On the derivation of Fourier’s law for coupled anharmonic oscillators

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Abstract

We study the Hamiltonian system made of weakly coupled anharmonic oscillators arranged on a three dimensional lattice $\mathbb{Z}_{2N} \times \mathbb{Z}^2$, and subjected to a stochastic forcing mimicking heat baths of temperatures $T_1$ and $T_2$ on the hyperplanes at 0 and $N$. We introduce a truncation of the Hopf equations describing the stationary state of the system which leads to a nonlinear equation for the two-point stationary correlation functions. We prove that these equations have a unique solution which, for $N$ large, is approximately a local equilibrium state satisfying Fourier law that relates the heat current to a local temperature gradient. The temperature exhibits a nonlinear profile.

1 Introduction

Fourier’s law states that a local temperature gradient is associated with a flux of heat $J$ which is proportional to the gradient:

$$J(x) = -k(x)\nabla T(x)$$

where the heat conductivity $k(x)$ is a function of the temperature at $x$: $k(x) = \tilde{k}(T(x))$.

Fourier’s law is experimentally observed in a variety of materials from gases to solids at low and at high temperatures. It also belongs to basic textbook material. However, a first principle derivation of the law is missing and, many would say, is not even on the horizon.

The quantities $T$ and $J$ in (1.1) are macroscopic variables, statistical averages of the variables describing the microscopic dynamics of matter. A first principle derivation of (1.1) entails a definition of $T$ and $J$ in terms of the microscopic variables and a proof of the law in some appropriate limit.

An example of an idealized physical situation would be a crystal occupying the region $[0, N] \times \mathbb{R}^2$ in $\mathbb{R}^3$. The crystal is heated at the two boundaries by uniform temperatures, $T_1$ on

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\( \{0\} \times \mathbb{R}^2 \) and \( T_2 \) on \( \{N\} \times \mathbb{R}^2 \). Then, for large \( N \), the relation (1.1) should hold, with corrections \( o(1/N) \) and \( \nabla T = \mathcal{O}(1/N) = J \).

The proper description of the crystal would be quantum mechanical, but, being a macroscopic law, (1.1) is expected to hold for a classical system as well and the quantum corrections are expected to be small except at low temperatures. An example of a classical toy model of such a system is given by coupled oscillators organized on a \( d \)-dimensional lattice \( \mathbb{Z}^d \).

Consider a subset \( \Lambda = [0, N] \times \mathbb{Z}^{d-1} \) of \( \mathbb{Z}^d \) and let \( x \in \Lambda \) index the dynamical variables, coordinates \( q_x \) and momenta \( p_x \). The dynamics in the phase space \( \mathbb{R}^{2\Lambda} = \{(q_x, p_x) \mid x \in \Lambda\} \) is defined in terms of the Hamiltonian

\[
H(q, p) = \frac{1}{2} \sum_x p_x^2 + V(q)
\]

i.e. the Hamiltonian flow is given by the system

\[
\begin{align*}
\dot{q}_x &= p_x \quad (1.2) \\
\dot{p}_x &= -\frac{\partial V}{\partial q_x}. \quad (1.3)
\end{align*}
\]

We can think of this model as describing atoms with unit mass with equilibrium positions at \( x \in \Lambda \) and \( q_x \) being the deviation of the position of the atom indexed by \( x \) from its equilibrium \( (q_x \) should of course be in \( \mathbb{R}^d \) but we simplify and take \( q_x \in \mathbb{R} \), while \( p_x \) is the momentum (velocity) of the atom.

The potential \( V \) should describe the forces between the atoms and possible restoring or pinning forces pulling \( q_x \) to the equilibrium \( q_x = 0 \). Let, for unit lattice vectors \( e_\alpha \)

\[
\nabla_\alpha q(x) = q(x + e_\alpha) - q(x)
\]

denote the discrete derivative. Then, the interactions are described by a potential \( U(\nabla q) \) whereas the pinning is described by \( W(q) \), and \( V = U + W \). In the simplest case, \( U \) and \( W \) are local:

\[
\begin{align*}
W(q) &= \sum_{x \in \Lambda} w(q_x), \\
U(\nabla q) &= \sum_{x, x + e_\alpha \in \Lambda} u(\nabla_\alpha q(x)),
\end{align*}
\]

and, to model small oscillations, \( w \) and \( u \) are given by a low order polynomial:

\[
w(q) = aq^2 + bq^3 + cq^4,
\]

and \( u \) similarly.

In addition to the Hamiltonian dynamics of the system, we want to model the heating of the system at the boundary. The simplest way to model this is to add stochastic forces to (1.3) for \( x \in \partial \Lambda \), see Section 2 for details. Then the deterministic flow (1.2, 1.3) is replaced by a Markov process \( (q(t), p(t)) \) and the Fourier law will be a statement about the stationary state of this Markov process.

The first question one would like to answer is the existence and uniqueness of the stationary state. When \( T_1 = T_2 = T \) such a state exists and is a Gibbs state, given formally by

\[
\frac{1}{Z} e^{-\beta H(q, p)} dq \, dp \quad (1.4)
\]
(see Sect 2), with $\beta = 1/T$. This is customarily referred to as the *equilibrium* state in contrast to the non-equilibrium situation $T_1 \neq T_2$. In the latter case, there is no simple formula like (1.4) and, indeed, in our setup, even the existence of a stationary state is an open problem. In the $d = 1$ case of a finite chain of $N + 1$ oscillators the existence is proved provided the interaction potential $U$ dominates the pinning one $W$ (see [9, 10, 11, 12, 17, 13, 24, 25, 26]). In this case, uniqueness is also proved, i.e. the Markov process converges to this state as $t \to \infty$.

Supposing that we have a stationary state $\mu$, let us formulate the statement (1.1). The Hamiltonian flow (1.2, 1.3) preserves the total energy $H$. In particular, if we write $H$ as a sum of local terms, each one pertaining to a single oscillator:

$$H = \sum_{x \in \Lambda} H_x,$$

then, under the flow (1.2, 1.3),

$$\dot{H}_x = \sum_{\alpha} \nabla_j j_\alpha(x) = \nabla \cdot \vec{j}(x),$$

where the *microscopic heat current* $\vec{j}(x)$ will depend on $p_y$ and $q_y$ for $y$ near $x$ (see (7.2) for a concrete expression). Let also $t(x) = \frac{1}{2}p_x^2$ be the kinetic energy of the oscillator indexed by $x$. Then, the macroscopic temperature and heat current in eq. (1.1) are defined by

$$T(x) = E_\mu t(x)$$

$$J(x) = E_\mu j(x)$$

where $E_\mu$ denotes expectation in the stationary state.

We do not attempt here to give a comprehensive review of the status of (1.1), but refer the reader to the reviews [6], [20] and [29]. There is also substantial amount of work on Fourier’s law for fully stochastic models (i.e. where there is noise in the bulk too), going back to [16], [18], see eg. [7], [3], [4], [5], [14], [15].

The only rigorous results in our model are for the harmonic case where $U$ and $W$ are quadratic [27, 28]. In that case, Fourier’s law does not hold: the current $\vec{j}$ is $O(1)$ as $N \to \infty$ whereas $\nabla T = 0$ except near the boundary. If the model has pinning, i.e. $W \neq 0$ and $U$ or $W$ are not harmonic, the law seems to hold in simulations in all dimensions [1]. In the unpinned $W = 0$ anharmonic case, conductivity seems anomalous in low dimensions: $k$ in (1.1) depends on $N^\alpha$ in $d = 1$ and logarithmically in $d = 2$. It is a major challenge to explain the $\alpha$ which, numerically, seems to be in the interval $[1/3, 2/5]$ (see [21], [22], [23] for theories on $\alpha$).

One way to try to get hold of the stationary state $\mu$ is via its correlation functions. Denote $u_x = (q_x, p_x)^T$ and choose $U, W$ even for simplicity. Then, the Hamiltonian vector field (1.2, 1.3) is a sum of a linear and a cubic term in $u$. Therefore, the correlation functions

$$G_n(x_1, x_2, ..., x_n) = E_\mu u_{x_1} \otimes ... \otimes u_{x_n}$$

satisfy a linear set of equations

$$\partial_t G_n = A_n G_n + V_n G_{n+2} + C_n G_{n-2},$$

(1.5)

where $A_n, V_n, C_n$ are linear operators coming respectively from the linear and cubic terms of (1.2, 1.3), and the term coming from the noise via Ito’s formula. See Section 3 for a detailed derivation of these equations.
Such equations for the correlation functions are known as the BBGKY hierarchy in particle systems or the Hopf equations in turbulence. Although linear, the system (1.5) is intractable due to the appearance of $G_{n+2}$ in the equation for $G_n$.

In this paper we will consider the situation where the equilibrium $T_1 = T_2$ Gibbs measure is close to a Gaussian measure. This holds if the anharmonicity in $u$ and $w$ (the coefficients $b$ and $c$) is weak and the harmonic part in $w$ (i.e. the pinning) is large (i.e. the Gibbs measure is far from critical), as we will assume. In such cases, we expect that the non-equilibrium measure is also close to a Gaussian. In such a situation, one can attempt a closure of the Hopf equations, i.e. to express the higher order correlation functions $G_{n+2}$ in terms of $G_m$ with $m \leq n$, thereby obtaining a finite set of equations for $G_m$, $m \leq n$.

We will introduce such a closure and solve the closed equations. The simplest closure would be to write, for $n = 2$, the equation

$$G_4 = \sum G_2 \otimes G_2 + G_4^c,$$

and set $G_4^c = 0$, thereby obtaining a closed quadratic equation for $G_2$. It turns out that this is too simple: the solution will be qualitatively similar to the one of the harmonic case. Our closure is done to the $G_4$-equation by setting the connected 6-point function to zero. This is an uncontrolled approximation that we do not know how to justify rigorously. An analogous approximation was studied in both classical and quantum systems in [29] and has been used in [19] in a model similar to ours, but in a translation invariant setting and in one-dimension; our model, in one-dimension, was further studied, theoretically and numerically, in [2].

Our motivation for studying the closure equations in detail is on the one hand in the interesting picture of the local equilibrium state that emerges and on the other hand in building approaches that go beyond this approximation. Traditionally one arrives to such a closure in an appropriate limit, the "kinetic limit", which in our case means taking the anharmonicity proportional to $N^{-\frac{1}{2}}$ and rescaling distances by $N$, see [29]. One then arrives to a Boltzman equation and after a further limit [29] to the Fourier law. Most of the structure of the stationary state correlations disappear in these limits.

In our case we arrive to approximate expressions of stationary correlation functions whithout any limits (admittedly with no control of the corrections!) and then can study how the Fourier law emerges from these expressions. Other interesting phenomena emerge too. In particular, one expects the presence of very long range spatial correlations in the stationary state even though the equilibrium state has exponential decay of correlations. Although we do not demonstrate the presence of these long range correlations, because we obtain only upper bounds on the decay rates, not lower ones, we show how to handle the technical problems caused by these slowly decaying correlations, and that could be useful in other contexts. Finally, we believe some of the methods developed in this paper could be of use in trying to prove the existence of the kinetic limit and Fourier’s law therein.

The outline of the paper (to which the reader can return later) is as follows: in the next section, we define our model and we derive the Hopf equation (or BBGKY hierarchy) in section 3. In section 4, we explain the particular closure that we will study, leading to our final equations (see (4.18)–(4.20) below). Section 5 is devoted to several changes of variables: we first apply a Fourier transformation, and then we introduce variables that could be called slowly and fastly varying, namely the one of which the non translation invariant part of the correlation functions depends (slow) and the one related to the translation invariant part (fast). Next, we outline our arguments and state qualitatively our main result (section 6). To prove the latter, we first derive (in section 7) identities satisfied by our equations, which take the form of current conservation equations, consisting of an energy conservation law and a number conservation
law (the presence of the latter being, to some extent, a consequence of our approximations, i.e. of our closure). We also write down the stationary states in the translation invariant case. These do not reduce, for our closed equations, to the usual Gibbs states, but depend on two parameters, the temperature, as one would expect, but also a "chemical potential", corresponding to the number conservation law. These conservation laws are related to the presence of zero modes in the linearization of our equations, which are discussed also in section 6. In fact, the current conservation equations coincide with the projection of the full equations on the zero modes. In section 8, we define precisely the spaces in which our equations are solved and we state our main result in a more technical form. The solution that we shall construct is the sum of a modified stationary state, with coefficients (temperature and chemical potential) slowly varying in space, and a perturbation.

The main technical problem that we face is that the nonlinear terms in our equations involve collision kernels that are delta functions (or principal values) (see (5.12)-(5.14) below). Since we want to solve our equations by using a fixed point theorem, we need to show that the nonlinear terms belong to the space that we introduced, and, because of the presence of the delta functions, this is rather technical. Section 9 is devoted to solving those problems, but most of the proofs of that section are given in Appendix B. Another problem is that the linear operator in our equations is not invertible, because of the zero modes. In section 10, we show that our linear operator can be inverted in the complement of the zero modes. This uses the fact that this operator is a sum of a multiplication operator and a convolution. To show invertibility, it is useful to know that the convolution operator is compact and this in turn follows from Hölder regularity properties that are proven in section 9. Finally, in section 11, we prove our main result, which consist in using the result of section 10 to solve the equations in the complement of the zero modes with the solution to the current conservation equations, i.e. of the projection of the full equations onto the zero modes. The first equations lead to Fourier’s law, namely an expression of the conserved currents in terms of the parameters of the modified stationary state (temperature and chemical potential). And the second equations determine the spatial dependence of those parameters.

2 Lattice dynamics with boundary noise

Let us define the model we consider in more detail. Instead of working in the strip $[0, N] \times \mathbb{Z}^{d-1}$ it is convenient to double it to the cylinder $V = \mathbb{Z}_{2N} \times \mathbb{Z}^{d-1}$ where $\mathbb{Z}_{2N}$ are the integers modulo $2N$. The noise is put on the "boundary" $\{0\} \times \mathbb{Z}^{d-1} \cup \{N\} \times \mathbb{Z}^{d-1}$. We consider the phase space $(q, p) \in \mathbb{R}^{2N}$ i.e. $q = (q_x)_{x \in V}$, $p_x = p_{x+y}$ for $y = (2N, 0)$, and similarly for $p_x$. The dynamics is given by the stochastic differential equations

$$
\begin{align*}
dq_x &= p_x dt \\
dp_x &= \left(-\frac{\partial H}{\partial q_x} - \gamma_x p_x\right) dt + d\xi_x
\end{align*}
$$

where

$$
H(q, p) = \frac{1}{2} \sum_{x \in V} p_x^2 + \frac{1}{2} (q, \omega^2 q) + \frac{\lambda}{4} \sum_{x \in V} q_x^4,
$$

$$
\gamma_x = \gamma (\delta_{x,0} + \delta_{x, N})
$$
and the random variables $\xi_x(t)$ are Brownian motions with covariance
\[ E\xi_x\xi_y = 4\gamma\delta_{xy}(T_1\delta_{x10} + T_2\delta_{x1N})t. \] (2.4)
The Hamiltonian (2.2) describes a system of coupled anharmonic oscillators with coupling matrix $\omega^2$:
\[ (q,\omega^2q) = \sum_{x,y \in V} q_x q_y \omega^2(x - y) \]
Our analysis requires that the Fourier transform $\omega^2(k)$ of $\omega^2$ is smooth and
\[ \omega^2(k) = m^2 + \rho(k) \]
with $\rho(k) = \mathcal{O}(k^2)$ as $k \to 0$, and $m^2 > 2||\rho||_\infty$. Moreover we will need some regularity properties that will be checked explicitly for
\[ \omega^2 = (-\Delta + m^2)^2, \]
i.e.
\[ \omega(k) = 2 \sum_{\alpha=1}^d (1 - \cos k_\alpha) + m^2, \] (2.5)
see the proof of Proposition 9.3.

As explained in the Introduction, the equations (2.1) describe a Hamiltonian dynamics subjected to stochastic heat baths on the “boundaries” of $V$, with temperature $T_1$ on the hyperplane $x_1 = 0$, and temperature $T_2$ on the hyperplane $x_1 = N$. This defines a Markov process $(q(t),p(t))$ in the phase space $\mathbb{R}^{2V}$ and we are interested in the stationary states for this process.

If the temperatures are equal, $T_1 = T_2 = T$, an explicit stationary state is given by the Gibbs state at temperature $T$ of the Hamiltonian $H$. This probability measure is given as a weak limit
\[ \nu_T = \lim_{M \to \infty} \frac{1}{Z_M} \exp \left[ -\frac{\lambda}{T} \sum_{x \in V_M} q_x^4 \right] \mu_T(dp,dq) \] (2.6)
where $V_M$ is defined by $|x_i| < M$, $i = 2, \ldots, d$, and $\mu_T$ is the Gaussian measure with covariance
\[ Ep_xp_y = T\delta_{xy}, \quad Ep_xq_y = 0 \] (2.7)
\[ Eq_xq_y = T\omega^{-2}(x - y) \] (2.8)
For small $\lambda$, the Gibbs measure $\nu_T$ is nearly Gaussian, with (2.7) still true and small $\mathcal{O}(\lambda)$ corrections in (2.8). It is very well understood via cluster expansions. Physically, the fact that the Markov process reaches the stationary distribution $\nu_T$ means that the heat introduced at the boundary spreads inside the system, which reaches equilibrium at temperature $T$.

When $T_1 \neq T_2$, things are very different. Even the existence of a stationary state is not known rigorously (not even in finite volume, $M < \infty$). However, physically, one expects a unique stationary state $\nu$ to exist. In this paper we assume this and inquire about the properties of $\nu$. In particular one would like to understand how the heat from the boundary now spreads inside the system: what is its stationary temperature distribution and what sort of flux of heat exists in it.
3 Hopf equations

Let us introduce a more compact notation for the stochastic differential equation (2.1). Denote $(q,p)^T = u$, $\Lambda(u) = -\lambda(0,q^3)^T$, $(\Gamma u)_x = (0,\gamma_x p_x)^T$ and $\eta = (0,\xi)^T$. Then (2.1) becomes

$$du(t) = \left((A - \Gamma)u + \Lambda(u)\right)dt + d\eta(t)$$

(3.1)

where

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

Define the correlation functions

$$G_n(x_1,\ldots,x_n,t) = E\ u_{x_1}(t) \otimes \ldots \otimes u_{x_n}(t) \in \mathbb{R}^{2nV}$$

By Ito’s formula, we get

$$\dot{G}_n = (A_n - \Gamma_n)G_n + \Lambda_n G_{n+2} + C_n G_{n-2},$$

(3.2)

where

$$A_n = A \otimes 1 \otimes \ldots \otimes 1 + \ldots 1 \otimes 1 \otimes \ldots \otimes A$$

and $\Gamma_n$ is defined similarly. Moreover,

$$\Lambda_n G_{n+2} = \sum_{i=1}^n E u_{x_1} \otimes \ldots \otimes \Lambda(u)_{x_i} \otimes \ldots \otimes u_{x_n},$$

$$C_n G_{n-2} = \sum_{i<j} C_{x_i x_j} G_{n-2}(x_1,\ldots,\hat{x}_i,\ldots,\hat{x}_j\ldots x_n)$$

where the arguments $\hat{x}_i, \hat{x}_j$ are missing. This defines linear operators from $\mathbb{R}^{2(n+2)V} \to \mathbb{R}^{2nV}$ and $\mathbb{R}^{2(n-2)V} \to \mathbb{R}^{2nV}$ respectively. $C$ equals one-half the time derivative of the covariance of $\eta$, i.e.

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C \end{pmatrix},$$

with

$$C_{xy} = 2\gamma \delta_{xy}(T_1 \delta_{x_0} + T_2 \delta_{x_N}).$$

(3.3)

Suppose $\nu$ is a stationary state of the process $u(t)$. Then equation (3.2) leads to a linear set of equations for the stationary correlation functions

$$G_n(x_1,\ldots,x_n) = \int \otimes_{i=1}^n u_{x_i} \nu(du)$$

(3.4)

$$(A_n - \Gamma_n)G_n + \Lambda_n G_{n+2} + C_n G_{n-2} = 0.$$
The equations (3.5) have the drawback that they do not “close”: to solve for \( G_n \), we need to know \( G_{n+2} \).

For \( \lambda \) small, the equilibrium \( T_1 = T_2 \) measure is close to Gaussian. When \( T_1 \neq T_2 \) we expect this to remain true; however, the measure will not satisfy \( E p_x q_y = 0 \).

We look for a Gaussian approximation to equation (3.5) for small \( \lambda \) by means of a closure, i.e. expressing the \( G_n \) in terms of \( G_2 \). Since \( G_n \neq 0 \) only for \( n \) even, the first equation in (3.5) reads:

\[
(A_2 - \Gamma_2) G_2 + \Lambda_2 G_4 + C = 0
\] (3.6)

The simplest closure would be to replace \( G_4 \) in (3.6) by the Gaussian expression

\[
\sum_p G_2(x_i, x_j) \otimes G_2(x_k, x_l)
\] (3.7)

where the sum runs over the pairings of \( \{1, 2, 3, 4\} \). Equations (3.6) and (3.7) lead to a nonlinear equation for \( G_2 \). It turns out that the solution to this equation is qualitatively similar to the \( \lambda = 0 \) case, i.e. \( G_2 \) does not exhibit a temperature profile nor a finite conductivity. The only effect of the nonlinearity is a renormalization of \( \omega \). We will therefore not discuss this closure any further.

The next equation is

\[
(A_4 - \Gamma_4) G_4 + \Lambda_4 G_6 + C_4 G_2 = 0
\]

Write

\[
G_4 = \sum_p G_2 \otimes G_2 + G_4^c
\]

and

\[
G_6 = \sum_p G_2 \otimes G_2 \otimes G_2 + \sum_{p'} G_2 \otimes G_4^c + G_6^c
\]

where \( G_4^c \) and \( G_6^c \) are the connected correlation functions describing deviation from Gaussianity and the sums run over the usual partitions of indices. After some algebra, we may write the first two Hopf equations in the following form:

\[
(A_2 - \Gamma_2 + \Sigma_2) G_2 + \Lambda_2 G_4^c + C = 0
\] (3.8)

\[
(A_4 - \Gamma_4 + \Sigma_4) G_4^c + b(G_2) + \Lambda_4 G_6^c = 0,
\] (3.9)

where the operators \( \Sigma_2 \) and \( \Sigma_4 \) are

\[
\Sigma_2(G_2)G_2 = \Lambda_2 \sum_p G_2 \otimes G_2
\] (3.10)

\[
\Sigma_4(G_2)G_4^c = \sum_p \Lambda_4' G_2 \otimes G_4^c
\]

and \( \Lambda_4' \) means the following: \( G_2 \otimes G_4^c \) belongs to \( (R^{2V})^{\otimes 2} \otimes (R^{2V})^{\otimes 4} \); \( \Lambda_4' \) is a sum of terms

\[
\Lambda_4' = \sum_{i<j<k} \Lambda_{ij}^{ik}
\]
where $\Lambda_{ijk}^4$ acts with $\Lambda_4$ in the spaces $i, j, k$ and as identity in the rest. $\Lambda_4'$ then has at least one of the indices $i, j, k$ equal to either 1 or 2.

Finally,

$$b(G_2) = \sum_p \Lambda''_4 G_2 \otimes G_2 \otimes G_2,$$

where $\Lambda''_4$ is similar to $\Lambda_4'$, but with $\Lambda_{ijk}^4$ acting on all the three factors $G_2$. Explicitely, denote $i = (\alpha, x) \in \{1, 2\} \times \mathbb{R}^V$, so that $u = (u_i)$, $u_{(1, x)} = q_x$, $u_{(2, x)} = p_x$. Then,

$$b(G_2)_{i_1i_2i_3i_4} = 6 \sum_{j_2j_3j_4} \Lambda_{i_1j_2j_3j_4}^4 \prod_{a=2}^4 (G_2)_{i_aj_a} + (i_1 \rightarrow i_b, b = 2, 3, 4)$$

with

$$\Lambda_{j_1j_2j_3j_4} = -\lambda \delta_{\alpha_12} \prod_{a=2}^4 \delta_{\alpha a1} \delta_{x_1x_a}. \quad (3.11)$$

Equation (3.9) may be solved for $G_4^c$:

$$G_4^c = -(A_4 - \Gamma_4 - \Sigma_4)^{-1} \left( b(G_2) + \Lambda_4 G_6^c \right), \quad (3.12)$$

provided that $A_4 - \Gamma_4 - \Sigma_4$ is invertible. Substitution of (3.12) in (3.8) yields a nonlinear equation for $G_2$ with dependence on $G_6^c$:

$$(A_2 - \Gamma_2 + \Sigma_2)G_2 + \mathcal{N}(G_2, G_6^c) + \mathcal{C} = 0 \quad (3.13)$$

with

$$\mathcal{N}(G_2, G_6^c) = -\Lambda_2 (A_4 - \Gamma_4 + \Sigma_4)^{-1} \left( b(G_2) + \Lambda_4 G_6^c \right) \quad (3.14)$$

If we set $G_6^c = 0$ in (3.13) we get a closed equation for $G_2$. However, before defining the closure equation to be studied, let us discuss in more detail the term $b(G_2)$.

### 4 Closure

The leading term (in powers of $\lambda$ and $\gamma$) in (3.14) is

$$\mathcal{N}' = -\Lambda_2 A_4^{-1} b(G), \quad (4.1)$$

where we write, as we shall do from now on, $G$ for $G_2$, since we shall only deal with $G_2$. Let us write this more concretely.

To define the inverse of $A_4$ we define the stationary state correlation functions $G_n$ as limits $\epsilon \rightarrow 0$ of the stationary state $G^\epsilon_n$, where, in equation (2.1), we add a term $\epsilon p_x dt$, and in (2.4) a term $4\epsilon t \delta_{xy}$, i.e. we put a noise and a friction of size $\epsilon$ everywhere. Then, the matrix $A$ becomes

$$A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\epsilon \end{pmatrix} \quad (4.2)$$
and, letting
\[ R(t) = e^{tA}, \] (4.3)
we have,
\[ -A_4^{-1} = \int_0^\infty e^{tA} dt = \int_0^\infty R(t)^{\otimes 4} dt. \]
Thus,
\[ -\left(A_4^{-1}b(G)\right)_{i_1i_2i_3i_4} = 6 \sum_j \int_0^\infty dt \left( R(t) \Lambda \right)_{i_1j_2j_3j_4} \prod_{a=2}^4 \left( R(t)G \right)_{i_aoa} + (3 \text{ permutations}), \]
with \( j = (j_2, j_3, j_4) \) and, then, with \( i = (i_2, i_3, i_4) \),
\[ \left(-\Lambda_2 A_4^{-1}b(G)\right)_{i'i} = 6 \sum_j \int_0^\infty dt \left( R(t) \Lambda \right)_{i_1j_2j_3j_4} \prod_{a=2}^4 \left( R(t)G \right)_{i_aoa} \]
\[ +3 \sum_j \sum_{i_1i_2i_3} \int_0^\infty dt \Lambda_{i_1j_2j_3} \left( R(t) \Lambda \right)_{i_1j_3} \prod_{a=2}^3 \left( R(t)G \right)_{i_aoa} \left( R(t)G \right)_{i_3o3} + (i \leftrightarrow i') \]
Recalling the expression (3.11), some calculation gives
\[ N'_{\alpha\beta}(x, y) = 6\lambda^2 \delta_{\alpha2} \int_0^\infty dt \sum_z \left( R(t)G \right)_{11}(x, z)^2 \]
\[ \cdot \left[ 3R_{12}(t, x - z) \left( R(t)G \right)_{\beta 1}(y, z) + \left( R(t)G \right)_{11}(x, z)R_{\beta 2}(t, z - y) \right] + tr \] (4.4)
where tr means that both \( \alpha, \beta \) and \( x, y \) are interchanged and where the translation invariance of \( R \) was used. Since \( \partial_t R = AR \) we have, from (4.2),
\[ \partial_t R_{1\alpha} = R_{2\alpha} \] (4.5)
and, using \( \partial_t R = RA \), we get:
\[ R_{\beta 2} = -\partial_t (RG_0)_{\beta 1}, \] (4.6)
where
\[ G_0 = \begin{pmatrix} \omega^{-2} & 0 \\ 0 & 1 \end{pmatrix}. \] (4.7)
Inserting (4.6) in (4.4), and integrating by parts, (4.4) becomes:
\[ N'_{\alpha\beta}(x, y) = N_{\alpha\beta}(x, y) + 6\lambda^2 \delta_{\alpha2} \delta_{\beta1} \sum_z Q(x, z)^3 \omega^{-2}(z - y) + tr \] (4.8)
where we denote \( G_{11} = Q \) and
\[ N_{\alpha\beta}(x, y) = 18\lambda^2 \delta_{\alpha2} \lim_{\epsilon \to 0} \int_0^\infty dt \sum_z \left[ \left( R(t)G \right)_{11}(x, z) \right]^2 \]
\[ \cdot \left[ R_{12}(t, x - z) \left( R(t)G \right)_{\beta 1}(y, z) + \left( R(t)G \right)_{21}(x, z) \left( R(t)G_0 \right)_{\beta 1}(z - y) \right] + tr \] (4.9)
The closure equation that we will study is the replacement of the exact equation (3.13) by

\[(A_2 - \Gamma_2)G + N(G) + C = 0 \quad (4.10)\]

i.e. we drop the terms \(\Sigma_2, \Sigma_4, \Gamma_4, G_\text{c}_6\), as well as the second term in (4.8). Before discussing the motivation for this approximation, let us write (4.10) explicitly. Let

\[G = \begin{pmatrix} Q & H \\ H^T & P \end{pmatrix} \quad (4.11)\]

so, e.g. \(H(x, y) = \text{Eq}_x p_y\) and \(H^T(x, y) = H(y, x)\). Then, (4.10) can be written as:

\[H + H^T = 0 \quad (4.12)\]

\[P - \omega^2 Q - HT\Gamma + N_{12} = 0 \quad (4.13)\]

\[-\omega^2 H - H^T\omega^2 - PT\Gamma - \Gamma P + N_{22} + C = 0 \quad (4.14)\]

where

\[\Gamma_{xy} = \gamma \delta_{xy}(\delta_{x1} + \delta_{x1,N}) \quad (4.15)\]

and \(C\) was defined in (3.3).

Let

\[J = \frac{1}{2}(H - H^T) \quad (4.16)\]

so by (4.12), \(H = J, H^T = -J\), and we write:

\[G = \begin{pmatrix} Q & J \\ -J & P \end{pmatrix}. \quad (4.17)\]

Then (4.13) and (4.14) can be written as:

\[2P = \omega^2 Q + Q\omega^2 + J\Gamma - \Gamma J - N_{12} - N_{12}^T \quad (4.18)\]

\[\omega^2 Q - Q\omega^2 + J\Gamma + N_{12}^T - N_{12} = 0 \quad (4.19)\]

\[\omega^2 J - J\omega^2 + PT\Gamma + \Gamma P - N_{22} - C = 0 \quad (4.20)\]

(4.18) - (4.20) is a system of nonlinear equations for the correlation functions \(Q, J, P\).

An important property of \(N\) in equation (4.9) is that, for all \(T\),

\[N(TG_0) = 0 \quad (4.21)\]
for

\[ G_0 = \begin{pmatrix} \omega^{-2} & 0 \\ 0 & 1 \end{pmatrix} \]  

(4.22)

Indeed, \((R(t)G_0)_{21} = R(t)_{21}\omega^{-2} = -R(t)_{12}\) (see formula (5.2) below). Thus, at \(\gamma = 0\), our set of equations (4.18)-(4.20) has a 1-parameter family of solutions

\[ Q = T\omega^{-2}, \quad P = T, \quad J = 0 \]

and, for the equilibrium case, with \(\gamma \neq 0\), \(T_1 = T_2\), only one of these persists, namely the one with \(T = T_1\).

Note that the true equilibrium Gibbs state has \(P = T, \quad J = 0\), and

\[ \hat{Q}(k) = T\left(\omega(k)^2 + \sigma(k, \lambda, T)\right)^{-1} \]

with \(\sigma = \mathcal{O}(\lambda)\). The terms \(\Sigma_2, \Sigma_4\) and the second term in (4.8) would contribute to changing the \(\sigma = 0\) of our closure solution to a \(\sigma\) which would agree with the true \(\sigma\) to \(\mathcal{O}(\lambda^2)\). Dropping these terms, as well as \(\Gamma_4\), is done for convenience and should not change our analysis qualitatively. The uncontrolled approximation consists in dropping \(G_6^c\). Presumably, for small \(\lambda\), this would not make a qualitative difference. It would be interesting to try to prove this in the kinetic limit \(\lambda = \mathcal{O}(1/\sqrt{N})\). In the rest of this paper, we will discuss only equations (4.18)-(4.20).

5 Changes of coordinates

For large \(N\) we expect the solution to the equations (4.18)-(4.20) to be translation invariant in the directions perpendicular to the 1-direction with a slowly varying dependence of the first coordinate \(x_1\). It is therefore convenient to represent \(G\) in coordinates that are suited to such behaviour.

We first write eqs. (4.18)-(4.20) in terms of the Fourier transform of \(G\). Recall that \(x \in \mathbb{Z}^d\) with \(x_1 \in \mathbb{Z}_{2N}\). Introduce momentum variables:

\[ q = (q, \mathbf{q}), \]

and write

\[ G(x, y) = \int e^{i(qx+q'y)}\hat{G}(q, q')dq dq' \]

with the shorthand notation

\[ \int dq = \sum_q \frac{1}{2N} \int_I \frac{dq}{(2\pi)^{d-1}} \]  

(5.1)

where \(q \in \frac{\pi}{N}\mathbb{Z}_{2N}\) and \(I = [0, 2\pi]^{d-1}\). Then \(R\) becomes a Fourier multiplier

\[ \hat{R}(t, q) = \begin{pmatrix} \partial_t + \epsilon & 1 \\ \partial_t(\partial_t + \epsilon) & \partial_t \end{pmatrix} \frac{\sin \tilde{\omega}(q)t}{\tilde{\omega}(q)}e^{-\alpha/2}, \]
where \( \tilde{\omega}(q) = (\omega^2(q) - \frac{t^2}{4})^{1/2} \). Then, letting \( \epsilon \to 0 \) in the matrix, we get

\[
\hat{R}(t, q) = \begin{pmatrix}
\cos \omega(q)t & \frac{1}{\omega(q)} \sin \omega(q)t \\
-\omega(q) \sin \omega(q)t & \cos \omega(q)t
\end{pmatrix} e^{-\epsilon t/2}
\]

\[
= \frac{1}{2} \sum_{s=\pm 1} e^{i(s\omega(q)-\epsilon/2)t} \left( \begin{pmatrix} 1 & 0 \\ is\omega(q) & 1 \end{pmatrix} \right),
\]

so,

\[
\left( R(t) G \right)_1(q, q') = \frac{1}{2} \sum_{s=\pm 1} e^{i(s\omega(q)-\epsilon/2)t} W_s(q, q') \left( \begin{pmatrix} 1 & 0 \\ is\omega(q) & 1 \end{pmatrix} \right)
\]

where

\[
W_s(q, q') = \hat{Q}(q, q') + is\omega(q)^{-1} \tilde{J}(q, q').
\]

Thus, (4.9) becomes

\[
\hat{N}^{\alpha\beta}(q, q') = \frac{9}{8} (2\pi)^d \lambda^2 (N^{\alpha\beta}(q, q') + N^{\beta\alpha}(q', q))
\]

with

\[
N(q, q') = i \sum_s \int d\mu \left( \sum_{i=1}^4 s_i \omega(q_i) + i\epsilon \right)^{-1} \prod_{i=1}^2 W_{s_i}(q_i, q_i') \cdot \begin{pmatrix} 0 & 0 \\ is_4\omega(q') \end{pmatrix} \\
[ -is_3\omega(q_3)^{-1}\delta(q_3 + q_3') W_{s_4}(q_4, q_4') + is_3\omega(q_3) W_{s_3}(q_3, q_3') \omega(q_4)^{-2}\delta(q_4 + q_4') ]
\]

where

\[
d\mu = \delta \left( q - \sum_{i=1}^3 q_i \right) \delta \left( \sum_{i=1}^4 q_i \right) \delta(q' - q) \prod_{i=1}^4 dq_i dq_i',
\]

and we replaced \( 2\epsilon \) by \( \epsilon \). We got in (5.4) a factor \( (2\pi)^d \) for each lattice sum in the defintion of \( \hat{N}^{\alpha\beta}(q, q') \), and a factor \( \frac{1}{2} \) for each of the four sums over \( s_i \). Note that, because of the sum over \( s \) in (5.5), only terms that are even in \( s \) contribute, which means that the factor \( \left( \sum_{i=1}^4 s_i \omega(q_i) + i\epsilon \right)^{-1} \) gives rise to a delta function if the integrand is even in \( s \), and a principal value if it is odd.

We will look for solutions to (4.18)-(4.20) which are translation invariant in the directions orthogonal to the 1-direction. Thus, we look for solutions of the form

\[
\hat{G}(q, q') = (2\pi)^{d-1} \delta(q + q') g(q, q', q)
\]

and

\[
G(x, y) = \int e^{i(q(x+y) + i\epsilon(q(x-y))} g(q, q', q) dq dq' dq,
\]

13
where we write $x = (x, x)$, and similarly for $y$.

It will be convenient to change coordinates in the 1-direction. Let $g$ be a $2\pi$ periodic function on $\frac{\pi}{N}Z_{2N} \times \frac{\pi}{N}Z_{2N}$. Define

$$\tilde{g}(p, k) = g(p + k, p - k)$$

on the set

$$\{p, k \in \frac{\pi}{2N}Z_{4N} \mid p + k \in \frac{\pi}{N}Z_{2N}\}$$

Then,

$$\sum_{q, q'} \left(\frac{1}{2N}\right)^2 e^{i(qx + iq'y)} \tilde{g}(q, q') = \sum_{p, k} \left(\frac{1}{2\sqrt{2}N}\right)^2 e^{i(p(x+y) + ik(x-y))} \tilde{g}(p, k),$$

(5.7)

Indeed, each pair $(q, q')$ gets counted exactly twice in the $(p, k)$-sum (because the pair $(p + 2N, k)$ gives the same contribution as the pair $(p + 2N, k + 2N)$, with addition modulo $4N$), which accounts for the factor $\sqrt{2}$. Note that, in the $N \to \infty$ limit, both sides tend to $\int_{[-\pi, \pi]^2} (2\pi)^{-2}$ since the lattice spacing in the RHS of (5.7) is $\sqrt{2} \cdot \frac{\pi}{2N}$.

Let $p = (p, 0)$. We have then

$$G(x, y) = \int e^{i(p(x+y) + ik(x-y))} \tilde{G}(p, k) dp dk$$

(5.8)

where $\tilde{G}(p, k) = g(p + k, p - k, k)$ and the integrals over the first components are Riemann sums, with the same convention as in (5.7). The function $\tilde{G}$ is $2\pi$ periodic in all the variables and, moreover,

$$\tilde{G}(p + \tilde{\pi}, k - \tilde{\pi}) = \tilde{G}(p, k)$$

(5.9)

where $\tilde{\pi} = (\pi, 0)$.

We will, from now on, work in the $p, k$ variables, drop, in general, the tilde in $\tilde{G}(p, k)$, and we shall not distinguish between $p$ and $p$, unless we need to stress that $p$ is a number. Since the components of $p$ (unlike those of $k$) other than the first one are always 0, this abuse of notation is harmless.

Equations (5.5) then becomes

$$N(p, k) = \sum_s \int d\nu(s_i \omega(p_i + k_i) + i\epsilon)^{-1} \prod_{i=1}^2 W_{s_i}(p_i, k_i) \cdot \begin{pmatrix} 0 & 0 \\ 1 & i\omega(p_4 + k_4) \end{pmatrix} s_3 \omega(p_3 + k_3) \left[ \omega(p_3 + k_3)^{-2} \delta(2p_3)W_{s_4}(p_4, k_4) - \omega(p_4 + k_4)^{-2} \delta(2p_4)W_{s_3}(p_3, k_3) \right]$$

(5.10)

where

$$d\nu = \delta(2p - \sum(p_i + k_i))\delta(\sum(p_i - k_i))\delta(p - k - p_4 - k_4) dp dk.$$
and \( k = (k_i)_{i=1}^4, P = (p_i)_{i=1}^4 \) and where we used the identity (see (5.6)):

\[
q - \sum_{i=1}^{3} q_i = p + k - \left( \sum_{i=1}^{4} (p_i + k_i) - q_4 \right) = p + k - \left( \sum_{i=1}^{4} (p_i + k_i) - q' \right)
\]

\[
= p + k - \sum_{i=1}^{4} (p_i + k_i) + p - k = 2p - \sum_{i=1}^{4} (p_i + k_i)
\]

We can then write

\[
\mathcal{N}_{12}(p, -k) \equiv \frac{9}{8} (2\pi)^{3d} \lambda^2 n_1(p, k), \quad \mathcal{N}_{22}(p, k) \equiv \frac{9}{8} (2\pi)^{3d} \lambda^2 (n_2(p, k) + n_2(p, -k))
\]

with

\[
n(W)(p, k) = \left( \frac{n_1}{n_2} \right) = \sum_n \int \prod_{i=1}^{2} W_s_i(p_i, k_i - p_i) s_3 \omega(k_3) \left( \frac{1}{i s_4 \omega(k_4)} \right) \cdot \left[ \omega(k_3)^{-2} \delta(2p_3) W_{34}(p_4, k_4 - p_4) - (3 \leftrightarrow 4) \right] \nu_{spk}(dp \, dk)
\]

where

\[
\nu_{spk}(dp \, dk) = \left( \sum_i s_i \omega(k_i) + i \epsilon \right)^{-1} \delta \left( 2(p - \sum_i p_i) \right) \delta \left( 2(p - \sum_i k_i) \right) \delta(p - k - k_4) dp \, dk
\]

In (5.13) we have shifted the \( k_i \)-integrals by \(-p_i\) compared to (5.10).

Let us introduce the convenient notation

\[
\omega(p, k) = \left( \frac{1}{2}(\omega(p + k)^2 + \omega(p - k)^2) \right)^{\frac{1}{2}}
\]

and

\[
\delta \omega^2(p, k) = \omega(p + k)^2 - \omega(p - k)^2
\]

Then the equations (4.18)-(4.20) read as follows in the \( p, k \) variables:

\[
\omega(p, k)^2 Q + \frac{1}{2}((J \Gamma - \Gamma J)^\sim - \mathcal{N}_{12}(p, k) - \mathcal{N}_{12}(p, -k)) = P
\]

\[
\delta \omega^2(p, k) Q + (J \Gamma + \Gamma J)^\sim + \mathcal{N}_{12}(p, -k) - \mathcal{N}_{12}(p, k) = 0
\]

\[
\delta \omega^2(p, k) J + (P \Gamma + \Gamma P)^\sim - \mathcal{N}_{22}(p, k) - \tilde{C}(p, k) = 0
\]

We look for a solution that satisfies in \( x \)-space \( G(x, y) = G(-x, -y), Q(x, y) = Q(y, x) \) and \( J(x, y) = -J(y, x) \) i.e.

\[
Q(p, k) = Q(p, -k) = Q(-p, k)
\]

\[
J(p, k) = -J(p, -k) = -J(-p, k)
\]

This is consistent since \( \mathcal{N}_{\alpha\beta}(p, k, W) = \mathcal{N}_{\alpha\beta}(-p, -k, W(-\cdot, -\cdot)) \).

We will see below that the \( \lim_{\epsilon \to 0} \mathcal{N} \) is well defined for bounded \( W(p, k) \). Before that, let us outline our arguments.
6 Heuristics

Before getting into the details we give an outline of the argument. Since the nonlinear term \( N \) depends only on \( Q \) and \( J \), \( P \) can be solved from (5.17) in terms of them. Then (5.18) and (5.19) are two equations for the two unknown functions \( Q \) and \( J \) which we write as

\[
\delta \omega^2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} J \\ Q \end{pmatrix} + N(Q, J) + N_\Gamma(Q, J) = \begin{pmatrix} 0 \\ C \end{pmatrix}
\]

where

\[
N(Q, J) = \begin{pmatrix} N_{12}(p, -k) - N_{12}(p, k) \\ -N_{22}(p, k) \end{pmatrix}
\]

and

\[
N_\Gamma(Q, J) = \begin{pmatrix} J\Gamma + \Gamma J \\ P\Gamma + \Gamma P \end{pmatrix}
\]

where \( P \) is expressed in terms of \( J \) and \( Q \) in (5.17). \( N_\Gamma \) collects the friction terms that have both linear and nonlinear contributions but will be treated as a perturbation.

For large \( N \) the solution to this equation will be locally in the \( x_1 \) coordinate a perturbation of the equilibrium state (4.22) given in the new variables by

\[
Q_T(p, k) = T\omega(p + k)^{-2}\delta(2p)
\]

where

\[
\delta(p) = 2N\delta_{p,0}.
\]

Hence the importance to study the linearization of \( N \) around that state. It turns out (see Section 9.2) that it is given by an operator which is a multiplier in the variable \( p \):

\[
DN(Q_T, 0) \begin{pmatrix} J \\ Q \end{pmatrix}(p, k) = \mathcal{L}_p \begin{pmatrix} J(p, \cdot) \\ Q(p, \cdot) \end{pmatrix}(k).
\]

\( \mathcal{L}_p \) is a matrix of operators

\[
\mathcal{L}_p = \begin{pmatrix} \mathcal{L}_{11}(p) & \mathcal{L}_{12}(p) \\ \mathcal{L}_{21}(p) & \mathcal{L}_{22}(p) \end{pmatrix}.
\]

The translation invariant equilibrium (6.4) has support at \( p = 0 \) (and due to the periodicity (5.20) at \( p = \pi \)). The nonequilibrium solution will also have most of its mass in the neighborhood of these points. Thus it is important to understand \( \mathcal{L}_0 \). It will turn out that

\[
\mathcal{L}_{ij}(0) = \mathcal{L}_{ij}(\pi) = 0, \quad i \neq j
\]

whereas \( \mathcal{L}_{11}(0) \) is invertible. Invertibility of \( \mathcal{L}_0 \) would then follow from invertibility of \( \mathcal{L}_{22}(0) \). This, however, is not the case: \( \mathcal{L}_{22}(0) \) has two zero modes.
One of them is easy to understand. Eq. (6.4) is a one-parameter family of solutions to the \( \gamma = 0 \) equations. Hence the derivative with respect to the parameter \( T \) is a zero eigenvalue eigenvector of the linearization \( \mathcal{L}_{22}(0) \), i.e.

\[
\mathcal{L}_{22}(0) \omega^{-2} = 0.
\]  

(6.9)

There is, however, a second zero mode for \( \mathcal{L}_{22}(0) \):

\[
\mathcal{L}_{22}(0) \omega^{-3} = 0.
\]  

(6.10)

While the first zero mode has to persist for the full Hopf equations due to the one parameter family of Gibbs states that solve them for \( \gamma = 0 \), the second one is an artifact of the closure approximation. The nonlinear terms in the closure equations can be interpreted as describing phonon scattering and in our case only processes where two phonons scatter occur. These processes conserve phonon energy leading the the first zero mode and also phonon number, leading to the second one. The connected six-point correlation function which was neglected in the closure approximation would produce terms that violate phonon conservation and remove the second zero mode. However, for weak anharmonicity its eigenvalue would be close to zero and should be treated with some perturbation of the present analysis.

The second zero mode leads one to expect that our equations have in the \( \gamma = 0 \) limit a two parameter family of stationary solutions which indeed is the case. These are given by (see Section 7.3)

\[
Q_{T,A}(p,k) = T(\omega(p+k)^2 - A\omega(p+k))^{-1}\delta(2p).
\]  

(6.11)

with \( J = 0 \) and \( P \) given by (4.18). For (6.11) to be a well defined covariance we need positivity of the denominator which holds if \( A < m^2 \). The zero-mode (6.10) is proportional to the derivative of \( Q_{T,A} \) with respect to \( A \), at \( A = 0 \).

These considerations lead to the following ansatz for the solution

\[
Q(x, y) = Q_0(x-y) + r(x, y),
\]  

(6.12)

where

\[
Q_0(x-y) = Q_{T(x),A(x)}(x-y) + Q_1(x, y)
\]  

(6.13)

The first term here is of local equilibrium form with slowly varying temperature and “chemical potential” profiles \( T(x) \) and \( A(x) \) (\( A \), or more precisely \( T \cdot A \), can be considered as being related to a chemical potential, because it arises from the conservation of a number current, see section 7.2), and the second is a small perturbation (see (8.16) below and the remark following it). \( T(x) \) and \( A(x) \) are determined from the current conservation laws which are projections of the equations (6.1) as follows. The operator \( \mathcal{L}_{22}(0) \) has two left zero modes. Projection onto them the equation (6.1) yields two nonlinear elliptic equations for the functions \( T(x) \) and \( A(x) \) coupled to the rest of the variables i.e. \( J \) and \( r \). For the latter taken in a subspace complementary to the zero modes the linear operator \( \delta \omega^2 + \mathcal{L}_p \) is invertible and \( J \) and \( r \) can be determined by fixed point arguments in a suitable Banach space as functionals of \( T(x) \) and \( A(x) \). The projected equations then allow us to determine the latter.

Our main result, stated more precisely in Section 8, says that \( Q(x, y) \) is as above, while the currents corresponding to the two conservation laws discussed above and defined in section 7, are linearly related, to leading order in \( |T_1 - T_2| \), to the gradients of \( T(x) \) and \( A(x) \). \( T(x) \) is, to leading order in \( |T_1 - T_2| \), linear in \( x \), and therefore the currents are, to leading order, \( \mathcal{O}(\frac{|T_1 - T_2|}{N^{\frac{3}{2}}}) \).
7 Current conservation

In the Introduction we recalled that the Hamiltonian structure of the dynamics leads to a local conservation law. We will show that our closure too has such a conservation law.

7.1 Heat-current

We start with a simple identity

**Lemma 7.1** \( f N_{22}(p, k)dk = 0 \) for all (bounded) \( W \).

**Proof** From (5.11) we see that \( f dk \nu_{p,k} \) is symmetric under the simultaneous interchanges \( s_3 \leftrightarrow s_4 \), \((p_3, k_3) \leftrightarrow (p_4, k_4)\). Hence, the integral over \( k \) of the second term in (5.12) vanishes, because the integrand in (5.13) is antisymmetric and the rest of the integrand is symmetric under those simultaneous interchanges. \( \square \)

Note that, by (5.8),

\[
N_{22}(x, x) = \int e^{2ipx} N_{22}(p, k) dp dk \tag{7.1}
\]
i.e., by Lemma 7.1, \( N_{22}(x, x) = 0 \), \( \forall x \). This follows also by inspection from (4.9) since \( \left(R(t)G_0\right)_{21} = -R(t)_{12} \) (by (5.2)) and \( R(t)_{12}(z - y) = R(t)_{12}(x - z) \) if \( x = y \).

Consider now equation (4.20) restricted to the diagonal \( x = y \). Let \( J' = \omega J + J \omega \). Then, \( \omega^2 J - J \omega^2 = \omega J' - J' \omega \). Defining, for \( e_\mu \), the unit vector in the \( \mu \)-direction,

\[
j_\mu(x) = J'(x - e_\mu, x), \tag{7.2}
\]
we have, for \( \omega \) given in (2.5):

\[
(\omega^2 J - J \omega^2)(x, x) = (-\Delta J' + J' \Delta)(x, x) = \sum_\mu \left(-J'(x - e_\mu, x) - J'(x + e_\mu, x) + J'(x, x - e_\mu) + J'(x, x + e_\mu)\right) = 2 \sum_\mu (J'(x, x + e_\mu) - J'(x - e_\mu, x)) = 2 \nabla \cdot \vec{j}(x),
\]
where \( (\nabla_\mu f)(x) = f(x + e_\mu) - f(x) \). Note that, for other functions \( \omega \), there will be also a current, but its form will depend on \( \omega \). In view of Lemma 7.1, (4.20), for \( x = y \), reads

\[
\nabla \cdot \vec{j}(x) + \gamma P(x, x)(\delta_{x0} + \delta_{xN}) = \gamma (T_1 \delta_{x0} + T_2 \delta_{xN})
\]
or, since \( P(x, x) \) depends only on \( x \),

\[
\nabla \cdot \vec{j}(x) = \gamma((T_1 - P(0, 0))\delta_{x0} + (T_2 - P(N, N))\delta_{xN}), \tag{7.3}
\]
which is a current conservation equation; the heat current \( \vec{j} \) has sources on the boundary. Since \( \vec{j} \) depends only on \( x \), we define \( j(x) = j_1(x) \) and, so,

\[
\nabla \cdot \vec{j}(x) = j(x + 1) - j(x). \tag{7.4}
\]

It is a useful exercise to rewrite this in the \((p,k)\)-variables. We have, see (2.5),

\[
\omega(p + k) - \omega(p - k) = 2\left(\cos(p - k) - \cos(p + k)\right) = 4 \sin p \sin k
\]
and so
\[ \delta \omega^2(p, k) = 4 \sin p \sin k (\omega(p + k) + \omega(p - k)). \] (7.5)

Thus
\[ \int \delta \omega^2(p, k) J(p, k) \, dk = 2(e^{2ip} - 1) j(2p) \] (7.6)

where
\[ j(p) = -i \int dk e^{-ip/2} \sin k (\omega(p/2 + k) + \omega(p/2 - k)) J(p/2, k) \] (7.7)

for \( p \in \frac{\pi}{N}\mathbb{Z}_{2N} \). In (7.6) we used the fact that the \( k \) integral is \( \pi \)-periodic in \( p \), due to (5.9), to write it as a function of \( 2p \). It may be checked directly that this is the Fourier transform of \( j_1(x) \) given by (7.2).

### 7.2 Number current

As stated in Section 6, the closure equations possess another, approximate, conservation law which we will derive now. Let \( \rho(p, k) \) be given by
\[ \rho(p, k) = \omega(p, k)^{-1}, \] (7.8)

with \( \omega(p, k) \) given by (5.15), and project \( \mathcal{N}_{22} \) now onto \( \rho \) instead of the function 1 as in Lemma 7.1. Let
\[ \theta(p) = \int \rho(\frac{p}{2}, k) \mathcal{N}_{22}(\frac{p}{2}, k) \, dk = \frac{9}{4} (2\pi)^{3d} \lambda^2 \int \rho(\frac{p}{2}, k) n_2(\frac{p}{2}, k) \, dk \] (7.9)

Then \( \theta \) is \( 2\pi \) periodic. Unlike what happened in Lemma 7.1 \( \theta \) is not zero, but it will turn out to be very regular, see Proposition 9.7 and Appendix B below, due to the fact that (7.8) at \( p = 0 \) is a left zero eigenvector of the linearization of \( \mathcal{N}_{22} \).

We will now integrate equation (5.19) multiplied by a linear combination of \( \rho \) and 1. For this, write it in the \( (p, k) \) representation. The covariance is
\[ \tilde{C}(p, k) = 2\gamma(T_1 + T_2 e^{-2iNp}) \] (7.10)

and the friction term
\[ (\Gamma P + P \Gamma)(q_1, q_2) = \gamma \int dq \left[(1 + e^{i(q-q_1)N}) \tilde{P}(q, q_2) + (1 + e^{i(q-q_2)N}) \tilde{P}(q_1, q)\right]. \]

So, after shifting \( q \) in the first integral by \( \frac{q_1}{2} \) and in the second by \( \frac{q_2}{2} \), we get:
\[ (\Gamma P + P \Gamma)\sim(p, k) = \gamma \int dq \left[\tilde{P}\left(\frac{q}{2} + k - p\right) + \tilde{P}\left(\frac{q}{2} - p + k\right)\right] (1 + e^{i(q-2p)N}). \] (7.11)

Let
\[ \eta(p, k) = \rho(p, k) - \int \rho(p, k) \, dk, \] (7.12)
with \( \rho \) given by (7.8). Integrating equation (5.19) multiplied by \( \eta \), we get

\[
j'(x + 1) - j'(x) - \theta(x) + \gamma(x) = 0 \quad (7.13)
\]

where

\[
j'(p) = -i \int dk e^{-ip/2} \eta(p/2, k) \sin k \left( \omega(p/2 + k) + \omega(p/2 - k) \right) J(p/2, k) \quad (7.14)
\]

and

\[
\gamma(p) = \gamma \int dk d\xi \psi(p, k, q) \tilde{P} \left( \frac{\xi}{2}, k \right) \left( 1 + e^{i(q-p)N} \right) \quad (7.15)
\]

where

\[
\psi(p, k, q) = \eta(p, k + \frac{p}{2} - \frac{q}{2}) + \eta(p, k - \frac{p}{2} + \frac{q}{2}) \quad (7.16)
\]

is a smooth function. Note that the covariance dropped out from (7.13) because \( \tilde{C}(p, k) \) is independent of \( k \), and we subtract from \( \rho \) its average in the definition of \( \eta \). Also, the latter does not contribute to \( \theta \) due to Lemma 7.1.

Equation (7.13) is again a conservation law: \( \gamma \) is a boundary term and, as we will see, \( \theta \) vanishes in the limit of translation invariance. We call the current \( j' \) the particle number current.

### 7.3 Generalized Gibbs states

We finish this section by checking that the states (6.11), with \( J = 0 \), are indeed solutions to the \( \gamma = 0 \) equations. Indeed, now,

\[
W_s(p, k - p) = Q_{T,A}(k) \delta(2p),
\]

with \( Q_{T,A} = T(\omega(k)^2 - A\omega(k))^{-1} \). So, from (5.13), we get:

\[
n(p, k) = \delta(2p) \sum_s \int \prod\limits_{i=1}^{2} Q_{T,A}(k_i) s_3\omega(k_3) \left( \frac{1}{i\omega(k_1)} \right) \left[ \omega(k_2)^{-2} Q_{T,A}(k_4) - \omega(k_4)^{-2} Q_{T,A}(k_3) \right] \cdot \left( \sum s_i \omega(k_i) + i\epsilon \right)^{-1} \delta(2p - \sum k_i) \delta(p - k - k_4) dk
\]

\[
(7.17)
\]

\( n_1 \) is obviously even in \( k \), thus (5.18) holds. As for (5.19), write the \([-\cdot]\) in (7.17) as

\[-AT^{-1}Q_{T,A}(k_3)Q_{T,A}(k_4) \left[ \omega(k_3)^{-1} - \omega(k_4)^{-1} \right],
\]

and use \((x + i\epsilon)^{-1} = \mathcal{P} \left( \frac{x}{\epsilon} \right) - 2\pi i \delta(x)\) to get:

\[
n_2(p, k) = -2\pi \delta(2p) AT^{-1} \tilde{n}_2(p, k), \quad (7.18)
\]

\[
\tilde{n}_2(p, k) = \sum_s \int \prod\limits_{i=1}^{4} Q_{T,A}(k_i) s_3 s_4 \left[ \omega(k_4) - \omega(k_3) \right] \cdot \left( \sum s_i \omega(k_i) \right) \delta(2p - \sum k_i) \delta(p - k - k_4) dk
\]

\[
(7.19)
\]
(By $s \to -s$ symmetry only the delta function contributes). The integral in (7.17) is supported on
\[ \sum s_i \omega(k_i) = 0. \] (7.20)
We will choose $m^2$ in (2.4) large enough so that (7.20) forces
\[ \sum s_i = 0 \] (7.21)
By symmetry, we may replace $s_3$ by $\frac{1}{3} \sum_{i=1}^{3} s_i$ and, by (7.21) also by $-\frac{1}{3} s_4$. Again, by symmetry, $s_3 \omega(k_3)$ may be replaced by $\frac{1}{3} \sum_{i=1}^{3} s_i \omega(k_i)$ and by (7.20) by $-\frac{1}{3} s_4 \omega(k_4)$. Doing the first replacement for $s_3 \omega(k_4)$ in the first term in the $[\cdot]$ in (7.19), and the second for $s_3 \omega(k_3)$ in the second term, we see that (7.19) vanishes. Note that $\tilde{n}_2$ vanishes for all $p$ and not only for $2p = 0 \mod 2\pi$. We shall need this later (see the derivation of (9.19) below). Hence (6.11) solves (5.17)-(5.19) if $\gamma = 0$.

Note however that these are not equilibrium $T_1 = T_2$ solutions, unless $A = 0$. This is because for $A \neq 0$, the noise and friction terms in (4.20) will not any more balance each other.

8 The space of local equilibrium solutions

We will now describe the space where (4.18)-(4.20) are solved. To motivate our choice, consider the current conservation equation (7.3). Summing over $x$, we get, since $j(x)$ is periodic:
\[ T_1 - P(0,0) = -(T_2 - P(N,N)) \equiv j_0 \] (8.1)
and so
\[ j(x + 1) - j(x) = j_0 (\delta_{x0} - \delta_{xN}) \] (8.2)
which is solved by
\[ j(x) = \begin{cases} j(0) + j_0 & x \in [1, N] \\ j(0) & x \in [-N+1, 0] \end{cases} \] (8.3)
Since $j(-x) = -j(x)$ we get $j(0) = -\frac{1}{2} j_0$. The Fourier transform of this is
\[ j(p) = \begin{cases} j_0 \frac{1 - e^{ipN}}{e^{ip} - 1} & p \in \pi \mathbb{N}, p \neq 0 \\ j_0 & p = 0 \end{cases} \] (8.4)
(note that $e^{ipN}$ takes only the values $\pm 1$). The current $j_0$ will turn out to be $O(1/N)$.

We now describe the space to which functions such as $J(p,k)$ belong. This space has to encode the $1/p$ singularity at origin in the equation (8.4) as well as the factor $e^{ipN}$ coming from the fact that in $x$ space there are two special points in the first coordinate, the origin and $N$. 21
Let $\mathcal{H}$ be the space of continuous functions $f(p,k)$ on
\[ \Omega = \{(p,k) \mid p, k \in \frac{\pi}{2N} \mathbb{Z}_{4N}, \ p + k \in \frac{\pi}{N} \mathbb{Z}_{2N}, \ k \in [-\pi, \pi]^{d-1}\} \] (8.5)
that are $2\pi$-periodic in all the variables, and invariant under $(p,k) \to (p + \pi, k + \pi)$. We denote by $\|f\|_{\infty}$ the sup norm in $\mathcal{H}$.

Let $\Omega_+ = \{(p,k) \in \Omega \mid p \in \frac{\pi}{N} \mathbb{Z}_{2N}\}$ and $\Omega_- = \Omega \setminus \Omega_+$. Then
\[ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \]
with
\[ f(p,k) = f_+(p,k)\sigma_+(2p) + f_-(p,k)\sigma_-(2p) \] (8.6)
with
\[ \sigma_\pm(p) = 1 \pm e^{ipN} \] (8.7)
and $f_\pm = \pm f|_{\Omega_\pm}$.

Define the discrete $C^\alpha$-norm for $f \in \mathcal{H}_\pm$ as
\[ \|f\|_\alpha = \sup_{p,k,\lambda,\mu} \left( |f(p,k)| + |\lambda|^{-\alpha}|f(p + \lambda e_1, k) - f(p,k)| + |\mu|^{-\alpha}|f(p,k + \mu e_1) - f(p,k)| \right) \] (8.8)
where $\lambda, \mu \in \frac{\pi}{N} \mathbb{Z}_{2N} \setminus 0$ (note that $p + \lambda \in \Omega_\pm$ if $p \in \Omega_\pm$). Set then, for $f \in \mathcal{H}$,
\[ \|f\|_\alpha \equiv \|f_+\|_\alpha + \|f_-\|_\alpha. \]

Let
\[ d(p) = \begin{cases} (e^{ip} - 1)^{-1} & p \neq 0 \\ 0 & p = 0, \end{cases} \] (8.9)
and let, with some abuse of notation,
\[ d(p) = e^{ip} - 1. \] (8.10)

We will then consider a space $\mathcal{S}$ of functions $J(p,k)$ of the form
\[ J = N^{-1}\delta(2p)J_0 + (N(d(2p))^{-1}J_1 + N^{\alpha/2-1}J_2 + (Nd(2p))^{-3/2}J_3 \] (8.11)
where $J_i(p,k)$ are in $\mathcal{H}$. Define
\[ \|J\|_S = \max\{\|J_i\|_\alpha, \|J_3\|_{\infty}, \ i = 0, 1, 2\}. \] (8.12)

More properly, $\mathcal{S}$ is the space $\mathcal{H}^{\oplus 4}$ of 4-tuples $(J_0, \ldots, J_3) \equiv \mathbf{J}$ which is a Banach space with this norm. We identify $J$ and $\mathbf{J}$ when no confusion arises.
Remarks. 1. Due to the friction and the noise terms (see (7.10) and (7.11)) we end up with the factors $e^{2ipN}$. This leads to a factor $e^{ipN}$ in functions such as $j(p)$ and $j'(p)$. A H"older continuous function $f$, of exponent $\alpha$, decays in $x$-space at least as $|x|^{-\alpha}$ whereas $e^{ipN}f$ produces decay away from $x = N$ i.e. $|x - N|^{-\alpha}$. Thus $\|f\|_{\alpha} \leq O(1)$ gives rise to functions of $x$ that are localized near the boundaries, $x = 0$ or $N$.

2. The Fourier transform of (8.11) is as follows. The first term is constant in the 1-direction and $O(1/N)$. The second one is also of $O(1/N)$ because it involves a (discrete) Hilbert transform of a H"older continuous function. However, e.g. for the current it can produce opposite constant values for positive and negative $x$. The third term is also of this order whereas the second is $O(N^{-1+\alpha/2}(|x|^{-\alpha} + |N - x|^{-\alpha}))$ i.e. $O(N^{-1-\alpha/2})$ for $x$ far away from the boundaries. The $J_2$ and $J_3$ terms will be subleading corrections to the current and the profile. Note that no smoothness in the transverse variable $k$ is assumed so that correlations will have a very slow decay in $x$ space. This is due to the poor regularity properties of the "collision kernel" that enters the nonlinear terms, see Proposition 9.3.

3. The $J$ correlation function is odd in $p$ and hence $J_0 = 0$. However, with $Q$, we will encounter functions with a nonzero first term in (8.11).

Finally, we will extend these definitions to the functions like $j(p)$ defined on $\frac{\pi}{N}Z_{2N}$, the only difference being the occurrence of $\sigma_{\pm}(2p)$ in (8.6) and $d(2p)$ in (8.11), instead of $d(p)$ here. We denote the space now by $S$. Hence $j \in S$ is of the form

$$j(p) = N^{-1}\delta(p)j_0 + (Nd(p))^{-1}j_1(p) + N^{\alpha/2-1}j_2(p) + (Nd(p))^{-3/2}j_3(p) \quad (8.13)$$

with $j_0$ a constant. Of course, here, since $j$ is defined on $\frac{\pi}{N}Z_{2N}$, the space is finite dimensional and all norms are equivalent, but (8.13) is a convenient way to record the dependence on $N$ of various terms.

We will look for solutions to (4.18)-(4.20) with $J \in S$ and $Q$ as follows. $Q$ will have a leading “local equilibrium” term which we now describe. Recall that our equations have the 2-parameter family (6.11) of solutions when the friction vanishes. The leading term in $Q$ will be of this form, where the constants $T$ and $A$ will be replaced by $p$-dependent functions. More precisely let $T(p), A(p)$ be functions on $\frac{\pi}{N}Z_{2N}$, of the form

$$T(p) = T_0\delta(p) + (d(-p))^{-1}t(p) \quad (8.14)$$

$$A(p) = A_0\delta(p) + (d(-p))^{-1}a(p) \quad (8.15)$$

where $\delta(p)$ is defined in (6.5) and $t, a \in S$. Define

$$Q_0(p, k) = \sum_{n=0}^{\infty} (T* A^n)(2p)\omega(p, k)^{-2-n} \quad (8.16)$$

where $*$ is the convolution and $\omega(p, k)$ is defined in (5.15).

Remark Let $Q_{T,A}$ be the generalized Gibbs state in (6.11), at $p = 0$,

$$Q_{T,A}(x - y) = T \int dke^{ik(x-y)}(\omega(k)^2 - A\omega(k))^{-1}. \quad (8.17)$$
Comparing (8.16) with
\[
T(\omega^2 - A\omega)^{-1} = \sum_{n=0}^{\infty} TA^n \omega^{-2-n}
\] (8.18)

it is not hard to show that the Fourier transform of (8.16) is
\[
Q_0(x, y) = Q_{T(x), A(x)}(x - y) + Q_1(x, y)
\] (8.19)

where (the Fourier transform of) \( Q_1 \) is in \( S \), and, as it will turn out,
\[
Q_1(x, y) = \mathcal{O}(\tau/N).
\] (8.20)

The first term is of a local (generalized) equilibrium form.

We make now the following assumptions. We take
\[
\gamma = N^{-1+\alpha/4}
\] (8.21)

and \( m^2 \) large enough that
\[
\sum s_i \omega(k_i) = 0 \Rightarrow \sum s_i = 0
\]
(recall (7.20) and (7.21)). We shall use below the set of \( p' \)'s that are close to singularities:
\[
E_0 = [-p_0, p_0] \cup [\pi - p_0, \pi + p_0]
\] (8.22)

with \( p_0 = B\lambda^2 \), where \( B \) is a number that will be chosen large below (see (10.16)). Then, we have the

**Theorem**  There exist \( \lambda_0 \) such that for \( 0 < \lambda < \lambda_0 \), \( |T_1 - T_2| < \tau \), with \( \tau = \tau(\lambda) \), and for \( N > N(\lambda) \) the equations (4.18)-(4.20) have a unique solution with \( J \in S \) and
\[
Q = Q_0 + r
\]

with \( r \in S \). \( Q_0 \) is given by (8.14)-(8.16) with \( T_0 = \Phi(T_1 + T_2) + \mathcal{O}(\lambda^2\tau) \),
\[
t(p) = t_0(p) + \tilde{t}(p),
\] (8.23)

where, in \( x \) space,
\[
\frac{1}{2}(T_1 + T_2) + t_0(x) = T_1 + \frac{|x|}{N}(T_2 - T_1),
\] (8.24)

and \( \tilde{t} \in S \), with
\[
\|\tilde{t}\|_S = \mathcal{O}(\lambda^2\tau).
\]

Moreover, \( A_0 = \mathcal{O}(\tau^2) \), \( a \in S \), and
\[
\|a\|_S = \mathcal{O}(\tau^2).
\]
Finally, the currents $j$ and $j'$ are given by

$$(j(p), j'(p))^T = \kappa(p)(t(p), s(p))^T + O(\tau^2).$$

where $s(p) = d(-p)T * A(p)$, the conductivity matrix $\kappa$ is Hölder continuous in $p$ of exponent $\alpha$, for some $\alpha > 0$, and is invertible and $O(\lambda^{-2})$ for $p \in E_0$.

Remarks

1. The shape of the profile, to leading order, is actually given by

$$\beta(x) \equiv T(x) = T_0 \delta(p) + d(-p)^{-1}t(p),$$

This can be seen, from the proofs, as follows: The full relation between $J$ and $t(p)$, which, in $x$ space, is equal to $-\nabla T(x)$, is given by (11.34) with $D_p$ replaced by the full $DN(q_0) = \mathcal{L} + \mathcal{L}'$, see (9.22). The leading term in $\mathcal{L}'$ comes from the $n = 0$ term in (8.4) inserted in (9.21, 9.22). Going back to $x$-space, we get, within those approximations: $T(x)j(x) = -\frac{1}{\lambda} \nabla T(x)$, which, since $j(x)$ is constant, implies $\nabla (T^{-1})(x)$ constant, i.e. (8.25). Of course, for $\tau$ small, we can expand (8.25) in $\tau$ and obtain (8.24), to leading order in $\tau$.

2. The choice of constants. There are four parameters in this model: $m^2$, the parameter in $\omega(k)$, see (2.5), $\lambda$ the strength of the nonlinearity, $\tau$ that bounds the temperature difference and $N$, the size of the system in the direction where there is a temperature difference. We choose $m^2$ enough so that (7.20) implies (7.21). Next we choose $\tau$ small compared to $\lambda$^2, say $O(\lambda^3)$ and we choose $N$ large so that we can use bounds like $N^{-\alpha} \leq \tau$, for any $\alpha > 0$.

In the proofs, $C$ or $c$ will denote constants that can change from place to place. We also use in various proofs auxiliary functions denoted $f, g$, whose meaning is given in the proofs where they are used.

Since $\lambda^2$ enters as a multiplication factor in the nonlinear terms, see (5.12), it will be convenient to discuss in the next section the $\lambda$ independent nonlinearities $n(W)$, and introduce explicitly $\lambda^2$ in section 10 (because only part of the linear operator there is multiplied by $\lambda^2$) and in section 11, where $\lambda^2$ is used to make some Lipschitz constants less than 1.

The value of $\alpha$ is determined by the degree of Hölder continuity that one obtains in Proposition 9.3 below. We do not try to optimize that value (any $\alpha > 0$ suffices); in fact, it will convenient sometimes to assume that $\alpha$ is not too close to one, so that, e.g. the power on $N$ in (11.21) is negative, and we shall implicitly assume that.

9 Nonlinear terms

In this Section we study the nonlinear terms of our equations given in (5.12) and (5.13). Our goal is to show that their linearization defines a bounded operator on $S$ and that the remaining nonlinearities define suitable Lipschitz functions on $S$.

It is convenient to introduce a space for the functions $T, A$, analogous to $S$ but stronger singularity at $p = 0$. We let $E$ denote the pairs $(T_0, t)$ with $T_0 \in \mathbb{R}$ and $t \in S$ with $t_0 = 0$ in (8.13). They parametrize functions

$$T(p) = T_0 \delta(p) + d(-p)^{-1}t(p),$$

(from now on, we shall identify $p$ and $p$). We use the norm

$$\|T\|_E = |T_0| + \|t\|_S,$$
which has the convenient property, used often below, that \( \|dT\|_S \leq \|T\|_E \). In this section, we shall work with \((T, A) \in \mathcal{B}_\epsilon\), where

\[
\mathcal{B}_\epsilon = \{(T, A) \in E \times E | \|T\|_E \leq C, \|A\|_E \leq \epsilon\}
\] (9.3)

where \(C\) is arbitrary and \(\epsilon\) is chosen small enough so that various series below converge.

We expand (5.13) around \(Q_0\) defined by (8.16). Let \(W = Q_0 + w\)

\[
n(W) = n(Q_0) + Dn(Q_0)w + \tilde{n}(w)
\] (9.4)

We discuss the three terms on the right hand side in turn.

### 9.1 \(n(Q_0)\)

Consider first \(n(Q_0)\). Let us start with a lemma, whose proof is given in Appendix B, and which controls convolutions between elements of \(E, S\) and among themselves:

**Lemma 9.1.** (a) Let \(T \in E\) and \(j \in S\). Then \(T * j \in S\) and \n
\[
\|T * j\|_S \leq C \|T\|_E \|j\|_S
\] (9.5)

(b) Let \(T, A \in E\). Then \(T * A \in E\) and

\[
\|T * A\|_E \leq C \|T\|_E \|A\|_E
\] (9.6)

(c) Let \(j, k \in S\). Then \(j' \equiv j * k \in S\) with

\[
|j'_0| \leq CN^{-1} \|j\|_S \|k\|_S,
\]

\[
|j'_1|_\alpha \leq CN^{-1} \|j\|_S \|k\|_S,
\]

\[
|j'_2|_\alpha \leq CN^{-1+\alpha/2} \|j\|_S \|k\|_S,
\]

\[
|j'_3(p)| \leq CN^{-1} \log(N|p|) \|j\|_S \|k\|_S,
\]

(9.7)

(9.8)

(9.9)

(9.10)

and

\[
\|j'\|_S \leq CN^{-1+\alpha/2} \|j\|_S \|k\|_S
\] (9.11)

Using this Lemma, we get:

**Proposition 9.2** The function \(n(Q_0)\) satisfies the following bounds, for \((T, A) \in \mathcal{B}_\epsilon\).

\[
n_1(Q_0) = \sum_{n=1}^{\infty} T^{n+3} * A^n(2p)g_n(p, k) + m
\] (9.12)

with \(g_n\) smooth functions bounded together with their derivatives (to any given order) by \(C^n\) and \(m \in S\). Moreover

\[
\|m\|_S \leq C(\|t\|_S + \|A\|_E)
\] (9.13)

\[
\|n_1(Q_0)(p, k) - n_1(Q_0)(-p, k)\|_S \leq C(\|t\|_S + \|A\|_E)
\] (9.14)

\[
\|n_2(Q_0)\|_S \leq C(\|t\|_S + \|A\|_E)
\] (9.15)
Furthermore the functions on the LHS of (9.13), (9.14) and (9.15) are uniformly Lipschitz in \( T \) and \( A \), for \((T, A) \in \mathcal{B}_x\).

**Proof.** From (5.13) and (8.16), we get

\[
n(Q_0)(p, k) = \sum_{n} \sum_{s} \int_{i=1}^{2} \prod_{i=1}^{2} T \ast A^{n_{i}}(2p_i) \omega(p_i, k_i - p_i)^{-2 - n_{i}} s_3 \omega(k_3) \left( \frac{1}{i s_4 \omega(k_4)} \right) \cdot [\omega(k_3)^{-2} \delta(2p_3) T \ast A^{n_{4}}(2p_4) \omega(p_4, k_4 - p_4)^{-2 - n_{4}} - (3 \leftrightarrow 4)] \nu_{spk}(dp \, dk)_k
\]

(9.16)

with \( \nu_{spk} \) given by (5.14), \( n = (n_i)^4 \) and \( \omega(p, k) \) given in (5.15). We have, see (5.15),

\[
\omega(p, k - p) = \omega(k) + (e^{2p} - 1) O(1),
\]

(9.17)

where the \( O(p) \) term, is written in an unusual form, which records the fact that it vanishes also at \( p = \pi \), and which will be convenient later. Inserting this into (9.16), the leading term is

\[
\sum_{n} \sum_{s} \int_{i=1}^{2} \prod_{i=1}^{2} T \ast A^{n_{i}}(2p_i) \omega(k_i)^{-2 - n_{i}} s_3 \omega(k_3) \left( \frac{1}{i s_4 \omega(k_4)} \right) \cdot [\omega(k_3)^{-2} \omega(k_4)^{-2 - n_{3}} - (3 \leftrightarrow 4)] \nu_{spk}(dp \, dk)_k
\]

We may do the \( p_i \)-integrals to get

\[
= \sum_{n=0}^{\infty} T^{n_3} \ast A^n(2p) \sum_{s} \sum_{s_3 = n} \int \prod_{i=1}^{2} \omega(k_i)^{-2 - n_i} s_3 \omega(k_3) \left( \frac{1}{i s_4 \omega(k_4)} \right) \cdot \nu_{spk}(dk_k)
\]

(9.18)

where \( \tilde{\nu}_{spk} \) is like \( \nu_{spk} \) above, but without \( \delta(2(p - \sum p_i)) \), and where, in the last equality, we used the fact that the \( n = 0 \) term has \( n_3 = 0 \) and therefore the \( [\cdot] \) factor vanishes. Using (8.18), we see that \( f_n(p, k) \) are the Taylor coefficients of the expansion in \( A \) of \( \bar{n}_2(p, k) \), given by (7.18, 7.19). Since \( \bar{n}_2 \) vanishes identically, we get

\[
f_n(p, k) = 0.
\]

(9.19)

For \( g_n \) we need to study \( n_1 \) in (7.17) in more detail. Proceeding as with \( n_2 \), we see that \( g_n \) are the Taylor coefficients of the expansion in powers of the constant \( A \) of

\[
\bar{n}_1(p, k) = \sum_{s} \int \prod_{i=1}^{4} Q_0(k_i) \mathcal{P} \left( \sum_{i=1}^{4} s_i \omega(k_i) \right)^{-1} \cdot s_3 \left[ 1 - \omega(k_3) \omega(k_4)^{-1} \right] \delta(2p - \sum k_i) \delta(p - k - k_4) dk_k.
\]

Here, again because of the \( s \rightarrow -s \) symmetry, only the principal value contributes. Consider first those terms with \( \sum s_i = 0 \). We may replace \( s_3 \) by \( -\frac{3}{2} s_4 \) and \( s_3 \omega(k_3) \) by

\[
\frac{1}{3} \sum_{i=1}^{3} s_i \omega(k_i) = -\frac{1}{3} s_4 \omega(k_4) + \frac{1}{3} \sum_{i=1}^{4} s_i \omega(k_i).
\]

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Thus these terms give

\[
\frac{1}{3} \sum_{s_i = 0} \int \prod_{i=1}^{4} Q_{T,A}(k_i) \delta \left(2p - \sum k_i\right) \delta(p - k - k_4) dk
\]

which is smooth in \(p\). For the terms with \(\sum s_i \neq 0\), \(P(\cdot)\) has no singularity, (see (7.20), 7.21)), and they are smooth. Thus, the functions \(g_n\) are smooth and bounded together with their derivatives (to any given order) by \(C^n\).

The contribution to (9.16) of the \((e^{2ip} - 1)O(1)\) term in (9.17) is of form

\[
m = \sum_{i} \sum_{s} \int \prod_{i=1}^{2} T^{*} A^{*n_i}(2p_i)(e^{2ip_3} - 1)T^{*} A^{*n_4}(2p_4) \delta(2p_4)
\]

\[
\cdot G_s(p, k, l, k) \delta \left(2p - 2 \sum p_3\right) \nu_{s p k}(dp \, dk)
\]

(9.20)

where \(G_s\) is smooth. By an easy extension of Lemma 9.1 to convolutions of functions in \(E\) and \(S\), each summand is in \(S\) and (9.13) follows easily. The series over \(n\) converge for \(\|A\|_E \leq \epsilon\) small enough.

To get (9.14) we use the smoothness and periodicity to get

\[g_n(p, k) - g_n(-p, k) = (e^{2ip} - 1)O(1),\]

and hence each summand is now in \(S\). The bound (9.15) is obtained in the same way as the bound on \(m\) (by (9.19), the only contribution to \(n_2(Q_0)\) is similar to \(m\)).

\[
\square
\]

### 9.2 Linearization

Let us turn to the second term in (9.4) i.e. the linearization of \(n\) at \(Q_0\). We wish to separate a leading term when \(T_2 - T_1\) is small. Therefore, we write, using (8.14) and (8.16),

\[
Q_0 = T_+ \delta(2p)\omega^{-2}(p, k) + \tilde{Q}_0
\]

(9.21)

with \(\omega\) given by (5.15), and \(T_+ = \frac{1}{2}(T_1 + T_2)\) being the average temperature. Insert this into (5.13). The \([-\cdot]\) term vanishes for the first term in (9.21), so

\[
Dn(Q_0)w = \mathcal{L}w + \mathcal{L}'w,
\]

(9.22)

with

\[
(\mathcal{L}w)(p, k) = 2T_0^2 \sum s \int \left(\omega(k_1)\omega(k_2)\right)^{-2}s_3\omega(k_3) \left(1_{s_4}\omega(k_4)\right)\]

\[
\left[\omega(k_3)^{-2}w_{s_4}(p, k_4 - p) - \omega(k_4)^{-2}w_{s_3}(p, k_3 - p)\right]
\]

\[
\left(\sum s_i\omega(k_i) + ie\right)^{-1} \delta \left(2p - \sum k_i\right) \delta(p - k - k_4) dk
\]

(9.23)

(9.23)

(9.24)
and \( \tilde{L}_p \) in turn is a sum of an operator which acts as a multiplication operator in the subspace of even or of odd functions, and an integral operator:

\[
\tilde{L}_p = \tilde{M}_p + \tilde{K}_p, \tag{9.25}
\]

where, after shifting \( k_3 \) by \( p \),

\[
(\tilde{K}_p w)(k) = -2T_0^2 \omega(p - k)^{-2} \sum_{s} w_{s4}(p, k_3) \frac{s_3 \omega(k_3 + p)}{\omega(k_1) \omega(k_2)^2} \left( i s_4 \omega(p - k) \right) d\mu_{spk}, \tag{9.26}
\]

with

\[
d\mu_{spk} = \left( \sum_{i=1}^{2} s_i \omega(k_i) + s_3 \omega(k_3 + p) + s_4 \omega(p - k) + i \epsilon \right)^{-1} \delta \left( k - \sum_{i=1}^{3} k_i \right) \prod_{i=1}^{3} dk_i \tag{9.27}
\]

and (remember that \( k_4 - p = -k \) so that the operator acts as a multiplication operator only on functions of definite parity):

\[
\tilde{M}_p(k) = -\omega(p - k)^2 \left( \tilde{K}_p \omega^{-2}(p + \cdot) \right)(k), \tag{9.28}
\]

when it acts on even functions, and

\[
\tilde{M}_p(k) = \omega(p - k)^2 \left( \tilde{K}_p \omega^{-2}(p + \cdot) \right)(k), \tag{9.29}
\]

when it acts on odd functions. Here, we integrated over \( k_4 = p - k \), and we used:

\[
2p - \sum_{i=1}^{4} k_i = 2p - \sum_{i=1}^{3} k_i - p + k,
\]

which, after shifting \( k_3 \) by \( p \) equals \( k - \sum_{i=1}^{3} k_i \). We shall discuss \( \mathcal{L}' \) in the next subsection, but let now analyze further \( \tilde{L} \).

Separating the real and imaginary parts of \( w_s \) (see (5.3)), one can also view the operator \( \mathcal{L}_p \) as a \( 2 \times 2 \) matrix of operators the \( (J, Q) \) variables, where here \( Q \) denotes an arbitrary even function, and \( J \) an arbitrary odd one. We write it as:

\[
\tilde{L}_p \begin{pmatrix} J \\ Q \end{pmatrix} = \begin{pmatrix} \tilde{L}_{11}(p)J + \tilde{L}_{12}(p)Q \\ \tilde{L}_{21}(p)J + \tilde{L}_{22}(p)Q \end{pmatrix}; \tag{9.30}
\]

this defines the operators \( \tilde{L}_{ij}(p) \), \( i, j = 1, 2 \). From (9.25), we see that each \( \tilde{L}_{ij}(p) \) is a sum of a multiplication operator \( \tilde{M}_{ij}(p) \) and an integral operator \( \tilde{K}_{ij}(p) \).

The operators here are acting on functions defined on \( \Omega \), see (8.5). However, it is convenient to consider them as acting on functions defined on \( [-\pi, \pi]^d \), which is always possible, by extending, say in a piecewise linear way, function on \( \Omega \) to functions on \( [-\pi, \pi]^d \). With that identification, we may consider these operators to be acting, for all \( N \), on the same space.
Moreover, a discrete Hölder continuous function (see (8.8)) becomes, with such an extension, an ordinary Hölder continuous function. The main property of these operators is:

**Proposition 9.3.** \( \tilde{M}_p(k) \) is \( C^\alpha \) in \( p,k \), for some \( \alpha > 0 \), and the operators \( \tilde{K}_p \) are compact operators mapping \( C^\alpha (\Omega_p(k)) \), where \( \Omega_p(k) = \{ k(p,k) \in \Omega \} \), into itself. Moreover, in the norm of bounded operators on \( C^\alpha \), they are uniformly bounded in \( N \) and \( C^\alpha \) in \( p \), uniformly in \( N \). \( \tilde{M}_p(k) \) converges as \( N \to \infty \) to a \( C^\alpha \) function, while \( \tilde{K}_p \) converges to a compact operator mapping \( C^\alpha([-\pi,\pi]^d) \) into itself.

Obviously, this proposition implies that the operators \( \mathcal{L}_{ij}(p) \) define also bounded operators from \( S \) into itself.

**Proof.** Looking at (9.26-9.27, 9.30), we see that \( \tilde{K}_p \) is a two by two matrix of integral operators whose kernel is a sum of terms of the form

\[
A_p(k,k') = \int \Delta \left( \sum_{i=1}^{2} s_i \omega(k_i) + s_3 \omega(k' + p) + s_4 \omega(p - k) \right) \cdot \delta(k - k_1 - k_2 - k') \rho_\alpha(k_1, k_2, k', k, p) dk_1 dk_2 \tag{9.31}
\]

where \( \Delta(x) = \delta(x) \) or \( \mathcal{P}\left(\frac{1}{x}\right) \), \( \rho_\alpha \) is \( C^\infty \) in all its arguments and we write \( k' \) for \( k_3 \).

Let us consider first \( \Delta(x) = \delta(x) \), integrate over \( k_2 \), and choose \( s_1 = 1, s_2 = -1 \) (all other terms can be treated similarly). We obtain the integral

\[
\int \delta \left( \omega(k_1) - \omega(k - k_1 - k') + s_3 \omega(k' + p) + s_4 \omega(p - k) \right) \rho_\alpha(k_1, k - k_1 - k', k', k, p) dk_1. \tag{9.32}
\]

Now, by (5.1), \( \int dk_1 \) is actually a discrete sum over the first component of \( k_1 \) and an integral over the last two components. Fix a value of \( k_1 \), replace \( k' - k \) by \( k' \), write \( k_2' = -\frac{x}{2} + q_2, k_3' = -\frac{x}{2} + q_3 \) and use lower indices, \( k_1', k_2', k_3' \) for the components of \( k' \). We get, using the explicit formula (2.5) for \( \omega \),

\[
(9.32) = \int \delta \left( \sin q_2 - \sin(q_2 - k_2') + \sin q_3 - \sin(q_3 - k_3') + f(k_2', k_3', \lambda) \right) \rho_\alpha(q_2, q_3, \lambda) dq_2 dq_3 \tag{9.33}
\]

where we write \( \lambda \) for the set of variables \( (k_1, k_1', k, p) \), the integral is an ordinary, not discrete, one and

\[
f(k_2', k_3', \lambda) = \sin q_1 - \sin(q_1 - k_1') + s_3 \omega(k' + p) + s_4 \omega(p - k) \tag{9.34}
\]

We shall now study the singularities of (9.33) in \( k_2', k_3' \) and the smoothness of (9.33) in \( \lambda \) away from the singularities. For notational simplicity, we set \( \rho_\alpha = 1 \); since \( \rho_\alpha \) is smooth, this does not affect our arguments. Let us denote by \( \Omega \) the argument of the delta function and shift the integration variables: let \( q_i = r_{i-1} + y_{i-1} \) where \( y_{i-1} = \frac{1}{2} k'_i \). Then

\[
\Omega = \sum_{i=1}^{2} \left( \sin(r_i + y_i) - \sin(r_i - y_i) \right) + f(2y, \lambda) = 2 \sum_{i=1}^{2} \cos r_i \sin y_i + f(2y, \lambda). \tag{9.35}
\]
Next, change variables to \(2 - 2 \cos r_i = x_i^2\). Our integral becomes

\[
I(y, \lambda) = \int \delta(x_1^2 \sin y_1 + x_2^2 \sin y_2 - g(y, \lambda)) \prod_{i=1}^2 h(x_i) dx_i
\]  

(9.36)

with \(h(x) = (1 - \frac{1}{2}x^2)^{\frac{1}{2}}\), \(x_i^2 \in [0, 2]\) and

\[
g(y, \lambda) = f(2y, \lambda) + 2(\sin y_1 + \sin y_2).
\]  

(9.37)

For \(y_i \neq 0\) and \(g \neq 0\) \(I\) is bounded by

\[
|I(y, \lambda)| \leq C|\sin y_1 \sin y_2|^{-\frac{1}{2}}(1 + |\log |g(y, \lambda)||)
\]  

(9.38)

(the log term is absent when \(y_1y_2 > 0\)). From (9.34) and (9.37) we have

\[
g(y, \lambda) = 2s_3(\cos(2y_1) + \cos(2y_2) + 2(\sin y_1 + \sin y_2) + g'(\lambda)
\]

with \(g'\) smooth. Thus the bound (9.38) is integrable in \(y\), uniformly in \(\lambda\). Moreover, the Hölder derivative of order \(\alpha\) in \(\lambda\) also remains integrable for \(\alpha\) small enough. Hence for all bounded functions \(f\), \(\int A_p(k, k')f(k')dk'\) is \(C^\alpha\) in \(\lambda = (k_1, k_1', k, p)\). This in turn implies that \(\int A_p(k, k')f(k')dk'dk_1\) (where now the integral includes a Riemann sum over \(k_1, k_1', k, p\), since each term in the Riemann sum is \(C^\alpha\) in \(k, p\). This means that each matrix element of \(K_p\) maps bounded functions into Hölder continuous ones. Moreover, all the bounds are uniform in \(N\), since the bound (9.38) is independent of \(N\), and taking the Riemann sums preserves that property.

To obtain compactness, let \(\alpha'\) the Hölder exponent obtained above. Using the Arzelà-Ascoli’s theorem, one easily shows that the unit ball in \(C^{\alpha'}\) is compactly embedded in \(C^\alpha\) for \(\alpha' > \alpha\). Since \(A_p\) is bounded from \(C^\alpha\) into itself and maps \(C^\alpha\) into \(C^{\alpha'}\), it a compact operator from \(C^\alpha\) into itself.

Obviously \(\int A_p(k, k')dk'\) is also \(C^\alpha\) in \(p, k\), so that each matrix element of \(M_p(k)\) is \(C^\alpha\) in \(p, k\).

Next, we get:

\[
|A_p f(k) - A_p f(k') - (A_p f(k') - A_p d f(k'))| \leq C\|f\|_\infty \min(|k - k'|^{\alpha'}, |p - p'|^{\alpha'})
\]

\[
\leq C\|f\|_\infty |k - k'|^{\alpha'/2} |p - p'|^{\alpha'/2},
\]  

(9.39)

so that, choosing \(\alpha = \alpha'/2\), we get:

\[
\|A_p - A_p\|_\alpha \leq C|p - p'|^\alpha
\]  

(9.40)

where \(\| \cdot \|_\alpha\) is the operator norm on bounded operators of \(C^\alpha\) into itself.

Finally, it is easy to see that the Riemann sum of a \(C^\alpha\) function converges to the corresponding integral, with, for the sum in (5.1), an error \(O(N^{-\alpha})\). This then implies the claims on convergence as \(N \to \infty\) made in the Proposition. \(\square\)

In Appendix A, we will extend this result.

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9.3 Nonlinearities

Let us now turn to the $n_i$'s defined in (5.13). They are linear combinations of functionals of the following form

\[
u(f_1, f_2, f_3)(p, k) = \int G \prod_{i=1}^3 f_i(p_i, k_i - p_i)\delta(2p - \sum_{i=1}^3 2p_i)dp\mu(dk) \quad (9.41)\]

\[
v(f_1, f_2, f_3)(p, k) = \int G \prod_{i=1}^2 f_i(p_i, k_i - p_i)f_3(p_3, p - k - p_3)\delta(2p - \sum_{i=1}^3 2p_i)dp\mu(dk) \quad (9.42)\]

where $G$ is some smooth function of all the variables (defined in the continuum ($N = \infty$) and restricted to the discrete) and

\[
\mu_{p,k}(dk) = (\sum_1^3 s_i\omega(k_i) + s_4\omega(p - k) + i\epsilon)^{-1} \delta(\sum_1^3 k_i - p - k)dk_1dk_2dk_3. \quad (9.43)\]

Indeed, (9.41) corresponds to the second term in the bracket in (5.13), after integrating over $k_4$, $p_4$, while (9.42) corresponds to the first term, after integrating over $k_4$ (i.e. replacing $k_4$ by $p - k$) and $p_3$, and relabelling $p_4$ as $p_3$. Since $Q = Q_0 + r$, $r \in S$ and $Q_0$ is given by the series (8.16) we only need to consider $f_i(p, k) = F_0(2p)$ for $F \in E$ or $f_i \in S$. We have then the

Proposition 9.4. Let $m$ be of the form (9.41) or (9.42). Then

(a) Let $f_i(p, k) = F_i(2p)$ with $F_i \in E$. Then

\[
\|m(f_1, f_2, f_3)\|_E \leq C \prod_{i=1}^3 \|F_i\|_E \quad (9.44)\]

(b) Let $f_i \in S$, $f_j(p, k) = F_j(2p)$ with $F_j \in E$ for $j \neq i$. Then

\[
\|m(f_1, f_2, f_3)\|_S \leq C\|f_i\|_S \prod_{j \neq i} \|F_j\|_E \quad (9.45)\]

(c) Let $f_i, f_j \in S$, $f_k \in E$. Then

\[
\|m(f_1, f_2, f_3)\|_S \leq C\, N^{-\frac{1}{2} + \alpha}\|f_k\|_E \prod_{l \neq k} \|f_l\|_S \quad (9.46)\]

For the proof, see Appendix B.

The operators $L'$ in (9.22) are not multiplication operators in $p$. Inserting the expansion (8.16) in (5.13), we see that $L'w$ is a sum of terms of the form discussed in Proposition 9.4.(b) with $F_i$ of form $T*A^n$ with $n > 0$ or $d(-2p)^{-1}t(2p)$. Proposition 9.4.(b) then gives the

Proposition 9.5. The operator $L': S \to S$ is bounded, for $(T, A) \in B_e$, in operator norm by

\[
\|L'\| \leq C(\|t\|_S + \|A\|_E), \quad (9.47)\]

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and it is uniformly Lipschitz in $T$ and $A$ for $(T, A) \in B_\varepsilon$.

Proposition 9.4.(c) gives immediately for the term $\tilde{n}(w)$ in (9.4):

**Proposition 9.6.** For $\|w\|_S \leq O(1)$, and $(T, A) \in B_\varepsilon \|\tilde{n}(w)\|_S \leq CN^{-\frac{1}{2}+\alpha}\|w\|_S^2$. and it is Lipschitz in $w$, $T$ and $A$ with constant $CN^{-\frac{1}{2}+\alpha}$.

We still need to discuss the function $\theta$ given by (7.9). Its main property, proven in Appendix B, is:

**Proposition 9.7.** $d^{-1}\theta$ is in $S$, for $(T, A) \in B_\varepsilon$, with, for $\|w\|_S \leq O(1)$,

$$
\|d^{-1}\theta\|_S \leq C\lambda^2(\|t\|_S + \|A\|_E + \|r\|_J + \|J\|_S^2).
$$

(9.48)

and it is uniformly Lipschitz in $T$ and $A$ for $(T, A) \in B_\varepsilon$, and in $J$ and $r$, with constants $C\lambda^2$. Moreover,

$$
|\theta(0)| \leq CN^{-1}.
$$

(9.49)

10 Solution of the linear problem

In order to solve equations (5.18), (5.19) we need to study the invertibility of the linear operator

$$
\delta \omega^2(p, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + L_p
$$

(10.1)

where $\delta \omega^2(p, k) = \omega^2(p + k) - \omega^2(p - k)$ and $L_p$ denotes the linearization around the first term in (9.21) of the nonlinear terms in (5.18, 5.19). $L_p$ is given explicitly by adding or subtracting to (9.23) a term with $k \rightarrow -k$, see (5.18, 5.12), and multiplying it by $\frac{2}{9}(2\pi)^{3d}\lambda^2$, see (5.12). The fact that $L'$ in (9.22) is a small perturbation of $L$ follows from Proposition 9.5, for $\tau$ small enough, since, as we shall see in the next section, we shall solve our equations in a space where the RHS of (9.47) is of order $\tau$. So, to invert $\delta \omega^2(p, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + D N(Q_0)$, it is enough to concentrate our attention to (10.1), where $L_p$ is written as a two by two matrix as in (9.30).

Because of the zero modes, in order for (10.1) to be invertible, it needs to be restricted to the orthogonal complement of the zero modes (which occur at $p = 0, \pi$). Let $H_p$ be the Hilbert space $L^2\left(\frac{\pi}{N}\mathbb{Z}_N \times [-\pi, \pi]^{d-1}, \omega(p, k)^2dk\right)$ and let $P^\perp$ be the projection to the orthogonal complement of $\{\omega(p, k)^{-2}, \omega(p, k)^{-3}\}$ in $H_p$, and $P = 1 - P^\perp$. Note that the scalar product of $f(k)$ with $\omega(p, k)^{-2}$ equals $\int f(k)dk$, which implies

$$
P^\perp 1 = 0.
$$

(10.2)

We shall study the operator $D_p$, defined as:

$$
D_p = \Pi \left(\delta \omega^2\sigma_1 + L_p\right)\Pi,
$$

(10.3)

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where we use the shorthand notations \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), 
\[ \Pi = \begin{pmatrix} 1 & 0 \\ 0 & P^\perp \end{pmatrix}, \]
for \( p \in E_0 \), with \( E_0 \) defined in (8.22).

\[ D_p = \left( \delta \omega^2 \sigma_1 + \mathcal{L}_p \right), \quad (10.4) \]
for \( p \notin E_0 \).

We shall make the following assumptions, that we shall verify later for the operator \( \mathcal{L}_p \):

1. \( \mathcal{L}_{ij}(0) = \mathcal{L}_{ij}(\pi) = 0, \quad i \neq j \) \quad (10.5)

2. \( \exists \ c > 0 \) such that
\[ \mathcal{L}_{11}(0) < -c\lambda^2 \]
\[ P^\perp \mathcal{L}_{22}(0) P^\perp > c\lambda^2 \quad (10.6) \]

where the inequalities holds for operators in \( L^2 \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1}, \omega^2(k)dk \right) \) restricted to functions that are odd in \( k \) (for \( \mathcal{L}_{11} \)) or even in \( k \) (for \( \mathcal{L}_{22} \)). The same inequalities hold for \( \mathcal{L}_{11}(\pi), \mathcal{L}_{22}(\pi) \).

3. \( \exists \ C < \infty \), independent of \( N \), such that, \( \forall p,p', \forall i,j = 1,2 \),
\[ \| \mathcal{L}_{ij}(p) - \mathcal{L}_{ij}(p') \| \leq C|p - p'|^\alpha \quad (10.8) \]

where \( \| \cdot \| \) is the operator norm of bounded operators mapping \( C^\alpha \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1} \right) \) into itself.

4. \( \exists \ c > 0 \) such that, \( \forall p, \forall k, \forall f \in \mathbb{R}^2 \),
\[ \left| \left( \delta \omega^2 \sigma_1 + M(p,k) \right)f \right| \geq c(\lambda^2 + |\sin p| |\sin k|)|f| \quad (10.9) \]

where \( |f| = |f_1| + |f_2| \)

5. The kernels \( K_{ij}(k, \cdot) \in L^{1+\eta} \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1} \right) \), for some \( \eta > 0 \), and a norm \( \mathcal{O}(\lambda^2) \).

**Proposition 10.1** Under assumptions (1-5) above, for any \( B \) in (8.22), \( \exists \lambda_0 \) such that, for \( \lambda \leq \lambda_0 \) and for all \( p \in \frac{\pi}{N} \mathbb{Z}_{2N} \), \( D_p \) is invertible, and \( \exists C < \infty \) such that \( \forall \lambda \leq \lambda_0 \),
\[ \| D_p^{-1} \| \leq \frac{C}{\lambda^2}, \quad (10.10) \]

where the norm is the one of operators in \( C^\alpha \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1} \right) \oplus C^\alpha \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1} \right) \).

**Proof.** We consider \( |p| \leq \frac{\pi}{2} \). For \( |p| > \frac{\pi}{2} \) the proof below can be repeated with 0 replaced by \( \pi \).
Let us first consider $|p| \leq B\lambda^2$, where the constant $B$, independent of $\lambda$, will be specified later. Write $p = a\lambda^2$, with $|a| \leq B$. We have, for $|p| \leq B\lambda^2$,

$$\delta\omega^2(k, p) = a\lambda^2 \varphi(k) + \mathcal{O}(\lambda^4)$$  \hspace{1cm} (10.11)

where we expand $\delta\omega^2(k, p)$ (which vanishes at $p = 0$) in $p$, the first term in (10.11) is the one linear in $p$, with $\varphi$ an odd function of $k$ alone, and the second is $\mathcal{O}(p^2)$ for $|p| \leq B\lambda^2$. Moreover, due to (10.8),

$$\mathcal{L}_{ij}(p) = \mathcal{L}_{ij}(0) + \mathcal{O}(\lambda^{2+\alpha})$$  \hspace{1cm} (10.12)

where $\mathcal{O}(\lambda^{2+\alpha})$ is a bound on the operator norm $(\mathcal{L}_{ij}(p)$ has a factor $\lambda^2$). So, we have

$$\delta\omega^2 \sigma_1 + \mathcal{L}_p = \lambda^2 (a\varphi(k)\sigma_1 + \hat{\mathcal{L}}_0) + \mathcal{O}(\lambda^{2+\alpha})$$  \hspace{1cm} (10.13)

with $\hat{\mathcal{L}}_0 = \mathcal{L}_0/\lambda^2$, which is $\lambda$-independent. Thus if we show that $\exists C(B)$ such that $\forall a, |a| \leq B$,

$$\| \prod \left( a\varphi(k)\sigma_1 + \hat{\mathcal{L}}_0 \right) \| \leq C(B) < \infty,$$  \hspace{1cm} (10.14)

we obtain, from (10.13) and a resolvent expansion the bound (10.10) for $|p| \leq B\lambda^2$, provided that $\lambda$ is small enough, given $B$.

To prove (10.14), we observe that the spectrum of the multiplication part of $a\varphi(k) \sigma_1 + \hat{\mathcal{L}}_0$, i.e. $a\varphi(k)\sigma_1 + \frac{M(0,k)}{\lambda^2}$ lies outside a ball of fixed radius around zero. This follows from (10.9), using the approximations (10.11) and (10.12) for $\delta\omega^2$ and $M(p,k)$. Hence, since adding to this a compact operator does not change the essential spectrum (note that the projection operator $P^\perp$ adds a rank 2 operator), (10.14) will hold, provided that we show that $a\sigma_1 + \hat{\mathcal{L}}(0)$ has no zero eigenvalue. But solving for $f_2$ the first of the two equations

$$\prod \left( \hat{\mathcal{L}}_{11}(0) \quad a\varphi(k) \right) \prod \left( f_1(k) \quad f_2(k) \right) = 0$$

(we use the fact that, by (10.5), $\hat{\mathcal{L}}_{ij}(0) = 0, i \neq j$), substituting in the second equation, and taking a scalar product in $L^2\left( \frac{2N}{\pi} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1}, \omega^2(k)dk \right)$, with $f_1$, we get:

$$(P^\perp f_1, \hat{\mathcal{L}}_{11}(0)P^\perp f_1) - a^2 \left( P^\perp f_1, \varphi\hat{\mathcal{L}}_{22}^{-1}(0)\varphi P^\perp f_1 \right) = 0$$

which is impossible (for all $a$’s) because of (10.6) and (10.7). This finishes the proof of (10.14) and therefore of (10.10) for $|p| \leq B\lambda^2$.

Consider now $|p| > B\lambda^2$. Since adding a compact operator does not change the essential spectrum, which is bounded away from zero by (10.9), it is enough to show that the equation

$$(\delta\omega^2 \sigma_1 + \mathcal{L}_p)f = \mu f$$  \hspace{1cm} (10.15)

with $f = \left( \begin{array}{c} f_1(k) \\ f_2(k) \end{array} \right)$, does not have non zero solutions for $|\mu| \leq c\lambda^2$, and $c > 0$. Let $E = \{k| \sin k| \leq b\}$ for some constant $b$ to be specified later. Then, from (10.9), (10.15) and the fact that $\mathcal{L}_p = M_p + K_p$, we get, for $k \not\in E$:

$$c'Bb\lambda^2|f(k)| \leq |(\delta\omega^2 \sigma_1 + M_p)f(k)| \leq \|K_p\| \|f\|_\infty + |\mu| \|f\|_\infty$$

$$\leq (C\lambda^2 + |\mu|)\|f\|_\infty,$$
using the fact (see the proof of Proposition 9.3) that $K_p$ maps bounded functions into $C^\alpha$ ones which, in particular, are bounded, and that $\|K_p\| \leq C\lambda^2$. Given $\beta > 0$, $c > 0$, $c' > 0$ $C < \infty$ and $b > 0$, we can choose $B = B(\beta, c, c', C, b)$, independent of $\lambda$, so that this implies, if $|\mu| \leq c\lambda^2$,

$$|f(k)| \leq \beta \|f\|_\infty$$

(10.16)

for $k \notin E$.

Consider now $k \in E$. Then, we get from (10.9), (10.15):

$$c\lambda^2|f(k)| \leq |(\delta\omega^2\sigma_1 + M_p) f(k)|$$

$$\leq \left| \int_{k' \in E} K(k, k') f(k') \right| + \left| \int_{k' \notin E} K(k, k') f(k') \right| + |\mu| \|f\|_\infty$$

(10.17)

We bound, using (10.16),

$$\left| \int_{k' \notin E} K(k, k') f(k') \right| \leq C\lambda^2 \beta \|f\|_\infty$$

(10.18)

and since $k' \in E$ means, for $b$ small, that $k'$ must be close to zero or close to $\pi$, we get, using assumption 5 and Hölder’s inequality

$$\left| \int_{k' \in E} K(k, k') f(k') \right| \leq \|K(k, \cdot)\|_{1+\eta}(cb)^{\eta/1+\eta} \|f\|_\infty \leq C\lambda^2 b^{\eta/1+\eta} \|f\|_\infty.$$  

(10.19)

Inserting (10.18) and (10.19) in (10.17), we get $|f(k)| \leq \left( C'\beta + \frac{|\mu|}{\|f\|_\infty} \right) \|f\|_\infty$, for $k \in E$, by choosing $b$ small enough. This, combined with (10.16) implies $f = 0$, for $\beta$ small enough, if, say, $|\mu| < \frac{c\lambda^2}{2}$. Thus, there is no non-zero solution of (10.15) in that ball and (10.10) holds.

Let us now check that the operator $L_p$ has the properties (1-5) above.

To do that, we must first write explicitly the operators $L_{ij}(0)$, $L_{ij}(\pi)$. Let us start with $p = 0$. Using (9.23, 9.30), we get, since only terms that are even in $s$ give a non-zero contribution (see (5.5) and comments afterwards):

$$L_{12}(0) Q(k) = 2 T_0^2 \sum_s \int \left( \omega(k_1) \omega(k_2) \right)^{-2} s_3 \omega(k_3) \left[ \omega(k)^{-2} Q(k) - \omega(k_3)^{-2} Q(-k) \right]$$

$$\cdot \mathcal{P} \left( \left( \sum s_i \omega(k_i) \right)^{-1} \right) \delta \left( \sum k_i \right) \delta(k + k_4) dk - (k \to -k),$$

(10.20)

where the $(k \to -k)$ term comes from (5.18); (10.20) vanishes because $Q$ and $\omega$ are even in $k$, see (5.20).

$$L_{11} J(k) = 2 T_0^2 \sum_s \int \left( \omega(k_1) \omega(k_2) \right)^{-2} s_3 \omega(k_3) \left[ \omega(k)^{-2} s_3 J(k_3) \omega(k_3) - \omega(k_3)^{-2} s_4 J(-k) \right]$$

$$\cdot \delta \left( \sum s_i \omega(k_i) \right) \delta \left( \sum k_i \right) \delta(k + k_4) dk - (k \to -k)$$

(10.21)

which equals twice the first term, since $J$ is odd in $k$ (see (5.21)) and $\omega$ even.
\( \mathcal{L}_{21}(0)J \) vanishes by symmetry, like (10.20) (there is a + \((k \to -k)\) term in (5.12) and \(J\) is odd). This proves property 1, for \(p = 0\).

To prove point 2, write:

\[
(\mathcal{L}_{22}Q)(k) = 4T_0^2 \sum_s \int \omega(k_1)\omega(k_2) s_3\omega(k_3)s_4\omega(k_4) \left[ \omega(k_4)^{-2}Q(k) - \omega(k_3)^{-2}Q(k) \right] \\
\quad \cdot \delta \left( \sum s_i\omega(k_i) \right) \delta \left( \sum k_i \right) \delta(k + k_4)dk.
\]

(10.22)

Taking the scalar product of (10.22) with \(Q(k)\) in \(L^2 \left( \frac{\pi}{N} \mathbb{Z}_{2N} \times [-\pi, \pi]^{d-1}, \omega^2(k)dk \right)\), and replacing in the second term in \([-], s_3\omega(k_3)\) by \(\left( \frac{1}{3} \sum s_i\omega(k_i) \right)\), using symmetry, and then by \(\left( -\frac{1}{3} s_4\omega(k_4) \right)\), using the delta function, we get:

\[
(Q, \mathcal{L}_{22}Q) = 4T_0^2 \sum_s \int \prod_{i=1}^4 \omega(k_i)^{-2} \left[ s_3\omega^3(k_3)s_4\omega^3(k_4)Q(k_3)Q(k) + \frac{1}{3} \omega^6(k_4)\right] \left| Q(k) \right|^2 \\
\quad \cdot \delta \left( \sum s_i\omega(k_i) \right) \delta \left( \sum k_i \right) \delta(k + k_4)dk.
\]

(10.23)

Using \(k = -k_4\), the symmetry between the \(k_i\)’s and the evenness of \(\omega, Q\), we can write (10.23) as:

\[
\frac{T_0^2}{3} \sum_s \int \prod_{i=1}^4 \omega(k_i)^{-2} \left| \sum s_i\omega(k_i)^3Q(k) \right|^2 \delta \left( \sum s_i\omega(k_i) \right) \delta \left( \sum k_i \right) \delta(k + k_4)dk.
\]

(10.24)

A similar computation, starting with (10.21), leads to

\[
(J, \mathcal{L}_{11}J) = -\frac{T_0^2}{3} \sum_s \int \prod_{i=1}^4 \omega(k_i)^{-2} \left| \sum \omega(k_i)^2J(k) \right|^2 \delta \left( \sum s_i\omega(k_i) \right) \delta \left( \sum k_i \right) \delta(k + k_4)dk.
\]

(10.25)

To conclude the proof of (10.6), (10.7), we need the following

**Lemma 10.2.** Let \(f\) be a Hölder continuous function from \(\mathbb{R}^d\) into \(\mathbb{R}\), with \(d \geq 3\), satisfying

\[
f(k_1) + f(k_2) = f(k_3) + f(k_4)
\]
on the set

\[
\{(k_i)_{i=1}^4 \mid k_i \in \mathbb{T}^d, \ i = 1, ..., 4, \ \ k_1 + k_2 = k_3 + k_4, \ \omega(k_1) + \omega(k_2) = \omega(k_3) + \omega(k_4)\},
\]

then,

\[
f(k) = a\omega(k) + b.
\]

(10.26)

Now, consider first (10.24) and (10.25) in the \(N \to \infty\) limit, i.e. with the sum in (5.1) replaced by an integral.
To apply the Lemma, use the fact that, by (5.21) and relabelling indices, we may assume that \( s_1 = s_2 = 1, s_3 = s_4 = -1 \) and change \( k_3 \rightarrow -k_3, k_4 \rightarrow -k_4 \). Then, the Lemma applied to \( f = \omega(k)^3 Q(k) \) or \( f = \omega(k)^2 J(k) \) implies that (10.24) and (10.25) cannot equal zero unless \( J = 0 \), since \( J(k) \) is odd, or unless \( Q(k) = a\omega(k)^{-2} + b\omega(k)^{-3} \) which, for \( Q = P^\perp Q \), i.e. for \( Q \) orthogonal to \( \omega(k)^{-2} \) and to \( \omega(k)^{-3} \) implies \( a = b = 0 \). Since each \( \mathcal{L}_n(0) \) is the sum of a multiplication operator which is bounded away from zero (see (10.28), (10.29) below for \( f \) to hold. Since we showed that there cannot be a zero eigenvalue (10.6)-(10.7) hold for \( N = \infty \) in Proposition 9.3, which imply the convergence of (10.24) and (10.25) to their limit. Finally (10.9) holds because, for given \( p, k \), \( \left( \delta \omega^2 \sigma_1 + M(p, k) \right) \) is a \( 2 \times 2 \) matrix, and the lower bound (10.9) holds if the eigenvalues of that matrix satisfy

\[
|\mu_i(p, k)| \geq c(\lambda^2 + |\sin p||\sin k|), \quad i = 1, 2.
\]

To prove this lower bound, it is enough to prove it for the square root of the absolute value of the determinant of the matrix \( \delta \omega^2 \sigma_1 + M(p, k) \), which equals

\[
M_{11}(k, p)M_{22}(k, p) - \left( \delta \omega^2 + M_{12}(k, p) \right)\left( \delta \omega^2 + M_{21}(k, p) \right)
\]

(10.27)

We can check, from the explicit formulas (10.21), (10.22), in which the multiplication operator corresponds to the last term in the \([-] \), that, for all \( p \),

\[
M_{11}(k, p) < -c\lambda^2
\]

(10.28)

\[
M_{22}(k, p) > c\lambda^2
\]

(10.29)

for \( c > 0 \). The signs and the factor \( \lambda^2 \) are obvious, and to get a non zero contribution, we need only to check that \( \sum s_i \omega(k_i) \) vanishes for some \( k_1, k_2, k_3, k_4 \), with the constraints \( k_4 = p - k \), \( \sum_{i=1}^3 k_i = p + k \). Choosing \( s_1 = +1, s_2 = -2, s_3 = +1, s_4 = -1 \) (which can always be obtained by relabelling indices), and inserting the constraints, this means

\[
\omega(k_1) - \omega(k_2) + \omega(p + k - k_1 - k_2) - \omega(p - k)
\]

which vanishes for \( k_1 = k_2 = k \).

Now, since \( M_{ij}(k, p) \) for \( i \neq j \) vanishes at \( p = 0 \) or \( \pi \) and at \( k = 0 \) or \( \pi \), we get, using Proposition 9.3,

\[
|M_{ij}(k, p)| \leq C\lambda^2\left(|\sin k| |\sin p|\right)^{\alpha/2}
\]

(10.30)

for \( i \neq j \).

Inserting (10.28), (10.29), (10.30) into (10.27), using \( |\delta \omega^2| = 4|\sin k| |\sin p| \), we get that \( M_{ij}, i \neq j \) is small compared to \(|\delta \omega^2| \) if \( |\sin k| |\sin p| \geq c'\lambda^2 \), for \( c' \) small (in which case,
both terms in (10.27) are negative), and that \( \left( \delta \omega^2 + M_{ij}(k, p) \right) \) is small compared to \( \lambda^2 \), i.e.

compared to \( M_{ii} \), otherwise.

We are left with the

**Proof of Lemma 10.2.** The proof follows closely the one of Proposition 12.1 in [29], which itself is inspired by [8].

Let us first assume that \( f \) is twice continuously differentiable. The hypothesis of the Lemma imply that \( f(k) + f(k') \) is constant \( \forall k, k' \in \mathbb{T}^d \), with \( k + k' \) constant and \( \omega(k) + \omega(k') \) constant. Therefore, there exists \( g : \mathbb{R} \times \mathbb{T}^d \to \mathbb{R} \), of class \( C^2 \), such that

\[
    f(k) + f(k') = g(\omega(k) + \omega(k'), k + k').
\]

Writing \( k = (k^\alpha)_{\alpha=1}^d, k^\alpha \in \mathbb{T} \), \( \omega = \omega(k') \), \( \omega' = \omega(k) \), we get

\[
    \partial_\alpha f(k) = \partial_\omega g(\omega + \omega', k + k') \partial_\alpha \omega + \partial_\alpha g(\omega + \omega', k + k'),
\]

\[
    \partial_\alpha f(k') = \partial_\omega g(\omega' + \omega, k + k'), \partial_\omega' + \partial_\omega g(\omega + \omega', k + k'),
\]

where \( \partial_\alpha = \frac{\partial}{\partial k^\alpha}, \partial_\omega = \frac{\partial}{\partial \omega} \). Subtracting these two equations, we get

\[
    \left( \partial_\alpha f(k) - \partial_\alpha f(k') \right) = \partial_\omega g(\omega + \omega', k + k') \left( \partial_\omega \omega(k) - \partial_\omega \omega(k') \right)
\]

\[(10.31)\]

Multiplying first (10.31) by \( \left( \partial_\beta \omega(k) - \partial_\beta \omega(k') \right) \), then rewriting the resulting equation by exchanging \( \alpha \) and \( \beta \) we get, for all \( \alpha, \beta \):

\[
    \left( \partial_\alpha f(k) - \partial_\alpha f(k') \right) \left( \partial_\beta \omega(k) - \partial_\beta \omega(k') \right) = \left( \partial_\beta f(k) - \partial_\beta f(k') \right) \left( \partial_\alpha \omega(k) - \partial_\alpha \omega(k') \right).
\]

If we differentiate this identity with respect to \( k_\gamma \), we get:

\[
    \partial_\alpha \partial_\gamma f(k) \left( \partial_\beta \omega(k) - \partial_\beta \omega(k') \right) + \left( \partial_\alpha f(k) - \partial_\alpha f(k') \right) \partial_\beta \partial_\gamma \omega(k) = \left( \partial_\beta f(k) - \partial_\beta f(k') \right) \partial_\alpha \partial_\gamma \omega(k).
\]

Differentiating now this with respect to \( k'_\delta \), we get:

\[
    \partial_\alpha \partial_\gamma f(k) \partial_\beta \partial_\delta \omega(k') + \partial_\alpha \partial_\delta f(k') \partial_\beta \partial_\gamma \omega(k') = \partial_\beta \partial_\delta f(k) \partial_\alpha \partial_\gamma \omega(k') + \partial_\beta \partial_\delta f(k') \partial_\alpha \partial_\gamma \omega(k). \tag{10.32}
\]

Now, for \( \omega(k) \) as in (2.5), we have \( \partial_\alpha \partial_\delta \omega(k) = \delta_{\alpha\beta} \cos k_\alpha \). Using this and choosing in (10.32) \( \alpha \neq \gamma \neq \beta = \delta \), we get \( \partial_\alpha \partial_\gamma f(k) = 0 \) for \( \alpha \neq \gamma \). This holds first on a dense set \( k_\gamma \neq \pm \frac{\pi}{2} \) (\( \cos k_\gamma \neq 0 \)) and then, by continuity, everywhere on \( \mathbb{T}^d \). Then, choosing \( \alpha = \gamma \neq \beta = \delta \), we get

\[
    \partial_\beta^2 f(k) \partial_\alpha^2 \omega(k') = \partial_\beta^2 f(k') \partial_\alpha^2 \omega(k).
\]

which implies that \( \partial_\alpha^2 f(k) = a \partial_\alpha^2 \omega(k) \) for a constant \( a \). Integrating, we get

\[
    f(k) = a \omega(k) + b + ck,
\]

for \( a, b, c \in \mathbb{R} \), and we get \( c = 0 \) from the fact that \( f \) is a continuous function on \( \mathbb{T}^d \).

This finishes the proof of \( f \) of class \( C^2 \). For \( f \) merely Hölder continuous, we interpret all the (linear) identities above, and all the derivatives in the sense of distributions, and we obtain the same conclusion.

\[\square\]

**Remark.** The Lemma holds by assuming only that \( f \) is a distribution, but we do not need this. It is crucial here that the dimension \( d \geq 3 \) in order to be able to choose \( \alpha \neq \gamma \neq \beta \). For counterexamples in \( d = 1 \), see [19].
11 Proof of the Theorem

In this section we solve equation (6.1). Due to the presence of zero modes in the operator $L_{22}(0)$ we need to consider separately (6.1) in the complement of these zero modes, at least for $p \in E_0$, where

$$E_0 = [-p_0, p_0] \cup [\pi - p_0, \pi + p_0],$$

with $p_0 = B\lambda^2$, was defined in (8.22), and the projection of (6.1) onto the zero modes. The solution to the complementary equations leads to the Fourier Law, i.e. an expression of the currents $(j, j')$ in terms of the temperature $T$ and chemical potential $A$. The solution to the projected equation determines finally $T$ and $A$.

We look for a solution of the form

$$W = Q_0 + w,$$

where $Q_0 = Q_0(T, A)$, for some functions $T, A$, is given by (8.16), and $w$, also written as a pair $(J, r)$, is as follows. Let

$$w_s = w \chi(p \in E_0)$$

Let $P$ be the projection in $L^2\left(\omega(p, k)^2 dk\right)$ to the span of $\{\omega(p, k)^{-2}, \omega(p, k)^{-3}\}$ and $P^\perp = 1 - P$. We demand

$$P^\perp r_s(p, \cdot) = r_s(p, \cdot), \quad p \in E_0$$

Given a function $f$ that is Hölder continuous in $p$ on $E_0$, let $\tilde{f}$ denote a linear extension of $f$ to $\frac{N}{N} \mathbb{Z}_{2N}$. We have $\|\tilde{f}\|_\alpha \leq C\|f\|_\alpha$. We proceed similarly with elements of $S$ or of $E$. Now write

$$w = \tilde{w}_s + w_\ell$$

where $\tilde{w}_s$ is the extension of $w_s$ defined above. The function $w_\ell$ satisfies $w_\ell(p) = 0$ for $p \in E_0$.

Since from (7.10, 10.2) we have $PC = C$, eq. (6.1) can be written, for $p \in E_0$, as a pair of equations:

$$\Pi \left(\begin{array}{cc}
0 & \delta \omega^2 \\
\delta \omega^2 & 0
\end{array}\right) \left(\begin{array}{c}
J \\
Q
\end{array}\right) + N(Q, J) + N_\Gamma(Q, J) = 0$$

(11.1)

where

$$\Pi = \left(\begin{array}{cc}
1 & 0 \\
0 & P^\perp
\end{array}\right)$$

and

$$P(\delta \omega^2 J + N_2 + N_\Gamma) = PC.$$  

(11.2)

For $p \notin E_0$, we will solve directly (6.1):

$$\left(\begin{array}{cc}
0 & \delta \omega^2 \\
\delta \omega^2 & 0
\end{array}\right) \left(\begin{array}{c}
J \\
Q
\end{array}\right) + N(Q, J) + N_\Gamma(Q, J) = \left(\begin{array}{c}
0 \\
C
\end{array}\right)$$

(11.3)
We look for solutions where \((T, A)\), defined for \(p \in E_0\), is such that \((\tilde{T}, \tilde{A}) \in B_1\), where \(B_1 \subset E \times E\) is the ball

\[
\|T - T_+\delta(p)\|_E \leq B_1 \tau
\]

\[
\|A\|_E \leq B_1 \tau^2
\]

with

\[
T_+ = \frac{1}{2}(T_1 + T_2)
\]

and \(B_1\) will be fixed later to be \(O(1)\). For \(\tau\) small enough, we have \(B_1 \subset B_2\), defined in (9.3), so that the estimates of section 9 can be used. As for \(w\), we look for \(w_s\) with \(\tilde{w}_s\) in the ball \(B_2 \subset S \oplus S\), given by

\[
B_2 = \{(J, r) \mid \| (\tilde{J}, \tilde{r}) - (J_0, r_0) \|_S \leq B_2 \tau \}
\]

where \((J_0, r_0)\) is given in (11.36) and bounded in (11.37). Finally, we shall choose \(w_{\ell} \in B_3\), where

\[
B_3 = \{(J, r) \mid \| J \|_S + \| r \|_S \leq B_3 \lambda^{-2} \tau, \quad J_p = r_p = 0 \text{ for } p \in E_0 \}
\]

Our Theorem is a consequence of the Proposition below and the Remark following it.

**Proposition 11.1.** Let \(\lambda, \tau, N\) be as in the Theorem.

(a) Given \((T, A)\) defined for \(p \in E_0\), such that \((\tilde{T}, \tilde{A}) \in B_1\), and given \(w_{\ell} \in B_3\), there exists a unique \(w_s\), such that \(\tilde{w}_s \in B_2\) and such that \(Q_0(\tilde{T}, \tilde{A}) + \tilde{w}_s + w_{\ell}\) solves (11.1) for \(p \in E_0\). Moreover, \(\tilde{w}_s\) is Lipschitz in \((T, A, w_{\ell})\) with Lipschitz constant \(O(\lambda^2)\).

(b) Given \(w_{\ell} \in B_3\), there exists a unique \((T, A)\) defined for \(p \in E_0\), such that \((\tilde{T}, \tilde{A}) \in B_1\), and such that \(Q_0(\tilde{T}, \tilde{A}) + \tilde{w}_s + w_{\ell}\) solves (11.2), for \(p \in E_0\), where \(w_s(T, A, w_{\ell})\) is the solution obtained in (a). The pair \((T, A)\) is Lipschitz in \(w_{\ell}\) with Lipschitz constant \(O(\tau)\).

(c) There exists a unique \(w_{\ell} \in B_3\) such that \(Q_0(\tilde{T}, \tilde{A}) + \tilde{w}_s(T, A, w_{\ell}) + w_{\ell}\) solves (11.3), for \(p \notin E_0\), where \((T, A) = (T, A)(w_{\ell})\) is the solution obtained in (b).

**Remark.** Moreover, the precise bounds stated in the Theorem will be given in the course of the proof: see (11.81), (11.86), for the statements about \(t\), (11.5) for the ones on \(A\), and, for the currents, see (11.7) the Remark at the end of subsection 11.1, specially (11.43).

### 11.1 Fourier’s Law

Let us prove first part (a) of Proposition 11.1. For simplicity of notation, we shall write \(Q_0\) for \(Q_0(\tilde{T}, \tilde{A})\), and drop the tilde on \(T, A\). Recall that \(Q = Q_0 + r\). The leading inhomogeneous term in (11.1) is \(-\delta \omega^2 Q_0\). Using (8.16), we write:

\[
\delta \omega^2 Q_0 = t(2p)\rho_1(p, k) + d(-2p)(T * A)(2p)\rho_2(p, k) + \rho_3(p, k)
\]

(11.9)
where \( t(p) = d(-p)T(p) \),

\[
\begin{align*}
\rho_1(p, k) &= d(-2p)^{-1} \delta \omega^2(p, k) \omega(p, k)^{-2} \\
\rho_2(p, k) &= d(-2p)^{-1} \delta \omega^2(p, k) \omega(p, k)^{-3}
\end{align*}
\]

are smooth functions and

\[
\rho_3(p, k) = \delta \omega^2(p, k) \sum_{n=2}^{\infty} T * A^n \omega(p, k)^{-2-n}
\]

is in \( S \) with

\[
\|\rho_3\|_S \leq C\|A\|_E^2.
\]

The nonlinear term \( \mathcal{N}(Q, J) \) in (11.1, 11.3) was studied in Section 9. It is given by (see eq. (5.12, 5.18, 5.19))

\[
\mathcal{N}(p, k) = \frac{9}{8}(2\pi)^3 \lambda^2 \left( \begin{array}{c}
n_1(p, k) - n_1(p, -k) \\
n_2(p, k) + n_2(p, -k)
\end{array} \right)
\]

and \( n \) is the sum (9.4). From Proposition 9.2, we infer

\[
\|\mathcal{N}(Q_0, 0)\|_S \leq C\lambda^2(\|t\|_S + \|A\|_E).
\]

From (9.22), (9.24), the definition of \( \mathcal{L}_p \) in section 10 and Proposition 9.5, we get

\[
\|D\mathcal{N}(Q_0, 0) - \mathcal{L}_p\| \leq C\lambda^2(\|t\|_S + \|A\|_E)
\]

where on the LHS the norm is the operator norm in \( S \oplus S \). Finally, Proposition 9.6 gives

\[
\|\mathcal{N} - \mathcal{N}(Q_0, 0) - D\mathcal{N}(Q_0, 0)(J, r)^T\|_S \leq C\lambda^{2+\alpha/2}(\|J\|_S + \|r\|_S)^2.
\]

So, combining those estimates, we get that, for \( T, A, J, r \), as in the theorem:

\[
\|\mathcal{N}(p, k) - \mathcal{L}_p(J, r)^T\|_S \leq C\lambda^2 \tau
\]

and is Lipschitz as a function of \( T, A \) with constant \( C\lambda^2 \), and as a function of \( J, r \), with a constant \( CN^{-4+\alpha} \). Consider next the function \( \mathcal{N}_r \), defined in (6.3):

\[
\mathcal{N}_r = (\Gamma J + J\Gamma, \Gamma P + P\Gamma)^T.
\]

Let us start with \( \Gamma J + J\Gamma \), given by (7.11) with \( P \) replaced by \( J \). Recalling the definition (8.11), using

\[
\int |d(p)|^{-1} dp \leq C\log N,
\]

that follows from \( |d(p)|^{-1} \leq C|p|^{-1} \), for \( p \neq 0 \) and \( |p| \geq cN^{-1} \), we get

\[
\int |J(q, q + k - p)| dq \leq CN^{-1+\alpha/2}\|J\|_S,
\]

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uniformly in $k, p$, since the singularities in (8.11) affect only the first variable and $\log N \leq N^{\alpha/2}$.

Now, use the fact that, for functions on the $\pi/2N$ lattice,

$$\|f\|_\alpha \leq CN^\alpha \|f\|_\infty,$$

and identify $\Gamma J + J\Gamma$ as an element of $S$ of the form $\Gamma J + J\Gamma = (0, 0, *, 0)$, to get:

$$\|\Gamma J + J\Gamma\|_S \leq C\gamma N^\alpha N^{1-\alpha/2} \int |J(q, q + k - p)| dq \|_\infty \leq C\gamma N^\alpha \|J\|_S \leq CN^{-1+5\alpha/4} \|J\|_S (11.21)$$

where in the last inequality, we use the definition (8.21) of $\gamma$. $\Gamma J + J\Gamma$ is of course Lipschitz, with constant $CN^{-1+5\alpha/4}$. For the term $\Gamma P + P\Gamma$, we need to study $P$, given in (5.17):

$$P = \omega(p, k)^2 Q + \frac{1}{2}(J\Gamma - \Gamma J) - \frac{9}{8}(2\pi)^3d\lambda^2(n_1(p, k) + n_1(p, -k)). (11.22)$$

We have

$$\omega(p, k)^2 Q = T(2p) + (T * A)(2p)\omega(p, k)^{-1} + p_1 + \omega(p, k)^2 r$$

where $p_1$ collects the $n \geq 2$ terms coming from the expansion (8.16). From Proposition 9.2, the last term in (11.22) at $J = r = 0$ is given by (9.12). Propositions 9.3, 9.5 and 9.6 control the corrections. The $(J\Gamma - \Gamma J)$ term is bounded as in (11.21). To summarize, let us call $P_S$ the sum of $\frac{1}{2}(J\Gamma - \Gamma J) + \omega(p, k)^2 r$, of the corrections to the $J = r = 0$ term in $-\frac{9}{8}(2\pi)^3d\lambda^2(n_1(p, k) + n_1(p, -k))$ and of the term corresponding to $m$ in (9.12). Let $P_E$ be the rest, i.e. $T(2p) + (T * A)(2p)\omega(p, k)^{-1} + p_1$, and what corresponds to the sum in (9.12). Collecting the bounds established for these various terms, we get (remember that inverse powers of $N$ are small compared to $\lambda^2$):

**Proposition 11.2.** $P$ can be written as

$$P = P_E + P_S$$

with

$$P_E(p, k) = T(2p) + (T * A)(2p)\omega(p, k)^{-1} + \overline{P}_E(p, k) (11.23)$$

where

$$\overline{P}_E(p, k) = \sum_{n=1}^\infty F_n(2p)h_n(p, k) (11.24)$$

where $F_n \in E$, with $\|F_n\|_E \leq C^n\|A\|_E^n$, for $n \geq 2$, $\|F_1\|_E \leq C\lambda^2\|A\|_E$, and where the functions $h_n$ are smooth with

$$\|h_n\|_\infty \leq C^n, (11.25)$$

$P_S \in S$ and

$$\|P_S\|_S \leq C(\|r\|_S + \lambda^2(\|J\|_S + \|t\|_S + \|A\|_E)). (11.26)$$
We may now bound $P^\perp (P\Gamma + \Gamma P)$. The $P_S$ term is in $\mathcal{S}$, like $J$ and can be bounded, as in (11.21), by the RHS of (11.26) times $N^{-1+5\alpha/4}$. The $T(2p)$ in (11.23) drops out from $P^\perp$, since it is constant in $k$ and we use (10.2). The other terms in (11.23) are bounded using

$$\int dq |F(q)| \leq C\|F\|_E,$$  \hspace{1cm} (11.27)

which follows easily from:

$$|F(p)| \leq \left( \delta(p) + \frac{1}{N|d|^2} + \frac{1}{N^{1-\alpha/2}|d|} + \frac{1}{N^{3/2}|d|^{5/2}} \right) \|F\|_E,$$  \hspace{1cm} (11.28)

using (11.18) and:

$$\int |d(p)|^{-k} dp \leq CN^{k-1} \quad k > 1,$$  \hspace{1cm} (11.29)

which is proven like (11.18). Thus $P^\perp (\Gamma P_E + P_E\Gamma)(p, k)$ is smooth, since, looking at (7.11) we see that the dependence on $p, k$ in $P^\perp (\Gamma P_E + P_E\Gamma)(q, q + k - p)$ is only through the second argument of $P_E$, in which $P_E$ is smooth, by Proposition 11.2. Moreover, using (11.27) and (11.23), (11.24),

$$|\int P^\perp (\Gamma P_E + P_E\Gamma)(q, q + k - p)| \leq C\gamma \|A\|_E.$$  \hspace{1cm} (11.30)

Since $P^\perp (\Gamma P_E + P_E\Gamma)(p, k)$ is smooth, let us identify it with an element of $\mathcal{S}$ of the form $(0, 0, *, 0)$. Then, by (8.21), $P^\perp (\Gamma P_E + P_E\Gamma) \in \mathcal{S}$ and

$$\|P^\perp (\Gamma P_E + P_E\Gamma)\|_S \leq CN^{-1+\alpha/4}N^{1-\alpha/2} \|A\|_E = CN^{-\alpha/4} \|A\|_E.$$  \hspace{1cm} (11.31)

Combining the above bounds, we get:

$$\|P^\perp (\Gamma P + \Gamma P)\|_S \leq CN^{-\alpha/4} \|A\|_E + CN^{-1+5\alpha/4}(\|r\|_S + (\|J\|_S + \|t\|_S))$$  \hspace{1cm} (11.32)

and $P^\perp (\Gamma P_E + P_E\Gamma)$ is Lipschitz in $A, E, J, r$, with constants $O(N^{-\alpha/4})$.

We may summarize this discussion by rewriting equation (11.1); consider $T, A$ given, for $p \in E_0$ and $w_\ell$ given for $p \notin E_0$. Let us denote

$$s(p) = d(-p)(T * A)(p) \equiv d(-p)S(p).$$  \hspace{1cm} (11.33)

Then

$$(J_s, r_s)^T = -\mathcal{D}_p^{-1} \Pi \left[ (\rho_1 t(2p) + \rho_2 s(2p), 0)^T + \mathcal{R} \right]$$  \hspace{1cm} (11.34)

for $p \in E_0$, using the fact that $r_s = P^\perp r_s$, and therefore, $(J_s, r_s)^T = -\Pi (J_s, r_s)^T$. Here, $\mathcal{R} \in \mathcal{S}$ includes all the terms in (11.1), apart from the first two in (11.9). Combining (11.12), (11.16), (11.21), (11.32), $\|\mathcal{R}\|_S$ is bounded, for $T, A, J, r$ as in the Theorem (using the fact that $\tau \lambda^{-2} \leq 1$), by:

$$\|\mathcal{R}\|_S \leq C\lambda^2 B_1 \tau$$  \hspace{1cm} (11.35)
and is Lipschitz \( J, r \) with constant \( \mathcal{O}(\tau) \), and in \( T, A \) with constant \( \mathcal{O}(\lambda^2) \). \( \mathcal{D}_p^{-1} \) is the operator defined by (10.3), and bounded in Proposition 10.1.

Let, for \( p \in E_0 \),

\[
(J_0, r_0)^T = -\mathcal{D}_p^{-1}\Pi(\rho_1 t(2p) + \rho_2 s(2p), 0)^T
\]

(11.36)

Then, by Proposition 10.1 and Lemma 9.1.b,

\[
\| (\tilde{J}_0, \tilde{r}_0) \|_S \leq C\lambda^{-2}(\| t \|_S + \| s \|_S) \leq C\lambda^{-2}(\| t \|_S + \| A \|_E).
\]

(11.37)

Now, given \( \tilde{w}_s \in B_2 \) and \( w_\ell \in B_3 \) we have from Proposition 10.1. and (11.35)

\[
\| \mathcal{D}_p^{-1}\Pi \mathcal{R} \|_S \leq CB_1\tau,
\]

(11.38)
i.e. \( \mathcal{D}^{-1}\Pi \mathcal{R} \in B_2 \) for \( B_2 > CB_1 \). It is Lipschitz in \( J, r, w_\ell \) with constant \( \mathcal{O}(\tau) \), and in \( T, A \) with constant \( \mathcal{O}(\lambda^2) \). Thus we get, from the contraction mapping principle,

**Proposition 11.3.** Given \( T, A, w_\ell \) as above, equation (11.34) has a unique solution \((\tilde{J}_s, \tilde{r}_s) \in B_2, \) which is Lipschitz in \((T, A)\) with Lipschitz constant \( \mathcal{O}(\lambda^2) \) and in \( w_\ell \) with Lipschitz constant \( \mathcal{O}(\tau) \).

This proves part (a) of Proposition 11.1. Now assume that we have proven part (b) of that Proposition. This will be done in the next subsection. We want to prove here part (c), since it is done in the same spirit as part (a).

Thus, consider equation (11.3), for \( p \notin E_0 \). Following the argument that led from (11.1) to (11.34), we may rewrite it as:

\[
(J, r)^T = -\mathcal{D}_p^{-1}\left[ (\rho_1 t(2p) + \rho_2 s(2p), 0)^T + \mathcal{R} \right]
\]

(11.39)

where \( \mathcal{D}_p \) was defined in (10.4), for \( p \notin E_0 \), and is invertible by Proposition 10.1. We can write (11.39) as:

\[
(J_\ell, r_\ell)^T = -(\tilde{J}_s, \tilde{r}_s)^T - \mathcal{D}_p^{-1}\left[ (\rho_1 t + \rho_2 s, 0)^T + \mathcal{R} \right]
\]

(11.40)

where \((\tilde{J}_s, \tilde{r}_s) \) is the Lipschitz function of \( w_\ell \), given by Proposition 11.3. The \( S \)-norm of the RHS is bounded by \( C\lambda^{-2}B_1\tau \) and so, taking \( B_3 = CB_1 \), (11.40) is in \( B_3 \), provided we show it vanishes for \( p \in E_0 \).

By assumption, \( T, A \) here is such that (11.2) holds for \( p \in E_0 \) (part (b) of the Proposition, to be proven below), and, by part (a), we know that (11.1) holds for \( p \in E_0 \). Putting (11.1) and (11.2) together, we see that the full equation, (11.3), is satisfied by \( Q_{T,A} + \tilde{w}_s + w_\ell \) for \( p \in E_0 \). But (11.39) is merely a rewriting of (11.3), whenever \( \mathcal{D}_p \) is invertible. Since we can assume, by choosing, if necessary, \( B \) in Proposition 10.1 larger than here, that \( \mathcal{D}_{\pm p} \) and \( \mathcal{D}_{\pi \pm p} \) are invertible, and since, by assumption,

\[
Q_{T,A} + \tilde{w}_s + w_\ell = Q_{T,A} + \tilde{w}_s
\]

for \( p \in E_0 \), we know that \( Q_{T,A} + \tilde{w}_s \) solves (11.39) for \( p = \pm p_0 \) or \( p = \pi \pm p_0 \). But that means that the RHS of (11.40) vanishes for those values of \( p \). We can define both sides to be zero for other values of \( p \in E_0 \), if we want, without affecting the fact that \( J_\ell, r_\ell \) are in \( S \), and thus we
obtain part (c) of Proposition 11.1, by applying the contraction mapping principle to the fixed point equation (11.40) to \((J_\ell, r_\ell) \in B_3\).

**Remark.** The function \(J(T, A)\) is a general form of the *Fourier Law*, expressing the \(E_{q_x}p_y\) correlation function as a function of the local temperature and chemical potential. In particular, see (11.36) and remember that \(\Pi\) is the identity of the first component,

\[
J_0(p, k) = \kappa_1(p, k)t(2p) + \kappa_2(p, k)s(2p)
\]

with

\[
\kappa_j = -\mathcal{D}_p^{-1}(\rho_j, 0)^T.
\]

Inserting to (7.7) and (7.14) we get

\[
\left( \dot{j}_0(p), j'_0(p) \right)^T = \kappa_0(p) \left( t(p), s(p) \right)^T
\]

where \(\kappa_0(p)\) is \(C^\alpha\) in \(p\). Since \(\mathcal{D}_p\) is of order \(\lambda^2\), \(\kappa_0(p)\) is of order \(\lambda^{-2}\). The full \(\kappa(p)\) introduced in the Theorem has corrections \(\mathcal{O}(1)\) coming from \(\mathcal{D}_p^{-1}\) applied to the parts of \(\mathcal{R}\) in (11.34) that are linear in \(t, s\). Since \(\mathcal{R}\) is of order \(\lambda^2\), the result is \(\mathcal{O}(1)\), i.e. small compared to \(\kappa_0\), at least if the latter does not vanish. In the next subsection, we shall need the fact that \(\kappa(p)\) is invertible for \(p \in E_0\). To show that, let us first compute \(\kappa_0(0)\), which we shall denote for simplicity \(\kappa^0\).

From (10.3) and (6.8), we get

\[
\mathcal{D}_0 = \begin{pmatrix} \mathcal{L}_{11}(0) & 0 \\ 0 & \mathcal{L}_{22}(0) \end{pmatrix}.
\]

Inserting (7.5) to (11.10) and (11.11) we have

\[
\rho_j(0, k) = 4i \sin k \omega(k)^{-j}
\]

for \(j = 1, 2\), and so, defining

\[
\psi_j(k) = 4 \sin k \omega(k)^{-i}, \quad j = 1, 2,
\]

we have

\[
\kappa_j(0, k) = -i \left( \mathcal{L}_{11}(0)^{-1} \psi_j \right)(k).
\]

Inserting to (7.7) we have, for \(j = 1, 2\),

\[
\kappa^0_{ij} = -2 \int dk \sin k \omega(k)(\mathcal{L}_{11}(0)^{-1} \psi_j)(k).
\]

From (7.14) we get

\[
\kappa^0_{2j} = -2 \int dk \sin k \eta(0, k) \omega(k)(\mathcal{L}_{11}(0)^{-1} \psi_j)(k).
\]

where \(\eta(0, k)\) equals \(\omega(k)^{-1} - \int dk \omega(k)^{-1}\). Now, note that

\[
\kappa^0_{ij} = -\frac{1}{2} \left( \psi_1, \mathcal{L}_{11}^{-1}(0) \psi_j \right)
\]
where $(\cdot, \cdot)$ is the scalar product in $H_0$, and that
\[
\kappa^0_{2j} = -\frac{1}{2}(\psi_2, L^{-1}_{11}(0)\psi_j) + \beta_0 \frac{1}{2}(\psi_1, L^{-1}_{11}(0)\psi_j),
\]
(11.47)
where $\beta_0 = \int dk \omega(k)^{-1}$. Computing the determinant of the $2 \times 2$ matrix $\kappa^0$, we see that it equals the one with $\beta_0 = 0$, and the latter does not vanish because $L_{11}(0)$ is a strictly negative operator (see (10.6)). Now, from Hölder continuity, we get that $\kappa_0(p)$, and therefore $\kappa(p)$, is invertible for $p$ small, i.e. for $p \in E_0$, for $\lambda$ small enough.

Finally, observe that, since $L_{11}(0)$ is a strictly negative operator, $\kappa(p)$, is a positive matrix, for $p \in E_0$. To understand the connection with the Fourier law (1.1), note that $t(p) = d(-p)T(p)$, and, in $x$-space, $d(-p)$ is $-\nabla$.

### 11.2 Solving the conservation laws

We are left with the proof of part (b) of Proposition 11.1. This reduces to solving the two conservation laws, equations (7.3) and (7.13), which are equivalent to (11.2). Indeed, (7.3) is (4.20) for $x = y$, which is the same as integrating (5.19) with $dk$, or taking the scalar product of (5.19) with $\omega(p, k)^{-2}$ in $H_0$. For (7.13), it amounts to taking the scalar product of (5.19) with a linear combination of $\omega(p, k)^{-2}$ and $\omega(p, k)^{-3}$.

Let us introduce a more compact notation. We set $J = (j, j')$ and write (7.3) and (7.13) as
\[
d(p)J(p) + F(p) - \Theta(p) = 2\gamma(T_1 + e^{iNp}T_2, 0)^T
\]
(11.48)
for $p \in E_0$ with $E_0$ given by (8.22), where we can assume that $\kappa(p)$ is invertible. Equation (11.48) has the friction term, see (7.15)
\[
F(p) = \gamma \int dk dq P(q, k)(1 + e^{i(q+p)N})(2, \psi(p, k, q))^T,
\]
(11.49)
where we use $e^{ipN} = e^{-ipN}$, and the projection of $N_{22}$:
\[
\Theta(p) = (0, \theta(p))^T,
\]
(11.50)
where $\theta$ is given by (7.9).

$J$ is given by the Fourier law i.e. the solution of (11.39) with leading term (11.43) which we may write as (see the Remark at the end of subsection 11.1):
\[
J(p) = \kappa(p)[(t(p), s(p))^T + z(p)],
\]
(11.51)
where $s$ is given in eq. (11.33). $J, F$ and $\Theta$ are functions of $T, A \in E$, and $w_l \in B_3$ (including, indirectly, as functions of $w_s$, which is a function of $T, A, w_l$, by part a of Proposition 11.1). So, for $w_l$ fixed (11.48) is a nonlinear, nonlocal elliptic equation for $T$ and $A$. We look for a solution to (11.48) in the ball $B_1 \subset E \times E$, defined in (11.4, 11.5). For such $T$ the map $A \to T * A \equiv S$ is invertible (because $T$ in (11.4) is close to a delta function, which is the identity for the convolution) and
\[
\|S - T+A\|_E \leq C\tau^3.
\]
Thus we can use $T, S$ as the unknowns, in the ball $\mathcal{B}_1$. The following Proposition collects the properties of the functions $z$ and $\theta$, studied in Propositions 11.3 and 9.7 (since $|\theta(0)| \leq CN^{-1}$; we have that $d^{-1}(\theta - \theta(0)) \in S$):

**Proposition 11.4.** $z$ and $d^{-1}(\Theta - \Theta(0))$ are Lipschitz functions $\mathcal{B}_1 \times \mathcal{B}_3 \to S \oplus S$ with

$$\|z\|_S \leq C\lambda^2 \tau, \quad \|d^{-1}(\Theta - \Theta(0))\|_S \leq C\lambda^2 \tau, \quad (11.52)$$

$$|\Theta(0)| \leq CN^{-1}. \quad (11.53)$$

and Lipschitz constants bounded by $C\lambda^2$.

Recall next that $f \in S$ is decomposed as (see (8.6))

$$f(p, k) = f_+(p, k)\sigma_+(2p) + f_-(p, k)\sigma_-(2p),$$

with $\sigma_\pm(p) = 1 \pm e^{iNp}$, and $f_\pm = \frac{1}{2} f$. Insert

$$1 + e^{i(q+p)N} = \frac{1}{2} \left( \sigma_+(p)\sigma_+(q) + \sigma_-(p)\sigma_-(q) \right) \quad (11.54)$$

into eq. (11.49) and use $\sigma_+(q)\sigma_-(q) = 0$ for $q \in \frac{N}{N}Z_{2N}$ to get:

$$\mathcal{F}(p) = \mathcal{F}_+(p)\sigma_+ + \mathcal{F}_-(p)\sigma_-,$$

with

$$\mathcal{F}_\pm(p) = \gamma \int_\pm dq \int dk P_\pm(q/2, k)(2, \psi(p, k, q))^T \quad (11.55)$$

where $f_\pm dq = \frac{1}{2} f dq\sigma_\pm(q)$ i.e. the $q$-sum in $f_\pm$ runs over the odd (for $-$) or even, non zero (for $+$), multiples of $\frac{\pi}{N}$. We used here the fact that $\sigma_\pm(q)^2 = 0, 4$ to cancel the factors of $\frac{1}{2}$ in (11.54) and in $P_\pm = \frac{1}{2}P$.

Thus, (11.48) becomes two equations, one $(+)$ valid on the even sublattice, and the other one $(-)$ on the odd sublattice:

$$d \mathcal{J}_\pm + \mathcal{F}_\pm - \Theta_\pm = 2\gamma T_\pm(1, 0)^T \quad (11.56)$$

with $T_\pm$ the average temperature (11.6) and

$$T_\pm = \frac{1}{2}(T_1 - T_2) \quad (11.57)$$

It is useful to separate from (11.55) the part which is constant in $p$:

$$\mathcal{F}_\pm(p) = \mathcal{F}_\pm(0) + f_\pm(p), \quad (11.58)$$

and similarly for $\Theta_\pm(p)$:

$$\Theta_\pm(p) = \Theta_\pm(0) + \Theta'_\pm(p), \quad (11.59)$$

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Since $\psi$ in (7.16) is smooth we may write:

$$f_\pm(p) = d(p)g_\pm(p).$$  \hspace{1cm} (11.60)

and we may estimate $g_\pm$ as in the derivation of (11.31), and obtain (there is no $P^\perp$ here, so we have a term linear in $T$):

**Proposition 11.5** The functions $g_\pm \in S$ are Lipschitz in $(T, S) \in \mathcal{B}_1$ with $\|g_\pm\|_S$ and the Lipschitz constants $O(N^{-\alpha/4})$.

Using (11.51), we may write (11.56) as

$$d(p)\kappa(p) \left((t_\pm(p), s_\pm(p))^T + W_\pm(p)\right) = 2\gamma T_\pm(1, 0)^T - F_\pm(0) + \Theta_\pm(0)$$  \hspace{1cm} (11.61)

where $W_\pm(p) = z_\pm(p) + \kappa(p)^{-1}(g_\pm(p) - d(p)^{-1}\Theta'_\pm(p))$, is defined for $p \in E_0$, where $\kappa(p)$ is invertible (see Remark at the end of subsection 11.1). So, combining Propositions 11.4 and 11.5, $\tilde{W}_\pm \in S$, with

$$\|\tilde{W}_\pm\|_S \leq C\lambda^2 \tau,$$  \hspace{1cm} (11.62)

and the functions $\tilde{W}_\pm$ are Lipschitz in $(T, S)$, with constants $C\lambda^2$.

Let us next analyze $F_\pm(0)$. Using Proposition 11.2, $S = T \ast A$ and (7.8), we write

$$F_\pm(0) - \Theta_\pm(0) = \phi_\pm + \psi_\pm$$

$$\phi_\pm = \gamma \int_{\pm} dq \int dk \left(T(q) + \rho(q/2,k)S(q)\right)(2, \psi(0, k, q))^T$$

$$\psi_\pm = \gamma \int_{\pm} dq \int dk \left(P_E(q/2, k) + P_S(q/2, k)\right)(2, \psi(0, k, q))^T - \Theta_\pm(0).$$ \hspace{1cm} (11.64)

For $\psi_\pm$, we proceed as in the proof of (11.31). The contribution coming from $P_S$ is small, see (11.21), while the one coming from $P_E$ is to leading order, see (11.24), $O(\gamma^2 \lambda^2 \tau^2)$. The contribution of $\Theta_\pm(0)$ is $O(N^{-1})$, by Proposition 9.7. So, $\psi_\pm$ are bounded by

$$|\psi_\pm| \leq C\gamma\lambda^2 \tau^2$$  \hspace{1cm} (11.65)

and are Lipschitz in $(T, S)$ with constant $C\gamma\lambda^2$.

It is instructive to solve first the simplified equation (11.61) with $W_\pm$ dropped and $F_\pm(0)$ replaced by $\phi_\pm$. Then, for $p \neq 0$,

$$\kappa(p) \left((t_\pm(p), s_\pm(p))^T\right) = d(p)^{-1}\xi_\pm$$  \hspace{1cm} (11.66)

with

$$\xi_\pm = 2\gamma T_\pm(1, 0)^T - \phi_\pm$$  \hspace{1cm} (11.67)

and, for $p = 0$ (where obviously only the equation with index $+$ holds),

$$2\gamma T_+(1, 0)^T = \phi_+.$$  \hspace{1cm} (11.68)
Equations (11.67) and (11.68) imply
\[ \xi_+ = 0 \]  \hspace{1cm} (11.69)
so, by (11.66),
\[ (t_+, s_+) = 0 \]
and, from (11.66) for \(-\), we can write
\[ \left( t_-(p), s_-(p) \right) = d(p)^{-1} \left( \tau(p), \zeta(p) \right) \]  \hspace{1cm} (11.70)
with
\[ \kappa(p) \begin{pmatrix} \tau(p) \\ \zeta(p) \end{pmatrix}^T = \xi_- \]  \hspace{1cm} (11.71)
constant in \( p \), i.e.:
\[ \left( \tau(p), \zeta(p) \right)^T = \kappa(p)^{-1} \kappa(0) \begin{pmatrix} \tau(0) \\ \zeta(0) \end{pmatrix}^T, \]  \hspace{1cm} (11.72)
where \( \begin{pmatrix} \tau(0) \\ \zeta(0) \end{pmatrix} \) is defined by extending (11.71) (which was derived on the odd sublattice) to \( p = 0 \). Hence, since \( s(p) = d(-p)S(p) \), \( t(p) = d(-p)T(p) \),
\[ U(p) \equiv \begin{pmatrix} T(p) \\ S(p) \end{pmatrix}^T = U_0 \delta(p) + |d(p)|^{-2} \begin{pmatrix} \tau(p) \\ \zeta(p) \end{pmatrix}^T \sigma_-(p). \]  \hspace{1cm} (11.73)
The unknowns are \( U_0 = (T_0, S_0)^T, \tau(0) \) and \( \zeta(0) \) and they will be determined from (11.67) and (11.68), where the functions \( \phi_\pm \), given by (11.63), are functions of \( U_0, \tau(0), \zeta(0) \) via (11.73):
\[ \phi_+ = \gamma \int dk \begin{pmatrix} T_0 + \rho(0, k)S_0 \\ 2, \psi(0, k, 0) \end{pmatrix}^T \]  \hspace{1cm} (11.74)
\[ \phi_- = 2\gamma \int dq \int dk |d(q)|^{-2} \begin{pmatrix} \tau(q) + \rho(q/2, k)\zeta(q) \\ 2, \psi(0, k, q) \end{pmatrix}^T, \]  \hspace{1cm} (11.75)
where the prefactor of 2 comes from the fact that, in (11.73), \( \sigma_-(p) = 0, 2 \).
Equations (7.8), (7.12) and (7.16) imply \( \psi(0, k, 0) = 2(\rho(0, k) - \int dk \rho(0, k)) \), with \( \rho(0, k) = \omega(k)^{-1} \). So,
\[ \int \psi(0, k, 0) dk = 0, \]
\[ \int \rho(0, k) dk \equiv \beta_1 > 0, \]
\[ \frac{1}{2} \int \rho(0, k) \psi(0, k, 0) dk = \int \omega(k)^{-2} dk - \left( \int \omega(k)^{-1} dk \right)^2 \equiv \beta_2 > 0. \]
Hence (11.74) gives \( \phi_+ = 2\gamma(T_0 + \beta_1 S_0, \beta_2 S_0)^T \) and, from (11.68), we get:
\[ S_0 = 0 \quad , \quad T_0 = T_+ = \frac{1}{2}(T_1 + T_2). \]  \hspace{1cm} (11.76)
To analyze (11.75), we need the straightforward lemma.

**Lemma 11.6** Let $a$ be a function on $\mathbb{Z}_2$. Then

$$2\int_{\pm} |d(q)|^{-2}a(q) dq = N \left( I_\pm a(0) + O(N^{-\alpha}) \right)$$

where $I_* = \frac{1}{12}$ and $I_- = 3I_+ = \frac{1}{4}$.

The constants $I_+, I_-$ follow from the fact that (see (5.1)) $2\int_{\pm} |d(q)|^{-2} dq$ equals, to leading order in $N$, $N$ times the sum over even or odd non-zero integers $n$ (positive and negative) of $\frac{1}{\pi^2 n^2}$. The even sum equals a quarter of the sum over all integers, and the latter equals $\frac{1}{3}$.

Inserting (11.72) into (11.75), and applying Lemma 11.6 to the result, we obtain

$$\phi_- = 2\gamma NI_- \left[ \left( \begin{array}{c} 1 \\ 0 \\ \beta_1 \\ \beta_2 \end{array} \right) + O(N^{-\alpha}) \right] \left( \begin{array}{c} \tau(0) \\ \zeta(0) \end{array} \right).$$

Recalling that $\xi_- = \kappa(0) \left( \tau(0), \zeta(0) \right)^T = 2\gamma T_-(1,0)^T - \phi_-$ (see (11.71), (11.67)), we can write:

$$2\gamma NI_- \left[ \left( \begin{array}{c} 1 \\ 0 \\ \beta_1 \\ \beta_2 \end{array} \right) + O(N^{-\alpha}) \right] \left( \begin{array}{c} \tau(0) \\ \zeta(0) \end{array} \right) + \kappa(0) \left( \begin{array}{c} \tau(0) \\ \zeta(0) \end{array} \right) = 2\gamma T_- \left( \begin{array}{c} 1 \\ 0 \end{array} \right).$$

Since $\gamma = N^{-1+\alpha/4}$, we get:

$$\tau(0) = (NI_-)^{-1}(T_- + O(N^{-\alpha/4}))$$
$$\zeta(0) = O(N^{-1-\alpha/4}).$$

So, the simplified problem is solved by

$$(T^0, S^0) \equiv (T_+, 0)\delta(p) + \frac{T_-}{NI_-} |d(p)|^{-2} \sigma_- \left[ (1,0) + O(N^{-\alpha/4}) \right].$$

In $x$-space, this is a linear profile: use, for $p \neq 0$ the explicit formula (8.4) with $j_0$ replaced by $\frac{T_+ T_-}{N}$, and observe that, in (8.3), $j(x) = \frac{N}{2}$ for $x \in [1, N]$; remember also that $t(p) = d(-p)T(p)$, and, in $x$-space, $d(-p) = -\nabla$. So, we get:

$$T^0(x) = T_1 + \frac{|x|}{N}(T_2 - T_1)$$

plus a $O(N^{-1-\alpha/4})$ correction to the slope $1/N$. The first term comes from integrating (11.80) over $p$, using (11.77) for the second term, and $T_1 = T_+ + T_-.$

Let us now return to the full eq. (11.61) and incorporate the corrections $W_\pm$ in (11.61) and $\psi_\pm$ given by (11.64). Let

$$u_\pm(p) = (t_\pm(p), s_\pm(p))^T$$
\[ U_\pm(p) = U_0 \delta(p) + d(-p)^{-1}u_\pm(p). \]

Write
\[ u_\pm(p) = \kappa(p)^{-1} \left( d(p)^{-1} \xi_\pm \right) + v_\pm(p). \tag{11.82} \]

In the absence of \( W_\pm, \psi_\pm, v_\pm = 0 \) and \( \xi_\pm \) are as above. Consider the equation
\[ v_\pm(p) = -W_\pm(p). \tag{11.83} \]

Since the functions \( W_\pm \) are Lipschitz in \( U \), (11.83) has a unique solution \( v_\pm \in S \), with
\[ \|v_\pm\|_S \leq C \lambda^2 \tau, \tag{11.84} \]
and \( v_\pm \) is Lipschitz in \( U_0, \xi_\pm \). With this \( v_\pm \), (11.61) becomes, as in (11.67) and (11.68) and (11.69):
\[ \begin{align*}
\xi_+ &= 0, \quad 2 \gamma T_+(1,0)^T = \phi_+ + \psi_+ , \\
\xi_- &= 2 \gamma T_-(1,0)^T - \phi_ - \psi_- .
\end{align*} \tag{11.85} \]

Proceeding as in the simple case,
\[ \begin{align*}
\phi_+ &= 2 \gamma (T_0, \beta_1 S_0)^T + O(\gamma \tau \lambda^2) \\
\phi_- &= 2 \gamma N \left( \begin{array}{cc}
\beta_1 \\
\beta_2
\end{array} \right) + O(N^{-\alpha}) \right) \kappa^{-1}(0) \xi_- + O(\gamma \tau \lambda^2)
\end{align*} \]

where the term \( O(\gamma \tau \lambda^2) \) collects the contributions from \( v_\pm \) and \( \psi_\pm \) that are bounded by (11.65), (11.84), and is Lipschitz in \( T_0, S_0, \xi_\pm \). Thus \( T_0, S_0, N \xi_- \) have \( O(\tau \lambda^2) \) corrections and
\[ \|(T, S) - (T^0, S^0)\|_E \leq C \tau \lambda^2 \tag{11.86} \]
which, combined with (11.80), yields the claim of Proposition 11.1 (b). \( \square \)

A H"older regularity.

In this Appendix, we prove some refinements of the H"older continuity of the kernels proven in Section 9. We start with a corollary of Proposition 9.3 that will be needed in Appendix B. For this, let \( G \in C^0(\mathbb{T}^{3d}) \) and consider the function
\[ g(p, k) = \int G(k) \mu_{p,k}(dk) \tag{A.1} \]
on \( \mathbb{T}^{1+d} \), where \( \mu_{p,k}(dk) \) is defined by (9.43):
\[ \mu_{p,k}(dk) = \left( \sum_{i=1}^{3} s_i \omega(k_i) + s_4 \omega(p - k) + ik \right)^{-1} \delta(\sum_{i=1}^{3} k_i - p - k) dk_1 dk_2 dk_3 . \tag{A.2} \]
Denote
\[ g = I(G) \]  
(A.3)

Then we have

**Corollary A.1.**

(a) If \( G \) is smooth, \( I(G) \) is in \( C^\alpha(\mathbb{T}^{1+d}) \).

(b) Let \( G(k) = H(k_1, k_2)h(k_3) \) with \( H \) smooth and \( h \in C^0(\mathbb{T}^d) \). Then \( I(G) \) is in \( C^\alpha(\mathbb{T}^{1+d}) \) and
\[ \| I(G) \|_\alpha \leq C(H) \| h \|_\infty. \]  
(A.4)

(c) \( I \) is a bounded map from \( C^0(\mathbb{T}^{3d}) \) to \( C^0(\mathbb{T}^{1+d}) \).

Furthermore, if \( G \) depends smoothly on some parameters so does \( g \).

**Proof.** By comparing (A.2) and (9.31), we see that, if we replace \( k_3 \) in (A.2) by \( k_3 + p \), we may identify \( k_3 \) here and \( k' \) in (9.31). But then, the function \( G \) in (a) only changes \( \rho_s \) in (9.31), and the statement (a) can be proven just as the one on \( M(p, k) \) in Proposition 9.3. In (b), the function \( H \) again affects only \( \rho_s \), and \( h \) is the function on which the operator \( K_p \) in Proposition 9.3 acts. Hence, (b) follows from the claims made on \( K_p \). Finally (c) follows because we can bound the integral \( I(G) \) by the sum norm of \( G \) and the resulting integral is continuous on \( \mathbb{T}^{1+d} \), for the same reason that \( M(p, k) \) is continuous.

We need one more regularity result for the analysis of the \( \theta \) in (7.9) that will be made in the Appendix B. For this, define the function
\[ I(p) = \int \delta(\omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4)) \delta \left( p - \sum_{i=1}^{4} k_i \right) \chi(k) dk. \]  
(A.5)

Then

**Lemma A.2.** Let \( \chi \) in (A.5) be smooth and let \( I'(p) \) be the discrete derivative on \( \mathbb{Z}^{2N} \). Then, for some \( \alpha > 0 \),
\[ \| I' \|_\alpha \leq C \]  
(A.6)

uniformly in \( N \).

For the proof we need a further Lemma:

**Lemma A.3.** Let \( \chi(x) \) be smooth on \( \mathbb{T}^3 \) and \( F(x) = \sin(x_1 + x_2 + x_3) - \sin x_1 - \sin x_2 - \sin x_3 \).

Then \( g(p) \equiv \int \delta(F(x) - p)\chi(x)dx \) is smooth if \( p \neq 0 \) and
\[ |g(p)| \leq C(\log p)^2, \]
\[ |g'(p)| \leq C \left| \frac{\log p}{p} \right|. \]
Proof of Lemma A.2. In (A.5) write \( k_i = \frac{\pi}{2} + k'_i, i = 1, 2 \) and \( k_3 = -\frac{\pi}{2} + k'_3 \) which imply \( k_4 = p - k'_1 - k'_2 - k'_3 - \frac{\pi}{2} \) and the argument of the \( \delta \)-function equals

\[
\sum_{\alpha=2}^{3} \sin(k_1^\alpha + k_2^\alpha + k_3^\alpha) - \sin k_1^\alpha - \sin k_2^\alpha - \sin k_3^\alpha + P,
\]

where

\[
P = \sin(k_1 + k_2 + k_3 - p) - \sin k_1 - \sin k_2 - \sin k_3.
\]

Thus,

\[
I(p) = \int d\mathbf{k} J(P(k, p), k),
\]

with

\[
J(\lambda, k) = \int d\mathbf{x} dy \delta(F(\mathbf{x}) + F(\mathbf{y}) + \lambda) \chi(k, \mathbf{x}, \mathbf{y})
\]

\[
= \int dt \int d\mathbf{x} g(t, k, \mathbf{x}) \delta(F(\mathbf{x}) + t + \lambda),
\]

where

\[
g(t, k, \mathbf{x}) = \int \delta(F(\mathbf{y}) - t) \chi(k, \mathbf{x}, \mathbf{y}) dy,
\]

which by Lemma A.3. is smooth in all variables, for \( t \neq 0 \), and is bounded by \( |g| \leq C(\log t)^2 \).

Similarly the \( \mathbf{x} \)-integral yields

\[
J(\lambda, k) = \int dt ds h(t, s, k) \delta(t + s + \lambda),
\]

with \( h \) smooth if \( t, s \neq 0 \) and

\[
|\partial_t h| \leq C \left| \frac{\log |t|}{t} (\log |s|)^2 \right|
\]

and similarly for \( \partial_s h \). Then one gets

\[
|\partial_\lambda J| \leq C(\log |\lambda|)^4.
\]

Consider first the \( N \to \infty \) limit of (A.8). Then by (A.9) \( |I'(p)| \leq C \int dk |\log P|^4 \).

Since \( P \) is an analytic function, \( |\log P|^4 \) is integrable. It is easy to do the argument for the Riemann sum. In the same vein, one can extract a little Hölder continuity for \( I'(p) \). We leave the details for the reader.

\( \square \)

Proof of Lemma A.3. a) We have

\[
\partial_\alpha F = \cos(x_1 + x_2 + x_3) - \cos x_\alpha,
\]

so \( \nabla F = 0 \iff \cos x_1 = \cos x_2 = \cos x_3 = \cos(x_1 + x_2 + x_3) \) and thus \( x_1 = x_2 = -x_3 \) and permutations thereof. At these points \( F = 0 \). Hence \( F(x) \neq 0 \iff \nabla F(x) \neq 0 \). Thus, given
$x \in F^{-1}(p)$ there is a neighbourhood $U$ of $x$ and a smooth diffeomorphism: $\phi_p : U_0(0) \to U$ such that $(F \circ \phi_p)(y) = p + y_1$ and $\phi_p$ is smooth in $p$. Given $\psi \in C^\infty_0(U)$,

$$g_\psi(p) = \int \delta(F - p) \psi \, dx = \int dy_2 \ldots dy_d(\psi \circ \phi^{-1}_p)(0, y_2, \ldots, y_d) \det D\phi_p$$

is smooth. By a partition of unity, we conclude that $g$ is smooth for $p \neq 0$.

b) The Hessian of $F$ is

$$\partial_\alpha \partial_\beta F = \delta_{\alpha \beta} \sin x_\alpha - \sin(x_1 + x_2 + x_3).$$

At the critical points $x_1 = x_2 = -x_3$ it equals

$$H = -\sin x_1 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$ 

$H$ has eigenvalues $0, \sin x_1, -3 \sin x_1$. By a partition of unity argument it suffices to study two cases:

$1^\circ$ $\chi$ has support in a small ball around a critical point with $x_1 \neq 0, x_1 \neq \pi$.

$2^\circ$ supp $\chi$ is a small ball around the origin, around $(\pi, \pi, -\pi)$ or permutations thereof.

Case 1° By scaling and rotation, there exists a local coordinate $z = (u, v, w)$ such that

$$F = uv + f(u, v, w) \quad (A.10)$$

with $f$ analytic, $O(z^3)$ and $f(0, 0, w) \equiv 0 \equiv D_{uvw}f(0, 0, w)$. Indeed, set $x_1 = x + u + w, x_2 = x + v + w$ and $x_3 = x - w$, then

$$F = \sin x (\cos(u + v + w) - \cos(u + w) - \cos(v + w) + \cos w) + \\
\cos x (\sin(u + v + w) - \sin(u + v) - \sin(v + w) + \sin w),$$

which, upon scaling, is of the form (A.10). Writing $(u, v) \equiv y, f(u, v, w) = (y, A(z)y)$ with $A = O(z)$. Since $uv$ is nondegenerate, there exist an analytic diffeomorphism $g$ close to identity such that $F \circ g = uv$.

Hence

$$g(p) = \int \delta(uv - p) \tilde{\chi}(u, v, w) \, du \, dv \, dw,$$

with $\tilde{\chi} \in C^\infty_0(B_c(0))$. Let $\phi = \int \tilde{\chi} \, dw$, then,

$$g(p) = -\int_0^\infty du \log u \partial_u(\phi(u, p/u) + \phi(-u, -p/u))$$

$$= -\log |p| \int_0^\infty du \partial_u(\psi(pu, 1/u) - \int_0^\infty du \log u \partial_u(\psi(pu, 1/u)),$$

with $\psi(u, v) = \phi(u, v) + \phi(-u, -v)$. Thus $g(p) = a(p) \log |p| + b(p)$ with $a$ and $b$ smooth. The claim follows.

Case 2° We may suppose $x = 0$, the other points being similar. We have:

$$F = s_1 c_2 c_3 + c_1 s_2 c_3 + c_1 c_2 s_3 - s_1 s_2 s_3 - s_1 - s_2 - s_3$$
where \( s_i = \sin x_i, \ c_i = \cos x_i \). The diffeomorphism near the origin \( y_i = \sin x_i \) leads to
\[
F = (y_1 + y_2)(y_2 + y_3)(y_1 + y_3) + \mathcal{O}(\epsilon^5),
\]
and letting \( z_1 = y_1 + y_2, \ z_2 = y_2 + y_3, \ z_3 = y_1 + y_3, \) we get:
\[
F = z_1 z_2 z_3 + f(\zeta),
\]
with \( f \) analytic in \( B_\epsilon(0), \ f = \mathcal{O}(\zeta^5) \), and symmetric under permutations of coordinates. We want to bound
\[
\delta(p) = \int \delta(F(\zeta) - p)\tilde{\psi}(\zeta)\,d\zeta,
\]
for \( \psi \in C_0^\infty(B_\epsilon(0)) \) and \( \epsilon \) small enough. By symmetry we may insert into (A.11)
\[
6\chi(|z_1| \leq |z_2| \leq |z_3|).
\]
Expand
\[
f(\zeta) = f_0(z_2, z_3) + f_1(z_2, z_3)z_1 + f_2(\zeta)z_1^2.
\]
Since \( f_1 = \mathcal{O}(z_2, z_3)^4 \), by a diffeomorphism \( \phi \) close to identity we have \((z_2 z_3 + f_1) \circ \phi = z_2 z_3 \) i.e. may assume \( f_1 = 0 \) and bound \( \chi \) from above by \( \chi(|z_1| \leq 2|z_2| \leq 3|z_3|) \):
\[
|h(p)| \leq \int \delta(z_{12} z_3 + f_0(z_2, z_3) + f_2(\zeta)z_1^2 - p)\tilde{\psi}(\zeta)\cdot \chi(|z_1| \leq 2|z_2| \leq 3|z_3|)\,d\zeta.
\]
with \( \tilde{\psi} \in C_0^\infty(B_\epsilon(0)) \). On the support of \( \chi \) and \( \tilde{\psi} \cdot |f_2| \leq C|z_3|^3 \leq C\epsilon^2|z_3| \) and thus \( |z_{23} + f_2 z_1| \geq (1 - \mathcal{O}(\epsilon^2))|z_2 z_3| \), since \( |z_1| \leq 2|z_2| \). So, given \( z_2, z_3 \) and \( p \), the argument of the \( \delta \) function in (A.12) either is nonzero for all \( z_1 \in B_\epsilon(0) \) or vanishes at \( z_1(z_2, z_3, p) \) satisfying
\[
\frac{1}{2} \left| \frac{p - f_0}{z_2 z_3} \right| \leq |z_1| \leq 2 \left| \frac{p - f_0}{z_2 z_3} \right|.
\]
Moreover \( |z_{23} + \partial_1(f_2 z_1^2)| \geq (1 - \mathcal{O}(\epsilon^2))|z_2 z_3| \). Thus
\[
|h(p)| \leq C \int |z_2 z_3|^{-1} \chi(|z_2| \leq 2|z_3|) \chi \left( \left| \frac{p - f_0}{z_2 z_3} \right| < C\epsilon \right) \tilde{\psi}(z_1, z_2, z_3)\,dz_2 dz_3.
\]
Since \( f_0(0, z_3) = f_0(0, z_3) + z_2 f_3(z_2, z_3) \) where \( |f_3| \leq C z_3^4 \leq C\epsilon^3|z_3| \), the second factor \( \chi \) in (A.13) is bounded by
\[
\chi \left( \left| \frac{p - f_0(0, z_3)}{z_2 z_3} \right| < C\epsilon \right).
\]
Now, \( f_0(0, z_3) \) is analytic, so
\[
f_0(0, z_3) = az_3^n(1 + \mathcal{O}(z_3)),
\]
for some \( a \neq 0, n < \infty \) (actually \( n = 5 \)).

Insert this into (A.13), write \( \chi = \chi_p(z_2) + (\chi - \chi_p(z_2)) \), with
\[
\chi_p(z_2) = \chi \left( |z_2| > C|p|^{\frac{n}{n-1}} \right).
\]

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Then \( h = h_1 + h_2 \) with
\[
|h_1(p)| \leq \int \frac{dz_2}{|z_2|} \chi(z_2) \int \frac{dz_3}{|z_3|} \chi(|z_3| > \frac{1}{2}|z_2|) \chi_p(\hat{z} \in B_c(0)) \\
\leq C(\log p)^2.
\]

For \(|z_2| \leq C|p|^{\frac{n-1}{n}}\), we note that, since, in the support of \( \chi \left( \left| \frac{p-f_0(0,z_3)}{z_2 z_3} \right| < C\epsilon \right) \)
\[
|p - a z_3^n| \leq C\epsilon |z_2 z_3|,
\]
we may write \( z_3 = (p/a)^{1/n} + x \), with
\[
\left| \frac{p}{a} \right|^{\frac{n-2}{n}} |x| \leq C\epsilon |z_2| |p|^{1/n},
\]
where \( a \) is absorbed into \( C \) in the RHS, or
\[
|x| \leq C\epsilon |p|^{\frac{n-2}{n}} |z_2|.
\]

So, doing the \( z_3 \) integral, we get:
\[
|h_2(p)| \leq C\epsilon \int \frac{dz_2}{|z_2|} \chi \left( |z_2| < C |p|^{\frac{n-1}{n}} \right) |p|^{-1/n} |z_2| |p|^{-\frac{n-2}{n}}
\leq C\epsilon.
\]

We conclude that \(|g(p)| \leq C(\log p)^2 \) i.e. the claim holds for \( g \). The claim for \( g' \) is similar: \( \partial_p \) brings an extra \(|z_2 z_3|^{-1}\) and
\[
\int \frac{dz_2 dz_3}{(z_2 z_3)^2} \chi \left( |z_2 z_3| > \epsilon(p) \right) \leq \frac{C}{\epsilon} \left| \log p \right|.
\]

\[\square\]

**Remark** \( I(p) \) is not smooth for \( p \neq 0 \). By some algebra one can show
\[
\nabla_k P = 0, \quad P = 0 \iff k_1 = k_2 = k_3, \quad p = 2k_1,
\]
and permutations. Thus zeros of \( P \) and \( \nabla_k P \) occur for nonzero \( p \) too.

### B Nonlinear estimates

Here we prove the estimates on the nonlinear terms that were made in Section 10. They are based mostly on an analysis of convolutions of functions in \( E \) or \( S \), whose results were stated in Lemma 9.1. We start with the proof of this Lemma.

**Proof of Lemma 9.1** Recall first that \( j = j_+ \sigma_+ + j_+ \sigma_+ \) and \( T \) similarly. Let \( T \ast j = j' \). Then
\[
\begin{align*}
  j'_+ &= T_+ \ast j_+ + T_- \ast j_- \\
  j'_- &= T_+ \ast j_- + T_- \ast j_+.
\end{align*}
\]
Consider eg. $T_+ * j_+$ and drop the $+$.

Recall that $j$ is defined through the 4-tuple $j$ in (8.13). We need to define the 4-tuple $j'$.

We set

$$j'_0 = \frac{1}{2} (T * j)(0),$$

(B.1)

where the factor $\frac{1}{2}$ follows from our conventions (8.13) and (6.5). Let, for $p \neq 0$,

$$j'_1 = T * j_1,$$

(B.2)

$$j'_2 = T * j_2$$

(B.3)

$$\frac{j'_3}{(N \delta)^{3/2}} = T * \frac{j_3}{(N \delta)^{3/2}} + T * \frac{j_1}{N \delta} - \frac{1}{N \delta} (T * j_1).$$

(B.4)

Let us estimate these in turn, using two simple observations:

$$\| \int f(p - p')g(p') dp' \|_\alpha \leq \| f \|_\alpha \| g \|_{L^1}$$

(B.5)

where $\| - \|_{L^1}$ is the $L^1$ norm, and

$$\| \int f(p - p')d(p')^{-1} dp' \|_\alpha \leq \| f \|_\alpha$$

(B.6)

which holds because the Hilbert transform of a Hölder continuous function is Hölder continuous (for $\alpha < 1$, see [30], Theorem 106), and this is true also when $p^{-1}$ is replaced by $d(p)^{-1}$, and when we have discrete sums as here.

Then, using $\| T \|_{L^1} \leq C \| T \|_E$, (see (11.27)), and (B.5), we get:

$$\| j'_1 \|_\alpha \leq C \| T \|_E \| j_1 \|_\alpha$$

(B.7)

$$\| j'_2 \|_\alpha \leq C \| T \|_E \| j_2 \|_\alpha.$$  

(B.8)

For $j'_3$, we use the identity

$$d(p) = d(p') + d(p - p') + d(p')d(p - p'),$$

(B.9)

to write:

$$\left( T * \frac{j_1}{N \delta} - \frac{1}{N \delta} (T * j_1) \right)(p) = \frac{1}{N \delta} \left( Nd(T * \frac{j_1}{N \delta}) - (T * j_1) \right)(p)$$

$$= \frac{1}{N \delta} \left( \int j_1(p - p')(d(p - p')^{-1} d(p')T(p') dp' + \int j_1(p - p')d(p')T(p') dp' \right)$$

(B.10)

Hence, using (11.28), (B.10) is bounded by

$$C \| T \|_E \| j \|_{S(N|p|)^{-1}} \int (\frac{1}{N|p'|} + N^{-1+\alpha/2} + \frac{1}{(N|p'|)^{3/2}})(1 + |p - p'|)^{-1} dp'$$

(B.11)

The integral is bounded by

$$C \log |Np| (\frac{1}{N|p|} + N^{-1+\alpha/2}).$$

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Combining with (B.11) we arrive at the bound
\[ |(B.10)| \leq C\|T\|_E \|j\|_S \frac{1}{|Nd(p)|^{3/2}} \] (B.12)

Similar calculations give the bound
\[ |T \ast \frac{1}{|Nd|^{3/2}}| \leq \|T\|_E \frac{1}{|Nd|^{3/2}} \]

Combining with (B.12) we get
\[ \|j'_3(p)\|_\infty \leq C\|T\|_E \|j\|_S \] (B.13)

Finally (B.1) is bounded by
\[ \frac{1}{2} \int |T(p)j(p)| dp \leq C\|T\|_E \|j\|_S \] (B.14)

using (11.29), e.g.
\[ \int (N^2|p|^3)^{-1} dp \leq C \]

Estimates (B.7), (B.8), (B.13) and (B.14) give the claim (9.5).

Let us next consider part (c). We set
\[ j'_0 = \frac{1}{2} (j \ast k)(0) \] (B.15)

and, for \( p \neq 0 \),
\[ j'_1 = Nd \left( j_1 \ast \frac{k_1}{Nd} + H_1 \right) + \frac{1}{N} (j_0 k_1 + k_0 j_1) \] (B.16)
\[ j'_2 = j \ast k_2 + k \ast j_2 - N^{-1+\alpha/2} j_2 \ast k_2 \] (B.17)
\[ \frac{j'_3}{(Nd)^{3/2}} = \frac{j_3}{(Nd)^{3/2}} \ast \frac{k_3}{(Nd)^{3/2}} + H_2 + \frac{1}{N} (j_0 k_3 + k_0 j_3), \] (B.18)

where we write \( \frac{j_1}{(Nd)^{3/2}} \ast \frac{k_1}{(Nd)^{3/2}} + \frac{j_2}{(Nd)^{3/2}} \ast \frac{k_2}{(Nd)^{3/2}} = H_1 + H_2 \), and the \( H_i \)'s are defined after (B.23) below.

To proceed, we need the following bounds: using (B.5), (B.6), we get that,
\[ h_1(p) \equiv \int f(p-p') g(p') d(p')^{-1} dp', \] (B.19)

satisfies
\[ \|h_1\|_\alpha \leq C\|f\|_\alpha \|g\|_\alpha. \] (B.20)

This holds because we can write in (B.19) \( g(p') = g(p') - g(0) + g(0) \); since \( g \) is in \( C^\alpha \), \( |g(p') - g(0)||d(p')^{-1}| \leq \|g\|_\alpha |p'|^{-1+\alpha} \), i.e. \( (g(p') - g(0))d(p')^{-1} \) is integrable; so (B.20) follows from (B.6) applied to the \( g(0) \) term above and (B.5) to the \( g(p') - g(0) \) term.

Using (B.9), we then get that
\[ h_2(p) \equiv d(p) \int f(p-p') g(p') d(p')^{-1} g(p') d(p')^{-1} dp' \] (B.21)
also satisfies
\[ \|h_2\|_\alpha \leq C\|f\|_\alpha\|g\|_\alpha. \] (B.22)

Now, consider (B.16). The bound (B.22) applied to the first term implies that its $C^\alpha$ norm is $O(N^{-1})\|j_1\|_\alpha\|k_1\|_\alpha$.

Now, use (B.9) to write
\[ Nd\left(\frac{j_1}{Nd} * \frac{k_3}{(Nd)^{3/2}}\right) = j_1 * \frac{k_3}{(Nd)^{3/2}} + \frac{j_1}{Nd} * \frac{k_3}{(Nd)^{3/2}} + \frac{j_1}{N} * \frac{k_3}{(Nd)^{1/2}}. \] (B.23)

Let the sum of the first and third term in (B.23), plus the corresponding terms in $NdH_1$ and let $H_2 = (Nd)^{-1}(\frac{j_1}{Nd} * \frac{k_3}{(Nd)^{3/2}})$ plus the corresponding terms in $\frac{j_1}{Nd} * \frac{k_3}{(Nd)^{3/2}}$.

Since $\|\frac{k_3}{(Nd)^{3/2}}\|_{L^1} \leq CN^{-1}\|k_3\|_\infty$, we get from (B.5) that
\[ \|j_1 * \frac{k_3}{(Nd)^{3/2}}\|_\alpha \leq O(N^{-1})\|j_1\|_\alpha\|k_3\|_\infty, \]
which controls the contribution of the first term in (B.23) to $NdH_1$. This bound holds also for the last term in (B.23), of course (which is even $O(N^{-3/2})\|j_1\|_\alpha\|k_3\|_\infty$). The terms $\nabla (j_0k_1 + k_0j_1)$ are trivially bounded, so we get:
\[ \|j'_1\|_\alpha \leq CN^{-1}\|j\|_S\|k\|_S. \] (B.24)

Using (B.5) and the fact that $\|\frac{\partial}{\partial \tau'}\|_{L^1} \leq N^{-1}\log N\|j_1\|_\infty$ for all terms in (B.17), we get
\[ \|j'_2\|_\alpha \leq CN^{-1+\alpha/2}. \] (B.25)

Consider now (B.18). Going back to the discrete sum (see (5.1)), it is easy to see that the first term is bounded by $CN^{-1}|Np|^{-3/2}\|j_3\|_\infty\|k_3\|_\infty$, i.e. that the contribution of this term to $|j'_3(p)|$ is less than $CN^{-1}\|j\|_S\|k\|_S$. Consider now $(Nd)^{-1}(\frac{\partial}{\partial \tau'} * \frac{k_3}{(Nd)^{3/2}})$, contributing to $H_2$.

Going back to the discrete sum, it easy to bound it by $CN^{-1}|Np|^{-3/2}\log |Np|$. Finally, the bound on $\nabla (j_0k_3 + k_0j_3)$ is trivial and we get:
\[ |j'_3(p)| \leq CN^{-1}\log(N|p|)\|j\|_S\|k\|_S. \] (B.26)

Finally, using repeatedly (11.29), we get
\[ |j'_0| \leq CN^{-1}\|j\|_S\|k\|_S. \] (B.27)

Combining (B.24), (B.25), (B.26), (B.27), proves part c) of the Lemma, since the bound on $\|j'_0\|_S$ follows from the previous ones.

Let us finally turn to (9.6). Note that, using (B.9),
\[ d(T * A) = dT * A + T * dA + dT * dA \]
Since $dT \in S$ if $T \in E$ the claim, for $p \neq 0$, follows from (a) and (c). For $p = 0$, we simply apply (11.29) to all the terms in $(T * A)(0)$.

Using this Lemma, it is rather easy to give the proof of the main estimate of section 9.

**Proof of Proposition 9.4.** (a) Apply (a) of Corollary A.1 to (9.41) and (9.42) to get
\[ m(p, k) = \int g(p, k, p) \prod_{i=1}^{3} F_i(2p_i)\delta(2p - \sum_{i=1}^{3} 2p_i)dp. \] (B.28)
where \( g \) is smooth in \( p \) and \( C^\alpha \) in \( p, k \). \( m \) may now be defined as an element of \( E \) as in Lemma 9.1.b, the smooth function not affecting the bounds.

(b) If \( m = u \) or if \( m = v \) and \( i \neq 3 \) (in which case \( i = 1, 2 \), and, by symmetry, we can choose \( i = 1 \)) we have

\[
m(p, k) = \prod_{i=2}^{3} F_1(2p_i) \left( \int G f_1(p_1, k_1 - p_1) \mu_{p,k}(dk) \right) \delta(2p - \sum_{i=1}^{3} 2p_i) dp
\]

with \( G \) smooth, and \( f_1 \) in \( S \). Using the representation (8.11) for \( f_1 \) we write \( G f_1(p_1, k_1 - p_1) \) as a sum of terms, with singularities in \( p_1 \); by Corollary A.1.b, we get that integrating each of those terms with \( \mu_{p,k}(dk) \) gives rise to a function that is \( C^\alpha \) in \( p, k \). Thus,

\[
\int G f_1(p_1, k_1 - p_1) \mu_{p,k}(dk) = f(p_1, k; p, p_2, p_3)
\]

where \( f \) is in \( S \) in the variables \( p_1, k \) depending smoothly on \( p_2, p_3 \) and \( C^\alpha \) in \( p \). The convolutions with the \( F_j \)'s can then be estimated as in Lemma 9.1.

If \( m = v \) and \( i = 3 \) we need to study

\[
\int \prod_{i=1}^{2} F_i(2p_i) f(p_3, p - k - p_3) \left( \int G \mu_{p,k}(dk) \right) \delta(2p - \sum_{i=1}^{3} 2p_i) dp,
\]

where \( f \) is in \( S \). Shifting \( p_3 \) by \( p \), this becomes

\[
\int g(p, k, p) \prod_{i=1}^{2} F_i(2p_i) \delta(\sum_{i=1}^{3} 2p_i) f_3(p + p_3, -k - p_3) dp
\]

where \( g \), defined in (A.1) is, by Corollary A.1.a, \( C^\alpha \) in \( p, k \) and smooth in \( p \). Performing the \( p_i \) integrals for \( i = 1, 2 \), we get

\[
\int F(p, k, p_3) f(p + \frac{1}{2} p_3, k - \frac{1}{2} p_3) dp_3
\]

where \( F \) is \( C^\alpha \) in the first two arguments and belongs to \( E \) as a function of the third. We may proceed now as in Lemma 9.1.(a) to define and estimate the quadruple \( g \) in \( S \) corresponding to (B.30). E.g. the component of index 1, see (B.2), is given by

\[
g_1(p, k) = \int F(p, k, p_3) f_1(p + \frac{1}{2} p_3, k - \frac{1}{2} p_3) dp_3
\]

where \( f_1 \) is \( C^\alpha \) in both arguments. Again, since \( F \) is integrable in the third argument and \( C^\alpha \) in the others the integral is \( C^\alpha \) in \( p, k \). The other components of \( g \) can be bounded as in Lemma 9.1.a.

(c) We have to specify again the quadruple \( m \in S \) corresponding to \( m \). We define \( m_i = 0 \) and \( m_3(p, k) \) to be the integral (9.41) or (9.42) corresponding to \( m \). We use Corollary A.1.c to do the \( k_i \) integrals and estimate the \( p_i \) integrals by brute force. Since only the singularity at \( p_i = 0 \) (or, by periodicity at \( \pi \)) matters, we need the easy bounds (remember that the variable \( p \) is discrete!)

\[
|Nd(p)|^{-3/2} \leq CN^{-1/2} \log N |Nd(p)|^{-3/2}
\]

\[
N^{-1+\alpha/2} \cdot |Nd(p)|^{-1} \leq C \log N N^{-2+\alpha/2} \leq CN^{-1+\alpha/2} \log N |Nd(p)|^{-3/2}
\]

\[
N^{-1+\alpha/2} \cdot N^{-1+\alpha/2} \leq CN^{-1+\alpha} |Nd(p)|^{-3/2}
\]
together with $|Nd(p)|^{-3/2} \leq |Nd(p)|^{-1}$ (since $N|p| \geq \pi$), which allows to use the first inequality here in order to bound the convolutions with the last term in (8.11), to conclude

$$\|m(p, k)\|_\infty \leq C \ N^{-\frac{1}{3}+\alpha} \|f_k\|_E \prod_{l \neq k} \|f_l\|_S$$

which is the claim (9.46) since we defined $m = (0, 0, 0, m)$. \hfill \Box

Finally, we prove the estimates on the function $\theta$ defined in (7.9).

**Proof of Proposition 9.7**

By the (3 ↔ 4) symmetry in (5.13), we get (leaving out the factor $\frac{9}{4}(2\pi)^3 \lambda^2$),

$$\theta(p) = i \sum_s \int \prod_{i=1}^{3} \omega_s(p_i - k_i) w_s(\omega(k_i))^{-1} \left[ \rho\left(\frac{p_i}{2}, k_i\right) - \rho\left(p_i, \frac{p_i}{2} - k_i\right) \right] \nu_{spk}(dp \, dk)$$

(B.32)

and

$$\nu_{spk} = \left( \sum_i s_i \omega(k_i) + i\epsilon \right)^{-1} \delta(2p_4) \delta(p - \sum_i p_i) \delta(p - \sum_i k_i) dp \, dk \nu'_{spk}$$

Then the $[\ ]$ in (B.32) equals

$$2 \left( \frac{1}{\omega(k_3)} - \frac{1}{\omega(k_4)} \right) + (e^{ip} - 1) \ r(p, k)$$

(B.33)

with $r$ smooth. The integral in (B.32) has singularities when

$$\sum_s s_i \omega(k_i) = 0.$$  (B.34)

Recall that (B.34) forces

$$\sum_s s_i = 0.$$  (B.35)

Consider the $s$ such that (B.35) holds in (B.32), and, replace $[\ ]$ in (B.32) by the first term of (B.33). We define

$$\theta_1(p) \equiv 2i \sum_{s_i=0} \int \prod_{i=1}^{3} W_{s_i} \left( \frac{s_3 s_4}{\omega(k_3)} - \frac{s_3 \omega(k_3) s_4}{\omega(k_4)^2} \right) \cdot \nu'_{spk}(dp \, dk).$$

(B.36)

By symmetry, we may replace $s_3$ by $\frac{1}{3} \sum_{i=1}^{3} s_i$ and, by (B.35) also by $-\frac{1}{3} s_4$. Again, by symmetry, $s_3 \omega(k_3)$ may be replaced by $\frac{1}{3} \sum_{i=1}^{3} s_i \omega(k_i)$. So, the parenthesis equals $-\frac{s_4}{3 \omega(k_4)^2} \sum_{i=1}^{4} s_i \omega(k_i)$, and the sum cancels the factor $(\sum_i s_i \omega(k_i) + i\epsilon)^{-1}$ in $\nu'$. Hence, (B.36) equals

$$\theta_1(p) = -\frac{2i}{3} \sum_{s_i=0} \int \prod_{i=1}^{3} W_{s_i} \frac{s_4}{\omega(k_4)^2} \delta(2p_4) \delta(p - \sum_i p_i) \delta(p - \sum_i k_i) dp \, dk$$

(B.37)
We decompose \( \theta \) as

\[
\theta(p) = \theta_1(p) + \theta_2(p) + p\theta_3(p)
\]  

(B.38)

where \( \theta_2 \) has the terms of (B.32) with \( \sum s_i \neq 0 \) and the first term of (B.33) inserted, while \( \theta_3 \) corresponds to the insertion of the second term of (B.33).

Consider first \( \theta_1 \) given by (B.37). Remember that

\[
W_s(p, k - p) = Q(p, k - p) + is\omega(k)^{-1}J(p, k - p).
\]

The terms with an odd number of \( Q \) factors vanish by the \( s \to -s \) symmetry. Consider then the term linear in \( J \). We insert \( Q = Q_0 + r \) and start with the term with no \( r \). After shifting the \( k_i \) variables by \( 2p_i \) (and using \( 2p_4 = 0 \)), we obtain a sum of terms of the form

\[
\int T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)J(p_3, k_3 + p_3) \prod_{i=1}^{2} \left( \omega(k_i + 2p_i) + \omega(k_i) \right)^{-2-n_i} \cdot \\
\omega(k_1)^{-2}\omega(k_3 + 2p_3)^{-3}\delta(2p_4)\delta \left( p - 2 \sum p_i \right) \delta \left( \sum k_i \right) dk dp
\]

(B.39)

Now, use the fact that \( \omega(k_i + 2p_i) - \omega(k_i) = (e^{2ip} - 1)O(1) \), which implies

\[
\left( \omega(k_i + 2p_i) + \omega(k_i) \right)^{-2-n_i} = \left( 2\omega(k_i) \right)^{-2-n_i} + n_i(e^{2ip} - 1)O(1).
\]

(B.40)

For the second term on the RHS of (B.40), let us choose \( i = 1 \), which we can do by symmetry, and insert it in (B.39), to obtain, after integrating over \( k_1, k_2, k_4 \):

\[
I_1(p) = n_1 \int (e^{2ip_1} - 1)T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)J(p_3, k_3 + p_3) \cdot \\
f(p_3, k_3)\delta(2p_4)\delta \left( p - 2 \sum p_i \right) dpdk_3,
\]

with \( f \) smooth. For the first term on the RHS of (B.40), we obtain:

\[
\tilde{I}_1(p) = \int T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)J(p_3, k_3 + p_3) \cdot \\
\tilde{f}(p, k_3)\delta(2p_4)\delta \left( p - 2 \sum p_i \right) dpdk_3,
\]

(B.42)

with \( \tilde{f} \) smooth and even in \( k_3 \). Doing the \( k_3 \)-integral, we get, for (B.41),

\[
\int J(p_3, k_3 + p_3) f(p, k_3)dk_3 = g(2p_3, 2p)
\]

(B.43)

where we used the \( \pi \)-periodicity of the result. \( g \) is in \( S \) as a function of \( p_3 \), depending smoothly on \( p \), with \( \|g\|_S \leq C\|J\|_S \). For (B.42), we get, since \( J \) is odd in \( k_3 \) and \( \tilde{f} \) even, that the integral vanishes if \( J(p_3, k_3 + p_3) \) is replaced by \( J(p_3, k_3) \), and thus, since \( \tilde{f} \) is smooth, the integral can be written as:

\[
\int J(p_3, k_3 + p_3) \tilde{f}(p, k_3)dk_3 = \int J(p_3, k_3) \tilde{f}(p, k_3 - p_3)dk_3 = (e^{2ip_3} - 1)\tilde{g}(2p_3, 2p),
\]

(B.44)
using again the \( \pi \)-periodicity of the result. \( \tilde{g} \) is in \( S \) as a function of \( p_3 \), depending smoothly on \( \underline{p} \), with \( \| \tilde{g} \|_S \leq C\|J\|_S \). We write, using the constraints \( \delta(2p_4), \delta \left( p - 2 \sum p_i \right) \),
\[
e^{2ip_3} - 1 = (e^{ip} - 1 + 1)(e^{-2ip_1} - 1 + 1)(e^{-2ip_2} - 1 + 1) - 1. \tag{B.45}
\]
Expanding the product we see that the integral (B.42) equals the sum of terms of the form \( I_1 \) and of the form:
\[
I_2 = d(p) \int T * A^{n_1}(2p_1)T * A^{n_2}(2p_2)\tilde{g}(2p_3, 2p)\delta(2p_4)\delta \left( p - 2 \sum p_i \right) \, dp \, dk_3. \tag{B.46}
\]
The integral in (B.46) is a convolution of two functions in \( E \) with one in \( S \), hence, by Lemma 9.1, it is in \( S \). The prefactor \( d(p) \) cancels the \( d^{-1} \), so that this contribution satisfies
\[
\|d^{-1}I_2\|_S \leq (C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S, \tag{B.47}
\]
and also \( I_2(0) = 0 \).

Going back to \( I_1 \), see (B.41), (B.43), we obtain:
\[
I_1(p) = n_1 \int f(p_1)F(p_2)g(p_3, p)\delta(2p_4)\delta \left( p - \sum p_i \right) \, dp \tag{B.48}
\]
with \( f \in S \) and \( F \in E \). So, we have a convolution of two elements of \( S \) and one of \( E \), i.e. the convolution of two elements of \( S \). Going back to the definition (8.11), we see that we can write
\[
I_1 = I_1' + I_1'' + I_1''' \tag{B.49}
\]
corresponding to the \( j_i' \), \( i = 1, 2, 3 \) terms in the convolution of two elements of \( S \) that are bounded in part c) of Lemma 9.1. From that Lemma, we get:
\[
|I_1'(p)| \leq \frac{1}{N^2|d(p)|}(C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S, \tag{B.50}
\]
If we identify \( d(p)^{-1}I_1' \) with an element of \( S \) of the form \((0,0,0,*))\), we get from (B.50) and \((N|d(p)|)^{-1/2} \leq C\),
\[
\|d^{-1}I_1'\|_S \leq (C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S. \tag{B.51}
\]
Next, we get, also from Lemma 9.1.c, together with the definition (8.11):
\[
\|I_1''\|_a \leq N^{-2+\alpha}(C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S, \tag{B.52}
\]
Now, identify \( d(p)^{-1}I_1'' \) with an element of \( S \) of the form \((0,*,0,0))\), we get, writing \( d(p)^{-1}I_1'' = (Nd(p))^{-1}NI_1'' \), that
\[
\|d^{-1}I_1''\|_S \leq N^{-1+\alpha}(C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S. \tag{B.53}
\]
For \( I_1''' \), we use (9.10), and identify \( d(p)^{-1}I_1''' \) with an element of \( S \) of the form \((0,0,0,*))\). Since \((N|d(p)|)^{-1}\log(N|p|) \leq C\), we get
\[
\|d^{-1}I_1'''\|_S \leq (C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S. \tag{B.54}
\]
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Combining these estimates, we get:
\[ \|d^{-1}I_1\|_S \leq (C\delta_{n_1+n_2,0}\|t\|_S + (C\|A\|_E)^{n_1+n_2})\|J\|_S. \]  
(B.55)

We also get, by (9.7):
\[ |I_1(0)| \leq CN^{-1}. \]  
(B.56)

The terms with one \( J \) and one \( r \) are sums of terms of the form:
\[
\int T \ast A^{n_1}(2p_1)r(p_2,k_2-p_2)J(p_3,k_3-p_3)\left(\omega(2p_1-k_1) + \omega(k_1)\right)^{-2-n_1} \omega(k_3)^{-1}\omega(k_4)^{-2}\delta(2p_4)\delta\left(p-2\sum p_i\right)\delta\left(p-\sum k_i\right)dpdk
\]

Doing the \( k_1 \) and \( k_4 \) integrals, and shifting \( k_2, k_3 \), this equals
\[
I_3 = \int T \ast A^{n_1}(2p_1)r(p_2,k_2)J(p_3,k_3)f(p,k_2,k_3,p)\delta(2p_4)\delta\left(p-\sum p_i\right)dpdkdk
\]
with \( f \) smooth. We have \( T \ast A^{n_1} \in E, r, J \in S \). Proceeding as with \( I_1 \), we get
\[ \|d^{-1}I_3\|_S \leq (C\|A\|_E)^{n_1}\|r\|_S \|J\|_S. \]  
(B.58)

We also have, by (9.7):
\[ |I_3(0)| \leq CN^{-1}. \]  
(B.59)

In a similar way, we may analyse \( \theta_2 \) i.e. (B.32) with \( \sum s_i \neq 0 \) and the first term in (B.33) inserted. We note that the terms that are odd in \( Q \) vanish since, for those terms, because of the \( s \rightarrow -s \) symmetry, the measure is proportional to \( \delta\left(\sum s_i\omega(k_i)\right) \), which vanishes for \( \sum s_i \neq 0 \).

Starting again with the term linear in \( J \) and with \( r = 0 \), it is given by
\[
\int T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)J(p_3,k_3)h(p_3+k_3,k_3)\delta(2p_4)\delta\left(p-2\sum p_i\right)dpdk
\]
with \( h(p,k) = h(-p,-k) \) smooth. By oddness of \( J \) in \( k \), we may replace \( h \) by \( h(k_3+p_3,k_3) - h(k_3-p_3,k_3) \) i.e., near \( p_3 = 0, h \) is \( O(p_3) \). Similarly, using \( J(p+\pi,k+\pi) = J(p,k) \) and the \( 2\pi \)-periodicity of \( h \),
\[
\int J(p_3,k_3)h(p_3+k_3)dk_3 = \frac{1}{2}\int J(p_3-\pi,k_3)\left[h(k_3+p_3-\pi,k_3+\pi) - h(k_3-(p_3-\pi),k_3+\pi)\right]
\]
i.e. (B.60) may be written as
\[
\int T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)(e^{2ip_3}-1)J(p_3,k_3)h(p_3,k_3)\delta(2p_4)\delta\left(p-2\sum p_i\right)dpdk
\]
(B.61)
Writing \( e^{2ip_3} - 1 \) as in (B.45), and expanding the product, we see that the integral (B.61) equals the sum of terms of the form \( I_1 \) and of the form \( I_2 \), i.e.
\[
\tilde{I}_2 = d(p)\int T \ast A^{n_1}(2p_1)T \ast A^{n_2}(2p_2)J(p_3,k_3)h(p_3,k_3)\delta(2p_4)\delta\left(p-2\sum p_i\right)dpdk
\]
(B.62)
with \( \tilde{h} \) smooth. The integral in (B.62) is in \( S \) and the prefactor \( d(p) \) cancels the \( d^{-1} \), so that \( I_2 \) has the same bound as in (B.47). Finally, the term with one \( J \) and one \( r \) is again of the form (B.57).

The remaining terms in \( \theta_1 \) and \( \theta_2 \) are of type \( J^3 \) and \( Jr^2 \). These are bounded by brute force by

\[
\frac{\log N}{N^2} \| J \|(\| r \|^2 + \| J \|^2),
\]

and considered as elements of \( S \) of the form \((0,0,0,\ast)\). Since \( |d|^{-1} \leq CN \) we obtain by combining eqs. (B.55), (B.58), (B.62) and (B.63)

\[
\|d^{-1}(\theta_1 + \theta_2)\|_S \leq C(\| t \|_S + \| A \|_E + \| r \|_S + \| J \|_S^2)\| J \|_S.
\]

We still need to estimate \( p\theta_3(p) \) in (B.38), i.e. the contribution to (B.32) of the second term in (B.33), which we can write as:

\[
\omega^{-1}(k_3 - p) - \omega^{-1}(k_3) - (3 \leftrightarrow 4) = pf(k_3, k_4) + O(p^2),
\]

with \( f \) odd. Consider the terms where \( W_{s_i} = Q_0, \ \forall i \).

We get a sum of terms of the form

\[
\int \prod_{i=1}^{3} F_{n_i}(2p_i)\omega^{-2-n_i}(p_i, k_i - p_i) s_3 s_4 \omega(k_3)\omega(k_4)^{-1}.
\]

\[
[\omega(k_3 - p)^{-1} - \omega(k_3)^{-1} + \omega(k_4 - p)^{-1} - \omega(k_4)^{-1}] \nu'(dp\,dk).
\]

Write:

\[
\omega^{-2-n_i}(p_i, k_i - p_i) = \omega(k_i)^{-2-n_i} + (e^{2ip_i} - 1)n_iO(1).
\]

Therefore (B.66) gives rise to two contributions: the one coming from \( \omega(k_i)^{-2-n_i} \); after inserting (B.65) in the \([\cdot]\) in (B.66), and writing \( p = -id(p) + O(p^2) \), this contribution can be written as: 

\[
\frac{1}{2} d(p)(F_{n_1} * F_{n_2} * F_{n_3})(p)(I(p) - I(-p)) + O(p^2)(F_{n_1} * F_{n_2} * F_{n_3})(p),
\]

with \( F_n = T * A^n \),

\[
I(p) = \int \delta(\omega(k_1) + \omega(k_2) - \omega(k_3) - \omega(k_4))\delta\left(p - \sum_{i=1}^{4} k_i\right) \phi(k)dk,
\]

where \( \phi \) is smooth, and \( I \) is odd (since \( f(k_3, k_4) \) above, and hence \( \phi \), is odd). By Lemma 9.1, \( (F_{n_1} * F_{n_2} * F_{n_3})(p) \) is in \( E \), so \( d(p)(F_{n_1} * F_{n_2} * F_{n_3})(p) \) is in \( S \). By Lemma A.2., the first term in the RHS of (B.67) multiplied by \( d^{-1} \) is in \( S \) and so is the last one, since \( O(p^2)(F_{n_1} * F_{n_2} * F_{n_3})(p) \) is \( N^{-1} \) times a \( C^\alpha \) function. The second contribution, coming from \( (e^{2ip_i} - 1)n_iO(1) \), is of the form:

\[
d(p)n_i \int \prod_{i=1}^{3} F_{n_i}(2p_i)(e^{2ip_i} - 1)\psi(p,k,p)\nu'(dp\,dk),
\]

(69)
where $\psi$ is smooth. The integral in (B.69) is in $S$ with norm bounded by $(C\|A\|_E)\sum^{n_i}$. The other terms in $\theta_3$ are simpler to bound, and we get:

$$\|d^{-1}\theta_3\|_S \leq C(\|t\|_S + \|A\|_E).$$

(B.70)

Of course, $\theta_3(0) = 0$. (B.64) and (B.70) together with (B.46), (B.56), (B.59), yield the claims. □

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