On the Navier-Stokes Equations with Stochastic Lie Transport

Daniel Goodair        Dan Crisan

November 3, 2022

Abstract

We prove the existence and uniqueness of maximal solutions to the 3D SALT (Stochastic Advection by Lie Transport, [28]) Navier-Stokes Equation in velocity and vorticity form, on the torus and the bounded domain respectively. The current work partners the paper [27] as an application of the abstract framework presented there, justifying the results first announced in [26]. In particular this represents the first well-posedness result for a fluid equation perturbed by a general transport type noise on a bounded domain.

Contents

Introduction

1 Preliminaries
  1.1 Elementary Notation ......................................................... 2
  1.2 Stochastic Framework ..................................................... 4
  1.3 Functional Framework .................................................... 5
  1.4 The SALT Operator ....................................................... 13

2 Analysis of the Velocity Equation ........................................ 16
  2.1 The Itô Form .............................................................. 16
  2.2 Existence, Uniqueness and Maximality ................................ 17

3 Analysis of the Vorticity Equation ....................................... 28
  3.1 Deriving the Equation ................................................... 28
  3.2 Existence, Uniqueness and Maximality ................................ 31

4 Appendices
  4.1 Appendix I: Proofs from Subsection 1.3 ............................. 35
  4.2 Appendix II: Proofs from Subsection 1.4 ............................ 41
  4.3 Appendix III: A Conversion from Stratonovich to Itô ............ 51
  4.4 Appendix IV: Abstract Solution Criterion, Part I ................ 52
  4.5 Appendix V: Abstract Solution Criterion, Part II ................ 55
Introduction

The theoretical analysis of Stochastic Navier-Stokes Equations dates back to the work of Bensoussan and Temam in 1973, where the problem of existence of solutions is addressed in the presence of a random forcing term. The well-posedness question for additive and multiplicative noise has since seen significant developments, for example through the works and references therein. The interest in this problem has seen developments into analytical properties of these solutions, particularly along the lines of ergodicity, which can be seen in references. In the present work our concern is the Navier-Stokes Equations with Stochastic Lie Transport, derived through the principle of Stochastic Advection by Lie Transport (SALT) introduced in. We consider the equation

\[ u_t - u_0 + \int_0^t L_{u_s} u_s ds - \nu \int_0^t \Delta u_s ds + \int_0^t B u_s \circ dW_s + \int_0^t \nabla \rho_s ds = 0 \]  \hspace{1cm} (1)

where \( u \) represents the fluid velocity, \( \rho \) the pressure, \( W \) is a Cylindrical Brownian Motion, \( L \) represents the nonlinear term and \( B \) is a first order differential operator (the SALT Operator) formally addressed in Subsection 1.4. Intrinsic to this stochastic methodology is that \( B \) is defined relative to a collection of functions \( (\xi_i) \) which physically represent spatial correlations. These \( (\xi_i) \) can be determined at coarse-grain resolutions from finely resolved numerical simulations, and mathematically are derived as eigenvectors of a velocity-velocity correlation matrix (see references). We pose the equation (1) in \( N \) dimensions for \( N = 2, 3 \) and impose the divergence free constraint on \( u \). We shall consider the problem both over the torus \( \mathbb{T}^N \) and a smooth bounded domain \( \mathcal{O} \subset \mathbb{R}^N \). In the case of the torus we supplement the equation with the zero-average condition (as is classical), whilst for the bounded domain we impose the boundary condition

\[ u \cdot n = 0, \quad w = 0 \]  \hspace{1cm} (2)

where \( n \) represents the outwards unit normal at the boundary, and \( w \) the fluid vorticity. These are the so called Lions boundary conditions, considered in and shown to be a particular case of the Navier boundary conditions in (note that this is done in 2D and our analysis will take place in 3D, though with some clarification it continues to hold in 2D). The significance of such a boundary condition is well documented in that work by Kelliher, and can be seen in other works such as where the boundary layer is explicitly addressed. The precise mathematical interpretation of these conditions, and the operators of (1), are explicated in Subsection 1.3. A complete derivation of this equation can be found in.

This work continues the theoretical development of fluid models perturbed by a transport type noise, which has seen significant developments since the seminal works. This paper partners that of where we showed the existence and uniqueness of maximal solutions to Stochastic Partial Differential Equations satisfying an abstract framework, built to cope with a general transport type noise as we see in (1). The significance of such equations in modelling, numerical schemes and data assimilation is reviewed there, along with the theoretical developments of these equations. We only draw particular attention here to the Navier-Stokes Equations, and results on a bounded domain. The Navier-Stokes Equations have been studied with transport type noise, for example in the works, though typically solutions are analytically weak and where strong solutions are considered major concessions in the noise are made. In these cases a cancellation property is evident in the noise term, so that in energy methods the differential operator is not felt. These difficulties have been addressed on the torus, in the likes of the papers and those further
addressed in [27], but extending a control of this noise term to a bounded domain remains open. Indeed the situation becomes more complex in the presence of viscosity, as energy methods require non-standard Sobolev inner products to conduct the required integration by parts in the bounded domain, rendering control on the noise terms completely out of familiarity. The problem of analytically strong solutions to fluid equations perturbed by a transport type noise in the bounded domain has been considered in [5], though the authors assume that the gradient dependency is of a small enough size to be directly controlled and that the noise terms are traceless under Leray Projection; such assumptions are designed to circumvent the technical difficulties of a first order noise operator on a bounded domain. Our results of Section 3 pertaining to a bounded domain thus represent a substantial improvement on the literature.

The goal of this paper is to apply the abstract framework established in [27], together providing a rigorous justification of the results first announced in [26] and extending them to the vorticity form of equation (1) on a bounded domain. In Section 1 we establish the stochastic and functional framework necessary to understand (1), along with fundamental properties of the operators involved. In Section 2 we make precise how equation (1) fits into the framework of [27], as a problem posed on the torus \( T^N \). In Section 3 we consider the vorticity form of equation (1) as a problem posed on a bounded domain of \( \mathbb{R}^3 \). We again justify the assumptions in [27] to prove existence and uniqueness of maximal solutions to this equation. Additional details for the proofs are given in the Appendices Section 4, along with the results of the partnering paper [27].

1 Preliminaries

1.1 Elementary Notation

In the following \( \mathcal{O} \) represents a subset of \( \mathbb{R}^N \), and will be used throughout the paper to represent either the \( N \)-dimensional torus \( T^N \) or a smooth bounded domain \( \mathcal{O} \subset \mathbb{R}^N \) for \( N = 2, 3 \). In the paper’s duration we consider Banach Spaces as measure spaces equipped with the Borel \( \sigma \)-algebra, and will on occasion use \( \lambda \) to represent the Lebesgue Measure.

**Definition 1.1.** Let \((\mathcal{X}, \mu)\) denote a general measure space, \((\mathcal{Y}, \| \cdot \|_\mathcal{Y})\) and \((\mathcal{Z}, \| \cdot \|_\mathcal{Z})\) be Banach Spaces, and \((\mathcal{U}, \langle \cdot, \cdot \rangle_\mathcal{U}), (\mathcal{H}, \langle \cdot, \cdot \rangle_\mathcal{H})\) be general Hilbert spaces. \( \mathcal{O} \) is equipped with Euclidean norm.

- \( L^p(\mathcal{X}; \mathcal{Y}) \) is the usual class of measurable \( p \)-integrable functions from \( \mathcal{X} \) into \( \mathcal{Y} \), \( 1 \leq p < \infty \), which is a Banach space with norm

\[
\| \phi \|_{L^p(\mathcal{X}; \mathcal{Y})} := \int_\mathcal{X} \| \phi(x) \|_{\mathcal{Y}}^p \mu(dx).
\]

The space \( L^2(\mathcal{X}; \mathcal{Y}) \) is a Hilbert Space when \( \mathcal{Y} \) itself is Hilbert, with the standard inner product

\[
\langle \phi, \psi \rangle_{L^2(\mathcal{X}; \mathcal{Y})} = \int_\mathcal{X} \langle \phi(x), \psi(x) \rangle_\mathcal{Y} \mu(dx).
\]

In the case \( \mathcal{X} = \mathcal{O} \) and \( \mathcal{Y} = \mathbb{R}^N \) note that

\[
\| \phi \|_{L^2(\mathcal{O}; \mathbb{R}^N)}^2 = \sum_{l=1}^N \| \phi^l \|_{L^2(\mathcal{O}; \mathbb{R})}^2
\]

for the component mappings \( \phi^l : \mathcal{O} \rightarrow \mathbb{R} \).
• $L^\infty(\mathcal{X};\mathcal{Y})$ is the usual class of measurable functions from $\mathcal{X}$ into $\mathcal{Y}$ which are essentially bounded, which is a Banach Space when equipped with the norm

$$\|\phi\|_{L^\infty(\mathcal{X};\mathcal{Y})} := \inf\{C \geq 0 : \|\phi(x)\|_Y \leq C \text{ for \mu-a.e. } x \in \mathcal{X}\}.$$

• $L^\infty(\mathcal{O};\mathbb{R}^N)$ is the usual class of measurable functions from $\mathcal{O}$ into $\mathbb{R}^N$ such that $\phi^l \in L^\infty(\mathcal{O};\mathbb{R})$ for $l = 1, \ldots, N$, which is a Banach Space when equipped with the norm

$$\|\phi\|_{L^\infty}\ := \sup_{l \leq N} \|\phi^l\|_{L^\infty(\mathcal{O};\mathbb{R})}.$$

• $C(\mathcal{X};\mathcal{Y})$ is the space of continuous functions from $\mathcal{X}$ into $\mathcal{Y}$.

• $C^m(\mathcal{O};\mathbb{R})$ is the space of $m \in \mathbb{N}$ times continuously differentiable functions from $\mathcal{O}$ to $\mathbb{R}$, that is $\phi \in C^m(\mathcal{O};\mathbb{R})$ if and only if for every $N$ dimensional multi index $\alpha = \alpha_1, \ldots, \alpha_N$ with $|\alpha| \leq m$, $D^\alpha \phi \in C(\mathcal{O};\mathbb{R})$ where $D^\alpha$ is the corresponding classical derivative operator $\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}$.

• $C^\infty(\mathcal{O};\mathbb{R})$ is the intersection over all $m \in \mathbb{N}$ of the spaces $C^m(\mathcal{O};\mathbb{R})$.

• $C^m_0(\mathcal{O};\mathbb{R})$ for $m \in \mathbb{N}$ or $m = \infty$ is the subspace of $C^m(\mathcal{O};\mathbb{R})$ of functions which have compact support.

• $C^m(\mathcal{O};\mathbb{R}^N), C^m_0(\mathcal{O};\mathbb{R}^N)$ for $m \in \mathbb{N}$ or $m = \infty$ is the space of functions from $\mathcal{O}$ to $\mathbb{R}^N$ whose $N$ component mappings each belong to $C^m(\mathcal{O};\mathbb{R}), C^m_0(\mathcal{O};\mathbb{R})$.

• $W^{m,p}(\mathcal{O};\mathbb{R})$ for $1 \leq p < \infty$ is the sub-class of $L^p(\mathcal{O};\mathbb{R})$ which has all weak derivatives up to order $m \in \mathbb{N}$ also of class $L^p(\mathcal{O};\mathbb{R})$. This is a Banach space with norm

$$\|\phi\|_{W^{m,p}(\mathcal{O};\mathbb{R})} := \sum_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^p(\mathcal{O};\mathbb{R})}.$$

where $D^\alpha$ is the corresponding weak derivative operator. In the case $p = 2$ the space $W^{m,2}(\mathcal{O},\mathbb{R})$ is Hilbert with inner product

$$\langle \phi, \psi \rangle_{W^{m,2}(\mathcal{O},\mathbb{R})} := \sum_{|\alpha| \leq m} \langle D^\alpha \phi, D^\alpha \psi \rangle_{L^2(\mathcal{O};\mathbb{R})}.$$

• $W^{m,\infty}(\mathcal{O};\mathbb{R})$ for $m \in \mathbb{N}$ is the sub-class of $L^\infty(\mathcal{O};\mathbb{R})$ which has all weak derivatives up to order $m \in \mathbb{N}$ also of class $L^\infty(\mathcal{O};\mathbb{R})$. This is a Banach space with norm

$$\|\phi\|_{W^{m,\infty}(\mathcal{O};\mathbb{R})} := \sup_{|\alpha| \leq m} \|D^\alpha \phi\|_{L^\infty(\mathcal{O};\mathbb{R}^N)}.$$

• $W^{m,p}(\mathcal{O};\mathbb{R}^N)$ for $1 \leq p < \infty$ is the sub-class of $L^p(\mathcal{O};\mathbb{R}^N)$ which has all weak derivatives up to order $m \in \mathbb{N}$ also of class $L^p(\mathcal{O};\mathbb{R}^N)$. This is a Banach space with norm

$$\|\phi\|_{W^{m,p}(\mathcal{O};\mathbb{R}^N)} := \sum_{l=1}^N \|\phi^l\|_{W^{m,p}(\mathcal{O};\mathbb{R})}.$$
• $W^{m,\infty}(O;\mathbb{R}^N)$ for is the sub-class of $L^{\infty}(O,\mathbb{R}^N)$ which has all weak derivatives up to order $m \in \mathbb{N}$ also of class $L^{\infty}(O,\mathbb{R}^N)$. This is a Banach space with norm

$$\|\phi\|_{W^{m,\infty}(O,\mathbb{R}^N)} := \sup_{l \leq N} \|\phi^l\|_{W^{m,\infty}(O,\mathbb{R})}.$$ 

• $\dot{L}^2(\mathbb{T}^N;\mathbb{R}^N)$ is the subset of $L^2(\mathbb{T}^N;\mathbb{R}^N)$ of functions $\phi$ such that

$$\int_{\mathbb{T}^N} \phi \, d\lambda = 0.$$ 

• $\dot{W}^{m,2}(\mathbb{T}^N;\mathbb{R}^N)$ is simply the intersection $W^{m,2}(\mathbb{T}^N;\mathbb{R}^N) \cap \dot{L}^2(\mathbb{T}^N;\mathbb{R}^N)$.

• $W^{m,p}(O;\mathbb{R}), W^{m,p}_0(O;\mathbb{R}^N)$ for $m \in \mathbb{N}$ and $1 \leq p \leq \infty$ is the closure of $C^\infty_0(O;\mathbb{R}), C^\infty_0(O;\mathbb{R}^N)$ in $W^{m,p}(O;\mathbb{R}), W^{m,p}(O;\mathbb{R}^N)$.

• $\mathcal{L}(\mathcal{Y}; Z)$ is the space of bounded linear operators from $\mathcal{Y}$ to $Z$. This is a Banach Space when equipped with the norm

$$\|F\|_{\mathcal{L}(\mathcal{Y}; Z)} = \sup_{\|y\|_\mathcal{Y} = 1} \|Fy\|_Z$$

and is simply the dual space $\mathcal{Y}^*$ when $Z = \mathbb{R}$, with operator norm $\|\cdot\|_{\mathcal{Y}^*}$.

• $\mathcal{L}^2(\mathcal{U}; \mathcal{H})$ is the space of Hilbert-Schmidt operators from $\mathcal{U}$ to $\mathcal{H}$, defined as the elements $F \in \mathcal{L}(\mathcal{U}; \mathcal{H})$ such that for some basis $(e_i)$ of $\mathcal{U}$,

$$\sum_{i=1}^{\infty} \|Fe_i\|_{\mathcal{H}}^2 < \infty.$$ 

This is a Hilbert space with inner product

$$\langle F, G \rangle_{\mathcal{L}^2(\mathcal{U}; \mathcal{H})} = \sum_{i=1}^{\infty} \langle Fe_i, Ge_i \rangle_{\mathcal{H}}$$

which is independent of the choice of basis.

We will consider a partial ordering on the $N-$dimensional multi-indices by $\alpha \leq \beta$ if and only if for all $l = 1, \ldots, N$ we have that $\alpha_l \leq \beta_l$. We extend this to notation $< \alpha$ if and only if $\alpha \leq \beta$ and for some $l = 1, \ldots, N$, $\alpha_l < \beta_l$.

1.2 Stochastic Framework

We work with a fixed filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ satisfying the usual conditions of completeness and right continuity. We take $\mathcal{W}$ to be a cylindrical Brownian Motion over some Hilbert Space $\mathcal{H}$ with orthonormal basis $(e_i)$. Recall ([25], Subsection 1.4) that $\mathcal{W}$ admits the representation $W_t = \sum_{i=1}^{\infty} e_i W_t^i$ as a limit in $L^2(\Omega; \mathcal{U})$ whereby the $(W_t^i)$ are a collection of i.i.d. standard real valued Brownian Motions and $\mathcal{U}$ is an enlargement of the Hilbert Space $\mathcal{H}$ such that the embedding $J : \mathcal{U} \to \mathcal{U}'$ is Hilbert-Schmidt and $\mathcal{W}$ is a $JJ^*$-cylindrical Brownian Motion over $\mathcal{U}'$. Given a process $F : [0, T] \times \Omega \to \mathcal{L}^2(\mathcal{U}; \mathcal{H})$ progressively measurable and such that $F \in L^2(\Omega \times [0, T]; \mathcal{L}^2(\mathcal{U}; \mathcal{H}))$, for any $0 \leq t \leq T$ we define the stochastic integral

$$\int_0^t F_s dW_s := \sum_{i=1}^{\infty} \int_0^t F_s(e_i) dW_s^i.$$
where the infinite sum is taken in $L^2(\Omega; \mathcal{H})$. We can extend this notion to processes $F$ which are such that $F(\omega) \in L^2([0,T]; L^2(\Omega; \mathcal{H}))$ for $P - a.e. \omega$ via the traditional localisation procedure. In this case the stochastic integral is a local martingale in $\mathcal{H}$.  

1.3 Functional Framework

We now recap the classical functional framework for the study of the deterministic Navier-Stokes Equation, using freely the notation introduced in 1.1. As promised we now formally define the operator $L$ as well as the divergence-free and Lions boundary conditions. Firstly though we briefly comment on the pressure term $\nabla \rho$, which will not play any role in our analysis. $\rho$ does not come with an evolution equation and is simply chosen to ensure the incompressibility (divergence-free) condition; moreover we will ignore this term via a suitable projection (in Section 3 we even consider a different form of the equation) and treat the projected equation, with the understanding to append a pressure to it later. This procedure is well discussed in [38] Section 5 and [33], and an explicit form for the pressure for the SALT Euler Equation is given in [39] Subsection 3.3.

The mapping $L$ is defined for sufficiently regular functions $f, g : \mathcal{O} \to \mathbb{R}^N$ by

$$Lfg := \sum_{j=1}^{N} f^j \partial_j g.$$  

Here and throughout the text we make no notational distinction between differential operators acting on a vector valued function or a scalar valued one; that is, we understand $\partial_j g$ by its component mappings

$$(\partial_i g)^l := \partial_j g^j.$$  

We can immediately give some clarification as to 'sufficiently regular'.

**Lemma 1.2.** For any $m \in \mathbb{N}$, the mapping $L : W^{m+1,2}(\mathcal{O}; \mathbb{R}^N) \to W^{m,2}(\mathcal{O}; \mathbb{R}^N)$ defined by

$$f \mapsto Lf$$

is continuous.

**Proof.** See Appendix I, 4.1. \qed

**Lemma 1.3.** There exists a constant $c$ such that for any $f \in W^{2,2}(\mathcal{O}; \mathbb{R}^N)$ and $g \in W^{1,2}(\mathcal{O}; \mathbb{R}^N)$, we have the bounds

$$\|Lfg\| + \|Lf\| \leq c\|g\|_{W^{1,2}} \|f\|_{W^{2,2}}.$$  

If $f$ has the additional regularity $f \in W^{3,2}(\mathcal{O}; \mathbb{R}^N)$ then

$$\|Lfg\|_{W^{1,2}} \leq c\|g\|_{W^{1,2}} \|f\|_{W^{3,2}}$$  

whilst for $g \in W^{2,2}(\mathcal{O}; \mathbb{R}^N)$ we have that

$$\|Lf\|_{W^{1,2}} \leq c\|g\|_{W^{2,2}} \|f\|_{W^{2,2}}.$$  

[1] A complete, direct construction of this integral, a treatment of its properties and the fundamentals of stochastic calculus in infinite dimensions can be found in [25] Section 1.
Proof. See Appendix I, 4.1.

As for the divergence-free condition we mean a function \( f \) such that the property
\[
\text{div} f := \sum_{j=1}^{N} \partial_j f^j = 0
\]
holds. We require this property and the boundary condition to hold for our solution \( u \) at all times for which it is a solution, though there is some ambiguity in how we understand these conditions for a solution \( u \) which need not be defined pointwise everywhere on \( \partial \). Moreover we shall understand these conditions in their traditional weak sense, that is for weak derivatives \( \partial_j \) so \( \sum_{j=1}^{N} \partial_j f^j = 0 \) holds as an identity in \( L^2(\partial; \mathbb{R}) \) whilst the boundary condition \( w = 0 \) is understood as each component mapping \( w^j \) having zero trace (recall e.g. [16] that \( f^j \in W^{1,2}(\partial; \mathbb{R}) \cap C(\partial; \mathbb{R}) \) has zero trace if and only if \( f^j(x) = 0 \) for all \( x \in \partial \)). The boundary condition \( u \cdot n = 0 \) is a little more technical and not particularly relevant for our analysis, so we defer to [38] Lemma 2.12 for a precise formulation in the weak sense. We impose these conditions by incorporating them into the function spaces where our solution takes value, and give separate definitions for the cases of the Torus and the bounded domain.

**Definition 1.4.** We define \( \Sigma_{0,\sigma}(\partial; \mathbb{R}^N) \) as the subset of \( \Sigma_{0}(\partial; \mathbb{R}^N) \) of functions which are divergence-free. \( L^2_{\sigma}(\partial; \mathbb{R}^N) \) is defined as the completion of \( \Sigma_{0,\sigma}(\partial; \mathbb{R}^N) \) in \( L^2(\partial; \mathbb{R}^N) \), whilst we introduce \( W^{1,2}_{\sigma}(\partial; \mathbb{R}^N) \) as the intersection of \( W^{1,2}(\partial; \mathbb{R}^N) \) with \( L^{2}_{\sigma}(\partial; \mathbb{R}^N) \) and \( W^{2,2}_{\sigma}(\partial; \mathbb{R}^N) \) as the intersection of \( W^{2,2}(\partial; \mathbb{R}^N) \) with \( L^{2}_{\sigma}(\partial; \mathbb{R}^N) \).

For the definitions on the Torus we make explicit reference to the Fourier decomposition. We do this just to define the spaces, after which we make an analogy with the corresponding spaces on the bounded domain and need not rely on any techniques exploiting the Fourier decomposition. To this end recall that any function \( f \in L^2(\mathbb{T}^N; \mathbb{R}^N) \) admits the representation
\[
f(x) = \sum_{k \in \mathbb{Z}^N} f_k e^{ikx}
\]
whereby each \( f_k \in \mathbb{C}^N \) is such that \( f_k = \overline{f_{-k}} \) and the infinite sum is defined as a limit in \( L^2(\mathbb{T}^N; \mathbb{R}^N) \), see e.g. [38] Subsection 1.5 for details.

**Definition 1.5.** We define \( L^2_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \) as the subset of \( \hat{L}^2(\mathbb{T}^N; \mathbb{R}^N) \) of functions \( f \) whereby for all \( k \in \mathbb{Z}^N \), \( k \cdot f_k = 0 \) with \( f_k \) as in (7). For general \( m \in \mathbb{N} \) we introduce \( W^{m,2}_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \) as the intersection of \( W^{m,2}(\mathbb{T}^N; \mathbb{R}^N) \) respectively with \( L^2_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \).

**Remark.** \( L^2_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \) is closed in the \( L^2(\mathbb{T}^N; \mathbb{R}^N) \) norm. The same is trivially true of \( L^2_{\sigma}(\partial; \mathbb{R}^N) \). Furthermore as the \( W^{1,2}(\partial; \mathbb{R}^N) \) norm induces a coarser topology, the space \( W^{1,2}_{\sigma}(\partial; \mathbb{R}^N) \) is closed in the \( W^{1,2}(\partial; \mathbb{R}^N) \) norm and similarly for \( W^{2,2}_{\sigma}(\partial; \mathbb{R}^N) \). Thus \( L^2_{\sigma}(\partial; \mathbb{R}^N), W^{1,2}_{\sigma}(\partial; \mathbb{R}^N) \) and \( W^{2,2}_{\sigma}(\partial; \mathbb{R}^N) \) are Hilbert Spaces in their respective inner products.

**Lemma 1.6.** \( W^{1,2}_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \) is precisely the subspace of \( W^{1,2}(\mathbb{T}^N; \mathbb{R}^N) \) consisting of zero-average divergence free functions.

**Proof.** Note that every \( f \in W^{1,2}(\mathbb{T}^N; \mathbb{R}^N) \) admits the representation (7) but for convergence holding in \( W^{1,2}(\mathbb{T}^N; \mathbb{R}^N) \). Moreover
\[
f^j(x) = \sum_{k \in \mathbb{Z}^N} f^j_k e^{ikx}
\]
with convergence in $W^{1,2}(T^N; \mathbb{R})$ so
\[
\partial_j f^j(x) = \sum_{k \in \mathbb{Z}^N} \partial_j \left( f_k^j e^{ik \cdot x} \right) = \sum_{k \in \mathbb{Z}^N} i f_k^j k^j e^{ik \cdot x}
\]
where the infinite sum is in $L^2(T^N; \mathbb{R})$; with the property $f_k^j = \int_{-k^j}$, a direct computation shows that
\[
if_k^j k^j = if_{-k}^j(-k^j)
\]
verifying once more that this is indeed real valued. Furthermore the divergence of $f$ is given by
\[
\sum_{k \in \mathbb{Z}^N} i(f_k \cdot k)e^{ik \cdot x}
\]
so we see that this is zero if and only if $f_k \cdot k = 0$ for all $k \in \mathbb{Z}^N$. The result follows.

**Lemma 1.7.** $W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ is precisely the subspace of $W^{1,2}_0(\mathcal{O}; \mathbb{R}^N)$ consisting of divergence free functions. Moreover, $W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ is the completion of $C^\infty_0(\mathcal{O}; \mathbb{R}^N)$ in $W^{1,2}(\mathcal{O}; \mathbb{R}^N)$.

**Proof.** We claim at first that any given $f \in W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ is divergence-free. In [38] Lemma 2.11 it is shown that $f$ satisfies the property
\[
\langle f, \nabla \phi \rangle = 0
\]
for every $\phi \in C^\infty_0(\mathcal{O}; \mathbb{R})$, or equivalently that
\[
\sum_{j=1}^N \langle f^j, \partial_j \phi \rangle_{L^2(\mathcal{O}; \mathbb{R})} = 0
\]
for such $\phi$. We can use the compact support of $\phi$ and $W^{1,2}(\mathcal{O}; \mathbb{R}^N)$ regularity of $f$ to carry out an integration by parts, taking the sum through the inner product to see that
\[
\left\langle \sum_{j=1}^N \partial_j f^j, \phi \right\rangle_{L^2(\mathcal{O}; \mathbb{R})} = 0.
\]
Observing that $\phi \in C^\infty_0(\mathcal{O}; \mathbb{R})$ was arbitrary and using the density of this space in $L^2(\mathcal{O}; \mathbb{R})$, we deduce that the divergence of $f$ is zero as an element of $L^2(\mathcal{O}; \mathbb{R})$ and so $f$ is divergence-free. Working in the reverse direction now, we suppose that $g \in W^{1,2}_0(\mathcal{O}; \mathbb{R}^N)$ is divergence-free with the intent to show that $g \in W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$, which is to say $g \in L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$. For this we simply refer to Lemma 2.14 again of [38], noting that $g$ is of zero trace so satisfies the boundary condition and the divergence-free property implies all 'weak divergence-free' notions of [38].

The final statement to prove is the characterisation of $W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ as the completion of $C^\infty_0(\mathcal{O}; \mathbb{R}^N)$ in $W^{1,2}(\mathcal{O}; \mathbb{R}^N)$. We defer this to [41] Theorem 1.6, using the representation of $W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ just proved.

**Remark.** The space $W^{1,2}_\sigma(\mathcal{O}; \mathbb{R}^N)$ thus incorporates the divergence-free and zero-average/zero-trace condition.

With these spaces in place we can define the Leray Projector, which is the projection alluded to when discussing the pressure term.
**Definition 1.8.** The Leray Projector $P$ is defined as the orthogonal projection in $L^2(\mathcal{O}; \mathbb{R}^N)$ onto $L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$.

We immediately give two important properties of this projection.

**Proposition 1.9.** For any $m \in \mathbb{N}$, $P$ is continuous as a mapping $P : W^{m,2}(\mathcal{O}; \mathbb{R}^N) \to W^{m,2}(\mathcal{O}; \mathbb{R}^N)$.

**Proof.** See [41] Remark 1.6. \hfill \square

In fact, the complement space of $L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$ can be characterised. This is the so-called Helmholtz-Weyl decomposition.

**Proposition 1.10.** Define the space $L^2_\perp(\mathcal{O}; \mathbb{R}^N) := \left\{ \psi \in L^2(\mathcal{O}; \mathbb{R}^N) : \psi = \nabla g \text{ for some } g \in W^{1,2}(\mathcal{O}; \mathbb{R}) \right\}$.

Then indeed $L^2_\perp(\mathcal{O}; \mathbb{R}^N)$ is orthogonal to $L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$ in $L^2(\mathcal{O}; \mathbb{R}^N)$, i.e. for any $\phi \in L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$ and $\psi \in L^2_\perp(\mathcal{O}; \mathbb{R}^N)$ we have that $\langle \phi, \psi \rangle = 0$.

Moreover, every $f \in L^2(\mathcal{O}; \mathbb{R}^N)$ has the unique decomposition

$$ f = \phi + \psi $$

for some $\phi \in L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$, $\psi \in L^2_\perp(\mathcal{O}; \mathbb{R}^N)$ and every $f \in L^2(\mathbb{T}_N; \mathbb{R}^N)$ has the unique decomposition

$$ f = \phi + \psi + c $$

where $\phi \in L^2_\sigma(\mathbb{T}_N; \mathbb{R}^N)$, $\psi \in L^2_\perp(\mathbb{T}_N; \mathbb{R}^N)$ and $c$ is a constant function: that is, there exists $k \in \mathbb{R}^N$ such that each component mapping $c^j$ is identically equal to $k^j$, $j = 1, \ldots, N$.

**Proof.** See [41] Theorems 1.4, 1.5 and [38] Theorem 2.6. \hfill \square

**Corollary 1.10.1.** Every $f \in L^2(\mathcal{O}; \mathbb{R}^N)$ admits the representation

$$ f = Pf + \nabla g $$

for some $g \in W^{1,2}(\mathcal{O}; \mathbb{R})$.

**Corollary 1.10.2.** Every $f \in L^2(\mathbb{T}_N; \mathbb{R}^N)$ admits the representation

$$ f = Pf + \nabla g + c $$

for some $g \in W^{1,2}(\mathbb{T}_N; \mathbb{R})$ and constant function $c$.

**Proof.** It is an immediate property of the orthogonal projection that $Pf$ is the unique element $\phi \in L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$ of (8),(9). \hfill \square

This representation allows us to establish important relations between the Leray Projector and the operators involved in (1), and gives rise to a new operator that will be fundamental in defining the spaces used in the abstract formulations of [25, 27, 26].

**Definition 1.11.** The Stokes Operator $A : W^{2,2}(\mathcal{O}; \mathbb{R}^N) \to L^2_\sigma(\mathcal{O}; \mathbb{R}^N)$ is defined by $A := -P\Delta$. 

8
Remark. Once more we understand the Laplacian as an operator on vector valued functions in the sense (3).

Remark. From Proposition 1.9 we have immediately that for $m \in \{0\} \cup \mathbb{N}$, $A : W^{m+2,2}(\mathcal{O}; \mathbb{R}^N) \to W^{m,2}(\mathcal{O}; \mathbb{R}^N)$ is continuous.

Lemma 1.12. $AP$ is equal to $A$ on $W^{2,2}(\mathcal{O}; \mathbb{R}^N)$.

Proof. We call upon the representations (10) and (11), so briefly distinguish between the two domains. In the case of $T^N$ with representation (11), Proposition 1.9 ensures that for $f \in W^{2,2}(T^N; \mathbb{R}^N)$, $Pf \in W^{2,2}(T^N; \mathbb{R}^N)$ and hence so too is $\nabla g$. We then have

$$Af = APf + A\nabla g + Ac = APf + A\nabla g$$

where the Laplacian of the constant is trivially zero, and this same expression is achieved directly for $f \in W^{2,2}(\mathcal{O}; \mathbb{R}^N)$ from (10) so we work now with general $f \in W^{2,2}(\mathcal{O}; \mathbb{R}^N)$. The result would thus be true if $A\nabla g = 0$. This is not difficult to see however, as

$$A\nabla g = -P\Delta \nabla g = -P\nabla \Delta g = 0$$

owing to the fact that $\Delta g \in W^{1,2}(\mathcal{O}; \mathbb{R}^N)$ from $\nabla g \in W^{2,2}(\mathcal{O}; \mathbb{R}^N)$ so $\nabla \Delta g \in L^2_{\sigma}(\mathcal{O}; \mathbb{R}^N)$ and $P$ projects to an orthogonal space. \hfill $\square$

As we look to exploit properties of the Stokes Operator, we will rely heavily on the following Proposition.

Proposition 1.13. There exists a collection of functions $(a_k)$, $a_k \in W^{1,2}_{\sigma}(\mathcal{O}; \mathbb{R}^N) \cap C^\infty(\overline{\mathcal{O}}; \mathbb{R}^N)$ such that the $(a_k)$ are eigenfunctions of $A$, are an orthonormal basis in $L^2_{\sigma}(\mathcal{O}; \mathbb{R}^N)$ and an orthogonal basis in $W^{1,2}_{\sigma}(\mathcal{O}; \mathbb{R}^N)$. The eigenvalues $(\lambda_k)$ are strictly positive and approach infinity as $k \to \infty$.

Proof. See [38] Theorem 2.24. \hfill $\square$

Remark. Every $f \in L^2_{\sigma}(\mathcal{O}; \mathbb{R}^N)$ admits the representation

$$f = \sum_{k=1}^{\infty} f_k a_k$$

where $f_k = \langle f, a_k \rangle$, as a limit in $L^2(\mathcal{O}; \mathbb{R}^N)$.

Lemma 1.14. For $f \in L^2(\mathcal{O}; \mathbb{R}^N)$,

$$P f = \sum_{k=1}^{\infty} \langle f, a_k \rangle a_k.$$

Proof. This is immediate from the fact that the $(a_k)$ form an orthogonal basis of $L^2_{\sigma}(\mathcal{O}; \mathbb{R}^N)$, $P f \in L^2_{\sigma}(\mathcal{O}; \mathbb{R}^N)$ and $P$ is an orthogonal projection:

$$P f = \sum_{k=1}^{\infty} \langle Pf, a_k \rangle a_k = \sum_{k=1}^{\infty} \langle f, a_k \rangle a_k.$$
Definition 1.15. For \( m \in \mathbb{N} \) we introduce the spaces \( D(A^{m/2}) \) as the subspaces of \( L^2_\sigma(O; \mathbb{R}^N) \) consisting of functions \( f \) such that
\[
\sum_{k=1}^{\infty} \lambda_k^m f_k^2 < \infty
\]
for \( f_k \) as in (12). Then \( A^{m/2} : D(A^{m/2}) \to L^2_\sigma(O; \mathbb{R}^N) \) is defined by
\[
A^{m/2} : f \mapsto \sum_{k=1}^{\infty} \lambda_k^{m/2} f_k a_k.
\]

Proposition 1.16. \( D(A^{m/2}) \subset W^{m,2}(O; \mathbb{R}^N) \cap W^{1,2}_\sigma(O; \mathbb{R}^N) \) and the bilinear form
\[
\langle f, g \rangle_m := \langle A^{m/2} f, A^{m/2} g \rangle
\]
is an inner product on \( D(A^{m/2}) \). For \( m \) even the induced norm is equivalent to the \( W^{m,2}(O; \mathbb{R}^N) \) norm, and for \( m \) odd we have the relation
\[
\|\cdot\|_{W^{m,2}} \leq c\|\cdot\|_m
\]
for some constant \( c \).

Proposition 1.17. \( D(A) = W^{2,2}_\sigma(O; \mathbb{R}^N) \cap W^{1,2}_\sigma(O; \mathbb{R}^N) \) with the additional property that \( \|\cdot\|_1 \) is equivalent to \( \|\cdot\|_{W^{1,2}} \) on this space.

Proof of 1.16, 1.17: See [6] Proposition 4.12, [38] Exercises 2.12, 2.13 and the discussion in Subsection 2.3.

Proposition 1.18. For any \( p, q \in \mathbb{N} \) with \( p \leq q \), \( p + q = 2m \) and \( f \in D(A^{m/2}) \), \( g \in D(A^{q/2}) \) we have that
\[
\langle f, g \rangle_m = \langle A^{p/2} f, A^{q/2} g \rangle.
\]
Proof. The proof is direct:
\[
\begin{align*}
\langle f, g \rangle_m &= \langle A^{m/2} f, A^{m/2} g \rangle \\
&= \left\langle \sum_{k=1}^{\infty} \lambda_k^{m/2} f_k a_k, \sum_{j=1}^{\infty} \lambda_j^{m/2} g_j a_j \right\rangle \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k^{m/2} f_k \lambda_j^{m/2} g_j \langle a_k, a_j \rangle \\
&= \sum_{k=1}^{\infty} \lambda_k^{m} f_k g_k \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \lambda_k^{p/2} f_k \lambda_j^{q/2} g_j \langle a_k, a_j \rangle \\
&= \left\langle \sum_{k=1}^{\infty} \lambda_k^{p/2} f_k a_k, \sum_{j=1}^{\infty} \lambda_j^{q/2} g_j a_j \right\rangle \\
&= \langle A^{p/2} f, A^{q/2} g \rangle.
\end{align*}
\]
Lemma 1.19. The collection of functions \((a_k)\) form an orthogonal basis of \(W^{1,2}_\sigma(O;\mathbb{R}^N)\) equipped with the \(\langle\cdot,\cdot\rangle_1\) inner product.

Proof. The completeness follows from that in \(W^{1,2}_\sigma(O;\mathbb{R}^N)\) for the equivalent \(\|\cdot\|_{W^{1,2}}\). As for orthogonality, observe that

\[
\langle a_j, a_k \rangle_1 = \langle A^{1/2}a_j, A^{1/2}a_k \rangle = \lambda_j^{1/2} \lambda_k^{1/2} \langle a_j, a_k \rangle = 0
\]

from the orthogonality in \(L^2(O;\mathbb{R}^N)\). \(\square\)

In addition to using these spaces defined by powers of the Stokes Operator, we also use the basis provided in Proposition 1.13 to consider finite dimensional approximations of these spaces.

Definition 1.20. We define \(P_n\) as the orthogonal projection onto \(\text{span}\{a_1, \ldots, a_n\}\) in \(L^2(O;\mathbb{R}^N)\). That is \(P_n\) is given by

\[
P_n : f \mapsto \sum_{k=1}^n \langle f, a_k \rangle a_k
\]

for \(f \in L^2(O;\mathbb{R}^N)\).

Lemma 1.21. The restriction of \(P_n\) to \(D(A^{m/2})\) is self-adjoint for the \(\langle\cdot,\cdot\rangle_m\) inner product, and there exists a constant \(c\) independent of \(n\) such that for all \(f \in D(A^{m/2})\),

\[
\|P_n f\|_{W^{m,2}} \leq c \|f\|_{W^{m,2}}. \tag{13}
\]

Proof. For \(f, g \in D(A^{m/2})\) (and thus admitting the representation (12)), we have that

\[
\langle P_n f, g \rangle_m = \left\langle \sum_{j=1}^n \langle f, a_j \rangle a_j, \sum_{k=1}^\infty \langle g, a_k \rangle a_k \right\rangle_m
\]

\[
= \left\langle \sum_{j=1}^n \langle f, a_j \rangle \lambda_j^{m/2} a_j, \sum_{k=1}^\infty \langle g, a_k \rangle \lambda_k^{m/2} a_k \right\rangle_m
\]

\[
= \left\langle \sum_{j=1}^n \langle f, a_j \rangle \lambda_j^{m/2} a_j, \sum_{k=1}^n \langle g, a_k \rangle \lambda_k^{m/2} a_k \right\rangle_m
\]

\[
= \left\langle \sum_{j=1}^\infty \langle f, a_j \rangle \lambda_j^{m/2} a_j, \sum_{k=1}^n \langle g, a_k \rangle \lambda_k^{m/2} a_k \right\rangle_m
\]

\[
= \langle f, P_n g \rangle_m
\]

as required for the first statement. For the second see [38] Lemma 4.1. \(\square\)

Lemma 1.22. There exists a constant \(c\) such that for all \(f \in W^{1,2}_\sigma(O;\mathbb{R}^N), g \in W^{2,2}_\sigma(O;\mathbb{R}^N)\) we have that

\[
\| (I - P_n) f \|^2 \leq \frac{1}{\lambda_n} \|f\|^2_1
\]

\[
\| (I - P_n) g \|^2 \leq \frac{1}{\lambda_n} \|g\|^2_2
\]

where \(I\) represents the identity operator in the relevant spaces.
Proof. For the first result, note that
\[
\|(I - P_n)f\|_2^2 = \sum_{k=n+1}^{\infty} \langle f, a_k \rangle^2 \\
= \sum_{k=n+1}^{\infty} \frac{\lambda_k}{\lambda_k} \langle f, a_k \rangle^2 \\
\leq \frac{1}{\lambda_n} \sum_{k=n+1}^{\infty} \lambda_k \langle f, a_k \rangle^2 \\
\leq \frac{1}{\lambda_n} \sum_{k=1}^{\infty} \lambda_k \langle f, a_k \rangle^2 \\
= \frac{1}{\lambda_n} \|f\|_1^2
\]
and the second follows identically. \qed

Remark. Evidently from the proof, the stronger property
\[
\|(I - P_n)f\|_2^2 \leq \frac{1}{\lambda_n} \|(I - P_n)f\|_1^2 \\
\|(I - P_n)g\|_1^2 \leq \frac{1}{\lambda_n} \|(I - P_n)g\|_2^2
\]
is true.

To conclude this subsection we discuss briefly bounds related to the nonlinear term, which will be used in our analysis.

Lemma 1.23. For every \(\phi \in W^{1,2}_\sigma(O;\mathbb{R}^N)\) and \(f, g \in W^{1,2}(O;\mathbb{R}^N)\), we have that
\[
\langle L_\phi f, g \rangle = -\langle f, L_\phi g \rangle.
\tag{14}
\]
Proof. See Appendix I, 4.1. \qed

Corollary 1.23.1. For every \(\phi \in W^{1,2}_\sigma(O;\mathbb{R}^N)\) and \(f \in W^{1,2}(O;\mathbb{R}^N)\), we have that
\[
\langle L_\phi f, f \rangle = 0.
\]

Proposition 1.24. There exists a constant \(c\) such that for any \(f \in W^{1,2}_\sigma(T^N;\mathbb{R}^N)\) and \(g \in W^{2,2}_\sigma(T^N;\mathbb{R}^N)\),
\[
\|L_f g\| \leq c \|f\|_1 \|g\|_1^{1/2} \|g\|_2^{1/2}.
\tag{15}
\]
Meanwhile for \(f \in W^{1,2}_\sigma(O;\mathbb{R}^N)\) and \(g \in W^{2,2}_\sigma(O;\mathbb{R}^N)\) we have that
\[
\|L_f g\| \leq c \|f\|_1 \left(\|g\|_1^{1/2} \|g\|_2^{1/2} + \|g\|_1\right).
\tag{16}
\]
Proof. See Appendix I, 4.1. \qed
1.4 The SALT Operator

Having established the relevant function spaces and some fundamental properties of the operators involved in the deterministic Navier-Stokes Equation, we now address the operator $B$ appearing in the Stratonovich integral of (1). As in Subsection 2.2, the operator $B$ is defined by its action on the basis vectors $(e_i)$ of $\mathcal{U}$. We shall show in Subsection 2.1 that $B$ does indeed satisfy Assumption 2.2.2 of [25] for the spaces to $V, H, U, X$ to be defined. With the notation of [25], each $B_i$ is defined relative to the correlations $\xi_i$ for sufficiently regular $f$ by the mapping

$$B_i : f \mapsto \mathcal{L}_\xi f + \mathcal{T}_\xi f$$

where $\mathcal{L}$ is as before, and $\mathcal{T}$ is a new operator that we introduce defined by

$$\mathcal{T}_g f := \sum_{j=1}^{N} f^j \nabla g^j.$$

We shall assume throughout that each $\xi_i$ belongs to the space $W^{1,2}_{\sigma}(O; \mathbb{R}^N)$, and for the meantime that there is some fixed $m \in \mathbb{N}$ with $\xi_i \in W^{m+2,\infty}(O; \mathbb{R}^N)$.

**Lemma 1.25.** There exists a constant $c$ such that for each $i$ and for all $f \in W^{k,2}(O; \mathbb{R}^N)$ with $k = 0, \ldots, m+1$, we have the bound

$$\|\mathcal{T}_\xi f\|_{W^{k,2}}^2 \leq c \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k,2}}^2.$$  \hspace{1cm} (17)

Therefore $\mathcal{T}_\xi$ is a bounded linear operator on $L^2(O; \mathbb{R}^N)$ so has an everywhere defined adjoint $\mathcal{T}_\xi^*$ in this space.

**Proof.** See Appendix II, 4.2. \qed

**Lemma 1.26.** There exists a constant $c$ such that for each $i$ and for all $f \in W^{k+1,2}(O; \mathbb{R}^N)$ with $k = 0, \ldots, m+1$, we have the bound

$$\|\mathcal{L}_\xi f\|_{W^{k,2}}^2 \leq c \|\xi_i\|_{W^{k,\infty}}^2 \|f\|_{W^{k+1,2}}^2.$$  \hspace{1cm} (18)

Therefore $\mathcal{L}_\xi$ is a densely defined operator in $L^2(O; \mathbb{R}^N)$ with domain of definition $W^{1,2}(O; \mathbb{R}^N)$, and has adjoint $\mathcal{L}_\xi^*$ in this space given by $-\mathcal{L}_\xi$, with same dense domain of definition.

**Proof.** See Appendix II, 4.2 for the proof of (18). The fact that $-\mathcal{L}_\xi$ is the adjoint for $\mathcal{L}_\xi$ on $W^{1,2}(O; \mathbb{R}^N)$ follows immediately from Lemma 1.23. \qed

**Corollary 1.26.1.** There exists a constant $c$ such that for each $i$ and for all $f \in W^{k+1,2}(O; \mathbb{R}^N)$ with $k = 0, \ldots, m+1$, we have the bound

$$\|B_i f\|_{W^{k,2}}^2 \leq c \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k+1,2}}^2.$$  \hspace{1cm} (19)

Moreover $B_i$ is a densely defined operator in $L^2(O; \mathbb{R}^N)$ with domain of definition $W^{1,2}(O; \mathbb{R}^N)$, and has adjoint $B_i^*$ in this space given by $-\mathcal{L}_\xi + \mathcal{T}_\xi^*$ with same dense domain of definition.

Our techniques all centre around energy estimates, where the key idea as to how we preserve these estimates in the case of a transport type noise owes to the following proposition.
Proposition 1.27. There exists a constant c such that for each f and all f ∈ W^{k+2,2}(O; \mathbb{R}^N) with k = 0, ..., m, we have the bounds
\begin{align}
\langle B_i^2 f, f \rangle_{W^{k,2}} + \|B_i f\|^2_{W^{k,2}} &\leq c\|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|^2_{W^{k,2}}, \\
\langle B_i f, f \rangle_{W^{k,2}} &\leq c\|\xi_i\|_{W^{k+1,\infty}} \|f\|^2_{W^{k,2}}.
\end{align}

Proof. See Appendix II, 4.2. □

Remark. Proposition 1.27 has been understood to hold in the case O = T^N, for example in the works [11, 30], though perhaps not for the bounded domain.

We will also use the corresponding result to Lemma 1.12 for the case of the operator B_i. This holds true only in the presence of the additional T_{\xi_i} term in the operator, highlighting the significance of considering a noise which is not purely transport.

Lemma 1.28. We have that
B_i : L^{2,\perp}_\sigma(O; \mathbb{R}^N) \cap W^{1,2}(O; \mathbb{R}^N) \to L^{2,\perp}_\sigma(O; \mathbb{R}^N)
and moreover that PB_i = PB_iP on W^{1,2}(O; \mathbb{R}^N).

Proof. For \nabla g \in L^{2,\perp}_\sigma(O; \mathbb{R}^N) \cap W^{1,2}(O; \mathbb{R}^N),
\begin{align}
B_i(\nabla g) &= L_{\xi_i}(\nabla g) + T_{\xi_i}(\nabla g) \\
&= \sum_{j=1}^{N} \xi_i^j \partial_j (\nabla g) + \sum_{j=1}^{N} \partial_j g \nabla \xi_i^j \\
&= \sum_{j=1}^{N} \xi_i^j (\nabla \partial_j g) + \sum_{j=1}^{N} (\nabla \xi_i^j) \partial_j g \\
&= \nabla \sum_{j=1}^{N} \xi_i^j \partial_j g \\
&\in L^{2,\perp}_\sigma(O; \mathbb{R}^N)
\end{align}
using in the last line the assumption that \nabla g \in W^{1,2}(O; \mathbb{R}^N) and that \xi_i \in W^{1,\infty}(O; \mathbb{R}^N). Just as seen in Lemma 1.12 we now make a distinction between the settings of T^N and \mathcal{O}. For the bounded domain \mathcal{O}, we take any f \in W^{1,2}(\mathcal{O}; \mathbb{R}^N) and use the representation (10) to see that
PB_i f = PB_i (P f + \nabla g) = PB_i P f + P(B_i \nabla g) = PB_i P f
as required, using again the W^{1,2}(\mathcal{O}; \mathbb{R}^N) regularity of both components of the decomposition (10). In the case of the Torus we must address the constant term in the decomposition (11), appreciating that
B_i c = T_{\xi_i} c = \sum_{j=1}^{N} c^j \nabla \xi_i^j = \nabla \sum_{j=1}^{N} c^j \xi_i^j \in L^{2,\perp}_\sigma(T^N; \mathbb{R}^N)
so the result follows in the same manner. □

This relationship between B_i and the Leray Projector sets up an analysis of the B_i terms in the \langle \cdot, \cdot \rangle_m inner product spaces, to which we are further interested in the commutation relationship between B_i and the Laplacian \Delta. This is addressed here.
**Proposition 1.29.** There exists a constant $c$ such that for every $f \in W^{3,2}(O; \mathbb{R}^N)$,

$$\|[[\Delta, B_i]f]\|_2 \leq c\|\xi_i\|^2_{W^{3,\infty}}\|f\|^2_{W^{2,2}}$$

where $[\Delta, B_i]$ is the commutator

$$[\Delta, B_i] := \Delta B_i - B_i \Delta.$$

**Proof.** See Appendix II, 4.2. \qed
2 Analysis of the Velocity Equation

In this section we restrict ourselves to the Torus $\mathbb{T}^N$, leaving a treatment of the bounded domain to Section 3. We also now fix our assumptions on the $(\xi_i)$, assuming that each $\xi_i \in W^{1,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N) \cap W^{3,\infty}(\mathbb{T}^N;\mathbb{R}^N)$ and they collectively satisfy

$$\sum_{i=1}^\infty ||\xi_i||_{W^{3,\infty}}^2 < \infty. \quad (22)$$

2.1 The Itô Form

To facilitate our analysis we work not with the equation (1), but instead one written in terms of an Itô integral and projected by the Leray Projector as discussed at the start of Subsection 1.3. As a first step then we consider the new equation

$$u_t - u_0 + \int_0^t \mathcal{P}L u_s \, ds + \nu \int_0^t A u_s \, ds + \int_0^t \mathcal{P}B u_s \, dW_s = 0 \quad (23)$$

obtained at a heuristic level by projecting all terms of (1). Having not defined solutions of (1) we cannot be too formal here, but the idea is that we required solutions in $L^2(\mathcal{O};\mathbb{R}^N)$ with initial condition also in this space so they are invariant under $\mathcal{P}$, and $\mathcal{P}$ is a bounded linear operator so can be taken through the integrals (see [25] Corollary 1.6.9.1 for this result in stochastic integration). We then look to convert (23) into Itô form via the abstract procedure used in [25] Subsections 2.2 and 2.3, the result of which is stated in Appendix III, 4.3.

Towards this goal we define the quartet of spaces

$$V := W^{3,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N), \quad H := W^{2,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N),$$

$$U := W^{1,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N), \quad X := L^2(\mathbb{T}^N;\mathbb{R}^N).$$

We equip $L^2(\mathbb{T}^N;\mathbb{R}^N)$ with the usual $\langle \cdot, \cdot \rangle$ inner product, but then equip $W^{1,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ and $W^{2,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ with the $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ inner products respectively, recalling propositions 1.16 and 1.17. In fact we also have that $D(A^{3/2}) = W^{3,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ and that the $\langle \cdot, \cdot \rangle_3$ inner product is equivalent to the usual $\langle \cdot, \cdot \rangle_{W^{3,2}}$ one on this space (see [38] Theorem 2.27), so we endow $W^{3,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ with $\langle \cdot, \cdot \rangle_3$.

Our SPDE (23) takes the form of (70) for the operators

$$\mathcal{Q} := -(\mathcal{P}L + \nu A)$$

and

$$\mathcal{G} := -\mathcal{P}B.$$

We now check the Assumptions 4.3, 4.4. Starting with 4.3, we first of all have that $\nu A$ is continuous from $W^{3,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ into $W^{2,2}_\sigma(\mathbb{T}^N;\mathbb{R}^N)$ as the Laplacian is continuous from $W^{3,2}(\mathbb{T}^N;\mathbb{R}^N)$ into $W^{1,2}(\mathbb{T}^N;\mathbb{R}^N)$, Proposition 1.9 and the equivalence of norms. Therefore it is measurable and as a linear operator too satisfies the boundedness. As for $\mathcal{P}L$, measurability is satisfied in the same way (recall Lemma 1.2) and for the boundedness we have that

$$||\mathcal{P}L f||_1 \leq c ||\mathcal{P}L f||_{W^{1,2}} \leq c ||L f||_{W^{1,2}} \leq c ||f||_{W^{1,2}} ||f||_{W^{3,2}} \leq c ||f||_1 ||f||_3$$

16
for any $f \in W_{\sigma}^{3,2}(\mathbb{T}^N; \mathbb{R}^N)$ where $c$ is a generic constant, critically applying (5). This verifies Assumption 4.3 so we move on to 4.4, which is immediate from (19) and the linearity of $PB$ to show continuity in all relevant spaces. We note now though that the Leray Projector does not preserve the space $W_{\sigma}^{3,2}(\mathbb{T}^N; \mathbb{R}^N)$, and so we cannot say that $PB_i : W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N) \to W_{\sigma}^{1,2}(\mathbb{T}^N; \mathbb{R}^N)$. The issues arising from this operator not satisfying the zero-trace property are fundamentally why we only treat the Torus in this context.

With Theorem 4.5 in mind, we move instead to an analysis of the Itô Form

$$u_t = u_0 - \int_0^t PLu_s u_s \, ds - \nu \int_0^t Au_s \, ds + \frac{1}{2} \int_0^t \sum_{i=1}^\infty PB_i^2 u_s \, ds - \int_0^t PBu_s dW_s$$

(24)

where we rewrite $(-PB_i)^2$ as $PB_i^2$ firstly from the linearity of $PB_i$ to deal with the minus and secondly using Lemma 1.28. It is worth appreciating here that we chose to project the equation and then convert it into Itô Form, but we may equally have chosen to convert the unprojected Stratonovich Form and then project the resulting Itô Equation. Without addressing the conversion of the unprojected equation in complete detail, we would directly arrive at (24) taking this approach as our correction term would simply be $PB \sum_{i=1}^\infty B_i^2$ which is just $\sum_{i=1}^\infty PB_i^2$ from the linearity and continuity. Therefore Lemma 1.28 is necessary in ensuring a consistency between these approaches.

### 2.2 Existence, Uniqueness and Maximality

We now state and prove the existence, uniqueness and maximality results for our Navier-Stokes Equation, the idea being to use the abstract results of [27] stated in Appendices IV and V, 4.4 and 4.5. We first give the result pertaining directly to the Stratonovich form.

**Theorem 2.1.** For any given $F_0$–measurable $u_0 : \Omega \to W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)$ there exists a pair $(u, \tau)$ such that: $\tau$ is a $\mathbb{P}$ – a.s. positive stopping time and $u$ is a process whereby for $\mathbb{P}$ – a.e. $\omega$, $u(\omega) \in C \left([0, T]; W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)\right)$ and $u(\omega)1_{\tau(\omega)} \in L^2 \left([0, T]; W_{\sigma}^{3,2}(\mathbb{T}^N; \mathbb{R}^N)\right)$ for all $T > 0$ with $u, 1_{\tau} \text{ progressively measurable in } W_{\sigma}^{3,2}(\mathbb{T}^N; \mathbb{R}^N)$, and moreover satisfying the identity

$$u_t = u_0 - \int_0^{t\land\tau} PLu_s u_s \, ds - \nu \int_0^{t\land\tau} Au_s \, ds - \int_0^{t\land\tau} PBu_s \circ dW_s$$

$\mathbb{P}$ – a.s. in $L^2_{\sigma}(\mathbb{T}^N; \mathbb{R}^N)$ for all $t \geq 0$.

The idea behind Theorem 2.1 is of course to analyse the Itô Form and apply Theorem 4.5 as justified in Subsection 2.1. Once we convert to the Itô Form though starting with an initial condition in $W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)$ is not optimal in the sense that, at least according to the deterministic theory, we should be able to construct a solution (satisfying the identity in $L^2_{\sigma}(\mathbb{T}^N; \mathbb{R}^N)$ as is natural) for only a $W_{\sigma}^{1,2}(\mathbb{T}^N; \mathbb{R}^N)$ initial condition. To this end we give the following definitions and the main result of this subsection. Definition 2.2 is stated for an arbitrary $F_0$–measurable $u_0 : \Omega \to W_{\sigma}^{1,2}(\mathbb{T}^N; \mathbb{R}^N)$.

**Definition 2.2.** A pair $(u, \tau)$ where $\tau$ is a $\mathbb{P}$ – a.s. positive stopping time and $u$ is a process such that for $\mathbb{P}$ – a.e. $\omega$, $u(\omega) \in C \left([0, T]; W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)\right)$ and $u(\omega)1_{\tau(\omega)} \in L^2 \left([0, T]; W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)\right)$ for all $T > 0$ with $u, 1_{\tau} \text{ progressively measurable in } W_{\sigma}^{2,2}(\mathbb{T}^N; \mathbb{R}^N)$, is said to be a local strong solution of the equation (24) if the identity

$$u_t = u_0 - \int_0^{t\land\tau} PLu_s u_s \, ds - \nu \int_0^{t\land\tau} Au_s \, ds + \frac{1}{2} \int_0^{t\land\tau} \sum_{i=1}^\infty PB_i^2 u_s \, ds - \int_0^{t\land\tau} PBu_s dW_s$$

(25)
holds \( P \) – a.s. in \( L^2_0(T^N; \mathbb{R}^N) \) for all \( t \geq 0 \).

**Remark.** If \((u, \tau)\) is a \( V\)-valued local strong solution of the equation (71), then \( u = u_{\land \tau} \). A justification that the integrals are well defined is given in the abstract case in [27].

**Definition 2.3.** A pair \((u, \Theta)\) such that there exists a sequence of stopping times \((\theta_j)\) which are \( P \) – a.s. monotone increasing and convergent to \( \Theta \), whereby \((u_{\land \theta_j}, \theta_j)\) is a local strong solution of the equation (24) for each \( j \), is said to be a maximal strong solution of the equation (71) if for any other pair \((v, \Gamma)\) with this property then \( \Theta \leq \Gamma \) \( P \) – a.s. implies \( \Theta = \Gamma \) \( P \) – a.s.

**Definition 2.4.** A maximal strong solution \((u, \Theta)\) of the equation (24) is said to be unique if for any other such solution \((v, \Gamma)\), then \( \Theta = \Gamma \) \( P \) – a.s. and for all \( t \in [0, \Theta) \),
\[
P(\{\omega \in \Omega : u_t(\omega) = v_t(\omega)\}) = 1.
\]

**Theorem 2.5.** For any given \( F_0 \) – measurable \( u_0 : \Omega \to W^{1,2}_\sigma(T^N; \mathbb{R}^N) \), there exists a unique maximal strong solution \((u, \Theta)\) of the equation (24). Moreover at \( P \) – a.e. \( \omega \) for which \( \Theta(\omega) < \infty \), we have that
\[
\sup_{r \in [0, \Theta(\omega))] \|u_r(\omega)\|_2^2 + \int_0^{\Theta(\omega)} \|u_r(\omega)\|_2^2 dr = \infty.
\] (26)

Definitions 2.2, 2.3, 2.4 and Theorem 2.5 are precisely Definitions 4.19, 4.20, 4.21 and Theorem 4.22 for the equation (24) with respect to the spaces \( V, H, U, X \) as defined in Subsection 2.1. Indeed we would also prove Theorem 2.1 through Theorem 4.5 by showing the existence of a local solution with the regularity of Definition 4.12. Therefore we prove both Theorem 2.1 and 2.5 by showing that the assumptions of Appendices IV and V, 4.4 and 4.5, are satisfied.

We work with the operators
\[
A := -(PL + \nu A) + \frac{1}{2} \sum_{i=1}^{\infty} PB_i^2
\]
\[
G := -PB
\]
which were addressed to be measurable mappings into the required spaces in Subsection 2.1. We now proceed to justify the assumptions of Appendix IV. First note that the density of the spaces is immediately inherited from the density of the usual Sobolev Spaces and the equivalence of the norms. The bilinear form satisfying (72) is chosen to be
\[
(f, g)_{U \times V} := \langle A^{1/2} f, A^{3/2} g \rangle
\]
which reduces to the \( \langle \cdot, \cdot \rangle_2 \) inner product from Proposition 1.18. In the following \( c \) will represent a generic constant changing from line to line, \( c(\varepsilon) \) will be a generic constant dependent on a fixed \( \varepsilon \), \( f \) and \( g \) will be arbitrary elements of \( W^{3,2}_\sigma(T^N; \mathbb{R}^N) \) and \( f_n \in \text{span}\{a_1, \ldots, a_n\} \).

**Assumption 4.6:** We use the system \((a_k)\) of eigenfunctions of the Stokes Operator given in Proposition 1.13, satisfying (73) and (74) from Lemmas 1.21 and 1.22 respectively.

\[\square\]
**Assumption 4.7:** Once more (76) follows from the discussion in Subsection 2.1. For (77) we treat the different operators in \( A \) individually, starting from the nonlinear term:

\[
\| \mathcal{P} \mathcal{L} f - \mathcal{P} \mathcal{L} g \|_1 = \| \mathcal{P} \mathcal{L} (f - g) + \mathcal{P} \mathcal{L} f - \mathcal{P} \mathcal{L} g \|_1 \\
\leq \| \mathcal{P} \mathcal{L} (f - g) \|_1 + \| \mathcal{P} \mathcal{L} f - \mathcal{P} \mathcal{L} g \|_1 \\
\leq c \| \mathcal{L} (f - g) \|_1 W^{1,2} + c \| \mathcal{L} (f - g) \|_1 W^{1,2} \\
\leq c \| f \|_1 W^{1,2} \| f - g \|_1 W^{3,2} + c \| f \|_1 W^{1,2} \| g \|_1 W^{3,2} \\
\leq c ( \| f \|_1 W^{1,2} + \| g \|_1 W^{3,2} ) \| f - g \|_3 \\
\leq c ( \| f \|_1 + \| g \|_3 ) \| f - g \|_3
\]

having applied (5). From the linearity of \( \nu A \) and \( \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{P} B_i^2 \) then the corresponding result follows immediately from (76) and this subsequently justifies (77). Additionally (78) follows immediately from the already justified Assumption 4.4.

\[ \square \]

For the justification of Assumption 4.8 we call upon some intermediary results.

**Lemma 2.6.** For any \( \varepsilon > 0 \), we have that

\[
| \langle \mathcal{P}_n \mathcal{L} f_n, f_n \rangle_2 | \leq c(\varepsilon) \| f_n \|_2^4 + \varepsilon \| f_n \|_3^2.
\]

**Proof.** We use the established result that for \( k = 2 \) then the Sobolev Space \( W^{k,2}(\mathcal{O}; \mathbb{R}) \) is an algebra (a result first presented in [40]), to deduce that

\[
\| \mathcal{L}_n f_n \|_{W^{2,2}} = \left\| \sum_{j=1}^{N} f_n^j \partial_j f_n \right\|_{W^{2,2}} \\
\leq \sum_{j=1}^{N} \left\| f_n^j \partial_j f_n \right\|_{W^{2,2}} \\
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} \left\| f_n^j \partial_j f_n^l \right\|_{W^{2,2}(\mathbb{T}^N; \mathbb{R})} \\
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} \left\| f_n^j \right\|_{W^{2,2}(\mathbb{T}^N; \mathbb{R})} \left\| \partial_j f_n^l \right\|_{W^{2,2}(\mathbb{T}^N; \mathbb{R})} \\
\leq c \| f_n \|_{W^{2,2}} \| f_n \|_{W^{3,2}} \\
\leq c \| f_n \|_2 \| f_n \|_3.
\]

From here we simply use that \( \mathcal{P}_n \) is self-adjoint (Lemma 1.21) and Young’s Inequality to see that

\[
| \langle \mathcal{P}_n \mathcal{L} f_n, f_n \rangle_2 | = | \langle \mathcal{L} f_n, f_n \rangle_2 | \\
\leq \| \mathcal{L} f_n \|_2 \| f_n \|_2 \\
\leq c \| f_n \|_{W^{2,2}} \| f_n \|_2 \\
\leq c \| f_n \|_2 \| f_n \|_3 \| f_n \|_2 \\
\leq c(\varepsilon) \| f_n \|_2^4 + \varepsilon \| f_n \|_3^2.
\]

\[ \square \]
Remark. The algebra property when \( k = 2 \) is fundamental to this result; we are prevented from applying Theorem 4.15 in the case \( H := W^{1,2}_{\sigma}(\mathbb{T}^N; \mathbb{R}^N) \) as we would have no algebra property to apply the same method.

Lemma 2.7. For any \( \varepsilon > 0 \) we have the bound
\[
\langle P_n P B_i^2 f_n, f_n \rangle_1 + \langle P_n P B_i f_n, P_n P B_i f_n \rangle_1 \leq c(\varepsilon) \|\xi_i\|_{W^{3,\infty}}^2 \|f_n\|_1^2 + \varepsilon \|\xi_i\|_{W^{3,\infty}}^2 \|f_n\|_2^2.
\]

Proof. From Lemma 1.21 we can readily justify the inequality
\[
\langle P_n P B_i^2 f_n, f_n \rangle_1 + \langle P_n P B_i f_n, P_n P B_i f_n \rangle_1 \leq \langle P B_i^2 f_n, f_n \rangle_1 + \langle P B_i f_n, P B_i f_n \rangle_1
\]
and moreover from Proposition 1.18 that this is just
\[
\langle P B_i^2 f_n, A f_n \rangle + \langle P B_i f_n, A P B_i f_n \rangle.
\]
Using Lemmas 1.28 and 1.12, we rewrite this as
\[
\langle (P B_i)^2 f_n, A f_n \rangle + \langle P B_i f_n, A B_i f_n \rangle
\]
and further as
\[
\langle P B_i f_n, B_i^* A f_n \rangle - \langle P B_i f_n, \Delta B_i f_n \rangle
\]
(27)
for the adjoint \( B_i^* \) specified in Corollary 1.26.1. We look to commute the Laplacian and \( B_i \), using Proposition 1.29 and subsequently the cancellation of the derivative in \( B_i \) when considered with the adjoint. Indeed,
\[
-\langle P B_i f_n, \Delta B_i f_n \rangle = -\langle P B_i f_n, ([\Delta, B_i] + B_i \Delta) f_n \rangle
\]
\[
= -\langle P B_i f_n, [\Delta, B_i] f_n \rangle - \langle P B_i f_n, B_i \Delta f_n \rangle
\]
\[
= -\langle P B_i f_n, [\Delta, B_i] f_n \rangle - \langle P B_i f_n, P B_i \Delta f_n \rangle
\]
\[
= -\langle P B_i f_n, [\Delta, B_i] f_n \rangle + \langle P B_i f_n, P B_i A f_n \rangle
\]
\[
= -\langle P B_i f_n, [\Delta, B_i] f_n \rangle + \langle P B_i f_n, B_i A f_n \rangle
\]
using Lemma 1.28 again. Thus (27) becomes
\[
\langle P B_i f_n, B_i^* A f_n \rangle - \langle P B_i f_n, [\Delta, B_i] f_n \rangle + \langle P B_i f_n, B_i A f_n \rangle
\]
or simply
\[
\langle P B_i f_n, (\mathcal{T}_\xi + \mathcal{T}_\xi^*) A f_n - [\Delta, B_i] f_n \rangle
\]
(28)
which we look to bound through Cauchy-Schwartz and the results of (19), Lemma 1.25 and Proposition 1.29 to see that
\[
(28) \leq \|P B_i f_n\| \left( \|\mathcal{T}_\xi + \mathcal{T}_\xi^*\| A f_n \| + \|[\Delta, B_i] f_n\| \right)
\]
\[
\leq c \|\xi\|_{W^{1,\infty}} \|f_n\|_{W^{1,2}} (\|\xi\|_{W^{1,\infty}} \|A f_n\| + \|\xi\|_{W^{3,\infty}} \|f_n\|_{W^{1,2}})
\]
\[
\leq c \|\xi\|_{W^{1,\infty}} \|f_n\|_1 (\|\xi\|_{W^{1,\infty}} \|f_n\|_2 + \|\xi\|_{W^{3,\infty}} \|f_n\|_2)
\]
\[
\leq c \|\xi\|_{W^{3,\infty}} \|f_n\|_1 \|f_n\|_2
\]
\[
\leq c(\varepsilon) \|\xi\|_{W^{3,\infty}} \|f_n\|_1^2 + \varepsilon \|\xi\|_{W^{3,\infty}} \|f_n\|_2^2
\]
as required. \(\square\)
Lemma 2.8. For any $\varepsilon > 0$, we have that
\[
\langle P_n P B_i^2 f_n, f_n \rangle + \langle P_n P B_i f_n, P_n P B_i f_n \rangle \leq c(\varepsilon) \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^2 + \varepsilon \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^2.
\]

Proof. As with Lemma 2.7 we can immediately say that
\[
\langle P_n P B_i^2 f_n, f_n \rangle + \langle P_n P B_i f_n, P_n P B_i f_n \rangle \leq \langle P B_i^2 f_n, f_n \rangle + \langle P B_i f_n, P B_i f_n \rangle
\]
which we again manipulate with Lemmas 1.12 and 1.28 to give
\[
(29) = \langle A P B_i^2 f_n, A f_n \rangle + \langle A P B_i f_n, A P B_i f_n \rangle
\]
\[
= \langle A B_i^2 f_n, A f_n \rangle + \langle A f_n, A B_i f_n \rangle
\]
\[
= -\langle P \Delta B_i^2 f_n, A f_n \rangle - \langle A B_i f_n, P \Delta B_i f_n \rangle
\]
\[
= -\langle P [\Delta, B_i] B_i f_n + P B_i \Delta B_i f_n, A f_n \rangle - \langle A B_i f_n, P [\Delta, B_i] f_n + P B_i \Delta f_n \rangle
\]
\[
= -\langle P [\Delta, B_i] B_i f_n, A f_n \rangle + \langle P B_i A B_i f_n, A f_n \rangle - \langle A B_i f_n, P [\Delta, B_i] f_n \rangle - \langle A B_i f_n, P B_i A f_n \rangle
\]
\[
= \langle B_i A B_i f_n, A f_n \rangle - \langle A B_i f_n, P [\Delta, B_i] f_n \rangle
\]
\[
= \langle A B_i f_n, (B_i + B_i^*) A f_n \rangle - \langle P [\Delta, B_i] B_i f_n, A f_n \rangle - \langle A B_i f_n, P [\Delta, B_i] f_n \rangle
\]
\[
= \langle A B_i f_n, (T_{\xi_i}^* + T_{\xi_i}^*) A f_n \rangle - \langle P [\Delta, B_i] B_i f_n, A f_n \rangle - \langle A B_i f_n, P [\Delta, B_i] f_n \rangle.
\]

We shall treat each term individually using Cauchy-Schwartz and Young’s Inequality in conjunction with the relevant bounds as seen in Lemma 2.7:
\[
\langle A B_i f_n, (T_{\xi_i} + T_{\xi_i}^*) A f_n \rangle \leq \| A B_i f_n \| \| (T_{\xi_i} + T_{\xi_i}^*) A f_n \|
\]
\[
\leq c \| \xi_i \|_{W^{1,\infty}} \| B_i f_n \|_{W^{2,2}} \| A f_n \|
\]
\[
\leq c \| \xi_i \|_{W^{1,\infty}} \| \xi_i \|_{W^{3,\infty}} \| f_n \|_{W^{3,2}} \| f_n \|^2
\]
\[
\leq c(\varepsilon) \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^2 + \frac{\varepsilon}{3} \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^3
\]

as well as
\[
-\langle P [\Delta, B_i] B_i f_n, A f_n \rangle \leq \| P [\Delta, B_i] B_i f_n \| \| A f_n \|
\]
\[
\leq c \| [\Delta, B_i] B_i f_n \| \| f_n \|_2
\]
\[
\leq c \| \xi_i \|_{W^{3,\infty}} \| f_n \|_{W^{2,2}} \| f_n \|_2
\]
\[
\leq c(\varepsilon) \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^2 + \frac{\varepsilon}{3} \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^3
\]

and finally
\[
-\langle A B_i f_n, P [\Delta, B_i] f_n \rangle \leq \| A B_i f_n \| \| P [\Delta, B_i] f_n \|
\]
\[
\leq c \| B_i f_n \|_{W^{2,2}} \| [\Delta, B_i] f_n \|
\]
\[
\leq c \| \xi_i \|_{W^{3,\infty}} \| f_n \|_{W^{3,2}} \| f_n \|_{W^{2,2}}
\]
\[
\leq c(\varepsilon) \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^2 + \frac{\varepsilon}{3} \| \xi_i \|^2_{W^{3,\infty}} \| f_n \|^3
\]

Summing these up completes the proof. 

$\square$
**Assumption 4.8:** Lemmas 2.6 and 2.8 will be our basis of showing (79). The task is to control
\[
2 \left\langle \mathcal{P}_n \left( -\mathcal{P} \mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{P} B_i^2 \right) f_n, f_n \right\rangle + \sum_{i=1}^{\infty} \left\| \mathcal{P}_n \mathcal{P} B_i f_n \right\|^2
\]
which we rewrite as
\[
-2 \left\langle \mathcal{P}_n \mathcal{P} \mathcal{L} f_n, f_n \right\rangle - 2 \nu \left\langle \mathcal{P}_n A f_n, f_n \right\rangle + \sum_{i=1}^{\infty} \left( \left\langle \mathcal{P}_n \mathcal{P} B_i^2 f_n, f_n \right\rangle + \left\| \mathcal{P}_n \mathcal{P} B_i f_n \right\|^2 \right).
\]
(30)

Recalling the assumption (22) and Lemmas 2.6, 2.8, we have that for any \( \varepsilon > 0 \),
\[
-2 \nu \left\langle \mathcal{P}_n A f_n, f_n \right\rangle + c(\varepsilon) \left\| f_n \right\|^2 + \varepsilon \left\| f_n \right\|^3 + \sum_{i=1}^{\infty} \left( c(\varepsilon) \left\| \xi_i \right\|^2_{W^{3,\infty}} \left\| f_n \right\|^2 + \varepsilon \left\| \xi_i \right\|^2_{W^{3,\infty}} \left\| f_n \right\|^3 \right)
\]
\[
= -2 \nu \left\langle A f_n, f_n \right\rangle + \left( c(\varepsilon) \left\| f_n \right\|^4 + c(\varepsilon) \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}} \left\| f_n \right\|^2 \right) + \varepsilon \left[ 1 + \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}} \right] \left\| f_n \right\|^3
\]
\[
\leq -2 \nu \left\langle A^{1/2} B f_n, A^{3/2} f_n \right\rangle + c(\varepsilon) \left[ 1 + \left\| f_n \right\|^2 \right] \left\| f_n \right\|^3 + \varepsilon \left[ 1 + \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}} \right] \left\| f_n \right\|^3
\]
\[
= -2 \nu \left\| f_n \right\|^3 + c(\varepsilon) \left[ 1 + \left\| f_n \right\|^2 \right] \left\| f_n \right\|^3 + \varepsilon \left[ 1 + \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}} \right] \left\| f_n \right\|^3
\]

where we have embedded the \( \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}} \) into the constant \( c(\varepsilon) \). Therefore by choosing
\[
\varepsilon := \frac{\nu}{1 + \sum_{i=1}^{\infty} \left\| \xi_i \right\|^2_{W^{3,\infty}}}
\]
then (79) is satisfied for \( \kappa := \nu \). Moving on to (80), we are interested in the term
\[
\sum_{i=1}^{\infty} \left\langle \mathcal{P}_n \mathcal{P} B_i f_n, f_n \right\rangle^2.
\]

Using Lemmas 1.12 and 1.28 once more, observe that
\[
\left\langle \mathcal{P}_n \mathcal{P} B_i f_n, f_n \right\rangle^2 = \left\langle \mathcal{P} B_i f_n, f_n \right\rangle^2
\]
\[
= \left\langle A \mathcal{P} B_i f_n, A f_n \right\rangle^2
\]
\[
= \left\langle A \mathcal{P} B_i f_n, A f_n \right\rangle^2
\]
\[
= \left\langle \mathcal{P} \left[ \Delta, B_i \right] f_n + \mathcal{P} B_i \Delta f_n, A f_n \right\rangle^2
\]
\[
\leq 2 \left\langle \mathcal{P} \left[ \Delta, B_i \right] f_n, A f_n \right\rangle^2 + 2 \left\langle \mathcal{P} B_i \Delta f_n, A f_n \right\rangle^2
\]
\[
= 2 \left\langle \mathcal{P} \left[ \Delta, B_i \right] f_n, A f_n \right\rangle^2 + 2 \left\langle \mathcal{P} B_i A f_n, A f_n \right\rangle^2
\]
\[
= 2 \left\langle \Delta, B_i \right\rangle f_n, A f_n \right\rangle^2 + 2 \left\langle B_i A f_n, A f_n \right\rangle^2.
\]
The first of these terms is dealt with through a simple Cauchy-Schwartz, as
\[
\left\langle \left\langle \left[ \Delta, B_i \right], A f_n \right\rangle \right\rangle \leq \left\| \left\langle \left[ \Delta, B_i \right] f_n \right\rangle \right\| \left\| A f_n \right\|^2
\]
\[
\leq c \left\| \xi_i \right\|^2_{W^{3,\infty}} \left\| f_n \right\|^2_{W^{2,2}} \left\| f_n \right\|^2
\]
\[
\leq c \left\| \xi_i \right\|^2_{W^{3,\infty}} \left\| f_n \right\|^4
\]
using Proposition 1.29, and the second comes directly from (21) as
\[
\langle B_i A f_n, A f_n \rangle^2 \leq c \| \xi_i \|^2 W_{1,\infty} A f_n \|^4 \leq c \| \xi_i \|^2 W_{3,\infty} f_n \|^4.
\]

Summing up the two terms and over all \(i\) gives that
\[
\sum_{i=1}^{\infty} \langle P_i PB_i f_n, f_n \rangle \leq \left( \sum_{i=1}^{\infty} \| \xi_i \|^2 W_{3,\infty} \right) f_n \|^2
\]
which justifies (80) and Assumption 4.8.

Towards Assumption 4.9 we again state an intermediary lemma.

**Lemma 2.9.** For any \(\varepsilon > 0\), we have that
\[
\langle P L f, f - P L g, f - g \rangle \leq c(\varepsilon) \left( \| g \|^4 + \| f \|^2 \right) \| f - g \|^2 + \varepsilon \| f - g \|^2
\]

**Proof.** Observe that
\[
\langle P L f, f - P L g, f - g \rangle = \langle P L f, f - L g, A(f - g) \rangle = \langle L f, f - L g, A(f - g) \rangle = \langle L f - L f g + L f g - L g, A(f - g) \rangle
\]

and so it is sufficient to control the terms
\[
\langle L f - g, A(f - g) \rangle, \quad (31)
\]
\[
\langle L g(f - g), A(f - g) \rangle \quad (32)
\]

We bound each item individually, using (15):
\[
(31) \leq \| L f - g \| A(f - g)
\leq c \| f - g \|^2 f \| f \|^2 f - g \| f - g \|_2
\leq c \| f - g \|^2 f \| f - g \|_2
\leq c(\varepsilon) \| f \|^2 f - g \|_2^2 + \varepsilon \| f - g \|^2
\]

and
\[
(32) \leq \| L g(f - g) \| A(f - g)
\leq c \| g \|^2 f - g \|^2 f - g \| f - g \|_2
\leq c \| g \|^2 f - g \|^2 f - g \|_2
\leq c(\varepsilon) \| g \|^2 f - g \|^2 f - g \|_2^2 + \varepsilon \| f - g \|^2
\]
using Young’s Inequality with conjugate exponents 4 and 4/3.
Assumption 4.9: For (81) we must bound the term

\[ 2 \left( \langle \mathcal{P} \mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{P} B_i^2 \rangle f - \langle \mathcal{P} \mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{P} B_i^2 \rangle g, f - g \right) + \sum_{i=1}^{\infty} \| \mathcal{P} B_i f - \mathcal{P} B_i g \|_1^2 \]

which we simply rewrite as

\[ -2 \langle \mathcal{P} \mathcal{L} f - \mathcal{P} \mathcal{L} g, f - g \rangle_1 - 2\nu \langle A(f - g), f - g \rangle_1 + \sum_{i=1}^{\infty} \left( \langle \mathcal{P} B_i^2(f - g), f - g \rangle_1 + \| \mathcal{P} B_i(f - g) \|_1^2 \right) \]

and inspect the distinct items individually. Firstly from Lemma 2.9 we have that for any \( \varepsilon > 0 \),

\[ -2 \langle \mathcal{P} \mathcal{L} f - \mathcal{P} \mathcal{L} g, f - g \rangle_1 \leq c(\varepsilon) \left( \| g \|_1^4 + \| f \|_2^4 \right) \| f - g \|_1^2 + \varepsilon \| f - g \|_2^2. \]  

(33)

Similarly to the justification of Assumption 4.8 we also see that

\[ -2\nu \langle A(f - g), f - g \rangle_1 = -2\nu \| f - g \|_2^2. \]  

(34)

Shifting focus to the final term, note that in Lemma 2.7 we in fact showed that

\[ \langle \mathcal{P} B_i^2 f_n, f_n \rangle_1 + \langle \mathcal{P} B_i f_n, \mathcal{P} B_i f_n \rangle_1 \leq c(\varepsilon) \| \xi_i \|_{W^{3,\infty}} \| f_n \|_1^2 + \varepsilon \| \xi_i \|_{W^{3,\infty}} \| f_n \|_2^2 \]

and scanning the proof we see that all arguments hold for arbitrary \( f_n \in W_2^{3,2}(T^N; \mathbb{R}^N) \) so we can deduce directly the bound

\[ \langle \mathcal{P} B_i^2(f - g), f - g \rangle_1 + \| \mathcal{P} B_i(f - g) \|_1^2 \leq c(\varepsilon) \| \xi_i \|_{W^{3,\infty}} \| f - g \|_1^2 + \varepsilon \| \xi_i \|_{W^{3,\infty}} \| f - g \|_2^2. \]  

(35)

Summing over (33), (34) and all \( i \) in (35), we deduce a bound by

\[ -2\nu \| f - g \|_2^2 + c(\varepsilon) \left[ \| g \|_1^4 + \| f \|_2^4 + \sum_{i=1}^{\infty} \| \xi_i \|_{W^{3,\infty}}^2 \right] \| f - g \|_1^2 + \varepsilon \left[ 1 + \sum_{i=1}^{\infty} \| \xi_i \|_{W^{3,\infty}}^2 \right] \| f - g \|_2^2 \]

so again a choice of

\[ \varepsilon := \frac{\nu}{1 + \sum_{i=1}^{\infty} \| \xi_i \|_{W^{3,\infty}}^2} \]  

(36)

ensures (81) is satisfied for \( \kappa := \nu \). Moving on to (82), we are interested in the term

\[ \sum_{i=1}^{\infty} \langle \mathcal{P} B_i(f - g), f - g \rangle_1^2, \]  

(37)

noting that

\[ \langle \mathcal{P} B_i(f - g), f - g \rangle_1^2 = \langle A \mathcal{P} B_i(f - g), f - g \rangle_1^2 \]
\[ = \langle A \mathcal{P} B_i(f - g), f - g \rangle_1^2 \]
\[ = \langle \Delta \mathcal{B}_i(f - g), f - g \rangle_1^2 \]
\[ = \left( \sum_{k=1}^{N} \partial_k^2 \mathcal{B}_i(f - g), f - g \right)^2 \]
\[ = \left( \sum_{k=1}^{N} \langle \partial_k \mathcal{B}_i(f - g), \partial_k(f - g) \rangle \right)^2 \]
\[ \leq N \sum_{k=1}^{N} \langle \partial_k \mathcal{B}_i(f - g), \partial_k(f - g) \rangle^2 \]
using Lemma 1.12. Combining the ideas of Lemmas 1.25 and 1.26, we have
\[ \partial_k B_i(f - g) = B_{\partial_k \xi_i}(f - g) + B_i \partial_k (f - g) \]
so
\[ \langle \partial_k B_i(f - g), \partial_k (f - g) \rangle \leq 2 \langle B_{\partial_k \xi_i}(f - g), \partial_k (f - g) \rangle^2 + 2 \langle B_i \partial_k (f - g), \partial_k (f - g) \rangle^2. \]

Now from (19),
\[ \langle B_{\partial_k \xi_i}(f - g), \partial_k (f - g) \rangle \leq \|B_{\partial_k \xi_i}(f - g)\|^2 \|\partial_k (f - g)\|^2 \]
\[ \leq c \|\partial_k \xi_i\|^2_{W^{1,\infty}} \|f - g\|^2_{W^{1,2}} \|\partial_k (f - g)\|^2 \]
\[ \leq c \|\xi_i\|^2_{W^{3,\infty}} \|f - g\|^2 \|\partial_k (f - g)\|^2 \]
\[ = c \|\xi_i\|^2_{W^{3,\infty}} \|f - g\|^4. \]

and from Corollary 1.23.1,
\[ \langle B_i \partial_k (f - g), \partial_k (f - g) \rangle = \langle T_{\xi_i} \partial_k (f - g), \partial_k (f - g) \rangle \]
\[ \leq \|T_{\xi_i} \partial_k (f - g)\|^2 \|\partial_k (f - g)\|^2 \]
\[ \leq c \|\xi_i\|^2_{W^{1,\infty}} \|\partial_k (f - g)\|^2 \|\partial_k (f - g)\|^2 \]
\[ \leq c \|\xi_i\|^2_{W^{3,\infty}} \|f - g\|^4. \]

By summing both terms, over all \( k = 1, \ldots, N \) and \( i \in \mathbb{N} \), we have shown that
\[ \sum_{i=1}^{\infty} \langle PB_i(f - g), f - g \rangle^2 \leq \left( c \sum_{i=1}^{\infty} \|\xi_i\|^2_{W^{3,\infty}} \right) \|f - g\|^4 \quad (38) \]
demonstrating (82) and hence Assumption 4.9. \( \square \)

**Assumption 4.10:** For (83) we must bound the term
\[ 2 \left( \left( -\mathcal{P} \mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{P} B_i^2 \right) f, f \right) + \sum_{i=1}^{\infty} \|PB_i f\|^2 \]
which we simply rewrite as
\[ -2 \langle \mathcal{P} L f, f \rangle_1 - 2\nu \langle Af, f \rangle_1 + \sum_{i=1}^{\infty} \left( \langle PB_i^2 f, f \rangle_1 + \|PB_i f\|^2_1 \right). \quad (39) \]

The nonlinear term can be controlled precisely as done for (32) to deduce that for any \( \varepsilon > 0, \)
\[ |\langle \mathcal{P} L f, f \rangle_1| \leq c(\varepsilon) \|f\|^6_1 + \varepsilon \|f\|^2_2. \]

Meanwhile across (34) and (35) we have that
\[ -2\nu \langle Af, f \rangle_1 + \sum_{i=1}^{\infty} \left( \langle PB_i^2 f, f \rangle_1 + \|PB_i f\|^2_1 \right) \]
\[ \leq -2\nu \|f\|^2_2 + c(\varepsilon) \left( \sum_{i=1}^{\infty} \|\xi_i\|^2_{W^{3,\infty}} \right) \|f\|^2_2 + \varepsilon \left( \sum_{i=1}^{\infty} \|\xi_i\|^2_{W^{3,\infty}} \right) \|f\|^2_2 \]

25
so with the familiar choice of $\varepsilon$ (36) we see that
\[
(39) \leq c \left( 1 + \|f\|_1^6 \right) - \nu \|f\|_2^2
\]
which is more than sufficient to show (83). (84) follows immediately from (38), concluding the proof.

**Assumption 4.11:** For any $\eta \in W^{2,2}_\sigma(T^N;\mathbb{R}^N)$ we must bound the term
\[
\left\langle \left( -\mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^\infty \mathcal{P}B_i^2 \right) f - \left( -\mathcal{L} - \nu A + \frac{1}{2} \sum_{i=1}^\infty \mathcal{P}B_i^2 \right) g, \eta \right\rangle_1
\]
which we simply rewrite as
\[
-2\langle \mathcal{P}\mathcal{L} f - \mathcal{P}\mathcal{L} g, \eta \rangle_1 - 2\nu \langle A(f - g), \eta \rangle_1 + \sum_{i=1}^\infty \langle \mathcal{P}B_i^2(f - g), \eta \rangle_1
\]
and further by
\[
-2\langle \mathcal{P}\mathcal{L} f - \mathcal{P}\mathcal{L} g, A\eta \rangle - 2\nu \langle A(f - g), A\eta \rangle + \sum_{i=1}^\infty \langle \mathcal{P}B_i^2(f - g), A\eta \rangle.
\]
Through Cauchy-Schwartz this is controlled by
\[
\|\eta\|_2 \left( 2\|\mathcal{P}\mathcal{L} f - \mathcal{P}\mathcal{L} g\| + 2\nu\|A(f - g)\| + \sum_{i=1}^\infty \|\mathcal{P}B_i^2(f - g)\| \right).
\]
so our problem is reduced to bounding the bracketed terms. The linear terms are trivial when recalling (19), and for the nonlinear term we revert back to (4) to see that
\[
\|\mathcal{P}\mathcal{L} f - \mathcal{P}\mathcal{L} g\| \leq \|f - g\| + \|\mathcal{L}(f - g)\| \\
\leq c (\|f - g\|_1 \|f\|_2 + \|g\|_1 \|f - g\|_2) \\
\leq c (\|f\|_2 + \|g\|_1) \|f - g\|_2
\]
comfortably justifying the assumption.

Moving on now to the setting of Appendix V, 4.5, the new space $X$ will again be $L^2_\sigma(T^N;\mathbb{R}^N)$ as laid out in Subsection 2.1. We choose the bilinear form $\langle \cdot, \cdot \rangle_{X \times H}$ to be given by
\[
\langle f, g \rangle_{X \times H} := \langle f, Ag \rangle
\]
noting that the property (88) follows from Proposition 1.18. Noting also that the system $\{a_k\}$ given in Proposition 1.13 is an orthogonal basis of $W^{1,2}_\sigma(T^N;\mathbb{R}^N)$, and that the operators were shown to be measurable into the relevant spaces in Subsection 2.1, we are in the setting of Appendix V. We now proceed to justify the assumptions of this appendix.

**Assumption 4.16:** This follows identically to Assumption 4.7, referring again to Subsection 2.1 and Lemma 1.3.

The process to justify Assumption 4.17 is the same as for 4.9, and we introduce the corresponding Lemma to 2.9.
Lemma 2.10. For any $\varepsilon > 0$, we have that
\[ |\langle P_L f - P_L g, f - g \rangle| \leq c(\varepsilon)\|f\|_2^2\|f - g\|_1^2 + \varepsilon\|f - g\|_1^2. \]

Proof. As in Lemma 2.9, we use the inequality
\[ |\langle P_L f - P_L g, f - g \rangle| \leq |\langle L f - g, f - g \rangle| + |\langle L_g (f - g), f - g \rangle|. \]

For the first term, appealing to (4), observe that
\[ |\langle L f - g, f - g \rangle| \leq \|L f - g\|\|f - g\| \leq \|f - g\|_1\|f\|_2\|f - g\|_1 \leq c(\varepsilon)\|f\|_2^2\|f - g\|_1^2 + \varepsilon\|f - g\|_1^2. \]

In fact the second term is null due to Corollary 1.23.1, which concludes the proof.

Assumption 4.17: Continuing to consider the distinct terms, we have that
\[ -2\nu\langle A(f - g), f - g \rangle = -2\nu\|f - g\|_1^2 \]
and
\[ \langle P_B^2_i (f - g), f - g \rangle + \|P_B_i (f - g)\|_1^2 \leq \langle B_i^2 (f - g), f - g \rangle + \|B_i (f - g)\|_1^2 \leq c\|\xi_i\|_{W^{2,\infty}}\|f - g\|_1^2. \]

from (20). With these components in place, the proof of (91) then follows identically to that of (81). (92) is a direct consequence of (21), concluding the justification.

Assumption 4.18: This stronger Assumption was in fact already verified in the address of Assumption 4.10.
3 Analysis of the Vorticity Equation

In order to address the well-posedness problem of the SALT Navier-Stokes Equation on the bounded domain, we now pose it in vorticity form. The analysis conducted in Subsection 2.2 was done with reference to the properties derived across Subsections 1.3 and 1.4, applicable to the bounded domain as well as the torus. The issue in studying the velocity form is that our operators do not map into the correct spaces in order to use these properties: in particular, the Leray Projector does not preserve the zero trace property and so the operators do not map into the necessary $W^{k,2}_σ(Ω;\mathbb{R}^N)$ spaces. The motivation behind the vorticity form is to circumvent the necessity of Leray Projection.

Throughout this section we shall specifically work in the case $N = 3$, understanding that the results similarly hold for $N = 2$. We do this to work with an explicit form of the curl as seen below.

Similarly our attentions shall be decidedly on the bounded domain $Ω$, though the results carry over seamlessly to the torus $\mathbb{T}^N$. For this section we impose new constraints on the $ξ_i$, which are such that for each $i \in \mathbb{N}$, $ξ_i \in W^{1,2}_σ(Ω;\mathbb{R}^3) \cap W^{2,2}_0(Ω;\mathbb{R}^3) \cap W^{3,∞}(Ω;\mathbb{R}^3)$ and they collectively satisfy

$$\sum_{i=1}^{∞} ||ξ_i||^2_{W^{3,∞}} < ∞.$$ (41)

3.1 Deriving the Equation

The vorticity form of the equation is derived through taking the curl of the velocity form, where the curl operator is defined for $f \in W^{1,2}(Ω;\mathbb{R}^3)$ by

$$\text{curl} f := \begin{pmatrix} \partial_2 f^3 - \partial_3 f^2 \\ \partial_3 f^1 - \partial_1 f^3 \\ \partial_1 f^2 - \partial_2 f^1 \end{pmatrix}.$$  

We introduce the operator $\mathcal{L}$ defined on sufficiently regular functions $f, g : Ω \to \mathbb{R}^3$ by

$$\mathcal{L}_fg := \mathcal{L}_f g - \mathcal{L}_g f.$$ (42)

In [38] Subesction 12.1 it is shown that, with notation $φ := \text{curl} f$,

$$\text{curl} (\mathcal{L}_f f) = \mathcal{L}_f φ$$

where it is also observed that the curl of elements of $W^{1,2}(Ω;\mathbb{R}^3) \cap L^{2,1}_σ(Ω;\mathbb{R}^3)$ is null. It is clear that the Laplacian commutes with the curl operation, and so in looking to take the curl of equation (1) it only remains to consider the SALT Operator $B$.

**Proposition 3.1.** We have that

$$\text{curl}(B_i f) = \mathcal{L}_{ξ_i} φ$$ (43)

where once more $φ := \text{curl} f$.

**Proof.** We shall show only that the identity (43) holds in its first component, with the others
following similarly. To this end we calculate the first component of the left hand side of (43):

\[
\text{curl}(B_i f) = \partial_2 [B_i f]^3 - \partial_3 [B_i f]^2
\]

\[
= \partial_2 \left( \sum_{j=1}^{3} \xi_j \partial_j f^3 + f^3 \partial_3 \xi_i \right) - \partial_3 \left( \sum_{j=1}^{3} \xi_j \partial_j f^2 + f^2 \partial_3 \xi_i \right)
\]

\[
= \sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^3 + \xi_j \partial_3 \partial_j f^2 + \partial_3 f^3 \partial_3 \xi_i + f^3 \partial_3 \partial_3 \xi_i \right)
\]

\[
- \sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^2 + \xi_j \partial_3 \partial_j f^2 + \partial_3 f^2 \partial_3 \xi_i + f^2 \partial_3 \partial_3 \xi_i \right)
\]

\[
= \sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^3 + \xi_j \partial_3 \partial_j f^2 + \partial_3 f^3 \partial_3 \xi_i + f^3 \partial_3 \partial_3 \xi_i \right)
\]

\[
= \sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^3 + \partial_3 f^3 \partial_3 \xi_i - \partial_3 \xi_j \partial_3 \partial_j f^2 + f^3 \partial_3 \partial_3 \xi_i \right)
\]

\[
= [L_{\xi_i} \phi]^1 + \sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^3 + \partial_3 f^3 \partial_3 \xi_i - \partial_3 \xi_j \partial_3 \partial_j f^2 + f^3 \partial_3 \partial_3 \xi_i \right).
\]

Therefore it remains to show that

\[
\sum_{j=1}^{3} \left( \partial_3 \xi_j \partial_j f^3 + \partial_3 f^3 \partial_3 \xi_i - \partial_3 \xi_j \partial_3 \partial_j f^2 - f^3 \partial_3 \partial_3 \xi_i \right) = -[L_{\phi \xi_i}]^1. \tag{44}
\]

We expand the sum in (44) to

\[
(\partial_2 \xi^1_1 \partial_1 f^3 + \partial_2 f^1 \partial_3 \xi^1_i - \partial_3 \xi^1_i \partial_1 f^2 - \partial_3 f^1 \partial_2 \xi^1_i) + (\partial_2 \xi^2_1 \partial_2 f^3 + \partial_2 f^2 \partial_3 \xi^2_i - \partial_3 \xi^2_i \partial_2 f^2 - \partial_3 f^2 \partial_2 \xi^2_i)
\]

+ (\partial_2 \xi^3_1 \partial_3 f^3 + \partial_2 f^3 \partial_3 \xi^3_i - \partial_3 \xi^3_i \partial_3 f^2 - \partial_3 f^3 \partial_2 \xi^3_i)

achieving some immediate cancellation in the second two brackets to

\[
(\partial_2 \xi^1_1 \partial_1 f^3 + \partial_2 f^1 \partial_3 \xi^1_i - \partial_3 \xi^1_i \partial_1 f^2 - \partial_3 f^1 \partial_2 \xi^1_i) + (\partial_2 \xi^2_1 \partial_2 f^3 - \partial_3 f^3 \partial_2 \xi^2_i) + (\partial_2 f^3 \partial_3 \xi^3_i - \partial_3 f^3 \partial_3 \xi^3_i).
\]

We now simply rewrite the above by combining like terms, into

\[
\partial_2 \xi^1_i (\partial_1 f^3 - \partial_3 f^1) + \partial_3 \xi^1_i (\partial_2 f^1 - \partial_1 f^2) + (\partial_2 \xi^2_1 \partial_2 f^3 - \partial_3 f^3 \partial_2 \xi^2_i) + (\partial_2 f^3 \partial_3 \xi^3_i - \partial_3 f^3 \partial_3 \xi^3_i).
\]

or more succinctly as

\[
-\partial_2 \xi^1_i \phi^3 - \partial_3 \xi^1_i \phi^3 + (\partial_2 \xi^2_1 + \partial_3 \xi^3_i) \phi^1.
\]

to which we add and subtract \(\partial_1 \xi^1_i \phi^1\) to arrive at

\[
-\sum_{j=1}^{3} \phi^j \partial_j \xi^1_i + \sum_{j=1}^{3} \left( \partial_j \xi^1_i \right) \phi^1.
\]

The first term is precisely \(-[L_{\phi \xi_i}]^1\) as we wished to show, appreciating that the second term is zero given the divergence free condition on \(\xi_i\).
From this point forwards we adopt familiar notation of $L_i := L_\xi_i$. Writing the Stratonovich integral of $u$ in its component form over the basis vectors of $U$, and introducing the notation $w := \text{curl} u$, at a heuristic level we can take the curl of (1) to obtain

$$w_t - w_0 + \int_0^t L_u w_s ds - \nu \int_0^t \Delta w_s ds + \sum_{i=1}^\infty \int_0^t L_i w_s dW^i_s = 0. \quad (45)$$

Having already braved the rigorous Itô conversion of (23) we shall make the conversion without explicit reference to the conditions of Appendix III (4.3), for reasons elucidated at the start of Subsection 3.2. So at least again then at the heuristic level, the Itô form is

$$w_t = w_0 - \int_0^t L_u w_s ds + \nu \int_0^t \Delta w_s ds + \frac{1}{2} \int_0^t \sum_{i=1}^\infty L_i^2 w_s ds - \sum_{i=1}^\infty \int_0^t L_i w dW^i_s$$

which can again be projected to the equation

$$w_t = w_0 - \int_0^t \mathcal{P} L_u w_s ds - \nu \int_0^t \mathcal{P} w_s ds + \frac{1}{2} \int_0^t \sum_{i=1}^\infty \mathcal{P} L_i^2 w_s ds - \sum_{i=1}^\infty \int_0^t \mathcal{P} L_i w dW^i_s. \quad (46)$$

Having motivated this section by an avoidance of the Leray Projection this may seem counter intuitive, however we shall shortly show that the projection is not felt in the noise (where it becomes problematic in velocity form). The goal is to deduce the existence of a unique maximal solution of (46), a task which requires some clarification. Having reached (46) from the velocity form, we now look to solve the equation for vorticity which demands a representation of the velocity $u$ in terms of the vorticity $w$. For this we quote Theorem 1 of [15] (or in fact, a slightly relaxed version).

**Theorem 3.2.** There exists a mapping $K : \mathcal{O} \times \mathcal{O} \to \mathbb{R}^3$ such that for every $\phi \in W^{1,2}_0(\mathcal{O}; \mathbb{R}^3) \cap W^{k,p}(\mathcal{O}; \mathbb{R}^3)$ where $k \in \mathbb{N} \cup \{0\}$, $1 < p < \infty$, the function $f : \mathcal{O} \to \mathbb{R}^3$ defined $\lambda$-a.e. by

$$f(x) = \int_\mathcal{O} K(x,y) \phi(y) dy \quad (47)$$

is such that:

1. $f \in L^2_0(\mathcal{O}; \mathbb{R}^3) \cap W^{k+1,p}(\mathcal{O}; \mathbb{R}^3)$;
2. $\text{curl} f = \phi$;
3. There exists a constant $C$ independent of $\phi$ (but dependent on $k, p$) such that

$$\|f\|_{W^{k+1,p}} \leq C \|\phi\|_{W^{k,p}}.$$

It should immediately be noted that such a $K$ is not claimed to be unique, and in [15] is explicitly shown to be non-unique, however it does allow us to identify a velocity from a given vorticity satisfying the divergence-free and boundary conditions (2). From this point forwards we fix a specific $K$ from the class of admissible integral kernels postulated in Theorem 3.2. We thus understand the nonlinear term as a mapping

$$\phi \mapsto \mathcal{L} f \phi$$

where $f$ is defined as in (47). This mapping shall at times be simply denoted by $\mathcal{L}$. The equation (46) is thus closed in $w$. 

30
3.2 Existence, Uniqueness and Maximality

We now state and prove the existence, uniqueness and maximality results for (46). We recall that to solve the velocity form (24) we used the extended criterion of Appendix V (4.5), requiring the space \( V := W^{3,2}_\sigma (\mathbb{T}^N; \mathbb{R}^N) \) to prove Theorem 2.5. This arose naturally in first showing Theorem 2.1, where we considered solutions explicitly in terms of the original Stratonovich form. For (46), however, solutions can be obtained for the natural choice of \( w_0 \in W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^3) \) (so the equation satisfies its identity in \( L^2(\mathcal{O}; \mathbb{R}^3) \)) with an application only of Theorem 4.15 in Appendix IV (4.4).

We thus do not take the detour of considering a fourth Hilbert Space to rigorously define solutions of the Stratonovich form (45), although this can be done similarly.

**Definition 3.3.** A pair \((w, \tau)\) where \(\tau\) is a \(\mathbb{P}\) – a.s. positive stopping time and \(w\) is a process such that for \(\mathbb{P}\) – a.e. \(\omega\), \(w(\omega) \in C \left( [0, T]; W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^3) \right)\) and \(w(\omega)\mathbb{1}_{\tau(\omega) < T} \in L^2 \left( [0, T]; W^{2,2}_\sigma (\mathcal{O}; \mathbb{R}^3) \right)\) for all \(T > 0\) with \(\mathbb{1}_{\tau(\omega) < T}\) progressively measurable in \(W^{2,2}_\sigma (\mathcal{O}; \mathbb{R}^3)\), is said to be a local strong solution of the equation (46) if the identity

\[
 w_t = w_0 - \int_0^{t \wedge \tau} \mathbb{P} \mathcal{L}_w w_s ds - \nu \int_0^{t \wedge \tau} A w_s ds + \frac{1}{2} \int_0^{t \wedge \tau} \sum_{i=1}^{\infty} \mathbb{P} \mathcal{L}_i^2 w_s ds - \sum_{i=1}^{\infty} \int_0^{t \wedge \tau} \mathbb{P} \mathcal{L}_i w_s dW_s^i
\]

holds \(\mathbb{P}\) – a.s. in \(L^2_\sigma (\mathcal{O}; \mathbb{R}^N)\) for all \(t \geq 0\).

**Definition 3.4.** A pair \((w, \Theta)\) such that there exists a sequence of stopping times \((\theta_j)\) which are \(\mathbb{P}\) – a.s. monotone increasing and convergent to \(\Theta\), whereby \((w, \wedge \theta_j, \theta_j)\) is a local strong solution of the equation (46) for each \(j\), is said to be a maximal strong solution of the equation (46) if for any other pair \((\eta, \Gamma)\) with this property then \(\Theta \leq \Gamma\ \mathbb{P}\) – a.s. implies \(\Theta = \Gamma\ \mathbb{P}\) – a.s..

**Definition 3.5.** A maximal strong solution \((w, \Theta)\) of the equation (46) is said to be unique if for any other such solution \((\eta, \Gamma)\), then \(\Theta = \Gamma\ \mathbb{P}\) – a.s. and for all \(t \in [0, \Theta)\),

\[
 \mathbb{P} \left( \{\omega \in \Omega : w_t(\omega) = \eta_t(\omega)\} \right) = 1.
\]

**Theorem 3.6.** For any given \(\mathcal{F}_0\) – measurable \(w_0 : \Omega \to W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^N)\), there exists a unique maximal strong solution \((w, \Theta)\) of the equation (46). Moreover at \(\mathbb{P}\) – a.e. \(\omega\) for which \(\Theta(\omega) < \infty\), we have that

\[
 \sup_{r \in [0, \Theta(\omega))] \|w_r(\omega)\|_1^2 + \int_0^{\Theta(\omega)} \|w_r(\omega)\|_2^2 dr = \infty.
\]

As discussed the idea is to apply Theorem 4.15, which we look to do for the spaces

\[
 V := W^{2,2}_\sigma (\mathcal{O}; \mathbb{R}^3), \quad H := W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^3), \quad U := L^2_\sigma (\mathcal{O}; \mathbb{R}^3).
\]

The density relations are clear as \(C^\infty_{0,\sigma}(\mathcal{O}; \mathbb{R}^3) \subset W^{2,2}_\sigma (\mathcal{O}; \mathbb{R}^3)\) is dense in both \(W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^3)\) and \(L^2_\sigma (\mathcal{O}; \mathbb{R}^3)\). The bilinear form (72) is simply again (40). Now we shift attentions to checking that the operators are measurable into the correct spaces. We note that \(\mathcal{L}\) has improved regularity properties over \(L\) given item 3 of Theorem 3.2, so retains the continuity observed in Lemma 1.2 with measurability following. There is no change to the Stokes Operator from Section 2. As for \(\mathcal{P}\mathcal{L}_i\), we in fact first show that for \(\mathcal{L}_i \in C \left( W^{2,2}_\sigma (\mathcal{O}; \mathbb{R}^3); W^{1,2}_\sigma (\mathcal{O}; \mathbb{R}^3) \right)\) (and hence is invariant under \(\mathcal{P}\)). This consists of three parts: showing that it is continuous as a mapping into \(W^{1,2}(\mathcal{O}; \mathbb{R}^3)\), showing the divergence free property and then the zero trace property. In fact with the appropriate regularity, it follows identically to Corollary 1.26.1 that we again have

\[
 \|\mathcal{L}_i \phi\|_{2W^{k,2}}^2 \leq c \|\xi_i\|_{W^{k,1,\infty}}^2 \|\phi\|_{W^{k+1,2}}^2 \quad (48)
\]
which addresses the continuity. The fact that \( \mathcal{L}_i \phi \) is divergence free comes immediately from the relation \( \mathcal{L}_i \phi = \text{curl}(B_i f) \) and the well-established fact the divergence of a curl is zero. For the zero trace property it is sufficient to show the existence of a sequence of compactly supported \( \eta_n \in W^{1,2}(\Omega;\mathbb{R}^3) \) which converge to \( \mathcal{L}_i \phi \) in \( W^{1,2}(\Omega;\mathbb{R}^3) \). By definition of the property that \( \xi_i \in W^{2,2}_0(\Omega;\mathbb{R}^3) \) there is a sequence \( (\gamma_n) \), \( \gamma_n \in C_0(\Omega;\mathbb{R}^3) \) such that \( \gamma_n \to \xi_i \) in \( W^{2,2}(\Omega;\mathbb{R}^3) \). Evidently \( \eta_n := \mathcal{L}_n \phi \) is again compactly supported, and observe that

\[
\| \mathcal{L}_{\gamma_n} \phi - \mathcal{L}_i \phi \|_{W^{1,2}} = \| \mathcal{L}_{\gamma_n} - \xi_i \|_{W^{1,2}} \leq c \| \gamma_n - \xi_i \|_{W^{2,2}} \| \phi \|_{W^{2,2}}
\]

from (6), which converges to zero as required to justify the zero trace property.

**Remark.** As \( \mathcal{P} \mathcal{L}_i \) is equal to \( L_i \) on \( W^{2,2}_0(\Omega;\mathbb{R}^3) \), then \( \mathcal{P} \mathcal{L}_i \) is equal to \( \mathcal{P} \mathcal{L}_i^2 \) on this space too, justifying the consistency between taking the Leray Projector and then converting from Stratonovich to Itô and vice versa as seen for (24).

The fact that \( \mathcal{P} \mathcal{L}_i \in C \left( W^{2,2}_0(\Omega;\mathbb{R}^3); L^2_\sigma(\Omega;\mathbb{R}^3) \right) \) again follows from the linearity, (48) and Proposition 1.9. We now proceed to justify the assumptions of Appendix IV, 4.4.

**Assumption 4.6:** We use the system \((a_k)\) of eigenfunctions of the Stokes Operator given in Proposition 1.13, satisfying (73) and (74) from Lemmas 1.21 and 1.22 respectively.

**Assumption 4.7:** Items (76), (77) follow identically to the justification of (89), (90) for (24) given the increased regularity of \( \mathcal{L} \) over \( \mathcal{L} \) and the corresponding boundedness of the noise term (48). With the linearity of \( \mathcal{L}_i \) then (78) follows trivially from (48).

In the following \( c \) will represent a generic constant changing from line to line, \( c(\varepsilon) \) will be a generic constant dependent on a fixed \( \varepsilon \), \( \phi \) and \( \psi \) will be arbitrary elements of \( W^{2,2}_0(\Omega;\mathbb{R}^3) \) and \( \phi_n \in \text{span}\{a_1, \ldots, a_n\} \). Towards Assumption 4.8, we appreciate some further properties of \( \mathcal{L}_i \) which were previously shown for \( B_i \). Of course the mapping \( \mathcal{L}_i \), given by \( \phi \mapsto \mathcal{L}_i \phi - L_\phi \xi_i \), is the sum of \( \mathcal{L}_i \) and a bounded linear operator on any \( W^{k,2}(\Omega;\mathbb{R}^3) \) as was the case for \( B_i \) (where the bounded linear operator was \( \mathcal{T}_\xi \)). Due to this structure, the results of Corollary 1.26.1, Proposition 1.27 and Proposition 1.29 continue to hold for \( \mathcal{L}_i \). Deduced from this is the following lemma.

**Lemma 3.7.** For any \( \varepsilon > 0 \) we have the bound

\[
\langle \mathcal{P}_n \mathcal{P} \mathcal{L}_i^2 \phi_n, \phi_n \rangle_1 + \langle \mathcal{P}_n \mathcal{P} \mathcal{L}_i \phi_n, \mathcal{P}_n \mathcal{P} \mathcal{L}_i \phi_n \rangle_1 \leq c(\varepsilon) \| \xi_i \|_{W^{3,\infty}}^2 \| \phi_n \|_1^2 + \varepsilon \| \xi_i \|_{W^{3,\infty}} \| \phi_n \|_2^2.
\]

**Proof.** This now follows precisely as for Lemma 2.7. \( \Box \)

**Lemma 3.8.** For any \( \varepsilon > 0 \) we have the bound

\[
| \langle \mathcal{P}_n \mathcal{P} \mathcal{L}_f \phi_n, \phi_n \rangle_1 | \leq c(\varepsilon) \| \phi_n \|_4^4 + \varepsilon \| \phi_n \|_2^2.
\]

**Proof.** Writing \( f_n \) corresponding to \( \phi_n \) as in Theorem 3.2, note that

\[
| \langle \mathcal{P}_n \mathcal{P} \mathcal{L}_f \phi_n, \phi_n \rangle_1 | = | \langle \mathcal{P}_n \mathcal{P} \mathcal{L}_f \phi_n, A \phi_n \rangle | \leq c(\varepsilon) \| \mathcal{L}_f \phi_n \|_2^2 + \varepsilon \| \phi_n \|_2^2
\]

and

\[
\| \mathcal{L}_f \phi_n \|_2^2 \leq 2 \left( \| \mathcal{L}_f \phi_n \|_2^2 + \| \mathcal{L}_\phi \phi_n \|_2^2 \right)
\]

32
Lemma 3.9. For any \( \varepsilon > 0 \), we have that
\[
|\langle \mathcal{P}_f \phi - \mathcal{P}_g \psi, \phi - \psi \rangle| \leq c(\varepsilon) \left( \|\phi\|_1^2 + \|\psi\|_1^2 \right) \|\phi - \psi\|^2 + \varepsilon \|\phi - \psi\|_1^2
\]

Proof. We write out the left hand side of the above in full:
\[
|\langle \mathcal{P}_f \phi - \mathcal{P}_g \psi, \phi - \psi \rangle| = |\langle \mathcal{L}_f \phi - \mathcal{L}_\phi f - \mathcal{L}_g \psi + \mathcal{L}_\psi y, \phi - \psi \rangle|
\]
\[
= |\langle \mathcal{L}_{f-g} \phi + \mathcal{L}_g (\phi - \psi) - \mathcal{L}_{\phi-f} \psi - \mathcal{L}_\psi (f - g), \phi - \psi \rangle|
\]
from which we shall split up the terms and control them individually. Firstly,
\[
|\langle \mathcal{L}_{f-g} \phi, \phi - \hat{\psi} \rangle| \leq \| \mathcal{L}_{f-g} \phi \| \| \phi - \hat{\psi} \|
\leq c \| f-g \|_2 \| \phi \|_1 \| \phi - \hat{\psi} \|
\leq c \| \phi - \hat{\psi} \|_1 \| \phi \|_1 \| \phi - \hat{\psi} \|
\leq c(\varepsilon) \| \phi \|_2^2 \| \phi - \hat{\psi} \|^2 + \frac{\varepsilon}{3} \| \phi - \hat{\psi} \|_1^2
\]
using \((4)\) and that \([f-g](x) = \int_\phi K(x,y)[\phi - \hat{\psi}](y)dy\) is the solution specified in Theorem 3.2 for \(\phi - \hat{\psi}\). Even more directly we have that
\[
\langle \mathcal{L}_g(\phi - \hat{\psi}), \phi - \hat{\psi} \rangle = 0
\]
owing to \((1.23.1)\), and for the final two terms the bounds
\[
|\langle \mathcal{L}_{\phi - \psi} f, \phi - \hat{\psi} \rangle| \leq c \| \phi - \hat{\psi} \|_1 \| f \|_2 \| \phi - \hat{\psi} \| \leq c(\varepsilon) \| \phi \|_2^2 \| \phi - \hat{\psi} \|^2 + \frac{\varepsilon}{3} \| \phi - \hat{\psi} \|_1^2
\]
and
\[
|\langle \mathcal{L}_{\phi}(f - g), \phi - \hat{\psi} \rangle| \leq c \| \phi \|_1 \| f - g \|_2 \| \phi - \hat{\psi} \| \leq c(\varepsilon) \| \phi \|_2^2 \| \phi - \hat{\psi} \|^2 + \frac{\varepsilon}{3} \| \phi - \hat{\psi} \|_1^2.
\]
Summing these terms concludes the proof. \(\square\)

**Assumption 4.9:** The justification now comes together exactly as in the proof for Assumption 4.17 in the velocity case, noting again that \((20)\) holds for \(\mathcal{L}_i\) as well, and using Lemma 3.9. \(\square\)

**Assumption 4.10:** There is very little to demonstrate here, as the linear terms follow from Assumption 4.9 so we just briefly address the nonlinear term. Through the same process as in Lemma 3.9, we have that
\[
|\langle \mathcal{P} \mathcal{L}_f \phi, \phi \rangle| \leq |\langle \mathcal{L}_f \phi, \phi \rangle + \langle \mathcal{L}_g f, \phi \rangle|
\leq c \| f \|_2 \| \phi \|_1 + \| \phi \|_1 \| f \|_2 \| \phi \|
\leq c \| \phi \| \| \phi \|_2^2
\]
where the rest simply follows as in Assumption 4.9. \(\square\)

**Assumption 4.11:** We consider the different operators in turn, starting with the nonlinear term and using that
\[
|\langle \mathcal{P} \mathcal{L}_f \phi - \mathcal{P} \mathcal{L}_g \phi - \mathcal{L}_g \phi - \mathcal{L}_g \phi \rangle| = |\langle \mathcal{L}_f \phi - \mathcal{L}_g f + \mathcal{L}_g \phi - \mathcal{L}_g \phi \rangle|
\leq |\langle \mathcal{L}_{f-g} \phi + \mathcal{L}_g (\phi - \hat{\psi}) - \mathcal{L}_{\phi - \psi} f - \mathcal{L}_g (f - g), \eta \rangle|
\]
where exactly as in Lemma 3.9 we have that
\[
\| \mathcal{L}_{f-g} \phi + \mathcal{L}_g (\phi - \hat{\psi}) - \mathcal{L}_{\phi - \psi} f - \mathcal{L}_g (f - g) \| \leq c \| \| \phi \|_1 + \| \psi \|_1 \| \phi - \hat{\psi} \|_1
\]
so in particular
\[
|\langle \mathcal{P} \mathcal{L}_f \phi - \mathcal{P} \mathcal{L}_g \phi, \eta \rangle| \leq \| \eta \| (c \| \phi \|_1 + \| \psi \|_1 \| \phi - \hat{\psi} \|_1). \quad (49)
\]
For the Stokes Operator we simply apply Proposition 1.18 to see that
\[
|\langle \mathcal{A} \phi - \mathcal{A} \psi, \eta \rangle| = |\langle \mathcal{A} (\phi - \psi), \eta \rangle| = |\langle \phi - \psi, \eta \rangle| \leq \| \phi - \psi \|_1 \| \eta \|_1 \quad (50)
\]
and for the \(\mathcal{L}_i^2\) term we use Corollary 1.26.1 to observe that
\[
|\langle \mathcal{L}_i^2 \phi - \mathcal{L}_i^2 \psi, \eta \rangle| = |\langle \mathcal{L}_i^2 (\phi - \psi), \eta \rangle| = |\langle \mathcal{L}_i (\phi - \psi), \mathcal{L}_i \phi \rangle| \leq c \| \xi_i \|_{W^{1,\infty}} \| \phi - \psi \|_1 \| \eta \|_1. \quad (51)
\]
Combining \((49),(50)\) and \((51)\) gives the result. \(\square\)
4 Appendices

4.1 Appendix I: Proofs from Subsection 1.3

Before proving the promised results, we state the Sobolev Embeddings which are vital for our analysis. These can be found in [1] Section 4.

**Theorem 4.1.** Letting \( k \in \{0\} \cup \mathbb{N} \) be arbitrary and \( ' \hookrightarrow ' \) denote 'embeds into', we have the following results:

- For \( mp > N \), or \( m = N \) and \( p = 1 \),
  \[
  W^{m+k,p}(O; \mathbb{R}) \hookrightarrow C^k_b(O; \mathbb{R})
  \]  
  and further if \( p \leq r \leq \infty \), then
  \[
  W^{m+k,p}(O; \mathbb{R}) \hookrightarrow W^{k,r}(O; \mathbb{R})
  \]
  and in particular
  \[
  W^{m,p}(O; \mathbb{R}) \hookrightarrow L^r(O; \mathbb{R}).
  \]

- For \( mp = N \), if \( p \leq r < \infty \) then (53) and (54) again hold.

- For \( mp < N \), if \( p \leq r \leq Np/(N - mp) \) then (53) and (54) again hold.

The embedding is a continuous linear operator, where the boundedness constant is dependent only on the choices of \( m, p, N, r, k \).

**Corollary 4.1.1.** We have the embeddings

\[
W^{2,2}(O; \mathbb{R}) \hookrightarrow L^\infty(O; \mathbb{R})
\]
\[
W^{1,2}(O; \mathbb{R}) \hookrightarrow L^6(O; \mathbb{R}).
\]

**Proof.** Immediate from (54), from the first and third bullet points respectively.

We also state the following classical result.

**Theorem 4.2** (Gagliardo-Nirenberg Inequality). Let \( p, q, \alpha \in \mathbb{R} \), \( m \in \mathbb{N} \) be such that \( p > q \geq 1 \), \( m > N(\frac{1}{2} - \frac{1}{p}) \) and \( \frac{1}{p} = \frac{\alpha}{q} + (1 - \alpha)(\frac{1}{2} - \frac{m}{N}) \). Then there exists a constant \( c \) (dependent on the given parameters) such that for any \( f \in L^p(T^N; \mathbb{R}) \cap W^{m,2}(T^N; \mathbb{R}) \), we have

\[
\| f \|_{L^p(T^N; \mathbb{R})} \leq c \| f \|_{L^q(T^N; \mathbb{R})}^{\alpha} \| f \|_{W^{m,2}(T^N; \mathbb{R})}^{1-\alpha}.
\]  

In the case \( f \in L^p(\mathcal{O}; \mathbb{R}) \cap W^{m,2}(\mathcal{O}; \mathbb{R}) \) we have instead

\[
\| f \|_{L^p(\mathcal{O}; \mathbb{R})} \leq c \left( \| f \|_{L^q(\mathcal{O}; \mathbb{R})}^{\alpha} \| f \|_{W^{m,2}(\mathcal{O}; \mathbb{R})}^{1-\alpha} + \| f \|_{L^2(\mathcal{O}; \mathbb{R})} \right).
\]

**Proof.** See [37] pp.125-126.
Proof of 1.2: We shall directly jump to the continuity of the mapping before showing it is in fact well defined, as the latter argument will be contained in the former. We proceed by sequential continuity, so we take an arbitrary sequence \((f_n)\) convergent to some \(f\) in \(W^{m+1,2}(\mathcal{O}; \mathbb{R}^N)\) and deduce the convergence of \((\mathcal{L}_{f_n} f_n)\) to \(\mathcal{L}_f f\). We have for any \(n\)

\[
\|\mathcal{L}_{f_n} f_n - \mathcal{L}_f f\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)} = \|\mathcal{L}_{f_n} f_n - \mathcal{L}_{f_n} f + \mathcal{L}_{f_n} f - \mathcal{L}_f f\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)} \\
\leq \|\mathcal{L}_{f_n} f_n - \mathcal{L}_{f_n} f\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)} + \|\mathcal{L}_{f_n} f - \mathcal{L}_f f\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)} \\
= \left\| \sum_{j=1}^{N} f_n^j (\partial_j f_n - \partial_j f) \right\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)} + \left\| \sum_{j=1}^{N} (f_n^j - f^j) \partial_j f \right\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)}
\]

(57)

The norms are defined by the \(W^{m,2}(\mathcal{O}; \mathbb{R})\) norms of the component mappings, so we consider the \(i\)th component. Looking at the first term, we have

\[
\left\| \sum_{j=1}^{N} f_n^j (\partial_j f_n - \partial_j f^i) \right\|^2_{W^{m,2}(\mathcal{O}; \mathbb{R})} = \sum_{0 \leq |\alpha| \leq m} \left\| \sum_{j=1}^{N} \mathcal{D}^\alpha \left( f_n^j (\partial_j f_n^i - \partial_j f^i) \right) \right\|^2_{L^2(\mathcal{O}; \mathbb{R})}
\]

which we further split up by considering each \(\alpha\) in the sum, introducing the notation for a multi-index \(\alpha'\) whereby \(\alpha' \leq \alpha\) mean \(\alpha'_l \leq \alpha_l\) all \(1 \leq l \leq N\) \((\alpha'_l \in \mathbb{N} \cup \{0\})\). Further by \(\alpha - \alpha'\) we mean the multi-index with components \((\alpha - \alpha')_l = \alpha_l - \alpha'_l\). Then by the Leibniz Rule, for each \(j\),

\[
\mathcal{D}^\alpha \left( f_n^j (\partial_j f_n^i - \partial_j f^i) \right) = \sum_{\alpha' \leq \alpha} \mathcal{D}^{\alpha - \alpha'} f_n^j \left( \mathcal{D}^{\alpha'} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha'} \mathcal{D}_j f^i \right)
\]

(58)

so

\[
\| (58) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 \leq 2^{|\alpha|} \sum_{\alpha' \leq \alpha} \| \mathcal{D}^{\alpha - \alpha'} f_n^j \left( \mathcal{D}^{\alpha'} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha'} \mathcal{D}_j f^i \right) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 \\
\leq 2^m \sum_{\alpha' \leq \alpha} \| \mathcal{D}^{\alpha - \alpha'} f_n^j \left( \mathcal{D}^{\alpha'} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha'} \mathcal{D}_j f^i \right) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 \\
= 2^m \left( \sum_{\alpha' \leq \alpha} \| \mathcal{D}^{\alpha - \alpha'} f_n^j \left( \mathcal{D}^{\alpha'} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha'} \mathcal{D}_j f^i \right) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 + \| f_n^j \left( \mathcal{D}^{\alpha} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha} \mathcal{D}_j f^i \right) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 \right) \\
\leq 2^m \left( \sum_{\alpha' \leq \alpha} \| \mathcal{D}^{\alpha - \alpha'} f_n^j \|_{L^1(\mathcal{O}; \mathbb{R})}^2 \| \mathcal{D}^{\alpha'} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha'} \mathcal{D}_j f^i \|_{L^1(\mathcal{O}; \mathbb{R})}^2 + \| f_n^j \|_{L^\infty(\mathcal{O}; \mathbb{R})} \| \left( \mathcal{D}^{\alpha} \mathcal{D}_j f_n^i - \mathcal{D}^{\alpha} \mathcal{D}_j f^i \right) \|_{L^2(\mathcal{O}; \mathbb{R})}^2 \right)
\]

having applied Hölder type inequalities and invoking two different Sobolev embeddings; the first is the embedding of \(W^{1,2}(\mathcal{O}; \mathbb{R})\) into \(L^4(\mathcal{O}; \mathbb{R})\), noting of course that the operators \(\mathcal{D}^{\alpha - \alpha'}\) and \(\mathcal{D}^{\alpha'} \mathcal{D}_j\) are of order no greater than \(m\) so the terms are indeed in \(W^{1,2}(\mathcal{O}; \mathbb{R})\). The second is the embedding of \(W^{2,2}(\mathcal{O}; \mathbb{R})\) into \(L^\infty(\mathcal{O}; \mathbb{R})\) (and hence the embedding of \(W^{m+1,2}(\mathcal{O}; \mathbb{R})\) into \(L^\infty(\mathcal{O}; \mathbb{R})\)). Returning now to (57),

\[
\left\| \sum_{j=1}^{N} f_n^j (\partial_j f_n - \partial_j f) \right\|_{W^{m,2}(\mathcal{O}; \mathbb{R}^N)}^2 = \sum_{i=1}^{N} \sum_{|\alpha| \leq m} \left\| \sum_{j=1}^{N} \mathcal{D}^\alpha \left( f_n^j (\partial_j f_n^i - \partial_j f^i) \right) \right\|^2_{L^2(\mathcal{O}; \mathbb{R})} \\
\leq N \sum_{i=1}^{N} \sum_{|\alpha| \leq m} \left\| \sum_{j=1}^{N} \mathcal{D}^\alpha \left( f_n^j (\partial_j f_n^i - \partial_j f^i) \right) \right\|^2_{L^2(\mathcal{O}; \mathbb{R})}
\]

36
at which point we use the bound derived on (58) to deduce further that this is

\[
\leq 2^m N \sum_{i=1}^N \sum_{|\alpha| \leq m} \sum_{j=1}^N \left( \sum_{\alpha' < \alpha} \|D^{\alpha-\alpha'} f_n^i\|_{L^4(O; \mathbb{R})}^2 \|D^{\alpha'} \partial_j f_n^i - D^{\alpha'} \partial_j f_n\|_{L^2(O; \mathbb{R})}^2 \right).
\]

Now the quoted embeddings are continuous, so we can deduce convergence of \((D^{\alpha'} \partial_j f_n^i)\) to \(D^{\alpha'} \partial_j f^i\) in \(L^4(O; \mathbb{R})\) from the convergence in \(W^{1,2}(O; \mathbb{R})\), which is itself inherited from the assumed convergence of \((f_n)\) to \(f\) in \(W^{m+1,2}(O; \mathbb{R}^N)\). A similar argument affords us control of \(\|D^{\alpha'-\alpha} f_n^j\|_{L^4(O; \mathbb{R})}\), as the sequence \((D^{\alpha'-\alpha} f_n^j)\) converges in \(W^{1,2}(O; \mathbb{R})\) and hence \(L^4(O; \mathbb{R})\). Thus we can bound this term uniformly, say by \(\|D^{\alpha'-\alpha} f_n^j\|_{L^4(O; \mathbb{R})} < 1\) for sufficiently large \(n\). We implor the same idea to bound \(\|f_n^j\|_{L^\infty(O; \mathbb{R})}\) noting also that \((D^\alpha \partial_j f_n^i)\) converges to \(D^\alpha \partial_j f^i\) in \(L^2(O; \mathbb{R})\) from the assumed \(W^{m+1,2}(O; \mathbb{R}^N)\) convergence. Thus all terms in the finite sum above converge to 0 as \(n \to \infty\), which deals with the first term in (57). In fact the second term is treated near identically so we conclude the proof here, noting of course that these Hölder type inequalities are what justifies the mapping to be well defined.

\[\square\]

**Proof of 1.3:** We use the same ideas as seen in the proof of Lemma 1.2. For (4) starting with the term \(L_f g\):

\[
\|L_f g\| \leq \sum_{j=1}^N \|f^j \partial_j g\|
\leq \sum_{j=1}^N \sum_{l=1}^N \|f^j \partial_j g^l\|_{L^2(O; \mathbb{R})}
\leq \sum_{j=1}^N \sum_{l=1}^N \|f^j\|_{L^\infty(O; \mathbb{R})} \|\partial_j g^l\|_{L^2(O; \mathbb{R})}
\leq c \sum_{j=1}^N \sum_{l=1}^N \|f^j\|_{W^{2,2}(O; \mathbb{R})} \|\partial_j g^l\|_{L^2(O; \mathbb{R})}
\leq c \|f\|_{W^{2,2}} \|g\|_{W^{1,2}}
\]

using that \(\left(\sum_{i=1}^N |a_i|^2\right)^{1/2} \leq \sum_{i=1}^N |a_i|\) and the Sobolev Embedding \(W^{2,2}(O; \mathbb{R}) \hookrightarrow L^\infty(O; \mathbb{R})\).
Similarly comes the bound
\[
\|\mathcal{L}_g f\| \leq \sum_{j=1}^{N} \sum_{l=1}^{N} \|g^j \partial_j f^l\|_{L^2(\Omega; \mathbb{R})}
\]
\[
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} \|g^j\|_{L^4(\Omega; \mathbb{R})} \|\partial_j f^l\|_{L^4(\Omega; \mathbb{R})}
\]
\[
\leq c \sum_{j=1}^{N} \sum_{l=1}^{N} \|g^j\|_{W^{1,2}(\Omega; \mathbb{R})} \|\partial_j f^l\|_{W^{1,2}(\Omega; \mathbb{R})}
\]
\[
\leq c \sum_{j=1}^{N} \sum_{l=1}^{N} \|g^j\|_{W^{1,2}(\Omega; \mathbb{R})} \|f^l\|_{W^{2,2}(\Omega; \mathbb{R})}
\]
\[
\leq c \|g\|_{W^{1,2}} \|f\|_{W^{2,2}}
\]

applying the embedding of \(W^{1,2}(\Omega; \mathbb{R}) \hookrightarrow L^4(\Omega; \mathbb{R})\). This justifies (4). As for (5),
\[
\|\mathcal{L}_g f\|_{W^{1,2}}^2 = \|\mathcal{L}_g f\|^2 + \sum_{k=1}^{N} \|\partial_k \mathcal{L}_g f\|^2
\]
where
\[
\|\mathcal{L}_g f\|^2 \leq c \|g\|_{W^{1,2}}^2 \|f\|_{W^{2,2}}^2 \leq c \|g\|_{W^{1,2}}^2 \|f\|_{W^{3,2}}^2
\]
having applied (4), and
\[
\sum_{k=1}^{N} \|\partial_k \mathcal{L}_g f\|^2 \leq c \sum_{j=1}^{N} \sum_{k=1}^{N} (\|\partial_k g^j \partial_j f^l\|^2 + \|g^j \partial_k \partial_j f^l\|^2)
\]
\[
= c \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \left(\|\partial_k g^j \partial_j f^l\|_{L^2(\Omega; \mathbb{R})}^2 + \|g^j \partial_k \partial_j f^l\|_{L^2(\Omega; \mathbb{R})}^2\right)
\]
\[
\leq c \sum_{k=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{N} \left(\|\partial_k g^j \partial_j f^l\|_{L^2(\Omega; \mathbb{R})}^2 + \|g^j \partial_k \partial_j f^l\|_{L^2(\Omega; \mathbb{R})}^2\right)
\]
\[
\leq c \sum_{j=1}^{N} \sum_{l=1}^{N} \left(\|\partial_k g^j \partial_j f^l\|_{W^{2,2}(\Omega; \mathbb{R})}^2 + \|g^j \partial_k \partial_j f^l\|_{W^{2,2}(\Omega; \mathbb{R})}^2\right)
\]
\[
\leq c \|g\|_{W^{1,2}}^2 \|f\|_{W^{3,2}}^2.
\]
Combining these justifies (5). Indeed (6) is attained by the same procedure, just instead passing to the bounds
\[
\|\partial_k g^j \partial_j f^l\|_{L^2(\Omega; \mathbb{R})} + \|g^j \partial_k \partial_j f^l\|_{L^2(\Omega; \mathbb{R})} \leq \|\partial_k g^j\|_{L^4(\Omega; \mathbb{R})}^2 \|\partial_j f^l\|_{L^4(\Omega; \mathbb{R})}^2 + \|g^j\|_{L^\infty(\Omega; \mathbb{R})} \|\partial_k \partial_j f^l\|_{L^4(\Omega; \mathbb{R})}^2
\]
and proceeding in the same manner.

**Proof of 1.23:** We prove this result in the case of the bounded domain \(\Omega\), noting that the Torus follows in the same way just without the detour to compactly supported functions. The first task is to show that these inner products are well defined, which immediately prompts clarification as to how the \(L^2(\mathcal{T}; \mathbb{R}^N)\) inner product is being used here. Formally this should be understood
as the duality bracket between $L^{6/5}(\mathcal{O}; \mathbb{R}^N)$ and $L^6(\mathcal{O}; \mathbb{R}^N)$, noting that the reciprocal of these components sum to 1 from which the duality follows. Of course, this duality is prescribed by the $L^2(\mathcal{O}; \mathbb{R}^N)$ inner product. Without loss of generality we treat the left hand side of (14) for arbitrarily chosen $\phi, f, g$, and consider the $l^{th}$ component. We have that

$$
\| (\mathcal{L}_\phi f)^l \|_{L^{6/5}(\mathcal{O}; \mathbb{R})} \leq \sum_{j=1}^{N} \| \phi^j \partial_j f^l \|_{L^{6/5}(\mathcal{O}; \mathbb{R})}
$$

$$
\leq \sum_{j=1}^{N} \| \phi^j \|_{L^3(\mathcal{O}; \mathbb{R})} \| \partial_j f^l \|_{L^2(\mathcal{O}; \mathbb{R})}
$$

having applied a general Hölder Inequality, noting that

$$
\frac{5}{6} = \frac{1}{3} + \frac{1}{2}
$$

Indeed this expression is finite from the embedding $W^{1,2}(\mathcal{O}; \mathbb{R}) \hookrightarrow L^3(\mathcal{O}; \mathbb{R})$. Subsequently

$$
\| \mathcal{L}_\phi f \|_{L^{6/5}(\mathcal{O}; \mathbb{R}^N)} = \left( \sum_{l=1}^{N} \| (\mathcal{L}_\phi f)^l \|_{L^{6/5}(\mathcal{O}; \mathbb{R})} \right)^{5/6} < \infty
$$

as required. Showing that $g \in L^6(\mathcal{O}; \mathbb{R}^N)$ is simply immediate from the stronger embedding $W^{1,2}(\mathcal{O}; \mathbb{R}) \hookrightarrow L^6(\mathcal{O}; \mathbb{R})$. To show the desired equality (14) consider a sequence $(\phi_n)$ in $C^\infty_{0,\sigma}(\mathcal{O}; \mathbb{R}^N)$ convergent to $\phi$ in $W^{1,2}(\mathcal{O}; \mathbb{R}^N)$, which certainly exists from Lemma 1.7. Then

$$
\langle \mathcal{L}_{\phi_n} f, g \rangle = \sum_{j=1}^{N} \phi_n^j \partial_j f^l, g^l \rangle_{L^2(\mathcal{O}; \mathbb{R})}
$$

$$
= \sum_{l=1}^{N} \sum_{j=1}^{N} \langle \phi_n^j \partial_j f^l, g^l \rangle_{L^2(\mathcal{O}; \mathbb{R})} = \left( \sum_{l=1}^{N} \langle f^l, \phi_n^j g^l \rangle_{L^2(\mathcal{O}; \mathbb{R})} + \langle f^l, \phi_n^j \partial_j g^l \rangle_{L^2(\mathcal{O}; \mathbb{R})} \right)
$$

having applied integration by parts and calling upon the compact support of $\phi_n$. Now note that upon summation over $j$, the first inner product is nullified thanks to the divergence free property of $\phi_n$. We can now collapse the second inner product back:

$$
- \sum_{l=1}^{N} \sum_{j=1}^{N} \langle f^l, \phi_n^j \partial_j g^l \rangle_{L^2(\mathcal{O}; \mathbb{R})} = - \sum_{j=1}^{N} \langle f, \phi_n^j \partial_j g \rangle = - \langle f, \mathcal{L}_{\phi_n} g \rangle.
$$

It only remains to show that the equality holds in the limit, so let’s again treat the LHS of (14) for the $l^{th}$ component. Using the Hölder Inequality which defines the duality bracket,

$$
\| (\mathcal{L}_\phi f - \mathcal{L}_{\phi_n} f)^l \|_{L^2(\mathcal{O}; \mathbb{R})} \leq \| (\mathcal{L}_\phi f - \mathcal{L}_{\phi_n} f)^l \|_{L^{6/5}(\mathcal{O}; \mathbb{R})} \| g^l \|_{L^6(\mathcal{O}; \mathbb{R})}
$$

$$
\leq \left( \sum_{j=1}^{N} \| (\phi - \phi_n)^j \|_{L^3(\mathcal{O}; \mathbb{R})} \| \partial_j f^l \|_{L^2(\mathcal{O}; \mathbb{R})} \right) \| g^l \|_{L^6(\mathcal{O}; \mathbb{R})}
$$

39
demonstrating the required convergence in the limit from the continuity of the embedding \( W^{1,2}(\mathcal{O}; \mathbb{R}) \) into \( L^3(\mathcal{O}; \mathbb{R}) \). Thus we have the convergence for each component in the \( L^2(\mathcal{O}; \mathbb{R}^N) \) inner product, from which we conclude the result.

\[ \square \]

**Proof of 1.24**: We prove this in the more involved case of the bounded domain \( \mathcal{O} \), calling upon Theorem 4.2 and using (56) in this case instead of the simpler (55). For such \( f \) and \( g \) observe that

\[
\| L_{fg} \| = \left\| \sum_{j=1}^{N} f^j \partial_j g \right\|
\leq \sum_{j=1}^{N} \| f^j \partial_j g \|
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} \| f^j \partial_j g^l \|_{L^2(\mathcal{O}; \mathbb{R})}
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} \| f^j \|_{L^6(\mathcal{O}; \mathbb{R})} \| \partial_j g^l \|_{L^3(\mathcal{O}; \mathbb{R})}
\]

having just used a Hölder Inequality. We now apply Theorem 4.2 for the values \( p = 3, q = 2, m = 1 \) and \( \alpha = \frac{1}{2} \) to bound this further by

\[
\sum_{j=1}^{N} \sum_{l=1}^{N} c \| f^j \|_{L^6(\mathcal{O}; \mathbb{R})} \left( \| \partial_j g^l \|_{L^2(\mathcal{O}; \mathbb{R})}^{1/2} \| \partial_j g^l \|_{W^{1,2}(\mathcal{O}; \mathbb{R})}^{1/2} + \| \partial_j g^l \|_{L^2(\mathcal{O}; \mathbb{R})} \right)
\]

We now use the Sobolev Embedding \( W^{1,2}(\mathcal{O}; \mathbb{R}) \hookrightarrow L^6(\mathcal{O}; \mathbb{R}) \) from the third bullet point of Theorem 4.1 and some coarse bounds to control this further by

\[
\sum_{j=1}^{N} \sum_{l=1}^{N} c \| f^j \|_{W^{1,2}(\mathcal{O}; \mathbb{R})} \left( \| \partial_j g^l \|_{L^2(\mathcal{O}; \mathbb{R})}^{1/2} \| \partial_j g^l \|_{W^{1,2}(\mathcal{O}; \mathbb{R})}^{1/2} + \| \partial_j g^l \|_{L^2(\mathcal{O}; \mathbb{R})} \right)
\leq \sum_{j=1}^{N} \sum_{l=1}^{N} c \| f \|_{W^{1,2}} \left( \| g \|_{W^{1,2}}^{1/2} \| g \|_{W^{2,2}}^{1/2} + \| g \|_{W^{1,2}} \right)
\leq c \| f \|_{1} \left( \| g \|_{1}^{1/2} \| g \|_{2}^{1/2} + \| g \|_{1} \right)
\]

using the norm equivalences from Proposition 1.17.

\[ \square \]
4.2 Appendix II: Proofs from Subsection 1.4

In the following proofs we will track the constant \( c \) stated in the results, though we make no attempt to optimise the tracked constant that we give. \( \eta_k \) will represent the number of distinct multi-indices \( \alpha \) such that \(|\alpha| \leq k\).

Proof of 1.25: We fix any such \( k \), and consider differential operators \( D^\alpha \) for \(|\alpha| \leq k\). From the definition of the operator \( T_{\xi} \) and the Leibniz Rule, it is clear that

\[
D^\alpha T_{\xi_i} f = \sum_{\alpha' \leq \alpha} T_{D^{\alpha - \alpha'} \xi_i} D^{\alpha'} f.
\]

Now

\[
\|D^\alpha T_{\xi_i} f\|_2^2 \leq 2^{|\alpha|} \sum_{\alpha' \leq \alpha} \|T_{D^{\alpha - \alpha'} \xi_i} D^{\alpha'} f\|_2^2
\]

\[
\leq 2^k \sum_{\alpha' \leq \alpha} \|T_{D^{\alpha - \alpha'} \xi_i} D^{\alpha'} f\|_2^2
\]

\[
= 2^k \sum_{\alpha' \leq \alpha} \left\| \sum_{j=1}^N D^{\alpha'} f^j (\nabla D^{\alpha - \alpha'} \xi_i^j) \right\|_2^2
\]

\[
\leq 2^k \sum_{\alpha' \leq \alpha} \|D^{\alpha - \alpha'} \xi_i\|_{W^{1,\infty}} \left\| \sum_{j=1}^N D^{\alpha'} f^j \right\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq 2^k \|\xi_i\|_{W^{k+1,\infty}} \sum_{\alpha' \leq \alpha} \left\| D^{\alpha'} f \right\|_{L^2(O;\mathbb{R})}^2
\]

\[
= 2^k N \|\xi_i\|_{W^{k+1,\infty}} \sum_{\alpha' \leq \alpha} \|D^{\alpha'} f\|_2^2
\]

\[
\leq 2^k N \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k,2}}^2
\]

so ultimately

\[
\|T_{\xi_i} f\|_{W^{k,2}}^2 = \sum_{|\alpha| \leq k} \|D^\alpha T_{\xi_i} f\|_2^2
\]

\[
\leq 2^k N \eta_k \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k,2}}^2.
\]

Proof of 1.26, (18): We fix any such \( k \), and similarly to 1.25 it is clear that

\[
D^\alpha \mathcal{L}_{\xi_i} f = \sum_{\alpha' \leq \alpha} \mathcal{L}_{D^{\alpha - \alpha'} \xi_i} D^{\alpha'} f
\]
again for any $|\alpha| \leq k$. Looking at each term in this sum,

\[
\|L_{D^{\alpha'\xi_i}} D^{\alpha'} f\|^2 = \left\| \sum_{j=1}^{N} D^{\alpha_\xi_i \partial_j D^{\alpha'} f} \right\|^2
\]

\[
\leq N \sum_{j=1}^{N} \sum_{l=1}^{N} \|D^{\alpha_\xi_i \partial_j D^{\alpha'} f}\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq N \sum_{j=1}^{N} \sum_{l=1}^{N} \|D^{\alpha_\xi_i \partial_j D^{\alpha'} f}\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq N \sum_{j=1}^{N} \sum_{l=1}^{N} \|D^{\alpha_\xi_i \partial_j D^{\alpha'} f}\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq N \sum_{j=1}^{N} \sum_{l=1}^{N} \|D^{\alpha_\xi_i \partial_j D^{\alpha'} f}\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq N \|\xi_i\|_{W^{k,\infty}} \sum_{j=1}^{N} \|D^{\alpha_\xi_i \partial_j D^{\alpha'} f}\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq N \|\xi_i\|_{W^{k,\infty}} \|D^{\alpha} f\|_{W^{k+1,2}}^2
\]

moreover

\[
\|L_{\xi_i} f\|_{W^{k,2}}^2 = \sum_{|\alpha| \leq k} \|D^{\alpha} L_{\xi_i} f\|_{L^2(O;\mathbb{R})}^2
\]

\[
= \sum_{|\alpha| \leq k} \left\| \sum_{\alpha' \leq \alpha} L_{D^{\alpha'\xi_i}} D^{\alpha'} f \right\|^2
\]

\[
\leq \sum_{|\alpha| \leq k} \sum_{\alpha' \leq \alpha} \|L_{D^{\alpha'\xi_i}} D^{\alpha'} f\|_{L^2(O;\mathbb{R})}^2
\]

\[
\leq 2^{k} N \eta_k \|\xi_i\|_{W^{k,\infty}} \|D^{\alpha} f\|_{W^{k+1,2}}^2.
\]

\[\square\]

In the following proofs we generalise the definition of $B$ to a mapping

\[B_{fg} := L_{fg} + \mathcal{T}_{fg}.\]

In particular, $B_{i} := B_{\xi_i}$.

**Proof of 1.27, (20):** We consider terms

\[
\langle B^2_i f, f \rangle_{W^{k,2}} + \|B_i f\|_{W^{k,2}}^2
\]

and each derivative in the sum for the inner product: that is, we are looking at

\[
\langle D^\alpha B^2_i f, D^\alpha f \rangle + \langle D^\alpha B_i f, D^\alpha B_i f \rangle
\]

where the only starting place here is to simplify these derivatives. Combining the arguments in 1.25 and 1.26 then evidently

\[
D^\alpha B_{\xi_i} f = \sum_{\alpha' \leq \alpha} B_{D^{\alpha'\xi_i}} D^{\alpha'} f
\]

\[
= \sum_{\alpha' < \alpha} B_{D^{\alpha'\xi_i}} D^{\alpha'} f + B_{\xi_i} D^\alpha f.
\]

42
Plugging this result in, we also see that
\[
D^\alpha B_{\xi_i}^2 f = D^\alpha B_{\xi_i} (B_{\xi_i} f) = \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f + B_{\xi_i} D^\alpha B_{\xi_i} f
\]
which will use in our analysis of (60), reducing the expression to
\[
\left\langle \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right\rangle + \left\langle D^\alpha B_{\xi_i} f, B_{\xi_i}^* D^\alpha f \right\rangle + \left\langle D^\alpha B_{\xi_i} f, D^\alpha B_{\xi_i} f \right\rangle
\]
which we further break up in terms of the $L^2(\mathcal{O}; \mathbb{R}^N)$ adjoint $B_{\xi_i}^*$:
\[
\left\langle \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right\rangle + \left\langle D^\alpha B_{\xi_i} f, B_{\xi_i}^* D^\alpha f + \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} f + B_{\xi_i} D^\alpha f \right\rangle
\]
requiring that $D^\alpha f \in W^{1,2}$, which is satisfied by the assumption $f \in W^{k+2,2}$. By combining the second and third inner products and using (61), this becomes
\[
\left\langle \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right\rangle + \left\langle D^\alpha B_{\xi_i} f, (B_{\xi_i} + \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i}) D^\alpha f \right\rangle
\]
which we look to simplify by combining $B_{\xi_i}^*$ and $B_{\xi_i}$, noting that
\[
B_{\xi_i}^* + B_i = L_{\xi_i}^* + T_{\xi_i}^* + L_{\xi_i} + T_i = T_{\xi_i}^* + T_i.
\]
Indeed this arrives us at the expression
\[
\left\langle \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right\rangle + \left\langle D^\alpha B_{\xi_i} f, (T_{\xi_i} + T_{\xi_i}^*) D^\alpha f + \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} f \right\rangle.
\]
As we are looking to achieve control with respect to the $W^{k,2}(\mathcal{O}; \mathbb{R}^N)$ norm of $f$, then it is the terms with differential operators of order greater than $k$ that concern us. Of course this was the motivating factor behind combining $B_{\xi_i}$ and its adjoint, nullifying the additional derivative coming from $L_{\xi_i}$. There are more higher order terms to go through, and the strategy will be to write these in terms of commutators with a differential operator of controllable order. This will involve considering different aspects of our sum in tandem, which will be helped by calling (61) into action once more to reduce our expression again to
\[
\left\langle \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right\rangle + \left\langle \sum_{\beta < \alpha} B_{D_{\alpha - \beta \xi_i}} D^{\beta} f + B_{\xi_i} D^\alpha f, (T_{\xi_i} + T_{\xi_i}^*) D^\alpha f + \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} f \right\rangle.
\]
Ultimately the terms in the summand are split up into
\[
\left\langle B_{\xi_i} D^\alpha f, (T_{\xi_i} + T_{\xi_i}^*) D^\alpha f \right\rangle + \left\langle \sum_{\beta < \alpha} B_{D_{\alpha - \beta \xi_i}} D^{\beta} f, (T_{\xi_i} + T_{\xi_i}^*) D^\alpha f + \sum_{\alpha' < \alpha} B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} f \right\rangle
\]
\[
+ \sum_{\alpha' < \alpha} \left( B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} B_{\xi_i} f, D^\alpha f \right) + \left\langle B_{\xi_i} D^\alpha f, B_{D_{\alpha - \alpha'} \xi_i} D^{\alpha'} f \right\rangle.
\]
with the intention of controlling each one individually. Firstly for a treatment of (63),

\[
(63) = \langle (\mathcal{L}_{\xi_i} + \mathcal{T}_{\xi_i}) D^\alpha f, (\mathcal{T}_{\xi_i}^* + \mathcal{T}_{\xi_i}) D^\alpha f \rangle = \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle + \langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle + \langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle = \left( \langle \mathcal{T}_{\xi_i}^2 D^\alpha f, D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle \right) + \left( \langle \mathcal{T}_{\xi_i}^2 D^\alpha f, D^\alpha f \rangle + \langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle \right).
\]

We now bound the brackets in terms of \(\|D^\alpha f\|^2\) separately, starting with the latter one as

\[
\langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle \leq \|\mathcal{T}_{\xi_i}^2 D^\alpha f\| \|\mathcal{T}_{\xi_i} D^\alpha f\| \leq \sqrt{N} \|\xi_i\|_{W^{1,\infty}} \|\mathcal{T}_{\xi_i} D^\alpha f\| \|\mathcal{T}_{\xi_i} D^\alpha f\| \leq N \|\xi_i\|^2_{W^{1,\infty}} \|D^\alpha f\|^2
\]

from 1.25 for the result in \(L^2(\mathcal{O}; \mathbb{R}^N)\), and similarly

\[
\langle \mathcal{T}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle \leq \|\mathcal{T}_{\xi_i} D^\alpha f\| \|\mathcal{T}_{\xi_i} D^\alpha f\| \leq N \|\xi_i\|^2_{W^{1,\infty}} \|D^\alpha f\|^2.
\]

Now for the first bracket, we add and subtract a term to have an expression through the commutator of the operators:

\[
\langle \mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle = \langle \mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle = \langle \mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle = \langle \mathcal{T}_{\xi_i} D^\alpha f, D^\alpha f \rangle + \langle \mathcal{L}_{\xi_i} D^\alpha f, \mathcal{T}_{\xi_i} D^\alpha f \rangle.
\]

The commutator term is given explicitly through

\[
\mathcal{T}_{\xi_i} \mathcal{L}_{\xi_i} D^\alpha f = \mathcal{T}_{\xi_i} \left( \sum_{j=1}^{N} \xi_j^i \partial_j D^\alpha f \right) = \sum_{k=1}^{N} \left( \sum_{j=1}^{N} \xi_j^i \partial_j D^\alpha f \right)^k \nabla \xi_i^k
\]

and

\[
\mathcal{L}_{\xi_i} \mathcal{T}_{\xi_i} D^\alpha f = \mathcal{L}_{\xi_i} \left( \sum_{k=1}^{N} D^\alpha f^k \nabla \xi_i^k \right) = \sum_{j=1}^{N} \xi_j^i \partial_j \left( \sum_{k=1}^{N} D^\alpha f^k \nabla \xi_i^k \right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_j^i \partial_j \left( D^\alpha f^k \nabla \xi_i^k \right) = \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \xi_j^i \partial_j D^\alpha f^k \nabla \xi_i^k + \xi_j^i D^\alpha f^k \partial_j \nabla \xi_i^k \right)
\]

44
such that
\[(T_{\xi_i}L_{\xi_i} - L_{\xi_i} T_{\xi_i}) D^\alpha f = \sum_{j=1}^{N} \sum_{k=1}^{N} \xi_{i}^j D^\alpha f^k \partial_j \nabla \xi_{i}^k \]
therefore
\[
\| (T_{\xi_i} L_{\xi_i} - L_{\xi_i} T_{\xi_i}) D^\alpha f \|^2 \leq N^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \| \xi_{i}^j D^\alpha f^k \partial_j \xi_{i}^k \|^2_{L^2(\mathcal{O}; \mathbb{R})}
\]
\[
\leq N^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \| \xi_{i}^j \partial_j \partial_l \xi_{i}^k \|^2_{L^\infty(\mathcal{O}; \mathbb{R})} \| D^\alpha f^k \|^2_{L^2(\mathcal{O}; \mathbb{R})}
\]
\[
\leq N^2 \| \xi_{i} \|_{W^{2,\infty}} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \| D^\alpha f^k \|^2_{L^2(\mathcal{O}; \mathbb{R})}
\]
\[
= N^4 \| \xi_{i} \|_{W^{2,\infty}} \| D^\alpha f \|^2
\]
and
\[
\langle (T_{\xi_i}L_{\xi_i} - L_{\xi_i} T_{\xi_i}) D^\alpha f, D^\alpha f \rangle \leq \| (T_{\xi_i} L_{\xi_i} - L_{\xi_i} T_{\xi_i}) D^\alpha f \| \| D^\alpha f \| \leq N^2 \| \xi_{i} \|_{W^{2,\infty}} \| D^\alpha f \|^2.
\]
Combining these inequalities we determine the bound
\[(63) \leq (N^2 \| \xi_{i} \|^2_{W^{2,\infty}} + 2N \| \xi_{i} \|^2_{W^{2,\infty}}) \| D^\alpha f \|^2
\]
\[
\leq 3N^2 \| \xi_{i} \|^2_{W^{2,\infty}} \| D^\alpha f \|^2.
\]
As for (64) we look to use Cauchy-Schwartz and bound each item in the inner product. Indeed straight from (19) in the $L^2(\mathcal{O}; \mathbb{R}^N)$ setting, by simply replacing $\xi_i$ with $D^\alpha - \beta \xi_i$, we have that
\[
\| B_{D^\alpha - \beta \xi_i} D^\beta f \|^2 \leq 4N \| D^\alpha - \beta \xi_i \|^2_{W^{1,\infty}} \| D^\beta f \|^2_{W^{1,2}}
\]
\[
\leq 4N \| \xi_{i} \|^2_{W^{k+1,\infty}} \| f \|^2_{W^{k,2}}
\]
Moreover,
\[
\left| \sum_{\beta < \alpha} B_{D^\alpha - \beta \xi_i} D^\beta f \right|^2 \leq 2^{k-1} \sum_{\beta < \alpha} \| B_{D^\alpha - \beta \xi_i} D^\beta f \|^2
\]
\[
\leq 2^{k+1} N \| \xi_{i} \|^2_{W^{k+1,\infty}} \| f \|^2_{W^{k,2}}.
\]
In addition to this, we have again the result from 1.25 in the $L^2(\mathcal{O}; \mathbb{R}^N)$ setting that
\[
\| (T_{\xi_i} + T_{\xi_i}^*) D^\alpha f \| \leq \| T_{\xi_i} D^\alpha f \| + \| T_{\xi_i}^* D^\alpha f \| \leq 2\sqrt{N} \| \xi_{i} \|_{W^{k+1,\infty}} \| D^\alpha f \|
\]
using the equivalence in operator norm of the adjoint. Together this amounts to

\[ (64) \leq \left\| \sum_{\beta \leq \alpha} B_{D^\alpha - \beta, \xi_i} D^\beta f \right\| \cdot \left\| (\mathcal{T}_{\xi_i} + \mathcal{\mathcal{T}}_{\xi_i}^*) D^\alpha f + \sum_{\alpha' < \alpha} B_{D^{\alpha' - \beta}, \xi_i} D^{\alpha'} f \right\| \\
\leq \left\| \sum_{\beta \leq \alpha} B_{D^\alpha - \beta, \xi_i} D^\beta f \left( \left\| (\mathcal{T}_{\xi_i} + \mathcal{\mathcal{T}}_{\xi_i}^*) D^\alpha f \right\| + \left\| \sum_{\alpha' < \alpha} B_{D^{\alpha' - \beta}, \xi_i} D^{\alpha'} f \right\| \right) \right. \\
\leq \sqrt{2k+1} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| f \right\|_{W^{k, 2}} \left( 2\sqrt{N} \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| D^\alpha f \right\| + \sqrt{2k+1} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| f \right\|_{W^{k, 2}} \right) \\
\leq \sqrt{2k+2} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| f \right\|_{W^{k, 2}} \left( \sqrt{2k+2} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| D^\alpha f \right\| + \sqrt{2k+2} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| f \right\|_{W^{k, 2}} \right) \\
\leq 2^{k+3} N \left\| \xi_i \right\|_{W^{k+1, \infty}} \left\| f \right\|_{W^{k, 2}}^2.
\]

Let’s now turn our attentions to (65), which for each \( \alpha' \) in the sum we rewrite as

\[ \langle D^\alpha f, B_{D^\alpha - \alpha', \xi_i} D^{\alpha'} f + B_{\xi_i}^* B_{D^\alpha - \alpha', \xi_i} D^{\alpha'} f \rangle \tag{66} \]

and employing (61) again we see this becomes

\[ \left\langle D^\alpha f, B_{D^\alpha - \alpha', \xi_i} \left( \sum_{\beta \leq \alpha'} B_{D^\alpha - \beta, \xi_i} D^\beta f + B_{\xi_i} D^{\alpha'} f \right) + B_{\xi_i}^* B_{D^\alpha - \alpha', \xi_i} D^{\alpha'} f \right\rangle \\
= \left\langle D^\alpha f, \sum_{\beta \leq \alpha'} B_{D^\alpha - \alpha', \xi_i} B_{D^\alpha - \beta, \xi_i} D^\beta f \right\rangle + \left\langle D^\alpha f, B_{D^\alpha - \alpha', \xi_i} B_{\xi_i} D^{\alpha'} f + B_{\xi_i}^* B_{D^\alpha - \alpha', \xi_i} D^{\alpha'} f \right\rangle.
\]

We have split up these terms to make our approach clearer, as the two right hand sides of the inner products will be considered separately. For the first inner product, two applications of (19) give that

\[ \left\| B_{D^\alpha - \alpha', \xi_i} B_{D^\alpha - \beta, \xi_i} D^\beta f \right\|^2 \leq 4N \left\| D^\alpha - \alpha', \xi_i \right\|^2_{W^{1, \infty}} \left\| B_{D^\alpha - \beta, \xi_i} D^\beta f \right\|^2_{W^{1, 2}} \\
\leq 4N \left\| D^\alpha - \alpha', \xi_i \right\|^2_{W^{1, \infty}} (8N(N+1)) \left\| D^\alpha - \beta, \xi_i \right\|^2_{W^{2, \infty}} \left\| D^\beta f \right\|^2_{W^{2, 2}} \\
\leq 32N^2 (N+1) \left\| \xi_i \right\|^4_{W^{k+1, \infty}} \left\| f \right\|^2_{W^{k, 2}}.
\]

noting that the \( N+1 \) comes from the number of distinct multi-indices of order up to 1. Moreover,

\[ \left\| \sum_{\beta \leq \alpha'} B_{D^\alpha - \alpha', \xi_i} B_{D^\alpha - \beta, \xi_i} D^\beta f \right\|^2 \leq 2^{k-2} \left\| B_{D^\alpha - \alpha', \xi_i} B_{D^\alpha - \beta, \xi_i} D^\beta f \right\|^2 \\
\leq 2^{k+3} N^2 (N+1) \left\| \xi_i \right\|^4_{W^{k+1, \infty}} \left\| f \right\|^2_{W^{k, 2}}.
\]

As for the second inner product, we rewrite the right side as

\[ B_{D^\alpha - \alpha', \xi_i} \left( (\mathcal{L}_{\xi_i} + \mathcal{T}_{\xi_i}) D^\alpha f \right) + (\mathcal{\mathcal{L}}_{\xi_i}^* + \mathcal{\mathcal{T}}_{\xi_i}^*) B_{D^\alpha - \alpha', \xi_i} D^\alpha f 
\]

and further

\[ (B_{D^\alpha - \alpha', \xi_i} \mathcal{L}_{\xi_i} - \mathcal{\mathcal{L}}_{\xi_i} B_{D^\alpha - \alpha', \xi_i}) D^\alpha f + B_{D^\alpha - \alpha', \xi_i} \mathcal{T}_{\xi_i} D^\alpha f + \mathcal{\mathcal{T}}_{\xi_i}^* B_{D^\alpha - \alpha', \xi_i} D^\alpha f. \tag{67} \]
Starting with the latter two terms, we use the familiar (19) and 1.25 for the bound
\[
\|B_{D^{\alpha'-\alpha'}_i} T_{\xi_i} D^{\alpha'} f\|^2 \leq 4N \|D^{\alpha-\alpha'}_i \xi_i\|^2_{W^{1,\infty}} \|T_{\xi_i} D^{\alpha'} f\|^2_{W^{1,2}} \\
\leq 4N \|D^{\alpha-\alpha'}_i \xi_i\|^2_{W^{1,\infty}} \left(2N(N+1)\|\xi_i\|^2_{W^{2,\infty}} \|D^{\alpha'} f\|^2_{W^{1,2}} \right) \\
\leq 8N^2(N+1)\|\xi_i\|^4_{W^{k+1,\infty}} \|f\|^2_{W^{k,2}}
\]
and likewise
\[
\|T_{\xi_i} B_{D^{\alpha'-\alpha'}_i} D^{\alpha'} f\|^2 \leq N\|\xi_i\|^2_{W^{1,\infty}} \|B_{D^{\alpha'-\alpha'}_i} D^{\alpha'} f\|^2 \\
\leq N\|\xi_i\|^2_{W^{1,\infty}} \left(4N \|D^{\alpha'-\alpha'}_i \xi_i\|^2_{W^{1,\infty}} \|D^{\alpha'} f\|^2_{W^{1,2}} \right) \\
\leq 4N^2 \|\xi_i\|^4_{W^{k+1,\infty}} \|f\|^2_{W^{k,2}}
\]
Now we show explicitly that the commutator
\[
(B_{D^{\alpha'-\alpha'}_i} L_{\xi_i} - L_{\xi_i} B_{D^{\alpha'-\alpha'}_i}) D^{\alpha'} f
\] (68)
given in (67) is of first order (so of \(k^{th}\) order when composed with \(D^{\alpha'}\)), through the expressions
\[
B_{D^{\alpha'-\alpha'}_i} L_{\xi_i} D^{\alpha'} f = \sum_{j=1}^{N} \left(D^{\alpha'-\alpha'}_i \xi_j \partial_j \left( \sum_{k=1}^{N} \xi_k \partial_k D^{\alpha'} f \right) + \left( \sum_{k=1}^{N} \xi_k \partial_k D^{\alpha'} f \right) \nabla D^{\alpha'-\alpha'}_i \xi_i \right) \\
= \sum_{j=1}^{N} \sum_{k=1}^{N} \left(D^{\alpha'-\alpha'}_i \xi_j \partial_j \xi_k \partial_k D^{\alpha'} f + D^{\alpha'-\alpha'}_i \xi_j \xi_k \partial_j \partial_k D^{\alpha'} f + \xi_k \partial_k D^{\alpha'} f \nabla D^{\alpha'-\alpha'}_i \xi_i \right)
\]
and
\[
L_{\xi_i} B_{D^{\alpha'-\alpha'}_i} D^{\alpha'} f = \sum_{k=1}^{N} \xi_k \partial_k \left( \sum_{j=1}^{N} D^{\alpha'-\alpha'}_i \xi_j \partial_j D^{\alpha'} f + D^{\alpha'} f \nabla D^{\alpha'-\alpha'}_i \xi_i \right) \\
= \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \xi_k \partial_k D^{\alpha'-\alpha'}_i \xi_j \partial_j D^{\alpha'} f + \xi_k D^{\alpha'-\alpha'}_i \xi_j \partial_j \partial_k D^{\alpha'} f \\
+ \xi_k \partial_k D^{\alpha'} f \nabla D^{\alpha'-\alpha'}_i \xi_i \right)
\]
such that
\[
(68) = \sum_{j=1}^{N} \sum_{k=1}^{N} \left(D^{\alpha'-\alpha'}_i \xi_j \partial_j \xi_k \partial_k D^{\alpha'} f - \xi_k \partial_k D^{\alpha'-\alpha'}_i \xi_j \partial_j D^{\alpha'} f - \xi_k D^{\alpha'} f \nabla D^{\alpha'-\alpha'}_i \xi_i \right)
\]
and in particular

\[
\| (68) \| \leq 3 N^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \left( \| D^{\alpha - \alpha'} \xi_{j} \partial_{j} \xi_{k} \partial_{k} D^{\alpha'} f \|^{2} + \| \xi_{j} \partial_{j} D^{\alpha - \alpha'} \xi_{j} \partial_{j} D^{\alpha'} f \|^{2} + \| \xi_{k} D^{\alpha'} f^{j} \partial_{k} \nabla D^{\alpha - \alpha'} \xi_{j} \|^{2} \right)
\]

\[
= 3 N^2 \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{l=1}^{N} \left( \| D^{\alpha - \alpha'} \xi_{j} \partial_{j} \xi_{k} \partial_{k} D^{\alpha'} f \|^{2} \right)
\]

\[
+ \| \xi_{j} \partial_{j} D^{\alpha - \alpha'} \xi_{j} \partial_{j} D^{\alpha'} f \|^{2} \leq \left( \sum_{j=1}^{N} \| f \|^{2}_{W^{k+2, \infty}} + \sum_{k=1}^{N} \| f \|^{2}_{W^{k+2, \infty}} + \sum_{j=1}^{N} \sum_{k=1}^{N} \| D^{\alpha'} f \|^{2} \right)
\]

\[
\leq 9 N^4 \| \xi \|^{2}_{W^{2+k, \infty}} \| f \|^{2}_{W^{k+2, \infty}} .
\]

Finally now we can piece these four inequalities together to produce a bound on (66):

\[
(66) \leq \| D^{\alpha} f \| \left( \sum_{\beta < \alpha} B_{D^{\alpha - \alpha'} \xi_{j}} B_{D^{\alpha' - \beta} \xi_{j}} D^{\beta} f + (67) \right)
\]

\[
\leq \| D^{\alpha} f \| \left( \left( \sum_{\beta < \alpha} B_{D^{\alpha - \alpha'} \xi_{j}} B_{D^{\alpha' - \beta} \xi_{j}} D^{3} f \right) + \| B_{D^{\alpha - \alpha'} \xi_{j}} \partial_{j} D^{\alpha'} f \| + \| D^{\alpha'} f \| + \| (68) \| \right)
\]

\[
\leq \| D^{\alpha} f \| \left( \sqrt{2^{k+3}(N+1)N} \| \xi \|^{2}_{W^{k+1, \infty}} \| f \|_{W^{k+2, \infty}} + \sqrt{8(N+1)N} \| \xi \|^{2}_{W^{k+1, \infty}} \| f \|_{W^{k+2, \infty}}^{} + 2N \| \xi \|_{W^{k+1, \infty}} \| f \|_{W^{k+2, \infty}}^{} + 3N^2 \| \xi \|^{2}_{W^{k+2, \infty}} \| f \|_{W^{k+2, \infty}}^{} \right)
\]

\[
\leq \| D^{\alpha} f \| \left( (2^{k+8} + 3)N^2 \| \xi \|^{2}_{W^{k+2, \infty}} \| f \|_{W^{k+2, \infty}}^{} \right)
\]

\[
\leq (2^{k+8} + 3)N^2 \| \xi \|^{2}_{W^{k+2, \infty}} \| f \|^{2}_{W^{k+2, \infty}}
\]

using that for \( N \geq 2 \) we have \( \sqrt{N+1} \leq N \), and subsequently of (65):

\[
(65) = \sum_{\alpha' < \alpha} (66)
\]

\[
\leq \sum_{\alpha' < \alpha} (2^{k+8} + 3)N^2 \| \xi \|^{2}_{W^{k+2, \infty}} \| f \|^{2}_{W^{k+2, \infty}}
\]

\[
\leq 2^k (2^{k+8} + 3)N^2 \| \xi \|^{2}_{W^{k+2, \infty}} \| f \|^{2}_{W^{k+2, \infty}}
\]
We can now conclude the proof by observing that

\[
(59) = \sum_{|\alpha| \leq k} (60)
\]
\[
= \sum_{|\alpha| \leq k} (63) + (64) + (65)
\]
\[
\leq \sum_{|\alpha| \leq k} \left( 3N^2 \|\xi_i\|_{W^{2,\infty}}^2 \|D^\alpha f\|^2 + 2^{k+3}N \|\xi_i\|_{W^{k+1,\infty}}^2 \|f\|_{W^{k,2}}^2 \right.
\]
\[
+ 2^k \left( 2^{k+4} + 3 \right) N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \bigg) \]
\[
\leq \sum_{|\alpha| \leq k} \left( 3N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 + 2^{k+3}N \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \right.
\]
\[
+ \left( 2^{k+4} + 2 \right) N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \bigg) \]
\[
\leq \sum_{|\alpha| \leq k} \left( 3 \cdot 2^{k+4}N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 + 2^{k+4}N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \right.
\]
\[
+ \left( 2^{k+4} + 2 \cdot 2^{k+4} \cdot 3 \right) N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \bigg) \]
\[
= \sum_{|\alpha| \leq k} 2^{k+7}N^2 \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2 \]
\[
= 2^{k+7}N^2 \eta_k \|\xi_i\|_{W^{k+2,\infty}}^2 \|f\|_{W^{k,2}}^2.
\]

\[\square\]

**Proof of 1.27, (21):** Using (61) once more, we see that for each \(\alpha\) in the sum for the inner product,

\[
|\langle D^\alpha B f, D^\alpha f \rangle| = \left| \left\langle \sum_{\alpha' < \alpha} B_{D^\alpha - \alpha', \xi} D^{\alpha'} f + B_{\xi}, D^\alpha f, D^\alpha f \right\rangle \right|
\]
\[
= \left| \left\langle \sum_{\alpha' < \alpha} B_{D^\alpha - \alpha', \xi} D^{\alpha'} f, D^\alpha f \right\rangle + \langle B_{\xi}, D^\alpha f, D^\alpha f \rangle \right|
\]
\[
\leq \left| \left\langle \sum_{\alpha' < \alpha} B_{D^\alpha - \alpha', \xi} D^{\alpha'} f, D^\alpha f \right\rangle \right| + |\langle \xi, D^\alpha f, D^\alpha f \rangle|
\]

using the cancellation from 1.23 to dispose of the order \(k + 1\) term. In our treatment of (64) in 20, we showed the bound

\[
\left| \sum_{\beta < \alpha} B_{D^{\alpha - \beta}, \xi} D^\beta f \right| \leq \sqrt{2^{k+4}N} \|\xi\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}}
\]

and therefore

\[
\left| \sum_{\alpha' < \alpha} B_{D^\alpha - \alpha', \xi} D^{\alpha'} f, D^\alpha f \right| \leq \sqrt{2^{k+4}N} \|\xi\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}} \|D^\alpha f\|
\]
\[
\leq \sqrt{2^{k+4}N} \|\xi\|_{W^{k+1,\infty}} \|f\|_{W^{k,2}}^2
\]

49
whilst in the treatment of (63) we noted
\[ \langle T^\alpha, D^\alpha f, T^\alpha, D^\alpha f \rangle \leq N\|\xi_i\|^2_{W^{1,\infty}}\|D^\alpha f\|^2 \leq N\|\xi_i\|^2_{W^{k+1,\infty}}\|f\|^2_{W^{k,2}}. \]

Combining these terms, we loosen to the bound
\[ |\langle D^\alpha B_i f, f \rangle| \leq (\|\xi_i\|_{W^{k+1,\infty}} + \sqrt{2^{k+1}})N\|\xi_i\|_{W^{k+1,\infty}}\|f\|^2_{W^{k,2}} \]
and see therefore that
\[ |\langle B_i f, f \rangle\rangle_{W^{k,2}} \leq (\|\xi_i\|_{W^{k+1,\infty}} + \sqrt{2^{k+1}})N\|\xi_i\|_{W^{k+1,\infty}}\|f\|^2_{W^{k,2}} \]
which readily gives the result. \(\square\)

**Proof of 1.29.** We fix any such \(f\) and first show that
\[ [\Delta, B_i]f = \sum_{k=1}^{N} \sum_{j=1}^{N} \left( \partial_k^2 \xi_i^j \partial_j f + 2 \partial_k \xi_i^j \partial_k \partial_j f + 2 \partial_k f \partial_k \nabla \xi_i^j + f^j \partial_k^2 \nabla \xi_i^j \right). \tag{69} \]

Indeed
\[
\Delta B_i f = \sum_{k=1}^{N} \partial_k^2 \left( \sum_{j=1}^{N} \left( \xi_i^j \partial_j f + f^j \nabla \xi_i^j \right) \right) \\
= \sum_{k=1}^{N} \partial_k \left( \sum_{j=1}^{N} \left( \partial_k \xi_i^j \partial_j f + \xi_i^j \partial_k \partial_j f + \partial_k f \partial_k \nabla \xi_i^j + f^j \partial_k \nabla \xi_i^j \right) \right) \\
= \sum_{k=1}^{N} \sum_{j=1}^{N} \left( \partial_k^2 \xi_i^j \partial_j f + 2 \partial_k \xi_i^j \partial_k \partial_j f + \xi_i^j \partial_k^2 \partial_k f + \partial_k^2 f \partial_k \nabla \xi_i^j + 2 \partial_k f \partial_k \nabla \xi_i^j + f^j \partial_k^2 \nabla \xi_i^j \right)
\]

and
\[
B_i \Delta f = \sum_{j=1}^{N} \left( \xi_i^j \partial_j \left( \sum_{k=1}^{N} \partial_k^2 f \right) + \left( \sum_{k=1}^{N} \partial_k^2 f \right) \nabla \xi_i^j \right) \\
= \sum_{k=1}^{N} \sum_{j=1}^{N} \left( \xi_i^j \partial_k \partial_j f + \partial_k^2 f \partial_k \nabla \xi_i^j \right)
\]

therefore
\[
[\Delta, B_i]f = \Delta B_i f - B_i \Delta f \\
= \sum_{k=1}^{N} \sum_{j=1}^{N} \left( \partial_k^2 \xi_i^j \partial_j f + 2 \partial_k \xi_i^j \partial_k \partial_j f + 2 \partial_k f \partial_k \nabla \xi_i^j + f^j \partial_k^2 \nabla \xi_i^j \right)
\]

50
where the operators $Q$ defined by Assumption 4.3.

G is understood as a measurable operator

\[ G : V \to \mathcal{L}^2(\Omega; H) \]
\[ G : H \to \mathcal{L}^2(\Omega; U) \]
\[ G : U \to \mathcal{L}^2(\Omega; X) \]
defined over $\mathcal{U}$ by its action on the basis vectors

\[ G(\cdot, e_i) := G_i(\cdot). \]

In addition each $G_i$ is linear and there exists constants $c_i$ such that for all $\phi \in V$, $\psi \in H$, $\eta \in U$:

\[ \|G_i \phi\|_H \leq c_i \|\phi\|_V \]
\[ \|G_i \psi\|_U \leq c_i \|\psi\|_H \]
\[ \|G_i \eta\|_X \leq c_i \|\eta\|_U \]
\[ \sum_{i=1}^{\infty} c_i^2 < \infty. \]

In this setting, we have the following result ([25] Theorem 2.3.1 and Corollary 2.3.1.1).

**Theorem 4.5.** Suppose that $(\Psi, \tau)$ are such that: $\tau$ is a $\mathbb{P} - \text{a.s.}$ positive stopping time and $\Psi$ is a process whereby for $\mathbb{P} - \text{a.e. } \omega$, $\Psi(\omega) \in C([0,T];H)$ and $\Psi(\omega)_{1 \leq \tau(\omega)} \in L^2([0,T];V)$ for all $T > 0$ with $\Psi_{1 \leq \tau}$ progressively measurable in $V$, and moreover satisfy the identity

\[ \Psi_t = \Psi_0 + \int_0^{t \wedge \tau} \left( Q + \frac{1}{2} \sum_{i=1}^{\infty} G_i^2 \right) \Psi_s ds + \int_0^{t \wedge \tau} G \Psi_s dW_s \]

$\mathbb{P} - \text{a.s. in } U$ for all $t \geq 0$. Then the pair $(\Psi, \tau)$ satisfies the identity

\[ \Psi_t = \Psi_0 + \int_0^{t \wedge \tau} Q \Psi_s ds + \int_0^{t \wedge \tau} G \Psi_s \circ dW_s \]

$\mathbb{P} - \text{a.s. in } X$ for all $t \geq 0$.

The operator $\frac{1}{2} \sum_{i=1}^{\infty} G_i^2$ is understood as a pointwise limit, which is justified in [25] Subsection 2.3.

### 4.4 Appendix IV: Abstract Solution Criterion, Part I

The result is given in the context of an Itô SPDE

\[ \Psi_t = \Psi_0 + \int_0^t A(s, \Psi_s) ds + \int_0^t G(s, \Psi_s) dW_s. \] (71)

We state the assumptions for a triplet of embedded Hilbert Spaces

\[ V \hookrightarrow H \hookrightarrow U \]

and ask that there is a continuous bilinear form $\langle \cdot, \cdot \rangle_{U \times V} : U \times V \rightarrow \mathbb{R}$ such that for $\phi \in H$ and $\psi \in V$,

\[ \langle \phi, \psi \rangle_{U \times V} = \langle \phi, \psi \rangle_H. \] (72)

The operators $A, G$ are such that for any $T > 0$, $A : [0,T] \times V \rightarrow U, G : [0,T] \times V \rightarrow L^2(\mathcal{U};H)$ are measurable. We assume that $V$ is dense in $H$ which is dense in $U$.

**Assumption 4.6.** There exists a system $(a_n)$ of elements of $V$ which are such that, defining the spaces $V_n := \text{span} \{a_1, \ldots, a_n\}$ and $P_n$ as the orthogonal projection to $V_n$ in $U$, then:
1. There exists some constant $c$ independent of $n$ such that for all $\phi \in H$,
\[ \| P_n \phi \|^2_H \leq c \| \phi \|^2_H. \] (73)

2. There exists a real valued sequence $(\mu_n)$ with $\mu_n \to \infty$ such that for any $\phi \in H$,
\[ \| (I - P_n) \phi \|_U \leq \frac{1}{\mu_n} \| \phi \|_H \] (74)

where $I$ represents the identity operator in $U$.

These assumptions are of course supplemented by a series of assumptions on the operators. We shall use general notation $c_t$ to represent a function $c_\cdot : [0, \infty) \to \mathbb{R}$ bounded on $[0, T]$ for any $T > 0$, evaluated at the time $t$. Moreover we define functions $K_t, \tilde{K}$ relative to some non-negative constants $p, \bar{p}, q, \bar{q}$. We use a generic notation to define the functions $K : U \to \mathbb{R}$, $K : U \times U \to \mathbb{R}$, $\tilde{K} : H \to \mathbb{R}$ and $\tilde{K} : H \times H \to \mathbb{R}$ by
\[
K(\phi) := 1 + \| \phi \|^p_U,
K(\phi, \psi) := 1 + \| \phi \|^p_U + \| \psi \|^q_U,
\tilde{K}(\phi) := K(\phi) + \| \phi \|^\bar{p}_H,
\tilde{K}(\phi, \psi) := K(\phi, \psi) + \| \phi \|^\bar{p}_H + \| \psi \|^\bar{q}_H.
\]

Distinct use of the function $K$ will depend on different constants but in no meaningful way in our applications, hence no explicit reference to them shall be made. In the case of $\tilde{K}$, when $\bar{p}, \bar{q} = 2$ then we shall denote the general $\tilde{K}$ by $\tilde{K}_2$. In this case no further assumptions are made on the $p, q$. That is, $\tilde{K}_2$ has the general representation
\[
\tilde{K}_2(\phi, \psi) = K(\phi, \psi) + \| \phi \|^2_H + \| \psi \|^2_H
\] (75)

and similarly as a function of one variable.

We state the subsequent assumptions for arbitrary elements $\phi, \psi \in V$, $\phi^n \in V_n$, $\eta \in H$ and $t \in [0, \infty)$, and a fixed $\kappa > 0$. Understanding $G$ as an operator $G : [0, \infty) \times V \times \mathcal{U} \to H$, we introduce the notation $G_t(\cdot, \cdot) := G(\cdot, \cdot, e_t)$.

**Assumption 4.7.**

\[
\| A(t, \phi) \|^2_U + \sum_{i=1}^{\infty} \| G_i(t, \phi) \|^2_H \leq c_t K(\phi) \left[ 1 + \| \phi \|^2_U \right],
\] (76)

\[
\| A(t, \phi) - A(t, \psi) \|^2_U \leq c_t \left[ K(\phi, \psi) + \| \phi \|^2_U + \| \psi \|^2_U \right] \| \phi - \psi \|^2_U,
\] (77)

\[
\sum_{i=1}^{\infty} \| G_i(t, \phi) - G_i(t, \psi) \|^2_H \leq c_t K(\phi, \psi) \| \phi - \psi \|^2_H.
\] (78)

**Assumption 4.8.**

\[
2 \langle P_n A(t, \phi^n), \phi^n \rangle_H + \sum_{i=1}^{\infty} \| P_n G_i(t, \phi^n) \|^2_H \leq c_t \tilde{K}_2(\phi^n) \left[ 1 + \| \phi^n \|^2_H \right] - \kappa \| \phi^n \|^2_V,
\] (79)

\[
\sum_{i=1}^{\infty} \langle P_n G_i(t, \phi^n), \phi^n \rangle_H \leq c_t \tilde{K}_2(\phi^n) \left[ 1 + \| \phi^n \|^2_H \right].
\] (80)
Assumption 4.9.

\[
2 \langle A(t, \phi) - A(t, \psi), \phi - \psi \rangle_U + \sum_{i=1}^{\infty} \| G_i(t, \phi) - G_i(t, \psi) \|^2_U \\
\leq c_1 K_2(\phi, \psi) \| \phi - \psi \|^2_U - \kappa \| \phi - \psi \|^2_H ,
\]

Assumption 4.10.

\[
2 \langle A(t, \phi), \phi \rangle_U + \sum_{i=1}^{\infty} \| G_i(t, \phi) \|^2_U \leq c_1 K(\phi) [1 + \| \phi \|^2_H] ,
\]

\[
\sum_{i=1}^{\infty} \| G_i(t, \phi) \|^2_U \leq c_1 K(\phi) [1 + \| \phi \|^4_H] .
\]

Assumption 4.11.

\[
\langle A(t, \phi) - A(t, \psi), \eta \rangle_U \leq c_1 (1 + \| \eta \|^2_H) [K(\phi, \psi) + \| \phi \|_V + \| \psi \|_V] \| \phi - \psi \|_H .
\]

With these assumptions in place we state the relevant definitions and results, first announced in [26] and proven in [27]. Definition 4.12 is stated for an \( \mathcal{F}_0 \)- measurable \( \Psi_0 : \Omega \to H \).

Definition 4.12. A pair \( (\Psi, \tau) \) where \( \tau \) is a \( \mathbb{P} \)-a.s. positive stopping time and \( \Psi \) is a process such that for \( \mathbb{P} \)-a.e. \( \omega \), \( \Psi(\omega) \in C([0, T]; H) \) and \( \Psi(\omega) 1_{\leq \tau(\omega)} \in L^2([0, T]; V) \) for all \( T > 0 \) with \( \Psi 1_{\leq \tau} \) progressively measurable in \( \tau \), is said to be an \( H \)-valued local strong solution of the equation (71) if the identity

\[
\Psi_t = \Psi_0 + \int_0^{t \wedge \tau} A(s, \Psi_s) ds + \int_0^{t \wedge \tau} G(s, \Psi_s) d \mathcal{W}_s
\]

holds \( \mathbb{P} \)-a.s. in \( U \) for all \( t \geq 0 \).

Definition 4.13. A pair \( (\Phi, \Theta) \) such that there exists a sequence of stopping times \( (\theta_j) \) which are \( \mathbb{P} \)-a.s. monotone increasing and convergent to \( \Theta \), whereby \( (\Phi, \Theta, \theta_j) \) is a \( V \)-valued local strong solution of the equation (71) for each \( j \), is said to be an \( H \)-valued maximal strong solution of the equation (71) if for any other pair \( (\Phi, \Gamma) \) with this property then \( \Theta \leq \Gamma \mathbb{P} \)-a.s. implies \( \Theta = \Gamma \mathbb{P} \)-a.s..

Definition 4.14. An \( H \)-valued maximal strong solution \( (\Psi, \Theta) \) of the equation (71) is said to be unique if for any other such solution \( (\Phi, \Gamma) \), then \( \Theta = \Gamma \mathbb{P} \)-a.s. and for all \( t \in [0, \Theta) \),

\[
\mathbb{P} \left\{ \omega \in \Omega : \Psi_t(\omega) = \Phi_t(\omega) \right\} = 1.
\]

The theorem below is stated on the condition that the assumptions of this subsection are met.

Theorem 4.15. For any given \( \mathcal{F}_0 \)- measurable \( \Psi_0 : \Omega \to H \), there exists a unique \( H \)-valued maximal strong solution \( (\Psi, \Theta) \) of the equation (71). Moreover at \( \mathbb{P} \)-a.e. \( \omega \) for which \( \Theta(\omega) < \infty \), we have that

\[
\sup_{r \in [0, \Theta(\omega))} \| \Psi_r(\omega) \|^2_H + \int_0^{\Theta(\omega)} \| \Psi_r(\omega) \|^2_V dr = \infty .
\]
4.5 Appendix V: Abstract Solution Criterion, Part II

We extend the framework of Appendix IV, 4.4, introducing now another Hilbert Space $X$ which is such that $U \hookrightarrow X$. We ask that there is a continuous bilinear form $\langle \cdot, \cdot \rangle_{X \times H} : X \times H \rightarrow \mathbb{R}$ such that for $\phi \in U$ and $\psi \in H$,

$$\langle \phi, \psi \rangle_{X \times H} = \langle \phi, \psi \rangle_U. \tag{88}$$

Moreover it is now necessary that the system $(a_n)$ forms an orthogonal basis of $U$. We state the remaining assumptions now for arbitrary elements $\phi, \psi \in H$ and $t \in [0, \infty)$, and continue to use the $c, K, \tilde{K}, \kappa$ notation of Assumption Set 1. We now further assume that for any $T > 0$, $A : [0, T] \times H \rightarrow X$ and $G : [0, T] \times H \rightarrow \mathcal{L}^2(\Omega; U)$ are measurable.

**Assumption 4.16.**

$$\|A(t, \phi)\|_X \leq \sum_{i=1}^{\infty} \|G_i(t, \phi)\|_X^2 \leq c_1K(\phi) \left[ 1 + \|\phi\|_H^2 \right], \tag{89}$$

$$\|A(t, \phi) - A(t, \psi)\|_X \leq c_2 [K(\phi, \psi) + \|\phi\|_H + \|\psi\|_H] \|\phi - \psi\|_H \tag{90}$$

**Assumption 4.17.**

$$2\langle A(t, \phi) - A(t, \psi), \phi - \psi \rangle_X \leq \sum_{i=1}^{\infty} \|G_i(t, \phi) - G_i(t, \psi)\|_X^2 \leq c_1K_2(\phi, \psi) \|\phi - \psi\|_X^2, \tag{91}$$

$$\sum_{i=1}^{\infty} \langle G_i(t, \phi) - G_i(t, \psi), \phi - \psi \rangle_X^2 \leq c_1K_2(\phi, \psi) \|\phi - \psi\|_X^2 \tag{92}$$

**Assumption 4.18.** With the stricter requirement that $\phi \in V$ then

$$2\langle A(t, \phi), \phi \rangle_U + \sum_{i=1}^{\infty} \|G_i(t, \phi)\|_U^2 \leq c_1K(\phi) - \kappa\|\phi\|_H^2, \tag{93}$$

$$\sum_{i=1}^{\infty} \langle G_i(t, \phi), \phi \rangle_U^2 \leq c_1K(\phi). \tag{94}$$

**Remark.** This is a stronger assumption than Assumption 4.10.

Analogously to Appendix IV, 4.4, we state the relevant definitions and the resulting theorem in this context (again proved in [27]). Definition 4.19 is stated for a $\mathcal{F}_0-$ measurable $\Psi_0 : \Omega \rightarrow U$.

**Definition 4.19.** A pair $(\Psi, \tau)$ where $\tau$ is a $\mathbb{P}$-a.s. positive stopping time and $\Psi$ is a process such that for $\mathbb{P}$-a.e. $\omega$, $\Psi_0(\omega) \in C([0, T]; U)$ and $\Psi_0(\omega)1_{\leq \tau(\omega)} \in L^2([0, T]; H)$ for all $T > 0$ with $\Psi 1_{\leq \tau}$ progressively measurable in $H$, is said to be a $U$-valued local strong solution of the equation (71) if the identity

$$\Psi_t = \Psi_0 + \int_0^{\tau_{\wedge \tau}} A(s, \Psi_s)ds + \int_0^{\tau_{\wedge \tau}} G(s, \Psi_s)dW_s \tag{95}$$

holds $\mathbb{P}$-a.s. in $X$ for all $t \geq 0$.

**Definition 4.20.** A pair $(\Psi, \Theta)$ such that there exists a sequence of stopping times $(\theta_j)$ which are $\mathbb{P}$-a.s. monotone increasing and convergent to $\Theta$, whereby $(\Psi_{\Lambda_{\theta_j}}, \theta_j)$ is a $U$-valued local strong solution of the equation (71) for each $j$, is said to be an $H$-valued maximal strong solution of the equation (71) if for any other pair $(\Phi, \Gamma)$ with this property then $\Theta \leq \Gamma \implies \Theta = \Gamma$ $\mathbb{P}$-a.s.
Definition 4.21. A $U$-valued maximal strong solution $(\Psi, \Theta)$ of the equation (71) is said to be unique if for any other such solution $(\Phi, \Gamma)$, then $\Theta = \Gamma \, \mathbb{P} - \text{a.s.}$ and for all $t \in [0, \Theta)$,

$$\mathbb{P} \left( \{ \omega \in \Omega : \Psi_t(\omega) = \Phi_t(\omega) \} \right) = 1.$$

The theorem below is stated on the condition that the assumptions of this subsection and 4.4 are met.

Theorem 4.22. For any given $\mathcal{F}_0$-measurable $\Psi_0 : \Omega \to U$, there exists a unique $U$-valued maximal strong solution $(\Psi, \Theta)$ of the equation (71). Moreover at $\mathbb{P}$-a.e. $\omega$ for which $\Theta(\omega) < \infty$, we have that

$$\sup_{r \in [0, \Theta(\omega))] \| \Psi_r(\omega) \|^2_U + \int_0^{\Theta(\omega)} \| \Psi_r(\omega) \|^2_{H} dr = \infty. \quad (96)$$
References

[1] Adams, R.A. and Fournier, J.J., 2003. Sobolev spaces. Elsevier.

[2] Agresti, A. and Veraar, M., 2021. Stochastic Navier-Stokes equations for turbulent flows in critical spaces. arXiv preprint arXiv:2107.03953.

[3] Bensoussan, A. and Temam, R., 1973. Equations stochastiques du type Navier-Stokes. Journal of Functional Analysis, 13(2), pp.195-222.

[4] Brzezniak, Z. and Peszat, S., 1999. Strong local and global solutions for stochastic Navier-Stokes equations. Infinite dimensional stochastic analysis, pp.85-98.

[5] Brzeźniak, Z. and Slavík, J., 2021. Well-posedness of the 3D stochastic primitive equations with multiplicative and transport noise. Journal of Differential Equations, 296, pp.617-676.

[6] Constantin, P. and Foias, C., 2020. Navier-stokes equations. University of Chicago Press.

[7] Cotter, C., Crisan, D., Holm, D., Pan, W. and Shevchenko, I., 2020. Data assimilation for a quasi-geostrophic model with circulation-preserving stochastic transport noise. Journal of Statistical Physics, 179(5), pp.1186-1221.

[8] Cotter, C., Crisan, D., Holm, D., Pan, W. and Shevchenko, I., 2020. Modelling uncertainty using stochastic transport noise in a 2-layer quasi-geostrophic model. Foundations of Data Science, 2(2), p.173.

[9] Cotter, C., Crisan, D., Holm, D.D., Pan, W. and Shevchenko, I., 2019. Numerically modeling stochastic Lie transport in fluid dynamics. Multiscale Modeling and Simulation, 17(1), pp.192-232.

[10] Crisan, D. and Mensah, P.R., 2022. Spatial analyticity and exponential decay of Fourier modes for the stochastic Navier-Stokes equation. arXiv preprint arXiv:2209.14862.

[11] Crisan, D., Flandoli, F. and Holm, D.D., 2019. Solution properties of a 3D stochastic Euler fluid equation. Journal of Nonlinear Science, 29(3), pp.813-870.

[12] Da Prato, G. and Debussche, A., 2003. Ergodicity for the 3D stochastic Navier–Stokes equations. Journal de mathématiques pures et appliquées, 82(8), pp.877-947.

[13] Debussche, A., 2013. Ergodicity results for the stochastic Navier–Stokes equations: an introduction. In Topics in mathematical fluid mechanics (pp. 23-108). Springer, Berlin, Heidelberg.

[14] Debussche, A., Hug, B. and Mémin, E., 2022. A consistent stochastic large-scale representation of the Navier-Stokes equations. arXiv preprint arXiv:2207.07472.

[15] Enciso, A., Garcia-Ferrero, M.A. and Peralta-Salas, D., 2018. The Biot–Savart operator of a bounded domain. Journal de Mathématiques Pures et Appliquées, 119, pp.85-113.

[16] Evans, L.C., 2010. Partial differential equations (Vol. 19). American Mathematical Soc..

[17] Flandoli, F., 1994. Dissipativity and invariant measures for stochastic Navier-Stokes equations. Nonlinear Differential Equations and Applications NoDEA, 1(4), pp.403-423.
[18] Flandoli, F. and Gatarek, D., 1995. Martingale and stationary solutions for stochastic Navier-Stokes equations. Probability Theory and Related Fields, 102(3), pp.367-391.

[19] Flandoli, F. and Luo, D., 2021. High mode transport noise improves vorticity blow-up control in 3D Navier–Stokes equations. Probability Theory and Related Fields, 180(1), pp.309-363.

[20] Flandoli, F. and Romito, M., 2008. Markov selections for the 3D stochastic Navier–Stokes equations. Probability Theory and Related Fields, 140(3), pp.407-458.

[21] Flandoli, F. and Schmalfuss, B., 1996. Random attractors for the 3D stochastic Navier-Stokes equation with multiplicative white noise. Stochastics: An International Journal of Probability and Stochastic Processes, 59(1-2), pp.21-45.

[22] Galdi, G., 2011. An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems. Springer Science and Business Media.

[23] Gie, G.M. and Kelliher, J.P., 2012. Boundary layer analysis of the Navier–Stokes equations with generalized Navier boundary conditions. Journal of Differential Equations, 253(6), pp.1862-1892.

[24] Glatt-Holtz, N. and Ziane, M., 2009. Strong pathwise solutions of the stochastic Navier-Stokes system. Advances in Differential Equations, 14(5/6), pp.567-600.

[25] Goodair, D., 2022. Stochastic Calculus in Infinite Dimensions and SPDEs. arXiv preprint arXiv:2203.17206.

[26] Goodair, D., 2022. Existence and Uniqueness of Maximal Solutions to a 3D Navier-Stokes Equation with Stochastic Lie Transport. arXiv preprint arXiv:2202.09242v2.

[27] Goodair, D., Crisan, D. and Lang, O., 2022. Existence and Uniqueness of Maximal Solutions to SPDEs with Applications to Viscous Fluid Equations. arXiv preprint arXiv:2209.09137.

[28] Holm, D.D., 2015. Variational principles for stochastic fluid dynamics. Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 471(2176), p.20140963.

[29] Kelliher, J.P., 2006. Navier–Stokes equations with Navier boundary conditions for a bounded domain in the plane. SIAM journal on mathematical analysis, 38(1), pp.210-232.

[30] Lang, O. and Crisan, D., 2022. Well-posedness for a stochastic 2D Euler equation with transport noise. Stochastics and Partial Differential Equations: Analysis and Computations, pp.1-48.

[31] Lions, P.L., 1996. Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models (Vol. 1). Oxford University Press on Demand.

[32] Lototsky, S.V. and Rozovsky, B.L., 2017. Stochastic partial differential equations. New York: Springer.

[33] Majda, A.J., Bertozzi, A.L. and Ogawa, A., 2002. Vorticity and incompressible flow. Cambridge texts in applied mathematics. Appl. Mech. Rev., 55(4), pp.B77-B78.

[34] Mémin, E., 2014. Fluid flow dynamics under location uncertainty. Geophysical and Astrophysical Fluid Dynamics, 108(2), pp.119-146.
[35] Mikulevicius, R. and Rozovskii, B.L., 2004. Stochastic Navier–Stokes equations for turbulent flows. SIAM Journal on Mathematical Analysis, 35(5), pp.1250-1310.

[36] Mikulevicius, R. and Rozovskii, B.L., 2005. Global L2-solutions of stochastic Navier–Stokes equations. The Annals of Probability, 33(1), pp.137-176.

[37] Nirenberg, L., 2011. On elliptic partial differential equations. In Il principio di minimo e sue applicazioni alle equazioni funzionali (pp. 1-48). Springer, Berlin, Heidelberg.

[38] Robinson, J.C., Rodrigo, J.L. and Sadowski, W., 2016. The three-dimensional Navier–Stokes equations: Classical theory (Vol. 157). Cambridge university press.

[39] Street, O.D. and Crisan, D., 2021. Semi-martingale driven variational principles. Proceedings of the Royal Society A, 477(2247), p.20200957.

[40] Strichartz, R.S., 1967. Multipliers on fractional Sobolev spaces. Journal of Mathematics and Mechanics, 16(9), pp.1031-1060.

[41] Temam, R., 1977. Navier-Stokes equations: Theory and numerical analysis(Book). Amsterdam, North-Holland Publishing Co.(Studies in Mathematics and Its Applications, 2, p.510.)