Computational search of small point sets with small rectilinear crossing number

Ruy Fabila-Monroy$^1$  Jorge López$^2$

$^1$Departamento de Matemáticas, Cinvestav. México.
$^2$Escuela de Física y Matemáticas, Instituto Politécnico Nacional. México.

Abstract

Let \( \text{cr}(K_n) \) be the minimum number of crossings over all rectilinear drawings of the complete graph on \( n \) vertices in the plane. In this paper we prove that \( \text{cr}(K_n) < 0.380473(n^4) + \Theta(n^3) \); improving thus on the previous best known upper bound. This is done by obtaining new rectilinear drawings of \( K_n \) for small values of \( n \), and then using known constructions to obtain arbitrarily large good drawings from smaller ones. The “small” sets were found using a simple heuristic detailed in this paper.
1 Introduction

A rectilinear drawing of a graph is a drawing of the graph in the plane in which all the edges are drawn as straight line segments. For a set $S$ of $n$ points in general position in the plane, let $\text{cr}(S)$ be the number of (interior) edge crossings in a rectilinear drawing of the complete graph $K_n$ with vertex set $S$. The rectilinear crossing number of $K_n$, denoted by $\text{cr}(K_n)$, is the minimum of $\text{cr}(S)$ over all sets of $n$ points in general position in the plane. The problem of bounding the rectilinear crossing number of $K_n$ is an important problem in combinatorial geometry. Most of the progress has been made in the last decade, for a state-of-the-art survey see [4]. Since two edges cross if and only if their endpoints span a convex quadrilateral, $\text{cr}(S)$ is equal to the number $\Box(S)$, of convex quadrilaterals spanned by $S$. We use this equality extensively throughout the paper. The current best bounds for $\text{cr}(K_n)$ are $[3, 1]$: $0.379972 \left(\frac{n}{4}\right) < \text{cr}(K_n) < 0.380488 \left(\frac{n}{4}\right) + \Theta(n^3)$

Our main result is the following improvement of the upper bound.

Theorem 1

$$\text{cr}(K_n) \leq \frac{9363184}{24609375} \left(\frac{n}{4}\right) + \Theta(n^3) < 0.380473 \left(\frac{n}{4}\right) + \Theta(n^3)$$

Although it is a modest improvement, we note that the gap between the lower and upper bound is already quite small and that actually the lower bound is conjectured to be at least $0.380029 \left(\frac{n}{4}\right) + \Theta(n^3)$. In [1] the authors conjecture that every optimal set is 3-decomposable$^1$ and show that every 3-decomposable set contains at least $0.380029 \left(\frac{n}{4}\right) + \Theta(n^3)$ crossings. The current general approach to produce rectilinear drawings of $K_n$ with few crossings, is to start with a drawing with few crossings (for a small value of $n$), and use it to recursively obtain drawings with few number of crossings for arbitrarily large values of $n$. This approach has been refined and improved over the years [10, 7, 5, 2, 1].

The upper bound provided by the best recursive construction to this date is expressed in Theorem 2.

Theorem 2 (Theorem 3 in [1]) If $S$ is an $m$-element point set in general position, with $m$ odd, then

$$\text{cr}(K_n) \leq \frac{24\text{cr}(S) + 3m^3 - 7m^2 + (30/7)m}{m^3} \left(\frac{n}{4}\right) + \Theta(n^3)$$

Given these recursive constructions, there is a natural interest in finding sets with few crossings for small values of $n$. The use of computers to aid this search was initiated in [6].

$^1$S is 3-decomposable if there is a triangle $T$ enclosing $S$, and a balanced partition $(A, B, C)$ of $S$, such that the orthogonal projections of $S$ onto the sides of $T$ show $A$ between $B$ and $C$ on one side, $B$ between $A$ and $C$ on another side, and $C$ between $A$ and $B$ on the third side.
2 Results

For $n \leq 100$, we improved many of best known point sets of $n$ points with few crossings using the following simple heuristic.

Given a starting set $S$ of $n$ points in general position in the plane, do:

- **Step 1.** Choose randomly a point $p \in S$.
- **Step 2.** Choose a random point $q$ in the plane “close” to $p$.
- **Step 3.** If $\text{cr}(S \setminus \{p\} \cup \{q\}) \leq \text{cr}(S)$, then update $S$ to $S := S \setminus \{p\} \cup \{q\}$.
- **Step 4.** Go to Step 1.

For each $n = 3, \ldots, 100$, the starting set was taken from Oswin Aichholzer’s homepage. These are available at: www.ist.tugraz.at/aichholzer/research/rp/triangulations/crossing/

Some of the best known examples come from [1], rather than from this page. However, they provide explicit coordinates only for a few of their point sets. In many instances we managed to improve the previous best examples. In many cases we improved the examples from [1], even though we started from a worse point set. For $n = 54, 96$ and 99 we failed to improve upon [1]. Our results are shown in Table 1. Theorem 1 now follows directly from Theorem 2 using the set of 75 points we found with 450492 crossings.

3 The Algorithm

In this section we describe an $O(n^2)$ time algorithm used to compute $\text{cr}(S)$ in step 3 of the heuristic. Recall that $\text{cr}(S)$ is equal to $\square(S)$. We compute this number instead. Quadratic time algorithms for computing $\square(S)$ have been known for a long time [8, 9]. We learned of these algorithms after we finished the implementation of our algorithm. We present our algorithm nevertheless, since in the process we obtained an equality (Theorem 3) between certain substruc-tures of $S$ and $\text{cr}(S)$, which may be of independent interest. We also think that given that the main aim of this paper is to communicate the method by which we obtained these sets, it is important to provide as many details as possible so that an interested reader can obtain similar results.

We compute $\square(S)$ by computing the number of certain subconfigurations of $S$ which determine $\square(S)$. Let $(p, q)$ be an ordered pair of distinct points in $S$, and let $\{r, s\}$ be a set of two points of $S \setminus \{p, q\}$. We call the tuple $((p, q), \{r, s\})$ a pattern. We say that $((p, q), \{r, s\})$ is of type $A$ if $q$ lies in the convex cone with apex $p$ and bounded by the rays $\overrightarrow{pr}$ and $\overrightarrow{ps}$, otherwise it is of type $B$. Let $A(S)$ be the number of type $A$ patterns in $S$, and $B(S)$ the number of its type $B$ patterns. Note that every choice of $((p, q), \{r, s\})$ is either an $A$ pattern or a $B$ pattern. The number of these patterns determine $\square(S)$ as the following theorem shows.
Table 1: Improvements on the starting point sets. Starred numbers come from [1].

| n   | # of crossings in the best point set obtained | # of crossings in the previous best point set obtained | n   | # of crossings in the best point set obtained | # of crossings in the previous best point set obtained |
|-----|---------------------------------------------|------------------------------------------------------|-----|---------------------------------------------|------------------------------------------------------|
| 46  | 59463                                       | 59464                                                | 76  | 47593                                       | 475849                                               |
| 47  | 65059                                       | 65061                                                | 77  | 502021                                      | 502079                                               |
| 49  | 77428                                       | 77430                                                | 78  | 529291                                      | 529332                                               |
| 50  | 84223                                       | 84226                                                | 79  | 557745                                      | 557849                                               |
| 52  | 99169                                       | 99170                                                | 80  | 587289                                      | 587367                                               |
| 53  | 107347                                      | 107355                                               | 81  | 617958                                      | 618018                                               |
| 54  | 115979                                      | 115977                                               | 82  | 649900                                      | 649983                                               |
| 56  | 134917                                      | 134930                                               | 83  | 682986                                      | 683096                                               |
| 57  | 145174                                      | 145176                                               | 84  | 717280                                      | 717360                                               |
| 58  | 156049                                      | 156058                                               | 85  | 753013                                      | 753079                                               |
| 59  | 167506                                      | 167514                                               | 86  | 789960                                      | 790038                                               |
| 61  | 192289                                      | 192293                                               | 87  | 828165                                      | 828225                                               |
| 63  | 219659                                      | 219681                                               | 88  | 867911                                      | 868023                                               |
| 64  | 234464                                      | 234470                                               | 89  | 908972                                      | 909128                                               |
| 65  | 249962                                      | 249988                                               | 90  | 951418                                      | 951459                                               |
| 66  | 266151                                      | 266181                                               | 91  | 995486                                      | 995678                                               |
| 67  | 283238                                      | 283286                                               | 92  | 1040954                                     | 1041165                                              |
| 68  | 301057                                      | 301098                                               | 93  | 1087981                                     | 1088055                                              |
| 69  | 319691                                      | 319731                                               | 94  | 1136655                                     | 1136919                                              |
| 70  | 339254                                      | 339297                                               | 95  | 1187165                                     | 1187263                                              |
| 71  | 359645                                      | 359695                                               | 96  | 1238918                                     | 1238646                                              |
| 72  | 380926                                      | 380964                                               | 97  | 1292796                                     | 1292802                                              |
| 73  | 403180                                      | 403234                                               | 98  | 1348070                                     | 1348072                                              |
| 74  | 426419                                      | 426466                                               | 99  | 1405096                                     | 1404552                                               |
| 75  | 450492                                      | 450540                                               |      |                                             |                                                       |

Theorem 3

\[ \Box(S) = \frac{3A(S) - B(S)}{4} \]

**Proof:** Let \( X \) be a subset of \( S \), of 4 points. Simple arithmetic shows that if \( X \) is not in convex position then it determines 3 patterns of type \( A \) and 9 patterns of type \( B \); on the other hand if \( X \) is in convex position then it determines 4 patterns of type \( A \) and 8 patterns of type \( B \). Assume that we assign a value of 3 to type \( A \) patterns and a value of -1 to type \( B \) patterns. If \( X \) is not in convex position its total contributed value would be zero and if it is convex position it would be 4. Thus \( 4\Box(S) = 3A(S) - B(S) \), and the result follows. □

Note that the total number of patterns is \( n(n-1)(n-2)/2 \). Thus by Theorem 3 to compute \( \Box(S) \) it is sufficient to compute \( A(S) \). Let \( p \) be a point in \( S \). We
now show how to count the number of type $A$ patterns in which $p$ is the apex
of the corresponding wedge.

Sort the points in $S \setminus \{p\}$ counterclockwise by angle around $p$. Let
$y_1, y_2, \ldots, y_{n-1}$ be these points in such an order. For $1 \leq i \leq n - 1$, starting
from $y_i$ and going counterclockwise, let $k(i)$ be the first index (modulo $n$) such
that the angle $\angle y_i py_{k(i)}$ is more than $\pi$. Let $m_i := k(i) - i \mod (n - 1)$.
Note that for $1 \leq j < m_i$ there are exactly $j - 1$ type $A$ patterns of the form
$(p,q), \{y_i, y_{i+j}\}$ for some $q \in S$. In total, summing over all such $j$’s, this
amounts to $\sum_{j=1}^{m_i-1} (j-1) = (m_i-1)$. Thus the total number of type $A$ patterns
in which $p$ is the apex of the corresponding wedge is equal to $\sum_{i=1}^{n-1} (m_i-1)$.

Compute $y_{k(1)}$ and $m_1$ from scratch in linear time. For $2 \leq i \leq n - 1$, to
calculate $y_{k(i+1)}$ and $m_{i+1}$, assume that we have computed $y_{k(i)}$ and $m_i$. Start
from $y_{k(i)}$ and go counterclockwise until the first $y_{k(i+1)}$ is found such that the
angle $\angle y_i py_{k(i+1)}$ is more than $\pi$; then $m_{i+1} = k(i+1) - (i + 1)$. Since one
pass is done over each $y_{k(i)}$, this is done in $O(n)$ total time. Finally, sorting
$S \setminus p$ by angle around $p$, for all $p \in S$, can be done in $O(n^2)$ total time. This is
done by dualizing $S$ to a set of $n$ lines. The corresponding line arrangement can
be constructed in time $O(n^2)$ with standard algorithms. The orderings around
each point can then be extracted from the line arrangement in $O(n^2)$ time.

4 Implementation

In this section we provide relevant information of the implementation of the
algorithm described in Section 3 and of the searching heuristic we used to obtain
the point sets of Table 1.

Instead of sorting in $O(n^2)$ time the points by angle around each point of
$S$, we used standard sorting functions. This was done because these functions
have been quite optimized, and the known algorithms to do it in $O(n^2)$ time
are not straightforward to implement. Thus our implementation actually runs in
$O(n^2 \log n)$ time.

All our point sets have integer coordinates. This was done to ensure the
correctness of the computation. The only geometric primitive involved in the
algorithm is to test whether certain angles are greater than $\pi$; this can be done
with a determinant. Therefore as long as all the points have integer coordinates,
the result is an integer as well. We did two implementations of our algorithm,
one in Python and the other in C. In Python, integers have arbitrarily large
precision, so the Python implementation is always correct. In the C implement-
tion we used 128-bit integers. Here, we have to establish a safety margin—as
long as the absolute value of the coordinates is at most $2^{62}$, the C implement-
tion will produce a correct answer. Empirically we observed a $30 \times$ speed up
of the C implementation over the Python implementation. At each step of the
heuristic we checked if it was safe to use the (faster) C implementation.

To find the point $q$ replacing $p = (x, y)$ in Step 2, we first chose two natural
numbers $t_x$ and $t_y$. These number were distributed exponentially with
a prespecified mean $M$ and rounded to the nearest integer. Afterwards with
probability 1/2 they were replaced by their negative. Point $q$ was then set to $(x + t_x, y + t_y)$. We should note that the exponential distribution was chosen only to ensure that $q$ can be arbitrarily far away from $p$. It is possible that other distributions yield better results.

After choosing an initial mean, the heuristic was left to run for some time, if no improvement was found by then, the mean was halved (or rather the point set was doubled by multiplying each of its points by two). Many attempts varying the amount of time spent waiting for an improvement were done; we kept the best point sets we found. This was done over the course of several months. We also focused our computing resources on those points sets with a better chance of improving the upper bound. As a result some sets were processed for a far longer time. We also mention that the computational resources used were quite modest—only 3 personal computers were used in total.

All the code used in this paper is available upon request from the first author. The point sets obtained can be downloaded from the sources of the arXiv version of this paper.

**Set of 75 points with 450492 crossings**

\[
\begin{align*}
(4473587539, 8674070321), & \quad (2195118038, 12138376293), & \quad (3359570710, 10389672946), \\
(2067188794, 12364750532), & \quad (3798074340, 9176659177), & \quad (495951185, 16620108498), \\
(1133302075, 13923635114), & \quad (1044168259, 7662169961), & \quad (31149395, 16314077753), \\
(2027617092, 3459524378), & \quad (4601468259, 7662169961), & \quad (4601078091, 766213857), \\
(1413182939, 7619250591), & \quad (4116054424, 7605654413), & \quad (3570685582, 9808713565), \\
(3722340414, 916231785), & \quad (4112078622, 7625130881), & \quad (4107912992, 7542476726), \\
(4106745227, 7355480343), & \quad (3189483730, 5743999450), & \quad (3168421193, 5701152359), \\
(894439519, 7965414411), & \quad (3955068845, 6639763085), & \quad (4012346331, 6733970340), \\
(3648786718, 6305728855), & \quad (3653540692, 6310524663), & \quad (3253435351, 5873175144), \\
(2113073755, 12281280867), & \quad (-1364755153, -2899618565), & \quad (1679455404, 2812631891), \\
(1549775961, 2575539287), & \quad (2154725117, 3676030999), & \quad (2297590336, 3930708704), \\
(1474528964, 2436685704), & \quad (1293365372, 2995165431), & \quad (5207789612, 7710691788), \\
(1889666524, 3220648103), & \quad (1902363904, 3245131307), & \quad (4899124137, 8126829846), \\
(4897948559, 8128714256), & \quad (5216754785, 7718023020), & \quad (1683153691, 13003463181), \\
(5202684700, 7706307614), & \quad (5277878757, 7741749531), & \quad (5279252153, 7742686707), \\
(7370957968, 7863465953), & \quad (7493305742, 7871610457), & \quad (3571434484, 9806112525), \\
(6168237700, 8065376208), & \quad (6032867454, 8075892871), & \quad (598119867, 8072572208), \\
(6888712646, 7936512772), & \quad (6851478487, 7943849321), & \quad (3214953430, 10665338217), \\
(7338699912, 7861922951), & \quad (9000883017, 7965096231), & \quad (4059850707, 6811671897), \\
(880690260, 796353399), & \quad (3839573186, 9100031657), & \quad (4471841261, 8674882244), \\
(1504159073, 8118237065), & \quad (10588618608, 8009247798), & \quad (1017492870, 7993197449), \\
(1902291407, 12661152660), & \quad (1811935937, 12802330604), & \quad (1118524774, 8018462436), \\
(10634751909, 8004278071), & \quad (9630956054, 7968154616), & \quad (9359003224, 7955792213), \\
(4338851382, 8157414467), & \quad (4338568456, 8157953847), & \quad (4520171274, 8637506721), \\
(4532317105, 8633237970), & \quad (4538889274, 8639906861), & \quad (3400009645, 10327277784)
\end{align*}
\]

**Acknowledgements**

We thank Jesús Leanos and Gelasio Salazar for various helpful discussions.
References

[1] B. M. Ábrego, M. Cetina, S. Fernández-Merchant, J. Leaños, and G. Salazar. 3-symmetric and 3-decomposable geometric drawings of $K_n$. *Discrete Applied Mathematics*, 158(12):1240–1258, 2010. Traces from LAGOS07 IV Latin American Algorithms, Graphs, and Optimization Symposium Puerto Varas - 2007. doi:10.1016/j.dam.2009.09.020

[2] B. M. Ábrego and S. Fernández-Merchant. Geometric drawings of $K_n$ with few crossings. *J. Combin. Theory Ser. A*, 114(2):373–379, 2007. doi:10.1016/j.jcta.2006.05.003

[3] B. M. Ábrego, S. Fernández-Merchant, J. Leaños, and G. Salazar. A central approach to bound the number of crossings in a generalized configuration. *Electronic Notes in Discrete Mathematics*, 30(0):273–278, 2008. The IV Latin-American Algorithms, Graphs, and Optimization Symposium. doi:10.1016/j.endm.2008.01.047

[4] B. M. Ábrego, S. Fernández-Merchant, and G. Salazar. The rectilinear crossing number of $K_n$: Closing in (or are we?). In J. Pach, editor, *Thirty Essays on Geometric Graph Theory*, pages 5–18. Springer New York, 2013. doi:10.1007/978-1-4614-0110-0_2

[5] O. Aichholzer, F. Aurenhammer, and H. Krasser. On the crossing number of complete graphs. *Computing*, 76(1-2):165–176, 2006. doi:10.1007/s00607-005-0133-3

[6] O. Aichholzer and H. Krasser. Abstract order type extension and new results on the rectilinear crossing number. *Comput. Geom.*, 36(1):2–15, 2007. doi:10.1016/j.comgeo.2005.07.005

[7] A. Brodsky, S. Durocher, and E. Gethner. Toward the rectilinear crossing number of $K_n$: new drawings, upper bounds, and asymptotics. *Discrete Math.*, 262(1-3):59–77, 2003. doi:10.1016/S0012-365X(02)00491-0

[8] G. Rote, G. Woeginger, and B. Zhu. Counting $k$-subsets and convex $k$-gons in the plane. *Information Processing Letters*, 38:149–151, 1991. doi:10.1016/0020-0190(91)90237-C

[9] G. Rote, G. Woeginger, and B. Zhu. Counting convex $k$-gons in planar point sets. *Information Processing Letters*, 41:191–194, 1992. doi:10.1016/0020-0190(92)90178-X

[10] D. Singer. Rectilinear crossing numbers. Manuscript, 1971.