Stability of the inverse resonance problem on the line

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Abstract

In the absence of a half-bound state, a compactly supported potential of a Schrödinger operator on the line is determined up to a translation by the zeros and poles of the meromorphically continued left (or right) reflection coefficient. The poles are the eigenvalues and resonances, while the zeros also are physically relevant. We prove that all compactly supported potentials (without half-bound states) that have reflection coefficients whose zeros and poles are ε-close in some disc centered at the origin are also close (in a suitable sense). In addition, we prove the stability of small perturbations of the zero potential (which has a half-bound state) from only the eigenvalues and resonances of the perturbation.

1. Introduction

The inverse resonance problem for the Schrödinger equation

\[-y'' + q(x)y = \lambda y, \quad x \in \mathbb{R},\]

seeks to determine a compactly supported potential \(q\) from the eigenvalues and resonances which are fundamental objects in quantum mechanics. Physically, eigenvalues represent energies for which a particle is permanently trapped by the potential, while resonances are related to energies for which the particle is temporarily trapped, but eventually escapes.¹

Classically, one needs the left (or right) reflection coefficient (as a function on \(\mathbb{R}\)), the eigenvalues, and the norming constants to solve the Gel’fand–Levitan–Marchenko equation for the potential (see [13] or [14]). However, if the potential is known to have support on a left (or right) half-line, then the right (or left) reflection coefficient is sufficient to recover the potential [15, 2, 7, 9]. In this case, the reflection coefficient can be meromorphically extended to the upper half-plane (see also [5]) with poles at the eigenvalues and residues equal to the norming constants modulo a factor of \(i\). Therefore, either of the reflection coefficients is sufficient to determine a compactly supported potential uniquely. The question then arises: The question becomes whether the eigenvalues and resonances can determine a reflection coefficient as a function on \(\mathbb{R}\).

¹ See [21] for an expositional introduction to resonances.
When the potential is compactly supported, the reflection coefficients can be meromorphically continued to the entire complex plane. The eigenvalues and resonances are the (squares of) poles of the reflection coefficients in the upper and lower half-planes, respectively. When the potential is real-valued, these data are sufficient to determine the modulus of the reflection coefficient on the line. However, they cannot determine the phase. Therefore, more data are needed to determine the reflection coefficient. To date, only Korotyaev [12] has addressed uniqueness and characterization for this problem by adding additional data (see section 2), although Zworski pointed out earlier [22] that even symmetric potentials may not be determined by their eigenvalues and resonances.

The zeros of the reflection coefficient on the real line are (the square roots of) the energies for which an incoming particle will pass through the potential unreflected. The physical meaning of the non-real zeros of the reflection coefficient is less obvious. However, they must be at least as physical as the resonances for the following reason. The wavefunctions (the solutions of the Schrödinger equation) associated with resonances are not square integrable at either plus or minus infinity. On the other hand, the wavefunctions associated with non-real zeros of the reflection coefficient are square integrable at one infinity, but not at the other depending on whether the zero is in the upper or lower half-plane. In this sense, then, the non-real zeros of the reflection coefficient are physical.

Since there are infinitely many zeros and poles, a natural question arises: What happens if we know only a finite subset of the data? Specifically, if two potentials have reflection coefficients whose zeros and poles are, respectively, close to each other in a large disc centered at the origin, then how ‘close’ are the potentials? That is, we are interested in a finite data stability problem. This problem is physically and computationally significant: since only finitely many data can ever be measured or input into an inversion algorithm, one needs to know how close one can get to the ‘true’ potential.

Stability of the inverse scattering problem on the line has previously received some attention, but little when compared to uniqueness. Aktosun [1] considers stability in the case of no eigenvalues and when the reflection coefficient is known in some interval. Aslanov [3] considers a similar problem, but allows eigenvalues. Dorren et al [6] consider a perturbation of the Fourier transform of the reflection coefficient as data and allow only rational reflection coefficients. Finally, Hitrik [9] considers a finite data stability problem with data consisting of discrete values of the reflection coefficient on the positive imaginary axis; he also does not allow for eigenvalues. The use of resonances as data in a stability problem has not been considered previously. We also mention some other stability results in one dimension: for bounded intervals where the data consist of two (infinite) sets of eigenvalues see [20, 18, 17, 10] and for the half-line inverse resonance problem see [11].

Our method is an extension of the one used by Marletta et al in [16]. There the authors treat the finite data inverse resonance problem on the half-line \([0, \infty)\) with the Dirichlet condition at zero. It uses Hadamard’s factorization theorem, some simple properties of the Fourier transform, and an estimate of the solution of an integral equation based on iteration. Let us briefly compare the half-line and full-line cases. In the half-line problem, the Jost function is the key, and its zeros in the upper and lower half-planes are the eigenvalues and resonances, respectively. The Jost function is an entire function of growth order at most one and so can be factored by Hadamard’s theorem.

The outline of the paper is as follows. In section 2, we review scattering theory on the line, set notation and present the uniqueness result, theorem 2.2, upon which our stability analysis

\[2\] With symmetric potentials, the square of the reflection coefficient can be determined from the eigenvalues and resonances, but not necessarily the sign.
is based. The final four sections are devoted to stability. Our main results are theorem 5.3 and its corollary 5.4. The final section presents a special case of stability in the presence of a half-bound state—namely, when one of the potentials is identically zero.

2. Scattering theory

We begin with the Schrödinger equation

\[ -y'' + qy = z^2 y, \]  

(2.1)

where \( q \) satisfies the following.

**Assumption 1.** The potential is real-valued, integrable and compactly supported.

Let \( q \) satisfy assumption 1 and suppose \( \text{supp } q \subset [c, d] \). Then for every \( z \in \mathbb{C} \), there exist unique solutions, \( f^\pm(z, z) \), of (2.1) such that \( f^+(x, z) = e^{izx} \) for \( x \geq d \) and \( f^-(x, z) = e^{-izx} \) for \( x \leq c \). These solutions are called the Jost solutions and have the following representations:

\[ f^+(x, z) = e^{izx} + \int_x^{2d-x} K^+(x, t) e^{it} \, dt, \]  

(2.2)

\[ f^-(x, z) = e^{-izx} + \int_{2c-x}^x K^-(x, t) e^{-it} \, dt, \]  

(2.3)

for \( x \in \mathbb{R} \). The functions \( K^\pm \) are the kernels of the transformation operators. These kernels are real-valued, supported in the triangles \( \{(x, t) : x \leq t \leq 2d - x\} \) and \( \{(x, t) : 2c - x \leq t \leq x\} \), respectively, and satisfy

\[ |K^\pm(x, t)| \leq \frac{1}{2} \|q\|_1 \exp(\|q\|_1 (d - c)), \quad (x, t) \in \text{supp } K^\pm. \]  

(2.4)

We also note that \( K^+(x, t) = 0 \) on the line \( x + t = 2d \), while \( K^-(x, t) = 0 \) along \( x + t = 2c \) (see [4]). Moreover, in the interior of their supports, \( K^\pm \) have first-order partial derivatives such that

\[ |K_t^\pm(x, t)| \leq \frac{1}{4} q \left( \frac{x + t}{2} \right) \leq \frac{1}{2} \|q\|_1^2 \exp(\|q\|_1 (d - c)), \]  

(2.5)

where \( t \) stands for \( x \) or \( t \). We refer the reader to [14] or [4] for details about these transformation operators.

Let \( [f, g] = fg' - f'g \) be the Wronskian of \( f \) and \( g \). Since the Wronskian of two solutions of (2.1) is constant, we find \([f^\pm(\cdot, z), f^\pm(\cdot, -z)] = \mp 2iz\). Furthermore, we define the functions \( w \) and \( s^\pm \) by

\[ w(z) = [f^-(\cdot, z), f^+(\cdot, z)] \]  

(2.6)

and

\[ s^\pm(z) = [f^+(\cdot, \mp z), f^-(\cdot, \pm z)]. \]  

(2.7)

As a non-zero Wronskian implies linear independence of solutions of (2.1), we easily deduce that

\[ f^\pm(\cdot, z) = \frac{w(z)}{2iz} f^\pm(\cdot, -z) + \frac{s^\pm(z)}{2iz} f^\mp(\cdot, z) \]  

(2.8)

for every \( z \neq 0 \). The scattering matrix is given by

\[ S(z) = \begin{pmatrix} \Im(z) & \Re^{-}(z) \\ \Re^+(z) & \Im(z) \end{pmatrix}, \]

where \( \Im(z) = 2iz/w(z) \) is the transmission coefficient and \( \Re^\pm(z) = s^\pm(z)/w(z) \) are the right and left reflection coefficients, respectively. We will use the notations \( w_q \) and \( s^\pm_q \), when necessary, to make the dependence upon the potential explicit.
Lemma 2.1. Let \( q \) satisfy assumption 1. The functions \( w \) and \( s^\pm \) are entire, have growth order at most one and satisfy:

(i) \( w(z) = w(-\overline{z}) \), \( s^\pm(z) = s^\pm(-\overline{z}) \);
(ii) \( s^-(z) = s^+(z) \);
(iii) \( w(z)w(-z) - 4z^2 = s^+(z)s^-(z) \);
(iv) \( w(0) = -s^+(0) \).

Proof. From (2.2) and (2.3) and the estimates (2.4) and (2.5) it is clear that for each \( x \), \( f^\pm(x, \cdot) \) and \( f^\pm(x, \cdot) \) are entire functions of growth order at most 1. Since \( K^\pm \) are real-valued, \( f^\pm(x, \cdot) \) and \( f^\pm(x, \cdot) \) have the property that \( f(z) = f(-\overline{z}) \). Therefore, \( w \) and \( s^\pm \) are entire, of growth order at most 1, and satisfy (i). Property (ii) is a direct consequence of (2.7). Applying the identity

\[
[g_1, g_2][g_3, g_4] = [g_1, g_3][g_2, g_4] = [g_1, g_2][g_3, g_4]
\]

and using \([f^+(x, z), f^+(x, -z)] = \pm 2\lambda z\) establishes (iii). Because \( [f, g] = -[g, f] \), the final property is true.

Lemma 2.1 part (iii) shows that \( w \) and \( s^\pm \) cannot be zero simultaneously except at \( z = 0 \). So the poles of the reflection coefficients are precisely the zeros of \( w \). The zeros of \( K^\pm \) are, of course, the zeros of \( s^\pm \). When \( \text{Im} \ z > 0 \), the Jost solutions, \( f^+(\cdot, z) \) and \( f^-(\cdot, z) \), are exponentially decreasing on the right and left half-lines, respectively. Therefore, if \( w(z) = 0 \) and \( \text{Im} \ z > 0 \), then \( f^+(\cdot, z) \in L^2(\mathbb{R}) \), since it is proportional to \( f^-(\cdot, z) \) and \( z^2 \) is an eigenvalue of the operator generated by \( -\frac{d^2}{dx^2} + q \). The eigenvalues must be real, so the zeros of \( w \) in the upper half-plane must be on the positive imaginary axis. When \( w(z) = 0 \) and \( \text{Im} \ z < 0 \), \( z^2 \) is called a resonance. Lemma 2.1 parts (i) and (iii) show that \( w \) cannot vanish on \( \mathbb{R} \) except at \( z = 0 \). If \( w(0) = 0 \), then we say that there is a half-bound state. Combining parts (i), (iii) and (iv) of Lemma 2.1 shows that this zero is at most simple. However, the zero of \( s^\pm \) at \( z = 0 \) need not be simple. We now state the main result of this section.

Theorem 2.2. A real-valued, integrable and compactly supported potential is determined by the zeros and poles of one of its reflection coefficients up to a shift when there is no half-bound state or there is at least one eigenvalue.

The proof of this theorem will be given at the end of the section after a series of lemmas. We note that Korotyaev, in [12] theorem 1.2 part (i), has a result in much the same vein as theorem 2.2. First, he excludes the possibility of a shift by requiring that the potential is supported in \([0, 1]\) and for every \( \varepsilon > 0 \), the sets \( \text{supp} \ q \cap (0, \varepsilon) \) and \( \text{supp} \ q \cap (1 - \varepsilon, 1) \) have positive measure. Next, the data given for the inversion are the eigenvalues, resonances and a sequence, \( \sigma \), whose values are taken from the set \([-1, 0, 1]\) (subject to some characterization constraints). The eigenvalues and resonances determine \( w \) as in the proof below. Then, the function on the left-hand side of (2.13) is also determined. Its zeros are either zeros of \( s^- \) or \( s^- \). Then \( \sigma \) is used to separate these zeros into those of \( s^- \) and \( s^- \) and to determine the sign of \( \exp(b_0) \). Therefore, the left reflection coefficient is determined. Knowing the sequence \( \sigma \) is the same as knowing the zeros of \( s^- \) and \( s^- \). So, in this sense, our result is not new and we do not claim originality. Our goal is to achieve a finite data stability result, and the zeros of \( s^- \) are easier to work with than \( \sigma \) in this context.

Since a potential satisfying assumption 1 is determined by one of its reflection coefficients, in order to prove the theorem, we need to show that a reflection coefficient is determined (up to a certain factor) by its zeros and poles. To this end, we will utilize two different representations of \( w \) and \( s^\pm \). The first is in terms of the zeros and poles of \( K^\pm \), i.e. the
zeros of \( s^\pm \) and \( w \). The second is in terms of the transformation operators. For the first, let \( \{ w_n : 0 < |w_1| \leq |w_2| \leq \cdots, n \in \mathbb{N} \} \) be the zeros of \( w \) (the square roots of the eigenvalues and resonances) listed according to multiplicity. By Hadamard’s factorization theorem and lemma 2.1, we have

\[
w(z) = z^m e^{s^+(z)} \prod_{n=1}^{\infty} \left( 1 - \frac{z}{w_n} \right) e^{z/w_n}, \quad m \in \{0, 1\},
\]

(2.9)

where \( g(z) = a_1 z + a_0 \). Likewise, let \( \{ s_n : n \in \mathbb{N} \} \) be the set of zeros of \( s^- \) listed according to multiplicity and by increasing modulus; we have by part (ii) of lemma 2.1

\[
s^\pm(z) = (\mp 1)^{\delta} z^\delta e^{h(\mp \delta)} \prod_{n=1}^{\infty} \left( 1 \mp \frac{z}{s_n} \right) e^{z/s_n}, \quad \ell \geq m,
\]

(2.10)

where \( h(z) = b_1 z + b_0 \).

For the second representation, suppose, again, that \( \text{supp } q \subset [c, d] \). Then a straightforward calculation from (2.2), (2.3), (2.6) and (2.7) shows

\[
w(z) = 2iz - \int_c^d q(s) ds + \int_c^d [K^+(c, t) - K^+(c, t)] e^{z(t-c)} dt
\]

(2.11)

and

\[
s^-(z) = -\int_c^{2d-c} [K_+^+(c, t) + K^-_-(c, t)] e^{z(t+c)} dt,
\]

(2.12)

where we have used \( 2K^+(c, c) = \int_c^d q \) and \( K^-(c, c) = 0 \). Note that these representations are the Fourier transforms of the distributions \( X(t) \) and \( Y(t) \) found in equations (5.11) and (5.12) of Melin’s paper [19]. We gain from (2.11) and (2.12) that \((2iz)^{-1}w(z) \to 1\) as \( z \to \infty \) in the closed upper half-plane and the ability to estimate \( w \) and \( s^- \) from estimates on \( K^+ \) (see lemma 3.1 below).

**Proposition 2.3.** In the factorization (2.10), \( b_1 \) is purely imaginary.

**Proof.** By part (i) of lemma 2.1, \( s^- \) is real and \( \delta^- \) is purely imaginary on the imaginary axis with the dot representing differentiation with respect to \( z \). Let \( S(z) = z^{-\delta}s^-(z) \). Differentiating, we find

\[
\frac{\dot{S}(z)}{S(z)} = \frac{\delta^- (z) - \ell}{z}
\]

which is purely imaginary on the imaginary axis. Because \( S(0) \neq 0 \), \( \dot{S}/S \) is analytic (in particular, continuous) near zero. Therefore, \( b_1 = S(0)/S(0) \) is purely imaginary.

The final ingredient shows the effect of a shift of the potential on \( w \) and \( s^\pm \).

**Lemma 2.4.** Let \( q \) and \( \tilde{q} \) satisfy assumption 1. Then, \( \tilde{q}(x) = q(x - \alpha) \) for some real number \( \alpha \) if and only if \( w_{\tilde{q}} = w_q \) and \( s^\pm_{\tilde{q}}(z) = e^{\pm 2\alpha i z}s^\pm_q(z) \).

**Proof.** Let \( f^\pm \) be the Jost solutions associated with \( q \). Suppose \( \alpha \in \mathbb{R} \) and \( \tilde{q}(x) = q(x - \alpha) \). Then the Jost solutions associated with \( \tilde{q} \) are given by \( \tilde{f}^+(x, z) = e^{i\alpha f^+(x - \alpha)} \) and \( \tilde{f}^+(x, z) = e^{-i\alpha f^+(x - \alpha)} \). Substituting these into (2.6) and (2.7) gives the necessary form for \( w_{\tilde{q}} \) and \( s^\pm_{\tilde{q}} \).

On the other hand, suppose \( w_{\tilde{q}} = w_q \) and \( s^\pm_{\tilde{q}}(z) = e^{\pm 2\alpha i z}s^\pm_q(z) \). Since a compactly supported potential is determined by either of the reflection coefficients, we must have that \( \tilde{q}(x) = q(x - \alpha) \).
Proof of theorem 2.2. The poles of the reflection coefficient and the asymptotics of $w$ determine all the necessary quantities in (2.9). Thus, we need to determine $s^-$ or $s^+$ from their zeros. We only give the proof for $s^-$, since the proof for $s^+$ is similar. Recall from part (iii) of lemma 2.1 that

$$w(z)w(-z) - 4z^2 = s^-(z)s^-(z).$$

(2.13)

Since the left-hand side above is known and the right-hand side has a zero of order 2 by writing next two sections. Our assumption is that because $w(0) = -s^-(0) \neq 0$. On the other hand, if there is a half-bound state, then we can only determine $\exp(2b_0)$ from (2.13). However, if there is an eigenvalue corresponding to $z = ik$, the sign of $\exp(b_0)$ is determined by the following standard fact:

$$\int_{-\infty}^{\infty} |f^+ (x, ik)|^2 dx = i \frac{\dot{s}^+(ik) \psi(ik)}{4k^2} > 0,$$

where the dot denotes differentiation with respect to $z$.

In both cases, $s^-$ is determined up to the factor $\exp(b_1z)$. Applying proposition 2.3 and lemma 2.4 completes the proof. □

3. The effect of perturbing the zeros of $w$ and $s$ in a disc

3.1. Outline of the stability proof

We now turn to the issue of stability. For convenience (and without loss of generality), we assume that all potentials are supported in $[-1, 1]$. Let us first fix some notation. Since we will be dealing with different transformation operators (we identify the operator with its kernel), we write

- $K^+$ and $\tilde{K}^+$ transform from the zero potential to potentials $q$ and $\tilde{q}$, respectively;
- $L^+$ transforms from the potential $q$ back to the zero potential;
- $B^+$ transforms from $q$ to $\tilde{q}$.

We put $\tilde{w} = w_\tilde{q}$ and $s^\pm = s^\pm_{\tilde{q}}$. We use $D(r, z_0)$ and $\overline{D}(r, z_0)$ for the open and, respectively, closed discs with radius $r$ and center $z_0$. When $z_0 = 0$, we write $D(r) = D(r, 0)$ and $\overline{D}(r) = \overline{D}(r, 0)$. Finally, we denote the dependence of a constant on the various parameters by writing $C = C(Q)$, for example.

The proof of theorem 5.3 has three main steps which are the contents of this and the next two sections. Our assumption is that $q$ and $\tilde{q}$ are two potentials without half-bound states ($q \equiv 0$ is, thus, excluded) for which the zeros and poles of their left reflection coefficients (i.e. the zeros of $w$ and $s^-$) are $\varepsilon$-close, respectively, in $D(R)$. The first step is to obtain estimates of $|w(z) - \tilde{w}(z)|$ and $|s^-(z) - \tilde{s}^-(z)|$ in an interval of the real line using the factorizations (2.9) and (2.10). These estimates will result in a bound on $|f^+(-1, z) - \tilde{f}^+(-1, z)|$

$$f^+(-1, z) - \tilde{f}^+(-1, z) = \frac{e^{iz}}{2iz} (w(z) - \tilde{w}(z)) + \frac{e^{iz}}{2iz} (s^-(z) - \tilde{s}^-(z)), \quad (3.1)$$

by (2.8).

The next step is to use the bound on $|f^+(-1, z) - \tilde{f}^+(-1, z)|$ and properties of the Fourier transform to estimate $|K^+(-1, \cdot) - \tilde{K}^+(-1, \cdot)|$ via

$$f^+(-1, z) - \tilde{f}^+(-1, z) = \int_{-1}^{1} [K^+(-1, t) - \tilde{K}^+(-1, t)] e^{itz} \, dt. \quad (3.2)$$

(3.1)
Since
\[ B^+(x, t) = \tilde{K}^+(x, t) - K^+(x, t) + \int_x \tilde{K}^+(x, s)L^+(s, t) \, ds, \]
and \( L^+ \) is bounded by a constant depending only on \( \|q\|_1 \), we obtain a bound on \( B^+(-1, \cdot) \) from one on \( K^+(-1, \cdot) - \tilde{K}^+(-1, \cdot) \).

Finally, we use an integral equation and our estimate of \( B^+(-1, \cdot) \) to bound \( B^+(x, t) \) in the triangle \( \{(x, t) \in \mathbb{R}^2 : -1 \leq x \leq t \leq 2 - x \} \). In particular, we bound
\[
2|B^+(x, x)| = \left| \int_x q(s) - \tilde{q}(s) \, ds \right|
\]
for \( x \in [-1, 1] \).

### 3.2. Preliminary estimates and some properties of \( w \) and \( s \)

The remainder of this section will be dedicated to obtaining estimates on the differences of \( w \) and \( \tilde{w} \) and \( s^- \) and \( \tilde{s}^- \). Since we will exclusively focus on \( s^- \) in the following, we drop the superscript, i.e. \( s = s^- \) and \( \tilde{s} = \tilde{s}^- \).

Since the value of \( s(0) \) (and \( w(0) \)) can be arbitrarily small in the absence of a half-bound state, we will need to assume a uniform lower bound on \( |s(0)| \) for every potential under consideration. This fact leads us to the definition of the class of potentials in which we will work.

**Definition 1.** Let \( Q \) and \( \delta \) be positive numbers. The set \( B_\delta(Q) \) consists of functions \( q \in L^1(\mathbb{R}) \) with \( \text{supp } q \subset [-1, 1] \) such that

(i) \( \|q\|_1 \leq Q \);
(ii) \( \delta \leq |s(0)| = |w(0)| \).

We begin with \textit{a priori} estimates on \( w \) and \( s \) for potentials in \( B_\delta(Q) \).

**Lemma 3.1.** For every \( Q > 0 \) there is a constant \( \kappa = \kappa(Q) > 0 \) such that for any \( q \) whose support is contained in \([-1, 1]\) and whose \( L^1 \)-norm is bounded by \( Q \), the associated functions \( w \) and \( s \) satisfy the following.

(i) For \( z \in \mathbb{C} \) with \( \text{Im } z \geq 0 \), \( |w(z) - 2iz| \leq \kappa \);
(ii) for \( z \in \mathbb{C} \), \( |w(z)| \leq \kappa e^{4|z|} \);
(iii) for any positive \( \rho \), \( |w(z)/(2iz) - 1| \leq \kappa/\rho \) for all \( z \in \overline{D}(\rho; 3i\rho) \);
(iv) for \( z \in \mathbb{R} \), \( |s(z)| \leq \kappa \);
(v) for \( z \in \mathbb{C} \), \( |s(z)| \leq \kappa e^{2|z|} \).

**Proof.** From representations (2.11) and (2.12) and estimates (2.4) and (2.5), we have
\[
|w(z) - 2iz| \leq Q + \int_{-1}^3 Q^2 e^{2Q} e^{-\text{Im } z(t+1)} \, dt \tag{3.4}
\]
\[
|s(z)| \leq \int_{-1}^3 \left[ 2Q^2 e^{2Q} + \frac{1}{2} \left| q \left( t - \frac{1}{2} \right) \right| \right] e^{\text{Im } z(t-1)} \, dt. \tag{3.5}
\]
When \( \text{Im } z \geq 0 \), we have \( |w(z) - 2iz| \leq Q + 4Q^2 e^{2Q} \). On the other hand, if \( z \in \mathbb{R} \), then \( |s(z)| \leq 8Q^2 e^{2Q} + Q/2 \). Choosing the larger of the two bounds as \( \kappa \) proves (i) and (iv). Bounding \( -\text{Im } z \) by \( |z| \) shows that the exponential in (3.4) is bounded by \( \exp(4|z|) \) and the
one in (3.5) of s by $\exp(2|z|)$ which proves (ii) and (v). Finally, multiplying both sides of the inequality in (i) by $|2iz|^{-1}$ and using $0 < 2\rho \leq |z|$ for all $z \in \overline{D}(\rho; 3i\rho)$ proves (iii). □

An entire function, $f$, is said to be of exponential type if there exist constants $c_1$ and $c_2$ such that $|f(z)| \leq c_1 \exp(c_2|z|)$ for every $z \in \mathbb{C}$. From parts (ii) and (v) of the above lemma, we see that $w$ and $s$ are of exponential type. We will need the following lemma about this class of functions.

**Lemma 3.2.** Suppose $f$ is an entire function of exponential type with the associated constants, $c_1$ and $c_2$, establishing that fact. Let $a_n$, $n \in \mathbb{N}$, be the non-zero zeros of $f$ listed according to multiplicity and by increasing modulus. Let $N_f(r, z_0)$ be the number of zeros of $f$ in $D(r, z_0)$ and define

$$\Pi_f(R, z) = \prod_{|a_n| \geq R} \left(1 - \frac{z}{a_n}\right) e^{z/a_n}.$$ 

If $f(z_0) \neq 0$, then

$$N_f(r, z_0) \leq c_2 (er + |z_0|) + \log(c_1/|f(z_0)|).$$

(3.6)

for any $r > 0$.

In addition, if $R \geq 3|z_0|$, then

$$|\Pi_f(R, z) - 1| \leq 18(c_2 + \log(c_1/|f(z_0)|)) \left|\frac{|z|^2}{R}\right| \exp\left[18(c_2 + \log(c_1/|f(z_0)|)) \left|\frac{|z|^2}{R}\right|\right]$$

for all $z \in D(R/2)$.

**Proof.** Jensen's formula implies

$$N_f(r, z_0) \leq \int_0^r \frac{N_f(t, z_0)}{t} \, dt = \frac{1}{2\pi} \int_0^{2\pi} \log |f(z_0 + ere^{i\theta})| \, d\theta - \log |f(z_0)|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} \log c_1 + c_2 |z_0 + ere^{i\theta}| \, d\theta - \log |f(z_0)|.$$

Applying the triangle inequality proves (3.6).

Let $R \geq 3|z_0|$. Then, for every $z \in \overline{D}(R/2, z_0)$, we know $|z/a_n| \leq 1/2$, so $u = \log \Pi_f(R, z)$ is well defined. To prove the final statement of the lemma, it suffices to show

$$|u| \leq 18(c_2 + \log(c_1/|f(z_0)|)) \left|\frac{|z|^2}{R}\right|,$$

(3.7)

since $|e^u - 1| \leq |u| e^{|u|}$. The elementary factor $E(z/a_n)$ satisfies $|\log E(z/a_n)| \leq 2|z/a_n|^2$. Hence, we must bound

$$S = \sum_{|a_n| \geq R} |a_n|^{-2}.$$

To that end, observe

$$S \leq 2 \sum_{|a_n - z_0| \geq 2R/3} |a_n - z_0|^{-2} \leq 2 \int_{2R/3}^{\infty} \frac{dN_f(t, z_0)}{t^2} \leq 4 \int_{2R/3}^{\infty} N_f(t, z_0) \, dt \frac{1}{t^3},$$

since $|a_n| \geq R \geq 3|z_0|$ implies $|a_n| \geq |a_n - z_0|/4$. Now, we use (3.6) to see that

$$4 \int_{2R/3}^{\infty} N_f(t, z_0) \, dt \frac{1}{t^3} \leq 9c_2 + \log(c_1/|f(z_0)|) \frac{|z|^2}{R}.$$

This inequality implies (3.7), so we are finished. □
Assume $\rho \geq \max(1, 2\kappa)$; then part (iii) of lemma 3.1 shows that $3 \leq |w(3i\rho)|$. Therefore, by lemma 3.2

\begin{equation}
N_w(r, 3i\rho) \leq 4er + 12\rho + \log(5\kappa);
\end{equation}

and

\begin{equation}
|\Pi_w(R, z) - 1| \leq \frac{C_1|z|^2}{R} \exp\left(\frac{C_1|z|^2}{R}\right)
\end{equation}

for $R \geq 3\rho$, $z \in \overline{D}(R/2)$ and $C_1 = 18(4 + \log[(\kappa + 2)/3])$. Furthermore, the same lemma gives

\begin{equation}
N_s(r, 0) \leq 2er + \log(\kappa/\delta);
\end{equation}

\begin{equation}
|\Pi_s(R, z) - 1| \leq C_2 \frac{|z|^2}{R} \exp\left(\frac{C_2|z|^2}{R}\right)
\end{equation}

for $R \geq 0$, $z \in \overline{D}(R/2)$ and $C_2 = 18[2 + \log(\kappa/\delta)]$.

We now give two propositions that reveal regions where $w$ and $s$ cannot have zeros. The first gives a neighborhood of the origin in which neither $w$ nor $s$ vanish in terms of $Q$ and $\delta$. The second, due to Hitrik, supplies a resonance-free strip for a potential based upon the length of its support and its $L^1$-norm. We omit the proof of this proposition due to its length and refer the interested reader to [8] for the details.

**Proposition 3.3.** There exists a constant $\gamma = \gamma(Q, \delta) > 0$ such that for every $q \in B_3(Q)$, the functions $w_q$ and $s_q$ do not vanish in the disc $D(\gamma)$.

**Proof.** Let $q \in B_3(Q)$ and set $A(t) = K^+_n(-1, t) - K^+_n(-1, t)$. Using (2.11), we expand $w$ as

\[ w_q(z) = w_q(0) + iz \left(2 + \int_{-1}^{3} (t + 1)A(t) \, dt\right) + \sum_{n=2}^{\infty} \frac{(iz)^n}{n!} \int_{-1}^{3} (t + 1)^n A(t) \, dt. \]

By (2.5), lemma 2.1(iv), and definition 1, there is a constant $C = C(Q)$ such that

\[ |w(z)| \geq \delta - C|z|e^{4|z|}. \]

Therefore, there exists a constant $\gamma_w = \gamma_w(Q, \delta)$ such that $\delta - C|z|\exp(4|z|) > 0$ for $|z| \leq \gamma_w$.

On the other hand, we have

\[ s_q(z) = -\sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-1}^{3} (t - 1)^n[K^+_n(c, t) + K^+_n(c, t)] \, dt z^n. \]

Again, we apply (2.5) and definition 1 to find that there exists a constant $\gamma_s = \gamma_s(Q, \delta)$ such that $|s_q(z)| > 0$ for $|z| \leq \gamma_s$. The smaller of $\gamma_w$ and $\gamma_s$ is the desired constant. $\square$

**Proposition 3.4.** Let $p \in L^1(\mathbb{R})$ be supported on an interval of length $d > 0$. Assume $k$ is (the square root of) a resonance of $p$ with $\Re k \neq 0$. Then,

\[ d|\Im k| \geq \frac{1}{4} \exp(-2d\|p\|_1). \]
3.3. The difference of \( w \) and \( \tilde{w} \) when their small zeros are approximately the same

We will focus now on the difference of \( w \) and \( \tilde{w} \). Our goal is to prove the following.

**Theorem 3.5.** Let \( Q \) and \( \delta \) be positive and \( q, \tilde{q} \in B_3(Q) \). Then there exist positive constants \( R_0 = R_0(Q) \) and \( C = C(Q, \delta) \) such that the following statement is true for every \( R \geq R_0 \). There exists an \( E = E(Q, R) > 0 \) such that for every \( 0 < \varepsilon \leq E \) it is the case that if the zeros of \( w \) and \( \tilde{w} \) are \( \varepsilon \)-close in the disc \( \mathcal{D}(R) \), then for every \( z \in [-R^{1/3}, R^{1/3}] \), we have

\[
|w(z) - \tilde{w}(z)| \leq C(1 + |z|)(R^{-1/3} + \varepsilon N),
\]

where \( N = N_w(R, 0) = N_{\tilde{w}}(R, 0) \).

The proof of this theorem will be given after the following lemmas which contain the main steps. First, we redefine \( g \), so that we may rewrite (2.9) as

\[
w(z) = e^{\zeta(z)} \prod_{|w_n| < R} (w_n - z).
\]

Set \( N = N_w(R, 0) = N_{\tilde{w}}(R, 0) \) and define

\[
W(z) = \prod_{n=1}^{N} \frac{z - w_n}{z - \tilde{w}_n}.
\]

Since \( \tilde{w} \) does not vanish on the real line, dividing by \( z - \tilde{w}_n \) poses no problem for \( z \in \mathbb{R} \). Note we have rewritten the indices so that \( \tilde{w}_n \) is the one close to \( w_n \). From (3.13), we have

\[
w(z) = \tilde{w}(z) e^{\zeta(z) - \tilde{\zeta}(z)} W(z) \frac{\Pi_n(R, z)}{\Pi_n(R, \tilde{z})},
\]

after also redefining \( \tilde{g} \).

**Lemma 3.6.** Suppose \( |w_n - \tilde{w}_n| \leq \varepsilon \leq 9\kappa/4 \) for \( 1 \leq n \leq N \). Then, there exist constants \( R_0 = R_0(Q) \geq \max(2\kappa, 1) \) and \( C = C(Q) \) such that the following statement is true for every \( R \geq R_0 \). If \( |z - 3\i R^{1/3}| \leq R^{1/3} \), then

\[
|e^{\zeta(z) - \tilde{\zeta}(z)} - 1| \leq C \left[ R^{-1/3} + \varepsilon N \exp \left( \frac{2}{\kappa \varepsilon N} \right) \right].
\]

**Proof.** From (3.14),

\[
|e^{\zeta(z) - \tilde{\zeta}(z)} - 1| \leq \left| \frac{w(z)}{\tilde{w}(z)} \right| \left| \frac{W(z)}{\Pi_n(R, z)} \right| + |W(z)| \left| \frac{w(z)}{\tilde{w}(z)} - 1 \right| + |W(z)|^{-1} - 1. \tag{15.13}
\]

We begin by estimating the final term on the right hand side above. By lemma 3.1(iii),

\[
|w(z)| \geq |z| \geq 2\rho \geq 4\kappa
\]

whenever \( \rho \geq 2\kappa \) and \( |z - 3\i \rho| \leq \rho \). Furthermore, by lemma 3.1(i), we know \( |w(z)| \geq 2|z| - \kappa \) when \( \text{Im} z \geq 0 \). So, we obtain \( w(z) \neq 0 \) for those \( z \) in the closed upper half-plane for which \( 2|z| > \kappa \). Hence, for every \( n \in \mathbb{N} \), the imaginary part of \( w_n \) cannot exceed \( \kappa/2 \). We conclude that

\[
|z - w_n| \geq \text{Im}(z - w_n) \geq 2\rho - \kappa/2 \geq 3\kappa,
\]

when \( |z - 3\i \rho| \leq \rho \). Therefore, for \( 1 \leq n \leq N \)

\[
\left| \frac{w_n - \tilde{w}_n}{z - w_n} \right| \leq \frac{\varepsilon}{3\kappa} \leq \frac{3}{4}.
\]
Since 
\[
\frac{z - \tilde{w}_n}{z - w_n} = 1 + \frac{w_n - \tilde{w}_n}{z - w_n},
\]
and \(\log(1 + t) \leq 2|t|\) for \(|t| \leq 3/4\), we have 
\[
|\log(W(z)^{-1})| \leq \sum_{n=1}^{N} \left| \log \left( 1 + \frac{w_n - \tilde{w}_n}{z - w_n} \right) \right| \leq \frac{2}{3\kappa} \varepsilon N.
\]
Using \(|e^u - 1| \leq |u| e^{|u|}\) with \(u = \log(W^{-1})\), we obtain 
\[
|(W(z))^{-1} - 1| \leq \frac{2}{3\kappa} \varepsilon N \exp \left( \frac{2}{3\kappa} \varepsilon N \right). \tag{3.17}
\]

For the middle term of (3.15), we apply lemma 3.1(iii) and (3.16) (which is valid for \(\tilde{w}\) as well) to obtain 
\[
\left| \frac{u(z)}{\tilde{w}(z)} - 1 \right| \leq 4\kappa R^{-1/3}. \tag{3.18}
\]

As for the first term, inequality (3.9) implies that there are constants \(C = C(Q) > 0\) and \(R_0 = R_0(Q) > 0\) so that for every \(R \geq R_0\), 
\[
\left| \prod_{\alpha} (R, z) \right| \leq CR^{-1/3}. \tag{3.19}
\]
Applying (3.17), (3.18) and (3.19) to (3.15) yields the desired inequality. \(\square\)

**Corollary 3.7.** There exist positive constants \(R_0 = R_0(Q)\) and \(C = C(Q)\) such that for every \(R \geq R_0\) the following statement is true. There exists an \(E = E(Q, R) > 0\) such that for every \(0 \leq \varepsilon \leq E\) we have 
\[
|e^{\hat{g}(z)} - \hat{g}(z) - 1| \leq C(R^{-1/3} + \varepsilon N)
\]
when \(|z| \leq R^{1/3}\) and \(|w_n - \tilde{w}_n| \leq \varepsilon\) for \(1 \leq n \leq N\).

**Proof.** Let \(R_0\) and \(C\) be the constants given by lemma 3.6. Increase them (if needed) so that \(C > 1/2\) and \(R_0^{-1/3} < \exp(-2/(3\kappa))/2C\). Choose \(E\) so that 
\[
EN \leq \frac{1}{2C} e^{-2/(3\kappa)} - R_0^{-1/3}.
\]
Then for \(R \geq R_0\) and \(\varepsilon \leq E\), lemma 3.6 implies 
\[
|e^{\hat{g}(z)} - \hat{g}(z) - 1| \leq C e^{2/(3\kappa)} (R^{-1/3} + \varepsilon N)
\]
for \(|z - 3i R^{1/3}| \leq R^{1/3}\).

We are finished after applying the following claim. If \(F(z) = \exp(a_1 z + a_0) - 1\) and \(|F(z)| \leq \alpha < 1\) for \(z \in D(\rho, 3i\rho)\), then 
\[
|F(z)| \leq \frac{6\alpha}{1 - \alpha} \exp \left( \frac{6\alpha}{1 - \alpha} \right)
\]
for \(|z| \leq \rho\). Indeed, for \(z \in D(\rho, 3i\rho)\), we have 
\[
|a_1 z + a_0| = |\log(F(z) + 1)| \leq \alpha (1 - \alpha)^{-1}.
\]
Then, Cauchy’s estimate yields 
\[
|a_1| = \left| \frac{d}{dz} \log(F(z) + 1) \right|_{z=3i\rho} \leq \frac{\alpha}{\rho}.
\]
Thus, we find \(|a_0| \leq 5\alpha (1 - \alpha)^{-1}\) because \(|z| \leq 4\rho\). Applying the inequality \(|\exp(z) - 1| \leq |z| \exp |z|\) and the bounds on \(a_1\) and \(a_0\) for \(z \in D(\rho)\) completes the proof of our claim. \(\square\)
Lemma 3.8. Suppose $|w_n - \tilde{w}_n| \leq \varepsilon \leq 3 \min \{ \gamma, \exp(-4Q)/8 \}/4$ for $1 \leq n \leq N$ where $\gamma$ is the number given in proposition 3.3. Then, there exists a $C = C(Q, \delta)$ such that for every $z \in \mathbb{R}$

$$|W(z) - 1| \leq C \varepsilon N \exp(C \varepsilon N).$$

In particular, there exists an $E = E(Q, R)$ such that for all resonances $\varepsilon N < 1$ whenever $\varepsilon \in [0, E]$ implying

$$|W(z) - 1| \leq C E \varepsilon N$$

for $z \in \mathbb{R}$.

Proof. Let $z \in \mathbb{R}$. By proposition 3.4 we have

$$|z - \tilde{w}_n| \geq -\text{Im} \tilde{w}_n \geq \frac{3}{8} \exp(-4Q)$$

for all resonances $\tilde{w}_n$ whose real part is non-zero. For the eigenvalues and those resonances on the imaginary axis, we apply lemma 3.3 to find

$$|z - \tilde{w}_n| \geq |\text{Im} \tilde{w}_n| = |w_n| \geq \gamma.$$

Therefore, $|z - \tilde{w}_n| \geq \min \{ \gamma, \exp(-4Q)/8 \}$ for every $n$.

The inequality $|z - \tilde{w}_n| \geq \min \{ \gamma, \exp(-4Q)/8 \}$ for every $n$. The inequality $|z - \tilde{w}_n| \geq |\text{Im} \tilde{w}_n| = |w_n| \geq \gamma.$ completes the proof if we can show $|z - \tilde{w}_n| \geq |\text{Im} \tilde{w}_n|$ for every $n$. To that end, let $2C^{-1} = \min \{ \gamma, \exp(-4Q)/8 \}$, and note that $|w_n - \tilde{w}_n| \leq 3|z - \tilde{w}_n|/4$ for $1 \leq n \leq N$ by assumption. Then,

$$|u| \leq \sum_{n=1}^{N} \left| \log \left( 1 + \frac{\tilde{w}_n - w_n}{z - \tilde{w}_n} \right) \right| \leq C \varepsilon N,$$

since $\log(1 + t) \leq 2|t|$ for $|t| \leq 3/4$. \hfill \Box

We have all we need to prove the theorem.

Proof of theorem 3.5. From (3.18) we have

$$|w(z) - \tilde{w}(z)| \leq |\tilde{w}(z)| \left[ |e^{\theta(z)} - \tilde{\theta}(z)| \left| \frac{\Pi_w(R, z)}{\Pi_\tilde{w}(R, z)} - 1 \right| + |W(z) - 1| \right].$$

Applying (3.9), lemmas 3.1 and 3.8, and corollary 3.7 finishes the proof. \hfill \Box

3.4. The difference of $s$ and $\tilde{s}$ when finitely many of their zeros are approximately the same

We now move on to the difference of $s$ and $\tilde{s}$. Due to lemma 2.4, changing the number $b_1 = \tilde{s}(0)/s(0)$ in the factorization (2.10) will only result in a shift of the potential. Since such a shift does not affect the zeros of $s$ (or $w$) we make the following assumption.

Assumption 2. The functions $s$ and $\tilde{s}$ satisfy $\tilde{s}(0)/s(0) = \tilde{s}(0)/\tilde{s}(0)$.

Set $N' = N_s(R, 0) = N_{\tilde{s}}(R, 0)$. As we did for $w$, we rewrite (2.10) as

$$s(z) = e^{h(z)} \left( \prod_{n=1}^{N} \frac{1}{s_n - z} \right) \left( \prod_{n=1}^{N} (s_n - z) \Pi_s(R, z) \right),$$

after redefining $h$ properly. We need to be more careful with $s$, because it may have real zeros. Due to these zeros, our estimates do not come out as neatly as they did for $w - \tilde{w}$. We will prove the following.
Let $Q$ and $\delta$ be positive, and suppose $q, \tilde{q} \in B_\delta(Q)$. Assume that $s$ and $\tilde{s}$ satisfy assumption 2. Then there are positive constants $R'_0 = R'_0(Q, \delta)$ and $C' = C'(Q, \delta)$ such that for every $R \geq R'_0$, the following statement is true. There exists a constant $E' = E(Q, R) > 0$ such that for every $\varepsilon' \in [0, E']$ and every $\eta \in (\varepsilon', 1)$ we have

$$|s(z) - \tilde{s}(z)| \leq C[\eta^{-1} \varepsilon' N' + R'^{-1/3} + (1 + |z|)\varepsilon' N' + |s(0) - \tilde{s}(0)|]$$

whenever the zeros of $s$ and $\tilde{s}$ are $\varepsilon'$-close in the disc $D(R)$ and $z \in \{z \in \mathbb{R} : |z| \leq R^{1/3}, |z - s_n| \geq \eta, 1 \leq n \leq N'\}$.

As we did for $w - \tilde{w}$, we will break the proof into a few lemmas.

**Lemma 3.10.** Let $R > 0$. Suppose $s$ and $\tilde{s}$ satisfy assumption 2 and $|s_n - \tilde{s}_n| \leq \varepsilon' \leq 3\gamma/4$ for $1 \leq n \leq N'$. Then, we have

$$\left| e^{h(z) - \tilde{h}(z)} \sum_{n=1}^{N} \frac{\tilde{s}_n}{s_n} - 1 \right| \leq \frac{2\kappa}{\gamma^2} \varepsilon' N' \exp(2\gamma^{-1} \varepsilon' N') + \frac{\kappa}{\delta \gamma^2} \varepsilon' |N'z| + \frac{1}{\delta} |s(0) - \tilde{s}(0)|$$

for every $z \in \mathbb{R}$.

**Proof.** Using (3.20) and $b_1 = \tilde{b}_1$ from assumption 2, we see that

$$h(z) - \tilde{h}(z) = \sum_{n=1}^{N} \left( \frac{1}{s_n} - \frac{1}{\tilde{s}_n} \right) z + b_0 - \tilde{b}_0,$$

meaning

$$\left| e^{h(z) - \tilde{h}(z)} \prod_{n=1}^{N} \frac{\tilde{s}_n}{s_n} - 1 \right| \leq \left| e^{h(z) - \tilde{h}(z)} \right| \left| \prod_{n=1}^{N} \frac{\tilde{s}_n}{s_n} - 1 \right| + \left| e^{b_0 - \tilde{b}_0} \right| \left| \prod_{n=1}^{N} \frac{1}{s_n} - \frac{1}{\tilde{s}_n} \right| \left| \sum_{n=1}^{N} \left( \frac{1}{s_n} - \frac{1}{\tilde{s}_n} \right) z \right| + \left| e^{b_0 - \tilde{b}_0} - 1 \right|.$$  \hspace{1cm} (3.21)

Recall that $e^{b_0} = s(0)$ and similarly for $\tilde{s}$. Therefore, the final term on the right-hand side above is bounded by $\delta^{-1} |s(0) - \tilde{s}(0)|$.

For the middle term on the right-hand side of (3.21), we begin by claiming that $\sum s_n^{-1} z$ is purely imaginary. Indeed by lemma (2.1)(i), $s_n$ is a zero of $s$ in $D(R)$ with $\text{Re} s_n \neq 0$ if and only if $-\bar{s_n}$ is a zero of $s$ in the same disc. Summing over all zeros in $D(R)$ yields

$$\sum_{n} \frac{1}{s_n} = \sum_{s_n \in \mathbb{R}} \frac{1}{s_n} + \sum_{\text{Re} s_n > 0} \frac{1}{s_n} - \frac{1}{\bar{s_n}} = \sum_{s_n \in \mathbb{R}} \frac{1}{s_n} + 2i \sum_{\text{Re} s_n > 0} \frac{1}{s_n},$$

which is purely imaginary. The inequality $|e^{\theta}-1| \leq |\theta|$ for $\theta \in \mathbb{R}$, the assumption $|s_n - \tilde{s}_n| \leq \varepsilon'$, proposition 3.3, and lemma 3.1(iv) imply

$$\left| e^{b_0 - \tilde{b}_0} \right| \left| \prod_{n=1}^{N} \frac{1}{s_n} - \frac{1}{\tilde{s}_n} \right| \left| \sum_{n=1}^{N} \left( \frac{1}{s_n} - \frac{1}{\tilde{s}_n} \right) z \right| \leq \frac{\kappa \varepsilon' N' \eta}{\delta \gamma^2} |z|.$$
since $|\log(1 + t)| \leq 2|t|$ when $|t| \leq 3/4$. Thus, 
\[ |u| \leq \frac{2}{\gamma} e^{\varepsilon N'}. \]

Applying $|e^u - 1| \leq |u| e^{\varepsilon u}$, we find that the first term is bounded by
\[ 2\kappa \varepsilon' N' \exp(2\varepsilon'/\gamma^2). \]

Putting all the estimates we obtained in (3.21), we are done as long as $u$ is well defined. To make sure it is, we must verify that $s_n/\bar{s}_n$ is never a negative real number for $1 \leq n \leq N'$. Suppose that there is a $k$ between 1 and $N'$ such that $\bar{s}_k/\bar{s}_k < 0$. Then, there is a positive number $y$ such that $\bar{s}_k = -y\bar{s}_k$. However, $|\bar{s}_k(1 + y)| = |\bar{s}_k - s_k| \leq \varepsilon'$ implies that $1 + y \leq \varepsilon'/\gamma < 1$ which contradicts $y > 0$. Therefore, $u$ is well defined.

Define for $z \notin \{\bar{s}_n : 1 \leq n \leq N'\}$,
\[ S(z) = \prod_{n=1}^{N'} \frac{z - \bar{s}_n}{z - s_n}. \]
(3.22)

**Lemma 3.11.** Let $R > 0$, and suppose $|s_n - \bar{s}_n| \leq \varepsilon'$ for $1 \leq n \leq N'$. Then, for every $\eta$ such that $\varepsilon' \leq 3\eta/4$, we have
\[ |S(z) - 1| \leq \frac{2}{\eta} \varepsilon' N' \exp(2\eta^{-1}\varepsilon' N') \]
whenever $z \in \mathbb{R}$ and $|z - s_n| \geq \eta$ for every $n$ such that $1 \leq n \leq N'$.

**Proof.** Set $u = \log S$. Then,
\[ |u| \leq \sum_{n=1}^{N'} \left| \log \left(1 + \frac{s_n - \bar{s}_n}{z - s_n}\right) \right| \leq 2 \sum_{n=1}^{N'} \left| \frac{s_n - \bar{s}_n}{z - s_n} \right| \leq \frac{2}{\eta} \varepsilon' N' \]
because $\log(1 + t) \leq 2|t|$ for $|t| \leq 3/4$. Applying $|e^u - 1| \leq |u| e^{\varepsilon u}$ completes the proof. □

**Proof of theorem 3.9.** By inequality (3.11) we may choose an $R_0$ large enough so that for every $R \geq R_0$, the estimate
\[ \left| \frac{\Pi_y(R, z)}{\Pi_y(R, \bar{z})} - 1 \right| \leq c R^{-1/3} \]
(3.23)
holds for some $c = c(Q, \delta)$ and for every $z \in [-R^{1/3}, R^{1/3}]$. By (3.20), we have
\[ |s(z) - \bar{s}(z)| \leq |s(z)||S(z) - 1| + |\bar{s}(z)||e^{\tilde{h}(z) - \bar{h}(z)}| \left| \prod_{n=1}^{N'} \frac{\bar{s}_n}{\bar{s}_n} \left( \frac{\Pi_y(R, z)}{\Pi_y(R, \bar{z})} - 1 \right) \right| \]
\[ + |\bar{s}(z)| e^{\tilde{h}(z) - \bar{h}(z)} \prod_{n=1}^{N'} \frac{\bar{s}_n}{s_n} - 1. \]
(3.24)

Let $E' = \min\{N'^{-1}, 3\gamma/4\}$ where $\gamma < 1$ is given by lemma 3.3. Suppose $0 \leq \varepsilon' \leq E'$ and $\varepsilon' < \eta < 1$. Applying (3.23) and lemmas 3.1, 3.10 and 3.11, we have the required estimate for a $C' = C'(Q, \delta)$ and for all $z \in [-R^{1/3}, R^{1/3}]$ which are at least a distance $\eta$ away from a zero of $s$. □
Corollary 3.12. Let \( Q \) and \( \delta \) be positive, and suppose \( q, \tilde{q} \in B_2(Q) \). Assume \( s \) and \( \tilde{s} \) satisfy assumption 2. Then there are positive constants \( R_0 = R_0(Q, \delta) \geq 1 \) and \( C = C(Q, \delta) \) such that for every \( R \geq R_0 \) the following statement is true. There exists a constant \( E = E(Q, R) > 0 \) such that for every \( \varepsilon \in [0, E] \) and every \( \eta \in (\varepsilon, 1) \) we have

\[
|f^+(-1, z) - \tilde{f}^+(-1, z)| \leq \frac{C}{|z|} \left( R^{-1/3} + \varepsilon N + \frac{\varepsilon}{\eta} \right)
\]

whenever the zeros of \( w \) and \( \tilde{w} \) and \( s \) and \( \tilde{s} \) are, respectively, \( \varepsilon \)-close in \( D(R) \) and \( z \in [-R^{1/3}, R^{1/3}] \), but \( z \) is at least \( \eta \) away from a zero of \( s \).

**Proof.** By lemmas 2.1(iv) and 3.2 and definition 1, we find that there are positive constants \( \varepsilon, R, \eta \) such that \( f^+(-1, z) - \tilde{f}^+(-1, z) = -\eta \) for every \( \varepsilon \in [0, E] \) and every \( \eta \in (\varepsilon, 1) \) we have

\[
|f^+(-1, z) - \tilde{f}^+(-1, z)| \leq \frac{C}{|z|} \left( R^{-1/3} + \varepsilon N + \frac{\varepsilon}{\eta} \right)
\]

for some \( C = C(Q, \delta) \) such that for every \( R \geq R_0 \) the following statement is true. There exists a constant \( E = E(Q, R) > 0 \) such that for every \( \varepsilon \in [0, E] \) and every \( \eta \in (\varepsilon, 1) \) we have

\[
|f^+(-1, z) - \tilde{f}^+(-1, z)| \leq \frac{C}{|z|} \left( R^{-1/3} + \varepsilon N + \frac{\varepsilon}{\eta} \right)
\]

whenever the zeros of \( w \) and \( \tilde{w} \) and \( s \) and \( \tilde{s} \) are, respectively, \( \varepsilon \)-close in \( D(R) \) and \( z \in [-R^{1/3}, R^{1/3}] \), but \( z \) is at least \( \eta \) away from a zero of \( s \).

**Proof.** By lemmas 2.1(iv) and 3.2 and definition 1, we find that \( N_w(R, 0) \leq N_s(R, 0) \) for any \( R > 0 \). Therefore, we obtain the desired inequality by theorems 3.5 and 3.9. \( \square \)

4. The difference of transformation operators along a line

In order to obtain a rate of convergence, we assume that \( q - \tilde{q} \in L^p(\mathbb{R}) \) for some \( p \in (1, 2] \), in addition to \( q, \tilde{q} \in B_2(Q) \). Now we wish to bound the difference of \( K^+ \) and \( \tilde{K}^+ \) along the line \( x = -t \). We begin with (3.2) and invert the Fourier transform to find

\[
K^+(-1, t) - \tilde{K}^+(-1, t) = \lim_{A \to \infty} \frac{1}{2\pi} \int_{-A}^{A} (f^+(-1, z) - \tilde{f}^+(-1, z)) e^{-i\tau z} \, dz.
\]

Because the supports of \( K^+(-1, \cdot) \) and \( \tilde{K}^+(-1, \cdot) \) are contained in the interval \([-1, 3]\), we only need to be concerned with \( t \) in that interval.

Let \( R \geq R_0 \) and assume \( A \geq R^{1/9} \). We break the interval \([-A, A]\) into four parts:

\[
X_1 = [-R^{-1/9}, R^{-1/9}],
\]

\[
X_2 = \{ z \in \mathbb{R} : R^{-1/9} \leq |z| \leq R^{1/9}, |z - s_n| \geq \eta \text{ for every } |s_n| < R \},
\]

\[
X_3 = [-R^{1/9}, R^{1/9}] \setminus (X_1 \cup X_2),
\]

and

\[
X_4 = \{ z : R^{1/9} < |z| \leq A \}.
\]

We define the corresponding integrals:

\[
I_1(t) = \frac{1}{2\pi} \int_{X_1} (f^+(-1, z) - \tilde{f}^+(-1, z)) e^{-i\tau z} \, dz,
\]

\[
I_2(t) = \frac{1}{2\pi} \int_{X_2} (f^+(-1, z) - \tilde{f}^+(-1, z)) e^{-i\tau z} \, dz,
\]

\[
I_3(t) = \frac{1}{2\pi} \int_{X_3} (f^+(-1, z) - \tilde{f}^+(-1, z)) e^{-i\tau z} \, dz,
\]

and

\[
I_4(t, A) = \frac{1}{2\pi} \int_{X_4} (f^+(-1, z) - \tilde{f}^+(-1, z)) e^{-i\tau z} \, dz.
\]

Equation (2.2) implies that \( f^+(-1, \cdot) \) and \( \tilde{f}^+(-1, \cdot) \) are bounded by a constant, \( C_1 = C_1(Q) \). Hence, we conclude

\[
|I_1(t)| \leq C_1 R^{-1/9},
\]

for every real \( t \). Corollary 3.12 and (3.6) imply that there is a \( C_2 = C_2(Q, \delta) \) such that

\[
|I_2(t)| \leq C_2 \left( R^{-1/9} + \varepsilon R^{11/9} + \frac{\varepsilon}{\eta} R^{11/9} \right).
\]
As for $I_3$, we bound the integrand by a constant and multiply by the measure of $X_3$. Because $|z - s| > \eta$ when $z \in [-\nu^{1/9}, \nu^{1/9}]$ and $|s| \geq \nu^{1/9} + \eta$, the zero of $s$ which is $\eta$-close to $z \in X_3$ must be in the disc $D(\nu^{1/9} + \eta)$. Thus, by (3.10) there exists a $C_3 = C_3(Q, \delta)$ such that
\[
m(X_3) \leq 2\eta N_\nu(\nu^{1/9} + \eta, 0) \leq C_3 \eta^{1/9}.
\] (4.5)
Hence, the same bound applies to $|I_3|$ but with a different constant.

We are left with estimating $I_4$. Going back to (2.2), integrating by parts and using $K^+(1, 3) = 0$ (see section 2), we obtain
\[
f^+(1, z) = e^{-iz} + i \frac{K^+(1, -1)}{z} e^{-iz} + i \frac{\hat{g}}{z}
\] where $g(t) = K_t(1, t)$. Recall from section 2 that the support of $g$ is contained in $[-1, 3]$. Thus,
\[
f^+(1, z) - \hat{g}^+(1, z) = \frac{i}{z} (K^+(1, -1) - \hat{K}^+(1, -1)) e^{-iz} + i \frac{\hat{g} - \hat{\hat{g}}}{z}.
\] (4.6)
Inequality (2.5) implies that $|g(t) - \hat{g}(t)|$ is bounded by the sum of a constant depending only on $Q$ and the difference of $\eta i(t - 1/2)$ and $\hat{g}(t - 1/2)$. Therefore, because $q - \hat{q} \in L^p(\mathbb{R})$, we have that $g - \hat{g} \in L^p(\mathbb{R})$ and
\[
\|g - \hat{g}\|_p \leq c(1 + \|q - \hat{q}\|_p)
\] for a constant $c$ depending only on $Q$.

Substituting (4.6) into (4.2) gives
\[
I_4(t, A) = \frac{i}{2\pi} \int_{R^{1/9} \leq |z| \leq A} \frac{K^+(1, -1) - \hat{K}^+(1, -1)}{z} e^{-iz(t + 1)} \, dz
\]
\[
- \frac{i}{2\pi} \int_{R^{1/9} \leq |z| \leq A} \frac{(\hat{g} - \hat{\hat{g}})(z)}{z} e^{-iz} \, dz.
\] By the inequalities of Hölder and Hausdorff–Young, we obtain for every real $t$,
\[
\left| \int \frac{i}{2\pi} \int_{R^{1/9} \leq |z| \leq A} \frac{(\hat{g} - \hat{\hat{g}})(z)}{z} e^{-iz} \, dz \right| \leq C_4(p - 1)^{-1/p}(1 + \|q - \hat{q}\|_p)R^{(1-p)/9}p
\] for every large $A$ and, thus, for the limit $A \to \infty$.

On the other hand, assume $-1 \leq t \leq 3$. Then, after the change $y = (t + 1)z$,
\[
\frac{i}{2} \int_{R^{1/9} \leq |z| \leq A} \frac{e^{-i(t+1)z}}{z} \, dz = \int_{R^{1/9}(t+1)} \frac{\sin(y)}{y} \, dy.
\] If $(t + 1)R^{1/9} \geq 1$, we integrate by parts to find
\[
\int_{(t+1)R^{1/9}}^{(t+1)A} \frac{\sin(y)}{y} \, dy = \frac{\cos((t+1)R^{1/9})}{(t+1)R^{1/9}} - \frac{\cos((t+1)A)}{(t+1)A} - \int_{(t+1)R^{1/9}}^{(t+1)A} \frac{\cos y}{y^2} \, dy
\] which is $O((t + 1)R^{1/9})^{-1}$ as $A \to \infty$. Otherwise, we use that $\int_0^x \sin(t) / t \, dt \leq \pi$ for all $x > 0$ to see
\[
\int_{(t+1)R^{1/9}}^{(t+1)A} \frac{\sin(y)}{y} \, dy \leq \int_{0}^{(t+1)A} \frac{\sin y}{y} \, dy \leq \pi.
\] Therefore, there exists a numerical constant, $C_5$, such that
\[
\left| \int \frac{i}{2\pi} \int_{R^{1/9} \leq |z| \leq A} \frac{e^{-i(t+1)z}}{z} \, dz \right| \leq C_5 \min \left( 1, \frac{1}{(t + 1)R^{1/9}} \right)
\] (4.9) for every large $A$.
Combining estimates (4.3), (4.4), (4.5), (4.8) and (4.9), we obtain this result.
Theorem 4.1. Suppose $q$ and $\tilde{q}$ satisfy the hypotheses of corollary 3.12 and $q - \tilde{q} \in L^p[-1, 1]$ for some $p \in (1, 2]$; then there is a constant $C = C(Q, \delta)$ such that for every $R \geq R_0$ and $\varepsilon \in [0, E]$, we have
\[ |K^+(-1, t) - \tilde{K}^+(-1, t)| \leq C(p - 1)^{-1/p}(1 + \|q - \tilde{q}\|_p)\min \left(1, \frac{1}{(t + 1)R^\alpha} + \varepsilon^{1/2}R^{2/3}\right) \]
where $\mu = (p - 1)/(9p)$.

Proof. If needed, decrease $E$ so that $\eta := \varepsilon^{1/2}R^{5/9} < R^{-1/9} < 1$. The theorem follows from the aforementioned estimates.

Corollary 4.2. Under the assumptions of theorem 4.1 and with $\lambda = \varepsilon^{1/2}R^{2/3}$ we have the estimate
\[ |B^+(1, t)| \leq C(p - 1)^{-1/p}(1 + \|q - \tilde{q}\|_p)\min \left(1, \frac{1}{(t + 1)R^\alpha} + \lambda\right) \]
for some $C = C(Q, \delta)$.

Proof. Substituting the estimate of theorem 4.1 into (3.3) and recalling that $L(s, t)$ is bounded by a constant depending only on $Q$, we find that
\[ |B^+(1, t)| \leq C(p - 1)^{-1/p}(1 + \|q - \tilde{q}\|_p)\min \left(1, \frac{1}{(t + 1)R^\alpha} + \lambda\right) + \int_0^{t+1} \min \left(1, \frac{1}{uR^\alpha} + \lambda\right) \, du. \]
Integrating completes the proof.

5. The difference of two potentials

We wish to estimate $B^+(x, x), x \in [-1, 1]$, from corollary 4.2. Given $B^+(x, t) = 0$ for $x + t \geq 2$ and assuming $B^+(1, t)$ is known for $-1 \leq t \leq 3$, we can determine $B^+(x, t)$ in the triangle bounded by $x = -1, x = t$ and $x + t = 2$ using the integral equation
\[ B^+(x, t) = B^+(-1, 1 + x + t) + \int_1^t \int_{(x+t)/2}^{1+(x+t)/2} (q(\alpha + \beta) - \tilde{q}(\alpha - \beta))B^+(\alpha - \beta, \alpha + \beta) \, d\beta. \]
The derivation of this integral equation can be found in [16]. Iteration shows that the solution is given by
\[ B^+(x, t) = \sum_{n=0}^{\infty} B_n^+(x, t), \]
where
\[ B_n^+(x, t) = B^+(-1, x + t + 1) \]
and
\[ B_{n+1}^+(x, t) = \int_1^t \int_{(x+t)/2}^{1+(x+t)/2} (q(\alpha + \beta) - \tilde{q}(\alpha - \beta))B_n(\alpha - \beta, \alpha + \beta) \, d\beta. \]
Lemma 5.1. Suppose that \( q \) and \( \tilde{q} \) are in \( B_1(Q) \) and that there exist positive constants \( C, R_1 \geq 1 \) and \( \lambda < 1 \) such that for all \( t \in (-1, 3] \).
\[
|B^+(-1, t)| \leq C \min \left( 1, \frac{1}{(t+1)R_1} + \lambda \right). 
\] (5.2)

Then
\[
|B^+_n(x, t)| \leq 2C \left( \frac{\log(4R_1)}{(1-\lambda)R_1} + \lambda \right) \frac{(2Q)^n}{(n-1)!} \left( 1 - \frac{x+t}{2} \right)^{n-1} 
\] (5.3)
for \( n \in \mathbb{N} \) and \(-2 \leq x + t \leq 2\).

**Proof.** The proof is by induction. For \( n = 1 \), we have
\[
|B^+_1(x, t)| \leq \int_{(x+1)/2}^{1} 2QC \min \left( 1, \frac{1}{2(\alpha+1)R_1} + \lambda \right) d\alpha
\]
\[
\leq C(2Q) \left( \int_{-1}^{1/2(1-\lambda)R_1-1} d\alpha + \int_{1/(2(1-\lambda)R_1)-1}^{1} \frac{1}{2(\alpha+1)R_1} + \lambda d\alpha \right) .
\]
Integrating, we obtain the required estimate.
Taking (5.3) and substituting into the right-hand side of (5.1), we obtain
\[
|B^+_n(x, t)| \leq 2C \left( \frac{\log(4R_1)}{(1-\lambda)R_1} + \lambda \right) \frac{(2Q)^n+1}{(n-1)!} \int_{(x+t)/2}^{1} (1-\alpha)^{n-1} d\alpha
\]
which completes the proof. \( \square \)

Lemma 5.2. Under the hypotheses of lemma 5.1 the estimate
\[
|B^+(x, t)| \leq C(1 + 16Q)e^{4Q} \left( \frac{\log(4R_1)}{(x+t+2)(1-\lambda)R_1} + \lambda \right) 
\] (5.4)
holds for all \((x, t)\) in the triangle bounded by \(x = -1, x = t\) and \( t + x = 2\).

**Proof.** The claim is an immediate consequence of \( B^+(x, t) = B^+(-1, x+t+1) + \sum B^+_n(x, t) \) and the estimates (5.2) and (5.3). \( \square \)

Theorem 5.3. Let \( Q_1, Q_p \) and \( \delta \) be positive constants and \( p \in (1, 2] \). Then there are positive numbers \( C = C(Q_1, Q_p, \delta) \) and \( R_0 = R_0(Q_1, Q_p, \delta, p) \) so that the following is true for any \( R \geq R_0 \). There is a constant \( E = E(Q_1, R, \delta) \) such that when \( q, \tilde{q} \in B_1(Q_1) \) are two potentials for which \( s \) and \( \tilde{s} \) satisfy assumption 2, the zeros of \( w \) and \( \tilde{w} \) and \( s \) and \( \tilde{s} \) are, respectively, \( \varepsilon \)-close in the disc \( D(R) \) with \( \varepsilon \in [0, E] \), and \( \|q - \tilde{q}\|_p \leq Q_p \), then
\[
\sup_{x \in [-1, 1]} \left| \int_{x}^{1} q(s) - \tilde{q}(s) \, ds \right| \leq C(\log R)^{2(p-1)/2}R^{-p(p-1)/2} + C(p - 1)^{-1/p}e^{1/2}R^{2/3}.
\]

**Proof.** Choose \( R_0 \) and \( E \) as in corollary 3.12. Then corollary 4.2 and lemma 5.2 show, with \( \mu = (p-1)/(9p) \) and \( \lambda = e^{1/2}R^{2/3} \),
\[
|B^+(x, t)| \leq C(p - 1)^{-1/p}Q_p \left[ \frac{(\log R)^2}{(2+t+s)(1-\lambda)R^n} + \lambda \right] .
\]
Let \( 0 < \eta < 1 \). For \( \eta - 1 \leq x \leq 1 \),
\[
\left| \int_x^1 q(s) - \tilde{q}(s) \, ds \right| \leq C(p - 1)^{-1/p}(1 + Q_p) \left[ \frac{(\log R)^2}{\eta(1 - \lambda)R^\mu + \lambda} \right]
\]
\[= \frac{M_1}{\eta} + M_2 \lambda.\]

However, for \(-1 \leq x < \eta - 1\), Hölder’s inequality yields
\[
\left| \int_x^1 q(s) - \tilde{q}(s) \, ds \right| \leq \left| \int_{x}^{\eta - 1} q - \tilde{q} \right| + \left| \int_{\eta - 1}^{1} q - \tilde{q} \right| < \frac{M_1}{\eta} + M_2 \lambda,
\]
where \( p' = p/(p - 1) \). Increasing \( R_0 \) so that \( M_1 p' < Q_p \), the first two terms of the error balance when \( \eta = (M_1 p'/Q_p)^{p'/p+1} < 1 \). Finally, we decrease \( \lambda \), so that \( \lambda \leq 1/2 \) to arrive at the desired inequality. \( \square \)

The proof of theorem 5.3 shows \( C = O(\tilde{Q}_p^{p/(2p-1)}) \) as \( Q_p \to 0 \).

Since we could have carried out our analysis with \( s^+ \) instead of \( s^- \), we arrive at the following corollary.

**Corollary 5.4** (Conditional stability). Let \( q \) and \( \tilde{q} \) be two real-valued potentials with support in \([-1, 1]\); let \( s_q \) and \( s_{\tilde{q}} \) stand for either \( s_q^+ \) and \( s_{\tilde{q}}^+ \) or \( s_q^- \) and \( s_{\tilde{q}}^- \). Suppose \( \|q\|_p \) and \( \|\tilde{q}\|_p \) are bounded by \( Q_p \) for some \( p > 1 \), and the moduli of \( s_q(0) \) and \( s_{\tilde{q}}(0) \) are no less than \( \delta \), and \( \ddot{s}_q(0)/s_q(0) = \ddot{s}_{\tilde{q}}(0)/s_{\tilde{q}}(0) \). Then for any \( \alpha > 0 \), there exists a pair \( (R, \varepsilon) \), depending only on \( \delta, Q_p, p \) and \( \alpha \), such that if the corresponding (left or right) reflection coefficients have zeros and poles differing by at most \( \varepsilon \), respectively, in a disc of radius \( R \), then
\[
\sup_{x \in [-1, 1]} \left| \int_x^1 (q - \tilde{q}) \right| \leq \alpha.
\]

### 6. Stability of small perturbations of the zero potential

In this section, we prove stability when \( \tilde{q} \equiv 0 \). Recall that this potential was previously excluded since it has a resonance at zero. Indeed, since \( \tilde{w}(z) = 2iz \), the zero potential only has a resonance at zero. Furthermore, we have \( s \equiv 0, \tilde{K}^+ \equiv 0 \), and \( \tilde{f}^+(x, z) = \exp(izx) \). Thus, we have \( B^+ = K^+ \) and
\[
\left( \frac{w(z)}{2iz} - 1 \right) e^{-iz} + \frac{s(z)}{2iz} e^{iz} = \tilde{f}^+(-1, z) - e^{-iz} = \int_{-1}^3 K^+(-1, t) e^{izt} \, dt.
\]

The proof will go much the same as it did before. First, we estimate
\[
|w(z)(2iz)^{-1} - 2iz|
\]
assuming that \( w \) has a zero in a small neighborhood of the origin and all its other zeros are large. We then estimate \( |s(z)(2iz)^{-1}| \) for \( z \in \mathbb{R} \) from
\[
|s(z)|^2 = |w(z)|^2 - 4z^2
\]
which is a consequence of lemma 2.1 parts (i) and (iii). Note that we do not need the zeros of \( s \) in this case. We bound \( |f^+(-1, z) - \exp(-iz)| \) from the previous estimates and then apply the results of sections 4 and 5 almost without change to arrive at this result.
Theorem 6.1. Let $Q_1$ and $Q_2$ be positive constants and $p \in (1, 2]$. Then there are positive numbers $C = C(Q_1, Q_2, p)$ and $R_0 = R_0(Q_1, Q_2, p)$, so that the following is true for any $R > R_0$.

There is a constant $E = E(Q_1, R)$ such that if $q \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ is supported in $[-1, 1]$, $\|q\|_1 \leq Q_1$, $\|q\|_p \leq Q_2$, and has exactly one eigenvalue or resonance in the disc $D(\varepsilon)$ with $\varepsilon \in [0, E]$ and no others in the disc $D(R)$, then

$$\sup_{x \in [-1, 1]} \int_1^1 q(s) ds \leq C(\log R)^{2(p-1)/(2p-1)} R^{-(p-1)/2(2p-1)} + C(p - 1)^{-1/\sqrt{p}} \sqrt{\varepsilon} R^{1/2} \log R.$$ 

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