Convergence - Divergence of the Finite Volume Method for the Fluid and Heat Flow in Heterogeneous Porous Rocks

Mario-César Suárez-Arriaga1,* and Mahendra Pal Verma2
1International Geothermal Association - Mexican Geothermal Association, Mexico
2Instituto Nacional de Electricidad y Energías Limpias, Palmira, Mor., Mexico

*Corresponding author email: mcsa50@gmail.com

Abstract. Fluid velocity has an important impact in the Finite Volume numerical method (FVM) used to model fluid and energy flows in heterogeneous porous media. This formulation is based on the energy flow conservation equations under local thermal non-equilibrium conditions (LTNE). The heat transfer from the solid matrix to the moving non-isothermal fluid inside the pores is simulated for several velocities and global different heat transfer coefficients. During the numerical simulation of coupled heat/mass flow in a heterogeneous porous system it is necessary to average highly variable physical parameters at the boundaries between different domains represented geometrically in the computational mesh of the FVM. This can be effectively achieved using appropriate average techniques at the contact interfaces of each domain. The averaging process should correctly represent the behaviour of the fluid velocity crossing different areas of the reservoir. Many numerical divergence problems arise from the fluid velocity value and the interfacial interactions at the boundaries of different continua. The averages have a decisive influence on the numeric results of the simulation. A very stable, convergent averaging scheme for the FVM is presented herein and compared to new analytical diffusion-convection models. Especial attention is paid to the dynamic process of cold water (50°C) injected into a geothermal reservoir at higher temperature (350°C). The fluid temperature profile is calculated when the fluid migrates at constant speed through a permeable corridor from a fluid injection point to an extraction zone. The physical reason for the divergence of a numerical model, and the exploration of the range of validity of the LTNE hypothesis during the injection of cold water into a hot reservoir are both explained.

Keywords: Finite volume method, numerical divergence, fluid and heat flow in porous rocks.

1. Introduction

Numerical simulations of coupled heat and mass flows in heterogeneous porous systems, need to average highly variable parameters at the boundaries between different media. Averaging has a decisive influence on the results of a heterogeneous porous media simulation in any numerical method. The averaging process should correctly represent the behaviour of the fluid crossing areas of the system with different physical properties. A frequent assumption made in porous medium simulation is that both phases, solid and fluid, are locally at the same temperature, which is called Local Thermal Equilibrium hypothesis (LTE). In this work the attention is focused only on the fluid velocity averaging process when the skeleton of the solid and the fluid are at different temperatures. Both phases interact through a volumetric heat transfer mechanism under local thermal non-equilibrium conditions (LTNE). Depending on the average formula used, the behaviour of mass and energy flows will be different producing frequent numerical divergences. This phenomenon is independent of the numerical method used to simulate the physical processes involved. The Finite Volume Method (FVM)
is a generalization of the finite difference method. The FVM has the intrinsic ability to represent the main conservation laws of flow in geothermal reservoirs for example [1]. In this paper it is numerically shown that one of the main divergence problems arises when the flow velocity is higher than a threshold value. The average formula used at the interfaces among neighbour elements will determine the numerical stability and convergence of the method to compute accurately and correctly the fluid and heat flow. To compute the flow at the boundaries of the reservoir, several averaging formulas are introduced and tested. As a general trend, when the flow velocity increases, the numerical procedure diverges and/or oscillates at high velocities. The final objective is to produce efficient differentiation schemes that are physically realistic, do not produce oscillations and are numerically stable during the simulation of complex flows in any highly heterogeneous porous system. This technique is general and applicable to any heterogeneous porous medium; to illustrate it with a concrete example, attention is focused on rocks of geothermal reservoirs.

2. The Conduction-convection Local Thermal Nonequilibrium Mathematical Models

The general energy flow equation in non-isothermal porous systems includes conduction and convection through an advective term related to the transport of heat by the fluid phase moving inside the pores and heat conduction in the solid phase, plus an extra term for the energy transfer between the solid and the fluid phases that can be at different temperatures. Under these conditions there is no longer local thermal equilibrium (LTE) and $T_s \neq T_f$, except at the solid-fluid common boundary $\Gamma_{sf}$. In local thermal non-equilibrium (LTNE) state, when dispersion, viscous dissipation and radiative effects are negligible, this combined energy transfer process is modelled by the following system of two partial differential equations, one for the solid phase ($s$) and one for the fluid phase ($f$) [2], [3]:

\[
\frac{\partial}{\partial t} \left( (1-\phi) c_s \rho_s T_s \right) - (1-\phi) \vec{V} \cdot \left( \mathbf{k}_s \cdot \vec{V} T_s \right) = (1-\phi) q_s - (1-\phi) q_{sf} \rightarrow \left[ \frac{W}{m^3} \right] \quad (1)
\]

\[
\frac{\partial}{\partial t} \left( \phi c_f \rho_f T_f \right) - \phi \vec{V} \cdot \left( \mathbf{k}_f \cdot \vec{V} T_f \right) + \vec{V} \cdot \left( c_f \rho_f T_f \vec{v}_f \right) = \phi q_{sf} \rightarrow \left[ \frac{W}{m^3} \right] \quad (2)
\]

where $t$, $\phi$, $c$, $\rho$, $T$, $\mathbf{k}$, $\vec{v}_0$, $q_s$, and $q_{sf}$ are time, porosity, heat capacity, density, temperature, conductivity tensor, Darcy velocity and volumetric heat generation of the solid ($s$) and fluid ($f$) phases respectively. The function $q_{sf}$ (W/m$^3$) is the amount of volumetric heat transferred from the solid matrix to the fluid and vice versa, and is proportional to the solid-fluid temperature difference $\Delta T = (T_s - T_f)$. The volumetric heat $q_{sf}$ depends also on thermal coefficients and geometric variables discussed in the next subsection. The conservation of the fluid mass in the porous medium is:

\[
\frac{\partial}{\partial t} \left( \phi \rho_f \right) + \vec{V} \cdot \left( \rho_f \vec{v}_f \right) = 0 \rightarrow \left[ \frac{kg}{s \ m^3} \right], \quad \vec{v}_f = \phi \vec{v}_f = -\frac{K}{\mu_f} \left( \vec{V} p - \rho_f \vec{g} \right) \quad (3)
\]

where $K$, $\mu_f$, $p$, $g$ are absolute permeability tensor, fluid viscosity, fluid pressure and gravity acceleration, respectively. The microscopic fluid velocity in the pores and fractures $\vec{v}_f$ is related to the Darcy flux $\vec{v}_0$ by the Dupuit-Forchheimer relation $\vec{v}_f = \phi \vec{v}_f$ [1]. Assuming ($\partial/\partial t = 0$) for steady state, replacing the mass conservation (3) into equation (2), within volumes $V_s$ and $V_f$ of the solid and fluid phases respectively, a steady state LTNE coupled model is obtained:

\[
\vec{V} \cdot \left( \frac{\mathbf{k}_s}{c_s \rho_s} \cdot \vec{V} T_s \right) = -\frac{q_s}{c_s \rho_s} + \frac{q_{sf}}{c_s \rho_s} = Q_s \rightarrow \left[ \frac{^\circ C}{s} \right] \quad (a)
\]

\[
\vec{V} \cdot \left( \frac{\mathbf{k}_f}{c_f \rho_f} \cdot \vec{V} T_f \right) - \frac{\vec{v}_f}{\phi} \cdot \vec{V} T_f = -\frac{q_{sf}}{c_f \rho_f} = -Q_f \rightarrow \left[ \frac{^\circ C}{s} \right] \quad (b)
\]

Where $q_{sf} = h_f (T_s - T_f) \ [W/m^3]$, $h_f = \gamma_{sf} h_{sf}$, $\gamma_{sf} = \Gamma_{sf} / V_s$, $h_f$ (W/m$^3/^\circ$C) is the volumetric heat transfer coefficient, $Q_s$ and $Q_f$ are the symbols to quantify the global solid-fluid heat transfers; $h_{sf}$
(W/m²/°C) is the interfacial heat transfer coefficient, coupling the heat equation of the skeleton (1) to the fluid heat equation (2); \( \gamma_{sf} \) is a geometric factor, which is discussed later, and \( \Gamma_{sf} \) (m²) represents the surface of the common solid/fluid boundary (Figure 1).

\[ 3 \]

Figure 1. Physical macroscale models of porous rocks illustrating the skeleton, the pores, the heat flow \( q_{sf} \) and other geometric elements, the fluid-solid contact surface \( \Gamma_{sf} \), the particle diameter \( d_p \) and the pore diameter \( d_p \). Left: wall made with volcanic rocks \( (A \sim 1.2\text{ m}^2, d_p \sim 35\text{ cm}, d_o \sim 4\text{ cm}) \). Right: floor with rounded stones \( (A \sim 0.25\text{ m}^2, d_p \sim 5\text{ cm}, d_o \sim 1.5\text{ cm}) \).

2.1. The Solid-fluid Heat transfer Coefficient \( h_{sf} \) and Other Parameters

There are two types of heat transfer coefficients, the volumetric coefficient \( h_V \) (W/m³/°C), and the interfacial coefficient \( h_{sf} \) (W/m²/°C), for the energy transfer between the solid matrix and the fluid. This last coefficient can be determined experimentally only in some simple cases [2], [3]. Under the same temperature difference, thermal energy flows along paths of high conductivity and least resistance. Low conductivity regions act as a retardant to heat flow.

Examples of heat parameters: acidic rocks present a volumetric heat generation around \( 2 \times 10^{-6} \text{ W/m}^3 \); basic rocks produce heat in the range \( [10^{-8}, 10^{-7}] \text{ W/m}^3 \), [1]. For temperature differences \( (T_s - T_f) \) between rock - fluid in the range \( [1, 300] \text{ °C} \), the corresponding heat transfer term \( h_V (T_s - T_f)/\rho_f/c_f \) can vary between \( 10^{-2} \) and \( 10^{-7} \text{ °C/s} \), depending on the value of the factor \( \gamma_{sf} \). The two temperatures can vary within the horizontal distance; increases in velocity generally result in increases in heat transfer coefficient \( h_{sf} \). The physical meaning of \( h_V \) and \( h_{sf} \) correspond to very complex phenomena, which depend on many factors such as fluid and rock properties, distribution and topology of pores, geometry of both, the solid skeleton and the fractures, dimensions and contact area between pores-fractures and matrix (Figure 1). The area \( \Gamma_{sf} \) is extremely difficult to compute exactly in real systems, because unknown geometrical factors such as \( \gamma_{sf} \), \( d_p \) and \( d_o \) are needed. The unknown geometric factor \( \gamma_{sf} = \Gamma_{sf}/V_s \) can be computed exactly only in artificial porous media for very simple contact surfaces in idealized cases; for spheres the ratio is \( \gamma_{sf} = 6/d_p \); for cylinders \( \gamma_{sf} = 4/d_p \), in rectangles \( \gamma_{sf} = 1/d_p \), etc. Because of the complicated and almost unknown internal geometry of the pores and fractures themselves, and dimensions of the contact surface \( \Gamma_{sf} \) (Figure 1) and the lack of experimental data, other types of approach need to be used in geothermal porous rocks to estimate correctly the solid-fluid heat transfer in the LTNE model. These approaches could come from numerical deduction in physical-mathematical models such as the one presented herein.

3. The Finite Volume Method to Solve the Conduction-convection Model

The FVM or Method of Integrated Finite Volumes [4], [5], [6] is a powerful numerical technique that can be used to solve the partial differential equations of the general LTNE equations (1) and (2). The key factor is the approximation of flows at the boundary of each control volume. In the next section the FVM is formulated to solve approximately equations (4a) and (4b) using this technique. Assuming homogeneous properties in the fluid-rock system, let \( V_s \) and \( V_f \) be the volumes occupied by the solid and the fluid phase respectively in an elementary volume \( V_n \). Integrating the model represented by equation (4a) and applying the divergence theorem over each volume \( V_n \):
Integrating in a similar way the model represented by equation (4b):

\[
\int_{V_n} \left( \frac{k_f}{\rho_f \phi_f} \nabla T_f \right) dV_f = \int_{\Gamma_n} \eta_f \left( \frac{\partial T_f}{\partial n} - \frac{\nabla T_f}{\phi_f} \right) \cdot \hat{n} d\Gamma_n = \left( \frac{\nabla T_f}{\phi_f} \right) dV_f - \int_{V_n} Q_f dV_f, \quad n = 1, N
\]  

(6)

Where \( \eta_{s,f} \) are solid, fluid diffusivities, \( A \) is the area of the surface \( \Gamma_n \) overlapping the finite volume \( V_n \). The terms inside the parentheses are average values over the whole surface \( \Gamma_n \) and over each volume \( V_n \), where \( N \) is the total number of finite volumes composing the computational mesh. A simple one-dimensional mesh is shown in figure 2. Inside the 1D domain local thermal equilibrium is assumed; the temperature of the rock above and below this domain is larger than the fluid temperature in the mesh and there is no local thermal equilibrium between both zones.

**Figure 2.** A uniform 1D mesh and main elements of the FVM: Volumes \( V_n \), internal boundaries \( \Gamma_n \), internodal distances \( \delta x_n = x_n - x_{n+1} \), solid and fluid temperatures \( (T_f \neq T_s) \) in the LTNE model, the heat transferred from the solid matrix to the fluid is \( q_{sf} \); the Darcy velocity \( v_D \) goes in the positive direction.

### 3.1. The Local Thermal Non-Equilibrium Model formulated using Finite Volumes in 1D

After integrating equations (5) and (6) and applying the divergence theorem to the first integral over \( V_n \), two systems of algebraic equations are obtained that approximates the LTNE model:

\[
\left\{ \eta_s \frac{\partial T_s}{\partial n} A \right\}_{\Gamma_n} = \left( Q_s, V_s \right); \quad \left\{ \eta_f \frac{\partial T_f}{\partial n} A \right\}_{\Gamma_n} = \left( \frac{\nabla T_f}{\phi_f} \cdot \hat{n} - \frac{\nabla T_f}{\phi_f} \right), \quad \left( Q_f, V_s \right), \quad \{ n = 1, N \}
\]  

(7)

where \( A \) is the total area of the surface \( \Gamma_n \) overlapping the finite volume \( V_n \) of porous rock. The terms inside the parentheses are average values over the whole surface \( \Gamma_n \) and over each volume \( V_n = A \delta x_n \) respectively; \( Q_s, Q_f \) are the heats exchanged between the solid particles and the fluid in the volume \( V_n \), and \( N \) is the total number of finite volumes or nodes composing the computational mesh. Inside the \( \{V_n\} \) domain, local thermal equilibrium is assumed; the temperature of the rock above and below this domain is larger than the fluid temperature in the mesh and there is local thermal non-equilibrium between both zones. The \( q_{sf} \) transfer occurs only at this common interface. Computing averages for the temperature and fluid properties in equation (7) and assuming equal lateral areas \( A \) (right \( R \) and left \( L \)) for the heat and mass flows between both lateral boundaries \( \Gamma_R \) and \( \Gamma_L \):

\[
\begin{align*}
\eta_s \left( \frac{\partial T_s}{\partial n} A \right)_R - \eta_s \left( \frac{\partial T_s}{\partial n} A \right)_L &= \eta_f \left( \frac{\partial T_f}{\partial n} A \right)_R - \left( v_f T_f A \right)_R + \left( v_f T_f A \right)_L + Q_f A \delta x_n = 0 \Rightarrow \\
\eta_{s+1} T_{s+1} - T_s - \frac{\eta_s - \eta_{s+1}}{\delta x_{s+1}} &= - v_f \left( T_f \right)_R + v_f \left( T_f \right)_L + Q_f \delta x_n = 0, \quad \{ n = 2, N-1 \}
\end{align*}
\]  

(8)

where \( \eta_{s,n} \) represent average diffusivity values at the corresponding interface \( \{ n = 1, N \} \), the boundary conditions at \( \Gamma_R \) and \( \Gamma_L \) are included; \( \delta x_n = x_{n+1} - x_n \) is the distance between the center of \( V_n \) and the center of \( V_{n+1} \), \( (n = 1, N-1) \). The symbols \( (T_f)_R, (T_f)_L \) represent average values for the changing temperature at each internal boundary \( \Gamma_R \) and \( \Gamma_L \) of each finite volume \( V_n \).
3.2. The Weighted Average in the Finite Volume Method

The common weighted general average is:

\[
(T_f)_n = \alpha_R T_{n+1} + (1 - \alpha_R) T_n , \quad (T_f)_L = \alpha_L T_n + (1 - \alpha_L) T_{n-1} ; \quad \alpha_L, \alpha_R \in [0, 1]
\]

(9)

where \(\alpha_L = 1 - \alpha_R\), and \(0 \leq \alpha_R \leq 1\), are the weight coefficients representing the influence of the fluid velocity at each boundary. Other type of averages can be used, and the corresponding numerical results can be different as is shown in next section. Using explicitly this average, developing and factorizing the last expression in equation (8), a tridiagonal system of algebraic equations is obtained:

\[
\eta_{n+1} - \eta_n = \frac{\eta_{n+1} - \eta_{n-1}}{\alpha_{n+1}} - \frac{\eta_{n+1} - \eta_{n-1}}{\alpha_n} - \nu f (\alpha_R T_{n+1} + (1 - \alpha_R) T_n) + \nu f (\alpha_L T_n + (1 - \alpha_L) T_{n-1}) + Q_n \delta x_n = 0
\]

\[
\iff \quad -a_{n+1} T_{n+1} + b_n T_n - c_{n-1} T_{n-1} = Q_n \delta x_n , \quad \{ n = 2, N-1 \}
\]

(10)

where: \(a_{n+1} = \frac{\eta_{n+1}}{\alpha_{n+1}} \nu f\), \(b_n = \frac{\eta_{n+1} + \eta_{n-1}}{\alpha_n} - (1 - \alpha_R - \alpha_L) \nu f\), \(c_{n-1} = \frac{\eta_{n-1}}{\alpha_{n-1}} + (1 - \alpha_L) \nu f\)

The boundary conditions, \(T = T_0\) for \(n = 1\), and \(T = T_L\) for \(n = N\) are known and computed separately.

3.3. A Diffusion Exact Analytical Model in 1D

To compare the convergence of the FVM, an analytical solution is needed for an idealized case. Two exact solutions are presented, one here and another one in next subsection. Let \(V_S\) be the skeleton elementary volume. Thermal energy is diffused by conduction through a two-dimensional rock domain with low porosity (Figure 3), where the matrix temperature only depends on the vertical coordinate \(z\) (m) and the boundary conditions are constant and fixed at the bottom \((T_B)\) and the top \((T_A)\) respectively, \(T_B > T_A\). In steady-state conditions and homogeneous rock, under appropriate assumptions in 1D, the diffusion model described by equation (4a), with constant boundary conditions in \(V_S\), is:

\[
\frac{d}{dz} \left( \eta_s \frac{dT_s(z)}{dx} \right) = Q_s, \quad T_s(z=0) = T_B, \quad T_s(z=100) = T_A ; \quad \eta_s = \frac{k_s}{c_s \rho_s}
\]

(11)

where \(\eta_s\) [m²/s] is the thermal diffusivity of the solid matrix. Integrating twice equation (11):

\[
T_s(z) = T_B + \frac{T_A - T_B}{L} z + \frac{Q_s (z-L) z}{2 \eta_s}
\]

(12)

Where \(L_x\) is an arbitrary horizontal length. Figure 4 shows the vertical distribution of temperature obtained with this pure diffusion model in the range \(0 \leq Z \leq 100\) m, in the interval \(350°C \leq T \leq 370°C\). The continuous curve is the exact solution (12), the red points are the approximate solution obtained with the FVM, which converges to the exact solution even using a coarse mesh. Fluid thermal parameters needed to compute the graphics of this and of next section are shown in Table 1.
Figure 4. Vertical distribution of temperatures obtained from the analytic solution (12) and the Finite Volume Method numerical solution of equation (5) using a coarse mesh of only 12 nodes.

Table 1. Rock and Fluid Properties for different pressures and temperatures.

| Rock Props. | φ (ad) | (1 - φ) | ρs (kg/m³) | ks (W/m/°C) | cs (J/kg/°C) | ηs (m²/s) | d0 (m) |
|-------------|--------|---------|------------|-------------|-------------|-----------|--------|
| φ (ad)      | 0.116  | 0.884   | 2370       | 2.14        | 1130.4      | 7.99×10⁻⁷ | 0.001  |

| Fluid Props. | p (bar) | T (°C) | ρf (kg/m³) | kf (W/m/°C) | cf (J/kg/°C) | ηf (m²/s) | µf (Pa.s) |
|--------------|---------|--------|------------|-------------|-------------|-----------|----------|
|              | 1       | 50     | 990.2      | 0.6459      | 4170        | 15.64×10⁻⁸ | 54.75×10⁻⁵ | 3.54    |
|              | 2       | 170    | 579.0      | 0.4498      | 9683        | 8.02×10⁻⁸  | 6.64×10⁻⁵  | 1.43    |
|              | 3       | 220    | 493.0      | 0.4105      | 18,190      | 4.58×10⁻⁸  | 5.67×10⁻⁵  | 2.51    |

3.4. A Conduction-convection Exact Model in 1D

Assuming heterogeneous properties in the fluid-rock system, let \( V_s \) and \( V_f \) be the elementary volumes representing the solid matrix and the fluid respectively; \( V_B = V_s + V_f \) is the bulk volume. Thermal energy is transported by conduction and convection through a one-dimensional domain of porous rock of volume \( V_B \), as illustrated in figure 2. It is assumed that below and above this domain there is a very low porosity impervious rock of volume \( V_S \), which transfers constant volumetric heat to the upper part of the 1D corridor with fluid flowing with constant velocity \( v_f \). This model idealizes a situation occurring when colder water at \( T_0 \) °C is injected into a geothermal reservoir at \( T_L \) °C, with an important temperature difference (\( T_0 \ll T_L \)). The injection well is located at the origin \( 0 \) and the fluid migrates to the right a distance \( L \). In steady state the model described by equation (4b) is:

\[
\frac{d}{dx} \left( \eta_f \frac{dT_f(x)}{dx} \right) - \frac{v_f}{\phi} \frac{dT_f(x)}{dx} = -Q_f, \quad \left\{ \begin{array}{l} T_f(x = 0) = T_0 \\ T_f(x = L) = T_L \end{array} \right., \quad \eta_f = \frac{k_f}{c_f \rho_f} \tag{13}
\]

where \( \eta_f [m^2/s] \) is the fluid thermal diffusivity. This ordinary linear differential equation for \( T_f(x) \) has an exact solution \( \tau_f(x) \) in terms of the boundary conditions \( T_0 \) and \( T_L \), integrating twice (13):

\[
\tau_f(x) = \left( \phi Q_f L + (T_0 - T_L) v_f \right) e^{\phi x} - \left( T_0 v_D + \phi Q_f x \right) e^{\phi x} - \frac{\phi Q_f (x - L) + T_L v_D}{1 - e^{\phi L}} e^{\phi x} \tag{14}
\]
Where \( v_f \) is the fluid velocity in the pores \( (v_f = v_D/\phi) \). Darcy’s law is valid for laminar flows where the viscous forces opposing the fluid flow prevail over the inertial forces related to the fluid acceleration. Solution (14) is also valid for non-Darcy fluxes. Figure 5 shows the non-convergence behaviour of the FVM solution for different fluid velocities, using the weighted average formula with two different coefficients \( \{\alpha_L, \alpha_R\} \) and the same solid-fluid heat transfer. It is evident that in higher flow dimensions 2D or 3D, this divergence becomes even worse because heat flow directions are more complex, and the fluid paths can be more intricate because of tortuosity and other heterogeneities in a real reservoir.

Using weighted averages, the FVM solution produces oscillations and inaccurate approximations specifically at the end of the 1D domain. When the fluid velocity increases from 5 to \( 40 \times 10^{-8} \text{ m/s} \), the oscillations increase in number and the situation becomes worse for larger velocities. The failure of the weighted average in the FVM to simulate combined convection and diffusion flows comes from the fact that this simple average does not recognize the flow direction \( (v_f > 0) \) in the OX axis nor the strength of convection relative to diffusion as indicated by the Peclet number \( P_e \) (Table 1). The following technique, called QUICK (Quadratic Upstream Interpolation for Convective Kinetics), [7], [8], computes non-linear averages of the fluid velocity that can be used in equation (8) to improve the accuracy reducing or eliminating the oscillations.

3.5. The QUICK - Leonard Differencing Algorithm in the Finite Volume Method
This is a powerful and stable average upstream-weighted quadratic interpolation formula to approximate the values at each finite volume interface; it uses the information of three points at the boundaries of each \( V_n \).
\[-\left( \frac{8\eta_{n+1}}{\delta x_{n+1}} - 3v_f \right) T_{n+1} + \left( \frac{8\eta_{n+1}}{\delta x_{n+1}} + 8\eta_n + 3v_f \right) T_n - \left( \frac{8\eta_n}{\delta x_n} + 7v_f \right) T_{n-1} + v_f T_{n-2} = 8Q_n \delta x_n \]

\[ \leftrightarrow -\alpha_{n+1} T_{n+1} + \beta_n T_n - \gamma_{n-1} T_{n-1} + v_f T_{n-2} = q_n \geq 0, \quad \{n = 2, N-1\} \]

Equation (15) is a tetra diagonal system of algebraic equations, which needs an unknown value $T_1$ for $n = 1$ at the $\Gamma_0$ boundary, which can be obtained by interpolation. Figure 6 shows the convergence of the corresponding FVM solution for different fluid velocities, using this average formula with different solid-fluid heat transfers. The numerical scheme converges at all tested fluid speeds in every volume $V_n$. The total number $N$ of nodes and finite volumes can be increased to improve the accuracy of this algorithm for exceedingly high fluid speeds (Figures 6).

![Figure 6](image_url)

**Figure 6.** Analytical (curves) and numerical (red points) solutions for the fluid temperature profile using the QUICK-Leonard algorithm and different $Q_f$ (5 x 10^{-8} and 2 x 10^{-7} °C/s), with $N = 12, 15, \text{ and } 25$ internal nodes respectively; units of $v_f$ [10^{-8} m/s] in all cases.

The QUICK-Leonard algorithm does not produce oscillations or inaccuracies at low fluid velocities. In coarse meshes of only 10 nodes; it reproduces the exact solution when fluid velocity increases. For higher fluid velocities ($v_f \geq 4 \times 10^{-7} \text{ m/s}$) it exhibits some oscillations at the end of the 1D domain because of the abrupt temperature jump, but the scheme can be corrected easily by increasing moderately the number of finite volumes, as is shown in figure 7. This numerical scheme always works correctly for the LTNE model at extremely high fluid velocities in the porous medium.
The non-isothermal groundwater fluid flow illustrated in figure 7 (right) is particularly difficult to simulate for any numerical scheme. The temperature jumps abruptly at the end of the interval for any fluid velocity larger than $10^{-6}$ m/s, which is a common velocity in porous rocks. The numerical problem occurs because this jump represents an almost mathematical singularity with discontinuous derivatives at the point $Z = 100$ m. The QUICK-Leonard algorithm in the Finite Volume method, reproduces exactly the analytic solution of the conduction-convection model in equation (14) without production of oscillations at the discontinuity point. The numerical scheme converges in this critical situation using an appropriate number of nodes. It provides an efficient differentiation scheme that is physically realistic and numerically stable, which is very appropriate to simulate heat and mass flows in any type of geothermal reservoirs and in other heterogeneous porous systems.

4. Conclusion
The main objective of this work was to provide a physical reason and explanation for the divergence of a numerical model using the finite volume method. Using a one-dimensional diffusion-convection model, the objective was achieved. A numerical model based on the FVM can diverge because of the inappropriate average formula used to compute the fluid velocity at the internal interfaces of the computational mesh. Besides a sustainability factor, any discretisation or numerical scheme must satisfy some fundamental properties to simulate the physics of heat and mass flow in a geothermal reservoir. These properties are synthesized in three parts: conservativeness of the physical laws, transportiveness of the fluid flow, and boundedness of the system of algebraic equations. The QUICK-Leonard algorithm possesses these three properties, it is very accurate and does not produce oscillations at normal Darcy velocities. For higher fluid speeds this scheme is also convergent by increasing the number of finite volumes. It also works correctly even in the case of an abrupt temperature jump or mathematical singularity with discontinuous derivatives, as is the case during the injection of cold water into a hot reservoir; this scheme converges in this critical situation with an appropriate larger number of nodes. For any other conditions, the FVM can be used to solve general geothermal reservoir engineering problems.

Porous rocks are coexistent in multiple length scales. Experimental data reveals that there is no unique characteristic length for scaling the relevant dimensionless parameters; more fundamental and experimental research is needed to establish interfacial heat transfer coefficient correlations. The key point in LTNE models is the correct calculation of $q_{sf}$, $h_{sf}$ and $T_{sf}$. The physical theory and laws that are the basis of the flows in geothermal reservoirs and other heterogeneous porous systems, are supported by both, experiments and equations. The numerical model representing these equations should be coherent with the same physical laws that is supposed to represent in terms of numbers. Therefore, it must be possible to deduce unknown values of a physical property from a pure numerical outcome, based on this assumed coherence [9], [10].
References

[1] Bundschuh J and Suarez-Arriaga M C 2010 Introduction to the Numerical Modeling of Groundwater and Geothermal Systems – Fundamentals of mass, energy and solute transport in poroelastic rocks Vol 2 Multiphysics Modelling Series (London: CRC Press – Taylor & Francis Group)

[2] Nield D A and Bejan A 2013 Convection in Porous Media (New York: Springer Science)

[3] Vafai K 2015 Handbook of Porous Media (London: CRC Press – Taylor & Francis Group)

[4] Narasimhan T and Witherspoon P 1976 An integrated finite difference method for analysing fluid flow in porous media (Water Resources Res 12:1) pp 57–64

[5] Patankar S V 1980 Numerical Heat Transfer and Fluid Flow (New York: Hemisphere Publishing Corporation Taylor & Francis Group)

[6] Pruess K 1988 SHAFT, MULKOM, TOUGH: A set of numerical simulators for multiphase fluid and heat flow (Mexico: Geotermia vol 4/1) pp 185–202

[7] Leonard B P 1979 A Stable and Accurate Convective Modeling Procedure Based on Quadratic Upstream Interpolation (Computational Methods Appl. Mechanics Engineering vol 19 pp 59-98

[8] Hayase T Humprey J A C and Greif R 1992 A Consistently Formulated QUICK Scheme for Fast and Stable Convergence Using Finite-Volume Iterative Calculation Procedures, (Journal of Computational Physics Vol 98) pp 108-118

[9] Suárez Arriaga MC 2019 An uncoupled radial thermoporoelastic model with local thermal non-equilibrium IOP Conference Series: Earth and Environmental Science Vol 367 (Petropavlovsk, Kamchatka, Russia: Proc. 3rd International Geothermal Conference GEOHEAT2019)

[10] Suárez Arriaga MC 2003 The interfacial interaction problem in complex multiple porosity fractured reservoirs Seventh Granada Lectures, AIP Conference Proceedings (New York: American Institute of Physics 661)