THE MEAN CURVATURE MEASURE

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ABSTRACT. We assign a measure to an upper semicontinuous function which is subharmonic with respect to the mean curvature operator, so that it agrees with the mean curvature of its graph when the function is smooth. We prove that the measure is weakly continuous with respect to almost everywhere convergence. We also establish a sharp Harnack inequality for the minimal surface equation, which is crucial for our proof of the weak continuity. As an application we prove the existence of weak solutions to the corresponding Dirichlet problem when the inhomogeneous term is a measure.

1. Introduction

Notions of curvature measures arise in convex geometry, (see for example [S]), and were extended to general surfaces by Federer [F1] under a hypothesis of positive reach. For graphs of functions, this condition is equivalent to semi-convexity and implies twice almost everywhere differentiability by virtue of the well-know theorem of Aleksandrov. The development of a corresponding theory of curvature measures on more general sets is an open problem. Without any assumption such a theory seems impossible as the second derivative of a nonsmooth function is usually a distribution but not a measure. In this paper we consider the mean curvature and restrict ourselves to graphs of functions defined over domains Ω in Euclidean n-space, \( \mathbb{R}^n \). The mean curvature has been the most extensively studied geometric quantity but usually it is regarded as a distribution when the function is not twice differentiable, such as in the case when its graph is a rectifiable set.

In particular in this paper we assign a measure to an upper semicontinuous function which is subharmonic with respect to the mean curvature operator, so that it agrees with the mean curvature of its graph when the function is smooth. We prove that the measure is weakly continuous with respect to almost everywhere convergence (Theorem 6.1). We also establish a sharp Harnack inequality for the minimal surface equation (Theorem 2.1), which is crucial for our proof of weak continuity. As an application we prove the existence of weak solutions to the Dirichlet problem of the mean curvature equation when the right hand side is a measure (Theorem 7.1).

This work was supported by ARC grants DP0664517 and DP0879422; and NSFC grants 10428103 and 10671064.
We say an upper semi-continuous function \( u : \Omega \to [-\infty, +\infty) \) is subharmonic with respect to the mean curvature operator \( H_1 \), or \( H_1 \)-subharmonic in short, if the set \( \{ u = -\infty \} \) has measure zero and \( H_1[u] \geq 0 \) in the viscosity sense. That is for any open set \( \omega \subset \Omega \) and any smooth function \( h \in C^2(\omega) \) with \( H_1[h] \leq 0 \), \( h \geq u \) on \( \partial \omega \), one has \( h \geq u \) in \( \omega \). We say a function \( u \) is \( H_1 \)-harmonic if it is \( H_1 \)-subharmonic and for any open set \( \omega \subset \Omega \) and any \( H_1 \)-subharmonic function \( h \) in \( \omega \) with \( h \leq u \) on \( \partial \omega \), one has \( h \leq u \) in \( \omega \). This definition does not imply directly that an \( H_1 \)-harmonic function is bounded from below, but we will prove in Section 4 it is the case, and so is smooth. We denote the set of all \( H_1 \)-subharmonic functions in \( \Omega \) by \( \text{SH}_1(\Omega) \).

A main result of the paper is the weak continuity of the mean curvature operator. That is if \( \{ u_k \} \) is a sequence of smooth \( H_1 \)-subharmonic functions which converges a.e. to \( u \in \text{SH}_1(\Omega) \), then \( H_1[u_k] \) converges weakly to the density of a measure \( \mu \). The measure \( \mu \) depends only on \( u \) but not on the sequence \( \{ u_k \} \), so that we can assign a measure, called the mean curvature measure and denoted by \( \mu_1[u] \), to the function \( u \). Note that our measure \( \mu_1 \) is defined on \( \Omega \) but Federer’s measure \( \nu_1 \) is defined on the graph of \( u \).

As an application, we study the existence of solutions to the Dirichlet problem of the mean curvature equation

\[
H_1[u] = \text{div}\left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = 0. \tag{1.1}
\]

Namely

\[
\sup_{B_r} u \leq C \inf_{B_r} u \tag{1.2}
\]

for nonnegative solution of (1.1) in \( B_{2r} \). The Harnack inequality for the mean curvature equation has been studied in several works [FL, Lia, PS1, T1]. We prove that the constant \( C \) depends on the decay rate of \( \{|x \in B_{2r} : u(x) > t|\}_{n-1} \), or \( \{|x \in \partial B_{2r} : u(x) > t|\}_{n-1} \), as \( t \to \infty \), where \( | \cdot |_k \) denotes the \( k \)-dimensional Hausdorff measure. This is indeed the best possibility one can expect. A similar Harnack inequality also holds for the non-homogeneous equation, see Remark 2.4.

As an application, we study the existence of solutions to the Dirichlet problem of the mean curvature equation

\[
H_1[u] = \nu \quad \text{in} \quad \Omega, \tag{1.3}
\]

\[
u = \phi \quad \text{on} \quad \partial \Omega,
\]

where \( \nu \) is the density of a nonnegative measure, with respect to Lebesgue measure.

For the Dirichlet problem of the mean curvature equation, it is usually assumed that the right hand side \( \nu \) is a Lipschitz function, so that the interior gradient estimate holds and the solution is smooth, in \( C^{2,\alpha}(\Omega) \) for \( \alpha \in (0, 1) \) [GT]. If \( \nu \) is not Lipschitz continuous, the solution may not be \( C^2 \) smooth even if \( \nu \) is Hölder continuous; (see
the example in §8). In [Gia, G2] it was proved that when \( \nu \) is a measurable function satisfying a necessary condition, equation (1.3) has a weak solution which is a minimizer of an associated functional. Through the mean curvature measure introduced above, we introduce a notion of weak solution and prove its existence when \( \nu \) is a nonnegative measure.

This paper is arranged as follows. In Section 2 we establish the Harnack inequality (1.2) for the minimal surface equation. In Section 3 we establish an integral gradient estimate and a uniform estimate for \( H_1 \)-subharmonic functions. In these two sections we assume that the functions are smooth. But the assumption can be removed by an approximation result proved in Section 5.

In Section 4 we introduce the Perron lifting and prove some basic properties for \( H_1 \)-subharmonic functions. In Section 5 we prove that every \( H_1 \)-subharmonic function can be approximated by a sequence of smooth, \( H_1 \)-subharmonic functions. Section 6 is devoted to the proof of the weak continuity of the mean curvature operator. The Dirichlet problem is discussed in Section 7. The final Section 8 contains some remarks.

In recent years it was proved that for several important homogeneous elliptic operators, such as the \( p \)-Laplace operator and the \( k \)-Hessian operator, one can assign a measure to a function which is subharmonic with respect to the operators, and as applications various potential theoretical results have been established. See [HKM, Lab, TW1-TW4]. Our treatment of the weak continuity of the mean curvature operator was inspired by the earlier works [TW1-TW4]. However as the mean curvature operator is non-homogeneous, the situation is much more delicate.

### 2. The Harnack inequality

In this section we prove a Harnack inequality for the minimal surface equation, which will be used for the Perron liftings process in Section 4 and the study of the Dirichlet problem in Section 7. We also establish a weak Harnack inequality for \( H_1 \)-subharmonic functions, which will be used in the proof of Lemma 4.3.

First we quote the basic existence and regularity result for the mean curvature equation [GT]. The regularity of the mean curvature equation is based on the interior gradient estimate (see Theorem 16.5 in [GT]).

**Lemma 2.1.** Let \( u \leq 0 \) be a \( C^3 \) solution to the mean curvature equation

\[
H_1[u] = f(x) \quad \text{in} \ B_r(0).
\]  

Then

\[
|Du(0)| \leq C_1 e^{C_2 \|u(0)\|},
\]

where \( C_1, C_2 \) depend only on \( n \) and \( \|f\|_{C^{0,1}} \).
Simpler proofs of the interior gradient estimate, with $|u(0)|_r$ replaced by $|u(0)|_{r^2}$, was given in [K1, Wan]. The proofs also applies to the $k$-th mean curvature equation and more general Weingarten curvature equations [K2, Wan].

From the gradient estimate, the mean curvature equation becomes uniformly elliptic and one has local uniform estimate in $C^{2,\alpha}$ for the equation, for any $\alpha \in (0, 1)$.

By the regularity, one has the existence of solutions to the Dirichlet problem (see Theorem 16.8 in [GT]).

**Lemma 2.2.** Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^n$. Suppose the mean curvature of $\partial \Omega$ is positive. Then for any continuous function $\varphi$ on $\partial \Omega$, there is a unique solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to $H_1[u] = 0$ such that $u = \varphi$ on $\partial \Omega$.

Lemma 2.2 also holds for the inhomogeneous equation $H_1[u] = f$ with $f \in C^{0,1}$, under certain conditions on $f$ and $\partial \Omega$, see Theorem 16.10 in [GT].

In this section we prove the following Harnack inequality. Here we consider smooth solutions only. In Section 4 we will show that an $H_1$-harmonic function must be smooth.

**Theorem 2.1.** Let $u \geq 0$ be a smooth solution to the minimal surface equation

$$H_1[u] = 0 \quad \text{in } B_r(0). \quad (2.3)$$

Let

$$\psi(t) = |\{x \in \partial B_r(0): u(x) > t\}|_{n-1},$$

where $| \cdot |_{n-1}$ denotes the $(n-1)$-dim Hausdorff measure. Suppose $\psi(t) \to 0$ as $t \to \infty$. Then there exists a constant $C > 0$ depending only on $n$, $r$, and $\psi$ such that

$$\sup_{B_{r/2}(0)} u \leq C \inf_{B_{r/2}(0)} u. \quad (2.4)$$

**Remark 2.1**

(i) The Harnack inequality (2.4) was also established in [T1], but the constant $C$ depends on $\sup u$. The main point in [T1] is a positive lower bound of $u(0)$ for the mean curvature equation and more general elliptic equations satisfying certain structural conditions. The paper [T1] also includes the following weak Harnack inequality for the upper bound for $u(0)$: if $u \in W^{2,n}(B_r(0))$ is a subsolution, then for any $p \in (0, n],$

$$\sup_{B_{r/2}} u \leq \frac{C}{r^{(n+2)/p}} \left( \int_{B_r} (u^+)^{p+2} \right)^{1/p}, \quad (2.5)$$

where $C$ is a constant depending only on $n$ and $p$. We also refer the reader to [FL, Lia, PS1] for discussions of the Harnack inequality.
(ii) Recall that in the Harnack inequality for the Laplace equation, the constant $C$ depends only on $n$. But this is impossible for the minimal surface equation. One can construct a positive solution of (2.3) in $B_1(0)$ such that $u(0) \leq 1$ but $\int_{B_1} u^p$ can be as large as we want, for any $p > 0$. To see this, let $\varphi(x_1)$ be a positive, convex function defined for $x_1 \in (-1, 1)$ such that $\varphi(x_1)$ is small when $x_1 < \frac{1}{4}$ and $\varphi(x_1) \to \infty$ as $x_1 \to 1$. Let $u$ be the solution of (2.3) with the Dirichlet condition $u = \varphi$ on $\partial B_1$. Then by the convexity of $\varphi$, $H_1[\varphi] \geq 0$. Hence by the comparison principle, we have $u \geq \varphi$ in $B_1$.

Hence $\int_{B_1} u^p$ can be as large as we want provided $\varphi$ is sufficiently large near $x_1 = 1$. On the other hand, by constructing a suitable upper barrier one has $u(0) \leq 1$.

To prove Theorem 2.1, we start with some technical lemmas.

Let $\Omega$ be an open set contained in $B_r(0)$. For $s \in (0, r]$, denote

$$
\Gamma^\text{int}_s = \bar{\Omega} \cap \partial B_s(0),
\Gamma^\text{bdy}_s = \partial \Omega \cap B_s(0),
$$

so that

$$
\Gamma^\text{bdy}_s \cup \Gamma^\text{int}_s = \partial (\Omega \cap B_s(0)).
$$

Let $\tilde{\Gamma}^\text{int}_s$ be a geodesic ball in $\partial B_s$, with center at $(s, 0, \cdots, 0)$, such that $|\tilde{\Gamma}^\text{int}_s|_{n-1} = |\Gamma^\text{int}_s|_{n-1}$, where $| \cdot |_k$ denotes the $k$-dimensional Hausdorff measure. Denote by $\rho(s)$ the geodesic radius of $\tilde{\Gamma}^\text{int}_s$. Then

$$
(1 - \varepsilon) \alpha_{n-1} \rho^{n-1}(s) \leq |\Gamma^\text{int}_s|_{n-1} \leq \alpha_{n-1} \rho^{n-1}(s)
$$

with $\varepsilon \to 0$ as $\rho(s) \to 0$, where $\alpha_{n-1}$ is the volume of the unit ball in $\mathbb{R}^{n-1}$. The second inequality is due to the positive curvature of the sphere, and the first one can be obtained easily by representing $\tilde{\Gamma}^\text{int}_s$ as a graph.

Let $\rho_1$ be the constat such that $|\Gamma^\text{int}_s|_{n-1} \geq \frac{1}{3} \alpha_{n-1} \rho^{n-1}(s)$ for any $\rho(s) \leq \rho_1$. We also denote

$$
b_n = 2^{-4n} c_{n-1} \alpha_{n-1}^{-1/(n-1)},
$$

where $c_{n-1}$ is the best constant in the isoperimetric inequality, see (2.9) below.

**Lemma 2.3.** Let $\Omega$ be an open set in $B_r(0)$ for some $\frac{1}{4} \leq r \leq 1$. Suppose $\Gamma^\text{bdy}_r$ is smooth, $\rho(r) \leq \rho_1$, and

$$
\rho(s) \geq \frac{1}{4} \rho(r) \quad \forall \ s \in (r', r),
$$

where $r' = r - \rho(r)/2b_n$. Then

$$
|\Gamma^\text{bdy}_r|_{n-1} \geq 2|\Gamma^\text{int}_r|_{n-1}. \quad (2.7)
$$
Proof. We claim that
\[ |\Gamma_{r}^{\text{bdy}}|_{n-1} \geq \int_{r'}^{r} |\partial \Gamma_{s}^{\text{int}}|_{n-2} ds. \] (2.8)

In the following we will drop the subscripts \( k \) in the Hausdorff measure \(|\cdot|_{k} (k = 1, \cdots, n)\) if no confusion arises.

Formula (2.8) can be derived as follows. For any point \( x_{0} \in \partial \Gamma_{r}^{\text{int}} \), by a rotation of axes we assume that \( x_{0} = (r, 0, \cdots, 0) \) such that \((0, \cdots, 0, 1)\) is the normal of \( \partial \Gamma_{r}^{\text{int}} \) at \( x_{0} \). Then near \( x_{0} \), \( \Gamma_{r}^{\text{bdy}} \) can be represented as
\[ x_{n} = \psi(x') \] such that \( \partial_{x_{i}} \psi(x_{0}) = 0 \) for \( i = 2, \cdots, n-1 \), where \( x' = (x_{1}, \cdots, x_{n-1}) \). Hence at \( x_{0} \) the area element is
\[
d\sigma = \sqrt{1 + |D\psi|^{2}} dx' = \sqrt{1 + \psi_{x_{1}}^{2}} d\sigma' \geq d\sigma'
\]
Hence
\[
|\Gamma_{r}^{\text{bdy}}| = \int_{\Gamma_{r}^{\text{bdy}}} d\sigma \geq \int_{\Gamma_{r}^{\text{bdy}}} dx' = \int_{0}^{r} |\partial \Gamma_{s}^{\text{int}}| ds
\]
and we obtain (2.8).

By the isoperimetric inequality,
\[
\int_{0}^{r} |\partial \Gamma_{s}^{\text{int}}| ds \geq c_{n-1} \int_{0}^{r} |\Gamma_{s}^{\text{int}}|_{n-2}^{\frac{n-2}{n-1}} ds \geq c_{n-1} \int_{r'}^{r} |\Gamma_{s}^{\text{int}}|_{n-2}^{\frac{n-2}{n-1}} ds.
\] (2.9)

Since \( \rho(s) \geq \frac{1}{4} \rho(r) \) for any \( s \in (r', r) \),
\[
|\Gamma_{s}^{\text{int}}| \geq 4^{-n} |\Gamma_{r}^{\text{int}}| \forall s \in (r', r).
\]
We obtain
\[
\int_{0}^{r} |\partial \Gamma_{s}^{\text{int}}| ds \geq \frac{c_{n-1}}{4^{n}} (r - r') |\Gamma_{r}^{\text{int}}|_{\frac{n-2}{n-1}}.
\]
Therefore
\[
|\Gamma_{r}^{\text{bdy}}| \geq \frac{c_{n-1}}{4^{n}} \frac{r - r'}{\Gamma_{r}^{\text{int}} |\Gamma_{r}^{\text{int}}|_{1/(n-1)}} \Gamma_{r}^{\text{int}} \geq \frac{c_{n-1} \alpha_{n-1}}{4^{n+1} b_{n}} |\Gamma_{r}^{\text{int}}|.
\]
The Lemma holds by our choice of \( b_{n} \). □
Lemma 2.4. Let $\Omega$ be an open set in $B_r(0)$ for some $\frac{1}{4} \leq r \leq 1$. Suppose $\rho(r) \leq \rho_1$ and $\Gamma_r^{\text{bdy}}$ is smooth. Then

$$|\Gamma_r^{\text{bdy}}| \geq (1 - 4^{-n+1} - \varepsilon)|\Gamma_r^{\text{int}}|,$$

(2.10)

with $\varepsilon \to 0$ as $\rho(r) \to 0$. In particular there exists a constant $\rho_2 > 0$ such that when $\rho(r) \leq \rho_2$,

$$|\Gamma_r^{\text{bdy}}| \geq (1 - 4^{-n+5/4})|\Gamma_r^{\text{int}}|$$

(2.10)'

Proof. If $\rho(s) \geq \frac{1}{4}\rho(r)$ for all $s \in (r', r)$, where $r' = r - \rho(r)/2b_n$, then (2.10) follow from (2.7).

Hence we may assume that $\rho(s) < \frac{1}{4}\rho(r)$ for some $s \in (r', r)$. Let

$$G' = \{x \in \partial B_r(0) : \exists t \in (\frac{s}{r}, 1) \text{ such that } tx \in \Gamma_r^{\text{bdy}} - \Gamma_s^{\text{bdy}}\},$$

(2.11)

$$G'' = \{x \in \partial B_r(0) : \frac{s}{r}x \in \Gamma_s^{\text{int}}\}$$

be respectively the radial projection of $\Gamma_r^{\text{bdy}} - \Gamma_s^{\text{bdy}}$ and $\Gamma_s^{\text{int}}$ on $\partial B_r(0)$. Then $\Gamma_s^{\text{int}} \subset G' \cup G''$. But since $\rho(s) < \frac{1}{4}\rho(r)$, we have

$$|G''| = (\frac{r}{s})^{n-1}|\Gamma_s^{\text{int}}| \leq 4^{-n+1}(\frac{r}{r'})^{n-1}|\Gamma_r^{\text{int}}|$$

Hence we obtain

$$|G'| \geq (1 - 4^{-n+1}(\frac{r}{r'})^{n-1})|\Gamma_r^{\text{int}}|. \quad (2.12)$$

Regard $\Gamma_r^{\text{bdy}} - \Gamma_s^{\text{bdy}}$ as a (multi-valued) radial graph over $G'$. For any point $y \in \Gamma_r^{\text{bdy}} - \Gamma_s^{\text{bdy}}$, let $x$ be the projection of $y$ on $\partial B_r$. Then through the projection, the area element of $\Gamma_r^{\text{bdy}}$ at $y$ is greater than $(\frac{r}{r'})^{n-1}$ times the area element of $\partial B_r$ at $x$. Hence we have

$$|\Gamma_r^{\text{bdy}}| \geq (\frac{r}{r'})^{n-1}|G'| \geq (1 - 4^{-n+1}(\frac{r}{r'})^{n-1})|\Gamma_r^{\text{int}}|.$$ 

Note that $r' = r - \rho(r)/2b_n \to r$ as $\rho(r) \to 0$. We obtain (2.10). \(\Box\)

Remark 2.2. The above proof implies that if the volume $|\Omega|$ is small, then we have

$$|\Gamma_r^{\text{bdy}}| \geq \frac{1}{2}|\Gamma_r^{\text{int}}|,$$

(2.13)

where $r \in [\frac{1}{4}, 1]$. Note that in (2.13) we do not assume that $\rho(r)$ is small. Indeed, if $\rho(r) < \min(\rho_1, \rho_2)$ is small, (2.13) is proved in Lemma 2.4. Otherwise, let $s = r - |\Omega|^{1/n}$. 7
Define $G'$ as in (2.11) and let $G'' = \Gamma^\text{int}_r - G'$. We have $|\Omega| \geq (r - s)(\frac{s}{r})^{n-1}|G''|$. Hence $|G''| \leq 2|\Omega|^{1-1/n}$ and so $|G'| \geq |\Gamma^\text{int}_r| - 2|\Omega|^{1-1/n}$. The proof of Lemma 2.4 then implies that

$$|\Gamma^\text{bdy}_r| \geq \left(\frac{s}{r}\right)^{n-1}|G'| \geq \frac{3}{4}(|\Gamma^\text{int}_r| - 2|\Omega|^{1-1/n}).$$

Hence (2.13) follows if $\rho(r) \geq \min(\rho_1, \rho_2)$ and $|\Omega| < \frac{1}{16}\alpha_{n-1}[\min(\rho_1, \rho_2)]^n$.

**Lemma 2.5.** Let $u$ be a smooth $H_1$-subharmonic function in $B_1(0)$. Suppose $\sup_{B_{1/2}(0)} u > 1$. Then

$$|\{x \in B_1(0) : u(x) > 0\}| \geq C,$$

where the constant $C > 0$ depends only on $n$.

**Proof.** We prove by contradiction, assuming that $|\Omega_{1,0}| < \delta_0^{4n}$ for some small positive constant $\delta_0$ depending on $\rho_1$ and $\rho_2$. We divide the proof into three steps.

**Step 1.** For $r \in (0, 1]$ and $t \geq 0$, denote

$$\Omega_{r,t} = \{x \in B_r(0) : u(x) > t\},$$

$$\Gamma^\text{bdy}_{r,t} = \partial \Omega_{r,t} \cap B_r,$$

$$\Gamma^\text{int}_{r,t} = \Omega_{r,t} \cap \partial B_r,$$

so that $\partial \Omega_{r,t} = \Gamma^\text{bdy}_{r,t} \cup \Gamma^\text{int}_{r,t}$. By Sard’s lemma, $\Gamma^\text{bdy}_{r,t}$ is smooth for almost all $t$. Note that

$$\delta_0^{4n} \geq |\Omega_{1,0}| \geq \int_{7/8}^{1} |\Gamma^\text{int}_{r,0}|dr.$$  

Hence there exists $r \in [7/8, 1]$ such that $|\Gamma^\text{int}_{r,0}| < 8\delta_0^{4n} < \delta_0^{4(n-1)}$. Without loss of generality we may also assume that $|\Gamma^\text{int}_{1,0}| < \delta_0^{4(n-1)}$. Note that $\Omega_{r',t'} \subset \Omega_{r,t}$ for any $r' < r, t' > t$. Hence for all $r \in (0, 1]$ and $t \geq 0$,

$$|\Omega_{r,t}| < \delta_0^{4n} \text{ and } |\Gamma^\text{int}_{1,t}| < \delta_0^{4(n-1)}.$$  

To apply the previous Lemmas, we assume that $\delta_0 < \frac{1}{32}\min\{\rho_1, \rho_2\}$.

Consider the integration

$$0 \leq \int_{\Omega_{r,t}} H_1[u] = -\left(\int_{\Gamma^\text{bdy}_{r,t}} + \int_{\Gamma^\text{int}_{r,t}}\right) \frac{\gamma \cdot Du}{\sqrt{1 + |Du|^2}},$$

where $\gamma$ is the unit inner normal of $\Omega_{r,t}$. We have

$$\left|\int_{\Gamma^\text{int}_{r,t}} \frac{\gamma \cdot Du}{\sqrt{1 + |Du|^2}}\right| \leq |\Gamma^\text{int}_{r,t}|,$$

$$\int_{\Gamma^\text{bdy}_{r,t}} \frac{\gamma \cdot Du}{\sqrt{1 + |Du|^2}} = 8 \int_{\Gamma^\text{bdy}_{r,t}} \frac{|Du|}{\sqrt{1 + |Du|^2}}.$$
Suppose there exist \( r \) and \( t \) such that
\[
|\Gamma_{r,t}^{\text{bdy}}| \geq (1 + \delta)|\Gamma_{r,t}^{\text{int}}| \tag{2.19}
\]
for some small constant \( \delta > 0 \) (we can fix \( \delta = 4^{-n} \)), and there exists a subset \( \hat{\Gamma}_{r,t}^{\text{bdy}} \subset \Gamma_{r,t}^{\text{bdy}} \) such that
\[
|Du| > 2\delta^{-1/2} \quad \text{on} \quad \hat{\Gamma}_{r,t}^{\text{bdy}},
\]
\[
|\hat{\Gamma}_{r,t}^{\text{bdy}}| > (1 - \delta/4)|\Gamma_{r,t}^{\text{bdy}}|. \tag{2.20}
\]
Then
\[
\int_{\hat{\Gamma}_{r,t}^{\text{bdy}}} \frac{|Du|}{\sqrt{1 + |Du|^2}} \geq \int_{\hat{\Gamma}_{r,t}^{\text{bdy}}} \frac{2\delta^{-1/2}}{\sqrt{1 + 4\delta^{-1}}}
\]
\[
\geq \frac{1 - \delta/4}{\sqrt{1 + \delta/4}}|\Gamma_{r,t}^{\text{bdy}}|
\]
\[
> |\Gamma_{r,t}^{\text{int}}|.
\]
We reach a contradiction.

In the following we prove there exists \( r, t \) such that (2.19) and (2.20) hold (so we reach a contradiction and Lemma 2.5 is proved). Accordingly we introduce the sets
\[
P = \{(r, t) \in [\frac{1}{2}, 1] \times [0, 1] : |\Gamma_{r,t}^{\text{bdy}}| \leq (1 + \delta)|\Gamma_{r,t}^{\text{int}}|\},
\]
\[
Q = \{(r, t) \in [\frac{1}{2}, 1] \times [0, 1] : |\Gamma_{r,t}^{*}\} \geq \frac{\delta}{4}|\Gamma_{r,t}^{\text{bdy}}|\},
\]
where
\[
\Gamma_{r,t}^{*} = \{x \in \Gamma_{r,t}^{\text{bdy}} : |Du|(x) \leq 2\delta^{-1/2}\}.
\]
If there exists \( (r, t) \in [\frac{1}{2}, 1] \times [0, 1] \) such that \( (r, t) \notin P \cup Q \), then (2.19) and (2.20) hold and the lemma is proved. In the following we show that both sets \( P \) and \( Q \) have small Lebesgue measure.

**Remark 2.3.** We remark that (2.19) may not hold if the shape of \( \Omega_{r,t} \) is like a thumbtack, namely a flat cap with a thin cylinder.

**Step 2.** Estimate of \(|Q|\). For any fixed \( r \in [\frac{1}{2}, 1] \), denote
\[
Q_r = \{t \in [0, 1] : (r, t) \in Q\}
\]
a slice of \( Q \) at \( r \), and denote \( \varphi(t) = |\Omega_{r,t}| \). By the co-area formula, we have, for a.e. \( t \),
\[
\varphi'(t) = -\int_{\Omega_{r,t} \cap \{u(t) = t\}} \frac{1}{|Du|} = -\int_{\hat{\Gamma}_{r,t}^{\text{bdy}}} \frac{1}{|Du|}, \tag{2.21}
\]
as \( u \) is smooth and \( r \) is fixed. Hence for any \( t \in Q_r \),

\[
\varphi'(t) \leq - \int_{\Gamma_{r,t}^*} \frac{1}{|Du|} \leq - \frac{1}{2\delta^{-1/2}} |\Gamma_{r,t}^*| \\
\leq - \frac{1}{8} \delta^{3/2} |\Gamma_{r,t}^{\text{bdy}}|.
\]

By the isoperimetric inequality,

\[
|\partial \Omega_{r,t}| \geq c_n |\Omega_{r,t}|^{1-\frac{1}{n}}, \tag{2.22}
\]

where the best constant \( c_n \) is attained when the domain is a ball. Similar to (2.16),

\[
\delta_0^{3n} \geq |\Omega_{1,t}| \geq \int_0^1 |\Gamma_{r,t}^{\text{int}}| \, dr.
\]

Hence the Lebesgue measure of the set \( \hat{I} := \{ t \in [0, 1] : |\Gamma_{r,t}^{\text{int}}| > \delta_0^{4(n-1)} \} \) is less than \( \delta_0^4 \). For any \( t \notin \hat{I} \), by Lemma 2.4,

\[
|\Gamma_{r,t}^{\text{int}}| \leq 2|\Gamma_{r,t}^{\text{bdy}}|.
\]

Hence from (2.22) and noting that \( \partial \Omega_{r,t} = \Gamma_{r,t}^{\text{bdy}} \cup \Gamma_{r,t}^{\text{int}} \),

\[
|\Gamma_{r,t}^{\text{bdy}}| \geq \frac{c_n}{3} |\Omega_{r,t}|^{1-\frac{1}{n}}.
\]

We obtain

\[
\varphi'(t) \leq - \frac{c_n}{24} \delta^{3/2} \varphi^{1-\frac{1}{n}}(t). \tag{2.23}
\]

Namely \((\varphi^{\frac{1}{n}})'(t) \leq - \frac{c_n}{24n} \delta^{3/2} \) when \( t \in \hat{Q}_r := Q_r - \hat{I} \). Note that \((\varphi^{\frac{1}{n}})'(t) \leq 0 \) for any \( t \in [0, 1] - \hat{Q}_r \). Hence

\[
\varphi^{\frac{1}{n}}(0) - \varphi^{\frac{1}{n}}(1) = - \int_{\hat{Q}_r} (\varphi^{\frac{1}{n}})'(t) - \int_{[0,1]-\hat{Q}_r} (\varphi^{\frac{1}{n}})'(t)
\geq \frac{c_n}{24n} \delta^{3/2} |\hat{Q}_r|.
\]

We get the estimate

\[
|\hat{Q}_r| \leq \frac{24n}{c_n} \delta^{-3/2} \varphi^{\frac{1}{n}}(0).
\]

By assumption, \( \varphi(0) = |\Omega_{1,0}| \leq \delta_0^{2n} \). Hence when \( \delta_0 \) is small (recall that \( \delta = 4^{-n} \)), we obtain \( |\hat{Q}_r| < \frac{1}{2} \delta_0^2 \). Hence \( |Q_r| \leq |\hat{Q}_r| + |\hat{I}| \leq \delta_0^2 \). It follows that

\[
|Q| = \int_{1/2}^1 |Q_r| \leq \frac{1}{2} \delta_0^2.
\]
That is, $Q$ is a small set.

**Step 3.** Estimate of $|P|$. For any fixed $t \in [0,1]$, denote $P_t = \{r \in [\frac{1}{2}, 1] : (r, t) \in P\}$ a slice of $P$ at height $t$. We prove that $P_t$ has small Lebesgue measure, so that $|P| = \int_0^1 |P_t|$ is also small.

Denote by $\rho(r)$ the geodesic radius of $\Gamma_{r,t}^{\text{int}}$, as introduced before Lemma 2.3. Namely, we define $\rho(r)$ such that a geodesic ball of radius $\rho(r)$ in $\partial B_r$ has the volume $|\Gamma_{r,t}^{\text{int}}|$.

We first consider the case when $\rho$ is increasing in $r$. In this case, by (2.17) we have

$$\rho(r) < \rho(1) < \delta_0^4$$

for any $r < 1$. Let $\bar{r}_1 = \sup r : r \in [\frac{1}{2}, 1]$ and there exists $\bar{r} < r$ such that

$$\rho(\bar{r}) = \frac{1}{2} \rho(r),$$

$$\frac{\rho(\bar{r}) - \rho(r)}{r - \bar{r}} \geq b_n.$$  \hspace{1cm} (2.24)

We obtain an interval $I_1 = [\bar{r}_1, \bar{r}]$, where $\bar{r}_1$ is the largest $\bar{r}$ satisfying (2.24). Next let $\bar{r}_2 = \sup r \in [0, \bar{r}_1]$ such that the above formulae hold, and we obtain an interval $I_2 = [\bar{r}_2, \bar{r}_1]$. Continue the process we obtain a sequence of intervals $\{I_k\}$, $I_k = [\bar{r}_k, \bar{r}_k]$. By the monotonicity of $\rho$, we have

$$\sum_k |I_k| \leq \rho(1)/b_n \leq \delta_0^4/b_n.$$  \hspace{1cm} (2.25)

For any $r \not\in \bigcup_k I_k$ and $r \in [\frac{1}{2}, 1]$, by our definition of $I_k$ we have

$$\rho(s) \geq \frac{1}{2} \rho(r) \quad \forall s \in (r', r),$$

where $r' = r - \rho(r)/2b_n$. By Lemma 2.3, $|\Gamma_{r,t}^{\text{bdy}}| \geq 2|\Gamma_{r,t}^{\text{int}}|$. Hence (2.19) holds and $r \not\in P_t$. It follows that $P_t \subset \bigcup_k I_k$. By (2.25), $|P_t|$ is small.

Next we consider the case $\rho$ is not monotone increasing. In this case, we may also assume that $\sup_{r \in [\frac{1}{4}, 1]} \rho(r)$ is small. For if there exists $r_0 \in [\frac{1}{4}, 1]$ such that $\rho(r_0) \geq \delta_0^3$, we choose $r_0 = \inf\{r \in [\frac{1}{4}, 1] : \rho(r) \geq \delta_0^3\}$. By the argument below, the set $P'_t = \{r \in [\frac{1}{2}, r_0] : (r, t) \in P\}$ is a small set. If $r_0 > 1 - \delta_0^2$, then $P_t \subset P'_t \cup [r_0, 1]$ is also small. If $r_0 < 1 - \delta_0^2$, recall that $\delta_0^4 \geq |\Omega_{1,t}| \geq \int_0^1 |\Gamma_{r,t}^{\text{int}}| dr$. Hence the Lebesgue measure of the set $I' := \{r \in [r_0, 1] : |\Gamma_{r,t}^{\text{int}}| > \delta_0^{4(n-1)}\}$ is less than $\delta_0^4$. For any $r \in [r_0, 1] - I'$, we have $|\Gamma_{r,t}^{\text{int}}| \leq \delta_0^{4(n-1)}$ but by Remark 2.2,

$$|\Gamma_{r,t}^{\text{bdy}}| \geq |\Gamma_{r_0,t}^{\text{bdy}}| \geq \frac{1}{2} |\Gamma_{r_0,t}^{\text{int}}| \geq \frac{1}{2} \alpha_n - 1 \delta_0^{3(n-1)}$$

Hence $|\Gamma_{r,t}^{\text{bdy}}| \geq 2|\Gamma_{r,t}^{\text{int}}|$ and so $r \not\in P_t$. Again $|P_t| \leq |P'_t| + |I'|$ is small. In the following we assume directly that $\sup_{r \in [1/4, 1]} \rho(r) < \delta_0^3$.\[11]
Let
\[ \hat{\rho}(r) = \sup \{ \rho(s) : \frac{1}{4} < s < r \}, \quad r \in [\frac{1}{4}, 1]. \] (2.26)

Then \( \hat{\rho} \) is increasing in \( [\frac{1}{4}, 1] \). Similarly we define the sequence of intervals \( I_k = [\mathcal{R}_k, \mathcal{T}_k] \) in terms of \( \hat{\rho} \). Then
\[ \sum_k |I_k| \leq \hat{\rho}(1)/b_n \leq \delta_0^3/b_n. \]

We claim that \( P_t \subset \bigcup_k I_k \). Indeed, for any \( r \not\in \bigcup_k I_k \) and \( r \in [\frac{1}{2}, 1] \), let \( r' = r - \rho(r)/2b_n \). If \( \rho(s) \geq \frac{1}{4} \rho(r) \quad \forall \quad s \in (r', r) \), the claim follows from Lemma 2.3. If there exists an \( s \in (r', r) \) such that \( \rho(s) < \frac{1}{4} \rho(r) \), note that \( r' \geq r - \hat{\rho}(r)/2b_n \), by our definition of \( I_k \),
\[ \hat{\rho}(s) \geq \frac{1}{2} \hat{\rho}(r) \geq \frac{1}{2} \rho(r) \quad \forall \quad s \in [r', r]. \]

Hence there exists \( \tau \in [\frac{1}{4}, s] \) such that \( \rho(\tau) \geq \frac{1}{2} \rho(r) \). We divide \( \Gamma_{r,t}^{\text{bdy}} \) into three pieces, \( \Gamma_{r,t}^{\text{bdy}} = \Gamma_a \cup \Gamma_b \cup \Gamma_c \), where
\begin{align*}
\Gamma_a &= \Gamma_{r,t}^{\text{bdy}} \cap \{ s < |x| < r \}, \\
\Gamma_b &= \Gamma_{r,t}^{\text{bdy}} \cap \{ \tau < |x| < s \}, \\
\Gamma_c &= \Gamma_{r,t}^{\text{bdy}} \cap \{ 0 < |x| < \tau \}.
\end{align*}

By Lemma 2.4, we have
\begin{align*}
|\Gamma_a| &\geq (1 - 4^{-n+\frac{5}{4}})|\Gamma_{r,t}^{\text{int}}|, \\
|\Gamma_c| &\geq (1 - 4^{-n+\frac{5}{4}})|\Gamma_{\tau,t}^{\text{int}}|.
\end{align*}

By projecting \( \Gamma_b \) to \( \partial B_{\tau} \) and noticing that \( \rho(s) \leq \frac{1}{2} \rho(\tau) \), we have, similarly to the proof of Lemma 2.4,
\[ |\Gamma_b| \geq (1 - 2^{-n+\frac{5}{4}})|\Gamma_{\tau,t}^{\text{int}}|. \]

Recall that \( \sup_{r \in [\frac{1}{4}, 1]} \rho(r) \leq \delta_0^3 \), we have \( |\Gamma_{r,t}^{\text{int}}| \approx \alpha_{n+1}\rho^{n-1}(\tau) \) and \( |\Gamma_{r,t}^{\text{int}}| \approx \alpha_{n+1}\rho^{n-1}(r) \). Hence by \( \rho(\tau) \geq \frac{1}{2} \rho(r) \) we have \( |\Gamma_{r,t}^{\text{int}}| \geq (1 - \varepsilon)2^{-n+1}|\Gamma_{r,t}^{\text{int}}| \), where \( \varepsilon \to 0 \) as \( \delta_0 \to 0 \). Assume \( \delta_0 \) small such that \( \varepsilon < 4^{-n} \). Then we obtain
\begin{align*}
|\Gamma_{r,t}^{\text{bdy}}| &= |\Gamma_a| + |\Gamma_b| + |\Gamma_c| \\
&\geq (1 + 4^{-n+1})|\Gamma_{r,t}^{\text{int}}|. \tag{2.27}
\end{align*}

The claim \( P_t \subset \bigcup_k I_k \) is proved and hence \( P_t \) is a small set. \( \square \)

From Lemma 2.5, we have the following weak Harnack inequality, which is an improvement of (2.5).
Corollary 2.1. Let \( u \) be an \( H_1 \)-subharmonic function in \( B_r(0) \). Then for any constant \( p > 0 \), there exists a constant \( C \) depending on \( n \) and \( p \) such that

\[
\sup_{B_{r/2}} u \leq \frac{C}{r^{n/p}} \left( \int_{B_r} (u^+)^p \right)^{1/p},
\]

(2.28)

where \( u^+ = \max(u, 0) \).

**Proof.** It suffices to prove that

\[
u(0) \leq \frac{C}{r^{n/p}} \left( \int_{B_r} (u^+)^p \right)^{1/p}.
\]

(2.29)

We will prove it for smooth \( H_1 \)-subharmonic functions. In the general case it follows from the approximation in §5.

If \( \sup_{B_{r/2}} u \leq r \), then (2.29) follows from (2.5). In the following we assume that \( \sup_{B_{r/2}} u \geq r \). By the transformation \( u \to u/r \) and \( x \to x/r \), we may assume that \( r = 1 \).

If \( u(0) \geq 1 \), applying Lemma 2.5 to the function \( (u - \frac{1}{2}u(0))^+ \), we see that \( |\{x \in B_1(0) : u(x) > \frac{1}{2}u(0)\}| \geq C \). Hence we obtain (2.29).

If \( u(0) \leq 1 \), assume \( \sup_{B_{1/2}(0)} u \) is attained at \( x_0 \). Then \( \sup_{B_{1/2}(x_0)} u \geq 1 \). Applying Lemma 2.5 to \( (u - \frac{1}{2}u(x_0))^+ \) in \( B_{1/2}(x_0) \), we also obtain (2.29). \( \square \)

**Proof of Theorem 2.1.** Let \( u \) be a nonnegative solution to the minimal surface equation (2.3) in \( B_r(0) \). It suffices to show that \( \sup_{B_{r/2}(0)} u \) is bounded from above by a constant \( C \) depending only on \( n, r \) and \( \psi \). Once \( u \) is bounded from above, by the interior gradient estimate, equation (2.3) becomes uniformly elliptic and the full Harnack inequality follows [GT]. Alternatively we may also use the estimates for \( \inf_{B_{1/2}(0)} u \) in [T1] or [PS2].

By a scaling we may assume that \( r = 1 \). Denote \( \Omega_t = \{x \in B_1(0) : u(x) > t\} \) and \( \Gamma_t^{\text{int}} = \Omega_t \cap \partial B_1(0) \) and \( \Gamma_t^{\text{bdy}} = \partial \Omega_t \cap B_1(0) \). If \( \sup_{B_{1/2}(0)} u \) is sufficiently large, by (2.14) we have \( |\Omega_t| \geq C \) for some \( C > 0 \) independent of \( t \). Hence by the assumption \( \lim_{t \to \infty} \psi(t) = 0 \), we have \( |\Gamma_t^{\text{bdy}}| > 2|\Gamma_t^{\text{int}}| \) for all large \( t \). Namely (2.19) (with \( r = 1, \delta = 1 \)) is satisfied for all large \( t \).

Let \( \varphi(t) = |\Omega_t| \) and denote \( Q = \{t \geq 0 : |\Gamma_t^{\text{int}}| \geq \frac{1}{4} |\Gamma_t^{\text{bdy}}| \} \), where \( \Gamma_t^{\text{int}} = \{x \in \Gamma_t^{\text{bdy}} : |Du|(x) \leq 2\} \). Then from the proof of Step 2 above, \( \varphi \) satisfies (2.23). Hence

\[
\varphi(t) - \varphi(0) \geq \frac{c_1}{24n} |Q|.
\]

(2.30)

Hence (2.20) (with \( r = 1, \delta = 1 \)) is satisfies for most large \( t \). Choosing a \( t \notin Q \), we reach a contradiction as in Step 1 of the proof of Lemma 2.5. \( \square \)

**Remark 2.4.** From the proof of Lemma 2.5 (see (2.18)), one sees that if for any \( \omega \subset \Omega \),

\[
\int_{\omega} H_1[u] \geq -\nu(\omega)
\]

(2.31)
for some nonnegative measure $\nu$ satisfying $\frac{\nu(\cdot)}{|\partial \omega|} \to 0$ as $|\omega| \to 0$, then estimate (2.14) holds, with the constant $C$ depending also on $\nu$. This estimate, combined with Theorem 3.1 in [T1], implies a Harnack inequality for solutions $u \in W^{2,n}(\Omega)$ to the non-homogeneous mean curvature equation.

3. Gradient and uniform estimates

First we establish an integral gradient estimate.

**Theorem 3.1.** Let $u \in C^2(\Omega)$ be a non-positive $H_1$-subharmonic function. Then for any open set $\omega \subset \Omega$,
\[
\int_\omega |Du_t| \leq C,
\]
where $u_t = \max(u, -t)$, $t$ is a constant, and $C > 0$ depends on $\omega, t$, but is independent of $u$.

**Proof.** Let $\varphi(x) \in C_0^\infty(\Omega)$ be a smooth function with support in $\Omega$ such that $0 \leq \varphi(x) \leq 1$ and $\varphi(x) \equiv 1$ on $\omega$. We may assume that $|\partial \Omega|$, the area of $\partial \Omega$, is bounded, otherwise we may restrict to a subdomain of $\Omega$ which contains $\omega$. Then
\[
\int_\Omega \varphi(-u_t)H_1[u] = \int_\Omega \frac{\varphi|Du_t|^2}{\sqrt{1 + |Du_t|^2}} + \int_\Omega \frac{u_tD_{u_t} \cdot D\varphi}{\sqrt{1 + |Du_t|^2}}
\]
\[
\geq \int_\omega \frac{|Du_t|^2}{\sqrt{1 + |Du_t|^2}} + \int_\Omega \frac{u_tD_{u_t} \cdot D\varphi}{\sqrt{1 + |Du_t|^2}}
\]
\[
\geq \int_\omega |Du_t| - |\omega| + \int_\Omega \frac{u_tD_{u_t} \cdot D\varphi}{\sqrt{1 + |Du_t|^2}}.
\]

Note that
\[
\int_\Omega \varphi(-u_t)H_1[u] \leq t \int_\Omega H_1[u] \leq t|\partial \Omega|
\]
and
\[
\int_\Omega \frac{u_tD_{u_t} \cdot D\varphi}{\sqrt{1 + |Du_t|^2}} \leq Ct|\Omega|.
\]

We obtain
\[
\int_\omega |Du_t| \leq C(1 + t)(|\Omega| + |\partial \Omega|).
\]

Hence (3.1) is proved. $\Box$

In the next section we will prove that every $H_1$-subharmonic function can be approximated by smooth ones. Note that if $u \in SH_1(\Omega)$, then $u_t \in SH_1(\Omega)$. Hence by Theorem 3.1 we have
Corollary 3.1. For any \( u \in SH_1(\Omega) \) bounded from above and any \( \Omega' \Subset \Omega \), \( u_t \in BV(\Omega') \).

In particular if \( u \) is bounded from below, then \( u \in BV(\Omega') \).

By the example in §8, \( u \not\in W^{1,1}(\Omega') \) in general.

Next we consider the \( \mathcal{L}^\infty \) estimate for \( H_1 \)-subharmonic functions. We say a set \( A \) is Caccioppoli if it is a Borel set with characteristic function \( \varphi_A \) whose distributional derivatives \( D\varphi_A \) are Radon measures [G3]. If \( A \) is Caccioppoli, we have

\[
|\partial A| = \int_{\mathbb{R}^n} |D\varphi_A|.
\]  

(3.2)

Theorem 3.2. Assume that \( u \in SH_1(\Omega) \cap C^2(\Omega) \) is bounded from below on \( \partial \Omega \). Assume that there is a positive constant \( \eta \) such that for any Caccioppoli set \( A \subset \Omega \),

\[
\int_A H_1[u] \leq (1 - \eta)|\partial A|.
\]  

(3.3)

Then there is a constant \( C > 0 \) such that

\[
\inf_{x \in \Omega} u \geq -C.
\]  

(3.4)

Proof. For any \( t > 0 \), denote \( \Omega_t = \{ x \in \Omega : u(x) \leq -t \} \) and \( \partial_1 \Omega_t = \{ x \in \partial \Omega : |Du| \leq t^{2/3} \} \). Since \( u \) is bounded from below on \( \partial \Omega \), we may choose a large \( T \) such that \( \Omega_T \Subset \Omega \) and

\[
\frac{T^{2/3}}{\sqrt{1 + T^{4/3}}} \geq 1 - \eta/2.
\]  

(3.5)

We claim that for any \( t > T \),

\[
|\partial_1 \Omega_t| \geq \frac{\eta}{2} |\partial \Omega_t|.
\]  

(3.6)

Indeed, if there exists a \( t \geq T \) such that \( |\partial_1 \Omega_t| < \frac{\eta}{2} |\partial \Omega_t| \), we have

\[
\int_{\Omega_t} H_1[u] = \int_{\partial \Omega_t} \frac{|Du|}{\sqrt{1 + |Du|^2}} \\
\geq \int_{\partial \Omega_t - \partial_1 \Omega_t} \frac{|Du|}{\sqrt{1 + |Du|^2}} \\
\geq (1 - \eta/2)(1 - \eta/2)|\partial \Omega_t| \\
> (1 - \eta)|\partial \Omega_t|,
\]

which is in contradiction with the assumption (3.3).

Let \( \varphi(t) = |\Omega_t| \). If \( t > -\inf_{\partial \Omega} u \), then \( \Omega_t \Subset \\subset \Omega \). Hence by the co-area formula,

\[
\varphi'(t) = -\int_{\partial \Omega_t} \frac{1}{|Du|} \leq -\int_{\partial_1 \Omega_t} \frac{1}{|Du|}.
\]

15
When \( t > T \),
\[
\varphi'(t) \leq -\frac{\eta}{2t^{2/3}}|\partial\Omega_t|.
\]
By the isoperimetric inequality,
\[
\varphi^{1-1/n}(t) \leq C|\partial\Omega_t|,
\]
we obtain
\[
\varphi'(t) \leq -\frac{Cn}{t^{2/3}}\varphi^{1-1/n}(t). \tag{3.7}
\]
Namely \([\varphi^{1/n}(t)]' \leq -C\eta t^{-2/3}\). Taking integration from \( T \) to \( t \), we obtain
\[
\varphi^{1/n}(t) \leq \varphi^{1/n}(T) + C\eta(T^{1/3} - t^{1/3}) \tag{3.8}
\]
for a different \( C \). Hence \( \varphi \) vanishes when \( t > C\left[T + (\frac{|\Omega|^{1/n}}{\eta})^{3/2}\right] \). This completes the proof. \( \square \)

**Remark 3.1.** Condition (3.3) was introduced in [Gia], in which it is proved that (1.3) is necessary and sufficient for the existence of a minimizer of an associated functional. From the proof of Theorem 3.2 one sees that the condition (3.3) can be weakened to
\[
|G(\bar{t})| \to \infty \quad \text{as} \quad \bar{t} \to \infty, \tag{3.9}
\]
where \( G(\bar{t}) \) is the set of \( t \in (0, \bar{t}) \) such that
\[
\int_{\Omega_t} H_1[u] \leq (1 - \eta)|\partial\Omega_t|. \tag{3.10}
\]
This is because (3.6) and (3.7) hold for any \( t \in G(\bar{t}) \). Furthermore, as the co-area formula holds for BV functions [G3], the above argument applies to BV functions.

**Remark 3.2.** From the proof, the constant \( C \) in Theorem 3.2 depends only on \( n, \Omega, \eta, \) and \( \inf_{\partial\Omega} u \). Hence Theorem 3.2 also holds for non-smooth \( H_1 \)-subharmonic functions, by the approximation in Section 5.

**Remark 3.3.** A similar estimate for the prescribing \( k \)-curvature equation was established in [T2]. We include a direct proof for the mean curvature case (namely the case \( k = 1 \)) here for completeness.

### 4. Perron lifting

Let \( u \) be an \( H_1 \)-subharmonic function in \( \Omega \) and let \( \omega \subset \Omega \) be an open, precompact set in \( \Omega \). The Perron lifting of \( u \) in \( \omega \), \( u^\omega \), is defined as the upper semicontinuous regularization of
\[
u^* = \sup\{v \mid v \text{ is } H_1\text{-subharmonic in } \Omega \text{ and } v \leq u \text{ in } \Omega - \omega\}, \tag{4.1}
\]
namely
\[ u^\omega(x) = \lim_{r \to 0} \sup_{B_r(x)} u^*. \tag{4.2} \]

**Remark 4.1.** Obviously we have \( u^\omega \geq u \) on \( \partial \omega \). However for general open set \( \omega \), it may occur that \( u^\omega > u \) on part of the boundary \( \partial \omega \), even if \( u \) is a smooth function. This is easily seen by considering the Perron lifting in \( \omega = B_R - \overline{B}_r \) of a radial function \( u \), where \( R > r \). Then in general one has \( u^\omega > \neq u \) on the inner boundary \( \partial B_r \). But if \( u \) is continuous, by Lemma 2.2 one has \( u^\omega = u \) on the outer boundary \( \partial B_R \).

First we prove the following basic result for \( H_1 \)-harmonic functions. Note that our definition of \( H_1 \)-harmonic functions does not imply they are bounded from below.

**Lemma 4.1.** Let \( u \) be an \( H_1 \)-harmonic function in \( \Omega \). Then \( u \) is locally bounded and smooth in \( \Omega \), and satisfies the equation \( H_1[u] = 0 \) in \( \Omega \).

**Proof.** Assume that \( B_1(0) \subseteq \Omega \). By definition, an \( H_1 \)-harmonic function is \( H_1 \)-subharmonic. The \( n \)-dimensional Hausdorff measure \( |\{x \in \Omega : u < -t\}| \to 0 \) as \( t \to \infty \). Hence we may assume that the \((n - 1)\)-dimensional Hausdorff measure \( |\{x \in \partial B_1 : u < -t\}| \to 0 \) as \( t \to \infty \).

Since \( u \) is upper semicontinuous, there exists a sequence of smooth functions \( \{v_j\} \) in \( \Omega \) such that \( v_j \searrow u \), namely \( v_j \) converges to \( u \) monotone decreasingly. By Lemma 2.2, there is a solution \( \hat{v}_j \in C^2(B_1) \cap C^0(\overline{B}_1) \) to

\[
\begin{cases} 
H_1[v] = 0 & \text{in } B_1(0), \\
v = v_j & \text{on } \partial B_1.
\end{cases} \tag{4.3}
\]

Since \( \hat{v}_j \) is monotone decreasing and \( \hat{v}_j > u \), it is convergent. We may assume that \( \hat{v}_j \searrow \hat{v} \). Obviously \( \hat{v} \geq u \) in \( B_1 \).

Next we show that \( \hat{v} \leq u \) on \( \partial B_1 \), namely for any given \( x_0 \in \partial B_1 \),

\[
\lim_{x \to x_0} \hat{v}(x) \leq u(x_0), \tag{4.4}
\]

so that

\[ \hat{v} \equiv u \quad \text{in } B_1. \]

Indeed, since \( u \) is upper semicontinuous on \( \partial B_1 \), there is a continuous function \( w \) on \( \partial B_1 \) such that \( w(x_0) = u(x_0) \) and \( w \geq u \) on \( \partial B_1 \). By the monotonicity of \( v_j \) on \( \partial B_1 \), it is easy to show that for any \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that for sufficiently large \( j \), \( v_j(x) < u(x) + \varepsilon \) in \( \{x \in \partial B_1 : |x - x_0| \leq \delta\} \). Hence by adding \( C|x - x_0|^2 \) to \( w \) for some large \( C \), we may assume that \( w > v_j - \varepsilon \) on \( \partial B_1 \) when \( j \) is sufficiently large. Let \( \hat{w} \in C^2(B_1) \cap C^0(\overline{B}_1) \) be the solution of \( H_1[\hat{w}] = 0 \) in \( B_1(0) \), satisfying the boundary condition \( \hat{w} = w \) on \( \partial B_1 \). Then \( \hat{w} \geq \hat{v}_j - \varepsilon \geq \hat{v} - \varepsilon \). Since \( \varepsilon > 0 \) is arbitrary, we obtain

\[ u(x_0) = \hat{w}(x_0) \geq \hat{v}(x_0), \]

namely (4.4) holds.
If \( \inf_{B_{1/2}} \hat{v}_j \to -\infty \) as \( j \to \infty \), by the Harnack inequality (Theorem 2.1), we see that \( \hat{v}_j \to -\infty \) uniformly in \( B_{1/2} \). Recall that \( \hat{v}_j \geq u \). We obtain \( u = -\infty \) in \( B_{1/2} \). But by the definition of subharmonic functions, the set \( \{ u = -\infty \} \) has measure zero. We reach a contradiction. Hence \( \hat{v}_j \) is locally uniformly bounded, and so \( u \) is locally uniformly bounded and smooth. Note that to apply Theorem 2.1 we need the condition \( \| u(x) < -t : x \in \partial B_t \|_{n-1} \to 0 \) as \( t \to \infty \), which is satisfied as noted at the beginning of the proof. \( \square \)

**Remark 4.2.** The function \( \hat{v} \) is independent of the sequence \( v_j \). Indeed, let \( w_j \) be another sequence of smooth functions on \( \partial B_1 \) such that \( w_j \searrow u \). Let \( \hat{w}_j \) be the solution of (4.3) with boundary condition \( \hat{w}_j = w_j \) on \( \partial B_1 \) and let \( \hat{w} = \lim \hat{w}_j \). Then by (4.4), we have \( \hat{w}_j \geq \hat{v} \). Hence \( \hat{w} \geq \hat{v} \). Similarly we have \( \hat{v} \geq \hat{w} \). Therefore we may regard \( \hat{v} \) as the solution of the Dirichlet problem \( H_1[v] = 0 \) in \( B_1 \) with \( v = u \) on \( \partial B_1 \).

**Lemma 4.2.** Let \( u \in SH_1(\Omega) \). Then for any open set \( \omega \subseteq \Omega \), the Perron lifting \( u^\omega \) is \( H_1 \)-harmonic in \( \omega \) and \( H_1 \)-subharmonic in \( \Omega \).

**Proof.** The property that \( u^\omega \) is \( H_1 \)-subharmonic in \( \Omega \) follows by definition. Indeed, let \( E \subset \Omega \) be an open set and \( h \in C^2(\overline{E}) \) be an \( H_1 \)-harmonic function satisfying \( h \geq u^\omega \) on \( \partial E \). Then for any \( H_1 \)-subharmonic function \( v \) in (4.1), \( h \geq v \) on \( \partial E \). Hence \( h \geq v \) in \( E \). By the definition of \( u^\omega \) in (4.1) and (4.2) and note that \( h \in C^2(E) \), it follows that \( h \geq u^\omega \) in \( E \). That is, \( u^\omega \) is \( H_1 \)-subharmonic.

To show that \( u^\omega \) is \( H_1 \)-harmonic in \( \omega \), let \( B_r \subseteq \omega \) and let \( v \) be the solution of the Dirichlet problem \( H_1[v] = 0 \) in \( B_r \) with \( v = u^\omega \) on \( \partial B_r \) (see Remark 4.2). Then \( v \geq u^\omega \) in \( B_r \). Let \( \hat{u} = v \) in \( B_r \) and \( \hat{u} \) be upper semicontinuous and \( H_1 \)-subharmonic. It follows by (4.1) that \( \hat{u} \leq u^\omega \). Hence \( u^\omega = v \) in \( B_r \). Namely \( u^\omega \) is \( H_1 \)-harmonic in \( B_r \). \( \square \)

**Lemma 4.3.** Suppose \( \{ u_j \} \subset SH_1(\Omega) \) such that \( u_j \) converges to a measurable function \( u \) a.e. with \( \| \{ u = -\infty \} \| = 0 \). Let \( \tilde{u} \) be the upper semicontinuous regularization of \( u \). Then \( \tilde{u} = u \) a.e. and \( \tilde{u} \) is \( H_1 \)-subharmonic.

**Proof.** Let \( x_0 \) be a Lebesgue point of \( u \). By adding a constant we assume that \( u(x_0) = 0 \). Then Lemma 2.5 implies that \( \sup_{B_r(x_0)} u \to 0 \) as \( r \to 0 \). Hence \( u = \tilde{u} \) at all Lebesgue points, namely \( u = \tilde{u} \) a.e..

To prove that \( \tilde{u} \) is \( H_1 \)-subharmonic, let \( \omega \subseteq \Omega \) be an open set and \( h \in C^2(\overline{\omega}) \) be an \( H_1 \)-harmonic function with \( h \geq \tilde{u} \) on \( \partial \omega \). If \( u_j \) is monotone decreasing, then for any \( \varepsilon > 0 \), by the monotonicity and the upper semicontinuity of \( u_j \), \( h \geq u_j - \varepsilon \) on \( \partial \omega \) provided \( j \) is sufficiently large. It follows that \( h \geq u_j - \varepsilon \) in \( \omega \) for all large \( j \). Hence \( h \geq \tilde{u} \) in \( \omega \) and so \( \tilde{u} \) is \( H_1 \)-subharmonic. If \( u_j \) is monotone increasing, obviously \( h \geq u_j \) on \( \partial \omega \) for all \( j \). Hence \( h \geq \tilde{u} \) in \( \omega \) and so \( \tilde{u} \) is \( H_1 \)-subharmonic.
For general \( \{ u_j \} \), let \( w_{k,j} = \max\{ u_k, \cdots, u_j \} \). Then for fixed \( k \), \( w_{k,j} \not\to w_k \) a.e., as \( j \to \infty \), for some \( w_k \in SH_1(\Omega) \), and \( w_k \not\to u \) a.e. as \( k \to \infty \). Hence \( u \) is \( H_1 \)-subharmonic.

For \( u \in SH_1(\Omega) \), the Perron lifting \( u^{B_t} \) is monotone increasing in \( t \),

\[
\lim_{t \to \delta^-} u^{B_t} \leq u^{B_\delta}(x) \leq \lim_{t \to \delta^+} u^{B_t} \quad \forall \ x \in \Omega. \tag{4.5}
\]

This implies that \( \|u^{B_t}\|_{L^1(\Omega)} \) as a function of \( t \), is monotone and bounded. Hence, \( \|u^{B_t}\|_{L^1(\Omega)} \) is continuous for almost all \( t \). Since \( u^{B_t} \) is continuous in \( B_t \), it follows that

\[
\lim_{t \to r} u^{B_t}(x) = u^{B_r}(x) \text{ for a.e. } r > 0. \tag{4.6}
\]

Similar to Lemma 3.6 in [TW4], we have the following

**Lemma 4.4.** Suppose \( u_j, \ u \in SH_1(\Omega) \) and \( u_j \to u \) a.e. in \( \Omega \). Then for any \( B_r \in \Omega \) such that (4.6) holds, we have \( u_j^{B_r} \to u^{B_r} \) a.e. in \( \Omega \) as \( j \to \infty \).

**Proof.** Since \( u_j^{B_r} \) and \( u^{B_r} \) are locally uniformly bounded in \( C^2_{loc}(B_r) \), by passing to a subsequence, we may assume that \( u_j^{B_r} \) is convergent. Let \( w' = \lim u_j^{B_r} \) and \( w \) be the upper semicontinuous regularization of \( w' \) (note that \( w \) and \( w' \) can differ only on \( \partial B_r \)). Then \( w \in SH_1(\Omega) \) and \( w = u \) in \( \Omega - \overline{B}_r \). Hence by the definition of the Perron lifting, we have \( u^{B_r} \geq w \).

Next we prove that for any \( \delta > 0 \), \( w \geq u^{B_r-\delta} \). Once this is proved, we have \( u^{B_r} \geq w \geq u^{B_r-\delta} \). Sending \( \delta \to 0 \), we obtain \( u^{B_r} = w \) by (4.6).

To prove \( w \geq u^{B_r-\delta} \), it suffices to prove that for any \( \varepsilon > 0 \), \( u_j^{B_r} \geq u - \varepsilon \) on \( \partial B_r - \delta \) for sufficiently large \( j \). By the interior gradient estimate, \( u_j^{B_r} \) is uniformly bounded in \( C^2(B_{r-\delta/4}) \). If there exists a point \( x_0 \in \partial B_{r-\delta} \) such that \( u(x_0) > u_j^{B_r}(x_0) + \varepsilon \) for all large \( j \), by Lemma 2.5, there is a Lebesgue point \( x_1 \in B_{\delta/4}(0) \) of \( u \) such that \( u(x_1) > u_j^{B_r}(x_1) + \frac{1}{2} \varepsilon \) for all large \( j \). It follows that the limit function \( w = \lim_{j \to \infty} u_j^{B_r} \) is strictly less than \( u \) a.e. near \( x_1 \). We reach a contradiction as \( w = \lim_{j \to \infty} u_j^{B_r} \geq \lim_{j \to \infty} u_j = u. \)

\[ \]

5. Approximation by smooth functions

We prove that every \( H_1 \)-subharmonic function can be approximated by a sequence of smooth, \( H_1 \)-subharmonic functions.

**Theorem 5.1.** For any \( u \in SH_1(\Omega) \), there is a sequence of smooth functions \( \{ u_j \} \subset SH_1(\Omega) \) such that \( u_j \to u \) a.e. on \( \Omega \).

**Proof.** For each \( j = 1, 2, \cdots \), let \( \{ B_{j,k} \} \) be a family of finitely many balls of radius \( 2^{-j} \), contained in \( \overline{\Omega} \), such that \( \Omega_{2^{-j-1}} \subset \bigcup_{k=1}^{k_j} B_{j,k} \), where \( \Omega_\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \).
Let $u_{j,0} = u$. For $m = 1, \ldots, k_j$, define $u_{j,m}$ such that $u_{j,m} = u_{j,m-1}$ in $\Omega - \overline{B}_{j,m}$ and $u_{j,m}$ is the solution of
\begin{align}
\begin{cases}
H_1[v] = 0 & \text{in } B_{j,m}, \\
v = u_{j,m-1} & \text{on } \partial B_{j,m}.
\end{cases}
\end{align}
(5.1)
and denote $u_j = u_{j,k_j}$. Then $u_j$ is a sequence of piecewise smooth $H_1$-subharmonic functions and
\begin{equation}
\label{eq:5.2}
u_j \geq u.
\end{equation}

To show that $u_j \to u$ a.e., recall that every upper semi-continuous function $u$ can be approximated by a sequence of smooth, monotone decreasing functions $\{v^m\}$, namely $v^m \searrow u$. For each $m$, define $v_j^m$ as above. Then we have $v_j^m \to v^m$ as $j \to \infty$. Hence we may choose $j = j_m$ large such that $v_j^m \to u$ a.e.. Note that $v_j^m \geq u_{j,m}$. Hence $u_j \to u$ a.e..

In the above proof we obtain a sequence of piece-wise smooth functions $\{u_j\} \subset SH_1(\Omega)$ which converges to $u$. To prove the theorem we make certain mollification of $u_{j,k}$. A simple way is to replace $u_{j,k}$ by the convolution $u_{j,k} * \rho_\varepsilon$ ($\varepsilon$ depends on $j, k$, and $\varepsilon \to 0$ sufficiently fast as $j \to \infty$), where $\rho_\varepsilon = \varepsilon^{-n} \rho(\varepsilon^{-1})$ and $\rho$ is a mollifier. Namely $\rho$ is a nonnegative function satisfying $\rho \in C^\infty_0(B_1(0))$ and $\int_{B_1} \rho = 1$. Specifically we may choose
\begin{equation}
\label{eq:5.3}
\rho(x) = \begin{cases}
C \exp(-\frac{1}{|x|^2 - 1}) & \text{for } |x| \leq 1, \\
0 & \text{for } |x| \geq 1,
\end{cases}
\end{equation}
where $C$ is chosen such that $\int_{\mathbb{R}^n} \rho(x) dx = 1$.

The function $u_{j,k} * \rho_\varepsilon$ may not be $H_1$-subharmonic. But we have
\begin{equation}
\label{eq:5.4}
H_1[u_{j,k} * \rho_\varepsilon] \geq -\delta
\end{equation}
with $\delta \to 0$ as $\varepsilon \to 0$. This is fine for our treatment, as the mean curvature operator is elliptic for any smooth functions.

We can also mollify $u_{j,k}$ in the following way to get a sequence of $C^{1,1}$ smooth, $H_1$-subharmonic functions which converges to $u$. For a fixed $j$, recall that we first get the function $u_{j,1}$, which is smooth in $B_{j,1}$. We then get $u_{j,2}$, which is the Perron lifting of $u_{j,1}$ in $B_{j,2}$. The function $u_{j,2}$ is piece-wise smooth in $B_{j,1} \cup B_{j,2}$, its gradient may have a jump across the boundary $\Gamma =: B_{j,1} \cap \partial B_{j,2}$. If $Du_{j,2}$ has a jump at some point on $\Gamma$, then by the maximum principle, we have $u_{j,2} > u_{j,1}$ in $B_{j,2} - B_{j,1}$. By the Hopf lemma, $Du_{j,2}$ has a jump at every point on $\Gamma$.

Let us indicate the mollification of $u_{j,2}$ near $\Gamma$. By a proper choice of the axes, we assume that $B_{j,2}$ is centered at $(0, 2^{-j})$ and $B_{j,1}$ is centered at $(0, c)$ for some $c < 2^{-j}$. Then $\Gamma$ is given by
\begin{equation}
x_n = g(x') = 2^{-j} - \sqrt{2^{-2j} - |x'|^2},
\end{equation}
(5.5)
where \( x' = (x_1, \ldots, x_{n-1}) \). Let

\[
\begin{aligned}
a(x') &= \lim_{t \to +0} \frac{1}{t} [u_{j,2}(x', g(x') + t) - u_{j,2}(x', g(x'))] \\
&- \lim_{t \to +0} \frac{1}{t} [u_{j,2}(x', g(x')) - u_{j,2}(x', g(x') - t)] \\
&= \partial_{x_n} u_{j,2}(x', g(x')) - \partial_{x_n} u_{j,1}(x', g(x')).
\end{aligned}
\] (5.6)

By the Hopf lemma, \( a(x') > 0 \) for all \( x' \) near 0. Let

\[
\phi(x) = a(x') \frac{x_n - g(x') + \varepsilon}{4\varepsilon},
\] (5.7)

where \( \varepsilon << 2^{-j} \) is a small constant. Now let

\[
\tilde{u}_{j,2}(x) = \begin{cases} 
  u_{j,2}(x) & \text{if } |x_n - g(x')| \geq \varepsilon, \\
  u_{j,2}(x) + \phi(x) & \text{if } g(x') - \varepsilon \leq x_n \leq g(x') \\
  u_{j,2}(x) + \phi(x) - a(x')(x_n - g(x')) & \text{if } g(x') \leq x_n \leq g(x') + \varepsilon,
\end{cases}
\] (5.8)

It is obvious that \( \tilde{u}_{j,2} \in C^{1,1} \). When \( g(x') - \varepsilon \leq x_n \leq g(x') + \varepsilon \),

\[
D^2\phi = \frac{a}{2\varepsilon}(Dg, -1) \otimes (Dg, -1) + O(1).
\] (5.9)

Note that

\[
H_1[u] = \text{trace of } (1 - \frac{u_iu_j}{1 + |Du|^2})(D^2u)
\] (5.10)

and the matrix \( (1 - \frac{u_iu_j}{1 + |Du|^2}) \) is positive definite (since \( |Du| \leq C \)). Hence \( \tilde{u}_{j,2} \) is \( \text{SH}_1 \)-subharmonic when \( \varepsilon \) is sufficiently small.

After the modification, \( u_{j,2} \) is smooth in \( B_{j,1} \cup B_{j,2} \). Next we can modify \( u_{j,k} \), for \( k = 3, 4, \ldots \), in the same way, but the constant \( \varepsilon \) will be chosen smaller and smaller. □

We note that by choosing the function \( \phi \) in (5.7) more carefully, one can make the function \( \tilde{u}_{j,2} \) in (5.8) \( C^{2,1} \)-smooth.

6. Weak convergence

For \( u \in \text{SH}_1 \cap C^2 \), denote \( \mu_1[u] = H_1[u]dx \) the associated measure. In this section, we prove the following weak convergence result for \( H_1[u] \).

**Lemma 6.1.** Let \( u_j \in C^2(\Omega) \) be a sequence of \( \text{H}_1 \)-subharmonic functions which converges to \( u \in \text{SH}_1(\Omega) \) a.e. in \( \Omega \). Then \( \{\mu_1[u_j]\} \) converges to a measure \( \mu \) weakly.

**Proof.** For any open set \( \omega \subset \Omega \),

\[
\mu_1[u_j](\omega) \leq \mu_1[u_j](\Omega) \leq |\partial\Omega|
\] (6.1)
is uniformly bounded. Hence there is a subsequence of $\mu_1[u_j]$ which converges weakly to a measure $\mu$. We need to prove that $\mu$ is independent of the choice of subsequences of \{u_j\}.

Let \{u_j\}, \{v_j\} $\subset SH_1(\Omega) \cap C^2(\Omega)$. Suppose both sequences converge to u a.e. in $\Omega$ and

$$\mu_1[u_j] \to \mu, \quad \mu_1[v_j] \to \nu$$

weakly as measures. We claim that for any ball $B_r(x_0)$ such that $B_{2r}(x_0) \subset \Omega$,

$$\mu(B_r) = \nu(B_r), \quad (6.3)$$

or equivalently, for any $t > 0$,

$$\mu(B_r) \leq \nu(B_{r+t}), \quad (6.4a)$$
$$\nu(B_r) \leq \mu(B_{r+t}). \quad (6.4b)$$

We choose finitely many small balls \{B_l\}_{l=1}^k contained in $B_{r+4t/5} - B_{r+t/5}$ such that the center of each ball is on $\partial B_{r+t/2}$ and $\overline{B_{r+3t/4} - B_{r+t/4}} \subset \bigcup_{l=1}^k B_l$. Now let $u_{j,1}$ be the Perron lifting of $u_j$ on $B_1$, and let $u_{j,2}$ be the Perron lifting of $u_{j,1}$ on $B_2$, ..., and let $u_{j,k}$ be the Perron lifting of $u_{j,k-1}$ on $B_k$. Denote $u^t_j = u_{j,k}$. Similarly we obtain $v^t_j$ and $u^t$. Then $u^t_j, v^t_j$ and $u^t$ are piece-wise smooth in $B_{r+3t/4} - B_{r+t/4}$, and $u^t_j = u_j$, $v^t_j = v_j$ in $B_r$, and so are smooth in $B_r$. By Lemma 4.4, we have

$$u^t_j, v^t_j \to u^t \quad \text{in } \Omega \text{ a.e.} \quad (6.5)$$

and

$$Du^t_j, Dv^t_j \to Du^t \quad \text{on } \partial B_{r+t/2} \text{ a.e.}$$

Let $u^t_{j,\varepsilon} = u^t_j * \rho_{\varepsilon}$ and $v^t_{j,\varepsilon} = v^t_j * \rho_{\varepsilon}$ be the mollifications of $u^t_j$ and $v^t_j$, where $\rho_{\varepsilon} = \varepsilon^{-n} \rho(\frac{\cdot}{\varepsilon})$ and $\rho$ is a mollifier, as was given in (5.3). Then $H_1[u^t_{j,\varepsilon}] \geq -\delta_{\varepsilon}$ with $\delta_{\varepsilon} \to 0$ as $\varepsilon \to 0$. Noting that $u^t_j$ is independent of $t$ in $B_r$, we have

$$\int_{B_r} H_1[u_j] = \int_{B_r} H_1[u^t_j] = \lim_{\varepsilon \to 0} \int_{B_r} H_1[u^t_{j,\varepsilon}] \quad (6.6)$$

$$\leq \lim_{\varepsilon \to 0} \int_{B_{r+t/2}} H_1[u^t_{j,\varepsilon}]$$
$$= \lim_{\varepsilon \to 0} \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Du^t_{j,\varepsilon}}{\sqrt{1 + |Du^t_{j,\varepsilon}|^2}}$$
$$= \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Du^t_j}{\sqrt{1 + |Du^t_j|^2}},$$

22
where $\gamma$ denotes the unit outer normal. Recall that $u^t_j, v^t_j$ and $u^t$ are piece-wise smooth in $B_{r+3t/4} - B_{r+t/4}$, we have

$$\lim_{j \to \infty} \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Du^t_j}{\sqrt{1 + |Du^t_j|^2}} = \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Du^t}{\sqrt{1 + |Du^t|^2}}$$

(6.7)

Similarly we have

$$\int_{\partial B_{r+t/2}} \frac{\gamma \cdot Du^t}{\sqrt{1 + |Du^t|^2}} = \lim_{j \to \infty} \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Dv^t_j}{\sqrt{1 + |Dv^t_j|^2}}$$

(6.8)

Note that

$$\int_{\partial B_{r+t}} \frac{\gamma \cdot Dv^t_{j,\varepsilon}}{\sqrt{1 + |Dv^t_{j,\varepsilon}|^2}} - \int_{\partial B_{r+t/2}} \frac{\gamma \cdot Dv^t_{j,\varepsilon}}{\sqrt{1 + |Dv^t_{j,\varepsilon}|^2}} = \int_{B_{r+t} - B_{r+t/2}} H_1[v^t_{j,\varepsilon}]$$

and $H_1[v^t_{j,\varepsilon}] \geq -\delta_\varepsilon$ with $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence the right hand side of (6.8)

$$\leq \lim_{j \to \infty} \lim_{\varepsilon \to 0} \int_{\partial B_{r+t}} \frac{\gamma \cdot Dv^t_{j,\varepsilon}}{\sqrt{1 + |Dv^t_{j,\varepsilon}|^2}}.$$

Note that $v^t_{j,\varepsilon}$ is independent of $t$ on $\partial B_{r+t}$. The above formula

$$\lim_{j \to \infty} \lim_{\varepsilon \to 0} \int_{B_{r+t}} H_1[v^t_{j,\varepsilon}] = \lim_{j \to \infty} \int_{B_{r+t}} H_1[v^t_j].$$

Hence we obtain $\mu(B_r) \leq \nu(B_{r+t})$. Similarly, we can prove $\nu(B_r) \leq \mu(B_{r+t})$. This completes the proof. □

From the above lemma, we can assign a measure $\mu$ to $u$ for any $u \in SH_1(\Omega)$, and obtain the following weak convergence theorem.

**Theorem 6.1.** For any $u \in SH_1(\Omega)$, there exists a Radon measure $\mu_1[u]$ such that

(i) $\mu_1[u] = H_1[u]dx$ if $u \in C^2(\Omega)$,

(ii) if $\{u_j\} \subset SH_1(\Omega)$ is a sequence which converges to $u$ a.e., then $\mu_1[u_j] \to \mu_1[u]$ weakly as measure.

Note that in (ii) above, we need to use the approximation in Section 5.

**Remark 6.1.** If $\{u_j\}$ is a sequence of semi-convex functions converging to $u$, then the weak convergence $\mu_1[u_j] \to \mu_1[u]$ is a special case of the weak continuity of Federer [F1].
7. Existence of weak solution

In this section we consider the Dirichlet problem

\[ H_1[u] = \nu \quad \text{in } \Omega, \quad \tag{7.1} \]

\[ u = \varphi \quad \text{on } \partial \Omega, \]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^n \), \( \varphi \) is a continuous function on \( \partial \Omega \), and \( \nu \) is a nonnegative measure. Here we also use \( \nu \) to denote its density with respect to the Lebesgue measure.

For the Dirichlet problem of the mean curvature equation, usually one assumes that the right hand side \( \nu \) is Lipschitz continuous so that the solution is smooth [GT, G1]. When \( \nu \in L^n(\Omega) \), the existence of a generalized solution, introduced in [Mi], was investigated in [Gia, G2]. Here we consider solutions in \( SH_1(\Omega) \). We say \( u \in SH_1(\Omega) \) is a weak solution of (1.3) if \( \mu_1[u] = \nu \).

Assume that for any Caccioppoli set \( \omega \subset \Omega \) with nonzero measure,

\[ \nu(\omega) < |\partial \omega|. \quad \tag{7.2} \]

This is also a necessary condition for the existence of smooth solutions to the mean curvature equation (7.1), which can be verified easily by taking integration by parts of the equation.

Let \( \rho \) be a mollifier, as was given in (5.3). Let \( g_\varepsilon(x) \) be the mollification of \( \nu \), namely

\[ g_\varepsilon(x) = \int_\Omega \rho_\varepsilon(x-y)d\nu. \]

Extend \( \nu \) to \( \mathbb{R}^n \) such that \( \nu = 0 \) outside \( \Omega \). Then \( g_\varepsilon \in C^\infty(\mathbb{R}^n) \) and \( g_\varepsilon dx \) converges to \( \nu \) weakly.

**Lemma 7.1.** For any open set \( \omega \subset \Omega \), we have

\[ \int_\omega g_\varepsilon dx < |\partial \omega|. \quad \tag{7.3} \]

**Proof.** We have

\[ \int_\omega g_\varepsilon dx = \int_\omega dx \int_\Omega \rho_\varepsilon(x-y)d\nu \]

\[ = \int_{|z|\leq1} \nu(\omega-\varepsilon z)\rho(z)dz, \]

where \( \omega-\varepsilon z = \{x \in \mathbb{R}^n : x+\varepsilon x \in \omega\} \). By (7.2), \( \nu(\omega-\varepsilon z) < |\partial \omega| \). Hence we obtain (7.3). \( \square \)

Consider the approximation problem

\[ H_1[u] = g_\varepsilon(x) \quad \text{in } \Omega, \quad \tag{7.4} \]

\[ u = \varphi \quad \text{on } \partial \Omega. \]

For equation with smooth right hand side, we quote the following result [Gia].
Lemma 7.2. Under condition (7.3), there is a minimizer $u_{\varepsilon}$ of the functional
\[
\mathcal{F}(u) = \int_{\Omega} \sqrt{1 + |Du|^2} - \int_{\Omega} g_{\varepsilon} u + \int_{\partial\Omega} |u - \varphi|.
\] (7.5)
If $\varphi \in C^0(\partial\Omega)$, the minimizer is a smooth solution to the mean curvature equation (7.4). If the mean curvature $H'$ of $\partial\Omega$ (with respect to the inner normal) satisfies
\[
H'(x) > \frac{n}{n-1} g_{\varepsilon}(x) \quad \forall \quad x \in \partial\Omega,
\] (7.6)
then $u_{\varepsilon} = \varphi$ on $\partial\Omega$.

Remark 7.1. By our Harnack inequality, the minimizer is a smooth solution to the mean curvature equation $H_1[u] = g_{\varepsilon}$ if $\varphi \in L^1(\partial\Omega)$.

Theorem 7.1. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with $C^2$ boundary. Let $\nu$ be a non-negative measure which satisfies (7.2) and can be decomposed as $\nu = \nu_1 + f$ for some nonnegative measure $\nu_1$ with compact support in $\Omega$ and some Lipschitz function $f \geq 0$. Suppose the boundary mean curvature satisfies
\[
H'(x) > \frac{n}{n-1} f(x) \quad \forall \quad x \in \partial\Omega.
\] (7.7)
Then (7.1) has a weak solution.

Proof. We divide the proof into two steps.

Step 1. First we prove the theorem under the additional assumption that there exists a positive constant $\eta > 0$ such that for any Caccioppoli set $\omega \subset \Omega$,
\[
\nu(\omega) \leq (1 - \eta)|\partial\omega|.
\] (7.8)
Let $g_{\varepsilon}$ be the mollification of $\nu$ as above. Note that (7.7) implies (7.6) for small $\varepsilon > 0$. Hence by Lemma 7.2, there is a solution $u_{\varepsilon}$ to (7.4). By Theorem 3.2, $u_{\varepsilon}$ is uniformly bounded,
\[
\sup_{\partial\Omega} \varphi \geq u_{\varepsilon} \geq -C
\] (7.9)
for some $C > 0$ independent of $\varepsilon$. By assumption, $\nu$ is given by a Lipschitz continuous function $f$ in $\Omega - S$, where $S = \text{supp} \nu_1$. Hence $u_{\varepsilon}$ is locally uniformly bounded in $C^2(\Omega - S)$. By Theorem 3.1, $u_{\varepsilon}$ is uniformly bounded in $W^{1,1}(\Omega')$ for any $\Omega' \Subset \Omega$. Hence we may assume that $u_{\varepsilon}$ converges in $L^1$ to a limit function $u$. Note that $g_{\varepsilon}dx$ converges weakly to $\nu$. By Theorem 6.1, $u$ is a weak solution of (7.1). By Corollary 3.1 and since $\nu = f$ is Lipschitz continuous in $\Omega - S$, $u \in BV(\Omega)$.

Step 2. Next we remove the assumption (7.8). For any small constant $\delta \in (0,1)$, from Step 1 there is a solution $u_\delta \in BV(\Omega)$ to
\[
H_1[u] = (1 - \delta)\nu \quad \text{in} \quad \Omega,
\] (7.10)
\[
u_{\delta} = \varphi \quad \text{on} \quad \partial\Omega.
\]
Then $u_\delta$ is monotone, namely $u_{\delta_1} \geq u_{\delta_2}$ if $\delta_1 > \delta_2$; and $u_\delta$ is smooth near $\partial \Omega$. We wish to prove that $u_\delta$ converges to a solution of (7.1) as $\delta \to 0$. Since $u_\delta$ is monotone, we may define

$$
u = \lim_{\delta \to 0} u_\delta. \quad (7.11)$$

Denote $N =: \{ x \in \Omega :\; u(x) = -\infty \}$. If $N$ has measure zero, then by Lemma 4.3, $u \in SH_1(\Omega)$, and by Theorem 6.1, $\mu_1[u] = \nu$. To see that $u$ satisfies the boundary condition $u = \varphi$ on $\partial \Omega$, note that $\nu = f$ is a Lipschitz function near $\partial \Omega$ and recall that Lemma 2.5 holds for functions satisfying $H_1[u] \geq f$, see Remark 2.4. Hence $u_\delta$ is locally uniformly bounded and smooth near $\partial \Omega$. Hence the boundary condition $u = \varphi$ is satisfied and so $u$ is a weak solution of (7.1).

It remains to prove that Lebesgue measure $|N| = 0$. Suppose to the contrary that

$$|N| > \sigma > 0. \quad (7.12)$$

We claim that there exists a positive constant $\eta > 0$ such that

$$\nu(\Omega_t) < (1 - \eta)|\partial \Omega_t| \quad (7.13)$$

for all large $t$, where $\Omega_t = \{ x \in \Omega :\; u(x) \leq -t \}$, so that $N = \Omega_\infty$. (7.13) can be proved by a compactness argument. Indeed, if it is not true, there is a sequence of $\{t_j\}$, $t_j \to t_\infty \leq \infty$, such that

$$\nu(\Omega_{t_j}) \geq (1 - 2^{-j})|\partial \Omega_{t_j}|.$$

Let $\varphi_j$ be the characteristic function so that

$$|\partial \Omega_{t_j}| = \int_{\mathbb{R}^n} |D\varphi_j|.$$

Since $\nu(\Omega_{t_j}) \leq \nu(\Omega)$ is uniformly bounded, $\varphi_j$ converges in $L^1$ to the characteristic function $\varphi$ of $\Omega_{t_\infty}$ and

$$\int_{\mathbb{R}^n} |D\varphi| \leq \lim_{j \to \infty} \int_{\mathbb{R}^n} |D\varphi_j|.$$

Since $\Omega_t$ is monotone, we have $\nu(\Omega_{t_j}) \to \nu(\Omega_{t_\infty}) > \sigma$. Hence we obtain

$$\nu(\Omega_{t_\infty}) \geq \int_{\mathbb{R}^n} |D\varphi|,$$

which is in contradiction with (7.2). Hence (7.13) holds.

Denote $\Omega_{\delta,t} = \{ x \in \Omega :\; u_\delta(x) \leq -t \}$. Recall that $u_\delta$ is monotone. Hence for any $t > 0$, $|\Omega_{\delta,t}| > \sigma$ provided $\delta$ is sufficiently small. For any fixed $t$, by a compactness argument as above, we also have

$$\nu(\Omega_{\delta,t}) < (1 - \eta)|\partial \Omega_{\delta,t}| \quad (7.14)$$
when \( \delta \) is sufficiently small. Let \( \delta_t > 0 \) be the sup of all such \( \delta \). Then again by a similar compactness argument, we have

\[
\lim_{t \to t_0} \delta_t \geq \delta_{t_0}.
\]  

Therefore for any \( T > 0 \), we can choose \( \delta > 0 \) sufficiently small such that (7.14) holds for all \( t \in (0, T] \). Now we fix \( T \) as in (3.5). By Step 1 above, \( u_\delta \) is a bounded function and \( u_\delta \in BV(\Omega) \). Hence the proof of Theorem 3.2 is valid (see Remark 3.1) and we obtain

\[
\inf u_\delta \geq -C
\]

for some \( C > 0 \) depending on \( n, |\Omega|, \inf_{\partial\Omega} u_\delta \), and \( \eta \), but is independent of \( \delta \). Sending \( \delta \to 0 \), we find that \( u \) is bounded from below, a contradiction. \( \square \)

**Remark 7.2.** Condition like (7.2) was included in [Gia, G1, G2]. When \( \nu \) (more precisely its density) is a bounded function, (7.2) implies (7.8) for a small \( \eta \) [G1].

**Remark 7.3.** A weak solution is usually not \( C^2 \) smooth if \( \nu \) is not Lipschitz continuous. This is easily seen by considering functions of one variable, \( u = u(x_1) \). However, if \( n \leq 7 \) and \( \nu \) is a bounded function and the weak solution is a minimizer of the functional (7.5), then the graph of the solution is a \( C^{2,\alpha} \) hypersurface if \( \nu \) is a Hölder continuous function; or \( C^{1,\alpha} \) if \( \nu \) is a bounded nonnegative function [Ma].

### 8. Remarks

We include an example showing that some potential theoretical properties which hold for the \( p \)-Laplace equation and the \( k \)-Hessian equation [HKM, Lab, TW1-TW4] may not hold for curvature equations.

Let

\[
u_c(x) = \begin{cases} 
a(r - 1)^\delta & \text{if } r \geq 1, \\
-b(1 - r)^\sigma - c & \text{if } 0 \leq r < 1,
\end{cases}
\]

where \( r = |x|, a, b > 1, c \geq 0, \delta, \sigma \in (0, \frac{1}{2}) \) are positive constants. Then \( H_1[u_0] \) is positive and Hölder continuous near \( \partial B_1 \), but \( u_0 \notin C^1 \), since \(|Du| = \infty\) on the sphere \( \{ |x| = 1 \} \). As remarked at the end of last section, the graph of \( u_0 \) is \( C^{2,\alpha} \) for some \( \alpha > 0 \).

If \( c > 0 \), \( u_c \) is \( H_1 \)-subharmonic, and can be approximated by smooth \( H_1 \)-subharmonic functions. Therefore weak solutions to the Dirichlet problem (7.1), without the restriction (7.2), is not unique in general. We note that the corresponding uniqueness problem for the \( p \)-Laplace equation and the \( k \)-Hessian equations remains open.

When \( c > 0 \), we also see that the Wolff potential estimate (see, e.g., [L, TW4]) does not hold for the mean curvature equation, and an \( H_1 \)-subharmonic function may not be quasi-continuous, as the capacity of \( \partial B_1 \) is positive.
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