Achievement Stability Set for Parametric Rough Linear Goal Programming Problem

F. A. Farahat and M. A. ElSayed

Higher Technological Institute, 10th of Ramadan city, Cairo, Egypt; Department of Basic Engineering Sciences, Faculty of Engineering, Benha University ElQaiyoubia, Egypt

ABSTRACT

This paper introduces an algorithm for researching the achievement stability set for parametric rough linear goal programming (PRLGP) issues with parameters in the achievement function and roughness in goal and system constraints. The proposed goal programming model has two types of uncertainty. We transform the PRLGP into an upper approximation model and a lower approximation model. Then, the Lexicographical goal programming method is utilized to solve such upper and lower approximation models iteratively to avoid the complexity of the attainment issue. This model has been applied as it enables the decision-maker to articulate the weights for goals however, for the sub-goals it shall be complicated as they possess the same measure. Finally, to clarify how to investigate the achievement stability set for the PRLGP, an algorithm and a numerical illustration was given.

ARTICLE HISTORY

Received 20 October 2019
Revised 12 September 2020
Accepted 26 September 2020

KEYWORDS

Goal programming; lexicographical approach; rough set; matching extreme points; parametric analysis

1. Introduction

For many real-world problems, it may not be possible to satisfy certain specified goals within given constraints. The problem then becomes one of maximizing the degree of attainment of these goals. Goal programming (GP) is designed to solve this problem of satisfying (possibly conflicting) goals as well as possible when some of them have a higher priority than others [1,2]. GP is an effective methodology for the modeling, solving, and analysis of multiple objective programming issues. GP was firstly introduced by Charnes and Cooper [1]. GP for stochastic systems was examined by Contini [3]. For solving GP problem, it is necessary to establish a hierarchy of importance among these goals so that lower priority goals are considered only after higher priority ones are satisfied [2,4,5]. Since it is not constantly conceivable to achieve each goal as indicated by the decision-maker (DM) wants, GP tries to attain a satisfying solution. The objective function of GP problem focused on minimizing the deviation between the available resources and the required goals [2,4,6,7].

The lexicographical methodology is an amazing strategy for solving the linear, nonlinear, integer, and other GP issues [2]. GP in a fuzzy environment has been introduced by Narasimhan [7]. Multi-choice mixed-integer GP optimization for real problems in a sugar
and ethanol milling company has been applied by DaSilva [6]. Fuzzy goal programming (FGP) via a parametric approach was exhibited by Li [8]. FGP has been used to solve bi-level and multi-level optimization issues as in [9,10]. Also, FGP was applied to a stochastic fuzzy multi-level fractional optimization issue [11].

Parametric programing has a remarkable role in mathematical optimization models. It is mainly based on detecting the variation on the behavior of an efficient set or the optimum value whether the entered data are varied. So, it might be viewed as a technique for researching uncertainty. The basis and theorems of parametric convex programing, the stability set of the first and second kinds have been constructed by Osman [12,13]. Parametric research for integer MONPP has been shown by Emam [14]. The stability of Multi-objective nonlinear programing problems (MONPP) with an uncertain domain was considered by Kassem and Ammar [15]. The procedures of the stability set of the first kind for the integer GP issue has been introduced by Saad and Sharif [16]. The stability of proper efficient solutions in multi-objective fractional programing problems under fuzziness has been studied by Saad [17]. The achievement stability set for parametric linear GP has been introduced by Osman et al. [18]. Parametric multi-level multi-objective fractional programing problems with fuzziness in the constraints have been exhibited by Osman et al. [19]. Recently, Elsayed and Farahat studied the achievement stability set for parametric linear FGP problems [20].

The newest theory for the joint management of vagueness and uncertainty is that of Rough Sets Theory (RST) proposed by Pawlak [21]. It is an effective theoretical framework for discussion about the knowledge that can classify objects. An object in a crisp and ordinary set is completely determined, while in a rough set, it is approximately determined based on partial knowledge. In RST, any vague concept is replaced by a pair of precise concepts called the lower and the upper approximations of the vague concept. For a vague concept \(X\), a lower approximation is contained in all objects which surely belong to the concept \(X\) and an upper approximation contains all objects which possibly belong to the concept \(X\) [22].

In recent decades, RST has been used as the fundamental tool in many applications including optimization problems [22,23]. For the mathematical programing problems in the crisp environment, the aim is to maximize (minimize) an objective function over a certain set of feasible solutions. However, in many problems, the DM may not be able to specify the objective and/or the feasible set, exactly. One of the ways by which DM can specify them by using RST. The RST has a significant role in the following applications artificial intelligence, machine learning, and decision analysis, knowledge discovery from databases, expert systems, and pattern recognition [22].

Osman et al. classified the rough programing problems into three classes according to the place of the roughness and a theoretical framework of convexity in rough programing has been presented [24]. Characterizing the solutions of rough programing problems was investigated by Youness [25]. Rough multiple objective programing from a new point of view has been introduced by Atteya [26]. MONPP with rough intervals in the constraints has been exhibited by Khalifa [27]. The duality of multi-objective rough convex programing problems was setup by Elsisy et al. [28]. A class of multi-objective mathematical programing problems in a rough environment has been developed by Hamzehee et al. [22]. Elsisy et al. presented a qualitative analysis of parametric rough convex programing when parameters in the objective function and roughness in the constraints [29]. The solution of rough goal bi-level multi-objective linear programing problem was studied by Sultan et al. [30]. The
linear GP problem in a rough environment has been discussed by Maaty and Farahat [31]. Rough multi-level linear programming issue has been exhibited by Emam et al. [32].

1.1. Motivation and Contribution

The previous studies presented in this era focused on making the solution stable. Besides, the achievement stability set introduced by Osman et al. [18] assumed a deterministic case even though real-world problems possess uncertain natures which led us to present the current study. In this study, the achievement stability set is proposed which makes the achievement function for rough GP problem stable. The current model also possesses two types of uncertainty the first type is the parameters which represent the weights in the achievement function. The second type of uncertainty is roughness in both goal and system constraints. Moreover, we introduce some new definitions for the achievement stability set of rough GP problem. This model has been handled because it is very easy for the DM to put the weights for goals but, for the sub-goals, it will be difficult as they have the same measure.

In this paper, we present an algorithm for determining the achievement stability set for the PRLGP issue with parameters in the achievement function and roughness in both goal and system constraints. Firstly, we transform the PRLGP into an upper approximation model (UAM) and a lower approximation model (LAM). Then, the Lexicographical GP method is utilized to solve such UAM and LAM iteratively. The procedures for obtaining the achievement stability set for the PRLGP issue is introduced in detail. An illustrative example was given to assure the applicability and efficiency of the suggested algorithm.

The rest of this article is organized as follows: Section 2 presents some preliminaries. In the next section, the general model for the PRLGP problem was introduced. Section 4 incorporates basic notions and an algorithm for determining the achievement stability set for PRLGP. A numerical example was presented in Section 5. Results and discussion are exhibited in section 6. Finally, some conclusions are incorporated.

2. Preliminaries

In this section, a lexicographical GP approach and the definition of a rough set are reviewed.

In the Lexicographic GP approach, DM must specify a lexicographic order for the goals in addition to the aspiration levels. The goal at the highest priority level is supposed to be infinitely more important than the goal at the second priority level. When the objective functions $f_1(x), \ldots, f_k(x)$ are divided into $L$ ordinal ranking classes, $1 \leq L \leq k$, having the preemptive priorities $P_1, \ldots, P_L$ in decreasing order, it may be convenient to write $P_l \gg P_{l+1}, l = 1, \ldots, L - 1$. to mean that no real number $t$, however large, can produce $tP_{l+1} \gg P_l, l = 1, \ldots, L - 1$. By incorporating such preemptive priorities $P_l$ together with over – and under-achievement weights $w_i^+$ and $w_i^-$, the general GP formulation takes on the following form [6,33,34]:

$$\min \sum_{i=1}^{L} P_l \left( \sum_{i \in I_l} (w_i^- d_i^- + w_i^+ d_i^+) \right)$$  \hspace{1cm} (1)

subject to  \hspace{1cm} $f_i(x) + d_i^+ - d_i^- = \bar{z}_i$ \hspace{0.5cm} $\forall i = 1, 2, \ldots, k$,  \hspace{1cm} (2)
\[ d_i - d_i^+ = 0 \quad d_i^+ \geq 0, \quad \forall i = 1, 2, \ldots, k, \quad (3) \]

\[ x \in S, \quad x \geq 0 \quad (4) \]

where \( I \neq \emptyset \) is the index set of objective functions in the \( \ell^{th} \) priority class.

**Definition 2.1:** [29,35,37]: Let \( \Theta \) be the universal set, \( \mathcal{R} \) be the equivalence relation on \( \Theta \), \( [\theta]_\mathcal{R} \) be the set of an equivalence class of \( \mathcal{R} \), and \( \Omega \) be a non-empty subset of \( \Theta \). The upper and lower approximations of the set \( \Omega \) are defined as:

\[ \overline{\mathcal{R} \Omega} = \{ \theta \in \Theta : [\theta]_\mathcal{R} \cap \Omega \neq \emptyset \} \]

\[ \mathcal{R} \Omega = \{ \theta \in \Theta : [\theta]_\mathcal{R} \subseteq \Omega \} \]

\[ \Delta \Omega = \overline{\mathcal{R} \Omega} - \mathcal{R} \Omega \]

If \( \Delta \Omega \neq \emptyset \) then set \( \Omega \) is called rough set.

**Definition 2.2:** [29,35,38]: The collection of all sets having the same upper and lower approximations is called a rough set, denoted by \( (\overline{\mathcal{R} \Omega}, \mathcal{R} \Omega) \).

### 3. General Model for PRLGP Problem

In this section, we introduce the general model of the PRLGP problem. In this article, we utilize GP to solve this model due to the following merits. GP is a tool that has been proposed for modeling and approach for the analysis of problems involving multiple, conflicting objectives with different measures units. For solving GPP, it is necessary to establish a hierarchy of importance among these goals so that lower priority goals are considered only after higher priority ones are satisfied. Since it is not always possible to achieve every goal according to the DM desires, GP attempts to reach a satisfactory solution. So that GP is a good tool to enable the DM to realize the balance between the available resources and the desired goals. The objective function of GP focused on minimizing the deviation between the available resources and the required goals. GP is one of the main approaches to solve multicriteria decision-making problems.

Consider the following general linear rough GP problem:

\[ G_{ij}(x) \leq g_{ij}, \quad i = 1, 2, \ldots, I_j, \quad j = 1, \ldots, m, \quad (5) \]

\[ \overline{G_{ij}(x)} \leq \overline{G_{ij}(x)} \leq \overline{G_{ij}(x)}, \quad i = 1, 2, \ldots, I_j, \quad j = 1, \ldots, m, \quad (6) \]

\[ x \in S, \quad x \geq 0, \quad (7) \]

\[ S \subseteq S \subseteq \overline{S}, \quad (8) \]

where \( \overline{G_{ij}(x)} \) and \( \overline{G_{ij}(x)} \) represents the upper and lower approximation functions for \( G_{ij}(x) \) respectively. Moreover, \( \overline{S} \) and \( S \) are the upper and lower approximation sets for the feasible domain \( S \). The current model also possesses two types of uncertainty the first type is the parameters which represent the weights in the achievement function. The second type of uncertainty is roughness in both goal and system constraints.
Based on the lexicographical GP methodology the previous model is formulated as [2,18]:

\[
\min Z = \sum_{j=1}^{m} \sum_{i=1}^{l_j} P_j [\beta_{ij}(d_{ij}^+)] \tag{9}
\]

Subject to

\[
G_{ij}(x) + d_{ij}^- - d_{ij}^+ = b_{ij}, \quad i = 1, 2, \ldots, l_j, \quad j = 1, \ldots, m, \tag{10}
\]

\[
x \in S \tag{11}
\]

\[
\sum_{i=1}^{l_j} \gamma_{ij} = 1, \quad \gamma_{ij} \geq 0, \quad j \in \{1, 2, \ldots, m\} \tag{12}
\]

where \(G_{ij}(x)\) represent the sub-goal number \(i\) in the priority level \(j\) which are linear functions, \(\beta_{ij}\) is the importance of the sub-goals. Moreover, the under and over achievement denoted by \(d_{ij}^-, d_{ij}^+\). \(P_j\) is the priority level and \(b_{ij}\) denotes the aspiration level.

We will introduce the following models and some basic notions to determine the achievement stability set for the PRLGP problem. Firstly, we transform the PLRGP into the following UAM and LAM. Then the attainment problem for each model is formulated.

The UAM for PLRGP is defined as:

\[
\min Z = \sum_{j=1}^{m} \sum_{i=1}^{l_j} P_j [\beta_{ij}(d_{ij}^+)] \tag{14}
\]

Subject to

\[
G_{ij}(x) + d_{ij}^- - d_{ij}^+ = g_{ij}, \quad i = 1, 2, \ldots, l_j, \quad j = 1, \ldots, m, \tag{15}
\]

\[
x \in \bar{S} \tag{16}
\]

\[
\sum_{i=1}^{l_j} \beta_{ij} = 1, \quad j \in \{1, 2, \ldots, m\}, \tag{17}
\]

\[
x, d_{ij}^-, d_{ij}^+, \beta_{ij} \geq 0, \quad d_{ij}^-, d_{ij}^+ = 0 \quad \forall \ i,j \tag{18}
\]

The LAM for PLRGP is defined as:

\[
\min Z = \sum_{j=1}^{m} \sum_{i=1}^{l_j} P_j [\beta_{ij}(d_{ij}^+)] \tag{19}
\]

Subject to

\[
G_{ij}(x) + d_{ij}^- - d_{ij}^+ = g_{ij}, \quad i = 1, 2, \ldots, l_j, \quad j = 1, \ldots, m, \tag{20}
\]

\[
x \in S \tag{21}
\]

\[
\sum_{i=1}^{l_j} \beta_{ij} = 1, \quad j \in \{1, 2, \ldots, m\}, \tag{22}
\]
\[ x, d_{ij}^-, d_{ij}^+, \beta_{ij} \geq 0, \quad d_{ij}^-d_{ij}^+ = 0, \quad \forall i,j \]  

Consequently, the entire parametric attainment issue \( P_m \) for the UAM may be characterized as:

\[
\min z_m = \sum_{i=1}^{l_m} \beta_{im}(d_{im}^- + d_{im}^+) \tag{24}
\]

Subject to

\[
\bar{G}_{ij}(x) + d_{ij}^- - d_{ij}^+ = g_{ij}, \quad i = 1, 2, \ldots, l_j, \quad j = 1, \ldots, m \tag{25}
\]

\[
\sum_{i=1}^{l_j} \beta_{ij}(d_{ij}^- + d_{ij}^+) \leq Z_j^+, \quad j = 1, 2, \ldots, m, \tag{26}
\]

\[ x \in \bar{S} \tag{27} \]

\[
\sum_{i=1}^{l_j} \beta_{ij} = 1, \quad j \in \{1, 2, \ldots, m\}, \tag{28}
\]

\[ x, d_{ij}^-, d_{ij}^+, \beta_{ij} \geq 0, \quad d_{ij}^-d_{ij}^+ = 0, \quad \forall i,j \tag{29} \]

Let the \( \tilde{y}^* = (\tilde{x}^*, \tilde{d}_{ij}^-, \tilde{d}_{ij}^+) \) be the optimal solution for the UAM, and the corresponding vector of the optimum values is \( \tilde{Z}^* = (\tilde{z}_1^*, \tilde{z}_2^*, \ldots, \tilde{z}_m^*)^T \) \[18,20\].

Also, the entire parametric attainment issue \( P_m \) for the LAM may be characterized as:

\[
\min z_m = \sum_{i=1}^{l_m} \beta_{im}(d_{im}^- + d_{im}^+) \tag{30}
\]

Subject to

\[
\bar{G}_{ij}(x) + d_{ij}^- - d_{ij}^+ = g_{ij}, \quad i = 1, 2, \ldots, l_j, \quad j = 1, \ldots, m \tag{31}
\]

\[
\sum_{i=1}^{l_j} \beta_{ij}(d_{ij}^- + d_{ij}^+) \leq Z_j^+, \quad j = 1, 2, \ldots, m, \tag{32}
\]

\[ x \in \bar{S} \tag{33} \]

\[
\sum_{i=1}^{l_j} \beta_{ij} = 1, \quad j \in \{1, 2, \ldots, m\}, \tag{34}
\]

\[ x, d_{ij}^-, d_{ij}^+, \beta_{ij} \geq 0, \quad d_{ij}^-d_{ij}^+ = 0, \quad \forall i,j \tag{35} \]

Let the \( y^* = (x^*, d^-_{ij}^*, d^+_{ij}^*) \) be the optimal solution for the LAM, and the corresponding vector of the optimum values is \( Z^* = (z_1^*, z_2^*, \ldots, z_m^*)^T \).

4. Basic Notions for PRLGP

In this section, we are intended to introduce some basic definitions of the UAM and LAM to help us in determining the achievement stability set for the PLRGP. The current model is
very easy for the DM to put the weights for goals but, for the sub-goals, it will be difficult as
they have the same measure.

**Definition 4.1:** Matching extreme points

Let \( y_1, y_2 \) be any two extreme points of a set \( S \) such that \( y_1 = (x_1, d_{ij1}^-, d_{ij1}^+) \), \( y_2 = (x_2, d_{ij2}^-, d_{ij2}^+) \) then they are matching with one another concerning sub-goal \( r \) of the
goal \( v \) if and only if \( d_{rv1}^- \times d_{rv1}^+ + d_{rv2}^- \times d_{rv2}^+ = 0 \) \[18,20\].

If \( y_1, y_2 \) are matching to all sub-goals for all goals they are said to be matching one
another. For more characteristics of matching relation see \[18,20\].

**Definition 4.2:** The goal achievement stability set for UAM

Let \( \bar{\beta} = (\bar{\beta}_{11}, \bar{\beta}_{21}, \ldots, \bar{\beta}_{11}, \bar{\beta}_{12}, \bar{\beta}_{22}, \ldots, \bar{\beta}_{1m}) \) corresponds to the optimal extreme point
\((\bar{x}^*, d_{ij1}^-, d_{ij1}^+)\) then,

\[
\bar{A}_j = \left\{ (\bar{\beta}_{1j}, \bar{\beta}_{2j}, \ldots, \bar{\beta}_{lj}) \in R^l : \sum_{i=1}^{l} \bar{\beta}_{ij} (d_{ij}^+) \leq z_j^* \right\}, \quad j \in \{1, 2, \ldots, m\} \tag{36}
\]
is called the goal achievement stability set of goal \( j \).

**Definition 4.3:** The achievement stability set for UAM

The achievement stability set denoted by \( \bar{A} \) is defined as: \( \bar{A} = \bar{A}_1 \times \bar{A}_2 \times \ldots \times \bar{A}_m \)

The achievement stability set for the UAM can be obtained by the following algorithm I:

**Algorithm I:**

1. **Step 1:** Formulate the UAM (14)-(18).
2. **Step 2:** Set \( j = 1 \).
3. **Step 3:** Construct and solve the problem \( P_j \) model (24)-(29) at certain \( \beta_j \) and find \( z_j^* \).
4. **Step 4:** Set \( j = j + 1 \) and go to Step 3 till exhausting all goals.
5. **Step 5:** Create all the extreme points by utilizing Balinski procedures \[36\].
6. **Step 6:** Apply the matching extreme points definition 1.
7. **Step 7:** Specify the convex combination among matching points to find \( \bar{A} \).
8. **Step 8:** End.

Thus, after determining the achievement stability set for the UAM utilizing Alg. I, we
are going to obtain the achievement stability set for the LAM by introducing the following
definitions. Then, Alg. I is applied to obtain the achievement stability set for the LAM.

**Definition 4.4:** The goal achievement stability set for LAM

Let \( \beta = (\beta_{11}, \beta_{21}, \ldots, \beta_{11}, \beta_{12}, \beta_{22}, \ldots, \beta_{1m}) \) corresponds to the optimal extreme point
\((\bar{x}^*, d_{ij1}^-, d_{ij1}^+)\) then,

\[
\bar{A}_j = \left\{ (\beta_{1j}, \beta_{2j}, \ldots, \beta_{lj}) \in R^l : \sum_{i=1}^{l} \beta_{ij} (d_{ij}^+) \leq z_j^* \right\}, j \in \{1, 2, \ldots, m\} \tag{37}
\]
is called the goal achievement stability set of goal \( j \).
**Definition 4.5:** The achievement stability set for LAM
The achievement stability set denoted by $A$ is defined as: $A = A_1 \times A_2 \times \ldots \times A_m$

**Definition 4.6:** The achievement stability set for PLRGP.
The achievement stability set for PLRGP denoted by $A$ is defined as: $A = \bar{A} \cap \tilde{A}$

### 4.1. An Algorithm for Determining the Achievement Stability Set for PRLGP

Following the previous discussion, an algorithm to determine the achievement stability set for PRLGP can be given as:

1. **Step 1:** Apply the lexicographical GP method to formulate model (5)-(8).
2. **Step 2:** Go to Algorithm I, to determine $\bar{A}$.
3. **Step 3:** Apply Algorithm I, to obtain $\tilde{A}$.
4. **Step 4:** Determine $A = \bar{A} \cap \tilde{A}$.
5. **Step 5:** Stop.

### 5. Illustrative Example

Consider the following rough GP problem:

$G_{11}(x, y) \leq 15,$
$G_{21}(x, y) \geq 32,$
$G_{12}(x, y) \leq 12,$
$G_{22}(x, y) \geq 40,$
$x + y \leq G_{11}(x, y) \leq 5x + 3y,$
$4x + y \leq G_{21}(x, y) \leq 8x + 5y,$
$x + 2y \leq G_{12}(x, y) \leq 4x + 3y,$
$3x + y \leq G_{22}(x, y) \leq 10x + 8y,$
$x \in S$

Where

$S \subseteq \bar{S} \subseteq \tilde{S}$

$\tilde{S} = \{x \in R^2 | x + y \leq 6, 2x + y \leq 6, x, y \geq 0\},$

$\bar{S} = \{x \in R^2 | 2x + y \leq 6, x + y \leq 2, x, y \geq 0\}.$

Firstly, the attainment problem will take the following equivalent mathematical form:

$$\min Z = P_1[\beta_1 d_{11}^+ + (1 - \beta_1)d_{11}^{-}] + P_2[\beta_2 d_{12}^+ + (1 - \beta_2)d_{12}^{-}]$$
Subject to
\[ G_{11}(x, y) + d_{11}^- - d_{11}^+ = 15, \]
\[ G_{21}(x, y) + d_{21}^- - d_{21}^+ = 32, \]
\[ G_{12}(x, y) + d_{12}^- - d_{12}^+ = 12, \]
\[ G_{22}(x, y) + d_{22}^- - d_{22}^+ = 40, \]
\[ x + y \leq G_{11}(x, y) \leq 5x + 3y, \]
\[ 4x + y \leq G_{21}(x, y) \leq 8x + 5y, \]
\[ x + 2y \leq G_{12}(x, y) \leq 4x + 3y, \]
\[ 3x + y \leq G_{22}(x, y) \leq 10x + 8y, \]
\[ x \in S \]
\[ x, y, d_{ij}^-, d_{ij}^+, \beta_1, \beta_2 \geq 0, d_{ij}^- . d_{ij}^+ = 0 \quad \forall i, j \]

where
\[ S \subseteq S \subseteq \tilde{S}, \]
\[ \tilde{S} = \{x \in R^2 | x + y \leq 4, 2x + y \leq 6, x, y \geq 0\}, \]
\[ S = \{x \in R^2 | 2x + y \leq 3, x + y \leq 2, x, y \geq 0\}. \]

Thus, for solving the previous problem, we start by solving the UAM:

\[ \min Z = P_1[\beta_1 d_{11}^+ + (1 - \beta_1) d_{21}^-] + P_2[\beta_2 d_{12}^+ + (1 - \beta_2) d_{22}^-] \]

Subject to
\[ x + y + d_{11}^- - d_{11}^+ = 15, \]
\[ 8x + 5y + d_{21}^- - d_{21}^+ = 32, \]
\[ x + 2y + d_{12}^- - d_{12}^+ = 12, \]
\[ 10x + 8y + d_{22}^- - d_{22}^+ = 40, \]
\[ x + y \leq 4, \]
\[ 2x + y \leq 6, \]
\[ x, y, d_{ij}^-, d_{ij}^+ \geq 0, d_{ij}^- . d_{ij}^+ = 0 \quad i = 1, 2, j = 1, 2 \]

Based on the lexicographical GP method, the first priority problem has the form:

\[ P_1 : \min Z_1 = \beta_1 d_{11}^+ + (1 - \beta_1) d_{21}^- \]

Subject to
\[ x + y + d_{11}^- - d_{11}^+ = 15, \]
\[ 8x + 5y + d_{21}^- - d_{21}^+ = 32, \]
\[ x + 2y + d_{12}^+ - d_{12}^- = 12, \]
\[ 10x + 8y + d_{22}^+ - d_{22}^- = 40, \]
\[ x + y \leq 4, \]
\[ 2x + y \leq 6, \]
\[ x, y, d_{ij}^-, d_{ij}^+ \geq 0, d_{ij}^-d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2 \]

Let \( \beta_1 = 0.3 \), then the optimal solution is \((x, y, d_{11}^-, d_{21}^-, d_{11}^+, d_{21}^+) = (2, 2, 11, 6, 0, 0)\) and \( Z_1^* = 4.2 \).

The second priority problem has the form

\[ P_2 : \min Z_2 = \beta_2 d_{12}^+ + (1 - \beta_2)d_{22}^- \]

Subject to
\[ x + y + d_{11}^- - d_{11}^+ = 15, \]
\[ 8x + 5y + d_{21}^- - d_{21}^+ = 32, \]
\[ x + 2y + d_{12}^- - d_{12}^+ = 12, \]
\[ 10x + 8y + d_{22}^- - d_{22}^+ = 40, \]
\[ \beta_1 d_{11}^+ + (1 - \beta_1)d_{21}^- \leq 4.2, \]
\[ x + y \leq 4, \]
\[ 2x + y \leq 6, \]
\[ x, y, d_{ij}^-, d_{ij}^+, \alpha_1, \alpha_2 \geq 0, d_{ij}^-d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2 \]

Let \( \beta_1 = 0.3, \beta_2 = 0.4 \), then the optimal solution for the second priority problem is \((x, y, d_{11}^-, d_{21}^-, d_{11}^+, d_{21}^+, d_{12}^+, d_{22}^+) = (2, 2, 11, 6, 6, 4, 0, 0, 0, 0)\), and \( Z_1^* = 4.2, Z_2^* = 2.4 \).

Then we will generate extreme points for the following problem:

\[ \min Z_2 = \beta_2 d_{12}^+ + (1 - \beta_2)d_{22}^- \]

Subject to
\[ x + y + d_{11}^- - d_{11}^+ = 15, \]
\[ 8x + 5y + d_{21}^- - d_{21}^+ = 32, \]
\[ x + 2y + d_{12}^- - d_{12}^+ = 12, \]
\[ 10x + 8y + d_{22}^- - d_{22}^+ = 40, \]
\[ \beta_1 d_{11}^+ + (1 - \beta_1)d_{21}^- \leq 4.2, \]
\[ \beta_2 d_{12}^+ + (1 - \beta_2)d_{22}^- \leq 2.4, \]
\[ x + y + \eta_1 = 4, \]
\[ 2x + y + \eta_2 = 6, \]
\[ x, y, d_{ij}^-, d_{ij}^+, \eta_1, \eta_2, \beta_1, \beta_2 \geq 0, d_{ij}^-d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2 \]
By applying the Balinski algorithm [36] we obtain the following extreme points:

\[(x, y, d_{11}^-, d_{21}^-, d_{12}^-, d_{22}^-, d_{11}^+, d_{21}^+, d_{12}^+, d_{22}^+) = (0, 0, 15, 32, 12, 40, 0, 0, 0, 0)\]

\[(x, y, d_{11}^-, d_{21}^-, d_{12}^-, d_{22}^-, d_{11}^+, d_{21}^+, d_{12}^+, d_{22}^+) = (0, 4, 11, 12, 4, 8, 0, 0, 0, 0)\]

\[(x, y, d_{11}^-, d_{21}^-, d_{12}^-, d_{22}^-, d_{11}^+, d_{21}^+, d_{12}^+, d_{22}^+) = (2, 2, 11, 6, 6, 4, 0, 0, 0, 0)\]

\[(x, y, d_{11}^-, d_{21}^-, d_{12}^-, d_{22}^-, d_{11}^+, d_{21}^+, d_{12}^+, d_{22}^+) = (3, 0, 12, 8, 9, 10, 0, 0, 0, 0)\]

Since these extreme points are matching with each other then, the convex combination between them is defined as:

\[(d_{11}^-, d_{21}^-) = \lambda_1(15, 32) + \lambda_2(11, 12) + \lambda_3(11, 6) + \lambda_4(12, 8),\]

\[\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \quad \lambda_j \geq 0, \quad j = 1, 2, 3, 4, \text{ and } d_{11}^+ = d_{21}^+ = 0, \text{ then}\]

\[d_{11}^- = 15\lambda_1 + 11\lambda_2 + 11\lambda_3 + 12\lambda_4, \quad d_{21}^- = 32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4. \quad \text{Thus}\]

\[(1 - \beta_1)(32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4) \leq 4,2,\]

\[32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4 - \beta_1(32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4) \leq 4,2, 0 \leq \beta_1 \leq 1,\]

So, the goal achievement stability set of goal 1 for the UAM is:

\[\bar{A}_1 = \{\beta_1 \in \mathbb{R} | 32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4 - \beta_1(32\lambda_1 + 12\lambda_2 + 6\lambda_3 + 8\lambda_4) \leq 4,2, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_j \geq 0, j = 1, 2, 3, 4, 0 \leq \beta_1 \leq 1\}\]

Similarly, \((d_{12}^-, d_{22}^-) = \mu_1(12, 40) + \mu_2(4, 8) + \mu_3(6, 4) + \mu_4(9, 10)\).

\[\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1, \quad \mu_j \geq 0, \quad j = 1, 2, 3, 4, \text{ and Then},\]

\[d_{12}^- = 12\mu_1 + 4\mu_2 + 6\mu_3 + 9\mu_4, \quad d_{22}^- = 40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4.\]

\[d_{12}^+ = d_{22}^+ = 0.\quad (1 - \beta_2)(40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4) \leq 2,4, \quad \text{Thus},\]

\[40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4 - \beta_2(40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4) \leq 2,4, 0 \leq \alpha_2 \leq 1.\]

So, the goal achievement stability set of goal 2 for the UAM is:

\[\bar{A}_2 = \{\beta_2 \in \mathbb{R} | 40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4 - \beta_2(40\mu_1 + 8\mu_2 + 4\mu_3 + 10\mu_4) \leq 2,4, \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1, \mu_j \geq 0, j = 1, 2, 3, 4, 0 \leq \beta_2 \leq 1\}\]

Then the achievement stability set for the UAM is:

\[\bar{A} = \bar{A}_1 \times \bar{A}_2 = \{(\beta_1, \beta_2) \in \mathbb{R}^2 | \beta_1 \in \bar{A}_1, \beta_2 \in \bar{A}_2\}\]

Formulate and solve the LAM as follows

\[
\min Z = P_1[\beta_1(1 - \beta_1)d_{11}^+] + P_2[\beta_2d_{12}^+ + (1 - \beta_2)d_{22}^-]
\]

Subject to

\[5x + 3y + d_{11}^- - d_{11}^+ = 15,\]
Based on the lexicographical GP method, the first priority problem has the form:

\[ P_1 : \min Z_1 = \beta_1 d_{11}^+ + (1 - \beta_1) d_{21}^- \]

Subject to

\[ 5x + 3y + d_{11}^- - d_{11}^+ = 15, \]
\[ 4x + y + d_{21}^- - d_{21}^+ = 32, \]
\[ 2x + y \leq 3, \]
\[ x + y \leq 2, \]
\[ x, y, d_{ij}^-, d_{ij}^+, \beta_1, \beta_2 \geq 0, d_{ij}^-, d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2 \]

Let \( \beta_1 = 0.3 \), then the optimal solution is \((x, y, d_{11}^-, d_{11}^+, d_{21}^-, d_{21}^+ \) = \((1.5, 0, 7.5, 26, 0, 0)\) and \( Z_1^* = 18.2 \)

The second priority problem has the form:

\[ P_2 : \min Z_2 = \beta_2 d_{12}^+ + (1 - \beta_2) d_{22}^- \]

Subject to

\[ 5x + 3y + d_{11}^- - d_{11}^+ = 15, \]
\[ 4x + y + d_{21}^- - d_{21}^+ = 32, \]
\[ 4x + 3y + d_{12}^- - d_{12}^+ = 12, \]
\[ 3x + y + d_{22}^- - d_{22}^+ = 40, \]
\[ \beta_1 d_{11}^+ + (1 - \beta_1) d_{21}^- \leq 18.2, \]
\[ 2x + y \leq 3, \]
\[ x + y \leq 2, \]
\[ x, y, d_{ij}^-, d_{ij}^+, \beta_1, \beta_2 \geq 0, d_{ij}^-, d_{ij}^+ = 0 \quad i = 1, 2, \quad j = 1, 2 \]

Let \( \beta_1 = 0.3, \beta_2 = 0.4 \), then the optimal solution is \((x, y, d_{11}^-, d_{11}^+, d_{12}^-, d_{12}^+, d_{21}^-, d_{21}^+, d_{22}^-, d_{22}^+ \) = \((1.5, 0, 7.5, 26, 6, 35.5, 0, 0, 0, 0)\) And \( Z_2^* = 21.3 \)

Then we will generate extreme points for the following:

\[ \min Z_2 = \beta_2 d_{12}^+ + (1 - \beta_2) d_{22}^- \]
Using the Baliniski algorithm [36] the following extreme points were obtained as:

\[ (x, y, d_{11}^-, d_{12}^-, d_{21}^-, d_{22}^-, d_{11}^+, d_{12}^+, d_{21}^+, d_{22}^+) = (0, 0, 15, 32, 12, 40, 0, 0, 0, 0) \]

\[ (x, y, d_{11}^-, d_{12}^-, d_{21}^-, d_{22}^-, d_{11}^+, d_{12}^+, d_{21}^+, d_{22}^+) = (0, 2, 9, 30, 6, 38, 0, 0, 0, 0) \]

\[ (x, y, d_{11}^-, d_{12}^-, d_{21}^-, d_{22}^-, d_{11}^+, d_{12}^+, d_{21}^+, d_{22}^+) = (1.5, 0, 7.5, 26, 6, 35.5, 0, 0, 0, 0) \]

\[ (x, y, d_{11}^-, d_{12}^-, d_{21}^-, d_{22}^-, d_{11}^+, d_{12}^+, d_{21}^+, d_{22}^+) = (1, 1, 7, 27, 5, 36, 0, 0, 0, 0) \]

Since the previous extreme points are matching with each other so the convex combination between them is defined as.

\[ (d_{11}^-, d_{21}^-) = \lambda_1 (15, 32) + \lambda_2 (9, 30) + \lambda_3 (7, 27) + \lambda_4 (7.5, 26), \]

\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_j \geq 0, j = 1, 2, 3, 4 \text{ and } d_{11}^+ = d_{21}^- = 0, \text{ Thus} \]

\[ d_{11}^- = 15\lambda_1 + 9\lambda_2 + 7\lambda_3 + 7.5\lambda_4, \text{ and } d_{21}^- = 32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4. \text{ Thus,} \]

\[ (1 - \beta_1) (32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4) \leq 18.2, \]

\[ 32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4 - \beta_1 (32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4) \leq 18.2, \]

Then the goal achievement stability set of goal 1 for the LAM is:

\[ A_1 = \{ \beta_1 \in \mathbb{R} | 32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4 - \beta_1 (32\lambda_1 + 30\lambda_2 + 27\lambda_3 + 26\lambda_4) \leq 18.2, \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1, \lambda_j \geq 0, j = 1, 2, 3, 4, 0 \leq \beta_1 \leq 1 \} \]

Similarly

\[ (d_{12}^-, d_{22}^-) = \mu_1 (12, 40) + \mu_2 (6, 38) + \mu_3 (5, 36) + \mu_4 (6, 35.5). \]

\[ \mu_1 + \mu_2 + \mu_3 + \mu_4 = 1, \mu_j \geq 0, j = 1, 2, 3, 4 \text{ and } d_{12}^+ = d_{22}^- = 0. \text{ Then} \]

\[ d_{12}^- = 12\mu_1 + 6\mu_2 + 5\mu_3 + 6\mu_4, \text{ and } d_{22}^- = 40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4. \text{ Thus} \]

\[ (1 - \beta_2) (40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4) \leq 21.3, \]

\[ 40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4 - \beta_2 (40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4) \leq 21.3, \]
Then the goal achievement stability set of goal 2 for the LAM is:

\[
A_2 = \left\{ \beta_2 \in \mathbb{R} | 40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4 \\
-\beta_2(40\mu_1 + 38\mu_2 + 36\mu_3 + 35.5\mu_4) \leq 21.3, \\
\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1, \mu_j \geq 0, j = 1, 2, 3, 4, 0 \leq \beta_2 \leq 1 \right\}
\]

So, the achievement stability set for the LAM is:

\[
A = A_1 \times A_2 = \{ (\beta_1, \beta_2) \in \mathbb{R}^2 | \beta_1 \in A_1, \beta_2 \in A_2 \}
\]

Then the achievement stability set for the rough GP problem is:

\[
A = \bar{A} \cap \bar{A}
\]

6. Results and Discussion

In this article, we exhibit procedures for determining the achievement stability set for PRLGP which is a new and novel type of parametric analysis in GP. The novel advantage is to make the achievement function of the rough goals be settled on the rough feasible domain. We determine the achievement stability set for the UAM, so also are the LAM then we obtain the intersection of these stability sets which is the overall stability set of the problem. Another advantage is that the proposed model has two types of uncertainty the parameter in the achievement function and roughness in both goals and system constraints.

7. Conclusion

An algorithm for investigating the achievement stability set for the PRLGP problem is presented in this study. The suggested PRLGP model has two types of uncertainty the parameter in the achievement function and roughness in both goals and system constraints. Solution algorithms and an illustrative example are also introduced. The current model can be applied in real-world problems to give an analysis of the results due to the change of the parameter in the achievement function or the feasible domain. Moreover, this model has been handled because it is very easy for the DM to put the weights for goals but, for the sub-goals, it will be difficult as they have the same measure. Several open points for research in uncertain optimization, from our point of view, to be studied in the future. Some of these points are given in the following:

(1) Achievement stability set for a parametric linear fractional fuzzy goal programing problem.

(2) Achievement stability set for parametric bi-level linear fractional fuzzy goal programing problem.

(3) Different types of stability set for a linear fractional rough programing problem.

Disclosure statement

No potential conflict of interest was reported by the author(s).
Notes on contributors

F. A. Farahat is a lecturer at the Department of Basic Science, Higher Technological Institute, 10th of Ramadan city. His research interest is in multi-level multi-objective rough fractional programming problems and goal programming problems.

M. A. ElSayed is a lecturer in the Department of Basic Engineering Sciences, Faculty of Engineering, Benha University. His research interest is in bi-level and multi-level fractional optimization field, parametric analysis, goal programming, fuzzy and stochastic programming, transportation and assignment problems.

ORCID

M. A. ElSayed http://orcid.org/0000-0001-8256-7833

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