A note on the ergodicity of Fokker–Planck flows in $L^1(\mathbb{R}^d)$

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Abstract

One proves that the nonlinear semigroups in $L^1(\mathbb{R}^d)$, $d \geq 3$, associated with the nonlinear Fokker–Planck equation $u_t - \Delta \beta(u) + \text{div}(Db(u)u) = 0$ in $(0, \infty) \times \mathbb{R}^d$ is ergodic under suitable conditions on the coefficients $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$ and $b : \mathbb{R} \to \mathbb{R}$. In particular, this implies the ergodicity of probabilistically weak solutions to the corresponding McKean–Vlasov stochastic differential equations. Such a result completes the asymptotic results established in [7] in the case where the Fokker–Planck flow $S(t)$ in $L^1(\mathbb{R}^d)$ has not a fixed point.

MSC: 60H15, 47H05, 47J05.

Keywords: nonlinear Fokker–Planck equation, mild solution, stochastic differential equation. ergodic.

1 Introduction

Consider the nonlinear Fokker–Planck equation

$$
\begin{align*}
    u_t - \Delta \beta(u) + \text{div}(Db(u)u) &= 0, \quad \text{in } (0, \infty) \times \mathbb{R}^d, \\
    u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d,
\end{align*}
$$

(1.1)

where $\beta : \mathbb{R} \to \mathbb{R}$, $D : \mathbb{R}^d \to \mathbb{R}^d$, $d \geq 3$, and $b : \mathbb{R} \to \mathbb{R}$ are given functions to be made precise in the following.

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This equation describes, in statistical physics and mean field theory, the dynamics of a set of interacting particles or of many body systems in disordered media (the so-called anomalous diffusion). (See, e.g., [13].) In such a situation, for each \( t \geq 0 \), \( u = u(t, \cdot) \) is a probability density for each probability density \( u_0 \). Another source for equation (1.1) is the description of the dynamics of Itô stochastic processes \( X(t) \) in terms of their probability densities \( u = u(t, x) \). Namely, if \( u \in L^1_{\text{loc}}((0, \infty) \times \mathbb{R}^d) \) is a Schwartz distributional solution to (1.1) such that \( t \rightarrow u(t, \cdot)dx \) is weakly continuous and \( u(t, \cdot) \) is a probability density, then there is a probabilistically weak solution \( X \) to the McKean–Vlasov stochastic differential equation in \( \mathbb{R}^d \)

\[
\begin{align*}
    dX(t) &= D(X(t))b \left( \frac{d\mathcal{L}_{X(t)}}{dx}(X(t)) \right) dt + \sqrt{2\beta \left( \frac{d\mathcal{L}_{X(t)}}{dx}(X(t)) \right)} dW(t), \\
    \mathcal{L}_{X(t)}(dx) &= u(t, x)dx, 
\end{align*}
\]

on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with the filtration \((\mathcal{F}_t)_{t\geq0}\), where \( W \) is a \( d \)-dimensional \((\mathcal{F}_t)\)-Brownian motion. Here, \( \mathcal{L}_{X(t)} \) is the law of the process \( X(t) \) under \( \mathbb{P} \) and \( \mathcal{L}_{X_0} = u_0dx \) (see [3] for details).

A function \( u : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R} \) is called a mild solution to (1.1) if \( u \in C([0, \infty); L^1(\mathbb{R}^d)) \) and

\[
    u(t) = \lim_{h \rightarrow 0} u_h(t) \text{ strongly in } L^1(\mathbb{R}^d), \quad \forall t \geq 0, \tag{1.3}
\]

where \( u_h : [0, \infty) \rightarrow L^1(\mathbb{R}^d) \) is the solution to the equation

\[
\begin{align*}
    \frac{1}{h} (u_h(t) - u_h(t - h)) + A_0u_h(t) &= 0 \quad \text{for } t \geq 0, \\
    u_h(t) &= u_0 \quad \text{for } t < 0, \tag{1.4}
\end{align*}
\]

and \( A_0 \) is the operator in \( L^1(\mathbb{R}^d) \) defined by

\[
\begin{align*}
    A_0(y) &= -\Delta \beta(y) + \text{div}(Db(y)y) \text{ in } \mathcal{D}'(\mathbb{R}^d), \quad \forall y \in D(A_0), \tag{1.5} \\
    D(A_0) &= \{ y \in L^1(\mathbb{R}^d); \ \beta(y) \in L^1_{\text{loc}}(\mathbb{R}^d), \ Db(y)y \in L^1_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d); \tag{1.6} \\
    &\quad -\Delta \beta(y) + \text{div}(Db(y)y) \in L^1(\mathbb{R}^d) \}. \end{align*}
\]

The existence of a mild solution \( u \) to (1.1) was studied under various hypotheses on \( \beta, d \) and \( b \) in the works [3]–[7]. The idea, previously used by
M.G. Crandall in the existence theory of entropy solutions to a nonlinear conservation law equation [11], is to represent (1.1) as a Cauchy problem in $L^1(\mathbb{R}^d)$

$$\frac{du}{dt} + Au = 0, \quad \forall \ t \geq 0; \quad u(0) = u_0,$$

(1.7)

where $A$ is an $m$-accretive operator in $L^1(\mathbb{R}^d)$ such that $(I + \lambda A_0)^{-1} f \in (I + \lambda A_0)^{-1} f, \forall f \in L^1(\mathbb{R}^d), \lambda > 0$. Then, by the Crandall & Liggett generation theorem (see, e.g., [1], [2]) there exists

$$S(t)u_0 = u(t, u_0) = \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^{-n} u_0, \quad \forall t \geq 0, \ u_0 \in \overline{D(A)},$$

(1.8)

strongly in $L^1(\mathbb{R}^d)$, uniformly in $t$ on bounded intervals. The function $u = u(t, u_0)$ is a mild solution to equation (1.1) in the sense of (1.3)–(1.4) and the mapping $S(t) : \overline{D(A)} \to \overline{D(A)}, \ t \geq 0$, is a continuous semigroup of contractions in $L^1(\mathbb{R}^d)$ on $\overline{D(A)}$ – the closure of the domain $D(A)$ of $A$ in $L^1(\mathbb{R}^d)$. We call such a semigroup of contraction a nonlinear Fokker–Planck flow. It should be emphasized that, in general, such a semigroup $S(t)$ is not unique because its generator $A$ is constructed from $A_0$ by

$$A(y) = A_0(J_\lambda(y)), \ \forall y \in D(A) = \{ u = J_\lambda(f); \ f \in L^1(\mathbb{R}^d), \ \lambda > 0 \},$$

(1.9)

where $J_\lambda : L^1(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$ is a family of contractions such that $J_\lambda(f) \in (I + \lambda A_0)^{-1} f, \forall f \in L^1(\mathbb{R}^d), \lambda > 0$. Since $I + \lambda A_0$ is, in general, not one-to-one, hence $(I + \lambda A_0)^{-1}$ is multivalued, the family $\{ J_\lambda \}_{\lambda > 0}$ is not unique, hence so is the operator $A$. There is an alternative approach to existence theory for nonlinear Fokker–Planck equations developed by J.A. Carrillo [9], G.Q. Chen and B. Perthame [10] in the context of entropy and kinetic solutions, but we shall not pursue this approach in this paper. (In fact, a mild solution to (1.1) is a weaker concept of solution than that of entropy solutions as developed in [9], [10].) The semigroups $S(t)$ represents a section in the class of mild solutions.

Here, we shall consider equation (1.1) under the following hypotheses:

\begin{itemize}
  \item[(H1)] $\beta \in C^1(\mathbb{R}), \ \beta'(r) > 0, \ \forall r \in \mathbb{R} \setminus \{0\}, \ \beta(0) = 0$, and
  \begin{equation}
  \mu_1 \min\{|r|^{\nu}, |r|\} \leq |\beta(r)| \leq \mu_2 |r|, \ \forall r \in \mathbb{R},
  \end{equation}
  \end{itemize}

(1.10)

for $\mu_1, \mu_2 > 0$ and $\nu > \frac{d-1}{d}, \ d \geq 3$. 

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(H2) $D \in L^\infty(\mathbb{R}^d; \mathbb{R}^d) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d)$, $\text{div} D \in (L^2(\mathbb{R}^d) + L^\infty(\mathbb{R}^d))$, $D = -\nabla \Phi$, where $\Phi \in C(\mathbb{R}^d) \cap W^{1,1}_{\text{loc}}(\mathbb{R}^d)$ and

$$
\Phi(x) \geq 1, \forall x \in \mathbb{R}^d, \lim_{|x| \to \infty} \Phi(x) = +\infty,
$$

$$
\Phi^{-m} \in L^1(\mathbb{R}^d) \text{ for some } m \geq 2,
$$

$$
\mu_2 \Delta \Phi(x) - b_0 |\nabla \Phi(x)|^2 \leq 0, \text{ a.e. } x \in \mathbb{R}^d.
$$

(H3) $b \in C^1(\mathbb{R}) \cap C_b(\mathbb{R})$, $b(r) \geq b_0 > 0$ for all $r \in [0, \infty)$.

For instance, any continuous, increasing function $\beta : \mathbb{R} \to \mathbb{R}$ of the form

$$
\beta(r) = \begin{cases} 
\mu_1 r |r|^{d-1} & \text{for } |r| \leq r_0, \\
\mu_2 h(r) & \text{for } |r| > r_0,
\end{cases}
$$

where $r_0 > 0$, $\mu_1, \mu_2 > 0$, $|h(r)| \leq L |r|$, $\forall r \in \mathbb{R}$, $L > 0$, satisfies (1.10). As regards Hypothesis (H2), an example of such a function $\Phi$ is (see [5])

$$
\Phi(x) = \begin{cases} 
|x|^2 \log |x| + \mu & \text{for } |x| \leq \delta = \exp \left(-\frac{d+2}{2d}\right), \\
\varphi(|x|) + \eta |x| + \mu & \text{for } |x| > \delta,
\end{cases}
$$

where $\mu, \eta > 0$ are sufficiently large and $\varphi$ is as in [5, Appendix]. As a matter of fact, in this case (see [4], [7]) the family $\{J_\lambda\}_{\lambda > 0}$ of resolvents which defines the operator $A$ is given by the viscosity approximation scheme

$$
J_\lambda(f) = \lim_{\varepsilon \to 0} y_\varepsilon \text{ in } L^1(\mathbb{R}^d),
$$

(1.11)

where $y_\varepsilon$ is the solution to the equation

$$
y_\varepsilon - \lambda \Delta (\beta_\varepsilon(y_\varepsilon) + \varepsilon y_\varepsilon) + \lambda \text{div}(D_\varepsilon b_\varepsilon(y_\varepsilon)y_\varepsilon) = f,
$$

(1.12)

where $\beta_\varepsilon, D_\varepsilon$ and $b_\varepsilon$ are smooth approximations of $\beta, D$ and $b$.

It turns out, however (see [6]), that if one further assumes that, for some $\alpha > 0,$

$$
|b(r)r - b(\bar{r})\bar{r}| \leq \alpha |\beta(r) - \beta(\bar{r})|, \forall r, \bar{r} \in \mathbb{R},
$$

(1.13)

then $(I + \lambda A_0)^{-1}$ is single-valued and so $A$ is uniquely defined, more precisely $A = A_0$. In the following, we shall consider the semigroup $S(t)$ generated by $A$, which is given by (1.9), and we shall call it the nonlinear Fokker–Planck flow. This semigroup leaves invariant the set $\mathcal{P}$ of all the probability densities $\rho$ on $\mathbb{R}^d$, that is,
\[ P = \left\{ \rho \in L^1(\mathbb{R}^d); \, \rho \geq 0, \text{ a.e. in } \mathbb{R}^d; \, \int_{\mathbb{R}^d} \rho \, dx = 1 \right\}. \]

Now, consider the orbit \( \gamma(u_0) = \{ S(t)u_0, \, t \geq 0 \} \) where \( u_0 \in C := \overline{D(A)} = L^1(\mathbb{R}^d) \) if \( \beta \in C^2(\mathbb{R}) \). We associate to \( u_0 \) the \( \omega \)-limit set
\[
\omega(u_0) = \{ u_\infty = \lim_{n \to \infty} S(t_n)u_0 \in L^1 \text{ for some } \{ t_n \} \to \infty \}
\]
\[
= \bigcap_{s \geq 0} \bigcup_{t \geq s} S(t)u_0.
\]

In particular, if \( \omega(u_0) \neq \emptyset \) and consists of one element \( u_\infty \) only, then we have
\[
\lim_{t \to \infty} S(t)u_0 = u_\infty \text{ in } L^1.
\]

In [5], it was proved that, if \( \beta \) is not degenerate in the origin, that is,
\[
0 < \gamma_0 \leq \beta'(r) \leq \gamma_1, \quad \forall r \in \mathbb{R},
\]
(which also implies that \( C = L^1 \)), then, for each \( u_0 \in P \), such that
\[
u \ln(u_0) \in L^1(\mathbb{R}^d), \, \| u_0 \| = \int_{\mathbb{R}^d} u_0(x)\Phi(x)dx < \infty,
\]
one has \( \omega(u_0) = \{ u_\infty \} \), where \( u_\infty \) is the unique solution in \( (L^1 \cap L^\infty)(\mathbb{R}^d) \) to the stationary equation
\[
-\Delta \beta(u) + \text{div}(Db(u)u) = 0 \quad \text{in } D'(\mathbb{R}^d).
\]

This is an \( H \)-theorem type result for the Fokker–Planck equation (1.1) (see, e.g., [14] for physical significance and examples). In [7], the nondegeneracy condition was relaxed to (H1), which along with (H2)–(H3) leads to the conclusion that, if \( u_0 \in P \cap C \) and \( \| u_0 \| \leq \eta \) for some \( \eta > 0 \), then \( \omega(u_0) \) is nonempty, invariant under \( S(t), \, t \geq 0 \), and compact in \( L^1(\mathbb{R}^d) \) which implies that it is an attractor for the trajectory \( \gamma(u_0) \). If, in addition, there is \( a \in P \cap C \) such that \( \| a \| \leq \eta \) and \( S(t)a = a \) for \( t > 0 \), then \( \omega(u_0) \) lies on a ball \( \{ y \in L^1(\mathbb{R}^d); \ |y - a|_{L^1(\mathbb{R}^d)} = r \} \). In [7], sufficient conditions on \( \beta \) and \( b \) for the existence of such a fixed point for \( S(t) \) were given. For instance, this happens if
\[
\lim_{r \to +\infty} \int_{1}^{r} \frac{\beta'(s)}{sb(s)} \, ds = +\infty \quad \text{if } \nu \in \left( 1 - \frac{1}{d}, 1 \right]
\]
\[ \lim_{r \to 0} \int_1^r \frac{\beta'(s)}{sb(s)} \, ds = -\infty \text{ if } \nu > 1. \]

Here, no such condition will be imposed and so it is not clear whether the semigroup \( S(t) \) has a fixed point \( a \) in \( \mathcal{P} \cap C \), but the nature of the omega-limit set \( \omega(u_0) \) will be made clear from the asymptotic properties of the semigroup \( S(t) \). Namely, we shall prove that the flow \( t \to S(t) \) is ergodic in \( L^1(\mathbb{R}^d) \) (Theorem 2.1).

**Notation.** \( L^p(\mathbb{R}^d) = L^p, \ 1 \leq p \leq \infty \), is the space of real-valued Lebesgue measurable, \( p \)-integrable functions on \( \mathbb{R}^d \). The space \( L^p(\mathbb{R}^d; \mathbb{R}^d) \) is analogously defined and \( W^{1,1}(\mathbb{R}^d; \mathbb{R}^d) \) is the Sobolev space \( \{ u \in L^1(\mathbb{R}^d; \mathbb{R}^d); D_i u_j \in L^1(\mathbb{R}^d), \ i = 1, \ldots, d; u = (u_j)_{j=1}^d \} \). By \( W^{1,1}_{loc}(\mathbb{R}^d; \mathbb{R}^d) \) we denote the corresponding local space. Let \( C_b(\mathbb{R}) \) denote the space of continuous and bounded functions on \( \mathbb{R} \) and \( C^1(\mathbb{R}) \) the space of continuously differentiable functions on \( \mathbb{R} \).

An operator \( A : \mathcal{X} \to \mathcal{X} \), where \( \mathcal{X} \) is a Banach space, is called \( m \)-accretive if \( R(I + \lambda A) = \mathcal{X}, \ \forall \lambda > 0 \), and
\[
\|u_1 - u_2 + \lambda (Au_1 - Au_2)\|_X \geq \|u_1 - u_2\|_X, \ \forall \lambda > 0, \ u_1, u_2 \in D(A),
\]
where \( D(A) \) is the domain of \( A \) and \( R(I + \lambda A) \) is the range of \( I + \lambda A \). (See, e.g., [1], [2].) For each \( \eta > 0 \), we set
\[
\mathcal{M}_\eta := \left\{ u \in L^1; \|u\| = \int_{\mathbb{R}^d} |u(x)|\Phi(x)dx \leq \eta \right\},
\]
where \( \Phi \) is the potential of \( D \) defined in (H2).

### 2 The main results

Let \( S(t) : C \to C, \ C = \overline{D(A)} \), be the semigroup generated by the operator \( A \) given above by (1.9) and let
\[
\mathcal{K} := \mathcal{M}_\eta \cap C \cap \mathcal{P}
\]
for a given \( \eta > 0 \). We shall assume that hypotheses (H1)–(H3) hold. Then we have
**Theorem 2.1.** Let $\mathcal{X}$ be a real Banach space and let $F : \mathcal{K} \to \mathcal{X}$ be a uniformly continuous mapping. Then, for each $u_0 \in \mathcal{K}$, the set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$ and

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T F(S(t)u_0)dt = \int_{\omega(u_0)} F(\xi) d\xi,$$

(2.1)

where $\omega(u_0)$ is endowed with its natural commutative group structure (recalled below in the proof of the theorem) and where $d\xi$ is the normalized Haar measure on it.

We recall that a Haar measure $\mu$ on a locally compact topological commutative group $G$ is a nonzero Borel measure $\mu$ which is invariant on $G$, that is, $\mu(gS) = \mu(Sg) = \mu(S)$ for any Borel subset $S \subset G$.

An example covered by Theorem 2.1 is $\mathcal{X} = \mathbb{R}$ and

$$F(u) = \int_{\mathbb{R}^d} g(x)u(x)dx, \quad \forall u \in L^1(\mathbb{R}^d),$$

(2.2)

where $g \in L^\infty(\mathbb{R}^d)$. Then, by Theorem 2.1, we obviously have

**Corollary 2.2.** Under hypotheses (H1)–(H3), for each $u_0 \in \mathcal{K}$ and $g \in L^\infty(\mathbb{R}^d)$, we have

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x)dx = \int_{\omega(u_0)} \int_{\mathbb{R}^d} g(x)\xi(x)dx d\xi.$$ (2.3)

Furthermore, the semigroup $S(t)$ is mean-ergodic, that is,

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T S(t)u_0 dt = \int_{\omega(u_0)} \xi d\xi \text{ in } L^1(\mathbb{R}^d),$$

(2.4)

where $d\xi$ is, as above, the normalized Haar measure on $\omega(u_0)$.

By Corollary 2.2 it follows in particular that, under hypotheses (H1)–(H3) it is satisfied for the nonlinear Fokker–Planck flow $t \to S(t)u_0$ where $u_0 \in \mathcal{K}$ the classical *Boltzmann hypothesis* (see, e.g., [16], p. 389) is satisfied with the time average $\int_{\omega(u_0)} \xi d\xi$ which is the mean of the Haar measure $d\xi$.

Now, coming back to the McKean–Vlasov equation (1.2), we get by Corollary 2.2 the following ergodic result for solutions $X(t)$ to (1.2).
Corollary 2.3. Let $u_0 \in \mathcal{K}$. Then, under hypotheses (H1)–(H3) there is a probabilistically weak solution $X$ to (1.2), where $\mathcal{L}_{X_0} = u_0 dx$, such that

$$
\lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbb{E}[g(X(t))] dt = \int_{\omega(u_0)} \int g(x) \xi(x) dx \, d\xi, \quad \forall g \in L^\infty.
$$

(2.5)

Furthermore, we have

$$
\frac{1}{T} \int_0^T \mathcal{L}_{X(t)}(dx) dt \to \int_{\omega(u_0)} \xi(x) dx \, d\xi
$$

in the weak topology on $\mathcal{P}$ a.s. $T \to \infty$.

Remark 2.4. The case $d = 2$ is singular for the semigroup approach of equation (1.1), namely for the existence of an $m$-accretive realization $A$ of the operator $A_0$ and this is the principal motivation to avoid it (see [4], [7]). However, the case $d = 1$ could be treated in a similar way following [4], but we omit the details.

3 Proofs

As shown in [7], Lemma 4.3, under hypotheses (H1)–(H3) for each $u_0 \in \mathcal{K}$, the orbit $\gamma(u_0)$ of $S(t)u_0$ is precompact in $L^1$. This implies that the $\omega$-limit set $\omega(u_0)$ is compact in $L^1(\mathbb{R}^d)$. We recall that, for each $t \geq 0$, $\omega(u_0)$ is invariant under $S(t)$ which is an homeomorphism of $\omega(u_0)$ onto $\omega(u_0)$, that is, $S(t)$ is a group on $\omega(u_0)$. Then, $\omega(u_0)$ can be endowed with a topological commutative group structure with the product $g_1 \circ g_2 = S(t_1 + t_2)u_0$, $g_1, g_2 \in \omega(u_0)$, where $g_1 = S(t_1)u_0$, $g_2 = S(t_2)u_0$. Then, by the classical A. Weil theorem (see [15]), there is a unique normalized Haar measure on $\omega(u_0)$ and so, by Birkhoff’s ergodic theorem (see [8] and Theorem 1 in [12]) it follows that (2.1) holds for each uniformly continuous mapping $F : \mathcal{K} \to \mathcal{X}$ and so Theorem 2.1 follows.

As regards Corollary 2.3, the existence of a weak solution to (1.2) follows from [3]. Furthermore, formula (2.5) follows by (2.3) taking into account that

$$
\mathbb{E}[g(X(t))] = \int_{\mathbb{R}^d} g(x)(S(t)u_0)(x) dx, \quad \forall t \geq 0.
$$

Acknowledgement. This work was supported by the DFG through SFB 1283/2 2021-317210226 and by a grant of the Ministry of Research, Innovation and Digitization, CNCS–UEFISCDI project PN-III-P4-PCE-2021-0006, within PNCDI III.
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