Heisenberg quantization for the systems of identical particles and the Pauli exclusion principle in noncommutative spaces

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1 Abstract. We study the Heisenberg quantization for the systems of identical particles in noncommutative spaces. We get fermions and bosons as a special cases of our argument, in the same way as commutative case and therefore we conclude that the Pauli exclusion principle is also valid in noncommutative spaces.

1 Introduction.

Recently there have been notable studies on the formulation and possible experimental consequences of extensions of the standard (usual) quantum mechanics in the noncommutative spaces [1-13]. Many physical problems have been studied in the framework of the noncommutative quantum mechanics (NCQM), see e.g. [1-4]. NCQM is formulated in the same way as the standard quantum mechanics SQM (quantum mechanics in commutative spaces), that is in terms of the same dynamical variables represented by operators in a Hilbert space and a state vector that evolves according to the Schroedinger equation:

\[ i \frac{d}{dt} |\psi> = H_{nc} |\psi> \]  

(1)
we have taken in to account $\hbar = 1$ and $H_{nc} \equiv H_\theta$ denotes the Hamiltonian for a given system in the noncommutative space. In the literatures two approaches have been considered for constructing the NCQM:

a) $H_\theta = H$, so that the only difference between SQM and NCQM is the presence of a nonzero $\theta$ in the commutator of the position operators:

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{x}_i, \hat{p}_i] = i\delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (2)$$

b) By deriving the Hamiltonian from the Moyal analogue of the standard Schroedinger equation:

$$i\frac{\partial}{\partial t}\psi(x,t) = H(p = \frac{1}{i}\nabla, x) \star \psi(x,t) \equiv H_\theta \psi(x,t), \quad (3)$$

where $H(p,x)$ is the same Hamiltonian as in the standard theory, and as we observe the $\theta$ - dependence enters now through the star product [11]. In [13], it has been shown that these two approaches lead to the same physical theory.

For the Hamiltonian of the type:

$$H(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (4)$$

the modified Hamiltonian $H_\theta$ can be obtained by a shift in the argument of the potential [11,11] :

$$x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}\hat{p}_j \quad \quad \hat{p}_i = p_i, \quad (5)$$

which lead to

$$H_\theta = \frac{\hat{p}^2}{2m} + V(x_i - \frac{1}{2}\theta_{ij}p_j). \quad (6)$$

The variables $x_i$ and $p_i$ now, satisfy in the same commutation relations as the usual case:

$$[x_i, x_j] = [p_i, p_j] = 0 \quad \quad [x_i, p_j] = \delta_{ij}. \quad (7)$$

In this paper we study one of the most important problems in the quantum mechanics namely the Heisenberg quantization for the systems of identical particles and the Pauli exclusion principle in noncommutative spaces. It appears that the main difference is between one and two dimensions not between two and three dimensions and therefore we study only one and two dimensional spaces.

Heisenberg quantization for the systems of identical particles in the framework of SQM has been studied in detailed in [14]. The process of quantization in SQM consists of:

a) Identifying the classical observables and the Lie algebra which they form under poisson bracket.

b) Looking for the linear representations of this lie algebra such that classical
observables $A$ and $B$ are represented by Hermitian operators $\hat{A}$ and $\hat{B}$ on some Hilbert space.

c) Replacing the Poisson bracket $C = \{\hat{A}, \hat{B}\}$ by the commutator:

$$\hat{C} = \frac{1}{i}[\hat{A}, \hat{B}] = \frac{1}{i}(\hat{A}\hat{B} - \hat{B}\hat{A}).$$

The classical observables of a system of $N$ identical particles are the real valued functions on the $N$ particles phase space. Every observable must be symmetric as a function of the $N$ one-particle phase space variables, because if there would exist a nonsymmetric observable, it can be used to distinguish between particles [15, 16]. Boson and fermion quantization are separate possibilities, because the symmetry of the quantum mechanical observables ensures that they do not mix symmetric and antisymmetric wave functions. In order to apply the Heisenberg quantization to a system of identical particles in NCQM, we follow the same way as SQM but we replace the commutator relations (7) by their noncommutative counterparts i.e. equ.(2).

### 2 Two particles in one dimensional spaces

In this section we apply the Heisenberg quantization to systems of two identical particles in commutative (usual) space. The fundamental relations are the canonical commutation relations:

$$[x, x] = [p, p] = 0 \quad [x, p] = i.$$  \hspace{1cm} (9)

To describe the two-particles systems, we introduce the relative coordinate and momentum [14]:

$$x = x^{(1)} - x^{(2)} \quad p = \frac{1}{2}(p^{(1)} - p^{(2)}),$$ \hspace{1cm} (10)

where the subscript $\ell = 1, 2$ refers to the particles. If the two particles are not identical, the relative $x$ and $p$ of the two-particles system are observables and satisfy the same canonical relations as (9), but if particles are identical, then $x$ and $p$ are not observables any more, because they are antisymmetric under exchange of the particles indices. The observables can be [14]:

$$A = \frac{1}{4}(p^2 + x^2) \quad B = \frac{1}{4}(p^2 - x^2) \quad C = \frac{1}{4}(px + xp),$$ \hspace{1cm} (11)

which are symmetric under the exchange of the particles indices, i.e. $x \rightarrow -x$ and $p \rightarrow -p$. The observables $A$, $B$ and $C$ satisfy the $sp(1, R)$ algebra:

$$[A, B] = iC \quad [A, C] = -iB \quad [B, C] = -iA.$$ \hspace{1cm} (12)
We can transform the above commutation relations into the more familiar form by introducing the operators $B_{\pm}$ as follows:

$$B_{\pm} = B \pm iC.$$  

(13)

Then we have:

$$[A, B_{\pm}] = \pm B_{\pm}$$  

and

$$[B_{+}, B_{-}] = -2A.$$  

(14)

We can define the irreducible representation of the algebra by introducing an orthonormal vector $|\alpha_0, n\rangle$, $n = 0, 1, 2, ...$ as follows [14] :

$$\Gamma |\alpha_0, n\rangle = \alpha_0(\alpha_0 - 1)|\alpha_0, n\rangle,$$  

(15)

$$A |\alpha_0, n\rangle = (\alpha_0 + n)|\alpha_0, n\rangle,$$  

(16)

$$B_{\pm} |\alpha_0, n\rangle = \sqrt{(n + 1)(n + 2\alpha_0)}|\alpha_0, n + 1\rangle,$$  

(17)

$$B_{-} |\alpha_0, n\rangle = \sqrt{n(n - 1 + 2\alpha_0)}|\alpha_0, n - 1\rangle,$$  

(18)

where $\Gamma = A^2 - B^2 - C^2$ is the Casimir operator and $\alpha_0$ is the minimum eigenvalue of $A$ (an arbitrary constant which defines the representation). The cases $\alpha_0 = \frac{1}{2}$ and $\alpha_0 = \frac{3}{2}$ correspond to bosons and fermions respectively.

3 Two identical particles in two dimensional non-commutative spaces.

We first consider the problem in commutative spaces [14]. We can describe the two particles systems by relative coordinates in the same way as in one dimensional case. We define complex quantities $a_{j\pm}$ as follows:

$$a_{j\pm} = \frac{1}{\sqrt{2}}(p_{j\pm} + ix_{j\pm}), \quad j = 1, 2.$$  

(19)

They satisfy in the following commutation relations:

$$[a_{j+}, a_{k+}] = [a_{j-}, a_{k-}] = 0 \quad [a_{j-}, a_{k+}] = \delta_{jk}.$$  

(20)

As we mentioned before, $x$ and $p$ are not observables. The generalized one-dimensional observables $A, B$ and $C$ are:

$$A_j = \frac{1}{4}(a_{j+} + a_{j-} + a_{j-} + a_{j+}), \quad B_{j\pm} = B_j \pm iC_j = \frac{1}{2}(a_{j\pm})^2.$$  

(21)

In addition we have two-dimensional observables which are the real and imaginary parts of:

$$D_{\pm} = D_{re} \pm iD_{im} = a_{1\pm}a_{2\pm} \quad E_{\pm} = E_{re} \pm iE_{im} = a_{1\pm} \mp a_{2\pm}.$$  

(22)
There are two \( sp(1, R) \) algebras \( A_1, B_{1\pm} \) and \( A_2, B_{2\pm} \) [14] :

\[
[A_j, B_{j\pm}] = \pm B_{j\pm} \quad [B_{j-}, B_{j+}] = 2A_j \quad j = 1, 2. \tag{23}
\]

There are also two other algebras, one \( sp(1, R) \) algebra :

\[
[A_1 + A_2, D_{\pm}] = \pm D_{\pm} \quad [D_+, D_-] = -2(A_1 + A_2). \tag{24}
\]

and one \( su(2) \) algebra :

\[
[A_2 - A_1, E_{\pm}] = \pm E_{\pm} \quad [E_+, E_-] = 2(A_2 - A_1). \tag{25}
\]

Now we study the same problem in two dimensional noncommutative spaces. The operators \( a_{j\pm} \) are given by :

\[
a_{j\pm} = \frac{1}{\sqrt{2}}(\hat{p}_j \pm i\hat{x}_j), \tag{26}
\]

where \( \hat{p}_{j\pm} \) and \( \hat{x}_j \) satisfy in the Heisenberg commutation relations in noncommutative spaces i.e. equ.(2). We have the following commutation relations :

\[
[a_{j+}, a_{k+}] = [a_{j-}, a_{k-}] = -\frac{i}{2}\theta_{jk}, \quad [a_{j-}, a_{k+}] = \delta_{ij} + \frac{i}{2}\theta_{jk}. \tag{27}
\]

The two \( sp(1, R) \) algebras (23) are also valid in this case, but the commutation relations (24) and (25) are no longer valid and in addition \( A_1 \) and \( A_2 \) don’t commute with each other and can not have common eigenvectors. Now we use the variables \( x_i \) and \( p_i \) introduced in equ.(5), then we have :

\[
a_{j\pm} = \frac{1}{\sqrt{2}}(p_j \pm ix_j), \quad j = 1, 2. \tag{28}
\]

where \( x_i \) and \( p_i \) satisfy in (7). One can show that \( a_{j\pm} \) satisfy the same commutation relations as commutative case :

\[
[a_{j+}, a_{k+}] = [a_{j-}, a_{k-}] = 0 \quad [a_{j-}, a_{k+}] = \delta_{ij}, \tag{29}
\]

and therefore the operators \( A_j, B_{j\pm}, A_2 \pm A_1, E_{\pm} \) and \( D_{\pm} \) satisfy in the same \( sp(1, R) \) and \( su(2) \) algebras as commutative space :

\[
[A_j, B_{j\pm}] = \pm B_{j\pm} \quad [B_{j-}, B_{j+}] = 2A_j \quad j = 1, 2, \tag{30}
\]

\[
[A_1 + A_2, D_{\pm}] = \pm D_{\pm} \quad [D_+, D_-] = -2(A_1 + A_2), \tag{31}
\]

\[
[A_2 - A_1, E_{\pm}] = \pm E_{\pm} \quad [E_-, E_+] = -2(A_2 - A_1). \tag{32}
\]

We also have the following commutation relations :

\[
[A_1, A_2] = 0, \tag{33}
\]

5
\[ [E_-, D_+] = 2B_{1+} \quad [D_-, E_+] = 2B_{1-}, \tag{34} \]
\[ [E_+, D_+] = 2B_{2+} \quad [D_-, E_-] = 2B_{2-}, \tag{35} \]
\[ [E_-, B_{2+}] = [E_+, B_{1+}] = D_+, \tag{36} \]
\[ [E_-, B_{1-}] = [E_+, B_{2-}] = -D_-, \tag{37} \]
\[ [D_-, B_{1+}] = -[D_+, B_{2-}] = E_-, \tag{38} \]
\[ [D_-, B_{2+}] = -[D_+, B_{1-}] = E_+. \tag{39} \]

We observe that by using the variables \( x_i \) and \( p_i \) defined by equ.(5), all of the commutation relations are the same as commutative case and the algebra governs physics of the system of identical particles and its representation remain unchanged.

The operators \( A_1 \) and \( A_2 \) together with the raising and lowering operators \( B_{1\pm}, B_{2\pm}, D_{\pm} \) and \( E_{\pm} \) form a cartan basis for \( sp(2, R) \) algebra.

The "symplectic" algebra \( sp(2, R) \) has 10 generators \( a_i \), which are \( 4 \times 4 \) matrices and satisfy in the following relation:

\[ \tilde{a}_i J = -J a_i, \tag{40} \]

where \( \tilde{a}_i \) is the transpose of the \( a_i \), and the matrix \( J \) is defined by:

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \]

where 1 is the \( 2 \times 2 \) unit matrix. The graphic representation of the root vectors of \( sp(2, R) \) algebra can be drawn by introducing the angle \( \phi \) between the root vectors \( \alpha \) and \( \beta \):

\[ \cos(\phi) = \frac{(\alpha, \beta)}{\sqrt{(\alpha, \alpha)(\beta, \beta)}} \tag{41} \]

where \( (\alpha, \beta) = \alpha^i \beta_i \) is the scalar product. Fig.1 shows the graphic representation of the root vectors.

Like commutative case we can construct an arbitrary representation of the \( sp(2, R) \) by a common eigenvector of \( A_1 \) and \( A_2 \):

\[ A_1 |a> = a_1 |a> \quad A_2 |a> = a_2 |a>, \tag{42} \]

where \( (a_1, a_2) \) is a weight of the representation. All of the operators \( B_{1-}, B_{2-}, D_- \) and \( E_{\pm} \) lower the eigenvalues of at least one of the \( A_1 \) or \( A_2 \) by either \( \frac{1}{2} \) or 1.
and we can use this fact to obtain the eigenvectors and eigenvalues of \( A_1 \) and \( A_2 \), we will have [14] :

\[
A_1 |jkmn> = (a_{10} + j + \frac{m-n}{2})|jkmn>,
\]

(43)

\[
A_2 |jkmn> = (a_{20} + k + \frac{m+n}{2})|jkmn>,
\]

(44)

where \((a_{10}, a_{20})\) are the weights of the lowest leading weight vector \(|0>\).

Like in one dimension we can get bosons and fermions as special cases of the Heisenberg quantization in two dimensions :

a) If \(a_{10} = a_{20} = \frac{1}{4}\), the eigenfunction is symmetric in both \(x_1\) and \(x_2\) or antisymmetric in both and this corresponds to the bosons.

b) If \(a_{10} = \frac{3}{4}, a_{20} = \frac{1}{4}\), the eigenfunction has opposite symmetry in \(x_1\) and \(x_2\) which corresponds to fermions, thus :

\[
2(a_{10} - a_{20}) = K = \begin{cases} 0 & \text{for Bosons} \\ 1 & \text{for Fermions} \end{cases}
\]

This means that a system of two identical fermions or bosons in two dimensional noncommutative spaces has a definite symmetry in each of the relative coordinates \(x_1\) and \(x_2\) (like commutative case) and therefore the Pauli exclusion principle is also valid in noncommutative spaces.

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