A framework for topological music analysis (TMA)

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In the present article we describe and discuss a framework for applying different topological data analysis (TDA) techniques to a music fragment given as a score in traditional Western notation. We first consider different sets of points in Euclidean spaces of different dimensions that correspond to musical events in the score, and obtain their persistent homology features. Then we introduce two families of simplicial complexes that can be associated with chord sequences, and leverage homology to compute their salient features. Finally, we show the results of applying the described methods to the analysis and stylistic comparison of fragments from three Brandenburg Concertos by J.S. Bach and two pieces from the Graffiti series by Mexican composer Armando Luna.

Keywords: Topological data analysis (TDA); simplicial complexes; persistent homology; music analysis

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Introduction

In this work we present several ways of modelling data extracted from a digital music score, and discuss the results of applying some tools and techniques from algebraic topology (mainly simplicial homology) to music analysis. Our motivation is to incorporate new scopes and computational tools to music analysis, hoping they will contribute in establishing a theoretical and practical framework suitable for analysing music in a wide variety of languages and styles (ideally, in any language or style).

Usually, music is analysed through the lens of some very specific framework, such as the traditional Western tonal theory, the jazz modal harmonic setting, the combinatorics of dodecaphonic technique, the classical Indian music tradition, etc. There is no doubt they provide valuable and useful analytical techniques and terms to deal with the elements and processes which occur in their respective musical systems. Nevertheless, when trying to describe, within a single framework, musical objects and phenomena found in a diversity of repertoire or musical cultures, musicians and musicologists often find less complete and consistent methodological resources. Thus, for certain analytical and musicological purposes, it is desirable to work in a more general framework within which to speak about many types and styles of music, in equal terms, though this might necessarily lead to losing some of the fine details given by more particular analytical scopes.

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In recent years, there has been a wide range of works pointing in this direction, especially coming from applied mathematics, introducing general theoretical and methodological frameworks that include the use of concepts and tools from very different mathematical areas (see, for example: Agustín-Aquino et al. 2009; Andreatta 2003; Beran and Mazzola 1999; Bergomi 2015; Estrada 2011; Mazzola 2012; Padilla et al. 2017; Pareyón 2011; Szeto and Wong 2006; Tymoczko 2010). In the interest of contributing to this task, we focus on some mathematical models and tools that seem pertinent to describe data codified (or codable) in musical scores (computationally we will constrain ourselves to working with fragments written in traditional Western music notation, though this does not restrict us to Western music tradition or repertoire exclusively). Of particular interest to us is the idea of applying techniques of data analysis to musical information, and for the present article we work with techniques from topological data analysis (TDA). The application of algebraic topology concepts and techniques to describe harmonic structure in music has been treated in several works, for example: Bigo, Giavitto, and Spicher (2011), Sethares and Budney (2014), Bigo et al. (2013), Bigo and Andreatta (2016), Giavitto and Spicher (2016), Bergomi (2015), Bigo and Andreatta (2019), Bergomi and Baratè (2020), Sassone et al. (2022), and Jen-Yu, Shyh-Kang, and Yi-Hsuan (2016).

Through the application of these techniques, we seek to directly deal with the events noted in a score, without assuming any given system of relations between pitches or pitch class sets. In contrast with classical harmonic or Schenkerian analysis, there is no assumption of a particular or pre-established hierarchical harmonic nor formal system. Also, we may state that our scope differs from other methodologies which include some “geometrization” of musical data, such as the Tonnetz and its generalizations. Unlike these models, we do not assume any fixed structure of chords, and do not deal with voice-leading or chord-generation processes. We also do not properly deal with chords as defined in traditional music theory. Instead, we consider vertical events, that is, sets of pitches sounding simultaneously (as encoded in the score), which may actually incorporate two or more chords overlapping (as seen under a particular analytical framework such as tonality). These vertical events are determined by the appearance, disappearance or prolongation of notes in the score.

On one hand, we study the persistent homology (for a presentation on the subject, the reader may refer to the Appendix of this paper) of different sets of points formed from pitch and time data in a score. We consider vertical events both with and without their rhythm and onset position in the score, and continue to plot the homological features of all such simplicial complexes associated with fragments of different scores as barcodes and persistence diagrams (see the Appendix). We then compare such objects by using the bottleneck distance (also refer to the Appendix), which is a fairly standard way of measuring difference between persistence diagrams. Informally, we may say the distance between two diagrams $D, D'$ is the measure of the minimum “gap” through which all pairs of nearest corresponding points of $D$ and $D'$ can “pass through.” Finally, we summarize our results in dendrograms that show distances between diagrams obtained from several samples.

On the other hand, we propose two different ways of constructing sequences of simplicial complexes from chord progressions. One of these constructions coincides with similar scopes in the way of modelling a chord as a simplex on vertices corresponding to pitches or pitch classes (see, for example, Bigo et al. 2013), thus associating simplices on $n$ vertices with chords containing $n$ pitch classes. Yet a novelty introduced in this article is the homological description of complexes assembled with simplices obtained from different sequences of events, which may consist of any number of pitch classes. We also describe another way of associating a simplicial complex to a pitch class set, which encodes not only pitches but also intervals. Simplicial complexes associated with sequences of chords have been treated in a few previous works, such as the ones by Bergomi (2015) and Bergomi and Baratè (2020).
For the sake of space, we assume the reader is familiar with the basic post-tonal theory concepts (pitch and interval classes; normal form or order, and interval vector of a chord; see, for example Straus 2016). We present the basics of simplicial and persistent homology at the end of the paper, in the Appendix. For a deeper treatment of homology (homology groups, Betti numbers, Euler characteristic) the reader can refer to, for example, the classic books of Hatcher (2002) or Rotman (2013). Persistent homology concepts and methods (Vietoris-Rips complexes, barcodes and persistence diagrams, bottleneck distance) may be consulted in the survey by Edelsbrunner and Harer (2008).

1. Definitions and methods

In this section, some fundamental concepts and terminology are established and exemplified. Also, an outline of the general methodology is given.

1.1. Events and sequences of events

We consider music scores written in traditional Western notation. We define vertical events in a music score as tuples containing information of synchronous sounds, usually pitches or pitch classes, possibly together with some other features, such as its onset, duration (rhythm), dynamics (loudness) or timbre (instrumentation). In this work we focus on events given as tuples of pitch classes, with or without their duration and onset.

A vertical event given only by pitch information will be called a chord, and a chord formed by \( n \) different pitch classes will be referred to as an \( n \)-chord. A music fragment is a (usually assumed finite) sequence (ordered set) of vertical events. Given a fragment \( M = \{e_0, \ldots, e_N\} \), the interval of events \([e_i, e_j]\) is the sequence \( \{e_i, e_{i+1}, \ldots, e_j\} \).

This segmentation of events in the score does not take into account any information other than their sequential order. In the case of “continuous” (smooth) music passages or textures, one could identify starting points without the need of counting beats, and smooth fade-in/fade-out elements by a gradual discrete approximation. In any case, a digital measurable score is not absolutely necessary to apply the present model (though it actually could be produced). One may directly define the sequence of events \( e_0, e_1, \ldots \).

Working on sequences of events indexed by their order of occurrence allows us to easily focus merely in harmonic changes, and, when necessary, take into account time information (duration and onset, encoded as separate coordinates of an event). This implies that there may be chords appearing more than once in \( \mathcal{M} \), when they occur at different times (except in cases such as the example below, when a distinguished coordinate such as the onset is included). We will always take into account the index of a chord or event in the sequence being considered. Thus, technically we should write \( \mathcal{M} = \{(e_0, 0), (e_1, 1), \ldots, (e_N, N)\} \). This notation attempts to show the possible time-dependence of the consecutive occurrence of two or more chords, and helps us grasp harmonic progressions independently of the rhythmic values involved.

Given a music fragment \( \mathcal{M} = \{e_0, \ldots, e_N\} \), we define \( \mathcal{A}(\mathcal{M}) = \{a_0, a_1, \ldots, a_N\} \) as the sequence (ordered set) whose \( a_i \) term is the chord corresponding to the \( i \)th vertical event of \( \mathcal{M} \), \( e_i \). Chords in the set \( \mathcal{A}(\mathcal{M}) \) may be expressed in different ways: by their common name (C Major, d minor, F\# diminished, etc.), as a tuple of pitch classes (for example, in ascending order, or following the normal form of the chord), or as a tuple of intervals or interval classes.

\[1\] Here we mean smooth in the sense Boulez talks about smooth time in music, which he opposes to striated (“discrete”) time (see Boulez 1986), developed later by Deleuze and Guattari: “Boulez says that in a smooth space-time one occupies without counting, whereas in a striated space-time one counts in order to occupy” (see Guattari and Deleuze 1987, 477).
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(For example each chord may be expressed as a sequence of intervals starting at the lowest pitch (as in figured bass notation), or as an interval vector as defined in classic post-tonal theory).

1.2. Delimiting a particular musical setting for computation

For now we restrict ourselves to considering pitches in 12-tone equal temperament and more specifically, their corresponding pitch classes represented as elements of the set of integers modulo 12, \( \mathbb{Z}_{12} = \{ \bar{0}, \bar{1}, \ldots, \bar{11} \} \). This assumption allows us to show how these techniques can work in a fairly standard setting. Yet, the same methods can be applied in any other tempered, non-tempered or microtonal setting, making the proper straightforward adjustments to definitions dependant on the number of pitches or pitch classes considered (for example, the mappings defined in Section 1.4). We may think of our particular setting as a projection of the one involving all 1200 cents per octave. It is also worth mentioning that the possibility of dealing with microvalues of pitch and duration implies that the methodology described may be useful to analyse continuous music, including electronic music, as we could analyse discrete, measurable transcriptions of continuous musical textures.

The algorithm currently used to parse digital scores does not capture certain musical notations, such as grace notes, slurs (prolongation and phrasing), glissandi (unless explicitly written), tempo, metre, measure, and in general all text indications such as dynamics, playing mode, expressive marks, etc.

In order to conduct our harmonic analysis, all staves containing unpitched percussion instruments are removed from the scores. Time durations and positions are given in decimal quarter note values. This does not imply restricting ourselves to notes on integer quarter beats.

The quarter note value is only taken as a time unit; we could well establish the use of milliseconds, for instance. The specific measure units chosen either for time or pitch will determine the point clouds and simplicial complexes associated with musical events. Nevertheless, homeomorphic data encodings (change of units) lead to similar shape features under TDA analyses, being a topological invariant.

Let us exemplify the given definitions within the musical context established above:

Example 1.1 Let us consider the first two measures (see Figure 1) of Mexican composer Armando Luna’s Graffiti: Hommage to Franz Joseph Haydn (G.H. to F.J.H.), belonging to his Graffiti, a series of miniatures written for ensemble in 2006 as musical hommages dedicated to different composers. Each Graffiti is built upon the motif corresponding to the musical translation of the name of the composer to whom it is dedicated, according to the following letter equivalences:

\[
A=la, B=si, C=do, D=re, E=mi, F=fa, G=sol, H=si, S=mi.
\]

These pieces incorporate some of the most characteristic elements in the styles of composers Johann Sebastian Bach, Franz Joseph Haydn, Bela Bartók, Dave Brubeck, and seven others.

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2 In general, notes representing unpitched sounds could be included in the analysis, for instance assigning a numerical “pitch” value sufficiently distant from the ones representing actual pitches.
3 Armando Luna Ponce (1964–2015). Mexican composer born in the city of Chihuahua. He studied at the National Conservatory of Music of Mexico (where he later taught composition and music analysis) and the Carnegie Mellon University in the U.S.A. He mainly produced chamber and symphonic music for acoustic instruments, in what he came to name a ludic-eclectic-neorampageous style. Many of his pieces take the suite structure of brief movements as a model, incorporating Renaissance and Baroque dances from the European tradition as well as many other genres from both academic and folk origins. Also, a considerable amount of his works are hommages dedicated to different composers of the Western academic music canon, whose language and style are synthesized and reinterpreted.

4 The equivalences of letters A,B,…,H correspond to the usual German letter system for the notes of the diatonic scale. The equivalence S = mi also comes from the German “Es,” which stands for E and is pronounced like the letter “s.” It was used for example by Shostakovich to musically encode his name as the motif re-mi-do-si (D-S-C-H). See, for example DSCH motif in Wikipedia.
as seen and condensed by Luna through his own language. You may listen to the recording of *Graffiti* by the Present Music ensemble [here](#).

Below (see next page) is the sequence of the 28 events corresponding to the first two measures of Armando Luna’s *G.H. to F.J.H* (see the score in Figure 1), given as triads of the form

(normal-form vector, duration, onset).

We remind the reader that the normal form considered in this work is determined according to the 12-tone equal temperament, though it may be taken in a more general context. Time coordinates are expressed in decimal quarter note values.

\[
\mathcal{M} = \{(5, 9, 0), 0.5, 0.0), ((4, 8, 9, \Pi), 0.5, 0.5),
\]
\[
((10, \Pi, 3, 5, 6), 0.25, 1.0), ((10, \Pi, 3, 6), 0.25, 1.25),
\]
\[
((8, 9, 10, \Pi, 1, 2, 4, 5), 0.25, 1.5), ((9, 10, \Pi, 2, 5), 0.25, 1.75),
\]
We may associate several music fragments with the same score, as events may be described and encoded in several different ways, for example as collections containing only pitches, or adding some other relevant information encoded in the digital score.

An event is generated every time there is a change in the notes sounding simultaneously, as notated in the score. Vertical events are taken as sets of notes all with the same rhythmic value, and articulation (namely staccato) is not taken into account. If pitches last longer than the vertical event’s associated rhythmic value, some pitch classes will appear in the subsequent events in order to fill their actual duration according to the score.

Note for example that events 9 and 10, respectively starting at time positions 2.5 and 2.75, show the fact that chord \((9, \bar{6}, 3, \bar{4})\), where pitch class \(\bar{6}\) has rhythmic value 0.25 (a sixteenth note) in the score while the others last 0.5 quarters, is followed by \((9, 1, \bar{3}, \bar{4})\), with a rhythmic value of 0.25. Besides that, grace notes are considered as part of the chord they precede. We point out that the chord \((9, \bar{10}, \bar{1}, \bar{2}, \bar{5})\) is the only repeated chord in this sequence, and it appears twice: at the 6th and then at the 25th positions. Periodic decimal temporal values have been truncated.

From the above fragment \(\mathcal{M}\), we get

\[
\mathcal{A}(\mathcal{M}) = \{(\bar{9}, \bar{0}, \bar{1}, \bar{3}, \bar{4}), (\bar{9}, \bar{0}, \bar{1}, \bar{3}, \bar{4}), (\bar{0}, \bar{1}, \bar{3}, \bar{7}, \bar{10}), (\bar{0}, \bar{1}, \bar{3}, \bar{7}, \bar{10}), (\bar{6}, \bar{8}, \bar{9}, \bar{11}, \bar{2}), (\bar{6}, \bar{8}, \bar{9}, \bar{11}, \bar{2}), (\bar{5}, \bar{7}, \bar{10}), (\bar{5}, \bar{7}, \bar{10}), (\bar{4}, \bar{6}, \bar{8}, \bar{9}), (\bar{4}, \bar{6}, \bar{8}, \bar{9}), (\bar{3}, \bar{5}, \bar{6}), (\bar{3}, \bar{5}, \bar{6}), (\bar{2}, \bar{3}, \bar{6}, \bar{7}, \bar{10}), (\bar{2}, \bar{3}, \bar{6}, \bar{7}, \bar{10}), (\bar{1}, \bar{2}, \bar{4}, \bar{5}), (\bar{1}, \bar{2}, \bar{4}, \bar{5}), (\bar{0}, \bar{1}, \bar{3}, \bar{7}, \bar{10}), (\bar{0}, \bar{1}, \bar{3}, \bar{7}, \bar{10}), \}\)

In the above example we attached subscripts to pitch class tuples in order to keep track of their position in the corresponding sequence of events. Throughout the rest of the text, we will be omitting subindices of chords and events.

In the present work we describe several ways of encoding chords as tuples in different Euclidean spaces, for example

(interval vector of chord, duration, onset).
In general, we consider events given as tuples of the form

(chord as vector in some Euclidean space \( \mathbb{R}^n \), duration, onset),

or simply as vectors representing chords (without duration and onset coordinates). Observe that in order to use the Vietoris-Rips filtration for computing the persistent homology of a set of points, these must belong to the same metric space. Hence, tuples representing chords in each encoding must belong to the same \( \mathbb{R}^n \). This fact influences our definition of mappings associating Euclidean vectors to events.

1.3. General strategy

Given a music score, we extract its vertical events as tuples containing pitch classes in normal form, together with their duration and onset in the score (both in quarter notes). We also consider similar tuples in which chords are encoded as interval vectors (as defined in post-tonal theory). We point out that any of these representations of chords are a useful abstract standard for dealing with harmony in general (that is, outside the tonal context), but imply some loss of information, such as voicings and inversions of chords, as well as voice leading. To incorporate such aspects, we need some particular representation encoding them, for instance modelling chord connections rather than chords themselves. This leads to other embeddings and associated spaces, such as the connection simplicial complex mentioned in Section 3 (“Conclusions and future work”). All such events will be analysed both with and without time values (duration and onset); that is to say, we will be analysing chords with and without their rhythm and position in time.

One part of our analysis proposal involves classic TDA: musical events are encoded as points in some Euclidean space \( \mathbb{R}^n \) by means of different embeddings (which we describe in the next subsection), to later calculate their persistent homology under the Euclidean metric (using the Vietoris-Rips filtration; see Appendix).

On the other hand, we compute the homology of simplicial complexes directly associated with event intervals, without considering a metric among data points. We exhibit some cases in which these simplicial complexes actually form a filtration (though not associated with a metric).

A note about computations

All computations presented here were done in Python. We have used the Music21 library for parsing and extracting data from digital symbolic music files. For constructing simplicial complexes, computing their homologies, generating the corresponding barcodes and persistence diagrams, as well as for calculating the bottleneck distances between them, we employed algorithms from MoguTDA, Ripser, and Persim (the latter two incorporated as modules of the Scikit-TDA library)\(^5\). We also made use of some standard libraries for mathematics and plotting, namely NumPy, SciPy, and Matplotlib. The scripts used for computations in this paper may be consulted in this github project.

We work with digital symbolic music files corresponding to musical scores. The filetypes acceptable are those supported by Music21 and include MIDI, .XML and .MXL files (Music21 supports many other formats). After parsing these digital scores in Python, different lists of meaningful musical data are generated for a given fragment.

Each of these lists defines a musical data mapping. In our case, we obtain lists of vertical events given as tuples consisting of the normal form or interval vector of a chord, possibly

\(^5\) It has been suggested by one of the reviewers of the present paper that perhaps the Cgal C++ library might be computationally more efficient for persistent homology calculations. This computational geometry project is at the time of submission unknown to the authors, who shall definitely try it out for future work.
including its duration (given in quarter notes) and onset (position in time from the beginning of
the score, also given in quarter notes). These tuples are the raw data analysed by the persistent
homology algorithms, after which we generate and plot the corresponding persistence diagrams
for different encodings of data. Besides calculating the persistent homology of the Vietoris-Rips
filtration constructed from a point cloud in some $\mathbb{R}^n$, we also compute and plot the homological
features of two families of simplicial complexes associated with chord sequences in the score.

1.4. Persistent homology on various musical data mappings

In this section we describe several sets of data points (in different Euclidean spaces) associated
with a given music fragment $\mathcal{M}$. Concretely, we distinguish six different ways of generating
vectors in Euclidean spaces of different dimensions, from the vertical events in $\mathcal{M}$. We refer
to these sets of tuples and the functions generating them as data mappings (as not to confuse
them with data sets, which in the data science context would usually be understood as the sets of
samples analysed). We enumerate mappings from I through VI.

The first two mappings we consider (I and II) consist of points that stand for vertical events in
a fragment $\mathcal{M}$ and incorporate time data: each event is translated as a tuple of pitch or interval
classes, together with a rhythm (duration) and an onset coordinates, both measured in quarter
notes (see Example 1.1). Such encodings of events exemplify a plausible general way of treating
music information from the viewpoint of data analysis: assigning numerical coordinates to data
features, which may include variables in different scales. The rest of the mappings (III, IV, V,
and VI) contain only pitch and/or interval information from each event, that is, they focus only
on harmonic aspects of $\mathcal{M}$. It is worth pointing out that embeddings of pitch sets without time
coordinates do not keep track of repeated chords, as these are mapped to the same point in the
corresponding $\mathbb{R}^n$.

After generating data point clouds with each mapping, we run a persistent homology analysis
upon them, and plot the resulting persistence diagrams or barcodes. Then we compare persistence
diagrams coming from different scores in each mapping by using the bottleneck distance, which
is a standard tool for such task. Thus, we are able to depict in dendrograms these distances
among corresponding data mappings from such examples. This lets us establish a certain notion
of closeness between pieces and styles. It is important to note that the persistent homology of
all mappings except I and V remains invariant under transposition of the score by any interval;
that is, the bottleneck distance between persistence diagrams of a fragment and its transpositions
is 0 in all other mappings (II, III, IV, and VI). So data mappings I and V are the ones actually
measuring the “tonality,” “tonic” or current transposition of the score in question.

To illustrate the persistence diagrams obtained from each mapping, we work upon the same
fragment treated in Example 1.1, consisting of the two measures shown in Figure 1. Colors in
persistence diagrams correspond to different dimensions (computed here up to dimension 3, due
to computational time$^6$). For barcode plots we use two different colors: teal for data mappings
which include time information (mappings I and II), and dark purple for the rest of mappings.

- **Data mapping I**: We begin considering a set of points or vectors in $\mathbb{R}^{14}$ whose first 12 coor-
dinates correspond to pitch classes, followed by rhythm and an onset values expressed in
quarter notes. In these vectors, coordinates representing pitch classes of a chord are repre-
sented by the integers 12, 13, . . . , 23, according to their ordering in the chord’s normal form.
We choose these representatives for each pitch class so we are able to embed any $n$-chord,
with $n = 0, \ldots, 11$. The remaining two coordinates (rhythm and onset) are given as a decimal

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$^6$ For some tests including higher dimensions in homology, ran over 32-bar samples ($\sim$500 event-points), there were
overflow problems in the execution of the script. From the documentation of the Ripser library: “[I]t [sic] practice,
anything above $H_1$ is very slow.”
value representing length and onset in quarter notes. For example, the first three events of the score in Figure 1 yield the following associated tuples in \( R^{14} \):

\[
((-5, -9, 0), 0.5, 0) \mapsto (17, 21, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.5, 0)
\]
\[
((-4, -8, -9, 11), 0.5, 0.5) \mapsto (16, 20, 21, 23, 0, 0, 0, 0, 0, 0, 0, 0.5, 0.5)
\]
\[
((-10, -11, -3, -5, -6), 0.25, 1) \mapsto (22, 23, 15, 17, 18, 0, 0, 0, 0, 0, 0, 0, 0.25, 1)
\]

Applying this mapping to the 28-event fragment in Figure 1 (mm. 1–2 from Luna’s Graffiti: Hommage to F.J.H.), we obtain the persistence and barcode diagrams shown in Figure 2.

Picturing events as pitch-rhythm-onset vectors, this mapping lets us get a notion of their general distribution over time, as well as identify the presence of distinguished harmonic regions. This way, together with an overview of the score, we can have a general impression of the harmonic-rhythmic texture of the fragment in question.

- **Data mapping II**: Similarly, we consider each event as a point in \( \mathbb{R}^8 \) whose first six coordinates are the integers forming the interval vector of its corresponding chord, followed by its duration and onset in time. This mapping reflects similarity in the chord structures present in each event, together with their distribution in time and rhythm. In this case, for the same three events as above, we get:

\[
((-5, -9, -0), 0.5, 0) \mapsto (0, 0, 1, 1, 0, 0, 0.5, 0)
\]
\[
((-4, -8, -9, -11), 0.5, 0.5) \mapsto (1, 1, 1, 2, 0, 0.5, 0.5)
\]
\[
((-10, -11, -3, -5, -6), 0.25, 1) \mapsto (2, 1, 1, 2, 3, 1, 0.25, 1)
\]

The resulting diagrams for Example 1.1 are shown in Figure 3.

This data mapping, together with mapping III focus on the types of chords or different chord structures (modulo interval inversion) present in the fragment analysed, as they deal with interval vectors rather than the chords themselves.

Parallel to the above, we also work on sets obtained only from pitch data. That is, we focus especially on harmony, by considering data points containing only pitches in vertical events, forgetting about their distribution along a timeline, duration, etc. Points generated for this analysis consist of tuples of pitch classes or interval classes. We propose several ways of analysing the
same data, by generating the following sets of points from a given fragment, upon which we run a persistent homology analysis algorithm (under the Euclidean distance):

- **Data mapping III**: Projection of data mapping II on its first six components. That is, we get interval vectors of chords, which are integer tuples in $\mathbb{R}^6$:

$$\mathcal{A}_{\text{int.vect.}}(\mathcal{M}) = \{(0, 0, 1, 1, 1, 0), (1, 1, 1, 1, 2, 0), (2, 1, 1, 2, 3, 1), (1, 0, 1, 2, 2, 0), (5, 4, 6, 5, 5, 3), (2, 1, 2, 2, 2, 1), (1, 0, 1, 2, 2, 0), (2, 0, 2, 4, 2, 0), (2, 1, 3, 2, 1, 1), (1, 1, 1, 1, 1, 1), (1, 2, 3, 1, 2, 1), (2, 1, 1, 2, 3, 1), (1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1), (2, 2, 1, 3, 1, 1), (2, 1, 1, 2, 3, 1), (1, 0, 1, 2, 2, 0), (5, 3, 3, 4, 4, 2), (2, 3, 3, 3, 3, 1), (1, 2, 2, 2, 3, 0), (2, 3, 3, 2, 4, 1), (3, 2, 2, 3, 3, 2), (2, 2, 2, 1, 2, 1), (2, 1, 2, 2, 2, 1), (2, 1, 1, 2, 3, 1), (2, 2, 1, 3, 1, 1), (2, 1, 3, 2, 1, 1)\}.$$

Diversity in the harmonic content (modulo inversion) of a music fragment, understood as the number of chord types involved in it, can be grasped from diagrams arising from mapping III. Barcodes for this mapping somehow summarize the ones obtained from all other mappings, as images of events under any of them can be “projected” onto the set of their interval vectors. Figure 4 shows persistence and barcode diagrams for the set $\mathcal{A}_{\text{int.vect.}}(\mathcal{M})$. These diagrams give us a hint of the sample’s harmonic diversity.

This representation of events is the coarsest we consider. It is based purely on interval content modulo inversions (without duration or onset). This has the effect of both reducing dimensionality (only half the intervals are considered, due to inversion equivalence), and constraining possible values of coordinates (a chord may only contain so many intervals of each kind).

On the other hand, entries of interval vectors are always integers, even within a non-equally-tempered or microtonal context. Thus, (Euclidean) distances among vectors produced by mapping III will always be square roots of integers.

The above observations imply that persistence diagrams for this mapping will show less homological features, and these will collapse at very specific values.

- **Data mapping IV**: Chords mapped as vectors in $\{0, 1\}^{12} \subset I^{12} \subset \mathbb{R}^{12}$, where $I = [0, 1] \subset \mathbb{R}$, as follows:
Given a chord \( a \) in normal form, we define \( a_{12} = (r_0, \ldots, r_{11}) \) as:

\[
    r_i = \begin{cases} 
        0 & \bar{i} \not\in a \\
        1 & \bar{i} \in a
    \end{cases}
\]

for \( i \in \{0, 1, \ldots, 11\} \) and \( \bar{i} \in \{\bar{0}, \bar{1}, \ldots, \bar{11}\} \), the set of pitch classes represented by the smallest possible non-negative integers. For example, for the first chord of score in Example 1.1, \((\bar{5}, \bar{9}, \bar{0})\) (F Major vectorized following its normal form), we have

\[
(\bar{5}, \bar{9}, \bar{0})_{12} = (1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0).
\]

This association yields the sequence

\[
A_{12}(\mathcal{M}) = \{a_{i_{12}} \mid a_i \in A(\mathcal{M})\},
\]

upon which we perform a persistent homology analysis, getting the plots shown in Figure 5.

In this setting, each coordinate corresponds to a pitch class, and so the images of two events are close exactly when they share most of their pitches. More precisely, a given chord \( a \) contains \( n \) different pitch classes if and only if \( \|a_{12}\| = \sqrt{n} \). Furthermore, for chords \( a, b \) we have \( \|a_{12} - b_{12}\| = \sqrt{k} \) if and only if \( a \) and \( b \) differ in exactly \( k \) pitch classes. This way, from the diagrams we can measure how close chords are among themselves, in terms of common/distinct pitches.

This mapping yields homological features in higher dimensions (up to \( H_3 \)) than mappings III, V, and VI. As with mapping III, these features persist only during specific intervals, determined by the square roots of integers, and bars in barcodes appear to form “blocks.” More generally, we may take chords \( a = \{x_1, x_2, \ldots, x_l\} \) without octave equivalence of pitches. In this case, we may codify each chord as a vector with integer coordinates in \( \mathbb{R}^{12} \) taking this time

\[
r_i = \begin{cases} 
        0 & \bar{i} \not\in a \\
        k & \bar{i} \text{ appears exactly } k \text{ times in } a
    \end{cases}
\]

\textbf{Data mapping V}: Projection of data mapping I on its first 12 components. In this case we get tuples of pitch classes as vectors in \( \mathbb{R}^{12} \): for \( n \in \{1, \ldots, 12\} \), \( n \)-chords are mapped to tuples with non-zero integer values between 12 and 23 in the first \( n \) entries, and 0 in all the rest. As a consequence, a chord is an \( n \)-chord (\( 1 \leq n \leq 12 \)) if and only if its associated vector belongs to the subspace spanned by the first \( n \) canonical basis vectors of \( \mathbb{R}^{12} \), \( \hat{e}_1, \ldots, \hat{e}_n \). Thus, through this mapping, samples produce similar diagrams if and only if their events are similar in pitch and number of harmonic voices. For example, all triads belong to the 3-dimensional linear space spanned by the first three standard basis vectors in \( \mathbb{R}^{12} \):

\[
\hat{e}_1 = (1, 0, 0, 0 \ldots, 0) \\
\hat{e}_2 = (0, 1, 0, 0 \ldots, 0) \\
\hat{e}_3 = (0, 0, 1, 0 \ldots, 0).
\]

To associate such a tuple to a chord \( a \), we choose integers 12, 13, \ldots, 23 as representatives of pitch classes \( \bar{0}, \bar{1}, \ldots, \bar{11} \), and set each coordinate following the order given by the chord’s normal form, adding the necessary 0s after the last pitch class. This way, to chord \( a = (\bar{5}, \bar{9}, \bar{0}) \) we associate the tuple

\[
a(12, \ldots, 23) = a_{12} = (17, 21, 12, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in \mathbb{R}^{12}.
\]

In this mapping chords are embedded into \( \mathbb{R}^{12} \) as the vectors obtained from those in mapping I, without their last two (time) coordinates.
Also, this mapping is one of the only two (the other one being mapping I, of which mapping V is a projection) that is sensitive to transposition of the fragment in question by a given interval, that is, the bottleneck distance between persistence diagrams associated with a fragment and its transpositions is not always 0.

Through this mapping, we capture closeness of vertical events in terms of pitch content. As we said before, it is sensitive to transpositions, due to the embedding of chords with $k$ pitch classes in the subspace generated by the first $k$ canonical basis vectors. The corresponding persistence diagram and barcode are shown in Figure 6.

• **Data mapping VI:** Finally, we codify not only the pitches in a chord, but also the intervals between two consecutive pitch classes in a chord’s normal form (me other order for pitches). To achieve this, we propose the association

$$a \mapsto a_{\text{p,int}} = (r_0, \ldots, r_{11}) \in \mathbb{R}^{12},$$

where $0 \leq r_i < 12$ stands for the interval class ($r_i > 0$) between pitch class $\bar{i}$ and the next pitch class in the normal form of $a$, if $\bar{i}$ belongs to $a$, and $r_i = 0$ otherwise. That is,

$$r_i = \begin{cases} 0 & \bar{i} \notin a \\ k & k \text{ is the interval following } \bar{i} \text{ in } a \end{cases}$$

From our example we get, for chord $(\bar{5}, \bar{9}, \bar{0})$, vector $(0, 0, 0, 0, 4, 0, 0, 0, 3, 0, 0)$. Under this mapping, points are close to each other if and only if their corresponding events involve similar intervals over the same pitches.

Persistent homology analysis of data under mapping VI usually shows non-trivial homology cycles in higher dimensions than mappings III or V, sometimes in agreement with dimensions of features detected by mapping IV. Diagrams obtained from our sample fragment under mapping VI are depicted in Figure 7.

### 1.5. Comparing persistence diagrams from different mappings

Looking at persistence diagrams and barcodes arising from different mappings (Figures 2–7), we immediately observe some overall differences among them. For example, the first two appear to have a smoother shape. This is consequence of the fact that mappings I and II yield a larger
Figure 5. Persistence and barcode diagrams from data mapping IV for mm. 1–2 of Luna’s Graffiti: Hommage to F.J.H.

Figure 6. Persistence and barcode diagrams from data mapping V for mm. 1–2 of Luna’s Graffiti: Hommage to F.J.H.

Figure 7. Persistence and barcode diagrams from data mapping VI for mm. 1–2 of Luna’s Graffiti: Hommage to F.J.H.
number of points than the rest, due to the presence of the onset coordinate, which implies injec-
tivity (all events have a unique onset position). These diagrams let us grasp rhythmic variety and
regularity of events in time (in our example, every 0.25 quarter notes with only a few rhythmic
figures).

On the other hand, mappings not involving onset of events may produce fewer points, as
repeated chords and interval vectors are indentified. Especially in the case of mapping III, which
is a sort of “projection” of the other mappings. This translates into simpler diagrams consisting
of block-shaped groups of persistent features.

From Figures 2–7 one may also notice bars in barcodes corresponding to mappings V and VI seem more scattered than the ones for mappings III and IV. Instead of forming “blocks,”
persistent homology features appear staggered.

Also noteworthy is the fact that mapping IV is the result of projecting the image of mapping VI
onto the unit hypercube, $I^2$. This fact is reflected in the different shapes of their corresponding
diagrams: under this projection, persistence bars become delimited by squares of positive inte-
gers, and tend to collapse together in groups. In dimension 0, these groups correspond to chords
that differ by a specific number of notes (see description of mapping IV in the previous section).
This gives us some insight into the pitch variance among events and the level of chromatism of
the fragment. Differences in harmonic density of events are also captured by this mapping, since
events with few pitches will be distant from big chords.

Mappings IV, V and VI are different encodings of the normal form of chords. They are related
to each other, since we may recover the pitch classes of a vertical event from any of them
(and thus obtain their normal form). Nevertheless, they yield diagrams with different levels of
detail and varying homological features. We hope these variations will lead to a more complete
perspective on the harmonic data.

As we said before, we will use the resulting homological features of our data mappings as
stylistic descriptors. For instance, in the case of the fragment we have chosen to illustrate our
proposal (see Example 1.1), we identify by sight a certain general pattern in the shapes of all
the six pairs of diagrams and barcodes: one or two connected components ($H_0 / \beta_0$ values) that
are present in most of Vietoris-Rips complexes (see, for instance Edelsbrunner and Harer 2008)
associated with the corresponding cloud of data points, and a few briefly present hollowed circles
($H_1 / \beta_1$ values).

Following a standard TDA procedure, after calculating the persistent homology of these
six sets of points for different examples, we compute the bottleneck distance (again, refer to
Edelsbrunner and Harer 2008) between their corresponding $H_0$-diagrams, and plot a dendrogram
showing distances between them. We focus on $H_0$-diagrams, as only for a few samples and data
mappings we obtained diagrams in higher dimensions (nevertheless we include dendrograms for
$H_1$-diagrams when available). For this discussion, see Section 2.

1.6. Two harmonic simplicial complexes

We propose the construction of two different families of simplicial complexes to describe the
harmonic structure and evolution of a music fragment. In a subsequent paper we will also develop
the construction of a family of simplicial complexes describing harmonic connections. It is worth
mentioning that these two constructions do not consider data to be embedded in some metric
space, but are the result of assigning a (combinatorial) simplicial complex to each chord in the
score. With respect to the temporal parameters considered (duration and onset) when identifying
data points as vectors in a Euclidean space (which were treated as coordinates additional to
pitch coordinates), defining simplicial complexes from chords in a purely combinatorial way has
the disadvantage that it no longer makes sense to encode time values along with pitch classes to
conform a simplex. Nevertheless we may still incorporate temporal data in this model by filtering chord sequences according to their total cumulative duration or variable onset thresholds.

Simplicial complexes described here are formed of simplices or simplicial complexes representing individual chords, which are then combined into a bigger simplicial complex which somehow captures the harmonic structure of the given interval of events. From all the tests run so far, we can remark that for similar score samples, the associated complexes introduced here have similar Betti numbers. This will become more clear from the examples developed below. Thus, these mathematical objects may be useful for musical style identification and classification.

1.6.1. Simplicial complexes of cumulative chords by pitch

This representation captures the pitches of chords as vertices of simplices which are “added” together as events occur through the score. Simplices corresponding to repeated chords will only appear once in the complex of cumulative events.

Given an \((n + 1)\)-chord \(a\) with normal form vector \((\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n)\), we define its associated \(n\)-simplex as \(s(a) = \{\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n\}\). This association coincides with the one presented for example in Bigo, Giavitto, and Spicher (2011), Bigo et al. (2013), and Bigo and Andreatta (2016), though it is treated differently. Given a music fragment \(\mathcal{M} = \{e_0, \ldots, e_N\}\), we consider its sequence of chords \(\mathcal{A}(\mathcal{M}) = \{a_0, a_1, \ldots, a_N\}\), which yields the sequence

\[
s(a_0), s(a_1), \ldots, s(a_N)
\]

of associated simplices. For any integers \(0 \leq i \leq j \leq N\), we define the simplicial complex of cumulative chords by pitch in the interval of chords \([a_i, a_j]\), denoted by \(K(i, j)\), as the simplicial complex on simplices \(s(a_i), s(a_{i+1}), \ldots, s(a_j)\), together with all their faces. This is easily seen to comply with the definition of a simplicial complex. Somehow, \(K(i, j)\) codifies the “shape” of the harmonic sequence or path from chord \(a_i\) to \(a_j\). Nevertheless, this codification is not sensitive to the order of appearance of chords. For taking into account sequential order of events, we propose the study of the following sequence of complexes \(K(i, j)\), and some of its subsequences:

\[
S_a(\mathcal{M}) = \{K(m, n)\}_{0 \leq m, n \leq N} = K(0, 0), K(0, 1), \ldots, K(0, N), K(1, 1), K(1, 2), \ldots, K(1, N), K(2, 2), \ldots, K(2, N), \ldots, K(N - 1, N - 1), K(N - 1, N),
\]

We have defined simplices and simplicial complexes from ordered sets of pitch classes as non-oriented objects. The order we have chosen on pitch-class sets is given their normal form. However, the order of pitch classes in a chord does not change the simplex obtained: no matter the ordering, we get the abstract (non-oriented) simplex on the same set of pitch classes taken as vertices. However, the structure of the simplicial complex by pitches and intervals associated with a chord (defined below), will vary according to the order of pitches, thus yielding a different topological encoding of the same data. This could become useful when trying to preserve information of the actual intervals appearing in the score (chord voicing). In that case it may be congruent to drop the octave-equivalence hypothesis and work directly with pitches rather than pitch classes. On the other hand, we emphasize that each of the simplices and simplicial complexes associated with vertical events do not represent a sequence of notes, but a set of notes vertically coincident at some point. Oriented simplices and simplicial complexes associated with

\[7\] So far, we have run the present methods on over 100 fragments, from 13 classical, baroque, renaissance composers, as well as from traditional Indian and Mexican music.
vertical events could be considered in this framework, being interpreted as encoding the position (voicing) of chords, from lowest to highest, for example.

For now, we work with the main homological descriptors of each $K(i,j)$: its Betti numbers and Euler characteristic (refer to the Appendix). Since in our context we are considering chords consisting of up to 12 equally tempered pitch classes, the maximum dimension of the associated simplices and thus of the complexes $K(i,j)$ is 11, and so it suffices to compute their first 12 Betti numbers $\beta_0, \beta_1, \ldots, \beta_{11}$. We focus on the sequence of simplicial complexes $K(0,0), K(0,1), \ldots, K(0,N)$, which cover the full fragment, to get a picture of the change in the topology of these accumulated successive harmonic events. Note that with every step in this sequence, we add a simplicial complex to the one we have so far built, namely in step $i$ we merge complex $K(i,i)$ with $K(0,i-1)$.

Visualizing simplicial complexes

In order to visualize an abstract simplicial complex of any dimension (particularly $>3$), we may draw a graph whose vertices are its 0-faces (also called vertices) and whose edges are its 1-faces. 2-faces will then be represented as closed 3-paths, 3-faces as closed 4-paths, and so on, though not every closed path in the graph will correspond to a simplex (we use the term closed path instead of cycle to avoid confusion between homology cycles and cycles in a graph). As an example, for the simplex described above we get the graph shown in Figure 8. It is important to note that since this graph represents a 2-simplex together with its faces, it must be interpreted as a full triangle, i.e. vertices, perimeter, and area. We would get the same graph for the sequence of chords $(\bar{0}, \bar{5}), (\bar{0}, \bar{9}), (\bar{5}, \bar{9})$, but in such a case we would have to picture it as the vertices and perimeter only, without the triangle’s inscribed area. So it is always important to keep in mind what the picture of the graph is actually representing. Another disadvantage of this graphic representation is that sometimes when adding new simplices and their faces we may get cycles which are actually voids in the simplicial complex, and not faces (see Figure 10). Nevertheless, since simplices are grouped properly in the plot, we still get a very good picture of our simplicial complexes.

We now give an example to show how these simplicial complexes are built:

**Example 1.2** Consider the fragment in Example 1.1. Let us show how we build $K(0,0), K(0,1)$ and $K(0,2)$ from simplices on vertices in the set of pitch classes $\{\bar{0}, \ldots, \bar{11}\}$. The first three chords (according to their normal form) in this fragment are

$$(\bar{5}, \bar{9}, \bar{0}), (\bar{4}, \bar{8}, \bar{9}, \bar{11}), (\bar{10}, \bar{11}, \bar{3}, \bar{5}, \bar{6}).$$
Figure 9. Simplicial complex $K(0, 1)$, corresponding to progression $(\bar{5}, \bar{9}, \bar{0}), (\bar{4}, \bar{8}, \bar{9}, \bar{11})$ (see Example 1.1).

Figure 10. Simplicial complex $K(0, 2)$, corresponding to progression $(\bar{5}, \bar{9}, \bar{0}), (\bar{4}, \bar{8}, \bar{9}, \bar{11}), (\bar{10}, \bar{11}, \bar{3}, \bar{5}, \bar{6})$ (see Example 1.1).

$K(0, 0)$ is just the simplex $s((\bar{5}, \bar{9}, \bar{0})) = \{\bar{5}, \bar{9}, \bar{0}\}$ (see Figure 8), which is the 2-simplex on the three vertices $\bar{5}, \bar{9}, \bar{0}$, together with all its 1- and 0-faces. That is,

$$K(0, 0) = \{\{\bar{0}\}, \{\bar{5}\}, \{\bar{9}\}, \{\bar{0}, \bar{5}\}, \{\bar{0}, \bar{9}\}, \{\bar{5}, \bar{9}\}, \{\bar{5}, \bar{9}, \bar{0}\}\}.$$  

$K(0, 1)$ is the simplicial complex whose simplices are the simplices of $K(0, 0)$ together with simplex $s((\bar{4}, \bar{8}, \bar{9}, \bar{11}))$ and its 15 faces. So we get the following simplicial complex on the six vertices $\bar{5}, \bar{9}, \bar{0}, 4, 8, 11$:

$$K(0, 1) = \{\{\bar{0}\}, \{\bar{5}\}, \{\bar{9}\}, \{\bar{4}\}, \{\bar{8}\}, \{\bar{11}\}, \{\bar{0}, \bar{5}\}, \{\bar{0}, \bar{9}\}, \{\bar{5}, \bar{9}\}, \{\bar{4}, \bar{8}\}, \{\bar{4}, \bar{9}\},$$

$$\{\bar{4}, \bar{11}\}, \{\bar{8}, \bar{9}\}, \{\bar{8}, \bar{11}\}, \{\bar{9}, \bar{11}\}, \{\bar{5}, \bar{9}, \bar{0}\}, \{\bar{4}, \bar{8}, \bar{9}\},$$

$$\{\bar{4}, \bar{8}, \bar{11}\}, \{\bar{4}, \bar{9}, \bar{11}\}, \{\bar{8}, \bar{9}, \bar{11}\}, \{\bar{4}, \bar{8}, \bar{9}, \bar{11}\}\}.\$$

This complex features 21 simplices: the seven simplices from $K(0, 0)$, together with the 15 faces of $s((\bar{4}, \bar{8}, \bar{9}, \bar{11}))$, out of which one is already in $K(0, 0)$ (the 0-face $\{\bar{9}\}$).

We get a picture of simplicial complex $K(0, 1)$, corresponding to the sequence of chords $(\bar{5}, \bar{9}, \bar{0}), (\bar{4}, \bar{8}, \bar{9}, \bar{11})$ in Figure 9. In this case, we can clearly see the simplicial complex $K(0, 0)$, which stands for chord $(\bar{5}, \bar{9}, \bar{0})$, and the new added simplices forming simplicial complex $K(1, 1)$, associated with chord $(\bar{4}, \bar{8}, \bar{9}, \bar{11})$, which again we must picture as a full tetrahedron containing all its vertices, edges, faces, and volume. These two simplicial complexes are joined together by pitch class $\bar{9}$, which is the only one common to both chords.

Then we take a look at $K(0, 2)$, which is in this case the simplicial complex formed by all simplices in $K(0, 1)$, together with all 31 faces of simplex $\{\bar{10}, \bar{11}, \bar{3}, \bar{5}, \bar{6}\}$ (that accounts
for five 0-faces, ten 1-faces, ten 2-faces, five 3-faces and one 4-face). From these, two have already appeared in \( K(0, 1) \), namely the two 0-faces \( \{\overline{5}\} \) and \( \{11\} \). Thus, \( K(0, 2) \) consists of \( 21 + 29 = 50 \) simplices: nine 0-simplices, nineteen 1-simplices, fifteen 2-simplices, six 3-simplices, and one 4-simplex.

Figure 10 shows a 3-cycle in its centre, formed by vertices labelled 5, 9, 11, which does not correspond to a simplex in \( K(0, 2) \), as it is not a face a simplex associated with any of the chords considered. So actually this simplicial complex has a 1-dimensional hole given by this 3-cycle.

Finally, Figure 11 depicts the barcode plot of Betti numbers for simplicial complexes \( K(0, i) \) associated with fragment in Example 1.1. It is noteworthy that this barcode actually shows persistent homology features, since complexes \( K(0, i) \) do form a filtration.

1.6.2. Simplicial complexes of cumulative events of radius \( r \)

As a special case of the above, given an integer \( r \in \{0, \lfloor \frac{N}{2} \rfloor \} \) and a sequence of chords \( A(M) = \{a_0, \ldots, a_N\} \), we consider the complex \( K_r(i) = K(i - r, i + r) \), corresponding to events in the interval

\[
[a_{i-r}, a_{i+r}] = \{a_{i-r}, a_{i-r+1}, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{i+r-1}, a_{i+r}\}.
\]

We call the complex \( K_r(i) \) the simplicial complex of events of radius \( r \) around \( e_i \) in \( M \). These complexes give us local information about harmonic sequences. Note that the resulting simplicial complexes for a fixed radius \( r \) are not contained into one another. Thus, this sequence of simplicial complexes does not define a filtration of the complete complex \( K(0, N) \), but only a cover of it. As a consequence of this, we cannot strictly speak of persistent homology, and so the corresponding barcodes depict the values of Betti numbers of a sequence of simplicial complexes, without representing persistent homology cycles. Focusing on the subsequence of cumulative
Figure 12. Barcode plot for Betti numbers of complexes for intervals of events of radius 4, $K_4(i)$ for $4 \leq i \leq 24$, from fragment associated with Luna, *Graffiti: Homage to F.J.H.*, mm. 1–2 (see Figure 1 and Example 1.1).

events of varying radii around a fixed event $e_{i_0}$,

\[ K_0(i_0), K_1(i_0), K_2(i_0), K_3(i_0), \ldots, \]

we obtain an actual filtration. As an example, from the score in Figure 1 we get the barcode plot shown in Figure 12, showing Betti numbers of cumulative events of radius 4, $K_4(i)$.

In subsequent work we will focus on studying the results of calculating these homological descriptors for all possible radii $r$. Note that all simplicial complexes $\tilde{K}(i, j)$ (including $\tilde{K}_r(i)$) contain information about pitch classes common to chords as well as the number of pitch classes that constitute them. However, they do not capture intervals in chords, which is a crucial stylistic feature. In order to catch intervals in building simplicial complexes from chords in a music fragment, we propose the construction described in the next subsection.

1.6.3. Simplicial complexes of cumulative events by pitch and interval

We also consider another family of simplicial complexes on subsets of pitch classes in the 12-tone equally tempered system. We associate with a given chord $a$ with normal form vector $(\overline{x}_0, \overline{x}_1, \ldots, \overline{x}_n)$, the simplicial complex whose simplices are $\sigma_i = \{\overline{x}_i, \overline{x}_{i+1}, \overline{x}_{i+2}, \ldots, \overline{x}_{i+1}\}$, $i \in \{0, \ldots, n - 1\}$, together with their faces, where pitch classes $\overline{x}_j \in \mathbb{Z}_{12}$ are always assumed to be represented by the smallest possible non-negative integer. Given a sequence of chords $\mathcal{A} = \{a_0, a_1, \ldots, a_N\}$, we denote the simplicial complex associated in this form with chord $a_i$ by $\tilde{K}(i, i)$, and proceed to define $\tilde{K}(i, j)$ for $0 \leq i \leq j \leq N$ similarly to how we defined $\tilde{K}(i, j)$ (see the previous section). Thus, from this construction we get a sequence of simplicial complexes representing the harmonic subsequences of $\mathcal{A}$, upon which we can run a homology analysis.
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Figure 13. Simplicial complex of chord $\overline{5}$, $\overline{9}$, $\overline{0}$ by pitch class and interval.

To illustrate this construction, take again chord $\overline{5}$, $\overline{9}$, $\overline{0}$ for example. With this construction we get a simplicial complex made up from simplices $\{5, 6, 7, 8, 9\}$ and $\{9, 10, 11, 0\}$, which represent the major third interval between pitch classes $\overline{5}$ and $\overline{9}$, and the minor third between $\overline{9}$ and $\overline{0}$, respectively, together with all their faces. Considering this chord is the beginning of fragment from Example 1.1, we get $\tilde{K}(0, 0)$ for that score (see Figure 13).

2. Results

In order to test our proposal as a way for describing and comparing symbolic music data, we present and discuss persistence diagrams arising from the application of different mappings to the first four bars of the following scores: *Graffiti: Hommage to J.S.B.* (from *Graffiti* by Armando Luna; see the first paragraph of Example 1.1 for a brief commentary on this work, and Figure 14 for the score), dedicated to J.S. Bach, and three actual pieces by J.S. Bach: *Brandenburg Concertos nos. 1–3 BWV 1046–1048* (*B.C. 1–3*). Such pieces share many common stylistic elements while showing a distinctive harmonic treatment.

For demonstration purposes we consider only the first four measures of each piece. We compare harmonic data (with and without temporal data) contained in these examples by using the bottleneck distance calculated between their $H_0$- and $H_1$-diagrams for all six data mappings (see Tables 1 and 2). Diagrams for homology in dimensions greater than 0 ($H_1$, $H_2$, ...) arise jointly only for a few mappings and samples\(^8\), so it is possible to compare these diagrams in only a few cases. For the particular samples analysed, we get $H_1$-diagrams for almost all mappings (except for mapping III). Thus, we restrict ourselves to discussing persistent homology in dimensions 0 and 1, since no $H_2$- or higher dimension diagrams were jointly generated for any pair of samples (refer to Table 2). We do not reproduce here all persistence and barcode diagrams associated with our examples, but rather present dendrograms comparing distances among $H_0$- and $H_1$-diagrams.

\(^8\) $H_0$-diagrams are necessarily non-empty, as a finite data set is a bounded set in some $\mathbb{R}^n$, and so in the successive construction of Vietoris-Rips complexes eventually at least one connected component is always persistent.
2.1. Analysis of dendrograms for samples applying mappings I–VI

In the following, we compare and discuss bottleneck distances between $H_0$-persistence diagrams arising from all six data mappings. Such distances are summarized in several dendrograms (hierarchical clustering plots). Samples used correspond to the first four measures of Luna’s Graffiti: Hommage to J.S. Bach (see Figure 14) and J.S. Bach’s first three Brandenburg Concertos. For the sake of completeness, we also include dendrograms for bottleneck distances between $H_1$-diagrams from data mappings I, II, IV, V, and VI (for which $H_1$-diagrams actually exist for all samples; see Table 2). Bottleneck distances between diagrams are represented on the vertical
axis of these plots, while numbers above the samples’ identification on the horizontal axis are only labels.

We will also contrast our approach with traditional harmonic analysis. In Table 3 we briefly summarize chord progressions for Brandenburg Concertos 1–3 in traditional tonal nomenclature, together with a tonal interpretation of Luna’s Graffiti. When comparing tonal progressions against bottleneck distance measurements, one must keep in mind that harmonic ornaments (passing notes, auxiliary notes, retardations, etc.) are usually left out of traditional harmonic notation, but not from the encodings of chords in this work.

- In the plot shown in Figure 15 we see the distances between the $H_0$-persistence diagrams describing the clouds of data points resulting from mapping chords with their rhythms and onsets into $\mathbb{R}^{14}$, as tuples which include pitch information in their first 12 entries as integers from 12 to 23 (instead of 0 through 12). We observe that according to this embedding of events into $\mathbb{R}^{14}$, the first four bars of the first movements of Brandenburg Concertos 1 and 2 are the closest in shape among all four samples. This is compatible with the traditional tonal analysis of these pieces, since the first movements of Brandenburg Concertos 1 and 2 are both in the key of F Major (as we said before, this data mapping is sensitive to transpositions), and have a very similar distribution of rhythms (mostly eights and sixteenths). Nevertheless, these two movements differ notably in their harmonic progression, which will be discussed below. On the other hand, Luna’s Graffiti and Brandenburg Concerto no. 3 appear as the next pair of closest samples, although B.C.3 is the furthest. This may be explained by the fact that Brandenburg 3 is written in G Major, while Luna’s Graffiti includes many instances of D minor (the relative minor of F) and D Major (the dominant of G).

- In Figure 16, in which interval vectors instead of normal form vectors are considered (together with durations and onsets of events), we observe that Brandenburg Concertos 2 and 3 are the most similar samples, with Brandenburg 1 being the next one closest to them, and Luna’s Graffiti as substantially dissimilar to the rest of the samples. This may be explained by the auxiliary notes present over the triads in these pieces, which are mostly fourths, minor sevenths, and major seconds in Bach, and minor seconds in Luna. Thus, it makes sense that, beyond the tonality of the fragments, harmonic, and rhythmic content is fairly distinguishable.

- The third data mapping, formed by interval vectors only, produces the distances shown in Figure 17 among $H_0$-diagrams of the analysed fragments. We can conclude from it almost
the same relations as from the plot for mapping II, except that in this case Brandenburg Concerto 1 and Graffiti: Hommage to J.S.B. are equidistant to the cluster formed by Brandenburg Concertos 1 and 3. This tells us that vertical intervalic relations are similar in Brandenburg Concerto 1 and Graffiti: Hommage to J.S.B., showing similarity in their harmonic styles.

- The present samples are indistinguishable by comparing their $H_0$-diagrams associated with data mapping IV (see figure See Figure 18). Nevertheless, this is not the case for $H_1$-diagrams; see Figure 23 ahead). This points towards a similar structure among sets of vertical pitch sets.

- As with diagrams coming from data mapping I, in data mapping V (see Figure 19) the three Brandenburg Concertos are closer among themselves than they are to Luna’s Graffiti: Hommage to J.S.B., Brandenburg Concertos 1 and 2 being the most similar fragments.

- Finally, for chords mapped as tuples in which the $i$th coordinate represents the interval class in the chords’ normal form above pitch class $\bar{i}$, we get Figure 20. For this mapping, the clustering of the given samples coincides with that of mappings II and III.
Figure 17. Dendrogram for bottleneck distances between $H_0$-diagrams for data mapping III from mm. 1–4 of Luna’s Graffiti: Hommage to J.S.B. and J.S. Bach’s Brandenburg Concertos BWV 1046–1048.

Figure 18. Dendrogram for bottleneck distances between $H_0$-diagrams for data mapping IV from mm. 1–4 of Luna’s Graffiti: Hommage to J.S.B. and J.S. Bach’s Brandenburg Concertos BWV 1046–1048.

Figure 19. Dendrogram for bottleneck distances between $H_0$-diagrams for data mapping V from mm. 1–4 of Luna’s Graffiti: Hommage to J.S.B. and J.S. Bach’s Brandenburg Concertos BWV 1046–1048.

Figure 20. Dendrogram for bottleneck distances between $H_0$-diagrams for data mapping VI from mm. 1–4 of Luna’s Graffiti: Hommage to J.S.B. and J.S. Bach’s Brandenburg Concertos BWV 1046–1048.
Now we present dendrograms corresponding to $H_1$-diagrams from data mappings I, II, IV, V, and VI ($H_1$-diagrams do not exist for all of the samples for mapping III). The corresponding plots are shown in Figures 21–25. From these we may draw the following:

- For mapping I we get a kind of inversion in the “closeness” of samples: for $H_1$-diagrams (see Figure 21), the closest samples are *Graffiti: Hommage to J.S.B.* and *Brandenburg 3*, while *Brandenburg 1* and *2* generate clusters within a small distance. This suggests some measure of similarity between data from a piece by Bach and Luna’s hommage.

- Mapping II remains consistent in the closest pair of samples (though not in the furthest one), as can be seen in Figure 22.

- Clustering in mapping IV is disambiguated, as the $H_0$-diagrams of all fragments were equidistant for this mapping, while the distance between $H_1$-diagrams of *Brandenburg Concertos 1* and *2* is 0 (see Figure 23 in the next page).

- $H_1$-diagrams for mappings V and VI point towards similarity between data from *Graffiti: Hommage to J.S.B.* and *Brandenburg Conerto 2* (refer to Figures 24 and 25). Although from Table 4 the results obtained for this two mappings seem redundant, they are not necessarily so for our analysis (let us not forget that mapping V is sensitive to transpositions, while mapping VI is not; also mapping VI incorporates intervals, while mapping V does not).

In Table 4 we summarize the results of our hierarchical clustering. From this table we can conclude that *Brandenburg Conerto no. 2* lies “between” *Brandenburg Concertos nos. 1* and *3*, which sometimes appear almost as opposites. Also, we observe that Luna’s *Graffiti* is clearly the most distinguishable among all four sample pieces, though it shares some common harmonic features with the rest, as witnessed by distances between the persistence diagrams of certain
mappings: it is clustered together with Brandenburg 3 in mappings I and IV, Brandenburg 1 in mapping III, and Brandenburg 2 in mapping IV (for $H_1$-diagrams).

3. Conclusions and future work

Through the study of topological features of “clouds” of events represented in a music score, we get a plausible way of describing and comparing musical features, for example, the harmonic structure of a fragment. We believe the combination of homological descriptors of different sets of data points (mappings) associated with a given music fragment, may lead to a homological fingerprint of symbolic music scores, perhaps focusing specifically in some aspect, such as harmony, or encompassing a wide range of music parameters (rhythm, timbre, dynamics, etc.) simultaneously.
Table 4. Closest pairs (<) and most distant samples (>) according to the bottleneck distance between their $H_0$-and $H_1$-persistent diagrams, by mapping.

| Data mapping | $H_0$—diagrams | $H_1$—diagrams |
|--------------|-----------------|-----------------|
|              | I               | II              | III             | IV              | V               | VI              |
| $<$ B.C.1    | B.C.2           | B.C.2           |                 | B.C.1           | B.C.1           |
| B.C.2        | B.C.3           | B.C.3           |                 |                 |                 |
| $>$ B.C.1    | G.H. to J.S.B.  | B.C.1           |                 | G.H. to J.S.B.  | G.H. to J.S.B.  |
| G.H. to J.S.B. | B.C.3          |                 |                 | G.H. to J.S.B.  |                 |
| $<$ B.C.2    | B.C.1           |                 |                 |                 |                 |
| B.C.1        |                 |                 |                 |                 |                 |

Even beyond a concrete musical or musicological interpretation, by means of the (persistent) homological invariants of musical data, evidence in other works suggests we may be able to classify and relate musical styles and features, particularly by using them as training data for machine learning models (see, for example Deng and Duzhin 2022; Park, Hwang, and Yang 2022).

Our immediate goal is to run a variety of data analysis algorithms on results obtained from a large collection of samples, including some standard statistical analyses and recent TDA methods such as persistence landscapes (Bubenik 2015; Bubenik and Dłotko 2017; Beltramo et al. 2021), and the Euler characteristic curve (Beltramo et al. 2021). This will give us robust mathematical descriptors for style classification, all suitable for machine learning. In the way, we will record our results under the several proposed mappings, in order to unveil which of them seem to detect certain harmonic features, or if some are disposable or redundant. We also aim to work with microtonal scores.

Parting from the framework discussed in this paper, we will devote some work to describing the dynamical properties of sequences of simplicial complexes dealt with herein. Some work already done in studying time-varying simplicial complexes can be seen in Bergomi, Baratè, and Fabio (2016), Bergomi (2015), and Bergomi and Baratè (2020). We will attempt to approach this task by modelling harmonic progressions through dynamical systems defined on the families of simplicial complexes constructed in Sections 1.6.1 and 1.6.3. Such dynamical systems may be used to study change in musical structure, and thus may lead to, for example, a general way of describing musical change in texture and form. On the other hand, we will incorporate a dynamical perspective on persistence diagrams and landscapes.

Last, given that simplicial complexes introduced in this article represent the agglomeration of chords in a given event interval, but not the way they are linked from one to the next, certain musical aspects such as voice leading is left out of the present analysis algorithm. To address this issue, in a subsequent paper we will introduce two other families of simplicial complexes to model chord connections.

Disclosure statement

No potential conflict of interest was reported by the author(s).
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Definition A.1 An (abstract) simplicial complex is a finite collection of sets $K$ such that $\sigma \in K$ and $\tau \subseteq \sigma$ implies $\tau \in K$. The sets in $K$ are referred to as its simplices, the union of which makes up its set of vertices. To explicitly refer to the vertices of a simplex we write $\sigma[x_0, \ldots, x_n]$, and we say that $x_0, \ldots, x_n$ span $\sigma$. The dimension of a simplex $\sigma$ is given by $|\sigma| - 1$, where $|\sigma|$ denotes the cardinality of $\sigma$. A simplex on $n+1$ vertices has dimension $n$, and is called an $n$-simplex. The dimension of a simplicial complex is the maximum dimension of any of its simplices. A non-empty subset $\tau \subseteq \sigma$ is called a face of $\sigma$. Note that from the definition, a simplicial complex contains all the faces of its simplices. In the case of geometric simplicial complexes, in addition to this, simplices must be assembled together along their faces.

For simplicial complexes, we have a way of algebraically encoding its shape, by describing its boundary as a formal sum of its simplices and their faces. A simplex together with a fixed order of its vertices is called an oriented simplex. We introduce the boundary operator, denoted by $\partial$, which is defined on every oriented simplex and then linearly extended to the graded abelian group (or vector space) of formal sums of simplices

$$\dim(K) \oplus C_n,$$

where $C_n$ is the group with basis $S_n$, the set of simplices of dimension $n$. The boundary operator is defined as follows, for each $n$-simplex $\sigma$:

$$\sigma[x_0, \ldots, x_n] \mapsto \sum_{k=0}^n (-1)^k \sigma[x_0, \ldots, \hat{x}_k, \ldots, x_n].$$

Appendix

For the sake of fluency, as well as to facilitate citation and reference for the reader, we have dedicated this section to present a brief summary of simplicial and persistent homology. The basic objects of study in simplicial homology are simplicial complexes, which are abstract (either geometrical or merely combinatorial) equivalents of $n$-dimensional triangles (that is, vertices, edges, triangles, tetrahedrons, etc., respectively in dimensions 0, 1, 2, 3, etc.). Simplices are building blocks for the actual objects of study in this area: simplicial complexes, which are the topological spaces obtained by assembling simplices together.

As well as simplices, simplicial complexes can also be actual geometric objects in some Euclidean space, or more abstract, purely combinatorial objects. These two conceptions (geometric vs. combinatorial or abstract) are equivalent, both connected by the notions of geometric realization and scheme: for every abstract $n$-dimensional simplicial complex we may build a geometric simplicial complex (its geometric realization) in $\mathbb{R}^{2n+1}$, and from every geometric simplicial complex we get the abstract simplicial complex (its scheme) given by the sets of vertices of its simplices. We will be focusing on abstract simplicial complexes only. We recall that throughout the whole text, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, in which we consider the usual (Euclidean) distance between two points, given by the length of the straight line segment joining them.

Simplicial complexes are important in our context because of two main reasons:

- We can encode chords (vertical events) as simplicial complexes in different ways (see Section 1.6) that let us describe a fragment of a music score in terms of its homological invariants (see below).
- Simplicial complexes are the basis for topological data analysis, in which simplicial complexes are built (in different ways) from points representing data in a metric space, in relation to their distance (in our case, the Euclidean distance in some $\mathbb{R}^n$). This construction yields a filtration of simplicial complexes (in our case, the Vietoris-Rips filtration) whose vertices are these data points. This filtration lets us have a homological description of the topological “shape” of the given representation of the data through different “levels”: its persistent homology.

Formally, we have the following (most is taken from Edelsbrunner and Harer 2008):

For simplicial complexes, we have a way of algebraically encoding its shape, by describing its boundary as a formal sum of its simplices and their faces. A simplex together with a fixed order of its vertices is called an oriented simplex. We introduce the boundary operator, denoted by $\partial$, which is defined on every oriented simplex and then linearly extended to the graded abelian group (or vector space) of formal sums of simplices

$$\dim(K) \oplus C_n,$$

where $C_n$ is the group with basis $S_n$, the set of simplices of dimension $n$. The boundary operator is defined as follows, for each $n$-simplex $\sigma$:

$$\sigma[x_0, \ldots, x_n] \mapsto \sum_{k=0}^n (-1)^k \sigma[x_0, \ldots, \hat{x}_k, \ldots, x_n].$$

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where \(\sigma[x_0, \ldots, \hat{x}_k, \ldots, x_n]\) denotes the \((n-1)\)-simplex on vertices \(\{x_0, \ldots, x_n\} - \{x_k\}\). By linear extension, this defines a sequence of homomorphisms (respectively, linear functions)

\[
\cdots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots
\]

which has the property that \(\partial_n \circ \partial_{n+1} = 0\) for every \(n\). Thus, we may define the \(n\)th homology group of simplicial complex \(K\) as the quotient group (resp. vector space)

\[
H_n(K) = \text{Ker}(\partial_n)/\text{Im}(\partial_{n+1}).
\]

Homology groups somehow capture the way simplices are “glued” together to form a simplicial complex. They give a rough description of the shape of a space by measuring how many \(n\)-dimensional “holes” or “voids” are enclosed by it. This is summarized by the ranks or dimensions of the homology groups, called the Betti numbers of \(K\), and denoted by \(\beta_0(K), \beta_1(K), \beta_2(K), \ldots\). These numbers constitute a family of important homotopical invariants of a topological space, and are summarized in the Euler characteristic, given by their alternated sum:

\[
E(K) = \sum_{k=0}^{\dim(K)} (-1)^k \beta_k(K).
\]

Betti numbers are central in topological data analysis, particularly in persistent homology. Persistent homology is the computation of homological features (namely, Betti numbers) at different “levels,” “scales,” or “resolutions” to get an algebraic description of the shape of a set of points \(S\) in a space. This leads to consider filtrations (another crucial concept in persistent homology). A filtration of a simplicial complex \(K\) is a sequence of simplicial subcomplexes

\[
F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = K.
\]

We restrict ourselves to finite filtrations.

We consider a one-parameter filtration of simplicial complexes built upon the points in \(S\) (taken as vertices). This parameter, usually \(\varepsilon\), establishes the level or scale of scope on the data.

\[
S = F_{\varepsilon_0} \subset F_{\varepsilon_1} \subset F_{\varepsilon_2} \subset \cdots \subset F_{\varepsilon_\ell} = K,
\]

where \(K\) is the simplex on all points in \(S\) and \(\varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_\ell\). Simplicial complexes in the filtration can be defined according to different constructions. There are different ways to associate a filtration of simplicial complexes with a set of discrete points in a metric space. Persistent homology consists of computing the Betti numbers (which count \(n\)-dimensional holes or voids) of the simplicial complexes in the filtration considered.

Now we present one particular filtration that can be associated with any cloud of data points in a metric space, which is one of the most commonly used for computing persistent homology, and the one incorporated by the Python library used in this work.

Given a metric space \((X, d)\) and a finite set of points \(S = \{x_0, x_1, \ldots, x_N\}\) in \(X\), for each given distance \(\varepsilon\) we consider the Vietoris-Rips simplicial complex

\[
\mathcal{VR}(S, \varepsilon) = \{\sigma \subseteq S \mid \text{diam}(\sigma) \leq 2\varepsilon\}.
\]

\(\mathcal{VR}(S, \varepsilon)\) is the simplicial complex whose simplices consist of those subsets of points in \(S\) which are not further than \(2\varepsilon\) among themselves. So simplices in this case represent “closeness” of points: a \(k\)-simplex is formed whenever \(k + 1\) points can be enclosed together in a \(k\)-ball of radius \(\varepsilon\). As the parameter \(\varepsilon\) varies, we obtain a filtration of simplicial complexes, called the Vietoris-Rips filtration, in which the last element is the \(N\)-dimensional simplex on all vertices of \(S\). Figure A3 illustrates this definition.

The Betti numbers of the simplicial complexes in this filtration are computed to obtain a homological description of the “shape” of our cloud of points \(S\) in space \((X, d)\) at different scales. They are usually summarized in two graphic ways: persistent barcode graphs and persistent diagrams. These barcodes and diagrams (see figures in Section 1.4) record the changing values of the Betti numbers throughout the Vietoris-Rips filtration, and so give us a way of visualizing at which scales connected components and \(n\)-dimensional voids appear (their birth, \(b\)) and disappear (their death, \(d\)). Barcodes are plotted as a set of line segments in the plane \(\varepsilon \times \text{Betti numbers}\), starting and ending at the values of \(\varepsilon\) for which each homological feature persists. Similarly, persistence diagrams show the points \((b, d)\) in the plane \(\mathbb{R}^2 = \text{Birth} \times \text{Death}\), usually plotted alongside the diagonal (all points sit above this line). From this information, certain conclusions about the general distribution of the given points can be drawn, which help in understanding and classifying large collections of data. For example, the longest barcodes are interpreted as the most relevant (persistent) features of the point cloud’s shape.

Barcodes and persistence diagrams corresponding to different sets of points can be compared in several ways. One of the most common is the bottleneck distance, usually denoted \(W_\infty\) (in relation to the family of Wasserstein metrics). This consists of finding a pairing between points \(x = (b, d)\) in both diagrams, minimizing the maximum possible \(L_\infty\)-distance
between corresponding points in the pairing. Formally, the bottleneck distance between diagrams $D, D'$ is given by

$$W_{\infty}(D, D') = \inf_{\varphi : D \rightarrow D'} \sup_{x \in D} \|x - \varphi(x)\|_{\infty},$$

where $\varphi$ ranges over all bijections between $D$ and $D'$, and

$$\|(x_1, x_2) - (y_1, y_2)\|_{\infty} = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

Whenever there is no bijection between $D$ and $D'$, a partial pairing is considered, and those remaining points are paired with their projection over the diagonal. This metric has the property that it is the least number such that one can draw on the plane squares of side $2W_{\infty}(D, D')$ centred at the elements of $D$, and these will also contain the corresponding points of $D'$ under the matching that defines $W_{\infty}(D, D')$. Figure A4 shows how points in two different persistence diagrams are matched in order to compute their bottleneck distance. Distances taken among several persistence diagrams can be plotted in a dendrogram (see Section 2), which facilitates visualization and comparison of the values obtained for several samples.

Finally, we include the plots of barcodes for mappings III–VI (only harmonic data), arising from fragments of several scores (Figures A5–A9). From looking at these plots we can tell there are some distinctive features in the barcode diagrams of samples from different music styles. We can also see the difference in the overall shape of barcodes resulting from mappings III and IV (forming straight blocks), against barcodes corresponding to mappings V and VI (irregular staggered and jagged shapes). In addition, we point out how barcodes for mappings III and V seem to simplify features arising from mappings IV and VI, respectively. In-depth comparison and analysis of particular examples belonging to diverse genres, styles, or traditions will be discussed thoroughly in future works.
Figure A4. Matching for the bottleneck distance between persistence diagrams in dimensions 0 (left) and 1 (right) for measures 1–16 of J. S. Bach’s *Brandenburg Concertos* 1 and 3, under mapping V.

Figure A5. Barcodes for the first phrase in Dave Brubeck’s version of “My Favourite Things,” by Rodgers and Hammerstein: (a) Persistent homology barcodes for interval vectors, (b) Persistent homology barcodes for binary pitch vectors, (c) Persistent homology barcodes for embedded normal form vectors, and (d) Persistent homology barcodes for pitch-interval vectors.
Figure A6. Barcodes for mm. 1–16 from J. S. Bach’s *Brandenburg Concerto* no. 1, I: (a) Persistent homology barcodes for interval vectors, (b) Persistent homology barcodes for binary pitch vectors, (c) Persistent homology barcodes for embedded normal form vectors, and (d) Persistent homology barcodes for pitch-interval vectors.

Figure A7. Barcodes for mm. 1–16 from J. S. Bach’s *Brandenburg Concerto* no. 2, I: (a) Persistent homology barcodes for interval vectors, (b) Persistent homology barcodes for binary pitch vectors, (c) Persistent homology barcodes for embedded normal form vectors, and (d) Persistent homology barcodes for pitch-interval vectors.
Figure A8. Barcodes for mappings III–VI of mm. 17–44 from F. J. Haydn’s Symphony no. 88, I. Allegro: (a) Persistent homology barcodes for interval vectors, (b) Persistent homology barcodes for binary pitch vectors, (c) Persistent homology barcodes for embedded normal form vectors, and (d) Persistent homology barcodes for pitch-interval vectors.

Figure A9. Barcodes for mappings III–VI of a single phrase from Raga Asavari: (a) Persistent homology barcodes for interval vectors, (b) Persistent homology barcodes for binary pitch vectors, (c) Persistent homology barcodes for embedded normal form vectors, and (d) Persistent homology barcodes for pitch-interval vectors.