Structure of the field equations
in $N = 1$ chiral supergravity

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Abstract. We study the structure of linearized field equations in $N = 1$ chiral supergravity (SUGRA) with a complex tetrad, as a preliminary to introducing additional auxiliary fields in order that the supersymmetry (SUSY) algebra close off shell. We follow the first-order formulation we have recently constructed using the method of the usual $N = 1$ SUGRA. In particular, we see how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. Starting from the linearized (free) theory of $N = 1$ chiral SUGRA, we then construct a Lagrangian which is invariant under local SUSY transformations to zeroth order of the gravitational constant, and compare the results with the linearized field equations.
1. Introduction

Right- and left-handed supersymmetry (SUSY) transformations in $N = 1$ chiral supergravity (SUGRA) with a complex tetrad were introduced by Jacobson [1, 2], and their first-order formulation was then constructed using the two-form gravity [3, 4]. In a previous paper [5], following the method used in the usual $N = 1$ SUGRA [6, 7], we presented the explicit form of the first-order SUSY transformations in $N = 1$ chiral SUGRA for complex field variables; a complex tetrad $e^i_\mu$, a self-dual connection $A^{(+)}_{ij\mu} = A^{(+)}_{[ij\mu]}$ which satisfies $(1/2)\varepsilon^{ijkl}A^{(+)}_{kl\mu} = iA^{(+)}_{ij\mu}$, and two independent (Majorana) Rarita-Schwinger fields $\psi_{R\mu}$ and $\tilde{\psi}_{R\mu}$. The SUSY transformation parameters are not constrained at all in contrast with the method of the two-form gravity. We showed that the SUSY algebra for $N = 1$ chiral SUGRA closes only on shell.

The sum of the right- and left-handed SUSY transformations, however, is not twice the usual $N = 1$ SUGRA in the following sense: The sum of the right- and left-handed SUSY transformations for the complex tetrad is

$$\delta e^i_\mu = -i(\bar{\psi}_{R\mu} \gamma^i \alpha_R + \bar{\psi}_{L\mu} \gamma^i \tilde{\alpha}_L)$$

(1.1)

with $\alpha$ and $\tilde{\alpha}$ being two anticommuting Majorana spinor parameters. Here both $\bar{\psi}_{R\mu} \gamma^i \alpha_R$ and $\bar{\psi}_{L\mu} \gamma^i \tilde{\alpha}_L$ are complex. The real and imaginary parts of (1.1) cannot be written as the form

$$\text{Re}(\delta e^i_\mu) = -i\Phi^1_\mu \gamma^i \beta^1, \quad \text{Im}(\delta e^i_\mu) = -i\Phi^2_\mu \gamma^i \beta^2$$

(1.2)

by field redefinition from $\psi_\mu$ and $\tilde{\psi}_\mu$ to appropriate Majorana spinors $\Phi^1_\mu$ and $\Phi^2_\mu$, accompanied by corresponding change of $\alpha$ and $\tilde{\alpha}$ to Majorana spinor parameters $\beta^1$ and $\beta^2$. Instead, we have

$$\text{Re}(\delta e^i_\mu) = -\frac{i}{2}(\bar{\psi}_\mu \gamma^i \alpha + \bar{\psi}_\mu \gamma^i \tilde{\alpha}),$$

(1.3)

* We assume $\psi_\mu$ and $\tilde{\psi}_\mu$ to be two independent (Majorana) Rarita-Schwinger fields, and define the right-handed spinor fields $\psi_{R\mu} := (1/2)(1 + \gamma_5)\psi_\mu$ and $\tilde{\psi}_{R\mu} := (1/2)(1 + \gamma_5)\tilde{\psi}_\mu$. The $\psi_{R\mu}$ and $\tilde{\psi}_{R\mu}$ relate to the left-handed spinor fields $\psi_{L\mu}$ and $\tilde{\psi}_{L\mu}$ respectively, because $\psi_\mu$ and $\tilde{\psi}_\mu$ are Majorana spinors. The antisymmetrization of a tensor with respect to $i$ and $j$ is denoted by $A_{[ij]\cdots[j]} := (1/2)(A_{i\cdots j} - A_{j\cdots i})$. We shall follow the notation and convention of [4].
\[
\text{Im}(\delta e^i_\mu) = -\frac{i}{2}(\overline{\psi}_\mu \gamma^i \alpha + \overline{\psi}'_\mu \gamma^i \hat{\alpha}),
\]

(1.4)

where \( \overline{\psi}'_\mu := \overline{\psi}_\mu \exp\left\{ (-i\pi \gamma_5)/2 \right\} \) and \( \overline{\psi}'_\mu := \overline{\psi}_\mu \exp\left\{ (i\pi \gamma_5)/2 \right\} \). Both (1.3) and (1.4) contain two kinds of Majorana spinor parameters.

Therefore it seems non-trivial to introduce additional auxiliary fields which will make the SUSY algebra of \( N = 1 \) chiral SUGRA closed off shell. The full non-linear theory with auxiliary fields of the usual \( N = 1 \) SUGRA can be constructed from its linearized theory, making the rigid SUSY transformations local and adding appropriate terms to the free Lagrangian order-by-order in the gravitational constant \( \kappa \). This suggests that if we can introduce additional auxiliary fields at linearized level, the full nonlinear theory with auxiliary fields will be constructed also for \( N = 1 \) chiral SUGRA. Motivated by this expectation, we consider the structure of the linearized field equations in \( N = 1 \) chiral SUGRA. In particular, we see how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. We shall then modify the linearized (free) Lagrangian so that it be invariant under local SUSY transformations up to order \( \kappa^0 \), and show that the modified Lagrangian correctly reproduces the field equations of \( N = 1 \) chiral SUGRA in the weak field approximation.

This paper is organized as follows. In section 2 we define a real Lagrangian from chiral one which is assumed to be analytic in the complex field variables, and derive the field equations for the real and imaginary parts of the complex tetrad. The chiral Lagrangian of matter fields includes massless Majorana spin-1/2 and spin-3/2 fields. In section 3 we apply the weak field approximation to the field equations derived from the real Lagrangian. The explicit form of energy-momentum tensors is calculated for (Majorana) Rarita-Schwinger fields. In section 4 we construct a local SUSY invariant Lagrangian to order \( \kappa^0 \), and compare the resultant field equations with the linearized field equations in \( N = 1 \) chiral SUGRA. In section 5 we present our conclusion. We summarize the identities derived from general coordinate and local Lorentz invariances of the chiral Lagrangian in the appendix.

\[\text{† The } \kappa^2 \text{ is the Einstein constant: } \kappa^2 = 8\pi G/c^4. \text{ Unless stated otherwise, we use units } c = 1 = \kappa^2.\]
2. Real Lagrangian

In local field theory, the spinor field $\psi$ usually appears in the kinetic Lagrangian forming a bilinear product with its own Dirac conjugate $\bar{\psi}$. However, in $N = 1$ chiral SUGRA, we must use the kinetic Lagrangian formed of the bilinear product of the two independent spinor fields $\psi_R\mu$ and $\bar{\psi}_{R\mu}$ in order to make the SUSY transformations compatible with the complex tetrad. Therefore, to recover the Hermiticity of the kinetic Lagrangian, we add the complex conjugate, $L^{(+)}$, to the chiral Lagrangian density $L^{(+)}$: Namely, we define the real Lagrangian density,

$$L := L^{(+)} + \text{c.c.},$$

(2.1)

where $L^{(+)}$ is the sum of the chiral gravitational Lagrangian density and the chiral Lagrangian density of matter fields, and “c.c.” means “the complex conjugate of the preceding term”. The chiral gravitational Lagrangian density constructed from the complex tetrad and the self-dual connection is

$$L^{(+)}_G = -\frac{i}{2} e^{\mu\nu\rho\sigma} e_i \mu e_j \nu R^{(+)}_{ij\rho\sigma},$$

(2.2)

where $e$ denotes det($e^i_\mu$) and the curvature of self-dual connection $R^{(+)}_{ij\mu\nu}$ is

$$R^{(+)}_{ij\mu\nu} := 2(\partial_{[\mu} A^{(+)}_{ij\nu]} + A^{(+)}_{[i\mu} A^{(+)}_{kj\nu]}).$$

(2.3)

In order to discuss the structure of linearized field equations as generally as possible, we suppose that the chiral Lagrangian density of matter fields take the form

$$L^{(+)}_M := L^{(+)}_M [e, A^{(+)}, \bar{\Psi}_R, D^{(+)}_\mu \Psi_R],$$

(2.4)

which is analytic in the complex field variables, $e^i_\mu$, $A^{(+)}_{ij\mu}$, $\Psi_R$ and $\bar{\Psi}_R$. Here $\Psi_R$ and $\bar{\Psi}_R$ are independent of each other, and denote collectively the matter fields; the $D^{(+)}_\mu$ means the covariant derivative with respect to $A^{(+)}_{ij\mu}$.

$$D^{(+)}_\mu := \partial_\mu + \frac{i}{2} A^{(+)}_{ij\mu} S^{ij}. $$

(2.5)

\(^4\) For pure gravity case, Peldán \[9\] used the real Lagrangian to derive the real Ashtekar-like theory based on the Lie-algebra $SO(3,1)$, denoting the real Lagrangian density by $L^{\text{tot}}$. 
The $\mathcal{L}_M^{(+)}$ of (2.4) is invariant under general coordinate and local Lorentz transformations, but its SUSY invariance is not necessarily satisfied.

The matter fields under consideration include massless Majorana spin-1/2 and spin-3/2 fields. For massless Majorana spin-1/2 fields, the chiral Lagrangian density is

$$\mathcal{L}_M^{(+)} = \frac{e}{6} \varepsilon^{\mu\nu\rho\sigma} \overline{\psi}_R^\dagger \gamma_{\nu\rho\sigma} D_\mu \psi_R$$

with $\gamma_{\nu\rho\sigma} := \gamma_{[\nu} \gamma_\rho \gamma_{\sigma]}$. For (Majorana) Rarita-Schwinger fields, we have

$$\mathcal{L}_{RS}^{(+)} = -e \varepsilon^{\mu\nu\rho\sigma} \overline{\psi}_R^\dagger \gamma_\rho D_\sigma \psi_R.$$ (2.7)

Note that the $\mathcal{L}_M^{(+)}$ of (2.4) changes by phase under global chiral transformations,

$$\Psi_R(x) \rightarrow \exp(i \gamma_5 \theta) \Psi_R(x) = \exp(i \theta) \Psi_R(x),$$

$$\overline{\Psi}_R(x) \rightarrow \overline{\Psi}_R(x) \exp(i \gamma_5 \theta') = \overline{\Psi}_R(x) \exp(-i \theta'),$$ (2.9)

where $\theta$ and $\theta'$ are real constant parameters. Accordingly, we define the chiral Lagrangian density $\mathcal{L}^{(+)}$ as

$$\mathcal{L}^{(+)} = \mathcal{L}_G^{(+)} + e^{i \varphi} \mathcal{L}_M^{(+)}$$, (2.10)

with $\varphi$ being a real constant parameter. Then $\varphi$ is changed like $\varphi \rightarrow \varphi + (\theta - \theta')$ under the chiral transformations (2.8) and (2.9).

Let us express the real gravitational Lagrangian density,

$$\mathcal{L}_G := \mathcal{L}_G^{(+)} + \text{c.c.},$$ (2.11)

by using only real variables. We define the real $SO(3,1)$ connection by

$$\omega_{ij\mu} := A_{ij\mu}^{(+)} + \text{c.c.}$$ (2.12)

Then the $R^{(+)}ij_{\mu\nu}$ of (2.3) becomes

$$R^{(+)}ij_{\mu\nu} = \frac{1}{2} \left( R^{ij}_{\mu\nu}[\omega] - \frac{i}{2} \varepsilon_{ijkl} R^{kl}_{\mu\nu}[\omega] \right),$$ (2.13)

§ The totally antisymmetrization of a tensor with respect to $i,j$ and $k$ is denoted by $A_{[ijk]} := (1/3)(A_{ij} + A_{kj} + A_{ji})$. 

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where \( R^{ij}_{\mu\nu} [\omega] \) is the curvature of the real connection \( \omega_{ij\mu} \). Further we decompose the complex tetrad into the real and imaginary parts:

\[
e^i_\mu = V^i_\mu + iW^i_\mu. \tag{2.14}
\]

The \( \mathcal{L}_G \) of (2.11) can then be written as

\[
\mathcal{L}_G = -\frac{e}{4} \epsilon^\mu_{\nu\rho\sigma} (V^i_\mu V^j_\nu - W^i_\mu W^j_\nu) \epsilon^{ijkl} R_{kl\rho\sigma} [\omega] + e \epsilon^\mu_{\nu\rho\sigma} V^i_\mu W^j_\nu R_{ij\rho\sigma} [\omega]. \tag{2.15}
\]

The real Lagrangian density of matter fields, \( \mathcal{L}_M \), can be written by using \( V^i_\mu \), \( W^i_\mu \) and \( \omega_{ij\mu} \) as

\[
\mathcal{L}_M := e^{i\varphi} \mathcal{L}_M^{(\pm)} + c.c.
\]

\[
= \mathcal{L}_M [V, W, \omega, \overline{\Psi}, D_\mu [\omega] \Psi], \tag{2.16}
\]

where \( D_\mu [\omega] \) denotes the covariant derivative with respect to \( \omega_{ij\mu} \):

\[
D_\mu [\omega] := \partial_\mu + \frac{i}{2} \omega_{ij\mu} S^{ij}. \tag{2.17}
\]

Note that \( \overline{\Psi} \) and \( \Psi \) appear in (2.16) instead of \( \overline{\Psi}_R \) and \( \Psi_R \). In fact, the real Lagrangian density of massless Majorana spin-1/2 fields can be expressed as

\[
\mathcal{L}_{1/2} = \frac{e}{6} \epsilon^\mu_{\nu\rho\sigma} \{ V_{ijk}^{\nu\rho\sigma} (e^{i\varphi} \overline{\Psi}_R \gamma_{ijk} D_\mu [\omega] \Psi_R - e^{-i\varphi} \overline{\Psi}_L \gamma_{ijk} D_\mu [\omega] \Psi_L ) + W_{ijk}^{\nu\rho\sigma} (e^{i(\varphi + \frac{\pi}{2})} \overline{\Psi}_R \gamma_{ijk} D_\mu [\omega] \Psi_R - e^{-i(\varphi + \frac{\pi}{2})} \overline{\Psi}_L \gamma_{ijk} D_\mu [\omega] \Psi_L ) \}. \tag{2.18}
\]

with \( V_{ijk}^{\nu\rho\sigma} \) and \( W_{ijk}^{\nu\rho\sigma} \) being defined by

\[
V_{ijk}^{\nu\rho\sigma} := V^{[i}_\nu (V^j_\rho V^k]_\sigma - 3W^j_\rho W^k]_\sigma), \tag{2.19}
\]

\[
W_{ijk}^{\nu\rho\sigma} := W^{[i}_\nu (3V^j_\rho V^k]_\sigma - W^j_\rho W^k]_\sigma). \tag{2.20}
\]

For massless (Majorana) Rarita-Schwinger fields, the real Lagrangian density is

\[
\mathcal{L}_{RS} = -e \epsilon^\mu_{\nu\rho\sigma} \{ V^i_\rho (e^{i\varphi} \overline{\Psi}_R \gamma_i D_\sigma [\omega] \Psi_R - e^{-i\varphi} \overline{\Psi}_L \gamma_i D_\sigma [\omega] \Psi_L ) + W^i_\rho (e^{i(\varphi + \frac{\pi}{2})} \overline{\Psi}_R \gamma_i D_\sigma [\omega] \Psi_R - e^{-i(\varphi + \frac{\pi}{2})} \overline{\Psi}_L \gamma_i D_\sigma [\omega] \Psi_L ) \}. \tag{2.21}
\]

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From (2.15) and (2.16), we can derive the field equations for $V^i_\mu$, $W^i_\mu$ and $\omega^\mu_{ij}$. Varying $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_M$ with respect to $V^i_\mu$ and $W^i_\mu$ yields

$$e^\mu_{\rho\sigma} (W^j_\nu R_{ij\rho\sigma} [\omega] - \frac{1}{2} V^j_\nu \epsilon_{ij}^{kl} R_{kl\rho\sigma} [\omega]) + e \ T^{(1)}_i \mu = 0,$$

$$e^\mu_{\rho\sigma} (V^j_\nu R_{ij\rho\sigma} [\omega] + \frac{1}{2} W^j_\nu \epsilon_{ij}^{kl} R_{kl\rho\sigma} [\omega]) + e \ T^{(2)}_i \mu = 0,$$

respectively. We note that there appear two energy-momentum tensors of matter fields, $T^{(1)}_i \mu$ and $T^{(2)}_i \mu$, defined by

$$T^{(1)}_i \mu := e^{-1} \frac{\delta \mathcal{L}_M}{\delta V^i_\mu}, \quad T^{(2)}_i \mu := e^{-1} \frac{\delta \mathcal{L}_M}{\delta W^i_\mu}.$$

These are related to the complex energy-momentum tensor,

$$T^{(+)}_i \mu := e^{-1} \frac{\delta \mathcal{L}_M^{(+)}}{\delta e^i_\mu},$$

by

$$T^{(1)}_i \mu - iT^{(2)}_i \mu = 2T^{(+)}_i \mu,$$

due to the Cauchy-Riemann relation. For $\omega^\mu_{ij}$, we have the field equation,

$$e^\mu_{\rho\sigma} D^\rho [\omega] \text{Im} H^{(+)}_i e^\mu_{ij} + \frac{e}{2} S^{ij} = 0,$$

where the spin tensor of matter fields is defined by

$$S^{ij} := -2e^{-1} \frac{\delta \mathcal{L}_M}{\delta \omega^\mu_{ij}},$$

and $\text{Im} H^{(+)}_i e^\mu_{ij}$ is the imaginary part of $H^{(+)}_i e^\mu_{ij} := e^{i [\mu} e^{j \nu]} - (i/2) \epsilon^{ijkl} e^k_{[\mu} e^l_{\nu]}$. Note also that the $S^{ij}$ of (2.28) is connected with the self-dual spin tensor,

$$S^{(+)ij} := -2e^{-1} \frac{\delta \mathcal{L}_M^{(+)}}{\delta A^{(+)}}.$$

by

$$S^{ij} = S^{(+)ij} + \text{c.c.}.$$
The relation between $T^{(+)}_{\mu}^{\mu}$ and $S^{(ij\mu)}_{\mu}$ is shown in the appendix (see (A.5)).

If we impose the reality condition, $W_{\mu}^{i} = 0$ and $\overline{\Psi} = \overline{\Psi}$, then the two real constant parameters $\theta$ and $\theta'$ in (2.3) and (2.9) are equal to each other, i.e., $\theta = \theta'$, so the $\mathcal{L}_M^{(+)}$ of (2.4) becomes strictly invariant under the global chiral transformations. Thus we choose the phase factor $\varphi = 0$. Then, in the case of spin-1/2 and spin-3/2 fields, substituting the solution of (2.27) into (2.22) and (2.23) yields the ordinary Einstein equation and the Bianchi identity respectively.

3. Weak field approximation

To see how the real and imaginary parts of the complex tetrad are coupled to matter fields, we apply the weak field approximation to the field equations (2.22), (2.23) and (2.27), assuming that $V_{\mu}^{i}$ and $W_{\mu}^{i}$ satisfy

$$V_{\mu}^{i} = \delta_{\mu}^{i} + a_{\mu}^{i}, \quad |a_{\mu}^{i}| \ll 1,$$
$$W_{\mu}^{i} = b_{\mu}^{i}, \quad |b_{\mu}^{i}| \ll 1.$$  \hfill (3.1) \hfill (3.2)

It is convenient to decompose the self-dual connection $A_{\muij}^{(+)}$ as

$$A_{\muij}^{(+)} = A_{\muij}^{(+)}[a] + K_{\muij}^{(+)}$$  \hfill (3.3)

and to take $K_{\muij}^{(+)}$ as an independent variable instead of $A_{\muij}^{(+)}$. Here $A_{\muij}^{(+)}[a]$ is the self-dual part of the Ricci rotation coefficients $A_{\muij}[a]$. Since we need not distinguish Latin indices from Greek ones in the weak field approximation, we use Greek indices in the rest of this paper, which are raised and lowered with the Minkowski metric tensor $\eta_{\mu\nu}$. Now the real $SO(3,1)$ connection $\omega_{\mu\nu\lambda}$ can be expressed in terms of $a_{\mu\nu}$, $b_{\mu\nu}$ and $K_{\mu\nu\lambda}^{(+)}$ as follows:

$$\omega_{\mu\nu\lambda} = \omega_{\mu\nu\lambda}^{[a]} + \frac{1}{2} \epsilon_{\rho\sigma\tau\mu\nu} \omega_{\rho\sigma\tau\lambda}^{[b]} + K_{\mu\nu\lambda}^{(+)}$$  \hfill (3.4)

where

$$\omega_{\mu\nu\lambda}^{[a]} := -\partial_{\lambda}a_{\mu\nu} - (\partial_{\mu}a_{\nu\lambda} - \partial_{\nu}a_{\mu\lambda}),$$
$$\omega_{\mu\nu\lambda}^{[b]} := -\partial_{\lambda}b_{\mu\nu} - (\partial_{\mu}b_{\nu\lambda} - \partial_{\nu}b_{\mu\lambda}),$$  \hfill (3.5) \hfill (3.6)
and $K_{\mu\nu\lambda}$ is defined by

$$K_{\mu\nu\lambda} := K_{\mu\nu\lambda}^{(+)} + \text{c.c.} \quad (3.7)$$

Therefore, the curvature $R_{\mu\nu\rho\sigma}[\omega]$ can be written as

$$R_{\mu\nu\rho\sigma}[\omega] = R_{\mu\nu\rho\sigma}[a] + \frac{1}{2} \epsilon^{\alpha\beta} R_{\alpha\beta\rho\sigma}[b] + 2 \partial_{[\rho} K_{\mu\nu\sigma]} , \quad (3.8)$$

where

$$R_{\mu\nu\rho\sigma}[a] := 2 \partial_{[\rho} \omega_{\mu\nu\sigma]} [a] , \quad (3.9)$$

$$R_{\mu\nu\rho\sigma}[b] := 2 \partial_{[\rho} \omega_{\mu\nu\sigma]} [b] , \quad (3.10)$$

which are linear in $a_{(\mu\nu)}$ and $b_{(\mu\nu)}$ respectively.

Substituting (3.8) into (2.22) and (2.23), we can get the field equations in the weak field approximation. In (2.22), the term of $\epsilon^{\mu\nu\rho\sigma} W^j_{\nu} R^i_{\nu\rho\sigma}[\omega]$ can be neglected, and the term proportional to $R_{\mu\nu\rho\sigma}[b]$ vanishes due to the Bianchi identity. Therefore (2.22) becomes

$$G_{\mu\nu}[a] + \partial^\rho K_{\mu\nu\rho} + \partial_{\mu} v_{\nu} - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} T^{(1)}_{\mu\nu} \quad (3.11)$$

with $v_\mu := K_{\mu\nu}$. Here $G_{\mu\nu}[a]$ is the linearized Einstein tensor for $a_{\mu\nu}$,

$$G_{\mu\nu}[a] = - \{ \Box a_{(\mu\nu)} - \partial^\rho (\partial_{\mu} a_{(\nu\rho)} + \partial_{\nu} a_{(\mu\rho)} ) + \eta_{\mu\nu} \partial^\rho \partial^\sigma a_{(\rho\sigma)} \} \quad (3.12)$$

with

$$a_{(\mu\nu)} := a_{(\mu\nu)} - \frac{1}{2} \eta_{\mu\nu} a , \quad a := \eta^{\mu\nu} a_{(\mu\nu)} , \quad (3.13)$$

and the d’Alembertian $\Box$ being defined by $\Box := \partial^\mu \partial_\mu$. Similarly, from (2.23) we have

$$G_{\mu\nu}[b] + \epsilon_{\lambda\rho\sigma\nu} \partial^\lambda K_{\mu\nu}^{\rho\sigma} = - \frac{1}{2} T^{(2)}_{\mu\nu} , \quad (3.14)$$

where $G_{\mu\nu}[b]$ is the linearized Einstein tensor for $b_{\mu\nu}$. Further, the field equation (2.27) becomes

$$- K_{\rho[\mu\nu]} + \eta_{\rho[\mu} v_{\nu]} = \frac{1}{4} S_{\mu\nu\rho} . \quad (3.15)$$

We take the energy-momentum tensors, $T^{(1)}_{\mu\nu}$ and $T^{(2)}_{\mu\nu}$, and the spin tensor $S_{\mu\nu\rho}$ to lowest order in $a_{\mu\nu}$, $b_{\mu\nu}$ and $K_{\mu\nu\rho}$: Namely, they are independent of $a_{\mu\nu}$, $b_{\mu\nu}$ and $K_{\mu\nu\rho}$, and satisfy the conservation law,

$$\partial^\nu T^{(1)}_{\mu\nu} = 0 , \quad \partial^\nu T^{(2)}_{\mu\nu} = 0 . \quad (3.16)$$
and the Tetrode formula in special relativity,

\[ T^{(1)}_{\mu\nu} - \frac{1}{2} \epsilon_{\mu\nu}^{\rho\sigma} T^{(2)}_{\rho\sigma} = \partial^\rho S_{\mu\nu\rho}, \]  

(3.17)
as is shown in the appendix (see (A.8) and (A.9)). Note that the antisymmetric parts of \( T^{(1)}_{\mu\nu} \) and \( T^{(2)}_{\mu\nu} \) are related with each other by

\[ 2T^{(1)}_{\mu\nu} = -\epsilon_{\mu\nu}^{\rho\sigma} T^{(2)}_{\rho\sigma}. \]  

(3.18)

The linearized field equations (3.11), (3.14) and (3.15) are invariant under the gauge transformations,

\[ a_{(\mu\nu)} \to a'_{(\mu\nu)} = a_{(\mu\nu)} + \partial_\mu \Lambda_1^{\nu}\]  

(3.19)\n
\[ a_{[\mu\nu]} \to a'_{[\mu\nu]} = a_{[\mu\nu]} + \epsilon_{\mu\nu}^1, \]  

(3.20)\n
and

\[ b_{(\mu\nu)} \to b'_{(\mu\nu)} = b_{(\mu\nu)} + \partial_\mu \Lambda_2^{\nu} + \partial_\nu \Lambda_2^{\mu}, \]  

(3.21)\n
\[ b_{[\mu\nu]} \to b'_{[\mu\nu]} = b_{[\mu\nu]} + \epsilon_{\mu\nu}^2, \]  

(3.22)

where \( \Lambda_a^{\mu} \) and \( \epsilon^{a}_{\mu\nu} = \epsilon^{a}_{[\mu\nu]} (a = 1, 2) \) are arbitrary four and six functions, respectively. These transformations are the linearized version of complex general coordinate and complex local Lorentz transformations. Equations (3.21) and (3.22) mean that \( a_{[\mu\nu]} \) and \( b_{[\mu\nu]} \) can be eliminated. By means of the gauge freedom we can put the harmonic condition,

\[ \partial^\rho \overline{a}_{(\mu\nu)} = 0, \quad \partial^\rho \overline{b}_{(\mu\nu)} = 0. \]  

(3.23)

Then the remaining degrees of freedom are 6 for each \( a_{\mu\nu} \) and \( b_{\mu\nu} \), and the linearized Einstein tensors for \( a_{\mu\nu} \) and \( b_{\mu\nu} \) are written as

\[ G_{\mu\nu}[a] = -\Box \overline{a}_{(\mu\nu)}, \quad G_{\mu\nu}[b] = -\Box \overline{b}_{(\mu\nu)}. \]  

(3.24)

Equation (3.11) is decomposed into the symmetric and antisymmetric parts as

\[ G_{\mu\nu}[a] + \partial^\rho K_{\rho(\mu\nu)} + \partial_{(\mu} v_{\nu)} - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} T^{(1)}_{(\mu\nu)}, \]  

(3.25)\n
\[ -\partial^\rho K_{\rho[\mu\nu]} + \partial_{[\mu} v_{\nu]} = \frac{1}{2} T^{(1)}_{[\mu\nu]} \]  

(3.26)
and similarly (3.14) into
\[ G_{\mu\nu}[b] + \epsilon_{\lambda\rho\sigma(\mu} \partial^{\lambda} K_{\nu)^\rho\sigma} = -\frac{1}{2} T_{(\mu\nu)}^{(2)}, \]
(3.27)
\[ -\epsilon_{\lambda\rho\sigma(\mu} \partial^{\lambda} K_{\nu)^\rho\sigma} = -\frac{1}{2} T_{[\mu\nu]}^{(2)}, \]
(3.28)
Both sides of (3.11) and (3.14) are divergenceless with respect to \( \nu \), and the divergence of (3.15) with respect to \( \rho \) yields (3.26) and (3.28) due to the Tetrode formula (3.17). Therefore, there are \( 16 + 16 + 24 - (4 + 4 + 6 + 6) = 36 \) independent equations for \( 6 + 6 + 24 = 36 \) independent variables, \( a_{(\mu\nu)}, b_{(\mu\nu)} \) and \( K_{\mu\nu\rho} \).

We can rewrite (3.25) and (3.27) with the help of (3.15). Taking the divergence of (3.15) with respect to \( \mu \) yields
\[ \partial^\rho K_{\rho(\mu\nu)} + \partial_{(\mu} v_{\nu)} - \eta_{\mu\nu} \partial^\rho v_\rho = \frac{1}{2} \partial^\rho S_{(\rho(\mu\nu)}, \]
(3.29)
or equivalently
\[ \epsilon_{\lambda\rho\sigma(\mu} \partial^{\lambda} K_{\nu)^\rho\sigma} = -\frac{1}{2} \partial^\rho S_{\ast(\rho(\mu\nu)}, \]
(3.30)
with the \ast \) operation being the duality operation
\[ S_{\ast\mu\nu\rho} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} S_{\alpha\beta\rho}. \]
(3.31)
Using (3.29) and (3.30) in (3.25) and (3.27) respectively, we get
\[ G_{\mu\nu}[a] = \frac{1}{2} T_{\mu\nu}^{(1)(\text{sym})}, \]
(3.32)
\[ G_{\mu\nu}[b] = -\frac{1}{2} T_{\mu\nu}^{(2)(\text{sym})}, \]
(3.33)
where \( T_{\mu\nu}^{(1)(\text{sym})} \) and \( T_{\mu\nu}^{(2)(\text{sym})} \) are the symmetrized energy-momentum tensors:
\[ T_{\mu\nu}^{(1)(\text{sym})} = T_{\mu\nu}^{(1)} - \frac{1}{2} \partial^\rho (S_{\mu\nu\rho} + S_{\rho\mu\nu} + S_{\rho\nu\mu}), \]
(3.34)
\[ T_{\mu\nu}^{(2)(\text{sym})} = T_{\mu\nu}^{(2)} - \frac{1}{2} \partial^\rho (S_{\mu\nu\rho} + S_{\rho\mu\nu} + S_{\rho\nu\mu}). \]
(3.35)

Now we restrict ourselves to the case of (Majorana) Rarita-Schwinger fields. From the real Lagrangian density (2.21), \( T_{\mu\nu}^{(1)} \) and \( T_{\mu\nu}^{(2)} \) can be obtained as
\[ \begin{pmatrix} T_{\mu\nu}^{(1)} \\ T_{\mu\nu}^{(2)} \end{pmatrix} = \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} T_{\mu\nu}^{(1)} \big|_{\varphi=0} \\ T_{\mu\nu}^{(2)} \big|_{\varphi=0} \end{pmatrix}, \]
(3.36)
where

\[
T^{(1)}_{\mu\nu}|_{\varphi=0} = -\epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}^R\gamma^\mu\partial^\rho\psi^R_R - \bar{\psi}^L\gamma^\mu\partial^\rho\psi^L_L)
= -\epsilon_{\lambda\rho\sigma\nu}\bar{\psi}\gamma_5\gamma^\mu\partial^\rho\psi^\sigma,
\]

\[
T^{(2)}_{\mu\nu}|_{\varphi=0} = -i\epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}^R\gamma^\mu\partial^\rho\psi^R_R + \bar{\psi}^L\gamma^\mu\partial^\rho\psi^L_L)
= -i\epsilon_{\lambda\rho\sigma\nu}\bar{\psi}\gamma^\mu\partial^\rho\psi^\sigma.
\]  

(3.37)

Thus, \(T^{(1)}_{\mu\nu}\) and \(T^{(2)}_{\mu\nu}\) are mixed as the parameter \(\varphi\) changes. Since both of \(\psi_\mu\) and \(\bar{\psi}_\mu\) are Majorana spinors, the \(T^{(1)}_{\mu\nu}|_{\varphi=0}\) of (3.37) can be diagonalized as

\[
T^{(1)}_{\mu\nu}|_{\varphi=0} = \epsilon_{\lambda\rho\sigma\nu}(\bar{\psi}^1\gamma_5\gamma^\mu\partial^\rho\psi_{1\sigma} - \bar{\psi}^2\gamma_5\gamma^\mu\partial^\rho\psi_{2\sigma}),
\]

(3.39)

where \(\psi^1_\mu := (1/2)(\psi_\mu + \bar{\psi}_\mu)\) and \(\psi^2_\mu := (1/2)(\psi_\mu - \bar{\psi}_\mu)\). The minus sign in (3.39) means the appearance of negative energy: Namely, the positivity of \(T^{(1)}_{\mu\nu}|_{\varphi=0}\) is not guaranteed. On the other hand, the \(T^{(2)}_{\mu\nu}|_{\varphi=0}\) of Eq. (3.38) is not diagonalized by using \(\psi^1_\mu\) and \(\psi^2_\mu\). However we have

\[
T^{(2)}_{\mu\nu}|_{\varphi=0}[\psi, \bar{\psi}] = T^{(1)}_{\mu\nu}|_{\varphi=\pi/2}[\psi, \bar{\psi}] = T^{(1)}_{\mu\nu}|_{\varphi=0}[\psi', \bar{\psi}],
\]

\[
T^{(1)}_{\mu\nu}|_{\varphi=0}[\psi, \bar{\psi}] = -T^{(2)}_{\mu\nu}|_{\varphi=\pi/2}[\psi, \bar{\psi}] = -T^{(2)}_{\mu\nu}|_{\varphi=0}[\psi', \bar{\psi}],
\]  

(3.40)

(3.41)

where we define \(\psi'_\mu := \exp\{(i\pi\gamma_5)/2\}\psi_\mu\). Therefore \(T^{(2)}_{\mu\nu}|_{\varphi=0}[\psi, \bar{\psi}]\) is not positive definite, either.

If we impose the reality condition, \(b_{(\mu\nu)} = 0\) and \(\bar{\psi}_\mu = \psi_\mu\), together with \(\varphi = 0\), then the \(S^{*}_{\mu\nu\rho}\) of (3.31) can be calculated as

\[
S^{*}_{\mu\nu\rho} = -i\epsilon^{\alpha\beta}_{\rho\mu\nu}\bar{\psi}\gamma^{[\alpha}\gamma^{\beta]}\psi_\rho,
\]

(3.42)

by virtue of which the symmetrized energy-momentum tensor \(T^{(2)(sym)}_{\mu\nu}\) of (3.35) vanishes and the positivity of \(T^{(1)(sym)}_{\mu\nu}\) is recovered.
4. Local SUSY invariant Lagrangian to order $\kappa^0$

In the usual $N = 1$ SUGRA, the full nonlinear theory can be constructed from the linearized (free) theory, making the rigid SUSY transformations local and adding appropriate terms to the free Lagrangian order-by-order in the gravitational constant $\kappa$ \[8, 10\]. For example, one can construct a local SUSY invariant Lagrangian to order $\kappa^0$ by adding an interaction term proportional to the energy-momentum tensor. Similarly, we can obtain a local SUSY invariant Lagrangian to order $\kappa^0$ also in $N = 1$ chiral SUGRA as we shall explain below. Hereafter we shall write the $\kappa$ explicitly. In the linearized theory, the free Lagrangian of $N = 1$ chiral SUGRA is twice the usual $N = 1$ SUGRA: Namely, taking $\varphi = 0$ for simplicity, the free field limit of the real Lagrangian density, $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{RS}$, is

$$L^0 = L^0_G + L^0_{RS},$$

(4.1)

where the linearized gravitational Lagrangian, $L^0_G$, is

$$L^0_G = -(a^{(\mu\nu})G_{\mu\nu}[a] - b^{(\mu\nu})G_{\mu\nu}[b])$$

(4.2)

and the free Lagrangian of (Majorana) Rarita-Schwinger fields, $L^0_{RS}$, is

$$L^0_{RS} = \epsilon^{\mu\nu\rho\sigma} \bar{\psi}_\mu \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu,$$

(4.3)

which can be diagonalized as

$$L^0_{RS} = \epsilon^{\mu\nu\rho\sigma} (\bar{\psi}_\mu^1 \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu^1 - \bar{\psi}_\mu^2 \gamma_5 \gamma_\rho \partial_\sigma \psi_\nu^2)$$

(4.4)

up to a total divergence term, where $\psi^1_\mu$ and $\psi^2_\mu$ are defined below (3.39).

The linearized theory of $N = 1$ chiral SUGRA possesses local gauge invariance and rigid SUSY invariance just like the usual $N = 1$ SUGRA: Indeed, the $L^0$ of (4.1) is invariant under local gauge transformations,

$$\delta a^{(\mu\nu)} = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x),$$

$$\delta b^{(\mu\nu)} = \partial_\mu \eta_\nu(x) + \partial_\nu \eta_\mu(x),$$

$$\delta \psi_\mu = \partial_\mu \epsilon(x),$$

$$\delta \bar{\psi}_\mu = \partial_\mu \bar{\epsilon}(x).$$

(4.5)
In contrast with the usual $N = 1$ SUGRA, on the other hand, $L^0$ is invariant under two kinds of rigid SUSY transformations. One is the following rigid SUSY transformations with supersymmetric partners being $(a(\mu\nu), \psi_1^{\mu})$ and $(b(\mu\nu), \psi_2^{\mu})$:

$$
\delta a(\mu\nu) = -i \overline{\psi}_1^{\mu} \gamma_\nu \alpha^1, \quad \delta \psi_1^{\mu} = -2i S^{\rho\sigma} \partial_\rho a(\sigma\mu) \alpha^1,
$$

$$
\delta b(\mu\nu) = -i \overline{\psi}_2^{\mu} \gamma_\nu \alpha^2, \quad \delta \psi_2^{\mu} = -2i S^{\rho\sigma} \partial_\rho b(\sigma\mu) \alpha^2,
$$

(4.6)

where $\alpha^1$ and $\alpha^2$ are constant Majorana spinor parameters. In this case, if we make $\alpha^1$ and $\alpha^2$ local and add appropriate terms to the $L^0$ of (4.1) order-by-order in $\kappa$, then the resultant Lagrangian density is twice that of the usual $N = 1$ SUGRA: Namely, we have

$$
\mathcal{L}^1 = \mathcal{L}_{N=1\text{SUGRA}}^1 - \mathcal{L}_{N=1\text{SUGRA}}^2.
$$

The other is the rigid SUSY transformations,

$$
\delta a(\mu\nu) = -i \overline{\psi}_2^{\mu} \gamma_\nu \alpha^1, \quad \delta \psi_1^{\mu} = -2i S^{\rho\sigma} \partial_\rho a(\sigma\mu) \alpha^1,
$$

$$
\delta b(\mu\nu) = \frac{1}{2} (\overline{\psi}_1^{\mu} (\gamma_5) \gamma_\nu \alpha - \overline{\psi}_2^{\mu} (\gamma_5) \gamma_\nu \alpha^1),
$$

$$
\delta \psi_2^{\mu} = -2i S^{\rho\sigma} \partial_\rho a(\sigma\mu) + i \gamma_5 \partial_\rho b(\sigma\mu) \alpha,
$$

$$
\delta \psi_1^{\mu} = -2i S^{\rho\sigma} \partial_\rho b(\sigma\mu) - i \gamma_5 \partial_\rho a(\sigma\mu) \bar{\alpha},
$$

(4.7)

with $\alpha$ and $\bar{\alpha}$ being constant Majorana spinor parameters. Although the gauge and SUSY transformations (4.5) and (4.7) form a closed algebra on shell, the rigid SUSY transformations (4.7) are not twice the usual $N = 1$ SUGRA.

Let us construct a local SUSY invariant Lagrangian to order $\kappa^0$. If the spinor parameters $\alpha$ and $\bar{\alpha}$ in (4.7) become spacetime dependent, i.e., $\alpha = \alpha(x)$ and $\bar{\alpha} = \bar{\alpha}(x)$, then the $L^0$ of (4.1) is no longer invariant: The variation of $L^0$ can be expressed as

$$
\delta L^0 = (\partial_\mu \bar{\alpha})(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]) + (\partial_\mu \alpha)(\bar{J}^{1\mu}[a, \bar{\psi}] + \bar{J}^{2\mu}[b, \bar{\psi}]),
$$

(4.8)

up to total divergence, because $L^0$ is invariant when $\alpha$ and $\bar{\alpha}$ are constant. However, $J^{1\mu}[a, \psi]$, $J^{2\mu}[b, \psi]$, $\bar{J}^{1\mu}[a, \bar{\psi}]$ and $\bar{J}^{2\mu}[b, \bar{\psi}]$ are not uniquely fixed: As example, we shall explain how the ambiguity arises in $J^{1\mu}[a, \psi]$. The variation of $L^0$ with respect to the transformations (4.7) is

$$
\delta L^0 = (\partial_\mu a(\lambda\nu)) \epsilon^{\mu\nu\rho\sigma} \bar{\alpha} \gamma_5 \gamma^\lambda \partial_\rho \psi_\sigma + \cdots.
$$

(4.9)

If the spinor parameter $\bar{\alpha}$ is constant, the first term of (4.9) is just a total divergence. However, when $\bar{\alpha}$ is spacetime dependent, this term can be rewritten as

$$
(\partial_\mu a(\lambda\nu)) \epsilon^{\mu\nu\rho\sigma} \bar{\alpha} \gamma_5 \gamma^\lambda \partial_\rho \psi_\sigma\]
satisfies (4.15), we modify the Lagrangian as
\[ \psi \] 

the gauge transformations with the identification, \( \epsilon \), which is invariant to order \( N \)
\[ \text{Note that similar ambiguity also appears in the usual manner, we get} \]
\[ J^{1\mu}[a, \psi] = -pa_{(\lambda \nu)}\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \partial_\rho \psi_\sigma + (1 - p)(\partial_\mu a_{(\lambda \nu)})\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \psi_\sigma + \cdots \] (4.11)
\]

with \( p \) being an arbitrary real constant. Although the two terms proportional to \( p \)
in (4.10) are combined to a total divergence, they do lead to non-trivial ambiguity in \( J^{1\mu}[a, \psi] \). We can explicitly show that the remaining terms of (4.9) do not lead to any ambiguity in \( J^{1\mu}[a, \psi] \). Thus we have
\[ J^{1\mu}[a, \psi] = -pa_{(\lambda \nu)}\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \partial_\rho \psi_\sigma + (1 - p)(\partial_\mu a_{(\lambda \nu)})\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \psi_\sigma + \cdots \] (4.11)
Note that similar ambiguity also appears in the usual \( N = 1 \) SUGRA. In the same manner, we get
\[ J^{2\mu}[b, \psi] = iq_b(\lambda \nu)\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \partial_\rho \psi_\sigma - i(1 - q)(\partial_\mu b_{(\lambda \nu)})\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \psi_\sigma + \cdots \] (4.12)
\[ \tilde{J}^{1\mu}[a, \tilde{\psi}] = -p' a_{(\lambda \nu)}\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \partial_\rho \tilde{\psi}_\sigma + (1 - p')(\partial_\mu a_{(\lambda \nu)})\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \tilde{\psi}_\sigma + \cdots \] (4.13)
\[ \tilde{J}^{2\mu}[b, \tilde{\psi}] = -iq' b_{(\lambda \nu)}\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \partial_\rho \tilde{\psi}_\sigma + i(1 - q')(\partial_\mu b_{(\lambda \nu)})\epsilon^{\mu\rho\sigma\gamma_5 \gamma^\lambda} \tilde{\psi}_\sigma + \cdots \] (4.14)

with \( p', q \) and \( q' \) being arbitrary real constants.

Inspection of (4.8) shows that the invariance under transformations with local spinor parameter \( \bar{\alpha}(x) \) is recovered to order \( \kappa^0 \) if the interaction term, \((-\kappa/2)\bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi])\), is added to \( L^0 \), and if we simultaneously make the gauge transformation of \( \tilde{\psi}_\mu \) with \( \bar{\epsilon}(x) = (2/\kappa)\bar{\alpha}(x) \). On the other hand, the interaction term, \((-\kappa/2)\bar{\psi}_\mu(\tilde{J}^{1\mu}[a, \tilde{\psi}] + \tilde{J}^{2\mu}[b, \tilde{\psi}])\), is needed if the spinor parameter \( \alpha \) becomes spacetime dependent. Since these two interaction terms must be the same, we require
\[ \bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]) = \bar{\psi}_\mu(\tilde{J}^{1\mu}[a, \tilde{\psi}] + \tilde{J}^{2\mu}[b, \tilde{\psi}]). \] (4.15)
Then the constants \( p, p', q \) and \( q' \) are uniquely determined as \( p = p' = 2 \) and \( q = q' = 2 \). Since \( J^{1\mu}[a, \psi], J^{2\mu}[b, \psi], \tilde{J}^{1\mu}[a, \tilde{\psi}] \) and \( \tilde{J}^{2\mu}[b, \tilde{\psi}] \) are now fixed and satisfies (4.13), we modify the Lagrangian as
\[ L^1 = L^0 - \frac{\kappa}{2}\bar{\psi}_\mu(J^{1\mu}[a, \psi] + J^{2\mu}[b, \psi]), \] (4.16)
which is invariant to order \( \kappa^0 \) under the local SUSY transformations combined with the gauge transformations with the identification, \( \epsilon(x) = (2/\kappa)\alpha(x) \) and \( \bar{\epsilon}(x) = (2/\kappa)\bar{\alpha}(x) \).
The interaction term in the $L^1$ of (4.16) can be rewritten as $\kappa (a^{(\mu \nu)} T^{(1)(\text{sym})}_{\mu \nu} + b^{(\mu \nu)} T^{(2)(\text{sym})}_{\mu \nu})$. Thus (3.32) and (3.33) are derived from $L^1$ by taking variation with respect to $a^{(\mu \nu)}$ and $b^{(\mu \nu)}$ respectively. It should also be noted that the correction term of $L^1$ is not twice the usual $N=1$ SUGRA, because $\psi_\mu$ and $\bar{\psi}_\mu$ are contained both in $T^{(1)}_{\mu \nu}$ and $T^{(2)}_{\mu \nu}$ as shown in (3.37) and (3.38).

5. Conclusion

We have studied the structure of linearized field equations in the chiral formulation of gravity with the complex tetrad, and seen how the real and imaginary parts of the complex tetrad are coupled to matter fields in the weak field approximation. As an example, the explicit form of energy-momentum tensors has been calculated for (Majorana) Rarita-Schwinger fields. Starting from the linearized (free) theory of $N=1$ chiral SUGRA, we have then obtained the Lagrangian which is invariant under local SUSY transformations to order $\kappa^0$. The resultant Lagrangian just reproduces the field equations in the weak field approximation. We expect that the full nonlinear theory of $N=1$ chiral SUGRA can be constructed by adding appropriate terms order-by-order in $\kappa$ to the first-order Lagrangian we have obtained. We are also trying to introduce additional auxiliary fields into the linearized theory of $N=1$ chiral SUGRA as the preliminary step to construct the full nonlinear theory with auxiliary fields of $N=1$ chiral SUGRA.

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Appendix

In this appendix, we derive identities from general coordinate and local Lorentz invariances of the chiral Lagrangian density

$$\mathcal{L}^{(+)} = \mathcal{L}^{(+)}[q_\mu; q_{\mu,\nu}; e_{k\mu}; e_{k\mu,\nu}; A^{(+)}_{ij\mu}; A^{(+)}_{ij\mu,\nu}], \quad (A.1)$$

where matter fields are collectively denoted by $q_\mu$. An arbitrary variation of $\mathcal{L}^{(+)}$ is

$$\delta \mathcal{L}^{(+)} = \frac{\delta \mathcal{L}^{(+)}}{\delta F} \delta F + \partial_\nu \left( \frac{\partial \mathcal{L}^{(+)}}{\partial F_{\nu}} \delta F + \mathcal{L}^{(+)} \delta x^\nu \right), \quad (A.2)$$

where $F$ denotes $q_\mu, e_{k\mu}$ and $A^{(+)}_{ij\mu}$ collectively, and $\delta$ is the Lie derivative defined by $\delta F = \delta F - F_\mu \delta x^\nu$.

If the $\mathcal{L}^{(+)}$ is invariant under general coordinate and local Lorentz transformations, we have Noether's identity,

$$\frac{\delta \mathcal{L}^{(+)}}{\delta F} \delta F + \partial_\nu \left( \frac{\partial \mathcal{L}^{(+)}}{\partial F_{\nu}} \delta F + \mathcal{L}^{(+)} \delta x^\nu \right) \equiv 0. \quad (A.3)$$

For complex local Lorentz transformations, variation of the fields is given by

$$\overline{\delta} q_\mu = \frac{i}{2} \epsilon_{ij} S^{ij}_q q_\mu, \quad \overline{\delta} e_{k\mu} = \epsilon_{k}^{\prime} e_{l\mu}$$

$$\overline{\delta} A^{(+)}_{ij\mu} = \epsilon_{i}^{\prime} A^{(+)}_{kji\mu} + \epsilon_{j}^{\prime} A^{(+)}_{ikj\mu} - \epsilon_{ij}^{(+)} A^{(+)}_{k\mu} \quad (A.4)$$

where $\epsilon_{ij} = \epsilon_{[ij]}$ is an arbitrary complex parameter. Here $\epsilon_{ij}^{(+)} := (1/2)\{\epsilon_{ij} - (i/2)\epsilon_{ij}^{kl}\epsilon_{kl}\}$, and it is independent of $\epsilon_{ij}^{(-)} := (1/2)\{\epsilon_{ij} + (i/2)\epsilon_{ij}^{kl}\epsilon_{kl}\}$. Using $(A.4)$ in $(A.3)$ with $\mathcal{L}^{(+)} = \mathcal{L}^{(+)}_M$ yields

$$\frac{1}{2} \left( T^{(+)[ij]} - \frac{i}{2} \epsilon_{ij}^{kl} T^{(+)[kl]} \right) - \frac{1}{2} D^{(+)}_\nu S^{(+)[ij\nu]} + \frac{\delta \mathcal{L}^{(+)}}{\delta q_\mu} \frac{i}{2} S^{(+)[ij]} q_\mu \equiv 0 \quad (A.5)$$

and

$$\frac{1}{2} \left( T^{(+)[ij]} + \frac{i}{2} \epsilon_{ij}^{kl} T^{(+)[kl]} \right) + \frac{\delta \mathcal{L}^{(+)}}{\delta q_\mu} \frac{i}{2} S^{(-)[ij]} q_\mu \equiv 0 \quad (A.6)$$
with $T_{ij}^{(+)} = e_{ij} T_{i}^{(+)}{}_{\mu}$. Here $T_{i}^{(+)}{}_{\mu}$ and $S_{ij}^{(+)}{}_{\mu}$ denote $e T_{i}^{(+)}{}_{\mu}$ and $e S_{ij}^{(+)}{}_{\mu}$ with $T_{i}^{(+)}{}_{\mu}$ and $S_{ij}^{(+)}{}_{\mu}$ being defined by (2.25) and (2.29) respectively. Similarly, the identity for general coordinate invariance of $L_{M}^{(+)}$ can be written as

$$\partial_{\nu} T_{\mu}^{(+)} + e^{i}_{\nu, \mu} T_{i}^{(+)} + A_{ij}^{(+)} T_{ij}^{(+)} + 1 \frac{R_{ij}^{(+)} S_{ij}^{(+)} \mu}{2} + \frac{\delta L_{M}^{(+)} D_{\mu}^{(+)} q_{\nu}}{\delta q_{\nu}} + \partial_{\nu} \left( \frac{\delta L_{M}^{(+)} q_{\nu}}{\delta q_{\nu}} \right) \equiv 0,$$

(A.7)

by using (A.5).

With the help of matter field equations, the identities (A.5) and (A.6) become, in the special relativistic limit,

$$T_{[\mu \nu]}^{(+)} + i \frac{\epsilon_{\rho \sigma} T_{\rho \sigma}^{(+)} \equiv \partial^{\rho} S_{\mu \rho}^{(+)} + \partial_{[\mu}^{\rho} T_{\nu]}^{(+)} \equiv 0, \quad (A.8)$$

while (A.7) gives the conservation law,

$$\partial^{\nu} T_{\mu \nu}^{(+)} \equiv 0.$$

(A.9)

Since $T_{i}^{(+)}{}_{\mu}$ and $S_{ij}^{(+)}{}_{\mu}$ are related to $T_{i}^{(1)}{}_{\mu}$, $T_{i}^{(2)}{}_{\mu}$ and $S_{ij}{}_{\mu}$ by (2.26) and (2.30), we obtain the conservation law (3.16) from (A.9), and further the relations (3.17) and (3.18) from (A.8).
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