Bad News
on the
Brane

by

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ABSTRACT

There has been substantial interest in obtaining a quantum-gravitational description of de Sitter space. However, any such attempts have encountered formidable obstacles, and new philosophical directions may be in order. One possibility, although somewhat speculative, would be to view the physical universe as a timelike hypersurface evolving in a higher-dimensional bulk spacetime; that is, the renowned brane-world scenario. In this paper, we extend some recent studies along this line, and consider a non-critical 3-brane moving in the background of an anti-de Sitter Reissner-Nordstrom-like black hole. Interestingly, even an arbitrarily small electrostatic charge in the bulk can induce a singularity-free “bounce” universe on the brane, whereas a vanishing charge typically implies a singular cosmology. However, under closer examination, from a holographic (dS/CFT) perspective, we demonstrate that the charge-induced bounce cosmologies are not physically viable. This implies the necessity for censoring against charge in a bulk black hole.
1 Introduction

Recent empirical evidence of an accelerating universe \[1\] is most easily explained in terms of a positive vacuum energy density or cosmological constant. This observation has triggered substantial interest in de Sitter (dS) space, which happens to be the maximally symmetric solution of Einstein gravity when this constant takes on a positive, non-vanishing value.

In particular, one hopes that a quantum theory of dS gravity can eventually be realized. Work along this line has been inspired by significant achievements in the realm of anti-dS space (i.e., maximally symmetric Einstein gravity with a negative cosmological constant). For instance, higher-dimensional spacetimes in string theory (our best candidate for a fundamental theory of quantum gravity) can often be decomposed into anti-dS space times a simple compact manifold. (For a review on string theory, see \[2\].) Furthermore, there is a well-known correspondence between any anti-dS spacetime and a conformal field theory (CFT) that “lives” at spatial infinity \[3, 4, 5\]. Significantly, this duality is a direct manifestation of the holographic principle \[6, 7, 8\], which is expected to be a fundamental constituent in the ultimate theory of quantum gravity.

Unfortunately, the remarkable achievements of anti-dS space have not translated well into a dS setting. For example, attempts at compactifying string theory into a stable vacuum with a positive cosmological constant have been plagued by pathologies; including singularities, wrong-sign kinetic terms and non-compact “compactification” manifolds. (See \[9\] and references therein for further discussion.) Meanwhile, from a holographic perspective, there has, actually, been significant progress in establishing a dS analogue to the anti-dS/CFT correspondence. However, the status of this conjectured dS/CFT duality still remains a very open question \[22\].

Progress along the desired lines has been stymied by various “detrimental” features of dS space; including the lack of an asymptotic spatial infinity, the absence of an objective observer, the existence of an observer-dependent event horizon, the lack of a global timelike killing vector, the eternally thermal background, \textit{etcetera}. (For further discussion on the properties of dS space, see \[23, 24\].) However, it is the finite entropy of dS space \[25\] that appears to

\[1\] A dS/CFT holographic duality was first rigorously formulated in \[10\]. For both earlier and later studies on this topic, one can consult the bibliographies of \[11, 12, 13\]. More recent studies of interest include \[14, 15, 16, 17, 18, 19, 20, 21\].
be the main impediment towards a quantum gravity realization. Significantly, this finite bound on the entropy implies that any dS theory of quantum gravity should be described by a finite-dimensional Hilbert space. Alas, this deduction is in conflict with the infinite-dimensional Hilbert space associated with string theory, any of its related conceptualizations (such as matrix theory) and, from a holographic perspective, any CFT. That is to say, the finite entropy of dS space appears to sabotage any prospective quantum-gravity framework at a very fundamental level.

With regard to the above conundrum, it is of further interest that the finite entropy of dS space serves as an upper bound on the observable entropy of a large class of spacetimes with a positive cosmological constant; including any spacetime with an asymptotically dS future.

It should be quite clear that any assimilation of quantum physics and dS gravity will have to overcome some very formidable obstacles. Although speculative, one possible road to salvation may yet arise out of the realm of brane-world scenarios. For sake of definiteness, let us now outline a particular form of brane-world model, as inspired by Randall and Sundrum, which will be adhered to throughout the paper. In the scenario of interest, our physical universe is regarded as a 3+1-dimensional hypersurface, or 3-brane, immersed in a 4+1-dimensional anti-dS bulk spacetime (with any additional dimensions having been compactified to string-length scale). We also follow the usual convention of confining all standard-model particles to the brane except for the graviton. In compliance with the observed cosmological constant, we further stipulate a non-critical brane with positive curvature. Furthermore, following Kraus, we take the viewpoint of a brane that flows through an otherwise static, black hole bulk.

Given the above framework, there are a couple of points of interest that should be kept in mind. Firstly, the brane dynamics will effectively describe the cosmological evolution of the universe. That is to say, from the perspective of an observer, the motion of the brane will appear as either a cosmological contraction or expansion. Secondly, the (effective) cosmological constant on the brane can be expressed in terms of a pair of bulk parameters; namely, the anti-dS curvature radius and the brane tension. In this sense, the cosmological constant can be regarded as an input parameter, rather than a dynamical parameter of the brane. Although the graviton propagates freely into the bulk, it has been shown that gravity will typically remain localized even when the brane is curved.

\[ \text{2} \] Although the graviton propagates freely into the bulk, it has been shown that gravity will typically remain localized even when the brane is curved.\[ \text{3} \]
than a variable quantity of the lower-dimensional gravity theory.

One might well query as to how this brane-world interpretation could possibly address the difficulties associated with a quantum formulation of dS gravity. The answer lies in the notion of a physical universe that can now be holographically viewed in an anti-dS framework \[36, 37\]. As discussed earlier, the quantum-gravitational aspects of anti-dS spacetimes are much better understood than their dS counterparts. Hypothetically speaking, one could obtain a string-theoretical description of the 5-dimensional bulk spacetime and then apply the anti-dS/CFT duality to translate this theory onto the brane \[38\]. Although, admittedly, such a prospect remains somewhat speculative at the present time.

In the current treatment, we investigate the above brane-world scenario for the intriguing case of a charged (anti-dS) black hole in the bulk. Before discussing the content of this paper, let us take note of some recent works of relevance. First of all, Petkou and Siopsis \[39\] considered the cosmological and holographic implications of a brane world evolving in the background of an anti-dS Schwarzschild black hole. This study was subsequently generalized by the present author \[13\] to include “topological” anti-dS black holes \[40\]. That is, Schwarzschild-like black holes (static with a constant-curvature horizon), but having an arbitrary horizon topology (flat, hyperbolic or spherical).

More recently, Mukherji and Peloso \[41\] have considered topological anti-dS black holes with a finite electrostatic charge (i.e., Reissner-Nordstrom-like) \[4\]. These authors focused their analysis on a critical brane (i.e., the effective cosmological constant is fine tuned to vanish); although they briefly touched upon the qualitative features of a non-critical brane scenario as well. In the program to follow, we elaborate on this last case of a non-critical (positively curved) brane evolving in a charged black hole background. Along with the switch in emphasis (from critical to non-critical), the current study substantially deviates from \[41\] in the following manner: we come here not to praise charged black hole bulks but, rather, to bury them.

The remainder of the paper is organized as follows. In Section 2, we introduce the relevant formalism; including the bulk solutions of interest (topological anti-dS black holes with charge) and the equation of motion

\[\text{For some earlier studies and discussion on brane cosmology with a bulk charge, see 42, 43, 44, 46, 47, 48, 49.}\]
for the brane. Notably, this equation mimics the Friedmann equation for radiative matter, along with an additional exotic form of matter. This latter contribution, which only arises when the bulk solution has a finite charge, can be identified as stiff matter with a negative-energy density [43].

In Section 3, we consider the cosmological implications for a certain class of solutions of the priorly mentioned Friedmann equation. Although explicit analytical expressions are out of our reach, it is still possible to describe the cosmologies of interest. Here, we appeal to asymptotic regimes for which the relevant solutions have already been formulated [13, 11]. In this process, we observe that a non-vanishing charge in the bulk (even an arbitrarily small one) generally induces a FRW “bounce” universe; that is, asymptotically dS and devoid of singularities.

Conversely, for the class of solutions under investigation, a vanishing charge implies a singular spacetime; either a “big bang” or a “big crunch”.

In Section 4, we consider the holographic implications of the charged-induced bounce cosmologies. More precisely, we calculate their generalized c-functions [24, 50] (as prescribed for renormalization group flow in an asymptotically dS spacetime [22, 53, 54]) and then determine how these functions evolve in time. On the basis of this analysis, we deduce that any such bounce cosmology, although free of singularities, is of an unphysical nature. We attribute this failure to the exotic, negative-energy matter that is induced by the bulk charge. Here, we note that similar failures have been observed when negative-energy matter is introduced into an otherwise purely dS spacetime [24, 12].

Finally, Section 5 contains a summary and considers future prospects.

2 Brane-World Scenario

The formal interest of this paper is a Randall-Sundrum [32] type of brane-world scenario; more specifically, a 3+1-dimensional brane (of positive tension) moving in a 4+1-dimensional bulk spacetime that is otherwise static. Furthermore, we will assume that the bulk geometry is described by an anti-dS black hole with an electrostatic charge and a constant-curvature horizon.

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4 For general discussion on Friedmann-Robertson-Walker (FRW) bounce cosmologies, see [24, 50]. For a recent, topical study, from both a brane-world and string-theoretical perspective, see [51].
The relevant black hole solutions can be regarded as “Reissner-Nordstrom-like”, but with an arbitrary horizon topology.

The solutions in the anti-dS bulk can thus be expressed as follows \[56\]:

\[
ds^2 = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 d\Omega^2_{k,3},
\]

where:

\[
f(r) = \frac{r^2}{L^2} + k - \frac{\omega M}{r^2} + \frac{3\omega^2 Q^2}{16r^4},
\]

\[
\omega = \frac{16\pi G_5}{3V}.
\]

Here, \(d\Omega^2_{k,3}\) denotes the line element of a 3-dimensional constant-curvature (Euclidean) hypersurface, \(V\) is the dimensionless volume of this hypersurface, \(L\) is the curvature radius of the anti-dS bulk (i.e., \(\Lambda_5 = -3/L^2\) is the bulk cosmological constant) and \(G_5\) is the 5-dimensional Newtonian constant. There are also three constants of integration in this solution: \(k\), \(M\) and \(Q^2\). Without loss of generality, \(k\) can be set equal to +1, 0 or -1; describing a horizon geometry that is respectively spherical (i.e., the anti-dS Reissner-Nordstrom case), flat or hyperbolic. Meanwhile, \(M\) and \(Q^2\) respectively measure the conserved mass and the electrostatic charge (squared) of the black hole; with these being regarded as strictly non-negative quantities.

Note that the existence of a pair of positive, real horizons or non-extremal black hole solution (as we assume to be the case in this analysis) depends on, in general, a sufficiently large value of \(L^2\) and, for \(k = +1\), a sufficiently small value of \(Q^2\) (in fact, \(Q^2 < 4M^2/3\))\[1\]. To be definite, we will assume that \(M\) is “large” (with \(L^2\) always taking on a large enough value to compensate) and \(Q^2\) is “small”. More precisely, we henceforth limit the following dimensionless parameters: \(\tilde{M} > 1\) (where \(\tilde{M} \sim M\) is defined below) and \(\epsilon^2 \equiv 3Q^2/4M^2 < 1\) (and, typically, \(\epsilon^2 << 1\)). That is, the black hole of interest is appropriate for a semi-classical regime and has a small (but non-vanishing) electrostatic charge.

\[5\] It should be noted that the \(k = -1\) hyperbolic solution supports a negative value for the mass \[4\]. However, this controversial scenario will be disregarded, on the grounds that a negative-mass black hole probably induces a non-unitary boundary theory \[7\].

\[6\] One can verify these claims by solving for the roots of Eq.\[3\]. These roots, of course, locate the positions of the horizons.
As demonstrated elsewhere (for instance, [35]), if one considers an evolving brane in an otherwise static bulk, the dynamics of the brane will mimic that of a FRW universe [58]. Here, we will skip the gory details (see [13] for a recent presentation) and simply quote the results.

As it so happens, the induced metric on the brane can be expressed in the following FRW form:

$$ds^2 = -d\tau^2 + r^2(\tau)d\Omega^2_{k,3},$$

where $\tau$ measures the physical time from the point of view of a brane observer and $r$ is the time-dependent cosmological scale factor. Moreover, the corresponding Friedmann equation is found to be as follows:

$$H^2 = \frac{\Lambda_4}{3} - \frac{k}{r^2} + \frac{\omega M}{r^4} - \frac{\omega^2 M^2 c^2}{4r^6},$$

where $H \equiv \dot{r}/r$ is the usual Hubble “constant” (with a dot denoting differentiation with respect to $\tau$) and $\Lambda_4$ is the effective cosmological constant on the brane. In achieving this form, we have invoked the following defining relation:

$$\Lambda_4 \equiv 3 \left[ \left( \frac{\sigma}{3} \right)^2 - \frac{1}{L^2} \right],$$

where $\sigma$ represents the tension of the brane (see [13] for a precise definition).

To touch base with the status of our own universe [1], we will assume that $\Lambda_4$ takes on a relatively small, positive value. From a brane-world standpoint, this translates into a non-critical, positively curved brane universe.\footnote{Traditionally, the Randall-Sundrum brane-world scenario [32] assumes a fine tuning of the brane tension so that $\Lambda_4$ vanishes; that is, a critical brane. However, there is no a priori rationale for this choice of brane tension, and we note some prior studies [59, 60, 39, 61, 62, 63, 64, 13] that have considered the dynamics of a non-critical brane.}

The above form (5) can, in fact, be identified with the 4-dimensional Friedmann equation for radiative matter (since $\rho_{rad} \sim r^{-4}$, where $\rho$ denotes energy density) along with an exotic matter contribution by virtue of the $r^{-6}$ term. This latter contribution can be identified with stiff matter (which is defined by $p_{sti} = \rho_{sti}$, where $p$ denotes pressure) [13]. What makes this holographic stiff matter all the more intriguing is its negative prefactor in
Eq. (5); thus implying that $\rho_{sti} < 0$. Although negative-energy matter is typically frowned upon, as it directly violates any number of positive-energy conditions [65], we note that such conditions have their foundation in technical convenience rather than fundamental arguments [66]. That is to say, it is still too early in the game to base the physical status of a theory on this criteria alone. Hence, we will keep an open mind (for the time being) and go on to consider the implications of including this exotic matter.

Although we have clearly established a brane-world pedigree for the relevant cosmological equation (5), one can bypass the brane description altogether and view this picture from a strictly 4-dimensional sense. That is to say, one may regard Eq. (5) as the Friedmann equation for a toy cosmological model; namely, a model universe that contains radiative matter, exotic stiff matter and a positive cosmological constant. From this perspective, one can view the current work as another chapter in “How not to construct an asymptotically de Sitter universe” [12].

3 FRW Cosmology on the Brane

Ideally, we would like to obtain an analytical solution to Eq. (5) for reasonably generic circumstances. Although this does not seem to be possible, it is still instructive to consider the asymptotic limits (i.e., very large or very small $r$) for which (approximate) analytic expressions are indeed obtainable. As it turns out, the asymptotic behavior of the scale factor is, by itself, sufficient for our current purposes.

First, let us consider the solutions for the special circumstance of vanishing charge (i.e., $Q^2$ or $\epsilon^2 = 0$). For this special case, we turn to a recent work [13] (also see [39]) and quote the relevant expressions for all three choices of $k$:

(i) $k = +1$:

$$r^2 = \frac{1}{2\mathcal{H}^2} \left( 1 + \sqrt{\mathcal{M}} - 1 \sinh [\pm 2\mathcal{H}(\tau - \tau_o)] \right).$$

(ii) $k = 0$:

$$r^2 = \frac{\sqrt{\mathcal{M}}}{2\mathcal{H}^2} \sinh [\pm 2\mathcal{H}\tau].$$

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(iii) $k = -1$:

$$r^2 = \frac{1}{2 \mathcal{H}^2} \left( \sqrt{\tilde{M} - 1} \sinh [\pm 2 \mathcal{H}(\tau - \tau_o)] - 1 \right). \quad (9)$$

Here, we have incorporated the following definitions: $\mathcal{H}^2 \equiv \Lambda_4 / 3$, $\tilde{M} \equiv 4 \omega M \mathcal{H}^2$, and $\tau_o$ is a constant of integration that can always be chosen so that $r^2$ vanishes at $\tau = 0$ (for sake of convenience). Keep in mind that we have assumed $\mathcal{H}^2$ to be positive and $\tilde{M} > 1$. (Solutions for other choices of $\tilde{M}$ are given in [13].)

Some discussion on the above solutions is in order. Each solution describes a FRW universe that either begins at a “big bang” or ends at a “big crunch”, depending on the choice of sign in the hyperbolic argument (positive or negative, respectively). These two scenarios are effectively equivalent, by way of time-reversal symmetry, so it is sufficient to illustrate just the big bang universe. In this case, the brane world “comes into existence” at $\tau = 0$, where $r = 0$ and the curvature can be shown to diverge. As time proceeds, the scale factor monotonically increases and ultimately describes an asymptotically dS universe ($r \sim e^{\mathcal{H} \tau}$) in the far future ($\tau \to \infty$). It is interesting to note that the asymptotic solution is essentially ignorant of the details of the bulk geometry.

What makes the above solutions particularly significant is their general validity for very large values of $r$ (or, equivalently, large values of $|\tau|$). In this regime, we can neglect the effects of charge, given that the charge term in Eq.(5) falls off rapidly ($\sim r^{-6}$) as the scale factor increases. Hence, as long as there is no singularity in the spacetime (which is always, as shown below, to be true for non-vanishing charge), the scale factor must asymptotically approach the above forms in either the distant past or far future.\footnote{Note that the sign in the hyperbolic functions should always be suitably chosen to ensure $r^2 > 0$.}

Next, let us consider the solutions for another special case; namely, $\Lambda_4 = 0$ or a critical brane. For these solutions, we turn to a recent analysis by Mukherji and Peloso [41]. These authors obtained the following results:

(i) $k = +1$:

$$r^2 = \frac{\omega M}{2} \left( 1 - \sqrt{1 - e^2 \cos [2\eta]} \right). \quad (10)$$
(ii) $k = 0$:
\[
r^2 = \frac{\omega M}{4} \left( \epsilon^2 + 4\eta^2 \right).
\]
(11)

(iii) $k = -1$:
\[
r^2 = \frac{\omega M}{2} \left( \sqrt{1 + \epsilon^2} \cosh(2\eta) - 1 \right).
\]
(12)

Note that the above have been expressed in terms of conformal time, $\eta$, which is defined according to $d\tau = r d\eta$. Also note that $\epsilon^2 > 0$ has been assumed.

A very interesting feature of these critical solutions is that they all have a non-vanishing minimum value for the scale factor (which occurs in the above forms, by way of arbitrary convention, precisely at $\eta = 0$). That is to say, as long as the charge is non-vanishing, a singularity will certainly be avoided. To put it another way, these critical solutions all “bounce” (at least) at the surface of vanishing time. In fact, the $k = +1$ solution bounces, by virtue of its periodicity, an infinite number of times.

As for the prior case of vanishing charge, the special nature of a critical brane belies the generality of the above solutions. On inspecting the Friedmann equation (5), we see that, for small enough values of the scale factor, the $\Lambda_4$ term will make a negligible contribution. More to the point, $\Lambda_4$ is a finite-valued constant, whereas all the other terms in Eq.(5) increase rapidly as the scale factor decreases. Hence, any relevant solution must asymptotically approach the above forms (11-12) as $\eta$ goes to zero. Therefore, given a bulk black hole with a non-vanishing charge, there will always be a non-vanishing minimum for the scale factor on the brane. In fact, this singularity-free status holds up for arbitrarily small values of $\epsilon^2 \sim Q^2$.

In spite of the conspicuous absence of a generic analytical solution, we are now in a position to say something about the brane cosmologies of interest (i.e., those with $\Lambda_4 > 0$, $1 > \epsilon^2 > 0$, and $\tilde{M} > 1$). It is clear that, for at least the cases of a flat ($k = 0$) and open ($k = -1$) brane universe, the scale factor will have an asymptotically dS form in the distant past, contract to a non-vanishing minimum at $\tau = 0$, and then expand towards an asymptotically dS future. Such a cosmological scenario is often said to describe a FRW bounce universe. The closed-universe ($k = +1$) case is, however, not so clear. It is not evident whether the scale factor, after its initial bounce, will oscillate between a maximal and minimal value (as suggested by Eq.(11)), or manage to “escape” towards an asymptotically dS future (as implied by Eq.(12)). The ultimate outcome depends, in all likelihood, on the relative values of
the model-dependent parameters; in particular, $M$ and $\Lambda_4$. One can imagine that, for a sufficiently small value of $M$, the vacuum energy will dominate and an exponential expansion will ensue. Conversely, if $M$ is above some threshold value, one might expect that the radiative matter wins out and the brane universe eternally oscillates. It would be interesting to verify the existence (or lack thereof) of such a threshold value; however, a numerical analysis is almost certainly required.

It is quite interesting that even an arbitrarily small amount of negative-energy matter can have such a dramatic effect on the underlying cosmology. When the bulk charge is precisely vanishing (i.e., no exotic matter on the brane), a bounce cosmology will never be possible; assuming that $\tilde{M} > 1$ has also been enforced. That is, under these conditions, a singular surface (either a big bang or big crunch) is an inevitable feature of the spacetime manifold. Conversely, after introducing arbitrarily small amounts of negative energy onto the brane, we find that the universe is regular throughout and (in most cases) has both an asymptotically dS past and future.

On some intuitive level, the above outcome is not particularly surprising. Regardless of how small (a non-vanishing) $\epsilon^2$ is chosen to be, it is clear from the Friedmann equation (5) that the negative-energy term will dominate at sufficiently small values of the scale factor (thanks to the $r^{-6}$ factor in this stiff-matter term). Hence, as the brane universe contracts towards $r = 0$, the negative-energy matter must, at some point, take over and create a significant repulsive force; thus preventing the otherwise inevitable collapse of the spacetime.

In spite of the ambiguous status of the various positive-energy conditions, one might be troubled by a universe that is dominated by negative-energy matter at some time in its evolution. Moreover, the resilient nature of these bounce cosmologies is most puzzling, insofar as any asymptotically dS spacetime should have an upper bound on its observable entropy ([27] and see Section 1). That is to say, there is clearly no upper limit on the size of $M$ and, hence, presumably no bound on the amount of entropy that can be “pumped” onto the brane. Therefore, at least naively, one would expect that the dS entropy bound can easily be violated.

Fortunately, these unsettling matters can still be resolved without bring-

\footnote{In fact, for vanishing charge, a closed brane universe can only bounce if $\tilde{M} < 1$ [3] and an open/flat universe will only bounce if $\tilde{M} = 0$ [13].}
ing the positive-energy conditions or entropic upper bound directly into play. For this purpose, we next call on our old friend, holography.

4 Holographic Interpretation

Essential to the following analysis is the concept of holographically induced renormalization group (RG) flows. Let us, therefore, give a brief account of this phenomenon.

Firstly, we consider RG flows as they apply to the well-known anti-dS/CFT correspondence [3, 4, 5]. In this anti-dS context, it has been established that the monotonic evolution of a relevant bulk parameter induces a “flow” in the renormalization scale of the dual boundary theory, and vice versa. (Keep in mind that this renormalization scale determines the ultraviolet cutoff or, equivalently, the lattice spacing of the boundary theory.) This framework follows, in large part, from the so-called ultraviolet/infrared correspondence [67]. That is, high (low) energies in the boundary theory translate into large (small) distances in the anti-dS bulk.

Significant to any RG flow is the existence of a generalized $c$-function. Such a function follows, by way of analogy, with the $c$-function of a 2-dimensional CFT [68]. For anti-dS holography in particular, any such $c$-function is expected to exhibit various monotonicity properties that are reflective of the ultraviolet/infrared duality. For further pertinent discussion, see (for instance) [69, 70, 71, 72].

Next, we will extend the above discussion to a dS holographic framework. It is first worth recalling that, in analogy to anti-dS holography, a dS spacetime has been conjectured to have a dually related boundary theory [10]. Such a boundary theory, if it does exist, would presumably be a Euclidean CFT that lives at one or both of past and future infinity. On the other hand, the dS/CFT correspondence is not without its detractors (most notably, [22]). However, in spite of the unclear status of this duality, there is little doubt that many aspects of anti-dS holography do indeed translate over to a dS setting; including RG flows.

Let us now (finally!) focus on RG flows in a dS context. As observed by Strominger [52] (also see [53]), time evolution in a purely dS spacetime

\footnote{For the pathway to other literature on this subject matter, see Section 1.}

\footnote{And for the inevitable counter-argument, see [22].}
will generate conformal-symmetry transformations on its asymptotic space-like boundaries. However, this conformal symmetry will be broken if the spacetime is “demoted” to being only asymptotically dS. On the basis of these observations, the author went on to argue that time evolution (in an asymptotically dS universe) will naturally induce a RG flow between a pair of conformal fixed points. These fixed points occur in the asymptotic past and future when the symmetries of pure dS space ultimately re-emerge. Moreover, in analogy to anti-dS holography, Strominger proposed an associated $c$-function of the form:

$$c \sim \left( \frac{\dot{r}(\tau)}{r(\tau)} \right)^{(n-1)},$$

(13)

where $r$ is the FRW scale factor, a dot denotes differentiation with respect to cosmological time ($\tau$), and $n + 1$ is the spacetime dimensionality.

Significantly to its status as a legitimate $c$-function, the above form does indeed exhibit appropriate monotonicity properties [52, 53]. In particular, $c$ flows to the ultraviolet (infrared) - that is, increases (decreases) - for an expanding (contracting) universe.

One issue of concern is that Eq.(13) can only be applied to a FRW space-time with flat spatial slices (i.e., $k = 0$). Nonetheless, the above form of $c$ has since been generalized so that all choices of spatial slicing can indeed be utilized [24, 50]. This generalization can be expressed as follows:

$$c \sim \left( \frac{r^2(\tau)}{k + \dot{r}^2(\tau)} \right)^{(n-1/2)},$$

(14)

where $k$ is the usual FRW topological parameter. It is easy to see that this form reduces to the prior one when $k = 0$.

To reclarify, $c$ is expected to increase if the universe is expanding and decrease if the universe is contracting. An interesting technical issue is the application of this “$c$-theorem” to a bounce universe, as such a cosmology obviously has both an expanding and a contracting phase. In this sense, one is forced to abandon the conventional wisdom that a $c$-function should evolve in a strictly monotonic fashion. (For further discussion on this caveat, see [24].)

\[12\] In particular, Leblond et al [24] proposed the generalized form and verified the appropriate monotonic behavior, while Kristjansson and Thorlacius [50] provided a geometrical interpretation by identifying $c$ with the area of an apparent horizon.
By this time, the reader may be wondering as to the point of this minor dissertation on holographic RG flows and the like. Well, as it so happens, RG flows have recently been utilized (specifically by McInnes [55]; also see [12]) to expose certain types of bounce cosmologies as being physically unacceptable. Interestingly, these charlatan cosmologies also contain negative-energy matter. However, unlike the current study, no other form of matter was included (besides a cosmological constant). In spite of this fundamental difference, we will proceed to show that the same debilitating argument applies here as well.

For sake of calculational simplicity, let us consider the case of an open brane universe or flat-horizon black hole (i.e., $k = 0$). We will restrict considerations to the expanding phase ($\tau$ or $\eta > 0$), as time-reversal symmetry enables one to choose either expansion or contraction without loss of generality. First of all, let us focus on the large $r$ (or late time) solution, as presented in Eq.(8). Substituting this result into the above form of the $c$-function (14), we find that:

$$c \sim \frac{\sinh^2[2\mathcal{H}(\tau - \tau_o)]}{\cosh^2[2\mathcal{H}(\tau - \tau_o)]},$$

(15)

where we have set $n = 3$ and neglected any irrelevant constant factors.

Our primary objective is to ascertain how $c$ responds to temporal variations. Given the generic positivity of the $c$-function, it is sufficient (and considerably simpler) to consider variations in $\ln[c]$. Again neglecting constant (positive) factors, we have:

$$\frac{\partial \ln[c]}{\partial \tau} = \frac{1}{\sinh[4\mathcal{H}(\tau - \tau_o)]} > 0.$$

(16)

Evidently, this expression satisfies the expectations of the $c$-theorem; namely, $c$ should monotonically increase with time during an expanding phase of the universe.

Next, let us consider the small $r$ (or small $|\tau|$) solution, as documented in Eq.(11). Before proceeding, we should first re-express the $c$-function (14) in a form that is directly compatible with the conformal-time variable, $\eta$. That is:

$$c \sim \left( \frac{\eta^4(\eta)}{k r^2(\eta) + [r'(\eta)]^2} \right)^{(n-1/2)},$$

(17)

where a prime indicates differentiation with respect to conformal time.
Substituting Eq.(11) into the above, we obtain:

\[ c \sim \frac{1}{\eta^2} \left( \epsilon^2 + 4\eta^2 \right)^3. \tag{18} \]

Again varying the logarithm (and recalling that \( \eta > 0 \) has, without loss of generality, been assumed), we find that:

\[ \frac{\partial \ln[c]}{\partial \eta} = \frac{12\eta}{\epsilon^2 + 4\eta^2} - \frac{1}{\eta}. \tag{19} \]

It is difficult to make sense of this result, until we remember that this asymptotic solution (11) is only strictly valid for vanishingly small values of \( \eta \). Hence, it is most appropriate to consider the limit of \( \eta \to 0_+ \), in which case:

\[ \frac{\partial \ln[c]}{\partial \eta} \to -\frac{1}{0_+} < 0. \tag{20} \]

This time around, we observe a direct violation of the relevant \( c \)-theorem. That is, for small values of \( r \), \( c \) monotonically decreases with time during an expanding phase of the universe. This outcome implies that time flows in the “wrong” direction when the brane universe is small; assuming that Strominger’s interpretation - time evolution is dual with a RG flow to the ultraviolet [52] - should be taken literally.

Before elaborating on this curious behavior, we point out that the above qualitative features will naturally persist for both the closed \( (k = +1) \) and open \( (k = -1) \) brane-universe scenarios. This realization follows from an inspection of Eq.(5), from which it is clear that the topological \( (k) \) term will not play a significant role in either asymptotic regime.

Generally speaking, a violation in the \( c \)-theorem, although definitely a “red-flag”, should not necessarily be regarded as an unphysical outcome. In support of this claim, we point out that dS holography is not particularly well understood (not in comparison to its anti-dS analogue), and it remains uncertain as to how literally its predictions should be interpreted. Moreover, the validity of the \( c \)-theorem depends on the assumption of the “weak energy condition” [53, 52, 53, 24], which will certainly be violated whenever negative-energy matter is dominant. (For our brane model, this domination occurs when the universe is small.) With this caveat in mind, one could well have anticipated the observed failure of the \( c \)-theorem at small values of \( r \). To put
it another way, the preferred direction of time flow is not particularly clear until the second law of thermodynamics can somehow be invoked.

In spite of this attempt at spin-doctoring, there remains a pressing concern that cannot, quite so easily, be argued away: not only does time apparently flow in the wrong direction, but it actually reverses direction at some point in the evolution of the universe. That is, the $c$-function flows to the infrared when the universe is small but ends up flowing to the ultraviolet at some later time. Such a “phase transition” in the $c$-function does not appear to be compatible with any interpretation of holographic RG flows; taken literally or otherwise. That is to say, we can perhaps tolerate a RG flow that reverses direction at a bounce, but not one that reverses direction while the universe is in the process of expanding (or contracting).

In view of the preceding discussion, we are forced to conclude that these bounce cosmologies with exotic stiff matter are physically unacceptable on holographic grounds. Moreover, from a brane-world perspective, this failure implies that a bulk black hole should be prohibited from acquiring an electrostatic charge.

5 Conclusion

In summary, we have been studying the cosmological and holographic implications of a certain class of brane-world scenarios. To elaborate, we have focused on a non-critical (positively curved) 3-brane moving in an anti-dS background that is described by a Reissner-Nordstrom-like black hole. It should be noted that our analysis incorporated the topological variants of the “standard” (anti-dS) Reissner-Nordstrom solution; thus allowing for spherical, flat and hyperbolic horizon geometries.

We began the analysis by introducing the relevant bulk solutions and the equation of motion for the brane. It is significant that the form of the latter essentially mimics the Friedmann equation for radiative matter. Along with the radiative contribution (which is proportional to the mass of the bulk black hole) and an effective cosmological constant, this induced Friedmann equation contains an additional exotic form of matter. This exotic contribu-

\[\text{13}^{\text{This behavior seems oddly reminiscent of the Hawking-Page anti-dS phase transition }}\]

\[\text{\textsuperscript{74, 8}}. \text{ It may be of interest to investigate this possible analogy; however, such a line is impeded by our lack of a solution for the non-asymptotic universe.}\]
tion, which is holographically induced by the bulk electrostatic charge, can be identified as stiff matter with a negative energy density. Also of interest, the horizon geometry of the black hole fixes the spatial topology of the brane universe.

In the next section of the analysis, we specifically considered bulk black holes with a relatively large mass and a relatively small (but non-vanishing) charge. The interest in this regime follows from prior studies on uncharged black holes in the bulk \[39, 13\]. These earlier treatments have indicated that a cosmological singularity is inevitable when the black hole mass exceeds a certain (topology-dependent) value. That is, under these conditions, the brane universe must either begin with a big bang or end in a big crunch.

Although the induced Friedmann equation is not generally solvable, we were still able to proceed by focusing on a pair of asymptotic regimes. In fact, the solutions for very large and very small values of the scale factor are known (\[13\] and \[41\], respectively), and these could be used to deduce the gross features of the relevant cosmologies (up to some ambiguity in the case of a closed brane universe). In precisely this manner, we have demonstrated that even an arbitrarily small charge (in the bulk) is sufficient to avoid a singularity in the brane universe. Rather than collapsing to a singularity (as is inevitable in the uncharged scenario), the brane universe either bounces towards an asymptotically dS future (as is always the case for a flat or open universe) or eternally oscillates between a maximal and minimal size.

In the final phase of our study, we investigated the charge-induced bounce cosmologies from a holographic perspective. More to the point, we calculated the generalized c-functions (as prescribed for dS-holographic RG flows \[52, 53, 24, 50\]) in the asymptotic regimes and then tested these functions against the recommended c-theorem. Notably, this methodology was inspired by prior treatments \[55, 12\] that revealed a violation of the c-theorem for a special class of bounce cosmologies. Significantly, these cosmologies also contain negative-energy matter.

Ultimately, in applying this litmus test, we have demonstrated that the charge-induced bounce cosmologies are of an unphysical nature. More precisely, when the universe is very small (and the negative-energy matter dominates), the associated c-function decreases as the universe expands; an outcome which is in direct violation of the c-theorem. Although one could still argue on how literally the c-theorem should be interpreted (given that the direction of time is perhaps ambiguous in a universe with built-in time-reversal
symmetry), we pointed out an even more disturbing consequence of this calculation. Namely, the RG flow must reverse its direction, at some point, while the universe is expanding. (This follows from the behavior of the $c$-function in the other asymptotic regime, where it clearly increases as the universe expands.) Such a reversal is decidedly in conflict with the philosophical premise of holographic RG flows.

In view of the above arguments, we conclude that the charge-induced bounce universes are, indeed, unphysical cosmologies. Moreover, assuming the feasibility of brane-world scenarios in general, we suggest the necessity for censoring against charge in a bulk black hole.

Of course, the last declaration may be somewhat premature, inasmuch as we still have no direct knowledge of the non-asymptotic evolution. It would be interesting to identify the “reversal point” and better understand whatever mechanism is deviously at work. However, it would appear that such an investigation would require a numerical analysis. As an alternative to this unsettling proposition, one might consider the scenario of a 2-brane moving in a 4-dimensional charged black hole bulk. Although unphysical, this simpler model may have an analytical solution with similar features to that of the priorly studied case.

One might also extend the prior treatment by considering other FRW cosmologies that contain negative-energy matter (as well as a positive cosmological constant). Clearly, a pattern is arising: the presence of such exotic matter consistently leads to violations in the holographic $c$-theorem. On the other hand, the models so-far studied have a common feature; namely, the negative-energy matter dominates when the universe is very small. It may be of interest to construct an analytically solvable model in which this is not the case. Given the ambiguous status of the various positive-energy conditions [66], it would certainly be useful to ascertain their feasibility via such independent means. Naturally, we defer the above prospects to a future (cosmological or conformal) time.

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