Equivalents of the finitary non-deterministic inductive definitions

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Abstract

We present statements equivalent to some fragments of the principle of non-deterministic inductive definitions (NID) by van den Berg (2013), working in a weak subsystem of constructive set theory CZF. We show that several statements in constructive topology which were initially proved using NID are equivalent to the elementary and finitary NIDs. We also show that the finitary NID is equivalent to its binary fragment and that the elementary NID is equivalent to a variant of NID based on the notion of biclosed subset. Our result suggests that proving these statements in constructive topology requires genuine extensions of CZF with the elementary or finitary NID.

Keywords: Constructive set theory; Non-deterministic inductive definition; Set-generated class; Basic pair; Formal topology

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1 Introduction

Many objects in mathematics are defined as subsets of some given set, e.g., an open set of a topological space; a prime ideal of a commutative ring; a subalgebra of a certain algebraic structure. The totality of these objects, however, does not necessarily form a set in predicative constructive foundations such as Martin-Löf’s type theory [11] or Aczel’s constructive set theory CZF [3]. In particular, the lack of power-sets in these foundations makes some of the standard constructions in general topology substantially difficult to carry out, which requires a certain amount of ingenuity [1,10,12,13].

The crucial element of these predicative results consists in constructing a subset of the totality of a certain type of objects, called a generating subset, in such a way that every object of that type can be expressed as the union of the elements of the generating subset. The problems of constructing such generating subsets in constructive topology motivated van den Berg [17] and Aczel, Ishihara, Nemoto, and Sangu [2] to independently introduce principles of CZF which allow us to show that a wide range of collections of mathematical objects are set-generated.

The focus of this paper is on the principle introduced by van den Berg [17], called non-deterministic inductive definitions (NID), or more specifically, its
elementary and finitary fragments. The NID principle asserts that the class of models of an infinitary propositional theory consisting of formulas (or rules) of the form $\land U \rightarrow \lor V$ has a generating subset, where $U$ and $V$ are subsets of the set of propositional variables. The elementary and finitary NID principles are obtained by restricting the propositional theory to those rules whose premise is singleton and finitely enumerable, respectively. In fact, van den Berg [17] showed that the finitary NID is equivalent to the principle introduced by Aczel et al. [2], called the set generation axiom (SGA).

The purpose of this paper is to extend the scope of reverse mathematics, classical [16] or constructive [7, 18], in a set-theoretic foundation. Here, we develop the reverse mathematics of the NID principle initiated in [9] much further by showing that several statements in constructive topology which were initially proved using NID or SGA are in fact equivalent to the elementary NID or the finitary NID. Specifically, we show that the elementary NID is equivalent to (1) completeness and cocompleteness of the category of basic pairs by Sambin [15] and (2) the existence of weak equalisers in the category of sets and relations. Moreover, the finitary NID is equivalent to (1) completeness and cocompleteness of the category of concrete spaces and (2) set-generation of the class of formal points of an inductively generated formal topology. We also show that the finitary NID is equivalent to its binary fragment and that the elementary NID is equivalent to a symmetric variant of NID, called NID_{bi}, formulated with respect to the notion of biclosed subset.

Our result suggests that proving these statements in constructive topology requires genuine extensions of CZF with the elementary NID or the finitary NID, which are thought to be independent of CZF (cf. van den Berg [17, Section 8]). It remains to settle the exact relation between CZF, and its extensions with the elementary NID or the finitary NID, which would also settle the relative strength of the equivalents of these principles.

Organisation Section 2 introduces the base set theory for our work; Section 3 recalls the NID principle and its relation to SGA; Section 4 gives equivalents of the elementary NID; and Section 5 gives equivalents of the finitary NID.

2 Elementary constructive set theory

We work in a weak subsystem of CZF, called the elementary constructive set theory ECST [4], where none of the known fragments of the NID principle seems to be derivable.

The language of ECST contains variables for sets and binary predicates $\equiv$ and $\in$. The axioms and rules of ECST are the axioms and rules of intuitionistic predicate logic with equality, and the following set-theoretic axioms:

Extensionality: $\forall a \forall b (\forall x (x \in a \leftrightarrow x \in b) \rightarrow a = b)$.

Paring: $\forall a \forall b \exists y \forall u (u \in y \leftrightarrow u = a \lor u = b)$.

Union: $\forall a \exists y \forall x (x \in y \leftrightarrow \exists u (x \in u))$.

Restricted Separation: $\forall a \exists b \forall x (x \in b \leftrightarrow x \in a \land \varphi(x))$

where $\varphi(x)$ is restricted. Here, a formula is said to be restricted if all quantifiers in the formula occur in the forms $\forall x \in a$ or $\exists x \in a$. 

2
Replacement: \( \forall a (\forall x \in a \exists y \varphi(x, y) \rightarrow \exists b \forall y (y \in b \leftrightarrow \exists x \in a \varphi(x, y))) \)

where \( \varphi(x, y) \) is any formula.

Strong Infinity: \( \exists a [0 \in a \land \forall x (x \in a \rightarrow x + 1 \in a) \land \forall y (0 \in y \land \forall x (x \in y \rightarrow x + 1 \in y) \rightarrow a \subseteq y)] \)

where \( x + 1 \) denotes \( x \cup \{x\} \) and 0 is the empty set \( \emptyset \). The set \( a \) asserted to exist will be denoted by \( \omega \).

This completes the description of ECST.

The constructive Zermelo–Fraenkel set theory CZF [3] is obtained from ECST by substituting Strong Collection for Replacement and adding Subset Collection and \( \in \)-Induction. For the details of these axioms, the reader is referred to Aczel and Rathjen [3, 4].

In ECST, Subset Collection implies

Fullness: \( \forall a \forall b \exists c [c \subseteq \operatorname{mv}(a, b) \land \forall s \in \operatorname{mv}(a, b) \exists r \in c (r \subseteq s)] \),

where \( \operatorname{mv}(a, b) \) is the class of total relations from \( a \) to \( b \). In practice, Fullness is very important as it implies Exponentiation, which asserts that the class \( B^A \) of functions between sets \( A \) and \( B \) is a set. In particular, Fullness implies the following weak form of Exponentiation:

Finite Powers Axiom (FPA): For any set \( S \), the class \( S^n \) of functions from \( \{0, \ldots, n - 1\} \) to \( S \) is a set for all \( n \in \omega \).

The extension of ECST with FPA is denoted by ECST + FPA.

A notable consequence of FPA is that the class \( \operatorname{Fin}(S) \) of finitely enumerable subsets of a set \( S \) is a set. Here, a set \( A \) is finitely enumerable if there is a surjection \( f: \{0, \ldots, n - 1\} \rightarrow A \) for some \( n \in \omega \). Note that we can decide whether a finitary enumerable set is empty or inhabited by inspecting the domain of \( f \).

3 NID principles

The subjects of our investigation are classes closed under some sets of rules on a set. NID principles say that such a class has a generating subset \( G \) so that every member of the class arises as the union of elements of \( G \).

Definition 1. A rule on a set \( S \) is a pair \((a, b)\) of subsets of \( S \). A rule \((a, b)\) is said to be nullary if \( a \) is empty, elementary if \( a \) is singleton, and finitary if \( a \) is finitely enumerable. A subset \( \alpha \subseteq S \) is said to be closed under a rule \((a, b)\) if

\[ a \subseteq \alpha \implies b \upharpoonright \alpha, \]

where \( b \upharpoonright \alpha \) means that \( b \cap \alpha \) is inhabited. If \( R \) is a set of rules on \( S \), then a subset \( \alpha \subseteq S \) is said to be \( R \)-closed if it is closed under every rule in \( R \).

Definition 2. Let \( S \) be a set and \( \operatorname{Pow}(S) \) be the class of subsets of \( S \). A subclass \( \mathcal{C} \) of \( \operatorname{Pow}(S) \) is said to be set-generated if there exists a subset \( G \subseteq \mathcal{C} \), called a generating subset, such that

\[ \forall \alpha \in \mathcal{C} \forall x \in \alpha \exists \beta \in G (x \in \beta \subseteq \alpha). \]

The principle NID reads:
NID: For each set $S$ and a set $R$ of rules on $S$, the class of $R$-closed subsets of $S$ is set-generated.

The nullary, elementary, and finitary NID are the principles obtained from NID by restricting $R$ to nullary, elementary, and finitary rules, respectively. These principles are denoted by NID$_0$, NID$_1$, and NID$_{<\omega}$.

Remark 3. NID$_{<\omega}$ clearly implies NID$_1$. Moreover, Ishihara and Nemoto [9] showed that NID$_1$ implies NID$_0$ and that NID$_0$ is equivalent to Fullness over ECST. In particular, NID$_0$ implies FPA.

We recall the connection between NID$_{<\omega}$ and the set generation axiom (SGA) introduced by Aczel et al. [2].

Definition 4. For any set $S$, a subclass $C$ of $\text{Pow}(S)$ is said to be strongly set-generated if there exists a subset $G \subseteq C$ such that

$$\forall \alpha \in C \forall \sigma \in \text{Fin}(\alpha) \exists \beta \in G (\sigma \subseteq \beta \subseteq \alpha).$$

The principle SGA reads:

SGA: For each set $S$ and each subset $Z \subseteq \text{Fin}(S) \times \text{Pow}(\text{Pow}(S))$, the class

$$\mathcal{M}(Z) = \{ \alpha \in \text{Pow}(S) \mid \forall (\sigma, \Gamma) \in Z (\sigma \subseteq \alpha \rightarrow \exists U \in \Gamma (U \subseteq \alpha)) \}$$

of models of $Z$ is strongly set-generated.

A subset $Z \subseteq \text{Fin}(S) \times \text{Pow}(\text{Pow}(S))$ is called a generalised geometric theory over $S$ (of rank 1), and its element is written $\bigwedge \sigma \vdash \bigvee U \in \Gamma \bigwedge U$ instead of $(\sigma, \Gamma)$.

The principle SGA looks stronger than NID$_{<\omega}$, but they are equivalent.

Proposition 5 (van den Berg [17, Theorem 7.3]). SGA and NID$_{<\omega}$ are equivalent over ECST.

Remark 6. In SGA, we can take $Z$ to be a set of elements of the form

$$\bigwedge \sigma \vdash \bigvee_{U_{n-1} \in \Gamma_{n-1}} \bigwedge_{U_{n-2} \in U_{n-1}} \cdots \bigwedge_{U_0 \in \Gamma_0} U_0$$

where $\sigma \in \text{Fin}(S), U_0 \in \text{Pow}(S), U_1 \in \text{Pow}(\text{Pow}(\text{Pow}(S))), \ldots$, and the right-hand-side is a finite nesting of $\bigvee \bigwedge$ pairs. The resulting principle is still equivalent to NID$_{<\omega}$ over ECST. See van den Berg [17, Theorem 4.2].

4 Elementary NID

In this section, we give some statements equivalent to NID$_1$.

4.1 NID$_0$ principle

We introduce a symmetric variant of NID, which seems to be quite natural and useful in practice (cf. Section 4.2 and Section 4.3 below).
Definition 7. Let \((a, b)\) be a rule on a set \(S\). A subset \(\alpha \subseteq S\) is said to be \textit{biclosed} under \((a, b)\) if
\[
a \circ \alpha \iff b \circ \alpha.
\]
If \(R\) is a set of rules on \(S\), then a subset \(\alpha \subseteq S\) is said to be \(R\)-biclosed if \(\alpha\) is biclosed under every rule in \(R\). Then, the principle \(\text{NID}_{bi}\) reads:

\(\text{NID}_{bi}\): For each set \(S\) and a set \(R\) of rules on \(S\), the class of \(R\)-biclosed subsets of \(S\) is set-generated.

Proposition 8. \(\text{NID}_1\) is equivalent to \(\text{NID}_{bi}\).

Proof. Assume \(\text{NID}_1\), and let \(R\) be a set of rules on a set \(S\). Define a set \(R'\) of elementary rules on \(S\) by
\[
R' \defeq \{(\{x\}, b) \mid (a, b) \in R, x \in a\} \cup \{(\{y\}, a) \mid (a, b) \in R, y \in b\}.
\]
Then, a subset \(\alpha \subseteq S\) is \(R\)-biclosed if and only if it is \(R'\)-closed. This proves one direction.

Conversely, assume \(\text{NID}_{bi}\), and let \(R\) be a set of elementary rules on a set \(S\). Define another set \(R'\) of rules on \(S\) by
\[
R' \defeq \{(a \cup b, b) \mid (a, b) \in R\}.
\]
Then, a subset \(\alpha \subseteq S\) is \(R\)-closed if and only if it is \(R'\)-biclosed.

4.2 Weak equalisers in the category of sets and relations

We show that \(\text{NID}_1\) is equivalent to the existence of weak equalisers in the category of sets and relations.

Let \(\text{Rel}\) be the category of sets and relations between them: the identity on a set \(X\) is a diagonal relation
\[
\Delta_X \defeq \{(x, x) \mid x \in X\},
\]
and the composition of morphisms is the relational composition.

Definition 9. Let \(f, g: A \to B\) be a parallel pair of morphisms in a category \(\mathcal{C}\). A \textit{weak equaliser} of \(f\) and \(g\) is an object \(E\) together with a morphism \(e: E \to A\) such that
1. \(f \circ e = g \circ e\),
2. for any morphism \(h: C \to A\) such that \(f \circ h = g \circ h\), there exists a morphism \(\overline{h}: C \to E\) such that \(e \circ \overline{h} = h\).

Proposition 10. The following are equivalent over \(\text{ECST}\):
1. \(\text{NID}_{bi}\);
2. \(\text{Rel}\) has weak equalisers.
Proof. (1 \rightarrow 2) Assume NIDbi, and let \( r_1, r_2 \subseteq X \times Y \) be a parallel pair of relations. Consider a class

\[ \mathcal{E} \overset{\text{def}}{=} \{ U \in \text{Pow}(X) \mid r_1U = r_2U \}, \]

where \( r_iU \overset{\text{def}}{=} \{ y \in Y \mid \exists x \in U (x r_i y) \} \) \((i = 0, 1)\). Define a set of rules on \( X \) by

\[ R \overset{\text{def}}{=} \{(r_1^\alpha y, r_2^\alpha y) \mid y \in Y \}, \]

where \( r_i^\alpha y \overset{\text{def}}{=} \{ x \in X \mid x r_i y \} \) \((i = 0, 1)\). Since

\[ U \in \mathcal{E} \iff \forall y \in Y (y \in r_1U \iff y \in r_2U) \iff \forall y \in Y (r_1^\alpha y \uplus U \iff r_2^\alpha y \uplus U), \]

\( \mathcal{E} \) is the class of \( R \)-biclosed subsets. Thus \( \mathcal{E} \) has a generating subset \( G \) by NIDbi. Define a relation \( r \subseteq G \times X \) by

\[ U \; r \; x \iff x \in U. \]

Clearly, we have \( r_1 \circ r = r_2 \circ r \). We show that \( r \) is a weak equaliser of \( r_1 \) and \( r_2 \). Let \( s \subseteq Z \times X \) be a relation such that \( r_1 \circ s = r_2 \circ s \). Define a relation \( \overline{s} \subseteq Z \times G \) by

\[ z \; \overline{s} \; U \overset{\text{def}}{=} U \subseteq sz, \]

where \( sz \overset{\text{def}}{=} \{ x \in X \mid z s x \} \). Obviously, we have \( r \circ \overline{s} \subseteq s \). Conversely, suppose that \( z s x \). Since \( sz \) is an \( R \)-biclosed subset of \( X \), there exists a \( U \in G \) such that \( x \in U \subseteq sz \). Thus \( z (r \circ \overline{s}) x \), and so \( s \subseteq r \circ \overline{s} \). Hence \( s = r \circ \overline{s} \).

(2 \rightarrow 1) Assume that \( \text{Rel} \) has weak equalisers, and let \( R \) be a set of rules on a set \( S \). Then, \( R \) corresponds to two relations \( r_1, r_2 \subseteq S \times R \) given by

\[ x \; r_1 (a, b) \iff x \in a, \]

\[ x \; r_2 (a, b) \iff x \in b. \]

Let \( r \subseteq E \times S \) be a weak equaliser of \( r_1 \) and \( r_2 \) in \( \text{Rel} \), and put

\[ G \overset{\text{def}}{=} \{ re \mid e \in E \}. \]

Since \( r_1 \circ r = r_2 \circ r \), elements of \( G \) are \( R \)-biclosed subsets of \( S \). We show that \( G \) generates the class of \( R \)-biclosed subsets. Let \( \alpha \subseteq S \) be an arbitrary \( R \)-biclosed subset, and let \( z \in \alpha \). Define a relation \( r_{\alpha} \subseteq \{\} \times S \) by

\[ \ast r_{\alpha} x \overset{\text{def}}{=} x \in \alpha, \]

where \( \{\} \) is a fixed one-element set. Since \( r_{\alpha} \ast = \alpha \) and \( \alpha \) is \( R \)-biclosed, we have \( r_1 \circ r_{\alpha} = r_2 \circ r_{\alpha} \). Thus, \( r_{\alpha} \) factors through \( r \) via some relation \( \overline{r_{\alpha}} \subseteq \{\} \times E \). Since \( \ast r_{\alpha} z \), there exists an \( e \in E \) such that \( \ast \overline{r_{\alpha}} e \) and \( e \; r \; z \). Then, for any \( y \in re \), we have \( \ast (r \circ \overline{r_{\alpha}}) y \), i.e., \( \ast r_{\alpha} y \), and so \( y \in \alpha \). Hence \( z \in re \subseteq \alpha \). \( \square \)
4.3 Equalisers in the category of basic pairs

We show that NID is equivalent to the existence of equalisers in the category of basic pairs described in the forthcoming book by Sambin [15]. The result in this subsection refines the result by Ishihara and Kawai [8, Proposition 3.8], where they showed that the category of basic pairs has coequalisers using SGA (cf. Remark 13).

**Definition 11.** A *basic pair* is a triple \((X, \models, S)\) where \(X\) and \(S\) are sets, and \(\models\) is a relation from \(X\) to \(S\). A *relation pair* between basic pairs \(X_1 = (X_1, \models_1, S_1)\) and \(X_2 = (X_2, \models_2, S_2)\) is a pair \((r, s)\) of relations \(r \subseteq X_1 \times X_2\) and \(s \subseteq S_1 \times S_2\) such that \(\models_2 \circ r = s \circ \models_1\), i.e., the following diagram commutes in \(\text{Rel}\):

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\models_1} & S_1 \\
\downarrow r & & \downarrow s \\
X_2 & \xrightarrow{\models_2} & S_2
\end{array}
\]

Two relation pairs \((r_1, s_1)\) and \((r_2, s_2)\) between basic pairs \(X_1\) and \(X_2\) are said to be *equivalent* (or *equal*) if

\[
\models_2 \circ r_1 = \models_2 \circ r_2.
\]  \hspace{1cm} (4.1)

In this case, we write \((r_1, s_1) \sim (r_2, s_2)\).

Basic pairs and relation pairs with equality defined by (4.1) form a category \(\text{BP}\): the identity on a basic pair \(X = (X, \models, S)\) is \((\Delta_X, \Delta_S)\) and the composition of two relation pairs \((r, s)\): \(X_1 \to X_2\) and \((u, v)\): \(X_2 \to X_3\) is \((u \circ r, v \circ s)\), where each component is a relational composition. It is easy to check that compositions respect equality of relation pairs; see [8, Proposition 2.3] for more details.

**Proposition 12.** The following are equivalent over \(\text{ECST}\):

1. \(\text{Rel}\) has weak equalisers;

2. \(\text{BP}\) has equalisers.

*Proof.* \((1 \rightarrow 2)\) This follows from a general result on the Freyd completion of categories [6, Section 2.5 (b)]. We recall the proof for the particular case of \(\text{Rel}\). Assume that \(\text{Rel}\) has weak equalisers. Let \((r_1, s_1)\), \((r_2, s_2)\): \(X_1 \to X_2\) be relation pairs between basic pairs \(X_1 = (X_1, \models_1, S_1)\) and \(X_2 = (X_2, \models_2, S_2)\). Put

\[
u_1 \overset{\text{def}}{=} \models_2 \circ r_1, \hspace{1cm} \nu_2 \overset{\text{def}}{=} \models_2 \circ r_2.
\]

Let \(e: E \to X_1\) be a weak equaliser of \(\nu_1\) and \(\nu_2\) in \(\text{Rel}\). Consider a basic pair \(E = (E, \models_1 \circ e, S_1)\). Then \((e, \Delta_{S_1})\) is a relation pair from \(E\) to \(X_1\), and we have \((r_1, s_1) \circ (e, \Delta_{S_1}) \sim (r_2, s_2) \circ (e, \Delta_{S_1})\). We show that \((e, \Delta_{S_1})\): \(E \to X_1\) is an equaliser of \((r_1, s_1)\) and \((r_2, s_2)\). Let \(Z = (Z, \models, T)\) be a basic pair, and let \((u, v)\): \(Z \to X_1\) be a relation pair such that \((r_1, s_1) \circ (u, v) \sim (r_2, s_2) \circ (u, v)\). Then \(u_1 \circ u = u_2 \circ u\) in \(\text{Rel}\), so there exists a relation \(\varpi \subseteq Z \times E\) such that \(e \circ \varpi = u\). Then, \((\varpi, v)\) is a relation pair from \(Z\) to \(E\). It is also straightforward to check that \((e, \Delta_{S_1}) \circ (\varpi, v) \sim (u, v)\) and that \((\varpi, v)\) is a unique relation pair from \(Z\) to \(E\) with this property.
In the following, we write $S_\Delta$ for the basic pair $(S, \Delta S, S)$ on a set $S$. Assume that $\text{BP}$ has equalisers, and let $r_1, r_2 \subseteq X \times Y$ be a parallel pair of relations. Then, $(r_1, r_1)$ and $(r_2, r_2)$ are relation pairs from $X_\Delta$ to $Y_\Delta$. Let $(r, s): \mathcal{E} \to X_\Delta$ be an equaliser of $(r_1, r_1)$ and $(r_2, r_2)$ in $\text{BP}$, and write $\mathcal{E} = (E, \sqsubseteq, S)$. We show that $r$ is a weak equaliser of $r_1$ and $r_2$ in $\text{Rel}$. First, since $(r_1, r_1) \circ (r, s) \sim (r_2, r_2) \circ (r, s)$, we have $r_1 \circ r = r_2 \circ r$. Next, let $u \subseteq Z \times X$ be a relation such that $r_1 \circ u = r_2 \circ u$. Then, $(u, u)$ is a relation pair from $Z_\Delta$ to $X_\Delta$, and we have $(r_1, r_1) \circ (u, u) \sim (r_2, r_2) \circ (u, u)$. Thus, there exists a unique relation pair $(v, w): Z_\Delta \to \mathcal{E}$ such that $(r, s) \circ (v, w) \sim (u, u)$. Then, $r \circ v = u$.

**Remark 13.** The categories $\text{Rel}$ and $\text{BP}$ are self-dual, i.e., $\text{Rel}$ is equivalent to its opposite $\text{Rel}^{op}$ (and similarly for $\text{BP}$). Thus, $\text{Rel}$ has weak equalisers if and only if it has weak coequalisers, and $\text{BP}$ has equalisers if and only if it has coequalisers. Moreover, since $\text{Rel}$ has small products and hence small coproducts as well, $\text{BP}$ has small products and coproducts [6, Section 2.5 (a)] (see also [8, Proposition 3.2]). Hence, the following are equivalent over $\text{ECST}$:

1. $\text{BP}$ has (co)equalisers;
2. $\text{BP}$ is (co)complete.

We summarise the equivalents of $\text{NID}_1$.

**Theorem 14.** The following are equivalent over $\text{ECST}$:

1. $\text{NID}_1$;
2. $\text{NID}_{\text{bi}}$;
3. $\text{Rel}$ has weak (co)equalisers;
4. $\text{BP}$ has (co)equalisers.
5. $\text{BP}$ is complete and cocomplete.

## 5 Finitary NID

In this section, we give some statements equivalent to $\text{NID}_{<\omega}$.

### 5.1 Models of geometric theories

The connection between NID principles and set-generation of the class of models of a game theory was studied by van den Berg [17, Section 4]. In particular, he showed that $\text{NID}_{<\omega}$ is equivalent to the statement that the class of models of any finitary game theory is set-generated [17, Corollary 4.4]. Since geometric theories form a subclass of finitary game theories, the result in this subsection follows from his result. However, we provide a small refinement, which serves as a stepping-stone for Section 5.2.

**Definition 15.** A (propositional) geometric theory over a set $S$ is a set of axioms of the form

$$\bigwedge_{i \in I} A_i \vdash \bigvee_{i \in I} B_i$$


where \( I \) is a set and \( A, B_i \) are finitely enumerable subsets of \( S \). If \( T \) is a geometric theory over \( S \), a model of \( T \) is a subset \( m \subseteq S \) such that
\[
A \subseteq m \implies \exists i \in I \ (B_i \subseteq m)
\]
for each axiom \( \bigwedge A \vdash \bigvee_{i \in I} \bigwedge B_i \) in \( T \). The class of models of \( T \) is denoted by \( \mathcal{M}(T) \).

**Definition 16.** Let \( \text{NID}_{\leq 2} \) be the principle obtained from \( \text{NID} \) by restricting the set \( R \) to those rules \( (a, b) \) where \( a \) is a surjective image of \( \{0, \ldots, n-1\} \) for some \( n \leq 2 \).

**Proposition 17.** The following are equivalent over \( \text{ECST} \):

1. \( \text{NID}_{\leq 2} \);
2. \( \text{NID}_{<\omega} \);
3. The class of models of any geometric theory is set-generated.

**Proof.** Clearly 2 implies 1. The equivalence of 2 and 3 is a corollary of van den Berg [17, Corollary 4.4]. We give a proof for the sake of completeness.

(1 \( \rightarrow \) 3) Assume \( \text{NID}_{\leq 2} \). Let \( T \) be a geometric theory over a set \( S \). Define a set \( R \) of rules on \( \text{Fin}(S) \) by
\[
R \overset{\text{def}}{=} \{(\emptyset, \{\emptyset\})\} \quad (5.1)
\]
\[
\cup \{(\{A\}, \{B\}) \mid B \subseteq A\} \quad (5.2)
\]
\[
\cup \{(\{A, B\}, \{A \cup B\}) \mid A, B \in \text{Fin}(S)\} \quad (5.3)
\]
\[
\cup \left\{\left. (\{A\}, \{B_i \mid i \in I\}) \right| \bigwedge_{i \in I} A \vdash \bigvee B_i \in T \right\}. \quad (5.4)
\]
Note that \( \text{Fin}(S) \) is a set since \( \text{NID}_{\leq 2} \) implies \( \text{NID}_0 \) (cf. Remark 3). Note also that \( R \) consists of nullary and binary rules.

Let \( C \) be the class of \( R \)-closed subsets. Define functions \( \Phi : C \to \mathcal{M}(T) \) and \( \Psi : \mathcal{M}(T) \to C \) by
\[
\Phi(\alpha) \overset{\text{def}}{=} \bigcup \alpha, \quad \Psi(m) \overset{\text{def}}{=} \text{Fin}(m).
\]
It is straightforward to show that these functions are well-defined. We show that they are inverses of each other. First, we have
\[
\Phi(\Psi(m)) = \bigcup \Psi(m) = \bigcup \text{Fin}(m) = m
\]
for each \( m \in \mathcal{M}(T) \). Next, for each \( \alpha \in C \), we have \( \alpha \subseteq \text{Fin}(\bigcup \alpha) = \Psi(\Phi(\alpha)) \).

Conversely, let \( A \in \text{Fin}(\bigcup \alpha) \), and write \( A = \{x_0, \ldots, x_{n-1}\} \). For each \( i < n \), there is a \( B_i \in \alpha \) such that \( x_i \in B_i \), so \( \{x_i\} \in \alpha \) by (5.2). Then \( A = \{x_0, \ldots, x_{n-1}\} \in \alpha \) by (5.3) and induction. Note that if \( A = \emptyset \), then \( A \in \alpha \) by (5.1). Thus, \( \Psi(\Phi(\alpha)) = \alpha \) for each \( \alpha \in C \). By \( \text{NID}_{\leq 2} \), the class \( C \) has a generating subset \( G \). Then, \( \{\Phi(\alpha) \mid \alpha \in G\} \) is a generating subset of \( \mathcal{M}(T) \).

(3 \( \rightarrow \) 2) Assume that the class of models of any geometric theory is set-generated, and let \( R \) be a set of finitary rules on a set \( S \). We can identify each rule \( (a, b) \) in \( R \) with the following geometric axiom over \( S 
\]
\[
\bigwedge a \vdash \bigvee_{y \in b} y.
\]
Let $T_R$ be the geometric theory over $S$ with the axioms of the above form for each rule in $R$. Then, a model of $T_R$ is just an $R$-closed subset of $S$. Hence, the class of $R$-closed subsets is set-generated.

5.2 $n$-ary NID

By setting a uniform bound on the size of the premise of finitary rules, we have countably many fragments of $\text{NID}_{<\omega}$.

**Definition 18.** Let $n \in \omega$. A rule $(a, b)$ on a set $S$ is said to be $n$-ary if there exists a surjection $f : \{0, \ldots, n - 1\} \to a$. The $n$-ary NID, denoted by $\text{NID}_n$, is the principle obtained from NID by restricting the set $R$ to $n$-ary rules.

**Lemma 19.** $\text{NID}_{<\omega}$ and $\text{NID}_2$ are equivalent over ECST.

**Proof.** It suffices to show that $\text{NID}_2$ implies $\text{NID}_{<\omega}$. Assume $\text{NID}_2$, and let $R$ be a set of rules on $S$ consisting of nullary and binary rules. Choose any set $\star S$ not in $S$, and define a set $R'$ of binary rules on $S \cup \{\star S\}$ by

$$R' \overset{\text{def}}{=} \{(\star S), b\} \cup \{(a, b) \in R \mid a \not\subseteq a\}.$$

By NID$_2$, the class of $R'$-closed subsets of $S \cup \{\star S\}$ has a generating subset $G$.

Put

$$H \overset{\text{def}}{=} \{\alpha \cap S \mid \alpha \in G, \star S \in \alpha\}.$$

We show that $H$ generates the class of $R$-closed subsets of $S$. First, let $\alpha \in G$ such that $\star S \in \alpha$. Consider any $(a, b) \in R$ such that $a \subseteq a \cap \alpha$. If $a = \emptyset$, then $b \not\subseteq (a \cap \alpha)$ because $\star S \in \alpha$. If $a$ is inhabited, then obviously $b \not\subseteq (a \cap \alpha)$. Thus, the elements of $H$ are $R$-closed. Let $\beta$ be any $R$-closed subset of $S$, and let $x \in \beta$. Then, $\beta \cup \{\star S\}$ is an $R'$-closed subset of $S \cup \{\star S\}$. Thus, there exists an $\alpha \in G$ such that $x \in \alpha \subseteq \beta \cup \{\star S\}$. Then, $x \in \alpha \cap S \subseteq \beta$. 

Lemma 19, together with Proposition 17, yields the following.

**Proposition 20.** The following are equivalent over ECST:

1. $\text{NID}_{<\omega}$;
2. $\text{NID}_n$ ($n \geq 2$).

5.3 Formal points of formal topologies

The initial motivation of the NID principle comes from the problem of constructing generating subsets in constructive point-free topology, where some of its results require various extensions of CZF. Van den Berg [17, Section 5] and Aczel et al. [2, Section 7.2] illustrated the power of NID by providing a uniform solution to these problems using NID and SGA, respectively. With some adjustments to the setting of ECST, we turn some of their results into equivalents of $\text{NID}_{<\omega}$.

We adopt the following definition of formal topology [14], which is the notion of point-free topology in constructive and predicative foundations.

**Definition 21** (Coquand et al. [5, Definition 2.1]). A formal topology is a triple $S = (S, \leq, <)$ where $(S, \leq)$ is a preordered set and $<$ is a relation from $S$ to $\text{Pow}(S)$ such that $\{a \in S \mid a < U\}$ is a set for each $U \subseteq S$ and
1. $a \in U \implies a \triangledown U$, 
2. $a \triangledown U \land U \triangledown V \implies a \triangledown V$, 
3. $a \triangledown U \land a \triangledown V \implies a \triangledown U \downarrow V$, 
4. $a \leq b \implies a \triangledown \{b\}$,

where $U \triangledown V \overset{\text{def}}{\iff} \forall a \in U (a \triangledown V)$,

$U \downarrow V \overset{\text{def}}{=} \{c \in S \mid \exists a \in U \exists b \in V (c \leq a \land c \leq b)\}$.

A subset $\alpha \subseteq S$ is a formal point of $S$ if

(P1) $\alpha$ is inhabited,

(P2) $a, b \in \alpha \implies \alpha \upharpoonright (a \downarrow b)$,

(P3) $a \in \alpha \land a \triangledown U \implies \alpha \upharpoonright U$,

where $a \downarrow b \overset{\text{def}}{=} \{a\} \downarrow \{b\}$. The class of formal points of $S$ is denoted by $Pt(S)$.

Our main interest is in inductively generated topologies, which allow us to reason about formal topologies using selected sets of axioms.

**Definition 22.** An axiom-set on a set $S$ is a pair $(I, C)$, where $(I(a))_{a \in S}$ is a family of sets indexed by $S$, and $C$ is a family $(C(a, i))_{a \in S, i \in I(a)}$ of subsets of $S$ indexed by $\sum_{a \in S} I(a)$. A formal topology $(S, \leq, \triangledown)$ is inductively generated by $(I, C)$ if $\triangledown$ is the smallest among the relations $\triangledown'$ such that

1. $a \leq b \triangledown' U \implies a \triangledown' U$, 
2. $a \triangledown' C(a, i)$ for each $i \in I(a)$,

and which makes $(S, \leq, \triangledown')$ a formal topology.

A formal point of an inductively generated formal topology can be characterised by an axiom-set, where condition (P3) is replaced by

1. $a \leq b \land a \in \alpha \implies b \in \alpha$, 
2. $a \in \alpha \implies \alpha \upharpoonright C(a, i)$ for each $i \in I(a)$.

**Remark 23.** The construction of an inductively generated formal topology requires CZF extended with the Regular Extension Axiom [3], which is much stronger than ECST. However, in Proposition 24 and Proposition 28, all we need is a preorder equipped with an axiom-set. Hence, in this paper, we identify inductively generated formal topologies with preorders equipped with axiom-sets, and adopt the notions of formal point and formal topology map formulated in terms of axiom-sets.

**Proposition 24.** The following are equivalent over ECST + FPA:

1. $\text{NID}_{<\omega}$;

2. The class of formal points of any inductively generated formal topology is set-generated.
Proof. (1 → 2) Assume $\text{NID}_{<\omega}$. Let $S = (S, \leq, \triangleleft)$ be a formal topology inductively generated by an axiom-set $(I, C)$ on $S$. Then, formal points of $S$ are closed subsets of the following set of finitary rules on $S$:

$$
\begin{align*}
\&\left\{ (\emptyset, S) \right\} \\
\&\cup \left\{ \left( \{a, b\}, a \downarrow b \right) \right\} \\
\&\cup \left\{ \left( \{a\}, \{b\} \right) | a \leq b \right\} \\
\&\cup \left\{ \left( \{a\}, C(a, i) \right) | a \in S, i \in I(a) \right\} .
\end{align*}
$$

Thus $\text{Pt}(S)$ is set-generated.

(2 → 1) Assume that the class of formal points of any inductively generated formal topology is set-generated. Let $R$ be a set of finitary rules on a set $S$. Using FPA, define an axiom-set $(I, C)$ on $\text{Fin}(S)$ by

$$
\begin{align*}
I(a) &\overset{\text{def}}{=} \left\{ (b, c) \in R | b = a \right\}, \\
C(a, (b, c)) &\overset{\text{def}}{=} \left\{ y \in c \right\}.
\end{align*}
$$

Let $S = (\text{Fin}(S), \supseteq, \triangleleft)$ be a formal topology inductively generated by $(I, C)$ using the reverse inclusion order on $\text{Fin}(S)$, and let $C$ be the class of $R$-closed subsets of $S$. As in the proof of $(1 \rightarrow 3)$ in Proposition 17, one can show that the mappings

$$
\alpha \mapsto \cup \alpha : \text{Pt}(S) \rightarrow C, \\
\beta \mapsto \text{Fin}(\beta) : C \rightarrow \text{Pt}(S)
$$

are well-defined and that they are inverses of each other. By the assumption, $\text{Pt}(S)$ has a generating subset $G$. Then, $H = \{ \cup \alpha | \alpha \in G \}$ is a generating subset of the class of $R$-closed subsets of $S$.

Note that FPA is needed only in the direction $(2 \rightarrow 1)$. \hfill \square

A formal point is an instance of morphisms between formal topologies.

**Definition 25.** Let $S = (S, \leq, \triangleleft)$ and $S' = (S', \leq', \triangleleft')$ be formal topologies. A relation $r \subseteq S \times S'$ is a formal topology map from $S$ to $S'$ if

$$
\begin{align*}
(\text{FTM1}) &\ S \triangleleft r^- S', \\
(\text{FTM2}) &\ r^- a \downarrow r^- b \triangleleft r^- (a \downarrow' b), \\
(\text{FTM3}) &\ a \triangleleft' U \implies r^- a \triangleleft r^- U.
\end{align*}
$$

Two formal topology maps $r, s : S \rightarrow S'$ are equal if $r^- a \triangleleft s^- a$ and $s^- a \triangleleft r^- a$ for all $a \in S'$.

If $r : S \rightarrow S'$ is a formal topology map and $S'$ is inductively generated by an axiom-set $(I, C)$ on $S'$, then the condition (FTM3) can be replaced by

$$
\begin{align*}
(\text{FTM3a}) &\ a \leq' b \implies r^- a \triangleleft r^- b, \\
(\text{FTM3b}) &\ r^- a \triangleleft r^- C(a, i) \text{ for each } i \in I(a).
\end{align*}
$$

**Remark 26.** Let $1$ denote the formal topology $(\{\ast\}, =, \in)$. Then, a formal point $\alpha$ of a formal topology $S$ corresponds to a formal topology map $r_\alpha : 1 \rightarrow S$ given by $\ast r_\alpha a \overset{\text{def}}{\iff} a \in \alpha$. 

12
Set-presentations of formal topologies provide a stronger notion of inductive generation.

**Definition 27.** A formal topology $S = (S, \leq, \triangleright)$ is *set-presented* if there exists an axiom-set $(I, C)$ on $S$ such that
\[
a \triangleright U \iff \exists i \in I(a) (C(a, i) \subseteq U).
\]
In this case, $(I, C)$ is called a *set-presentation* of $S$.

**Proposition 28.** The following are equivalent over ECST + FPA:

1. $\text{NID}_{\omega}$;
2. The class of formal topology maps from a set-presented formal topology to an inductively generated formal topology is set-generated.

**Proof.** Since 1 is set-presented, 2 implies 1 by Proposition 24 and Remark 26. Note that we need FPA in this direction.

Conversely, assume $\text{NID}_{\omega}$. Let $S = (S, \leq, \triangleright)$ be a formal topology with a set-presentation $(I, C)$, and let $T = (T, \leq', \triangleright')$ be a formal topology inductively generated by an axiom-set $(J, D)$. Then, a formal topology map $r : S \to T$ is a model of the following generalised geometric theory over $S \times T$ (cf. Aczel et al. [2, Proposition 7.8]):

\[
(\forall i \in I(a) \forall a' \in C(a, i) \exists b' \in T) (a' \in S) \quad (5.5)
\]
\[
(\forall i \in I(a) \forall a' \in C(a, i) \exists b' \in D(b, j)) (a, b) \vdash \forall a' \in C(a, i) \forall b' \in D(b, j) (a', b') \quad (5.8)
\]

where $(5.5)$, $(5.6)$, $(5.7)$, and $(5.8)$ are derived from (FTM1), (FTM2), (FTM3a), and (FTM3b), respectively, using the fact that $S$ is set-presented by $(I, C)$. The disjunctions such as $\bigvee_{b \in T} \{a', b\}$ must be read as $\bigwedge_{b \in T} \bigvee \{a', b\}$. Then, the required conclusion follows from Remark 6.

\[\square\]

### 5.4 Equalisers in the category of concrete spaces

We show that $\text{NID}_{\omega}$ is equivalent to the existence of equalisers in the category of concrete spaces, a predicative notion of point-set topology by Sambin [15]. Ishihara and Kawai [8, Section 4] have already shown that the category is complete and cocomplete using SGA. Hence, the essence of this subsection is that the converse holds.

**Definition 29.** A *concrete space* is a basic pair $(X, \mathcal{R}, S)$ such that

1. $X = \text{ext } S$,
2. $\text{ext } a \cap \text{ext } b = \text{ext } (a \downarrow b)$
for all $a, b \in S$, where
\[ \text{ext } a \overset{\text{def}}{=} \{ x \in X \mid x \Vdash a \} , \quad \text{ext } U \overset{\text{def}}{=} \bigcup_{a \in U} \text{ext } a, \quad (5.9) \]
\[ a \downarrow b \overset{\text{def}}{=} \{ c \in S \mid \text{ext } c \subseteq \text{ext } a \cap \text{ext } b \} . \]

We also define $U \downarrow V \overset{\text{def}}{=} \bigcup_{a \in U, b \in V} a \downarrow b$ for $U, V \in \text{Pow}(S)$.

Let $X_1 = (X_1, \Vdash_1, S_1)$ and $X_2 = (X_2, \Vdash_2, S_2)$ be basic pairs. A relation pair $(r, s) : X_1 \to X_2$ is said to be convergent if

1. $\text{ext }_1 S_1 = r^{-1} \text{ext }_2 S_2$,
2. $\text{ext }_1 (s^{-1} a \downarrow s^{-1} b) = r^{-1} \text{ext }_2 (a \downarrow b)$,

where $\text{ext }_i (i = 1, 2)$ is the operator given by (5.9) associated with $\Vdash_i$ $(i = 1, 2)$.

Concrete spaces and convergent relation pairs form a subcategory $\text{CSpa}$ of $\text{BP}$. Specifically, $\text{CSpa}$ is a coreflective subcategory of $\text{BP}$.

**Proposition 30** (Ishihara and Kawai [8, Theorem 7.1]). For any basic pair $X$, there exist a concrete space $\tilde{X}$ and a relation pair $(r, s) : \tilde{X} \to X$ such that for any concrete space $Y$ and a relation pair $(u, v) : Y \to X$ there exists a unique convergent relation pair $(\tilde{u}, \tilde{v}) : Y \to \tilde{X}$ which makes the following diagram commute:

\[ Y \xrightarrow{(\tilde{u}, \tilde{v})} \tilde{X} \xrightarrow{(r, s)} X. \quad (5.10) \]

**Proof.** See Ishihara and Kawai [8, Theorem 7.1] for the details. Their proof can be carried out in $\text{ECST} + \text{FPA}$.

The construction of an equaliser of $\text{CSpa}$ uses the notion of convergent subset.

**Definition 31.** Let $X = (X, \Vdash, S)$ be a basic pair. A subset $D \subseteq X$ is said to be convergent if

1. $D \nvdash \text{ext } S$,
2. $D \nvdash \text{ext } a \& D \nvdash \text{ext } b \implies D \nvdash (a \downarrow b)$.

The class of convergent subsets of a basic pair $X$ is denoted by $\text{Conv}(X)$.

An equaliser in $\text{CSpa}$ is constructed from a generating subset of a certain class. In the lemma below, $\odot_2 U$ denotes the set $\{ a \in S_2 \mid \exists x \in U \ (x \Vdash_2 a) \}$ for each subset $U \subseteq X_2$.

**Lemma 32** (Ishihara and Kawai [8, Proposition 6.3]). Let $X_1 = (X_1, \Vdash_1, S_1)$ and $X_2 = (X_2, \Vdash_2, S_2)$ be concrete spaces and $(r_1, s_1), (r_2, s_2) : X_1 \to X_2$ be a parallel pair of morphisms in $\text{CSpa}$. If the class defined by
\[ E \overset{\text{def}}{=} \{ D \in \text{Conv}(X_1) \mid \odot_2 r_1 D = \odot_2 r_2 D \} \quad (5.11) \]
is set-generated, then the parallel pair has an equaliser.
Proposition 33. The following are equivalent over ECST + FPA:

1. NID_{<\omega};

2. CSpa has equalisers.

Proof. (1 \rightarrow 2) This was proved by Ishihara and Kawai [8, Proposition 6.3] using SGA. We reformulate the proof using NID_{<\omega}. Assume NID_{<\omega}, and let \((r_1, s_1), (r_2, s_2) : X_1 \rightarrow X_2\) be a parallel pair of morphisms in CSpa. It suffices to show that the class \(E\) given in (5.11) is set-generated. Define a set of finitary rules on \(X_1\) by

\[
R \overset{\text{def}}{=} \{(\emptyset, \text{ext}_1 S_1)\} \cup \{(\{x, y\}, \text{ext}_1 (a \downarrow b)) \mid x \not\vdash a, y \not\vdash b\} \\
\cup \{(\{x\}, \text{ext}_2 c) \mid c \in \Diamond_{\exists} r_1 \{x\}\} \\
\cup \{(\{x\}, \text{ext}_2 c) \mid c \in \Diamond_{\exists} r_2 \{x\}\}.
\]

Then, \(E\) is the class of \(R\)-closed subsets of \(X_1\), and thus it is set-generated.

(2 \rightarrow 1) Assume that CSpa has equalisers. In the following, we write \(S_{\Delta}\) for the basic pair \((S, \Delta_S, S)\) on a set \(S\) and \(\text{Fin}(S)_{\supseteq}\) for the concrete space \((\text{Fin}(S), \supseteq, \text{Fin}(S))\).

Let \(R\) be a set of finitary rules on a set \(S\). Define relations \(r_1, r_2 \subseteq \text{Fin}(S) \times R\) by

\[
A \ r_1 (a, b) \overset{\text{def}}{=} A \supseteq a, \\
A \ r_2 (a, b) \overset{\text{def}}{=} \exists y \in b (A \supseteq \{y\} \cup a).
\]

Obviously, \((r_1, r_1)\) and \((r_2, r_2)\) are relation pairs from \(\text{Fin}(S)_{\supseteq}\) to \(R_{\Delta}\). By Proposition 30, \((r_1, r_1)\) and \((r_2, r_2)\) determine unique convergent relation pairs \((\tilde{r}_1, \tilde{r}_1)\) and \((\tilde{r}_2, \tilde{r}_2)\) from \(\text{Fin}(S)_{\supseteq}\) to \(R_{\Delta}\), respectively, which make the diagram (5.10) commute. Let \((p, q) : X \rightarrow \text{Fin}(S)_{\supseteq}\) be an equaliser of \((\tilde{r}_1, \tilde{r}_1)\) and \((\tilde{r}_2, \tilde{r}_2)\) in CSpa, and write \(X = (X, \vdash, K)\). Define a set \(G\) of subsets of \(S\) by

\[
G \overset{\text{def}}{=} \{\alpha_x \mid x \in X\},
\]

where

\[
\alpha_x \overset{\text{def}}{=} \{z \in S \mid \exists A \in \text{Fin}(S) (x \ p \ A \land A \supseteq \{z\})\}.
\]

Let \(x \in X\) and \((a, b) \in R\) such that \(a \subseteq \alpha_x\). Since \((p, q)\) is convergent, there exists a \(B \in \text{Fin}(S)\) such that \(x \ p \ B \land B \supseteq a\). Thus, \(x \ (r_1 \circ p)\) \(a\). Since \((r_1, r_1)\) and \((r_2, r_2)\) are relation pairs from \(\text{Fin}(S)_{\supseteq}\) to \(R_{\Delta}\), respectively, which make the diagram (5.10) commute. Let \((p, q) : X \rightarrow \text{Fin}(S)_{\supseteq}\) be an equaliser of \((\tilde{r}_1, \tilde{r}_1)\) and \((\tilde{r}_2, \tilde{r}_2)\) in CSpa, and write \(X = (X, \vdash, K)\). Define a set \(G\) of subsets of \(S\) by

\[
G \overset{\text{def}}{=} \{\alpha_x \mid x \in X\},
\]

where

\[
\alpha_x \overset{\text{def}}{=} \{z \in S \mid \exists A \in \text{Fin}(S) (x \ p \ A \land A \supseteq \{z\})\}.
\]

Let \(x \in X\) and \((a, b) \in R\) such that \(a \subseteq \alpha_x\). Since \((p, q)\) is convergent, there exists a \(B \in \text{Fin}(S)\) such that \(x \ p \ B \land B \supseteq a\). Thus, \(x \ (r_1 \circ p)\) \(a\). Since \((r_1, r_1)\) and \((r_2, r_2)\) are relation pairs from \(\text{Fin}(S)_{\supseteq}\) to \(R_{\Delta}\), respectively, which make the diagram (5.10) commute. Let \((p, q) : X \rightarrow \text{Fin}(S)_{\supseteq}\) be an equaliser of \((\tilde{r}_1, \tilde{r}_1)\) and \((\tilde{r}_2, \tilde{r}_2)\) in CSpa, and write \(X = (X, \vdash, K)\). Define a set \(G\) of subsets of \(S\) by

\[
G \overset{\text{def}}{=} \{\alpha_x \mid x \in X\},
\]

where

\[
\alpha_x \overset{\text{def}}{=} \{z \in S \mid \exists A \in \text{Fin}(S) (x \ p \ A \land A \supseteq \{z\})\}.
\]
It is easy to see that \((p_{\beta}, q_{\beta})\) is convergent. Then, for any \((a, b) \in R\)
\[
* (r_1 \circ p_{\beta}) (a, b) \iff \exists A \in \text{Fin}(S) (a \subseteq A \subseteq \beta) \\
\iff a \subseteq \beta \\
\iff \exists y \in b (a \cup \{y\} \subseteq \beta) \\
\iff * (r_2 \circ p_{\beta}) (a, b).
\]
Thus \((r_1, r_1) \circ (p_{\beta}, q_{\beta}) \sim (r_2, r_2) \circ (p_{\beta}, q_{\beta})\), and so \((\tilde{r}_1, \tilde{r}_1) \circ (p_{\beta}, q_{\beta}) \sim (\tilde{r}_2, \tilde{r}_2) \circ (p_{\beta}, q_{\beta})\) by Proposition 30. Hence, \((p_{\beta}, q_{\beta})\) factors uniquely through \((p, q)\) via a convergent relation pair \((u, v)\): \(*\Delta \rightarrow X\). Since \(* p_{\beta} \{z\}\), there exist \(x \in X\) and \(B \in \text{Fin}(S)\) such that \(* u x, x p B, \) and \(B \supseteq \{z\}\). Thus \(z \in \alpha_x\). Moreover, for any \(y \in \alpha_x\), there exists an \(A \in \text{Fin}(S)\) such that \(\{y\} \subseteq A \subseteq \beta\). Hence \(\alpha_x \subseteq \beta\). Therefore, \(G\) generates the class of \(R\)-closed subsets.

Note that FPA is needed only in the direction \((2 \rightarrow 1)\).

Remark 34. Ishihara and Kawai [8, Proposition 6.4] showed that \(\text{CSpa}\) has small products using SGA. Thus, if \(\text{CSpa}\) has equalisers, then \(\text{CSpa}\) is complete under FPA. Moreover, coequalisers in \(\text{CSpa}\) can be constructed exactly as in BP [8, Lemma 5.2]. Thus, \(\text{CSpa}\) is cocomplete under \(\text{NID}_1\) and hence under \(\text{NID}_{<\omega}\) as well. Therefore, the following are equivalent over \(\text{ECST} + \text{FPA}\):

1. \(\text{CSpa}\) has equalisers;
2. \(\text{CSpa}\) is complete and cocomplete.

We summarise the equivalents of \(\text{NID}_{<\omega}\).

**Theorem 35.** The following are equivalent over \(\text{ECST}\):

1. \(\text{NID}_{<\omega}\);
2. SGA;
3. \(\text{NID}_n (n \geq 2)\);
4. The class of models of any geometric theory is set-generated.

Moreover, each of the following statement together with FPA is equivalent to \(\text{NID}_{<\omega}\) over \(\text{ECST}\):

5. The class of formal points of any inductively generated formal topology is set-generated;
6. The class of formal topology maps from a set-presented formal topology to an inductively generated formal topology is set-generated;
7. \(\text{CSpa}\) has equalisers;
8. \(\text{CSpa}\) is complete and cocomplete.

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