DESINGULARIZATION ALGORITHMS
I. ROLE OF EXCEPTIONAL DIVISORS

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Abstract. The article is about a “desingularization principle” (Theorem 1.14) common to various canonical desingularization algorithms in characteristic zero, and the roles played by the exceptional divisors in the underlying local construction. We compare algorithms of the authors and of Villamayor and his collaborators, distinguishing between the fundamental effect of the way the exceptional divisors are used, and different theorems obtained because of flexibility allowed in the choice of “input data”. We show how the meaning of “invariance” of the desingularization invariant, and the efficiency of the algorithm depend on the notion of “equivalence” of collections of local data used in the inductive construction.

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1. INTRODUCTION

This article is about (1) a “desingularization principle” (Theorem 1.14 below) that is common to various algorithms for canonical resolution of singularities in characteristic zero (embedded desingularization,
principalization of an ideal, etc.); (2) the roles played by the exceptional divisors arising from blowings-up in the local inductive construction at the heart of the desingularization principle. The exceptional divisors play two roles – one local and one in the passage from local to global – that are reflected in different ways in the invariant whose maximum loci provide the centres of blowing up.

This paper is to be followed by “Desingularization algorithms II. Locally binomial varieties”.

The main results on resolution of singularities in characteristic zero originate in the towering work of Hironaka [Hi1]. Among our aims here is to compare the various algorithms for canonical desingularization developed by the authors [BM2, BM3, BM4, BM5], and by Villamayor and his collaborators [V1, V2, EV1, EV3, BrV]. (Lipman [L] has raised the question of such a comparison.) The approaches have much in common but, apart from different applications of the general principle to particular results, there are important differences due to the ways that the exceptional divisors are used, affecting the invariant, the meaning of “invariance”, and the choice of centre of blowing up.

The general desingularization principle (Theorem 1.14) can be stated roughly as follows: An initial choice of invariant that distinguishes between “general” and “special” points and that satisfies certain basic properties, can be extended to a desingularization invariant defined over sequences of “admissible” blowings-up, satisfying several simple properties which show that special points can be eliminated by successively blowing up the maximum loci of the invariant.

The desingularization principle depends on a local inductive construction (Section 2). We distinguish between the more fundamental effect of the way that the exceptional divisors are used in the local construction, and different theorems that can be obtained because of the flexibility allowed in the choice of “input data” and “when we stop running the algorithm” (cf. §1.2). For example, “embedded desingularization” (Example 1.19(2) below, [Hi1, Main Thm. 1], [BM5, Thms. 1.6, 11.14], [V2]) and “principalization of an ideal” (Example 1.19(1) below, [Hi1, Main Thm. 2], [BM5, Thm. 1.10]) are both applications of Theorem 1.14, the only differences being in the notion of transformation used and the choice of initial invariant (“strict” transform and the Hilbert-Samuel function for embedded desingularization, or “weak” transform and the order of an ideal for principalization; see §1.4 and Examples 1.8). Likewise, the “weak embedded desingularization” theorem of [EV3] (Example 1.19(4) below) is obtained by stopping the principalization algorithm early. (See also §6.2.)
The preceding point of view is not always clear in the literature. It is further developed in Section 6, where we treat universal embedded desingularization of (not necessarily embedded) spaces (§6.1), the relationship between weak and strict transform (§6.2), and an extension of the general desingularization principal to parametrized families (§6.3; cf. [ENV]).

A second related aim is an understanding of the meaning of “invariance”. The desingularization invariants are invariants of what? In other words, on what do the invariants and therefore the algorithms depend? These questions are closely connected to the notion of “equivalence” of collections of local data used in the inductive construction. The collection of local data is called a “presentation” in [BM5]. A presentation is not invariant or canonical. The philosophy behind [BM5] is to introduce an equivalence relation on presentations (using sequences of “test blowings-up” that depend on the accumulating exceptional divisors) so that the corresponding equivalence class is an invariant and certain natural numerical characters of a presentation depend only on the equivalence class. A presentation is similar to the notion of “basic object” or “idealistic space” used in [V2, EV1]. Both originate in Hironaka’s idea of an “idealistic exponent” [H2]. But the equivalence class of a presentation is strictly smaller than that of an idealistic exponent or basic object used in these articles; the “residual multiplicities” appearing in the desingularization invariants of [BM5] or even those of [V2, EV1] depend only on the equivalence class as a presentation, but not only on the equivalence class as a basic object. (See Remarks 2.5 and §§3.5, 5.3 below.)

Our third aim is to show that for certain natural classes of algebraic or analytic varieties (e.g., locally toric or toric), a combinatorial structure can be used to simplify one or both of the roles played by the exceptional divisors. This is the subject of “Desingularization algorithms II. Locally binomial varieties”. A locally binomial variety is a variety defined in local coordinates by systems of binomial equations. (A binomial means a difference of two monomials with no common factor.) A locally toric variety is simply a locally binomial variety that is normal. Locally binomial varieties are a very natural class, both as a testing-ground for general conjectures in algebraic or analytic geometry, and because many general questions and computational problems can be reduced to the binomial case.

The ideas above will be made more precise in the remainder of this introductory section. The notion of a presentation and the desingularization algorithms of [BM5], [V2] and [EV1] are recalled in Sections 2
and 3, where we also outline the proof of the desingularization principle Theorem 1.14. Section 4 presents a worked example (the details of Example 1.2 below), and Section 5 deals with the idea of equivalence of presentations. In comparing our approach to desingularization with that of Villamayor et al, we nevertheless use the more analytic language of [BM5, BM8]. We treat equivalence of presentations (Section 5) using transformation formulas for differential operators developed by Hironaka [Hi2, Sect. 8], Giraud [Gi] and Encinas and Villamayor [EV1, Sect. 4].

1.1. Role of exceptional divisors. The desingularization invariant $\text{inv}(\cdot) = \text{inv}_X(\cdot)$ (or $\text{inv}_J(\cdot)$) will be described precisely below. The invariant is defined recursively over a sequence of “admissible” blowings-up of a singular space $X$ (or a coherent ideal sheaf $J$). Let $X_j$ (or $J_j$) denote the transform of $X$ (or $J$) in “year” $j$ (i.e., after the first $j$ blowings-up; see §1.5). If $a \in X_j$ (or $\text{supp} J_j$), then inv$(a)$ is a finite sequence $(\iota(a), s_1(a); \nu_2(a), s_2(a); \ldots)$, where $\iota(\cdot)$ is the initial invariant and the successive pairs are themselves defined using data (“presentations”) that involve “maximal contact” subspaces $N_{r-1}(a)$ of increasing codimension in an ambient manifold. For each $r$, the truncated invariant $\text{inv}_{r-1/2}(a) := (\iota(a), s_1(a); \ldots; \nu_r(a))$ determines a block $E_r(a)$ of “old” exceptional divisors passing through $a$ (those which do not necessarily have normal crossings with respect to $N_r(a)$ and which do not appear in a previous block $E_q(a)$, $1 \leq q < r$; see Definitions 1.15), and a block $E_r(a)$ of “new” exceptional divisors (those exceptional divisors passing through $a$, not in $E_1(a) \cup \cdots \cup E_r(a)$). The exceptional divisors are used in two ways; they are:

1. Counted in. The term $s_r(a)$ of inv$(a)$ is simply the number of elements of $E_r(a)$. This is the local role of the exceptional divisors – the old exceptional divisors are counted in by $s_r(a)$ in order to guarantee that the centre of blowing up lies inside each of them.

2. Factored out. The new exceptional divisors (those in $E_r(a)$) are factored out from the resolution data on $N_r(a)$ and the next term $\nu_{r+1}(a)$ is the “residual multiplicity” – the minimal order of the resolution data (with respect to suitable weighting) after this factorization. Factoring of the new exceptional divisors guarantees that the centre of blowing up (defined a priori by a local construction) extends to a global smooth subspace – see Corollary 1.17 below.

Remark 1.1. The invariants of Bierstone-Milman and Villamayor. The fact that the equivalence class of a presentation is strictly smaller than that as a basic object has an important consequence for the definition
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of the invariant (due to the second role of the exceptional divisors): The equivalence relation used in [V2, EV4, EV3] does not identify the powers of the new exceptional divisors as invariants (§5.3 below). Thinking of a sequence of blowings-up as a “history”, the residual multiplicities are defined in [BM5] using only the “future” (see §2.4) and in the articles of Villamayor et al by a calculation involving the “past” (Lemma 3.7 below). But the latter applies to only certain subblocks of the “more recent” new exceptional divisors, so only these subblocks are factored out to define the residual multiplicities in [V2], etc.

Each approach has certain advantages: The notion of invariance introduced in [BM5] is stronger (see Remarks 1.16 and §3.5) and the desingularization algorithm is in general faster (cf. Example 1.2). But we need Villamayor’s invariant in the desingularization principle for families (Theorem 6.14), which involves a comparison of the desingularization invariants for a fibre and for the total space, inductively over the previous history.

In “Desingularization algorithms II”, we will show that, for locally binomial varieties, the desingularization algorithm can be greatly simplified: The local role of the exceptional divisors (1) above is unnecessary because the successive maximal contact subspaces and exceptional divisors are coordinate subspaces in suitable local charts; we can use a simpler invariant of the form \((\nu(a), \nu_2(a), \nu_3(a), \ldots)\). For affine binomial varieties, there is a purely combinatorial algorithm (cf. [BM5, Theorem 1.13]) in which the second role of the exceptional divisors (2) above is equally superfluous. The general desingularization principle has a combinatorial component that is closely related to the algorithms of “Desingularization algorithms II”; we will show in the latter that the techniques involved lead to several natural questions about the efficiency of desingularization algorithms, in general.

Example 1.2. Consider the hypersurface \(X\) in 4-space defined by
\[z^d w^{d-1} - x^{d-1} y^d = 0,\]
where \(d\) is a positive integer \(\geq 2\). The maximum order \(2d - 1\) is taken only at the origin – this is the first centre of blowing up in desingularizing \(X\). The strict transform \(X_1\) of \(X\) is defined by
\[z^d - x^{d-1} y^d = 0\]
(in a local coordinate chart where the blowing-up is given by substituting \((xw, yw, zw, w)\) for the original variables \((x, y, z, w)\). See Section 4. For simplicity of notation, we use the same variables before and after blowing up.) In Section 4, we estimate the number of blowings-up needed to reduce the maximum order \(d\) of \(X_1\). Let \(n(BM), n(V)\),...
n(EV) and n(LB) denote the number of blowings-up prescribed by the algorithms of [BM5], [V2], [EV1] and the locally binomial algorithm of “Desingularization algorithms II”, respectively. Also write n(AB) for the number of blowings-up given by the affine binomial algorithm. Then
\[
\begin{align*}
n(BM) & \leq 2d + j \\
n(V) & = n(EV) \geq 9d + k \\
n(LB) & \leq d + l \\
n(AB) & = 1,
\end{align*}
\]
where \(j, k\) and \(l\) are independent of \(d\). The striking difference between AB and the other algorithms reflects the fact that a local invariant that produces a global centre of blowing up, as in BM, V, EV or LB, necessarily has some inefficiency from a purely local point of view.

**Example 1.3.** Let \(X\) denote the surface
\[z^2 - x^2y^3 = 0.\]

The singular locus of \(X\) is the union of the \(x\)- and \(y\)-axes. The algorithms of [BM5] or [V2] prescribe the origin as the centre \(C_0\) of the first blowing-up \(\sigma_1\). The blowing-up \(\sigma_1\) is given by the substitution \((xy, y, yz)\) in one of three coordinate charts, so that the strict transform \(X_1\) of \(X\) by \(\sigma_1\) is given in this chart by the same equation \(z^2 - x^2y^3 = 0\) as before; we seem to have accomplished nothing! But \(y = y_{\text{exc}}\) now defines the exceptional hypersurface \(H_1 = \sigma_1^{-1}(C_0)\). The next blowing-up prescribed by the algorithms again has centre \(C_1 = \{0\}\); \(\sigma_2\) is given by the substitution \((x, xy, xz)\) in one of the charts, so the strict transform \(X_2\) of \(X_1\) in this chart is defined by the equation \(z^2 - x^3y^3 = 0\). Here \(x\) and \(y\) are both exceptional divisors: \(\{y = 0\}\) is the strict transform of \(H_1\) above and \(\{x = 0\} = H_2 := \sigma_2^{-1}(C_1)\). The two blowings-up \(\sigma_1\) and \(\sigma_2\) have not simplified the equation, but serve to re-mark the variables \(x\) and \(y\) as exceptional divisors. The exceptional divisors can be thought of as global coordinates.

In the general algorithm, when a suitable re-marking of variables as new exceptional divisors has been completed, the simple combinatorial part of the algorithm takes over to reduce the orders of functions that are part of the data in a presentation. In general, when the re-marking has been completed for the algorithm of [BM5], the exceptional variables may not be new in the sense of [V2] or [EV1], so further blowings-up are needed before the combinatorial part kicks in. This is the difference in the algorithms highlighted by the example in Section 4.
1.2. **Input data.** The desingularization invariant is a sequence \( \text{inv}(\cdot) = \text{inv}_X(\cdot) \) (or \( \text{inv}_J(\cdot) \)) as above, beginning with a local invariant \( \iota(\cdot) = \iota_X(\cdot) \) (or \( \iota_J(\cdot) \)) of a space \( X \) (or an ideal sheaf \( J \)) that distinguishes between general and special points (e.g., between smooth and singular points of \( X \)). (For simplicity of exposition, we sometimes write \( \iota(\cdot) \) as \( \nu_1(\cdot) \), though, strictly speaking, we reserve \( \nu_1(\cdot) \) for the case that \( \iota(\cdot) \) is the order of an ideal at a point.) The residual multiplicity \( \nu_{r+1}(\cdot), r \geq 1 \), is an invariant of the equivalence class of a local presentation of the truncated invariant \( \text{inv}_r(\cdot) = (\iota(\cdot), s_1(\cdot), \ldots, \nu_r(\cdot), s_r(\cdot)) \).

A presentation includes a collection of data defined on a maximal contact subspace \( N_r(\cdot) \) of a certain codimension \( q \) – the **codimension** of the presentation. The maximum locus of the invariant is a union of global smooth subspaces having only normal crossings. The desingularization algorithm is given by choosing as each successive centre of blowing up a component of the maximum locus (or of the maximum locus in a suitable subspace). The local inductive construction allows us to define \( (\nu_{r+1}(\cdot), s_{r+1}(\cdot)) \) using \( \text{inv}_r(\cdot) \) and a local presentation (of codimension \( q \), say), and to pass to a local presentation of \( \text{inv}_{r+1}(\cdot) \) of codimension \( q + 1 \). The possibility of recognizing invariant characteristics of resolution of singularities by inductive steps of codimension +1 appears already in \([BM2]\) and is one of the main features distinguishing \([BM4, BM5]\) and \([V1, V2]\) from the work of Hironaka and Abhyankar.

Apart from the way that the exceptional divisors are used in the local construction, there is great flexibility in the desingularization algorithm depending on choices of the following:

1. A notion of transformation by blowing up – usually strict or weak transform. (See §1.4 below.)
2. An initial invariant \( \iota(\cdot) \); for example, the order of an ideal \( J \) or a space \( X \) at a point \( a \), or the Hilbert-Samuel function of \( X \) at \( a \) (see Examples 1.8).
3. The codimension of a presentation of \( \iota(a) \).
4. When we “stop running” the algorithm.

This flexibility is illustrated by the results in Section 6.

1.3. **The category of spaces.** \( \mathbb{N} \) denotes the nonnegative integers. Throughout this article, \( K \) denotes a field of characteristic zero, and \( X \) denotes an algebraic variety or a scheme of finite type over \( K \), or an analytic space over \( K \) (in the case that \( K \) is locally compact). For simplicity, we will always assume that our analytic spaces are compact, or relatively compact in an ambient space (so that \( \text{inv}_X(\cdot) \) will always have maximum values), but these assumptions are not necessary (see \([BM5, Section 13]\)). We will usually assume that \( X \) is embedded as a
closed subspace of a smooth ambient space ("manifold") $M$. (See §6.1, however.) Let $\mathcal{O}_M$ denote the structure sheaf of $M$; $\mathcal{O}_M$ is a coherent sheaf of rings. To be precise, $M$ is a local-ringed space $(|M|, \mathcal{O}_M)$, where $|M|$ denotes the underlying topological space of $M$. A closed subspace $X = (|X|, \mathcal{O}_X)$ of $M = (|M|, \mathcal{O}_M)$ is a local-ringed space corresponding to an ideal (i.e., a sheaf of ideals) $\mathcal{J} = \mathcal{I}_X$ of finite type in $\mathcal{O}_M$: $|X| = \text{supp} \mathcal{O}_M/\mathcal{J}$ and $\mathcal{O}_X = (\mathcal{O}_M/\mathcal{J})|_{|X|}$.

We say that $X$ is a hypersurface if $\mathcal{I}_X$ is principal.

A function in our category (e.g., an element of $\mathcal{O}_X,a$, where $a \in X$, or of $\mathcal{O}_X(U)$, where $U \subset |X|$ is open) will be called a "regular function".

Our desingularization algorithms hold much more generally than in the categories above, at least in the hypersurface case (see [BM5, BM8]), but we will not pursue this point here. We do, however, recommend the proof given in [BM8] as a motivation for many of the ideas in this article.

1.4. Weak and strict transform. Consider an ideal of finite type $\mathcal{J}$ in $\mathcal{O}_M$.

Definitions 1.4. Let $a \in M$. The order $\mu_a(\mathcal{J})$ of $\mathcal{J}$ at $a$ is defined as

$$\mu_a(\mathcal{J}) := \max\{\mu \in \mathbb{N} : \mathcal{J} \subset m_{M,a}^{\mu}\},$$

where $m_{M,a}$ denotes the maximal ideal of $\mathcal{O}_{M,a}$. ($\mu_a(\mathcal{J})$ generalizes the order of a (germ of a) function $f \in \mathcal{O}_{M,a}$,

$$\mu_a(f) := \max\{\mu \in \mathbb{N} : f \in m_{M,a}^{\mu}\}.$$)

Let $C$ denote a (smooth) subspace of $M$. The order of $\mathcal{J}$ along $C$ at $a$,

$$\mu_{C,a}(\mathcal{J}) := \max\{\mu \in \mathbb{N} : \mathcal{J} \subset I_{C,a}^{\mu}\}.$$

Thus, $\mu_{C,a}(\mathcal{J})$ is the generic value of $\mu_x(\mathcal{J})$, for $x \in C$ near $a$.

Let $\sigma : M' \rightarrow M$ denote a blowing-up (or a local blowing-up) with smooth centre $C$. Given an ideal $\mathcal{J}$ in $\mathcal{O}_M$ (or a closed subspace $X$ of $M$, with ideal sheaf $\mathcal{I} = \mathcal{J}$), we define weak and strict transforms of $\mathcal{J}$ (or $X$) by $\sigma$ as follows:

Definition 1.5. The weak transform $\mathcal{J}'$ of $\mathcal{J}$ by $\sigma$ is the ideal sheaf $\mathcal{J}' \subset \mathcal{O}_{M'}$ such that, for all $a' \in M'$, $\mathcal{J}'_{a'}$ is the ideal of $\mathcal{O}_{M',a'}$ generated by

$$\{y_{\text{exc}}^{-\mu} f \circ \sigma : f \in \mathcal{J}_a\},$$

where $a = \sigma(a')$, $\mu = \mu_{C,a}(\mathcal{J})$ and $y_{\text{exc}}$ denotes a generator of the principal ideal $\mathcal{I}_{\sigma^{-1}(C),a'}$. 
The weak transform $X'$ of $X$ is the subspace of $M'$ defined by the weak transform $\mathcal{J}'$ of $\mathcal{J} := \mathcal{I}_X$.

Clearly, the weak transform $\mathcal{J}'$ of $\mathcal{J}$ by $\sigma$ is an ideal of finite type.

**Definition 1.6.** If $X$ is a hypersurface (i.e., $\mathcal{J} := \mathcal{I}_X$ is principal), then, by definition, the strict transform $X'$ of $X$ (or the strict transform $\mathcal{J}'$ of $\mathcal{J}$) by $\sigma$ coincides with the weak transform. (If $f \in \mathcal{O}_{M,a}$ and $a' \in \sigma^{-1}(a)$, then we will say that $f' := \mu^{-1}_{C,a}(f) \circ \sigma$ is the “strict transform” of $f$ at $a'$, though of course this is only well-defined up to an invertible factor.

Now let $X$ be an arbitrary closed subspace of $M$ (i.e., $\mathcal{J} = \mathcal{I}_X$ is an arbitrary ideal of finite type in $\mathcal{O}_M$). The strict transform $X'$ of $X$ by $\sigma$ is defined locally, at each $a' \in M'$, as the intersection of the strict transforms of all hypersurfaces containing $X$ at $a = \sigma(a')$ (i.e., $\mathcal{J}'_{a'} = \mathcal{I}_{X',a'} \subset \mathcal{O}_{M',a'}$ is generated by the strict transforms $f'$ of all $f \in \mathcal{J}_a$).

It is not trivial that the strict transform $X'$ of $X$ is a closed subspace of $M'$ (i.e., that $\mathcal{J}' \subset \mathcal{O}_{M'}$ is an ideal of finite type). This depends on Noetherian and coherence properties of our category of spaces. (For this reason, it is easier to prove versions of our desingularization theorems that require only the weak transform, or the strict transform in the hypersurface case, in categories more general than schemes of finite type or analytic spaces; cf. [BM8].)

**Remark 1.7.** Let $\mathcal{J} \subset \mathcal{O}_M$ denote an ideal of finite type. Let $\mathcal{J}^s$, $\mathcal{J}^w$ and $\mathcal{J}^t$ denote the strict, weak and total transforms of $\mathcal{J}$ (respectively) by a local blowing-up $\sigma$. (By definition, $\mathcal{J}^t := \sigma^{-1}(\mathcal{J})$.) Then

$$\mathcal{J}^s \subset \mathcal{J}^w \subset \mathcal{J}^t$$

and $\mathcal{J}^s = \mathcal{J}^t$ if and only if $C \not\subset \text{supp} \mathcal{J}$.

Although the notions of weak and strict transform both apply to ideals $\mathcal{J}$ or subspaces $X$, we will usually refer to the “strict transform $X'$ of $X$” or the “weak transform $\mathcal{J}'$ of $\mathcal{J}$” in order to economize notation and to emphasize the role of the strict transform in embedded resolution of singularities and of the weak transform in principalization of an ideal.

**1.5. Desingularization invariants.** Given a closed subspace $X$ of $M$, or an ideal of finite type $\mathcal{J} \subset \mathcal{O}_M$, we will consider sequences of
transformations

\[ M_{j+1} \xrightarrow{\sigma_{j+1}} M_j \xrightarrow{} \cdots \xrightarrow{} M_1 \xrightarrow{\sigma_1} M_0 = M \]

\[ X_{j+1} \xrightarrow{} X_j \xrightarrow{} X_1 \xrightarrow{} X_0 = X \]

\[ E_{j+1} \xrightarrow{} E_j \xrightarrow{} E_1 \xrightarrow{} E_0 = E \]

or sequences of transformations

\[ J_{j+1} \xrightarrow{\sigma_{j+1}} J_j \xrightarrow{} \cdots \xrightarrow{} J_1 \xrightarrow{\sigma_1} J_0 = J \]

\[ \mathcal{J}_{j+1} \xrightarrow{} \mathcal{J}_j \xrightarrow{} \cdots \xrightarrow{} \mathcal{J}_1 \xrightarrow{\sigma_1} \mathcal{J}_0 = \mathcal{J} \]

where, in each case, \( E \) is a finite collection of smooth hypersurfaces in \( M \) having only normal crossings (usually \( E = \emptyset \)) and, for each \( j \):

\( \sigma_{j+1} \) is a (local) blowing-up of \( M_j \) with smooth centre \( C_j \) such that \( E_j \) and \( C_j \) simultaneously have only normal crossings.

\( E_{j+1} := E_j' \cup \{ \sigma_{j+1}^{-1}(C_j) \} \), where \( E_j' \) denotes the collection of strict transforms \( H' \) of all hypersurfaces \( H \in E_j \). (Thus, \( E_{j+1} \) has only normal crossings.)

\( X_{j+1} \) denotes the strict transform \( X'_j \) of \( X_j \) by \( \sigma_{j+1} \), or

\( \mathcal{J}_{j+1} \) denotes the weak transform \( \mathcal{J}'_j \) of \( \mathcal{J}_j \) by \( \sigma_{j+1} \).

Let \( \Sigma \) denote a partially ordered set. Let \( \iota_X \) denote a local invariant of \( X \) with values in \( \Sigma \); i.e., a function \( \iota_X : X \ni a \mapsto \iota_X(a) \in \Sigma \) such that \( \iota_X(a) \) depends only on the local isomorphism class of \( X \) at \( a \). (Or let \( \iota_J : M \to \Sigma \) denote a local invariant of \( J \). We can also consider \( \iota_X \) to be defined on \( M \).) Write \( \iota = \iota_X \) or \( \iota_J \), to cover both cases.

**Examples 1.8.**

1. \( \iota_J(a) = \mu_a(J) \), the order of \( J \) at \( a \).
2. \( \iota_X(a) = \nu_X(a) \), where \( \nu_X(a) := \mu_a(I_X) \), the order of \( X \) at \( a \).
3. \( \iota_X(a) = H_{X,a} \), the Hilbert-Samuel function of \( X \) at \( a \); i.e., the function

\[ H_{X,a}(l) = \dim_k \frac{O_{X,a}}{m_{X,a}^{l+1}}, \quad l \in \mathbb{N}, \]

where \( m_{X,a} \) denotes the maximal ideal of \( O_{X,a} \). (In the case of schemes, this definition is correct as stated only at a \( \mathbb{K} \)-rational point \( a \); we should otherwise replace \( \dim_k \) by length.) The order \( \nu_X(a) \) is determined by \( H_{X,a} \); if \( X \) is a hypersurface, then \( H_{X,a} \) is determined by \( \nu_X(a) \). See [BM5, Rmks. 1.3] for details of these remarks.

**Definition 1.9.** A local blowing-up \( \sigma : M' \to M \) with centre \( C \) is \( \iota \)-admissible if \( \iota \) is locally constant on \( C \).
Hypotheses 1.10. We will assume that \( \iota \) satisfies the following three properties:

1. *Semicontinuity.*
   (a) \( \iota \) is upper-semicontinuous in the Zariski topology.
   (b) \( \iota \) is infinitesimally upper-semicontinuous in the sense that, if \( \sigma : M' \to M \) is an \( \iota \)-admissible local blowing-up, then \( \iota_{a'} \leq \iota_a \), for all \( a' \in \sigma^{-1}(a) \).
2. *Stabilization.* Every decreasing sequence in the value set of \( \iota \) stabilizes.
3. \( \iota \) admits a semicoherent presentation at every point, in the sense of 1.12 following.

Remark 1.11. In an analytic category, where we can distinguish between the classical and Zariski topologies, we will always understand by (1)(a) above, the following somewhat weaker property: Every point admits a classical neighbourhood in which \( \iota \) is Zariski upper-semicontinuous. Neighbourhoods and germs, in the analytic case, will be understood to be in the Zariski sense, but within some classical coordinate neighbourhood of a given point. This is important, for example, in 1.12, §2.5 and Section 3 below (see Remark 3.2), but we will not labour the point here; we refer to [BM5] for more details.

Definitions and Remarks 1.12. Let

\[ S_\iota(a) := \{ x \in M : \iota(x) \geq \iota(a) \} \]

as a germ at \( a \) (so that \( S_\iota(a) := \{ x \in M : \iota(x) = \iota(a) \} \), as germs, by property (1)(a) above). We call \( S_\iota(a) \) (the germ of) the constant locus of \( \iota \) at \( a \). Let

\[ G(a) = (N(a), G(a)) \]

where \( N(a) \) denotes a germ of a submanifold of \( M \) at \( a \), of codimension \( p \), say, and \( G(a) \) is a finite collection of pairs \( (g, \mu_g) \), where \( g \in \mathcal{O}_{N(a)} \), \( \mu_g \in \mathbb{Q} \), and \( \mu_{\iota_\mu g} \geq \mu_g \). We define the equimultiple locus of \( G(a) \),

\[ S_{G(a)} := \{ x \in N(a) : \mu_{\iota_\mu g} \geq \mu_g, \text{ for all } (g, \mu_g) \in G(a) \} \].

This makes sense as a germ at \( a \).

We say that \( G(a) \) is a presentation of \( \iota \) of codimension \( p \) at \( a \) if:

1. \( S_{G(a)} = S_\iota(a) \).
2. If \( \sigma : M' \to M \) is an \( \iota \)-admissible local blowing-up (with centre \( C' \)) and \( a' \in \sigma^{-1}(a) \), then \( \iota(a') = \iota(a) \) if and only if \( a' \in N(a') \) and
   \[ \mu_{a'}(y_{\mu g} g \circ \sigma) \geq \mu_g, \text{ for all } (g, \mu_g) \in G(a) \].
where $N(a')$ denotes the germ at $a'$ of the strict transform $N(a)'$ of $N(a)$, and $y_{\text{exc}}$ denotes a local generator of the ideal of the exceptional hypersurface $\sigma^{-1}(C)$.

3. The preceding properties (1) and (2) are stable (i.e., continue to hold) after suitable finite sequences of three kinds of morphisms (admissible blowing-up, product with a line, and exceptional blowing-up) and corresponding transformations of $X$ (or $\mathcal{J}$).

The notion of presentation will be made more precise in Section 2. Exceptional blowings-up are defined in terms of the new exceptional divisors (§2.2; cf. §1.1 above). Property (3) concerns sequences of “test blowings-up” used to prove invariance of inv, and is used to define equivalence of presentations (§2.3). The corresponding stability property for idealistic exponents or basic objects involves only admissible blowings-up and product with a line.

We will usually identify a germ with a representative in a small neighbourhood. Then $\mathcal{G}(a) = (N(a), \mathcal{G}(a))$ induces a pair $\mathcal{G}(x) = (N(x), \mathcal{G}(x))$, at any $x \in S_{\mathcal{G}(a)}$ near $a$. A presentation $\mathcal{G}(a)$ of $\mathcal{I}$ at $a$ is called semicoherent if $\mathcal{G}(x)$ is a presentation of $\mathcal{I}$ at $x$, for each $x$ in a neighbourhood of $a$ in $S_{\mathcal{I}}(a) = S_{\mathcal{G}(a)}$. The notion of semicoherent presentation at $a$ clearly depends only on $\mathcal{G}(a)$ and the germ of $X$ (or the stalk of $\mathcal{J}$) at $a$.

**Examples 1.13.** (1) *The order $\mu_{\mathcal{J}}$ of an ideal $\mathcal{J}$ satisfies Hypotheses 1.10:* Properties (1)(a) and (2) are obvious. (1)(b) is an elementary Taylor series computation (cf. [BM5, lemma 5.1]). Let $\mathcal{G}(a) = \{(g, \mu_g)\}$, where the $g$ form any finite set of generators of $\mathcal{J}_a$, and $\mu_g = \mu_a(\mathcal{J})$. Then $\mathcal{G}(a)$ provides a codimension zero semicoherent presentation of $\mu(\mathcal{J})$ at $a$. (See [BM5, Prop. 6.5].)

(2) *The Hilbert-Samuel function satisfies Hypotheses 1.10:* Properties (1) and (2) are due to Bennett [Be]. See [BM3, Thms. 9.2, 7.20] and [BM3, Thm. 5.2.1] for elementary proofs of (1)(a),(b) and (2), respectively. [BM3, Thms. 9.4, 9.6] provide a semicoherent presentation of the Hilbert-Samuel function.

In Theorem 6.17, we show that the Hilbert-Samuel function satisfies stronger hypotheses (6.12 below) that are needed to extend the desingularization principle to parametrized families.

The purpose of the local inductive construction is to extend $\iota(a) = \iota_X(a)$ or $\iota_{\mathcal{J}}(a)$, $a \in M_0 = M$, to an “invariant” $\text{inv}(a) = \text{inv}_X(a)$ or $\text{inv}_{\mathcal{J}}(a)$, where $a \in M_j$, $j = 0, 1, \ldots$, defined recursively over a
sequence of (local) blowings-up (1.1) or (1.2) provided that, for all \( i \leq j \), \( \sigma_i \) is inv-admissible; i.e., \( \text{inv}(\cdot) \) is locally constant on \( C_i \).

In other words, \( X \) or \( J \) determines \( \text{inv}_X(a) \) or \( \text{inv}_J(a) \) for \( a \in M_0 = M \), and thus the first centre of blowing up \( C_0 \subset M_0 \); then \( \text{inv}(a) = \text{inv}_X(a) \) or \( \text{inv}_J(a) \) can be defined on \( M_1 \) and determines \( C_1 \), etc.

The notation \( \text{inv}_X(a) \) or \( \text{inv}_J(a) \), where \( a \in M_j \), indicates a dependence on the original space \( X_0 = X \) or ideal \( J_0 = J \), and not simply on \( X_j \) or \( J_j \). Some dependence on the history of the desingularization process (1.1) or (1.2) is necessary in order to determine a global centre of blowing up using a local invariant – see Example 1.3 above and [BM6, Example 1.9]. Corollary 1.17 below shows how \( \text{inv} \) is used to determine a global centre.

The invariant \( \text{inv}(a) = \text{inv}_X(a) \) or \( \text{inv}_J(a) \), \( a \in M_j \), is a finite sequence

\[
\text{inv}(a) = (\iota(a), s_1(a), \nu_2(a), s_2(a), \ldots, \nu_{t+1}(a))
\]

beginning with \( \iota(a) = \iota_X(a) \) or \( \iota_J(a) \). The terms \( s_r(a) \in \mathbb{N} \) are defined in 1.15 below. For each \( r = 1, \ldots \), the residual multiplicity \( \nu_{r+1}(a) \in \mathbb{Q} \) is defined using a presentation of \( \text{inv}_r \) of codimension \( q + r \) at \( a \) (where we begin with a presentation of \( \text{inv}_{1/2} = \iota \) of codimension \( q + 1 \), say). We can then define a presentation of \( \text{inv}_{r+1} \) of codimension \( q + r \), and the local inductive contraction allows us to pass to an equivalent presentation of codimension \( q + r + 1 \), to complete a cycle in the inductive definition. (See §§3.2 and 6.1 below.) The construction terminates by exhaustion of variables; if \( n = \dim_a M_j \), then \( t \leq n \) and \( \nu_{t+1}(a) = 0 \) or \( \infty \). Sequences in the value set of \( \text{inv} \) can be compared lexicographically.

Although the residual multiplicities are rational, their denominators are controlled in the following way: There is \( e_1 \in \mathbb{N} \) (for example, \( e_1 = \iota(a) \) in the case that \( \iota(a) \) is the order of an ideal) such that, for all \( r > 0 \), \( e_r \nu_{r+1}(a) \in \mathbb{N} \), where \( e_{r+1} = \max\{e_r!, e_r! \nu_{r+1}(a)\} \). (See §3.3(2).)

**Theorem 1.14.** Let \( \iota = \iota_X \) or \( \iota_J \) denote an invariant of \( X \) or \( J \), satisfying the hypotheses 1.10 above. Then \( \iota \) extends to an invariant \( \text{inv} = \text{inv}_X \) or \( \text{inv}_J \) which is defined over any sequence of transformations (1.1) or (1.2), where the successive (local) blowings-up are inv-admissible, having the following properties:

1. **Semicontinuity.**
   
   (a) \( \text{inv} \) is Zariski upper-semicontinuous.
   
   (b) \( \text{inv} \) is infinitesimally upper-semicontinuous; i.e., \( \text{inv}(a) \leq \text{inv}(\sigma_j(a)) \), for all \( a \in M_j \), \( j \geq 1 \).
2. Stabilization. If \( a_j \in M_j \) and \( a_j = \sigma_{j+1}(a_{j+1}) \), \( j = 0, 1, \ldots \), then there exists \( j_0 \) such that \( \text{inv}(a_j) = \text{inv}(a_{j+1}) \), \( j \geq j_0 \).

3. Normal crossings. Let \( a \in M_j \). Then \( S_{\text{inv}}(a) \) and \( E(a) \) simultaneously have only normal crossings. If \( \text{inv}(a) = \text{inv}_{t+1/2}(a) = (\ldots , 0) \), then \( \text{inv}(a) \) is smooth. If \( \text{inv}(a) = \text{inv}_{t+1/2}(a) = (\ldots , 0) \), then each component \( Z \) of \( S_{\text{inv}}(a) \) is of the form

\[
Z = S_{\text{inv}}(a) \cap \bigcap \{ H \in E(a) : Z \subset H \}. \tag{1.4}
\]

4. Decrease. Let \( a \in M_j \). If \( \text{inv}(a) = (\ldots , \infty) \) and \( \sigma \) is the local blowing-up of \( M_j \) with centre \( S_{\text{inv}}(a) \), then \( \text{inv}(a') < \text{inv}(a) \) for all \( a' \in \sigma^{-1}(a) \). On the other hand, suppose that \( \text{inv}(a) = \text{inv}_{t+1/2}(a) = (\ldots , 0) \). Then there is an additional invariant \( \mu(a) = \mu_X(a) \) or \( \mu_{\mathcal{S}}(a) \in \mathbb{Q}, \mu(a) \geq 1 \), such that, if \( Z \) denotes any component of \( S_{\text{inv}}(a) \) and \( \sigma \) is the local blowing-up with centre \( Z \), then

\[
(\text{inv}(a'), \mu(a')) < (\text{inv}(a), \mu(a)),
\]

for all \( a' \in \sigma^{-1}(a) \). (e.g., \( \mu(a) \in \mathbb{N} \).

Old exceptional divisors. The terms \( s_r(a) \) in \( \text{inv}(a) \) can be defined immediately, in an invariant way: If \( a \in M_j \), write \( a_i \) to denote the image of \( a \) in \( M_i \), where \( i \leq j \); i.e., \( a_j = a \) and \( a_i = (\sigma_{i+1} \circ \cdots \circ \sigma_j)(a) \) if \( i < j \).

Definitions 1.15. Let \( a \in M_j \). Let \( i \) denote the earliest year in the resolution history (1.1) or (1.2) where \( \iota(a) = \iota(a_i) \) (i.e., \( i \) is the smallest index \( i' \) such that \( \iota(a) = \iota(a_i) \)). We sometimes call \( i \) the “year of birth” of \( \iota(a) \). Set

\[
E^1(a) := \{ H \in E(a) : H \text{ is transformed from } E(a_i) \}, \quad s_1(a) := \#E^1(a).
\]

To define \( s_{r+1}(a) \) in general, let \( i \) be the earliest year where \( \text{inv}_{r+1/2}(a) = \text{inv}_{r+1/2}(a_i) \). Set

\[
E^{r+1}(a) := \{ H \in E(a) \setminus (E^1(a) \cup \cdots \cup E^r(a)) : H \text{ is transformed from } E(a_i) \}, \quad s_{r+1}(a) := \#E^{r+1}(a).
\]

Meaning of invariance. \( \text{inv}(a) \) depends only on the local isomorphism class of \( (X_{j,a} \text{ or } \mathcal{J}_{j,a}), E(a), E^1(a), \ldots, E^l(a) \) — see §§3.2, 3.5.

Remarks 1.16. The proof of \cite{BM5} or \cite{EV} gives a weaker sense of invariance; see §3.5. Although \cite{BM5} and \cite{V2, EV} count in old exceptional divisors in the same way, the latter factor out only a subset of the new exceptional divisors \( E^r(a) \) (those accumulating after the birth of \( \text{inv}_{r}(a) \)) to define the residual multiplicities \( \nu_{r+1}(a) \). The resulting
difference in the invariants can first show up in the $\nu_2$-terms. But $s_2$ depends on $\operatorname{inv}_{3/2}$, $\nu_3$ depends of $\operatorname{inv}_2$ and the next block of new exceptional divisors, etc., so the difference is magnified in each of the following terms.

1.6. Desingularization algorithm. The following corollary of Theorem 1.14 above captures the role of the exceptional divisors in the passage from local to global. (The exceptional divisors involved in (1.4) above belong to the new block $\mathcal{E}'(a)$.)

Corollary 1.17. Let $a \in M_j$, and consider an open neighbourhood $U$ of $a$ in $M_j$ such that $\operatorname{inv}(a)$ is a maximum value of $\operatorname{inv}(\cdot)$ on $U$. Then each component $Z$ of $S_{\operatorname{inv}}(a)$ extends to a global smooth closed subspace of $U$.

Proof. Consider any total order on $\{I : I \subseteq E_j\}$. Let $a \in M_j$. We label each component $Z$ of $S_{\operatorname{inv}}(a)$ as $Z_I$, where $I := \{H \in E(a) : Z \subseteq H\}$. Define

$$J(a) := \max\{I : Z_I \text{ is a component of } S_{\operatorname{inv}}(a)\},$$
$$\operatorname{inv}^e(a) := (\operatorname{inv}(a), J(a)).$$

It is easy to see that $\operatorname{inv}^e(\cdot)$ is Zariski upper-semicontinuous on $M_j$ and its maximum locus in any open subspace of $M_j$ is smooth.

Given $a \in M_j$ and a component $Z_I$ of $S_{\operatorname{inv}}(a)$, we can choose the order above so that $I = J(a) = \max\{J : J \subseteq E_j\}$. Then $Z_I$ extends to a smooth closed subspace of the open set \{x \in M_j : \operatorname{inv}(x) \leq \operatorname{inv}(a)\}.

We can order $\{I : I \subseteq E_j\}$ using the resolution history (1.1) or (1.2) as follows: (Assuming $E_0 = \emptyset$), write $E_j = \{H^j_1, \ldots, H^j_j\}$, where $H^j_i$ is the strict transform of $H^i_{j-1}$ by $\sigma_j$, $i = 1, \ldots, j - 1$, and $H^j_j = \sigma^{-1}_J(C_{j-1})$. Associate to each $I \subseteq E_j$ the sequence $(\delta_1, \ldots, \delta_j)$, where $\delta_i = 0$ if $H^j_i \notin I$ and $\delta_i = 1$ if $H^j_i \in I$, and use the lexicographic ordering of such sequences, for all $j$ and $I \subseteq E_j$. Consider the extended invariant $\operatorname{inv}^e(\cdot) := (\operatorname{inv}(\cdot), J(\cdot))$ defined using this ordering. The desingularization principle Theorem 1.14 above can then be applied as follows:

Canonical desingularization algorithm 1.18. Blow up with each successive centre given by the maximum locus of $\operatorname{inv}^e$ (or the maximum locus within a suitable closed subspace, depending on the problem). Stop when $\operatorname{inv}$ is locally constant (on a suitable subspace).
Examples 1.19. (1) Principalization of an ideal $\mathcal{J}$ (cf. [Hi1, Main Thm. II], [BM5, Thm. 1.10]). Take $\iota(a) = \iota_{\mathcal{J}}(a) = \mu_a(\mathcal{J})$. Then there is a finite sequence of blowings-up (1.2) with $\text{inv}_{\mathcal{J}}$-admissible centres, such that, if $\mathcal{J}' \subset \mathcal{O}_{M'}$ denotes the final weak transform of $\mathcal{J}$, then $\mathcal{J}' = \mathcal{O}_{M'}$ and $\sigma^{-1}(\mathcal{J}) = \sigma^*(\mathcal{J}) \cdot \mathcal{O}_{M'}$ is a normal-crossings divisor, where $\sigma : M' \to M$ denotes the composite of the sequence of blowings-up.

Algorithm: Blow up with successive centres given by the maximum locus of $\text{inv}_{\mathcal{J}}$ in $M_j$. Stop when $\text{supp} \mathcal{O}_{M_j} / \mathcal{J}_j = \emptyset$; i.e., $\mathcal{J}_j = \mathcal{O}_{M_j}$.

(2) Embedded desingularization (cf. [Hi1, Main Thm. I], [BM5, Theorem 1.6], [V2, EV1]). The following “geometric version” is meaningful if the set of smooth points of $X$ is not empty. Take $\iota(X) = \iota_X = H_X$. Then there is a finite sequence of blowings-up (1.1) with $\text{inv}_X$-admissible centres $C_j$, such that:

(a) For each $j$, either $C_j \subset \text{Sing } X_j$ or $X_j$ is smooth and $C_j \subset X_j \cap E_j$.

(b) Let $X'$ and $E'$ denote the final strict transform of $X$ and exceptional set, respectively. Then $X'$ is smooth and $X', E'$ simultaneously have only normal crossings.

Algorithm:

Step 1. Blow up with successive centres given by the maximum locus of $\text{inv}_X$ in $X_j$. Stop when $\text{Sing } X_j = \emptyset$.

Step 2. Continue to blow up with successive centres given by the maximum locus of $\text{inv}_X$ in $S_j := \{x \in M_j : s_1(x) > 0\}$. Stop when $S_j = \emptyset$.

(3) Embedded desingularization in the nonreduced case. See [BM5, Section 11].

(4) Weak embedded desingularization (cf. [EV3]). Take $\iota(a) = \iota_{\mathcal{J}}(a) = \mu_a(\mathcal{J})$, where $\mathcal{J} = \mathcal{I}_X$. Then there is a finite sequence of blowings-up (1.2) with $\text{inv}_{\mathcal{J}}$-admissible centres $C_j$, such that:

(a) Each $C_j \subset \pi_j^{-1}(\text{Sing } X)$, where $\pi_j$ denotes the composite of the blowings-up to year $j$.

(b) $X'$ is smooth and $X', E'$ simultaneously have only normal crossings (in the notation of (2)(b)).

Algorithm: (Assume that $X$ is pure-dimensional.) Apply the algorithm for principalization of $\mathcal{J} = \mathcal{I}_X$, but stop early – when the maximum value of $\text{inv}_{\mathcal{J}}$ becomes equal to the generic value of $\text{inv}_{\mathcal{J}}$ on $X_0 = X$. See §6.2.

Other applications of the desingularization principle are given in [BM5, Chapter IV] and in Section 6 below.
**Remark 1.20.** “Weak embedded desingularization” (4) above is weaker than “embedded desingularization” (2) because, in (4), it is not in general true that the successive centres $C_j$ lie in the strict transforms $X_j$, nor that the $C_j \cap X_j$ are smooth. (Any birational projective morphism of quasiprojective varieties is a blowing-up with (not necessarily smooth) centre.)

2. **Idea of a presentation**

The desingularization invariant $\text{inv} = \text{inv}_X$ or $\text{inv}_J$ is to be defined recursively over a sequence of admissible blowings-up (1.1) or (1.2). The successive pairs $(\nu_r, s_r)$ in the sequence $\text{inv}$ will themselves be defined inductively. Given $\text{inv}_{r-1/2}$, $r \geq 1$, $s_r$ and $\nu_{r+1}$ can be defined recursively over any sequence of $(r - 1/2)$-admissible (i.e., $\text{inv}_{r-1/2}$-admissible) (local) blowings-up.

Consider a sequence of $(r - 1/2)$-admissible (local) blowings-up (1.1) or (1.2). Let $a \in M_j$. Assume that $\nu_r(a) \neq 0, \infty$. We have defined $s_r(a)$ in Definitions 1.15. The following term $\nu_{r+1}(a)$ of $\text{inv}(a)$ can be defined using a “presentation of $\text{inv}_r$ at $a$”. A presentation of codimension $p$ involves a collection of regular functions (i.e., functions in our category) with “assigned multiplicities” on a “maximal contact subspace” of codimension $p$. A presentation has an “equimultiple locus” (as a germ at $a$); cf. 1.12.

A presentation is not an invariant. We introduce a notion of equivalence of presentations; The equivalence class of a presentation of $\text{inv}_r$ at $a$ does have an invariant meaning, and $\nu_{r+1}(a)$ depends only on the equivalence class. See Theorems 2.3, 2.4 and §3.2. It is convenient, in fact, to consider two notions of equivalence:

1. A purely local notion; see §2.3. The corresponding equivalence class will be denoted $\llbracket \cdot \rrbracket$. Two presentations at $a$ are equivalent in this sense if they have the same equimultiple locus, both at $a$ and also after certain sequences of transformations (cf. 1.12).

2. A stronger notion of “semicoherent equivalence”. Two presentations at $a$ are semicoherent equivalent if they induce presentations that are equivalent in the sense of (1) at each point of the equimultiple locus of $a$. (This again makes sense as a notion about germs.)

See §2.5. The semicoherent equivalence class will be denoted $\llbracket \llbracket \cdot \rrbracket \rrbracket$.

We differentiate between (1) and (2) in order to explain precisely on what the successive terms of $\text{inv}$ depend.

The idea of a presentation is treated in an abstract way in this section. In Section 3 below, we show how the idea is used to define
desingularization invariants and to prove the desingularization principle Theorem 1.14.

2.1. Definition of a presentation (cf. [BM3, (4.1)]. Let $M$ denote a manifold and let $a \in M$. A (local) presentation of codimension $p$ at $a$ is a triple

$$\mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a)),$$

where:

$N(a)$ is a germ of a submanifold of codimension $p$ at $a$;

$\mathcal{H}(a) = \{(h, \mu_h)\}$ is a finite collection of pairs $(h, \mu_h)$, where $h \in \mathcal{O}_{N(a)}$, $\mu_h \in \mathbb{Q}$ and $\mu_a(h) \geq \mu_h$.

$\mathcal{E}(a)$ is a finite collection of smooth hypersurfaces such that $N(a), \mathcal{E}(a)$ simultaneously have only normal crossings, and $N(a) \not\subset H$, for all $H \in \mathcal{E}(a)$.

We define the equimultiple locus $S_{\mathcal{H}(a)}$ of $\mathcal{H}(a)$ (as a germ at $a$) by

$$S_{\mathcal{H}(a)} := \{x \in N(a) : \mu_x(h) \geq \mu_h, \text{ for all } (h, \mu_h) \in \mathcal{H}(a)\}.$$

2.2. Transforms of a presentation. Our notion of equivalence of presentations is given by stability of the condition that their equimultiple loci coincide, after sequences of transformations by three types of morphisms [BM3, Section 4]. Let $\mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a))$ denote a local presentation of codimension $p$ at $a$. We will consider transformations of $\mathcal{H}(a)$ by morphisms $\sigma$ of the following three types.

(i) Admissible blowing-up: a local blowing-up $\sigma$ with centre $C$ such that $C$ and $\mathcal{E}(a)$ simultaneously have only normal crossings, and $a \in C \subset S_{\mathcal{H}(a)}$.

(ii) Product with a line: $\sigma$ is a projection $M' = W \times \mathbb{A}^1 \to W \hookrightarrow M$ over a neighbourhood $W$ of $a$.

(iii) Exceptional blowing-up: a local blowing-up $\sigma$ at $a$ with centre $C = H_0 \cap H_1$, where $H_0, H_1 \in \mathcal{E}(a)$.

For any of the morphisms $\sigma$ above, we define a transform

$$\mathcal{H}(a') = (N(a'), \mathcal{H}(a'), \mathcal{E}(a'))$$

of $\mathcal{H}(a)$ at certain points $a' \in \sigma^{-1}(a)$:

(i) Admissible blowing-up: Suppose that $a' \in \sigma^{-1}(a)$ is a point such that

$$\mu_{a'}(y_{\text{exc}}^{-\mu_h} \circ \sigma) \geq \mu_h, \text{ for all } (h, \mu_h) \in \mathcal{H}(a)$$

(where $y_{\text{exc}}$ denotes a local generator of the ideal of $\sigma^{-1}(C)$). Then we define:

$N(a') := \text{germ at } a' \text{ of the strict transform } N(a')' \text{ of } N(a)$,
\[ \mathcal{H}(a') := \{(h', \mu_{h'}) : (h, \mu_h) \in \mathcal{H}(a)\}, \text{ where } h' := \text{germ at } a' \text{ of } y_{\text{exc}}^{\mu_h}h \circ \sigma \text{ and } \mu_{h'} := \mu_h, \]

\[ \mathcal{E}(a') := \{H' : H \in \mathcal{E}(a), a' \in H'\} \cup \{\sigma^{-1}(C)\} \text{ (where } H' \text{ means the strict transform of } H). \]

(ii) Product with a line. Let \( a' = (a, 0) \). Then we define:

\[ N(a') := \text{germ at } a' \text{ of } \sigma^{-1}(N(a)), \]

\[ \mathcal{H}(a') := \{(h \circ \sigma, \mu_h) : (h, \mu_h) \in \mathcal{H}(a)\}, \]

\[ \mathcal{E}(a') := \{\sigma^{-1}(H) : H \in \mathcal{E}(a)\} \cup \{W \times 0\}. \]

(iii) Exceptional blowing-up. Let \( a' \) be the unique point of \( \sigma^{-1}(a) \cap H'_1 \).

Then we define \( N(a'), \mathcal{H}(a') \) as in (ii), and \( \mathcal{E}(a') \) as in (i).

2.3. Equivalence of presentations [BM5, Section 4].

Definitions 2.1. Two presentations

\[ \mathcal{H}(a) = (N = N(a), \mathcal{H}(a), \mathcal{E}(a)) \]
\[ \mathcal{F}(a) = (P = P(a), \mathcal{F}(a), \mathcal{E}(a)) \]

perhaps of different codimension but with common \( \mathcal{E}(a) \), are equivalent with respect to transformations of types (i), (ii) if:

1. \( S_{\mathcal{H}(a)} = S_{\mathcal{F}(a)} \).
2. If \( \sigma \) is an admissible blowing-up and \( a' \in \sigma^{-1}(a) \), then \( a' \in N' \) and \( \mu_{a'}(y_{\text{exc}}^{\mu_h}h \circ \sigma) \geq \mu_h \), for all \( (h, \mu_h) \in \mathcal{H}(a) \), if and only if \( a' \in P' \) and \( \mu_{a'}(y_{\text{exc}}^{\mu_f}f \circ \sigma) \geq \mu_f \), for all \( (f, \mu_f) \in \mathcal{F}(a) \).
3. Conditions (1) and (2) continue to hold for \( \mathcal{H}(a') \), \( \mathcal{F}(a') \) obtained by any sequence of transformations by morphisms of types (i), (ii).

We likewise define equivalence with respect to transformations of types (i), (ii), (iii) (by simply using all three types of morphisms in (3) above). We write \([\mathcal{H}(a)]_{(i,ii)}\) and \([\mathcal{H}(a)]_{(i,ii,iii)}\) for the corresponding equivalence classes.

Definition 2.2. We introduce an intermediate notion of equivalence by allowing only certain sequences of morphisms (i), (ii) and (iii) in Definitions 2.1 above; namely,

\[ \cdots \xrightarrow{\sigma_j} M_i \xrightarrow{\sigma_{i-1}} \cdots \xrightarrow{\sigma_2} M_1 \xrightarrow{\sigma_1} M_0 = M \]

where, if \( \sigma_{i+1}, \ldots, \sigma_j \) are exceptional blowings-up, then \( i \geq 1 \) and \( \sigma_i \) is either of type (iii) or (ii). In the latter case, \( \sigma_i : M_i = M_{i-1} \times \mathbb{A}^1 \to M_{i-1} \) is the projection and each \( \sigma_{k+1}, k = i, \ldots, j-1 \), is the blowing-up with centre \( C_k = H_0^k \cap H_1^k \), where \( H_0^k, H_1^k \in \mathcal{E}(a_k), a_{k+1} = \sigma_{k+1}^{-1}(a_k) \cap H_1^{k+1} \), and the \( H_0^k, H_1^k \) are determined by some fixed \( H \in \mathcal{E}(a_{i-1}) \) inductively.
in the following way: \( H_i^j := M_{i-1} \times \{0\} \), \( H_i^k := \sigma_i^{-1}(H) \), and, for each \( k = i + 1, \ldots, j-1 \) \( H_i^k := \sigma_k^{-1}(C_{k-1}) \), \( H_i^1 := \) the strict transform of \( H_i^{k-1} \) by \( \sigma_k \).

Let \([\mathcal{H}(a)]\) denote the equivalence class corresponding to Definition 2.2. Clearly,

\[
[H(a)](i,ii,iii) \subset [\mathcal{H}(a)] \subset [H(a)](i,ii).
\]

2.4. Invariants of a presentation. Let \( \mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a)) \) be a presentation at \( a \); say \( \mathcal{H}(a) = \{(h, \mu_h)\} \). We define

\[
\mu(a) = \mu_{\mathcal{H}(a)} := \min_{\mathcal{H}(a)} \frac{\mu_a(h)}{\mu_h},
\]

\[
\mu_H(a) = \mu_{\mathcal{H}(a),H} := \min_{\mathcal{H}(a)} \frac{\mu_H(a)(h)}{\mu_h}, \quad H \in \mathcal{E}(a),
\]

\[
\nu(a) = \nu_{\mathcal{H}(a)} := \mu_{\mathcal{H}(a)} - \sum_{H \in \mathcal{E}(a)} \mu_{\mathcal{H}(a),H}.
\]

(Recall Definitions 1.4.) In particular, \( 0 \leq \nu(a) < \infty \) if \( \mu(a) < \infty \).

We set \( \nu(a) = \infty \) if \( \mu(a) = \infty \).

**Theorem 2.3.** Let \( \mathcal{H}(a) \) and \( \mathcal{F}(a) \) denote presentations that are equivalent with respect to transformations of types (i) and (ii). If \( \mathcal{H}(a) \) and \( \mathcal{F}(a) \) have the same codimension, then

\[
\mu_{\mathcal{H}(a)} = \mu_{\mathcal{F}(a)}.
\]

If \( \mathrm{codim} \mathcal{H}(a) > \mathrm{codim} \mathcal{F}(a) \), then \( \mu_{\mathcal{F}(a)} = 1 \).

**Theorem 2.4.** If \( \mathcal{H}(a) \) and \( \mathcal{F}(a) \) are presentations of the same codimension that are equivalent (i.e., \([\mathcal{H}(a)] = [\mathcal{F}(a)]\)), then, for all \( H \in \mathcal{E}(a) \),

\[
\mu_{\mathcal{H}(a),H} = \mu_{\mathcal{F}(a),H}.
\]

See [BM5, Propositions 4.8, 4.11] or [BM6, Propositions 4.4, 4.6] for proofs of these assertions. (The second assertion of Theorem 2.3 is not stated explicitly in these references, but is clear from the proof of [BM5, Proposition 4.8] or [BM6, Proposition 4.4], and is worth noting – see §6.1.1 below.)

**Remarks 2.5.** As we have remarked in Section 1, our presentations \( \mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a)) \) are similar to Villamayor’s basic objects. In the latter \( \mathcal{H}(a) \) is replaced by an idealistic exponent \((J,b)\) in the sense of Hironaka [Hi2]: \( J \) is an ideal in \( \mathcal{O}_{N(a)} \) and \( b \in \mathbb{N} \). For example, choose \( q \in \mathbb{N} \) such that \( q \cdot \mu_h \in \mathbb{N} \), for all \( (h, \mu_h) \in \mathcal{H}(a) \). Then we can take \( b = \max(q\mu_h)! \) and \( J = \) ideal generated by the \( h^{b/\mu_h} \), for all \( (h, \mu_h) \in \mathcal{H}(a) \). (Each \( b/\mu_h \in \mathbb{N} \).)
Our notion of equivalence of presentations, however, is stronger than the notions of equivalence of idealistic exponents or of basic objects used by Hironaka \cite{Hi2} and Villamayor \cite{V2,EV1}. The latter involve stability under transformations of types (i), (ii) alone, so the corresponding equivalence classes (essentially \([i, ii]\)) are larger.

Example 5.14 below shows that the conclusion of Theorem 2.4 is not necessarily true if \(\mathcal{H}(a)\) and \(\mathcal{F}(a)\) are merely equivalent with respect to transformations of types (i) and (ii); in particular, \([\cdot]\) is in general a strictly smaller class of equivalence than \([i, ii]\). Example 5.14 shows that even the variant of \(\nu(a)\) used by Villamayor is not an invariant of the equivalence class of an idealistic exponent. It is for this reason that the definitions of \(\text{inv}\), the resolution algorithms and the meanings of invariance are not the same in \cite{BM5} and \cite{V2,EV1}. The proofs in the latter show that the underlying invariant \(\text{inv}(a)\), \(a \in M_j\) depends on the previous history of the resolution process, but does not show that it depends only on \(X_{j,a}, E(a),\) and the \(E^q(a)\).

On the other hand, we need to use \([\cdot]\) rather than the smaller equivalence class \([i, ii, iii]\) – see Remark 2.7 below.

2.5. **Semicoherent equivalence.** Let \(\mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a))\) be a presentation. Say \(\mathcal{H}(a) = \{(h, \mu_h)\}\). We identify \(N(a)\) (respectively, each \(h\)) with a submanifold (respectively, a function) in some neighbourhood of \(a\). Let \(\mathcal{H}(x) = (N(x), \mathcal{H}(x), \mathcal{E}(x))\) denote the presentation induced by \(\mathcal{H}(a)\) at each \(x \in N(a)\). \((\mathcal{E}(x) := \{H \in \mathcal{E}(a) : x \in H\}\).)

Let \(\mathcal{F}(a)\) be another presentation at \(a\). We say that \(\mathcal{F}(a)\) and \(\mathcal{H}(a)\) are semicoherent equivalent if \(\mathcal{F}(x)\) and \(\mathcal{H}(x)\) are equivalent (in the sense of Definition 2.2) at each \(x\) in a neighbourhood of \(a\) in \(S_{\mathcal{F}(a)} = S_{\mathcal{H}(a)}\). This notion of semicoherent equivalence clearly depends only on \(\mathcal{F}(a)\) and \(\mathcal{H}(a)\). Write \([\mathcal{H}(a)]\) for the semicoherent equivalence class.

The stronger notion of semicoherent equivalence is important in the local construction \(\S 2.6\) below that will be used in Section 3 in the inductive definition of inv. The following theorem is the basis of the induction on the codimension of a presentation.

**Theorem 2.6.** Let \(\mathcal{G}(a) = (N(a), \mathcal{G}(a), \emptyset)\) be a presentation of codimension \(p\). If \(\mu_{\mathcal{G}(a)} = 1\), then \(\mathcal{G}(a)\) is semicoherent equivalent to a presentation \(\mathcal{G}(a) = (N_{+1}(a), \mathcal{C}(a), \emptyset)\) of codimension \(p + 1\).

We will prove Theorem 2.6 in Section 5 below; see also \cite[Proposition 4.12]{BM5}.
2.6. **Local construction.** We use the notation of §2.4. Suppose that \( \mu(a) < \infty \). Define

\[
D(a) = D_{\mathcal{H}(a)} := \prod_{H \in \mathcal{E}(a)} x_H^{\mu_H(a)},
\]

where \( x_H \in \mathcal{O}_{N(a)} \) denotes a generator of the ideal of \( N(a) \cap H \), for each \( H \in \mathcal{E}(a) \). If \( \nu(a) = 0 \), we define

\[
G(a) = G_{\mathcal{H}(a)} := \{(D(a), 1)\}.
\]

If \( 0 < \nu(a) < \infty \), then, for each \( (h, \mu_h) \in \mathcal{H}(a) \), we can write

\[
h = D(a)^{\mu_h} \cdot g_h
\]

(see Remark 2.7 following), and we define

\[
G(a) = G_{\mathcal{H}(a)} := \{(h, \mu_h \nu(a)) : (h, \mu_h) \in \mathcal{H}(a)\} \cup \{(D(a), 1 - \nu(a))\}.
\]

(The element \( (D(a), 1 - \nu(a)) \) plays no part and can be deleted unless \( \nu(a) < 1 \).) Set

\[
G(a) = G_{\mathcal{H}(a)} := \{(h, \mu_h \nu(a)) : (h, \mu_h) \in \mathcal{H}(a)\} \cup \{(D(a), 1 - \nu(a))\}.
\]

(2.3)

\[
S_G(a) = \{x \in \mathcal{H}(a) : \nu(x) \geq \nu(a)\},
\]

where \( \nu(x) = \nu_{\mathcal{H}(x)} = \min_{\mathcal{H}(a)} \mu_x(h)/\mu_h \). ((2.4) makes sense as an equality of germs at \( a \).

**Remark 2.7.** Using (2.4), it is easy to see that \([G(a)]\) depends only on \([\mathcal{H}(a)]\) \([BM5, \text{Proposition 4.24}]\). We need to use \([\cdot]\) rather than the smaller equivalence class \([\cdot]_{(i,ii,iii)}\) because it is not \textit{a priori} true that the semicoherent class \([G(a)]_{(i,ii,iii)}\) corresponding to \([\cdot]_{(i,ii,iii)}\) depends only on \([\mathcal{H}(a)]_{(i,ii,iii)}\). We do not know whether \([\cdot] = [\cdot]_{(i,ii,iii)}\).

**Remark 2.8.** The factors appearing in (2.2) are (perhaps rational) powers of elements of \( \mathcal{O}_{N(a)} \). We can avoid non-integral powers by replacing \( \mathcal{H}(a) \) by an equivalent presentation where all \( \mu_h = d \), for some \( d \in \mathbb{N} \): Choose \( q \in \mathbb{N} \) such that \( q \mu_h \in \mathbb{N} \), for all \( h \); let \( d = \max(q \mu_h) \) ! and replace each \( (h, \mu_h) \) by \( (h^{d/\mu_h}, d) \) to get an equivalent presentation as claimed. If \( \mathcal{H}(a) = \{(h, d)\} \) with common \( d \in \mathbb{N} \), then \( D(a)^d \) is a monomial in \( x_H \), \( H \in \mathcal{E}(a) \) (with integral powers) and (2.2) becomes \( h = D(a)^d g_h \), so that each \( g_h \in \mathcal{O}_{N(a)} \); \( D(a)^d \) is the greatest common factor of the \( h \) which is monomial in \( x_H \), \( H \in \mathcal{E}(a) \).
3. THE INVARIANT AND THE DESINGULARIZATION PRINCIPLE

In this section, we give the local inductive construction needed to define the desingularization invariant \( \text{inv} = \text{inv}_X \) or \( \text{inv}_J \) (§3.2), and to prove the desingularization principle, Theorem 1.14 (see §3.3). We compare our local construction with that used in Villamayor’s algorithm \([V2]\) (§3.4), as well as with the variant of Encinas and Villamayor \([EV1]\) (§3.6). We answer the question, “\( \text{inv} \) is an invariant of what?”; see §3.5.

3.1. Presentation of a local invariant. Let \( \iota = \iota_X \) (or \( \iota_J \)) denote a local invariant of spaces \( X \) (or ideals of finite type \( J \)). (We assume that \( X \) is a closed subspace of a manifold \( M \), or that \( J \) is a subsheaf of \( \mathcal{O}_M \)).

Let us first be more precise about the Definitions 1.12 used in the Hypotheses 1.10 that we will impose on \( \iota \). Assume that \( \iota \) satisfies the semicontinuity hypotheses 1.10(1). We will use the following transforms of \( X \) (or \( J \)) by the three types of morphisms listed in §2.2: If \( \sigma : M' \to M \) is an \( \iota \)-admissible (local) blowing-up with smooth centre \( C \), we consider the strict transform \( X' \) of \( X \) (or the weak transform \( J' \) of \( J \)). On the other hand, if \( \sigma : M' \to M \) is a morphism of either type (ii) (product with a line) or type (iii) (exceptional blowing-up, in the presence of exceptional divisors), we transform \( X \) (or \( J \)) simply by inverse image: \( X' := \sigma^{-1}(X) \) (or \( J' := \sigma^{-1}(J) \)).

Let \( \mathcal{G}(a) = (N(a), G(a), \emptyset) \) denote a presentation at \( a \), as in §2.1. Of course, if \( S_{\mathcal{G}(a)} = S_{\iota}(a) \), then a local blowing-up at \( a \) is \( \iota \)-admissible if and only if \( \mathcal{G}(a) \) is admissible for \( \mathcal{G}(a) \). (See Definitions 1.9 and §2.2.)

Definitions 3.1. \( \mathcal{G}(a) \) is a presentation of \( \iota \) at \( a \) if conditions (1)-(3) of Definitions 1.12 hold, where (3) refers to any finite sequence of morphisms allowed by Definition 2.2.

A presentation \( \mathcal{G}(a) \) of \( \iota \) at \( a \) is semicoherent if it induces a presentation of \( \iota \) at each \( x \) in a neighbourhood of \( a \) in \( S_{\iota}(a) = S_{\mathcal{G}(a)} \) (cf. §2.5).

The Hypothesis 1.10(3) means that \( \iota \) admits a semicoherent presentation \( \mathcal{G}(a) = (N(a), G(a), \emptyset) \) at each point \( a \).

Remark 3.2. In general (e.g., in an analytic category), it is necessary to be somewhat more precise about the meaning of a presentation associated to an invariant (as above or as in §3.2 below): We assume that \( M \) can be covered by coordinate charts \( U \) such that, for each \( a \in U \), the functions involved in the presentation at \( a \) (e.g., the functions in \( \mathcal{G}(a) \) above) are quotients of elements of \( \mathcal{O}(U) \) with denominators not
vanishing at \( a \) (likewise for a collection of functions defining the maximal contact submanifold, e.g., \( N(a) \) above); cf. \([\text{BM3}]\), Definition and remarks 4.14. (The resulting definition is identical to the preceding in the case of algebraic varieties or schemes.) The equimultiple locus of a presentation at \( a \in U \), and the ideas of semicoherent presentation of an invariant or of semicoherent equivalence of presentations (§2.5) involve germs with respect to the Zariski topology of \( U \). Presentations in this more precise sense exist in Examples 1.13 (according to the references given), and are needed to prove upper semicontinuity of \( \text{inv} \) in Theorem 1.14 (§3.3 below).

**Remarks 3.3.** (1) Suppose that \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \mathcal{E}(a)) \) is a (semicoherent) presentation of \( \iota \) at \( a \). If \( \mathcal{H}(a) = (P(a), \mathcal{H}(a), \mathcal{E}(a)) \) is a presentation at \( a \), then \( \mathcal{H}(a) \) is a (semicoherent) presentation of \( \iota \) at \( a \) if and only if \( \mathcal{H}(a) \) is (semicoherent) equivalent to \( \mathcal{G}(a) \).

(2) Consider any sequence of \( \iota \)-admissible transformations (1.1) (or (1.2)). Let \( a \in M_j \) and let \( a_i \) denote the image of \( a \) in \( M_i \), for all \( i \leq j \). Suppose that \( \iota(a) = \iota(a_0) \). If \( \mathcal{G}(a_0) = (N(a_0), \mathcal{G}(a_0), \emptyset) \) is a (semicoherent) presentation of \( \iota \) at \( a_0 \), then we can consider the successive transforms \( \mathcal{G}(a_i) = (N(a_i), \mathcal{G}(a_i), \mathcal{E}(a_i)) \) of \( \mathcal{G}(a_0) \) at \( a_i \), \( i = 1, \ldots, j \). It follows from Definitions 3.1 that \( \mathcal{G}(a) \) is a (semicoherent) presentation of \( \iota \) at \( a \).

(3) Suppose that \( \iota \) satisfies the Hypotheses 1.10(1) and (3) for any \( X \) (or for any \( J \)). Consider any sequence of \( \iota \)-admissible transformations (1.1) (or (1.2)). It follows that, for all \( j \) and all \( a \in M_j \), there is a semicoherent presentation \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \mathcal{E}_1(a)) \) of \( \iota \) at \( a \), where \( \mathcal{E}_1(a) := E(a) \setminus E^1(a) \). (Recall Definitions 1.15. We obtain \( \mathcal{G}(a) \) simply by transforming a presentation \( \mathcal{G}(a_i) = (N(a_i), \mathcal{G}(a_i), \emptyset) \) of \( \iota \) at \( a_i \), where \( i \leq j \) is the year of birth of \( \iota(a) \).

By (1) above, if \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \mathcal{E}_1(a)) \) is a presentation (respectively, semicoherent presentation) of \( \iota \) at \( a \), then the equivalence class \( [\mathcal{G}(a)] \) (respectively, \( [\mathcal{G}(a)] \)) depends only on \( X_{j,a} \) (or \( J_{j,a} \)) and \( \mathcal{E}_1(a) \).

The following is a corollary of Theorem 2.6.

**Corollary 3.4.** Suppose that \( \iota \) satisfies the Hypotheses 1.10(1) and (3). Let \( \mathcal{G}(a_0) = (N(a_0), \mathcal{G}(a_0), \emptyset) \) be a (semicoherent) presentation of \( \iota \) at \( a_0 \in M \), of codimension \( p(a_0) \), say. Suppose that \( \mu_{\mathcal{G}(a_0)} = 1 \). Consider any sequence of \( \iota \)-admissible transformations (1.1) (or (1.2)). Then, for all \( j \) and all \( a \in M_j \) lying over \( a_0 \), if \( \iota(a) = \iota(a_0) \), \( \iota \) admits a (semicoherent) presentation \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \mathcal{E}_1(a)) \) of codimension \( p(a_0) + 1 \) at \( a \).
3.2. The local inductive construction and the desingularization invariant. Consider \( \iota = \iota_X \) (or \( \iota_J \)) satisfying the Hypotheses 1.10. Our aim is to extend \( \iota \) to an invariant \( \inv = \inv_X \) (or \( \inv_J \)) defined recursively over a sequence of admissible transformations (1.1) (or (1.2)). (See the introductory paragraph of Section 2 above.)

For simplicity in this section, we will assume that \( \iota \) admits a semicoherent presentation \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \emptyset) \) of codimension 0 at every point \( a \), where \( \mu_{\mathcal{G}(a)} = 1 \). (For example, if \( \iota(a) = \iota_J(a) \) denotes the order \( \mu_{a}(\mathcal{J}) \) of an ideal of finite type \( \mathcal{J} \subset \mathcal{O}_M \) at \( a \in M \), then we can take \( N(a) = \text{the germ of } M \text{ at } a \), and \( \mathcal{G}(a) = \{(g, \mu_g(\mathcal{J}))\} \), where the \( g \) form any finite set of generators of \( \mathcal{J}_a \) – cf. Examples 1.13, 1.19.) But this simplifying assumption is not necessary; see \( \S 6.1 \) below (in particular, for the Hilbert-Samuel function).

First consider a sequence of \( \iota \)-admissible transformations (1.1) (or (1.2)). Recall that, if \( a \in M_j \), then \( \iota(a) \) denotes \( \iota_{X_j}(a) \) (or \( \iota_{\mathcal{J}_j}(a) \)), and \( E(a) \) denotes \( \{ H \in E_j : a \in H \} \). We write \( \inv_{1/2} := \iota \).

Let \( a \in M_j \). Define \( E^1(a) \) as in Definitions 1.15. Set \( \mathcal{E}_1(a) := \mathcal{E}(a) \setminus E^1(a) \). By Corollary 3.4, \( \inv_{1/2} = \iota \) admits a semicoherent presentation

\[
\mathcal{C}_1(a) = (N_1(a), \mathcal{C}_1(a), \mathcal{E}_1(a))
\]

of codimension 1 at \( a \). By Remarks 3.3, the equivalence classes \( [\mathcal{C}_1(a)] \) and \( [\mathcal{C}_1(a)] \) depend only on \( X_{j,a} \) (or \( \mathcal{J}_{j,a} \)) and \( \mathcal{E}_1(a) \). Define \( \inv_1(a) := (\iota(a), s_1(a)) \), where \( s_1(a) := \#E^1(a) \), and set

\[
\mathcal{H}_1(a) = (N_1(a), \mathcal{H}_1(a), \mathcal{E}_1(a)),
\]

where \( \mathcal{H}_1(a) := \mathcal{C}_1(a) \cup \left( E^1(a) \big|_{N_1(a)} \right) \) and

\[
\left( E^1(a) \big|_{N_1(a)} \right) := \left\{ (x_H \big|_{N_1(a)}, 1) : H \in E^1(a) \right\}
\]

(and \( x_H \) denotes a generator of \( \mathcal{I}_{H,a} \)).

Clearly, \( \mathcal{H}_1(a) \) is a semicoherent presentation of \( \inv_1 \) at \( a \), and the equivalence class \( [\mathcal{H}_1(a)] \) (respectively, \( [\mathcal{H}_1(a)] \)) depends only on \( [\mathcal{C}_1(a)] \) and \( E^1(a) \) (respectively, on \( [\mathcal{C}_1(a)] \) and \( E^1(a) \)).

Remark 3.5. By “semicoherent presentation of \( \inv_1 \) at \( a \)”, we mean the analogue of Definitions 3.1 for \( \inv_1 \), but where the condition (3) (of Definitions 1.12) is replaced by the weaker condition of stability after finite sequences of transformations of type (i) (admissible blowings-up) only.

We use this weaker version of semicoherent presentation of an invariant here (and below) because, together with the fact that \( [\mathcal{H}_1(a)] \) depends only on \( [\mathcal{C}_1(a)] \) and \( E^1(a) \), it suffices to continue the induction, and avoids the necessity of defining \( \inv \) over sequences of all three
types of transformations in §2.2. The stronger version Definition 3.1 for inv$_{1/2} = \iota$ is needed only to start the induction.

Define $\nu_2(a) := \nu_2^j(a)$ and inv$_{3/2}(a) := (\text{inv}_1(a); \nu_2(a))$. Then $\nu_2(a)$ depends only on $[\mathcal{H}^j(a)]$ (hence only on $[\mathcal{H}_1(a)]$); thus inv$_{3/2}(a)$ depends only on $X_{j,a}, E(a)$ and $E^1(a)$.

If $\nu_2(a) = \infty$, we write $\text{inv}(a) := (\nu_1(a), s_1(a); \infty)$; then $S_{\text{inv}}(a) = N_1(a)$. If $\nu_2(a) = \infty$ only if $s_1(a) = 0$. If $\nu_2(a) = 0$, we write $\text{inv}(a) := (\nu_1(a), s_1(a); 0)$ and define $\mathcal{G}_2(a) := \mathcal{G}_2^j(a) = (N_1(a), \mathcal{G}_2(a), \mathcal{E}_1(a))$, where $\mathcal{G}_2(a) = \{(D_2(a), 1)\}$, and $D_2(a) := D_{\mathcal{H}^j(a)}$ (see §2.6). Then $\mathcal{G}_2(a)$ is a codimension 1 presentation of inv at $a$, and

$$S_{\text{inv}}(a) = \{x \in N_1(a) : \mu_a(D_2(a)) \geq 1\}.$$ 

On the other hand, suppose that $0 < \nu_2(a) < \infty$. Then

$$\mathcal{G}_2^j(a) = (N_1(a), \mathcal{G}_2(a), \mathcal{E}_1(a)),$$ 

where $\mathcal{G}_2(a) := \mathcal{G}_2^j(a)$ is given by (2.3). Then $\mathcal{G}_2^j(a)$ is a codimension 1 semicoherent presentation of inv$_{3/2}$ at $a$, the semicoherent equivalence class $[\mathcal{G}_2^j(a)]$ depends only on $[\mathcal{H}_1(a)]$ (Remark 2.7), and $\mu_{\mathcal{G}_2^j(a)} = 1$. Define $\mathcal{E}_2(a) := \mathcal{E}_1(a) \setminus E^2(a)$ (see Definitions 1.15) and put

$$\mathcal{G}_2(a) := (N_1(a), \mathcal{G}_2(a), \mathcal{E}_2(a)).$$

Then $\mathcal{G}_2(a)$ is a semicoherent presentation of inv$_{3/2}$ at $a$, and $[\mathcal{G}_2(a)]$ (respectively, $[\mathcal{G}_2^j(a)]$) depends only on $[\mathcal{G}_2^j(a)]$ and $E^2(a)$ (respectively, on $[\mathcal{G}_2^j(a)]$ and $E^2(a)$). It follows from Theorem 2.6 (cf. Corollary 3.4) that $\mathcal{G}_2(a)$ is semicoherent equivalent to a presentation

$$\mathcal{C}_2(a) = (N_2(a), \mathcal{C}_2(a), \mathcal{E}_2(a))$$

de of codimension 2 (where $N_2(a)$ is a submanifold of $N_1(a)$). This completes one cycle in the inductive definition of inv.

In general, let $r \geq 1$ and suppose that we have introduced inv$_{r-1/2}(a) = (\text{inv}_{r-1}(a); \nu_r(a))$. Consider a sequence of $(r - 1/2)$-admissible transformations (1.1) (or (1.2)). Let $a \in M_j$. Assume that $0 < \nu_r(a) < \infty$. By induction, we can assume that inv$_{r-1/2}$ admits a semicoherent presentation

$$\mathcal{C}_r(a) = (N_r(a), \mathcal{C}_r(a), \mathcal{E}_r(a))$$

of codimension $r$ at $a$, where $\mathcal{E}_r(a) := \mathcal{E}_{r-1}(a) \setminus E^r(a)$, and that the semicoherent equivalence class of $\mathcal{C}_r(a)$ depends only on $X_{j,a}$ (or $\mathcal{J}_{j,a}$), $E(a), E^1(a), \ldots, E^r(a)$. (We can assume, inductively, that the semicoherence class of $\mathcal{C}_r(a)$ depends only on $\mathcal{E}_r(a)$ and the semicoherence class of $\mathcal{H}_{r-1}(a)$.)
Define $\text{inv}_r(a) := (\text{inv}_{r-1/2}(a), s_r(a))$, where $s_r(a) := \# E^r(a)$, and set

$$\mathcal{H}_r(a) := (N_r(a), \mathcal{H}_r(a), \mathcal{E}_r(a)),$$

where $\mathcal{H}_r(a) := \mathcal{C}_r(a) \cup \left( E^r(a)|_{N_r(a)} \right)$. Then $\mathcal{H}_r(a)$ is a codimension $r$ semicoherent presentation of $\text{inv}_r$ at $a$, and its equivalence class (or semicoherent equivalence class) depends only on $E^r(a)$ and that of $\mathcal{C}_r(a)$.

Define $\mu_{r+1}(a) := \mu_{\mathcal{H}_r(a)}$, $\mu_{r+1,H}(a) := \mu_{\mathcal{H}_r(a),H}$, for all $H \in \mathcal{E}_r(a)$, and

$$\nu_{r+1}(a) := \nu_{\mathcal{H}_r(a)} = \mu_{r+1}(a) - \sum_{H \in \mathcal{E}_r(a)} \mu_{r+1,H}(a).$$

If $0 \leq \nu_{r+1}(a) < \infty$, define

$$D_{r+1}(a) := D_{\mathcal{H}_r(a)} = \prod_{H \in \mathcal{E}_r(a)} x_H^{\mu_{r+1,H}(a)},$$

and introduce $\mathcal{G}_{\mathcal{H}_r(a)}$ as in §2.6. Then $\mathcal{G}_{\mathcal{H}_r(a)} = (N_r(a), \mathcal{G}_{r+1}(a), \mathcal{E}_{r+1}(a))$ is a semicoherent codimension $r$ presentation of $\text{inv}_{r+1/2} := (\text{inv}_r; \nu_{r+1})$ at $a$, and $[\mathcal{G}_{\mathcal{H}_r(a)}]$ depends only on $[\mathcal{H}_r(a)]$. If $0 < \nu_{r+1}(a) < \infty$, then

$$\mathcal{G}_{r+1}(a) := (N_r(a), \mathcal{G}_{r+1}(a), \mathcal{E}_{r+1}(a)),$$

where $\mathcal{E}_{r+1}(a) := \mathcal{E}_r(a) \setminus E^{r+1}(a)$, is semicoherent equivalent to a presentation

$$\mathcal{C}_{r+1}(a) := (N_{r+1}(a), \mathcal{C}_{r+1}(a), \mathcal{E}_{r+1}(a))$$

of codimension $r + 1$. Clearly $[\mathcal{C}_{r+1}(a)]$ depends only on $[\mathcal{H}_r(a)]$ and $\mathcal{E}_{r+1}(a)$. Etc.

Eventually, we find $t \leq n := \dim_a M_j$ such that $\nu_{t+1}(a) = 0$ or $\infty$, and we set $\text{inv}(a) := \text{inv}_{t+1/2}(a)$. Suppose that $\nu_{t+1}(a) = \infty$. Then $S_{\text{inv}}(a) = N_t(a)$. On the other hand, if $\nu_{t+1}(a) = 0$, then

$$S_{\text{inv}}(a) = \{ x \in N_t(a) : \mu_x(D_{t+1}(a)) \geq 1 \},$$

where $D_{t+1}(a) = \prod_{\mathcal{E}_t(a)} x_H^{\mu_{t+1,H}(a)}$.

Remark 3.6. In the local inductive construction above, we pass from $\mathcal{G}_r(a) = (N_{r-1}(a), \mathcal{G}_r(a), \mathcal{E}_r(a))$ to an equivalent presentation $\mathcal{C}_r(a) = (N_r(a), \mathcal{C}_r(a), \mathcal{E}_r(a))$ in codimension $+1$, and then to $\mathcal{H}_r(a)$ by adjoining $\left( E^r(a)|_{N_r(a)} \right)$. The construction of $[\mathcal{B} \mathcal{M} 5]$ follows a slightly different route - from $\mathcal{G}_r(a)$ to $\mathcal{F}_r(a) = (N_{r-1}(a), \mathcal{F}_r(a), \mathcal{E}_r(a))$, where $\mathcal{F}_r(a) = \mathcal{G}_r(a) \cup \left( E^r(a)|_{N_{r-1}(a)} \right)$, and then to an equivalent presentation $\mathcal{H}_r(a)$ in codimension $+1$. The latter gives a little more flexibility in the choice
of the maximal contact submanifold \( N_r(a) \), but the construction above makes for a clearer parallel treatment of alternative initial invariants \( \iota \) — see, for example, §6.1.

3.3. **Proof of the desingularization principle** Theorem 1.14. 

(1) Semicontinuity. The semicontinuity properties (a) and (b) can be proved for the truncated invariants \( \text{inv}_{r-1/2} \) and \( \text{inv}_r \) by induction on \( r \). By the hypothesis 1.10(1), \( \text{inv}_{1/2} \) satisfies (a) and (b). Assume that \( \text{inv}_{r-1/2} \) satisfies (a) and (b), where \( r \geq 1 \). The properties (a) and (b) for \( \text{inv}_r \) are then consequences of the following semicontinuity assertion for \( E^r(\cdot) \): If \( a \in M_j \), then \( E^r(x) = E(x) \cap E^r(a) \), where \( x \in S_{\text{inv}_{r-1/2}}(a) \) \[BM3\], Proposition 6.6]. To prove (a) and (b) for \( \text{inv}_{r+1/2} \), consider a semicoherent presentation \( \mathcal{H}_r(a) := (N_r(a), \mathcal{H}_r(a), \mathcal{E}_r(a)) \) of \( \text{inv}_r \) at \( a \in M_j \); say \( \mathcal{H}_r(a) = \{(h, \mu_h)\} \). If \( x \in S_{\text{inv}_r}(a) \), then

\[
\mu_h \nu_{r+1}(x) = \min_{\mathcal{H}_r(a)} \mu_x \left( \frac{h}{D_{r+1}(a)^{\mu_h}} \right).
\]

(See §§2.4, 2.6.) The semicontinuity properties for \( \text{inv}_{r+1/2} \) are consequences of the analogous properties for the order of an element \( g = h/D_{r+1}(a)^{\mu_h} \) such that \( \mu_a(g) = \mu_h \nu_{r+1}(a) \). (This is where Remark 3.2 is relevant, in general.)

(2) Stabilization. Suppose that \( C_1(a) \) is a presentation of \( \text{inv}_{1/2} = \iota \) at \( a \in M_j \), as above, and that \( C_1(a) = \{(c, \mu_c)\} \). Choose \( q \in \mathbb{N} \) such that \( q \cdot \mu_c \in \mathbb{N} \), for all \( (c, \mu_c) \). Let \( e_1 = e_1(a) := \max q \cdot \mu_c \). Then, for all \( r > 0 \), \( e_r! \nu_{r+1}(a) \in \mathbb{N} \), where \( e_{r+1} = \max \{ e_r!, e_r! \nu_{r+1}(a) \} \). The assertion follows from Hypotheses 1.10 for \( \iota \) (using infinitesimal semicontinuity ((1) above) and the stabilization property 1.12(3) of a presentation of \( \iota \) at \( a \)).

(3) Normal crossings. Say \( \text{inv}(a) = \text{inv}_{t+1/2}(a) \). The assertion is an immediate consequence of the fact that \( S_{\text{inv}}(a) = N_t(a) \) in the case that \( \nu_{t+1}(a) = \infty \), and of (3.2) in the case that \( \nu_{t+1}(a) = 0 \). (See also (4) below).

(4) Decrease. Say \( \text{inv}(a) = \text{inv}_{t+1/2}(a) \). First suppose that \( \nu_{t+1}(a) = \infty \). Then \( S_{\text{inv}}(a) = N_t(a) \). If \( \sigma \) is the local blowing-up with centre \( N_t(a) \), then the strict transform \( N_t(a)' = \emptyset \), so that \( \text{inv}(a') < \text{inv}(a) \), for all \( a' \in \sigma^{-1}(a) \). (See Definitions 1.12 and 3.1.)

On the other hand, suppose that \( \nu_{t+1}(a) = 0 \). Then \( h = D_{t+1}(a)^{\mu_h} \), for some \( (h, \mu_h) \in \mathcal{H}_t(a) \), and \( S_{\text{inv}}(a) = S_{\text{inv}_t}(a) = \{ x \in N_t(a) : \mu_x(D_{t+1}(a)) \geq 1 \} \).

(We can choose coordinates \( x = (x_1, \ldots, x_{n-t}) \) for \( N_t(a) \) such that \( D_{t+1}(a) \) is a monomial \( x_1^{\Omega_1} \cdots x_{n-t}^{\Omega_{n-t}} \) with rational exponents and, if \( \Omega_t \neq 0 \), then \( x_1 = x_H \), for some \( H \in \mathcal{E}_t(a) \), and \( \Omega_t = \mu_{t+1,H}(a) \). Thus \( \mu_x(D_{t+1}(a)) \) makes sense as a rational number.) Therefore, \( S_{\text{inv}_t}(a) \) is
a union of smooth components $\bigcup_l Z_l$, where $Z_l = \{ x \in N(a) : x_l = 0, l \in I \}$ and the union is over the minimal subsets $I$ of $\{1, \ldots, n-t\}$ such that $\sum_{l \in I} \Omega_l \geq 1$; equivalently, over the subsets $I$ such that

$$0 \leq \sum_{k \in I} \Omega_k - 1 \leq \Omega_l, \quad \text{for all } l \in I.$$

Set $\mu(a) := \mu_{t+1}(a)$. Consider a local blowing-up $\sigma$ with centre $Z_l$, for some $I$ as above. Suppose that $a' \in \sigma^{-1}(a)$ and $\inv_l(a') = \inv_l(a)$. Then $a' \in N(a')$. $N(a')$ is a union of coordinate charts $\bigcup_{l \in I} U'_l$ such that $\sigma|_U'$ is given by the substitution $x_l = y_l$, $x_k = yKy_k$ if $k \in I \setminus \{l\}$, and $x_k = y_k$ if $k \notin I$. Consider $h = D_{t+1}(a)^\mu$. Suppose that $a' \in U'_l$. Write $d = \mu_h$. Then $(h', d) \in \mathcal{H}(a')$, where $h' := y_i^{-d} (D_{t+1}(a)^d \circ \sigma) = \left( y_1^{\nu_1} \cdots y_{n-t}^{\nu_{n-t}} \right)^d$, and

$$\Omega_k' = \Omega_k, \quad k \neq l; \quad \Omega'_l = \sum_{k \in I} \Omega_k - 1.$$

Therefore,

$$1 \leq \mu_{t+1}(a') \leq \sum_{k=1}^{n-t} \Omega'_k < \sum_{k=1}^{n-t} \Omega_k = \mu_{t+1}(a),$$

as required.

3.4. Villamayor’s algorithm. Villamayor’s algorithm can be described in the framework of §3.2 above, by making a simple modification in the construction and the corresponding invariant. This modification corresponds to using only a certain subset $\mathcal{E}_r'(a) \subset \mathcal{E}_r(a)$ when we define $\nu_r(a)$ according to the formula (3.1); equivalently, to factoring from $\mathcal{H}_r(a)$ only a part of the exceptional monomial $D_{r+1}(a)$ in order to define $\mathcal{G}_r(a)$. Such a modification for given $r$ will, in general, change all subsequent terms of the invariant and the corresponding presentations.

Consider $r \geq 1$. Suppose (inductively) that we have introduced $\inv_{r-1/2}(a) = (\inv_{r-1}(a); \nu_r(a))$. As before, we define $E'(a) := \{ H \in \mathcal{E}_{r-1}(a) : H$ is transformed from $E(a_i) \}$, where $i$ denotes the year of birth of $\inv_{r-1/2}(a)$ (see Definitions 1.15), and we set $s_r(a) := \#E'(a), \mathcal{E}_r(a) := \mathcal{E}_{r-1}(a) \setminus E'(a)$. But now let $\mathcal{E}_r'(a) \subset \mathcal{E}_r(a)$ denote the subset consisting of only those exceptional hypersurfaces passing through $a$ that have accumulated since the year of birth of $\inv_r(a) = (\inv_{r-1}(a); \nu_r(a), s_r(a))$. In other words, if $i'$ denotes the year of birth of $\inv_r(a)$, then $\mathcal{E}_r'(a) := \mathcal{E}_{r-1}(a) \setminus E'(a)$, where $E'(a) := \{ H \in \mathcal{E}_{r-1}(a) : H$ is transformed from $E(a'_i) \}$. 
We define \( \nu_2(a) := \mu_2(a) - \sum_{H \in E_2'(a)} \mu_{2,H}(a) \),

where \( \mu_2(a) := \mu_{H_1(a)}(a) \) and \( \mu_{2,H}(a) := \mu_{H_1(a),H} \) for all \( H \in E_2'(a) \). (See §2.4.) Then \( \text{inv}_{3/2} = (\text{inv}_1; \nu_2) \) has a codimension 1 presentation

\[ \mathcal{G}_2(a) = (N_1(a), \mathcal{G}_2(a), E_2'(a)) \]

where \( \mathcal{G}_2(a) = \mathcal{G}_{H_1(a)} \). We pass to an equivalent presentation

\[ \mathcal{C}_2(a) = (N_2(a), \mathcal{C}_2(a), E_2'(a)) \]

in codimension 2, and adjoin \( (E_2(a)|_{N_2(a)}, 1) \) to get a presentation of \( \text{inv}_2 = (\text{inv}_{3/2}, s_2) \).

(Although we are using the same notation for presentations as before), \( \nu_2 \) and all subsequent terms of the invariant will, in general, be different, as will the corresponding presentations: The change in the local construction is repeated for each successive \( r \); when we pass from from \( H_r(a) \) to \( \mathcal{G}_{H_r(a)} = (N_r(a), \mathcal{G}_{r+1}(a), E_r'(a)) \) in order to obtain a presentation of \( \text{inv}_{r+3/2} \) at \( a \), we factor from \( H_r(a) \) not \( D_{r+1}(a) = \prod_{E_r'(a)} \mu_{H_r(a)}^{a_H} \) as before, but only the product

\[ D_{r+1}'(a) := \prod_{H \in E_r'(a)} \mu_{H,r+1,H}^{a_H} \]

(i.e., the product over those \( H \) accumulated since the birth of \( \text{inv}_r(a) \)). This change will be magnified in all subsequent terms of \( \text{inv} \) because \( s_{r+1} \) depends on \( \text{inv}_{r+1/2} \), etc.

### 3.5. Meaning of invariance.

Consider \( \text{inv} = \text{inv}_X \) (or \( \text{inv}_J \)) defined over a sequence of admissible blowings-up (1.1) (or (1.2)) according to the construction of §3.2 above. We have shown in §3.2 that, if \( a \in M_j \), then \( \text{inv}(a) \) depends only on the local isomorphism class of \( (X_{j,a}(a) \text{ (or } J_{j,a}, E(a), E_1(a), E_2(a), \ldots) \). The key point is that, for each \( r \), \( \nu_{r+1}(a) \) depends only on the equivalence class \([H_r(a)]\) (by Theorems 2.3 and 2.4).

According to these theorems, the version of \( \nu_{r+1}(a) \) defined in §3.4 is also an invariant of the corresponding equivalence class \([H_r(a)]\). But it is not an invariant with respect to the weaker notion of equivalence of idealistic exponents \([H_2] \) or equivalence of basic objects \([V_2, EV_1] \) – see Example 5.14 below.
This is the reason for our introducing the idea of equivalence of presentations involving exceptional blowings-up. The result is not only a proof of invariance in a stronger sense, but also an invariant whose maximum locus, in general, provides a larger centre of blowing up because Theorem 2.4 shows that the $\mu_{r+1,H}(a)$ are invariants for the larger block $E_r(a)$ of exceptional divisors $H$.

The difference in the algorithms is thus not accidental: for the smaller block $E'_r(a)$ of exceptional divisors $H$ used by Villamayor, there is another approach to invariance of $\mu_{r+1,H}(a)$ in terms of the previous history:

**Lemma 3.7.** Suppose that $a \in M_j$ and that $\text{inv}_r(a) = \text{inv}_r(a_{j-1})$. Let $H \in E'_r(a)$. (Note that $E'_r(a) = \emptyset$ unless $\text{inv}_r(a) = \text{inv}_r(a_{j-1})$.) If $H = \sigma^{-1}_j(C_j)$, then

$$\mu_{r+1,H}(a) = \sum_{K \in J} \mu_{r+1,K}(a_{j-1}) + \nu_{r+1}(a_{j-1}) - 1,$$

where $J := \{K \in E'_r(a_{j-1}) : C_j \subset K\}$. Otherwise, $H$ is the strict transform of an element $K \in E'_r(a_{j-1})$, and

$$\mu_{r+1,H}(a) = \mu_{r+1,K}(a_{j-1}).$$

The proof is a simple calculation. (Cf. [V2, §5.4].) The transformation formulas of Lemma 3.7 are in general not valid for the exceptional divisors $H \in E_r(a) \setminus E'_r(a)$.

Desingularization as realized by either the algorithm of the authors or that of Villamayor is canonical; in fact, local isomorphisms between open subsets of $X$ lift throughout the sequence of blowings-up determined by the algorithm. But the notion of invariance provided by Lemma 3.7 is weaker than that determined by equivalence of presentations because it depends in a stronger way on the history. Of course, the fact that $\nu_{r+1}(a)$ as defined in §3.4 is also an invariant of the corresponding equivalence class $[\mathcal{H}_r(a)]$ shows that Villamayor’s version of $\text{inv}_X$ is an invariant in a stronger sense than shown by Lemma 3.7; at $a \in X_j$, it depends on $X_{j,a}$, $E(a)$, the $E^r(a)$ and the $E'^r(a)$.

### 3.6. The variant of Encinas and Villamayor

Encinas and Villamayor [EV] have modified Villamayor’s algorithm to avoid blowings-up that are superfluous in a certain situation based on Abhyankar’s idea of “good points” [A]. [EV] suggests a substantial modification of Villamayor’s construction; in particular, each pair $(\nu_r, s_r)$ in the invariant as described above is replaced by a triple. But the inductive construction described in §3.2 above is flexible, and [EV] can be understood in the following way.
We make a modification in the definition of $\text{inv}_r$ by induction on $r$:
Suppose that we have defined $\text{inv}_{r-1}$ and an associated codimension $r-1$ presentation $\mathcal{H}_{r-1}(a) = (N_{r-1}(a), \mathcal{H}_{r-1}(a), \mathcal{E}_{r-1}(a))$ at $a \in M_j$, as above. (We are using the notation above. The modification of [EV1] can be applied equally to the construction of the authors (§3.2 above) or that of Villamayor (§3.4), so that $\text{inv}_{r-1}$ and $\mathcal{H}_{r-1}(a)$ here mean the notions defined for either algorithm. For Villamayor’s version, therefore, $\mathcal{E}_{r-1}(a)$ here is to be understood as $\mathcal{E}'_{r-1}(a)$ in the notation of §3.4.)

Define $\nu_r(a), \mu_r(a), \mu_{rH}(a)$ for all $H \in \mathcal{E}_{r-1}(a)$, $\nu_r(a), D_r(a) = \prod x_H^{\nu_{rH}(a)}$ and $\mathcal{G}_r(a)$ as before. (In the case of Villamayor’s construction again, $D_r(a)$ here means $D'_r(a)$ in the notation of §3.4.) Of course, $\mu_r(a) = \sum_{H \in \mathcal{E}_r(a)} \mu_{rH}(a) + \nu_r(a) \geq 1$.

But now, we make a modification of the definition of $\nu_r$ in the case that

$$\sum (\mu_{rH}(a) - [\mu_{rH}(a)]) + \nu_r(a) < 1$$

(where $[\cdot]$ denotes the integral part): In this case, we redefine $\nu_r(a)$ as $\nu^*_r(a) := 0$, and $\mathcal{G}_r(a) := \{(D_r(a), 1)\}$. This is the only change.

Like $\nu_r(a)$, the modified version $\nu^*_r(a)$ depends only on the equivalence class $[\mathcal{H}_{r-1}(a)]$ since it is defined in terms of $\mu_r(a)$ and the $\mu_{rH}(a)$.

It follows that the corresponding modified invariant $\text{inv}^*$ can be defined inductively over a sequence of transformations (1.1) (or (1.2)), provided we assume that each successive centre of blowing up is a component of the maximum locus of $\text{inv}^*$. Theorem 1.14 holds for such sequences; our proof as sketched above applies with no change. (When $\nu^*_r(a) = 0$, the $\text{inv}^*$-stratum of $a$ coincides with the $\text{inv}_{r-1}$-stratum and has only normal crossings as in Theorem 1.14(3), where each component has codimension 1 in $N_{r-1}(a)$ – this special situation is analogous to the idea of a “good point”. The blowing-up of $N_{r-1}(a)$ with centre such a component is the identity.) This is the variant of Encinas and Villamayor; the situation in which it applies does not occur in Example 1.2 – see Section 4 following.

4. Worked example

We illustrate the desingularization principle in this section by working out Example 1.2 above (except for the estimates on $n(AB)$ and $n(LB)$ given by the affine and locally binomial algorithms. These will be computed in Desingularization algorithms II.) Let $X$ denote the
hypersurface in 4-dimensional affine space $M$ given by $g_0 = 0$, where
\begin{equation}
    g_0(x, y, z, w) = z^d w^{d-1} - x^{d-1} y^d,
\end{equation}
for any natural number $d \geq 2$. The hypersurface $X$ takes its maximum order $2d - 1$ precisely at the origin, so in any case we take $C_0 = \{0\}$ as the centre of the first blowing-up $\sigma_1$: $M_1 \to M_0 = M$. Then $M_1$ can be covered by four affine coordinate charts, corresponding to the four variables. For example, $\sigma_1$ is given in the “$w$-chart” $U_w$ by substituting $(xw, yw, zw, w)$ for the original variables $(x, y, z, w)$. The strict transform $X_1 = X'_0$ of $X_0 = X$ is given in $U_w$ by $g_1 = 0$, where
\begin{equation}
    g_1(x, y, z, w) = z^d - x^{d-1} y^d.
\end{equation}
(For economy of notation, we are using the same letters for the variables before and after blowing up, and we are “bookkeeping” by writing $U_w$ for the “$w$-chart” of $M_1$; $U_w$ is the complement in $M_1$ of the strict transform by $\sigma_1$ of the coordinate subspace $\{w = 0\}$ of $M_0 = M$.)

We will estimate the number of blowings-up needed to reduce the maximum order $d$ of $X_1$. Let $n(BM)$ and $n(V)$ denote the number of blowings-up determined by the algorithms of [BM3] and [V2], respectively. We will show that
\begin{align*}
n(BM) &\leq 2d + j, \\
n(V) &\geq 9d + k,
\end{align*}
where $j, k$ are independent of $d$. It is easy to check in the calculations below that the variant of Encinas-Villamayor [EV1] (see §3.6) makes no difference to either algorithm when applied to (4.1).

4.1. **Year one.** $X_1 \cap U_w$ is given by $g_1 = 0$ as above. (Note that the order at 0 of a binomial majorizes its orders at points of the chart. In particular, $X_1$ has order $< d$ throughout the charts $U_z, U_y$.) Let $a_1 = 0$. Then $E(a_1) = \{H_1\}$, where $H_1$ denotes the exceptional hypersurface $\{w = 0\}$. We have $v_1(a_1) = d$, $E^{1}(a_1) = E(a_1) = \{H_1\}$, $s_1(a_1) = 1$, and $E_1(a_1) := E(a_1) \setminus E^{1}(a_1) = \emptyset$. Then $\text{inv}_{1/2} = v_1$ has a (semicoherent) codimension 0 presentation at $a_1$ given by $G_1(a_1) = \{(g_1, d)\}$; therefore, $\text{inv}_{1/2}$ and $\text{inv}_1 = (v_1, s_1)$ have (semicoherent) codimension 1 presentations
\begin{align*}
    \mathcal{C}_1(a_1) &= (N_1(a_1), \mathcal{C}_1(a_1), E_1(a_1)), \\
    \mathcal{H}_1(a_1) &= (N_1(a_1), \mathcal{H}_1(a_1), E_1(a_1)),
\end{align*}
(respectively), where $N_1(a_1) = \{z = 0\}$, $\mathcal{C}_1(a_1) = \{(x^{d-1}y^d, d)\}$ and
\begin{equation}
    \mathcal{H}_1(a_1) = \{(x^{d-1}y^d, d), (w, 1)\}.
\end{equation}
Therefore, 
\[ \nu_2(a_1) = \mu_2(a_1) := \min_{(h, \mu_h) \in H_1(a_1)} \frac{\mu_2(h)}{\mu_h} = 1 \]
and \( s_2(a_1) = 0 \). Then \( \text{inv}_{3/2} \) has presentations 
\[ G_2(a_1) = (N_1(a_1), G_2(a_1) = H_1(a_1), \emptyset), \]
\[ \mathcal{C}_2(a_1) = (N_2(a_1), C_2(a_1), \emptyset), \]
of codimensions 1 and 2 (respectively), where \( N_2(a_1) = \{ z = w = 0 \} \) and \( C_2(a_1) = \{(x^{d-1}y^d, d)\} \), and \( \text{inv}_2 \) has a codimension 2 presentation 
\[ H_2(a_1) = (N_2(a_1), H_2(a_1) = C_2(a_1), \emptyset). \]
(There is no change from \( H_1(a_1) \) to \( G_2(a_1) \) because \( \nu_2(a_1) = 1 \), and no change from \( C_2(a_1) \) to \( H_2(a_1) \) because \( s_2(a_1) = 0 \).) Therefore, 
\[ \nu_3(a_1) = \mu_3(a_1) = \frac{2d - 1}{d} \]
and \( s_3(a_1) = 0 \). Then \( \text{inv}_{5/2} \) admits a codimension 2 presentation at \( a_1 \), 
\[ G_3(a_1) = (N_2(a_1), G_3(a_1), \emptyset), \]
where \( G_3(a_1) = \{(x^{d-1}y^d, 2d - 1)\} \), and we can replace the latter by 
\[ G_3(a_1) = \{(x, 1), (y, 1)\} \]
to get an equivalent presentation (cf. Corollary 5.10 below). Therefore, \( \text{inv}_{5/2} \) has a codimension 3 presentation at \( a_1 \), 
\[ G_3(a_1) = (N_3(a_1), C_3(a_1), \emptyset), \]
where \( N_3(a_1) = \{ z = w = y = 0 \} \) and \( C_3(a_1) = \{(x, 1)\} \), and \( \text{inv}_3 \) has the same codimension 3 presentation \( H_3(a_1) = C_3(a_1) \). Hence \( \nu_4(a_1) = 1 \), \( s_4(a_1) = 0 \), and \( \text{inv}_4 \) has a codimension 4 presentation 
\[ H_4(a_1) = (N_4(a_1), \emptyset, \emptyset) \] at \( a_1 \), where \( N_4(a_1) = \{ z = w = y = x = 0 \} = \{0\} \). Thus 
\[ \text{inv}_X(a_1) = \left( d, 1; 1, 0; \frac{2d-1}{d}, 0; 1, 0; \infty \right) \];
the preceding presentation of \( \text{inv}_4 \) is also a presentation of \( \text{inv}_X \), and, in the chart \( U_w \), the centre of the next blowing-up \( \sigma_2 \) is \( C_1 = N_4(a_1) = \{0\} \). By symmetry, as a global centre of the blowing-up \( \sigma_2 \): \( M_2 \to M_1 \) we would take \( C_2 := \text{union of the origins in the charts} \ U_w, U_x \).
4.2. Year two. Let $X_2$ denote the strict transform $X_1'$ of $X_1$ by $\sigma_2$. Consider the chart $U_{wy}$ of $M_2$, in which $\sigma_2$ is given by the substitution $(x,y,yz,wy)$. Then $X_2 \cap U_{wy} = \{ g_2 = 0 \}$ where
$$g_2(x, y, z, w) = z^d - x^d - 1.$$ 

Let $a_2 = 0$. Then $E(a_2) = \{ H_1, H_2 \}$, where $H_1$ and $H_2$ denote the exceptional hypersurfaces \{w = 0\} and \{y = 0\}, respectively; $H_2 = \sigma_2^{-1}(C_1)$ and $H_1$ is the strict transform of the hypersurface in year one that we also denoted $H_2$ (to economize notation). We have $v_1(a_2) = d$, $E_1(a_2) = \{ H_1 \}$, $s_1(a_2) = 1$, and $E_1(a_2) := E(a_2) \setminus E_1(a_2) = \{ H_2 \}$. At $a_2$, $\text{inv}_1$ admits a codimension 1 presentation
$$\mathcal{H}_1(a_2) = (N_1(a_2), \mathcal{H}_1(a_2), \mathcal{E}_1(a_2)),$$
where $N_1(a_2) = \{z = 0\}$ and
$$\mathcal{H}_1(a_2) = \{(x^d, y^d, d), (w, 1)\}.$$

We compute
$$\mu_2(a_2) := \min_{\mathcal{H}_1(a_2)} \frac{\mu_{a_2}(h)}{\mu_h} = 1,$$
$$\mu_{2H_2}(a_2) := \min_{\mathcal{H}_1(a_2)} \frac{\mu_{H_2,a_2}(h)}{\mu_h} = 0,$$
$$\nu_2(a_2) := \mu_2(a_2) - \sum_{H \in \mathcal{E}_1(a_2)} \mu_{2H}(a_2) = 1 - 0 = 1.$$

Therefore, $E^2(a_2) = \emptyset$ and $s_2(a_2) = 0$; $\text{inv}_2$ has a codimension 2 presentation
$$\mathcal{H}_2(a_2) = (N_2(a_2), \mathcal{H}_2(a_2), \mathcal{E}_2(a_2)),$$
where $N_2(a_2) = \{z = w = 0\}$ and $\mathcal{H}_2(a_2) = \{(x^d, y^d, d)\}$. Therefore,
$$\mu_3(a_2) = \frac{2d - 2}{d}, \quad \mu_{3H_2}(a_2) = \frac{d - 1}{d},$$
$$\nu_3(a_2) = \mu_3(a_2) - \mu_{3H_2}(a_2) = \frac{d - 1}{d},$$
and $s_3(a_2) = 1$. At $a_2$, $\text{inv}_{5/2}$ has a codimension 2 presentation
$$\mathcal{G}_3(a_2) = (N_2(a_2), \mathcal{G}_3(a_2), \mathcal{E}_3(a_2)),$$
where $\mathcal{E}_3(a_2) = \emptyset$ and
$$\mathcal{G}_3(a_2) = \left\{(x^d, d - 1), \left( y^d, 1 - \frac{d - 1}{d} \right) \right\}.$$
or, equivalently,
\[ G_3(a_2) = \left\{ (x, 1), \left( y, \frac{1}{d-1} \right) \right\}; \]

\[ \text{inv}_{5/2} \] has a codimension 3 presentation
\[ C_3(a_2) = (N_3(a_2), C_3(a_2), E_3(a_2)), \]
where \( N_3(a_2) = \{ z = w = x = 0 \} \) and \( C_3(a_2) = (y, 1/(d - 1)) \). Then \( \text{inv}_3 \) has a codimension 3 presentation
\[ H_3(a_2) = (N_3(a_2), H_3(a_2), E_3(a_2)), \]
where \( H_3(a_2) = \{(y, 1)\} \). Clearly,
\[ \text{inv}_X(a_2) = \left( d, 1; 1, 0; \frac{d-1}{d}, 1, 0; \infty \right) \]
and the centre of the next blowing-up \( \sigma_3 \) is \( C_2 = \{0\} \), in this chart.

Of course, over \( U_w \), \( X_2 \) lies in the union of the following charts, for each of which we give a defining equation \( g_2 = 0 \) and equations of the exceptional hypersurfaces \( H_1, H_2 \):

| Chart  | Equation of \( X_2 \) | Exceptional hypersurfaces |
|--------|------------------------|---------------------------|
| \( U_{wy} \) | \( z^d - x^{d-1}y^{d-1} = 0 \) | \( H_1 : w = 0, \ H_2 : y = 0 \) |
| \( U_{wx} \) | \( z^d - x^{d-1}y^d = 0 \) | \( H_1 : w = 0, \ H_2 : x = 0 \) |
| \( U_{ww} \) | \( z^d - x^{d-1}y^d w^{d-1} = 0 \) | \( H_1 = \emptyset, \ H_2 : w = 0 \) |

A calculation parallel to that above shows that, in each of these charts, the centre of the next blowing-up is the origin.

To estimate \( n(BM) \), we have to follow the branching of the coordinate charts, as above, after each subsequent blowing-up. We give the calculation in detail for two particular branches and leave it to the reader to check (exploiting the similar form of the data in the various charts) that for no branch do we need a number of blowings-up > 2d + j.

4.3. **First branch.** Charts \( U_w, U_{wy}, U_{wx}, U_{ww}, \ldots \).

**Year three.** Let \( X_3 \) denote the strict transform \( X'_2 \) of \( X_2 \) by \( \sigma_3 \). In the chart \( U_{wyx} \) of \( M_3 \), \( \sigma_3 \) is given by the substitution \( (x, xy, xz, xw) \). Therefore \( X_3 \cap U_{wyx} = \{ g_3 = 0 \} \), where
\[ g_3(x, y, z, w) = z^d - x^{d-2}y^{d-1}. \]

Let \( a_3 = 0 \). Then \( E(a_3) = \{ H_1, H_2, H_3 \} \), where
\[ H_1 : w = 0, \ H_2 : y = 0, \ H_3 : x = 0 \]
The first two pairs can be computed as before; inv at $a_3$ has a codimension 1 presentation $H_1(a_3) = (N_1(a_3), H_1(a_3), E_1(a_3))$, where $N_1(a_3) = \{ z = 0 \}$, $H_1(a_3) = \{(x^{d-2}y^{d-1}, d), (w, 1)\}$, $E_1(a_3) = \{H_2, H_3\}$, and inv has a codimension 2 presentation $H_2(a_3)$ given by $N_2(a_3) = \{ z = w = 0 \}$, $H_2(a_3) = \{(x^{d-2}y^{d-1}, d)\}$, $E_2(a_3) = E_1(a_3) = \{H_2, H_3\}$. Therefore, 

$$\nu_3(a_3) = \mu_3(a_3) - \mu_3H_2(a_3) - \mu_3H_3(a_3) = 0,$$

and $G_3(a_3) = H_2(a_3)$ is a codimension 2 presentation of inv$_5/2$ or inv$_X$ at $a_3$; in particular, $D_3(a_3)^d = x^{d-2}y^{d-1}$, in the notation of §3.2. Since $D_3(a_3)^d$ has order at least $d$ only at the origin, the centre of the next blowing-up $\sigma_4$ in this chart is $C_3 = \{0\}$.

**Year four.** Let $X_4 := X_3'$. In the chart $V := U_{wzv}$, $\sigma_4$ is given by the substitution $(xw, yw, zw, w)$. Therefore, $X_4 \cap V$ is defined by

$$g_4(x, y, z, w) = z^d - x^{d-2}y^{d-1}w^{d-3}.$$  

Let $a_4 = 0$. Then $E(a_4) = \{H_2, H_3, H_4\}$, where

$$H_2 : y = 0, \quad H_3 : x = 0, \quad H_4 : w = 0.$$  

($H_1 \cap V = \emptyset$; the variable $w$ has been “re-marked” as $H_4$!). Now inv$_1(a_4) = (d, 0)$ and inv$_1$ at $a_4$ has a codimension 1 presentation $H_1(a_4)$ given by $N_1(a_4) = \{ z = 0 \}$, $H_1(a_4) = \{(x^{d-2}y^{d-1}w^{d-3}, d)\}$ and $E_1(a_4) = E(a_4) = \{H_2, H_3, H_4\}$. Therefore,

$$\text{inv}_X(a_4) = (d, 0; 0)$$

and $G_2(a_4) = H_1(a_4)$ is a codimension 1 presentation of inv$_3/2$ or of inv$_X$ at $a_4$; in particular, $G_2(a_4) = (D_2(a_4)^d, d)$, where $D_2(a_4)^d = x^{d-2}y^{d-1}w^{d-3}$. Then $S_{\text{inv}_X}(a_4)$ has three components as follows; we order these components using the lexicographic ordering of the quadruples shown (cf. §1.6):

$$\{ z = w = x = 0 \} \quad (0, 0, 1, 1)$$
$$\{ z = w = y = 0 \} \quad (0, 1, 0, 1)$$
$$\{ z = y = x = 0 \} \quad (0, 1, 1, 0)$$

The centre $C_4$ of the next blowing-up is given by the maximum order; i.e., $C_5 = \{ z = y = x = 0 \}$. We are now in the combinatorial situation of §3.3(4). We need a number of blowings-up of the order of $2d$ (i.e., $2d + j$ blowings-up, where $j$ is independent of $d$) to decrease the order $d$ of the strict transform of $X$ along any branch of coordinate charts over $V$; for example:
| Year $j$ | Chart | Strict transform $g_j$ | Exceptional divisors | Centre $C_j$ |
|---------|-------|------------------------|----------------------|-------------|
| 5       | $V_x$ | $z^d - x^{d-3}y^{d-1}w^{d-3}$ | $x : H_5, y : H_2, w : H_4$ | $\{z = w = y = 0\}$ |
| 6       | $V_{xw}$ | $z^d - x^{d-3}y^{d-1}w^{d-4}$ | $x : H_5, y : H_2, w : H_6$ | $\{z = y = x = 0\}$ |
| 7       | $V_{xwy}$ | $z^d - x^{d-4}y^{d-1}w^{d-4}$ | $x : H_7, y : H_2, w : H_6$ | $\{z = w = y = 0\}$ |
|         |       | etc.                   |                      |             |

4.4. **Second branch.** We will briefly make a calculation analogous to the preceding or the branch of charts $U_w, U_{wy}, U_{wyw}, U_{wywy}, \ldots$. Years one and two are the same as in the branch above.

**Year three.** In the chart $U_{wyw}$, $\sigma_3$ is given by the substitution $(xw, yw, zw, w)$. Therefore, $X_3 \cap U_{wyw}$ is defined by $g_3 = z^d - x^{d-1}y^{d-1}w^{d-2}$. Let $a_3 = 0$. Then $E(a_3) = \{H_2, H_3\}$ where $H_2$ and $H_3$ are defined by $y$ and $w$, respectively ($H_1 \cap U_{wyw} = \emptyset$). Then $\text{inv}_1(a_3) = (d, 0)$. We calculate $N_1(a_3) = \{z = 0\}$, $\mathcal{H}_1(a_3) = \{(x^{d-1}y^{d-1}w^{d-2}, d)\}$, $\nu_2(a_3) = (d - 1)/d$, $s_2(a_3) = 2$, $N_2(a_3) = \{z = x = 0\}$, and $\mathcal{H}_2(a_3) = \{(y, 1), (w, 1)\}$. Therefore,  
\[
\text{inv}_X(a_3) = \left(d, 0; \frac{d-1}{d}, 2, 1, 0; 1, 0; \infty\right)
\]
and $C_3 = \{0\}$.

**Year four.** In $U_{wywy}$, $\sigma_4$ is given by $(xy, yz, yw)$; therefore $X_4 \cap U_{wywy}$ is defined by $g_4 = z^d - x^{d-1}y^{2d-4}w^{d-2}$. Let $a_4 = 0$. Then $H_3, H_4$ are defined by $w, y$, respectively ($H_1$ and $H_2$ do not intersect this chart). We calculate  
\[
\text{inv}_X(a_4) = \left(d, 0; \frac{d-1}{d}, 1; 1, 0; \infty\right)
\]
with $N_1(a_4) = \{z = 0\}$, $\mathcal{H}_1(a_4) = \{(x^{d-1}y^{2d-4}w^{d-2}, d)\}$, $N_2(a_4) = \{z = x = 0\}$, and $\mathcal{H}_2(a_4) = \{(w, 1)\}$. Therefore, $C_4 = \{z = w = x = 0\}$.

**Year five.** In $U_{wywyw}$, $\sigma_5 = (xw, zw, w)$. Therefore $X_5 \cap U_{wywyw}$ is given by $g_5 = z^d - x^{d-1}y^{2d-4}w^{d-3}$ and $y, w$ are exceptional divisors representing $H_4, H_5$ (respectively). At $a_5 = 0$, we calculate  
\[
\text{inv}_X(a_5) = \left(d, 0; \frac{d-1}{d}, 0; 0\right)
\]
together with \(N_1(a_5) = \{z = 0\}\), \(\mathcal{H}_1(a_5) = \{(x^{d-1}y^{2d-4}w^{d-3}, d)\},\)
\(N_2(a_5) = \{z = x = 0\}\), \(\mathcal{H}_2(a_5) = \{(y^{2d-4}w^{d-2}, 1)\} = \mathcal{G}_3(a_5)\). Therefore, \(S_{inv_X}(a_5)\) has two components, ordered as follows:
\[
\begin{align*}
\{z = y = x = 0\} & \quad (0, 0, 0, 1, 0) \\
\{z = w = x = 0\} & \quad (0, 0, 0, 0, 1)
\end{align*}
\]
(c.f. \(\S\) 4.3, year four). The next blowing-up \(\sigma_6\) has centre \(C_5 = \{z = y = x = 0\}\).

Over \(U_{wywyw}\), the strict transform \(X_6\) of \(X_5\) by \(\sigma_6\) lies entirely in two charts, given as follows:
\[
\begin{align*}
U_{wywyw} & \quad (x, xy, xz, w) \\
U_{wywyw} & \quad (xy, y, yz, w)
\end{align*}
\]
\(g_6 = z^d - x^{2d-5}y^{2d-4}w^{d-3}\)
\(g_6 = z^d - x^{d-1}y^{2d-5}w^{d-3}\)

It is easy to check that, following either branch, the order \(d\) is decreased by a number of blowings-up of the order of \(2d\).

4.5. Villamayor’s algorithm. We now reconsider our example (4.1) above using Villamayor’s algorithm. We will exhibit a sequence of points over which a number of blowings-up of the order \(9d\) is needed to decrease the order \(d\) of \(X_1\) at \(a_1\): Following the first branch above, there is no change until year four, so we reconsider our calculation from that point: In each year \(j\) below, \(a_j\) denotes the origin of the chart we study.

Year four. Let \(V := U_{wywyw}\) as in \(\S\) 4.3, year four, above. Then \(X_4 \cap V\) is defined by \(g_4(x, y, z, w) = z^d - x^{d-2}y^{d-1}w^{d-3}\) (4.2), and \(E(a_4) = \{H_2, H_3, H_4\}\), where \(H_2, H_3\) and \(H_4\) are defined by \(y, x\) and \(w\), respectively (4.3) (and \(H_1 \cap V = \emptyset\)). As before, we have \(inv_X(a_4) = (d, 0)\).

(We use an asterisk to distinguish the invariant corresponding to Villamayor’s algorithm.) But, following Villamayor’s algorithm, we take
\[
\mathcal{H}_1(a_4) = (N_1(a_4), \mathcal{H}_1(a_4), \mathcal{E}_1^*(a_4)),
\]
where \(N_1(a_4) = \{z = 0\}\), \(\mathcal{H}_1(a_4) = \{(x^{d-2}y^{d-1}w^{d-3}, d)\}\), and \(\mathcal{E}_1^*(a_4) = \emptyset\) (the latter because this year four is the year of birth of the value \((d, 0)\) of \(inv^*_1(a_4)\)). Then \(E^2(a_4) = E(a_4) = \{H_2, H_3, H_4\}\) and \(s_2(a_4) = 3\). Therefore, we take \(\mathcal{H}_2(a_4) = (N_2(a_4), \mathcal{H}_2(a_4), \emptyset)\), where \(N_2(a_4) = \{z = w = 0\}\), \(\mathcal{H}_2(a_4) = \{(x, 1), (y, 1)\}\), and get
\[
\begin{align*}
inv_X^*(a_4) & = \left(\frac{3d - 6}{d}, 3; 1, 0; 1, 0; \infty\right);
\end{align*}
\]
the centre of the next blowing-up \(\sigma_5\) in this chart is \(C_4 = \{0\}\).
Year five. In $V_w$, $g_5 = z^d - x^{d-2}y^{d-1}w^{2d-6}$. In a fashion similar to year four, we compute

$$\text{inv}_X^*(a_5) = \left(d, 0; \frac{2d-3}{d}, 3; 1, 0; 1, 0; \infty \right).$$

(Here $E_1(a_5) = \{H_5\}$, where $H_5 = \{w = 0\}$, so we factor $w^{2d-6}$ from $H_1(a_5)$ to get $\nu_2(a_5) = (2d-3)/d$.) Therefore, $C_5 = \{0\}$.

Year six. In $V_{ww}$, $g_6 = z^d - x^{d-2}y^{d-1}w^{3d-9}$. We compute

$$\text{inv}_X^*(a_6) = \left(d, 0; \frac{2d-3}{d}, 2; 1, 1; 1, 0; \infty \right).$$

(where $s_2(a_6) = 2$ counts $H_2$: $y = 0$ and $H_3$: $x = 0$, and $s_3(a_6) = 1$ counts $H_6$: $w = 0$), together with $N_1(a_6) = \{z = 0\}$, $H_1(a_6) = \{(x^{d-2}y^{d-1}w^{3d-9}, d)\}$, $N_2(a_6) = \{z = y = 0\}$, $H_2(a_6) = \{(x, 1)\}$, $N_3(a_6) = \{z = y = x = 0\}$, and $H_3(a_6) = \{(w, 1)\}$. In particular, $C_6 = \{0\}$.

Year seven. In $V_{www}$, $g_7 = z^d - x^{d-2}y^{d-1}w^{4d-12}$. Then

$$\text{inv}_X^*(a_7) = \left(d, 0; \frac{2d-3}{d}, 2; 1, 0; \infty \right),$$

and $N_2(a_7) = \{z = y = 0\}$, $G_3(a_7) = H_2(a_7) = \{(x, 1)\}$, so that $C_7 = \{z = y = x = 0\}$.

We go on as follows. (In this table, “chart $x$” means $U_x$, where $U$ denotes the chart in the previous year. For example, the chart in year eight is $V_{wwwx}$.)

| Year $j$ | Chart $x$ | Strict transform $g_j$ | $\text{inv}_X^*(a_j)$ | Centre $C_j$ |
|----------|-----------|-------------------------|-----------------------|-------------|
| 8        | $x$       | $z^d - x^{d-3}y^{d-1}w^{4d-12}$ | $(d, 0; \frac{d-1}{d}, 3; 1, 0; 1, 0; \infty)$ | $\{0\}$ |
| 9        | $w$       | $z^d - x^{d-3}y^{d-1}w^{5d-16}$ | $(d, 0; \frac{d-1}{d}, 2; 1, 1; 1, 0; \infty)$ | $\{0\}$ |
| 10       | $w$       | $z^d - x^{d-3}y^{d-1}w^{6d-20}$ | $(d, 0; \frac{d-1}{d}, 2; 1, 0; \infty)$ | $\{z = y = x = 0\}$ |
| 11       | $x$       | $z^d - x^{d-4}y^{d-1}w^{6d-20}$ | $(d, 0; \frac{d-1}{d}, 1; 7d - 24, 2; 1, 0; \infty)$ | $\{0\}$ |
| 12       | $w$       | $z^d - x^{d-4}y^{d-1}w^{7d-25}$ | $(d, 0; \frac{d-1}{d}, 1; d - 4, 2; 1, 0; \infty)$ | $\{z = y = x = 0\}$ |
| 13       | $w$       | $z^d - x^{d-4}y^{d-1}w^{8d-30}$ | $(d, 0; \frac{d-1}{d}, 1; d - 4, 4; 1, \infty)$ | $\{z = y = x = 0\}$ |
| 14       | $x$       | $z^d - x^{d-5}y^{d-1}w^{8d-30}$ | $(d, 0; \frac{d-1}{d}, 1; 0)$ | $\{z = w = y = 0\}$ |

As an aid to computing the entries in the above table, we note that, in year eleven, we can take $G_2(a_{11}) = \{(y, 1), (x^{d-4}w^{2d-20}, 1)\}$, and in year thirteen, we can take $N_2(a_{13}) = \{z = y = 0\}$, $H_2(a_{13}) = \{(x^{d-4}w^{8d-30}, 1)\}$.
In year fourteen, we can take $N_2(a_{14}) = \{ z = y = 0 \}$ and $H_2(a_{14}) = \{ (x^{d-5}u^{8d-30}, 1) \}$; $S_{inv \chi}(a_{14})$ has two components $\{ z = y = x = 0 \}$ and $\{ z = w = y = 0 \}$; $C_{14}$ is the latter.

We now follow alternately the “$w$-chart” or “$x$-chart”, finding alternately $C = \{ z = y = x = 0 \}$ or $C = \{ z = w = y = 0 \}$ until, in year $2d + 4$, we have $g_{2d+4} = z^d - y^{d-1}w^{d-25}$ and $C_{2d+4} = \{ z = w = y = 0 \}$. Now following the “$w$-charts”, we reduce the order $d$ after $7d - 25$ further blowings-up (i.e., in year $9d - 21$). (Throughout the calculation above, we assume that $d \geq 5$.)

5. Equivalence of presentations

Our purpose in this section is to elucidate the ideas of equivalence of presentations introduced in Section 2. In particular, Corollaries 5.12 and 5.13 below imply Theorem 2.6, which is used in the inductive definition of $\text{inv}$ in §3.2, to pass from a codimension $r$ presentation of $\text{inv}_{r+1/2}$ to a presentation in codimension $r + 1$. It is convenient to use transformation formulas for differential operators introduced by Hironaka [Hi2, Sect. 8] and developed by Giraud [Gi] and Villamayor and Encinas [V2, EV1]. (An alternative approach is given in [BM5, Propositions 4.12, 4.19].) The formulas describe the way that the partial derivatives of a regular function transform by admissible blowings-up. The treatment below differs from that of [V2, EV1] in our use of coordinate charts (as in [BM5, BM7]) and of transformation formulas that account also for the effect of exceptional blowings-up.

5.1. Transformation of differential operators. Let $U$ denote a coordinate chart in a manifold $M$, with coordinate system $(x_1, \ldots, x_n)$. (The $x_i$ are regular functions on $U$. In the case of schemes of finite type, “coordinate chart” means “étale (or regular) coordinate chart”, as defined in [BM5, §3]. See also [BM7, §2] for coordinate charts, more generally.) Let $a = 0$. Consider a blowing-up $\sigma$ with centre

$$C = Z_I := \{ x_i = 0, \ i \in I \},$$

where $I \subset \{ 1, \ldots, n \}$. If $i \in I$, then $\sigma$ is given in the chart $U_i = U_{x_j} \subset \sigma^{-1}(U)$ (cf. Section 4) by the formulas $x_i = y_i = y_{exc}$, $x_j = y_iy_j$ if $j \in I \setminus \{ i \}$, and $x_j = y_j$ if $j \notin I$. (For $j \in I \setminus \{ i \}$, $y_j$ does not necessarily vanish at $a' \in \sigma^{-1}(a)$.) The following lemma is a simple calculation.
Lemma 5.1. Let $f \in \mathcal{O}_a = \mathcal{O}_{M,a}$. Suppose that $d \leq \mu_{C,a}(f)$. Let $i \in I$. Then

\[
\frac{1}{y_i^{d-1}} \left( \frac{\partial f}{\partial x_j} \circ \sigma \right) = y_i \frac{\partial}{\partial y_j} \left( \frac{f \circ \sigma}{y_i^d} \right), \quad \text{if } j \not\in I;
\]

\[
\frac{1}{y_i^{d-1}} \left( \frac{\partial f}{\partial x_j} \circ \sigma \right) = \frac{\partial}{\partial y_j} \left( \frac{f \circ \sigma}{y_i^d} \right), \quad \text{if } j \in I \setminus \{i\};
\]

\[
\frac{1}{y_i^{d-1}} \left( \frac{\partial f}{\partial x_i} \circ \sigma \right) = df \circ \sigma + y_i \frac{\partial}{\partial y_i} \left( \frac{f \circ \sigma}{y_i^d} \right) - \sum_{j \in I \setminus \{i\}} y_j \frac{\partial}{\partial y_j} \left( \frac{f \circ \sigma}{y_i^d} \right).
\]

Let $\mathcal{E}(a)$ denote a collection of smooth hypersurfaces passing through $a$. Suppose that $\mathcal{E}(a)$ and $C$ simultaneously have only normal crossings. Then we can choose coordinates so that $C$ has the form $Z_I$ as above, and also, for each $H \in \mathcal{E}(a)$, $\mathcal{I}_{H,a}$ is generated by $x_j$, for some $j$; we will write $x_j = x_H$.

The following lemma will be used to study the effect of an exceptional blowing-up. Let $H_1, H_2 \in \mathcal{E}(a)$. Suppose $x_1 = x_{H_1}$, $x_2 = x_{H_2}$. Let $\sigma$ denote the blowing-up with centre $C = H_1 \cap H_2$. In the chart $\hat{U}_1 = U_{x_1}$, $\sigma$ is given by $x_1 = y_1$, $x_2 = y_1 y_2$, and $x_j = y_j$ if $j > 2$.

Lemma 5.2. Let $f \in \mathcal{O}_a$, $\mu_a(f) \geq 1$. Then

\[
\left( \begin{array}{c} x_2 \frac{\partial f}{\partial x_2} \\ x_1 \frac{\partial f}{\partial x_1} \end{array} \right) \circ \sigma = \left( \begin{array}{c} \frac{\partial (f \circ \sigma)}{\partial y_2} \\ \frac{\partial (f \circ \sigma)}{\partial y_1} \end{array} \right);
\]

\[
\frac{\partial f}{\partial x_j} \circ \sigma = \frac{\partial (f \circ \sigma)}{\partial y_j}, \quad \text{if } j > 2.
\]

Consider a coordinate system of the form

\[
x = (\xi, u) = (\xi_1, \ldots, \xi_r, u_1, \ldots, u_s),
\]

where the $\xi_i$ are precisely the $x_H$, $H \in \mathcal{E}(a)$. We will say that the variables $u_1, \ldots, u_s$ are complementary to $\mathcal{E}(a)$.

Lemma 5.3. Let $f \in \mathcal{O}_a$. Then the ideal generated by

\[
\frac{\partial f}{\partial x_H}, \quad H \in \mathcal{E}(a),
\]

\[
\frac{\partial f}{\partial u_j}, \quad 1 \leq j \leq s,
\]

is independent of the choice of generators $x_H$ of the ideals of $H \in \mathcal{E}(a)$ and the choice of complementary variables $u$. 

Proof. If we change the generators \( \xi_i = x_H \) of the ideals of the \( H \in \mathcal{E}(a) \), say to \( \eta_i \), then we have \( \eta_i = \lambda_i \xi_i \), where each \( \lambda_i = \lambda_i(\xi, u) \) is a unit. Consider such a change of generators, as well as new complementary variables \( v = (v_1, \ldots, v_s) \). Then we can write
\[
(\eta, v) = (\lambda_1(\xi, u)\xi_1, \ldots, \lambda_r(\xi, u)\xi_r, v_1(\xi, u), \ldots, v_s(\xi, u)).
\]
Therefore, for each \( i = 1, \ldots, r \),
\[
\xi_i \frac{\partial}{\partial \xi_i} = \frac{1}{\lambda_i} \frac{\partial \eta_i}{\partial \eta_i} \frac{\partial}{\partial \eta_i} + \sum_{k \neq i} \frac{\xi_i}{\lambda_k} \frac{\partial k}{\partial \xi_i} \frac{\partial}{\partial \eta_k} + \sum_{\ell=1}^s \xi_i \frac{\partial v_{\ell}}{\partial \xi_i} \frac{\partial}{\partial v_{\ell}},
\]
and, for each \( j = 1, \ldots, s \),
\[
\frac{\partial}{\partial u_j} = \sum_{k=1}^r \frac{1}{\lambda_k} \frac{\partial k}{\partial u_j} \frac{\partial}{\partial \eta_k} + \sum_{\ell=1}^s \frac{\partial v_{\ell}}{\partial u_j} \frac{\partial}{\partial v_{\ell}}.
\]
\[\square\]

5.2. Passage to codimension +1. Let \( M \) denote a manifold and let \( a \in M \). Let \( N(a) \) denote a germ of a submanifold of \( M \) at \( a \); say \( p = \text{codim} \ N(a) \).

Lemma 5.4. Let \( z_1, \ldots, z_p \in \mathcal{O}_a = \mathcal{O}_{M,a} \) denote generators of \( \mathcal{I}_N(a) \) (so that \( z_1, \ldots, z_p \) have linearly independent gradients). Let \( f \in \mathcal{O}_a \) and let \( d \in \mathbb{N} \). Then \( \mu_a(f) \geq d \) if and only if
\[
\mu_a \left( \frac{\partial^{|\beta|} f}{\partial z^\beta} \bigg|_{N(a)} \right) \geq d - |\beta|, \quad \text{for all } \beta \in \mathbb{N}^p, \ |\beta| \leq d - 1.
\]

Let \( \mathcal{E}(a) \) denote a collection of smooth hypersurfaces passing through \( a \), such that \( N(a) \) and \( \mathcal{E}(a) \) simultaneously have only normal crossings, and \( N(a) \not\subset H \), for all \( H \in \mathcal{E}(a) \). Then we can choose a coordinate system of the form (5.1) for \( N(a) \).

Definitions 5.5. Choose a coordinate system \( x = (\xi, u) \) for \( N(a) \) as in (5.1). Let \( f \in \mathcal{O}_{N(a)} \) and let \( \mu_f \in \mathbb{Q} \) such that \( \mu_a(f) \geq \mu_f \). Set
\[
\Delta(f, \mu_f) := \left\{ (f, \mu_f - 1), \left( x_H \frac{\partial f}{\partial x_H}, \mu_f - 1 \right) \right\}, \text{for all } H \in \mathcal{E}(a),
\]
\[
\left( \frac{\partial f}{\partial u_j}, \mu_f - 1 \right), \text{for all } j = 1, \ldots, s \right\},
\]
\[
(\Delta(f, \mu_f)) := \text{the ideal in } \mathcal{O}_{N(a)} \text{ generated by}
\]
\[
\{ h : (h, \mu_h) \in \Delta(f, \mu_f) \},
\]
\[
\mathcal{D}(f, \mu_f) := \{ (f, \mu_f) \} \cup \Delta(f, \mu_f);
\]
\[
S(f, \mu_f) := \{ x \in N(a) : \mu_x(f) \geq \mu_f \};
\]
\[
S_{\mathcal{D}(f, \mu_f)} := \{ x \in N(a) : \mu_x(h) \geq \mu_h \}, \text{for all } (h, \mu_h) \in \mathcal{D}(f, \mu_f) \}
(\(S_{(f,\mu_f)}\) and \(S_{D(f,\mu_f)}\) make sense as germs at \(a\)). Note that an element \((h,\mu_h)\) in \(D(f,\mu_f)\) with \(\mu_h \leq 0\) imposes no condition in \(S_{D(f,\mu_f)}\).

**Corollary 5.6.** Let \(f \in \mathcal{O}_{N(a)}\) and \(\mu_f \in \mathbb{Q}\) such that \(\mu_a(f) \geq \mu_f\). Then

\[
S_{(f,\mu_f)} = S_{D(f,\mu_f)}.
\]

Now let

\[
\mathcal{F}(a) = (N(a), F(a), \mathcal{E}(a))
\]
denote a presentation (of codimension \(p\)) at \(a\); say \(\mathcal{F}(a) = \{(f,\mu_f)\}\).

**Definitions 5.7.** Choose coordinates as in Definitions 5.5 above. Set

\[
\Delta(\mathcal{F}(a)) := \bigcup_{\mathcal{F}(a)} \Delta(f,\mu_f),
\]

\[
\mathcal{D}(\mathcal{F}(a)) := \mathcal{F}(a) \cup \Delta(\mathcal{F}(a)) = \bigcup_{\mathcal{F}(a)} \mathcal{D}(f,\mu_f),
\]

\[
\mathcal{D}(\mathcal{F}(a)) := (N(a), \mathcal{D}(\mathcal{F}(a)), \mathcal{E}(a)).
\]

Let \(\sigma\) be a morphism of type (i), (ii) or (iii) (cf. §2.2) and let \(\mathcal{F}(a') = (N(a'), F(a'), \mathcal{E}(a'))\) or \(\mathcal{F}(a') = (N(a'), F(a'), \mathcal{E}(a'))\) denote the transform of \(\mathcal{F}(a)\) by \(\sigma\) at a point \(a' \in \sigma^{-1}(a)\) as in §2.2. (It will be convenient to use both notations for the transform.) We will also write \(\Delta(f,\mu_f)', \Delta(\mathcal{F}(a))'\) and \(\mathcal{D}(\mathcal{F}(a))'\) for the analogous transforms of \(\Delta(f,\mu_f), \Delta(\mathcal{F}(a))\) and \(\mathcal{D}(\mathcal{F}(a))\), respectively; for example,

\[
\Delta(f,\mu_f)' := \{(h',\mu_{h'}) : (h,\mu_h) \in \Delta(f,\mu_f)\},
\]

where \((h',\mu_{h'})\) is given by the transformation rules in §2.2.

**Lemma 5.8.** \(\Delta(f,\mu_f)' \subset (\Delta(f',\mu_{f'}))\).

(This is a minor abuse of notation; we mean that, for all \((h',\mu_{h'}) \in \Delta(f,\mu_f)'\), \(h' \in (\Delta(f',\mu_{f'}))\)).

**Proof.** For transformations of types (i) or (iii) (admissible or exceptional blowings-up), this follows from the formulas in Lemmas 5.1 and 5.2. For a transformation of type (ii) (product with a line), it is trivial. \(\square\)

**Theorem 5.9.** Let \((f,\mu_f) \in \mathcal{F}(a)\). Then

\[
S_{(f',\mu_{f'})} = S_{D(f,\mu_f)}
\]

after any transformation of type (i), (ii) or (iii) (and, in fact, after any sequence of transformations of types (i), (ii) and (iii)).
Proof. For a given sequence of transformations of types (i), (ii) and (iii), write $f^{(0)} = f$, $\Delta f^{(0)} = \Delta f$, $D f^{(0)} = D f$, and recursively define $f^{(k+1)} := f^{(k)}', \mu f^{(k+1)} = \mu f^{(k)}$, $\Delta f^{(k+1)} := (\Delta f^{(k)})'$ and $D f^{(k+1)} := (D f^{(k)})'$, for all $k \geq 0$.

We have $(f, \mu f) \in D(f, \mu f)$, so that for all $k$, $(f^{(k+1)}, \mu f^{(k+1)}) \in D(f, \mu f)$ and $S D f^{(k+1)} \subset S f^{(k+1)}$. By Lemma 5.8, $\Delta f^{(k)}' \subset (\Delta f^{(k)})'$. Assume that $\Delta f^{(k)} \subset (\Delta f^{(k)})$, by induction. Consider $(h, \mu h) \in \Delta f^{(k)}$. Then $h' \in (\Delta f^{(k+1)}, \mu f^{(k+1)})$, again by Lemma 5.8. Therefore, for all $k$,

$$\Delta f^{(k+1)} \subset (\Delta f^{(k+1)}, \mu f^{(k+1)})$$

and hence

$$S D f^{(k+1)} \subset S f^{(k+1)}.$$ 

But

$$S D f^{(k+1)} = S f^{(k+1)},$$

by Corollary 5.6, so the result follows. 

**Corollary 5.10.** $F(a)$ and $D(F(a))$ are equivalent with respect to transformations of types (i), (ii) and (iii). (See Definitions 2.1.)

**Definitions 5.11.** Write $D^0 F(a) = \Delta^0 F(a) = F(a)$, and, for all $d = 0, 1, 2, \ldots$, set

$$\Delta^{d+1} F(a) := \Delta (\Delta^d F(a)),$$

$$D^{d+1} F(a) := \bigcup_{q=0}^{d+1} \Delta^q F(a) = D (D^d F(a)).$$

**Corollary 5.12.** Let $z \in O N(a)$ and let $d$ be a positive integer. Suppose that $\mu a(z) = 1$ and that $z \in (\Delta^{d-1} F(a))$. Then, after any sequence of transformations of types (i), (ii) or (iii), $z' \in (\Delta^{d-1} F(a'))$, and $S F a' \subset V z' \subset N(a')$.

The first assertion is a consequence of Lemma 5.8 and the second is a consequence of Corollary 5.10. From Corollaries 5.10 and 5.12, we deduce:
Corollary 5.13. Under the hypotheses of Corollary 5.12, set
\[ N_{+1}(a) := V(z) \subset N(a), \]
\[ \mathcal{H}(a) := \mathcal{D}^{d-1}(\mathcal{F}(a)|_{N_{+1}(a)}), \]
\[ \mathcal{H}(a) := (N_{+1}(a), \mathcal{H}(a), \mathcal{E}(a)). \]
Then the presentations \( \mathcal{F}(a) \) and \( \mathcal{H}(a) \) are equivalent with respect to transformations of types (i), (ii) and (iii).

It is clear that \( \mathcal{F}(a) \) and \( \mathcal{H}(a) \) are, in fact, semicoherent equivalent. (See §2.5). Theorem 2.6 follows from Corollaries 5.12 and 5.13 – this is the basis of our constructive definition of \( \text{inv}(\cdot) \) by induction on codimension (§3.2).

5.3. On the notion of equivalence. We conclude this section by showing that, in contrast to Theorem 2.4, it is not true that the \( \mu_{r+1,H}(a) \) and \( \nu_{r+1}(a) \), \( r > 0 \) (even as occurring in Villamayor’s invariant) in general depend only on the equivalence class of a presentation of \( \text{inv} \) at \( a \) with respect to transformations of types (i) and (ii). (See Remarks 2.5.)

Example 5.14. Let \( X \) denote the surface in affine 3-space \( U \) defined by
\[ z^d - x^{d-1}y^d = 0, \]
where \( d \geq 2 \). We will use the notational conventions of Section 4. Let \( \sigma_1 \) denote the blowing-up of \( U \) with centre \( C_0 = \{0\} \). In the chart \( U_x \), the strict transform \( X_1 \) of \( X \) is given by \( g_1 = 0 \), where
\[ g_1(x, y, z) = z^d - x^{d-1}y^d. \]
Let \( \sigma_2 \) be the blowing-up of \( U_x \) with centre \( C_1 = \{0\} \). In the chart \( U_{xy} \), the strict transform \( X_2 \) of \( X_1 \) is given by \( g_2 = 0 \), where
\[ g_2(x, y, z) = z^d - x^{d-1}y^{d-1}. \]

Let \( a = 0 \) in \( U_{xy} \). Following either the algorithm of the authors or that of Villamayor, we have \( E^1(a) = \emptyset \) and \( E_1(a) = E(a) = \{H_1, H_2\} \), where the exceptional hyperplanes \( H_1 \) and \( H_2 \) are given by \( x = 0 \) and \( y = 0 \), respectively. Then
\[ \mu_2(a) = \frac{2d - 2}{d}, \quad \mu_{2H_1}(a) = \frac{d - 1}{d}, \quad \mu_{2H_2}(a) = \frac{d - 1}{d}, \]
and
\[ \text{inv}_X(a) = (d, 0; 0). \]

At \( a \), \( \text{inv}_1 \) has a codimension 1 presentation \( \mathcal{H}_1(a) = (N_1(a), \mathcal{H}_1(a), \mathcal{E}_1(a)) \), where \( N_1(a) = \{z = 0\} \) and \( \mathcal{H}_1(a) = \{(x^{d-1}y^{d-1}, d)\} \). By Theorem 2.4,
\( \mu_{2H_1}(a) \) and \( \mu_{2H_2}(a) \) depend only on \([H_1(a)]\). But it is not true that they depend only on \([H_1(a)]\):(i,ii):

Recall that, for each \( H \in \mathcal{E}_1(a) \), \( \mu_{2H}(a) = \mu_{H_1}(a) \) where

It follows from Lemmas 5.1 and 5.8 above that \( H_1(a) \) is equivalent with respect to transformations of types (i) and (ii) to a presentation

where \((x_1, \ldots, x_m)\) denotes the coordinates of \( N_1(a) \). In our example,

so that

The main topics of this section – universal embedded desingularization (of spaces that are not necessarily embedded), comparison of weak and strict transforms, and simultaneous desingularization of parametrized families – are introduced individually in the subsections below. The point of view is that of the general desingularization principle, Theorem 1.14. Particular results concerning embedded desingularization (e.g., Theorems 6.1 and 6.8) depend on the fact that the Hilbert-Samuel function satisfies the Hypotheses 1.10 that are needed to apply the desingularization principle (see Example 1.13(2)). In Theorem 6.18, we show that the Hilbert-Samuel function also satisfies the stronger Hypotheses 6.13 below that are needed to extend the desingularization principle to parametrized families (Theorem 6.15). Theorems 6.8 and 6.15 thus generalize results of [BrV] and [ENV], respectively, which are tied to the weak embedded desingularization algorithm of [EV3] (Example 1.19(4)).

6.1. Universal desingularization. Let \( \iota = \iota_X \) (or \( \iota_J \)) denote a local invariant of spaces \( X \) (or ideals of finite type \( J \); see §1.3) satisfying the Hypotheses 1.10. The invariant \( \text{inv}(\cdot) = \text{inv}_X(\cdot) \) or \( \text{inv}_J(\cdot) \), and therefore the desingularization algorithm depend \( a \) \text{priori} on the codimension of a presentation of \( \text{inv}_{1/2} = \iota \) (see Hypotheses 1.10(3)) and, in particular, on the dimension of the ambient manifold \( M \). We avoided these issues in §3.2 by making the following simplifying assumptions in Hypothesis 1.10(3): (1) \( \iota \) admits a semicoherent presentation \( \mathcal{G}(a) = (N(a), \mathcal{G}(a), \emptyset) \) of codimension 0 at every point \( a \in M \); (2) \( \mu_{\mathcal{G}(a)} = 1 \). (To begin the inductive construction, we then showed
that, for any sequence of $\iota$-admissible transformations (1.1) (or (1.2)), and all $a \in M_j$, $j = 0, 1, \ldots$, the invariant $\iota$ admits a semicoherent presentation $\mathcal{C}(a) = (N_1(a), \mathcal{C}(a), \mathcal{E}_1(a))$ of codimension 1 at $a$, where $\mu_{\mathcal{C}(a)} \geq 1$ (Corollary 3.4.)

In §6.1.1 below, we show how to modify the local inductive construction (§3.2) so that it applies without the simplifying assumptions above. We assume only that $\iota$ admits a semicoherent presentation $\mathcal{C}(a) = (N(a), \mathcal{C}(a), \emptyset)$ at every point $a \in M$, of codimension $p(a) \geq 1$ (Hypothesis 1.10(3); see Remark 6.2 below).

For example, [BM5, Theorems 9.4, 9.6] provide such a semicoherent presentation of the Hilbert-Samuel function with variable codimension $p(a)$. The largest codimension of a presentation of the Hilbert-Samuel function at $a$ is not determined uniquely by $H_{X,a}$ [BM5, Remarks 9.15(1)]. This is why it is important to modify the local inductive construction to ensure that $\text{inv}(\cdot)$ does not depend on the codimension of a presentation of $\iota$ at $a$.

The invariant $\text{inv}(\cdot)$ and therefore the desingularization algorithm, as described in §6.1.1, nevertheless still depend on the dimension of the ambient manifold $M$. In §6.1.2, we show that $\text{inv}_X(\cdot)$ can be made independent of the embedding space of $X$, by a simple variation in the definition. Given a scheme of finite type or an analytic space $X$ (not necessarily globally embedded), then $\text{inv}_X(\cdot)$ can be defined in this way using any local embedding $X\mid_U \hookrightarrow M$ over an open subset $U$ of $X$. As a result, we obtain the following universal embedded desingularization theorem (cf. [BM5, Theorem 13.2]) for (not necessarily embedded) spaces $X$. We include the argument here both as an illustration of the desingularization principle and to correct an error in [BM5, Remarks 9.15(3)] that is illustrated by an example of Encinas [E].

**Theorem 6.1.** (1) There is a finite sequence of blowings-up $\sigma_{j+1} : X_{j+1} \rightarrow X_j$, where $X_0 = X$, such that, for any local embedding $X\mid_U \hookrightarrow M$ (over an open subset $U$ of $|X|$), the sequence of blowings-up $\sigma_{j+1}$ restricted to the inverse images of $U$ is induced by embedded desingularization of $X\mid_U$ in the sense of Example 1.19(2).

(2) The desingularization is universal in the sense that, to each $X$ we associate a morphism $\sigma_X : X' \rightarrow X$ such that

(i) $\sigma_X$ is a composite of a finite sequence of blowings-up as in (1).

(ii) If $\varphi : X\mid_U \rightarrow Y\mid_V$ is an isomorphism over open subsets $U,V$ of two spaces $X,Y$ (respectively), then there is an isomorphism $\varphi' : X'\mid_{\sigma_X^{-1}(U)} \rightarrow Y'\mid_{\sigma_Y^{-1}(V)}$ such that $\sigma_Y \circ \varphi' = \varphi \circ \sigma_X$. (The lifting $\varphi'$ of $\varphi$ is necessarily unique.) In fact, $\varphi$ lifts to isomorphisms throughout the desingularization towers.
6.1.1. Independence of the codimension of a presentation. Assume that \( \iota = \iota_X \) (or \( \iota, \tau \)) admits a semicoherent presentation \( \mathcal{C}(a) = (N(a), \mathcal{C}(a), \emptyset) \) at every point \( a \in M \), of codimension \( p(a) \geq 1 \). Consider any sequence of \( \iota \)-admissible transformations (1.1) (or (1.2)). We use the notation of §3.2. Let \( a \in M_j \). Then \( \text{inv}_{1/2} = \iota \) admits a semicoherent presentation

\[
\mathcal{C}_1(a) = (N_1(a), \mathcal{C}_1(a), \mathcal{E}_1(a))
\]

at \( a \), of codimension \( p(a) \geq 1 \). As before, we define \( \text{inv}_1(a) := (\iota(a), s_1(a)) \) and set

\[
\mathcal{H}_1(a) = (N_1(a), \mathcal{H}_1(a), \mathcal{E}_1(a)),
\]

where

\[
\mathcal{H}_1(a) := \mathcal{C}_1(a) \cup \left( E^1(a) \big|_{N_1(a)}, 1 \right).
\]

Then \( \mathcal{H}_1(a) \) is a semicoherent presentation of \( \text{inv}_1 \) at \( a \), of codimension \( p(a) \).

If \( p(a) = 1 \), we continue the inductive definition of \( \text{inv}(a) \) exactly as in §3.2. If \( p(a) > 1 \), however, we consider the following variation of the definition given in §3.2. Recursively, for each \( r = 1, \ldots, p(a) - 1 \), we define

\[
\nu_{r+1}(a) \quad := \quad 1,
\]

\[
\text{inv}_{r+1/2}(a) \quad := \quad (\text{inv}_r(a); \nu_{r+1}(a)),
\]

\( E^{r+1}(a) \) and \( \mathcal{E}_{r+1}(a) \) as before, \( s_{r+1}(a) := \#E^{r+1}(a) \), and

\[
\text{inv}_{r+1}(a) \quad := \quad (\text{inv}_{r+1/2}(a), s_{r+1}(a)).
\]

We set \( N_{r+1}(a) := N_r(a) = N_1(a), \mathcal{C}_{r+1}(a) := \mathcal{H}_r(a), \)

\[
\mathcal{H}_{r+1}(a) \quad := \quad \mathcal{C}_{r+1}(a) \cup \left( E^{r+1}(a) \big|_{N_{r+1}(a)}, 1 \right),
\]

\[
\mathcal{C}_{r+1}(a) \quad := \quad (N_{r+1}(a), \mathcal{C}_{r+1}(a), \mathcal{E}_{r+1}(a)),
\]

\[
\mathcal{H}_{r+1}(a) \quad := \quad (N_{r+1}(a), \mathcal{H}_{r+1}(a), \mathcal{E}_{r+1}(a));
\]

then \( \mathcal{C}_{r+1}(a) \) (respectively, \( \mathcal{H}_{r+1}(a) \)) is a codimension \( p(a) \) presentation of \( \text{inv}_{r+1/2} \) (respectively, of \( \text{inv}_{r+1} \) at \( a \)). Finally,

\[
\text{inv}_{p(a)}(a) = (\iota(a), s_1(a); 1, s_2(a); \ldots; 1, s_{p(a)}(a))
\]

and

\[
\mathcal{H}_{p(a)}(a) = (N_{p(a)}(a), \mathcal{H}_{p(a)}(a), \mathcal{E}_{p(a)}(a))
\]
is a codimension $p(a)$ presentation of $\text{inv}_{p(a)}$ at $a$, where $N_{p(a)}(a) = N_1(a)$,

$$\mathcal{H}_{p(a)}(a) = \mathcal{C}_1(a) \cup \bigcup_{r=1}^{p(a)} \left( E_r(a) \big|_{N_1(a)}, 1 \right),$$

$$\mathcal{E}_{p(a)}(a) = E(a) \setminus \bigcup_{r=1}^{p(a)} E_r(a).$$

The definitions of $\text{inv}(a)$ and an associated presentation now proceed as in §3.2. The resulting definition of $\text{inv}(a)$ is independent of the choice of a presentation of $\iota$ at $a$ (in particular, independent of its codimension $p(a)$ – see Theorem 2.3). We thus obtain the desingularization principle Theorem 1.14 in the more general setting.

**Remark 6.2.** We have assumed that $\iota$ admits a semicoherent presentation of codimension $p(a) \geq 1$ at each point $a$ in order to make the above generalization of §3.2 consistent with the latter (and because we know of no interesting example where we need to use a presentation $G(a)$ with $p(a) = \text{codim } G(a) = 0$ and $\mu_G(a) > 1$). The same local construction can, of course, be used if we merely assume that $p(a) \geq 0$, but the codimension of the presentation of $\text{inv}_{r+1}$ will be shifted by 1. ($\text{inv}_{r+1}$ will have a presentation $\mathcal{H}_{r+1}(a)$ of codimension $r$ at $a$, when $r > p(a)$.)

6.1.2. Independence of the embedding dimension. Consider $X \subset M$ and $\text{inv} = \text{inv}_X$, where $\iota_X(a)$ is the Hilbert-Samuel function $H_{X,a}$. If $X \subset M \subset M'$, where $\dim M' > \dim M$, then $\text{inv}_X$ as defined above would not be the same for $X$ as a subspace of $M$ or $M'$ (cf. examples of [E]). We can resolve this problem by a simple variation of §6.1.1: The Hilbert-Samuel function $H_{X,a}$ determines the minimal embedding dimension $e_{X,a}$ of $X$ at $a \in M$:

$$e_{X,a} = H_{X,a}(1) - 1.$$  

Let $n = \dim_a M$ and $e(a) = e_{X,a}$. There is a semicoherent presentation $\mathcal{G}(a) = (N(a), \mathcal{C}(a), \emptyset)$ of $\iota_X(\cdot) = H_{X,a}$ at $a$, where $N(a)$ lies in a minimal embedding submanifold for $X$ at $a$; in particular, $\mathcal{G}(a)$ has codimension $p(a) \geq n - e(a)$. Consider a sequence of $\iota_X$-admissible transformations (1.1). (We use the notation above.) Let $a \in M_j$. Then $\text{inv}_{1/2} = \iota_X$ has a semicoherent presentation $\mathcal{G}_1(a) = (N_1(a), \mathcal{C}_1(a), \mathcal{E}_1(a))$ at $a$, of codimension $p(a) \geq n - e(a)$, where $e(a) = e_{X_j,a}$.

We use the construction of §6.1.1, but where we now think of $\mathcal{G}_1(a)$ as a presentation of codimension $e(a) - (n - p(a))$ in a minimal embedding
space for $X_j$ at $a$. So, if $p(a) > n - e(a)$ and $q(a)$ denotes $e(a) - (n - p(a))$, then

$$\text{inv}_{q(a)}(a) = (H_{X_j,a}, s_1(a); 1, s_2(a); \ldots; 1, s_{q(a)}(a)),$$

with a semicoherent presentation

$$H_{q(a)}(a) = (N_{q(a)}(a), H_{q(a)}(a), E_{q(a)}(a))$$

at $a$, of codimension $p(a)$ in $M_j$, or of codimension $q(a)$ in a minimal embedding submanifold, and we now proceed as in §3.2.

If $p(a) = n - e(a)$, then $\text{inv}_1(a) = (H_{e(a)}, s_1(a))$ has a presentation

$$H_1(a) = (N_1(a), H_1(a), E_1(a))$$

at $a$, where $H_1(a) = C_1(a) \cup \left( E_1(a) \big|_{N_1(a)}, 1 \right)$, of codimension $p(a)$ in $M_j$, or of codimension 0 in a minimal embedding submanifold. In this case, it is easy to see that $\mu_{H(a)} = 1$, so we can find an equivalent presentation in codimension +1, and proceed as usual. For example, $X_j$ is smooth at $a$ if and only if $\dim_a X_j = e(a)$; in this case, $\text{inv}_1(a) = (H_e(a), s_1(a))$, where $H_e$ denotes the Hilbert-Samuel function of a smooth space of dimension $e$:

$$H_e(k) = \left( e + k \right) \div e, \quad k \in \mathbb{N}.$$  

The resulting desingularization invariant $\text{inv}_X$ is independent of the dimension of a smooth embedding space $M$ for $X$, and the desingularization principle Theorem 1.14 applies to give the universal embedded desingularization Theorem 6.1. The embedded desingularization algorithm stops over a neighbourhood of $a$ when $\text{inv}_X(a) = (H_e(a), 0; \infty)$.

### 6.2. Comparison of weak and strict transforms

Let $X$ denote a closed subspace of $M$ and let $J = I_X \subset O_M$. (See §1.3.) The purpose of this subsection is to study the strict transforms of $X$ by the sequence of blowings-up involved in principalization of $J$ according to the desingularization principle Theorem 1.14. (See Example 1.19(1).) We show, in particular, that the theorems of [EV3] (cf. Example 1.19(4)) and [BrV] are direct consequences of the desingularization principle (Corollaries 6.5 and 6.7 below). Theorem 6.8 following generalizes these results by using the desingularization principle in a novel way.

Consider an $\text{inv}_J$-admissible sequence of transformations (1.2), where $\text{inv}_{1/2}(a) = \mu_a(J_j), a \in M_j$. (See §3.2.) The following lemma is the key point of this subsection.

**Lemma 6.3.** Let $a \in M_j$. Then

$$(6.1) \quad \text{inv}_J(a) = (1, 0; \ldots; 1, 0; \infty)$$
(where, let us say, there are \( t \) pairs \((1,0)\)) if and only if there are local coordinates \((x_1, \ldots, x_{n-t}, x_{n-t+1}, \ldots, x_n)\) for \( M_j \) at \( a = 0 \) in which:

1. \( E(a) = \{ H \} \), where each \( x_H = x_i \), for some \( i = 1, \ldots, n-t \).
2. Set \( \bar{x} = (x_1, \ldots, x_{n-t}) \). Then \( \mathcal{J}_{j,a} \) is generated by

\[
x_n, \bar{x}_1 x_{n-1}, \ldots, \bar{x}_{t-1} x_{n-t+1},
\]

where each \( \bar{x}_k^{\theta_k} \) denotes a monomial

\[
\bar{x}_k^{\theta_k} = x_1^{\theta_{k1}} \cdots x_{n-t}^{\theta_{k,n-t}},
\]

and

(a) each \( \bar{x}_k^{\theta_k} \) is a monomial in the \( x_H \), \( H \in E(a) \); i.e., \( \theta_{ki} = 0 \) unless \( x_i = x_H \), for some \( H \);

(b) \( \theta_1 \leq \theta_2 \leq \cdots \leq \theta_{t-1} \) (where \( \leq \) means componentwise inequality).

Proof. The assertion can be seen by following the local construction in §3.2. In the language of the latter, \( \bar{x}_1^{\theta_1} = D_2(a) \) and

\[
\bar{x}_k^{\theta_k+1-\theta_k} = D_{k+2}(a), \quad k = 1, \ldots, t-2.
\]

(This is where (2)(b) comes from.) Moreover, \( \{ x_n = \cdots = x_{n-t+1} = 0 \} \) is a maximal contact subspace \( N_t(a) \) (in the notation of §3.2), and \( \bar{x} \) restricts to a coordinate system on \( N_t(a) \). \( \square \)

Now set \( X_0 = X \) and, for each \( j \), let \( X_{j+1} \) denote the strict transform of \( X \) by the blowing-up \( \sigma_{j+1} \). (We are not assuming that the blowings-up \( \sigma_{j+1} \) are in any sense “admissible” for the strict transforms.)

**Corollary 6.4.** If \( a \in M_j \) and \( \text{inv}_{\mathcal{J}}(a) = (1,0; \ldots; 1,0; \infty) \), then \( a \in X_j \).

Proof. Let \( \pi_j : M_j \to M_0 = M \) denote the composite of the blowings-up \( \sigma_1, \ldots, \sigma_j \). By Lemma 6.3, \( \pi_j|_{N_t(a) \cup \{ H \in E(a) \}} \) is an isomorphism of \( N_t(a) \setminus \bigcup \{ H \in E(a) \} \) with an open set of smooth points of \( X \). The claim follows. \( \square \)

**Corollary 6.5 (\[EV3\]; see Example 1.19(4)).** Suppose that \( X \) is pure-dimensional (of dimension \( n-t \), say, where \( n = \dim M \)). Let \( \mathcal{J} = \mathcal{I}_X \). Then there is a finite sequence of \( \text{inv}_{\mathcal{J}} \)-admissible blowings-up (1.2) with smooth centres in the successive inverse images of \( \text{Sing} X \), such that:

1. The final strict transform \( X' = X_{j_0} \) is smooth.
2. \( X' \) and \( E' = E_{j_0} \) simultaneously have only normal crossings.
Proof. We simply apply the algorithm for principalization of $\mathcal{J} = \mathcal{I}_X$, stopping when the maximum value of the $\text{inv}_{\mathcal{J}}(\cdot)$ becomes $(1, 0; \ldots; 1, 0; \infty)$ (where there are $t$ pairs $(1, 0)$).

The precise statement of Lemma 6.3 immediately provides stronger versions of this result. Corollary 6.7 below, for example, is the theorem of [BrV]. The point is that, assuming (6.1), we can use Lemma 6.3 to completely describe the stratification by values of $\text{inv}_{\mathcal{J}}$, in a neighbourhood of $a$. We will need to work only with the largest value of $\text{inv}_{\mathcal{J}}$, apart from $\text{inv}_{\mathcal{J}}(a)$.

Lemma 6.6. Assume (6.1). Suppose that

$$\theta_{t-q-1} < \theta_{t-q} = \cdots = \theta_{t-1},$$

where $q \leq t - 1$ (and where $\theta_0$ means $0 \in \mathbb{N}^{n-t}$; we are using the notation of Lemma 6.3). Let $\tau(a)$ denote the largest value of $\text{inv}_{\mathcal{J}}$, apart from $\text{inv}_{\mathcal{J}}(a)$, in a small neighbourhood of $a$. Then

$$\tau(a) = (1, 0; \ldots; 1, 0; 0),$$

(6.2)

where there are $t - q$ pairs $(1, 0)$ in (6.2). The locus of points $x$ near $a$ where $\text{inv}_{\mathcal{J}}(x) = \tau(a)$ is given by

$$x_n = x_{n-1} = \cdots = x_{n-t+q+1} = 0,$$

$$x_{n-k} \neq 0, \quad \text{for some } k = t - q, \ldots, t - 1,$$

$$D_{t-q+1}(a) = \tilde{x}^{\theta_{t-q} - \theta_{t-q-1}} = 0.$$

(6.3)

This is a simple consequence of Lemma 6.3. Let $\Sigma = \{x : \text{inv}_{\mathcal{J}}(x) \geq \tau(a)\}$ (i.e., the closure of the locus (6.3)). Then $\Sigma$ is given by

$$x_n = x_{n-1} = \cdots = x_{n-t+q+1} = 0,$$

$$D_{t-q+1}(a) = 0.$$

Consider the local blowing-up $\sigma$ with centre given by any component of $\Sigma$; i.e., with centre

$$x_n = x_{n-1} = \cdots = x_{n-t+q+1} = 0,$$

$$x_i = 0, \quad \text{for some } x_i \text{ occurring in } D_{t-q+1}(a).$$

In the “$x_i$-chart” $U_{x_i}$ of this blowing-up, the weak transform $\mathcal{J}'_j$ of $\mathcal{J}_j$ is generated by

$$x_n, \tilde{x}^{\theta_1} x_{n-1}, \ldots, \tilde{x}^{\theta_{t-q-1}} x_{n-t+q+1}, \tilde{x}^{\theta_{t-q} - (i)} x_{n-t+q}, \ldots, \tilde{x}^{\theta_{t-q} - (i)} x_{n-t},$$

where (i) denotes the multiindex of length $n - t$ with 1 in the $i$'th place and 0 elsewhere. The strict transform $X'_j$ of $X_j$ is still given by

$$x_n = x_{n-1} = \cdots = x_{n-t+1} = 0$$

in the chart $U_{x_i}$. 
Corollary 6.7 (BrV). Suppose that $X$ is a closed subspace of $M$. Let $\mathcal{J} = \mathcal{I}_X$. Then there is a finite sequence of $\text{inv}_\mathcal{J}$-admissible blowings-up (1.2) with smooth centres in the successive inverse images of $\text{Sing} X$, such that:

1. The final strict transform $X' = X_{j_1}$ is smooth.
2. $X'$ and $E' = E_{j_1}$ simultaneously have only normal crossings.
3. The final total transform $\mathcal{J}_{\pi_{j_1}^{-1}(X)}$ is the product of $\mathcal{I}_{X_{j_1}}$ with a normal crossings divisor supported on the exceptional locus. ($\pi_{j_1}$ denotes the composite of the sequence of blowings-up.)

Proof. Let us first suppose that $X$ is pure-dimensional (of dimension $n - t$, say, where $n = \dim M$. We apply the algorithm for principalization of $\mathcal{J}$ as in Corollary 6.5, stopping (in year $j_0$, say) when the maximum value of $\text{inv}_\mathcal{J}(\cdot)$ becomes $(1, 0; \ldots; 1, 0; \infty)$ (where there are $n$ pairs $(1, 0)$).

We can now simply continue to blow up with centre determined by the maximum value of the extended invariant $\text{inv}_\mathcal{J}^e (\cdot)$ (cf. proof of Corollary 1.17) outside the strict transform of $X$. (For example, if the maximum stratum in $M_{j_0 \setminus X_{j_0}$ is not closed in $M_{j_0}$, then its closure is smooth and has only normal crossings with respect to $X_{j_0}$, by Lemma 6.6.) We continue to blow up with centre given by the closure of the locus of maximum values of $\text{inv}_\mathcal{J}^e (\cdot)$ outside the strict transform of $X$, until the support of the weak transform equals the support of the strict transform, say in year $j_1$: i.e., until at any point $a \in X_{j_1}$, $\mathcal{J}_{j_1, a}$ is generated by $x_n, x_{n-1}, \ldots, x_{n-t+1}$ (in suitable coordinates as in Lemma 6.3). In particular, $\mathcal{J}_{j_1} = \mathcal{I}_{X_{j_1}}$ and the assertion follows.

We can of course continue in this way to prove Corollary 6.7 (and also Corollary 6.5) without the assumption of pure-dimensionality. Suppose that $X$ has smooth parts of codimensions $t_1, t_2, \ldots, t_q$. We blow up first until the maximum value of the invariant is $(1, 0; \ldots; 1, 0; \infty)$ (with $t_1$ pairs $(1, 0)$), say on $Z(t_1)$. We then continue as above until the maximum value of the invariant outside the strict transform of $Z(t_1)$ is $(1, 0; \ldots; 1, 0; \infty)$ (with $t_2$ pairs $(1, 0)$), say on $Z(t_2)$. We now continue using the maximum value of the (extended) invariant on the complement of the strict transforms of $Z(t_1)$ and $Z(t_2)$, etc. \qed

The assertion of Corollary 6.7 is not restricted to the weak form of desingularization given by Corollary 6.5. Beginning with the conclusion of the embedded desingularization theorem (Example 1.19(1) above), we can transform to the product condition (3) (in Corollary 6.7) using
the following theorem (applied with \(X\) the final strict transform of our original closed subspace \((Y, \text{say})\) and with \(J\) the final weak transform of \(I_Y\)).

**Theorem 6.8.** Let \(X\) denote a smooth closed subspace of \(M\), and let \(E\) be a collection of smooth hypersurfaces in \(M\) such that \(X\) and \(E\) simultaneously have only normal crossings. Let \(J\) be an ideal of finite type in \(\mathcal{O}_M\) such that \(\text{supp} \mathcal{O}_M/J \subset X \cup E\) and \(J = I_X\) on \(X \setminus E\). Then there exists a finite sequence of transformations

\[
\begin{array}{cccc}
M_{i+1} & \xrightarrow{\sigma_{j+1}} & M_i & \cdots & \xrightarrow{} & M_0 = M \\
X_{j+1} & \xrightarrow{J_{j+1}} & X_j & \cdots & \xrightarrow{} & E_0 = E \\
\end{array}
\]

(6.4)

where, for each \(j\),

- \(\sigma_{j+1}\) is a blowing-up of \(M_j\) with smooth centre \(C_j\) that is \(\text{inv}_J\)-admissible, is supported in \(E_j\), and simultaneously has only normal crossings with respect to \(X_j\) and \(E_j\),
- \(X_{j+1}\) denotes the strict transform of \(X_j\),
- \(J_{j+1}\) denotes the weak transform of \(J_j\),

and such that the final transforms \(X'\) and \(J'\) satisfy

\[J' = I_{X'};\]

thus, the final total transform \(\pi^{-1}(I_X)\) is the product of \(I_{X'}\) with a normal crossings divisor supported in \(E'\).

**Proof.** We define a new invariant \(\text{inv}_{J,X}(\cdot)\) inductively over a sequence (6.4), in the following way. Let \(a \in M_j\). Set \(\text{inv}_{1/2}(a) = \iota(a)\), where

\[\iota(a) := (\text{inv}_J(a), \delta_{X_j}(a));\]

and

\[\delta_{X_j}(a) = \begin{cases} 
1, & a \in X_j \\
0, & a \notin X_j.
\end{cases}\]

The \(\iota_{1/2} = \iota\) satisfies Hypotheses 1.10(1) and (2), as well as (a slight variant of) (3): If \(\mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a))\) is a semicoherent presentation of \(\text{inv}_J\) at \(a\) (the term \(\mathcal{E}(a)\) is irrelevant here), then \(\iota_{1/2}\) has a semicoherent presentation at \(a\) given by

\[(N(a), \mathcal{H}(a) \cup \{(g_k, 1)\}, \mathcal{E}_1(a));\]

where \(\{g_k\}\) is the set of restrictions to \(N(a)\) of a finite set of generators of \(\mathcal{I}_{X_j,a}\) with linearly independent gradients (and \(\mathcal{E}_1(a) = E(a) \setminus E^1(a)\), where \(E^1(a)\) is determined using \(\text{inv}_{1/2}\) according to Definitions 1.15, as usual). (Although this semicoherent presentation of \(\iota_{1/2}\) is in the
weaker sense of Remark 3.5, it suffices to begin the inductive construction as in §3.2 because its semicoherent equivalence class depends only on $J_{j,a}, X_{j,a}$ and the various blocks of exceptional divisors involved in defining $\text{inv}_J(a)$.

Then $\text{inv}_{1/2}$ extends to an invariant $\text{inv}_J, X$ defined inductively over a sequence of blowings-up (6.4) with $\text{inv}_J, X$-admissible centres, according to the desingularization principle Theorem 1.14. If $a \in X_j \setminus E_j$, then

$$\text{inv}_J(a) = (1,0; \ldots ; 1,0; \infty)$$

$((1,0) occuring t times, where $t = \text{codim}_a X_j$), and

$$\text{inv}_{J,X}(a) = ((\text{inv}_J(a), 1), 0; \infty).$$

On the other hand, if $a \in X_j$ and

$$\text{inv}_{J,X}(a) = (((1,0; \ldots ; 1,0; \infty), 1), 0; \infty),$$

then $\text{inv}_J(a) = (1,0; \ldots ; 1,0; \infty)$, so that Lemma 6.3 applies. Hence we can blow up the maximum locus of $\text{inv}_{\text{J,X}}^e$ until $\text{inv}_{J,X}$ is constant on $X_j$ (at least when $X$ is pure-dimensional), and then argue as in the proof of Corollary 6.7; the general (not necessarily pure-dimensional) case follows as in the latter. 

6.3. Simultaneous desingularization of parametrized families.

A family of spaces parametrized by a space $T$ means a morphism $p : X \to T$ (e.g. a morphism of schemes of finite type over a field of characteristic zero, or a morphism of analytic spaces). The fibres $X_t$, $t \in T$, form a family of closed subspaces of $X$.

Definitions 6.9. Let $p : M \to T$ be a morphism of smooth spaces (manifolds). We say that $p$ is smooth at $a \in M$ if there is a system of local coordinates $(x_1, \ldots, x_n)$ in a neighbourhood of $a$, in which $p$ is a projection onto a coordinate subspace $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k)$ (where $k \leq n$). We say that $p$ is smooth if it is smooth at every point of $M$.

Let $E$ denote a configuration (finite collection) of smooth closed subspaces of $M$ that simultaneously have only normal crossings. We say that the morphism $p$ restricts to a smooth projection from $E$ to $T$ if $p$ restricts to a smooth morphism of every element of $E$ and of every intersection of elements of $E$. (There is a more general notion of “smooth morphism” between spaces that are not necessarily smooth [Ha, Chapt. III, Sect. 10] which we will not need; it is equivalent to the preceding when the target space is smooth and the source has only normal crossings.)
**Definitions 6.10.** Simultaneous resolution of singularities of a family of spaces $X \to T$ means (some version of) resolution of singularities of $X$ by blowings-up with smooth centres such that all centres and also the final strict transform of $X$ project smoothly to $T$.

Simultaneous embedded desingularization means simultaneous resolution of singularities with the additional property that, at each step, the entire configuration of exceptional divisors together with the centre of blowing up (or together with the final strict transform of $X$) projects smoothly to $T$.

A simultaneous resolution of singularities (respectively, simultaneous embedded resolution of singularities) of $X \to T$ restricts to a resolution of singularities (respectively, embedded resolution of singularities) of every fibre $X_t$.

Any theorem of resolution of singularities by blowings-up with smooth centres immediately implies that the parameter-space of a proper family can be stratified so that the fibres can be simultaneously desingularized over every stratum (cf. [PS, Thm. 4, Ref. BM99], [ENV, Sect. 4]):

**Theorem 6.11.** Let $p : X \to T$ denote a proper morphism. Then there is a finite filtration by closed subsets,

$$T = T_0 \supset T_1 \supset \cdots \supset T_l = \emptyset,$$

such that, for all $k$, $U_k := T_k \setminus T_{k+1}$ is smooth and the family $X_k|_{U_k} \to U_k$ admits simultaneous (embedded) desingularization.

**Proof.** We can assume the $p$ is surjective. Consider (embedded) resolution of singularities of $X$. By the “generic smoothness theorem” (cf. [Ha, Cor. 10.7]), there is a proper closed subspace $T_1$ of $T$ such that $T \setminus T_1$ is smooth and (over $T \setminus T_1$) all centres of blowing up, as well as the final strict transform of $X$, project smoothly onto $T \setminus T_1$ (and, at every step, the collection of exceptional divisors together with the centre, or together with the final strict transform of $X$, projects smoothly onto $T \setminus T_1$). Thus the family of fibres admits simultaneous (embedded) desingularization over $T \setminus T_1$. The result follows by induction.

**Remark 6.12.** We have not stated this theorem with the precisions that are evidently needed to cover all categories. For example, in the complex-analytic category, we should either apply the statement to a relatively compact open subset of $T$, or use a locally finite filtration; in the real-analytic case, we should either assume that $p$ admits a proper complexification, or use a semianalytic filtration of $T$. 

Our main purpose in this section is to give a more precise version of Theorem 6.11 by extending Theorem 1.14 to a “desingularization principle for families” (Theorem 6.15 below). The Hilbert-Samuel function of \( X \), as well as the order of the ideal \( \mathcal{I}_X \) satisfy the Hypotheses 6.13 below that are needed for Theorem 6.15. (See Lemma 6.14 and Theorem 6.17). The theorem of [ENV] uses the order of \( \mathcal{I}_X \), so Theorems 6.15 and 6.17 extend the latter to the stronger form of embedded desingularization (cf. Examples 1.19).

**Notation.** Let \( X \to T \) be a family of spaces, as above. Let \( t \in T \). We let \( X_t \) denote the fibre over \( t \), and set
\[
X_{(t)} := X_t \times (\text{germ of } T \text{ at } t).
\]
(It is more convenient to use \( X_{(t)} \) rather than \( X_t \) in comparing the Hilbert-Samuel functions (or other invariants) of \( X \) and of \( X_t \) at a given point in \( X_t \).) If \( a \in X \), let \( t(a) \) denote the image of \( a \) in \( T \).

If \( X \to T \), then, locally in \( X \), there is an embedding \( X \hookrightarrow M \) in a manifold \( M \), together with a smooth morphism \( M \to T \) that restricts to the given projection \( X \to T \). In Hypotheses 6.13 and Theorem 6.15 following, we will therefore assume that \( X \hookrightarrow M \) and that \( X \to T \) is induced by a smooth projection \( p : M \to T \). As in §6.1, however, our results will be independent of the embedding.

Let \( \mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a)) \) denote a local presentation of codimension \( q \) at \( a \in M \). If \( \mathcal{E}(a) \) and \( N(a) \) together map smoothly to \( T \), then we can define the **restriction of** \( \mathcal{H}(a) \) **to the fibre** \( M_{t(a)} \) in an obvious way: Let \( t = t(a) \) and let \( \mathcal{H}(a)_t := (N(a)_t, \mathcal{H}(a)_t, \mathcal{E}(a)_t) \), where \( N(a)_t \) is the fibre of \( N(a) \to T, \mathcal{E}(a)_t := \{ H_t : H \in \mathcal{E}(a) \} \), where \( H_t \) denotes the fibre of \( H \to T \), and
\[
\mathcal{H}(a)_t = \{ (h_t, \mu_{h_t}) : (h, \mu_h) \in \mathcal{H}(a) \},
\]
where \( h_t := h|_{N(a)_t} \).

**Hypotheses 6.13.** We will assume that \( \iota_X \) satisfies Hypotheses 1.10 (1) and (2), together with the following additional property:

(3) Let \( a \in X \) and let \( t = t(a) \). Then:
   
   (a) \( \iota_{X_{(t)}}(a) \geq \iota_X(a) \).
   
   (b) If \( \iota_{X_{(t)}}(a) = \iota_X(a) \), then there is a semicoherent presentation
   
   \[
   \mathcal{H}(a) = (N(a), \mathcal{H}(a), \emptyset)
   \]
   
   of \( \iota_X \) at \( a \), of codimension \( q = q(a) \), say, such that \( N(a) \to T \) is smooth, \( \mathcal{H}(a)_t \) is a semicoherent presentation (of codimension \( q \)) of \( \iota_{X_{(t)}} \) at \( a \), and \( \mu_{\mathcal{H}(a)_t} = \mu_{\mathcal{H}(a)} \).
(c) If there is a (germ of a) smooth subspace $S \subset S_{\iota_X}$ such that $S \to T$ is smooth, then $\iota_{X(t)}(a) = \iota_X(a)$.

**Lemma 6.14.** let $\iota_X(a)$ denote the order $\mu_a(J)$ of $J := I_X$ (cf. Definitions 1.4 and Examples 1.8). Then $\iota_X$ satisfies Hypotheses 6.13.

**Proof.** By Example 1.13(1), $\iota_X$ satisfies Hypotheses 1.10; we have to verify 6.13(3). Choose coordinates $(x_1, \ldots, x_n)$ for $M$ in a neighbourhood of $a = 0$, in which $p$ is a projection $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-r})$; write $u := (x_1, \ldots, x_{n-r}), \ v := (x_{n-r+1}, \ldots, x_n)$. If $\{g_i(u, v) : i = 1, \ldots, q\}$ is a set of generators of $J_a$, then $J_{(t)}$ is generated by $\{g_i(0, v)\}$.

Therefore, properties (a) and (b) are obvious.

To prove (c): Consider a smooth germ $S \subset S_{\mu,(J)}(a)$ such that $S \to T$ is smooth. Then $\mu_{S,a}(J) = \mu_a(J)$ (because $\mu(J)$ is constant on $S$; cf. Definitions 1.4). Let $d = \mu_a(J)$ and write $g_i(u, v) = \sum_{\alpha \in \mathbb{N}^r, |\alpha| \geq d} g_{i,\alpha}(u)v^\alpha, \ i = 1, \ldots, q$.

Then $g_{i,\alpha}(0) \neq 0$, for some $i$ and some $\alpha$ such that $|\alpha| = d$. Therefore, $\mu_a(J_{(t)}) = d$. \qed

**Theorem 6.15.** Suppose that $\iota_X$ satisfies Hypotheses 6.13. Consider any sequence of inv-admissible local blowings-up (1.1). Let $a \in X_j$ and let $a_i$ denote the image of $a$ in $X_i$, $i \leq j$. Assume that, for all $i < j$, $E_i$ together with $C_i$ maps smoothly to $T$ at $a_i$, and $\text{inv}_{X(t)}(a_i) = \text{inv}_X(a_i)$. (In particular, the sequence up to year $j$ induces an inv-admissible sequence (locally at the points $a_i$) on the fibres over $t$.) Then

$$\text{inv}_{X(t)}(a) \geq \text{inv}_X(a),$$

and the following conditions are equivalent:

1. $\text{inv}_{X(t)}(a) = \text{inv}_X(a)$.
2. There is a smooth subspace (germ) $S$ of $S_{\text{inv}_X}(a)$ such that $S \to T$ is smooth.
3. $S_{\text{inv}_X}(a)$ maps smoothly to $T$.

By $\text{inv}_{X(t)}$, we mean the invariant for the fibre $X_t$ given by the desingularization principle Theorem 1.14, beginning with $\text{inv}_{1/2} = \iota_{X(t)}$. The assumption that $\text{inv}_{X(t)}(a_i) = \text{inv}_X(a_i), \ i < j$, is, in fact, redundant – it follows by induction from the conclusion of the theorem.

**Proof of Theorem 6.15.** We will follow the local inductive construction of §3.2 (or, more generally, §6.1), but we will use Villamayor's variant of this construction (as described in §3.4), to extend the properties of
Hypotheses 6.13 to the truncated invariants $\text{inv}_{r-1/2}(a)$ and $\text{inv}_r(a)$, for each successive $r$. (See Remark 6.16.)

Note first that, by the hypotheses, $E(a) \rightarrow T$ is smooth and, if $\iota_{X(t)}(a) = \iota_X(a)$, then there is a semicoherent presentation

$$\mathcal{H}(a) = (N(a), \mathcal{H}(a), \mathcal{E}(a))$$

of $\iota_X$ at $a$, satisfying the obvious generalization of property 6.13(3)(b): $\mathcal{H}(a)_t = (N(a)_t, \mathcal{H}(a)_t, \mathcal{E}(a)_t)$ is a presentation of $\text{inv}_{1/2} = \iota_{X(t)}$ at $a$ (for the fibre $X_{s,t}$).

Now consider $r \geq 1$, and assume that $\text{inv}_{r-1/2}$ satisfies the analogues of Hypotheses 1.10(1) and 6.13(3) (where, in (3)(b), there is a semicoherent presentation)

$$\mathcal{H}_r(a) = (N_r(a), \mathcal{H}_r(a), \mathcal{E}_r(a))$$

of $\text{inv}_{r-1/2}$ at $a$, that restricts to a semicoherent presentation

$$\mathcal{H}_r(a)_t = (N_r(a)_t, \mathcal{H}_r(a)_t, \mathcal{E}_r(a)_t)$$

of $\text{inv}_{r-1/2,t}$ (the invariant for the fibre) at $a$). We then have to show that $\text{inv}_r$ satisfies the analogues of 6.13(3)(a)-(c). (For this, it is enough to assume that the previous centres of blowing up are $(r - 1/2)$-admissible and map smoothly to $T$.)

First consider (a) and (b). If $\text{inv}_{r-1/2,t}(a) > \text{inv}_{r-1/2}(a)$, then there is nothing more to do. Assume $\text{inv}_{r-1/2,t}(a) = \text{inv}_{r-1/2}(a)$. Then automatically $E^r_t(a) = E^r(a)$ (more precisely, $E^r_t(a) = \{ H|_{X_t} : H \in E^r(a) \}$, where $E^r_t(a)$ denotes the analogue of $E^r(a)$ for the fibre $X_t$), and $s_{r,t}(a) = s_r(a)$ (where $s_{r,t}(a)$ is again the analogue of $s_r(a)$ for the fibre). (In fact, these equalities hold at $a_i$, for all $i \leq j$.) It follows that $\text{inv}_{r,t}(a) = \text{inv}_r(a)$, and there is a semicoherent presentation

$$\mathcal{C}_r(a) = (N_r(a), \mathcal{C}_r(a), \mathcal{E}_r(a))$$

of $\text{inv}_r$ at $a$, with the properties required for (b).

Property (c) is obvious because, if $S \subset S_{\text{inv}_r}(a)$ is a smooth germ such that $S \rightarrow T$ is smooth, then $\text{inv}_{r-1/2,t}(a) = \text{inv}_{r-1/2}(a)$, by property (c) for $\text{inv}_{r-1/2}$, so that $s_{r,t}(a) = s_r(a)$, as above. This completes the step from $\text{inv}_{r-1/2}$ to $\text{inv}_r$.

Now assume that $\text{inv}_r$ satisfies the analogues of Hypotheses 1.10(1) and 6.13(3); in particular, in (b), there is a semicoherent presentation $\mathcal{C}_r(a)$ of $\text{inv}_r$ at $a$, as above, that restricts to a semicoherent presentation of $\text{inv}_{r,t}$ at $a$. It is enough to assume that the previous centres of blowing up are $\text{inv}_{r+1/2}$-admissible and map smoothly to $T$. 

If \( \text{inv}_{r,t}(a) > \text{inv}_r(a) \), then again there is nothing to do. Assume that \( \text{inv}_{r,t}(a) = \text{inv}_r(a) \). Then

\[
\mu_{r+1,t}(a) \geq \mu_{r+1}(a),
\]

by property (b) for \( \text{inv}_r \), because \( \mu_{r+1}(a) \) and \( \mu_{r+1,t}(a) \) are realized as multiplicities of a given function, before and after restriction to the fibre \( X_t \) (cf. Lemma 6.14). On the other hand, for every \( H \in \mathcal{E}_r(a) \),

\[
\mu_{r+1,H,t}(a) = \mu_{r+1,H}(a),
\]

by induction over the sequence of blowings-up, using Lemma 3.7 and the assumption on the previous centres. Therefore,

\[
\nu_{r+1,t}(a) \geq \nu_{r+1}(a),
\]

so we have proved (a).

If \( \text{inv}_{r,t}(a) = \text{inv}_r(a) \) and \( \nu_{r+1,t}(a) = \nu_{r+1}(a) \), then the presentation \( G_{r+1}(a) \) of \( \text{inv}_{r+1/2} \) at \( a \) constructed as in §§3.2, 3.4, restricts to a presentation \( G_{r+1,t}(a) \) of \( \text{inv}_{r+1/2,t}(a) \). Thus (b) is satisfied.

To prove (c), let \( S \subset S_{\text{inv}_{r+1/2}}(a) \) be a smooth germ such that \( S \to T \) is smooth. By property (c) for \( \text{inv}_r \), \( \text{inv}_{r,t}(a) = \text{inv}_r(a) \). Then, by (6.5), in order to prove that \( \nu_{r+1,t}(a) = \nu_{r+1}(a) \), it is enough to prove that \( \mu_{r+1,t}(a) = \mu_{r+1}(a) \). The latter is a statement about the order of a function (or an ideal), established by Lemma 6.14. This completes the step from \( \text{inv}_r \) to \( \text{inv}_{r+1/2} \).

Finally, the full invariant \( \text{inv}_X \) satisfies the analogue of property 6.13(3), and the conclusion of the theorem follows immediately.

Remark 6.16. Recall, from §§3.2, 3.3, that, in order to define \( \nu_{r+1} \) and prove the semicontinuity properties of \( \text{inv}_{r+1/2} \), it is enough to assume that the previous centres of blowing up are \((r - 1/2)\)-admissible. We need the stronger \((r + 1/2)\)-admissibility assumption on the previous centres in the step from \( \text{inv}_r \) to \( \text{inv}_{r+1/2} \) in the proof of Theorem 6.15 in order to prove the semicontinuity condition (6.6) using (6.5). This is the reason for using Villamayor’s smaller block of exceptional divisors \( \mathcal{E}'(a) \) – Lemma 3.7 shows that, for \( H \) in the smaller block, \( \mu_{r+1,H}(a) \) can be computed using the values in previous years. It would seem that (6.5) need not hold for the additional exceptional divisors that are factored out to define the residual multiplicities according to the inductive construction in the Bierstone-Milman algorithm.

Corollary 6.17. Suppose that \( \iota_X \) satisfies Hypotheses 6.13. Consider a desingularization of \( X \) by a finite sequence of \( \text{inv}_X \)-admissible blowings-up (1.1). Then the following conditions are equivalent:
1. All centres of blowing up \( C_i \) (as well as the final strict transform of \( X \)) project smoothly to \( T \).
2. \( \text{inv}_{X(t(a))}(a) = \text{inv}_{X}(a) \) for all \( a \in M_{i}, i = 0, 1, \ldots \).

This is an immediate consequence of Theorem 6.15. Corollary 6.17 provides a more precise version of Theorem 6.11 (by using the generic smoothness theorem as in the proof of the latter).

**Theorem 6.18.** Let \( \iota_{X}(a) \) denote the Hilbert-Samuel function \( H_{X,a} \) (cf. Examples 1.8). Then \( \iota_{X} \) satisfies Hypotheses 6.13.

**Proof.** By Examples 1.13(2), \( \iota_{X} \) satisfies Hypotheses 1.10. We have to verify properties (3)(a)-(c) of Hypotheses 6.13. Let \( a \in X \) and let \( t = t(a) \).

(a) \( H_{X(t),a} \geq H_{X,a} \), by an elementary lemma [BM3, Lemma 7.5].

We will establish properties (b) and (c) (in fact, we will first prove (c)) using a semicoherent presentation of the Hilbert-Samuel function constructed in [BM3, Theorems 9.4, 9.6] (so an understanding of the following argument requires some familiarity with the latter). We will use the notation of [BM3, (7.1)]; \( \mathfrak{N}(\cdot) \) will denote the diagram of initial exponents of an ideal in a ring of formal power series, as defined, for example, in [BM3, Sect. 3]. Let \( K \) denote the underlying ground field of our space. (In the following arguments in the case of schemes, \( K \) should really be understood as the residue field of the point \( a \); see [BM3, Remark 3.8] for an explanation of this point.)

(c) Let \( S \subset S_{H_{X,a}}(a) \) denote a germ of a smooth subset such that \( S \rightarrow T \) is smooth. Consider a presentation \( \mathfrak{H}(a) = (N(a), \mathcal{H}(a), \emptyset) \) of \( H_{X,a} \) at \( a \), satisfying the hypotheses of [BM3, Theorem 9.4]. Then \( N(a) \rightarrow T \) is smooth, since \( S \subset N(a) \) and \( S \rightarrow T \) is smooth.

Claim. If \( N(a) \rightarrow T \) is smooth, then \( H_{X(t),a} = H_{X,a} \).

To prove this claim: We use the notation of [BM3, Theorem 9.4]. In particular, \( w = (w_{1}, \ldots, w_{n-r}) \) represents a coordinate system on \( N(a), \hat{\mathcal{I}}_{X,a} \subset \hat{O}_{M,a} = K[W,Z] \), where \( W = (W_{1}, \ldots, W_{n-r}) \) and \( Z = (Z_{1}, \ldots, Z_{r}) \), and the vertices of \( \mathfrak{N}(\hat{\mathcal{I}}_{X,a}) \subset \mathbb{N}^{n} \) depend only on the \( Z \)-coordinates; i.e., \( \mathfrak{N}(\hat{\mathcal{I}}_{X,a}) = \mathbb{N}^{n-r} \times \mathfrak{N}^{*} \), where \( \mathfrak{N}^{*} \subset \mathbb{N}^{r} \). Moreover, since \( N(a) \rightarrow T \) is smooth, we can assume that the coordinates \( w \) split as \( w = (u, v) \), where \( (u, v) \mapsto u \) is the projection \( N(a) \rightarrow T \), and that \( \hat{\mathcal{I}}_{X,a} \subset K[W,Z] = K[U,V,Z] \). Since \( \hat{\mathcal{I}}_{X(t),a} \) is obtained from \( \hat{\mathcal{I}}_{X,a} \) by setting \( U = 0 \),

\[
\mathfrak{N}(\hat{\mathcal{I}}_{X(t),a}) = \mathfrak{N}(\hat{\mathcal{I}}_{X,a}).
\]
Then, by [BM5, Lemma 7.5],
\[ H_{X(t),a} = H_{X,a} . \]

(b) Assume that \( H_{X(t),a} = H_{X,a} \). Choose local coordinates \((u, v) = (u_1, \ldots, u_{n-s}, v_1, \ldots, v_s)\) for \( M \) at \( a \) such that \((u, v) \mapsto u\) is the projection to \( T \). Set \( J_1 := \hat{\mathcal{I}}_{X,a} \subset \mathbb{K}[u, v] \) and \( J_0 := \hat{\mathcal{I}}_{X(t),a} \); thus \( J_0 \) is the ideal in \( \mathbb{K}[u, v] \) generated by the evaluations at \( u = 0 \) of the elements of \( J_1 \). Write \( H_{J_1} \) and \( H_{J_0} \) for the Hilbert-Samuel functions of the quotients of \( \mathbb{K}[u, v] \) by \( J_1 \) and \( J_0 \) (respectively); i.e., for \( H_{X,a} \) and \( H_{X(t),a} \) (respectively). Let \( m \) denote the maximal ideal of \( \mathbb{K}[u, v] \).

We will first show that
\[ N(\hat{\mathcal{I}}_{X(t),a}) = N(\hat{\mathcal{I}}_{X,a}) . \]

Let \( \mathfrak{N}^* \subset N^s \) denote the diagram of initial exponents of \( \hat{\mathcal{I}}_{X,t,a} \subset \mathbb{K}[v] \), so that the diagram \( \mathfrak{N} = \mathfrak{N}(J_0) \) is
\[ \mathfrak{N} = N^{n-s} \times \mathfrak{N}^* \subset N^n . \]
Then
\[ \mathbb{K}[u, v] = J_0 \oplus \mathbb{K}[u, v]^{\mathfrak{N}} , \]
where \( \mathbb{K}[u, v]^{\mathfrak{N}} \) denotes the subset of formal power series supported in the complement of \( \mathfrak{N} \), by Hironaka’s formal division theorem [BM5, Theorem 3.17], and
\[ \mathbb{K}[u, v] = J_1 + \mathbb{K}[u, v]^{\mathfrak{N}} \]
(where the sum is not necessarily direct), by the formal division theorem with parameters. (See [BM1, Theorem 3.1], [Ga].) By the assumption,
\[ H_{J_1} = H_{J_0} = H_{\mathfrak{N}} , \]
where
\[ H_{\mathfrak{N}}(k) := \#\{\alpha \in N^n : \alpha \notin \mathfrak{N}, |\alpha| \leq k\}, \quad k \in \mathbb{N} . \]

Claim. \( J_1 \cap \mathbb{K}[u, v]^{\mathfrak{N}} = \{0\} \). Moreover, if \( f = g + h \), where \( f \in m^k \), \( g \in J_1 \) and \( h \in \mathbb{K}[u, v]^{\mathfrak{N}} \), then \( g \in m^k \) and \( h \in m^k \).

The claim implies (6.7); i.e., \( \mathfrak{N}(J_1) = \mathfrak{N}(J_0) \): Consider any vertex \( \alpha \) of \( \mathfrak{N}^* \). Using (6.9) and the claim, we can write
\[ v^\alpha = f_\alpha(u, v) + r_\alpha(u, v) , \]
where \( f_\alpha(u, v) \in J_1, r_\alpha(u, v) \in \mathbb{K}[u, v]^{\mathbb{N}}, \) and the order of any monomial in \( r_\alpha(u, v) \) is at least \(|\alpha|\). Set \( u = 0 \) in (6.11). Then

\[
v^\alpha = f_\alpha(0, v) + r_\alpha(0, v).
\]

Thus, \( f_\alpha(0, v) \) is the element of the standard basis of \( J_0 \) representing the vertex \( \alpha \); in particular, every monomial in \( r_\alpha(0, v) \) has exponent \( (0, \alpha) > (0, \alpha) \), according to the formal division theorem (where the elements \( \delta \in \mathbb{N}^n \) are ordered according to the lexicographic order of \((|\delta|, \delta)\). See [BM5, Corollary 3.19].) It follows that any monomial in \( r_\alpha(u, v) \) has exponent \( > (0, \alpha) \): Consider a monomial \( u^\beta v^\gamma \in r_\alpha(u, v) \), where \(|\beta| + |\gamma| = |\alpha|\). If \( \beta \neq 0 \), then \((\beta, \gamma) > (0, \alpha)\), by the definition of the ordering. On the other hand, if \( \beta = 0 \), then \((0, \gamma) > (0, \alpha)\), since all exponents of \( r_\alpha(0, v) \) are \( > (0, \alpha) \). Therefore, the initial exponent (the smallest exponent) of \( f_\alpha(u, v) \) is \((0, \alpha)\). Hence

\[
\mathfrak{N}(J_1) \supset \mathfrak{N}(J_0) = \mathfrak{N}.
\]

and it follows from (6.10) that \( \mathfrak{N}(J_1) = \mathfrak{N}(J_0) \).

Proof of the preceding claim. For each \( k \in \mathbb{N} \), there is a surjective homomorphism

\[
\begin{align*}
\mathbb{K}[u, v]^{\mathbb{N}} & \twoheadrightarrow \mathbb{K}[u, v]^{\mathbb{N}} \cap (J_1 + m^{k+1}) \\
& \hookrightarrow \mathbb{K}[u, v]^{\mathbb{N}} \cap (J_1 + m^{k+1}).
\end{align*}
\]

Therefore,

\[
H_{\mathfrak{N}}(k) = \dim \frac{\mathbb{K}[u, v]^{\mathbb{N}}}{\mathbb{K}[u, v]^{\mathbb{N}} \cap m^{k+1}} \\
\geq \dim \frac{\mathbb{K}[u, v]^{\mathbb{N}}}{\mathbb{K}[u, v]^{\mathbb{N}} \cap (J_1 + m^{k+1})} \\
\geq \dim \frac{\mathbb{K}[u, v]}{J_1 + m^{k+1}} \\
= H_{J_1}(k) = H_{\mathfrak{N}}(k),
\]

so all terms are equal. Hence, for every \( k \), (6.12) is an isomorphism, and

\[
\mathbb{K}[u, v]^{\mathbb{N}} \cap (J_1 + m^{k+1}) = \mathbb{K}[u, v]^{\mathbb{N}} \cap m^{k+1}.
\]

So

\[
J_1 \cap \mathbb{K}[u, v]^{\mathbb{N}} \subset \mathbb{K}[u, v]^{\mathbb{N}} \cap m^{k+1},
\]

for all \( k \); therefore \( J_1 \cap \mathbb{K}[u, v]^{\mathbb{N}} = 0 \); i.e.,

\[
\mathbb{K}[u, v] = J_1 \oplus \mathbb{K}[u, v]^{\mathbb{N}}.
\]
Moreover, if \( f = g + h \), where \( f \in \mathcal{m}^k \), \( g \in J_1 \) and \( h \in \mathbb{K}[u,v]^{\mathcal{m}} \cap \mathcal{m}^k \), by (6.13); i.e., \( h \in \mathcal{m}^k \), so \( g \in \mathcal{m}^k \). This proves the claim.

We have proved that the standard basis of \( J_1 \) gives the standard basis of \( J_0 \) when we set \( u = 0 \). This gives the variant of (b) where the presentation is merely \textit{formal}. We can, in fact, prove that the semicoherent presentation of the Hilbert-Samuel function constructed in \([BM5, \text{Theorem 9.6}]\) satisfies property (b) as stated:

We have shown that \( \mathfrak{N}(\widehat{I}_{X,a}) = \mathfrak{N} \) is a product \( \mathbb{N}^{n-s} \times \mathfrak{N}^* \), where \( \mathfrak{N}^* \subset \mathbb{N}^s \) (corresponding to the \( v \)-coordinates). Let \( \mathcal{H}(a) = (N(a), \mathcal{H}(a), \emptyset) \) denote the semicoherent presentation constructed in \([BM5, \text{Theorem 9.6}]\); then the maximal contact submanifold \( N(a) \) has coordinates \( w = (u, w_1) \) such that \( (u, w_1) \mapsto u \) is the mapping \( N(a) \rightarrow T \) (in particular, this mapping is smooth). Because of this and the structure of \([BM5, \text{Theorem 9.4}]\), we get a semicoherent presentation of \( H_{X(t)} \) at \( a \) by setting \( u = 0 \). To see this, it is enough to observe that the properties of \([BM5, \text{Theorem 9.4}]\) survive on setting certain of the \( w \)-coordinates in the latter equal to 0. (In particular, the formal properties \([BM5, (7.2)(1)-(5)]\) survive on setting certain of the formal \( W \)-coordinates in \([BM5, \text{Theorem 9.4}]\) equal to 0.) This completes the proof. \( \square \)

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