CONFINEMENT-DECONFINEMENT ORDER PARAMETER AND
DIRAC’S QUANTIZATION CONDITION
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We describe a monopole-like order parameter for the confinement-deconfinement transition in gauge theories where dynamical charges and monopoles coexist. It has been recently proposed in a collaboration with J. Fröhlich. It avoids an inconsistency in the treatment of small scales present in earlier definitions of monopole fields by respecting Dirac’s quantization condition for electromagnetic fluxes. An application to SU(2) lattice Yang-Mills theory is outlined, naturally fitting in the ’t Hooft scenario for confinement.

1. Monopoles and confinement

’t Hooft proposed to explain confinement in SU(2) Yang-Mills theory as a consequence of condensation of magnetic monopoles defined as follows. He suggested 1 to construct a scalar field \( X(U) \) with values in \( su(2) \), as a function of the gauge field \( U \) and transforming in the adjoint representation of the gauge group \( SU(2) \). By requiring that \( X(U) \) be diagonal one then fixes a gauge ("Abelian projection"). The resulting theory exhibits a residual \( U(1) \) gauge invariance.

The argument of the diagonal component of the SU(2) gauge field in this “Abelian projection gauge” plays the role of a compact \( U(1) \) “photon” field, \( A_\mu \), with range \((-2\pi, 2\pi)\), and the off-diagonal components are described by a complex field, \( c \), charged with respect to the residual \( U(1) \) gauge group. The points in space-time where the two eigenvalues of the matrix \( X \) coincide identify the positions of the monopoles in this gauge. Confinement is believed to emerge as a consequence of monopole–condensation in the form of a “dual Meissner effect”.

This characterization of confinement needs a charged order parameter in presence of both dynamical charges and monopoles, in the dual Higgs model.

2. Monopoles in Abelian gauge theories

In scalar electrodynamics, or non-compact Higgs model, with charged scalar field \( \phi \) one can construct gauge–invariant charged fields adapting the Dirac recipe 2, dressing the local non–gauge invariant field \( \phi(x) \) with a cloud of soft photons described by a phase factor with argument given by the photon field \( \vec{A} \) weighted by a classical Coulomb field \( \vec{E}_\mu \) generated by a charge at \( \vec{x} \). More precisely, we define an electric distribution \( E_{\mu}(y) = \{0, \vec{E}(\vec{y})\delta(x^0 - y^0)\} \) satisfying \( \partial^\mu E_{\mu}(y) = \delta(x - y) \) and the non-local gauge-invariant charged field is given by \( \phi(x)e^{i\int A^\mu E_{\mu}} \). One can prove rigorously in the lattice approximation that the vacuum expectation value \( \langle \phi(x)e^{i\int A^\mu E_{\mu}} \rangle \neq 0 \) in the Higgs phase and it vanishes in the Coulomb phase 3.

In Abelian gauge theories there is a natural notion of duality exchanging the rôle of charges and monopoles 4. In particular, one can obtain monopole correlation functions from gauge–invariant charged correlation functions by a duality transformation. The non-compact Higgs model \((n.c.H.)\) with charges and photons (described by \( A_\mu \)) is mapped by duality to the \( U(1) \) gauge theory with monopoles and dual photons (described by \( \tilde{A}_\mu \)) and the charged correlator

\[ \langle \phi(x)e^{i\int A^\mu E_{\mu}} \rangle \neq 0 \]
\[ \langle \phi(x) e^{-i \int A^\mu E_\mu} \phi(x') e^{i \int A^\mu E_\mu} \rangle_{n.c.H.} \] (1)

is mapped into the disorder field correlator
\[ \langle e^{-i S_{U(1)}(\partial_\mu A_\mu) + \partial^\rho \Delta^{-1} B_{\mu\nu}^x} - S_{U(1)}(\partial_\mu A_\mu) \rangle_{U(1)} \] (2)

where \( B_{\mu\nu}^x = B_{\mu\nu} - B^x + \omega^{xx'} \) with \( \Delta \) denoting the Laplacian, \( B_{\mu\nu}^x = \epsilon_{\mu\nu\rho\sigma} E_\sigma^x \), and \( \omega^{xx'} \) is an integer 3-current from \( x \) to \( x' \) required by magnetic flux conservation in \( U(1) \) gauge theory and in this respect playing the role of the role of \( \langle \phi(x') \phi(x) \rangle_{n.c.H.} \) in eq. (1).

In the models previously considered there were only dynamical charges or monopoles. Let us consider the changes needed in models where dynamical charges and monopoles coexist, like the compact Higgs model. In this model, the Dirac surfaces, \( S \), swept by the Dirac strings of monopoles are described by integer-valued surface currents, \( n^{\mu\rho} \) Hodge dual to \( S \). A change of Dirac surfaces, \( S \rightarrow S' \), for a fixed configuration of monopole worldlines, corresponds to the shift
\[ n^{\mu\rho} \rightarrow n^{\mu\rho} + \partial^\rho V^\mu - \partial^\mu V^\rho, \] (3)

where \( V^\mu \) is the integer current Hodge dual to the volume whose boundary is the closed surface \( S' - S \). In the partition function, the interaction of the electric currents generated by the charged particles, \( j_\mu \), with the Dirac surfaces of the monopoles is of the form
\[ ie g \int j_\mu \partial_\rho \Delta^{-1} n^{\rho\mu} \] (4)

where \( e \) is the electric charge of the matter field and \( g \) the magnetic charge of the monopole field. The change (3) induces a shift of (4) by
\[ ie g \int j_\mu V^\mu \] (5)

which when exponentiated is unity, as physically required, provided (5) is an integer multiple of \( 2\pi \) [Dirac quantization condition for fluxes]. This happens in the partition function if Dirac’s quantization condition for charges holds, i.e. \( eg = 2\pi q, q \) an integer, because \( j_\mu \) and \( V^\mu \) are integer currents. In the Dirac ansatz for the 2-point function of the charged field, however, \( j_\mu \) acquires additional Coulomb-like terms, \( E_\mu \), which are real-valued. The action then acquires a monopole-charged field interaction term
\[ eg \int E^\mu V^\mu \notin 2\pi Z \] (6)

even if \( eg \in 2\pi Z \), and the Dirac strings of monopoles become unphysically ‘visible’. An obvious cure for this inconsistency would be to replace the Coulomb field \( E^\mu_\mu \) by a “Mandelstam string” \( j_\mu^x \) [5], squeezing the entire flux of \( E^x \) into a single line from \( x \) to \( x' \) at fixed time (and adding suitable b.c.).

However, this squeezing of the flux is so strong that it produces IR divergences [with a lattice UV cutoff \( \int (E^\mu_\mu - E_\mu^x) \Delta^{-1} (E_\mu^\nu - E_\nu^x) < \infty \) but \( \int (j_\mu^x - j_\mu^x) \Delta^{-1} (j_\mu^\nu - j_\nu^x) = \infty \)].

To avoid these divergences, we propose (6) to replace a fixed Mandelstam string by a sum over fluctuating Mandelstam strings \( j_\mu^x \) weighted by a measure \( \mathcal{D}v_q(j_\mu^x) \) with the property that in the scaling limit,
\[ \int \mathcal{D}v_q(j_\mu^x) e^{i e \int j_\mu^x A_\mu} \sim e^{i e \int E_\mu V^\mu}. \] (7)

[The integer \( q \) in the measure \( \mathcal{D}v_q \) is the one appearing in the Dirac quantization condition \( eg = 2\pi q \). A measure with such property is the measure over \( Z/q \)-valued currents appearing in the Fourier representation in terms of the gauge field \( A_\mu \) of the spin correlator of a 3D gauged Villain model with period \( 2\pi q \), in the broken symmetry phase, with a point removed at infinity. This measure is supported on currents \( j_\mu^x \) associated with \( q \) paths in a 3-plane at a fixed time, starting at the site \( x \) and reaching a common point at infinity. From (7) we see that the measure \( \mathcal{D}v_q(j_\mu^x) \) is peaked at \( E_\mu^x \) at large scales.]

The 2-point correlation function for the gauge-invariant charged field in the compact Abelian Higgs (c.H.) model has then the form
\[ \int \mathcal{D}v_q(j_\mu^x) \int \mathcal{D}v_q(j_\mu^x) \langle \phi(x) \phi(x') \rangle e^{i e \int (j_\mu^x - j_\mu^x) A_\mu} \rangle_{c.H.}. \] (8)
replacing the correlator \((1)\) of the non-compact model. This definition respects Dirac’s quantization condition for fluxes and, as a consequence, it is independent of the Dirac strings of the magnetic monopoles of the compact Higgs model. In \([9]\) one finds a numerical evidence for the validity of an order parameter for the Coulomb-Higgs transition in this model, based on the above correlation function (using the formalism of effective potential).

The 2-point monopole correlation function obtained by duality from \([8]\) is given by

$$\int \mathcal{D} \nu_q(j^x_\mu) \int \mathcal{D} \nu_q(j^{x'}_\mu) \langle D(\Sigma(j^x - j^{x'} + j^{xx'})) \rangle^\prime \mu (j^x_\mu) (j^{x'}_\mu) \rangle$$

where \(j^{xx'}\) is the dual of \(\omega^{xx'}\) (see eq.(9)). Here \(D(\Sigma)\) is the ’t Hooft loop \([4]\) in the dual of the compact Higgs model \((\prime)\). The surface \(\Sigma\) has boundary given by the support of \(\mathcal{A}_\mu\) by \(2\pi q \ast \mathcal{A}_{\mu\nu}\), with \(q\) the \(Z\)-valued surface current supported on \(\Sigma\) and \(*\) the Hodge dual. Since \(j^x_\mu\) is supported on \(q\) paths, \(\Sigma\) is a \(q\)-sheet surface with the \(q\) sheets having a common boundary given by the single line support of \(j^{xx'}\).

3. Monopoles in Yang-Mills theory

We wish to export the above ideas to \(SU(2)\) lattice Yang-Mills theory, with action defined by

$$S_{YM}(U_{\mu\nu}) = -\beta \sum_{y, \mu, \nu} Tr U_{\mu\nu}(y). \quad (10)$$

where \(U_{\mu\nu}(y)\) is the Wilson plaquette with initial point the site \(y\). Firstly one remarks \([5]\) that, in an Abelian projection gauge, there appear a charged field, \(c\), of electric charge 1 and regular \(\mathbb{Z}\) monopoles with magnetic charge \(q = 4\pi\), whose condensation should be responsible for confinement, and for them Dirac’s quantization condition for charges is satisfied with \(q = 2\).

Integrating out \(c\) in the partition function one obtains an effective action \(U(1)-\) gauge invariant \(S_{U(1)}^X(\partial_\mu A_\nu)\). Since \(A\) is \(4\pi\) periodic, a Fourier expansion yields:

$$e^{-S_{U(1)}^X(\partial_\mu A_\nu)} = \int \mathcal{D} \ell^{\mu\nu} e^{\int \ell^{\mu\nu} \partial_\mu A_\nu F(\ell^{\mu\nu})} \quad (11)$$

where \(\ell^{\mu\nu}\) is an integer surface current, for a suitable functional \(F\). Integrating out \(A_\nu\) one obtains \(\partial_\mu \ell^{\mu\nu} = 0\). We replace the integer currents \(\ell^{\mu\nu}\) by a real gauge field strength \(e^{\mu\nu\sigma\rho} \partial_\mu A_\sigma\) solving the above constraint and use a Fourier representation of the integrality condition (Poisson formula) in terms of integer currents \(\rho^\mu\), describing the Abelian projection monopole worldlines. This yields a representation of the partition function of the dual theory:

$$\hat{Z} = \int \mathcal{D} \mathcal{A}_\mu F(e^{\mu\nu\sigma\rho} \partial_\mu \mathcal{A}_\sigma) \int \mathcal{D} \rho^\mu e^{i4\pi \int \rho^\mu \mathcal{A}_\mu},$$

with \(\partial_\mu \rho^\mu = 0\), by \(U(1)\) gauge invariance. The 2-point correlator for the gauge-invariant charged field in the dual model is given by:

$$\int \mathcal{D} \nu_2(j^x_\mu) \int \mathcal{D} \nu_2(j^{x'}_\mu) \langle e^{i4\pi \int (j^x - j^{x'} + j^{xx'}) \mu \mathcal{A}^\mu} \rangle$$

where \(j^x\) is a 2-path half-integer current at constant time and \(j^{xx'}\) an integer current required by flux conservation as \(\omega^{xx'}\) in eq.(2). Applying backward the duality transformation one obtains eq.(9) for \(q = 2\) and action \(S_{U(1)}^X\). Reexpressing this correlator in terms of the original \(SU(2)\) gauge field \(U\) yields:

$$\int \mathcal{D} \nu_2(j^x_\mu) \int \mathcal{D} \nu_2(j^{x'}_\mu) \langle D(\Sigma(j^x - j^{x'} + j^{xx'})) \rangle_{YM} = \langle \mathcal{M}(x) \mathcal{M}(x') \rangle_{YM}. \quad (12)$$

Here \(D(\Sigma)\) is the ’t Hooft loop in \(SU(2)\), obtained substituting in the Yang-Mills action \((10)\) \(U_{\mu\nu}\) by \(U_{\mu\nu} e^{i2\pi q_3 \Sigma_{\mu\nu}}\) and \(x, x'\) are the creation and annihilation points for the monopole field \(\mathcal{M}\). The definition in \((12)\) is independent of the choice of an Abelian projection, whereas the initial definition of monopole currents \(\rho\) was projection-dependent. Hence the position where the monopoles are created or annihilated are intrinsic to the \(SU(2)\) Yang-Mills theory, but to define the trajectories of the monopoles one needs...
an Abelian projection; these monopole do not appear to have a semiclassical limit. We propose the following criterion for confinement based on monopole condensation:

\[
\langle M(x)M(x') \rangle_{YM} \underset{|x-x'| \to \infty}{\to} c > 0. \tag{13}
\]

A justification for our criterion is based upon the following considerations:

1) since \(2\Sigma_{\mu\nu}\) is integer valued in the 't Hooft loop we can substitute

\[
e^{i2\pi \Sigma_{\mu\nu}} \to e^{i2\pi X \Sigma_{\mu\nu}} \tag{14}
\]

for any choice of \(X\) selecting an Abelian projection

2) since the measure \(D\nu_2(j^x_{\mu})\) is peaked at large scales around \(E^x_\mu\), in a mean-field approximation with respect to \(\int D\nu_2(j^x_{\mu}) \int D\nu_2(j^{x'}_{\mu})\) we have in the scaling limit

\[
\langle \Sigma_{\mu\nu} \rangle_{j^x_{\mu}, j^{x'}_{\mu}} \simeq \partial^\rho \Delta^{-1} B^{xx'}_{\mu\nu}. \tag{15}
\]

Hence in the above mean field we have:

\[
\langle M(x)M(x') \rangle_{YM} \simeq \langle e^{-[S_M(U_{\mu\nu}) + i2\pi \rho] - 1 \Sigma_{\mu\nu} \delta^{xx'}_{\mu\nu}} - S_M(U_{\mu\nu}) \rangle_{YM}
\]

\[
= \langle M_{MF}(x)M_{MF}(x') \rangle_{YM}. \tag{15}
\]

This is the order parameter proposed by Di Giacomo et al. [10] (see also [11] for a variant). The “Mean-Field” v.e.v. \(\langle M_{MF} \rangle_{YM}\) is numerically a good order parameter for the confinement-deconfinement transition [12]. Nevertheless the definition [15] is inconsistent in the treatment of small scales because it violates Dirac’s quantization for fluxes and therefore it depends on the choice of Dirac strings [at order \(\epsilon^2\), \(\epsilon\) lattice spacing, assuming good continuum limit for the fields of the Abelian projection]. However in our approach it is simply the Mean-Field of a correlator well defined even at small scales, strictly independent of Dirac strings and Abelian projection.

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