Wrapped D–Branes as BPS Monopoles:  
The Moduli Space Perspective  

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Abstract  
We study the four dimensional effective action of a system of D6–branes wrapped on the K3 manifold times a torus, allowing the volume of the internal manifolds to remain dynamical. An unwrapped brane is at best a Dirac monopole of the dual R–R sector field to which it couples. After wrapping, a brane is expected to behave as a BPS monopole, where the Higgs vacuum expectation value is set by the size of the K3. We determine the moduli space of an arbitrary number of these wrapped branes by introducing a time dependent perturbation of the static solution, and expanding the supergravity equations of motion to determine the dynamics of this perturbation, in the low velocity limit. The result is the hyper–Kähler generalisation of the Euclidean Taub–NUT metric presented by Gibbons and Manton. We note that our results also pertain to the behavior of bound states of Kaluza–Klein monopoles and wrapped NS5–branes in the \(T^4\) compactified heterotic string.
1 Introduction and Motivation

D–branes are the smallest sources of Ramond–Ramond (R–R) sector charge in types I and II string theory \[1\]. A D\(p\)–brane has an electric coupling to the \((p+1)\)–form Ramond–Ramond sector potential \(C_{(p+1)}\). By Hodge duality, it couples magnetically to the potential \(C_{(7−p)}\). As magnetic monopoles in the full ten dimensions, the D–branes may be said to be monopoles of “Dirac” type. They have topology, and a completely fixed spherically symmetric transverse geometry as ten dimensional objects in perturbative string theory: They are point–like (on scales coarser than the string length \(\ell_s\)) in their transverse dimensions while at stringy scales they are only slightly more interesting, having an effective “halo” of interaction radius of \(\ell_s = 2\pi\sqrt{\alpha'}\), which is still effectively spherical.

These simple properties are in contrast to those of a BPS magnetic monopole \[2, 3, 4, 5\], obtained from the non–trivial dynamics and topology of spontaneously broken non–Abelian gauge symmetry. They again have gauge bundle topology which supplies them with magnetic charge, but they have an adjustable transverse size set by the physical parameters of the dynamical symmetry breaking. When endowed with multiples of the basic unit charge, they also have shapes more interesting than simple spherical geometry.

Perhaps one of the most interesting and rewarding features of D–branes is the fact that while they are extremely simple objects with somewhat boring features (such as those listed in the first paragraph) in the overall scheme of things, they are readily amenable to being endowed with interesting properties. This is achieved in a variety of ways, such as allowing them to interact with other branes (of the same or different type and dimension) by immersion or intersection, putting them in background fields, or embedding them in non–trivial geometry.

In particular, D–branes can behave as BPS monopoles, and therefore are endowed with properties which are sensitive to the non–Abelian gauge theory dynamics of the string theory vacuum in which they find themselves. This is intriguing for a number of reasons. BPS monopoles have a size and shape, both of which are tunable by adjusting the asymptotic values of the extra scalars which appear in the theory. At the same time, supersymmetry is preserved, which allows for a clean study of the properties of such objects. So we are in a position to study, with the aid of supersymmetry if we wish, the dynamics of D–branes which have interesting transverse structure and size. A whole host of highly instructive examples (various gauge duals, holographic correspondences, etc.,) which have been excavated over the last five years or so have shown that such regimes —where branes stop being simply point–like in their transverse directions—are likely to prove the highly instructive in uncovering the next levels of understanding of the
fundamental physics to which brane dynamics seem to be guiding us.

In this paper we would like to further examine the story of how D–branes behave precisely like BPS monopoles as a result of being wrapped on the four dimensional manifold K3. The problem that we tackle is to take the explicit effective action of the superstring theory, and the static solutions for the D–branes in this K3–wrapped situation, and derive the effective Lagrangian for the slow relative motion of an arbitrary number of such branes in arbitrary positions. Even though the solution for the fields around the branes (involving gravity, the dilaton, and various higher rank forms) are very different from those of a BPS monopole, the result for the effective Lagrangian should be identical to that of the standard BPS monopole result presented by Gibbons and Manton [6]. We show that this is the case explicitly. We notice along the way that the effective action within which we work is also appropriate to the $T^4$ compactified heterotic string. The reduced action has an elegant symmetry which takes us from one setup to another. The wrapped D6–brane under this duality becomes [7] a bound state of an H–monopole (wrapped NS5–brane) and a Kaluza–Klein monopole, which also is expected to behave as a BPS monopole [8, 9]. Our result proves that this is the case for that dual picture also.

2 BPS Monopoles from wrapping branes

Realising D–branes as BPS monopoles can be achieved, for example, by wrapping D–branes on K3, as discussed in ref. [10]. In type IIA superstring theory, for example, a compactification on K3 yields\(^1\) a gauge group of $U(1)^{24}$ in the six non–compact dimensions, generically. This arises from the R–R three–form $C_{(3)}$ reduced on the 22 two–cycles of K3, from the R–R one–form $C_{(1)}$, and from the R–R five–form $C_{(5)}$ wrapped on the four–cycle that is the entire K3. There is a large family of scalars in the $\mathcal{N} = 2$ gauge multiplets (counting in four dimensional units), which have the geometrical interpretation as representing the volume of the cycles on which these forms are reduced, together with the flux of the Neveu-Schwarz–Neveu-Schwarz (NS–NS) two form $B_{(2)}$ through them. The Abelian gauge group can be enhanced at special points on the moduli space of these scalar vacuum expectation values (vevs), corresponding to the vanishing of the $B$–flux [12], and at the same time where the characteristic length scales of these volumes, $\ell_{\text{cycle}}$, reach the value $\ell_s = 2\pi\sqrt{\alpha'}$. In this special situation, new string theory physics appears. In particular, the D2– and D4–branes (which couple electrically to the reduced forms) are wrapping the cycles, appearing as particles in the non–compact directions.

\(^1\)See for example, ref. [11] for a review.
These particles become massless at the special points, and are the W–bosons of the enhanced non–Abelian gauge symmetry.

The BPS monopoles of this pattern of spontaneously broken gauge groups are constructed as D–branes too. In six dimensions, they must have two extended directions (i.e., they are membranes), and they are formed by wrapping D4–branes on the two–cycles, or by wrapping a D6–brane on the entire K3. These cycle–wrapped D–branes carry magnetic charges of the $U(1)$ which arose from direct reduction on the cycle in question. To couple magnetically to a $q$–form potential in $D$ dimensions is to couple electrically to its $(D − 2 − q)$–form potential arising by Hodge duality. In six dimensions therefore, our putative monopoles must have an electric coupling to $C_{(3)}$. This is achieved for a wrapped D4–brane because the flux through the two–cycle induces a single unit of (negative) D2–brane charge, since there is a world–volume coupling of lower rank R–R forms to the Chern characteristic $e^{i\mathcal{F}}$ of the mixed “gauge/two–form bundle”, with field strength two–form $\mathcal{F} = B + 2\pi \alpha' F$. A somewhat different mechanism achieves the same thing for a wrapped D6–brane. There is a world–volume coupling of lower rank R–R forms to the square root of the Dirac genus $\hat{\mathcal{A}}(\ell_2^2 R)$, where $R$ is the Ricci two–form. Since K3 has non–vanishing Pontryagin class, it induces $C_{(3)}$–form charge in six dimensions.

Notice that the facts that D6–branes are magnetic sources of $C_{(1)}$, and that D4–branes are magnetic sources of $C_{(3)}$ in ten dimensions are red herrings, from the point of view of turning them into BPS monopoles under wrapping. If this were not the case, one could produce BPS monopoles by wrapping on a torus, $T^4$, for which in superstring theory (as opposed to heterotic string theory of course) there are no patterns of spontaneous gauge symmetry breaking. Instead, it is the collection of subtle features of D–branes mentioned above which endow the wrapped D–branes with properties which are intimately related to the geometrical properties of the cycles upon which they are wrapped. Hence, their size and shape are controlled by the moduli of the cycles. Correspondingly, as the moduli of the cycles translate into Higgs vevs in the reduced model, the branes’ properties will be controlled by the Higgs vevs, as should be the case for BPS monopoles.

The above reasoning is satisfying, but certainly not enough to show beyond doubt that these wrapped branes are indeed BPS monopoles of the type we know and love. They could carry the same asymptotic charges, have the same number of moduli, but behave quite differently in detail. The next step is to demonstrate that the metric on the space of moduli is the same as for BPS monopoles\(^2\). This is what we will show explicitly in the rest of the paper, at least

\(^2\)In the original enhàcon paper, the metric on the moduli space of a single constituent wrapped brane,
in the limit where we do not allow the objects to approach each other too closely. For closer separations there are instanton corrections to our computation which we leave for further study. However, note that supersymmetry ensures that our moduli space metric is hyper–Kähler, and it is believed that additional assumptions of smoothness and the presence of certain isometries of the metric completes the asymptotic answer into a unique non–perturbative result. This is known explicitly for the two–monopole case, with the complete metric being the Atiyah–Hitchin manifold [18, 19].

3 Wrapped Brane Solutions

We will focus on the case of wrapping D6–branes. We do not have to do this. However, the case of the wrapping of D4–branes is equivalent to this discussion by T–duality. There is a large $O(20, 4)$ symmetry at our disposal [11], with which we can write the problem in a variety of ways, and the wrapped D6–brane approach of ref. [10] is the one we choose. It would be very instructive to make the duality of the situation explicit, since K3 is an interesting manifold to study, but this is not the goal of this paper.

The wrapping of the D6–branes on K3, which we assume has a volume $V$, induces a negative amount of D2–brane charge [17, 16]. Therefore in the limit where we have a large number, $N$ of D6–branes present, the metric, dilaton and Ramond–Ramond potentials which correspond to this scenario are expected to be [10]:

$$
\begin{align*}
    ds_S^2 &= Z_2^{-1/2} Z_6^{-1/2} \eta_{\mu \nu} dx^\mu dx^\nu + Z_2^{1/2} Z_6^{1/2} dx^i dx^i + V^{1/2} Z_2^{1/2} Z_6^{-1/2} dS_{K3}^2 \\
    e^{2\Phi} &= g_s^2 Z_2^{1/2} Z_6^{-3/2} \\
    C^{(3)} &= (Z_2 g_s)^{-1} dx^0 \wedge dx^4 \wedge dx^5 \\
    C^{(7)} &= V (Z_6 g_s)^{-1} dx^0 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9,
\end{align*}
$$

where $\mu, \nu = 0, 4, 5$ are the directions tangent to all branes, and $i, j = 1, 2, 3$ are the directions transverse to all branes, with

$$
\begin{align*}
    Z_2 &= 1 + \frac{r_2}{r}, \quad r_2 = -\frac{(2\pi)^4 g_s N \alpha'^5/2}{2 V}, \\
    Z_6 &= 1 + \frac{r_6}{r}, \quad r_6 = \frac{g_s N \alpha'^{1/2}}{2}.
\end{align*}
$$

This solution has $N$ units of D6–brane charge, and consequently $N$ units of negative D2–brane charge. Notice that as one moves in from $r = +\infty$ toward smaller $r$, the volume of K3 reduces moving in the background produced by the others, was derived. Our result extends that case considerably.
from $V$. This is the result of the solution for the wrapped brane back–reacting on the geometry. In fact, the geometry apparently has a repulsive singularity at $r = |r_2|$, where the effective K3 volume vanishes. However, in ref. [10] it was shown that for sufficiently large $N$ the geometry given by equations (1) is only correct down to the “enhâncion radius”:

$$r_e = \frac{2V}{V - V^*} |r_2| > |r_2|,$$

where $V^* \equiv \ell_s^4 = (2\pi)^4(\alpha')^2$, and we assume that $V > V^*$. In fact, all the D6–branes live on the sphere at $r = r_e$, inside of which the geometry is flat.

Inside the enhâncion radius there is an enhanced $SU(2)$ gauge symmetry (hence the name enhâncion). These facts were deduced in part by appealing to a probe computation: A single D6–brane was used to probe the background due to all of the others. The brane becomes unphysical for motion in the background as written, once one proceeds inside the enhâncion radius. This is because a constituent brane’s tension would become negative in that region due to the stringy effects (induced charges) arising from wrapping. The simplest solution is to declare the geometry in that region to be unphysical, and replace it by flat space, since there are no sharp sources in the interior. This is not unreasonable, since the geometry was written down in supergravity with no reference to possible stringy effects in the first place, being designed to match only the asymptotic charges.

For a large number, $N$, of D6–branes sourcing the background geometry, it is consistent to neglect the back–reaction of the probe on the background and do a probe computation as in ref. [10], if one assumes that one can freeze the moduli of all of the other branes. The result was the Euclidean Taub–NUT metric, with negative mass parameter set by $r_e$, and hence proportional to $N$.

In the computation on which we report here, we unfreeze all of the moduli. We will put the branes in arbitrary positions, not all clumped together. Furthermore we will let the branes explore all of the moduli available to them, and show that in the limit where we restrict to slow relative motion, the problem is tractable, and yields a simple result, the multi–Taub–NUT generalisation written down by Gibbons and Manton [6]. In ref. [23], it was argued that when we arrange all the moduli (except four of them) as in ref. [10], placing $N - 1$ of the branes on a sphere of radius $r_e$, we should expect to recover the moduli space result derived there, Taub–NUT with negative mass $r_e$. This should be accurate for large $N_a$, a limit where the necessary deviations away from spherical symmetry for a multi–monopole configuration should be suppressed.

\[\text{This aspect of the construction found further support in the supergravity computations presented in ref. [8].}\]
In deriving the result for the $4N$ dimensional moduli space of BPS monopoles [24], our technique follows closely that of Ferrell and Eardley in refs. [25,26], who calculated the metric on moduli space for slowly moving Reissner–Nordström black holes, using the ideas pioneered by Manton in ref. [27]. To make contact with those techniques, we first dimensionally reduce our action to lower dimensions. We reduce not just on K3, but on an additional $T^2$ as well, making the monopoles localised objects. The torus produces no additional subtleties, since it has no non–trivial structure. In the next section we perform the dimensional reduction.

4 Dimensional Reduction to Four Dimensions

In our calculations we will assume that the D6–branes are reasonably far apart from one another. We dimensionally reduce over the D6–brane directions, chosen to be $x^4, \ldots, x^9$, to obtain a four–dimensional picture in which they behave like distinct localised objects. From the work of ref. [8], we know that if there are $N_a$ coincident D6–branes at the $a$th position, the lower limit on its effective size is set by the enhançon radius given in equation (3). We should assume that they are separated by at least that size, but since $N_a$ will be taken to be unity, the corrections to the enhançon radius are significant (it is no longer a sharp radius, as it is for large $N$ (see discussions in ref. [23]), and so we expect each brane to be separated from each other by distances significantly greater than that, to avoid needing to worry about instanton corrections to our result. The typical separation at which instanton corrections become significant can be estimated by comparing the Atiyah–Hitchin manifold to the four–dimensional Euclidean negative mass Taub–NUT solution to which it reduces for large separations. This is the effective core or enhançon radius for a single D6–brane [23].

4.1 The Type IIA Action in the Einstein Frame

We start with the type IIA ten dimensional supergravity action in the string frame with a D6–brane field strength and a D2–brane field strength excited:

$$S_{IIA} = \frac{1}{2\kappa_0^2} \int d^{10}x \sqrt{-g^S} \left\{ e^{-2\Phi}[R^S + 4(\nabla^S \Phi)^2] - \frac{1}{2.4!}(F^S_4)^2 - \frac{1}{2.8!}(F^S_8)^2 \right\}.$$  \hspace{1cm} (4)

We will work mostly in the Einstein frame, defined by setting:

$$g^S_{\mu\nu} = e^{\frac{\Phi - \Phi_0}{2}} g_{\mu\nu}^E,$$  \hspace{1cm} (5)
where $\Phi_0$ is the reference value of the dilaton field $\Phi$, and $g_s = e^{\Phi_0}$ is the string coupling. Let $\tilde{\Phi} = (\Phi - \Phi_0)$. Then, after some algebra, the Einstein frame action is:

$$S_{\text{IIA}}^E = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g_E} \left\{ R^E - \frac{1}{2}(\nabla^E \tilde{\Phi})^2 - \frac{1}{2.4!} e^{\frac{\Phi}{2}} (F_E^{(4)})^2 - \frac{1}{2.8!} e^{-\frac{3\Phi}{2}} (F_E^{(8)})^2 \right\},$$

where $2\kappa^2 g_s^2 = 16\pi G_N = (2\pi)^7 (\alpha')^4$ sets the ten dimensional Newton’s constant $G_N$. In what follows we will relabel $\tilde{\Phi}$ as $\Phi$, and we have absorbed a factor of $g_s$ into the R–R potentials in going to the Einstein frame.

### 4.2 Reduction on K3

Here we dimensionally reduce on K3. After the dimensional reduction it will be necessary to perform a conformal transformation so that the gravity part of the dimensionally reduced action is of the canonical Einstein–Hilbert form.

We rewrite the action in equation (6) as

$$S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-\hat{g}} \left\{ \hat{R} - \frac{1}{2}(\nabla^E \Phi)^2 - \frac{1}{2.4!} e^{\frac{\Phi}{2}} (\hat{F}^{(4)})^2 - \frac{1}{2.8!} e^{-\frac{3\Phi}{2}} (\hat{F}^{(8)})^2 \right\},$$

where we have relabeled all the fields with hats to indicate that they are 10-dimensional fields. The hat in $(\hat{F}^{(4)})^2$ also indicates that the metric $\hat{g}_{\mu\nu}$ is used to compute the square, i.e.,

$$ (\hat{F}^{(4)})^2 = \hat{g}^{\mu_1\nu_1} \hat{g}^{\mu_2\nu_2} \hat{g}^{\mu_3\nu_3} \hat{g}^{\mu_4\nu_4} \hat{F}_\mu^{(4)} \hat{F}_{\mu_1\mu_2\mu_3\mu_4} \hat{F}_{\nu_1\nu_2\nu_3\nu_4}^{(4)}. $$

$\hat{F}^{(4)}$ is a 4-form field strength with potential $\hat{C}^{(3)} = d\hat{F}^{(4)}$. Similar remarks apply to $\hat{C}^{(7)} = d\hat{F}^{(8)}$. We wish to calculate the dimensionally reduced version of equation (6) when the directions $x^6, x^7, x^8, x^9$ are compactified on a K3 manifold. We set

$$ \hat{g}_{MN} = \begin{pmatrix} \bar{g}_{\mu\nu} & 0 \\ 0 & V^{1/2} e^{\beta/2} g_{ij}^{K3} \end{pmatrix}, $$

where $M, N = 0, \ldots, 9, \mu, \nu = 0, \ldots, 5, i, j = 6, \ldots, 9$ and $V_{K3} = V e^\beta$ is the volume of the K3 manifold, with $V$ constant.

Since we are assuming that the wrapped branes are localised in the four dimensions, we can take $\bar{g}_{\mu\nu}, \beta, \Phi, \hat{F}^{(8)}$ to be independent of the compactified directions $x^i$. We also take the metric on K3, $g_{ij}^{K3}$, to depend only on the $x^i$. Then

$$ \hat{R} = \bar{R} - \frac{5}{4} (\nabla \beta)^2 - 2 (\nabla^2 \beta), $$
where we have used that K3 is Ricci flat. Also, we have

\[ \sqrt{-\hat{g}} = V e^\beta \sqrt{-\bar{g}} . \]  \hspace{1cm} (11)

We make the choice that each non-vanishing component of \( \hat{C}^{(7)} \) contains the indices 6,7,8,9. This is consistent with the form of the solution given in equation (1). We set

\[ \bar{C}^{(3)}_{\mu_1 \mu_2 \mu_3} = V^{-1} \hat{C}^{(7)}_{\mu_1 \mu_2 \mu_3 6789} , \]  \hspace{1cm} (12)

and we have used a prime to distinguish the dimensionally reduced D6–brane potential \( \bar{C}^{(3)} \) from the D2–brane potential, which in the dimensionally reduced action we will denote \( \hat{C}^{(3)} \). Then

\[ (\hat{F}^{(8)})^2 = 8.7.6.5. e^{-2\beta} (\bar{F}^{(4)})^2 \]  \hspace{1cm} (13)

Also

\[ \hat{C}^{(3)}_{\mu_1 \mu_2 \mu_3} = \bar{C}^{(3)}_{\mu_1 \mu_2 \mu_3} , \]  \hspace{1cm} (14)

since \( \hat{C}^{(3)} \) does not have components in the K3 directions.

Substituting (10) - (14) into the action (7), we get the six dimensional action

\[ S_{\text{IIA}} = \frac{1}{2\kappa^2} \int d^6x \ V e^\beta \sqrt{-g} \left[ \bar{R} - \frac{5}{4} (\bar{\nabla} \beta)^2 - 2(\bar{\nabla}^2 \beta) - \frac{1}{2} (\bar{\nabla} \Phi)^2 \\
- \frac{1}{2.4!} e^\Phi (\bar{F}^{(4)})^2 - \frac{1}{2.4!} e^{-3\beta} e^{-2\beta} (\bar{F}^{(4)})^2 \right] . \]  \hspace{1cm} (15)

This action is not of the standard Einstein form since there is a factor of \( e^\beta \) multiplying \( \bar{R} \). In order to remove this factor we perform the following conformal transformation:

\[ g_{\mu\nu} = e^{-\beta/2} \bar{g}_{\mu\nu} . \]  \hspace{1cm} (16)

After some algebra, we find that the action is:

\[ S_{\text{IIA}} = \frac{V}{2\kappa^2} \int d^6x \sqrt{-g} \left[ \tilde{R} - \frac{1}{2} (\tilde{\nabla} \beta)^2 - \frac{1}{2} (\tilde{\nabla} \Phi)^2 \\
- \frac{1}{2.4!} e^\Phi e^{3\beta} (\tilde{F}^{(4)})^2 - \frac{1}{2.4!} e^{-3\beta} e^{-\beta/2} (\tilde{F}^{(4)})^2 \right] . \]  \hspace{1cm} (17)

where we have discarded total derivative terms.
4.3 Reduction on $T^2$

Next, we dimensionally reduce the action of equation (17) on the directions $x^4, x^5$, which we will compactify on a $T^2$. We set

$$\tilde{g}_{MN} = \begin{pmatrix} \tilde{g}_{\mu\nu} & 0 \\ 0 & e^\rho \delta_{ij} \end{pmatrix}$$  \hspace{1cm} (18)$$

where $M, N = 0, \ldots, 5$, $\mu, \nu = 0, \ldots, 3$, $i, j = 4, 5$. As in section 4.2 we assume that all fields are independent of the compactified directions $x^4, x^5$. Then:

$$\tilde{R} = \tilde{R} - 2\tilde{\nabla}^2 \rho - \frac{3}{2} (\tilde{\nabla} \rho)^2, \quad \text{and} \quad \sqrt{-\tilde{g}} = e^\rho \sqrt{-\bar{g}}. \hspace{1cm} (19)$$

We make the choice that the non–vanishing components of $\tilde{C}^{(3)}$ and $\tilde{C}'^{(3)}$ contain the indices 4,5, which is consistent with the solutions (11). We set

$$\tilde{C}^{(1)}_\mu = \bar{C}^{(3)}_{\mu 45}, \quad \tilde{C}'^{(1)}_\mu = \bar{C}'^{(3)}_{\mu 45}. \hspace{1cm} (20)$$

Then

$$\left( \tilde{F}^{(4)} \right)^2 = 4.3 e^{-2\rho} \bar{F}^{(2)}, \quad \left( \bar{F}'^{(4)} \right)^2 = 4.3 e^{-2\rho} \bar{F}'^{(2)}. \hspace{1cm} (21)$$

Substituting the above into the action (17) gives the four-dimensional action:

$$S_{\text{IIA}} = \frac{L^2 V}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} \left[ \tilde{R} - 2\tilde{\nabla}^2 \rho - \frac{3}{2} (\tilde{\nabla} \rho)^2 - \frac{1}{2} (\tilde{\nabla} \beta)^2 - \frac{1}{2} (\tilde{\nabla} \Phi)^2 - \frac{1}{4} e^{\frac{\rho}{2}} e^{3\beta/2} e^{-\rho} (\tilde{F}^{(2)})^2 - \frac{1}{4} e^{-\frac{\rho}{2}} e^{-\beta/2} e^{-\rho} (\tilde{F}'^{(2)})^2 \right], \hspace{1cm} (22)$$

where $L$ is the length of the compactified dimensions. To restore the action to canonical form, we make the conformal transformation:

$$\bar{g}_{\mu\nu} = e^{-\rho} g_{\mu\nu},$$  \hspace{1cm} (23)$$

and the resulting action is:

$$S_{\text{IIA}} = \frac{L^2 V}{2\kappa^2} \int d^4x \sqrt{-g} \left[ R - (\nabla \rho)^2 - \frac{1}{2} (\nabla \beta)^2 - \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{4} e^{\frac{\rho}{2}} e^{3\beta/2} e^{-\rho} (F^{(2)})^2 - \frac{1}{4} e^{-\frac{\rho}{2}} e^{-\beta/2} e^{-\rho} (F'^{(2)})^2 \right], \hspace{1cm} (24)$$

where we have dropped all adornments on the four dimensional quantities, for clarity.
4.4 A Symmetry of the Action and the Dual Heterotic String

Note that the six dimensional action (17) is invariant under the transformation
\[ \Phi \leftrightarrow -\beta, \quad \text{and} \quad F^{(4)} \leftrightarrow F^{(4)}, \] (25)
and the four dimensional action is invariant under
\[ \Phi \leftrightarrow -\beta, \quad \text{and} \quad F^{(2)} \leftrightarrow F^{(2)}. \] (26)
That is, the dilaton is exchanged with the volume of K3, while the D2–brane and D6–brane potentials are interchanged.

This symmetry is a consequence of, or consistent with (depending upon taste), the duality between Type IIA strings compactified on K3 and heterotic strings compactified on $T^4$. Under this duality, the dilaton field $e^\Phi$, which plays the role of the Type IIA string coupling in the Type IIA string action, becomes the volume of the $T^4$ in the heterotic string action. Conversely, the field $e^\beta$ plays the role of the volume of K3 in the Type IIA action, and the role of the heterotic string coupling in the heterotic string action.

In terms of the field strengths, the $F^{(8)}$ field strength in the ten-dimensional Type IIA action is Hodge dual to a $F^{(2)}$ field strength. The $F^{(2)}$ field strength is not wrapped in the six–dimensional Type IIA theory. Under the heterotic–Type IIA duality this $F^{(2)}$ field strength becomes an $F^{(2)}$ field strength in the heterotic theory, which is wrapped on the $T^4$ directions. Therefore in the ten–dimensional heterotic string theory the corresponding field strength is $F^{(6)}$, which is Hodge dual to $F^{(4)}$. So the $F^{(8)}$ in the ten dimensional string theory becomes $F^{(4)}$ in the heterotic theory, and vice versa, as the transformation requires. In other words, the D6–brane charges in the Type IIA string theory are transformed \[7\] into Kaluza–Klein monopole charges in the heterotic string theory, and the D2–brane charges are transformed into H–monopole (wrapped NS5–brane) charges. Our results in this paper therefore pertain to the bound state of these two objects in the heterotic theory, which also is expected to behave as a BPS monopole \[8,9\].

4.5 The Static Solution

We take as our static solution the form we displayed in equations (1), with the assumption being that $Z_2$ and $Z_6$ are general harmonic functions of the $x^i$. We convert the string frame solution to the Einstein frame using
\[ g_{\mu\nu}^E = e^{-(\Phi-\Phi_0)/2} g_{\mu\nu}^S = Z_2^{-1/8} Z_6^{3/8} g_{\mu\nu}^S. \]
Then the ten dimensional solution in the Einstein frame is
\begin{align*}
\tilde{ds}^2 &= Z^{-5/8} Z_6^{-1/8} \eta_{\mu\nu} dx^\mu dx^\nu + Z_2^{3/8} Z_6^{7/8} dx^i dx^i + V^{1/2} Z_2^{3/8} Z_6^{-1/8} dS^2_{K3}, \\
\tilde{e}^{2\Phi} &= Z_2^{1/2} Z_6^{-3/2}, \\
C^{(3)} &= (Z_2)^{-1} dx^0 \wedge dx^4 \wedge dx^5, \\
C^{(7)} &= V (Z_6)^{-1} dx^0 \wedge dx^4 \wedge dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8 \wedge dx^9, 
\end{align*}

where again we have relabeled $\tilde{\Phi}$ as $\Phi$. We wish to compactify this ten dimensional solution following the same steps as in sections 4.2 and 4.3 to obtain the four dimensional solution to the action (24). After some computation, the solution to the action (17) is
\begin{align*}
\tilde{ds}^2 &= Z^{-1/4} Z_6^{-1/4} \eta_{\mu\nu} dx^\mu dx^\nu + Z_2^{3/4} Z_6^{-1/4} dx^i dx^i, \\
\tilde{e}^{\rho} &= Z_2^{3/4} Z_6^{-1/4}, \\
\tilde{e}^{2\Phi} &= Z_2^{1/2} Z_6^{-3/2}, \\
\tilde{C}^{(3)} &= (Z_2)^{-1} dx^0 \wedge dx^4 \wedge dx^5, \\
\tilde{C}^{(3)}' &= (Z_6)^{-1} dx^0 \wedge dx^4 \wedge dx^5.
\end{align*}

After compactifying on $T^2$, some computation gives the four dimensional solution to the action (24) as:
\begin{align*}
\tilde{ds}^2 &= -Z_2^{-1/2} Z_6^{-1/2} dx^0 dx^0 + Z_2^{1/2} Z_6^{1/2} dx^i dx^i, \\
\tilde{e}^{\rho} &= Z_2^{-1/4} Z_6^{-1/4}, \\
\tilde{e}^{2\Phi} &= Z_2^{1/2} Z_6^{-3/2}, \\
\tilde{C}^{(1)} &= (Z_2)^{-1} dx^0, \\
\tilde{C}^{(1)}' &= (Z_6)^{-1} dx^0.
\end{align*}

5 The Multi–Wrapped–Brane Moduli Space

We observe that the dimensionally reduced action in equation (24) is identical to the action for four-dimensional gravity, with two $U(1)$ gauge potentials, coupled to three scalar fields, as we would expect. Moreover, the form of the four–dimensional enhançon solution (29) is very similar to that of Reissner–Nordström black holes in the Einstein–Maxwell–dilaton system, which is given by:
\begin{align*}
\tilde{ds}^2 &= -U^{-2} (\vec{x}) dt^2 + U^2 (\vec{x}) d\vec{x}^2, \\
\tilde{e}^{-2\Phi} &= F(\vec{x}), \\
A &= (F(\vec{x}))^{-1} dt,
\end{align*}

with
\[ U(\vec{x}) = (F(\vec{x}))^{1/2} \quad \text{and} \quad F(\vec{x}) = 1 + \sum_a \frac{\mu_a}{|\vec{x} - \vec{x}_a|}. \]
where $\vec{x}_a$ is the position of the $a$th black hole, and $\mu_a$ is its mass, and $A$ is the $U(1)$ potential. The metric on moduli space for Reissner–Nordström black holes in the Einstein–Maxwell system was calculated by Ferrell and Eardley in ref. [25], (see also ref. [28]) using Manton’s technique for slowly moving solitons, outlined in [27]. This work was extended to the Einstein–Maxwell–dilaton system by Shiraishi in [29], and first applied to the study of strings in ref. [30] and to branes in ref. [31]. Here we will follow the procedure of the Ferrell and Eardley papers, this time for a system of $N$ K3–wrapped D6–branes.

5.1 The Static Solution

We start with the static four–dimensional solution given by equations (29), where we will henceforth relabel $C'(1)$ as $\tilde{C}(1)$. The solutions for the metric, $e^\Phi$, $C(1)$ and $\tilde{C}(1)$ are in agreement with the Reissner–Nordström solution given in equations (30) and (31) if we take $U = Z_2^{1/4} Z_6^{1/4}$ and $Z_2 = Z_6$. The main schematic difference between the black hole solution and our solution is the extra scalar fields $\beta$ and $\rho$. The results for the low energy scattering will be very different however, as we shall see.

5.2 Point Sources and a Regulator

Since we are taking the limit where the branes are kept a reasonable distance apart, and since we are in the supergravity approximation with a small number of branes at each core, we will be able to legitimately treat them as point–like. Therefore the source terms for the $U(1)$ charges in the action have the form of $\delta$–functions. Then the equations of motion for $C(1)$ and $\tilde{C}(1)$ imply that $Z_2$ and $Z_6$ obey the equations:

$$\nabla^2 Z_2 = \sum_{a=1}^N (r_2)_a \delta^3(\vec{x} - \vec{x}_a), \quad \text{and} \quad \nabla^2 Z_6 = \sum_{a=1}^N (r_6)_a \delta^3(\vec{x} - \vec{x}_a), \quad (32)$$

where the positions of the branes are denoted $\vec{x}_a$. The $\vec{x}_a$ are the modular parameters of the solution. Also, we have

$$(r_2)_a = -\frac{(2\pi)^4 g_s Q_a \alpha'^{5/2}}{2V}, \quad (r_6)_a = \frac{g_s Q_a \alpha'^{1/2}}{2},$$

where $Q_a$ is the number of D6–branes at the $a$th position. We will ultimately set this number to unity. Equations (32) have the solutions:

$$Z_2 = 1 + \sum_{a=1}^N \frac{(r_2)_a}{|\vec{x} - \vec{x}_a|}, \quad \text{and} \quad Z_6 = 1 + \sum_{a=1}^N \frac{(r_6)_a}{|\vec{x} - \vec{x}_a|}. \quad (33)$$
As in ref. [25], the form of these functions will produce infinities at various points in the computations. These divergences hide a lot of interesting physics, in fact, and must be regularised. We regularise by assuming a general charge density $\tilde{Q}$, in effect smearing out the branes. Then equations (32) become:

$$\nabla^2 Z_2 = -\frac{(2\pi)^4 g_s \alpha'^{5/2}}{2V} \tilde{Q}(\vec{x}) \quad \text{and} \quad \nabla^2 Z_6 = \frac{g_s \alpha'^{1/2}}{2} \tilde{Q}(\vec{x}),$$

and we will take the limit

$$\tilde{Q} \to \sum_{a=1}^{N} Q_a \delta^{(3)}(\vec{x} - \vec{x}_a) \quad (35)$$
in the final stages of the calculation, as in ref. [25]. Later on, it will be clear that this scheme actually corresponds to regularising correctly in order to extract what is in effect, in the dual [32,33] 2+1 dimensional $U(N)$ gauge theory, the one–loop quantum corrections to the classical physics. In the monopole moduli space picture, the classical physics is simply a flat moduli space, and the deviations from this to give a non–trivial metric is what the regularisation scheme is able to capture in a controlled manner.

5.3 Perturbing the Static Solution

In the low–velocity approximation we can make the static solutions time dependent by allowing the moduli to depend on time $\vec{x}_a \to \vec{x}_a(t)$. We define $\vec{u}_a$ to be the velocity of the $a$th centre, so that $\vec{u}_a = \vec{\dot{x}}_a(t)$. For the general charge density we define $\vec{u} = \vec{\dot{x}}(t)$ to be the velocity of a charged particle of dust.

We perturb the solution (29) (recall we have relabeled $C^{(1)}$ as $\tilde{C}^{(1)}$) to take into account the effects of the time dependence. Since we are assuming that $u = |\vec{u}|$ is small we only need calculate the perturbed fields to linear order. As in ref. [25], first–order perturbations in quantities which are even under time reversal vanish. Therefore the perturbed solution can be written in the form (we perform a simple gauge transformation on the R–R potentials for later convenience):

$$ds^2 = -Z_2^{-1/2}Z_6^{-1/2}dt^2 + Z_2^{1/2}Z_6^{1/2}d\vec{x}^2 + 2\vec{N} \cdot d\vec{x} dt,$$

$$C^{(1)} = (1 - (Z_2)^{-1}) dt + \vec{A} \cdot d\vec{x},$$

$$\tilde{C}^{(1)} = (1 - (Z_6)^{-1}) dt + \vec{\tilde{A}} \cdot d\vec{x}, \quad (36)$$

and the scalar fields $\Phi$, $\beta$ and $\rho$ remain unperturbed. The perturbations $\vec{A}$, $\vec{\tilde{A}}$ and $\vec{N}$ depend on time through $\vec{x}(t)$. 


According to standard lore \cite{27}, we can neglect radiation effects, because these effects are of higher order than $u^2$. Therefore we can assume that the energy in the system remains in the zero modes; the non–zero modes are not excited. This means that the motion takes the form of a geodesic in moduli space.

### 5.4 The Action in the Slow Motion Limit

We wish to find the equations of motion for the perturbations $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ in order to express these fields as functions of $\tilde{Q}$ and $u$. We need expressions for $\vec{N}$, $\vec{A}$ and $\vec{\tilde{A}}$ to $O(u)$, so we must calculate the perturbed action to $O(u^2)$. Therefore we substitute the perturbed solutions (36) into the action, neglecting terms $O(u^3)$, whence we derive equations of motion from the resulting approximate action.

In section 4 we found that the ten dimensional IIA supergravity action with four dimensions compactified on K3, and two dimensions compactified on $T^2$ reduces to the following four–dimensional action

$$S_{\text{IIA}} = S_{\text{gravity}} + S_{\text{Maxwell}} + S_{\text{scalar}},$$

where

$$S_{\text{gravity}} = k \int d^4x \sqrt{-g} R,$$

$$S_{\text{Maxwell}} = k \int d^4x \sqrt{-g} \left( -\frac{1}{4} e^{\Phi/2} e^{\beta/2} e^{-\rho} (F^{(2)})^2 - \frac{1}{4} e^{-3\Phi/2} e^{-\beta/2} e^{-\rho} (\tilde{F}^{(2)})^2 \right),$$

$$S_{\text{scalar}} = k \int d^4x \sqrt{-g} \left( - (\nabla \rho)^2 - \frac{1}{2} (\nabla \beta)^2 - \frac{1}{2} (\nabla \Phi)^2 \right),$$

where we have defined the useful constant, $k$, as

$$k = L^2 V/2\kappa^2.$$  

Substituting the perturbed solutions (36) into the action (37), and integrating by parts several
We convert to the Einstein frame, then compactify on K3 and follow the same steps as in section 4 to reduce the ten dimensional action (41) to a four-dimensional metric in equation (36). Substituting the rest of the perturbed solutions of equations (36) into the action (43), we find:

\[
S_{\text{IIA}}^{\text{approx}} = k \int d^4x \left\{ -\frac{1}{2} \left| \tilde{\nabla} \times (\tilde{A} + Z_2^{-1/2} Z_6^{1/2} \tilde{N}) \right|^2 \frac{Z_2^{-1} Z_6}{Z_2 Z_6^{-1}} + \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{-1/2} Z_6^{1/2} \tilde{N})}{Z_2} \cdot \left( \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \right) \right. \\
\left. + \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{1/2} Z_6^{-1/2} \tilde{N})}{Z_2} \cdot \left( \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \right) - \frac{1}{2} \left| \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \right|^2 - \tilde{Z}_2 \tilde{Z}_6 - \tilde{\nabla} \cdot \left( \tilde{A} + Z_2^{-1/2} Z_6^{1/2} \tilde{N} \right) - \tilde{\nabla} \cdot \left( \tilde{A} + Z_2^{1/2} Z_6^{-1/2} \tilde{N} \right) \right\} \quad (40)
\]

We also need to include source terms in the action for the matter density and for the current. To find the matter source terms we need to dimensionally reduce the Born–Infeld action for the D6–branes and the Born–Infeld action for the D2–branes. These are given by:

\[
S_{\text{matter}} = -\int d^7\xi \ e^{-\Phi} T_6 \sqrt{-\det \tilde{G}^S} + \int d^3\xi \ e^{-\Phi} T_2 \sqrt{-\det \tilde{G}^S} , \quad (41)
\]

where \(\xi^a\) are world–volume coordinates. Also \(\tilde{G}^S_{\alpha\gamma}\) and \(\tilde{G}^S_{\alpha\gamma}\) are the induced or “pulled–back” metrics on the D6–brane world–volume and the D2–brane world–volume respectively, e.g.

\[
\tilde{G}_{\alpha\gamma} = G_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\gamma} , \quad (42)
\]

and \(T_6\) and \(-T_2\) are the D6–brane tension and the (negative) D2–brane tension respectively. We follow the same steps as in section 4 to reduce the ten dimensional action (41) to a four-dimensional one; we convert to the Einstein frame, then compactify on K3\(\times T^2\) to get

\[
S_{\text{matter}} = -L^2 \int dt \ e^{-\Phi/4} e^{-3\beta/4} e^{\beta/2} (e^{\Phi} e^\beta V \tau_6 - \tau_2) \sqrt{-\det \tilde{G}} \quad (43)
\]

where \(\tau_p = T_p g_s^{-1}\) is the physical tension for \(p = 2, 6\). Also, \(G_{\alpha\gamma}\) is the metric induced from the four–dimensional metric in equation (36). Substituting the rest of the perturbed solutions of equations (36) into the action (43), we find

\[
S_{\text{matter}}^{\text{approx}} = -L^2 \int dt \ Z_2^{-1} (V \tilde{Z}_2 \tilde{Z}_6^{-1} \tau_6 - \tau_2) \left( 1 - Z_2^{1/2} Z_6^{1/2} \tilde{N} \cdot \tilde{u} - \frac{1}{2} Z_2 \tilde{Z}_6 \tilde{u}^2 \right) \quad (44)
\]

The BPS bounds give \(\tau_6 = q_6 \mu_6 g_s^{-1}\), and \(\tau_2 = q_2 \mu_2 g_s^{-1}\), where \(q_6\) is the D6–brane charge and \(q_2\) is the D2–brane charge and \(\mu_2 = (2\pi)^{-2} \alpha'^{-3/2}\) and \(\mu_6 = (2\pi)^{-6} \alpha'^{-7/2}\). In terms of the current density \(\tilde{Q}(\tilde{x})\) we have

\[
q_6 = -q_2 = \int d^3\tilde{x} \ \tilde{Q}(\tilde{x}) \quad (45)
\]

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So we may write a four dimensional action:

\[ S_{\text{matter}}^{\text{approx}} = -L^2 \int d^4x \frac{\tilde{Q}}{g_s} (VZ_6^{-1}\mu_6 - Z_2^{-1}\mu_2) \left( 1 - Z_2^{1/2}Z_6^{1/2}\tilde{N} \cdot \tilde{u} - \frac{1}{2}Z_2Z_6\tilde{u}^2 \right), \quad (46) \]

The source terms in the action for \( C^{(3)} \) and \( C^{(7)} \) are given by the integrated pulls–back:

\[ S_{\text{current}}^{(3)} = \frac{\mu_2q_2}{g_s} \int C^{(3)}, \quad \text{and} \quad S_{\text{current}}^{(7)} = \frac{\mu_6q_6}{g_s} \int C^{(7)}. \quad (47) \]

Compactifying the actions in \( (47) \) on \( T^2 \) and on \( T^2 \times K3 \), respectively, and substituting in the perturbed solution \( (36) \) gives

\[
S_{\text{current}}^{\text{approx}} = L^2 \int dt \left( (Z_2^{-1} - 1) + \tilde{A} \cdot \tilde{u} \right) \frac{q_2\mu_2}{g_s} + L^2 \int dt \left( (Z_6^{-1} - 1) + \tilde{A} \cdot \tilde{u} \right) V\frac{g_6\mu_6}{g_s}
\]

\[
= L^2 \int d^4x \left( (Z_2^{-1} - 1) + \tilde{A} \cdot \tilde{u} \right) \frac{\tilde{Q}\mu_2}{g_s} + L^2 \int d^4x \left( (Z_6^{-1} - 1) + \tilde{A} \cdot \tilde{u} \right) \tilde{Q}V\frac{\mu_6}{g_s}. \quad (48)
\]

Altogether we have

\[ S^{\text{approx}} = S_{\text{IIA}}^{\text{approx}} + S_{\text{matter}}^{\text{approx}} + S_{\text{current}}^{\text{approx}}. \quad (49) \]

Substituting equations \( (40) \), \( (46) \) and \( (48) \) into \( (49) \) we get the expression

\[
S^{\text{approx}} = k \int d^4x \left\{ -\frac{1}{2} \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{-1/2}Z_6^{1/2}\tilde{N})^2}{Z_2Z_6^{-1}} - \frac{1}{2} \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{1/2}Z_6^{-1/2}\tilde{N})^2}{Z_2Z_6^{-1}} \right. \\
+ \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{-1/2}Z_6^{1/2}\tilde{N}) \cdot (\tilde{\nabla} \times (Z_2^{1/2}Z_6^{1/2}\tilde{N}))}{Z_2} \\
+ \frac{\tilde{\nabla} \times (\tilde{A} + Z_2^{1/2}Z_6^{-1/2}\tilde{N}) \cdot (\tilde{\nabla} \times (Z_2^{1/2}Z_6^{1/2}\tilde{N}))}{Z_2} - \frac{1}{2} \frac{\tilde{\nabla} \times (Z_2^{1/2}Z_6^{1/2}\tilde{N})^2}{Z_2Z_6} \\
- \dot{Z}_2 \dot{Z}_6 + \left\{ \tilde{Q} \left( \frac{1}{g_s} \frac{\mu_6}{V} + \frac{1}{g_s} \frac{\mu_2}{\tilde{u}} \right) + \frac{1}{2} \tilde{Q} \left( V\frac{\mu_6Z_2}{g_s} - \frac{\mu_2Z_6}{g_s} \right) \right\} \frac{L^2}{k} \\
- \left( \tilde{A} + Z_2^{-1/2}Z_6^{1/2}\tilde{N} \right) \cdot \left( \tilde{\nabla}(\dot{Z}_2) + \tilde{Q}\frac{L^2\mu_2}{g_sk} \tilde{u} \right) \\
- \left( \tilde{A} + Z_2^{1/2}Z_6^{-1/2}\tilde{N} \right) \cdot \left( \tilde{\nabla}(\dot{Z}_6) - \tilde{Q}\frac{L^2V\mu_6}{g_sk} \tilde{u} \right) \right\}. \quad (50)
\]
5.5 Perturbation Equations of Motion

Since we have calculated $S^{\text{approx}}$ up to $O(u^2)$, we can derive equations of motion from it which are correct to $O(u)$. The equations of motion for $\tilde{A}$ and $\tilde{A}$ are

$$-\vec{\nabla} \times \left( \frac{\vec{\nabla} \times (\tilde{A} + Z_2^{-1/2} Z_6^{1/2} \tilde{N})}{Z_2^{-1} Z_6} \right) + \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_6} \right) - \vec{\nabla} \dot{Z}_2 - \frac{\tilde{Q} \mu_2}{g_s k} L^2 \ddot{u} = 0 , (51)$$

$$-\vec{\nabla} \times \left( \frac{\vec{\nabla} \times (\tilde{A} + Z_2^{1/2} Z_6^{-1/2} \tilde{N})}{Z_2 Z_6^{-1}} \right) + \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_2} \right) - \vec{\nabla} \dot{Z}_6 - \frac{\tilde{Q} V \mu_6}{g_s k} L^2 \ddot{u} = 0 , (52)$$

and the equation of motion for $\tilde{N}$ is

$$-Z_2^{-1/2} Z_6^{1/2} \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (\tilde{A} + Z_2^{-1/2} Z_6^{1/2} \tilde{N})}{Z_2^{-1} Z_6} \right) - Z_2^{1/2} Z_6^{-1/2} \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (\tilde{A} + Z_2^{1/2} Z_6^{-1/2} \tilde{N})}{Z_2 Z_6^{-1}} \right)$$

$$+ Z_2^{1/2} Z_6^{1/2} \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_6} \right) + Z_2^{1/2} Z_6^{-1/2} \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (\tilde{A} + Z_2^{1/2} Z_6^{-1/2} \tilde{N})}{Z_2} \right)$$

$$+ Z_2^{1/2} Z_6^{-1/2} \vec{\nabla} \times \left( \frac{\vec{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_2 Z_6^{-1}} \right) \right) - Z_2^{-1/2} Z_6^{1/2} \left( \vec{\nabla} \dot{Z}_2 + \frac{\tilde{Q} \mu_2}{g_s k} L^2 \ddot{u} \right)$$

$$- Z_2^{1/2} Z_6^{-1/2} \left( \vec{\nabla} \dot{Z}_6 - \frac{\tilde{Q} V \mu_6}{g_s k} L^2 \ddot{u} \right) = 0 . (53)$$

If we define

$$\vec{K} = c \vec{\nabla}^{-2} (\tilde{Q} \ddot{u}) , (54)$$

for some constant $c$, then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{K}) = -\frac{c V}{(2\pi)^{3} \alpha^{5/2} g_s} \vec{\nabla} \dot{Z}_2 - c \tilde{Q} \ddot{u} , (55)$$

where we have used the expression for $Z_2$ in equation (53) and also charge conservation in the form

$$\partial_0 (\tilde{Q}) + \vec{\nabla} \cdot (\tilde{Q} \ddot{u}) = 0 . (56)$$

Comparing (53) to the equation of motion (51), we find that with $c = \mu_2 L^2 / g_s k$ we get:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{K}) = -\vec{\nabla} \dot{Z}_2 - \frac{\tilde{Q} \mu_2}{g_s k} \ddot{u} L^2 . (57)$$
Similarly we can define
\[ \tilde{K} = -c \nabla^{-2}(\tilde{Q} \tilde{u}) \],
with \( c = V \mu_0 L^2 / g_k \), giving
\[ \nabla \times (\nabla \times \tilde{K}) = -\nabla \tilde{Z}_6 + \frac{V \tilde{Q} \mu_0}{g_k} \tilde{u} L^2 \].

Taking linear combinations of the equations of motion (51), (52) and (53), we get
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2^{-1} Z_6} \right) - \frac{\tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_6} = \tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K}) \],
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2 Z_6^{-1}} \right) - \frac{\tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_2} = \tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K}) \],
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_6} \right) + \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2} = \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \]
\[ = 0 \].

5.6 The Effective Action

We can integrate the equations of motion (60) - (62) to get
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2^{-1} Z_6} \right) - \frac{\tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_6} = \tilde{\nabla} \times \tilde{K} + \tilde{\nabla} \alpha \],
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2 Z_6^{-1}} \right) - \frac{\tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N})}{Z_2} = \tilde{\nabla} \times \tilde{K} + \tilde{\nabla} \alpha \],
\[ \tilde{\nabla} \times \left( \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_6} \right) + \frac{\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K})}{Z_2} = \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \]
\[ = \tilde{\nabla} \nu \],
where \( \nu, \alpha \) and \( \tilde{\alpha} \) are functions of integration. Taking divergences of the equations (63), we can show that it is consistent to set \( \alpha = \tilde{\alpha} = \nu = 0 \). (See e.g. ref. [26] for a discussion.) Then linear combinations of equations (63) give:
\[ \tilde{\nabla} \times (\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K}) \right) = -\tilde{\nabla} \times \tilde{K} \]
\[ \tilde{\nabla} \times (\tilde{\nabla} \times (\tilde{\nabla} \times \tilde{K}) \right) = -\tilde{\nabla} \times \tilde{K} \]
\[ \tilde{\nabla} \times (Z_2^{1/2} Z_6^{1/2} \tilde{N}) \]
Substituting equations (64) into the action (50) and integrating by parts gives:

$$S_{\text{approx}} = \int d^4x \left\{ k \left\{ -\dot{Z}_2\dot{Z}_6 + \frac{1}{2} \left( \vec{\nabla} \times (\vec{\nabla} + Z_2^{-1/2} Z_6^{1/2} \vec{N}) \right) \cdot (\vec{\nabla} \times \vec{K}) + \frac{1}{2} \left( \vec{\nabla} \times (\vec{\nabla} + Z_2^{1/2} Z_6^{-1/2} \vec{N}) \right) \cdot (\vec{\nabla} \times \vec{K}) \right\} + L^2 \left\{ -\frac{\vec{Q}}{g_s} (\mu_6 V - \mu_2) + \frac{1}{2} \frac{\vec{Q}}{g_s} (\mu_6 V Z_2 - \mu_2 Z_6) u^2 \right\} \right\}. \quad (65)$$

We now take the point–like limit, in which the charge density is written as in equation (35). Then $Z_2$ and $Z_6$ are given by the equations (33). Also the equations (54) and (58) for $K$ and $\vec{K}$ have the solutions

$$\vec{K} = -\frac{1}{4\pi} \frac{\mu_2 L^2}{g_s k} \sum_a \frac{Q_a}{r_a} \vec{u}_a, \quad \text{and} \quad \vec{\nabla} \times \vec{K} = -\frac{1}{4\pi} \frac{V \mu_6 L^2}{g_s k} \sum_a \frac{Q_a}{r_a} \vec{u}_a, \quad (66)$$

where $\vec{r}_a = \vec{x} - \vec{x}_a$.

Then the first term in the action (65) becomes:

$$Z_2 \dot{Z}_6 = \sum_{a,b} \frac{(r_2)_a (r_6)_b}{r_3^3 r_3^3} \left\{ (\vec{r}_a \cdot \vec{u}_a)(\vec{r}_b \cdot \vec{u}_b) \right\}, \quad (67)$$

and the second term:

$$(\vec{\nabla} \times \vec{K}) \cdot (\vec{\nabla} \times \vec{K}) = -\frac{1}{(4\pi)^2} \frac{V \mu_2 \mu_6 L^4}{g_s^2 k^2} \sum_{a,b} \frac{Q_a Q_b}{r_3^3 r_3^3} \left\{ (\vec{r}_a \cdot \vec{r}_b)(\vec{u}_a \cdot \vec{u}_b) - (\vec{r}_a \cdot \vec{u}_a)(\vec{r}_b \cdot \vec{u}_a) \right\}. \quad (68)$$

Consider the fifth term in the action (65). Writing the delta function in $\vec{Q}$ of equation (83) as

$$\delta^{(3)}(\vec{x} - \vec{x}_a) = \frac{1}{4\pi} \vec{\nabla}^2 \left( \frac{1}{r_a} \right), \quad (69)$$

then integrating by parts, we find

$$\int d^3x \frac{L^2 \vec{Q} u^2}{2g_s} (V \mu_6 Z_2 - \mu_2 Z_6) = \sum_a \frac{L^2 Q_a u_a^2}{2g_s} (V \mu_6 - \mu_2) - \frac{1}{4\pi} \sum_{a,b} \int d^3x \frac{L^2 u_a^2}{2\alpha'(2\pi)^2} \frac{Q_a Q_b}{r_3^3 r_3^3} (\vec{r}_a \cdot \vec{r}_b). \quad (70)$$

Substituting equations (67), (68) and (70) into the action (65), and rearranging, and defining:

$$S_{\text{eff}} = \int L_{\text{eff}} dt, \quad (71)$$

we find that (using equation (39) for $k$):

$$L_{\text{eff}} = -\frac{L^2}{g_s} (\mu_6 V - \mu_2) \sum_a Q_a + \frac{L^2}{g_s} (\mu_6 V - \mu_2) \sum_a \frac{Q_a u_a^2}{2} + \frac{L^2}{4(2\pi)^3 \alpha'} \int d^3x \sum_{a,b} \frac{Q_a Q_b}{r_3^3 r_3^3} \left\{ (\vec{r}_a \times \vec{r}_b) \cdot (\vec{u}_a \times \vec{u}_b) - \frac{1}{2} |\vec{u}_a - \vec{u}_b|^2 (\vec{r}_a \cdot \vec{r}_b) \right\}. \quad (72)$$
5.7 Extracting the Metric

For two wrapped branes, one of charge $Q_2$ and the other of charge $Q_1$, equation (72) reduces to

\[
L_{\text{eff}} = -\frac{L^2}{g_s} (\mu_0 V - \mu_2) (Q_1 + Q_2) + \frac{L^2}{g_s} (\mu_0 V - \mu_2) \left( \frac{Q_1 u_1^2}{2} + \frac{Q_2 u_2^2}{2} \right) + \frac{L^2}{4(2\pi)^3\alpha'} \int d^3x \frac{Q_1 Q_2}{r_1^3 r_2^3} \left\{ (\vec{r}_1 \times \vec{r}_2) \cdot (\vec{u}_1 \times \vec{u}_2) - \frac{1}{2} |\vec{u}_1 - \vec{u}_2|^2 (\vec{r}_1 \cdot \vec{r}_2) \right\}, \tag{73}
\]

where $r_1 = |\vec{x} - \vec{x}_1|$, and $r_2 = |\vec{x} - \vec{x}_2|$.

We can unpack this expression considerably. Consider the integral

\[
I = \int d^3x \frac{(\vec{r}_1 \cdot \vec{r}_2)}{r_1^3 r_2^3}. \tag{74}
\]

We can introduce a Feynman parameter $\omega$ using the formula

\[
\frac{1}{A^\alpha B^\beta} = \int_0^1 d\omega \frac{\omega^{\alpha-1}(1-\omega)^{\beta-1} \Gamma(\alpha+\beta)}{[\omega A + (1-\omega)B]^{\alpha+\beta} \Gamma(\alpha) \Gamma(\beta)}. \tag{75}
\]

Then (74) becomes

\[
I = \int d^3x \int_0^1 d\omega \frac{\omega^{1/2}(1-\omega)^{1/2}(\vec{r}_1 \cdot \vec{r}_2)}{[\omega(x^2 - 2\vec{x} \cdot \vec{x}_1 + x_1^2) + (1-\omega)(x^2 - 2\vec{x}_2 \cdot \vec{x}_2 + x_2^2)]^{3/2}} \frac{\Gamma(3)}{\Gamma(2) \Gamma(2)}. \tag{76}
\]

Completing the square in the denominator in (76), and substituting $\vec{y} = \vec{x}_1 - \omega \vec{x}_1 - (1-\omega)\vec{x}_2$ gives

\[
I = \frac{\Gamma(3)}{\Gamma(2) \Gamma(2)} \int_0^1 d\omega \omega^{1/2}(1-\omega)^{1/2} \int d^3y \frac{y^2 + (2\omega - 1)\vec{y} \cdot (\vec{x}_1 - \vec{x}_2) - \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2}{[y^2 + \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2]^{3/2}}. \tag{77}
\]

Now

\[
\int d^3y \frac{(2\omega - 1)\vec{y} \cdot (\vec{x}_1 - \vec{x}_2)}{[y^2 + \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2]^{3/2}} = 0, \tag{78}
\]

since the integrand is the sum of odd functions of the $y_i$. Therefore we can write

\[
I = \frac{\Gamma(3)}{\Gamma(2) \Gamma(2)} \int_0^1 d\omega \omega^{1/2}(1-\omega)^{1/2} \int d\Omega_2 dy \frac{y^2(a^2)}{(y^2 + a^2)^3}, \tag{79}
\]

where $a^2 = \omega(1-\omega)(\vec{x}_1 - \vec{x}_2)^2 > 0$. We can do the $y$ integral using contour integration to get

\[
I = \frac{\Gamma(3)}{\Gamma(2) \Gamma(2)} \int_0^1 d\omega \int d\Omega_2 \frac{\pi}{8|\vec{x}_1 - \vec{x}_2|} = \frac{4\pi}{|\vec{x}_1 - \vec{x}_2|}. \tag{80}
\]

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Using Feynman parameters again we can show that
\[
\int d^3 x \frac{(\vec{r}_1 \times \vec{r}_2) \cdot (\vec{u}_1 \times \vec{u}_2)}{r_1^3 r_2^3} = 0.
\] (81)

Substituting (80) and (81) into (73), and setting \( Q_1 = Q_2 = 1 \), we find that we can write the metric as:
\[
L_{\text{eff}} = \left( C - \frac{D}{|\vec{x}_1 - \vec{x}_2|} \right) (\vec{u}_1^2 + \vec{u}_2^2) + \frac{2D}{|\vec{x}_1 - \vec{x}_2|} \vec{u}_1 \cdot \vec{u}_2,
\] (82)
where
\[
C = \frac{2L^2}{g_s} (\mu_6 V - \mu_2), \quad \text{and} \quad D = \frac{2L^2}{(4\pi)^2 \alpha'}.
\] (83)
The two terms in (82) controlled by the constant \( C \) describe the motion of the centre of mass moduli, for which the metric on moduli space is flat. However, the terms controlled by \( D \) determine the relative motion of the branes. Writing everything in terms of the relative coordinates, \( r = |\vec{x}_1 - \vec{x}_2|, \vec{u} = d\vec{r}/dt \), we can write:
\[
ds^2 = \left\{ 1 - \ell_e \right\} (dr^2 + r^2 d\Omega_2^2), \quad \ell_e = \frac{D}{C} = \frac{\alpha' e^2}{4}, \quad \frac{1}{e^2} = \frac{\alpha'^{1/2}}{g_s} \left( \frac{V}{V^*} - 1 \right),
\] (84)
where we have used standard coordinates \((r, \theta, \phi)\) on \( \mathbb{R}^3 \), and \( d\Omega_2^2 \) is the metric on a round two–sphere, with coordinates \((\theta, \phi)\). Note the satisfying relation between \(1/e^2\) and the enhançon radius for \( N = 1 \), from equation (3).

In fact, the result in equation (82) is the essential content of the computation so far. The full result in equation (72) is really only a sum of two–body interactions. In view of the work of Shiraishi for the “\( a = 1 \)” Einstein–Maxwell–Dilaton system \([29]\), this is to be expected\(^4\). The structure of our computations is in that class. So our result for an arbitrary number of branes can be obtained by summing over the result for the two body case and so we can write the metric on moduli space in the following form:
\[
ds^2 = g_{ab} d\bar{x}_a d\bar{x}_b, \quad \text{where}
\]
\[
g_{ab} = \begin{cases} \ell_e & a \neq b, \\ \frac{1}{|\bar{x}_a - \bar{x}_b|} & a = b, \end{cases}
\]
\[
g_{aa} = 1 - \sum_{b \neq a} \frac{\ell_e}{|\bar{x}_a - \bar{x}_b|}, \quad \text{no sum on } a.
\] (85)

This is not quite the final result. Throughout, we have neglected \( N \) parameters of the solution, one for each of the wrapped D–branes. We should take these into account, since there is non–trivial structure on their moduli space. These parameters are, in the monopole language, the
\[^4\text{For more on the many–body interpretation of the interaction terms for various types of dilaton black holes, see ref. [34].}\]
canonical conjugates to the electric charges that each monopole can be endowed with by a duality transformation, as originally shown by Julia and Zee in ref. [35]. There is also an excellent discussion of these parameters by Zee in ref. [36], and by Gibbons and Manton in ref. [37]. The point is that these parameters appear as natural physical phases of the monopole solution. It is interesting to see how these extra degrees of freedom are included in our computation here.

5.8 The Phases

In the wrapped D6–brane language, there is an elegant description of the extra parameters (or internal phases) corresponding to the electric charges of the monopoles. The key piece of the construction was presented in ref. [10], but here we will need to augment it considerably. To recapitulate, the D6–brane part of the solution contributes a Dirac magnetic monopole source, which we will call $C_D^{(1)}$ which comes from Hodge–dualising $C^{(7)}$ in ten dimensions. In the world–volume action of the brane, $C_D^{(1)}$ couples, through its pull–back, to the world–volume gauge field strength $F_{\alpha\gamma} = t^a F_{\alpha\gamma}^a$ as follows [13, 14, 15]:

$$-2\pi\alpha'\mu_2 \int \text{Tr} \left[ C_D^{(1) \wedge F} \right].$$

Of course, $F$ is the field strength for a gauge group as large as $U(N)$, but in the present context of separated branes, we are working in the Abelian case where only the maximal $U(1)^N$ subgroup survives. In the non–Abelian case, we would also have to include contributions arising from the self–interactions of the (adjoint) scalar fields corresponding to the positions of the individual wrapped branes, as in ref. [38]. The field $C_D^{(1)}$ itself would be sensitive to this non–Abelian physics, since it depends on the brane positions. In the Abelian limit, however, things are much simpler, at least in the limit of extreme monopole separations i.e. , the classical limit in the dual $U(N)$ gauge theory. There is no interaction between the different $U(1)$s. A natural basis for the generators $t^a$ in the $N\times N$ fundamental representation is:

$$t^a = \text{diag}\{\ldots 0, \ldots, 1, \ldots, 0, \ldots\},$$

where there is a single entry of unity in the $a$th position along the diagonal. The trace in the above equation (86) will give zero for all of the $U(1)$ generators except the diagonal one, and so we end up with a coupling only to the diagonal $\sum_{a=1}^N F_{\alpha\gamma}^a$.

This is not the only modification, however. In the Dirac–Born–Infeld action we treated earlier in section 5.3 we did not include the coupling to the gauge fields $F_{\alpha\gamma}^a$. The modification is
S_{\text{matter}} = - \int d^3 \xi \, e^{-\Phi} (T_6 V e^\beta - T_2) \, \text{Tr} \left[ -\det (G_{\alpha \gamma} + 2\pi \alpha' t^a F_{\alpha \gamma}^a) \right]^{\frac{1}{2}}, \quad (88)

and since we can write

\begin{align*}
-\det (G_{\alpha \gamma} + 2\pi \alpha' t^a F_{\alpha \gamma}^a) &= (-\det G) \left(1 + \frac{1}{2} t^a t^b G^{\alpha \beta} G^{\gamma \delta} F_{\alpha \gamma}^a F_{\delta \beta}^b \right), \\
\end{align*}

(89)

and recalling that \( \text{Tr} \left( t^a t^b \right) = \delta_{ab} \), we see that the trace gives us \( N \) independent contributions from each of the \( N \) gauge fields.

The point is that in this three–dimensional world–volume, the \( a \)th gauge field \( A_{\alpha}^a \), can be dualised to give a scalar, \( s^a \). These \( N \) scalars are the phases of the monopoles in the reduced action. We can write an equivalent action for them by following a slight generalisation of ref. [39], introducing a family of auxiliary fields \( f_\alpha^a \), and writing a slightly different action. The term \( 2\pi \alpha' F_{\alpha \gamma}^a \) in the Dirac–Born–Infeld action is replaced by \( e^{2\Phi} (\tau_6 V e^\beta - \tau_2)^{-2} f_\alpha^a f^\alpha_a \) (no sum on \( a \) here), and the terms \( \sum_a F^a \wedge f^a \) are added to the Lagrangian. Integrating out the \( f^a \) will give us the action we had previously, while integrating out the potential \( A_{\delta}^a \) instead will give us the equations:

\begin{align*}
\epsilon^{\alpha \gamma \delta} \partial_\gamma ( (C_D^{(1)})_\delta - f^a_\delta ) &= 0 , \\
\end{align*}

(90)

where the world–volume index on \( C_D^{(1)} \) arises from the pull–back, \textit{i.e.} absorbing the spacetime index with a factor \( \partial x^\mu / \partial \xi^\delta \), where \( \xi^\delta \) are world–volume coordinates. The solution of the equation (90) defines a family of scalars, \( s^a \):

\begin{align*}
\partial_\alpha s^a &= (C_D^{(1)})_\alpha - f^a_\alpha .
\end{align*}

(91)

We can now eliminate \( F_{\alpha \gamma}^a \) from our action and substitute for the \( f^a_\alpha \) using equation (91), with result:

\begin{align*}
S_{\text{matter}} &= - \int d^3 \xi \, e^{-\Phi} (T_6 V e^\beta - T_2) \times \\
&\quad \text{Tr} \left[ -\det \{ G_{\alpha \gamma} + e^{2\Phi} (\tau_6 V e^\beta - \tau_2)^{-2} t^a (\partial_\alpha s^a - (C_D^{(1)})_\alpha) (\partial_\gamma s^a - (C_D^{(1)})_\gamma) \} \right]^{\frac{1}{2}}.
\end{align*}

(92)

To compute \( C_D^{(1)} \), we must use its definition via ten dimensional Hodge duality:

\begin{align*}
dC_D^{(1)} &= e^{-\frac{1}{2} \Phi} * F^{(8)} = e^{-\frac{1}{2} \Phi} * dC^{(7)},
\end{align*}

(93)

which gives, after some algebra that \( C_D^{(1)} \) is given by:

\begin{align*}
\tilde{\nabla} (Z_6) &= \tilde{\nabla} \times C_D^{(1)} .
\end{align*}

(94)
For example, if we use $Z_6$ as given in equation (33), and we choose to use coordinates $(r, \theta, \phi)$ on the three–dimensions transverse to the branes, then

\[
C_D^{(1)} = r^2 Z_6' \cos \theta \, d\phi = - \sum_a \left\{ \frac{(r_a)_a}{|\vec{r} - \vec{r}_a|^3} r(r^2 - \vec{r} \cdot \vec{r}_a) \right\} \cos \theta \, d\phi ,
\]

which for $\vec{r}_a = 0$, reduces to the familiar charge $N$ monopole potential:

\[
C_D^{(1)} = -r_6 N \cos \theta \, d\phi .
\]

In principle, we must redo the computations we carried out in the previous sections, inserting the perturbed fields into the modified “matter” and “current” actions that we have written above. However, it is important to note that the method will miss some crucial parts of the computation unless we augment the procedure somewhat. The crucial point is that the supergravity techniques that we have used so far are insensitive to the $N$ different $U(1)$s which reside on the D–branes. They are only sensitive to the overall diagonal $U(1)$ which refers to the center of mass of the system.

In particular this means that the system will not be able to generate terms which couple different $U(1)$s together, which is a non–trivial feature of the moduli space, arising from terms such as the off–diagonal $g_{ab}$ derived in the previous section. From the point of view of the world–volume $U(N)$ 2+1 dimensional gauge theory, such coupling corresponds to important one–loop corrections to the classical decoupled result for the Coulomb branch moduli space [40, 32, 33]. As may have been observed in the previous sections, the crucial tool which generates such terms is the $\delta$–function regulator (discussed in section 5.2) which converts the smeared charge/mass distribution to the point–like ones at the end of the computation. In section 5.6 it is this technique that extracts the terms that allow the $a$th centre to interact with the $b$th one. We now need this to be correlated with the individual $U(1)$s we are studying in this section, correctly tying the $a$th $U(1)$ (and hence the phase $s^a$) with the $a$th position $\vec{x}^a$. For the regulator tool to be sensitive to the different $U(1)$s, it needs to be modified to something like:

\[
\tilde{Q} = \sum_a t^a \delta^{(3)}(x - x_a) ,
\]

where it is to be understood this operator is to be used under the gauge trace in the actions written here. The facility of this modification is that on objects which are charged under the diagonal $U(1)$ it will count the total R–R charge in the usual way, since to be diagonally charged is to carry the identity matrix as the generator. Everything from the previous section falls into this category. However, our object is now more refined, since it can also be sensitive to the
individual $U(1)$s. Our supergravity computations now have a chance of uncovering the subtle terms.

Having noted the shortcomings of the direct supergravity computation, we can proceed more rapidly as follows. The brane labeled “$a$” has coordinate $\vec{x}^a$, and we have already computed the metric $g_{ab}$ on these coordinates in the last section. This result clearly tells us what the metric for the kinetic terms of the $U(1)$s is directly, telling us precisely how the $a$th $U(1)$ is coupled.

The coordinates $\vec{x}^a$ are simply the adjoint scalars in the $a$th $U(1)$ gauge supermultiplet. So the output of expanding out the $U(1)^N$ sector to leading order (quadratic in $F_{\alpha\gamma}$) is easy to write:

$$S_{YM} = \int d^3\xi \left(-\frac{1}{4e^2}g_{ab}F^a_{\alpha\gamma}F^{b\alpha\gamma}\right), \quad (98)$$

where $g_{ab}$ is given in equations [33], and we’ve used the standard value of the bare Yang–Mills coupling on the brane, taking into account the wrapping on a K3 of asymptotic volume $V$. The constant part of $g_{ab}$ (for $a=b$), which is $C$ given in equation [33], is the classical result giving the basic coupling of each individual $U(1)$, while the rest of the metric is one–loop, from the point of view of the three–dimensional $U(N)$ gauge theory [40, 32, 33].

There is another set of terms that we can deduce. These terms again follow in principle from the fact that we know that the $U(1)$ on each brane must couple to the pull–back of the dual of the R–R scalar produced by all the other branes. This is again something that the supergravity expressions need some encouragement to produce, since they are naturally adapted to producing only the diagonal $U(1)$ data. With care, however, we can see how it must work. The $a$th brane has a Chern–Simons coupling of the gauge field $\sum_a t^a F^a_{\alpha\gamma}$ to the Dirac monopole R–R field produced by the brane at $\vec{x}^b$. In the supergravity computation, that field descends from $C^{(1)}_D$, whose curl is given by the gradient of $|\vec{x}–\vec{x}_a|^{-1}$, as we have discussed above. In the computation, a regularising $\delta$–function will produce a $|\vec{x}_b–\vec{x}_a|^{-1}$ from this. Morally speaking, the supergravity field should contain the information that there is a generator $t^b$ associated to this term as well.

This survives in the remnant, denoted $\vec{\omega}_{ab}$, of $C^{(1)}_D$ which is the field at $a$ due to $b$. Its curl is, up to a numerical factor, the gradient of $|\vec{x}_b–\vec{x}_a|^{-1}$, and the dependence on the generator $t^b$ picks out the component $F^{b}_{\alpha\gamma}$ on $a$’s world–volume after taking the trace. We must pull the spacetime index of the vector $\vec{\omega}_{ab}$ back to a world–volume index using the coordinate on the $a$th world–volume, and we do this by forming the dot product with $\partial \vec{x}_a/\partial \xi^\gamma$. In summary, the term is simply:

$$S_{FRR} = -\frac{1}{8\pi} \int d^3\xi \varepsilon^{\alpha\gamma\kappa} F^b_{\alpha\gamma} \vec{\omega}_{ab} \cdot \partial_k \vec{x}^a. \quad (99)$$

Finally, in three dimensions we can introduce the analogue of four dimensional $\theta$–angles for each
gauge group, since there is a natural topological invariant measuring the analogue of instanton number (winding at infinity) in the gauge field [41, 36]:

\[ S_{\text{wind}} = \frac{1}{8\pi} \int d^3 \xi \epsilon^{\alpha\gamma\kappa} \partial_\kappa F_{\alpha\gamma s_a}, \] (100)

where the \( s_a \) are \( 2\pi \) periodic, since \( e^{iS_{\text{wind}}} = e^{in^a s_a} \) where \( n^a \) is an integer denoting the amount of “instanton number” in the \( a \)th \( U(1) \). Clearly, if we treat the \( s_a \) as dynamical variables in the full problem \( S_{\text{total}} = S_{\text{YM}} + S_{\text{FRR}} + S_{\text{wind}} \), they are simply Lagrange multipliers which set the \( n^a \) to zero. Alternatively, we can integrate out the \( F_{\alpha\beta} \) to which they couple [42,40]. Their equation of motion is simply:

\[ F_{\alpha\gamma} = -\frac{e^2}{4\pi} (g^{-1})^{ab} \epsilon_{\alpha\gamma}^\kappa (\partial_\kappa s_b + \vec{\omega}_{bc} \cdot \partial_\kappa \vec{x}_c). \] (101)

Substituting this back into the action, we get the simple result:

\[ S_{\text{phases}} = \frac{1}{2} \int d^3 \xi \frac{e^2}{(4\pi)^2} (g^{-1})^{ab} (\partial_\kappa s_a + \vec{\omega}_{ac} \cdot \partial_\kappa \vec{x}_c)(\partial_\kappa s_b + \vec{\omega}_{bc} \cdot \partial_\kappa \vec{x}_c). \] (102)

### 5.9 The Final Metric

So upon reducing the result (102) on \( T^2 \), the \( s^a \) are just functions of \( t \), and including the result (85) from the previous section, we see that our complete metric on the \( 4N \) dimensional moduli space of our \( N \) wrapped D6–branes is:

\[ ds^2 = g_{ab} dx^a \cdot dx^b + (g^{-1})^{ab} (ds_a + \vec{\omega}_{ac} \cdot dx^c)(ds_b + \vec{\omega}_{bc} \cdot dx^c), \] (103)

where \( g_{ab} \) is given in equation (85), and \( \vec{\nabla} \times \vec{\omega}_{ab} = \vec{\nabla} (g_{ab}) \). This is the Gibbons–Manton metric [6] for \( N \) well–separated BPS monopoles, demonstrating that our K3–wrapped D6–branes are indeed behaving as non–coincident BPS monopoles.

### 6 Summary

Our result is satisfying. We have confirmed the expectation that K3 wrapped D6–branes in type IIA string theory behave like BPS monopoles [10], by showing that the metrics on moduli spaces for \( N \) objects exactly match, at least in perturbation theory. We expect that exactly the same arguments that the non–perturbative corrections complete the monopole moduli space into a unique \( 4N \) dimensional hyper–Kähler manifold generalising the Atiyah–Hitchin manifold
will apply to our case as well. This also fits well with the fact that this is isomorphic to the geometry of the Coulomb branch of 2+1 dimensional pure $U(N)$ gauge theory \[40,32,33\]. The presence of this gauge theory is extremely natural in this picture. It is simply the gauge theory on the world–volume of the K3–wrapped D6–branes.

We also notice that we have essentially indirectly computed the same result for $N$ of the objects made by binding together a Kaluza–Klein monopole and an H–monopole (wrapped NS5–brane) for the case of the heterotic string compactified on $T^4$, since our basic action has a simple symmetry which performs the strong/weak coupling duality transformation between the two string theories. These objects are confirmed as BPS monopoles as well \[8, 9\].

As mentioned in the introduction, the kinds of study we have presented here are very interesting and worthwhile to explore in the detail that we have carried out here. D–branes in flat spacetime are already interesting and instructive, having fueled most of the discoveries since (and been used to check the consistency of) the Second Revolution. We know that they get much more structure when put into more interesting situations, such as wrapping, intersecting, coupling to background fields, etc. We have learned much from these new situations already, and further studies will certainly teach us about even more new phenomena which will augment our understanding of the physics of gauge theories, spacetime, geometry, and perhaps even more.

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