EXPANSIONS OF THE REAL FIELD BY DISCRETE SUBGROUPS OF $\text{GL}_n(\mathbb{C})$

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Abstract. Let $\Gamma$ be an infinite discrete subgroup of $\text{GL}_n(\mathbb{C})$. Then either $(\mathbb{R}, <, +, \cdot, \Gamma)$ is interdefinable with $(\mathbb{R}, <, +, \cdot, \lambda\mathbb{Z})$ for some $\lambda \in \mathbb{R}$, or $(\mathbb{R}, <, +, \cdot, \Gamma)$ defines the set of integers. When $\Gamma$ is not virtually abelian, the second case holds.

1. Introduction

Let $\bar{\mathbb{R}} = (\mathbb{R}, <, +, 0, 1)$ be the real field. For $\lambda \in \mathbb{R}_{>0}$, set $\lambda\mathbb{Z} := \{\lambda^m : m \in \mathbb{Z}\}$. Throughout this paper $\Gamma$ denotes a discrete subgroup of $\text{GL}_n(\mathbb{C})$, and $G$ denotes a subgroup of $\text{GL}_n(\mathbb{C})$. We identify the set $M_n(\mathbb{C})$ of $n$-by-$n$ complex matrices with $\mathbb{C}^{n^2}$ and identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way. Our main result is the following classification of expansions of $\bar{\mathbb{R}}$ by a discrete subgroup of $\text{GL}_n(\mathbb{C})$.

**Theorem A.** Let $\Gamma$ be an infinite discrete subgroup of $\text{GL}_n(\mathbb{C})$. Then either

- $(\mathbb{R},\Gamma)$ defines $\mathbb{Z}$ or
- there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}},\Gamma)$ is interdefinable with $(\bar{\mathbb{R}},\lambda\mathbb{Z})$.

If $\Gamma$ is not virtually abelian, then $(\bar{\mathbb{R}},\Gamma)$ defines $\mathbb{Z}$.

By Hieronymi [11, Theorem 1.3], the structure $(\bar{\mathbb{R}},\lambda\mathbb{Z},\mu\mathbb{Z})$ defines $\mathbb{Z}$ whenever $\log_\lambda \mu \not\in \mathbb{Q}$, and is interdefinable with $(\bar{\mathbb{R}},\lambda\mathbb{Z})$ otherwise. Therefore Theorem A extends immediately to expansions of $\bar{\mathbb{R}}$ by multiple discrete subgroups of $\text{GL}_n(\mathbb{C})$.

**Corollary A.** Let $\mathcal{G}$ be a collection of infinite discrete subgroups of various $\text{GL}_n(\mathbb{C})$. Then either

- $(\bar{\mathbb{R}},(\Gamma)_{\Gamma \in \mathcal{G}})$ defines $\mathbb{Z}$ or
- there is $\lambda \in \mathbb{R}_{>0}$ such that $(\bar{\mathbb{R}},(\Gamma)_{\Gamma \in \mathcal{G}})$ is interdefinable with $(\bar{\mathbb{R}},\lambda\mathbb{Z})$.

The dichotomies in Theorem A and Corollary A are arguably as strong as they can be. An expansion of the real field that defines $\mathbb{Z}$, has not only an undecidable theory, but also defines every real projective set in sense of descriptive set theory (see Kechris [16, 37.6]). From a model-theoretic/geometric point of view such a structure is a wild as can be. On the other hand, by van den Dries [4] the structure $(\bar{\mathbb{R}},\lambda\mathbb{Z})$ has a decidable theory whenever $\lambda$ is recursive, and admits quantifier-elimination in a suitably extended language. It satisfies combinatorical model-theoretic tameness conditions such as NIP and distality (see [9, 13]). Furthermore, it follows...
from these results that every subset of $\mathbb{R}^n$ definable in $(\bar{\mathbb{R}}, \lambda^2)$ is a boolean combination of open sets, and thus $(\bar{\mathbb{R}}, \lambda^2)$ defines only sets on the lowest level of the Borel hierarchy. See Miller [18] for more on tameness in expansions of the real field.

Our proof of Theorem A relies crucially on the following two criteria for the definability of $\mathbb{Z}$ in expansions of the real field.

**Fact 1.1.** Suppose $D \subseteq \mathbb{R}^k$ is discrete.

1. If $(\bar{\mathbb{R}}, D)$ defines a subset of $\mathbb{R}$ that is dense and co-dense in a nonempty open interval, then $(\bar{\mathbb{R}}, D)$ defines $\mathbb{Z}$.
2. If $D$ has positive Assouad dimension, then $(\bar{\mathbb{R}}, D)$ defines $\mathbb{Z}$.

The first statement is [12] Theorem E, a fundamental theorem on first-order expansions of $\bar{\mathbb{R}}$, and the second claim is proven using the first in Hieronymi and Miller [14, Theorem A]. We recall the definition of Assouad dimension in Section 5. This important metric dimension bounds more familiar metric dimensions (such as Hausdorff and Minkowski dimension) from above. We refer to [14] for a more detailed discussion of Assouad dimension and its relevance to definability theory.

The outline of our proof of Theorem A is as follows. Let $\Gamma$ be a discrete, infinite subgroup of $\text{Gl}_n(\mathbb{C})$. Using Fact 1.1(1), we first show that $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ whenever $\Gamma$ contains a non-diagonalizable matrix. It follows from a theorem of Mal’tsev that $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ when $\Gamma$ is virtually solvable and not virtually abelian. In the case that $\Gamma$ is not virtually solvable, we prove using Tits’ alternative that $\Gamma$ has positive Assouad dimension, and hence $(\bar{\mathbb{R}}, \Gamma)$ defines $\mathbb{Z}$ by Fact 1.1(2). We conclude the proof of Theorem A by proving that whenever $\Gamma$ is virtually abelian and $(\bar{\mathbb{R}}, \Gamma)$ does not define $\mathbb{Z}$, then $(\bar{\mathbb{R}}, \Gamma)$ is interdefinable with $(\bar{\mathbb{R}}, \lambda^2)$ for some $\lambda \in \mathbb{R}_{>0}$. Along the way we give (Lemma 3.4) an elementary proof showing that a torsion free non abelian nilpotent subgroup of $\text{Gl}_n(\mathbb{C})$ has a non-diagonalizable element. As every finitely generated subgroup of $\text{Gl}_n(\mathbb{C})$ is either virtually nilpotent or has exponential growth, this yields a more direct proof of Theorem A in the case when $\Gamma$ is finitely generated.

We want to make an extra comment about the case when $\Gamma$ is a discrete, virtually solvable, and not virtually abelian subgroup of $\text{Gl}_n(\mathbb{C})$. The Novosibirsk theorem [22] of Noskov (following work of Mal’tsev, Ershov, and Romanovskii) shows that a finitely generated, virtually solvable and non-virtually abelian group interprets $(\mathbb{Z}, +, \cdot)$. It trivially follows that if $G$ is finitely generated, virtually solvable, and non-virtually abelian, then $(\bar{\mathbb{R}}, G)$ interprets $(\mathbb{Z}, +, \cdot)$. However, it does not directly follow that $(\bar{\mathbb{R}}, G)$ defines $\mathbb{Z}$. We use an entirely different method below to show that if $G$ is in addition discrete, then $(\bar{\mathbb{R}}, G)$ defines $\mathbb{Z}$. Our method also applies when $G$ is not finitely generated, but relies crucially on the discreteness of $G$.

This paper is by no means the first paper to study expansions of the real field by subgroups of $\text{Gl}_n(\mathbb{C})$. Indeed, there is a large body of work on this subject, often not explicitly mentioning $\text{Gl}_n(\mathbb{C})$. Because we see this paper as part of a larger investigation, we survey some of the earlier results and state a conjecture. It is convenient to consider three distinct classes of such expansion. By Miller and Speissegger [20] every first-order expansion $\mathcal{R}$ of $\bar{\mathbb{R}}$ satisfies at least one of the following:
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1. $\mathcal{R}$ is o-minimal,
2. $\mathcal{R}$ defines an infinite discrete subset of $\mathbb{R}$,
3. $\mathcal{R}$ defines a dense and co-dense subset of $\mathbb{R}$.

The open core $\mathcal{R}^o$ of $\mathcal{R}$ is the expansion of $(\mathbb{R}, <)$ generated by all open $\mathcal{R}$-definable subsets of all $\mathbb{R}^k$. By [20], if $\mathcal{R}$ does not satisfy (2), then $\mathcal{R}^o$ is o-minimal.

The case when $\mathcal{R}$ is o-minimal, is largely understood. Wilkie’s famous theorem [28] that $(\overline{\mathbb{R}}, \exp)$ is o-minimal is crucial. This shows the expansion of $\overline{\mathbb{R}}$ by the subgroup

$$\left\{ \begin{pmatrix} 1 & 0 & t \\ 0 & \lambda^t & 0 \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

for $\lambda \in \mathbb{R}_{>0}$, and so is the expansion of $\overline{\mathbb{R}}$ by any subgroup of the form

$$\left\{ \begin{pmatrix} t^s & 0 \\ 0 & t^r \end{pmatrix} : t \in \mathbb{R}_{>0} \right\}$$

for $s, r \in \mathbb{R}_{>0}$. Indeed, by Peterzil, Pillary, and Starchenko [24], whenever an expansion $(\overline{\mathbb{R}}, G)$ by a subgroup $G$ of $\text{GL}_n(\mathbb{R})$ is o-minimal, then $G$ is already definable in $(\overline{\mathbb{R}}, \exp)$. Furthermore, note that by a classical theorem of Tamagaki and Chevalley [3] every compact subgroup of $\text{GL}_n(\mathbb{C})$ is the group of real points on an algebraic group defined over $\mathbb{R}$. Thus every compact subgroup of $\text{GL}_n(\mathbb{C})$ is $\overline{\mathbb{R}}$-definable, and therefore the case of expansions by compact subgroups of $\text{GL}_n(\mathbb{C})$ is understood as well.

We now consider the case when infinite discrete sets are definable. Corollary A for discrete subgroups of $\mathbb{C}^\times$ follows easily from the proof of [11, Theorem 1.6]. While Corollary A handles the case of expansions by discrete subgroups of $\text{GL}_n(\mathbb{C})$, there are examples of subgroups of $\text{GL}_n(\mathbb{C})$ that define infinite discrete sets, but fail the conclusion of Theorem A. Given $\alpha \in \mathbb{R}^\times$ the logarithmic spiral

$$S_\alpha = \{(\exp(t)\sin(\alpha t), \exp(t)\cos(\alpha t)) : t \in \mathbb{R}\}$$

is a subgroup of $\mathbb{C}^\times$. Let $s$ and $e$ be the restrictions of sin and exp to $[0, 2\pi]$, respectively. Then $(\overline{\mathbb{R}}, S_\alpha)$ is a reduct of $(\overline{\mathbb{R}}, s, e, \lambda^\mathbb{Z})$ when $\lambda = \exp(2\pi\alpha)$, as was first observed by Miller and Speissegger. As $(\overline{\mathbb{R}}, s, u)$ is $\overline{\mathbb{R}}$-minimal with field of exponents $\mathbb{Q}$, the structure $(\overline{\mathbb{R}}, S_\alpha)$ is $d$-minimal by Miller [18, Theorem 3.4.2] and thus does not define $\mathbb{Z}$. It can be checked that $(\overline{\mathbb{R}}, S_\alpha)$ defines a analytic function that is not semi-algebraic, and thus is not interdefinable with $(\overline{\mathbb{R}}, \lambda^\mathbb{Z})$ for any $\lambda \in \mathbb{R}_{>0}$.

Most work in the case of expansions that define dense and co-dense sets, concerns expansions by finite rank subgroups of $\mathbb{C}^\times$ (see introduction of [2] for a thorough discussion of expansions by subgroups of $\mathbb{C}^\times$). In [5] van den Dries and Günaydın

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1. A **d-minimal** expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$ is d-minimal if every definable unary set in every model of the theory of $\mathcal{R}$ is a union of an open set and finitely many discrete sets.

2. By induction on the complexity of terms it follows easily from [Theorem II, vdD] that the definable functions in $(\overline{\mathbb{R}}, \lambda^\mathbb{Z})$ are given piecewise by a finite compositions of $x \mapsto \max \left(\{0\} \cup (\lambda^\mathbb{Z} \cap [-\infty, x])\right)$ and functions definable in $\overline{\mathbb{R}}$. From this one can deduce that every definable function in this structure is piecewise semi-algebraic.
showed that an expansion of \( \mathbb{R} \) by a finitely generated dense subgroup of \((\mathbb{R}_{>0},\cdot)\) admits quantifier-elimination in a suitably extend language. Günaydın \[8\] and Belegradek and Zilber \[1\] proved similar results for the expansion of \( \mathbb{R} \) by a dense finite rank subgroup of the unit circle \( U := \{ a \in \mathbb{C}^\times : |a| = 1 \} \). This covers the case when \( G \) is the group of roots of unity. In all these cases the open core of the resulting expansion is interdefinable with \( \mathbb{R} \). This does not always have to be the case. In Caulfield \[?\] studies expansions by subgroups of \( \mathbb{C}^\times \) of the form

\[
\{ \lambda^k \exp(i \alpha l) : k, l \in \mathbb{Z} \} \quad \text{where} \quad \lambda \in \mathbb{R}_{>0} \text{ and } \alpha \in \mathbb{R} \setminus \pi \mathbb{Q}.
\]

Such an expansions obviously defines a dense and co-dense subset of \( \mathbb{R} \), but by \[?\] its open core is interdefinable with \((\mathbb{R}, \lambda^\mathbb{Z})\). Furthermore, even if the open core is \( \omega \)-minimal, it does not have to be interdefinable with \( \mathbb{R} \). By \[13\] there is a co-countable subset \( \Lambda \) of \( \mathbb{R}_{>0} \) such that if \( r \in \Lambda \) and \( H \) is a finitely generated dense subgroup of \((\mathbb{R}_{>0},\cdot)\) contained in the algebraic closure of \( \mathbb{Q}(r) \), then the open core of the expansion of \( \mathbb{R} \) by the subgroup

\[
\left\{ \begin{pmatrix} t & 0 \\ 0 & t^r \end{pmatrix} : t \in H \right\}
\]

is interdefinable with the expansion of \( \mathbb{R} \) by the power function \( t \mapsto t^r : \mathbb{R}_{>0} \to \mathbb{R}_{>0} \).

All these previous results suggest that the next class of subgroups of \( \text{GL}_n(\mathbb{C}) \) for which we can hope to prove a classification comparable to Theorem A, is the class of finitely generated subgroups. Here the following conjecture seems natural, but most likely very hard to prove. Let \( \mathbb{R}_{\text{Pow}} \) be the expansion of \( \mathbb{R} \) by all power functions \( \mathbb{R}_{>0} \to \mathbb{R}_{>0} \) of the form \( t \mapsto t^r \) for \( r \in \mathbb{R}^\times \).

**Conjecture.** Let \( G \) be a finitely generated subgroup of \( \text{GL}_n(\mathbb{C}) \) such that \((\mathbb{R}, G)\) does not define \( \mathbb{Z} \). Then the open core of \((\mathbb{R}, G)\) is a reduct of \( \mathbb{R}_{\text{Pow}} \) or of \((\mathbb{R}, \text{Pow} \alpha)\) for some \( \alpha \in \mathbb{R}_{>0} \).

Even when the statement “\((\mathbb{R}, G)\) does not define \( \mathbb{Z} \)” is replaced by “\((\mathbb{R}, G)\) does not interpret \((\mathbb{Z}, +, \cdot)\)”, the conjecture is open. However, this weaker conjecture might be easier to prove, because the Novosibirsk theorem can be used to rule out the case when \( G \) is virtually solvable and non-virtually abelian. It is worth pointing out that Caulfield conjectured that when \( G \) is assumed to be a subgroup of \( \mathbb{C}^\times \), then the open core \((\mathbb{R}, G)\) is either \( \mathbb{R} \) or a reduct of \((\mathbb{R}, \text{Pow} \alpha)\) for some \( \alpha \in \mathbb{R}_{>0} \). See \[?\] for progress towards this later conjecture.

2. **Notation and Conventions**

Throughout \( m, n \) range over \( \mathbb{N} \) and \( k, l \) range over \( \mathbb{Z} \), \( G \) is a subgroup of \( \text{GL}_n(\mathbb{C}) \), and \( \Gamma \) is a discrete subgroup of \( \text{GL}_n(\mathbb{C}) \). Let \( \mathbb{R}_\Gamma \) be the expansion of \( \mathbb{R} \) by a \((2n)^2\)-ary predicate defining \( \Gamma \). We set \( \mathbb{R}_\Lambda := \mathbb{R}_\chi^\mathbb{C} \). A subset of \( \mathbb{R}^k \) is **discrete** if every point is isolated. We let \( UT_n(\mathbb{C}) \) be the group of \( n \)-by-\( n \) upper triangular matrices, \( D_n(\mathbb{C}) \) be the group of \( n \)-by-\( n \) diagonal matrices, and \( U \) be the multiplicative group of complex numbers with norm one.

All structures considered are first-order, “definable” means “definable, possibly with parameters”. Two expansions of \((\mathbb{R}, <)\) are **interdefinable** if they define the same subsets of \( \mathbb{R}^k \) for all \( k \). If \( P \) is a property of groups then a group \( H \) is **virtually \( P \)** if there is finite index subgroup \( H' \) of \( H \) that is \( \mathbb{P} \).
3. Linear Groups

We gather some general facts on groups. Throughout this section $H$ is a finitely generated group with a symmetric set $S$ of generators. Let $S_m$ be the set of $m$-fold products of elements of $S$ for all $m$. If $S'$ is another symmetric set of generators then there is a constant $k \geq 1$ such that

$$k^{-1}|S_m| \leq |S'_m| \leq k|S_m|$$

for all $m$.

Thus the growth rate of $m \mapsto |S_m|$ is an invariant of $H$. We say $H$ has exponential growth if there is a $C \geq 1$ such that $|S_m| \geq Cm$ for all $m$ and $H$ has polynomial growth if there are $k, t \in \mathbb{R}_{>0}$ such that $|S_m| \leq tm^k$ for all $m$. Note finitely generated non-abelian free groups are of exponential growth.

Gromov’s theorem [7] says $H$ has polynomial growth if and only if it is virtually nilpotent. Gromov’s theorem for subgroups of $\text{GL}_n(\mathbb{C})$ is less difficult and may be proven using the following two theorems:

**Fact 3.1.** If $G$ does not contain a non-abelian free subgroup, then $G$ is virtually solvable.

Fact 3.1 is Tits’ alternative [26]. Fact 3.2 is due to Milnor [21] and Wolf [29].

**Fact 3.2.** Suppose $H$ is virtually solvable. Then $H$ either has exponential or polynomial growth. If the latter case holds then $H$ is virtually nilpotent.

Note Fact 3.1 and Fact 3.2 imply every finitely generated subgroup of $\text{GL}_n(\mathbb{C})$ is of polynomial or exponential growth. This dichotomy famously does not hold for finitely generated groups in general, see for example [6].

The Heisenberg group $\mathbb{H}$ is presented by generators $a, b, c$ and relations

$$[a, b] = c, \quad ac = ca, \quad bc = cb.$$

The following fact is folklore; we include a proof for the reader.

**Fact 3.3.** Let $E$ be a nilpotent, torsion-free, and non-abelian group. Then there is a subgroup of $E$ isomorphic to $\mathbb{H}$.

**Proof.** Let $e$ be the identity element of $E$. We define the lower central series $(E_k)_{k \in \mathbb{N}}$ of $E$ by declaring $E_0 = E$ and $E_k = [E_{k-1}, E]$ for $k \geq 1$. Nilpotency means there is an $m$ such that $E_m \neq \{e\}$ and $[E_m, E] = \{e\}$. Moreover $m \geq 1$ as $E$ is not abelian.

On one hand, $[E_{m-1}, E] = E_m \neq \{e\}$ and so $E_{m-1}$ is not contained in $Z(E)$. Thus, there exists $a \in E_{m-1} \setminus Z(E)$ and $b \in E_m$ that does not commute with $a$. On the other hand, $[E_m, E] = \{e\}$ implies $E_m$ is contained in the center $Z(E)$ of $E$ and is thus abelian. So, $c := [a, b]$ is an element of $Z(E)$ and commutes with both $a$ and $b$.

Finally, $a, b, c$ have infinite order because $E$ is torsion-free. So, $a, b, c$ generate a subgroup of $E$ isomorphic to the Heisenberg group.

3.1. Non-diagonalizable elements. We show certain linear groups necessarily contain non-diagonalizable elements.

**Lemma 3.4.** If $G$ is nilpotent, torsion-free, and not abelian, then $G$ contains a non-diagonalizable element.

Lemma 3.4 follows from Fact 3.3 above and Lemma 3.5 below.
Lemma 3.5. Suppose $a, b, c \in \text{GL}_n(\mathbb{C})$ satisfy

$$[a, b] = c, \quad ac = ca, \quad bc = cb,$$

and $c$ is not torsion. Then either $a$ or $c$ is not diagonalizable.

Proof. Suppose $a, c$ are both diagonalizable. As $a, c$ commute, they are simultaneously diagonalizable and share a basis $\mathcal{B}$ of eigenvectors. As $c$ is not torsion, there is $\lambda_c \in \mathbb{C}^\times$ which is not a root of unity and $v \in \mathcal{B}$ such that $cv = \lambda_c v$. Let $\lambda_a \in \mathbb{C}^\times$ be such that $av = \lambda_a v$.

By way of contradiction, we will show $a(b^k v) = (\lambda_a \lambda_c^k)(b^k v)$ for all $k \geq 1$. As $\lambda_c$ is not a root of unity, this implies $a$ has infinitely many eigenvalues, which is impossible for an $n \times n$ matrix. The base case holds as

$$a(bv) = bacv = (\lambda_a \lambda_c)(bv).$$

Let $k \geq 2$ and suppose $a(b^{k-1} v) = (\lambda_a \lambda_c^{k-1})(b^{k-1} v)$. As $c$ commutes with $b$,

$$a(b^k v) = ab(b^{k-1} v) = bac(b^{k-1} v) = bab^{k-1}cv = (\lambda_c)(bab^{k-1}v).$$

Applying the inductive assumption,

$$(\lambda_c)(bab^{k-1}v) = (\lambda_c)b(\lambda_a \lambda_c^{k-1}b^{k-1}v) = (\lambda_a \lambda_c^k)(b^k v).$$

We now prove a slight weakening of Lemma 3.4 for solvable groups. Recall $a \in \text{GL}_n(\mathbb{C})$ is unipotent if some conjugate of $a$ is upper triangular with every diagonal entry equal to one. The only diagonalizable unipotent matrix is the identity. We recall a theorem of Mal’tsev [17].

Fact 3.6. Suppose $G$ is solvable. Then there is a finite index subgroup $G'$ of $G$ such that $G'$ is conjugate to a subgroup of $\text{UT}_n(\mathbb{C})$.

We now derive an easy corollary from Fact 3.6.

Lemma 3.7. Suppose $G$ is solvable and not virtually abelian. Then $G$ contains a non-diagonalizable element.

Proof. Suppose every element of $G$ is diagonalizable. After applying Fact 3.6 and making a change of basis if necessary we suppose $G' = G \cap \text{UT}_n(\mathbb{C})$ has finite index in $G$. Let $\rho : \text{UT}_n(\mathbb{C}) \to \text{D}_n(\mathbb{C})$ be the natural quotient map; that is the restriction to the diagonal. Every element of the kernel of $\rho$ is unipotent. Thus the restriction of $\rho$ to $G'$ is injective, and so $G'$ is abelian. □

4. Non-diagonalizable matrices

Lemma 4.1. Suppose $G$ contains a non-diagonalizable matrix. Then there is a rational function $h$ on $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ such that $h(G \times G) \subseteq \mathbb{C}$ is dense in $\mathbb{R}_{>0}$.

Proof. Suppose $a \in G$ is non-diagonalizable. Let $b \in \text{GL}_n(\mathbb{C})$ be such that $bab^{-1}$ is in Jordan form, i.e.

$$bab^{-1} = 
\begin{pmatrix}
A_1 & 0 & \cdots & 0 \\
0 & A_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_l
\end{pmatrix}$$
where each $A_i$ is a Jordan block and each $O$ is a zero matrix of the appropriate dimensions. We have

$$ba^k b^{-1} = \begin{pmatrix} A_1^k & O & \ldots & O \\ O & A_2^k & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & A_1^k \end{pmatrix}$$

for all $k$.

As $a$ is not diagonalizable, $A_k$ has more than one entry for some $k$. We suppose $A_1$ is $m$-by-$m$ with $m \geq 2$. For some $\lambda \in \mathbb{C}^\times$ we have

$$A_1 = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ 0 & 0 & \lambda & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

It is well-known and easy to show by induction that for every $k \geq 1$:

$$A_1^k = \begin{pmatrix} \lambda^k & (k_1)(\lambda^{k-1}) & (k_2)(\lambda^{k-2}) & (k_3)(\lambda^{k-3}) & \ldots & (k_m)(\lambda^{k-m}) \\ 0 & \lambda^k & (k_1)(\lambda^{k-1}) & (k_2)(\lambda^{k-2}) & \ldots & (k_{m-1})(\lambda^{k-m+1}) \\ 0 & 0 & \lambda^k & (k_1)(\lambda^{k-1}) & \ldots & (k_{m-2})(\lambda^{k-m+2}) \\ 0 & 0 & 0 & \lambda^k & \ldots & (k_{m-3})(\lambda^{k-m+3}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda^k & (k_1)(\lambda^{k-1}) \\ 0 & 0 & 0 & \ldots & 0 & \lambda^k \end{pmatrix}.$$

Let $g_{ij}$ be the $(i,j)$-entry of $g \in \text{GL}_n(\mathbb{C})$. Thus, for each $k \geq 1$,

$$(ba^k b^{-1})_{01} = k\lambda^{k-1} \quad \text{and} \quad (ba^k b^{-1})_{11} = \lambda^k.$$

We define a rational function $h'$ on $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ by declaring

$$h'(g, g') := \frac{g_{01}g'_{11}}{g_{01}g_{11}}$$

for all $g, g' \in \text{GL}_n(\mathbb{C})$ such that $g_{11}, g'_{01} \neq 0$. Then define $h$ by declaring

$$h(g, g') := h'(bg b^{-1}, bg' b^{-1})$$

We have

$$h(a^i, a^j) = \frac{(i\lambda^{i-1})(\lambda^j)}{(j\lambda^{j-1})(\lambda^i)} = \frac{i}{j}$$

for all $i, j \geq 1$.

Thus $\mathbb{Q}_{>0}$ is a subset of the image of $G \times G$ under $h$. \qed

**Corollary 4.2.** If $\Gamma$ contains a non-diagonalizable matrix, then $\mathbb{R}_\Gamma$ defines $\mathbb{Z}$. In particular, if $\Gamma$ is either

- solvable and not virtually abelian, or
• torsion-free, nilpotent and non-abelian,
then \( \mathbb{R}_\Gamma \) defines \( \mathbb{Z} \).

Proof. Applying Lemma 4.1, suppose \( h \) is a rational function on \( \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \) such that the image of \( \Gamma \times \Gamma \) under \( h \) is dense in \( \mathbb{R}_{>0} \). Note \( \Gamma \) is countable as \( \Gamma \) is discrete. It follows that the image of \( \Gamma \times \Gamma \) under any function is co-dense in \( \mathbb{R}_{>0} \).

Fact 4.1(1) implies that \( \mathbb{R}_\Gamma \) defines \( \mathbb{Z} \). The second claim follows from the first by applying Lemma 3.4 and Lemma 3.7.

Corollary 4.3. If \( a \in \text{GL}_n(\mathbb{C}) \) is non-diagonalizable, then \( (\mathbb{R}, \{ ak : k \in \mathbb{Z} \}) \) defines \( \mathbb{Z} \).

Proof. Set \( G := \{ ak : k \in \mathbb{Z} \} \). The proof of Lemma 4.1 shows that in this case \( \mathbb{Q}_{>0} \) is the intersection of \( h(G \times G) \) and \( \mathbb{R}_{>0} \). Thus the corollary follows by Julia Robinson’s classical theorem of definability of \( \mathbb{Z} \) in \( (\mathbb{Q}, +, \cdot) \) in [25]. □

5. The case of exponential growth

We recall the Assouad dimension of a metric space \( (X, d) \). See Heinonen [10] for more information. The Assouad dimension of a subset \( Y \) of \( \mathbb{R}^k \) is the Assouad dimension of \( Y \) equipped with the euclidean metric induced from \( \mathbb{R}^k \).

Suppose \( A \subseteq X \) has at least two elements. Then \( A \) is \( \delta \)-separated for \( \delta \in \mathbb{R}_{>0} \) if \( d(a, b) \geq \delta \) for all distinct \( a, b \in A \), and \( A \) is \( \delta \)-separated if \( A \) is \( \delta \)-separated for some \( \delta > 0 \). Let \( S(A) \in \mathbb{R} \) be the supremum of all \( \delta \geq 0 \) for which \( A \) is \( \delta \)-separated. Let \( \mathcal{D}(A) \) be the diameter of \( A \); that is the infimum of all \( \delta \in \mathbb{R} \cup \{ \infty \} \) such that \( d(a, b) < \delta \) for all \( a, b \in A \), and \( A \) is bounded if \( \mathcal{D}(A) < \infty \). Note \( S(A) \leq \mathcal{D}(A) \).

The Assouad dimension of \( (X, d) \) is the infimum of the set of \( \beta \in \mathbb{R}_{>0} \) for which there is a \( C > 0 \) such that

\[
|A| \leq C \left( \frac{\mathcal{D}(A)}{S(A)} \right)^\beta \quad \text{for all bounded and separated } A \subseteq X.
\]

The proof of Fact 5.1 is an elementary computation which we leave to the reader.

Fact 5.1. Suppose there is a sequence \( \{ A_m \}_{m \in \mathbb{N}} \) of bounded separated subsets of \( X \) with cardinality at least two, and \( B, C, t > 1 \) are such that

\[
|A_m| \geq C^m \quad \text{and} \quad \frac{\mathcal{D}(A_m)}{S(A_m)} \leq tB^m \quad \text{for all } m
\]

then \( (X, d) \) has positive Assouad dimension.

Let \( |v| \) be the usual euclidean norm of \( v \in \mathbb{C}^n \). Given \( g \in \text{M}_n(\mathbb{C}) \) we let

\[
\|g\| = \inf \{ t \in \mathbb{R}_{>0} : |gv| \leq t|v| \quad \text{for all } v \in \mathbb{C}^n \}
\]

be the operator norm of \( g \). Then \( \| \| \) is a linear norm on \( \text{M}_n(\mathbb{C}) \) and satisfies \( \| gh \| \leq \| g \| \| h \| \) for all \( g, h \in \text{M}_n(\mathbb{C}) \). As any two linear norms on \( \text{M}_n(\mathbb{C}) \) are bi-Lipschitz equivalent the metric induced by \( \| \| \) is bi-Lipschitz equivalent to the usual euclidean metric on \( \mathbb{R}^{n^2} \).

Proposition 5.2. Suppose \( \Gamma \) contains a finitely generated subgroup \( \Gamma' \) of exponential growth. Then \( \Gamma \) has positive Assouad dimension.
Proof. Because Assouad dimension is a bi-Lipschitz invariant (see [10]), it suffices to show that \( \Gamma \) has positive Assouad dimension with respect to the metric induced by \( \| \cdot \| \). We let \( I \) be the \( n \times n \) identity matrix. Let \( S \) be a symmetric generating set of \( \Gamma' \), and let \( S_m \) be the set of \( m \)-fold products of elements of \( S \) for \( m \geq 2 \). Set

\[
B := \max\{\|g\| : g \in S\} \quad \text{and} \quad D := \min\{\|g - I\| : g \in \Gamma\}.
\]

Note that \( D > 0 \), as \( \Gamma \) is discrete, and that \( B > 0 \), as \( \Gamma \neq \{I\} \). Induction shows that \( \|g\| \leq B^m \) when \( g \in S_m \). The triangle inequality directly yields \( \mathcal{D}(S_m) \leq 2B^m \). Each \( S_m \) is symmetric as \( S \) is symmetric. Therefore \( \|g^{-1}\| \leq B^m \) for all \( g \in S_m \).

Let \( g, h \in \Gamma \). We have

\[
\|I - g^{-1}h\| \leq \|g^{-1}\|\|g - h\|.
\]

Equivalently,

\[
\frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \leq \|g - h\|.
\]

Suppose \( g, h \in S_m \) are distinct. Then \( g^{-1}h \neq I \), and hence \( \|I - g^{-1}h\| \geq D \). So

\[
\|g - h\| \geq \frac{\|I - g^{-1}h\|}{\|g^{-1}\|} \geq \frac{D}{B^m}.
\]

Hence \( S(S_m) \geq D/B^m \). Thus

\[
\frac{\mathcal{D}(S_m)}{S(S_m)} \leq \frac{2B^m}{D/B^m} = \frac{2}{D}B^{2m}.
\]

As \( \Gamma' \) has exponential growth, there is a \( C > 0 \) such that \( |S_m| \geq C^m \) for all \( m \). An application of Fact 5.1 shows that \( \Gamma \) has positive Assouad dimension. \( \square \)

**Proposition 5.3.** Suppose \( \Gamma \) is not virtually abelian. Then \( \mathbb{R}_\Gamma \) defines \( \mathbb{Z} \).

**Proof.** By Corollary 4.2, we can assume that \( \Gamma \) is solvable. Thus by Fact 3.1, the group \( \Gamma \) contains a non-abelian free subgroup. Therefore \( \Gamma \) has positive Assouad dimension by Proposition 5.2. We conclude that \( \mathbb{R}_\Gamma \) defines \( \mathbb{Z} \) by Fact 1.1(2). \( \square \)

### 6. The virtually abelian case

We first reduce the virtually abelian case to the abelian case.

**Lemma 6.1.** Suppose \( G \) is virtually abelian and every element of \( G \) is diagonalizable. Then there is a finite index abelian subgroup \( G' \) of \( G \) such that \((\mathbb{R}, G)\) and \((\mathbb{R}, G')\) are interdefinable.

**Proof.** Let \( G'' \) be a finite index abelian subgroup of \( G \). As every element of \( G'' \) is diagonalizable, \( G'' \) is simultaneously diagonalizable. Fix \( g \in \text{Gl}_n(\mathbb{C}) \) such that \( gag^{-1} \) is diagonal for all \( a \in G'' \). Let \( G' \) be the set of \( a \in G \) such that \( gag^{-1} \) is diagonal, i.e., \( G' \) is the intersection of \( G \) and \( g^{-1}\text{D}_n(\mathbb{C})g \). Then \( G' \) is abelian, \((\mathbb{R}, G)\)-definable, and is of finite index in \( G \) as \( G'' \subseteq G' \). Because \( G' \) has finite index in \( G \), we have

\[
G = g_1G' \cup \ldots \cup g_mG' \text{ for some } g_1, \ldots, g_m \in G.
\]

So \( G \) is \((\mathbb{R}, G')\)-definable. \( \square \)

Proposition 6.2 finishes the proof of Theorem A.

**Proposition 6.2.** Suppose \( \Gamma \) is abelian and \( \mathbb{R}_\Gamma \) does not define \( \mathbb{Z} \). Then there is \( \lambda \in \mathbb{R}_{>0} \) such that \( \mathbb{R}_\Gamma \) is interdefinable with \( \mathbb{R}_\Lambda \).
Let \( u : \mathbb{C}^\times \to U \) be the argument map and \( || : \mathbb{C}^\times \to \mathbb{R}_{>0} \) be the absolute value map. Thus \( z = u(z)||z| \) for all \( z \in \mathbb{C}^\times \). Let \( U_m \) be the group of \( m \)th roots of unity for all \( m \geq 1 \). In the following proof of Proposition 5.2 we will use the immediate corollary of [11, Theorem 1.3] that the structure \((\mathbb{R}, \lambda^z, \mu^z)\) defines \( \mathbb{Z} \) whenever \( \log \lambda \mu \not\in \mathbb{Q} \), and is is interdefinable with \((\mathbb{R}, \lambda^z)\) otherwise.

**Proof.** Fact [11, 1) implies every countable \( \mathbb{R}_\Gamma \)-definable subset of \( \mathbb{R} \) is nowhere dense. It follows that every \( \mathbb{R}_\Gamma \)-definable countable subgroup of \( U \) is finite and every \( \mathbb{R}_\Gamma \)-definable countable subgroup of \((\mathbb{R}_{>0}, \cdot)\) is of the form \( \lambda^z \) for some \( \lambda \in \mathbb{R}_{>0} \).

Every element of \( \Gamma \) is diagonalizable by Corollary [12]. Thus \( \Gamma \) is simultaneously diagonalizable. After making a change of basis we suppose \( \Gamma \) is a subgroup of \( D_n(\mathbb{C}) \). We identify \( D_n(\mathbb{C}) \) with \((\mathbb{C}^\times)^n \). Let \( \Gamma_i \) be the image of \( \Gamma \) under the projection \((\mathbb{C}^\times)^n \to \mathbb{C}^\times \) onto the \( i \)th coordinate for \( 1 \leq i \leq n \).

Each \( u(\Gamma_i) \) is finite. Fix an \( m \) such that \( u(\Gamma_i) \) is a subgroup of \( U_m \) for all \( 1 \leq i \leq n \). For each \( 1 \leq i \leq n \), \( |\Gamma_i| \) is a discrete subgroup of \( \mathbb{R}_{>0} \) and is thus equal to \( \alpha_i^z \) for some \( \alpha_i \in \mathbb{R}_{>0} \). By [11, Theorem 1.3] each \( \alpha_i \) is a rational power of \( \alpha_1 \). Let \( \lambda \in \mathbb{R}_{>0} \) be a rational power of \( \alpha_1 \) such that each \( \alpha_i \) is an integer power of \( \lambda \). We show \( \mathbb{R}_\Gamma \) and \( \mathbb{R}_\lambda \) are interdefinable. Note that \( \lambda^z \) is \( \mathbb{R}_\Gamma \)-definable; so it suffices to show \( \Gamma \) is \( \mathbb{R}_\lambda \)-definable.

Every element of \( \Gamma_i \) is of the form \( \sigma \lambda^k \) for some \( \sigma \in U_m \) and \( k \in \mathbb{Z} \). Thus \( \Gamma \) is a subgroup of

\[
\Gamma' = \left\{ \begin{pmatrix}
\sigma_1 \lambda^{k_1} & 0 & \ldots & 0 \\
0 & \sigma_2 \lambda^{k_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \sigma_n \lambda^{k_n}
\end{pmatrix} : \sigma_1, \ldots, \sigma_n \in U_m, k_1, \ldots, k_n \in \mathbb{Z} \right\}.
\]

Note \( \Gamma' \) is \( \mathbb{R}_\lambda \)-definable. Abusing notation we let \( u : (\mathbb{C}^\times)^n \to U^n \) and we let \( || : (\mathbb{C}^\times)^n \to (\mathbb{R}_{>0})^n \) be given by

\[
 u(z_1, \ldots, z_n) = (u(z_1), \ldots, u(z_n)) \quad \text{and} \quad ||(z_1, \ldots, z_n)| = (|z_1|, \ldots, |z_n|).
\]

Then the map \((\mathbb{C}^\times)^n \to U^n \times (\mathbb{R}_{>0})^n \) given by \( \bar{z} \mapsto (u(\bar{z}), ||\bar{z}|) \) restricts to a \( \mathbb{R}_\lambda \)-definable isomorphism between \( \Gamma' \) and \( U^n_m \times (\lambda^z)^n \). Lemma 6.3 below implies any subgroup of \( U^n_m \times (\lambda^z)^n \) is \( \mathbb{R}_\lambda \)-definable. \( \square \)

We consider \((\mathbb{Z}/m\mathbb{Z}, +)\) to be a group with underlying set \( \{0, \ldots, m-1\} \) in the usual way so that \((\mathbb{Z}/m\mathbb{Z}, +)\) is a \((\mathbb{Z}, +)\)-definable group. Lemma 6.3 is folklore. We include a proof for the sake of completeness.

**Lemma 6.3.** Every subgroup \( H \) of \((\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n \) for \( l \geq 0 \) is \((\mathbb{Z}, +)\)-definable.

**Proof.** We first reduce to the case \( l = 0 \). The quotient map \( \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \) is \((\mathbb{Z}, +)\)-definable, it follows that the coordinate-wise quotient \( \mathbb{Z}^l \times \mathbb{Z}^n \to (\mathbb{Z}/m\mathbb{Z})^l \times \mathbb{Z}^n \) is \((\mathbb{Z}, +)\)-definable. It suffices to show the preimage of \( H \) in \( \mathbb{Z}^{l+n} \) is \((\mathbb{Z}, +)\)-definable.
Suppose $H$ is a subgroup of $\mathbb{Z}^n$. Then $H$ is finitely generated with generators $\beta_1, \ldots, \beta_k$ where $\beta_i = (b_{i1}, \ldots, b_{in})$ for all $1 \leq i \leq k$. Then

$$H = \left\{ \sum_{i=1}^{k} c_i \beta_i : c_1, \ldots, c_k \in \mathbb{Z} \right\} = \left\{ \left( \sum_{i=1}^{k} c_i b_{i1}, \ldots, \sum_{i=1}^{k} c_i b_{in} \right) : c_1, \ldots, c_n \in \mathbb{Z} \right\}.$$ 

Thus $H$ is $(\mathbb{Z}, +)$-definable.

### 7. Countable $(\mathbb{R}, \lambda^\mathbb{Z})$-definable groups

Fix $\lambda \in \mathbb{R}_{>0}$ and an $\alpha$-minimal $\mathbb{R}$ with field of exponents $\mathbb{Q}$. Since $(\mathbb{R}, \lambda^\mathbb{Z})$ does not define $\mathbb{Z}$ by [18, Theorem 3.4.2], Theorem A implies every $(\mathbb{R}, \lambda^\mathbb{Z})$-definable discrete subgroup of $\text{GL}_n(\mathbb{C})$ is virtually abelian. We extend this result to all countable interpretable groups.

**Proposition 7.1.** Every countable $(\mathbb{R}, \lambda^\mathbb{Z})$-interpretable group is virtually abelian.

Proposition 7.1 follows directly from several previous results. Every d-minimal expansion of $\mathbb{R}$ admits definable selection by Miller [19]. Therefore an $(\mathbb{R}, \lambda^\mathbb{Z})$-interpretable group is isomorphic to an $(\mathbb{R}, \lambda^\mathbb{Z})$-definable group. We now recall two results of Tychonievich. The first is a special case of [27, 4.1.10].

**Fact 7.2.** If $X \subseteq \mathbb{R}^k$ is $(\mathbb{R}, \lambda^\mathbb{Z})$-definable and countable, then there is an $\lambda^\mathbb{Z}$-definable surjection $f : (\lambda^\mathbb{Z})^m \to X$ for some $m$.

Fact 7.2 is a minor rewording of [27, 4.1.2].

**Fact 7.3.** Every $(\mathbb{R}, \lambda^\mathbb{Z})$-definable subset of $(\lambda^\mathbb{Z})^m$ is $(\lambda^\mathbb{Z}, <, \cdot)$-definable.

Facts 7.2 and 7.3 together imply that every countable $(\mathbb{R}, \lambda^\mathbb{Z})$-definable group is isomorphic to a $(\mathbb{Z}, <, +)$-definable group. Now apply the following result of Onshuus and Vicaria [23] to complete the proof of Proposition 7.1.

**Fact 7.4.** Every $(\mathbb{Z}, <, +)$-definable group is virtually abelian.

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