LOW-DISCREPANCY SEQUENCES FOR PIECEWISE SMOOTH FUNCTIONS ON THE TWO-DIMENSIONAL TORUS

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ABSTRACT. We produce explicit low-discrepancy infinite sequences which can be used to approximate the integral of a smooth periodic function restricted to a convex domain with positive curvature in \(\mathbb{R}^2\). The proof depends on simultaneous diophantine approximation and a general version of the Erdős-Turán inequality.

Keywords: Koksma-Hlawka inequality, piecewise smooth functions, discrepancy, diophantine approximation, Erdős-Turán inequality.

1. INTRODUCTION

Let \(f\) be a suitable function on the \(d\)-dimensional torus \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\), and let \(\{t(j)\}_{j=1}^N\) be a distribution of points on \(\mathbb{T}^d\). The quality of the approximation of \(\int_{\mathbb{T}^d} f(t) \, dt\) by the Riemann sum \(\frac{1}{N} \sum_{j=1}^N f(t(j))\) is a basic problem with applications in 2D or 3D computer graphics, and also with applications when \(d\) is large (and the curse of dimensionality appears). See e.g. [9]. Any bound of the form

\[
\left| \frac{1}{N} \sum_{j=1}^N f(t(j)) - \int_{\mathbb{T}^d} f(t) \, dt \right| \leq D\left(\{t(j)\}_{j=1}^N\right) V(f)
\]

can be termed a **Koksma-Hlawka type inequality**, provided the RHS is a variation \(V(f)\) of the function \(f\) times a discrepancy \(D\left(\{t(j)\}_{j=1}^N\right)\) of the finite set \(\{t(j)\}_{j=1}^N\) with respect to a reasonably simple family of subsets of \(\mathbb{T}^d\).

The case \(d = 1\) is the amazingly simple Koksma inequality, where \(\mathbb{T}\) is replaced by the unit interval, \(V(f)\) is the usual total variation and \(D\left(\{t(j)\}_{j=1}^N\right)\) is the *-discrepancy

\[
\sup_{0<\alpha\leq1} \left| \frac{1}{N} \sum_{j=1}^N \chi_{(0,\alpha)}(t(j)) - \alpha \right|
\]

that is the discrepancy measured on the family of all intervals anchored at the origin.

See [3], [9], [14], [15], [21], [22], [28] as general references.

The term Koksma-Hlawka inequality properly refers to E. Hlawka’s generalization of Koksma inequality to several variables, where \(f\) is required to have bounded variation in the sense of Hardy and Krause. In one variable, many familiar bounded functions have bounded variation, but, in several variables, the Hardy-Krause condition cannot be applied to most functions with simple discontinuities. For example: the characteristic function of a polyhedron has bounded Hardy-Krause variation if and only if the polyhedron is a \(d\)-dimensional interval.

We recall some of the variants of the Koksma-Hlawka inequality which have appeared in the literature so far. F. Hickernell [20] has proposed Koksma-Hlawka type inequalities

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for reproducing kernel Hilbert spaces. J. Dick [13] has used fractional calculus to prove a Koksm-Hlawka type inequality for functions with relaxed smoothness assumptions. G. Harman [19] has considered a geometric approach and measured the variation by counting the convex sets needed to describe super-level sets of the function \( f \). In [6] the authors of the present paper proposed a Koksm-Hlawka type inequality especially tailored for simplices, while in [7] they have introduced a Koksm-Hlawka type inequality for piecewise smooth functions. Analogues of the above problem in more general settings can be found e.g. in [4] and [5].

K. Basu and A. Owen [1] have recently produced low-discrepancy sequences for a triangle, where the discrepancy is the one considered in [6]. In this paper we propose a particular two-dimensional case of the statement therein.

Theorem 1 (7). Let \( h(t) = f(t) \chi_\Omega(t) \), where \( f \) is a smooth \( \mathbb{Z}^2 \)-periodic function on \( \mathbb{R}^2 \) and \( \chi_\Omega \) is the characteristic function of a bounded Borel set in \( \mathbb{R}^2 \). Let

\[
V(f) := 4 \| f \|_{L^1(\mathbb{T}^2)} + 2 \left\| \frac{\partial f}{\partial t_1} \right\|_{L^1(\mathbb{T}^2)} + 2 \left\| \frac{\partial f}{\partial t_2} \right\|_{L^1(\mathbb{T}^2)} + \left\| \frac{\partial^2 f}{\partial t_1 \partial t_2} \right\|_{L^1(\mathbb{T}^2)}.
\]

Let \( \{t(j)\}_{j=1}^N \subset \mathbb{R}^2 \), for any \( s \in (0,1)^2 \) and for any \( x \in \mathbb{R}^2 \) let

\[
I(s,x) = \bigcup_{m \in \mathbb{Z}^2} \left( [0,s_1] \times [0,s_2] + x + m \right),
\]

and let

\[
D \left( \{t(j)\}_{j=1}^N \right) := \sup_{s \in (0,1)^2, x \in \mathbb{R}^2} \left| \frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^2} \chi_{I(s,x) \cap \Omega}(t(j) + m) - |I(s,x) \cap \Omega| \right|.
\]

Then

\[
\left| \frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^2} h(t(j) + m) - \int_{\mathbb{R}^2} h(t) \, dt \right| \leq V(f) \cdot D \left( \{t(j)\}_{j=1}^N \right).
\]

Observe that if a set \( K \in \mathbb{R}^2 \) does not intersect any of its integer translates, then it can be thought of as a subset of \( \mathbb{T}^2 \), and in that case the expression

\[
\frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^2} \chi_K(t(j) + m) - |K|
\]

compares the measure of \( K \) with the share of points in \( K \) of the collection obtained by projecting \( \{t(j)\}_{j=1}^N \) onto \( \mathbb{T}^2 \). It follows that the above theorem includes, but is slightly more general than the analogous theorem where not only the function \( f \) but also the set \( \Omega \) and the point collection \( \{t(j)\} \) in \( \mathbb{T}^2 \), and the quantity \( D(\{t(j)\}_{j=1}^N) \) is just the discrepancy with respect to the intersection of \( \Omega \) with all the rectangles in \( \mathbb{T}^2 \).

We are therefore interested in choices of the set \( \{t(j)\}_{j=1}^N \) which give satisfactory upper bounds for the discrepancy \( D \).

An interesting result in this direction is due to J. Beck [2]: for every positive integer \( N \) there is a collection of \( N \) points in the unit square with isotropic discrepancy (that is, the discrepancy with respect to all convex sets) bounded by \( cN^{-2/3} \log^4 N \). Since the discrepancy \( D \) is smaller than the isotropic discrepancy, Beck's result gives a sequence that can be used in the Koksm-Hlawka type inequality in Theorem 1 when \( \Omega \) is convex. On the other hand, Beck’s construction is somewhat intricate, and is obtained partly by random and partly by deterministic methods.
A more explicit extensible construction comes from a result of H. Niederreiter (see [23] or [21] page 129 and page 132, Exercise 3.17): if $1, \alpha, \beta$ are algebraic linearly independent on $\mathbb{Q}$, then the discrepancy of $\{(j\alpha, j\beta)\}_{j=1}^{N}$ with respect to all axis parallel rectangles contained in the unit square is bounded by $cN^{-1+\epsilon}$. This immediately implies that the isotropic discrepancy of this sequence is bounded by $cN^{-1/2+\epsilon}$ (see [21] Theorem 1.6, page 95), an estimate that is far from Beck’s result.

Our main result is the following.

**Theorem 2.** Assume that $\alpha, \beta$ are real algebraic numbers and that $1, \alpha, \beta$ is a basis of a number field on $\mathbb{Q}$ of degree $3$. For all integers $j \geq 0$, let $t(j) = (j\alpha, j\beta)$. Let $\Omega$ be a convex domain contained in $\mathbb{R}^2$ with $\mathbb{S}^2$ boundary having strictly positive curvature. Then the discrepancy defined in (1) satisfies

\[
D\left(\{t(j)\}_{j=1}^{N}\right) \leq cN^{-2/3}\log N.
\]

The above constant $c$ depends on the minimum and the maximum of the curvature of $\partial\Omega$, on its length, and on the numbers $\alpha, \beta$.

For example, one can take $\alpha = \xi, \beta = \xi^2$, where $\xi$ is a real root of a third degree irreducible polynomial in $\mathbb{Z}$.

In other words, Theorem 2 says that a regularity assumption on the convex set $\Omega$ suffices for the sequence in Niederreiter’s result to improve Beck’s estimate $N^{-2/3}\log^4 N$. This can be obtained by estimating directly the discrepancy $D(\{t(j)\}_{j=1}^{N})$, and avoiding the isotropic discrepancy. The main tool that will allow us to do it is a version of the Erdős-Turán inequality essentially contained in [11].

2. PROOFS AND AUXILIARY RESULTS

Let us begin by recalling the above mentioned general form of the Erdős-Turán inequality.

**Theorem 3.** There exists a positive function $\psi(u)$ on $[0, +\infty)$ with rapid decay at infinity such that for every collection of points $\{t(j)\}_{j=1}^{N} \subset \mathbb{R}^d$, for every bounded Borel set $D \subset \mathbb{R}^d$, and for every $R > 0$,

\[
\left| \frac{1}{N} \sum_{j=1}^{N} \sum_{m \in \mathbb{Z}^d} \chi_{D}(t(j) + m) - |D| \right| 
\leq \left| \hat{H}_R(0) \right| + \sum_{n \in \mathbb{Z}^d, 0 < |n| < R} \left( |\hat{\chi}_D(n)| + |\hat{H}_R(n)| \right) \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i n \cdot t(j)} \right|.
\]

Here $H_R(x) = \psi(R \text{dist}(x, \partial D))$, where $\text{dist}$ is the Euclidean distance in $\mathbb{R}^d$.

**Proof.** Take a smooth radial function $m(\xi)$ supported in $|\xi| < 1/2$ and with $\int_{\mathbb{R}^d} m^2(\xi) d\xi = 1$, and define

\[
K(x) = \int_{\mathbb{R}^d} \left(1 + |\xi|^2\right)^{-(d+1)/2} (m \cdot m)(\xi) e^{2\pi i \xi \cdot x} d\xi,
\]

\[
\psi(u) = e^{2\pi} \left( \int_{|y| \leq 1} K(y) dy \right)^{-1} \int_{|y| \geq u} K(y) dy.
\]
Since $\tilde{K}(\xi) = 0$ if $|\xi| \geq 1$, it follows from the Paley-Wiener theorem that $K(x)$ is an entire function of exponential type smaller than 1, positive with mean 1, all its derivatives have rapid decay at infinity, and $|\tilde{K}(\xi)| \leq 1$ for every $\xi \in \mathbb{R}^d$. If we set $K_R(x) = R^d K(Rx)$, then the functions

$$A(x) = \int_{\mathbb{R}^d} K_R(y) (\chi_D(x-y) - H_R(x-y)) \, dy$$

$$B(x) = \int_{\mathbb{R}^d} K_R(y) (\chi_D(x-y) + H_R(x-y)) \, dy,$$

are entire functions of exponential type smaller than $R$ and

$$A(x) \leq \chi_D(x) \leq B(x), \quad |B(x) - A(x)| \leq 4 \psi(R \text{dist}(x, \partial D)/2)$$

(see [11] for the details). Periodization gives

$$\sum_{m \in \mathbb{Z}^d} A(x + m) \leq \sum_{m \in \mathbb{Z}^d} \chi_D(x + m) \leq \sum_{m \in \mathbb{Z}^d} B(x + m),$$

and, by the Poisson summation formula,

$$\sum_{m \in \mathbb{Z}^d} A(x + m) = \sum_{n \in \mathbb{Z}^d} \tilde{K}(R^{-1}n) \left( \tilde{\chi}_D(n) - \tilde{H}_R(n) \right) e^{2\pi in \cdot x},$$

$$\sum_{m \in \mathbb{Z}^d} B(x + m) = \sum_{n \in \mathbb{Z}^d} \tilde{K}(R^{-1}n) \left( \tilde{\chi}_D(n) + \tilde{H}_R(n) \right) e^{2\pi in \cdot x}$$

are trigonometric polynomials of degree at most $R$. It now follows that

$$\frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^d} \chi_D(t(j) + m) - |D|$$

$$\leq \frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^d} B(t(j) + m) - |D|$$

$$= \frac{1}{N} \sum_{j=1}^N \sum_{n \in \mathbb{Z}^d} \tilde{K}(R^{-1}n) \left( \tilde{\chi}_D(n) + \tilde{H}_R(n) \right) e^{2\pi i n \cdot (j)} - |D|$$

$$= \tilde{H}_R(0) + \sum_{n \in \mathbb{Z}^d, 0 < |n| < R} \tilde{K}(R^{-1}n) \left( \tilde{\chi}_D(n) + \tilde{H}_R(n) \right) \frac{1}{N} \sum_{j=1}^N e^{2\pi i n \cdot j}$$

$$\leq \tilde{H}_R(0) + \sum_{n \in \mathbb{Z}^d, 0 < |n| < R} \left( |\tilde{\chi}_D(n)| + |\tilde{H}_R(n)| \right) \frac{1}{N} \sum_{j=1}^N e^{2\pi i n \cdot j}.$$
the Fourier transform is bounded by
\[ |\hat{\gamma}(\xi)| \leq \min \left( \ell, c \frac{1 + \kappa_{\min}}{|\xi|^{1/2}} \right). \]

Here \( \ell \) is the length of the arc and \( c \) is a universal constant.

**Proof.** Let \( r(\tau) \) be the parametrization of \( \gamma \) with respect to arclength, so that
\[ \hat{\gamma}(\xi) = \int_0^\ell e^{-2\pi ir(\tau)\xi} d\tau. \]

For any \( \xi \) we have the trivial estimate
\[ \left| \int_0^\ell e^{-2\pi ir(\tau)\xi} d\tau \right| \leq \ell. \]

Assume \( \xi \neq 0 \) and let
\[ \xi = \rho \eta \]
where \( |\eta| = 1 \) and \( \rho > 0 \). First consider the (at most) three intervals \( I_1, I_2 \) and \( I_3 \) where \( |r'(\tau) \cdot \eta| > 2^{-1/2} \). By Van der Corput’s lemma, since \( |r'(\tau) \cdot \eta| > 2^{-1/2} \) and the expression \( r''(\tau) \cdot \eta = -\kappa(\tau) \nu(\tau) \cdot \eta \) changes sign at most once (here \( \nu(\tau) \) and \( \kappa(\tau) \) are respectively the outer normal and the curvature of \( \gamma \) at a point \( r(\tau) \)), then
\[ \left| \int_{I_i} e^{-2\pi i r(\tau) \eta} d\tau \right| \leq \frac{c_1}{\rho} \]
for \( i = 1, 2, 3 \). The constant \( c_1 \) is universal. If \( |r'(\tau) \cdot \eta| \leq 2^{-1/2} \) we have \( |\nu(\tau) \cdot \eta| \geq 2^{-1/2} \) so that
\[ |r''(\tau) \cdot \eta| = \kappa(\tau) |\nu(\tau) \cdot \eta| > \kappa_{\min} 2^{-1/2}. \]

Thus, by Van der Corput’s lemma, for the at most three intervals \( J_1, J_2 \) and \( J_3 \) where \( |r'(\tau) \cdot \eta| \leq 2^{-1/2} \), we have
\[ \left| \int_{J_j} e^{-2\pi i r(\tau) \eta} d\tau \right| \leq \frac{c_2}{(\kappa_{\min} \rho)^{1/2}} \]
for \( j = 1, 2, 3 \). Again, \( c_2 \) is a universal constant. Finally,
\[ \left| \int_0^\ell e^{-2\pi i r(\tau) \eta} d\tau \right| \leq \min \left( \ell, \frac{3c_1}{\rho} + \frac{3c_2}{(\kappa_{\min} \rho)^{1/2}} \right). \]

When \( \rho \geq 1 \), this gives
\[ \left| \int_0^\ell e^{-2\pi i r(\tau) \eta} d\tau \right| \leq \min \left( \ell, \frac{1 + \kappa_{\min}^{-1/2}}{\rho^{1/2}} \right). \]

**Lemma 5.** The Fourier transform of the arclength measure on the segment \( \gamma \) joining two points \( x \) and \( y \) in \( \mathbb{R}^2 \) is
\[ \hat{\gamma}(\xi) = |x - y| \frac{\sin(\pi (x - y) \cdot \xi)}{\pi (x - y) \cdot \xi} e^{-2\pi i \frac{|x + y|}{|x - y|} \cdot \xi}. \]

In particular, calling \( \ell = |x - y| \) and \( \theta = \frac{x + y}{|x - y|} \), we have
\[ |\hat{\gamma}(\xi)| \leq \min \left( \ell, \frac{1}{\pi |\xi \cdot \theta|} \right). \]
Proof. This is just an explicit calculation. □

Before we proceed with the proof of Theorem 2, we need a few results on convex sets in \( \mathbb{R}^d \). Let us begin with some terminology.

**Definition 6.** Let \( K \) be a non-empty compact convex subset (a “convex body”) of \( \mathbb{R}^d \). The signed distance function \( \delta_K \) is defined by

\[
\delta_K(x) = \begin{cases} 
\operatorname{dist}(x, \partial K) & \text{if } x \in K \\
-\operatorname{dist}(x, \partial K) & \text{if } x \notin K.
\end{cases}
\]

For any real number \( u \), define

\[
K^u = \{ x \in \mathbb{R}^d : \delta_K(x) \geq u \}
\]

and

\[
K_u = \{ x \in \mathbb{R}^d : \delta_K(x) = u \}
\]

The signed distance function is Lipschitz continuous with constant 1, and \( |\nabla \delta_K| = 1 \) almost everywhere (see [17, Section 14.6]).

**Definition 7.** Let \( B \) be the closed unit ball centered at the origin. If \( K \) is a convex body in \( \mathbb{R}^d \), then the outer parallel body of \( K \) at distance \( r \) is defined as the Minkowski sum of \( K \) and \( rB \),

\[
K + rB = \{ x + y : x \in K, |y| \leq r \}
\]

while the inner parallel body of \( K \) at distance \( r \) is defined as the Minkowski difference of \( K \) and \( rB \),

\[
K \div rB = \{ x : x + rB \subset K \}
\]

**Lemma 8.** Let \( K \) be a convex body in \( \mathbb{R}^d \).

(i) For any real number \( u \), the set \( K^u \) is the outer or the inner parallel body of \( K \) at distance \( |u| \), according to whether \( u \) is negative or positive, that is

\[
K^u = K + |u|B, \quad \text{if } u \leq 0,
\]

\[
K^u = K \div uB, \quad \text{if } u > 0.
\]

(ii) For any real number \( u \), the set \( K^u \) is convex (possibly empty).

(iii) If \( M \) is a convex body too, then for every \( u \geq 0 \),

\[
(M \cap K)_u = (M_u \cap K^u) \cup (M^u \cap K_u).
\]

**Proof.** Point (i) follows easily from the definitions, while the proof of (ii) can be found in [26, Chapter 3]. As for point (iii), we sketch a proof, highlighting the main steps. First observe that \( \partial (K^u) = K_u \) and that \( (M \cap K)^u = M^u \cap K^u \) when \( u \geq 0 \). The thesis now follows after the observation that for any two compact sets \( A \) and \( B \) one has

\[
\partial (A \cap B) = (\partial A \cap B) \cup (A \cap \partial B).
\]

□

**Lemma 9.** Let \( K \) be a convex body in \( \mathbb{R}^d \) with \( C^2 \) boundary and let \( \kappa_{\max} \) be the maximum of all the principal curvatures of \( \partial K \). Finally, let

\[
\Gamma = \Gamma(K, \kappa_{\max}) = \{ x : -(2\kappa_{\max})^{-1} < \delta_K(x) < (2\kappa_{\max})^{-1} \}.
\]
Then $\delta_K \in C^2(\Gamma)$. Furthermore, the level set $K_u$ is $C^2$ whenever $|u| < (2\kappa_{\text{max}})^{-1}$ and its principal curvatures at a point $x$ are given by
\[
\kappa_j(x) = \frac{\kappa_j(y)}{1 - u\kappa_j(y)}, \quad j = 1, \ldots, d - 1,
\]
where $y$ is the unique point of $\partial K$ such that $\text{dist}(x, y) = |u|$ and $\kappa_j(y)$ are the principal curvatures of $\partial K$ at $y$.

**Proof.** This is essentially a reformulation of Lemmas 14.16 and 14.17 in [17] for the case of convex bodies.

Let us now move back to the two-dimensional case. In the next two lemmas we estimate the Fourier transforms of the functions $\chi_D$ and $H_\Omega$ in Theorem 3 for the specific type of sets $D$ that one needs in the proof of Theorem 2.

**Lemma 10.** Let $\Omega$ be a convex body in $\mathbb{R}^2$ with $C^2$ boundary with non-vanishing curvature and let $\kappa_{\text{min}}$ and $\kappa_{\text{max}}$ be the minimum and the maximum of the curvature of $\partial \Omega$. Let $I$ be a rectangle contained in a unit square with sides parallel to the axes, and call $K = \Omega \cap I$. Then there exists a constant $c$ depending only on $\kappa_{\text{min}}$ such that for all $R \geq 4\kappa_{\text{max}}^2$ and for every $n = (n_1, n_2) \in \mathbb{Z}^2$ with $0 < |n| < R$,
\[
|\hat{H}_R(n)| \leq c \left( \frac{1}{|n|^{3/2}} + c \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|} \right).
\]
Here, $H_R(x)$ is the function defined in Theorem 3 by $H_R(x) = \psi(R|\delta_x(\kappa)||)$. Finally, there is a universal constant $c > 0$ such that for all $R \geq 1$,
\[
|\hat{H}_R(0)| \leq \frac{c}{R}.
\]

**Proof.** By the coarea formula (see [16] Theorem 2, page 117), since $|\nabla \delta_x(\kappa)| = 1$ almost everywhere,
\[
\hat{H}_R(n) = \int_{\mathbb{R}^2} \psi(R|\delta_x(\kappa)||) e^{-2\pi i n \cdot x} dx = \int_{-\infty}^{\infty} \psi(|Ru|) \int_{K_u} e^{-2\pi i n \cdot x} dx du.
\]
where $K_u = \{ x : \delta_x(\kappa) = u \}$ as in the above Definition 5 and the integration on the level set $K_u$ is with respect to the Hausdorff measure. Thus
\[
|\hat{H}_R(n)| \leq \int_{|u| < R^{-1/2}} \psi(R|u|) du \sup_{|u| < R^{-1/2}} \left| \int_{K_u} e^{-2\pi i n \cdot x} dx \right|
+ \int_{|u| \geq R^{-1/2}} \psi(R|u|) |K_u| du
\leq \frac{c_1}{R} \sup_{|u| < R^{-1/2}} \left| \int_{K_u} e^{-2\pi i n \cdot x} dx \right| + \frac{c_2}{R^{10}}.
\]
The constant $c_1$ is just the integral of $2\psi$ on $[0, +\infty)$, while $c_2$ depends on the rapid decay of $\psi$ and the slow growth of $|K_u|$ (recall that $K^u$ is convex and contained in a square of side $1 + 2|u|$, and therefore the Hausdorff measure of $K_u$ is smaller than $4(1 + 2|u|)$). In particular, $c_1$ and $c_2$ are universal constants and we immediately have that for any $R \geq 1$
\[
|\hat{H}_R(0)| \leq \frac{c}{R},
\]
where $c$ is a universal constant.
Now assume $n \neq 0$, $R^{-1/2} \leq 1/(2\kappa_{\text{max}})$ and $0 \leq u \leq R^{-1/2}$. Then, by the above Lemma 8 and Lemma 9, $K_u$ consists of at most four smooth convex curves with curvature bounded below by $\kappa_{\text{min}}$, and at most four segments of length at most 1 parallel to the axes. By Lemma 4 and Lemma 5 this gives

$$\sup_{0 \leq u \leq R^{-1/2}} \left| \int_{K_u} e^{-2\pi i x \cdot n} \, dx \right| \leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1 + |n_i|},$$

where the constant $c$ depends only on the curvature $\kappa_{\text{min}}$. On the other hand, if $R^{-1/2} \leq 1/(2\kappa_{\text{max}})$ and if $-R^{-1/2} \leq u < 0$, then $K_u$ is composed by at most four smooth convex curves with curvature greater than or equal to $2\kappa_{\text{min}}/3$, at most four segments parallel to the axes and of length at most 1, and at most eight arcs of circles of radius $|u|$. In order to better understand this, observe (see Figure 1) that one can divide the complement of $K$ into at most sixteen regions by taking the two normals to $\partial K$ at each "vertex" of $K$ (there are at most eight "vertices"). The part of $K_u$ that intersects a region attached to a straight line is a parallel straight line of length at most 1. The part of $K_u$ that intersects a region attached to a curve coming from $\partial \Omega$ is a part of $\Omega_u$. Finally, the part of $K_u$ that intersects a region attached to a vertex of $K$ is an arc of circle of radius $|u|$. It follows that

$$\sup_{-R^{-1/2} \leq u < 0} \left| \int_{K_u} e^{-2\pi i x \cdot n} \, dx \right| \leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1 + |n_i|} + c \frac{|u|^{1/2}}{|n|^{1/2}}$$

$$\leq c \frac{1}{|n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{1 + |n_i|}.$$
where the constant $c$ depends only on the minimal curvature $\kappa_{\text{min}}$. Therefore, when $0 < |n| < R$ we have

$$\left| \hat{H}(n) \right| \leq c \frac{1}{R |n|^{1/2}} + c \sum_{i=1}^{2} \frac{1}{R} \frac{1}{1 + |n_i|} + c \frac{1}{R^{10}}$$

$$\leq c \frac{1}{|n|^{3/2}} + c \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|}.$$

\[ \square \]

**Lemma 11.** Let $\Omega$ be a convex body in $\mathbb{R}^2$ with $C^2$ boundary with non-vanishing curvature and let $\kappa_{\text{min}}$ be the minimum of the curvature of $\partial \Omega$. Let $I$ be a rectangle contained in a unit square with sides parallel to the axes, and call $K = \Omega \cap I$. Then there exists a constant $c$ depending only on $\kappa_{\text{min}}$ such that for every $n = (n_1, n_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$

$$\left| \hat{\gamma}_R(n) \right| \leq c \frac{1}{|n|^{3/2}} + c \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|}.$$

**Proof.** An application of the divergence theorem gives

$$\left| \hat{\gamma}_R(n) \right| = \left| \int_K e^{-2\pi in \cdot x} dx \right| = \left| \int_{\partial K} \frac{\nabla(x) \cdot n}{2\pi |n|^2} e^{-2\pi in \cdot x} dx \right|$$

$$= \frac{1}{2\pi |n|} \left| \int_{\partial K} \nabla(x) \cdot n e^{-2\pi in \cdot x} dx \right|.$$

Here $\nabla(x)$ is the outer normal to $\partial K$ at the point $x$. This oscillatory integral can be estimated by means of standard techniques. We include the details for the sake of completeness. The boundary of $K = \Omega \cap I$ is composed of at most four smooth convex curves with curvature bounded below by $\kappa_{\text{min}}$, coming from $\partial \Omega$, and at most four segments of length at most 1 parallel to the axes, coming from $\partial I$. We therefore split the above integral into a sum of integrals over the components of $\partial K$ described above. When integrating over a segment, the quantity $\nabla(x) \cdot n/|n|$ remains constant and an immediate application of Lemma 10 gives the estimate

$$c \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|},$$

with $c$ a universal constant. Let us now estimate the integral over an arc of $\partial \Omega$, call it $\gamma$. If $r(\tau)$ is a parametrization of $\gamma$ with respect to arclength, integration by parts gives

$$\left| \int_0^\ell \nabla(x) \frac{n}{|n|} e^{-2\pi in \cdot x} dx \right| = \left| \int_0^\ell \nabla(r(\tau)) \cdot n e^{-2\pi in \cdot r(\tau)} d\tau \right|$$

$$= \left| \int_0^\ell e^{-2\pi in \cdot r(u)} du - \int_0^\ell \frac{d}{d\tau} (\nabla(r(\tau))) \cdot n \int_0^\tau e^{-2\pi in \cdot r(u)} du d\tau \right|$$

$$\leq \left| \int_0^\ell e^{-2\pi in \cdot r(u)} du \right| \leq 2\pi \sup_{0 \leq \tau \leq \ell} \int_0^\tau e^{-2\pi in \cdot r(u)} du \leq \frac{c}{|n|^{1/2}}.$$

Here $\kappa(\tau)$ is the curvature of $\gamma$ at the point $r(\tau)$ and $\int_0^\ell \kappa(\tau) d\tau$ is the total curvature of $\gamma$. Since $\gamma$ is an arc of $\partial \Omega$, the total curvature of $\gamma$ is smaller than the total curvature of $\partial \Omega$, that is $2\pi$. The last inequality is just an immediate application of Lemma 10, where the constant $c$ above depends only on the minimal curvature $\kappa_{\text{min}}$ of $\partial \Omega$. 
We are now ready to proceed with the proof of the main result of the paper.

**Proof of Theorem 2** Let \( \kappa_{\text{min}} \) and \( \kappa_{\text{max}} \) be the minimum and the maximum of the curvature of \( \partial \Omega \). If we call \( m_1, \ldots, m_q \) the lattice points for which the sets

\[
([0,s_1] \times [0,s_2] + x + m_i) \cap \Omega
\]

are nonempty, and let

\[
K_i = ([0,s_1] \times [0,s_2] + x + m_i) \cap \Omega,
\]

then of course

\[
\bigcup_{m \in \mathbb{Z}^2} ([0,s_1] \times [0,s_2] + x + m) \cap \Omega = \bigcup_{i=1}^q K_i.
\]

The number \( q \) is bounded by the maximum number of unit squares with integer vertices that intersect any given translate of \( \Omega \) in \( \mathbb{R}^2 \). This number is of course bounded by \( (\text{diam}(\Omega) + 2)^2 \). We recall that we need a uniform estimate with respect to \( s \) and \( x \).

![Figure 2](image_url)

**Figure 2.** The intersection of a convex set \( \Omega \) with smooth boundary having non-vanishing curvature with the integer translates of a fixed rectangle.

The sets \( K_i \) are as in Figure 2, at most four sides are parallel to the coordinate axes, while the curved parts come from \( \partial \Omega \). The discrepancy

\[
\left| \frac{1}{N} \sum_{j=1}^N \sum_{m \in \mathbb{Z}^2} \chi_{I(s,x) \cap \Omega}(t(j) + m) - |I(s,x) \cap \Omega| \right|
\]
is clearly bounded by the sum of the discrepancies of the sets $K_i$,

$$\sum_{i=1}^{q} \left| \frac{1}{N} \sum_{j=1}^{N} \sum_{t \in \mathbb{Z}^2} \zeta_{K_i}(t \cdot j + m) - |K_i| \right|,$$

and we shall therefore study the discrepancy of a single piece $K_i$. Let us call $K$ one such set.

By the general form of the Erdős-Turán inequality in Theorem 3, the discrepancy of a single piece $K$ is bounded by the quantity

$$\left| \hat{H}(0) \right| + \sum_{n \in \mathbb{Z}^2, 0 < |n| < R} \left( |\hat{\varphi}(n)| + \left| \hat{H}(n) \right| \right) \left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i n \cdot (j \cdot t)} \right|,$$

We recall that $R > 0$ is a number that we can choose at our convenience, $H(x) = \varphi(R |\delta_K(x)|)$ and $\varphi(u)$ is a properly chosen function on $[0, +\infty)$ with rapid decay at infinity.

The estimates of $\hat{\varphi}(n)$ and $\hat{H}(n)$ are contained in the above Lemmas 10 and 11, while the estimate of the exponential sums follows a standard argument,

$$\left| \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i n \cdot (j \cdot \alpha \cdot \beta)} \right| = \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i n \cdot (\alpha \cdot \beta)} = \frac{1}{N \cdot \sin (\pi n \cdot (\alpha \cdot \beta))} \leq \frac{1}{N \cdot \|n \cdot (\alpha \cdot \beta)\|},$$

where $\|u\|$ is the distance from $u$ to the closest integer.

Overall, the goal estimate (3) becomes

$$\frac{1}{R} + \sum_{0 < |n| < R} \left( \frac{1}{|n|^2} + \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|} \right) \frac{1}{N \cdot \|n \cdot (\alpha \cdot \beta)\|}.$$ 

Observe now that

$$\sum_{0 < |n| < R} \frac{1}{1 + |n_1|} \frac{1}{1 + |n_2|} \frac{1}{\|n \cdot (\alpha \cdot \beta)\|} \leq c \sum_{i=0}^{\log R \log R} \sum_{j=0}^{\log R} \frac{1}{2^j \cdot \sum_{n_1=2^i} 2^{j+1}} \frac{1}{\sum_{n_2=2^j} 2^{j+1}} \frac{1}{\|n_1 \cdot \alpha + n_2 \cdot \beta\|},$$

Let us study the sum $\sum_{i=1}^{2^{i+1}-1} \sum_{j=1}^{2^{j+1}-1} \frac{1}{|n_1 \cdot \alpha + n_2 \cdot \beta|}$ first. By the celebrated result of W. M. Schmidt [24], see also [25, Theorem 7C], since $1, \alpha, \beta$ are linearly independent on $\mathbb{Q}$, for any $\varepsilon > 0$ there is a constant $\gamma > 0$ such that for any $n \neq 0$,

$$\|n_1 \cdot \alpha + n_2 \cdot \beta\| > \frac{\gamma}{(1 + |n_1|)^{1+\varepsilon}(1 + |n_2|)^{1+\varepsilon}}.$$ 

Then, following [12], in any interval of the form

$$\frac{(k-1) \gamma}{(1 + 2^{j+1})^{1+\varepsilon}(1 + 2^{j+1})^{1+\varepsilon}} \frac{k \gamma}{(1 + 2^{j+1})^{1+\varepsilon}(1 + 2^{j+1})^{1+\varepsilon}},$$

where $k$ is a positive integer, there are at most two numbers of the form $|n_1 \cdot \alpha + n_2 \cdot \beta|$, with $2^j \leq |n_1| < 2^{j+1}$ and $2^j \leq |n_2| < 2^{j+1}$. Indeed, assume by contradiction that there are three such numbers. Then for two of them, say $|n_1 \cdot \alpha + n_2 \cdot \beta|$ and $|m_1 \cdot \alpha + m_2 \cdot \beta|$, the fractional
parts of $n_1 \alpha + n_2 \beta$ and $m_1 \alpha + m_2 \beta$ belong either to $(0, 1/2]$ or to $(1/2, 1)$. Assume without loss of generality that they belong to $(0, 1/2]$. Then
\[
\frac{\gamma}{(1 + 2^{i+1})(1 + \varepsilon)(1 + 2^{j+1})^{1+\varepsilon}} > \left\| n_1 \alpha + n_2 \beta - \| m_1 \alpha + m_2 \beta \| \right\|
\]
\[
= \left\| n_1 \alpha + n_2 \beta - p - (m_1 \alpha + m_2 \beta - q) \right\|
\]
\[
\geq \| (n_1 - m_1) \alpha + (n_2 - m_2) \beta \|
\]
\[
> \frac{\gamma}{(1 + 2^{i+1})(1 + \varepsilon)(1 + 2^{j+1})^{1+\varepsilon}}.
\]

By the same type of argument, in the first interval $\left[0, \frac{\gamma}{(1 + 2^{i+1})(1 + \varepsilon)(1 + 2^{j+1})^{1+\varepsilon}} \right)$, there are no points of the form $\| n_1 \alpha + n_2 \beta \|$. It follows that
\[
\sum_{n_1 = 2^i}^{2^{i+1} - 1} \sum_{n_2 = 2^j}^{2^{j+1} - 1} \frac{1}{\| n_1 \alpha + n_2 \beta \|} \leq c \sum_{k=1}^{2^{i+j}} \frac{2^{i+j}(1+\varepsilon)}{k \gamma} \leq c 2^{i+j}(1+\varepsilon) (i+j).
\]

Similarly,
\[
\sum_{n_1 = 2^i}^{2^{i+1} - 1} \frac{1}{\| n_1 \alpha \|} \leq c 2^{i(1+\varepsilon)} i, \quad \sum_{n_2 = 2^j}^{2^{j+1} - 1} \frac{1}{\| n_2 \beta \|} \leq c 2^{j(1+\varepsilon)} j.
\]

Finally,
\[
\sum_{0 < |n| < R} \frac{1}{\| n \|} \frac{1}{\| n \cdot (\alpha, \beta) \|} \leq c \sum_{i=0}^{\log R} \log R \frac{1}{2^i} 2^{i+j}(1+\varepsilon) (i+j) + c \sum_{i=0}^{\log R} \frac{1}{2^i} 2^{i(1+\varepsilon)} i + c \sum_{j=0}^{\log R} \frac{1}{2^j} 2^{j(1+\varepsilon)} j
\]
\[
\leq c \sum_{i=0}^{\log R} 2^{i+1} R^2 \log R + c R^2 \log R \leq c R^{2\varepsilon} \log R.
\]

Finally, we use the hypothesis that $1, \alpha, \beta$ are a basis of a number field in $\mathbb{Q}$. By a simple argument in number field theory, there is a constant $\eta$ such that for any $n \neq 0$,
\[
\| n_1 \alpha + n_2 \beta \| > \frac{\eta}{(\max(|n_1|, |n_2|))^2}.
\]

See for example [25] Theorem 6F. By a similar argument as before, this implies that
\[
\sum_{\max(|n_1|, |n_2|) = 2^i}^{2^{i+1} - 1} \frac{1}{\| n \cdot (\alpha, \beta) \|} \leq c \sum_{k=1}^{2^i} \frac{2^{2i}}{k} \leq c i 2^{2i}.
\]

Thus,
\[
\sum_{0 < |n| < R} \frac{1}{\| n \|^{3/2}} \frac{1}{\| n \cdot (\alpha, \beta) \|} \leq c \sum_{i=0}^{\log R} \sum_{\max(|n_1|, |n_2|) = 2^i}^{2^{i+1} - 1} \frac{1}{|n|^{3/2} \| n \cdot (\alpha, \beta) \|} \leq c \sum_{i=0}^{\log R} \frac{1}{2^{3i/2}} i 2^{2i} \leq c R^{1/2} \log R.
\]

Setting $R = N^{2/3}$ gives the desired estimate $N^{-2/3} \log N$, as long as $N \geq 8 \kappa_{\max}^3$. \qed
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