GLOBAL DYNAMICS OF A HUANGLONGBING MODEL WITH A PERIODIC LATENT PERIOD

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Abstract. Huanglongbing (HLB) is a disease of citrus that caused by phloem-restricted bacteria of the Candidatus Liberibacter group. In this paper, we present a HLB transmission model to investigate the effects of temperature-dependent latent periods and seasonality on the spread of HLB. We first establish disease free dynamics in terms of a threshold value $R_p^0$, and then introduce the basic reproduction number $R_0$ and show the threshold dynamics of HLB with respect to $R_p$ and $R_0$. Numerical simulations are further provided to illustrate our analytic results.

1. Introduction. Huanglongbing (HLB) is one of the most serious diseases of citrus trees caused by a vector-transmitted pathogen. It affects all citrus types and causes rapid decline of trees. Mature trees if infected will become non-productive and young trees once infected will never come into fruit production [4]. HLB is associated with Candidatus Liberibacter asiaticus (CLas), Ca.L. americanus (CLam), and Ca.L. africanus (CLaf) [1]. The species, CLaf causing African HLB and CLas causing Asian HLB, can be transmitted respectively by two species of citrus psyllids: African citrus psyllid and Asian citrus psyllid [1]. When the amount of psyllid is abundant and environmental conditions are suitable, HLB can rapidly destroy existing groves and prevent commercial production of oranges and other citrus cultivars [4]. In short, HLB is difficult to manage in areas where it widely spreads.

Many mathematical models of HLB have been proposed to understand epidemiology of vector-transmitted plant pathogens. Jacobsen, Stupiansky and Pilyugin [9] used a model to understand the spread of citrus greening disease within a single grove of citrus trees, which it specially allowed for a newly planted tree to become infected immediately due to the possibility that the soil might be a reservoir for the disease. Parry et al. [22] formulated a spatially explicit disease model where trees were either susceptible, exposed, infectious, detected or removed, and used data from Southern Garden’s citrus groves. Lee et al. [10] combined experiments and individual-based mathematical models, illustrated that host plants could become infectious in a shorter time than previously thought. Chiyaka et al. [3] developed a...
mathematical model to describe and analyze the transmission of HLB between its psyllid vector and citrus host.

Many plants and animals are sensitive to the seasonal variations. Different temperatures have been shown to have significant impacts on the development, survivorship, longevity, reproduction, and life table of the Asian citrus psyllid in [13]. There is a growing body of experimental data to demonstrate that temperature affected the development of psyllids. For instance, the study in [20] indicated that temperature condition affects the mortality of psyllid development on various hosts. Munir et al. [19] observed the seasonal variation of HLB-causing pathogen in citrus varieties in China, and concluded that the abundance of HLB-causing pathogens changes several times during different seasons and less pathogens were observed in warm summer seasons.

To explore the effect of temperature, more recent theoretical studies took the temperature into account in spread of HLB. A malaria model was adapted to HLB [25], by including temperature-dependent psyllid traits, flushing of trees, and economic costs. They analyzed how HLB spreads within individual trees, within a citrus grove, or from grove to grove and provided the model predictions for spread of HLB within a single grove with only roguing. A compartmental deterministic model with periodic parameters, including constant delay in the plants and a delay period on the nymphal stage of Asian citrus psyllid, was formulated in [26]. Wang, Gao, Luo, and Xie [29] proposed an $SI$ model about HLB transmission with the logistic growth in psyllid insect vectors and periodic coefficients to investigate the impact of seasonal activity of psyllid on the dynamics of HLB infection. Moreover, there exists a latency period between infection by a pathogen and the onset of infectivity tree [29]. Therefore, Gao, Yu, Meng and Zhang [6] derived a stage-structured HLB model with constant time delay.

Many population models with temperature-dependent delays and seasonality have been developed (see, e.g., [5, 21, 30, 14, 28, 27, 11]). Among these works, Freedman and Wu [5] studied the existence of a periodic solution of a single species population model with periodic delays in periodic environments. Omori and Adams [21] proposed a mathematical model with time-dependent delays to investigate the effect of seasonally-varying temperatures on koi herpes virus in common carp due to the delays depending on water temperature. Recently, Lou and Zhao [14] proposed a host-macroparasite model with time-varying developmental durations, and explored its global dynamics by considering a periodic semiflow on a suitable phase space. Later, Wang and Zhao [28] studied the threshold dynamics of a malaria disease model with a time-changing incubation period and seasonality. As introduced in [19, 20], the development of psyllid is highly sensitive to temperature and seasonal variation. Therefore, it is more reasonable to incorporate temperature-dependent latent periods and seasonality into HLB transmission models, and investigate their combined effects in HLB spread.

Motivated by these pioneer works, we extend the model framework of [6] to seasonally changing environments, and develop a periodic compartment disease model with temperature-dependent latent period in the psyllid population to fully consider the effects of seasonal variations. We first study the disease free dynamics, and then define the basic reproduction number $R_0$ and show that $R_0$ acts as a threshold parameter for the uniform persistence and global extinction of the disease. Mathematically, it is not straightforward to show the uniform persistence of the full system due to the model formulation and the time periodicity of coefficients,
we manage to perform a fundamental analysis to handle those issues with an additional assumption that the loss rate of psyllid population is greater than that of tree population, which would make sense biologically.

The rest of paper is organized as follows. We propose the model system and discuss its well-posedness in section 2. In section 3, we study the disease free dynamics and introduce the basic reproduction number $R_0$. The threshold dynamics for the model in terms of basic reproduction number $R_0$ is established in section 4. In section 5, we present numerical simulations. Finally, we have a brief discussion.

2. The model. This section aims to formulate a mathematical model of HLB transmission that incorporates a temperature-dependent delay and investigate its well-posedness. Motivated by [6], we first divide citrus tree population into susceptible, exposed, and infective classes at time $t$, denoted by $S_h(t)$, $E_h(t)$ and $I_h(t)$, respectively; and the psyllid vector population is divided into susceptible ($S_v(t)$), exposed ($E_v(t)$), and infective ($I_v(t)$) classes, respectively. The total number of citrus tree is $N_h(t) = S_h(t) + E_h(t) + I_h(t)$. The total number of citrus psyllid is $N_v(t) = S_v(t) + E_v(t) + I_v(t)$. Assume that the infection rates of susceptible trees and susceptible psyllid are described by

$$k(t)\sigma_1(t) \frac{S_h(t)I_v(t)}{N_h(t)} \quad \text{and} \quad k(t)\sigma_2(t) \frac{I_h(t)S_v(t)}{N_h(t)},$$

respectively, where $k(t)$ is the mean feeding rate of the psyllid on the trees, $\sigma_1(t)$ is the probability that a susceptible tree becomes infected from the contact with an infected psyllid, $\sigma_2(t)$ is the probability that a susceptible psyllid becomes infected from the contact with an infected tree, and they are time $t$-dependent. The length of latency period is $\tau$, thus $e^{-\mu_1\tau}$ denotes the survival rate from susceptible to exposed trees as the time goes from $t - \tau$ to $t$. We use $\alpha(t)N_v(t)(1 - \frac{N_v(t)}{mN_h(t)})$ to describe the logistic growth of susceptible psyllid, where $\alpha(t)$ denotes the intrinsic growth rate of psyllid, and $m$ is the maximum abundance of psyllid per tree. $\mu_1$ and $d_v(t)$ are natural death rates of citrus tree population and psyllid population, respectively; $K$ represents maximum citrus tree population size, which is defined in terms of the area of grove; $\delta_1$ is the infection-induced death rate of infected trees; $\delta_2$ represents the rate of removal of infected trees; $\theta(t)$ is the killing rate of psyllid with spraying insecticide over entire grove to killing psyllid; $r$ is a replanting rate proportional of recruitment to the citrus tree population, which is between the actual number of trees present $N_h$ and maximum population size $K$. Since the impact of varying temperature on tree population is considerably smaller than that on psyllid population, we simply assume all coefficients of tree population are temperature-independent. The model is governed by the following form:

$$
\begin{cases}
S_h'(t) = r(K - N_h(t)) - \beta_1(t)\frac{S_h(t)I_v(t)}{N_h(t)} - \mu_1S_h(t), \\
E_h'(t) = \beta_1(t)\frac{S_h(t)I_v(t)}{N_h(t)} - \beta_1(t - \tau)e^{-\mu_1\tau}\frac{S_h(t-\tau)I_v(t-\tau)}{N_h(t-\tau)} - \mu_1E_h(t), \\
I_h'(t) = \beta_1(t - \tau)e^{-\mu_1\tau}\frac{S_h(t-\tau)I_v(t-\tau)}{N_h(t-\tau)} - \mu_1I_h(t) - \delta_1I_h(t) - \delta_2I_v(t), \\
S_v'(t) = \alpha(t)N_v(t)(1 - \frac{N_v(t)}{mN_h(t)}) - \beta_2(t)\frac{I_h(t)S_v(t)}{N_v(t)} - d_v(t)S_v(t) - \theta(t)S_v(t), \\
E_v'(t) = \beta_2(t)\frac{I_h(t)S_v(t)}{N_v(t)} - \delta_1(t)E_v(t), \\
I_v'(t) = \delta_1(t)E_v(t) - \delta_2(t)E_v(t) - \theta(t)I_v(t) + \Psi(t)N_h(t),
\end{cases}
$$

(1)
and since

\[ J(t) = \text{newly infectious psyllids per unit time at time } t. \]

The biological interpretations for parameters in system (1) are listed in Table 1.

Next, we focus on the derivation of the expression for \( \mathcal{M}_V(t) \). The temperature \( T \) is assumed to vary as a function of time \( t \), that is, \( T = T(t) \). Let \( q \) be the development level of infection such that \( q \) increases at a temperature-dependent rate \( \gamma_v(T(t)) = \gamma_v(t), q = q_{E_v} = 0 \) at the transition from \( S_v \) to \( E_v \), and \( q = q_{I_v} \) at the transition from \( E_v \) to \( I_v \). Let \( f(q,t) \) be the density of psyllid with infection development level \( q \) at time \( t \). Consequently, \( \mathcal{M}_V(t) = \gamma_v(t)f(q_{E_v},t) \). Let \( J(q,t) \) be the flux, in the direction of increasing \( q \) of psyllid with infection development level \( q \). Then we have the following equation:

\[
\frac{\partial f(q,t)}{\partial t} = - \frac{\partial J}{\partial q} - (d_v(t) + \theta(t))f. \tag{2}
\]

Substituting \( J(q,t) = \gamma_v(t)f(q,t) \) into the above equation, we obtain

\[
\frac{\partial f(q,t)}{\partial t} = - \frac{\partial}{\partial q} [\gamma_v(t)f] - (d_v(t) + \theta(t))f \tag{3}
\]

and since \( J(q_{E_v},t) = \gamma_v(t)f(q_{E_v},t) = \beta_2(t) \frac{I_v(t)S_v(t)}{N_h(t)} \), we impose the following boundary condition on (3):

\[
f(q_{E_v},t) = \beta_2(t) \frac{I_v(t)S_v(t)}{N_h(t)\gamma_v(t)}. \tag{4}
\]

In order to solve system (1) with the boundary condition (4), we introduce a new variable

\[
\eta = h(t) := q_{E_v} + \int_0^t \gamma_v(s)ds. \tag{5}
\]

Let \( h^{-1}(\eta) \) be the inverse function of \( h(t) \), and define

\[
\hat{f}(q,\eta) = f(q,h^{-1}(\eta)), \hat{d}_v(\eta) = d_v(h^{-1}(\eta)), \hat{\theta}(\eta) = \theta(h^{-1}(\eta)), \hat{\gamma}_v(\eta) = \gamma_v(h^{-1}(\eta)). \tag{6}
\]

| Parameter       | Description                                      |
|-----------------|--------------------------------------------------|
| \( r \)         | Rate of replanting citrus tree                   |
| \( K \)         | Maximum citrus tree population size              |
| \( \beta_1(t) \) | Infection rate of susceptible trees              |
| \( \mu_1 \)     | Natural death rate of citrus tree population     |
| \( \delta_1 \)  | Death rate of infected trees                     |
| \( \delta_2 \)  | Rate of removal of infected trees                |
| \( \sigma(t) \) | Intrinsic growth rate of psyllid                 |
| \( m \)         | Maximum abundance of psyllid per tree            |
| \( d_v(t) \)    | Natural death rate of psyllid population         |
| \( \beta_2(t) \) | Infection rate of susceptible psyllid            |
| \( \theta(t) \) | Killing rate of psyllid with spraying insecticide|
| \( \tau \)      | Incubation period in trees                       |
| \( \tau_v(t) \) | Extrinsic incubation period(EIP) of psyllid      |
According to equations (5) and (11), we get
\[
\frac{\partial \hat{f}(q, \eta)}{\partial \eta} = -\frac{\partial \hat{f}(q, \eta)}{\partial q} - \frac{\hat{d}_v(\eta) + \hat{\theta}(\eta)}{\gamma_v(\eta)} \hat{f}(q, \eta).
\] (7)

Let \( V(s) = \hat{f}(s + q - \eta, s) \), and hence system (7) becomes
\[
\frac{dV(s)}{ds} = -\frac{\hat{d}_v(s) + \hat{\theta}(s)}{\gamma_v(s)} V(s).
\] (8)

As \( \eta - (q - q_{E_v}) \leq \eta \), we have
\[
V(\eta) = V(\eta - (q - q_{E_v})) e^{-\int_{0}^{\eta-q+q_{E_v}} \frac{\hat{d}_v(s)+\hat{\theta}(s)}{\gamma_v(s)} ds}
\] (9)

which implies
\[
\hat{f}(q, \eta) = \hat{f}(q_{E_v}, \eta - q + q_{E_v}) e^{-\int_{0}^{\eta-q+q_{E_v}} \frac{\hat{d}_v(s)+\hat{\theta}(s)}{\gamma_v(s)} ds}.
\] (10)

Let \( \tau(q, t) \) be the time taken to grow from infection development level \( q_{E_v} \) to level \( q \) by a psyllid who arrives at infection development level \( q \) at time \( t \), since \( \frac{dq}{dt} = \gamma_v(t) \), we have
\[
q - q_{E_v} = \int_{t-\tau(q,t)}^{t} \gamma_v(s) ds.
\] (11)

According to equations (5) and (11), we get
\[
h(t - \tau(q, t)) = h(t) = \int_{0}^{t} -\gamma_v(s) ds = -\int_{0}^{t} \gamma_v(s) ds = -(q - q_{E_v}),
\] (12)

and hence \( h(t - \tau(q, t)) = \eta - q + q_{E_v} \). It easily follows from the fact that \( f(q, t) = \hat{f}(q, h(t)) \) that
\[
f(q, t) = f(q_{E_v}, t - \tau(q, t)) e^{-\int_{t-\tau(q,t)}^{t} \frac{\hat{d}_v(s)+\hat{\theta}(s)}{\gamma_v(s)} ds}
\] (13)

where
\[
f(q_{E_v}, t - \tau(q, t)) = \beta_2(t - \tau(q, t)) \frac{I_h(t - \tau(q, t)) S_v(t - \tau(q, t))}{N_h(t - \tau(q, t)) \gamma_v(t - \tau(q, t))}.\]

Let \( \tau_v(t) = \tau_{I_v}(t) \). Then (13), together with \( M_V(t) = \gamma_v(t) f(q_{I_v}, t) \), implies that
\[
M_V(t) = \frac{\gamma_v(t)}{\gamma_v(t - \tau_v(t))} \beta_2(t - \tau_v(t)) \frac{I_h(t - \tau_v(t)) S_v(t - \tau_v(t))}{N_h(t - \tau_v(t))} e^{-\int_{t-\tau_v(t)}^{t} \frac{\hat{d}_v(s)+\hat{\theta}(s)}{\gamma_v(s)} ds}.
\] (14)

Substituting \( q = q_{I_v} \) into (11), we have
\[
q_{I_v} - q_{E_v} = \int_{t-\tau_v(t)}^{t} \gamma_v(s) ds.
\] (15)

Taking the derivative with respect to \( t \) on both sides of (15), we obtain
\[
1 - \tau_v'(t) = \frac{\gamma_v(t)}{\gamma_v(t - \tau_v(t))},
\] (16)

which means that \( 1 - \tau_v'(t) > 0 \). This condition is biologically significant. Recall that \( \tau_v(t) \) represents the time taken by the psyllid who arrives at the infectious stage at time \( t \) from the infected stage. For sufficiently small \( s > 0 \), the psyllid is still infectious at time \( t + s \) when it was infected at time \( t \). Thus, \( \tau_v(t + s) < \tau_v(t) + s \).
and hence $\tau'_v(t) = \lim_{s \to 0} \frac{\tau_v(t+s) - \tau_v(t)}{s} \leq 1$. According to (15), it easily follows that if $\gamma_v(t)$ is periodic function, then $\tau_v(t)$ is also a periodic function with the same period. With (14)and (16), we have

$$\mathcal{M}_V(t) = (1 - \tau'_v(t))\beta_2(t - \tau_v(t)) \frac{I_h(t - \tau_v(t))S_v(t - \tau_v(t))}{N_h(t - \tau_v(t))} e^{-\int_{t - \tau_v(t)}^{t} \gamma(t) dt} d\xi.$$  

(17)

Since the temperature $T$ can be assumed to vary as a function of time $t$, we let $\tau_v(t)$ be the length of the EIP. Substituting (17) into system (1), we can obtain the following system with temperature-dependent incubation period:

$$\begin{cases}
S'_h(t) = r(K - N_h(t)) - \beta_1(t) \frac{S_h(t)I_v(t)}{N_h(t)} - \mu_1 S_h(t), \\
n'_{h}(t) = \beta_1(t) \frac{S_h(t)I_v(t)}{N_h(t)} - \beta_1(t - \tau)e^{-\mu_1 \tau} \frac{S_h(t - \tau)I_v(t - \tau)}{N_h(t - \tau)} - \mu_1 n(t), \\
I'_h(t) = \beta_1(t - \tau)e^{-\mu_1 \tau} \frac{S_h(t - \tau)I_v(t - \tau)}{N_h(t - \tau)} - \mu_1 I_h(t) - \delta_1 I_h(t) - \delta_2 I_h(t), \\
S'_v(t) = \alpha(t) S_v(t) \left(1 - \frac{N_v(t)}{m N_h(t)}\right) - \beta_2(t) \frac{I_h(t)S_v(t)}{N_h(t)} - \gamma(t) S_v(t), \\
n'_{v}(t) = \beta_1(t) \frac{I_h(t)S_v(t)}{N_h(t)} - (1 - \tau'_v(t))\beta_2(t - \tau_v(t)) \frac{I_h(t - \tau_v(t))S_v(t - \tau_v(t))}{N_h(t - \tau_v(t))}
\times e^{-\int_{t - \tau_v(t)}^{t} \gamma(t) dt} d\xi - \gamma(t) E_v(t), \\
I'_v(t) = (1 - \tau'_v(t))\beta_2(t - \tau_v(t)) \frac{I_h(t - \tau_v(t))S_v(t - \tau_v(t))}{N_h(t - \tau_v(t))} e^{-\int_{t - \tau_v(t)}^{t} \gamma(t) dt} d\xi \\
- \gamma(t) I_v(t),
\end{cases}$$

(18)

where $\beta_1(t) = \sigma_1(t)k(t)$, $\beta_2(t) = \sigma_2(t)k(t)$ and $\gamma(t) = d_v(t) + \theta(t)$. Moreover, we assume these psyllids-related parameters $\beta_1(t)$, $\beta_2(t)$, $\gamma(t)$ are $\omega$-periodic in time $t$. As a matter of convenience, we denote $(u_1, u_2, u_3, u_4, u_5, u_6) = (S_h, E_h, I_h, S_v, E_v, I_v)$. Therefore, we have the following system:

$$\begin{cases}
u'_1(t) = r(K - \sum_{i=1}^{3} u_i(t)) - \beta_1(t) \frac{u_1(t)u_6(t)}{\sum_{i=1}^{3} u_i(t)} - \mu_1 u_1(t), \\
u'_2(t) = \beta_1(t) \sum_{i=1}^{3} u_i(t) - \beta_1(t - \tau)e^{-\mu_1 \tau} \frac{u_1(t - \tau)u_6(t - \tau)}{\sum_{i=1}^{3} u_i(t - \tau)} - \mu_1 u_2(t), \\
u'_3(t) = \beta_1(t - \tau)e^{-\mu_1 \tau} \frac{u_1(t - \tau)u_6(t - \tau)}{\sum_{i=1}^{3} u_i(t - \tau)} - \mu_1 u_3(t) - \delta_1 u_3(t) - \delta_2 u_3(t), \\
u'_4(t) = \alpha(t) \left(1 - \frac{\sum_{i=4}^{6} u_i(t)}{m \sum_{i=1}^{3} u_i(t)}\right) \frac{\sum_{i=1}^{3} u_i(t) - \beta_2(t) \sum_{i=1}^{3} u_i(t) - \gamma(t) u_4(t)}{\sum_{i=1}^{3} u_i(t)} - \gamma(t) u_4(t), \\
u'_5(t) = \beta_2(t) \frac{\sum_{i=1}^{3} u_i(t)u_4(t)}{\sum_{i=1}^{3} u_i(t)} - (1 - \tau'_v(t))\beta_2(t - \tau_v(t)) \frac{u_3(t - \tau_v(t))u_4(t - \tau_v(t))}{\sum_{i=1}^{3} u_i(t - \tau_v(t))}
\times e^{-\int_{t - \tau_v(t)}^{t} \gamma(t) dt} d\xi - \gamma(t) u_5(t), \\
u'_6(t) = (1 - \tau'_v(t))\beta_2(t - \tau_v(t)) \frac{u_3(t - \tau_v(t))u_4(t - \tau_v(t))}{\sum_{i=1}^{3} u_i(t - \tau_v(t))} e^{-\int_{t - \tau_v(t)}^{t} \gamma(t) dt} d\xi \\
- \gamma(t) u_6(t).
\end{cases}$$

(19)
In view of biological meaning of \( \tau \) and \( \tau_c(t) \), we impose the following compatibility conditions:

\[
    u_2(0) = \int_{-\tau}^{0} \beta_1(s) e^{\mu_1 \varepsilon} \frac{u_1(s) u_6(s)}{\sum_{i=1}^{3} u_i(s)} \, ds \tag{20}
\]

and

\[
    u_5(0) = \int_{-\tau_c(0)}^{0} \beta_2(s) \frac{u_3(s) u_4(s) e^{\tau_c(s)}}{\sum_{i=1}^{3} u_i(s)} \, ds. \tag{21}
\]

Let \( \hat{\tau} = \max_{t \in [0, w]} \{ \tau, \tau_c(t) \} \), \( C := C([-\hat{\tau}, 0], \mathbb{R}^6) \), \( C^+ := C([-\hat{\tau}, 0], \mathbb{R}^6_+) \). Then \( (C, C^+) \) is an ordered Banach space equipped with the maximum norm. For any given continuous function \( u : [-\hat{\tau}, \eta) \to \mathbb{R}^6 \) with \( \eta > 0 \), define \( u_t \in C \) by \( u_t(\theta) = u(t + \theta) \), for each \( t \in [0, \eta) \) and \( \theta \in [-\hat{\tau}, 0] \). Define

\[
    D_\varepsilon = \left\{ \phi \in C^+ : \sum_{i=1}^{3} \phi_i(s) \geq \varepsilon \text{ for each } s \in [-\hat{\tau}, 0] \right\},
\]

and

\[
    X_\varepsilon(t) = \left\{ \phi \in D_\varepsilon : \sum_{i=1}^{3} \phi_i(s) \leq K \text{ for each } s \in [-\hat{\tau}, 0], \phi_2(0) = \int_{-\tau}^{0} \beta_1(t + s) \right. \\
    \times \frac{\phi_1(s) \phi_6(s)}{\sum_{i=1}^{3} \phi_i(s)} e^{\mu_1 \varepsilon} \left. ds, \phi_5(0) = \int_{-\tau_c(t)}^{0} \beta_2(t + s) \frac{\phi_3(s) \phi_4(s)}{\sum_{i=1}^{3} \phi_i(s)} e^{-\int_{s}^{0} \gamma(t + \xi) \, d\xi} ds \right\} \tag{22}
\]

for any \( \varepsilon \in \left( 0, \frac{r K}{r + \mu_1 + \delta_1 + \delta_2} \right) \) and \( t \geq 0 \).

**Theorem 2.1.** For any \( \phi \in X_\varepsilon(0) \), there is a unique nonnegative solution \( u(t, \phi) \) on \([0, \infty)\) with initial value \( u_0 = \phi \), and \( u_t(\phi) \in X_\varepsilon(t) \) for any \( t \geq 0 \). Furthermore, system (19) generates an \( \omega \)-periodic semiflow \( Q(t) = u_t : X_\varepsilon(0) \to X_\varepsilon(t) \) for any \( t \geq 0 \) in sense that \( Q(0) = I \), \( Q(t + \omega) = Q(t) \circ Q(\omega) \) for any \( t \geq 0 \), \( Q(t) \phi \) is continuous in \((t, \phi) \in [0, \infty) \times X_\varepsilon(0)\) and \( Q := Q(\omega) \) has a strong global attractor.

**Proof.** For any \( t \geq 0 \) and \( \phi \in X_\varepsilon(t) \), define

\[
    G(t, \phi) := (G_1(t, \phi), G_2(t, \phi), G_3(t, \phi), G_4(t, \phi), G_5(t, \phi), G_6(t, \phi)),
\]
where

\[ G_1(t, \phi) = r(K - \sum_{i=1}^{3} \phi_i(0)) - \beta_1(t) \frac{\phi_1(0)u_6(0)}{\sum_{i=1}^{3} \phi_i(0)} - \mu_1 \phi_1(0), \]

\[ G_2(t, \phi) = \beta_1(t) \frac{\phi_1(0)u_6(0)}{\sum_{i=1}^{3} \phi_i(0)} - \beta_1(t-\tau)e^{-\mu_1 \tau} \frac{\phi_1(t-\tau)\phi_6(t-\tau)}{\sum_{i=1}^{3} \phi_i(t-\tau)} - \mu_1 \phi_2(0), \]

\[ G_3(t, \phi) = \beta_1(t-\tau)e^{-\mu_1 \tau} \frac{\phi_1(-\tau)\phi_6(-\tau)}{\sum_{i=1}^{3} \phi_i(-\tau)} - \mu_1 \phi_3(0) - \delta_1 \phi_3(0) - \delta_2 \phi_3(0), \]

\[ G_4(t, \phi) = \alpha(t) \frac{6}{\mu} \sum_{i=1}^{3} \phi_i(0) \left(1 - \sum_{i=1}^{3} \phi_i(0)\right) - \beta_2(t) \frac{\phi_3(0)\phi_4(0)}{\sum_{i=1}^{3} \phi_i(0)} - \gamma(t) \phi_4(0), \]

\[ G_5(t, \phi) = \beta_2(t) \frac{\phi_3(0)\phi_4(0)}{\sum_{i=1}^{3} \phi_i(0)} - (1 - \tau_0'(t))\beta_2(t-\tau_0(t)) \phi_3(-\tau_0(t)) \phi_4(-\tau_0(t)) \frac{e^{-\int_{-\tau_0(t)}^{s} \gamma(s)ds} - \gamma(t)\phi_5(0)}{\sum_{i=1}^{3} \phi_i(-\tau_0(t))}, \]

\[ G_6(t, \phi) = (1 - \tau_0'(t))\beta_2(t-\tau_0(t)) \phi_3(-\tau_0(t)) \phi_4(-\tau_0(t)) e^{-\int_{-\tau_0(t)}^{s} \gamma(s)ds} - \gamma(t)\phi_5(0), \]

\( D_e \) is closed in \( C \), for any \( \phi \in X_e(0) \subset D_e \), \( G(t, \phi) \) is continuous and Lipschitz in \( \phi \) in each compact set \( R_+ \times D_e \). It is easy to see that system (19) has a unique solution \( u(t, \phi) \) satisfying \( \sum_{i=1}^{3} u_i(t, \phi) > 0 \) on \( [0, t_\phi) \) with \( u_0 = \phi \) and \( t_\phi < \infty \).

By the uniqueness of solutions of system (19) and the compatibility conditions (20) and (21), it follows that

\[ u_2(t) = \int_{t-\tau}^{t} \beta_1(s) e^{-\mu_1 \tau(s-t)} \frac{u_1(s)u_6(s)}{\sum_{i=1}^{3} u_i(s)} ds \tag{23} \]

and

\[ u_5(t) = \int_{t-\tau_0(t)}^{t} \beta_2(s) \frac{u_3(s)u_4(s)}{\sum_{i=1}^{3} u_i(s)} e^{-\int_{t-\tau_0(t)}^{s} \gamma(s)ds} ds. \tag{24} \]

And hence, we observe that if all \( u_i(t) \geq 0, i \neq 2, 5 \) are non-negative on \( [0, s) \subset [0, t_\phi) \), then so are \( u_2(t) \geq 0, u_5(t) \geq 0 \) on \( [0, s) \). Note that \( G_1(t, \phi) \geq 0, i \neq 2, 5 \) provided \( \phi_i(0) = 0 \) and \( \phi \in X_e(0) \), we utilize [24, Theorem 5.2.1] and its proof to obtain that for the above \( \phi \in X_e(t), u_i(t, \phi) \geq 0, i \neq 2, 5, \forall t \in [0, t_\phi) \). It immediately follows from the above observation that \( u_2(t, \phi) \geq 0, u_5(t, \phi) \geq 0, \forall t \in [0, t_\phi) \).

Next, we prove that \( u_i \in X_e(t) \) for any \( t \geq 0 \). Observe that the total tree population \( (N_h(t) = \sum_{i=1}^{3} u_i(t)) \) satisfies

\[ \frac{dN_h(t)}{dt} = r(K-N_h(t)) - \mu_1 N_h(t) - (\delta_1 + \delta_2) u_3(t) \geq rK - (r + \mu_1 + \delta_1 + \delta_2)N_h(t), \tag{25} \]

this implies \( N_h(t) \) is an upper solution of

\[ x'(t) = rK - (r + \mu_1 + \delta_1 + \delta_2)x(t). \tag{26} \]

Denote \( x(t; N_h(0)) \) be the unique solution of (26) with \( x(0) = N_h(0) \), then \( N_h(t) \geq x(t; N_h(0)) \) for any \( t \in [0, t_\phi) \). It then follows that for any \( 0 < \varepsilon < \frac{rK}{r + \mu_1 + \delta_1 + \delta_2} \), \( N_h(t) \geq x(t; N_h(0)) \geq \varepsilon \) for any \( t \in [0, t_\phi) \) provided \( N_h(0) \geq \varepsilon \). Furthermore,

\[ \frac{dN_h(t)}{dt} \leq rK - (r + \mu_1)N_h(t), \tag{27} \]

which yields that if \( \varepsilon \leq \sum_{i=1}^{3} u_i(0) \leq K, \) then \( \varepsilon \leq \sum_{i=1}^{3} u_i(t) \leq K \) for any \( t \in [0, t_\phi) \).
Hence, we know that follows from [32, Theorem 1.1.3] that

\[ Q \]

Therefore, \( u \in X_\varepsilon(0) \), \( \forall t \in [0, t_\phi) \). Since (23) can be rewritten as

\[ u_2(t) = \int_{-\tau}^{\tau} \beta_1(t + s) e^{\mu_0 s} \frac{u_1(t + s) u_6(t + s)}{\sum_{i=1}^{3} u_i(t + s)} ds, \]

which is equivalent to

\[ u_{2t}(0, \phi) = \int_{-\tau}^{\tau} \beta_1(t + s) e^{\mu_0 s} \frac{u_{1t}(s, \phi) u_{6t}(s, \phi)}{\sum_{i=1}^{3} u_{it}(s, \phi)} ds. \]

Similarly, (24) is equivalent to

\[ u_{3t}(0, \phi) = \int_{-\tau}^{\tau} \beta_2(t + s) e^{\mu_0 s} \frac{u_{3t}(s, \phi) u_{4t}(s, \phi) - f_n^{\alpha(t)}(d_v(t + \xi) + \theta(\xi)) ds}{\sum_{i=1}^{3} u_{it}(s, \phi)}, \]

Hence, we know \( u_i(\phi) \in X_\varepsilon(t), \forall t \in [0, t_\phi) \).

Note that \( \varepsilon \leq N_h(t) \leq K \) and \( N_v(t) \geq 0 \) for any \( t \geq 0 \). Adding the last three equations of (18), the total number of psyllid population \( N_v(t) \) satisfies

\[ \frac{dN_v(t)}{dt} = \alpha(t) N_v(t) \left( 1 - \frac{N_v(t)}{mN_h(t)} \right) - (d_v(t) + \theta(t)) N_v(t) \]

\[ \leq \alpha(t) N_v(t) \left( 1 - \frac{N_h(t)}{mK} \right). \]

This gives \( \limsup_{t \to \infty} N_v(t) \leq mK \). In view of (27), we have \( \limsup_{t \to \infty} N_h(t) \leq \frac{rK}{r + \mu_1} = N_h^*. \) Therefore,

\[ \limsup_{t \to \infty} N_h(t) \leq \frac{rK}{r + \mu_1} = N_h^*, \quad \limsup_{t \to \infty} N_v(t) \leq mK. \]

Thus the solutions of system (19) with initial data in \( X_\varepsilon(0) \) exist globally on \([0, \infty)\) and ultimately bounded. By [7, Theorem 3.6.1], we see that for each \( t > \tau \), \( Q(t) \) is compact, and hence, \( Q^{\omega_0} := Q(n_0 \omega) \) is compact for some large \( n_0 > \frac{\tau}{\omega} \). It then follows from [32, Theorem 1.1.3] that \( Q : X_\varepsilon(0) \to X_\varepsilon(0) \) admits a strong global attractor.

3. Basic reproduction number \( R_0 \). In this section, we will first discuss the existence of non-negative \( \omega \)-periodic solutions for the disease free system and their global stability. Then we define the basic reproduction number \( R_0 \) for the model system.

3.1. Disease free dynamics. Letting \( u_3 = u_6 = 0 \) in (19), we then obtain the disease-free system

\[ \begin{align*}
&u_1'(t) = r(K - u_1(t) - u_2(t)) - \mu_1 u_1(t), \\
&u_2'(t) = -\mu_1 u_2(t), \\
&u_4'(t) = \alpha(t)(u_4(t) + u_5(t)) \left( 1 - \frac{u_4(t) + u_5(t)}{m(u_1(t) + u_2(t))} \right) - \gamma(t) u_4(t), \\
&u_5'(t) = -\gamma(t) u_5(t).
\end{align*} \]

Denote \( R_0^P := \frac{\int_0^\tau \alpha(t) dt}{\int_0^\tau \gamma(t) dt} \). The next result is on the threshold dynamics of (29) in terms of \( R_0^P \).

**Proposition 1.** The following statements on system (29) are valid.
(i) If $R_0^p > 1$, then $(N_h^*, 0, S_v^*(t), 0)$ is a globally asymptotically stable periodic solution of system (29) in $\{\mathbb{R}^2 \setminus \{0\}\} \times \{\mathbb{R}_+ \setminus \{0\}\}$. Here $S_v^*(t)$ is a unique positive $\omega$-periodic solution of

$$w'(t) = w(t) \left(\alpha(t) - \gamma(t) - \frac{\alpha(t)w(t)}{mN_h^*}\right).$$

(ii) If $R_0^p \leq 1$, then $(N_h^*, 0, 0, 0)$ is globally asymptotically stable in $\{\mathbb{R}^2 \setminus \{0\}\} \times \mathbb{R}_+^2$.

**Proof.** From the second and last equations in system (29), it is easy to see that $u_2(t) \to 0$ and $u_5(t) \to 0$ as $t \to \infty$, respectively. Clearly, $v_1 := u_1 + u_2$ and $v_2 := u_4 + u_5$ satisfy

$$v_1'(t) = rK - (r + \mu_1)v_1(t),$$

$$v_2'(t) = \alpha(t)v_2(t) \left(1 - \frac{v_2(t)}{mv_1(t)}\right) - \gamma(t)v_2(t),$$

and hence, $v_1(t) \to N_h^*$ as $t \to \infty$. It then follows that $v_2$ is asymptotic to (30). In view of [32, Theorem 5.2.1], we can infer that

(a) If $R_0^p > 1$, then $|v_2(t) - S_v^*(t)| \to 0$ as $t \to \infty$, for any $v_2(0) \in \mathbb{R}_+ \setminus \{0\}$;

(b) If $R_0^p \leq 1$, then $v_2(t) \to 0$ as $t \to \infty$, for any $v_2(0) \in \mathbb{R}_+$.

(i) In the case that $R_0^p > 1$, by the theories of asymptotically periodic semiflows and internally chain transitive sets (see, e.g., [32]), together with the fact that system (31) is cooperative, we get $(N_h^*, S_v^*(t))$ is a globally asymptotically stable periodic solution of system (31) in $\text{Int}(\mathbb{R}_+^2)$. Therefore, $(N_h^*, 0, S_v^*(t), 0)$ is a globally asymptotically stable periodic solution of system (29) in $\{\mathbb{R}^2 \setminus \{0\}\} \times \{\mathbb{R}_+ \setminus \{0\}\}$.

(ii) In the case that $R_0^p \leq 1$, we could argue analogously to those in (i) to obtain $(N_h^*, 0, 0, 0)$ is globally asymptotically stable in $\{\mathbb{R}^2 \setminus \{0\}\} \times \mathbb{R}_+^2$. \hfill \Box

3.2. $R_0$ for the whole system. To define the disease reproduction number for system (19), we follow the procedure in [33]. By Proposition 1, when $R_0^p > 1$, there are two infection-free equilibria

$$E_0 = (N_h^*, 0, 0, 0, 0, 0) \quad \text{and} \quad E_1 = (N_h^*, 0, 0, S_v^*(t), 0, 0).$$

Linearizing system (19) at $E_1$, we obtain the following periodic linear system for the infectious compartments:

$$
\begin{align*}
\begin{cases}
\dot{w}_3(t) &= \beta_1(t - \tau)e^{-\mu_1\tau}w_6(t - \tau) - (\mu_1 + \delta_1 + \delta_2)w_3(t), \\
\dot{w}_6(t) &= (1 - \gamma(t))(\beta_1(t - \tau)e^{-\mu_1\tau}w_6(t - \tau) - \mu_1w_3(t)) - \gamma(t)w_6(t),
\end{cases}
\end{align*}
$$

Assume that $C_{\omega}(\mathbb{R}, \mathbb{R}^2)$ is the Banach space consisting of all $\omega$-periodic and continuous functions from $\mathbb{R}$ to $\mathbb{R}^2$, where $\|\varphi\|_{C_{\omega}(\mathbb{R}, \mathbb{R}^2)} = \max_{\theta \in [0, \omega]} \|\varphi(\theta)\|$ for any $\varphi \in C_{\omega}(\mathbb{R}, \mathbb{R}^2)$. Recall that $\hat{\tau} = \max_{\xi \in [0, \omega]} \{\tau, \tau_v(t)\}$. From system (32), we define $F(t) : C([-\hat{\tau}, 0], \mathbb{R}^2) \to \mathbb{R}^2$ by $F(t)$

$$F(t) \left(\begin{array}{c}
\varphi_3 \\
\varphi_6
\end{array}\right) = \left(\begin{array}{c}
\beta_1(t - \tau)e^{-\mu_1\tau}\varphi_6(-\tau) \\
(1 - \gamma(t))(\beta_1(t - \tau)e^{-\mu_1\tau}\varphi_6(-\tau) - \mu_1\varphi_3) - \gamma(t)\varphi_6
\end{array}\right)$$

for $t \geq 0$ and $(\varphi_3, \varphi_6) \in C([-\hat{\tau}, 0], \mathbb{R}^2)$, and $-V(t) : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$-V(t) \left(\begin{array}{c}
w_3 \\
w_6
\end{array}\right) = \left(-\mu_1 + \delta_1 + \delta_2\right)w_3 - \gamma(t)w_6.$$
for \( t \geq 0 \) and \((w_3, w_6) \in \mathbb{R}^2\). Then system (32) can be written as
\[
\frac{dv(t)}{dt} = F(t) v_t - V(t) v(t).
\]
It is obvious that \( F(t) : C([-\bar{\tau}, 0], \mathbb{R}^2) \to \mathbb{R}^2 \) is positive in the sense that
\[
F(t) C([-\bar{\tau}, 0], \mathbb{R}^2_+) \subset \mathbb{R}^2_+,
\]
and hence the condition (H1) in [33] holds. Obviously, \(-V(t)\) is cooperative. Let \( Z(t, s), t \geq s \) be the evolution family on \( \mathbb{R}^2 \) associated with the system \( \frac{dv(t)}{dt} = -V(t) v(t) \). Supposed that \( \Omega(Z) \) stands for the exponential growth bound of the evolution family \( Z(t, s), t \geq s \), which is defined by
\[
\Omega(Z) := \inf \{ \bar{\omega} : \exists M \geq 1 \text{ such that } \| Z(t + s, s) \| \leq Me^{\bar{\omega}t}, \forall s \in \mathbb{R}, t \geq 0 \}.
\]
Note that
\[
Z(t, s) = \begin{pmatrix}
e^{-\int_s^t (\mu_1 + \delta_1 + \delta_2) d\xi} & 0 \\
e^{-\int_s^t \gamma(\xi) d\xi} & e^{-\int_s^t (\mu_1 + \delta_1 + \delta_2) d\xi}
\end{pmatrix},
\]
we infer the exponential growth bound of the evolution family \( Z(t, s), t \geq s \) is negative, and hence the condition (H2) in [33] holds.

After the above discussions, we can use the theory and method in [33] to define the reproduction number for system (19). We assume that \( \nu \in C_\omega(\mathbb{R}, \mathbb{R}^2) \) and \( \nu(t) \) is the initial distribution of infectious citrus trees and psyllid at time \( t \in \mathbb{R} \). For any \( s \geq 0 \), \( F(t - s) \nu_{t-s} \) represents the density distribution of newly infected trees and psyllid at time \( t - s \), and \( Z(t, t - s) F(t - s) \nu_{t-s} \) represents the distribution of those infected citrus trees and psyllid who were newly infected at time \( t - s \) and still survive in the environment at time \( t \) for \( t \geq s \). Hence,
\[
\int_0^\infty Z(t, t - s) F(t - s) \nu_{t-s} ds = \int_0^\infty Z(t, t - s) F(t - s) \nu(t - s + \cdot) ds
\]
is the distribution of accumulative infective citrus trees and psyllid at time \( t \) produced by those infectious trees and psyllid introduced at all previous time to \( t \). Define a linear operator on \( C_\omega(\mathbb{R}, \mathbb{R}^2) \) by
\[
[L\nu](t) := \int_0^\infty Z(t, t - s) F(t - s) \nu(t - s + \cdot) ds, \forall t \in \mathbb{R}, \nu \in C_\omega(\mathbb{R}, \mathbb{R}^2),
\]
Then we define the spectral radius of \( L \) as the basic reproduction number for system (19), that is, \( R_0 := r(L) \).

Let \( \hat{P}(t) \) be the solution maps of system (32) and \( \hat{P}(t) = \omega_t(\phi), t \geq 0 \), where \( w(t, \phi) \) is the unique solution of system (32) with \( w_0 = \phi \in C([-\bar{\tau}, 0], \mathbb{R}^2) \). Then \( \hat{P}(\omega) \) is the Poincaré map associated with linear system (32). Let \( r(\hat{P}(\omega)) \) be the spectral radius of \( \hat{P}(\omega) \). By [33, Theorem 2.1], we have the following statement.

**Lemma 3.1.** \( R_0 - 1 \) has the same sign as \( r(\hat{P}(\omega)) - 1 \).

Let \( \mathcal{W} := C([-\tau_v(0), 0], \mathbb{R}) \times C([-\tau, 0], \mathbb{R}) \) and \( \mathcal{W}^+ := C([-\tau_v(0), 0], \mathbb{R}_+) \times C([-\tau, 0], \mathbb{R}_+) \). We see that \( (\mathcal{W}, \mathcal{W}^+) \) is an ordered Banach space. Given a function \( w : [-\tau_v(0), \infty) \times [-\tau, \infty) \to \mathbb{R}^2 \), we define \( w_t \in \mathcal{W} \) by \( w_t(\theta) = (w_3(t + \theta_3), w_6(t + \theta_6)) \) for any \( t \geq 0, \theta := (\theta_3, \theta_6) \in [-\tau_v(0), \infty) \times [-\tau, \infty) \). We use method of steps and have the following result which is similar to that in [14, Lemma 3.3].

**Lemma 3.2.** For any \( \varphi = (\varphi_3, \varphi_6) \in \mathcal{W}^+ \), system (32) admits a unique solution \( w(t, \varphi) \) on \([0, \infty) \) with \( w_0 = \varphi \).
Proof. Let \( \bar{\tau} = \min \{ \tau, \min_{t \in [0, \tau]} \tau_v(t) \} \). Then for any \( t \in [0, \bar{\tau}] \), it follows from the strict monotonicity of \( t - \tau_v(t) \) that
\[
- \tau_v(0) = 0 - \tau_v(t) \leq t - \tau_v(t) \leq \bar{\tau} - \tau_v(\bar{\tau}) = \bar{\tau} - \bar{\tau} = 0,
\]
and hence, \( w_3(t - \tau_v(t)) = \varphi_3(t - \tau_v(t)) \). Similarly, we have \( u_6(t - \tau) = \varphi_6(t - \tau) \).

Therefore, when \( t \in [0, \bar{\tau}] \) there holds
\[
\begin{align*}
\frac{dw_3(t)}{dt} &= \beta_1(t - \tau) e^{-\mu_1 \tau} \varphi_6(t - \tau) - (\mu_1 + \delta_1 + \delta_2) w_3(t), \\
\frac{dw_6(t)}{dt} &= (1 - \tau_v(t)) \beta_2(t - \tau_v(t)) \left( \frac{S^*_v(t - \tau_v(t)) \varphi_3(t - \tau_v(t))}{S^*_h(t - \tau_v(t))} \right) e^{-\int_{t - \tau_v(t)}^{\bar{\tau}} \gamma(\xi) d\xi} - \gamma(t) w_6(t).
\end{align*}
\]

Given \( \varphi \in W^+ \), the solution \((w_3(t), w_6(t))\) of the above system exists uniquely for \( t \in [0, \bar{\tau}] \). In other words, we obtain the values of \( u_3(\theta_3) = w_3(\theta_3) \) for \( \theta_3 \in [-\tau_v(0), \bar{\tau}] \), and \( u_6(\theta_6) = w_6(\theta_6) \) for \( \theta_6 \in [-\tau, \bar{\tau}] \). For any \( t \in [\bar{\tau}, 2\bar{\tau}] \), we have
\[
t - \tau - \tau_v(t) \leq 2\bar{\tau} - \tau_v(t) \leq 2\bar{\tau} - \bar{\tau} = \bar{\tau}.
\]

Hence, by the previous step, we can obtain that \( w_3(t - \tau(t)) = w_3(t - \tau(t)) \), \( w_6(t - \tau) = u_6(t - \tau) \). Solving the following system for \( t \in [\bar{\tau}, 2\bar{\tau}] \) with \( w_3(\bar{\tau}) = u_3(\bar{\tau}) \) and \( w_6(\bar{\tau}) = u_6(\bar{\tau}) \),
\[
\begin{align*}
\frac{dw_3(t)}{dt} &= \beta_1(t - \tau) e^{-\mu_1 \tau} u_6(t - \tau) - (\mu_1 + \delta_1 + \delta_2) w_3(t), \\
\frac{dw_6(t)}{dt} &= (1 - \tau_v(t)) \beta_2(t - \tau_v(t)) \left( \frac{S^*_v(t - \tau_v(t)) u_3(t - \tau_v(t))}{S^*_h(t - \tau_v(t))} \right) e^{-\int_{t - \tau_v(t)}^{\bar{\tau}} \gamma(\xi) d\xi} - \gamma(t) w_6(t).
\end{align*}
\]

We then get the unique solution \((w_3(t), w_6(t))\) on the interval \([\bar{\tau}, 2\bar{\tau}]\). We can extend this procedure to \([n\bar{\tau}, (n+1)\bar{\tau}]\) for all \( n \in \mathbb{N} \). It then follows that for any \( \varphi \in W^+ \), the solution exists uniquely for all \( t \geq 0 \). This completes the proof. \( \square \)

For any given \( t \geq 0 \), let \( P(t) \) be the solution map of system (32) on \( W \) defined by
\[
P(t) \varphi = w_6(t) \varphi \text{ for any } \varphi \in W^+.
\]
Next, we will illustrate that the periodic semiflow \( P(t) \) is eventually strongly positive on \( W^+ \).

Lemma 3.3. \( P(t) \varphi \gg 0 \) for any \( t > 3\bar{\tau} \) provided \( \varphi = (\varphi_3, \varphi_6) \in W^+ \) with \( \varphi \neq 0 \).

Proof. As in the proof of Lemma (3.2), a simple comparison argument on each interval \([n\bar{\tau}, (n+1)\bar{\tau}]\), \( n \in \mathbb{N} \), implies that \( w_i(t) \geq 0 \) \( \forall t \geq 0 \), \( i = 3, 6 \). Next, we choose a large \( C > 0 \) such that \( h_3(t, w_3) := Cw_3 - (\mu_1 + \delta_1 + \delta_2)w_3 \) is strictly increasing in \( w_3 \), and \( h_6(t, w_6) := Cw_6 - \gamma(t)w_6 \) is strictly increasing in \( w_6 \). For convenience, we assume that
\[
b_3(t) = \beta_1(t - \tau)e^{-\mu_1 \tau}, \quad b_6(t) = (1 - \tau_v(t)) \beta_2(t - \tau_v(t)) \left( \frac{S^*_v(t - \tau_v(t))}{S^*_h(t - \tau_v(t))} \right) e^{-\int_{t - \tau_v(t)}^{\bar{\tau}} \gamma(\xi) d\xi}.
\]

Then system (32) can be rewritten as
\[
\begin{align*}
\frac{dw_3(t)}{dt} &= b_3(t) w_6(t - \tau) + h_3(t, w_3) - Cw_3(t), \\
\frac{dw_6(t)}{dt} &= b_6(t) w_3(t - \tau_v(t)) + h_6(t, w_6) - Cw_6(t).
\end{align*}
\]

It then follows that, for any given \( \varphi := (\varphi_3, \varphi_6) \in W^+ \), the solution \((w_3(t), w_6(t))\) satisfies the following system of integral equations for \( t \geq 0 \):
\[
\begin{align*}
w_3(t) &= e^{-Ct} \varphi_3(0) + \int_0^t e^{-C(t - s)}h_3(s, w_3(s)) ds + \int_0^t e^{-C(t - s)}b_3(s)w_6(s - \tau) ds, \\
w_6(t) &= e^{-Ct} \varphi_6(0) + \int_0^t e^{-C(t - s)}h_6(s, w_6(s)) ds + \int_0^t e^{-C(t - s)}b_6(s)w_3(s - \tau_v(s)) ds.
\end{align*}
(33)
In the case that $\varphi_3 \not\equiv 0$. It then follows that there exists $\theta^0_3 \in [-\tau_v(0), 0)$ such that $\varphi_3(\theta^0_3) > 0$. Note that $[-\tau_v(0), 0] \subset [-\tau_v(0), \hat{\tau} - \tau_v(\hat{\tau})]$.

The second equation in system (33) implies that
\[
 w_6(\tau, \varphi) \geq \int_0^\tau e^{-C(\tau - s)} b_6(s) w_3(s - \tau_v(s)) ds > 0,
\]
and hence, $w_6(t, \varphi) > 0$ for any $t \geq \hat{\tau}$. Since $s - \tau \geq 2\hat{\tau} - \tau \geq \hat{\tau}$ for any $t > s \geq 2\hat{\tau}$, now the first equation in system (33) implies that $w_3(t, \varphi) \geq \int_{2\hat{\tau}}^t e^{-C(t - s)} b_3(s) w_6(s - \tau) ds > 0$ for any $t > 2\hat{\tau}$.

In the case that $\varphi_6 \not\equiv 0$, we can argue similarly to show $w_i(t, \varphi) > 0$ for any $t > 2\hat{\tau}$, $i = 3, 6$.

Therefore, we see that $w_i(t, \varphi) > 0$ for any $t > 2\hat{\tau}$, $i = 3, 6$. As a consequence, $P(t)$ is strongly positive on $W^+$ for any $t > 3\hat{\tau}$.

Fix an $n_0$ such that $n_0 \omega > 3\hat{\tau}$. Then it follows from the above lemma which prove that the periodic semiflow $P(t)$ is eventually strongly positive and the arguments in [14, Lemma 3.8] that $[P(\omega)]^{n_0} = P(n_0 \omega)$ is strongly positive. Moreover, we have the following result.

**Lemma 3.4.** Two Poincaré maps $\hat{P}(\omega) : C([\hat{\tau}, 0], \mathbb{R}^2) \to C([\hat{\tau}, 0], \mathbb{R}^2)$, and $P(\omega) : W \to W$ have the same spectral radius, that is, $r(\hat{P}(\omega)) = r(P(\omega))$.

Further, by the argument similar to that in [27], we have the following observation.

**Lemma 3.5.** Let $\mu = \frac{\ln r(P(\omega))}{\omega}$. Then there exists a positive $\omega$-periodic function $w^*(t)$ such that the following statements hold:

(i) $e^{\mu t} w^*(t)$ is a solution of system (32) with the feasible domain $W^+$ for any $t \geq 0$;

(ii) $e^{\mu t} w^*(t)$ is a solution of system (32) with the feasible domain $C([\hat{\tau}, 0], \mathbb{R}^2)$ for any $t \geq 0$.

4. **Threshold dynamics.** This section is devoted to prove a threshold-type result on the global dynamics of system (19) in terms of $R_0^p$ and $R_0$.

**Theorem 4.1.** Assume that $u(t, \phi)$ is the unique solution of system (19) with $u_0 = \phi \in X_\omega(0)$. Then the following statements are valid:

(i) If $R_0^p \leq 1$, then $\lim_{t \to \infty} (u(t, \phi) - (N_0^p, N_0^c, 0, 0, 0, 0)) = 0$.

(ii) Suppose $\delta_1 = \delta_2 = 0$ in system (19) and $V_0(0) := \phi_4(0) + \phi_5(0) + \phi_6(0) > 0$.

If $R_0^p > 1$ and $R_0 < 1$, then $\lim_{t \to \infty} (u(t, \phi) - (N_0^p, N_0^p, S^*(t), 0, 0)) = 0$.

**Proof.** (i) In the case that $R_0^p \leq 1$, in view of system (19), we have (recall that $N_h := u_1 + u_2 + u_3$ and $N_v := u_4 + u_5 + u_6$)

\[
\begin{align*}
N_0^c(t) &\leq rK - (r + \mu_1)N_h(t), \\
N_0^c(t) &\geq \alpha(t)N_v(t) \left(1 - \frac{N_v(t)}{mN_h(t)}\right) - \gamma(t)N_v(t).
\end{align*}
\]

By the comparison argument and the proof in Proposition 1, we see that $N_v(t) \to 0$ as $t \to \infty$, and hence, $u_3$ in system (19) is asymptotic to $u_3(t) = -(\mu_1 + \delta_1 + \delta_2)u_3(t)$, which yields $u_3(t) \to 0$ as $t \to \infty$. Similarly, $u_2(t) \to 0$ as $t \to \infty$. Finally, $u_1(t)$ is asymptotic to $u_1(t) = rK - (r + \mu_1)u_1$, this implies $u_1(t) \to N_0^p$ as $t \to \infty$. 

Then the following statements hold:

(i) If there exists some \( \delta_1 = \delta_2 = 0 \) and \( R_0^p > 1 \), for any \( \phi \in X_\varepsilon(0) \) with \( N_\varepsilon(0) > 0 \), we have \((N_h(t), N_\varepsilon(t))\) satisfies system (31), then there holds \( \lim_{t \to \infty} \left( (N_h(t), N_\varepsilon(t)) - (N_h^*, S_\varepsilon^*) \right) = 0 \).

Since \( R_0 < 1 \), it follows from (3.1) and (3.4) that \( r(P(\omega)) < 1 \). By continuity, there exists a \( \rho > 0 \) such that \( r(P_\rho(\omega)) < 1 \), where \( P_\rho(\omega) : W \to W \) is the Poincaré map associated with the following system:

\[
\begin{align*}
    w_3'(t) &= \beta_1(t-\tau)e^{-\mu_1\tau}w_6(t-\tau) - \mu_1w_3(t), \\
    w_6'(t) &= (1 - \tau_\nu(t))\beta_2(t-\tau_\nu(t)) \frac{S_\varepsilon^*(t-\tau_\nu(t)) + \rho}{N_h^* - \rho}w_3(t-\tau_\nu(t))e^{-\beta_2(t-\tau_\nu(t))\gamma(\xi)d\xi} \\
    &\quad - \gamma(t)w_6(t).
\end{align*}
\]

For the above \( \rho \), there exists a positive integers \( M \) such that

\( N_\varepsilon^* - \rho < N_h(t) < N_h^* + \rho, \quad S_\varepsilon^* - \rho < N_\varepsilon(t) < S_\varepsilon^* + \rho \) for any \( t - \hat{\tau} \geq M\omega \).

From the fact \( \frac{\sum_{i=1}^{u_1(t-\tau)} - u_1(t-\tau)}{\sum_{i=1}^{u_3(t-\tau)} - u_3(t-\tau)} \leq 1 \), it is easy to see that

\[
    u_3'(t) \leq \beta_1(t-\tau)e^{-\mu_1\tau}u_6(t-\tau) - \mu_1u_3(t)
\]

and

\[
    \begin{align*}
        u_6'(t) &\leq (1 - \tau_\nu(t))\beta_2(t-\tau_\nu(t)) \frac{N_\varepsilon(t-\tau_\nu(t))}{N_h(t-\tau_\nu(t))}u_3(t-\tau_\nu(t))e^{-\beta_2(t-\tau_\nu(t))\gamma(\xi)d\xi} - \gamma(t)u_6(t), \\
        &\leq (1 - \tau_\nu(t))\beta_2(t-\tau_\nu(t)) \frac{C(t-\tau_\nu(t))}{N_h^* - \rho}u_3(t-\tau_\nu(t))e^{-\beta_2(t-\tau_\nu(t))\gamma(\xi)d\xi} - \gamma(t)u_6(t),
    \end{align*}
\]

for \( t - \hat{\tau} \geq M\omega \).

Let \( \mu_0 = \frac{\ln r(p_\rho(\omega))}{\omega} \). By Lemma 3.5, there exists a positive \( \omega \)-periodic function \( \omega^*_\rho(t) \) such that \( e^{\mu_0\omega^*_\rho(t)} \) is solution of (35) with the feasible domain \( C([-\hat{\tau}, 0], \mathbb{R}^2) \) for any \( t \geq 0 \). So we could choose a large \( K > 0 \), such that \( (u_3(t), u_6(t)) \leq Ke^{\mu_0\omega^*_\rho(t)} \) for any \( t \in [M\omega - \hat{\tau}, M\omega] \). According to the comparison theorem for delay differential equations, we get \( (u_3(t), u_6(t)) \leq Ke^{\mu_0\omega^*_\rho(t)} \) for any \( t - \hat{\tau} \geq M\omega \). Since \( \mu_\rho < 0 \), it follows that \( \lim_{t \to \infty} (u_3, u_6) = (0, 0) \). In view of the compatibility conditions (20) and (21), we could directly argue that \( (u_2, u_4) \to (0, 0) \) as \( t \to \infty \).

Finally, Since \( \lim_{t \to \infty} \left( (N_h(t), N_\varepsilon(t)) - (N_h^*, S_\varepsilon^*) \right) = 0 \), it immediately follows that

\[
    \lim_{t \to \infty} \left( u_1(t), u_4(t) \right) - (N_h^*, S_\varepsilon^*) = 0.
\]

In the remaining of this section, we investigate the uniform persistence of HLB.

In order to give the uniform persistence of the full system, we have the following assumption:

(A) \( \gamma(t) > \mu_1 + \delta_1 + \delta_2 \) for any \( t \in [0, \omega] \).

**Lemma 4.2.** Let \( u(t, \phi) \) be the unique solution of system (19) with \( u_0 = \phi \in X_\varepsilon(0) \). Then the following statements hold:

(i) If there exists some \( t_0 \geq 0 \) such that \( u_i(t_0, \phi) \neq 0 \) for some \( i \in \{3, 6\} \), then \( u_i(t, \phi) > 0 \) for any \( t > t_0 \).

(ii) Assume that (A) is valid. Then there exists a positive number \( A_0 \in (0, 1) \) and \( T_0 > 0 \) such that \( N_\varepsilon(t) \leq A_0N_\varepsilon(t) \) and \( N_h(t) \leq A_0K \) for any \( t \geq T_0 \).

(iii) If \( \phi_i(0) \neq 0 \) for \( i = 3, 6 \), then \( u_i(t, \phi) > 0 \) for any \( t > T_0, \; 1 \leq i \leq 6 \), where \( T_0 \) is given in (ii).

(iv) There exists some \( \rho > 0 \) such that \( \liminf_{t \to \infty} u_1(t) \geq \rho \).
(v) Assume that (A) is valid, $R_0^p > 1$ and $\phi_i(0) \neq 0$ for $i = 3, 6$. If there exists a $\rho^* > 0$ such that $\liminf_{t \to \infty} u_i(t, \phi) \geq \rho^*$ for $i = 3, 6$, then there exists a $\rho_0 \in (0, \min(\rho, \rho^*))$ such that $\liminf_{t \to \infty} u_i(t, \phi) \geq \rho_0$ for any $1 \leq i \leq 6$.

Proof. (i) The positivity of solutions (see Theorem 2.1) implies that $u_3(t), u_6(t)$ satisfy
\[ u_3'(t) \geq -\mu_1 u_3(t), \quad u_6'(t) \geq -\gamma(t) u_6(t). \tag{36} \]
If there exists some $t_0 > 0$ such that $u_i(t_0, \phi) = 0$ for some $i \in \{3, 6\}$, the standard comparison principle for the scalar equations gives that $u_i(t, \phi) > 0$ for any $t > t_0$.

(ii) Let $x(t) = \frac{N_{2}(t)}{N_{6}(t)}$. Note that $N_{2}(t) \leq r(K - N_{2}) - (\mu_1 + \delta_1 + \delta_2)N_2 \geq -(\mu_1 + \delta_1 + \delta_2)N_{2}$.

\[ x'(t) \leq \alpha(t)x(t) \left(1 - \frac{x(t)}{m}\right) - (\gamma(t) - \mu_1 - \delta_1 - \delta_2)x(t). \]

In view of (A), there exists some $a_0 \in (0, 1)$ such that $(1 - a_0)\alpha(t) - (\gamma(t) - \mu_1 - \delta_1 - \delta_2) < 0$ for any $t \in [0, \omega]$. Now one could directly verify that for any $a \in [a_0, \infty)$, $a m$ is an upper solution of
\[ y'(t) = \alpha(t)y(t) \left(1 - \frac{y(t)}{m}\right) - (\gamma(t) - \mu_1 - \delta_1 - \delta_2)y(t), \]
we can infer that
\[ \limsup_{t \to \infty} x(t) \leq \limsup_{t \to \infty} y(t) \leq a_0 m < m. \]
Since $\limsup_{t \to \infty} N_{2}(t) \leq N_{6}^* < K$, it then follows that there exist some $T_0 > 0$ and $A_0 \in \left(\max\{a_0, \frac{r}{r+\mu_1}\}, 1\right)$ such that $x(t) \leq A_0 m$ and $N_{2}(t) \leq A_0 K$ for any $t \geq T_0$.

(iii) By (i), it is easy to see that $u_i(t, \phi) > 0$ holds for $i = 3, 6$. We will show that $u_1(t, \phi) > 0$ for any $t \geq T_0$. Assume, by contradiction, that there exists a $t_1 \geq T_0$ such that $u_1(t_1, \phi) = 0$. Then $u_1'(t_1, \phi) = 0$ and the first equation in (19) ensures that
\[ r(K - N_{2}(t_1)) = 0. \]
which contradicts with statement (ii). And hence $u_i(t, \phi) > 0$ for any $t \geq T_0$. Similarly, we can show that $u_i(t, \phi) > 0$ for any $t \geq T_0$. By (20) and (21), it is easy to see that $u_i(t, \phi) > 0$ holds for $i = 2, 5$. We complete the proof of (iii).

(iv) By Theorem 2.1 and Statement (ii), we know there exists $T_1 > 0$ such that $N_{2}(t) \leq 2mK$ and $N_{6}(t) \leq A_0 K$ for any $t \geq T_1$, and hence, we have
\[ u_1'(t) \geq r(1 - A_0)K - (2\beta(t)mK + m)u_1(t), \quad t \geq T_1. \]
It then follows that there exists $\rho > 0$ such that $\liminf_{t \to \infty} u_1(t) \geq \rho$.

(v) Since there exists a $\rho^* > 0$ such that $\liminf_{t \to \infty} u_6(t, \phi) \geq \rho^*$, we see that there exists a $T_1 > T_0$ such that
\[ u_6(t, \phi) \geq \frac{1}{2}\rho^*, \quad t \geq T_1. \]
From the fourth equation of system (19) and by Statement (ii), it follows that
\begin{align*}
\frac{du_4(t)}{dt} & \geq \alpha(t)N_{6}(t)(1 - A_0 m) - (\beta_2(t) + \gamma(t))u_4(t) \\
& \geq \alpha(t)u_6(t)(1 - A_0 m) - (\beta_2(t) + \gamma(t))u_4(t) \\
& \geq \alpha(t)\frac{1}{2}\rho^*(1 - A_0 m) - (\beta_2(t) + \gamma(t))u_4(t), \tag{37}
\end{align*}
for all $t \geq T_1$. It then follows that there exists $\rho^*_1$ such that $\liminf_{t \to \infty} u_4(t) \geq \rho^*_1$. By the above discussions together with the integral forms (20) and (21), it follows that there exists $\rho^*_2$ such that $\liminf_{t \to \infty} u_i(t) \geq \rho^*_2$, $i = 2, 5$. Letting $\rho^*_0 = \min\{\rho, \rho^*, \rho^*_1, \rho^*_2\} > 0$, we then obtain the desired result.

**Theorem 4.3.** Assume that (A) holds. If $R^p_0 > 1$ and $R_0 > 1$, then there is a $\varrho > 0$ such that any solution $u(t, \phi)$ of system (19) with $u_0 = \phi \in X_\epsilon(0), \phi_3(0) \neq 0$, and $\phi_6(0) \neq 0$ satisfies $\liminf_{t \to \infty} u_i(t, \phi) \geq \varrho$, $\forall i \leq 6$.

In addition, if $\delta_1 = \delta_2 = 0$, then system (19) admits a positive $\omega$-periodic solution $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t))$. 

**Proof.** Let $\mathbb{C} = X_\epsilon(0), \mathbb{C}_0 = \{\phi \in \mathbb{C} : \phi_3(0) \neq 0$ and $\phi_6(0) \neq 0\}$ and 

$$\partial \mathbb{C}_0 := \mathbb{C} \setminus \mathbb{C}_0 = \{\phi \in \mathbb{C} : \phi_3(0) = 0$ or $\phi_6(0) = 0\}.$$ 

For any $\phi \in \mathbb{C}_0$, it follows from Lemma 4.2 that $u_i(t, \phi) > 0$ for any $t > 0, i = 3, 6$. In other words, $Q(\omega)^n\mathbb{C}_0 \subset \mathbb{C}_0$ for any $n \in \mathbb{N}$. Moreover, it follows from (2.1) that $Q(\omega)$ admits a strong global attractor in $\mathbb{C}$. Define 

$$\mathcal{M}_0 = \{\phi \in \partial \mathbb{C}_0 : Q(\omega)^n\phi \in \partial \mathbb{C}_0$ for any $n \in \mathbb{N}\},$$ 

$$\tilde{\omega}(\psi)$$ be the omega limit set of the forward orbit $\Gamma = \{Q(\omega)^n\psi : n \in \mathbb{N}\}$, and 

$$\mathcal{M}_0 = \{(N^*_h, 0, 0, 0, 0, 0)\}, \quad \mathcal{M}_1 = \{(N^*_h, 0, 0, S^*_h(0), 0, 0)\}.$$

**Claim 1.** For any $\psi \in \mathcal{M}_0$, $\tilde{\omega}(\psi) = \mathcal{M}_0 \cup \mathcal{M}_1$ and $\mathcal{M}_1$ is locally stable in $\mathcal{M}_0$.

For any given $\psi \in \mathcal{M}_0$, we can obtain that $Q(\omega)^n(\psi) \in \partial \mathbb{C}_0, \forall n \in \mathbb{N}$. Consequently, it follows that for each $n \in \mathbb{N}$, $u_3(n\omega, \varphi) = 0$ or $u_6(n\omega, \psi) = 0$. By Lemma 4.2, either $u_5(t, \psi) \equiv 0$ or $u_6(t, \psi) \equiv 0$ for any $t \geq 0$.

In the case that $u_3(t, \psi) = 0$ for each $t \geq 0$, we see from the third equation of (19) that $u_1(t - \tau)u_6(t - \tau) = 0$ for any $t > 0$. By Lemma 4.2, we know there exists $T_1 > 0$, such that $u_1(t) > 0$ for any $t \geq T_1$, and hence $u_6(t) = 0$ for any $t \geq T_1$. By the similar argument to that in Proposition 1, we have $\tilde{\omega}(\psi) = \mathcal{M}_0 \cup \mathcal{M}_1$.

In the case that $u_6(t, \psi) = 0$ for any $t \geq 0$, $\lim_{t \to \infty} u_3(t, \psi) = 0$ from the second equation of (19). Moreover, it follows from the third equations of system (19) that $\lim_{t \to \infty} u_4(t, \psi) = 0$. Thus, $u_5$ is asymptotic to $u_4^*(t) = -\gamma(t)u_5(t)$, this implies $\lim_{t \to \infty} u_5(t, \psi) = 0$. In view of the theories of asymptotically periodic semiflows and internally chain transitive sets, it follows that $\lim_{t \to \infty} u_1(t, \psi) = N^*_h$ and $\lim_{t \to \infty} (u_4(t, \psi) - S^*_h(t)) = 0$ or $\lim_{t \to \infty} u_4(t, \psi) = 0$. This proves the first part of Claim 1.

It remains to verify the local stability of $\mathcal{M}_1$. It suffices to check the case that $u_3(t) \equiv 0$ for any $t \geq 0$, then we consider the following system which is equivalent to system (19) with $u_3 \equiv 0$

$$\begin{cases} 
    u_1(t) = r(K - N_h) - \beta_1(t)\frac{u_1(t)u_6(t)}{N_h(t)} - \mu_1 u_1(t), \\
    N_h(t) = rK - (r + \mu_1)N_h(t), \\
    u_4(t) = \alpha(t) \left(1 - \sum_{i=4}^{6} \frac{u_i(t)}{mN_h(t)}\right) \sum_{i=4}^{6} u_i(t) - \gamma(t) u_4(t), \\
    u_5(t) = -\gamma(t)u_5(t), \\
    u_6(t) = -\gamma(t)u_6(t). 
\end{cases} \quad (38)$$
Linearizing (38) at \((N^*_h, N^*_h, S^*_v(t), 0, 0)\), we have
\[
\begin{align*}
\dot{w}_1'(t) &= -rw_2(t) - \beta_1(t)w_5(t) - \mu_1w_1(t), \\
\dot{w}_2'(t) &= -(r + \mu_1)w_2(t), \\
\dot{w}_3'(t) &= \alpha(t)\frac{S^*_v(t)}{mN^*_h}w_2(t) + \alpha(t)\left(1 - \frac{2S^*_v(t)}{mN^*_h(t)}\right)\sum_{i=4}^6 w_i(t) - \gamma(t)w_3(t), \\
\dot{w}_4'(t) &= -\gamma(t)w_4(t), \\
\dot{w}_5'(t) &= -\gamma(t)w_5(t).
\end{align*}
\]
One could directly check that its exponential growth bound is negative (see e.g., [27, Lemma 7]), and hence, it is the locally stable. Claim 1 is verified.

We easily get that
\[
\begin{align*}
&D_s > 0, \\
&\forall t \in [0, \omega],
\end{align*}
\]
which implies (39).

We see that \(\lim_{\delta \to 0} r(P(\omega)) = r(P(\omega)) > 1\) by continuity. Thus, Choose a sufficiently small number \(\delta > 0\) such that
\[
\delta < \min\{N^*_h, \min_{t \in [0, \omega]} S^*_v(t)\} \text{ and } r(P(\omega)) > 1.
\]
Now the continuous dependence of solutions on the initial value implies for the chosen \(\delta > 0\), there exists \(\delta^* > 0\) such that \(\|Q(t)\phi - Q(t)M_1\| < \delta\) for any \(t \in [0, \omega]\) provided \(\|\phi - M_1\| \leq \delta^*\). Next we prove the following claim.

Claim 2. For all \(\phi \in C_0\), there holds \(\lim \sup_{n \to \infty} \|Q(\omega)^n(\phi) - M_1\| \geq \delta^*\).

Assume that \(\lim \sup_{n \to \infty} \|Q(\omega)^n(\phi_0) - M_1\| < \delta^*\) for some \(\phi_0 \in C_0\) by contradiction. Then there exists \(n_1 \geq 1\) such that \(\|Q^n(\omega)(\phi) - M_1\| < \delta^*\) for \(n \geq n_1\). For any \(t \geq n_1\omega\), letting \(n = [t/\omega]\) and \(t' \in [0, \omega]\), we have
\[
\|Q(t)\phi_0 - Q(t)M_1\| = \|Q(t')(Q^n(\omega)(\phi_0)) - Q(t')M_1\| < \delta.
\]
We easily get that \(Q(t)M_1 = (N^*_h, 0, 0, S^*_v(t), 0, 0)\). Then it follows from (40) that for any \(t \geq n_1\omega - \tau\), we have \(0 < u_j(t, \phi_0) < \delta, j = 2, 3, 5, 6\) and \(N^*_h - \delta < u_1(t, \phi_0) < N^*_h + \delta, u_4(t, \phi_0) > S^*_v(t) - \delta\). Thus, the equations of \(u_3(t, \phi_0)\) and \(u_6(t, \phi_0)\) in (19) satisfy
\[
\begin{align*}
\dot{u}_3'(t) &= \beta_1e^{-\mu_1r}N^*_h - \mu_1u_3(t), \\
\dot{u}_6'(t) &= (1 - \tau'_1(t))\beta_2(t - \tau_0(t))u_3(t - \tau_0(t))e^{-\int_{t-\tau_0(t)}^{t}(d_*(\xi) + \theta(\xi))d\xi} - \gamma(t)u_6(t),
\end{align*}
\]
for \(t \geq n_1\omega\). Let \(\mu_\delta = \frac{\ln r(P(\omega))}{\omega}\). Then it follows from (3.5) that there exists a positive \(\omega\)-periodic function \(w^*_\delta\) such that \(e^{\mu_\delta t}w^*_\delta\) is a solution of (4) with feasible domain \(C([-\hat{\tau}, 0], \mathbb{R}^2), \forall t \geq 0\). In view of (4.2), we see that \((u_3(t, \phi_0), u_6(t, \phi_0)) \gg (0, 0)\). Thus, we may choose a \(U_\delta > 0\) such that
\[
(u_3(t, \phi_0), u_6(t, \phi_0)) \gg U_\delta e^{\mu_\delta t}w^*_\delta, \quad \forall t \in [n_1\omega - \hat{\tau}, n_1\omega].
\]
By the comparison theorem for delay differential equations, it implies that
\[
(u_3(t, \phi_0), u_6(t, \phi_0)) \gg U_\delta e^{\mu_\delta t}w^*_\delta, \quad \forall t \geq n_1\omega.
\]
Since \( \mu_3 > 0 \), it follows that \( u_i(t, \phi_0) \to \infty \) for any \( i = 3, 6 \) as \( t \to \infty \). This leads to a contradiction.

We also have the following statement which is similar to the argument in Claim 2.

**Claim 3.** There exists a \( \delta_1^* > 0 \) such that \( \limsup_{n \to \infty} \| Q(\omega)^n(\phi) - M_0 \| \geq \delta_1^* \) for any \( \phi \in \mathbb{C}_0 \).

The above claims indicate that \( M_i, i = 0, 1 \) cannot form a cycle for \( Q(\omega) \) in \( \mathbb{C} \), and \( W^s(M_i) \cap \mathbb{C}_0 = \emptyset \), where \( W^s(M_i) \) is the stable set of \( M_i \) for \( Q(\omega) \). It follows from [18, Theorem 3.7] that \( Q(\omega) \) has a global attractor \( \mathbb{S}_0 \) in \( \mathbb{C}_0 \). By [32, Theorem 1.3.1], we can get that \( Q(\omega) \) is uniformly persistent in regard to \( (\mathbb{C}_0, \partial \mathbb{C}_0) \) in terms of existing \( \bar{\sigma} > 0 \) such that

\[
\liminf_{n \to \infty} d(Q^n(\phi), \partial \mathbb{C}_0) \geq \bar{\sigma}, \quad \forall \phi \in \mathbb{C}_0.
\]

Since \( S_0 = Q(\omega)S_0 \), we have \( \phi_i(0) > 0, \forall \phi \in S_0, i = 3, 6 \). Let \( Y_0 := \bigcup_{t \in [0, \omega]} Q(\omega)S_0 \). Then \( Y_0 \subset \mathbb{C}_0 \) and \( \lim_{t \to \infty} d(Q(t)\phi, Y_0) = 0 \) for all \( \phi \in \mathbb{C}_0 \). Define a continuous function \( q : \mathbb{C} \to \mathbb{R}_+ \) by

\[
q(\phi) = \min\{\phi_3(0), \phi_6(0)\}, \quad \forall \phi \in \mathbb{C}.
\]

Since \( Y_0 \) is compact subset of \( \mathbb{C}_0 \), it follows that \( \inf_{\phi \in Y_0} = \min_{\phi \in Y_0} q(\phi) > 0 \). Then there exists a \( \sigma^* > 0 \) such that

\[
\liminf_{t \to \infty} q(Q(t)\phi) = \liminf_{t \to \infty} \min\{u_3(t, \phi), u_6(t, \phi)\} \geq \sigma^*, \forall \phi \in \mathbb{C}_0.
\]

Furthermore, by Lemma 4.2 (v), there exists a \( \varrho \in (0, \sigma^*) \) such that

\[
\liminf_{t \to \infty} u_i(t, \phi) \geq \varrho, \quad \forall \phi \in \mathbb{C}_0(1 \leq i \leq 6).
\]

In the remaining, we show the existence of positive periodic solution in the case that \( \delta_1 = \delta_2 = 0 \).

Note that when \( R_0^p > 1 \), for any \( \phi \in X_\epsilon(0) \) with \( N_\epsilon(0) > 0 \), we have \( (N_h(t), N_v(t)) \) satisfies system (31), then there holds \( \lim_{t \to \infty} \left( (N_h(t), N_v(t)) - (N_h^*, S_v^*(t)) \right) = 0 \).

Now we could look at the following limiting system

\[
\begin{align*}
\dot{u}_1'(t) &= r(K - N_h^*) - \beta_1(t)\frac{u_1(t)u_6(t)}{N_h^*} - \mu_1u_1(t), \\
\dot{u}_3'(t) &= \beta_1(t - \tau)e^{-\mu_1\tau}u_1(t - \tau)\frac{u_6(t - \tau)}{N_h^*} - \mu_1u_3(t), \\
\dot{u}_4'(t) &= \alpha(t) \left( 1 - \frac{S_v^*(t)}{mN_h^*} \right) S_v^*(t) - \beta_2(t)\frac{u_3(t)u_4(t)}{N_h^*} - \gamma(t)u_4(t), \\
\dot{u}_6'(t) &= (1 - \tau_v'(t))\beta_2(t - \tau_v'(t))\frac{u_3(t - \tau_v(t))u_4(t - \tau_v(t))}{N_h^*} e^{-k_1^*\tau_v(t)} - \gamma(t)u_6(t).
\end{align*}
\]

Set \( Y := C([-\hat{\tau}, 0], \mathbb{R}_+^k) \), \( Y_0 = \{ \phi \in Y : \phi_3(0) \neq 0 \) and \( \phi_6(0) \neq 0 \} \) and \( \partial Y_0 = Y \setminus Y_0 \). In view of Theorem 2.1 (see, e.g., [32, Page 299]), we could infer that system (41) admits a unique non-negative solution \( w(t, \phi) \) with \( \phi \in Y \) for any \( t \in [0, \infty) \) and \( w(t, \phi) \) are ultimately bounded. Now define \( \tilde{Q}(t) : Y \to Y \) by \( \tilde{Q}(t) = w(t, \phi) \) for any \( t \geq 0 \). When \( R_0 > 1 \), we could perform similar analysis to that in the above, it then follows that \( \tilde{Q}(\omega) \) is uniformly persistent with respect to \( (Y_0, \partial Y_0) \). Note that for any \( n \) with \( nw \geq \hat{\tau} \), \( \tilde{Q}(nw) \) is compact. By [32, Theorem 3.5.1], \( \tilde{Q}(t) \) is an \( \alpha \)-contraction with respect to an equivalent norm in \( C([-\hat{\tau}, 0], \mathbb{R}^k) \) for any \( t > 0 \).
Table 2. Parameter values in simulation

| Parameter | Value | Unit      | Reference |
|-----------|-------|-----------|-----------|
| $r$       | 0.05  | month$^{-1}$ | [17]      |
| $K$       | 2000  | -         | [17]      |
| $\beta_1(t)$ | to be estimated | month$^{-1}$ | see text |
| $\mu_1$   | 0.00333 | month$^{-1}$ | [29]      |
| $\delta_1$| 0.0015 | month$^{-1}$ | [25]      |
| $\delta_2$| 0.02   | month$^{-1}$ | [25]      |
| $\alpha(t)$ | $18.120952 + 14.45466475 \cos(2\pi t/12)$ | month$^{-1}$ | [29]      |
| $m$       | $1 \times 10^6$ | -         | [29]      |
| $d_v(t)$  | to be estimated | month$^{-1}$ | see text |
| $\beta_2(t)$ | to be estimated | month$^{-1}$ | see text |
| $\theta(t)$ | to be estimated | month$^{-1}$ | see text |
| $\tau$    | 6     | month     | [25]      |
| $\tau_v(t)$ | assumed | month     | see text |

Table 3. Monthly mean temperature in the South of Jiangxi, 1961-2016($^\circ$C)

| Month | Jan | Feb | Mar | Apr | May | June |
|-------|-----|-----|-----|-----|-----|------|
| temperature | 5.7 | 7.5 | 11.6 | 17.3 | 22.4 | 25.0 |

| Month | Jul | Aug | Sep | Oct | Nov | Dec |
|-------|-----|-----|-----|-----|-----|-----|
| temperature | 28.3 | 27.6 | 24.5 | 19.2 | 13.2 | 7.4 |

Now [32, Theorem 1.3.10] implies $\tilde{Q}(\omega) : Y_0 \rightarrow Y_0$ admits a fixed point $\phi^* \in Y_0$. It then follows that $w(t, \phi^*) = (u_1^*(t), u_2^*(t), u_3^*(t), u_6^*(t))$ is an $\omega$-periodic solution of system (41) with $u_i^*(t) > 0$, $\forall t \geq 0$, $i = 3, 6$. Moreover, in view of equations for $u_1$ and $u_4$, we obtain that $u_1^*(t) > 0$ and $u_4^*(t) > 0$ thanks to $K > N_h^*$ and $1 - S^*_v(t)/mN_h^* > 0$ for any $t \geq 0$.

Finally, set

$$u_2^*(t) = \int_{t-\tau}^{t} \beta_1(s) e^{-\mu_1(t-s)} \frac{u_1^*(s)u_6^*(s)}{N_h^*} \, ds$$

and

$$u_5^*(t) = \int_{t-\tau_v(t)}^{t} \beta_2(s) \frac{u_4^*(s)u_6^*(s)}{N_h^*} e^{-\int_s^t \gamma(\xi) \, d\xi} \, ds.$$

It is easy to see that both are positive and $\omega$-periodic. And hence, $(u_1^*(t), u_2^*(t), u_3^*(t), u_4^*(t), u_5^*(t), u_6^*(t))$ is a positive $\omega$-periodic of system (19) with $\sum_{i=1}^{3} u_i^*(t) = N_h^*$ and $\sum_{i=4}^{6} u_i^*(t) = S^*_v(t)$.

5. **Numerical simulations.** In this section, we study how temperature affects the dynamics of HLB transmission in the South of Jiangxi, China. Values for parameters in model (18) are listed in Table 2, and Table 3 shows monthly mean temperature in Jiangxi Province from 1961 to 2016 [16]. By [25], the temperature-dependent natural death rates of psyllid population can be expressed as

$$d_v(T) = \frac{365}{12(0.14221 \times T^2 + 4.31998 \times T + 31.25498)}.$$
where $T$ is the temperature in °C. Then the natural death rate of psyllid can be fitted by

$$d_v(t) = 4.54 + 0.3748 \cos\left(\frac{\pi t}{6}\right) + 0.3615 \sin\left(\frac{\pi t}{6}\right) - 0.1964 \cos\left(\frac{\pi t}{3}\right) - 0.0681 \sin\left(\frac{\pi t}{3}\right) - 0.0942 \cos\left(\frac{\pi t}{2}\right) - 0.07901 \sin\left(\frac{\pi t}{2}\right) + 0.03812 \cos\left(\frac{2\pi t}{3}\right) - 0.02288 \sin\left(\frac{2\pi t}{3}\right) - 0.6966 \cos\left(\frac{5\pi t}{6}\right) - 0.003061 \sin\left(\frac{5\pi t}{6}\right).$$

From [29], we obtain $\beta_1(t) = a_0 + p \cos\left(\frac{\pi t}{6}\right), \beta_2(t) = a_1 + q \cos\left(\frac{\pi t}{6}\right)$, where $a_0 = \frac{1}{12} \int_0^{12} \beta_1(t)dt$ and $a_1 = \frac{1}{12} \int_0^{12} \beta_2(t)dt$. Moreover, we assume

$$\tau_v(t) = 0.47(1 + 0.5 \cos\left(\frac{\pi t}{6}\right)).$$

According to some data from [15], we fit the curve and obtain

$$\theta(T) = -0.003551T^3 + 0.1592T^2 - 2.34T + 12.11.$$  

Then the estimation of $\theta(t)$ can be written as

$$\theta(t) = 0.01 \times (3.723 + 0.4814 \cos\left(\frac{\pi t}{6}\right) + 3.215 \sin\left(\frac{\pi t}{6}\right) - 0.5598 \cos\left(\frac{\pi t}{3}\right) + 1.593 \sin\left(\frac{\pi t}{3}\right) - 0.7504 \cos\left(\frac{\pi t}{2}\right) + 0.9303 \sin\left(\frac{\pi t}{2}\right) - 0.8139 \cos\left(\frac{2\pi t}{3}\right) + 0.5391 \sin\left(\frac{2\pi t}{3}\right) - 0.8389 \cos\left(\frac{5\pi t}{6}\right) + 0.2506 \sin\left(\frac{5\pi t}{6}\right)).$$

Then

$$\gamma(t) = \theta(t) + d_v(t)$$

$$= 0.01 \times (8.263 + 0.8562 \cos\left(\frac{\pi t}{6}\right) + 3.5765 \sin\left(\frac{\pi t}{6}\right) - 0.7562 \cos\left(\frac{\pi t}{3}\right) + 0.9112 \sin\left(\frac{\pi t}{3}\right) - 0.8446 \cos\left(\frac{\pi t}{2}\right) + 0.8512 \sin\left(\frac{\pi t}{2}\right) - 0.6175 \cos\left(\frac{2\pi t}{3}\right) + 0.51622 \sin\left(\frac{2\pi t}{3}\right) + 1.5355 \cos\left(\frac{5\pi t}{6}\right) + 0.247539 \sin\left(\frac{5\pi t}{6}\right)).$$

Now we will make numerical simulations to illustrate our theoretical results. We set initial value as $S_h(\theta) = 500, E_h(\theta) = 1, I_h(\theta) = 0, S_v(\theta) = 1 \times 10^3, I_v(\theta) = 100, E_v(\theta) = 0, \forall \theta \in [-\hat{\tau}, 0]$. By the algorithm [12], we can compute $R_0$ numerically and simulate the long-term behaviors of the disease transmission, where we can see that the number of psyllid presents an obvious seasonal fluctuation. We observe that the tendency of the infected citrus trees and infected psyllid, the disease will disappear when $R_0 = 0.5657 < 1$ (Figures 1(a) and 2(a)), which is consistent with Theorem 4.1. Meanwhile, the disease will persist when $R_0 = 5.6575 > 1$.

Moreover, from Figures 1(b) and 2(b), we can see that the epidemic of HLB increased rapidly before 25 months, while there is tendency to a stable periodic solution in a long time when $R_0 > 1$.

Next, we consider the combined effects of $\beta_1(t)$ and $\beta_2(t)$ on $R_0$. The increasing contact rate leads to more infected psyllids (Figure 3) and hence makes the HLB transmission more intensive. Figure 4 shows the relationship between $r$ and $R_0$. From Figure 4, we observe that $R_0$ decreases as $r$ increases in sense that increasing recruitment rate of citrus tree can control HLB.
(a) $R_0 = 0.5657 < 1$  
(b) $R_0 = 5.6575 > 1$

**Figure 1.** Tendency of citrus trees when $R_0 < 1$ and $R_0 > 1$

(a) $R_0 = 0.5657 < 1$  
(b) $R_0 = 5.6575 > 1$

**Figure 2.** Behaviors of psyllid when $R_0 < 1$ and $R_0 > 1$

(a) $q = 0.4838437$  
(b) $p = 0.3543$

**Figure 3.** Infectious psyllid under different $p$ and $q$
6. Discussion. In this paper, we have formulated and analyzed a HLB transmission model with periodic delay in periodic environments. When $R_0 > 1$, it is proved that $R_0$ is the threshold for distinguishing the disease extinction or permanence. The disease-free periodic solution is globally asymptotically stable and the disease dies out when $R_0 < 1$. When $R_0 > 1$, the disease persists. Our numerical simulations indicate that a decrease in the infection rate and replanting rate of the citrus trees could reduce the value of $R_0$. Control strategies focus mainly on the eradication of psyllid such as rouging infected trees and spraying insecticide over entire grove. On the other hand, our numerical results suggest that there exists a positive stable periodic solution when $R_0 > 1$ (see, Figures 1-2), we only prove the existence for the case that $\delta_1 = \delta_2 = 0$ in Theorem 4.3. For the general case, because our feasible domain $X_r(0)$ in (22) is non-convex, we could not directly utilize [32, Theorem 1.3.10] to verify the existence of the positive periodic solution, we will leave the general existence and its stability for the further investigation.

Our model takes account into the influence of seasonal changes on population dynamics. However, it is difficult for natural environment changes to show accurate periodicity. The almost periodic function is a generalization of periodic function, which is more suitable for describing natural periodic fluctuations. The changes described by the quasi periodic function are approximately periodic, but not necessarily accurate periodic changes. Each cycle change is not necessarily a common cycle. In particular, if the coefficients of a periodic system have no common period, the model becomes an almost periodic system. Therefore, we can consider using the almost periodic model to study the population development of psyllid population and the spread of Huanglong disease on the basis of the original model.

In fact, studies have shown that in different citrus planting areas, the use of natural or artificial forests for ecological isolation can control the spread of psyllid to a certain extent, and provide a living environment for the natural enemies of psyllid. In addition, biological control can effectively manage the field population density of psyllid. We can explore how natural enemies affect the spread of HLB for future investigation. Furthermore, we could follow the idea in [2] to introduce the distributed incubation delays and study the global dynamics of the model system.

![Figure 4. $R_0$ vs $r$](image-url)
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