Summing large-$N$ towers in colour flow evolution.

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DESY

based on SP – arXiv:1312.2448
Putting it into context.

- Soft gluons in the colour flow basis
  - Understanding of colour reconnection
  - Resummation in presence of many legs (⇒ efficient amp. generators)
  - Improved parton showers
    - Simples NLO matching
Outline.

– Colour flow bases.
– Soft anomalous dimensions.
– Summing large-$N$ towers.
– Conclusions & outlook.
Colour flow bases.

Physically transparent QCD amplitudes:
Track flow of colour charge.

**Not** tied to large-$N$ limit.

Basis for highly efficient, recursive amplitude evaluation.
Comix, OpenLoops, Weinzierl et al., also see Reuschle’s talk.

Very close connection to parton showers.
Initial conditions driven by large-$N$ flows.

Rarely used in the context of soft-gluon evolution.

- Properties of soft anomalous dimensions?
- Insight into colour evolution dynamics?

\[
A_a^\mu \rightarrow A_a^\mu(t^a)^i_j \equiv A^{\mu i}_j
\]

\[
(t^a)^i_j(t^a)^j_i = TR \left( \delta^i_j \delta^j_i - \frac{1}{N} \delta^i_i \delta^j_j \right)
\]

Note: ‘Singlet’ gluon decouples from pure-gluon vertices.

Full set of Feynman rules e.g. in
[Maltoni et al. – Phys.Rev.D 67, 014026 (2003)]
**Colour flow bases.**

Amplitude decomposition:

$$|\mathcal{M}_n\rangle = \sum_\sigma \mathcal{M}_{n,\sigma} |\sigma\rangle$$

‘Normal ordered’ numbering of (outgoing) legs $\alpha$:

$$1_N, 2_{\bar{N}}, \ldots, n_q, \bar{N}, (n_q + 1)_A, \ldots, (n_q + n_g)_A$$

translated to (anti-)fundamental indices:

$$k \leftrightarrow \alpha = k_N$$

$$\frac{k - 1}{2} \leftrightarrow \alpha = k_{\bar{N}}$$

$$\frac{k - n_q/2}{2} \leftrightarrow \alpha = k_A$$

**Basis tensors labelled by $\sigma \in S_m$, $m = n_q/2 + n_g$:**

$$|\sigma\rangle = \delta^{i_1}_{\sigma(1)} \cdots \delta^{i_m}_{\sigma(m)}$$

Cannot restrict the basis to the tree-level subspace, even when only considering evolution of tree-level amplitudes only!

Partial amplitudes in this basis used to calculate weights for large-$N$ colour flow assignment.
Soft anomalous dimensions.

\[ |\mathcal{M}'_n\rangle = e^\Gamma |\mathcal{M}_n\rangle \quad \Gamma = \sum_{\alpha \neq \beta} \Gamma^{\alpha\beta} \mathbf{T}_\alpha \cdot \mathbf{T}_\beta \]

Translate colour correlations: Legs \( \alpha \leftrightarrow \) (anti-)colour indices \( \bar{i}, j \).

\[
\begin{align*}
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_l \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} + \mathbf{T}_k \cdot \mathbf{T}_l \\
\mathbf{T}_\alpha \cdot \mathbf{T}_\beta &= \mathbf{T}_k \cdot \mathbf{T}_{\bar{l}} + \mathbf{T}_k \cdot \mathbf{T}_l + \mathbf{T}_l \cdot \mathbf{T}_{\bar{k}} + \mathbf{T}_{\bar{k}} \cdot \mathbf{T}_{\bar{l}}
\end{align*}
\]

Express colour correlators by Fierz identities:

\[
\begin{align*}
\mathbf{T}_i \cdot \mathbf{T}_j &= \frac{1}{2} \left( \delta_j^i \delta_i^j - \frac{1}{N} \delta_i^i \delta_j^j \right) \\
\mathbf{T}_i \cdot \mathbf{T}_{\bar{j}} &= -\frac{1}{2} \left( \delta_i^i \delta_{\bar{j}}^j - \frac{1}{N} \delta_i^i \delta_{\bar{\bar{j}}}^{\bar{j}} \right)
\end{align*}
\]
Soft anomalous dimensions.

\[ \Gamma = \sum_{i<j} (\gamma_{ij} T_i \cdot T_j + \gamma_{ij} \tilde{T}_i \cdot \tilde{T}_j) + \sum_{i,j} \gamma_{ij} T_i \cdot T_j \]

Anomalous dimension in the flow basis:

\[ T_i \cdot T_j = \frac{1}{2} \left( \delta_{ij}' \delta_{ij}' - \frac{1}{N} \delta_{ij}' \delta_{ij}' \right) \]

\[ [\tau | \Gamma | \sigma] = \left( -N \Gamma_\sigma + \frac{1}{N} \rho \right) \delta_{\tau\sigma} + \Sigma_{\tau\sigma} \]

Colour 'reconnectors'
→ genuinely 1/N supressed.

\[ \Sigma_{\tau\sigma} \neq 0 \text{ only if } \sigma = \text{transposition}_{ij}(\tau) \]
Soft anomalous dimensions.

Transparent organization of powers of $N$:

$$\Gamma \equiv N\Gamma + \Sigma + \frac{1}{N} \rho 1$$

**Leading-$N$ → diagonal**

**Next-to-leading-$N$ → colour reconnections, diagonal elements vanish**

**Next-to-next-to-leading-$N$ → trivial**

Can we learn something on exponentiation?

What is the ‘leading-$N$’ structure of the evolution matrix $e^{\Gamma}$?

NB: (Integrated) subtraction terms at NLO exhibit the same structure as $\Gamma$. 
Large-$N$ towers.

What is the ‘leading-$N$’ structure of the evolution matrix $e^\Gamma$?

‘Never use brute force to fight an exponential!’

Perturbation theory in $1/N$ won’t work.

Need resummation to get at least close to the exact result.

$$\exp \Gamma \equiv \exp \left( N\Gamma + \Sigma + \frac{1}{N} \rho 1 \right)$$

Power counting: Need to resum if $\gamma N = \mathcal{O}(1)$.

Treat remainder as higher-order effects/perturbation, but on same footing.

at LC : $1 + \gamma N + \gamma^2 N^2 + ...$

at NLC : $\left( \gamma + \frac{\gamma}{N} \right) \left( 1 + \gamma N + \gamma^2 N^2 + ... \right)$

at NNLC : $\left( \gamma + \frac{\gamma}{N} \right)^2 \left( 1 + \gamma N + \gamma^2 N^2 + ... \right)$

Will also consider a ‘primed’ resummation: Absorb $\rho$ contribution into redifinition of $\Gamma$. 
How to get there?

Work out the exponential ...

\[
[\tau | e^\Gamma | \sigma] = \sum_{l=0}^{\infty} \left( \frac{-1}{N^l} \right)^l \frac{(-\rho)^k}{k!} \sum_{\sigma_0, \ldots, \sigma_{l-k}} \delta_{\tau \sigma_0} \delta_{\sigma_{l-k} \sigma} \left( \prod_{\alpha=0}^{l-k-1} \Sigma_{\sigma_{\alpha+1}} \right) R(\{\sigma_0, \ldots, \sigma_{l-k}\}, \{\Gamma\})
\]

\[
R(\sigma, \Gamma) = \left[ \prod_{\alpha=0}^{\#\text{uniq}(\sigma)-1} \frac{1}{d_{\alpha}(\sigma)!} \frac{\partial^{d_{\alpha}(\sigma)}}{\partial d_{\alpha}(\sigma) \Gamma_{\sigma_{\alpha}}} \right] \sum_{\alpha=0}^{\#\text{uniq}(\sigma)-1} e^{-N\Gamma_{\sigma_{\alpha}}} \prod_{\beta=0, \beta \neq \alpha}^{\#\text{uniq}(\sigma)-1} \frac{(\Gamma_{\sigma_{\beta}} / \Gamma_{\sigma_{\alpha}})^{d_{\beta}(\sigma)}}{\Gamma_{\sigma_{\alpha}} - \Gamma_{\sigma_{\beta}}}
\]

(next-to-)\textsuperscript{d}-leading color is truncating the series at \textit{d}.
Note that the leading colour contributions are exponentiated.
Some first observations.

Next-to-leading color:

\[
[\tau|e^\Gamma|\sigma] = \delta_{\tau\sigma} e^{-N\Gamma_\sigma} \left( 1 + \frac{\rho}{N} \right) - \sum_{\tau\sigma} \frac{1}{N} \frac{e^{-N\Gamma_\tau} - e^{-N\Gamma_\sigma}}{\Gamma_\tau - \Gamma_\sigma} + \text{NNLC}
\]

At \(N^d\text{LC}\) need \(d\) matrix multiplications.

Or not even this: \(\Sigma\) is very sparse, with easily calculable non-vanishing elements.

\(\rightarrow\) Think of a Monte Carlo over colour structures.

In a resummation context, to be matched to NLO:
NLC is in principle sufficient for the matching.

\[
[\tau|e^\Gamma|\sigma] \bigg|_{\text{NLC}} = \delta_{\tau\sigma} + [\tau|\Gamma|\sigma] + \mathcal{O}(\gamma^2)
\]

For a large number of legs, a sufficiently high order is required to get at least a lowest order for transition elements involving many transpositions.

\(\rightarrow\) Impact of these contributions to be checked at the level of amplitudes squared.
Higher orders & numerics.

E.g. $N^3$LC ingredients:

\[
R(\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}, \Gamma) = \frac{e^{-N\Gamma\sigma_0}}{(\Gamma_{\sigma_0} - \Gamma_{\sigma_1})(\Gamma_{\sigma_0} - \Gamma_{\sigma_2})(\Gamma_{\sigma_0} - \Gamma_{\sigma_3})} + (0 \leftrightarrow 1) + (0 \leftrightarrow 2) + (0 \leftrightarrow 3)
\]

\[
R(\{\sigma_0, \sigma_0, \sigma_1, \sigma_2\}, \Gamma) = -N \frac{e^{-N\Gamma\sigma_0}}{(\Gamma_{\sigma_0} - \Gamma_{\sigma_1})(\Gamma_{\sigma_0} - \Gamma_{\sigma_2})} + \frac{(\Gamma_{\sigma_1} + \Gamma_{\sigma_2} - 2\Gamma_{\sigma_0}) e^{-N\Gamma\sigma_0}}{(\Gamma_{\sigma_0} - \Gamma_{\sigma_1})^2(\Gamma_{\sigma_0} - \Gamma_{\sigma_2})^2} + \left( \frac{e^{-N\Gamma\sigma_1}}{(\Gamma_{\sigma_0} - \Gamma_{\sigma_1})^2(\Gamma_{\sigma_1} - \Gamma_{\sigma_2})} + (1 \leftrightarrow 2) \right)
\]

\[
R(\{\sigma_0, \sigma_0, \sigma_0, \sigma_0\}, \Gamma) = -\frac{N^3}{6} e^{-N\Gamma\sigma_0}
\]

CVolver C++ library implements all required ingredients at arbitrary $N^d$LC order.

- Transparent interface to implement anomalous dimensions and hard amplitudes.
- Prepared to perform Monte Carlo over colour flows.
Squaring evolved amplitudes.

Colour bases are typically not orthogonal:

\[ |\mathcal{M}|^2 = \langle \mathcal{M} | \mathcal{M} \rangle = \mathcal{M}_{\sigma}^\dagger S_{\sigma \tau} \mathcal{M}_\tau \]

Colour flow basis for \( m \) colour flows:

\[ S_{\sigma \tau} = \langle \sigma | \tau \rangle = N^{m-\#\text{transpositions}(\sigma, \tau)} \]

Approximate squared amplitudes on large-\( N \) approximation?
\( \rightarrow \) No. Unrelated to approximation of soft-gluon evolution, will obscure convergence.

Again, amazingly simple structure:
Don’t even need to do linear algebra, but Monte Carlo over colour flows.
Numerical results.

Testbed: (Dipole) shower emission veto in dijets $p_1, p_2 \rightarrow p_3, p_4$. Each of the possible II, IF, FF dipoles can radiate with $\mu^2 < p_\perp^2 \lesssim 2p_i \cdot p_j$.

$$\Gamma^{12} = \Gamma^{34} = \frac{\alpha_s}{4\pi} \left( \frac{1}{2} \ln^2 \frac{s}{\mu^2} - i\pi \ln \frac{s}{\mu^2} \right)$$

$$\Gamma^{13} = \Gamma^{24} = \frac{\alpha_s}{8\pi} \ln^2 \frac{|t|}{\mu^2}$$

$$\Gamma^{14} = \Gamma^{23} = \frac{\alpha_s}{8\pi} \ln^2 \frac{|u|}{\mu^2}$$

Exact solution for $q\bar{q}q\bar{q}$ amplitudes can be obtained straightforwardly. Now check convergence of the $N^d$LC approximations.
Numerical results.

C++ library CVolver will be public at some point.
Numerical results: Primed resummation.

C++ library CVolver will be public at some point.
Another observation.

Look at the exact solution for two colour flows:

\[ e^\Gamma \equiv e^{\frac{1}{N}\rho} e^{-\frac{N}{2}(\Gamma_{12} + \Gamma_{21})} \left( \begin{array}{cc} -\Delta \sinh \frac{\kappa}{2} + \kappa \cosh \frac{\kappa}{2} & 2\Sigma_{1212} \sinh \frac{\kappa}{2} \\ 2\Sigma_{1221} \sinh \frac{\kappa}{2} & \Delta \sinh \frac{\kappa}{2} + \kappa \cosh \frac{\kappa}{2} \end{array} \right) \]

\[ \Delta = N(\Gamma_{12} - \Gamma_{21}), \quad \kappa = \sqrt{\Delta^2 + 4\Sigma_{1212} \Sigma_{1221}} \]

Then find:

\[ e^\Gamma \rightarrow e^\Gamma \bigg|_{\text{NLC'}} \quad \Delta^2 \gg 4\Sigma_{1212} \Sigma_{1221} \]

Delicate correlation between kinematic limits and large-\(N\) limit.

Hint why large-\(N\) for showers seems to work quite well for a lot of observables?

Note showers do primed evolution: Primed vs. non-primed is essentially \(C_F\) vs \(C_A/2\).
Applications.

Soft gluon evolution, exact (few legs):
- Colour structure of anomalous dimension in flow basis.
- Use powerful colour flow based ME generators.

Soft gluon evolution, approximate (many legs):
- Check convergence of successive approximations.
- No matrix exponentiation, only (sparse) matrix multiplication needed.
- Ultimately: Do this within shower algorithms $\rightarrow$ NLL @ N$^d$LC

No need for linear algebra operations by performing a Monte Carlo over colour flows.

Not covered here: Insight into colour reconnection models.
So far only based on simple phenomenological reasoning.
Conclusions.

Soft gluon evolution in colour flow basis has a very transparent structure.

Exponentiation of the anomalous dimension can be performed approximately, by resumming large-$N$ enhanced terms.

Checking successive approximations for convergence, evolution of a large number of legs is feasible.

Outlook:
Study more complicated colour correlations.
Identify phase space regions where subleading-$N$ is (un)important.