AN EXAMINATION OF THE MULTI-PEAKED ANALYTICALLY EXTENDED FUNCTION FOR APPROXIMATION OF LIGHTNING CHANNEL-BASE CURRENTS

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Abstract: A multi-peaked version of the analytically extended function (AEF) intended for approximation of multi-peaked lightning current waveforms will be presented along with some of its basic properties. A general framework for estimating the parameters of the AEF using the Marquardt least-squares method (MLSM) for a waveform with an arbitrary (finite) number of peaks as well as a given charge transfer and specific energy will also be described. This framework is used to find parameters for some common single-peak waveforms and some advantages and disadvantages of the approach will be discussed.

Keywords: Analytically extended function, Electromagnetic compatibility, Electrostatic discharge current, Lightning discharge, Marquardt least-squares method.

INTRODUCTION

Many different types of systems, objects and equipment are susceptible to damage from lightning discharge. Lightning effects are usually analysed using lightning discharge models. Most of these models imply channel-base current functions. Various single and multi-peaked functions have been proposed in the literature, e.g. [1]-[5]. For engineering and electromagnetic models, a general function that would be able to reproduce desired waveshapes is needed, such that analytical solutions for its derivatives, integrals, and integral transformations exist. A multi-peaked channel-base current function has been proposed in [4] as a generalization of the so-called TRF (two-rise front) function from [5], which possesses such properties.

In this paper we analyse a proposed multi-peaked function, the so-called $p$-peak analytically extended function (AEF), giving explicit expressions for a number of basic properties such as derivatives and integrals.

Some discussion on how to fit a $p$-peak AEF will be done and a general framework for how parameters can be estimated using the Marquardt least-squares method is presented. Some numerical results are presented, including those for the Standard IEC62305 currents of the first negative stroke ([10]), and an example of a fast-decaying lightning current waveform. Based on presented results, corresponding conclusions are made, and further research ideas are discussed.

Fig.1 – Illustration of the power exponential function given by (1) for a few different values of $\beta$.

THE MULTI-PEAKED AEF

The $p$-peaked AEF is constructed using the function

$$x(\beta; t) = \left( e^{1-t} \right)^{\beta}, 0 \leq t,$$

which we will refer to as the power exponential function. The power exponential function is qualitatively similar to the desired waveforms in the sense that it has a steeply rising initial part followed by a more slowly decaying part. The steepness of both the rising and decaying part is determined by the $\beta$-parameter. This is illustrated in Fig. 1.

In order to get a function with multiple peaks and where the steepness of the rise between each peak as well as the slope of the decaying part is not dependent on each other, we define the analytically extended function (AEF) as a function that consist of piecewise linear combinations of the power exponential function that have been scaled and translated so that the resulting function is continuous. Given the difference in height between each pair of peaks $I_{m_1}, I_{m_2}, \ldots, I_{m_p}$, the corresponding times $t_{m_1}, \ldots, t_{m_p}$, integers $n_q > 0$, real values $\beta_{q,k}, n_{q,k}, 1 \leq q \leq p+1, 1 \leq k \leq n_q$ such that the sum over $k$ of $n_{q,k}$ is equal to unit, the $p$-peaked AEF $i(t)$ is given by Eqn. (2).

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are diagonal matrices, the integral of the rising part of the AEF is given by Eqn. (6) and

\[
\int_{t_a}^{t_b} i(t) = (t_{m_a} - t_a) \left( \sum_{k=1}^{a-1} m_k \sum_{k'=1}^n \eta_{a,k} \delta_{a} t_a t_{m_a} \right) + \sum_{q=1}^{b-1} \left( \Delta t_{m_q} \sum_{k=1}^n \eta_{q,k} \beta \beta \beta + 1 \right) + (t_{m_a} - t_a) \sum_{k=1}^{a-1} m_k \sum_{k'=1}^n \eta_{q,k} \beta \beta \beta + 1 b_b + 1 \right) + (7)
\]

Introducing the vectors
\[
\eta_q = [\eta_{q,1} \eta_{q,2} \cdots \eta_{q,n}]^T,
\]
the \( p \)-peaked AEF can be written in a more compact form
\[
i(t) = \left\{ \begin{array}{l}
\sum_{k=1}^{a-1} m_k + m_q \eta_q^T x_q(t), \\
\sum_{k=1}^p m_k \eta_{p+1} x_{p+1}(t), \\
\end{array} \right.
\]

where \( \gamma(\beta,t) = \int_0^t t(t-1)e^{-\gamma}dt \) is the incomplete gamma function.

Using Eqn. (6) and Eqn. (2) together with some standard results for integration gives that for \( 0 \leq t_a \leq t_b \leq t_{m_p} \), \( t_{m_{a-1}} \leq t_a \leq t_{m_a} \), \( t_{m_{b-1}} \leq t_b \leq t_{m_b} \), the integral of the rising part of the AEF is

\[
\int_{t_a}^{t_b} i(t) = (t_{m_a} - t_a) \left( \sum_{k=1}^{a-1} m_k \sum_{k'=1}^n \eta_{a,k} \delta_{a} t_a t_{m_a} \right) + \sum_{q=1}^{b-1} \left( \Delta t_{m_q} \sum_{k=1}^n \eta_{q,k} \beta \beta \beta + 1 \right) + (t_{m_a} - t_a) \sum_{k=1}^{a-1} m_k \sum_{k'=1}^n \eta_{q,k} \beta \beta \beta + 1 b_b + 1 \right) + (7)
\]

Similarly for the decaying part, \( t_{m_p} \leq t_0 \leq t_1 < \infty \), the integration formula is

\[
\int_{t_0}^{t_1} i(t) dt = \left[ \sum_{k=1}^p m_k \sum_{k'=1}^n \eta_{p+1,k} \delta_{p+1,1} \delta_{p+1,2} \cdots \delta_{p+1,n_{p+1}} \right] + (8)
\]

### FITTING THE AEF TO DATA USING THE MARQUARDT LEAST SQUARE METHOD

Here we will attempt to fit the AEF to some different current waveshapes. We will use the Marquardt least square method (MLSM) to estimate the \( \beta \)-parameters, and from these calculate the corresponding \( \eta \)-parameters. We will not give a detailed description of MLM, instead we point to [9] for a description of how to apply it in a similar situation. Here we will only supply the parts of the method specific to the use of the AEF.

The MLM uses a Jacobian matrix that contains the partial derivatives of the residuals, here we denote this matrix with \( J \). Suppose that we want to find the least square fit of the AEF to a set of data points. Then the fitting can be done separately between each peak (and after the final peak). The \( J \) matrix in this case is

\[
J = \left[ \begin{array}{cccc}
p_{q,1}(t_{q,1}) & p_{q,2}(t_{q,1}) & \cdots & p_{q,n_{q}}(t_{q,1}) \\
p_{q,1}(t_{q,2}) & p_{q,2}(t_{q,2}) & \cdots & p_{q,n_{q}}(t_{q,2}) \\
\vdots & \vdots & \ddots & \vdots \\
p_{q,1}(t_{q,k_q}) & p_{q,2}(t_{q,k_q}) & \cdots & p_{q,n_{q}}(t_{q,k_q}) \\
\end{array} \right],
\]

where \( k_q \) is the number of data points between the \( q \)-th and \((q-1)\)-th peak, \( t_{q,r} \) are the corresponding times for these.
data points, and \( p_{q,r}(t_{q,s}) = \frac{\partial I}{\partial \beta_{q,r}} \bigg|_{t=t_{q,r}} \) more explicitly
given by

\[
p_{q,r}(t_{q,s}) = 2I_{m_q} n_{q,k} \beta_{q,k} h_q(t) x(\beta_{q,k}^2 + 1), 1 \leq q \leq p,
\]

\[
p_{p+1,r}(t_{q,s}) = I_{m_{p+1}} n_{d+1,k} \beta_{d+1,k} h_q(t) x(\beta_{p+1,k}^2 + 1).
\]

where

\[
h_q(t) = \ln \left( \frac{t - t_{m_q}}{\Delta t_{m_q}} \right) + \frac{t - t_{m_q}}{\Delta t_{m_q}} + 1, 1 \leq q \leq p,
\]

\[
h_{p+1}(t) = \ln \left( \frac{t}{t_{m_{p+1}}} \right) + \frac{t}{t_{m_{p+1}}} + 1.
\]

The MLSM is an iterative method, and in order to find a new set of \( \beta \)-parameters in each iteration we also need to
find the \( \eta \)-parameters. This is done by using the regular
least square method since for fixed \( \beta \) the AEF is linear in \( \eta \).

Often we also wish to take the charge transfer, \( Q_0 \), and
specific energy, \( W_0 \), of the lightning discharge into account.
These two quantities are given by

\[
Q_0 = \int_0^\infty i(t) \, dt,
\]

\[
W_0 = \int_0^\infty i(t)^2 \, dt.
\]

Using Eqns. (7) and (8) the corresponding quantities for
the AEF are found to be

\[
Q = \sum_{q=1}^p \Delta t_{m_q} \left( \sum_{k=1}^{q-1} I_{m_k} + I_{m_q} \sum_{k=1}^q n_{q,k} \tilde{\gamma}(\beta_{q,k}^2 + 1) \right) + \left( \sum_{k=1}^p I_{m_k} \right) \sum_{k=1}^{p+1} n_{p+1,k} \tilde{\gamma}(\beta_{p+1,k}^2 + 1),
\]

\[
W = \sum_{q=1}^p W_q + \left( \sum_{k=1}^p I_{m_k} \right) W_{p+1},
\]

where

\[
\tilde{\gamma}(\beta) = \frac{e^{-\beta - 1}}{\beta^2} (1 - \gamma(1 + \beta, 1, \beta)),
\]

\[
W_q = \Delta t_{m_q} \left( \sum_{k=1}^{q-1} I_{m_k} \right)^2 + I_{m_q}^2 \sum_{k=1}^q n_{q,k} \tilde{\gamma}(\beta_{q,k}^2 + 1) + \left( \sum_{k=1}^p I_{m_k} \right) \sum_{k=1}^{p+1} n_{p+1,k} \tilde{\gamma}(\beta_{p+1,k}^2 + 1) + 2I_{m_q} \Delta t_{m_q} \left( \sum_{k=1}^{q-1} I_{m_k} \right) \sum_{k=1}^q n_{q,k} \tilde{\gamma}(\beta_{q,k}^2 + 1) + 2I_{m_q} \Delta t_{m_q} \left( \sum_{k=1}^{p+1} n_{p+1,k} \tilde{\gamma}(\beta_{p+1,k}^2 + 1) + 2. \right.
\]

\[
W_{p+1} = \sum_{k=1}^{p+1} n_{p+1,k} \tilde{\gamma}(\beta_{p+1,k}^2 + 1).
\]

Since the charge transfer and specific energy place condi-
tions on the entire function it is no longer possible to fit
each interval separately. Instead the entire function is fitted
at once using the \( J \) matrix

\[
J = \begin{bmatrix}
J_1 & 0 & \cdots & 0 \\
J_2 & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\hat{W}_1 & \hat{W}_2 & \cdots & \hat{W}_{p+1}
\end{bmatrix},
\]

where \( J_q, 1 \leq q \leq p + 1 \), is the matrix for the corresponding
interval and \( \hat{W}_q \), \( 1 \leq q \leq p + 1 \), are matrices with
the partial derivatives of the \( Q \) and \( W \) with respect to the \( \beta \)-
parameters.

The expressions for the partial derivatives are somewhat
complicated and have been omitted to save space. Full
expressions can be found in [12].

Note that the residuals for the charge transfer and specif-
ic energy can be on a very different scale from the residuals
of the current values. Therefore some sort of weighted least
square fitting might be more appropriate in many situations.

RESULTS AND DISCUSSION

Here we will show some results of the fitting for 1-peak
and 2-peak AEF functions to data either given by Heidler
functions (Fig. 2, Fig. 3 and Fig. 4) or experimental data
(Fig. 5).

In Fig. 2 and Fig. 3 monotonicity of the functions have
been guaranteed by forcing all \( \eta \)-parameters to be positive.
This was not necessary for Fig. 4 and Fig. 5.

![Fast-decaying waveshape, \( T_1/T_2=8/20 \)](image)

Fig. 2 – Illustration of the 1-peak AEF fitted to four data-points
generated by the Heidler function, note the good fit.
In Fig. 2 the fitting for fast-decaying waveform is shown, and in Fig. 3 the fitting for rising part of the first negative stroke. The reason that the fit in Fig. 3 is so poor is due to the fact that increasing the steepness of the AEF also moves the transition forward in time. Thus we would need either to move the time for the peak or complement the AEF with some function that can have a steep rise in the middle of the interval.

In Fig. 4 the quality of the fit varies between the interval. In the first interval the fit is not very good but in the second interval the AEF approximates the waveform well. We can even use the AEF to approximate the peak in the middle of the interval without need to specify its position. Since the data points were chosen randomly for this fitting and the results varied considerably depending on chosen points some strategy for choosing points should be devised.

In Fig. 5 the AEF fits the data well for the most part and here further experimentation with the number of terms in each interval could be fruitful.

It should be noted that in all cases MLSM showed some tendencies to find local minima in the objective function instead of the global and therefore other methods of fitting should be explored, such as genetic algorithms or more specialised methods, for instance methods used for \( L \)-splines that the AEF is conceptually similar to.

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