ENERGY OF SOLENOIDAL VECTOR FIELDS ON SPHERICAL DOMAINS

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Abstract. We present a “boundary version” of a theorem about solenoidal unit vector fields with minimum energy on a spherical domain of an odd dimensional Euclidean sphere.

1. Introduction.

Let \((M, g)\) be a closed, \(n\)-dimensional Riemannian manifold and \(T^1M\) the unit tangent bundle of \(M\) considered as a closed Riemannian manifold with the Sasaki metric. Let \(X: M \rightarrow T^1M\) be a unit vector field defined on \(M\), regarded as a smooth section of the unit tangent bundle \(T^1M\). Using an orthonormal local frame \(\{e_1, e_2, \ldots, e_{n-1}, e_n = X\}\), the energy of the unit vector field \(X\) is given by

\[
E(X) = \frac{n}{2} \text{vol}(M) + \frac{1}{2} \int_M \sum_{a=1}^n \|\nabla e_a X\|^2 \nu_M \langle g \rangle
\]

The Hopf vector fields on \(S^{2k+1}\) are unit vector fields tangent to the classical Hopf fibration \(S^1 \hookrightarrow S^{2k+1}\). The following theorem gives a characterization of Hopf flows as absolute minima of energy functional among all solenoidal (that is, divergence free) unit vector fields on \(S^{2k+1}\).

**Teorema 1.1.** [3] The Hopf vector fields has minimum energy among all solenoidal unit vector fields on the sphere \(S^{2k+1}\).

We prove in this paper the following boundary version for this Theorem:

**Teorema 1.2.** Let \(U\) be an open set of the \((2k + 1)\)-dimensional unit sphere \(S^{2k+1}\) and let \(K \subset U\) be a connected \((2k + 1)\)-submanifold with boundary of the sphere \(S^{2k+1}\). Let \(\vec{v}\) be a solenoidal unit vector field on \(U\) which coincides with a Hopf flow \(H\) along the boundary of \(K\). Then

\[
E(\vec{v}) \geq \left(\frac{2k + 1}{2} + k\right) \text{vol}(K) = E(H)
\]

2. Preliminaries.

Let \(U \subset S^{2k+1}\) be an open set of the unit sphere and let \(K \subset U\) be a connected \((2k + 1)\)-submanifold with boundary of \(S^{2k+1}\). Let \(H\) be a Hopf vector field on \(S^{2k+1}\) and let \(\vec{v}\) be an unit vector field defined on \(U\). We also consider the map \(\varphi_{\vec{v}}: U \rightarrow S^{2k+1}(\sqrt{1+t^2})\) given by \(\varphi_{\vec{v}}(x) = x + t\vec{v}(x)\). This map was introduced in [1] and [9].
Lemma 2.1. For \( t > 0 \) sufficiently small, the map \( \varphi_t^\theta \) is a diffeomorphism.

Proof. A simple application of the identity perturbation method □

From now on, we assume that \( t > 0 \) is small enough so that the map \( \varphi_t^\theta \) is a diffeomorphism. In order to find the Jacobian matrix of \( \varphi_t^\theta \), we define the unit vector field \( \vec{u} \) on \( \varphi_t^\theta(U) \subset S^{2k+1}(\sqrt{1+\vec{u}}^2) \) by

\[
\vec{u}(x) := \frac{1}{\sqrt{1+t^2}} \vec{v}(x) - \frac{t}{\sqrt{1+t^2}} x
\]

Using an adapted orthonormal frame \( \{e_1, \ldots, e_{2k}, \vec{u}\} \) on a neighborhood \( V \) of \( U \), we obtain an adapted orthonormal frame on \( \varphi_t^\theta(V) \) given by \( \{\bar{e}_1, \ldots, \bar{e}_{2k}, \vec{u}\} \), where \( \bar{e}_i = e_i \) for all \( i \in \{1, \ldots, 2k\} \).

In this manner, we can write

\[
\begin{align*}
\frac{d}{dt} \varphi_t^\theta(e_1) &= \langle d\varphi_t^\theta(e_1), e_1 \rangle e_1 + \cdots + \langle d\varphi_t^\theta(e_{2k}), e_{2k} \rangle e_{2k} + \langle d\varphi_t^\theta(e_1), \vec{u} \rangle \vec{u} \\
\frac{d}{dt} \varphi_t^\theta(e_2) &= \langle d\varphi_t^\theta(e_2), e_1 \rangle e_1 + \cdots + \langle d\varphi_t^\theta(e_{2k}), e_{2k} \rangle e_{2k} + \langle d\varphi_t^\theta(e_2), \vec{u} \rangle \vec{u} \\
&\vdots \\
\frac{d}{dt} \varphi_t^\theta(e_{2k}) &= \langle d\varphi_t^\theta(e_{2k}), e_1 \rangle e_1 + \cdots + \langle d\varphi_t^\theta(e_{2k}), e_{2k} \rangle e_{2k} + \langle d\varphi_t^\theta(e_{2k}), \vec{u} \rangle \vec{u} \\
\frac{d}{dt} \varphi_t^\theta(\vec{u}) &= \langle d\varphi_t^\theta(\vec{u}), e_1 \rangle e_1 + \cdots + \langle d\varphi_t^\theta(\vec{u}), e_{2k} \rangle e_{2k} + \langle d\varphi_t^\theta(\vec{u}), \vec{u} \rangle \vec{u}
\end{align*}
\]

Now, by Gauss' equation of the trivial immersion \( S^{2k+1} \hookrightarrow \mathbb{R}^{2k+2} \), we have

\[
\nabla_Y \vec{v} = d\vec{v}(Y) = \nabla_Y \vec{v} - \langle \vec{v}, Y \rangle x
\]

for every vector field \( Y \) on \( S^{2k+1} \), and then

\[
\langle d\varphi_t^\theta(e_1), e_1 \rangle = \langle e_1 + td\vec{v}(e_1), e_1 \rangle = 1 + t \langle \nabla_{e_1} \vec{v}, e_1 \rangle
\]

Analogously, we can conclude that

\[
\begin{align*}
\langle d\varphi_t^\theta(e_i), e_i \rangle &= 1 + t \langle \nabla_{e_i} \vec{v}, e_i \rangle, \forall i \in \{1, \ldots, 2k\} \\
\langle d\varphi_t^\theta(e_i), e_j \rangle &= t \langle \nabla_{e_i} \vec{v}, e_j \rangle, \forall i, j \in \{1, \ldots, 2k\}, (i \neq j) \\
\langle d\varphi_t^\theta(e_i), \vec{u} \rangle &= 0, \forall i \in \{1, \ldots, 2k\} \\
\langle d\varphi_t^\theta(\vec{u}), \vec{u} \rangle &= \sqrt{1+t^2}
\end{align*}
\]

By employing the notation \( h_{ij}(\vec{v}) := \langle \nabla_{e_i} \vec{v}, e_j \rangle \) (where \( i, j \in \{1, \ldots, 2k\} \)), we can express the determinant of the Jacobian matrix of \( \varphi_t^\theta \) in the form

\[
\det(d\varphi_t^\theta) = \sqrt{1+t^2}(1 + \sum_{i=1}^{2k} \sigma_i(\vec{v})t^2)
\]

where, by definition, the functions \( \sigma_i \) are the \( i \)-symmetric functions of the \( h_{ij} \). For instance, if \( k = 1 \), we have

\[
\begin{align*}
\sigma_1(\vec{v}) &:= h_{11}(\vec{v}) + h_{22}(\vec{v}) \\
\sigma_2(\vec{v}) &:= h_{11}(\vec{v})h_{22}(\vec{v}) - h_{12}(\vec{v})h_{21}(\vec{v})
\end{align*}
\]
3. PROOF OF THE THEOREM.

The energy of the vector field \( \vec{v} \) (on \( K \)) is given by

\[
E(\vec{v}) := \frac{1}{2} \int_K \|d\vec{v}\|^2 = \frac{2k + 1}{2} \text{vol}(K) + \frac{1}{2} \int_K \|\nabla \vec{v}\|^2
\]

Using the notation above, we have

\[
E(\vec{v}) = \frac{2k + 1}{2} \text{vol}(K) + \frac{1}{2} \int_K \sum_{i,j=1}^{2k} (h_{ij}(\vec{v}))^2 + \sum_{i=1}^{2k} (\langle \nabla \vec{v}, e_i \rangle)^2
\]

and then

\[
E(\vec{v}) \geq \frac{2k + 1}{2} \text{vol}(K) + \frac{1}{2} \int_K \sum_{i,j=1}^{2k} (h_{ij}(\vec{v}))^2
\]

(3.1)

Now observe that

\[
\sum_{i<j} (h_{ii} - h_{jj})^2 = (2k - 1) \sum_i h_{ii}^2 - 2 \sum_{i<j} h_{ii}h_{jj}
\]

(3.2)

and as \( \vec{v} \) is a solenoidal vector field

\[
0 = [\text{div}(\vec{v})]^2 = [\sigma_1(\vec{v})]^2 = (\sum_i h_{ii})^2 = \sum_i h_{ii}^2 + 2 \sum_{i<j} h_{ii}h_{jj}
\]

(3.3)

in other words

\[
-2 \sum_{i<j} h_{ii}h_{jj} = \sum_i h_{ii}^2
\]

(3.4)

Substituting equation (3.4) in (3.2) we obtain

\[
\sum_{i<j} (h_{ii} - h_{jj})^2 = -4k \sum_{i<j} h_{ii}h_{jj}
\]

(3.5)

Further, we also have the following equation

\[
\sum_{i<j} (h_{ij} + h_{ji})^2 = \sum_{i<j} h_{ij}^2 + 2 \sum_{i<j} h_{ij}h_{ji}
\]

(3.6)

and then

\[
2k \sum_{i<j} (h_{ij} + h_{ji})^2 = 2k \sum_{i<j} h_{ij}^2 + 4k \sum_{i<j} h_{ij}h_{ji}
\]

(3.7)

Adding equations (3.5) and (3.7), we have

\[
\sum_{i<j} h_{ii}^2 \geq 2\sigma_2
\]

(3.8)

and

\[
\sum_{i,j=1}^{2k} h_{ij}^2 = \sum_i h_{ii}^2 + \sum_{i<j} h_{ij}^2 \geq 2\sigma_2
\]

(3.9)

Using the inequalities (3.1) and (3.9), we find

\[
E(\vec{v}) \geq \frac{2k + 1}{2} \text{vol}(K) + \int_K \sigma_2(\vec{v})
\]

(3.10)
On the other hand, by change of variables theorem, we obtain
\[
\text{vol}[\varphi^t_H(K)] = \int_K \sqrt{1 + t^2}(1 + \sum_{i=1}^{2k} \sigma_i(H)t^i)
\]
By a straightforward computation shown in [4], we have \(\sigma_i(H) = \eta_i\) for all index \(i \in \{1, \ldots, 2k\}\), where the numbers \(\eta_i\) are defined by
\[
\eta_i = \begin{cases} 
\left( \frac{k}{i/2} \right) & \text{if } i \text{ is even} \\
0 & \text{if } i \text{ is odd}
\end{cases}
\]
We know that the vector fields \(\vec{v}\) and \(H\) are the same on \(\partial K\). Thus, \(\varphi^t_v(K)\) and \(\varphi^t_H(K)\) are \((2k + 1)\)-submanifolds of \(S^{2k+1}(\sqrt{1 + t^2})\) with the same boundary. We claim that \(\varphi^t_v(K) = \varphi^t_H(K)\) for all \(t\) sufficiently small. In fact, if \(p\) is an interior point of \(K\), \(\lim_{t \to 0} \varphi^t_v(p) = \lim_{t \to 0} \varphi^t_H(p) = p\) and then we have necessarily
\[
\text{vol}[\varphi^t_v(K)] = \text{vol}[\varphi^t_H(K)]
\]
for all \(t\) sufficiently small, or equivalently,
\[
\int_K \sqrt{1 + t^2}(1 + \sum_{i=1}^{2k} \sigma_i(\vec{v})t^i) = \int_K \sqrt{1 + t^2}(1 + \sum_{i=1}^{2k} \eta_i t^i)
\]
for all \(t > 0\) sufficiently small. Consequently, after cancelling the factor \(\sqrt{1 + t^2}\) and rearranging the terms, we obtain
\[
\left(\int_K [\sigma_1(\vec{v}) - \eta_1] \right) t + \left(\int_K [\sigma_2(\vec{v}) - \eta_2] \right) t^2 + \ldots + \left(\int_K [\sigma_{2k}(\vec{v}) - \eta_{2k}] \right) t^{2k} = 0
\]
for all sufficiently small \(t\). By identity of polynomials, we conclude
\[
(3.11) \quad \int_K \sigma_i(\vec{v}) = \int_K \eta_i = \eta_i \text{vol}(K), \quad \forall i \in \{1, \ldots, 2k\}
\]
and then, using the inequality (3.10) and the equality (3.11) (for \(i = 2\)), we have
\[
\mathcal{E}(\vec{v}) \geq \frac{2k+1}{2} \text{vol}(K) + \eta_2 \text{vol}(K) = \left(\frac{2k+1}{2} + k\right) \text{vol}(K)
\]

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