Quantum machine language and quantum computation with Josephson junctions

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I. INTRODUCTION

In classical computing the programming is based on commands written in the machine language. Each command is translated into manipulations of the considered device, obtained by electronic switches. In quantum computation quantum mechanics is employed to process information. Therefore the conception of the quantum computer programming, the structure of the quantum machine language and the commands are expected to be quite different from the classical case. Although there are differences between classical and quantum computers, the programming in both cases should be based on commands, and a part of these commands is realized by quantum gates. Initially, one of the leading ideas in quantum computation was the introduction of the notion of the universal gate [1]. Given the notion of a universal set of elementary gates, various physical implementations of a quantum computer have been proposed [2]. Naturally, in order for an implementation to qualify as a valid quantum computer, all the set of elementary gates have to be implemented by the proposed system. This property is related to the problem of controllability of the quantum computer. The controllability of quantum systems is an open problem under investigation [3]. Recently, attention was focused on the notion of encoded universality, which is a different functional approach to the quantum computation. Instead of forcing a physical system to act as a predetermined set of universal gates, which will be connected by quantum connections, the focus of research is proposed to be shifted to the study of the intrinsic ability of a given physical system, to act as a quantum computer using only its natural available interactions. Therefore the quantum computers are rather a collection of interacting cells (e.g. quantum dots, nuclear spins, Josephson junctions etc). These cells are controlled by external classical switches and they evolve in time by modifying the switches. The quantum algorithms are translated into time manipulations of the external classical switches which control the system. This kind of quantum computer does not have connections, which is the difficult part of a physical implementation. Any device operating by external classical switches has an internal range of capabilities, i.e. it can manipulate the quantum information encoded in a subspace of the full system of Hilbert space. This capability is called encoded universality of the system [1]. The notion of the encoded universality is identical to the notion of the controllability of the considered quantum device. The controllability on Lie groups from a mathematical point of view was studied in [4, 5]. The controllability of atomic and molecular systems was studied by several authors see review article [6] and the special issue of the Chemical Physics vol. 267, devoted to this problem. In the case of laser systems the question of controllability was studied by several authors and recently in [7, 8].

In this paper we investigate the conditions in order to obtain the full system of Hilbert space by a finite number of choices of the values of the classical switches. As a working example we use the Josephson junction devices in their simplest form [7, 8, 9, 10], but this study can be extended in the case of quantum dots or NMR devices.

In this paper the intrinsic interaction of a system operating as a universal computer is employed instead of forcing the system to enact a predetermined set of universal gates [1]. As a basic building block we use a system of two identical Josephson junctions coupled by a mutual inductor. The values of the classical control parameters (the charging energy $E_C$ and the inductor energy $E_L$) are chosen in such a way that four basic Hamiltonians, $H_i (i = 1, \ldots, 4)$ are created by switching on and off the bias voltages and the inductor, where the tunneling amplitude $E_J$ is assumed to be fixed. Our procedure allows the construction of any one-qubit and two-qubit gate, through a finite number of steps evolving in time according to the four basic Hamiltonians. Using the two Josephson junctions network a construction scheme of
II. ONE-QUBIT DEVICES

![Fig. 1: One qubit device](image)

The simplest Josephson junction one qubit device is shown in Fig. 1. In this section we give a summary of the considered device. The detailed description and the complete list of references can be found in the detailed review paper [16], section II. The device consists of a small superconducting island ("box"), with $n$ excess Cooper pair charges connected by a tunnel junction with capacitance $C_J$ and Josephson coupling energy $E_J$ to a superconducting electrode. A control gate voltage $V_g$ (ideal voltage source) is coupled to the system via a gate capacitor $C_g$.

The chosen material is such that the superconducting energy gap is the largest energy in the problem, larger even than the single-electron charging energy. In this case quasi-particle tunneling is suppressed at low temperatures, and a situation can be reached where no quasi-particle excitation is found on the island. Under special condition described in ref [16], only Cooper pairs tunnel coherently in the superconducting junction.

The voltage $V_g$ is constrained in a range interval where the number of Cooper pairs takes the values 0 and 1, while all other coherent charge states, having much higher energy, can be ignored. These charge states correspond to the spin basis states: $|\uparrow\rangle$ corresponding to 0 Cooper-pair charges on the island, and $|\downarrow\rangle$ corresponding to 1 Cooper-pair charges. In this case the superconducting charge box reduces to a two-state quantum system, *qubit*, with Hamiltonian (in spin 1/2 notation):

$$H = \frac{1}{2}E_c \sigma_z - \frac{1}{2}E_J \sigma_x, \quad \sigma_z |\uparrow\rangle = |\uparrow\rangle \quad \text{and} \quad \sigma_z |\downarrow\rangle = -|\downarrow\rangle.$$  \hspace{1cm} (1)

In this Hamiltonian there are two parameters the bias energy $E_c$ and the tunneling amplitude $E_J$. The bias energy $E_c$ is controlled by the gate voltage $V_g$ of Fig 1, while the tunneling amplitude $E_J$ here is assumed to be constant i.e. it is a constant system parameter. The tunneling amplitude can be controlled in the case of the tunable effective Josephson junction, where the single Josephson junction is replaced by a flux-threaded SQUID [16], but this device is more complicated than the one considered in this paper.

The Hamiltonian is written as:

$$H = \frac{1}{2} \Delta E(\eta)(\cos \eta \sigma_z - \sin \eta \sigma_x)$$  \hspace{1cm} (2)

where $\eta$ is the mixing angle

$$\eta = \tan^{-1} \frac{E_J}{E_c}.$$  

The energy eigenvalues are

$$E_{\pm} = \pm \Delta E(\eta) \frac{1}{2}$$  

and the splitting between the eigenstates is:

$$\Delta E(\eta) = \sqrt{E_J^2 + E_c^2}.$$  

The eigenstates provided by the Hamiltonian (2), are denoted in the following as $|+\rangle$ and $|\rangle$:

$$|+\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$$

$$|\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle - |\downarrow\rangle)$$  \hspace{1cm} (3)

To avoid confusion we introduce a second set of Pauli matrices $\bar{\sigma} = (\rho_x, \rho_y, \rho_z)$, which operate in the basis $|+\rangle$, $|\rangle$, while reserving the $\sigma$ operators for the basis of $|\uparrow\rangle$ and $|\downarrow\rangle$: 

$$\rho_x = |+\rangle \langle +| - |\rangle \langle \rangle,$$

$$\rho_y = |+\rangle \langle +| - |+\rangle \langle +|,$$

$$\rho_z = |\uparrow\rangle \langle \uparrow| - |\downarrow\rangle \langle \downarrow|.$$  

In the proposed model we assume that the device of Fig 1 has a switch taking two values 1 and 0, corresponding to the switch states ON and OFF. This switch controls the gate voltage $V_g$, which takes only two values either
$V_{\text{id}}$ or $V_{\text{deg}}$, where the first one corresponds to the *idle Hamiltonian*, while the second one corresponds to the *degenerate Hamiltonian*.

The *idle* point can be achieved for a characteristic value of the control gate voltage $V_g = V_{\text{id}}$, corresponding to a special value of the bias energy and to the phase parameter $\eta = \eta_{\text{id}}$. At this point the energy splitting $\Delta E(\eta)$ achieves its maximum value, which is denoted by $\Delta E$. For simplicity reasons we reserve the symbol $E_c$ for the bias energy corresponding to the idle point and by definition, the Hamiltonian at the *idle point* then becomes:

$$H_{\text{id}} = \frac{E_c}{2} \sigma_z - \frac{E_J}{2} \sigma_x = \frac{1}{2} \Delta E \rho_z$$  \hspace{1cm} (4)

At the *degeneracy* point $\eta = \frac{\pi}{2}$ the energy splitting reduces to $E_J$, which is the minimal energy splitting. This point is characteristic for the material of the Josephson junction and corresponds to a special characteristic choice of the control gate voltage $V_g = V_{\text{deg}}$.

$$H_{\text{deg}} = -\frac{E_J}{2} \sigma_x = -\frac{E_J}{2} \left( \sin \eta_{\text{deg}} \rho_z - \cos \eta_{\text{deg}} \rho_x \right) \hspace{1cm} (5)$$

The system is switched in the state OFF (or 0) corresponding to the *degenerate* Hamiltonian (6) during a time interval $t_1$, then the system is switched to the state ON (or 1) i.e. the *idle* Hamiltonian (4) during a time interval $t_2$ and it comes back to the initial *degenerate* Hamiltonian during the time $t_3$. The general form of the evolution operator is:

$$U = e^{-it_3 H_{\text{deg}}} e^{-it_2 H_{\text{id}}} e^{-it_1 H_{\text{deg}}}$$  \hspace{1cm} (6)

The operators $H_{\text{id}}$ and $H_{\text{deg}}$ and their commutator

$$[H_{\text{id}}, H_{\text{deg}}] = \frac{E_c E_J}{2} \sigma_y$$

form a (non-orthogonal) basis of the algebra $su(2)$. Therefore the pair $H_{\text{id}}, H_{\text{deg}}$ generates $su(2)$ by taking these elements and all their possible commutators and their linear combinations. That means that the combination of three terms as in equation (1) for all the triples $(t_1, t_2, t_3)$ cover all the matrices belonging in $SU(2)$. Thus we conclude that every $2 \times 2$ matrix $U$ in $SU(2)$ can be achieved by a device as in Fig 1, with manipulation of the binary switch permitting to the Hamiltonian two possible states i.e. the *idle* one and the *degenerate* one. The above described manipulations can be codified by a rudimentary Quantum Machine Language (QML) for the one qubit device. In that elementary language the gate $U$ corresponds to a *command* of the language, each command is constituted by (three) letters, each of them having the form of a pair

$\{e, t\}$,  \hspace{0.5cm} $e = 0$ (OFF), or 1 (ON), and $0 \leq t < \infty$

i.e. the command corresponding to equation (6) is analyzed in the following (at most three) letters:

$$\begin{align*}
U & \begin{cases}
0, t_1 \\
1, t_2 \\
0, t_3
\end{cases}
\end{align*}$$

The one qubit gates are $2 \times 2$ unitary matrices belonging to the group $U(2)$. Each element in the group $U(2)$ can be projected up to one multiplication constant to an element of the language, but the similar construction is far from evident and quite complicated in the N-qubit case.
In order to perform one and two qubit quantum gate manipulations in the same device, we need to couple pairs of qubits together and to control the interaction between them. For this purpose identical Josephson junctions are coupled by one mutual inductor $L$ as shown in Fig. 2. The physics and the detailed description of the coupled Josephson junctions are discussed and reviewed in [16].

For $L = 0$ the system reduces to a series of uncoupled, single qubits, while for $L \to \infty$ they are coupled strongly. The ideal system would be one where the coupling between different qubits could be switched in the state $ON$ (or state ‘1′) by applying an induction via a constant value inductor $L$ and in the state $OFF$ (or state ‘0′) corresponding to $L = 0$ and leaving the qubits uncoupled in the $idle$ state. The Hamiltonian for a general two-qubit system is written:

$$H = H_1 + H_2 + H_{int} =$$

$$= \frac{1}{2} E_{c1} \sigma_z^{(1)} + \frac{1}{2} E_{1} \sigma_x^{(1)} + \frac{1}{2} E_{c2} \sigma_z^{(2)} - \frac{1}{2} E_{2} \sigma_x^{(2)} - \frac{1}{2} E_{L} \sigma_y^{(1)} \sigma_y^{(2)}$$

For an explanation of the formalism used in this section see Appendix A. In the case of two identical junctions we have $E_{J1} = E_{J2} = E_J$, since the tunneling amplitude of the junction is a system parameter, depending on the material. Under these conditions the two coupled Josephson junctions Hamiltonian will be controlled by the following control parameters: $E_{c1}$, $E_{c2}$, $E_L$, which will be called switches. The first two parameters are controlled by the gate voltages $V_{g1}$, $V_{g2}$, while the last parameter is related to the inductor switch $L$. In the proposed model each of the parameters $E_{c1}$, $E_{c2}$ can have two values 0 or $E_c$. The first is the state ‘0′ (or $OFF$) corresponding to the $degenerate$ one qubit state, while the other one is equal to $E_c$ (or $ON$ or ‘1′ state) corresponding to the one qubit $idle$ state. Also the parameter $E_L$ takes two values. The one is $E_L = 0$ ($OFF$ or ‘0′ state) corresponding to an uncoupled two qubit state and the other one has a fixed value ($ON$ or ‘1′ state). For the sake of simplicity we use the symbol $E_L$ for this induction amplitude. Using this combination of parameter values or binary switches values, we can obtain the following four fundamental states of the Hamiltonian (5):

$$H_1:$$

where both of the junctions are in the idle state ($E_{c1} = E_{c2} = E_c$), while they are uncoupled ($E_L = 0$).

$$H_1 = \frac{1}{2} E_{c} (\sigma_z^{(1)} + \sigma_z^{(2)}) - \frac{1}{2} E_{J} (\sigma_x^{(1)} + \sigma_x^{(2)})$$

This Hamiltonian corresponds to the switches choice ($E_{c1}, E_{c2}, E_{L} \rightarrow (1, 1, 0)$).

$$H_2:$$

where both of the junctions are in the degenerate state ($E_{c1} = E_{c2} = 0$), while the two qubits are coupled.

$$H_2 = -\frac{1}{2} E_{J} (\sigma_x^{(1)} + \sigma_x^{(2)}) - \frac{1}{2} E_{L} \sigma_y^{(1)} \sigma_y^{(2)}$$

corresponding to the switch choice $(0, 0, 1)$.

$$H_3:$$

where the first junction is in degeneracy ($E_{c1} = 0$), the second is in the idle state ($E_{c2} = E_c$) and they are uncoupled ($E_L = 0$).

$$H_3 = \frac{1}{2} E_{c} \sigma_z^{(2)} - \frac{1}{2} E_{J} (\sigma_x^{(1)} + \sigma_x^{(2)})$$

corresponding to the switch choice $(0, 1, 0)$.

$$H_4:$$

where the first junction is in the idle state ($E_{c1} = E_c$), the second is in the degeneracy ($E_{c2} = 0$) and they are uncoupled ($E_L = 0$).

$$H_4 = \frac{1}{2} E_{c} \sigma_z^{(1)} - \frac{1}{2} E_{J} (\sigma_x^{(1)} + \sigma_x^{(2)})$$

corresponding to the switch choice $(1, 0, 0)$.

These four Hamiltonian forms are linearly independent and they can generate the $su(4)$ algebra by repeated commutations and linear combinations. For a detailed discussion see the discussion in Appendix A, proposition A.2. any elementary two-qubit gate, which is represented by a unitary $4 \times 4$ matrix $U \in SU(4)$, can be constructed as follows:

$$U = e^{-iH_{t1}t_{1}}e^{-iH_{t2}t_{2}} \cdots e^{-iH_{tk}t_{k}}$$

$$= e^{-iH_{t1}t_{1}}e^{-iH_{t2}t_{2}} \cdots e^{-iH_{t5}t_{5}}e^{-iH_{t6}t_{6}}$$

From a physical point of view any two qubit quantum gate can be obtained by 15 time steps. At the k-th step the device is put at one of the Hamiltonian states $H_1$, $H_2$, $H_3$ or $H_4$ by appropriate manipulations of the

| NOT | √NOT | Had | PhS |
|-----|------|-----|-----|
| $\{0, \pi/E_J\}$ | $\{0, \pi/2E_J\}$ | $\{0, t^b_1\}$ | $\{0, t^{ph}_1\}$ |
| $\{1, t^b_2\}$ | $\{1, t^{ph}_2\}$ | $\{0, t^b_3\}$ | $\{0, t^{ph}_3\}$ |

TABLE I: Letter Analysis of One-qubit gates

![Two qubit device](image-url)

FIG. 2: Two qubit device
switches during a time interval $t_k$, $k = 1, 2, \ldots, 15$. Using this fact a Quantum Machine Language (QML) can be defined as in section II. Each gate, corresponding to the $4 \times 4$ matrix $U$, is associated to a command of the QML. Each command consists of at most 15 letters or steps, each of them being a collection of 4 numbers \( \{c_1, c_2, \ell, t\} \) of the following form:

\[
\begin{align*}
\text{letter} & \equiv \{c_1, c_2, \ell, t\} \\
& \quad \text{subject to } 0 \leq t < \infty \\
& \quad \text{and } c_1 = 0 \text{ if } V_{g_1} = V_{\text{deg}} \Rightarrow E_{c_1} = 0 \\
& \quad \text{and } c_1 = 1 \text{ if } V_{g_1} = V_{	ext{id}} \Rightarrow E_{c_1} = E_x \\
& \quad \text{and } c_2 = 0 \text{ if } V_{g_2} = V_{\text{deg}} \Rightarrow E_{c_2} = 0 \\
& \quad \text{and } c_2 = 1 \text{ if } V_{g_2} = V_{\text{id}} \Rightarrow E_{c_2} = E_z \\
& \quad \ell = 0 \text{ if } L = 0 \Rightarrow E_L = 0 \\
& \quad \ell = 1 \text{ if } L \neq 0 \Rightarrow E_L \neq 0.
\end{align*}
\]

The gate $U$ given by equation \((12)\) corresponds to one command, which contains 15 letters and is presented in Table~I. Each letter is the codified command which indicates the state of the binary switches and the time interval.

| \(U\) | \{(1, 0, 0, \ell_1)\} | \{(0, 0, 1, \ell_2)\} | \{(1, 0, 0, \ell_3)\} | \{(1, 1, 0, \ell_4)\} | \{(0, 0, 1, \ell_5)\} | \{(0, 1, 0, \ell_6)\} | \{(1, 0, 0, \ell_7)\} | \{(1, 1, 0, \ell_8)\} | \{(0, 0, 1, \ell_9)\} | \{(0, 1, 0, \ell_{10})\} | \{(1, 0, 0, \ell_{11})\} | \{(1, 0, 0, \ell_{12})\} | \{(1, 1, 0, \ell_{13})\} | \{(0, 0, 1, \ell_{14})\} | \{(0, 1, 0, \ell_{15})\} |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|

TABLE II: Letter analysis of a command (gate) $U$

We should notice that the succession of switch states follows a cyclic pattern. This regularity might facilitate the manipulation of the coupled junction device.

Let us now give some numerical simulations of the proposed model for some fundamental quantum gates essential for the quantum computation (we use these gates multiplied with a proper constant because the corresponding matrices should be elements of the $SU(4)$ group). These gates are:

1. The CNOT gate. Probably the most important gate in quantum computation:

\[
\text{CNOT} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\Rightarrow \frac{i \pi}{2} \left( -\sigma_z^{(1)} \sigma_x^{(2)} + \sigma_z^{(1)} \sigma_x^{(2)} + I \otimes I \right) \in SU(4)
\]

2. The SWAP gate, which interchanges the input qubits:

\[
\text{SWAP} = \begin{pmatrix}1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1\end{pmatrix}
\]

\[
\Rightarrow \frac{i \pi}{2} \left( \sigma_x^{(1)} \sigma_z^{(2)} + \sigma_x^{(1)} \sigma_z^{(2)} + I \otimes I \right)
\]

3. The QFT$_4$ gate, the gate of the Quantum Fourier Transform (the quantum version of the Discrete Fourier Transform), for 2 qubits. A very useful gate for the implementation of several quantum algorithms (e.g. Shor’s algorithm):

\[
\text{QFT}_4 = \frac{1}{2} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & -1 & i
\end{pmatrix}
\]

\[
\Rightarrow \frac{i \pi}{2} \left( \sigma_x^{(1)} \sigma_x^{(2)} + I \otimes I \right) + \frac{e^{-i \pi}}{2 \sqrt{2}} \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) + \frac{e^{-i \pi}}{2} \left( \sigma_y^{(1)} \sigma_y^{(2)} + \sigma_z^{(1)} \sigma_z^{(2)} \right)
\]

4. A conditional Phase Shift. This gate provides a conditional phase shift $e^{i \phi}$ on the second qubit (adds a phase to the second qubit):

\[
\text{PhShift} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & e^{i \phi}
\end{pmatrix}
\]

\[
\Rightarrow \frac{i \sin \frac{\phi}{2}}{2} \left( \sigma_z^{(1)} \sigma_x^{(2)} - \sigma_z^{(1)} \sigma_x^{(2)} \right) + \frac{1}{4} \left( 3 e^{-i \frac{\phi}{2}} + e^{i \frac{\phi}{2}} \right) I \otimes I
\]

In our analysis we consider this phase to be $\phi = \frac{\pi}{2}$.

In the simulations of this paper, the energies are assumed to take the following values:

\[
E_c = 2.5 K = 3.45 \times 10^{-23} J,
\]

\[
E_J = 0.1 K = 0.138 \times 10^{-23} J,
\]

\[
E_L = 0.1 K = 0.138 \times 10^{-23} J
\]

i.e. the time scale is of the order of $10^{-11}$ sec. The numerical value corresponding to the idle state is chosen to conform to the available experimental data \cite{20}, and to keep in the range of different experimental propositions \cite{15, 16, 17, 18, 19}. Each gate or command $U_{\text{gate}}$ is approximated by an evolution operator $U(t_1, t_2, \ldots, t_{15})$ of the form \((12)\). This simulation is equivalent to the analysis of the command $U_{\text{gate}}$ to letters in conformity with
Table II. The efficiency of our simulation is defined by a test function, $f_{test}$. It is a function of 15 time variables:

$$f_{test}(t_1, t_2, \ldots, t_{15}) = \sum_{i,j=1}^{4} |(U_{gate})_{ij} - (U(t_1, t_2, \ldots, t_{15}))_{ij}|^2$$

where $U_{gate}$ is the matrix depending on 3 independent time parameters $t_1, t_2, t_3$.

Actually, $f_{test}$ is the norm deviation of our simulation. The optimum, is obviously the nullification of this norm, $f_{test} = 0$. In fact we apply a minimization procedure and we calculate the time values, which minimize $f_{test}$. The numerical results are shown in Table II.

In our numerical examples we use an approximation of the time parameters to the fourth decimal digit. Respectively we calculate the value of the test function. Taking into consideration three more decimal digits the test function $f_{test}$ attains values of the order of $10^{-10}$. It is a matter of intensity of the numerical algorithms used to find the minimum of the test function ($f_{test} = 0$) and convention of the number of decimal digits of the time parameters to succeed the optimal simulation. Indeed time parameters can not be determined with absolute precision in an implementation scheme for quantum computation.

The construction of the one-qubit gates with the two-qubit Josephson device of Fig 3 is possible. That is gates of the form $I \otimes W$ and $W \otimes I$, where $W \in SU(2)$, Fig 3, which are simulated by the same device.

![FIG. 3: Two-qubit and one-qubit gates in two qubit networks](image)

The construction scheme comes as follows:

$$U = e^{-it_4 H_4} e^{-it_3 H_3} e^{-it_2 H_2} e^{-it_1 H_1}$$

where $H_1$ and $H_4$ are special forms of the Hamiltonian $H$, rewritten in the idle basis and $t_1, \ldots, t_4$ the time duration of each step. Obviously:

$$H_1 = \left( \frac{1}{2} E_c \sigma_z^{(1)} - \frac{1}{2} E_J \sigma_x^{(1)} \right) + \left( \frac{1}{2} E_c \sigma_z^{(2)} - \frac{1}{2} E_J \sigma_x^{(2)} \right) = \frac{\Delta E}{2} (\rho_z^{(1)} + \rho_z^{(2)})$$

$$H_4 = \left( \frac{1}{2} E_c \sigma_z^{(1)} - \frac{1}{2} E_J \sigma_x^{(1)} \right) - \frac{1}{2} E_J \sigma_x^{(2)} = \frac{\Delta E}{2} (\rho_z^{(1)} + \rho_z^{(2)})$$

where

$$\tau = -\frac{E_J}{\Delta E} \sigma_x$$

It can easily be shown that:

$$U = e^{-it_4 \Delta E \rho_z} \otimes e^{-it_3 \Delta E \rho_z} e^{-it_2 \Delta E \rho_z} e^{-it_1 \Delta E \rho_z}$$

Setting the total time

$$t_{tot} = \sum_{i=1}^{4} t_i = \frac{4 k \pi}{\Delta E}, k \in \mathbb{N}$$

the previous relation is written as:

$$U = I \otimes e^{-it_4 \Delta E \rho_z} e^{-it_3 \Delta E \rho_z} e^{-it_2 \Delta E \rho_z} e^{-it_1 \Delta E \rho_z}$$

The right hand side of the last relation is a $2 \times 2$ SU(2) matrix depending on 3 independent time parameters $t_1, t_2, t_3$, since the fourth time parameter $t_4$ is specified from the demand that total time is assumed to be fixed. By an appropriate choice of these three time parameters any gate $U$ of the above form can be constructed. So any command of the form $U = I \otimes W$, which corresponds to an one qubit gate can be constructed by at most four steps. Therefore any command $I \otimes W$ can be analyzed in at most four letters. We simulate numerically the proposed model for the following one-qubit gates, $I \otimes (NOT), I \otimes (\text{Had}), I \otimes (\sqrt{NOT})$ and $I \otimes \text{PhS}$:

$$I \otimes \text{NOT} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \mapsto I \otimes (i \sigma_x) \in SU(4)$$

$$I \otimes h = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \mapsto I \otimes \left( i \sqrt{2} \sigma_x + \sigma_z \right)$$

$$I \otimes \sqrt{\text{NOT}} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-i \frac{\pi}{4}} & e^{i \frac{\pi}{4}} & 0 & 0 \\ e^{i \frac{\pi}{4}} & e^{-i \frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i \frac{\pi}{4}} & e^{i \frac{\pi}{4}} \\ 0 & 0 & e^{i \frac{\pi}{4}} & e^{-i \frac{\pi}{4}} \end{pmatrix} \mapsto I \otimes \left( \frac{1}{\sqrt{2}} (I + i \sigma_x) \right)$$

$$I \otimes \text{PhS} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i \phi} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i \phi} \end{pmatrix} \mapsto I \otimes \left( \cos \frac{\phi}{2} - i \sin \frac{\phi}{2} \sigma_z \right)$$

The case of the gates of the form $U = W \otimes I$ can be similarly treated by using basic Hamiltonians $H_1$ and $H_3$.

$$H_1 = \left( \frac{1}{2} E_c \sigma_z^{(1)} - \frac{1}{2} E_J \sigma_z^{(1)} \right) + \left( \frac{1}{2} E_c \sigma_z^{(2)} - \frac{1}{2} E_J \sigma_z^{(2)} \right) = \frac{\Delta E}{2} (\rho_z^{(1)} + \rho_z^{(2)})$$

$$H_3 = -\frac{1}{2} E_J \sigma_x^{(1)} \left( \frac{1}{2} E_c \sigma_z^{(2)} - \frac{1}{2} E_J \sigma_z^{(2)} \right) = \frac{\Delta E}{2} (\tau^{(1)} + \rho_z^{(2)})$$
where

\[ \tau = -\frac{E_J}{\Delta E} \sigma_x \]

The equivalent construction scheme is:

\[ U = e^{-it_3 H_3} e^{-it_2 H_2} e^{-it_1 H_1} \]

(15)

Obviously,

\[ U = e^{-it_3 \frac{\Delta \phi}{\tau} \rho_3} e^{-it_2 \frac{\Delta \phi}{\tau} \rho_2} e^{-it_1 \frac{\Delta \phi}{\tau} \rho_1} \otimes e^{-it_{tot} \frac{\Delta \phi}{\tau} \rho_0} \]

Setting the total time

\[ t_{tot} = \sum_{i=1}^{4} t_i = \frac{4k\pi}{\Delta E}, \quad k \in \mathbb{N} \]

the previous relation is written as:

\[ U = e^{-it_3 \frac{\Delta \phi}{\tau} \rho_3} e^{-it_2 \frac{\Delta \phi}{\tau} \rho_2} e^{-it_1 \frac{\Delta \phi}{\tau} \rho_1} \otimes I \]

Table III: Letter analysis of Two-Qubit Gates

| CNOT       | SWAP       | QFT \_1 | Phase Shift (ϕ = 1/2) |
|------------|------------|---------|-----------------------|
| \{1, 1, 0, 102,775\} | \{1, 1, 0, 700,587\} | \{1, 1, 0, 710,258\} | \{1, 1, 0, 110,711\} |
| \{0, 0, 1, 158,893\} | \{0, 0, 1, 205,239\} | \{0, 0, 1, 084,863\} | \{0, 0, 1, 920,613\} |
| \{1, 1, 0, 009,967\} | \{1, 1, 0, 139,745\} | \{1, 0, 1, 159,439\} | \{1, 0, 1, 120,308\} |
| \{1, 1, 0, 130,050\} | \{1, 1, 0, 199,388\} | \{1, 0, 1, 142,568\} | \{1, 0, 1, 397,824\} |
| \{1, 1, 0, 130,720\} | \{1, 1, 0, 130,496\} | \{1, 0, 1, 133,276\} | \{1, 1, 0, 109,399\} |
| \{0, 0, 1, 143,987\} | \{0, 1, 115,879\} | \{0, 1, 1, 653,350\} | \{0, 1, 1, 175,819\} |
| \{0, 1, 0, 101,050\} | \{0, 1, 110,958\} | \{0, 1, 1, 133,482\} | \{0, 1, 1, 521,326\} |
| \{1, 0, 0, 000,569\} | \{1, 0, 200,550\} | \{1, 0, 1, 924,288\} | \{1, 0, 1, 122,503\} |
| \{1, 1, 0, 151,529\} | \{1, 1, 0, 120,279\} | \{1, 1, 1, 173,805\} | \{1, 1, 1, 102,230\} |
| \{0, 0, 1, 108,408\} | \{0, 0, 1, 784,009\} | \{0, 1, 1, 633,452\} | \{0, 1, 1, 795,321\} |
| \{0, 1, 0, 161,083\} | \{0, 1, 798,072\} | \{0, 1, 0, 701,262\} | \{0, 1, 0, 108,907\} |
| \{0, 0, 0, 901,955\} | \{1, 0, 129,138\} | \{1, 0, 1, 131,304\} | \{1, 0, 1, 198,717\} |
| \{1, 1, 0, 699,397\} | \{1, 1, 0, 501,678\} | \{1, 1, 1, 849,656\} | \{1, 1, 1, 159,051\} |
| \{0, 0, 1, 191,427\} | \{0, 0, 1, 130,044\} | \{0, 0, 1, 128,843\} | \{0, 0, 1, 888,000\} |
| \{0, 1, 0, 101,208\} | \{0, 1, 160,221\} | \{0, 1, 0, 150,028\} | \{0, 1, 0, 100,713\} |

\[ f_{text} = 2.9 \times 10^{-6} \quad f_{text} = 9.5 \times 10^{-8} \quad f_{text} = 1.0 \times 10^{-8} \quad f_{text} = 1.2 \times 10^{-10} \]

The traditional approach to quantum computing is the construction of elementary one-qubit and two-qubit gates (universal set of quantum gates) which are connected by quantum connections and can represent any quantum algorithm [23]. A different view is employed in the present paper, proposed in [4, 5, 7] under the name of encoded universality. According to this, we do not force the system to act as a predetermined set of universal gates connected by quantum connections, but we exploit its intrinsic ability to act as a quantum computer employing its natural available interaction.

Thus, any one-qubit and two-qubit gate can be expressed by two identical Josephson junctions coupled by a mutual inductor. This can be realized by a finite number of time steps evolving according to a restricted collection of basic Hamiltonians. These Hamiltonians are implemented using the above system of junctions by choosing suitably the control parameters, by switching on and off the bias voltages and the mutual inductor. The interaction times of the steps are calculated numerically.

The values of the switches together with the values of the time steps may constitute the quantum machine language. Each command of the language consists of a series of letters and each letter of a binary part (the values of the switch characterizing the Hamiltonian) and a numerical part (the interaction time).

The generalization to N-qubit gates is currently under investigation. In this case we need N + 2 basic Hamiltonians in order to represent the corresponding N-qubit gate. The structure of commands is an open problem. Each command can be obtained by \(2^N - 1\) letters see Proposition A.3 in Appendix A. The mathematical foundation of this conjecture will be studied in another specialized paper. However by using the techniques described in [23].
the number of letters can be reasonably reduced in the N-qubit case. The application of the same methodology for other devices as quantum dots and NMR are under investigation.

APPENDIX A: MINIMAL GENERATING SET OF THE SU(2^N) - ALGEBRA

Let us consider the su(2) algebra in the adjoint representation. This algebra representation is a three dimensional vector space with basis the 2 x 2 Pauli matrices:

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \]

therefore

\[ su(2) = \text{span}(\sigma_z, \sigma_x, \sigma_y) \]

This su(2) algebra in the adjoint representation is generated by the following 2 x 2 hermitian matrices:

\[ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]

because

\[ [\sigma_z, \sigma_x] = 2i\sigma_y \]

The adjoint representation of the algebra su(2^2) is the vector space spanned by the 15 matrices

\[ su(2^2) = \text{span}(\sigma_i^{(1)}, \sigma_i^{(2)}, \sigma_i^{(1)}\sigma_j^{(2)}, i, j = 1, 2, 3) \] (A1)

where

\[ \sigma_i^{(1)} = \sigma_i \otimes \mathbb{I}, \quad \sigma_i^{(2)} = \mathbb{I} \otimes \sigma_i, \quad \sigma_i^{(1)}\sigma_j^{(2)} = \sigma_i \otimes \sigma_j \]

the adjoint representation of the su(2^2) algebra can be generated by linear combinations and successive commutations of the following 4 elements:

\[ \sigma_z^{(1)}, \quad \sigma_z^{(2)}, \quad \sigma_x^{(1)} + \sigma_x^{(2)}, \quad \text{and} \quad \sigma_y^{(1)}\sigma_y^{(2)} \] (A2)

This is indeed true because all the elements of the basis [A1] can be generated by repeated commutations of the elements [A2], because for \( k = 1, 2, 3 \):

\[
\begin{align*}
\sigma_y^{(k)} &= \frac{1}{2} \left[ \sigma_z^{(1)} + \sigma_z^{(2)}, \sigma_y^{(k)} \right], \\
\sigma_x^{(k)} &= \frac{1}{2} \left[ \sigma_z^{(k)}, \sigma_y^{(k)} \right] \\
&= \frac{1}{4} \left[ \sigma_z^{(k)}, \sigma_x^{(1)} + \sigma_x^{(2)} \right] \\
\end{align*}
\]

The elements \( \sigma_i^{(1)} \sigma_j^{(2)} \) can be generated by commutating the generators \( \sigma_i^{(1)} \) with \( \sigma_j^{(2)} \). One illustrative example is the construction of the element \( \sigma_x^{(1)}\sigma_x^{(2)} \), by using the generators [A3]:

\[
\begin{align*}
\sigma_x^{(1)}\sigma_x^{(2)} &= \frac{1}{4} \left[ \sigma_x^{(2)}, \sigma_x^{(1)} \right] \\
&= \frac{1}{16} \left[ \sigma_z^{(k)}, \sigma_x^{(1)} + \sigma_x^{(2)} \right] \\
&= \frac{1}{16} \left[ \sigma_z^{(k)}, \sigma_x^{(1)} \right] \\
\end{align*}
\]

In the case of the adjoint representation of the algebra su(2^3) we can work following a similar methodology. The adjoint representation algebra su(2^3) is a vector space spanned by the following 63 matrices:

\[ su(2^3) = \text{span}(\sigma_i^{(1)}, \sigma_i^{(2)}, \sigma_i^{(3)}, \sigma_i^{(1)}\sigma_j^{(2)}, \sigma_i^{(1)}\sigma_j^{(2)}\sigma_k^{(3)}, i, j, k = 1, 2, 3) \] (A5)

where

\[ \sigma_i^{(1)} = \sigma_i \otimes \mathbb{I} \otimes \mathbb{I}, \quad \sigma_i^{(2)} = \mathbb{I} \otimes \sigma_i \otimes \mathbb{I}, \quad \sigma_i^{(3)} = \mathbb{I} \otimes \mathbb{I} \otimes \sigma_i \]

The above elements can be generated by repeated commutations of the following 5 matrices:

\[
\begin{align*}
\sigma_z^{(1)}, \quad \sigma_z^{(2)}, \quad \sigma_z^{(3)}, \\
\sigma_x^{(1)} + \sigma_x^{(2)} + \sigma_x^{(3)}, \quad \text{and} \quad \sigma_y^{(1)}\sigma_y^{(2)} + \sigma_y^{(1)}\sigma_y^{(3)} + \sigma_y^{(2)}\sigma_y^{(3)} \\
\end{align*}
\]

The linear terms \( \sigma_i^{(k)} \) can be easily generated by formulas as in equation [A3]. The quadratic terms \( \sigma_i^{(k)} \sigma_j^{(l)} \) are
generated by manipulations slightly more complicated than in the case of equation (A4). Let us take the example of the generation of the element $\sigma_x^{(1)} \sigma_x^{(2)}$, then we must perform the following commutation actions:

$$
\sigma_y^{(1)} \sigma_x^{(2)} + \sigma_x^{(2)} \sigma_y^{(1)} = \frac{i}{2} \left[ \sigma_x^{(1)} \sigma_y^{(2)} + \sigma_y^{(2)} \sigma_x^{(1)} \right]
$$

Therefore all the quadratic terms can be generated by the elements (A6). Let us now generate a cubic term of the algebra as the element $\sigma_x^{(1)} \sigma_y^{(2)} \sigma_z^{(3)}$. This element is generated by the commutation elements $\sigma_z^{(1)} \sigma_x^{(2)}$ and $\sigma_x^{(2)} \sigma_z^{(3)}$, which are generated previously:

$$
\sigma_z^{(1)} \sigma_y^{(2)} \sigma_z^{(3)} = \frac{i}{2} \left[ \sigma_z^{(1)} \sigma_y^{(2)} \sigma_z^{(3)} - \sigma_z^{(1)} \sigma_z^{(3)} \sigma_y^{(2)} \right]
$$

By induction we can prove the following proposition:

**Proposition A.1** The adjoint hermitian representation of the algebra su(4), i.e. the set of hermitian traceless $2^N \times 2^N$ can be generated by the algebra of Lie-polynomials of the set:

$$
A_N = \left\{ \sigma_x^{(1)}, \sigma_x^{(2)}, \ldots, \sigma_z^{(N)}, N \sum_{k=1}^{N} \sigma_y^{(k)}, N \sum_{i<j}^{N} \sigma_y^{(i)} \sigma_y^{(j)} \right\}
$$

The set $A_N$ of the generators has $N + 1$ elements, we should point out that this number is very small than the number $4^N - 1$, which is the dimension of the algebra su(2^N). Therefore, large Lie algebras can be generated by using a relatively small number of elements.

Let us now construct the group SU(2^N). For the sake of simplicity we start the discussion with the SU(4) case, i.e. with the set of unitary $4 \times 4$ matrices with determinant equal to 1.

Let us consider four linearly independent elements, which are given by the formulas:

$$
\begin{align*}
H_1 &= \frac{1}{2} E_c \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) - \frac{1}{2} E_j \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) \\
H_2 &= -\frac{1}{2} E_j \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) - \frac{1}{2} E_c \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) \\
H_3 &= \frac{1}{2} E_c \sigma_x^{(1)} - \frac{1}{2} E_j \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right) \\
H_4 &= \frac{1}{2} E_c \sigma_x^{(1)} - \frac{1}{2} E_j \left( \sigma_x^{(1)} + \sigma_x^{(2)} \right)
\end{align*}
$$

Starting from this system we can reconstruct the elements (A3) because:

$$
\begin{align*}
\sigma_x^{(1)} &= \frac{2}{E_c} (H_1 - H_3) \\
\sigma_x^{(2)} &= \frac{2}{E_c} (H_1 - H_4) \\
\sigma_y^{(1)} &= \frac{2}{E_c} (H_2 - H_3) \\
\sigma_y^{(2)} &= \frac{2}{E_c} (H_2 - H_4)
\end{align*}
$$

These relations prove that the su(4) algebra can be generated by combinations and successive commutations of the four elements $\{H_1, H_2, H_3, H_4\}$. Starting from this fact we can construct all the elements of the form:

$$
U = e^{-iH_1 t_1} e^{-iH_2 t_2} \cdots e^{-iH_4 t_4} e^{-iH_5 t_5} e^{-iH_6 t_6} \cdots e^{-iH_1 t_1},
$$

From the Baker-Campbell-Hausdorff (BCH) formula [22, Sec. 2.15]:

$$
e^{-iA} e^{-iB} = e^{-i[A,B]} - \frac{1}{12} \left( [A,B] - [A,B], A \right) - \frac{1}{48} \left( [B, [A,B]] - [A, [B, [A,B]]] \right) + \ldots
$$

one can calculate $U$ in (A11) by successive applications of the the above BCH formula starting from the left to the right, i.e.

$$
U = \exp[-iu_{15}] \quad \text{where} \quad u_1 = t_1 H_1, \\
u_2 = h(t_2 H_2, u_1), \\
u_3 = h(t_3 H_3, u_2) \\
\ldots \\
u_{15} = h(t_{15} H_4, u_{14})
$$

The elements $u_k$ are complicated combinations of the elements $H_1, H_2, H_3, H_4$, bracketed inside commutators, i.e are Lie-polynomials of the free associative algebra $\mathbb{C} \{H_1, H_2, H_3, H_4\}$. By definition the combinations and all the successive commutators of these elements generate the algebra su(4), which has as a linear basis the 15 elements given by equation (A3). Thus the general form of $u_{15}$ is given by:

$$
u_{15} = \sum_{i=1}^{3} \left( f_i^{(1)} \sigma_i^{(1)} + f_i^{(2)} \sigma_i^{(2)} \right) + \sum_{i,j=1}^{3} \frac{3}{i,j=1} f_i^{(1,2)} \sigma_i^{(1)} \sigma_j^{(2)}
$$

where the 15 coefficients

$$
f_i^{(k)} = f_i^{(k)} (t_1, t_2, \ldots, t_{15}) \\
f_i^{(1,2)} = f_i^{(1,2)} (t_1, t_2, \ldots, t_{15})
$$

are complicated functions of the parameters $t_1, t_2, \ldots, t_{15}$. Any element of the algebra su(4) is written as a linear combination of the elements of the basis (A3) and from the known functions (A12) we can find the values of the finite time series $t_1, t_2, \ldots, t_{15}$. Then we have proved the following Proposition

**Proposition A.2** The group SU(4) is given by the elements of the form:

$$
U = e^{-iH_1 t_1} e^{-iH_2 t_2} \cdots e^{-iH_4 t_4} e^{-iH_5 t_5} e^{-iH_6 t_6} e^{-iH_1 t_1},
$$

where $H_1, H_2, H_3, H_4$ are some special elements of the algebra su(4). The combinations and the successive commutations of these elements generate su(4).
This proposition is a special form of the bang-bang controllability theory for SU(4) matrices. The above decomposition of the SU(4) matrices is a generalization of the Euler decomposition of the SU(2) matrices. In another decomposition is proposed based on the Cartan decomposition of SU(2^n) matrices. In the Cartan decomposition a choice of orthogonal basic Hamiltonians is used. In the present decomposition we do not consider an orthogonal set of basic Hamiltonians but the form of the Hamiltonians is imposed by the physical system under consideration, fulfilling the conditions of encoded universality.

Proposition A.3 Let $H_1$, $H_2$, $H_3$, $H_4$, $H_5$ are elements of the algebra $su(2^3)$ such that the successive commutations of these elements generate $su(2^3)$. Then any element $U$ belonging to the group SU(4) is given by the relation:

$$U = e^{-iH_5 t_5}e^{-iH_4 t_4}e^{-iH_3 t_3}e^{-iH_2 t_2}e^{-iH_1 t_1}$$

A similar Proposition can be stated in the case of the general problem related to the SU(2^n) group. The above Proposition concerns the problem of controllability of spin systems, in the context of the Cartan decomposition technique. This problem was solved and other studies of the same problem by different techniques have been recently proposed. The general problem can be formulated in a different way. From the theory of universal gates and the papers on the control of the molecular systems it is well known that the SU(2^n) can be decomposed into simpler matrix factors with SU(2) and SU(4) structure. That means that the one and two qubit gates are universal ones. A systematic study of this technique is under current investigation.
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