NONLINEAR STABILITY OF RAREFACTION WAVES FOR A HYPERBOLIC SYSTEM WITH CATTANEO’S LAW

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Dedicated to Professor Shuxing Chen on the occasion of his 80th birthday

Abstract. This paper is concerned with the time-asymptotically nonlinear stability of rarefaction waves to the Cauchy problem and the initial-boundary value problem in the half space with impermeable wall boundary condition for a scalar conservation laws with an artificial heat flux satisfying Cattaneo’s law. In our results, although the $L^2 \cap L^\infty$-norm of the initial perturbation is assumed to be small, the $H^1$-norm of the first order derivative of the initial perturbation with respect to the spatial variable can indeed be large. Moreover the far fields of the artificial heat flux can be different. Our analysis is based on the $L^2$ energy method.

1. Introduction. In this paper, we consider the precise description of large time behaviors of solutions to a scalar conservation laws with an artificial heat flux:

$$u_t + f(u)_x + q_x = 0 \quad (1.1)$$

over an open interval $\Omega \subset \mathbb{R}$. Here $t \geq 0$, $x \in \Omega$, $u(t,x) \in \mathbb{R}$ is an unknown function, $f(u) \in \mathbb{R}$ is a flux function which is assumed to be sufficiently smooth strictly convex function and $q = q(t,x) \in \mathbb{R}$ is an artificial heat flux which satisfies one of the following two types of relations:

- Fourier’s laws:

$$q_x = -\mu u_x, \quad (1.2)$$

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• Cattaneo’s law (cf. [1,17,19,22]):

\[
\kappa q_t + \mu u_x + q = 0,
\]

where \(\kappa > 0\) is a relaxation time and \(\mu > 0\) is viscosity coefficient.

In the rest of this paper, we will focus on the case when \(q(t,x)\) satisfies the Cattaneo’s law (1.3) and without loss of generality, we assume in the rest of this paper that \(\kappa = \mu = 1\).

As for \(\Omega\), two cases will be discussed in this paper:

• The first case is \(\Omega = \mathbb{R}\) and in such a case, we supplement the equations (1.1), (1.3) with the initial condition

\[
(u(0,x), q(0,x)) = (u_0(x), q_0(x)), \quad x \in \mathbb{R},
\]

where \(u_\pm\) and \(q_\pm\) are given constants;

• The second case is \(\Omega = \mathbb{R}^+ = (0, \infty)\) and for the initial-boundary value problem in the half space \(\mathbb{R}^+\), we prescribe the boundary and initial conditions as follows:

\[
u(t,0) = u_b, \quad t > 0,
\]

\[
(u(0,x), q(0,x)) = (u_0(x), q_0(x)), \quad x \geq 0,
\]

where the boundary datum \(u_b\) is the constant state determined uniquely by

\[
f'(u_b) = 0
\]

satisfies the compatibility condition \(u_b = u_0(0)\).

It is well-known, cf. [13] and the references cited therein, that the large time behaviors of the unique global solution \((u(t,x), q(t,x))\) of the Cauchy problem (1.1), (1.3), (1.4) can be completely determined by the unique global entropy solution \(u_R(t,x)\) of the resulting Riemann problem of the following scalar conservation laws

\[
u_t + f(u)_x = 0, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R},
\]

where \(u = u_0(x)\) is the initial data.

Under the assumption that \(f(u)\) is a sufficiently smooth convex function, one can get that if \(u_- < u_+\), the Riemann problem (1.6) admits a unique global rarefaction wave solution \(u^r(t,x)\) given by (cf. [24])

\[
u^r(t,x) = \begin{cases} 
  u_-, & x \leq f'(u_-)t, \\
  (f')^{-1}(\frac{x}{t}), & f'(u_-)t \leq x \leq f'(u_+)t, \\
  u_+, & x \geq f'(u_+)t,
\end{cases}
\]

while if \(u_- > u_+\), (1.6) admits a unique global shock wave solution \(u^s(t,x)\) given by

\[
u^s(t,x) = \begin{cases} 
  u_-, & x < st, \\
  u_+, & x > st
\end{cases}
\]

with \(s = \frac{f(u_-) - f(u_+)}{u_- - u_+}\). Then the large time behaviors of the global solutions \((u(t,x), q(t,x))\) of the Cauchy problem (1.1), (1.3), (1.4) is expected to be described by \((u^r(t,x), 0)\) for \(u_- < u_+\) or \((u^s\text{SW}(x-st), 0)\) with a suitable shift for \(u_- > u_+\).
Here $u^{VSW}(x - st)$ is a travelling wave solution of the scalar viscous conservation laws

$$-su_V^{VSW} + f(u^{VSW})_x = u^{VSW}_x, \quad \xi = x - st \in \mathbb{R},$$

$$\lim_{\xi \to \pm \infty} u^{VSW}(\xi) = u_{\pm}.$$

For the initial-boundary value problem (1.1), (1.3), (1.5), to describe its large time behavior, generally speaking, in addition to the rarefaction waves $u^r(t, x)$ and viscous shock profiles $u^{VSW}(x - st)$, another type of nonlinear waves, i.e. the so-called boundary layer $u^{BL}(x)$ satisfying

$$\frac{df(u^{BL}(x))}{dx} = \frac{d^2 u^{BL}(x)}{dx^2}, \quad x > 0,$$

$$u^{BL}(0) = u_b, \quad \lim_{x \to +\infty} u^{BL}(x) = u_+,$$  \hspace{1cm} (1.8)

should be introduced which is due to the occurrence of the boundary condition (1.5). Former results available up to now on the rigorously mathematical justifications of the above expectations can be summarized as follows:

- When $u_- < u_+$, the nonlinear stability of rarefaction waves $u^r(t, x)$ for both the Cauchy problem (1.1), (1.3), (1.4) and the initial-boundary value problem (1.1), (1.3), (1.5) is studied in [17];
- When $u_- > u_+$, the nonlinear stability of viscous shock profiles $u^{VSW}(x - st)$ under suitable shift for the Cauchy problem (1.1), (1.3), (1.4) is treated in [19].

In the above two results, both the initial perturbations and the strengths of the rarefaction waves $u^r(t, x)$ and the viscous shock profiles $u^{VSW}(x - st)$ are asked to be small and the far fields of the artificial heat flux, i.e. $q_{\pm}$ for the Cauchy problem (1.1), (1.3), (1.4) (or $q_+$ for the initial-boundary value problem (1.1), (1.3), (1.5)), are assumed to satisfy $q_- = q_+ = 0$ for the Cauchy problem (1.1), (1.3), (1.4) and $q_+ = 0$ for the initial-boundary value problem (1.1), (1.3), (1.5).

This paper focuses on the case $u_- < u_+$ for both the Cauchy problem (1.1), (1.3), (1.4) and the initial-boundary value problem (1.1), (1.3), (1.5) and our main purpose is two-fold:

- Try to relax the smallness conditions imposed in [17] on the initial perturbation;
- To deal with the case when the far-fields of the artificial heat flux are not equal to zero.

Before stating our main result, we first point out that, to simplify the presentation, we will assume in the rest of this paper that $f(u) = \frac{1}{2}u^2$. Such a simplification is without loss of generality since similar result still holds provided that $\frac{df(u)}{du}$ are assumed to satisfy certain polynomial-order growth conditions for large $|u|$ and for $i = 0, 1, 2$. In such a case, we have $u_b = 0$ and

$$u^r(t, x) = \begin{cases} 
    u_-, & x \leq u_- t, \\
    \frac{x}{t}, & u_- t \leq x \leq u_+ t, \\
    u_+, & x \geq u_+ t,
\end{cases}$$  \hspace{1cm} (1.9)
Now we turn to state our result for the Cauchy problem (1.1), (1.3), (1.4). To do so, for some positive constant $\epsilon > 0$ and any given $m(x) \in C_0^\infty(\mathbb{R})$ satisfying, cf. [3]
\[
\int_{-\infty}^{+\infty} m(x)dx = 1,
\]
if we define $\bar{q}(t, x)$ by
\[
\bar{q}(t, x) = e^{-t} \left[ q_+ + (q_+ - q_-) \int_{-\infty}^{x} m(y)dy \right],
\]
then it is easy to see that $\bar{q}(x, t)$ solves
\[
\bar{q}_t + \bar{q} = 0, \quad \lim_{x \to \pm \infty} \bar{q}(0, x) = q_{\pm}.
\]
With the above preparations in hand, we have the following result for the Cauchy problem (1.1), (1.3), (1.4)

**Theorem 1.1** (Stability in the whole space). Let $\epsilon = \delta$ in (1.10) with $\delta := u_+ - u_- > 0$ being the strength of the rarefaction waves $u^r(t, x)$ given by (1.9) and assume that the initial data $(u_0(x), q_0(x))$ satisfies
\[
\|u_0(x) - u_0^r(x)\|_{L^2(\mathbb{R})} , \quad \|u_0(x)\|_{H^1(\mathbb{R})} , \quad \|q_0(x) - \bar{q}(0, x)\|_{H^2(\mathbb{R})},
\]
then, there exists a sufficiently small positive constant $\varepsilon_0$ such that if
\[
\|(u_0(x) - u_0^r(x), q_0(x) - \bar{q}(0, \cdot))\|_{L^2(\mathbb{R})}^{\frac{3}{2}} \left( 1 + \|u_0'(x), q_0'(x))\|_{H^1(\mathbb{R})}^2 \right) \leq \varepsilon_0,
\]
then the Cauchy problem (1.1), (1.3), (1.4) admits a unique global solution $(u(t, x), q(t, x))$ satisfying
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |(u(t, x) - u^r(t, x), q(t, x) - \bar{q}(t, x))| = 0.
\]

Now we turn to the initial-boundary value problem (1.1), (1.3), (1.5). In such a case, we have $\delta = u_+ > 0$, $u_b = 0$ and the corresponding rarefaction wave $u^r(t, x)$ can be given explicitly as
\[
\begin{cases}
\frac{x}{t}, & 0 \leq x \leq u_+ t, \\
u_+, & x \geq u_+ t.
\end{cases}
\]
It’s easy to see that $u^r(t, x)$ is compatible with the boundary condition, namely, $u^r(t, 0) = u_b = 0$ for $t > 0$. Meanwhile, $\bar{q}(t, x)$ can be constructed as
\[
\bar{q}(t, x) = e^{-t} q_+ \int_{0}^{x} m(y)dy,
\]
where $\epsilon > 0$ is a constant and $m(x) \in C_0^\infty(\mathbb{R}^+)$ satisfying
\[
\int_{0}^{+\infty} m(x)dx = 1.
\]
Similar to the Cauchy problem case, we can get that
\[
\bar{q}_t + \bar{q} = 0, \quad \bar{q}(t, 0) = 0, \quad \lim_{x \to +\infty} \bar{q}(0, x) = q_+.
\]
Moreover, for the half space case, we need to consider the corresponding compatibility condition. If we assume that \((u(t, x), q(t, x))\) is a \(C^1\)–solution of the initial-boundary value problem \((1.1), (1.3), (1.5)\) over \((t, x) \in [0, +\infty) \times [0, +\infty)\), then the initial data \((u_0(x), q_0(x))\) should satisfies
\[
u_0(0) = 0, \quad u_1(0) = 0, \tag{1.14}
\]
where \(u_1(x) := u_t(0, x)\) with \(u_t = -(\frac{u^2}{2})_x - q_x\). That is \(u_1(x) = -u_0(x)u_0'(x) - q_0'(x)\). The condition (1.14) is called the compatibility condition up to order 1 for the initial-boundary value problem \((1.1), (1.3), (1.5)\).

Under the above preparations, we have the following result for the initial-boundary value problem \((1.1), (1.3), (1.5)\):

**Theorem 1.2** (Stability in the half space). Let \(\epsilon = \delta \) in \((1.13)\) with \(\delta := u_+ > 0\) being the strength of the rarefaction waves \(u^r(t, x)\) given by \((1.12)\) and assume that the initial data \((u_0(x), q_0(x))\) satisfies
\[
(u_0(x) - u_+, q_0(x) - q(0, x)) \in H^2(\mathbb{R}^+),
\]
and also we assume the compatibility conditions (1.14) holds. Then there exists a sufficiently small positive constant \(\epsilon_1\) such that if
\[
\| (u_0(x) - u_+, q_0(x) - q(0, x)) \|_{L^2(\mathbb{R}^+)}^\frac{1}{2} \left( 1 + \sum_{1 \leq i, j \leq 2} \| \partial_x^i \partial_t^j (u_0(x), q_0(x)) \|^2 \right) \leq \epsilon_1,
\]
\(u_+ \leq \epsilon_1, \tag{1.15}\)
the initial-boundary value problem \((1.1), (1.3), (1.5)\) has a unique global solution \((u(t, x), q(t, x))\) satisfying
\[
\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^+} | (u(t, x) - u^r(t, x), q(t, x) - q(t, x)) | = 0.
\]

**Remark 1.** Several remarks concerning Theorem 1.1 and Theorem 1.2 are given below:

- In [17], both the \(H^2\)–norm of the initial perturbation and \(\delta\), the strength of the rarefaction wave \(u^r(t, x)\), are assumed to small, while in our main result Theorem 1.1 and Theorem 1.2, although we still need to assume that \(\delta\) is small enough, the initial perturbation \((\phi_0(x), r_0(x))\) is asked to satisfy the somewhat weak smallness assumptions \((1.11)\) and \((1.15)\). As discussed in the Remark 2 immediately after Theorem 3.1, such assumptions do imply that the \(L^2 \cap L^\infty\)–norm of the initial perturbation is small, but the \(H^1\)–norm of the first order derivative of the initial perturbation with respect to the spatial variable can indeed be large;

- In our results, the far fields of the artificial heat flux, i.e. \(q_{\pm}\) for the Cauchy problem \((1.1), (1.3), (1.4)\) (or \(q_+\) for the initial-boundary value problem \((1.1), (1.3), (1.5)\)), are not necessary to be assumed to be equal to zero;

- In Theorem 1.2, \(\partial_x^i \partial_t^j (u_0(x), q_0(x)) \) \((i + j \leq 2, j \geq 1)\) is understood as
\[
\partial_x^i \partial_t^j (u_0(x), q_0(x)) = \lim_{t \to u_+} \partial_x^i \partial_t^j (u(t, x), q(t, x)),
\]
thus
\[
\partial_t u_0(x) = -u_0(x)u_0'(x) - q_0'(x),
\]
\[
\partial_x \partial_t u_0(x) = -(u_0'(x))^2 - u_0(x)u_0''(x) - q_0''(x),
\]
\[ \partial_t^2 u_0(x) = 2u_0(x)(u_0'(x))^2 + (1 + u_0^2(x))u_0''(x) + (1 + u_0'(x))q_0'(x) + u_0(x)q_0''(x), \]
\[ \partial_t q_0(x) = -q_0(x) - u_0'(x), \]
\[ \partial_x \partial_t q_0(x) = -q_0'(x) - u_0''(x), \]
\[ \partial_t^2 q_0(x) = q_0(x) + q_0'(x) + (1 + u_0'(x))u_0'(x) + u_0(x)u_0''(x): \]

- For the initial-boundary value problem (1.1), (1.3), (1.5), to describe its large time behaviors, it is unnecessary to introduce the boundary layer \( u^{BL}(x) \) solving (1.8), while for the corresponding initial-boundary value problem with inflow (i.e. \( f'(u_b) > 0 \)) or outflow (i.e. \( f'(u_b) < 0 \)) boundary conditions, such a new type of nonlinear wave should be introduced.

We now sketch the main ideas used in proving our main results. Since the Cauchy problem (1.1), (1.3), (1.4) and the initial-boundary value problem (1.1), (1.3), (1.5) are nonlinear, the main difficulty in yielding the desired global solvability results is to control the possible growth of their solutions induced by the nonlinearities of the system (1.1), (1.3) under consideration and the interactions between the rarefaction waves and the solutions themselves. Compared with [17], the new ingredients in our analysis lie in the following:

- Compared with that of [17], the first is that we use a different way to construct the smooth approximation \( U(t, x) \) of rarefaction waves \( u^r(t, x) \), cf. (2.1), which was first introduced in [15]. An advantage of such a smooth approximation is that \( \|\partial_t^k U(t)\|_{L^p} \in L^1([0, +\infty)) \) for each \( p > 1 \) and \( k \geq 2 \). Based on such a property, we can first yield a closed basic energy type estimates, cf. Lemma 4.1 and Lemma 5.1. With such an estimate in hand, we can then use the smallness of both \( \|(u_0(x) - u_+, q_0(x) - \bar{q}(0, x))\|_{L^2(\Omega)} \) and \( \delta \) to control the possible growth of the solutions of the Cauchy problem (1.1), (1.3), (1.4) and the initial-boundary value problem (1.1), (1.3), (1.5) induced by the nonlinearities of the system (1.1), (1.3) under consideration and the interactions between the rarefaction waves and the solutions themselves;

- The correcting function \( \bar{q}(t, x) \) defined by (1.10) and (1.13) are constructed to overcome the difficulty caused by the case when the far fields of the artificial heat flux are not equal to zero.

Before concluding this section, it is worth to pointing out the following two facts:

- Cattaneo’s law (1.3), compared to Fourier’s law (1.2), has been widely used to describing the finite speed of heat conduction. Some models with Cattaneo’s law was studied by Racke and his cooperators, cf. [1, 4, 22] and the references cited therein;

- Many results have been obtained on the time-asymptotically nonlinear stability of basic wave patterns to hyperbolic conservation laws with dissipative terms. To closely related to the theme of this paper, we will only review some former results closely related on the scalar conservation laws with dissipation: For the nonlinear stability of rarefaction waves, cf. [5–7, 10, 17, 18] and references cited therein; For the nonlinear stability of viscous shock profiles, cf. [2, 7, 11, 12, 16, 19–21, 26] and the references cited therein; For the nonlinear stability of boundary layer or the superpositon of boundary layers and rarefaction waves, cf. [8, 10, 27] and references cited therein.

The rest of this paper is arranged as following: In Section 2, we will firstly construct smooth approximation \( \hat{U}(t, x) \) of the rarefaction waves \( u^r(t, x) \) given by
(1.9) and (1.12), respectively, and some properties of \( U(t, x) \) will be given in this section. In Section 3, we will reformulate the problem for the whole space and for the half space, respectively, and two theorems (Theorem 3.1 and Theorem 3.2) about the reformulated problems will be given, based on which the main theorems (Theorem 1.1 and Theorem 1.2) will be proved. In Section 4 and Section 5, some a priori estimates on both cases will be obtained and Theorem 3.1 and Theorem 3.2 will be proved respectively.

**Notations.** We take \( \Omega = \mathbb{R} \) for the Cauchy problem case and \( \Omega = \mathbb{R}^+ \) for the initial-boundary value problem case. \( L^p \) and \( H^s \) denote the usual Lebesgue space and Hilbert space over \( \Omega \) with norm \( \| \cdot \|_{L^p} \) and \( \| \cdot \|_s \), respectively. For simplify, we take \( \| \cdot \| := \| \cdot \|_{L^2} \). \( c_i, C_i (i \in \mathbb{Z}_+) \) stand for some generic positive constants independent of \( \delta \). \( C \) represents some generic positive constants which may changes line to line without confusion but independent of \( \delta \). For two quantities \( A \) and \( B \), \( A \lesssim B \) means that there exists a generic \( \delta \)-independent positive constant \( C \) such that \( A \leq CB \). \( A \sim B \) if \( A \lesssim B \) and \( B \lesssim A \).

### 2. Smooth approximation of rarefaction waves

Since the rarefaction waves \( u^r(t, x) \) given by (1.9) and (1.12) are only Lipschitz continuous, as in [14, 15], we need first to construct their smooth approximation \( U(t, x) \).

For the Cauchy problem (1.1), (1.3), (1.4), \( U(t, x) \) is constructed as the smooth solution of the following Cauchy problem, cf. [15]

\[
U_t + UU_x = 0, \quad U(0, x) = U_0(x) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2} \kappa_0 \int_0^x (1 + y^2)^{-q} dy, \quad (2.1)
\]

where \( \kappa_0 \) is a constant such that \( \kappa_0 \int_0^\infty (1 + y^2)^{-q} dy = 1 \) for each \( q > \frac{3}{2} \). Some properties about the smooth approximation \( U(t, x) \) can be summarized as below:

**Lemma 2.1** ([15, 17]). Let \( u_- < u_+ \) and \( \delta = u_+ - u_- \), the Cauchy problem (2.1) has a unique smooth solution \( U(t, x) \) satisfying the following properties holds:

(i). \( \frac{\partial U(t, x)}{\partial x} > 0, \quad u_- < U(t, x) < u_+, \quad \forall x \in \mathbb{R}, \quad t \geq 0; \)

(ii). For each \( 1 \leq p \leq \infty \) and \( t \geq 0 \), there exists a positive constant \( C_{p, q} \) such that

\[
\| \partial_x U(t) \|_{L^p} \leq C_{p, q} \min \left\{ \epsilon^{1 - \frac{1}{2}} \delta^{\frac{1}{2}} (1 + t)^{-(1 - \frac{1}{2})} \right\};
\]

(iii). \( \| \partial_t^k U(t) \|_{L^p} \leq C_{p, q} \min \{ \epsilon^{k - \frac{1}{2}} \delta, \epsilon^{(k - 1 - \frac{1}{2})(1 - \frac{1}{2}) + \frac{1}{2p} \frac{1}{2} - \frac{q - 1}{q} \frac{1}{2} (1 + t)^{-(1 - \frac{1}{2}) - \frac{q - 1}{q}} \} \) holds for \( t \geq 0, \quad 1 \leq p \leq \infty, \quad k = 2, 3, 4 \) and some positive constant \( C_{p, q} \);

(iv). \( \lim_{t \to +\infty} \sup_{x \in \mathbb{R}} |U(t, x) - u^r(t, x)| = 0 \), where \( u^r(t, x) \) is defined in (1.9).

For the initial-boundary value problem (1.1), (1.3), (1.5), we take \( u_- = -u_+ \) in (2.1), then the corresponding initial condition is

\[
U(0, x) = U_0(x) = u_+ \kappa_0 \int_0^x (1 + y^2)^{-q} dy,
\]

and, corresponding to Lemma 2.1, we can get that the resulting smooth approximation \( U(t, x) \) of the rarefaction wave \( u^r(t, x) \) defined by (1.12) in the half space that:

**Lemma 2.2** ([14, 17]). Let \( 0 < u_+, \quad u_- = -u_+ \) and \( \delta = u_+ \), the Cauchy problem (2.1) has a unique smooth solution \( U(t, x) \) and the following properties holds:
Here $\Omega = \mathbb{R}$, then in the same interval such that $\epsilon$ theorems about the reformulated problems will be stated. In the rest of this paper, problems for the whole space case and the half space case respectively. And two

3. Reformulation of the problems. In this section, we will reformulate the problems for the whole space case and the half space case respectively. And two theorems about the reformulated problems will be stated. In the rest of this paper, we take $\epsilon = \delta$.

Let

\begin{equation}
\phi(t, x) = u(t, x) - U(t, x), \quad r(t, x) = q(t, x) - \bar{q}(t, x) + U_x(t, x),
\end{equation}

then it is easy to see that $(\phi(t, x), r(t, x))$ satisfies

\begin{align*}
\phi_t + \left( U + \phi \right)^2 - U^2 \quad &+ \bar{q}_x + r_x = U_{xx}, \quad t > 0, \ x \in \Omega, \\
\phi_t + r + \phi_x = -(U_x^2 + UU_{xx}), \quad t > 0, \ x \in \Omega.
\end{align*}

Here $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}^+$ and recall that $f(u) = \frac{1}{2} u^2$.

For the Cauchy problem (1.1), (1.3), (1.4), the initial data is taken as

\begin{equation}
(\phi(0, x), r(0, x)) = (\phi_0(x), r_0(x)), \quad x \in \mathbb{R},
\end{equation}

\begin{equation}
\lim_{|x| \to +\infty} (\phi_0(x), r_0(x)) = (0, 0).
\end{equation}

For the Cauchy problem (3.2), (3.3), we have the following result:

**Theorem 3.1** (Global existence and stability in the whole space). Assume the initial data $(\phi_0(x), r_0(x)) \in H^2(\mathbb{R})$ and there exists a sufficiently small constant $\varepsilon'_0$ such that

\begin{equation}
\| (\phi_0, r_0) \|_{2} \left( 1 + \| (\phi_0, r_0) \|_{2}^2 \right) < \varepsilon'_0, \quad 0 < \delta < \varepsilon'_0,
\end{equation}

A brief proof of Lemmas 2.1 and 2.2 will be given in Appendix.
then the Cauchy problem (3.2), (3.3) has a unique global solution \((\phi(t,x), r(t,x))\) satisfying
\[
(\phi(t,x), r(t,x)) \in C([0, \infty); H^2(\mathbb{R})) \cap C^1([0, \infty); H^1(\mathbb{R})),
\phi_x(t,x) \in L^2([0, \infty); H^1(\mathbb{R})),
r(t,x) \in L^2([0, \infty); H^2(\mathbb{R})).
\]
Furthermore, we have
\[
\|(\phi(t), r(t))\|_2^2 + \int_0^\infty \left( \|r(\tau)\|_2^2 + \|\sqrt{U_x(\tau)}\phi(\tau)\|^2 + \|\phi_x(\tau)\|^2 \right) d\tau 
\leq \|(\phi_0, r_0)\|_2^2 + \delta^{1-\frac{1}{s}}.
\tag{3.5}
\]

Remark 2. At first glance, it seems that the smallness condition (3.4) only ask that \(|\|(\phi_0, r_0)\|)| is small, but from Sobolev’s inequality
\[
\|(\phi_0, r_0)\|_{L^\infty(\mathbb{R})} \leq \|(\phi_0, r_0)\|_2^\frac{s}{2} \|(\phi'_0, r'_0)\|_2^\frac{1}{2} \leq \|(\phi_0, r_0)\|_2^\frac{s}{2} \|(\phi'_0, r'_0)\|_2^\frac{1}{2},
\]
one can easily deduce that \(|\|(\phi_0, r_0)\|_{L^\infty(\mathbb{R})}| is also small. Even so, one can easily construct initial perturbation \((\phi_0(x), r_0(x))\) such that \(|\|(\phi'_0, r'_0)\|_{H^1(\mathbb{R})}| can indeed be large while the assumption (3.4) holds. It would interesting to see whether the smallness assumption (3.4) can be relaxed further or not.

With Theorem 3.1 in hand, now we prove the Theorem 1.1. And Theorem 3.1 will be proved in Section 4.

By direct calculations, we can get from (3.1), (2.12) and Lemma 2.1 that
\[
\|(\phi_0, r_0)\| \leq \|(u_0 - u_{r0}, q_0 - \bar{q}(0, \cdot))\| + \|(u_{r0} - U_0, U_{0x})\|
\leq \|(u_0 - u_{r0}, q_0 - \bar{q}(0, \cdot))\| + C\delta^\frac{1}{2},
\]
\[
\|(\phi_{0x}, r_{0x})\|_1 \leq \|(u_{0x}, q_{0x})\|_1 + \|(U_{0x}, \bar{q}_x(0, x) - U_{0x})\| \leq \|(u_{0x}, q_{0x})\|_1 + C\delta^\frac{1}{2},
\]
consequently
\[
\|(\phi_0, r_0)\|_2^\frac{s}{2} \left( 1 + \|(\phi_0, r_0)\|_2^\frac{1}{2} \right)
\lesssim \left( \|(u_0 - u_{r0}, q_0 - \bar{q}(0, \cdot))\| + \delta^\frac{1}{2} \right)^\frac{s}{2} \left( \|(u_0 - u_{r0}, q_0 - \bar{q}(0, \cdot))\| + \|(u_{0x}, q_{0x})\|_1 + 1 \right)^2.
\]

Thus under the assumption (1.11) imposed on the initial data \((u_0(x), q_0(x))\) and the strength of the rarefaction waves \(u^r(t, x)\) defined by (1.9), we can deduce that the assumption (3.4) holds and consequently by applying Theorem 3.1, the Cauchy problem (3.2), (3.3) admits a unique global solution \((\phi(t,x), r(t,x))\) and if we let \((u(t,x), q(t,x)) := (\phi(t,x) + U(t,x), r(t,x) + \bar{q}(t,x) - U_x(t,x))\), it is easy to see that \((u(t,x), q(t,x))\) is a solution of the Cauchy problem (1.1), (1.3), (1.4) which satisfies the estimate (3.5).

With the estimate (3.5), we can deduce by repeating the argument used in [17] that
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(\phi(t,x), r(t,x))| = 0,
\]
from which and Lemma 2.1, we have
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(u(t,x) - u^r(t,x), q(t,x) - \bar{q}(t,x))| 
\leq \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(U(t,x) - u^r(t,x), -U_x(t,x))| + \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |(\phi(t,x), r(t,x))| = 0.
\]
Thus we complete the proof of Theorem 1.1.

For the initial-boundary value problem (1.1), (1.3), (1.5) in the half space \( \mathbb{R}^+ \), the boundary condition and initial data imposed on \((\phi(t, x), r(t, x))\) are

\[
\phi(t, 0) = 0, \quad t > 0, \\
(\phi(0, x), r(0, x)) = (\phi_0(x), r_0(x)), \quad x \geq 0.
\]

Here \(\phi_0(x) = u_0(x) - U_0(x), r_0(x) = q_0(x) - \bar{q}(0, x) + U_0x(x)\), the associated compatibility conditions are

\[
\phi_0(0) = 0, \quad \phi_1(0) = 0,
\]

where \(\phi_1(x) := \phi_1(0, x)\), which is similar to (1.14). From (3.2) we have

\[
\phi_1(x) = - \left( \frac{(U_0(x) + \phi_0(x))^2 - U_0(x)^2}{2} \right) x - (\bar{q}(0, x))_x - (r_0(x))_x + (U_0(x))_{xx}.
\]

For the reformulated problem (3.2), (3.6), we have

**Theorem 3.2** (Global existence and stability in the half space). *Assume the compatibility condition (3.7) holds, the initial data \((\phi_0(x), r_0(x))\) \(\in H^2(\mathbb{R}^+)\) and \(\delta > 0\) is small enough such that

\[
\left( \sum_{0 \leq i+j \leq 2} \left\| \partial^i_x \partial^j_t (\phi_0, r_0) \right\|^2 + 1 \right) x \left( \delta^{\frac{1}{2} - \frac{i}{2}} + \left\| (\phi_0, r_0) \right\|_{2}^{\frac{1}{2}} \right) \ll 1. \quad (3.8)
\]

Then the initial boundary value problem (3.2), (3.6) has a unique global solution \((\phi(t, x), r(t, x))\) satisfies

\[
(\phi(t, x), r(t, x)) \in \bigcap_{j=0}^{2} C^j \left( [0, \infty); H^{2-j}(\mathbb{R}^+) \right), \quad r(t, x) \in L^2 \left( [0, \infty); H^2(\mathbb{R}^+) \right),
\]

\[
\partial_x \phi(t, x) \in L^2 \left( [0, \infty); H^1(\mathbb{R}^+) \right), \quad \partial_t^j (\phi(t, x), r(t, x)) \in L^2 \left( [0, \infty); H^{2-j}(\mathbb{R}^+) \right), \quad j = 1, 2.
\]

Moreover, we have

\[
\sum_{j=0}^{2} \left\| \partial_t^j (\phi(t), r(t)) \right\|_{2-j}^2 \\
+ \int_{0}^{\infty} \left( \| r(r) \|^2 + \| U_x(r) \phi(r) \|^2 + \| \phi_x(r) \|^2 \right) + \sum_{j=1}^{2} \left\| \partial_t^j (\phi(r), r(r)) \right\|_{2-j}^2 \quad d\tau
\lesssim \sum_{j=0}^{2} \left\| \partial_t^j (\phi_0, r_0) \right\|_{2-j}^2 + \delta^{1 - \frac{i}{4}}.
\]

Having obtained Theorem 3.2, Theorem 1.2 can be proved as in Theorem 1.1, we omit the details for brevity.

4. Proof of Theorem 3.1. In this section, we devote our efforts to prove the Theorem 3.1. For this purpose, for some positive constant \(T > 0\), we first define the following set of functions \( X(T) \) for which we seek the solution \((\phi(t, x), r(t, x))\) of the Cauchy problem (3.2), (3.3):

\[
X(T) := \left\{ (\phi(t, x), r(t, x)) \mid (\phi(t, x), r(t, x)) \in C([0, T]; H^2(\mathbb{R})) \cap C^1([0, T]; H^1(\mathbb{R})), \right. \\
\left. \phi_x(t, x) \in L^2([0, T]; H^1(\mathbb{R})), \ r(t, x) \in L^2([0, T]; H^2(\mathbb{R})) \right\}.
\]
For \((\phi_0(x), r_0(x)) \in H^2(\mathbb{R})\), since (3.2) is a symmetric hyperbolic system, we can get from the well-established result on the local solvability of the Cauchy problem of symmetric hyperbolic systems that there exists a sufficiently small positive constant \(t_1\) depending on \(\|\phi_0, r_0\|_2\) such that the Cauchy problem (3.2), (3.3) admits a unique solution \((\phi(t, x), r(t, x)) \in X(t_1)\).

Now suppose that such a solution has been extended to the time step \(t = T \geq t_1\) and satisfies \((\phi(t, x), r(t, x)) \in X(T)\), then to prove Theorem 3.1, we only need to deduce the following a priori estimates:

**Proposition 1** (A priori estimate for whole space). Suppose that \((\phi(t, x), r(t, x)) \in X(T)\) is a solution of the Cauchy problem (3.2), (3.3) defined on \([0, T] \times \mathbb{R}\), if the assumption (3.4) holds, then \((\phi(t, x), r(t, x))\) satisfies the following estimates

\[
\|\phi(t), r(t)\|^2_2 + \int_0^T \left(\|r(\tau)\|^2_2 + \|\sqrt{U_x(\tau)}\phi(\tau)\|^2 + \|\phi_x(\tau)\|^2_2\right) d\tau
\leq \|\phi_0, r_0\|^2_2 + \delta^{1 - \frac{1}{2}}, \quad 0 \leq t \leq T.
\]

Having obtained Proposition 1, Theorem 3.1 can be proved by the standard continuation argument which is based on the local existence result combined with the a priori estimates deduced in Proposition 1. We omit the details for brevity.

Now we turn to prove the Proposition 1. The proof of Proposition 1 consists of the following several lemmas. Firstly, we show the basic energy estimate as below.

**Lemma 4.1.** Suppose that \((\phi(t, x), r(t, x)) \in X(T)\) is a solution of the Cauchy problem (3.2), (3.3) defined on \([0, T] \times \mathbb{R}\), we have for all \(0 \leq t \leq T\) that

\[
\|\phi(t), r(t)\|^2_2 + \int_0^T \left(\|\sqrt{U_x(\tau)}\phi(\tau)\|^2 + \|r(\tau)\|^2\right) d\tau \leq \|\phi_0, r_0\|^2_2 + \delta^{1 - \frac{1}{2}}.
\]

**Proof.** Multiply (3.2)_1, (3.2)_2 by \(\phi\) and \(r\) respectively, and add the results together to get

\[
\frac{1}{2} (\phi^2 + r^2)_t + \left[\frac{1}{3} \phi^3 + \frac{1}{2} U \phi^2 + \phi r\right]_x + r^2 + \frac{1}{2} U_x \phi^2 = U_{xx} \phi - (U^2)_x r - U U_{xx} r - \delta e^{-\tau} (q_+ - q_-) m(\delta x) \phi.
\]

Noticing that

\[
\int_0^t \int_\mathbb{R} U_{xx} \phi \, dx \, d\tau \leq \int_0^t \|\phi\| \|U_{xx}\| \, d\tau \leq \int_0^t \delta^{\frac{1}{2}} - \frac{1}{4} (1 + \tau)^{-\frac{1}{4}} \|\phi\| d\tau
\]

\[
\leq \int_0^t (1 + \tau)^{-\frac{1}{4}} \|\phi\|^2 d\tau + \delta^{1 - \frac{1}{2}}, \quad \text{(4.2)}
\]

\[
\int_0^t \int_\mathbb{R} (U_{xx} r + U U_{xx} r) \, dx \, d\tau \leq \frac{1}{2} \int_0^t \|r\|^2 \, d\tau + \int_0^t \left(\|U_{xx}\|_{L^2}^2 + \|U_x\|_{L^4}^4\right) \, d\tau
\]

\[
\leq \frac{1}{2} \int_0^t \|r\|^2 d\tau + \delta^{1 - \frac{1}{2}} + \delta \int_0^t (1 + \tau)^{-3} d\tau \quad \text{(4.3)}
\]

and

\[
\int_0^t \int_\mathbb{R} \delta e^{-\tau} (q_+ - q_-) m(\delta x) \phi \, dx \, d\tau \leq \delta^{\frac{1}{2}} \int_0^t \|\phi\| \|m\| e^{-\tau} \, d\tau \leq \int_0^t \|\phi\|^2 e^{-\tau} \, d\tau + \delta.
\]

(4.4)
Thus Lemma 4.1 can be proved by integrating (4.1) with respect to $t$ and $x$ over $[0, t] \times \mathbb{R}$, by making use of the estimates (4.2), (4.3), (4.4) and Gronwall’s inequality.

For the first order energy type estimates, we can get that

**Lemma 4.2.** Under the conditions listed in Lemma 4.1, we have for $0 \leq t \leq T$ that

\[
\|(\phi(t), r(t))\|^2 + \int_0^t \|r(\tau)\|^2 d\tau + \int_0^t \int_{\mathbb{R}} U_x(\tau, x) \left( \phi_x^2(\tau, x) + \phi_x(\tau, x)^2 \right) dx d\tau \\
\lesssim \|(\phi_0, r_0)\|^2 + \delta^{\frac{1}{4}} + \delta^2 - \frac{1}{\tau} \int_0^t \|\phi_x(\tau)\|^2 d\tau + N(T) \int_0^t (\|\phi_{xx}(\tau)\|^2 + \|\phi_x(\tau)\|^2) d\tau.
\]

Here $N(T) := \sup_{0 \leq \tau \leq T} \{\|(\phi(t), r(t))\|\}$ and we can get from Lemma 4.1 that

\[
N(T) \lesssim \|(\phi_0, r_0)\| + \delta^{\frac{1}{2}} - \frac{1}{\tau}.
\]

**Proof.** Differentiating (3.2) _1 and (3.2) _2 with respect to $x$ once, and then multiplying the results by $\phi_x$ and $r_x$ respectively, we have

\[
\frac{1}{2} \left( \phi_x^2 + r_x^2 \right)_t + \left[ \frac{1}{2} u \phi_x^2 + U_x \phi_x + \phi_x r_x \right]_x + r_x^2 + \frac{1}{2} U_x \phi_x^2 \\
= U_x \phi_{xx} - \frac{1}{2} \phi_x^2 + U_{xxx} \phi_x - (3U_x U_{xx} + U U_{xxx}) r_x - \delta^2 e^{-t}(q_+ - q_-) m'(\delta x) \phi_x.
\]

Integrating the above identity with respect to $t$ and $x$ over $[0, t] \times \mathbb{R}$ yields

\[
\|(\phi_x(t), r_x(t))\|^2 + \int_0^t \|r_x(\tau)\|^2 d\tau + \int_0^t \int_{\mathbb{R}} U_x(\tau, x) \phi_x^2(\tau, x) dx d\tau \\
= 2 \int_0^t \int_{\mathbb{R}} \left( U_x \phi_{xx} - \frac{1}{2} \phi_x^2 + U_{xxx} \phi_x \\
- (3U_x U_{xx} + U U_{xxx}) r_x - \delta^2 e^{-t}(q_+ - q_-) m'(\delta x) \phi_x \right) dx d\tau.
\]

Now we turn to estimate the terms in the right hand side of (4.6) term by term. For the first term, due to

\[
U_x \phi_{xx} = \left( U_x \phi_x - \frac{1}{2} U_{xx} \phi^2 \right)_x + \frac{1}{2} U_{xxx} \phi^2 - U_x \phi_x^2,
\]

noticing that $-U_x \phi_x^2$ is a good term and since

\[
\int_0^t \int_{\mathbb{R}} U_{xxx} \phi^2 dx d\tau \lesssim \int_0^t \|\phi\|_{L^\infty} \|U_{xxx}\| d\tau \lesssim \delta^3 - \frac{1}{\tau} \int_0^t \|\phi\| \|\phi_x\| \left( 1 + \tau \right)^{-1 - \frac{1}{4\tau}} d\tau \\
\lesssim \delta^2 - \frac{1}{\tau} \int_0^t \|\phi_x\|^2 + (1 + \tau)^{-\frac{3}{2}(1 + \frac{1}{\tau})} \|\phi\|^2 d\tau,
\]

we can get that

\[
2 \int_0^t \int_{\mathbb{R}} U_x \phi_{xx} dx d\tau \\
\lesssim -2 \int_0^t \int_{\mathbb{R}} U_x \phi_x^2 dx d\tau + \delta^2 - \frac{1}{\tau} \int_0^t \|\phi_x\|^2 + (1 + \tau)^{-\frac{3}{2}(1 + \frac{1}{\tau})} \|\phi\|^2 d\tau.
\]
The second term in the right hand side of (4.6) can be estimated as follows:
\[
- \int_0^t \int_{\mathbb{R}} \phi_x^2 dx d\tau \lesssim \int_0^t \|\phi_x\| \|\phi_x\|^2 d\tau \lesssim \int_0^t \|\phi_x\|^\frac{3}{2} \|\phi_{xx}\|^\frac{1}{2} d\tau \quad (4.8)
\]
and similarly the other terms in the right hand side of (4.6) can be estimated as in the following:
\[
2 \int_0^t \int_{\mathbb{R}} \phi_{xxx} \phi_x dx d\tau
\lesssim \int_0^t \|\phi_x\| \|U_{xxx}\| d\tau \lesssim \delta^2 \int_0^t \|\phi_x\|(1 + \tau)^{-1} \frac{1}{2} d\tau
\lesssim \delta^2 \int_0^t \|\phi_x\|^2 (1 + \tau)^{-1} \frac{1}{2} d\tau + \delta^2 \int_0^t \|\phi_x\|^2 d\tau
\]
\[
-2 \int_0^t \int_{\mathbb{R}} (3U_x U_{xx} + U U_{xxx}) r_x dx d\tau
\lesssim \frac{1}{2} \int_0^t \|r_x\|^2 d\tau + \int_0^t \left( \|U_{xxx}\|^2 + \|U_x\|^2 \|U_{xx}\|^2 \right) d\tau \lesssim \frac{1}{2} \int_0^t \|r_x\|^2 d\tau + \delta^3 \frac{1}{2},
\]
\[
-2 \delta^2 \int_0^t \int_{\mathbb{R}} e^{-\tau} (q_+ - q_-) m'(\delta x) \phi_x dx d\tau
\lesssim \delta^2 \int_0^t e^{-\tau} \|\phi_x\|^2 d\tau + \delta^2.
\]
Substituting the estimates (4.7),(4.8),(4.9) into (4.1), we have by employing Gronwall’s inequality that
\[
\|(\phi_x, r_x)\|^2 + \int_0^t \|r_x\|^2 d\tau + \int_0^t \int_{\mathbb{R}} U_x \phi_x^2 dx d\tau
\lesssim \|(\phi_0, r_0)\|^2 + N(T) \int_0^t (\|\phi_{xx}\|^2 + \|\phi_x\|^2) d\tau + \delta^3 \int_0^t \|\phi_x\|^2 d\tau \quad (4.10)
\]
Combining the estimate (4.10) with the result obtained in Lemma 4.1, we can prove Lemma 4.2 immediately by using Gronwall’s inequality again.

Now we turn to deal with the term \(\int_0^t \|\phi_x\|^2 d\tau\) and we can get that

**Lemma 4.3.** Under the conditions listed in Lemma 4.1, if we assume further that \(\|(\phi_0, r_0)\|\) and \(\delta\) are sufficiently small, we can get for \(0 \leq t \leq T\) that
\[
\|(\phi(t), r(t))\|^2 + \int_0^t \left( \|r(\tau)\|^2 + \|\phi_x(\tau)\|^2 \right) d\tau \lesssim \|(\phi_0, r_0)\|^2 + \delta^3 + N(T) \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau.
\]

**Proof.** Multiplying (3.2)\(_1\) and (3.2)\(_2\) by \(-r_x\) and \(\phi_x\) respectively, adding the results together and noticing \(r_t \phi_x = (r\phi_x)_t - (r\phi)_x + r_x \phi_t\), we have
\[
\phi_x^2 - (r\phi_t)_x = r_x^2 - (r\phi_x)_t + \phi_x r_x + U_x \phi r_x + U \phi_x r_x - r \phi_x
\]
\[
- (U_x^2 + U U_{xx}) \phi_x - U_{xx} r_x - \delta e^{-t}(q_+ - q_-) m(\delta x) r_x. \quad (4.11)
\]
Since
\[
\int_0^t \int_\mathbb{R} (r \phi_x) dx \, d\tau = \int_\mathbb{R} r \phi_x - r_0 \phi_0 x dx \lesssim \| (r, \phi_x) \|^2 + \| (r_0, \phi_0 x) \|^2,
\]
\[
\int_0^t \int_\mathbb{R} \phi_x r x \, dx \, d\tau \lesssim \int_0^t \| r_x \|^2 d\tau + \int_0^t \| \phi_x \|^2 d\tau
\lesssim \int_0^t \| r_x \|^2 d\tau + N(T)^2 \int_0^t \| \phi_x \|^2 d\tau,
\]
\[
\int_0^t \int_\mathbb{R} U_x \phi r x \, dx \, d\tau \lesssim \int_0^t \| r_x \|^2 d\tau + \int_0^t \| U_x \|^2 \| \phi_x \|^2 d\tau
\lesssim \int_0^t \| r_x \|^2 d\tau + \int_0^t (1 + \tau)^{-2} \| \phi_x \|^2 d\tau,
\]
\[
(4.12)
\]
we can get by integrating (4.11) with respect to \( t \) and \( x \) over \((0, t) \times \mathbb{R}\) and by employing the estimates (4.12) and (4.13) that
\[
\int_0^t \| \phi_x \|^2 d\tau \lesssim \| (r, \phi_x) \|^2 + \| (r_0, \phi_0 x) \|^2 + \int_0^t \| r_x \|^2 d\tau + N(T)^2 \int_0^t \| \phi_x \|^2 d\tau
+ \int_0^t (1 + \tau)^{-2} \| \phi_x \|^2 d\tau + \delta^{1 - \frac{1}{2}}.
\]
\[
(4.14)
\]
Having obtained the estimate (4.14) and noticing that the estimate (4.5) obtained in Lemma 4.1 tells us that \( N(T) \) can be chosen as small as we wanted provided that \( \| (\phi_0, r_0) \| \) is assumed to be small, thus the estimate stated in Lemma 4.3 can be proved by multiplying the estimate obtained in Lemma 4.2 by a sufficiently large \( \delta \)-independent positive constant, by taking \( \delta \) to be suitably small and Gronwall’s inequality.

The next lemma is concerned with the desired second order energy type estimates.

**Lemma 4.4.** Under the conditions listed in Lemma 4.1, if we assume further that \( \delta \) is sufficiently small, we can get for all \( 0 \leq t \leq T \) that
\[
\| (\phi(t), r(t)) \|^2_2 + \int_0^t \left( \| r(\tau) \|^2_2 + \| \phi_x(\tau) \|^2 \right) d\tau
\]
\begin{equation}
\lesssim \|(\phi_0, r_0)\|_2^2 + \delta^{1-\frac{1}{q}} + (\delta^2 + N(T)) \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau + N(T)^{\frac{1}{q}} \int_0^t \|\phi_{xx}(\tau)\|^{\frac{4}{3}} d\tau.
\end{equation}

**Proof.** Differentiating (3.2)\(_1\) and (3.2)\(_2\) with respect to \(x\) twice, then multiplying the results by \(\phi_{xx}\) and \(r_{xx}\) respectively and adding the resulting two identities together, we have

\begin{align}
\frac{1}{2} \left( \phi_{xx}^2 + r_{xx}^2 \right) + \left[ \frac{1}{2} u \phi_{xx}^2 - \phi_x^2 \phi_{xx} + \phi_{xx} r_{xx} \right] + r_{xx}^2 + \frac{1}{2} U_x \phi_{xx}^2 \\
= 2U_x \phi_x \phi_{xxx} + U_{xx} \phi \phi_{xxx} - \frac{5}{2} \phi_x^2 \phi_{xx} + U_{xxxx} \phi_{xx}
\end{align}

\begin{equation}
(4.15)
\end{equation}

To estimate the corresponding terms in the right hand side of (4.15), we first get from

\begin{equation}
2U_x \phi_x \phi_{xxx} + U_{xx} \phi \phi_{xxx}
= \left( 2U_x \phi_x \phi_{xx} + U_{xx} \phi \phi_{xx} - \frac{3}{2} U_x \phi_x^2 \right) + \frac{3}{2} U_{xxx} \phi_x^2 - 2U_x \phi_x^2 - U_{xx} \phi \phi_{xx}
\end{equation}

and

\begin{equation}
\int_0^t \int_{\mathbb{R}} \left( \frac{3}{2} U_{xxx} \phi_x^2 - 2U_x \phi_x^2 - U_{xx} \phi \phi_{xx} \right) dx d\tau
\lesssim \delta^2 \int_0^t \left( \|\phi_x\|^2 + \|\phi_{xx}\|^2 \right) d\tau + \delta^{2-\frac{1}{q}} \int_0^t (1 + \tau)^{-1-\frac{1}{q}} \|\phi\|^2 d\tau
\end{equation}

that

\begin{equation}
\int_0^t \int_{\mathbb{R}} \left( 2U_x \phi_x \phi_{xxx} + U_{xx} \phi \phi_{xxx} \right) dx d\tau
\lesssim \delta^2 \int_0^t \left( \|\phi_x\|^2 + \|\phi_{xx}\|^2 \right) d\tau + \delta^{2-\frac{1}{q}} \int_0^t (1 + \tau)^{-1-\frac{1}{q}} \|\phi\|^2 d\tau.
\end{equation}

For other terms, we can deduce also that

\begin{equation}
\int_0^t \int_{\mathbb{R}} \left( \phi_x \phi_{xx}^2 \right) dx d\tau \lesssim \int_0^t \|\phi_x\|_{L^\infty} \|\phi_{xx}\|^2 d\tau \lesssim N(T)^{\frac{1}{2}} \int_0^t \|\phi_{xx}\|^{\frac{4}{3}} d\tau,
\end{equation}

\begin{equation}
\int_0^t \int_{\mathbb{R}} U_{xxx} \phi_{xx} dx d\tau \lesssim \int_0^t \|\phi_{xx}\| \|U_{xxx}\| d\tau \lesssim \delta^{\frac{3}{2}} \int_0^t (1 + \tau)^{-1-\frac{1}{q}} \|\phi_{xx}\| d\tau
\end{equation}

\begin{equation}
\lesssim \delta^{\frac{3}{2}} \int_0^t (1 + \tau)^{-1-\frac{1}{q}} \|\phi_{xx}\|^2 d\tau + \delta^{\frac{5}{2}},
\end{equation}

and

\begin{equation}
\int_0^t \int_{\mathbb{R}} \left( 3U_{xx}^2 + 4U_x U_{xxx} + UU_{xxxx} \right) r_{xx} dx d\tau
\lesssim \frac{1}{2} \int_0^t \|r_{xx}\|^2 d\tau + \int_0^t \left( \|U_{xx}\|_{L^2}^2 + \|U_x\|_{L^2}^2 \|U_{xxx}\|^2 + \|UU_{xxx}\|^2 + \|U_{xxxx}\|^2 \right) d\tau
\end{equation}

\begin{equation}
(4.19)
\end{equation}

\begin{equation}
\lesssim \frac{1}{2} \int_0^t \|r_{xx}\|^2 d\tau + \delta^{3-\frac{3}{2}}.
\end{equation}
Thus, from the estimates (4.16)-(4.19), if we integrate (4.15) with respect to \( t \) and \( x \) over \( (0, t) \times \mathbb{R} \), we can get by Gronwall’s inequality that

\[
\|(\phi_{xx}, r_{xx})\|^2 + \int_0^t \|r_{xx}\|^2 d\tau + \int_0^t \int_{\mathbb{R}} U_x \phi_{xx}^2 dx d\tau \\
\lesssim \|(\phi_{0xx}, r_{0xx})\|^2 + N(T) \frac{1}{2} \int_0^t \|\phi_{xx}\|^2 d\tau + \delta^2 \int_0^t \left(\|\phi_x\|^2 + \|\phi_{xx}\|^2\right) d\tau \\
+ \delta^{\frac{5}{2}} - \frac{1}{3} \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\phi\|^2 d\tau. \tag{4.20}
\]

Having obtained (4.20), Lemma 4.4 can be validated immediately by a suitable linear combination of (4.20) and the estimate obtained in Lemma 4.3 and by Gronwall’s inequality provided that we take \( \delta \) small enough.

For the estimate of \( \int_0^t \|\phi_{xx}\|^2 d\tau \), we can get that

**Lemma 4.5.** Under the conditions listed in Lemma 4.1, if we assume further that \( \|(\phi_0, r_0)\| \) and \( \delta \) are sufficiently small, we have for \( 0 \leq t \leq T \) that

\[
\|(\phi(t), r(t))\|^2_0 + \int_0^t \left(\|r(\tau)\|^2_0 + \|\phi_x(\tau)\|^2_0\right) d\tau \\
\lesssim \|(\phi_0, r_0)\|^2_0 + \delta^{1-\frac{1}{2}} + N(T) \frac{1}{2} \int_0^t \|\phi_{xx}(\tau)\|^0 d\tau.
\]

**Proof.** Differentiating (3.2) and (3.2) with respect to \( x \) twice, multiplying the results by \( -r_{xx} \) and \( \phi_{xx} \) respectively, we can get by summing the results together that

\[
\phi_{xx}^2 + (r_x \phi_{tx})_x = r_{xx}^2 - (r_x \phi_{xx})_t - r_x \phi_{xx} + 2U_x \phi_x r_{xx} + \phi_{xx}^2 + \phi_{xx} r_{xx} + U_x \phi_x^2 + U \phi_{xx} r_{xx} - U_{xxx} r_{xx} \tag{4.21}
\]

where we have used the fact that

\[
r_{xx} \phi_{xx} = (r_x \phi_{xx})_t - (r_x \phi_{tx})_x + r_{xx} \phi_{tx}.
\]

Since

\[
\int_0^t \int_{\mathbb{R}} (r_x \phi_{xx}) dx d\tau \lesssim \|(r_x, \phi_{xx})\|^2 + \|(r_{0x}, \phi_{0xx})\|^2,
\]

\[
\int_0^t \int_{\mathbb{R}} r_x \phi_{xx} dx d\tau \lesssim \frac{1}{4} \int_0^t \|\phi_{xx}\|^2 d\tau + \int_0^t \|r_{xx}\|^2 d\tau,
\]

\[
\int_0^t \int_{\mathbb{R}} U_x \phi_x r_{xx} dx d\tau \lesssim \delta^2 \int_0^t \|\phi_x\|^2 + \|r_{xx}\|^2 d\tau,
\]

\[
\int_0^t \int_{\mathbb{R}} \left(2r_{xx} + \phi_{xx} r_{xx}\right) dx d\tau \\
\lesssim \int_0^t \|r_{xx}\|^2 d\tau + \int_0^t \left(\|\phi_x\|^2 \|\phi_x\|^2 + \|\phi\|^2 \|\phi_{xx}\|^2\right) d\tau \\
\lesssim \int_0^t \|r_{xx}\|^2 d\tau + N(T) \frac{1}{2} \int_0^t \|\phi_{xx}\|^2 d\tau, \tag{4.22}
\]

\[
\int_0^t \int_{\mathbb{R}} U_x \phi_{xx} r_{xx} dx d\tau \lesssim \int_0^t \|r_{xx}\|^2 d\tau + \delta^2 \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\phi\|^2 d\tau,
\]

\[
\int_0^t \int_{\mathbb{R}} U_x \phi_{xx} r_{xx} dx d\tau \lesssim \int_0^t \|r_{xx}\|^2 d\tau + \delta^{\frac{5}{2}} - \frac{1}{3} \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\phi\|^2 d\tau,
\]
\[
\int_0^t \int_{\mathbb{R}} U_{xxx} r_{xx} \, dx \, dt \lesssim \int_0^t \|r_{xx}\|^2 \, dt + \delta^{3 - \frac{1}{4}},
\]
\[
\int_0^t \int_{\mathbb{R}} (3U_x U_{xx} + U U_{xxx}) \phi_{xx} \, dx \, dt \lesssim \frac{1}{4} \int_0^t \|\phi_{xx}\|^2 \, dt + \delta^{3 - \frac{1}{4}},
\]
\[
\delta^2 \int_0^t \int_{\mathbb{R}} e^{-\tau(q_+ - q_-)m'(\delta x)} r_{xx} \, dx \, dt \lesssim \delta^2 \int_0^t \|r_{xx}\|^2 \, dt + \delta^3.
\]
we can get by integrating (4.21) with respect to \( t \) and \( x \) over \((0, t) \times \mathbb{R}\) that
\[
\int_0^t \|\phi_{xx}\|^2 \, dt \lesssim \left( \|r_x, \phi_{xx}\|^2 + \|r_0 x, \phi_{0 xx}\|^2 \right) + \int_0^t \left( \|r_x\|^2 + \|\phi_x\|^2 \right) \, d\tau \tag{4.23}
\]
\[
+ N(T)^{\frac{1}{2}} \int_0^t \|\phi_{xx}\|^{\frac{5}{2}} \, d\tau + \delta^{2 - \frac{1}{4}} \int_0^1 (1 + \tau)^{-2 - \frac{3}{4}} \|\phi\|^2 \, d\tau + \delta^{rac{3}{2}}.
\]
A suitable linear combination of (4.23) and the estimate obtained in Lemma 4.4 together with Gronwall’s inequality, we can prove Lemma 4.5 provided that \( \|(\phi_0, r_0)\| \) and \( \delta \) are chosen sufficiently small.

With the Lemma 4.5 in hand, we can prove our Proposition 1 now.

In fact, Lemma 4.5 tells us that if \( \|(\phi_0, r_0)\| \) and \( \delta \) are assumed to be suitably small such that
\[
\|(\phi_0, r_0)\| + \delta \leq \eta_1 \tag{4.24}
\]
holds for some suitably small \( \eta_1 > 0 \), then we can get from Lemma 4.5 and the estimate (4.5) that
\[
\|(\phi(t), r(t))\|^2 + \int_0^t \left( \|r(\tau)\|^2 + \|\phi_x(\tau)\|^2 \right) \, d\tau \leq \left( \|\phi_0, r_0\| + \delta^{\frac{1}{2}} \right)^{\frac{1}{2}} \int_0^t \|\phi_{xx}(\tau)\|^{\frac{11}{10}} \, d\tau. \tag{4.25}
\]
If we set
\[
M(t) := \|\phi(t), r(t)\|^2 + \int_0^t \left( \|r(\tau)\|^2 + \|\phi_x(\tau)\|^2 \right) \, d\tau,
\]
\[
c_1 := C \left( \|\phi_0, r_0\| + \delta^{\frac{1}{2}} \right), \tag{4.26}
\]
\[
c_2 := C \left( \|\phi_0, r_0\| + \delta^{\frac{1}{2}} \right)^{\frac{1}{2}},
\]
where \( C \) is some positive constant independent of \( \delta \) whose precise range can be specified from the proof of Lemma 4.5, then we have from (4.25) that
\[
M(t) \leq c_1 + c_2 M(t)^{\frac{11}{10}},
\]
thus one can get from the Strauss Lemma 2.3 that if both \( \|\phi_0, r_0\| \) and \( \delta \) are assumed to be small further such that
\[
c_1 c_2^{\frac{5}{4}} = C^{\frac{11}{4}} \left( \|\phi_0, r_0\|^{\frac{11}{10}} + \delta^{\frac{1}{2}} \right) \left( \|\phi_0, r_0\| + \delta^{\frac{1}{2}} \right)^{\frac{1}{2}} < \frac{3}{11} \left( \frac{8}{11} \right)^{\frac{5}{2}}, \tag{4.27}
\]
we can get that
\[
\|(\phi(t), r(t))\|^2 + \int_0^t \left( \|r(\tau)\|^2 + \|\phi_x(\tau)\|^2 \right) \, d\tau < \frac{8}{3} c_1 = \frac{8C}{3} \left( \|\phi_0, r_0\|^{\frac{11}{10}} + \delta^{\frac{1}{2}} \right). \tag{4.28}
\]
The smallness assumptions (4.24) and (4.27) we imposed are equivalent to the assumption that there exists a sufficiently small positive constant \(\eta_2\) such that
\[
\left( \| (\phi_0, r_0) \|_2^2 + 1 \right) \left( \| (\phi_0, r_0) \| + \delta^{\frac{1}{2} - \frac{1}{r}} \right)^{\frac{3}{2}} < \eta_2. \tag{4.29}
\]
It is easy to see that if the assumption (3.4) we imposed in Theorem 3.1 is satisfied, then (4.24) and (4.29) hold also, thus we have proved the Proposition 1.

5. Proof of Theorem 3.2. In this section, we prove Theorem 3.2. To this end, for some positive constant \(T > 0\), similar to the proof of Theorem 3.1, we define the following set of functions \(X_1(T)\) for which we seek the solution \((\phi(t, x), r(t, x))\) of the initial-boundary value problem (3.2), (3.6):
\[
X_1(T) := \left\{ \begin{array}{l}
(\phi(t, x), r(t, x)) \mid \\
(\phi(t, x), r(t, x)) \in \bigcap_{j=0}^{2} C^j \left( [0, T]; H^{2-j}(\mathbb{R}^+) \right), \\
r(t, x) \in L^2 \left( [0, T]; H^1(\mathbb{R}^+) \right), \\
\partial_x \phi(t, x) \in L^2 \left( [0, T]; H^1(\mathbb{R}^+) \right), \\
\partial^2 \phi(t, x) \in L^2 \left( [0, T]; H^{2-j}(\mathbb{R}^+) \right), \\
\partial^2 \phi(t, x) \in L^2 \left( [0, T]; H^{2-j}(\mathbb{R}^+) \right), \\
\partial^2 \phi(t, x) \in L^2 \left( [0, T]; H^{2-j}(\mathbb{R}^+) \right), \\
\end{array} \right. \}
\]

For \((\phi_0(x), r_0(x)) \in H^2(\mathbb{R}^+)\) satisfying the compatibility condition (3.7), since the initial-boundary value problem (3.2), (3.6) under our consideration is an initial-boundary value problem for a symmetric hyperbolic system with non-singular boundary matrix, therefore from the well-established result, cf. [9, 23], we can deduce that there exists a sufficiently small positive constant \(t_1\) depending on \(\| (\phi_0, r_0) \|_2\) such that the initial-boundary value problem (3.2), (3.6) admits a unique solution \((\phi(t, x), r(t, x)) \in X_1(t_1)\).

Now suppose that such a solution has been extended to the time step \(t = T \geq t_1\) and satisfies \((\phi(t, x), r(t, x)) \in X_1(T)\), then to prove Theorem 3.2, we only need to deduce the following a priori estimates:

**Proposition 2.** Let \((\phi(t, x), r(t, x)) \in X_1(T)\) be a solution to the initial-boundary value problem (3.2),(3.6) defined on \([0, T] \times \mathbb{R}\), if the assumption (3.8) holds, then \((\phi(t, x), r(t, x))\) satisfies the following estimates
\[
\sum_{j=0}^{2} \left\| \partial^2 \phi(t, r(t)) \right\|_{2-j}^2 + \int_0^T \left( \| \tau \|_2^2 + \left\| \sqrt{U_x(\tau)} \phi(\tau) \right\|_2^2 + \| \phi_x(\tau) \|_1^2 + \sum_{j=1}^{2} \left\| \partial^2 \phi(t, r(t)) \right\|_{2-j}^2 \right) d\tau \leq \sum_{j=0}^{2} \left\| \partial^2 (\phi_0, r_0) \right\|_{2-j}^2 + \delta^{1 - \frac{1}{2}}, \quad 0 \leq t \leq T.
\]

The proof of Proposition 2 will be carried out by the following seven lemmas. To derive our energy estimates, as in [17], we need the additional regularity for the solution \((\phi(t, x), r(t, x))\) that
\[
(\phi(t, x), r(t, x)) \in \bigcap_{j=0}^{2} C^j \left( [0, T]; H^{2-j}(\mathbb{R}^+) \right),
\]
this can be realized by make use of the Friedrichs mollifier with respect to \(t\) for initial-boundary value problem (3.2), (3.6). With this additional regularity we have \(\partial^2 \phi(t, 0) = 0, \quad j = 0, 1, 2, \quad \text{for } t > 0\).
The first lemma is on the basic energy estimate, whose proof is the same as in Lemma 4.1.

**Lemma 5.1.** Under the assumptions listed in Proposition 2, we have for $0 \leq t \leq T$ that

$$
\|(\phi(t), r(t))\|^2 + \int_0^t \left( \left\| \sqrt{U_x(x, t)} \phi(t) \right\|^2 + \|r(t)\|^2 \right) dt \lesssim \|(\phi_0, r_0)\|^2 + \delta^{1 - \frac{1}{4}}.
$$

Consequently, if we set $N(T) := \sup_{0 \leq t \leq T} \{(\phi(t), r(t))\}$, we can get from Lemma 5.1 that

$$
N(T) \lesssim \|(\phi_0, r_0)\| + \delta^{1 - \frac{1}{4}}.
$$

(5.1)

The next lemma is concerned with the estimates of $\|(\phi_t, r_t)\|$, which, compared to the whole space case, is the main difference for the half space problem.

**Lemma 5.2.** Under the assumptions listed in Proposition 2, we have for $0 \leq t \leq T$ that

$$
\|(\phi(t), r(t), \phi_t(t), r_t(t))\|^2 + \int_0^t \left( \left\| \sqrt{U_x(x, t)} (\phi(t), \phi_t(t)) \right\|^2 + \|r(t), r_t(t)\|^2 \right) dt \lesssim \|(\phi_0, r_0, \phi_{t0}, r_{t0})\|^2 + N(T)^2 \int_0^t \|\phi_{xx}(\tau)\|^2 d\tau + \delta^{1 - \frac{1}{4}}
$$

$$
+ (\delta^2 + N(T)) \int_0^t \left( \left\| \phi_t(\tau) \right\|^2 + \|\phi_{xx}(\tau)\|^2 + \|\phi_x(\tau)\|^2 + \|\phi_{xx}(\tau)\|^2 + \|r_{xx}(\tau)\|^2 \right) d\tau.
$$

Proof. Differentiating (3.2)$_1$ and (3.2)$_2$ with respect to $t$ once, multiplying the results by $\phi_t$ and $r_t$ respectively, we have by adding the final results together that

$$
\frac{1}{2} \left( \phi_t^2 + r_t^2 \right)_t + \left[ \frac{1}{2} u \phi_t^2 - \frac{1}{2} \phi^2 + U \phi \right] \phi_t + \phi_t r_t - U \phi_t \phi_t + \phi_t^2 + \frac{1}{2} U_x \phi_t^2
$$

$$
= -(U_t \phi_x) \phi_t - \frac{1}{2} \phi_x \phi_t^2 + U_{xx} \phi_t - (U U_x)_{x} r_t + \delta e^{-t}(q_+ - q_0) m(\delta x) \phi_t.
$$

Integrating the above identity with respect to $t$ and $x$ over $[0, t] \times \mathbb{R}^+$, we can get by using the boundary conditions $\partial_t \phi(t, 0) = 0$ that

$$
\|(\phi_t(t), r_t(t))\|^2 + \int_0^t \left( \left\| \sqrt{U_x(x, \tau)} \phi_t(\tau) \right\|^2 + \|r_t(\tau)\|^2 \right) d\tau \lesssim \|(\phi_0, r_0)\|^2 + 2 \int_0^t \int_{\mathbb{R}^+} \left( -(U_t \phi_x) \phi_t - \frac{1}{2} \phi_x \phi_t^2 + U_{xx} \phi_t \right)
$$

$$
- (U U_x)_{x} r_t + \delta e^{-t}(q_+ - q_0) m(\delta x) \phi_t dx d\tau.
$$

(5.2)

To control the terms in the right hand side of (5.2), we first get from Lemma 2.2 and

$$
-(U_t \phi_x) \phi_t = -U_{xx} \phi_t - U_t \phi_x \phi_t
$$

that

$$
- \int_0^t \int_{\mathbb{R}^+} (U_t \phi_x) \phi_t dx d\tau \lesssim \int_0^t \|U_{xx}\|_{L^\infty} \|\phi_t\| d\tau + \int_0^t \left\| U_t \right\|_{L^\infty} \|\phi_t\| d\tau
$$

(5.3)
Similarly, we can get from Lemma 2.2 that
\[
\int_0^t \int_{\mathbb{R}^+} U_{xxt} \phi_t dx dt \lesssim \int_0^t \|\phi_t\| \|U_{xxt}\| dx dt \lesssim \delta^2 - \frac{1}{4} \int_0^t \|\phi_t\| (1 + \tau)^{-1} d\tau \\
\lesssim \delta^2 - \frac{1}{4} \int_0^t \|\phi_t\|^2 (1 + \tau)^{-1} d\tau + \delta^3 - \frac{1}{2}, \tag{5.4}
\]
and
\[
\delta \int_0^t \int_{\mathbb{R}^+} e^{-\tau}(q_t - q_0) m(\delta x) \phi_t dx dt \lesssim \int_0^t \|\phi_t\|^2 e^{-\tau} d\tau + \delta. \tag{5.5}
\]
To control the term related to \(-\frac{1}{2} \phi_x \phi_t^2\), we can first deduce from (3.2) that
\[-\frac{1}{2} \phi_x \phi_t^2 = -\frac{1}{2} \phi_x (U_{xx} - r_x - U_x \phi - U_x \phi_x - \phi \phi_x) \phi_t,
\]
then we can get from Lemma 2.2 that
\[
\int_0^t \int_{\mathbb{R}^+} U_{xxt} \phi_t dx dt \lesssim \delta^3 \int_0^t \|\phi_t\|^2 + \|\phi_x\|^2 dx d\tau,
\]
\[
\int_0^t \int_{\mathbb{R}^+} U_x \phi_x \phi_t dx dt \lesssim \int_0^t \|U_x\|_L^\infty \|\phi\|_L^\infty \|\phi_x\|_L^\infty \|\phi_x\| \|\phi_t\|^2 dx d\tau \\
\lesssim \delta^2 N(T) \int_0^t \|\phi_t\|^2 + \|\phi_x\|^2 + \|\phi_{xx}\|^2 dx d\tau,
\]
\[
\int_0^t \int_{\mathbb{R}^+} \phi_x \phi_t^2 dx dt \lesssim \int_0^t \|\phi_t\| L^s \|\phi_x\| L^s \|\phi_x\|^2 dx d\tau \\
\lesssim \int_0^t \|\phi\|_L^s \|\phi_x\|_L^s \|\phi_t\|_L^s \|\phi_x\|_L^s \|\phi_x\|_L^s \|\phi_t\|_L^s \|\phi_x\|_L^s dx d\tau \\
\lesssim N(T) \int_0^t \|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|\phi_{xx}\|^2 dx d\tau,
\]
and
\[
\int_0^t \int_{\mathbb{R}^+} (\phi_x r_x + \phi_x^2) \phi_t dx dt \\
\lesssim \int_0^t \|\phi_t\| L^\infty (\|\phi_x\|_L^s \|\phi_x\|_L^s + \|\phi_x\|^2) dx d\tau \\
\lesssim \int_0^t \|\phi_t\|_L^s \|\phi_x\|_L^s \left( \|\phi\|_L^s \|r_x\|_L^s \|\phi_x\|_L^s \|\phi_x\|_L^s \|\phi_x\|_L^s \|\phi\|_L^s \|\phi_{xx}\| \right) dx d\tau \\
\lesssim N(T) \int_0^t \|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|\phi_{xx}\|^2 + \|r_{xx}\|^2 dx d\tau,
\]
thus we have
\[
-\frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \phi_x \phi_t^2 dx dt \lesssim (\delta^2 + N(T)) \int_0^t \left( \|\phi_t\|^2 + \|\phi_{xt}\|^2 + \|\phi_{xx}\|^2 \right)
\]
Next we estimate Lemma 5.2 that together with (5.8) and Lemma 5.2 imply
\[ \int_0^t \| \phi_x \|^2 + \| r_{xx} \|^2 \, d\tau + N(T)^{\frac{7}{2}} \int_0^t \| \phi_{xx} \|^2 \, d\tau. \] (5.6)

Substituting the above estimates (5.3), (5.4), (5.5) and (5.6) into (5.2), we can prove Lemma 5.2 directly by Lemma 5.1 and Gronwall’s inequality.

The next lemma is to deduce an estimate on the term \( \| (\phi_x, r_x) \| \). Unlike the case for the whole space, in order to avoid the difficulties caused by the boundary condition, we will use different approaches.

**Lemma 5.3.** Under the assumptions listed in Proposition 2, we have for \( 0 \leq t \leq T \) that
\[ \| (\phi_x(t), r_x(t)) \|^2 + \int_0^t (\| \phi_x(\tau) \|^2 + \| r_x(\tau) \|^2) \, dxd\tau \leq \| (\phi_0, r_0, \phi_{t0}, r_{t0}, \phi_{xx0}) \|^2 + N(T)^{\frac{7}{2}} \int_0^t \| \phi_{xx}(\tau) \|^2 \, d\tau \]
\[ + \| \phi_x(t) \|^2 (1 + t)^{-\frac{3}{2}} d\tau + N(T)^2 \| \phi_{xx}(t) \|^2 + \delta^{1 - \frac{1}{2}}. \] (5.7)

**Proof.** From (3.2) we have
\[ \phi_x^2 = - (U_x^2 + UU_{xx}) \phi_x - r_t \phi_x - q \phi_x, \] (5.8)
and can get from (5.8) and Lemma 5.2 imply
\[ \| r_x \|^2 \leq (\| \phi_x(t) \|^2 + \| \phi_x(t) \|^2 + \| U_x \|^2 + \| U_t \|^2 + \| \phi_x(t) \|^2 + \| \phi_x(t) \|^2 + \| \phi_{xx}(t) \|^2) + \delta(1 + t)^{-\frac{3}{2}} \| \phi \|^2 + \theta^{-t}. \] (5.9)

Next we estimate \( \int_0^t \| \phi_x \|^2 d\tau \) and \( \int_0^t |r_x|^2 d\tau \). To this end, we first integrate the first inequality of (5.8) with respect to \( t \) and \( x \) over \( (0, t) \times \mathbb{R}^1 \) and can get from Lemma 5.2 that
\[ \int_0^t \| \phi_x \|^2 d\tau \leq \int_0^t (\| U_x \|^4 + \| U_{xx} \|^2 + \| r_t \|^2 + \| r \|^2) \, d\tau. \]
Lemma 5.4. Under the conditions listed in Proposition 2, we have for $0 \leq t \leq T$
that

\[
\| (\phi_{tt}(t), r_{tt}(t)) \|^2 + \int_0^t \left( \| r_{tt}(\tau) \|^2 + \| U_{xx}(\tau) \phi_{tt}(\tau) \|^2 \right) d\tau \\
\leq \| (\phi_{tt}, r_{tt}) \|^2 + \left( \delta^2 + \frac{1}{4} \right) \int_0^t (\| \phi_{tt} \|^2 + \| \phi_{xx} \|^2 + \| r_{xx} \|^2) d\tau + \delta
\]

(5.14)

To yield an estimate on the term $\int_0^t \| r_x \|^2 d\tau$, we can get by performing $(5.7)_2 - (5.7)_1$ that

\[
r_x^2 + (r \phi_t)_x
\]

(5.12)

Integration (5.12) with respect to $t$ and $x$ over $(0, t) \times \mathbb{R}^+$ and by Hölder’s inequality, we can obtain that

\[
\int_0^t \| r_x \|^2 d\tau \lesssim \| (r, \phi_x) \|^2 + \| (r_0, \phi_{0x}) \|^2 + \int_0^t (\| r \|^2 + \| \phi_x \|^2) d\tau + \delta^{1 - \frac{1}{4}}
\]

(5.13)

where we have used the following estimate

\[
\int_0^t \int_{\mathbb{R}^+} \phi^2 \phi_x^2 dxd\tau \lesssim \int_0^t \| \phi \|_{L^\infty} \| \phi_x \|^2 d\tau \lesssim \int_0^t \| \phi \|^2 \| \phi_x \| \| \phi_{xx} \| d\tau
\]

\[
\lesssim N(T)^2 \int_0^t \left( \| \phi_x \|^2 + \| \phi_{xx} \|^2 \right) d\tau.
\]

Thus we get from the estimates (5.13), (5.8), (5.11) and Lemma 5.2 that

\[
\int_0^t \| r_x \|^2 d\tau \lesssim \| (\phi_0, r_0, \phi_{0x}, r_{0x}, \phi_{0xx}) \|^2 + N(T)^{\frac{1}{2}} \int_0^t \| \phi_{xx} \|^2 d\tau + \frac{\delta}{4} \int_0^t \| \phi \|^2 (1 + \tau)^{-\frac{1}{2}} d\tau
\]

(5.14)

\[
+ \left( \delta^2 + N(T) \right) \int_0^t \left( \| \phi_t \|^2 + \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 + \| r_{xx} \|^2 \right) d\tau,
\]

and Lemma 5.3 follows immediately from (5.8), (5.10), (5.11) and (5.14).

Next, we turn to estimate the terms $\| (\phi_{tt}, r_{tt}) \|$ and for result in this direction, we have

Lemma 5.4. Under the conditions listed in Proposition 2, we have for $0 \leq t \leq T$
that

\[
\| (\phi_{tt}(t), r_{tt}(t)) \|^2 + \int_0^t \left( \| r_{tt}(\tau) \|^2 + \| U_{xx}(\tau) \phi_{tt}(\tau) \|^2 \right) d\tau
\]

\[
\lesssim \| (\phi_{tt}, r_{tt}) \|^2 + \delta^{1 - \frac{1}{4}} \int_0^t (1 + \tau)^{-2 - \frac{1}{4}} \| \phi_t(\tau) \|^2 d\tau + \delta
\]

\[
+ \left( \delta^{\frac{1}{2}} + N(T)^{\frac{1}{4}} \right) \int_0^t \left( \| \phi_t(\tau) \|^3 + \| \phi_{xt}(\tau) \|^3 + \| \phi_{xx}(\tau) \|^3 + \| r_{xx}(\tau) \|^3 \right) d\tau
\]

\[
+ \| r_{tt}(\tau) \|^3 \right) d\tau + \left( \delta^{\frac{1}{2}} + N(T)^{\frac{1}{4}} \right) \int_0^t \left( \| \phi_t(\tau) \|^2 + \| \phi_{xt}(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \right)
\]
we can get from (5.18), (5.17) and Sobolev’s inequality that
\[ \| \phi_t(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 + \| r_{xx}(\tau) \|^2 \] \[ d\tau. \]

**Proof.** Differentiating the equations (3.2)\(_1\) and (3.2)\(_2\) with respect to \( t \) twice, multiplying the results by \( \phi_t \) and \( r_{tt} \) respectively and adding the results together, we have
\[ \frac{1}{2} \left( \phi_{tt}^2 + r_{tt}^2 \right)_t + \left[ \frac{1}{2} u \phi_{tt}^2 + \phi_t r_{tt} \right]_x + r_{tt}^2 + \frac{1}{2} U_x \phi_{tt}^2 \]
\[ = - \frac{1}{2} \phi_x \phi_{tt}^2 - 2 \phi_t \phi_x \phi_{tt} - 2 (U_t \phi_t)_x \phi_{tt} -(U_t \phi)_x \phi_{tt} - (U U)_x r_{tt} + U_{xxt} \phi_{tt} - q_{xxt} \phi_{tt}. \]

Firstly, Lemma 2.2 together with Cauchy’s inequality tell us that
\[ -2 \int_0^t \int_{\mathbb{R}^+} (U_t \phi)_x \phi_{tt} dx d\tau \leq \int_0^t \left( \| U_{xt} \|_{L^\infty} \| \phi_t \| \| \phi_{tt} \| + \| U_t \|_{L^\infty} \| \phi_{xt} \| \| \phi_{tt} \| \right) d\tau \]
\[ \leq \delta^2 \int_0^t \left( \| \phi_t \|^2 + \| \phi_{xt} \|^2 + \| \phi_{tt} \| \right) d\tau \]
\[ - \int_0^t \int_{\mathbb{R}^+} (U U)_{xtt} r_{tt} dx d\tau \leq \frac{1}{2} \int_0^t \| r_{tt} \|^2 d\tau + \delta^{\frac{3}{2}}, \] (5.16)
\[ \int_0^t \int_{\mathbb{R}^+} U_{xxtt} \phi_{tt} dx d\tau \leq \int_0^t \| \phi_{tt} \| \| U_{xxtt} \| d\tau \]
\[ \leq \delta^{\frac{1}{2} - \frac{\delta}{2}} \int_0^t \| \phi_{tt} \|^2 (1 + \tau)^{-1 - \frac{\delta}{2}} d\tau + \delta^{1 - \frac{\delta}{2}}, \]
\[ - \int_0^t \int_{\mathbb{R}^+} q_{xxtt} \phi_{tt} dx d\tau \leq \int_0^t e^{-\tau} \| \phi_{tt} \|^2 d\tau + \delta. \]

Secondly, we estimate the terms related to \(-2 \phi_t \phi_{xt} \phi_{tt}\) and \(-\frac{1}{2} \phi_x \phi_{tt}^2\). For the first term, due to
\[ -2 \int_0^t \int_{\mathbb{R}^+} \phi_t \phi_{xt} \phi_{tt} dx d\tau \leq \int_0^t \| \phi_t \|_{L^\infty} \| \phi_{xt} \| \| \phi_{tt} \| d\tau \leq \int_0^t \| \phi_t \|^{\frac{1}{2}} \| \phi_{xt} \|^{\frac{1}{2}} \| \phi_{tt} \| d\tau \]
\[ \leq N(T) \frac{1}{2} \left( \| \phi_t \|^3 + \| \phi_{xt} \|^3 + \| \phi_{xx} \|^3 \right) d\tau \] (5.17)
and
\[ \| \phi_t \| \leq \| \phi_x \| + \| \phi_{xx} \| + \| U_x \phi \| + \| r_x \| + \| U_{xx} \| \]
\[ \leq \| \phi \|^{\frac{1}{2}} \| \phi_{xx} \|^{\frac{1}{2}} + \| \phi \|^{\frac{1}{2}} \| \phi_{xx} \|^{\frac{1}{2}} + \delta^2 \| \phi \| + \| r \|^{\frac{1}{2}} \| r_{xx} \|^{\frac{1}{2}} + \delta^{\frac{3}{2}}, \]
we can get from (5.18), (5.17) and Sobolev’s inequality that
\[ -2 \int_0^t \int_{\mathbb{R}^+} \phi_t \phi_{xt} \phi_{tt} dx d\tau \]
\[ \approx N(T) \frac{1}{2} \left( \| \phi_{tt} \|^3 + \| \phi_{xt} \|^3 + \| \phi_{xx} \|^3 \right) d\tau \]
Integrating (5.15) with respect to $t$

\[ + N(T)^{\frac{1}{2}} \int_0^t \left( \|\phi_{tt}\|^3 + \|\phi_{xt}\|^2 + \|\phi_{xx}\|^3 + \|r_{xx}\|^3 \right) d\tau \]

\[ + \left( \delta N(T)^{\frac{1}{2}} + \delta^2 \right) \int_0^t \left( \|\phi_{tt}\|^2 + \|\phi_{xt}\|^2 \right) d\tau \]

(5.19)

\[ \lesssim \left( \delta^2 + N(T)^{\frac{1}{2}} \right) \int_0^t \left( \|\phi_{xx}\|^2 + \|\phi_{xt}\|^2 + \|r_{xx}\|^2 \right) d\tau \]

For the estimate of $\|\phi_{tt}(t)\|$,

\[- \frac{1}{2} \phi_x \phi_{tt} = \frac{1}{2} \left( U_x^2 + U_{xx} \right) \phi_{tt} + \frac{1}{2} r \phi_{tt} + \frac{1}{2} r \phi_{tt}, \]

thus

\[ - \frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \phi_x \phi_{tt}^2 dxd\tau \]

\[ \lesssim \int_0^t \left( \|U_x^2, U_{xx}, r_t, r\|_{L^\infty} \|\phi_{tt}\|^2 \right) d\tau \]

(5.20)

\[ \lesssim \delta^2 \int_0^t \|\phi_{tt}\|^2 d\tau + \int_0^t \left( \|r_t\|^2 \|r_x\|^2 + \|r\|^2 \|r_{xx}\|^2 \right) \|\phi_{tt}\|^2 d\tau. \]

Moreover, (3.2) implies

\[ \|r_t\| \lesssim \|r\| + \|\phi_x\| + \|U_{x}^2\| + \|U_{xx}\| \]

(5.21)

\[ \lesssim \|U_t\| + \|\phi_x\|^2 \|\phi_{xx}\|^2 + \|U_x L^\infty \|U_x\| + \|U_{xx}\|. \]

we can get by substituting (5.21) into (5.20) and by young’s inequality that

\[ - \frac{1}{2} \int_0^t \int_{\mathbb{R}^+} \phi_x \phi_{tt}^2 dxd\tau \]

\[ \lesssim \delta^2 \int_0^t \|\phi_{tt}\|^2 d\tau + \left( \delta^2 + \delta^2 \right) \int_0^t \left( \|r_{xx}\|^2 + \|r_{tt}\|^2 \right) d\tau \]

\[ + N(T)^{\frac{1}{2}} \int_0^t \left( \|\phi_{xx}\|^3 + \|r_{xt}\|^2 \right) d\tau + N(T)^{\frac{1}{2}} \int_0^t \left( \|\phi_{tt}\|^2 + \|r_{xx}\|^2 \right) d\tau \]

\[ \lesssim \left( \delta^2 + N(T)^{\frac{1}{2}} \right) \int_0^t \left( \|\phi_{xx}\|^2 + \|r_{xx}\|^2 \right) d\tau \]

(5.22)

Integrating (5.15) with respect to $t$ and $x$ over $[0, t] \times \mathbb{R}^+$, we can prove Lemma 5.4 from the estimates (5.16), (5.19), (5.22) and by employing Gronwall’s inequality. \( \square \)

For the estimate of $\|(\phi_{xx}(t), r_{xx}(t))\|$, we have

**Lemma 5.5.** Under the conditions listed in Proposition 2, we have for $0 \leq t \leq T$ that

\[ \|(\phi_{xx}(t), r_{xx}(t))\|^2 \]

\[ \lesssim \left( \|\phi_0, r_0, \phi_{0t}, r_{0t}, \phi_{0x}, r_{0t}\| + \delta \int_0^t (1 + \tau)^{\frac{3}{2}} \|\phi(\tau)\|^2 d\tau + \delta^1 \right. \]
Proof. From (3.2), we have

\[ \phi_{xt} = - (U U_x)_{xt} - r_{tt} - r_t, \]  

Thus

\[ \| \phi_{xt} \|^2 \lesssim \| (U U_x)_{xt} \|^2 + \| r_{tt} \|^2 + \| r_t \|^2 \]  

and

\[ \| r_{xt} \|^2 \lesssim \| (U \phi_x)_{xt} \|^2 + \| (U_x \phi)_{xt} \|^2 + \| (\phi_{xx})_{xt} \|^2 + \| U_{xxt} \|^2 + \| \phi_{tt} \|^2 + \| q_{xt} \|^2 \]

\[ \lesssim \frac{\delta^2}{2} \left( \| \phi_t \|^2 + \| \phi_{xx} \|^2 + \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \right) \]  

\[ + \frac{3}{2} \| \phi_{xx} \|^2 + \frac{3}{2} \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \]  

\[ + (N(T)^{\frac{3}{2}} + \delta^2 N(T)) \left( \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 \right) \]  

Moreover, we can get from (5.18) that

\[ \| \phi_t \|^2 \lesssim \frac{\delta^2}{2} \left( \| \phi_x \|^2 + \| \phi_{xx} \|^2 + \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \right) \]  

\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \left( \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 \right) \]

\[ \lesssim \frac{\delta^2}{2} \left( \| \phi_x \|^2 + \| \phi_{xx} \|^2 + \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \right) \]  

\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \left( \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 \right) \]

consequently, we can deduce from (5.25) and (5.26) that

\[ \| r_{xt} \|^2 \lesssim \frac{\delta^2}{2} \left( \| \phi_x \|^2 + \| \phi_{xx} \|^2 + \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \right) \]  

\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \left( \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 \right) \]

Thus

\[ \| (\phi_{xt}, r_{xt}) \|^2 \lesssim \frac{\delta^2}{2} \left( \| \phi_x \|^2 + \| \phi_{xx} \|^2 + \| \phi_{xx} \|^2 + \frac{3}{2} \| r_{xx} \|^2 \right) \]  

\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \left( \| \phi_{xt} \|^2 + \| \phi_{xx} \|^2 \right) \]

By virtue of Lemma 5.2-5.4 and from (5.28), we can prove the Lemma 5.5 immediately.

For the estimate of \( \| (\phi_{xx}(t), r_{xx}(t)) \| \), we have
Lemma 5.6. Under the conditions listed in Proposition 2, we have for $0 \leq t \leq T$ that
\[
\| (\phi_{xx}(t), r_{xx}(t)) \|^2 \\
\lesssim \| (\phi_0, r_0, \phi_{tt}, \phi_{xt}, \phi_{xx}, r_{tt}, r_{xt}, r_{xx}) \|^2 + \delta \int_0^t (1 + \tau)^{-\frac{3}{2}} \| \phi(\tau) \|^2 d\tau + \delta^{1 - \frac{3}{2}} \\
+ \left( \delta^{\frac{3}{4}} + N(T)^{\frac{1}{4}} \right) \int_0^t \left( \| \phi_\tau(\tau) \|^2 + \| \phi_x(\tau) \|^2 + \| \phi_{xt}(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 + \| \phi_x(\tau) \|^2 + \| \phi_{xx}(\tau) \|^2 \right) d\tau \\
+ \| r_{xx}(\tau) \|^2 \right) d\tau + \left( \delta^{\frac{3}{4}} + N(T)^{\frac{1}{4}} \right) \int_0^t \left( \| \phi_{xt}(\tau) \|^3 + \| \phi_{xx}(\tau) \|^3 + \| r_{xx}(\tau) \|^3 \right) d\tau.
\]

Proof. Differentiating the equations (3.2) and (3.2) with respect to $x$ once to obtain
\[
\begin{align*}
\phi_{xx} &= U_{xxx} - \phi_{xt} - \left( \frac{1}{2} \phi^2 + U \phi \right)_x - q_{xx}, \\
\phi_{xx} &= -(UU_x)_{xx} - r_{xt} - r_x,
\end{align*}
\]
from which one can deduce that
\[
\begin{align*}
\phi_{xx}^2 &= -(UU_x)_{xx} \phi_{xx} - r_{xt} \phi_{xx} - r_x \phi_{xx}, \\
r_{xx}^2 &= U_{xxx} \phi_{xx} - \phi_{xt} r_{xx} - \left( \frac{1}{2} \phi^2 + U \phi \right) r_{xx} - q_{xx} r_{xx}.
\end{align*}
\]
Consequently
\[
\begin{align*}
\| (\phi_{xx}, r_{xx}) \|^2 \\
\lesssim \| (U_x U_x, U_{xxx}, q_{xx}) \|^2 + \| (\phi_{xt}, r_{xt}) \|^2 + \| (\phi_{xx}, r_{xx}) \|^2 + \| \phi \|^2 + \| U \|^2 \\
\lesssim \delta^{3 - \frac{3}{2}} (1 + t)^{-\frac{3}{2}} + \delta^3 e^{-2t} + \| (\phi_{xt}, r_{xt}) \|^2 + \| (\phi_{xx}, r_{xx}) \|^2 \\
+ N(T)^2 \| \phi_{xx} \|^2 + \delta (1 + t)^{-\frac{3}{2}} \| \phi \|^2.
\end{align*}
\]
Having obtained (5.31), we can prove Lemma 5.6 immediately from Lemma 5.1-5.5 provided that we take $\| (\phi_0, r_0) \|$ small enough. \hfill \Box

The last lemma is concerned with the estimate of $\int_0^t \| (\phi_t, \phi_{xt}, r_t, \phi_{tt}, \phi_{xx}, r_{xx}) \|^2 d\tau$.

Lemma 5.7. Under the conditions listed in the Proposition 2, we have for $0 \leq t \leq T$ that
\[
\begin{align*}
\int_0^t \| (\phi_t, \phi_{xt}, r_t, \phi_{tt}, \phi_{xx}, r_{xx}) (\tau) \|^2 d\tau \\
\lesssim \| r_t(t) \|^2 + \| \phi_{xt}(t) \|^2 + \| r_{xt}(t) \|^2 + \| \phi_{xx}(t) \|^2 + \| \phi_x(t) \|^2 + \delta \int_0^t (1 + \tau)^{-\frac{3}{2}} \| \phi(\tau) \|^2 d\tau \\
+ \int_0^t \left( \| \phi_x(\tau) \|^2 + \| r_x(\tau) \|^2 + \| r_{tt}(\tau) \|^2 \right) d\tau \\
+ N(T)^\frac{1}{2} \int_0^t \left( \| \phi_{xx}(\tau) \|^3 + \| \phi_{xt}(\tau) \|^3 \right) d\tau.
\end{align*}
\]
Proof. To yield an estimate on \( \int_0^t \| \phi_t(\tau) \|^2 d\tau \), we have by multiplying the equation (3.2) by \( \phi_t \) and integrating the result with respect to \( t \) over \((0, t) \times \mathbb{R}^+ \) that
\[
\int_0^t \| \phi_t \|^2 d\tau \lesssim \int_0^t \left( \| \phi_t \|^2 + \| \phi_x \|^2 + \| r_x \|^2 + \| U_t \|^2 \right) d\tau + \int_0^t (\| q_t \|^2 + \| U_{tt} \|^2) d\tau
\]
\[
\lesssim \int_0^t (\| \phi_x \|^2 + \| r_x \|^2) d\tau + \delta \int_0^t (1 + \tau)^{-\frac{1}{2}} \| \phi \|^2 d\tau + \delta^{1 - \frac{1}{2}}
\]
\[
+ N(T)^2 \int_0^t \| \phi_{xx} \|^2 d\tau.
\]
For the term \( \int_0^t \| \phi_{xx} \|^2 d\tau \), we can get by integrating (5.24) with respect to \( x \) over \( \mathbb{R}^+ \) that
\[
\int_0^t \| \phi_{xx} \|^2 d\tau \lesssim \int_0^t \| r_t \|^2 + \| r_{xt} \|^2 d\tau + \delta^{3 - \frac{1}{2}}.
\]
Now for the term \( \int_0^t \| r_{xt} \|^2 d\tau \), similar to (5.12), we have by multiplying (5.23) by \( -\phi_t \) and \( r_{xt} \) respectively and then adding the results together that
\[
r_{xt}^2 - (\phi_t r_t)_x = (\phi_{xt} r_t)_t + r_t \phi_{xt} + (U U_t)_{xt} \phi_t + U_{xt} r_{xt} - (\phi_t \phi_t + U + U_t \phi_t) t r_{xt} - q_{xt} r_{xt}.
\]
Integrating (5.34) with respect to \( t \) and \( x \) over \((0, t) \times \mathbb{R}^+ \), we have
\[
\int_0^t \| r_{xt} \|^2 d\tau
\]
\[
= \int_0^t \| \phi_{xt} \|^2 d\tau + \int_0^t \int_{\mathbb{R}^+} (\phi_{xxt} r_t)_\tau + r_t \phi_{xxt} + (U U_t)_{xt} \phi_{xxt} + U_{xtxt} r_{xt}) dx d\tau
\]
\[
- \int_0^t \int_{\mathbb{R}^+} (\phi_{xt} + U_x \phi + U \phi_x) r_{xt} dx d\tau - \int_0^t \int_{\mathbb{R}^+} \bar{q}_{xt} r_{xt} dx d\tau.
\]
The terms in the right side of (5.35) can be estimated as follows:
\[
\int_0^t \int_{\mathbb{R}^+} (\phi_{xxt} r_t)_\tau dx d\tau = \int_0^t \int_{\mathbb{R}^+} (\phi_{xxt} r_t - \phi_{xxt} r_{xt}) dx \lesssim \| r_t \|^2 + \| \phi_{xxt} \|^2 + \| r_{xt} \|^2 + \| \phi_{xt} \|^2,
\]
\[
\int_0^t \int_{\mathbb{R}^+} (\phi_{xt} r_t + r_t \phi_{xt}) dx d\tau \lesssim \int_0^t \| r_t \|^2 d\tau + \int_0^t \| \phi_{xt} \|^2 d\tau,
\]
\[
\int_0^t \int_{\mathbb{R}^+} (U U_t)_{xt} \phi_{xxt} dx d\tau \lesssim \int_0^t \| \phi_{xxt} \|^2 d\tau + \delta^{3 - \frac{1}{2}},
\]
\[
\int_0^t \int_{\mathbb{R}^+} \bar{q}_{xt} r_{xt} dx d\tau \lesssim \frac{1}{8} \int_0^t \| r_{xt} \|^2 d\tau + \delta,
\]
\[
\int_0^t \int_{\mathbb{R}^+} (U_x \phi)_t r_{xt} dx d\tau
\]
\[
\leq \frac{1}{8} \int_0^t \| r_{xt} \|^2 d\tau + \int_0^t \left( \| U_{xt} \|^2 \| \phi_t \|^2 + \| U_{xxt} \|^2 \| \phi_{xt} \|^2 \right) d\tau
\]
\[
\leq \frac{1}{8} \int_0^t \| r_{xt} \|^2 d\tau + \delta \int_0^t (1 + \tau)^{-\frac{1}{2}} (\| \phi_t \|^2 + \| \phi_{xt} \|^2) d\tau.
\]
\[ \int_0^t \int_{\mathbb{R}^+} (U \phi_x)_\tau r_{xt} dx d\tau \]
\[ \lesssim \frac{1}{8} \int_0^t \|r_{xt}\|^2 d\tau + \delta \int_0^t (1 + \tau)^{-\frac{3}{2}} \|\phi_{xt}\|^2 d\tau + \int_0^t \|\phi_{xt}\|^2 d\tau, \]
and
\[ \int_0^t \int_{\mathbb{R}^+} (\phi \phi_x)_\tau r_{xt} dx d\tau \lesssim \frac{1}{8} \int_0^t \|r_{xt}\|^2 d\tau + \int_0^t (\|\phi_t\|^2 \|\phi_{xt}\|^2 + \|\phi\|^2 \|\phi_{xt}\|^2) d\tau \]
\[ \lesssim \frac{1}{8} \int_0^t \|r_{xt}\|^2 d\tau + N(T)^{\frac{3}{2}} \int_0^t (\|\phi_{xx}\|^3 + \|\phi_{xt}\|^3) d\tau \]
\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \int_0^t (\|\phi_{xx}\|^2 + \|\phi_{xt}\|^2) d\tau, \quad (5.38) \]
where we have used the inequality (5.18).

From (5.36)-(5.38) and (5.33), we have
\[ \int_0^t \|r_{xt}\|^2 d\tau \lesssim \|r_t\|^2 + \|\phi_{xt}\|^2 + \|r_{0xt}\|^2 + \|\phi_{0xt}\|^2 + \delta + \int_0^t (\|r_{tt}\|^2 + \|r_{t}\|^2) d\tau \]
\[ + \delta \int_0^t (1 + \tau)^{-\frac{3}{2}} (\|\phi_t\|^2 + \|\phi_{t}\|^2 + \|\phi_{xt}\|^2) d\tau \]
\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \int_0^t \phi_{xx}\|^2 d\tau + N(T)^{\frac{3}{2}} \int_0^t (\|\phi_{xx}\|^3 + \|\phi_{xt}\|^3) d\tau. \quad (5.39) \]

To deal with the term \( \int_0^t \|\phi_{tt}\|^2 d\tau \), we have by differentiating the (3.2) with respect to \( t \) and multiplying the result by \( \phi_{tt} \) that
\[ \phi_{tt}^2 = - (\phi \phi_x + U \phi_x + U \phi_x) + \phi_{tt} - \phi_{xt} \phi_{tt} - r_{xt} \phi_{tt} + U_{xx} \phi_{tt}, \quad (5.40) \]
and similar to the estimate of \( \int_0^t \|r_{xt}\|^2 d\tau \), we can get that
\[ \int_0^t \|\phi_{tt}\|^2 d\tau \lesssim \delta + \int_0^t (\|r_{xt}\|^2 + \|\phi_{tt}\|^2) d\tau + \delta \int_0^t (1 + \tau)^{-\frac{3}{2}} (\|\phi_t\|^2 + \|\phi_{t}\|^2 + \|\phi_{xt}\|^2) d\tau \]
\[ + \left( N(T)^{\frac{3}{2}} + \delta^2 N(T) \right) \int_0^t \phi_{xx}\|^2 d\tau + N(T)^{\frac{3}{2}} \int_0^t (\|\phi_{xx}\|^3 + \|\phi_{xt}\|^3) d\tau. \quad (5.41) \]

Finally, for the estimate on \( \int_0^t (\phi_{xx}, r_{xx})^2 d\tau \), we can deduce by integrating (5.31) with respect to \( t \) over \((0, t)\) that
\[ \int_0^t (\phi_{xx}, r_{xx})^2 d\tau \lesssim \delta^{3 - \frac{1}{2}} + \int_0^t \left( (\phi_{xt}, r_{xt})^2 + (\phi_t, r_t)^2 \right) d\tau \]
\[ + \delta \int_0^t (1 + t)^{-\frac{3}{2}} \|\phi_t\|^2 d\tau. \quad (5.42) \]

Having obtained (5.32), (5.33), (5.39), (5.41) and (5.42), Lemma 5.7 can be proved directly provided that \( \|\phi(0, r_0)\| \) is chosen sufficiently small. \[ \square \]
With the Lemmas 5.1-5.7 in hand, we now turn to prove the Proposition 2 by making use of the Strauss Lemma 2.3.

In fact, if \( \|(\phi_0, r_0)\| \) and \( \delta \) are assumed to be suitably small such that

\[
\|(\phi_0, r_0)\| + \delta \leq \eta_2
\]  \hspace{1cm} (5.43)

holds for some suitably small \( \eta_2 > 0 \), then we can get from Lemmas 5.1-5.7 and by virtue of Gronwall’s inequality that

\[
\sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi(t), r(t)) \right\|^2 \\
+ \int_0^t \left[ \left\| (\sqrt{U_x} \phi)(\tau), r(\tau) \right\|^2 + \sum_{1 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi(\tau), r(\tau)) \right\|^2 \right] d\tau \leq \sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi_0, r_0) \right\|^2 + \delta^{1-\frac{1}{4}} + N(T)^{\frac{2}{3}} \left( \|(\phi_{xt}(t))\|^3 + \|(\phi_{xx}(t))\|^3 + \|r_{xx}(t)\|^3 \right) \\
+ \left( \delta^{\frac{2}{3}} + N(T)^{\frac{4}{3}} \right) \int_0^t \left( \|\phi_{xt}(\tau)\|^3 + \|\phi_{xx}(\tau)\|^3 + \|r_{xx}(\tau)\|^3 \right) d\tau.
\]  \hspace{1cm} (5.44)

Recall that (5.1) tells us that

\[ N(T) \lesssim \|(\phi_0, r_0)\| + \delta^{\frac{1}{2}} - \frac{1}{\eta_2}, \]

thus if we set

\[
M(t) := \sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi(t), r(t)) \right\|^2 \\
+ \int_0^t \left[ \left\| (\sqrt{U_x} \phi)(\tau), r(\tau) \right\|^2 + \sum_{1 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi(\tau), r(\tau)) \right\|^2 \right] d\tau,
\]

\[
c_1 := C \left( \sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi_0, r_0) \right\|^2 + \delta^{1-\frac{1}{2}} \right), \hspace{1cm} (5.45)
\]

\[
c_2 := C \left( \|(\phi_0, r_0)\| + \delta^{\frac{1}{2}} - \frac{1}{\eta_2} \right)^{\frac{1}{2}},
\]

where \( C \) is some positive \( \delta \)-independent constant whose range can be determined by Lemmas 5.1-5.7, then from (5.44), we have

\[
M(t) \leq c_1 + c_2 M(t)^{\frac{3}{2}}.
\]  \hspace{1cm} (5.46)

If both \( \|(\phi_0, r_0)\| \) and \( \delta \) are chosen sufficiently small such that

\[
c_1c_2 = C^3 \left( \sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j (\phi_0, r_0) \right\|^2 + \delta^{1-\frac{1}{4}} \right) \times \left( \|(\phi_0, r_0)\| + \delta^{\frac{1}{2}} - \frac{1}{\eta_2} \right)^{\frac{1}{2}} \\
< \left( 1 - \frac{2}{3} \right) \left( \frac{3}{2} \right)^{-2} = \frac{4}{27},
\]  \hspace{1cm} (5.47)
then, we can deduce from Strauss’ Lemma 2.3 that
\[ M(t) \lesssim \sum_{0 \leq i+j \leq 2} \left\| \partial_x^i \partial_t^j \left( \phi_0, r_0 \right) \right\|^2 + \delta^{1 - \frac{1}{q}}. \]  
(5.48)

Notice that a sufficient condition to guarantee that the assumptions (5.43) and (5.47) hold is the assumption (3.8) imposed in Theorem 3.2, thus we have proved the Proposition 2.

6. Appendix. In this section, we prove Lemma 2.1 and Lemma 2.2.

6.1. The proof of Lemma 2.1. Since the proof of (i), (ii) and (iv) in Lemma 2.1 can be found in [14,15], we only prove (iii) here.

Following the argument as in [14, 17], let \( U(t, x) \) be the solution of initial value problem (2.1). By the characteristic curve method, we can deduce that
\[ U(t, x) = U_0(x_0(t, x)), \]
(6.1)
\[ x = x_0(t, x) + U_0(x_0(t, x))t, \]
where \( x_0 = x_0(t, x) \) is a point in \( \mathbb{R} \).

From (6.1) we have
\[ \partial_x x_0 = \frac{1}{1 + U'_0(x_0)t}, \]
(6.2)
where \( U'_0(x_0) = \frac{d}{dx_0} U_0(x_0) \).

Also, from (6.1) we have
\[ \partial_x U = U''_0(x_0) \cdot \partial_x x_0 = \frac{U''_0(x_0)}{1 + U'_0(x_0)t}, \]
\[ \partial^2_x U = \frac{U'''_0(x_0)}{(1 + U'_0(x_0)t)^3}, \]
(6.3)
\[ \partial^2_x U = \frac{U''_0(x_0)(1 + U'_0(x_0)t)^3 - 3U''_0(x_0)^2 t}{(1 + U'_0(x_0)t)^5}, \]
\[ \partial^3_x U = \frac{U'_0(x_0)(1 + U'_0(x_0)t)^5 - 10U''_0(x_0)U'''_0(x_0)t}{(1 + U'_0(x_0)t)^6} + \frac{15U'''_0(x_0)^3 t^2}{(1 + U'_0(x_0)t)^7}, \]
where \( U^{(k)}_0(x_0) = \frac{d^k U_0(x_0)}{dx_0^k}, k \in \mathbb{N} \).

(2.1) tells us that
\[ U'_0(x_0) = \frac{\epsilon \delta \kappa_q}{2(1 + (\epsilon x_0)^2)^q} > 0, \]
\[ U''_0(x_0) = \frac{-q \epsilon^5 \delta \kappa_q x_0}{(1 + (\epsilon x_0)^2)^{q+1}}, \]
\[ U^{(3)}_0(x_0) = \frac{-q \epsilon^5 \delta \kappa_q x_0^2}{(1 + (\epsilon x_0)^2)^{q+1}} + \frac{2q(q + 1)\epsilon^5 \delta \kappa_q x_0^2}{(1 + (\epsilon x_0)^2)^{q+2}}, \]
\[ U^{(4)}_0(x_0) = \frac{6q(q + 1)\epsilon^5 \delta \kappa_q x_0 - 2q(q + 1)(2q + 1)\epsilon^7 \delta \kappa_q x_0^3}{(1 + (\epsilon x_0)^2)^{q+3}}, \]
from which and the fact that \( \left| \epsilon x_0 \right| \leq \sqrt{1 + (\epsilon x_0)^2} \), we can deduce that
\[ |U''_0(x_0)| \leq C_q \frac{\epsilon^2 \delta \kappa_q}{(1 + (\epsilon x_0)^2)^{q+1}} \leq C_q \epsilon^{1 - \frac{3}{q}} \delta^{-\frac{1}{q}} \left| U'_0(x_0) \right|^{1 + \frac{1}{q}}, \]
\[ |U_0^{(3)}(x_0)| \leq C_q \epsilon^{2 - \frac{1}{4p} \delta - \frac{1}{4} |U_0'(x_0)|^{1 + \frac{1}{4p}}, \]
\[ |U_0^{(4)}(x_0)| \leq C_q \epsilon^{2 - \frac{1}{4p} \delta - \frac{1}{4} |U_0''(x_0)| |U_0'(x_0)|^{\frac{1}{2}} \leq C_q \epsilon^{3 - \frac{3}{4p} \delta - \frac{3}{4} |U_0'(x_0)|^{1 + \frac{3}{4p}}. \]

Substituting (6.5) into (6.3) yields
\[
|U_{xx}| \leq C_q \epsilon^{1 - \frac{3}{4p} \delta - \frac{1}{4} |U_0'(x_0)|^{1 + \frac{1}{4p}},
\]
\[
|\partial_2^2 U| \leq C_q \epsilon^{2 - \frac{1}{4p} \delta - \frac{1}{4} |U_0'(x_0)|^{1 + \frac{1}{4p}} / (1 + U_0'(x_0)t)^2,
\]
\[
|\partial_1^2 U| \leq C_q \epsilon^{3 - \frac{3}{4p} \delta - \frac{3}{4} |U_0'(x_0)|^{1 + \frac{3}{4p}} / (1 + U_0'(x_0)t)^3. \]

With the above calculations in hand, we now turn to prove (iii) of Lemma 2.1. In fact, for \( k = 2, 3, 4 \) and \( p \in [1, +\infty) \), on one hand, we can get from (6.6) and (6.4) that
\[
\|\partial_x^k U(t)\|_{L^p} \leq C_{p,q} \epsilon^{(k-1) - \frac{k-1}{2p} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R |U_0'(x_0)|^{p+\frac{k-1}{2p}} \, dx_0
\leq C_{p,q} \epsilon^{k-1} \delta \int_R \left( \frac{1}{(1 + (\epsilon x_0)^2)^\frac{1}{2}} \right)^{p+\frac{k-1}{2p}} \, dx_0
\leq C_{p,q} \epsilon^{k-1} \delta, \]
where we have used the fact that \( dx = (1 + U_0'(x_0)t) \, dx_0 \), while for \( p = \infty \) we have
\[
\|\partial_x^k U(t)\|_{L^\infty} \leq C_{p,q} \epsilon^{k-1} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \leq C_{p,q} \epsilon^k. \]

On the other hand, we have for \( k = 3, 4 \) and \( p \in [1, +\infty) \) that
\[
\|\partial_x^k U(t)\|_{L^p} \leq C_{p,q} \epsilon^{(k-1) - \frac{k-1}{2p} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R |U_0'(x_0)|^{p+\frac{k-1}{2p}} \, dx_0
\leq C_{p,q} \epsilon^{(k-1) - \frac{k-1}{2p} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R \left( \frac{1}{(1 + U_0'(x_0)t)^\frac{1}{2}} \right)^{p-\frac{k-1}{2p}} \, dx_0
\leq C_{p,q} \epsilon^{(k-1) - \frac{k-1}{2p} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R \frac{1}{(1 + (\epsilon x_0)^2)^\frac{1}{2}} \, dx_0
\leq C_{p,q} \epsilon^{(k-1)p-1} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R \frac{1}{(1 + (\epsilon x_0)^2)^\frac{1}{2}} \, dx_0
\leq C_{p,q} \epsilon^{(k-1)p-1} \delta - \frac{k-1}{2} |U_0'(x_0)|^{1 + \frac{k-1}{2p}} \int_R \frac{1}{(1 + (\epsilon x_0)^2)^\frac{1}{2}} \, dx_0,
\]
while if \( k = 2 \) and \( p \in [2, +\infty) \) (cf. [14, 15]), let \( y = U_0'(x_0)t \), then \( dy = U_0''(x_0)t \, dx_0 = \frac{|U_0'(x_0)|^{-1} t}{tdx_0} \) for \( x_0 < 0 \), thus we have
\[
\|U_{xx}(t)\|_{L^p} = \int_R \frac{|U_0''(x_0)|^p}{(1 + U_0'(x_0)t)^{1-p}} \, dx_0 = 2 \int_{-\infty}^0 \frac{|U_0''(x_0)|^p}{(1 + U_0'(x_0)t)^{1-p}} \, dx_0
\leq C \int_0^{U_0'(x_0)t} \frac{|U_0''(x_0)|^{p-1} t^{-1}}{(1 + y)^{3p-1}} \, dy
\leq C \int_0^{U_0'(x_0)t} \frac{|U_0''(x_0)|^{p-1} t^{-1}}{(1 + y)^{3p-1}} \, dy \]
and similarly, by tedious calculations, we have

\[ C_{p,q} \chi^{(1-\frac{1}{p}) (p-1)} \delta^{-\frac{k+1}{p} t^{-1}} \int_0^{U_0'(0) t} \frac{t^{-\left(1+\frac{1}{p}\right)(p-1)}}{(1+y)^{p-\frac{k+1}{p}}} dy \]

\[ \leq C_{p,q} \chi^{(1-\frac{1}{p}) (p-1)} \delta^{-\frac{k+1}{p}} t^{-p-\frac{k+1}{p}}, \]

while for \( p = \infty \), we have

\[ \left\| \partial_x^k U(t) \right\|_{L^\infty} \leq C_0 \frac{e^{k-1-\frac{k+1}{p} \delta^{-\frac{k+1}{p}} t^{-1}}}{U_0'(x_0) t^{k+1}} \leq C_0 e^{k-1-\frac{k+1}{p} \delta^{-\frac{k+1}{p}} t^{-1}}. \]

Putting (6.7)-(6.11) together, we finished the proof of (iii) of Lemma 2.1.

6.2. The proof of Lemma 2.2. By directly calculation, we have

\[
\begin{align*}
\partial_t x_0 &= -\frac{U_0(x_0)}{1+U_0'(x_0)t}, & U_t &= U_0'(x_0) \cdot \frac{\partial x_0}{\partial t} = -\frac{U_0 U_0''}{1+U_0'}, \\
U_{tt} &= \frac{2U_0 U_0'' + U_0'^2 U_0'''}{(1+U_0')^2} - \frac{U_0}{1+U_0'} U_0'' + U_0 U_0''', & U_{xt} &= -\frac{U_0^4 + U_0'' U_0'''}{1+U_0'} + U_0 U_0''' t, \\
U_{ttx} &= -\frac{6U_0 U_0''}{(1+U_0'(x_0)t)^3} - \frac{9U_0^3 U_0'''}{(1+U_0'(x_0)t)^4} + \frac{3U_0^3 U_0'''}{1+U_0'(x_0)t}, & U_{xxt} &= -\frac{U_0 U_0'' + 3U_0'' U_0'''}{(1+U_0'(x_0)t)^4} + \frac{3U_0 U_0'''}{1+U_0'(x_0)t},
\end{align*}
\]

It’s easy to check that

\[
\begin{align*}
|U_{tt}| + |U_{xt}| &\leq \frac{|U_0|^2}{(1+U_0')^2} + \frac{|U_0''|^2}{(1+U_0')^3}, \\
|U_{ttx}| &\leq \frac{|U_0'|^3}{(1+U_0')^3} + \frac{|U_0'' U_0'''|}{(1+U_0')^4} + \frac{|U_0''|^2 t}{(1+U_0')^5}, \\
|U_{xxt}| &\leq \frac{|U_0'|^3}{(1+U_0')^3} + \frac{|U_0'' U_0'''|}{(1+U_0')^4} + \frac{|U_0''|^2 t}{(1+U_0')^5}, \\
|U_{xxt}| &\leq \frac{|U_0'' U_0'''|}{(1+U_0')^4} + \frac{|U_0''|^2 t}{(1+U_0')^5},
\end{align*}
\]

and similarly, by tedious calculations, we have

\[
\begin{align*}
U_{tttx} &= \frac{24U_0 U_0''}{(1+U_0')^4} + \frac{72 U_0^2 |U_0''|^2 U_0''' + 12 U_0^3 U_0'' U_0^{(3)} + U_0^4 U_0^{(4)}}{(1+U_0')^5} \\
&\quad - \frac{60 U_0^5 U_0'' |U_0''|^2 t + 10 U_0^4 U_0'' U_0^{(3)} t}{(1+U_0')^6} + \frac{15 U_0^4 |U_0''|^3 t^2}{(1+U_0')^7}, \\
U_{xxtt} &= -\frac{6 U_0' U_0''}{(1+U_0')^4} - \frac{36 U_0 U_0''^2 U_0''' + 9 U_0^2 |U_0'''|^2 + 12 U_0^2 U_0'' U_0^{(3)} + U_0''}{} \frac{|U_0''|^3 U_0^{(4)}}{(1+U_0')^5},
\end{align*}
\]
\[
\frac{45 [U_0^3]^2 U_0' [U_0'']^2 t + 10 [U_0^4] U_0' U_0^{(3)} t}{(1 + U_0^4)^6} - \frac{15 [U_0^3] [U_0'']^3 t^2}{(1 + U_0^4)^7},
\]

\[
U_{xxtt} = \frac{12 [U_0'^2] U''_0 + 6 U_0 [U''_0]^2 + 8 U_0' U_0' U_0^{(3)} + [U_0']^2 U_0^{(4)}}{(1 + U_0^4)^7}
- \frac{30 U_0 [U_0']^2 t + 10 [U_0']^2 U_0' U_0^{(3)} + [U_0']^2 U_0^{(4)}}{(1 + U_0^4)^7} + \frac{15 [U_0]^2 [U_0'']^2 t^2}{(1 + U_0^4)^7},
\]

\[
U_{xttt} = -\frac{4U_0' U_0^{(3)} + U_0 U_0''^2 + 3 [U_0'']^2 + 15 U_0' [U_0'']^2 t + 10 U_0 U_0'' U_0^{(3)} t}{(1 + U_0^4)^7} - \frac{15 U_0 [U_0'']^3 t^2}{(1 + U_0^4)^7}.
\]

Having obtained the above identities, we can prove Lemma 2.2 directly by employing the arguments used to deal with (6.7)–(6.11) and by using the results obtained in Lemma 2.1, we omit the details for brevity.

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