THE TORSION SUBGROUP OF THE ADDITIVE GROUP
OF A LIE NILPOTENT ASSOCIATIVE RING OF CLASS 3

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ABSTRACT. Let $\mathbb{Z}\langle X \rangle$ be the free unital associative ring freely generated by an infinite countable set $X = \{x_1, x_2, \ldots \}$. Define a left-normed commutator $[x_1, x_2, \ldots, x_n]$ by $[a, b] = ab - ba$, $[a, b, c] = [[a, b], c]$. For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $\mathbb{Z}\langle X \rangle$ generated by all commutators $[a_1, a_2, \ldots, a_n]$ ($a_i \in \mathbb{Z}\langle X \rangle$). Let $T^{(3,2)}$ be the two-sided ideal of the ring $\mathbb{Z}\langle X \rangle$ generated by all elements $[a_1, a_2, a_3, a_4]$ and $[a_1, a_2][a_3, a_4, a_5]$ ($a_i \in \mathbb{Z}\langle X \rangle$).

It has been recently proved in [22] that the additive group of $\mathbb{Z}\langle X \rangle/T^{(4)}$ is a direct sum $A \oplus B$ where $A$ is a free abelian group isomorphic to the additive group of $\mathbb{Z}\langle X \rangle/T^{(3,2)}$ and $B = T^{(3,2)}/T^{(4)}$ is an elementary abelian 3-group. A basis of the free abelian summand $A$ was described explicitly in [22]. The aim of the present article is to find a basis of the elementary abelian 3-group $B$.

1. Introduction

Let $\mathbb{Z}\langle X \rangle$ be the free unital associative ring freely generated by an infinite countable set $X = \{x_i \mid i \in \mathbb{N}\}$. Then $\mathbb{Z}\langle X \rangle$ is the free $\mathbb{Z}$-module with a basis $\{x_{i_1}x_{i_2}\ldots x_{i_k} \mid k \geq 0, i_j \in \mathbb{N}\}$ formed by the non-commutative monomials in $x_1, x_2, \ldots$. Define a left-normed commutator $[x_1, x_2, \ldots, x_n]$ by $[a, b] = ab - ba$, $[a, b, c] = [[a, b], c]$. For $n \geq 2$, let $T^{(n)}$ be the two-sided ideal in $\mathbb{Z}\langle X \rangle$ generated by all commutators $[a_1, a_2, \ldots, a_n]$ ($a_i \in \mathbb{Z}\langle X \rangle$). Note that the quotient ring $\mathbb{Z}\langle X \rangle/T^{(n)}$ is the universal Lie nilpotent associative ring of class $n - 1$ generated by $X$.

It is clear that the quotient ring $\mathbb{Z}\langle X \rangle/T^{(2)}$ is isomorphic to the ring $\mathbb{Z}[X]$ of commutative polynomials in $x_1, x_2, \ldots$. Hence, the additive group of $\mathbb{Z}\langle X \rangle/T^{(2)}$ is free abelian and its basis is formed by the (commutative) monomials. Recently Bhupatiraju, Etingof, Jordan, Kusznial and Li [5] have proved that the additive group of $\mathbb{Z}\langle X \rangle/T^{(3)}$ is also free abelian and found explicitly its basis [5, Prop. 3.2]. Very recently in [22] the second author of the present article has proved that the additive group of $\mathbb{Z}\langle X \rangle/T^{(4)}$ is a direct sum $A \oplus B$ of a free abelian group $A$ and an elementary abelian 3-group $B$.

More precisely, let $T^{(3,2)}$ be the two-sided ideal of the ring $\mathbb{Z}\langle X \rangle$ generated by all elements $[a_1, a_2, a_3, a_4]$ and $[a_1, a_2][a_3, a_4, a_5]$ where $a_i \in \mathbb{Z}\langle X \rangle$. Clearly, $T^{(4)} \subseteq T^{(3,2)}$. It has been proved in [22] that $T^{(3,2)}/T^{(4)}$ is a non-trivial elementary abelian 3-group and the additive group of $\mathbb{Z}\langle X \rangle/T^{(3,2)}$ is
free abelian. It follows that the additive group of $\mathbb{Z}(X)/T^{(4)}$ is a direct sum $A \oplus B$ where $B = T^{(3,2)}/T^{(4)}$ is an elementary abelian 3-group and $A$ is a free abelian group isomorphic to the additive group of $\mathbb{Z}(X)/T^{(3,2)}$. A $\mathbb{Z}$-basis of the additive group of $\mathbb{Z}(X)/T^{(3,2)}$ was explicitly written in [22] Lemma 5.6; this basis can be easily deduced from the results of either [12] or [15] or [27]. The aim of the present article is to find a basis over $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ of the elementary abelian 3-group $T^{(3,2)}/T^{(4)}$.

Let $v_k = [x_1, x_2] \ldots [x_{2k-1}, x_{2k}][x_{2k+1}, x_{2k+2}, x_{2k+3}]$ ($k \geq 1$). One can deduce from the results of [12, 15, 27] that $3v_1 \in T^{(4)}$; on the other hand, it was proved in [22, Theorem 1.1] that $v_1 \notin T^{(4)}$. Since $3v_1 \in T^{(4)}$, we have $3v_k \in T^{(4)}$ for all $k$. Our first main result is as follows.

**Theorem 1.1.** For all $k \geq 1$, $v_k \notin T^{(4)}$.

Our second main result describes the 3-torsion subgroup $T^{(3,2)}/T^{(4)}$ of the additive group of $\mathbb{Z}(X)/T^{(4)}$. Let

$$
\mathcal{E} = \left\{ x_{j_1} \ldots x_{j_l} [x_{i_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}][x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}] \mid l \geq 0, k \geq 1, j_1 \leq j_2 \leq \cdots \leq j_l; i_1 < i_2 < \cdots < i_{2k+3} \right\}.
$$

**Theorem 1.2.** The set $\{e + T^{(4)} \mid e \in \mathcal{E}\}$ is a basis of the elementary abelian 3-group $T^{(3,2)}/T^{(4)}$ over $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$.

This theorem confirms the conjecture concerning a basis of $T^{(3,2)}/T^{(4)}$ over $\mathbb{F}_3$ made in [22, Remark 1.7].

To prove Theorem 1.1 we use the following result that may be of independent interest.

**Theorem 1.3.** Let $K$ be an arbitrary unital associative and commutative ring and let $K\langle Y \rangle$ be the free associative $K$-algebra on a non-empty set $Y$ of free generators. Let $T^{(4)}$ be the two-sided ideal in $K\langle Y \rangle$ generated by all commutators $[a_1, a_2, a_3, a_4] (a_i \in K\langle Y \rangle)$. Then the ideal $T^{(4)}$ is generated by the polynomials

1. $[y_1, y_2, y_3, y_4]$ \ (y_i \in Y),
2. $[y_1, y_2, y_3, y_4, y_5]$ \ (y_i \in Y),
3. $[y_1, y_2][y_3, y_4, y_5] + [y_1, y_5][y_3, y_4, y_2]$ \ (y_i \in Y),
4. $[y_1, y_2][y_3, y_4, y_5] + [y_1, y_4][y_3, y_2, y_5]$ \ (y_i \in Y),
5. $([y_1, y_2][y_3, y_4] + [y_1, y_3][y_2, y_4])[y_5, y_6]$ \ (y_i \in Y).

A generating set of the ideal $T^{(4)}$ can be rewritten in the following more convenient way:
Corollary 1.4. The ideal $T^{(4)}$ is generated (as a two-sided ideal in $K\langle Y \rangle$) by the polynomials (1), (2) together with the polynomials

\begin{align*}
(6) \quad & [y_1, y_2] [y_3, y_4, y_5] - \text{sgn}(\sigma) [y_{\sigma(1)}, y_{\sigma(2)}] [y_{\sigma(3)}, y_{\sigma(4)}, y_{\sigma(5)}] \\
& (y_i \in Y, \sigma \in S_5),

(7) \quad & [y_1, y_2] [y_3, y_4, y_5, y_6] - \text{sgn}(\sigma) [y_{\sigma(1)}, y_{\sigma(2)}] [y_{\sigma(3)}, y_{\sigma(4)}] [y_{\sigma(5)}, y_{\sigma(6)}] \\
& (y_i \in Y, \sigma \in S_6).
\end{align*}

Note that $3 [y_1, y_2] [y_3, y_4, y_5] \in T^{(4)}$ for all $y_i \in Y$; one can deduce this from the results of [12, 15, 27], see also [22, Corollary 2.4]. Hence, if $\frac{1}{3} \in K$ then all polynomials

\begin{align*}
(8) \quad & [y_1, y_2] [y_3, y_4, y_5] \\
& (y_i \in Y)
\end{align*}

belong to $T^{(4)}$. Since the polynomials (2)–(4) belong to the ideal generated by the polynomials (8), Theorem 1.3 implies the following corollary that has been proved in [12, 27].

Corollary 1.5 (see [12, 27]). If $\frac{1}{3} \in K$ then $T^{(4)}$ is generated as a two-sided ideal of $K\langle Y \rangle$ by the polynomials (1), (2) and (8).

Remarks. The study of Lie nilpotent associative rings and algebras was started by Jennings [18] in 1947. Jennings’ work was motivated by the study of modular group algebras of finite $p$-groups. Since then Lie nilpotent associative rings and algebras were investigated in various papers from various points of view; see, for instance, [1, 15, 16, 17, 21, 23, 24, 25, 27] and the bibliography there.

Recent interest in Lie nilpotent associative algebras was motivated by the study of the quotients $L_i/L_{i+1}$ of the lower central series

$L_1 > L_2 > \cdots > L_n > \cdots$

of the associated Lie algebra of a free associative algebra $A$; here $L_n$ is the linear span in $A$ of the set of all commutators $[a_1, a_2, \ldots, a_n]$ ($a_i \in A$). The study of these quotients $L_i/L_{i+1}$ was initiated in 2007 in a pioneering article of Feigin and Shoikhet [13]; further results on this subject can be found, for example, in [2, 3, 4, 8, 9, 12, 19, 20]. Since $T^{(n)}$ is the ideal in $A$ generated by $L_n$, some results about the quotients $T^{(i)}/T^{(i+1)}$ were obtained in these articles as well; in [12, 20] the latter quotients were the primary objects of study.

In 2012 Bhupatiraju, Etingof, Jordan, Kuszmaul and Li [5] started the study of the quotients $L_i/L_{i+1}$ for free associative rings $A$; that is, for free associative algebras $A$ over $\mathbb{Z}$. In this case the quotients in question (may) develop torsion and one of the objects of study is the pattern of this torsion. In [5] many results concerning torsion in $L_i/L_{i+1}$ were obtained and various open problems concerning this torsion were posed. Various results about the quotients $T^{(i)}/T^{(i+1)}$ and $A/T^{(i+1)}$ were obtained also in [5]; in particular, as mentioned above, it was proved that the additive group of $\mathbb{Z}\langle X \rangle/T^{(3)}$ is
free abelian and a basis of this group was exhibited \[5\) Prop. 3.2]. Some further results concerning the quotients \(T^{(i)}/T^{(i+1)}\) for various associative rings \(A\) were obtained by Cordwell, Fei and Zhou in \[7\].

It was observed in \[5\] that in the quotients \(T^{(i)}/T^{(i+1)}\) there is no torsion in the (additive) subgroups generated by the polynomials of small degree. However, in \[22\] the second author of the present article has proved that the image of \(v_1 = [x_1, x_2][x_3, x_4, x_5]\) in \(\mathbb{Z}\langle X\rangle/T^{(4)}\) is an element of order 3, that the torsion subgroup of \(\mathbb{Z}\langle X\rangle/T^{(4)}\) coincides with \(T^{(3,2)}/T^{(4)}\) and that the latter is an elementary abelian 3-group. To find a basis of this group is the aim of the present article.

Interesting computational data about the torsion subgroup of \(T^{(i)}/T^{(i+1)}\) for various \(i\) was presented in \[7\]. In particular, this data suggests that the additive group of \(\mathbb{Z}\langle X\rangle/T^{(5)}\) may have no torsion. Whether this group is torsion-free indeed is still an open problem. In \[6\] Costa and the second author of the present article have found generators for the ideal \(T^{(5)}\) of the free associative algebra \(K\langle X\rangle\). This result is similar to Theorem 1.3, which gives such generators for the ideal \(T^{(4)}\). The result of \[6\] might be the first step in proving that the additive group of \(\mathbb{Z}\langle X\rangle/T^{(5)}\) is torsion-free.

2. Preliminaries

Let \(K\) be an arbitrary associative and commutative unital ring. An ideal \(T\) of the free \(K\)-algebra \(K\langle X\rangle\) is called \(T\)-ideal if \(\phi(T) \subseteq T\) for all endomorphisms \(\phi\) of \(K\langle X\rangle\). One can easily see that \(T^{(4)}\) and \(T^{(3,2)}\) are \(T\)-ideals. We refer to \[10, 14, 26\] for terminology, basic facts and references concerning \(T\)-ideals and polynomial identities in associative algebras.

Let \(R^{(i_1,i_2,\ldots)}\) denote the \(K\)-linear span in \(K\langle X\rangle\) of all monomials of multidegree \((i_1, i_2, \ldots)\), that is, of the monomials of degree \(i_1\) in \(x_1\), \(i_2\) in \(x_2\) etc. Let \(f \in K\langle X\rangle\) be a polynomial, \(f = f^{(i_1,i_2,\ldots)} + \cdots + f^{(i_s,i_1,i_2,\ldots)}\) where \(f^{(i_1,i_2,\ldots)} \in R^{(i_1,i_2,\ldots)}\) for all \(t\). We say that the polynomials \(f^{(i_1,i_2,\ldots)}, \ldots, f^{(i_s,i_1,i_2,\ldots)}\) are multihomogeneous components of the polynomial \(f\). For instance, if \(f = x_1^3 + 2x_1x_2x_1 - x_1^2x_2\) then \(x_1^3\) and \(2x_1x_2x_1 - x_1^2x_2\) are the multihomogeneous components of \(f\) of multidegrees \((3,0,\ldots)\) and \((2,1,0,\ldots)\), respectively.

We say that an ideal \(I\) of \(K\langle X\rangle\) is multihomogeneous if, for each \(f \in I\), all multihomogeneous components of \(f\) also belong to \(I\). The ideal \(T^{(4)}\) is multihomogeneous because it is spanned by multihomogeneous polynomials \(a_0[a_1,a_2,a_3,a_4]a_5\) where all \(a_i\) are monomials. Similarly, the ideal \(T^{(3,2)}\) is also multihomogeneous.

Let \(\Gamma\) be the unital subalgebra of the free \(K\)-algebra \(K\langle X\rangle\) generated by all (left-normed) commutators \([x_{i_1}, x_{i_2}, \ldots, x_i]\) where \(l \geq 2\) and \(x_i \in X\) for all \(i\). The following assertion is well-known for an arbitrary \(T\)-ideal \(T\) in \(K\langle X\rangle\) if \(K\) is an infinite field (see, for instance, \[10\] Proposition 4.3.3)]. However, its proof remains valid over an arbitrary associative and commutative unital ring \(K\) if the \(T\)-ideal \(T\) is multihomogeneous.
Proposition 2.1 (see [10]). Let $K$ be an associative and commutative unital ring and let $T$ be a multihomogeneous $T$-ideal in the free unital associative algebra $K\langle X \rangle$. Then $T$ is generated by the set $T \cap \Gamma$ as a two-sided ideal of $K\langle X \rangle$.

Note that if $I$ is a two-sided ideal of $K\langle X \rangle$ then, for all $i$,
\[ [(I \cap \Gamma), x_i] \subseteq (I \cap \Gamma). \]
It follows that if the set $I \cap \Gamma$ generates $I$ as a two-sided ideal of $K\langle X \rangle$ then $I \cap \Gamma$ generates $I$ as a left ideal as well. Thus, we have

Corollary 2.2. Let $K$ be an associative and commutative unital ring and let $T$ be a multihomogeneous $T$-ideal in the free unital associative algebra $K\langle X \rangle$. Then $T$ is generated by the set $T \cap \Gamma$ as a left ideal of $K\langle X \rangle$.

The following lemma is well-known.

Lemma 2.3. The additive group of the ring $\mathbb{Z}\langle X \rangle$ is a direct sum of the subgroups $x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma$ where $s \geq 0$, $i_1 \leq i_2 \leq \cdots \leq i_s$,
\[ \mathbb{Z}\langle X \rangle = \bigoplus_{s \geq 0; \ i_1 \leq \cdots \leq i_s} x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma. \]
This equality can be rewritten as follows:
\[ (9) \quad \mathbb{Z}\langle X \rangle = \Gamma \oplus \bigoplus_{s \geq 1; \ i_1 \leq \cdots \leq i_s} x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma. \]

Lemma 2.4. Let $I$ be a left ideal of $\mathbb{Z}\langle X \rangle$ generated (as a left ideal in $\mathbb{Z}\langle X \rangle$) by a set $S \subset \Gamma$. Then
\[ I = \Gamma \cdot S \oplus \bigoplus_{s \geq 1; \ i_1 \leq \cdots \leq i_s} x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma \cdot S \]
where $\Gamma \cdot S$ is the left ideal of the ring $\Gamma$ generated by $S$. In particular, $I \cap \Gamma = \Gamma \cdot S$, that is, $I \cap \Gamma$ is the left ideal of $\Gamma$ generated by $S$.

Proof. We have
\[ \mathbb{Z}\langle X \rangle \cdot S = \Gamma \cdot S + \sum_{s \geq 1; \ i_1 \leq \cdots \leq i_s} x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma \cdot S. \]
Since $S \subset \Gamma$, we have
\[ x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma \cdot S \subset x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma \]
for all $s \geq 0; \ i_1 \leq \cdots \leq i_s$. The result follows from (9). \hfill \Box

Lemma 2.4 immediately implies the following:

Corollary 2.5. Under the hypothesis of Lemma 2.4, the (additive) group $\mathbb{Z}\langle X \rangle/I$ is isomorphic to the direct sum of the groups
\[ (x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma)/(x_{i_1}x_{i_2} \ldots x_{i_s}\Gamma \cdot S) \quad (s \geq 0; \ i_1 \leq \cdots \leq i_s), \]
that is,
\[
\mathbb{Z}\langle X \rangle / I \cong \Gamma / (\Gamma \cdot S) \oplus \bigoplus_{s \geq 1; \, i_1 \leq \cdots \leq i_s} (x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma) / (x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma \cdot S).
\]

For all \( s \geq 1, \ i_1 \leq \cdots \leq i_s \), the group \((x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma) / (x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma \cdot S)\) is isomorphic to \( \Gamma / (\Gamma \cdot S) \) with an isomorphism induced by the mapping \( \Gamma \to x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma \) that maps \( f \in \Gamma \) to \( x_{i_1} x_{i_2} \ldots x_{i_s} f \).

Let \( P_n \) (\( n \geq 1 \)) be the subgroup of the additive group of \( \mathbb{Z}\langle X \rangle \) generated by all monomials which are of degree 1 in each variable \( x_1, \ldots, x_n \) and do not contain any other variable. Then \( P_n \) is a free abelian group of rank \( n! \).

The following well-known theorem (see, for example, [10, Theorem 4.3.9]) describes a certain basis \( S \) (called Specht basis) of the free abelian group \( P_n \cap \Gamma \). Note that in [10] this theorem is stated for the vector subspace \( P_n \cap \Gamma \) of the free algebra \( K(X) \) over a field \( K \). However, it remains valid for the additive group \( P_n \cap \Gamma \) of the free ring \( \mathbb{Z}\langle X \rangle \). Indeed, on one hand, it is easy to see that the elements of the set \( S \) generate the additive group \( P_n \cap \Gamma \). On the other hand, the set \( S \) is linearly independent over \( \mathbb{Q} \) in \( \mathbb{Q}\langle X \rangle \), therefore, it is linearly independent over \( \mathbb{Z} \) in \( \mathbb{Z}\langle X \rangle \subset \mathbb{Q}\langle X \rangle \).

**Theorem 2.6** (see [10]). A basis \( S \) of the free abelian group \( P_n \cap \Gamma \) (\( n \geq 2 \)) consists of all products \( c_1 c_2 \ldots c_m \) (\( m \geq 1 \)) of commutators \( c_i \) (\( 1 \leq i \leq m \)) such that

(i) Each product \( c_1 c_2 \ldots c_m \) is multilinear in the variables \( x_1, \ldots, x_n \);

(ii) Each factor \( c_i = [x_{p_1}, x_{p_2}, \ldots, x_{p_s}] \) is a left-normed commutator of length \( \geq 2 \) and the maximal index is in the first position, that is, \( p_1 > p_j \) for all \( j, \ 2 \leq j \leq s \);

(iii) In each product the shorter commutators precede the longer, that is, the length of \( c_k \) is smaller than or equal to the length of \( c_{k+1} \) (\( 1 \leq k \leq m - 1 \));

(iv) If two consecutive factors are commutators of equal length then the first variable of the first commutator is smaller than the first variable in the second one, that is, if

\[
c_k = [x_{p_1}, x_{p_2}, \ldots, x_{p_s}], \quad c_{k+1} = [x_{q_1}, x_{q_2}, \ldots, x_{q_s}]
\]

then \( p_1 < q_1 \).

For example, the Specht basis \( S \) of \( P_4 \cap \Gamma \) is as follows: \( S = S_1 \cup S_2 \) where

\[
S_1 = \{ [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \mid \{i_1, i_2, i_3\} = \{1, 2, 3\} \},
\]

\[
S_2 = \{ [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \mid \{i_1, i_2, i_3\} = \{1, 2, 3\}, \ i_1 > i_2 \}.
\]

The basis \( S \) of \( P_5 \cap \Gamma \) is as follows: \( S = S_1 \cup S_2 \cup S_3 \) where

\[
S_1 = \{ [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\} \},
\]

\[
S_2 = \{ [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, \ i_2 > i_3, i_4 \},
\]

\[
S_3 = \{ [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}] \mid \{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}, \ i_1 > i_2 \}.
\]
3. Proof of Theorem 1.1

In this section we will prove Theorem 1.1 assuming that Corollary 1.4 holds. If $K = \mathbb{Z}$ and $Y = X = \{x_i \mid i \in \mathbb{N}\}$ then Corollary 1.4 can be rewritten as follows:

**Corollary 3.1.** The ideal $T^{(4)}$ is generated as a two-sided ideal in $\mathbb{Z}\langle X \rangle$ by the polynomials

\begin{align*}
(10) & \quad [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}] \quad (i_t \in \mathbb{N}), \\
(11) & \quad [x_{i_1}, x_{i_2}, x_{i_3}][x_{i_4}, x_{i_5}, x_{i_6}] \quad (i_t \in \mathbb{N}), \\
(12) & \quad [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}] - \text{sgn}(\sigma)[x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}][x_{i_{\sigma(3)}}, x_{i_{\sigma(4)}}, x_{i_{\sigma(5)}}] \\
& \quad \quad (i_t \in \mathbb{N}, \, \sigma \in S_6), \\
(13) & \quad [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}][x_{i_5}, x_{i_6}] - \text{sgn}(\sigma)[x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}][x_{i_{\sigma(3)}}, x_{i_{\sigma(4)}}, x_{i_{\sigma(5)}}] \\
& \quad \quad \quad \quad [x_{i_{\sigma(5)}}, x_{i_{\sigma(6)}}] \quad (i_t \in \mathbb{N}, \, \sigma \in S_6).
\end{align*}

We need the following:

**Lemma 3.2.** The ideal $T^{(4)}$ is generated as a left ideal in $\mathbb{Z}\langle X \rangle$ by the polynomials

\begin{equation}
[\mathbb{Z}\langle X \rangle : x_{i_1}, x_{i_2}, \ldots, x_{i_k}] \quad (k \geq 4, \, i_t \in \mathbb{N})
\end{equation}

together with the polynomials (11)–(13).

**Proof.** Let $I_1$ be the two-sided ideal of $\mathbb{Z}\langle X \rangle$ generated by all polynomials (10). It is clear that as a left ideal $I_1$ is generated by the polynomials (14) and (11). One can easily check that, modulo $I_1$, each polynomial of the form (11) is central in $\mathbb{Z}\langle X \rangle$. It follows that $I_2/I_1$ is generated as a left ideal in $\mathbb{Z}\langle X \rangle/I_1$ by the images of the polynomials (11). Hence, $I_2$ is generated as a left ideal in $\mathbb{Z}\langle X \rangle$ by the polynomials (14) and (11).

Let $I_3$ be the two-sided ideal of $\mathbb{Z}\langle X \rangle$ generated by all polynomials (10)–(12). Since the polynomials (12) are central modulo $I_2$, $I_3/I_2$ is generated as a left ideal of $\mathbb{Z}\langle X \rangle/I_2$ by all polynomials (12). It follows that $I_3$ as a left ideal of $\mathbb{Z}\langle X \rangle$ is generated by the polynomials (14), (11), (12).

Now to complete the proof of Lemma 3.2 that is, to prove that $T^{(4)}$ is generated as a left ideal of $\mathbb{Z}\langle X \rangle$ by the polynomials (14) together with the polynomials (11)–(13), it suffices to check that the polynomials (13) are central modulo $I_3$. Since each polynomial of the form (13) is, modulo $I_1$, a linear combination of polynomials of the form

\begin{equation}
([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] + [x_{i_1}, x_{i_3}][x_{i_2}, x_{i_4}])[x_{i_5}, x_{i_6}] \quad (i_t \in \mathbb{N}),
\end{equation}
it suffices to check that each polynomial of the form \([15]\) is central modulo \(I_3\). We have

\[
(16) \quad \left[ ([x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}] + [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}] ) [x_{i_5}, x_{i_6}, x_{i_7}] \right]
\]

\[
= [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}] [x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}]
\]

\[
= f_1 + f_2 + f_3
\]

where

\[
f_1 = [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}] [x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_3}, x_{i_4}] [x_{i_2}, x_{i_4}] [x_{i_5}, x_{i_6}, x_{i_7}],
\]

\[
f_2 = [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}],
\]

\[
f_3 = [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}].
\]

One can easily check that \(f_1, f_2 \in I_3\). Further,

\[
f_3 = \left( [x_{i_1}, x_{i_2}] [x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] \right)
\]

\[
- \left( [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_3}] \right)
\]

\[
+ \left( [x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] + [x_{i_1}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}, x_{i_7}] \right)
\]

where each sum in parenthesis belong to \(I_3\). Hence, \(f_3\) belongs to \(I_3\) and so does the polynomial \([15]\). It follows that each polynomial \([15]\) is central modulo \(I_3\) so \(T(4)\) is generated as a left ideal in \(\mathbb{Z}[X]\) by the polynomials \([14]\) and \((11)\)–\((13)\). The proof of Lemma 3.2 is completed. \(\square\)

Let \(\Lambda\) be the set of all (left-normed) commutators \([x_{i_1}, \ldots, x_{i_k}]\) where \(k \geq 2\) and \(i_l \in \mathbb{N}\) for all \(l\). Let \(\ell([x_{i_1}, \ldots, x_{i_k}])\) denote the length of the commutator \([x_{i_1}, \ldots, x_{i_k}]\), \(\ell([x_{i_1}, \ldots, x_{i_k}]) = k\). Recall that \(\Gamma\) is the unital subring of \(\mathbb{Z}(X)\) generated by \(\Lambda\).

Let \(\Gamma(4) = T(4) \cap \Gamma\); note that \(\Gamma(4)\) is an ideal of the ring \(\Gamma\). By Lemmas 2.4 and 3.2 we have

**Corollary 3.3.** The ideal \(\Gamma(4)\) is generated as a **left** ideal of \(\Gamma\) by the polynomials \([11], [12], [13]\) and \([14]\).

Let \(\mathcal{P}\) be the set of all products \(c_1 \ldots c_m\) where \(m \geq 0\), \(c_1, \ldots, c_m \in \Lambda\), \(\ell(c_1) \leq \cdots \leq \ell(c_m)\). We assume that if \(m = 0\) then the product above is equal to 1 so \(1 \in \mathcal{P}\). One can easily check that the set \(\mathcal{P}\) generates the additive group of \(\Gamma\).

The following proposition can be deduced easily from Corollary 3.3.
Proposition 3.4. The additive group of the ideal $\Gamma^{(4)}$ of the ring $\Gamma$ is generated by the following polynomials:

\begin{align*}
(17) & \quad c_1 \ldots c_m \in \mathcal{P} \text{ where } m \geq 1, \ell(c_m) \geq 4, \\
(18) & \quad c_1 \ldots c_m \in \mathcal{P} \text{ where } m \geq 2, \ell(c_{m-1}) = \ell(c_m) = 3, \\
(19) & \quad f \cdot ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}] - \text{sgn} (\tau)[x_{i_{\tau(1)}}, x_{i_{\tau(2)}}][x_{i_{\tau(3)}}, x_{i_{\tau(4)}}, x_{i_{\tau(5)}}]) \\
& \quad \text{ where } f \in \mathcal{P}, i_l \in \mathbb{N}, \tau \in S_5, \\
(20) & \quad f \cdot ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}][x_{i_5}, x_{i_6}] - \text{sgn} (\tau)[x_{i_{\tau(1)}}, x_{i_{\tau(2)}}][x_{i_{\tau(3)}}, x_{i_{\tau(4)}}, x_{i_{\tau(5)}}, x_{i_{\tau(6)}}]) \\
& \quad \text{ where } f \in \mathcal{P}, i_l \in \mathbb{N}, \tau \in S_6.
\end{align*}

Note that the elements (17)–(20) are multihomogeneous so the additive group $P_n \cap \Gamma^{(4)}$ is generated by all elements (17)–(20) that belong to $P_n$.

Let $W = P_{2k+3} \cap \Gamma$. Then $W$ is generated (as an additive group) by the products $c_1 \ldots c_m$ ($m \geq 1$) of commutators $c_i \in \Lambda$ $(1 \leq i \leq m)$ such that $c_1 \ldots c_m \in P_{2k+3}$. Note that at least one of the commutators $c_i$ is of odd length.

Let $W'$ be the subgroup of $W$ generated by those products $c_1 \ldots c_m \in P_{2k+3}$ of commutators that contain either a commutator $c_i$ of length at least 4 or at least 2 commutators $c_i, c_j$ ($i \neq j$) of length 3. One can easily check that $W'$ can be generated by the products $c_1 \ldots c_m \in \mathcal{P}$ that belong to $P_{2k+3}$ and have $\ell(c_m) \geq 4$ or $\ell(c_{m-1}) = \ell(c_m) = 3$. Note that the latter set of generators of $W'$ is the set of all elements of the forms (17)–(18) that belong to $P_{2k+3}$. It follows that $W' \subset (P_{2k+3} \cap \Gamma^{(4)})$.

Note that, by Theorem 2.6, $W$ is a free abelian group with the Specht basis $S$ consisting of certain products $c_1c_2\ldots c_m$ of commutators; these products $c_1c_2\ldots c_m \in S$ were described in Theorem 2.6. One can easily check that a basis of $W'$ is formed by the products $c_1c_2\ldots c_m \in S$ such that either the commutator $c_m$ is of length at least 4 or the commutators $c_{m-1}, c_m$ are of length 3. It follows that $W/W'$ is a free abelian group whose basis is formed by the images of the products $c_1c_2\ldots c_m \in S$ such that $c_1, c_2, \ldots, c_{m-1}$ are commutators of length 2 and $c_m$ is a commutator of length 3.

Let

\[ c_{1,12,\ldots,i_{2k+3}} = [x_{i_1}, x_{i_2}] \ldots [x_{i_{2k-1}}, x_{i_{2k}}][x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}]. \]

Let

\[ C = \left\{ c_{1,12,\ldots,i_{2k+3}} \mid \{i_1, i_2, \ldots, i_{2k+3}\} = \{1, 2, \ldots, 2k + 3\}; \ i_1 > i_2, \ldots, \ i_{2k-1} > i_{2k}; \ i_{2k+1} > i_{2k+2}, i_{2k+3}; \ i_1 < i_3 < \cdots < i_{2k-1} \right\}. \]

Then $C$ coincides with the set of the products $c_1c_2\ldots c_m \in S$ above whose images form a basis of $W/W'$. Hence, we have

Lemma 3.5. The set $C = \{ c + W' \mid c \in C \}$ is a basis of $W/W'$. 

Hence, each element of the form (21) is, modulo elements of the form (19). Since \( P \) are commutators of length \( g \) elements

Note that \( P \) belonging to the elements (19) belonging to \( P \) that belong to \( P \). Indeed, if suffices to check that the elements

\[
(21) \quad [x_{j_1}, x_{j_2}] \ldots [x_{j_{2k-7}}, x_{j_{2k-6}}][x_{j_{2k-5}}, x_{j_{2k-4}}, x_{j_{2k-3}}] \\
\times ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}) [x_{i_5}, x_{i_6}]
\]

belonging to \( P \) are contained in \( U + W' \) because all elements of the form (20) belonging to \( P \) are, modulo \( W' \), linear combinations of the elements above.

Recall that \( I_1 \) is the two-sided ideal of \( Z(X) \) generated by all polynomials of the form (10), that is, by all commutators of length 4 in \( x_l (l \in \mathbb{N}) \). Since all commutators in variables \( x_l (l \in \mathbb{N}) \) of length at least 2 commute modulo \( I_1 \), we have

\[
[x_{j_{2k-5}}, x_{j_{2k-4}}, x_{j_{2k-3}}] = [x_{j_{2k-5}}, x_{j_{2k-4}}, x_{j_{2k-3}}] + [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}]
\]

\[
\equiv ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}]) [x_{i_5}, x_{i_6}]
\]

Hence, each element of the form (21) is, modulo \( I_1 \), a linear combination of elements of the form (19). Since \( (P_{2k+3} \cap I_1) \subset W' \), the claim follows.

It follows that the group \( (P_{2k+3} \cap \Gamma(^4)) / W' \) is generated by the images of the elements (19) belonging to \( P_{2k+3} \), that is, by the elements \( g + W' \) where (22)

\[
g = c_1 \ldots c_m ([x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}, x_{i_5}]) - \text{sgn} (\tau) [x_{i_1}(1), x_{i_2}(2), x_{i_3}(3), x_{i_4}(4), x_{i_5}(5)),
\]

g \in \bigcap_{k+3}, c_s \in \Lambda, \tau \in S_5. \text{ It is easy to check that if for some } s, 1 \leq s \leq m, \text{ the commutator } c_s \text{ is of length at least 3 then the product } g \text{ of the form (22) belongs to } W'. \text{ Hence, we have}

**Lemma 3.6.** The (additive) group \( (P_{2k+3} \cap \Gamma(^4)) / W' \) is generated by the elements \( g + W' \) where \( g \) is of the form (22), \( m = k - 1 \) and \( c_1, \ldots, c_{k-1} \in \Lambda \) are commutators of length 2.

Let \( H \) be the free abelian group on a basis

\[
\{h_{i_1i_2i_3} | \{i_1, i_2, \ldots, i_{2k+3} = \{1, 2, \ldots, 2k + 3} \}
\]

Note that \( H \) is isomorphic to the additive group of the group ring \( \mathbb{Z}S_{2k+3} \) of the symmetric group \( S_{2k+3} \) with an isomorphism

\[
\eta : h_{i_1i_2i_3} \rightarrow \begin{pmatrix} 1 & 2 & \ldots & 2k + 3 \\ i_1 & i_2 & \ldots & i_{2k+3} \end{pmatrix} \in S_{2k+3}.
\]
Define a homomorphism $\psi : H \rightarrow W/W'$ by
$$\psi(h_{i_1\ldots i_{2k+3}}) = c_{i_1\ldots i_{2k+3}} + W'.$$
Let $Q$ be the subgroup of $H$ generated by all elements

\begin{align*}
(23) & \quad h_{i_1\ldots i_{2l-2}i_{2l-1}i_{2l+1}\ldots i_{2k+3}} + h_{i_1\ldots i_{2l-2}i_{2l+1}i_{2l+3}\ldots i_{2k+3}} & (1 \leq l \leq k+1),
(24) & \quad h_{i_1\ldots i_{2l-4}i_{2l-3}i_{2l-1}i_{2l+1}i_{2l+3}i_{2k+3}} - h_{i_1\ldots i_{2l-4}i_{2l-3}i_{2l+1}i_{2l+3}i_{2k+3}} & (2 \leq l \leq k)
\end{align*}

and

\begin{align*}
(25) & \quad h_{i_1\ldots i_{2k}i_{2k+1}2i_{2k+3}i_{2k+3} + h_{i_1\ldots i_{2k}i_{2k+3}i_{2k+3}} + h_{i_1\ldots i_{2k}i_{2k+3}i_{2k+3}}.
\end{align*}

**Lemma 3.7.** $\ker \psi = Q$.

**Proof.** It is clear that the elements of the forms (23), (24) and (25) belong to $\ker \psi$. Hence, $Q \subseteq \ker \psi$. On the other hand, one can easily check that $H/Q$ is generated by the set $D = \{d + Q \mid d \in D\}$ where

$$D = \left\{ h_{i_1\ldots i_{2k+3}} \mid i_1 > i_2, \ldots, i_{2k-1} > i_{2k};
\quad i_{2k+1} > i_{2k+2}, i_{2k+3} \colon i_1 < i_3 < \cdots < i_{2k-1} \right\}.$$ 

Clearly, $\mu(D) = C$ so, by Lemma 3.5, $\mu(D)$ is a basis of $W/W'$. It follows that $D$ is a basis of $H/Q$ and $\ker \psi = Q$, as required. $\square$

Let $P$ be the subgroup of $H$ generated by all elements

$$h_{i_1\ldots i_{2k+3}} - \text{sgn} (\sigma) h_{i_{\sigma(1)}\ldots i_{\sigma(2k+3)}}$$

and

$$h_{i_1\ldots i_{2k}i_{2k+1}i_{2k+3}i_{2k+3} + h_{i_1\ldots i_{2k}i_{2k+3}i_{2k+3}} + h_{i_1\ldots i_{2k}i_{2k+3}i_{2k+3}}$$

where $\{i_1, \ldots, i_{2k+3}\} = \{1, \ldots, 2k+3\}, \sigma \in S_{2k+3}$. Note that $Q \subset P$ because a generating set of $Q$ is a subset of a generating set of $P$. By Lemma 3.6, the (additive) group $(P_{2k+3} \cap \Gamma(4))/W'$ is generated by the elements

$$c_{j_1\ldots j_{2k-2}i_5} - \text{sgn} (\tau) c_{j_1\ldots j_{2k-2}i_5} + W'$$

where $\{j_1, \ldots, j_{2k-2}, i_1, \ldots, i_5\} = \{1, \ldots, 2k+3\}$ and $\tau \in S_5$. It follows that $\psi^{-1}((P_{2k+3} \cap \Gamma(4))/W')$ is generated by the elements

$$h_{j_1\ldots j_{2k-2}i_5} - \text{sgn} (\tau) h_{j_1\ldots j_{2k-2}i_5}$$

together with $\ker \psi = Q$ so

\begin{equation}
(26) \quad \psi^{-1}((P_{2k+3} \cap \Gamma(4))/W') \subseteq P.
\end{equation}

(In fact, one can check that $\psi^{-1}((P_{2k+3} \cap \Gamma(4))/W') = P$ but for our purpose (26) is sufficient.)

Note that $v_k = c_{12\ldots(2k+3)} \in P_{2k+3} \cap \Gamma$ so to prove that $v_k \notin T(4)$ (that is, to prove Theorem 1.1) it suffices to prove that $v_k \notin (P_{2k+3} \cap \Gamma \cap T(4)) = (P_{2k+3} \cap \Gamma(4))$. To prove the latter it suffices to check that
Lemma 4.2 proved in [22, Lemma 6.1].

We claim that the set $\Gamma$.

Proof. By Corollary 2.5, to prove Theorem 1.2 it suffices to prove the following:

Lemma 3.8. $h_{12\ldots(2k+3)} \notin P$.

Proof. Let $\mu : H \to Z$ be the homomorphism of $H$ into $Z$ defined by

$$\mu(h_{i_1i_2\ldots i_{2k+3}}) = \text{sgn}(\rho)$$

where $\rho = \begin{pmatrix} 1 & 2 & \ldots & 2k + 3 \\ i_1 & i_2 & \ldots & i_{2k+3} \end{pmatrix}$. Then

$$\mu(h_{i_1i_2\ldots i_{2k+3}} - \text{sgn}(\sigma) h_{i_{\sigma(1)}i_{\sigma(2)}\ldots i_{\sigma(2k+3)}}) = \text{sgn}(\rho) - \text{sgn}(\sigma) \text{sgn}(\sigma\rho) = 0$$

and

$$\mu(h_{i_1i_2\ldots i_{2k+3}} - h_{i_1i_2i_3i_{k+1}i_{2k+3}} + h_{i_1i_2i_3\ldots i_{2k+3}i_{k+1}i_{2k+2}}) = \pm 3$$

so $\mu(P) = 3Z$. On the other hand, $\mu(h_{12\ldots(2k+3)}) = 1 \notin 3Z$, therefore $h_{12\ldots(2k+3)} \notin P$.

This completes the proof of Lemma 3.8 and hence of Theorem 1.1.

4. Proof of Theorem 1.2

Recall that $T^{(4)}$ and $T^{(3,2)}$ are multihomogeneous $T$-ideals of $Z(X)$. Hence, by Corollary 2.2, $T^{(4)}$ and $T^{(3,2)}$ are generated, as left ideals of $Z(X)$, by the sets $\Gamma^{(4)} = T^{(4)} \cap \Gamma$ and $\Gamma^{(3,2)} = T^{(3,2)} \cap \Gamma$, respectively. By Lemma 2.4

$$T^{(4)} = \Gamma^{(4)} \oplus \bigoplus_{s \geq 1; \ i_1 \leq \ldots \leq i_s} x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma^{(4)},$$

$$T^{(3,2)} = \Gamma^{(3,2)} \oplus \bigoplus_{s \geq 1; \ i_1 \leq \ldots \leq i_s} x_{i_1} x_{i_2} \ldots x_{i_s} \Gamma^{(3,2)}.$$ 

Let

$$\mathcal{E}' = \left\{ \left[x_{i_1}, x_{i_2} \right] \ldots \left[x_{i_{2k-1}}, x_{i_{2k}} \right] \left[x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}} \right] \mid k \geq 1, \ i_1 < i_2 < \ldots < i_k \right\}.$$ 

By Corollary 2.5, to prove Theorem 1.2 it suffices to prove the following lemma.

Lemma 4.1. The set $\{e' + \Gamma^{(4)} \mid e' \in \mathcal{E}'\}$ is a basis of the elementary abelian 3-group $\Gamma^{(3,2)}/\Gamma^{(4)}$ over $F_3 = \mathbb{Z}/3\mathbb{Z}$.

Proof. We claim that the set $\{e' + \Gamma^{(4)} \mid e' \in \mathcal{E}'\}$ generates the (additive) group $\Gamma^{(3,2)}/\Gamma^{(4)}$. Indeed, let

$$S = \left\{ \left[x_{i_1}, x_{i_2}, x_{i_3} \right] \left[x_{i_4}, x_{i_5} \right] \mid i_1 < i_2 < i_3 < i_4 < i_5 \right\}$$

and let $I^{(3,2)}$ be the ideal in $Z(X)$ generated by $S$. We need the following lemma proved in [22, Lemma 6.1].

Lemma 4.2 (see [22]). $T^{(3,2)} = T^{(4)} + I^{(3,2)}$. 

It follows from Lemma 1.2 that $T^{(3, 2)}$ is generated as a two-sided ideal of $\mathbb{Z}(X)$ by the set $\Gamma^{(4)} \cup S$. Hence, $\Gamma^{(3, 2)}$ is generated as a two-sided ideal of $\Gamma$ by the same set $\Gamma^{(4)} \cup S$. In other words, $\Gamma^{(3, 2)}/\Gamma^{(4)}$ is the ideal of the (commutative) algebra $\Gamma/\Gamma^{(4)}$ generated by the image of $S$. Since, for all $j_t \in \mathbb{N}$,

$$[x_{j_1}, x_{j_2}, x_{j_3}] [x_{j_4}, x_{j_5}, x_{j_6}] \in \Gamma^{(4)},$$

the (additive) group $\Gamma^{(3, 2)}/\Gamma^{(4)}$ is generated by the products

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] [x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}] + \Gamma^{(4)} \quad (k \geq 1, i_t \in \mathbb{N}).$$

It follows easily from (12) that

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] [x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}]$$

$$\equiv [x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}] \cdots [x_{i_{\sigma(2k-1)}}, x_{i_{\sigma(2k)}}, x_{i_{\sigma(2k+1)}}, x_{i_{\sigma(2k+2)}}, x_{i_{\sigma(2k+3)}}] \pmod{\Gamma^{(4)}}$$

for all permutations $\sigma \in S_{2k+3}$ so the group $\Gamma^{(3, 2)}/\Gamma^{(4)}$ is generated by the elements

$$[x_{i_1}, x_{i_2}] \cdots [x_{i_{2k-1}}, x_{i_{2k}}] [x_{i_{2k+1}}, x_{i_{2k+2}}, x_{i_{2k+3}}] + \Gamma^{(4)}$$

such that $k \geq 1$ and $i_1 \leq i_2 \leq \cdots \leq i_{2k+3}$. Thus, the set $\{e' + \Gamma^{(4)} \mid e' \in \mathcal{E}'\}$ generates the group $\Gamma^{(3, 2)}/\Gamma^{(4)}$, as claimed.

It follows easily from Theorem 1.1 that $e' \notin \Gamma^{(4)}$ for all $e' \in \mathcal{E}'$. Since $3 \cdot e' \in \Gamma^{(4)}$ ($e' \in \mathcal{E}'$), $\Gamma^{(3, 2)}/\Gamma^{(4)}$ is a non-trivial elementary abelian 3-group generated by the image of $\mathcal{E}'$.

It remains to check that the image of the set $\mathcal{E}'$ is linearly independent in $\Gamma^{(3, 2)}/\Gamma^{(4)}$ over $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$. Suppose that $\alpha_1 e'_1 + \cdots + \alpha_n e'_n \equiv 0 \pmod{\Gamma^{(4)}}$ ($\alpha_i \in \mathbb{F}_3, e'_i \in \mathcal{E}'$), that is,

$$(27) \quad \alpha_1 e'_1 + \cdots + \alpha_n e'_n \in T^{(4)} \quad (\alpha_i \in \mathbb{F}_3, e'_i \in \mathcal{E}').$$

Note that the polynomials $e'_1, \ldots, e'_n$ belong to distinct multihomogeneous components of the ring $\mathbb{Z}(X)$. Since $T^{(4)}$ is a multihomogeneous $T$-ideal, it follows from (27) that $\alpha_i e'_i \in T^{(4)}$ for each $i$, $1 \leq i \leq n$. Since $e'_i \notin T^{(4)}$, we have $\alpha_i = 0$ for all $i$ ($\alpha_i \in \mathbb{F}_3$). Thus, the set $\{e' + \Gamma^{(4)} \mid e' \in \mathcal{E}'\}$ is linearly independent in $\Gamma^{(3, 2)}/\Gamma^{(4)}$ over $\mathbb{F}_3$.

This completes the proof of Lemma 1.1 and of Theorem 1.2. \hfill $\blacksquare$

5. Proof of Theorem 1.3 and of Corollary 1.4

Let $I$ be the ideal of $K\langle Y \rangle$ generated by the polynomials (11)-(13). We will prove Theorem 1.3 by showing that $I = T^{(4)}$.

First we prove that $I \subseteq T^{(4)}$. It is clear that the elements (1) belong to $T^{(4)}$. Further, it is well-known (see, for instance, [12, Theorem 3.4], [13, Lemma 1], [22, Lemma 2.1], [23, Lemma 2]) that, for all $a_1, \ldots, a_5 \in K\langle Y \rangle$,
we have
\[(28) \quad [a_1, a_2, a_3][a_4, a_5] + [a_1, a_2, a_4][a_3, a_5] \in T^{(4)}, \]
\[(29) \quad [a_1, a_2, a_3][a_4, a_5] + [a_1, a_4, a_3][a_2, a_5] \in T^{(4)}. \]
It follows that the elements (28) and (29) belong to $T^{(4)}$.

It is also well-known (see, for example, [17, Theorem 3.2], [23, Lemma 1]) that, for all $a_1, \ldots, a_6 \in K(Y)$, we have
\[(30) \quad [a_1, a_2, a_3][a_4, a_5, a_6] \in T^{(4)}. \]
Indeed, by (28) we have
\[[a_1, a_2, a_3] [a_4, a_5, a_6] + [a_1, a_2, [a_4, a_5]] [a_3, a_6] \in T^{(4)}. \]
Since
\[[a_1, a_2, [a_4, a_5]] = [a_1, a_2, a_4, a_5] - [a_1, a_2, a_5, a_4] \in T^{(4)}, \]
we have $[a_1, a_2, a_3][a_4, a_5, a_6] \in T^{(4)}$, as claimed. In particular, the elements (2) belong to $T^{(4)}$.

Now to prove that $I \subseteq T^{(4)}$ it remains to check that the elements (5) belong to $T^{(4)}$. By (29), for all $a_1, \ldots, a_6 \in K(Y)$ we have
\[[a_1, a_2 a_3, a_4][a_5, a_6] + [a_1, a_5, a_4][a_2 a_3, a_6] \in T^{(4)} \]
so
\[[a_1, a_2 a_3, a_4][a_5, a_6] + [a_1, a_5, a_4][a_2 a_3, a_6] \]
\[= ([a_2 [a_1, a_3] + [a_1, a_2] a_3, a_4][a_5, a_6] + [a_1, a_5, a_4][a_2 [a_3, a_6] + [a_1, a_5, a_4][a_2, a_6]a_3 \\
= a_2 [a_1, a_3, a_4][a_5, a_6] + [a_2, a_4][a_1, a_3][a_5, a_6] + [a_1, a_5, a_4][a_2, a_6]a_3 \\
+ [a_1, a_2, a_4] a_3[a_5, a_6] + [a_1, a_5, a_4] a_2[a_3, a_6] + [a_1, a_5, a_4] a_2[a_6]a_3 \in T^{(4)} . \]
Note that
\[a_2 [a_1, a_3, a_4][a_5, a_6] + [a_1, a_5, a_4] a_2[a_3, a_6] \]
\[= a_2 ([a_1, a_3, a_4][a_5, a_6] + [a_1, a_5, a_4][a_3, a_6]) + [a_1, a_5, a_4, a_2][a_3, a_6] \in T^{(4)} \]
by (29) and
\[[a_1, a_2, a_4] a_3[a_5, a_6] + [a_1, a_5, a_4] a_2[a_6]a_3 \]
\[= ([a_1, a_2, a_4][a_5, a_6] + [a_1, a_5, a_4][a_2, a_6]) a_3 - [a_1, a_2, a_4][a_5, a_6, a_3] \in T^{(4)} \]
by (29) and (30) so $[a_2, a_4][a_1, a_3][a_5, a_6] + [a_1, a_2][a_3, a_4][a_5, a_6] \in T^{(4)}$. It follows that
\[([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, a_6] \]
\[= [a_1, a_2][a_3, a_4][a_5, a_6] + [a_2, a_4][a_1, a_3][a_5, a_6] + [a_1, a_3][a_2, a_4][a_5, a_6] \in T^{(4)} . \]
In particular, the elements (5) belong to $T^{(4)}$.

Thus, all the generators (1)–(5) of the ideal $I$ belong to $T^{(4)}$. Hence, $I \subseteq T^{(4)}$. 
Now we prove that \( T^{(4)} \subseteq I \). Since the ideal \( T^{(4)} \) is generated by the polynomials \([a_1, a_2, a_3, a_4] \) \((a_i \in K(Y))\), it suffices to check that, for all \( a_i \in K(Y) \), the polynomial

\[
[a_1, a_2, a_3, a_4]
\]

belongs to \( I \). Clearly, one can assume that all \( a_i \) are monomials.

In order to prove that each polynomial of the form \((31)\) belongs to \( I \) we will prove that, for all monomials \( a_i \in K(Y) \), the following polynomials belong to \( I \) as well:

\[
\begin{align*}
(32) & \quad ([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, a_6]; \\
(33) & \quad [a_1, a_2, a_3][a_4, a_5, a_6]; \\
(34) & \quad [a_1, a_2, a_3][a_4, a_5] + [a_1, a_2, a_4][a_3, a_5]; \\
(35) & \quad [a_1, a_2, a_3][a_4, a_5] + [a_1, a_4, a_3][a_2, a_5].
\end{align*}
\]

The proof is by induction on the degree \( m = \deg f \) of a polynomial \( f \) that is of one of the forms \((31)–(35)\). It is clear that \( m \geq 4 \). If \( m = 4 \) then \( f \) is of the form \((31)\) with all monomials \( a_i \) of degree 1. Hence, \( f \) is of the form \((1)\) so \( f \in I \). This establishes the base of the induction.

To prove the induction step suppose that \( m = \deg f > 4 \) and that all polynomials of the forms \((31)–(35)\) of degree less than \( m \) belong to \( I \).

**Case 1.** Suppose that \( f \) is of the form \((32)\),

\[
f = f(a_1, \ldots, a_6) = ([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, a_6].
\]

If \( \deg f = 6 \) then \( f \) is of the form \((33)\) so \( f \in I \). If \( \deg f > 6 \) then, for some \( i, 1 \leq i \leq 6 \), we have \( a_i = a_i' a_i'' \) where \( \deg a_i' < \deg a_i \). We claim that to check that \( f \in I \) one may assume without loss of generality that \( i = 1 \) or \( i = 6 \).

Indeed, we have \( f(a_1, \ldots, a_4, a_5, a_6) = -f(a_1, \ldots, a_4, a_6, a_5) \). Further, by the induction hypothesis, \([a_i, a_j], [a_k, a_l] \in I \) for all distinct \( i, j, k, l, 1 \leq i, j, k, l \leq 6 \). It follows that

\[
f(a_1, a_2, a_3, a_4, a_5, a_6) \equiv f(a_2, a_1, a_4, a_3, a_5, a_6) \equiv f(a_4, a_2, a_3, a_1, a_5, a_6) \quad (\text{mod } I).
\]

The claim follows.

Suppose that \( a_6 = a_6' a_6'' \). We have

\[
f = ([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, a_6'
\]

\[
= ([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])(a_6'[a_5, a_6'' + [a_5, a_6''] a_6'').
\]

By the induction hypothesis, \(([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, b] \in I \) where \( b \in \{a_6', a_6''\} \). It follows that

\[
([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])([a_5, a_6''] a_6' + [a_5, a_6'] a_6'') \in I.
\]
On the other hand, by the induction hypothesis, we have
\[
[a_3, a_4][a_5, a_6', a_6'] + [a_5, a_6', a_3] \in I,
\]
\[
[a_1, a_2][a_5, a_6', a_3] + [a_1, a_3][a_5, a_6', a_2] \in I,
\]
\[
[a_6', a_4][a_5, a_6', a_2] + [a_2, a_4][a_5, a_6', a_6] \in I
\]
so
\[
([a_1, a_2][a_3, a_4] + [a_1, a_3][a_2, a_4])[a_5, a_6', a_6]
\]
\[
= [a_1, a_2]([a_3, a_4][a_5, a_6', a_3] + [a_6', a_4][a_5, a_6', a_3])
\]
\[
- [a_6', a_4]([a_1, a_2][a_5, a_6', a_3] + [a_1, a_3][a_5, a_6', a_2])
\]
\[
+ [a_1, a_3]([a_6', a_4][a_5, a_6', a_2] + [a_2, a_4][a_5, a_6', a_6]) \in I.
\]

It follows that \( f \in I \) if \( a_6 = a_6'.a_6' \).

Suppose that \( a_1 = a_1'a_1'' \). We have
\[
f = ([a_1'a_1'', a_2][a_3, a_4] + [a_1'a_1', a_3][a_2, a_4])[a_5, a_6]
\]
\[
= (a_1'([a_1'', a_2][a_3, a_4] + [a_1', a_3][a_2, a_4]) + [a_1', a_2]a_1''[a_3, a_4] + [a_1', a_3]a_1'[a_2, a_4])
\]
\[
\times [a_5, a_6] = (a_1'([a_1'', a_2][a_3, a_4] + [a_1', a_3][a_2, a_4]) + [a_1', a_3]a_1'[a_2, a_4]
\]
\[
+ [a_1', a_3][a_2, a_4]) + [a_1', a_2, a_1''][a_3, a_4] + [a_1', a_3, a_1''][a_2, a_4]) [a_5, a_6].
\]

By the induction hypothesis, \( ([b, a_2][a_3, a_4] + [b, a_3][a_2, a_4])[a_5, a_6] \in I \) where \( b \in \{a_1', a_1''\} \). It follows that
\[
(a_1'([a_1'', a_2][a_3, a_4] + [a_1', a_3][a_2, a_4])
\]
\[
+ a_1''([a_1', a_2][a_3, a_4] + [a_1', a_3][a_2, a_4])[a_5, a_6] \in I.
\]

On the other hand, by the induction hypothesis,
\[
[a_1', a_2, a_1''][a_3, a_4] + [a_1', a_3, a_1''][a_2, a_4] \in I
\]
so
\[
([a_1', a_2, a_1''][a_3, a_4] + [a_1', a_3, a_1''][a_2, a_4])[a_5, a_6] \in I.
\]

It follows that \( f \in I \) if \( a_1 = a_1'a_1'' \).

Thus, if \( f \) is a polynomial of the form \( \text{(32)} \) of degree \( m \) then \( f \in I \).

**Case 2.** Suppose that \( f \) is of the form \( \text{(33)} \).

If \( \deg f = 6 \) then \( f \) is of the form \( \text{(2)} \) so \( f \in I \). If \( \deg f > 6 \) then, for some \( i, 1 \leq i \leq 6 \), we have \( a_i = a_i'a_i'' \) where \( \deg a_i', \deg a_i'' < \deg a_i \). We claim that to check that \( f \in I \) one may assume without loss of generality that \( i = 1 \) or \( i = 3 \).
Indeed, $f(a_1, a_2, a_3, \ldots) = -f(a_2, a_1, a_3, \ldots)$. Further, by the induction hypothesis, $[a_1, a_2, a_3, a_i] \in I$ for $i = 4, 5, 6$; it follows that $[a_1, a_2, a_3]$ commutes with $[a_4, a_5, a_6]$ modulo $I$ so

$$f(a_1, a_2, a_3, a_4, a_5, a_6) \equiv f(a_4, a_5, a_6, a_1, a_2, a_3) \pmod{I}.$$ 

The claim follows.

Suppose that $a_3 = a_3' a_3''$. Then

$$f = [a_1, a_2, a_3'] [a_4, a_5, a_6] = a_3' [a_1, a_2, a_3'][a_4, a_5, a_6] + [a_1, a_2, a_3'] a_3''[a_4, a_5, a_6]$$

$$= a_3' [a_1, a_2, a_3'][a_4, a_5, a_6] + a_3'' [a_1, a_2, a_3'][a_4, a_5, a_6] + [a_1, a_2, a_3', a_3''] [a_4, a_5, a_6].$$

By the induction hypothesis, $[a_1, a_2, b][a_4, a_5, a_6] \in I$ where $b \in \{a_3', a_3''\}$ and $[a_1, a_2, a_3', a_3''] \in I$ so in this case $f \in I$, as required.

Suppose that $a_1 = a_1'a_1''$. We have

$$f = [a_1'a_1'', a_2, a_3][a_4, a_5, a_6] = [(a_1'a_1'', a_2) + [a_1'a_1'', a_2]]a_3][a_4, a_5, a_6]$$

$$= (a_1'[a_1'', a_2, a_3] + [a_1', a_3][a_1'', a_2] + [a_1', a_2][a_1'', a_3] + [a_1', a_2, a_3][a_1'', a_3]) [a_4, a_5, a_6]$$

$$= (a_1'[a_1'', a_2, a_3] + [a_1', a_3][a_1'', a_2] + [a_1', a_2][a_1'', a_3] + [a_1', a_2, a_3][a_1'', a_3])$$

$$+ [a_1', a_2, a_3, a_1''][a_4, a_5, a_6].$$

By the induction hypothesis, $[b, a_2, a_3][a_4, a_5, a_6] \in I$ where $b \in \{a_1', a_1''\}$ and $[a_1', a_2, a_3, a_1''] \in I$. It follows that

$$([a_1'a_1'', a_2, a_3] + [a_1', a_2, a_3])[a_4, a_5, a_6] \in I.$$ 

On the other hand, by the induction hypothesis,

$$[a_1'a_1'', a_2][a_4, a_5, a_6] + [a_6, a_2][a_4, a_5, a_1''] \in I,$$

$$[a_1', a_3][a_4, a_5, a_1''] + [a_1', a_2][a_4, a_5, a_1'] \in I,$$

$$[a_6, a_2][a_4, a_5, a_1'] + [a_1', a_2][a_4, a_5, a_6] \in I$$

so

$$([a_1', a_3][a_1'a_1'', a_2] + [a_1', a_2][a_1'a_1'', a_3])[a_4, a_5, a_6]$$

$$= [a_1', a_3]([a_1'a_1'', a_2][a_4, a_5, a_6] + [a_6, a_2][a_4, a_5, a_1''])$$

$$- [a_6, a_2]([a_1', a_3][a_4, a_5, a_1''] + [a_1', a_3][a_4, a_5, a_1'])$$

$$+ [a_1'a_1'', a_3][a_6, a_2][a_4, a_5, a_1'] + [a_1', a_2][a_4, a_5, a_6] \in I.$$ 

It follows that $f \in I$ if $a_1 = a_1'a_1''$.

Thus, if $f$ is a polynomial of the form (33) of degree $m$ then $f \in I$.

**Case 3.** Suppose that $f$ is of the form (34). If deg $f = 5$ then $f$ is of the form (33) so $f \in I$. If deg $f > 5$ then, for some $i$, $1 \leq i \leq 5$, we have $a_i = a_i'a_i''$ where deg $a_i'$, deg $a_i'' < $ deg $a_i$. 


Suppose first that \( a_5 = a''_5 a''_5 \). Then
\[
 f = \frac{1}{2} [a_1, a_2, a_3][a'_4, a''_4 a''_5] + [a_1, a_2, a_4][a_3, a''_5 a''_5] \\
= \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 \\
+ \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 \\
= a''_5 ( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] ) \\
+ \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 \\
+ [a_1, a_2, a_3, a''_5][a_4, a''_5] + [a_1, a_2, a_4, a''_5][a_3, a''_5]
\]
so in this case \( f \in I \) by the induction hypothesis.

Suppose that \( a_4 = a'_4 a''_4 \). Then
\[
 f = \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 \\
= \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 \\
+ [a_1, a_2, a_3][a'_4, a''_4] a''_5 + [a_1, a_2, a_4][a_3, a''_5] \\
= a''_5 \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) + [a_1, a_2, a_3, a''_5][a_4, a''_5] \\
+ \left( [a_1, a_2, a_3][a'_4, a''_4] + [a_1, a_2, a_4][a_3, a''_5] \right) a''_5 - [a_1, a_2, a_4'][a_3, a''_5] a''_5.
\]

Since each polynomial of the form \( [a_3, a_5] \) of degree \( m \) belongs to \( I \), we have \( [a_1, a_2, a'_4][a_3, a_5, a''_4] \in I \). On the other hand, by the induction hypothesis,
\[
 [a_1, a_2, a_3][a'_4, a''_5] + [a_1, a_2, a_4'][a_3, a_5] \in I, \\
 [a_1, a_2, a_3][a'_4, a''_5] + [a_1, a_2, a'_4][a_3, a_5] \in I
\]
and \( [a_1, a_2, a_3, a'_4] \in I \). It follows that in this case \( f \in I \). Similarly, \( f \in I \) if \( a_3 = a'_3 a''_3 \).

Suppose that \( a_1 = a'_1 a''_1 \). Then
\[
 f = \left( [a''_1, a_2, a_3][a_4, a_5] + [a'_1 a''_1, a_2, a_4][a_3, a_5] \right) a''_5 \\
= \left( [a''_1, a_2, a_3][a_4, a_5] + [a'_1 a''_1, a_2, a_4][a_3, a_5] \right) a''_5 \\
+ [a''_1, a_2, a_3][a_4, a_5] + [a'_1, a_2, a_4][a_3, a_5] + [a''_1, a_2, a_4][a_3, a_5] \\
+ [a'_1, a_2, a_4][a'_1, a_4][a_3, a_5] + [a'_1, a_2, a_4][a''_1, a_4][a_3, a_5] \\
= a''_5 \left( [a''_1, a_2, a_3][a_4, a_5] + [a'_1 a''_1, a_2, a_4][a_3, a_5] \right) \\
+ a'_1 \left( [a'_1, a_2, a_3][a_4, a_5] + [a'_1, a_2, a_4][a_3, a_5] \right) + [a''_1, a_2, a_4][a_3, a_5] \\
+ [a'_1, a_2, a_4][a''_1, a_4][a_3, a_5] - [a''_1, a_3][a_2, a_4][a_3, a_5] \\
- [a''_1, a_3][a_2, a''_4][a_3, a_5].
\]

Note that
\[
([a'_1, a_3][a_2, a''_4] + [a'_1, a_2][a_3, a''_4])[a_4, a_5] \in I, \\
([a'_1, a_4][a_2, a''_4] + [a'_1, a_2][a_4, a''_4])[a_3, a_5] \in I
\]
because the polynomials of the form \((32)\) of degree \(m\) belong to \(I\). On the other hand, by the induction hypothesis,
\[
[a''_1, a_2, a_3][a_4, a_5] + [a''_1, a_2, a_4][a_3, a_5] \in I,
\]
\[
[a'_1, a_2, a_3][a_4, a_5] + [a'_1, a_2, a_4][a_3, a_5] \in I
\]
and \([a'_1, a_2, a_4, a''_1] \in I\) so in this case \(f \in I\). Similarly, \(f \in I\) if \(a_2 = a'_2 a''_2\).

Thus, if \(f\) is a polynomial of the form \((34)\) of degree \(m\) then \(f \in I\).

**Case 4.** Suppose that \(f\) is of the form \((35)\). If \(\deg f = 5\) then \(f\) is of the form \((1)\) so \(f \in I\). If \(\deg f > 5\) then, for some \(i, 1 \leq i \leq 5\), we have \(a_i = a'_i a''_i\) where \(\deg a'_i, \deg a''_i < \deg a_i\).

Suppose first that \(a_5 = a'_5 a''_5\). Then
\[
f = [a_1, a_2, a_3][a_4, a'_5 a''_5] + [a_1, a_4, a_3][a_2, a'_5 a''_5]
\]
\[
= [a_1, a_2, a_3][a'_5 a''_5][a_4, a_5] + [a_1, a_2, a_3][a_4, a'_5][a''_5]
\]
\[
+ [a_1, a_4, a_3][a'_5 a''_5][a_2, a''_5] + [a_1, a_4, a_3][a_2, a'_5][a''_5]
\]
\[
= a'_5([a_1, a_2, a_3][a_4, a_5] + [a_1, a_4, a_3][a_2, a'_5]) + ([a_1, a_2, a_3][a_4, a'_5] + [a_1, a_2, a_3][a_4, a''_5])
\]
\[
+ [a_1, a_4, a_3][a_2, a'_5][a''_5] + [a_1, a_2, a_3, a'_5][a_4, a''_5] + [a_1, a_4, a_3, a''_5][a_2, a''_5]
\]
so in this case \(f \in I\) by the induction hypothesis.

Suppose that \(a_4 = a'_4 a''_4\). Then
\[
f = [a_1, a_2, a_3][a'_4 a''_4, a_5] + [a_1, a'_4 a''_4, a_3][a_2, a_5]
\]
\[
= [a_1, a_2, a_3][a''_4][a_4, a_5] + [a_1, a_2, a_5][a'_4 a''_4]
\]
\[
+ ([a_1, a'_4]a''_4 + a'_4[a_1, a''_4]), a_3][a_2, a_5]
\]
\[
= [a_1, a_2, a_3][a'_4][a''_4, a_5] + [a_1, a_2, a_5][a'_4, a_5]a''_4 + [a_1, a'_4][a''_4, a_3][a_2, a_5]
\]
\[
+ [a_1, a'_4, a_3][a''_4][a_2, a_5] + [a_1, a'_4, a_3][a_2, a_5] + [a'_4, a_3][a_2, a''_4][a_5]
\]
\[
= a'_4([a_1, a_2, a_3][a'_4, a_5] + [a_1, a'_4, a_3][a_2, a_5]) + [a_1, a_2, a_3, a'_4][a''_4, a_5]
\]
\[
+ ([a_1, a_2, a_3][a'_4, a_5] + [a_1, a'_4, a_3][a_2, a_5])a''_4 - [a_1, a'_4, a_3][a_2, a''_4]
\]
\[
+ ([a_1, a'_4][a''_4, a_3] + [a_1, a'_4][a''_4, a_3])a_2, a_5 + [a'_4, a_3, a_4][a'_4, a_5], a_2, a_5]
\]
It follows that in this case \(f \in I\) by the induction hypothesis and because the polynomials of the forms \((32)\)–\((33)\) of degree \(m\) belong to \(I\). Similarly, \(f \in I\) if \(a_2 = a'_2 a''_2\).

Suppose that \(a_3 = a'_3 a''_3\). We have
\[
f = [a_1, a_2, a'_3 a''_3][a_4, a_5] + [a_1, a_4, a'_3 a''_3][a_2, a_5]
\]
\[
= a'_3[a_1, a_2, a''_3][a_4, a_5] + [a_1, a_2, a'_3][a''_3][a_4, a_5]
\]
\[
+ [a_1, a_4, a''_3][a_2, a_5] + [a_1, a_4, a'_3][a''_3][a_2, a_5]
\]
\[
= a'_3([a_1, a_2, a''_3][a_4, a_5] + [a_1, a_4, a''_3][a_2, a_5])
\]
\[
+ ([a_1, a_2, a''_3][a_4, a_5] + [a_1, a_4, a''_3][a_2, a_5])a''_3
\]
\[
- [a_1, a_2, a'_3][a_4, a_5, a''_3] - [a_1, a_4, a'_3][a_2, a_5, a''_3]
\]
so in this case \(f \in I\) as above.
Suppose that $a_1 = a_1'' a_1'''$. Then

$$f = [a_1'' a_1''', a_2, a_3] [a_4, a_5] + [a_1'' a_1'', a_4, a_3] [a_2, a_5]$$

$$= [(a_1'' [a_1'', a_2] + [a_1', a_2] a_1''')] [a_4, a_5] + [(a_1'' [a_4])$$

$$= a_1'' [a_1'', a_2, a_3] [a_4, a_5] + [a_1', a_3] [a_1'', a_2] [a_4, a_5] + [a_1', a_2] [a_1'', a_3] [a_4, a_5]$$

$$+ [a_1', a_2, a_3] a_1'' [a_4, a_5] + [a_1' a_1', a_1', a_3] [a_2, a_5] + [a_1', a_1', a_3] [a_4', a_4] [a_2, a_5]$$

$$+ [a_1', a_4] [a_1'', a_3] [a_2, a_5] + [a_1', a_4, a_3] a_1'' [a_2, a_5]$$

$$= a_1' ([a_1'', a_2, a_3] [a_4, a_5] + [a_1'', a_4, a_3] [a_2, a_5]) + a_1'' ([a_1', a_2, a_3] [a_4, a_5]$$

$$+ [a_1', a_4, a_3] [a_2, a_5]) + [a_1', a_2, a_3, a_1''] [a_4, a_5] + [a_1' a_1', a_4, a_3, a_1''] [a_2, a_5]$$

$$+ [a_1', a_3] ([a_1'', a_2, a_4, a_5] + [a_1'', a_4, a_3] [a_2, a_5]) + ([a_1', a_2, a_4, a_5] + [a_1', a_4] [a_2, a_5])$$

$$\times [a_1'' a_3] + [a_1', a_2] [a_1'', a_3, a_4, a_5] + [a_1', a_4] [a_1'', a_3, a_2, a_5]$$

so in this case $f \in I$ by the induction hypothesis and because the polynomials of the form (35) of

Thus, if $f$ is a polynomial of the form (35) of degree $m$ then $f \in I$.

**Case 5.** Finally, suppose that $f$ is of the form (31). Since $\deg f > 4$, we have $a_i = a_i'' a_i'''$ for some $i$, $1 \leq i \leq 4$ where $\deg a_i', \deg a_i'' < \deg a_i$. Suppose that $a_4 = a_4'' a_4'''$. Then

$$f = [a_1, a_2, a_3, a_4' a_4''] = a_4' [a_1, a_2, a_3, a_4'] + [a_1, a_2, a_3, a_4'] a_4''$$

so, by the induction hypothesis, $f \in I$.

Now suppose that $a_3 = a_3' a_3''$. Then

$$f = [a_1, a_2, a_3' a_3'', a_4] = [a_3'][a_1, a_2, a_3''] + [a_1, a_2, a_3'] a_3''$$

$$= a_3' [a_1, a_2, a_3', a_4] + [a_3, a_4] [a_1, a_2, a_3''] + [a_1, a_2, a_3'] [a_3'', a_4] + [a_1, a_2, a_3''] a_3''$$

$$= a_3' [a_1, a_2, a_3', a_4] + [a_1, a_2, a_3''] [a_3', a_4] + [a_1, a_2, a_3''] [a_3', a_4]$$

$$+ [a_1, a_2, a_3', a_4] a_3'' + [a_3', a_4, [a_1, a_2, a_3'']]$$

By the induction hypothesis, $[a_1, a_2, b, a_4] \in I$ if $b \in \{a_3', a_3''\}$ and

$$[a_3', a_4, [a_1, a_2, a_3']] = -[[a_1, a_2, a_3'], [a_3', a_4]]$$

$$= -[a_1, a_2, a_3', a_3', a_4] + [a_1, a_2, a_3', a_4, a_3'] \in I.$$
Finally suppose that either $a_1 = a''_1 a''_2$, or $a_2 = a''_2 a''_3$. It is clear that without loss of generality we may assume $a_2 = a''_2 a''_3$. Then

$$f = [a_1, a''_2 a''_3, a_3, a_4] = [(a''_2)[a_1, a''_3] + [a_1, a''_2][a''_3] + [a_1, a''_2, a_3][a''_3], a_4]$$

$$= [(a''_2)[a_1, a''_3] + [a_2, a_3][a_1, a''_2] + [a_1, a''_2][a_3, a_4] + [a_1, a''_2, a_3][a''_3], a_4]$$

$$= a''_2[a_1, a''_3, a_4] + [a_2, a_4][a_1, a''_2, a_3] + [a_2, a_3][a_1, a''_2, a_4]$$

$$+ [a_2, a_3, a_4][a_1, a''_2] + [a_1, a''_2][a''_3, a_4] + [a_1, a''_2, a_3][a''_3, a_4] + [a_1, a''_2, a_3][a''_3, a_4]$$

$$+ [a_1, a''_2, a_3][a''_2, a_4] + [a_1, a''_2, a_3][a''_2, a_4]$$

We have $a''_2[a_1, a''_3, a_4], [a_1, a''_2, a_3, a_4]a''_3 \in I$ by the induction hypothesis and

$$[a_1, a''_2, a_3, a_4][a''_2, a_3] + [a_1, a''_2, a_3][a''_2, a_4] = -([a_1, a''_2, a_3][a_3, a_4][a_2] + [a_1, a''_2, a_3][a_4, a''_2]) \in I$$

because each polynomial of the form (33) of degree $m$ belong to $I$. Further,

$$[a''_2, a_4][a_1, a''_3, a_3] + [a''_2, a_3][a_1, a''_2, a_4]$$

$$= [a_1, a''_2, a_3, a_4] + [a_1, a''_2, a_4][a''_3, a_4] - ([a_1, a''_2, a_3], [a_2, a_4]) - ([a_1, a''_2, a_4], [a''_2, a_3])$$

where $[a_1, a''_2, a_3, a_4] + [a_1, a''_2, a_4][a''_3, a_4] \in I$ as above.

$$[a_1, a''_2, a_3, a_4] + [a_1, a''_2, a_4][a''_3, a_4] \in I$$

by the induction hypothesis and, similarly, $[a_1, a''_2, a_4], [a''_2, a_3] \in I$. Finally,

$$[a''_2, a_3, a_4][a_1, a''_2] + [a_1, a''_2][a''_3, a_4]$$

$$= [a_3, a''_2, a_4][a''_3, a_1] + [a_3, a''_2, a_4][a''_3, a_1] - ([a_3, a''_2, a_4], [a''_2, a_1])$$

where $[a_3, a''_2, a_4][a''_3, a_1] + [a_3, a''_2, a_4][a''_3, a_1] \in I$ and $([a_3, a''_2, a_4], [a''_2, a_1]) \in I$ as above. It follows that $f = [a_1, a''_2, a_3, a_4] \in I$, as required.

Thus, $[a_1, a''_2, a_3, a_4] \in I$ for all monomials $a_i \in K\langle X \rangle$ such that $[a_1, a''_2, a_3, a_4]$ is of degree $m$. This establishes the induction step and proves that $T^{(4)} \subseteq I$.

The proof of Theorem 1.3 is completed.

Now we prove Corollary 1.4.

Let $I$ be the two-sided ideal of $K\langle Y \rangle$ generated by the polynomials (1), (2), (3) and (7). We will prove that $I = T^{(4)}$. Since the ideal $T^{(4)}$ is generated by the polynomials (1), (2), (3) and (5) where polynomials (3) and (4) are of the form (6) and polynomials (5) are of the form (7), we have $T^{(4)} \subseteq I$. To prove that $I \subseteq T^{(4)}$ it suffices to check that all monomials of degree less than $m$ belong to $T^{(4)}$.

The following lemma is well-known (see, for instance, [12, 15 Lemma 1], [22, 24 Lemma 2.3][24, Lemma 1]).

**Lemma 5.1.** For all $a_1, \ldots, a_5 \in Z\langle X \rangle$ and all $\sigma \in S_5$, we have

$$(36) \quad [a_1, a_2][a_3, a_4, a_5] \equiv \text{sgn}(\sigma)[a_{\sigma(1)}, a_{\sigma(2)}][a_{\sigma(3)}, a_{\sigma(4)}, a_{\sigma(5)}] \pmod{T^{(4)}}.$$
Proof. It is clear that the congruence (36) holds if \( \sigma = (12) \) or \( \sigma = (34) \). If \( \sigma = (25) \) or \( \sigma = (24) \) then, by Theorem 1.3, the congruence (36) holds as well. Since the transpositions (12), (34), (25) and (24) generate the entire group \( S_5 \) of the permutations of the set \( \{1, 2, 3, 4, 5\} \), the result follows. □

It follows from Lemma 5.1 that the polynomials (6) belong to \( T^{(4)} \). One can prove in a similar way that the polynomials (7) belong to \( T^{(4)} \) as well. Thus, \( I \subseteq T^{(4)} \), as required.

The proof of Corollary 1.4 is completed.

References

[1] Bernhard Amberg, Yaroslav Sysak, Associative rings whose adjoint semigroup is locally nilpotent, Archiv der Mathematik (Basel) 76 (2001), 426–435.
[2] Noah Arbesfeld, David Jordan, New results on the lower central series quotients of a free associative algebra, Journal of Algebra 323 (2010), 1813–1825. arXiv:0902.1899
[3] Martina Balagović, Anirudha Balasubramanian, On the lower central series quotients of a graded associative algebra, Journal of Algebra 328 (2011), 287–300.
[4] Asilata Bapat, David Jordan, Lower central series of free algebras in symmetric tensor categories, Journal of Algebra 373 (2013), 299–311. arXiv:1001.1375
[5] Surya Bhupatiraju, Pavel Etingof, David Jordan, William Kuszmaul and Jason Li, Lower central series of a free associative algebra over the integers and finite fields, Journal of Algebra 372 (2012), 251–274. arXiv:1203.1893
[6] Eudes Antonio da Costa, Alexei Krasilnikov, Relations in universal Lie nilpotent associative algebras of class 4, arXiv:1306.4294
[7] Katherine Cordwell, Teng Fei, Kathleen Zhou, On lower central series quotients of finitely generated algebras over \( \mathbb{Z} \), arXiv:1309.1237
[8] Galyna Dobrovolska, Pavel Etingof, An upper bound for the lower central series quotients of a free associative algebra, International Mathematics Research Notices 2008, no 12, Art. ID rnm039, 10 pp. arXiv:0801.1997
[9] Galyna Dobrovolska, John Kim, Xiaoguang Ma, On the lower central series of an associative algebra (with an appendix by Pavel Etingof), Journal of Algebra 320 (2008), 2132–237. arXiv:0709.1905
[10] V. Drensky, Free algebras and PI-algebras. Graduate course in algebra, Springer, Singapore, 1999.
[11] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, Journal of Algebra 91 (1984), 1–17.
[12] Pavel Etingof, John Kim, Xiaoguang Ma, On universal Lie nilpotent associative algebras, Journal of Algebra 321 (2009), 697–703. arXiv:0805.1999
[13] Boris Feigin, Boris Shoikhet, On \([A,A]/[A, [A,A]]\) and on a \(W_n\)-action on the consecutive commutators of free associative algebras, Mathematical Research Letters 14 (2007), 781–795. arXiv:math/0610410
[14] A. Giambruno, M. Zaicev, Polynomial identities and asymptotic methods. Mathematical Surveys and Monographs, 122. American Mathematical Society, Providence, RI, 2005.
[15] A.S. Gordienko, Codimensions of commutators of length 4, Russian Mathematical Surveys 62 (2007), 187–188.
[16] A.V. Grishin, L.M. Tsybulya, A.A. Shokola, On T-spaces and relations in relatively free Lie nilpotent associative algebras, Journal of Mathematical Sciences (New York) 177 (2011), 868–877.
[17] Narain Gupta, Frank Levin, On the Lie ideals of a ring, Journal of Algebra 81 (1983), 225–231.
[18] S.A. Jennings, *On rings whose associated Lie rings are nilpotent*, Bulletin of the American Mathematical Society 53 (1947), 593–597.

[19] David Jordan, Hendrik Orem, *An algebro-geometric construction of lower central series of associative algebras*, [arXiv:1302.3992](http://arxiv.org/abs/1302.3992).

[20] George Kerchev, *On the filtration of a free algebra by its associative lower central series*, Journal of Algebra 375 (2013), 322–327. [arXiv:1101.5741](http://arxiv.org/abs/1101.5741).

[21] A.N. Krasil’nikov, *On the semigroup nilpotency and the Lie nilpotency of associative algebras*, Mathematical Notes 62 (1997), 426–433.

[22] Alexei Krasilnikov, *The additive group of a Lie nilpotent associative ring*, Journal of Algebra 392 (2013), 10–22. [arXiv:1204.2674](http://arxiv.org/abs/1204.2674).

[23] V.N. Latyshev, *On finite generation of a T-ideal with the element \([x_1, x_2, x_3, x_4]\)*, Siberian Mathematical Journal 6 (1965), 1432–1434. (in Russian)

[24] V.M. Petrogradsky, *Codimension growth of strong Lie nilpotent associative algebras*, Communications in Algebra 39 (2011), 918–928.

[25] D.M. Riley, V. Tasić, *Mal’cev nilpotent algebras*, Archiv der Mathematik (Basel) 72 (1999), 22–27.

[26] L.H. Rowen, *Polynomial Identities in Ring Theory*. Pure and Applied Mathematics, 84. Acad. Press, New York-London, 1980.

[27] I.B. Volichenko, *The T-ideal generated by the element \([x_1, x_2, x_3, x_4]\)*, Preprint no. 22, Institute of Mathematics of the Academy of Sciences of the Belorussian SSR, 1978. (in Russian)