OPERATOR-VALUED FOURIER MULTIPLIERS IN BESOV SPACES AND ITS APPLICATIONS

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Abstract. The present paper is devoted to investigation of operator-valued Fourier multiplier theorems from $B_{q_1}^{s,r}$ to $B_{q_2}^{s,r}$, optimal embedding of Besov spaces, the separability and positivity of differential operators. Here, we show that these differential operators generate analytic semigroup.

1. Introduction, notations and background

In recent years, Fourier multiplier theorems in vector-valued function spaces have found many applications in embedding theorems of abstract function spaces and in theory of differential operator equations, especially in maximal regularity of parabolic and elliptic differential-operator equations. Operator-valued multiplier theorems in Banach-valued function spaces have been discussed extensively in [3,8,12,15,17]. Boundary value problems (BVPs) for differential-operator equations (DOEs) in $H$-valued (Hilbert valued) function spaces and parabolic type convolution operator equations (COEs) with bounded operator coefficient have been studied in [1,2,6,7,9,13,14], and [3] respectively.

Let $E$ be a Banach space, $x = (x_1, x_2, \ldots, x_n) \in \Omega \subset R^n$. $L_r(\Omega; E)$ denotes the space of all strongly measurable $E$-valued functions that are defined on the measurable subset $\Omega \subset R^n$, equipped with its usual topology generated by semi-norms. $L_\infty(\Omega; E)$ denotes the space of all continuous linear operators $L : S \to E$, equipped with the bounded convergence topology. Recall $S(R^n; E)$ is norm dense in $B_{p,r}^{s}(R^n; E)$ when $1 \leq p, r < \infty$.

Let $S = S(R^n; E)$ denote a Schwartz class i.e. the space of $E$-valued rapidly decreasing smooth functions on $R^n$, equipped with its usual topology generated by semi-norms. Let $S'(R^n; E)$ denote the space of all continuous linear operators $L : S \to E$, equipped with the bounded convergence topology. Recall $S'(R^n; E)$ is norm dense in $B_{p,r}^{s}(R^n; E)$ when $1 \leq p, r < \infty$.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\alpha_i$ are integers. An $E$-valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwartz distributions, if the equality

$$<D^\alpha f, \varphi> = (-1)^{|\alpha|} <f, D^\alpha \varphi>$$

holds for all $\varphi \in S$.

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The Fourier transform $F : S(X) \to S(X)$ is defined by

$$(Ff)(t) = \hat{f}(t) = \int_{R^N} \exp(-its)f(s)ds$$

is an isomorphism whose inverse is given by

$$(F^{-1}f)(t) = \check{f}(t) = (2\pi)^{-N} \int_{R^N} \exp(its)f(s)ds,$$

where $f \in S(X)$ and $t \in R^N$.

It is known that

$$F(D_{x}^{\alpha} f) = (i\xi_1)^{\alpha_1} \cdots (i\xi_n)^{\alpha_n} \hat{f},$$
$$D_{\xi}^{\alpha}(F(f)) = F((-ix_1)^{\alpha_1} \cdots (-ix_n)^{\alpha_n} f)$$

for all $f \in S'(R^n; E)$.

Let $\mathbb{C}$ be the set of complex numbers and

$$S_{\varphi} = \{\lambda; \lambda \in \mathbb{C}, |\arg \lambda| \leq \varphi \} \cup \{0\}, 0 \leq \varphi < \pi.$$

A linear operator $A$ is said to be a $\varphi$–positive in a Banach space $E$, if $D(A)$ dense in $E$, and

$$\|(A + \lambda I)^{-1}\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$$

with $M > 0, \lambda \in S_{\varphi}, \varphi \in [0, \pi)$; here $I$ is the identity operator in $E$, $B(E)$ is the space of all bounded linear operators in $E$. Sometimes instead of $A + \lambda I$, we will write $A + \lambda$ and denote it by $A_{\lambda}$. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^{p} + \|A^\theta u\|^{p}\right)^{\frac{1}{p}}, 1 \leq p < \infty, -\infty < \theta < \infty.$$

**Definition 1.** Let $A = A(t)$ be a uniformly positive operator in $E$, $u \in E(A)$ and $Au \in S(R; E)$. Then, the Fourier transformation of $A(t)$ is defined by

$$((FA)u)(\xi) = (Au)(\xi) = (2\pi)^{-\frac{1}{2}} \int_{R} e^{-it\xi} A(t)u \, dt.$$  

**Definition 2.** Let $A = A(t)$ be a uniformly positive operator in $E$. Then, it is differentiable if for all $u \in E(A)$

$$\left(\frac{d}{dt}A\right)u = A'(t)u = \lim_{h \to 0} \frac{\|A(t+h)u - A(t)u\|_{E}}{h} < \infty.$$

**Definition 3.** Let $A = A(t)$ be a uniformly positive operator in $E$ and $u \in S(R; E(A))$ and

$$(A \ast u)(t) = \int_{R} A(t-y)u(y) \, dy.$$
Let $y \in R$, $m \in N$ and $e_i$, $i = 1, 2, \cdots, n$ be standard unit vectors of $R^n$. Let
\[
\Delta_i(y)f(x) = f(x + ye_i) - f(x),
\]
\[
\Delta^m_i(y)f(x) = \Delta_i(y)\left[\Delta^{m-1}_i(y)f(x)\right] = \sum_{k=0}^{m}(-1)^{m+k}C_{m,k}f(x + kye_i).
\]
Let
\[
\Delta_i(\Omega, y) = \begin{cases} \Delta_i(y)f(x), & \text{for } [x, x + mye_i] \subset \Omega \\ 0, & \text{for } [x, x + mye_i] \notin \Omega. \end{cases}
\]
Let $m$ be integer, $s$ be positive number, and
\[
m > s, 1 \leq p \leq \infty, 1 \leq q \leq \infty, y_0 > 0.
\]
The space $B^s_{p,q}(\Omega; E)$ is $E$–valued Besov space, i.e.,
\[
B^s_{p,q}(\Omega; E) = \left\{ f : f \in L_p(\Omega; E), \|f\|_{B^s_{p,q}(\Omega; E)}^q \right\},
\]
\[
\|f\|_{B^s_{p,q}(\Omega; E)}^q = \sum_{i=1}^{n} \left( \int_{0}^{y_0} \gamma^{(sq+1)} \|\Delta^m_i(y, \Omega)f(x)\|_{L_p(\Omega; E)}^q dy \right)^{\frac{1}{q}} < \infty.
\]
Let,
\[
D^\alpha = D_1^{\alpha_1}D_2^{\alpha_2}\cdots D_n^{\alpha_n}, D_k^i = \left( \frac{\partial}{\partial x_k} \right)^i.
\]
and
\[
B^s_{p,q}(\Omega; E_0, E) = \{ u : u \in B^s_{p,q}(\Omega; E_0), D_k^iu \in B^s_{p,q}(\Omega; E), k = 1, 2, \ldots, n \},
\]
\[
\|u\|_{B^s_{p,q}(\Omega; E_0, E)} = \|u\|_{B^s_{p,q}(\Omega; E_0)} + \sum_{k=1}^{n} \|D_k^iu\|_{B^s_{p,q}(\Omega; E)} < \infty.
\]
$C^{(m)}(\Omega; E)$ denotes the space of $E$–valued uniformly bounded and $m$–times continuously differentiable functions on $\Omega$.

[7, Lemma 2.3]. Let $\lambda \in S_\varphi$ and $\mu \in S_\psi$, where $\varphi + \psi < \pi$. Then there exist $M > 0$ such that $|\lambda + \mu| \geq M \left( |\lambda| + |\mu| \right)$.

2. Youngs type Fourier multipliers

Here, we shall study Fourier multiplier theorems from $B^s_{q_1, r}$ to $B^s_{q_2, r}$ for $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{r}$ and $1 < q_1 < q' \leq \infty$ in the highlight of [10]. In this section, $X$ and $Y$ are Banach spaces over the field $C$ and $X^*$ is the dual space of $X$. The space $B(X, Y)$ of bounded linear operators from $X$ to $Y$ is endowed with the usual uniform operator topology. $N_0$ is the set of natural numbers containing zero.

All the basic properties of $F$ and $F^{-1}$ that hold in the scalar–valued case also hold in vector–valued case; however, the Housdorff–Young inequality need not hold. Therefore, we need to define Banach spaces that was introduced by Peetre [11].
Definition 2.1. Let $X$ be a Banach space and $1 \leq p \leq 2$. We say $X$ has Fourier type $p$ if
\[
\|Ff\|_{L^p_p(R^n,X)} \leq C\|f\|_{L^p_p(R^n,X)} \text{ for each } f \in S(R^n, X),
\]
where $\frac{1}{p} + \frac{1}{p'} = 1$, $F_{p,N}(X)$ is the smallest $C \in [0, \infty]$.

[10. Proposition 2.3] Let $X$ be a Banach space with Fourier type $p \in [1, 2]$ and $p \leq q \leq p'$. Then $X^*$ and $L_q(R^n, X)$ also have Fourier type provided both are with the same constant $F_{p,N}(X)$.

We shall use Fourier analytic definition of Besov spaces in this section. Therefore, we need to consider some subsets $\{J_k\}_{k=0}^\infty$ and $\{I_k\}_{k=0}^\infty$ of $R^n$, where
\[
J_0 = \{ t \in R^n : |t| \leq 1 \}, \quad J_k = \{ t \in R^n : 2^{k-1} \leq |t| \leq 2^k \} \quad \text{for } k \in N
\]
and
\[
I_0 = \{ t \in R^n : |t| \leq 2 \}, \quad I_k = \{ t \in R^n : 2^{k-1} \leq |t| \leq 2^{k+1} \} \quad \text{for } k \in N.
\]
Next, we define the unity $\{ \varphi_k \}_{k \in N_0}$ of functions from $S(R^n, R)$. Let $\psi \in S(R, R)$ be nonnegative function with support in $[2^{-1}, 2]$, which satisfies
\[
\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \quad \text{for } s \in R \setminus \{0\}
\]
and
\[
\varphi_k(t) = \psi(2^{-k}|t|), \quad \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \quad \text{for } t \in R^n.
\]
Later, we will need the following useful properties:
\quad \supp \varphi_k \subset I_k \text{ for each } k \in N_0,
\quad \varphi_k \equiv 0 \text{ for each } k < 0,
\quad \sum_{k=0}^{\infty} \varphi_k(s) \text{ for each } s \in R^n,
\quad J_m \cap \supp \varphi_k = \emptyset \text{ if } |m - k| > 1,
\quad \varphi_{k-1}(s) + \varphi_k(s) + \varphi_{k+1}(s) = 1 \quad \text{for each } s \in \supp \varphi_k, k \in N_0.

Definition 2.3. Let $1 \leq q \leq r \leq \infty$ and $s \in R$. The Besov space is the set of all functions $f \in S'(R^n, X)$ for which
\[
\|f\|_{B^s_{r,q}(R^n,X)} : = \left\{ 2^{ks} \left\{ \|\varphi_k * f\|_{L^r_q(R^n,X)} \right\}_{k=0}^\infty \right\}_{r \neq \infty}^* + \sup_{k \in N_0} \left[ 2^{ks} \|\varphi_k * f\|_{L^r_q(R^n,X)} \right] \quad \text{if } r = \infty
\]
is finite; here $q$ and $s$ are main and smoothness indexes respectively.

To prove multiplier theorems, we will later need following results.
[10, **Theorem 3.1**] Let $X$ be a Banach space with the Fourier type $p \in [1, 2]$. Let $1 \leq q < p'$, $1 \leq r \leq \infty$ and $s \geq N \left( \frac{1}{q'} - \frac{1}{p'} \right)$. Then, there exists a constant $C$ depending only on $F_{p,N}(X)$, so that if $f \in B_{p,r}^{s}(R^{N}, X)$,
\[
\left\| \left\{ \hat{f} \chi_{m} \right\}_{k=0}^{\infty} \right\|_{L_{q}(R^{N}, X)} \leq C \| f \|_{B_{p,r}^{s}(R^{N}, X)}.
\]

Note that [10, **Theorem 3.1**] remains valid if Fourier transform is replaced by the inverse Fourier transform. Choosing $r = 1$ and $s = N \left( \frac{1}{q} - \frac{1}{p'} \right)$ we obtain the following corollary.

**Corollary 2.5.** Let $X$ be a Banach space having Fourier type $p \in [1, 2]$ and $1 \leq q < p'$. Then the Fourier transform defines bounded operator
\[
F : B_{p,1}^{N} \left( \frac{1}{q} - \frac{1}{p'} \right)(R^{N}, X) \to L_{q}(R^{N}, X).
\]

For a bounded measurable function $m : R^{N} \to B(X, Y)$, its corresponding Fourier multiplier operator $T_m$ is defined as follows
\[
T_m(f) = F^{-1}[m(\cdot)(Ff)(\cdot)].
\]

In this section, we identify conditions on $m$, extending those of [10], that
\[
\|T_m f\|_{B_{q,r}^{s}} \leq C \| f \|_{B_{q,r}^{s}} \text{ for each } f \in S(X).
\]

**Definition 2.6.** Let \( \left( E_1(R^N, X), E_2(R^N, Y) \right) \) be one of the following systems, where $1 \leq q_1, q_2, \, r \leq \infty$ and $s \in \mathbb{R}$
\[
(L_{q_1}(X), L_{q_2}(Y)) \text{ or } (B_{q_1,r}^{s}(X), B_{q_2,r}^{s}(Y)).
\]
A bounded measurable function $m : R^{N} \to B(X, Y)$ is called a Fourier multiplier from $E_1(X)$ to $E_2(Y)$ if there is a bounded linear operator
\[
T_m : E_1(X) \to E_2(Y)
\]
such that
\[
T_m(f) = F^{-1}[m(\cdot)(Ff)(\cdot)] \text{ for each } f \in S(X),
\]
\[
T_m \text{ is } \sigma(E_1(X), E_1^{\ast}(X^{\ast})) \text{ to } \sigma(E_2(Y), E_2^{\ast}(Y^{\ast})) \text{ continuous. (3)}
\]
The uniquely determined operator $T_m$ is the Fourier multiplier operator induced by $m$.

**Remark 2.7.** If $T_m \in B(E_1(X), E_2(Y))$ and $T_m^{\ast}$ maps $E_2^{\ast}(Y^{\ast})$ into $E_1^{\ast}(X^{\ast})$ then $T_m$ satisfies the continuity condition (3).

**Lemma 2.8.** Let $k \in L_{q_1}(R^N, B(X, Y))$, $\frac{1}{q_1} = \frac{1}{q} - \frac{1}{q'}$ and $\frac{1}{q} + \frac{1}{q'} = 1$ for $1 < q_1 < q' \leq \infty$. Assume there exist $C_0$ so that for all $x \in X$ and $y^{\ast} \in Y^{\ast}$
\[
\|k x\|_{L_{q_1}(Y)} \leq C_0 \| x \|_{X} \text{ (4)}
\]
and $C_1$ so that
\[
\|k^{\ast} y^{\ast}\|_{L_{q_1}(X^{\ast})} \leq C_1 \| y^{\ast} \|_{Y^{\ast}} \text{ (5)}
\]
Then the convolution operator
\[
K : L_{q_1}(R^N, X) \to L_{q_2}(R^N, Y)
\]
defined by
\[(Kf)(t) = \int_{\mathbb{R}^N} k(t-s)f(s)ds \quad \text{for} \quad t \in \mathbb{R}^N\]
satisfies \(\|K\|_{L_{q_1} \to L_{q_2}} \leq C\).

**Proof.** Taking into account that \(1 \leq \eta\) and by using general Minkowski-Jessen inequality and (4) we obtain
\[
\|(Kf)(t)\|_{L^\eta(Y)} \leq \left[ \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \|k(t-s)f(s)\|^\eta_Y ds \right)^{\frac{1}{\eta}} dt \right]
\leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \|k(t-s)f(s)\|_{Y}^\eta dt \right)^{\frac{1}{\eta}} ds
\leq \int_{\mathbb{R}^N} \|kf(s)\|_{L^\eta(Y)} ds
\leq C_0 \int_{\mathbb{R}^N} \|f(s)\|_X ds \leq C_0 \|f\|_{L^1(X)}.
\]
for all \(f \in L^1(X)\). Thus, \(\|K\|_{L_{1} \to L_{\eta}} \leq C_0\). Now, assume \(f \in L_{\eta'}(X)\), \(y^* \in Y^*\) and \(t \in \mathbb{R}^N\). Then, by using Hölder inequality and (5) we get
\[
|\langle y^*, (Kf)(t) \rangle_Y | \leq \int_{\mathbb{R}^N} |k(t-s)^*y^*, f(s) >_X | ds
\leq \int_{\mathbb{R}^N} |k(t-s)^*y^*f(s)| ds
\leq \|k^*y^*\|_{L^\eta(X^*)} \|f(s)\|_{L_{\eta'}(X)}
\leq C_1 \|y^*\| \|f\|_{L_{\eta'}(X)}.
\]
Hence, \(\|K\|_{L_{\eta'} \to L^\infty} \leq C_1\). Now, Riesz-Thorin theorem implies that
\[\|K\|_{L_{q_1} \to L_{q_2}} \leq C\]
for
\[\frac{1}{q_1} = 1 - \theta + \frac{\theta}{\eta'} \quad \text{and} \quad \frac{1}{q_2} = 1 - \frac{\theta}{\eta}
\]
where \(0 < \theta < 1\). Solving these equation we obtain
\[\|K\|_{L_{q_1} \to L_{q_2}} \leq C\]
for
\[\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta'}.
\]
Remark 2.9. Let us note that if we choose \( \eta' = \infty \) in Lemma 2.8, we obtain Lemma 4.5 of [10].

**Theorem 2.10.** Let \( X \) and \( Y \) be Banach spaces having Fourier type \( p \in [1, 2] \). Then there is a constant \( C \) depending only on \( F_{p,N}(X) \) and \( F_{p,N}(Y) \) so that if

\[
m \in B_{p,1}^N \left( \frac{1}{p} - \frac{1}{\eta}, (R^N, B(X, Y)) \right)
\]

then \( m \) is a Fourier multiplier from \( L_{q_1}(R^N, X) \) to \( L_{q_2}(R^N, Y) \) with

\[
\| T_m \|_{L_{q_1}(R^N, X) \to L_{q_2}(R^N, Y)} \leq CM_{\eta, \eta}(m) \quad \text{for each}
\]

where \( \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta}, \frac{1}{p} + \frac{1}{\eta} = 1, 1 < q_1 < \eta' \leq \infty \) and

\[
M_{\eta, \eta}(m) = \inf \left\{ \| m(a \cdot) \|_{B_{p,1}^N \left( \frac{1}{p} - \frac{1}{\eta}, (R^N, B(X, Y)) \right)} : \ a > 0 \right\}
\]

**Proof.** The key point in this proof is the fact (1). As in the proof of [10,Theorem 4.3] we assume in addition that \( m \in S(B(X, Y)) \). Hence, \( \tilde{m} \in S(B(X, Y)) \). Since, \( F^{-1} [m(a \cdot) x] (s) = a^{-N} \tilde{m} \left( \frac{s}{a} \right) x \) choosing an appropriate \( a \) and using (1) we obtain

\[
\| m(\cdot) x \|_{L_p(Y)} = \| [m(a \cdot) x] \|_{L_p(Y)} \leq C_1 \| m(a \cdot) \|_{B_{p,1}^N \left( \frac{1}{p} - \frac{1}{\eta}, (R^N, B(X, Y)) \right)} \| x \|_X
\]

\[
\leq 2C_1 M_{\eta, \eta}(m) \| x \|_X
\]

where \( C_1 \) depends only on \( F_{p,N}(Y) \). If \( m \in S(B(X, Y)) \) then \( [m(\cdot)]^* \in S(B(Y^*, X^*)) \) and \( M_{\eta, \eta}(m) = M_{\eta, \eta}(m^*) \). Thus, in a similar manner as above, we have

\[
\| m(\cdot) y^* \|_{L_p(Y)} \leq 2C_2 M_{\eta, \eta}(m) \| y^* \|_{Y^*}
\]

for some constant \( C_2 \) depends on \( F_{p,N}(X^*) \). Since, we have

\[
\| m(\cdot) x \|_{L_p(Y)} \leq 2C_1 M_{\eta, \eta}(m) \| x \|_X
\]

\[
\| [m(\cdot)]^* y^* \|_{L_p(Y)} \leq 2C_2 M_{\eta, \eta}(m) \| y^* \|_{Y^*}
\]

by Lemma 2.8 we can conclude

\[
(T_m f)(t) = \int_{R^N} \tilde{m}(t - s) f(s) \, ds
\]

satisfies

\[
\| T_m f \|_{L_{q_1}(R^N, Y)} \leq CM_{\eta, \eta}(m) \| f \|_{L_{q_1}(R^N, X)}
\]

for \( \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta} \) where \( 1 < q_1 < \eta' \leq \infty \). Now, taking into account the fact that \( S(B(X, Y)) \) continuously embedded to \( B_{p,1}^N \left( \frac{1}{p} - \frac{1}{\eta}, (B(X, Y)) \right) \) and using the same reasoning as in the proof of [10,Theorem 4.3] one can easily prove for the general case \( m \in B_{p,1}^N \left( \frac{1}{p} - \frac{1}{\eta}, (B(X, Y)) \right) \) and that \( T_m \) satisfies \((2), (3)\).

Let us note that choosing \( \eta = 1 \) in the Theorem 2.10, we obtain [10,Theorem 4.3]. The next theorem is the extension of [10,Theorem 4.8].
**Theorem 2.11.** Let $X$ and $Y$ be Banach spaces having Fourier type $p \in [1, 2]$ and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta}, \frac{1}{\eta} + \frac{1}{q_1} = 1$ for $1 < q_1 < \eta' \leq \infty$. Then, there exist a constant $C$ depending only on $F_{p,N}(X)$ and $F_{p,N}(Y)$ so that if $m : R^N \rightarrow B(X,Y)$ satisfy

$$\varphi_k \cdot m \in B_{p,1}^N \left( R^N, B(X,Y) \right) \text{ and } M_{p,\eta}(\varphi_k \cdot m) \leq A$$

(7)

then $m$ is Fourier multiplier from $B_{q_1,r}^s(R^N, X)$ to $B_{q_2,r}^s(R^N, Y)$ and $\|m\| \leq CA$ for each $s \in R$ and $r \in [1, \infty]$.

**Proof.** Since $\varphi_k \cdot m \in B_{p,1}^N \left( R^N, B(X,Y) \right)$, Theorem 2.10 ensures that

$$\|T_{m\varphi_k}f\|_{L_{q_2}(R^N)} \leq C M_{p,\eta}(\varphi_k \cdot m) \|f\|_{L_{q_1}(R^N, X)} \leq CA \|f\|_{L_{q_1}(R^N, X)}.$$ 

In the introduction we defined function $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$ that is equal to 1 on supp$\varphi_k$. Thus,

$$\|T_{m\psi_k}f\|_{L_{q_2}(R^N)} \leq \|T_{m\varphi_{k-1}}f\|_{L_{q_2}(R^N)} + \|T_{m\varphi_k}f\|_{L_{q_2}(R^N)} + \|T_{m\varphi_{k+1}}f\|_{L_{q_2}(R^N)} \leq 3CA \|f\|_{L_{q_1}(R^N, X)}.$$ 

Let $T_0 : S(X) \rightarrow S'(Y)$ be defined as $T_0f = F^{-1} [m(\cdot)(Ff)(\cdot)].$ From the proof of [10, Theorem 4.3] we know that

$$\varphi_k * T_0f = T_{m\psi_k}(\varphi_k * f).$$

Hence,

$$\|\varphi_k * T_0f\|_{L_{q_2}(Y)} = \|T_{m\psi_k}(\varphi_k * f)\|_{L_{q_2}(Y)} \leq 3CA \|f\|_{L_{q_1}(R^N, X)}$$

and

$$\sum_{k=0}^{\infty} 2^{kqs} \|\varphi_k * T_0f\|_{L_{q_2}(Y)} \leq 3CA \sum_{k=0}^{\infty} 2^{kqs} \|f\|_{L_{q_1}(R^N, X)}.$$ 

Thus, we obtain that

$$\|T_0f\|_{B_{q_2,r}^s(R^N, Y)} \leq 3CA \|f\|_{B_{q_1,r}^s(R^N, X)},$$ 

for $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta}$ where $1 < q_1 < \eta' \leq \infty$. If $q, r < \infty$ then $\dot{B}_{q,r}^s = B_{q,r}^s$. Therefore, it remains only to show the weak continuity condition (3). Proof of the case $r = \infty$ and the weak continuity condition (3) follows trivially from the proof of [10, Theorem 4.3].

To establish maximal regularity of DOEs and abstract embeddings we shall use Theorem 2.11 in the next sections. However, it is not easy to check assumptions of the Theorem 2.11 for multiplier functions. Therefore, we prove a lemma that makes Theorem 2.11 more applicable.
Lemma 2.12. Let $N \left( \frac{1}{q} - \frac{1}{p} \right) < l \in \mathbb{N}$ and $u \in [p, \infty]$. If $m \in C^l(R^N, B(X, Y))$ with

$$
\|D^\alpha m|_{L_\infty(B(X,Y))}\leq A, \|D^\alpha m_k|_{L_\infty(B(X,Y))}\leq A,
m_k(\cdot) = m(2^{k-1}),
$$

for each $\alpha \in N_0^N$, $|\alpha| \leq l$ and $k \in \mathbb{N}$ then $m$ satisfies conditions of Theorem 2.11.

Proof. Using the fact $W^l_p(R^N, B(X, Y)) \subset B^{N\left( \frac{1}{q} - \frac{1}{p} \right)}_{p,1}(R^N, B(X, Y))$ for $N \left( \frac{1}{q} - \frac{1}{p} \right) < l$ (see [2]) one can prove this lemma in a similar manner as [10, Lemma 4.10].

The following corollary shows that classical Hörmander condition implies assumptions (7) of Theorem 2.11.

Corollary 2.13. Let $X$ and $Y$ be Banach spaces having Fourier type $p \in [1,2]$ and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{p}$, $\frac{1}{q} + \frac{1}{r} = 1$ for $1 < q_1 < q' \leq \infty$. If $m \in C^l(R^N, B(X, Y))$ satisfies

$$
\left\| (1 + |t|)^{|\alpha|} D^\alpha m(t) \right\|_{L_\infty(R^N, B(X,Y))} \leq A \tag{8}
$$

for each multi-index $\alpha$ with $|\alpha| \leq l = \left\lceil N \left( \frac{1}{q} - \frac{1}{p} \right) \right\rceil + 1$ then $m$ is Fourier multiplier from $B^{q_1,r}(R^N, X)$ to $B^{q_2,r}(R^N, Y)$ for each $s \in R$ and $r \in [1, \infty]$.

Proof. The proof follows from [10, Corollary 4.11]. Actually, choosing $u = \infty$ in the Lemma 2.12 we get assertions of Corollary.

Remark 2.14. It is known that, any Banach space has Fourier type 1. Therefore, in the Corollary 2.13 if $X$ and $Y$ are arbitrary Banach spaces then $l = \left\lceil N \left( \frac{1}{q} - \frac{1}{p} \right) \right\rceil + 1$.

The next corollary shows that classical Hörmander condition implies assumptions (7) of Theorem 2.11.

Corollary 2.15. Assume $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{p}$ for $\frac{1}{q} + \frac{1}{r} = 1$ and $1 < q_1 < q' \leq \infty$. Suppose $X$ and $Y$ have Fourier type $p \in [1,2]$ and $l = \left\lceil N \left( \frac{1}{q} - \frac{1}{p} \right) \right\rceil + 1$.

If $m \in C^l(R^N, B(X, Y))$ satisfies

$$
\left\lfloor \int_{|t| \leq 2} \|D^\alpha m(t)\|^\frac{1}{p} \right\rfloor \leq A
$$

and

$$
\left\lfloor R^{-N} \int_{R \leq |t| \leq 4R} \|D^\alpha m(t)\|^\frac{1}{p} \right\rfloor \leq AR^{-|\alpha|}
$$

for each multi-index $\alpha$ with $|\alpha| \leq l$ then $m$ is Fourier multiplier from $B^{q_1,r}(R^N, X)$ to $B^{q_2,r}(R^N, Y)$ for each $s \in R$ and $r \in [1, \infty]$.

Proof. Choosing $u = p$ in the Lemma 2.12 we get assertions of corollary.
In the present section, using Corollary 2.13 we shall prove continuity of the following embedding
\[ D^\alpha : B^{l,s}_{q_1,r} (R^N, E (A)) \subset B^{s}_{q_2,r} (R^N, E). \]

These type of embeddings very often used to establish maximal regularity for differential operator equations see e.g [14], [15].

**Proposition 3.1.** Let \( A \) be a positive operator in Banach space \( E \), \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \) and \( x = \left\{ \alpha \right\} \leq 1 \). Then, operator-function
\[ \Psi (\xi) = (i\xi)^\alpha A^{1-x} [A + \theta (\xi)]^{-1} \]
is uniformly bounded and satisfies
\[ \| \Psi (\xi) \|_{B(E)} \leq C \] (10)
for all \( \xi \in R^N \), where
\[ \theta (\xi) = N \sum_{k=1}^{N} |\xi_k|^l \in S (\varphi). \]

**Proof.** Since \( \theta (\xi) \in S (\varphi) \), for all \( \varphi \in (0, \pi] \) and \( A \) is \( \varphi \)-positive in \( E \), the operator \( A + \theta (\xi) \) is invertible. Let
\[ u = [A + \theta (\xi)]^{-1} f. \] (11)
Then
\[ \| \Psi (\xi) f \|_E \leq \| A^{1-x} u \|_E |\xi_1|^{\alpha_1} \cdots |\xi_N|^{\alpha_N}. \]
Using the Moment inequality for powers of positive operators, we get a constant \( C \) such that
\[ \| \Psi (\xi) f \|_E \leq C \| Au \|^{1-x} \| u \|^{x} |\xi_1|^{\alpha_1} \cdots |\xi_N|^{\alpha_N}. \]
Then, applying Young inequality, which states that \( ab \leq \frac{a^{k_1}}{k_1} + \frac{b^{k_2}}{k_2} \) for any positive real numbers \( a, b \) and \( k_1, k_2 \) with \( \frac{1}{k_1} + \frac{1}{k_2} = 1 \), to the product
\[ \| Au \|^{1-x} \| u \|^{x} |\xi_1|^{\alpha_1} \cdots |\xi_N|^{\alpha_N} \]
we obtain
\[ \| \Psi (\xi) f \|_E \leq C \{ (1 - x) \| Au \| \]
\[ \quad + x \| \xi_1 \|^{\alpha_1} \cdots |\xi_N|^{\alpha_N} \| u \| \} \]
(12)
Since,
\[ \sum_{i=1}^{N} \frac{\alpha_i}{x} = l \]
there exists a constant \( M_0 \) independent of \( \xi \), such that
\[ |\xi_1|^{\frac{x}{\alpha_1}} \cdots |\xi_N|^{\frac{x}{\alpha_N}} \leq M_0 \left( 1 + \sum_{k=1}^{N} |\xi_k|^l \right) \]
for all \( \xi \in \mathbb{R}^N \). Combining above estimate with inequality (12) and using the fact that \( x = \frac{|\alpha|}{l} \leq 1 \), we obtain

\[
\| \psi(\xi)f \| \leq C \left[ \| Au \| + \sum_{k=1}^{N} |\xi_k|^l \| u \| + \| u \| \right].
\]

Then, with the help of (11) we get

\[
\| \psi(\xi)f \| \leq C \left[ \| A \| \| A + \theta(\xi) \|^{-1} f \| + \sum_{k=1}^{N} |\xi_k|^l \| A + \theta(\xi) \|^{-1} f \| + \| A + \theta(\xi) \|^{-1} f \| \right].
\]

Finally, using resolvent properties of positive operator \( A \) we conclude

\[
\| \Psi(\xi)f \|_{E} \leq C_0 \| f \|_{E}
\]

for all \( f \in E \).

**Proposition 3.2.** Let \( A \) be a positive operator in Banach space \( E \), \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \) and \( x = \frac{|\alpha| + \sigma}{l} \leq 1 \) for \( \sigma = \left[ N(1 + \frac{1}{q_2} - \frac{1}{q_1}) \right] + 1 \). Then, operator-function

\[
\Psi(\xi) = |\xi|^{|\alpha|} (i\xi)^{\alpha} A^{1-x} D^\sigma [A + \theta(\xi)]^{-1}
\]

is uniformly bounded.

**Proof.** Since, \( \frac{|\alpha| + \sigma}{l} \leq 1 \) we have

\[
|\xi|^{|\alpha|} |i\xi|^\alpha \leq C \sum_{j=1}^{N} |\xi_j|^{|\alpha_j|} |\xi_1|^{|\alpha_1|} |\xi_2|^{|\alpha_2|} \cdots |\xi_N|^{|\alpha_N|}
\]

\[
\leq C_\alpha \left( 1 + \sum_{k=1}^{N} |\xi_k|^l \right).
\]

Thus, using above estimate and proof of Proposition 3.1 one can easily prove assertion of this theorem.

**Theorem 3.3.** Let \( E \) be a Banach space and \( A \) be a \( \varphi \)-positive operator in \( E \), where \( \varphi \in (0, \pi] \). If \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \) and \( x = \frac{|\alpha| + \sigma}{l} \leq 1 \) for \( \sigma = \left[ N(1 + \frac{1}{q_2} - \frac{1}{q_1}) \right] + 1 \) then the following embedding

\[
D^\alpha : B^{q_1,s}_{q_2,r}(\mathbb{R}^N; E(A), E(1-x)) \subset B^{q_2,s}_{q_2,r}(\mathbb{R}^N; E(A^{1-x}))
\]

is continuous for \( \frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{q_1'}, \frac{1}{\eta} + \frac{1}{\eta'} = 1, 1 < q_1 < \eta' \leq \infty \) and there exists a positive constant \( C \), such that

\[
\| D^\alpha u \|_{B^{q_2,s}_{q_2,r}(\mathbb{R}^N; E(A^{1-x}))} \leq C \| u \|_{B^{q_2,s}_{q_2,r}(\mathbb{R}^N; E(A), E)}
\]

for all \( u \in B^{q_1,s}_{q_1,r}(\mathbb{R}^N; E(A), E) \).
Proof. Since $A$ is constant and closed operator, we have
\[
\| D^{\alpha} u \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E(A^{1-z})))} = \| A^{1-z} D^{\alpha} u \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)} \sim \| F^{-1} (i\xi)^{\alpha} A^{1-z} Fu \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)}.
\]  
(The symbol $\sim$ indicates norm equivalency). In a similar manner, from definition of $B_{q_1,r}^{l,s}(\mathbb{R}^N; E_0, E)$ we have
\[
\| u \|_{B_{q_1,r}^{l,s}(\mathbb{R}^N; E_0, E)} \sim \| Au \|_{B_{q_1,r}^{l,s}(\mathbb{R}^N; E)} + \sum_{k=1}^{N} \| F^{-1} \xi_k u \|_{B_{q_1,r}^{l,s}(\mathbb{R}^N; E)} \].
\]
By virtue of above relations, it is sufficient to prove
\[
\| F^{-1} \left[ (i\xi)^{\alpha} A^{1-z} u \right] \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)} \leq C \left[ \| F^{-1} A u \|_{B^{s,r}_{q_1,r}(\mathbb{R}^N; E)} + \sum_{k=1}^{N} \| F^{-1} \left( \xi_k^l \hat{u} \right) \|_{B^{s,r}_{q_1,r}(\mathbb{R}^N; E)} \right].
\]
Hence, the inequality (13) will be followed if we can prove the following estimate
\[
\| F^{-1} \left[ (i\xi)^{\alpha} A^{1-z} u \right] \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)} \leq C \| F^{-1} \left( [A + I\theta] \hat{u} \right) \|_{B^{s,r}_{q_1,r}(\mathbb{R}^N; E)}
\]
for all $u \in B_{q_1,r}^{l,s}(\mathbb{R}^N; E(A), E)$, where
\[
\theta = \theta(\xi) = \sum_{k=1}^{N} |\xi_k|^l \in S(\varphi).
\]
Let us express the left hand side of (16) as follows
\[
\| F^{-1} \left[ (i\xi)^{\alpha} A^{1-z} u \right] \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)} = \| F^{-1} (i\xi)^{\alpha} A^{1-z} \left( [A + I\theta]^{-1} [A + I\theta] \right) \hat{u} \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)}
\]
(Since $A$ is the positive operator in $E$ and $\theta(\xi) \in S(\varphi)$, $[A + I\theta]^{-1}$ exists ). From Corollary 2.13 we know that
\[
\| F^{-1} (i\xi)^{\alpha} A^{1-z} \left( [A + I\theta]^{-1} [A + I\theta] \right) \hat{u} \|_{B^{s,r}_{q_2,\infty}(\mathbb{R}^N; E)} \leq C \| F^{-1} \left( [A + I\theta] \hat{u} \right) \|_{B^{s,r}_{q_1,r}(\mathbb{R}^N; E)}
\]
holds if $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta}$ and operator-function $(i\xi)^{\alpha} A^{1-z} \left( [A + \theta]^{-1} \right) \hat{u}$ satisfies (8) for each multi-index $\beta$, $|\beta| \leq \left[ \frac{N}{\eta} \right] + 1$. It is clear that
\[
\| (1 + |\xi|) |\beta| D^{\beta} \tilde{\Psi}(\xi) \|_{L^\infty(B(E))} \leq \sum_{k=0}^{N} \| |\xi|^k D^{\beta} \tilde{\Psi}(\xi) \|_{L^\infty(B(E))}
\]
(18)
Therefore, it is enough to show
\[
\| |\xi|^k D^{\beta} \tilde{\Psi}(\xi) \|_{L^\infty(B(E))} \leq C
\]
for $k = 0, 1, \cdots |\beta|$ and $|\beta| \leq \left[ N(1 + \frac{1}{q_2} - \frac{1}{\eta}) \right] + 1 \leq N + 1$. The case $k = 0$ and $|\beta| = 0$ follows from Proposition 3.1 and the principal case $k = |\beta| = \left[ N(1 + \frac{1}{q_2} - \frac{1}{\eta}) \right] + 1$ follows from Proposition 3.2. For the sake of simplicity of calculations, we shall
prove only for the cases $|\beta| = 1$, $k = 0$ and $k = 1$. Taking derivative of operator function $\Psi (\xi)$ with respect to $\xi_1$, we get

$$\left\| \frac{\partial}{\partial \xi_1} \Psi (\xi) \right\|_{L^\infty (B(E))} \leq \|I_1\|_{B(E)} + \|I_2\|_{B(E)} + \|I_3\|_{B(E)}$$

where

$$I_1 = |\xi_1|^{\alpha_1} \cdots |\alpha_i \xi_i|^{\alpha_i} \cdots |\xi_N|^{\alpha_N} A^{1-x} [(A + \theta)^{-1}$$

$$I_2 = |\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} \cdots |\xi_N|^{\alpha_N} A^{1-x} \sum_{k=1, k\neq i}^N (\xi_k)^i [(A + \theta)^{-1}$$

$$I_3 = |\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} \cdots |\xi_N|^{\alpha_N} A^{1-x} l (\xi_i)^{-1} [(A + \theta)^{-2}.$$}

Since, the first term $I_1$ is almost same with $\Psi (\xi)$ one can easily show that it is uniformly bounded. By virtue of the facts that $\Psi (\xi)$ is uniformly bounded, $A$ is closed and positive operator in $E$ and $\theta (\xi) \in S (\varphi)$ we obtain

$$\|I_2\|_{B(E)} = \left\| |\xi_1|^{\alpha_1} |\xi_2|^{\alpha_2} \cdots |\xi_N|^{\alpha_N} A^{1-x} \sum_{k=1, k\neq i}^N (\xi_k)^i [(A + \theta)^{-1}$$

$$= \left\| \Psi (\xi) \sum_{k=1, k\neq i}^N (\xi_k)^i [(A + \theta)^{-1} \leq C \sum_{k=1, k\neq i}^N (\xi_k)^i \left( 1 + \sum_{k=1}^N |\xi_k|^i \right)^{-1}$$

$$\leq C.$$}

Taking into account

$$l (|\xi_i|)^{-1} \leq C \sum_{k=1}^N |\xi_k|^i$$

and using the same reasoning as above one can easily prove that $I_3$ is uniformly bounded. Hence,

$$\left\| \frac{\partial}{\partial \xi_1} \Psi (\xi) \right\|_{L^\infty (B(E))} \leq C.$$}

In order to show

$$\left\| |\xi_1| \frac{\partial}{\partial \xi_1} \Psi (\xi) \right\|_{L^\infty (B(E))} \leq C$$

it is sufficient to prove $|\xi| I_j$ are bounded for $j = 1, 2, 3.$ It is clear that,

$$\| |\xi| I_j \|_{B(E)} \leq \sum_{j=1}^N \| |\xi_1|^{\alpha_1} \cdots |\alpha_i \xi_i|^{\alpha_i} \cdots |\xi_N|^{\alpha_N} A^{1-x} [(A + \theta)^{-1}\|_{B(E)}$$

Since, there exist a constant $M_0$ independent of $\xi$ such that

$$|\xi_j| |\xi_1|^{\alpha_1} \cdots |\xi_i|^{\alpha_i} \cdots |\xi_N|^{\alpha_N} \leq M_0 \left( 1 + \sum_{k=1}^N |\xi_k|^i \right)^{\alpha_N}$$

for all $j = 0, 1, \cdots, N,$ using same arguments as in Proposition 3.1 one can easily show that

$$\| |\xi| I_j \|_{B(E)} \leq C.$$}

Similarly, by virtue of Proposition 3.1 and using the same techniques, one can establish uniformly boundedness of $|\xi| I_2$ and $|\xi| I_3.$ Hence, operator functions $\frac{\partial}{\partial \xi_1} \Psi (\xi)$
and $|\xi| \frac{\partial}{\partial \xi} \Psi(\xi)$ are uniformly bounded multipliers. Other cases can be proved analogously.

The next embedding theorem arises in the investigation of COE’s.

**Theorem 3.5.** Let $E$ be a Banach space and $A$ be a $\varphi$-positive operator in $E$, where $\varphi \in (0, \pi]$. If $a_\alpha \in L_1(R)$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_N)$ and $x = \frac{|\alpha|+\sigma}{1+\frac{1}{\eta}+\frac{1}{\eta'}} \leq 1$ for $\sigma = \left[N(1+\frac{1}{\eta}+\frac{1}{\eta'})\right] + 1$ then the following embedding

$$a_\alpha \ast D^\alpha : B^l_{q_1,r}(R^N; E(A), E) \subset B^s_{q_2,r}(R^N; E(A^{1-x}))$$

is continuous for $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta'}$, $\left(\frac{1}{\eta} + \frac{1}{\eta'} = 1, 1 < q_1 < \eta' \leq \infty\right)$ and there exists a positive constant $C$, such that

$$\|a_\alpha \ast D^\alpha u\|_{B^l_{q_1,r}(R^N; E(A^{1-x}))} \leq C a_\alpha \|u\|_{B^s_{q_1,r}(R^N; E(A), E)} \tag{19}$$

for all $u \in B^l_{q_1,r}(R^N; E(A), E)$.

**Proof.** Really, using Theorem 3.3 and Youngs inequality we get

$$\|a_\alpha \ast D^\alpha u\|_{B^l_{q_1,r}(R^N; E(A^{1-x}))} \leq \|a_\alpha\|_{L_1(R)} \|D^\alpha u\|_{B^s_{q_2,r}(R^N; E(A^{1-x}))} \leq C a_\alpha \|u\|_{B^s_{q_1,r}(R^N; E(A), E)}.$$

Theorem 3.5.

**Result 3.6.** Let all conditions of Theorem 3.3 hold and $|\alpha| = l - \sigma$. Then for all $u \in B^l_{q_1,s}(R^N; E(A), E)$ we have

$$D^\alpha : B^l_{q_1,s}(R^N; E(A), E) \subset B^s_{q_2,r}(R^N; E) .$$

Indeed, choosing $|\alpha| = l - \sigma$ in Theorem 3.3 we get $x = 1$ that implies desired result.

**Result 3.7.** Let all conditions of Theorem 3.5 hold and $|\alpha| = l - \sigma$. Then for all $u \in B^l_{q_1,s}(R^N; E(A), E)$ we have

$$a_\alpha \ast D^\alpha : B^l_{q_1,s}(R^N; E(A), E) \subset B^s_{q_2,r}(R^N; E) .$$

The main aim of this section is to establish $B^s_{q_1,r} \to B^s_{q_2,r}$ regularity of the following elliptic DOE

$$(Q + \lambda)u = \sum_{|\alpha| \leq 2l} a_\alpha D^\alpha u + A_\lambda u = f,$$  \tag{20}$$

where $A_\lambda = A + \lambda$ is a possible unbounded operator in $E$.

**Theorem 4.1.** Suppose $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{\eta'} \left( \text{for } \frac{1}{\eta} + \frac{1}{\eta'} = 1 \text{ and } 1 < q_1 < \eta' \leq \infty \right)$ and the following conditions hold:

1. $E$ is a Banach space;
(2) $A$ is a $\varphi$–positive operator in $E$ with $\varphi \in [0, \pi)$ and

$$L(\xi) = \sum_{|\alpha| \leq 2l} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_N)^{\alpha_N} \in S(\varphi_1),$$

$$|L(\xi)| \geq K \sum_{k=1}^{N} |\xi_k|^{2l}, \xi \in \mathbb{R}^N, \varphi_1 + \varphi < \pi.$$ 

Then for all $f \in B_{q_1, r}^s (\mathbb{R}^N; E), \ r \in [1, \infty], \lambda \in S(\varphi)$ the equation (20) has a unique solution $u(x) \in B_{q_1, r}^s (\mathbb{R}^N; E(A), E)$ and coercive uniform estimate

$$\sum_{|\alpha| \leq 2l} |\lambda|^{1 - \frac{|\alpha|}{q}} \|D^\alpha u\|_{B_{q_2, r}^s (\mathbb{R}^N; E)} + \|Au\|_{B_{q_2, r}^s (\mathbb{R}^N; E)} \leq C\|f\|_{B_{q_1, r}^s (\mathbb{R}^N; E)}$$

holds.

**Proof.** Applying Fourier transform to equation (20), we obtain

$$[L(\xi) + A + \lambda] \hat{u}(\xi) = \hat{f}(\xi).$$

Since $L(\xi) \in S(\varphi_1)$ for all $\xi \in \mathbb{R}^N$ and $A$ is a positive operator, solutions are of the form

$$u(x) = F^{-1} [A + \lambda + L(\xi)]^{-1} \hat{f}.$$ 

By using (23), we have

$$\|Au\|_{B_{q_2, r}^s (\mathbb{R}^N; E)} = \left\| F^{-1} A [A + (\lambda + L(\xi))]^{-1} \hat{f} \right\|_{B_{q_2, r}^s (\mathbb{R}^N; E)}$$

$$\|D^\alpha u\|_{B_{q_2, r}^s (\mathbb{R}^N; E)} = \left\| F^{-1} [L_1 \hat{f}] \right\|_{B_{q_2, r}^s (\mathbb{R}^N; E)},$$

where

$$L_1(\xi) = (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_N)^{\alpha_N} [A + (\lambda + L(\xi))]^{-1}.$$ 

Hence, it suffices to show that operator–functions

$$\sigma_{1\lambda}(\xi) = A[A + (\lambda + L(\xi))]^{-1}, \ \sigma_{2\lambda}(\xi) = \sum_{|\alpha| \leq 2l} |\lambda|^{1 - \frac{|\alpha|}{q}} L_1(\xi)$$

are uniformly bounded multipliers from $B_{q_1, r}^s (\mathbb{R}^N; E)$ to $B_{q_2, r}^s (\mathbb{R}^N; E)$. In order to use Corollary 2.13, we have to show that $\sigma_{j\lambda} \in C^s(\mathbb{R}^N, B(E))$ and there exists a constant $C > 0$ such that

$$\left\| (1 + |\xi|)^{\beta} D^\beta \sigma_{j\lambda}(\xi) \right\|_{L_\infty(\mathbb{R}^N, B(E))} \leq C$$

for each multi–index $\beta$ with $|\beta| \leq \left\lfloor \frac{N}{q} \right\rfloor + 1 \leq N + 1$. For this, from the resolvent property of positive operator $A$, we have

$$\|\sigma_{1\lambda}(\xi)\|_{B(E)} = \left\| A[A + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}$$

$$= \left\| I - (\lambda + L(\xi))[A + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}$$

$$\leq 1 + |\lambda + L(\xi)| \left\| [A + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}$$

$$\leq 1 + M |\lambda + L(\xi)| (1 + |\lambda + L(\xi)|)^{-1} \leq 1 + M.$$ 

(24)

Now let us consider $\sigma_{2\lambda}$. It is clear that

$$\|\sigma_{2\lambda}(\xi) f\|_E = \left\| |\lambda|^{1 - \frac{|\alpha|}{q}} L_1(\xi) f \right\|_E \leq |\lambda| (\left[ \xi_1 |\lambda|^{-\frac{1}{q}} \right]^{\alpha_1} \cdots (\left[ \xi_N |\lambda|^{-\frac{1}{q}} \right]^{\alpha_N}) \left\| f \right\|_{L_\infty(\mathbb{R}^N, B(E))}.$$ 

(25)
Then, by using the well–known inequality
\[
|\xi_1|^\alpha_1 |\xi_2|^\alpha_2 \cdots |\xi_N|^\alpha_N \leq C \left( 1 + \sum_{k=1}^N |\xi_k|^{2l} \right), \quad |\alpha| \leq 2l
\]  
(26)

[7, Lemma 2.3], (25), the positivity of operator $A$ and ellipticity of $L$, we obtain that
\[
||\sigma_{2\lambda}(\xi)f||_E \leq CM \left( |\lambda| + \sum_{k=1}^N |\xi_k|^{2l} \right) (1 + |\lambda + L(\xi)|)^{-1} ||f||_E
\]
\[
\leq C_0 \left( |\lambda| + \sum_{k=1}^N |\xi_k|^{2l} \right) (1 + |\lambda| + |L(\xi)|)^{-1} ||f||_E \leq C_1 ||f||_E
\]
for all $\lambda \in S(\varphi_1)$ and $\xi \in R^N$.

Without any loss of generality, we shall prove (28) for $\sigma_{1\lambda}(\xi)$. For the sake of simplicity of calculations, we shall prove for cases $|\beta| = 1$, $k = 0$ and $k = 1$. Let us first recall the following well–known inequality
\[
\prod_{k=1}^N |\xi_k|^\alpha_k \leq M \left[ 1 + \sum_{k=1}^N |\xi_k|^{2l} \right] \quad \text{for } |\alpha| \leq 2l.
\]  
(29)

Since
\[
\frac{\partial}{\partial \xi_k} \sigma_{1\lambda}(\xi) = -A [A + (\lambda + L(\xi))]^{-2} \sum_{|\alpha| \leq 2l} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_k)^{\alpha_k-1} \cdots (i\xi_N)^{\alpha_N}
\]
by using (29), we obtain
\[
\left\| \frac{\partial}{\partial \xi_k} \sigma_{1\lambda}(\xi) \right\|_{B(E)} \leq C \sum_{|\alpha| \leq 2l} |\xi_1|^\alpha_1 |\xi_2|^\alpha_2 \cdots |\xi_k|^\alpha_k-1 \cdots |\xi_N|^\alpha_N \left\| A [A + (\lambda + L(\xi))]^{-2} \right\|_{B(E)}
\]
\[
\leq C \left[ 1 + \sum_{k=1}^N |\xi_k|^{2l} \right] \left\| A [A + (\lambda + L(\xi))]^{-2} \right\|_{B(E)}
\]
where $|\alpha| - 1 \leq 2l$. Then by using [7, Lemma 2.3], resolvent properties of positive operator $A$ and ellipticity of $L$, for all $\lambda \in S(\varphi_1)$, $\xi \in R^N$, we have
\[
\left\| \frac{\partial}{\partial \xi_k} \sigma_{1\lambda}(\xi) \right\|_{B(E)} \leq C \left\| (A + (\lambda + L(\xi))^{-1} \right\|_{B(E)} \left\| A [A + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}
\]
\[
\leq C \left[ 1 + \sum_{k=1}^N |\xi_k|^{2l} \right] (1 + |\lambda| + |L(\xi)|)^{-1}
\]
\[
\leq C \left[ 1 + \sum_{k=1}^N |\xi_k|^{2l} \right] \left( 1 + |\lambda| + \sum_{k=1}^N |\xi_k|^{2l} \right)^{-1} \leq C.
\]
Now let us consider the case $|\beta| = 1$ and $k = 1$. Similarly, we have

$$
\left\| \xi \frac{\partial}{\partial \xi_k} \sigma_{1\lambda}(\xi) \right\|_{B(E)} \leq C|\xi| \sum_{|\alpha| \leq 2l} |\xi_1|^{\alpha_1} \cdots |\xi_k|^{\alpha_k-1} \cdots |\xi_N|^{\alpha_N} \times \|A[A + (\lambda + L(\xi))]^{-2}\|_{B(E)}
$$

$$
\leq C \sum_{j=1}^{N} |\xi_j| \sum_{|\alpha| \leq 2l} |\xi_1|^{\alpha_1} \cdots |\xi_k|^{\alpha_k-1} \cdots |\xi_N|^{\alpha_N} \|A[A + (\lambda + L(\xi))]^{-2}\|_{B(E)}
$$

$$
\leq C \sum_{j=1}^{N} |\xi_j| \sum_{|\alpha| \leq 2l} |\xi_1|^{\alpha_1} \cdots |\xi_k|^{\alpha_k-1} \cdots |\xi_N|^{\alpha_N} \|A[A + (\lambda + L(\xi))]^{-1}\|_{B(E)}
$$

$$
\leq C \left[ 1 + \sum_{k=1}^{N} |\xi_k|^{2l} \right] \left( 1 + |\lambda| + \sum_{k=1}^{N} |\xi_k|^{2l} \right)^{-1} \leq C.
$$

Hence,

$$
\left\| \xi \frac{\partial}{\partial \xi_k} \sigma_{1\lambda}(\xi) \right\|_{B(E)} \leq C \left[ 1 + \sum_{k=1}^{N} |\xi_k|^{2l} \right] \left( 1 + |\lambda| + \sum_{k=1}^{N} |\xi_k|^{2l} \right)^{-1} \leq C.
$$

Other cases can be proved analogously. Therefore, we obtain

$$
\left\| \xi \right\|^{\beta} D^\beta \sigma_{1\lambda}(\xi) \right\|_{L_\infty(B(E))} \leq C \quad (30)
$$

and

$$
\left\| \xi \right\|^{\beta} D^\beta \sigma_{2\lambda}(\xi) \right\|_{L_\infty(B(E))} \leq K. \quad (31)
$$

for each multi-index $\beta$ with $|\beta| \leq \left[ \frac{N}{\eta} \right] + 1 \leq N+1$. Taking into account assumptions of the theorem and using (30), (31) and Corollary 2.13, we find $\sigma_{j\lambda}(\xi)$ are Fourier multipliers from $B^{q_1}_{\eta\tau}(R^N; E)$ to $B^{q_2}_{2\tau}(R^N; E)$. Hence, for all $f \in B^{q_2}_{2\tau}(R^N; E)$ there is a unique solution of the equation (20) in the form $u(x) = F^{-1}[A + (\lambda + L(\xi))]^{-1}\hat{f}$ and the estimate (21) holds.

Let $B$ denote the space $B(B^{q_1}_{\eta\tau}(R^N; E), B^{q_2}_{2\tau}(R^N; E))$ and $Q$ be an operator generated by the problem (21), i.e.,

$$
D(Q) = B^{2l}_{q_2\tau}(R^N; E(A); E), \quad Qu = \sum_{|\alpha| \leq 2l} a_\alpha D^\alpha u + Au. \quad (32)
$$

**Result 4.2.** Assume all conditions of the Theorem 4.1 hold. Then for all $\lambda \in S(\varphi)$ the resolvent of operator $Q$ exist and the following estimate holds

$$
\sum_{|\alpha| \leq 2l} |\lambda|^{1-|\alpha|} \left\| D^\alpha (Q + \lambda)^{-1} \right\|_{B} + \left\| A(Q + \lambda)^{-1} \right\|_{B} \leq C.
$$

**Remark 4.3.** The Result 4.2 particularly, implies that the operator $Q + a$, $a > 0$ is positive ($B^{2l}_{q_2\tau}(R; E(A); E) \rightarrow B^{q_1}_{\eta\tau}(R; E)$). I.e. if $A$ is strongly positive for $\varphi \in \left( \frac{\pi}{2}, \pi \right)$ then (see e.g. [3, Theorem 8.8]) the operator $Q + a$ is a generator of analytic semigroup.
5. Convolution operator equations

In this section we shall investigate an ordinary convolution operator equation

\[(L + \lambda) u = \sum_{k=0}^{l} a_k \frac{d^k u}{dx^k} + A_\lambda u = f(t) \tag{33}\]

in \(B_{q_1}^*(R; E)\), where \(A_\lambda = A_\lambda(x) = A(x) + \lambda\) is a possible unbounded operator in \(E\) and \(a_k = a_k(x)\) are complex valued functions.

**Condition 5.1.** Suppose \(a_k \in L_1(R)\), \(L(\xi) = \sum_{k=0}^{l} \hat{a}_k(\xi)(i\xi)^k \in S(\phi_1), \phi_1 + \phi < \pi\).

and there is a positive constant \(C\) so that

\[|L(\xi)| > C |\xi|^{l} \sum_{k=0}^{l} |\hat{a}_k|.\]

**Lemma 5.2.** Let the Condition 5.1 be satisfied and \(A(\xi)\) be a uniformly \(\varphi\)-positive \((\varphi \in [0, \pi))\) operator in a Banach space \(E\), \(\lambda \in S(\varphi)\). Then, operator functions

\[
\begin{align*}
\sigma_0(\xi, \lambda) &= \lambda [A(\xi) + (\lambda + L(\xi))]^{-1}, \\
\sigma_1(\xi, \lambda) &= A(\xi) [A(\xi) + (\lambda + L(\xi))]^{-1} \\
\sigma_2(\xi, \lambda) &= \sum_{k=0}^{l} |\lambda|^{1-k} \hat{a}_k(\xi)(i\xi)^k [A(\xi) + (\lambda + L(\xi))]^{-1}
\end{align*}
\]

are uniformly bounded.

**Proof.** Let us note that for the sake of simplicity we shall not change constants in every step. By using the resolvent properties of positive operators we obtain

\[
\|\sigma_0(\xi, \lambda)\|_{B(E)} \leq M |\lambda| (1 + |\lambda + L(\xi)|)^{-1} \leq M,
\]

\[
\|\sigma_1(\xi, \lambda)\|_{B(E)} = \left\| A(\xi) [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}
\]

\[
= \left\| I - (\lambda + L(\xi)) [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}
\]

\[
\leq 1 + |\lambda + L(\xi)| \left\| [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}
\]

\[
\leq 1 + M |\lambda + L(\xi)| (1 + |\lambda + L(\xi)|)^{-1} \leq 1 + M.
\]

Next, let us consider \(\sigma_2\). It is clear to see that

\[
\|\sigma_2(\xi)\|_{B(E)} = \left\| \sum_{k=0}^{l} |\lambda|^{1-k} \hat{a}_k(\xi)(i\xi)^k [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}
\]

\[
\leq C \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\lambda| \left[ |\xi| |\lambda|^{-1} \right]^k \left\| [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}.
\]
Therefore, $\sigma_2 (\xi, \lambda)$ is bounded if
\[
\|I\|_{B(E)} = \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\lambda| |\xi| k |\lambda|^{-\frac{k}{2}} \left\| [A(\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)} \leq C.
\]
Since $A$ is a uniformly $\varphi$-positive and $L(\xi) \in S(\varphi_1)$ for all $\xi \in R$ then
\[
\|I\|_{B(E)} \leq C \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\lambda| \left[ 1 + |\xi| l |\lambda|^{-1} \right] [1 + |\lambda + L(\xi)|]^{-1}.
\]
Taking into account $\hat{a}_k(\xi)(i\xi)^k \in S(\varphi_2), \varphi_2 < \frac{1}{2}, \hat{a}_i(\xi) \neq 0$ and by using [7, Lemma 2.3], Condition 5.1 and the fact $\hat{a}_k(\xi) \in L_\infty(R)$ (Housdorff-Youngs inequality) we get
\[
\|\sigma_2 (\xi)\|_{B(E)} \leq C \|I\|_{B(E)} \leq C \sum_{k=0}^{l} |\hat{a}_k(\xi)| \left[ |\lambda| + |\xi| l \right] [1 + |\lambda| + |L(\xi)|]^{-1}
\leq C \sum_{k=0}^{l} |\hat{a}_k(\xi)| \left[ |\lambda| + |\xi| l \right] \left[ 1 + |\lambda| + \sum_{k=0}^{l} |\hat{a}_k(\xi)||\xi|^k \right]^{-1} \leq C.
\]

**Proposition 5.3.** Let the Condition 5.1 be satisfied and $A(\xi)$ be a uniformly $\varphi$-positive ($\varphi \in [0, \pi]$) operator in a Banach space $E$ and
\[
a_k \in C^{(m)}(R), k = 0, 1, ..., I, m = 1, 2, A(\xi) A^{-1} (\xi_0) \in C^{(m)}(R; B(E)), \xi_0 \in R.
\]
Suppose there are positive constants $C_1, i = 1, ..., 4$ so that
\[
\left\| A^{(m)} (\xi) A^{-1} (\xi) \right\|_{B(E)} \leq C_1, \left\| \xi^m A^{(m)} (\xi) A^{-1} (\xi) \right\|_{B(E)} \leq C_2 \quad (35)
\]
\[
|\xi^m \hat{a}_k(\xi)| \leq M, \left| \frac{d^m}{d\xi^m} \hat{a}_k(\xi) \right| \leq C_3, \left| \xi^m \frac{d^m}{d\xi^m} \hat{a}_k(\xi) \right| \leq C_4. \quad (36)
\]
Then, operator functions $\frac{d^m}{d\xi} \sigma_i(\xi), i = 0, 1, 2$ are uniformly bounded for $\lambda \in S_\varphi$ with $0 < \lambda_0 \leq |\lambda|$.

**Proof.** Let us first prove for $\frac{d^m}{d\xi} \sigma_i(\xi)$. Really,
\[
\left\| \frac{d}{d\xi} \sigma_i(\xi) \right\|_{B(E)} \leq \|I_1\|_{B(E)} + \|I_2\|_{B(E)} + \|I_3\|_{B(E)}
\]
where
\[
I_1 = A'(\xi) [A(\xi) + \lambda + L(\xi)]^{-1}, \quad I_2 = A(\xi) A'(\xi) [A(\xi) + \lambda + L(\xi)]^{-2}
\]
and
\[
I_3 = A(\xi) L'(\xi) [A(\xi) + \lambda + L(\xi)]^{-2}.
\]
By using (35) we get
\[
\|I_1\|_{B(E)} = \left\| A'(\xi) A^{-1} (\xi) A(\xi) [A(\xi) + \lambda + L(\xi)]^{-1} \right\|_{B(E)} \leq C.
\]

Taking into account the fact $A$ is closed and linear operator and by using (35) we obtain
\[
\|I_2\|_{B(E)} \leq \left\| A' (\xi) \left[ A (\xi) + \lambda + L(\xi) \right]^{-1} \right\|_{B(E)}
\cdot \left\| A (\xi) [A (\xi) + \lambda + L(\xi)]^{-1} \right\|_{B(E)} \leq C.
\]
Since, $A(\xi)$ is a uniformly $\varphi$-positive, $L (\xi) \in S(\varphi_1)$ for $\varphi_1 + \varphi < \pi$, we get
\[
\|I_2\|_{B(E)} \leq |L' (\xi)| \left\| A (\xi) [A (\xi) + \lambda + L(\xi)]^{-1} \right\|_{B(E)}
\cdot \| A (\xi) [A (\xi) + \lambda + L(\xi)]^{-1} \|_{B(E)} \leq C |L' (\xi)| \| 1 + |\lambda + L(\xi)| \|^{-1}.
\]
Thus, by using [7, Lemma 2.3] we have
\[
\|I_3\|_{B(E)} \leq C |L' (\xi)| \left[ 1 + |\lambda| + \sum_{k=0}^{l} |a_k(\xi)| |\xi|^k \right]^{-1}.
\]
It is clear to see that
\[
|L' (\xi)| \leq \sum_{k=0}^{l} \left| \frac{d}{d\xi} a_k(\xi) \right| |\xi|^k + \sum_{k=1}^{l} |\hat{a}_k(\xi)| |\xi|^{k-1}.
\]
By using (36), (37) we obtain
\[
\sum_{k=0}^{l} \left| \frac{d}{d\xi} a_k(\xi) \right| |\xi|^k \leq C \left[ 1 + |\lambda| + \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\xi|^k \right]
\]
and
\[
\sum_{k=1}^{l} |\hat{a}_k(\xi)| |\xi|^{k-1} \leq C \left[ 1 + |\lambda| + \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\xi|^k \right]
\]
that implies
\[
\|I_3\|_{B(E)} \leq C |L' (\xi)| \left[ 1 + |\lambda| + \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\xi|^k \right]^{-1} \leq C. \tag{38}
\]
Next we shall prove uniformly boundedness of $\frac{d}{d\xi} \sigma_2 (\xi, \lambda)$. Similarly,
\[
\left\| \frac{d}{d\xi} \sigma_2 (\xi) \right\|_{B(E)} \leq \| J_1 \|_{B(E)} + \| J_2 \|_{B(E)} + \| J_3 \|_{B(E)} + \| J_4 \|_{B(E)},
\]
where
\[
J_1 = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{2}} \frac{d}{d\xi} a_k(\xi) (i\xi)^k [A(\xi) + L(\xi)]^{-1},
\]
\[
J_2 = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{2}} \hat{a}_k(\xi)ik (i\xi)^{k-1} [A(\xi) + L(\xi)]^{-1},
\]
\[
J_3 = \sum_{k=0}^{l} |\lambda|^{1-\frac{k}{2}} a_k(\xi) (i\xi)^k L' (\xi) [A(\xi) + L(\xi)]^{-2}.
\]
and
\[ J_4 = \sum_{k=0}^{l} |\lambda|^{1-k} \hat{a}_k(\xi) (i\xi)^k A'(\xi) [A(\xi) + L(\xi)]^{-2}. \]

Let us first show \( J_1 \) is uniformly bounded. Since,
\[ \|J_1\|_{B(E)} \leq \sum_{k=0}^{l} \left| \frac{d}{d\xi} \hat{a}_k(\xi) \right| \left| |\lambda|^{1-k} (i\xi)^k [A(\xi) + (\lambda + L(\xi))]^{-1} \right|_{B(E)} \]
by virtue of (34) and (37) we obtain \( \|J_1\|_{B(E)} \leq C \). Then, with the help of (34), (37), Condition 5.1 and the fact \( \hat{a}_k(\xi) \in L_\infty(R) \) we get
\[ ||J_2||_{B(E)} \leq \sum_{k=1}^{l} |\hat{a}_k(\xi)| \left| |\lambda|^{1-k} (i\xi)^k - 1 \right| \left| A(\xi) + (\lambda + L(\xi))^{-1} \right|_{B(E)} \]
\[ \leq C \sum_{k=1}^{l} |\hat{a}_k(\xi)| |\lambda| \left( 1 + |\lambda|^{1-k} \right) \left( 1 + |\lambda| + |\xi|^l \sum_{k=0}^{l} |\hat{a}_k(\xi)| \right)^{-1} \]
\[ \leq C_0 \sum_{k=1}^{l} |\hat{a}_k(\xi)| \left( |\lambda| + |\xi|^l \right) \left( 1 + |\lambda| + |\xi|^l \sum_{k=0}^{l} |\hat{a}_k(\xi)| \right)^{-1} \]
\[ \leq C. \]

Next, by means of (34), (35), (38) and the facts \( \hat{a}_k(\xi) \in L_\infty(R) \), \( A(\xi) \) is a uniformly \( \varphi \)-positive, \( L(\xi) \in S(\varphi_1) \) for \( \varphi_1 + \varphi < \pi \), we obtain
\[ ||J_3||_{B(E)} \leq C |L'(\xi)| \left| |A(\xi) + (\lambda + L(\xi))|^{-1} \right|_{B(E)} \]
\[ \cdot \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\lambda| |\xi|^k |\lambda|^{-\frac{k}{2}} \left| A(\xi) + (\lambda + L(\xi))^{-1} \right|_{B(E)} \]
\[ \leq C \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\xi|^k \left[ 1 + |\lambda| + \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\xi|^k \right]^{-1} \]
\[ \leq C. \]

Finally, by virtue of (34) and (35) we obtain
\[ ||J_4||_{B(E)} \leq C \left\| \frac{d}{d\xi} A(\xi) A^{-1}(\xi) [A(\xi) + (\lambda + L(\xi))^{-1}] B(E) \right\| \]
\[ \cdot \sum_{k=0}^{l} |\hat{a}_k(\xi)| |\lambda| |\xi|^k |\lambda|^{-\frac{k}{2}} \left| A(\xi) + (\lambda + L(\xi))^{-1} \right|_{B(E)} \]
\[ \leq C \left\| A'(\xi) A^{-1}(\xi) \right\|_{B(E)} \left\| [A(\xi) + (\lambda + L(\xi))^{-1}] \right\|_{B(E)} \]
\[ \leq C. \]

Hence, operator functions \( \frac{d}{d\xi} \sigma_i(\xi, \lambda) \) are uniformly bounded. Hence, operator functions \( \frac{d^2}{d\xi^2} \sigma_i(\xi) \) are uniformly bounded. Using the same techniques one can easily establish boundedness of \( \frac{d^2}{d\xi^2} \sigma_i(\xi) \).
Proposition 5.4. Let all conditions of Proposition 5.3 are satisfied. Then the following estimates hold
\[ \left\| \xi^m \frac{d^m}{d\xi^m} \sigma_i (\xi, \lambda) \right\|_{L_\infty(B(E))} \leq A_i, \ m, i = 0, 1, 2. \]

Proof. As a matter of fact, it is enough to prove
\[ |\xi| \| I_1 \|_{B(E)} \leq C_1 \quad \text{and} \quad |\xi| \| J_j \|_{B(E)} \leq D_j \]
for some constant \( C_1 \) and \( D_j \), \( i = 1, 2, 3 \), \( j = 1, 2, 3, 4 \). It is easy to see from the proof of Proposition 5.3
\[ |\xi| \| I_1 \|_{B(E)} \leq C_1 \left\| \xi A' (\xi) A^{-1} (\xi) \right\|_{B(E)} \]
\[ \cdot \left\| A (\xi) [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}^2 \]
\[ |\xi| \| I_2 \|_{B(E)} \leq C_2 \left\| \xi A' (\xi) A^{-1} (\xi) \right\|_{B(E)} \]
\[ \cdot \left\| A (\xi) [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)} \]
\[ |\xi| \| I_3 \|_{B(E)} \leq C_3 |\xi| L'(\xi) \left[ 1 + |\lambda| + \sum_{k=0}^l |\hat{a}_k (\xi)| |\xi|^k \right]^{-1}. \]

From resolvent properties of positive operators, it follows \( \xi I_1 \) and \( \xi I_2 \) are uniformly bounded. By using (37) and (38) we obtain
\[ |\xi| \| I_3 \|_{B(E)} \leq C_3 \sum_{k=0}^l |\hat{a}_k (\xi)| |\xi|^k \left[ 1 + |\lambda| + \sum_{k=0}^l |\hat{a}_k (\xi)| |\xi|^k \right]^{-1} \leq C_3. \]

Similarly, from the proof of Lemma 5.3 it follows
\[ |\xi| \| J_1 \|_{B(E)} \leq \sum_{k=0}^l \left| \frac{d}{d\xi} \hat{a}_k (\xi) \right| \left\| |\lambda|^{-\frac{k}{2}} (i\xi)^k [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}, \]
\[ |\xi| \| J_2 \|_{B(E)} \leq \sum_{k=0}^l \left| \hat{a}_k (\xi) \right| \left\| |\lambda|^{-\frac{k}{2}} (i\xi)^k [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}, \]
\[ |\xi| \| J_3 \|_{B(E)} \leq C \left| \xi L'(\xi) \right| \left\| |\lambda|^{-\frac{k}{2}} (i\xi)^k [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)} \]
\[ \cdot \left\| A (\xi) + (\lambda + L(\xi)) \right\|_{B(E)}, \]
and
\[ |\xi| \| J_4 \|_{B(E)} \leq C \left\| \xi A' (\xi) A^{-1} (\xi) \right\|_{B(E)} \left\| A (\xi) [A (\xi) + (\lambda + L(\xi))]^{-1} \right\|_{B(E)}. \]

Using (34), (35), (36), (38) and the fact \( \hat{a}_k (\xi) \in L_\infty(R) \) it is easy to show that \( |\xi| \| J_1 \|_{B(E)} \), \( |\xi| \| J_2 \|_{B(E)} \), \( |\xi| \| J_3 \|_{B(E)} \) and \( |\xi| \| J_4 \|_{B(E)} \) are uniformly bounded. By virtue of the same techniques one can easily establish uniformly boundedness of \( |\xi|^m \frac{d^m}{d\xi^m} \sigma_i (\xi) \) for \( m = 1, 2 \).

Corollary 5.5. Assume all conditions of Proposition 5.4 are satisfied. Then, operator-functions \( \sigma_i (\xi) \) are Fourier multipliers from \( B_{q_1, r}^s (\mathbb{R}^n; E) \) to \( B_{q_2, r}^s (\mathbb{R}^n; E) \).
Proof. To prove $\sigma_i(\xi)$ are uniformly bounded multipliers from $\mathcal{B}_{q_1,r}^{s}(\mathbb{R}^{n};\mathcal{K})$ to $\mathcal{B}_{q_2,r}^{s}(\mathbb{R}^{n};\mathcal{E})$, we need to show $\sigma_i \in C^{(1)}(\mathbb{R}^{n};\mathcal{K})$ and there exists a constant $K > 0$ such that

$$\left\| (1 + |\xi|)^{\beta} \frac{d^\beta}{d\xi^\beta} \sigma_i(\xi) \right\|_{L^\infty(\mathbb{R};\mathcal{K})} \leq K$$

for each multi-index $\beta$ with $|\beta| \leq \left\lfloor \frac{1}{\gamma} \right\rfloor + 1 \leq 2$. From the Proposition 5.2, Proposition 5.3 and Proposition 5.4 it follows $\sigma_i \in C^{(1)}(\mathbb{R};\mathcal{K})$ and

$$\left\| \frac{d^m}{d\xi^m} \sigma_i(\xi) \right\|_{L^\infty(\mathbb{R};\mathcal{K})} \leq A_1; \left\| |\xi|^m \frac{d^m}{d\xi^m} \sigma_i(\xi) \right\|_{L^\infty(\mathbb{R};\mathcal{K})} \leq A_2$$

for every $i, m = 0, 1, 2$. Hence, $\sigma_i(\xi)$ are Fourier multipliers from $\mathcal{B}_{q_1,r}^{s}(\mathbb{R}^{n};\mathcal{K})$ to $\mathcal{B}_{q_2,r}^{s}(\mathbb{R}^{n};\mathcal{E})$.

Theorem 5.6. Let $f \in \mathcal{B}_{q_1,r}^{s}(\mathbb{R};\mathcal{K})$ and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{r'}$, $1 < q_1 < q' \leq \infty$. Then, (33) has a unique solution $u \in \mathcal{B}_{q_2,r}^{s}(\mathbb{R};\mathcal{E}(A),\mathcal{E})$ and the following coercive uniform estimate holds

$$\left\| \lambda u \right\|_{\mathcal{B}_{q_2,r}^{s}(\mathbb{R};\mathcal{E}(A),\mathcal{E})} + \sum_{k=0}^{\ell} \left| \lambda \right|^{1 - \frac{k}{l}} \left\| a_k \star \frac{d^k}{d\xi^k} u \right\|_{\mathcal{B}_{q_2,r}^{s}(\mathbb{R};\mathcal{K})}$$

$$+ \left\| \hat{A} \ast u \right\|_{\mathcal{B}_{q_2,r}^{s}(\mathbb{R};\mathcal{K})} \leq C \left\| f \right\|_{\mathcal{B}_{q_2,r}^{s}(\mathbb{R};\mathcal{K})}$$

for all $\lambda \in \mathcal{S}(\varphi)$ with $0 < \lambda_0 \leq |\lambda|$, provided the below conditions satisfied:

1. $\mathcal{E}$ is a Banach space;
2. Condition 5.1 holds and
3. $\hat{A}(\xi)$ is a uniformly $\varphi$-positive ($\varphi \in [0, \pi]$) operator in $\mathcal{E}$. Moreover, there are positive constants $C_i$, $i = 1, \ldots, 4$ so that for $m = 0, 1, 2$

$$|\xi^m \hat{a}_k(\xi)| \leq M, \left\| \frac{d^m}{d\xi^m} \hat{a}_k(\xi) \right\| \leq C_1, \left\| \xi^m \frac{d^m}{d\xi^m} \hat{a}_k(\xi) \right\| \leq C_2;$$

$$\left\| \hat{A}(\xi) A^{-1}(\xi) \left\|_{\mathcal{B}(\mathcal{E})} \leq C_3, \left\| \xi^m \hat{A}(\xi) A^{-1}(\xi) \left\|_{\mathcal{B}(\mathcal{E})} \leq C_4. $$

Proof. Applying Fourier transform to equation (37) we get

$$\hat{A}(\xi) + L(\xi) \hat{\lambda}(\xi) = \hat{f}(\xi).$$

Since $L(\xi) \in \mathcal{S}(\varphi_1)$ for all $\xi \in \mathbb{R}$ and $\hat{A}$ is positive, the operator $\hat{A}(\xi) + L(\xi)$ is invertible in $\mathcal{E}$. Thus, we obtain that the solution of equation (33) can be represented in the following form

$$u(x) = F^{-1} \left[ \hat{A}(\xi) + \lambda + L(\xi) \right]^{-1} \hat{f}. $$

By using (41) we get
\[ \|A * u\|_{B^{s}_{q_2}, (R; E)} = \left\| F^{-1} \left[ \sigma_1 (\xi) \hat{f} \right] \right\|_{B^{s}_{q_2}, (R; E)} \]

\[ \sum_{k=0}^{l} |\lambda|^{1 - \frac{k}{q}} \left\| a_k * \frac{d^k u}{dx^k} \right\|_{B^{s}_{q_2}, (R; E)} = \left\| F^{-1} \left[ \sigma_2 (\xi) \hat{f} \right] \right\|_{B^{s}_{q_2}, (R; E)}, \]

where

\[ \sigma_0 (\xi) = \left[ \hat{A} (\xi) + \lambda + L (\xi) \right]^{-1}, \quad \sigma_1 (\xi) = \hat{A} (\xi) \left[ \hat{A} (\xi) + \lambda + L (\xi) \right]^{-1}, \]

\[ \sigma_2 (\xi) = \sum_{k=0}^{l} |\lambda|^{1 - \frac{k}{q}} \hat{a}_k (\xi) (i \xi)^k \left[ \hat{A} (\xi) + \lambda + L (\xi) \right]^{-1}. \]

From Corollary 5.5 we know that operator-functions \( \sigma_i (\xi) \) are uniformly bounded multipliers from \( B^{s}_{q_1}, (R^n; E) \) to \( B^{s}_{q_2}, (R^n; E) \).

Since,

\[ \|A * u\|_{B^{s}_{q_2}, (R; E)} \leq C_1 \|f\|_{B^{s}_{q_1}, (R; E)}, \]

\[ \sum_{k=0}^{l} |\lambda|^{1 - \frac{k}{q}} \left\| a_k * \frac{d^k u}{dx^k} \right\|_{B^{s}_{q_2}, (R; E)} \leq C_2 \|f\|_{B^{s}_{q_1}, (R; E)} \]

we obtain that there is a unique solution of the equation (33) in the form \( u (x) = F^{-1} [A + \lambda + L (\xi)]^{-1} \hat{f} \) and the estimate (40) holds for all \( f \in B^{s}_{q_1}, (R; E). \)

Let \( Q \) be an operator that generates the problem (33) i. e.

\[ D (Q) = B^{l,s}_{q_2}, (R; E (A), E), \quad Qu = \sum_{k=0}^{l} a_k * \frac{d^k u}{dx^k} + A \lambda * u. \]

**Result 5.7.** Assume all conditions of the Theorem 5.6 hold. Then for all \( \lambda \in S_{\phi} \) with \( 0 < \lambda_0 \leq |\lambda| \) the resolvent of operator \( Q \) exist and the following estimate holds

\[ \sum_{k=0}^{l} |\lambda|^{1 - \frac{k}{q}} \left\| a_k * \left[ \frac{d^k}{dx^k} (Q + \lambda)^{-1} \right] \right\|_{B(B^{q_1,r}, B^{q_2,r})} + \left\| |\lambda| (Q + \lambda)^{-1} \right\|_{B(B^{q_1,r}, B^{q_2,r})} + \left\| A * (Q + \lambda)^{-1} \right\|_{B(B^{q_1,r}, B^{q_2,r})} \leq C. \]

**Remark 5.8.** The Result 5.7 particularly, implies that the operator \( Q + a, \quad a > 0 \) is positive \( (B^{q_2,r}_2, (R; E) \rightarrow B^{q_1,r}_1, (R; E)). \) i.e. if \( A \) is strongly positive for \( \phi \in \left( \frac{\pi}{2}, \pi \right) \) then ( see e.g. [3, Theorem 8.8] ) the operator \( Q + a \) is a generator of analytic semigroup.

6. **INFINITE SYSTEMS OF QUASI ELLIPTIC EQUATIONS**

Consider the following infinite system

\[ \sum_{|\alpha| \leq 2l} a_{\alpha} D^\alpha u_m + \sum_{j=1}^{\infty} (d_j + \lambda) u_j (x) = f_m (x), \quad x \in R^N, \quad m = 1, 2, \cdots, \infty. \quad (41) \]
Let \( D = \{d_m\}, \) \( d_m > 0, \) \( u = \{u_m\}, \) \( Du = \{d_m u_m\}, \) \( m = 1, 2, \cdots, \infty, \)
\[ l_q(D) = \left\{ u : u \in l_q, \|u\|_{l_q(D)} = \|Du\|_{l_q} = \left( \sum_{m=1}^{\infty} |d_m u_m|^q \right)^{\frac{1}{q}} < \infty \right\}, \ 1 < q < \infty. \]

Let \( Q \) be a differential operator generating the boundary value problem (41).

**Theorem 6.1.** Suppose \( \frac{1}{q_1} = \frac{1}{q} - \frac{1}{q'}, \) \( 1 < q_1 < q' \leq \infty \) and the following conditions hold:
\[ L(\xi) = \sum_{|\alpha| \leq 2l} a_\alpha (i\xi_1)^{\alpha_1} (i\xi_2)^{\alpha_2} \cdots (i\xi_N)^{\alpha_N} \in S(\varphi), \]
\[ \sum_{m=1}^{\infty} d_m^{-1} < \infty, \ |L(\xi)| \geq K \sum_{k=1}^{n} |\xi_k|^{2l}, \xi \in R^N, \ \varphi_1 + \varphi < \pi. \]

Then,
(a) For all \( f(x) = \{f_m(x)\}_1^{\infty} \in B_{q_1,r,(R^N;l_q(D))}^s \) and \( \lambda \in S(\varphi), \ \varphi \in [0, \pi) \)
the problem (41) has a unique solution \( u = \{u_m(x)\}_1^{\infty} \) that belongs to space \( B_{q_2,r,(R^N;l_q(D))}^{2l,s} \) and the coercive estimate
\[ \sum_{|\alpha| \leq 2l} \|D^\alpha u\|_{B_{q_2,r,(R^N;l_q(D))}^s} + \|Au\|_{B_{q_2,r,(R^N;l_q(D))}^s} \leq C \|f\|_{B_{q_1,r,(R^N;l_q(D))}^s} \tag{42} \]
hold for the solution of the problem (41).

(b) There exists a resolvent \( (Q + \lambda)^{-1} \) of operator \( Q \) and
\[ \sum_{|\alpha| \leq 2l} \|D^\alpha (Q + \lambda)^{-1}\| + \|A(Q + \lambda)^{-1}\| \leq C. \tag{43} \]

**Proof.** Let \( E = l_q \) and \( A \) be an infinite matrix such that
\[ A = [d_m(x)\delta_{jm}], \ m, j = 1, 2, \cdots, \infty. \]
It is clear to see that the operator \( A \) is positive in \( l_q. \) Therefore, using Theorem 4.1 we get that the problem (41) has a unique solution \( u \in B_{q_2,r,(R^N;l_q(D))}^{2l,s} \) and estimates (42) and (43) holds for all \( f \in B_{q_1,r,(R^N;l_q(D))}^s. \)

**Remark 6.2.** There are lots of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of \( E \) and concrete positive differential, pseudo differential operators, or finite, infinite matrices, etc. instead of operator \( A \) in (20), we can obtain the maximal regularity of different classes of BVPs for partial differential equations or system of equations by Theorem 4.1.

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