Fast Exact Matrix Completion: A Unifying Optimization Framework

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Abstract

We consider the problem of matrix completion of rank $k$ on an $n \times m$ matrix. We show that both the general case and the case with side information can be formulated as a combinatorial problem of selecting $k$ vectors from $p$ column features. We demonstrate that it is equivalent to a separable optimization problem that is amenable to stochastic gradient descent. We design fastImpute, based on projected stochastic gradient descent, to enable efficient scaling of the algorithm of sizes of $10^5 \times 10^5$. We report experiments on both synthetic and real-world datasets that show fastImpute is competitive in both the accuracy of the matrix recovered and the time needed across all cases. Furthermore, when a high number of entries are missing, fastImpute is over 75% lower in MAPE and 10x faster than current state-of-the-art matrix completion methods in both the case with side information and without.

Keywords: Matrix Completion, Projected Gradient Descent, Stochastic Approximation

1. Introduction

Low-rank matrix completion is one of the most studied problems after its successful application in the Netflix Competition. It is utilized in areas including computer vision (Candes and Plan (2010)), signal processing (Ji et al. (2010)), and control theory (Boyd et al. (1994)) to generate a completed matrix from partially observed entries. Given a data matrix $A \in \mathbb{R}^{n \times m}$, the low-rank assumption assumes that $\text{rank}(A)$ is small - in other words there are only a few, but still unknown, common linear factors that affect $A_{ij}$.
In recent years, as noted by Nazarov et al. (2018), there has been a rise in interest for inductive matrix completion, where the common linear factors are chosen from a set of given vectors in the form of side information.

In this paper, we present an optimization based approach that improves upon the state of the art across these two subfields of matrix completion. Next we review the literature in both the general matrix completion area, and also the inductive matrix completion area.

Literature

General Matrix Completion

Matrix completion has been applied successfully for many tasks, including recommender systems Koren et al. (2009), social network analysis Chiang et al. (2014) and clustering Chen et al. (2014b). After Candès and Tao (2010) proved a theoretical guarantee for the retrieval of the exact matrix under the nuclear norm convex relaxation, a lot of methods have focused on the nuclear norm problem (see Mazumder et al. (2010), Beck and Teboulle (2009), Jain et al. (2010), and Tanner and Wei (2013) for examples). There has also been work where the uniform distributional assumptions required by the theoretical guarantees are violated, such as Negahban and Wainwright (2012) and Chen et al. (2014a).

Alternative methods include alternating projections by Recht and Ré (2013) and Grassmann manifold optimization by Keshavan et al. (2009). Despite the non-convexity of the problem, many gradient-descent based approaches have also been proposed. This includes lifting the matrix into a positive semi-definite matrix before conducting gradient descent, as explored in Zheng and Lafferty (2016). Jain and Netrapalli (2015) details an algorithm based on projected gradient descent onto the set of rank-$r$ matrices. Works including Koren et al. (2009) and Jin et al. (2016) are based on the popular factorization formulation of $A = UV$, where $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{k \times m}$, and perform gradient updates for $U$ and $V$.

Our work differs from previous work as we consider the factorization formulation of $A = UV$, but derive $U$ as a function of $V$, $g(V)$, first, and then directly perform gradient updates for $U$, with a projection to a fixed-norm hypersphere $\|U\|_2 = 1$ to ensure scaling invariance. This allows us to achieve near-exact retrieval with faster speeds than the top performing methods discussed here.

Inductive Matrix Completion

Interest in inductive matrix completion intensified after Xu et al. (2013) showed that given predictive side information, one only needs $O(\log n)$ samples to retrieve the full matrix. Thus, most of this work (see Xu et al. (2013), Jain and Dhillon (2013), Farhat et al. (2013), Natarajan and Dhillon (2014)) have focused on the case in which the side information is
assumed to be perfectly predictive so that the theoretical bound of $O(\log n)$ sample complexity Xu et al. (2013) can be achieved. Chiang et al. (2015) explored the case in which the side information is corrupted with noise, while Shah et al. (2017) and Si et al. (2016) incorporated nonlinear combination of factors into the side information. As pointed out by a recent article Nazarov et al. (2018), there is a considerable lack of effort to introduce sparsity into inductive matrix completion, with Lu et al. (2016), Soni et al. (2016), Nazarov et al. (2018) as examples. Recently, in Bertsimas and Li (2018), the authors introduced a convex binary formulation of the sparse inductive matrix completion problem, and constructed randomized algorithms for scaling.

Our work differs from the attempts in Lu et al. (2016), Soni et al. (2016), and Nazarov et al. (2018) as it does not consider the heuristic convex relaxation of sparsity in the nuclear norm, but rather the exact sparse problem. Different to Bertsimas and Li (2018) where the underlying rank vectors need to be an exact vector given in the side information, we relax such assumption and allow the rank vectors to be formed by any linear combination of the vectors contained in the side information. This greatly increases the flexibility of the algorithm, provides much more modeling power, and leads to stronger matrix recovery.

Contributions and Structure

Our contributions in this paper are as follows:

1. We reformulate the low-rank matrix completion problem, both with side information and without, as a separable optimization problem. We show that the general matrix completion problem is in fact a special case of the matrix completion problem with side information.

2. We propose a novel algorithm using projected gradient descent and Nesterov’s accelerated gradient to solve the reformulated matrix completion problem.

3. We present computational results on both synthetic and real-world datasets with side information that shows the algorithm outperforms current state-of-the-art methods in scalability and accuracy.

4. We introduce further optimization in the special case of the general matrix completion problem that allows further reduction in complexity.

5. We present computational results on both synthetic and real-world datasets for the general matrix completion problem that shows the algorithm is about 10x faster than the fastest algorithm available, while achieving on average a 75% decrease in error of retrieval on synthetic datasets.

The structure of the paper is as follows. In Section 2, we introduce the separable reformulation of the low-rank matrix completion problem. In Section 3, we introduce the base projected gradient descent method, projImpute. In Section 4, we introduce fastImpute-S,
the stochastic version of projImpute designed for use with side information. We report on its computational complexity and compare it with other algorithms. In Section 5, we report results on both synthetic and real-world datasets with side information using fastImpute-S and compare its performance with other algorithms. In Section 6, we introduce fastImpute, a further optimized version of fastImpute-S to use under the case of no side information. In Section 7, we report results on both synthetic and real-world datasets without side information using fastImpute and compare its performance with other algorithms. In Section 8 we provide our conclusions.

2. Reformulation of Matrix Completion

The classical matrix completion problem considers a matrix $A \in \mathbb{R}^{n \times m}$ in which $\Omega = \{(i,j) \mid A_{ij} \text{ is known}\}$ is the set of known values. We aim to recover a matrix $X = UV$ of rank $k$ that minimizes the distance between $X$ and $A$ on the known entries $A$:

$$\min_X \frac{1}{n} \sum_{(i,j) \in \Omega} (X_{ij} - A_{ij})^2 \quad \text{subject to} \quad \text{Rank}(X) = k.$$ 

The problem we consider here is that for every column $j = 1, \ldots, m$, we have a given $p$-dimensional feature vector $B_j$ with $p \geq k$ that contains the information we have on column $j$. In the Netflix example, column $j$ corresponds to movie $j$, and thus the feature vector $B_j$ includes information about the $j$th movie: Budget, Box Office revenue, IMDB rating, etc. We represent all this side information with a matrix $B \in \mathbb{R}^{p \times m}$. Given side data $B$ we postulate that $X = USB$, where $U \in \mathbb{R}^{n \times k}$ is the matrix of feature exposures, and $S \in \mathbb{R}^{s \times p}$ is the matrix formulation of the rank $k$ constraint, as shown below:

$$S = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ s_{k1} & s_{k2} & \cdots & s_{kp} \end{pmatrix} \in \mathbb{R}^{k \times p}.$$

Then the matrix completion problem with side data $B$ can be written as:

$$\min_S \min_U \frac{1}{n} \sum_{(i,j) \in \Omega} (X_{ij} - A_{ij})^2 \quad \text{subject to} \quad X = USB, \|S\|_2 = 1.$$ 

The norm constraint on $S$ is needed for uniqueness of the solution, as the problem is invariant under the transformation $U \rightarrow UD^{-1}$ and $S \rightarrow DS$ for any diagonal matrix $D \in \mathbb{R}^{k \times k}$ that is invertible. We note that since $S$ is a $k \times p$ matrix, the rank of matrix $X$ is indeed $k$. Further, we note that if $p = m$, and $B = I_m$, then the problem is reduced back to the general low-rank matrix completion problem with no feature information.

Similar to linear regression and for robustness purposes as shown in Bertsimas and Copenhaver (2018), we address in this paper the problem with a Tikhonov regularization term. Specifically, the matrix completion problem with side information and regularization we address
is
\[
\min_{\|S\|_2=1} \min_{U} \frac{1}{n} \left( \sum_{(i,j) \in \Omega} (X_{ij} - A_{ij})^2 + \frac{1}{\gamma} \|U\|_2^2 \right) \quad \text{subject to} \quad X = USB, 
\]
where \( \gamma > 0 \) is a given parameter that controls the strength of the regularization term. We can reformulate Problem (1) as followed:

**Theorem 1.** Problem (1) can be reformulated as a separable optimization problem:
\[
\min_{\|S\|_2=1} c(S) = \frac{1}{n} \sum_{i=1}^n \left( X_i - a_i \right)^T W_i \left( (I_m + \gamma B^T S B W_i^T)^{-1} a_i \right)
\]
\[
= \frac{1}{n} \sum_{i=1}^n \left( X_i - a_i \right)^T \left( I_m - V \left( \frac{I_k}{\gamma} + V^T V \right)^{-1} V^T \right) a_i,
\]
where \( V = SB, W_1, \ldots, W_n \in \mathbb{R}^{m \times m} \) are diagonal matrices:
\[
(W_i)_{jj} = \begin{cases} 
1, & (i, j) \in \Omega, \\
0, & (i, j) \notin \Omega,
\end{cases}
\]
\( \overline{a}_i = a_i W_i, i = 1 \ldots, n, \) where \( a_i \) is the \( i \)th row of \( A \) with unknown entries taken to be 0.

**Proof.** With the diagonal projection matrices \( W_i \) defined above, we can rewrite the sum in (1) over known entries of \( A \), \( \sum_{(i,j) \in \Omega} (X_{ij} - A_{ij})^2 \), as a sum over the rows of \( A \):
\[
\sum_{i=1}^n \|(x_i - a_i) W_i\|_2^2,
\]
where \( x_i \) is the \( i \)th row of \( X \). Using \( X = USB \), then \( x_i = u_i SB \) where \( u_i \) is the \( i \)th row of \( U \). Moreover,
\[
\|U\|_2^2 = \sum_{i=1}^n \|u_i\|_2^2.
\]
Then, Problem (1) becomes:
\[
\min_{\|S\|_2=1} \min_{U} \frac{1}{n} \left( \sum_{i=1}^n \left( (u_i SB - a_i) W_i \right)^2 + \frac{1}{\gamma} \|u_i\|_2^2 \right).
\]
We then notice that within the sum \( \sum_{i=1}^n \) each row of \( U \) can be optimized separately, leading to:
\[
\min_{\|S\|_2=1} \frac{1}{n} \left( \sum_{i=1}^n \min_{u_i} \left( (u_i SB - a_i) W_i \right)^2 + \frac{1}{\gamma} \|u_i\|_2^2 \right). 
\]
The inner optimization problem \( \min_{u_i} \|u_i SB - a_i\|_2^2 + \frac{1}{\gamma} \|u_i\|_2^2 \) can be solved in closed form given \( S \), as it is a weighted linear regression problem with Tiknonov regularization. The closed form solution is:
\[
a_i^T W_i (I_m + \gamma W_i^T S B W_i)^{-1} W_i a_i = \overline{a}_i^T (I_m + \gamma W_i^T S B W_i)^{-1} \overline{a}_i.
\]
So Problem (4) can be simplified to:

$$\min \frac{1}{n} \left( \sum_{i=1}^{n} \pi_i^T (I_m + \gamma W_i B^T S B W_i^T)^{-1} \pi_i \right).$$

Which is the form in (2). To reach the form in (3), take \( V = SB \), and apply the matrix inversion lemma, as derived in Woodbury (1950):

**Lemma 1.** For matrices \( U \in \mathbb{R}^{n \times k} \) and \( V \in \mathbb{R}^{k \times n} \), we have the following equivalence:

$$\left( I_n + UV \right)^{-1} = I_n - U \left( I_k + VU \right)^{-1} V.$$

\[\square\]

3. A Gradient Descent Algorithm

In this section, we describe a gradient descent algorithm to solve the separable optimization problem (2). We first introduce some terminology.

$$\alpha_i(S) = \pi_i^T (I_m + \gamma W_i B^T S B W_i^T)^{-1} \pi_i$$  \hspace{1cm} (5)

$$= \pi_i^T \left( I_m - V \left( \frac{I_k}{\gamma} + V^T W_i V \right)^{-1} V^T \right) \pi_i, \ i = 1, \ldots, n,$$  \hspace{1cm} (6)

where again \( V = SB \). We further define:

$$\gamma_i(S) = (I_m + \gamma W_i B^T S B W_i^T)^{-1} \pi_i,$$  \hspace{1cm} (7)

$$= \left( I_m - V \left( \frac{I_k}{\gamma} + V^T W_i V \right)^{-1} V^T \right) \pi_i, \ i = 1, \ldots, n.$$  \hspace{1cm} (8)

Here \( \alpha_i(S) \) is a scalar and \( \gamma_i(S) \) is a \( m \times 1 \) vector. Then the function \( c(S) \) in (2) can be expressed as

$$c(S) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i(S) = \frac{1}{n} \sum_{i=1}^{n} \pi_i^T \gamma_i(S).$$

We have the following result on the gradient of \( \alpha_i(S) \):

**Theorem 2.**

$$\nabla \alpha_i(S) = -2\gamma V \gamma_i(S) \gamma_i(S)^T B^T.$$  \hspace{1cm} (9)

**Proof.** By the standard result of the derivative of the inverse matrix, we have:

$$\nabla \alpha_i(S) = \nabla \pi_i^T \gamma_i(S)$$

$$= -2\gamma S B \gamma_i(S) \pi_i^T (I_m + \gamma \pi_i^T W_i B^T S B W_i^T)^{-1} B^T$$

$$= -2\gamma V \gamma_i(S) \gamma_i(S)^T B^T.$$  \hspace{1cm} \( \square \)
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This is a surprising result as the additional complexity of calculating the derivative of \( \alpha_i(S) \) is only \( O(mp + mk + k^2) \). Then calculating \( c(S) \) is just:

\[
\nabla c(S) = \frac{1}{n} \sum_{i=1}^{n} -2\gamma V \gamma_i(S) \gamma_i(S)^T B^T.
\]

In this form, calculating the entire derivative only requires \( O(nmp + nmk + nk^2) \) additional multiplications beyond those needed in calculating the objective.

Then applying Theorem 2, we apply a projected gradient descent algorithm (as discussed in Bertsekas (1997)) on the hypersphere \( \|S\|_2 = 1 \) as followed: (\( 0^{k \times p} \) denotes a zero matrix of dimension \( k \times p \)). Candès and Tao (2010) showed that top-k SVD is a good approximation

Algorithm 1 Gradient Descent algorithm for matrix completion with side information.

1: procedure PROJIMPUTE(\( A, B, k \)) \( \triangleright \) \( A \in \mathbb{R}^{n \times m} \) the masked matrix, \( B \in \mathbb{R}^{p \times m} \) the feature matrix, \( k \) the desired rank
2: \( t \leftarrow 1 \)
3: if \( \alpha > 0.5 \) then \( \triangleright \) If existing percentage is high
4: \( S_1 \leftarrow \text{TopkSVD}(A) \triangleright \text{Top-k singular vectors of } A \text{ with zeros for missing entries} \)
5: else
6: \( S_1 \leftarrow \text{random initial matrix with } \|S_1\|_2 = 1 \) \( \triangleright \) Randomized Start
7: end if
8: \( \eta_0 \leftarrow \infty \) \( \triangleright \) Initialize objective value
9: \( \eta_1, G_1 \leftarrow c(S_1), \nabla c(S_1) \) \( \triangleright \) Initialize objective value and gradient
10: \( \theta \leftarrow 0 \) \( \triangleright \) Initialize step size
11: \( t_{\text{max}} \leftarrow 50 \) \( \triangleright \) Initialize number of gradient descent steps
12: \( \nabla \tilde{S}_1 \leftarrow 0^{k \times p} \) \( \triangleright \) Initialize accelerated gradient
13: while \( t < t_{\text{max}} \) do \( \triangleright \) While we have not reached \( t_{\text{max}} \) iterations
14: \( \nabla \tilde{S}_{t+1} = G_t + \frac{1}{t+2} \nabla \tilde{S}_t \) \( \triangleright \) Nesterov accelerated gradient update step
15: \( \nabla S_{t+1} = -\nabla \tilde{S}_{t+1} + (\nabla \tilde{S}_{t+1} \cdot S_t) S_t \) \( \triangleright \) Project gradient to tangent plane of \( S_t \)
16: \( S_{t+1} \leftarrow S_t \cos \theta + \frac{\nabla S_{t+1}}{\|\nabla S_{t+1}\|_2} \sin \theta \) \( \triangleright \) Update \( S_t \) by projected gradient
17: \( \eta_{t+1}, G_{t+1} \leftarrow c(S_{t+1}), \nabla c(S_{t+1}) \) \( \triangleright \) Update the new cost and derivative.
18: \( t \leftarrow t + 1 \)
19: end while
20: \( S^* \leftarrow S_t \)
21: \( i \leftarrow 1 \)
22: for \( i < n \) do
23: \( a_i \leftarrow B^T S^* S^T (S^* B W_i B^T S^* S^T)^{-1} S^* B \bar{a}_i \) \( \triangleright \) Calculate the final \( A \) matrix
24: end for
25: return \( A \) \( \triangleright \) Return the filled matrix \( A \)
26: end procedure

if the matrix is mostly filled (as top-k SVD is the optimal solution when the matrix is completely filled), so we use a random warmstart when most entries are missing, and SVD warmstart when most entries are filled. We explain some key steps of this algorithm:
Nesterov Step  This is Step 14 of Algorithm 1. We update the gradient with the accelerated formula in Nesterov (1983) by adding the gradient of the current step to \( \frac{t - 1}{t + 2} \) times the previous gradient, which introduces damping in the resulting gradient and enables faster convergence.

Projection Step  This is Step 15 of Algorithm 1. Because we are optimizing on the hypersphere \( \|S\|_2 = 1 \), our gradient updates need to be projected to the tangent plane of the sphere at the current point \( S_t \). Thus, we project the raw gradient, \( \nabla S_{t+1} \) onto the tangent plane to get \( \nabla S_{t+1} \).

Update Step  This is step 16 of Algorithm 1. To update \( S_{t+1} \), we add \( S_t \) to the projected gradient \( \frac{\nabla S_{t+1}}{\|\nabla S_{t+1}\|_2} \) (normalized to ensure that the resulting \( S_{t+1} \) still has norm of 1) over a step size of \( \frac{\pi}{64} \). This step size is selected through experimentation, along with the number of gradient descent steps \( t_{\text{max}} = 50 \). We would show in the experiments that this fixed step size and number of gradient descent steps works well across different-sized problems.

Note that updating the gradient on the hypersphere is a rotation on the great circle formed by \( S_t \) and \( \nabla S_{t+1} \), so the update formula is \( S_t \cos \theta + \frac{\nabla S_{t+1}}{\|\nabla S_{t+1}\|_2} \sin \theta \).

3.1 Discussion on Computational Complexity

There are 2 key computational steps of the algorithm - the first is Step 17, where the cost and the derivative is calculated in every gradient update. The second step is Step 23, where the final imputed matrix \( A \) is derived with the imputed \( S \). We have the following theorem on the asymptotic complexity of both steps:

**Theorem 3.** The computation complexity of Step 17 in projImpute is \( O(|\Omega|(p + m \alpha k + \frac{pk^3}{mn})) \), where \( |\Omega| \) is the number of samples given in the original matrix \( A \) and \( \alpha = \frac{|\Omega|}{mn} \) is the percentage not missing. The computation complexity of Step 23 in projImpute is \( O \left(|\Omega| \left(k^2 + \frac{k}{\alpha} + \frac{k^3}{m \alpha}\right)\right) \).

*Proof.* We separate the proof for the two steps into two separate sections below.

### Computational Complexity of Step 17

Recall we have that:

\[
c(S) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i(S) = \frac{1}{n} \sum_{i=1}^{n} \pi_i^T \gamma_i(S),
\]  

(11)
where
\[ \alpha_i(S) = \overline{\alpha}_i^T (I_m + \gamma W_i B^T S^T S B W_i^T)^{-1} \overline{\alpha}_i, \]
\[ = \overline{\alpha}_i^T \left( I_m - V \left( \frac{I_k}{\gamma} + V^T W_i V \right)^{-1} V^T \right) \overline{\alpha}_i, \quad i = 1, \ldots, n, \]
(12)
where again \( V = SB \). Moreover,
\[ \gamma_i(S) = (I_m + \gamma W_i B^T S^T S B W_i^T)^{-1} \overline{\alpha}_i, \]
\[ = \left( I_m - V \left( \frac{I_k}{\gamma} + V^T W_i V \right)^{-1} V^T \right) \overline{\alpha}_i, \quad i = 1, \ldots, n. \]
(13)
First let us denote \( m_i \) as the number of non-zero entries of \( W_i \). This is the number of known entries per row. Then define \( V W_i \in \mathbb{R}^{k \times m_i} \) as the matrix of \( V W_i \) after removing the all-zero columns, as illustrated below:
\[ V W_i = \begin{pmatrix} v_1 & v_2 & v_3 & \cdots & v_k \end{pmatrix} \times \text{Diag}(1, 0, 1, \cdots, 0) \]
\[ = \begin{pmatrix} v_1 & 0 & v_3 & 0 & \cdots \end{pmatrix} \sim \begin{pmatrix} v_1 & v_3 & \cdots \end{pmatrix} := V_{W_i}. \]
(16)
Note that \( V_{W_i} \) can be created efficiently through subsetting, and its creation does not impact the asymptotic running time. Then similarly, we denote \( a_{W_i} \in \mathbb{R}^{m_i} \) as \( a_{W_i} \) with all the zero elements removed. Then we have the following equality, using (11) and (15):
\[ c(S) = \frac{1}{n} \sum_{i=1}^{n} \overline{\alpha}_i^T \gamma_i(S) \]
\[ = \frac{1}{n} \sum_{i=1}^{n} \overline{\alpha}_i^T \left( I_m - V \left( \frac{I_k}{\gamma} + V^T W_i V \right)^{-1} V^T \right) \overline{\alpha}_i \]
\[ = \frac{1}{n} \sum_{i=1}^{n} a_{W_i}^T \left( I_{m_i} - V_{W_i} \left( \frac{I_k}{\gamma} + V_{W_i}^T V_{W_i} \right)^{-1} V_{W_i}^T \right) a_{W_i}. \]
(17)
The final equality (17) follows directly from the definition of \( V W_i \) and \( a_{W_i} \) as all the removed elements/columns are zero.

Then, the complexity of each term in the sum (17) is \( O(m_i^2 k + k^3) \). Summing over \( n \) terms and noting that:
\[ \sum_{i=1}^{n} n m_i^2 \approx \left( \sum_{i=1}^{n} n m_i \right) \left( \frac{1}{n} \sum_{i=1}^{n} m_i \right) \approx |\Omega| m \alpha. \]
The full complexity of the cost function is roughly
\[ O(|\Omega| m \alpha k + nk^3) = O \left( |\Omega| \left( m \alpha k + \frac{1}{m \alpha} k^3 \right) \right). \]
Now let us further define:

\[
\gamma_{W_i}(S) = \left( I_{m_i} - V_{W_i} \left( \frac{I_k}{\gamma} + V_{W_i}^T V_{W_i} \right)^{-1} V_{W_i}^T \right) a_{W_i}, \tag{18}
\]

and \( B_{W_i} \) in the same fashion as defined in (16). Then we can similarly show that the derivative (10) is equivalent to the following expression:

\[
\nabla c(S) = \frac{1}{n} \sum_{i=1}^{n} -2\gamma V_{W_i}(S) \gamma_{W_i}(S)^T B_{W_i}^T
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} -2\gamma V_{W_i}(S) \gamma_{W_i}(S)^T B_{W_i}^T. \tag{19}
\]

The matrix multiplication in the final expression (20) has computational complexity of \( O(m_i(p + k) + pk) \) (every term has been previously calculated), so summing over \( n \) terms the computational complexity is

\[
O \left( |\Omega| \left( p + \frac{pk}{ma} \right) \right).
\]

Thus, the computational complexity of the entire step is:

\[
O \left( |\Omega| \left( p + m\alpha k + \frac{pk + k^3}{ma} \right) \right).
\]

**Computational Complexity of Step 23**

Recall that Step 23 of projImpute does the following calculation for every \( i \in \{1, \cdots, n\} \):

\[
a_i \leftarrow B_{i}^T S^*^T (S^* B W_i B_i^T S^*^T)^{-1} S^* B \bar{a}_i, \tag{21}
\]

where \( S^* \) is the imputed \( S \). Now define \( V^* = S^* B \), and define \( V_{W_i}^* \) in the same fashion as defined in (16). Then we have that:

\[
B_{i}^T S^*^T (S^* B W_i B_i^T S^*^T)^{-1} S^* B \bar{a}_i = V^* (V^* W_i V^*^T)^{-1} V^* \bar{a}_i
\]

\[
= V^* (V_{W_i}^* V_{W_i}^*^T)^{-1} V_{W_i}^* a_{W_i}. \tag{22}
\]

The final expression (22) has a matrix multiplication complexity of \( O(km_i + k^2 m_i + k^3 + mk) \), so summing over \( n \) samples the total computational complexity of Step 23 is:

\[
O \left( |\Omega| \left( k^2 + \frac{k}{\alpha} + \frac{k^3}{ma} \right) \right).
\]

Using these results we can easily arrive at the following corollary:
Corollary 1. The projImpute algorithm terminates in \( t_{\text{max}} \) number of steps. Furthermore, it has complexity
\[
O \left( |\Omega| \left( p + m a k + k^2 + \frac{k^3 + pk}{m a} \right) \right).
\]

Proof. The computational complexity follows immediately from Theorem 3 and the fact that Step 17 and 23 are the only two non-trivial calculation steps in the algorithm.

With this result, we first note that any algorithm that solves (1) and returns the filled matrix \( \mathbf{A} \) must have a complexity of at least \( O(n m k + k p) = O(|\Omega| (\frac{k^2}{\alpha}) + k p) \) by the number of degrees of freedom. Thus, we can see that projImpute has the additional terms \( (|\Omega| (p + m a k + k^2 + k^3 + pk) \). In typical applications \( m, n \gg p \gg k \), implying that the most computationally intensive step is the \( O(|\Omega| m a k) \) appearing in the cost and gradient calculation step (Step 17). Thus, in the next section, we would introduce an adaptive stochastic version of projImpute that removes this term in complexity.

4. fastImpute-S: An Adaptive Stochastic Projected Gradient Descent Algorithm for Matrix Completion

In the last section, we explored the projImpute algorithm, which conducted full projected gradient updates for every step \( t \in \{1, \ldots, t_{\text{max}}\} \) using all \( n \) rows and \( m \) columns. In this section, we introduce fastImpute-S that uses \( n_t \) rows and \( m_t \) columns to estimate the gradient update at step \( t \), where \( n_t \) and \( m_t \) are selected through a procedure AdaptS in response to the performance of the gradient descent over time. Let \( [n_i] \) denote a set of numbers from \( \{1, \ldots, n\} \) of size \( n_i \), and \( [m_i] \) denote a set of numbers from \( \{1, \ldots, m\} \) of size \( m_i \). Then the cost function evaluated with columns \( [m_t] \) and rows \( [n_t] \) is:
\[
c_t(S) = \frac{1}{n_t} \sum_{i \in [n_t]} \bar{a}_{imt}^T \left( I_{m_t} - V_{m_t} \left( I_{k} \gamma + V_{mt}^T W_{imt} V_{mt} \right)^{-1} V_{mt}^T \right) \bar{a}_{imt},
\]
where \( V_{m_t} \) is the submatrix of \( V \) formed with \( [m_t] \) columns, \( W_{imt} \) the submatrix of \( W_i \) formed with \( [m_t] \) columns, and \( \bar{a}_{imt} \) the subvector of \( \bar{a}_i \) formed with \( [m_t] \) elements. Similarly, using Theorem 2, the derivative evaluated with columns \( [m_t] \) and rows \( [n_t] \) is:
\[
\nabla c_t(S) = \frac{1}{n_t} \sum_{i \in [n_t]} -2 \gamma V_{m_t} \left( I_{m_t} - V_{m_t} \left( I_{k} \gamma + V_{mt}^T W_{imt} V_{mt} \right)^{-1} V_{mt}^T \right) \bar{a}_{imt}
\times \bar{a}_{imt}^T \left( I_{m_t} - V_{m_t} \left( I_{k} \gamma + V_{mt}^T W_{imt} V_{mt} \right)^{-1} V_{mt}^T \right)^T B_{m_t}.
\]

We then present the algorithm as Algorithm 2. It is the same procedure as projImpute, except with stochastic gradient updates of \( n_t \) rows and \( m_t \) columns. The procedure to update \( n_t \) and \( m_t \) is called AdaptS, presented in Algorithm 3.
Algorithm 2 Gradient Descent algorithm for matrix completion with side information.

1: procedure fastImpute-S(A, B, k) \> A ∈ \mathbb{R}^{n \times m} the masked matrix, \(B \in \mathbb{R}^{p \times m}\) the feature matrix, \(k\) the desired rank
2: \> t ← 1
3: \> \(\alpha \leftarrow \frac{\|\Omega\|}{mn}\) \> Define existing percentage of \(A\). \(|\Omega|\) is the set of non-zero entries in \(A\)
4: \if \alpha > 0.5 \> If existing percentage is high
5: \> \(S_1 \leftarrow \text{TopkSVD}(A)\) \> Top-k singular vectors of \(A\) with zeros for missing entries
6: \> else
7: \> \(S_1 \leftarrow \text{random initial matrix with } \|S_1\|_2 = 1\) \> Randomized Start
8: \end if
9: \> \(\eta_0 \leftarrow \infty\) \> Initialize objective value
10: \> \(q_1 \leftarrow 0\) \> Initialize counter for non-improving steps
11: \> \(m_0 \leftarrow \min(2p, m)\) \> Define initial gradient update size for columns
12: \> \(n_0 \leftarrow \frac{k\sqrt{mn \log(\sqrt{mn})}}{8mn\alpha}\) \> Define initial gradient update size for rows
13: \> \([m_0] \leftarrow \{1, \cdots, m_0\}\) \> Initialize columns selected
14: \> \([n_0] \leftarrow \{1, \cdots, n_0\}\) \> Initialize rows selected
15: \> \(\eta_1, G_1 \leftarrow c_t(S_1), \nabla c_t(S_1)\) \> Initialize objective value and gradient
16: \> \(\theta \leftarrow \frac{\pi}{2}\) \> Initialize step size
17: \> \(t_{\text{max}} \leftarrow 50\) \> Initialize number of gradient descent steps
18: \> \(\nabla \bar{S}_t \leftarrow 0^{k \times p}\) \> Initialize accelerated gradient
19: \> while \(t < t_{\text{max}}\) do \> While we have not reached \(t_{\text{max}}\) iterations
20: \> \([m_t], [n_t], m_t, n_t, q_{t+1} \leftarrow \text{AdaptS}(m_{t-1}, n_{t-1}, \eta_t, \eta_*, q_t)\) \> We select the set of rows and columns used to update
21: \> \(\nabla \bar{S}_{t+1} = G_t + \frac{t-1}{t+2} \nabla \bar{S}_t\) \> Nesterov accelerated gradient update step
22: \> \(\nabla S_{t+1} = -\nabla \bar{S}_{t+1} + (\nabla \bar{S}_{t+1} \cdot S_t) S_t\) \> Project gradient to the tangent plane of \(S_t\)
23: \> \(S_{t+1} = S_t \cos \theta + \frac{\nabla S_{t+1}}{\|\nabla S_{t+1}\|_2} \sin \theta\) \> Update \(S_t\) based on projected gradient
24: \> \(\eta_{t+1}, G_{t+1} \leftarrow c_t(S_{t+1}), \nabla c_t(S_{t+1})\) \> Update the cost and derivative.
25: \> \(t \leftarrow t + 1\)
26: \> end while
27: \> \(S^* \leftarrow S_t\)
28: \> \(i \leftarrow 1\)
29: \> for \(i < n\) do
30: \> \(a_i \leftarrow B^T S^* T(S^* BW_i B^T S^* T)^{-1} S^* B \bar{a}_i\) \> Calculate the final \(A\) matrix
31: \> end for
32: return \(A\) \> Return the filled matrix \(A\)
33: end procedure

AdaptS is a very simple routine - if during the gradient descent there has been 5 consecutive steps in which the objective has not improved from the previous best, we double the number of rows. This adaptive procedure follows suggestions as illustrated in Smith et al. (2017) about increasing the samples used in stochastic gradient descent. These use of 5 steps as a threshold was experimentally determined and such value performs well across different
Algorithm 3 Update routine for $m_t$ and $n_t$.

1: **procedure** ADAPTS($m_{t-1}, n_{t-1}, \eta_t, \eta_*, q_t$)
2:     **if** $\eta_t \geq \eta_*$ **then**
3:         $q_{t+1} \leftarrow q_t + 1$  
4:     **else**
5:         $\eta_* \leftarrow \eta_t$  
6:         $q_{t+1} \leftarrow 0$  
7:     **end if**
8:     **if** $q_t \geq 5$ **then**
9:         $q_{t+1} \leftarrow 0$  
10:        $m_t \leftarrow m_{t-1}$  
11:        $n_t \leftarrow 2n_{t-1}$  
12:     **end if**
13:     $[m_t] \leftarrow \text{randomly sample } m_t \text{ elements from } \{1, \cdots m\}$
14:     $[n_t] \leftarrow \text{randomly sample } n_t \text{ elements from } \{1, \cdots n\}$
15:     **return** $[m_t], [n_t], m_t, n_t, q_t+1$
16: **end procedure**

In practice, we found that the following constants seem to be minimally needed for the algorithm to perform well:

$$m_0 = O(p), \quad n_0 = O\left(\frac{\sqrt{mn} \log(\sqrt{mn})}{m_0^\alpha}\right).$$

And thus we would select this sample size as our starting default.
With $m_0$ and $n_0$ selected, we can now calculate the complexity of fastImpute-S. We arrive at the following result for computational complexity:

**Corollary 2.** The computational complexity of Algorithm 2 is

$$O \left( |\Omega| \left( k^2 + \frac{k}{\alpha} + \frac{k^3}{m \alpha} \right) + \frac{\sqrt{nm} \log(\sqrt{nm})pk}{\alpha} \right).$$

**Proof.** Note that Step 23 of projImpute did not change in fastImpute-S, and thus the complexity to calculate the final matrix $A$ is still at $O \left( |\Omega| \left( k^2 + \frac{k}{\alpha} + \frac{k^3}{m \alpha} \right) \right)$. For the gradient update step, we replace $m$ with $\min(p,m)$ and $n$ with $O \left( \frac{k \sqrt{nm} \log(\sqrt{nm})}{\min(m,p) \alpha} \right)$ in the original formula for Step 17 in projImpute to get that its computational complexity is:

$$O \left( \frac{k \sqrt{nm} \log(\sqrt{nm})}{\alpha} \left( p + \min(m,p)\alpha k + k^2 + \frac{k}{\alpha} + \frac{k^3 + pk}{\min(m,p) \alpha} \right) \right)$$

$$= O \left( \frac{pk \sqrt{nm} \log(\sqrt{nm})}{\alpha} \right).$$

We suppress the complexity terms not involving $p$ in the final expression, as they are strictly dominated by the complexity of the final step of calculating the matrix. So conservatively the entire gradient update adds $O \left( \frac{\log(\sqrt{nm})pk}{\alpha} \right)$ to the complexity of calculating $A$, so the final complexity is:

$$O \left( |\Omega| \left( k^2 + \frac{k}{\alpha} + \frac{k^3}{m \alpha} \right) + \frac{\sqrt{nm} \log(\sqrt{nm})pk}{\alpha} \right),$$

as required. \qed

If we compare the new asymptotic complexity with the theoretical limit of $O \left( |\Omega| \left( \frac{k}{\alpha} \right) + kp \right)$, we are only a $\sqrt{nm}$ factor away from optimality on $p$ (compared to a factor of $nm$ for projImpute), while our dependence on $k$ is cubic compared with the theoretical limit of linear. The cubic term is due to the inverse calculation in calculating $A$. Theoretically there are more asymptotically efficient methods of calculating the inverse, but the $k$ term is usually very small compared to $m, n, p$ and thus not worth it.

In the next section we would look at experiments conducted for fastImpute-S.

**5. Experiments on fastImpute-S**

In this section, we would compare fastImpute-S with other inductive matrix completion algorithms on both synthetic datasets and real-world datasets to explore its performance and scaling behavior.
5.1 Synthetic Data Experiments

For synthetic data experiments, we assume that the underlying matrix satisfies the form $A = USB$, where $U \in \mathbb{R}^{n \times k}$, $S \in \mathbb{R}^{k \times p}$, and $B \in \mathbb{R}^{p \times m}$. The elements of $U$, $S$, and $B$ are selected from a uniform distribution of $[0, 1]$, where a fraction $\mu$ is missing. We report statistics on various combinations of $(m, n, p, k, \mu)$.

All algorithms are tested on a server with 16 CPU cores. For each combination $(m, n, p, k, \mu)$, we ran 10 tests and report the average value for every statistic. The algorithms tested are:

- **fastImpute**: We use the default parameters as defined in the algorithm above, with regularization parameter $\gamma = 10^6$. We explicitly stress here that no parameter tuning is done on fastImpute as we intend to show that the algorithm is not parameter sensitive. We implement our algorithm in Julia 0.6 with only the base packages.

- **IMC**: This algorithm is a well-accepted benchmark for testing Inductive Matrix Completion algorithms developed by Natarajan and Dhillon (2014). For each combination of $(m, n, p, k, \mu)$, we tune the regularization parameter $\lambda$ by imputing random matrices with such combination and find the $\lambda$ that gives the best results. We utilize the implementation provided by the authors in Matlab.

We report the following statistics for each algorithm:

- $n, m$ - the dimensions of $A$.
- $p$ - the number of features in the feature matrix.
- $k$ - the true number of features.
- $\mu$ - The fraction of missing entries in $A$.
- $T$ - the total time of algorithm execution.
- MAPE - the Mean Absolute Percentage Error (MAPE) for the retrieved matrix $\hat{A}$:

$$
MAPE = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\hat{A}_{ij} - A_{ij}}{A_{ij}}.
$$

We have the following observations:

- for $n, m$, fastImpute-S scales as $O(nm)$ as predicted. IMC scales similarly with slightly worse accuracy.

- Both algorithms roughly exhibit linear scaling in $p$, but IMC retrieves a matrix that has much higher MAPE than fastImpute-S.
|    | n     | m     | p     | k   | µ% | fastImpute-S T | MAPE | IMC T | MAPE |
|----|-------|-------|-------|-----|----|---------------|------|-------|------|
|    | 10^3  | 10^3  | 100   | 5   | 95%| 4.3s          | 0.3% | 4.4s  | 2.9% |
|    | 10^4  | 10^3  | 100   | 5   | 95%| 21.0s         | 0.1% | 18s   | 2.5% |
|    | 10^5  | 10^3  | 100   | 5   | 95%| 163s          | 0.2% | 173s  | 1.0% |
|    | 10^6  | 10^3  | 100   | 5   | 95%| 1210s         | 0.2% | 1400s | 0.3% |
|    | 10^4  | 10^4  | 100   | 5   | 95%| 21.0s         | 0.1% | 18s   | 2.5% |
|    | 10^5  | 10^4  | 100   | 5   | 95%| 160s          | 0.3% | 144s  | 0.8% |
|    | 10^6  | 10^4  | 100   | 5   | 95%| 1070s         | 0.2% | 1265s | 0.4% |
|    | 10^4  | 10^5  | 100   | 5   | 95%| 7600s         | 0.2% | 13891s| 0.2% |
|    | 10^5  | 10^4  | 1000  | 5   | 95%| 21.0s         | 0.1% | 18s   | 2.5% |
|    | 10^6  | 10^4  | 5000  | 10   | 95%| 92s           | 0.1% | 87s   | 2.1% |
|    | 10^4  | 10^5  | 1000  | 5   | 95%| 130s          | 0.03%| 160s  | 1.9% |
|    | 10^4  | 10^5  | 1000  | 5   | 20%| 32.1s         | 0.01%| 6.2s  | 0.01%|
|    | 10^4  | 10^4  | 100   | 5   | 50%| 26.2s         | 0.02%| 8.9s  | 0.03%|
|    | 10^4  | 10^4  | 100   | 5   | 80%| 22.3s         | 0.2% | 14s   | 0.7% |
|    | 10^5  | 10^4  | 100   | 5   | 95%| 21.0s         | 0.1% | 18s   | 2.5% |
|    | 10^5  | 10^4  | 100   | 10  | 95%| 61.6s         | 0.3% | 32s   | 2.1% |
|    | 10^5  | 10^4  | 100   | 20  | 95%| 140s          | 0.3% | 47s   | 2.2% |
|    | 10^5  | 10^4  | 100   | 30  | 95%| 260s          | 0.4% | 55s   | 2.6% |
|    | 10^4  | 10^5  | 100   | 5   | 95%| 21.0s         | 0.1% | 18s   | 2.5% |
|    | 10^5  | 10^4  | 100   | 10  | 95%| 130s          | 0.1% | 203s  | 0.8% |
|    | 10^5  | 10^4  | 200   | 10  | 95%| 709s          | 0.2% | 2430s | 0.4% |
|    | 10^6  | 10^4  | 200   | 10  | 95%| 6841s         | 0.2% | 25707s| 0.2% |
|    | 10^6  | 10^4  | 500   | 20  | 95%| 14790s        | 0.2% | N/A   | N/A  |

Table 1: Comparison of fastImpute-S and IMC on synthetic data. N/A means the algorithm did not complete running in 20 hours, corresponding to 72000 seconds.

- fastImpute-S roughly shows the $O(k^2)$ dominant behavior as expected, while IMC seems to scale linearly in $k$.

- IMC’s running time increases with more missing data while it decreases for fastImpute-S. Both algorithms achieve roughly the same performance, with IMC dropping significantly as the number of missing entries increased. IMC performs on a similar scale when $\mu \leq 50\%$, as in these cases, the top-k SVD warmstart (which is also utilized in IMC) is very close to the true solution.
5.2 Real-World Experiments

For real-world experiments, we utilize the Netflix Prize Dataset. This dataset was released in a competition to predict ratings of customers on unseen movies, given over 10 million ratings scattered across 500,000 people and 16,000 movies. Thus, when presented in a matrix $A$ where $A_{ij}$ represents the rating of individual $i$ on movie $j$, the goal is to complete the matrix $A$ under a low-rank assumption.

For this experiment, we included movies where people who had at least 5 ratings present. This gives a matrix of 471,268 people and 14,538 movies. To observe the scalability of fastImpute, we created five data sets (in similar format to Bertsimas and Li (2018)):

1. Base - $A_1$ has dimensions $3,923 \times 103$.
2. Small - $A_2$ has dimensions $18,227 \times 323$.
3. Medium - $A_3$ has dimensions $96,601 \times 788$.
4. Large - $A_4$ has dimensions $471,268 \times 1760$.
5. Full - $A$ has dimensions $471,268 \times 14,538$.

These sizes are constructed such that the total number of elements in $A$ in the successive sizes are approximately different by approximately an order of magnitude.

The feature matrix $B$ is constructed using data from the TMDB Database, and covers 59 features that measure geography, popularity, top actors/actresses, box office, runtime, genre and more. The full list of 59 features is contained in Appendix A (Section 9.1).

For comparison, we test against IMC. We split the training set in 80%/20%, where the latter group is used for validation of the rank in IMC and fastImpute. We then report the time taken, $T$, the MAPE, and the optimal chosen rank $k^*$ for each algorithm: We see that

| $n$   | $m$   | $p$ | $\mu\%$ | fastImpute | IMC          |
|-------|-------|-----|----------|-------------|--------------|
|       |       |     |          | $T$  | $k^*$ | MAPE | $T$  | $k^*$ | MAPE |
| 3,923 | 103   | 59  | 92.6%    | 3s   | 5     | 33.0% | 0.8s | 5     | 34.1% |
| 18,227| 323   | 59  | 94.8%    | 21s  | 6     | 27.6% | 7.5s | 6     | 29.0% |
| 96,601| 788   | 59  | 94.2%    | 120s | 6     | 25.5% | 49s  | 8     | 28.5% |
| 471,268| 1,760 | 59  | 93.6%    | 845s | 8     | 22.6% | 870s | 10    | 24.1% |
| 471,268| 14,538| 59  | 94.1%    | 5902s| 8     | 20.5% | 7605s| 10    | 21.0% |

Table 2: Comparison of methods on Netflix data for fastImpute-S.

fastImpute-S is able to outperform IMC on the Netflix dataset across the different $n$ and $m$ values, while enjoying competitive scalability.
6. fastImpute: Further Optimizations for Problems without Side Information

In this section, we consider further optimizations to the algorithm when \( p = m \) and the feature matrix is the identity \( B = I_m \). We first note that (1) reduces to:

\[
\min_{\|S\|_2 = 1} \min_U \frac{1}{n} \left( \sum_{(i,j) \in \Omega} (X_{ij} - A_{ij})^2 + \frac{1}{\gamma} \|U\|_2^2 \right) \quad \text{subject to} \quad X = US, \tag{23}
\]

where \( U \in \mathbb{R}^{n \times k} \) and \( S \in \mathbb{R}^{k \times m} \). This is thus the general problem of matrix completion under no side information.

The key property that we are going to further exploit for \( B = I_m \) is that it is fully diagonal. For any fixed integer \( \ell \), that means the first \( \ell \) columns of \( B \) can only affect the first \( \ell \) columns of \( A \). Thus we could form \( \left\lceil \frac{m}{\ell} \right\rceil \) equal submatrix pairs of \( A \) and \( B \), denoted \((A^1, B^1), (A^2, B^2), \ldots, (A^{\left\lceil \frac{m}{\ell} \right\rceil}, B^{\left\lceil \frac{m}{\ell} \right\rceil})\), where the \( k \)th pair are:

- \( A^k \) is the \( k\ell \)th to \( k\ell + 1 \)th column of \( A \)
- \( B^k \) is a \( \ell \times \ell \) sized identity matrix. (The final one is a \( (m - \ell \left\lfloor \frac{m}{\ell} \right\rfloor) \times (m - \ell \left\lfloor \frac{m}{\ell} \right\rfloor) \) sized identity matrix.)

If we solve (1) with each of the pairs, then we would retrieve the optimal \( \hat{A} \) for the problem with \((A, B)\): Solution to the problem using \((A^1, B^1)\) would be the first \( \ell \) columns of the solution to the problem using \((A, B)\). The reason for splitting the problem up like this is because the gradient descent step (Step 17 of projImpute) scales quadratically with \( m \), while if we fix \( \ell \), then solving the problem this way would let the algorithm scale linearly in \( m \).

This would not change the asymptotic computational complexity of the algorithm (as the step to calculate \( A \) dominates, and in that step both \( n \) and \( m \) have linear dependence), but it provides a considerable speedup even for large problems as most often the gradient descent takes the bulk of the time.

The full algorithm for fastImpute is provided below: Here \( \hat{A}[k\ell : \min(k(\ell + 1) - 1, m)] \) means the \( k\ell \)th to \( \min(k(\ell + 1) - 1, m) \)th column of \( \hat{A} \). Experimentally we found that \( \ell = 1000 \) works the best, but the algorithm is not very sensitive to this parameter, and \( \ell \in (300, 2000) \) all works well.

6.1 Computational Complexity of fastImpute

We arrive at the following result for computational complexity:
Algorithm 4 Gradient Descent algorithm for matrix completion without side information.

\begin{algorithm}
\begin{algorithmic}[1]
\Procedure{fastImpute}{$A, B = I_m, k$} \Comment{$A \in \mathbb{R}^{n \times m}$ the masked matrix, $k$ the desired rank}
\State $\ell \leftarrow 1000$
\State $A^1, A^2, \ldots, A^{\left\lceil \frac{m}{\ell} \right\rceil} \leftarrow A$ \Comment{Split $A$ into $\left\lceil \frac{m}{\ell} \right\rceil$ matrices of size $\mathbb{R}^{n \times \ell}$}
\State $i \leftarrow 1$ \Comment{Initialize counter}
\State $\hat{A} \leftarrow 0^{n \times m}$ \Comment{Initialize final output matrix}
\For{$i \leq \left\lceil \frac{m}{\ell} \right\rceil$}
\State $\hat{A}[k\ell: \min(k(\ell + 1) - 1, m)] \leftarrow \text{fastImpute-S}(A^i, I_\ell, k)$ \Comment{We slot the $i$th batch’s solution to the correct columns $i \leftarrow i + 1$}
\EndFor
\State \Return $\hat{A}$
\EndProcedure
\end{algorithmic}
\end{algorithm}

Corollary 3. The asymptotic computational complexity of Algorithm (4) is

$$O \left( |\Omega| \left( k^2 + \frac{k}{\alpha} + \frac{k^3}{m\alpha} \right) \right).$$

\textit{Proof.} The proof follows exactly that of Theorem 3 and thus it is omitted. The improvement in asymptotic complexity comes as do not need to count the multiplications done on $B$ since it is the identity matrix. \hfill \Box

Now the theoretical complexity limit of general matrix completion is $O(|\Omega|(\frac{k}{\alpha}))$, so we only differ from the optimal limit with roughly a $O(|\Omega|k^2)$ factor (the $\frac{k^3}{m\alpha}$ is almost always dominated by $k^2$ as $m \gg k$). This favorable computational complexity would also reflect in the experiments we conduct below.

7. Experiments on fastImpute

In this section, we would compare fastImpute with multiple matrix completion algorithms on both synthetic datasets and real-world datasets to explore its performance and scaling behavior.

7.1 Synthetic Data Experiments

For synthetic data experiments, we assume that the underlying matrix satisfies the form $A = UV$, where $U \in \mathbb{R}^{n \times k}$, $V \in \mathbb{R}^{k \times m}$. Then the elements of $U$ and $V$ are selected from a uniform distribution of $[0, 1]$, where a fraction $\mu$ is missing. We report statistics on various combinations of $(m, n, k, \mu)$. 
The algorithms tested are:

- **fastImpute**: We use the default parameters as defined in the algorithm above, with the regularization parameter $\gamma = 10^6$. We explicitly stress here that no parameter tuning is done on fastImpute as we intend to show that the algorithm is not parameter sensitive. We implement our algorithm in Julia 0.6 with only the base packages.

- **softImpute-ALS (SIALS)**: Developed by Hastie et al. (2015), this is widely recognized as a state-of-the-art matrix completion method without feature information. It has among the best scaling behavior across all classes of matrix completion algorithms as it utilizes fast alternating least squares to achieve scalability. For each combination of $(m, n, \mu, k)$, we tune the regularization parameter $\lambda$ by imputing random matrices with such combination and find the $\lambda$ that gives the best results. We utilize the implementation in the softImpute package in R for testing.

- **softImpute-SVD (SISVD)**: Developed by Mazumder et al. (2010), this is the original softImpute algorithm that utilizes truncated SVDs and spectral regularization to impute the matrix. This method is used as a fast benchmark for SVD-type methods. For each combination of $(m, n, \mu, k)$, we tune the regularization parameter $\lambda$ by imputing random matrices with such combination and find the $\lambda$ that gives the best results. We utilize the implementation in the softImpute package in R for testing.

- **Matrix Factorization Stochastic Gradient Descent (MFSGD)**: This is a popular stochastic gradient descent algorithm (discussed in Jin et al. (2016)) which separates $A = UV$, where $U \in \mathbb{R}^{n \times k}$ and $V \in \mathbb{R}^{k \times m}$, and perform gradient updates for $U$ and $V$. We utilize the implementation in the Fancyimpute package of python that utilizes Tensorflow and the latest available speed optimizations for such algorithm.

All of the algorithms are executed on a server with 16 CPU cores. Each combination $(m, n, k, \mu)$ was ran 10 times, and we report the average value of every statistic. The statistics reported are as followed:

- **$n, m$** - the dimensions of $A$.
- **$k$** - the true number of features.
- **$\mu$** - The fraction of missing entries in $A$.
- **$T$** - the total time of algorithm execution.
- **MAPE** - the Mean Absolute Percentage Error (MAPE) for the retrieved matrix $\hat{A}$:

$$
MAPE = \frac{1}{nm} \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{|\hat{A}_{ij} - A_{ij}|}{A_{ij}}.
$$
The results are separated into sections below. The first four sections investigate fastImpute’s scalability with respect to each of the 4 parameters \( m, n, \mu, k \), with the parameter under investigation denoted in the leftmost column. The final section of the results compares the different algorithms’ performance on large realistic combinations of \((m, n, p, k)\).

Table 3: Comparison of fastImpute, SIALS, SISVD, and MFSGD on synthetic data. \( N/A \) means the algorithm did not complete running in 20 hours, corresponding to 72000 seconds.

We see that on the final set of large realistic combinations, fastImpute outperforms all comparison algorithms in all cases. The following table records the average difference in time and MAPE between fastImpute and the other algorithms, on the final set of combinations:
Table 4: Average performance of fastImpute, SIALS, SISVD, and MFSGD on synthetic trials. Percentages are computed by averaging over the set of realistic combinations.

|          | SIALS | SISVD | MFSGD |
|----------|-------|-------|-------|
| fastImpute vs. | ΔT  | ΔMAPE | ΔT  | ΔMAPE | ΔT  | ΔMAPE |
|          | −75% | −71%  | −99% | −84%  | −88% | −45%  |

On average fastImpute takes 20% of the time of comparison while achieving ~ 40 – 70% reduction in MAPE at the same time.

For scaling behavior, we have the following observations:

- **n** - We see that fastImpute scales as $O(n^{1/2})$ for low $n$ as the gradient descent step dominates, and as we move to $n \sim 10^6$ it starts to scale as $O(n)$ as the step of completing the final matrix starts to dominate. The MAPE steadily decreases as we have more entries. In contrast, SIALS, SISVD, and MFSGD all roughly scale linearly with $n$ from the start. Interestingly, the MFSGD algorithm has increasing error with increasing number of entries - we hypothesize this may be due to the gradient descent in factorized form $A = UV$ failing to capture non-linear dynamics of the interactions between $U$ and $V$ at high levels.

- **m** - From the derivation we know that fastImpute scales as $O(m)$ for $n > m$. For $n < m$, we transpose the matrix before conducting fastImpute, as the algorithm is not symmetric (and is biased to be faster in $n$ rather than $m$). SIALS, SISVD, and MFSGD all roughly scale linearly with $n$ from the start.

- **k** - Somewhat surprisingly, fastImpute scales as $O(k)$ even though theoretically it scales at $O(k^3)$. We believe this is due to the small constant factor in front of the $k^2$ and $k^3$ terms.

- **µ** - We see that in accordance to the linear dependence on $|\Omega|$, the number of known entries, fastImpute runs slower as we have more filled elements, while in contrast SIALS and SISVD both run faster. MFSGD is similar to fastImpute in that it runs slower with more filled elements (a construct of the gradient descent method), and all of the comparison methods seem to match fastImpute at low $\mu$. This is due to the high accuracy of the top-k SVD approximation used as warmstarts across all methods - when most of the entries are filled, this is an extremely well approximation.

### 7.2 Real-World Experiments

For real-world experiments, we would again utilize the Netflix datasets created in Section 5.2 without the feature matrix.
Fast Exact Matrix Completion

We test against SIALS as it is the only algorithm capable of scaling to such size. We split the training set in 80%/20%, where the latter group is used for validation of the rank in SIALS and fastImpute. We then report the time taken, $T$, the MAPE, and the optimal chosen rank $k^*$ for each algorithm:

| $n$  | $m$  | $\mu\%$ | fastImpute | SIALS |
|------|------|----------|------------|-------|
|      |      |          | $T$ | $k^*$ | MAPE | $T$ | $k^*$ | MAPE |
| 3,923| 103  | 92.6%    | 8s  | 5    | 23.5% | 4s  | 5    | 30.6% |
| 18,227| 323  | 94.8%    | 49s | 8    | 19.8% | 47s | 7    | 27.5% |
| 96,601| 788  | 94.2%    | 312s| 10   | 18.2% | 620s| 10   | 24.0% |
| 471,268| 1,760| 93.6%    | 940s| 10   | 16.5% | 2837s| 12   | 22.5% |
| 471,268| 14,538| 94.1%   | 4708s| 12   | 13.8% | 38256s| 14   | 20.1% |

Table 5: Comparison of methods on Netflix data for fastImpute

We see that fastImpute is able to outperform SIALS on the Netflix dataset across the different $n$ and $m$ values, while enjoying superior scalability especially as we approach the full matrix.

8. Conclusion

In conclusion, we have designed a unified optimization framework that is able to conduct state-of-the-art matrix completion with and without side information. Using the factorization approach $A = USB$, we wrote $U = f(S)$ as a function of $S$, and derived the cost and gradient expressions with respect to $S$ through a separable reformulation of the problem. By then conducting non-convex gradient descent on $S$, our synthetic and real-world data experiments show the competitiveness of the method in both scalability and accuracy against a multitude of comparison algorithms.
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9. Appendix

9.1 Appendix A: Reference Feature List

- 24 Indicator Variables for Genres: Action, Adventure, Animation, Biography, Comedy, Crime, Documentary, Drama, Family, Fantasy, Film Noir, History, Horror, Music, Musical, Mystery, Romance, Sci-Fi, Short, Sport, Superhero, Thriller, War, Western
- 5 Indicator Variables for Release Date: Within last 10 years, Between 10-20 years, Between 20-30 years, Between 30-40 years, Between 40-50 Years
- 6 Indicator Variables for Top Actors/Actresses defined by their Influence Score at time of release: Top 100 Actors, Top 100 Actresses, Top 250 Actors, Top 250 Actresses, Top 1000 Actors, Top 1000 Actresses
- IMDB Rating
- Number of Reviews
- Total Production Budget
- Total Runtime
- Total Box Office Revenue
• Indicator Variable for whether it is US produced
• 11 Indicator Variables for Month of Year Released (January removed to prevent multicollinearity)
• Number of Original Music Score
• Number of Male Actors
• Number of Female Factors
• 3 Indicator Variables for Film Language: English, French, Japanese
• Constant