Generalized Grassmann variables for quantum kit ($k$-level) systems and \nBarut-Girardello coherent states for $su(r+1)$ algebras

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Abstract

This paper concerns the construction of $su(r+1)$ Barut–Girardello coherent states in term of \ngeneralized Grassmann variables. We first introduce a generalized Weyl-Heisenberg algebra $\mathcal{A}(r)$ \n($r \geq 1$) generated by $r$ pairs of creation and annihilation operators. This algebra provides a \nuseful framework to describe qubit and qukit ($k$-level) systems. It includes the usual Weyl-Heisenberg \nand $su(2)$ algebras. We investigate the corresponding Fock representation space. The \ngeneralized Grassmann variables are introduced as variables spanning the Fock–Bargmann space associated \nwith the algebra $\mathcal{A}(r)$. The Barut–Girardello coherent states for $su(r+1)$ algebras are explicitly \nderived and their over-completion properties are discussed.

Key words: Grassmann variables; Generalized Weyl-Heisenberg algebra; quantum bit and quantum kit \nsystems; Barut–Girardello coherent states.
1 Introduction

The coherent states formalism has been widely used in several areas of quantum physics \[1, 2, 3, 4, 5, 6\]. The coherent states for quantum harmonic oscillator were initially introduced by Schrödinger \[7\] and were extended to other exactly solvable quantum systems (see for instance \[8, 9, 10, 11, 12, 13, 14\] and references therein). The generalized coherent states associated with any irreducible unitary representation of any Lie group were also investigated \[2, 1, 3, 4, 5, 6\]. The construction of coherent states has been proceeded along three (not equivalent in general) lines (see for instance \[15\]). The first line, due to Perelomov, generates the set of generalized coherent states by the action of an unitary displacement operator on a reference state of a group representation Hilbert space. The representation space might be finite or infinite dimensional \[11, 16\]. The second line, proposed by Barut and Girardello, defines the generalized coherent states as the eigenstates of the lowering generators \[17\]. This approach applies only in the case of non-compact groups with infinite dimensional representation spaces. The third line is based on the minimization of the Robertson-Schrödinger uncertainty relations for the hermitian generators of the group symmetry \[7, 18, 19\] (see also \[20, 21\]).

The Perelomov and minimum Robertson-Schrödinger uncertainty approaches apply for Lie symmetries with infinite as well finite dimensional representation spaces. However, the Barut–Girardello approach can be employed only for non-compact Lie groups (\(SU(1, 1)\) for instance) \[17\]. Recently, the construction of \(SU(2)\) coherent states of Barut–Girardello type has been considered in \[37\] and it has been shown that the eigenstates of the lowering angular momentum operator exist if the eigenvalues are no longer complex numbers but are variables generalizing the usual Grassmann variables. One of the purposes of the present work is to continue the study reported in \[37\] in order to derive the Barut–Girardello coherent states for \(su(r+1)\) algebras \((r = 1, 2, \cdots)\) using the formalism of generalized Grassmann variables. We note that the mathematical structures and the properties of this kind of variables were investigated in the context of non commutative geometry in several works (see for instance \[22, 23, 24, 25, 26, 27, 28, 29\] ). They were employed to formulate the coherent states of quantum systems with finite dimensional Hilbert space such as deformed harmonic oscillators \[30, 31, 32, 33\] and pseudo-Hermitian quantum systems \[34, 35, 36\].

The essential algebraic tool of this paper is the generalized Weyl-Heisenberg algebra \(A(r)\) \((r \geq 1)\) generated by \(r\) creation, \(r\) annihilation and \(r\) number operators. This algebra covers the generalized Weyl-Heisenberg algebras \(A(1)\) and \(A(2)\) introduced in \[38, 39, 40\] and provides the appropriate algebraic tools to describe finite quantum systems such as qubits and qudits. Furthermore, the generalized Grassmann variables emerge naturally in the analytical representations of these generalized oscillator algebras and provide us with the adequate ingredient to construct the Barut–Girardello coherent states for \(su(r+1)\) algebras. It must be emphasized that the idea of this work originates from the recent developments discussed in \[37\] to determine the Barut–Girardello coherent states for \(su(2)\) algebra.

This paper is organized as follows. The first section introduces the generalized Weyl-Heisenberg algebra \(A(r)\). The corresponding finite dimensional Fock representations are also presented. In section 3, we develop the relation between the Weyl-Heisenberg algebra \(A(1)\) and qukit systems \((k\text{-level}...\} \)...

systems). We also discuss the correspondence between Dicke states and the Fock vectors generating the representation space for the algebra $\mathcal{A}(1)$. This correspondence is useful to provide the realization of the generalized Grassmann variables in terms of the usual Grassmann variables and subsequently to define the generalized Grassmann derivative and integration. Section 4 deals with the Fock–Bargmann representation of the algebra $\mathcal{A}(r)$. In this representation the creation operators acts as multiplication by a generalized Grassmann variables. In section 5, we give the $su(r+1)$ coherent states of Barut–Girardello labelled by $r$ complex variable and a generalized Grassmann variable. The over-completion property of the obtained sets of coherent states is examined. Concluding remarks close this paper.

2 Generalized Weyl-Heisenberg algebras

2.1 The algebra $\mathcal{A}(r)$

We begin by introducing the generalized Weyl-Heisenberg algebra $\mathcal{A}(r)$. This algebra is generated by $3r$ operators $a_i^-, a_i^+$ and $N_i$ ($i = 1, 2, \ldots, r$). They satisfy the commutation relations

$$[a_i^-, a_i^+] = k I - \left( \sum_{j=1}^{r} N_j + N_i \right), \quad [N_i, a_j^\pm] = \pm \delta_{i,j} a_i^\pm, \quad i, j = 1, 2, \ldots, r,$$

and

$$[a_i^\pm, a_j^\pm] = 0, \quad i \neq j,$$

complemented by the triple relations

$$[a_i^\pm, [a_i^\pm, a_j^\mp]] = 0, \quad i \neq j.$$  \hfill (3)

In Eq. (1), $I$ denotes the identity operator and $k \in \mathbb{N}^*$. This algebra extends the generalized Weyl-Heisenberg algebras $\mathcal{A}(1)$ and $\mathcal{A}(2)$ introduced in [39, 40]. Indeed, for $r = 1$, the algebra $\mathcal{A}(1)$ is spanned by the three linear operators $a_-, a_+$ and $N$ satisfying the following relations

$$[a_-, a_+] = k I - 2N \quad [N, a_\pm] = \pm a_\pm. \hfill (4)$$

For $r = 2$, the algebra $\mathcal{A}(2)$ is generated by six linear operators $a_i^-, a_i^+$ and $N_i$ ($i = 1, 2$) satisfying the structure relations

$$[a_i^-, a_i^+] = k I - (N_1 + N_2 + N_i), \quad [N_i, a_j^\pm] = \pm \delta_{i,j} a_i^\pm, \quad i, j = 1, 2 \hfill (5)$$

and

$$[a_i^\pm, a_j^\pm] = 0, \quad i \neq j, \quad [a_i^\pm, [a_i^\pm, a_j^\mp]] = 0, \quad i \neq j.$$  \hfill (6)

Remark that the $\mathcal{A}(r)$ algebra is similar to the bosonic algebra introduced by Palev [41] which describes a collection of identical particles obeying the so-called $A_r$–statistics. This algebraic description was further investigated from the microscopic point of view byPalev and Van der Jeugt [42]. The creation and annihilation of particles obeying the $A_r$–statistics are identified with the $2r$ Weyl generators of the $su(r+1)$ Lie algebra [41, 42]. In this sense, it is interesting to note that the $su(r+1)$ generators can
be realized in terms of the creation and the annihilation operators of the generalized Weyl-Heisenberg algebra $\mathcal{A}(r)$ as follows

$$E_{+\alpha} = a_\alpha^+, \quad E_{-\alpha} = a^\alpha_, \quad \alpha = 1, 2, \cdots, r$$

$$H_i = \frac{1}{2} \left( k \mathbb{I} - \sum_{j=1}^r (N_j + N_i) \right), \quad i = 1, 2, \cdots, r.$$ 

The creation and annihilation operators coincide with the Weyl generators $E_{\pm\alpha}$. The Cartan generators $H_i$ are expressed in terms of the number operators $N_i$ ($i = 1, 2, \cdots, r$). The remaining $r^2 - r$ generators of $su(r+1)$ algebra are realized as the commutators between the annihilation and creation operators as

$$E_{+\alpha,-\beta} = [a_\alpha^+, a^\beta_\alpha], \quad E_{+\beta,-\alpha} = [a_\beta^+, a^\alpha_\beta] \quad (\alpha < \beta, \quad \alpha, \beta = 1, 2, \cdots, r).$$

We stress that the algebra $\mathcal{A}(r)$ is defined by means of $3r$ generators, satisfying commutation relations (1), (2) and the triple commutation relations (3), rather than the usual $r(r+2)$ generators for the $su(r+1)$ Lie algebra.

### 2.2 The Fock representation

Let us denote the Hilbert-Fock space of the algebra $\mathcal{A}(r)$ by $\mathcal{F}$. It is defined by

$$\mathcal{F} = \bigoplus_{n=0}^\infty \mathcal{H}^n,$$  

where $\mathcal{H}^n \equiv \{|n_1, n_2, \cdots, n_r\}, n_i \in \mathbb{N}, \sum_{i=1}^r n_i = n > 0\}$ and $\mathcal{H}^0 \equiv \mathbb{C}$. The Fock states $|n_1, n_2, \cdots, n_r\rangle$ are the eigenvectors of the operators number $N_i$:

$$N_i|n_1, n_2, \cdots, n_r\rangle = n_i|n_1, n_2, \cdots, n_r\rangle.$$ 

The action of the creation and annihilation operators $a_i^\pm$, on $\mathcal{F}$, are defined by

$$a_i^\pm|n_1, n_2, \cdots, n_r\rangle = \sqrt{F_i(n_1, \cdots, n_i \pm 1, \cdots, n_r, n_1, \cdots, n_i \pm 1, \cdots, n_r)}|n_1, \cdots, n_r\rangle, \quad i = 1, 2, \cdots, r.$$  

where the structure functions $F_i$ can be determined by employing the structure relations (1), (2) and (3). They should be non-negatives so that all states are well defined. We assume that $|0, 0, \cdots, 0\rangle$ is the vacuum from which the states $|n_1, n_2, \cdots, n_r\rangle$ are generated by repeated applications of the raising operators $a_i^+$. The condition $a_i^-|0, 0, \cdots, 0\rangle = 0$ implies that the functions $F_i(n_1, \cdots, n_i, \cdots, n_r)$ satisfy

$$F_i(n_1, \cdots, 0, \cdots, n_r) = 0$$

for any mode $i$ ($i = 1, 2, \cdots, r$). Considering the commutation rules (1), one gets the following recurrence relations

$$F_i(n_1, \cdots, n_i + 1, \cdots, n_r) - F_i(n_1, \cdots, n_i + 1, \cdots, n_r) = k - (n_i + n_2 + \cdots + n_r) - n_i,$$

from which one obtains

$$F_i(n_1, \cdots, n_i, \cdots, n_r) = n_i(k + 1 - (n_1 + \cdots + n_i + \cdots + n_r)).$$
The actions of the raising and lowering operators on the Hilbert-Fock space $\mathcal{F}$ are thus given by

$$a_i^{-}|n_1,\ldots,n_i,\ldots,n_r\rangle = \sqrt{n_i(k+1-(n_1+n_2+\cdots+n_r))}|n_1,\ldots,n_i-1,\ldots,n_r\rangle,$$

$$a_i^{+}|n_1,\ldots,n_i,\ldots,n_r\rangle = \sqrt{(n_i+1)(k+1-(n_1+n_2+\cdots+n_r+1))}|n_1,\ldots,n_i+1,\ldots,n_r\rangle.$$  

The positivity condition of the structure functions $F_i$ given by

$$k+1-(n_1+n_2+\cdots+n_r) > 0.$$  

determines the dimension of the irreducible representation space $\mathcal{F}$. Indeed, there exists a finite number of states satisfying this condition and the Fock space dimension is given by $\frac{(k+r)!}{k!r!}$. We note that the operators defined by (7) act in this representation as

$$[a_i^{-},a_j^{-}]|n_1,\ldots,n_i,\ldots,n_j,\ldots,n_r\rangle = \sqrt{n_j(n_i+1)}|n_1,\ldots,n_i+1,\ldots,n_j-1,\ldots,n_r\rangle,$$

$$[a_j^{+},a_i^{+}]|n_1,\ldots,n_i,\ldots,n_j,\ldots,n_r\rangle = \sqrt{n_i(n_j+1)}|n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_r\rangle.$$  

In the situation where $k$ is large, the algebra $A(r)$ reduces to $r$ commuting copies of the usual harmonic oscillator algebra. Indeed, using the equations (13), (14), (16) and (17) one has

$$\left[a_i^{-},a_j^{-}\right] \sim I \quad \text{and} \quad \left[a_i^{-},a_j^{+}\right] \sim 0 \quad \text{for} \quad i \neq j,$$

which describes a $r$-dimensional quantum harmonic oscillator.

### 3 Qukits and generalized Weyl-Heisenberg algebra

Dealing with bosonic and fermionic many particles states is simplified by considering the algebraic structures of the corresponding raising and lowering operators. For bosons the creation and annihilation operators satisfy the commutations relations

$$[b_i^{-},b_j^{+}] = \delta_{ij}I, \quad [b_i^{-},b_j^{-}] = [b_i^{+},b_j^{+}] = 0,$$

where the unit operator $I$ commute with the creation and annihilation operators $b_i^{+}$ and $b_i^{-}$. On the hand, fermions are specified by the following anti-commutation relations

$$\{f_i^{-},f_j^{+}\} = \delta_{ij}I, \quad \{f_i^{+},f_j^{+}\} = \{f_i^{-},f_j^{-}\} = 0.$$  

The properties of Fock states follow from the commutation and anti-commutation relations which imposes only one particle in each state for fermions (two dimensions) and multiple particles for bosons (infinite dimension). Following Wu and Vidal [43] there is a crucial difference between fermions and qubits (two level systems). In fact, a qubit is a vector in a two dimensional Hilbert space like fermions and the Hilbert space of a multi-qubit system has a tensor product structure like bosons. In this respect, the raising and lowering operators commutation rules for qubits are neither specified by relations of bosonic type (18) nor of fermionic type (19).
3.1 Qubit algebra from generalized Weyl-Heisenberg algebra

The qubits appear like objects which exhibit both bosonic and fermionic properties so that they cannot be described by Fermi-like or Bose-like operators. An alternative way for the algebraic description of qukits \( (k+1)\)-level quantum systems) is possible by resorting to the formalism of generalized Weyl-Heisenberg algebras. We denote by \(|-\rangle\) the ground state and \(|+\rangle\) the excited state of a two-level system (qubit) so that the lowering, raising and number operators are defined by

\[
q^- = |-\rangle\langle+|, \quad q^+ = |+\rangle\langle-|, \quad N_q = |-\rangle\langle+|.
\]  

They satisfy the commutation relations

\[
[q^-, q^+] = I_2 - 2N_q, \quad [N_q, q^+] = -q^+, \quad [N_q, q^-] = +q^-,
\]  

where \(I_2\) is the \(2 \times 2\) identity matrix. In this scheme, the qubit is described by the modified bosonic algebra (21) and the creation and the annihilation operators satisfy the nilpotency condition

\[
(q^+)^2 = (q^-)^2 = 0
\]  

like Fermi operators. This representation turns out to be a particular case of the finite dimensional representations of the generalized Weyl-Heisenberg algebra \(A(1)\) corresponding to the situation where \(r = 1\) with \(k = 1\). We note that the commutation relations (21) coincide with ones defining the algebra introduced in [44] to provide an alternative algebraic description of qubits instead of the parafermionic formulation considered in [43].

3.2 Qukit algebra and Dicke states

To extend the above qubit description to qukits \((k+1)\)-dimensional quantum systems \((k \in \mathbb{N}^*)\), we consider a collection of \(k\) copies of the algebra (21) generated by the raising and lowering operators \(q_i^+\) and \(q_i^-\), the number operators \(N_{q_i}\) and the unit operator \(I_2\) satisfying the relations

\[
[q_i^-, q_j^+] = (I_2 - 2N_{q_i}) \delta_{ij}, \quad [N_{q_i}, q_j^+] = -\delta_{ij} q_j^+, \quad [N_{q_i}, q_j^-] = +\delta_{ij} q_j^- \quad [q_i^-, q_j^-] = [q_i^+, q_j^+] = 0,
\]  

where \(i = 1, 2, \cdots, k\). Let denote by \(H_i = \{|m_i\}, \quad m_i = -, +\) the Hilbert space for the qubit \(i\). In view of the relations \([q_i^-, q_j^-] = 0\) for \(i \neq j\), the multi-qubit Hilbert space has the following tensor product structure

\[
H(k) = \bigotimes_{i=1}^{k} H_i = \{|m_1, m_2, \cdots, m_k\}, \quad m_i = -, +\}
\]

like bosons. We define the collective lowering and raising operators in the Hilbert space \(H(k)\) as follows

\[
a^- = \sum_{i=1}^{k} q_i^- \quad a^+ = \sum_{i=1}^{k} q_i^+
\]  

in terms of the creation and annihilation operators \(q_i^+\) and \(q_i^-\). Here and in the following the index \(i\) refers to the system the operator is acting on, e.g.

\[
q_i^\pm \equiv I_2 \otimes \cdots I_2 \otimes q_i^\pm \otimes I_2 \otimes \cdots I_2.
\]
It is simple to see that the state $|-, -, \cdots, -\rangle \equiv |0\rangle$ satisfies $a^-|0\rangle = 0$. Furthermore, using the commutation relations (23), one gets the nilpotency relations

$$(a^-)^{k+1} = 0 \quad (a^+)^{k+1} = 0$$

(25)

which extends the Pauli exclusion principle for ordinary qubits (i.e., $k = 1$) described by the conditions (22). The actions of the operators $a^-$ and $a^+$ on the Hilbert space $H(k)$ can be determined from the standard actions of the fermionic operators $q_i^-$ and $q_i^+$ (cf. equations (20)). Using a recursive procedure, one verifies that repeated applications of the raising operator $a^+$ on the vacuum $|-, -, \cdots, -\rangle \equiv |0\rangle$ gives

$$(a^+)^n|0\rangle = \sqrt{\frac{n!k!}{(k-n)!}} |n\rangle$$

(26)

where the vectors $|n\rangle$ are the symmetric Dicke states with $n$ excitations ($n = 0, 1, 2, \cdots, k$). They are defined by

$$|n\rangle = \sqrt{\frac{n!(k-n)!}{k!}} \sum_{\sigma \in S_k} |-, -, \cdots, -, +, \cdots, +\rangle$$

(27)

where $S_k$ is the permutation group of $k$ objects. The Dicke states generate an orthonormal basis of the symmetric Hilbert subspace of dimension $k+1$. The explicit expressions of the actions of the ladder operators $a^\pm$ can be written using the structure function $F(n) = n(k+1-n)$ (12). The equation (26) rewrites as

$$(a^+)^n|0\rangle = \sqrt{F(n)!} |n\rangle$$

(28)

where $F(n)! = F(n)F(n-1)\cdots F(1)$ and $F(0) = 1$. After some algebra, it is simple to verify that

$$a^+|n\rangle = \sqrt{F(n+1)} |n+1\rangle, \quad a^-|n\rangle = \sqrt{F(n)} |n-1\rangle$$

(29)

and the action of the creation and annihilation operators on the vectors $|k, 0\rangle$ and $|k, k\rangle$ gives

$$a^-|0\rangle = 0 \quad a^+|k\rangle = 0.$$ 

(30)

The number operator $N$ is defined as

$$N|n\rangle = n |n\rangle.$$

(31)

The qubit operators $a^+, a^-$ and $N$ satisfy the commutation rules

$$[a^+, a^-] = k\mathbb{I} - 2N, \quad [a^+, N] = a^+, \quad [a^-, N] = -a^-,$$

(32)

which reflects that the algebra $\mathcal{A}(1)$ can be realized in terms of an ensemble of identical qubits. Using the commutation relation $[q_i^+, q_j^-] = 0$ for $i \neq j$, it is simple to verify that

$$[a^+, a^-] = \sum_{i,j} [q_i^+, q_j^-] = \sum_{i} [q_i^+, q_i^-],$$

and the operator $N$ can be expressed as

$$N = \sum_{i=1}^{k} N_{q_i}$$
where \( N_q_i \) is the single qubit number operator \( (N_q_i|\rangle_i = 0 \text{ and } N_q_i|\rangle_i = |\rangle_i) \). It is remarkable that the creation and annihilation operator \( a^+ \) and \( a^- \) close the following trilinear relation commutation

\[
[a^-, [a^+, a^-]] = 2a^-, \quad [a^+, [a^+, a^-]] = -2a^-
\]

characterizing a parafermion \([41]\). Note also that the definition \([24]\) is similar to Green decomposition in the construction of parafermions from ordinary fermions. Therefore, the operators \( a^+, a^- \) and \( N \) satisfying the relations \([32]\) provide a simple algebraic description of \((k + 1)\)-level quantum systems (qukit). This result shows the relevance of generalized Weyl-Heisenberg algebras in describing qukit systems. In particular, this realization expresses the Hilbert states of a qukit system in terms of Dicke states of \( k \) qubits. In this way, the global properties of the qukit system are encoded in an ensemble of \( k \) identical qubits. To close this section we note that the algebraic realization of qukit systems provides a natural way to define the generalized Grassmann variables associated with the generalized Weyl-Heisenberg algebras possessing finite dimensional representation spaces.

### 3.3 Generalized Grassmann variables

We consider the algebra \( \mathcal{G} \) generated by the identity \( 1 \) and \( k \) commuting Grassmann variables \( \theta_i \) \((i = 1, 2, \cdots k)\) obeying the usual nilpotency conditions:

\[
\theta_i^2 = 0, \quad [\theta_i, \theta_j] = 0.
\]  

It is important to mention that in order to simplify our purpose, we consider in this work a set of commuting Grassmann variables. We denote by \( \bar{\theta}_i \) the complex conjugate of the element \( \theta_i \). The algebra \( \mathcal{G} \) is spanned by \( 2^k \) linearly independent elements of the form \( \theta_{i_1}\theta_{i_2}\cdots\theta_{i_n} \) with \( i_1 < i_2 < \cdots < i_n \) for \( n = 0, 1, \cdots, k \). For \( n = 0 \), the corresponding element is the identity. The \( \theta \)-derivative \( \partial_{\theta_i} = \frac{\partial}{\partial \theta_i} \) satisfies

\[
\partial_{\theta_i}\theta_j = \delta_{ij}, \quad \partial_i1 = 0, \quad \partial_{\theta_i}\partial_{\theta_j} = \partial_{\theta_j}\partial_{\theta_i}.
\]

We define the generalized Grassmann variable as

\[
\eta = \sum_{i=1}^{k} \theta_i, \quad \bar{\eta} = \sum_{i=1}^{k} \bar{\theta}_i
\]

in terms of the nilpotent variables \( \theta_i \) and \( \bar{\theta}_i \). We define the following symmetric \( \theta \)-polynomials

\[
e_n(\bar{\theta}) = \sum_{i_1 < i_2 < \cdots < i_n} \theta_{i_1}\theta_{i_2}\cdots\theta_{i_n}, \text{ for } n = 1, 2, \cdots, k \text{ and } e_0(\bar{\theta}) = 1.
\]

where \( \bar{\theta} = (\theta_1, \theta_2, \cdots, \theta_n) \). Explicitly, we have

\[
e_1(\bar{\theta}) = \theta, \quad e_2(\bar{\theta}) = \sum_{i<j} \theta_i\theta_j, \quad e_3(\bar{\theta}) = \sum_{i<j<l} \theta_i\theta_j\theta_l, \cdots, \quad e_k(\bar{\theta}) = \theta_1\theta_2\cdots\theta_k.
\]

The \( n \)-th power of the variable \( \eta \) \([35]\) writes in term of the symmetric \( \theta \)-polynomials \([36]\) as

\[
\eta^n = n!e_n(\bar{\theta}) \text{ for } n = 1, 2, \cdots, k
\]
and the nilpotency conditions (33) for ordinary Grassmann numbers imply

\[ \eta^{k+1} = 0. \]

Furthermore, we define the \( \eta \)-derivative as follows

\[ \frac{\partial}{\partial \eta} = \sum_{i=1}^{k} \frac{\partial}{\partial \theta_i}, \quad \frac{\partial}{\partial \bar{\eta}} = \sum_{i=1}^{k} \frac{\partial}{\partial \bar{\theta}_i}. \]

Using the properties of \( \theta \)-derivatives (34), one shows

\[ \partial^n \eta = n! g_n \text{ for } n = 1, 2, \cdots, k \text{ and } \partial^{k+1} \eta = 0 \]

where the differential operator \( g_n \) is given by

\[ g_n = \sum_{i_1 < i_2 < \cdots < i_n} \partial_{\theta_{i_1}} \partial_{\theta_{i_2}} \cdots \partial_{\theta_{i_n}}, \text{ for } n = 1, 2, \cdots, k \text{ and } g_0 = 1. \]  

Using the symmetric \( \theta \)-polynomials (36), we define the Grassmann analogue of Dicke states (27) as

\[ D_n(\vec{\theta}) = \sqrt{\frac{n!(k-n)!}{k!}} e_n(\vec{\theta}). \]

It is interesting to note that the \( n \)-th power of the generalized Grassmann variable (35) express as

\[ \eta^n = \sqrt{\frac{n!k!}{(k-n)!}} D_n(\vec{\theta}) = n! e_n(\vec{\theta}) \]

in term of the functions \( D_n(\vec{\theta}) \) from which one shows

\[ \eta D_n(\vec{\theta}) = \sqrt{(n+1)(k-n)} D_{n+1}(\vec{\theta}). \]

This relation is similar to the action of the creation operator given by (29). To obtain the derivative of the functions \( D_n(\vec{\theta}) \), we employ first the definition (37) to get the derivative of \( \theta \)-polynomials (36):

\[ \frac{\partial e_n(\vec{\theta})}{\partial \eta} = (k-n) e_{n-1}(\vec{\theta}) \text{ for } n = 1, 2, \cdots, k \text{ and } \frac{\partial e_0(\vec{\theta})}{\partial \eta} = 0, \]

from which one gets

\[ \frac{\partial D_n(\vec{\theta})}{\partial \eta} = \sqrt{n(k+1-n)} D_{n-1}(\vec{\theta}). \]

This result is similar to the action of the annihilation operation of the qukit algebra \( A(1) \) given by (29). In this scheme, the \( \eta \)-integral can be derived from the Berezin integral of Grassmann variables given by

\[ \int \theta_i d\theta_j = \delta_{ij} \quad \int d\theta_i = 0. \]

Using the Berezin integration formula, it is simple to verify that the symmetric \( \theta \)-polynomials (36) satisfy

\[ \int e_n(\vec{\theta}) d\eta = 0 \quad (n = 0, 1, \cdots, k - 2) \quad \text{and} \quad \int e_k(\vec{\theta}) d\eta = 1. \]
where \( d\eta = d\theta_1 d\theta_2 \cdots d\theta_k \). This gives the following \( \eta \)-integral formulas

\[
\int \eta^n \, d\eta = 0 \quad (n = 0, 1, \cdots, k-1) \quad \text{and} \quad \int \eta^k \, d\eta = k!.
\]

The usual Berezin integration for ordinary Grassmann variables is recovered for \( k = 1 \). Similarly, for the conjugate generalized Grassmann variables we have the following integration rules

\[
\int \bar{\eta}^n \, d\bar{\eta} = 0 \quad (n = 0, 1, \cdots, k-1), \quad \int \bar{\eta}^k \, d\bar{\eta} = k!.
\]

The generalized integration formulas \((45)\) and \((46)\) are of especially important in deriving the over-completion property of the Barut–Girardello coherent of \( su(r+1) \). This issue is discussed in Section 5.

4 Fock–Bargmann realization of generalized Weyl-Heisenberg algebra

In the Fock–Bargmann representation of the usual Heisenberg-Weyl algebra, the creation and annihilation operators are respectively realized as multiplication and derivation with respect a complex variable \([45, 46]\). This representation is widely used in several problems of quantum physics and mathematics \([4]\). Thus, it is natural to determine the Fock–Bargmann spaces for algebras of \( A(r) \) type. In this sense, the main goal of this section concerns the characterization of any vector state in the Hilbert space of the generalized algebra \( A(r) \) by an analytic function expressed in terms of generalized Grassmann variables.

4.1 The analytical representation of the generalized algebra \( A(1) \)

We shall first discuss the analytical realization of the algebra \( A(1) \) in which creation operator \( a^+ \) acts as multiplications by the variable \( \eta \). We realize the Fock space basis and the creation operation as

\[
|n\rangle \rightarrow f_n(\eta) = c_n \eta^n, \quad a^+ \rightarrow \eta.
\]

The nilpotency relation \((25)\) implies that the variable \( \eta \) satisfies

\[
\eta^{k+1} = 0
\]

From the correspondence \((47)\), we write

\[
|0\rangle \rightarrow 1
\]

such that \( c_0 = 1 \). Using the actions of the creation and annihilation operators \((29)\), one verifies

\[
|n\rangle = \sqrt{\frac{(k-n)!}{k!n!}} (a^+)^n |0\rangle \quad n \leq k,
\]

and using \((47)\), one gets \( c_n = \sqrt{(k-n)!/k!n!} \). therefore, the analytical functions \( f_n(\eta) \) read as

\[
f_n(\eta) = \sqrt{\frac{(k-n)!}{k!n!}} \eta^n.
\]
It is simple to check that the multiplication and the derivative of the functions \( f_n(\eta) \) with respect to the variable \( \eta \) leads to

\[
\eta f_n(\eta) = \sqrt{(n + 1)(k - n)} f_{n-1}(\eta), \quad \frac{\partial}{\partial \eta} f_n(\eta) = \sqrt{n(k + 1 - n)} f_{n-1}(\eta).
\] (52)

From the last equation, one verifies that the \( \eta \)-derivative satisfy the condition

\[
\left( \frac{\partial}{\partial \eta} \right)^{k+1} = 0.
\] (53)

For \( k = 1 \), the qubit becomes a qubit system and the relations (48) and (53) reduces to the nilpotency conditions for the usual Grassmann variables.

### 4.2 The analytical representation of the generalized algebra \( \mathcal{A}(2) \)

The \( \mathcal{A}(2) \) algebra is spanned by two pairs of creation and annihilation operators \( a_1^+, a_2^+ \) with \( i = 1, 2 \) and two number operators \( N_i \). They satisfy the structures relations (1), (2) and (3). The dimension of the Fock space \( \mathcal{F}_k = \{ |n_1, n_2 \rangle : n_1 \in \mathbb{N}, n_2 \in \mathbb{N}; n_1 + n_2 \leq k \} \) is \( d = (k + 1)(k + 2)/2 \). The vectors \( |n_1, n_2 \rangle \) are the eigenstates of the number operators \( N_1 \) and \( N_2 \) \( (N_1|n_1, n_2 \rangle = n_1|n_1, n_2 \rangle, \ i = 1, 2 \).

From the equations (13) and (14), the raising and lowering operators \( a_1^\pm \) and \( a_2^\pm \) act as

\[
a_1^+ |n_1, n_2 \rangle = \sqrt{(n_1 + 1)(k - n_1 - n_2)} |n_1 + 1, n_2 \rangle, \quad a_1^- |n_1, n_2 \rangle = \sqrt{n_1(k + 1 - n_1 - n_2)} |n_1 - 1, n_2 \rangle \] (54)

and

\[
a_2^+ |n_1, n_2 \rangle = \sqrt{(n_2 + 1)(k - n_1 - n_2)} |n_1, n_2 + 1 \rangle, \quad a_2^- |n_1, n_2 \rangle = \sqrt{n_1(k + 1 - n_1 - n_2)} |n_1, n_2 - 1 \rangle. \] (55)

Using the actions (54) and (55), one verifies that the creation and annihilation operators satisfy the conditions

\[
(a_1^+)^{k+1-l}(a_2^+)^l = 0, \quad (a_1^-)^{k+1-l}(a_2^-)^l = 0, \quad \text{for } l = 0, 1, 2, \ldots, k + 1.
\] (56)

To discuss the analytical realization of the algebra \( \mathcal{A}(2) \), in which creation operator \( a_i^+ \) acts as multiplications by the generalized Grassmann variables \( \eta_i \) with \( i = 1, 2 \), we realize the Fock space basis as

\[
|n_1, n_2 \rangle \rightarrow f_{n_1, n_2}(\eta_1, \eta_2) = c_{n_1, n_2} \eta_1^{n_1} \eta_2^{n_2} \quad a_i^+ \rightarrow \eta_i.
\] (57)

Using the relations (56), the variables \( \eta_1 \) and \( \eta_2 \) satisfy

\[
(\eta_1)^{k+1-l}(\eta_2)^l = 0, \quad (\eta_1)^{k+1-l}(\eta_2)^l = 0,
\] (58)

for \( l = 0, 1, 2, \ldots, k + 1 \). From the correspondence (56), one can write

\[
|0, 0 \rangle \rightarrow 1
\] (59)

where we set \( c_{0,0} = 1 \). Using the actions of the creation and annihilation operators (54) and (55), one verifies

\[
|n_1, n_2 \rangle = \frac{(k - n_1 - n_2)!}{k! n_1! n_2!} (a_1^+)^{n_1} (a_2^+)^{n_2} |0, 0 \rangle \quad n_1 + n_2 \leq k,
\] (60)
from which one gets the expressions of the analytical functions \( f_{n_1,n_2}(\eta_1, \eta_2) \) associated with the Fock states \( |n_1,n_2\rangle \)

\[
f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{\frac{(k-n_1-n_2)!}{k!n_1!n_2!}} \eta_1^{n_1} \eta_2^{n_2}.
\]

(61)

The derivative with respect to the variables \( \eta_1 \) and \( \eta_2 \) gives

\[
\frac{\partial}{\partial \eta_1} f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{n_1(k+1-(n_1+n_2))} f_{n_1-1,n_2}(\eta_1, \eta_2),
\]

(62)

\[
\frac{\partial}{\partial \eta_2} f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{n_2(k+1-(n_1+n_2))} f_{n_1,n_2-1}(\eta_1, \eta_2).
\]

(63)

It follows that the derivatives satisfy the conditions

\[
\left( \frac{\partial}{\partial \eta_1} \right)^{k+1-l} \left( \frac{\partial}{\partial \eta_2} \right)^l = 0; \quad l = 0, 1, 2, \ldots, k + 1,
\]

(64)

which gives the analytical analogue of the hybrid nilpotency relations satisfied by the annihilation operators \( a_1^+ \) and \( a_2^+ \) given by the equations \( \text{(56)} \).

4.3 The analytical representation of the generalized algebra \( \mathcal{A}(r) \)

The Fock–Bargmann representations corresponding to \( \mathcal{A}(1) \) and \( \mathcal{A}(2) \) algebras can be easily extended to higher ranks \( r \). Indeed, the representation space basis of the algebra \( \mathcal{A}(r) \) can be realized as

\[
|n_1,n_2,\cdots,n_r\rangle \rightarrow f_{n_1,n_2,\cdots,n_r}(\eta_1,\eta_2,\cdots,\eta_r) = c_{n_1,n_2,\cdots,n_r} \eta_1^{n_1} \eta_2^{n_2} \cdots \eta_r^{n_r} \quad a_i^+ \rightarrow \eta_i.
\]

(65)

In particular, we assume that the ground state is represented as

\[
|0,0,\cdots,0\rangle \rightarrow 1,
\]

(66)

by considering \( c_{0,0,\cdots,0} = 1 \). Using the actions of the creation and annihilation operators \( \text{(13)} \) and \( \text{(14)} \), one verifies

\[
|n_1,n_2,\cdots,n_r\rangle = \sqrt{\frac{(k-n_1-n_2-\cdots-n_r)!}{k!n_1!n_2!\cdots n_r!}} (a_1^+)^{n_1} (a_2^+)^{n_2} \cdots (a_r^+)^{n_r} |0,0,\cdots,0\rangle \quad n_1+n_2+\cdots+n_r \leq k,
\]

(67)

and the function \( f_{n_1,n_2,\cdots,n_r}(\eta_1,\eta_2,\cdots,\eta_r) \), representing an arbitrary Fock vector \( |n_1,n_2,\cdots,n_r\rangle \), is given by

\[
f_{n_1,n_2,\cdots,n_r}(\eta_1,\eta_2,\cdots,\eta_r) = \sqrt{\frac{(k-n_1-n_2-\cdots-n_r)!}{k!n_1!n_2!\cdots n_r!}} \eta_1^{n_1} \eta_2^{n_2} \cdots \eta_r^{n_r}.
\]

(68)

From the equation \( \text{(67)} \), one can see that the creation operators satisfy the conditions

\[
(a_1^+)^{l_1} (a_2^+)^{l_2} \cdots (a_r^+)^{l_r} = 0 \quad l_1 + l_2 + \cdots l_r = k + 1.
\]

(69)

Hence, the variables \( \eta_i \) satisfy the following generalized nilpotency conditions

\[
(\eta_1)^{l_1} (\eta_2)^{l_2} \cdots (\eta_r)^{l_r} = 0 \quad l_1 + l_2 + \cdots l_r = k + 1.
\]

(70)
Using the derivative formula with respect a generalized Grassmann variable discussed in the previous section, one obtains the derivative of the functions $f_{n_1, n_2, \ldots, n_r}(\eta_1, \eta_2, \ldots, \eta_r)$

$$\frac{\partial}{\partial \eta_i} f_{n_1,\ldots,n_{i-1},n_i\ldots,n_r}(\eta_1,\ldots,\eta_{i-1},\eta_i,\ldots,\eta_r) = \sqrt{n_i(k + 1 - (n_1 + \cdots + n_i + \cdots + n_r))} f_{n_1,\ldots,n_{i-1},n_i\ldots,n_r}(\eta_1,\ldots,\eta_{i-1},\eta_i,\ldots,\eta_r)$$

(71)

From this result, one has the following nilpotency conditions

$$\left( \frac{\partial}{\partial \eta_1} \right)^{l_1} \left( \frac{\partial}{\partial \eta_2} \right)^{l_2} \cdots \left( \frac{\partial}{\partial \eta_r} \right)^{l_r} = 0, \quad l_1 + l_2 + \cdots + l_r = k + 1,$$

(72)

which reduces to the conditions (53) and (64) for $r = 1$ and $r = 2$, respectively.

### 5 Barut-Girardello coherent states of $su(r + 1)$ algebras

#### 5.1 The spin coherent states à la Barut-Girardello

The irreducible set associated with a spin $j (2j \in \mathbb{N})$ is spanned by the basis

$$B_{2j+1} = \{|j, m\rangle : m = j, j-1, \ldots, -j\},$$

(73)

where $|j, m\rangle$ is a common eigenvector of the Casimir operator $j^2$ and of the Cartan operator $j_z$ of the Lie algebra $su(2)$:

$$j^2 |j, m\rangle = j(j + 1)|j, m\rangle, \quad j_z |j, m\rangle = m |j, m\rangle.$$  

(74)

The raising and lowering operators are given by

$$j_+ = \sum_{m=-j}^{j} \sqrt{(j + m + 1)(j - m)} |j, m + 1\rangle \langle j, m|, \quad j_- = \sum_{m=-j}^{j} \sqrt{(j + m)(j - m + 1)} |j, m - 1\rangle \langle j, m|.$$  

(75)

They satisfy the structure relations

$$[j_z, j_+] = +j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = 2j_z.$$  

(76)

In what follows, we make the identifications

$$|j, m\rangle \leftrightarrow |n\rangle \quad j + m \leftrightarrow n,$$

so that the Hilbert space $B_{2j+1}$ is given by

$$B_{2j+1} = \{|n\rangle : n = 0, 1, \ldots, 2j\}.$$  

To construct the $su(2)$ coherent states à la Barut-Girardello, one has to determine the eigenstates of the lowering operator $j_-$ by solving the following eigenvalue equation

$$j_- |\lambda\rangle = \lambda |\lambda\rangle \quad |\lambda\rangle = \sum_{n=0}^{2j} C_n \lambda^n |n\rangle.$$  

(77)

Using the action of $j_-$ (75), one gets the following recurrence relation

$$C_{n+1} \sqrt{(n+1)(2j-n)} = C_n, \quad \text{for} \quad n = 0, 1, \ldots, 2j - 1,$$

(78)
with the extremal condition

$$C_{2j} \lambda^{2j+1} = 0. \tag{79}$$

From the equations (78) and (79), it is clear that the eigenvalue equation admits a solution if the variable $\lambda$ satisfies

$$\lambda^{2j+1} = 0. \tag{80}$$

The eigenvalue $\lambda$ can be written

$$\lambda = \eta z, \tag{81}$$

where $\eta$ is a generalized Grassmann variable of order $2j + 1$ and $z$ is an arbitrary complex variable which commute with $\eta$. The coefficients in the expansion of the states $|\eta\rangle$ (77) writes

$$C_n = \sqrt{\frac{(2j-n)!}{n!(2j)!}} C_0, \tag{82}$$

where $C_0$ can be fixed from the normalization condition of the states $|\lambda\rangle \equiv |\eta, z\rangle$. As result one obtains

$$|\eta, z\rangle = \mathcal{N}_2 \sum_{n=0}^{2j} \sqrt{\frac{(2j-n)!}{n!(2j)!}} \eta^n z^n |n\rangle, \tag{83}$$

where the normalization factor is given by

$$|\mathcal{N}_2|^{-2} = \sum_{n=0}^{2j} \frac{(2j-n)!}{n!(2j)!} \bar{\eta}^n \eta^n \bar{z}^n z^n. \tag{84}$$

It is remarkable that the states $|\eta, z\rangle$ reduce for $z = 1$ to the so-called $(2j + 1)$-fermionic coherent states introduced in [37]. In addition, they constitute an over-complete set

$$\int |\eta, z\rangle d\mu(\eta, \bar{\eta}, z, \bar{z}) \langle \eta, z| = \sum_{n=0}^{2j} |n\rangle \langle n|, \tag{85}$$

where the measure $d\mu(\eta, \bar{\eta}, z, \bar{z})$ takes the form

$$d\mu(\eta, \bar{\eta}, z, \bar{z}) = |\mathcal{N}_2|^{-2} \sigma_{2j}(\eta, \bar{\eta}) d\eta d\bar{\eta} d\mu dm(z, \bar{z}), \tag{86}$$

where the measure involving the complex variable $z$

$$dm(z, \bar{z}) = e^{-|z|^2} \frac{d^2z}{\pi},$$

coincides with the standard one with respect which the Glauber coherent states of the usual Weyl-Heisenberg (quantum harmonic oscillator) satisfy the identity resolution property. Reporting (85) in (84) implies that the function $\sigma_{2j}(\eta, \bar{\eta})$ must verify the integral formula

$$\int \sigma_{2j}(\eta, \bar{\eta}) d\eta d\bar{\eta} \bar{\eta}^n \eta^n = \frac{(2j)!}{(2j-n)!}. \tag{87}$$
Expanding the function $\sigma_{2j}(\eta, \bar{\eta})$ as

$$\sigma_{2j}(\eta, \bar{\eta}) = \sum_{n=0}^{2j} a_n \eta^{2j-n} \bar{\eta}^{2j-n},$$

and using the generalized Berezin integration (45) and (46), one checks that the integral equation (86) holds for $a_n = \frac{1}{(2j)!} \frac{1}{(2j-n)!}$ and the explicit form of the function $\sigma_{2j}(\eta, \bar{\eta})$ is then given by

$$\sigma_{2j}(\eta, \bar{\eta}) = \sum_{n=0}^{2j} \frac{1}{(2j)!} \frac{1}{(2j-n)!} \eta^{2j-n} \bar{\eta}^{2j-n}. \quad (87)$$

### 5.2 The $su(3)$ coherent states à la Barut-Girardello

Inspired by the Barut–Girardello coherent states for spin systems, we continue with the $SU(3)$ symmetry. The generators of the $su(3)$ algebra are denoted by $j_i^-$, $j_i^+$ and $j_i^0$ with $i = 1, 2$. They satisfy the following commutation relations

$$[j_i^+, j_j^-] = 2j_i^0, \quad [j_i^0, j_j^\pm] = \pm\delta_{i,j} j_i^\pm, \quad i, j = 1, 2, \quad [j_1^\pm, j_2^\pm] = 0, \quad (88)$$

complemented by the triple relations

$$[j_1^\pm, [j_1^\pm, j_2^\mp]] = 0, \quad [j_2^\pm, [j_2^\pm, j_1^\mp]] = 0. \quad (89)$$

The remaining $su(3)$ operators are defined by

$$j_3^- = [j_2^+, j_1^-], \quad j_3^- = [j_1^+, j_2^-], \quad (90)$$

in terms of the $j_i^-$, $j_i^+$ ($i = 1, 2$) and the corresponding commutation rules are encoded in the triple structure relations (89). The dimension $d(s, t)$ of the irreducible representation $(s, t)$ of $SU(3)$ is given by

$$d(s, t) = \frac{1}{2} (s+1)(t+1)(s+t+2), \quad s \in \mathbb{N}, \quad t \in \mathbb{N}. \quad (91)$$

For the irreducible representation $(0, k)$, or its adjoint $(k, 0)$, the dimension of representation space is given by

$$d = \frac{1}{2} (k+1)(k+2), \quad k \in \mathbb{N}^*. \quad (92)$$

The associated Hilbert-Fock space is

$$B_k = \{ |n_1, n_2\rangle : n_1 + n_2 = 0, 1, \ldots, k \}$$

where the vectors generating the orthonormal basis of $B_k$ are the eigenstates of the the number operators $N_1$ and $N_2$:

$$N_i |n_1, n_2\rangle = n_i |n_1, n_2\rangle, \quad i = 1, 2. \quad (92)$$

The action of the raising and lowering operators $j_1^+$ and $j_2^+$ are given by

$$j_1^+ |n_1, n_2\rangle = \sqrt{(n_1+1)[k-(n_1+n_2)]} |n_1+1, n_2\rangle, \quad j_1^- |n_1, n_2\rangle = \sqrt{n_1[k+1-(n_1+n_2)]} |n_1-1, n_2\rangle. \quad (93)$$
and

\[ j^n_1 |n_1, n_2\rangle = \sqrt{(n_2 + 1)[k - (n_1 + n_2)]} |n_1, n_2 + 1\rangle, \quad j^-_1 |n_1, n_2\rangle = \sqrt{n_2[k + 1 - (n_1 + n_2)]} |n_1, n_2 - 1\rangle. \tag{94} \]

The actions of the Cartan generators read as

\[ j^0_1 |n_1, n_2\rangle = \left[ n_1 + \frac{1}{2}(n_2 - k) \right] |n_1, n_2\rangle, \quad j^0_2 |n_1, n_2\rangle = \left[ n_2 + \frac{1}{2}(n_1 - k) \right] |n_1, n_2\rangle. \tag{95} \]

The actions of the operators \((j^+_3, j^-_3)\) defined by (91) in terms of the generators \((j^+_1, j^-_1)\) and \((j^+_2, j^-_2)\) can be determined from (93)-(94). One obtains

\[ j^+_3 |n_1, n_2\rangle = \sqrt{n_1(n_2 + 1)} |n_1 - 1, n_2 + 1\rangle, \quad j^-_3 |n_1, n_2\rangle = \sqrt{(n_1 + 1)n_2} |n_1 + 1, n_2 - 1\rangle. \tag{96} \]

The \(su(3)\) lowering operators \(j^-_1\) and \(j^-_2\) commute and therefore can be diagonalized simultaneously. To find the common set of eigenstates we consider the solutions of the following eigenvalue equations

\[ j^+_1 |\lambda_1, \lambda_2\rangle = \lambda_1 |\lambda_1, \lambda_2\rangle, \quad j^-_2 |\lambda_1, \lambda_2\rangle = \lambda_2 |\lambda_1, \lambda_2\rangle. \tag{97} \]

Expanding the state \(|\lambda_1, \lambda_2\rangle\) as

\[ |\lambda_1, \lambda_2\rangle = \sum_{l=0}^{k} \sum_{n=0}^{k-l} C_{n,l} \lambda_1^l \lambda_2^l |n, l\rangle, \tag{98} \]

the eigenvalues equation (97) of the first mode gives the following recurrence relations

\[ C_{n+1,l} \sqrt{(n + 1)[k - (n + l)]} = C_{n,l}, \quad n = 0, 1, 2, \ldots k - l - 1, \tag{99} \]

with the following conditions that must be satisfied by the eigenvalues \(\lambda_1\) and \(\lambda_2\)

\[ \lambda_1^{k-l+1} \lambda_2^l = 0, \tag{100} \]

for \(l = 0, 1, \ldots, k+1\). Similarly using the action of the operator \(j^-_2\) in Eq.(94), the eigenvalues equation (97) for the second mode gives

\[ C_{n,l+1} \sqrt{(l + 1)[k - (n + l)]} = C_{n,l}, \quad l = 0, 1, 2, \ldots k. \tag{101} \]

The conditions (100) are satisfied simultaneously for the variables \(\lambda_1\) and \(\lambda_2\) given by

\[ \lambda_1 = \eta z_1, \quad \lambda_2 = \eta z_2, \quad (z_1, z_2) \in \mathbb{C}^2 \]

in terms of the generalized Grassmann variable \(\eta\) of order \(k+1\) \((\eta^{k+1} = 0)\). From the recurrence relation (101), one shows

\[ C_{n,l} = C_{0,l} \sqrt{(k - n - l)!} / (k! n!) \tag{102} \]

On the other hand, using the recurrence relation (101), it is simple to verify that

\[ C_{0,l} = C_{0,0} \sqrt{(k - l)!} / (k! l!). \tag{103} \]
It follows that the expansion coefficients $C_{n,l}$ are given by

$$C_{n,l} = C_{0,0} \sqrt{\frac{(k - n - l)!}{k! \ n! \ l!}}. \quad (104)$$

Finally the eigenstates $|\lambda_1, \lambda_2 \rangle \equiv |\eta, z_1, z_2 \rangle$ write

$$|\eta, z_1, z_2 \rangle = \mathcal{N}_3 \sum_{l=0}^{k} \sum_{n=0}^{k-l} \sqrt{\frac{(k - n - l)!}{k! \ n! \ l!}} \eta^{n+l} z_1^n z_2^l |n, l\rangle. \quad (105)$$

where $\mathcal{N}_3$ is fixed from the normalization condition of the states $|\theta_1, \theta_2 \rangle$. Indeed, one gets

$$|\mathcal{N}_3|^{-2} = \sum_{l=0}^{k} \sum_{n=0}^{k-l} \sqrt{\frac{(k - n - l)!}{k! \ n! \ l!}} \eta^{n+l} \eta^{n+l} |z_1|^{2n} |z_2|^{2l}. \quad (106)$$

The coherent states $|\eta, z_1, z_2 \rangle$ satisfy the identity resolution

$$\int |\eta, z_1, z_2 \rangle d\mu(\eta, \bar{\eta}, z_1, z_2) \langle \eta, z_1, z_2 | = \sum_{l=0}^{k} \sum_{n=0}^{k-l} |n, l\rangle \langle n, l| \quad (107)$$

where the measure $d\mu(\eta, \bar{\eta}, z_1, z_2, \bar{z}_2)$ is obtained as for $su(2)$ case. It can written as

$$d\mu(\eta, \bar{\eta}, z_1, z_2, \bar{z}_2) = |\mathcal{N}_3|^{-2} \sigma(\eta, \bar{\eta}) d\eta d\bar{\eta} e^{-(|z_1|^2+|z_2|^2)} \frac{d^2 z_1 d^2 z_2}{\pi^2}. \quad (108)$$

The sum over the complex variables implies that the function $\sigma(\eta, \bar{\eta})$ must satisfy the integral equation

$$\int \sigma(\eta, \bar{\eta}) d\eta d\bar{\eta} \eta^{n+l} \bar{\eta}^{n+l} = \frac{k!}{(k-n-l)!} \quad (109)$$

which coincides with (5.2) for $k = 2j$. Indeed, one gets

$$\sigma(\eta, \bar{\eta}) = \sum_{n=0}^{k} \frac{1}{k!(k-n)!} \eta^{k-n} \bar{\eta}^{k-n}. \quad (110)$$

### 5.3 The Barut-Girardello coherent states for $su(r+1)$ algebra

The algebra $su(r+1)$ is defined by the generators $e_i, f_i, h_i$ ($i = 1, 2, \ldots, r$) and the relations

$$[e_i, f_j] = \delta_{ij} h_j, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \quad (111)$$

$$[e_i, e_j] = 0 \quad [f_i, f_j] = 0 \quad \text{for} \quad |i-j| > 1 \quad (112)$$

$$e_i^2 e_{i+1} - 2e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0, \quad f_i^2 f_{i+1} - 2f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0 \quad (113)$$

where $(a_{ij})_{i,j=1,2,\ldots,r}$ is the Cartan matrix of $su(r+1)$, i.e. $a_{ii} = 2$, $a_{i,i+1} = -1$ and $a_{ij} = 0$ for $|i-j| > 1$. To construct the Barut–Girardello coherent states of $su(r+1)$ algebras, we first recall some necessary elements related to irreducible unitary representations for $SU(r+1)$. We denote $D_r^k$ as a representation of $SU(r+1)$ where $k$ determines the dimension of the representation. The matrix representation is easily obtained by employing the bosonic realization in which an adapted
basis is given in term of \((r + 1)\) bosonic pairs of creation and annihilation operators; They satisfy the commutation relations
\[
[b^+_l, b^-_l] = \delta_{kl},
\]
where \(k, l = 0, 1, 2, \ldots, r\). The simultaneous eigenstates \(||n_0, n_1, \ldots, n_r\rangle\rangle \ (n_i \in \mathbb{N})\) are the tensorial product of the eigenstates of the occupation numbers \(b^+_l b^-_l\). The Fock space is then generated by the states
\[
||n_0, n_1, \ldots, n_r\rangle\rangle = \frac{(b^+_0)^{n_0}}{\sqrt{n_0!}} \frac{(b^+_1)^{n_1}}{\sqrt{n_1!}} \cdots \frac{(b^+_r)^{n_r}}{\sqrt{n_r!}} \langle 0, 0, \ldots, 0\rangle. \tag{112}
\]
In this bosonic representation, the Weyl generators \(e_i, f_i\) and the Cartan operators \(h_i\) of the algebra \(su(r + 1)\) are realized as follows
\[
e_i = b^+_i b^-_{i-1}, \quad f_i = b^-_i b^+_{i-1}, \quad h_i = b^+_i b^-_{i-1} - b^+_i b^-_i, \quad i = 1, 2, \ldots, r. \tag{113}
\]
The irreducible unitary representation \(D^k_r\) is obtained within this realization by considering the subspace
\[
E^k_r = \{||n_0, n_1, \ldots, n_r\rangle\rangle; n_0 + n_1 + \ldots + n_r = k\}.
\]
of dimension \(\binom{k + r - 1}{r - 1}\). In this symmetric representation, the state of highest weight is \(||k, 0, \ldots, 0\rangle\rangle\).

For the Lie algebra \(su(r + 1)\), the generators defined in terms of the Weyl operators \(f_i\) as
\[
j^+_i \equiv f_i, \quad j^+_i \equiv [f_i, j^-_{i-1}], \quad i = 2, 3, \ldots, r, \tag{114}
\]
are mutually commuting. Similarly, the generators \(j^-_i = (j^+_i)^\dagger \ (i = 1, 2, \ldots, r)\) defined as
\[
j^-_i \equiv f_i, \quad j^-_i \equiv [j^-_{i-1}, e_i], \quad i = 2, 3, \ldots, r, \tag{115}
\]
commute. The bosonic realization of operators \(j^+_i\) \(\text{(114)}\) and \(j^-_i\) \(\text{(115)}\) writes as
\[
j^+_i = b^+_0 b^-_i, \quad j^-_i = b^+_i b^-_0, \quad i = 1, 2, 3, \ldots, r.
\]
in terms of the oscillators \(b^-_i\) and \(b^+_i\). Using the condition \(k = n_0 + n_1 + \ldots + n_r\), one identifies \(||n_0, n_1, \ldots, n_r\rangle\rangle = ||k - (n_1 + \ldots + n_r), n_1, n_2, \ldots, n_r\rangle\rangle \equiv |n_1, n_2, \ldots, n_r\rangle\rangle\) so that the representation space is given by
\[
E^k_r = \{|n_1, n_2, \ldots, n_r\rangle; n_1 + n_2 + \ldots + n_r \leq k\}.
\]
Using the usual action of the lowering and raising harmonic oscillator operators on the Fock number states, one gets
\[
j^+_i |n_1, n_2, \ldots, n_i, \ldots, n_{r+1}\rangle = \sqrt{(n_i + 1)(k - (n_1 + n_2 + \cdots + n_r))} |n_1, n_2, \ldots, n_i + 1, \ldots, n_{r+1}\rangle \tag{116}
\]
\[
j^-_i |n_1, n_2, \ldots, n_i, \ldots, n_{r+1}\rangle = \sqrt{n_i(k + 1 - (n_1 + n_2 + \cdots + n_r))} |n_1, n_2, \ldots, n_i - 1, \ldots, n_{r+1}\rangle. \tag{117}
\]
We note that the operators \(j^+_i\) and \(j^-_i\) satisfy the following nilpotency relations
\[
(j^+_1)^{n_1} (j^+_2)^{n_2} \cdots (j^+_r)^{n_r} = 0, \quad (j^-_1)^{n_1} (j^-_2)^{n_2} \cdots (j^-_r)^{n_r} = 0, \quad \text{for} \quad n_1 + n_2 + \cdots + n_r = k + 1. \tag{118}
\]
To get the explicit expressions of the Barut–Giraredello coherent states associated with $\text{su}(r+1)$, one has to determine the common eigenvectors of the ladder generators $j_i^- (i = 1, 2, \ldots, r)$. In this respect, we consider the eigenvalues equations

$$j_i^- |\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_r\rangle = \lambda_i |\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_r\rangle. \quad (119)$$

To find the corresponding solutions, we expand the state $|\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_r\rangle$ as follows

$$|\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \theta_r\rangle = \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} \sum_{n_r=0}^{k-(n_1+\cdots+n_{r-1})} C_{n_1,n_2,\ldots,n_r} \lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_r^{n_r} |n_1, n_2, \ldots, n_i, \ldots, n_r\rangle. \quad (120)$$

Using the nilpotency conditions (118), it is simple to see from the eigenvalues equations (119) that the variables $\lambda_i (i = 1, 2, \ldots, r)$ satisfy the conditions

$$\lambda_1^{n_1} \lambda_2^{n_2} \cdots \lambda_r^{n_r} = 0 \quad \text{for} \quad n_1 + n_2 + \cdots + n_r = k + 1. \quad (121)$$

As discussed previously for $r = 1$ and $r = 2$, we factorize the variables $\lambda_i (i = 1, 2, \ldots, r)$

$$\theta_i = \eta z_i$$

in terms of one generalized Grassmann variable $\eta$ and the complex variable $z_i$ so that the conditions (121) are fulfilled. Reporting the expression (120) in the eigenvalue equation (119) for the mode $i$ ($i = 1, 2, \ldots, r$), we obtain the recurrence relation

$$C_{n_1,n_2,\ldots,n_i-1,\ldots,n_r} = \sqrt{n_i(k - (n_1 + n_2 + \cdots + n_i + \cdots + n_r))} C_{n_1,n_2,\ldots,n_i,\ldots,n_r},$$

which gives

$$C_{n_1,n_2,\ldots,n_i,\ldots,n_r} = \sqrt{\frac{(k - (n_1 + n_2 + \cdots + n_i + \cdots + n_r))!}{n_i!(k - (n_1 + n_2 + \cdots + n_i + \cdots + n_r) + n_i)!}} C_{n_1,n_2,\ldots,0,\ldots,n_r}$$

for $n_i = 0, 1, \ldots, k - (n_1 + n_2 + \cdots + n_{i-1})$ when $i = 1, 2, \ldots, r$ with $n_{-1} \equiv 0$. Using this result, the explicit form of the expansion coefficients $C_{n_1,n_2,\ldots,n_i,\ldots,n_r}$ is given by

$$C_{n_1,n_2,\ldots,n_i,\ldots,n_r} = \sqrt{\frac{(k - (n_1 + n_2 + \cdots + n_r))!}{n_1!n_2!\cdots n_r!k!}} C_{0,0,\ldots,0}, \quad (122)$$

where the factor $C_{0,0,\ldots,0}$ is determined from the normalization condition of the eigenstates $|\lambda_1, \lambda_2, \ldots, \lambda_r\rangle \equiv |\eta, z_1, z_2, \ldots, z_r\rangle$. This gives

$$|\eta, z_1, z_2, \ldots, z_r\rangle = \mathcal{N}_r \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} \cdots \sum_{n_r=0}^{k-(n_1+\cdots+n_{r-1})} \sqrt{\frac{(k - (n_1 + n_2 + \cdots + n_r))!}{n_1!n_2!\cdots n_r!k!}} \times \eta^{n_1+n_2+\cdots+n_r} \tilde{z}_1^{n_1} \tilde{z}_2^{n_2} \cdots \tilde{z}_r^{n_r} |n_1, n_2, \ldots, n_r\rangle. \quad (123)$$

where the normalization factor writes

$$|\mathcal{N}_r|^2 = \sum_{n_1=0}^{k} \sum_{n_2=0}^{k-n_1} \cdots \sum_{n_r=0}^{k-(n_1+\cdots+n_{r-1})} \frac{(k - (n_1 + n_2 + \cdots + n_r))!}{n_1!n_2!\cdots n_r!k!} |\eta|^{2(n_1+n_2+\cdots+n_r)} |z_1|^{2n_1} |z_2|^{2n_2} \cdots |z_r|^{2n_r}. \quad (124)$$
The vectors $|\eta, z_1, z_2, \ldots, z_r\rangle$ constitutes an over-complete set of states labelled continuously by one Grassmann variable $\eta$ and $r$ complex (bosonic) variables $z_i$. The computation of the measure, with respect which the identity resolution is ensured, works like the $su(3)$ case discussed in the previous section. Indeed, we have

$$\int |\eta, z_1, z_2, \ldots, z_r\rangle d\mu_r(\eta, \bar{\eta}, \{z_i\}, \{\bar{z}_i\})(\eta, z_1, z_2, \ldots, z_r| = \text{Identity}. \quad (125)$$

The measure $d\mu_r(\eta, \bar{\eta}, \{z_i\}, \{\bar{z}_i\})$, is given by

$$d\mu(\eta, \bar{\eta}, \{z_i\}, \{\bar{z}_i\}) = |N_r|^{-2} \sigma(\eta, \bar{\eta}) d\eta d\bar{\eta} e^{-\sum_{i=1}^{r} |z_i|^2} \frac{d^2z_1 d^2z_2 \cdots d^2z_r}{\pi^r},$$

where the function $\sigma(\eta, \bar{\eta})$ is given by (107). The coherent states (123) reduces to $su(2)$ and $su(3)$ Barut–Girardello coherent states given by (83) and (105) for $r = 1$ and $r = 2$ respectively.

6 Closing remarks

The Barut–Girardello coherent states, labelled by complex variables, have been initially proposed by Barut and Girardello as the eigenstates of the lowering generators in the representation spaces of Lie algebras [17]. This procedure is only possible for non-compact Lie Groups, as for instance $SU(1,1)$. Most of the works presented in the literature on the Barut–Girardello coherent states were done for infinite-dimensional representation algebras. Recently, the Barut–Girardello coherent states for the Lie algebra $su(2)$ algebra, the Lie algebra $su(1,1)$ with truncated representation space and Pegg–Barnett oscillator algebra were developed in [37] using the formalism of generalized Grassmann variables [22].

In this paper we developed further the idea proposed in [37]. We define first the generalized Weyl-Heisenberg algebra by mean of $r$ pairs of creation annihilation operators and the associated Fock–Hilbert space. This algebra extends the generalized oscillator algebras introduced in [39] for $r = 1$ and in [40] for $r = 1$. The Hilbertian and analytical representation were investigated. In the Fock-Bargmann representation, we employed the generalized Grassmann variables. A special attention were devoted to the definition of this kind of exotic variables in terms of the ordinary Grassmann commonly used in the formulation of super-symmetric models. This connection provides us with the appropriate scheme to define the derivative and integration operations by exploiting the properties of the ordinary Grassmann-valued variables. The essential observation arising from this realization is the algebraic description of qukits in terms of a symmetric ensemble of qubits. In this sense, the generalized Weyl-Heisenberg algebras provide the algebraic framework to describe qubits and qukits which are the basic ingredients in quantum information theory. Furthermore, the method, based on generalized Weyl–Heisenberg algebras, that we adopted in this work is an alternative way to define the generalized Grassmann variables without resorting to the analysis developed in the context of quantum algebras (see [37] and references therein). This simplifies the construction of the coherent states à la Barut–Girardello associated with the $su(r + 1)$ Lie algebra and their over–completion properties. We hope that this novel construction of generalized Grassmann variables for qukit systems and Barut-Girardello coherent states for $su(r + 1)$ algebras will be of interest in the field of quantum systems with finite dimensional Hilbert space, especially for pseudo-Hermitian quantum systems. Also, it is
interesting to investigate the relation between the generalized Weyl–Heisenberg algebras discussed in this paper and the formalism of nonlinear fermions discussed in [47].

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