Modified scaling in \( k \)-essence model in interacting dark energy -
dark matter scenario

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November 24, 2022

Abstract

It has been shown by Scherrer and Putter et.al. in [1, 2] that, when dynamics of dark energy is driven by a homogeneous \( k \)-essence scalar field \( \phi \), with a Lagrangian of the form 
\[
L = V_0 F(X) \text{ with a constant potential } V_0 \text{ and } X = \frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi = \frac{1}{2} \dot{\phi}^2, \]
one obtains a scaling relation
\[
X \left( \frac{dF}{dX} \right)^2 = Ca^{-6}, \]
where \( C \) is a constant and \( a \) is the FRW scale factor of the universe. The separate energy conservation in the dark energy sector and the constancy of \( k \)-essence potential are instrumental in obtaining such a scaling. In this paper we have shown that, even when considering time-dependent interactions between dark energy and dark matter, the constancy of \( k \)-essence potential may lead to a modified form of scaling. We have obtained such a scaling relation for a particular class of parametrization of the source term occurring in the continuity equation of dark energy and dark matter in the interacting scenario. We used inputs from the JLA analysis of luminosity distance and redshift data from Supernova Ia observations, to obtain the modified form of the scaling.

1 Introduction

Several cosmological observations and surveys revealed that our universe is presently undergoing an accelerated phase of expansion and a transition from decelerated phase to this accelerated phase happened during the late time phase of cosmic evolution. The luminosity distance and redshift measurements of type Ia Supernova [3, 4, 5] are instrumental in establishing this fact. Independent observations like Baryon Acoustic Oscillation [6, 7], Cosmic Microwave Background Radiation [8] and Observed Hubble Data (OHD) [9] also reinforce this conclusion. The cause of acceleration of the present-day cosmic expansion still remains a mystery and has been presented in literature [3, 4, 5] as an effect due to presence of an unclustered form of energy with negative pressure - dubbed ‘Dark Energy’ (DE). On the other hand, observed rotation curves of spiral galaxies [10], observation of gravitational lensing [11], bullet and other colliding clusters provide evidences for existence of non-luminous matter in the universe, called ‘Dark Matter’ (DM), which reveals its presence only through gravitational effects detected in above mentioned observations. Results from satellite borne experiments like WMAP [12] and Planck [13] have

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arXiv:2207.00888v3 [gr-qc] 23 Nov 2022
estimated that DE and DM together contribute \( \sim 96\% \) of total energy density of the present-day universe, with \( \sim 69\% \) and \( \sim 27\% \) as their respective shares. The rest \( 4\% \) contribution comes from radiation and baryonic matter. Though a physical theory of dark energy is still lacking, there exist diverse theoretical approaches aiming construction of models for DE leading to present-day cosmic acceleration. These include the \( \Lambda \)-CDM model, where \( \Lambda \) refers to cosmological constant and ‘CDM’ corresponds to cold (non-relativistic) dark matter. This model, though fits well with the present-day cosmological observations, is associated with the problems of cosmological coincidence [14] and fine tuning [15] which motivate construction of alternative DE models. One approach of constructing such models, called modified gravity models [16], involves modification of Einstein tensor in the geometric part of the Einstein field equations. Another class of models treats DE to be driven by (scalar) fields with suitably chosen field theoretic Lagrangians contributing to energy-momentum tensor in Einstein equations. The second kind of models, as widely discussed in literature, includes the Quintessence [17, 18, 19, 20, 21, 22, 23, 24, 25] and \( k \)-essence models [26, 27, 28, 29, 30, 31, 32, 33]. A sub-class of such models involves consideration of interaction between DE and DM [34, 35, 36, 37, 38, 39, 40, 41, 42, 43] to explain the present-day observed features of cosmic expansion. We have considered such a model in the context of this paper.

To investigate the interacting scenario of DE and DM, we neglect the radiation and baryonic matter contribution to the total energy density of universe during its late time phase of cosmic evolution due to their small share (\( \sim 4\% \)) in the present-day energy-content as estimated from WMAP [12] and Planck [13] observations. In this paper, we consider DE to be represented by a homogeneous scalar field \( \phi = \phi(t) \) whose dynamics is driven by a purely kinetic (\( k \)-essence) Lagrangian with a constant potential. Purely kinetic scalar field models have been widely discussed in [1, 2, 27, 28, 31, 41, 43, 44, 45, 46, 47] and references therein. The DM component of the universe, on the other hand, is chosen to be a non-relativistic pressure-less fluid (dust). We consider the DE-DM interaction to be time-dependent and introduce it through a source term \( Q(t) \) in the non-conserving continuity equations for DM and DE (see Eqs. (17) and (18) in Sec. 4). The function \( Q(t) \) provides a measure of instantaneous rate of energy transfer between DE and DM components of the universe.

We consider spacetime geometry of the expanding universe, at large scales, to be isotropic and homogeneous and flat which is described by the Friedmann - Robertson - Walker (FRW) metric characterised by a time dependent scale factor \( a(t) \) and zero curvature constant. The evolution of such a universe with its content modelled as an ideal fluid (characterised by its energy density \( \rho(t) \) and pressure \( p(t) \)) is governed by the Friedmann equations which connect \( a(t) \) and its derivatives with \( \rho(t) \) and \( p(t) \). Using the luminosity distance vs. redshift measurements in Supernova Ia (SNe Ia) observations, the temporal profile of the scale factor, the Hubble parameter \( H = \dot{a}/a \) may be extracted. From this knowledge, the time profile of total energy density and pressure of the universe may as well be computed, exploiting the Friedmann equations.

When we consider DM and DE to be non-interacting, both the components of total dark fluid separately satisfy respective continuity equations (Eqs. (17) and (18) with \( Q(t) = 0 \)) implying separate energy conservation in each individual sector. In this non-interacting DE-DM scenario, if the dynamics of DE is considered to be driven by a scalar field \( \phi \) governed by a \( k \)-essence Lagrangian of form \( L = V_0 F(X) \) with a constant potential \( V_0 \) and \( X = \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi = \frac{1}{2} \phi^2 \), one obtains a scaling of the form: \( X(dF/dX)^2 = Ca^{-6} \) (\( C \) is a constant) [11, 2]. Separate energy conservation in the DE sector and the constancy of \( k \)-essence potential are instrumental in obtaining such a scaling. This scaling relation connects the scale factor \( a \) and \( k \)-essence scalar field \( \phi \) through its time derivatives appearing in \( X \) and \( dF/dX \).
In this paper we have shown that, even in presence of time-dependent interactions between 
DE and DM ($Q(t) \neq 0$) implying continual exchange of energy between the two sectors, the 
constancy of $k$–essence potential may lead to a modified form of scaling. We have obtained 
such a scaling, by parametrising the dependence of source term $Q(t)$ on $a(t)$ as a power law: 
$Q(t) = Q_0[a(t)]^k$, where $k$ is a constant and $Q_0$ is the value of $Q(t)$ at present epoch ($a(t)$ is normalised to 1 at present epoch). In obtaining the form of scaling relation with above 
parametrisation of DE-DM interactions, we have taken into consideration the observed feature 
of temporal behaviour of the FRW scale factor $a(t)$, probed in the measurement of luminosity 
distance and redshift of SNe Ia events. The observational ingredient enters into the explicit form 
of scaling relation at different levels of its derivation, through various constants which encode 
in them features of the observational data. The obtained scaling depends on $k$ and $Q_0$, which 
are the two parameters of the model. For computational convenience, in stead of $Q_0$, we used a 
dimensionless parameter $\beta_0 \equiv Q_0/\rho_{\text{de}} + \rho_{\text{dm}}$ all through the work. We also find the region of the 
corresponding $k – \beta_0$ parameter space that is allowed from SNe Ia data, in the context of 
interacting DE-DM model considered here.

The paper is organised as follows. In Sec. 2, we briefly discussed Joint Light curve Analysis 
(JLA) of Supernova Ia data using which we obtain the temporal behaviour of scale factor and 
its derivatives. In Sec. 3 we gave a brief outline of the framework of $k$-essence model of dark 
energy and discussed its aspects for a constant potential scenario, which is relevant in the context 
of the paper. In Sec. 4 we presented the theoretical framework of interacting DE-DM scenario 
in FRW universe. We obtained temporal behaviour of equation of state parameter ($\omega = p/\rho$) of 
the total dark fluid and the total energy density $\rho$ of the universe using obtained time profile of 
the scale factor and its derivatives obtained in Sec. 2. In the context of this interacting DE-DM 
scenario, temporal behaviour of energy density of individual DE and DM components ($\rho_{\text{de}}$ and 
$\rho_{\text{dm}}$) have also been obtained in terms of model-parameters $k$ and $\beta_0$. In Sec. 5 we have shown 
how we derived the corresponding modified scaling relation for $k$–essence model of DE with a 
constant potential. The sensitivity of the modified scaling relation on the parameters $k$ and $\beta_0$ 
are also graphically represented for different chosen benchmark values of the parameters within 
their allowed region. We summarised the results in the concluding Sec. 6.

2 Cosmological parameters from analysis of SNe Ia data

As discussed in Sec. 1 the luminosity distance and redshift measurements of type Ia Supernova is 
the key observational ingredient in establishing the transition from decelerated to an accelerated 
phase of cosmic expansion during its late time phase of evolution. We have used the observational 
data as input, in estimating the behaviour of modified scaling, on the parameters $k$, $\beta_0$ which 
parametrises the time dependent interaction $Q(t)$ between DE and DM. In this section we 
describe how we extract the relevant cosmological parameters from the SNe Ia data to use 
them as direct inputs into this estimation. There exist different compilations of the SNe Ia 
data corresponding to supernova surveys in different redshift region using diverse probes and 
measurements. Small redshift ($z > 0.1$) projects comprise Harvard-Smithsonian Center for 
Astrophysics survey (cFa) [48], the Carnegie Supernova Project (CSP) [49, 50, 51], the Lick 
Observatory Supernova Search (LOSS) [52] and the Nearby Supernova Factory (SNF) [53]. 
SDSS-II supernova surveys [54, 55, 56, 57, 58] are mainly focused on the redshift region of 
$(0.05 < z < 0.4)$. Programmes like Supernova Legacy Survey (SNLS) [59, 60] the ESSENCE 
project [61], the Pan-STARRS survey [62, 63, 64] correspond to high redshift regime. More 
than one thousand SNe Ia events have been discovered through all surveys. However, in the 
range between $z \sim 0.01$ and $z \sim 0.7$, luminosity distances have been measured with a very high
statistical precision. ‘Joint Light-curve Analysis (JLA) data’ ([63], [65], [66]) contains a total of 740 SNe Ia events from full three years of SDSS survey, first three seasons of the five-year SNLS survey and 14 very high redshift \( 0.7 < z < 1.4 \) SNe Ia from space-based observations with the HST [67]. For the present work, we consider this data set for analysis where the different systematic uncertainties are taken care of by compiling the data with flux-averaging technique whose technical details have been comprehensively described in [68, 69, 70]. We briefly outline the methodology of analysis here. The \( \chi^2 \) function corresponding to JLA data is defined as

\[
\chi^2 = \sum_{i,j=1}^{740} (\mu^{(i)}_{\text{obs}} - \mu^{(i)}_{\text{th}})(\sigma^{-1})_{ij}(\mu^{(j)}_{\text{obs}} - \mu^{(j)}_{\text{th}}),
\]

where \( \mu^{(i)}_{\text{obs}} \) and \( \mu^{(i)}_{\text{th}} \) respectively denote the observed value and theoretical expression for distance modulus at red-shift \( z_i \). In a flat FRW spacetime, \( \mu^{(i)}_{\text{th}} \) is given by

\[
\mu^{(i)}_{\text{th}} = 5 \log_{10}[d_L(z_{\text{hel}}, z_{\text{CMB}})/\text{Mpc}] + 25
\]

where \( d_L \) is the luminosity distance it’s given by

\[
d_L(z_{\text{hel}}, z_{\text{CMB}}) = (1 + z_{\text{hel}})r(z_{\text{CMB}}) \quad \text{with} \quad r(z) = cH_0^{-1} \int_0^z \frac{dz'}{E(z')} ,
\]

and \( z_{\text{CMB}} \) and \( z_{\text{hel}} \) respectively refer to CMB rest frame and heliocentric frame value of SNe Ia redshifts. The value of Hubble parameter at present epoch is denoted by \( H_0 \). The observed distance modulus \( \mu^{(i)}_{\text{obs}} \) is expressed in terms of the observed peak magnitude \( m_B^\star \), the time stretching parameter of the light-curve \( X_1 \) and supernova color at maximum brightness, \( C \) as

\[
\mu^{(i)}_{\text{obs}} = m_B^\star(z_i) - M_B + \alpha X_1(z_i) - \beta C(z_i)
\]

\( \alpha, \beta \) being the nuisance parameters and the absolute magnitude \( M_B \) is kept fixed at the value -19. \( \sigma_{ij} \) in Eq. [1] is the covariant matrix as given in Eq. (2.16) of [69]. Wang in [69] proposed a flux averaging technique to reduce the effect of systematic uncertainties involved in the covariant matrix owing to weak lensing of SNe Ia data. We take the result of redshift\((z)\)-dependence of the function \( E(z) = H(z)/H_0 \) (corresponding to a zero red-shift cut-off \( z = 0 \)) obtained in [69] from the \( \chi^2 \)-marginalisation with respect to \( (M_B, \alpha, \beta) \). The 1\( \sigma \) range of \( E(z) \) resulting from the analysis is shown in Fig. [1]. The mean of \( E(z) \) values for every \( z \) in this 1\( \sigma \) range is shown by the dashed line in Fig. [1]. The temporal behaviour of relevant cosmological quantities are obtained using this mean \( E(z) \) vs \( z \) curve as benchmark.
Figure 1: Plots of $E$ vs $z$ (Dashed line correspond to the average value of $E(z)$ over the $1\sigma$ range of $E(z)$) depicted by shaded region.

The scale factor $a$, which we have taken to be normalised to $a = 1$ at present epoch, is related to redshift $z$ by the relation $1/a = 1 + z$. Using $H = \dot{a}/a$ we can write

$$dt = -\frac{dz}{(1+z)H(z)} = -\frac{dz}{(1+z)H_0E(z)}$$

which on integration gives

$$\frac{t(z)}{t_0} = 1 - \frac{1}{H_0t_0} \int_z^0 \frac{dz'}{(1+z')E(z')}$$

where $t_0$ denotes present epoch. Using $E(z)$ vs $z$ profile as shown in Fig. 1 we perform the above integration numerically to obtain $z$ dependence of $t(z)$. Using Eq. (6) and the relation $1/a = 1 + z$, simultaneous values of $a$ and $t$ at any given redshift $z$ are computed. This amounts to elimination of $z$ from them to obtain the time profile of $a(t)$ from the analysis of observational data. The obtained profile is shown in left panel of Fig. 2 and this corresponds to the redshift range $0 \leq z \leq 1$ or to the equivalent $t$-range: $0.44 \leq t(z) \leq 1$ as obtained from Eq. (6). $t$ is also normalised to unity ($t_0 = 1$) at present epoch $z = 0$. 
3 \(k\)-essence scalar field model of dark energy with field-independent potential

The field theoretic models of dark energy involve suitably chosen Lagrangians contributing to energy-momentum tensor in the Einstein’s field equations. In this paper, we consider dark energy described by a homogeneous scalar field \(\phi\) whose dynamics is governed by a \(k\)-essence Lagrangian with a field-independent (constant) potential. Such models involve actions with non-canonical kinetic terms and have phenomenologically rich properties. Theories with a non-canonical kinetic term was first proposed by Born and Infeld in [71] to get rid of the infinite self-energy of the electron and were also investigated by Heisenberg in [72] in the context of physics of cosmic rays and meson productions and further developed by Dirac in [73]. Such non-canonical Lagrangians have also been investigated in low energy effective string theory and in D-branes models [74, 75, 76]. In cosmology, the kinetic energy driven acceleration was first proposed in the context of cosmic inflation [27] and also applied to explain the late time cosmic acceleration \(i.e.,\) dark energy, in [28, 29, 30, 31, 32, 33]. In this section, we present a brief outline of the general framework and features of the \(k\)-essence model that would be relevant in the context of this paper.

The dynamics of the \(k\)-essence scalar field \(\phi\), which is minimally coupled to the gravitational field \(g_{\mu\nu}\) is driven by the action

\[
S = \int d^4x \sqrt{-g} \mathcal{L}(X, \phi)
\]

where \(X \equiv \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi\), \(g\) is determinant of the metric \(g_{\mu\nu}\) and \(\nabla_\mu\) denotes covariant derivative. Variation of the action with respect to the field \(g_{\mu\nu}\) gives the energy momentum tensor for the \(k\)-essence field as

\[
T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{\partial \mathcal{L}}{\partial X} \nabla_\mu \phi \nabla_\nu \phi - g_{\mu\nu} \mathcal{L}
\]

which when compared with the form of that of perfect fluid (characterised by energy density \(\tilde{\rho}\) and pressure \(\tilde{p}\)): \(T_{\mu\nu} = (\tilde{\rho} + \tilde{p}) u_\mu u_\nu - \tilde{p} g_{\mu\nu}\) gives

\[
\tilde{p} = \mathcal{L}(\phi, X), \quad \tilde{\rho} = 2X \frac{\partial \tilde{\rho}}{\partial X} - \tilde{p}
\]
where \( u_\mu \) is the effective four-velocity given by \( u_\mu = \text{sgn}(\partial_\mu \phi) \frac{\partial \phi}{\partial X} \). We take the form of the non-canonical Lagrangian in \( k \)-essence model as

\[
\mathcal{L}(\phi, X) = V(\phi)F(X) \tag{10}
\]

Motivations for consideration of such restricted form of the \( k \)-essence Lagrangian density also comes from low energy effective string theory with a suitable redefinition of the field discussed comprehensively in [77, 78, 79]. For such \( k \)-essence Lagrangians in Eq. (10), the pressure and energy density as given in Eq. (9) may be written as

\[
\rho = V(\phi)(2XF_X - F), \quad p = V(\phi)(2XF_X - F) \tag{11}
\]

where \( F_X = dF/dX \). In an expanding FRW universe, described by the FRW scale factor \( a(t) \), when the dark energy component does not interact with any other component of the universe, the energy density and pressure due to this component satisfies the continuity equation

\[
\dot{\rho} + 3H(\rho + p) = 0, \quad \text{which using with Eq. (11) one obtains}
\]

\[
\left[ \frac{d}{dt}V(\phi) \right] (2XF_X - F) + V(\phi) \left[ \frac{d}{dt}(2XF_X - F) \right] + 6V(\phi)HF_X = 0 \tag{12}
\]

Taking the \( k \)-essence field in the FRW spacetime background to be homogeneous: \( \phi(t, \vec{x}) \equiv \phi(t) \), we have \( X = (1/2)\dot{\phi}^2, \quad \left[ \frac{d}{dt}V(\phi) \right] = \left[ \frac{d}{d\phi}V(\phi) \right] \dot{\phi} \) and \( \left[ \frac{d}{dt}(2XF_X - F) \right] = \left[ \frac{d}{dX}(2XF_X - F) \right] \dot{\phi} \).

From Eq. (12), we then have the equation of motion for the \( k \)-essence field in the evolving in FRW background spacetime as

\[
(F_X + 2XF_{XX})\ddot{\phi} + 3HF_X \dot{\phi} + (2XF_X - F) \frac{V_\phi}{V} = 0 \tag{13}
\]

where \( F_{XX} = d^2F/dX^2 \) and \( V_\phi = dV/d\phi \).

A simple class of such \( k \)-essence models, called ‘purely’ kinetic \( k \)-essence models was investigated in [1], where the Lagrangian involves only the kinetic factor - a function of \( X \), without having any explicit dependence on the field \( \phi \). This corresponds to \( k \)-essence models with Lagrangians given by Eq. (10) with a constant potential: \( V(\phi) \equiv V_0 \), a constant. Such models have been shown to generate an exponential cosmic inflation but remains unable to explain the graceful exit from the phase of inflation. However, in the context of dark energy, these purely kinetic \( k \)-essence models have been shown to generate the observed transition from the phase of decelerated expansion to a phase of accelerated expansion, during late time evolution of the universe.

For such models with constant \( V(\phi) \), \( V_\phi = 0 \) and using \( X = \dot{\phi}^2/2, \quad \dot{X} = \dot{\phi} \dot{\phi} \), the equation of motion (13) for the \( k \)-essence field can be written as

\[
(F_X + 2XF_{XX})\ddot{\phi} + 6HF_X = 0 \tag{14}
\]

Writing \( H = \dot{a}/a \) and changing the independent variable from \( t \) to the scale factor \( a \), the above equation can be expressed in a differential form as

\[
d\ln(XF_X^2) + d\ln a^6 = 0 \tag{15}
\]

which on integration gives a scaling relation:

\[
XF_X^2 = Ca^{-6} \tag{16}
\]

where \( C \) is the constant of integration. In the context of a field independent potential, such a form of scaling relation holds only when the dark energy component does not interact with dark matter and continuity equation in each of these sectors are satisfied without any source term. In the context of interacting DE-DM scenario, the Eq. (16) is no longer valid. In the following sections, we obtained the modified scaling relation, in the interacting DE-DM scenario for \( k \)-essence model of Dark energy with a constant potential.
4 Framework of interactive Dark energy - Dark matter model

In this section we describe the theoretical framework of interacting DE-DM scenario in FRW flat spacetime background filled with an ideal fluid with its components as DE and DM. We focus only on the late time era of cosmic evolution where we neglect the contribution to energy density due to baryonic matter and radiation, as supported by small estimated value of their combined share (~ 4%) in present universe based on measurements from WMAP [12] and Planck [13] experiments. The conservation of energy momentum tensor for the total dark fluid with interactions between its components may be expressed through the non-conserving continuity equations,

\[ \dot{\rho}_{\text{dm}} + 3H\rho_{\text{dm}} = Q(t) \]  
\[ \dot{\rho}_{\text{de}} + 3H(\rho_{\text{de}} + p_{\text{de}}) = -Q(t) \]

where, \( \rho_{\text{de}} \) and \( \rho_{\text{dm}} \) denote the instantaneous energy densities of DE and DM components respectively and \( p_{\text{de}} \) is the pressure of DE fluid. Dark matter is considered to be (non-relativistic) pressure-less dust. The function \( Q(t) \) in the source term of the above equation gives a measure of instantaneous rate of energy transfer between DE and DM components. Eqs. (17) and (18) imply

\[ \dot{\rho} + 3H(\rho + p) = 0 \]

which implies conservation of energy momentum tensor of the total dark fluid with energy density \( \rho = \rho_{\text{de}} + \rho_{\text{dm}} \) and pressure \( p = p_{\text{de}} \). The dot (\( \cdot \)) represents derivatives with respect to the dimensionless time parameter \( t \) which is normalised to \( t = 1 \) at present epoch. As discussed in Sec. 2, the time profile of \( a(t) \) has been extracted from analysis of observational data over the late time domain \( 0.44 \lesssim t \lesssim 1 \) accessible in SNe Ia observations. We express the time dependence of the source term \( Q(t) \) through the scale factor \( a(t) \) and a constant parameter \( k \) as

\[ Q(t) = Q_0 [a(t)]^k \]

Where \( Q_0 \) is the value of \( Q(t) \) at present epoch since we used the normalisation \( a(t) = 1 \) at present epoch. For convenience, we express time dependences of various quantities in terms of a time parameter \( \eta \) defined as \( \eta \equiv \ln a(t) \) (\( \eta = 0 \) then corresponds to the present epoch and \( a = e^{\eta} \)). Eqs. (17) and (18) then take the form

\[ \rho'_{\text{dm}} + 3H\rho_{\text{dm}} = Q_0 \frac{e^{kn}}{H} \]  
\[ \rho'_{\text{de}} + 3(\rho_{\text{de}} + p_{\text{de}}) = -Q_0 \frac{e^{kn}}{H} \]

where all the time-dependent quantities involved, are regarded as functions of \( \eta \) and \( ' \) denotes derivative with respect to \( \eta \). Multiplying both sides of Eq. (21) by \( e^{3\eta} \) and writing the right hand side as a total derivative we get

\[ \frac{d}{d\eta} \left[ e^{3\eta} \rho_{\text{dm}} \right] = Q_0 \frac{e^{(k+3)\eta}}{H} \]

which on integration between limits \( \eta = 0 \) and \( \eta = \eta \) gives

\[ \rho_{\text{dm}}(\eta) = e^{-3\eta} \left[ \rho_{\text{dm}}^0 + Q_0 \int_0^\eta d\eta_1 \frac{e^{(k+3)\eta_1}}{H(\eta_1)} \right] \]
where, $\rho_{\text{dm}}^0$ denotes the DM density at present epoch. Note that, in absence of any interaction between DE and DM ($Q_0 = 0$), we get $\rho_{\text{dm}}(\eta) = e^{-3\eta}\rho_{\text{dm}}^0 = a^{-3}\rho_{\text{dm}}^0$, which is as expected from $\Lambda$-CDM model. Dividing both sides by the total dark fluid density at present epoch, $(\rho_{\text{de}}^0 + \rho_{\text{dm}}^0)$, we have

$$\frac{\rho_{\text{dm}}(\eta; k, \beta_0)}{(\rho_{\text{dm}}^0 + \rho_{\text{de}}^0)} = e^{-3\eta} \left[ \Omega_{\text{dm}}^0 + \beta_0 \int_0^\eta d\eta_1 e^{(k+3)\eta_1} \right]$$

(25)

Here, $\beta_0 \equiv Q_0/(\rho_{\text{dm}}^0 + \rho_{\text{de}}^0)$ and $\Omega_{\text{dm}}^0(\equiv \rho_{\text{dm}}^0/(\rho_{\text{dm}}^0 + \rho_{\text{de}}^0))$ denotes fractional content of DM in the present universe whose value is $\sim 0.268$ from WMAP and PLANCK observations [12, 13]. $k$ and $\beta_0$ are also put in the argument of $\rho_{\text{dm}}$, to emphasize the point that temporal behaviour of the DM energy density, in this scenario, depends on values of these parameters.

The time profile of the Hubble parameter $H(t) = \dot{a}/a$ may be obtained from the temporal profile of $a$ and $\dot{a}$ as shown in Fig. 2. Numerically eliminating $t$ from the $a(t)$ and $H(t)$ we may express the Hubble parameter as a function of $a$ or subsequently as a function of $\eta$ as $H(\eta)$ which appear in the right hand side of Eq. (25). We find that the quantity $1/H(\eta)$, thus obtained, corresponding to the central line in Fig. 1 may be fitted with a polynomial of order 3, which we express as

$$\frac{1}{H(\eta)} = \sum_{m=0}^3 A_m \eta^m$$

where, $A_0 = 0$, $A_1 = 0.45$, $A_2 = -0.32$, $A_3 = -0.15$. (26)

Using Eq. (26) in the integral appearing in right hand side of Eq. (25) we have

$$\frac{\rho_{\text{dm}}(\eta; k, \beta_0)}{(\rho_{\text{dm}}^0 + \rho_{\text{de}}^0)} = e^{-3\eta} \left[ \Omega_{\text{dm}}^0 + \beta_0 \sum_{m=0}^3 A_m I_{k,m}(\eta) \right]$$

(27)

where

$$I_{k,m}(\eta) \equiv \int_0^\eta d\eta_1 e^{(k+3)\eta_1}(\eta_1)^m$$

(28)

For $k \neq -3$, the function $I_{k,m}(\eta)$ satisfies the recursion relation

$$I_{k,m}(\eta) = \frac{\eta^m e^{(k+3)\eta}}{k+3} - \frac{m}{k+3} I_{k,m-1}(\eta)$$

(29)

with $I_{k,0}(\eta) = \int_0^\eta d\eta_1 e^{(k+3)\eta_1} = e^{(k+3)\eta}/(k+3)$

However, for $k = -3$, the exponential term in Eq. (28) becomes unity and we can easily compute the integral as

$$I_{-3,m}(\eta) = \int_0^\eta d\eta_1 (\eta_1)^m = \frac{\eta^{m+1}}{m+1}$$

(30)

where $m$ can take 4 values viz. 0, 1, 2, 3 as evident from Eq. (26). Using Eq. (29) or (30), according as $k \neq -3$ or $k = -3$, we can write the term $\sum_{m=0}^3 A_m I_{k,m}(\eta)$ appearing in right hand side of Eq. (27) as

$$\sum_{m=0}^3 A_m I_{k,m}(\eta) = \begin{cases} (B_{k0} + B_{k1}\eta + B_{k2}\eta^2 + B_{k3}\eta^3)e^{(k+3)\eta} & \text{for } k \neq -3 \\ A_0\eta + \frac{A_1\eta^2}{2} + \frac{A_2\eta^3}{3} + \frac{A_3\eta^4}{4} & \text{for } k = -3 \end{cases}$$

(31)
where, the constants $B_{ki}$’s can be expressed as

\[
B_{k0} = \left( \frac{A_0}{k+3} - \frac{A_1}{(k+3)^2} + \frac{2A_2}{(k+3)^3} - \frac{6A_3}{(k+3)^4} \right)
\]
\[
B_{k1} = \left( \frac{A_1}{k+3} - \frac{2A_2}{(k+3)^2} + \frac{6A_3}{(k+3)^3} \right)
\]
\[
B_{k2} = \left( \frac{A_2}{k+3} - \frac{3A_3}{(k+3)^2} \right)
\]
\[
B_{k3} = \left( \frac{A_3}{k+3} \right)
\]

With the aid of Eq. (31) the $\eta$-dependence appearing in right hand side of Eq. (27) can be expressed in an algebraic form in terms of parameters $k$ and $\beta_0$, with known values of all other factors involved in the expression, e.g $A_i$’s and $\Omega^0_{\text{dm}} \sim 0.268$. This allows us to numerically compute the time($\eta$)-profile of DM energy density term for any chosen benchmark values of the parameters: $k$ and $\beta_0$. Note that, determination of values of constants $A_i$’s (and so also $B_{ki}$’s for any given $k$) uses time profile of scale factor as extracted from analysis of SNe Ia data. The obtained $\eta$-dependence of DM energy density term in DE-DM interacting scenario is, therefore, consistent with the SNe Ia data. The features of the data are encoded in the corresponding expression through the constants $A_i$’s. We may now also use the substitution $\eta = \ln a$ in Eqs. (27) and (31) to express the time dependence in terms of scale factor $a$ itself as

\[
\frac{\rho_{\text{dm}}(a; k, \beta_0)}{(\rho^0_{\text{dm}} + \rho^0_{\text{de}})} = a^{-3} \left[ \Omega^0_{\text{dm}} + \beta_0 \sum_{m=0}^{3} A_m I_{k,m}(\ln a) \right]
\]

The total energy density $\rho = \rho_{\text{dm}} + \rho_{\text{de}}$ and pressure $p = p_{\text{de}}$ of the dark fluid, on the other hand, are independent of the parameters $\beta_0$ and $k$, as the continuity equation of the total dark fluid involves no source term. We can obtain their temporal behaviour, directly from the time profile of the scale factor obtained from the analysis discussed in Sec. 2. To see this, we write the Friedmann equations governing late-time cosmic evolution, which in a flat FRW spacetime background with DE and DM as its primary contents take the forms

\[
H^2 = \frac{\dot{a}^2}{a} = \frac{\kappa^2}{3} (\rho_{\text{dm}} + \rho_{\text{de}})
\]
\[
\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6} \left[ (\rho_{\text{dm}} + \rho_{\text{de}}) + 3p_{\text{de}} \right]
\]

where $\kappa^2 \equiv 8\pi G$ ($G$ is the Newton’s Gravitational constant). Using Eqs. (33) and (34), we may express the equation of state $w$ of the total dark fluid in terms of the scale factor and its time derivatives as

\[
w \equiv \frac{p_{\text{de}}}{\rho_{\text{de}} + \rho_{\text{dm}}} = -\frac{2}{3} \frac{a\ddot{a}}{\dot{a}^2} - \frac{1}{3}
\]

From the obtained time profile of scale factor $a(t)$ and its derivatives as shown in Fig. 2 we can obtain $t$—dependence of $w(t)$. Numerically eliminating $t$ from the $a(t)$ and $w(t)$ we may express the equation of state parameter $w$ as a function of $a$ or $\eta$, making use of the substitution $\eta = \ln a$. We find that, $w(\eta)$ thus obtained, corresponding to the central line in Fig. 1 may be fitted with a polynomial of order 5. This we express as

\[
w(\eta) = -1 + \sum_{i=0} W_i \eta^i
\]
with values of the coefficients \( W_i \) at best-fit are given by

\[
W_0 = -0.70, W_1 = -0.61, W_2 = -0.49, W_3 = -2.29, \\
W_4 = -2.81, W_5 = -0.92, \text{ and } W_i = 0 \text{ for } i > 5
\]  

(37)

In terms of the parameter \( \eta \) the continuity equation \([19]\) for the total dark fluid takes the form

\[
\frac{d}{d\eta} \ln \left( \rho_{\text{dm}} + \rho_{\text{de}} \right) = -3 \left( 1 + w(\eta) \right)
\]  

(38)

which on integration between the limits \( \eta = 0 \) and \( \eta = \eta \) gives

\[
\frac{(\rho_{\text{de}} + \rho_{\text{dm}})_{\eta}}{(\rho_{\text{de}} + \rho_{\text{dm}})_0} = \exp \left[ -3 \int_0^{\eta} (1 + w(\eta_1)) d\eta_1 \right]
\]  

(39)

The integral within exponent can be performed by using Eq. (36), with \( W_i \)'s as given in Eq. (37). This gives

\[
\frac{\rho_{\text{de}}(\eta) + \rho_{\text{dm}}(\eta)}{(\rho_{\text{de}} + \rho_{\text{dm}})_0} = \exp \left[ -3 \int_0^{\eta} \left( \sum_{i=0}^{5} W_i \eta_i^i \right) d\eta_1 \right] = \exp \left[ -3 \left( \sum_{i=0}^{5} \frac{W_i \eta^{i+1}}{i+1} \right) \right]
\]  

(40)

We may again use the substitution \( \eta = \ln a \) in the above to express the energy density of the total dark fluid \( \rho = \rho_{\text{de}} + \rho_{\text{dm}} \) as a function of scale factor \( a \) as,

\[
\frac{\rho(a)}{\rho_{\text{de}} + \rho_{\text{dm}}^0} = \exp \left[ -3 \left( \sum_{i=0}^{5} \frac{W_i (\ln a)^{i+1}}{i+1} \right) \right]
\]  

(41)

We find that this scale factor dependence of the total energy density, thus obtained, can be fitted best with a fourth order polynomial expressed as,

\[
\frac{\rho(a)}{\rho_{\text{de}} + \rho_{\text{dm}}^0} = \sum_{m=0}^{4} R_m a^m
\]  

(42)

with the best-fit values of coefficients as

\[
R_0 = 29.3, \quad R_1 = -120.3, \quad R_2 = 199.96, \quad R_3 = -151.75, \quad R_4 = 43.8
\]  

(43)

Note that, the constants \( W_i \)'s and \( R_i \)'s encode the inputs from the observational data used here.

Similarly we may also obtain temporal behaviour of the pressure \( (p_{\text{de}}) \) of DE. Since dark matter dust has zero pressure, the equation of state parameter of the dark fluid can be written as

\[
w(\eta) = \frac{p_{\text{de}}(\eta)}{\rho_{\text{de}}(\eta) + \rho_{\text{dm}}(\eta)},
\]  

(44)

from which we may write

\[
\frac{p_{\text{de}}(\eta)}{\rho_{\text{de}} + \rho_{\text{dm}}^0} = w(\eta) \frac{\rho_{\text{de}}(\eta) + \rho_{\text{dm}}(\eta)}{\rho_{\text{de}} + \rho_{\text{dm}}^0}
\]  

(45)

Using Eqs. (36) and (40) in Eq. (45) and by making the substitution \( \eta = \ln a \) we may express this functional behaviour of the \( p_{\text{de}} \) in terms of \( a \) as

\[
\frac{p_{\text{de}}(a)}{\rho_{\text{de}} + \rho_{\text{dm}}^0} = \left[ -1 + \sum_{i=0}^{5} W_i (\ln a)^i \right] \cdot \exp \left[ -3 \left( \sum_{i=0}^{5} \frac{W_i (\ln a)^{i+1}}{i+1} \right) \right]
\]  

(46)
We find that, this scale factor dependence of $p_{de}$ can be fitted best with a polynomial of order 6 expressed as,

$$\frac{p_{de}(a)}{\rho_{de}^0 + \rho_{dm}^0} = \sum_{m=0}^{6} P_m a^m$$

(47)

with the best-fit values of coefficients as

$$P_0 = -11.84, \quad P_1 = 90.04, \quad P_2 = -297.92,$$

$$P_3 = 515.97, \quad P_4 = -495.22, \quad P_5 = 250.83, \quad P_6 = -52.57$$

(48)

The scale factor dependence of $\rho(a)$ and $p_{de}(a)$, thus obtained using the temporal profile of $a(t)$ from SNe Ia data, are shown in left panel and right panel of Fig. 3 respectively.

Note that, the density of DE as a function of the scale factor $a$ may now be expressed as

$$\frac{\rho_{de}(a; k, \beta_0)}{\rho_{dm}^0 + \rho_{de}^0} = \frac{\rho(a)}{(\rho_{dm}^0 + \rho_{de}^0)} - \frac{\rho_{dm}(a; k, \beta_0)}{(\rho_{dm}^0 + \rho_{de}^0)},$$

(49)

which involves the parameters $\beta_0$ and $k$, due to their appearance in $\rho_{dm}(a; k, \beta_0)$. Since energy density is always a positive quantity and the scale factor or time dependence of energy density of total dark fluid has already been obtained directly from observation (left panel of Fig. 3), the estimated profile of the dark matter density $\rho_{dm}(a; k, \beta_0)$ computed from Eq. (25) for given values of $k$ and $\beta_0$ are subject to the constraint

$$0 < \frac{\rho_{dm}(a; k, \beta_0)}{(\rho_{de}^0 + \rho_{dm}^0)} < \frac{\rho(a)}{(\rho_{de}^0 + \rho_{dm}^0)},$$

(50)
Figure 4: Range of parameter space of model parameters $k$ vs. $\beta_0$ for which the condition in Eq. (51) is satisfied. A black line is drawn at $\beta_0 = 0$ to reflect the fact that for $\beta_0 = 0$, the parameter $k$ loses its relevance in the context (described in text).

Figure 5: Variation of energy density of dark energy and dark matter with scale factor for chosen benchmark values of the parameters ($k$ and $\beta_0$).

for the accessible domain of $a$ ($0.5 \lesssim a < 1$) in SNe Ia observations. Using Eqs. (32) and (42), we may express the above condition as

$$0 < a^{-3} \left[ \Omega^0_{dm} + \beta_0 \sum_{m=0}^{3} A_m I_{k,m}(\ln a) \right] < \sum_{m=0}^{4} R_m a^m$$  \hspace{1cm} (51)$$

where the function $I_{k,m}(\ln a)$ contains the parameter $k$ as seen from its explicit form in Eq. (31) with $\eta = \ln a$. Using the form of $I_{k,m}(\ln a)$ for $k \neq -3$, we find the range in the parameter space spanned by $k$ and $\beta_0$, every point $(k, \beta_0)$ of which satisfies the condition in Eq. (51). This allowed region in parameter space has been shown by the shaded region in Fig. 4. However, for $k = -3$ (when $Q(t) = Q_0 a^{-3}$), we find the range of $\beta_0$ for which the condition in Eq. (51) is satisfied, is $-0.2 \leq \beta_0 \leq 0.42$.

For some benchmark values of $k$ and $\beta_0$ chosen within this allowed region we have also depicted the variation of the dark energy and dark matter densities with the scale factor in Fig. 5.
from Eq. (20) that for $\beta_0 = 0 = Q_0$, the parameter $k$ becomes irrelevant and corresponding continuity equation for DM (Eq. (17)) has no source term ($Q(t) = 0$). This implies $\rho_{\text{dm}}(a) = \rho_{\text{dm}}^0 a^{-3}$ with $\rho_{\text{de}}(a) = \rho(a) - \rho_{\text{dm}}^0 a^{-3}$, where $\rho(a)$ as directly obtained from observation has been expressed thorough a fitted polynomial in Eq. (42). These profiles for $\beta_0 = 0$ (which corresponds to ‘non interacting DE-DM’ scenario) are presented in left panel of Fig. [5] and we find that DM energy density falling as $\sim 1/a^3$ falls below dark energy density at an epoch corresponding to $a \sim 0.71$. For $\beta_0 = 0.5$ and $k = 1$ (middle panel of Fig. [5]) the source term $Q(t)$ grows with time having the same profile as that $a(t)$ itself, and the corresponding DM and DE density profiles evaluated using Eqs. (32) and (49) show that DM energy density falls below that of DE at an epoch when $a \sim 0.75$. For $\beta_0 = 0.1$ and $k = -3$ the source term $Q(t)$ falls as $\sim 1/a(t)^3$ as the universe expands, and the corresponding plots presented in right panel of Fig. [5] show beginning of dominance of DE density over that of DM at a relatively earlier epoch marked by $a \sim 0.63$.

5 k-essence model with constant potential and modification of scaling relation in presence of DE-DM interaction

We assume dynamics of dark energy of universe to be driven by a homogeneous scalar field $\phi = \phi(t)$ governed by non-canonical k–essence Lagrangian of the form $L = V(\phi)F(X)$ where the potential $V(\phi) = V_0$ is considered to be constant and the dynamical term $F(X)$ is a function of $X \equiv (1/2)g_{\mu\nu}\nabla^\mu \phi \nabla^\nu \phi = (1/2)\phi^2$. The energy density and pressure of DE are identified with the corresponding quantities in the context of the k–essence model which may be expressed as

$$\rho_{\text{de}} = V_0 (2XF_X - F)$$

$$\rho_{\text{de}} = V_0 F(X)$$

where, $F_X \equiv dF/dX$. In the context of this constant potential k–essence model, it has been shown in [12] that, in the non-interacting DE-DM scenario when the energy conservation is separately conserved in the DE sector (and also in DM sector) implying corresponding continuity equation [18] being satisfied with $Q(t) = 0$, we have a scaling relation of the form $XF_X^2 = Ca^{-6}$ where $C$ is a constant. In this section we show that, even in presence of time-dependent interactions implying continual exchange of energy between the two sectors ($Q(t) \neq 0$), the constancy of k–essence potential may lead to a modified form of the scaling. Below we discuss how we obtain the modified form of scaling taking into consideration observational inputs from SNe Ia data.

To obtain this we use Eqs. (52) and (53) in Eq. (18) with $Q(t)$ parametrised as $Q(t) = Q_0[a(t)]^k$ (Eq. (20)) to get

$$\frac{d}{dt} \left[ V_0 (2XF_X - F) \right] + 3H (2V_0XF_X) = -Q_0 a^k.$$  \hfill (54)

Writing $\frac{d}{dt} = \dot{a} \frac{d}{da}$ and $H = \dot{a}/a$, after some rearrangements the above equation takes the form

$$d \left[ \ln \left( a^6 (XF_X^2) \right) \right] = -\frac{Q_0 a^{k-1}}{V_0} \frac{da}{H (XF_X)}.$$  \hfill (55)

where $F_{XX} = d^2F/dX^2$ and the quantities $H, X, F$ and its derivatives are regarded as functions of scale factor $a$. Integrating both sides of Eq. (55) between the limits $a = 1$ (present epoch) and $a = a$ we have,

$$\left[ \ln \left( a^6 (XF_X^2) \right) \right] - \left[ \ln \left( (XF_X^2) \right) \right]_{a=1} = -\frac{Q_0}{V_0} \int_1^a \frac{f(a')}{(XF_X)} da'$$

or,

$$XF_X^2 = Ca^{-6} \exp \left( \frac{Q_0}{V_0} \int_1^a \frac{f(a';k)}{(XF_X)} \right) \hfill (56)$$
where, \( C \equiv XF_X^2 |_{a=1} \) is a constant and \( f(a; k) \equiv \frac{a^{k-1}}{H(a)} \). In Eq. (26) we have shown how \( H^{-1} \)
depends on \( \eta = \ln a \), with the help of a polynomial of \( \ln a \), with given of values of coefficients \( A_i \)’s encoding the observational inputs extracted from the analysis of JLA data. We may, equivalently, fit this scale factor dependence of \( H^{-1} \) by a polynomial in \( a \) (instead of \( \ln a \)) and found that, the corresponding best-fit curve may be given by a second order polynomial of scale factor \( a \) as

\[
\frac{1}{H(a)} = \sum_{n=0}^{2} D_n a^n \quad \text{with} \quad D_0 = -0.272, \; D_1 = 2.174, \; D_2 = -0.9. \quad (57)
\]

To perform the integration involved in Eq. (56), we can then express the function \( f(a, k) \) as a polynomial

\[
f(a; k) = a^{k-1}(D_0 + D_1 a + D_2 a^2),
\]

and we also need to express \( (XF_X) \) as function of \( a \). To do so, we eliminate \( F(X) \) from Eqs. (52) and (53) and write

\[
XF_X = \frac{1}{2V_0} \left[ \rho_{de} + p_{de} \right] \quad (58)
\]

Using Eq. (49) and writing all variables and parameters explicitly, we may write the above equation as

\[
XF_X = \left( \frac{\rho_0^{dm} + \rho_0^{de}}{2V_0} \right) \left[ \frac{\rho(a)}{(\rho_0^{dm} + \rho_0^{de})} - \frac{\rho_{dm}(a; k, \beta_0)}{(\rho_0^{dm} + \rho_0^{de})} + \frac{p_{de}(a)}{(\rho_0^{dm} + \rho_0^{de})} \right] = \left( \frac{\rho_0^{dm} + \rho_0^{de}}{2V_0} \right) g(a; k, \beta_0) \quad (59)
\]

where

\[
g(a; k, \beta_0) = \left[ \frac{\rho(a)}{(\rho_0^{dm} + \rho_0^{de})} - \frac{\rho_{dm}(a; k, \beta_0)}{(\rho_0^{dm} + \rho_0^{de})} + \frac{p_{de}(a)}{(\rho_0^{dm} + \rho_0^{de})} \right] \quad (60)
\]

Finally, using Eq. (59) in Eq. (56) we obtain

\[
\frac{XF_X^2}{Ca^{-0}} = \exp \left( -2\beta_0 \int_1^a \frac{f(a_1; k)da_1}{g(a_1; k, \beta_0)} \right) \quad (61)
\]

All the three terms in the right hand side of Eq. (60) have been expressed in algebraic form in Eqs. (32), (42) and (47), using which we can numerically compute the function \( g(a; k, \beta_0) \) for any input values of \( a, k, \beta_0 \). With this and using the form of the function \( f(a; k) \) in Eq. (58), we may numerically evaluate the integral within the exponent appearing in Eq. (61). This is the modified scaling relation arising out of the constancy of the potential \( V = V_0 \) of \( k \)-essence model of dark energy, in presence of interaction between DE and DM parametrised in terms of \( \beta_0 \) and \( k \). The inputs from observational data are encoded in the form of the functions \( f(a; k) \) and \( g(a; k, \beta_0) \) through the various coefficients \( D_i \)'s, \( A_i \)'s, \( P_i \)'s, \( R_i \)'s etc. introduced in Secs. 4 and 5, while establishing connections with the temporal profile of the scale factor obtained from the SNe Ia data in Sec. 2. In the context of this constant potential \( k \)-essence model of dark energy in intersecting DE-DM scenario, the modified scaling relation (61) establishes a connection between the dynamical terms \( X, F(X) \) involved in the \( k \)-essence Lagrangian and the scale factor \( a(t) \) of FRW universe along with the parameters \( k \) and \( \beta_0 \). The constancy of the \( k \)-essence potential is instrumental in establishing the relation.
Figure 6: Plot of $XF_X^2/Ca^{-6}$ as a function of scale factor ($a$) for some benchmark values of the model parameters $k$ and $\beta_0$. The plots within the small boxes drawn on the top of each panel depict the same plots of the panel in an appropriately zoomed domain $0.7 < a < 1$ to bring out a better resolution of each of the distinct curves corresponding to different $\beta_0$ values in that domain.

Note that, in absence of any interaction ($\beta_0 = Q_0/(\rho_{dm}^0 + \rho_{de}^0) = 0$), the exponential term in Eq. (61) becomes unity and the modified scaling relation reduces to the usual form $XF_X^2 = Ca^{-6}$ as obtained in [1, 2]. Therefore, the deviation from unity, of the quantity $\exp \left(-2\beta_0 \int_a^1 f(a, k)da \right)$ in right hand side of (Eq. 61), evaluated for any parameter set $(k, \beta_0)$ gives the extent of modification in the scaling relation due to presence of (time-dependent) interaction between DE and DM parametrised in terms of $(k, \beta_0)$. The behaviour of modifications in the scaling relation are shown in three panels Fig. 6 for three chosen benchmark values of $k$ viz. $k = 0$ (left panel), $k = 1$ (middle panel) and $k = -3$ (right panel). In each of the panels, corresponding to a given value of $k$, we have plotted the quantity $XF_X^2/Ca^{-6}$ as a function of the scale factor, for few different values of the parameter $\beta_0$ chosen from the corresponding allowed domain of $k - \beta_0$ parameter space discussed (and also shown in Fig. 4) in the end of Sec. 4. The benchmark cases: $k = 0$, $k = 1$ and $k = -3$ respectively correspond to $Q(t) = \text{constant}$, $Q(t) = Q_0 a(t)$ and $Q(t) = Q_0/[a(t)]^3$ in the non-conserving continuity equations (17) and (18) of DM and DE sectors. The plots shown in Fig. 6 show exponential behaviour for $\beta_0 \neq 0$ as evident from Eq. (61). Since we have assigned the value $(XF_X^2)_{a=1}$ to the constant $C$ the plots of $XF_X^2/Ca^{-6}$ approach unity as the scale factor approaches to its (normalised) value $a = 1$ (at present epoch). The $\beta_0 = 0$ case, corresponds to non-interacting DE-DM scenario and is represented by the $XF_X^2/Ca^{-6} = 1$ line in Fig. 6.

6 Conclusion

In this paper we considered a scenario of interacting dark matter and dark energy during the late time evolution of cosmic evolution, neglecting the contribution due to radiation and
baryonic matter to the total energy density. We describe the dynamics of DE to be driven by a homogeneous $k$—essence field ($\phi$) driven by a non-canonical Lagrangian of the form $L = V(\phi)V(X)$ with a constant potential $V(\phi) = V_0$, where the dynamical term $F(X)$ is a function of $X = 1/2\nabla_\mu \phi \nabla^\mu \phi = \phi^2$ (for homogeneous field). Under such considerations, we showed existence of a scaling relation in the theory, which connects the dynamical quantities $X, dF/dX$ (i.e. $\dot{\phi}$) to the FRW scale factor $a(t)$ of the universe along with two relevant parameters ($\beta_0$ and $k$) of the model. The time-dependent interaction between DM and DE has been incorporated through a function $Q(t)$ in the continuity equations ((17) and (18)) of the two fluids. The source term $Q(t)$ is parametrised in terms of the dimensionless parameters $\beta_0$ and $k$ as $Q(t) = Q_0[a(t)]^k$ where $Q_0 \equiv \beta_0(\rho_{dm}^0 + \rho_{de}^0)$, $\rho_{dm}^0$ and $\rho_{de}^0$ being the present-day observed energy densities of the dark matter and dark energy respectively. The constancy of the $k$—essence potential is instrumental in proving the scaling relation (Eq. (61)) in interacting DE-DM scenario. For $\beta_0 = 0 = Q(t)$, the obtained scaling relation reduces to the usual scaling relation $XF_X = Ca^{-6}$ obtained in [1, 2] for non-interacting DE-DM scenario.

We have expressed the modification to the usual scaling relation due to the effect of DE-DM interaction in terms of an exponential term of form: $\exp \left(-2\beta_0 \int_1^a \frac{f(a;k)d\alpha}{g(a;k,\beta_0)} \right)$ (see Eq. (61)). In obtaining such a form we have taken into consideration the observed feature of temporal behaviour of the FRW scale factor $a(t)$, probed in the measurement of luminosity distance and redshift of SNe Ia events. This key observational ingredient enters into the above exponential form at different levels of its derivation, through various constants which finally got twined together in the obtained expressions for the functions $f(a;k)$ and $g(a; k, \beta_0)$. The modified scaling expressed in Eq. (61), in the form of the exponential function, thus encodes in it the features of the SNe Ia data. The values of parameters $\beta_0$ and $k$, involved in the scaling, are also restricted from observed data. This has been imposed by the condition $0 < \rho_{dm}(a; k, \beta_0) < \rho(a)$, where $\rho_{dm}(a; k, \beta_0)$ is the dark matter density at an epoch corresponding to scale factor value $a$ in the interacting DE-DM scenario and is expressed by Eq. (62). $\rho(a)$ is the profile of energy density of the total dark fluid extracted from observation and expressed thorough a fitted polynomial in Eq. (42). This constraint puts a bound in the $k - \beta_0$ parameter space as shown in Fig. 4. We also observe that, the values of parameters $\beta_0$ and $k$ which determines the time dependence of the source term $Q(t)$ responsible for DE-DM interactions, decide the epoch in the past where density of DE starts dominating over that of DM. This has been demonstrated in Fig. 5.

This modified form of scaling relation in Eq. (61), obtained in the context of DE-DM interacting DE-DM scenario, may also be used to eliminate $F_X$ from Eqs. (59), to obtain $X$ in terms of the functions $f(a; k)$ and $g(a; k, \beta_0)$, along with the parameters $k$ and $\beta_0$. Since for homogeneous field $\phi$, we have $X = 1/2\phi^2$, one may thus obtain the scale factor dependence or the temporal profile of the $k$—essence scalar field in the context of DE-DM interacting scenario using the modified scaling.

Finally, we also note that, for a given set of values of the parameters $k$ and $\beta_0$, eliminating $F_X$ from Eq. (58) and the modified scaling relation (61), we may obtain the scale factor dependence of the term $X$ as

$$X = \left[\frac{(\rho_{dm}^0 + \rho_{de}^0)^2}{4CV_0^2}\right] a^6 \left[g(a; k, \beta_0)\right]^2 \exp \left(2\beta_0 \int_1^a \frac{f(a'; k)da'}{g(a'; k, \beta_0)} \right).$$  \hspace{1cm} (62)

Up to the constant multiplicative factor $\frac{(\rho_{dm}^0 + \rho_{de}^0)^2}{4CV_0^2}$, the right hand side of the Eq. (62) is numerically computable at different values of the scale factor $a$, with the knowledge of the functions $g(a; k, \beta_0)$ and $f(a; k)$ as extracted from observation, for any given choice of values of the parameters $\beta_0$ and $k$. Also computing $F(X) = \rho_{de}(a)/V_0$ from Eq. (53), upto the factor $1/V_0$,
at different values of the scale factor $a$, we may eliminate $a$ from these two dependencies to obtain $V_0 F(X)$ as a function of $-\frac{4C V_0^2}{(\rho_{dm} + \rho_{de})} X$. Such a dependence, as extracted from observation, using the modified scaling in the constant potential $k$—essence model contains the information in the form of $F(X)$ for given choices of the parameters $\beta_0$ and $k$ in the context of interacting DE-DM scenario. A given choice of the constant $C$ (appearing in scaling relation), the value of the constant potential $V_0$ and the total density of dark matter and dark energy ($\rho_{dm}^0 + \rho_{de}^0$) would uniquely determine the form of the function $F(X)$.

Acknowledgement We thank the honourable referee for valuable suggestions. A.C. would like to thank Indian Institute of Technology, Kanpur for supporting this work by means of Institute Post-Doctoral Fellowship (Ref.No.DF/PDF197/2020-IITK/970).

Data Availability Statement No Data associated in the manuscript.

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