HAUSDORFF MEASURE OF ESCAPING SETS ON CERTAIN MEROMORPHIC FUNCTIONS

WENLI LI

ABSTRACT. We consider transcendental meromorphic function for which the set of finite singularities of its inverse is bounded. Bergweiler and Kotus gave bounds for the Hausdorff dimension of the escaping sets if the function has no logarithmic singularities over \( \infty \), the multiplicities of poles are bounded and the order is finite. We study the case of infinite order and find gauge functions for which the Hausdorff measure of escaping sets is zero or \( \infty \).

1. Introduction and main results

Suppose that \( f \) is a meromorphic function on the whole complex plane. Denote by \( f^n = f(f^{n-1}) \) the \( n \)-th iterate of \( f \), for a natural number \( n \). The Fatou set \( F(f) \) is defined as the set of all points with a neighborhood where the iterates \( f^n \) of \( f \) are defined and form a normal family. The Julia set \( J(f) \) is the complement of \( F(f) \), that is \( J(f) = \hat{\mathbb{C}} \setminus F(f) \), where \( \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) and the escaping set of \( f \) is

\[ I(f) = \{ z \in \mathbb{C} : f^n(z) \to \infty, \text{ as } n \to \infty \}. \]

It was shown that \( I(f) \neq \emptyset \) and \( J(f) = \partial I(f) \) by Eremenko [9] for entire \( f \) and by Domínguez [5] for meromorphic \( f \). We say that a meromorphic function \( f \) is in the Eremenko-Lyubich class \( \mathcal{B} \) if the set of finite singularities of its inverse function \( f^{-1} \) is bounded. The result \( I(f) \subset J(f) \) was proved for entire \( f \in \mathcal{B} \) by Eremenko-Lyubich [7] and by Rippon-Stallard [13] for meromorphic \( f \in \mathcal{B} \). The Hausdorff dimension of Julia sets and related sets are studied in many papers, see e.g. [11, 21] for surveys. As a comprehensive introduction to iteration theory of meromorphic functions we refer the readers to [2].

The order \( \rho(f) \) of a meromorphic function is defined by

\[ \rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}, \]

where \( T(r, f) \) denotes the Nevanlinna characteristic of \( f \), see [9, 14, 25]. Denote the Hausdorff dimension of a set \( A \subset \mathbb{C} \) by \( \dim(A) \) and the two-dimensional Lebesgue measure of \( A \) by \( \text{area}(A) \). For a subset \( A \subset \mathbb{C} \) and a gauge function \( h \), we denote by \( \mu_h(A) \) the Hausdorff measure of \( A \) with respect to \( h \). The specific definition is given by (2.1) in the next section, where we also give more information about the gauge function.

Barański [1] and Schubert [20] proved that if an entire function \( f \in \mathcal{B} \) and \( \rho(f) < \infty \), then \( \dim(J(f)) = 2 \). Actually they proved that \( \dim(I_R(f)) = 2 \) for all \( R > 0 \), where

\[ I_R(f) = \{ z \in \mathbb{C} : \liminf_{n \to \infty} |f^n(z)| \geq R \} \].

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and \( I_R(f) \subset J(f) \) for large \( R \). It was pointed out by Bergweiler and Kotus in \cite{3} that for meromorphic functions in \( B \) which have finite order and for which \( \infty \) is an asymptotic value the same conclusion holds. Assume that \( \infty \) is not an asymptotic value and that there exists \( M \in \mathbb{N} \) such that the multiplicity of all poles, except finitely many, is at most \( M \). In the same paper they proved that for such a function, the Hausdorff dimension of its escaping set is no more than \( 2M\rho/(2 + M\rho) \), where \( \rho \) is the order of \( f \).

If \( f \) is as above but of infinite order, then the area of \( I(f) \) is zero, yet there is an example \cite{3 section 6} with \( \text{dim}(I(f)) = 2 \). McMullen \cite{12} proved that the Julia set of \( \lambda e^z \) has Hausdorff dimension two but in the presence of an attracting periodic cycle its area is zero. He further remarked that \( \mu_h(J(\lambda e^z)) = \infty \) for \( h(r) = r^2 \log^n(1/r) \), for arbitrary \( n \in \mathbb{N} \). Peter \cite{17} gave a fairly precise description of the gauge functions \( h \) for which \( \mu_h(J(\lambda e^z)) = \infty \).

An example is constructed in section 5 to prepare for the proof of Theorem 1.2 in section which play an important role in our proof. Section 4 is to give the proof of Theorem 1.1.

Essential properties related to the gauge function. Afterwards we recall several lemmas which play an important role in our proof. Section 3 is to give the proof of Theorem 1.1. An example is constructed in section 5 to prepare for the proof of Theorem 1.2 in section 6.

2. Hausdorff Measure and Gauge Function

For \( \alpha > 0 \) we say that \( h: (0, \alpha] \to (0, +\infty) \) is a gauge function if it is continuous, increasing and satisfies \( \lim_{t \to 0} h(t) = 0 \). An example is the function that we defined in
For a set $A \subset \mathbb{C}$, we call a sequence $(A_j)_{j \in \mathbb{N}}$ of sets $A_j \subset \mathbb{C}$ a $\delta$-cover of $A$ if

$$A \subset \bigcup_{j=1}^{\infty} A_j,$$

and

$$\text{diam}(A_j) \leq \delta$$

for all $j \in \mathbb{N}$, where $\text{diam}(A) = \sup_{x,y \in A} |x-y|$ denotes the diameter. The diameter with respect to the spherical metric will be denoted by $\text{diam}_\chi(A)$.

Let $h$ be a gauge function. The measure $\mu_h$ defined by

$$(2.1) \quad \mu_h(A) = \lim_{\delta \to 0} \inf \left\{ \sum_{j=1}^{\infty} h(\text{diam} A_j) : (A_j)_{j \in \mathbb{N}} \text{ a } \delta\text{-cover of } A \right\},$$

is called the Hausdorff measure corresponding to the function $h$. For more details about Hausdorff measure we refer to Rogers [19] and Falconer [8, chapter 2].

We are going to show some properties of interest for the gauge function that we choose, which also take part in our following proofs. First we prove the following results.

**Lemma 2.1.** Let $c > 1$ and $h(t)$ be defined as in (1.1), then

$$(2.2) \quad h(ct) \leq c^2 h(t).$$

**Proof.** With the definition we have

$$h(ct) = (ct)^2 \left( \log^n \frac{1}{ct} \right)^\gamma \leq c^2 t^2 \left( \log^n \frac{1}{t} \right)^\gamma = c^2 h(t).$$

□

**Lemma 2.2.** Let $n \in \mathbb{N}$, $l \geq 1$ be a positive integer and $t_1, t_2, \ldots, t_l$ be real numbers. If $0 < t_j \leq 1/\exp^n 2$, $j = 1, 2, \ldots, l$, then we have

$$(2.3) \quad \log^n \left( \frac{1}{t_1 t_2 \cdots t_l} \right) \leq \left( \log^n \frac{1}{t_1} \right) \left( \log^n \frac{1}{t_2} \right) \ldots \left( \log^n \frac{1}{t_l} \right).$$

**Proof.** We denote $t_{j_0} = \min\{t_j, 1 \leq j \leq l\}$. Consider first the case that $n = 1$ and let $u_1 = \log(1/t_{j_0})$. Then we have

$$(2.4) \quad \log \frac{1}{t_1 t_2 \cdots t_l} = \sum_{j=1}^{l} \log \frac{1}{t_j} \leq u_1 l.$$

On the other hand noting that $1/t_j \geq \exp^n 2 = \exp 2$,

$$(2.5) \quad \left( \log \frac{1}{t_1} \right) \left( \log \frac{1}{t_2} \right) \ldots \left( \log \frac{1}{t_l} \right) \geq 2^{l-1} \log \frac{1}{t_{j_0}} = u_1 2^{l-1}.$$

Since $l \leq 2^{l-1}$ for $l \geq 1$, we deduce from (2.4) and (2.5) that (2.3) holds for $n = 1$ and $t_j \leq e^{-2}$, $j = 1, 2, \ldots, l$, that is,

$$(2.6) \quad \log \left( \frac{1}{t_1 t_2 \cdots t_l} \right) \leq \left( \log \frac{1}{t_1} \right) \left( \log \frac{1}{t_2} \right) \ldots \left( \log \frac{1}{t_l} \right).$$
We now prove the conclusion by induction. Suppose that (2.3) holds for \( n = k \) and \( t_j \leq 1/\exp^k 2 \) for \( j = 1, 2, \ldots, l \), which is,

\[
(2.7) \quad \log^k \left( \frac{1}{t_1 t_2 \cdots t_l} \right) \leq \left( \log^k \frac{1}{t_1} \right) \left( \log^k \frac{1}{t_2} \right) \cdots \left( \log^k \frac{1}{t_l} \right).
\]

Suppose that \( t_j \leq 1/\exp^{k+1} 2 \) for \( j = 1, 2, \ldots, l \). Therefore \( 1/\log^k (1/t_j) \leq e^{-2} \). Then from (2.7) and (2.6) we obtain

\[
\log^{k+1} \left( \frac{1}{t_1 t_2 \cdots t_l} \right) = \log \log^k \left( \frac{1}{t_1 t_2 \cdots t_l} \right)
\leq \frac{1}{(\log^{k+1} \frac{1}{t_1}) \left( \log^{k+1} \frac{1}{t_2} \right) \cdots \left( \log^{k+1} \frac{1}{t_l} \right)},
\]

from which we see that (2.3) holds for \( n = k + 1 \) if \( t_j \leq 1/\exp^{k+1} 2, j = 1, 2, \ldots, l \). \( \square \)

**Lemma 2.3.** Let \( n, l \in \mathbb{N} \) and \( l \geq 1 \). Set \( h(t) \) as in (1.1). Suppose that \( t_j \leq 1/\exp^n 2 \), for \( 1 \leq j \leq l \). Then we have

\[
(2.8) \quad h(t_1 t_2 \cdots t_j) \leq \prod_{j=1}^{l} h(t_j).
\]

**Proof.** Since \( t_j \leq 1/\exp^n 2 \) for \( 1 \leq j \leq l \) we deduce from (1.1) and (2.3) that

\[
h(t_1 t_2 \cdots t_l) = (t_1 t_2 \cdots t_l)^2 \left( \log^n \frac{1}{t_1 t_2 \cdots t_l} \right)\gamma
\leq (t_1 t_2 \cdots t_l)^2 \left( \log^n \frac{1}{t_1} \right) \left( \log^n \frac{1}{t_2} \right) \cdots \left( \log^n \frac{1}{t_l} \right)\gamma
= t_1^2 \left( \log^n \frac{1}{t_1} \right)\gamma t_2^2 \left( \log^n \frac{1}{t_2} \right)\gamma \cdots t_l^2 \left( \log^n \frac{1}{t_l} \right)\gamma
= h(t_1) h(t_2) \cdots h(t_l).
\]

Therefore we obtain (2.8). \( \square \)

**Lemma 2.4.** Suppose that \( h(t) \) is defined as in (1.1) for \( \gamma > 0 \) and \( n \in \mathbb{N} \). Define the function \( G(t) = h(\sqrt{t}) \). Then \( G(t) \) is increasing and concave on \( (0, \delta_n^2] \), where \( \delta_n = 1/\exp^n \gamma \).

**Proof.** According to the definition and (1.1) we have

\[
G(t) = t \left( \log^n \frac{1}{\sqrt{t}} \right)\gamma.
\]
Thus
\[ G'(t) = \left( \log^n \frac{1}{\sqrt{t}} \right)^\gamma + t \left( \left( \log^n \frac{1}{\sqrt{t}} \right)^\gamma \right)'. \]

\[ = \left( \log^n \frac{1}{\sqrt{t}} \right)^\gamma \frac{1}{2} \gamma \left( \frac{\log^n \frac{1}{\sqrt{t}}}{\log^{n-1} \frac{1}{\sqrt{t}}} \right) \cdots \left( \frac{\log \frac{1}{\sqrt{t}}}{\log 1} \right) \]

\[ = \left( \log^n \frac{1}{\sqrt{t}} \right)^\gamma \left( 1 - \frac{\gamma}{2} \frac{\log^n \frac{1}{\sqrt{t}}}{\log^{n-1} \frac{1}{\sqrt{t}}} \cdots \log \frac{1}{\sqrt{t}} \right). \]

(2.9)

If \( t \leq \delta_n^2 = (1/\exp^n(\gamma))^2 \) then
\[ \left( \log^n \frac{1}{\sqrt{t}} \right) \cdots \log \frac{1}{\sqrt{t}} \geq \gamma \exp(\gamma) \cdots \exp^{n-1}(\gamma) \geq \gamma > \frac{\gamma}{2}, \]
which yields with (2.9) that
\[ G'(t) \geq 0. \]

Hence \( G(t) \) is increasing. One may also find that \( G'(t) \) is decreasing on \( (0, \delta_n^2] \) with a short observation of (2.9). And therefore \( G(t) \) is a concave function on \( (0, \delta_n^2] \).

\[ \Box \]

3. Notations and lemmas

The following lemma is known as Iversen’s theorem, see e.g. [14, chapter 5].

**Lemma 3.1.** Let \( f \) be a transcendental meromorphic function for which \( \infty \) is not an asymptotic value. Then \( f \) has infinitely many poles.

We recall Koebe’s theorem, which is usually stated only for univalent functions defined in the open unit disk, see [18, Theorem 1.6], but the following version follows immediately from this special case, see [3, Lemma 2.1].

For \( a \in \mathbb{C} \) and \( r > 0 \) we use the notation \( D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \} \).

**Lemma 3.2.** Let \( g : D(a, r) \to \mathbb{C} \) be univalent, \( 0 < \lambda < 1 \) and \( z \in D(a, \lambda r) \). Then

\[ \frac{\lambda}{(1 + \lambda)^2} \leq \frac{|g(z) - g(a)|}{|(z - a)g'(a)|} \leq \frac{\lambda}{(1 - \lambda)^2}, \]

(3.1)

\[ \frac{1 - \lambda}{(1 + \lambda)^3} \leq \frac{|g'(z)|}{|g'(a)|} \leq \frac{1 + \lambda}{(1 - \lambda)^3}, \]

(3.2)

and

\[ g(D(a, r)) \supset D \left( g(a), \frac{1}{4} |g'(a)| r \right). \]

(3.3)

Rippon-Stallard [13, Lemma 2.1] proved the following result while Bergweiler-Kotus [3, Lemma 2.2] made a supplement.

Designate \( B(R) = \{ z \in \mathbb{C} : |z| > R \} \cup \{ \infty \} \).

**Lemma 3.3.** Let \( f \in \mathcal{B} \) be transcendental. If \( R > 0 \) such that \( \text{sing}(f^{-1}) \subset D(0, R) \), then all components of \( f^{-1}(B(R)) \) are simply-connected. Moreover, if \( \infty \) is not an asymptotic value of \( f \) then all components of \( f^{-1}(B(R)) \) are bounded and contain exactly one pole of \( f \).
We continue with Jensen’s inequality [16, p.12], one of the crucial tools used in our proof.

**Lemma 3.4.** Suppose that $I$ is an interval and the function $f: I \to \mathbb{R}$ is concave. For any points $x_1, x_2, \cdots, x_n \in I$ and any real nonnegative numbers $r_1, r_2, \cdots, r_n$ such that $r_1 + r_2 + \cdots r_n = 1$, we have

$$f \left( \sum_{j=1}^{n} r_j x_j \right) \geq \sum_{j=1}^{n} r_j f(x_j).$$

The next lemma from Jank-Volkmann [10, p.103] shows the relation between the $n$-th order and its number of poles for a meromorphic function.

**Lemma 3.5.** Suppose that $f$ is a meromorphic function and its $n$-th order is defined as in Section 1 and that $n(r)$ denotes the number of the poles contained in the closed disc $D(0, r)$. Then we have

$$\rho_n(f) \geq \limsup_{r \to \infty} \frac{\log^{n+1} n(r)}{\log r}.$$

For $k \in \mathbb{N}$, let $N_k$ be a collection of disjoint compact sets in $\mathbb{R}^n$ such that

(a) every element of $N_k$ contains an element of $N_{k+1}$,

(b) every element of $N_{k+1}$ is contained in an element of $N_k$.

Let $\mathcal{N}_k = \bigcup_{A \in N_k} A$ and $\mathcal{N} = \bigcap_{k=1}^{\infty} \mathcal{N}_k$.

McMullen [12] gave a lower bound for the Hausdorff dimension of a set $N$ constructed this way. Peter [17, p.33] used McMullen’s method to obtain a sufficient condition for the set $N$ to have infinite Hausdorff measure with respect to some gauge function $h$. We mention that they both worked with the Euclidean distance but the following lemma follows directly from the original one.

For measurable subsets $X, Y$ of the plane (or sphere) we define the Euclidean and the spherical density of $X$ in $Y$ by

$$\text{dens}(X, Y) = \frac{\text{area}(X \cap Y)}{\text{area}(Y)} \quad \text{and} \quad \text{dens}_\chi(X, Y) = \frac{\text{area}_\chi(X \cap Y)}{\text{area}_\chi(Y)}.$$  

Note that

$$\left( \frac{1 + R^2}{1 + S^2} \right)^2 \text{dens}(X, Y) \leq \text{dens}_\chi(X, Y) \leq \left( \frac{1 + S^2}{1 + R^2} \right)^2 \text{dens}(X, Y),$$

if $Y$ is a subset of the annulus $\{ z \in \mathbb{C} : R < |z| < S \}$.

With this terminology Peter’s result takes the following form.

**Lemma 3.6.** For $k \in \mathbb{N}$, let $N_k$, $N$ be as above. Suppose that $\Delta_k > 0$, $d_k > 0$, $d_k \to 0$, such that if $B \in N_k$, then

$$\text{dens}_\chi(N_{k+1}, B) \geq \Delta_k \quad \text{and} \quad \text{diam}_\chi B \leq d_k.$$  

Set $h(t) = t^2 g(t)$ for $t > 0$, where $g(t)$ is a decreasing continuous function such that $h(t)$ is increasing and satisfies $\lim_{t \to 0} t^2 g(t) = 0$. Then we have $\mu_h(N) = \infty$ if

$$\lim_{k \to \infty} g(d_k) \prod_{j=1}^{k} \Delta_j = \infty.$$
4. Proof of theorem 1.1

We follow the method used in [3, Section 3] with some modifications.

With the assumption and Lemma 3.1, $f$ has infinitely many poles, say denoted by $a_j$ and ordered such that $|a_j| \leq |a_{j+1}|$ for all $j \in \mathbb{N}$. Let $m_j$ be the multiplicity of $a_j$. Thus for some $b_j \in \mathbb{C} \setminus \{0\}$,

$$f(z) \sim \left(\frac{b_j}{z-a_j}\right)^{m_j} \text{ as } z \to a_j.$$  

We may assume that $|a_j| \geq 1$ for all $j$. Choose $R_0 > 1$ such that $\text{sing}(f^{-1}) \subset D(0, R_0)$ and $|f(0)| < R_0$.

If $R \geq R_0$, then all the components of $f^{-1}(B(R))$ are bounded and simply-connected and each component contains exactly one pole by Lemma 3.3. Let $U_j$ be the component containing $a_j$. By the Riemann mapping theorem we may choose a conformal map

$$\phi_j: U_j \to D(0, R^{-1/m_j})$$

satisfying the normalization $\phi_j(a_j) = 0$ and $\phi'_j(a_j) = 1/b_j$, see [3] for the details.

Denote the inverse function of $\phi_j$ by $\psi_j$. Since $\psi_j(0) = a_j$ and $\psi'_j(0) = b_j$ we can deduce from (3.3) that

$$U_j = \psi_j \left(D(0, R^{-1/m_j})\right) \supset D \left(a_j, \frac{1}{4} |b_j| R^{-1/m_j}\right) \supset D \left(a_j, \frac{1}{4R} |b_j|\right).$$

Since $|f(0)| < R$ we have $0 \notin U_j$. Then (4.1) implies that

$$\frac{1}{4R} |b_j| \leq |a_j|$$

for all $R \geq R_0$. Hence

$$|b_j| \leq 4R_0 |a_j|.$$  

Note that $\psi_j$ actually extends to a map univalent in $D(0, R^{-1/m_j})$. Choosing $R \geq 2^M R_0$ we can apply (3.3) with

$$\lambda = (R/R_0)^{-1/m_j} = (R_0/R)^{1/m_j} \leq \frac{1}{2}$$

and obtain

$$U_j \subset D \left(a_j, 2|b_j| R^{-1/M}\right),$$

provided $j$ is so large that $m_j \leq M$. With (4.1) and (4.3) we see that

$$D \left(a_j, \frac{1}{4R} |b_j|\right) \subset U_j \subset D \left(a_j, 2R^{-1/M} |b_j|\right)$$

for large $j$. Combining (4.2) and (4.3) and choosing $R \geq (16R_0)^M$ we have

$$U_j \subset D \left(a_j, \frac{1}{2} |a_j|\right) \subset D \left(0, \frac{3}{2} |a_j|\right).$$

Let $n(r)$ denote the number of $a_j$ contained in the closed disc $\overline{D(0, r)}$. Since the $U_j$ are pairwise disjoint we see with (4.1) and (4.4) that

$$\sum_{j=1}^{n(r)} |b_j|^2 \leq 36 R^2 r^2,$$
Let $D \subset B(R) \setminus \{\infty\}$ be a simply connected domain. Then any branch of the inverse of $f$ defined in a subdomain of $D$ can be continued analytically to $D$. Let $g_j$ be a branch of $f^{-1}$ that maps $D$ to $U_j$. Thus

\begin{equation}
(4.6) \quad g_j(z) = \psi_j \left( \frac{1}{z^m} \right),
\end{equation}

for some branch of the $m_j$-th root. Since we assumed that $R \geq 2^M R_0$ we deduce from (3.1) with $\lambda = 1/2$ that

\begin{equation}
(4.7) \quad |g_j'(z)| \leq \frac{12|b_j|}{m_j |z|^{1 + \frac{1}{m_j}}} \leq \frac{12|b_j|}{|z|^{1 + \frac{1}{M}}},
\end{equation}

for $z \in D \subset B(R) \setminus \{\infty\}$, provided $j$ is so large that $m_j \leq M$. Moreover, if $U_k \subset B(R)$ with (4.3) and (4.4) we have

\begin{equation}
\text{diam} g_j(U_k) \leq \sup_{z \in U_k} |g_j'(z)| \text{diam } U_k \leq 2^{1 + \frac{1}{M}} 12 \frac{4}{R^{\frac{1}{M}}} \frac{|b_j|}{|a_j|^{1 + \frac{1}{M}}}|b_k|.
\end{equation}

By induction if $U_{j_1}, U_{j_2}, \ldots, U_{j_l} \subset B(R)$, then with (4.3) and transferring to the spherical distance we have (see [3 equation (3.10)])

\begin{equation}
(4.8) \quad \text{diam}_{\chi}(g_{j_1} \circ g_{j_2} \circ \ldots \circ g_{j_{l-1}})(U_{j_l})) \leq (2^{1 + \frac{1}{M}} 12)^{l-1} \frac{32}{R^{\frac{1}{M}}} \prod_{k=1}^{l} \frac{|b_{j_k}|}{|a_{j_k}|^{1 + \frac{1}{M}}}.\end{equation}

Before we continue we shall prove the following result. This corresponds to [3 lemma 3.1], dealing with gauge functions of the form $h(t) = t^\alpha$. These gauge functions are estimated using Hölder’s inequality. Instead, here we consider the gauge functions defined by (1.1) and use the results of section 2 to estimate them.

**Lemma 4.1.** Let $h$ be defined as in (1.1). If $\gamma < 2/(M \rho)$ then

\begin{equation}
(4.9) \quad \sum_{j=1}^{\infty} h \left( \frac{|b_j|}{|a_j|^{1 + \frac{1}{M}}} \right) < \infty.
\end{equation}

**Proof.** For $l \geq 0$, we put

\[ P_l = \left\{ j \in \mathbb{N}: n(2^l) \leq j < n(2^{l+1}) \right\} = \left\{ j \in \mathbb{N}: 2^l \leq |a_j| < 2^{l+1} \right\}. \]

Denote by Card $P_l$ the cardinality of $P_l$ and put

\[ c_j = \left( \frac{|b_j|}{|a_j|^{L}} \right)^2, \]
where \( L = 1 + 1/M \). With (4.5) we obtain
\[
\sum_{j \in P_l} c_j = \sum_{j \in P_l} \frac{|b_j|^2}{|a_j|^{2L}} \\
\leq 2^{-2lL} \sum_{j \in P_l} |b_j|^2 \\
\leq 2^{-2lL} \sum_{j=1}^{n(2^{l+1})} |b_j|^2 \\
\leq 2^{-2lL} 36R^2 2^{2(l+1)}
\]

Thus
\[
\sum_{j \in P_l} c_j \leq K2^{-\frac{2l}{M}},
\]
where \( K = 144R^2 \). Set
\[
S_l = \sum_{j \in P_l} h \left( \frac{|b_j|}{|a_j|^{1+\frac{1}{M}}} \right)
\]
and
\[
G(t) = h(\sqrt{t}).
\]
Then
\[
S_l = \sum_{j \in P_l} G(c_j).
\]

Let \( \delta_n = 1/\exp^n(\gamma) \) as in Lemma 2.4

**Case 1.** Suppose that
\[
K2^{-\frac{2l}{M}} \geq \delta_n^2.
\]
Then
\[
\text{Card } P_l \leq \delta_n^{-2} K2^{-\frac{2l}{M}} < 1, \quad \text{as } l \to \infty.
\]
Thus \( P_l = \emptyset \) for large \( l \). For such \( l \) we have \( S_l = 0 \).

**Case 2.** Suppose that
\[
K2^{-\frac{2l}{M}} < \delta_n^2.
\]
Then from (4.10) we have
\[
\frac{\sum_{j \in P_l} c_j}{\text{Card } P_l} \leq \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l}.
\]

Hence by Lemma 2.4
\[
(4.12) \quad G \left( \frac{\sum_{j \in P_l} c_j}{\text{Card } P_l} \right) \leq G \left( \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l} \right).
\]

Applying Lemma 3.4 to \( G(t) \) with \( r_j = 1/\text{Card } P_l \) and \( x_j = c_j \) for \( j \in P_l \) we obtain
\[
G \left( \frac{\sum_{j \in P_l} c_j}{\text{Card } P_l} \right) \geq \frac{\sum_{j \in P_l} G(c_j)}{\text{Card } P_l}.
\]
This together with (4.10), (4.11) and (4.12) give,
\[
S_l \leq (\text{Card } P_l) G \left( \frac{\sum_{j \in P_l} c_j}{\text{Card } P_l} \right) \\
\leq (\text{Card } P_l) G \left( \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l} \right) \\
= (\text{Card } P_l) \frac{K2^{-\frac{2l}{M}}}{\text{Card } P_l} \left( \log^n 1 \left( \sqrt{n} \sqrt{K2} - \frac{2l}{M} \right) \right)^\gamma \\
= K2^{-\frac{2l}{M}} \left( \log^n \frac{\sqrt{\text{Card } P_l}}{\sqrt{K2} - \frac{2l}{M}} \right)^\gamma.
\]
(4.13)

Lemma 3.5 implies for \( \varepsilon > 0 \) that
\[
\text{(4.14) Card } P_l \leq n(2l+1) \leq \exp^n \left( (2l+1)^{\rho+\varepsilon} \right),
\]
for large \( l \). Then (4.13) and (4.14) give,
\[
S_l \leq K2^{-\frac{2l}{M}} \left( \log^n \frac{\sqrt{\text{Card } P_l}}{\sqrt{K2} - \frac{2l}{M}} \right)^\gamma \\
= K2^{-\frac{2l}{M}} \left( \log^n \frac{\sqrt{\text{Card } P_l}}{\sqrt{K2} - \frac{2l}{M}} \right)^\gamma \\
\leq K2^{-\frac{2l}{M}} \left( \log^n \left( \exp^n \left( 2^{\rho+\varepsilon} \gamma \right) \right) \right)^\gamma \\
\leq K2^{-\frac{2l}{M}} 2^{(\rho+\varepsilon)\gamma} \gamma 2^{-l(\frac{2l}{M}-(\rho+\varepsilon)\gamma)} \\
= K2^{(\rho+\varepsilon+1)\gamma} - (\rho + \varepsilon)\gamma > 0,
\]
for \( n \geq 2 \) and \( l \) large. If \( \gamma < 2/(M(\rho + \varepsilon)) \), then

\[
\frac{2}{M} - (\rho + \varepsilon)\gamma > 0,
\]
which implies the series \( \sum_{l=0}^{\infty} S_l \) converges. The conclusion follows as \( \varepsilon \to 0 \). \[ \square \]

We continue the proof by denoting \( E_l \) the collections of all components \( V \) of \( f^{-l}(B(R)) \) for which \( f^k(V) \subset U_{j_{k+1}} \subset B(R) \) for \( k = 0, 1, ..., l - 1 \). For \( V \in E_l \), there exist \( j_1, j_2, \cdots, j_l \geq n(R) \) such that
\[
V = (g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_{l-1}})(U_{j_l}).
\]

From (4.8) we have
\[
\text{diam}_r(V) \leq (2^{1+\frac{2l}{M}} 12)l^{-1} \frac{32}{R^{1/M}} \prod_{k=1}^{l} \frac{|b_{j_k}|}{|a_{j_k}|^{l+\frac{1}{M}}}. \tag{4.15}
\]
It is easy to see from (4.2) that for \( R \) large,
\[
2^{1+\frac{2l}{M}} 12 \frac{|b_{j_k}|}{|a_{j_k}|^{l+\frac{1}{M}}} \leq \frac{1}{\exp^n 2}
\]
and
\[ \frac{32}{R^\pi} \leq 2^{1+\frac{\gamma}{\pi}} 12. \]

Since there are \( m_{jk} \) branches of \( f^{-1} \) mapping \( U_{jk+1} \) into \( U_{jk} \) for \( k = 1, 2, \ldots, l - 1 \), we conclude that there are
\[ \prod_{k=1}^{l-1} m_{jk} \leq M^{l-1} \]
sets of diameters bounded as in (4.8) which cover all those components \( V \) of \( f^{-l}(B(R)) \) for which \( f^k(V) \subset U_{jk+1} \subset B(R) \) for \( k = 0, 1, \ldots, l - 1 \).

Now we may apply Lemma 2.3 which together with (4.15) gives,
\[ \sum_{V \in E_l} h(\text{diam}_V) \leq M^{l-1} \sum_{j_1=\nu(R)}^{\infty} \ldots \sum_{j_l=\nu(R)}^{\infty} h\left(2^{1+\frac{\gamma}{\pi}} 12^{l-1} \frac{32}{R^{1+\gamma}} \prod_{k=1}^{l} \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{\gamma}{\pi}}} \right) \]
\[ \leq M^{l-1} \sum_{j_1=\nu(R)}^{\infty} \ldots \sum_{j_l=\nu(R)}^{\infty} \prod_{k=1}^{l} h\left(2^{1+\frac{\gamma}{\pi}} 12 \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{\gamma}{\pi}}} \right) \]
\[ = \frac{1}{M} \left(M \sum_{j=\nu(R)}^{\infty} h\left(2^{1+\frac{\gamma}{\pi}} 12 \frac{|b_{j_k}|}{|a_{j_k}|^{1+\frac{\gamma}{\pi}}} \right) \right)^l \]
for \( R \) large enough.

We can get from (2.2) and Lemma 2.1 that if \( \gamma < 2/(M\rho) \),
\[ M \sum_{j=\nu(R)}^{\infty} h\left(2^{1+\frac{\gamma}{\pi}} 12 \frac{|b_{j}|}{|a_{j}|^{1+\frac{\gamma}{\pi}}} \right) \leq M \left(2^{1+\frac{\gamma}{\pi}} 12 \right)^2 \sum_{j=\nu(R)}^{\infty} h\left(\frac{|b_{j}|}{|a_{j}|^{1+\frac{\gamma}{\pi}}} \right) < 1, \]
for \( R \) large. For such \( R \) we find that
\[ \lim_{l \to \infty} \sum_{V \in E_l} h(\text{diam}_V) = 0, \quad \text{if } \gamma < \frac{2}{M\rho}. \]
We deduce from (4.4) that if \( U_j \cap B(3R) \neq \emptyset \), then \( |a_j| > 2R \) and \( U_j \subset B(R) \). It follows that \( E_l \) is a cover of the set
\[ \{ z \in B(3R) : f^k(z) \in B(3R) \text{ for } 1 \leq k \leq l - 1 \}. \]
Therefore
\[ \mu_h(I_{3R}(f)) = 0, \quad \text{for } \gamma < \frac{2}{M\rho}. \]
The conclusion follows since \( I(f) = \bigcap_{R>0} I_{3R}(f) \).

5. Construction of examples

Let \( 0 < \rho < \infty \) and \( n \in \mathbb{N} \). We introduce the following function
\[ q : [2^{1/\rho}, \infty) \to [\exp^n 2, \infty), \quad q(r) = \exp^n(r^\rho) \]
and the inverse function
\[ (5.1) \quad p : [\exp^n 2, \infty) \to [2^{1/\rho}, \infty), \quad p(t) = (\log^n t)^{1/\rho}. \]
We put \( k_0 = \lfloor \exp^n 2 \rfloor + 1 \). For \( k \geq k_0, k \in \mathbb{N} \) set
\[
(5.2) \quad n_k = \left\lfloor \frac{p(k)}{p'(k)} \right\rfloor.
\]
The next lemmas are giving some essential features of these functions, which help us to construct the function in Theorem 5.1.

**Lemma 5.1.**
\[
(5.3) \quad \frac{d}{dr} \left( \frac{q(r)}{q'(r)} \right) \to 0, \quad \frac{q(r)}{rq'(r)} \to 0,
\]
as \( r \to \infty \).

**Proof.** By differentiation,
\[
(5.4) \quad q'(r) = (\exp^n r^\rho)(\exp^{n-1} r^\rho) \cdots (\exp r^\rho) \rho r^{\rho-1}.
\]
Thus for \( n = 1 \),
\[
\frac{q(r)}{rq'(r)} = \frac{1}{\rho r^\rho} \to 0,
\]
and clearly
\[
\frac{q(r)}{rq'(r)} \to 0, \quad r \to \infty.
\]
For \( n \geq 2 \),
\[
(5.5) \quad \frac{q(r)}{q'(r)} = \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho)(\exp r^\rho) \rho r^{\rho-1}},
\]
and clearly
\[
\frac{q(r)}{rq'(r)} \to 0, \quad r \to \infty.
\]
Differentiating (5.5) we obtain
\[
\frac{d}{dr} \left( \frac{q(r)}{q'(r)} \right) = -\frac{1}{\exp^{n-1} r^\rho} - \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho)} - \cdots - \frac{1}{(\exp^{n-1} r^\rho)(\exp^{n-2} r^\rho) \cdots (\exp r^\rho)} \left( 1 + \frac{\rho - 1}{\rho r^\rho} \right).
\]
It is easy to see that
\[
\frac{d}{dr} \left( \frac{q(r)}{q'(r)} \right) \to 0, \quad r \to \infty.
\]

**Lemma 5.2.**
\[
(5.6) \quad p \left( t + \frac{1}{2} \right) - p(t) \sim \frac{1}{2} p'(t), \quad t \to \infty.
\]

**Proof.** From (5.1) we have
\[
(5.6) \quad p'(t) = \frac{1}{\rho} \left( \log^n t \right)^{\frac{1}{2}} \frac{1}{t(\log t)(\log^2 t) \cdots (\log^n t)}
\]
from which we can deduce that there exists \( t_0 \) such that \( p'(t) \) is nonincreasing on \( (t_0, +\infty) \).
Therefore
\[
(5.7) \quad \int_t^{t+\frac{1}{2}} p'(s) ds \leq \frac{1}{2} \max_{t \leq s \leq t+\frac{1}{2}} p'(s) = \frac{1}{2} p'(t),
\]
and

\[
\int_{t}^{t+1/2} p'(s)ds \geq \frac{1}{2} \min_{t \leq s \leq t+1/2} p'(t) = \frac{1}{2} p' \left( t + \frac{1}{2} \right)
\]

for \( t \geq t_0 \). Note that

\[
\frac{p'(t + \frac{1}{2})}{p'(t)} = \left( \frac{\log^n (t + \frac{1}{2})}{\log^n t} \right)^{\frac{1}{n}} \frac{t(n) \cdots (\log^n t)}{(t + \frac{1}{2}) \log(t + \frac{1}{2}) \cdots \log^n(t + \frac{1}{2})} \to 1
\]
as \( t \to \infty \). Together with (5.8) and (5.7) we have

\[
p \left( t + \frac{1}{2} \right) - p(t) = \int_{t}^{t+1/2} p'(s)ds \sim \frac{1}{2} p'(t), \quad t \to \infty.
\]

\[\square\]

**Lemma 5.3.** For \( l \in \mathbb{R} \) with \( \lfloor l \rfloor \geq k_0 + 1 \),

\[
\sum_{k=k_0+1}^{\lfloor l \rfloor} n_k \sim \int_{k_0+1}^{l} \frac{p(t)}{p'(t)} dt
\]
as \( l \to \infty \).

**Proof.** Denote

\[
P(t) = \frac{p(t)}{p'(t)} \quad \text{and} \quad \int_{k_0+1}^{l} P(t)dt = I.
\]

With (5.1) and (5.2) we get

\[
n_k \sim P(k) = \rho k \log k \cdots \log^n k,
\]

for \( k \geq k_0 \). Since \( P(t) \) is increasing with \( t \) we have for \( k_0 \leq k \leq l \),

\[
P(k) = P(k) \int_{k}^{k+1} dt \leq \int_{k}^{k+1} P(t)dt.
\]

Therefore

\[
\sum_{k=k_0+1}^{\lfloor l \rfloor} P(k) \leq \int_{k_0+1}^{l+1} P(t)dt = I + \int_{l}^{l+1} P(t)dt \leq I + P(l + 1).
\]

Similarly we obtain

\[
\sum_{k=k_0+1}^{\lfloor l \rfloor} P(k) \geq \int_{k_0+1}^{l+1} P(t)dt \geq I + c_0 - P(l),
\]

where \( c_0 = \int_{k_0+1}^{l+1} P(t)dt \). From (5.9) we may take \( l \geq k_1 > k_0 + 1 \) so large that \( P(t) \geq t \) for all \( t > k_1 \). Thus

\[
I \geq \int_{k_1}^{l} t dt \geq \frac{1}{2} l^2 - \frac{1}{2} k_1^2.
\]

Since

\[
P(l) = \rho l \log(l) \cdots \log^n(l) = o(l^2) \quad \text{as} \quad l \to \infty,
\]

we have

\[
P(l) = o(I), \quad P(l + 1) = o(I)
\]
as \( l \to \infty \). Together with (5.9), (5.10) and (5.11) we have our conclusion. \( \square \)

**Lemma 5.4.** For \( k_0 \leq k < l \), \( k \in \mathbb{N} \) and \( l \in \mathbb{R} \),

\[
(5.12) \quad \left( \frac{p(l)}{p(k)} \right)^{n_k} \geq \exp(c \min\{k, l - k\}),
\]

where \( c = (\log 2)^{n+1}/2 \).

**Proof.** With (5.1), (5.2) and (5.9) we have

\[
\left( \frac{p(l)}{p(k)} \right)^{n_k} = \exp \left( n_k \log \frac{p(l)}{p(k)} \right) \\
\geq \exp \left( \frac{1}{2} p(k) \log \frac{p(l)}{p(k)} \right) \\
= \exp \left( \frac{1}{2} k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^n l}{\log^n k} \right),
\]

(5.13)

We claim that

\[
(5.14) \quad k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^n l}{\log^n k} \geq (\log 2)^{n+1} \min\{k, l - k\},
\]

which is verified as follows by induction to \( n \). We first consider that \( n = 0 \).

**Case 1.** If \( k < \frac{l}{2} \) then

\[
(5.15) \quad k \log \frac{l}{k} > k \log 2.
\]

**Case 2.** If \( \frac{l}{2} \leq k < l \), then \( (l - k)/k \leq 1 \) and thus

\[
k \log \frac{l}{k} = k \log \left( 1 + \frac{l - k}{k} \right) \\
\geq k \frac{l - k}{k} \log 2 \\
= (l - k) \log 2.
\]

(5.16)

Together (5.15) and (5.16) give

\[
(5.17) \quad k \log \frac{l}{k} \geq (\log 2) \min\{k, l - k\},
\]

which is (5.14) for \( n = 0 \). Suppose that (5.14) holds for some \( n \).

**Case 1.** If \( \log^{n+1} l > 2 \log^{n+1} k \) and since \( k \geq k_0 \) then

\[
k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k} > k(\log k) \cdots (\log^{n+1} k) \log 2 \\
> k(\log 2)^{n+2}.
\]
Case 2. If \( \log^{n+1} l \leq 2 \log^{n+1} k \) then
\[
k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k}
= k(\log k) \cdots (\log^{n+1} k) \log (1 + \frac{\log^{n+1} l - \log^{n+1} k}{\log^{n+1} k})
\geq k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} l - \log^{n+1} k}{\log^{n+1} k} \log 2
= k(\log k) \cdots (\log^{n} k) \log \frac{\log^{n} l}{\log^{n} k} \log 2
\geq (\log 2)^{n+1} \min\{k, l - k\} \log 2
= (\log 2)^{n+2} \min\{k, l - k\}.
\]

Therefore
\[
(5.18) \quad k(\log k) \cdots (\log^{n+1} k) \log \frac{\log^{n+1} l}{\log^{n+1} k} \geq (\log 2)^{n+2} \min\{k, l - k\}.
\]

From (5.17) and (5.18) we see that (5.14) holds with \( n \) replaced by \( n + 1 \). Together with (5.13) this gives (5.12), by taking \( c = (\log 2)^{n+1}/2 \).

\[\square\]

Lemma 5.5. For \( k \in \mathbb{N}, l \in \mathbb{R} \) with \( k > l \geq k_0 \),
\[
(5.19) \quad \left( \frac{p(k)}{p(l)} \right)^{n_k} \geq \exp(c(k - l)),
\]
where \( c = (\log 2)^{n+1}/2 \).

Proof. We prove along the same path as in Lemma 5.4. For \( k > 2l \), instead of (5.13) and (5.14) we have
\[
(5.20) \quad \left( \frac{p(k)}{p(l)} \right)^{n_k} \geq \exp \left( \frac{1}{2} k(\log k) \cdots (\log^{n} k) \log \frac{\log^{n} k}{\log^{n} l} \right)
\]
and
\[
(5.21) \quad k(\log k) \cdots (\log^{n} k) \log \frac{\log^{n} k}{\log^{n} l} > (k - l)(\log 2)^{n+1}.
\]

We first consider that \( n = 0 \).

Case 1. If \( k > 2l \), then
\[
k \log \frac{k}{l} > k \log 2 > (k - l) \log 2.
\]

Case 2. If \( l < k \leq 2l \), then
\[
k \log \frac{k}{l} = k \log \left( 1 + \frac{k - l}{l} \right) \geq k \frac{k - l}{l} \log 2 > (k - l) \log 2.
\]

Therefore we have (5.21) for \( n = 0 \). Next we suppose that (5.21) holds for some \( n \in \mathbb{N} \).

Case 1. If \( \log^{n+1} k > 2 \log^{n+1} l \), then
\[
k(\log k) \cdots \log \frac{\log^{n+1} k}{\log^{n+1} l} > (k - l)(\log 2)^{n+2}.
\]
Case 2. If \( \log^{n+1} l < \log^{n+1} k \leq 2 \log^{n+1} l \), then with the assumption,

\[ k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} k}{\log^{n+1} l} = k(\log k) \cdots (\log^{n+1} k) \log \left( 1 + \frac{\log^{n+1} k - \log^{n+1} l}{\log^{n+1} l} \right) \]
\[ \geq k(\log k) \cdots (\log^{n+1} k) \frac{\log^{n+1} k - \log^{n+1} l}{\log^{n+1} l} \log 2 \]
\[ > k(\log k) \cdots (\log^n k) \log \frac{\log^n k}{\log^n l} \log 2 \]
\[ > (\log 2)^{n+1}(k - l) \log 2 \]
\[ = (k - l)(\log 2)^{n+2}. \]

Hence we have (5.21) by induction. Together with (5.20) and \( c = (\log 2)^{n+1}/2 \) we have (5.19).

**Theorem 5.1.** Let \( p(k) \) and \( n_k \) be defined by (5.1) and (5.2). Put

\[ g(z) = 2 \sum_{k=k_0+1}^{\infty} \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)2^{2n_k}} \]

Then \( g \in \mathcal{B} \) and \( \infty \) is not an asymptotic value of \( g \).

**Remark.** Bergweiler and Kotus [3] gave an example for the case of infinite order,

\[ f(z) = 2 \sum_{k=2}^{\infty} \frac{\log^n k z^{n_k}}{z^{2n_k} - (\log k)^{2n_k}} \]

where \( n_k = \lfloor k \log k \rfloor \). Here we take \( n_k = \lfloor \frac{p(k)}{p'(k)} \rfloor \) instead. If we let \( n = 1 \) and \( \rho = 1 \), then (5.22) is essentially the above function.

**Proof.** If \( |z| \leq p(k)/2 \), then

\[ \left| \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)2^{2n_k}} \right| \leq \frac{|z|^{n_k} p(k)^{n_k}}{p(k)^{2n_k} - |z|^{2n_k}} \leq 2 \frac{|z|^{n_k}}{p(k)^{n_k}} \leq 2^{1-n_k}. \]

From (5.9) we see that \( n_k \geq k \) for large \( k \). Thus the series in (5.22) converges locally uniformly and hence it defines a function \( g \) meromorphic in \( \mathbb{C} \).

Note that

\[ u_{k,l} = p(k) \exp \left( \frac{\pi il}{n_k} \right) \]

are the poles of \( g \), where \( k \in \mathbb{N} \) and \( 0 \leq l \leq 2n_k - 1 \). With \( u_{k,l} \) we rewrite \( g(z) \) as follows

\[ g(z) = 2 \sum_{k=k_0+1}^{\infty} \sum_{l=0}^{2n_k-1} \frac{u_{k,l}}{z - u_{k,l}}. \]
where

\[ \nu_{k,l} = \lim_{z \to u_{k,l}} (z - u_{k,l})^2 \frac{p(k)^{n_k} z^{n_k}}{z^{2n_k} - p(k)^{2n_k}} \]

\[ = 2p(k)^{n_k} u_{k,l}^{n_k} \lim_{z \to u_{k,l}} \frac{z - u_{k,l}}{z^{2n_k} - p(k)^{2n_k}} \]

\[ = 2p(k)^{n_k} u_{k,l}^{n_k} \lim_{z \to u_{k,l}} \frac{1}{2n_k z^{2n_k - 1}} \]

\[ = p(k)^{n_k} \frac{1}{np(k)^{n_k - 1} \left( \exp \left( \frac{\pi i l}{n_k} \right) \right)^{n_k - 1}} \]

(5.24)

\[ \frac{1}{2^n} |g(z)| \leq \sum_{k=k_0+1}^{m} \frac{p(k)^{n_k} |z|^{n_k}}{|z|^{2n_k} - p(k)^{2n_k}} + \sum_{k=m+1}^{\infty} \frac{p(k)^{n_k} |z|^{n_k}}{p(k)^{2n_k} - |z|^{2n_k}} \]

\[ \leq \sum_{k=k_0+1}^{m} \frac{p(k)^{n_k}}{|z|^{n_k} - p(k)^{n_k}} + \sum_{k=m+1}^{\infty} \frac{|z|^{n_k}}{p(k)^{n_k} - |z|^{n_k}} \]

\[ = \sum_{k=k_0+1}^{m} \left( \frac{1}{p(k)} \right)^{n_k} + \sum_{k=m+1}^{\infty} \left( \frac{1}{p(k)} \right)^{n_k - 1} \]

(5.25)

\[ = \Sigma_{1,m} + \Sigma_{2,m}. \]

From Lemma 5.4 with \( l = m + \frac{1}{2} \) we have, for \( k_0 \leq k \leq m \),

\[ \left( \frac{p(m + \frac{1}{2})}{p(k)} \right)^{n_k} \geq \exp \left( c \min\{k, m + \frac{1}{2} - k\} \right), \]
where \( c = (\log 2)^{n+1}/2 \). Thus

\[
\Sigma_{1,m} \leq \sum_{k=k_0+1}^{m} \frac{1}{\exp \left( c \min \{k, m + \frac{1}{2} - k\} \right) - 1} = \sum_{k=k_0+1}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{1}{\exp(ck) - 1} + \sum_{k=\left\lfloor \frac{m}{2} \right\rfloor+1}^{m} \frac{1}{\exp \left( c (m + \frac{1}{2} - k) \right) - 1} = \sum_{k=k_0+1}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{1}{\exp(ck) - 1} + \sum_{j=0}^{m-\left\lfloor \frac{m}{2} \right\rfloor-1} \frac{1}{\exp \left( c (j + \frac{1}{2}) \right) - 1} \leq \sum_{k=1}^{\infty} \frac{1}{\exp(ck) - 1} + \sum_{j=0}^{\infty} \frac{1}{\exp \left( c (j + \frac{1}{2}) \right) - 1} =: C.
\]

Similarly with Lemma 5.3 and \( l = m + \frac{1}{2} \), we obtain

\[
\Sigma_{2,m} \leq \sum_{k=m+1}^{\infty} \frac{1}{\exp \left( c \left( k - (m + \frac{1}{2}) \right) \right) - 1} = \sum_{j=0}^{\infty} \frac{1}{\exp \left( c (j + \frac{1}{2}) \right) - 1} \leq C.
\]

Therefore

\[(5.26) \quad |g(z)| \leq 2(\Sigma_{1,m} + \Sigma_{2,m}) \leq 4C,\]

for \( |z| = p \left( m + \frac{1}{2} \right), m \in \mathbb{N} \). Next we consider \( z \in W_2 \). Then \( z = r \exp(i\pi(2\eta - 1)/(2n_m)) \), where \( p(m - \frac{1}{2}) \leq r \leq p(m + \frac{1}{2}) \) and \( \eta \in \mathbb{N}, 1 \leq \eta \leq 2n_m \). Thus

\[
z^{2n_m} = \left( r \exp \left( i\pi \frac{2\eta - 1}{2n_m} \right) \right)^{2n_m} = -r^{2n_m}
\]

With this, \((5.25)\) and since \( p(k) \) is increasing with \( k \), we have

\[
\frac{1}{2} |g(z)| \leq \sum_{k=k_0+1}^{m-1} \frac{p(k)^{n_k}r^{n_k}}{r^{2n_k} - p(k)^{2n_k}} + \frac{p(m)^{n_m}r^{n_m}}{r^{2n_m} + p(m)^{2n_m}} + \sum_{k=m+1}^{\infty} \frac{p(k)^{n_k}r^{n_k}}{r^{2n_k} - p(k)^{2n_k}} \leq \Sigma_{1,m-1} + 2 + \Sigma_{2,m} \leq 2C + 2.
\]

Combining with \((5.26)\) it follows that

\[(5.27) \quad |g(z)| \leq 4C + 4 \text{ for } z \in W.
\]

Actually \( g \) is bounded on a larger set, which we want to show next.

From Lemma 5.2 we have

\[(5.28) \quad p \left( m + \frac{1}{2} \right) - p(m) = \int_{m}^{m+\frac{1}{2}} p'(t)dt \sim \frac{1}{2} p'(m), \quad m \to \infty.
\]

And note that by \((5.2)\)

\[(5.29) \quad |u_{m,\eta} - u_{m,\eta+1}| = p(m) \left| \exp \left( \frac{i\pi}{n_m} \right) - 1 \right| \sim p(m) \frac{\pi}{n_m} \sim \pi p'(m), \quad m \to \infty.
\]
If $W_{m,\eta}$ denotes the component of $\mathbb{C} \setminus W$ that contains $u_{m,\eta}$, we find that there exists $\lambda > 0$ such that

\begin{equation}
\text{dist} (u_{m,\eta}, \partial W_{m,\eta}) \geq 2\lambda p'(m),
\end{equation}

for $m$ large and $\eta \in \{0, 1, \ldots, 2n_m - 1\}$.

Consider the function

\begin{equation}
\zeta(z) = g(z) - \frac{\nu_{m,\eta}}{z - u_{m,\eta}},
\end{equation}

which is holomorphic in the closure of $W_{m,\eta}$. For $z \in \partial W_{m,\eta}$ with (5.24), (5.27), (5.2) and $n_m \geq p(m)/(2p'(m))$ we have

\[ |\zeta(z)| \leq |g(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|} \leq 4C + 4 + \frac{p(m)}{2\lambda n_m p'(m)} \leq 4C + 4 + \frac{1}{\lambda}, \]

for $m$ large. By the maximum principle,

\[ |\zeta(z)| \leq 4C + 4 + \frac{1}{\lambda} \text{ for } z \in W_{m,\eta}. \]

We put $r_m = \lambda p'(m)$ and deduce that if $z \in W_{m,\eta} \setminus D(u_{m,\eta}, r_m)$, then

\[ |g(z)| \leq |\zeta(z)| + \frac{|\nu_{m,\eta}|}{r_m} \leq 4C + 4 + \frac{1}{\lambda} + \frac{p(m)}{\lambda n_m p'(m)} \leq 4C + 4 + \frac{3}{\lambda}. \]

This means that $g$ is large only in small neighborhoods of the poles.

On the other hand we will show that the set of critical values of $f$ is bounded by verifying that there are no critical points of $g$ in these small neighborhoods of the poles.

Assume that $z \in \partial W_{m,\eta}$ and $m', \eta'$ are such that $z \in \partial W_{m',\eta'}$. Then $|m - m'| \leq 1$ and so $r_m \leq 2r_{m'}$ by (5.30). Therefore $D\left(z, \frac{1}{2}r_m\right) \cap D\left(u_{m',\eta'}, r_{m'}\right) = \emptyset$. Thus

\[ |g'(z)| = \frac{1}{2\pi} \left| \int_{|\xi - z| = \frac{1}{2}r_m} \frac{g(\xi)}{(\xi - z)^2} d\xi \right| \leq \frac{r_m}{2} \max_{|\xi - z| = \frac{1}{2}r_m} |g(\xi)| \leq \frac{2}{r_m} \left( 4C + 4 + \frac{3}{\lambda} \right). \]
Since $n_m > p(m)/(2p'(m))$ for $m$ large, from (5.31), (5.24) and (5.30) we have
\[ |\zeta'(z)| \leq |g'(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|^2} \]
\[ \leq \frac{2}{r_m} \left( 4C + 4 + \frac{3}{\lambda} \right) + \frac{p(m)}{n_m(2\lambda p'(m))^2} \]
\[ = \frac{2}{r_m} \left( 4C + 4 + \frac{3}{\lambda} \right) + \frac{p(m)}{4\lambda n_m r_m p'(m)} \]
\[ \leq \frac{2}{r_m} \left( 4C + 4 + \frac{3}{\lambda} \right) + \frac{1}{2\lambda r_m} \]
\[ \leq \frac{2}{r_m} \left( 4C + 4 + \frac{13}{4\lambda} \right) \]
for $z \in \partial W_{m,\eta}$. It implies by maximum principle that
\[ |\zeta'(z)| \leq \frac{2}{r_m} \left( 4C + 4 + \frac{13}{4\lambda} \right) \text{ for } z \in W_{m,\eta}. \]
Choose $\delta > 0$ sufficiently small and $z \in D(u_{m,\eta}, \delta r_m)$. Since $n_m < 2p(m)/p'(m)$ for $m$ large we have
\[ |g'(z)| \geq \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|^2} - |\zeta'(z)| \]
\[ \geq \frac{p(m)}{\delta^2 n_m r_m \lambda p'(m)} - \frac{2}{r_m} \left( 4C + 4 + \frac{13}{4\lambda} \right) \]
\[ \geq \frac{2}{r_m} \left( \frac{1}{4\delta^2 \lambda} - 4C - 4 - \frac{13}{4\lambda} \right) > 0. \]
Hence if $g'(z) = 0$ for some $z \in W_{m,\eta}$ then $|z - u_{m,\eta}| \geq \delta r_m$. Therefore
\[ (5.32) \quad |g(z)| \leq |\zeta(z)| + \frac{|\nu_{m,\eta}|}{|z - u_{m,\eta}|} \leq 4C + 4 + \frac{1}{\lambda} + \frac{p(m)}{\delta \lambda n_m r_m} \leq 4C + 4 + \frac{1}{\lambda} + \frac{2}{\delta \lambda} \]
as claimed. The same is true for the set of asymptotic values of $g$ with (5.27). Hence $g \in B$.

**Theorem 5.2.** Let $g$ be defined as in (5.22). Then $\rho_n(g) = \rho$.

**Proof.** From Lemma 5.3 the number $n(r, g)$ of poles of $g$ in $D(0, r)$ satisfies
\[ n(r, g) = \sum_{k=k_0+1}^{q(r)} 2n_k \sim 2 \int_{k_0+1}^{q(r)} \frac{p(t)}{p'(t)} dt. \]
Now let $t = q(s)$ and $r_0 = p(k_0 + 1)$. Then
\[ \int_{k_0+1}^{q(r)} \frac{p(t)}{p'(t)} dt = \int_{r_0}^{r} \frac{p(q(s))}{p'(q(s))} q'(s) ds \]
\[ = \int_{r_0}^{r} \frac{s}{p'(q(s))} q'(s) ds \]
\[ = \int_{r_0}^{r} q'(s)^2 ds. \]
Hence
\begin{equation}
(5.33) \quad n(r, g) \sim 2 \int_{r_0}^{r} q'(s)^2 sds
\end{equation}
for \( r \to \infty \). By Lemma 5.1 we have
\begin{equation}
(5.34) \quad \frac{q(r)q'(r)}{q'(r)^2 r} = \frac{q(r)}{r q'(r)} \to 0, \quad r \to \infty.
\end{equation}
On the other hand,
\begin{equation*}
\frac{q''(r)q(r)}{q'(r)^2} = 1 - \frac{q'(r)q'(r) - q(r)q''(r)}{q'(r)^2}
\end{equation*}
\begin{equation*}
= 1 - \frac{d}{dr} \left( \frac{q(r)}{q'(r)} \right).
\end{equation*}
Again with lemma 5.1 we have
\begin{equation}
(5.35) \quad \frac{q(r)q''(r)}{q'(r)^2} \to 1, \quad r \to \infty,
\end{equation}
We claim that
\begin{equation}
(5.36) \quad 2 \int_{r_0}^{r} q'(s)^2 sds \sim q(r)q'(r)r \text{ as } r \to \infty.
\end{equation}
In fact with (5.34) and (5.35) by l’Hospital’s rule we have
\begin{equation*}
\lim_{r \to \infty} \frac{q(r)q'(r)r}{\int_{r_0}^{r} q'(s)^2 sds} = \lim_{r \to \infty} \frac{\frac{d}{dr} \left( q(r)q'(r)r \right)}{\frac{d}{dr} \int_{r_0}^{r} q'(s)^2 sds}
\end{equation*}
\begin{equation*}
= \lim_{r \to \infty} \frac{q'(r)^2 r + q(r)q''(r)r + q(r)q'(r)}{q'(r)^2 r}
\end{equation*}
\begin{equation*}
= \lim_{r \to \infty} \left( 1 + \frac{q(r)q''(r)}{q'(r)^2} + \frac{q(r)}{q'(r)r} \right)
\end{equation*}
\begin{equation*}
= 2.
\end{equation*}
Therefore from (5.33), (5.36) and the definition of counting function,
\begin{equation*}
N(r, g) \sim \frac{1}{2} q^2(r)
\end{equation*}
as \( r \to \infty \).

Suppose that \( r \) has the form \( r = p(k + \frac{1}{2}) \) for \( k_0 \leq k \in \mathbb{N} \) large. From (5.27) we have \( m(r, g) \leq 4C + 4 \). Since
\begin{equation*}
T(r, g) = N(r, g) + m(r, g)
\end{equation*}
we obtain
\begin{equation*}
T(r, g) \sim \frac{1}{2} q(r)^2, \quad r \to \infty.
\end{equation*}
It yields that
\begin{equation*}
\log T(r, g) = (1 + o(1))2 \log q(r)
\end{equation*}
\begin{equation*}
= (1 + o(1))2 \exp^{n-1}(r^\rho),
\end{equation*}
and thus
\begin{equation}
(5.37) \quad \log^{n+1} T(r, g) \sim \rho \log r
\end{equation}
as $r \to \infty$ through $r$-values of the form $r = p \left( k + \frac{1}{2} \right)$. It follows that (5.37) holds for all $r$ since $T(r, g)$ is increasing with $r$. Hence

$$
\rho_n(g) = \limsup_{r \to \infty} \frac{\log^{n+1} T(r, g)}{\log r} = \rho.
$$

\[ \square \]

6. PROOF OF THEOREM 1.2

Let $g$ be the function constructed in section 5 and put $f(z) = g(z)^M$. Hence the multiplicity of all poles of $f$ is $M$, $f \in B$ without $\infty$ as its asymptotic value and $\rho_n(f) = \rho$.

As in section 4 we denote the sequence of poles by $a_j$, ordered such that $|a_j| \leq |a_{j+1}|$ for all $j \in \mathbb{N}$. Choose $b_j$ as in section 4 so that

$$
f(z) \sim \left( \frac{b_j}{z - a_j} \right)^M \quad \text{as } z \to a_j,
$$

for each $j \in \mathbb{N}$. We thus have $a_j = u_{m, \eta}$ and $b_j = \nu_{m, \eta}$ for some $m, \eta \in \mathbb{N}$ and $0 \leq \eta \leq 2m - 1$.

Choose $R_0 \geq 4C + 4 + \frac{1}{\lambda} + \frac{2}{\delta}$, where $\lambda$, $\delta$ are as in (5.32) and $R_l = R_0 \exp(2^l)$ for $l \in \mathbb{N}$. We denote by $E_l$ the collections of all components $V$ of $f^{-l}(B(R_l))$ which satisfy $f^k(V) \subset U_{j_{k+1}} \subset B(R_k)$ for $0 \leq k \leq l - 1$ and $E_l = \bigcup_{A \in E_l} A$. It follows that $E = \bigcap_{l=1}^{\infty} E_l \subset I(f)$.

The estimates obtained in section 4 also hold with $R$ replaced by $R_l$. So we may use them for the map $g_j$ that maps $D \subset B(R_l) \setminus \{ \infty \}$ to $U_j$, the component of $f^{-1}(B(R_l))$ containing $a_j$. From (1.8) we deduce that if $V \in E_l$ such that $f^k(V) \subset U_{j_{k+1}} \subset B(R_k)$ for $0 \leq k \leq l - 1$, then (4.15) holds.

Here $a_{jk}$ is a pole of $f$ that is contained in $U_{j_k}$ for $k = 1, 2, \ldots, l$. From (5.23) and (5.24) we know that $|a_{jk}| = : r_{jk} = p(l)$ for some $l \geq k_0 + 1$ and accordingly $|b_{jk}| = r_{jk}/n_{jk}$. With the definition of $p$, $q$ and $n_{jk}$ we have

$$
|b_{jk}| \sim \frac{p(l)}{p(l)/p'(l)} = p'(l) = \frac{1}{q'(p(l))}.
$$

Therefore

$$
(6.1)
\frac{|b_{jk}|}{|a_{jk}|^{1 + \frac{1}{M}}} \sim \frac{1}{q'(r_{jk})^{\frac{1}{M}}}. \quad \text{Recall that } q(r) = \exp^n(r^o) \text{ is convex and thus } q'(r) \text{ is increasing. Moreover } r_{jk} \geq R_{k-1} \text{ for } k = 1, 2, \ldots, l. \text{ It follows from (4.15) and (6.1) that}
$$

$$
(6.2)
diam \chi(V) \leq \prod_{k=1}^{l} \frac{A}{q'(R_{k-1}) R_{k-1}^{1 + \frac{1}{M}}} =: d_l,
$$

where $A \neq 0$ is a constant.

With $d_l$ we intend to apply Lemma 3.6. In order to do so we are estimating $\Delta_l$. From (4.11) and (5.27) we deduce that

$$
(6.3)
D \left( a_j, \frac{|b_j|}{4R_l^{1 + \frac{1}{M}}} \right) \subset U_j = W_{m, \eta} \cap f^{-1}(B(R_l)).
$$
Meanwhile (5.28) and (5.29) imply that

\[(6.4) \quad W_{m, \eta} \subset D(u_{m, \eta}, \tau |\nu_{m, \eta}|) = D(a_j, \tau |b_j|),\]

where \(\tau = 1/2 + \pi/2\), \(a_j = u_{m, \eta}\), and \(m\) large. For \(\varepsilon > 0\) small set

\[A(S) = \{z \in \mathbb{C} : S < |z| < 2S\},\]

\[A_{\varepsilon}(S) = \{z \in \mathbb{C} : (1 + \varepsilon)S < |z| < (1 - \varepsilon)2S\}.
\]

Then from (6.3) we have

\[
\text{area} \left( \bigcup_{j \in \mathbb{N}} U_j \cap A(S) \right) \geq \text{area} \left( \bigcup_{a_j \in A_{\varepsilon}(S)} U_j \right) \\
\geq \sum_{a_j \in A_{\varepsilon}(S)} \pi \left( \frac{1}{4R_1^{1/2}} |b_j| \right)^2 \\
= \pi \frac{1}{16R_1^{1/2}} \sum_{a_j \in A_{\varepsilon}(S)} |b_j|^2.
\]

On the other hand with (6.4) there exists a \(\delta > 0\) such that

\[
\text{area} A(S) \leq (1 + \delta) \text{area} A_{\varepsilon}(S) \\
\leq (1 + \delta) \text{area} \left( \bigcup_{a_j \in A_{\varepsilon}(S)} W_{m, \eta} \right) \\
\leq \pi (1 + \delta) \tau^2 \sum_{a_j \in A_{\varepsilon}(S)} |b_j|^2.
\]

We conclude that

\[(6.5) \quad \text{dens} \left( \bigcup_{j \in \mathbb{N}} U_j, A(S) \right) \geq \frac{1}{16(1 + \delta) \tau^2 R_1^{1/2}}.
\]

Now consider \(g_j\) as defined in (4.6), which is a branch of \(f^{-1}\) mapping

\[A'(S) = A(S) \setminus (-2S, -S)
\]

into \(U_j\). With \(\lambda = 1/2\) in (3.2) and (4.7) we obtain

\[
\frac{4|b_j|}{27m_j(2S)^{1 + \frac{1}{m_j}}} \leq |g_j'(z)| \leq \frac{12|b_j|}{m_j S^{1 + \frac{1}{m_j}}},
\]

for \(z \in A'(S)\). Then

\[
\sup_{u, v \in A'(S)} \frac{|g_j'(u)|}{|g_j'(v)|} \leq 324,
\]

for \(S\) large enough. Hence with (6.5) it yields

\[(6.6) \quad \text{dens} \left( g_j \left( E_1 \right), g_j \left( A'(S) \right) \right) \geq \frac{1}{324^2} \text{dens} \left( E_1, A'(S) \right) \geq \frac{\alpha}{R_1^{2/2M}},\]

where \(\alpha = 1/(16(1 + \delta)324^2 \tau^2)\).
Now we let $S = 2^k R_0$ with $k \geq 0$. Applying the above for all such $S$ and for all branches $g_j$ mapping $A'(S)$ to $U_j$ from (6.6) we deduce that

\[(6.7) \quad \text{dens}(E_2, U_j) \geq \frac{\alpha}{R_j^{2/M}},\]

for each $U_j$ in $E_1$.

Suppose that $V \subset U_{j_1}$ is a component of $E_1$. Let $j_2, \ldots, j_l$ be such that $f^m(V) \subset U_{j_{m+1}}$ for $m \in \mathbb{N}$, $0 \leq m \leq l - 1$. Then $f^{l-1}(V) = U_{j_l}$ and

\[(6.8) \quad f^{l-1}(E_{l+1} \cap V) = E_2 \cap U_{j_l}.\]

Denote by $g_{j_l}$ a branch of $f^{-1}$ that maps $U_{j_l}$ into $U_{j_{l-1}}$. For $l$ large $g_{j_l}$ extends univalently to a map from $D(a_{j_l}, \frac{5}{6}|a_{j_l}|)$ into $B(R_l)$. It implies that the branch of the inverse of $f^{l-1}$ which maps $U_{j_l}$ to $V$ extends univalently to $D(a_{j_l}, \frac{5}{6}|a_{j_l}|)$.

Noting that $U_{j_l} \subset D(a_{j_l}, \frac{1}{2}|a_{j_l}|)$ by (4.4), we can now apply Koebe’s distortion theorem with $\lambda = \frac{2}{5}$. From (6.7), (6.8) and (3.2) we obtain

\[
\text{dens}(E_{l+1}, V) \geq \frac{1}{256} \text{dens}(E_2, U_{j_l}) \geq \frac{1}{256} \frac{\alpha}{R_l^{2/M}}.
\]

Together with (3.4) and $B = \alpha/(9^2 \cdot 256)$ we conclude that

\[(6.9) \quad \text{dens}_\chi(E_{l+1}, V) \geq \frac{B}{R_l^{2/M}} =: \Delta_l.\]

Next set $h(t)$ be as in (1.1) and $g(t) = (\log^n \tfrac{1}{t})^\gamma$. It is easy to see that $g(t)$ is a decreasing continuous function and $\lim_{t \to 0} t^2 g(t) = \lim_{t \to 0} h(t) = 0$. Now we shall apply Lemma 3.6 with $h(t)$ and $g(t)$.

From (6.2) we have

\[
\log^n \frac{1}{d_l} = \log^n \left( \prod_{k=1}^{l} \frac{1}{A} q'(R_{k-1}) R_{k-1}^{1 + \frac{1}{M}} \right)
\]

\[
= \log^{n-1} \log \left( \prod_{k=1}^{l} \frac{1}{A} q'(R_{k-1}) R_{k-1}^{1 + \frac{1}{M}} \right)
\]

\[
= \log^{n-1} \sum_{k=1}^{l} \left( \log \frac{1}{A} + \log q'(R_{k-1}) + \left( 1 + \frac{1}{M} \right) \log R_{k-1} \right).
\]

Noting that by (5.4),

\[
\log q'(r) = \log \left( (\exp^n r^\rho) (\exp^{n-1} r^\rho) \cdots (\exp r^\rho) r^{\rho - 1} \right) \sim \exp^{n-1} (r^\rho)
\]

as $r \to \infty$, we have

\[
\log^n \frac{1}{d_l} \sim \log^{n-1} \sum_{k=1}^{l} \log q'(R_{k-1}) \sim \log^{n-1} \sum_{k=1}^{l} \exp^{n-1} (R_{k-1}^\rho) \sim R_{l-1}^\rho
\]

as $l \to \infty$. Since $R_l = R_0 \exp(2^l)$ we obtain

\[
g(d_l) = \left( \log^n \frac{1}{d_l} \right)^\gamma \sim R_0^{\gamma \rho} \exp \left( \rho \gamma 2^{l-1} \right), \quad l \to \infty.
\]
Thus there exists a constant $K > 0$ such that

$$g(d_l) \geq KR_0^{\gamma \rho} \exp \left( \frac{\rho \gamma 2^l - 1}{M} \right)$$

for $l$ large.

On the other hand since $R_l$ is nondecreasing it follows that

$$
\prod_{k=1}^{l} \Delta_k = B^l \prod_{k=1}^{l} R_k^{-2/M} = B^l R_0^{-2l/M} \prod_{k=1}^{l} \exp \left( -\frac{2}{M} 2^l \right) \\
= B^l R_0^{-2l/M} \exp \left( -\frac{2}{M} \sum_{k=1}^{l} 2^k \right) \\
= B^l R_0^{-2l/M} \exp \left( -\frac{8}{M} 2^{l-1} + \frac{4}{M} \right).
$$

Together with (6.10) we have

$$g(d_l) \prod_{k=1}^{l} \Delta_k \geq K B^l R_0^{\gamma \rho - 2l/M} \exp \left( \frac{4}{M} \right) \exp \left( 2^{l-1} \left( \rho \gamma - \frac{8}{M} \right) \right) \to \infty$$
as $l \to \infty$ if $\gamma > 8/(M \rho)$.

With Lemma 3.6 and (6.11) we complete the proof.

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Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, 24098 Kiel, Germany
E-mail address: w.li@math.uni-kiel.de