Nonlinear integral equations for an inverse electromagnetic scattering problem

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Abstract. We present a new method to solve an inverse scattering problem for time-harmonic electromagnetic waves. The method is based on an equivalence relation of the inverse problem to a system of two integral equations. This system can be solved numerically by a recently developed spectral method for the direct scattering problem.

1. Introduction
In many practical applications such as medical imaging, non-destructive testing, geophysical exploration and radar and sonar obstacle detection, inverse scattering of acoustic and electromagnetic waves is used to detect unknown obstacles. In general one wants to find the shape of the unknown obstacle from a knowledge of the scattered wave in the far field. For the sake of simplicity we confine ourselves to the case of time-harmonic electromagnetic waves and assume that the unknown obstacle is a perfect conductor.

We begin with the formulation of the direct scattering problem. Given an obstacle $D \subset \mathbb{R}^3$, that is a bounded domain with a connected $C^2$ boundary $\partial D$, and an incident field $E^i$ and $H^i$, we want to find the total field $E = E^i + E^s$ and $H = H^i + H^s$ as a solution to the reduced Maxwell equations

$$\text{curl}E - i\kappa H = 0 \quad \text{and} \quad \text{curl}H + i\kappa E = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bar{D},$$

satisfying the perfect conductor boundary condition

$$\nu \times E = 0 \quad \text{on} \quad \partial D.$$  

By $\nu$ we denote the outward unit normal to the boundary $\partial D$. Further the scattered field satisfies one of the Silver-Müller radiation conditions

$$\lim_{r \to \infty} (H^s \times x - rE^s) = 0 \quad \text{or} \quad \lim_{r \to \infty} (E^s \times x + rH^s) = 0$$

uniformly with respect to all directions.
The scattered field has an asymptotic behavior of the form

\[ E_s(x) = e^{-i\kappa |x|} \left\{ E_\infty(\hat{x}) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \quad (4) \]

\[ H_s(x) = e^{-i\kappa |x|} \left\{ H_\infty(\hat{x}) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \quad (5) \]

uniformly in all directions. The far field patterns \( E_\infty \) and \( H_\infty \) are defined on the unit sphere \( \partial B \) and form our measured data for the inverse problem.

The inverse problem under consideration is to reconstruct the unknown shape \( \partial D \) of the perfect conducting obstacle \( D \) from a knowledge of the incident field \( E^i \) and \( H^i \) and the corresponding far field pattern \( E_\infty \) and \( H_\infty \), respectively. We remark that up to now it is not clear, if the considered inverse problem is uniquely solvable, although there are some recent results on this topic for the special case of polyhedral domains.

To reconstruct the unknown shape of the obstacle, we use a method which is based on integral equations arising from the Stratton-Chu formulas. The basic concept of this method was first introduced by Kress and Rundell (2005, [11]) for an inverse problem for the Laplace equation and transferred to inverse acoustic scattering problems by Ivanyshyn and Kress (2006, [9]) and Johansson and Sleeman (2007, [10]). A similar equation was used by Hassen, Erhard and Potthast (2006, [8]). Their method belongs to the class of decomposition methods, however the method described in this paper is a hybrid method, i.e. it is something between decomposition and iterative methods. The results in this paper originate from the author’s PhD thesis [12].

The plan of the paper is as follows. In section 2, we prove that our inverse problem is equivalent to a system of two integral equations that are nonlinear with respect to the unknown boundary. After that we discuss how to solve these equations iteratively. In the last section we give an initial description of the numerical discretization. Unfortunately there are still no numerical results which confirm that our method works in practice.

2. Formulation as system of integral equations

In this section we want to show that the above inverse problem is equivalent to a system of two integral equations. We begin with the derivation of this system and state the main result at the end of the section in Theorem 2.1.

In contrast to Ivanyshyn and Kress, who use Huygen’s principle, we use the Stratton-Chu formulas as starting point for our analysis. This is, because we want to avoid the occurrence of the hypersingular electric dipole operator, that causes numerical difficulties.

Any interior solution \( E^i, H^i \in C^1(D) \cap C(\hat{D}) \) to the Maxwell equations can be represented as follows ([1], Thm. 6.2):

\[ E^i(x) = -\text{curl} \int_{\partial D} \nu(y) \times E^i(y) \Phi(x, y) \, ds(y) \]

\[ + \frac{1}{i\kappa} \text{curl} \text{curl} \int_{\partial D} \nu(y) \times H^i(y) \Phi(x, y) \, ds(y), \quad x \in D \]

\[ H^i(x) = -\text{curl} \int_{\partial D} \nu(y) \times H^i(y) \Phi(x, y) \, ds(y) \]

in the case of spherical outgoing waves.

\[ E^s(x) = \frac{e^{-i\kappa |x|}}{|x|} \left\{ E_\infty(\hat{x}) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \quad (4) \]

\[ H^s(x) = \frac{e^{-i\kappa |x|}}{|x|} \left\{ H_\infty(\hat{x}) + O \left( \frac{1}{|x|} \right) \right\}, \quad |x| \to \infty, \quad (5) \]

in the case of spherical outgoing waves.
Analogously for radiating solutions $E^s, H^s \in C^1(R^3 \setminus \bar{D}) \cap C(R^3 \setminus D)$ we have that ([1], Thm. 6.6)

$$E^s(x) = \text{curl} \int_{\partial D} \nu(y) \times E^s(y) \Phi(x, y) \, ds(y)$$

$$H^s(x) = \text{curl} \int_{\partial D} \nu(y) \times H^s(y) \Phi(x, y) \, ds(y)$$

If we use the jump relations for the magnetic dipole operator

$$\mathcal{(Ma)}(x) := 2 \int_{\partial D} \nu(x) \times \text{curl}_x \{a(y) \Phi(x, y)\} \, ds(y), \quad x \in \partial D,$$

and the electric dipole operator

$$\mathcal{(Nb)}(x) := 2 \nu(x) \times \text{curl} \int_{\partial D} \nu(y) \times b(y) \Phi(x, y) \, ds(y), \quad x \in \partial D,$$

and the Stratton-Chu formulas, we obtain the so-called Calderon projections ([14], Thm. 5.4, Sec. 1.4.1):

For solutions $E^i, H^i \in C^1(D) \cap C^{0,\alpha}(\bar{D})$ to the Maxwell equations we have that

$$\begin{pmatrix} \nu \times E^i \\ \nu \times H^i \end{pmatrix} = \begin{pmatrix} -\mathcal{M} & \frac{1}{ik\nu} \mathcal{N} \mathcal{Q} \\ \frac{1}{ik\nu} \mathcal{N} \mathcal{Q} & -\mathcal{M} \end{pmatrix} \begin{pmatrix} \nu \times E^i \\ \nu \times H^i \end{pmatrix}$$

(12)

and for radiating solutions $E^s, H^s \in C^1(R^3 \setminus \bar{D}) \cap C^{0,\alpha}(R^3 \setminus D)$ we obtain

$$\begin{pmatrix} \nu \times E^s \\ \nu \times H^s \end{pmatrix} = \begin{pmatrix} \mathcal{M} & \frac{1}{ik\nu} \mathcal{N} \mathcal{Q} \\ \frac{1}{ik\nu} \mathcal{N} \mathcal{Q} & \mathcal{M} \end{pmatrix} \begin{pmatrix} \nu \times E^s \\ \nu \times H^s \end{pmatrix}$$

(13)

Here, the operator $Q$ is defined by

$$Qa := a \times \nu.$$

(14)

If we build the vector product of equation (12) and (13) with the normal $\nu$ and combine, we get the magnetic field equation

$$b + \mathcal{M}'b = 2(\nu \times H^i) \times \nu$$

(15)

for the tangential component $b := (\nu \times H) \times \nu$ of the total magnetic field $H$. Here the a priori knowledge that $D$ is a perfect conductor, i.e. $\nu \times E = 0$ on $\partial D$, was used. $\mathcal{M}'$ is the adjoint to $\mathcal{M}$ with respect to the bilinear form

$$B(a, b) := \int_{\partial D} a \cdot b \, ds.$$
We have the following representation for $\mathcal{M}'$ ([1], (6.44)):

$$\mathcal{M}' = QM.$$  (16)

Now we come to the second equation. We consider the corresponding magnetic far field pattern $H_\infty$ and use Huygens principle ([1], Thm. 6.22) to get:

The magnetic far field pattern for an entire solution $E$ and $H$ to the Maxwell equations has the representation

$$H_\infty(\xi) = \frac{ik}{4\pi} \xi \times \int_{\partial D} \nu(y) \times H(y) e^{-ik\xi \cdot y} \, ds(y), \quad \xi \in \partial B,$$  (17)

where $\partial B$ denotes the unit sphere in $\mathbb{R}^3$. If we define the magnetic far field operator by

$$(S_\infty f)(\xi) := \frac{ik}{4\pi} \xi \times \int_{\partial D} f(y) e^{-ik\xi \cdot y} \, ds(y), \quad \xi \in \partial B, f \in c(\partial D),$$  (18)

we find that every solution $\partial D$ to the inverse scattering problem and the tangential component of the total magnetic field $b := (\nu \times H) \times \nu$ satisfy the following system of integral equations:

$$b + \mathcal{M}' b = 2(\nu \times H^i) \times \nu \big|_{\partial D}$$  (19)

$$S_\infty (\nu \times b) = H_\infty.$$  (20)

Notice that both equations depend nonlinearly on the unknown boundary $\partial D$.

In order to show the equivalence of this system and the inverse problem, we have to prove that every solution to (19) and (20) solves the inverse problem. We first remark that $b$ is a tangential vector field because of the right hand side of (19) and the mapping properties of $\mathcal{M}'$. Since the incident wave $H^i$ is known we also know the tangential component

$$\nu \times H^i \big|_{\partial D},$$

thus we can define $E^s$ and $H^s$ as the solution to the exterior problem with boundary condition

$$\nu \times H^s \big|_{\partial D} = \nu \times b - \nu \times H^i \big|_{\partial D}.$$  (21)

Then again the Stratton-Chu formulas are valid for $E^s$ and $H^s$ in $\mathbb{R}^3 \setminus D$ and the same is true for the incident field $E^i$ and $H^i$ in $D$. If we proceed similar as above and use the Calderon projections, we obtain that

$$b + \mathcal{M}' b = 2(\nu \times H^i) \times \nu + \frac{1}{ik} NQ(\nu \times E) \times \nu.$$

Here we have used

$$b = \left(\nu \times \left(H^s + H^i\right) \big|_{\partial D}\right) \times \nu,$$

which is true because of (21) and $E := E^s + E^i$ on $\partial D$. As $b$ solves (19) we find that

$$\frac{1}{ik} NQ(\nu \times E) \times \nu = 0.$$  (22)
If we now use that \( \mathcal{N} \) maps tangential vector fields onto tangential vector fields ([1], Thm. 6.17), we see that (22) is satisfied if and only if

\[
\mathcal{N}Q(\nu \times E) = 0.
\]

Under the condition that \( \kappa \) is a regular wave number the operator \( \mathcal{N}Q \) is injective ([14], Sec. 1.4.2, §2). In this case we finally obtain from (23) that

\[
\nu \times E = 0 \text{ on } \partial D,
\]

which is the perfect conductor boundary condition for \( E \). From equation (20) we see that \( E \) and \( H := H^s + H^i \) have the correct far field patterns, so we finally have shown that the solutions \( \partial D \) and \( b \) of the system (19) and (20) solve our inverse scattering problem, provided \( \kappa \) is a regular wave number.

We summarise our results in the following Theorem:

**Theorem 2.1** *The inverse electromagnetic scattering problem under consideration is equivalent to the system of integral equations (19) and (20), provided \( \kappa \) is a regular wave number.*

As remarked before, using the Stratton-Chu formulas avoids the occurrence of the hypersingular electric dipole operator \( \mathcal{N} \). In our equations (19) and (20) only the adjoint \( M' \) of the magnetic field operator \( M \) and the far field operator \( S_\infty \) occur and these operators can be handled numerically without difficulties.

3. **Iterative solution method**

Now in order to get a solution to the inverse scattering problem, we use the equivalence in Theorem 2.1 and solve the system (19) and (20) for the unknown boundary \( \partial D \) and the density \( b \). For this we suggest to use an iterative procedure as described in the following.

Let us first compare the two equations (19) and (20). We see that for a fixed boundary \( \partial D \) the first equation (19) is of the second kind, i.e., it is well posed and uniquely solvable for the density \( b \) if \( \kappa \) is regular. On the other hand, the second equation (20) is an integral equation of the first kind and consequently ill posed. As the kernel (the exponential function) is very smooth, it actually is severely ill posed. Additionally it is nonlinear with respect to \( \partial D \).

If we now start with an initial guess for the unknown boundary and solve equation (19) only for the density \( b \), we have preserved the nice properties of this equation. This is the reason that we solve the equations separately in our method instead of simultaneous, as it is done in [9]. After obtaining an approximate solution of the first equation, we use this density in a linearized version of the second equation to get a better approximation to the unknown boundary \( \partial D \). After that we iterate this procedure.

Before we come to the numerical discretization, we develop the linearized version of the second equation. For this we restrict ourselves to starlike domains with global parametrization

\[
q(\theta, \phi) := r(\theta, \phi) p(\theta, \phi).
\]

Here \( r \) is the unknown radius function that we want to determine and

\[
p(\theta, \phi) := (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)^T
\]

is the usual parametrization of the unit sphere in polar coordinates.
If we use the density
\[ \psi := (\nu \times b \circ q)J , \]
where \( J \) denotes the Jacobian of \( q \), we obtain the following parametrized version of the far field operator
\[ (S_\infty \psi)(\tilde{\theta}, \tilde{\phi}) := \frac{i\kappa}{4\pi} p(\tilde{\theta}, \tilde{\phi}) \times \int_0^{2\pi} \int_0^{\pi} \psi(\theta, \phi) e^{-i\kappa p(\tilde{\theta}, \tilde{\phi}) \cdot q(\theta, \phi)} \sin \theta \, d\theta \, d\phi . \] (26)

To emphasize the dependence of \( S_\infty \) on the radius function \( r \) we write \( S_\infty(r, \psi) \). In order to get a linearized equation with respect to the boundary, we first have to calculate the Fréchet derivative of \( S_\infty \) (see [15]):
\[ S'_\infty(r, \psi, s) = \frac{i\kappa}{4\pi} p(\tilde{\theta}, \tilde{\phi}) \times \int_0^{2\pi} \int_0^{\pi} \psi(\theta, \phi) e^{-i\kappa p(\tilde{\theta}, \tilde{\phi}) \cdot q(\theta, \phi)} \cdot \zeta(\theta, \phi) \sin \theta \, d\theta \, d\phi . \] (27)

Here \( s \) denotes the desired update to \( r \) and further on we used the notation
\[ \zeta(\theta, \phi) := s(\theta, \phi) \cdot p(\theta, \phi) . \]

Then we finally obtain the linearized version of (20)
\[ S'_\infty(r, \psi, s) = H_\infty - S_\infty(r, \psi) . \] (28)

We now have to solve this linear equation for the radius update \( s \) to get a new approximation \( r + s \) to the unknown boundary.

### 4. Numerical discretization
In this last section we want to discuss the discretization of the integral operators. Here we follow different strategies. As the operators \( S_\infty \) and \( S'_\infty \) have very smooth kernels, it simply suffices to apply a suitable quadrature rule to get a discrete version of these two operators. On the other hand \( \mathcal{M}' \) has a weakly singular kernel, hence we suggest to apply a spectral method to solve equation (19). Such spectral methods are well known in acoustic scattering ([7],[3]) and were recently extended to electromagnetic scattering. In the following we just want to give a brief idea of these spectral methods. For more details we refer to [4], [5], [6] and [12].

For simplicity we assume that our operator \( \mathcal{M}' \) is decomposed as follows:
\[ (\mathcal{M}'b)(x) := \int_{\partial D} \left[ \frac{1}{|x - y|} M_1(x, y) + M_2(x, y) \right] b(y) \, ds(y) , \] (29)

where \( M_1(x, y) \) and \( M_2(x, y) \) are matrices that map tangential vectors with respect to \( y \in \partial D \) onto tangential vectors with respect to \( x \in \partial D \):
\[ \nu(y) \cdot a = 0 \quad \Rightarrow \quad \nu(x) \cdot (M_1(x, y)a) = 0 \quad \text{and} \quad \nu(x) \cdot (M_2(x, y)a) = 0 , a \in \mathbb{R}^3 . \]

We first transform the integral to the unit sphere \( \partial B \). This is possible because we have assumed that \( D \) is starlike. In order to preserve the tangential nature of the equation, we also add the Jacobian matrix \( D_q \) of the parametrization \( q \) and obtain:
\[ D_{q(\xi)}^{-1} b(q(\xi)) + \int_{\partial B} D_{q(\eta)}^{-1} M(q(\xi), q(\eta)) D_{q(\eta)}^{-1} b(q(\eta)) J(\eta) \, ds(\eta) = D_{q(\xi)}^{-1} c(q(\xi)) , \]
where we have used $c$ as abbreviation for the right hand side of (19). Further on we denote vectors on the sphere by Greek letters and assume that $x \in \partial D$ is parametrized by $\xi \in \partial B$ and $y \in \partial D$ by $\eta \in \partial B$. The inverse $D_q^{-1}$ always exists, because $q$ is a global parametrization. If we then use the function

$$R(\xi, \eta) := \frac{|\xi - \eta|}{|q(\xi) - q(\eta)|}$$

(30)

and the definitions

$$\tilde{b}(\xi) := D_{q(\xi)}^{-1} b(q(\xi))$$

(31)

$$\tilde{c}(\xi) := D_{q(\xi)}^{-1} c(q(\xi))$$

(32)

$$\tilde{M}_1(\xi, \eta) := D_{q(\xi)}^{-1} R(\xi, \eta) M_1(q(\xi), q(\eta)) J(\eta) D_{q(\eta)}$$

(33)

$$\tilde{M}_2(\xi, \eta) := D_{q(\xi)}^{-1} M_2(q(\xi), q(\eta)) J(\eta) D_{q(\eta)}$$

(34)

we obtain the following equation on the sphere:

$$\tilde{b}(\xi) + \int_{\partial B} \begin{bmatrix} \tilde{M}_1(\xi, \eta) & \tilde{M}_2(\xi, \eta) \end{bmatrix} \tilde{b}(\eta) \, ds(\eta) = \tilde{c}(\xi), \quad \xi \in \partial B.$$  

(35)

The new kernels $\tilde{M}_1$ and $\tilde{M}_2$ map tangential vectors with respect to $\eta \in \partial B$ onto tangential vectors with respect to $\xi \in \partial B$. Now equation (35) is ready for our discretization. For this, we approximate the smooth parts of the kernel by tangential vector spherical harmonics (see [2]):

**Definition 4.1** Let $Y_{l,k}$ ($l = 1, 2, \ldots, |k| \leq l$) be an orthonormal system of scalar spherical harmonics. Then we define the tangential vector spherical harmonics:

$$y_{l,k}^{(2)}(\xi) := \nabla_\xi Y_{l,k}(\xi) \quad \text{and} \quad y_{l,k}^{(3)}(\xi) := \xi \times \nabla_\xi Y_{l,k}(\xi).$$

(36)

By truncation at order $n \in N$ and by applying the Gauss-rectangular rule to the Fourier coefficients of the Fourier expansion in vector spherical harmonics we obtain the following approximation operator:

$$L_n f := \sum_{i=2}^3 \sum_{l=1}^n \sum_{|k| \leq l} (f, y_{l,k}^{(i)})_{n} y_{l,k}^{(i)}.$$  

(37)

For more theoretical background on this hyperinterpolation operator and the vector Gauss-rectangular rule we refer to [13].

In order to be able to approximate the smooth integral parts, we have to rotate the singularity to the north pole $\hat{n} := (0, 0, 1)^T$. To achieve this, we define a transformation on scalar-, vector- and matrixvalued functions:

$$T_\xi F(\eta) := F(T_\xi^{-1} \eta)$$

$$T_\xi f(\eta) := T_\xi f(T_\xi^{-1} \eta)$$

$$T_\xi A(\eta) := T_\xi A(T_\xi^{-1} \eta) T_\xi^{-1}.$$
The matrix $T_{\xi}$ maps $\xi \in \partial B$ onto the north pole. With this transformation we get the following integral operator with north pole singularity:

$$
(\tilde{M}^{n'}\tilde{b})(\xi) = T_{\xi}^{-1} \int_{\partial B} \left[ \frac{T_{\xi}M_{1}(\hat{n}, \hat{\eta})}{[\hat{n} - \hat{\eta}]} + T_{\xi}M_{2}(\hat{n}, \hat{\eta}) \right] T_{\xi}\tilde{b}(\hat{\eta}) \, d\hat{\eta}, \quad \xi \in \partial B. \tag{38}
$$

The integral is tangential with respect to $\hat{n}$, so we can apply the orthogonal projection $P(\hat{n}) := 1 - \hat{n} \circ \hat{n}$. This is necessary to preserve the tangential character after the approximation by vector spherical harmonics. Finally we get the discrete operator by applying $L_{n}$:

$$
(\tilde{M}^{n'}\tilde{b})(\xi) := T_{\xi}^{-1}P(\hat{n}) \int_{\partial B} \frac{1}{[\hat{n} - \hat{\eta}]} L_{n}\left\{ T_{\xi}M_{1}(\hat{n}, \cdot)T_{\xi}\tilde{b}(\cdot) \right\}(\hat{\eta}) \, d\hat{\eta} \tag{39}
$$

$$
+ T_{\xi}^{-1}P(\hat{n}) \int_{\partial B} L_{n}\left\{ T_{\xi}M_{2}(\hat{n}, \cdot)T_{\xi}\tilde{b}(\cdot) \right\}(\hat{\eta}) \, d\hat{\eta}.
$$

The resulting integrals

$$
\int_{\partial B} \frac{1}{[\hat{n} - \hat{\eta}]} y_{l,k}^{(i)}(\hat{\eta}) \, d\hat{\eta} \quad \text{and} \quad \int_{\partial B} y_{l,k}^{(i)}(\hat{\eta}) \, d\hat{\eta}
$$

are well known and can be calculated by using the second Funck-Hecke formula (see [2]). This leads to the fully discrete Galerkin scheme:

**We seek an approximate solution** $a_{n} \in \text{span}\{y_{l,k}^{(i)}\}$, **which satisfies**

$$
a_{n} + L_{n}\tilde{M}^{n'}a_{n} = L_{n}c. \tag{40}
$$

In order to solve the linearized equation (28) we use again a fully discrete Galerkin scheme and the following ansatz for the required radius update

$$
s(\theta, \phi) := \sum_{l=0}^{n} \sum_{|k| \leq l} s_{l,k} Y_{l,k}(\theta, \phi), \tag{41}
$$

where $s_{l,k} \in R$ are the unknown coefficients. As we have pointed out before, for the discretization of $S'_{\infty}$ we use a suitable quadrature rule. The kernel of this operator is very smooth, hence again the Gauss-rectangular rule provides good results. Thus we obtain the following discrete operator:

$$
S_{\infty,N}^{r}(r, \psi, s) := \frac{4\pi}{4\pi} p(\theta, \phi) \times \sum_{r=0}^{2N+1} \mu_{r} \nu_{s} \psi_{rs} e^{-ikp(\theta, \phi)\cdot q_{rs}} p(\theta, \phi) \cdot p_{rs} s(\theta_{s}, \phi_{r}), \tag{42}
$$

where we have used the abbreviations

$$
q_{rs} := q(\theta_{s}, \phi_{r}), \quad p_{rs} := p(\theta_{s}, \phi_{r}) \quad \text{and} \quad \psi_{rs} := \psi(\theta_{s}, \phi_{r}).
$$

$\theta_{s}$ and $\phi_{r}$ are the Gauss-rectangular nodes and $\mu_{r}$ and $\nu_{s}$ denote the corresponding quadrature weights (see [13]). If we write

$$
c := H_{\infty} - S_{\infty}(r, \psi)
$$

for the right handside of the linearized equation and use

$$
(f, g)_{N} := \sum_{r=0}^{2N+1} \sum_{s=1}^{N+1} \mu_{r} \nu_{s} f(\theta_{s}, \phi_{r}) g(\theta_{s}, \phi_{r}), \tag{43}
$$

for the right handside of the linearized equation and use
as discrete inner product, we obtain the fully discrete Galerkin scheme:

We seek an approximate solution \( s_n \in \text{span}\{Y_{l,k}\} \), which satisfies

\[
(S'_{\infty,N}(r,\psi,s_n), w_n)_N = (c, w_n)_N
\]

for all \( w_n \in \text{span}\{Y_{l,k}\} \).

\( s_n \) is a scalar function, hence (44) is overdetermined, so we use a least squares method to find a solution. Furthermore the discrete system inherits the ill posedness of equation (20), therefore we suggest to apply also Tichonov regularization.

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