Relativity of representations in quantum mechanics

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Abstract

Only the position representation is used in introductory quantum mechanics and the momentum representation is not usually presented until advanced undergraduate courses. To emphasize the relativity of the representations of the abstract formulation of quantum mechanics, two examples of representations related to the operators $\alpha X + (1 - \alpha)P$ and $\frac{1}{2}(XP + PX)$ are presented.

I. INTRODUCTION

The position representation is adopted in every introductory text on quantum mechanics. In this representation, the position observable $X$ for a particle in one dimension is associated with multiplication by a real variable $x$ and the momentum observable $P$ with the derivative operator $-i\partial_x$. The time evolution is described by Schrödinger’s equation whose solution determines the general state of the system $\psi(x, t)$ or the stationary states $\psi(x)$ associated with a fixed value of the energy. (We will use the convention $\hbar = 1$ and denote the derivative operator $d/dx$ by $\partial_x$. We will also consider only one spatial dimension with a trivial generalization to three dimensions.)

When students reach an advanced undergraduate quantum mechanics course, they may arrive with the misconception that the position representation is the only one or that it is a privileged one. Then they encounter, as the second choice, the momentum representation where the observables $(X, P)$ are represented by $(i\partial_p, p)$. The students soon learn that this choice is fully equivalent, not secondary, to the position representation. Other possible representations are usually ignored. To fully appreciate the beauty of the mathematical
formalism of quantum mechanics in abstract Hilbert spaces, it is convenient to present the
position and momentum representations as just two particular representations among an
infinite number of choices associated with all possible observables that can be constructed
as functions of position and momentum. Of course, in practical cases where the potential
depends only on position, the position representation is more convenient because it leads
to simpler differential equations. And in many cases the momentum representation is more
convenient for stating the initial conditions for the system. The position and momentum
representations are preferred for practical simplicity, but they are not essential choices of
the theory. An analogous situation occurs when a reference frame, for example, the center
of mass or rest frame, is chosen for simplicity although any other choice is equally valid.

In this paper we will review how the position and momentum representations emerge
from the abstract formulation of quantum mechanics, and we will see some examples of
other representations that present some interesting physical and mathematical features. The
representations discussed here can be used to emphasize the \textit{relativity of representations} in
the teaching of quantum mechanics. Many exercises are suggested although not explicitly
stated.

\section{II. ABSTRACT FORMALISM}

The state of a particle in a one-dimensional space is an element $\psi$ of an abstract Hilbert
space $\mathcal{H}$ of infinite dimensions. In addition, the position and momentum observables
are associated with hermitian operators with continuous spectra $X$ and $P$. The physical
requirement that the momentum operator be the generator of translations, that is
$X + a 1 = \exp(i a P)X \exp(-i a P)$, leads to the mathematical requirement that these opera-
tors satisfy the commutation relation $[X, P] = i$. Let $\{\varphi_x\}$ and $\{\phi_p\}$ denote the two Hilbert
space bases associated with the position and momentum operators, that is, their eigenvectors
correspond to the eigenvalues $x$ and $p$ respectively. The physical requirement that position
and momentum be independent, in the sense that any momentum is compatible with any
position, requires that these two bases should be \textit{unbiased}, that is, any element $\varphi_x$ has an
equal “projection” along every element $\phi_p$. Stated precisely, the norm of the inner product
$|\langle \varphi_x, \phi_p \rangle|$ should be a constant independent of $x$ and $p$ and can depend only on the dimension
of the Hilbert space (actually, this constant is undetermined because the basis elements are
not normalizable. This difficulty is related to the rigorous treatment that will be suggested
in Sec. [V]).

Any state of the system can be expanded with respect to one of the bases discussed.
However, besides the bases associated with the position and momentum operators, we can define other bases associated with any observable \( F(X,P) \) that depends on position and momentum and is described by a properly defined hermitian operator. In the following section we will see how the bases \( \{ \varphi_x \} \) and \( \{ \phi_p \} \) lead to the position and momentum representations respectively, and how any other basis can define a different representation of quantum mechanics.

### III. POSITION, MOMENTUM, AND THE RELATIVITY OF REPRESENTATIONS

Let us consider the expansion of a state \( \psi \) in the basis \( \{ \varphi_x \} \) associated with the position operator,

\[
\psi = \int_{-\infty}^{\infty} dx \, \langle \varphi_x, \psi \rangle \varphi_x .
\]  

Due to the required normalization of \( \psi \), the coefficients of the expansion, given by the function \( \psi(x) = \langle \varphi_x, \psi \rangle \), must belong to the Hilbert space \( L_2(\mathbb{R}) \) of all square integrable complex functions of a real variable \( x \). The function \( \psi(x) \) is then the position representation of the state. The position representation results from the isomorphism between \( \mathcal{H} \) and \( L_2(\mathbb{R}) \), defined by the basis \( \{ \varphi_x \} \). One can easily determine that in this representation the eigenvectors of the position and momentum operators are

\[
\varphi_a(x) = \delta(x - a) \\
\phi_g(x) = \frac{1}{\sqrt{2\pi}} \exp(igx).
\]  

It is important to emphasize to students that in Eqs. (2) and (3), the physically relevant quantities are \( a \) and \( g \), whereas \( x \) is just a mathematical variable for the functions in \( L_2(\mathbb{R}) \).

In an equivalent way we obtain the momentum representation from the isomorphism between \( \mathcal{H} \) and \( L_2(\mathbb{R}) \), defined by the basis \( \{ \phi_p \} \). In this representation, where the state \( \psi(p) = \langle \phi_p, \psi \rangle \) is an element of \( L_2(\mathbb{R}) \), the eigenvectors of the position and momentum operators are given by

\[
\varphi_a(p) = \frac{1}{\sqrt{2\pi}} \exp(-iap) \\
\phi_g(p) = \delta(p - g).
\]  

Here again, it is important to point out that the physically relevant quantities are \( a \) and \( g \), whereas \( p \) is just a mathematical variable.
These two representations arise from two isomorphisms of the abstract Hilbert space \( \mathcal{H} \), and the isomorphism between them is defined by the Fourier transformation. This subject is treated with more or less detail in all advanced books of quantum mechanics but, in many cases, without reference to the general abstract Hilbert space. It is however convenient to make this reference in order to place both representations on an equal footing and to suggest the existence of many other, equally valid, possible representations. The relativity of representation implies some sort of completeness of quantum mechanics in the sense that it guarantees that the probability distribution for every observable \( F(X, P) \), represented by a properly defined hermitian operator, can be obtained from \( \psi \in \mathcal{H} \). To extract this information, encoded in \( \psi \), we must express the state in the \( F \) representation, that is \( \psi(f) = \langle \chi_f, \psi \rangle \), where \( \{ \chi_f \} \) is the basis associated with the operator \( F(X, P) \).

We will present here two additional representations that turn out to be interesting from the physical and mathematical point of view. However, before presenting them, it may be useful to mention a mathematical difficulty that is often ignored in undergraduate courses, but that should be presented more rigorously. This difficulty is sketched in the next section, but can be skipped if no mathematical rigor is desired.

IV. RIGGED HILBERT SPACE

It can be proven that the commutation relation \([X, P] = i\) implies that the position and momentum operators are unbound and that they do not have eigenvectors in the Hilbert space. It is a simple exercise to prove that the assumption of the existence of eigenvectors of, say \( X \), leads to a contradiction when we calculate the expectation value of the commutator \([X, P]\). Indeed, the functions given in Eqs. (2) and (3) or those of Eqs. (4) and (5) clearly do not belong to \( L_2(\mathbb{R}) \) because they are not square integrable. The bases do not belong to the Hilbert space \( \mathcal{H} \), but we can anyway expand any element of the Hilbert space in these bases. In order to achieve this expansion we must extend the Hilbert space, \( \mathcal{H} \rightarrow \mathcal{H}' \) to include all such bases. The space so obtained is called a rigged Hilbert space or Gelfand triplet \( \mathcal{H}^0 \subseteq \mathcal{H} \subseteq \mathcal{H}' \) and is presented in some advanced texts.\(^1\) A rigorous but very clear exposition of the rigged Hilbert space is given in Ref.\(^2\).

V. INTERPOLATING REPRESENTATION

As an example of another possible representation, we consider the isomorphism defined by the basis \( \{ \eta_\lambda \} \) of the eigenvectors corresponding to the eigenvalue \( \lambda \) of a family of operators
S(α) that is defined to linearly interpolate between position and momentum:

\[ S(\alpha) = \alpha X + (1 - \alpha)P, \quad \alpha \in [0, 1]. \]  \tag{6}

In Eq. (6) we have ignored scale factors that make \( X \) and \( P \) dimensionless. We have then

\[ S(\alpha) \eta_\lambda = \lambda \eta_\lambda. \] \tag{7}

Using this basis, any state can be expanded as

\[ \psi = \int_{-\infty}^{\infty} d\lambda \langle \eta_\lambda, \psi \rangle \eta_\lambda = \int_{-\infty}^{\infty} d\lambda \psi(\lambda) \eta_\lambda. \] \tag{8}

In order to have an expression for \( \eta_\lambda \) in the position representation, we must write and solve Eq. (7) in \( L_2(\mathbb{R}) \). That is,

\[ [\alpha x - i(1 - \alpha) \partial_x] \eta^\alpha_\lambda(x) = \lambda \eta^\alpha_\lambda(x), \] \tag{9}

where we have written explicitly the parameter \( \alpha \). It is not difficult to find that the solution of this equation is \( K(\alpha, \lambda) \exp \left[ -\frac{i}{2} \alpha (x - \lambda/\alpha)^2 \right] \), where the constant \( K(\alpha, \lambda) \) is independent of \( x \) but may depend on \( \lambda \) and \( \alpha \). We can now choose \( K \) such that the eigenvector \( \eta^\alpha_\lambda(x) \) tends to \( \exp(i\lambda x) \) when \( \alpha \to 0 \) and to \( \delta(x - \lambda) \) when \( \alpha \to 1 \) as required by Eqs. (2) and (3).

The appropriate choice for \( K \) yields

\[ \eta^\alpha_\lambda(x) = \frac{1}{\sqrt{2\pi}} \exp \left[ i \left( \frac{x^2}{2\alpha} + \frac{\pi}{4} \right) \right] \exp \left[ -\frac{i}{2} \frac{\alpha}{1 - \alpha} \left( x - \frac{\lambda}{\alpha} \right)^2 \right]. \] \tag{10}

Indeed, the limit \( \alpha \to 0 \) leads to

\[ \eta^0_\lambda(x) = \frac{\exp(i\pi/4)}{\sqrt{2\pi}} \exp(i\lambda x). \] \tag{11}

For \( \alpha \to 1 \), we must use (prove) the unusual expression for the Dirac delta function

\[ \delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \frac{\exp(i\pi/4)}{\sqrt{\pi}} \exp \left( -\frac{i}{\varepsilon} x^2 \right), \] \tag{12}

which results in

\[ \eta^1_\lambda(x) = \exp \left( \frac{i\lambda^2}{2} \right) \delta(x - \lambda). \] \tag{13}

These eigenfunctions are delta function normalized as is usual for operators with continuous spectra, that is, \( \langle \eta^\alpha_\lambda, \eta^\alpha_{\lambda'} \rangle = \delta(\lambda - \lambda') \). There are many possible exercises in this representation. In particular, it is interesting to study the mathematical transformation between \( \psi(x) \) and
ψ(λ) as function of the continuous parameter α that interpolates smoothly between the identity and the Fourier transformation.

Instead of the linear interpolation of Eq. (7), we may consider a phase space rotation and define the operator

\[ S(\theta) = X \cos \theta + P \sin \theta . \tag{14} \]

The treatment for this case is identical to the case just presented, with the replacement \( \alpha \to \cos \theta \) and \((1 - \alpha) \to \sin \theta\). One possible interest in this family of operators follows from the commutation relation \([S(\theta), S(\theta')] = i \sin(\theta' - \theta)\), indicating that for \(\theta' = \theta + \pi/2\), we have a pair of canonical conjugate observables that play the same role as position and momentum.

VI. CORRELATION REPRESENTATION

Another representation of quantum mechanics arises when we build an isomorphism with the basis \(\{\xi_\gamma\}\) associated with the eigenvectors of the correlation operator defined as the symmetrized product of position and momentum.

\[ C = \frac{1}{2} \{XP\} = \frac{1}{2}(XP + PX) . \tag{15} \]

The eigenvalue equation

\[ C \; \xi_\gamma = \gamma \; \xi_\gamma \tag{16} \]

can be written in the position or momentum representation and solved to find the associated eigenfunctions. Notice however that the correlation operator commutes with the parity operator \(P\) which changes \(X \to -X\) and \(P \to -P\). Students can easily prove that this property implies that the eigenvectors \(\{\xi_\gamma\}\) must have definite parity, either even \(\{\xi_\gamma^e\}\) or odd \(\{\xi_\gamma^o\}\) (the upper index stands for gerade (even) or ungerade (odd) under the parity transformation).

The explicit treatment of the above equation in the position representation provides the two degenerate solutions.

\[ \xi_\gamma^e(x) = K(\gamma) \; |x|^{-\frac{1}{2} + i\gamma} = K(\gamma) \frac{\exp(i\gamma \ln |x|)}{\sqrt{|x|}} \tag{17} \]

\[ \xi_\gamma^o(x) = K(\gamma) \; \text{sign}(x)|x|^{-\frac{1}{2} + i\gamma} = K(\gamma) \text{sign}(x) \frac{\exp(i\gamma \ln |x|)}{\sqrt{|x|}} , \tag{18} \]

where \(K(\gamma)\) is an arbitrary constant that can be fixed by requiring the delta function normalization of the eigenvectors. The momentum representation of the eigenfunctions can be
obtained in the same way, that is, by writing Eq. (16) in terms of $p$ and $\partial_p$, or by taking the Fourier transform of Eqs. (17) and (18) or, most easily, by noticing that the operator $C$ in the momentum representation is obtained from the position representation by replacing $x \rightarrow p$ and taking the complex conjugate. Therefore, if $\xi_\gamma(x)$ is an eigenfunction in the position representation, then $\xi_\gamma^*(p)$ is the corresponding eigenfunction in the momentum representation. These eigenfunctions have the interesting property that their Fourier transformation is equal to their complex conjugate.

The correlation operator discussed here has been ignored in most text books although it is relevant, because it corresponds to an extra contribution to the uncertainty relations in the improved version given by Schrödinger.

$$\Delta_x^2 \Delta_p^2 \geq \frac{\hbar^2}{4} + (\langle C \rangle - \langle X \rangle \langle P \rangle)^2.$$  

(19)

Another interesting property of the correlation operator is that the term due to the correlation in the inequality (19) (for general observables) has been related to nonseparability in compound systems.

\section*{VII. CONCLUSION}

Two possible representations have been sketched in addition to the position and momentum representations. Many other examples of representations can be produced and they all illustrate the importance of the relativity of representations in the abstract formulation of quantum mechanics. From the mathematical point of view, this work presents a didactic approach to a general theory of transformations, because any pair of representations define a transformation of which, the Fourier transformation is just one example corresponding to two representations related to two unbiased bases.

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