Improved bounds for covering hypergraphs

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Abstract

The Graham-Pollak theorem states that at least \( n - 1 \) bicliques are required to partition the edge set of the complete graph on \( n \) vertices. In this paper, we provide improvements for the generalizations of coverings of graphs and hypergraphs for some specific multiplicities. We study an extension of Katona Szemerédi theorem to \( r \)-uniform hypergraphs. We also discuss the \( r \)-partite covering number and matching number and how large the \( r \)-partite partition number would be in terms of \( r \)-partite covering number for \( r \)-uniform hypergraphs.

1 Introduction

An \( r \)-uniform hypergraph \( H \) (also referred to as an \( r \)-graph) is said to be \( r \)-partite if its vertex set \( V(H) \) can be partitioned into sets \( V_1, V_2, \cdots, V_r \), so that every edge in the edge set \( E(H) \) of \( H \) intersects \( V_i \) in one vertex. An \( r \)-partite cover of an \( r \)-uniform hypergraph \( H(V, E) \) is a collection of complete \( r \)-partite \( r \)-graphs such that each edge in the edge set \( E \) is contained in some complete \( r \)-partite \( r \)-graphs. An \( r \)-partite partition of an \( r \)-uniform hypergraph \( H(V, E) \) is a pair-wise disjoint collection of complete \( r \)-partite \( r \)-graphs such that each edge in the edge set \( E \) is present in some complete \( r \)-partite \( r \)-graph. The minimum size of the collection of complete \( r \)-partite \( r \)-graphs that partitions the edge set of an \( r \)-uniform hypergraph \( H \) is represented by \( f_r(H) \). The complete \( r \)-uniform hypergraph with \( n \) vertices has an edge set consisting of all \( r \)-sized subsets of \( [n] \). For a complete \( r \)-uniform hypergraph on \( n \) vertices, the minimum size of the collection of complete \( r \)-partite \( r \)-graphs that partitions their edge set is denoted by \( f_r(n) \).

The problem of determining \( f_r(n) \) for \( r > 2 \) was proposed by Aharoni and Linial [1]. For \( r = 2 \), \( f_2(n) \) is the minimum number of biclique subgraphs required to partition the edge set of the complete graph on \( n \) vertices. Graham and Pollak [17, 18] see also [3] and [15] proved that at least \( n - 1 \) bicliques are required to partition the edge set of the complete graph \( K_n \). Since the edges of the complete graph \( K_n \) can be partitioned into \( n - 1 \) disjoint bicliques, this shows that \( f_2(n) = n - 1 \). The original proof by Graham and Pollak uses Sylvester’s law of inertia [18]. Other proofs of the same
were found by Tverberg [35], Peck [31] and Vishwanathan [36] using linear algebraic methods. A combinatorial proof was given by Vishwanathan [37]. The generalisations of Graham-Pollak theorem were provided by Alon [1] and showed that \( f_3(n) = n - 2 \) and \( f_r(n) = \theta(n^{r/2}) \). Cioabă, Kündgen and Verstraëte [6] and later Cioabă and Tait [7] provided improvements in the lower order terms of \( f_r(n) \). Later Leader, Milčević and Tan [27] provided improved bounds on \( f_r(n) \). Further improvements on \( f_r(n) \) are provided in [28] [4].

Alon [2] provided a one to one correspondence between \( p \)-neighborly family of standard boxes in \( \mathbb{R}^d \) and the bipartite covering of a complete graph of cardinality \( d \) such that each edge in the edge set of the complete graph is contained in at least one biclique and at most \( p \) bicliques. Alon also provided bounds for the minimum number of bicliques required to cover the edges of a complete graph such that the edges are covered at least once and at most \( p \) times. Multicovering the edge set in complete \( r \)-uniform hypergraphs is discussed in [5].

The order of a hypergraph \( H(V, E) \) is the number of vertices present in the hypergraph \( H \). For an \( r \)-uniform hypergraph \( H(V, E) \), the minimum of the sum of the orders of the complete \( r \)-partite \( r \)-graphs in a collection over all the collections of complete \( r \)-partite \( r \)-graphs that cover the edge set \( E \) is denoted by \( \zeta_r(H) \). A classical theorem relating the biclique partition and the sum of the orders of the biclique cover is due to Katona and Szemerédi theorem [26]. The Katona-Szemerédi theorem states that the minimum of the sum of the orders of a collection of bicliques that cover the edge set of a complete graph on \( n \) vertices is \( n \log n \). The generalisation of the theorem for graphs with chromatic number \( \chi \) was provided by Mubayi and Vishwanathan [29].

In this paper, we describe some generalisations of Graham-Pollak theorem and an extension of the Katona Szemerédi theorem. In section 2, we provide improved bounds for the minimum number of bicliques required to cover the edges of a complete graph such that the number of times the edges get covered is contained in some specific list of positive integers. Similar results on the minimum number of complete 3-partite 3-graphs required to cover the edge set of a complete 3-uniform hypergraph such that the number of occurrences of the edges belongs to specific list are also provided.

In section 3, we provide an extension of Katona Szemerédi theorem to \( r \)-uniform hypergraphs, by providing a lower bound for the sum of the orders of the collection of complete \( r \)-partite \( r \)-graphs that cover the edge set of an \( r \)-uniform hypergraph in terms of the chromatic number of the \( r \)-uniform hypergraph.

In section 4, we provide lower bounds for complete \( r \)-partite covering number in terms of the matching number in \( r \)-uniform hypergraphs.

In section 5, we discuss how large the complete \( r \)-partite partition number can be in terms of the complete \( r \)-partite covering number for \( r \)-uniform hypergraphs.
2 Multicovering graphs

Let $L = \{l_1, l_2, \ldots, l_t\}$ where $l_1, l_2, \ldots, l_t$ are positive integers. An $L$-biclique covering of graph $G$ is a collection of bicliques such that every edge of $G$ is contained in $l_j$ of the bicliques for $l_j \in L$. The minimum number of bicliques in an $L$-biclique covering is termed the $L$-biclique covering number and is denoted by $b_{pL}(G)$. This definition can be extended to hypergraphs in a natural way. Here, the definition is specified for complete $r$-uniform hypergraphs for specified lists only. Let $[p] = 1, 2, \cdots, p$. An $r$-partite $p$-multicover of a complete $r$-uniform hypergraph $K^r_p$ is a collection of complete $r$-partite $r$-graphs such that every hyperedge of $K^r_p$ is contained in $t$ of the $r$-partite $r$-graphs for some $t \in [p]$. The minimum size of such a covering is called the $r$-partite $p$-multicovering number and is denoted by $f_r(n, p)$. Note that $b_{pL}(K_n)$ where the list $L = \{1, 2, \cdots, p\}$ is same as $f_2(n, p)$.

The problem of bipartite $p$-multicovering of the complete graph $K_n$ on $n$ vertices was first studied by Alon [2]. Alon proved that $(1 + o(1)) \left( \frac{m}{2n} \right)^{1/p} n^{1/p} \leq f_2(n, p) \leq (1 + o(1)) pm^{1/p}$. Though the bounds are asymptotically tight there is still a constant gap between the bounds. Huang and Sudakov [22] improved the lower bound for $f_2(n, p)$ to $(1 + o(1)) \left( \frac{p^1}{2n^1} \right)^{1/p} n^{1/p} \leq f_2(n, p)$. Cioabă and Tait [7] provided a lower bound for $b_{pL}(G)$ for any list $L$ and graph $G$. They also provided constructive $L$-partite covering of $K_n$ for some specified lists, like $L = \{1, 2\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 4, \cdots, 2i\}$. Despite being asymptotically tight, not many lists are known for which the values are exactly known. It is part of folklore that for $L = \{1, 2, \cdots, \lceil \log n \rceil\}$, $b_{pL}(K_n)$ is $\lceil \log_2 n \rceil$. Radhakrishnan, Sen and Vishwanathan [34] gives another list for which the $b_{pL}(K_n)$ is exactly known. They show that $b_{pL}(K_n) = \frac{4}{p}$ for infinitely many values of even $n$ when $L$ is the list of odd numbers less than $n$. They also give some similar results for $b_{pL}(K_n)$ for list of numbers congruent $1(modp)$ where $p$ is prime. For $L = \{\lambda\}$ for a constant $\lambda$, De Caen, Gregory, and Pritikin conjectured that $b_{pL}(K_n) = n - 1$, for large $n$. It is known to be true for $\lambda \leq 18$. Bounds on $r$-partite $p$-multicovering of complete $r$-uniform hypergraphs are provided in [5].

2.1 Constructive upper bound for $b_{p\{2,3\}}(K_n)$

In this subsection, we give an improved upper bound for $b_{p\{2,3\}}(K_n)$. This improves on the previous bound of $3\sqrt{2n}$ by Cioabă and Tait [7].

For $L = \{2,3\}$ order the $n$ vertices into a hexagonal grid with $m$ vertices on each side as in figure 1. A hexagonal grid has three sides that are pairwise non-parallel. The rows parallel to each such side of the grid is denoted by $A_i$, $B_i$ and $C_i$ for $1 \leq i \leq 2m$ respectively. Consider the following collection of complete bipartite graphs with parts $A_i$ and $\cup_{j=i+1}^{2m} A_j$, the collection of complete bipartite graphs with parts $B_i$ and $\cup_{j=i+1}^{2m} B_j$ and the collection of complete bipartite graphs with parts $C_i$ and $\cup_{j=i+1}^{2m} C_j$. This forms
a \{2,3\} covering of \(K_n\) since the edge whose vertices both lie in some row which is parallel to the sides of the hexagonal grid are covered exactly twice and the rest of the vertices are covered thrice.

The total number of vertices in a hexagonal grid is given by

\[
n = 2(m) + 2(m + 1) + \cdots + 2(m + m - 2) + (m + m - 1)
\]

\[
= 2[m(m - 1) + (m - 2)(m - 1)/2] + (m + m - 1)
\]

\[
= 2m^2 - 2m + m^2 - 3m + 2 + 2m - 1
\]

\[
= 3m^2 - 3m + 1
\]

So for \(n = 3(m^2 - m + 1), \) \(bp_{\{2,3\}}(K_n) \leq 6m - 3 < 2\sqrt{3}\sqrt{n}.

### 2.2 Constructive Upper bound for \(f_3(n,4)\)

In this subsection, we give a constructive bound for \(f_3(n,4).\) This bound is better than the general bound described in [5].

For \(L = \{1,2,3,4\}\) order the \(n\) vertices into a square grid with \(m\) rows and \(m\) columns as in figure 2. The rows are denoted by \(R_i\) for \(1 \leq i \leq m\) and the columns are denoted by \(C_i\) for \(1 \leq i \leq m.\) The diagonal rows are the lines that are parallel to the main diagonal of the square grid. The counter diagonal rows are the lines that are parallel to the counter diagonal. The diagonal rows are denoted by \(M_i\) and counter diagonal rows are denoted by \(N_i\) for \(1 \leq i \leq 2m - 1.\) Few of all the different types of rows in the square grid are depicted in figure (3.2). Consider the following collection of complete 3-partite 3-graphs.
1. Complete 3-partite 3-graphs with parts $R_i$, $\bigcup_{j=i+1}^m R_j$ and $\bigcup_{k=1}^{i-1} R_k$ for $2 \leq i \leq m - 1$.

2. Complete 3-partite 3-graphs with parts $C_i$, $\bigcup_{j=i+1}^m C_j$ and $\bigcup_{k=1}^{i-1} C_k$ for $2 \leq i \leq m - 1$.

3. Complete 3-partite 3-graphs with parts $M_i$, $\bigcup_{j=i+1}^m M_j$ and $\bigcup_{k=1}^{i-1} M_k$ for $2 \leq i \leq 2m - 2$.

4. Complete 3-partite 3-graphs with parts $N_i$, $\bigcup_{j=i+1}^m N_j$ and $\bigcup_{k=1}^{i-1} N_k$ for $2 \leq i \leq 2m - 2$.

Observe that for any three vertices in the square grid there is always a row or a column or a diagonal row or a counter diagonal row that passes through one of these vertices such that the other two vertices are on either side of it. The number of occurrences of an edge in the collection of 3-partite 3-graphs depends upon the number of such lines that pass through one of the vertices such that the other two vertices lie on either side of it.

There are $n = m^2$ vertices in the square grid. The number of complete 3-partite 3-graphs is given by

$$m - 2 + m - 2 + 2m - 3 + 2m - 3 = 6m - 10.$$ 

So for $n = m^2$, $f_3(n, 4) \leq 6m - 10 \leq 6\sqrt{n}$. This shows that the general upper bound in [5] is not tight.
3 Cover Order and the Chromatic Number

In this section we provide the extension of the generalization of the Katona Sze-merédi Theorem by Mubayi and Vishwanathan [29] to $r$-uniform hypergraphs.

A hypergraph vertex coloring assigns $s$ colors to the vertices of the hypergraph. Such a coloring of a hypergraph $H$ is said to be a proper coloring if every edge in the edge set of the hypergraph $H$ contains at least two distinct colors. A hypergraph $H$ is said to be $k$-colorable if there exists a proper coloring of the hypergraph $H$ using at most $k$ distinct colors. The chromatic number of a hypergraph $H$, denoted by $\chi(H)$ is the smallest $k$ for which the hypergraph $H$ is $k$-colorable.

The minimum number of bipartite graphs required to cover the edge set of any graph $H$ with chromatic number $\chi(H)$ is $\lceil \log \chi(H) \rceil$ [20]. This result is part of folklore. One of the classical theorems that study an associated problem is the Katona-Szemerédi theorem [26]. Katona-Szemerédi theorem [26] states that the minimum of the sum of the orders of a collection of bipartite graphs that cover the edge set of a complete graph on $n$ vertices is $n \log n$. Hansel [19] provides an alternate proof for Katona-Szemerédi theorem. Mubayi and Vishwanathan [29] provides a generalization of the theorem by showing that the sum of the orders of any collection of complete bipartite graphs that cover the edge set of $G$ is at least $k \log k - k \log \log k - k \log \log \log k$ for any graph $G$ with chromatic number $k$. For extending this theorem to $r$-uniform hypergraphs we require Jensen’s inequality. Jensen’s inequality states that

\begin{align*}
\text{Lemma 1.} \quad & \text{Let } f \text{ be a convex function of one real variable. Let } x_1, \ldots, x_n \in \mathbb{R} \text{ and let } c_1, \ldots, c_n \geq 0 \text{ satisfy } c_1 + \cdots + c_n = 1. \text{ Then } f(c_1 x_1 + \cdots + c_n x_n) \leq c_1 f(x_1) + \cdots + c_n f(x_n). \\
\text{Lemma 2.} \quad & \text{Let } H \text{ be an } r\text{-uniform hypergraph with } n \text{ vertices with independence number } \alpha. \text{ Then the sum of the orders of any collection of complete } r\text{-partite } r\text{-graphs that cover the edge set of } H, b_r(H) \geq (r-1) n \log \left( \frac{n}{\alpha} \right).
\end{align*}

Proof. Consider a collection of complete $r$-partite $r$-graphs that cover the edge set of the $r$-uniform hypergraph. Let vertex $i$ be present in $a_i$ complete $r$-partite $r$-graphs in the collection. Uniformly at random pick one part each from every complete $r$-partite $r$-graph and remove all the vertices present in the part. The probability that a vertex $i$ is not removed from any of the complete $r$-partite $r$-graphs is $(\frac{1}{r} - \frac{1}{r})^{a_i}$. Hence the expected size of the resulting subset of vertices is $\sum_{i=1}^n \left(1 - \frac{1}{r}\right)^{a_i}$. Note that this is an independent set. Since the independence number is $\alpha$, we have

\begin{equation}
\sum_{i=1}^n \left(1 - \frac{1}{r}\right)^{a_i} \leq \alpha
\end{equation}

$F(x) = (1 - \frac{1}{r})^x$ is a convex function for fixed $r$, since $F'(x) = -(1 + \frac{1}{r-1})^{-2} \ln(1 + \frac{1}{r-1})$ and $F''(x) = (1 + \frac{1}{r-1})^{-3} \ln^2(1 + \frac{1}{r-1}) > 0$. Applying Jensen’s Inequality with $c_1 = \alpha$,...
\[ c_2 = \cdots = c_n = \frac{1}{n} \text{ and since } \sum_{i=1}^{n} a_i = b_r(H) \text{ we have,} \]
\[
\frac{1}{1 + \frac{1}{r-1}} \sum_{i=1}^{n} \frac{1}{a_i} \leq \sum_{i=1}^{n} \frac{1}{1 + \frac{1}{r-1}} a_i
\]
\[
\frac{n}{1 + \frac{1}{r-1}} \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} \frac{1}{1 + \frac{1}{r-1}} a_i
\]
\[
n \left[ 1 - \frac{1}{r} \right] \frac{b_r(H)}{n} \leq \sum_{i=1}^{n} \frac{1}{1 + \frac{1}{r-1}} a_i = \sum_{i=1}^{n} \left[ 1 - \frac{1}{r} \right] a_i
\]

From equations (1) and (2) we have
\[
n \left[ 1 - \frac{1}{r} \right] \frac{b_r(H)}{n} \leq \alpha
\]
Taking log and using \((1 + \frac{1}{r})^x \leq e\) we have,
\[
\log n \leq \log \alpha + \frac{b_r(H)}{n} \log \left( \frac{r}{r-1} \right)
\]
\[
n \log n - n \log \alpha \leq b_r(H) \log(1 + \frac{1}{r-1})
\]
\[
n \log n - n \log \alpha \leq b_r(H) \cdot \frac{1}{r-1}
\]
\[
b_r(H) \geq (r-1)n \log \left( \frac{n}{\alpha} \right)
\]

Equivalently,
\[
\alpha \geq \frac{n}{2^{(r-1)n} b_r(H)}
\]

**Theorem 1.** Let \( H = (V,E) \) be an \( r \)-uniform hypergraph on \( n \) vertices with chromatic number \( k \), where \( k \) is sufficiently large. The sum of the orders of any collection of complete \( r \)-partite \( r \)-graphs that cover the edge set of \( H \) is at least \((r-1)^2k \log k(1-o(1))\).

**Proof.** Let the chromatic number of \( H, \chi(H) = k \). For \( n \leq (r-1)k \log k \), we are done by Lemma (2). Hence we may assume that \( n \) is less than \((r-1)k \log k\). Let \( H = H_0 \). Starting with \( H_0 \), repeatedly remove independent sets of size given by Lemma (2), as long as the number of vertices is at least \((r-1)k \). Let \( H_i \) denote the \( i \)-th graph in the sequence and let \( H_t \) denote the last graph in the sequence. Let \(|V(H_i)| = n_i \) and \( (1 - \frac{1}{p}) \) denote the maximum rate at which \( n_i \) fall, over all \( i \). That is, \( \beta = \max \left[ 2^{(r-1)\log \beta} \right] \). Let this maximum be achieved for \( i = p \). From the definition, we see that \( n_{i+1} = n_i - \frac{n_i \beta}{2^{(r-1)\log \beta}} \).
\[ n_i \left[ 1 - \frac{1}{2^m} \right] \]. Hence \( n_i \leq n(1 - \frac{1}{2^m})^\ell < ne^{-\frac{\ell}{2}} \). Using the fact that \( n_i \geq (r-1)k \) and \( n < (r-1)k \log k \), we obtain \( (r-1)k < ne^{-\frac{\ell}{2}} \) that is \( t < \beta \log \left[ \frac{n}{(r-1)k} \right] \). Using the facts that \( n > (r-1)k \log k \) we have, 

\[ b_r(H_p) \geq (r-1)n_p \left[ \log k - \log k \log k - \log \log k \right] \]

Using the facts that \( n_p > (r-1)k \) we have, 

\[ b_r(H_p) \geq (r-1)^2 k \left[ \log k - \log k \log k - \log \log k \right] \]

We now consider the case that \( t \) is less than \( \frac{k}{\log k} \). Let \( H' \) be the hypergraph obtained after removing an independent set from \( H_t \) as mentioned in Lemma (2). By definition of \( t \), we have \( |V(H')| < (r-1)k \). Also \( \chi(H') \geq k - \frac{k}{\log k} \).

Consider an optimal coloring of the hypergraph \( H' \). Order the vertices such that the vertices in color class \( i \) precede the vertices in color class \( (i+1) \). Let \( C \) denote a greedy coloring of the vertices of \( H' \) considered in this order. Note that \( C \) has at most one color class with strictly less than \( (r-1) \) vertices since we color \( H' \) greedily.

Let \( p \) be the number of vertices present in the set of \( (r-1) \) sized color class and \( q \) be the rest of the elements. Counting the number of elements in \( H' \) we have, 

\[ p + q < (r-1)k \]

Counting the number of color classes we have, 

\[ 1 + \frac{p}{r-1} + \frac{q}{r} \geq \chi(H') \geq k - \frac{k}{\log k} \]

From the above two inequalities we have, 

\[ 1 + \frac{p}{r-1} + \frac{(r-1)k - p}{r} \geq k - \frac{k}{\log k} \]

\[ 1 + \frac{p}{r(r-1)} + \frac{(r-1)k}{r} \geq k - \frac{k}{\log k} \]

\[ 1 + \frac{p}{r(r-1)} \geq k - \frac{k}{\log k} \]

\[ r(r-1) + p \geq (r-1)k - \frac{r(r-1)k}{\log k} \]

\[ p \geq (r-1)k[1 - \frac{r}{\log k}] - r(r-1) \]
Consider the sub-hypergraph $H''$ spanned by the vertices of the $(r - 1)$ sized color classes in $H'$. Note that the color classes are ordered. Let $a_{i,j}$ be the $i$-th vertex in $j$-th $(r - 1)$ sized color class. Note that each vertex $a_{i,j}$ forms an edge with the $k$-th $(r - 1)$ sized color class for all $k$ where $k < j$ since coloring $C$ is a greedy coloring. Note that $H''$ has $\frac{p}{r-1}$ color classes in the optimal coloring. Suppose $H''$ has an independent set of size at least $2(r - 1)$, then color that independent set with one color and color the rest of the vertices with strictly less than $\frac{p}{r-1} - 2$ colors contradicting the assumption that $C$ is an optimal coloring. Therefore the size of the independent set in $H''$ is at most $2(r - 1)$. Applying lemma 2 for $H''$, we have

$$b_r(H'') \geq (r-1)p \log \left( \frac{p}{2(r-1)} \right)$$

Since $p \geq (r-1)k[1 - \frac{r}{\log k}] - r(r-1)$ we have,

$$p \geq (r-1)k - r(r-1)\frac{k}{\log k} - r(r-1)$$

$$\geq (r-1)k - r(r-1)\left[ \frac{k}{\log k} + 1 \right]$$

$$\geq (r-1)k - \frac{2r(r-1)k}{\log k}$$

Therefore,

$$b_r(H'') \geq (r-1)p \log \left( \frac{p}{2(r-1)} \right)$$

$$\geq (r-1)\left[ (r-1)k - \frac{2r(r-1)k}{\log k} \right] \log \left[ \frac{k}{2} - \frac{rk}{\log k} \right]$$

$$\geq (r-1)^2k - \frac{2r(r-1)^2k}{\log k} \log \left[ \frac{k}{2} - \frac{rk}{\log k} \right]$$

$$\geq (r-1)^2k \log \left[ \frac{k}{2} - \frac{rk}{\log k} \right] - 2r(r-1)^2k \log \left[ \frac{k}{2} - \frac{rk}{\log k} \right]$$

$$\geq (r-1)^2k \log \left[ k\left( \frac{1}{2} - \frac{r}{\log k} \right) \right] - 2r(r-1)^2k$$

$$\geq (r-1)^2k \left( \log k + \log \left( \frac{\log k - 2r}{2\log k} \right) \right) - 2r(r-1)^2k$$

$$\geq (r-1)^2k \left( \log k - \log \log k - 1 \right) - 2r(r-1)^2k$$

}\]
4 Covering number and Matching number

The covering number of an \( r \)-uniform hypergraph \( H \) is the minimum number of complete \( r \)-partite \( r \)-graphs required to cover all the edges of the hypergraph \( H \) at least once. The covering number of \( r \)-uniform hypergraph \( H \) is denoted by \( bc_r(H) \). A matching in a hypergraph \( H \) is a subset of edges of the edge set of the hypergraph \( H \) such that every two edges in the subset are disjoint. A matching of \( m \) vertex disjoint edges of a hypergraph \( H \) is called an \( m \)-matching. The matching number of a hypergraph \( H \), denoted by \( \nu(H) \) is the number of edges of a maximum matching in the hypergraph \( H \).

In this section we give results that relate the matching number and covering number of hypergraphs. This generalizes the results of Jukna and Kulikov [25]. For this we require Holder’s inequality. Holder’s inequality states that

Lemma 3. [21] Let \( a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \) be positive real numbers and \( p, q > 1 \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \), then \( \sum_{i=1}^{n} a_i b_i \leq (\sum_{i=1}^{n} a_i^p)^{\frac{1}{p}} (\sum_{i=1}^{n} b_i^q)^{\frac{1}{q}} \).

Lemma 4. For every non-empty hypergraph \( H = (V,E), bc_r(H) \geq \frac{\nu(H)^{1+\frac{1}{r}}}{|E|^{\frac{1}{r}}} \), where \( \nu(H) \) is the matching number of \( H \).

Proof. Let \( M \subseteq E \), be a matching with \( m \) vertex-disjoint edges. Let \( E = R_1 \cup R_2 \cup \cdots \cup R_t \) be a covering of the edges of \( H \) by \( t = bc_r(H) \) complete \( r \)-partite \( r \)-graphs of \( H \). Consider the function \( f : M \rightarrow \{1,2,\cdots,t\} \) by \( f(e) = \min \{ i \mid e \in R_i \} \), and let \( M_i = \{ e \in M \mid f(e) = i \} \). So \( M_i \) contains those edges of the matching \( M \) covered by the \( i \)th complete \( r \)-partite \( r \)-graph for the first time. Let \( F_i \subseteq R_i \) be the complete \( r \)-partite \( r \)-graph spanned by the vertices of matching \( M_i \), so each \( F_i \) is a complete \( r \)-partite \( r \)-graph such that each part contains \( r \) vertices, where \( r_i = |M_i| \) is the number of edges in the \( i \)th matching \( M_i \). Since the complete \( r \)-partite \( r \)-graphs are vertex disjoint we have, \( r_1 + r_2 + \cdots + r_t = |M| = m \) and \( r'_1 + r'_2 + \cdots + r'_t = |F| \).

By the Holder’s inequality we have,

\[
\sum_{k=1}^{t} r_k \leq (\sum_{k=1}^{t} r'_k)^{\frac{1}{r}} \cdot (\sum_{k=1}^{t} 1)^{1-\frac{1}{r}}
\]

\[
m \leq |F|^\frac{1}{r}, \quad \frac{1}{r} \leq 1
\]

\[
t \geq \left( \frac{m}{|F|} \right)^{\frac{1}{r} - \frac{1}{r}}
\]

\[
t \geq m^{\frac{1}{r}} \cdot \frac{1}{|F|^{\frac{1}{r}}}
\]

\[
\geq m^{\frac{1}{r}} \cdot \frac{1}{|E|^{\frac{1}{r}}}
\]

\[\square\]
For the complete $r$-partite $r$-graph $H = (V_1 \cup V_2 \cup \cdots \cup V_r, E)$ with unequal parts that is $V_1 < V_2 \leq \cdots \leq V_r$ or $V_1 = \cdots = V_i < \cdots \leq V_r$ the above bound can be improved. Suppose the subgraph spanned by the vertices of $M_1$ be $(U_1^{(1)} \cup U_2^{(1)} \cup \cdots \cup U_r^{(1)}, E')$ and that of $M_2$ be $(U_1^{(2)} \cup U_2^{(2)} \cup \cdots \cup U_r^{(2)}, E'')$. Two matchings $M_1, M_2$ are independent if $E' \cap E'' = \emptyset$.

**Lemma 5.** If a complete $r$-partite $r$-graph $H = (V_1 \cup \cdots \cup V_r, E)$ contains $k$ pairwise independent $m$-matchings, then $bc_r(H) \geq \frac{k \Gamma \left( \frac{1}{r} \right) m^{1-\frac{1}{r}}}{|E|^{1-\frac{1}{r}}}$. 

**Proof.** Suppose $M_1, \cdots, M_k \subseteq E$ are independent $m$-matchings. Consider the $k$ subgraphs $H_1, \cdots, H_k$ of $H$ induced by their vertex sets. Since the matchings are independent the induced subgraphs are edge disjoint. Applying Lemma 2, for each of the subgraphs we get $bc_r(H_i) \cdot |E_i|^{1-\frac{1}{r}} \geq m^{1-\frac{1}{r}}$, where $E_i$ is the set of all edges of $H_i$. Since each $H_i$ is an induced subgraph, its covering number is at most that of $H$. Summing over all $i$ we get $bc_r(H) \cdot \sum_{i} |E_i|^{1-\frac{1}{r}} k^{1-\frac{1}{r}} \geq bc_r(H) \left( |E_1|^{1-\frac{1}{r}} + \cdots + |E_k|^{1-\frac{1}{r}} \right) \geq \frac{k \cdot m^{1-\frac{1}{r}}}{\Gamma \left( \frac{1}{r} \right)}$. \(\square\)

## 5 Relation between covering number and partition number in $r$-uniform hypergraphs

The bi-clique covering number and bi-clique partition number have been studied extensively in [29] [25]. The Graham-Pollak theorem states that the bi-clique partition number of a complete graph on $n$ vertices is $n - 1$. It is folklore that the bi-clique covering number of a complete graph, $bc(K_n)$ is $\lceil \log n \rceil$. Since any bi-clique partition of a graph serves as a bi-clique cover of the graph, $bc(G) \leq f_r(G)$. For $K_n$, the bi-clique partition number is at least $2^{\log \left( K_n \right)} = 1$. So, it is natural to ask how large the bi-clique partition number would be compared to the bi-clique cover number. In fact, Pinto [33] proves that $f_2(G) \leq 2^{\log \left( K_n \right)} - \frac{1}{2}$ and also provides an example of a graph with the bi-clique partition number $\frac{2^{\log \left( K_n \right)} - 1}{2}$.

In this section, we generalise the result of Pinto, by providing bounds for the complete $r$-partite $r$-graph partition number in terms of the complete $r$-partite $r$-graph covering number for $r$-uniform hypergraphs. For this, we extend the definitions of the subcube intersection graph representation to hypergraphs.

We begin with a technical lemma.

**Lemma 6.** The solution for the recurrence relation $n_r = r \cdot n_{r-1} + 1$ with base condition $n_1 = 1$ is $n_r = e \cdot \Gamma(r + 1, 1) - \Gamma(r + 1)$, where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the upper incomplete gamma function and $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ is the gamma function.

**Proof.** The proof proceeds by induction. For the base case $n_2 = 2 \cdot n_1 + 1$. Using the fact that $n_1 = 1$ we have $n_2 = 3$. 

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By definition of the upper incomplete gamma function, we have

$$\Gamma(3, 1) = \int_1^\infty t^2 e^{-t} dt$$

Integrating $t^2 e^{-t} dt$ without the bounds, we have

$$= -t^2 e^{-t} - \int -2t e^{-t} dt \quad \text{(Integrating by parts)}$$

$$= -t^2 e^{-t} - \left[ -2(-te^{-t} - \int -e^{-t} dt) \right] \quad \text{(Integrating by parts)}$$

$$= -t^2 e^{-t} - \left[ -2(-te^{-t} - e^{-t}) \right]$$

$$= \left[ -(t^2 - t - 2)e^{-t} + c \right]$$

Reapplying the bounds, we have

$$\Gamma(3, 1) = \left[ -(t^2 - t - 2)e^{-t} + c \right]_1^\infty$$

$$= \frac{5}{e}.$$

Using $\Gamma(3, 1) = \frac{5}{e}$ and the identity $\Gamma(z) = (z - 1)!$ for positive integer $z$ we have $n_2 = e \cdot \Gamma(3, 1) - \Gamma(3) = 3$.

Assume the recurrence holds for $k < r$, we prove that the recurrence holds for $k = r$.

$$n_r = r \cdot n_{r-1} + 1$$

$$= r \left[ e \cdot \Gamma(r, 1) - \Gamma(r) \right] + 1$$

$$= r \cdot e \cdot \Gamma(r, 1) - r \cdot \Gamma(r) + 1 \quad \text{(Since $n_{r-1} = e \cdot \Gamma(r, 1) - \Gamma(r)$)}.$$

$$= r \cdot e \cdot \left[ \frac{\Gamma(r+1, 1)}{r} - \frac{1}{r^2} \right] - r \cdot \Gamma(r) + 1, \quad \text{(Using the identity $\Gamma(s+1, x) = s \cdot \Gamma(s, x) + x^s e^{-x}$)}$$

$$= e \cdot \Gamma(r+1, 1) - 1 - r \cdot \Gamma(r) + 1$$

$$= e \cdot \Gamma(r+1, 1) - \Gamma(r+1).$$

Consider a graph $G$ with $bc_r(G) = m$. Our aim is to produce an efficient partition of the edges of $G$. In order to do this we first define an $r$-uniform hypergraph $C_m^r$ and show how to efficiently partition its set. We then use this to show how to efficiently partition the hypergraph having $G$ as an induced hypergraph.
For a fixed $m$ and $r$, we will consider an $m$-tuple where each coordinate takes values from the set $\{0, 1, \cdots, r-1, *\}$. Fixed coordinates are those coordinates that take values from $\{0, 1, \cdots, r-1\}$ and free coordinates are those coordinates that take the value $*$. Note that we will use these $m$-tuple to represent the vertices of a hypergraph.

Consider an $r$-uniform hypergraph, $G_r^m$ with the vertex set $\{0, 1, \cdots, r-1, *\}^m$ and edge set consisting of all subsets of $r$ vertices such that there is at least one coordinate of the representation of vertices where it all differs and are fixed. Consider a complete $r$-partite $r$-graph $H$ with vertex classes $\alpha_1, \alpha_2, \cdots$ and $\alpha_r$. It is represented by $H(\alpha_1, \alpha_2, \cdots, \alpha_r)$. For $H$ a subhypergraph of $G$ define for $x \in \{0, 1, \cdots, r-1, *\}$, the following subsets of $\{0, 1, \cdots, r-1, *\}^{m+1}$:

$$x\alpha_i = \{(x, v_1, v_2, \cdots, v_m) | (v_1, v_2, \cdots, v_m) \in \alpha_i\}.$$ 

For a fixed complete $r$-partite $r$-graph $H(\alpha_1, \alpha_2, \cdots, \alpha_r)$ with vertex set subset of $\{0, 1, \cdots, r-1, *\}^m$, define the $r$-uniform hypergraphs $A_{m+1}^r$, $B^r_{m+1}$ and $C^r_{m+1}$ respectively as follows. Let $A_{m+1}^r(H)$ be the complete $r$-partite $r$-graph with the vertex set consisting of all vertices represented by $x\alpha_i$ for $x \in \{0, 1, \cdots, r-1, *\}$ and $1 \leq i \leq r$ such the $i$-th part consists of all vertices represented by $x\alpha_i$ for $x \in \{0, 1, \cdots, r-1, *\}$ for $1 \leq i \leq r$. Let $B^r_{m+1}(H)$ be the $r$-uniform hypergraph with the vertex set consisting of all vertices represented by $x\alpha_i$ for $x \in \{0, 1, \cdots, r-1, *\}$ and $1 \leq i \leq r$ and edge set containing all edges of $r$ vertices $(x_1, v_{1,1}, v_{1,2}, \cdots, v_{1,m}), (x_2, v_{2,1}, v_{2,2}, \cdots, v_{2,m}), \cdots, (x_r, v_{r,1}, v_{r,2}, \cdots, v_{r,m})$ for $(v_{i,1}, v_{i,2}, \cdots, v_{i,m}) \in \alpha_i$ and $x_j \in \{0, 1, \cdots, r-1\}$ such that $|\{x_1, x_2, \cdots, x_r\}| = r$. Let $C^r_{m+1}(H)$ be the $r$-uniform hypergraph with the vertex set consisting of all vertices represented by $x\alpha_i$ for $x \in \{0, 1, \cdots, r-1, *\}$ and $1 \leq i \leq r$ and the edge set $A_{m+1}^r(H) \setminus B^r_{m+1}(H)$.

**Claim 1.** For any fixed complete $r$-partite $r$-graph $H(\alpha_1, \alpha_2, \cdots, \alpha_r)$ with vertex classes $\alpha_1, \alpha_2, \cdots$ and $\alpha_r$ with vertex set subset of $\{0, 1, \cdots, r-1, *\}^m$, the edge set of the corresponding $r$-uniform hypergraph $C^r_{m+1}$ can be partitioned into a collection of $[(e-1)r!]$ complete $r$-partite $r$-graphs.

**Proof.** The proof proceeds by induction on $r$. We maintain a lexicographic order of $0 < 1 < 2 < \cdots < r-1 < *$. Let $n_i$ represent the number of complete $i$-partite $i$-graphs that partitions the edge set of $C_{m+1}^i$ with $x_j\alpha_i$ as their first part for $x_j \in \{0, 1, \cdots, i-1, *\}$.

For the base case consider any biclique $(\alpha_1, \alpha_2)$ with vertex classes $\alpha_1$ and $\alpha_2$. Consider the corresponding 2-uniform hypergraph $C_{m+1}^2$. The following collection of three bicliques partitions the edge set of $C_{m+1}^2$ with $x_j\alpha_1$ as their first part for $x_j \in \{0, 1, *\}$.

| 0\alpha_1 | 0\alpha_2 | 1\alpha_1 | 1\alpha_2 | *\alpha_1 | 0\alpha_2 | 1\alpha_2 | *\alpha_2 |
|------------|------------|------------|------------|------------|------------|------------|------------|
|            |            |            |            |            |            |            |            |

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Assuming there is a collection of \( n_{r-1} \) complete \((r-1)\)-partite \((r-1)\)-graphs that partitions the edge set of \( C_{m+1}^{r-1} \) with \( x_j \alpha_1 \) as their first part for \( x_j \in \{0, 1, 2, \cdots, r-2, *\} \), we construct the collection of \( n_r = r \cdot n_{r-1} + 1 \) complete \( r \)-partite \( r \)-graphs that partitions the edge set of \( C_{m+1}^r \) with \( x_j \alpha_1 \) as their first part for \( x_j \in \{0, 1, 2, \cdots, r-1, *\} \) using the collection of \( n_{r-1} \) complete \((r-1)\)-partite \((r-1)\)-graphs.

In order to obtain the the collection of complete \( r \)-partite \( r \)-graphs that partition the edge set of \( C_{m+1}^r \) containing \( 0 \alpha_1 \), we relabel the vertex classes and the set of values taken by \( x_j \) for the complete \((r-1)\)-partite \((r-1)\)-graphs that partition \( C_{m+1}^{r-1} \). The vertex classes of each of these complete \((r-1)\)-partite \((r-1)\)-graphs are relabelled to \((\alpha_2, \alpha_3, \cdots, \alpha_r)\) and the values taken by \( x_j \) to \( \{1, 2, \cdots, r-1, *\} \). After relabelling, for those complete \((r-1)\)-partite \((r-1)\)-graphs with \( x_j \alpha_2 \) as the first vertex class for \( 1 \leq x_j \leq r-1 \) add \( 0 \alpha_2 \) to \((i-1)\)-th vertex class for \( 3 \leq i \leq r \) and finally add \( 0 \alpha_1 \) as the new first part. For those complete \((r-1)\)-partite \((r-1)\)-graphs with \( x_j \alpha_2 \) as the first vertex class add \( 0 \alpha_2 \) to \( i \)-th vertex class for \( 2 \leq i \leq r \) and finally add \( 0 \alpha_1 \) as the new first part.

In a similar way the collection of complete \( r \)-partite \( r \)-graphs that partition the edge set of \( C_{m+1}^r \) containing \( x_j \alpha_1 \) can be obtained for \( 1 \leq x_j \leq r-1 \). In order to obtain the the collection of complete \( r \)-partite \( r \)-graphs that partition the edge set of \( C_{m+1}^r \) containing \( x_j \alpha_1 \) as their first part and \( x_j \alpha_2 \) for \( x_j \in \{0, 1, \cdots, r-1, *\} \) and \( 2 \leq i \leq r \). Note that the collection of complete \( r \)-partite \( r \)-graphs thus constructed partitions the edge set of \( C_{m+1}^r \) and the number of complete \( r \)-partite \( r \)-graphs in the collection is \( n_r = r \cdot n_{r-1} + 1 \). By lemma 6 the recurrence solves to \( n_r = e \cdot \Gamma(r+1, 1) - \Gamma(r+1) \). Using the identity \( \Gamma(r+1, 1) = \frac{[e \cdot r!]}{e} \), the recurrence simplifies to \( n_r = [e \cdot r!] - r! = [(e - 1) r!] \).

The collection of ten complete 3-partite 3-graphs produced by the inductive construction described above that partitions the edge set of \( C_{m+1}^3 \) with \( x_j \alpha_1 \) as their first part for \( x_j \in \{0, 1, 2, *\} \) is as follows.
subhypergraphs, we have

\[
\begin{array}{ccccccc}
2\alpha_1 & 0\alpha_2 & 0\alpha_3 & 2\alpha_1 & 1\alpha_2 & 1\alpha_3 & 2\alpha_1 \\
*\alpha_3 & 2\alpha_3 & *\alpha_3 & 2\alpha_3 & 1\alpha_3 & 2\alpha_3 & 2\alpha_3 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
*\alpha_1 & 0\alpha_2 & 0\alpha_3 & 2\alpha_1 & 1\alpha_2 & 1\alpha_3 & 2\alpha_1 \\
1\alpha_2 & 1\alpha_3 & 2\alpha_2 & 2\alpha_3 & *\alpha_2 & *\alpha_3 & 2\alpha_3 \\
\end{array}
\]

Theorem 2. If \( bc_r(G) = m \), then \( f_r(G) \leq \frac{1}{((r-1)!r-1)}((e-1)!)^m - 1 \).

Proof. Fix an \( m \)-cover of \( G \). Let this \( m \)-cover be \( D_1, D_2, \ldots, D_m \). For each complete \( r \)-partite \( r \)-graph \( D_i \), label the vertex classes as class 0, 1, 2, \ldots, \( r-1 \). Each vertex of \( G \) can now be represented as an element of \( \{0, 1, 2, \ldots, r-1, *\}^m \), based on the membership of a vertex in the \( m \)-cover. For \( v \in V(G) \), define \( \tilde{v} \in \{0, 1, 2, \ldots, r-1, *\}^m \) by

\[
\tilde{v}_i = \begin{cases} 
0 & \text{if } v \text{ is in class 0 of } D_i, \\
1 & \text{if } v \text{ is in class 1 of } D_i, \\
2 & \text{if } v \text{ is in class 2 of } D_i, \\
\vdots & \text{\ldots,} \\
r-1 & \text{if } v \text{ is in class } r-1 \text{ of } D_i, \\
* & \text{otherwise.}
\end{cases}
\]

If \( \tilde{u} = \tilde{v} \) then \( u \) and \( v \) have the same neighbours. Therefore, if \( u \neq v \), we can replace \( G \) with \( G - u \) since \( f_r(G) = f_r(G - u) \). Therefore we can assume that all vertices have different representations. Note that complete \( r \)-partite \( r \)-graph partition number of an \( r \)-uniform hypergraph is at least that of its induced subhypergraphs. Consider the \( r \)-uniform hypergraph with vertices \( V(G) = \{0, 1, 2, \ldots, r-1, *\}^m \). There is an edge containing \( r \) vertices iff there is some \( i \)-th coordinate of the representation of \( r \) vertices, say \( \bar{u}^1, \bar{u}^2, \ldots, \bar{u}^r \) such that \( \{u_1^i, u_2^i, \ldots, u_r^i\} = \{0, 1, 2, \ldots, r-1\} \). We represent this graph as \( G_m \). Since the complete \( r \)-partite \( r \)-graph partition number of a hypergraph is at least that of the complete \( r \)-partite \( r \)-graph partition number of the induced subhypergraphs, we have \( f_r(G'_m) \geq f_r(G) \).

Let \( \pi'_m \) be an optimal partition of \( G'_m \). So, \( \pi'_m \) is \( bc_r(G) \)-cover of \( G \) since \( G \) is an induced subhypergraph of \( G'_m \). Let \( W \) be the complete \( r \)-partite \( r \)-graph with part \( i \) consisting of all vertices subset of \( \{0, 1, \ldots, r-1\}^{m+1} \) represented by \( (i, w_1, w_2, \ldots, w_m) \) for \( 0 \leq i \leq r-1 \).
Add $W$ to $\pi'_{m+1}$. For every $(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \pi'_m$, add the \[((e-1)r)!\] complete $r$-partite $r$-graphs that partition the edge set of the corresponding $C^r_{m+1}$ to $\pi'_{m+1}$.

**Claim 2.** $\pi'_{m+1}$ is a partition of $G^r_{m+1}$ and contains \[((e-1)r)! \cdot f_r(G^r_{m}) + 1\] complete $r$-partite $r$-graphs.

**Proof.** Let $u^i = (u^i_1, u^i_2, \ldots, u^i_{m+1})$, for $i = 1$ to $r$, be the $r$ vertices that are not joined by an edge of $G^r_{m+1}$. Therefore, there is no $j$ such that \{$(u^i_1, u^i_2, \ldots, u^i_j)$\} $\subseteq \{(0, 1, \ldots, r-1)\}$. So $u^i_1 u^i_2 \cdots u^i_j$ is not an edge of $W$.

Define $\bar{u}^i = (\bar{u}^i_1, \bar{u}^i_2, \ldots, \bar{u}^i_{m+1})$ with the first coordinates removed from $u^i$ for $i = 1$ to $r$. So $\bar{u}^1 \bar{u}^2 \cdots \bar{u}^r$ is not an edge of $G^r_{m}$. So $\bar{u}^1 \bar{u}^2 \cdots \bar{u}^r$ is not an edge in any of the complete $r$-partite $r$-graphs of $\pi'_m$. So by construction $u^1 u^2 \cdots u^r$ is not in any complete $r$-partite $r$-graph of $\pi'_{m+1}$.

Let $u^1_j u^2_j \cdots u^r_j$ be an edge of $G^r_{m+1}$. If $u^1_j u^2_j \cdots u^r_j$ is also an edge of $W$, it is contained in no other complete $r$-partite $r$-graphs of $\pi'_{m+1}$ because the edges covered by other complete $r$-partite $r$-graphs does not differ in their first coordinate. If $u^1_j u^2_j \cdots u^r_j$ is not an edge of $W$, then remove the first coordinate from each of the vertices $u^1, u^2, \ldots, u^r$ from $G^r_{m}$. Since $\pi'_m$ is a partition of $G^r_{m}$, $u^1, u^2, \ldots, u^r$ lies in exactly one complete $r$-partite $r$-graph $(\alpha_1, \alpha_2, \ldots, \alpha_r) \in \pi'_m$. Therefore by the construction, the edge $u^1_j u^2_j \cdots u^r_j$ must lie in exactly one of the corresponding complete $r$-partite $r$-graphs in $\pi'_{m+1}$.

As $f_r(G_1^r) = 1$ and $f_r(G^r_{m+1}) \leq \frac{((e-1)r)!}{(e-1)!} \cdot f_r(G^r_{m}) + 1$, solving the recurrence we have $f_r(G^r_{m}) \leq \frac{1}{(e-1)!} \cdot f_r(G^r_{m}) + 1$. Since $f_r(G) \leq f_r(G^r_{m})$ as established earlier we have $f_r(G) \leq \frac{1}{(e-1)!} \cdot (\frac{1}{(e-1)!} - 1)$.

Let $r$ be even. Consider an $r$-uniform hypergraph $H$. Define the adjacency matrix of $H$, denoted by $A_H$ as an $\binom{n}{r/2} \times \binom{n}{r/2}$ matrix, with rows and columns indexed by $r/2$ sized subsets of $[n]$, as follows:

$$A_H(e_1, e_2) = \begin{cases} 1, & e_1 \cup e_2 = E(H) \\ 0, & \text{otherwise} \end{cases}$$

Suppose the edges of the $r$-uniform hypergraph on $n$ vertices are covered by $d$ complete $r$-partite $r$-graphs, $U_i \equiv (U^i_1, U^i_2, \ldots, U^i_{r/2})$ for $1 \leq i \leq d$. Here $U^i_j$ are the parts of the complete $r$-partite $r$-graph. The edges of the hypergraph $U_i$ are obtained by taking one vertex from each part.

For each $i$, $1 \leq i \leq d$ and each $L \in \binom{[r/2]}{\frac{r}{2}}$, define a matrix $M(U_i, L)$ whose rows and columns are indexed by $\frac{r}{2}$ sized subsets as follows:
For $e_1, e_2 \in \binom{|V|}{2}$,

$$M(e_1, e_2) = \begin{cases} 
1, & \text{if } e_1 \in \bigcap_{i \in L} U_1^i \text{ and } e_2 \in \bigcap_{i \in [r]-L} U_1^i \\
0, & \text{otherwise.}
\end{cases}$$

Here $\bigcap_{i \in L} U_1^i = \{ e \in \binom{|V|}{2} : |e \cap U_1^i| = 1, \text{for } l \in L \}$.

The adjacency matrix of the complete $r$-partite $r$-graph $U_1$ denoted by $N(U_1)$ is equal to $\sum \lambda_i M(U_1, L)$. Let $P = \sum_{d=1}^r \sum_{L:1 \in L} M(U_1, L)$ and $Q = \sum_{d=1}^r \sum_{L:1 \not\in L} M(U_1, L)$. Here, $P = Q^T$. Since the $d$ complete $r$-partite $r$-graphs partition the edge set of the $r$-uniform hypergraph, we have the adjacency matrix of the $r$-uniform hypergraph, $A = \sum_{d=1}^r N(U_1) = \sum_{d=1}^r \sum_{L} M(U_1, L) = P + Q$. Therefore, by subadditivity of ranks we have $\text{rank}(A) \leq d \cdot (\frac{r}{2}) \cdot \text{rank}(M(U_1, L))$. Since $\text{rank}(M(U_1, L))$ is 1 for fixed $L$, we have $\text{rank}(A) \leq d \cdot (\frac{r}{2})$.

Consider $r$-uniform hypergraph $G_m$ defined earlier. Let $AD_m'$ be the adjacency matrix of $G_m'$, for even $r$.

**Lemma 7.** [24] The $k$-disjointness matrix $D = D(n, k)$ has full rank over $\mathbb{F}_2$.

**Proof.** Refer Theorem 13.10 (Page 187). □

**Lemma 8.** The rank of the adjacency matrix $AD_m'$ of the $r$-uniform hypergraph $G_m'$ is at least $\left(\frac{r}{2}\right) + 1$ for even $r \geq 4$.

**Proof.** Consider the adjacency matrix $AD_m'$ of $G_m'$, for even $r \geq 4$. The rows and columns of $AD_m'$ are indexed by $\binom{n}{2}$ sized set of the vertices of $G_m'$. Recall that the vertex set of $G_m'$ is $\{0, 1, \ldots, r-1, *\}^n$. So, $AD_m'$ is an $\binom{n}{2} \times \binom{n}{2}$ matrix where $n = (r+1)^m$. Suppose $u' = (u'_1, u'_2, \ldots, u'_m)$, for $i = 1$ to $r$ represent any $r$ vertices of $G_m'$, then by definition of $G_m'$ for $e_1 = \{u'_1, u'_2, \ldots, u'_m\}$ and $e_2 = \{u'_1', u'_2', \ldots, u'_m'\}$, we have

$$AD_m'(e_1, e_2) = \begin{cases} 
1, & \text{if } \exists j \text{ such that } \{u'_j, u'_2, \ldots, u'_j\} = \{0, 1, \ldots, r-1\} \\
0, & \text{otherwise}
\end{cases}$$

We prove the theorem by induction on $m$. Let the set $S$ be $\{0, 1, \ldots, r-1\}$. We define the following lexicographic order: $0 < 1 < \cdots < r-1 < *$. For the base case i.e. $m = 1$, consider the submatrix $N_1$ of $AD_1'$, formed by selecting the rows and columns indexed by $\frac{r}{2}$ sized set $\{u^1, u^2, \ldots, u^2\}$ such that $\{u^1, u^2, \ldots, u^2\} \in \binom{\frac{r}{2}}{2}$ and $u^1 < u^2 < \cdots < u^2$ and also by selecting as the final row and final column, the row and column indexed by the ordered tuple $\{u^1, u^2, \ldots, u^1\} = \{0, 1, \ldots, \frac{r}{2}-2, *\}$ respectively. By Lemma[7] the rank of this submatrix is $\binom{\frac{r}{2}}{2}$. Let $N_1'$ be the submatrix of $AD_1'$, formed by replacing the final row of $N_1$ with only 1’s. By Lemma[7] the rank of this submatrix is $\binom{\frac{r}{2}}{2} + 1$. 

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Now consider the submatrix $N'_m$ of $AD'_m$, formed by selecting the rows and columns indexed by $\xi$ sized set $\{u^1, u^2, \ldots, u^7\}$ such that $\forall j$ either $\{u^1_j, u^2_j, \ldots, u^7_j\} \in \left(\frac{\xi}{2}\right)$ or the ordered tuple $\{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, \frac{\xi}{2} - 2, *\}$ and $u^1_j < u^2_j < \cdots < u^7_j$. The matrix entries with fixed values for the rows indexed by $\{u^1_j, u^2_j, \ldots, u^7_j\}$ and the columns indexed by $\{u^1_j + 1, u^2_j + 2, \ldots, u^1_j\}$ are grouped together into blocks. So, in block matrix notation,

$$N'_m = \begin{cases} 1, & \text{if } \{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, r - 1\} \\ N'_{m-1}, & \text{otherwise} \end{cases}$$

Note that 1 and $N'_{m-1}$ are blocks. By simple row operations on $N'_m$, we get the following matrix.

$$\tilde{N}'_m = \begin{cases} 1 - N'_{m-1}, & \text{if } \{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, r - 1\} \\ N'_{m-1}, & \text{if } \{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, \frac{\xi}{2} - 2, *\} \\ 0, & \text{otherwise} \end{cases}$$

Define $\tilde{N}_m$ as the matrix formed by replacing the last row of the above matrix by a row of 1’s.

$$\tilde{N}_m = \begin{cases} 1 - N'_{m-1}, & \text{if } \{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, r - 1\} \\ \tilde{N}'_{m-1}, & \text{if } \{u^1_j, u^2_j, \ldots, u^7_j\} = \{0, 1, \ldots, \frac{\xi}{2} - 2, *\} \\ 0, & \text{otherwise} \end{cases}$$

By inductive hypothesis, we have $\text{span}(\tilde{N}'_{m-1}) = \text{span}(1 - N'_{m-1}) = (\frac{\xi}{2} + 1)^{m-1}$. So, the rank $\text{rank}(\tilde{N}'_m) = (\frac{\xi}{2}) \left[\text{span}(1 - N'_{m-1})\right] + \text{span}(\tilde{N}'_{m-1}) = (\frac{\xi}{2} + 1)^m$. Since $N'_m$ is a submatrix of $AD'_m$ we have $\text{rank}(A'_m) \geq \text{rank}(N'_m) = (\frac{\xi}{2} + 1)^m - 1$. □

**Theorem 3.** There is an $r$-uniform hypergraph $G$ with $bc(G) = m$ and $f_r(G) \geq \frac{(\frac{\xi}{2} + 1)^m - 1}{(\frac{\xi}{2})}$ for even $r \geq 4$ and $f_r(G) \geq \frac{(\frac{\xi}{2} + 1)^{m-1}}{(\frac{\xi}{2})}$ for odd $r \geq 3$.

**Proof.** Suppose $U_1, U_2, \ldots, U_d$ be a collection of complete $r$-partite $r$-graphs that form a partition of the $r$-uniform hypergraph $G'_m$. Then as established earlier, we have $\text{rank}(AD'_m) \leq d \cdot (\frac{\xi}{2})$. So, by applying lemma we have $f_r(G'_m) \geq \frac{\text{rank}(AD'_m)}{(\frac{\xi}{2})} \geq \frac{(\frac{\xi}{2} + 1)^m - 1}{(\frac{\xi}{2})}$.

For odd $r$ Since $f_r(G'_m) \geq f_{r-1}(G'_m)$, we have $f_r(G'_m) \geq \frac{(\frac{\xi}{2} + 1)^{m-1}}{(\frac{\xi}{2})}$.
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