The Fritzsch Ansatz Revisited

W. K. Sze
Department of Physics, National Taiwan Normal University, Taipei, Taiwan 116
(Dated: June 29, 2018)

A modified Fritzsch ansatz for the quark mass matrices is proposed to account for the hierarchical structure of the CKM matrix. To allow for the observed CP asymmetry, restrictions have to be imposed on the relative phase degree of freedom among the weak eigenstates of the quark fields, as for example by certain additional symmetries. The ansatz can be accommodated in extensions to the Standard Model, such as the multiple Higgs doublets models.

PACS numbers: 12.15.Ff, 12.15.Hh, 12.60.Fr.
Keywords: Fritzsch ansatz, Quark masses and mixing, CKM matrix.

I. INTRODUCTION

It is an empirical fact that the left-handed quark mixing matrix (the CKM matrix) $V$ differs from the unit matrix only slightly. This is manifest in Wolfenstein’s parameterization of $V$:

$$
V = \begin{pmatrix}
1 - \frac{1}{2} |d| A^2 & \lambda A^3 (\rho - i\eta) \\
-\lambda & 1 - \frac{1}{2} |d| A^2 & A \lambda^2 \\
A \lambda^2 (1 - \rho - i\eta) & -A \lambda^2 & 1
\end{pmatrix},
$$

(1)

which is valid to leading orders of the small parameter $\lambda = 0.220$. Since it is known that the parameters $A$, $\rho$, and $\eta$ are all of order one, successive off-diagonal elements of $V$ are of ascending orders of $\lambda$. In this parameterization, the CP-violating phases are assigned to the elements $V_{ub}$ and $V_{cd}$, with their phases being denoted as $-\gamma$ and $-\beta$, respectively, where $\beta$ and $\gamma$ are angles appearing in the $d-b$ unitarity triangle.

One may speculate that this hierarchical structure of $V$ has a deeper meaning connected with the hierarchy of the quark masses, in which case one should be able to put $V$ in a form which reflects this connection. An early attempt to establish such a relation was the ansatz proposed by Fritzsch, in which it is assumed that only the $b$ and $t$ quarks have non-zero bare masses, the effect of which are cascaded through inter-generation couplings to give masses to the lighter quarks. In the context of the Standard Model (SM), various phases in the mass matrices can be absorbed into the quark fields. Hence one has to impose additional constraints among the quark phases, in order that the ansatz be consistent with the observed CP asymmetry. Even then, the numerical relations among the quark mass ratios and mixing angles as predicted by the ansatz enjoy only a partial success.

In this paper, we will show that with certain modification the difficulty of the ansatz can be circumvented, while maintaining its prediction of a hierarchical CKM matrix. To illustrate our point, let us put $V$ in a factorized form, expressing it as a product of successive rotations about the principal axes. We notice that $V_{\text{fac}}$ as given below is of a form similar to Eq. (1):

$$
V_{\text{fac}} = \begin{pmatrix}
1 - \frac{1}{2} |k_u| A^2 & k_u \lambda & 0 \\
-k_u^* \lambda & 1 - \frac{1}{2} |k_u| A^2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & A \lambda^2 \\
0 & -A \lambda^2 & 1
\end{pmatrix}
\begin{pmatrix}
1 - \frac{1}{2} |k_d| A^2 & k_d \lambda & 0 \\
-k_d^* \lambda & 1 - \frac{1}{2} |k_d| A^2 & 0 \\
0 & 0 & 1
\end{pmatrix},
$$

(2)

In the first and third factors of $V_{\text{fac}}$, phase factors are absorbed in the complex parameters $k_u$ and $k_d$, respectively. On the second line we have dropped terms of orders higher than $\lambda^2$ in the upper-left $2 \times 2$ block of $V_{\text{fac}}$, consistent with what is done in Wolfenstein’s parameterization. Furthermore we have denoted $k \equiv k_u + k_d$ and $\delta \equiv -\frac{1}{2} i (k_d k_u^* - k_u k_d^*) A^2 = \text{Im}(k_d k_u^*) A^2$. Since $k_u$ and $k_d$ are of order one (see below), the contributions of $\mp i \delta$ to the magnitudes $|V_{ud}|$ and $|V_{cs}|$ respectively cannot exceed $O(\lambda^4)$. If we ignore small phases of $V_{ud}$ and $V_{cs}$, as was done in Eq. (1), then these $i \delta$ terms can be dropped as well.

If we choose $k = k_u + k_d$ to be unity, which amounts to identifying the variables $\lambda$ and $A$ in Eq. (1) and (2),
then comparing the two equations we get $V = V_{\text{fac}}$ with:

\begin{align*}
k_u &= |k_u| e^{-i\gamma_u} = \rho - i\eta, \\
k_d &= |k_d| e^{i\beta_d} = 1 - \rho + i\eta.
\end{align*}

(3)

If the factorized form \[2\] of $V$ is of any physical significance, it provides a hint to the quark mass matrices in the weak basis. We will show that it is indeed consistent with a variant form of the Fritzsch ansatz.

A brief review of the Fritzsch ansatz, as well as a recapitulation of its short-comings, are given in the next section. In Section III we present a modified ansatz which is free of such difficulties and would yield $V$ as given by Eq. \[2\]. In Section IV the parameters in the quark mass matrix elements are estimated from empirical data. We give our conclusion in the final section.

II. A BRIEF REVIEW

In terms of the quark fields $\Psi_U \equiv (u, c, t)^T$ and $\Psi_D \equiv (d, s, b)^T$, the mass terms in the Lagrangian are:

\[ L_m = -(\Psi_{U,L} M_U \Psi_{U,R} + \Psi_{D,L} M_D \Psi_{D,R}) + \text{h.c.,} \]

(4)

where as usual $\Psi_{I,L}$ and $\Psi_{I,R} (I = U, D)$ denotes the left- and right-handed fields $\frac{1}{2}(1 + \gamma_5)\Psi_I$, respectively. If $\Psi_U$ and $\Psi_D$ are expressed in the weak basis, as is implicitly assumed above, the mass matrices $M_U$ and $M_D$ would in general not be diagonal.

The numerical coincidence $\sqrt{m_d/m_s} \simeq \lambda$ was noticed in the early days. As an endeavor to predict this relation, Fritzsch proposed an ansatz for $M_U$ and $M_D$.\[3\] Suppose these to be Hermitian matrices, for two generations of quarks the ansatz reads:

\[ M_U = \begin{pmatrix} 0 & A_U \\ A_U^* & B_U \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & A_D \\ A_D^* & B_D \end{pmatrix}, \]

(5)

where $B_I (I = U, D)$ are real and positive. If we assume that $|A_I| \ll B_I$, we would then obtain

\[ \lambda \simeq \sqrt{\frac{m_d}{m_s}} - \sqrt{\frac{m_u}{m_c}} \simeq 0.22, \]

(6)

which fits well with the experimental value for $\lambda$.

The interpretation for Eq. \[5\] is that only the heaviest quarks have non-zero bare masses, and there are couplings among quarks of neighboring generations. The lighter quarks get their masses only by cascading effects through these inter-generational couplings. Accordingly the Fritzsch ansatz for three generations of quarks is:

\[ M_I = \begin{pmatrix} 0 & A_I \\ A_I^* & 0 \\ 0 & B_I \end{pmatrix}, \quad \text{where } I = U \text{ or } D. \]

(7)

Here we have $|A_I| \ll |B_I| \ll C_I$, with the real parameters $C_I$ assumed to be real and positive.

The ansatz \[7\] did give predictions in qualitative agreement with empirical facts. But as eventually more experimental data for the heavy quark masses and mixing matrix elements such as $V_{ub}$ and $V_{cb}$ are gathered, it is realized that the success of the relation \[6\] does not extend to the three-generation case.

Another problem is with the observed $CP$-violation. With so many vanishing elements in $M_U$ and $M_D$, one can normally utilize the phase degree of freedom of the quark fields to put them in real forms, and the resulting quark mixing matrix would have been real as well. The observed $CP$ asymmetry would then have to be accounted for by other mechanism, which is hard to come by given the present experimental data.\[7\] Hence restrictions must be imposed on these phase ambiguities, as was implicitly stated in Fritzsch original ansatz.\[8\]

III. A MODIFIED FRITZSCH ANSATZ

Failure of the Fritzsch ansatz \[7\] can be traced to the Hermitian nature of the mass matrices, which is a very stringent condition on the magnitudes of the matrix elements. A natural extension would then be relaxing the Hermiticity assumption for $M_U$ and $M_D$, while retaining their general texture as given by Eq. \[4\].

As mentioned in the previous section, in order to get a complex CKM matrix, we should require that $M_U$ and $M_D$ be essentially complex, i.e., they cannot both be rendered real by a redefinition of the phases of the quark weak eigenstates. We assume that the only global phase symmetry among the quark weak eigenstates is:

\[ \Psi_L = \begin{pmatrix} \Psi_{U,L} \\ \Psi_{D,L} \end{pmatrix}, \quad \Psi_U \rightarrow P_U \Psi_U, \quad \Psi_D \rightarrow P_D \Psi_D, \]

(8a)

where the diagonal phase matrices $P_U$ and $P_D$ assume the following restrictive form:

\[ P_U = \text{diag}(e^{i\alpha}, e^{i\alpha'}, e^{i\alpha''}), \quad P_D = \text{diag}(e^{i\chi}, e^{i\chi'}, e^{i\chi''}). \]

(8b)

Briefly put, the transformation \[8\] is: $\Psi_U \rightarrow P_U \Psi_U, \quad \Psi_D \rightarrow P_D \Psi_D$.

In the SM, there is a much richer phase degree of freedom for the quark fields than that given by Eq. \[8\]. Its restriction to Eq. \[8\] therefore requires extensions to the SM. One example would be the Two Higgs Doublets Model, in which the up-type quarks get their masses from a scalar doublet $\phi_1$, while the down-type quarks get theirs from another scalar doublet $\phi_2$, and the two different sets of (quark and scalar) fields respect two distinct $U(1)$ or higher symmetries, with each field carrying non-trivial quantum number. Each of the scalar doublets $\phi_1$...
and $\phi_2$ may conceivably be replaced by a set of doublets, if the stringent constraints on FCNC can be met. We will not elaborate upon these possibilities beyond the requirement $[\mathbf{8}]$ in the following discussions.

A. The Two-generation Case

To begin with, we will illustrate our argument for the requirement (8) in the following discussions.

We will assume that only $B_I$ ($I = U, D$) are real and positive, while $A_I$ and $A'_I$ are distinct complex numbers. The Fritzsch-like condition $|A_I|, |A'_I| \ll B_I$ is still supposed to hold. The Eq. [\mathbf{8}] can be recast as the following transformation on these mass matrix elements:

\begin{align*}
A_U &\to e^{i\Delta\alpha} A_U, & A'_U &\to e^{-i\Delta\alpha} A'_U, \\
A_D &\to e^{i\Delta\chi} A_D, & A'_D &\to e^{-i\Delta\chi} A'_D, \\
B_U &\text{ and } B_D \text{ unchanged.}
\end{align*}

Here we have denoted $\Delta\alpha = \alpha' - \alpha$ and $\Delta\chi = \chi' - \chi$. In general we need to carry out bi-unitary transformations on $M_U$ and $M_D$ to put them in diagonal form.

Such diagonalization procedures are well known. We will nonetheless briefly re-state the steps to fix our phase convention. We know that, any $2 \times 2$ Hermitian matrix

\[M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}\]

can be diagonalized $M \to M_{\text{diag}} = O^\dagger M O$ by a unit-of-modular unitary matrix:

\[O = \begin{pmatrix}
\cos \theta & e^{i\varphi} \sin \theta \\
-e^{-i\varphi} \sin \theta & \cos \theta
\end{pmatrix}.
\]

The real parameters $\theta$ and $\varphi$ can be expressed in terms of elements of $M$:

\[
tan 2\theta = \frac{2|M_{12}|}{M_{22} - M_{11}}, \quad \varphi = \text{arg} M_{12},
\]

from the assumption that $O^\dagger M O$ is diagonal.

For $M = M_U$ or $M_D$ as given by Eq. [\mathbf{8}], the transformation matrices which bi-diagonalize $M$ are those which diagonalize the Hermitian matrices $M M^\dagger$ and $M^\dagger M$. Take $M_D$ for example. We have

\[
M_D M_D^\dagger = \begin{pmatrix}
|A_D|^2 & A_D B_D \\
A_D^* B_D & |A_D|^2 + B_D^2
\end{pmatrix},
\]

\[
M_D^\dagger M_D = \begin{pmatrix}
|A'_D|^2 & A'_D^* B_D \\
A'_D B_D & |A'_D|^2 + B_D^2
\end{pmatrix}.
\]

We assume that they are diagonalized by the unitary matrices $L_D$ and $R_D$, respectively, where $L_D$ and $R_D$ are both of forms similar to Eq. [\mathbf{10}]:

\[
L_D = \begin{pmatrix}
\cos \theta_d & e^{i\varphi_d} \sin \theta_d \\
-e^{-i\varphi_d} \sin \theta_d & \cos \theta_d
\end{pmatrix},
\]

\[
R_D = \begin{pmatrix}
\cos \theta'_d & e^{i\varphi'_d} \sin \theta'_d \\
-e^{-i\varphi'_d} \sin \theta'_d & \cos \theta'_d
\end{pmatrix}.
\]

Bearing in mind the Fritzsch condition $|A_D|, |A'_D| \ll B_D$, we obtain the expressions for the parameters in $L_D$ and $R_D$:

\[
\theta_d \simeq \frac{|A_D|}{B_D}, \quad \varphi_d = + \text{arg } A_D;
\]

\[
\theta'_d \simeq \frac{|A'_D|}{B_D}, \quad \varphi'_d = - \text{arg } A'_D.
\]

The matrix $L_D^\dagger M_D R_D$ will be diagonal as desired, but the lighter mass eigenvalue $-A_D A'_D / B_D$ will in general be complex. It is desirable to absorb the extra phases in the transformation matrices $L_D$ and $R_D$ so that all the mass eigenvalues are real and positive. Thus we take, instead of $L_D$ and $R_D$, the following transformation matrices:

\[
\tilde{L}_D \equiv L_D \begin{pmatrix}
e^{i\varphi_d} & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
e^{i\varphi_d} \cos \theta_d & e^{i\varphi_d} \sin \theta_d \\
-\sin \theta_d & \cos \theta_d
\end{pmatrix},
\]

\[
\tilde{R}_D \equiv R_D \begin{pmatrix}
e^{-i\varphi'_d} & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
-\cos \theta'_d & e^{-i\varphi'_d} \sin \theta'_d \\
\sin \theta'_d & \cos \theta'_d
\end{pmatrix}.
\]

With $\tilde{L}_D$ and $\tilde{R}_D$ as given above, $M_D$ is transformed to its diagonal form $\tilde{M}_D$:

\[
\tilde{M}_D = \tilde{L}_D^\dagger M_D \tilde{R}_D.
\]

Bearing in mind that from Eq. [\mathbf{11}] we have relations such as $e^{-i\varphi_d} A_D = |A_D|$, etc., and keeping only leading terms of the small parameters $\theta_d$ and $\theta'_d$, one verifies that $\tilde{M}_D$ is diagonal:

\[
\tilde{M}_D \simeq \begin{pmatrix}
|A_D A'_D| / B_D & 0 \\
0 & B_D
\end{pmatrix} = \begin{pmatrix}
m_d & 0 \\
0 & m_s
\end{pmatrix}.
\]

The corresponding down-quark mass eigenstates are then given by:

\[
\tilde{\Psi}_{D,L} = \tilde{L}_D^\dagger \Psi_{D,L}, \quad \tilde{\Psi}_{D,R} = \tilde{R}_D^\dagger \Psi_{D,R},
\]

where $\Psi_{D,L}$ and $\Psi_{D,R}$ are the weak eigenstates appearing in Eq. [\mathbf{11}].

The Cabibbo matrix $V_2$, which measures the mis-alignment between the up- and down-type left-handed
quark eigenstates, would then be given as:

\[ V_2 = L_D^\dagger L_D \]

\[ = \begin{pmatrix} c_u e^{-i\varphi_u} & -s_u \\ s_u e^{-i\varphi_u} & c_u \end{pmatrix} \begin{pmatrix} c_d e^{i\varphi_d} & s_d e^{i\varphi_d} \\ -s_d & c_d \end{pmatrix}, \]  

(18)

where as usual the abbreviations \( c_u \equiv \cos \theta_u \), \( c_d \equiv \cos \theta_d \), etc., have been used. We will make use of the phase degree of freedom as given by Eq. 3 or 10 to adjust the phase of \( A_U \) or \( A_D \) to get \( \varphi_d = \varphi_u \). Then we have

\[ V_2 = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \]  

(19)

where \( \theta \equiv \theta_d - \theta_u \). Hence \( V_2 \) can be put in a real form, as it should with two generations of quarks.

Relations between the mixing angle \( \theta \) and the quark mass ratios can be deduced from the preceding equations. For illustration we set \( m_u \sim 0 \), in which case we have \( \theta = \theta_d \). From Eq. 14 and 10 we get

\[ \left( \frac{m_d}{m_s} \right)^\dagger = \sqrt{\frac{|A_D A_D'|}{B_D}}, \quad \text{but} \quad \theta = \frac{|A_D|}{B_D}. \]

In the traditional Fritzsch scheme, we have \( A_D = A'_D \), and the equality \( \theta = \sqrt{m_d/m_s} \) results. Although for the \( d-s \) pair this is a success, such success does not survive in the three-generation case, and the Fritzsch ansatz fails consequently.

On the other hand, we see that in this modified ansatz, the mixing angle and the square-root of the mass ratio need not be equal anymore, and the data can again be fitted. The price to pay is that now the model loses predictive power.

### B. The Three-generation Case

We next proceed to the three generation case. Our modified ansatz for the mass matrices reads:

\[ M_U = \begin{pmatrix} 0 & A_U & 0 \\ A'_U & 0 & B_U \\ 0 & B'_U & C_U \end{pmatrix}, \quad M_D = \begin{pmatrix} 0 & A_D & 0 \\ A'_D & 0 & B_D \\ 0 & B'_D & C_D \end{pmatrix}. \]  

(20)

Again we will assume that only \( C_I (I = U, D) \) are real and positive; furthermore we have \(|A_I|, |A'_I| < |B_I|, |B'_I| \ll C_I \). The phase transformation 3 can be re-stated as:

\[ (A_U, B'_U) \rightarrow e^{i\Delta \alpha} (A_U, B'_U), \]

\[ (A'_U, B_U) \rightarrow e^{-i\Delta \alpha} (A'_U, B_U), \]

\[ (A_D, B'_D) \rightarrow e^{i\Delta \chi} (A_D, B'_D), \]

\[ (A'_D, B_D) \rightarrow e^{-i\Delta \chi} (A'_D, B_D), \]

\[ \theta_1 = \frac{|A_D C_D|}{|B_D B'_D|}, \quad \phi_1 = \arg \frac{A_D}{B_D} - \pi. \]  

Here \( \Delta \alpha = \alpha' - \alpha \), etc., as in the previous subsection. These mass matrices are diagonalized in two stages. Again take \( M_D \) as example. Let:

\[ \tilde{L}_{2D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\varphi_{2d}} \cos \theta_{2d} & e^{i\varphi_{2d}} \sin \theta_{2d} \\ 0 & -\sin \theta_{2d} & \cos \theta_{2d} \end{pmatrix}, \]  

(22)

wherein the parameters are

\[ \theta_{2d} = \frac{|B_D|}{C_D}, \quad \varphi_{2d} = +\arg B_D. \]  

(23)

The right-handed quark transformation matrix \( \tilde{R}_{2D} \) is similarly defined, its parameters being \( \theta'_{2d} = |B'_D|/C_D \) and \( \varphi'_{2d} = -\arg B'_D \). With \( \tilde{L}_{2D} \) and \( \tilde{R}_{2D} \), the matrix \( M_D \) is first transformed to \( M'_D \), in which the lower-right 2 \( \times \) 2 block is diagonal:

\[ M'_D \equiv \tilde{L}_{2D}^\dagger M_D \tilde{R}_{2D} \]

\[ \simeq \begin{pmatrix} 0 & -e^{i\varphi_{2d}} A_D' & 0 \\ 0 & |B_D B'_D| C_D & 0 \\ 0 & 0 & C_D \end{pmatrix}, \]  

(24)

where terms in higher orders of the small parameters \( \theta_{2d} \) and \( \theta'_{2d} \) have been dropped.

In the subsequent diagonalization of \( M'_D \), we would have used the matrices \( \tilde{L}_{1D} \) and \( \tilde{R}_{1D} \), which effect a rotation between the first and second generations, but are otherwise defined similarly as \( \tilde{L}_{2D} \) and \( \tilde{R}_{2D} \):

\[ \tilde{M}_D \equiv \tilde{L}_{1D}^\dagger M'_D \tilde{R}_{1D}. \]

But since we would like to have the phase assignment of the final CKM matrix agreeing with Wolfenstein’s, as given by Eq. 11 or 2, we will redefine the transformation matrices as follows:

\[ \tilde{L}_{1D} \rightarrow L_{1D} \equiv \tilde{L}_{1D} P_{1D}, \quad \tilde{R}_{1D} \rightarrow R_{1D} \equiv \tilde{R}_{1D} P_{1D}, \]

where the phase matrix \( P_{1D} \equiv \text{diag}(e^{-i\varphi_{1d}}, 1, 1) \), where \( \varphi_{1d} \) is defined in Eq. 26 below. Hence \( L_{1D} \) (but not \( R_{1D} \)) is uni-modular. The final diagonalized \( M_D \) is not altered if we use \( L_{1D} \) and \( R_{1D} \) instead of \( \tilde{L}_{1D} \) and \( \tilde{R}_{1D} \). For the pair \( L_{1D} \) and \( \tilde{L}_{1D} \), the above procedure is similar to the reverse of the redefinition from Eq. 13 to 14 in the previous subsection. The explicit form for \( L_{1D} \) is:

\[ L_{1D} = \begin{pmatrix} \cos \theta_{1d} & e^{i\varphi_{1d}} \sin \theta_{1d} & 0 \\ -e^{-i\varphi_{1d}} \sin \theta_{1d} & \cos \theta_{1d} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]  

(25)

From Eq. 24, and recalling that \( \varphi'_{2d} = -\arg B'_D \), we see that the parameters in \( L_{1D} \) are given by

\[ \theta_{1d} = \frac{|A_D C_D|}{|B_D B'_D|}, \quad \phi_{1d} = \arg \frac{A_D}{B_D} - \pi. \]  

(26)
The diagonal mass matrix $\hat{M}_D$ would then be

$$
\hat{M}_D = L_{1D}^\dagger M_D^r R_{1D}
\approx \begin{pmatrix}
|A_D A_D'| C_D & 0 & 0 \\
|B_D B_D'| & 0 & 0 \\
0 & 0 & C_D
\end{pmatrix}.
$$  (27a)

Again terms of higher order in the small parameters $\theta_{1d}$ and $\theta'_{1d}$ have been dropped.

The two-step process described above amounts to the following transformation of $M_D$ to $M_D$:

$$
\tilde{M}_D = L_{1D}^\dagger \tilde{L}_{2D}^\dagger M_D \tilde{R}_{2D} \tilde{R}_{1D}
= \text{diag}(m_d, m_s, m_b).
$$  (27b)

Correspondingly the down-type quark mass eigenstates are $\tilde{\Psi}_{D,L} = L_{1D}^\dagger \tilde{L}_{2D}^\dagger \Psi_{D,L}$ and $\tilde{\Psi}_{D,R} = \tilde{R}_{1D}^\dagger \tilde{R}_{2D}^\dagger \Psi_{D,R}$.

Exactly parallel procedures apply in the diagonalization of $M_U$. With similar notations, the transformation of $M_U$ to its diagonal form $\hat{M}_U$ reads:

$$
\tilde{M}_U = L_{1U}^\dagger \tilde{L}_{2U}^\dagger M_U \tilde{R}_{2U} \tilde{R}_{1U}
= \text{diag}(m_u, m_c, m_t).
$$  (28)

The matrices $L_{1U}$, etc., are defined similar to their counterparts for the down-type quarks. For example, the explicit form of $L_{1U}$ is:

$$
L_{1U} = \begin{pmatrix}
\cos \theta_{1u} & e^{i\varphi_{1u}} \sin \theta_{1u} & 0 \\
-e^{-i\varphi_{1u}} \sin \theta_{1u} & \cos \theta_{1u} & 0 \\
0 & 0 & 1
\end{pmatrix}.
$$  (29)

where the parameters $\theta_{1u}$ and $\varphi_{1u}$ are defined similarly as in Eq. [26], with appropriate change of suffices. Likewise $L_{2U}$ is given by:

$$
\tilde{L}_{2U} = \begin{pmatrix}
1 & 0 & 0 \\
e^{i\varphi_{2u}} \cos \theta_{2u} & e^{i\varphi_{2u}} \sin \theta_{2u} & 0 \\
0 & -\sin \theta_{2u} & \cos \theta_{2u}
\end{pmatrix},
$$  (30)

with $\theta_{2u}$ and $\varphi_{2u}$ defined similarly as in Eq. [23]. The up-type quark mass eigenstates would then be given by $\tilde{\Psi}_{U,L} = L_{1U}^\dagger \tilde{L}_{2U}^\dagger \Psi_{U,L}$ and $\tilde{\Psi}_{U,R} = \tilde{R}_{1U}^\dagger \tilde{R}_{2U}^\dagger \Psi_{U,R}$.

For evaluation of the CKM matrix, only the left-handed quark fields need be considered. For these fields, the expression of the weak eigenstates in terms of the mass eigenstates are:

$$
\Psi_{U,L} = \tilde{L}_{2U} L_{1U} \tilde{\Psi}_{U,L},
$$

$$
\Psi_{D,L} = \tilde{L}_{2D} L_{1D} \tilde{\Psi}_{D,L}.
$$

The CKM matrix is thus given by

$$
V = L_{1U}^\dagger \tilde{L}_{2U}^\dagger \tilde{L}_{2D} L_{1D}.
$$  (31)

From the expression for $\tilde{L}_{2D}$ and $\tilde{L}_{2U}$ as shown in Eq. [22] and [30], we get the middle factor on the right hand side of this equation:

$$
\tilde{L}_{2U}^\dagger \tilde{L}_{2D} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{2u} e^{-i\varphi_{2u}} - s_{2u} & 0 \\
0 & s_{2u} e^{-i\varphi_{2u}} & c_{2u}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{2d} e^{i\varphi_{2d}} & s_{2d} e^{i\varphi_{2d}} \\
0 & -s_{2d} & c_{2d}
\end{pmatrix},
$$  (32)

where the abbreviations $c_{2u} \equiv \cos \theta_{2u}$, $s_{2u} \equiv \sin \theta_{2u}$, etc., have been used. We then make use of the phase ambiguity as given by Eq. [15] or [24] to adjust the phase of $B_U$ or $B_D$, so that we have $\varphi_{2d} = \varphi_{2u}$. One then verifies that Eq. [32] reduce to the following real form:

$$
\tilde{L}_{2U}^\dagger \tilde{L}_{2D} =
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_2 & \sin \theta_2 \\
0 & -\sin \theta_2 & \cos \theta_2
\end{pmatrix},
$$  (33)

where we have denoted $\theta_2 \equiv \theta_{2d} - \theta_{2u}$. Substituting Eq. [25], [29], and [33] into Eq. [31], we finally get:

$$
V =
\begin{pmatrix}
c_{1u} & -s_{1u} e^{i\varphi_{1u}} & 0 \\
s_{1u} e^{-i\varphi_{1u}} & c_{1u} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_2 & s_2 \\
0 & -s_2 & c_2
\end{pmatrix}
\begin{pmatrix}
c_{1d} & s_{1d} e^{i\varphi_{1d}} & 0 \\
-s_{1d} e^{-i\varphi_{1d}} & c_{1d} & 0 \\
0 & 0 & 1
\end{pmatrix},
$$  (34)

where again we have denoted $c_{1u} \equiv \cos \theta_{1u}$, $s_{1u} \equiv \sin \theta_{1u}$, etc., for brevity.
We see that $V$ as given above would be equal to $V_{\text{fneq}}$ in Eq. (2), if we make the following identification:

$$k_u \lambda = e^{i\varphi_d} \sin \theta_{1d}, \quad k_u \lambda = -e^{i\varphi_u} \sin \theta_{1u},$$

$$A \lambda^2 = \sin \theta_2 \equiv \sin(\theta_{2d} - \theta_{2u}).$$

(35)

The relations listed above would be ‘natural’, yielding values of $|k_u|$, $|k_d|$, and $A$ of order unity, if the angles $\theta_{1u}$, $\theta_{1d}$, and $\theta_{2d}$ are all analytic functions of $\lambda$ with leading $O(\lambda)$ term, so that $\sin(\theta_{2d} - \theta_{2u}) \equiv \theta_{2d} - \theta_{2u} = O(\lambda^2)$.

If we compare $k_u$ and $k_d$ as given by Eq. (3) and (35), we get $\beta = \varphi_{1d}$ and $\gamma = -\varphi_{1u} - \pi$, modulo $2\pi$. The phase $\varphi_{1d}$ and $\varphi_{1u}$ are given by Eq. (20) and its counterpart for the up-type quarks. Hence we get, modulo $2\pi$:

$$\beta = \arg \frac{A_D}{B_D} - \pi, \quad \gamma = - \arg \frac{A_U}{B_U}. \quad (36)$$

We see that, in this ansatz, the CP violating phases $\beta$ and $\gamma$ are determined by the phase mismatches between $A_D$ and $B_D'$, and between $B_U'$ and $A_U$, respectively. Note that the ratios $A_U/B_U'$ and $A_D/B_D'$ are invariant under the transformation (21).

IV. THE QUARK MASS MATRICES

As mentioned in the previous section, empirical data indicate that the mixing angles $\theta_{1d}$, $\theta_{2d}$, etc., are of order $\lambda$. From Eq. (23), (26), etc., we would then deduce that both $|B_D/C_D|$ and $|A_D C_D|/|B_D B_D'|$ are of order $\lambda$. Similar conclusions hold for elements of $M_U$. Hence we rewrite the Fritzsch ansatz (20) in the following alternative form, where appropriate powers of $\lambda$ are extracted from the matrix elements:

$$M_U = m_t \begin{pmatrix} 0 & a_d \lambda^3 & 0 \\ a_d' \lambda^3 & 0 & b_d \lambda \\ 0 & b_d' \lambda & 1 \end{pmatrix},$$

$$M_D = m_b \begin{pmatrix} 0 & a_u \lambda^3 & 0 \\ a_u' \lambda^3 & 0 & b_u \lambda \\ 0 & b_u' \lambda & 1 \end{pmatrix}. \quad (37)$$

Presumably the complex parameters $a_d$, etc., should be of order unity. But as we will see below, most parameters in $M_U$ are significantly smaller. Hence Eq. (37) should only be taken as a formal expansion.

Substituting Eq. (37) into relations such as Eqs. (23) and (26), we have $\theta_{2d} = |b_d| \lambda$ and $\theta_{1d} = |a_d/b_d| \lambda$, and likewise for $\theta_{2u}$ and $\theta_{1u}$. Hence we get, from Eq. (35) and (36):

$$k_u = |k_u| e^{-i\gamma} = \frac{1}{|b_u|} \frac{a_u}{b_u'},$$

$$k_d = |k_d| e^{i\beta} = -\frac{1}{|b_d|} \frac{a_d}{b_d'},$$

$$A \lambda = |b_d| - |b_u|. \quad (38)$$

The parameters $a_u'$ and $a_d'$ are not constrained by the experimental data on elements of $V$.

The quark masses provide another set of data from which the parameters can be fitted. Substituting Eq. (37) into Eqs. (24) and (25), we get the quark mass ratios in terms of the parameters $a_u$, etc.:

$$\frac{m_u}{m_t} = \frac{|a_u|}{|b_u|} \lambda^2, \quad \frac{m_c}{m_t} = \frac{|b_u|}{|b_d|} \lambda^2,$$

$$\frac{m_d}{m_t} = \frac{|a_d| |b_d|}{|b_d'|} \lambda^2, \quad \frac{m_s}{m_b} = |b_d| \lambda^2. \quad (39)$$

From Eq. (38) and (39), values of $a_u$, $a_d$, etc., can be fitted from experimental data. The parameter $\lambda$ is known to a high precision:

$$\lambda = 0.220. \quad (40)$$

The other parameters in $V$ have also been determined, albeit with less accuracies:

$$A = 0.86, \quad \rho = 0.20, \quad \eta = 0.34. \quad (41)$$

Among these, the relative uncertainties of $\lambda$ is about 5%, while those of $\rho$ and $\eta$ are significantly higher, amounting to 30% or more.

Among the input data used for fitting the values of $\rho$ and $\eta$ above is the value of $\sin 2\beta \simeq 0.74$ obtained in recent $B^0\bar{B}^0$ experiments.\footnote{The more recent $B^0\bar{B}^0$ experiments have established $\sin 2\beta = 1.06 \pm 0.06$, which is consistent with the earlier results.}

As a result we can write $\beta \simeq 23.5^\circ$. Our knowledge on the other angle $\gamma$ is less precise, the central value being $\gamma \sim 60^\circ$.

We will adopt the central values of the quark masses as given by the Particle Data Group. For the light quarks, the running mass values at the scale $\mu_0 = 2$ GeV are listed:

$$m_u(\mu_0) = 3.3 \text{ MeV}, \quad m_d(\mu_0) = 7.0 \text{ MeV},$$

$$m_s(\mu_0) = 120 \text{ MeV}. \quad (42)$$

These are the current-algebra quark masses appearing in the mass Lagrangian. They have typically rather high uncertainties, ranging from 30% to more than 50%.

For the heavy quarks $q = c, b, t$, on the other hand, the values quoted are the physical masses $\hat{m}_q$, which are defined by $m_q(\mu = \hat{m}_q) = \hat{m}_q$:

$$\hat{m}_c = 1.2 \text{ GeV}, \quad \hat{m}_b = 4.2 \text{ GeV},$$

$$\hat{m}_t = 174 \text{ GeV}. \quad (43)$$

Since relations such as Eq. (39) are to be interpreted at some fixed energy scale, the heavy quark masses have to be run to the same scale $\mu_0 = 2$ GeV before they are substituted in Eq. (39). We will take $\Lambda_{\text{QCD}} = 220 \text{ MeV}$,\footnote{The recent improvements in lattice QCD calculations have raised the scale of QCD to $\Lambda_{\text{QCD}} = 280 \text{ MeV}$.} and the heavy quark masses at $\mu_0$ are:

$$m_c(\mu_0) = 1.06 \text{ GeV}, \quad m_b(\mu_0) = 4.83 \text{ GeV},$$

$$m_t(\mu_0) = 306 \text{ GeV}. \quad (44)$$

The uncertainties of the physical heavy-quark masses range from a few percent to around 15%. Nonetheless, taking into account that $\Lambda_{\text{QCD}}$ is only known to
TABLE I: Estimates of or constraints on parameters appearing in $M_U$ and $M_D$ as given by Eq. (37).

| Parameter | Value |
|-----------|-------|
| $|a_u|$ | $0.03$, $|a_d|$ $0.5$, $|a'_u|$ $0.01$, $|a'_d|$ $0.8$, $|b_u b'_u|$ $0.07$, $|b_d b'_d|$ $0.6$, $\arg \frac{b'_u}{a_u}$ $1.0$, $\arg \frac{b'_d}{a_d}$ $2.7$, (mod 2π) |
| $|b_u| - |b_u'|$ | $0.19$. |

around 10% precision, the uncertainties of the values given by Eq. (47) can be comparable to those of the light quark masses.

From the above considerations, estimates of $|a_u|, |a_d|$, etc., as well as other constraints on these parameters can be deduced. These are listed in Table I. From discussions on the uncertainties of the mixing matrix elements and quark masses in Eq. (41)-(43), it can be inferred that typical values given in the table are also subject to variation of a few tens of percent.

From the table we see that $|a_d|$ and $|a'_d|$ are of order one, and it is reasonable to assume that $|b_u|$ and $|b'_u|$ are also of order unity. Since $|b_u|$ are close in value to $|b_d|$, it is also of this order. On the other hand, the absolute values of the other parameters $|b'_u|, |a_u|$, and $|a'_u|$ in $M_U$ are significantly smaller, reflecting the larger discrepancy in mass values of the $u, c$, and $t$ quarks.

V. CONCLUSION

The observed CKM matrix $V$ is close to the identity matrix, and has a neat hierarchical form as manifested in Wolfenstein’s parameterization. One can accept the fact as it is, or try to explore for any possible structures behind this pattern.

In this paper a modified Fritzsch ansatz for the quark mass matrices, which is motivated by the factorized form of $V$ as given by Eq. (35), is proposed. The complex matrices $M_U$ and $M_D$ have the same general texture of those in the original Fritzsch ansatz, but are neither symmetric nor Hermitian. It was shown that, if $M_U$ and $M_D$ have the hierarchical pattern as given by Eq. (37), with values of parameters as listed on Table I, the resulting mixing matrix $V$ is in agreement with the observed one.

We see from Table I that $|a_d| = O(1)$, and it is also plausible that $|b_d|, |b'_d| = O(1)$. From Eq. (48) it then follows naturally that $|k_d|$ is of order unity. The fact that $|A| = O(1)$ is also easily acceptable, from the same equation, by noting that the two order-one parameters $b_d$ and $b'_d$ have a difference in magnitudes of order $\lambda$. On the other hand, the fact that $|k'_u| = |1 - k_d| = \sqrt{\rho^2 + \eta^2} = O(1)$ depends on the two small parameters $|a_u|, |b'_u| = O(10^{-2})$ having a ratio close to one. This may seem accidental, but we have noted that the phase of $a_u/b'_u$ is invariant under the transformation (8), and it is conceivable that there is some related mechanism which ensures that the magnitude $|a_u/b'_u|$ will not become too small, even when both $|a_u|$ and $|b'_u|$ do.

Within the context of the SM, one can utilize the phase degree of freedom of the weak eigenstates of the quark fields to remove all phases in $M_U$ and $M_D$ as given by Eq. (20) in the first place. It follows then that, in a certain sense, the ansatz mandates new physics beyond the SM, in which the quark phase ambiguities is more restrictive as in Eq. (8). As mentioned in the beginning of Section III, natural candidates which come to one’s mind are the multiple Higgs doublets models, although other possibilities should not be ruled out.

[1] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 652 (1973).
[2] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983).
[3] H. Fritzsch, Phys. Lett. B73, 317 (1978).
[4] The Particle Data Group: S. Eidelman et al., Phys. Lett. B592, 1 (2004).
[5] B. Aubert et al., Phys. Rev. Lett. 89, 201802 (2002); K. Abe et al., Belle-CONF-0353, LP ’03 (2003).