The strongly attached point topology of the abstract boundary for space-time

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Abstract

The abstract boundary construction of Scott and Szekeres provides a ‘boundary’ for any $n$-dimensional, paracompact, connected, Hausdorff, $C^\infty$ manifold. Singularities may then be defined as objects within this boundary. In a previous paper (Barry R A and Scott S M 2011 Class. Quantum Grav. 28 165003), a topology referred to as the attached point topology was defined for a manifold and its abstract boundary, thereby providing us with a description of how the abstract boundary is related to the underlying manifold. In this paper, a second topology, referred to as the strongly attached point topology, is presented for the abstract boundary construction. Whereas the abstract boundary was effectively disconnected from the manifold in the attached point topology, it is very much connected in the strongly attached point topology. A number of other interesting properties of the strongly attached point topology are considered, each of which support the idea that it is a very natural and appropriate topology for a manifold and its abstract boundary.

Keywords: spacetime singularities, spacetime topology, embeddings, boundary constructions

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1. Introduction

In the paper ‘the attached point topology of the abstract boundary for space-time’ [1], a topology for a manifold $\mathcal{M}$ and its collection of abstract boundary points $\mathcal{B}(\mathcal{M})$ was constructed. The topology, referred to as the attached point topology, represents one of the more natural topologies that can be placed upon the abstract boundary. It was produced via obvious extensions to the abstract boundary point definitions, and did not require any additional conditions to be placed upon the manifold or its boundary. However, it was demonstrated that it was possible to separate the manifold and its abstract boundary by disjoint open sets of the
attached point topology. The manifold and its abstract boundary were therefore disconnected from one another in some sense. Even so, the fact that the attached point topology was Hausdorff was a pleasing result and suggested that the attached point topology was a good starting point in producing a topology that is more descriptive of the topological relationship between a manifold and its abstract boundary.

Because the abstract boundary is produced via embeddings of the manifold, the abstract boundary exists in a space separate to that of the manifold. A topology on $\mathcal{M} \cup B(\mathcal{M})$ should therefore connect the abstract boundary to the underlying manifold $\mathcal{M}$. As noted previously, the attached point topology provided one such description of the topological relationship between $\mathcal{M}$ and $B(\mathcal{M})$. This topology relied on the notion of abstract boundary points being ‘close’, in some sense, to open sets of the manifold $\mathcal{M}$. Abstract boundary points that were close to an open set of $\mathcal{M}$ were said to be ‘attached’ to that open set, and thus the ‘location’ of an abstract boundary point could be described relative to the known topology of $\mathcal{M}$. Because it is possible to separate the manifold and its abstract boundary from one another by disjoint open sets of the attached point topology, it can be said that the attached point topology does not fully integrate $B(\mathcal{M})$ with the structure of $\mathcal{M}$. The fact that $\mathcal{M}$ and $B(\mathcal{M})$ can be separated in this way is a result of there being too many open sets in the attached point topology. In this paper, a new topology referred to as the strongly attached point topology is considered. The strongly attached point topology is defined similarly to the attached point topology but has one additional restriction. This restriction limits the ‘type’ of open set in $\mathcal{M}$ to which an abstract boundary point may be considered to be ‘close to’, and hence there are comparatively fewer open sets in the strongly attached point topology. The main consequence of this is that every open neighbourhood of an abstract boundary point necessarily contains some part of the manifold $\mathcal{M}$, i.e., the abstract boundary is topologically inseparable from the underlying manifold $\mathcal{M}$.

In section 2, the abstract boundary will again be defined as a matter of convenience. The strongly attached abstract boundary point definition is developed in section 3, which describes how an abstract boundary point may be related back to $\mathcal{M}$. The strongly attached point topology, which utilises the previously mentioned definition is presented in section 4. Various properties of the topology are then discussed in sections 5–8.

We refer the reader interested in the g-boundary, b-boundary and c-boundary to [2, 3] and [4]. For those interested in the more recent causal boundary, see [5–9] and [10].

Within this work, we use the following fact frequently and so formally present it here for ease of reference. Let $g$ be a Riemannian metric on a manifold $\mathcal{M}$, and let $\Omega_{p,q}$ denote the set of piecewise smooth curves in $\mathcal{M}$ from $p$ to $q$. For every curve $c \in \Omega_{p,q}$ with $c : [0, 1] \rightarrow \mathcal{M}$ there is a finite partition $0 = t_1 < t_2 < \cdots < t_k = 1$ such that $c \mid [t_i, t_{i+1}]$ is smooth for each $i$, $1 \leq i \leq k - 1$. The Riemannian arc length of $c$ with respect to $g$ is then defined to be $L(c) = \sum_{i=1}^{k-1} \int_{t_i}^{t_{i+1}} \sqrt{g(c(t), c'(t))} \, dt$, and the Riemannian distance function, $d(p, q)$, between $p$ and $q$ is then defined in terms of this by $d(p, q) = \inf\{L(c) : c \in \Omega_{p,q}\} \geq 0$. The most useful property of this distance function is that the open balls defined by $B_r(p) = \{q \in \mathcal{M} : d(p, q) < \epsilon\}$ form a basis for the manifold topology, and thus the topology induced by the Riemannian metric agrees with the manifold topology [11].

2. The abstract boundary

For the convenience of the reader, we will provide the definition of the a-boundary in this section. For a more complete discussion of the a-boundary, see [12–14] and [15]. It will be
assumed that all manifolds used in the following work will be n-dimensional, paracompact, connected, Hausdorff and smooth (i.e., $C^\infty$). The manifold topology will be employed throughout the paper unless explicitly stated otherwise. The principle feature of the a-boundary construction is that of an envelopment.

**Definition 1 (Embedding).** The function $\phi: M \to \hat{M}$ is an embedding if $\phi$ is a homeomorphism between $M$ and $\phi(M)$, where $\phi(M)$ has the subspace topology inherited from $\hat{M}$.

**Definition 2 (Envelopment).** An enveloped manifold is a triple $(M, \hat{M}, \phi)$ where $M$ and $\hat{M}$ are differentiable manifolds of the same dimension $n$ and $\phi$ is a $C^\infty$ embedding $\phi: M \to \hat{M}$. The enveloped manifold will also be referred to as an envelopment of $M$ by $\hat{M}$, and $\hat{M}$ will be called the enveloping manifold.

**Definition 3 (Boundary point).** A boundary point $p$ of an envelopment $(M, \hat{M}, \phi)$ is a point in the topological boundary of $\phi(M)$ in $\hat{M}$. The set of all such points $p$ is thus given by $\partial(\phi(M)) = \phi(M) \setminus \phi(M)$ where $\phi(M)$ is the closure of $\phi(M)$ in $\hat{M}$. The boundary points are then simply the limit points of the set $\phi(M)$ in $\hat{M}$ which do not lie in $\phi(M)$ itself.

The characteristic feature of a boundary point is that every open neighbourhood of it (in $\hat{M}$) has non-empty intersection with $\phi(M)$.

**Definition 4 (Boundary set).** A boundary set $B$ is a non-empty set of such boundary points for a given envelopment, i.e., a non-empty subset of $\partial(\phi(M))$.

It is important to note that different boundary points will arise with different envelopments of $M$. In order to continue, a notion of equivalence between boundary sets of different envelopments is required. This equivalence is defined in terms of a covering relation.

**Definition 5 (Covering relation).** Given a boundary set $B$ of one envelopment $(M, \hat{M}, \phi)$ and a boundary set $B'$ of a second envelopment $(M, \hat{M}', \phi')$, then $B$ covers $B'$, denoted $B \bowtie B'$, if for every open neighbourhood $U$ of $B$ in $\hat{M}$ there exists an open neighbourhood $U'$ of $B'$ in $\hat{M}'$ such that

$$\phi \circ \phi'^{-1}(U' \cap \phi'(M)) \subset U.$$  

In essence, this definition says that a sequence of points from within $M$ cannot get close to points of $B'$ without at the same time getting close to points of $B$. See figure 1.

**Definition 6 (Equivalent).** The boundary sets $B$ and $B'$ are equivalent (written $B \sim B'$) if $B \bowtie B'$ and $B' \bowtie B$. This definition produces an equivalence relation on the set of all boundary sets. An equivalence class is denoted by $[B]$, where $B$ is a representative of the set of equivalent boundary sets under the covering relation.
Definition 7 (Abstract boundary point and abstract boundary). An abstract boundary point is defined to be an equivalence class $[B]$ that has a singleton boundary point $\{p\}$ as a representative member. Such an equivalence class will then simply be denoted by $[p]$. The set of all such abstract boundary points of a manifold $\mathcal{M}$ will be denoted by $\mathcal{B}(\mathcal{M})$ and called the abstract boundary of $\mathcal{M}$. The union of all points of a manifold $\mathcal{M}$ and its collection of abstract boundary points $\mathcal{B}(\mathcal{M})$ may then be labelled as $\mathcal{M}$, i.e., $\mathcal{M} = \mathcal{M} \cup \mathcal{B}(\mathcal{M})$.

Definition 8 (Covered abstract boundary point). An abstract boundary point $[p]$ covers an abstract boundary point $[q]$, denoted $[p] \triangleright [q]$, if the representative singleton boundary point $\{p\}$ covers the representative singleton boundary point $\{q\}$.

3. Strongly attached boundary points and sets

The attached point topology was defined by topologically relating the abstract boundary points of a manifold $\mathcal{M}$ back to the points of $\mathcal{M}$ via the definition of an attached boundary point. We include this definition and the definition of an attached boundary set for the benefit of the reader.

Definition 9 (Attached boundary point). Given an open set $U$ of $\mathcal{M}$ and an envelopment $\phi : \mathcal{M} \rightarrow \hat{\mathcal{M}}$, then a boundary point $p$ of $\partial(\phi(\mathcal{M}))$ is said to be attached to $U$ if every open neighbourhood $N$ of $p$ in $\hat{\mathcal{M}}$ has non-empty intersection with $\phi(U)$, i.e., $N \cap \phi(U) \neq \emptyset$.

Definition 10 (Attached boundary set). Given an open set $U$ of $\mathcal{M}$ and an envelopment $\phi : \mathcal{M} \rightarrow \hat{\mathcal{M}}$, then a boundary set $B \subset \partial\phi(\mathcal{M})$ is said to be attached to $U$ if every open neighbourhood $N$ of $B$ in $\hat{\mathcal{M}}$ has non-empty intersection with $\phi(U)$, i.e., $N \cap \phi(U) \neq \emptyset$.

The strongly attached point topology also relies on the notion of an abstract boundary point being attached to an open set of $\mathcal{M}$, but the manner in which the abstract boundary point is attached is different.

Definition 11 (Strongly attached boundary point). Given an open set $U$ of $\mathcal{M}$ and an envelopment $\phi : \mathcal{M} \rightarrow \hat{\mathcal{M}}$, then a boundary point $p$ of $\partial(\phi(\mathcal{M}))$ is said to be strongly attached to $U$ if there exists an open neighbourhood $N$ of $p$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U)$. See figure 2.

Lemma 12. If the boundary point $p$ is strongly attached to the open set $U$ then $p$ is attached to $U$. 

![Figure 2. A boundary point $p$ strongly attached to the open set $U$.](image)
Figure 3. A boundary point $p$ attached, but not strongly attached, to an open set $U$ of $\mathcal{M}$.

Figure 4. The boundary point $p$ is strongly attached to $U_1 \cap U_2$.

**Proof.** Consider an envelopment $\phi : \mathcal{M} \to \hat{\mathcal{M}}$ and a boundary point $p \in \partial(\phi(\mathcal{M}))$. Suppose that $p$ is strongly attached to the open set $U \subset \mathcal{M}$. There therefore exists an open neighbourhood $N$ of $p$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U)$. Any other open neighbourhood of $p$ will have non-empty intersection with $N \cap \phi(\mathcal{M})$. This follows from the fact that the intersection of two open sets is another open set: $N'$ is an open set that contains $p$, and thus $N \cap N' = N'$ is an open set that also contains $p$. Because $N'$ is a neighbourhood of the boundary point $p$ we have that $N' \cap \phi(\mathcal{M}) \neq \emptyset$. This implies that $(N \cap N') \cap \phi(\mathcal{M}) \neq \emptyset$, and so $N' \cap \phi(U) \neq \emptyset$. This is a statement of the attached boundary point condition, i.e., $p$ is attached to $U$. $\square$

The requirement that there exists an open neighbourhood $N$ of $p$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U)$ removes the possibility of boundary points being strongly attached to open sets like those depicted in figure 3, i.e., open sets that are shaped more like a wedge and which have minimal ‘contact’ with the boundary of the particular envelopment. If a boundary point is strongly attached to an open set $U$ then that set $U$ will always be more ‘spread out’ along the boundary under the given envelopment.

**Lemma 13.** If a boundary point $p \in \partial(\phi(\mathcal{M}))$ is strongly attached to the open sets $U_1$ and $U_2$, then $U_1 \cap U_2 \neq \emptyset$. Furthermore, $p$ is strongly attached to $U_1 \cap U_2$. See figure 4.

**Proof.** The boundary point $p$ is strongly attached to $U_1$ and so there exists an open neighbourhood $N$ of $p$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U_1)$. It is also strongly attached to $U_2$ and so there exists an open neighbourhood $N'$ of $p$ in $\hat{\mathcal{M}}$ such that $N' \cap \phi(\mathcal{M}) \subseteq \phi(U_2)$. Now $N$ and $N'$ are both open neighbourhoods of $p$ and so their intersection is another open neighbourhood of $p$. In addition, $p$ is a boundary point, and so every open neighbourhood of
Figure 5. A boundary set $B$ is strongly attached to an open set $U$ of $\mathcal{M}$.

$p$ has non-empty intersection with $\phi(\mathcal{M})$. We therefore have $(N \cap N') \cap \phi(\mathcal{M}) \neq \emptyset$. Now, since $N \cap \phi(\mathcal{M}) \subseteq \phi(U_1)$ and $N' \cap \phi(\mathcal{M}) \subseteq \phi(U_2)$, we have that $(N \cap N') \cap \phi(\mathcal{M}) \subseteq \phi(U_1)$ and $(N \cap N') \cap \phi(\mathcal{M}) \subseteq \phi(U_2)$. This implies that $\phi(U_1) \cap \phi(U_2) \neq \emptyset$ and therefore that $U_1 \cap U_2 \neq \emptyset$. Moreover, $(N \cap N') \cap \phi(\mathcal{M}) \subseteq \phi(U_1) \cap \phi(U_2) = \phi(U_1 \cap U_2)$ from which it follows that $p$ is strongly attached to $U_1 \cap U_2$. □

**Definition 14** (Strongly attached boundary set). Given an open set $U$ of $\mathcal{M}$ and an envelopment $\phi : \mathcal{M} \to \hat{\mathcal{M}}$, then a boundary set $B \subset \partial(\phi(\mathcal{M}))$ is said to be strongly attached to $U$ if there exists an open neighbourhood $N$ of $B$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U)$. See figure 5.

**Lemma 15.** If $B \subset \partial(\phi(\mathcal{M}))$ is strongly attached to the open set $U \subset \mathcal{M}$ then $B$ is attached to $U$.

**Proof.** The proof of this is identical to the proof of lemma 12, except that we are dealing with open neighbourhoods of a boundary set rather than open neighbourhoods of a boundary point. □

**Lemma 16.** A boundary set $B \subset \partial(\phi(\mathcal{M}))$ is strongly attached to an open set $U \subset \mathcal{M}$ if and only if every boundary point $p \in B$ is strongly attached to $U$.

**Proof.** ($\Rightarrow$) Let $\phi : \mathcal{M} \to \hat{\mathcal{M}}$ be an envelopment, and $B \subset \partial(\phi(\mathcal{M}))$ a boundary set that is strongly attached to the open set $U \subset \mathcal{M}$. Because $B$ is strongly attached to $U$, there exists an open neighbourhood, $N$, of $B$ in $\hat{\mathcal{M}}$ such that $N \cap \phi(\mathcal{M}) \subseteq \phi(U)$. $N$ is an open neighbourhood of every boundary point $p \in B$. Clearly then, every $p \in B$ is strongly attached to $U$.

($\Leftarrow$) If every boundary point $p \in B$ is strongly attached to $U$, then there exists an open neighbourhood, $N_p$, of each $p$ such that $N_p \cap \phi(\mathcal{M}) \subseteq \phi(U)$. The union $N_B$ of every $N_p$, i.e., $N_B = \bigcup_{p \in B} N_p$, is an open neighbourhood of $B$ such that $N_B \cap \phi(\mathcal{M}) \subseteq \phi(U)$. The boundary set $B$ is therefore strongly attached to $U$. □

**Lemma 17.** If a boundary set $B$ is strongly attached to the open sets $U_1$ and $U_2$, then $U_1 \cap U_2 \neq \emptyset$. Furthermore, $B$ is strongly attached to $U_1 \cap U_2$.

**Proof.** The proof follows from lemma 13, except that we are dealing with open neighbourhoods of a boundary set, rather than open neighbourhoods of a boundary point. □
equivalence relation. More specifically, we wish to show that if a boundary set \( B \subset \partial(\phi(M)) \) is strongly attached to an open set \( U \subset M \) and there exists a boundary set \( B' \subset \partial(\psi(M)) \) that is equivalent to \( B \), then \( B' \) is also strongly attached to \( U \).

**Proposition 18.** Let \( B \subset \partial(\phi(M)) \) be strongly attached to an open set \( U \subset M \), and let \( B' \) be a boundary set of a second envelopment \( \phi' : M \to \hat{M} \). If \( B \triangleright B' \), then \( B' \) is also strongly attached to \( U \).

**Proof.** The boundary set \( B \subset \partial(\phi(M)) \) is strongly attached to the open set \( U \). There therefore exists an open neighbourhood \( N \) of \( B \) such that \( N \cap \phi(M) \subseteq \phi(U) \). Now, since \( B \triangleright B' \), we have that \( \phi \circ \phi'^{-1}(N' \cap \phi'(M)) \subseteq N' \), where \( N' \) is an open neighbourhood of \( B' \) in \( \hat{M} \). It follows that \( \phi \circ \phi'^{-1}(N' \cap \phi'(M)) \subseteq N \cap \phi(M) \subseteq \phi(U) \), and therefore \( N' \cap \phi'(M) \subseteq \phi'(U) \), i.e., \( B' \) is strongly attached to \( U \). \( \square \)

**Definition 19** (Strongly attached abstract boundary point). The abstract boundary point \([p]\) is said to be strongly attached to the open set \( U \) of \( M \) if the boundary point \( p \) is strongly attached to \( U \).

The abstract boundary point \([p]\) is an equivalence class of boundary sets which are equivalent to \([p]\). By proposition 18 the strongly attached abstract boundary point definition is well defined as any boundary set \( B \) such that \( B \sim p \) is also strongly attached to \( U \), i.e., all members of the equivalence class \([p]\) are strongly attached to \( U \).

Also, by lemma 15 it is clear that if an abstract boundary point \([p]\) is strongly attached to an open set \( U \) of \( M \), then \([p]\) is also attached to \( U \).

**Proposition 20.** Consider an open set \( U \) of \( M \) and an envelopment \( \phi : M \to \hat{M} \). Let \( B_U \) be the set of boundary points of \( \partial(\phi(M)) \) which are strongly attached to \( U \). The set \( B_U \) is closed in \( \hat{M} \) if and only if the limit points of \( B_U \) are strongly attached to \( U \).

**Proof.** Since \( B_U \subset \partial(\phi(M)) \), any limit point of \( B_U \) in \( \hat{M} \) will also lie in \( \partial(\phi(M)) \). By definition, \( B_U \) is closed in \( \hat{M} \) if and only if \( B_U \) contains all its limit points. Clearly, \( B_U \) contains all its limit points if and only if the limit points of \( B_U \) are strongly attached to \( U \), from which the result follows. \( \square \)

**Corollary 21.** Consider an open set \( U \) of \( M \) and an envelopment \( \phi : M \to \hat{M} \). Let \( B_U \) be the set of boundary points of \( \partial(\phi(M)) \) which are strongly attached to the \( U \). The set \( B_U \) is closed in the induced topology on \( \partial(\phi(M)) \) if and only if the limit points of \( B_U \) are strongly attached to \( U \).

**Proof.** Since \( \partial(\phi(M)) \) is closed in \( \hat{M} \) and \( B_U \subset \partial(\phi(M)) \), the set \( B_U \) is closed in the induced topology on \( \partial(\phi(M)) \) if and only if \( B_U \) is closed in \( \hat{M} \). The result then follows directly from proposition 20. \( \square \)

In general, \( B_U \) will not be closed in \( \partial(\phi(M)) \) or \( \hat{M} \) because not all the limit points of \( B_U \) are necessarily strongly attached to \( U \). See figures 6 and 7. As was shown in propositions 13 and 14 of [1], however, the limit points of \( B_U \) are always attached to \( U \).
4. The strongly attached point topology

Similarly to the attached point topology, a basis for a topology on $\mathcal{M} = \mathcal{M} \cup \mathcal{B}(\mathcal{M})$ may be constructed by defining the open sets in terms of the strongly attached abstract boundary point definition (definition 19). Again, in keeping with the notion of constructing a natural topology, the open sets of $\mathcal{M}$ to which the abstract boundary points are strongly attached are therefore taken to be the open sets of the manifold topology.

Consider the sets $A_i = U_i \cup B_i$, where $U_i$ is a non-empty open set of the manifold topology in $\mathcal{M}$ and $B_i$ is the set of all abstract boundary points which are strongly attached to $U_i$. $B_i$ may be the empty set if no abstract boundary points are strongly attached to $U_i$. Let $\mathcal{W}$ be the set comprised of every $A_i$ set. That is,

$$\mathcal{W} = \{ A_i = U_i \cup B_i \}.$$

Given that the strongly attached boundary point definition limits the ‘type’ of open set $U_i$ in $\mathcal{M}$ to which a boundary point may be strongly attached, it is important to know whether or not every abstract boundary point is strongly attached to an open set $U_i$ in $\mathcal{M}$. In other words, if a boundary point $p$ were attached to a ‘wedge’ shaped open set $U$ in $\mathcal{M}$ like that depicted in figure 3, does there always exist another open set $U'$ in $\mathcal{M}$ to which $p$ is strongly attached? A brief analysis reveals that the answer to this question is yes—every boundary point $p$ is strongly attached to an open set $U$ in $\mathcal{M}$.
Lemma 22. Every abstract boundary point \([p]\) is strongly attached to an open set \(U\) in \(\mathcal{M}\).

Proof. For the envelopment \(\phi : \mathcal{M} \rightarrow \overline{\mathcal{M}},\ p \in \partial(\phi(\mathcal{M}))\), let \(N\) be any open neighbourhood of \(p\) in \(\overline{\mathcal{M}}\). Since \(p\) is a boundary point, \(N \cap \phi(\mathcal{M})\) is non-empty. In addition, since \(\phi\) is an embedding, the non-empty set \(U = \phi^{-1}(N \cap \phi(\mathcal{M}))\) is an open set in \(\mathcal{M}\). We then have that \(p\) is strongly attached to \(U = \phi^{-1}(N \cap \phi(\mathcal{M}))\) because there exists an open neighbourhood \(N\) of \(p\) in \(\overline{\mathcal{M}}\) such that \(N \cap \phi(\mathcal{M}) \subseteq N \cap \phi(\mathcal{M})\). \(\square\)

Proposition 23. The elements of \(\mathcal{W}\) form a basis for a topology on \(\overline{\mathcal{M}}\).

Proof. By definition, \(\mathcal{M}\) is covered by the collection \(\{U_i\}\) of open sets in \(\mathcal{M}\). Also, by lemma 22 each abstract boundary point is strongly attached to an open set \(U_i\) in \(\mathcal{M}\). The set of open sets in \(\mathcal{M}\) and their strongly attached abstract boundary points, i.e., \(\{A_i\}\), therefore covers \(\overline{\mathcal{M}}\).

The intersection between two elements of \(\mathcal{W}\) must be examined. Consider the intersection between \(A_1 = U_1 \cup B_1\) and \(A_2 = U_2 \cup B_2\). In considering this intersection, there are three subcases to check:

1. \(U_1 \cap U_2 \neq \emptyset, B_1 \cap B_2 = \emptyset\) (this includes the cases when \(B_1 = \emptyset\) or \(B_2 = \emptyset\))
2. \(U_1 \cap U_2 \neq \emptyset, B_1 \cap B_2 \neq \emptyset\)
3. \(U_1 \cap U_2 = \emptyset, B_1 \cap B_2 \neq \emptyset\).

(i) In the first case we have that \(U_1 \cap U_2 \neq \emptyset\) and \(B_1 \cap B_2 = \emptyset\). Since \(B_1 \cap B_2 = \emptyset\), \(A_1 \cap A_2 = U_1 \cap U_2 = U_3\) which is an open set in \(\mathcal{M}\). If the abstract boundary point \([p]\) is strongly attached to \(U_3\), then \([p]\) is strongly attached to \(U_1 (\{p\} \in B_1)\) and \([p]\) is strongly attached to \(U_2 (\{p\} \in B_2)\) which would imply that \(B_1 \cap B_2 \neq \emptyset\). It follows that \(B_3 = \emptyset\), where \(B_3\) is the set of abstract boundary points that are strongly attached to \(U_3\), and thus \(A_1 \cap A_2 = U_3 \cup B_3 \in \mathcal{W}\).

(ii) For this case, \(A_1 \cap A_2 = (U_1 \cap U_2) \cup (B_1 \cap B_2) = U_3 \cup (B_1 \cap B_2)\). If \([p] \in B_1 \cap B_2\), it is strongly attached to both \(U_1\) and \(U_2\), so by lemma 13, \([p]\) is strongly attached to \(U_1 \cap U_2 = U_3\), i.e., \([p] \in B_3\). Thus \(B_1 \cap B_2 \subseteq B_3\). Now if \([p] \in B_3\), it is strongly attached to \(U_3 = U_1 \cap U_2\) and is strongly attached to both \(U_1\) and \(U_2\). That is, \([p] \in B_1 \cap B_2\) and so \(B_3 \subseteq B_1 \cap B_2\). It follows that \(A_1 \cap A_2 = U_3 \cup B_3 \in \mathcal{W}\).

(iii) This case cannot exist by lemma 13. Specifically, if \(B_1 \cap B_2 \neq \emptyset\), then there exist abstract boundary points that are strongly attached to both \(U_1\) and \(U_2\), and hence \(U_1 \cap U_2 \neq \emptyset\).

The intersection \(A_1 \cap A_2 = (U_1 \cap U_2) \cup (B_1 \cap B_2)\) is therefore always another element of \(\mathcal{W}\). Thus the elements of \(\mathcal{W}\) form a basis for a topology on \(\overline{\mathcal{M}}\). \(\square\)

Definition 24 (Strongly attached point topology). The strongly attached point topology on \(\overline{\mathcal{M}}\) is the topology which has the basis \(\mathcal{W}\).

The attached point topology required that sets of abstract boundary points be added to the collection of basis sets \(\mathcal{V}\). This was done to ensure that the sets of \(\mathcal{V}\) did in fact define a basis for a topology on \(\overline{\mathcal{M}}\). Because there exist basis sets \(A_i = U_i \cup B_i\) and \(A_j = U_j \cup B_j\) of the attached point topology such that \(A_i \cap A_j = B_i \cap B_j\), i.e., \(U_i \cap U_j = \emptyset\), the \(C_i\) sets which are collections of abstract boundary points must also be included in the collection \(\mathcal{V}\) of basis sets. The collection \(\mathcal{W}\) of basis sets for the strongly attached point topology, however, does not require the addition of such sets of abstract boundary points. This is a direct consequence of lemma 13. The non-empty intersection of any two \(A_i\) sets will necessarily contain points of
\( M \), and therefore it is impossible that a collection of abstract boundary points can be produced by considering intersections of \( A_i \) sets. It is for this reason that the topology is referred to as the strongly attached point topology—the abstract boundary \( B(M) \) is firmly affixed to the manifold \( M \) and has become, topologically speaking, an integral part of the larger space \( \tilde{M} \).

In other words, any open neighbourhood of an abstract boundary point \([p]\) will necessarily include some part of \( M \).

**5. Open and closed sets in the strongly attached point topology**

The open sets of \( \tilde{M} \) consist of arbitrary unions of the elements of \( W \). As in the case of the attached point topology, it may again seem that an arbitrary open set \((U_i \cup B_i) \cup (U_j \cup B_j) \cup \ldots\) is another basis element \( U_k \cup B_k \). Again, this is not true in general.

**Example 25.** Consider \( M = \{ (x, y) \in \mathbb{R}^2 : y < 0 \}, \tilde{M} = \mathbb{R}^2 \) and let \( \phi : M \to \tilde{M} \) be the inclusion map. Let \( p \in \partial \phi(M) \) be the boundary point \((0, 0)\); \([p]\) is the associated abstract boundary point. Define a sequence \( \{x_n\} \) in \( M \) by \( x_n = (0, -\frac{1}{n}) \). Around each \( x_n \), define an open set \( U_n = \{ (x, y) : -1 < x < 1, -\frac{3}{2} < y < -\frac{1}{n^2} \} \). See figure 8. By construction, in \( \tilde{M} \), for any \( n \), \( U_n \subset \tilde{M} \) and thus \( U_n \) has no strongly attached abstract boundary points, i.e., \( B_n = \emptyset \) and \( U_n = U_n \cup B_n = A_n \). Take an open ball \( B_\frac{1}{2}(p) \) of radius \( \frac{1}{2} \) around \( p \) and consider some \( a \in B_\frac{1}{2}(p) \cap \tilde{M} \). Because \( \{x_n\} \to p \), \( a \) will be contained in some \( U_n \). Since this is true for every \( a \in B_\frac{1}{2}(p) \cap \tilde{M} \), it follows that \( B_\frac{1}{2}(p) \cap \tilde{M} \subset \bigcup_n U_n \). The abstract boundary point \([p]\) is therefore strongly attached to the open set \( \bigcup_n U_n = O \) but \( O \) is the union of non-empty open sets \( U_n \) in \( M \), each of which does not have any strongly attached abstract boundary points, i.e., \( O \in \mathcal{M} \) and \([p] \notin O \). Since \( O \subset \mathcal{M} \) and \([p] \notin \mathcal{W} \).
Proposition 26. The set \( \mathcal{M} \) is open, and the set \( B(\mathcal{M}) \) is closed in the strongly attached point topology on \( \overline{\mathcal{M}} \).

Proof. For a manifold \( \mathcal{M} \), there exists a complete metric \( d \) on \( \mathcal{M} \) such that the topology induced by \( d \) agrees with the manifold topology of \( \mathcal{M} \) \([11]\). Choose \( \epsilon > 0 \), and for each \( x \in \mathcal{M} \), let \( U_x \) be the open ball \( U_x = \{ y \in \mathcal{M} : d(x, y) < \epsilon \} \). Now consider the envelopment \( \phi : \mathcal{M} \to \overline{\mathcal{M}} \) and a boundary point \( p \in \partial(\phi(\mathcal{M})) \). We know that \( p \notin \phi(U_x) \) since \( d \) is a complete metric on \( \mathcal{M} \) and so \( \phi(U_x) \subset \phi(\mathcal{M}) \). Thus the set \( \overline{\mathcal{M}} \setminus \phi(U_x) \) is an open neighbourhood of \( p \) in \( \overline{\mathcal{M}} \) which does not intersect \( \phi(U_x) \), and so \( p \) is not attached to \( U_x \). By lemma 12, \( p \) is also not strongly attached to \( U_x \). It follows that no boundary point \( p \) of any envelopment of \( \mathcal{M} \) is strongly attached to \( U_x \), which implies that \( U_x \) has no strongly attached abstract boundary points, i.e., \( B_x = \emptyset \).

Now
\[
\bigcup_{x \in \mathcal{M}} A_x = \bigcup_{x \in \mathcal{M}} (U_x \cup B_x) = \left( \bigcup_{x \in \mathcal{M}} U_x \right) \cup \left( \bigcup_{x \in \mathcal{M}} B_x \right) = \mathcal{M} \cup \emptyset = \mathcal{M}.
\]

It follows that \( \mathcal{M} \) is open in \( \overline{\mathcal{M}} \) and thus \( B(\mathcal{M}) \) is closed because \( \overline{\mathcal{M}} \setminus B(\mathcal{M}) = \mathcal{M} \) is open.

Proposition 27. The set \( B(\mathcal{M}) \) is not open, and the set \( \mathcal{M} \) is not closed in the strongly attached point topology on \( \overline{\mathcal{M}} \).

Proof. Consider any open neighbourhood of an abstract boundary point \( [p] \in B(\mathcal{M}) \) in \( \overline{\mathcal{M}} \). Every open set of \( \overline{\mathcal{M}} \) is a union of basis sets. Because every basis set contains a non-empty open subset of \( \mathcal{M} \), every open set in the strongly attached point topology will contain a non-empty open subset of \( \mathcal{M} \). Any open set that contains \( [p] \in B(\mathcal{M}) \) will therefore necessarily contain some open subset of \( \mathcal{M} \) as well, and thus \( B(\mathcal{M}) \) cannot be open. Since \( B(\mathcal{M}) = \overline{\mathcal{M}} \setminus \mathcal{M} \) is not open, \( \mathcal{M} \) is not closed.

Proposition 28. The manifold topology on \( \mathcal{M} \) and the topology induced on \( \mathcal{M} \) by the strongly attached point topology on \( \overline{\mathcal{M}} \) are the same.

Proof. Let \( U \neq \emptyset \) be an open set in \( \mathcal{M} \) in the manifold topology. The set \( A = U \cup B \), where \( B \) is the set of all abstract boundary points which are strongly attached to \( U \), is an open set in the strongly attached point topology of \( \overline{\mathcal{M}} \). Now \( U = A \cap \mathcal{M} \) and thus \( U \) is an open set in the topology induced on \( \mathcal{M} \) by the strongly attached point topology on \( \overline{\mathcal{M}} \). Now let \( U \neq \emptyset \) be an open set in the topology induced on \( \mathcal{M} \) by the strongly attached point topology on \( \overline{\mathcal{M}} \). So \( U = V \cap \mathcal{M} \) where \( V \) is an open set in the strongly attached point topology on \( \overline{\mathcal{M}} \). The set \( V \) can be expressed as a union of elements of \( W \), i.e., \( V = \bigcup_{i \in I} U_i \). Thus \( U = \bigcup_{i \in I} (U_i \cup B_i) \) \( \cap \mathcal{M} = \bigcup_{i \in I} U_i \cap \mathcal{M} = \bigcup_{i \in I} U_i \), and so \( U \) is a union of non-empty open sets of the manifold topology on \( \mathcal{M} \) and is, therefore, itself an open set of the manifold topology on \( \mathcal{M} \).

Corollary 29. If \( V \) is an open neighbourhood of the abstract boundary point \( [p] \) in \( \overline{\mathcal{M}} \), then \( V \cap \mathcal{M} \neq \emptyset \).
Corollary 30. If $V$ is an open neighbourhood of the abstract boundary point $[p]$ in $\overline{M}$, then $[p]$ is strongly attached to the open set $U \neq \emptyset$ of $M$, where $U \equiv V \cap M$.

Proof. $V$ is an open neighbourhood of $[p]$ in $\overline{M}$ and thus it may be written as a union of basis sets $U_i \cup B_i$, where $[p]$ is an element of at least one of the $B_i$ sets. It follows that $U = V \cap M = \bigcup U_i$. By proposition 28, we know that $U$ is an open set of $M$, and by corollary 29, we have that $\bigcup U_i \neq \emptyset$. Moreover, $[p]$ is strongly attached to one of the $U_i$ sets and thus $[p]$ is strongly attached to $U$ where $U \neq \emptyset$.

We will next consider if the singleton abstract boundary point sets $[[p]]$ are open or closed in the strongly attached point topology. Before addressing that question, however, three useful results are established.

Proposition 31. For abstract boundary points $[p]$ and $[q]$, $[p] \triangleright [q]$ if and only if $[q]$ is strongly attached to every open set, $U \subseteq M$, to which $[p]$ is strongly attached.

Proof. ($\Leftarrow$) Suppose that $[q]$ is strongly attached to every open set, $U \subseteq M$, to which $[p]$ is strongly attached. Consider the boundary point $p$ of the envelopment $\phi : M \rightarrow \overline{M}$, and the boundary point $q$ of the envelopment $\phi' : M \rightarrow \overline{M}$. Let $N$ be an open neighbourhood of $p$ in $\overline{M}$. It follows that $[p]$ is strongly attached to $\phi^{-1}(N \cap \phi(M))$. Because $[q]$ is strongly attached to every open set $U \subseteq M$ to which $[p]$ is strongly attached, $[q]$ is strongly attached to $\phi^{-1}(N \cap \phi(M))$, for all $N$. For every $N$ there therefore exists an open neighbourhood $W$ of $q$ in $\overline{M}$ such that $W \cap \phi'(M) \subseteq \phi' \circ \phi^{-1}(N \cap \phi(M))$, i.e., $[p]$ covers $[q]$, and thus $[p] \triangleright [q]$.

($\Rightarrow$) This follows immediately from proposition 18.

Corollary 32. For abstract boundary points $[p]$ and $[q]$, $[p] \triangleright [q]$ if and only if every open neighbourhood of $[p]$ in $\overline{M}$ also contains $[q]$, where $\overline{M}$ has the strongly attached point topology.

Proof. ($\Leftarrow$) Suppose that every open neighbourhood of $[p]$ in $\overline{M}$ also contains $[q]$, where $\overline{M}$ has the strongly attached point topology. Also suppose that $[p]$ is strongly attached to the open set $U \subseteq M$. Now define the set $V = U \cup B_U$, where $B_U$ is the set of all abstract boundary points which are strongly attached to $U$. It is clear that $V$ is an open set in $\overline{M}$ in the strongly attached point topology, $[p] \in V$, and thus $V$ is an open neighbourhood of $[p]$ in $\overline{M}$. By assumption, $[q] \in V$, and therefore $[q]$ is strongly attached to the open set $U \subseteq M$. It follows from proposition 31 that $[p] \triangleright [q]$.

($\Rightarrow$) Suppose now that $[p] \triangleright [q]$. Let $V$ be an open neighbourhood of $[p]$ in $\overline{M}$, where $\overline{M}$ has the strongly attached point topology. The set $V$ can be expressed as a union of elements
of $\mathcal{W}$, i.e., $V = \bigcup_{i \in I}(U_i \cup B_i)$. The abstract boundary point $[p]$ lies in at least one set $B_i$ and is therefore strongly attached to the open set $U_i$ of $\mathcal{M}$. It follows from proposition 31 that $[q]$ is also strongly attached to $U_i$, and so $[q] \in B_i$. Thus $V$ is also an open neighbourhood of $[q]$. \hfill \square

**Corollary 33.** The closure in $\overline{\mathcal{M}}$ of an abstract boundary point $[p]$ is $\overline{\{[p]\}} = \{[p]\} \cup \{[x] : [x] \in B(\mathcal{M}), [x] \triangleright [p]\}$.

**Proof.** From proposition 26, $B(\mathcal{M})$ is closed in the strongly attached point topology on $\overline{\mathcal{M}}$. Since $[p] \in B(\mathcal{M})$, it follows that $\overline{\{[p]\}} \subseteq B(\mathcal{M})$. Now consider $[x] \in B(\mathcal{M})$.

- $[x] \in \overline{\{[p]\}} \iff$ every closed subset of $B(\mathcal{M})$ that contains $[p]$ also contains $[x]$
  - there exists no closed subset of $B(\mathcal{M})$ that contains $[p]$ that does not contain $[x]$
  - there exists no open neighbourhood of $[x]$ in $\overline{\mathcal{M}}$ that does not contain $[p]$
  - every open neighbourhood of $[x]$ in $\overline{\mathcal{M}}$ contains $[p]$
  - $[x] \triangleright [p]$ (by proposition 32).

Thus $\overline{\{[p]\}} = \{[p]\} \cup \{[x] : [x] \in B(\mathcal{M}), [x] \triangleright [p]\}$. \hfill \square

We may now more readily consider the question of whether or not the singleton abstract boundary point sets $\{[p]\}$ are open or closed in the strongly attached point topology on $\overline{\mathcal{M}}$.

**Proposition 34.** The singleton abstract boundary point sets $\{[p]\}$ are not open in the strongly attached point topology on $\overline{\mathcal{M}}$. They are also not closed in the strongly attached point topology on $\overline{\mathcal{M}}$ if and only if there exists $[q] \in B(\mathcal{M})$, $[q] \neq [p]$, such that $[q] \triangleright [p]$.

**Proof.** Consider an abstract boundary point $[p]$. From corollary 29, any open set in $\overline{\mathcal{M}}$ that contains $[p]$ will necessarily have non-empty intersection with $\mathcal{M}$, and thus $\overline{\{[p]\}}$ is not an open set.

By corollary 33, the closure in $\overline{\mathcal{M}}$ of an abstract boundary point $[p]$ is $\overline{\{[p]\}} = \{[p]\} \cup \{[x] : [x] \in B(\mathcal{M}), [x] \triangleright [p]\}$. If there exists a $[q]$ such that $[q] \triangleright [p]$, $[q] \neq [p]$, then by corollary 33, $\overline{\{[p]\}}$ contains at least $[p]$ and $[q]$, and therefore $\overline{\{[p]\}}$ is not closed. Similarly, if $\overline{\{[p]\}}$ is not closed, then $\overline{\{[p]\}}$ contains at least 2 elements $[p]$ and $[q]$ such that $[p] \neq [q]$ and $[q] \triangleright [p]$. It follows that $\overline{\{[p]\}}$ is not closed in the strongly attached point topology on $\overline{\mathcal{M}}$ if and only if there exists $[q] \in B(\mathcal{M})$, $[q] \neq [p]$, such that $[q] \triangleright [p]$. \hfill \square

**Proposition 35.** The open sets of the induced topology on $B(\mathcal{M}) \subset \overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ has the strongly attached point topology, are arbitrary unions of the $B_i$ sets defined in the basis $\mathcal{W}$.

**Proof.** Let $\mathcal{T}_{\overline{\mathcal{M}}}$ be the strongly attached point topology on $\overline{\mathcal{M}}$. The subspace topology on $B(\mathcal{M})$ is the collection of sets $\mathcal{T}_{B(\mathcal{M})} = \{U \cap B(\mathcal{M}) : U \in \mathcal{T}_{\overline{\mathcal{M}}}\}$. The topology $\mathcal{T}_{\overline{\mathcal{M}}}$ is the collection of arbitrary unions of the $U_i \cup B_i$ sets of the basis $\mathcal{W}$. The intersection of these sets with $B(\mathcal{M})$ is therefore the collection of arbitrary unions of the $B_i$ sets. \hfill \square
6. The inclusion map from $\mathcal{M}$ to $\overline{\mathcal{M}}$

We now consider the inclusion map $i : \mathcal{M} \rightarrow \overline{\mathcal{M}} = \mathcal{M} \cup B(\mathcal{M}) | i(p) = p$. As in the case of the attached point topology, it can be shown that the inclusion map is an embedding.

**Proposition 36.** If $\overline{\mathcal{M}}$ has the strongly attached point topology, then the inclusion mapping $i : \mathcal{M} \rightarrow \overline{\mathcal{M}} | i(p) = p$ is an embedding.

**Proof.** The inclusion mapping $i$ is an embedding if it is a homeomorphism of $\mathcal{M}$ onto $i(\mathcal{M})$ in the subspace topology on $i(\mathcal{M}) \cap \overline{\mathcal{M}}$. Clearly $i$ is a bijection of $\mathcal{M}$ onto $i(\mathcal{M})$. Now let $T_M$ be the usual topology on $\mathcal{M}$ consisting of the collection of open sets $\{U_i\}$, $T_{\overline{\mathcal{M}}}$ the strongly attached point topology on $\overline{\mathcal{M}}$ as defined by the basis elements of $\mathcal{W}$, i.e., $T_{\overline{\mathcal{M}}}$ is the collection of arbitrary unions of the $U_i \cup B_i$ sets, and $T_{i(\mathcal{M})}$, the subspace topology on $i(\mathcal{M}) \cap \overline{\mathcal{M}}$. The subspace topology $T_{i(\mathcal{M})}$ is therefore the collection of sets $T_{i(\mathcal{M})} = \{U_k\}$. Clearly both $i$ and $i^{-1}$ are continuous with respect to $T_M$ and $T_{i(\mathcal{M})}$. It has thus been demonstrated that $i : \mathcal{M} \rightarrow \overline{\mathcal{M}} | i(p) = p$ is a homeomorphism onto its image in the induced topology and is therefore an embedding.  

Because it has been shown that $i : \mathcal{M} \rightarrow \overline{\mathcal{M}} | i(p) = p$ is an embedding, we may view $\overline{\mathcal{M}}$ as simply $\mathcal{M}$ with the addition of its abstract boundary points. This is a pleasing result as one would expect the nature of $\mathcal{M}$ to be preserved in $\overline{\mathcal{M}}$.

The following properties of $i(\mathcal{M})$ are readily obtained.

**Proposition 37.** For the inclusion mapping $i : \mathcal{M} \rightarrow \overline{\mathcal{M}} | i(p) = p$, $i(\mathcal{M})$ is open and not closed in the strongly attached point topology on $\overline{\mathcal{M}}$, $i(\mathcal{M}) = \overline{\mathcal{M}}$ and $\partial(i(\mathcal{M})) = B(\mathcal{M})$.

**Proof.** Since $i(\mathcal{M}) = \mathcal{M}$, it follows from propositions 26 and 27 that $i(\mathcal{M})$ is open and not closed in the strongly attached point topology on $\overline{\mathcal{M}}$. Because $i(\mathcal{M})$ is open, $\partial(i(\mathcal{M})) = \partial(\mathcal{M}) = \{y \in \overline{\mathcal{M}} \setminus \mathcal{M} : \text{every open neighbourhood of } y \text{ has non-empty intersection with } \mathcal{M}\}$. Consider an abstract boundary point $[p] \in B(\mathcal{M}) = \overline{\mathcal{M}} \setminus \mathcal{M}$. From corollary 29, every open neighbourhood of $[p]$ has non-empty intersection with $\mathcal{M}$, and so $[p] \in \partial(i(\mathcal{M}))$. Thus $\partial(i(\mathcal{M})) = B(\mathcal{M})$. Now $i(\overline{\mathcal{M}}) = i(\mathcal{M}) \cup \partial(i(\mathcal{M})) = \mathcal{M} \cup B(\mathcal{M}) = \overline{\mathcal{M}}$.  

7. Contact properties of the strongly attached point topology

A number of important properties of the strongly attached point topology will now be presented.

Due to the way that abstract boundary points are constructed, two abstract boundary points may share some of the same topological information. For example, if $[p] = [q]$ then any envelopment that produces a boundary set belonging to $[p]$ will also produce a boundary set belonging to $[q]$ and vice versa. Likewise, in the case that $[p]$ covers $[q]$ we have that $[p]$ contains $[q]$ in some sense. When $[p]$ and $[q]$ are realized as boundary sets $A \subseteq \partial(\phi(\mathcal{M}))$ and $B \subseteq \partial(\psi'(\mathcal{M}))$ respectively, the topological structure of $\phi(\mathcal{M})$ near $A$ incorporates the topological structure of $\psi'(\mathcal{M})$ near $B$. In this way, when we consider the abstract boundary point $[p]$ relative to $\mathcal{M}$, we are also considering the abstract boundary point $[q]$. Alternatively, we may have the case where $[p]$ and $[q]$ are not in contact at all, and are somehow ‘separate’ from each other.

A topology on $\overline{\mathcal{M}}$ should therefore be descriptive of the topological ‘contact’ properties between abstract boundary points. It can be seen that the strongly attached point topology describes the separation properties of abstract boundary points in a natural way in that greater levels of separation between abstract boundary points with respect to the covering relation...
correspond to greater levels of separation with respect to the usual topological separation axioms.

We begin by defining what it means for an abstract boundary point to be in contact with another abstract boundary point. In some sense the contact relation is a weaker form of the covering relation. If \([p] \) and \([q]\) are in contact, then they contain some of the same topological information, but not as much as if \([p]\) covered \([q]\) or \([q]\) covered \([p]\).

**Definition 38** (Contact \( \perp \)). Let \( p \in \partial(\phi(M)) \) and \( q \in \partial(\phi'(M)) \) be two enveloped boundary points of \( M \). They are said to be in contact (denoted \( p \perp q \)) if for all open neighbourhoods \( U \) and \( V \) of \( p \) and \( q \) respectively,

\[
U \cap V := \phi^{-1}(U \cap \phi(M)) \cap \phi'^{-1}(V \cap \phi'(M)) \neq \emptyset.
\]

**Definition 39** (Contact \( \perp \)). Two boundary points \( p \in \partial(\phi(M)) \) and \( q \in \partial(\phi'(M)) \) are in contact (denoted \( p \perp q \)) if there exists a sequence \( \{p_i\} \subseteq M \) such that \( \{\phi(p_i)\} \) has \( p \) as an endpoint and \( \{\phi'(p_i)\} \) has \( q \) as an endpoint.

Definitions 38 and 39 are equivalent. For a proof of this see lemma 6.3 of [14].

**Definition 40** (Abstract boundary points in contact). Two abstract boundary points \([p]\) and \([q]\) are in contact, denoted \([p] \perp [q] \), if \( p \perp q \) for boundary point representatives \( p \) and \( q \).

This definition can be shown to be well-defined. See theorem 3.10 of [14].

**Definition 41** (Separation of boundary points \( \parallel \)). Two boundary points \( p \in \partial(\phi(M)) \) and \( q \in \partial(\phi'(M)) \) are separate (denoted \( p \parallel q \)) if there is no sequence \( \{p_i\} \subseteq M \) for which \( \{\phi(p_i)\} \to p \) and \( \{\phi'(p_i)\} \to q \). Equivalently, the boundary points \( p \) and \( q \) are separate if there exist open neighbourhoods \( U \) and \( V \) of \( p \) and \( q \) respectively such that \( \phi^{-1}(U \cap \phi(M)) \cap \phi'^{-1}(V \cap \phi'(M)) = \emptyset \).

Equivalently, from definition 39, two boundary points \( p \in \partial(\phi(M)) \) and \( q \in \partial(\phi'(M)) \) are separate if they are not in contact.

**Definition 42** (Separation of abstract boundary points). Two abstract boundary points \([p]\) and \([q]\) are separate, denoted \([p] \parallel [q] \), if \( p \parallel q \) for boundary point representatives \( p \) and \( q \). Equivalently, \([p] \parallel [q] \) if they are not in contact.

Similar to definition 40, this definition can be shown to be well-defined. See theorem 3.10 of [14].

The results which follow relate to \( \overline{M} \) with the strongly attached point topology.

**Proposition 43.** Two abstract boundary points \([p]\) and \([q]\) are \( T_2 \) separable, i.e., they are Hausdorff separable, if and only if \([p] \parallel [q] \).

**Proof.** (\( \Leftarrow \)) If \([p] \parallel [q] \), then \([p] \) and \([q]\) are \( T_2 \) separable.

If \([p] \parallel [q] \) then there exists an open neighbourhood \( U \) of \( p \in \partial(\phi(M)) \) and an open neighbourhood \( V \) of \( q \in \partial(\phi'(M)) \) such that \( \phi^{-1}(U \cap \phi(M)) \cap \phi'^{-1}(V \cap \phi'(M)) = \emptyset \). We also have that \( p \) is strongly attached to \( \phi^{-1}(U \cap \phi(M)) \), \( q \) is strongly attached to \( \phi'^{-1}(V \cap \phi'(M)) \), and from lemma 13, \( p \) is not strongly attached to \( \phi'^{-1}(V \cap \phi'(M)) \) and \( q \) is not strongly attached to \( \phi^{-1}(U \cap \phi(M)) \). Furthermore, it also follows from lemma 13 that there exist no abstract boundary points which are strongly attached to both \( \phi^{-1}(U \cap \phi(M)) \) and \( \phi'^{-1}(V \cap \phi'(M)) \). There therefore exists an open neighbourhood of \([p] \), \( \phi^{-1}(U \cap \phi(M)) \cup B_U \), \([p] \in B_U \), and...
an open neighbourhood of \([q], \phi^{-1}(V \cap \phi'(\mathcal{M})) \cup B_V, [q] \in B_V\), such that their intersection is empty. The abstract boundary points \([p]\) and \([q]\) are therefore Hausdorff separated.

\((\Rightarrow)\) If \([p]\) and \([q]\) are \(T_2\) separable, then \([p] \parallel [q]\).

If \([p]\) and \([q]\) are Hausdorff separated then there exist open neighbourhoods \(U_p\) of \([p]\) and \(U_q\) of \([q]\) in the strongly attached point topology such that \(U_p \cap U_q = \emptyset\). Since \([p]\) is contained in \(U_p\), by corollary 30, \([p]\) is strongly attached to the open set \(V_p = U_p \cap \mathcal{M}\). Now, since \([p]\) is strongly attached to \(V_p\), there exists an open neighbourhood \(W_p\) in \(\mathcal{M}\) of \(p \in \partial(\phi(\mathcal{M}))\) such that \(W_p \cap \phi(\mathcal{M}) \subseteq \phi(V_p)\). Similarly, there exists an open neighbourhood \(W_q\) in \(\mathcal{M}\) of \(q \in \partial(\phi'(\mathcal{M}))\) such that \(W_q \cap \phi'(\mathcal{M}) \subseteq \phi'(V_q)\), where \(V_q\) is the open set \(V_q = U_q \cap \mathcal{M}\) in \(\mathcal{M}\). Now, since \(W_p \cap \phi(\mathcal{M}) \subseteq \phi(V_p)\) and \(W_q \cap \phi'(\mathcal{M}) \subseteq \phi'(V_q)\), and \(V_p \subseteq U_p\) and \(V_q \subseteq U_q\), where \(U_p \cap U_q = \emptyset\), it follows that \(\phi^{-1}(W_p \cap \phi(\mathcal{M})) \cap \phi'^{-1}(W_q \cap \phi'(\mathcal{M})) = \emptyset\), and so \([p] \parallel [q]\). \(\square\)

**Proposition 44.** Two abstract boundary points \([p]\) and \([q]\) are \(T_1\) separated if and only if \([p] \not\parallel [q]\) and \([q] \not\parallel [p]\).

**Proof.** \((\Leftarrow)\) If \([p] \not\parallel [q]\) and \([q] \not\parallel [p]\), then \([p]\) and \([q]\) are \(T_1\) separated.

If \([p] \not\parallel [q]\) and \([q] \not\parallel [p]\), then by corollary 32 there exists an open neighbourhood \(N_p\) of \([p]\) and an open neighbourhood \(N_q\) of \([q]\) such that \([q] \notin N_p\) and \([p] \notin N_q\). This is a statement of the \(T_1\) separation axiom.

\((\Rightarrow)\) If \([p]\) and \([q]\) are \(T_1\) separated, then \([p] \not\parallel [q]\) and \([q] \not\parallel [p]\).

If \([p]\) and \([q]\) are \(T_1\) separated, then there exists an open neighbourhood \(N_p\) of \([p]\) and an open neighbourhood \(N_q\) of \([q]\) such that \([q] \notin N_p\) and \([p] \notin N_q\). It then follows directly from corollary 32, that \([p] \not\parallel [q]\) and \([q] \not\parallel [p]\). \(\square\)

**Proposition 45.** Two abstract boundary points \([p]\) and \([q]\) are \(T_0\) separated if and only if \([p] \not\parallel [q]\) or \([q] \not\parallel [p]\).

**Proof.** \((\Leftarrow)\) If \([p] \not\parallel [q]\) or \([q] \not\parallel [p]\), then \([p]\) and \([q]\) are \(T_0\) separated.

By corollary 32, if \([q] \not\parallel [p]\), then there exists an open neighbourhood \(N_q\) of \([q]\) such that \([p] \notin N_q\), and so \([p]\) and \([q]\) are \(T_0\) separated. Likewise, if \([p] \not\parallel [q]\), then \([p]\) and \([q]\) are \(T_0\) separated.

\((\Rightarrow)\) If \([p]\) and \([q]\) are \(T_0\) separated, then \([p] \not\parallel [q]\) or \([q] \not\parallel [p]\).

If \([p]\) and \([q]\) are \(T_0\) separated, then there exists an open neighbourhood \(N_p\) of \([p]\) such that \([q] \notin N_p\), or there exists an open neighbourhood \(N_q\) of \([q]\) such that \([p] \notin N_q\). If \([p] \notin N_q\), then by corollary 32, \([q] \not\parallel [p]\). Likewise, if \([q] \notin N_p\), then \([p] \not\parallel [q]\). \(\square\)

The results of this section are summarized in table 1 which shows the correspondence between the contact properties of two enveloped boundary points \(p \in \partial(\phi(\mathcal{M}))\) and \(q \in \partial(\phi'(\mathcal{M}))\) and the topological relationship of the respective abstract boundary points \([p]\) and \([q]\) in \(\mathcal{M}\) with the strongly attached point topology. We provide examples of the second and third relationships in figures 9 and 10, respectively.

Hausdorff separability is lost between abstract boundary points which are in contact with each other, and therefore, also when one of the abstract boundary points covers the other. In many ways, this is an expected result. As has been stated previously, two abstract boundary points which are in contact with one another share a certain amount of topological information, and thus they do not represent two truly distinct points. This property is reflected in the loss of Hausdorff separation in the strongly attached point topology. And so, while it is desirable that a topology for \(\mathcal{M}\) be Hausdorff, it can be seen that the lack of separation between abstract
Figure 9. The boundary point $p \in \partial(\phi(M))$ is equivalent to the closed boundary set $B \subset \partial(\phi(M))$, where $q \in B$. It follows that $p \triangleright q$, but $q \not\triangleright p$.

Table 1. The left-hand column shows the possible relationships between boundary points $p$ and $q$ of envelopments $\phi$ and $\phi'$, respectively, of $M$; the right-hand column shows the corresponding topological relationships between the associated abstract boundary points $[p]$ and $[q]$ in $\mathcal{M}$ with the strongly attached point topology.

| Relationship between enveloped boundary points $p \in \partial(\phi(M))$ and $q \in \partial(\phi'(M))$ | Topological relationship of the abstract boundary points $[p]$ and $[q]$ |
|---|---|
| $p \sim q$, $q \not\triangleright p$ or $q \triangleright p$, $p \not\triangleright q$ | $[p] = [q]$ |
| $p \perp q$, $p \not\triangleright q$, $q \not\triangleright p$ | $[p]$ and $[q]$ are $T_0$ separated (proposition 45) $[p]$ and $[q]$ are not $T_1$ separated (proposition 44) |
| $p \parallel q$ | $[p]$ and $[q]$ are $T_1$ separated (proposition 44) $[p]$ and $[q]$ are not $T_2$ separated (proposition 43) |

boundary points actually provides us with information about the structure of the abstract boundary itself. Moreover, it can be argued that Hausdorff separation is not lost between truly distinct abstract boundary points (namely those which are separate). Instead, it is lost between abstract boundary points which represent different parts of some ‘larger’ entity.

We note that, in general, the strongly attached point topology on $\mathcal{M}$ will be $T_0$ separated only, as there will be occurrences of $p \triangleright q$, $q \not\triangleright p$ for boundary points $p \in \partial(\phi(M))$ and $q \in \partial(\phi'(M))$.

We will now determine if the strongly attached point topology is first countable.

**Proposition 46.** The strongly attached point topology on $\mathcal{M}$ is first countable.

**Proof.** A topological space $X$ is said to be first countable if, for each $x \in X$, there exists a sequence $U_1, U_2, \ldots$ of open neighbourhoods of $x$ such that for any open neighbourhood, $V$, of $x$, there exists an integer, $i$, such that $U_i \subseteq V$.

For $X = \mathcal{M}$ with the strongly attached point topology, we firstly consider the case where $x \in \mathcal{M}$. Given the existence of a complete metric $d$ on $\mathcal{M}$, we know from the proof of proposition 26, that for $n \in \mathbb{N}$, the open balls $U_n = \{ p \in \mathcal{M} : d(x, p) < 1/n \}$ based at the point $x$ have no attached abstract boundary points and therefore no strongly attached abstract
boundary points. The sets \( U_n \cup B_n = U_p \) are basis elements of \( \mathcal{W} \), and so \( U_1, U_2, \ldots \) is a sequence of open neighbourhoods of \( x \).

Let \( V \) be an open neighbourhood of \( x \). Thus \( V \) is an arbitrary union of basis elements \( A_i \), which implies that \( x \in A_1 = U_1 \cup B_1 \subseteq V \) for some \( A_1 \) in the union. It is possible to choose an \( n \in \mathbb{N} \), such that, for the open ball \( U_n \), \( \overline{U_n} \subseteq U_k \). Thus \( U_n \subseteq V \). We have therefore shown that \( \mathcal{M} \) is first countable at \( x \), for all \( x \in \mathcal{M} \).

Now we consider an abstract boundary point \( [p] \in \mathcal{B} (\mathcal{M}) \), where \( p \) is a boundary point of some envelopment \((\mathcal{M}, \mathcal{M}, \phi)\). Similarly to before, given the existence of a complete metric \( d \) on \( \mathcal{M} \), we can define a series of open balls of \( p \) in \( \mathcal{M} \) by \( O_n = \{ y \in \mathcal{M} : d(p, y) < 1/n \} \), \( n \in \mathbb{N} \). We therefore have a series of sets in \( \mathcal{M} \) that contain \([p]\) defined by \( [\phi^{-1}(O_n \cap \phi(\mathcal{M}))] \cup B_{O_n} \), where the \( B_{O_n} \) are the collections of abstract boundary points that are strongly attached to \( \phi^{-1}(O_n \cap \phi(\mathcal{M})) \). Clearly these sets are open in the strongly attached point topology on \( \mathcal{M} \) as they are elements of the collection of basis sets \( \mathcal{W} \). Every open set \( V \) in \( \mathcal{M} \) that contains \([p]\) is an arbitrary union of \( A_i = U_i \cup B_i \) sets. One of the \( B_j \) sets therefore contains \([p]\), and so \([p]\) is strongly attached to \( U_j \). There thus exists an open neighbourhood \( N \) of \( p \) in \( \mathcal{M} \) such that \( N \cap \phi(\mathcal{M}) \subseteq \phi(U_j) \). Now, there exists an \( n \in \mathbb{N} \), such that \( O_n \subset N \), and so \( [\phi^{-1}(O_n \cap \phi(\mathcal{M}))] \cup B_{O_n} \subseteq V \). This means that \( \mathcal{M} \) is first countable at \([p]\), for all \([p] \in \mathcal{B}(\mathcal{M}) \).

We have thereby shown that the strongly attached point topology for \( \mathcal{M} \) is first countable.\( \square \)

Figure 10. Two envelopments of the two-dimensional Misner space-time with respective metrics: \( ds^2 = 2d\psi dt + t(d\psi)^2 \) and \( ds^2 = -2d\psi dt' + t'(d\psi')^2 \). The curves \( \lambda_1 \) and \( \lambda_2 \) are null geodesics. We may construct a sequence along \( \phi(\lambda_1) \) that converges to \( p \). It follows that the image of this sequence under \( \phi' \) converges to \( q \). The boundary points \( p \) and \( q \) are therefore in contact. The curve \( \phi'(\lambda_2) \) is an element of a class of geodesics that spiral around the space-time and approach the waist. The image under \( \phi \) of each such geodesic is a straight vertical line similar to \( \phi(\lambda_2) \) that approaches some point of the boundary set \( \partial(\phi(\mathcal{M})) \) of which \( p \) is an element. It follows that \( p \not\in q \) as we can construct a sequence that converges to \( q \) along one of the spiraling geodesics in \( \phi'(\mathcal{M}) \) whose image under \( \phi \) does not converge to \( p \). By a similar argument it can be shown that \( q \not\in p \).
8. Optimal embeddings and partial cross sections

When presented with a solution to the Einstein field equations in a particular coordinate system, it is not necessarily the case that these coordinates properly display all of its global and physical properties. In practice, this often amounts to determining if the space-time is a proper subset of another, larger space-time. The abstract boundary is therefore the natural boundary construction to use when considering extensions to space-times, given its utility in dealing with multiple envelopments at once. An envelopment in which all of the global features of a space-time are evident may therefore be referred to as an optimal embedding.

In order to be able to choose an envelopment in which all of the global features of a space-time are properly displayed, the structure of the abstract boundary must be understood. If a boundary set of an abstract boundary point is present in an envelopment, then we would like to know how the abstract boundary point represented by this boundary set is related to other abstract boundary points. More specifically, we seek to know things like: is the abstract boundary point represented by that boundary set contained in some other abstract boundary point in some sense, i.e., is the abstract boundary point covered by some other abstract boundary point? And therefore, is there a better, more complete way of displaying the boundary of the space-time in an envelopment? If there exists an envelopment in which more topological and physical information can be displayed, then clearly we should choose that envelopment. Understanding the contact properties between abstract boundary points is therefore essential when considering optimal embeddings.

The contact properties that were defined earlier (definitions 40 and 42) may be used to define subsets of $B(\mathcal{M})$ referred to as partial cross sections. Partial cross sections provide us with a way of abstracting the idea of envelopments as pictures of the boundary. The abstract boundary is a very large object. In some sense, the complete abstract boundary of a manifold $\mathcal{M}$ contains too much information. As discussed previously, different abstract boundary points can share large amounts of the same topological information. It is therefore not necessary to consider every abstract boundary point in order to understand the structure of the abstract boundary. A partial cross section is a ‘slice’ through the abstract boundary containing only abstract boundary points which are topologically distinct from each other. Partial cross sections are therefore important because, ideally, they can be used to simplify the abstract boundary to something more manageable. In turn this can lead to the realisation of optimal embeddings. For further details on optimal embeddings see [14].

**Definition 47** (Partial cross section $\sigma$). Let $\sigma \subset B(\mathcal{M})$. $\sigma$ is a partial cross section if for every $[p], [q] \in \sigma$, $[p] \parallel [q]$ or $[p] = [q]$.

Of particular interest are partial cross sections of the following form:

**Example 48.** Each envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$ defines a partial cross section

$$\sigma_\phi := \{ [p] \mid p \in \partial (\phi(\mathcal{M})) \}.$$

These $\sigma_\phi$ sets are important because we know what the topology on these sets should look like. Each abstract boundary point in $\sigma_\phi$ has a boundary point representative in the topological boundary $\partial (\phi(\mathcal{M}))$. The topology of this set is well defined by the topology on $\widehat{\mathcal{M}}$ and agrees with the relative topology on $\phi(\mathcal{M})$, and hence it also agrees with the topology on $\mathcal{M}$ by virtue of the embedding $\phi$. Each $\sigma_\phi$ therefore has a natural topology defined on it by the given envelopment $(\mathcal{M}, \widehat{\mathcal{M}}, \phi)$.

**Definition 49.** Let $\phi : \mathcal{M} \to \widehat{\mathcal{M}}$ be an envelopment, and $\sigma_\phi$ the partial cross section induced by $\phi$. A natural topology $T_{\sigma_\phi}$ is defined upon $\sigma_\phi$ by the topology of $\widehat{\mathcal{M}}$. Let
Let $\phi : M \rightarrow \hat{M}$. We then take a set $U$ to be an open set of $\sigma_\phi$ $(U \in T_{\sigma_\phi})$ if and only if $U = \{p \in \sigma_\phi : \text{the singleton representative boundary point } p \in \partial(\phi(M))$ is an element of $N \cap \partial(\phi(M))\}$ for some open neighbourhood $N$.

As mentioned previously, the topology on $\partial(\phi(M))$ is that induced by the topology on $\hat{M}$. Because the elements of $\sigma_\phi$ and $\partial(\phi(M))$ are in one-to-one correspondence with each other, it follows that the collection $T_{\sigma_\phi}$ of open sets of $\sigma_\phi$ given by definition 49 is indeed a topology on $\sigma_\phi$.

**Lemma 50.** Let $\phi : M \rightarrow \hat{M}$ be an envelopment, and $\sigma_\phi$ the partial cross section induced by $\phi$. The topological space $(\sigma_\phi, T_{\sigma_\phi})$ is Hausdorff.

**Proof.** Let $[p], [q] \in \sigma_\phi$, $[p] \neq [q]$, where $p$ and $q$ are distinct boundary points of $\partial(\phi(M))$. Since the topology of $\hat{M}$ is Hausdorff, there exist disjoint open neighbourhoods $U$ and $V$ of $p$ and $q$, respectively, in $\hat{M}$. Now if we define $U^* = \phi^{-1}(U \cap \phi(M))$ and $V^* = \phi^{-1}(V \cap \phi(M))$, then $U^* \cap V^* = \emptyset$, and $[p]$ is strongly attached to $U^*$ and $[q]$ is strongly attached to $V^*$. Define $A_U^* = U^* \cup B_U$ and $A_V^* = V^* \cup B_V$, where $B_U$ is the set of all abstract boundary points in $\sigma_\phi$ which are strongly attached to $U^*$ (so $[p] \in B_U$) and $B_V$ is the set of all abstract boundary points in $\sigma_\phi$ which are strongly attached to $V^*$ (so $[q] \in B_V$). Thus $A_U^* \cap \sigma_\phi = B_U$ and $A_V^* \cap \sigma_\phi = B_V$ are open sets of $T_{\sigma_\phi}$ and open neighbourhoods of $[p]$ and $[q]$ respectively. Consider some $r \in \sigma_\phi$, where $r \in \partial(\phi(M))$ and $r \neq p$, such that $[r] \in B_V$. There therefore exists an open neighbourhood $W$ of $r$ in $\hat{M}$ such that $W \cap \phi(M) \subseteq \phi(V^*)$. Now assume that $[r] \in B_U$. This implies that there exists an open neighbourhood $X$ of $r$ in $\hat{M}$ such that $X \cap \phi(M) \subseteq \phi(U^*)$. Since $U^* \cap V^* = \emptyset$ and $W \cap \phi(M) \subseteq \phi(V^*)$ and $X \cap \phi(M) \subseteq \phi(U^*)$, it follows that $X \cap \phi(M) \cap [W \cap \phi(M)] = \emptyset$. We also have that $X$ and $W$ are both open neighbourhoods of $r$, and so $X \cap \phi(M) \cap [W \cap \phi(M)] \neq \emptyset$. We therefore have a contradiction. This implies that $B_U$ and $B_V$ are disjoint open neighbourhoods of $[p]$ and $[q]$ respectively thereby demonstrating that the topological space $(\sigma_\phi, T_{\sigma_\phi})$ is Hausdorff. It is clear from proposition 43 that $T_{\sigma_\phi,(\sigma_\phi)}$ is also Hausdorff.

In practice, the abstract boundary of a space-time is studied by considering its envelopments. It is therefore highly desirable that the natural topology $T_{\sigma_\phi}$ of a partial cross section $\sigma_\phi$ agrees with the topology on $\sigma_\phi$ induced by the strongly attached point topology. That way, the topological features of the boundary may be studied in the natural topology of an envelopment and any results shown to be true in that envelopment will also hold in the strongly attached point topology on the whole abstract boundary $B(M)$.

In the following proposition we show, assuming a condition holds, that the natural topology $T_{\sigma_\phi}$ of a partial cross section $\sigma_\phi$ agrees with the topology on $\sigma_\phi$ induced by the strongly attached point topology.

**Condition 51.** Consider an envelopment $(M, \hat{M}, \phi)$ with boundary $\partial(\phi(M)) \neq \emptyset$. There exists an open neighbourhood $V$ of $\partial(\phi(M))$ in $\hat{M}$ and a $C^2$ congruence of curves $\{\lambda_p\}$ on $V$ such that:

1. $\lambda_p \text{ passes through } p \in \overline{\partial(\phi(M))}$ from one side to the other side of $\overline{\partial(\phi(M))}$ where it exists as a surface
2. $\lambda_p \cap \overline{\partial(\phi(M))} = \{p\}$
3. $\{\lambda_p\}$ is non-intersecting
4. For each $p$, $\lambda_p : (-\alpha, \beta) \rightarrow V$, where $\alpha, \beta \in \mathbb{R}^+$, such that: $\lambda_p(0) = p$, $\lambda_p(-\alpha)$, $\lambda_p(\beta) \in V \setminus V$, $\lambda_p(-\alpha, 0) \subset \phi(M)$, $\lambda_p(0, \beta) \subset \hat{M} \setminus \phi(M)$ or $\lambda_p(0, \beta) \subset \phi(M)$.

20
Now define the set \(N_q\) of \(q\)-neighbourhood ball \(B\) with radius \(\epsilon\) about \(q\). By Proposition 52, the topology on \(\sigma_{\phi}\) is the topology on \(\hat{M}\), given in definition 52. Every open neighbourhood of \(\hat{M}\) is covered by \(\bigcap \hat{M}\). Let \(T\) be an open set of \(\sigma_{\phi}\). Suppose \(q \in \hat{M}\setminus \hat{N}\) which is an open set disjoint from the open set \(N\). Every open neighbourhood of \(q\) will have non-empty intersection with \(\hat{M}\setminus \hat{N}\) and so \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\) and hence \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\).

Now suppose that \(q \in N \cap \partial(\phi(M))\). The envelopment \(\sigma_{\phi}(\hat{M}, M, \phi)\) obeys condition 51, and so there exists an open neighbourhood \(V\) of \(\partial(\phi(M))\) in \(\hat{M}\) which satisfies condition 51. Consider the set \(Y = \bigcap_{p} \lambda_p\) where \(p \in N \cap \partial(\phi(M))\), and no curve \(\lambda_p\) enters \(B_{\epsilon}\), where \(p \in N \cap \partial(\phi(M))\). This set satisfies condition 51, and so there exists an open neighbourhood \(N^*\) of \(N\) such that \(N^* \cap \partial(\phi(M)) \subseteq N\). By condition 51, a small open ball \(B_{\epsilon}\) of radius \(\epsilon\) about \(q\) can be chosen such that \(B_{\epsilon} \subseteq N^*\) and no curve \(\lambda_p\) enters \(B_{\epsilon}\). It follows that \([q]\) is strongly attached to \(\phi^{-1}(N^*)\) for all \(q \in N \cap \partial(\phi(M))\). If \(q \in N \cap \partial(\phi(M))\), the curve \(\lambda_q\) enters every open neighbourhood of \(q\), and so \([q]\) is not strongly attached to \(\phi^{-1}(N^*)\). If \(q \in \partial(\phi(M))\), we know that \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\) and hence \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\) and so \(U\) is the open set of all abstract boundary points in \(\sigma_{\phi}\) which are strongly attached to the open set \(\phi^{-1}(N^*)\) of \(M\) and \(U\) is an open set of \(T_{\sigma_{\phi}}\).

See figure 11. The possibility that \(\lambda_p(0, \beta) \subseteq \phi(M)\) is included in (iv) to cover the case where, for a sufficiently small open neighbourhood \(U\) of \(p\) in \(\hat{M}\), \(U \cap \partial(\phi(M)) \subseteq \phi(M)\).

**Proposition 52.** Let \(T_{\sigma_{\phi}}\) be the topology on \(\sigma_{\phi}\), defined by the topology of \(\hat{M}\), given in definition 49, and let \(T_{\sigma_{\phi}(str)}\) be the topology on \(\sigma_{\phi}\) induced by the strongly attached point topology on \(\hat{M}\). If \((M, \hat{M}, \phi)\) obeys condition 51, then \(T_{\sigma_{\phi}} = T_{\sigma_{\phi}(str)}\).

**Proof.** (1) If \(U\) is an open set of \(T_{\sigma_p}\), then \(U\) is an open set of \(T_{\sigma_{\phi}(str)}\).

Let \(U\) be an open set of \(T_{\sigma_p}\). This means that for some open set \(N\) of \(\hat{M}\) such that \(N \cap \partial(\phi(M)) \neq \emptyset\), \(U = \{p \in \sigma_{\phi} : p \in N \cap \partial(\phi(M))\}\). It is clear that for each \(p \in N \cap \partial(\phi(M))\), \([p]\) is strongly attached to \(\phi^{-1}(N \cap \phi(M))\).

We now consider whether any other abstract boundary points in \(\sigma_{\phi}\) are strongly attached to \(\phi^{-1}(N \cap \phi(M))\). Suppose \(q \in \partial(\phi(M)) \cap \hat{N} \cap \partial(\phi(M))\). Thus \(q \in \hat{M} \setminus \hat{N}\) which is an open set disjoint from the open set \(N\). Every open neighbourhood of \(q\) will have non-empty intersection with \(\hat{M} \setminus \hat{N}\) and so \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\).

Now suppose that \(q \in N \cap \partial(\phi(M)) \cap N \cap \partial(\phi(M))\). The envelopment \((M, \hat{M}, \phi)\) obeys condition 51, and so there exists an open neighbourhood \(V\) of \(\partial(\phi(M))\) in \(\hat{M}\) which satisfies condition 51. Consider the set \(Y = \bigcap_{p} \lambda_p\) where \(p \in N \cap \partial(\phi(M))\), and no curve \(\lambda_p\) enters \(B_{\epsilon}\), where \(p \in N \cap \partial(\phi(M))\). This set satisfies condition 51, and so there exists a small open ball \(B_{\epsilon}\) of radius \(\epsilon\) about \(q\) that can be chosen such that \(B_{\epsilon} \subseteq N^*\) and no curve \(\lambda_p\) enters \(B_{\epsilon}\). It follows that \([q]\) is strongly attached to \(\phi^{-1}(N^*)\) for all \(q \in N \cap \partial(\phi(M))\). If \(q \in N \cap \partial(\phi(M))\), the curve \(\lambda_q\) enters every open neighbourhood of \(q\), and so \([q]\) is not strongly attached to \(\phi^{-1}(N^*)\). If \(q \in \partial(\phi(M))\), we know that \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\) and hence \([q]\) is not strongly attached to \(\phi^{-1}(N \cap \phi(M))\) and so \(U\) is the open set of all abstract boundary points in \(\sigma_{\phi}\) which are strongly attached to the open set \(\phi^{-1}(N^*)\) of \(M\) and \(U\) is an open set of \(T_{\sigma_{\phi}(str)}\).
(2) If $U$ is an open set of $T_{σφ}^{str}$, then $U$ is an open set of $T_{σφ}$.

Let $U$ be an open set of $T_{σφ}^{str}$. There therefore exists an open set $\bigcup A_i = \bigcup_i U_i \cup B_i$ of $\overline{M}$, where each $U_i$ is a non-empty open set of $M$ and $B_i$ is the set of all abstract boundary points which are strongly attached to $U_i$, such that $U = (\bigcup_i U_i \cup B_i) \cap σφ = (\bigcup_i B_i) \cap σφ$.

Consider $[p] ∈ (\bigcup B_i) \cap σφ$, where $p ∈ \partial(φ(M))$. There exists a $B_i$ such that $[p] ∈ B_i$, and thus $[p]$ is strongly attached to $U$. This means that there exists an open neighbourhood $N_p$ of $p$ in $\overline{M}$ such that $N_p \cap φ(M) \subseteq φ(U_i)$. Consider a boundary point $q ∈ \partial(φ(M))$ such that $q ∈ N_p$. It is clear that $[q] ∈ B_i$ and thus $[q] ∈ U$. The set $W = \bigcup[p] ∈ U N_p$ is an open set in $\overline{M}$. Consider the open set in $T_{σφ}$ defined by $A = \{[p] : p ∈ W \cap \partial(φ(M))\}$. It is clear that $U ⊆ A$ and, from the above argument, that $A ⊆ U$. Thus $U$ is an open set of $T_{σφ}$. □

Direction 2 of the previous proof is quite straightforward. Direction 1 on the other hand, is more complicated and requires us to invoke condition 51. We have to use this condition due to the existence of boundary points $p ∈ \partial(φ(M))$ that are strongly attached to $φ^{-1}(N \cap φ(M))$ but are not elements of $N$. The existence of these boundary points makes it difficult to construct an open neighbourhood of $T_{σφ}^{str}$ that doesn’t contain abstract boundary points additional to those contained in $U$. Even so, condition 51 is not very restrictive and may even hold in general. At the least, we have been unable to construct a space-time in which it does not hold.

9. Conclusion

There are many topologies that can be placed on $\overline{M}$. They will not all be physically useful, however. Ultimately, a topology should provide a structure for $\overline{M}$ which aids us in answering physical questions about $\overline{M}$. Ideally then, the topology should connect the abstract boundary to the manifold in a physically meaningful way, and the resulting structure on $\overline{M}$ should conform to many of our intuitive ideas regarding the behaviour of ‘missing points’, i.e., abstract boundary points, from the manifold $M$.

The strongly attached point topology was defined similarly to the attached point topology but with one important difference. This difference, related to the way in which abstract boundary points are ‘attached’ to open sets of $M$, means that the strongly attached point topology does not need to include collections of abstract boundary points. In the attached point topology these sets were necessary to ensure that the basis for the topology was well-defined. In some sense, because the abstract boundary points are more firmly connected to the manifold in the strongly attached point topology, we avoid having to add more open sets to the topology. It is interesting to note that as a consequence of this, every open neighbourhood of an abstract boundary point in the strongly attached point topology necessarily contains some part of the manifold $M$, thereby encapsulating the true essence of a boundary point.

Another consequence of the strongly attached point topology not containing collections of abstract boundary points is that there exist abstract boundary points which are not Hausdorff separated from each other. While it has been argued that a physically useful topology for a space-time should be Hausdorff [16], the lack of Hausdorff separation between abstract boundary points in the strongly attached point topology, nevertheless, contains useful information about the boundary. It was demonstrated that two abstract boundary points are Hausdorff separable if and only if they are not in contact. Intuitively, this makes sense as two abstract boundary points which are in contact share much of the same topological information, and therefore they do not represent two points which are distinct from each other. It therefore seems reasonable that abstract boundary points which are in contact with each other cannot be separated by disjoint open sets. It is also worth noting that there is a natural relationship between the separation axioms that two abstract boundary points obey, and how
much topological information they share. As propositions 43 through 45 show, as two abstract boundary points share more of the same topological information, they obey fewer separation axioms. For example, two abstract boundary points which are in contact are $T_1$ separable, but not $T_2$ separable, while if one abstract boundary point covers the other, they are $T_0$ separable, but not $T_1$ separable. Therefore, while separation is lost between abstract boundary points, it is lost in a way directly related to the amount of overlap between the abstract boundary points.

The strongly attached point topology possesses a number of other interesting properties which suggest that it is an appropriate topology for $\hat{M}$. One such property is that the description of $M$ and $B(M)$ in the strongly attached point topology agrees with many of our intuitive ideas about the nature of a space and its boundary. Traditionally, singularities are typically viewed as ‘points’ missing from a space-time. We can approach these missing points from within the space-time, becoming arbitrarily close to them, but we cannot reach them. In some sense then, these missing points make up the ‘closure’ of the space-time, and ideally, a topology on $\hat{M}$ should reflect this. The strongly attached point topology agrees with this notion in the sense that $M$ is open and not closed, $B(M)$ is closed and not open, and $M$ can be embedded identically into $\hat{M}$. The strongly attached point topology therefore provides a natural way of viewing the structure of $\hat{M}$ in that it can be seen as $M$ with the addition of a topological boundary made up of abstract boundary points.

Perhaps the most important property of the strongly attached point topology is that the topology induced by the strongly attached point topology on a partial cross section $\sigma_\phi$ associated with an envelopment $\phi : M \to \hat{M}$ generally agrees with the natural topology on $\sigma_\phi$. In practice, the abstract boundary is studied via envelopments of the manifold $M$. Consequently, the embedded manifold $\phi(M)$ and its topological boundary $\partial(\phi(M))$ already have a topology defined on them with which it is very easy to work. It is therefore very useful that the topologies agree on the partial cross sections $\sigma_\phi$ as it means that any topological result which holds in an envelopment (which, again, is where the abstract boundary is studied in practice) will also hold with respect to the larger topology on $B(M)$.

Without a topology on $\hat{M}$ we cannot say ‘where’ singular points are with respect to the manifold $M$. A topology on $\hat{M}$ should therefore relate the abstract boundary back to the manifold $M$. Moreover, it should ideally do so in a natural way, i.e., the topology should describe the singular points in a way that agrees with our intuitive ideas of how a singularity is related to the manifold. It has been shown that the strongly attached point topology does indeed relate the abstract boundary back to the manifold in a way that encompasses many of our intuitive notions of the nature of a topological boundary. For these reasons, the strongly attached point topology appears to be a particularly good choice for a topology on the set comprising a manifold and its abstract boundary.

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