ON HOLOMORPHIC CURVES INTO COMPLEX PROJECTIVE VARIETIES

GIANG LE

Department of Mathematics, Hanoi National University of Education, 136-Xuan Thuy, Cau Giay, Hanoi, Vietnam.
E-mail: legiang@hnue.edu.vn, legiang01@yahoo.com

ABSTRACT. In this paper, we study holomorphic curves satisfying the Fubini-Study derivative \(\|f'(z)\| = O(|z|^\sigma)\) for some \(\sigma > -1\) from the viewpoint of Nevanlinna theory.

1. Introduction

In this paper, we study holomorphic curves \(f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})\) satisfying the Fubini-Study derivative \(\|f'(z)\| = O(|z|^\sigma)\) for some \(\sigma > -1\). The explicit expression is:

\[
\|f'\|^2 = \frac{\sum_{i\neq j} |f'_i f_j - f_i f'_j|^2}{\|f\|^4},
\]

where \(f = (f_0, \ldots, f_N)\) is a homogeneous representation of \(f\) (that is, the \(f_j\) are entire functions which never simultaneously vanish) and

\[
\|f\|^2 = \sum_{j=0}^N |f_j|^2.
\]

When \(\sigma = 0\), \(f\) is called a Brody curve. The interest of these curves comes from Brody’s lemma. It is stated that for every non constant holomorphic curve \(f\), there exists a sequence of affine map \(a_k : \mathbb{C} \to \mathbb{C}\) such that the limit \(\int f \circ a_k\) exists and is a non constant Brody curves. Much attention has been given to study Brody curves from the viewpoint of Nevanlinna theory [3, 1, 13, 10]. In [3, 1, 12], some versions of Second Main Theorem have been established for Brody curves intersecting hypersurfaces located in general position. The first aim of this paper is to generalize such the results to the case of holomorphic curves satisfying the Fubini-Study derivative \(\|f'(z)\| = O(|z|^\sigma)\) for some \(\sigma > -1\) and arbitrary hypersurfaces.

We recall that the Nevanlinna characteristic is defined as follows

\[
T_f(r) = \int_0^r \frac{dt}{t} \left( \frac{1}{\pi} \int_{|z| \leq t} \|f'\|^2(z) dm_z \right),
\]

2010 Mathematics Subject Classification: 32H30 (14J70 32H04 32H25).
Key words and phrases: Second main theorem over an angular domain, Julia directions, Brody curves.
where \( dm \) is the area element in \( \mathbb{C} \).

Obviously, the condition \( \|f'(z)\| = O(|z|^\sigma) \) for some \( \sigma > -1 \) implies that
\[
T_f(r) = O(r^{2\sigma+2}).
\]

Clunie and Hayman \[2\] found that if the spherical derivative \( \|f'(z)\| \) of an entire function satisfies \( \|f'(z)\| = O(|z|^\sigma) \) for some \( \sigma > -1 \), then \( T_f(r) = O(r^{\sigma+1}) \). Barret and Eremenko \[3\] generalized this to the case of holomorphic curves in projective space of dimension \( n \) omitting \( n \) hyperplanes in general position. The first main result of this paper is to show that this phenomenon persists in complex projective variety omitting hypersurfaces satisfying the intersection of them consisting of finite points.

In this paper, we always assume that \( V \) be a complex projective algebraic variety in \( \mathbb{P}^N(\mathbb{C}) \).

**Theorem 1.** Let \( D_1, \ldots, D_q \) be \( q \) hypersurfaces such that \( \dim(\bigcap_{1 \leq i \leq q} \text{Supp} \ D_i \bigcap V) = 0 \). Let \( f : \mathbb{C} \to V \) satisfying \( \|f'(z)\| = O(|z|^\sigma) \) for some \( \sigma > -1 \). Assume that \( (\bigcup_{1 \leq i \leq q} \text{Supp} \ D_i) \bigcap f(\mathbb{C}) = \emptyset \). Then,
\[
T_f(r) = O(r^{\sigma+1}).
\]

Combined with a result of Tsukamoto \[13\, \text{Theorem 1.9}\], Theorem 1 implies that

**Corollary 2.** Mean dimension in the sense of Gromov \[6\] of the space of Brody curves in \( V \setminus \dim V \) hypersurfaces in general position is zero.

In \[1\], Da Costa and Duval proved a second main theorem for Brody curves in projective space intersecting hyperplanes. Later, Thai and Mai \[12\] established a second main theorem over an angular domain for Brody curves into a complex projective algebraic variety intersecting hypersurfaces in general position. Our second purpose is to generalize Thai-Mai’s results \[12\] to the case of arbitrary hypersurfaces and holomorphic curves satisfying \( \|f'(z)\| = O(|z|^\sigma) \) for some \( \sigma > -1 \). To state our result, we recall some notations.

**Definition 3.** Let \( D_1, \ldots, D_q \) be hypersurfaces in \( \mathbb{P}^N(\mathbb{C}) \).

\[ \text{a)} \quad \text{They are said to be in } m-\text{subgeneral position in } V (\dim V \leq m \in \mathbb{N}) \text{ if for any subset } I \subset \{1, \ldots, q\} \text{ with } \sharp I \leq m + 1, \text{ we have} \]
\[
\dim \bigcap_{i \in I} \text{Supp} \ D_i \bigcap V \leq m - \sharp I.
\]

\[ \text{b)} \quad \text{They are said to be in general position in } V \text{ if they are in } \dim V-\text{subgeneral position.} \]

Let \( f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C}) \) be a holomorphic curve. We choose a homogeneous representation \( f = (f_0, \ldots, f_N) \) of our curve. Set
\[
u(z) = \log \sqrt{|f_0(z)|^2 + \ldots + |f_N(z)|^2}.
\]
Obviously, \( u \) is a positive subharmonic function. Let \( \triangle u \) be the Riesz measure of \( u \), that is the measure with the density 

\[
\triangle u = \frac{1}{\pi} \| f' \|^2.
\]

Let 

\[
\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}, \quad r\mathbb{D} = \{ z : |z| < r \}
\]

throughout the paper. Let \( \Omega \) be a domain in \( \mathbb{C} \). The Ahlfors-Shimizu characteristic function of \( f \) is defined as follows

\[
T_f(r, \Omega) = \int_1^r \frac{\triangle u(\Omega \cap t\mathbb{D})}{t} dt + O(1).
\]

It is well-known that 

\[
T_f(r, \mathbb{C}) = T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} u(re^{i\theta}) d\theta + O(1).
\]

Let \( D \) be a hypersurface in \( \mathbb{P}^N(\mathbb{C}) \) defining by a homogeneous polynomial \( Q \). For a domain \( \Omega \) denote by \( n_f(r, \Omega, D) \) the number of zeros of \( Q \circ f \) in the domain \( \{ |z| < r \} \cap \Omega \), counting multiplicity. The counting function is defined as follows:

\[
N_f(r, \Omega, D) = \int_0^r \frac{n_f(r, \Omega, D) - n_f(0, \Omega, D)}{t} dt + n_f(0, \Omega, D) \log r.
\]

In this paper, we consider angular domains \( \Omega(\theta, \epsilon) \), \( 0 \leq \theta \leq 2\pi, 0 < \epsilon < \pi \) defined as follows

\[
\Omega(\theta, \epsilon) = \{ z : \theta - \epsilon < \arg z < \theta + \epsilon \}.
\]

Sometimes, we write simply \( \Omega \) instead of \( \Omega(\theta, \epsilon) \) if there is no confusion.

**Theorem 4.** Let \( f : \mathbb{C} \to V \) be a holomorphic curve satisfying \( \| f'(z) \| = O(|z|^\sigma) \) for some \( \sigma > -1 \). Let \( D_1, \ldots, D_q \) be hypersurfaces of degrees \( d_1, \ldots, d_q \) in \( \mathbb{P}^N(\mathbb{C}) \). Let \( \Omega = \Omega(\theta, \epsilon) \) be any angular domain in \( \mathbb{C} \).

(i) Let \( \dim V \leq m \in \mathbb{N} \). If \( D_1, \ldots, D_q \) are located in \( m \)-subgeneral position in \( V \) then we have

\[
(q - m + 1)T_f(r, \Omega) \leq \sum_{1 \leq i \leq q} \frac{1}{d_i}N_f(r, \Omega, D_i) + o(r^{2\sigma + 2}).
\]

(ii) If \( V \cap \text{Supp} \ D_1 \cap \ldots \cap \text{Supp} \ D_q = \emptyset \) then we have

\[
2T_f(r, \Omega) \leq \sum_{1 \leq i \leq q} \frac{1}{d_i}N_f(r, \Omega, D_i) + o(r^{2\sigma + 2}).
\]

(iii) If \( \dim(V \cap \text{Supp} \ D_1 \cap \ldots \cap \text{Supp} \ D_q) = 0 \) then we have

\[
T_f(r, \Omega) \leq \sum_{1 \leq i \leq q} \frac{1}{d_i}N_f(r, \Omega, D_i) + o(r^{2\sigma + 2}).
\]
Remark that, we obtain the part (ii) from the part (i) by observing that $D_1, \ldots, D_q$ satisfying the condition $V \cap \text{Supp} D_1 \ldots \cap \text{Supp} D_q = \emptyset$ could be considered to be located in $(q-1)$-subgeneral position. Similarly, the part (iii) comes from the part (i) by observing that $D_1, \ldots, D_q$ satisfying the condition $\dim(V \cap \text{Supp} D_1 \cap \ldots \cap \text{Supp} D_q) = 0$ could be considered to be located in $q$-subgeneral position.

As a consequence of Theorem 4, for holomorphic curve satisfying $\limsup_{r \to \infty} \frac{T_f(r)}{r^{2\alpha+2}} > 0$, we have the defect relation

$$\sum_{1 \leq i \leq q} \delta_{D_i}(f) \leq m - 1$$

for a family of hypersurfaces $D_1, \ldots, D_q$ located in $m$-subgeneral position. For a general holomorphic curve, the best result available at present is due to Quang [11], where the defect relation is $(m - n + 1)(n + 1)$ for the case of hypersurfaces $D_1, \ldots, D_q$ located in $m$-subgeneral position. For the case of arbitrary family of hypersurfaces, we do not know any result on that subject.

The last aim of this paper is to study singular directions of holomorphic curves. The existence of Julia directions was proved by G. Julia [8] in 1920 for all entire function and by Milloux in 1924 and Valiron in 1938 for most of meromorphic functions. For general holomorphic curves from $C$ to $P^N(C)$, it has been studied by Eremenko [4], Zheng [16], ... In [4], Eremenko showed the existence of Julia directions for some special classes of holomorphic curves into projective space and gave a definition of general Julia directions for the “subvariety” case.

**Definition 5.** [4, p.4] A number $\theta \in [0; 2\pi)$ is called a Julia direction for a holomorphic curve $f : C \to V$ if for every system of divisors $D_1, \ldots, D_q$ such that any $n + 1$ of them has empty intersection, and for any $\pi > \epsilon > 0$ all but at most $2n$ of these divisors have infinitely many preimages in the angle $\{z : |\arg z - \theta| < \epsilon\}$.

Recently, using the ideas of Eremenko [4], Zheng [16], p.867], Thai and Mai [12] gave the notion on singular directions for a holomorphic curve into $P^N(C)$ intersecting hypersurfaces in general position.

**Definition 6.** [16] Def 3.1, p. 867] A ray $J(\theta) = \{z \in C : \arg z = \theta\}$ is a $T$-direction for a holomorphic curve $f : C \to P^N(C)$ if for any $\epsilon(0 < \epsilon < \pi)$, we have

$$\limsup_{r \to \infty} \frac{N_f(r, \Omega, H)}{T_f(r)} = 0$$

for at most $2N$ hyperplanes $H$ in general position in $P^N(C)$.

**Definition 7.** [12] Let $V$ be a complex projective variety in $P^N(C)$ of dimension $n \geq 1$. Let $f : C \to V$ be a holomorphic curve. A ray $J(\theta)$ is called a $\bar{T}$-direction for $f$ if for any $\epsilon(0 < \epsilon < \pi)$, we have

$$\limsup_{r \to \infty} \frac{N_f(r, \Omega, D)}{T_f(r)} = 0$$
for at most \( n - 1 \) hypersurfaces \( D \) located in general position in \( V \).

Obviously, \( \bar{T} \)-direction is a \( T \)-direction and they are Julia directions in the sense of Eremenko \([4]\). However, this definition depends on the dimension of variety which is different from Eremenko’s notion. We now extend Thai-Mai’s notion on singular directions to not depend on the dimension of variety. Only the intersection pattern is relevant.

**Definition 8.** Let \( V \) be a complex projective variety in \( \mathbb{P}^N(\mathbb{C}) \). Let \( f : \mathbb{C} \to V \) be a holomorphic curve. A ray \( J(\theta) \) is called a \( \bar{T} \)-direction for \( f \) if for any \( \epsilon (0 < \epsilon < \pi) \) and for any hypersurfaces \( D_1, \ldots, D_q \) in \( \mathbb{P}^N(\mathbb{C}) \) satisfying \( \bigcap_{1 \leq i \leq q} \text{Supp} \, D_i \cap V \) consisting of finite points, we have

\[
\max_{1 \leq i \leq q} \limsup_{r \to \infty} \frac{N_f(r, \Omega, D_i)}{T_f(r)} > 0
\]

Based on Theorem 4, we state the third main result of this paper.

**Theorem 9.** Let \( f : \mathbb{C} \to V \) be a holomorphic curve satisfying \( \|f'(z)\| = O(|z|^\sigma) \) for some \( \sigma > -1 \). If \( \limsup_{r \to \infty} \frac{T_f(r)}{r^{2\sigma + 2}} > 0 \) then \( f \) has at least one \( \bar{T} \)-direction.

We would like to remark that Theorem 4 and Theorem 9 imply Theorem \([12, \text{Theorem 1.7}]\) and Theorem \([12, \text{Theorem 1.8}]\), respectively.

## 2. Some Lemmas

Let \( u \) be a subharmonic function in a domain \( \Omega \subset \mathbb{C} \). Let \( \nabla u \) be the Riesz measure of \( u \). We recall the following lemma due to Duval and Da Costa \([1, \text{p.1599}]\) reformulated by Thai-Mai \([12, \text{Lemma 2.3}]\).

**Lemma 10.** \([12, \text{Lemma 2.3}]\) \( \text{(see also}[1, \text{p.1599}]\) \) Let \( q, m \) be positive integers. Let \( \nu_i, 1 \leq i \leq q \) and \( \nu \) be subharmonic functions in \( \Omega \). Assume that \( \nabla \nu \) is \( L^\infty \) on \( \Omega \) and \( \nu = \max_I \nu_i \) for any subset \( I \subset \{1, \ldots, q\}, |I| = m \). Then \( \sum_{1 \leq i \leq q} \nu_i - (q - m + 1)\nu \) is subharmonic in \( \Omega \).

Let \( f : \mathbb{C} \to V \) be a holomorphic curve. We choose a homogeneous representation \( f = (f_0, \ldots, f_N) \) of our curve.

Let \( D_1, \ldots, D_q \) be hypersurfaces of degrees \( d_1, \ldots, d_q \) in \( \mathbb{P}^N(\mathbb{C}) \), defining by homogeneous polynomials \( Q_1, \ldots, Q_q \in \mathbb{C}[X_0, \ldots, X_N] \), located in \( m \)-subgeneral position in \( V \). Replacing \( Q_i \) by \( Q_i^{d/d_i} \) if necessary, where \( d \) is the l.c.m of \( d_i, 1 \leq i \leq q \), without loss of generality, we can assume that \( Q_i, 1 \leq i \leq q \) have the same degree \( d \). Define

\[
u = \log \|f\| = \log \sqrt{|f_0|^2 + \ldots + |f_N|^2}, \quad u_i = \frac{1}{d} \log |Q_i(f_0, \ldots, f_N)|, \quad 1 \leq i \leq q.
\]

We will prove the following.
**Lemma 11.** Let $f : \mathbb{C} \to V$ be a holomorphic curve satisfying $\|f'(z)\| = O(|z|^\sigma)$ for some $\sigma > -1$. Let $r_k$ be a sequence of positive real numbers which converges to $+\infty$. Let $h_k$ be the smallest harmonic majorant of $u$ in the disk $2r_k \mathbb{D}$ and \( \lambda_k : \mathbb{C} \to \mathbb{C} \) be the map given by $\lambda_k(z) = r_k z$. Then, there exist subsequences of the sequences $\frac{1}{r_k^{\sigma+2}}(u - h_k) \circ \lambda_k$ and $\frac{1}{r_k^{\sigma+2}}(u - h_k) \circ \lambda_k$ which converge to subharmonic functions $\nu_i$ and $\nu$, respectively, and $\nu = \max_i \nu_i$ for any subset $I \subset \{1, \ldots, q\}$ such that $|I| = m$.

**Proof.** For simplicity, we can assume that $I = \{1, \ldots, m\}$. If $q \geq m + 1$, from the fact that since $D_i, 1 \leq i \leq q$ are located in $m$–subgeneral position, we have $Q_i, 1 \leq i \leq m + 1$ have no common zeros in $V$. In the case of $q = m$, we can choose a hypersurface $D_{m+1}$ of $\mathbb{P}^N(\mathbb{C})$ defining by a homogeneous polynomial $Q_{m+1}$ such that $Q_i, 1 \leq i \leq m + 1$ have no common zeros in $V$. Therefore, there exist two positive constants $C_1, C_2$ such that

$$C_1 \leq \max_{1 \leq i \leq m+1} |Q_i(\omega)|^{1/d} \leq C_2, \forall \omega \in \pi^{-1}(V) \bigcap \{\|z\| \leq 1\},$$

where $\pi : \mathbb{C}^{N+1}\backslash\{0\} \to \mathbb{P}^N(\mathbb{C})$ be the standard projection. Hence,

$$C_1 \|f(z)\| \leq \max_{1 \leq i \leq m+1} \|Q_i \circ f(z)\|^{1/d} \leq C_2 \|f(z)\|, \forall z \in \mathbb{C}.$$

It follows that

$$u = \max_{1 \leq i \leq m+1} u_i + O(1)$$

Consider the sequence of non-positive subharmonic functions on $2 \mathbb{D}$ defined by $\frac{1}{r_k^{2\sigma+2}}(u - h_k) \circ \lambda_k$. Note that $h_k(0) = \frac{1}{2\pi} \int_0^{2\pi} u(2r_k e^{i\theta}) d\theta \leq O(r_k^{2\sigma+2})$ because $T_f(r) = O(r_k^{2\sigma+2})$. Hence $\liminf_{k \to \infty} \frac{1}{r_k^{2\sigma+2}}(u(0) - h_k(0)) > -\infty$. So the sequence $\frac{1}{r_k^{2\sigma+2}}(u - h_k) \circ \lambda_k$ does not converge locally uniformly to $-\infty$. By [7, Theorem 4.1.9], there exists a subsequence of the sequence $\frac{1}{r_k^{2\sigma+2}}(u - h_k) \circ \lambda_k$ (still denoted by $k$) such that

$$\frac{1}{r_k^{2\sigma+2}}(u - h_k) \circ \lambda_k \rightarrow \nu,$$

where $\nu$ is a subharmonic function in $2 \mathbb{D}$.

For each $i$, from (2.2), we have $u_i - h_k \leq u - h_k + O(1) < O(1)$. Therefore, $\frac{1}{r_k^{2\sigma+2}}(u_i - h_k) \circ \lambda_k$ is bounded above. Since $Q_i \circ f \neq 0$, there exists $y \in 2\mathbb{D}$ such that $u_i \circ \lambda_k(y) \neq -\infty$. For sufficiently large $k$, $u(2r_k e^{i\theta}) > 0$ for all $\theta$. By Poisson’s formula, we have

$$h_k \circ \lambda_k(y) = \frac{1}{2\pi} \int_0^{2\pi} \frac{4 - |y|^2}{|2e^{i\theta} - y|^2} u \circ \lambda_k(2e^{i\theta}) d\theta$$

$$\leq \frac{2 + |y|}{2\pi(2 - |y|)} \int_0^{2\pi} u(2r_k e^{i\theta}) d\theta \leq O(r_k^{2\sigma+2}).$$
Therefore, \( \liminf_{k \to \infty} \frac{1}{r_k^{2\sigma+2}} (u_i - h_k) \circ \lambda_k(y) > -\infty \). Hence, there exists a subsequence of the sequence \( \frac{1}{r_k^{2\sigma+2}} (u_i - h_k) \circ \lambda_k \) (still denoted by \( k \)) such that

\[
\frac{1}{r_k^{2\sigma+2}} (u_i - h_k) \circ \lambda_k \to \nu_i,
\]

where \( \nu_i \) is a subharmonic function in \( 2\mathbb{D} \). From (2.2), we have

\[
(2.3) \quad \nu = \max_{1 \leq i \leq m+1} \nu_i.
\]

We now prove that

\[
\nu \leq \max_{1 \leq i \leq m} \nu_i.
\]

Indeed, let \( Q : V \to \mathbb{P}^m(\mathbb{C}) \) be a morphism defined by

\[ z \mapsto Q(z) = (Q_{m+1}(z), Q_1(z), \ldots, Q_m(z)) \].

Since \( D_i, 1 \leq i \leq q \) are located in \( m \)-subgeneral position, \( Q_i, 1 \leq i \leq m + 1 \) have no common zeros in \( V \). Hence, \( Q \) is a holomorphic mapping on \( V \). Combining with the fact that \( V \) is a compact complex projective variety and \( \| f'(z) \| = O(|z|^\sigma) \), we have

\[ g = (g_0, \ldots, g_m) := Q \circ f : \mathbb{C} \to \mathbb{P}^m(\mathbb{C}) \]

satisfying \( \| g'(z) \| = O(|z|^\sigma) \). By composing \( g \) with an automorphism of \( \mathbb{P}^m(\mathbb{C}) \), for example replace \( g_0 \) by \( g_0 + cg_1, c \in \mathbb{C} \) and leave all other \( g_i \) unchanged, we can assume that \( g_0 \) has infinitely many zeros. Then, we can apply [3, Proposition 2] due to Barret and Eremenko [3] to \( g \). Then, for every \( \epsilon > 0 \), we have for \( z \) sufficiently large,

\[
(2.4) \quad \log \sqrt{|g_0|^2 + \ldots + |g_m|^2}(z) \leq \max_{1 \leq i \leq m} \log |g_i|(z) + K(2 + \epsilon)^{\sigma+1}(m + 1)|z|^\sigma + 1.
\]

where \( K \) is a constant depending only on \( f \) and \( Q \). Hence,

\[
(2.5) \quad \max_{1 \leq i \leq m+1} \log |Q_i \circ f(z)| \leq d \max_{1 \leq i \leq m} u_i(z) + K(2 + \epsilon)^{\sigma+1}(m + 1)|z|^\sigma + 1.
\]

From (2.5) and (2.1), we have for \( z \) sufficiently large

\[
(2.6) \quad u(z) \leq \max_{1 \leq i \leq m} u_i(z) + K(2 + \epsilon)^{\sigma+1}(m + 1)|z|^\sigma + 1.
\]

From (2.6), we have

\[
(2.7) \quad \nu \leq \max_{1 \leq i \leq m} \nu_i.
\]

From (2.3) and (2.7), we have \( \nu = \max_{1 \leq i \leq m} \nu_i \). \( \square \)

**Lemma 12.** \( \sum_{1 \leq i \leq q} \nu_i - (q - m + 1)\nu \) is subharmonic in \( 2\mathbb{D} \).

**Proof.** In view of Lemma 11 and Lemma 10, it suffices to prove that the Riesz measure \( \triangle \nu \) is \( L^\infty \) on \( 2\mathbb{D} \).
Indeed, since \( \frac{1}{r^2\sigma+2}(u - h_k) \circ \lambda_k \to \nu \), we have \( \frac{1}{r_k^2\sigma+2} \int \phi \Delta(u \circ \lambda_k) \to \int \phi \Delta \nu \) for any test function \( \phi \in C^\infty_0(2\mathbb{D}) \).

We denote by \( B(z, r) \) the open disc of radius \( r \) centered at \( z \). Hence, for any \( B(z, \delta) \subset 2\mathbb{D} \), we have

\[
\Delta \nu(B(z, \delta)) \leq \liminf_{r \to \infty} \frac{\Delta u(B(rz, r\delta))}{r^{2\sigma+2}} \leq \liminf_{r \to \infty} \frac{\delta^{2\sigma+2} r^{2\sigma+2} O(1)}{r^{2\sigma+2}} = O(1)\delta^{2\sigma+2},
\]

where \( O(1) \) is a constant depending only on \( f \). Hence \( \Delta \nu \) is \( L^\infty \) on \( 2\mathbb{D} \).

\[\square\]

3. Proof of main theorems

In this section, we provide the proofs of main theorems. More precisely, Theorem 1 follows from [3] and inequality 2.2. Theorem 4 is obtained from Lemma 12 and Theorem 9 is a consequence of Theorem 4. The details are given as follows.

**Proof of Theorem 4.**

We first prove that

\[
\limsup_{r \to \infty} \frac{1}{r^{2\sigma+2}} \left[ (q - m + 1) \Delta u(D_r) - \sum_{1 \leq i \leq q} \Delta u_i(D_r) \right] \leq 0,
\]

where \( D_r = \Omega \cap \{|z| < r\} \). Indeed, assume that \( r_k \to +\infty \) such that

\[
\frac{1}{r_k^{2\sigma+2}} \left[ (q - m + 1) \Delta u(D_{r_k}) - \sum_{1 \leq i \leq q} \Delta u_i(D_{r_k}) \right]
\]

converges.

Applying Lemma 12 we have \( \sum_{1 \leq i \leq q} \nu_i - (q - m + 1)\nu \) is subharmonic in \( 2\mathbb{D} \cap \Omega \). Fix \( \delta > 0 \) with \( 0 < \delta < \min\{1/2, \epsilon\} \). We choose a nonnegative smooth function \( \chi \) in \( \mathbb{D} \cap \Omega \) such that \( 0 \leq \chi \leq 1 \) and \( \chi = 1 \) in \( \mathbb{D}^\delta = (1 - \delta)\mathbb{D} \cap \Omega \cap \{\theta, \epsilon - \delta\} \cap \{|z| > \delta\} \) and \( \text{Supp} \chi \) is a compact subset in \( \mathbb{D} \cap \Omega \). Since \( \sum_{1 \leq i \leq q} \nu_i - (q - m + 1)\nu \) is subharmonic, we have

\[
(q - m + 1) \int \chi \Delta \nu \leq \sum_{1 \leq i \leq q} \int \chi \Delta \nu_i,
\]

where the integration is taken over \( \mathbb{D} \cap \Omega \). It follows that for \( k \) sufficient large, we get

\[
(q - m + 1) \frac{1}{r_k^{2\sigma+2}} \int \chi \Delta(u \circ \lambda_k) \leq \sum_{1 \leq i \leq q} \frac{1}{r_k^{2\sigma+2}} \int \chi \Delta(u_i \circ \lambda_k) + \delta.
\]

Therefore,

\[
(q - m + 1) \Delta u(r_k \mathbb{D}^\delta) \leq \frac{1}{d} \sum_{1 \leq i \leq q} \Delta u_i(r_k \mathbb{D} \cap \Omega) + \delta r_k^{2\sigma+2},
\]

(3.2)
Proof of Theorem 1. Without loss of generality, we can assume that

\[ r \]

points. Therefore, \( (3.1) \) holds. From \( (3.1) \), we have

\[ \theta \]

points. Hence, we can choose finite family of angles \( \Omega(\theta, \epsilon) \) for hypersurfaces \( (3.4) \) \( (3.5) \). From \( (3.4) \) and \( (3.7) \) and the fact that \( \lim \sup \frac{N_f(r, \Omega(\theta, \epsilon), D_i)}{T_f(r)} = 0 \), we have

\[ \frac{1}{r^{2\sigma+2}} \left[ (q - m + 1) \Delta u(r \mathbb{D}) - \sum_{1 \leq i \leq q} \Delta u_i(r \mathbb{D}) \right] \leq 0. \]

Hence, \( (3.1) \) holds. From \( (3.1) \), we have

\[ (q - m + 1) \Delta u(r \mathbb{D}) \leq \sum_{1 \leq i \leq q} \Delta u_i(r \mathbb{D}) + o(r^{2\sigma+2}) \]

\[ = n_f(r, r \mathbb{D}) \cap \Omega(D_i) + o(r^{2\sigma+2}), \]

where \( n_f(r, r \mathbb{D}) \cap \Omega(D_i) \) is the number of zeros of \( Q \circ f \) in the domain \( r \mathbb{D} \cap \Omega \), counting multiplicity. Dividing both sides of above inequality to \( r \) and integrating it, we get the conclusion.

Proof of Theorem 2. Suppose that \( f \) has no \( \overline{T} \)-direction. Then, for each ray \( J(\theta) \), we have an angle containing it such that

\[ \lim \sup_{r \to \infty} \frac{N_f(r, \Omega(\theta, \epsilon), D_i)}{T_f(r)} = 0 \]

for hypersurfaces \( D_1, \ldots, D_q \) in \( \mathbb{P}^N(\mathbb{C}) \) satisfying \( \bigcap_{1 \leq i \leq q} \text{Supp} \ D_i \cap V \) consisting of finite points. Hence, we can choose finite family of angles \( \Omega(\theta_i, \epsilon_i)(1 \leq i \leq p) \) such that

\[ C \subset \bigcup_{1 \leq i \leq p} \Omega(\theta_i, \epsilon_i) \]

and

\[ \lim \sup_{r \to \infty} \frac{N_f(r, \Omega(\theta_i, \epsilon_i), D_i)}{T_f(r)} = 0. \]

for hypersurfaces \( D_1, \ldots, D_q \) in \( \mathbb{P}^N(\mathbb{C}) \) satisfying \( \bigcap_{1 \leq i \leq q} \text{Supp} \ D_i \cap V \) consisting of finite points. Therefore,

\[ T_f(r) \leq \sum_{1 \leq i \leq p} T_f(r, \Omega(\theta_i, \epsilon_i)). \]

Applying Theorem 4 to \( D_i, 1 \leq i \leq q \), we have

\[ \sum_{1 \leq k \leq p} T_f(r, \Omega(\theta_k, \epsilon_k)) \leq \sum_{1 \leq k \leq p} \sum_{1 \leq i \leq q} \frac{1}{d_i} N_f(r, \Omega(\theta_k, \epsilon_k), D_i) + o(r^{2\sigma+2}). \]

From \( (3.5) \) and \( (3.6) \), we have

\[ T_f(r) \leq \sum_{1 \leq k \leq p} \sum_{1 \leq i \leq q} \frac{1}{d_i} N_f(r, \Omega(\theta_k, \epsilon_k), D_i) + o(r^{2\sigma+2}). \]

From \( (3.4) \) and \( (3.7) \) and the fact that \( \lim \sup_{r \to \infty} \frac{T_f(r)}{r^{2\sigma+2}} > 0 \), we get a contradiction.

Proof of Theorem 3. Without loss of generality, we can assume that \( D_1, \ldots, D_q \) have
the same degree $d$. Let $Q_1, \ldots, Q_q$ be the homogeneous polynomials of degree $d$ defining $D_1, \ldots, D_q$. Since $\bigcap_{1 \leq i \leq q} \{ Q_i = 0 \} \cap V$ consists of finite points, there exists a hypersurface $D_0$ of $\mathbb{P}^N(\mathbb{C})$ defining by a homogeneous polynomial $Q_0 \in \mathbb{C}[X_0, \ldots, X_N]$ such that $\bigcap_{0 \leq i \leq q} \{ Q_i = 0 \} \cap V = \emptyset$. Consider the map

$$Q = (Q_0, \ldots, Q_q) : V \to \mathbb{P}^q(\mathbb{C}), z \mapsto (Q_0(z), \ldots, Q_q(z)).$$

This map is a holomorphic map between two compact complex varieties. Together with the fact that $\| f'(z) \| = O(|z|^\sigma)$, we have $Q \circ f : \mathbb{C} \to \mathbb{P}^q(\mathbb{C})$ satisfies $\| Q' \circ f(z) \| = O(|z|^\sigma)$. Since $\bigcap_{1 \leq i \leq q} \{ Q_i = 0 \} \cap f(\mathbb{C}) = \emptyset$, $Q \circ f$ omits $q$ hyperplanes in general position. Hence, the main theorem in [3] is applicable to $Q \circ f$. Then, we have $T_{Q \circ f}(r) = O(r^{\sigma+1})$. Using the same arguments as in Lemma 11 to obtain inequality (2.2), we have $T_{Q \circ f}(r) = dT_f(r)$. This completes the proof of Theorem 1.

4. Acknowledgements

This work was supported by a NAFOSTED, grant of Vietnam (Grant No. 101.02-2021.16).

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