Cut Bounds for Some Weighted Graphs

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Abstract

In communication field, an important issue is to group users and base stations to as many as possible subnetworks satisfying certain interference constraints. These problems are usually formulated as a graph partition problems which minimize some forms of graph cut. Previous research already gave some results about the cut bounds for unweighted regular graph. In this paper, we prove a result about the lower bound for weighted graphs that have some regular properties and show similar results for more general case.

1 Introduction

In [1], Dai and Bai investigate the wireless network optimal decomposition problem. Ideal decomposition breaks a large-scale network into sufficiently large number of subnetworks with similar size with the interference among subnetworks being as small as possible. The network decomposition problem is formulated as follows. An corresponding graph decomposition should have as enough subgraphs with similar size and the cut between subgraphs should be small.

Let $G = (V, E)$ be a undirected weighted simple graph, where $V = \{v_1, v_2, \ldots, v_n\}$ is the vertex set of $G$, $E = \{e_{ij} = \{v_i, v_j\}|v_i$ is adjacent with $v_j\}$ is the edge set. The graphs we consider in this paper are undirected simple connected graph. The edge weight of $e_{ij}$ is $a_{ij} > 0$ and we define $a_{ij} = 0$ if $v_i$ is not adjacent with $v_j$. $A = \{a_{ij}\}_{n \times n}$ is the adjacency matrix of $G$. Vertex weight is a real-valued function defined on vertex set: $d : V \rightarrow \mathbb{R}$. For a vertex $v_i \in V, d(v_i) = \sum_{j=1}^{n} a_{ij}$ is the sum of $i$-th row or column of $A$.

A partition of $G$ is collection $\{V_1, V_2, \ldots, V_k\}$, where $V_i \cap V_j = \emptyset, \forall 1 \leq i < j \leq k$ and $\bigcup_{i=1}^{k} V_i = V$. For two disjoint subsets $A, B \subset V$ we define

$$\text{cut}(A, B) = \sum_{v_i \in A, v_j \in B} a_{ij}.$$
For a partition \( \{V_1, V_2, \ldots, V_k\} \) we define

\[
cut(\{V_1, V_2, \ldots, V_k\}) = \sum_{i=1}^{k} \cut(V_i, \bar{V}_i),
\]

where \( \bar{V}_i \) is the complement of \( V_i \subset V \).

To describe the above requirement about minimizing cut and balancing subgraphs, two objective functions \( \text{RatioCut} \) and the normalized cut \( \text{Ncut} \) are often used. In the definition of \( \text{RatioCut} \), the size of a subset \( V_i \) of a graph is measured by its number of vertices \( |V_i| \), while in \( \text{Ncut} \) the size is measured by the weights of its edges \( \text{vol}(V_i) = \sum_{v_i,v_j \in V_i} a_{ij} \).

\[
\text{RatioCut}(\{V_1, V_2, \ldots, V_k\}) = \sum_{i=1}^{k} \frac{\cut(V_i, \bar{V}_i)}{|V_i|},
\]

\[
\text{Ncut}(\{V_1, V_2, \ldots, V_k\}) = \sum_{i=1}^{k} \frac{\cut(V_i, \bar{V}_i)}{\text{vol}(V_i)}.
\]

Given \( \delta \geq 0 \), \( \Pi_M(G) \) represents \( G \)'s \( M \)-parts partition \( \{V_1, V_2, \ldots, V_M\} \). In [1], Optimal wireless network decomposition problem is formulated as the following constrained optimizing problem:

\[
\begin{align*}
\max & \quad M \\
\text{s.t.} & \quad \text{Ncut}(\Pi_M(G)) \leq \delta. \tag{1}
\end{align*}
\]

One similar format is:

\[
\begin{align*}
\max & \quad M \\
\text{s.t.} & \quad \text{RatioCut}(\Pi_M(G)) \leq \delta. \tag{2}
\end{align*}
\]

An approach to handle (1) and (2) is to consider the following two series optimizing problems:

\[
\begin{align*}
\min & \quad \text{Ncut}(\Pi_M(G)) \\
\text{s.t.} & \quad M \geq M_0. \tag{3}
\end{align*}
\]

and

\[
\begin{align*}
\min & \quad \text{RatioCut}(\Pi_M(G)) \\
\text{s.t.} & \quad M \geq M_0. \tag{4}
\end{align*}
\]

where \( M_0 = 2, 3, \ldots \).
Benefiting from monotonicity of $Ncut(\Pi_M(G))$ and $RatioCut(\Pi_M(G))$ about $M$ in minimal situation, (3) and (4) can be simplified as
\[
\min Ncut(\Pi_M(G)) \quad (5)
\]
and
\[
\min RatioCut(\Pi_M(G)) \quad (6)
\]
Unfortunately, it has been proven that solving (5) and (6) and even their constant approximation is NP-hard problem\[5\]. We may consider the lower bounds of $Ncut(\Pi_M(G))$ or $RatioCut(\Pi_M(G))$ for $M \geq 2$, which are helpful in practice. For example, optimal value of (6) and (4) is no less than a lower bound for $RatioCut(\Pi_{M_0}(G))$. So in (2), if $\delta$ is less than the above lower bound, the optimal value is at most $M_0 - 1$.

For a $2-$partition $\{S, T\}$ of $G$, we investigate bounds of $RatioCut(\{S, T\})$ and similar form $\frac{\text{cut}(S,T)}{|S||T|}$.
\[
RatioCut(\{S, T\}) = (\frac{1}{|S|} + \frac{1}{|T|}) \cdot \text{cut}(S,T),
\]
And by the mean inequality,
\[
\frac{1}{2} \left( \frac{1}{|S|} + \frac{1}{|T|} \right) \geq \frac{2}{\frac{1}{|S|} + \frac{1}{|T|}} = \frac{2}{|S| + |T|}
\]
So
\[
RatioCut(\{S, T\}) \geq \frac{\text{cut}(S,T)}{|S| + |T|} > \frac{\text{cut}(S,T)}{|S||T|}. \quad (7)
\]
(7) shows that the lower bound of $\frac{\text{cut}(S,T)}{|S||T|}$ is also a lower bound for $RatioCut(\{S, T\})$ although not a tight one. Inequalities like
\[
|\text{cut}(S,T) - k|S||T|| \leq L,
\]
where $k$ is depending on $G$ and $L$ is depending on $G$, $S$ and $T$ has been proved in \[\ref{6} \ref{7}\] for unweighted graphs. The inequality gives upper and lower bounds of $\frac{\text{cut}(S,T)}{|S||T|}$ and lower bound of $RatioCut(\{S, T\})$.

**Theorem 1** \[\ref{6} \ref{7}\] Let $X$ be a non-bipartite, $d-$regular graph of size $n$, and let $S$ and $T$ be vertex subsets, $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ are the eigenvalues of $X$’s adjacency matrix. Then:
\[
|\text{e}(S,T) - \frac{d}{n}|S||T|| \leq \frac{\alpha}{n} \sqrt{|S||T||S^C||T^C|}, \quad \alpha := \max\{\alpha_2, -\alpha_n\}.
\]
Theorem 2 Let $X$ be bipartite, $d$–regular graph of half-size $m$, and let $S$ and $T$ be vertex subsets lying in different sides of the bipartition. Then:

$$|e(S, T) - \frac{d}{m} |S||T|| \leq \frac{\alpha_2}{m} \sqrt{|S||T|(m - |S|)(m - |T|)}.$$ 

In this paper, we prove the following results of weighted version:

Theorem 3 Let $G = (V = X \cup Y, E)$ be a weighted bipartite graph with $n$ vertices. $X, Y$ are two parts of $V$ respectively. For every vertex $v_i \in V$, $d(v_i) = d$ is a constant. $\alpha_1, \alpha_2, \ldots, \alpha_n$ are eigenvalues of $A$ with descending order. Set $m = n/2$, for $S \subset X$ and $T \subset Y$,

$$|\text{cut}(S, T) - \frac{d}{m} |S||T|| \leq \frac{\alpha_2}{m} \sqrt{|S||T|(m - |S|)(m - |T|)}.$$ 

Theorem 4 Let $G = (V, E)$ be a weighted non-bipartite graph with $n$ vertices, for every vertex $v_i \in V$, $d(v_i) = d$ is a constant. $\alpha_1, \alpha_2, \ldots, \alpha_n$ are eigenvalues of $A$ with descending order and $\alpha = \max \{\alpha_2, -\alpha_n\}$. Then for disjoint $A, B \subset V$,

$$|\text{cut}(A, B) - \frac{d}{n} |A||B|| \leq \frac{\alpha}{n} \sqrt{|A||B||A^c||B^c|}.$$ 

For general graph, we prove the following results.

Theorem 5 Let $G = (V, E), G_0 = (V_0, E_0)$ be two weighted bipartite graph with $n$ vertices. The adjacency matrices of $G$ and $G_0$ are $A$ and $A_0$ respectively. For any vertex $v \in V_0$, $d_{G_0}(v) = d$ is a constant. $A = A_0 + X$, $\alpha_1, \alpha_2, \ldots, \alpha_n$ are eigenvalues of with descending order. $V = X \cup Y, X, Y$ are two parts of $V$ respectively, $S \subset X$ and $T \subset Y$, then

$$|\text{cut}(S, T) - \frac{2d}{(\sqrt{n} - \epsilon_2)^2} |S||T|| \leq (1 + \epsilon_1)\epsilon_2^2 \cdot O(n) + \frac{\alpha_2}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|((\sqrt{n} + \epsilon_2)^2 - 2|S|)|T|((\sqrt{n} + \epsilon_2)^2 - 2|T|)};$$

where $\epsilon_1 = \|X\|_2$, $\epsilon_2 = \frac{4\sqrt{n}(n-1)n}{(\alpha_2 - \alpha_1)^2} \|X\|_2^2$.

Theorem 6 Let $G = (V, E), G_0 = (V_0, E_0)$ be two weighted non-bipartite graph with $n$ vertices. The adjacency matrices of $G$ and $G_0$ are $A$ and $A_0$ respectively. For any vertex $v \in V_0$, $d_{G_0}(v) = d$ is a constant. $A = A_0 + X$, 

4
\(\alpha_1, \alpha_2, \ldots, \alpha_n\) are eigenvalues of with descending order, \(\alpha = \max\{\alpha_2, -\alpha_n\}\).

If \(S, T\) are two subsets of \(V\), then

\[
|\text{cut}(S, T) - \frac{2d}{(\sqrt{2n} - \epsilon_2)^2}|S||T| | \leq (1 + \epsilon_1)\epsilon_2^2 \cdot O(n)
\]

\[
\frac{\alpha}{(\sqrt{2n} - \epsilon_2)^2} \sqrt{|S|((\sqrt{2n} + \epsilon_2)^2 - 2|S||T|)((\sqrt{2n} + \epsilon_2)^2 - 2|T|)},
\]

where \(\epsilon_1 = \|X\|_2\), \(\epsilon_2 = \frac{4\sqrt{n}(n-1)n}{(\alpha_2 - \alpha_1)^2} \|X\|_2^2\).

In the following section, we will present the lemmas which play roles in the proof of the above results. The results of this paper will be proved in section 3 and the conclusion is drawn in section 4.

## 2 Lemmas

Theorem 4 can be derived from bipartite case. To prove Theorem 3 we need the follows lemmas.

**Lemma 1** Let \(A\) be an adjacent matrix of a weighted bipartite graph, then the eigenvalues of \(A\) is symmetrical about 0.

**Proof** \(A\) is an adjacent matrix of a bipartite graph, so we may arrange the vertices reasonably to get block matrix \(A = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}\).

Let \(\alpha\) is an eigenvalue of \(A\) with eigenvector \(v\), \(Av = \alpha v\). Partition \(v\) like \(A\) to get \(v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\).

\[
\begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = Wv_2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} v_1 \\ v_2 \end{bmatrix},
\]

so \(Wv_2 = \alpha v_1, W^Tv_1 = \alpha v_2\).

Consider vector \(v' = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}\), we have

\[
Av' = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix} \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = Wv_2 \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = \alpha \begin{bmatrix} -v_1 \\ -\alpha v_2 \end{bmatrix} = -\alpha \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = -\alpha v',
\]

it deduces \(-\alpha\) is an eigenvalue of \(A\) with eigenvector \(v'\). Since \(\alpha\) is chosen arbitrarily, the eigenvalues of \(A\) is symmetrical about 0. \(\square\)
Lemma 2  Let $A$ be an adjacent matrix of a weighted graph $G = (V, E, A)$, $\max_{v_i \in V} d(v_i) = D$, $\alpha$ is an eigenvalue of $A$, then $|\alpha| \leq D$.

Proof  Denote the corresponding eigenvector of $\alpha$ is $v = (u_1, u_2, \ldots, u_n)^T$. By $Av = \alpha v$,

\[
\begin{bmatrix}
  a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n \\
  a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n \\
  \vdots \\
  a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n
\end{bmatrix}
= \begin{bmatrix}
  \alpha u_1 \\
  \alpha u_2 \\
  \vdots \\
  \alpha u_n
\end{bmatrix},
\]

Without loss of generality, let $|u_1|$ be the maximum one in $\{|u_i|\}_{i=1}^n$. Suppose $|\alpha| > D$, in the first case that $\alpha > D$,

\[
|\alpha|u_1| = |a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n|
\leq a_{11}|u_1| + \cdots + a_{1n}|u_n|
\leq (a_{11} + a_{12} + \cdots + a_{1n})|v_1|
= d(v_1)|u_1|,
\]

Since $v \neq 0$, $|u_1| > 0$, it deduces $\alpha \leq d(v_1) \leq D$ and contradicts to $\alpha > D$. For the case that $\alpha < -D$, it also draws contradiction and completes the counter-proof.

Lemma 3  Let $G = (V, E)$ be a weighted graph with $n$ vertices, for every vertex $v_i \in V$, $d(v_i) = d$ is a constant. Then $\mathbf{1}_n$ is an eigenvector of $A$ with corresponding eigenvalue $d$.

Proof

\[
A\mathbf{1}_n = \begin{bmatrix}
  a_{11} + a_{12} + \cdots + a_{1n} \\
  a_{21} + a_{22} + \cdots + a_{2n} \\
  \vdots \\
  a_{n1} + a_{n2} + \cdots + a_{nn}
\end{bmatrix}
= \begin{bmatrix}
  d(v_1) \\
  d(v_2) \\
  \vdots \\
  d(v_n)
\end{bmatrix}
= d\mathbf{1}_n,
\]

so $\mathbf{1}_n$ is an eigenvector of $A$ with corresponding eigenvalue $d$.

To prove Theorem 5, we need to estimate the upper bound of eigenvalue and deviations of eigenvectors between $A$ and $A_0$.

By Weyl's theorem on matrix perturbation, the following lemma is deduced instantly:
Lemma 4 Let \( G = (V, E, A) \), \( G_0 = (V_0, E_0, A_0) \) be two weighted graph with \( n \) vertices. The adjacency matrices of \( G \) and \( G_0 \) are \( A \) and \( A_0 \) respectively. For any vertex \( v \in V_0 \), \( d_{G_0}(v) = d \) is a constant. \( A = A_0 + X, \alpha_1, \alpha_2, \ldots, \alpha_n \) are eigenvalues of with descending order. Then

\[
|\alpha_1 - d| \leq \|X\|_2.
\]

Lemma 5 Let \( G = (V, E), G_0 = (V_0, E_0) \) be two weighted graph with \( n \) vertices. The adjacency matrices of \( G \) and \( G_0 \) are \( A \) and \( A_0 \) respectively. For any vertex \( v \in V_0 \), \( d_{G_0}(v) = d \) is a constant. \( A = A_0 + X, \alpha_1, \alpha_2, \ldots, \alpha_n \) are eigenvalues of with descending order. \( e_1 \) is the eigenvector of \( A \) responding to \( \alpha_1 \), then

\[
\|e_1 - 1_n\| \leq \frac{4\sqrt{n(n-1)n}}{(\alpha_2 - \alpha_1)^2} \|X\|_2^2.
\]

Proof Let \( \alpha_1 = d + \mu, e_1 = 1_n + \eta \). By Lemma 4, \( \mu \leq \|X\|_2 \). \( Ae_1 = \alpha_1 e_1 \), thus

\[
(A_0 + X)(1_n + \eta) = (d + \mu)(1_n + \eta),
\]

\[
\|\eta\| \leq \|(A_0 + X - (d + \mu)I)^\dagger\|_F \cdot \|(\mu I - X)\|_F \cdot \|1_n\|. \tag{11}
\]

Let \( \chi_1 \geq \chi_2 \geq \cdots \geq \chi_n \) be eigenvalues of \( X \), then \( \chi_i \leq \|X\|_2 \) for \( 1 \leq i \leq n \). Easy to verify

\[
\|1_n\| = \sqrt{n}, \quad \|\mu I - X\|_F = \sum_{k=1}^{n} (\mu - \chi_k)^2 \leq 4n\|X\|_2^2, \tag{12}
\]

\[
\|(A_0 + X - (d + \mu)I)^\dagger\|_F \leq \sum_{k=2}^{n} \frac{1}{(\alpha_k - d - \mu)^2} = \sum_{k=2}^{n} \frac{1}{(\alpha_k - \alpha_1)^2} \leq \frac{n-1}{(\alpha_2 - \alpha_1)^2}. \tag{13}
\]

Combine (10), (11), (12) and (13), we have

\[
\|\eta\| \leq \frac{4\sqrt{n(n-1)n}}{(\alpha_2 - \alpha_1)^2} \|X\|_2^2.
\]

3 Proofs of theorems

Proof of Theorem 3
Denote the characteristic functions of $S, T$ by $1_S, 1_T$ respectively,
\[
cut(S, T) = \sum_{v_i, v_j \in V} a_{ij} 1_S(v_i) 1_T(v_j) = \langle A 1_S, 1_T \rangle.
\]

From Lemma 1, Lemma 2 and Lemma 3, $d, -d$ are the maximum and minimum eigenvalues $\alpha_1, \alpha_n$ respectively and $1 = (1, 1, \ldots, 1)^T$ is the eigenvector $e_1$ corresponding to $\alpha_1$ while $(1, 1, -1, \ldots, -1)^T$ is the eigenvector $e_n$ corresponding to $\alpha_n$, where number of components 1 of $e_n$ is exactly number of vertices of $G$ in one part and number of components $-1$ of $e_n$ is exactly number of vertices of $G$ in the other part. Orthonormalize eigenvectors $\{e_k\}_{k=1}^n$ of $A$ to get $\{f_k\}_{k=1}^n$. Thus
\[
f_1 = \left( \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)^T,
\]
\[
f_n = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}, \ldots, -\frac{1}{\sqrt{n}} \right)^T.
\]
Coordinate $1_S, 1_T$ as $1_S = \sum_{k=1}^n s_k f_k, 1_T = \sum_{k=1}^n t_k f_k$, then
\[
\langle A 1_S, 1_T \rangle = \sum_{k=1}^n \alpha_k s_k t_k.
\]
As $\{f_k\}_{k=1}^n$ is an orthonormal basis $s_k = \langle 1_S, f_k \rangle, t_k = \langle 1_T, f_k \rangle$, 
\[
 s_1 = \frac{1}{\sqrt{n}} |S| = s_n, \quad t_1 = \frac{1}{\sqrt{n}} |T| = -t_n,
\]
so
\[
\alpha_1 s_1 t_1 + \alpha_n s_n t_n = \frac{d}{m} |S||T|.
\]
For the intermediate coefficients, notice that
\[
\sum_{k=2}^{n-1} |s_k|^2 = \langle 1_S, 1_S \rangle - |s_1|^2 - |s_n|^2 = |S| - \frac{|S|^2}{m} = \frac{1}{m} |S|(m - |S|),
\]
and similarly for the coefficients of $T$. Therefore
\[
\sum_{k=2}^{n-1} |s_k| t_k \leq \frac{1}{m} \sqrt{|S||T|(m - |S|)(m - |T|)},
\]
\[|\text{cut}(S, T) - \frac{d}{m}|S||T| = | \sum_{k=2}^{n-1} \alpha_k s_k t_k | \leq \sum_{k=2}^{n-1} |\alpha_k||s_k||t_k| \leq \alpha_2 \sum_{k=2}^{n-1} |s_k||t_k| \]

and complete the proof. \[\square\]

**Proof of Theorem 4**

We construct a bipartite graph \(G'\) from \(G\) as following: If \(V(G) = \{v_1, v_2, \ldots, v_n\}\), \(V(G') = \{v'_1, v'_2, \ldots, v'_n\} \cup \{v_1, v_2, \ldots, v_n\}\). An edge in \(G'\) exits between \(v_i\) and \(v'_j\) if and only if \(v_i, v'_j\) are adjacent in \(G\) and its weight is equal to \(a_{ij}\). \(G'\) is called \(G\)'s bipartite double in some literatures. Denote the adjacent matrix of \(G'\) by \(A'\), the spectrum of \(A'\) by spec\((A')\), then

\[
A' = \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}, \quad A'^2 = \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix}.
\]

we have spec\((A')\) = spec\((A) \cup -\text{spec}(A)\), then \(\alpha = \max\{\alpha_2, -\alpha_n\}\) is the second largest eigenvalue of \(A'\). Viewing \(S\) and \(T\) as vertex subsets of \(G'\) in different parts, we may apply Theorem 3 to get the claimed bound. \[\square\]

Assume \(S, T \subset V\),

\[
\text{RatioCut}\bigl(\{S, T\}\bigr) = \bigl(\frac{1}{|S|} + \frac{1}{|T|}\bigr) \cdot \text{cut}(S, T).
\]

By the mean inequality,

\[
\frac{1}{2} \left(\frac{1}{|S|} + \frac{1}{|T|}\right) \geq \frac{2}{1/|S| + 1/|T|} = \frac{2}{|S| + |T|},
\]

so

\[
\text{RatioCut}\bigl(\{S, T\}\bigr) \geq 4 \frac{\text{cut}(S, T)}{|S| + |T|} = \frac{\text{cut}(S, T)}{(|S| + |T|)/4}.
\]

When \((S, T)\) is a 2-partition of \(V\), i.e.

\[S \cup T = V(G), \quad S \cap T = \emptyset.\]

Let \(|V(G)| = n = |S| + |T|\), we have

\[|S||T| = |S|(n - |S|), \quad 1 \leq |S| \leq n - 1,\]

it follows

\[n - 1 \leq |S||T| \leq n^2/4.\]

When \(n \geq \frac{4}{3}\), \(n/4 \leq n - 1\), thereby

\[(|S| + |T|)/4 \leq |S||T|,\]
and
\[ \text{RatioCut} \{S, T\} \geq \frac{\text{cut}(S, T)}{|S| + |T|}/4 \geq \frac{\text{cut}(S, T)}{|S||T|}, \]

It follows that the lower bound for \( \frac{\text{cut}(S, T)}{|S||T|} \) is also for \( \text{RatioCut} \{S, T\} \), but unfortunately the bound can’t be tight anyway. Because when \( n \geq \frac{4}{3} \) the equality cannot be established in any graph.

**Corollary 1** Let \( G = (V,E) \) be weighted graph with \( n \) vertices, for every vertex \( v_i \in V \), \( d(v_i) = d \) is a constant. \( \alpha_1, \alpha_2, \ldots, \alpha_n \) are eigenvalues of \( A \) with descending order and \( \alpha = \max \{ \alpha_2, -\alpha_n \} \). Then for a partition \( (S, T) \) for \( V \),
\[ \text{RatioCut} \{S, T\} > \frac{d - \alpha}{n}. \]

**Proof of Corollary 1** In Theorem 4, let \( A = S, B = T \). Because \((S, T)\) is a partition, \( S^c = T, T^c = S \). Expand the absolute value,
\[ \text{cut}(S, T) \geq \frac{d - \alpha}{n} |S||T|, \]
Divide both sides by \( |S||T| \),
\[ \text{RatioCut} \{S, T\} > \frac{\text{cut}(S, T)}{|S||T|} \geq \frac{d - \alpha}{n}, \]
which completes the proof. \( \square \)

**Proof of Theorem 5**

Denote the upper bounds of \( |\alpha_1 - d| \) and \( \|e_1 - 1_n\| \) by \( \epsilon_1 \) and \( \epsilon_2 \) respectively. By Lemma 4 and Lemma 5 let
\[ \epsilon_1 = \|X\|_2, \quad \epsilon_2 = \frac{4\sqrt{n}(n-1)n}{(\alpha_2 - \alpha_1)^2} \|X\|_2^2. \]

Let \( 1_S, 1_T \) be the indicator vectors of \( S, T \), \( \text{cut}(S, T) = \langle A1_S, 1_T \rangle \).
\[ d + \mu = \alpha_1 > \alpha_2 \geq \cdots \geq \alpha_n = -\alpha_1 = -d - \mu \]
are eigenvalues of \( A \). Denote the eigenvector responding to \( \alpha_1 \) as \( e_1 = 1 + v \). By proof of Lemma 1, \( e_n \) can be obtained by reversing the signs of vector components responding to \( X \) or \( Y \), supposing \( Y \). So \( \|e_n\| = \|e_1\| \).

Orthonormalize \( A \)'s eigenvectors \( \{e_k\}_{k=1}^n \) to get \( \{f_k\}_{k=1}^n \), and let \( 1_S = \sum_{k=1}^n s_k f_k, 1_T = \sum_{k=1}^n t_k f_k \), then \( \text{cut}(S, T) = \langle A1_S, 1_T \rangle = \sum_{k=1}^n \alpha_k s_k t_k \).

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Since \( f_1 = \frac{e_1}{\|e_1\|} = \frac{1}{\|e_1\|} + \frac{v}{\|e_1\|} \), \( s_1 = \langle 1_S, f_1 \rangle = \langle 1_S, \frac{1}{\|e_1\|} \rangle + \langle 1_S, \frac{v}{\|e_1\|} \rangle = \frac{1}{\|e_1\|} |S| + \langle 1_S, \frac{v}{\|e_1\|} \rangle \). Define \( m_1 = \langle 1_S, \frac{v}{\|e_1\|} \rangle \), \( |m_1| \leq \|1_S\| \cdot \|v\| \cdot \frac{1}{\|e_1\|} \leq \frac{\sqrt{n-\epsilon_2}}{\sqrt{n-\epsilon_2}} \sqrt{|S|} \). \( s_n = \langle 1_S, f_n \rangle = s_1, t_1 = \langle 1_T, f_1 \rangle = -t_n = \frac{1}{\|e_1\|} |T| + \langle 1_T, \frac{v}{\|e_1\|} \rangle \). Define \( m_2 = \langle 1_T, \frac{v}{\|e_1\|} \rangle \) and \( |m_2| \leq \frac{\sqrt{n-\epsilon_2}}{\sqrt{n-\epsilon_2}} \sqrt{|T|} \) similarly. So
\[
\alpha_1 s_1 t_1 + \alpha_n s_n t_n = 2(d + \mu) \left( \frac{|S|}{\|e_1\|} + m_1 \right) \left( \frac{|T|}{\|e_1\|} + m_2 \right) = \frac{2d}{\|e_1\|^2} |S||T| + R_1,
\]
where
\[
R_1 = 2dm_1 \left( \frac{|T|}{\|e_1\|} + 2dm_2 \frac{|S|}{\|e_1\|} + 2dm_1 m_2 + 2\mu \frac{|S||T|}{\|e_1\|^2} + 2\mu m_1 \frac{|T|}{\|e_1\|} + 2\mu m_2 \frac{|S|}{\|e_1\|} + 2\mu m_1 m_2 \right).
\]
So
\[
cut(S, T) = \alpha_1 s_1 t_1 + \alpha_n s_n t_n + \sum_{k=2}^{n-1} \alpha_k s_k t_k = \frac{2d}{\|e_1\|^2} |S||T| + R_1 + \sum_{k=2}^{n-1} \alpha_k s_k t_k,
\]
\[
\left| \cut(S, T) - \frac{2d}{\|e_1\|^2} |S||T| \right| = R_1 + \sum_{k=2}^{n-1} \alpha_k s_k t_k,
\]
\[
\left| \cut(S, T) - \frac{2d}{(\sqrt{n} - \epsilon_2)^2} |S||T| \right| \leq |R_1| + \sum_{k=2}^{n-1} |\alpha_k s_k t_k| \leq |R_1| + \alpha_2 \sum_{k=2}^{n-1} |s_k||t_k|.
\]
Next we will estimate \( |R_1| \) and \( \sum_{k=2}^{n-1} |s_k||t_k| \) respectively.
\[
|R_1| = |2dm_1 \left( \frac{|T|}{\|e_1\|} + 2dm_2 \frac{|S|}{\|e_1\|} + 2dm_1 m_2 + 2\mu \frac{|S||T|}{\|e_1\|^2} + 2\mu m_1 \frac{|T|}{\|e_1\|} + 2\mu m_2 \frac{|S|}{\|e_1\|} + 2\mu m_1 m_2 \right)|
\leq 2 \frac{\sqrt{|S||T|}}{\|e_1\|^2} \left[ d_2 \left( \sqrt{|T|} + \sqrt{|S|} \right) + d_2 \epsilon_1 + \epsilon_1 \sqrt{|S||T|} + \epsilon_1 \epsilon_2 \left( \sqrt{|T|} + \sqrt{|S|} \right) \right]
\leq 2 \frac{\sqrt{|S||T|}}{\|e_1\|^2} \left[ d_2 \left( d + \epsilon_2 \right) \left( \sqrt{|T|} + \sqrt{|S|} + \epsilon_2 \right) + \epsilon_1 \sqrt{|S||T|} \right].
\]
Since \(|S| + |T| \leq n, \sqrt{|S||T|} \leq \frac{n}{2}, \sqrt{|S|} + \sqrt{|T|} \leq \sqrt{2n}, \) we have
\[
|R_1| \leq \frac{n}{(\sqrt{n} - \epsilon_2)^2} \left[ d_2 \left( d + \epsilon_1 \right) \left( \sqrt{2n} + \epsilon_2 \right) + \frac{n}{2} \epsilon_1 \right].
\]
Afterwards, we exploit Cauchy-Schwartz’s inequality to estimate $\sum_{k=2}^{n-1} |s_k||t_k|$.

\[
\sum_{k=2}^{n-1} |s_k|^2 = \langle 1_S, 1_S \rangle - |s_1|^2 - |s_n|^2 = |S| - 2\left(\frac{|S|}{\|e_1\|} + m_1\right)^2 \tag{16}
\]

\[
\sum_{k=2}^{n-1} |t_k|^2 = \langle 1_T, 1_T \rangle - |t_1|^2 - |t_n|^2 = |T| - 2\left(\frac{|T|}{\|e_1\|} + m_1\right)^2 \tag{17}
\]

Let $R_2 = -4m_1\frac{|S|}{\|e_1\|} - 2m_1^2, R_3 = -4m_1\frac{|T|}{\|e_1\|} - 2m_1^2$, then

\[
\sum_{k=2}^{n-1} |s_k||t_k| \leq \sqrt{\sum_{k=2}^{n-1} |s_k|^2 \sum_{k=2}^{n-1} |t_k|^2}
\]

\[
= \sqrt{\left[\frac{|S|}{|v_1|^2}(|v_1|^2 - 2|S|) + R_2\right]\left[\frac{|T|}{|v_1|^2}(|v_1|^2 - 2|T|) + R_3\right]} \leq \frac{1}{|v_1|^2}\sqrt{|S|(|v_1|^2 - 2|S|)|T|(|v_1|^2 - 2|T|)}
\]

\[
+ \sqrt{R_2\frac{|T|}{|v_1|^2}(|v_1|^2 - 2|T|)} + \sqrt{R_3\frac{|S|}{|v_1|^2}(|v_1|^2 - 2|S|)} + \sqrt{R_2R_3}
\]

\[
\leq \frac{1}{(\sqrt{n} - \epsilon_2)^2}\sqrt{|S|(|v_1|^2 - 2|S|)|T|(|v_1|^2 - 2|T|)} + \frac{\sqrt{2}}{4}(\sqrt{R_2} + \sqrt{R_3}) + \sqrt{R_2R_3}. \tag{18}
\]

Since

\[
|R_2| = 2m\left(2\frac{|S|}{\|e_1\|} + m_1\right) \leq 2|m_1|\left(\frac{2|S|}{\|e_1\|} + m_1\right) \leq \frac{2|S|\epsilon_2}{(\sqrt{n} - \epsilon_2)^2}(2\sqrt{|S|} + \epsilon_2),
\]

\[
|R_3| \leq \frac{2|T|\epsilon_2}{(\sqrt{n} - \epsilon_2)^2}(2\sqrt{|T|} + \epsilon_2),
\]

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therefore,
\[
\sum_{k=2}^{n-1} |s_k||t_k| \leq \frac{1}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|(\|e_1\|^2 - 2|S|)|T|(\|e_1\|^2 - 2|T|)} + \frac{\sqrt{\epsilon_2}}{2} \frac{\sqrt{n} + \epsilon_2}{|\sqrt{n} - \epsilon_2|} (\sqrt{|S|(2\sqrt{|S|} + \epsilon_2) + \sqrt{|T|(2\sqrt{|T|} + \epsilon_2)}) + \frac{2\epsilon_2}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|(2\sqrt{|S|} + \epsilon_2)|T|(2\sqrt{|T|} + \epsilon_2)}.
\]
(19)

Substitute estimations of $|R_1|$ and $\sum_{k=2}^{n-1} |s_k||t_k|$ into
\[
|\text{cut}(S, T) - \frac{2d}{(\sqrt{n} - \epsilon_2)^2} |S||T| | \leq |R_1| + \alpha_2 \sum_{k=2}^{n-1} |s_k||t_k|,
\]
we have
\[
|\text{cut}(S, T) - \frac{2d}{(\sqrt{n} - \epsilon_2)^2} |S||T| | \leq \frac{n}{(\sqrt{n} - \epsilon_2)^2} [\epsilon_2(d + \epsilon_1)(\sqrt{2n} + \epsilon_2) + \frac{n}{2} \epsilon_1] + \alpha_2 [\frac{1}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|((\sqrt{n} + \epsilon_2)^2 - 2|S|)|T|((\sqrt{n} + \epsilon_2)^2 - 2|T|)} + \frac{\sqrt{\epsilon_2}}{2} \frac{\sqrt{n} + \epsilon_2}{|\sqrt{n} - \epsilon_2|} (\sqrt{|S|(2\sqrt{|S|} + \epsilon_2) + \sqrt{|T|(2\sqrt{|T|} + \epsilon_2)}) + \frac{2\epsilon_2}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|(2\sqrt{|S|} + \epsilon_2)|T|(2\sqrt{|T|} + \epsilon_2)}]
\leq \frac{\alpha_2}{(\sqrt{n} - \epsilon_2)^2} \sqrt{|S|((\sqrt{n} + \epsilon_2)^2 - 2|S|)|T|((\sqrt{n} + \epsilon_2)^2 - 2|T|)} + (1 + \epsilon_1)\epsilon_2^2 \cdot O(n).
\]
(20)

**Proof of Theorem 6**

Similar to the proof of Theorem 4, we construct a bipartite graph $G'$ from $G$, $\alpha = \max\{\alpha_2, -\alpha_n\}$ is the second largest eigenvalue of $A'$. Viewing $S$ and $T$ as vertex subsets of $G'$ in different parts, we may apply Theorem 5 to get the claimed bound.

**4 Conclusion**

In this paper, we give some cut bounds for weighted graphs with regular property and investigate the corresponding estimations for more general
cases. For further research, similar forms of bounds for $k-$partition ($k \geq 3$) are still unknown and worthy of study. Besides, when we have more information about the graph, how to improve those bounds is another research direction.

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