CROSSED HOMOMORPHISMS AND CARTIER-KOSTANT-MILNOR-MOORE THEOREM FOR DIFFERENCE HOPF ALGEBRAS

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ABSTRACT. The celebrated Milnor-Moore theorem and the more general Cartier-Kostant-Milnor-Moore theorem establish close relationships of a connected and a pointed cocommutative Hopf algebra with its Lie algebra of primitive elements and its group of group-like elements. Crossed homomorphisms for Lie algebras, groups and Hopf algebras have been studied extensively, first from a cohomological perspective and then more broadly, with an important case given by difference operators. This paper shows that the relationship among the different algebraic structures captured in the Milnor-Moore theorem can be strengthened to include crossed homomorphisms and difference operators. We give a graph characterization of Hopf algebra crossed homomorphisms which are also compatible with the Milnor-Moore relation. We further investigate derived actions from crossed homomorphisms on groups, Lie algebras and Hopf algebras, and establish their relationship. Finally we obtain a Cartier-Kostant-Milnor-Moore type structure theorem for pointed cocommutative difference Hopf algebras. Examples and classifications of difference operators are also provided for several Hopf algebras.

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1. Introduction

This paper studies the compatibility of crossed homomorphisms and in particular difference operators on Lie algebras, groups and Hopf algebras, in the contexts of the Milnor-Moore theorem and the Cartier-Kostant-Milnor-Moore theorem.

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Lie algebras, (Lie) groups and Hopf algebras are fundamental algebraic structures. Their studies have greatly benefited from their close interconnections. Most notably, the celebrated Milnor-Moore theorem \cite{22} provides a Hopf algebra isomorphism from a connected cocommutative Hopf algebra over a field of characteristic zero to the universal enveloping algebra of its Lie algebra of primitive elements, and the Cartier-Kostant-Milnor-Moore theorem \cite{3,6,11,15,21,31} determines a pointed cocommutative Hopf algebra over a field of characteristic zero by its Lie algebra of primitive elements and its group of group-like elements. Various generalizations of the theorems have been found over the years, including those for certain non-cocommutative Hopf algebras, generalized bialgebras, brace algebras, braided bialgebras and Hopf algebroids \cite{2,4,11,15,21,31}. Recently the higher homotopy algebra version of the Milnor-Moore theorem is also obtained in \cite{24}.

A common structure on these various algebraic structures is the crossed homomorphisms. Crossed homomorphisms on groups already appeared in Whitehead’s work \cite{42} in 1950 and were later applied to study non-abelian Galois cohomology \cite{34} and Banach modules of locally compact groups \cite{5}. The post-Lie Magnus expansion can be interpreted as a crossed homomorphism on (local) Lie groups \cite{20}. The concept of crossed homomorphisms on Lie algebras was introduced in \cite{18} in the study of non-abelian extensions of Lie algebras. Recently crossed homomorphisms on Lie algebras were used to construct actions of monoidal categories in the study of representations of Cartan type Lie algebras \cite{28}. By differentiation, crossed homomorphisms on Lie groups give rise to crossed homomorphisms on the corresponding Lie algebras.

Crossed homomorphisms on Hopf algebras \cite{36} can be interpreted as 1-cocycles of the Sweedler cohomology of cocommutative Hopf algebras with coefficients in commutative module algebras. In the study of non-abelian Hopf cohomology \cite{25,26,27}, 1-descent cocycles of a Hopf algebra $H$ with coefficients in a relative Hopf module $M$ serve the purpose of crossed homomorphisms on Hopf algebras. Among the recent studies, bijective Hopf algebra crossed homomorphisms were applied to construct solutions of the quantum Yang-Baxter equation \cite{1}, originated from the works of Etingof-Schedler-Soloviev and Lu-Yan-Zhu on the set-theoretical solutions of the quantum Yang-Baxter equation \cite{7,17}. Bijective crossed homomorphisms were also used to study Hopf-Galois structures on Galois field extensions \cite{38}.

Crossed homomorphisms with respect to the adjoint representation are in fact difference operators (also called differential operators of weight 1), abstracted from original instance in numerical analysis to algebraic settings of associative and Lie algebras \cite{9,14,37}. Differing by an affine transformation, another notion of difference algebra is also extensively studied in close connection with differential algebras \cite{12,13,20,11}. Recently, difference operators on groups were studied in \cite{10} as the inverse of Rota-Baxter operators on groups introduced there with motivation from integrable systems \cite{30,33}. The differentiation of a difference operator on a Lie group gives a difference operator on the corresponding Lie algebra.

The extensive study of crossed homomorphisms and difference operators on groups, Lie algebras and Hopf algebras naturally led us to investigate their interactions across the different algebraic structures. In doing so, we obtain a Milnor-Moore theorem for crossed homomorphisms on connected cocommutative Hopf algebras. More precisely, we show that a crossed
homomorphism on a connected cocommutative Hopf algebra with respect to a module bialgebra structure on another connected cocommutative Hopf algebra is isomorphic to the universal enveloping of the induced crossed homomorphism on the Lie algebra of primitive elements. Furthermore, we show that crossed homomorphisms can be characterized by their graphs and derived structures. Considering these properties on different algebraic structures give rise to other refinements of the Milnor-Moore theorem. We also give a Cartier-Kostant-Milnor-Moore theorem when difference operators are applied. More generally, we introduce the notion of a difference module bialgebra over a difference Hopf algebra, and construct the corresponding smash product Hopf algebra.

Overall, the organization of the paper is as follows.

In Section 2, we recall the notions of crossed homomorphisms and difference operators on Lie algebras, groups and Hopf algebras. We then show that crossed homomorphisms are compatible with the Milnor-Moore theorem for connected cocommutative Hopf algebras (Theorem 2.8), leading to a structure theorem of connected cocommutative difference Hopf algebras (Corollary 2.9).

In Section 3, we characterize crossed homomorphisms on Lie algebras, groups and Hopf algebras in terms of their graphs. We then show that taking the universal enveloping algebra of the graph of a Lie algebra crossed homomorphism give the graph of corresponding Hopf algebra crossed homomorphism (Theorem 3.4).

It is known that crossed homomorphisms on Lie algebras induce new Lie algebra actions. In Section 4, we show that this can also be done for crossed homomorphisms on (Lie) groups and Hopf algebras. Furthermore, the differentiation from a Lie group to its Lie algebra induces the differentiation from the derived actions on the Lie group to the derived actions on the corresponding Lie algebra (Theorem 4.2). In addition, there is a derived action of a Hopf algebra which restricts to derived actions of the Lie algebra of primitive elements and of the group of group-like elements (Theorem 4.5 and Corollary 4.6).

In Section 5, we first extend the notion of an $H$-module bialgebra from when $H$ is a bialgebra to when $H$ is a difference bialgebra. We then utilize this notion to obtain smash products of difference Hopf algebras. We prove the Cartier-Kostant-Milnor-Moore theorem for pointed cocommutative difference Hopf algebras (Theorem 5.10).

Finally in Section 6, we give examples and classifications of difference operators on some Hopf algebras, including the tensor Hopf algebra, Sweedler’s 4-dimensional Hopf algebra $H_4$ and the Kac-Paljutkin Hopf algebra $H_8$.

Conventions. In this paper, we fix a ground field $k$ of characteristic 0. All the objects under discussion, including vector spaces, algebras and tensor products, are taken over $k$ unless otherwise specified. We will use roman letters such as $G, H$ for associative algebras and Hopf algebras, Fraktur letters such as $g, h$ for Lie algebras, and calligraphic letters such as $G, H$ for groups. For a coalgebra $(C, \Delta, \varepsilon)$, we abbreviate the Sweedler notation of the comultiplication $\Delta$ to

$$\Delta(x) = x_1 \otimes x_2.$$

More generally, for $n \geq 1$ we write

$$\Delta^{(n)}(x) = (\Delta \otimes \text{id}^{\otimes (n-1)}) \cdots (\Delta \otimes \text{id})\Delta(x) = x_1 \otimes \cdots \otimes x_{n+1}.$$
We follow [23, 29] for other basic notions of Hopf algebras.

2. Milnor-Moore theorem for Hopf algebra crossed homomorphisms and difference Hopf algebras

The celebrated Milnor-Moore theorem establishes a close relationship between a connected cocommutative Hopf algebra and its Lie algebra of primitive elements. In this section, we extend this relationship to crossed homomorphisms on connected cocommutative Hopf algebras and Lie algebras. As a special case, we obtain an enriched Milnor-Moore theorem for connected cocommutative difference Hopf algebras.

2.1. Crossed homomorphisms and difference operators. The notion of crossed homomorphisms, often called 1-cocycles, have been defined in the context of non-abelian cohomology for various algebraic structures, including Lie algebras, groups and Hopf algebras [18, 54, 55]. An important special case is the difference operators in the corresponding categories [3, 10].

Definition 2.1. (i) Let $\phi : \mathfrak{g} \to \text{Der}(\mathfrak{h})$ be an action of a Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ on a Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_\mathfrak{h})$. A linear map $d : \mathfrak{g} \to \mathfrak{h}$ is called a crossed homomorphism on $\mathfrak{g}$ with respect to the action $(\mathfrak{h}, \phi)$ if

\[
d[x, y]_\mathfrak{g} = \phi(x)(d(y)) - \phi(y)(d(x)) + [d(x), d(y)]_\mathfrak{h}, \quad \forall x, y \in \mathfrak{g}.
\]

(ii) Let $\Phi : \mathcal{G} \to \text{Aut}(\mathcal{H})$ be an action of a group $\mathcal{G}$ on a group $\mathcal{H}$. A set map $D : \mathcal{G} \to \mathcal{H}$ is called a crossed homomorphism on $\mathcal{G}$ with respect to the action $(\mathcal{H}, \Phi)$ if

\[
D(gh) = D(g)\Phi(g)(D(h)), \quad \forall g, h \in \mathcal{G}.
\]

(iii) Let $H$ and $K$ be Hopf algebras such that $K$ is an $H$-module algebra in the sense that there is an algebra action $\cdot : H \to \text{End}(K)$ such that

\[
a \rightarrow 1 = \varepsilon(a)1, \quad a \rightarrow (xy) = (a_1 \rightarrow x)(a_2 \rightarrow y), \quad \forall a \in H, x, y \in K.
\]

A coalgebra homomorphism $\pi : H \to K$ is called a crossed homomorphism on $H$ with respect to the action $(K, \rightarrow)$ if

\[
\pi(ab) = \pi(a_1)(a_2 \rightarrow \pi(b)), \quad \forall a, b \in H.
\]

A homomorphism between crossed homomorphisms on Lie algebras, groups or Hopf algebras is defined as expected. For Hopf algebras, such a homomorphism from $\pi : H \to K$ to $\pi' : H' \to K'$ consists of a pair of Hopf algebra homomorphisms $f : K \to K'$ and $g : H \to H'$ such that

\[
f \pi = \pi' g, \quad f(a \rightarrow x) = g(a) \rightarrow f(x), \quad \forall x \in K, a \in H.
\]

When a crossed homomorphism on either a Lie algebra, a group or a Hopf algebra is defined with respect to the adjoint action, then the crossed homomorphism is called a difference operator on the corresponding structure [3, 10, 13]. More precisely,

(i) a difference operator on a Lie algebra $\mathfrak{g}$ is a linear operator $d$ on $\mathfrak{g}$ such that

\[
d([x, y])_\mathfrak{g} = [d(x), y]_\mathfrak{g} + [x, d(y)]_\mathfrak{g} + [d(x), d(y)]_\mathfrak{g}, \quad \forall x, y \in \mathfrak{g};
\]
(ii) a difference operator on a group $G$ is a map $D: G \to G$ such that

$$D(gh) = D(g)\text{Ad}_g D(h), \quad \forall g, h \in G;$$

(iii) a difference operator on a Hopf algebra $H$ is a coalgebra homomorphism $D : H \to H$ such that

$$D(xy) = D(x_1)\text{ad}_x D(y) = D(x_1)x_2 D(y)S(x_3), \quad \forall x, y \in H.$$

Here the (left) adjoint action of $H$ on itself is defined by

$$\text{ad}_x := x_1yS(x_2), \quad \forall x, y \in H.$$

The term difference operator comes from its origin of the difference operator on functions, defined by

$$D(f)(x) := f(x + 1) - f(x).$$

As shown in [1], the operator satisfies the operator identity

$$D(fg) = D(f)g + fD(g) + D(f)D(g),$$

as a special case of the differential operator of weight $\lambda$ when $\lambda = 1$. Another meaning of the term difference operator, commonly used in differential algebra and difference algebra [2, 3], is that the operator is an injective algebra homomorphism.

**Remark 2.2.**

(i) A crossed homomorphism on either a Lie algebra, a group or a Hopf algebra with respect to the trivial action is simply a homomorphism of the corresponding structure.

(ii) Bijective Hopf algebra crossed homomorphisms are the same as bijective 1-cocycles defined in [4, Definition 1.10] to construct Hopf braces.

We will show below that the notions of crossed homomorphisms and difference operators on Hopf algebras and Lie algebras are compatible in the context of the Milnor-Moore theorem.

**Lemma 2.3.** For a crossed homomorphism $\pi : H \to K$, we have

(i) $\pi(1) = 1$;

(ii) $S_K \pi(a) = a_1 \to \pi S_H(a_2), \quad \pi S_H(a) = S_H(a_1) \to S_K \pi(a_2), \quad \forall a \in H$;

(iii) the convolution inverse of $\pi$ in $\text{Hom}_H(H, K)$ is $S_K \pi$.

**Proof.** (i) Since $\pi$ is a coalgebra homomorphism, $\pi(1)$ has inverse $S_K \pi(1)$. By Eq. (i), we have $\pi(1) = \pi(1)^2$. Thus $\pi(1) = 1$.

(ii) Applying $\pi(1) = 1$ and Eq. (i), we first have

$$\varepsilon_H(a_1 S_H(a_2)) = \pi(a_1 a_2) \to \pi S_H(a_3), \quad \forall a \in H.$$

As $\pi$ is a coalgebra homomorphism, it implies that

$$S_K \pi(a) = S_K \pi(a_1) \varepsilon_H(a_2) = S_K \pi(a_1) \pi(a_2)(a_3 \to \pi S_H(a_4)) = a_1 \to \pi S_H(a_2).$$

In the same way,

$$\pi S_H(a) = S_H(a_1) a_2 \to \pi S_H(a_3) = S_H(a_1) \to (a_2 \to \pi S_H(a_3)) = S_H(a_1) \to S_K \pi(a_2).$$
(11) Since \( \pi \) is a coalgebra homomorphism, we have
\[
\pi(a_1)S_K \pi(a_2) = \pi(a_1)S_K (\pi(a)_2) = \varepsilon_K (\pi(a))1 = \varepsilon_H (a)1,
\]
\[
S_K \pi(a_1) \pi(a_2) = S_K (\pi(a)_1) \pi(a)_2 = \varepsilon_K (\pi(a))1 = \varepsilon_H (a)1, \quad \forall a \in H.
\]
This is what we need. \( \square \)

2.2. 

**Milnor-Moore theorem and crossed homomorphisms.** We next establish the relationship between the crossed homomorphisms on Lie algebras and Hopf algebras in the context of the Milnor-Moore theorem.

Let \( C \) be a Hopf algebra and \( c \in C \).

(i) \( c \) is called a **group-like** element, if \( \Delta(c) = c \otimes c \) and \( \varepsilon(c) = 1 \). Denote by \( G(C) \) the group of group-like elements in \( C \).

(ii) If \( \Delta(c) = c \otimes 1 + 1 \otimes c \), then \( c \) is called a **primitive element**. Let \( P(C) \) denote the set of primitive elements in \( C \), which is a Lie algebra.

**Proposition 2.4.** Let \( \pi : H \to K \) be a crossed homomorphism on a Hopf algebra \( H \) with respect to a module bialgebra action \( \to : H \to \text{End}(K) \), namely, the \( H \)-module algebra action \( \to \) is also compatible with the coalgebra structure of \( K \) as follows,
\[
\varepsilon_K (a \to x) = \varepsilon_H (a) \varepsilon_K (x), \quad \Delta_K (a \to x) = (a_1 \to x_1) \otimes (a_2 \to x_2), \quad \forall a \in H, x \in K.
\]
Then

(i) \( \pi|_{G(H)} \) is a group crossed homomorphism from \( G(H) \) to \( G(K) \);

(ii) \( \pi|_{P(H)} \) is a Lie algebra crossed homomorphism from \( P(H) \) to \( P(K) \).

**Proof.** Since \( \pi \) is a coalgebra homomorphism, \( \pi|_{G(H)} \) is a set map from \( G(H) \) to \( G(K) \), and \( \pi|_{P(H)} \) is a linear map from \( P(H) \) to \( P(K) \). Also, we have the restricted actions \( \to : G(H) \to \text{Aut}(G(K)) \) and \( \to : P(H) \to \text{Der}(P(K)) \), as \( K \) is an \( H \)-module bialgebra via \( \to \).

To obtain (i), just note that Eq. (3) indeed restricts to Eq. (4) when \( a, b \in G(H) \).

To obtain (ii), applying \( \pi(1) = 1 \), we first see that
\[
\pi(ab) = \pi(a) \pi(b) + a \to \pi(b) , \quad \forall a \in P(H), b \in H.
\]
Hence, for \( a, b \in P(H) \),
\[
\pi([a, b]) = \pi(ab - ba) \\
= (\pi(a) \pi(b) + a \to \pi(b)) - (\pi(b) \pi(a) + b \to \pi(a)) \\
= a \to \pi(b) - b \to \pi(a) + [\pi(a), \pi(b)],
\]
showing that \( \pi|_{P(H)} \) satisfies Eq. (4). \( \square \)

**Remark 2.5.** When \( K \) and \( H \) are cocommutative and the crossed homomorphism \( \pi : H \to K \) is bijective, Proposition 2.4 (1) recovers (1) Lemma 4.2. In fact, bijective 1-cocycles from \( H \) to \( K \) defined in (1) only require an \( H \)-module algebra action \( \to \) on \( K \). When \( K \) and \( H \) are cocommutative, such a module algebra action must be a module bialgebra action.

Since the adjoint action \( \text{ad} \) of \( H \) induces adjoint actions of \( G(H) \) and \( P(H) \) respectively, we obtain
Corollary 2.6. Let $D$ be a difference operator on a Hopf algebra $H$. Then
(i) $D|_{G(H)}$ is a difference operator on the group $G(H)$;
(ii) $D|_{P(H)}$ is a difference operator on the Lie algebra $P(H)$.

Going in the opposite direction, one can extend a crossed homomorphism on a Lie algebra to its universal enveloping algebra. See [1, Lemma 4.3, Lemma 4.4]. Let $\phi : g \rightarrow \text{Der}(h)$ be a Lie algebra action of $(g, [\cdot, \cdot], h)$ on $(h, [\cdot, \cdot], h)$. Then $\phi$ can be extended to a module bialgebra action $\bar{\phi} : U(g) \rightarrow \text{End}(U(h))$ defined by

$$
\bar{\phi}(x)(1) = 0, \quad \bar{\phi}(x)(y_1 \cdots y_r) = \sum_{i=1}^{r} y_1 \cdots y_{i-1} \phi(x)(y_i)y_{i+1} \cdots y_r, \quad \forall x \in g, y_1, \ldots, y_r \in h, r \geq 1.
$$

Proposition 2.7. A Lie algebra crossed homomorphism $\pi : g \rightarrow h$ with respect to the action $(h, \phi)$ can be extended to a unique Hopf algebra crossed homomorphism $\bar{\pi} : U(g) \rightarrow U(h)$ with respect to the extended module bialgebra action $\bar{\phi} : U(g) \rightarrow \text{End}(U(h))$. More precisely, $\bar{\pi} : U(g) \rightarrow U(h)$ is given by

$$
\bar{\pi}(x_1 \cdots x_n) = (\pi(x_1) + \bar{\phi}(x_1)) \cdots (\pi(x_n) + \bar{\phi}(x_n))(1), \quad \forall x_1, \ldots, x_n \in g, n \geq 1,
$$

where $\pi(x_k), 1 \leq k \leq n$ are left multiplications.

Recall that, under our running hypothesis of characteristic zero for the base field, for a connected cocommutative Hopf algebra $H$, the Milnor-Moore theorem gives a Hopf algebra isomorphism $\varphi_H : H \rightarrow U(P(H))$ of $H$ with the universal enveloping algebra of the Lie algebra $P(H)$ of primitive elements of $H$. We now give the following compatibility of the Milnor-Moore theorem for crossed homomorphisms.

Theorem 2.8. (Milnor-Moore Theorem for Crossed Homomorphisms) Let $H$ and $K$ be connected cocommutative Hopf algebras such that $K$ is an $H$-module bialgebra via an action $\rightarrow : H \rightarrow \text{End}(K)$. Then a Hopf algebra crossed homomorphism $\pi : H \rightarrow K$ with respect to $\rightarrow$ induces a unique Hopf algebra crossed homomorphism $\bar{\pi} : U(P(H)) \rightarrow U(P(K))$ such that the following diagram of crossed homomorphisms and inclusions is commutative:

![Diagram](image)

Moreover, the Milnor-Moore isomorphisms $\varphi_H$ and $\varphi_K$ give an isomorphism between the Hopf algebra crossed homomorphisms $\pi : H \rightarrow K$ and $\pi : U(P(H)) \rightarrow U(P(K))$.

Proof. First by Proposition [2,3], we obtain a crossed homomorphism $\pi|_{P(H)} : P(H) \rightarrow P(K)$ of Lie algebras by restricting $\pi : H \rightarrow K$ to the Lie algebra of primitive elements. Then by Proposition [2,4], it can be extended to a unique Hopf algebra crossed homomorphism $\bar{\pi} : U(P(H)) \rightarrow U(P(K))$.
\[ U(P(H)) \to U(P(K)) \]. This proves the commutativity of the two trapezoids in the middle of the diagram in Eq. (5).

On the other hand, by the classical Milnor-Moore theorem, the embedding \( i_H : P(H) \to H \) induces a Hopf algebra isomorphism \( \varphi_H : U(P(H)) \to H \), and the same holds for \( K \), giving the commutativity of the two triangles.

Before continuing, we verify the following compatibility condition:

\[ \varphi_K (a \to v) = \varphi_H(a) \to \varphi_K(v), \quad \forall v \in U(P(K)), a \in P(H). \]

Indeed, since \( K \) is an \( H \)-module bialgebra, and \( U(P(K)) \) is a \( U(P(H)) \)-module bialgebra, for all \( v = x_1 \cdots x_r \) with \( x_i \in P(K), 1 \leq i \leq r \), we derive

\[
\varphi_K (a \to v) = \varphi_K (a \to x_1 \cdots x_r)
= \varphi_K \left( \sum_{i=1}^{r} x_1 \cdots x_{i-1} (a \to x_i) x_{i+1} \cdots x_r \right)
= \sum_{i=1}^{r} \varphi_K(x_1) \cdots \varphi_K(x_{i-1}) \varphi_H(a) \varphi_K(x_{i+1}) \cdots \varphi_K(x_r)
= \varphi_H(a) \to \varphi_K(x_1) \cdots \varphi_K(x_r)
= \varphi_H(a) \to \varphi_K(v),
\]

as needed.

Next we prove the commutativity of the outer rectangle in the diagram (5), namely the equality

\[ \varphi_K(\bar{\pi}(u)) = \pi(\varphi_H(u)), \quad \forall u \in U(P(H)). \]

We apply induction on the degree of \( u \) via the coradical filtration \( \{U(P(H))_n \}_{n \geq 0} \) of \( U(P(H)) \).

First note that \( \varphi_K \bar{\pi}_H = \pi \varphi_H \bar{\pi}_H \) from the above discussion. So Eq. (7) clearly holds when \( u \in U(P(H))_1 \). Now suppose that Eq. (7) holds for all \( u \in U(P(H))_k \) with \( k \geq 1 \). To complete the inductive step, we only need to check elements of the form \( au \) with \( a \in P(H) \), as they span \( U(P(H))_{k+1} \). The left hand side of Eq. (7) for \( au \in U(P(H))_{k+1} \) is

\[
\varphi_K(\bar{\pi}(au)) = \varphi_K (\bar{\pi}(a)\bar{\pi}(u) + a \to \bar{\pi}(u))
= \varphi_K (\bar{\pi}(a)\bar{\pi}(u)) + \varphi_K (a \to \bar{\pi}(u))
= \varphi_K(\bar{\pi}(a)) \varphi_K(\bar{\pi}(u)) + \varphi_H(a) \to \varphi_K(\bar{\pi}(u))
= \pi(\varphi_H(a)) \pi(\varphi_H(u)) + \varphi_H(a) \to \pi(\varphi_H(u))
= \pi(\varphi_H(a) \varphi_H(u))
= \pi(\varphi_H(au)),
\]

where the first and the second to the last equalities use Eq. (5), the third equality is due to Eq. (7) and the fourth one is obtained by the induction hypothesis.
In order to show that $\varphi_H$ and $\varphi_K$ give an isomorphism between the Hopf algebra crossed homomorphisms $\pi$ and $\tilde{\pi}$, it remains to show

$$\varphi_K (u \rightarrow v) = \varphi_H (u) \rightarrow \varphi_K (v), \quad \forall v \in U(P(K)), \ u \in U(P(H)),$$

which we again prove by induction on the degree of $u$. When $u \in U(P(H))_1$, this is just due to Eq. (3). Now suppose that it holds for all $u \in U(P(H))_k$ with $k \geq 1$, then given any $a \in P(H)$, we only need to check $au \in U(P(H))_{k+1}$ as noted in the previous paragraph. In this case we have

$$\varphi_K (au \rightarrow v) = \varphi_K (a \rightarrow (u \rightarrow v))$$

$$= \varphi_H (a) \rightarrow \varphi_K (u \rightarrow v)$$

$$= \varphi_H (a) \rightarrow (\varphi_H (u) \rightarrow \varphi_K (v))$$

$$= \varphi_H (a) \varphi_H (u) \rightarrow \varphi_K (v)$$

$$= \varphi_H (au) \rightarrow \varphi_K (v),$$

where the second and the third equalities are obtained by the induction hypothesis. \hfill \Box

In particular, when the crossed homomorphisms are difference operators, we obtain the following difference enhancement of the Milnor-Moore isomorphism.

**Corollary 2.9.** Let $D$ be a difference operator on a connected cocommutative Hopf algebra $H$. Then there is an isomorphism from the difference operator $\tilde{D}$ on $U(P(H))$ to the difference operator $D$ on $H$.

At the end of this section, we establish a relation between difference cocommutative Hopf algebras and Rota-Baxter cocommutative Hopf algebras.

In [3], Goncharov defined a **Rota-Baxter operators of weight 1** on a cocommutative Hopf algebra $H$ as a coalgebra homomorphism $B$ satisfying

$$(8) \quad B(x)B(y) = B(x_1 \text{ad}_{B(x_2)}y) = B(x_1 B(x_2) y S(B(x_3))), \quad \forall x, y \in H.$$

**Proposition 2.10.** Let $H$ be a cocommutative Hopf algebra and let $D$ be a coalgebra automorphism on $D$. Then $D$ is a difference operator if and only if its inverse $D^{-1}$ is a Rota-Baxter operator of weight 1.

**Proof.** Write $B = D^{-1}$. Then $B$ is also a coalgebra homomorphism. If $D$ is a difference operator, then we use Eq. (3) to check that

$$B(x)B(y) = B(D(B(x)B(y)))$$

$$= B(D(B(x_1))B(x_2)D(B(y))S(B(x_3)))$$

$$= B(x_1 B(x_2) y S(B(x_3)))$$

for all $x, y \in H$. Hence, $B$ satisfies Eq. (3) and is a Rota-Baxter operator of weight 1.

Conversely, if $B$ is a Rota-Baxter operator, then by Eq. (3),

$$D(xy) = D(B(D(x))B(D(y)))$$

$$= D(B(x_1)B(x_2)D(y)S(B(D(x_3))))$$
for all \( x, y \in H \). So \( D \) is a difference operator. \( \square \)

3. Graph characterizations of crossed homomorphisms

In this section, we use the (left) graphs of maps to characterize crossed homomorphisms on Hopf algebras and study their relationship with the graphs of Lie algebra crossed homomorphisms.

Let \( \phi : \mathfrak{g} \to \text{Der}(b) \) be an action of a Lie algebra \((g, [\cdot, \cdot]_g)\) on a Lie algebra \((b, [\cdot, \cdot]_b)\). Denote by \( b \bowtie g \) the semi-direct product Lie algebra of \( b \) and \( g \) with respect to the action \((b, \phi)\). More precisely, the Lie bracket \([\cdot, \cdot]_b : \wedge^2 (b \oplus g) \to b \oplus g\) is given by

\[
[(u, x), (v, y)]_b = ([u, v]_b + \phi(x)(v) - \phi(y)(u), [x, y]_b), \quad \forall x, y \in g, \ u, v \in b.
\]

It is proved in [23] that a linear map \( \pi : g \to b \) is a Lie algebra crossed homomorphism with respect to the action \((b, \phi)\) if and only if the graph of \( \pi \),

\[
\text{Gr}_\pi := \{(\pi(x), x) \mid x \in g\}
\]

is a Lie subalgebra of \( b \bowtie g \). Similar graph characterizations can also be found for other algebraic structures. See [25] for example.

For Hopf algebras \( H \) and \( K \), let \( \rightarrow : H \to \text{End}(K) \) be an action such that \( K \) is an \( H \)-module algebra. The smash product of Hopf algebras \( K \) and \( H \) is defined to be \( K \otimes H \) with the multiplication

\[
(x \# a)(y \# b) = x(a_1 \to y) \# a_2 b, \quad \forall x, y \in K, \ a, b \in H,
\]

with \( x \otimes a \in K \otimes H \) denoted by \( x \# a \). We denote such a smash product algebra by \( K\#H \). If \( H \) is cocommutative and \( K \) is further an \( H \)-module bialgebra, then \( K\#H \) becomes a Hopf algebra with the usual tensor product comultiplication and the antipode defined by

\[
S(x \# a) := (S_H(a_1) \to S_K(x)) \# S_H(a_2).
\]

**Definition 3.1.** Given any coalgebra homomorphism \( \pi : H \to K \), the graph of \( \pi \), denoted by \( \text{Gr}_\pi \), is defined to be the subspace \( \text{im}(\pi \otimes \text{id})\Delta_H \) of \( K \otimes H \), that is,

\[
\text{Gr}_\pi := \{\pi(a_1) \otimes a_2 \mid a \in H\}.
\]

**Theorem 3.2.** Let \( H \) and \( K \) be Hopf algebras with an algebra action \( \rightarrow : H \to \text{End}(K) \) such that \( K \) is an \( H \)-module bialgebra. A coalgebra homomorphism \( \pi : H \to K \) is a Hopf algebra crossed homomorphism with respect to the action \( \rightarrow \) if and only if its graph \( \text{Gr}_\pi \) is a subalgebra of the smash product algebra \( K\#H \).

**Proof.** If \( \pi : H \to K \) is a crossed homomorphism, then

\[
(\pi(a_1) \# a_2)(\pi(b_1) \# b_2) = \pi(a_1)(a_2 \to \pi(b_1)) \# a_2 b_2 = \pi(a_1 b_1) \# a_2 b_2 \in \text{Gr}_\pi, \quad \forall a, b \in K.
\]

Also, \( 1\# 1 = \pi(1) \# 1 \in \text{Gr}_\pi \). Hence, \( \text{Gr}_\pi \) is a subalgebra of \( K\#H \).

Conversely, assume that \( \text{Gr}_\pi \) is a subalgebra of \( K\#H \). Then for any \( a, b \in H \), there exists \( w \in H \) such that

\[
\pi(w_1) \# w_2 = (\pi(a_1) \# a_2)(\pi(b_1) \# b_2) = \pi(a_1)(a_2 \to \pi(b_1)) \# a_2 b_2.
\]
In particular,
\[ w = \varepsilon_K(\pi(w_1))w_2 = \varepsilon_K(\pi(a_1)(a_2 \to \pi(b_1)))a_3b_2 = \varepsilon_K(a_1 \to \pi(b_1))a_2b_2 = ab, \]
as the counit \( \varepsilon_K \) of \( K \) is compatible with \( \to \). On the other hand,
\[ \pi(w) = \pi(w_1)\varepsilon_H(w_2) = \pi(a_1)(a_2 \to \pi(b_1))\varepsilon_H(a_3b_2) = \pi(a_1)(a_2 \to \pi(b)), \]
so \( \pi(ab) = \pi(a_1)(a_2 \to \pi(b)) \), showing that \( \pi : H \to K \) is a crossed homomorphism.

\[ \square \]

**Corollary 3.3.** Let \( K \) and \( H \) be as in Theorem 3.2 and with \( H \) cocommutative. A coalgebra homomorphism \( \pi : H \to K \) is a Hopf algebra crossed homomorphism with respect to the action \( \to \) if and only if \( \text{Gr}_x \) is a Hopf subalgebra of the smash product Hopf algebra \( K\#H \).

Moreover, a Hopf algebra crossed homomorphism induces a Hopf algebra isomorphism,
\[ \Psi : H \to \text{Gr}_x, \quad a \mapsto \pi(a)\#a_2. \]

**Proof.** By Theorem 3.2 we only need to show that \( \text{Gr}_x \) is a subcoalgebra of \( K\#H \), and \( S(\text{Gr}_x) \subseteq \text{Gr}_x \). These statements follow from the cocommutativity of \( H \):
\[ \Delta(\pi(a_1)\#a_2) = (\pi(a_1)\#a_3) \otimes (\pi(a_2)\#a_4) = (\pi(a_1)\#a_2) \otimes (\pi(a_3)\#a_4) \in \text{Gr}_x \otimes \text{Gr}_x, \]
and Lemma 2.3 (1):
\[ S(\pi(a_1)\#a_2) = (S_H(a_1) \to S_K\pi(a_2))\#S_H(a_3) = \pi S_H(a_1)\#S_H(a_2) \in \text{Gr}_x. \]

From the above discussion, one can easily see that \( \Psi \) as stated is a Hopf algebra isomorphism with its inverse \( \Psi^{-1} = \varepsilon_K \otimes \text{id} \).

We now show that the universal enveloping algebra \( U(\text{Gr}_x) \) of the graph of a Lie algebra crossed homomorphism \( \pi \) is given by the graph of the Hopf algebra crossed homomorphism \( \bar{\pi} \) lifted from \( \pi \).

**Theorem 3.4.** Let \( \pi : \mathfrak{g} \to \mathfrak{h} \) be a crossed homomorphism on a Lie algebra \( \mathfrak{g} \) with respect to an action \( (\mathfrak{h}, \phi) \) and let \( \bar{\pi} : U(\mathfrak{g}) \to U(\mathfrak{h}) \) be the lifted Hopf algebra crossed homomorphism given in Proposition 2.4.

(i) There is a Hopf algebra isomorphism \( U(\mathfrak{h})\#U(\mathfrak{g}) \cong U(\mathfrak{h} \rtimes \mathfrak{g}) \), that is, the universal enveloping algebra of a Lie algebra semidirect product is isomorphic to the corresponding Hopf algebra smash product;

(ii) There is a Hopf algebra isomorphism \( \text{Gr}_x \cong U(\text{Gr}_x) \), that is, the universal enveloping algebra of the graph of a Lie algebra crossed homomorphism is isomorphic to the graph of the lifted Hopf algebra crossed homomorphism.

**Proof.** Define a linear map \( \varphi : \mathfrak{h} \rtimes \mathfrak{g} \to U(\mathfrak{h})\#U(\mathfrak{g}) \) by
\[ \varphi(u, x) = u\#1 + 1\#x, \quad \forall x \in \mathfrak{g}, \ u \in \mathfrak{h}. \]

First we check that \( \varphi \) is a Lie algebra homomorphism, which follows from
\[ \varphi([[u, x], (v, y)]) = \varphi((u, v)_{\mathfrak{h}} + \phi(x)(v) - \phi(y)(u), [x, y]_{\mathfrak{g}}) \]
\[ = ([u, v]_{\mathfrak{h}} + \phi(x)(v) - \phi(y)(u))\#1 + 1\#[x, y]_{\mathfrak{g}}, \]
\[ [\varphi(u, x), \varphi(v, y)] = [u\#1 + 1\#x, v\#1 + 1\#y] \]
\[= uv\#1 + u\#y + \phi(x)(v)\#1 + v\#x + 1\#xy - (vu\#1 + v\#x + \phi(y)(u)\#1 + u\#y + 1\#yx) = ([u, v])_{\partial} + \phi(x)(v) - \phi(y)(u))\#1 + 1\#[x, y]_{\partial},\]

for all \(x, y \in g, u, v \in h\).

By the universal property of \(U(h \rtimes g)\), there exists a unique algebra homomorphism

\[\tilde{\phi} : U(h \rtimes g) \to U(h)\#U(g)\]

such that \(\varphi = \tilde{\phi} i_{\partial g}\), where \(i_{\partial g} : h \rtimes g \to U(h \rtimes g)\) is the natural embedding.

Since the image \(\text{Im} \varphi = \{u\#1, 1\#x | x \in g, u \in h\}\) generates \(U(h)\#U(g)\) as an algebra, \(\tilde{\phi}\) is surjective. On the other hand, one checks that

\[\Delta \tilde{\phi} = (\tilde{\phi} \otimes \tilde{\phi})\Delta\]

holds on \(\text{Im} \varphi\), thus it holds on \(U(h \rtimes g)\) as an equality of algebra homomorphisms. It is obvious that \(\varepsilon \tilde{\phi} = \varepsilon\). Therefore, \(\tilde{\phi}\) is a Hopf algebra surjection. By the Heyneman-Radford theorem (see [23, Theorem 5.3.1]), \(\tilde{\phi}\) is also injective, as \(U(h \rtimes g)_{\partial} = k \oplus (h \rtimes g)\) and \(\tilde{\phi}|_{U(h \rtimes g)_{\partial}}\) is clearly injective by the definition of \(\varphi\). In conclusion, \(\tilde{\phi}\) is the desired Hopf algebra isomorphism.

(\[\square\]) By Corollary [23], \(Gr_{\pi}\) is a Hopf subalgebra of \(U(h)\#U(g)\) isomorphic to \(U(g)\). Define \(\psi := \varphi|_{Gr_{\pi}}\). Then \(\psi\) is injective and

\[\psi(\pi(x), x) = \pi(x)\#1 + 1\#x = (\pi \otimes 1)\Delta(x) \in Gr_{\pi}, \quad \forall x \in g.\]

Also, note that \(\text{Im} \psi\) generates \(Gr_{\pi}\) as an algebra. Hence, \(\psi\) induces a Hopf algebra isomorphism

\[\tilde{\psi} : U(Gr_{\pi}) \to Gr_{\pi}\]

by the same argument as for \(\tilde{\phi}\).

4. Derived actions from crossed homomorphisms

In this section, we show that a crossed homomorphism on either a Lie algebra, a group or a Hopf algebra gives a derived action. Furthermore, these derived structures are related in the same way that the original structures are related.

4.1. Derived actions for Lie algebras and Lie groups. Let \(\phi : g \to \text{Der}(h)\) be an action of a Lie algebra \((g, [\cdot, \cdot]_{g})\) on a Lie algebra \((h, [\cdot, \cdot]_{h})\). By [23, Lemma 2.6], a crossed homomorphism \(d : g \to h\) gives rise to a derived action \(\phi_{d} : g \to \text{Der}(h)\) given by

\[\phi_{d}(x)(u) = \phi(x)(u) + [d(x), u]_{h}, \quad \forall x \in g, u \in h.\]

Furthermore, \(\bar{d}\) is a Lie algebra crossed homomorphism with respect to the derived action \(\phi_{d}\).

For group crossed homomorphisms, we have a similar result.

Lemma 4.1. Let \(D : G \to H\) be a group crossed homomorphism on \(G\) with respect to an action \((H, \Phi)\). There is a derived action \(\Phi_{D} : G \to \text{Aut}(H)\) by automorphisms given by

\[\Phi_{D}(g)h = Ad_{D(g)}(\Phi(g)(h)) = D(g)\Phi(g)(h)D(g)^{-1}, \quad \forall g, h \in G.\]

Moreover, the map

\[\overline{D} : G \to H, \quad \overline{D}(g) := D(g)^{-1}, \quad \forall g \in G\]

is a group crossed homomorphism, called the derived crossed homomorphism of \(D\) with respect to the derived action \((H, \Phi_{D})\).
Proof. First since $\Phi(g)$ and $\text{Ad}_{D(g)}$ are automorphisms, it is obvious that $\Phi_D(g)$ is an automorphism of $\mathcal{H}$ for all $g \in \mathcal{G}$. Also, by Eq. (3), we have

$$
\Phi_D(g_1g_2)(h) = \text{Ad}_{D(g_1g_2)}\Phi(g_1g_2)(h)
= \text{Ad}_{D(g_1)\Phi(g_1)(D(g_2))}\Phi(g_1)\Phi(g_2)(h)
= \text{Ad}_{\Phi(g_1)\text{Ad}_{D(g_2)}}\Phi(g_1)\Phi(g_2)(h)
= \text{Ad}_{D(g_1)}\Phi(g_1)\text{Ad}_{D(g_2)}\Phi(g_2)(h)
= \Phi_D(g_1)\Phi_D(g_2)(h).
$$

Thus $\Phi_D : \mathcal{G} \to \text{Aut}(\mathcal{H})$ is an action by automorphisms.

Furthermore, for all $g, h \in \mathcal{G}$, we have

$$
\overline{D}(g)\Phi_D(g)(\overline{D}(h)) = D(g)^{-1}D(g)\Phi(g)(D(h)^{-1})D(g)^{-1}
= (\Phi(g)(D(h)))^{-1}D(g)^{-1}
= (D(g)\Phi(g)(D(h)))^{-1}
= \overline{D}(gh),
$$

which implies that $\overline{D} : \mathcal{G} \to \mathcal{H}$ is a crossed homomorphism on $\mathcal{G}$ with respect to the action $(\mathcal{H}, \Phi_D)$.

We next establish a relationship between the induced action on a Lie algebra given by Eq. (11) and the one on a Lie group given by Eq. (10). Here a crossed homomorphism on a Lie group $\mathcal{G}$ with respect to an action $\Phi : \mathcal{G} \to \text{Aut}(\mathcal{H})$ is a smooth map $D : \mathcal{G} \to \mathcal{H}$ such that (4) holds.

Let $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ be the Lie algebra of the Lie group $\mathcal{G}$. Let $\exp : \mathfrak{g} \to \mathcal{G}$ denote the exponential map. Then the relation between the Lie bracket $[\cdot, \cdot]_\mathfrak{g}$ and the Lie group multiplication is given by the following fundamental formula:

$$
[u, v]_\mathfrak{g} = \left. \frac{d^2}{dt ds} \right|_{t,s=0} \exp(tu) \exp(sv) \exp(-tu), \quad \forall u, v \in \mathfrak{g}.
$$

For any $g \in \mathcal{G}$, since $\Phi_D(g)$ is in $\text{Aut}(\mathcal{H})$, it follows that $\Phi_D(g)(e_\mathcal{H}) = e_\mathcal{H}$ for the unit element $e_\mathcal{H}$ in the Lie group $\mathcal{H}$. By taking the differentiation, $\Phi_D(g)$ induces $\Phi_D(g)_* : \mathfrak{g} \to \text{Der}(\mathfrak{h})$. Consequently, we obtain a Lie group homomorphism, which is still denoted by $\Phi_D$, from $\mathcal{G}$ to $\text{Aut}(\mathfrak{h})$.

Again taking the differentiation, we obtain a Lie algebra homomorphism $(\Phi_D)_* : \mathfrak{g} \to \text{Der}(\mathfrak{h})$. The above process can be summarized in the following diagram:

$$
\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\Phi_D} & \text{Aut}(\mathfrak{h}) \\
\text{differentiation} & & \\
\mathfrak{g} & \xrightarrow{(\Phi_D)_*} & \text{Der}(\mathfrak{h}).
\end{array}
$$

On the other hand, it is obvious that the Lie group action $\Phi : \mathcal{G} \to \text{Aut}(\mathcal{H})$ induces a Lie algebra action of $\phi : \mathfrak{g} \to \text{Der}(\mathfrak{h})$, and the relation is given by

$$
\left. \frac{d^2}{dt ds} \right|_{t,s=0} \Phi(\exp_{\mathcal{G}}(tx))(\exp_{\mathcal{H}}(su)) = \phi(x)(u), \quad \forall x \in \mathfrak{g}, u \in \mathfrak{h}.
$$
By a discussion similar to the one in [11, Theorem 2.17], \( d \) := \( D \ast e \) is a crossed homomorphism on the Lie algebra \( (g, [\cdot, \cdot]_\mathfrak{g}) \) with respect to the action \( \phi : g \to \text{Der}(\mathfrak{h}) \), and therefore,

\[
\frac{d}{dt} \bigg|_{t=0} D(\exp_g(tx)) = \frac{d}{dt} \bigg|_{t=0} \exp_H(td(x)), \quad \forall x \in g.
\]

Now we show that the differentiation of the derived action from a Lie group crossed homomorphism is the derived action from the Lie algebra crossed homomorphism from differentiation.

**Theorem 4.2.** Let \( D : G \to \mathcal{H} \) be a crossed homomorphism on a Lie group \( G \) with respect to an action \( \Phi : G \to \text{Aut}(\mathcal{H}) \), and \( d = D \ast e \). Then we have

\[(\Phi_D)_* = \phi_d.\]

Moreover, the differentiation of the derived Lie group crossed homomorphism \( \overline{D} \) is exactly the derived Lie algebra crossed homomorphism \( -d \).

**Proof.** By the commutative diagram in Eq. (13), we have

\[
(\Phi_D)_*(x)(u) = \left. \frac{d}{dt} \right|_{t=0} \Phi_D(\exp_g(tx))(u) = \left. \frac{d^2}{dt ds} \right|_{t,s=0} \Phi_D(\exp_g(tx))(\exp_H(su)) = \left. \frac{d^2}{dt ds} \right|_{t,s=0} \text{Ad}_{\exp_g(tx)} \Phi(\exp_g(tx)) \exp_H(su) = \left. \frac{d^2}{dt ds} \right|_{t,s=0} \text{Ad}_{\exp_H(td(x))} \Phi(\exp_g(tx)) \exp(su) = \frac{d^2}{dt ds} \left|_{t,s=0} \Phi(\exp_g(tx))(\exp_H(su)) + \left. \frac{d^2}{dt ds} \right|_{t,s=0} \text{Ad}_{\exp_H(td(x))} \exp_H(su) \right.
\]

which implies that the differentiation of \( \Phi_D \) is exactly \( \phi_d \).

It is obvious that the differentiation of the derived Lie group crossed homomorphism \( \overline{D} \) is exactly the derived Lie algebra crossed homomorphism \( -d \). □

### 4.2. Module and module bialgebra characterizations of Hopf algebra crossed homomorphisms.

Under the cocommutativity condition, we give another characterization of Hopf algebra crossed homomorphisms utilizing derived module structures and module bialgebra structures.

**Theorem 4.3.** Let \( H \) and \( K \) be Hopf algebras such that \( K \) is an \( H \)-module algebra via an algebra action \( - : H \to \text{End}(K) \). A coalgebra homomorphism \( \pi : H \to K \) is a Hopf algebra crossed homomorphism if and only if the action

\[ a \cdot \pi x := \pi(a_1)(a_2 \to x), \quad \forall x \in K, \ a \in H, \]

defines an \( H \)-module structure on \( K \).
Proof. If $\pi : H \to K$ is a crossed homomorphism, then we have

\[
ab \cdot_\pi x = \pi(a_1 b_1)(a_2 b_2 \to x) = \pi(a_1)(a_2 \to \pi(b_1))(a_3 \to (b_2 \to x)) = \pi(a_1)(a_2 \to \pi(b_1)(b_2 \to x)) = a \cdot_\pi (b \cdot_\pi x), \quad \forall x \in K, a, b \in H.
\]

This means that the action $\cdot_\pi$ defines an $H$-module structure on $K$.

Conversely, if there is the stated $H$-module structure $\cdot_\pi$ on $K$, then

\[
\pi(ab) = \pi(a_1 b_1)(a_2 b_2 \to 1) = ab \cdot_\pi 1 = a \cdot_\pi (b \cdot_\pi 1) = \pi(a_1)(a_2 \to \pi(b_1)(b_2 \to 1)) = \pi(a_1)(a_2 \to \pi(b_1)\varepsilon_H(b_2)) = \pi(a_1)(a_2 \to \pi(b)), \quad \forall a, b \in H.
\]

Hence $\pi : H \to K$ is a crossed homomorphism. \hfill \Box

Note that in the above theorem, even though $K$ is an $H$-module via $\cdot_\pi$, it is in general not an $H$-module algebra action. In the sequel, we will give a new $H$-module algebra structure under the cocommutativity condition of $H$. The following result is straightforward to check.

Lemma 4.4. Let $K$ and $H$ be Hopf algebras such that $H$ is cocommutative and $K$ is an $H$-module bialgebra via an action $\to$. Then $K$ is a $K\#H$-module algebra with the action defined by

\[
(x\# a)y := \text{ad}_a(a \to y), \quad \forall x, y \in K, a \in H.
\]

Theorem 4.5. Let $K$ and $H$ be Hopf algebras with $H$ cocommutative. Suppose that $K$ is an $H$-module bialgebra via an action $\to$. Let $\pi : H \to K$ be a Hopf algebra crossed homomorphism. Then $K$ has another $H$-module bialgebra structure via the derived action given by

\[
\pi(ab) = \pi(a_1 b_1)(a_2 b_2 \to 1) = ab \cdot_\pi 1 = a \cdot_\pi (b \cdot_\pi 1) = \pi(a_1)(a_2 \to \pi(b_1)(b_2 \to 1)) = \pi(a_1)(a_2 \to \pi(b_1)\varepsilon_H(b_2)) = \pi(a_1)(a_2 \to \pi(b)), \quad \forall a, b \in H.
\]

Moreover, $S_K \pi : H \to K$ is a Hopf algebra crossed homomorphism with respect to the new action $\cdot_\pi$, called the derived crossed homomorphism of $\pi$.

Proof. By Corollary 4.3, the graph $\text{Gr}_\pi$ is a Hopf subalgebra of $K\#H$. Thanks to Lemma 4.4 and the cocommutativity of $H$, $K$ becomes a $\text{Gr}_\pi$-module bialgebra. Pulled back by the Hopf algebra isomorphism $\Psi : H \to \text{Gr}_\pi$ given in Eq. (1), $K$ becomes an $H$-module bialgebra via the desired action $\cdot_\pi$.

On the other hand, for any $a, b \in H$,

\[
S_K(\pi(ab)) = S_K(\pi(a_1)(a_2 \to \pi(b))) = (a_1 \to S_K(\pi(b)))S_K(\pi(a_2)) = S_K(\pi(a_1))\text{ad}_{\pi(a_2)}(a_3 \to S_K(\pi(b))) = S_K(\pi(a_1))(a_2 \to_\pi S_K(\pi(b))).
\]

Hence, $S_K \circ \pi : H \to K$ is a crossed homomorphism with respect to the action $\cdot_\pi$. \hfill \Box

From Theorem 4.5, we evidently have
Corollary 4.6. The above derived $H$-module bialgebra structure on $K$ is consistent with the derived actions on the corresponding Lie algebras and groups as follows.

(i) For $x \in P(K)$ and $a \in P(H)$, we have $a \rightarrow_{\pi} x = [\pi(a), y] + (a \rightarrow x)$, which is exactly the derived action given by Eq. (11) (see also [28, Lemma 2.6]).

(ii) For $x \in G(K)$ and $a \in G(H)$, we have $a \rightarrow_{\pi} x = \pi(a)(a \rightarrow x)\pi(a)^{-1}$, which is exactly the derived action given by Eq. (11).

By Proposition 2.4, the restriction $(S_{K \pi}|_{G(H)})$ of the derived crossed homomorphism is a crossed homomorphism on the group $G(H)$ with respect to the restriction of the derived action $\rightarrow_{\pi}$, which is exactly the derived crossed homomorphism $\pi_{|G(H)}$ of the restriction $\pi_{|G(H)}$ given in Lemma 4.1.

5. Structure theorem of pointed cocommutative difference Hopf algebras

In this section, we first give some characterizations of difference operators on a Hopf algebra. We then study difference operators on smash product Hopf algebras. In particular, we give the structure theorem of pointed cocommutative difference Hopf algebras.

Lemma 5.1. A coalgebra homomorphism $D : H \rightarrow H$ is a difference operator on $H$ if and only if

\begin{equation}
D(x_1 y)x_2 = D(x_1) x_2 D(y), \quad \forall x, y \in H.
\end{equation}

Proof. If Eq. (1) holds, then $D(x_1 y)x_2 = D(x_1) x_2 D(y) S(x_3) S(x_4) = D(x_1) x_2 D(y)$. Conversely, if Eq. (15) holds, then $D(xy) = D(x_1 y) x_2 S(x_3) = D(x_1) x_2 S(x_3)$.

Theorem 5.2. Let $H$ be a Hopf algebra and $D : H \rightarrow H$ a coalgebra homomorphism. Then $D$ is a difference operator on $H$ if and only if the convolution product $D \ast \text{id}$ is an algebra homomorphism.

Proof. For any $x, y \in H$, we have

\begin{align*}
(D \ast \text{id})(xy) &= D((xy)_1)(xy)_2 = D(x_1 y_1) x_2 y_2, \\
(D \ast \text{id})(x)(D \ast \text{id})(y) &= D(x_1) x_2 D(y_1) y_2.
\end{align*}

Hence, by Eq. (15), we find that

\begin{equation}
(D \ast \text{id})(xy) = (D \ast \text{id})(x)(D \ast \text{id})(y), \quad \forall x, y \in H,
\end{equation}

is equivalent to Eq. (1), giving the desired equivalence.

Corollary 5.3. There exists a set bijection between $\text{Diff}(G)$ of difference operators and $\text{End}(G)$ of group endomorphisms for every group $G$.

\[
\begin{array}{ccc}
\text{Diff}(G) & \rightarrow & \text{End}(G) \\
D & \mapsto & (g \mapsto D(g)g) \\
\end{array}
\quad
\begin{array}{ccc}
\text{End}(G) & \rightarrow & \text{Diff}(G) \\
F & \mapsto & (g \mapsto F(g)g^{-1})
\end{array}
\]

and a set bijection between $\text{Diff}(\mathfrak{g})$ of difference operators and $\text{End}(\mathfrak{g})$ of Lie algebra endomorphisms for every Lie algebra $\mathfrak{g}$.

\[
\begin{array}{ccc}
\text{Diff}(\mathfrak{g}) & \rightarrow & \text{End}(\mathfrak{g}) \\
d & \mapsto & (x \mapsto d(x) + x) \\
\end{array}
\quad
\begin{array}{ccc}
\text{End}(\mathfrak{g}) & \rightarrow & \text{Diff}(\mathfrak{g}) \\
f & \mapsto & (x \mapsto f(x) - x)
\end{array}
\]
Denote by Hopf($H$) the monoid of Hopf algebra endomorphisms of $H$ and Aut($H$) the group of Hopf algebra automorphisms of $H$, and denote by Diff($H$) the set of difference operators on $H$.

**Corollary 5.4.** Let $H$ be a cocommutative Hopf algebra. We have the following set bijection between Diff($H$) of difference operators and Hopf($H$) of Hopf algebra endomorphisms,

$$\begin{align*}
\text{Diff}(H) & \longrightarrow \text{Hopf}(H) \\
D & \mapsto D \ast \text{id} \\
\text{Hopf}(H) & \longrightarrow \text{Diff}(H) \\
F & \mapsto F \ast S
\end{align*}$$

Via such a bijection, we can define a monoid structure $(\text{Diff}(H), \star, u \circ \varepsilon)$ by

$$D \star D' := ((D \ast \text{id})D') \ast D = (D(D' \ast \text{id})) \ast D', \quad \forall D, D' \in \text{Diff}(H).$$

**Proof.** When $H$ is cocommutative, $D \ast \text{id}$ is also a coalgebra homomorphism for $D \in \text{Diff}(H)$. By Theorem 5.2, $D \ast \text{id}$ is a Hopf algebra homomorphism. Then the desired bijection follows from the fact that the antipode $S$ is the convolution inverse of the identity map $\text{id}$.

For any $D, D' \in \text{Diff}(H)$, applying this bijection we obtain

$$D \star D' = ((D \ast \text{id})(D' \ast \text{id})) \ast S = (((D \ast \text{id})D') \ast (D \ast \text{id})) \ast S = (((D \ast \text{id})D') \ast D) \ast (\text{id} \ast S) = ((D \ast \text{id})D') \ast D,$$

or

$$D \star D' = ((D \ast \text{id})(D' \ast \text{id})) \ast S = ((D(D' \ast \text{id})) \ast (D' \ast \text{id})) \ast S = ((D(D' \ast \text{id})) \ast (\text{id} \ast S) = (D(D' \ast \text{id})) \ast D',$$

and $u \circ \varepsilon$ serves as the unit of this monoid. \qed

**Proposition 5.5.** For a Hopf algebra $H$, Aut($H$) acts on Diff($H$) by conjugation.

**Proof.** Since End($H$) is an Aut($H$)-module by conjugation, it is enough to show that

$$\sigma D \sigma^{-1} \in \text{Diff}(H), \quad \forall \sigma \in \text{Aut}(H), D \in \text{Diff}(H).$$

Indeed, $\sigma D \sigma^{-1}$ is still a coalgebra homomorphism, and

$$(\sigma D \sigma^{-1}) \ast \text{id} = \sigma(D \ast \text{id})\sigma^{-1}$$

is an algebra homomorphism by Theorem 5.2. Thus $\sigma D \sigma^{-1}$ is in Diff($H$). \qed

The characterization of difference operators in Theorem 5.2 inspires us to give the following notion of difference module bialgebras over difference Hopf algebras.
**Definition 5.6.** Let \((K, D_K)\) and \((H, D_H)\) be cocommutative difference Hopf algebras. The pair \((K, D_K)\) is called a **difference \((H, D_H)\)-module bialgebra** via an action \(\rightarrow: H \rightarrow \text{End}(K)\), if \(K\) is an \(H\)-module bialgebra via \(\rightarrow\) such that

\[
(D_K \ast \text{id})(a \rightarrow x) = (D_H \ast \text{id})a \rightarrow (D_K \ast \text{id})x, \quad \forall x \in K, \ a \in H.
\]

Namely, the following equality holds.

\[
D_K(a_1 \rightarrow x_1)(a_2 \rightarrow x_2) = D_H(a_1)a_2 \rightarrow D_K(x_1)x_2, \quad \forall x \in K, \ a \in H.
\]

**Example 5.7.** (i) Any cocommutative difference Hopf algebra \((H, D_H)\) is a difference module bialgebra over itself via the adjoint action.
(ii) Let \((G, D_G)\) be a difference group and \((g, D_g)\) a difference Lie algebra. Let \(\rightarrow\) be an action of \(G\) on \(g\) as an automorphism group of Lie algebras. Then \(U(g)\) endowed with the extended difference operator, which is also denoted by \(D_g\), is a difference Hopf algebra, and \(kG\) endowed with the extended difference operator, which is also denoted by \(D_g\), is a difference Hopf algebra. Moreover, \((U(g), D_g)\) becomes a difference \((kG, D_g)\)-module bialgebra, if and only if

\[
(D_g + \text{id})(g \rightarrow x) = D_g(g)g \rightarrow (D_g + \text{id})x, \quad \forall x \in g, \ g \in G.
\]

Indeed, it is straightforward to verify Eq. (17) here by induction on the degree of any \(u \in U(g)\).

Next we study difference operators on the smash product of Hopf algebras, and give the following extension theorem of difference operators on cocommutative Hopf algebras.

**Theorem 5.8.** Let \((H, D_H)\) and \((K, D_K)\) be cocommutative difference Hopf algebras such that \(K\) is an \(H\)-module bialgebra via an action \(\rightarrow\). The smash product Hopf algebra \(K\#H\) has the unique difference operator \(D\) such that \(D|_K = D_K\) and \(D|_H = D_H\) if and only if \((K, D_K)\) is a difference \((H, D_H)\)-module bialgebra.

**Proof.** First suppose that the pair \((D_K, D_H)\) can be extended to a difference operator \(D\) on \(K\#H\). By Corollary 5.4, \(D_K \ast \text{id}, D_H \ast \text{id}\) and \(D \ast \text{id}\) are all Hopf algebra homomorphisms. Then for any \(x \in K, \ a \in H\),

\[
(D \ast \text{id})(1#a)(x#1)) = (D \ast \text{id})((a \rightarrow a)x\#1)
= (D \ast \text{id})((a \rightarrow x)\#1)(1#a)
= (D \ast \text{id})(a \rightarrow x)(1#D_H \ast \text{id})a
= (D_K \ast \text{id})(a \rightarrow x)(D_H \ast \text{id})a.
\]

On the other hand,

\[
(D \ast \text{id})(1#a)(x#1)) = (D \ast \text{id})(1#a)(D \ast \text{id})(x#1)
= (1#(D_H \ast \text{id})a)((D_K \ast \text{id})x\#1)
= ((D_H \ast \text{id})a \rightarrow (D_K \ast \text{id})x)(D_H \ast \text{id})a.
\]

Applying \(\text{id} \otimes \varepsilon_H\) to both equalities, we conclude that Eq. (17) holds, and \((K, D_K)\) is a difference \(H\)-module bialgebra.
Conversely, assume that Eq. (10) holds. We define a linear operator $D$ on $K\#H$ by

$$D(x\#a) := D_K(x_1)x_2(D_H(a_1) \rightarrow S_K(x_3))\#D_H(a_2), \quad \forall x \in K, a \in H.$$ 

Then clearly $D|_K = D_K$ and $D|_H = D_H$, and we have

$$(D \ast \text{id})(x\#a) = D(x_1\#a_1)(x_2\#a_2)$$

$$= (D_K(x_1)x_2(D_H(a_1) \rightarrow S_K(x_3))\#D_H(a_2))(x_3\#a_3)$$

$$= D_K(x_1)x_2(D_H(a_1) \rightarrow S_K(x_3))(D_H(a_2) \rightarrow x_4)\#D_H(a_4)$$

$$= D_K(x_1)x_2(D_H(a_1) \rightarrow S_K(x_3)x_4)\#D_H(a_2)a_3$$

$$= D_K(x_1)x_2\#D_H(a_1)a_2$$

$$= (D_K \ast \text{id})x\#(D_H \ast \text{id})a.$$ 

Also, since both $D_K$ and $D_H$ are coalgebra homomorphisms, $K$ is a cocommutative $H$-module bialgebra and $H$ is also cocommutative, we conclude that $D$ is a coalgebra homomorphism.

For $x, y \in K$ and $a, b \in H$, we have

$$(D \ast \text{id})(x\#a)(y\#b)) = (D \ast \text{id})(x(a_1 \rightarrow y)\#a_2b)$$

$$= (D_K \ast \text{id})(x(a_1 \rightarrow y))(D_H \ast \text{id})(a_2b)$$

$$= (D_K \ast \text{id})x(D_K \ast \text{id})(a_1 \rightarrow y)(D_H \ast \text{id})a_2(D_H \ast \text{id})b$$

$$= (D_K \ast \text{id})x((D_H \ast \text{id})a_1 \rightarrow (D_K \ast \text{id})y)(D_H \ast \text{id})a_2(D_H \ast \text{id})b$$

$$= (((D_K \ast \text{id})x(D_H \ast \text{id})a)((D_K \ast \text{id})y)(D_H \ast \text{id})b)$$

$$= (D \ast \text{id})(x\#a)(D \ast \text{id})(y\#b),$$

where the fourth equality uses Eq. (10). Hence, $D \ast \text{id}$ is an algebra homomorphism, and $D$ is a difference operator on $K\#H$ such that $D|_K = D_K$ and $D|_H = D_H$ by Theorem 5.4.

On the other hand, $D(x\#a) = D((x\#1)(1\#a))$ is determined by $D(x\#1) = D_K(x)\#1$ and $D(1\#a) = 1\#D_H(a)$ via Eq. (11), showing that such extension of difference operators is unique.

\[ \square \]

**Remark 5.9.** In the context of Theorem 5.8, when $H$ acts trivially on $K$, the compatibility condition in Eq. (10) holds automatically. Then the tensor product $(K\otimes H, D_K\otimes D_H)$ of difference Hopf algebras is also a difference Hopf algebra.

Let $\Phi : G \rightarrow \text{Aut}(H)$ be an action of a group $G$ on $H$. Then $\Phi$ can be linearly extended to a module bialgebra action $\tilde{\Phi} : kG \rightarrow \text{End}(kH)$ by

$$\tilde{\Phi}(\sum_{g \in G} a_g g)(\sum_{h \in H} b_h h) = \sum_{g \in G, h \in H} a_g b_h \Phi(g)(h).$$

Likewise, a group crossed homomorphism can be linearly extended to a Hopf algebra crossed homomorphism on the group Hopf algebras.

By Theorem 5.3, for cocommutative difference Hopf algebras $(K, D_K)$ and $(H, D_H)$ such that $(K, D_K)$ is a difference $(H, D_H)$-module bialgebra, one can define the smash product difference Hopf algebra $(K\#H, D)$ such that $D|_K = D_K$ and $D|_H = D_H$.  

CROSSED HOMOMORPHISMS AND CARTIER-KOSTANT-MILNOR-MOORE THEOREM 19
On the other hand, the structure theorem of pointed cocommutative Hopf algebras, called the Cartier-Kostant-Milnor-Moore theorem, states that such a Hopf algebra $H$ is isomorphic to the smash product Hopf algebra $U(P(H)) \# kG(H)$ of a universal enveloping algebra and a group algebra; see e.g. [20, Theorem 15.3.4].

Next we strengthen this theorem to a structure theorem of pointed cocommutative difference Hopf algebras.

**Theorem 5.10. (Difference Cartier-Kostant-Milnor-Moore Theorem)** A pointed cocommutative difference Hopf algebra $(H, D)$ is isomorphic to the smash product difference Hopf algebra $U(P(H)) \# kG(H)$, where $U(P(H))$ is the difference $kG(H)$-module bialgebra induced by the conjugation action of $G(H)$ on $P(H)$.

**Proof.** Let $G := G(H)$ with $D_G := D|_G$, and $g := P(H)$ with $D_g := D|_g$. By Corollary [2.4], $(G, D_G)$ is a difference group and $(g, D_g)$ is a difference Lie algebra.

By Proposition [2.7], the difference operator $D_g$ of $g$ can be uniquely extended to $U(g)$. On the other hand, the difference operator $D_G$ on $G$ is linearly extended to $kG$. We will use the same notation $D_g$ (resp. $D_G$) for such extended difference operator on $U(g)$ (resp. $kG$).

By the Cartier-Kostant-Milnor-Moore theorem, there exists a Hopf algebra isomorphism

$$\Phi : U(g) \# kG \to H,$$

where $U(g)$ is the $kG$-module bialgebra induced by the conjugation action of $G$ on $g$. We denote the extended action of $G$ on $U(g)$ by $\to$.

By Corollary [2.9], we have $\Phi D_g = D \Phi|_{U(g)}$. Also, it is clear that $\Phi D_G = D \Phi|_{kG}$. Then

$$(D_g \ast \text{id})(g \to u) = \Phi^{-1}(D \ast \text{id})\Phi(g \to u)$$

$$= \Phi^{-1}(D \ast \text{id})(\Phi(g)\Phi(u)\Phi(g)^{-1})$$

$$= \Phi^{-1}(\Phi(D_g \ast \text{id})g\Phi(D_g \ast \text{id})u(\Phi(D_g \ast \text{id})g)^{-1})$$

$$= \Phi^{-1}(\Phi((D_g \ast \text{id})g \to (D_g \ast \text{id})u))$$

$$= (D_g \ast \text{id})g \to (D_g \ast \text{id})u,$$

for any $u \in U(g)$ and $g \in G$. Here we identify $U(g)$ and $kG$ as subalgebras of $U(g) \# kG$. Hence, Eq. (11) holds, and $(U(g), D_g)$ is a difference $(kG, D_G)$-module bialgebra.

Therefore, we have the smash product difference Hopf algebra $(U(g) \# kG, D_g)$, where

$$D_g(u \# g) := D_g(u_1)u_2(D_g(g) \to S(g(u_2)))#D_g(g), \quad \forall u \in U(g), \ g \in G.$$ 

Now it remains to verify that $\Phi D_g = D \Phi$. Indeed,

$$\Phi(D_g(u \# g)) = \Phi(D_g(u_1))u_2(D_g(g) \to S(g(u_2)))\Phi(D_g(g))$$

$$= \Phi(D_g(u_1))\Phi(u_2)\Phi(D_g(g) \to S(g(u_2)))\Phi(D_g(g))$$

$$= D(\Phi(u_1))\Phi(u_2)D(\Phi(g))S(\Phi(u_2))D(\Phi(g))^{-1}D(\Phi(g))$$

$$= D(\Phi(u_1))\Phi(u_2)D(\Phi(g))S(\Phi(u_2))$$

$$= D(\Phi(u)\Phi(g)) = D(\Phi(u \# g)).$$

Hence, $\Phi$ gives the desired difference Hopf algebra isomorphism.  \[\square\]
6. Examples of difference operators on Hopf algebras

We end the paper with classifying difference operators on several Hopf algebras.

Example 6.1. Consider the difference operators on the tensor Hopf algebra \((TV, \cdot, \Delta^{\mathrm{cosh}})\) with the coshuffle coproduct \(\Delta^{\mathrm{cosh}}\). Since this is a connected cocommutative Hopf algebra and is the universal enveloping algebra of the free Lie algebra \(\text{Lie}(V) = P(TV)\), by Proposition 2.7 and Theorem 2.8, we see that difference operators on \((TV, \cdot, \Delta^{\mathrm{cosh}})\) are in one-to-one correspondence with difference operators on the Lie algebra \(\text{Lie}(V) = P(TV)\). By Corollary 5.3 (also see [14]), a linear map \(D : \text{Lie}(V) \rightarrow \text{Lie}(V)\) is a difference operator if and only if \(\text{id} + D\) is a Lie algebra endomorphism on \(\text{Lie}(V)\); while Lie algebra endomorphisms on \(\text{Lie}(V)\) are in one-to-one correspondence with \(\text{Hom}(V, \text{Lie}(V))\) by the universal property of \(\text{Lie}(V)\). Thus, we conclude that the set of difference operators on the tensor Hopf algebra \(TV\) is in bijection with \(\text{Hom}(V, \text{Lie}(V))\).

Next we give difference operators on two basic low-dimensional noncommutative and non-cocommutative Hopf algebras as examples of difference Hopf algebras.

Example 6.2. We classify difference operators on Sweedler’s 4-dimensional Hopf algebra

\[ H_4 = \mathbb{k}\{1, g, x, gx | g^2 = 1, x^2 = 0, gx = -xg\}, \]

with its coalgebra structure and its antipode given by

\[ \Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \epsilon(g) = 1, \quad \epsilon(x) = 0, \quad S(g) = g, \quad S(x) = -gx. \]

Further \(G(H_4) = \{1, g\}\) and \(P(H_4) = 0\).

Let \(D : H_4 \rightarrow H_4\) be a difference operator. Then it restricts to a difference operator \(D\) on the group \(G(H_4) = \{1, g\}\). Thus there are two cases to consider.

(i) Suppose \(D(g) = g\). Then as \(D\) is a coalgebra homomorphism, we have

\[ \Delta(D(x)) = D(x) \otimes 1 + g \otimes D(x). \]

That is, \(D(x) \in P_{1,g}(H_4) = \mathbb{k}(1 - g) \oplus \mathbb{k}x\). Thus there exist \(\alpha, \beta \in \mathbb{k}\) such that \(D(x) = \alpha(1 - g) + \beta x\).

Then Eq. (1) implies

\[ D(gx) = D(g)gD(x)g^{-1} = g^2(\alpha(1 - g) + \beta x)g = \alpha(g - 1) + \beta xg, \]

\[ D(xg) = D(x)D(g) + D(g)xD(g) + D(g)gD(g)(-gx) = (\alpha(1 - g) + \beta x)g - 2x. \]

So \(D(gx) \neq D(-xg)\), a contradiction. Therefore there is no difference operator in this case.

(ii) Suppose \(D(g) = 1\). Then

\[ \Delta(D(x)) = D(x) \otimes 1 + 1 \otimes D(x). \]

Thus \(D(x)\) is in \(P(H_4) = 0\), and hence \(D(x) = 0\). Then Eq. (2) implies that

\[ D(gx) = D(g)gD(x)g = 0, \]

\[ D(xg) = D(x)D(g) + D(g)xD(g) + D(g)gD(g)(-gx) = x - g^2x = 0. \]

Thus \(D\) defines a difference operator.

In summary, \(H_4\) only has one difference operator \(D = u \circ \epsilon\), namely \(D(g) = 1\) and \(D(x) = 0\).
Example 6.3. We finally determine bijective difference operators on the Kac-Paljutkin Hopf algebra \( H_8 \), the noncommutative and non-cocommutative semisimple Hopf algebra of dimension 8 which has been widely studied; see e.g. [15, 32, 35].

A basis for \( H_8 \) is given by \( \{1, x, y, xy, z, xz, yz, xyz \} \) with the relations
\[
x^2 = y^2 = 1, \quad z^2 = \frac{1}{2}(1 + x + y - xy), \quad xy = yx, \quad zx = yz, \quad zy = xz.
\]
The coalgebra structure and the antipode are defined by
\[
\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{2}(1 \otimes 1 + 1 \otimes x + y \otimes 1 - y \otimes x)(z \otimes z),
\]
\[
e(x) = e(y) = e(z) = 1, \quad S(x) = x, \quad S(y) = y, \quad S(z) = z.
\]
In particular, \( G(H_8) = \{1, x, y, xy\} \) and \( C_4 := \text{span}_k\{z, xz, yz, xyz\} \) is the unique simple subcoalgebra of \( H_8 \) of dimension > 1.

Now suppose that \( D : H_8 \rightarrow H_8 \) is a bijective difference operator, then \( D|_{G(H_8)} \) is an abelian group automorphism, and \( D(C_4) = C_4 \), since \( D(C_4) \neq 0 \) as \( e(D(z)) = e(z) = 1 \).

First let \( D \) be a difference operator such that \( D(x) = x \) and \( D(y) = y \).

As \( D(z) \in C_4 \), we can set \( D(z) = pz \) with nonzero \( p \in kG(H_8) \). For any \( g, h \in G(H_8) \), we obtain that \( g \sigma(h)z^2 = g \sigma(h)p^2z^2 \) by the equality \( D((gz)(hz)) = D((gz)1)D(hz) \), where \( \sigma \) denotes the linear operator on \( kG(H_8) \) defined by \( \sigma(1) = 1, \sigma(x) = y, \sigma(y) = x, \sigma(xy) = xy \).

Hence, we need \( p^2z^2 = z^2 \), which implies \( p^2 = 1 \) with further computations. There are 16 solutions of \( p \) as listed below:
\[
\pm 1, \ \pm x, \ \pm y, \ \pm xy, \ \pm \frac{1}{2}(1 + x + y - xy), \ \pm \frac{1}{2}(1 + x - y + xy), \ \pm \frac{1}{2}(1 - x + y + xy), \ \pm \frac{1}{2}(-1 + x + y + xy).
\]

On the other hand, we require
\[
\Delta D(z) = (D \otimes D)\Delta(z)
\]
\[
= \frac{1}{2}(D(z) \otimes D(z) + D(z) \otimes D(xz) + D(yz) \otimes D(z) - D(yz) \otimes D(xz))
\]
\[
= \frac{1}{2}(1 \otimes 1 + 1 \otimes y + x \otimes 1 - x \otimes y)(D(z) \otimes D(z)).
\]

Then the only choices for \( D(z) \) are
\[
\frac{1}{2}(1 + x + y - xy)z, \ \frac{1}{2}(1 + x - y + xy)z, \ \frac{1}{2}(1 - x + y + xy)z, \ \frac{1}{2}(-1 + x + y + xy)z.
\]

For every other standard basis element \( gz \in C_4 \) with \( g \in G(H_8) \), we correspondingly have
\[
\Delta D(gz) = \Delta(D(z)g) = ((D \otimes D)\Delta(z))(g \otimes g) = (D \otimes D)((g \otimes g)\Delta(z)) = (D \otimes D)\Delta(gz).
\]

So \( D \) is clearly a coalgebra homomorphism.

Now it is straightforward to check that the bijective difference operators \( D \) on \( H_8 \) such that \( D(x) = x \) and \( D(y) = y \) are given by
\[
\text{(1) } D_1(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, \frac{1}{2}(1 + x + y - xy)z, \frac{1}{2}(1 - x + y + xy)z, \frac{1}{2}(1 + x - y + xy)z, \frac{1}{2}(-1 + x + y + xy)z),
\]
\[
\text{(2) } D_2(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, \frac{1}{2}(1 + x - y + xy)z, \frac{1}{2}(-1 + x + y + xy)z, \frac{1}{2}(1 + x + y - xy)z, \frac{1}{2}(1 - x + y + xy)z),
\]
(3) $D_3(1, x, y, xy, z, xz, yz, xyz)$
   $= (1, x, y, xy, \frac{1}{2}(1-x+y+xy)z, \frac{1}{2}(1-x+y-xy)z, \frac{1}{2}(1-x-y+xy)z)$

(4) $D_4(1, x, y, xy, z, xz, yz, xyz)$
   $= (1, x, y, xy, \frac{1}{2}(1-x+y+xy)z, \frac{1}{2}(1-x-y+xy)z, \frac{1}{2}(1-x+y-xy)z)$

Next let $D$ be a difference operator such that $D(x) = y$ and $D(y) = x$. By a similar analysis, we find that all the bijective difference operators $D$ on $H_8$ with this property are given by

(1) $D_5(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, z, xz, yz, xyz)$
(2) $D_6(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, z, xz, yz, xyz)$
(3) $D_7(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, z, xz, yz, xyz)$
(4) $D_8(1, x, y, xy, z, xz, yz, xyz) = (1, x, y, xy, z, xz, yz, xyz)$

On the other hand, there is no bijective difference operator $D$ on $H_8$ such that $D(x) = xy$ or $D(y) = xy$. Hence, we have determined all bijective difference structures on $H_8$.

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