Characterization of stochastic spatially and spectrally partially coherent electromagnetic pulsed beams

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\textbf{Abstract.} The unified theory of coherence and polarization proposed by Wolf is extended from stochastic stationary electromagnetic beams to stochastic spatially and spectrally partially coherent electromagnetic pulsed beams. Taking the stochastic electromagnetic Gaussian Schell-model pulsed (GSMP) beam as a typical example of stochastic spatially and spectrally partially coherent electromagnetic pulsed beams, the expressions for the spectral density, spectral degree of polarization and spectral degree of coherence of stochastic electromagnetic GSMP beams propagating in free space are derived. Some special cases are analyzed. The illustrative examples are given and the results are interpreted physically.
1. Introduction

The coherence and polarization of light beams are two aspects of statistical optics that can be treated in a unified manner. In 2003, Wolf proposed the unified theory of coherence and polarization of stochastic electromagnetic beams [1]–[3]. Since then, a lot of work has been done on the propagation of stochastic electromagnetic beams in free space [4], through turbulent atmosphere [5], through a gradient-index fiber [6] and through various complex optical systems [7]–[9]. The theoretical predictions were confirmed experimentally [10, 11]. However, all the above investigations have been restricted to stationary beams.

On the other hand, statistical optical pulses represent a wide class of partially coherent fields [12]. Recently, a scalar model of partially coherent pulses, in which the correlation between different frequency components is taken into consideration, was introduced by Pääkkönen et al [13]. Lajunen et al [14] described the coherent-mode representation for spatially and spectrally partially coherent Gaussian Schell-model scalar pulses. In this paper, taking the stochastic electromagnetic Gaussian Schell-model pulsed (GSMP) beam as a typical example of stochastic spatially and spectrally partially coherent electromagnetic pulsed beams, we consider the characterization of stochastic electromagnetic GSMP beams. It is actually a generalization of Wolf’s unified theory from stochastic stationary electromagnetic beams to stochastic spatially and spectrally partially coherent electromagnetic pulsed beams. In section 2, the expression for the cross-spectral density matrix of stochastic electromagnetic GSMP beams propagating in free space is derived, and used to formulate the spectral density (spectrum), the spectral degree of polarization and the spectral degree of coherence of stochastic electromagnetic GSMP beams. Some special cases are given in section 3. In section 4, numerical calculation examples are presented to illustrate the changes in the spectral density, the spectral degree of polarization and the spectral degree of coherence of stochastic electromagnetic GSMP beams propagating in free space. Section 5 concludes the main results obtained in this paper.

2. Theoretical formulation

In the space-time domain, the electric mutual coherence matrix of a stochastic spatially and temporally partially coherent electromagnetic pulse at the source plane $z = 0$ is
components of the electric vector are given by

\[ \Gamma_0^0(\rho_1, \rho_2, t_1, t_2) = [\Gamma_{ij}^0(\rho_1, \rho_2, t_1, t_2)] \]

\[ = \left[ [E_i^e(\rho_1, t_1) E_j^e(\rho_2, t_2)] \right], \]

(i = x, y; j = x, y unless otherwise stated),

(1)

where \( E_i \) and \( E_j \) are the polarization components of the electric field \( E(\rho, t) \) at the plane \( z = 0 \), \( \rho_{i(j)} = (x_{i(j)}, y_{i(j)}) \) is the position vector, \( t_{1(2)} \) is the time. The asterisk denotes the complex conjugate and the angular brackets denote the ensemble average. \( \Gamma_0^0(\rho_1, \rho_2, t_1, t_2) \) can be expressed as

\[ \Gamma_0^0(\rho_1, \rho_2, t_1, t_2) = \mathcal{F}^{-1}[I_i^0(\rho_1, t_1) I_j^0(\rho_2, t_2) \mu_{ij}^0(\rho_1, \rho_2, t_1, t_2)]. \]

(2)

where spectral densities \( I_i^0(\rho_1, t_1) \) and the degree of coherence between the \( i \) and \( j \) polarization components of the electric vector are given by [13]

\[ I_i^0(\rho, t) = A_i \exp \left[ -\frac{t^2}{T_0^2} \right] \exp \left[ -\frac{\rho^2}{2\sigma_i^2} \right], \]

(3)

\[ \mu_{ij}^0(\rho_1, \rho_2, t_1, t_2) = B_{ij} \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}^2} \right] \exp \left[ -\frac{(t_1 - t_2)^2}{2T_c^2} \right] \exp[\imath \omega_0 (t_1 - t_2)], \]

(4)

where the coefficients \( A_i, B_{ij} \) and the variables \( \sigma_i, \delta_{ij} \) are independent of position but may depend on the frequency, \( \delta_{ij} \) is related to the spatial correlation length, which represents the spatial correlation between the \( i \) and \( j \) components of the electric field vector in the source plane. \( T_0 \) is the pulse duration. \( T_c \) describes the temporal coherence length of the pulse, which denotes the temporal correlation of the pulse. \( \omega_0 \) is the carrier frequency of the pulse.

By using the Fourier-transform

\[ \tilde{W}^0(\rho_1, \rho_2, \omega_1, \omega_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Gamma_0^0(\rho_1, \rho_2, t_1, t_2) \exp \left[ -\imath (\omega_1 t_1 - \omega_2 t_2) \right] dt_1 dt_2, \]

(5)

we can derive the cross-spectral density matrix at the plane \( z = 0 \)

\[ \tilde{W}^0(\rho_1, \rho_2, \omega_1, \omega_2) = [W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2)], \]

(6)

where

\[ W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2) = W_0 \sqrt{A_i A_j} B_{ij} \exp \left[ -\frac{(\rho_1^2 + \rho_2^2)}{4\sigma_i^2} \right] \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}^2} \right] \exp \left[ -\frac{(\omega_1 - \omega_2)^2 + (\omega_2 - \omega_0)^2}{2\Omega_0^2} \right] \exp \left[ -\frac{(\omega_1 - \omega_2)^2}{2\Omega_c^2} \right], \]

(7)

\[ \Omega_0 = \sqrt{\frac{1}{T_0^2} + \frac{2}{T_c^2}}, \quad \text{(spectral width)} \]

(8)

\[ \Omega_c = \frac{T_c}{T_0} \Omega_0, \quad \text{(spectral coherence width)} \]

(9)

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\[
W_0 = \frac{T_0}{2\pi \Omega_0}.
\]

Equations (8) and (9) give the relation of the pulse duration \( T_0 \), temporal coherence length \( T_c \), spectral width \( \Omega_0 \) and spectral coherence width \( \Omega_c \). The spectral coherence width \( \Omega_c \) is a measure of the correlation between different frequency components of the pulse [13]. It is obvious from equations (7)–(9) that the spectrally fully coherent electromagnetic pulsed beams are obtained in the limit \( T_c \to \infty \) (\( \Omega_c \to \infty \)). If \( \delta_{ij} \to \infty \), we obtain spatially fully coherent electromagnetic pulsed beams. In the limit \( T_0 \to \infty \) (\( \Omega_c = 0, \ T_c = \sqrt{2}/\Omega_0 \)) all frequency components become uncorrelated, thus we obtain stochastic stationary electromagnetic beams.

To simplify the analysis we take [3]
\[
\begin{align*}
B_{ij} &= 1, \quad (i = j), \\
B_{ij} &= 0, \quad (i \neq j), \\
\sigma_x &= \sigma_y = \sigma.
\end{align*}
\]

The elements of the cross-spectral density matrix of stochastic electromagnetic GSMP beams at the plane \( z = 0 \) are given by
\[
W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2) = A_j W_0 \exp \left[ -\frac{(\rho_1^2 + \rho_2^2)}{4\sigma^2} \right] \exp \left[ -\frac{1}{2\delta_{ij}} \left( (\omega_1 - \omega_0)^2 + (\omega_2 - \omega_0)^2 \right) \right] \times \exp \left[ -\frac{1}{2\Omega_0^2} \right] \exp \left[ -\frac{1}{2\Omega_c^2} \right],
\]

\[
W_{xy}^0(\rho_1, \rho_2, \omega_1, \omega_2) = W_{yx}^0(\rho_1, \rho_2, \omega_1, \omega_2) = 0.
\]

Therefore, the spectral density, the spectral degree of polarization and the spectral degree of coherence of stochastic electromagnetic GSMP beams at the plane \( z = 0 \) are expressed as [1]
\[
S^0(\rho, \omega) = \text{Tr}[\mathbf{W}^0(\rho, \rho, \omega, \omega)] = (A_x + A_y) W_0 \exp \left[ -\frac{\rho^2}{2\sigma^2} \right] \exp \left[ -\frac{1}{\Omega_0^2} \right] \exp \left[ -\frac{1}{\Omega_c^2} \right],
\]

\[
P^0(\rho, \omega) = \left[ 1 - \frac{4\text{Det}[\mathbf{W}^0(\rho, \rho, \omega, \omega)]}{\text{Tr} [\mathbf{W}^0(\rho, \rho, \omega, \omega)]^2} \right] = \frac{A_x - A_y}{A_x + A_y},
\]

\[
\eta^0(\rho_1, \rho_2, \omega_1, \omega_2) = \frac{\text{Tr}[\mathbf{W}^0(\rho_1, \rho_2, \omega_1, \omega_2)]}{\sqrt{S^0(\rho_1, \omega_1)S^0(\rho_2, \omega_2)}}
\]

\[
= \frac{1}{2} \left( \frac{A_x}{A_y} \exp \left[ -\frac{1}{2\delta_{xx}^2} \left( (\rho_1 - \rho_2)^2 \right) \right] + \frac{A_y}{A_x} \exp \left[ -\frac{1}{2\delta_{yy}^2} \left( (\rho_1 - \rho_2)^2 \right) \right] \right) \exp \left[ -\frac{1}{2\Omega_c^2} \right].
\]

The cross-spectral density matrix of stochastic electromagnetic GSMP beams at the plane \( z > 0 \) in the free-space propagation is expressed as [15]
\[
\mathbf{W}(r_1, r_2, z, \omega_1, \omega_2) = \frac{\omega_1 \omega_2}{4\pi c^2 z^2} \exp[i(\omega_2 - \omega_1)z/c] \int \int \mathbf{W}^0(\rho_1, \rho_2, \omega_1, \omega_2)
\]

\[
\times \exp \left[ \frac{i}{2c^2} \left[ (\omega_2 - \omega_1)^2 (r_1 - \rho_2)^2 - (\omega_1 - \omega_2)^2 (r_2 - \rho_1)^2 \right] \right] \, d^2\rho_1 \, d^2\rho_2.
\]

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The substitution of equations (12) and (13) into equation (17) yields

\[
W_{jj}(r_1, r_2, z, \omega_1, \omega_2) = \frac{A_j W_{0j} \delta_{jj}^4 \omega_1 \omega_2}{(4\alpha_j \omega_2 \delta_{jj}^4 - 1) c^2 z^2} \exp \left\{ \frac{i}{2c z} \left[ \omega_1 (x_1^2 + y_1^2) - \omega_2 (x_1^2 + y_1^2) \right] \right\} \\
\times \exp \left\{ \frac{i(\omega_2 - \omega_1) z}{c} \right\} \exp \left\{ \frac{- (\omega_1 - \omega_0)^2 + (\omega_2 - \omega_0)^2}{2\Omega_0^2} - \frac{(\omega_1 - \omega_2)^2}{2\Omega_c^2} \right\} \\
\times \exp \left\{ \frac{- [\alpha_{jj} \omega_1^2 (x_1^2 + y_1^2) + \alpha_{jj} \omega_2^2 (x_1^2 + y_1^2)] \delta_{jj}^4}{c^2 z^2 (4\alpha_j \omega_2 \delta_{jj}^4 - 1)} \right\},
\]

(18)

\[
W_{zz}(r_1, r_2, z, \omega_1, \omega_2) = W_{zz}(r_1, r_2, z, \omega_1, \omega_2) = 0,
\]

(19)

where

\[
\alpha_{jj} = \frac{1}{4\sigma_z^2} + \frac{1}{2\delta_{jj}^2} + \frac{i\omega_1}{2cz},
\]

(20)

\[
\alpha_{zz} = \frac{1}{4\sigma_z^2} + \frac{1}{2\delta_{zz}^2} + \frac{i\omega_2}{2cz}.
\]

(21)

In accordance with the unified theory [1], the spectral density, the spectral degree of polarization and the spectral degree of coherence of stochastic electromagnetic GSMP beams at the plane \( z > 0 \) are given by

\[
S(r, z, \omega) = \frac{\text{Tr}[\tilde{W}(r, r, z, \omega, \omega)]}{\sqrt{\text{Tr}[\tilde{W}(r, r, z, \omega, \omega)]}}
\]

\[
= \frac{T_0}{2\pi \Omega_0} \left[ \frac{A_{xx}}{Q_{xx}} \exp \left( -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right) + \frac{A_{yy}}{Q_{yy}} \exp \left( -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right) \right] \exp \left( -\frac{(\omega - \omega_0)^2}{2\Omega_0^2} \right),
\]

(22)

\[
P(r, z, \omega) = \left[ 1 - 4\text{Det}[\tilde{W}(r, r, z, \omega, \omega)] \right]^{1/2} = \frac{A_{xx} \exp \left( -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right) - A_{yy} \exp \left( -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right)}{\sqrt{\text{Tr}[\tilde{W}(r, r, z, \omega, \omega)]}}
\]

(23)

\[
\eta(r_1, r_2, z, \omega_1, \omega_2) = \frac{\text{Tr}[\tilde{W}(r_1, r_2, z, \omega_1, \omega_2)]}{\sqrt{S(r_1, z, \omega_1)S(r_2, z, \omega_2)}},
\]

(24)

where

\[
Q_{xx} = 1 + \frac{c^2 z^2}{2\sigma_x^2 \omega_1^2} \left( \frac{1}{2\sigma_x^2} + \frac{2}{\delta_{xx}^2} \right),
\]

(25)

\[
Q_{yy} = 1 + \frac{c^2 z^2}{2\sigma_y^2 \omega_2^2} \left( \frac{1}{2\sigma_y^2} + \frac{2}{\delta_{yy}^2} \right).
\]

(26)
Note that the definition of the spectral degree of coherence is not unique in the literature. Another definition is found, for example, in [16].

Equations (14)–(16) and (22)–(24) are the main analytical results obtained in this paper, and describe the changes in the spectral density, the spectral degree of polarization and the spectral degree of coherence of stochastic electromagnetic GSMP beams from the \( z = 0 \) plane to the \( z \)-plane in free space, which depend on the pulse duration \( T_0 \), temporal coherence length \( T_c \), spatial correlation length \( \delta_{jj} \), coefficients \( A_x, A_y \) and propagation distance \( z \).

### 3. Some special cases

#### 3.1. Stochastic stationary electromagnetic beams

Letting the pulse duration \( T_0 \to \infty \), from equations (8) and (9) we obtain \( \Omega_c = 0 \) and \( T_c = \sqrt{2}/\Omega_0 \). As a result, the spectral components are completely uncorrelated and the mutual coherence matrix of stochastic stationary electromagnetic beams at the plane \( z = 0 \) reduces to

\[
\Gamma_0^{ij}(\rho_1, \rho_2, t_1, t_2) = [\Gamma_0^{ij}(\rho_1, \rho_2, t_1, t_2)],
\]

where

\[
\Gamma_0^{ij}(\rho_1, \rho_2, t_1, t_2) = \sqrt{A_i A_j} B_{ij} \exp \left[ -\left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}^2} \right] \times \exp \left[ -\frac{\Omega_0^2(t_1 - t_2)^2}{4} \right] \exp \left[ i\omega_0(t_1 - t_2) \right].
\]

On substituting equation (27) into equation (5), the cross-spectral density matrix of stochastic stationary electromagnetic beams is expressed as

\[
\hat{\mathbf{W}}_0^0(\rho_1, \rho_2, \omega_1, \omega_2) = [W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2)],
\]

where

\[
W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2) = \frac{\sqrt{A_i A_j} B_{ij}}{\Omega_0 \sqrt{\pi}} \exp \left[ -\left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}^2} \right] \times \exp \left[ -\frac{(\omega - \omega_0)^2}{\Omega_0^2} \right] \delta(\omega_1 - \omega_2),
\]

with \( \delta(\cdot) \) being the Dirac delta function.

Substituting equation (29) into equation (17) and making use of equation (11), the spectral density, the spectral degree of polarization and the spectral degree of coherence of stochastic stationary electromagnetic beams at the plane \( z > 0 \) are given by

\[
S(r, z, \omega) = \text{Tr}[\hat{\mathbf{W}}(r, r, z, \omega)]
= \frac{1}{\Omega_0 \sqrt{\pi}} \left[ \frac{A_x}{Q_{xx}} \exp \left( -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right) + \frac{A_y}{Q_{yy}} \exp \left( -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right) \right] \exp \left[ -\frac{(\omega - \omega_0)^2}{\Omega_0^2} \right],
\]

\[\text{(31)}\]
electromagnetic pulsed beams at the plane \( z \), spectral degree of polarization and the spectral degree of coherence of spectrally fully coherent matrix of spectrally fully coherent electromagnetic GSMP beams at the plane \( z = 0 \) is given by

\[
P(r, z, \omega) = \sqrt{1 - \frac{4\text{Det}[\hat{W}(r, r, z, \omega)]}{\text{Tr} \left[ \hat{W}(r, r, z, \omega) \right]^2}} = \frac{A_x^* \exp \left[ -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right] - A_y \exp \left[ -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right]}{A_x^* \exp \left[ -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right] + A_y \exp \left[ -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right]},
\]

(32)

where \( \Omega_0 = \sqrt{2}/T_c \), equations (31)–(33) are consistent with equations (2)–(4) in [2].

3.2. Spectrally fully coherent electromagnetic pulsed beams

Letting the temporal coherence length \( T_c \rightarrow \infty \) (i.e. \( \Omega_c \rightarrow \infty \)), the electric mutual coherence matrix of spectrally fully coherent electromagnetic GSMP beams at the plane \( z = 0 \) simplifies to

\[
\hat{\Gamma}^0(\rho_1, \rho_2, t_1, t_2) = [\Gamma_{ij}^0(\rho_1, \rho_2, t_1, t_2)],
\]

(34)

where

\[
\Gamma^0_{ij}(\rho_1, \rho_2, t_1, t_2) = \sqrt{A_i A_j} B_{ij} \exp \left[ -\left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}} \right] \exp \left[ -\frac{t_1^2 + t_2^2}{2\tau_0^2} \right] \exp \left[ i\omega_0(t_1 - t_2) \right].
\]

(35)

From equations (5) and (34) the cross-spectral density matrix of spectrally fully coherent electromagnetic pulsed beams is given by

\[
\hat{W}^0(\rho_1, \rho_2, \omega_1, \omega_2) = [W_{ij}^0(\rho_1, \rho_2, \omega_1, \omega_2)],
\]

(36)

where

\[
W^0_{ij}(\rho_1, \rho_2, \omega_1, \omega_2) = \frac{T_0}{2\pi\Omega_0} \sqrt{A_i A_j} B_{ij} \exp \left[ -\left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \exp \left[ -\frac{(\rho_1 - \rho_2)^2}{2\delta_{ij}} \right] \times \exp \left[ -\frac{(\omega_1 - \omega_0)^2 + (\omega_2 - \omega_0)^2}{2\Omega_0^2} \right].
\]

(37)

Substituting equation (36) into equation (17) and using equation (11), the spectral density, the spectral degree of polarization and the spectral degree of coherence of spectrally fully coherent electromagnetic pulsed beams at the plane \( z > 0 \) are expressed as

\[
S(r, z, \omega) = \text{Tr}[\hat{W}(r, r, z, \omega)]
\]

\[
= \frac{T_0}{2\pi\Omega_0} \left[ A_x^* \exp \left( -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right) + A_y \exp \left( -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right) \right] \exp \left[ -\frac{(\omega - \omega_0)^2}{\Omega_0^2} \right],
\]

(38)

\[
P(r, z, \omega) = \sqrt{1 - \frac{4\text{Det}[\hat{W}(r, r, z, \omega)]}{\text{Tr} \left[ \hat{W}(r, r, z, \omega) \right]^2}} = \frac{A_x^* \exp \left[ -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right] - A_y \exp \left[ -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right]}{A_x^* \exp \left[ -\frac{r^2}{2\sigma_x^2 Q_{xx}} \right] + A_y \exp \left[ -\frac{r^2}{2\sigma_y^2 Q_{yy}} \right]},
\]

(39)
\[
\eta(r_1, r_2, z, \omega_1, \omega_2) = \frac{\text{Tr}[\tilde{W}(r_1, r_2, z, \omega_1, \omega_2)]}{\sqrt{S(r_1, z, \omega_1)} \sqrt{S(r_2, z, \omega_2)}}.
\]

where

\[
\Omega_0 = 1/T_0.
\]

### 3.3. Spatially fully coherent electromagnetic pulsed beams

Letting the spatial correlation length \(\delta_{ij} \to \infty\), the electric mutual coherence matrix of spatially fully coherent electromagnetic pulsed beams at the plane \(z = 0\) becomes

\[
\tilde{\Gamma}^0(\rho_1, \rho_2, t_1, t_2) = [\Gamma^0_{ij}(\rho_1, \rho_2, t_1, t_2)],
\]

where

\[
\Gamma^0_{ij}(\rho_1, \rho_2, t_1, t_2) = \sqrt{A_i A_j} B_{ij} \exp \left[ - \left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \\
\times \exp \left[ - \frac{(t_1 - t_2)^2}{2T_0^2} \right] \exp \left[ - \frac{(t_1 - t_2)^2}{2T_0^2} \right] \exp \left[ i\omega_0(t_1 - t_2) \right].
\]

The cross-spectral density matrix of spatially fully coherent electromagnetic pulsed beams reads as

\[
\tilde{W}^0(\rho_1, \rho_2, 0, \omega_1, \omega_2) = [W^0_{ij}(\rho_1, \rho_2, \omega_1, \omega_2)],
\]

where

\[
W^0_{ij}(\rho_1, \rho_2, \omega_1, \omega_2) = \frac{T_0}{2\pi\Omega_0} \sqrt{A_i A_j} B_{ij} \exp \left[ - \left( \frac{\rho_1^2}{4\sigma_i^2} + \frac{\rho_2^2}{4\sigma_j^2} \right) \right] \\
\times \exp \left[ - \frac{(\omega_1 - \omega_0)^2 + (\omega_2 - \omega_0)^2}{2\Omega_0^2} \right] \exp \left[ - \frac{(\omega_1 - \omega_0)^2}{2\Omega_0^2} \right].
\]

The spectral density, the spectral degree of polarization and the spectral degree of coherence of spatially fully coherent electromagnetic pulsed beams at the plane \(z > 0\) are given by

\[
S(r, z, \omega) = \text{Tr}[\tilde{W}(r, r, z, \omega)] \\
= \frac{A_x + A_y}{\Delta} \frac{T_0}{2\pi\Omega_0} \exp \left( - \frac{r^2}{2\sigma^2} \right) \exp \left[ - \frac{(\omega - \omega_0)^2}{\Omega_0^2} \right],
\]

\[
P(r, z, \omega) = \sqrt{1 - \frac{4\text{Det}[	ilde{W}(r, r, z, \omega)]}{\left\{ \text{Tr}[\tilde{W}(r, r, z, \omega)] \right\}^2}} = \left| \frac{A_x - A_y}{A_x + A_y} \right|,
\]

\[
\eta(r_1, r_2, z, \omega_1, \omega_2) = \frac{\text{Tr}[\tilde{W}(r_1, r_2, z, \omega_1, \omega_2)]}{\sqrt{S(r_1, z, \omega_1)} \sqrt{S(r_2, z, \omega_2)}} = \exp \left[ - \frac{(\omega_1 - \omega_2)^2}{2\Omega_0^2} \right],
\]

where

\[
\Delta = 1 + \frac{c^2 z^2}{4\sigma^2\omega^2}.
\]
Equation (47) implies the invariance of polarization of spatially fully coherent electromagnetic pulses in the free-space propagation [17]. Equations (31) and (38) are formally the same as equation (22), but the fields are physically different because of the different spectral widths $\Omega_0 = \sqrt{2}/T_c$, $\Omega_0 = 1/T_0$ and $\Omega_0 = \sqrt{1/T_0^2 + 2/T_c^2}$, respectively. Equations (32) and (39) are formally the same as equation (23), which implies that the temporal coherence length does not affect the spectral degree of polarization of stochastic electromagnetic GSMP beams provided that the temporal coherence length of the $i$ component of the electric vector are the same as that of the $j$ component of the electric vector, i.e. $T_{ci} = T_{cj} = T_c$. However, it can be shown that for the case of $T_{ci} \neq T_{cj}$ [18] the spectral degree of polarization depends on $T_{ci} \,(T_{cj})$.

4. Illustrative examples

To illustrate the applications of the theory, numerical calculation results for stochastic spatially and spectrally partially coherent electromagnetic pulses propagating in free space are presented. Figure 1(a) shows the on-axis relative spectral shift $\Delta \omega/\omega_0$ of stochastic spatially and spectrally partially coherent electromagnetic pulses versus the propagation distance $z$ for different values of the pulse duration $T_0$. The calculation parameters are $T_c = 7$ fs, $\sigma = 1$ mm, $\delta_{xx} = 1$ mm, $\delta_{yy} = 2\delta_{xx}$, $A_y = 1$, $P^0 = 0.2$ and $\omega_0 = 2.36 \text{ rad fs}^{-1}$. The relative spectral shift $\Delta \omega/\omega_0$ is defined as $\Delta \omega/\omega_0 = (\omega_{\text{max}} - \omega_0)/\omega_0$, where $\omega_{\text{max}}$ denotes the frequency at which the spectral density $S(r, z, \omega)$ takes the maximum value. It is shown that the spectrum is blue-shifted in free-space propagation. With increasing propagation distance $z$, the blue-shift increases and approaches an asymptotic value when $z$ is large enough. For example, depending on the pulse duration, the asymptotic value is equal to 0.651, 0.407 and 0.234 for $T_0 = 3$ fs, 5 fs and $\infty$, respectively. The results can be interpreted as follows.

For large enough $z$, equations (25) and (26) can be approximately expressed as

\[ Q_{jj} \approx \frac{R_{jj}}{\omega^2}, \tag{50} \]
where
\[ R_{jj} = \frac{e^2 z^2}{2\sigma^2} \left( \frac{1}{2\sigma^2} + \frac{2}{\delta^2_{jj}} \right), \quad (j = x, y). \] (51)

Thus, the on-axis spectral density is written as
\[ S(0, z, \omega) = \frac{T_0}{2\pi \Omega_0} \left( \frac{A_x}{R_{xx}} + \frac{A_y}{R_{yy}} \right) \omega^2 \exp \left[ -\frac{(\omega - \omega_0)^2}{\Omega_0^2} \right]. \] (52)

By letting \[ \frac{\partial S(0, z, \omega)}{\partial \omega} = 0, \] the on-axis relative spectral shift \[ \frac{\delta \omega}{\omega_0} \] is given by
\[ \frac{\delta \omega}{\omega_0} = \frac{1}{2} \left( 1 + \sqrt{\frac{4\Omega_0^2}{\omega_0^2} + 1} \right) - 1. \] (53)

On substituting \( T_0 = 3 \text{ fs}, 5 \text{ fs} \) and \( \infty \) into equation (53) with \( T_c = 7 \text{ fs} \) and \( \omega_0 = 2.36 \text{ rad fs}^{-1} \), we obtain \[ \frac{\delta \omega}{\omega_0} \] values of 0.651, 0.407 and 0.234, respectively, which are consistent with the above results in figure 1(a).

Figure 1(b) represents the on-axis relative spectral shift \( \frac{\delta \omega}{\omega_0} \) of stochastic spatially and spectrally partially coherent electromagnetic pulses versus the propagation distance \( z \) for different values of the temporal coherence length \( T_c = 5 \text{ fs}, 7 \text{ fs}, \infty \) and \( T_0 = 5 \text{ fs} \). The other calculation parameters are the same as those in figure 1(a). As can be seen, with increasing \( z \) the relative spectral shift \( \frac{\delta \omega}{\omega_0} \) approaches asymptotic values 0.549, 0.407 and 0.230 for \( T_c = 5 \text{ fs}, 7 \text{ fs} \) and \( \infty \), respectively. The physical explanation of figure 1(b) is similar to figure 1(a), because the substitution from \( T_c = 5 \text{ fs}, 7 \text{ fs} \) and \( \infty \) into equation (53) with \( T_0 = 5 \text{ fs} \) and \( \omega_0 = 2.36 \text{ rad fs}^{-1} \) yields \( \frac{\delta \omega}{\omega_0} \) values of 0.549, 0.407 and 0.230, respectively.

Figure 2(a) shows the changes of the on-axis spectral degree of polarization \( P \) of stochastic spatially and spectrally partially coherent electromagnetic pulses as a function of the propagation distance \( z \) and frequency \( \omega/\omega_0 \), and the color-coded plot corresponding to figure 2(a) is shown in figure 2(b). The calculation parameters are \( \delta_{xx} = 1 \text{ mm}, \delta_{yy} = 2\delta_{xx}, A_x = 1, \sigma = 1 \text{ mm}, P^0 = 0.2, T_0 = 5 \text{ fs} \) and \( T_c = 7 \text{ fs} \). It is seen that the frequency \( \omega \) influences the distribution of on-axis spectral degree of polarization \( P \) of stochastic spatially and spectrally partially coherent electromagnetic pulses, and the position of the on-axis spectral degree of polarization \( P = \text{const} \) increases linearly with increasing \( \omega \), the physical reason is as follows.

By letting on-axis spectral degree of polarization \( P = \text{const} \), from equation (23) we obtain
\[ z_1 = \omega \frac{2(P + P^0)}{F_{xx}(P - 1)(P - 1) - F_{yy}(P + 1)(P + 1)}, \] (54)
\[ z_2 = \omega \frac{2(P - P^0)}{F_{xx}(P + 1)(P + 1) - F_{yy}(P - 1)(P - 1)}. \] (55)

If \( P = 0 \), we have
\[ z_1 = z_2 = \omega \frac{2P^0}{F_{xx}(1 - P^0) - F_{yy}(1 + P^0)}, \] (56)

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Figure 2. (a) On-axis spectral degree of polarization $P$ of a stochastic electromagnetic GSMP beam as a function of $z$ and $\omega/\omega_0$ and (b) the color-coded plot corresponding to (a).

where

$$F_{jj} = \frac{c^2}{2\sigma^2} \left( \frac{1}{2\sigma^2} + \frac{2}{\delta_{jj}^2} \right).$$

Equations (54) and (55) indicate that $z$ is a linear function of $\omega$, provided that $\sigma$ and $\delta_{jj}$ are independent of the frequency, e.g. $\sigma = 1$ mm, $\delta_{xx} = 1$ mm and $\delta_{yy} = 2$ mm.

The variation of spectral degree of coherence $|\eta(x, -x, z, \omega_1, \omega_2)|$ of stochastic spatially and spectrally partially coherent electromagnetic pulses versus the transverse coordinate $x$ for

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different values of the spatial correlation length $\delta_{xx} = 1, 1.5$ and 2 mm is plotted in figure 3(a). The calculation parameters are $\delta_{yy} = 2\delta_{xx}$, $y = 1$ m, $y_1 = y_2 = 0$, $A_y = 1$, $P^0 = 0.2$, $\sigma = 1$ mm, $T_0 = 5$ fs, $T_c = 7$ fs, $\omega_1/\omega_0 = 1.2$ and $\omega_2/\omega_0 = 0.8$. It is seen that $|\eta(x, -x, z, \omega_1, \omega_2)|$ decreases with increasing $x$, and $\delta_{xx}$ affects the distribution of $|\eta(x, -x, z, \omega_1, \omega_2)|$ which has the same value for different $\delta_{xx}$ at $x = 0$ and for a fixed value $x |\eta(x, -x, z, \omega_1, \omega_2)|$ increases with increasing spatial correlation length $\delta_{xx}$. Therefore, the width at half maximum of $|\eta(x, -x, z, \omega_1, \omega_2)|$ increases with an increase of $\delta_{xx}$.

Figure 3(b) gives the spectral degree of coherence $|\eta(x, -x, z, \omega_1, \omega_2)|$ of the stochastic spatially and spectrally partially coherent electromagnetic pulses as a function of the propagation distance $z$ for different values of the temporal coherence length $T_c = 3$ fs, 7 fs, $\infty$ and $x = 1$ mm. The other calculation parameters are the same as those in figure 3(a). As can be seen, $|\eta(x, -x, z, \omega_1, \omega_2)|$ increases with increasing $z$ and approaches an asymptotic value, which increases with increasing $T_c$. The spectral degree of coherence $|\eta(x, -x, z, \omega_1, \omega_2)|$ at a pair of points $(x, z, \omega_1)$ and $(-x, z, \omega_2)$ is expressed as

\[ |\eta(x, -x, z, \omega_1, \omega_2)| = \frac{M_{xx} + M_{yy}}{\sqrt{N_1 N_2}} \exp \left[ -\frac{(\omega_1 - \omega_2)^2}{\Omega_c^2} \right], \]  \hspace{1cm} (58)

where

\[ M_{jj} = \frac{A_j \delta_{jj}^4}{4\alpha_1 \alpha_2 \delta_{jj}^4 - 1} \exp \left\{ -\frac{[(\alpha_2 \omega_1^2 + \alpha_1 \omega_2^2)\delta_{jj}^2 + \omega_1 \omega_2 \delta_{jj}^2]}{c^2 \Omega^2 (4\alpha_1 \alpha_2 \delta_{jj}^4 - 1)} \right\}, \]  \hspace{1cm} (59)

\[ N_{\xi} = \frac{A_x}{Q_{xx\xi}} \exp \left( -\frac{x^2}{2\sigma^2 Q_{xx\xi}} \right) + \frac{A_y}{Q_{yy\xi}} \exp \left( -\frac{x^2}{2\sigma^2 Q_{yy\xi}} \right), \]  \hspace{1cm} (60)

\[ Q_{jj\xi} = 1 + \frac{c^2 \Omega^2}{2\sigma^2 \Omega^2} \left( \frac{1}{2\sigma^2} + \frac{2}{\delta_{jj}^2} \right), \quad (\xi = 1, 2). \]  \hspace{1cm} (61)
\[ \Omega_c = \frac{1}{T_0} \sqrt{\frac{T_c^2}{T_0^2} + 2}. \] (62)

Equations (8), (9), (58)–(62) indicate that for fixed \( \sigma, \delta_{jj}, z, \omega_1, \omega_2, A_x, A_y \) and \( T_0, |\eta(x, -x, z, \omega_1, \omega_2)| \) increases with an increase of \( T_c \).

5. Conclusion

In this paper, the unified theory of coherence and polarization proposed by Wolf has been extended from stochastic stationary electromagnetic beams to stochastic spatially and spectrally partially coherent electromagnetic pulsed beams. Taking the stochastic electromagnetic GSMP beam as a typical example of stochastic spatially and spectrally partially coherent electromagnetic pulsed beams, the analytical expression for cross-spectral density matrix has been derived, and used to formulate the spectral density, spectral degree of polarization and spectral degree of coherence of stochastic electromagnetic GSMP beams propagating in free space. The stationary electromagnetic beams, spectrally fully coherent electromagnetic pulsed beams and spatially fully coherent electromagnetic pulsed beams can be regarded as special cases of stochastic spatially and spectrally partially coherent electromagnetic pulsed beams by letting \( T_0 \to \infty, T_c \to \infty \) and \( \delta_{jj} \to \infty \), respectively. Numerical calculation examples have been presented to illustrate the applications of the theory. Finally, we would like to point out that, although the theoretical formulation has been made for stochastic electromagnetic GSMP beams to clarify the main physical aspects, the extension to other types of stochastic electromagnetic pulsed beams, for example, of stochastic electromagnetic cosh–Gaussian, Bessel–Gauss pulsed beams, etc are straightforward. Therefore, the results obtained in this paper would be useful for the study of more general types of stochastic electromagnetic pulsed beams in a unique way.

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