THE q-WAKIMOTO REALIZATION OF
THE SUPERA LGERAS $U_q(\widehat{sl}(N|1))$ AND $U_{q,p}(\widehat{sl}(N|1))$

December 21, 2013

TAKEO KOJIMA

Department of Mathematics and Physics, Graduate School of Science and Engineering,
Yamagata University, Jonan 4-3-16, Yonezawa 992-8510, Japan
kojima@yz.yamagata-u.ac.jp

Abstract

We give bosonizations of the superalgebras $U_q(\widehat{sl}(N|1))$ and $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{C}$. We introduce the submodule by the $\xi$-$\eta$ system, that we call the $q$-Wakimoto realization.

1 Introduction

Bosonizations are known to be a powerful method to construct correlation functions in not only conformal field theory [1], but also exactly solvable lattice models [2]. The quantum algebra $U_q(g)$ and the elliptic algebra $U_{q,p}(g)$ play an important role in exactly solvable lattice models. The level parameter $k$ plays an important role in representation theory for $U_q(g)$ and $U_{q,p}(g)$. Bosonizations for an arbitrary level $k$ are completely different from those of level $k = 1$. In the case for level $k = 1$, bosonizations have been constructed for quantum algebra $U_q(g)$ in many cases $g = (ADE)^{(r)}$, $(BC)^{(1)}$, $G_2^{(1)}$, $\widehat{sl}(M|N)$, $osp(2|2)^{(2)}$ [3, 6, 4, 5, 7, 8, 9, 10]. Using the dressing method developed in non-twisted algebra [11] and twisted algebra $A_2^{(2)}$ [12], we have bosonizations of the elliptic algebra $U_{q,p}(g)$ for $g = (ADE)^{(1)}$, $(BC)^{(1)}$, $G_2^{(1)}$ and $A_2^{(2)}$. In the case of an arbitrary level $k$, bosonizations have been constructed only for $U_q(\widehat{sl}(N))$ [14, 15, 16], $U_q(\widehat{sl}(2|1))$ [17], $U_{q,p}(\widehat{sl}(N))$ [11], and $U_{q,p}(\widehat{sl}(2|1))$ [19]. In this paper we give a bosonization of the quantum superalgebra $U_q(\widehat{sl}(N|1))$ for an arbitrary level $k$ [20]. Using the dressing method developed in [19], we give a bosonization of the quantum superalgebra $U_{q,p}(\widehat{sl}(N|1))$ for an arbitrary level $k$. The level
k bosonizations on the boson Fock space of \( U_q(\hat{sl}(N)) \) and \( U_q(\hat{sl}(N|1)) \) \cite{16 17 20} are not irreducible realizations. The construction of the irreducible highest weight module \( V(\lambda) \) is nontrivial problem. We recall the non-quantum algebra \( \hat{sl}(2) \) case \cite{21}. The submodule \( \xi \) induced by the \( \xi \) highest weight module \( \hat{U}_q(\hat{sl}(2)) \) was constructed from the Wakimoto realization on the boson Fock space \cite{13} by the Felder complex. We recall the quantum algebra \( U_q(\hat{sl}(2)) \) case \cite{14 15 22}. The submodule \( \xi \) induced by the \( \xi \) bosonizations on the boson Fock space \cite{14 15} by two steps; the first step is the resolution by the \( \xi \)-\( \eta \) system, and the second step is the resolution by the Felder complex \cite{23 21}. The submodule \( \xi \) induced by the \( \xi \)-\( \eta \) system, plays the same role as the Wakimoto realization of the non-quantum algebra \( \hat{U}_q(\hat{sl}(2)) \). We would like to call this submodule induced by the \( \xi \)-\( \eta \) system "the \( q \)-Wakimoto realization". Constructions of the irreducible highest weight module \( V(\lambda) \) for \( U_q(\hat{sl}(N)) \) \((N \geq 3)\) and \( U_q(\hat{sl}(N|1)) \) \((N \geq 2)\) are still an open problem. In this paper we study the \( \xi \)-\( \eta \) system and introduce the \( q \)-Wakimoto realization for the superalgebra \( U_q(\hat{sl}(N|1)) \) and \( U_{q,p}(\hat{sl}(N|1)) \).

This paper is organized as follows. In section 2, after preparing notations, we give the definition of the quantum superalgebra \( U_q(\hat{sl}(N|1)) \) and the elliptic superalgebra \( U_{q,p}(\hat{sl}(N|1)) \). In section 3 we give bosonizations of the superalgebras \( U_q(\hat{sl}(N|1)) \) and \( U_{q,p}(\hat{sl}(N|1)) \) for an arbitrary level \( k \). In section 4 we introduce the \( q \)-Wakimoto realization of by the \( \xi \)-\( \eta \) system.

## 2 Superalgebra \( U_q(\hat{sl}(N|1)) \) and \( U_{q,p}(\hat{sl}(N|1)) \)

In this section we recall the definitions of the quantum superalgebra \( U_q(\hat{sl}(N|1)) \) \cite{18} and the elliptic deformed superalgebra \( U_{q,p}(\hat{sl}(N|1)) \) \cite{19} for \( N \geq 2 \). We fix a complex number \( q \neq 0, |q| < 1 \). We set

\[
[x, y] = xy - yx, \quad \{x, y\} = xy + yx, \quad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}.
\]

(2.1)

Let us fix complex numbers \( r, k \in \mathbb{C}, \) \( \text{Re}(r) > 0, \text{Re}(r - k) > 0 \). We use the abbreviation \( r^* = r - k \). We set \( p = q^{2r} \). We set the Jacobi theta functions

\[
[u] = q^{\frac{u^2}{2} - u} \frac{\Theta_{p^2r}(q^{2u})}{(q^{2r}; q^2)_\infty}, \quad [u]^* = q^{\frac{u^2}{2} - u} \frac{\Theta_{p^2r^*}(q^{2u})}{(q^{2r^*}; q^2)_\infty},
\]

(2.2)

where we have used

\[
\Theta_p(z) = (z;p)_\infty(pz^{-1}; p)_\infty(p;p)_\infty, \quad (z;p)_\infty = \prod_{m=0}^{\infty} (1 - p^m z).
\]

(2.3)

The Cartan matrix \( (A_{i,j})_{0 \leq i, j \leq N} \) of the affine Lie algebra \( \hat{sl}(N|1) \) is given by

\[
A_{i,j} = (\nu_i + \nu_{i+1}) \delta_{i,j} - \nu_i \delta_{i,j+1} - \nu_{i+1} \delta_{i+1,j}.
\]

(2.4)

Here we set \( \nu_1 = \cdots = \nu_N = +, \nu_{N+1} = \nu_0 = - \).

### 2.1 Quantum superalgebra \( U_q(\hat{sl}(N|1)) \)

In this section we recall the definition of the quantum affine superalgebra \( U_q(\hat{sl}(N|1)) \).
Definition 2.1 [18] The Drinfeld generators of the quantum superalgebra $U_q(\hat{sl}(N|1))$ are
\[ x_{i,m}^\pm, h_{i,m}, c, \quad (1 \leq i \leq N, m \in \mathbb{Z}). \] (2.5)

Defining relations are
\[ c : \text{central}, \quad [h_i, h_{j,m}] = 0, \] (2.6)
\[ [a_{i,m}, h_{j,n}] = \frac{[A_{i,j}m]_q [cm]_q}{m} q^{-c|m} \delta_{m+n,0} \quad (m, n \neq 0), \] (2.7)
\[ [h_i, x_j^+(z)] = \pm A_{i,j} x_j^+(z), \] (2.8)
\[ [h_{i,m}, x_j^+(z)] = \frac{[A_{i,j}m]_q}{m} q^{-c|m} z^m x_j^+(z) \quad (m \neq 0), \] (2.9)
\[ [h_{i,m}, x_j^-(z)] = \frac{[A_{i,j}m]_q}{m} z^m x_j^-(z) \quad (m \neq 0), \] (2.10)
\[ (z_1 - q^{\pm A_{i,j}} z_2) x_j^+(z_1) x_j^+(z_2) = (q^{\pm A_{i,j}} z_1 - z_2) x_j^+(z_2) x_j^+(z_1) \quad \text{for } |A_{i,j}| \neq 0, \] (2.11)
\[ x_j^+(z_1) x_j^+(z_2) = x_j^+(z_2) x_j^+(z_1) \quad \text{for } |A_{i,j}| = 0, (i, j) \neq (N, N), \] (2.12)
\[ \{x_N^+(z_1), x_N^-(z_2)\} = 0, \] (2.13)
\[ [x_i^+(z_1), x_j^-(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1/z_2) \psi_i^+(q z_2) - \delta(q^c z_1/z_2) \psi_i^-(q^{-1} z_2) \right), \] for $(i, j) \neq (N, N)$, (2.14)
\[ \{x_N^+(z_1), x_N^-(z_2)\} = \frac{1}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-c} z_1/z_2) \psi_N^+(q z_2) - \delta(q^c z_1/z_2) \psi_N^-(q^{-1} z_2) \right), \] (2.15)
\[ (x_i^+(z_1) x_i^+(z_2) x_j^+(z) - (q + q^{-1}) x_j^+(z_1) x_j^+(z) x_i^+(z_2) + x_j^+(z) x_i^+(z_1) x_i^+(z_2)) \] (2.16)
\[ + (z_1 \leftrightarrow z_2) = 0 \quad \text{for } |A_{i,j}| = 1, i \neq N,
\]
where we have used $\delta(z) = \sum_{m \in \mathbb{Z}} z^m$. Here we have used the abbreviation $h_i = h_{i,0}$. We have set the generating function
\[ x_j^+(z) = \sum_{m \in \mathbb{Z}} x_{j,m}^+ z^{-m-1}, \] (2.17)
\[ \psi_i^+(q^c z) = q^{h_i} \exp \left( (q - q^{-1}) \sum_{m > 0} h_{i,m} z^{-m} \right), \] (2.18)
\[ \psi_i^-(q^{-c} z) = q^{-h_i} \exp \left( -(q - q^{-1}) \sum_{m > 0} h_{i,-m} z^{m} \right), \] (2.19)

2.2 Elliptic Superalgebra $U_{q,p}(\hat{sl}(N|1))$

In this section we recall the definition of the elliptic superalgebra $U_{q,p}(\hat{sl}(N|1))$.

Definition 2.2 [17] The elliptic superalgebra $U_{q,p}(\hat{sl}(N|1))$ is the associative algebra generated by the currents $E_j(z), F_j(z), H_j^\pm(z) \quad (1 \leq j \leq N)$ and $B_{j,m}(1 \leq j \leq N, m \in \mathbb{Z}_{\neq 0}), h_j \quad (1 \leq j \leq N)$ that satisfy the following relations.
\[ [h_i, B_{j,m}] = 0, \quad [B_{i,m}, B_{j,n}] = \frac{[A_{i,j}m]_q [km]_q [r^*m]_q}{m} \delta_{m+n,0}. \] (2.20)
\[ [h_i, E_j(z)] = A_{i,j}E_j(z), \quad [h_i, F_j(z)] = -A_{i,j}F_j(z), \quad (2.21) \]
\[ [B_{i,m}, E_j(z)] = \frac{[A_{i,j}m]_q}{m} z^m E_j(z), \quad (2.22) \]
\[ [B_{i,m}, F_j(z)] = \frac{[A_{i,j}m]_q}{m} [r^*m]_q z^m F_j(z). \quad (2.23) \]

For \( 1 \leq i, j \leq N \) such that \((i, j) \neq (N, N)\) they satisfy
\[
\left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right]^* E_j(z_1)E_j(z_2) = \left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right]^* E_j(z_2)E_i(z_1), \quad (2.24) \\
\left[ u_1 - u_2 + \frac{A_{i,j}}{2} \right] F_i(z_1)F_j(z_2) = \left[ u_1 - u_2 - \frac{A_{i,j}}{2} \right] F_j(z_2)F_i(z_1), \quad (2.25) \\
[E_i(z_1), F_j(z_2)] = \frac{\delta_{i,j}}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-k} z_1 / z_2) H_i(q^r z_2) - \delta(q^k z_1 / z_2) H_i(q^{-r} z_2) \right), \quad (2.26) \\
\{ E_N(z_1), E_N(z_2) \} = 0, \quad \{ F_N(z_1), F_N(z_2) \} = 0, \quad (2.27) \\
\{ E_N(z_1), F_N(z_2) \} = \frac{1}{(q - q^{-1}) z_1 z_2} \left( \delta(q^{-k} z_1 / z_2) H_N(q^r z_2) - \delta(q^k z_1 / z_2) H_N(q^{-r} z_2) \right). \quad (2.28) \\
\]

For \( 1 \leq i, j \leq N \) they satisfy
\[
H_i(z_1)H_j(z_2) = \frac{[u_2 - u_1 - \frac{A_{i,j}}{2}]^*[u_2 - u_1 + \frac{A_{i,j}}{2}]}{[u_2 - u_1 + \frac{A_{i,j}}{2}]^*[u_2 - u_1 - \frac{A_{i,j}}{2}]} H_j(z_2)H_i(z_1), \quad (2.29) \\
H_i(z_1)E_j(z_2) = \frac{[u_1 - u_2 + \frac{A_{i,j}}{2}]^*[u_2 - u_1 + \frac{A_{i,j}}{2}]}{[u_1 - u_2 + \frac{A_{i,j}}{2}]^*[u_2 - u_1 - \frac{A_{i,j}}{2}]} E_j(z_2)H_i(z_1), \quad (2.30) \\
H_i(z_1)F_j(z_2) = \frac{[u_1 - u_2 + \frac{A_{i,j}}{2}]^*[u_2 - u_1 - \frac{A_{i,j}}{2}]}{[u_1 - u_2 + \frac{A_{i,j}}{2}]^*[u_2 - u_1 - \frac{A_{i,j}}{2}]} F_j(z_2)H_i(z_1). \quad (2.31) \\
\]

For \( 1 \leq i, j \leq N, (i \neq N) \) such that \( |A_{i,j}| = 1 \), they satisfy the Serre relations.
\[
\begin{align*}
\left\{ \begin{array}{l}
E_i(z_1)E_i(z_2)E_j(z) & \left( \frac{q^{2r^* + A_{i,j}} z_1}{q^{2r^* - A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* + A_{i,j}} z_2}{q^{2r^* - A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
E_i(z_1)E_j(z_2)E_i(z) & \left( \frac{q^{2r^* - A_{i,j}} z_1}{q^{2r^* + A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* - A_{i,j}} z_2}{q^{2r^* + A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
-(q + q^{-1})E_i(z_1)E_j(z_2)E_i(z) & \left( \frac{q^{2r^* + A_{i,j}} z_1}{q^{2r^* - A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* + A_{i,j}} z_2}{q^{2r^* - A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
E_j(z_1)E_i(z_2)E_i(z) & \left( \frac{q^{2r^* - A_{i,j}} z_1}{q^{2r^* + A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* - A_{i,j}} z_2}{q^{2r^* + A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
-(q + q^{-1})E_j(z_1)E_i(z_2)E_i(z) & \left( \frac{q^{2r^* - A_{i,j}} z_1}{q^{2r^* + A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* - A_{i,j}} z_2}{q^{2r^* + A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
The \end{array} \right. \\
\left\{ \begin{array}{l}
\left( \frac{q^{2r^* + A_{i,j}} z_1}{q^{2r^* - A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* + A_{i,j}} z_2}{q^{2r^* - A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
\left( \frac{q^{2r^* - A_{i,j}} z_1}{q^{2r^* + A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* - A_{i,j}} z_2}{q^{2r^* + A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
-(q + q^{-1}) \left( \frac{q^{2r^* + A_{i,j}} z_1}{q^{2r^* - A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* + A_{i,j}} z_2}{q^{2r^* - A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
-(q + q^{-1}) \left( \frac{q^{2r^* - A_{i,j}} z_1}{q^{2r^* + A_{i,j}} z_1}; q^{2r^*} \right)_\infty \left( \frac{q^{2r^* - A_{i,j}} z_2}{q^{2r^* + A_{i,j}} z_2}; q^{2r^*} \right)_\infty \left( \frac{z_1}{z_2} \right) \frac{1}{A_{i,j}} \\
0, \quad (2.32) \\
\end{align*} 
\]
We fix the level $k \leq 3.1$ Boson

We use the standard symbol of the normal orderings $::$. In what follows we use the abbreviations $z_3$ Bosonization

3.2 Quantum Superalgebra

In this section we give a bosonization of the quantum superalgebra $U_q(\hat{sl}(N|1))$ and $U_{q,p}(\hat{sl}(N|1))$ for an arbitrary level $k \in \mathbb{Z} [17, 19, 20].$

3.1 Boson

We fix the level $c = k \in \mathbb{C}$. We introduce the bosons and the zero-mode operators $a^i_m, Q^j_a (m \in \mathbb{Z}, 1 \leq j \leq N), b^{i,j}_m, Q^{i,j}_b (m \in \mathbb{Z}, 1 \leq i < j \leq N + 1), e^{i,j}_m, Q^{i,j}_c (m \in \mathbb{Z}, 1 \leq i < j \leq N)$. The bosons $a^i_m, b^{i,j}_m, c^{i,j}_m$, $m \in \mathbb{Z}_{\neq 0}$ and the zero-mode operators $a^i_0, Q^j_a, b^{i,j}_0, Q^{i,j}_b, c^{i,j}_0, Q^{i,j}_c$ satisfy

$$[a^i_m, a^j_n] = \frac{[(k + N - 1)m][A_{i,j}m]}{m} \delta_{m+n,0}, \quad [a^i_0, Q^j_a] = (k + N - 1)A_{i,j}, (3.1)$$

$$[b^{i,j}_m, b^{i',j'}_n] = -\nu_i \nu_j \frac{[m]^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [b^{i,j}_0, Q^{i',j'}_b] = -\nu_i \nu_j \delta_{i,i'} \delta_{j,j'}, (3.2)$$

$$[c^{i,j}_m, c^{i',j'}_n] = \frac{[m]^2}{m} \delta_{i,i'} \delta_{j,j'} \delta_{m+n,0}, \quad [c^{i,j}_0, Q^{i',j'}_c] = \delta_{i,i'} \delta_{j,j'}. (3.3)$$

We impose the cocycle condition on the zero-mode operator $Q^{i,j}_b$, $(1 \leq i < j \leq N + 1)$ by

$$[Q^{i,j}_b, Q^{i',j'}_b] = \delta_{j,N+1} \delta_{j',N+1} \pi \sqrt{-1} \quad \text{for} \ (i, j) \neq (i', j'). (3.4)$$

We have the following (anti) commutation relations

$$[e^{Q^{i,j}_b}, e^{Q^{i',j'}_b}] = 0 \quad (1 \leq i < j \leq N, 1 \leq i' < j' \leq N), (3.5)$$

$$\{e^{Q^{i,N+1}_b}, e^{Q^{i,N+1}_b} \} = 0 \quad (1 \leq i \neq j \leq N). (3.6)$$

We use the standard symbol of the normal orderings $::$. In what follows we use the abbreviations $b^{i,j}(z), c^{i,j}(z), b^{i,j}_\pm(z), a^{i}_\pm(z)$ given by

$$b^{i,j}(z) = -\sum_{m \neq 0} \frac{b^{i,j}_m}{[m]^q} z^{-m} + Q^{i,j}_b + b^{0,j}_0 \log z, \quad c^{i,j}(z) = -\sum_{m \neq 0} \frac{c^{i,j}_m}{[m]^q} z^{-m} + Q^{i,j}_c + c^{0,j}_0 \log z, (3.7)$$

$$b^{i,j}_\pm(z) = \pm (q - q^{-1}) \sum_{m \geq 0} b^{i,j}_m z^{-m} + b^{0,j}_0 \log q, \quad a^{i}_\pm(z) = \pm (q - q^{-1}) \sum_{m \geq 0} a^{i}_m z^{-m} + a^{0}_0 \log q. (3.8)$$

3.2 Quantum Superalgebra $U_q(\hat{sl}(N|1))$

In this section we give a bosonization of the quantum superalgebra $U_q(\hat{sl}(N|1))$ for an arbitrary level $k$.
Theorem 3.1 [20] The Drinfeld currents \(x_i^\pm(z), \psi_i^\pm(z), (1 \leq i \leq N)\) of \(U_q(\widehat{sl}(N|1))\) for an arbitrary level \(k\) are realized by the bosonic operators as follows.

\[
x_i^+(z) = \frac{1}{(q-q^{-1})z} \sum_{j=1}^{i} \exp \left( (b+c)^{j-i}(q^{-1}z) + \sum_{l=1}^{j-1} (b_{i,l}^+(q^{-1}z) - b_{i,l}^-(q^{-1}z)) \right) \times (3.9)
\]

\[
x_i^+(z) = \prod_{j=i+1}^{N} \exp \left( (b+c)^{j-i-N}(q^{-1}z) + b_{i,N}^+(q^{-1}z) - b_{i,N}^+(q^{-1}z) \right) \times (3.10)
\]

\[
x_i^-(z) = q^{k+N-1} \prod_{j=1}^{i} \exp \left( a_{i}^-(q^{k+N-1}z) - b_{i,N}^-(q^{k+N-1}z) - b_{i,N}^+(q^{k+N-1}z) + b_{i,N}^+(q^{k+N-1}z) \right) \times (3.11)
\]

\[
x_i^- (z) = q^{k+N-1} : \exp \left( a_{i}^-(q^{k+N-1}z) - b_{i,N}^-(q^{k+N-1}z) - b_{i,N}^+(q^{k+N-1}z) + b_{i,N}^+(q^{k+N-1}z) \right) : (3.12)
\]
Let us set the boson $B^k(z)$, where $k > 0$.

\[
\psi^+(q^z) = \exp\left(\sum_{l=1}^{N} (b_{+}^{l,1}(q^{z(l+1)}_{+}) - b_{+}^{l-1,1}(q^{z(l-1)}_{+}))\right),
\]

\[
\psi^-(q^z) = \exp\left(\sum_{l=1}^{N} (b_{-}^{l,1}(q^{z(l+1)}_{-}) - b_{-}^{l-1,1}(q^{z(l-1)}_{-}))\right).
\]

### 3.3 Elliptic Superalgebra $U_{q,p}(\hat{sl}(N|1))$

In this section we give a bosonization of the elliptic superalgebra $U_{q,p}(\hat{sl}(N|1))$ for an arbitrary level $k$, using the dressing deformation \([20]\). Let us introduce the zero-mode operators $P_i, Q_i$, ($1 \leq i \leq N$) by

\[
[P_i, Q_j] = -\frac{A_{i,j}}{2} \quad (1 \leq i, j \leq N),
\]

where $(A_{i,j})_{1 \leq i, j \leq N}$ is the Cartan matrix of the classical $sl(N|1)$. In \([20]\) the bosonization of the Drinfeld generator $h_{i,m}$ ($1 \leq i \leq N, m \in \mathbb{Z}$) is given by

\[
h_{i,m} = q^{-\frac{k-1}{2}|m|}a^i_m + i\sum_{l=1}^{N} (q^{-(k+1)|m|}b_{l,m}^{i,1} - q^{-(k+1)|m|}b_{l,m}^{i,1}),
\]

\[
h_{N,m} = q^{-\frac{k-1}{2}|m|}a^N_m - \sum_{l=1}^{N-1} (q^{-(k+1)|m|}b_{l,m}^{N,1} + q^{-(k+1)|m|}b_{l,m}^{N,1}).
\]

Let us set the boson $B_{j,m}$ ($1 \leq j \leq N, m \in \mathbb{Z}_{\geq 0}$) by

\[
B_{j,m} = \begin{cases} 
\frac{|r_m|^j}{|r_m|^j}h_{j,m} & (m > 0), \\
q^{j|m|}h_{j,m} & (m < 0).
\end{cases}
\]

**Theorem 3.2** \([15, 20]\) The currents $E_j(z), F_j(z), H_j^{\pm}(z)$ ($1 \leq j \leq N$) of the elliptic superalgebra $U_{q,p}(\hat{sl}(N|1))$ for an arbitrary level $k$ are realized by the bosonic operators as follows.

\[
E_j(z) = U_j^+(z)x_j^+(z)e^{2Q_jz^{-\frac{k}{r}}P_j},
\]

\[
F_j(z) = x_j^-(z)U_j^-(z)z^{\frac{k}{r}}(P_j + h_j),
\]

\[
H_j^{\pm}(z) = H_j(q^{\frac{r}{2}}z^{\frac{k}{r}}),
\]

\[
H_j(z) = \exp\left(-\sum_{m \neq 0} \frac{B_{j,m}}{|r_m|^j}z^{-m}\right) :e^{2Q_jz^{\frac{k}{r}}P_j + \frac{k}{r}h_j}.
\]

Here we have used the dressing operators $U_j^+(z), U_j^-(z)$ ($1 \leq j \leq N$) given by

\[
U_j^+(z) = \exp\left(\sum_{m \geq 0} \frac{|r_m|^j}{|r_m|^j}B_{j,m}z^m\right), \quad U_j^-(z) = \exp\left(-\sum_{m \geq 0} \frac{|r_m|^j}{|r_m|^j}B_{j,m}z^{-m}\right).
\]
4 $q$-Wakimoto Realization

In this section we introduce the $q$-Wakimoto realization by the $\xi, \eta$ system. We introduce the vacuum state $|0\rangle$ of the boson Fock space by

$$a_m^i |0\rangle = b_m^{i,j} |0\rangle = c_m^{i,j} |0\rangle = 0 \quad (m \geq 0).$$

(4.1)

For complex numbers $p_i^b \in \mathbb{C} \ (1 \leq i \leq N)$, $p_i^{i,j} \in \mathbb{C} \ (1 \leq i < j \leq N + 1)$, $p_c^{i,j} \in \mathbb{C} \ (1 \leq i < j \leq N)$, we set

$$|p_a, p_b, p_c\rangle = \exp \left( \sum_{i,j=1}^{N} \text{Min}(i,j)(N-1-\text{Max}(i,j)) \right) \prod_{1 \leq i < j \leq N} p_c^{i,j} Q_{c}^{i,j} \prod_{1 \leq i < j \leq N+1} p_b^{i,j} Q_{b}^{i,j} |0\rangle.

(4.2)

It satisfies

$$a_0^b |p_a, p_b, p_c\rangle = p_b^{i,j} |p_a, p_b, p_c\rangle, \quad b_0^{i,j} |p_a, p_b, p_c\rangle = p_b^{i,j} |p_a, p_b, p_c\rangle, \quad c_0^{i,j} |p_a, p_b, p_c\rangle = p_c^{i,j} |p_a, p_b, p_c\rangle.

(4.3)

The boson Fock space $F(p_a, p_b, p_c)$ is generated by the bosons $a_m^i, b_m^{i,j}, c_m^{i,j}$ on the vector $|p_a, p_b, p_c\rangle$. We set $U_q(\hat{sl}(N|1))$-module $F(p_a)$ by

$$F(p_a) = \bigoplus_{p_b^{i,j} = -p_c^{i,j} \in \mathbb{Z} \ (1 \leq i < j \leq N)} F(p_a, p_b, p_c).

(4.4)

We have imposed the restriction $p_c^{i,j} = -p_b^{i,j} \in \mathbb{Z}$, because the $x_{i,m}^z$ change $Q_{b}^{i,j} + Q_{c}^{i,j}$. The module $F(p_a)$ is not irreducible representation. For instance, the irreducible highest weight module $V(\lambda)$ for $U_q(\hat{sl}(2))$ was constructed from the similar space as $F(p_a)$ by two steps; the first step is the construction of the $q$-Wakimoto realization by the $\xi, \eta$ system, and the second step is the resolution by the Felder complex [22].

In this paper we study the $\xi, \eta$ system and introduce the $q$-Wakimoto realization for $U_q(\hat{sl}(N|1))$.

For $1 \leq i < j \leq N$ we introduce

$$\eta^{i,j}(z) = \sum_{m \in \mathbb{Z}} \eta_m^{i,j} z^{-m-1} = e^{e^{i,j}(z)} ; \quad \xi^{i,j}(z) = \sum_{m \in \mathbb{Z}} \xi_m^{i,j} z^{-m} = e^{-e^{i,j}(z)} .

(4.5)

The Fourier components $\eta_m^{i,j} = \oint \frac{dz}{2\pi i} z^m \eta^{i,j}(z)$, $\xi_m^{i,j} = \oint \frac{dz}{2\pi i} z^{m-1} \xi^{i,j}(z) \ (m \in \mathbb{Z})$ are well defined on the space $F(p_a)$. They satisfy

$$\{ \eta_m^{i,j}, \xi_n^{i,j} \} = \delta_{m+n, 0}, \quad \{ \eta_m^{i,j}, \eta_n^{i,j} \} = \{ \xi_m^{i,j}, \xi_n^{i,j} \} = 0 \quad (1 \leq i < j \leq N),

(4.6)

$$\{ \eta_m^{i,j}, \xi_n^{i,j'} \} = [\eta_m^{i,j}, \eta_n^{i',j'}] = [\xi_m^{i,j}, \xi_n^{i',j'}] = 0 \quad (i,j) \neq (i', j').

(4.7)

We focus our attention on the operators $\eta_0^{i,j}, \xi_0^{i,j}$ satisfying $(\eta_0^{i,j})^2 = 0, (\xi_0^{i,j})^2 = 0$ and $\text{Im}(\eta_0^{i,j}) = \text{Ker}(\eta_0^{i,j}), \text{Im}(\xi_0^{i,j}) = \text{Ker}(\xi_0^{i,j})$. The products $\eta_0^{i,j} \xi_0^{i,j}$ and $\xi_0^{i,j} \eta_0^{i,j}$ are the projection operators

$$\eta_0^{i,j} \xi_0^{i,j} + \xi_0^{i,j} \eta_0^{i,j} = 1.

(4.8)
We have a direct sum decomposition.

\[
F(p_a) = \eta_0^{i,j} \xi_0^{i,j} F(p_a) \oplus \xi_0^{i,j} \eta_0^{i,j} F(p_a), \\
(4.9)
\]

\[
\text{Ker}(\eta_0^{i,j}) = \eta_0^{i,j} \xi_0^{i,j} F(p_a), \quad \text{Coker}(\eta_0^{i,j}) = \xi_0^{i,j} \eta_0^{i,j} F(p_a).
(4.10)
\]

**Definition 4.1** We introduce the subspace \( F(p_a) \) that we call the \( q \)-Wakimoto realization.

\[
F(p_a) = \left( \prod_{1 \leq i < j \leq N} \eta_0^{i,j} \xi_0^{i,j} \right) F(p_a) = \bigcap_{1 \leq i < j \leq N} \text{Ker}(\eta_0^{i,j}),
(4.11)
\]

The dressing operators \( U_i^\pm(z) \) and the zero-mode operators \( P_i, Q_i \) commute with \( \eta_0^{i',j'} \). The bosonizations commute with the operators \( \eta_0^{i',j'}, \xi_0^{i',j'} \) up to sign \( \pm \).

**Proposition 4.2** The subspace \( F(p_a) \) is both \( U_q(\hat{sl}(N|1)) \) and \( U_{q,p}(\hat{sl}(N|1)) \) module.

Let \( \alpha_i, \Lambda_i, \ (1 \leq i \leq N) \) and \((\cdot|\cdot)\) be the simple roots, the fundamental weights, and the symmetric bilinear norm : \((\alpha_i, \alpha_j) = A_{i,j}, \ (\alpha_i, \Lambda_j) = \delta_{i,j}\). It is expected that we have the irreducible highest weight module \( V(\lambda) \) with the highest weight \( \lambda \), whose classical part \( \tilde{\lambda} = \sum_{j=1}^{N} p_0^i \Lambda_i \), by the Felder complex of the \( q \)-Wakimoto realization. We would like to report this problem for \( U_q(\hat{sl}(N)) \) and \( U_q(\hat{sl}(N|1)) \) in the future publication.

**Acknowledgements**

This work is supported by the Grant-in-Aid for Scientific Research C (21540228) from Japan Society for Promotion of Science. The author would like to thank the organizing committee of the 9-th International Workshop "Lie Theory and its application in Physics" for an invitation to Bulgaria.

**References**

[1] P.Bouwknegt, J.McCarthy and K.Pilch, *Prog.Theor.Phys.* **102** (1990) 67-135.

[2] M.Jimbo and T.Miwa, *Algebraic Analysis of Solvable Lattice Models*, CBMS Regional Conference Series in Mathematics **85** (American Mathematical Society), 1994.

[3] I.B.Frenkel and N.Jing, *Proc.Natl.Acad.Sci.* **85** (1988) 9373-9377.

[4] D.Bernard, *Lett.Math.Phys.* **17** (1989) 239-245.

[5] N.Jing, Y.Koyama and K.Misra, *Selecta Math.* **5** no. 2 (1999) 243-255.

[6] N.Jing, *Invent.Math.* **102** (1990) 663-690.

[7] N.Jing, *Proc.Amer.Math.Soc.* **127** no.1 (1999) 21-27.

[8] K.Kimura, J.Shiroishi and J.Uchiyama, *Comm. Math. Phys.* **188** no. 2 (1997) 367-378.
[9] Y.-Z. Zhang, *J. Math. Phys.* **40** no.11 (1999) 6110-6124.

[10] W.-L. Yang and Y.-Z. Zhang, *Phys. Lett.* **A261** (1999) 252-258.

[11] M. Jimbo, H. Konno, S. Odake and J. Shiraishi, *Commun. Math. Phys.* **199** (1999) 605-647.

[12] T. Kojima and H. Konno, *J. Math. Phys.* **45** (2004) 3146-3179.

[13] M. Wakimoto, *Commun. Math. Phys.* **104** no.4 (1986) 605-609.

[14] A. Matsuo, *Commun. Math. Phys.* **160** (1994) 33-48.

[15] J. Shiraishi, *Phys. Lett.* **A171** (1992) 243-248.

[16] H. Awata, S. Odake and J. Shiraishi, *Commun. Math. Phys.* **162** no. 1 (1994) 61-83.

[17] H. Awata, S. Odake and J. Shiraishi, *Lett. Math. Phys.* **42** no. 3 (1997) 271-279.

[18] H. Yamane, *Publ. Res. Inst. Math. Sci.* **35** (1999) 321-390.

[19] T. Kojima, *J. Phys.* **A44**: Math. Theor. (2011) 485205(23pp).

[20] T. Kojima, *J. Math. Phys.* **53** (2012) 013515(1-15).

[21] D. Bernard and G. Felder, *Commun. Math. Phys.* **127** (1990) 145-168.

[22] H. Konno, *Mod. Phys. Lett.* **A9** (1994) 1253-1265.

[23] Y.-Z. Zhang and M. D. Gould, *J. Math. Phys.* **41** (2000) 5577-5291.