Fluctuations in an Evolutional Model of Two-Dimensional Young Diagrams

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Abstract

We discuss the non-equilibrium fluctuation problem, which corresponds to the hydrodynamic limit established in [9], for the dynamics of two-dimensional Young diagrams associated with the uniform and restricted uniform statistics, and derive linear stochastic partial differential equations in the limit. We show that their invariant measures are identical to the Gaussian measures which appear in the fluctuation limits in the static situations.

1 Introduction

In our companion paper [9] we investigated the hydrodynamic limit for dynamics of two-dimensional Young diagrams associated with the grandcanonical ensembles determined from two types of statistics called uniform (or Bose) and restricted uniform (or Fermi) statistics introduced by Vershik [18]. The aim of the present paper is to study the corresponding non-equilibrium dynamic fluctuation problem. The theory of the equilibrium dynamic fluctuation around the hydrodynamic limit is well established based on the so-called Boltzmann-Gibbs principle, see [13]. However, the results on the non-equilibrium dynamic fluctuations are rather limited, cf. [3], [5] due to a special feature of the models and [2] in a more general setting. In the present case we are able to derive linear stochastic partial differential equations (SPDEs) in the limit. Also, the fluctuations can be studied in the static situations and these results are reinterpreted from the dynamic point of view

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by identifying the static fluctuation limits with the invariant measures of the limit SPDEs. See [16], [7], [20] for static fluctuations under canonical ensembles.

As shown in [9], the dynamics of the two-dimensional Young diagrams can be transformed into equivalent particle systems by considering their height differences. In fact, in the uniform statistics (short term U-case), the evolution of the height difference \( \xi_t = (\xi_t(x))_{x \in \mathbb{N}} \in \mathbb{Z}^\mathbb{N} \) of the Young diagrams’ height function \( \psi_t(u), u \in \mathbb{R}_+ \) defined by \( \xi_t(x) = \psi_t(x - 1) - \psi_t(x) \) and supplied with the condition \( \xi_t(0) = \infty \) performs a weakly asymmetric zero-range process on \( \mathbb{N} \) with a weakly asymmetric stochastic reservoir at \( \{0\} \). Here we denote \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \), \( \mathbb{N} = \{1, 2, \ldots\} \) and \( \mathbb{R}_+ = [0, \infty) \). Such a particle system is further transformed into a weakly asymmetric simple exclusion process \( \tilde{\eta}_t \in \{0, 1\}^\mathbb{Z} \) on the whole integer lattice \( \mathbb{Z} \) without any boundary conditions by rotating the \( xy \)-plane around the origin by 45 degrees counterclockwise and projecting the system to the \( x \)-axis rescaled by \( \sqrt{2} \). This involves quite a nonlinearity as observed in Section 4 of [9].

On the other hand, in the restricted uniform statistics (short term RU-case), the height difference \( \eta_t = (\eta_t(x))_{x \in \mathbb{N}} \in \{0, 1\}^\mathbb{N} \) of the Young diagrams’ height function supplied with the condition \( \eta_t(0) = \infty \) performs a weakly asymmetric simple exclusion process on \( \mathbb{N} \) with a weakly asymmetric stochastic reservoir at \( \{0\} \).

The hydrodynamic limit for the weakly asymmetric simple exclusion process \( \tilde{\eta}_t \) on the whole integer lattice was studied by [10] and [3], and the corresponding fluctuation limit by [3] and [5]. In these works the convergence of the density fluctuation fields was shown only in the space of processes taking values in generalized functions such as \( D([0, \infty), \mathcal{S}'(\mathbb{R})) \) or \( D([0, T], \mathcal{H}') \) for a kind of Sobolev space \( \mathcal{H}' \) with negative index. In the U-case it is indeed necessary to transform \( \tilde{\eta}_t \) back to \( \xi_t \) through a nonlinear map, so that these convergence results are too weak and it is necessary to establish the tightness of \( \tilde{\eta}_t \) (under scaling and linear interpolation) in the space \( D([0, T], \mathcal{D}(\mathbb{R})) \), i.e. a stronger topology. The boundary condition in the RU-case needs additional analysis. The fluctuations for the simple exclusion process with boundary conditions in a symmetric case (i.e. \( \varepsilon = 1 \)) were studied by [15]. The weakly asymmetric case was discussed by [4] but without mathematically rigorous proofs. The tightness in the space \( D([0, T], \mathcal{D}(\mathbb{R})) \) beyond the time scale of the hydrodynamic limit was established by [1] and they derived the KPZ equation in the limit. We follow their method with an adjustment concerning the boundary condition in the RU-case.

In Section 2 we first recall the results of [9] on the hydrodynamic limits and then formulate our main results on the fluctuations as Theorems 2.1 and 2.2 for the U-case and Theorem 2.3 for the RU-case. The proofs of these results are given in Sections 3 and 4. Finally, in Section 5 we discuss the invariant measures of the SPDEs obtained in the limit and their relations to those obtained in the static situations, see Theorems 5.2, 5.4 and Proposition 5.1.

In this paper, given a Banach space \( X \) and \( I \subseteq \mathbb{R} \), \( C(I, X) \) denotes the set of all continuous functions equipped with the locally uniform convergence, as well as \( D(I, X) \) the set of all càdlàg functions equipped with the Skorohod topology. Abbreviate \( C(I, \mathbb{R}) = C(I) \) and \( D(I, \mathbb{R}) = D(I) \). Furthermore define for each \( r > 0 \) the weighted \( L^2 \)- space \( L^2_r(I) \) equipped with the norm \( \|f\|_{L^2_r(I)} = \left\{ \int_I |f(u)|^2 e^{-2ru^2} du \right\}^{1/2} \) and set \( L^2_r(I) := \cap_{r > 0} L^2_r(I) \).
2 Main Results

We first recall the notation used in the paper [9] and briefly summarize the results obtained there. For each \( n \in \mathbb{N} \), let \( \mathcal{P}_n \) be the set of all sequences \( p = (p_i)_{i \in \mathbb{N}} \) satisfying \( p_1 \geq p_2 \geq \cdots \geq p_i \geq \cdots \), \( p_i \in \mathbb{Z}_+ \), and \( n(p) := \sum_{i \in \mathbb{N}} p_i = n \). Let \( \mathcal{Q}_n \) be the set of all sequences \( q = (q_i)_{i \in \mathbb{N}} \in \mathcal{P}_n \) satisfying \( q_i > q_{i+1} \) if \( q_i > 0 \). For \( n = 0 \), we define \( \mathcal{P}_0 = \mathcal{Q}_0 = \{0\} \), where 0 is a sequence such that \( p_i = 0 \) for all \( i \in \mathbb{N} \). The unions of \( \mathcal{P}_n \) and \( \mathcal{Q}_n \) in \( n \in \mathbb{Z}_+ \) are denoted by \( \mathcal{P} \) and \( \mathcal{Q} \), respectively. The height function of the Young diagram corresponding to \( p \in \mathcal{P} \) is defined by

\[
\psi_p(u) = \sum_{i \in \mathbb{N}} 1_{\{u < p_i\}}, \quad u \in \mathbb{R}_+,
\]

and its scaled height function by

\[
\tilde{\psi}_p^N(u) = \frac{1}{N} \psi_p(Nu), \quad u \in \mathbb{R}_+,
\]

for \( N \geq 1 \). Note that \( \psi_q \) and \( \tilde{\psi}_q^N \) are defined for \( q \in \mathcal{Q} \), since \( \mathcal{Q} \subset \mathcal{P} \). For \( 0 < \varepsilon < 1 \), the dynamics \( p_i := p_i^\varepsilon = (p_i(t))_{i \in \mathbb{N}} \) on \( \mathcal{P} \) and \( q_i := q_i^\varepsilon = (q_i(t))_{i \in \mathbb{N}} \) on \( \mathcal{Q} \) are introduced as Markov processes on these spaces having generators \( L_{\varepsilon,\mathcal{P}} \) and \( L_{\varepsilon,\mathcal{Q}} \), respectively, defined as follows. The operator \( L_{\varepsilon,\mathcal{P}} \) acts on functions \( f : \mathcal{P} \to \mathbb{R} \) as

\[
L_{\varepsilon,\mathcal{P}} f(p) = \sum_{i \in \mathbb{N}} \left[ \varepsilon 1_{\{p_i-1 > p_i\}} \{ f(p_i+1) - f(p_i) \} + 1_{\{p_i > p_{i+1}\}} \{ f(p_{i+1}) - f(p_i) \} \right],
\]

while the operator \( L_{\varepsilon,\mathcal{Q}} \) acts on functions \( f : \mathcal{Q} \to \mathbb{R} \) as

\[
L_{\varepsilon,\mathcal{Q}} f(q) = \sum_{i \in \mathbb{N}} \left[ \varepsilon 1_{\{q_i-1 > q_i+1\}} \{ f(q_i+1) - f(q_i) \} + 1_{\{q_i > q_{i+1}+1 \text{ or } q_i=1\}} \{ f(q_{i+1}) - f(q_i) \} \right],
\]

where \( p_i^\varepsilon = (p_j^\varepsilon)_{j \in \mathbb{N}} \in \mathcal{P} \) are defined for \( i \in \mathbb{N} \) and \( p \in \mathcal{P} \) by

\[
p_j^\varepsilon = \begin{cases} p_j & \text{if } j \neq i, \\ p_i \pm 1 & \text{if } j = i, \end{cases}
\]

and \( q_i^\varepsilon \in \mathcal{Q} \) similarly for \( q \in \mathcal{Q} \). In (2.1) and (2.2), take \( p_0 = \infty \) and \( q_0 = \infty \). These processes have the grandcanonical ensembles \( \mu_{\varepsilon,\mathcal{P}} \) on \( \mathcal{P} \) and \( \mu_{\varepsilon,\mathcal{Q}} \) on \( \mathcal{Q} \) as their invariant measures, respectively, where \( \mu_{\varepsilon,\mathcal{P}} \) and \( \mu_{\varepsilon,\mathcal{Q}} \) are probability measures on these spaces defined by

\[
\mu_{\varepsilon,\mathcal{P}}(p) = \frac{1}{Z_{\mathcal{P}}(\varepsilon)^{N(p)}} e^{n(p)}, \quad p \in \mathcal{P}, \quad \text{and} \quad \mu_{\varepsilon,\mathcal{Q}}(q) = \frac{1}{Z_{\mathcal{Q}}(\varepsilon)^{N(q)}} e^{n(q)}, \quad q \in \mathcal{Q},
\]

and \( Z_{\mathcal{P}}(\varepsilon) = \prod_{k=1}^{\infty} (1 - \varepsilon^k)^{-1} \) and \( Z_{\mathcal{Q}}(\varepsilon) = \prod_{k=1}^{\infty} (1 + \varepsilon^k) \) are the normalizing constants.

Choose \( \varepsilon = \varepsilon(N) = \varepsilon_U(N), \varepsilon_R(N) \) in such a way that in each case the averaged size of the Young diagrams under \( \mu_{\varepsilon,\mathcal{P}} \) or \( \mu_{\varepsilon,\mathcal{Q}} \) is equal to \( N^2 \), i.e.

\[
E_{\mu_{\varepsilon,\mathcal{P}}}(n(p)) = N^2 \quad \text{and} \quad E_{\mu_{\varepsilon,\mathcal{Q}}}(n(q)) = N^2.
\]
The asymptotic behavior $\varepsilon_U(N) = 1 - \alpha/N + O(\log N/N^2)$ and $\varepsilon_R(N) = 1 - \beta/N + O(\log N/N^2)$ as $N \to \infty$, with $\alpha = \pi/\sqrt{6}$ and $\beta = \pi/\sqrt{12}$ was shown in Lemmas 3.1 and 3.2 in [9]. Define the corresponding height functions diffusively scaled in space and time by

$$\tilde{\psi}_{N}^U(t,u) := \frac{1}{N} \psi_{N^2 t}^U(Nu)$$ and $$\tilde{\psi}_{N}^R(t,u) := \frac{1}{N} \psi_{N^2 t}^R(Nu),$$

with $\varepsilon = \varepsilon(N)$. The following results are obtained in [9], Theorems 2.1 and 2.2. Denote the partial derivative $\partial_u \psi$ by $\psi'$.

(1) If $\tilde{\psi}_{N}^U(0,u)$ converges to $\psi_0 \in X_U$ (see below) in probability as $N \to \infty$, then $\tilde{\psi}_{N}^U(t,u)$ converges to $\psi_U(t,u)$ in probability. Here $\psi(t,u) := \psi_U(t,u)$ is a unique solution of the following nonlinear partial differential equation (PDE):

$$\partial_t \psi = \left(\frac{\psi'}{1-\psi'}\right)' + \alpha \frac{\psi'}{1-\psi'}, \quad u \in \mathbb{R}_+^\circ,$$

with the initial condition $\psi(0,\cdot) = \psi_{U,0}(\cdot)$, where $\mathbb{R}_+^\circ = (0,\infty)$.

(2) If $\tilde{\psi}_{N}^R(0,u)$ converges to $\psi_0 \in X_R$ (see below) in probability as $N \to \infty$, then $\tilde{\psi}_{N}^R(t,u)$ converges to $\psi_R(t,u)$ in probability. Here $\psi(t,u) := \psi_R(t,u)$ is a unique solution of the following nonlinear PDE:

$$\partial_t \psi = \psi'' + \beta \psi'(1 + \psi'), \quad u \in \mathbb{R}_+,$$

with the initial condition $\psi(0,\cdot) = \psi_{R,0}(\cdot)$.

Consider these PDEs in the function spaces $X_U$ and $X_R$, respectively, and their solutions are unique in these classes:

$$X_U := \{ \psi : \mathbb{R}_+^\circ \to \mathbb{R}_+^\circ; \psi \in C^1, \psi' < 0, \lim_{u \downarrow 0} \psi(u) = \infty, \lim_{u \uparrow \infty} \psi(u) = 0 \},$$

$$X_R := \{ \psi : \mathbb{R}_+ \to \mathbb{R}_+; \psi \in C^1, -1 \leq \psi' \leq 0, \psi'(0) = -1/2, \lim_{u \uparrow \infty} \psi(u) = 0 \}.$$

Figure 1: A typical height function and its scaling limit
The aim of the present paper is to establish the corresponding fluctuation limits. Namely, we consider the fluctuations of \( \tilde{\psi}_U^N(t, u) \) and \( \tilde{\psi}_R^N(t, u) \) around their limits:

\[
\Psi_U^N(t, u) := \sqrt{N} (\tilde{\psi}_U^N(t, u) - \psi_U(t, u)) \quad \text{and} \quad \Psi_R^N(t, u) := \sqrt{N} (\tilde{\psi}_R^N(t, u) - \psi_R(t, u)),
\]

which are elements of \( D([0, T], D(\mathbb{R}_+)) \) and \( D([0, T], D(\mathbb{R}_+)) \), respectively.

A natural idea in the U-case is to investigate the fluctuation of the curve \( \tilde{\psi}_U^N(t) \) around \( \tilde{\psi}_U(t) \), which are obtained by rotating the original curves \( \tilde{\psi}_U^N(t) \) and \( \psi_U(t) = \{(u, y); y = \tilde{\psi}_U(t, u), u \in \mathbb{R}_+^+\} \) located in the first quadrant of the \( xy \)-plane by 45 degrees counterclockwise around the origin \( O \), respectively, where \( \tilde{\psi}_U^N(t) \) is a continuous indented curve obtained from the graph \( \{(u, y); y = \tilde{\psi}_U^N(t, u), u \in \mathbb{R}_+\} \) of the original function \( \tilde{\psi}_U^N(t, u) \) by filling all jumps by vertical segments. In particular, this contains a part of \( y \)-axis: \( \{(0, y); y \geq \tilde{\psi}_U^N(t, 0)\} \).

Figure 2: Rotating by 45° yields functions on \( \mathbb{R} \) and a particle system on \( \mathbb{Z} \).

Then, we consider

\[
\tilde{\Psi}_U^N(t, v) := \sqrt{N} (\tilde{\psi}_U^N(t, v) - \tilde{\psi}_U(t, v)), \quad v \in \mathbb{R},
\]

which is an element of \( D([0, T], C(\mathbb{R})) \). The fluctuation \( \tilde{\Psi}_U^N(t) \) defined as above is a natural object to study, since the Young diagrams corresponding to the class \( \mathcal{P} \) belong to the same class under the reflection with respect to the line \( \{y = u\} \), while those corresponding to \( \mathcal{Q} \) do not have such property in general.

We are now at the position to formulate our main theorems. In the U-case, we first state the result for \( \tilde{\Psi}_U^N(t) \) and then apply it to \( \Psi_U^N(t) \). We assume the following three conditions on the initial values \( \{\tilde{\Psi}_U^N(0, v)\}_N \) and \( \{\tilde{\psi}_U^N(0, 0)\}_N \):

**Assumption 1.** (1) For every \( \kappa \in \mathbb{N} \), the following holds:

(i) \( \sup_{N \in \mathbb{N}} E[\exp\{\kappa \tilde{\psi}_U^N(0, 0)\}] < \infty \),

(ii) \( \sup_{N \in \mathbb{N}} \sup_{v \in \mathbb{R}} E[|\tilde{\psi}_U^N(0, v)|^{2\kappa}] < \infty \),

(iii) for any \( v_1, v_2 \in \mathbb{Z}/N \sup_{N \in \mathbb{N}} E[|\tilde{\psi}_U^N(0, v_1) - \tilde{\psi}_U^N(0, v_2)|^{2\kappa}] \leq C|v_1 - v_2|^\kappa \) with \( C > 0 \).

(2) \( \{\tilde{\psi}_U^N(0, v)\}_N \) are independent of the noises determining the process \( \{p_t^{(N)}; t \geq 0\} \).
(3) \( \tilde{\Psi}^N_U(0, v) \) converges weakly to \( \tilde{\Psi}_{U,0}(v) \) in \( C(\mathbb{R}) \), and \( E[|\tilde{\Psi}_{U,0}|^2_{L^2(\mathbb{R})}] < \infty \) for all \( r > 0 \).

For every initial value \( \psi_U(0) \in X_U \), one can easily construct non-random or random sequences \( \{\tilde{\psi}^N_U(0)\}_N \) or \( \{\psi^N_U(0)\}_N \), which satisfy these three conditions.

**Theorem 2.1. (U-case under rotation)** Under Assumption 1, \( \tilde{\Psi}^N_U(t, v) \) converges weakly to \( \tilde{\Psi}_U(t, v) \) as \( N \to \infty \) on the space \( D([0, T], C(\mathbb{R})) \) for every \( T > 0 \). The limit \( \tilde{\Psi}_U(t, v) \) is in \( C([0, T], C(\mathbb{R})) \) (a.s.) and characterized as a solution of the following SPDE:

\[
\begin{aligned}
\frac{\partial_t}{\partial_t} \tilde{\Psi}_U(t, v) & = \frac{1}{2} \tilde{\Phi}_U''(t, v) + \rho(t, \sqrt{v}) \tilde{\Phi}_U''(t, v) + 2^{\frac{3}{2}} \rho(t, \sqrt{v})(1 - \rho(t, \sqrt{v})) \tilde{W}(t, v), \\
\tilde{\Psi}_U(0, v) & = \tilde{\Psi}_{U,0}(v),
\end{aligned}
\]

(2.3)

where \( \rho(t, \cdot) \) is the solution of the PDE (3.1) below, or equivalently \( \rho(t, \cdot) = \Phi_U(\psi_U(t))(\cdot) \) with the map \( \Phi_U : X_U \to Y_U \) defined in Proposition 4.4 of [9] (or given explicitly in the proof of Lemma 3.3 below), and \( \tilde{W}(t, v) \) is the space-time white noise on \([0, T] \times \mathbb{R}\).

The solution of (2.3) is defined in a weak sense: We call \( \tilde{\Psi}_U(t, v) \) a solution of the SPDE (2.3) if it is adapted with respect to the increasing \( \sigma \)-fields generated by \( \{W(s); s \leq t\} \), satisfies \( \tilde{\Psi}_U \in C([0, T], C(\mathbb{R})) \cap C([0, T], L^2(\mathbb{R})) \) (a.s.) and for every \( f \in C^1(\mathbb{R}) \),

\[
\langle \tilde{\Psi}_U(t), f(t) \rangle = \langle \tilde{\Psi}_{U,0}, f(0) \rangle + \int_0^t \langle \tilde{\Psi}_U(s), \frac{1}{2} f''(s) - \frac{\alpha}{2} (1 - 2 \rho(s, \sqrt{v})) f(s) \rangle + \partial_s f(s) \rangle ds \\
+ \int_0^t \int_{\mathbb{R}} f(s, v) 2^{\frac{3}{2}} \rho(s, \sqrt{v})(1 - \rho(s, \sqrt{v})) W(ds, dv) \text{ a.s.,}
\]

where \( \langle \cdot, \cdot \rangle = \int_{\mathbb{R}} \langle \cdot(v), \cdot(v) \rangle dv \). Similar to the SPDE (2.6) (with the boundary condition) stated below, one can show that the solution of (2.3) is equivalent to its mild form and unique in the above class.

Although the directions of the fluctuations are different in \( \tilde{\Psi}^N_U \) and \( \tilde{\Psi}_U^N \), we still are able to deduce the next theorem from Theorem 2.1. As pointed out before, the transformation is nonlinear, so it is important that the convergence in Theorem 2.1 is shown in a function space \( D([0, T], C(\mathbb{R})) \).

**Theorem 2.2. (U-case)** Under Assumption 1, \( \Psi^N_U(t, u) \) converges weakly to \( \Psi_U(t, u) \) as \( N \to \infty \) on the space \( D([0, T], D(\mathbb{R}^+)) \) for every \( T > 0 \). The limit \( \Psi_U(t, u) \) is in \( C([0, T], C(\mathbb{R}^+)) \) (a.s.) and a solution of the following SPDE:

\[
\begin{aligned}
\frac{\partial_t}{\partial_t} \Psi_U(t, u) & = \frac{\Psi_U''(t, u)}{(1 + \rho_U(t, u))^2} + \alpha \frac{\Psi_U'(t, u)}{(1 + \rho_U(t, u))^2} + \sqrt{\frac{2 \rho_U(t, u)}{1 + \rho_U(t, u)}} \tilde{W}(t, u), \\
\Psi_U(0, u) & = \Psi_{U,0}(u),
\end{aligned}
\]

(2.4)

where \( \rho_U(t, u) = -\psi_U'(t, u) \) and \( \tilde{W}(t, u) \) is the space-time white noise on \([0, T] \times \mathbb{R}^+\).

Let \( L^2_r(\mathbb{R}^+), r > 0 \) be the weighted \( L^2 \)-space of functions on \( \mathbb{R}^+ \) equipped with the following norm: Take a positive function \( g_r \in C^\infty(\mathbb{R}^+) \) such that \( g_r(u) = u^{1+2r/\alpha} \) for \( u \in \mathbb{R}^+ \).
(0, 1] and \( g_r(u) = e^{-2ru} \) for \( u \in [2, \infty) \), and define \( |\Psi|_{L_2^2(\mathbb{R}_+^\infty)} = \left\{ \int_{\mathbb{R}_+^\infty} |\Psi(u)|^2 g_r(u) du \right\}^{1/2} \). Again, we set \( \tilde{L}_2^2(\mathbb{R}_+^\infty) = \cap_{r>0} \tilde{L}_2^2(\mathbb{R}_+^\infty) \). The reason to introduce these spaces is explained in Remark 3.3 below.

The solution of the SPDE (2.4) is defined in a weak sense: We call \( \Psi_U(t, u) \) a solution of the SPDE (2.4) if it is adapted, satisfies \( \Psi_U \in C([0, T], C(\mathbb{R}_+^\infty)) \cap C([0, T], \tilde{L}_2^2(\mathbb{R}_+^\infty)) \) (a.s.) and for every \( f \in C_0^{1,2}([0, T] \times \mathbb{R}_+^\infty) \),

\[
\langle \Psi_U(t), f(t) \rangle = \langle \Psi_U(0), f(0) \rangle + \int_0^t \frac{1}{1 + \rho_U(s)} \psi(s) \left( f'(s) - \alpha f(s) \right) \, ds + \int_0^t \frac{2\rho_U(s) f(s) - \alpha f(s)}{1 + \rho_U(s)} \, ds \, \int_0^t \frac{1}{1 + \rho_U(s)} \, W(ds) \, a.s.,
\]

where \( \langle \Psi_U, f \rangle = \int_{\mathbb{R}_+^\infty} \Psi_U(u)f(u) du. \) The solution of the SPDE (2.4) is unique under condition (3.15), see Lemma 3.7 and Proposition 3.11 below.

**Remark 2.1.** The boundary condition for the SPDE (2.4) is unnecessary. Here this is seen at least under the equilibrium situation: \( \rho_U(t, u) = \rho_U(u) \). Consider the corresponding diffusion on \( \mathbb{R}_+^\infty \) to the linear differential operator appearing in (2.4) given by

\[
dX_t = b(X_t)dt + \sigma(X_t)dB_t
\]

with

\[
b(x) = \frac{\alpha}{(1 + \rho_U(x))^2} - 2 \frac{\rho_U(x)}{(1 + \rho_U(x))^2}, \quad \sigma(x) = \frac{1}{1 + \rho_U(x)},
\]

and \( B_t \) a 1-dimensional Brownian motion. Then we can show that the corresponding scale function defined on \( \mathbb{R}_+^\infty \) diverges to \(-\infty\) as \( u \downarrow 0 \). This means that \( 0 \) is a natural boundary for \( X_t \), see, e.g., Proposition 5.22 in [12]. Accordingly, we do not need any boundary condition at \( u = 0 \).

For the RU-case, we assume the following three conditions on the initial values \( \{\Psi^N_R(0, u)\}_N \) and \( \{\tilde{\Psi}^N_R(0, 0)\}_N \):

**Assumption 2.** (1) For any \( \kappa \in \mathbb{N} \), the following holds:

(i) \( \sup_{N \in \mathbb{N}} E[\exp\{\kappa \tilde{\Psi}^N_R(0, 0)\}] < \infty \),

(ii) \( \sup_{N \in \mathbb{N}} \sup_{u \in \mathbb{R}_+} E[|\Psi^N_R(0, u)|^{2\kappa}] < \infty \),

(iii) for any \( u_1, u_2 \in \mathbb{N}/\mathbb{N} \) \( \sup_{N \in \mathbb{N}} \sup_{N \in \mathbb{N}} E[|\Psi^N_R(0, u_1) - \Psi^N_R(0, u_2)|^{2\kappa}] \leq C|u_1 - u_2|^\kappa \) with \( C > 0 \).

(2) \( \{\Psi^N_R(0, u)\}_N \) are independent of the noises determining the process \( \{\eta^N_t; t \geq 0\} \).

(3) \( \Psi^N_R(0, u) \) converges weakly to \( \Psi_{R,0}(u) \) in \( D(\mathbb{R}_+^\infty) \). Moreover, we assume that \( \Psi_{R,0} \in C(\mathbb{R}_+^\infty) \) (a.s.) and \( E[|\Psi_{R,0}|_{L_2^2(\mathbb{R}_+^\infty)}^2] < \infty \) for all \( r > 0 \).

**Remark 2.2.** The scaled height \( \tilde{\Psi}^N_R(0, 0) \) at \( t = 0 \) and \( u = 0 \) appearing in Assumption 2-(1)(i) is equal to the initial particle number of the weakly asymmetric simple exclusion process \( \eta_t \) divided by \( N \), see Section 4.
\[ \Psi \text{ satisfying } f \quad \text{where} \quad (2.6) \]

where \( \rho_R(t,u) = -\psi_R(t,u) \) and \( \dot{W}(t,u) \) is the space-time white noise on \([0,T] \times \mathbb{R}_+\).

Again, we say \( \Psi_R(t,u) \) is a solution of the SPDE (2.6) if it is adapted, satisfies \( \Psi_R \in C([0,T], C(\mathbb{R}_+)) \cap C([0,T], L^2(\mathbb{R}_+)) \) (a.s.) and for every \( f \in C^{1,2}_0([0,T] \times \mathbb{R}_+) \) satisfying \( f'(t,0) = 0 \) the following holds:

\[ \langle \Psi_R(t), f(t) \rangle = \langle \Psi_R(0), f(0) \rangle + \int_0^t \langle \Psi_R(s), f''(s) - \beta((1 - 2\rho_R(s))f(s))' + \partial_s f \rangle ds \]

(2.7)

Similarly as in [11], one can show that the solution of (2.6) is equivalent to its mild form, that is, \( \Psi_R(t,u) \) is an \( L^2(\mathbb{R}_+) \)-valued adapted process and the following holds:

\[ \Psi_R(t,u) = \int_{\mathbb{R}_+} p(t,u,v)\Psi_R(0,v)dv + \int_0^t \int_{\mathbb{R}_+} 2\beta p(t-s,u,v)\rho_R(s,v)\Psi_R(s,v)dvds \]

\[ - \int_0^t \int_{\mathbb{R}_+} \frac{\partial}{\partial v} p(t-s,u,v)\beta(1 - 2\rho_R(s,v))\Psi_R(s,v)dvds \]

\[ + \int_0^t \int_{\mathbb{R}_+} p(t-s,u,v)\sqrt{2\rho_R(s,v)(1 - \rho_R(s,v))}W(dsdu) \text{ a.s.} \]

where \( p(t,u,v) \) is the fundamental solution to \( \partial_t \Psi(t,u) = \Psi''(t,u) \) with the homogeneous Neumann boundary condition at 0, that is \( p(t,u,v) = \frac{1}{\sqrt{4\pi t}} \left\{ e^{-\frac{(u-v)^2}{4t}} + e^{-\frac{(u+v)^2}{4t}} \right\}, u,v \in \mathbb{R}_+ \). The properties of \( \rho_R(t,u) \) and basic estimates for \( p(t,u,v) \) imply the existence and uniqueness of the solution to (2.6). On the other hand, one can also show the continuity of the trajectory of \( \Psi_R(t,\cdot) \) as an \( L^2(\mathbb{R}_+) \)-valued process and the joint continuity in \( t \) and \( u \). Since the arguments are standard, the details are omitted.

3 Proof of Theorems 2.1 and 2.2

3.1 Proof of Theorem 2.1

As already pointed out in Section 1 (or see Section 4 of [9] for more details), the height difference \( \xi_t = (\xi_t(x))_{x \in \mathbb{N}} \in (\mathbb{Z}_+)^\mathbb{N} \) of \( \psi_{p_t} \) can be transformed into a weakly asymmetric
simple exclusion process $\tilde{\eta}_t = (\tilde{\eta}_t(x))_{x \in \mathbb{Z}} \in \{0, 1\}^\mathbb{Z}$ on a whole integer lattice $\mathbb{Z}$. For further use, we introduce two functions $\zeta^-_{\tilde{\eta}}$ and $\zeta^+_{\tilde{\eta}}$ on $\mathbb{Z}$ by

$$
\zeta^-_{\tilde{\eta}}(x) := \sum_{z \leq x} (1 - \tilde{\eta}(z)) \quad \text{and} \quad \zeta^+_{\tilde{\eta}}(x) := \sum_{z \geq x + 1} \tilde{\eta}(z),
$$

which are the main parts of the transformation. The scaled empirical measures of the time accelerated process $\tilde{\eta}_t^N := \tilde{\eta}_{N\tilde{t}}$ of $\tilde{\eta}_t$ given by

$$
\pi_t^N(dv) := \frac{1}{N} \sum_{x \in \mathbb{Z}} \tilde{\eta}_t^N(x) \delta_{\frac{x}{N}}(dv), \quad v \in \mathbb{R},
$$

converge, as $N \to \infty$, to the unique classical solution $\rho(t, v)$ of

$$
\begin{cases}
\partial_t \rho(t, v) = \rho''(t, v) + \alpha(\rho(t, v)(1 - \rho(t, v)))', \quad t > 0, v \in \mathbb{R}, \\
\rho(0, v) = \rho_0(v).
\end{cases}
$$

See Proposition 4.2 of [9] for the precise statement and distinguish $\rho(t, v)$ from $\rho_U(t, u)$ in Theorem 2.2, though we use similar notation. Furthermore, the continuous version of the inverse transformation above leads to $\psi_U(t, u) = \zeta^+_{\rho(t, \cdot)}((\zeta^-_{\rho(t, \cdot)})^{-1}(u))$, with

$$
\zeta^+_{\rho(t, \cdot)}(v) := \int_v^\infty (1 - \rho(t, w))dw \quad \text{and} \quad \zeta^-_{\rho(t, \cdot)}(v) := \int_v^\infty \rho(t, w)dw.
$$

This is indeed defined via the inverse map of $\Phi_U$, see Proposition 4.4 of [9]. To shorten notation, we will use $\zeta_t(v) = \zeta^+_{\rho(t, \cdot)}(v)$ and $\zeta_t^N(x) = \zeta^-_{\rho(t, \cdot)}(x)$. The fluctuations for $\tilde{\eta}_t^N$ around $\rho(t, \cdot)$ by

$$
\tilde{\xi}^N_U(t, v) := \sqrt{N} \left( \pi_t^N(dv) - \rho(t, v)dv \right),
$$

are considered as distribution-valued processes in [5]. Due to the fact that we want to deal with the height function $\psi^N_U(t, v)$, we will look at an integrated version of $\tilde{\xi}^N_U(t, dv)$, namely

$$
\tilde{\Psi}_t^N(t, v) := \sqrt{N} \left( \pi_t^N([v, \infty)) - \int_v^\infty \rho(t, w)dw \right).
$$

The asymptotic properties of $\rho(t, \cdot)$ and of the tails of $\pi_t^N$ guarantee that the integrals are finite for all $v \in \mathbb{R}$, therefore $\tilde{\Psi}_t^N(t, v)$ is well-defined. There is an immediate result on the fluctuations following from Theorem 2.3 for the process with a stochastic reservoir at $\{0\}$.

**Assumption 3.** (1) For every $\kappa \in \mathbb{N}$, the following holds:

(i) $\sup_{N \in \mathbb{N}} E[\exp\{\kappa \pi_t^N([0, \infty))\}] < \infty$,  
(ii) $\sup_{N \in \mathbb{N}} \sup_{v \in \mathbb{R}} E[\|\tilde{\Psi}_t^N(0, v)\|^{2\kappa}] < \infty$,  
(iii) for any $v_1, v_2 \in \mathbb{Z}/N \sup_{N \in \mathbb{N}} E[\|\tilde{\Psi}_t^N(0, v_1) - \tilde{\Psi}_t^N(0, v_2)\|^{2\kappa}] \leq C|v_1 - v_2|^{\kappa}$ with $C > 0$.

(2) $\{\tilde{\Psi}_t^N(0, v)\}_N$ are independent of the noises determining the process $\{\xi_t; t \geq 0\}$.

(3) $\tilde{\Psi}_t^N(0, v)$ converges weakly to $\tilde{\Psi}_0(v)$ in $D(\mathbb{R})$, and $\tilde{\Psi}_0 \in C(\mathbb{R})$ (a.s.) such that for all $r > 0$ $E[\|\tilde{\Psi}_0\|_{L^2(\mathbb{R})}^r] < \infty$. 

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Proposition 3.1. Under Assumption 3, as $N \to \infty$, $\bar{\Psi}_U^N(t,v)$ converges weakly on the space $D([0,T], D(\mathbb{R}))$ to $\Psi_U(t,v)$. The limit is characterized as the unique (weak) solution of the following SPDE:

\begin{equation}
\partial_t \bar{\Psi}_U(t,v) = \frac{\partial^2}{N} \bar{\Psi}_U(t,v) + \alpha(1 - 2\rho(t,v)) \bar{\Psi}_U(t,v) + \sqrt{2\rho(t,v)(1 - \rho(t,v))} \dot{W}(t,v),
\end{equation}

i.e. $\bar{\Psi}_U \in C([0,T], C(\mathbb{R})) \cap C([0,T], L^2(\mathbb{R}))$ (a.s.) and for every $f \in C^{1,2}_0([0,T] \times \mathbb{R})$,

\begin{equation}
\langle \bar{\Psi}_U(t), f(t) \rangle = \langle \bar{\Psi}_{U,0}, f(0) \rangle + \int_0^t \langle \bar{\Psi}_U(s), f''(s) - \alpha((1 - 2\rho(s))f(s))' + \partial_s f(s) \rangle ds
\end{equation}

\begin{equation}
+ \int_0^t \int f(s,v) \sqrt{2\rho(s,v)(1 - \rho(s,v))} W(dsdv) \text{ a.s.},
\end{equation}

with $\rho(t)$ being the solution to (3.1), $\dot{W}(t,v)$ being the space-time white noise on $[0,T] \times \mathbb{R}$.

Remark 3.1. (1) In Assumption 3-(1)(i), $\pi^N_0([0,\infty))$ represents the initial particle number divided by $N$ of $\bar{\eta}_t$ on the positive side. Recalling the definition of the empirical measures of vacant sites of $\bar{\eta}_t$: $\pi^N_t(dv) = \frac{1}{N} \sum_{x \in Z}(1 - \bar{\eta}^N_t(x))\delta_{x/N}(dv)$ given in Lemma 4.3 of [9], this assumption implies a similar condition for the initial density $\pi^N_0((-\infty,0])$ of vacant sites divided by $N$ on the negative side by the symmetry in the state space $\mathcal{X}_U$ of $\bar{\eta}_t$ given in Section 4.1 of [9].

(2) The fluctuation limit for $\bar{\Psi}_U^N(t,v)$ on the positive side can be studied similarly to Theorem 2.3. To study it on the negative side, we note that $\bar{\Psi}_U^N(t,v)$ is equal to

$$
\bar{\Psi}_U^N(t,v) := \sqrt{N}\bar{\pi}^N_t((-\infty,v)) - \int_{-\infty}^v (1 - \rho(t,w)) dw
$$

with an error less than $\sqrt{N}/N$. The fluctuation limit for $\bar{\Psi}_U^N(t,v)$ (in particular, the tightness of the Hopf-Cole transformed process) on the negative side is shown similarly to Theorem 2.3 by looking at $1 - \bar{\eta}^N_t(x)$ instead of $\bar{\eta}^N_t(x)$.

Remark 3.2. Dittrich and Gärtner [5] proved the fluctuation results for $\bar{\xi}_U^N(t,dv)$ as a distribution-valued process. However, this is not sufficient for our purpose. Indeed, since we will apply a nonlinear transformation in the next stage, we need to establish the convergence in a usual function space formulated as in Proposition 3.1. This is essentially carried out in Section 4.

We now prepare two lemmas to deduce Theorem 2.1 from Proposition 3.1. Recall that $p \in \mathcal{P}$ determines $\psi_p$ and $\bar{\psi}_p^N$ as well as $\bar{\eta} = (\bar{\eta}(x))_{x \in Z}$ and the empirical measures $\pi^N(dv)$ on $\mathbb{R}$. For simplicity, we will write $\pi(dv)$ for $\pi^1(dv)$ in the following. The next lemma concerns the indented curves $\bar{\psi}_U^N$ and $\bar{\psi}_U^1$ (with $N = 1$), obtained by rotating $\bar{\psi}_p^N$ respectively $\bar{\psi}_p^1$ as described before.

Lemma 3.2. We number the set $\{x \in Z; \bar{\eta}(x) = 1\}$ from the right as $\{q_i\}_{i=1}^\infty$, that is, $q_1 = \max\{x \in Z; \bar{\eta}(x) = 1\}$, $q_2 = \max\{x < q_1; \bar{\eta}(x) = 1\}$ and so on. Then, we have that

\begin{equation}
\bar{\psi}_U^1(v) = \sqrt{2\pi}(\sqrt{2v}, \infty)) + v,
\end{equation}
for all \( v \in \cup_{i=0}^{\infty} [(\bar{q}_i + 1)/\sqrt{2}, \bar{q}_i/\sqrt{2}] \), where \( \bar{q}_0 = \infty \). In particular for arbitrary \( v \in \mathbb{R} \)
\begin{align}
|\hat{\psi}^j_U(v) - \{\sqrt{2}\pi([\sqrt{2}v, \infty)) + v\}| &\leq \sqrt{2}, \\
(3.6) |\hat{\psi}^\infty_U(v) - \{\sqrt{2}\pi([\sqrt{2}v, \infty)) + v\}| &\leq \frac{\sqrt{2}}{N}.
\end{align}

**Proof.** Set \( h(v) := 2\pi([v, \infty)) + v \). Since \( \pi([\bar{q}_i, \infty)) = \sum_j^{N} \{j; \bar{q}_j \geq \bar{q}_i\} = i \) for \( i \in \mathbb{N} \), we have \( h(\bar{q}_i) = 2i + \bar{q}_i = 2i + (p_i - i) = p_i + i \), which is equal to the height of the curve \( \hat{\psi}^j(\bar{q}_i) \) at \( v = \bar{q}_i/\sqrt{2} \) multiplied by \( \sqrt{2} \), i.e. \( h(\bar{q}_i) = \sqrt{2}\hat{\psi}^j(\bar{q}_i/\sqrt{2}) \). Therefore (3.4) holds for \( v = \bar{q}_i/\sqrt{2} \). The functions on both sides of (3.4) have slope 1 on the intervals \(((\bar{q}_{i+1} + 1)/\sqrt{2}, \bar{q}_i/\sqrt{2})\) which yields the first assertion. The function \( h(v)/\sqrt{2} \) has a jump with size \( \sqrt{2} \) at \( v = \bar{q}_i \) and this leads to (3.5). (3.6) follows from (3.5) by scaling. \( \square \)

The second lemma concerns the curve \( \tilde{\psi}_U \) obtained by rotating \( \psi_U \).

**Lemma 3.3.** It holds that
\begin{equation}
\tilde{\psi}_U(v) = \sqrt{2} \int_{\sqrt{2}v}^{\infty} \rho(w)dw + v, \quad v \in \mathbb{R},
\end{equation}
where \( \rho(v) := \Phi_U(\psi_U)(v) \), see Theorem 2.1 for the map \( \Phi_U \).

**Proof.** An explicit representation of the rotation via its rotation matrix yields
\[
\begin{pmatrix} v \\ \tilde{\psi}(v) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} G_\psi(u) \\ u + \psi(u) \end{pmatrix},
\]
where \( G_\psi(u) = u - \psi(u) \). This implies that \( \tilde{\psi}(v) = \{G_\psi^{-1}(\sqrt{2}v) + \psi(G_\psi^{-1}(\sqrt{2}v))\}/\sqrt{2} \).

Since \( (G_\psi^{-1})'(v) = 1/\{1 - \psi'(G_\psi^{-1}(v))\} \), this implies
\[
\tilde{\psi}'(v) = \frac{1 + \psi'(G_\psi^{-1}(\sqrt{2}v))}{1 - \psi'(G_\psi^{-1}(\sqrt{2}v))}.
\]

Together with \( \rho(v) = \Phi_U(\psi_U)(v) = -\psi'(G_\psi^{-1}(v))/\{1 - \psi'(G_\psi^{-1}(v))\} \) or written equivalently as \( \psi'(G_\psi^{-1}(v)) = -\rho(v)/(1 - \rho(v)) \) we obtain \( \tilde{\psi}(v) = 1 - 2\rho(\sqrt{2}v) \).

The derivative of the right hand side of (3.7) is given by \( 1 - 2\rho(\sqrt{2}v) \), which coincides with \( \tilde{\psi}'(v) \). Since both curves have \( \{y = v\} \) as an asymptotic line for \( v \to \infty \), (3.7) is proven for all \( v \in \mathbb{R} \). \( \square \)

There is an immediate corollary based on Lemmas 3.2 and 3.3.

**Corollary 3.4.** For all \( t \geq 0 \) and \( v \in \mathbb{R} \) the relation
\[
\tilde{\Psi}^N_U(t, v) = \sqrt{2}\tilde{\Psi}^N_U(t, \sqrt{2}v) + \tilde{R}^N(t, v)
\]
holds with an error term satisfying \( |\tilde{R}^N(t, v)| \leq \sqrt{2}/N \).

**Proof of Theorem 2.1.** By Corollary 3.4, the limits \( \hat{\Psi}_U(t, v) \) and \( \tilde{\Psi}_U(t, v) \) of \( \hat{\Psi}^N_U(t, v) \) and \( \tilde{\Psi}^N_U(t, v) \) are related by \( \hat{\Psi}_U(t, v) = \sqrt{2}\tilde{\Psi}_U(t, \sqrt{2}v) \). Therefore we can derive the SPDE (2.3) from the SPDE (3.2) in the weak formulation by replacing the space-time white noise properly. Corollary 3.4 also shows that Assumptions 1 and 3 are mutually equivalent. \( \square \)
3.2 Proof of Theorem 2.2

In order to derive the SPDE (2.4) for the limit $\Psi_U(t, u)$ of $\Psi^N_U(t, u)$, we are not able to apply the same transformation used to get $\psi_U(t, \cdot)$ from $\rho(t, \cdot)$ because the random noise certainly makes it impossible to extend it to the spaces containing $\Psi_U$ or equivalently $\tilde{\Psi}_U$ and $\Psi_U$. Instead, we exploit some of the calculations made in Section 4 of [9]. For every $f \in C_0(\mathbb{R}_+)$, we set $F(u) = \int_0^u f(v)dv$, and then we have that

\begin{align}
(3.8) & \quad \int_{\mathbb{R}_+^2} \psi^N_U(t, u)f(u)du = \frac{1}{N} \sum_{x \in \mathbb{Z}} F(\frac{1}{N} \zeta^N_t(x))\tilde{\eta}^N_t(x), \\
(3.9) & \quad \int_{\mathbb{R}_+^2} \psi_U(t, u)f(u)du = \int_{\mathbb{R}} F(\zeta_t(v))\rho(t, v)dv.
\end{align}

These are the key identities for our next proposition. We will employ Proposition 3.1 rather than Theorem 2.1, but which are actually equivalent as we observed above.

**Proposition 3.5.** The weak limit $\Psi_U(t, u)$ of $\Psi^N_U(t, u)$ as $N \to \infty$ exists and is given by the formula

$$
(3.10) \quad \Psi_U(t, u) = \frac{\tilde{\Psi}_U(t, \zeta_t^{-1}(u))}{1 - \rho(t, \zeta_t^{-1}(u))}.
$$

**Proof.** Since the convergence in Proposition 3.1 is only in a weak sense, we start by using Skorohod’s theorem and then assume that $\bar{\Psi}_N(t, u)$ converges almost surely to $\Psi_U(t, u)$ on $D([0, T], D(\mathbb{R}))$ by choosing a proper probability space. In order to simplify the notation, we still use the same name in the following. Then, by (3.8) and (3.9), for each function $f \in C_0(\mathbb{R}_+)$, we can compute

\begin{align*}
\int_{\mathbb{R}_+^2} \psi^N_U(t, u)f(u)du = & \frac{1}{N} \sum_{x \in \mathbb{Z}} F(\frac{1}{N} \zeta_t(x))\tilde{\eta}^N_t(x) - \int_{\mathbb{R}} F(\zeta_t(v))\rho(t, v)dv \\
& + \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} (F(\frac{1}{N} \zeta^N_t(x)) - F(\frac{1}{N} \bar{\eta}^N_t(x)))\tilde{\eta}^N_t(x) =: S^N_1 + S^N_2.
\end{align*}

Integration by parts and summation by parts yield

\begin{align*}
\int_{\mathbb{R}} F(\zeta_t(v))\rho(t, v)dv = & \int_{\mathbb{R}} f(\zeta_t(v))(1 - \rho(t, v))dv \int_{\mathbb{R}} \rho(t, w)dw, \\
\frac{1}{N} \sum_{x \in \mathbb{Z}} F(\frac{1}{N} \zeta_t(x))\tilde{\eta}^N_t(x) = & \int_{\mathbb{R}} f(\zeta_t(v))(1 - \rho(t, v))\pi^N_t([v, \infty))dv.
\end{align*}

Therefore $S^N_1$ can be written as an integral with respect to $\tilde{\Psi}^N_U$, and with the help of Proposition 3.1 and a simple substitution $u = \zeta_t(v)$, we have that

$$
(3.11) \quad \lim_{N \to \infty} S^N_1 = \lim_{N \to \infty} \int_{\mathbb{R}} f(\zeta_t(v))(1 - \rho(t, v))\tilde{\Psi}^N_U(t, v)dv = \int_{\mathbb{R}_+^2} f(u)\tilde{\Psi}_U(t, \zeta_t^{-1}(u))du.
$$

In the following, we are going to show

$$
(3.12) \quad \lim_{N \to \infty} S^N_2 = \int_{\mathbb{R}} f(\zeta_t(v))\rho(t, v)\tilde{\Psi}_U(t, v)dv \text{ a.s.}
$$
By Taylor’s formula, it holds
\[
F(\frac{1}{N}\zeta^N_1(x)) - F(\zeta_1(x)) = f(\zeta_1(x))\left(\frac{\zeta^N_1(x)}{N} - \zeta_1(x)\right) + f'(\zeta_{N,x})(\frac{\zeta^N_1(x)}{N} - \zeta_1(x))^2
\]
with \( \zeta_{N,x} \in \left[\min\left\{\frac{1}{N}\zeta^N_1(x), \zeta_1(x)\right\}, \max\left\{\frac{1}{N}\zeta^N_1(x), \zeta_1(x)\right\}\right] \).

The above appearing term \( \frac{1}{N}\zeta^N_1(x) - \zeta_1(x) \) is basically given by the process \( \bar{\Psi}^N_1 \). We show this using the asymmetry property (see Subsection 4.1 in [9]) which leads to
\[
\zeta_1(x) = \frac{x}{\sqrt{\pi}} + \int_{-\infty}^{\infty} \rho(t,v)dv \quad \text{and} \quad \frac{1}{N}\zeta^N_1(x) = \frac{x}{\sqrt{\pi}} + \pi^N_1(\{\frac{x}{\sqrt{\pi}}, \infty\}) - \frac{1}{\sqrt{N}} \bar{\Psi}^N_1(x).
\]

Thus, using these relations, we see that
\[
\left\|S_2^N - \int_{\mathbb{R}} f(\zeta(v))\rho(t,v)\bar{\Psi}_U(t,v)dv\right\| \leq
\]
\[
E_1 + |E_2| + \left|\frac{1}{N} \sum_{x \in \mathbb{Z}} f(\zeta_1(x))\left(\bar{\Psi}^N_1(t, \frac{x}{\sqrt{\pi}}) - \bar{\Psi}_U(t, \frac{x}{\sqrt{\pi}})\right)\bar{\Psi}^N_1(x)\right|
\]
\[
+ \left|\frac{1}{N} \sum_{x \in \mathbb{Z}} f(\zeta_1(x))\bar{\Psi}_U(t, \frac{x}{\sqrt{\pi}})\bar{\Psi}^N_1(x) - \int_{\mathbb{R}} f(\zeta(v))\rho(t,v)\bar{\Psi}_U(t,v)dv\right|
\]
where
\[
E_1 = \frac{-1}{N^{3/2}} \sum_{x \in \mathbb{Z}} f(\zeta_1(x))\bar{\Psi}^N_1(x) \quad \text{and} \quad E_2 = \frac{1}{N^{3/2}} \sum_{x \in \mathbb{Z}} \bar{\Psi}^N_1(x)f'(\zeta_{N,x})(\bar{\Psi}^N_1(t,u) - \bar{\Psi}^N_1(x))^2.
\]

Clearly, \( E_1 \to 0 \) a.s. because of the extra \( \sqrt{N} \) in the denominator. On the other hand, from Proposition 3.1, in particular, the fact that the limit \( \bar{\Psi}_U(t,u) \) of \( \bar{\Psi}^N_1(t,u) \) is in \( C([0,T],C(\mathbb{R})) \) a.s., we know that
\[
\sup_{t \in [0,T],N \in \mathbb{N},v \in [-K,K]} |\bar{\Psi}^N_1(t,v)| < \infty \text{ a.s.,}
\]
which implies \( E_2 \to 0 \) a.s. by recalling that \( f \in C^2_0(\mathbb{R}_+) \).

To conclude the proof of (3.12), let us now reformulate a result which follows from Proposition 4.2 of [9]. Under our assumptions, for any function \( g \in C_0(\mathbb{R}_+) \), as \( N \to \infty, \)
\[
\frac{1}{N} \sum_{x \in \mathbb{Z}} g(\zeta_1(x))\bar{\Psi}^N_1(x) \quad \text{converges to} \quad \int_{\mathbb{R}} g(\zeta(v))\rho(t,v)dv \quad \text{in probability.}
\]
Applying this result for \( g(\cdot) = f(\cdot)\bar{\Psi}_U(t, \zeta^{-1}(\cdot)) \) and using the compactness of the support of \( f \), we have that the last term on the right hand side of (3.13) converges to 0 a.s. In the end, using Proposition 3.1 again and recalling that we apply Skorohod’s theorem, we have
\[
\lim_{N \to \infty} \sup_{t \in [0,T],N \in \mathbb{N},v \in [-K,K]} |\bar{\Psi}^N_1(t,v) - \bar{\Psi}_U(t,v)| = 0 \text{ a.s.,}
\]
and thus applying the above result for \( g(\cdot) = f(\cdot) \), we also see that the third term on the right hand side of (3.13) converges to 0 a.s. So, the proof of (3.12) is completed.

Finally, we substitute with \( u = \zeta_1(v) \) and therefore the limit for \( S_2^N \) is given by
\[
\int_{\mathbb{R}} f(\zeta(v))\rho(t,v)\bar{\Psi}_U(t,v)dv = \int_{\mathbb{R}_+} f(u)\frac{\rho(t, \zeta^{-1}_1(u))}{1 - \rho(t, \zeta^{-1}_1(u))} \bar{\Psi}_U(t, \zeta^{-1}_1(u))du,
\]
which completes the proof (3.10) with the help of (3.11). \( \square \)
Now that we have a formula for the limit process the next step is to identify the corresponding SPDE. A direct computation with \( \psi_U(t, u) = \zeta_t^{-1}(u) \) leads to the following lemma, recall that \( \rho_U(t, u) = -\psi'_U(t, u) \) is defined in Theorem 2.2.

**Lemma 3.6.** We have that

\[
\rho_U(t, u) = \frac{\rho(t, \zeta_t^{-1}(u))}{1 - \rho(t, \zeta_t^{-1}(u))} \quad \text{and} \quad 1 + \rho_U(t, u) = \frac{1}{1 - \rho(t, \zeta_t^{-1}(u))}.
\]

We are at the position to give the proof of Theorem 2.2. We prove that the limit \( \Psi_U(t, u) \) of \( \rho_U(t, u) \) obtained in Proposition 3.5 satisfies the SPDE (2.4).

**Proof of Theorem 2.2.** Fix a function \( f \in C^{1,2}_0([0, T] \times \mathbb{R}^d) \) and consider the process \( \Psi_U(t, u) \) tested with \( f \). Then, by the representation formula (3.10) combined with Lemma 3.6 and the substitution \( v = \zeta_t^{-1}(u) \), we get

\[
\int_{\mathbb{R}_+} f(t, u) \Psi_U(t, u) du = \int_{\mathbb{R}_+} f(t, u) (1 + \rho_U(t, u)) \Psi_U(t, \zeta_t^{-1}(u)) du
\]

\[
= \int_{\mathbb{R}} f(t, \zeta_t(v)) \Psi_U(t, v) dv = \langle \Psi_U(t), f \rangle_\zeta.
\]

Since \( f(t) \circ \zeta_t \in C^{1,2}_0([0, T] \times \mathbb{R}) \), (3.3) rewrites the right hand side as

\[
\langle \Psi_U, 0 \circ \zeta_0 \rangle + \int_0^t \langle \Psi_U(s), (f(s) \circ \zeta_s)' - \alpha ((1 - 2\rho(s))f(s) \circ \zeta_s)' + \partial_s (f(s) \circ \zeta_s) \rangle ds
\]

\[
+ \int_0^t \int_{\mathbb{R}} f(s) \circ \zeta_s(v) \sqrt{2\rho(s, v)(1 - \rho(s, v))} W(dsv).
\]

Thus, for the initial condition, we get analogue to the above

\[
\langle \Psi_U, 0 \circ \zeta_0 \rangle = \int_{\mathbb{R}_+} f(0, u) \Psi_U(0, u) du.
\]

Let us consider the drift term. The relation \( \zeta_s'(v) = 1 - \rho(s, v) \) implies

\[
(f(s, \zeta_s(v)))'' - \alpha ((1 - 2\rho(s, v))f(s, \zeta_s(v))' + \partial_s (f(s, \zeta_s) = (1 - \rho(s, v))^2 f''(s, \zeta_s(v))
\]

\[
- \left( \frac{1}{2} \frac{d}{ds} \alpha(1 - 2\rho(s, v)(1 - \rho(s, v)) \right) f'(s, \zeta_s(v)) + 2\alpha \rho'(s, v) f(s, \zeta_s(v)),
\]

and by (3.1),

\[
\partial_s (f(s, \zeta_s(v)) = \partial_s f(s) \circ \zeta_s(v) - f'(s, \zeta_s(v))(\rho'(s, v) + \alpha \rho(s, v)(1 - \rho(s, v))).
\]

These yield that the drift term is equal to

\[
\int_0^t \langle \Psi_U(s), (1 - \rho(s))^2 f''(s) \circ \zeta_s - (2\rho'(s) + \alpha(1 - \rho(s))^2) f'(s) \circ \zeta_s
\]

\[
+ 2\alpha \rho'(s) f(s) \circ \zeta_s + \partial_s f(s) \circ \zeta_s \rangle ds.
\]
With the substitution $u = \zeta_t(v)$ and (3.10) combined with Lemma 3.6, we come back to an expression in $\Psi_U(t)$:

$$
\int_0^t \int_{\mathbb{R}_+^2} \Psi_U(s, u) \left( \left( \frac{f'(s, u) - \alpha f(s, u)}{(1 + \rho_U(s, u))^2} \right) + \partial_s f(s, u) \right) duds.
$$

The last task is to check the noise term. We consider the quadratic variation of the in the above appearing stochastic integral, which is given by

$$
\int_0^t \int_{\mathbb{R}} f^2(s, \zeta_s(v)) 2\rho(s, v)(1 - \rho(s, v))dvds = \int_0^t \int_{\mathbb{R}_+^2} f^2(s, u) 2\rho(s, \zeta_s^{-1}(u))duds = \int_0^t \int_{\mathbb{R}_+^2} f^2(s, u) \frac{2\rho_U(s, u)}{1 + \rho_U(s, u)} duds.
$$

This proves (2.5) with a suitably taken space-time white noise $\hat{W}(t, u)$ on $[0, T] \times \mathbb{R}_+$ (which is different from that in Proposition 3.1) as in Lemma 4.16 below.

For the proof of Theorem 2.2, the uniqueness of the solution to the SPDE (2.4) in the limit was unnecessary. Nevertheless, we show that uniqueness holds under the condition (3.15) stated below.

**Lemma 3.7.** The relation (3.10) for $\bar{\Psi}_U(t)$ and $\Psi_U(t)$ translates to

$$
\|\bar{\Psi}_U(t)\|_{L^2_t(\mathbb{R})}^2 = \int_{\mathbb{R}_+^2} \Psi_U^2(t, u) \frac{e^{-2r|u - \psi_U(t, u)|}}{1 + \rho_U(t, u)} du.
$$

If in addition $\rho_U(t, u)$ satisfies the condition

$$
(3.15) \quad c := \inf_{t \in [0, T], u \in [0, 1]} u \rho_U(t, u) > 0,
$$

then for every $r > 0$, there exists $C_r > 0$ such that

$$
(3.16) \quad \|\bar{\Psi}_U(t)\|_{L^2_t(\mathbb{R})} \leq C_r \|\Psi_U(t)\|_{L^2_t(\psi_U(t, u)}(\mathbb{R}_+^2)}.
$$

In particular, $\Psi_U(t) \in \tilde{L}^2(\mathbb{R}_+^2)$ implies $\bar{\Psi}_U(t) \in L^2_t(\mathbb{R})$ and then the solution $\Psi_U$ of the SPDE (2.4) is unique in the class $C([0, T], C(\mathbb{R}_+^2)) \cap C([0, T], \tilde{L}^2(\mathbb{R}_+^2))$.

**Proof.** By a change of variables, the left hand side of (3.14) can be rewritten as

$$
\int_{\mathbb{R}_+^2} \bar{\Psi}_U^2(t, \zeta_t^{-1}(u)) e^{-2r|\zeta_t^{-1}(u)|} (\zeta_t^{-1}(u))' du.
$$

It is easily seen that this integral is equal to the right hand side of (3.14) by applying Proposition 3.5, Lemma 3.6 and recalling that $\zeta_t^{-1}(u) = u - \psi_U(t, u)$ and $(\zeta_t^{-1}(u))' = 1 + \rho_U(t, u)$. This proves (3.14). To show (3.16), first note that $\psi_U(t) \in X_U$ behaves like

$$
\frac{e^{-2r|u - \psi_U(t, u)|}}{1 - \psi_U'(t, u)} \asymp e^{-2ru},
$$

15
for large enough \( u \) uniformly in \( t \in [0, T] \). On the other hand, condition (3.15) implies
\[
\rho_U(t, u) \geq \tilde{\rho}_U(u) := cu^{-1}
\]
on \([0, 1] \) and therefore
\[
u - \psi_U(t, u) \leq u - \tilde{\psi}_U(u) (< 0),
\]
near 0, where \( \tilde{\psi}_U(u) = -c \log u \). This results in a behavior like
\[
\frac{e^{-2|u - \psi_U(t, u)|}}{1 + \rho_U(t, u)} \leq \frac{e^{-2|u - \tilde{\psi}_U(u)|}}{1 + \tilde{\rho}_U(u)} = \frac{u^{2rc}e^{2ru}}{1 + cu^{-1}} \leq \frac{1}{c} u^{2rc+1},
\]
near 0. Applying these estimates to (3.14) yields (3.16). Finally, transform the solution \( \Psi_U(t) \) of the SPDE (2.4) in the class \( C([0, T], C(R^d_\alpha)) \cap C([0, T], L^2(R^d_\alpha)) \) into \( \tilde{\Psi}_U(t) \) by (3.10). By (3.16) \( \tilde{\Psi}_U(t) \) is a solution of the SPDE (3.2) in the class \( C([0, T], C(R)) \cap C([0, T], L^2(R)) \). Since \( \tilde{\Psi}_U(t) \) is uniquely determined in this class, uniqueness for \( \Psi_U(t) \) follows.

In the final part of this section, we give an example of a class of initial values \( \rho_U(0, u) \) for which the condition (3.15) is satisfied along the time evolution \( \rho_U(t, u) \). We first prepare a comparison theorem for solutions of PDE (3.1).

**Lemma 3.8.** If two initial values of (3.1) satisfy \( \rho^{(1)}(0, v) \leq \rho^{(2)}(0, v), v \in \mathbb{R} \), then the corresponding solutions satisfy \( \rho^{(1)}(t, v) \leq \rho^{(2)}(t, v) \) for every \( t > 0 \) and \( v \in \mathbb{R} \).

**Proof.** This is immediate by applying the Hopf-Cole transformation. Or since the underlying microscopic system, the weakly asymmetric simple exclusion process on \( \mathbb{Z} \), is attractive, by passing to the hydrodynamic limit we see the conclusion for the limit equation (3.1).

Let \( \rho_U^\infty(u) \) and \( \rho^\infty(v; C) \) be stationary solutions of the PDEs (3.22) and (3.1), respectively, with explicit formulas
\[
\rho_U^\infty(u) := \frac{1}{e^{\alpha u} - 1} \quad \text{and} \quad \rho^\infty(v; C) := \frac{C}{e^{\alpha v} + C}, \quad \text{for every} \quad C > 0.
\]

Note that \( \rho^\infty(v; C) \) are shifts of \( \rho^\infty(v; 1) \) and further recall that \( \rho(v) = \Phi_U(\psi_U)(v) \) is defined in Lemma 3.3.

**Lemma 3.9.** Assume that the derivative \( \rho_U'(u) = -\psi_U'(u) \) of \( \psi_U \in X_U \) satisfies
\[
C_2 \rho_U^\infty(u) \leq \rho_U(u) \leq C_1 \rho_U^\infty(u),
\]
for some \( C_1 \geq C_2 > 0 \) and
\[
\limsup_{u \downarrow 0} |\rho_U(u) - \rho_U^\infty(u)| < \infty.
\]

Then, there exist \( \hat{C}_1 \geq \hat{C}_2 > 0 \) such that
\[
\rho^\infty(v; \hat{C}_2) \leq \rho(v) \leq \rho^\infty(v; \hat{C}_1).
\]
Proof. Recall the definitions of $\Phi_U$ and $G_\psi$ given in the proof of Lemma 3.3 and note that what we only need to prove

$$C_2' e^{-av} \leq \rho_U((G_\psi)^{-1}(v)) \leq C_1' e^{-av},$$

for some $C_1' \geq C_2' > 0$. Under condition (3.18), we can reduce this to show that there exist $D_1 \geq D_2 > 0$ such that

$$D_2 e^{av} \leq e^{a(G_\psi)^{-1}(v)} - 1 \leq D_1 e^{av}.$$  

The condition (3.19) implies $A = \sup_{0 < u \leq 1} |\rho_U(u) - \rho_U^\infty(u)| < \infty$ and therefore

$$\psi_U(u) = \int_u^1 \rho_U'(u') du' + \psi_U(1) \leq -\frac{1}{\alpha} \log(1 - e^{-au}) + A + \psi_U(1), \text{ for any } 0 < u \leq 1.$$

Similarly, for any $0 < u \leq 1$, $\psi_U(u) \geq -\frac{1}{\alpha} \log(1 - e^{-au}) - A + \psi_U(1)$. Denote $A + \psi_U(1) = \tilde{C}_1$ and $-A + \psi_U(1) = \tilde{C}_2$. Then

$$\alpha u + \log(1 - e^{-av}) - \alpha \tilde{C}_1 \leq \alpha G_\psi(u) \leq \alpha u + \log(1 - e^{-av}) - \alpha \tilde{C}_2$$

and by taking the exponential, we obtain that for any $v$ satisfying $(G_\psi)^{-1}(v) \leq 1$,

$$(e^{a(G_\psi)^{-1}(v)} - 1)e^{-\alpha \tilde{C}_1} \leq e^{av} \leq (e^{a(G_\psi)^{-1}(v)} - 1)e^{-\alpha \tilde{C}_2}.$$  

For $v$ satisfying $(G_\psi)^{-1}(v) \geq 1$, we apply the same argument as above with $\psi_U(u) = \int_u^\infty \rho_U(u') du'$ to obtain the inequality

$$e^{a(G_\psi)^{-1}(v)} (1 - e^{-a(G_\psi)^{-1}(v)})^C \leq e^{av} \leq e^{a(G_\psi)^{-1}(v)} (1 - e^{-a(G_\psi)^{-1}(v)})^{1/C},$$

where $C = \max\{C_1, 1/C_2\} \geq 1$. Then, it is obvious that $(1 - e^{-a(G_\psi)^{-1}(v)})^{C-1} \geq 1 - e^{-\alpha C^{-1}}$ and $(1 - e^{-a(G_\psi)^{-1}(v)})^{1/C-1} \leq (1 - e^{-\alpha})^{1/C-1}$ for $v$ satisfying $(G_\psi)^{-1}(v) \geq 1$. In the end, for any $v \in \mathbb{R}$, we have

$$\min\{e^{-\alpha \tilde{C}_1}, (1 - e^{-\alpha})^{C-1}\} \leq e^{av} (e^{a(G_\psi)^{-1}(v)} - 1)^{-1} \leq \max\{e^{-\alpha \tilde{C}_2}, (1 - e^{-\alpha})^{1/C-1}\},$$

which concludes the proof. \qed

Lemma 3.10. Assume that $\rho(\cdot) \in Y_U$ satisfies the condition (3.20) for some $C_1 \geq C_2 > 0$ in place of $\tilde{C}_1 \geq \tilde{C}_2 > 0$. Then, $\rho_U(u) := -(\Psi_U(\rho(\cdot))(u))'$ satisfies

$$\frac{C_2}{C_1} \rho_U^\infty(u) \leq \rho_U(u) \leq \frac{C_1}{C_2} \rho_U^\infty(u),$$

where $\Psi_U : Y_U \to X_U$ is the inverse map of $\Phi_U$, see Proposition 4.4 of [9].

Proof. By definition $\rho_U(u) = (1 - \rho((\zeta_\rho)^{-1}(u)))^{-1} - 1$ and

$$(C_1 e^{-\alpha((\zeta_\rho)^{-1}(u))} + 1)^{-1} \leq 1 - \rho((\zeta_\rho)^{-1}(u)) \leq (C_2 e^{-\alpha((\zeta_\rho)^{-1}(u))} + 1)^{-1}$$
holds by assumption. Then, it is easy to see that
\[ C_2 e^{-\alpha(\zeta^{-1}_-)^{-1}(u)} \leq \rho_U(u) \leq C_1 e^{-\alpha(\zeta^{-1}_-)^{-1}(u)}. \]

On the other hand, since \( \zeta_+(v) = \int_0^v (1 - \rho(w))dw \) and \( \int_0^v \frac{1}{C_\alpha w+1}dw = \frac{1}{\alpha} \log \left( \frac{C+e^\alpha}{e^\alpha} \right), \)

\[ \frac{1}{\alpha} \log \left( \frac{C_1 + e^\alpha(\zeta^{-1}_-)^{-1}(u)}{C_1} \right) \leq u \leq \frac{1}{\alpha} \log \left( \frac{C_2 + e^\alpha(\zeta^{-1}_-)^{-1}(u)}{C_2} \right) \]

holds. Thus, we obtain
\[ C_1^{-1} \rho_\infty^\alpha(u) \leq e^{-\alpha(\zeta^{-1}_-)^{-1}(u)} \leq C_2^{-1} \rho_\infty^\alpha(u), \]

which concludes the proof. \( \square \)

**Proposition 3.11.** Assume that the derivative \( \rho_U(0, u) = -\psi'_U(0, u) \) of \( \psi_U(0, \cdot) \in X_U \) satisfies two conditions (3.18) and (3.19) in Lemma 3.9 with \( \rho_U(0, u) \) replaced by \( \rho_U(0, u) \). Then, there exist constants \( \bar{C}_1 > \bar{C}_2 > 0 \) such that for any \( t > 0 \), the solution \( \rho_U(t, u) \) of the PDE (3.22) below satisfies
\[ (3.21) \]
\[ \bar{C}_1 \rho_\infty^\alpha(u) \leq \rho_U(t, u) \leq \bar{C}_2 \rho_\infty^\alpha(u). \]

In particular the lower bound in (3.21) implies the condition (3.15).

**Proof.** First, note that the function \( \rho_\infty^\alpha(v; C) \) is a stationary solution of the PDE (3.1) for any \( C > 0 \). Then, with Lemma 3.8, the conclusion follows from Lemmas 3.9 and 3.10. \( \square \)

**Remark 3.3.** Under the equilibrium situation, that is, for \( \rho_\infty^\alpha(u) = \lim_{t \to \infty} \rho_U(t, u), \)
\( \zeta^{-1}(u) = \lim_{t \to \infty} \zeta^{-1}(t, u), \)
\( u \in \mathbb{R}_+^0 \) and \( \rho_\infty^\alpha(v; 1) = \lim_{t \to \infty} \rho(t, v), \)
\( \zeta(v) = \lim_{t \to \infty} \zeta(t, v), \)
\( v \in \mathbb{R}, \) we have explicit formulas:
\[ \zeta^{-1}(u) = \frac{1}{\alpha} \log(e^{\alpha u} - 1) \quad \text{and} \quad \zeta(v) = \frac{1}{\alpha} \log(e^{\alpha v} + 1). \]

From this, we see that the norm \( |\Psi|_{L^2(\mathbb{R})} \) is equivalent to \( |\Psi|_{L^2(\mathbb{R}_+^0)} \), if \( \Psi \) and \( \Psi \) are related with each other by the relation stated in Proposition 3.5: \( \Psi(u) = \Psi(\zeta^{-1}(u))/(1 - \rho(\zeta^{-1}(u))) \). This explains the reason for considering the norm \( |\Psi|_{L^2(\mathbb{R}_+^0)} \).

**Remark 3.4.** Similarly to Lemma 3.8, the attractiveness of the underlying weakly asymmetric zero-range process with stochastic reservoir leads to a comparison theorem for \( \rho_U(t, u) \). More precisely, the function \( \rho_U(t, u) = -\psi'_U(t, u) \), defined from a solution \( \psi_U(t, u) \) of the PDE in the statement (1) of Section 2, solves the nonlinear PDE:
\[ (3.22) \]
\[ \partial_t \rho_U = \left( \frac{\rho_U}{1 + \rho_U} \right)' + \alpha \left( \frac{\rho_U}{1 + \rho_U} \right)' \quad u \in \mathbb{R}_+^0. \]

If two initial values of (3.22) satisfy \( 0 < \rho_U^{(1)}(0, u) \leq \rho_U^{(2)}(0, u), u \in \mathbb{R}_+^0 \), then the corresponding solutions satisfy \( 0 < \rho_U^{(1)}(t, u) \leq \rho_U^{(2)}(t, u) \) for every \( t > 0 \) and \( u \in \mathbb{R}_+^0 \).
4 Proof of Theorem 2.3

Let \( q_t := q_t^\varepsilon = (q_t(t))_{t \in \mathbb{N}} \) be the Markov process on \( \mathcal{Q} \) introduced in Section 2 and let \( \eta_t = (\eta_t(x))_{x \in \mathbb{N}} \in \{0,1\}^\mathbb{N} \) be the height differences of the height function \( \psi_{q_t} \) determined from \( q_t \). The process \( \eta_t \) is also defined by \( \eta_t(x) = \mathbb{P}\{i; q_t(t) = x\} \), and set \( \eta_t(0) = \infty \) for convenience. As shown in Section 5.1 of [9], the process \( \eta_t \) is a weakly asymmetric simple exclusion process on \( \mathbb{N} \) with a weakly asymmetric stochastic reservoir at \( \{0\} \) and its generator is given at p. 353 in [9]. Here again, we apply the Hopf-Cole transformation for \( \eta_t \) at the microscopic level.

Section 4.1 essentially reduces the proof of Theorem 2.3 to a fluctuation result for a process on the whole lattice \( \mathbb{Z} \), which is related to the Hopf-Cole transformed process \( \zeta_t^N \) and is introduced mainly to avoid the boundary condition at \( \{0\} \) by a simple transformation, see Proposition 4.3. The proof of Theorem 2.3 is formulated in Section 4.1 based on Proposition 4.3, whose proof is given in Section 4.2.

4.1 Fluctuations for the Hopf-Cole Transformed Process

Let \( \eta_t^N = (\eta_t^N(x))_{x \in \mathbb{N}} := (\eta_{N2^i}(x))_{x \in \mathbb{N}} \) be the weakly asymmetric simple exclusion process speeded up by the factor \( N^2 \) in time with a stochastic reservoir at \( \{0\} \) and consider its microscopic Hopf-Cole transformation \( \zeta_t^N = (\zeta_t^N(x))_{x \in \mathbb{N}} \) defined by

\[
\zeta_t^N(x) := \exp\left\{- (\log \varepsilon) \sum_{y=x}^{\infty} \eta_t^N(y) \right\}, \quad \varepsilon = \varepsilon_R(N).
\]

Its interpolation \( \tilde{\zeta}^N(t,u), u \in \mathbb{R}_+ \) with the proper scaling in space is given by

\[
(4.1) \quad \tilde{\zeta}^N(t,u) := \exp\left\{- (\log \varepsilon) \left( \sum_{y=[Nu]+1}^{\infty} \eta_t^N(y) + 1_{\{u \geq 1/N\}} ([Nu] + 1 - Nu) \eta_t^N([Nu]) \right) \right\}.
\]

It is clear that for each \( t \geq 0 \), \( \tilde{\zeta}^N(t,\cdot) \) is a \( C(\mathbb{R}_+) \)-valued process. Theorem 5.2 of [9] states that, if the scaled empirical measure \( \pi_0^N \) of \( \eta_0^N \) converges to \( \rho_0(v)dv \) in probability as \( N \to \infty \) with \( \rho_0 \) satisfying \( \rho_0 \in C(\mathbb{R}_+, [0,1]) \) and \( \int_0^{\infty} \rho_0(v)dv < \infty \), then \( \tilde{\zeta}^N(t,u) \) converges to \( \omega(t,u) \) in probability, which is a unique bounded classical solution of the following linear diffusion equation:

\[
\begin{aligned}
\partial_t \omega &= \omega'' + \beta \omega', \quad u \in \mathbb{R}_+, \\
\omega(0,u) &= \exp\{\beta \int_u^{\infty} \rho_0(v)dv\}, \quad u \in \mathbb{R}_+, \\
2\omega'(t,0) + \beta \omega(t,0) &= 0, \quad \text{and} \quad \omega(t,\infty) = 1, \quad t > 0.
\end{aligned}
\]

Instead of immediately considering the fluctuations of \( \tilde{\zeta}^N(t,u) \) around its limit, the goal is to avoid the mixed boundary condition above. Therefore the next paragraph reduces the problem to another asymptotic problem on \( \mathbb{Z} \), formulated in Proposition 4.3 below.

At first recall from Section 5.3.3 of [9] that \( \zeta_t^N = (\zeta_t^N(x))_{x \in \mathbb{N}} \) satisfies the stochastic differential equation (SDE):

\[
d\zeta_t^N(x) = N^2 (\varepsilon \zeta_t^N(x-1) - (\varepsilon + 1) \zeta_t^N(x) + \zeta_t^N(x+1))dt + dM_t^N(x), \quad x \in \mathbb{N},
\]
where \( \zeta_t^N(0) := \epsilon^{-1}\zeta_t^N(2) \) and \( (M_t^N(x))_{x \in \mathbb{N}} \) are martingales with quadratic variations and covariances given as follows:

\[
\frac{d}{dt}(M_t^N(x))_t = \zeta_t^N(x)^2 \left\{ a_N c_+(x-1, \eta^N_1) + b_N c_-(x-1, \eta^N_1) \right\}, \quad x \geq 2,
\]

\[
\frac{d}{dt}(M_t^N(1))_t = \zeta_t^N(1)^2 \left\{ a_N c_1(\eta^N_1=0) + b_N c_1(\eta^N_1=1) \right\},
\]

\[
(M_t^N(x), M_N^N(y))_t = 0, \quad 1 \leq x \neq y.
\]

Here \( c_+(x, \eta) = 1_{\{\eta(x)=1, \eta(x+1)=0\}}, c_-(x, \eta) = 1_{\{\eta(x)=0, \eta(x+1)=1\}}, a_N = N^2(1-\epsilon)^2/\epsilon \) and \( b_N = N^2(1-\epsilon)^2 \). Note that \( \lim_{N \to \infty} a_N = \lim_{N \to \infty} b_N = \beta^2 \) and, in Lemma 5.6 of [9], \( c_\pm(x-1, \eta^N_1) \) are reversed.

Instead of dealing with the boundary condition \( \zeta_t^N(0) = \epsilon^{-1}\zeta_t^N(2) \) for \( x = 0 \), a simple transformation for \( \zeta_t^N \) and its extension to \( \mathbb{Z} \) makes the analysis easier.

**Lemma 4.1.** Let us consider the process \( \tilde{\zeta}_t^N = (\tilde{\zeta}_t^N(x))_{x \in \mathbb{Z}} \) defined by

\[
\tilde{\zeta}_t^N(x) = \exp\{-\log \epsilon x/2\} \zeta_t^N(x)
\]

for \( x \geq 1 \) and \( \tilde{\zeta}_t^N(x) = \tilde{\zeta}_t^N(2-x) \) for \( x \leq 0 \). Then, \( \tilde{\zeta}_t^N(x) \) satisfies the SDE:

\[
d\tilde{\zeta}_t^N(x) = N^2 \epsilon^{1/2} \Delta \tilde{\zeta}_t^N(x)dt + N^2 (2\epsilon^{1/2} - (\epsilon + 1)) \tilde{\zeta}_t^N(x)dt + d\tilde{M}_t^N(x),
\]

on the whole lattice space \( \mathbb{Z} \), where \( \tilde{M}_t^N(x) = e^{-(\log \epsilon x^2/2)\tilde{M}_t^N(x)} \) for \( x \geq 1 \), \( \tilde{M}_t^N(x) = \tilde{M}_t^N(2-x) \) for \( x \leq 0 \) and \( \Delta \zeta_t^N(x) = \zeta_t(x-1) - 2\zeta_t(x) + \zeta_t(x+1) \) for \( x \in \mathbb{Z} \).

The proof of this lemma is straightforward and omitted. The above transformation motivates the corresponding one for \( \omega(t, u) \), the solution of (4.2). In view of the scaling in \( \epsilon \), it is natural to set \( \tilde{\omega}(t, u) := e^{3|u|/2}\omega(t, |u|) \) and then, parallel to (4.4), to introduce its discretized equations with initial values \( \tilde{\omega}_0^N(x) = e^{-\log \epsilon |x|/2}\omega(0, |\frac{x}{N}|) \):

\[
d\tilde{\omega}_t^N(x) = N^2 \epsilon^{1/2} \Delta \tilde{\omega}_t^N(x)dt + N^2 (2\epsilon^{1/2} - (\epsilon + 1)) \tilde{\omega}_t^N(x)dt, \quad x \in \mathbb{Z}.
\]

It is known that the linear interpolation \( \tilde{\omega}_t^N(t, u) \) of \( (\tilde{\omega}_t^N(x))_{x \in \mathbb{Z}} \) converges to \( \tilde{\omega}(t, u) \), see p. 214 in [2] or [17]. More precisely, we have that

\[
\lim_{N \to \infty} \sup_{t \in [0, T]} \sup_{u \in [-K, K]} \sqrt{N} |\tilde{\omega}_t^N(t, u) - \tilde{\omega}(t, u)| = 0.
\]

**Lemma 4.2.** The process \( \Phi_t^N(x) \) defined by

\[
\Phi_t^N(x) := \sqrt{N} (\tilde{\zeta}_t^N(x) - \tilde{\omega}_t^N(x)), \quad x \in \mathbb{Z},
\]

satisfies the following SDE:

\[
d\Phi_t^N(x) = N^2 \epsilon^{1/2} \Delta \Phi_t^N(x)dt + N^2 (2\epsilon^{1/2} - (\epsilon + 1)) \Phi_t^N(x)dt + \sqrt{N}d\tilde{M}_t^N(x),
\]

which can be represented in its mild form:

\[
\Phi_t^N(x) = \sum_{y \in \mathbb{Z}} p^N(t, x, y)e^{\epsilon x y} \Phi_0^N(y) + \int_0^t \sum_{y \in \mathbb{Z}} p^N(t-s, x, y)e^{\epsilon x y(t-s)} \sqrt{N}d\tilde{M}_s^N(y).
\]
Here $p^N(t, x, y) = p(N^2 t^{1/2} x - y)$ and $p(t, x)$ is the (fundamental) solution of
\begin{equation}
\partial_t p(t, x) = \Delta p(t, x), \quad x \in \mathbb{Z}, \quad \text{with} \quad p(0, x) = \delta_0(x),
\end{equation}
and $c_N := N^2 (2^{1/2} - (\varepsilon + 1))^2 = -N^2 (\varepsilon^{1/2} - 1)^2$ behaves like $c_N \sim -\beta^2 / 4$.

The SDE (4.8) is an immediate consequence from (4.4) and (4.5). It is also easy to obtain (4.9). In fact, it is enough to apply integration by parts to the process $\{p^N(t - s, x, y)e^{CN(t-s)} \tilde{\Phi}_t(\cdot)\}_{s \in [0, t]}$ for each $t > 0$ and then integrate both sides from 0 to $t$.

Now let us consider the linear interpolation of $\tilde{\Phi}_N(x)$. More precisely, we deal with the following process with values in $D([0, T], C(\mathbb{R}))$:
\begin{equation}
\Phi^N(t, u) := ([Nu] + 1 - Nu)\Phi_t^N([Nu]) + (Nu - [Nu])\Phi_t^N([Nu] + 1).
\end{equation}

The next subsection is devoted to prove the following proposition.

**Proposition 4.3.** Suppose Assumption 2 is satisfied. Then, as $N \to \infty$, the transformed process $\Phi^N(t, u)$ converges weakly to $\Phi(t, u)$ on the space $D([0, T], C(\mathbb{R}))$. Moreover, the limit $\Phi(t, u)$ is in $C([0, T], C(\mathbb{R}))$ (a.s.) and it is a solution of the following SPDE:
\begin{equation}
\begin{cases}
\partial_t \Phi(t, u) = \Phi''(t, u) - \frac{\beta^2}{4} \Phi(t, u) \\
+ e^{\beta|u|/2} \omega(t, |u|) \sqrt{2\rho_R(t, |u|)}(1 - \rho_R(t, |u|)) \hat{W}(t, u), \quad u \in \mathbb{R}, \\
\Phi(0, u) = \Phi_0(u),
\end{cases}
\end{equation}
where $\hat{W}$ is a $Q$-cylindrical Brownian motion on $L^2(\mathbb{R})$ with the following covariance: for any test functions $\phi$ and $\psi$ on $\mathbb{R}$,
\begin{equation}
E[\hat{W}(t, \phi)\hat{W}(t, \psi)] = s \wedge t \langle \phi, Q\psi \rangle
\end{equation}
with $Q\psi(u) = \psi(u) + \psi(-u)$, and $\langle \cdot, \cdot \rangle$ denotes the inner product of $L^2(\mathbb{R})$. Furthermore, if $\Phi_0 \in \cap_{r > \beta/2} L^2_r(\mathbb{R})$ then there exists a unique weak solution $\Phi(t, \cdot)$ in $C([0, T], \cap_{r > \beta/2} L^2_r(\mathbb{R}))$.

**Remark 4.1.** The $Q$-cylindrical Brownian motion $\hat{W}$ can be easily constructed based on a Brownian sheet on $[0, \infty) \times \mathbb{R}_+$.

The weak and mild solutions of (4.12) are defined in similar ways to (2.6): $\tilde{\Phi}(t, u)$ is said to be a weak solution of the SPDE (4.12) with initial value $\Phi_0 \in \cap_{r \geq \beta/2} L^2_r(\mathbb{R})$ if $\tilde{\Phi} \in C([0, T], C(\mathbb{R})) \cap C([0, T], \cap_{r > \beta/2} L^2_r(\mathbb{R}))$ (a.s.) and for every function $f \in C^1([0, T] \times \mathbb{R})$,
\begin{equation}
\begin{aligned}
\langle \tilde{\Phi}(t), f(t) \rangle &= \langle \Phi_0, f(0) \rangle + \int_0^t \langle \Phi(s), f''(s) - \frac{\beta^2}{4} f(s) + \partial_s f \rangle ds \\
+ \int_0^t \int \langle f(s, u)e^{\beta|u|/2} \omega(s, |u|) \sqrt{2\rho_R(s, |u|)}(1 - \rho_R(s, |u|)) \hat{W}(ds du) \rangle \text{ a.s.}
\end{aligned}
\end{equation}
In particular, from its mild form
\begin{equation}
\tilde{\Phi}(t, u) = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{u^2}{4t} - \frac{\beta^2}{4t} - \frac{(u - v)^2}{4t}} \Phi_0(v) dv
\end{equation}
Taking such \( f \) satisfying
\[
2 \text{can be rewritten into conditions on } \bar{\Phi}
\]
Corollary 4.4. Under Assumption 2, \( \Phi^N(t, u) := \sqrt{N}(\zeta^N(t, u) - \omega(t, u)) \) converges weakly to \( \Phi(t, u) \) on the space \( D([0, T], C([\mathbb{R}_+])) \) as \( N \to \infty \). Moreover the limit \( \Phi(t, u) \) is in \( C([0, T], C(\mathbb{R}_+)) \) (a.s.) and characterized as a solution of the SPDE:
\[
\begin{aligned}
\partial_t \Phi(t, u) &= \Phi''(t, u) + \beta \Phi'(t, u) \\
+ \beta \omega(t, u)\sqrt{2\rho_R(t, u)(1 - \rho_R(t, u))}W(t), & \ u \in \mathbb{R}_+, \\
2\Phi'(t, 0) + \beta \Phi(0, 0) &= 0, \\
\Phi(0, u) &= \Phi_0(u),
\end{aligned}
\]
(4.15)
which has a unique weak solution in \( C([0, T], L^2_\mathcal{C}(\mathbb{R}_+)) \) for each \( \Phi_0 \in L^2_\mathcal{C}(\mathbb{R}_+) \).

Proof. Assume Proposition 4.3 is proved. Consider the even functions \( e^{\beta|u|/2}\Phi^N(t, |u|) \) on \( D([0, T], C(\mathbb{R})) \). We first show that for each \( K > 0 \)
\[
\lim_{N \to \infty} E\left[ \sup_{t \in [0, T]} \sup_{u \in [-K, K]} |e^{\beta|u|/2}\Phi^N(t, |u|) - \Phi^N(t, |u|)|^{2K} \right] = 0.
\]
The monotonicity of \( \zeta^N(x) \) in \( x \in \mathbb{N} \) yields
\[
\sqrt{N}(|[Nu]+1-Nu|\zeta^N_t([Nu])+(Nu-[Nu])\zeta^N_t([Nu]+1))-e^{\beta|u|/2}\zeta^N_t(1)\leq CN^{-\frac{1}{2}}\zeta^N_t(1).
\]
Now Lemma 4.7 below with (4.6) completes the proof of (4.16). Therefore, as \( N \to \infty \), \( e^{\beta|u|/2}\Phi^N(t, |u|) \) converges weakly on \( D([0, T], C(\mathbb{R})) \) to the same limit \( \Phi(t, u) \) as that of \( \Phi^N(t, u) \) and it immediately follows that \( \Phi^N(t, u) \) converges weakly on \( D([0, T], C(\mathbb{R}_+)) \) to \( \Phi(t, u) = e^{-\beta u/2}\Phi(t, u) \), \( u \in \mathbb{R}_+ \) and the limit \( \Phi(t, u) \) is in \( C([0, T], C(\mathbb{R}_+)) \) (a.s.).

To see that \( \Phi(t, u) \) is a solution of the SPDE (4.15), for a given \( g \in C^1_0([0, T] \times \mathbb{R}_+) \) satisfying \( 2g'(t, 0) - \beta g(t, 0) = 0 \), set \( f(t, u) = e^{-\beta|u|/2}g(t, |u|) \). Then \( f \in C^1_0([0, T] \times \mathbb{R}) \).

Taking such \( f \) in (4.14), a simple computation yields
\[
\langle \Phi(t), g(t) \rangle = \langle \Phi_0, g(0) \rangle + \int_0^t \langle \Phi(s), g'(s) - \beta g(s) + \partial_u g(s) \rangle ds \\
+ \beta \int_0^t \int_{\mathbb{R}_+} g(s, u)\omega(s, u)\sqrt{2\rho_R(s, u)(1 - \rho_R(s, u))}W(dsdu) \ \text{a. s.,}
\]
which completes the proof.

\[ \square \]

Remark 4.2. A microscopic interpretation of the mixed boundary condition at \( u = 0 \) in (4.15) is found in Lemma 5.8 of [9].

From now on, we formulate the proof of Theorem 2.3 based on Proposition 4.3, or more precisely Corollary 4.4, which is divided into two lemmas. First note that Assumption 2 can be rewritten into conditions on \( \Phi^N_0 \). This is mostly used later on but we state it here since the assertion (3) is needed.
Lemma 4.5. Under Assumption 2, the following holds:

1. For any $\kappa \in \mathbb{N}$, the following estimates hold:
   
   - $\sup_N E[\tilde{\zeta}_0^N(1)^{2\kappa}] < \infty$,
   - $E[|\Phi_0^N(x)|^{2\kappa}] \leq C e^{\frac{\kappa^2}{N}}$, $x \in \mathbb{Z}$,
   - $E \left[ \left| \Phi_0^N(x) - \tilde{\Phi}_0^N(y) \right|^{2\kappa} \right] \leq C \left( e^{-\frac{\kappa^2}{N}} + e^{-\frac{\kappa^2}{N^2}} \right) \left( \left| \sqrt{\kappa} \frac{x-y}{N} \right|^{2\alpha} + \left| \frac{x-y}{2N} \right|^{2\kappa} \right)$, for $x, y \in \mathbb{Z}$ and some $\kappa' > \kappa$ and any $\alpha \in (0, 1/2)$.

2. $\{\Phi_0^N(x)\}_N$ are independent of the noises determining the process $\{\eta_t^N; t \geq 0\}$.

3. $\Phi_0^N(0, u)$ converges weakly to $\tilde{\Phi}_0(0) = e^{\beta|u|/\sqrt{N}}$ in $C(\mathbb{R})$ as $N \to \infty$.

   In addition, for all $r > \frac{\beta}{2}$, $E[|\Phi_0^N|_{L_r^2(\mathbb{R})}^2] < \infty$.

Proof. Conditions (1)-(3) in Assumption 2 are referred to as (A2-1)-(A2-3), respectively. (2) is obviously implied by (A2-2). The next step is the condition (1). The properties of the transformation from $q \in \mathcal{Q}$ to $\eta \in \mathcal{X}_R$, see Lemma 5.1 of [9], yield for any $x \in \mathbb{N}$

$$\tilde{\zeta}_0^N(x) = e^{-\frac{\log(e)x}{2N}} \exp\left\{ -\left( \frac{x+1}{N} \right) \right\}.$$ 

Therefore, (i) follows directly from (A2-1)(i).

By the definition of $\Phi_0^N(x)$ it is enough to prove (ii) for $x \in \mathbb{N}$. Since $\omega(0, u) = \exp\{\beta\psi_{R,0}(u)\}$, $u \geq 0$, we deduce that

$$\Phi_0^N(x) = \sqrt{N} e^{-\frac{\log(e)x}{2}} \left[ \exp\left\{ -\left( \frac{x+1}{N} \right) \right\} \right] - \exp\left\{ \beta\psi_{R,0}(x) \right\}.$$ 

The mean value theorem implies the existence of a random variable $\theta^N(x)$ with values between $\beta\psi_{R,0}(x)$ and $-\left( \frac{x+1}{N} \right)$ such that

$$\tilde{\Phi}_0^N(x) = e^{-\frac{\log(e)x}{2} + \theta^N(x)} \left[ \beta\psi^N_{R,0}(x) - \sqrt{N} \left( \beta + \left( \log(e)N \right) \right) \tilde{\psi}^N_{R,0}(x) \right].$$

Note that $\sqrt{N} (\beta + \left( \log(e)N \right)) \to \infty$ with order $O\left( \frac{\log(N)}{\sqrt{N}} \right)$ and combine the monotonicity of $\tilde{\psi}^N_{R,0}(u)$ and $\psi_{R,0}(u)$ with (A2-1)(i) leads to the estimate

$$\sup_{x, N} E\left[ e^{4\beta\theta^N(x)} \right] \leq C \quad \text{and} \quad E \left[ \left| \sqrt{N} \left( \beta + \left( \log(e)N \right) \right) \tilde{\psi}^N_{R,0}(x) \right|^{4\alpha} \right] \leq C \left( \frac{\log(N)}{\sqrt{N}} \right)^{4\alpha}.$$ 

After applying Schwarz’s inequality together with (A2-1)(ii), we arrive at

$$E[|\Phi_0^N(x)|^{2\kappa}] \leq C e^{-2\alpha(\log(e)x)} \leq C e^{\frac{\kappa^2}{N}}.$$ 

As explained above, we assume $x, y \in \mathbb{N}$ in the proof of (iii). It follows from (4.17) that

$$\Phi_0^N(x) - \Phi_0^N(y) = A_1 + \sqrt{N} e^{-\frac{(\log(e)x)}{2}} \left( \exp\left\{ \beta\psi^N_{R,0}(x) \right\} - \exp\left\{ \beta\psi_{R,0}(x) \right\} \right)$$

$$- \sqrt{N} e^{-\frac{(\log(e)y)}{2}} \left( \exp\left\{ \beta\psi^N_{R,0}(y) \right\} - \exp\left\{ \beta\psi_{R,0}(y) \right\} \right).$$
where the term $A_1 := A_1^N(x, y)$ is estimated as above by

$$E[A_1^{2\kappa}] \leq C\left(e^{\frac{e^{2\kappa}}{N}} + e^{\frac{e^{2\kappa}}{N}}\right)N^{-2\kappa}$$

for every $\alpha < 1/2$.

Rewriting the two other summands on the right hand side of (4.19) yields

$$|\Phi_0^N(x) - \Phi_0^N(y)|^{2\kappa} \leq C[A_1^{2\kappa} + \left(e^{\frac{e^{2\kappa}}{N}} + e^{\frac{e^{2\kappa}}{N}}\right)|A_2 + A_3|^{2\kappa},$$

with

$$A_2 = N^{\frac{1}{2}} \exp\{\beta \psi_{R,0}(\frac{y}{N})\} \exp\{\beta (\tilde{\psi}_R^N(0, \frac{x}{N}) - \psi_{R,0}(\frac{y}{N}))\} - \exp\{\beta (\tilde{\psi}_R^N(0, \frac{x}{N}) - \psi_{R,0}(\frac{y}{N}))\} - 1).$$

and

$$A_3 = N^{\frac{1}{2}} (\exp\{\beta \psi_{R,0}(\frac{y}{N})\} - \exp\{\beta \psi_{R,0}(\frac{y}{N})\}) \exp\{\beta (\tilde{\psi}_R^N(0, \frac{x}{N}) - \psi_{R,0}(\frac{y}{N}))\} - 1).$$

However, with a similar approach to (4.18) one can derive

$$E[A_2^{2\kappa}] \leq C\left[\frac{y-x}{N}\right]^\kappa \quad \text{and} \quad E[A_3^{2\kappa}] \leq C\left[\frac{y-x}{N}\right]^\kappa.$$

This concludes the proof of (iii).

The final task is assertion (3). The proof of Corollary 4.4 suggests that it is enough to prove (3) for $e^{\beta |u|} \Phi^N(0, |u|)$ instead of $\Phi^N(t, u)$. It is easy to see that

$$\Phi^N(0, u) = \sqrt{N} \left[\exp\{\beta \psi_R^N(0, u)\} - \exp\{\beta \psi_{R,0}(u)\}\right]$$

(4.20)

$$+ \sqrt{N} \left[\exp\{-N \log \epsilon \psi_R^N(0, \frac{Nu}{N}) - (\log \epsilon)^2 N(0, u)\} - \exp\{\beta \psi_R^N(0, u)\}\right]$$

with $r^N(0, u) = 1_{\{u \geq 1/N\}} ([Nu] + 1 - Nu)\eta^N ([Nu]) \in [0, 1]$. By Taylor’s theorem, the first term on the right hand side of (4.20) is equal to

$$\beta \exp\{\beta \psi_{R,0}(u)\} \Psi_R^N(0, u) + \frac{1}{2} \beta^2 \exp\{\beta \theta^N(u)\} N^{-1/2} (\Psi_R^N(0, u))^2,$$

where $\theta^N(u)$ is a random variable with values between $\tilde{\psi}_R^N(0, u)$ and $\psi_{R,0}(u)$. From (A2-3), it follows that the first part converges weakly to $\beta \omega(0, u)\Psi_{R,0}(u)$ in $D(\mathbb{R}_+)$. In a similar way to the proof of (1)(ii), we see that

$$E[\exp\{\beta \theta^N(u)\} N^{-1/2} (\Psi_R^N(0, u))^2 ] \leq CN^{-\kappa/2} E[(\Psi_R^N(0, u))^2]^{1/2},$$

which, combined with the right continuity of $\Psi_R^N(0, u)$, implies that the second term of (4.21) converges to $0$ in probability in $D(\mathbb{R}_+)$. In addition, it is easy to check that the second term in (4.20) also converges to $0$ in probability in $D(\mathbb{R}_+)$. Because $e^{\beta |u|^2/2} \Phi^N(0, |u|)$ is even, we see that $e^{\beta |u|^2/2} \Phi^N(0, |u|)$ converges weakly to $e^{\beta |u|^2/2} \beta \omega(0, |u|)\Psi_{R,0}(|u|)$ on $D(\mathbb{R})$. On the other hand, since the Skorohod topology relativized to $C(\mathbb{R}_+)$ coincides with its locally uniform topology, the continuity of $\Phi^N(0, u)$ in $u$ completes the proof.

**Lemma 4.6.** Assume Proposition 4.3 is shown. Then as $N \to \infty$, $\Psi_R^N(t, u)$ converges weakly on the space $D([0, T], D(\mathbb{R}_+))$ to

$$\Psi_R(t, u) = \frac{\Phi(t, u)}{\beta \omega(t, u)}.$$

Moreover, the limit $\Psi_R(t, u)$ is in $C([0, T], C(\mathbb{R}_+))$ (a.s.) and satisfies the SPDE (2.6).
Proof. From Lemma 4.5-(3) the relation $\Psi_{R,0}(u) = \Phi_0(u)/(\beta \omega(0,u))$ is known. Due to Skorohod's representation theorem we may assume that $\Phi^N(t,u)$ converges to $\Phi(t,u)$ uniformly on $[0,T] \times [0,K]$ for every $K > 0$ (a.s.) on a properly changed probability space. The definitions of $\tilde{\zeta}^N(t,u)$ and $\psi^N_R(t,u)$ correspond to

$$\log \tilde{\zeta}^N(t,u) = -(N \log \varepsilon) \psi^N_R(t,\lfloor Nu \rfloor/N) - (\log \varepsilon) r^N(t,u),$$

where $r^N(t,u) = 1_{\{u \geq \frac{1}{N}\}}(\lfloor Nu \rfloor + 1 - Nu)\eta^N(\lfloor Nu \rfloor) \in [0,1]$. Since $\varepsilon = 1 - \frac{\beta}{N} + O\left(\frac{\log N}{N^2}\right)$, we have that

$$\tilde{\psi}^N_R(t,\lfloor Nu \rfloor/N) = \frac{1}{\beta} \{1 + O\left(\frac{\log N}{N}\right)\} \log \tilde{\zeta}^N(t,u) + O\left(\frac{1}{N}\right),$$

with an error $O\left(\frac{1}{N}\right)$ which is uniform in $(t,u)$. On the other hand, we know

$$\psi_R(t,u) = \frac{1}{\beta} \log \omega(t,u)$$

and $\inf_{t,u} \omega(t,u) \geq 1$, see p. 354 in [9]. Thus, we estimate the difference between $\Phi^N(t,u)$ and $\Phi^N(t,\lfloor Nu \rfloor/N)$ and due to the uniform convergence of $\Phi^N(t,\cdot)$ to $\Phi(t,\cdot)$, arrive at

$$\Psi^N_R(t,u) = \sqrt{N} \left[ \frac{1}{\beta} \left(1 + O\left(\frac{\log N}{N}\right)\right) \log (\omega(t,u) + \Phi^N(t,u)) \right] + o\left(\frac{1}{\sqrt{N}}\right) - \frac{1}{\beta} \log \omega(t,u),$$

which concludes the proof of the first part.

To complete the proof, it is enough to check the weak form (2.7) of the SPDE (2.6) for $f \in C^1_0([0,T] \times \mathbb{R}_+)$ satisfying $f'(t,0) = 0$. For such $f$, set $g(t,u) = f(t,u)/(\beta \omega(t,u))$. Then, we easily see that $g$ satisfies the condition: $2g'(t,0) - \beta g(t,0) = 0$ and, if we consider a weak solution of (4.15) for such $g$, a simple computation leads to (2.7) for $f$. \hfill \Box

## 4.2 Proof of Proposition 4.3

Subsection 4.2.1 concerns an important uniform estimate on $\zeta^N_t$. We then formulate some lemmas for the proof of the tightness of $\{\Phi^N(t,u)\}_N$ in Subsection 4.2.2 and finally give the proof of Proposition 4.3 in Subsection 4.2.3. To show the tightness of $\{\Phi^N(t,u)\}_N$ on the space $D([0,T], C(\mathbb{R}))$, we mainly mimic the approaches used in [1] and [5].

### 4.2.1 A Uniform Estimate

The following lemma is crucial and will be frequently used in the sequel.

**Lemma 4.7.** Let $\kappa \in \mathbb{N}$ as above. Under Assumption 2-(1)(i), for any $T > 0$, we have

$$\sup_{N \in \mathbb{N}} E \left[ \sup_{\kappa \in [0,T]} \zeta^N_\kappa(1) 2^\kappa \right] < \infty.$$

**Proof.** One can modify the proof of Proposition 5.4 in [9]. Let $\varphi \in C^2_0(\mathbb{R}_+)$ such that $\varphi' \geq 0$, $\varphi(u) = 0$ for $u \in (0,1]$ and $\varphi(u) = 1$ for $u \geq 2$, and for each $\kappa$ set

$$\Pi^N_R(\varphi) := \exp \left\{ -\kappa (\log \varepsilon) \sum_{x \in \mathbb{N}} \eta^N_x(\varphi(x)) \right\}.$$
Since \( \sup_{N \in \mathbb{N}, s \in [0, T]} s^N_\epsilon (1) / \Pi^N_s (\varphi) < \infty \), it is enough to show that

\[
(4.22) \quad \sup_{N \in \mathbb{N}} E \left[ \sup_{s \in [0, T]} \Pi^N_s (\varphi)^2 \right] < \infty.
\]

Consider the martingale \( M^N_t (\varphi) \) given by

\[
(4.23) \quad M^N_t (\varphi) = \Pi^N_t (\varphi) - \Pi^N_0 (\varphi) - \int_0^t \bar{L}^N_t (\varphi) ds,
\]

where \( \bar{L}^N = N^2 \bar{L}_{e,N,R} \). Here \( \bar{L}_{e,R} \) is the generator described at p. 353 of [9], so that

\[
\bar{L}^N \Pi^N_s (\varphi) = N^2 \Pi^N_s (\varphi) \sum_{x \in \mathbb{N}} \left( \varepsilon c_+ (x, \eta^N_s) + c_- (x, \eta^N_s) \right) \times \left[ \exp \left\{ -\kappa (\log \varepsilon) \left( \varphi \left( \frac{x + 1}{N} \right) - \varphi \left( \frac{x}{N} \right) \right) \right\} \right] - 1.
\]

By simple calculations the quadratic variation of \( M^N_t (\varphi) \) is given by the following relation:

\[
(4.24) \quad \bar{L}^N \Pi^N_s (\varphi) \leq C^1 \Pi^N_s (\varphi).
\]

In fact, note that \( c_- (x, \eta) - c_+ (x, \eta) = \eta (x + 1) - \eta (x) \) and after rearranging the sum, we can rewrite \( \bar{L}^N \Pi^N_s (\varphi) \) as follows:

\[
\bar{L}^N \Pi^N_s (\varphi) = N^2 \Pi^N_s (\varphi) \sum_{x \in \mathbb{N}} \left( \varepsilon c_+ (x, \eta^N_s) + c_- (x, \eta^N_s) \right) \times \left[ \exp \left\{ -\kappa (\log \varepsilon) \left( \varphi \left( \frac{x + 1}{N} \right) - \varphi \left( \frac{x}{N} \right) \right) \right\} \right] - 1.
\]

Thus, by Taylor’s formula and Lemma 3.2 of [9], we can show (4.24). As a consequence of (4.23), (4.24) and Gronwall’s inequality, we have that

\[
\Pi^N_t (\varphi) \leq e^{C^1 t} \left( \Pi^N_0 (\varphi) + \sup_{s \in [0, T]} M^N_t (\varphi) \right), \quad t \leq T,
\]

which implies

\[
(4.25) \quad E \left[ \sup_{s \in [0, T]} \Pi^N_s (\varphi)^2 \right] \leq 2e^{2C^1 t} \left( E \left[ \Pi^N_0 (\varphi)^2 \right] + E \left[ \sup_{s \in [0, t]} M^N_s (\varphi)^2 \right] \right), \quad t \leq T.
\]

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A similar approach to (4.24) yields that there exists $C_2 = C_2(\kappa, \|\varphi\|_\infty) > 0$ such that

$$\langle M^N(\varphi) \rangle_t \leq \frac{C_2}{N} \int_0^t \Pi_s^N(\varphi)^2 ds$$

and therefore Doob’s inequality implies

$$E \left[ \sup_{t \in [0, t]} M_s^N(\varphi)^2 \right] \leq \frac{4C_2}{N} E \left[ \int_0^t \Pi_s^N(\varphi)^2 ds \right], \ t \leq T.$$ 

In the end, Gronwall’s inequality, (4.25) and Lemma 4.5 conclude the proof of (4.22).

4.2.2 Tightness of $\Phi^N(t, u)$

A criterion for the tightness of on the space $D([0, T], C(\mathbb{R}))$ is given by the theorem due to Aldous and Kurtz (see [6], [14, Theorem 2.7] or [1, Proposition 4.9]). It states that it is sufficient to show the following estimates:

1. For every $t, K > 0$, there exist $\kappa \geq 1$, $C$ and $\alpha > 0$ such that

$$\sup_N E[|\Phi^N(t, 0)|^\kappa] < \infty,$$

$$\sup_N E[|\Phi^N(t, u_1) - \Phi^N(t, u_2)|^\kappa] \leq C|u_1 - u_2|^{1+\alpha}, \ |u_1|, |u_2| \leq K.$$

2. There exists a process $A^N(\delta), \delta > 0$ such that

$$E[d(\Phi^N(t + \delta, \cdot), \Phi^N(t, \cdot)) | \mathcal{F}_t] \leq E[A^N(\delta) | \mathcal{F}_t], \ t \in [0, T],$$

$$\lim_{\delta \downarrow 0} \limsup_{N \to \infty} E[A^N(\delta)] = 0.$$ 

Here $\mathcal{F}_t = \sigma\{\Phi^N(s, \cdot); 0 \leq s \leq t\}$ and $d(\cdot, \cdot)$ denotes a metric on $C(\mathbb{R})$ which determines the topology of the uniform convergence on each compact subset of $\mathbb{R}$.

Before we go to our main topic of this subsection, let us state Burkholder’s inequality according to Theorem 7.11 of [19].

Lemma 4.8. For any $L^{2\alpha}$-integrable real valued martingale $M_t$ and fixed $t > 0$, there exists a constant $C = C(\kappa, t) > 0$ such that

$$E \left[ \sup_{t \in [0, t]} |M_s|^{2\alpha} \right] \leq CE \langle M \rangle_t^\alpha + CE \left[ \sup_{t \in [0, t]} |M_s - M_{s-}|^{2\alpha} \right],$$

where $\langle M \rangle_t$ denotes the quadratic variational process of $M_t$.

As further preparations, we formulate some estimates for $(\Phi_t^N(x))_{x \in \mathbb{Z}}$ defined by (4.7). Let us first summarize some properties of $p(t, x)$ given by (4.10), see [1, 10] for their proofs.

Lemma 4.9. There exists a constant $C > 0$ such that the following holds:

(i) For any $t > 0, \sup_{x \in \mathbb{Z}} p(t, x) \leq C t^{-\frac{1}{2}},$

(ii) For any $\alpha \leq 1/2, \ |p(t, x) - p(t, y)| \leq C|x - y|^{2\alpha t^{-\frac{1}{2} - \alpha}}, \ x, y \in \mathbb{Z},$

$$|p(t + h, x) - p(t, x)| \leq C h^{2\alpha} t^{-\frac{1}{2} - 2\alpha}, \ x \in \mathbb{Z}, \ h > 0.$$ 

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Lemma 4.10. Under Assumption 2, there exist $C$ and $\kappa > 0$ such that

$$E[|\Phi_1^N(x)|^{2\kappa}] \leq Ce^{\frac{2\kappa}{N}}\left(1 + t^\frac{n}{2} + (\sqrt{N}(\varepsilon^{-1} - 1))^{2\kappa}\right), \ t \in [0, T].$$

Proof. We denote the first and second terms in the right hand side of (4.9) by $I^{(N,1)}(t, x)$ and $I^{(N,2)}(t, x)$, respectively. Then, Lemma 4.5 (1)(ii), Hölder’s inequality and the property $\sum_{y \in \mathbb{Z}} p^N(t, x, y) = 1$ result in

$$E[|I^{(N,1)}(t, x)|^{2\kappa}] \leq \sum_{y \in \mathbb{Z}} p^N(t, x, y)e^{2\kappa c_N} E[|\Phi_0^N(y)|^{2\kappa}] \leq \sum_{y \in \mathbb{Z}} p^N(t, x, y)e^{2\kappa c_N} Ce^{\frac{2\kappa |y|}{N}}.$$  

On the other hand, since for each $a \in \mathbb{R}$, the function $e^{ax}$ is the eigenfunction of the operator $\Delta$ corresponding to the eigenvalue $e^a$ and $e^{-a} - 2$

$$\sum_{y \in \mathbb{Z}} p^N(t, x, y)e^{ay} = e^{ax} \exp\left\{(e^a + e^{-a} - 2)N^2 \varepsilon^{1/2}\right\}$$

holds. Note that $e^{\frac{2\kappa |y|}{N}} \leq e^{\frac{\kappa |y|}{N}} + e^{-\frac{\kappa |y|}{N}}$ and it follows that

$$E[|I^{(N,1)}(t, x)|^{2\kappa}] \leq Ce^{\frac{\kappa |y|}{N}}, \ t \in [0, T],$$

by the behavior of $c_N$ and the convergence of $(e^{\frac{\kappa |y|}{N}} + e^{-\frac{\kappa |y|}{N}} - 2)N^2$ as $N \to \infty$.

In the following, we are going to estimate $E[|I^{(N,2)}(t, x)|^{2\kappa}]$ by using Burkholder’s inequality stated in Lemma 4.8. However, as a stochastic process, it is well-known that $I^{(N,2)}(t, x)$ is not a martingale. Instead of the direct disposal of $I^{(N,2)}(t, x)$, we will fix $t > 0$ and consider the process

$$I_t^{(N,2)}(r, x) = \int_0^r \sum_{y \in \mathbb{Z}} p^N(t - s, x, y)e^{c_N(t-s)}\sqrt{N}d\tilde{M}^N(y), \ r < t,$$

which is a real valued martingale in $r$ for each $x \in \mathbb{Z}$ with quadratic variation

$$\langle I_t^{(N,2)}(\cdot, x) \rangle_r = \int_0^r \sum_{y \in \mathbb{Z}} (p^N(t - s, x, y)e^{c_N(t-s)})^2 N d\langle \tilde{M}^N(y) \rangle_s, \ r < t.$$  

Since $\zeta_t^N(x) \leq \zeta_t^N(1)$ for any $x \geq 2$, $a_N$ and $b_N$ are both bounded in $N$, (4.3) yields that

$$d\langle \tilde{M}^N(y) \rangle_s \leq Ce^{-(\log \varepsilon)|y|}\zeta_s^N(1)^2 ds,$$

which implies

$$\langle I_t^{(N,2)}(\cdot, x) \rangle_r \leq C \int_0^r \sum_{y \in \mathbb{Z}} (p^N(t - s, x, y)e^{c_N(t-s)})^2 Ne^{-(\log \varepsilon)|y|}\zeta_s^N(1)^2 ds.$$  

Thus, by the above estimate and Lemma 4.7, we have

$$E[|I_t^{(N,2)}(\cdot, x)\rangle_r^\kappa] \leq C \left(\int_0^r \sup_{y \in \mathbb{Z}} |p^N(t - s, x, y)N|e^{-(\log \varepsilon)|y|} ds\right)^\kappa \leq Ce^{-\kappa(\log \varepsilon)|x|}t^\kappa/2,$$
where \( NP^N(t, x, y) \leq Ct^{-\frac{1}{2}} \) has been used, see Lemma 4.9 (i).

Finally, let us consider the jump size of \( I^{(N,2)}(r, x) \). By the definition, the jump size is determined by \( M^N_s(y) \), which inherits from \( \zeta^N_s(x) \). More precisely, we have that

\[
\sup_{r \in [0,t]} |I^{(N,2)}(r, x) - I^{(N,2)}(r-, x)| \leq C \sup_{r \in [0,t]} \sum_{y \in \mathbb{Z}} p^N(t - r, x, y) \sqrt{N} |M^N_r(y) - \bar{M}^N_r(y)|
\]

(4.30)

\[
= C \sup_{r \in [0,t]} \sum_{y \in \mathbb{Z}} p^N(t - r, x, y) \sqrt{N} |\bar{\zeta}^N_r(y) - \bar{\zeta}^N_{r-}(y)|.
\]

Since \( \eta^N_t(y) \) does not jump at same time for different \( y \), we see that

\[
|\bar{\zeta}^N_r(y) - \bar{\zeta}^N_{r-}(y)| \leq Ce^{-(\log \varepsilon)y/2} |\zeta^N_r(1) - \zeta^N_{r-}(1)| \leq Ce^{-(\log \varepsilon)y/2}(\varepsilon^{-1} - 1)\zeta^N_r(1).
\]

(4.31)

Consequently, Lemma 4.7 and (4.26) imply again that

\[
E \left[ \sup_{r \in [0,t]} |I_{t}^{(N,2)}(r, x) - I_{t}^{(N,2)}(r-, x)|^{2\kappa} \right] \leq Ce^{-\kappa(\log \varepsilon)x}(\sqrt{N}(\varepsilon^{-1} - 1))^{2\kappa}.
\]

Note that \( I_{t}^{(N,2)}(r, x) \) converges to \( I^{(N,2)}(t, x) \) in \( L^2(\Omega) \) as \( r \uparrow t \) and we can conclude the proof by Lemma 4.8, (4.27) and (4.29).

\[\square\]

**Lemma 4.11.** Under Assumption 2, the following estimates hold:

1. For each \( \alpha < 1/2 \), there exist \( C \) and \( \bar{\kappa} > 0 \) such that for any \( t \leq T \) and \( x, y \in \mathbb{Z} \),

\[
E \left[ \left| \bar{\Phi}^N_t(x) - \bar{\Phi}^N_t(y) \right|^{2\kappa} \right] \leq C \left( e^{\frac{\bar{\kappa} |x|}{N}} + e^{\frac{\bar{\kappa} |y|}{N}} \right) \times \left( \frac{|x - y|^{2\kappa}}{N^{2\kappa}} + \frac{|x - y|^{2\kappa}}{N^{2\kappa}} + \left( \sqrt{N}(\varepsilon^{-1} - 1) \right)^{2\kappa} \right).
\]

(4.32)

2. For each \( \alpha < 1/4 \), there exist \( C \) and \( \bar{\kappa} > 0 \) such that for any \( t_1, t_2 \leq T \) and \( x \in \mathbb{Z} \),

\[
E \left[ \left| \bar{\Phi}^N_{t_1}(x) - \bar{\Phi}^N_{t_2}(x) \right|^{2\kappa} \right] \leq Ce^{\frac{\bar{\kappa} |t_1 - t_2|}{N^{2\kappa}}} \left( |t_1 - t_2|^{2\kappa} + \left( \sqrt{N}(\varepsilon^{-1} - 1) \right)^{2\kappa} \right).
\]

(4.33)

**Proof.** The main idea to prove this lemma is similar to that for Lemma 4.10. We will only give some necessary explanations by using same notations. We begin with the proof of (4.32). The representation of \( I^{(N,1)}(t, x) \), change of variables and Lemma 4.5 yield that

\[
E \left[ \left( I^{(N,1)}(t, x) - I^{(N,1)}(t, y) \right)^{2\kappa} \right] \leq C \left( e^{\frac{\bar{\kappa} |x|}{N}} + e^{\frac{\bar{\kappa} |y|}{N}} \right) \times \left( \frac{|x - y|^{2\kappa}}{N^{2\kappa}} + \frac{|x - y|^{2\kappa}}{N^{2\kappa}} \right).
\]

(4.34)

For \( I_{t}^{(N,2)}(r, x) \), owing to Lemma 4.8, it is sufficient to deal with \( \langle I_{t}^{(N,2)}(\cdot, x) - I_{t}^{(N,2)}(\cdot, y) \rangle_r \) and the jump of \( I_{t}^{(N,2)}(r, x) - I_{t}^{(N,2)}(r, y) \) respectively.

By Lemma 4.9 (ii), \( N[p^N(s, x, z) - p^N(s, y, z)] \leq CN^{-2\alpha}s^{-1/2-\alpha}|x - y|^{2\alpha} \), \( \alpha < 1/2 \).

Now let us take \( r = t \) and we deduce that

\[
E \left[ \left( I_{t}^{(N,2)}(\cdot, x) - I_{t}^{(N,2)}(\cdot, y) \right)^{\kappa} \right] \leq CE \left[ \sup_{s \in [0,t]} |\zeta^N_s(1)|^{2\kappa} \right] \left( \int_0^t \sum_{z \in \mathbb{Z}} \left( p^N(s, x, z) - p^N(s, y, z) \right)^2 e^{2CNsN^e(\log \varepsilon)|z|} ds \right)^{\kappa}
\]

(4.35)
\[ \leq C \left( \frac{|x - y|^{2\alpha}}{N^{2\alpha}} \int_0^t s^{-1/2-\alpha} e^{2c_N s} \sum_{z \in \mathbb{Z}} (p^N(s, x, z) + p^N(s, y, z)) e^{-(\log \varepsilon)|z|} ds \right) \]

\[ \leq C \left( e^{-\kappa (\log \varepsilon)|x|} + e^{-\kappa (\log \varepsilon)|y|} \right) N^{-2\alpha} |x - y|^{2\alpha \kappa (1/2 - \alpha)}, \]

where Lemma 4.7 has been applied for the second inequality, and \( \alpha < 1/2 \) as well as \( (4.26) \) have been used for the last inequality. Using a similar approach to \( (4.30) \) and recalling \( (4.26) \) and \( (4.31) \), we have

\[ E \left[ \sup_{r \in [0,t]} \left( I^{(N,2)}_r(r, x) - I^{(N,2)}_r(r, y) \right) - \left( I^{(N,2)}_r(r-, x) - I^{(N,2)}_r(r-, y) \right) \right]^{2\kappa} \]

\[ \leq C \left( \sup_{r \in [0,t]} \sqrt{N} \sum_{z \in \mathbb{Z}} |p^N(t - r, x, z) - p^N(t - r, y, z)| e^{c_N (t-r)} e^{-\left(\log \varepsilon\right)|z|/2(\varepsilon - 1)} \right)^{2\kappa} \]

\[ \leq C \left( \sqrt{N} (\varepsilon - 1)^{2\kappa} \left( e^{-\kappa (\log \varepsilon)|x|} + e^{-\kappa (\log \varepsilon)|y|} \right) \right), \]

which yields \( (4.32) \) together with \( (4.34) \) and \( (4.35) \).

Now we show the second estimate \( (4.33) \) but only for \( I^{(N,2)}(t_2, x) - I^{(N,2)}(t_1, x) \), since that for \( I^{(N,1)}(t_2, x) - I^{(N,1)}(t_1, x) \) is easier. By a similar approach as above, for any \( 0 \leq t_1 < t_2 \leq T \), we can easily obtain

\[ E \left[ \left| \int_{t_1}^{t_2} \sum_{y \in \mathbb{Z}} p^N(t_2 - s, x, y) e^{c_N (t_2-s)} \sqrt{N} dM^N_s(y) \right|^{2\kappa} \right] \]

\[ \leq C e^{-\kappa (\log \varepsilon)|x|} \left( \int_{t_1}^{t_2} \sup_{y} p^N(t_2 - s, x, y) N e^{2c_N (t_2-s)} e^{(\varepsilon - 1)^2 N^2 \varepsilon^{-2/3}(t_2-s) ds} \right)^{\kappa} \]

\[ + C E \left[ \left( \sup_{r \in [t_1, t_2]} \left| \int_{r}^{t_2} \sum_{y \in \mathbb{Z}} p^N(t_2 - r, x, y) e^{c_N (t_2-s)} \sqrt{N} dM^N_s(y) \right|^{2\kappa} \right] \]

\[ \leq C e^{-\kappa (\log \varepsilon)|x|} (t_1 - t_2)^{2\kappa} + C e^{-\kappa (\log \varepsilon)|x|} \left( \sqrt{N} (\varepsilon - 1)^{2\kappa} \right). \]

Hereafter, for simplicity, to deal with the jump part, we write it in its integral form and consider the process directly. In fact, the following calculations are just formal, see Lemma 4.10 for concrete explanations.

By Lemma 4.8 and \( (4.28) \) and imitating the procedure used in the proof of \( (4.32) \), we can deduce that

\[ E \left[ \left| \int_{0}^{t_1} \sum_{y \in \mathbb{Z}} (p^N(t_1 - s, x, y) e^{c_N (t_1-s)} - p^N(t_2 - s, x, y) e^{c_N (t_2-s)}) \sqrt{N} dM^N_s(y) \right|^{2\kappa} \right] \]

\[ \leq C \left( \int_{0}^{t_1} \sum_{y \in \mathbb{Z}} (p^N(t_1 - s, x, y) e^{c_N (t_1-s)} - p^N(t_2 - s, x, y) e^{c_N (t_2-s)})^2 N e^{-(\log \varepsilon)|y| ds} \right)^{\kappa} \]

\[ + C E \left[ \sup_{r \in [0, t_1]} \left| \int_{r}^{t_1} \sum_{y \in \mathbb{Z}} (p^N(t_1 - s, x, y) e^{c_N (t_1-s)} - p^N(t_2 - s, x, y) e^{c_N (t_2-s)}) \sqrt{N} dM^N_s(y) \right|^{2\kappa} \right] \]

holds, where the second term on the right hand side denotes the corresponding jump part.

With the behavior of \( c_N \), the property

\[ N |p^N(t_1 - s, x, y) - p^N(t_2 - s, x, y)| \leq C(t_2 - t_1)^{2\alpha}(t_1 - s)^{-1/2 - 2\alpha}, \]

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and (4.26), we obtain that the first term on the right hand side is bounded from above by

\[ C e^{-\kappa (\log \varepsilon)|x|} (t_2 - t_1)^{2\alpha}, \]

where the restriction of \( \alpha < 1/4 \) has been used.

Relation (4.31) is used to bound the second term from above by

\[ 2e^{-\kappa (\log \varepsilon)|x|} (\sqrt{N}(\varepsilon^{-1} - 1))^{2\kappa} \sup_{s \in [0,t_1]} \left( c(\varepsilon + \varepsilon^{-1} - 2)N^{2\alpha}e^{1/2(t_1 - s)} + c(\varepsilon + \varepsilon^{-1} - 2)N^{2\alpha}e^{1/2(t_2 - s)} \right)^\kappa. \]

All the above estimates applied yield the upper bound

\[
E \left[ \left| \int_0^{t_1} \sum_{y \in \mathbb{Z}} (p_y(t_1 - s, x, y)e^{-\kappa N(t_1 - s)} - p_y(t_2 - s, x, y)e^{-\kappa N(t_2 - s)}) \sqrt{N} \sum_{y \in \mathbb{Z}} \left| \frac{\sum_{y \in \mathbb{Z}} (p_y(t_1 - s, x, y)e^{-\kappa N(t_1 - s)} - p_y(t_2 - s, x, y)e^{-\kappa N(t_2 - s)}) \sqrt{N} \sum_{y \in \mathbb{Z}}}{|u_1 - u_2|^{\alpha'}} \right|^{2\alpha} \right] < \infty.
\]

Proof. Let us first assume that \([Nu_1] = [Nu_2].\) Then by (4.32) we deduce that

\[
E \left[ \left| \Phi^N(t, u_1) - \Phi^N(t, u_2) \right|^{2\alpha} \right] \leq C N^{2\alpha} |u_1 - u_2|^{2\alpha} E \left[ \left| \Phi^N([Nu_1] + 1) - \Phi^N([Nu_1]) \right|^{2\alpha} \right] \leq C' |u_1 - u_2|^{2\alpha}. \]

Next we assume, without loss of generality, that \(Nu_1 < [Nu_1] + 1 \leq [Nu_2] \leq Nu_2.\) From the above estimate and again (4.32), we obtain

\[
E \left[ \left| \Phi^N(t, u_1) - \Phi^N(t, u_2) \right|^{2\alpha} \right] \leq C \left( \left( \frac{[Nu_1] + 1}{N} - u_1 \right)^{2\alpha} + \left( u_2 - \frac{[Nu_2]}{N} \right)^{2\alpha} \right) + C E \left[ \left| \Phi^N(t, \frac{[Nu_1] + 1}{N}) - \Phi^N(t, \frac{[Nu_2]}{N}) \right|^{2\alpha} \right] \leq C' |u_1 - u_2|^{2\alpha},
\]

which implies (4.36). On the other hand, the second assertion (4.37) is a direct consequence of (4.36) and Kolmogorov’s theorem, for example, see Proposition 4.4 of [1].

In fact, it is clear that \( \Phi^N(t, u) \) is not continuous in \( t, \) so Kolmogorov’s continuity theorem cannot be applied directly. To overcome this difficulty, we introduce the process \( \Phi^N(t, u), \) namely consider the linear interpolation in time \( t \) defined as

\[
\Phi^N(t, u) := ([N^2t] + 1 - N^2t)\Phi^N \left( \frac{[N^2t]}{N^2}, u \right) - (N^2t - [N^2t])\Phi^N \left( \frac{[N^2t]}{N^2}, u \right).
\]
Lemma 4.13. Let $\alpha < 1/4$. For any $t_1, t_2 \leq T$ and $u_1, u_2 \in [-K, K]$, there exists a constant $C > 0$ such that
\begin{equation}
E\left[\left|\Phi^N(t_1, u_1) - \Phi^N(t_2, u_2)\right|^{2\alpha}\right] \leq C \left(\left|t_2 - t_1\right|^{2\alpha} + \left|u_2 - u_1\right|^{2\alpha}\right) + \left(\sqrt{N}(\varepsilon^{-1} - 1)\right)^{2\alpha}
\end{equation}
and moreover for $\alpha' \in [0, \frac{2\alpha - 1}{2\alpha})$
\begin{equation}
E\left[\left|\Phi^N(t_1, u_1) - \Phi^N(t_2, u_2)\right|^{2\alpha'}\right] < \infty.
\end{equation}

Proof. Note that
\begin{align*}
E\left[\left|\Phi^N(t_1, u_1) - \Phi^N(t_1, u_2)\right|^{2\alpha}\right] & \leq C \left(\left|N^2 t_1 + 1 - N^2 t_1\right|^{2\alpha} E\left[\left|\Phi^N\left(\frac{[N^2 t_1]}{N^2}, u_1\right) - \Phi^N\left(\frac{[N^2 t_1]}{N^2}, u_2\right)\right|^{2\alpha}\right] \right. \\
& \quad + C \left(\left|N^2 t_1 - [N^2 t_2]\right|^{2\alpha} E\left[\left|\Phi^N\left(\frac{[N^2 t_1]+1}{N^2}, u_1\right) - \Phi^N\left(\frac{[N^2 t_1]+1}{N^2}, u_2\right)\right|^{2\alpha}\right] \right)
\end{align*}
and use Lemma 4.12 to obtain
\begin{equation}
E\left[\left|\Phi^N(t_1, u_1) - \Phi^N(t_1, u_2)\right|^{2\alpha}\right] \leq C\left|u_1 - u_2\right|^{2\alpha}.
\end{equation}

Let us now deal with the term $E\left[\left|\Phi^N(t_1, u_2) - \Phi^N(t_2, u_2)\right|^{2\alpha}\right]$. We mainly use a similar method to the proof of Lemma 4.12 and first assume that $[N^2 t_1] = [N^2 t_2]$. Then by (4.33) and $\alpha < 1/4$, we have that
\begin{equation}
E\left[\left|\Phi^N(t_1, u_2) - \Phi^N(t_2, u_2)\right|^{2\alpha}\right] \leq C(2\varepsilon^{-1})^{2\alpha}\left[N^{-4\alpha} + \left(\sqrt{N}(\varepsilon^{-1} - 1)\right)^{2\alpha}\right] \leq C\left|t_2 - t_1\right|^{2\alpha}.
\end{equation}
For general $t_1$ and $t_2$, without loss of generality, we may assume that $N^2 t_1 < [N^2 t_1] + 1 \leq [N^2 t_2] \leq N^2 t_2$. Use (4.33) and (4.40), to derive the estimate
\begin{align*}
E\left[\left|\Phi^N(t_1, u_2) - \Phi^N(t_2, u_2)\right|^{2\alpha}\right] & \leq C E\left[\left|\Phi^N(t_1, u_2) - \Phi^N\left(\frac{[N^2 t_1]+1}{N^2}, u_2\right)\right|^{2\alpha}\right] + C E\left[\left|\Phi^N\left(\frac{[N^2 t_1]+1}{N^2}, u_2\right) - \Phi^N(t_2, u_2)\right|^{2\alpha}\right] \\
& \quad + C E\left[\left|\Phi^N\left(\frac{[N^2 t_1]+1}{N^2}, u_2\right) - \Phi^N\left(\frac{[N^2 t_2]+1}{N^2}, u_2\right)\right|^{2\alpha}\right] \leq C\left|t_1 - t_2\right|^{2\alpha}.
\end{align*}

Now we obtain (4.38) by (4.39) and the above estimate. The last part of this lemma is trivial by Kolmogorov’s theorem.

Proposition 4.14. The process $\Phi^N(t, u)$ in Proposition 4.3 is tight in $D([0, T], C(\mathbb{R}))$.

Proof. Recall that $\Phi^N(t, 0) = \Phi^N(0)$ and the Lemmas 4.10 and 4.12 let us easily observe that for each $t \leq T$, $\Phi^N(t, \cdot)$ satisfies the estimates in (1) of Aldous-Kurtz’s conditions.

In the second step we have to show that condition (2) is also satisfied by $\Phi^N(t, u)$. To formulate our proof, we will consider the following metric on $C(\mathbb{R})$:
\begin{equation}
d(w_1, w_2) := \sum_{n \in \mathbb{N}} 2^{-n} \left(1 \wedge \sup_{u \in [-n, n]} |w_1(u) - w_2(u)|\right), \quad w_1, w_2 \in C(\mathbb{R}).
\end{equation}
It is clear that $C(\mathbb{R})$ equipped with the metric $d(\cdot, \cdot)$ is complete and separable. For each $\delta > 0$ define

$$A^N(\delta) := \sup_{t \in [0,T]} d(\Phi^N(t + \delta, \cdot), \Phi^N(t, \cdot)).$$

It is clear that $E[d(\Phi^N(t + \delta, \cdot), \Phi^N(t, \cdot))| \mathcal{F}_t] \leq E[A^N(\delta)| \mathcal{F}_t]$. Thus it is enough to show

$$(4.41) \lim_{\delta \downarrow 0} \limsup_{N \to \infty} E[A^N(\delta)] = 0$$

in order to complete the proof. For any $\delta' > 0$, we have that

$$E\left[ \sup_{t \in [0,T]} \left\{ 1 \wedge \sup_{u \in [-K,K]} | \Phi^N(t + \delta, u) - \Phi^N(t, u)| \right\} \right] \leq P(B^N_K(\delta')) + E\left[ \sup_{t \in [0,T]} \sup_{u \in [-K,K]} | \Phi^N(t, u) - \Phi^N(t, u)|, B^N_K(\delta')^c \right]$$

$$+ E\left[ \sup_{t \in [0,T]} \sup_{u \in [-K,K]} | \Phi^N(t + \delta, u) - \Phi^N(t, u)|, B^N_K(\delta')^c \right] + E\left[ \sup_{t \in [0,T]} \sup_{u \in [-K,K]} | \Phi^N(t + \delta, u) - \Phi^N(t, u)|, B^N_K(\delta')^c \right],$$

where $B^N_K(\delta')$ is defined in Lemma 4.15 below. Then, by Lemma 4.13, we see that

$$E\left[ \sup_{t \in [0,T]} \left\{ 1 \wedge \sup_{u \in [-K,K]} | \Phi^N(t + \delta, u) - \Phi^N(t, u)| \right\} \right] \leq P(B^N_K(\delta)) + \tilde{\delta} + C\sqrt{N}(\varepsilon^{-1} - 1)$$

with $\tilde{\delta} = 2\delta' + C\delta^\alpha$. This implies (4.41) because $\delta'$ and $K$ are arbitrary and $P(B^N_K(\delta)) \to 0$ by Lemma 4.15 (see below).

As a last step let us formulate the lemma, needed in the proof of Proposition 4.14 above. This lemma tells us that the processes $\Phi^N(t, u)$ and $\Phi^N(t, u)$ are uniformly close.

**Lemma 4.15.** For any $\delta' > 0$ and $K \in \mathbb{N}$, consider the following event:

$$B^N_K(\delta') := \left\{ \sup_{t \in [0,T]} \sup_{u \in [-K,K]} | \Phi^N(t, u) - \Phi^N(t, u)| \geq \delta' \right\}.$$

Then we have that $\lim_{N \to \infty} P(B^N_K(\delta')) = 0$.

**Proof.** Set

$$I = \{(k, x) : k = 0, 1, 2, \cdots, [N^2T], x \in \mathbb{Z} \text{ s.t. } \min_{u \in [-K,K]} |Nu - x| \leq 1\}.$$ 

It is easy to see that the number of the elements in $I$ is bounded from above by $CN^3$, that is, $\#I \leq CN^3$. Based on this observation, let us first show that for any $(k, x) \in I$

$$(4.42) E\left[ \sup_{N^2t \leq k+1} \sup_{Nu \in [x,x+1]} | \Phi^N(t, u) - \Phi^N(t, u)|^{2\alpha} \right] \leq CN^{-4\alpha}, \quad \alpha < 1/4,$$
where \( C \) is a generic constant and is independent of \( N, k \) and \( x \). From the definitions of \( \Phi_N(t, u) \) and \( \Phi^N(t, u) \), we easily see that

\[
\left| \Phi^N(t, u) - \Phi^N(t, u) \right| \leq \left| \Phi^N(t, x + 1) - \Phi^N(t, x + 1) \right| + \left| \Phi^N(t, x + 1) - \Phi^N(t, x + 1) \right|
\]

By the definition of \( \Phi^N(t, u) \) and (4.33), for some \( \alpha < 1/4 \), we observe that

\[
E \left[ \sup_{t \in I_N(k)} \left| \Phi^N(t, x + 1) - \Phi^N(t, x + 1) \right|^2 \right] \leq \frac{C}{N^{4\alpha}}.
\]

In the following, to conclude the proof of (4.42), we first show that

\[
E \left[ \sup_{t \in I_N(k)} \left| \Phi^N(t, x + 1) - \Phi^N(t, x + 1) \right|^2 \right] \leq C N^{-\kappa},
\]

where \( I_N(k) = \left[ \frac{k}{N}, \frac{k+1}{N} \right] \). By the definition of \( \Phi^N(t, u) \), it follows that the left side of (4.44) is bounded from above by

\[
CE \left[ \sup_{t \in I_N(k)} \left( \sqrt{N} \left( \frac{\xi_N}{N^2} (x + 1) - \xi_N^N (x + 1) \right) \right)^{2\kappa} + C \sup_{t \in I_N(k)} \left( \sqrt{N} \left( \omega_N^N (x + 1) - \omega_N^N (x + 1) \right) \right)^{2\kappa} \right].
\]

It is known that

\[
\sup_{t \in I_N(k)} \left| \omega_N^N (x + 1) - \omega_N^N (x + 1) \right| \leq N^{-1}.
\]

Hence, to show (4.44), it is enough to prove that there exists a constant \( C \) such that

\[
E \left[ \sup_{t \in I_N(k)} \left( \xi_N^N (x + 1) - \xi_N^N (x + 1) \right)^{2\kappa} \right] \leq C N^{-2\kappa}.
\]

To show this, we will use the martingale approach. For each \( x \in \mathbb{N} \), we have that

\[
\xi_N^N (x) = \xi_N^N (x) + \int_0^t L^N \xi_N^N (x) ds + M^N_t (x),
\]

where

\[
L^N \xi_N^N (x) = N^2 \left( \varepsilon c_+ (x - 1, \eta^N) + c_- (x - 1, \eta^N) \right) \xi_N^N (x) \left( e^{ - \log \varepsilon (\eta^N (x - 1) - \eta^N (x)) } - 1 \right), x \geq 2,
\]

\[
L^N \xi_N^N (1) = N^2 \left( \varepsilon 1_{\{\eta^N (1) = 0\}} + 1_{\{\eta^N (1) = 1\}} \right) \xi_N^N (1) \left( e^{ - \log \varepsilon (1 - 2\eta^N (1)) } - 1 \right).
\]

From the expression of \( L^N \xi_N^N (x) \), it is easy to deduce that there exists a constant \( C \) such that for any \( x \in \mathbb{N} \), \( L^N \xi_N^N (x) \leq C N \xi_N^N (x) \) and thus, it follows that

\[
E \left[ \sup_{t \in I_N(k)} \left| \int_0^t \xi_N^N (x + 1) ds \right|^{2\kappa} \right] \leq C N^{-2\kappa} E \left[ \sup_{t \in I_N(k)} \xi_N^N (1)^{2\kappa} \right].
\]

Since \( a_N \) and \( b_N \) converge to \( \beta^2 \) as \( N \to \infty \), (4.3) yields that for any \( x \in \mathbb{N} \)

\[
d(M^N(x))_{s} \leq C \xi_N^N (x)^2 ds,
\]

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Thus, we easily see that $\mathbb{R}$ above by where $E$

Therefore, by Lemma 4.7, we can show (4.45). A similar argument yields

So, by (4.43)-(4.46), we can complete the proof of (4.42).

Finally, the proof can be concluded by (4.42) and Chebyshev’s inequality. In fact,

which implies the result by taking $\kappa > \frac{3}{4\alpha}$ and then letting $N \to \infty$.

4.2.3 Derivation of the SPDE (4.12)

Taking a test function $g \in C^2_0(\mathbb{R})$ and by (4.8) and the definition of $\Phi^N(t, u)$, we arrive at

where $\langle \Phi^N(t, \cdot), g \rangle = \int_{\mathbb{R}} \Phi(t, u)g(u)du$,

and $R^N_t$ is an error term, i.e. $R^N_t = \langle \Phi^N(t, \cdot), g \rangle - \frac{1}{N} \sum_{x \in \mathbb{Z}} \Phi^N_t(x)g(\frac{x}{N})$.

We first deal with the error term $R^N_t$. It is easy to show that $R^N_t$ is bounded from above by

Thus, we easily see that $R^N_t$ converges to 0 in $L^{2\kappa}(\Omega)$. In fact, by Lemma 4.11, we have

$$E\left[ \left( \int_{\mathbb{R}} |\langle \Phi^N_t([Nu]) - \Phi^N_t([Nu] + 1) \rangle g(u) \rangle|du \right)^{2\kappa} \right] \leq C N^{-2\kappa}(\|g\|_{\infty}|\text{supp}(g)|)^{2\kappa},$$

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which goes to 0 as $N \to \infty$, and for the second term, we can apply Lemma 4.10.

Use (4.3) for the martingale term $\sqrt{N} \bar{M}_t^N(g)$ and observe that

$$
\frac{d}{dt}(\sqrt{N} \bar{M}_t^N(g))_t = \frac{1}{N} \zeta_t^N(1)^2 e^{-(\log \varepsilon)} \left(a_N 1_{\{t\}^N(1)=0} + b_N 1_{\{t\}^N(1)=1}\right) g\left(\frac{x}{N}\right)
$$

$$
+ \frac{1}{N} \sum_{x=2}^{\infty} \zeta_t^N(x)^2 e^{-(\log \varepsilon)x} \left(a_N c_+(x-1, \eta_N^x) + b_N c_-(x-1, \eta_N^x)\right) \times \left(g\left(\frac{x}{N}\right) + g\left(-\frac{x}{N}\right)\right)^2,
$$

which converges as $N \to \infty$ to

$$
\beta^2 \int_{\mathbb{R}_+} e^{\beta u} \omega(t,u)^2 \cdot 2\rho_R(t,u)(1 - \rho_R(t,u))(g(u) + g(-u))^2 du,
$$

by the hydrodynamic limit, i.e., by Corollary 5.3 of [9].

Now let us state the following lemma, which concludes the proof of Proposition 4.3.

**Lemma 4.16.** There exists a $Q$-cylindrical Brownian motion $\bar{W}$ with the covariance determined by (4.13) such that the weak limit of $\sqrt{N} \bar{M}_t^N(g)$ as $N \to \infty$ has the same law as that of the process

$$
\beta \int_0^t \int_{\mathbb{R}} e^{\beta u} \omega(s,u)^2 \sqrt{2\rho_R(s,u)(1 - \rho_R(s,u))} g(u) \bar{W}(dsdu).
$$

Therefore, the limit of $\bar{\Phi}^N(t,u)$ is characterized by the SPDE (4.12).

**Proof.** Let us consider

$$
\mathbf{M}_t^N(g) = \frac{1}{N} \sum_{x \in \mathbb{Z}} \bar{\Phi}_1^N(x) g\left(\frac{x}{N}\right) - \frac{1}{N} \sum_{x \in \mathbb{Z}} \bar{\Phi}_0^N(x) g\left(\frac{x}{N}\right) - \int_0^t b^N(\bar{\Phi}_s^N, g) ds,
$$

$$
\bar{\Phi}_t^N(g) = (\mathbf{M}_t^N(g))^2 - \langle \sqrt{N} \bar{M}_t^N(g) \rangle_t.
$$

Here, $\mathbf{M}_t^N(g)$ is nothing but $\sqrt{N} \bar{M}_t^N(g)$ appeared in (4.47). However, to make the explanation of the proof clear, we introduce this notation. From the definition of $\bar{\Phi}_t^N(x)$, we know that both of the above processes are martingales. Let $\mathcal{P}$ be a limit point of the sequence $\mathcal{P}^N$, the distribution of $\bar{\Phi}_t^N(t, \cdot)$ on $D([0, T], \mathcal{C}(\mathbb{R}))$. Then, it is clear that $\mathcal{P}$ is concentrated on $C([0, T], \mathcal{C}(\mathbb{R}))$ from Lemma 4.13. In the following, with some abuse of notations, we will use $\Phi(t)$ to denote the canonical coordinate process on $C([0, T], \mathcal{C}(\mathbb{R}))$. Assume $F$ denotes an arbitrary $D([0, s], \mathcal{C}(\mathbb{R}))$-measurable function defined on $D([0, T], \mathcal{C}(\mathbb{R}))$ with continuous and bounded restriction on $C([0, T], \mathcal{C}(\mathbb{R}))$. From the explanations at the beginning of this subsection, for $0 \leq s < t \leq T$, letting $N \to \infty$, we can show

$$
E^\mathcal{P}[\mathbf{M}_t(g) - \bar{\Phi}_s(g)] = \lim_{N \to \infty} E^{\mathcal{P}^N}[\mathbf{M}_t^N(g) - \bar{\Phi}_s^N(g)] = 0,
$$

$$
E^\mathcal{P}[(\mathbf{M}_t(g) - \bar{\Phi}_t(g))^2] = \lim_{N \to \infty} E^{\mathcal{P}^N}[(\mathbf{M}_t^N(g) - \bar{\Phi}_t^N(g))^2] = 0,
$$

where $E^\mathcal{P}$ denotes the expectation with respect to $\mathcal{P}$,

$$
\begin{equation}
\mathbf{M}_t(g) := \langle \Phi(t), g \rangle - \langle \Phi(0), g \rangle - \int_0^t \langle \Phi(s), g'' - \frac{\beta^2}{2} g \rangle ds,
\end{equation}
$$

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(4.49) \( M(t) := (M_t(g))^2 - \int_0^t \int_{\mathbb{R}} \psi^2(s, u)g(u) (g(u) + g(-u))du ds \),

and \( \psi(t, u) = \beta \omega(t, |u|) e^{\beta |u|^2/2} \cdot \sqrt{2\rho_R(t, |u|)} (1 - \rho_R(t, |u|)) \) in this part. Therefore, we deduce that both of the processes \( M_t(g) \) and \( \bar{M}_t(g) \) defined by (4.48) and (4.49), respectively, are \( \mathcal{P} \)-martingales.

Using a similar way to [1], we call that a probability measure \( \mathcal{P} \) on \( C([0, T], \mathbb{R}) \) is a martingale solution of (4.12) if the law of \( \Phi(0) \) under \( \mathcal{P} \) coincides with the law of \( \Phi_0 \) under \( \mathcal{P} \) and for any test function \( g \), \( \mathcal{M}_t(g) \) and \( \mathcal{M}_t(g) \) are \( \mathcal{P} \)-local martingales. We refer to [8] for another approach to study martingale problems for SPDEs.

In the following, we will show that the martingale solution of (4.12) is equivalent to its weak solution. To show this, we associate a martingale measure \( \mathcal{M}(t, A) \) on \([0, T] \times \mathbb{R} \) to \( M_t(g) \). In other words, we will assume that \( \mathcal{M}(t, A) \) is a continuous worthy martingale measure, see [19], with quadratic variational process

\[
\langle M \rangle (dt du) = \psi^2(t, u)dv dv,
\]

where \( \nu(A) = \mu(A) + | - A | \) for any Borel subset of \( \mathbb{R} \) and \( A := \{-x : x \in A \} \). Let us consider a \( Q \)-cylindrical Brownian motion \( \mathcal{W} \) with covariance defined by (4.13) such that it is independent of \( \mathcal{P} \). We remark that this can be realized by extending the probability space and the corresponding filtration. However, for the brevity of notation, we will still use \( \mathcal{P} \) to denote the extended probability measure. Now set

\[
W_t(g) = \int_0^t \int_{\mathbb{R}} \frac{g(u)}{\psi(s, u)} 1_{\{ \psi(s, u) \neq 0 \}} M(ds du) + \int_0^t \int_{\mathbb{R}} 1_{\{ \psi(s, u) = 0 \}} g(u) \mathcal{W}(ds du).
\]

From the symmetry of \( \psi(t, u) \) in \( u \) and the independence of \( \mathcal{M} \) and \( \mathcal{W} \), we see that

\[
E^\mathcal{P} [W_t^2(g)] = \int_0^t \int_{\mathbb{R}} g(u) (g(u) + g(-u)) du ds.
\]

Therefore, by Lévy’s martingale characterization theorem, we know that \( W_t \) is a \( Q \)-cylindrical Brownian motion with covariance characterized by (4.13) and

\[
\mathcal{M}_t(g) = \int_0^t \int_{\mathbb{R}} \psi(s, u) g(u) \mathcal{W}(ds du).
\]

In fact, by the definition of \( W_t \), see (4.50), we have that

\[
\int_0^t \int_{\mathbb{R}} \psi(s, u) g(u) \mathcal{W}(ds du) = \int_0^t \int_{\mathbb{R}} \psi(s, u) \frac{g(u)}{\psi(s, u)} 1_{\{ \psi(s, u) \neq 0 \}} M(ds du) + \int_0^t \int_{\mathbb{R}} 1_{\{ \psi(s, u) = 0 \}} \psi(s, u) g(u) \mathcal{W}(ds du) = \int_0^t \int_{\mathbb{R}} g(u) \mathcal{M}(ds du).
\]

Combining this with (4.48), we obtain that

\[
\langle \Phi(t), g \rangle = \langle \Phi(0), g \rangle + \int_0^t \langle \Phi(s), g'' - \frac{\beta^2}{4} g \rangle ds + \int_0^t \int_{\mathbb{R}} \psi(s, u) g(u) \mathcal{W}(ds du),
\]

which means that the martingale solution satisfies (4.12) in its weak sense with the \( Q \)-cylindrical Wiener process \( \mathcal{W}(t) \) constructed by (4.50) by the arbitrariness of \( g \). In the end, we remark that the martingale problem is well-posed, that is, the uniqueness holds, which is clear from the uniqueness of the weak solution. \( \square \)
5 Invariant Measures of the SPDEs

To compare our dynamic fluctuation results with the static fluctuations formulated in Proposition 5.1 below, we explicitly compute the invariant measures of the SPDEs (2.4) and (2.6).

5.1 Static Fluctuations

First, we state a result for the fluctuations under grandcanonical ensembles $\mu^{\epsilon(N)}_U$ and $\mu^{\epsilon(N)}_R$, which is in fact simpler than those under canonical ensembles, see [16], [7], [20]. Let $\psi_U$ and $\psi_R$ be the height functions of the Vershik curves:

$$
\psi_U(u) = -\frac{1}{\alpha} \log (1 - e^{-\alpha u}), \quad u \in \mathbb{R}_+^0,
$$

$$
\psi_R(u) = \frac{1}{\beta} \log (1 + e^{-\beta u}), \quad u \in \mathbb{R}_+.
$$

Then, for the static fluctuations $\Psi^N_U(u)$ and $\Psi^N_R(u)$ defined by

$$
\Psi^N_U(u) := \sqrt{N} (\psi^N(u) - \psi_U(u)), \quad u \in \mathbb{R}_+^0,
$$

$$
\Psi^N_R(u) := \sqrt{N} (\psi^N(u) - \psi_R(u)), \quad u \in \mathbb{R}_+,
$$

we have the following proposition.

**Proposition 5.1.** The fluctuation fields $\Psi^N_U(u)$ and $\Psi^N_R(u)$ weakly converge to $\Psi_U(u)$ and $\Psi_R(u)$ under $\mu^{\epsilon(N)}_U$ and $\mu^{\epsilon(N)}_R$, respectively, as $N \to \infty$, where $\Psi_U, \Psi_R$ are mean 0 Gaussian processes with covariance structures

$$
C_U(u, v) = \frac{1}{\alpha} \rho_U(u \lor v), \quad u, v \in \mathbb{R}_+^0
$$

$$
C_R(u, v) = \frac{1}{\beta} \rho_R(u \lor v), \quad u, v \in \mathbb{R}_+,
$$

and $\rho_U = -\psi'_U(= \rho^{\infty}_U$ in (3.17)), $\rho_R = -\psi'_R$ are slopes of the Vershik curves, respectively, with $u \lor v = \max\{u, v\}$.

**Proof.** The proof is not difficult by noting the following facts. Under $\mu^{\epsilon(N)}_U$, the height differences $\xi(x)(= \psi(x - 1) - \psi(x)$ or $\#\{i; p_i = x\}, x \in \mathbb{N}$, are independent random variables, which are geometrically distributed: $\mu^{\epsilon(N)}_U(\xi(x) = k) = a^k/(1 - a)$ for $k \in \mathbb{Z}_+$ with $a = \epsilon^x$. On the other hand, under $\mu^{\epsilon(N)}_R$, the height differences $\eta(x), x \in \mathbb{N}$, are independent and distributed as $\mu^{\epsilon(N)}_R(\eta(x) = k) = a^k/(1 + a)$ for $k = 0, 1$ with $a = \epsilon^x$. \hfill \square

**Remark 5.1.** (1) As shown in [20], [7], the CLT under canonical ensembles can be reduced from that under grandcanonical ensembles by removing the effect of fluctuations of area.

(2) The Gaussian process $\Psi_R$ satisfies $\Psi_R \in L^2_r(\mathbb{R}_+)$ a.s. for every $r > -\beta/2$ ($L^2_r$ is defined also for $r < 0$), since

$$
E[|\Psi_R|_{L^2_r(\mathbb{R}_+)}^2] = \int_0^\infty E[\Psi_R(u)^2] e^{-2ru} du = \frac{1}{\beta} \int_0^\infty \rho_R(u) e^{-2ru} du
$$

is finite if and only if $2r + \beta > 0$. 

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5.2 Uniform Case

Let $Q_U$ be the differential operator

$$Q_U = -\frac{\partial}{\partial u} \left\{ \frac{1}{\rho_U(u)(1+\rho_U(u))} \frac{\partial}{\partial u} \right\}$$

defined on $L^2(\mathbb{R}_+^0, du)$. Note that this operator does not require any boundary condition, see Remark 2.1.

**Theorem 5.2.** The Gaussian measure $N(0,Q_U^{-1})$ is the unique invariant measure of the SPDE $(2.4)$, which appeared in Theorem 2.2.

**Proof.** Since $\rho(t,u)$ in the SPDE $(2.4)$ converges as $t \to \infty$ to $\rho_U(u)$, we may study the invariant measure of the SPDE:

\[ \partial_t \Psi(t,u) = A_U \Psi(t,u) + \sqrt{2g_U(u)}W(t,u), \]

where

\[ A_U \Psi(u) := \left( \frac{\Psi'(u)}{(1+\rho_U(u))^2} \right)' + \alpha \frac{\Psi'(u)}{(1+\rho_U(u))^2} \quad \text{and} \quad g_U(u) = \frac{\rho_U(u)}{1+\rho_U(u)}. \]

Note that one can rewrite the operator $A_U$ as

\[ A_U \Psi(u) = -g_U(u)Q_U \Psi(u). \]

In particular, $A_U$ is symmetric in the space $L^2_U := L^2(\mathbb{R}_+^0,1/g_U(u)du)$. Let $e^{tA_U}$ be the semigroup generated by $A_U$ on $L^2_U$. Then, the solution of the SPDE (5.1) can be written in the mild form:

\[ \Psi_t = e^{tA_U} \Psi_0 + \int_0^t e^{(t-s)A_U} \sqrt{2g_U}dW_s. \]

In particular, for every $\psi \in L^2_U$, we have

\[ \langle \Psi_t, \psi \rangle_{L^2_U} = \langle e^{tA_U} \Psi_0, \psi \rangle_{L^2_U} + \int_0^t \langle dW_s, \frac{1}{g_U} \sqrt{2g_U}e^{(t-s)A_U} \psi \rangle_{L^2_U} = : m_t + I_t. \]

However, since $A_U$ on $L^2_U$ is unitary equivalent to $-Q_U$ on $L^2(\mathbb{R}_+^0)$, Lemma 5.3 below implies $A_U \leq -c$ with $c > 0$ and therefore $m_t \to 0$ as $t \to \infty$, while

\[ E[I_t^2] = \int_0^t \| \sqrt{\frac{2}{g_U}}e^{(t-s)A_U} \psi \|_{L^2_U}^2 ds = 2 \int_0^t \| e^{sA_U} \psi \|_{L^2_U}^2 ds \]

\[ = 2 \int_0^t \langle e^{2sA_U} \psi, \psi \rangle_{L^2_U} ds \to 2 \langle (-2A_U)^{-1} \psi, \psi \rangle_{L^2_U} = \langle (-A_U)^{-1} \psi, \psi \rangle_{L^2_U} \]

as $t \to \infty$. This proves that $\langle \Psi_t, \psi \rangle_{L^2_U}$ converges weakly to $N(0,\langle (-A_U)^{-1} \psi, \psi \rangle_{L^2_U})$ for every $\psi \in L^2_U$, which is an equivalent formulation to $\langle \Psi_t, \varphi \rangle_{L^2}$ converging weakly to $N(0,\langle (-A_U)^{-1} \varphi g_U, \varphi \rangle_{L^2})$ by taking $\varphi = \psi/g_U$. However, $(-A_U)^{-1} \varphi g_U = Q_U^{-1} \varphi$ and this implies the conclusion.

\[ \square \]
Remark 5.2. Since $C_U(u,v)$ is the Green kernel of $Q_U^{-1}$, this gives another proof of the static result in U-case.

Lemma 5.3. (Poincaré inequality; U-case) There exists $c > 0$ such that $(f, Q_U f) \geq c \|f\|^2$ holds for every $f \in C^1(\mathbb{R}_+^\circ) \cap L^2(\mathbb{R}_+^\circ, du)$, where the inner product and the norm are those of the space $L^2(\mathbb{R}_+^\circ, du)$.

Proof. We divide $\|f\|^2$ into a sum of integrals over $(1, \infty)$ and $(0, 1]$, and estimate them separately. We begin with the integral over $(1, \infty)$. Set

$$a_U(u) = \{u \rho_U(u)(1 + \rho_U(u))\}^{-1}, \quad u \in \mathbb{R}_+.$$  

Note that $a_U(u) > 0$ and $C = \int_1^\infty a_U(u)^{-1} du < \infty$ and by Schwarz’s inequality, we have for every $f \in C_0^1(\mathbb{R}_+^\circ)$ that

$$\int_1^\infty f^2(u) du = \int_1^\infty \left( \int_u^\infty f'(v) dv \right)^2 du \leq C \int_1^\infty du \int_u^\infty f'(v)^2 a_U(v) dv$$

$$\leq C \int_1^\infty f'(v)^2 v a_U(v) dv = C \int_1^\infty \frac{f'(u)^2}{\rho_U(u)(1 + \rho_U(u))} du.$$  

Next, we study the integral over $(0, 1]$. By Schwarz’s inequality,

$$\int_0^1 f^2(u) du = \int_0^1 \left( f(u) - f(1) \right)^2 du \leq 2 \int_0^1 (f(u) - f(1))^2 du + 2f(1)^2.$$  

We estimate two terms in the last expression separately. The first term is estimated by Schwarz’s inequality again as

$$f(1)^2 = \left( \int_1^\infty f'(u) du \right)^2 \leq \left( \int_1^\infty \frac{f'(u)^2}{\rho_U(u)(1 + \rho_U(u))} du \right) \left( \int_1^\infty \rho_U(u)(1 + \rho_U(u)) du \right)$$

$$\leq C \int_1^\infty \frac{f'(u)^2}{\rho_U(u)(1 + \rho_U(u))} du.$$  

To bound the remaining term, we need more detailed estimates. First, we obtain the following bound

$$\int_0^1 (f(u) - f(1))^2 du = \int_0^1 \left( \int_u^1 f'(v) dv \right)^2 du \leq \int_0^1 \left( \int_u^1 f'(v)^2 v^{\frac{1}{2}} dv \right) \left( \int_u^1 v^{-\frac{1}{2}} dv \right) du$$

$$\leq 2 \int_0^1 u^{-\frac{1}{2}} \left( \int_u^1 f'(v)^2 v^{\frac{1}{2}} dv \right) du = 2 \int_0^1 f'(v)^2 v^{\frac{1}{2}} \left( \int_0^v u^{-\frac{1}{2}} du \right) dv = 4 \int_0^1 f'(v)^2 v dv.$$  

Inserting the relation $\{\rho_U(u)(1 + \rho_U(u))\}^{-1} = (e^{\alpha u} - 1)^2 e^{-\alpha u} \geq \alpha^2 e^{-\alpha u} u^2, u \in [0, 1]$, into the last term of the above inequality, we have that

$$\int_0^1 (f(u) - f(1))^2 du \leq 4\alpha^2 e^\alpha \int_0^1 \frac{f'(u)^2}{\rho_U(u)(1 + \rho_U(u))} du.$$  

Combining inequalities obtained up to this point, we conclude that

$$\int_0^\infty f^2(u) du \leq \tilde{C} \int_0^\infty \frac{f'(u)^2}{\rho_U(u)(1 + \rho_U(u))} du = \tilde{C} (f, Q_U f)$$

where $\tilde{C} = 3C + 8\alpha^2 e^\alpha$. The last equality follows by integration by parts with $f \in C_0^1(\mathbb{R}_+^\circ)$ in mind. One can extend the class of functions $f$. \qed

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Remark 5.3. It is also possible to obtain the invariant measure for the U-case from the one for the RU-case in Theorem 5.4 by using the transformation used in Section 3.

5.3 Restricted Uniform Case

Let $Q_R$ be the differential operator

$$Q_R = -\frac{\partial}{\partial u} \left\{ \frac{1}{\rho_R(u)(1 - \rho_R(u))} \frac{\partial}{\partial u} \right\}$$

defined on $L^2(\mathbb{R}_+, du)$ with the Neumann boundary condition at $u = 0$.

**Theorem 5.4.** The Gaussian measure $N(0, Q_R^{-1})$ is the unique invariant measure of the SPDE (2.6), which appeared in Theorem 2.3.

**Proof.** Since $\rho(t, u)$ in the SPDE (2.6) converges as $t \to \infty$ to $\rho_R(u)$, we may study the invariant measure of the SPDE:

$$(5.2) \begin{cases} \partial_t \Psi(t, u) = A_R \Psi(t, u) + \sqrt{2g_R(u)} \dot{W}(t, u), \\ \Psi(t, 0) = 0, \end{cases}$$

where

$$A_R \Psi(u) := \Psi''(u) + \beta(1 - 2\rho_R(u))\Psi'(u) \quad \text{and} \quad g_R(u) = \rho_R(u)(1 - \rho_R(u)).$$

Note that one can rewrite the operator $A_R$ as

$$A_R \Psi(u) = -g_R(u)Q_R \Psi(u).$$

In particular, $A_R$ is symmetric in the space $L^2_R := L^2(\mathbb{R}_+, 1/g_R(u)du)$. Let $e^{tA_R}$ be the semigroup generated by $A_R$ on $L^2_R$. Then, the solution of the SPDE (5.2) can be written in the mild form:

$$\Psi_t = e^{tA_R} \Psi_0 + \int_0^t e^{(t-s)A_R} \sqrt{2g_R} \dot{W}_s.$$ 

In particular, for every $\psi \in L^2_R$, we have

$$\langle \Psi_t, \psi \rangle_{L^2_R} = \langle e^{tA_R} \Psi_0, \psi \rangle_{L^2_R} + \int_0^t \langle dW_s, \frac{1}{g_R} \sqrt{2g_R} e^{(t-s)A_R} \psi \rangle_{L^2_R} =: m_t + I_t.$$ 

However, since $A_R \leq -c$ from Lemma 5.5 below, $m_t \to 0$ as $t \to \infty$, while

$$E \left[ t^2 \right] = \int_0^t \left\| \frac{1}{g_R} e^{(t-s)A_R} \psi \right\|_{L^2}^2 ds = 2 \int_0^t \left\| e^{sA_R} \psi \right\|_{L^2}^2 ds = 2 \int_0^t \left\| e^{sA_R} \psi \right\|_{L^2}^2 ds \to 2 \langle (-2A_R)^{-1} \psi, \psi \rangle_{L^2_R} = \langle (-A_R)^{-1} \psi, \psi \rangle_{L^2_R}$$

as $t \to \infty$. This proves that $\langle \Psi_t, \psi \rangle_{L^2_R}$ converges weakly to $N(0, \langle (-A_R)^{-1} \psi, \psi \rangle_{L^2})$ for every $\psi \in L^2_R$, which is an equivalent formulation of $\langle \Psi_t, \varphi \rangle_{L^2}$ converging weakly to $N(0, \langle (-A_R)^{-1} \psi, \varphi \rangle_{L^2})$ by taking $\varphi = \psi/g_R$. However, $(-A_R)^{-1}(\varphi g_R) = Q_R^{-1}\varphi$ and this implies the conclusion. \qed
Remark 5.4. Since $C_R(u, v)$ is the Green kernel of $Q_R^{-1}$, this gives another proof of static result in RU-case.

Lemma 5.5. (Poincaré inequality; RU-case) There exists $c > 0$ such that $(f, Q_R f) \geq c \|f\|^2$ holds for every $f \in C^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+, du)$ satisfying $f'(0) = 0$, where the inner product and the norm are those of the space $L^2(\mathbb{R}_+, du)$.

Proof. Set

$$a_R(u) = \{u \rho_R(u)(1 - \rho_R(u))\}^{-1}, \quad u \in \mathbb{R}_+.$$  

Note that $a_R(u) > 0$ and $C = \int_0^\infty a_R(u)^{-1} du < \infty$ and by Schwarz’s inequality, we have for every $f \in C^1_0(\mathbb{R}_+)$ that

$$\int_0^\infty f^2(u) du = \int_0^\infty \left( \int_u^\infty f'(v) dv \right)^2 du \leq C \int_0^\infty du \int_u^\infty f'(v)^2 a_R(v) dv \int_0^\infty f'(v)^2 v a_R(v) dv = C(f, Q_R f) .$$

The last equality follows by integration by parts with $f'(0) = 0$ in mind. One can extend the class of functions $f$. □

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