A Joint Graph Based Coding Scheme for the Unsourced Random Access Gaussian Channel

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Abstract—This article introduces a novel communication paradigm for the unsourced, uncoordinated Gaussian multiple access problem. The major components of the envisioned framework are as follows. The encoded bits of every message are partitioned into two groups. The first portion is transmitted using a compressive sensing scheme, whereas the second set of bits is conveyed using a multi-user coding scheme. The compressive sensing portion is key in sidestepping some of the challenges posed by the unsourced aspect of the problem. The information afforded by the compressive sensing is employed to create a sparse random multi-access graph conducive to joint decoding. This construction leverages the lessons learned from traditional IDMA into creating low-complexity schemes for the unsourced setting and its inherent randomness. Under joint message-passing decoding, the proposed scheme offers superior performance compared to existing low-complexity alternatives. Findings are supported by numerical simulations.

Index Terms—Communication, unsourced multiple access, joint-Tanner graph, belief propagation, compressive sensing.

I. INTRODUCTION

Recently, there has been a lot of interest in the design of novel access paradigms for uplink data transfers in IoT scenarios [1], [2], [3], [4]. These paradigms envision a network with a very large number of devices, among which only a small subset (whose typical size is on the order of hundreds) are active at any given point in time.

In [1], Polyanskiy poses the unsourced multiple access (unsourced MAC) problem where each active devices wishes to transmit a $B$-bit message to a central base station and the base station is tasked with recovering the collection of $B$-bit messages transmitted by the active users, without regard to the identity of the senders. Therein, key finite block length (FBL) achievable bounds are derived for this setting. Since the publication of [1], there has been substantial interest in designing coding and decoding schemes with low complexity (polynomial in the number of message bits and the number of users) that perform close to the FBL bounds.

In [5], Ordentlich and Polyanskiy report that traditional MAC coding schemes like ALOHA and treating interference as noise (TIN) exhibit performance far away from these FBL bounds. They then introduce the first low-complexity algorithm tailored to the unsourced MAC setting. Subsequently, several practical coding schemes were proposed in [6], [7], [8], [9] for the unsourced and uncoordinated MAC. Other related works such as [10] propose coding schemes for the uncoordinated random access channel which are closely related to the unsourced MAC.

In [6], Vem et al. devise a coding scheme which uses a slotted structure. The information bits are encoded into codewords using a combination of compressed sensing and low density parity check (LDPC) codes and these codewords are repeated across several slots. The decoder uses message passing decoding within each slot and uses successive interference cancellation across slots. More recently, in [7], Amalladinne et al. cast the unsourced MAC as a very large-dimensional compressive sensing problem. They then adopt a divide-and-conquer approach to obtain a pragmatic, low-complexity solution. In [8], Fengler et al. propose using the approximate message passing (AMP) algorithm as the inner decoder in combination with the outer decoder found in [7]. This latter scheme represents the current state-of-the-art in terms of error performance.

A. Motivation and Contributions

In this paper, we propose a novel low-complexity solution that outperforms the state-of-the-art in [8]. Our proposed coding scheme is also a substantial enhancement over the scheme in [6]. We list below key features that distinguish our proposed scheme from prior art (details can be found in Section II).

i. In [6], messages are decoded on a per-slot basis, and copies are then peeled from other slots in the spirit of successive interference cancellation. In contrast, the approach we develop herein avoids the strategy of slotting-and-peeling altogether. A key contribution of this paper is to show that, when carefully designed, a single sparse joint Tanner graph that spans across all transmissions can provide substantial improvement in performance over the schemes in [6], [7], [8].

ii. The scheme in [6] relies on the existence of codes that achieve FBL capacity at the slot level. As the number of active users increases, the scheme in [6] warrants that the slot length decrease. Designing FBL capacity achieving multi-user LDPC codes for such short block lengths becomes very challenging.

iii. Our proposed scheme can be interpreted as a sparse version of interleave-division multiple access (IDMA) [11] adapted to the uncoordinated and unsourced MAC by using an additional compressed sensing part. Unlike traditional IDMA, we carefully control the multi-user interference by keeping the transmissions sparse. Such
sparsity is important in ensuring two key advantages: (a) the computational complexity of optimal soft-input soft-output demodulation is kept low, (b) the message passing decoding can perform efficiently for the large number of users and small message block lengths that are of interest in IoT. We derive the corresponding density evolution equations and optimize protograph based LDPC codes. Indeed, in the results section, we show that the proposed approach significantly outperforms traditional IDMA for large number of users.

Throughout, we employ the following notation. We use \([a : b]\) to denote the set of integers from \(a\) to \(b\), including end points. Vectors are denoted by underlined symbols.

II. System Model

Let \(K_{\text{tot}}\) and \(K_a\) denote the total number of users in the network and the number of active users, respectively. Each user has \(B\) bits of information (or one of \(M := 2^B\) indices) to be encoded and transmitted within a block of \(N_i\) uses of the channel. Let \(W_i \in [0 : M - 1]\) be a random variable that represents the message index of the \(i\)th user and let \(w_i\) be a realization of this random variable. We assume that \(W_i\) is uniformly distributed over the set \([0 : M - 1]\) and the messages are independent from one another.

The observed signal vector at the receiver corresponding to the \(N_i\) channel uses can be written as

\[
y = \sum_{i=1}^{K_{\text{tot}}} s_i \mathbf{z}(w_i) + \mathbf{z},
\]

where \(\mathbf{z}(w_i) \in \mathbb{R}^{N_i}\) is the signal transmitted by the user \(i\), and the additive noise is characterized by \(\mathbf{z} \sim \mathcal{N}(0, \mathbf{I}_{N_i})\). The Boolean indicator \(s_i\) is defined as, \(s_i = 1\) if user \(i\) is active and \(s_i = 0\) otherwise. We impose an average power constraint on the transmitted vectors when taken over all possible message indices, i.e., \(\frac{1}{N_i} \sum_w ||\mathbf{z}(w)||^2 \leq N_i P\). The energy-per-bit of the system is defined as \(\frac{E_b}{N_0} := \frac{1}{2N_i P}\). The receiver produces an estimate \(\hat{\mathbf{w}}(y)\) of the list of messages. As in \(\textbf{1}\), the probability of error is defined by

\[
P_e = \max_{(s_1, \ldots, s_{K_{\text{tot}}}) = K_a} \frac{1}{K_{\text{tot}}} \sum_{i=1}^{K_{\text{tot}}} s_i \Pr (w_i \notin \mathcal{L}(y))
\]

where \(| \cdot |\) denotes the Hamming weight. The objective is to design a low-complexity encoding and decoding scheme with the least possible \(\frac{E_b}{N_0}\) such that \(P_e \leq \varepsilon\), where \(\varepsilon\) is the target error probability.

III. Description of Proposed Scheme

The overall schematic of the proposed scheme is illustrated in Fig.\(\text{1}\) The parameters of encoding process in an unsourced setting are independent of the user identity. So, our description of the encoding process is solely based on the message index and the encoding process is identical at every active user.

A. Encoder

The encoder for the proposed scheme contains two components: a sensing matrix for a \(K_a\)-sparse robust compressed sensing (CS) problem, and a multi-user channel code for the binary-input real-adder multiple-access channel. The \(N_i\) channel uses available for communication are split between these two components: \(N_p\) channel uses for the compressed sensing part (\(p\) denotes preamble) and \(N_c := N_i - N_p\) channel uses for the channel coding part. The \(B\) bits to be transmitted are also split into two groups of \(B_p\) and \(B_c := B - B_p\) bits, respectively (\(B_p \ll B_c\)). For convenience, we define \(M_p := 2^{B_p}\) and \(M_c := 2^{B_c}\). Also, we denote the preamble and channel coding parts of the message index by \(w_p\) and \(w_c\).

For the CS part of the encoding process, we consider a sensing matrix of the form \(\mathbf{A} = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{M_p}] \in \mathbb{R}^{N_i \times M_p}\) normalized to meet the power constraint, i.e. \(||\mathbf{a}_j||_2 \leq \sqrt{N_i P_1}\) \(\forall 1 \leq j \leq M_p\). The active user encodes its preamble message \(w_p\) into the column \(\mathbf{a}_{w_p}\) of \(\mathbf{A}\).

The channel coding part of the message index \(w_c\) is first encoded into an \(N\)-bit codeword \(\mathbf{z}\) of an \((N, B_c)\) LDPC code \(\mathbf{C}_{\text{LDPC}}\) and modulated using binary phase shift keying (BPSK). The active user employs the many-to-one function \(l : \{1 : M_p\} \rightarrow \{1, 2, \ldots, L\}\) to generate an integer \(l(w_p)\) based on \(w_p\) and the LDPC codeword is repeated \(l(w_p)\) times. The vector thus constructed takes the form

\[
\mathbf{v} = \left[\mathbf{v}_l, \mathbf{v}_l, \ldots, \mathbf{v}_l\right].
\]

Vector \(\mathbf{v}'\) is then padded with \(N_c - NI(w_p)\) zeros to generate the \(N_c\) length vector

\[
\mathbf{v}'' = [\mathbf{v}', 0, \ldots, 0]
\]

and normalized to satisfy the power constraint \(||\mathbf{v}''||_2^2 \leq N_i P - N_p P_1\). At this stage, the preamble message \(w_p\) is again used to pick an interleaver \(\pi_{w_p}\) for the zero padded codeword \(\mathbf{v}''\).

Let \(\tilde{\mathbf{w}}_p\) be the codeword corresponding to message index \(w = (w_p, w_c)\). Then, \(\tilde{\mathbf{w}}_p\) is obtained by first permuting the zero-padded codeword \(\mathbf{v}''\) employing the permutation \(\pi_{w_p}\) and then inserting the \(w_p\)th column of the sensing matrix \(\mathbf{A}\) at the beginning of the permuted codeword, i.e.,

\[
\tilde{\mathbf{w}}_p = [\mathbf{a}_{w_p}^T, \pi_{w_p}(\mathbf{v}'')].
\]

The key idea of the proposed construction is that zero-paddling followed by interleaving the codeword \(\mathbf{v}'\) ‘sparsifies’ the transmissions and reduces the interference in each use of the channel significantly, especially when \(N \ll N_c\). Specifically, the average channel as seen by the receiver is (approximately) a \(N_c\) sum of \(N\)-user Gaussian MAC rather than a \(K_a\)-user Gaussian MAC, where \(N_t\) denotes the fraction of users that employ repetition factor \(l\). This results in a superior performance, and it enables us to design a computationally efficient decoding algorithm.
B. Decoder

The overall decoder has two components. The compressed sensing decoder recovers the preamble fragments, and concomitantly acquires the set of interleavers and repetition patterns picked by the active users. A low-complexity message passing decoder subsequently recovers the codewords sent over the $K_a$-user Gaussian multiple access channel.

1) Compressed Sensing Decoder: The first $N_p$ received symbols can be written in vector form as

$$y_p := y[1:N_p] = A\mathbf{b} + \mathbf{z}[1:N_p] \quad (4)$$

where $\mathbf{b} \in \{0,1\}^{M_p}$ is a $K_a$-sparse vector whose support indicates the set of transmitted preamble messages. We first run the non-negative least squares algorithm, which yields estimate $\hat{\mathbf{b}}$ of $\mathbf{b}$. Yet, we emphasize that this does not guarantee an output signal of the required sparsity or with elements strictly from the set $\{0,1\}$ (as we know a priori from the structure of the problem). To address this issue, we sort the candidates and choose the list of the top $K_b$ indices ($K_b \geq K_a$) as the output from the CS decoder.

2) Message Passing for Gaussian MAC: The compressed sensing decoder outputs a set of $K_b$ interleavers which is used as input by the message passing decoder. The channel coding part of the received signal can be expressed as

$$y_c := y[N_p + 1:N_t] = \sum_{k=1}^{K_b} \pi_{w_p}^b(c_k') + \sum_{k=K_b+1}^{K_c} \pi_{w_p}^b(0) + \mathbf{z}[N_p + 1:N_t]$$

Note that the received signal includes contributions from interleavers that were not employed by any of the $K_a$ active users. The $K_b - K_a$ additional interleavers can be viewed as the ones employed by the fictitious users, each transmitting a zero signal.

For ease of exposition, we describe the message passing rules for $K_b = K_a = 2$. It can be generalized to larger values of $K_b$, $K_a$ in a straightforward manner. Given the received signal $y_c$, the joint BP decoder proceeds iteratively passing messages along the edges of a Tanner graph that represents the coding scheme. Such a Tanner graph and the associated messages that are passed during the decoding are shown in Fig.2. The nodes marked $v$, $c$ and + represent variable nodes, check nodes and MAC nodes, respectively. Throughout this section, we use superscript to distinguish between users 1 and 2. The following messages are passed at every iteration along the edges in the Tanner graph.

- $m_{v-c}(e)$: Messages passed from bit node to check node along edge $e$ of user 1.
- $m_{v-c}(e)$: Messages passed from variable node to check node along edge $e$ of user 1.
- $m_{v-c}(e)$: Message passed from variable node of user 1 to MAC node along edge $e$.
- $m_{v-c}(e)$: Message passed from MAC node to variable node of user 1 along edge $e$.

The messages for user 2 are defined similarly. The rules for message passing are somewhat standard.

Given an edge $e$ between a variable node and a check node, let $v_c$ and $c_e$ denote the variable node and check node connected to $e$, respectively. Similarly, given an edge $e$ between a variable node and a MAC node, let $v_c$ and $c_e$ denote the variable node and MAC node connected to $e$, respectively. Let $\mathcal{N}_c(v_c)$ denote the set of edges connected to check node $c_e$. Let $\mathcal{N}_e(v_c)$ denote the set of edges that connect the variable node $v_c$ to check nodes. Let $\mathcal{N}_e(v_c)$ denote the set of edges that connect the variable node $v_c$ to MAC nodes. Let $\mathcal{N}_c(v_c)$ denote the set of edges connected to MAC node $+e$.

![Fig. 2: Tanner graph representation of the coding scheme](image)

At the bit node, we have

$$m_{v-c}(e) = \sum_{f \in \mathcal{N}_c(v_c)} m_{v-c}(f) + \sum_{e_i \in \mathcal{N}_c(v_c)} m_{v-c}(e_i)$$

$$m_{v-c}(e) = \sum_{e_i \in \mathcal{N}_e(v_c)} m_{v-c}(e_i) + \sum_{f \in \mathcal{N}_e(v_c)} m_{v-c}(f)$$

The LDPC check nodes implement

$$m_{v-c}(e) = 2 \tanh^{-1} \left( \prod_{e_i \in \mathcal{N}_c(v_c)} \tanh \left( \frac{m_{v-c}(e_i)}{2} \right) \right)$$.

As discussed earlier, the receiver sees a $\frac{1}{\sqrt{\nu N}} \sum_{l=1}^{\nu N} y_l Nl$-user Gaussian MAC because of the sparse nature of transmission, which enables the receiver to do optimal demodulation at
MAC nodes. The message at the MAC node corresponding to the jth use of the channel is updated by

\[ m^j_{V \to e}(e) = h(m^j_{V \to i}(f), y_i(j)), \]  

(5)

where \( f \in \mathbb{N}(+e) \setminus e \) is the neighboring edge of \( e \) at a MAC node and \( h(\ell, y; P) = \log \frac{1}{\sigma^2} e^{-\frac{(y-\ell)^2}{2\sigma^2}} \). The function \( h(\ell, y; P) \) can be viewed as the log-likelihood of variable \( x_2 \) when \( y = x_1 + x_2 + z \), \( x_1, x_2 \in \{ \pm \sqrt{P} \} \), the log-likelihood ratio of variable \( x_1 \) is known to be \( \ell \), and \( z \sim \mathcal{N}(0, 1) \).

IV. DENSITY EVOLUTION AND CODE CONSTRUCTION

A protograph \( G = (V \cup C, E) \) is a bipartite graph with the bipartition \( V \) and \( C \) called the set of variable or bit and check nodes, respectively. The set \( E \) of undirected edges specifies the connections between variable nodes in \( V \) and check nodes in \( C \). The \( i \)th variable node, check node and edge in the protograph are denoted, respectively, by \( v_i \), \( c_i \) and \( e_i \). An example of a protograph appears in Fig. 3. An LDPC code can be obtained from the protograph by copy-and-permute operation. Since the codes obtained form a multi-edge-type ensemble with \( |E| \) edge types, density evolution proceeds with \( |E| \) types of messages, one for each edge in the protograph [12].

Let \( \nu(x) := \sum_{l=1}^L \nu_l x^l \) denote the repetition degree distribution (d.d.) where \( \nu_l \) represents the fraction of active users who repeat their codewords \( l \) times. This structure induces a degree distribution on the MAC nodes given by \( G(x) := \sum_{i=1}^L G_i x^i \), where \( G_i \) is the fraction of time instants where \( i \) users transmit. When the interleavers in [4] are chosen uniformly at random from the set of all possible interleavers of length \( N_c \), it can be seen that \( G_i = \binom{N_c}{i} q^i (1-q)^{N_c-i} \) with \( q = \frac{\sum_{l=1}^L \nu_l N_l}{N_c} \). In the limit as \( N_c \) grows large, \( G(x) \) converges to \( \sum_{i=0}^\infty e^{-g} \gamma_i g^i + \frac{e^{-g}}{g} \).

The edge perspective MAC node degree distribution, denoted by \( \gamma(x) \), is given by \( G'(x)/G'(1) \). Now, we introduce notations required to describe the density evolution (DE). Without loss of generality, we consider a coded bit whose value is +1. Under the assumption that messages (log-likelihood ratios) along edges are Gaussian with mean \( \sigma^2/2 \) and variance \( \sigma^2 \), the mutual information (MI) between the message along an edge and the codeword bit associated with it is given by \( J(\sigma) \) [13], which is defined below

\[ J(\sigma) = 1 - \int_{-\infty}^{+\infty} \frac{1}{2\pi \sigma^2} e^{-\frac{(y-\sigma^2/2)^2}{2\sigma^2}} \log_2(1 + e^{-y})dy. \]

Note that \( (J^{-1}(I))^2 \) is the variance of the LLRs when the MI between the message and the corresponding variable is \( I \).

Consider the messages passed along the edges during the \( t \)th iteration for a user who repeats its bits \( l \) times. Let \( I_{v \to e}^t(e_i, l) \) represent the MI between the message from variable node to check node along the edge type \( e_i \) and the associated codeword bit. Similarly, define \( I_{e \to v}^t(e_i, l) \) as the MI between the message along the edge type \( e_i \) from check node to variable node and the associated codeword bit. Let \( I_{v \to v}^t(v_i, l) \) denote the MI between the message from variable node \( v_i \) to the MAC node and the codeword bit associated with \( v_i \). Let \( I_{e \to v}^t \) denote the average MI between from MAC node to variable node and the associated codeword bit. Let \( I_{v \to e}^t \) denote the average MI between from variable nodes to MAC nodes and the codeword bits. Finally, let \( I_{v \to v}^{\text{AP}} (v, l) \) denote the mutual information between the posterior log-likelihood-ratio (LLR) evaluated at variable node \( v \) and the associated codeword bit.

Consider a MAC node with two users and BPSK modulation without additive noise. Let \( \sigma^2 \) be the variance of a priori (incoming) LLRs at the MAC node. We assume that a MAC node performs soft interference cancellation and that the remaining interference at the MAC node is Gaussian. Let \( \phi(\sigma) \) denote the minimum mean squared error in the estimate of a variable after soft interference cancellation. Then, \( \phi(\sigma) \) is given by [14]

\[ \phi(\sigma) = 1 - \int_{-\infty}^{+\infty} \frac{e^{-2y}}{\sqrt{2\pi}} \tanh \left( \frac{\sigma^2}{4} - \frac{\sigma y}{2} \right) dy. \]

For a user whose codeword is repeated \( l \) times, we start the density evolution recursion by initializing \( I_{v \to e}^0 \) to zero.

\[ I_{v \to e}^t = \sum_k \gamma_k J \left( \frac{2}{\sqrt{\sigma_n^2 + (\sigma_{I,k}^t)^2}} \right), \]

(6)

where \( (\sigma_{I,k}^t)^2 \) is given by [14],

\[ (\sigma_{I,k}^t)^2 = (k-1) \phi \left( J^{-1}(I_{v \to e}^{t-1}) \right). \]

\[ I_{e \to v}^t(e_i, l) = J \left( \sum_{e \in \mathcal{N}_{(v_i)}(e_i)} [J^{-1}(I_{v \to e}^t(e_i, l)) - J^{-1}(I_{v \to e}^{t-1})]^2 \right). \]

(7)

\[ I_{v \to v}^t(v_i, l) = J \left( (l-1)[J^{-1}(I_{v \to e}^t)]^2 + \sum_{e \in \mathcal{N}_{(v_i)}(e_i)} [J^{-1}(I_{v \to e}^t(e_i, l))]^2 \right). \]

(8)

\[ I_{v \to e}^t(v_i, l) = \frac{1}{|V|} \sum_i I_{v \to e}^t(v_i, l), \]

where \( |V| \) is the number of variable nodes.
where \(|V|\) is the number the number of variable nodes in the protograph.

\[
I^t_{v \rightarrow +} = \sum_{l=1}^{L} \nu_l I^l_{v \rightarrow +}(l).
\]

\[
I_{\text{APP}}(v_i, l) = J \left( \sum_{e \in N_v(v_i)} \left[ J^{-1}(I^t_{e \rightarrow v}(e)) \right]^2 + \left[ J^{-1}(I^t_{v \rightarrow +}(v_i, l)) \right]^2 \right).
\]

Density evolution threshold is defined as the minimum \(E_b/N_0\) for which \(I_{\text{APP}}(v_i, u_t) \rightarrow 1\), as \(t \rightarrow \infty\), for all \(v_i\) and \(l \in \{1, 2, \ldots, L\}\).

We use differential evolution [15] to optimize the protographs and \(v(x)\) by using the density evolution threshold as the cost function. We lift optimized protographs to codes using the progressive edge growth algorithm. Even though DE thresholds are meaningful benchmarks only for asymptotic lengths, nevertheless designing codes based on DE thresholds offers a principled way to optimize the performance of our system. Simulation results will show that this approach is efficient even for short block lengths.

V. NUMERICAL RESULTS

The parameters we select for our numerical study are,

- Number of bits each user intends to transmit \(B = 100\)
- Total number of channel uses \(N_t = 30000\)
- Total number of active users \(K_a \in \{25 : 300\}\)
- Maximum per user error probability \(P_e \leq \varepsilon = 0.05\).

These values are chosen to match the parameters employed in [5] for ease of comparison.

We fix \(B_p = 15\) and \(N_p = 2000\). The sensing matrix for the CS encoder is constructed as follows. We pick \(N_p/2\) rows uniformly at random from the discrete Fourier transform (DFT) matrix of dimension \(M_p\). The real and imaginary parts of each row are then stacked to form a \(N_p \times M_p\) real sensing matrix \(A\); entries are normalized to meet the power constraint. This then yields the parameters for the channel coding part, with \(B_c = 85\) and \(N_c = 28000\).

For a fixed value of \(K_a\), computing the required SNR involves solving the optimization problem

\[
E_b = \min_{P_1, P_2, K_b} \frac{N_p P_1 + N_c P_2}{2B}
\]

such that \(P_r(E|P_1, P_2, K_b) \leq \varepsilon\).

\[
\begin{array}{c|c|c|c}
\text{Number of Users} & 25 - 125 & 150 - 200 & 225 - 300 \\
\hline
\text{Rate} & 0.125 & 0.25 & 0.4 \\
\end{array}
\]

TABLE I: Code rates corresponding to number of active users

The proposed scheme is evaluated as follows. For each \(K_a \in \{25, 50, \ldots, 300\}\), we use the optimization procedure described in Section [V] to optimize the protograph for the LDPC code and the repetition d.d. \(v(x)\). The function \(g(w_p)\) is then chosen to induce this degree distribution. Although, we need to solve the optimization problem in (11) to achieve the optimal SNR, this is computationally complex with the parameters space being huge. Using simulations, we found \(K_b = 110\) to be a suitable choice for \(K_a = 100\), and thus we fix \(K_b = [1.1 K_a]\). With \(K_b\) fixed, we sweep over all possible combinations of \(P_1, P_2\) in a two-dimensional grid of SNR values, with a resolution of 0.5 dB in each dimension, for the compressed sensing and the channel coding components. We emphasize that this only results in an approximate solution to the above optimization problem.

The rate of the protograph LDPC code has a significant effect on the required \(E_b/N_0\) for a fixed value of \(K_a\). For different rates, the minimum \(E_b/N_0\) required to achieve a probability error of 0.05 is plotted in Fig. 4 as a function of number of users. It can be seen that the optimal rate changes with the number of active users \(K_a\). For example, for \(P_e = 0.05\) and \(K_a = 125\), an \(E_b/N_0\) of 2.47 dB is required if a LDPC rate-0.125 code and \(v(x) = x^2\) is used, whereas an \(E_b/N_0\) of 3.24 dB is required if a rate-0.4 LDPC code with the same repetition pattern is employed. For a fixed number of users, we choose the rate through simulations to minimize the \(E_b/N_0\) required to achieve a target probability of error.

In Fig. 5, performance of the scheme developed herein is compared to the existing schemes. Rates of LDPC codes used for each value of \(K_a\) are given in Table III. In the simulation of proposed scheme, repetition pattern \(x^2\) is used for all values of \(K_a\). The obtained simulation results show that the proposed scheme performs better than existing alternatives. For example, at \(K_a = 175\), the proposed scheme outperforms the scheme in [8] by 1.5 dB. Performance can be further improved by using irregular repetition patterns across users. In Fig. 5, the red circles indicate the \(E_b/N_0\) required when optimized repetition d.d. given in Table III is used. A small improvement of about 0.2 dB results from using irregular repetition d.d.

We now present a comparison with conventional IDMA. Prior work has shown that IDMA is very effective when the number of users are small (less than 25-30) and block lengths (user bits) are relative large [11] and [16]. Designing very low rate iteratively-decodable multi-user codes (rate-1/300) with short block lengths and for a large number of users is a significant challenge that renders conventional IDMA inefficient for the block lengths and number of users considered in this paper. It is known that for the single user channel, generalized LDPC codes with Hadamard codes as check nodes (GHLDPC codes) can provide close-to-capacity performance at very low rates [17]. Motivated by this result, we tried to design rate-1/300 GHLDPC protograph codes for a multi-user channel using differential evolution; however, the optimization procedure did not iterate beyond initial population and the density evolution thresholds were poor. A better rate-1/300 code for IDMA was obtained by repeating each coded bit of a rate-1/4 LDPC code 75 times. The minimum \(E_b/N_0\) required to achieve a probability error of 0.05 for this code is plotted in Fig. 6. It can be seen that there is a significant gap between FBL and the performance of conventional IDMA and it can
It is an interesting open problem to determine if there are other codes of rate-1/300 codes that could work well with conventional IDMA and without sparse repetition even for a large number of users. Our proposed scheme circumvents this code design bottleneck by sparsifying the transmissions and controlling the interference and it provides significant performance improvement at low complexity for a large number of users. It can be seen that conventional IDMA scales very poorly with the number of users. Our proposed scheme circumvents this code design bottleneck by sparsifying the transmissions and controlling the interference and it provides significant performance improvement at low complexity for a large number of users. It is an interesting open problem to determine if there are other codes of rate-1/300 codes that could work well with conventional IDMA and without sparse repetition even for a large number of users; but it is left for future work.

| Number of Users | Repetition pattern $\nu(x)$ | Rate |
|-----------------|-----------------------------|------|
| 225             | $0.12x + 0.88x^2$          | 0.4  |
| 275             | $0.18x + 0.82x^2$          | 0.4  |
| 300             | $0.18x + 0.82x^2$          | 0.4  |

**TABLE II: Repetition patterns and code rates corresponding to number of active users**

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