SYMBOLIC POWERS OF IDEALS AND THEIR TOPOLOGY OVER A MODULE

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Abstract. Let I denote an ideal of a Noetherian ring R and N a non-zero finitely generated R-module. In the present paper, some necessary and sufficient conditions are given to determine when the I-adic topology on N is equivalent to the I-symbolic topology on N. Among other things, we shall give a complete solution to the question raised by R. Hartshorne in [Affine duality and cofiniteness, Invent. Math. 9(1970), 145-164], for a prime ideal p of dimension one in a local Noetherian ring R, by showing that the p-adic topology on N is equivalent to the p-symbolic topology on N if and only if for all z ∈ Ass_R N there exists q ∈ Supp(N*) such that z ⊆ q and q ∩ R = p. Also, it is shown that if for every p ∈ Supp(N) with dim R/p = 1, the p-adic and the p-symbolic topologies are equivalent on N, then N is unmixed and Ass_R N has only one element. Finally, we show that if Ass_R N* consists of a single prime ideal, for all p ∈ A*(I, N), then the I-adic and the I-symbolic topologies on N are equivalent.

1. Introduction

In this paper we continue the study of the equivalence between the I-adic topology and the I-symbolic topology on N. Let R denote a commutative Noetherian ring, I an ideal of R and let N be a finitely generated R-module. For a natural number n, the nth symbolic power of I with respect to N, denoted by (IN)^(n), is defined to be the intersection of those primary components of I^n N which correspond to the minimal primes of Ass_R N/IN. It is easy to see that the submodule (IN)^(n) is the set of all elements x in N for which there exists an element s in R \ ∪ p such that sx ∈ I^n N, where p runs over the set of minimal primes of Ass_R N/IN. This definition is inspired by the one given in [18, Remark, p. 233]. Symbolic powers of ideals are central objects in commutative algebra and algebraic geometry for their tight connection to primary decomposition of ideals and the order of vanishing of polynomials.

The I-adic filtration \{I^n N\}_{n≥0} and the I-symbolic filtration \{(IN)^(n)\}_{n≥0} induce topologies on N which are called the I-adic and the I-symbolic topology respectively. One readily sees from the definition that I^n N ⊆ (IN)^(n) for all natural numbers n, so that the I-adic topology on N is stronger than the I-symbolic topology on N, but there are not equivalent in general. Therefore, one would like to compare the I-adic topology and the I-symbolic topology on N and provide some criterions for equivalence. These two
topologies are said to be equivalent if for every natural number \( n \), there is a natural number \( m \) such that \( I^mN \) contains \( (IN)^{(m)} \). R. Hartshorne in [11 Proposition 7.1] proved that if \( \mathfrak{p} \) is a prime ideal of dimension one of a complete local ring \( R \), then the \( \mathfrak{p} \)-adic topology on \( N \) is equivalent to the \( \mathfrak{p} \)-symbolic topology on \( N \) if and only if every associated prime ideal of \( N \) is contained in \( \mathfrak{p} \). In this paper Hartshorne writes: “A general question, whose solution is quite complicated, is to determine when the \( \mathfrak{p} \)-adic topology is equivalent to the \( \mathfrak{p} \)-symbolic topology”. With respect to this question, P. Schenzel in [15, Theorem 1] gave a solution to this problem, in the case when \( R = N \), and later S. McAdam and L.J. Ratliff in [8] gave an elegant proof of Schenzel’s theorem. Finally, L.J. Ratliff in [14] and J.K. Verma in [17] generalized Schenzel’s theorem to primary and arbitrary ideals, respectively. The purpose of this paper is to give a generalization of Hartshorne’s result by removing the completeness condition on ring. Moreover, we prove some new results concerning on the equivalence of the \( I \)-adic topology and the \( I \)-symbolic topology on \( N \).

Namely, we show that these topologies are equivalent in the following cases:

(i) The \( \mathfrak{p} \)-adic and the \( \mathfrak{p} \)-symbolic topologies on \( N \) are equivalent, for all \( \mathfrak{p} \in \text{mAss}_R N/IN \).

(ii) \( N \) is locally unmixed and \( I = \text{Rad}(J + \text{Ann}_R N) \), where \( J \) is an \( N \)-proper ideal of \( R \) generated by \( \text{height}_N J \) elements.

(iii) \( \text{Ass}_R^* N^* \) consists of a single prime ideal, for all \( \mathfrak{p} \in A^*(I, N) \).

(iv) For every ideal \( J \) of \( R \) with \( \text{Ass}_R N/JN = \text{mAss}_R N/JN \) and \( I \subseteq J \), the \( I \)-symbolic topology is finer than the \( J \)-symbolic topology on \( N \).

In addition, we show that if \( R \) is local and for every \( \mathfrak{p} \in \text{Supp}(N) \) with \( \dim R/\mathfrak{p} = 1 \), the \( \mathfrak{p} \)-adic and the \( \mathfrak{p} \)-symbolic topologies are equivalent, then \( N \) is unmixed and \( \text{Ass}_R N \) has only one element.

Throughout this paper all rings are commutative and Noetherian, with identity, unless otherwise specified. We shall use \( R \) to denote such a ring, \( I \) an ideal of \( R \), and \( N \) a non-zero finitely generated module over \( R \). We denote by \( \mathcal{R} \) the Rees ring \( R[u, It] := \bigoplus_{n \in \mathbb{Z}} I^n t^n \) of \( R \) with respect to \( I \), where \( t \) is an indeterminate and \( u = t^{-1} \). Also, the graded Rees module \( N[u, It] := \bigoplus_{n \in \mathbb{Z}} I^n N \) over \( \mathcal{R} \) is denoted by \( \mathcal{N} \), which is a finitely generated \( \mathcal{R} \)-module.

If \((R, \mathfrak{m})\) is local, then \( R^* \) (resp. \( N^* \)) denotes the completion of \( R \) (resp. \( N \)) w.r.t. the \( \mathfrak{m} \)-adic topology. Then \( N \) is said to be an unmixed (resp. a quasi-unmixed) module if all the prime ideals (resp. all the minimal prime ideals) of \( \text{Ass}_{R^*} N^* \) have the same dimension. More generally, if \( R \) is not necessarily local, \( N \) is locally unmixed (resp. locally quasi-unmixed) module if for any \( \mathfrak{p} \in \text{Supp}(N) \), \( N_\mathfrak{p} \) is an unmixed (a quasi-unmixed) module over \( R_\mathfrak{p} \). We shall say that an ideal \( J \) of \( R \) is \( N \)-proper if \( N/JN \neq 0 \), and, when this is the case, we define the \( N \)-height of \( J \) (written \( \text{height}_N J \)) to be \( \inf \{ \text{height}_N \mathfrak{p} : \mathfrak{p} \in \text{Supp} N \cap V(J) \} \), where \( \text{height}_N \mathfrak{p} \) is defined to be the supremum of lengths of chains of prime ideals of \( \text{Supp}(N) \) terminating with \( \mathfrak{p} \). For any ideal \( \mathfrak{a} \) of \( R \), the radical of \( \mathfrak{a} \), denoted by \( \text{Rad}(\mathfrak{a}) \), is defined to be the set \( \{ x \in \mathfrak{a} : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N} \} \). For any \( R \)-module \( L \), we denote by \( \text{mAss}_R L \) the set of minimal prime ideals of \( \text{Ass}_R L \). For any unexplained notation and terminology we refer the reader to [3] or [9].
In the second section, we examine the equivalence between the \( I \)-adic and the \( I \)-symbolic topologies. In this section, we show that these topologies are equivalent, whenever for every \( p \in \text{mAss}_R N/IN \), the \( p \)-adic topology on \( N \) is equivalent to the \( p \)-symbolic topology on \( N \).

The main result of the third section is that if \( N \) is a non-zero finitely generated locally unmixed \( R \)-module and \( I \) an \( N \)-proper ideal of \( R \) generated by height \( N \) \( I \) elements, then the \( J \)-adic topology is equivalent to the \( J \)-symbolic topology, where \( J = \text{Rad}(I + \text{Ann}_R N) \). Also, in this section we shall extend a nice result of Hartshorne [4, Proposition 7.1].

Namely, we shall show the following result:

**Theorem 1.1.** If \((R, \mathfrak{m})\) is local and \( p \in \text{Supp}(N) \) with \( \dim R/p = 1 \), then the \( p \)-adic topology on \( N \) is equivalent to the \( p \)-symbolic topology on \( N \) if and only if for all \( z \in \text{Ass}_R N^* \) there exists \( q \in \text{Supp}(N^*) \) such that \( z \subseteq q \) and \( q \cap R = p \).

Finally, the main purpose of the Section 4 is to establish a connection between the unmixedness (resp. associated primes) of \( N \) and the comparison of the \( p \)-adic topology on \( N \) and the \( p \)-symbolic topology on \( N \) (resp. \( I \)-adic topology and the \( I \)-symbolic topology on \( N \)), for certain prime ideals \( p \) of \( R \). More precisely, we show that:

**Theorem 1.2.** Let \( N \) be a non-zero finitely generated module over a commutative Noetherian ring \( R \).

(i) If \( \text{Ass}_{R^p} N^* \) consists of a single prime, for all \( p \in A^*(I) \), then the \( I \)-adic topology on \( N \) is equivalent to the \( I \)-symbolic topology on \( N \).

(ii) If \((R, \mathfrak{m})\) is local and for every \( p \in \text{Supp}(N) \) with \( \dim R/p = 1 \), the \( p \)-adic topology on \( N \) is equivalent to the \( p \)-symbolic topology on \( N \), then \( N \) is unmixed and \( \text{Ass}_R N \) consists of a single prime.

2. Comparison of Topologies

The purpose of this section is to examine the equivalence between the \( I \)-adic and the \( I \)-symbolic topologies. The main goal of this section is Theorem 2.5, which shows that these topologies are equivalent, whenever for every \( p \in \text{mAss}_R N/IN \), the \( p \)-adic topology is equivalent to the \( p \)-symbolic topology. Before we state the main result of this section, let us give a definition:

**Definition 2.1.** A prime ideal \( p \) of \( R \) is called a *quitessential prime ideal* of \( I \) w.r.t. \( N \) precisely when there exists \( q \in \text{Ass}_{R^p} N^* \) such that \( \text{Rad}(IR^*_p + q) = pR^*_p \). The set of quitessential primes of \( I \) w.r.t. \( N \) is denoted by \( Q((u\mathcal{R}, N)) \). The concepts of the quitessential and essential prime ideals of \( I \) were introduced by McAdam [7], and Ahn in [1] extended them to modules.

**Proposition 2.2.** Let \( N \) be a finitely generated \( R \)-module and \( I \) an \( N \)-proper ideal of \( R \). Then the following conditions are equivalent:

(i) The \( I \)-symbolic topology on \( N \) is equivalent to the \( I \)-adic topology on \( N \).
(ii) For every $N$-proper ideal $J$ of $R$ with $\text{Ass}_R N/JN = \text{mAss}_R N/JN$ and $I \subseteq J$, the $I$-symbolic topology on $N$ is finer than the $J$-symbolic topology on $N$.

Proof. (i) $\implies$ (ii): Let $J$ be an $N$-proper ideal of $R$ such that $I \subseteq J$ and that $\text{Ass}_R N/JN = \text{mAss}_R N/JN$. For every integer $l \geq 0$, we need to show that there exists an integer $m \geq 0$ such that $(IN)^{(m)} \subseteq (JN)^{(l)}$. To this end, in view of assumption (i), there is an integer $m \geq 0$ such that $(IN)^{(m)} \subseteq I^lN$, and so $(IN)^{(m)} \subseteq J^lN \subseteq (JN)^{(l)}$, as required.

(ii) $\implies$ (i): In view of Corollary 3.7, it is enough for us to show that $Q(I, N) = \text{mAss}_R N/IN$. To achieve this, suppose the contrary is true. Then there is an element $p \in Q(I, N)$ such that $p \notin \text{mAss}_R N/IN$. (Note that $\text{mAss}_R N/IN \subseteq Q(I, N)$.) Hence, in view of Theorem 3.6, there exists an integer $k \geq 0$ such that $(IN)^{(m)} \nsubseteq (pN)^{(k)}$ for all integer $m \geq 0$. Now, because of $I \subseteq p$ and $\text{Ass}_R N/pN = \text{mAss}_R N/pN$, the assumption (ii) provides a contradiction. \hfill \Box

The following proposition and its corollary are quite useful in the proof of the main theorem.

Proposition 2.3. Let $N$ be a non-zero finitely generated $R$-module and $I_1, I_2$ be two $N$-proper ideals of $R$ such that

$$\text{mAss}_R N/(I_1I_2)N = \text{mAss}_R N/I_1N \cup \text{mAss}_R N/I_2N.$$ 

Suppose that the $I_i$-symbolic topology on $N$ is equivalent to the $I_i$-adic topology on $N$, for $i = 1, 2$. Then the $I_1I_2$-symbolic topology on $N$ is equivalent to the $I_1I_2$-adic topology on $N$ and $(I_1 \cap I_2)$-symbolic topology on $N$ is equivalent to the $(I_1 \cap I_2)$-adic topology on $N$.

Proof. As $\text{Rad}(I_1I_2) = \text{Rad}(I_1 \cap I_2)$ and $\text{mAss}_R N/(I_1I_2)N = \text{mAss}_R N/(I_1 \cap I_2)N$, it is enough, in view of Lemma 3.1 and Corollary 3.7, to show that $Q(I_1I_2, N) = \text{mAss}_R N/(I_1I_2)N$. To achieve this, suppose that $p \in Q(I_1I_2, N)$. Then there exists $z \in \text{Ass}_R N^*_p$ such that $\text{Rad}(I_1I_2 R^*_p + z) = pR^*_p$. As $I_1I_2 \subseteq p$, without loss of generality we may assume that $I_1 \subseteq p$. Then $\text{Rad}(I_1 R^*_p + z) = pR^*_p$, and so $p \in Q(I_1, N)$. Hence, in view of assumption and Corollary 3.7, $p \in \text{mAss}_R N/I_1N$. Therefore, it follows from

$$\text{mAss}_R N/(I_1I_2)N = \text{mAss}_R N/I_1N \cup \text{mAss}_R N/I_2N,$$

that $p \in \text{mAss}_R N/(I_1I_2)N$, as required. \hfill \Box

Corollary 2.4. Let $N$ be a non-zero finitely generated $R$-module and let $I_1, \ldots, I_n$ be $N$-proper ideals of $R$ such that $\text{mAss}_R N/(\prod_{i=1}^n I_i)N = \bigcup_{i=1}^n \text{mAss}_R N/I_iN$, and that the $I_i$-symbolic topology on $N$ is equivalent to the $I_i$-adic topology on $N$, for all $i = 1, \ldots, n$. Then the $\prod_{i=1}^n I_i$-symbolic topology on $N$ (resp. $\bigcap_{i=1}^n I_i$-symbolic topology on $N$) is equivalent to the $\prod_{i=1}^n I_i$-adic (resp. $\bigcap_{i=1}^n I_i$-adic) topology on $N$. 
Proof. The result follows from Proposition 2.3 and induction on \( n \).

We are now ready to state and prove the main theorem of this section, which gives us a criterion of the equivalence between the \( I \)-adic and the \( I \)-symbolic topologies.

**Theorem 2.5.** Let \( N \) be a non-zero finitely generated \( R \)-module and let \( I \) be an \( N \)-proper ideal of \( R \) such that the \( p \)-symbolic topology is equivalent to the \( p \)-adic topology on \( N \), for all \( p \in \text{mAss}_R N/IN \). Then the \( I \)-symbolic topology on \( N \) is equivalent to the \( I \)-adic topology on \( N \).

Proof. Let \( \text{mAss}_R N/IN = \{p_1, \ldots, p_n\} \). Then it is easy to see that

\[
\text{mAss}_R N/(\prod_{i=1}^n p_i)N = \bigcup_{i=1}^n \text{mAss}_R N/p_iN.
\]

Now, the assertion follows from Corollary 2.4 and [10, Lemma 3.1].

3. Locally Unmixed Modules and Comparison of Topologies

The main goal of this section is to prove the equivalence between the \( I \)-adic and the \( I \)-symbolic topologies on a finitely generated locally unmixed \( R \)-module \( N \) for certain ideals \( I \) of \( R \). Also, we explore an equivalence between the \( p \)-adic and the \( p \)-symbolic topologies and the associated primes of \( N \), for prime ideals \( p \) of dimension one. We begin with:

**Definition 3.1.** A prime ideal \( p \) of \( R \) is called a *quitasymptotic prime ideal of \( I \) w.r.t. \( N \)* precisely when there exists \( q \in \text{mAss}_R N/\mathfrak{p}N \) such that \( \text{Rad}(IR_p + q) = \mathfrak{p}R_p \). The set of quitasymptotic prime ideals of \( I \) w.r.t. \( N \), denoted by \( \overline{Q}^*_I(N) \), is defined to be the set \( \{q \cap R \mid q \in \overline{Q}^*(u\mathfrak{p}, N)\} \).

**Lemma 3.2.** Let \( N \) be a non-zero finitely generated locally unmixed \( R \)-module and let \( I \) be an ideal of \( R \). Then

\[
E(I, N) = \overline{A}^*(I, N).
\]

Proof. In view of [11, Corollary 3.7], it is enough for us to show that \( E(I, N) \subseteq \overline{A}^*(I, N) \).

To do this let \( p \in E(I, N) \). Since both \( E(I, N) \) and \( \overline{A}^*(I, N) \) behave well under localization, without loss of generality, we may assume that \((R, p)\) is local. Also, in view [11, Proposition 3.8], it is easy to see that we may assume in addition that \( R \) is complete. Now, according to [11, Proposition 3.6], there exists \( z \in \text{Ass}_R N \) such that \( z \subseteq p \) and \( p/z \in E(p + z) \). Since \( R/z \) is unmixed it follows from [5, Proposition 2.11] that \( p/z \in \overline{A}^*(I + z) \). Moreover, since by hypothesis \( z \in \text{mAss}_R N \), it follows from [11, Proposition 3.6] that \( p \in \overline{A}^*(I, N) \), as required.

Before we state the next lemma, let us recall the important notion *analytic spread of \( I \) with respect to \( N \)*, over a local ring \((R, \mathfrak{m})\), introduced by Brodmann in [3]:

\[
l(I, N) := \dim \mathcal{N}(I, N)/(\mathfrak{m}, u)\mathcal{N}(I, N),
\]
in the case \( N = R \), \( l(I,N) \) is the classical analytic spread \( l(I) \) of \( I \), introduced by Northcott and Rees (see [13]).

**Lemma 3.3.** Let \( N \) be a non-zero finitely generated locally quasi-unmixed \( R \)-module and let \( I \) be an \( N \)-proper ideal of \( R \) generated by \( \text{height}_N I \) elements. Then \( \bar{A}^*(I,N) = \text{mAss}_R N/IN \).

**Proof.** As \( \text{mAss}_R N/IN \subseteq \bar{A}^*(I,N) \), it will suffice for us to show that \( \bar{A}^*(I,N) \subseteq \text{mAss}_R N/IN \). To this end, let \( \mathfrak{p} \in \bar{A}^*(I,N) \). Since \( N_{\mathfrak{p}} \) is a quasi-unmixed \( \mathcal{R}_{\mathfrak{p}} \)-module, it follows from [12, Proposition 2.3] that \( \text{height}_N \mathfrak{p} = \ell(IR_{\mathfrak{p}}, N_{\mathfrak{p}}) \). Since at least \( \ell(\mathfrak{a}) \) elements are needed to generate \( \mathfrak{a} \), for any ideal \( \mathfrak{a} \) in a commutative Noetherian ring \( A \), it follows from [12, Lemma 2.2] that \( \ell(IR_{\mathfrak{p}}, N_{\mathfrak{p}}) \leq \text{height}_N I \), and so \( \text{height}_N \mathfrak{p} = \text{height}_N I \). Therefore \( \mathfrak{p} \in \text{mAss}_R N/IN \), as required. \( \square \)

The following theorem, which is one of our main results of this section, shows that for certain ideals \( I \), the \( I \)-symbolic topology on \( N \) is equivalent to the \( I \)-adic topology on \( N \), whenever \( N \) is a finitely generated locally unmixed \( R \)-module.

**Theorem 3.4.** Let \( N \) be a non-zero finitely generated locally unmixed \( R \)-module and let \( J \) be an \( N \)-proper ideal of \( R \) generated by \( \text{height}_N J \) elements. Then the \( I \)-symbolic topology on \( N \) is equivalent to the \( I \)-adic topology on \( N \), where \( I = \text{Rad}(J + \text{Ann}_R N) \).

**Proof.** In view of [11, Corollary 3.7 and Lemma 3.1], it will suffice to show that \( Q(I,N) \subseteq \text{mAss}_R N/IN \). For this let \( \mathfrak{p} \in Q(I,N) \). Then, it follows from [11, Lemma 3.1] that \( \mathfrak{p} \in Q(J,N) \). Thus, by [11 Corollary 3.7], \( \mathfrak{p} \in E(J,N) \), and so by virtue of Lemma [5.2] \( \mathfrak{p} \in \bar{A}^*(J,N) \). Therefore, in view of Lemma [3.3], \( \mathfrak{p} \in \text{mAss}_R N/JN \). Now, as \( \text{mAss}_R N/JN = \text{mAss}_R N/JN \), the desired result follows. \( \square \)

The next theorem, which is the final main result of this section, extends a nice result of Hartshorne [1] Proposition 7.1).

**Theorem 3.5.** Let \( (R, \mathfrak{m}) \) be a local ring and \( N \) a non-zero finitely generated \( R \)-module. Let \( \mathfrak{p} \in \text{Supp}(N) \) with \( \dim R/\mathfrak{p} = 1 \). Then the following conditions are equivalent:

(i) The \( \mathfrak{p} \)-symbolic topology on \( N \) is equivalent to the \( \mathfrak{p} \)-adic topology on \( N \).

(ii) For all \( z \in \text{Ass}_{R^*} N^* \) there exists \( \mathfrak{q} \in \text{Supp}(N^*) \) such that \( z \subseteq \mathfrak{q} \) and \( \mathfrak{q} \cap R = \mathfrak{p} \).

**Proof.** \( (i) \implies (ii) \): Let \( z \in \text{Ass}_{R^*} N^* \). In view of [11, Corollary 3.7] and the assumption (i), we have

\[
Q(\mathfrak{p}, N) = \text{mAss}_R N/\mathfrak{p}N = \{\mathfrak{p}\}.
\]

Therefore \( \mathfrak{m} \notin Q(\mathfrak{p}, N) \), and so \( \mathfrak{m}R^* \) is not minimal over \( \mathfrak{p}R^* + z \). Let \( \mathfrak{q} \) be a minimal prime over \( \mathfrak{p}R^* + z \). Then \( \mathfrak{q} \in \text{Supp}(N^*) \) and \( \mathfrak{p} \subseteq \mathfrak{q} \cap R \). Now, as \( \dim R/\mathfrak{p} = 1 \), one easily sees that \( \mathfrak{q} \cap R = \mathfrak{p} \) and \( z \subseteq \mathfrak{q} \), as required.

In order to prove \( (ii) \implies (i) \), in view of [11 Corollary 3.7], it is enough for us to show that \( Q(\mathfrak{p}, N) = \{\mathfrak{p}\} \). To do this, let \( \mathfrak{q} \in Q(\mathfrak{p}, N) \). Then \( \mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{m} \). Since \( \dim R/\mathfrak{p} = 1 \), we see that \( \mathfrak{q} = \mathfrak{p} \) or \( \mathfrak{q} = \mathfrak{m} \). If \( \mathfrak{q} = \mathfrak{m} \), then \( \mathfrak{m} \in Q(\mathfrak{p}, N) \), and so there exists \( z \in \text{Ass}_{R^*} N^* \) such that \( \text{Rad}(\mathfrak{p}R^* + z) = \mathfrak{m}R^* \). Hence, by the assumption (ii), there exists \( \mathfrak{q}' \in \text{Supp}(N^*) \) such that...
such that $z \subseteq q'$ and $q' \cap R = p$. Therefore, $q' \subseteq pR^*$, and so $\text{Rad}(pR^*) = mR^*$. Consequently, $\dim R^*/pR^* = 0$ which is a contradiction, because $\dim R^*/pR^* = \dim R/p = 1$. Whence $q = p$ and this completes the proof. \hfill $\square$

4. ASSOCIATED PRIMES AND UNMIXEDNESS

The main aim of this section shows that if $\text{Ass}_{R_p} N_p^*$ consists of a single prime, for all $p \in A^*(I, N)$, then the $I$-adic topology is equivalent to the $I$-symbolic topology on $N$. Furthermore, we show that, if $(R, \mathfrak{m})$ is local and for every $p \in \text{Supp}(N)$ with $\dim R/p = 1$, the $p$-adic topology is equivalent to the $p$-symbolic topology on $N$, then $N$ is unmixed and $\text{Ass}_R N$ has only one element. Following [2], we shall use $A^*(I, N)$ to denote the ultimately constant values of $\text{Ass}_R N/I^n N$ for all large $n$. The following theorem is the first main result of this section.

**Theorem 4.1.** Let $N$ be a non-zero finitely generated $R$-module and $I$ an $N$-proper ideal of $R$ such that $\text{Ass}_{R_p} N_p^*$ consists of a single prime ideal $z$, for all $p \in A^*(I, N)$. Then the $I$-symbolic topology on $N$ is equivalent to the $I$-adic topology on $N$.

**Proof.** In view of [11 Corollary 3.7], it will suffice to show that $Q(I, N) = \text{mAss}_R N/I N$. To do this, suppose the contrary is true. That is there exists $p \in Q(I, N)$ such that $p \notin \text{mAss}_R N/I N$. Since $p \in \text{Supp}(N/I N)$, it follows that there exists $q \in \text{mAss}_R N/I N$ such that $q \subseteq p$. Moreover, by virtue of [11 Theorem 3.17], $Q(I, N) \subseteq A^*(I, N)$, hence $\text{Ass}_{R_p} N_p^* = \{z\}$. Therefore, $\text{Rad}(IR_p^* + z) = pR_p^*$. Now, let $q^*$ be a minimal prime over $qR_p^*$. Then $IR_p^* \subseteq qR_p^* \subseteq q^*$. Furthermore, as $q \in \text{Supp}(N)$, it easily follows from [9 Theorem 18.1] that $q^* \in \text{Supp}(N_p^*)$, and so $z \subseteq q^*$. Consequently $pR_p^* \subseteq q^*$. On the other hand, since $q^*$ is a minimal prime over $qR_p^*$, we can therefore deduce from the Going-down Theorem (see [6 Theorem 9.5]) that $q^* \cap R_p = qR_p$. Hence $qR_p = pR_p$, and so $q = p$, which is a contradiction. \hfill $\square$

The following proposition is needed in the proof of the second main theorem.

**Proposition 4.2.** Let $(R, \mathfrak{m})$ be a local ring and $N$ a non-zero finitely generated $R$-module such that $\dim N > 0$ and that $\text{Ass}_R N$ has at least two elements. Then there exists an $N$-proper ideal $I$ of $R$ such that

$$m \in Q(I, N) \setminus \text{mAss}_R N/I N.$$ 

**Proof.** In view of assumption, there exist $z_1, z_2 \in \text{Ass}_R N$ such that $z_1 \neq z_2$. Without loss of generality, we may assume that $z_1 \in \text{mAss}_R N$. Let $n := \dim R/z_1 + z_2$. If $n = \dim N$, then there exists a minimal prime $p$ over $z_1 + z_2$ such that

$$\dim R/z_1 + z_2 = \dim R/p = n = \dim N.$$ 

Hence $p \in \text{mAss}_R N$ and $z_1 + z_2 \subseteq p$. Consequently, $z_1 = p = z_2$, which is a contradiction. Therefore, $n < \dim N$. Suppose $n = 0$. Then $z_1 + z_2$ is $m$-primary, and so in view of [11 Lemma 3.5], $m \in Q(z_1, N) \setminus \text{mAss}_R N/z_1 N$, as required.
Now, suppose \( n > 0 \). Then there exist elements \( a_1, \ldots, a_n \) of \( \mathfrak{m} \) such that their images in \( R/z_1 + z_2 \) form a system of parameters. Let \( J = (a_1, \ldots, a_n) \). Then \( \text{Rad}(J + z_1 + z_2) = \mathfrak{m} \), and so \( J + z_1 + z_2 \) is \( \mathfrak{m} \)-primary. Now, if \( \text{Rad}(J + z_1) = \mathfrak{m} \), then as \( z_1 \in \text{Ass}_R N \), it follows from [1, Lemma 3.5] that \( \mathfrak{m} \in Q(I, N) \). Moreover, \( \mathfrak{m} \not\in \text{mAss}_R N/JN \). Because, if \( \mathfrak{m} \in \text{mAss}_R N/JN \), then \( \mathfrak{m} = \text{Rad}(J + \text{Ann}_R N) \). Hence, 

\[
\text{height}(\mathfrak{m}/ \text{Ann} N) = \text{height}(J + \text{Ann}_R N/ \text{Ann}_R N),
\]

and so \( \text{dim} N = \text{height}_N J \leq n \), which is a contradiction. Also, if \( \text{Rad}(J + z_1) \neq \mathfrak{m} \), then \( \mathfrak{m} \not\in \text{mAss}_R N/(J + z_1)N \). Hence, using [1, Lemma 3.5] and \( \text{Rad}(J + z_1 + z_2) = \mathfrak{m} \), we obtain that \( \mathfrak{m} \in Q(J + z_1, N) \). This completes the proof. \( \square \)

Now, we can state and prove the second main theorem of this section.

**Theorem 4.3.** Let \((R, \mathfrak{m})\) be a local ring and let \( N \) a non-zero finitely generated \( R \)-module of positive dimension. Suppose that for every \( \mathfrak{p} \in \text{Supp}(N) \) with \( \text{dim} R/\mathfrak{p} = 1 \), the \( \mathfrak{p} \)-symbolic topology on \( N \) is equivalent to the \( \mathfrak{p} \)-adic topology on \( N \). Then \( N \) is unmixed and \( \text{Ass}_R N \) has exactly one element.

**Proof.** If \( \mathfrak{m}R^* \in \text{Ass}_{R^*} N^* \), then it follows easily from [6, Theorem 23.2] that \( \mathfrak{m} \in \text{Ass}_R N \), and so \( \mathfrak{m} \in Q(I, N) \) for every ideal \( I \) of \( R \). Now, as \( \text{dim} N > 0 \) there exists \( \mathfrak{p} \in \text{Supp}(N) \) such that \( \mathfrak{p} \not\subseteq \mathfrak{m} \) and \( \text{dim} R/\mathfrak{p} = 1 \). Hence

\[
\mathfrak{m} \in Q(\mathfrak{p}, N) = \text{mAss}_R N/\mathfrak{p}N = \{\mathfrak{p}\},
\]

which is a contradiction. Therefore \( \mathfrak{m}R^* \not\in \text{Ass}_{R^*} N^* \). Now, we show that \( N \) is unmixed. To do this, suppose the contrary, i.e., \( N \) is not unmixed. Then there exists \( z \in \text{Ass}_R N^* \) such that \( \text{dim} R^*/z < \text{dim} N \). Since \( \mathfrak{m}R^* \not\in \text{Ass}_{R^*} N^* \), we have \( \text{dim} R^*/z > 0 \). Therefore, in view of [10, Proposition 3.5] there exists an \( N \)-proper ideal \( I \) of \( R \) generated by \( \text{height}_N I \) elements, such that 

\[
\text{Rad}(IR^* + z) = \mathfrak{m}R^* \quad \text{and} \quad \text{height}_N I = \text{dim} R^*/z.
\]

Consequently, \( \mathfrak{m} \in Q(I, N) \), and as \( \text{height}_N I < \text{dim} N \), there exists \( \mathfrak{p} \in \text{Supp}(N) \) such that \( I \subseteq \mathfrak{p} \) and \( \text{dim} R/\mathfrak{p} = 1 \). Hence \( \mathfrak{p} \not\subseteq \mathfrak{m} \) and \( \mathfrak{m} \in Q(\mathfrak{p}, N) \). Now, since

\[
Q(\mathfrak{p}, N) = \text{mAss}_R N/\mathfrak{p}N = \{\mathfrak{p}\},
\]

it follows that \( \mathfrak{m} = \mathfrak{p} \), which is a contradiction, so \( N \) is unmixed. Now in order to complete the proof, we must show that \( \text{Ass}_R N \) consists of a single prime. To this end, suppose that the contrary is true. Then, by Proposition [4,2] there exists an ideal \( I \) of \( R \) such that 

\[
\mathfrak{m} \in Q(I, N) \setminus \text{mAss}_R N/IN.
\]

Since \( \mathfrak{m} \not\in \text{mAss}_R N/IN \), there exists \( \mathfrak{p} \in \text{Ass}_R N/IN \) such that \( \text{dim} R/\mathfrak{p} = 1 \). Hence, \( \mathfrak{m} \in Q(\mathfrak{p}, N) \). Now, because of

\[
Q(\mathfrak{p}, N) = \text{mAss}_R N/\mathfrak{p}N = \{\mathfrak{p}\},
\]

we see that \( \mathfrak{m} = \mathfrak{p} \). Therefore, we have arrived at a contradiction, and so \( \text{Ass}_R N \) has only one element, as required. \( \square \)
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