^2 = \frac{26}{3}, \quad \lambda(t) = \kappa_1 g^2(t), \quad \kappa_1 = \frac{97}{22}, \quad h^2(t) = \kappa_2 g^2(t), \]
\[ \kappa_2 = \frac{23}{24}, \quad \xi(t) = \frac{1}{6} + \left( \xi - \frac{1}{6} \right) (1 + l^2 t)^{-a^2/b^2}, \quad a^2 = 12 - \frac{5}{3} \kappa_1 - 8 \kappa_2 > 0, \]
\[ \eta(t) = \eta(1 + l^2 t)^{\kappa_2 a^2/b^2}, \quad \zeta(t) = \left( \zeta + \frac{32 \eta^2}{8 \kappa_2 + b^2} \right) (1 + l^2 t)^{-a^2/b^2} \]
\[ - \frac{32 \eta^2}{8 \kappa_2 + b^2} (1 + l^2 t)^{8 \kappa_1/b^2}, \quad f(t) = (1 + l^2 t)^{(a - 4 \kappa_2)/a^2}. \]  

Using the effective coupling constants \[9\], we obtain the renormalization-group improved effective potential

\[ V = \frac{1}{4!} \lambda(t) f^4(t) \phi^4 - \frac{1}{2} \xi(t) R f^2(t) \phi^2 - \frac{1}{2} \zeta(t) S_{\mu\nu} f^2(t) \phi^2, \]  

where \(\phi^2 = \phi^a \phi^a\) and \(t = \frac{1}{2} \ln(\phi^2/\mu^2)\). In the same way, one can easily get the effective potential for a variety of models (for example, GUTs, see ref. \[15\]), where the effective coupling constants are known. Let us now investigate the possibility of a first order phase transition induced by the external gravitational field (the possibility of a phase transition induced by torsion has been already pointed out in ref. \[16\]). A detailed analysis of curvature-induced phase transitions has been done in our previous work, ref. \[11\].
With torsion the situation changes as follows. For simplicity of the discussion, let us first restrict ourselves to the case of the $\lambda \varphi^4$ theory; although our considerations also apply to the SU(2) case and are indeed very general. As is manifest from the specific form of the potential (I) (and also from that of the potential (II)), the inclusion of torsion turns out, in the end, in the appearance of an additional term which has exactly the same $t$-dependence as the main term corresponding to non-zero curvature. This is to say, in the presence of both torsion and curvature, the analysis of phase transitions is basically the same as the analysis corresponding to curvature alone: no specifically new situation is created by the addition of torsion.

However, the remarkable thing is the fact that also in the absence of curvature, the presence of torsion still reproduces some of the behaviors typically induced by curvature. This is true both for the $\lambda \varphi^4$ theory and also for the SU(2) one. The potential (I) can be written in that case as

$$y = \frac{1}{2u(x)} \left[ \frac{\lambda}{12} x^2 - \frac{\zeta}{\mu^2} u(x)^{2/3} S^2 x \right], \quad (11)$$

where

$$x \equiv \frac{\varphi^2}{\mu^2}, \quad y \equiv \frac{V}{\mu^4}, \quad u(x) \equiv 1 - \frac{3\lambda \ln x}{2(4\pi)^2}. \quad (12)$$

A direct analysis of the parenthesis in (11) shows that it just has two different possible behaviors, as a function of $x$. Namely, for positive $\zeta$, it is a function that starts at the origin, goes down and reaches a minimum value at some $x_m \neq 0$, and ends as an increasing parabolic-like curve (Fig. 1a); on the other hand, for negative $\zeta$ the minimum is obtained at the origin and the curve is monotonically increasing all the time (Fig. 1b).

3 Two-loop effective potential in curved spacetime with torsion

We shall here develop a method for the calculation of the massless effective potential at any loop order. The method has its roots in the direct solution of the RG equations for a massless gauge theory in curved spacetime with torsion. (Again, we are going to work in the approximation where the invariants of the gravitational field —$R$ and $S_\mu S^\mu$— appear...
linearly). This method has been developed already for flat space [17] and generalized later to curved spacetime [18]. Here we will give the closed expressions corresponding to the two-loop effective potential. As an example of two-loop effective potential calculation, that for the $\lambda \varphi^4$-theory in curved spacetime with torsion will be carried out explicitly.

Starting once more with the effective potential (1), we consider again the RG equation (2) in the Landau gauge. Working in the same approximation as in sect. 2, we shall also assume that each of the $V_1$, $V_2$ and $V_3$ satisfy eq. (2) independently. (Then, of course, $V$ will also satisfy it). Using (1), we can write the following $n$-loop order RG equation for the effective potential

$$
\mu \frac{\partial}{\partial \mu} V^{(n)} + D_n V^{(0)} + D_{n-1} V^{(1)} + \cdots + D_1 V^{(n-1)} = 0,
$$

(13)

where $V^{(n)}$ is the $n$-loop correction to the effective potential,

$$
D_n = \beta_g^{(n)} \frac{\partial}{\partial g} + \beta_\xi^{(n)} \frac{\partial}{\partial \xi} - \gamma^{(n)} \varphi \frac{\partial}{\partial \varphi},
$$

and $\beta^{(n)}$ is the $n$-loop correction to the corresponding $\beta$-function. In accordance with our proposal, we have three equations (13) —for $V_1$, $V_2$ and $V_3$— and we can find the effective potential by using the recursion formula (13). For $V_1$ and $V_2$ this has been done already in refs. [17,18], respectively, where a closed expression up to two-loop order has been obtained. $V_3$ can be found in a similar way. Using the following renormalization conditions

$$
V_i^{(j)} \bigg|_{\mu=\varphi} = 0, \quad i = 1, 2, 3, \quad j = 1, 2,
$$

(14)

we get

$$
V = V^{(0)} + V^{(1)} + V^{(2)} = a \lambda \varphi^4 + A^{(1)} \varphi^4 \ln \frac{\varphi^2}{\mu^2} + \frac{1}{2} \left[ \left( \beta_\lambda^{(2)} - 4 \lambda \gamma^{(2)} \right) a - 2 \gamma^{(1)} A^{(1)} \right] \varphi^4 \ln \frac{\varphi^2}{\mu^2}
$$

$$
+ \frac{1}{4} \left[ \beta_g^{(1)} \frac{\partial A^{(1)}}{\partial g} - 4 \gamma^{(1)} A^{(1)} \right] \varphi^4 \left( \ln \frac{\varphi^2}{\mu^2} \right)^2 - b \xi R \varphi^2 - B^{(1)} R \varphi^2 \ln \frac{\varphi^2}{\mu^2}
$$

$$
- \frac{1}{2} \left[ \beta_\xi^{(2)} - 2 \gamma^{(2)} - 2 \gamma^{(1)} B^{(1)} b \right] b R \varphi^2 \ln \frac{\varphi^2}{\mu^2} - \frac{1}{4} \left[ \beta_\xi^{(1)} \frac{\partial B^{(1)}}{\partial \xi} + \beta_\xi^{(1)} \frac{\partial B^{(1)}}{\partial \xi} - 2 \gamma^{(1)} B^{(1)} \right]
$$

$$
\times \left( \ln \frac{\varphi^2}{\mu^2} \right)^2 - d \xi S^2 \varphi^2 - D^{(1)} S^2 \varphi^2 \ln \frac{\varphi^2}{\mu^2} - \frac{1}{2} \left[ \beta_\xi^{(2)} - 2 \gamma^{(2)} - 2 \gamma^{(1)} \frac{D^{(1)}}{d} \right] d S^2 \varphi^2 \ln \frac{\varphi^2}{\mu^2}
$$

$$
- \frac{1}{4} \left[ \beta_g^{(1)} \frac{\partial D^{(1)}}{\partial g} + \beta_\xi^{(1)} \frac{\partial D^{(1)}}{\partial \xi} - 2 \gamma^{(1)} D^{(1)} \right] R \varphi^2 \left( \ln \frac{\varphi^2}{\mu^2} \right)^2,
$$

(15)
where
\[ A^{(1)} = \frac{a}{2} \left( \beta^{(1)} - 4 \lambda \gamma^{(1)} \right), \quad B^{(1)} = \frac{b}{2} \left( \beta^{(1)} - 2 \xi \gamma^{(1)} \right), \quad D^{(1)} = \frac{d}{2} \left( \beta^{(1)} - 2 \zeta \gamma^{(1)} \right). \]

Using (13) one can immediately obtain the two-loop effective potential for any gauge theory in a spacetime with torsion.

As an example, we give now the result corresponding to the \( \lambda \varphi^4 \)-theory. The two-loop beta functions for flat spacetime have been obtained in ref. [19] already, while the two-loop \( \beta_\xi \) has been calculated in ref. [20]; moreover, up to two loops, we have \( \beta_\zeta = \zeta \gamma m^2 \) [15], and the two-loop \( \gamma \)-function for the scalar field mass is given, for instance, in [19]. Using these \( \beta \)-functions, we get

\[
V = \frac{\lambda}{24} \varphi^4 - \frac{\xi}{2} R \varphi^2 - \frac{\zeta}{2} S^2 \varphi^2 + \frac{\lambda^2}{16(4\pi)^2} \varphi^4 \ln \frac{\varphi^2}{\mu^2} - \frac{\lambda}{(8\pi)^2} \left( \xi - \frac{1}{6} \right) R \varphi^2 \ln \frac{\varphi^2}{\mu^2} - \frac{\lambda \zeta S^2}{(8\pi)^2} \varphi^2 \ln \frac{\varphi^2}{\mu^2}
\]

\[
- \frac{\lambda^3}{8(4\pi)^4} \varphi^4 \ln \frac{\varphi^2}{\mu^2} + \frac{3\lambda^3}{32(4\pi)^4} \varphi^4 \left( \ln \frac{\varphi^2}{\mu^2} \right)^2 - \frac{\lambda^2}{4(4\pi)^4} \left[ \left( \xi - \frac{1}{6} \right) + \frac{1}{36} \right] R \varphi^2 \ln \frac{\varphi^2}{\mu^2}
\]

\[
- \frac{\lambda^2}{4(4\pi)^4} \left( \xi - \frac{1}{6} \right) R \varphi^2 \left( \ln \frac{\varphi^2}{\mu^2} \right)^2 - \frac{\lambda^2}{4(4\pi)^4} \zeta S^2 \varphi^2 \ln \frac{\varphi^2}{\mu^2} - \frac{\lambda^2}{4(4\pi)^4} \zeta S^2 \varphi^2 \left( \ln \frac{\varphi^2}{\mu^2} \right)^2. \quad (16)
\]

In more realistic theories, as the SU(2) model, the analog of eq. (16) is actually more interesting because the contribution of the fermion coupling \( \eta \) will appear in terms connected with the torsion (even in the case of minimal coupling). The final expressions are indeed very complicated and, moreover, \( \beta_\zeta \) to two loops is not known in such theories. Notice also that, as it follows from (16), for zero curvature but non-zero torsion and \( \zeta \neq 0 \), we obtain spontaneous symmetry breaking induced by torsion. At tree level and for \( \zeta > 0 \), we get

\[
\varphi^2 = \frac{6 \zeta S^2}{\lambda}. \quad (17)
\]

Using (16) we can then obtain loop corrections to the minimum (17).

To summarize, we have proven in this section that the method developed allows us, in fact, to calculate the multiloop effective potential for a field theory in any spacetime with torsion.
4 Phase transitions induced by torsion in infrared quantum gravity

In this section we will study the effective potential which arises in the conformal dynamics of quantum gravity with torsion [22] (for a discussion of conformal dynamics of quantum gravity, see [21]).

Let us first briefly recall the construction of the trace anomaly induced dynamics of the conformal factor [22]. We start from the free, conformally invariant theory corresponding to $N_0$ scalars, $N_1/2$ spinors and $N_1$ vectors on a spacetime with torsion (see [15]). It is given by the action

\begin{equation}
S = S_0 + S_{1/2} + S_1,
\end{equation}

where

\begin{align*}
S_0 &= \frac{1}{2} \int d^4x \sqrt{-g} \left[ g^\alpha\beta \partial_\alpha \varphi \partial_\beta \varphi + \frac{1}{6} R \varphi^2 + \zeta S_\mu S^\mu \varphi^2 \right],
S_{1/2} &= i \int d^4x \sqrt{-g} \left[ \bar{\psi} \left( \gamma^\mu \nabla_\mu - \eta \gamma_5 \gamma^\mu S_\mu \right) \psi \right],
S_1 &= - \frac{1}{4} \int d^4x \sqrt{-g} G^2_{\mu\nu},
\end{align*}

being $\zeta$ and $\eta$ arbitrary coupling constants. The theory with the action (18) is conformally invariant for any value of $\zeta$ and $\eta$. The minimal coupling corresponds to $\zeta = 0$ and $\eta = 1/8$.

The trace anomaly for the theory (18) in curved spacetime with torsion is given by the following expression (see [15] for details and references):

\begin{align*}
T^\mu_\mu &= b C^{2}_{\mu\nu\alpha\beta} + b' \left( G - \frac{2}{3} \Box R \right) + \frac{1}{3} \left[ b'' + \frac{2}{3} (b + b') \right] \Box R + a_1 F^2_{\mu\nu}
&+ a_2 (S_\mu S^\mu)^2 + a_3 \Box (S_\mu S^\mu) + a_4 \nabla_\mu \left( S_\nu \nabla^\nu S^\mu - S^\mu \nabla^\nu S^\nu \right),
\end{align*}

where $G$ is the Gauss-Bonnet invariant, $F_{\mu\nu} = \nabla_\mu S_\nu - \nabla_\nu S_\mu$, and where the coefficients $b, b', \ldots, a_4$ are well known (see, for instance, [21,22]). In particular, the coefficients $a_1, \ldots, a_4$ relevant for non-zero torsion are

\begin{align*}
a_1 &= - \frac{2}{3(4\pi)^2} \Sigma \eta^2, \quad a_2 = \frac{1}{2(4\pi)^2} \Sigma \zeta^2,
& a_3 = \frac{1}{3(4\pi)^2} \Sigma \left( 2\eta^2 - \frac{1}{2}\zeta^2 \right), \quad a_4 = - \frac{2}{3(4\pi)^2} \Sigma \eta^2.
\end{align*}
Choosing the conformal parametrization

\[ g_{\mu\nu} = e^{2\sigma(x)} \eta_{\mu\nu}, \quad S_\mu = \bar{S}_\mu, \]  

(22)

where \( \sigma \) is the conformal factor, \( \eta_{\mu\nu} \) the Minkowski metric and \( \bar{S}_\mu \) an arbitrary constant torsion background, one can integrate over the trace anomaly in order to get the trace-anomaly-induced effective action, \( S_{anom} \) (see [15] for details). Adding the classical gravity action

\[ S_{cl} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left( R + h S_\mu S^\mu - 2\Lambda \right), \]  

(23)

with the parametrization (22) to \( S_{anom} \), we get the total effective action which describes the conformal factor dynamics [22]

\[ S_{eff} = S_{anom} + S_{cl} = -\frac{Q^2}{(4\pi)^2} \int d^4x (\Box \sigma)^2 - \zeta \int d^4x \left[ 2\alpha(\partial_\mu \sigma)^2 \Box \sigma + \alpha^2(\partial_\mu \sigma)^4 \right] 
+ \gamma \int d^4x e^{2\alpha \sigma}(\partial_\mu \sigma)^2 - \frac{\lambda}{\alpha^2} \int d^4x e^{4\alpha \sigma} + \int d^4x \left[ \left( a_3 + \frac{a_4}{2} \right) \bar{S}^2(\partial_\mu \sigma)^2 
+ a_4 \bar{S}^\mu \bar{S}^\nu \partial_\mu \sigma \partial_\nu \sigma + \frac{h}{2\kappa \alpha^2} e^{2\alpha \sigma} \bar{S}^2 \right], \]  

(24)

where the transformations \( \sigma \rightarrow \sigma\alpha \) and \( S_{eff} \rightarrow \alpha^{-2}S_{eff} \) have been performed, and

\[ \frac{Q^2}{(4\pi)^2} = 2b + 3b', \quad \zeta = 2b + 2b' + 3b'', \quad \gamma = \frac{3}{\kappa}, \quad \lambda = \frac{\Lambda}{\kappa}. \]  

(25)

In what follows we are going to work about the infrared stable fixed point \( \zeta = 0 \) (see [21,22]). Let us denote \( e^{a_\sigma} \equiv \Phi \). (Notice that \( \Phi \) is always positive).

The tree level effective potential is given by

\[ V(\Phi) = \frac{\lambda}{\alpha^2} \Phi^4 - \frac{h}{2\kappa \alpha^2} \Phi^2 S^2. \]  

(26)

The one-loop correction to this potential can be easily found using the general expression ([15]) (the one-loop beta functions are known from [21,22]). The result is (we use Coleman-Weinberg’s normalization conditions)

\[ V^{(1)}(\Phi) = \frac{\lambda}{\alpha^2} \Phi^4 - \left[ \frac{\gamma^2(4\pi)^2}{4Q^2} - \frac{4\lambda}{Q^2} \right] \Phi^4 \left( \ln \frac{\Phi^2}{\mu^2} - \frac{25}{6} \right) - \frac{h}{2\kappa \alpha^2} \Phi^2 S^2 
+ \left[ \frac{\gamma(4\pi)^2}{2Q^2} \left( a_3 + \frac{3}{4} a_4 \right) - \frac{h}{2\kappa Q^2} \right] \Phi^2 S^2 \left( \ln \frac{\Phi^2}{\mu^2} - 3 \right). \]  

(27)
As we can see, in the absence of torsion the minimum corresponding to the tree effective potential is obtained for
\[ \Phi = 0, \quad \sigma \rightarrow -\infty. \tag{28} \]
In terms of the original metric, this corresponds to the singularity. However, the remarkable point is the fact that symmetry breaking appears in two different ways: already at tree level as a result of adding torsion, or else at one-loop because of quantum corrections (see also [23]). Both lead to a non-zero vacuum, which in the second case is (for simplicity, we take \( S^2 = 0 \))
\[ \frac{\sigma}{\sigma_0} = -\frac{\lambda}{\alpha^2} \left[ \frac{\gamma^2(4\pi)^2}{4Q^4} - \frac{4\lambda}{Q^2} \right]^{-1} + \frac{25}{6} - \frac{11}{4a} \tag{29} \]
(the parameters should be chosen to have on the right-hand side the positive ones). Hence, the singular vacuum becomes the nonsingular one as a result of the Coleman-Weinberg symmetry breaking (the singularity is avoided). In principle one can construct the renormalization group improved effective potential as in sect. 2. Nevertheless, in the present case we have many coupling constants and the corresponding effective coupling constants do not show such a simple behavior as before (like asymptotic freedom, for example). For this reason we shall not discuss the RG improved effective potential here.

Let us now investigate the possibility of a phase transition induced by torsion in the effective conformal factor theory \( (25) \). It turns out that an exact analytical study can again be carried out. It leads to the following results. To start with, eq. \( (27) \) can be written in the shortened form
\[ y = F(x) = x^2 + ax^2 \left( \ln x - \frac{25}{6} \right) - bx + cx(\ln x - 3), \quad x > 0, \tag{30} \]
where the \( x, y \) and the constants \( a, b \) and \( c \) are immediately identified by simple inspection
\[
x = \frac{\Phi^2}{\mu^2}, \quad y = \frac{\alpha^2}{\lambda \mu^4} V(\Phi), \quad a = \frac{\alpha^2}{Q^2} \left[ \frac{\gamma^2(4\pi)^2}{4Q^2} - 4\lambda \right], \\
b = \frac{h}{2\lambda \kappa \mu^2} S^2, \quad c = \frac{3}{4} a_4 \left[ \frac{\gamma^2(4\pi)^2}{Q^2} \left( a_3 + \frac{3}{4} a_4 \right) - \frac{h}{\kappa} \right] S^2. \tag{31} \]
Notice again that \( x > 0 \) and that \( b \geq 0 \). Proceeding with the calculation of extrema, the first derivative yields
\[ y' = (2ax + c) \left[ \ln x + \left( \frac{1}{a} - \frac{11}{3} \right) + \frac{5}{3} \frac{c - a}{2ax + c} \right]. \tag{32} \]
Thus, the extrema are obtained as the crossing points of the two functions

\[ \ln x = G(x), \quad G(x) \equiv \left[ \left( \frac{11}{3} - \frac{1}{a} \right) + \frac{-\frac{5}{3}c + \frac{5}{a} + b}{2ax + c} \right]. \tag{33} \]

It is immediate that the function \( G(x) \) is either monotonically increasing in the whole range \( x > 0 \) or else monotonically decreasing in the whole range (its first derivative has a constant sign). For \( x \to \infty \) it goes asymptotically to the constant value \( G(\infty) = \frac{11}{3} - \frac{1}{a} \), unless \( a \equiv 0 \), in which case it is constantly equal to \( G(x) \equiv 2 + \frac{b}{c} \), and a single extremum is obtained. This last value is, in general, the one reached at the origin \( G(0) \). The convexity of \( G(x) \) has also a uniform sign in the whole range or can, at most, change sign once. All this leads to the conclusion that eq. (33) has at most two solutions. The two extrema can be obtained either both of them at the same side of the discontinuity of \( G(x) \) (i.e., \( x_0 = -c/2a \)) or each at one side of it. We shall be specially interested in the case \( a > 0 \) and \( c < 0 \), where the usual mexican-hat shape appears (Figs. 2a and 2b). However, the phase transition —induced by torsion— is obtained for a very wide range of values of the parameters. In fact, it already shows up for very small values of \( \alpha^2 \) and of the torsion \( S^2 \). Some results, with the typical form of the symmetry breaking potential, corresponding to several different values of the constants, are depicted in Fig. 2. The final stages (Figs. 2c and 2d) look the same as in Fig. 1. Similar questions have been investigated very recently for a different model in ref. [24].

5 Conclusions

We have developed in this paper a formalism which yields a well-defined procedure to study the effective potential, and the corresponding phase structure, of gauge theories in torsionful spacetime. In particular, the renormalization-group improved effective potential for any massless gauge theory has been thus found. That is, we have been able to extend the one-loop effective potential formalism to a spacetime with torsion, taking thereby into account all logarithmic corrections. There is no problem, in principle, to extend our approach to massive theories (except for some minor complications of technical nature). This will be done elsewhere.
Moreover, a method, based on the RG, for the calculation of the corresponding multiloop effective potential in a torsionful spacetime, has been also constructed, thus generalizing again the corresponding method valid for flat space.

Finally, the effective potential corresponding to the conformal sector of quantum gravity with torsion has been discussed. It has been shown that, for different values of the parameters of the theory, a phase transition induced by torsion may take place. In particular, this phase transition might lead to a removal of the original singularity. Such phenomenon could be very relevant to early Universe considerations.

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References

[1] M. Green, J. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press, 1987.

[2] F.H. Heyl, P. von der Heyde, G.D. Kerlick and J.M. Nester, Rev. Mod. Phys. **48** (1976) 393.

[3] V. De Sabbata, V.N. Melnikov and P.I. Pronin, Progr. Theor. Phys. **88** (1992) 623.

[4] J. Sherk and J.H. Schwarz, Phys. Lett. **B52** (1974) 347.

[5] D. Gross and J.H. Sloan, Nucl. Phys. **B291** (1987) 41; Z. Bern, T. Shimada and D. Hochberg, Phys. Lett. **B191** (1987) 267.

[6] M.J. Duncan, N. Kaloper and K.A. Olive, preprint UMN-TH.1019/92 (1992).

[7] M.J. Bowick et al., Phys. Rev. Lett. **61** (1988) 2823; B. Campbell, M. Duncan, N. Kaloper and K. Olive, Phys. Lett. **B251** (1990) 34; K. Lee and E. Weinberg, Phys. Rev. **D44** (1991) 3159.

[8] K. Hayashi and T. Shirafuji, Progr. Theor. Phys. **64** (1980) 866.

[9] T. Fujishiro, M.J. Hayashi and S. Takeshita, Mod. Phys. Lett. **A8** (1993) 491.

[10] S. Coleman and E. Weinberg, Phys. Rev. **D7** (1973) 1888.

[11] E. Elizalde and S.D. Odintsov, Phys. Lett B (1993), to appear.

[12] M.B. Einhorn and D.R.T. Jones, Nucl. Phys. **B211** (1983) 29; K. Yamagishi, Nucl. Phys. **B216** (1983) 508; G.B. West, Phys. Rev. **D27** (1983) 1402.

[13] M. Sher, Phys. Rep. **179** (1989) 273.

[14] B.L. Voronov and I.V. Tyutin, Yad. Fiz. (Sov. J. Nucl. Phys.) **23** (1976) 664; I.L. Buchbinder and S.D. Odintsov, Yad. Fiz. (Sov. J. Nucl. Phys.) **40** (1984) 1338; I.L. Buchbinder and I.L. Shapiro, Izv. VUZov. Fiz. (Sov. Phys. J.) **No8** (1985) 94.

[15] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, *Effective Action in Quantum Gravity*, IOP Publishing, Bristol and Philadelphia, 1992.
[16] I.L. Buchbinder, S.D. Odintsov and I.L. Shapiro, Izv. VUZov. Fiz. (Sov. Phys. J.) No3 (1987) 3; S.D. Odintsov, Fortschr. Phys. 39 (1991) 621.

[17] H. Alhendi, Phys. Rev. D37 (1988) 3749.

[18] S.D. Odintsov, preprint HUPD-9305 (1993).

[19] C. Ford and D.R.T. Jones, Phys. Lett. B 274 (1992) 409; C. Ford, D.R.T. Jones, P.W. Stephenson and M.B. Einhorn, preprint LTH288 (1992).

[20] T. Bunch and L. Parker, Phys. Rev. D20 (1979) 2499.

[21] I. Antoniadis and E. Mottola, Phys. Rev. D45 (1992) 2013; S.D. Odintsov., Z. Phys. C45 (1992) 531; I. Antoniadis, P.O. Mazur and E. Mottola, Nucl. Phys. B388 (1992) 627.

[22] I. Antoniadis and S.D. Odintsov, preprint CPTH-A213 (1992), Mod. Phys. Lett. A, to appear.

[23] E. Elizalde and S.D. Odintsov, preprints HUPD-92-10 and UB-ECM-PF 92/29 (1992).

[24] R. Percacci, preprint in preparation.
Figure captions

**Figure 1.** The renormalization-group improved effective potential \((y = V/\mu^4)\) corresponding to the \(\lambda\varphi^4\) theory in a curved spacetime with torsion, eqs. (7) and (11), as a function of \(x = \varphi^2/\mu^2\). In Fig. 1a, \(\zeta\) is positive. In Fig. 1b, \(\zeta\) is negative.

**Figure 2.** The function \(y = F(x)\), eqs. (30) and (27), corresponding to the effective potential for infrared quantum gravity with torsion for different values of the constants (31). All them have been taken between 0 and 1, except for \(c\) in Figs. 2a and 2b, which is negative and of order 10. When torsion varies, the typical evolution of the symmetry breaking potential is obtained.