An investigation for quantum qutrit entanglements through colour wheel compass

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Abstract: In this study qutrits which have 3-level quantum systems will be used. The investigation of quantum qutrit states was done in a mathematical thought experiment in which most of the quantum phenomena (such as decoherence as well as charge and mass) are neglected. Hence, the model based on the investigation of qutrit states by using colour wheel was given in terms of set theory. Then the results of quantum qutrit entanglements in a hypothetical entangled universe will be analysed through the colour wheel compass.

1. Introduction

Quantum entanglement plays a fundamental role in many applications of quantum information theory, such as quantum teleportation [1]. As the effects of quantum characteristics are taken into account, those classifications and analyses of those quantum entangled states becomes much more complicated [2]. Local quantum operations and classical communication cannot fundamentally change quantum entanglements without destroying it. Therefore, entanglement can be classified by dividing quantum states into equivalence classes [3]. Those studies are also done with different quantum systems. The most common and the simple one is qubits that refer to the state of a 2-level quantum systems. A qutrit is a 3-level quantum system which are labelled as $|0\rangle$, $|1\rangle$, $|2\rangle$ in Hilbert space [4]. A colour wheel or colour circle is an abstract illustrative organization of colour groups around a circle that shows relationships between primary colours, secondary colours, tertiary colours etc. RYB (red–yellow–blue) is the most common example of those kinds of illustrative maps [5].

In this study, first the qutrit states are introduced by using colour wheel in section 2. In section 3, a hypothetical device as a colour wheel compass is explained. Then, the quantum qutrit entanglements are also analysed by using colour wheel compass in section 3.

2. Theory

Set theory can be used to classify many mathematical patterns and mathematical calculations [6]. In this study, set theory is used for the classifications of qutrits. Therefore, set points and corresponding qutrit states are given in Table 1. These classifications can be represented as given in Figure 1. In this Figure, the set points and corresponding qutrit states are placed on the Fano plane. Those classifications can also be transformed into colour wheel in which $|0\rangle$, $|1\rangle$ and $|2\rangle$ can be represented by red, yellow and blue colours, respectively. Superposition states can be represented as the combination of those colours as shown in Figure 2. Negative qutrit states are represented in the outer part of the colour wheel as well as their superposition with positive quantum states. The superpositions of $|0\rangle + |1\rangle + |2\rangle$ and $-|0\rangle - |1\rangle - |2\rangle$ are represented by z coordinate passing through the centre of colour wheel.
| Set points | Corresponding qutrits |
|------------|----------------------|
| 100        | |0⟩          |
| 010        | |1⟩          |
| 001        | |2⟩          |
| 011        | |1⟩+|2⟩       |
| 101        | |0⟩+|1⟩       |
| 110        | |0⟩+|2⟩       |
| 111        | |0⟩+|1⟩+|2⟩   |

**Figure 1.** (a) Set points and (b) corresponding qutrit states in the Fano plane.

**Figure 2.** Colour Wheel Compass and corresponding qutrit states.

There exist many quantum qutrit gates. In this study, following ternary selective Hadamard gates are used [7]: 

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There are many mathematical methods for the analysis of entangled states. One simple method for the analysis is the scalar product of tensors in this respect. Gluing product can be defined as a scalar product in which two vector multiplied based on their main axis or regions. Let $i$ and $j = 0, 1, 2$ be the indices of quantum qutrit states, the quantum states $|\psi_i\rangle$ and $|\psi_j\rangle$ can be defined as following [8]:

$$|\psi_i\rangle \langle \psi_j| = |\psi_i\rangle \langle \psi_j| \delta_{ij}$$

For example let $u$ and $v$ are two vectors below:

$$|u\rangle = \alpha_{00} |0\rangle \hat{0} + \alpha_{11} |1\rangle \hat{1} + \alpha_{22} |2\rangle \hat{2}$$

They can be entangled by gluing product as below:

$$|u\rangle \Omega |v\rangle = \alpha_{00} |0\rangle |0\rangle + \alpha_{11} |1\rangle |1\rangle + \alpha_{22} |2\rangle |2\rangle$$

3. Results and Discussion

The route beginning with the initial qutrit $|0\rangle$ was created by the acting operators $H^{01}$ from the left $H^{02}$ from the right for each sub-cells in the same order represented as in Figure 3. The route restricted by ternary selective Hadamard gates finished with four quantum states $A= |0\rangle$, $B= |0\rangle + |1\rangle$, $C= |0\rangle + |2\rangle$ and $D= |0\rangle + |1\rangle + |2\rangle$ in the relation given by $A\rightarrow BC$, $B\rightarrow AD$, $C\rightarrow DA$, $D\rightarrow CB$. The route is finished when it reaches the loop points in which no new quantum states emerges and the points in route repeat itself. If the fixed states are represented by the coordinates of colour wheel, the symmetry axis passing through the line combines $|0\rangle$, and $|0\rangle + |1\rangle + |2\rangle$. It can be seen in Figure 4 that the centre of these coordinates is in the direction pointing red colour because there are more quantum states than the opposite symmetrical colour area direction.

![Figure 3](image-url)
Figure 4. The symmetry axis of the route beginning with $|0\rangle$ and its direction.

The results of the routes beginning with $|0\rangle$, $|1\rangle$, $|2\rangle$ restricted by $H^{01}$ from the left $H^{02}$ from the right can be given in Table 2 and the relationships among variables for each route is given in Table 3.

Table 2. The results of the routes beginning with $|0\rangle$, $|1\rangle$, $|2\rangle$ restricted by $H^{01}$ from the left $H^{02}$.

| The beginning of route | The main points of route |
|------------------------|-------------------------|
| $|0\rangle$            | A = $|0\rangle$, B = $|0\rangle+|1\rangle$, C = $|0\rangle+|2\rangle$ and D = $|0\rangle+|1\rangle+|2\rangle$ |
| $|1\rangle$            | E = $|1\rangle$ F = $|0\rangle+|1\rangle$, G = $|0\rangle-|1\rangle+|2\rangle$, H = $|1\rangle+|2\rangle$, K = $|0\rangle-|2\rangle$, L = $|0\rangle+|1\rangle-|2\rangle$, M = $|2\rangle$ |
| $|2\rangle$            | M = $|2\rangle$, K = $|0\rangle-|2\rangle$, L = $|0\rangle+|1\rangle-|2\rangle$, H = $|1\rangle+|2\rangle$, G = $|0\rangle-|1\rangle+|2\rangle$, F = $|0\rangle-|1\rangle$, E = $|1\rangle$ |

Table 3. The relationships among variables of the routes beginning with $|0\rangle$, $|1\rangle$, $|2\rangle$ restricted by $H^{01}$ from the left $H^{02}$.

| The beginning of route | The relationships among the main points of route |
|------------------------|-------------------------------------------------|
| $|0\rangle$            | A $\rightarrow$ BC, B $\rightarrow$ AD, D $\rightarrow$ CB, C $\rightarrow$ DA, |
| $|1\rangle$            | E $\rightarrow$ FE, F $\rightarrow$ EG, G $\rightarrow$ HF, K $\rightarrow$ LM, H $\rightarrow$ GL, L $\rightarrow$ KH, M $\rightarrow$ MK |
| $|2\rangle$            | M $\rightarrow$ MK, K $\rightarrow$ LM, L $\rightarrow$ KH, G $\rightarrow$ HF, H $\rightarrow$ GL, F $\rightarrow$ EG, E $\rightarrow$ FE |

The results of the routes can be glued in many ways. For example if the quantum states for all routes are glued based on their colour area, no entangled state can be observed. If the results of the qudrits states for the results of the routes beginning with $|1\rangle$, $|2\rangle$ can be glued based on their colour region, the results are given in Table 4. The entangled states can be also achieved by the rotation or changing the colour region in various ways. Then, they might be used for the classification of entangled states. Those states might be formulated based on the colour region area as well as indices and the angle between the colour wheels. If we modify the scalar product based on our colour wheel, that is vectors can be glued based on the colour areas rather their vector positions on the space, the result can be given as in Table 4. This can be regarded as the skeleton of our glued results which might be interpreted as the net result of entanglement of different vectors. Hence entangled systems can be
classified and formulized based the locations of vectors in each colour wheel which might physically the indicator of the extension of a multidimensional hologram.

### Table 4.
The gluing products of the fixed qutrit points for the routes beginning with $|1\rangle$, $|2\rangle$.

| Gluing region      | Results                                                                 |
|--------------------|-------------------------------------------------------------------------|
| Red Area           | $|0\rangle \Omega |0\rangle = |0\rangle$ |
| Upper Purple Area  | $(|0\rangle - |1\rangle + |2\rangle) \Omega(|0\rangle - |1\rangle + |2\rangle) = |0\rangle + |1\rangle + |2\rangle$ |
| Yellow Area        | $|1\rangle \Omega |1\rangle = |1\rangle$ |
| Green Area         | $(|1\rangle + |2\rangle) \Omega(|1\rangle + |2\rangle) = |1\rangle + |2\rangle$ |
| Upper Orange Area  | $(|0\rangle + |1\rangle - |2\rangle) \Omega(|0\rangle + |1\rangle - |2\rangle) = |0\rangle + |1\rangle + |2\rangle$ |
| Pink Area          | $(|0\rangle - |1\rangle) \Omega(|0\rangle - |1\rangle) = |0\rangle + |1\rangle$ |

### 4. Conclusion

In this study, first colour wheel compass was defined for qutrits based on set theory. Then the patterns beginning with qutrits $|0\rangle$, $|1\rangle$, $|2\rangle$ restricted by ternary Hadamard gates were analysed. The stable points were determined and located into colour wheel compass. Results for the route beginning with $|0\rangle$, $|1\rangle$, $|2\rangle$ indicates that the direction of the symmetry axis of each routes is the same which is in the direction in red area. And also the fixed points of routes beginning with qutrits $|1\rangle$, $|2\rangle$ are glued with each other and some entangled states for qutrits are obtained. As a suggestion multilayer colour wheel compass can be used as a map for the analysis of quantum systems. Further analysis can be done including the mass, charge and other classical variables as well as the quantum effects.

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