Black Holes and Super Black Holes
as
Chern Simons Theories in 2+1 Dimensions

Sharmanthie Fernando and Freydoon Mansouri

Physics Department, University of Cincinnati, Cincinnati, OH 45221

Abstract

We study anti-de Sitter black holes in 2+1 dimensions in terms of Chern Simons gauge theory of the anti-de Sitter group coupled to a source. Taking the source to be an anti-de Sitter state specified by its Casimir invariants, we show how all the relevant features of the black hole are accounted for. Enlarging the gauge symmetry to super AdS group, we obtain a supermultiplet of AdS black holes. We give explicit expressions for masses and the angular momenta of the members of the multiplet.

1 introduction

The AdS black hole in 2+1 dimensions is a solution of free Einstein’s equations with a negative cosmological constant [1]. It is well known that the free Einstein theory in 2+1 dimensions with or without a cosmological constant can be formulated as a free Chern Simons theory [2, 3]. In a free Chern Simons theory, the field strength vanishes identically, so that there are no local degrees of freedom. It is then somewhat surprising that a theory with no local degrees of freedom has a black hole solution which must have non-trivial degrees of freedom to account for its entropy! Presumably, M theory, once it is fully constructed, will provide us with the correct answer. In the meantime, one of the suggestions to resolve this issue is to modify the Chern Simons action with a WZW term [4, 5]. Another suggestion is to make use of the AdS/CFT correspondence [6]. More recently, it has been suggested that the answer lies in the coupling of the Chern Simons theory to a source [7]. In this approach, the free Chern

*email address: fernando@physung.phy.uc.edu
†email address: Mansouri@uc.edu
Simons theory is not only taken to be locally trivial. It is also taken to be globally
trivial in accord with Mach’s principle. Non-trivial topologies then arise as a result
of coupling to sources. This is one of the issues which we will address in this work.

One of the notable advantages of this approach is that it allows us to express the
asymptotic observables of the theory in terms of the properties of the sources. To
implement this idea, we must identify a localized source (particle) with an irreducible
representation of the gauge symmetry group \[8\]. For the present problem, this will
amount to relating the asymptotic observables of the BTZ black hole to the Casimir
invariants of an AdS state coupled to the Chern Simons action. We will show that
space-time will naturally emerge from such a gauge theory and will have all the
ingredients necessary for the AdS black hole [1, 9]. These include, in particular, the
discrete subgroup underlying the identifications.

A second issue which we will address in this work relates to the manner in which
black hole solutions fit into supersymmetric schemes. A conventional method of
searching for signs of supersymmetry in black hole solutions is to look for Killing
spinors. Many works along these lines already exist in the literature. We cite a
representative few here [10]- [18], from which more references can be traced. One
way to see whether a given black hole solution admits Killing spinors is identify
it with the bosonic part of an appropriate supergravity theory [10]- [18]. Then,
by requiring that the fermions in the theory as well as their variations vanish, one
arrives at Killing spinor equation(s). The asymptotic supersymmetries depend on
the number of non-trivial solutions of these equations consistent with the black hole
topology. For example, in asymptotically flat space-times, a typical supermultiplet
consists of a black hole and a number of ordinary particles all with the same mass. In
contrast to the familiar situation in particle physics, where we have Supermultiplets
consisting of particles only, in this approach there is no systematic way of looking for
supermultiplets consisting of black holes only. We will show that in 2 + 1 dimensions
it is possible to construct a theory which permits macroscopic solutions consisting of
all AdS black hole supermultiplets [19]. It involves the Chern Simons gauge theory
of the (1,1) super AdS group coupled to a super AdS state (source). As we shall
see below, to be able to accommodate the structure of the solution which emerges
from such a theory it becomes necessary to broaden the standard notions of classical
geometries to include some quantum mechanical elements.

2 Anti-de Sitter space and algebra

The anti-de Sitter space in 2+1 dimensions can be viewed as a subspace of a flat
4-dimensional space with the line element

\[ ds^2 = dX_A dX^A = dX_0^2 - dX_1^2 - dX_2^2 + dX_3^2 \] (1)
It is determined by the constraint
\[(X_0)^2 - (X_1)^2 - (X_2)^2 + (X_3)^2 = l^2\] (2)
where \(l\) is a real constant. The set of transformations which leave the line element invariant form the anti-de Sitter group \(SO(2, 2)\). It is locally isomorphic to \(SL(2, R) \times SL(2, R)\). From here on by anti-de Sitter group we shall mean its universal covering group.

With \(a = 0, 1, 2\), we can write the AdS algebra in two more convenient forms:
\[
\epsilon^{abc} J_c = \epsilon^{abc} (J^+_c + J^-_c)
\]
\[
\Pi^a = (J^+ - J^-)
\] (3)
where
\[
\epsilon^{012} = 1; \quad \eta^{ab} = (1, -1, -1)
\] (4)
Then, the commutation relations in, say, \(J^\pm_a\) basis take the form [7]
\[
[J^+_a, J^+_b] = -i\eta^{ab} J^+; \quad [J^-_a, J^-_b] = 0
\] (5)
The Casimir operators in this basis have the form
\[
j^2_\pm = \eta^{ab} J^\pm_a J^\pm_b
\] (6)
Alternatively, we can use a combination of these with eigenvalues corresponding to the parameters of the BTZ solution:
\[
M = l^2 (\Pi^a \Pi_a + l^{-2} J^a J_a) = 2(j^2_+ + j^2_-)
\]
\[
J/l = 2\Pi_a J^a = 2(j^2_+ - j^2_-)
\] (7)
Unless otherwise stated, we will use the same symbols for operators and their eigenvalues.

An irreducible representation of AdS group can be labeled by the eigenvalues of either the pair \((M, J)\) or the pair \((j_+, j_-)\). For application to black holes, it is also possible to label these representations by eigenvalues which are proportional to the horizon radii \(r_\pm\) of the AdS black hole [1, 7].

3 Connection and the Chern Simons action

To write down the Chern Simons action, we begin by writing the connection in \(SL(2, R) \times SL(2, R)\) basis.
\[
A_\mu = \omega^{AB}_\mu M^{AB} = \omega^a_\mu J_a + \epsilon^a_\mu \Pi_a = A^+_a J^+_a + A^-_a J^-_a
\] (8)
where
\[ A^{\pm a}_\mu = \omega^a_\mu \pm l^{-1}e^a_\mu \] (9)

Eq. 9 should be viewed as definitions of \( e \) and \( \omega \) in terms of the two \( SL(2, R) \) connections. The covariant derivative can be written as
\[ D_\mu = \partial_\mu - iA_\mu = \partial_\mu - iA^{+a}_\mu J^+_a - iA^{-a}_\mu J^-_a \] (10)

Then the components of the field strength are given by
\[ [D_\mu, D_\nu] = -iF^+_{\mu\nu}J^+_a - iF^-_{\mu\nu}J^-_a = -iF^+_{\mu\nu}[A^+] - iF^-_{\mu\nu}[A^-] \] (11)

For a simple or a semi-simple group, the Chern Simons action has the form
\[ I_{cs} = \frac{1}{4\pi} Tr \int_M A \wedge \left( dA + \frac{2}{3} A \wedge A \right) \] (12)

where \( Tr \) stands for trace and
\[ A = A_\mu dX^\mu = A^+ + A^- \] (13)

We require the 2+1 dimensional manifold \( M \) to have the topology \( R \times \Sigma \), with \( \Sigma \) a two-manifold. So, The Chern Simons action with \( SL(2, R) \times SL(2, R) \) gauge group will take the form
\[ I_{cs} = \frac{1}{4\pi} Tr \int_M \left[ \frac{1}{a_+} A^+ \wedge \left( dA^+ + \frac{2}{3} A^+ \wedge A^+ \right) + \frac{1}{a_-} A^- \wedge \left( dA^- + \frac{2}{3} A^- \wedge A^- \right) \right] \] (14)

Here the quantities \( a_\pm \) are, in general, arbitrary coefficients, reflecting the semisimplicity of the gauge group. Up to an overall normalization, only their ratio is significant. As explained elsewhere [7], in the presence of a source (or of sources), any \( \alpha \) priori choice of the coefficients \( a_\pm \) reduces the class of allowed holonomies, so that even the classical theory coupled to sources will be affected by such a choice. For this reason, we will keep the coefficients \( a_\pm \) as free parameters in the sequel, so that we can generate the correct holonomies for solutions both outside and inside the horizon.

Under infinitesimal gauge transformations
\[ u_\pm = \theta^{\pm a}_a J^\pm_a \] (15)

the gauge fields transform as
\[ \delta A_\mu = -\partial_\mu u - i[A_\mu, u] \] (16)

More specifically,
\[ \delta A^{\pm a}_\mu = -\partial_\mu \theta^{\pm a} - e^{a}_{bc} A^{\pm b} \theta^{\pm c} \] (17)
As we have stated, the manifold $M$ has the topology $R \times \Sigma$ with $R$ representing $x^0$. Then subject to the constraints

$$F^\pm_a[A^\pm] = \frac{1}{2} \eta_{ab} \epsilon^{ij} (\partial_i A_j^\pm b - \partial_j A_i^\pm b + \epsilon^{b}_{cd} A_i^\pm c A_j^\pm d) = 0$$

(18)

the Chern Simons action for $SO(2, 2)$ will take the form

$$2\pi I_{cs} = \frac{1}{a_+} \int_R dx^0 \int_\Sigma d^2 x \left( -\epsilon^{ij} \eta_{ab} A_i^\pm a \partial_0 A_j^\pm b + A_0^\pm a F^+_a \right)$$

$$+ \frac{1}{a_-} \int_R dx^0 \int_\Sigma d^2 x \left( -\epsilon^{ij} \eta_{ab} A_i^- a \partial_0 A_j^- b + A_0^- a F^-_a \right)$$

(19)

where $i, j = 1, 2$.

4 $SO(2, 2)$ States and Interaction with sources

Following the approach which has been successful in coupling sources to Poincaré Chern Simons theory [8], we take a source for the present problem to be an irreducible representation of anti-de-Sitter group characterized by Casimir invariants $M$ and $J$ (or $r_+$ and $r_-$). Within the representation, the states are further specified by the phase space variables of the source $\Pi^A$ and $q^A$, $A = 0, 1, 2, 3$, subject to anti-de Sitter constraint given by Eq. 2. To allow for the possibility of quantizing the Chern Simons theory consistently, we must require that our sources be represented by unitary representations of the AdS group. The choice of relevant representations from among these were discussed in reference [7]. Here we note that since the AdS group in 2+1 dimensions can be represented in the $SL(2, R) \times SL(2, R)$ form, the unitary representations of $SO(2, 2)$ can be constructed from those of $SL(2, R)$. The latter group has four series of unitary representations all of which are infinite dimensional. Of these, the appropriate representations turn out to be the discrete series bounded from below [4]. The states in an irreducible representation of $SL(2, R)$ are specified by the eigenvalues of its Casimir operator and one of the generators. In reference [4], the compact generator was chosen to label the states. This corresponds to inducing representations of $SL(2, R)$ via its maximal compact subgroup $SO(2)$. Thus, suppressing the superscripts $\pm$ which distinguish our two $SL(2, R)$’s, we can write

$$j^2|F, m > = F(F - 1)|F, m >$$

$$J_0|F, m > = (F + m)|F, m >$$

In these expressions

$$F = real \ number \geq 0; \quad m = 0, 1, 2, ...$$

(20)
So, for this series, the eigenvalues of the Casimir invariants of $SL(2, R) \times SL(2, R)$ can be written as,

$$j_{\pm}^2 = F_{\pm}^2 - F_{\mp} \tag{21}$$

It follows that the infinite set of states can, in a somewhat redundant notation, be specified as

$$|j_{\pm}^2, F_{\pm} + m_{\pm} >; \quad m_{\pm} = 0, 1, 2, ... \tag{22}$$

Clearly, the integers $m_{\pm}$ are not necessarily equal. Using these states, we can construct the discrete series of the unitary representations of $SO(2, 2)$. A typical state will have the following labels:

$$|M, J > = |j_{+}^2, j_{-}^2, F_+, F_- + m_+, F_- + m_- > \tag{23}$$

To be able to identify the labels $M$ and $J$ with the corresponding labels in the AdS black hole, we must require that $F_{\pm} \geq 1$ [7]. It would then follow that $|J/l| \leq |M|$, as required for having a black hole solution.

The main advantage of diagonalizing a compact generator is that the maximal compact subgroup $SO(2) \times SO(2)$ of $SO(2, 2)$ allows a parametrization of the AdS space in terms of circular functions so that the corresponding angular variable is periodic to begin with and will remain so in the black hole solution. As a result [7], the periodicity necessary to obtain the discrete identification group need not be imposed as an additional requirement as was done by BTZ [1]. However, to obtain the BTZ solution from this starting point, it becomes necessary [1] to perform a “Wick rotation” in the space of Casimir invariants to recover the hyperbolicity of the angular variable. An alternative to this procedure is to construct the $SL(2, R)$ representations by diagonalizing a non-compact generator. In that case, the irreducible representations of $SO(2, 2)$ can be constructed via induction from the subgroup $SO(1, 1) \times SO(1, 1)$.

One way to do this is to make the formal replacement $J_0 \rightarrow iJ_2$ to go from the formalism in which the compact generator $J_0$ is diagonal to one in which the non-compact generator $J_2$ is diagonal. Obviously, the structure and the classification of the unitary representations will remain the same. The only difference is that in the latter case the angular variables to be used to parametrize the AdS space are hyperbolic to begin with. Then, as we shall see below, the $2\pi$ periodicity of the hyperbolic angular variable will arise from the requirement of compatibility with the topological properties of the Chern Simons theory coupled to a source.

With these preliminaries, we can couple a source to the AdS Chern Simons theory in the following way:

$$I_s = \int_C d\tau \left[ \Pi_A \partial_\tau q^A - \left( A^{a+} J_a^+ + A^{-a} J_a^- \right) + \lambda \left( q^A q_A - l^2 \right) \right]$$

$$+ \int_C d\tau \left[ \lambda_+ \left( J^{a+} J_a^+ - l^2 j_+^2 \right) + \lambda_- \left( J^{-a} J_a^- - l^2 j_-^2 \right) \right] \tag{24}$$

In this expression, $C$ is a path in $M$, $\tau$ is a parameter along $C$, and $J_a^\pm$ play the role of c-number generalized angular momenta which transform in the same way as
the corresponding generators which label the source. The quantities $\lambda$ and $\lambda_{\pm}$ are Lagrange multipliers. The first constraint in this action ensures that $q^A(\tau)$ satisfy the AdS constraint. As explained in previous occasions [7, 8, 20], it is not the manifold $M$ over which the gauge theory is defined but the space of $q_A$’s which give rise to the classical space-time. The last two constraints identify the source being coupled to the Chern Simons theory as an anti-de Sitter state with invariants $j_+$ and $j_-$. These constraints are crucial in relating the invariants of the source to the asymptotic observables of the coupled theory via Wilson loops.

The total action for the theory is given by:

$$I = I_{cs} + I_s$$

(25)

It is easy to check that in this theory the components of the field strength still vanish everywhere except at the location of the sources. So, the analog of Eq. ? become

$$\epsilon^{ij} F_{ij}^\pm a = 2\pi a_{\pm} J_2^\pm (x, x_0)$$

(26)

In particular, fixing the gauge so that $SO(2,2)$ symmetry reduces to $SO(1,1) \times SO(1,1)$, we get

$$\epsilon^{ij} F_{ij}^\pm 2 = 2\pi a_{\pm} F_{\pm} \delta^2 (\bar{x}, \bar{x_0})$$

(27)

where $F_{\pm}$ are the invariant labels of the state as in Eq. 23, but now they are associated with the non-compact generator $J_2$. All other components of the field strength vanish. We thus see that because of the constraints appearing in the action given by Eq. 24, the strength of the sources become related to their Casimir invariants. These invariants, in turn, determine the asymptotic observables of the theory. Since such observables must be gauge invariant, they are expressible in terms of Wilson loops, and a Wilson loop about our source can only depend on, e.g., $j_+$ and $j_-$. From the data on the manifold $M$ given above, one can determine the properties of the emerging space-time by solving Eqs 27. The only non-vanishing components of the gauge potential are given by

$$A_\theta^{\pm} = 2a_{\pm} F_{\pm}$$

(28)

where $\theta$ is an angular variable. Although these are components of a connection which is pure gauge, they give rise to non-trivial holonomies around the source. More explicitly, we have

$$\omega[A^+] = exp^{\pi(r_-+r_+)} J_2^+$$

(29)

and

$$\omega[A^-] = exp^{\pi(r_--r_+)} J_2^-$$

(30)

Since we are diagonalizing $J_2^\pm$ operators, their matrix representation is given by

$$J_2^\pm(\alpha) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(31)
Then, the above holonomies will take the form

\[
\omega[A^+] = \begin{pmatrix}
\exp\pi(r_+ - r_-) & 0 \\
0 & \exp^{-\Pi(r_+ - r_-)}
\end{pmatrix}
\] (32)

\[
\omega[A^-] = \begin{pmatrix}
\exp\pi(r_+ - r_-) & 0 \\
0 & \exp^{-\Pi(r_+ - r_-)}
\end{pmatrix}
\] (33)

It was shown in reference [7] how these holonomies lead to a discrete identification subgroup of SO(2, 2), indicating that the manifold \(M_q\) of the 0 + 1 dimensional fields \(q^A\) has all the relevant features of the macroscopic AdS black hole solution. As we shall see below, the same holonomies, suitably interpreted, will play a crucial role in establishing the space-time structure of the supersymmetric theory discussed below.

5 The black hole space-time

To see how the space-time structure emerges from our anti-de Sitter gauge theory, we follow an approach which led to the emergence of space-time from Poincaré [8] and super Poincaré [20] Chern Simons gauge theories. We have emphasized that the manifold \(M\) is not to be identified with space-time. But the information encoded in \(M\) and discussed in the previous section is sufficient to fix the properties of the emerging space-time. To this end, let us consider a manifold \(\tilde{M}_q\) satisfying the AdS constraint

\[
\tilde{q}_0^2 - \tilde{q}_1^2 - \tilde{q}_2^2 + \tilde{q}_3^2 = l^2 = -\Lambda^{-1}
\] (34)

where \(\Lambda = \) cosmological constant. In fact, our \(SL(2, R) \times SL(2, R)\) formulation allows us to take \(\tilde{M}_q\) to be the universal covering space of the AdS space. As we shall see, the emerging space-time is the quotient of \(\tilde{M}_q\) by the discrete subgroup \(\Gamma\) discussed in the previous section. Moreover, the source coupled to the Chern Simons action is an AdS state characterized by the Casimir invariants \((M, J)\) or, equivalently, \(r_+, r_-)\).

To parametrize \(\tilde{M}_q\) consistent with the above constraint, consider a pair of 2-vectors,

\[
\tilde{q}_\phi = (q^0, q^1) = (f \cosh \phi, f \sinh \phi)
\] (35)

\[
\tilde{q}_t = (q^2, q^3) = \left(\sqrt{f^2 - l^2 \cosh(t/l)}, \sqrt{f^2 - l^2 \sinh(t/l)}\right)
\] (36)

where \(f = f(r)\), with \(r\) a radial coordinate which for an appropriate \(f(r)\) will become the radial coordinate appearing in the line element for the BTZ black hole. As far the constraint given by Eq. 34 is concerned, the functional form of \(f(r)\) is irrelevant. Computing the line element in terms of the parameters \((t/l, r, \phi)\), we get

\[
ds^2 = \left(\frac{f^2}{l^2} - 1\right) dt^2 - \frac{f^2 dr^2}{\left(\frac{f^2}{l^2} - 1\right)} - f^2 d\phi^2
\] (37)
where “prime” indicates differentiation with respect to \( r \).

Anticipating the results to be given below, let us compare this line element with that for the BTZ black hole \([1]\):

\[
ds^2 = \left[ \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right] dt^2 - \frac{dr^2}{\left[ \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \right]} - r^2 [d\phi - \frac{J^2}{2r^2} dt]^2 \tag{38}\]

If we identify the labels \( M \) and \( J \) with the Casimir invariants of an irreducible representation of the AdS group as discussed in the previous sections, we see that the line element given by Eq. 37 corresponds to an irreducible representation with \( J = 0 \) and \( M = 1 \). Such a parametrization cannot provide us with an AdS black hole with a continuous range of values for \( J \) and \( M \). What we need is a space-time manifold \( M_q \) in which the coordinates \( q^A \) carry an arbitrary irreducible representation of the AdS group. We will construct this manifold explicitly by performing appropriate gauge transformations on \( \hat{M}_q \). Although the original theory was invariant under \( SL(2, R) \times SL(2, R) \) gauge transformations, we have already reduced this symmetry by choosing to work in a gauge in which the left over symmetry is just \( SO(1, 1) \times SO(1, 1) \) generated, respectively, by \( J_2^\pm \). It turns out to be more convenient to work with generators

\[
J_2 = J_2^+ + J_2^-; \quad l\Pi_2 = J_2^+ - J_2^- \tag{39}\]

As was the case with the compact generators \( J_0 \) and \( \Pi_0 \), the non-compact operators \( J_2 \) and \( \Pi_2 \) generate a \( SO(1, 1) \times SO(1, 1) \) subgroup of the AdS group. We identifying the parameters \( \phi \) and \( t/l \), respectively, with each \( SO(1, 1) \) and proceed in the same manner as we did for the compact generators in reference \([1]\). In particular, consider the local gauge transformation \([1, 3, 20, 21, 22]\)

\[
\tilde{q}_{\phi}(\phi, t/l) = e^{\left( \frac{r_+ - r_-}{l^2} \right) J^2} \tilde{q}_{\phi}(\phi) \]

\[
\tilde{q}_t(t/l, \phi) = e^{\left( \frac{r_+ - r_-}{l^2} \right) \Pi^2} \tilde{q}_t(t/l) \tag{40}\]

It then follows that

\[
\tilde{q}_{\phi}(\phi + 2\pi, t/l) = e^{2\pi \frac{r_+ - r_-}{l^2} J^2} \tilde{q}_{\phi}(\phi, t/l) \]

\[
\tilde{q}_t(t/l, \phi + 2\pi) = e^{2\pi \frac{r_+ - r_-}{l^2} \Pi^2} \tilde{q}_t(t/l, \phi) \]

\[
\tilde{q}_{\phi}(\phi + 2\pi, t/l + 2\pi) = e^{2\pi \frac{r_+ - r_-}{l^2} J^2} \tilde{q}_{\phi}(\phi, t/l) \]

\[
\tilde{q}_t(t/l + 2\pi, \phi + 2\pi) = e^{2\pi \frac{r_+ - r_-}{l^2} \Pi^2} \tilde{q}_t(\phi, t/l) \tag{41}\]

The factors by which these quantities change as \( \phi \to (\phi + 2\pi) \) are reminiscent of the holonomies which we obtained in section 4. There we found that these holonomies in \( M \) led to a discrete identification group. To be consistent with this identification group, we must require that the hyperbolic variable \( \phi \) be periodic. Thus, the periodicity of \( \phi \) follows from the topology of \( M \) in the presence of a source. With this
provision, the vector \((\vec{q}', \vec{t}')\) transforms in the same way as the one which in section 4 was parallel transported around a loop in the manifold \(M\). Calling the manifold to which such vectors belong \(M_q\), we see that this manifold incorporates the same dynamics as the phase space variables in \(M\), and we are justified in using the same letter \(q\) for both. Thus, we can parametrize the manifold \(M_q\) as follows [1, 7, 9]:

\[
q^0 = f \cosh \left( \frac{r_+ - t}{l^2} \right)
\]

\[
q^1 = f \sinh \left( \frac{r_+ - t}{l^2} \right)
\]

\[
q^2 = \sqrt{f^2 - l^2} \cosh \left( \frac{r_+ - t}{l^2} \right)
\]

\[
q^3 = \sqrt{f^2 - l^2} \sinh \left( \frac{r_+ - t}{l^2} \right)
\]

From these we can compute the line element. It is easy to show that it is the same as that given by Eq. 38 for \(r > r_-\).

6 Supersymmetric Black Holes

The theory which we will describe below is the supersymmetric generalization of the theory which was discussed in the previous sections. We will see that the emerging macroscopic theory consists of a supermultiplet of ordinary space-times and, as a special case of this, a supermultiplet consisting of black holes only. The simplest way of obtaining a supersymmetric extension of the anti-de Sitter group is to begin with the AdS group in its \(SL(2, R) \times SL(2, R)\) basis. The \(N = 1\) supersymmetric extension of each \(SL(2, R)\) factor is the supergroup \(OSP(1|2; R)\). Thus, one arrives at the \((1,1)\) form of the \(N = 2\) super AdS group. Its algebra is given by

\[
[J^\pm_a, J^\pm_b] = -i \epsilon_{ab}^c J_c^\pm; \quad [J^\pm_a, Q^\pm_{a\alpha}] = -\sigma^a_{\alpha\beta} Q^\pm_{a\beta}; \quad \{Q^\pm_{a\alpha}, Q^\pm_{b\beta}\} = -\sigma^a_{\alpha\beta} J^\pm_a \\
\{Q^+_{a\alpha}, Q^-_{b\beta}\} = 0; \quad [J^+, J^-] = 0
\]

The Casimir invariants are given by

\[
C_\pm = j_\pm^2 + \epsilon^{\alpha\beta} Q^\pm_{a\alpha} Q^\pm_{a\beta}
\]

The spinor indices are raised and lowered by antisymmetric metric \(\epsilon^{\alpha\beta}\) defined by \(\epsilon^{12} = -\epsilon_{12} = 1\). The matrices \((\sigma^a)^\beta_{\alpha}, (a = 0, 1, 2)\), form a representation of \(SL(2, R)\) and satisfy the Clifford algebra

\[
\{\sigma^a, \sigma^b\} = \frac{1}{2} \eta^{ab}
\]

More explicitly, we can take them to be of the form:

\[
\sigma^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \sigma^1 = \frac{1}{2} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}; \quad \sigma^2 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
It is important to note that the supersymmetry generators of $OSp(1|2, R)$ do not commute with the Casimir invariant of its $SL(2, R)$ subgroup. That is,

$$[j^2, Q_\alpha] \neq 0 \quad (47)$$

Since super AdS group is semi-simple, we can construct its irreducible representations by first constructing the irreducible representations of $OSp(1|2, R)$. Depending on which $OSp(1|2, R)$ we are considering, the states within any such supermultiplet are the corresponding irreducible representations of $SL(2, R)$ characterized by the Casimir invariants $j_+$ or $j_-$, respectively. Based on the rationale given for the non-supersymmetric case [7], the irreducible representations of interest for the present case are those which can be obtained from the unitary discrete series of $SL(2, R)$ and which are bounded from below. To construct the supermultiplet corresponding to, say, the “plus” generators in Eq. 22, we can take the Clifford vacuum state $|\Omega^+\rangle$ to be the $SL(2, R)$ state with the lowest eigenvalue of $J^+_2$. In the notation of Eq. 22, this corresponds to an $m = 0$ state:

$$|F_+, m > = |F_+, m = 0 > = |F_+ >$$

Then, the superpartner of this state, again with $m = 0$, is the state $|F_+ + 1/2 >$ obtained by the application of one of the $Q$'s. The corresponding values of $j^2_+$ are $F_+(F_+ - 1)$ and $(F_+ + 1/2)(F_+ - 1/2)$, respectively. The supermultiplet for the second $OSp(1|2, R)$ can be constructed in a similar way.

We are now in a position to construct the (1,1) super AdS supermultiplet as a direct product of the two $OSp(1|2, R)$ doublets. Altogether, there will be four states in the supermultiplet. They will have the following labels:

$$|F_+, F_- >; \quad |F_+ + 1/2, F_- >; \quad |F_+, F_- + 1/2 >; \quad |F_+ + 1/2, F_- + 1/2 > \quad (48)$$

From these, we can also obtain the expressions for the eigenvalues $(M, J)$ of various states within the supermultiplet:

$$|M_1, J_1 > = |M, J >$$

$$|M_2, J_2 > = |M + 2F_+ - 1/2, J + 2F_+ - 1/2 >$$

$$|M_3, J_3 > = |M + 2F_- - 1/2, J - 2F_- + 1/2 >$$

$$|M_4, J_4 > = |M + 2(F_+ + F_-) - 1, J + 2(F_+ - F_-) > \quad (49)$$

these states transform into one another under supersymmetry transformations.

The Chern Simons action for simple and semisimple supergroups has the same structure as that for Lie groups. The only difference is that the trace operation is replaced by super trace (Str) operation. So, in the $OSp(1|2, R) \times OSp(1|2, R)$ basis the Chern Simons action for the super AdS group has the same form as that given by Eq. 14. But now the expression for connection is given by

$$A^\pm = \left[ A^\pm_\mu J^\pm_\alpha + \chi^\pm_\mu Q^\pm_\alpha \right] dx^\mu \quad (50)$$
Just as in the non-supersymmetric case, to have a non-trivial theory, we must couple sources to the Chern Simons action. To do this in a gauge invariant and locally supersymmetric fashion, we must take a source to be an irreducible representation of the super AdS group. As we saw above, such a supermultiplet consists of four AdS states. To couple it to the gauge fields, we must first generalize the canonical variables we used in the AdS theory to their supersymmetric forms \[13\]:

\[
\Pi_A \rightarrow (\Pi_A, \Pi_\alpha) \quad q_A \rightarrow (q_A, q_\alpha)
\] (51)

Then, the source coupling can be written as

\[
I_s = \int_C \left[ \Pi_A dq^A + \Pi_\alpha dq^\alpha + (A^+ + A^-) + \text{constraints} \right]
\] (52)

where again \(C\) is a path in \(M\). The constraints here include those discussed for the AdS group and, in addition, those which relate the AdS labels of the Clifford vacuum to the Casimir eigenvalues of the super AdS group. The combined action

\[
I = I_{cs} + I_s
\] (53)

leads to the constraint equations

\[
\epsilon^{ij} F^\pm_{ij} = 2\pi a_\pm \delta^2(\vec{x}, \vec{x}_0); \quad \epsilon^{ij} F^\pm_{ij} = 2\pi a_\pm Q^\pm_\alpha \delta^2(\vec{x}, \vec{x}_0)
\] (54)

Up to this point, everything proceeds in parallel with the non-supersymmetric case. Differences begin to show up when one attempts to solve these equations by choosing a gauge again so that the gauge symmetry is reduced to \(SO(1,1) \times SO(1,1)\):

\[
\epsilon^{ij} F^\pm = 2\pi a_\pm J^\pm_2 \delta^2(\vec{x}, \vec{x}_0)
\] (55)

Although this equation is identical in form to Eq. 27 for the non-supersymmetric case, there is an essential difference in the underlying physics. In the supersymmetric case, the supermultiplet which we couple to the Chern Simons action consist of four \(SO(2,2)\) states with different values of \(F^\pm\). As a result, in the parallel transport of \(q^A\) around a close path analogous to the non-supersymmetric case, there will be four sets of holonomies with different values of \((j_+, j_-)\) or, equivalently, \((r_+, r_-)\). Moreover, in the non-supersymmetric case, a single source with Casimir invariants \((r_+, r_-)\) or, equivalently, \((M, J)\) will give rise to an AdS black hole \[\text{[6]}\] for which the line element is characterized with the corresponding values of \(M\) and \(J\) and has the form given by Eq. \(38\). In the supersymmetric case, the source is a supermultiplet in which there are four states of differing \((M, J)\) values. Then, depending on which set \((M, J)\) that we choose, we will get a different BTZ solution. Since \(M\) and \(J\) are not invariant under supersymmetry transformations, these solutions are transformed into each other under supersymmetry. This makes it impossible for a single c-number line element of the type given by Eq. 38 to correspond to all the AdS states of a supermultiplet.
The situation here runs parallel to what was encountered in connection with super Poincaré Chern Simons theory [20]. There it was pointed out that standard classical geometries were not capable describing these structures and that one must make use of nonclassical geometries. Such geometries can be described in terms of three elements:

1. An algebra such as a Lie algebra or a Lie superalgebra.
2. A line element operator with values in this algebra.
3. A Hilbert space on which the algebra acts linearly.

For the problem at hand, the algebra of interest is the $N = (1, 1)$ super AdS algebra in $2 + 1$ dimensions. The corresponding Hilbert space is the representation space of the superalgebra given by Eq. 48. Then, instead of the BTZ line element given, we begin with a line element operator with values in the $N = (1, 1)$ superalgebra and assume that its diagonal elements depend on the algebra only through the Casimir operators $(\hat{M}, \hat{J})$ of its $SO(2, 2)$ subalgebra. The “hats” on top of $M$ and $J$ are meant to distinguish the operators from the corresponding eigenvalues. Thus, we have

$$ds^2 = ds^2(\hat{M}, \hat{J})$$

The matrix element of this operator for each state of the supermultiplet will produce a c-number line element:

$$< M_k, J_k | ds^2(\hat{M}, \hat{J}) | M_k, J_k > = ds^2(M_k, J_k)$$  \hspace{1cm} (56)

In other words, for each state of the supermultiplet, the nonclassical geometry produces a layer of classical space-time. The number of the layers is equal to the dimension of the supermultiplet. Supersymmetry transformations act as messengers linking different layers of this multilayered space-time.

An equivalent way of constructing the operator line element is to begin with the parametrization of $q^A$ given by Eq. 41. Then, replacing the Casimir eigenvalues in those expressions with the corresponding Casimir operators, we can proceed to compute the line element operator. The result will look like the line element given by Eq. 38, except that now Casimir eigenvalues are replaced with Casimir operators. This means that, for consistency, we must also interpret the quantities $J_\pm^2$ in Eq. 54 as operators. Acting on different states of the supermultiplet, they will give the corresponding $F_\pm$ eigenvalues. There will therefore be not one set but four sets of holonomies $W[A^+]$ and $W[A^-]$. Each set will produce the discrete identification subgroup in the corresponding layer of space-time.

Consider, next, the conditions under which every layer of the supermultiplet corresponds to an AdS black hole. For this to be true, we must have

$$M_k \geq 0; \quad |J_k| \leq lM_k$$

This, in turn implies that

$$F_+ \geq F_- \geq 1$$
In the notation of Eq. 48, for $|J| = lM$, two layers of the supermultiplet become extreme AdS black holes. The only exception is in the limiting case when $M = J = 0$, in which case there will be three extremal black holes in the supermultiplet. It is also interesting to note that for an appropriate choice of $M$ and $J$ or, equivalently, $F_+$ and $F_-$, the same supersource which generates a black hole in one layer can generate a naked singularity in another.

This work was supported, in part by the Department of Energy under the contract number DOE-FGO2-84ER40153. We would like to thank the Organizing Committee of Johns Hopkins Workshop for their hospitality and for the opportunity to present this material. We have also benefited from the hospitality of Aspen Center for Physics, where part of this work was carried out.

References

[1] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, Phys. Rev. D 48 (1993) 1506.

[2] A. Achucarro, P. Townsend, Phys. Lett.B 180 (1986) 35

[3] E. Witten, Nucl. Phys B311 (1988) 46; B323 (1989) 113

[4] S. Carlip, Phys. Rev. D51 (1995) 632

[5] A.P. Balachandran, L. Chandar, A. Momen, Nuc. Phys. B461 (1996) 581; eprint archive gr-qc/9506000

[6] A. Strominger

[7] S. Fernando, F. Mansouri, eprint archive hep-th/9804147, Int. Jour. Mod. Phys. A, in press

[8] F. Mansouri, M.K. Falbo-Kenkel, Mod. Phys Lett. A8 (1993) 2503; F. Mansouri, Proceedings of XIIth Johns Hopkins Workshop, ed. Z. Horwath, World Scientific, 1994

[9] S. Carlip, eprint archive gr-qc/9506079; Clas. Quan. Grav. 12 (1995) 2853

[10] P.C. Aichelburg, R. Guven, Phys. Rev. D 24 (1981) 2066

[11] R. Kallosh, A. Linde, T. Ortin, A. Peet, A. Van Proyen Phys. Rev. D 46 (1992) 5278
[12] O. Coussaert, M. Henneaux, eprint archive hep-th/9310194; Phys. Rev. Lett. 72 (1993) 183

[13] J.M. Izquierdo and P.K. Townsend, eprint archive gr-qc/9501018; Clas. Quan. Grav. 12 (1995) 895

[14] R. Kallosh, D. Kaster, T. Ortin, T. Torma, Phys. Rev. D 50 (1994) 6374

[15] M.J. Duff, J. Rahmfeld, Phys. Lett. B 345 (1995) 441; eprint archive hep-th/9605083; M.J. Duff, H. Lu, C.N. Pope, Phys. Lett. B 409 (1997) 136

[16] A.R. Steif, Phys. Rev. Lett. 69 (1995) 1849

[17] R. Kallosh, eprint archive hep-th/9503029; Phys. Rev. D 52 (1995) 1234

[18] T. Ortin, eprint archive hep-th/9705095

[19] S. Fernando, F. Mansouri, eprint archive hep-th/9809139

[20] Sunne Kim, F. Mansouri, Phys. Lett. B397 (1997) 81; F. Ardalan, S. Kim, F. Mansouri, Int. Jour. Mod. Phys., A12 (1997) 1183

[21] K. Koehler et al, Nucl. Phys. B348 (1990) 373

[22] C. Vaz and L. Witten, Phys. Lett. B327 (1994) 29