Reliability Estimation of a Component exposed to k Stresses for Gompertz-Fréchet distribution

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Abstract. In this paper, the reliability of the stress-strength model is derived for probability p (max(Y_1,Y_2,...,Y_k) < X) of a component having its strength X exposed to k independent stresses (Y_1,Y_2,...,Y_k) and also as a special case k=2, when X and Y_1,Y_2,...,Y_k flowing Gompertz-Fréchet distribution with unknown shape parameters \( \theta, \beta, \gamma \) and known parameters \( \alpha, \beta, \gamma \). Different methods used to estimate reliability R and Gompertz-Fréchet distribution parameters which are Maximum Likelihood, Least square, Weighted Least square, Regression and Ranked set sampling methods, and the comparison between these estimators by simulation study based on mean square error criteria (MSE). The comparison confirms that the performance of the maximum likelihood estimator works better than the other estimators.

1. Introduction

Based on the reliability formula \( R = p(\max(Y_1, Y_2) < X) \) in the case a component exposed to two independent stresses which are found by Karadyai, Saracoğlu and Pekgor [1], and \( R = p(\max(Y_1, Y_2, Y_3) < x) \) in the case a component exposed to three independent stresses which are found by Karam and Ranks [2], proposed a formula to find reliability \( R = p(\max(Y_1, Y_2, ..., Y_k) < X) \) in the case a component exposed to k from independent stresses \( Y_1, Y_2, ..., Y_k \).

Some attempts have been done to defined modern types of distributions to extend renewed families and at the same time provide big resilience in forming data in practice. Alizadeh et al in 2016 presented Gompertz-Fréchet, and some special models of the Go-G family Gompertz-Weibul, Gompertz-Gamma, Gompertz-beta, Gompertz-log logistic. Oguntunde et al was developed a new continuous distribution named the Gompertz-Fréchet distribution which extends the Fréchet distribution. Jabr and Karam in the year 2021 presented an estimate of reliability \( R = P(Y_1 < X < Y_2) \) using the Gompertz-Fréchet distribution [3].

Define the CDF of the Gompertz-G family by [4]:

\[
F(x) = \int_0^{-\log(1-G(x;\theta)}) \theta e^{\gamma t} e^{-\theta e^{\gamma t-1}} dt = 1 - e^{\frac{\theta}{\gamma}[1-1-G(x;\theta)]^{-\gamma}}
\]  

(1)

Where \( G(x;\theta) \) is the baseline CDF depending on a parameter vector \( \epsilon \) and \( \gamma > 0 \) and \( \theta > 0 \) are two shape parameter. and define the pdf of the Gompertz-G family by:

\[
f(x; \theta, \gamma, \epsilon) = \theta g(x; \epsilon) [1 - G(x; \epsilon)]^{\gamma-1} e^{-\theta [1-1-G(x;\epsilon)]^{-\gamma}}
\]

(2)

The CDF and pdf of Fréchet distribution are [5]:

\[
G(x; \alpha, \beta) = e^{-\left(\frac{x}{\beta}\right)^{\alpha}}
\]

(3)

and \( g(x, \alpha, \beta) = \beta \alpha^\beta \gamma x^{-\beta-1} e^{-\left(\frac{x}{\beta}\right)^{\alpha}} \)

(4)

respectively, where \( \alpha > 0 \) scale parameter and \( \beta > 0 \) shape parameter.
The CDF of the Gompertz-Frêchet (GF) distribution is obtained by substituting equation (3) in equation (1) given by [5]:

\[ F(x, \theta, \alpha, \beta, \gamma) = 1 - \exp \left[ \frac{\alpha}{\gamma} \left( 1 - e^{-\left( \frac{\gamma}{\alpha} \right)^\beta} \right)^\gamma \right] \]  

(5)

And pdf can be derived from equation (5) as:

\[ f(x, \theta, \alpha, \beta, \gamma) = \theta \alpha \beta x^{\beta - 1} \exp \left[ -\left( \frac{\gamma}{\alpha} \right)^\beta \right] \left( 1 - e^{-\left( \frac{\gamma}{\alpha} \right)^\beta} \right)^{\gamma - 1} \exp \left[ \frac{\alpha}{\gamma} \left( 1 - e^{-\left( \frac{\gamma}{\alpha} \right)^\beta} \right)^\gamma \right] \]  

(6)

where \( \theta > 0, \beta > 0 \) and \( \gamma > 0 \) are shape parameters and \( \alpha > 0 \) is the scale parameter.

The main aim of this paper is to obtain a mathematical formula of reliability \( R_1 \) and \( R_2 \) of probability \( p(\max(Y_1, Y_2, ..., Y_k) < X) \) and as a special case \( k=2 \), based on Gompertz-Frêchet distribution in section 2. In order to find the estimators of the shape parameters \( (\theta, \lambda) \) for the random variables, five different estimation methods (Maximum Likelihood, Least square Method, Weighted Least square Method, Regression Method and Ranked set sampling Method) are used and then the reliability parameters is estimated in section 3. A simulation study was conducted to compare the performance of the five different estimators of the reliability in section 4, based on six experiments for reliability \( R_1 \) and \( R_2 \) of shape parameter values and at different sample sizes of (15) for small, (30) for medium and (90) for large sample sizes. The comparison is made by the Mean Square Error (MSE), and the conclusions are discussed in section 5.

2. The Reliability expression

Let \( X \sim GF(\theta, \alpha, \beta, \gamma) \) be strength random variable and \( Y_1, Y_2, ..., Y_k \sim GF(\lambda, \alpha, \beta, \gamma) \) be stresses random variable.

Let \( u_x = 1 - e^{-\left( \frac{\gamma}{\alpha} \right)^\beta} \)

and we can write \( e^{-\left( \frac{\gamma}{\alpha} \right)^\beta} = 1 - u_x \)

Then \( F(x) \) and \( f(x) \) can be written as:

\[ F(x, \theta, \alpha, \beta, \gamma) = 1 - \exp \left[ \frac{\alpha}{\gamma} \left( 1 - u_x \right)^\gamma \right] \]

\[ f(x, \theta, \alpha, \beta, \gamma) = \theta \alpha \beta x^{\beta - 1} \left( 1 - u_x \right)^{\gamma - 1} \exp \left[ \frac{\alpha}{\gamma} \left( 1 - u_x \right)^\gamma \right] \]

\[ G(y, \lambda, \alpha, \beta, \gamma) = 1 - \exp \left[ \frac{\lambda}{\gamma} \left( 1 - u_y \right)^\gamma \right] \]

\[ g(y, \lambda, \alpha, \beta, \gamma) = \lambda \alpha \beta y^{\beta - 1} \left( 1 - u_y \right)^{\gamma - 1} \exp \left[ \frac{\lambda}{\gamma} \left( 1 - u_y \right)^\gamma \right] \]

The reliability given by:

\[ R_1 = p(\max(Y_1, Y_2, ..., Y_k) < X) \]

\[ R_1 = \int_{x=0}^{\infty} G_{y_1}(x) G_{y_2}(x) ... G_{y_k}(x) f(x) dx \]

\[ R_1 = \prod_{i=1}^{k} G_{y_i}(x) f(x) dx \]

Since the stress random variables are independent identically distributed with parameter \( \lambda_1 = \lambda_2 = ... = \lambda_k = \lambda \) then

\[ \prod_{i=1}^{k} G_{y_i}(x) = [G_y(x)]^k \]

So the reliability will be as

\[ R_1 = \int_{x=0}^{\infty} [G_y(x)]^k f(x) dx \]

We have Binomial expansion such that [6]:

\( (1 - x)^n = \sum_{i=0}^{n} C_n^i (-1)^i x^i \) then

\[ R_1 = \int_{x=0}^{\infty} \left( 1 - \exp \left[ \frac{\lambda}{\gamma} \left( 1 - u_x \right)^\gamma \right] \right)^k f(x) dx \]

\[ R_1 = \sum_{i=0}^{k} C_k^i \left( \exp \left[ \frac{\lambda}{\gamma} \left( 1 - u_x \right)^\gamma \right] \right)^i f(x) dx \]

\[ R_1 = \sum_{i=0}^{k} C_k^i (-1)^i \theta \int_{x=0}^{\infty} \alpha \beta x^{\beta - 1} [1 - u_x] u_x^{\gamma - 1} \exp \left[ \frac{\alpha}{\gamma} \left( 1 - u_x \right)^\gamma \right] \exp \left[ \frac{\lambda}{\gamma} \left( 1 - u_x \right)^\gamma \right] dx \]
\[ R_1 = \sum_{j=0}^{k} C_j^k (-1)^j \int_{x=0}^{\infty} \beta \alpha^\beta x^{-\beta-1} (1 - u_x) u_x^{-\gamma-1} \exp \left[ \frac{\theta + \lambda}{\gamma} (1 - u_x^{-\gamma}) \right] \, dx \]

Since \( \int_{x=0}^{\infty} f(x) \, dx = 1 \)

Then \( \int_{x=0}^{\infty} \beta \alpha^\beta x^{-\beta-1} (1 - u_x) u_x^{-\gamma-1} \exp \left[ \frac{\theta}{\gamma} (1 - u_x^{-\gamma}) \right] = \frac{1}{\theta} \)

There for

\[ R_1 = \sum_{j=0}^{k} C_j^k (-1)^j \frac{\theta}{\theta + \lambda} \]

When \( k = 2 \) we get

\[ R_2 = \sum_{i=0}^{2} C_i^2 (-1)^i \frac{\theta}{\theta + \lambda} = C_0^2 (-1)^0 \frac{\theta}{\theta + \lambda} + C_1^2 (-1)^1 \frac{\theta}{\theta + \lambda} + C_2^2 (-1)^2 \frac{\theta}{\theta + 2\lambda} \]

\[ R_2 = 1 - \frac{2\theta}{\theta + \lambda} + \frac{\theta}{\theta + 2\lambda} \] (9)

### 3. Estimation method

In this sub section, the shape parameters \( \lambda \) and \( \theta \) of the GF distribution and the reliability were are estimated using five methods of estimation: Maximum likelihood, Least square, Weighted least square, Regression and Rank set sampling methods.

#### 3.1. Maximum likelihood (MLE)

The MLE (Maximum Probability Estimation Method) is one of the most important and common parameter estimation methods. In 1922 R. A. Fisher introduced the method of maximum likelihood. He first presented the numerical procedure in 1912 [7]. Let \( x_1, x_2, \ldots, x_n \) be strength random sample of size \( n \) from GF(\( \theta, \alpha, \beta, \gamma \)) where \( \theta \) is unknown parameter and \( \alpha, \beta, \gamma \) are known.

Then the likelihood function given by [5]:

\[ L = \prod_{i=1}^{n} f(x_1, x_2, ..., x_n; \theta, \alpha, \beta, \gamma) \]

\[ L = \prod_{i=1}^{n} \left[ \theta \beta \alpha^\beta x_i^{-\beta-1} (1 - u_{x_i}) u_{x_i}^{-\gamma-1} \exp \left[ \frac{\theta}{\gamma} (1 - u_{x_i}^{-\gamma}) \right] \right] \]

\[ L = \theta^n \beta^n \alpha^n \prod_{i=1}^{n} x_i^{-\beta-1} \prod_{i=1}^{n} (1 - u_{x_i}) \prod_{i=1}^{n} u_{x_i}^{-\gamma-1} \exp \left( \sum_{i=1}^{n} \left[ \frac{\theta}{\gamma} (1 - u_{x_i}^{-\gamma}) \right] \right) \]

\[ \ln L = n \ln \theta + n \ln \beta + n \ln \alpha - (\beta + 1) \sum_{i=1}^{n} \ln x_i + \sum_{i=1}^{n} \ln (1 - u_{x_i}) - \left( \gamma + 1 \right) \sum_{i=1}^{n} \ln u_{x_i} + \frac{\theta}{\gamma} \sum_{i=1}^{n} (1 - u_{x_i}^{-\gamma}) \]

\[ \frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} + \frac{1}{\gamma} \sum_{i=1}^{n} (1 - u_{x_i}^{-\gamma}) = 0 \]

\[ \hat{\theta}_{MLE} = \frac{-ny}{\sum_{i=1}^{n} (1 - u_{x_i}^{-\gamma})} \] (10)

In the same way let \( y_1, y_2, ..., y_n \) be stress random sample of size \( m \) from GF(\( \lambda, \alpha, \beta, \gamma \)) where \( \lambda \) is unknown parameter and \( \alpha, \beta, \gamma \) are known.

Then \( \lambda_{MLE} \) given by:

\[ \hat{\lambda}_{MLE} = \frac{-my}{\sum_{j=1}^{m} (1 - u_{y_j}^{-\gamma})} \] (11)

Then by substitute equation (10) and (11) in equation (8) and in equation (9) we get :

\[ \hat{R}_{1MLE} = \sum_{j=0}^{k} C_j^k (-1)^j \frac{\hat{\theta}_{MLE}}{\hat{\theta}_{MLE} + \hat{\lambda}_{MLE}} \] (12)

\[ \hat{R}_{2MLE} = 1 - \frac{2\hat{\theta}_{MLE}}{\hat{\theta}_{MLE} + \hat{\lambda}_{MLE}} + \frac{\hat{\theta}_{MLE}}{\hat{\theta}_{MLE} + 2\hat{\lambda}_{MLE}} \] (13)

#### 3.2. Least Square Estimation Method (LS)

As early as (1794), the German mathematician Carl Friedrich Gauss investigated the least square method, until (1809) he published the method. This method of estimation is very common for model fitting, especially in linear regression and non-linear regression. The minimum square estimator scan method is generated by minimizing the number of squares between the value and the estimated value
of the error [8]. The combination of the parametric and the non-parametric distribution functions is the least square form. The minimizing following equation [9]:

$$S = \sum_{i=1}^{n}(F(x_{i}) - E(F(x_{i})))^2$$  \hspace{1cm} (14)

Suppose $x_1, x_2, ..., x_n$ be random sample have GF($\theta, \alpha, \beta, \gamma$) distribution with the sample size $n$. The procedure attempts to minimize the following function with respect to $\theta$ and $\alpha, \beta, \gamma$ will get as:

$$S(\theta, \alpha, \beta, \gamma) = \sum_{i=1}^{n} \left(1 - \exp\left[\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right]\right) - P_i \right)^2$$  \hspace{1cm} (15)

where $E(F(x_{i})) = P_i$ and $P_i$ is the plotting position, where $P_i = \frac{i}{n+1}$, $i = 1, 2, ..., n$

To obtain the formula of $F(x_{i})$ by equation (5):

$$F(x_{i}) = 1 - \exp\left[\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right]$$

$$1 - F(x_{i}) = \exp\left[\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right]$$

$$\ln(1 - F(x_{i})) = \left[\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right]$$  \hspace{1cm} (16)

Here $x_{i}$ is the $i$:th order statistics of the random sample of the size $n$ from GF hence for the GF, to obtain the LS estimation $\hat{\theta}_{LS}$ of the parameter $\theta$ can be define following equation (16)

$$S(\hat{\theta}, \alpha, \beta, \gamma) = \sum_{i=1}^{n} \left(\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right) - q_i \right)^2$$  \hspace{1cm} (17)

Where $q_i = \ln(1 - F(x_{i})) = \ln(1 - P_i)$

By taking derivative to equation (17) with respect to the parameter $\theta$ and equating result to the zero:

$$\frac{dS(\theta, \alpha, \beta, \gamma)}{d\theta} = \frac{d}{d\theta} \sum_{i=1}^{n} \left(\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right) - q_i \right) = 0$$

$$\frac{\theta}{\gamma} \sum_{i=1}^{n} \left(1 - u^{-\gamma}_{x(i)}\right)^2 - q_i \frac{1}{\gamma} \sum_{i=1}^{n} \left(1 - u^{-\gamma}_{x(i)}\right) = 0$$

$$\hat{\theta}_{LS} = \frac{\gamma}{\sum_{i=1}^{n} \left(1 - u^{-\gamma}_{x(i)}\right)^2} \sum_{i=1}^{n} q_i \left(1 - u^{-\gamma}_{x(i)}\right)$$  \hspace{1cm} (18)

In the same way above we can estimate $\lambda$ as bellow:

$$\hat{\lambda}_{LS} = \frac{\gamma}{\sum_{i=1}^{n} \left(1 - u^{-\gamma}_{x(i)}\right)^2} \sum_{i=1}^{n} \left(1 - u^{-\gamma}_{x(i)}\right)$$  \hspace{1cm} (19)

Then by substitute (18) and (19) in (8) and in (9), we get :

$$\hat{R}_{1,LS} = \frac{\sum_{i=0}^{k} C^i_{l}(-1)^i}{C^{l+1}_{l+1} \hat{\theta}_{LS}}$$  \hspace{1cm} (20)

$$\hat{R}_{2,LS} = 1 - \frac{2\hat{\theta}_{LS}}{\hat{\theta}_{LS}^2 + \hat{\lambda}_{LS}}$$  \hspace{1cm} (21)

### 3.3. Weighted Least Squares Estimation Method (WLS)

The weighted least-square method can be used to minimize the following equation [9]:

$$Q = \sum_{i=1}^{n} W_i (F(x_{i}) - E(F(x_{i})))^2$$  \hspace{1cm} (22)

Where $W_i = \frac{1}{\text{var}(F(x_{i}))} = \frac{(n+1)^2(n+2)}{i(n-i+1)}$, $i = 1, 2, ..., n$

Let a strength random sample $x_1, x_2, ..., x_n$ size $n$ take from have GF($\theta, \alpha, \beta, \gamma$) distribution. The procedure attempts to minimize the following function with respect to $\theta, \alpha, \beta$ and $\gamma$ will get as :

$$Q(\alpha, \beta, \gamma) = \sum_{i=1}^{n} W_i \left(1 - \exp\left[\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right]\right) - P_i \right)^2$$  \hspace{1cm} (23)

As steps in equations (15) and (17) will get as:

$$Q(\alpha, \beta, \gamma) = \sum_{i=1}^{n} W_i \left(\frac{\theta}{\gamma} (1 - u^{-\gamma}_{x(i)})\right) - q_i \right)^2$$  \hspace{1cm} (24)
By taking derivative equation (24) with respect to the parameter $\theta$ and equating result to the zero, we get $\hat{\theta}_{WLS}$ as:

$$
\hat{\theta}_{WLS} = \frac{\gamma \sum_{i=1}^{n} w_i q_i (1-u_{x_i}^{-\gamma})}{\sum_{i=1}^{n} w_i (1-u_{x_i}^{-\gamma})^2}
$$

(25)

In the same way, we can estimate $\lambda$ as bellow:

$$
\hat{\lambda}_{WLS} = \frac{\gamma \sum_{i=1}^{m} w_i q_i (1-u_{y_i}^{-\gamma})}{\sum_{i=1}^{m} w_i (1-u_{y_i}^{-\gamma})^2}
$$

(26)

Then by substitute (25) and (26) in (8) and in (9), we get:

$$
\hat{R}_{1WLS} = \sum_{k=0}^{r} C_k (-1)^{k} \frac{\partial_{WLS}}{\partial_{WLS} + \lambda_{WLS}}
$$

(27)

$$
\hat{R}_{2WLS} = 1 - \frac{2\hat{\theta}_{WLS}}{\hat{\theta}_{WLS} + \lambda_{WLS}} + \frac{\hat{\theta}_{WLS}}{\hat{\theta}_{WLS} + 2\lambda_{WLS}}
$$

(28)

3.4. Regression estimation method (RG)

Regression is one of the essential procedures that uses additional knowledge to create good efficiency estimators. Regression is the basic method of analyzing functional relations between variables conceptually. Relationships are represented as an equation or model relating the answer variable $Y$ and one or more expository variables to $X$. The simple true relations can be approximated by the standard regression equation [10]:

$$
Z_i = a + b u_i + e_i
$$

(29)

Where $Z_i$ is dependent variable, $u_i$ is independent variable and $e_i$ is the error random variable. $a$ and $b$ are called regression coefficients where $a$ is the intercept and $b$ is the slope [11]. Let $x_1, x_2, ..., x_n$ be random strength sample of size $(n)$ from GF($\theta, \alpha, \beta, \gamma$), then the GF estimators of the unknown parameter $\theta$ ,can be obtained by taking the natural logarithm to equation (5), we get:

$$
\ln(1 - F(x_i)) = \frac{\theta}{\gamma} (1 - u_{x_i}^{-\gamma})
$$

(30)

Substituted plotting position $p_i$ instead of $F(x_i)$ in equation (26), we get:

$$
\ln(1 - p_i) = \frac{\theta}{\gamma} (1 - u_{x_i}^{-\gamma})
$$

(31)

By comparison between equation (31) and equation (29), we can get:

$$
Z_i = \ln(1 - p_i) , a=0 , b=\theta , u_i = \frac{1}{\gamma} (1 - u_{x_i}^{-\gamma})
$$

(28)

Where $b$ can be estimated by the minimizing summation of the squared error with respect to $b$, then we get [12]:

$$
\hat{b} = \frac{n \sum_{i=1}^{n} u_i - \sum_{i=1}^{n} z_i \sum_{i=1}^{n} u_i}{n \sum_{i=1}^{n} u_i^2 - (\sum_{i=1}^{n} u_i)^2}
$$

(29)

By substatution (28) in (29) the GF estimator for the unknown parameter $\theta$ ,says $\hat{\theta}_{RG}$ is:

$$
\hat{\theta}_{RG} = E = \frac{n \sum_{i=1}^{n} \ln(1-p_i) (1-u_{x_i}^{-\gamma}) - \sum_{i=1}^{n} \sum_{i=1}^{n} \ln(1-p_j) \sum_{i=1}^{n} (1-u_{x_i}^{-\gamma})}{\frac{n}{\gamma} \sum_{i=1}^{n} \sum_{i=1}^{n} (1-u_{x_i}^{-\gamma})^2 - \left( \frac{1}{\gamma} \sum_{i=1}^{n} (1-u_{x_i}^{-\gamma}) \right)^2}
$$

(30)

In same way $\lambda$ can be estimate as bellow:

$$
\hat{\lambda}_{RG} = \frac{m \sum_{i=1}^{m} \ln(1-p_j) (1-u_{y_j}^{-\gamma}) - \sum_{i=1}^{m} \sum_{i=1}^{m} \ln(1-p_j) \sum_{i=1}^{m} (1-u_{y_j}^{-\gamma})}{\frac{m}{\gamma} \sum_{i=1}^{m} \sum_{i=1}^{m} (1-u_{y_j}^{-\gamma})^2 - \left( \frac{1}{\gamma} \sum_{i=1}^{m} (1-u_{y_j}^{-\gamma}) \right)^2}
$$

(31)

Then by substitute (30) and (31) in (8) and in (9), we get:

$$
\hat{R}_{1RG} = \sum_{k=0}^{r} C_k (-1)^{k} \frac{\hat{\theta}_{RG}}{\hat{\theta}_{RG} + \hat{\lambda}_{RG}}
$$

(32)

$$
\hat{R}_{2RG} = 1 - \frac{2\hat{\theta}_{RG}}{\hat{\theta}_{RG} + \hat{\lambda}_{RG}} + \frac{\hat{\theta}_{RG}}{\hat{\theta}_{RG} + 2\hat{\lambda}_{RG}}
$$

(33)

3.5. Ranked set sample method (RSS)
Let \((x_1, x_2, ..., x_n)\) be random sample from GF. assumed that \((x_1, x_2, ..., x_n)\) be order statistics obtained by ordering the sample in increasing order. The pdf of \(x(i)\) is [13]:

\[
f(x(i)) = \frac{n!}{(i-1)!(n-i)!} [F(x(i))]^{i-1} [1 - F(x(i))]^{n-i} f(x(i))
\]  

(34)

Then applying (5) and (6) in (34), we get:

\[
f(x(i)) = \theta \beta \alpha \beta x(i)^{B-1} \left(1 - u_x(i)\right) u_x^{-}\gamma 1 \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]
\]

Where \(u_x = 1 - e^{\frac{1}{\beta} x}\)

Suppose that \(Q = \frac{n!}{(i-1)!(n-i)!}\) then we get:

\[
f(x(i)) = Q \theta \beta \alpha \beta x(i)^{B-1} \left(1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]\right)^{i-1} \left(1 - u_x(i)\right) u_x^{-}\gamma 1 \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]^{n-i+1}
\]

The likelihood function of the order sample \(x_1, x_2, ..., x_n\) is:

\[
L(x_1, x_2, ..., x_n; \theta, \alpha, \beta, \gamma) = Q^n \theta^n \beta^n \alpha^n \beta^n \prod_{i=1}^{n} x(i)^{B-1} \prod_{i=1}^{n} (1 - u_x(i))
\]

\[
\prod_{i=1}^{n} \left[1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]\right] \prod_{i=1}^{n} u_x^{-}\gamma 1 \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]^{n-i+1}
\]

(35)

Then the natural logarithm function for the equation (35) can be written as:

\[
\ln L = n \ln Q + n \ln \theta + n \ln \beta + n \beta \ln \alpha - (\beta + 1) \sum_{i=1}^{n} \ln x(i) + \\
\sum_{i=1}^{n}(i - 1) \ln \left[1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]\right] + \sum_{i=1}^{n} \ln \left(1 - u_x(i)\right)
\]

\[
- (\gamma + 1) \sum_{i=1}^{n} u_x(i) + \sum_{i=1}^{n} (n - i + 1) \left(1 - u_x(i)\right)
\]

(36)

To minimize the equation (36), we must calculate the great endings by taking the partial derivative with respect to the unknown parameter \(\theta\), then we get:

\[
\frac{d \ln L}{d \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}(i - 1) \frac{-1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} + \frac{1}{\gamma} \sum_{i=1}^{n} (n - i + 1) \left(1 - u_x(i)\right)
\]

Equating the partial derivative to zero, thus the right hand side will be:

\[
\frac{n}{\theta} = \frac{\sum_{i=1}^{n}(i - 1) \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} + \frac{1}{\gamma} \sum_{i=1}^{n} (n - i + 1) \left(1 - u_x(i)\right)}{\sum_{i=1}^{n} \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} - \sum_{i=1}^{n} (n - i + 1) \left(1 - u_x(i)\right)}
\]

(37)

In the same way, can be estimation \(\lambda\) as follows:

\[
\frac{\sum_{i=1}^{m} \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} - \sum_{i=1}^{m} (m - i + 1) \left(1 - u_x(i)\right)}{\sum_{i=1}^{m} \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} - \sum_{i=1}^{m} (m - i + 1) \left(1 - u_x(i)\right)}
\]

(38)

Then by substitute (37) and (38) in (8) and in (9), we get:

\[
R_{1RSS} = \frac{\sum_{i=0}^{k} \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} - \sum_{i=1}^{k} (k - i + 1) \left(1 - u_x(i)\right)}{\sum_{i=1}^{k} \frac{1}{1 - \exp \left[\frac{\theta}{\gamma} (1 - u_x(i))\right]} - \sum_{i=1}^{k} (k - i + 1) \left(1 - u_x(i)\right)}
\]

(39)
\[ \hat{R}_{2RSS} = 1 - \frac{2\hat{\theta}_{RSS}}{\hat{\theta}_{RSS} + \lambda_{RSS}} + \frac{\hat{\theta}_{RSS}}{\hat{\theta}_{RSS} + 2\lambda_{RSS}} \]  

(40)

4. Simulation study

In this section, a simulation study is used to determine the best reliability estimate with unknown Gompertz-Fréchet distribution parameters, and to evaluate five different estimates from the Maximum Likelihood, Least Square, Weighted Least Square, Regression and Ranked Set Sampling Methods, where regression estimators were used as the initial value. The mean square error criteria (MSE) for various sample sizes (15,30,90) and \((\alpha=0.4, \gamma=0.6, \beta=0.9 \text{ and } \alpha=0.7, \gamma=0.7, \beta=0.2)\) are assessed for six different experiments for reliability \(R_1\) and \((\alpha=0.5, \gamma=0.9, \beta=1.5 \text{ and } \alpha=0.3, \gamma=0.3, \beta=0.7)\) are assessed for six different experiments for reliability \(R_2\), each with the parameters \(\alpha, \gamma \text{ and } \beta\). A simulation study is conducted using MATLAB 2020 for the six different experiments to compare the performance of reliability estimators using the following steps:

**Step1:** Generating the random values of the random variables by the inverse function according to the following formula:

\[ x = \alpha \left[ -\ln \left(1 - \left(1 - \frac{\gamma}{\theta} \ln(1 - F(x)) \right)^{-\frac{1}{\gamma}} \right) \right]^{-\frac{1}{\beta}} \]

**Step2:** Finding the MLE for reliability using equation (12) and (13), LS using equation (20) and (21), WLS using equation (27) and (28), Rg using equation (32) and (33) and RSS using equation (39) and (40).

**Step3:** Finding the mean by the equation:

\[ \text{Mean} = \frac{\sum_{i=1}^{N} \hat{R}_i}{N} \]

**Step4:** The estimation methods are compared using the mean square error criteria:

\[ \text{MSE} = \frac{1}{N} \sum_{i=1}^{N} \left( \hat{R}_i - R \right)^2 \]

where \(N\) is 500 in each experiment.

The results are recorded in the tables from 1 to 6 for reliabilities \(R_1\) and \(R_2\). The comparison of the performance of these estimators based on MSE values was noted as follows:

**In table (1) and (2)**
- For experiments (1),(2),(3) and (5) in the case of sample size (90,90) the best value of MSE is MLE, followed by LS, RSS, Rg, WLS.
- For experiments (2),(4),(5) and (6) in the case of sample size (15,15) and (30,30) and in experiment (1) in sample size (30,30) the best value of MSE is MLE, followed by LS, RSS, WLS ,Rg.
- For experiments (1),(3),(4),(5) and (6) in the case of sample size (15,30), (30,15) and (30,90) and in experiment (2) in sample size (30,15) and (30,90) the best value of MSE is MLE, followed by LS, WLS ,Rg, RSS.

**In table (1)**
- For experiments (1) and (3) in the case of sample size (15,15), in experiment (2) in sample size (15,30) and in experiment (3) in sample size (30,30) the best value of MSE is MLE, followed by LS, WLS, RSS, Rg.

**In table (2)**
- For experiments (4) and (6) in the case of sample size (90,90) the best value of MSE is MLE, followed by RSS, LS, Rg, WLS.

**In table (3) and (4)**
- For experiments (1) and (4) in the case of sample size (15,15) and (30,30) and in experiments (2),(3),(5) and (6) in sample size (30,30) the best value of MSE is MLE, followed by LS, RSS, WLS, Rg.
- For experiments (2),(3),(5) and (6) in the case of sample size (15,15) the best value of MSE is MLE, followed by LS, WLS, RSS, Rg.
- For experiments (1),(4) and (5) in the case of sample size (90,90) the best value of MSE is MLE, followed by LS, RSS, Rg, WLS.
• For experiments (2),(3) and (6) in the case of sample size (90,90) the best value of MSE is MLE, followed by RSS, LS, Rg, WLS.
• For experiments (1),(2),(3),(4),(5) and (6) in the case of sample size (15,30) and (30,15) the best value of MSE is MLE, followed by LS, WLS, Rg, RSS.
• For experiments (1),(2),(3),(4),(5) and (6) in the case of sample size (30,90) the best value of MSE is MLE, followed by LS, Rg, WLS, RSS.

5. Conclusion

In this paper, we presented five methods for estimating reliability \( p(\max(Y_1, Y_2, ..., Y_k) < X) \) and as a special case \( k=2 \) where the strength \( X \) and the stresses \( Y_1, Y_2, ..., Y_k \) follow Gompertz- Fréchet distribution with different parameters. Simulation results confirm that the performance of the maximum likelihood estimator is best for all experiments and for all sizes of samples, followed by the LS estimator is the second best.

**Table (1):** real Reliability \( R_1 \) values and its estimators performance for Exp. 1,2,3.

| \((n,m)\) | Exp1: \( R_1=0.1875 \) when \( \lambda=1.5, \beta=2.3 \) for \( \alpha=0.4, \gamma=0.6, \beta=0.9 \) | MLE | LS | WLS | Rg | RSS |
|-------|-----------------------------------------------------|-----|-----|-----|-----|-----|
| (15,15) | Mean 0.1914 | 0.1948 | 0.1977 | 0.2029 | 0.1930 | 0.1930 |
| | MSE 0.0075 | 0.0086 | 0.0104 | 0.0129 | 0.0107 | 0.0107 |
| (30,30) | Mean 0.1910 | 0.1909 | 0.1936 | 0.1946 | 0.1937 | 0.1937 |
| | MSE 0.0044 | 0.0050 | 0.0066 | 0.0071 | 0.0064 | 0.0064 |
| (90,90) | Mean 0.1892 | 0.1894 | 0.1915 | 0.1907 | 0.1897 | 0.1897 |
| | MSE 0.0014 | 0.0018 | 0.0037 | 0.0027 | 0.0019 | 0.0019 |
| (15,30) | Mean 0.1890 | 0.2032 | 0.2035 | 0.2101 | 0.0708 | 0.0708 |
| | MSE 0.0060 | 0.0079 | 0.0099 | 0.0119 | 0.0161 | 0.0161 |
| (30,15) | Mean 0.2033 | 0.1932 | 0.1991 | 0.1966 | 0.3935 | 0.3935 |
| | MSE 0.0066 | 0.0071 | 0.0092 | 0.0106 | 0.0559 | 0.0559 |
| (30,90) | Mean 0.1885 | 0.2026 | 0.2001 | 0.2083 | 0.0264 | 0.0264 |
| | MSE 0.0027 | 0.0042 | 0.0065 | 0.0068 | 0.0262 | 0.0262 |
| | Exp2: \( R_1=0.1486 \) when \( \lambda=1.5, \beta=2.7 \) for \( \alpha=0.4, \gamma=0.6, \beta=0.9 \) | MLE | LS | WLS | Rg | RSS |
|-------|-----------------------------------------------------|-----|-----|-----|-----|-----|
| (15,15) | Mean 0.1664 | 0.1690 | 0.1720 | 0.1759 | 0.1721 | 0.1721 |
| | MSE 0.0071 | 0.0087 | 0.0108 | 0.0133 | 0.0105 | 0.0105 |
| (30,30) | Mean 0.1564 | 0.1589 | 0.1639 | 0.1633 | 0.1579 | 0.1579 |
| | MSE 0.0034 | 0.0041 | 0.0062 | 0.0063 | 0.0049 | 0.0049 |
| (90,90) | Mean 0.1513 | 0.1521 | 0.1559 | 0.1542 | 0.1516 | 0.1516 |
| | MSE 0.0010 | 0.0013 | 0.0032 | 0.0022 | 0.0014 | 0.0014 |
| (15,30) | Mean 0.1570 | 0.1698 | 0.1694 | 0.1766 | 0.0535 | 0.0535 |
| | MSE 0.0052 | 0.0071 | 0.0087 | 0.0108 | 0.0106 | 0.0106 |
| (30,15) | Mean 0.1643 | 0.1562 | 0.1619 | 0.1612 | 0.3435 | 0.3435 |
| | MSE 0.0056 | 0.0060 | 0.0078 | 0.0088 | 0.0511 | 0.0511 |
| (30,90) | Mean 0.1550 | 0.1690 | 0.1699 | 0.1761 | 0.0187 | 0.0187 |
| | MSE 0.0023 | 0.0035 | 0.0055 | 0.0056 | 0.0170 | 0.0170 |
| | Exp3: \( R_1=0.2371 \) when \( \lambda=3.3, \beta=2.3 \) for \( \alpha=0.4, \gamma=0.6, \beta=0.9 \) | MLE | LS | WLS | Rg | RSS |
|-------|-----------------------------------------------------|-----|-----|-----|-----|-----|
| (15,15) | Mean 0.2460 | 0.2452 | 0.2450 | 0.2469 | 0.2535 | 0.2535 |
| | MSE 0.0099 | 0.0113 | 0.0132 | 0.0158 | 0.0138 | 0.0138 |
| (30,30) | Mean 0.2387 | 0.2406 | 0.2442 | 0.2451 | 0.2388 | 0.2388 |
| | MSE 0.0051 | 0.0063 | 0.0087 | 0.0093 | 0.0069 | 0.0069 |
| (90,90) | Mean 0.2402 | 0.2386 | 0.2376 | 0.2375 | 0.2435 | 0.2435 |
| | MSE 0.0020 | 0.0025 | 0.0052 | 0.0038 | 0.0026 | 0.0026 |
| (15,30) | Mean 0.2432 | 0.2541 | 0.2520 | 0.2557 | 0.1042 | 0.1042 |
| | MSE 0.0076 | 0.0096 | 0.0122 | 0.0141 | 0.0220 | 0.0220 |
| (30,15) | Mean 0.2384 | 0.2262 | 0.2318 | 0.2280 | 0.4382 | 0.4382 |
| | MSE 0.0075 | 0.0088 | 0.0113 | 0.0130 | 0.0539 | 0.0539 |
| (n,m) | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (30,90) | 0.2359 | 0.0031 | 0.2514 | 0.0043 | 0.2501 | 0.0067 | 0.2576 | 0.0069 | 0.0398 | 0.0393 |

Table (2) : real Reliability $R_1$ values and it’s estimators performance for Exp. 4, 5, 6.

Exp4: $R_1 = 0.1875$ when $\lambda = 2.5, \theta = 2.3$ for $\alpha = 0.7, \gamma = 0.7, \beta = 0.2$

| (n,m) | MLE    | LS     | WLS    | Rg    | RSS    |
|-------|--------|--------|--------|-------|--------|
| (15,15) | 0.1977 | 0.1982 | 0.1998 | 0.2024 | 0.2037 |
|       | 0.0077 | 0.0094 | 0.0113 | 0.0136 | 0.0104 |
| (30,30) | 0.1906 | 0.1921 | 0.1938 | 0.1953 | 0.1922 |
|       | 0.0043 | 0.0055 | 0.0080 | 0.0086 | 0.0057 |
| (90,90) | 0.1878 | 0.1886 | 0.1927 | 0.1905 | 0.1890 |
|       | 0.0015 | 0.0021 | 0.0046 | 0.0034 | 0.0019 |
| (15,30) | 0.1946 | 0.2117 | 0.2135 | 0.2211 | 0.0703 |
|       | 0.0057 | 0.0079 | 0.0102 | 0.0122 | 0.0161 |
| (30,15) | 0.2002 | 0.1881 | 0.1932 | 0.1894 | 0.3951 |
|       | 0.0075 | 0.0081 | 0.0103 | 0.0114 | 0.0579 |
| (30,90) | 0.1880 | 0.2038 | 0.2031 | 0.2108 | 0.0266 |
|       | 0.0028 | 0.0040 | 0.0062 | 0.0066 | 0.0261 |

Exp5: $R_1 = 0.1486$ when $\lambda = 2.5, \theta = 2.7$ for $\alpha = 0.7, \gamma = 0.7, \beta = 0.2$

Exp6: $R_1 = 0.2371$ when $\lambda = 3, \theta = 2.3$, for $\alpha = 0.7, \gamma = 0.7, \beta = 0.2$

| (n,m) | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    | Mean   | MSE    |
|-------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| (15,15) | 0.1655 | 0.1677 | 0.1708 | 0.1732 | 0.1710 |
|       | 0.0079 | 0.0095 | 0.0115 | 0.0137 | 0.0111 |
| (30,30) | 0.1530 | 0.1536 | 0.1561 | 0.1566 | 0.1552 |
|       | 0.0035 | 0.0041 | 0.0059 | 0.0061 | 0.0048 |
| (90,90) | 0.1507 | 0.1515 | 0.1553 | 0.1535 | 0.1509 |
|       | 0.0011 | 0.0014 | 0.0028 | 0.0021 | 0.0015 |
| (15,30) | 0.1504 | 0.1652 | 0.1674 | 0.1743 | 0.0496 |
|       | 0.0047 | 0.0066 | 0.0087 | 0.0107 | 0.0112 |
| (30,15) | 0.1655 | 0.1588 | 0.1652 | 0.1642 | 0.3422 |
|       | 0.0061 | 0.0069 | 0.0090 | 0.0104 | 0.0502 |
| (30,90) | 0.1486 | 0.1646 | 0.1663 | 0.1730 | 0.0173 |
|       | 0.0023 | 0.0037 | 0.0058 | 0.0062 | 0.0174 |

Table (3) : real Reliability $R_2$ values and it’s estimators performance for Exp. 1, 2, 3.

Exp1: $R_1 = 0.4808$ when $\lambda = 2, \theta = 1.2$ for $\alpha = 0.5, \gamma = 0.9, \beta = 1.5$

| (n,m) | MLE    | LS     | WLS    | Rg    | RSS    |
|-------|--------|--------|--------|-------|--------|
| (15,15) | 0.4742 | 0.4733 | 0.4728 | 0.4721 | 0.4751 |
|       | 0.0116 | 0.0138 | 0.0163 | 0.0191 | 0.0155 |
| (30,30) | 0.4801 | 0.4833 | 0.4847 | 0.4860 | 0.4752 |
|       | 0.0052 | 0.0067 | 0.0095 | 0.0100 | 0.0072 |
| (90,90) | 0.4805 | 0.4791 | 0.4775 | 0.4776 | 0.4817 |
|       | 0.0017 | 0.0022 | 0.0046 | 0.0034 | 0.0024 |
| (15,30) | 0.4816 | 0.4956 | 0.4925 | 0.4980 | 0.3000 |
Table (4) : real Reliability : $R_2$ values and it’s estimators performance for Exp. 4, 5, 6.

| (n, m) | MLE | LS | WLS | Rg   | RSS |
|-------|-----|----|-----|------|-----|
| (15, 15) | Mean 0.4792 0.4791 0.4790 0.4783 0.4795 | Mean 0.6610 0.6581 0.6566 0.6554 0.6562 |
| MSE 0.0110 0.0135 0.0162 0.0192 0.0142 | MSE 0.0081 0.0091 0.0108 0.0131 0.0117 |
| (30, 30) | Mean 0.4759 0.4732 0.4704 0.4707 0.4774 | Mean 0.6682 0.6672 0.6668 0.6662 0.6641 |
| MSE 0.0054 0.0068 0.0096 0.0101 0.0071 | MSE 0.0040 0.0046 0.0064 0.0068 0.0054 |
| (90, 90) | Mean 0.4812 0.4818 0.4835 0.4821 0.4816 | Mean 0.6647 0.6623 0.6572 0.6597 0.6655 |
| MSE 0.0018 0.0023 0.0044 0.0034 0.0022 | MSE 0.0014 0.0019 0.0040 0.0031 0.0017 |
| (30, 15) | Mean 0.5823 0.5935 0.5898 0.5932 0.4006 | Mean 0.6682 0.6564 0.6579 0.6535 0.7979 |
| MSE 0.0075 0.0092 0.0118 0.0135 0.0450 | MSE 0.0053 0.0072 0.0092 0.0109 0.0202 |
| (90, 90) | Mean 0.5847 0.5732 0.5781 0.5726 0.7360 | Mean 0.6601 0.6741 0.6694 0.6766 0.3646 |
| MSE 0.0071 0.0088 0.0111 0.0128 0.0281 | MSE 0.0027 0.0032 0.0052 0.0049 0.0973 |

Exp: $R_2 = 0.6689$ when $\lambda = 2$, $\theta = 0.6$ for $\alpha = 0.3$, $\gamma = 0.9$, $\beta = 1.5$.

Exp2: $R_2 = 0.6689$ when $\lambda = 2$, $\theta = 0.6$ for $\alpha = 0.5$, $\gamma = 0.9$, $\beta = 1.5$.

Exp3: $R_2 = 0.5861$ when $\lambda = 2.9$, $\theta = 1.2$ for $\alpha = 0.5$, $\gamma = 0.9$, $\beta = 1.5$.

Exp4: $R_2 = 0.4808$ when $\lambda = 2$, $\theta = 1.2$ for $\alpha = 0.3$, $\gamma = 0.3$, $\beta = 0.7$.

Exp5: $R_2 = 0.6689$ when $\lambda = 2$, $\theta = 0.6$ for $\alpha = 0.3$, $\gamma = 0.3$, $\beta = 0.7$.
### 6. References

[1] Karadayı, N., Saraçoğlu, B., & Pekgör, A. (2011). Stress-strength reliability and its estimation for a component which is exposed two independent stresses.

[2] Karam, N. S., & HG, R. (2016). Reliability Estimation for a Component Exposed Tow, Three Independent Stresses Based on Weibull and Inverse Lindly Distribution. *Mathematical Theory and Modeling*, 6(8), 2225-0522.

[3] Adnan, S., & Karam, N. S. (2021). Gompertz Fréchet Strength of a component between two stresses Reliability Estimation. *Al-Qadisiyah Journal Of Pure Science*, 26(2), 23-38.

[4] Alizadeh, M., Cordeiro, G. M., Pinho, L. G. B., & Ghosh, I. (2017). The Gompertz-G family of distributions. *Journal of Statistical Theory and Practice*, 11(1), 162-176.

[5] Oguntunde, P. E., Khaleel, M. A., Ahmed, M. T., & Okagbue, H. I. (2019). The Gompertz-fréchet distribution: properties and applications. *Cogent Mathematics & Statistics*, 6(1), 1568662.

[6] Guseinov, I. I., & Mamedov, B. A. (2007). Calculation of integer and noninteger n-dimensional debye functions using binomial coefficients and incomplete gamma functions. *International Journal of Thermophysics*, 28(4), 1420-1426.

[7] Aldrich, J. (1997). RA Fisher and the making of maximum likelihood 1912-1922. *Statistical science*, 12(3), 162-176.

[8] Aldrich, J. (1998). Doing least squares: perspectives from Gauss and Yule. *International Statistical Review/Revue Internationale de Statistique*, 61-81.

[9] Karam, N. S. (2016). One, Two and Multi–Component Gompertz Stress–Strength Reliability Estimation. *Mathematical Theory and Modeling*, 6(3), 77-92.

[10] Al-Nasser, A. D., & Radaideh, A. (2008). Estimation of simple linear regression model using L ranked set sampling. *Int. J. Open Problems Compt. Math.*, 1(1), 18-33.

[11] Montgomery, D. C., Peck, E. A., & Vining, G. G. (2012). Introduction to linear regression analysis (Vol. 821). John Wiley & Sons.

[12] Jha, M. K., Dey, S., & Tripathi, Y. M. (2019). Reliability estimation in a multicomponent stress–strength based on unit-Gompertz distribution. *International Journal of Quality & Reliability Management*.

[13] Chen, Z., Bai, Z., & Sinha, B. (2013). *Ranked set sampling: theory and applications* (Vol. 176). Springer Science & Business Media.

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### Table 1

| MSE       | 0.0055 | 0.0070 | 0.0089 | 0.0107 | 0.0203 |
|-----------|--------|--------|--------|--------|--------|
| Mean      | 0.6604 | 0.6741 | 0.6715 | 0.6773 | 0.3641 |
| MSE       | 0.0029 | 0.0032 | 0.0050 | 0.0047 | 0.0977 |

Exp6: R = 0.5861 when λ=2.9, θ=1.2, for α=0.3, γ=0.3, β=0.7