General-covariant constraint-free evolution system for Numerical Relativity

C. Bona, T. Ledvinka\textsuperscript{1}, C. Palenzuela and M. Žáček\textsuperscript{1}

\textit{Departament de Física, Universitat de les Illes Balears, Ctra de Valldemossa km 7.5, 07071 Palma de Mallorca, Spain}

\textsuperscript{1}\textit{Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 180 00 Prague 8, Czech Republic}

A general covariant extension of Einstein’s field equations is considered with a view to Numerical Relativity applications. The basic variables are taken to be the metric tensor and an additional four-vector. The extended field equations, when supplemented by suitable coordinate conditions, determine the time evolution of all these variables without any constraint. Einstein’s solutions are recovered when the additional four-vector vanishes, so that the energy and momentum constraints hold true. The extended system is well posed when the natural extension of either harmonic coordinates or the harmonic slicing condition in normal coordinates

I. INTRODUCTION

Einstein’s field equations can be understood as a set of ten second order partial differential equations on the ten unknown metric coefficients $g_{\mu\nu}$

$$R_{\mu\nu} = 8 \pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T)$$

But in General Relativity, like in other field theories, physical solutions are given instead by an equivalence class of mathematical expressions which are related one to another by gauge transformations. The peculiarity of General Relativity is that the gauge group is that of the general (smooth) coordinate transformations

$$g'_{\mu\nu} = f^\mu(x^{'})$$

This allows one to use the coordinate gauge freedom to fix up to four of the ten metric coefficients (kinematical degrees of freedom) so that only the remaining ones (dynamical degrees of freedom) can be actually determined by the field equations. It follows that four of these equations just provide additional constraints on the dynamical degrees of freedom: they can be understood as first integrals related with energy and momentum conservation.

This complex structure becomes more transparent in the context of the 3+1 formalism, where one considers the space-time sliced by $t = constant$ hypersurfaces. The line element can then be written as

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt) \quad i, j = 1, 2, 3$$

where the lapse $\alpha$ and the shift $\beta^i$ are the kinematical degrees of freedom, whereas the coefficients $\gamma_{ij}$ of the metric induced on the $t = constant$ hypersurfaces are the dynamical ones. One can use the extrinsic curvature (second fundamental form) of these slices

$$K_{ij} = -\frac{1}{2 \alpha} (\partial_t - \mathcal{L}_\beta) \gamma_{ij}$$

to decompose the field equations into six evolution equations

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -\nabla_i \alpha_j$$

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -\nabla_i \alpha_j + \alpha (3) R_{ij} - 2K^k_{ij} + tr K K_{ij}$$

(ADM evolution equations) and four constraints

$$(3) R - tr (K^2) + (tr K)^2 = 0$$

$$\nabla_k (K^k_i - tr K \delta^k_i) = 0$$

where we have restricted ourselves to the vacuum case for simplicity.

The 3+1 formalism is specially suited for Numerical Relativity applications. In this context, one usually takes advantage of the fact that the energy and momentum constraints are first integrals of the field equations. This allows one to impose them on the initial data only (‘free evolution’ approach), but not during time evolution. One is dealing then with an extended set of solutions. The resulting simulations will represent true Einstein’s solutions only to the extent to which the constraints are actually preserved during numerical evolution.

This approach raises some concerns (see Ref. \textsuperscript{[3]}). First of all, the set of ‘extended solutions’ is not determined in a general covariant way: equations just translate the space components of the original field equations. And the space components of four-dimensional tensors are covariant only under the restricted subgroup of coordinate transformations that preserve the time slicing (3+1 covariance):

$$t' = h(t), \quad y' = f^i(x^i, t)$$

Moreover, even in a given coordinate system, there are many ways of extending the space of solutions, because the set of six evolution equations is not univocally defined in that context: one can actually combine either the energy constraint or the momentum constraint with the original evolution equations to get a different set of evolution equations. This fact has been recently used by Kidder, Scheel and Teukolsky (KST) to obtain a multiparameter family of evolution systems.
This ambiguity was previously used in a different way by Bona and Mass \cite{5,6}, and also by Shibata and Nakamura \cite{7} and Baumgarte and Shapiro \cite{8} (BSSN system). The key idea in these works was to introduce three supplementary dynamical variables whose evolution equations were obtained by using the momentum constraint \cite{7}. As far as these works were focused on Numerical Relativity applications, the supplementary quantities were introduced in an ‘ad hoc’ way, breaking the 3+1 covariance of the formalism. Only very recently \cite{9} the same quantities could be then coupled to the other equations in many different ways so that many more new parameters will appear in the resulting evolution system (see for instance Ref. \cite{10} for a similar approach).

We will not go that way in the present work. We prefer to preserve general covariance in our extension of Einstein’s field equations. Instead of a separate three-vector \(Z_i\) and three-scalar \(\Theta\), we shall consider a covariant ‘zero’ four-vector \(Z_\mu\) such that their space components coincide with \(Z_i\) and

\[
\Theta \equiv n_\mu Z^\mu = \alpha Z^0
\]

where \(n_\mu\) is the unit normal to the \(t = \text{constant}\) slices. The purpose of this work is to use \(Z_\mu\) to extend in a covariant way Einstein’s equations so that the extended system contains only evolution equations, without any constraint. Einstein’s solutions will be recovered for \(Z_\mu = 0\). The four-vector \(Z_\mu\) will then provide a simple way to measure of the quality of numerical simulations: energy constraint violations can be monitored with \(\Theta^2\), whereas one can use the norm of \(Z_i\) to deal with momentum constraint deviations. These quantities can be combined to form a four-dimensional scalar:

\[
-Z_\mu Z^\mu = \Theta^2 - \gamma^{ij} Z_i Z_j
\]

where we have rescaled the four-vector \(Z_\mu\) to clean \(\Theta\) from any arbitrary parameter. The Cauchy problem of the extended equations \(\Theta\) is simpler than the corresponding one for \(\Theta\). The space components of \(\Theta\) provide again second order evolution equations for the dynamical degrees of freedom in the metric. In the 3+1 language, one has

\[
(\partial_t - \mathcal{L}_\beta) Z_i = - \nabla_i \alpha_j + \alpha \left( \partial_t R_{ij} + \nabla_i Z_j + \nabla_j Z_i \right) - 2 K^2_{ij} + (tr K - 2 \Theta) K_{ij}
\]

But now the remaining components can be combined with \(\mathcal{L}_{\mu} \Theta\) to provide, instead of constraints, first order evolution equations for the components of \(Z_\mu\), namely

\[
(\partial_t - \mathcal{L}_\beta) \Theta = \alpha \left( \partial_t R + (tr K - 2 \Theta) tr K - tr K^2 + 2 \nabla_i Z^i - 2 Z^k \alpha_k / \alpha \right)
\]

where here again we have restricted ourselves to the vacuum case.

The contracted Bianchi identities, when applied to eq. \(\Theta\), lead to a simple equation for \(Z_\mu\)

\[
L \Box Z_\mu \equiv g^{\rho\sigma} \nabla_\rho \nabla_\sigma Z_\mu + R_{\mu \nu} Z^\nu = 0
\]

where \(\Box\) stands for the four-dimensional Lichnerowicz Laplacian. This equation, of course, can be also derived from \(\Theta\), but the converse is not true: the first order evolution equations \(\Theta\) for \(Z_\mu\) can not be derived from \(\Theta\) and the second order equation \(\Theta\).

### III. COORDINATE CONDITIONS

In order to analyze the causal structure of the extended equations \(\Theta\), we will use de DeDonder \cite{11} expression of the Ricci tensor to write down the principal part, namely

\[
-\Box g_{\mu \nu} + \partial_\mu (\Gamma^\nu + 2 Z^\nu) + \partial_\nu (\Gamma^\mu + 2 Z^\mu) = \ldots
\]

where here the box symbol stands for the d’Alembert operator on functions and we have noted as usual \(\Gamma^\mu \equiv g^{\rho\sigma} \Gamma^\nu_{\rho \sigma}\). Comparing with the wave equation for \(g_{\mu \nu}\), it is clear that we can obtain a well posed system if we take out the additional terms in \(\Theta\) by using the following extension of the well known harmonic coordinate conditions:

\[
\Box x^\mu = -\Gamma^\mu = 2 Z^\mu
\]

so that the four-vector \(Z_\mu\) can be interpreted in this context as providing ‘gauge sources’, along the lines sketched in ref \cite{4}. Harmonic coordinates, including their extension \(\Theta\), are not flexible enough to be used in most Numerical
Relativity applications. A more suitable choice is the 'harmonic slicing', in which the time coordinate is again assumed to be an harmonic function, but the space coordinates are chosen so that the mixed components $g_{0i}$ vanish. We propose here to keep in the extended case the time component of (17), that is

$$\Box x^0 = -\Gamma^0 = 2 Z^0, \quad g_{0i} = 0 \tag{18}$$

which can be translated into the 3+1 language as

$$(\partial_i - L_\beta) \ln \alpha = -\alpha (tr K - 2 \Theta), \quad \beta^i = 0 \tag{19}$$

The mixed time-space components of (16) read now

$$\partial_0 (\Gamma_i + 2 Z_i) = ... \tag{20}$$

This means that in the harmonic slicing case (18) there are three degrees of freedom, directly related with the $t = \text{constant}$ slices. This 'standing' modes can be decoupled from the remaining ones by taking an extra time derivative of (14). This means that the resulting system will be of third order in $\alpha$ and $g_{ij}$ (second order in $\Theta$ and $K_{ij}$). Allowing for (16,19), one gets after some algebra

$$\Box \Theta = ... \quad \Box K_{ij} = ... \tag{21}$$

so that the corresponding characteristic lines are on the light cones. It follows from (21) that the resulting system (with that extra time derivative) is well posed (see Ref. [3] for a similar development, in the 3+1 language, from Einstein’s equations [I]).

Although numerical simulations are beyond the scope of the present letter, we will comment here some extensions of this work which are specially suited for Numerical Relativity. The first one is that harmonic slicing is often generalized in that context by including an extra factor $f(\alpha)$ in the right-hand-side of equation (19). The effect of this minor change is twofold: from the theoretical point of view, the gauge-related degrees of freedom no longer propagate with light speed (gauge speed) so that the first equation in (21) no longer holds true; from the numerical point of view, higher gauge speed increases the singularity avoidance properties of the gauge [13].

The second extension is to consider the first space derivatives

$$A_k \equiv \partial_k (\ln \alpha), \quad D_{kij} \equiv \frac{1}{2} \partial_k \gamma_{ij} \tag{22}$$

as independent dynamical variables with evolution equations

$$\partial_t A_k + \partial_i [\alpha f (tr K - 2 \Theta)] = 0 \tag{23}$$

$$\partial_t D_{kij} + \partial_i [\alpha K_{ij}] = 0 \tag{24}$$

so that the set (13-14)324 forms a first order hyperbolic evolution system, which is obviously much more adapted to Numerical Relativity applications than the above mentioned third order system. In addition, the causal structure of first order systems is much easier to analyze [3] and one can take advantage of the characteristic field decomposition to devise stable numerical algorithms.

Acknowledgements: This work has been supported by the EU Programme 'Improving the Human Research Potential and the Socio-Economic Knowledge Base' (Research Training Network Contract (HPRN-CT-2000-00137), by the Spanish Ministerio de Ciencia y Tecnologia through the research grant number BFM2001-0988 and by a grant from the Conselleria d’Innovacio i Energia of the Govern de les Illes Balears.

[1] Y. Choquet-Bruhat, J. Rat. Mec. Analysis 5, 951 (1956).
[2] R. Arnowit, S. Deser and C. W. Misner, Gravitation: an introduction to current research, ed. L. Witten, Wiley, New York (1962).
[3] O. Reula, Hyperbolic Methods for Einstein’s Equations, Living Reviews in Relativity.
[4] L. E. Kidder, M. A. Scheel and S. A. Teukolsky, Phys. Rev. D 64, 064007 (2001).
[5] C. Bona and J. Massó, Phys. Rev. Lett. 68 1097 (1992).
[6] C. Bona, J. Massó, E. Seidel and J. Stela, Phys. Rev. Lett. 75 600 (1995).
[7] M. Shibata and T. Nakamura, Phys. Rev. D 52 5428 (1995).
[8] T. W. Baumgarte and S. L. Shapiro, Phys. Rev. D 59 024007 (1999).
[9] C. Bona, T. Ledvinka and C. Palenzuela, gr-qc/0208087 (accepted at Phys. Rev. D).
[10] S. Frittelli and O. A. Reula, J. Math. Phys. 40, 5143 (1999).
[11] T. De. Donder, La Gravifique Einstenienne, Gauthier-Villars, Paris (1921).
[12] H. Friedrich and A. Rendall The Cauchy Problem for the Einstein Equations, Living Reviews in Relativity.
[13] Y. Choquet-Bruhat, Y. Ruggeri, Commun. Math. Physics 89, 269 (1983).
[14] A. Abrahams et al, Phys. Rev. D45, 3544 (1992).