Orbital angular momentum is dependent on the polarization

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Abstract

After presenting a new approach to separate the total angular momentum of an electromagnetic beam into the spin and orbital parts, which manifests that the spin angular momentum originates from that part of the linear momentum density the total amount of which is equal to zero, I show that the orbital angular momentum as well as the spin angular momentum of a non-paraxial monochromatic beam is dependent on the polarization ellipticity $\sigma$. The $\sigma$-dependent term of the orbital angular momentum is mediated by the recently advanced symmetry axis $I$; and the transverse component of the orbital angular momentum is consistent with the transverse displacement of the beam’s barycenter from the plane formed by $I$ and the propagation axis. For a beam of angular-spectrum scalar amplitude $f(k_{\rho}, \varphi) = f_0(k_{\rho}) \exp(il\varphi)$, the total angular momentum per unit energy in the propagation direction is simply $\frac{l + \sigma}{\omega}$ for $\Theta = \frac{\pi}{2}$ and $\frac{l}{\omega}$ for $\Theta = 0$, where $l$ is an integer, $\omega$ is the angular frequency, and $\Theta$ is the angle between $I$ and the propagation axis.

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I. INTRODUCTION

It is well established that electromagnetic beams can carry angular momenta. It was shown [1, 2] that the total angular momentum (TAM) could be separated into the spin and orbital parts. The spin angular momentum (SAM) [3] and the orbital angular momentum (OAM) [4, 5] have been experimentally measured. The transfer of SAM [6, 7, 8] and OAM [9, 10] to the matter has been observed. The distinction between SAM and OAM in the interaction with particles has also been demonstrated [11, 12]. The classical mechanics aspect of the angular momentum opens the door to the optical micromachines [8, 10, 13]. The quantum mechanics aspect opens the door to the quantum information [14, 15].

It is commonly believed on the basis of the knowledge of paraxial Laguerre-Gaussian beams [1] that the SAM and the OAM of free beams are independent of each other and are carried [2, 5, 16], respectively, by the polarization and the helical phase factor $\exp(il\phi)$, where $l$ is an integer. But a recent experiment [17] converted partly the SAM into the OAM by a high numerical aperture. This fact might signify some connection between the SAM and the OAM [18]. The purpose of this Letter is to investigate this connection for non-paraxial vector beams. The non-paraxial beam that I examine is characterized by the newly advanced symmetry axis $I$ [19] that makes an angle $\Theta$ with the propagation axis. The case of $\Theta = \frac{\pi}{2}$ [20, 21] represents the uniformly polarized beam (including the Laguerre-Gaussian beam) in the paraxial approximation. The case of $\Theta = 0$ [22] represents the cylindrical vector beam [23, 24]. And the beam of $\Theta$ that is neither equal to $\frac{\pi}{2}$ nor equal to 0 was transformed [25, 26] from a beam of $\Theta = \frac{\pi}{2}$ by the transmission at an interface between two different dielectric media. It will be shown that both the SAM and the OAM are dependent on the polarization. The polarization-dependent term of the OAM is mediated by $I$. It will also be shown that the transverse component of the OAM is consistent with the transverse displacement [19, 25, 26] of the beam’s barycenter away from the plane formed by $I$ and the propagation axis. To this end, I will first show in a new attempt to separate the TAM that the SAM originates from that part of the linear momentum density the total amount of which is equal to zero. It is this property that makes the SAM independent of the choice of the origin.
II. SEPARATION OF THE TAM

Consider an arbitrary electromagnetic beam in free space. Its electric vector in the position space can be written as the following integral over the plane wave,

\[ \mathcal{E}(x, t) = \frac{1}{2} \left\{ \frac{1}{(2\pi)^{3/2}} \int E(k) \exp[i(k \cdot x - \omega t)] d^3k + c.c. \right\}, \]

where

\[ k = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \]

is the wave vector each element of which is real, \( \varepsilon_0 \mu_0 \omega^2 = k^2 \), and \( E(k) \) is the electric vector in the wave-vector space. Obviously, integral expression (1) leads to the following transformation,

\[ \omega(-k) = -\omega(k), \quad E(-k) = E^*(k), \]

where the superscript * denotes the complex conjugate. The magnetic vector of the beam is derived from Eq. (1) and the Maxwell equation to be

\[ \mathcal{H}(x, t) = \frac{1}{2} \left\{ \frac{1}{(2\pi)^{3/2}} \int \frac{k \times E}{\omega} \exp[i(k \cdot x - \omega t)] d^3k + c.c. \right\}. \]

According to Eqs. (1) and (3), the linear momentum density \( p = \varepsilon_0 \mu_0 \mathcal{E} \times \mathcal{H} \) is separated into two parts, \( p = p_1 + p_2 \), where

\[ p_1 = \frac{\varepsilon_0}{4(2\pi)^3} \int \frac{E' \cdot E}{\omega} k e^{i(k' + k) \cdot x} e^{-i(\omega' + \omega)t} d^3k' d^3k + \text{c.c.}, \]

\[ p_2 = -\frac{\varepsilon_0}{4(2\pi)^3} \int \frac{E' \cdot k}{\omega} e^{i(k' + k) \cdot x} e^{-i(\omega' + \omega)t} d^3k' d^3k + \text{c.c.}, \]

\[ E = E(k), \quad E' = E(k'), \quad \omega = \omega(k), \quad \text{and} \quad \omega' = \omega(k'). \]

Correspondingly, the TAM \( J = \int x \times p d^3x \) is also separated into two parts, \( J = L + S \), where

\[ L = \int x \times p_1 d^3x = \int \frac{\varepsilon_0}{i\omega} E^1(k \times \nabla_k) E d^3k, \]

\[ S = \int x \times p_2 d^3x = \int \frac{\varepsilon_0}{i\omega} E^* \times E d^3k, \]
∇_k = e_x \frac{∂}{∂k_x} + e_y \frac{∂}{∂k_y} + e_z \frac{∂}{∂k_z}, \text{ and the superscript } \dagger \text{ stands for the conjugate transpose. The details to derive the above equations are given in Appendix. Eq. (7) shows that } S \text{ arises from that part of the linear momentum density the total amount of which is equal to zero, }
\int p_z d^3x = 0. \tag{8}

Consequently, the spatial translation } x \to x - x_0 \text{ does not change } S. \text{ It is thus tempting to regard } S \text{ and } L \text{ as the SAM and the OAM, respectively. Property } (8) \text{ also tells us that there is no paradox } [28, 29] \text{ about the SAM of the plane wave.}

III. ANGULAR MOMENTUM OF MONOCHROMATIC BEAMS

The electric vector of a monochromatic wave propagating in positive } z \text{ direction can be expressed in terms of the angular spectrum as }
\mathcal{E}(x, t) = \frac{1}{2} \left\{ \frac{1}{2\pi} \int \mathbb{E}(k_x, k_y) \exp[i(k \cdot x - \omega t)] dk_x dk_y + c.c. \right\}, \tag{9}

where } k_z^2 = k^2 - k_x^2 - k_y^2, \text{ and the integration limit } k_x^2 + k_y^2 \leq k^2 \text{ is omitted for brevity. Similarly, one has the following transformation from the integral expression (9),}
\begin{align*}
  k_z(-k_x, -k_y) &= -k_z(k_x, k_y), \\
  \omega(-k_x, -k_y) &= -\omega(k_x, k_y), \\
  \mathbb{E}(-k_x, -k_y) &= \mathbb{E}^*(k_x, k_y),
\end{align*} \tag{10}

which is consistent with transformation (2). In much the same way as before, one finds that the linear density of the TAM, the integral of the angular momentum density over the cross section } z = \text{constant}, \text{ is independent of the } z \text{ coordinate and is separated into the orbital and spin parts in a natural way, } J = \int x \times pdxdy = L + S, \text{ where}
\begin{align*}
  L &= \frac{\varepsilon_0}{i\omega} \int \left[ -e_x k_z \mathbb{E}^* \cdot \frac{∂\mathbb{E}}{∂k_y} + e_y k_z \mathbb{E}^* \cdot \frac{∂\mathbb{E}}{∂k_x} + e_z \left( k_x \mathbb{E}^* \cdot \frac{∂\mathbb{E}}{∂k_y} - k_y \mathbb{E}^* \cdot \frac{∂\mathbb{E}}{∂k_x} \right) \right] dk_x dk_y, \\
  S &= \frac{\varepsilon_0}{i\omega} \int \mathbb{E}^* \times E dk_x dk_y. \tag{11}
\end{align*}

It is easy to show that the above equations can be obtained directly from Eqs. (6) and (7) by noticing } \frac{∂\mathbb{E}}{∂k_z} = 0 \text{ and replacing the triple integral with the double integral.}
IV. PROPERTIES OF THE ANGULAR MOMENTUM OF NON-PARAXIAL BEAMS

A representation theory for non-paraxial vector beams was recently advanced in Ref. [19]. The electric vector in the position space is given by Eq. (9). The electric vector $E$ of the angular spectrum is factorized into three factors,

$$E = m\tilde{\alpha}f,$$

where $m$ is the $3 \times 2$ mapping matrix specified by a newly found symmetry axis $I$,

$$\tilde{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

is the Jones vector satisfying the normalization condition $\tilde{\alpha}^\dagger \tilde{\alpha} = 1$ and describing the polarization state of the angular spectrum through the polarization ellipticity,

$$\sigma = \tilde{\alpha}^\dagger \tilde{\sigma} \tilde{\alpha} = -i(\alpha_1^* \alpha_2 - \alpha_2^* \alpha_1),$$

$\tilde{\sigma} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the Pauli matrix, and $f$ is the scalar amplitude of the angular spectrum. Letting $I$ lie in the plane $zox$ and make an angle $\Theta$ with the propagation axis,

$$I = e_z \cos \Theta + e_x \sin \Theta,$$

the mapping matrix takes the following form [19],

$$m = \frac{1}{k|k \times I|} \begin{pmatrix} (k_y^2 + k_z^2) \sin \Theta - k_z k_x \cos \Theta & k k_y \cos \Theta \\ -k_y (k_z \cos \Theta + k_x \sin \Theta) & k (k_z \sin \Theta - k_x \cos \Theta) \\ (k_x^2 + k_y^2) \cos \Theta - k_x k_z \sin \Theta & -k k_y \sin \Theta \end{pmatrix},$$

where $|k \times I| = [(k_z^2 - (k_z \cos \Theta + k_x \sin \Theta)^2)^{1/2}$. The $\tilde{\alpha}$ is assumed to be the same to all the elements of the angular spectrum. In order that the Laguerre-Gaussian beams be included, the scalar amplitude $f$ is chosen to have the following form,

$$f(k\rho, \varphi) = f_0(k\rho) \exp(\imath l\varphi),$$

in the circular cylindrical system in which $k = k\rho + k_z e_z$, $k\rho = k\rho e_\rho = k_z e_x + k_y e_y$, $k_x = k\rho \cos \varphi$, $k_y = k\rho \sin \varphi$, $k_z = (k^2 - k_\rho^2)^{1/2}$, $e_\rho$ and $e_\varphi$ are, respectively, the unit vectors in the radial and azimuthal directions, where $f_0(k\rho)$ is square integrable.
Let us first look at the property of the SAM component in the propagation direction,

\[ S_z = \varepsilon_0 \frac{i\omega}{\int (E_x^* E_y - E_y^* E_x)k_\rho d\rho d\varphi}. \]

Substituting Eqs. (12)-(15), one has

\[ S_z = 2\pi \varepsilon_0 \frac{\sigma}{\omega} \int_0^k \frac{k_z}{k} |f_0(k_\rho)|^2 k_\rho dk_\rho, \tag{16} \]

which does not depend on the beam parameter \( \Theta \). This expression has the validity that does not rely on the paraxial approximation. In the linear paraxial approximation in which \( f_0(k_\rho) \) is sharply peaked at \( k_\rho = 0 \) and therefore the factor \( k_z \) in the integrand is approximated as \( k_z \approx k \), one arrives at

\[ S_z \approx \frac{\sigma}{\omega} W, \tag{17} \]

which was first observed in Ref. [1], where

\[ W = 2\pi \varepsilon_0 \int_0^k |f_0(k_\rho)|^2 k_\rho dk_\rho \tag{18} \]

is the linear density of the energy, the integral of the energy density over the cross section. Similar considerations lead to

\[ S_x = S_y = 0 \tag{19} \]

for the SAM components in the \( x \) and \( y \) directions.

Then let us investigate the OAM component in the propagation direction,

\[ L_z = \varepsilon_0 \frac{i\omega}{\int \mathbf{E}^\dagger (-i\frac{\partial}{\partial \varphi}) \mathbf{E} k_\rho d\rho d\varphi}. \]

Substituting Eqs. (12)-(15) and performing a lengthy calculation, one finds

\[ L_z = \frac{l}{\omega} W + 2\pi \varepsilon_0 \frac{\sigma}{\omega} \int_0^k \left\{ \frac{1}{2} \left( 1 + \frac{k_z - k \cos \Theta}{|k_z - k \cos \Theta|} - \frac{k_z}{k} \right) |f_0(k_\rho)|^2 k_\rho dk_\rho, \tag{20} \]

where \( |\Theta| \leq \frac{\pi}{2} \) is assumed [19]. This expression also has the validity that does not rely on the paraxial approximation. It shows an apparent dependence on the beam parameter \( \Theta \) as well as the polarization ellipticity \( \sigma \). If \( \Theta = \frac{\pi}{2} \) which describes the uniformly polarized beams (including the Laguerre-Gaussian beams) in the paraxial approximation, one has

\[ L_z = \frac{l}{\omega} W + 2\pi \varepsilon_0 \frac{\sigma}{\omega} \int_0^k \left( 1 - \frac{k_z}{k} \right) |f_0(k_\rho)|^2 k_\rho dk_\rho. \tag{21} \]
In the linear paraxial approximation, it reduces to

\[ L_z \approx \frac{l}{\omega} W. \]  \hspace{1cm} (22)

This relation was also obtained in Ref. [1]. If \( \Theta = 0 \) which describes the cylindrical vector beams [24], Eq. (20) becomes

\[ L_z = \frac{l}{\omega} W - 2\pi \varepsilon_0 \sigma \int_0^k \frac{k_z}{k} |f_0(k_\rho)|^2 k_\rho dk_\rho. \]  \hspace{1cm} (23)

In the linear paraxial approximation, it reduces to

\[ L_z \approx \frac{l - \sigma}{\omega} W, \]

which is \( \sigma \) dependent.

Eqs. (16) and (20) indicate that the TAM component in the propagation direction is

\[ J_z = \frac{l}{\omega} W + 2\pi \varepsilon_0 \sigma \int_0^k \frac{1}{2} \left( 1 + \frac{k_z - k \cos \Theta}{|k_z - k \cos \Theta|} \right) |f_0(k_\rho)|^2 k_\rho dk_\rho. \]  \hspace{1cm} (24)

If \( \Theta = \frac{\pi}{2} \), it reduces to

\[ J_z = \frac{l + \sigma}{\omega} W. \]  \hspace{1cm} (25)

If \( \Theta = 0 \), one simply has

\[ J_z = \frac{l}{\omega} W, \]  \hspace{1cm} (26)

which is \( \sigma \) independent.

With the above conclusions, it is ready to explain the spin-to-orbital angular momentum conversion through focusing a beam of \( \Theta = \frac{\pi}{2} \) [17]. Eq. (25) shows that the TAM per unit energy of this beam in the propagation direction is \( \frac{l + \sigma}{\omega} \). Before focusing, the SAM and OAM per unit energy of the paraxial beam in the propagation direction are approximately \( \frac{\sigma}{\omega} \) and \( \frac{l}{\omega} \), respectively, as Eqs. (17) and (22) show. After focusing, the SAM per unit energy of the non-paraxial beam is obtained from Eq. (16) to be

\[ \frac{\sigma \int_0^k (k_z/k)|f_0(k_\rho)|^2 k_\rho dk_\rho}{\omega \int_0^k |f_0(k_\rho)|^2 k_\rho dk_\rho}, \]

indicating that only a fraction of the incident SAM remains in the focused beam, where \( f_0(k_\rho) \) of the focused beam is not sharply peaked so that the linear paraxial approximation is no longer applicable. Since a lossless focusing system cannot change the TAM, the rest of
the incident SAM is converted into the OAM of the focused beam \[30\]. In fact, the OAM per unit energy of the focused beam is obtained from Eq. \[21\] to be

\[
\frac{l}{\omega} + \frac{\sigma}{\omega} \left[ 1 - \frac{\int_0^k (k_z/k) |f_0(k_\rho)|^2 k_\rho dk_\rho}{\int_0^k |f_0(k_\rho)|^2 k_\rho dk_\rho} \right].
\]

At last, let us discuss the OAM component in the \(x\) direction,

\[
L_x = \frac{\varepsilon_0}{\omega} \int k_z E^\dagger (i \frac{\partial}{\partial k_y}) E k_\rho dk_\rho d\psi.
\] (27)

Substituting Eqs. \(12\)-\(15\) and performing a similar calculation, one gets

\[
L_x = -2\pi \varepsilon_0 \frac{\sigma \cot \Theta}{\omega} \int_0^k \frac{1}{2} \left( 1 + \frac{k_z - k \cos \Theta}{|k_z - k \cos \Theta|} \right) |f_0(k_\rho)|^2 k_\rho dk_\rho,
\] (28)

which is also dependent on \(\Theta\) as well as \(\sigma\). This is surprisingly contrary to the common belief \[17\] that the OAM is carried only by the helical phase factor. If \(\Theta = 0\), one has \(L_x = 0\). On the other hand, if \(|\Theta| \gg \delta\Theta\) with \(\delta\Theta\) being the divergence angle of the beam, one approximately has \[19\]

\[
L_x \approx -\frac{\sigma \cot \Theta}{\omega} W.
\]

As a matter of fact, it can be seen from Eqs. \(27\) and \(18\) that in the linear paraxial approximation, the OAM per unit energy in the \(x\) direction, \(\frac{L_x}{W}\), is simply related to the transverse displacement \(y_b\) \[19\] of the beam’s barycenter from the plane formed by the symmetry axis \(I\) and the propagation axis in the following way,

\[
\frac{L_x}{W} = \frac{k}{\omega} y_b.
\] (29)

A similar calculation shows that the OAM component in the \(y\) direction vanishes,

\[
L_y = 0,
\] (30)

indicating together with Eqs. \(20\) and \(28\) that the OAM is located in the plane formed by the symmetry axis \(I\) and the propagation axis.

V. CONCLUDING REMARKS

In summary, I presented a new approach to the separation of the TAM into the spin and orbital parts and found that the spin originates from such a part of the linear momentum
density the total amount of which vanishes. Contrary to the common belief, I showed that both the OAM and the SAM are dependent on the polarization ellipticity $\sigma$. Their difference in respect of the $\sigma$-dependence lies in the different effects of the symmetry axis $I$. The $\sigma$-dependent term of the OAM is mediated by $I$ as Eqs. (20), (28), and (30) indicate, whereas the SAM is free of the effect of $I$ as Eqs. (16) and (19) indicate. The mediation of the $\sigma$-dependent term of the OAM by $I$ may offer further insights into the nature of the angular momentum of light.

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APPENDIX: DETAILS OF THE DERIVATION OF EQS. (6) AND (7)

Let us first derive Eq. (6). Substituting Eq. (4) into $L = \int \mathbf{x} \times \mathbf{p}_1 d^3x$, one has

$$L = L_1 + L_2 + c.c.,$$  \hspace{1cm} (A.1)

where

$$L_1 = \frac{\varepsilon_0}{4(2\pi)^3} \int d^3k' d^3k \int d^3x \frac{\mathbf{E}' \cdot \mathbf{E}}{\omega} \mathbf{x} \times \mathbf{k} e^{i(k'+\mathbf{k}) \cdot \mathbf{x} - i(\omega' + \omega)t},$$  \hspace{1cm} (A.2)

and

$$L_2 = \frac{\varepsilon_0}{4(2\pi)^3} \int d^3k' d^3k \int d^3x \frac{\mathbf{E}' \cdot \mathbf{E}^*}{\omega} \mathbf{x} \times \mathbf{k} e^{i(k'-\mathbf{k}) \cdot \mathbf{x} - i(\omega' - \omega)t}.$$  \hspace{1cm} (A.3)
Upon integrating Eq. (A.2) over the configuration space and noticing the following properties of Dirac’s δ function and its first-order derivative,

\[ \delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega t)d\omega, \quad -i\delta'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \omega \exp(i\omega t)d\omega, \quad (A.4) \]

one obtains

\[ L_1 = \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}' \cdot \mathbf{E}}{\omega} e^{-i(\omega' + \omega)t} \delta'(k_x' + k_y)\delta(k_y' + k_x)\delta(k_z' + k_z) d^3k d^3k' \right) \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}' \cdot \mathbf{E}}{\omega} e^{-i(\omega' + \omega)t} \delta(k_x' + k_y)\delta(k_y' + k_x)\delta(k_z' + k_z) d^3k d^3k' \right) \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}' \cdot \mathbf{E}}{\omega} e^{-i(\omega' + \omega)t} \delta(k_y' + k_x)\delta(k_y' + k_x)\delta(k_z' + k_z) d^3k d^3k' \right). \]

It is changed by eliminating the δ functions into

\[ L_1 = \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}(k_x', -k_y, -k_z) \cdot \mathbf{E}}{\omega} e^{-i[\omega(k_x', -k_y, -k_z) + \omega]t} \delta'(k_x' + k_y) d^3k d^3k' \right) \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}(-k_x, k_y', -k_z) \cdot \mathbf{E}}{\omega} e^{-i[\omega(k_x', -k_y, -k_z) + \omega]t} \delta'(k_y' + k_x) d^3k d^3k' \right) \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E}(-k_x, -k_y', k_z) \cdot \mathbf{E}}{\omega} e^{-i[\omega(k_x', -k_y, -k_z) + \omega]t} \delta'(k_z' + k_z) d^3k d^3k' \right). \]

Noticing the following property of the derivative of the δ function,

\[ \int_{t_1}^{t_2} f(t)\delta'(t - t_0)dt = -f'(t_0), \quad t_1 < t_0 < t_2, \quad (A.5) \]

and taking transformation (2) into account, the above equation is reduced to

\[ L_1 = \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E} \cdot \frac{\partial \mathbf{E}^*}{\partial k_x}}{\varepsilon_0\mu_0\omega} + i \frac{k_x t}{\varepsilon_0\mu_0\omega} \mathbf{E}^* \cdot \mathbf{E} \right) d^3k \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E} \cdot \frac{\partial \mathbf{E}^*}{\partial k_y}}{\varepsilon_0\mu_0\omega} + i \frac{k_y t}{\varepsilon_0\mu_0\omega} \mathbf{E}^* \cdot \mathbf{E} \right) d^3k \]

\[ + \frac{\varepsilon_0}{4i} \int \left( \frac{\mathbf{E} \cdot \frac{\partial \mathbf{E}^*}{\partial k_z}}{\varepsilon_0\mu_0\omega} + i \frac{k_z t}{\varepsilon_0\mu_0\omega} \mathbf{E}^* \cdot \mathbf{E} \right) d^3k \]

\[ = \frac{i\varepsilon_0}{4} \int \frac{1}{\omega} \mathbf{E}^T (\mathbf{k} \times \nabla_k) \mathbf{E}^* d^3k, \]

where the superscript \( T \) denotes the transpose. By making the variable replacement \( \mathbf{k} \to -\mathbf{k} \), it is changed into a familiar form,

\[ L_1 = \frac{1}{4} \int \frac{\varepsilon_0}{i\omega} \mathbf{E}^\dagger (\mathbf{k} \times \nabla_k) \mathbf{E} d^3k. \quad (A.6) \]

Since \(-i\nabla_k\) is a Hermitian operator, the \( L_1 \) given by Eq. (A.6) is real. A similar calculation produces from Eq. (A.3)

\[ L_2 = L_1. \quad (A.7) \]
It is clear that substituting Eqs. \((A.6)\) and \((A.7)\) into Eq. \((A.1)\) will yield Eq. \((6)\).

Then we derive Eq. \((7)\). Substituting Eq. \((5)\) into \(S = \int x \times p_2 d^3x\), one has

\[
S = S_1 + S_2 + c.c., \quad (A.8)
\]

where

\[
S_1 = -\frac{\varepsilon_0}{4(2\pi)^3} \int d^3k' d^3k \int d^3x \frac{E' \cdot k}{\omega} x \times E e^{i(k' + k) \cdot x} e^{-i(\omega' + \omega)t}, \quad (A.9)
\]

and

\[
S_2 = -\frac{\varepsilon_0}{4(2\pi)^3} \int d^3k' d^3k \int d^3x \frac{E' \cdot k}{\omega} x \times E^* e^{i(k' - k) \cdot x} e^{-i(\omega' - \omega)t}. \quad (A.10)
\]

Upon integrating Eq. \((A.9)\) over the configuration space and noticing Eq. \((A.4)\), one obtains

\[
S_1 = \frac{i\varepsilon_0}{4} \int \frac{(E_y e_z - E_z e_y)}{\omega} \frac{E' \cdot k}{\omega} e^{-i(\omega' + \omega)t} \delta(k'_x + k_x)\delta(k'_y + k_y)\delta(k'_z + k_z) d^3k' d^3k
\]

\[
+ \frac{i\varepsilon_0}{4} \int \frac{(E_z e_x - E_x e_z)}{\omega} \frac{E' \cdot k}{\omega} e^{-i(\omega' + \omega)t} \delta(k'_x + k_x)\delta(k'_y + k_y)\delta(k'_z + k_z) d^3k' d^3k
\]

\[
+ \frac{i\varepsilon_0}{4} \int \frac{(E_x e_y - E_y e_x)}{\omega} \frac{E' \cdot k}{\omega} e^{-i(\omega' + \omega)t} \delta(k'_x + k_x)\delta(k'_y + k_y)\delta(k'_z + k_z) d^3k' d^3k.
\]

It is changed into, by eliminating the \(\delta\) functions and taking Eqs. \((A.5)\) and \((2)\) into account,

\[
S_1 = \frac{i\varepsilon_0}{4} \int \frac{E_y e_z - E_z e_y}{\omega} k \cdot \frac{\partial E^*}{\partial k_x} d^3k + \frac{i\varepsilon_0}{4} \int \frac{E_z e_x - E_x e_z}{\omega} k \cdot \frac{\partial E^*}{\partial k_y} d^3k
\]

\[
+ \frac{i\varepsilon_0}{4} \int \frac{E_x e_y - E_y e_x}{\omega} k \cdot \frac{\partial E^*}{\partial k_z} d^3k.
\]

From the transversality condition \(k \cdot E^* = 0\), we know that

\[
k \cdot \frac{\partial E^*}{\partial k_x} = -E'_x, \quad k \cdot \frac{\partial E^*}{\partial k_y} = -E'_y, \quad k \cdot \frac{\partial E^*}{\partial k_z} = -E'_z.
\]

\(S_1\) thus takes the following form,

\[
S_1 = \frac{1}{4} \int \frac{\varepsilon_0}{i\omega} E^* \times E d^3k, \quad (A.11)
\]

which is clearly real. Similarly, \(S_2\) in Eq. \((A.10)\) is found to be real and is equal to \(S_1\),

\[
S_2 = S_1. \quad (A.12)
\]

Substituting Eqs. \((A.11)\) and \((A.12)\) into Eq. \((A.8)\) will yield Eq. \((7)\).