Possible confinement mechanisms for nonrelativistic particles on a line

P.C. Stichel
An der Krebskuhle 21
D-33619 Bielefeld, Germany
e-mail: pstichel@gmx.de

W.J. Zakrzewski
Department of Mathematical Sciences, University of Durham,
Durham DH1 3LE, UK
e-mail: W.J.Zakrzewski@durham.ac.uk
and
Center for Theoretical Physics
Massachusetts Institute of Technology, Cambridge, 02139, USA

Abstract

The gauge model of nonrelativistic particles on a line interacting with nonstandard gravitational fields [5] is supplemented by the addition of a (non)-Abelian gauge interaction. Solving for the gauge fields we obtain equations, in closed form, for a classical two particle system. The corresponding Schrödinger equation, obtained by the Moyal quantization procedure, is solved analytically. Its solutions exhibit two different confinement mechanisms - dependent on the sign of the coupling \( \lambda \) to the nonstandard gravitational fields. For \( \lambda > 0 \) confinement is due to a rising potential whereas for \( \lambda < 0 \) it is due to to the dynamical (geometric) bag formation. Numerical results for the corresponding energy spectra are given. For a particular relation between two coupling constants the model fits into the scheme of supersymmetrical quantum mechanics.

1 Introduction

Two dimensional models have often been considered as a testing laboratory for various ideas in elementary particle theory ([1]). One of the most outstanding problems in elementary particle theory is the nature of quark confinement, which is a non-perturbative phenomenon. Many models of it have been presented (see ([2]) and
the literature cited therein) but, so far, there is no understanding of it in terms of basic principles. In the framework of two-dimensional gauge theories \textit{i.e.} QED$_2$ (\cite{4}) and QCD$_2$ (\cite{3}) confinement is due to a linearly rising potential between two static fermions. This suggests that confinement should be considered a low-energy nonrelativistic phenomenon.

In a recent paper (\cite{5}) one of the present authors (PS), considered a theory invariant under local time-dependent nonrelativistic 1-d space translations:

$$x \rightarrow x'(x, t). \quad (1.1)$$

The corresponding particle action has been made invariant with respect to (1.1) by introducing two gauge fields $h(x, t)$ and $e(x, t)$ which couple via a Maxwell-like interaction (\cite{5})

$$L_{\text{field}} = \frac{1}{2\lambda} \int dx \ h(x, t) \ F^2(x, t), \quad (1.2)$$

where $F$ is an invariant field strength

$$F := \frac{1}{h} (\partial_t h - \partial_x e) \quad (1.3)$$

and $\lambda$ is a coupling constant.

Note that due to the similarity of this interaction to the zweibein formalism and torsion tensors in General Relativity (\cite{6}) we can consider this interaction as describing a nonstandard gravity.

The minimal coupling of the zweibein fields $h$ and $e$ to $N$ nonrelativistic classical particles on trajectories $x_\alpha(t), \ (\alpha = 1, ..N)$ is given, in the first order Lagrangian formalism (\cite{5}) by

$$L_{\text{part,0}}^{(N)} = \sum_{\alpha=1}^{N} \xi_\alpha (h_\alpha \dot{x}_\alpha + e_\alpha) - \frac{1}{2} \sum_{\alpha=1}^{N} \xi^2_\alpha. \quad (1.4)$$

In (\cite{5}) it was shown that the classical dynamics described by (1.2) and (1.4) leads, for $\lambda < 0$, to the phenomenon of geometric bag formation and, therefore, to confinement in the case of two or three particles ($N = 2$ or 3).

The corresponding stationary Schrödinger equation describing the relative motion of two such particles is given by (\cite{5})

$$E \xi_2(x) = \left[ -\hbar^2 \partial_x \left( 1 + \frac{\lambda}{2} |x| \right) \partial_x - \frac{\hbar^2}{4} \lambda \delta(x) \right] \xi_2(x) \quad (1.5)$$

exhibiting a singularity of the metric at $|x| = \frac{2}{|N|}$ for $\lambda < 0$, \textit{i.e.} confinement within the bag $[-\frac{2}{|N|}, \frac{2}{|N|}]$. Numerically determined values for the corresponding energy levels were given in (\cite{5}).

One of the aims of this paper is to study the extension of the theory described in (\cite{5}) by supplementing it with an additional (Non)-Abelian gauge interaction. To do this, in the gauge sector we consider the effects of the well-known Maxwellian actions:

\footnote{Throughout this paper we use the notation of [5].}
• Abelian case ($A_\mu$ - electromagnetic potential)

$$ S^A = \frac{1}{2} \int dt \, dx \, \frac{1}{\hbar} E^2 $$

(1.6)

with the electric field $E$ given by

$$ E := \partial_x A_0 - \partial_t A_1 $$

(1.7)

• Non-Abelian case ($A^i_\mu$-isospin gauge field potential; for simplicity we consider $SU(2)$ as the internal symmetry group)

$$ S^{NA} = \frac{1}{2\kappa} \int dt \, dx \, \frac{1}{\hbar} (E^a)^2 $$

(1.8)

with the non-Abelian electric field $E^a$ given by

$$ E^a := \partial_x A^a_0 - \partial_t A^a_1 + \epsilon_{abc} A^b_1 A^c_0. $$

(1.9)

Note that the factor $\frac{1}{\hbar}$ in front of $(E)^2$ in (1.6) and (1.8), arises from the requirement of the invariance of the action under local translations (1.1), when we have assumed that the gauge fields transform covariantly.

The plan of this paper is as follows: In section 2 we extend the model of ([5]) (given by (1.2) and (1.4)) by adding to it the coupling to the (1+1)-dimensional electrodynamics and discuss its classical dynamics. Corresponding results for the non-Abelian case are given in section 3. In section 4 we describe two-body quantum mechanics, discuss the confinement mechanisms for both signs of the coupling constant $\lambda$ and present some numerical results for the corresponding energy spectra. Section 5 contains some final remarks.

2 Classical Dynamics for the Abelian case

To describe $N$ nonrelativistic charged particles interacting with zweibein fields $e$ and $h$, and an electromagnetic field $A_\mu$ we consider the following action:

$$ S^{(N)}_{\text{part}} = S^{(N)}_{\text{part},0} + \int dt \sum_\alpha g_\alpha (\dot{x}_\alpha A_{1,\alpha} + A_{0,\alpha}) $$

(2.1)

where $g_\alpha$ is the electric charge of the $\alpha$-th particle. Clearly, $S^{(N)}_{\text{part}}$ is invariant under local translations (1.1).

The full action is given by

$$ S^{(N)} = S^{(N)}_{\text{field}} + S^{A} + S^{(N)}_{\text{part}}. $$

(2.2)
Variation of $S^{(N)}$ with respect to the zweibein fields $h$ and $e$ gives the equations of motion (EOM)

$$\frac{1}{\lambda} \partial_t F + \frac{1}{2\lambda} F^2 + \frac{1}{2} \frac{E^2}{h^2} = \sum_\alpha \dot{x}_\alpha \xi_\alpha \delta(x - x_\alpha)$$  \hspace{1cm} (2.3)$$
and the Gauss constraint

$$\partial_x F = -\lambda \sum_\alpha \xi_\alpha \delta(x - x_\alpha),$$  \hspace{1cm} (2.4)$$
respectively, where $\xi_\alpha$ is given by the constraint (2.5)

$$\xi_\alpha = \dot{x}_\alpha h_\alpha + e_\alpha.$$  \hspace{1cm} (2.5)$$

In this derivation, we have assumed that $e$ and $h$ are finite at spatial infinity and that $F$ vanishes there (3). This has assured vanishing boundary terms (in the variations of (1.2)), the finiteness of the integral (1.2) and it also leads to the constraint (3)

$$\sum_\alpha \xi_\alpha = 0.$$  \hspace{1cm} (2.6)$$

As discussed in (3) it is convenient to fix the gauge of the zweibein by imposing

$$h(x,t) = 1$$  \hspace{1cm} (2.7)$$
and so, in the remainder of this paper, we work in this gauge.

As shown in (3) the Gauss constraint (2.4) can then be solved and we have

$$e(x,t) = \frac{\lambda}{2} \sum_\alpha \xi_\alpha(t) |x - x_\alpha(t)| - v(t).$$  \hspace{1cm} (2.8)$$

The EOM, the Gauss constraint for the electric field $E$ and its solutions are well known from the (1+1) electrodynamics. Thus, in the axial gauge $A_1 = 0$, we have

$$A_0(x,t) = \frac{1}{2} \sum_\alpha g_\alpha |x - x_\alpha(t)|.$$  \hspace{1cm} (2.9)$$

Note that the existence of (1.6) imposes the condition that $E$ vanishes at spatial infinity, and so, due to (2.9) giving us the constraint

$$\sum_\alpha g_\alpha = 0.$$  \hspace{1cm} (2.10)$$

This corresponds to confinement of single charged states in $QED_2$ [3]. The electric field is non vanishing only in the space between the charged particles, i.e. it links them.
Finally, the variation of the action with respect to $x_\alpha$ gives the particle EOM, namely
\[ \dot{\xi}_\alpha + \xi_\alpha F_\alpha - g_\alpha E_\alpha = 0. \tag{2.11} \]

Note that when (2.6), (2.8) and (2.11) are satisfied, the EOM (2.3) is automatically satisfied too.

Applying the Legendre transformation to the Lagrangian (2.2) using (2.8), (2.9) and the constraints (2.6) and (2.10) we find that the two-body Hamiltonian $H$, describing the relative motion, is given by
\[ H = \frac{1}{4} \frac{x^2}{1 + \frac{\lambda}{2} |x|} + \frac{g^2}{2} |x|, \tag{2.12} \]
where we have defined
\[ x := x_1 - x_2 \tag{2.13} \]
and
\[ g := g_1 = -g_2. \tag{2.14} \]

We note that the electromagnetic interaction adds just the well known potential term $\frac{g^2}{2} |x|$ to the Hamiltonian given in (5).

From (2.12) we conclude that

- For $\lambda < 0$ the relative particle motion is confined to the bag $\left[ -\frac{2}{|\lambda|}, \frac{2}{|\lambda|} \right]$,
- For $\lambda > 0$ we have bounded motion due to the rising potential term.

3 Classical Dynamics in the Non-Abelian case

In the non-Abelian case the corresponding charge space coordinates are given by the isovectors $\{Q^i_\alpha\}_{i=1}^3$ which, after quantization, satisfy the equal-time commutation relations
\[ [\hat{Q}^i, \hat{Q}^j] = i \hbar \epsilon^{ijk} \hat{Q}^k. \tag{3.1} \]

Note that in the classical case the $\{Q^i\}$ can be considered as vectors on the sphere $S^2$ of radius $J$. So, taking on $S^2$ spherical coordinates $\theta$ and $\varphi$ we get, for the corresponding part of the particle action
\[ S^{(N)}_{SU(2)} = \int dt \sum_{\alpha=1}^N \cos \theta_\alpha(t) \dot{\varphi}_\alpha(t). \tag{3.2} \]

Thus, our particle action is now given by
\[ S^{(N)}_{part} = S^{(N)}_{part,0} + \int dt \sum_{\alpha=1}^N Q^a_\alpha (\dot{x}_\alpha A^a_{1,\alpha} + A^a_{0,\alpha}) + s^{(N)}_{SU(2)}. \tag{3.3} \]
and the total action (see (1.2), (1.8) and (3.3)) by

$$S^{(N)} = S_{\text{field}} + S^{NA} + S^{(N)}_{\text{part}}.$$  \hspace{1cm} (3.4)$$

Looking at the equations of motion we note that for the zweibein fields we obtain, again, the Gauss constraint (2.4) and the non-Abelian analogue of (2.3) \((E^2 \rightarrow (E^a)^2)\). We also have the corresponding requirements at spatial infinity (the vanishing of \(E^a\) and \(F\) and finiteness of \(e\) and \(h\)). Therefore, in the gauge (2.7), the zweibein field \(e(x,t)\) is again given by (2.8) with the constraint (2.6). Moreover, analogously to (2.9) and (2.10) we obtain in the axial gauge \(A^a_1 = 0\) that

$$A^a_0 = \frac{\kappa}{2} \sum_{\alpha} Q^a_{\alpha} |x - x_\alpha|$$  \hspace{1cm} (3.5)$$

with the constraint

$$\sum_{\alpha} Q^a_{\alpha} = 0,$$  \hspace{1cm} (3.6)$$

ie quantum mechanically we will have only isospin-singlet \(N\)-body states. This corresponds to the fact that only gauge invariant, \(i.e.\) singlet, states are elements of the physical Hilbert-space (cp. [7]).

Along the same lines as in the Abelian case we conclude that the non-Abelian analogue to (2.3) is satisfied identically.

Finally, applying the Legendre transformation to the Lagrangian (3.4), using (2.8), (3.5) and the constraints (2.6) and (3.6) we obtain for the two-body Hamiltonian \(H\)

$$H = \frac{1}{4} \frac{\dot{x}^2}{1 + \frac{\lambda}{2} |x|} + \frac{\kappa}{2} (Q^a)^2 |x|,$$  \hspace{1cm} (3.7)$$

where we have defined

$$Q^a := Q^a_1 = -Q^a_2.$$  \hspace{1cm} (3.8)$$

Let us note that the Hamiltonian (3.7) has the same structure as in the Abelian case (2.12). Therefore, we can give a common quantum mechanical treatment, for both cases, and the conclusions will be the same (apart from the change of parameters).

### 4 The Quantum Mechanical Two-Body Problem on a line

When expressed in terms of canonical variables the classical Hamiltonian \(H\) has the form

$$H = p^2(1 + \frac{\lambda}{2} |x|) + \frac{1}{2} q |x|$$  \hspace{1cm} (4.1)$$

where \(q\), a function of the particle charges, is given by

$$q := \begin{cases} g^2 & \text{Abelian case} \\ \frac{\kappa (Q^a)^2}{2} & \text{non-Abelian case} \end{cases}.$$  \hspace{1cm} (4.2)$$


In quantum mechanics the constraint (3.6) has to be considered as a subsidiary condition on the wave function \( \chi_2 \)

\[
(\hat{Q}_1^a + \hat{Q}_2^a) \chi_2 = 0 \quad (4.3)
\]

\( i e \) the two particle states belong to the same isospin multiplet and couple to produce the total isospin zero. Therefore, after quantization, we have in (4.2)

\[
(Q^a)^2 \rightarrow \hbar^2 \tau (\tau + 1) \quad (4.4)
\]

with \( \tau \) being the isospin of a single particle.

Solving the ordering problem involved in quantization of (4.1) by following the prescription given in ([5]) we obtain the following stationary Schrödinger equation

\[
E \chi_2(x) = \left[ -\hbar^2 \frac{\partial}{\partial x} (1 + \frac{\lambda}{2} |x|) \partial_x - \frac{\hbar^2 \lambda}{4} \delta(x) + \frac{1}{2} q|x| \right] \chi_2(x), \quad (4.5)
\]

which differs from (1.5) by just the potential term \( \frac{1}{2} q|x| \) - due to the additional gauge interaction.

Now we proceed in complete analogy to ([5]):

- With Bose-symmetry \( \chi_2(x) = \chi_2(-x) \) we obtain from (4.5) on \( R_+^1 \) the differential equation

\[
E \chi_2 = \left\{ -\hbar^2 \partial_x (1 + \frac{\lambda}{2} x) \partial_x + \frac{1}{2} q x \right\} \chi_2 \quad (4.6)
\]

with the boundary condition

\[
\partial_x \chi_2(0) = -\frac{\lambda}{8} \chi_2(0). \quad (4.7)
\]

- We perform the change of variables

\[
x \rightarrow y := \frac{4}{|\lambda|} \left( 1 + \frac{\lambda}{2} x \right)^{\frac{3}{2}} \quad (4.8)
\]

and define

\[
\tilde{\varphi}_2(y) := (2 + \lambda x)^{\frac{3}{2}} \chi_2(x) \quad (4.9)
\]

to obtain the Schrödinger equation for \( \tilde{\varphi}_2 \)

\[
E \tilde{\varphi}_2(y) = \left[ -\hbar^2 \partial_y^2 - \frac{\hbar^2}{4y^2} + \frac{q}{\lambda} \left( \frac{\lambda^2 y^2}{16} - 1 \right) \right] \tilde{\varphi}_2 \quad (4.10)
\]

together with the boundary condition

\[
\partial_y \tilde{\varphi}_2 \left( \frac{4}{|\lambda|} \right) = 0 \quad (4.11)
\]

In order to proceed further we have to consider separately the cases of \( \lambda < 0 \) and \( \lambda > 0 \).

\[\text{This prescription is in agreement with the Moyal quantization applied to the classical functions on phase space.}\]
4.1 $\lambda < 0$; Confinement by a Geometric Bag

From (4.8) we see that $y \in (0, \frac{4}{|\lambda|})$. The requirement of finiteness of $\chi_2$ at the edge of the bag ($x_0 = \frac{2}{|\lambda|}$) leads by (4.9) to

$$\bar{\varphi}_2(0) = 0. \quad (4.12)$$

The solution of (4.10) respecting the boundary condition (4.12), for a finite value of $\chi_2(\frac{2}{|\lambda|})$, is given in terms of the confluent hypergeometric function $\,_1F_1$ by

$$\bar{\varphi}_2 = z^{\frac{1}{4}} e^{-\frac{z}{2}} \,_1F_1(-A; 1; z), \quad (4.13)$$

where we have defined

$$z := \frac{\sqrt{\lambda q}}{4\hbar} y^2 \quad (4.14)$$

and

$$A := \frac{q}{\hbar \sqrt{\lambda q}} + \frac{E}{\hbar \sqrt{\lambda q}} - \frac{1}{2} \quad (4.15)$$
To determine the values of energies we have to resort to numerical methods. We have performed such calculations for the lowest values of energies (as a function of $\lambda$ and $q$). To do this we have first observed that (4.11) translates into the condition

$$ \_1F_1 (1 - 2z) + 4z \frac{\partial \_1F_1}{\partial z} = 0, $$ (4.16)

at $z = \frac{4\sqrt{|\lambda q|}}{h\lambda^2}$, where $\_1F_1 = \_1F_1(-A, 1; z)$. Thus defining $P$ as the left hand side of this formula we have varied $A$ and determined the values of $A$ for which $P$ vanishes. This, via (4.15) gives us the values of the energy.

Note that \[8\]

$$ \frac{d}{dz} \_1F_1(-A, 1; z) = -A \_1F_1(1 - A, 2; z) $$ (4.17)

and so $P$ is given by

$$ P = \_1F_1(-A, 1; z)(1 - 2z) - 4zA \_1F_1(1 - A, 2; z). $$ (4.18)

As $\lambda < 0$ we note that $z$ is real for $q < 0$. Of course, $q$ is given by (4.2) so it can be negative only in a nonabelian case.

Figure 2: Energy (in units of $h^2 \lambda^2$) as a function of $z = \frac{4\sqrt{|\lambda q|}}{h\lambda^2}$.
We have performed a series of numerical calculations - determining zeros of \( P \), as a function of \( A \) for many values of \( z \) and the results are presented in Fig. 1.

![Figure 3: Energy (in units of \( \hbar^2 \lambda^2 \)) as a function of \( z = \frac{4 \sqrt{|\lambda q|}}{\hbar \lambda} \) sign \((q)\).](image)

In Fig. 1 the vertical axis gives \( E \) (in units of \( \hbar^2 \lambda^2 \)), the horizontal \( z = \frac{4 \sqrt{|\lambda q|}}{\hbar \lambda} \) \( ie \) (4.14) at \( y = \frac{4}{|\lambda|} \). Looking at the figure we see that (for each \( z \)) the energy is quantised and we note a relatively weak dependence on \( z \) (with all values of \( E \) decreasing as \( z \) increases).

Note that when \( z \to 0 \) and \( A \to \infty \) \( ie \) \( q \to 0 \) our equation reduces to (cp. [5])

\[
2sJ_1(s) = J_0(s),
\]

(4.19)

where \( s = \frac{4\sqrt{E}}{\hbar |\lambda|} \) whose lowest solutions are

\[
E = 0.05531, \quad 0.9798, \quad 3.1385, \quad 6.531, ... \quad (4.20)
\]

where \( E \) is given in units of \( \hbar^2 \lambda^2 \).

For \( q > 0 \) \( z \) becomes complex and the calculations are more involved (as we need to use complex functions \( etc \)). However, we have managed to determine the dependence
of the few lowest energies on \( q \). Interestingly and reassuringly, both the real and imaginary parts of the complex function \( \psi F_1 \) vanished at the same value of \( A \). Our results are presented in Fig. 2, where on the horizontal axis we have put \( z = \frac{4\sqrt{|\lambda q|}}{\hbar \lambda} \).

We note that, as before, there is little dependence on \( z \) but, this time, the values of the energy increase with an increase of \( z \).

In Fig. 3 we put together 4 lowest energies as functions of \( z \) for both positive and negative values of \( q \); hence the horizontal axis now shows \( z = \frac{4\sqrt{|\lambda q|}}{\hbar \lambda} \) sign(\( q \)).

It is very interesting to note that for a particular value of \( z \), namely \( z = \frac{1}{2} \) which corresponds to \( q = \frac{\hbar^2 \lambda^3}{64} \), our model fits into the scheme of supersymmetric quantum mechanics\(^3\). At \( z = \frac{1}{2} \) the Hamiltonian (4.10) factorizes as

\[
H = B^\dagger B + E_0 = H_1 + E_0
\]

with

\[
B := \hbar \partial_y + W(y) , \quad B^\dagger := -\hbar \partial_y + W(y)
\]

and the superpotential

\[
W(y) := \frac{\hbar \lambda^2}{32} y - \frac{\hbar}{2y} ,
\]

where

\[
E_0 = \frac{3}{64} \hbar^2 \lambda^2
\]

is the ground state energy of \( H \).

The supersymmetric partner of \( H_1 \) is given by

\[
H_2 := BB^\dagger = -\hbar^2 \partial_y^2 + \left( \frac{\lambda^2 \hbar}{32} \right)^2 y^2 + \frac{3 \hbar^2}{4 y^2}.
\]

Note that the ground-state wave function of \( H_1 \)

\[
\Psi_0^{(1)} = \exp \left( -\frac{1}{\hbar} \int dy W(y) \right)
\]

satisfies the boundary condition (4.11) due to

\[
W \left( \frac{4}{|\lambda|} \right) = 0
\]

so the supersymmetry is unbroken. Thus \( H_2 \) has the same spectrum as \( H_1 \) with the exception of the ground state [9]:

\[
E_{n+1}^{(1)} = E_n^{(2)}, \quad n \in N ,
\]

with

\[
\Psi_n^{(2)} \sim B \Psi_{n+1}^{(1)}.
\]

\(^3\)see e.g the review article [9] and the literature cited therein.
Using (4.11) and (4.27) we obtain from (4.29) the boundary condition
\[ \Psi_n^{(2)} \left( \frac{4}{|\lambda|} \right) = 0. \]  
(4.30)

Therefore, the excitation energies of \( H \) can be read off from (4.30) which, due to (4.25), takes the form:
\[ _1F_1 \left( 1 - A, 2; \frac{1}{2} \right) = 0, \]  
(4.31)

which agrees with (4.18) taken at \( z = \frac{1}{2} \) for \( A \neq 0 \).

Note that the roots of (4.31) are approximately given [8] by
\[ A_n = \frac{\pi^2}{2} \left( n + \frac{1}{4} \right)^2, \quad n = 1, 2, 3, \ldots \]  
(4.32)
in good agreement with our, numerically determined, values.

Finally, let us note that our bag is impenetrable both classically and quantum mechanically. The boundary condition (4.12) allows no Schrödinger current at \( x_0 = \frac{2}{|\lambda|} \) and therefore no tunneling through the edge of the bag.

4.2 \( \lambda > 0 \); Confining oscillator potential

For \( \lambda > 0 \) \( y \) lies on the half-axis \((-\frac{1}{3}, \infty)\).

We look for solutions of the Schrödinger equation (4.10) for \( q > 0 \) with \( \varphi_2 \in L^2(\frac{1}{3}, \infty) \) obeying the boundary condition (4.11). This defines a self-adjoint eigenvalue problem leading to a discrete energy spectrum which is bounded from below.

To be more specific, we note that we have to replace \(_1F_1\) of section 4.1 by Kummer’s function \( U(-A, 1; z) \), which at infinity grows only as a power (\( \sim z^A \)) and so, due to (4.13) guarantees the square integrability. In this case the numerical calculations of zeros of \( P \) are somewhat cumbersome and so we have not carried them out.

Nevertheless we can obtain some energy values for a discrete set of \( z \) values by the following argument:

We start with the defining relation between the Kummer \( U \) function and the \(_1F_1\) functions [8]:
\[ U(-A, 1; z) = \lim_{b \to 1} \frac{\pi}{\sin \pi b} \left\{ \frac{_1F_1(-A, b; z)}{\Gamma(1 - b - A)\Gamma(b)} - z^{1-b} \frac{_1F_1(1 - b - A, 2 - b; z)}{\Gamma(-A)\Gamma(2 - b)} \right\}. \]  
(4.33)

Thus, for \( A \) equal to a positive integer \( n \) we obtain
\[ U(-n, 1; z) = \lim_{b \to 1} \frac{\Gamma(1 - b)}{\Gamma(1 - b - n)} _1F_1(-n, b; z) = (-1)^n n! _1F_1(-n, 1; z). \]  
(4.34)

But we have [8]
\[ _1F_1(-n, 1; z) = L_n(z), \]  
(4.35)
where $L_n(z)$ are the Laguerre polynomials.

Thus we see that for those values of $z$ which are roots of $P$ (4.18) and fulfill the relation

$$A(z) = n$$

(4.36)

our problem reduces mathematically to the well known case of a charged particle moving in a $s$-state on a plane and interacting with a constant magnetic field $B$ (i.e. - the Landau problem). The particular $z$ values satisfying (4.36) can be read off from Fig. 1

In Fig. 4 we plot some values of the energies determined this way. We plot only those corresponding to $z = \frac{4\sqrt{\lambda q}}{h\lambda^2} < 30$ and determined from $n \leq 13$. Assuming that the extrapolation to non-integer values of $A$ would give similar results (and there is no reason to expect it to be otherwise) we see that, at every value of $z$ there is a tower of energies. Moreover, these energies decrease as $z$ increases. The lowest energy, which is not really visible in Fig. 4) is plotted in Fig. 5. In this case, our approach gave us results in a very small range of $z$, and all the results show very little dependence on $z$. 

Figure 4: Some values of energy (in units of $h^2\lambda^2$) as a function of $z = \frac{4\sqrt{\lambda q}}{h\lambda^2}$
5 Conclusions

In the present paper we have considered the confinement mechanisms for two nonrelativistic particles on a line arising from the addition to the non-standard gravity \([5]\) of an additional (non)-Abelian gauge interaction. Our results show that for

- \(\lambda < 0\) and for any sign of the additional gauge coupling \(q\) we observe confinement by the geometric bag formation mechanism with only weak dependence of the energy spectrum on \(q\).

- \(\lambda > 0\) the confinement is due to the rising potential term.

Note that the confinement found for \(\lambda > 0\) is the well known confinement mechanism of two-dimensional gauge theories \([3],[4]\). Addition of nonstandard gravity only alters the energy spectrum as a function of the coupling constant \(\lambda\).

The confinement found for \(\lambda < 0\) is, however, of a completely different nature. It arises, selfconsistently, from a singularity of the nontrivial metric determined dynamically by the nonstandard gravity interaction of the confined particles. This is in contrast with the current treatments of \((3+1)\)-dimensional Yang-Mills theories for which it is only the gauge fields that form a geometric bag which confines test particles (cp. \([10]\) and the review \([11]\)). The results of the present paper, together with the corresponding results in \((2+1)\)-dimensions with \([12]\) or without \([6]\) an
additional (non)-Abelian gauge interaction, strengthen our feeling that nonstandard gravity might be of some relevance for the solution of the confinement problem in strong interactions. Thus, further research into nonstandard gravity, in particular in (3+1)-dimensions, is called for.

References

[1] E. Abdalla, Two-dimensional Quantum Field Theory, examples and applications, hep-th/9704192.

[2] A.V. Nesterenko, *Phys. Rev. D* 62, (2000) 094028; G. ‘t Hooft, Monopoles, Instantons and Confinement, hep-th/0010225; J.A. Magpantay, *Mod. Phys. Lett. A* 14, (1999), 447.

[3] P. Gaete and I. Schmidt, *Phys. Rev. D* 61, (2000), 125002.

[4] J. Pachos and A. Tsapalis, *Mod. Phys. Lett. A* 13, (1998) 1371.

[5] P.C. Stichel, *Ann. Phys. (NY)*, 285, (2000), 161; *Phys. Lett. B* 456, (1999), 129.

[6] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, *Ann. Phys. (NY)*, 288, (2001), 164.

[7] J. Hansson, A simple solution to color confinement, hep-ph/0011060; R.S. Wittman, Pedagogical Reflections on Color Confinement in Chromostatics, hep-th/0102189.

[8] L.J. Slater, “Confluent Hypergeometric Functions”, Cambridge Univ. Press, UK (1960).

[9] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* 251, (1995), 267.

[10] F.A. Lunev, *Phys. Lett. B* 311, (1993), 273.

[11] D. Singleton, *Theor. Math. Phys.* 117, (1998), 1351.

[12] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, Chern-Simons Particles with Nonstandard Gravitational Interaction, hep-th/0011214 (to appear in *Eur. Phys. Jour. C*).