INFLATIONARY SOLUTIONS IN QUANTUM COSMOLOGY

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Abstract. We prove that in the Hartle-Hawking approach to quantum cosmology the existence of an inflationary phase is a general property of minisuperspace models given by a closed Friedmann-Robertson-Walker universe containing a massless scalar field with a $\lambda \phi^n$ self-interaction. The evolution in time of the cosmic scale factor and of the scalar field in the very early universe is derived, together with the conditions to be satisfied in order to solve the horizon and flatness problems.
1. Introduction

The quantum state of the universe [1, 2] is described by a functional $\psi$ of the 3-geometry and of the matter field configuration on a compact spacelike 3-surface $S$. Any solution of the Wheeler-DeWitt equation and of the momentum constraints is a possible quantum state of the universe [1-3]. In the last few years there has been great interest in the Hartle-Hawking proposal for the boundary conditions of the universe [1, 2, 4, 5], which seems to predict a universe whose large-scale features are in agreement with observations [1, 6-9]. According to Hartle and Hawking it seems that $\psi$ is to be taken over all compact Euclidean 4-metrics which induce the given 3-metric on their boundary $S$, and over all field configurations which match the given value on $S$ and are regular on the compact 4-manifolds having $S$ as their only boundary.

In order to solve the Wheeler-DeWitt equation, most authors have considered the minisuperspace approximation [10]. In the by now well known Hawking massive scalar field model [1, 2, 11, 12], the universe begins its inflationary expansion from a non-singular state. At the end of the inflation, it undergoes a matter- or radiation-dominated expansion, reaches a maximum radius and recollapses to a singularity. However, in general it is not clear whether the solutions to the classical field equations corresponding to the Hartle-Hawking wavefunction describe a universe which will recollapse to a singularity. In fact, Louko [13] has pointed out that the method of providing initial conditions for purely Lorentzian solutions by analytically continuing purely Euclidean solutions in the matter-dominated and recollapsing phases may be erroneous. His calculations seem to suggest that the universe will not reach the singularity in the future.

After this short exposition of some ideas and problems of quantum cosmology, let us now turn our attention to the problem we try to solve in this paper. We study a minisuperspace model given by a closed Friedmann-Robertson-Walker (hereafter referred to as FRW) universe containing a massless scalar field with a $\lambda \phi^n$ self-interaction. A massless scalar field is considered for two main reasons:

(a) it is not clear whether a fundamental massive scalar field exists [7];
(b) one is still unable to say which theory of particle physics gives the best description of the very early universe [14].

In section 2 we write down the action integral and the field equations of our minisuperspace model, and we discuss in detail how the Hartle-Hawking proposal enables one to impose initial conditions for the solutions to these equations. We also endeavour to derive the oscillatory region for the wave function in view of its behaviour in the limit of small 3-geometry and by comparison with the massive scalar field model. In addition we discuss how the spacelike or timelike nature of the surfaces of constant potential of the Wheeler-De Witt equation may be useful in deriving the behaviour of the wavefunction.

In section 3 at first we focus our attention on the region in which the wavefunction oscillates. The phase $S$ of the WKB approximation is the one obtained by analytical continuation of the Euclidean action for compact metrics and regular matter fields. We are thus able to derive the evolution in time of the cosmic scale factor and of the matter field using an approximate form assumed by $S$. The Hartle-Hawking trajectories so derived are exact solutions to the Lorentzian field equations. They are distinguished with respect to the general set of solutions in that they are inflationary and singularity-free in the past.

In section 4 we derive the constraints to be satisfied in order to solve the horizon and flatness problems; in section 5 we summarize the results obtained and we mention some related problems to be studied.

2. Minisuperspace model and boundary conditions

Our model is based on the FRW metric which may be locally cast in the form: $ds^2 = \sigma^2(-N^2 dt^2 + a^2(t) d\Omega_3^2)$, where $\sigma = \sqrt{\frac{2G}{3\pi}}$, $N$ is the lapse function [15], $a(t)$ is the cosmic scale factor and $d\Omega_3^2$ is the metric on a unit 3-sphere.

Let $\phi'(t) = \frac{\phi(t)}{\pi \sigma \sqrt{2}}$ be the scalar field (in these units $\phi$ is dimensionless) having a potential of the form $V(\phi') = \frac{\lambda}{2} \phi'^{2n}$. The action integral containing a surface term [5]
becomes, in our case,

\[ I = -\frac{1}{2} \int Na^3 \left[ \frac{\dot{a}^2}{a^2 N^2} - \frac{1}{a^2} - \frac{\dot{\phi}^2}{N^2} + \lambda \phi^n \right] dt = \int L \, dt \quad (2.1) \]

where \( \lambda = \lambda' \sigma^{-n+4} (\pi \sqrt{2})^{-n+2} \).

Before commencing the calculations, it is worth emphasizing the main property of minisuperspace models containing scalar fields. The minisuperspace model given by a homogeneous isotropic universe without matter fields but with a positive cosmological constant (De Sitter space) did show that the Hartle-Hawking proposal leads to inflation if some mechanism is able to give rise to an effective cosmological constant in the early universe [1, 11]. But our universe is not expanding exponentially. This is why we require that the parameter we call the cosmological constant has finally to vanish as the time passes. The simplest model for generating a decaying cosmological constant is the one containing a massive scalar field. In fact, when \( \phi \) is initially very large and almost constant, the \( m^2 \phi^2 \) term in the action of the massive model acts as an effective cosmological constant. This implies that, when \( \phi \cong \text{constant} = \phi_0 \), all formulae for the massive model can be obtained from the De Sitter model by putting \( H = \sqrt{m^2 \phi_0^2} = m \phi_0 \), where \( H \) is the parameter of the De Sitter model related to the cosmological constant \( \Lambda_c \) by means of the relation [1] \( H^2 = \frac{2G \Lambda_c}{9\pi} \). If \( a < H^{-1} \), it is known that there are two solutions of the Euclidean field equations, and the dominant contribution to the semiclassical approximation of the wave function comes from the one having action [1] \( I_E = -\frac{1}{3H^2} \left[ 1 - (1 - H^2 a^2)^{\frac{3}{2}} \right] \), where \( I_E \) is the action of the smaller part of a 4-sphere of radius \( \frac{1}{H} \) bounded by a 3-sphere of radius \( a \). In our model, if the term \( \lambda \phi^n \) in (2.1) has to act as an effective cosmological constant when \( \phi \) is large and constant, the parameter \( H \) becomes \( H = \sqrt{\lambda \phi_0^2} = \sqrt{\lambda \phi_0^2} \). From now on, \( H \) will always denote \( \sqrt{\lambda \phi_0^2} \) in our equations.

We can now turn again our attention to (2.1). Putting \( N = 1 \), the Lorentzian field equations are

\[ a\ddot{a} + \frac{\dot{a}^2}{2} + \frac{1}{2} + \frac{3}{2} a^2 \dot{\phi}^2 - \frac{3}{2} \lambda a^2 \phi^n = 0 \quad (2.2) \]
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\[ \ddot{\phi} + 3\frac{\dot{a}}{a} \dot{\phi} + \frac{n}{2} \lambda \phi^{n-1} = 0. \]  

(2.3)

We now propose to derive the initial conditions for the solutions to (2.2) and (2.3) implied by the Hartle-Hawking proposal. To this purpose let us recall that the Hartle-Hawking ground state is a Euclidean functional integral taken over compact 4-metrics and regular matter fields. In the Euclidean regime one thus has [1]

\[
\begin{align*}
    a(\tau = 0) &= 0 \\
    \dot{a}(\tau = 0) &= 1 \\
    \phi(\tau = 0) &= \phi_0 \\
    \dot{\phi}(\tau = 0) &= 0.
\end{align*}
\]

(2.4)

(2.5)

In fact, having taken the path integral over compact 4-metrics, \(a(\tau)\) has to vanish at some value of \(\tau\) we can choose to be zero [1]. This implies that at large values of \(\phi\) one has \(a(\tau) = \frac{1}{H} \sin(\lambda \phi^{n-1})\), and this relation is in agreement with (2.4). In addition, in the Euclidean regime (2.3) can be written in a similar form, provided we take the derivatives with respect to \(\tau\) and we change the sign in front of \(\frac{n}{2} \lambda \phi^{n-1}\). Therefore, multiplying both sides of (2.3) by \(a\), and letting \(a\) go to zero, one has: \(a \ddot{\phi} \rightarrow 0, \frac{n}{2} a \phi^{n-1} \rightarrow 0\), in view of the regularity of \(\phi\). Therefore one has \(a(3\frac{\dot{a}}{a} \dot{\phi}) = 0\) if and only if \(\dot{\phi} = 0\), in view of (2.4).

In quantum cosmology one is mainly interested in the oscillatory regions for the wave-function, because such regions correspond to classically allowed regions [1]. One can then make the WKB ansatz: \(\psi = \sum Re[C_n \exp(iS_n)]\) (in what follows we shall consider a single component of the wave packet). The phase \(S\) is chosen to satisfy the Hamilton-Jacobi equation for general relativity, and the Hartle-Hawking proposal picks out the Hamilton-Jacobi function which is the analytic continuation of the Euclidean action. This implies that the Lorentzian Hartle-Hawking trajectories are obtained by analytic continuation of the compact Euclidean paths with regular matter fields. The application of this method yields, putting \(\tau = \frac{\pi}{2H} + it\) [12]:

\[ a(t) = \frac{1}{H} \cosh(Ht) \]  

(2.6)

\[ \phi(t) = \phi_0 \]  

(2.7)
at very small values of the time. Thus the minimum value of $a$ in the Lorentzian regime is equal to the maximum value of $a$ in the Euclidean regime, and the initial conditions for $a$ differ greatly from (2.4) because one has $a(t = 0) = \frac{1}{H}$, $\dot{a}(t = 0) = 0$. Therefore at large initial values of $\phi$, the initial conditions for the solutions to (2.2) and (2.3) are (see also (3.3) and the subsequent discussion):

$$\phi(t = 0) = \phi_0, \quad a(t = 0) = \frac{1}{\sqrt{\lambda\phi_0^2}} = a_0, \quad (2.8)$$

$$\dot{\phi}(t = 0) = 0, \quad \dot{a}(t = 0) = 0. \quad (2.9)$$

Moreover, putting $[8] \alpha = \log(a)$, $p_\alpha^2 = -\frac{1}{a} \frac{\partial a}{\partial \alpha} \frac{\partial}{\partial a}$, the Wheeler-De Witt equation for our model becomes

$$\frac{1}{2}Ne^{3\alpha} \left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} + (\lambda\phi^n e^{6\alpha} - e^{4\alpha}) \right] \psi(\alpha, \phi) = 0. \quad (2.10)$$

Now, the action $I$ of a compact solution of the field equations vanishes in the limit of small 3-geometries and one often argues that the prefactor $A$ of the semiclassical approximation of the wavefunction, $\psi \sim A \exp(-I)$ (this is the formula for the solution of the Wheeler-DeWitt equation to leading order), may be taken to be a constant, so as to normalize $\psi$ to 1 in the limit of small 3-geometries [8, 16]. This would be a boundary condition for (2.10) implied by the Hartle-Hawking proposal, and in numerical calculations it has been used so as to derive the oscillatory or exponential regions of the wavefunction [16]. Thus, from the knowledge of $\psi$ at small 3-geometries, one derives its behaviour at larger 3-geometries [16].

Strictly speaking, this is not completely true, because the prefactor of the semiclassical approximation is given by the path integral of $\exp(-I_2)$, where $I_2$ is the part of the action quadratic in the fluctuations about classical solutions. This calculation is not trivial and one can show, for example, that in the case of pure gravity the prefactor diverges in the limit of small 3-geometries in the Euclidean regime [17]. Thus, in general, the normalization of $\psi$ to 1 may not be sufficiently accurate. However, in our opinion one can still take, to a first approximation, the point of view of Hawking and Wu [16]. They point out that
a suitable choice (also made in equation (2.10) in this paper) for the factor ordering in the Wheeler-DeWitt equation is consistent with a constant behaviour of the prefactor and of the wavefunction in the limit of small \( a \), as can also be verified in our case. In fact
\[
\left( \frac{\partial^2 \psi}{\partial \alpha^2} - \frac{\partial^2 \psi}{\partial \phi^2} \right) \text{ vanishes if } \psi \text{ is a constant and the potential } V(\alpha, \phi) = e^{6\alpha} \lambda \phi^n - e^{4\alpha} = a^6 \lambda \phi^n - a^4, \text{ for a given value of } \phi, \text{ vanishes when } a \to 0. \]

The numerical integration of the Wheeler-DeWitt equation with this condition shows that in the massive model \( \psi \) starts to oscillate when \(| \phi |\) is greater than one and \( V(\alpha, \phi) \) is positive. In our case, it is clear that, even if one puts \( \lambda \frac{\dot{\phi}}{\dot{\phi}} = \tilde{m} \), \( \dot{\phi}^2 = \Phi \), one cannot cast (2.10) in a form which is formally identical to the massive case, because \( \frac{\partial^2 \psi}{\partial \phi^2} \) is very different from \( \frac{\partial^2 \psi}{\partial \Phi^2} \). However, in the case of a \( \lambda \phi^n \) theory, for a given value of \( \phi \) one has an even larger range of values of \( a \) for which the potential \( V(\alpha, \phi) \) in the Wheeler-DeWitt equation is positive, if \( \lambda \) is not too small. The sign of this potential plays a crucial role in determining the oscillatory behaviour of \( \psi \) and this is a quite general result not depending on the particular theory for the scalar field \[18\]. It is particularly useful to discuss this matter in detail. To begin with, we point out that the Wheeler-DeWitt equation (2.10) is a hyperbolic equation in the minisuperspace with metric \( ds^2 = e^{3\alpha}(-d\alpha^2 + d\phi^2) \), and we consider the regions in which the surfaces of constant \( V = e^{6\alpha} \lambda \phi^n - e^{4\alpha} \) are spacelike with respect to this metric. The idea is that in a local region we can perform a Lorentz transformation \[18\] to new coordinates \((\tilde{\alpha}, \tilde{\phi})\) so that the constant-\( V \) surfaces are parallel to the \( \tilde{\phi} \) axis. This implies that (2.10) takes the form
\[
\left[ \frac{\partial^2 \psi_1}{\partial \tilde{\alpha}^2} - \frac{\partial^2 \psi_1}{\partial \tilde{\phi}^2} + V(\tilde{\alpha}) \right] \psi_1(\tilde{\alpha}) = 0
\]
and can be solved by separation of variables, putting \( \psi = \psi_1(\tilde{\alpha}) \psi_2(\tilde{\phi}) \). One thus finds the differential equation for \( \psi_1(\tilde{\alpha}) \):
\[
\left[ \frac{d^2 \psi_1}{d\alpha^2} + \Lambda + V(\tilde{\alpha}) \right] \psi_1(\tilde{\alpha}) = 0,
\]
whose solution has oscillatory behaviour if \( \Lambda + V(\tilde{\alpha}) > 0 \), where \( \Lambda \) is a separation constant. It is also possible that the surfaces of constant \( V \) are timelike. In such a case they can be locally rotated so as to lie parallel to the \( \tilde{\alpha} \) axis, which in turn implies that \( V = V(\tilde{\phi}) \). In such a case one finds the following differential equation for \( \psi_2(\tilde{\phi}) \):
\[
\left[ \frac{d^2 \psi_2}{d\phi^2} + \Lambda' - V(\tilde{\phi}) \right] \psi_2(\tilde{\phi}) = 0,
\]
whose solution has oscillatory behaviour if \( \Lambda' - V(\tilde{\phi}) > 0 \), where \( \Lambda' \) is another separation constant. Thus \( V \) is expected to be negative in this case, if the separation constant \( \Lambda' \) does not assume too large a positive value.
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To sum up, if the $V = \text{constant}$ surfaces are spacelike in the metric of the minisuperspace, $\psi$ is expected to oscillate if $V > 0$ and to behave exponentially if $V < 0$, whereas the converse should hold true if the $V = \text{constant}$ surfaces are timelike. In our model, the nature of the surfaces of constant potential is determined by the quantity $\Sigma = g^{ab} V_a V_b$, where $g^{ab} = e^{3\alpha} \text{diag}(-1, 1)$. The sign of this quantity shows the timelike or spacelike nature of the vector normal to these surfaces. Computing $\Sigma$ when $V = k > 0$, one finds

$$\Sigma = e^{3\alpha} \left[ -(V,\alpha)^2 + (V,\phi)^2 \right]_{V=k>0}$$

$$= -e^{3\alpha} \left[ \left( 1 - \frac{n^2}{\phi^2} \right) (k + e^{4\alpha})^2 + 2(k + e^{4\alpha})(5k + e^{4\alpha}) + (5k + e^{4\alpha})^2 \right] < 0$$

if $\phi$ or $e^{\alpha}$ are sufficiently large. This implies that the surfaces of constant positive potential are spacelike for such values of $\phi$, which in turn implies that the wavefunction has to oscillate. Honestly speaking, in the massive scalar field model there are some regions in which this method disagrees with numerical results for the wavefunction derived under the assumption that $\psi = 1$ when $a \to 0$ [19]. In view of the fact that both these methods have flaws, the question seems to us to be an open one. We can thus show that the wavefunction of our model oscillates when $V(\alpha, \phi)$ is positive in one of the following two ways:

(a) by comparison with the results derived in the massive model, pointing out that also in our case one can take $\psi \to 1$ as $a \to 0$, and that in our case $V(\alpha, \phi)$ has an even stronger positivity property if the self-coupling parameter $\lambda$ is not too small;

(b) by studying the nature of the surfaces $V(\alpha, \phi) = \text{constant}$.

In the case of a $\lambda \phi^4$ theory, we are in agreement with the result derived in a paper by Zhuk [20]. However in this paper the normalizability of $\psi$ to 1 is not a boundary condition assumed from the beginning, but a consequence of the mathematical derivation, in which one requires that the solution of the Wheeler-De Witt equation has to vanish when the cosmic scale factor $\to \infty$.

So far we have emphasized that, at almost constant values of $\phi$, our model can be obtained from the massive one replacing the positive quantity $m^2 \phi^2$ by means of $\lambda \phi^n$. This is possible only when $\phi$ is positive, or when $\phi$ is negative but $n$ is even. In view of this result we shall not take into account the case when $\phi$ is negative and $n$ is odd. Of
course, the initial value of $|\phi|$ is not completely arbitrary, because the form of $S$ involving $\phi$ must be a solution of the Hamilton-Jacobi equation in the oscillatory region for the wavefunction. In our case $S$ satisfies the equation

$$
\left( \frac{\partial S}{\partial \phi} \right)^2 - \left( \frac{\partial S}{\partial \alpha} \right)^2 = e^{4\alpha} - \lambda \phi^n e^{6\alpha}
$$

where the right-hand side of (2.11) is negative. A careful examination of the quantity $\Sigma$ shows that this is possible only if $|\phi|$ is initially at least greater than a number of order one. We have not solved numerically the Wheeler-De Witt equation for our model, and this is an interesting problem to be studied for further research. However, it is to be remarked that the numerical techniques used so far have some limits. For example, the leapfrog algorithm is a valid approximation only when the potential $V$ is everywhere much smaller than the inverse of the square of the grid step size [19].

3. Lorentzian Hartle-Hawking trajectories and field equations

In this section we shall perform at first our calculations for positive values of $\phi$, and later on we shall also discuss briefly the case when $n$ is even and $\phi$ is negative.

The Euclidean action for compact metrics and regular matter fields is given in our model by $I_E = -\frac{1}{3\lambda\phi^n} \left[ 1 - (1 - e^{2\alpha} \lambda \phi^n)^{\frac{3}{2}} \right]$. The geometrical interpretation of $I_E$ is that it is the action of the smaller part of a 4-sphere of radius $\frac{1}{\sqrt{\lambda \phi^n}}$, bounded by a 3-sphere of radius $a = e^\alpha$; in so doing, we generalize what has been done in the case of the massive model [1, 11], as already emphasized in section 2. The solution of (2.11) is, in the Hartle-Hawking approach, the analytical continuation of $I_E$. At large values of $\phi$ it is given by

$$
S \simeq -\frac{1}{3\lambda\phi^n} \left( e^{2\alpha} \lambda \phi^n - 1 \right)^{\frac{3}{2}}.
$$

(3.1)
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In fact the insertion of (3.1) into the left-hand side of (2.11) yields

\[(\frac{\partial S}{\partial \phi})^2 - (\frac{\partial S}{\partial \alpha})^2 = \frac{n^2 e^{4\alpha}}{4\phi^2}(e^{2\alpha \lambda \phi^n} - 1) + \frac{n^2}{9\lambda^2 \phi^{2n+2}}(e^{2\alpha \lambda \phi^n} - 1)^3\]

\[-\frac{n^2 e^{2\alpha}}{3\lambda \phi^{n+2}}(e^{2\alpha \lambda \phi^n} - 1)^2 + e^{4\alpha} - \lambda \phi^n e^{6\alpha}\]

and we can easily recognize that the right-hand side of (3.2) reduces to \(e^{4\alpha} - \lambda \phi^n e^{6\alpha}\) at large initial \(\phi\) and small \(t\). The relation (3.1) shows again that \(\phi\) cannot be negative when \(n\) is odd, because in such a case \(S\) becomes complex, whereas by its very definition \(S\) is real. In addition we know that the function \(S\) in (2.11) and (3.1) defines a first integral of the system:

\[p_a = \frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}}\]

\[p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}\]

where \(L\) is the Lagrangian defined in (2.1). This system therefore becomes, putting \(N = 1\)

\[\dot{a} = (a^2 \lambda \phi^n - 1)^{\frac{1}{2}}\]  

(3.3)

\[\dot{\phi} = -\frac{n \dot{a}}{2a\phi} + \frac{n}{3\lambda \phi^{n+1}}\left(\frac{\dot{a}}{a}\right)^3.\]  

(3.4)

It is to be remarked that this system is in full agreement with the initial conditions (2.8) and (2.9) in the Lorentzian regime derived from the Hartle-Hawking proposal. In fact, for example, \(\dot{a}(t = 0) = 0\), when substituted into (3.3), yields \(a_0 = \frac{1}{\sqrt{\lambda \phi_0^2}}\). As the time passes, the dominant contribution to \(a(t)\) is given by the increasing exponential, so that \(a(t) \approx \frac{\exp(H_1 t)}{2H}\). Under such assumptions, (3.1) takes the approximate form:

\[S \approx -\frac{e^{3\alpha} \sqrt{\lambda \phi_0^2}}{3}.\]  

(3.5)
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The relation (3.1) enables us to also compute the first correction to (3.5) involving negative powers of \( \phi \). In fact, from (3.1) one has that

\[
S \approx -\frac{1}{3\lambda \phi^n} \left[ \sqrt{e^{2\alpha \lambda \phi^n}} \left( 1 - \frac{1}{\lambda e^{2\alpha \phi^n}} \right) \right]^3 \\
\approx -\frac{1}{3\lambda \phi^n} \left( e^{\alpha \sqrt{\lambda \phi^n}} \right)^3 \left( 1 - \frac{1}{2\lambda e^{2\alpha \phi^n}} \right)^3 \\
\approx -\frac{\sqrt{\lambda}}{3} e^{3\alpha \phi^n} \left( 1 - \frac{3}{2\lambda e^{2\alpha \phi^n}} \right). 
\]

(3.6)

In the derivation of (3.6), we have used the following approximate formulae which are valid when \( 0 < |x| \ll 1 \):

\[
\sqrt{1+x} \approx 1 + \frac{x}{2}; \quad (1+x)^n \approx 1 + nx 
\]

(3.7)

where in our case \( x = \frac{1}{\lambda \exp(2\alpha \phi^n)} \). The derivation of (3.6) shows that (3.5) holds true when \( \phi \) is much greater than one or when \( e^{\alpha} \) is very large.

Let \([0, t_a]\) be the time interval during which the Lorentzian Hartle-Hawking trajectories (hereafter referred to as LHH, see [13]) evolve according to (2.6), (2.7), (3.3) and (3.4). We are now interested in deriving solutions of the system

\[
p_a = \frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}} \\
p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}
\]

in the interval \([t_a, t_b]\) during which \( S \) can be approximated as in (3.5). Inserting (3.5) into this system, and putting \( N = 1 \), one finds that

\[
\dot{\phi} = -\frac{n\sqrt{\lambda}}{6} \phi^{n-1} 
\]

(3.8)

\[
\dot{\alpha} = \sqrt{\lambda} \phi^{\frac{n}{2}}. 
\]

(3.9)
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By integrating (3.8) and putting $\phi_a = \phi(t_a)$, where $\phi_a = \phi_0$, we get

$$\phi(t) = \left[ \phi_a^{(4-n) \over 2} - \left( 2 - \frac{n}{2} \right) \frac{n\sqrt{\lambda}}{6} (t - t_a) \right]^{(4-n) \over 2} \tag{3.10}$$

for every $n \neq 4$. This implies

$$a(t) = a_I \exp \left\{ \sqrt{\lambda} \int_{t_a}^{t} \left[ \phi_a^{(4-n) \over 2} - \left( 2 - \frac{n}{2} \right) \frac{n\sqrt{\lambda}}{6} (t' - t_a) \right]^{n \over (4-n)} \right\} \tag{3.11}$$

for every $n \neq 4$, where $a_I = \frac{1}{H_0} \cosh(H t_a) \approx \frac{1}{2H} \exp(H t_a)$. In order to study the case $n > 4$, we define $k = \phi_a^{(4-n) \over 2}$, $b = (n^2 - 2) n \sqrt{\lambda} \over 6$, $z = n \over n-4$. Therefore (3.11) becomes

$$a(t) = a_I \exp \left\{ \frac{\sqrt{\lambda}}{b(-z+1)} \left[ (k + b(t - t_a))^{(-z+1)} - k^{(-z+1)} \right] \right\}. \tag{3.12}$$

The second term in the curly brackets of (3.12) is

$$\frac{\sqrt{\lambda}}{b(z-1)} k^{(-z+1)} = \frac{3\phi_a^2}{n}. \tag{3.13}$$

Moreover, at the end of the era during which (3.12) holds, and putting $\phi_b = \phi(t_b)$, one has from (3.10) that

$$t_b - t_a = \frac{12}{n(n-4)\sqrt{\lambda}} \phi_a^{(4-n) \over 2} \left[ \left( \frac{\phi_b}{\phi_a} \right)^{4-n \over 2} - 1 \right] \approx \frac{12}{n(n-4)\sqrt{\lambda}} \phi_b^{(4-n) \over 2} \tag{3.14}$$

if $\phi_b \ll \phi_a$. Thus the first term in the curly brackets of (3.12) becomes

$$\frac{\sqrt{\lambda}}{b(-z+1)} (k + b(t_b - t_a))^{(-z+1)} \approx -\frac{3\phi_b^2}{n}. \tag{3.15}$$

From (3.13) and (3.15) we get

$$a(t_b) = a_I \exp \left[ \frac{3\phi_a^2}{n} \left( 1 - \left( \frac{\phi_b}{\phi_a} \right)^2 \right) \right] \approx a_I \exp \left( \frac{3\phi_a^2}{n} \right). \tag{3.16}$$
If \( n = 4 \), (3.10) and (3.11) are no longer valid; in such a case, the integration of (3.8) yields

\[
\phi = \phi_a \exp \left[ \frac{-2\sqrt{\lambda}}{3}(t - t_a) \right]
\]

(3.17)

so that (3.9) becomes \( \dot{\alpha} = \sqrt{\lambda} \phi_a^2 \exp(-\frac{4\sqrt{\lambda}}{3}(t - t_a)) \), which implies

\[
a(t) = a_I \exp \left\{ \frac{3\phi_a^2}{4} \left[ 1 - \exp \left( -\frac{4\sqrt{\lambda}}{3}(t - t_a) \right) \right] \right\}.
\]

(3.18)

Therefore the cosmic scale factor \( a(t_b) \) at the end of this era is

\[
a(t_b) = a_I \exp \left[ \frac{3\phi_a^2}{4} \left( 1 - \left( \frac{\phi_b}{\phi_a} \right)^2 \right) \right] \approx a_I \exp \left( \frac{3\phi_a^2}{4} \right)
\]

(3.19)

if \( \phi_b \ll \phi_a \), where

\[
t_b - t_a = \frac{3}{2\sqrt{\lambda}} \log \left( \frac{\phi_a}{\phi_b} \right).
\]

(3.20)

Furthermore, if \( n = 3 \) one finds that

\[
a(t) = a_I \exp \left[ \phi_a^2 - \left( \sqrt{\phi_a} - \sqrt{\lambda} \frac{t - t_a}{4} \right) \right].
\]

(3.21)

At the end of the era during which (3.21) holds one has

\[
t_b - t_a = 4\sqrt{\frac{\phi_a}{\lambda}} \left[ 1 - \sqrt{\frac{\phi_b}{\phi_a}} \right].
\]

(3.22)

Therefore \( a(t_b) \approx a_I \exp(\phi_a^2) \) if \( \phi_b \ll \phi_a \). When \( t_a < t \leq t_b \), and \( t - t_a \) is very small, one has for any \( n \) that

\[
a(t) \approx a_I \exp(\sqrt{\lambda} \phi_a^2 (t - t_a)) \approx \frac{a_0}{2} \exp(\sqrt{\lambda} \phi_a^2 t)
\]
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which is the correct inflationary formula one would have expected to hold true. It is to be emphasized that it would be wrong to put \( t_a = 0 \) for simplicity in formulae (3.10)-(3.22). In fact, \( t = 0 \) is the time at which \( \dot{\phi} \) and \( \dot{a} \) vanish according to (2.9), whereas \( t = t_a \) is the time at which \( S \) can be approximated by (3.5) up to \( t = t_b \). Therefore during the inflationary era \( \dot{\phi} \) and \( \dot{a} \) do not vanish, but can be computed by means of (3.10), (3.11), (3.17) and (3.18). This is what also happens in the massive scalar field model [8], in which the solutions corresponding to the oscillatory part of the wavefunction start out with \( \dot{\phi}(0) = \dot{a}(0) = 0 \), and expand exponentially with \( \dot{\phi} = -\frac{m}{3} \), \( \dot{a} = ma | \phi | \).

Finally, when \( \phi \) is negative and \( n \) is even, (3.5) becomes \( S \approx -\frac{e^{3\alpha \sqrt{2}\sqrt{|\phi|^n}}}{3} \). In fact, (3.5) is the approximate form of (3.1), which can also be written in the following way:

\[
S = -\frac{1}{3\lambda \phi^n} \left[ e^{2\alpha \lambda \phi^n} - 1 \right] \sqrt{e^{2\alpha \lambda \phi^n} - 1}.
\]

If we choose the square root on the right-hand side to be positive, as we already did in the case of positive values of \( \phi \), this can only be obtained introducing \( | \phi | \), because otherwise \( \phi^n \) could be negative when \( n \) is even but \( \frac{n}{2} \) is odd. Thus in equation (3.8) \( \dot{\phi} \) is to be replaced by the time derivative of \( | \phi | \) and in (3.9) \( \phi^{n/2} \) is to be replaced by \( | \phi |^{n/2} \). This implies that, for any even \( n \), when \( \phi \) is negative, \( | \phi | \) is a decreasing function of the time during the inflationary era. Such a behaviour is typical of a field \( \phi \) which starts from a large negative value and finally can reach zero before starting to oscillate. Thus the relations (3.10)-(3.22) hold true provided that one replaces \( \phi \) by \( | \phi | \).

So far we have derived solutions to the first-order system

\[
\frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}},
\]

\[
\frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}
\]

where \( S \) is obtained by analytical continuation of the Euclidean action for compact 4-metrics and regular matter fields, and \( L \) is the Lagrangian defined in (2.1). In particular, we have focused our attention on the case when \( S \) takes the approximate form (3.5).
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Some important comments are now in order. In our model the full field equations are the second-order differential equations (2.2) and (2.3) plus the Hamiltonian constraint:

$$a^2 \dot{\phi}^2 - \ddot{a}^2 + \lambda a^2 \phi^n - 1 = 0.$$  \hspace{1cm} (3.23)

Thus the solution to the full field equations involves three arbitrary constants. On the other hand, a wave function of the form $C e^{iS}$ is peaked about the first integral: $p_a = \frac{\partial S}{\partial a}$, $p_\phi = \frac{\partial S}{\partial \phi}$. This first integral consists of two first-order ordinary differential equations (see (3.3) and (3.4)) and so the solution involves just two arbitrary constants. The wave function is therefore peaked about a set of solutions which are a subset of the general solution.

Using the Hamilton-Jacobi equation for $S$, we shall show at first that the solutions of the above-mentioned first integral satisfy the full field equations exactly. Finally, we will briefly compare the set of solutions picked out by the Hartle-Hawking wave function with the general solution.

The Hamilton-Jacobi equation (2.11) satisfied by $S$ can be cast in the form:

$$g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial S}{\partial q^j} + V = 0.$$  \hspace{1cm} (3.24)

where $g^{ij} = \text{diag}(-1,1)$, $q^1 = \alpha$, $q^2 = \phi$, and $V(\alpha, \phi) = \lambda \phi^n e^{6\alpha} - e^{4\alpha}$ is the potential already defined in section 2. Following Halliwell [21], let us now differentiate (3.24) with respect to $q^k$. In so doing we get

$$2g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial^2 S}{\partial q^j \partial q^k} + \frac{\partial V}{\partial q^k} = 0.$$  \hspace{1cm} (3.25)

Let us now introduce, again according to [21], the vector

$$\frac{d}{d\tau} = g^{ij} \frac{\partial S}{\partial q^i} \frac{\partial}{\partial q^j}.$$  \hspace{1cm} (3.26)

The right choice for the parameter $\tau$ is its identification with proper time: $\tau = \int N dt$. In fact one can easily check that in so doing one obtains the well known relation between
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\(\dot{\alpha}, \dot{\phi}\), and the momenta \(p_\alpha\) and \(p_\phi\). Therefore \(\frac{d}{d\tau} = \frac{d}{dt}\) if \(N = 1\). The insertion of (3.26) into (3.25) yields finally

\[
\frac{d}{d\tau} p_k + \frac{1}{2} \frac{\partial V}{\partial q^k} = 0 \tag{3.27}
\]

and (3.27) is just the form assumed by the field equations. In fact, (2.2) and (2.3) may also be cast in the first-order form:

\[
\dot{p}_\phi = -\frac{n\lambda}{2} \phi^{n-1} e^{6\alpha} \tag{3.28}
\]

\[
\dot{p}_\alpha = 2e^{4\alpha} - 3\lambda \phi^n e^{6\alpha} \tag{3.29}
\]

where we have used the following formula for the Hamiltonian:

\[
H = \frac{1}{2} (p_\phi^2 - p_\alpha^2) - \frac{1}{2} e^{4\alpha} + \frac{\lambda}{2} \phi^n e^{6\alpha} \tag{3.30}
\]

If we now put \(q^k = \phi\) in (3.27) we obtain (3.28), and if we put \(q^k = \alpha\) we obtain (3.29).

The task remains now to examine the following point: what is so special about the solutions picked out by the Hartle-Hawking proposal in comparison to the general solution? In fact, if this question is not addressed, it is difficult to see what we have gained by computing the wavefunction for our model.

Indeed, the calculations and the arguments developed so far have shown that the solutions about which the Hartle-Hawking wavefunction is peaked are distinguished in that they are inflationary. They have this property in view of the fact that the scalar field is initially very large so that \(\lambda \phi^n\) acts like an effective cosmological constant in the early universe, whereas the initial value of \(\dot{\phi}\) is very small (see (2.9)). However, a member of the set of general solutions will not always be inflationary. This is easily understood looking more carefully at (2.2) and (3.23). Putting as usual \(\alpha = \ln(a)\), these relations imply that

\[
\ddot{\alpha} = -\dot{\alpha}^2 - 2\dot{\phi}^2 + \lambda \phi^n. \tag{3.31}
\]

Thus an inflationary solution must be such that \(\dot{\alpha} = \text{constant} = H > 0\), \(\ddot{\alpha} = 0 = -H^2 - 2\dot{\phi}^2 + \lambda \phi^n\). But a member of the set of general solutions might well have an initial
value of $\dot{\phi}$ which is very large. Under such a condition, the right-hand side of (3.31) does not vanish if $(\lambda \dot{\phi}^n - \dot{\alpha}^2) \ll 2\dot{\phi}^2$. This implies in turn that no inflation is possible if $\dot{\phi}$ is initially so large.

Therefore the Lorentzian Hartle-Hawking trajectories are very peculiar in that they are singularity free in the past (see (2.8)) and inflationary. The full comparison of these trajectories with the general solution may be done using the phase-plane method, generalizing the work done by Belinski et al [22]. This analysis may be an interesting problem for further research, but our analysis already shows the main difference between the two sets of solutions.

4. Minimal conditions for a sufficient inflation

During the inflationary era of the universe, (3.18) becomes

$$a(t) \cong a_I \exp[\sqrt{\lambda} \phi_a^2 (t - t_a)] \cong \frac{a_0}{2} \exp(\sqrt{\lambda} \phi_a^2 t).$$  \hspace{1cm} (4.1)

If we require that the inflationary formula (4.1) satisfies the condition $a(t) \geq a_0 10^{28} \cong a_0 \exp (65)$ in order to solve the horizon and flatness problems [23-25], and if we put $t = \beta t_b$, where $\beta \in [0, 1]$, we find the condition (see (3.20))

$$\sqrt{\lambda} \phi_a^2 t_a + \frac{3}{2} \phi_a^2 \log \left(\frac{\phi_a}{\phi_b}\right) \geq \frac{(65 + \log(2)) \beta}{\beta}$$ \hspace{1cm} (4.2)

in the case of a $\lambda \phi^4$ theory. In the same way, we find that the conditions to be satisfied in the cases $n = 3, n > 4$ are, respectively,

$$\sqrt{\lambda} \phi_a^2 t_a + 4 \phi_a^2 \left(1 - \sqrt{\frac{\phi_b}{\phi_a}}\right) \geq \frac{(65 + \log(2)) \beta}{\beta}$$ \hspace{1cm} (4.3)

$$\sqrt{\lambda} \phi_a^2 t_a + \frac{12}{n(n-4)} \left(\frac{\phi_a}{\phi_b}\right)^\frac{4}{n} \phi_b^2 \geq \frac{(65 + \log(2)) \beta}{\beta}.$$ \hspace{1cm} (4.4)
However, a thorough inflationary model also has to solve other problems such as, for example, the origin of the energy density fluctuations. Therefore the relations (4.2)-(4.4) are just a part of the minimal requirements to be satisfied by our model.

5. Conclusions

Many authors [20, 22, 24-28], by using a wide range of techniques, had already considered the effect produced in the early universe by a term in the scalar potential of the type $\lambda \phi^4$. In this paper we have studied the general case of a $\lambda \phi^n$ theory in a closed FRW minisuperspace model. We have mainly studied the case when $\phi$ is positive and we have shown that the case when $\phi$ is negative and $n$ is even can also be taken into account, provided that one replaces $\phi$ by $|\phi|$ in all the equations of the theory. But if $\phi$ is negative and $n$ is odd, this does not give rise to a positive effective cosmological constant in the early universe.

By applying the Hartle-Hawking proposal we have been able to derive the initial conditions for the solutions to the Lorentzian field equations (see (2.8) and (2.9)). We have also shown that a suitable choice of factor ordering in the Wheeler-DeWitt equation (see (2.10)) enables one to extend to our model a result that is already known in the case of massive scalar fields: namely, having taken the Euclidean functional integral over compact 4-metrics and regular matter fields, the wavefunction is normalizable to one in the limit of small 3-geometry and it starts to oscillate when the potential in the Wheeler-DeWitt equation is positive and $\phi$ is greater than a number of order one. However, we also emphasized that a more careful calculation and a deeper understanding of the semiclassical approximation in quantum cosmology are perhaps needed before taking for granted that, in general, the wavefunction is a constant on the boundary of minisuperspace.

Another interesting result is that, with respect to the massive model and for a given value of $\phi$, there may be an even larger range of values (i.e. also including smaller values) of the cosmic scale factor $a(t)$ for which the potential in the Wheeler-DeWitt equation is positive. The initial conditions (2.8) are indeed in agreement with this fact: the universe
starts off in a non-singular state and the initial value of $a(t)$ is smaller than the one for the massive scalar field model, if $\lambda$ is not too small. In the oscillatory region for the wavefunction, we have taken the phase $S$ of the WKB approximation to be the analytical continuation of the Euclidean action for compact 4-metrics and regular matter fields. The LHH trajectories are thus derived by solving the first-order system

$$p_a = \frac{\partial S}{\partial a} = \frac{\partial L}{\partial \dot{a}} \quad p_\phi = \frac{\partial S}{\partial \phi} = \frac{\partial L}{\partial \dot{\phi}}$$

where $L$ is the Lagrangian of the theory (see (2.1)). The solutions of this system are exact solutions of the field equations; they have vanishing time derivative in $t = 0$ and describe an exponentially expanding universe at small values of $(t - t_a)$, where $t > t_a$ and $[0, t_a]$ is the time interval during which the field $\phi$ has a constant large value. The Hartle-Hawking wavefunction is thus peaked about those classical solutions which are inflationary, whereas a member of the general set of solutions of the full field equations will not always be inflationary.

The duration of the inflationary era may still be of the order of $10^{-35}$ sec or $10^{-33}$ sec (which is very long if compared to the Planck time) as in the case of the massive scalar field model, provided that the parameter $\lambda$ is suitably chosen for any value of $n$. Other important conditions to be satisfied are the ones expressed in (4.2)-(4.4) in order to solve the horizon and flatness problems. Before going over the matter- or radiation-dominated phase, the universe may expand up to a maximum value of the order of $\exp\left(\frac{3\phi_0^2}{n}\right)$ for any $n$ with respect to the value of $a(t)$ at the beginning of the era when $\exp(-Ht)$ can be neglected in (2.6). As the time passes, $\phi$ decreases, but the approximate formula (3.5) from which we have derived most results in section 3 is still approximately valid as far as $e^\alpha$ is very large.

In our opinion it would be particularly interesting to try to solve numerically the field equations using the initial conditions derived from the Hartle-Hawking proposal for our model, along the lines of the work done by Laflamme and Shellard [12] for the massive model. We think that a thorough examination of this problem, both in the isotropic and the anisotropic case, could be of considerable importance in understanding whether or
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not the universe described by quantum cosmology will recollapse to a singularity after a maximum expansion.

Otherwise stated, it may seem quite reasonable that most (or all) minisuperspace toy models which are able to drive an inflationary era of the early universe lead to a singularity in the future apart from bouncing solutions, but so far this is an unproved conjecture. The proof of this result for $\lambda \phi^n$ theories and other models might show an intriguing link between inflationary models and the problem of the singularity in the future. In addition, it could be even more interesting to show that there are some physically meaningful minisuperspace models for which this result does not hold true.

Finally, we would like to address the attention of the reader to the following point. If we multiply (2.1) by $-i$ and if we put $N \to -iN$, we obtain a Euclidean action $I_E$ which, after the substitutions $N \to i$, $a \to ia$, $\phi \to i\phi$, becomes

$$I_E = \int \left[ \frac{a\dot{a}^2}{2} + \frac{a^3 \dot{\phi}^2}{2} + \frac{\lambda a^3 i^n \phi^n}{2} \right] d\tau$$

which is positive definite for all values of $n$ of the form $n = 4m$, where $m = 1, 2, 3, \ldots$. This seems to suggest that the most interesting $\lambda \phi^n$ models are just these ones, generalizing what was already known in the case of a $\lambda \phi^4$ theory [11, 20].

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