Cauchy problems of sublinear pseudoparabolic equations with unbounded, time-dependent potential

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Abstract

We consider non-negative solutions of the Cauchy problem: \(\partial_t u - \Delta \partial_t u = \Delta u + V(x, t)u^p\) in \(\mathbb{R}^n \times (0, \infty)\), \(u|_{t=0} = u_0\) in \(\mathbb{R}^n\), where \(0 < p < 1\) and \(V\) is an unbounded, time-dependent potential. By solving approximate problems and proving the corresponding comparison principles, we can establish, via a monotonicity argument, the global existence result of the Cauchy problem for a large class of potentials. Nontrivial solutions are shown to exhibit a certain grow-up rate (in time) and a lower bound for the growth (in space) of solutions are derived. For static radial potentials, we can prove the key comparison principle of the Cauchy problem. Then we can settle the uniqueness result when \(u_0 \not\equiv 0\) and classify all nontrivial solutions when \(u_0 \equiv 0\) in terms of the maximal solution.

1 Introduction

In this work, we consider the existence and uniqueness of non-negative solutions \(u = u(x, t)\) to the pseudoparabolic Cauchy problem

\[
\begin{cases}
\partial_t u - \Delta \partial_t u = \Delta u + V(x, t)u^p & x \in \mathbb{R}^n, t > 0, \\
u(x, 0) = u_0(x) & x \in \mathbb{R}^n,
\end{cases}
\]

where \(0 < p < 1\) is a constant and \(V, u_0\) are non-negative functions. The potential function \(V\) may be unbounded and time-dependent.

Pseudoparabolic equations are models of many nonlinear systems. We refer the reader to [1, 2, 3, 4, 5] and the references therein. Also, in recent years, driven by problems in stochastic differential equations, there has been a great interest in studying pseudoparabolic equations with unbounded, time-dependent coefficients [6].

When the viscosity term \(\Delta \partial_t u\) is dropped, the equation (1.1) becomes the heat equation which is closely related with the pseudoparabolic equation ([7, 8, 9, 10]). The study of sublinear heat equations (or systems) is quite well established now. Sublinear pseudoparabolic equations, however, have received very few results. To our knowledge, [11] is the only paper for this equation. In that paper, the authors studied the autonomous problem

\[
\partial_t u - \Delta \partial_t u = \Delta u + u^p, \quad u|_{t=0} = u_0 \geq 0
\]

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and they only obtained the existence result. Recently the first author has been able to prove, in [12], the uniqueness of solutions for non-zero initial and obtain the classification of solutions for zero-initial condition.

In this work, we solve the non-autonomous problem (1.1). We are inspired by the framework of J. Aguirre and M.A. Escobedo [13] on autonomous sublinear heat equation. This paper and most other studies on sublinear heat problems rely heavily on the explicit form of the heat semigroup. For the pseudoparabolic equation, however, the complicated Green operator prevents us from directly adopting the same arguments. Additional difficulty arises from the unboundedness and time-varying of the potential V which did not appear in [13]. We overcome these difficulties by analysing the problem in the continuous spaces

\[ BC_\alpha(\mathbb{R}^n) = \{ \varphi \in C(\mathbb{R}^n) : |\varphi(x)| \leq \text{const} \cdot |x|^{-\alpha} \text{ as } |x| \to \infty \}, \]

which is endowed with a weighted norm

\[ \| \varphi \|_{R,a} := \| (R^2 + |x|^2)^{\frac{\alpha}{2}} \varphi(x) \|_{L^\infty}, \]

where \( R > 0 \) is adjustable and \( a \leq 0 \). It turns out in our study that, by choosing suitable values of \( R \), the ingredient operators, the Bessel potential operator \( B \) and the pseudoparabolic Green operator \( G(t) \) behave ‘almost’ as contractions on \( BC_\alpha(\mathbb{R}^n) \) as \( t \to 0^+ \). This property, which is true for the heat semigroup (in a stronger form) on usual weighted norms \( \| \cdot \|_{1,a} \), plays an important role throughout this study.

The global existence result of the problem (1.1) is proved by introducing a new dependent variable \( w \) such that \( w(\cdot,t) \in BC_0(\mathbb{R}^n) \). Then we study approximate problems where the sublinear term is approximated by nice Lipschitz continuous functions. The global existence and comparison principle for these approximate problems can be proved by the standard contraction mapping and the Gronwall’s inequality. Then we prove the existence result for the original problem (1.1) using the monotonicity of solutions to the approximate problems. We have obtained the global existence result under a weak assumption that \( V = V(x,t) \geq 0 \) satisfies \( V(x,t) \lesssim |x|^\sigma \) as \( |x| \to \infty \) for all \( t > 0 \), where \( \sigma \in \mathbb{R} \) is a constant.

In order to determine the grow-up rate of solutions, the problem (1.1) will be expressed in both the original mild solution form:

\[ u(x,t) = \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B}\{Vu^p\}(x,s)ds, \]

and a modified form:

\[ u(x,t) = u_0 + \int_0^t \mathcal{B}u ds + \int_0^t \mathcal{B}\{Vu^p\}(x,s)ds \]

which is obtained formally by setting \( u = e^{-t}u \). By dropping the second integral on the right hand side of the modified equation, we can prove the positivity of solutions: \( u(t_0) \geq C(\delta, t_0, u_0)e^{-\delta|x|} > 0 \) for all \( t_0 > 0 \) whenever \( u_0 \neq 0 \). Applying this positivity estimate to the second integral of the modified equation and performing some iterations, we can derive a lower grow-up rate for the solutions provided \( V(x,t) \gtrsim |x|^\sigma \) (\( \sigma \geq 0 \)). It is important to note that we have obtained this lower grow-up rate through the knowledge
of the asymptotic behavior of the Bessel potential kernel as $|x| \to \infty$ and $|x| \to 0$. So we claim that our method can be used to obtain similar results for general Sobolev type equations:

$$\partial_t u - \mathcal{L} \partial_t u = \mathcal{L} u + V(x, t) u^p$$

where $\mathcal{L}$ is a second order uniformly elliptic operator. The only ingredient is the asymptotic behavior of the kernel of the operator $(I - \mathcal{L})^{-1}$ at infinity and at zeroes.

In this work, we have discovered a phenomenon induced by the unboundedness of the potential $V$. We have found that if $V \sim |x|^{\sigma}$ ($\sigma \geq 0$), then any solution $u \in BC_a(\mathbb{R}^n)$ ($a \leq -\frac{\sigma}{1-p}$) to the Cauchy problem (1.1) satisfies

$$|x|^{\frac{-\sigma}{1-p}} t^{1-p} \lesssim u(x, t) \lesssim |x|^{-a} t^{1-p} \quad \text{for all } t > 0.$$

Thus the solution grows up in time as $t^{1-p}$. This phenomenon, which has already been observed in the autonomous heat or pseudoparabolic equations, must be affected by sublinearity of the source. A new result is that the solution shows a growth in space at least as $|x|^{\frac{-\sigma}{1-p}}$, even if $u_0$ is bounded. It must be the effect of unbounded potential.

Finally, we prove that if $u_0 \equiv 0$ and $V = \vartheta_0 |x|^{\sigma}$, then the Cauchy problem (1.1) has a continuum family of nontrivial solutions. In fact, every nontrivial solution of (1.1) has the form

$$u(x, t) = U_{\sigma}(x, (t - \tau)_+) \quad \text{for some } \tau \geq 0,$$

where $U_{\sigma} > 0$ is the unique maximal solution of (1.1) with zero initial data.

We explain the plan of this paper. In Section 2, some preliminaries are given. We prove the existence of global solutions in Section 3. The grow-up both in space and time are proved in Section 4. In section 5, we settle the uniqueness result for the Cauchy problem when the initial data is nontrivial. Finally, we classify the family of nontrivial solutions to the Cauchy problem for zero-initial condition in Section 6.

## 2 Preliminaries

### 2.1 Function spaces, basic results, and notation.

In our study of pseudoparabolic equations with a sublinear source, we consider the spaces of weighted continuous functions:

$$BC_a(\mathbb{R}^n) = \{ \varphi \in C(\mathbb{R}^n) : |\varphi(x)| \leq \text{const} \cdot |x|^{-a} \text{ as } |x| \to \infty \},$$

where $a \in \mathbb{R}$. One can verify directly that $BC_a(\mathbb{R}^n)$ is a Banach space under any of the following norms: For each $R > 0$, we define the norm

$$\| \varphi \|_{R,a} := \|(R^2 + |x|^2)^{\frac{\sigma}{2}} \varphi(x)\|_{L^\infty_x}.$$

Furthermore, any two $\| \cdot \|_{R_{1,a}}$, $\| \cdot \|_{R_{2,a}}$ are equivalent norms.

We will need the following basic facts.

**Lemma 1** Let $0 < p < 1$.  

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(i) If \( r, s \geq 0 \) then \( |r^p - s^p| \leq |r - s|^p \).

(ii) If \( \psi \in C([0, T]; \mathbb{R}_+) \) and there is a constant \( \lambda > 0 \) such that
\[
\psi(t) \leq \lambda \int_0^t \psi(s)^p ds \quad (\forall t \in [0, T]),
\]
then \( \psi(t) \leq \{\lambda(1-p)t\}^{\frac{1}{1-p}} \) for all \( t \).

**Proof.** (i) This can be proved by elementary calculus.

(ii) The result follows from Bihari’s inequality, but for completeness sake, we provide our proof. Define \( g(t) = \sup_{\tau \in [0, t]} \psi(\tau) \). So \( g \) is non-decreasing, \( \psi(t) \leq g(t), \) and \( g(t) \leq \lambda \int_0^t g(s)^p ds \) for all \( 0 \leq t \leq T \). Then \( g(t) \leq \lambda g(t)^p \int_0^t ds = \lambda g(t)^p \) which implies \( g(t) \leq (\lambda t)^{\frac{1}{1-p}} =: J_0 \). By iteration,
\[
g(t) \leq \lambda \int_0^t J_0^p ds = (1-p)(\lambda t)^{\frac{1}{1-p}} =: J_1.
\]
So we obtain by induction that \( g(t) \leq J_k := (1-p)^{1+p+\cdots+p^k}(\lambda t)^{\frac{1}{1-p}} \) for all \( k \geq 0 \). Therefore \( \psi(t) \leq g(t) \leq \lim_{k \to \infty} J_k = \{\lambda(1-p)t\}^{\frac{1}{1-p}} \) which is the desired inequality. \( \square \)

**Notation.** Let \( 0 < T \leq \infty \). We will use the following notation.

- \( \mathcal{Z}_T^n = C([0, T]; BC_{c,0}(\mathbb{R}^n)), \|\varphi\|_{\mathcal{Z}_T^n} = \sup_{t \in [0, T]} \|\varphi(\cdot, t)\|_{R,a}; \)
- \( QT = \mathbb{R}^n \times [0, T), \mu = (R^2 + |x|^2)^{\frac{1}{2}} \) (when \( R \) is fixed).

### 2.2 Bessel potential and (pseudoparabolic) Green operators

For sufficiently regular data, (1.1) can be expressed in a nonlocal form as
\[
\partial_t u = B\triangle u + B\{Vu^p\}, \quad u|_{t=0} = u_0,
\]
where \( B = (I - \triangle)^{-1} \) is the Bessel potential operator. Then, by the Duhamel’s principle, \( u \) should satisfy the integral equation \( u(x, t) = \mathcal{M}u(x, t) \), where the mild operator is defined by
\[
\mathcal{M}u(x, t) := \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)B\{Vu^p\}(x, s)ds \tag{2.1}
\]
and \( \mathcal{G}(t) = e^{tB\triangle} = e^{-t} \sum_{k=0}^{\infty} (t^k/k!)B^k \) is the Green operator for (1.1).

By Fourier transform, \( B\varphi = B_1 * \varphi \) and \( \mathcal{G}(t)\varphi = G(\cdot, t) * \varphi \) where the kernel functions are given by
\[
G(x, t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k(x), \quad B_0 = I, \quad B_\alpha(x) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} |x|^{\alpha-\frac{2}{p}} K_{\frac{\alpha}{2}-\alpha}(|x|) \quad \text{for } \alpha > 0,
\]
and \( K_\alpha \) is the modified Bessel function of the second kind. For convenient, let \( B := B_1 \).
Lemma 2

(i) \( B \) and \( G(t) \) \((t > 0)\) are positive linear operators on \( BC_a(\mathbb{R}^n) \).

(ii) \( B(1) = 1 \) and \( G(t)(1) = 1 \) for all \( t \in \mathbb{R} \).

(iii) If \( \gamma \geq 0 \) then \( B|x|^\gamma \geq |x|^\gamma \) and \( G(t)|x|^\gamma \geq |x|^\gamma \).

Proof. (i) Obvious. To prove (ii), note that \( v = B(1), w = G(t)(1) \) solve the problems \( v_0 \triangle v = 1 \) in \( \mathbb{R}^n \), and \( \partial_tw - \triangle \partial_tw = \triangle w, w|_{t=0} = 1 \), respectively. The conclusion then follows by elliptic and pseudoparabolic comparison principles.

(iii) Let \( r = |x| \). Since \( \gamma \geq 0 \), we have by direct calculation that

\[
|x|^\gamma - \triangle |x|^\gamma = r^\gamma - (r^\gamma)'' - \frac{n-1}{r}(r^\gamma)' = r^{-2}(r^2 - \gamma(n + 2)) \leq r^\gamma.
\]

Applying \( B > 0 \), we obtain \( B|x|^\gamma \geq |x|^\gamma \). The second assertion is immediate. \( \square \)

Proposition 1 Let \( R > 0, \theta \in (0, 1) \), and \( \mu = (R^2 + |x|^2)^{\frac{1}{2}} \).

(i) If \( a \geq 0 \) and \( R \geq \left( \frac{\text{a} + (a - 2) + 1}{1 - \theta} \right)^{\frac{1}{2}} \), then \( B\mu^a \leq \theta^{-1} \mu^a \).

(ii) If \( a \leq 0 \) and \( R \geq \left( \frac{|a| + (a - 2) + 1}{1 - \theta} \right)^{\frac{1}{2}} \), then

\[
\|B\varphi\|_{R,a} \leq \theta^{-1}\|\varphi\|_{R,a},
\]

\[
\|G(t)\varphi\|_{R,a} \leq \beta(t)\|\varphi\|_{R,a} \quad \text{where} \quad \beta(t) = e^{\left(\frac{1-\theta}{\theta}\right)t}.
\]

Proof. (i) Let \( r = |x| \). We have \( (\mu^a)' = a\mu^{a-2} \) and \( (\mu^a)'' = a\mu^{a-2} + a(a - 2)r^2\mu^{a-4} \). So

\[
\mu^a - \triangle (\mu^a) = \mu^a - (\mu^a)'' - \frac{n-1}{r}(\mu^a)'
\]

\[
= \mu^{a-2} \left[ (R^2 + r^2) - na - a(a - 2) \frac{r^2}{R^2 + r^2} \right]
\]

\[
\geq \mu^{a-2} \left[ (R^2 + r^2) - a(n + (a - 2)) \right].
\]

Since \( a(n+(a-2)+1) \leq (1-\theta)R^2 \), it follows that \( \mu^a - \triangle (\mu^a) \geq \mu^{a-2} [(R^2 + r^2) - (1-\theta)R^2] \geq \theta\mu^a \). Taking the operator \( B > 0 \), we get \( \mu^a = B\{(I-\triangle)\mu^a \} \geq B\{\theta\mu^a \} = \theta B\mu^a \). Hence \( B\mu^a \leq \theta^{-1}\mu^a \).

(ii) Since \( \|\varphi\|_{R,a} = \|\mu^a\varphi\|_{L^\infty} \), we have

\[
B\varphi = B \left\{ \mu^{-a}\mu^a\varphi \right\} \leq B \left\{ \mu^{-a}\|\mu^a\varphi\|_{L^\infty} \right\} \leq B\{\mu^{-a}\} \cdot \|\varphi\|_{R,a}.
\]

Since \( -a = \|a\| \geq 0 \), we get by part (i) that \( B\{\mu^{-a}\} \leq \theta^{-1}\mu^{-a} \). Thus

\[
\|B\varphi\|_{R,a} = \|\mu^aB\varphi\|_{L^\infty} \leq \theta^{-1}\|\varphi\|_{R,a},
\]

which is the first statement of (ii). The second statement follows from the first one and that \( \|G(t)\varphi\|_{R,a} \leq e^{-t}e^{\|B\|}\|\varphi\|_{R,a} \). \( \square \)
Remark 1 In a recent work on superlinear pseudoparabolic equations, i.e., \( p > 1 \), we have established the boundedness of \( \mathcal{B} \) and \( \mathcal{G}(t) \) on weighted Lebesgue spaces:

\[
L^{q,a}(\mathbb{R}^n) = \{ \varphi \in L^1_{\text{loc}}(\mathbb{R}^n) : \|(1+|x|^2)^{\frac{a}{2}}\varphi(x)\|_{L^q} < \infty \}
\]

for any \( 1 \leq q \leq \infty \) and \( a \in \mathbb{R} \). The proof is rather involved. We showed that if \( \mathcal{B}^a \varphi := B_a \ast \varphi \), then

\[
\|\mathcal{B}^a \varphi\|_{L^{q,a}} \leq C_{n,a,a} \|\varphi\|_{L^{q,a}}, \quad \|\mathcal{G}(t)\varphi\|_{L^{q,a}} \leq C_{n,q,a}(t)^{\frac{a}{q}} \|\varphi\|_{L^{q,a}} \quad (t > 0).
\]

This result, of course, generalizes Proposition 1 (ii). For the present investigation of sublinear equations, \( BC_a(\mathbb{R}^n) \) is a more natural choice of function spaces. This happens as well in the study of sublinear heat equation (see [13]). A novel idea in this work is that using \( \|\cdot\|_{R,a} \) instead of the usual weighted norms \( \|\cdot\|_{1,a} \), we gain a control over the operator norms of \( \mathcal{B} \) and \( \mathcal{G}(t) \). They are almost contractions as \( \theta \to 1^- \) and \( t \to 0^+ \).

The following pointwise lower estimate for the kernel of \( \mathcal{B} \) will be crucial in our investigation of uniqueness.

Lemma 3 For all \( n \geq 1 \), there is a constant \( b_0 > 0 \) such that

\[
B(x) \geq b_0 \phi_0(x)e^{-|x|} \quad (x \in \mathbb{R}^n), \quad \text{where} \quad \phi_0(x) = \begin{cases} |x|^{\frac{1-n}{2}} & \text{if } n \neq 2 \text{ or } |x| \geq 1, \\ 1 - \ln |x| & \text{if } n = 2 \text{ and } |x| < 1. \end{cases}
\]

Proof. It is well-known (see [14]) that \( B \) is radial, strictly decreasing in \( |x| \), and

\[
B(x) \sim \begin{cases} |x|^{\frac{1-n}{2}}e^{-|x|} & |x| \to \infty, \\ \xi(x) & x \to 0 \end{cases}, \quad \text{where} \quad \xi(x) = \begin{cases} 1 & n = 1, \\ 1 - \ln |x| & n = 2, \\ |x|^{2-n} & n \geq 3. \end{cases}
\]

Thus we can choose \( C_1, C_2 > 0 \) so that \( B(x) \geq C_1 |x|^{\frac{1-n}{2}}e^{-|x|} \) for all \( |x| \geq 1 \) and \( B(x) \geq C_2 \xi(x) \) for all \( |x| < 1 \).

The desired estimate is trivial when \( n = 2 \).

In the case \( n \neq 2 \), one can verify directly that \( \xi(x) \geq |x|^{\frac{1-n}{2}}e^{-|x|} \) for all \( |x| < 1 \). So by taking \( b_0 = \min\{C_1, C_2\} \), we obtain the desired estimate as well. \( \square \)

2.3 Mild solutions, super/sub-solutions

We specify the notions of solutions considered in this work. Let \( 0 < T \leq \infty \) and \( a \leq 0 \).

Definition 1 A function \( u \geq 0 \) is said to be a mild solution of (1.1) if \( u \in \mathcal{Z}_T^a \) for some \( (T,a) \) and \( u \) satisfies the equation \( u = \mathcal{M}u \) on \( Q_T \), where \( \mathcal{M} \) is the integral operator defined by (2.1).

We will also need the following variant in the study of uniqueness.
Let $\text{Lemma 4}$ where $F$

Proof. (i) Setting $Y = B \{ Vu^p \} (x, s) ds$.

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From these calculations, we conclude that $u \geq 0$.
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Remark 2

(a) The above two notions of solutions coincide for differentiable solutions, i.e. $u \in C^1([0, T); C^2(\mathbb{R}^n))$. This is not true in the $Z_T^0$-setting however. In fact, it will be revealed that the latter is “stronger”.

(b) In view of (a), we will also use $M$-mild solutions and $P$-mild solutions to refer to solutions according to the above definitions.

(c) Whenever the dependence of the operators (2.1), (2.2) on $u_0$ is crucial, these operators will be denoted by $M_{u_0}$ and $P_{u_0}$.

(d) Mild super-solutions (respectively, mild sub-solutions) are defined by changing ‘$=$’ in the identities to ‘$\geq$’ (resp., ‘$\leq$’).

Lemma 4 Let $u, v \in Z_T^0$ and $u, v \geq 0$.

(i) If $u$ is a $P$-mild super-solution of (1.1), then $u \geq Mu$ on $Q_T$, i.e. $u$ is an $M$-mild super-solution.

(ii) If $v$ is a $P$-mild sub-solution of (1.1), then $v \leq Mv$ on $Q_T$, i.e. $u$ is an $M$-mild sub-solution.

Proof. (i) Setting $u = e^t u$ into the inequality $u \geq Pu$, then $u$ satisfies $u(x, t) \geq e^{-t} u_0 + \int_0^t e^{-(t-s)} B u(x, s) ds + \int_0^t e^{-(t-s)} B \{ Vu^p \} (x, s) ds =: X + F u + FY$,

where $X = e^{-t} u_0$, $Y = Vu^p$, and $F \varphi (\cdot, t) = \int_0^t e^{-(t-s)} B \varphi (\cdot, s) ds$. Since $u \geq 0$, we have $u \geq J_0 + F u \geq J_0$ where $J_0 := X + FY$. Then by iteration, we get $u \geq J_0 + F J_0 = (1 + F) X + (F^2 + F) Y =: J_1$ and generally $u(x, t) \geq J_k := (1 + F + \cdots + F^k) X + (F^{k+1} + \cdots + F) Y \quad \forall k \geq 0$.

We calculate $F^k X$ and $F^k Y$. We have $F X = e^{-t} Bu_0 \int_0^t ds = e^{-t} t Bu_0$, $F^2 X = e^{-t} B^2 u_0 \int_0^t ds = e^{-t} B^2 u_0$, and for any $k \geq 0$, $F^k X = e^{-t} B^k u_0$. Next, by Fubini’s theorem, we have $F^2 Y = \int_0^t (t-s) e^{-(t-s)} B^2 \{ Vu^p \} (x, s) ds$, and generally

$F^{k+1} Y = \int_0^t \frac{(t-s)^k}{k!} e^{-(t-s)} B^{k+1} \{ Vu^p \} (x, s) ds$.

From these calculations, we conclude that

$u(x, t) \geq \lim_{k \to \infty} J_k = \sum_{k=0}^\infty F^k X + \sum_{k=0}^\infty F^{k+1} Y = G(t) u_0 + \int_0^t G(t-s) B \{ Vu^p \} (x, s) ds$, 7
which is the desired result.

(ii) We perform an iteration almost entirely the same as (i). First, we express as above,
\[ v(x,t) \leq J_0 + \mathcal{F}v(x,t) \] where \( J_0 = X + \mathcal{F}Y, \mathcal{F} \) are the same. By the same iteration we get
\[ v \leq J_k + \mathcal{F}^{k+1}v \ (J_k \text{ is the same as above}) \] Thus it suffices to show that \( \mathcal{F}^k v \to 0 \) pointwise as \( k \to \infty \).

Fix \( t_0 \in (0, T) \). We choose \( \theta \in (0, 1) \) such that \( \theta^{-1}(1 - e^{-t_0}) =: \gamma < 1 \) and \( R > 0 \) sufficiently large according to Proposition 1 (ii). If \( 0 < t \leq t_0 \), then
\[
\| \mathcal{F}v(\cdot, t) \|_{R,a} \leq \int_0^t e^{-(t-s)\theta^{-1}} \| v(\cdot, s) \|_{R,a} ds \leq \gamma \sup_{[0,t_0]} \| v(\cdot, s) \|_{R,a} = \gamma \| v \|_{Z_{t_0}^a}.
\]
Thus we have
\[
\| \mathcal{F}v \|_{Z_{t_0}^a} \leq \gamma \| v \|_{Z_{t_0}^a}
\]
which implies
\[
\| \mathcal{F}^k v \|_{Z_{t_0}^a} \leq \gamma^k \| v \|_{Z_{t_0}^a} \to 0,
\]
as \( k \to \infty \). Now \( 0 \leq \mu(x)a^\sigma \mathcal{F}^k v(x,t) \leq \| \mathcal{F}^k v \|_{Z_{t_0}^a} \to 0 \) for all \( (x,t) \in Q_{t_0} \). Therefore, \( \mathcal{F}^k v \to 0 \) pointwise as \( k \to \infty \).

□

Corollary 1 If \( u \) is a \( \mathcal{P} \)-mild solution, then it is an \( \mathcal{M} \)-mild solution.

Finally, we present semigroup properties for the Cauchy problem (1.1). We use the following notation
\[
u(\cdot, t) = u(\cdot, t + \tau), \quad u(\tau) = u(\cdot, \tau), \quad u^* = \mathcal{M}u.
\]

Lemma 5 Let \( \tau \in (0, T) \) and \( u \in Z_{t}^a \) with \( u \geq 0 \).

(i) If \( u \) is a mild super-solution of (1.1), then \( u_\tau \) is an \( \mathcal{M}_{u^*(\tau)} \)-mild super-solution.

(ii) If \( u \) is a mild sub-solution of (1.1), then \( u_\tau \) is an \( \mathcal{M}_{u^*(\tau)} \)-mild sub-solution.

(iii) If \( u \) is a mild solution of (1.1), then \( u_\tau \) is an \( \mathcal{M}_{u(\tau)} \)-mild solution.

3 Global existence

We establish the global existence of solutions \( u \geq 0 \) for the Cauchy problem (1.1) under the main assumption that
\[ 0 \leq V \in C(Q_\infty) \quad \text{and} \quad V(x,t) \lesssim |x|^\sigma \quad \text{as} \ |x| \to \infty, \text{uniformly in} \ t, \quad (A1) \]
for some \( \sigma \in \mathbb{R} \).

Let \( a \leq -\sigma \frac{\mu}{1 - \mu} \). Fix \( \theta \in (0, 1) \) and \( R \geq 1 \) satisfying Lemma 1 (ii). Then the assumption on the potential above is equivalent to that \( V \in C(Q_\infty) \) and
\[ 0 \leq V(x,t) \leq c_0 (R^2 + |x|^2)^{\frac{\sigma}{2}} \quad \forall \ (x,t) \in Q_\infty, \quad (A1') \]
where \( c_0 > 0 \) is a constant. Let us denote \( \rho := \mu^a = (R^2 + |x|^2)^{\frac{\sigma}{2}} \).

We begin with the following simple but important lemma.
Lemma 6 Assume (A1′) and \(a \leq -\frac{\sigma_p}{1-p}\). Then

\[0 \leq \rho^{-p} V(x,t) \leq c_0 \rho^{-1}\]

or

\[0 \leq \rho V(x,t) \leq c_0 \rho^p\]

and, for any \(\varphi \in BC_a(\mathbb{R}^n)\),

\[\|G(t) B \{V \varphi^p\}\|_{R,a} \leq c_1 \beta(t) \|\varphi\|_{R,a}^p, \quad c_1 := \theta^{-1} c_0.\]

Proof. Since \(a \leq -\frac{\sigma_p}{1-p}\), we have \(\sigma - ap + a \leq \sigma_p + ap + a = (1 - p) \left(\frac{\sigma_p}{1-p} + a\right) \leq 0\). Hence \(\sigma - ap \leq -a\). It follows that

\[0 \leq \rho^{-p} V(x,t) \leq c_0 (R^2 + |x|^2)^{\frac{\sigma - ap}{2}} \leq c_0 (R^2 + |x|^2)^{-\frac{a}{2}} = c_0 \rho^{-1},\]

where we have used (A1′) and that \(R \geq 1\).

Next, by Proposition (1) and the preceding estimate, we have

\[\|G(t) B \{V \varphi^p\}\|_{R,a} \leq \theta^{-1} \beta(t) \|\rho V \varphi^p\|_{L^\infty} \leq c_1 \beta(t) \|\varphi\|_{R,a},\]

which is the desired estimate. \(\Box\)

Now we prepare to solve the main problem. Observe that a function \(u \in Z^a_\infty\) is a global mild solution of (1.1) if and only if the function

\[w := \rho u \in Z^0_\infty\]

satisfies the integral equation

\[w(x,t) = \rho G(t) u_0 + \rho \int_0^t G(t-s) B \{\rho^{-p} V w^p\}(x,s) ds.\] \hspace{1cm} (3.1)

Since \(w \mapsto w^p\) is not Lipschitz continuous near \(w = 0\) when \(0 < p < 1\), the contraction mapping approach cannot be employed directly to this equation to get the solutions. An approximation argument is needed (see for instance [13],[15]). So we choose \(\Phi(w) \approx w^p\) where \(\Phi : \mathbb{R} \rightarrow \mathbb{R}\) satisfies

\[\Phi \in \text{Lip}(\mathbb{R}), \quad \Phi' \geq 0, \quad \Phi(0) = 0.\] \hspace{1cm} (3.2)

Then we solve the (approximating) problem: Find \(0 \leq \omega \in Z^0_\infty\) satisfying

\[\omega(x,t) = Q_{w_0} \omega := \rho G(t) u_0 + \rho \int_0^t G(t-s) B \{\rho^{-p} V \Phi(\omega)\}(x,s) ds.\] \hspace{1cm} (3.3)

Proposition 2 Assume (A1), (3.2), and \(a \leq -\frac{\sigma_p}{1-p}\). If \(u_0 \in BC_a(\mathbb{R}^n)\) then the equation (3.3) admits a unique solution \(\omega \in Z^0_\infty\). Moreover, if \(u_0 \geq 0\) then \(\omega \geq 0\).

Proof. Since \(\Phi(0) = 0\) and \(\Phi\) is Lipschitz continuous, there is a constant \(L > 0\) such that \(|\Phi(s)| \leq L|s|\) for all \(s \in \mathbb{R}\).

Let \(T > 0\) and fix \(u_0 \in BC_a(\mathbb{R}^n)\). We equip the Banach space \(X := Z^0_T = C([0,T];C_b(\mathbb{R}^n))\) with the norm

\[\|\omega\|_X = \sup_{t \in [0,T]} \|\omega(\cdot,t)\|_{L^\infty} = \sup_{Q_T} |\omega(x,t)|.\]
Since $u_0$ is fixed, we denote for simplicity $Q := Q_{u_0}$. Thus $\omega$ is a solution of (3.3) if and only $\omega$ is a fixed point of $Q$.

We show that $Q$ is a self-map on $\mathcal{X}$. Let $\omega \in \mathcal{X}$. By Proposition 1 and Lemma 6, we have

$$
\|Q\omega(\cdot, t)\|_{L^\infty} \leq \|G(t)u_0\|_{R,a} + \int_0^t \|G(t-s)B\{\rho^{-p}V\Phi(\omega)\}(\cdot, s)\|_{R,a} ds
$$

$$
\leq \beta(t)\|u_0\|_{R,a} + \int_0^t \beta(t-s)\theta^{-1}\|\{\rho^{-p}V\Phi(\omega)\}(\cdot, s)\|_{R,a} ds
$$

$$
\leq \beta(t)\|u_0\|_{R,a} + \theta^{-1}c_0 \int_0^t \beta(t-s)\|\Phi(\omega)(\cdot, s)\|_{L^\infty} ds
$$

$$
\leq \beta(t)\|u_0\|_{R,a} + \theta^{-1}c_0L \int_0^t \beta(t-s)\|\omega(\cdot, s)\|_{L^\infty} ds
$$

$$
\leq \beta(T)(\|u_0\|_{R,a} + c_1LT\|\omega\|_{X}),
$$

where we have used that $\|\omega(\cdot, s)\|_{L^\infty} \leq \|\omega\|_X$. Thus $\omega(\cdot, s) \in L^\infty(\mathbb{R}^n)$ for all $t \in [0, T]$. Clearly, $Q\omega(\cdot, s) \in C(\mathbb{R}^n)$ so $Q\omega(\cdot, s) \in C_b(\mathbb{R}^n)$. For any $s, t \in (0, T]$, one can argue as above to get that

$$
\|Q\omega(\cdot, s) - Q\omega(\cdot, t)\|_{L^\infty} \leq \|(G(t) - G(s))u_0\|_{R,a} + c_1L\beta(T)\|\omega\|_X|s - t| \to 0
$$

as $|s - t| \to 0$. So $t \mapsto Q\omega(\cdot, t)$ is continuous, hence $Q\omega \in \mathcal{X}$. Also, note that $Q$ satisfies the estimate

$$
\|Q\omega\|_X \leq \beta(T)(\|u_0\|_{R,a} + c_1LT\|\omega\|_X).
$$

(3.4)

Let $D > 0$ be a large number to be specified and define

$$
\mathcal{X}_D = \{\omega \in \mathcal{X} : \|\omega\|_X \leq D\}.
$$

We show that $Q$ is a self-map on $\mathcal{X}_D$ provided $D$ is sufficiently large and $T$ is sufficiently small. Let $\omega \in \mathcal{X}_D$. By (3.4), we have

$$
\|Q\omega\|_X \leq D
$$

provided $T, D$ satisfy

$$
\beta(T)\|u_0\|_{R,a} \leq \frac{D}{2}
$$

and

$$
\frac{1}{2}c_1L\beta(T)T \leq \frac{1}{2}.
$$

It is clear that we can choose a sufficiently small $T > 0$ independent of $u_0$ to satisfy the second condition, then choose a large $D$ (which is allowed to depend on both $T$ and $u_0$) to meet the first condition. Now we fix such a choice of $D, T$.

We show that $Q$ is a contraction on $\mathcal{X}_D$. Let $\omega_1, \omega_2 \in \mathcal{X}_D$. By the Lipschitz property of $\Phi$ and Lemma 6, then

$$
\|Q\omega_1 - Q\omega_2\|_X \leq \sup_{t \in [0, T]} \int_0^t \|G(t-s)B\{\rho^{-p}V(\Phi(\omega_1) - \Phi(\omega_2))\}(\cdot, s)\|_{R,a} ds,
$$

$$
\leq \sup_{t \in [0, T]} \theta^{-1}\beta(t)c_0 \int_0^t \|\{\Phi(\omega_1) - \Phi(\omega_2)\}(\cdot, s)\|_{L^\infty} ds,
$$

$$
\leq c_1L\beta(T)T\|\omega_1 - \omega_2\|_X \leq \frac{1}{2}\|\omega_1 - \omega_2\|_X.
$$
Hence, $Q : X_{\delta} \to X_{\delta}$ is a contraction as desired.

By the Banach fixed point theorem, there is a unique mild solution $\omega$ to the Cauchy problem (3.3) which is defined on the time interval $[0, T]$. The semigroup property of the equation (3.3) is

$$\omega_t(x, t) = \tilde{Q}_{\rho^{-1}\omega(T)}\omega_T := \rho G(t) (\rho^{-1}\omega(T)) + \rho \int_0^t G(t-s) B \left\{ \rho^{-p} V T \Phi(\omega_T) \right\} (x, s)ds,$$

where $V_T, \omega_T$ are the translations by $T$ (forward in time) of $V, \omega$ respectively. Since the local existence time $T = T(\theta, c_0, L)$ is independent of $u_0$ and (A1′) holds independent of $t$, we can apply the preceding result to obtain a unique solution on $[0, T, 2T, 3T, \ldots]$ and so forth. Hence the solution can be extended globally in time. □

**Proposition 3** Assume (A1), (3.2), and $a \leq -\frac{\sigma_a}{1 - p}$. If $\omega$ and $\tilde{\omega}$ satisfy

$$\omega = Q_{u_0} \omega \quad \text{on } Q_T, \quad \tilde{\omega} = Q_{\tilde{u}_0} \tilde{\omega} \quad \text{on } Q_T,$$

and $u_0 \geq \tilde{u}_0$ on $\mathbb{R}^n$, then $\omega \geq \tilde{\omega}$ on $Q_T$. In particular, if $u_0 \geq 0$ then $\omega \geq 0$.

**Proof.** Since $\tilde{u}_0 \leq u_0$, we have $\rho G(t)(\tilde{u}_0 - u_0) \leq 0$. Then

$$(\tilde{\omega} - \omega)(x, t) \leq \int_0^t \rho G(t-s) B \left\{ \rho^{-p} V (\tilde{\omega} - \omega) \right\} (x, s)ds$$

$$\leq L \int_0^t \| G(t-s) B \left\{ \rho^{-p} V (\tilde{\omega} - \omega) \right\} (\cdot, s) \|_{R,a} ds$$

$$\leq c_1 L \int_0^t \beta(t-s) \| (\tilde{\omega} - \omega)_{+}(\cdot, s) \|_{L^\infty} ds$$

$$\leq c_1 L \beta(t) \int_0^t \| (\tilde{\omega} - \omega)_{+}(\cdot, s) \|_{L^\infty} ds.$$

Thus the continuous function $\zeta(t) := \| (\tilde{\omega} - \omega)_{+}(\cdot, t) \|_{L^\infty} \geq 0$ satisfies the inequality $\zeta(t) \leq c_1 L \beta(t) \int_0^t \zeta(s) ds$. By Gronwall’s inequality, we conclude that $\zeta \equiv 0$. Therefore $\omega \geq \tilde{\omega}$. □

The following elementary result will also be useful.

**Lemma 7** Let $a \in \mathbb{R}$. If $\varphi \in BC_{a}(\mathbb{R}^n)$ and $\varphi \geq 0$, then

$$G(t)\varphi(x) \geq e^{-t} \varphi(x) \quad \forall x \in \mathbb{R}^n.$$

**Proof.** The function $u = G(t)\varphi \geq 0$ solves the linear problem $\partial_t u - \Delta u = \Delta u, u|_{t=0} = \varphi$, in the sense of mild solutions. So the function $u = e^t u$ satisfies

$$u(x, t) = \varphi(x) + \int_0^t B u(x, s) ds.$$

Since $u \geq 0$, we have $u(x, t) \geq \varphi(x)$ for all $x \in \mathbb{R}^n$. Hence $G(t)\varphi = e^{-t} u(x, t) \geq e^{-t} \varphi(x)$ as needed.

We can now state and prove our main global existence result.
\textbf{Theorem 1} Assume (A1) and \( a \leq -\frac{\sigma}{1-p} \). If \( u_0 \in BC_a(\mathbb{R}^n) \) with \( u_0 \geq 0 \), then the Cauchy problem (1.1) admits a non-negative global mild solution \( u \in C([0, \infty); BC_a(\mathbb{R}^n)) \).

\textbf{Proof.} For each \( m \in \mathbb{N} \), let \( \Phi_m \) be a function satisfying (3.2) and \( \Phi_m(s) = s^p \) for all \( s \geq m^{-2} \). One may choose for example,

\[
\Phi_m(s) = \begin{cases} 
  m^{2(1-p)}s & \text{if } 0 \leq s \leq m^{-2}, \\
  s^p & \text{if } s \geq m^{-2}, \\
  -\Phi_m(-s) & \text{if } s < 0.
\end{cases}
\]

Fix \( u_0 \in BC_a(\mathbb{R}^n) \) and define \( \psi_m =: u_0 + (m\rho)^{-1} \in BC_a(\mathbb{R}^n) \). Let \( \omega_m \in Z^0_{\infty} \) be the unique mild solution of (3.3) having \( (\Phi, u_0) \rightarrow (\Phi_m, \psi_m) \). This means \( \omega_m \) satisfies the equation \( \omega_m = Q_{\psi_m}^n \omega_m \) on \( Q_{\infty} := \mathbb{R}^n \times [0, \infty) \), where

\[
Q_{\psi_m}^n \omega(x, t) := \rho \mathcal{G}(t)\psi_m + \rho \int_0^t \mathcal{G}(t-s) \mathcal{B} \{ \rho^{-p} V \Phi_m(\omega) \} (x, s) ds.
\]

Since \( \Phi_m(\omega_m) \geq 0 \) and \( \psi_m \geq (m\rho)^{-1} \), \( \omega_m \) has the uniform lower bound:

\[
\omega_m(x, t) = Q_{\psi_m}^n \omega_m(x, t) \geq \rho \mathcal{G}(t)\psi_m(x) \geq \rho \mathcal{G}(t) (m\rho)^{-1} \geq \rho e^{-t}(m\rho)^{-1} = \frac{e^{-t}}{m} \quad \forall (x, t) \in Q_{\infty},
\]

according to the preceding lemma.

\textbf{Claim 1.} For each fixed \( T > 0 \), there is \( N > 0 \) such that \( \{ \omega_m : m \geq N \} \) is pointwise non-increasing on \( Q_T \).

\textbf{Proof.} Choose \( N \geq e^T \). Then for all \( (x, t) \in Q_T, m \geq N \), we have

\[
\omega_m(x, t) \geq \frac{e^{-t}}{m} \geq \frac{1}{mN} \geq m^{-2}. \quad (3.5)
\]

Since \( \Phi_m(s) = s^p \) for all \( s \geq m^{-2} \) by the choice of \( \Phi_m \), it follows that

\[
\Phi_m(\omega_m) = \omega_m^p \quad \text{on } Q_T, \quad \text{for all } m \geq N. \quad (3.6)
\]

This implies that for \( \ell > m \geq N \) we have \( m^{-2} > \ell^{-2} \) hence

\[
\Phi_\ell(\omega_m) = \omega_m^p = \Phi_m(\omega_m) \quad \text{on } Q_T.
\]

Now \( \omega_m \) and \( \omega_\ell \), for \( \ell > m \geq N \), satisfy equations of type (3.3):

\[
\omega_m = Q_{\psi_m}^\ell \omega_m \quad \text{on } Q_T, \quad \omega_\ell = Q_{\psi_\ell}^t \omega_\ell \quad \text{on } Q_T.
\]

Since \( \psi_\ell \leq \psi_m \), it follows by Proposition 3 that \( \omega_\ell(x, t) \leq \omega_m(x, t) \) for all \( (x, t) \in Q_T \). Hence the claim is true. \( \square \)
In view of the claim, we can define the function \( w : \mathbb{R}^n \times [0, \infty) \to \mathbb{R} \) by the pointwise limit:

\[
w(x, t) = \lim_{m \to \infty} \omega_m(x, t).
\]

Clearly \( w \in L^\infty_{\text{loc}}([0, \infty); L^\infty(\mathbb{R}^n)) \) and \( w \geq 0 \).

**Claim 2.** The function \( w \) satisfies (3.1) on \( Q_\infty \).

**Proof.** Fix \( T > 0 \) and \( m \geq N \geq e^T \). Let \( (x, t) \in Q_T \). By (3.6) we have

\[
\omega_m(x, t) = \rho \mathcal{G}(t) u_0 + \frac{1}{m} \rho \mathcal{G}(t) \rho^{-1} + \int_0^t \rho \mathcal{G}(t-s) \mathcal{B} \{ \rho^{-p} V \omega_m^p \} (x, s) ds. 
\]

By (3.5), we have \( \omega_m \geq m^{-2} \) hence \( \omega_m^p = \omega_m^{p-1} \omega_m \leq (m^{-2})^{p-1} \omega_m \). This implies that for \( m = N \), the integral on the right hand side of (3.7) is convergent:

\[
\int_0^t \rho \mathcal{G}(t-s) \mathcal{B} \{ \rho^{-p} V \omega_N^p \} (x, s) ds \leq c_1 N^{2-2p} T \beta(T) \| \omega_N \|_{\mathcal{Z}_\infty^0} < \infty.
\]

We take \( m \to \infty \) in (3.7). The left hand side converges (pointwise) to \( w(x, t) \). On the right hand side, the first term is constant while the second term converges to zero. For the third term, we apply the monotone convergence theorem to conclude that this term converges to \( \int_0^t \rho \mathcal{G}(t-s) \mathcal{B} \{ \rho^{-p} V w^p \} (x, s) ds \). Thus, as \( m \to \infty \), \( w \) satisfies

\[
w(x, t) = \rho \mathcal{G}(t) u_0 + \rho \int_0^t \mathcal{G}(t-s) \mathcal{B} \{ \rho^{-p} V w^p \} (x, s) ds \quad \text{(on } Q_T),
\]

i.e. the equation (3.1). Since \( T > 0 \) is arbitrarily, the claim follows. \( \square \)

Finally, we set \( u = \rho^{-1} w \). We have shown that \( \rho u(\cdot, t) \in L^\infty(\mathbb{R}^n) \) and \( u \) satisfies \( u(x, t) = M u(x, t) \) for all \( x \in \mathbb{R}^n, t > 0 \). Using Proposition 1 (ii), one can readily check that \( M u(\cdot, t) \in BC_a(\mathbb{R}^n) \). Finally, one can argue as in Proposition 2 to find that \( t \mapsto M u(\cdot, t) \) is continuous. Therefore \( u = M u \in C([0, \infty); BC_a(\mathbb{R}^n)) \) is a mild solution of the Cauchy problem (1.1). \( \square \)

## 4 Grow-up rate

In this section, we prove a lower grow-up of solutions to the Cauchy problem (1.1) when \( u_0 \) is non-zero.

**Lemma 8** Assume that \( u_0 \in C(\mathbb{R}^n; \mathbb{R}_+) \setminus \{0\} \). If \( 0 \leq u \in \mathcal{Z}_T^\theta \) satisfies

\[
u(x, t) \geq u_0(x) + \int_0^t \mathcal{B} u(x, s) ds \quad \text{(on } Q_T),
\]

then, for \( \delta > 1, t_0 \in (0, T) \), there is a constant \( C_0 = C_0(\delta, t_0, u_0) \) such that

\[
\int_0^{t_0} \mathcal{B} u(x, s) ds \geq C_0 e^{-\delta|x|} \quad (\forall x \in \mathbb{R}^n).
\]
Proof. Since \( B > 0 \) and \( u \geq 0 \), we have \( u \geq u_0 \). This implies \( u \geq \int_0^t Bu_0 ds > 0 \) on \( Q_T \).
Let \( 0 < \varepsilon < t_0 \). We define

\[
m_0 := \min_{|x| \leq 2, x \in [0, t_0]} \int_0^t Bu(x, s) ds
\]

Since \( u > 0 \) and it is continuous, \( m_0 > 0 \).

If \( |x| > 2 \) and \( |y| < 1 \) then \( 1 \leq |x - y| \leq |x| + 1 \) hence, by Lemma 3,

\[
\int_0^{t_0} Bu(x, s) ds \geq \int_0^{t_0} \int_{|y| < 1} B(x - y) u(y, s) dy ds
\]

\[
\geq b_0 m_0 \int_0^{t_0} \int_{|y| < 1} |x - y| |\frac{1}{x} - \frac{1}{y}| dy ds
\]

\[
\geq b_0 m_0 e^{-\frac{|x|}{1}} (t_0 - \varepsilon) \int_{|y| < 1} |x - y| |\frac{1}{x} - \frac{1}{y}| dy
\]

\[
\geq Ce^{-|x|} (1 + |x|)^{-\frac{n-1}{2}} \geq C_1 e^{-\delta |x|}.
\]

If \( |x| \leq 2 \), then \( \int_0^{t_0} Bu(x, s) ds \geq m_0 \geq C_2 e^{-\delta |x|} \). Taking \( C_0 = C_1 \land C_2 \), the desired estimate then follows.

Remark 3 By this lemma, we find that if \( u \) is a \( P \)-mild super-solution of (1.1) and \( u_0 \neq 0 \), then for each \( \delta > 1 \), \( t_0 \in (0, T) \), we have \( u(x, t_0) \geq Pu(x, t_0) \geq C_0 e^{-|x|} \) on \( \mathbb{R}^n \), for a constant \( C_0 = C_0(\delta, t_0, u_0) \).

Now we assume the potential \( V \in C(Q_T) \) satisfies

\[
V(x, t) \geq \vartheta_0 |x|^\sigma \quad \text{(on } Q_T), \tag{A3}
\]

where \( \vartheta_0 > 0 \) and \( \sigma \geq 0 \) are constants.

Lemma 9 Assume \( (A3) \). Then there is a constant \( d_0 > 0 \) and a function

\[
\eta \in C(\mathbb{R}), \quad \eta > 0, \quad \eta \text{ is non-increasing}
\]

such that, for any \( P > 0 \), the following estimate holds:

\[
B\{Ve^{-P|\cdot|}\}(x, t) \geq d_0 \eta(P) e^{-P|x|} \quad \text{(on } Q_T).
\]

Proof. By Lemma 3, \( B(x) \geq b_0 \varphi_0(x) e^{-|x|} > 0 \). Then

\[
B\{Ve^{-P|\cdot|}\}(x, t) = \int B(y) V(x - y, t) e^{-P|x - y|} dy
\]

\[
\geq b_0 \varphi_0 \int \varphi_0(y) e^{-|y| |x - y|^\sigma} e^{-P(|x| + |y|)} dy
\]

\[
\geq b_0 \varphi_0 \omega_n e^{-P|x|} \int_0^\infty \varphi_0(r) e^{-(1 + P)r |x|^\sigma} r^{-n} dr := d_0 e^{-|x|} K,
\]

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where \( d_0 = b_0 \partial_0 \omega_n \). We estimate the integral \( K \). If \( |x| \leq 1 \) and \( r \geq 2|x| \), then \( |r - |x|| = r - |x| \geq \frac{r}{2} \); hence

\[
K \geq \int_{2|x|}^{\infty} \phi_0(r) e^{-(1+P)r} \left( \frac{r}{2} \right)^{\sigma} r^{n-1} dr \geq 2^{-\sigma} \int_{2}^{\infty} \phi_0(r) e^{-(1+P)r} r^{n+\sigma-1} dr =: \eta_1(P).
\]

If \( |x| > 1 \) and \( r < \frac{|x|}{2} \), then \( |r - |x|| = |x| - r > r \); hence

\[
K \geq \int_{0}^{x/2} \phi_0(r) e^{-(1+P)r} r^{n+\sigma-1} dr \geq \int_{0}^{1/2} \phi_0(r) e^{-(1+P)r} r^{n+\sigma-1} dr =: \eta_2(P).
\]

Both \( \eta_1, \eta_2 \) are finite because \( \int_{1}^{\infty} \phi(r) e^{-(1+P)r} r^{n+\sigma-1} dr < \infty \). Also, \( \eta_1, \eta_2 > 0 \). Setting \( \eta(P) = \min\{\eta_1(P), \eta_2(P)\} \), the desired estimate then follows. \( \square \)

**Remark 4** The results of the above lemma and the following theorem can be generalized to potentials \( V \) satisfying \( V(x,t) \geq \sum_{j=1}^{N} \psi_j |x-x_j|^\gamma \) on \( Q_T \), where \( \psi_j > 0 \), \( x_j \in \mathbb{R}^n \), \( \sigma_j \geq 0 \) are given parameters.

We prove a lower bound for *grow-up rate* of solutions of (1.1).

**Theorem 2** Assume (A3) and \( u_0 \in C(\mathbb{R}^n; \mathbb{R}_+) \setminus \{0\} \). If \( u \geq 0 \) is a \( \mathcal{P} \)-mild super-solution of (1.1), then \( u \) satisfies

\[
u(x,t) \geq \{\partial_0 |x|^\sigma (1-p)t\}^{\frac{1}{1-p}} \quad \text{(on } Q_T).\]

**Proof.** Let \( \delta > 1 \) be such that \( \tilde{p} := \delta p \in (0,1) \). We split the proof into several steps.

**Step 1.** We establish the theorem in the case that \( u_0(x) \geq C_0 e^{-\delta|x|} \) for some constant \( C_0 > 0 \). We claim that \( u(x,t) \geq \{\partial_0 |x|^\sigma (1-p)t\}^{\sigma} \) on \( Q_T \). It will be proved by repeatedly applying the fact that

\[
u \geq \mathcal{P}_t \nu := \int_{0}^{t} \mathcal{B}(V \nu^p) \nu \, ds
\]

which is true because \( \nu \) is a \( \mathcal{P} \)-mild super solution of (1.1) and \( B > 0 \), \( \nu \geq 0 \).

Since \( u \) is a super-solution, we have \( u(x,t) \geq \nu_0(x) \geq C_0 e^{-\delta|x|} \) on \( Q_T \). Using Lemma 9 and (A3), we have

\[
u(x,t) \geq \mathcal{P}_t \nu(x,t) \geq C_0^p \int_{0}^{t} \mathcal{B}(V e^{-\tilde{p}|x|}) \nu \, ds \quad (\tilde{p} = \delta p)
\]

\[
\geq C_0^p \partial_0 \eta(\tilde{p}) e^{-\tilde{p}|x|} \int_{0}^{t} ds =: C_1 \eta e^{-\tilde{p}|x|}, \quad \text{where}
\]

\[
C_1 = C_0^p \partial_0 \eta(\tilde{p}).
\]

Similarly, using this estimate and that \( 0 < p < \tilde{p} < 1 \) we iterate to get

\[
u(x,t) \geq \mathcal{P}_t \nu(x,t) \geq \int_{0}^{t} \mathcal{B}(C_0^p s^p V e^{-\tilde{p}|x|}) \nu \, ds
\]

\[
\geq C_1^p \partial_0 \eta(\tilde{p}^2) e^{-\tilde{p}^2|x|} \int_{0}^{t} s^p ds =: C_2 t^{1+p} e^{-\tilde{p}^2|x|}, \quad \text{where}
\]

\[
C_2 = C_1^p \partial_0 \eta(\tilde{p}^2)(1+p)^{-1}
\]

\[
= C_0^p \partial_0 \eta(\tilde{p})^{p} \eta(\tilde{p}^2)(1+p)^{-1}.
\]

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Applying the same reasoning, we get

\[ u(x, t) \geq \mathcal{P}_* u(x, t) \geq C_p^0 \int_0^t s^{p+2} B\{Ve^{-\tilde{p}^3|.|}\}(x, s)ds \]

\[ \geq C_p^0 d_0 \eta(\tilde{p}^3) e^{-\tilde{p}^3|x|} \int_0^t s^{p+2} ds =: C_3 t^{1+p+2} e^{-\tilde{p}^3|x|}, \]

where

\[ C_3 = C_p^0 d_0 \eta(\tilde{p}^3)(1 + p + p^2)^{-1} \]

\[ = C_p^0 d_0^{1+p+2} [\eta(\tilde{p})^p (\eta(\tilde{p}))^p (\eta(\tilde{p}))^p](1 + p)^{-p}(1 + p + p^2)^{-1}. \]

**Step 2.** By induction, we conclude that

\[ u(x, t) \geq C_{k+1} t^{1+p+\cdots+p^k} e^{-\tilde{p}^{k+1}|x|}, \]

where

\[ C_{k+1} = C_p^{k+1} d_0^{k+1+p+\cdots+p^k} \prod_{j=1}^{k+1} \eta(\tilde{p})^j \prod_{j=1}^{k} (1 + p + \cdots + p^j)^{-p^j}. \]

Since \( 0 < p, \tilde{p} < 1 \) and \( \eta \) is non-increasing, we have (i) \( p^k, \tilde{p}^k \to 0 \) as \( k \to \infty \), (ii) \( 1 + p + \cdots + p^j \leq q \), and (iii) \( \eta(\tilde{p})^j \geq \eta(\tilde{p}) \) for all \( j \geq 1 \). Thus

\[ C_{k+1} \geq C_p^{k+1} d_0^{k+1+p+\cdots+p^k} \eta(\tilde{p})^{\sum_{j=1}^{k+1} p^k} q - \sum_{j=1}^{k} p^j \]

\[ \geq C_p^{k+1} d_0^{k+1+p+\cdots+p^k} \eta(\tilde{p})^{1+p+\cdots+p^k} q - q \]

\[ = C_p^{k+1} [d_0 \eta(\tilde{p})]^1 \cdot \cdots \cdot p^k (1 - p)^q. \]

We now have for all \( k \geq 0 \) that

\[ u(x, t) \geq C_p^{k+1} [d_0 \eta(\tilde{p})] t^{1+p+\cdots+p^k} (1 - p)^q e^{-\tilde{p}^{k+1}|x|}. \]

By taking \( k \to \infty \) we obtain

\[ u(x, t) \geq \{d_0 \eta(\tilde{p})(1 - p)\}^q = (Dt)^q \quad \forall (x, t) \in Q_T. \]

**Step 3.** We perform a further iteration. This time we use Lemma 2 that \( B(1) = 1 \) and \( B(|x|^\gamma) \geq |x|^\gamma \) for \( \gamma \geq 0 \). Also note that \( pq = q - 1 \). By (A3), we have

\[ u(x, t) \geq \int_0^t B\{\vartheta_0 \cdot |\sigma| u^p\}(x, s)ds \geq \vartheta_0 |x|^\sigma D^{pq} \int_0^t s^{q-1}ds = \vartheta_0 |x|^\sigma D^{pq} q^{-1} t^q, \]

\[ u(x, t) \geq \int_0^t B\{\vartheta_0 \cdot |\sigma| u^p\}(x, s)ds \geq (\vartheta_0 |x|^\sigma)^{1+p} D^{pq} q^{-p} \int_0^t s^{q-1}ds \]

\[ = (\vartheta_0 |x|^\sigma)^{1+p} D^{pq} q^{-1} t^q. \]

By induction, we have

\[ u(x, t) \geq (\vartheta_0 |x|^\sigma)^{\sum_{j=0}^k p^j} D^{pq} q^{-\sum_{j=0}^k p^j} t^q. \]
Since \(0 < p < 1\), by taking \(k \to \infty\), we obtain that
\[
\mathbf{u}(x, t) \geq (\vartheta_0|\mathbf{x}|^\sigma)^q q^{-q} t^q = \{\vartheta_0|\mathbf{x}|^\sigma(1 - p)t\}^q,
\]
which proves the desired claim.

Step 4. We finish the proof in the case that \(u_0 \geq C_0 e^{-\delta|x|}\). Setting \(u = e^{-t} \mathbf{u}\), we have
\[
u(x, t) \geq e^{-t}\{\vartheta_0|x|^\sigma(1 - p)t\}^q
\]
and, by Lemma 4, \(u\) satisfies
\[
u(x, t) \geq \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B}\{Vu^p\}(x,s)\,ds.
\]
Employing Lemma 2 and (A3), we get
\[
u(x, t) \geq (\vartheta_0|x|^\sigma)^{pq+1}(1 - p)^p \int_0^t e^{-ps}s^{q-1} ds
\]
\[
\geq (\vartheta_0|x|^\sigma)^{pq+1}(1 - p)^p \int_0^t e^{-pt}s^{q-1} ds = \{\vartheta_0|x|^\sigma(1 - p)\}^{pq+1}e^{-pt} t^q,
\]
\[
u(x, t) \geq (\vartheta_0|x|^\sigma)^{pq+1}(1 - p)^p + \int_0^t e^{-ps}s^{q-1} ds
\]
\[
\geq \{\vartheta_0|x|^\sigma(1 - p)\}^{pq+1}e^{-pt} t^q.
\]
Generally, we have for any \(k \geq 0\) that
\[
u(x, t) \geq \{\vartheta_0|x|^\sigma(1 - p)\}^{pq+1}e^{-pt} t^q.
\]
Sending \(k \to \infty\) we obtain
\[
u(x, t) \geq \{\vartheta_0|x|^\sigma(1 - p)\}^{q} t^q = \{\vartheta_0|x|^\sigma(1 - p)t\}^q.
\]
This proves the theorem in the case \(u_0 \geq C_0 e^{-\delta|x|}\).

Step 5. Finally, we prove the theorem in the general case when \(u_0 \geq 0\), \(u_0 \neq 0\). Let \(0 < \tau < T\). It follows from the remark after Lemma 8 that \(\mathcal{P}\mathbf{u}(\tau) \geq C_0 e^{-\delta|x|}\) where \(C_0 = C_0(\delta, u_0, \tau) > 0\). Define \(u_\tau(x, t) = u(x, t + \tau)\) on \(Q_{T-\tau}\). Using the semigroup property (Lemma 5), \(u_\tau\) satisfies
\[
u_\tau(x, t) \geq \mathcal{G}(t)\mathcal{P}\mathbf{u}(\tau) + \int_0^t \mathcal{G}(t-s)\mathcal{B}\{Vu^p\}(x,s)\,ds.
\]
Since \(\mathcal{P}\mathbf{u}(\tau) \geq C_0 e^{-\delta|x|}\), it follows by the previous special case that
\[
u(x, t + \tau) = u_\tau(x, t) \geq \{\vartheta_0|x|^\sigma(1 - p)t\}^q \text{ on } Q_{T-\tau}.
\]
This is true for all \(0 < \tau < T\).

For each \(0 < t < T\) and any \(0 < \tau < t\) sufficiently small, we find that
\[
u(x, t) = u(x, (t - \tau) + \tau) \geq \{\vartheta_0|x|^\sigma(1 - p)(t - \tau)\}^q,
\]
for all \(x \in \mathbb{R}^n\). Letting \(\tau \to 0\), \(u\) satisfies the desired estimate. \(\square\)
Remark 5 From this theorem, we have discovered that the unbounded source term $Vu^p$ have induced (i) a grow-up in time:
\[ u(\cdot, t) \sim t^{1-p}, \]
(see also the next section) and a growth in space:
\[ u(x, \cdot) \gtrsim |x|^{\sigma}. \]

The grow-up in $t$ is affected by the sublinearity (by a recent work of the first author) while the growth in $x$ is caused by the unbounded potential. We note that this growth (in $x$) of solutions happen even when the initial condition $u_0$ is bounded.

5 Comparison theorem; uniqueness of solutions

We prove the key comparison theorem for (1.1) when $V = \vartheta_0 |x|^\sigma$ ($\sigma \geq 0$).

Theorem 3 Let $V = \vartheta_0 |x|^\sigma$ where $\vartheta_0 > 0$ and $\sigma \geq 0$, $a \leq -\frac{\sigma}{1-p}$. Assume that $u, v \in C([0, \infty); BC_a(\mathbb{R}^n))$ satisfy $u, v \geq 0$ and
\[
\begin{align*}
u(x, t) &\geq \mathcal{G}(t)u_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B} \{\vartheta_0 |\cdot|^\sigma u^p\} (x, s)ds \\
v(x, t) &\leq \mathcal{G}(t)v_0 + \int_0^t \mathcal{G}(t-s)\mathcal{B} \{\vartheta_0 |\cdot|^\sigma v^p\} (x, s)ds
\end{align*}
\]
for all $(x, t) \in \mathbb{R}^n \times [0, T)$ where $u_0, v_0 \in BC_a(\mathbb{R}^n)$ and $u_0(x) \geq v_0(x) \geq 0$ with $u_0 \neq 0$.

Then $u \geq v$ on $\mathbb{R}^n \times [0, \infty)$.

Proof. Let $\theta \in (0, 1)$ to be specified and $R > 0$ be sufficiently large according to Proposition 1 (ii). Denote $\rho = (R^2 + |x|^2)\frac{\sigma}{2}$ so that $\|\varphi\|_{R,a} = \|\rho \varphi\|_{L^\infty}$. Since $V = \vartheta_0 |x|^\sigma \leq \vartheta_0 (R^2 + |x|^2)^\frac{\sigma}{2}$, (A1') is true and we can apply Lemma 6.

We define
\[ w = (v - u)_+. \]

Since $v_0 \leq u_0$, we have
\[
\begin{align*}
(v-u)(x, t) &\leq \mathcal{G}(t)(v_0 - u_0) + \int_0^t \mathcal{G}(t-s)\mathcal{B} \{V(v^p - u^p)\} (x, s)ds \\
&\leq \int_0^t \mathcal{G}(t-s)\mathcal{B} \{V(v-u)^+_p\} (x, s)ds \quad \text{(by Lemma 1 (i))}
\end{align*}
\]
This implies
\[ w(x, t) \leq \int_0^t \mathcal{G}(t-s)\mathcal{B} \{Vw^p\}(x, s)ds. \]
Let $0 < T < \infty$. For $0 < t \leq T$, by taking the norm, we get
\[
\|w(\cdot, t)\|_{R,a} \leq \int_0^t \|G(t-s)B\{Vw^p\}(\cdot, s)\|_{R,a} \, ds
\]
\[
\leq \theta^{-1} \int_0^t \beta(t-s)\|\rho Vw^p(\cdot, s)\|_{L^\infty} \, ds
\]
\[
\leq \theta^{-1} \vartheta \beta(t) \int_0^t \|w(\cdot, s)\|_{L^\infty}^p \, ds \quad \text{by Lemma 6}
\]
\[
\leq \theta^{-1} \vartheta \beta(T) \int_0^t \|w(\cdot, s)\|_{R,a}^p \, ds.
\]

It follows by Lemma 1 (ii) that
\[
\|w(\cdot, t)\|_{R,a} \leq \left\{ \theta^{-1} \vartheta \beta(T)(1 - p) t \right\}^q \quad (0 < t \leq T). \tag{5.1}
\]

Claim. At each $x \in \mathbb{R}^n$, $t > 0$, we have
\[
(v^p(x, t) - u^p(x, t))_+ \leq pq(\vartheta |x|^\sigma)^{-1} t^{-1} w(x, t). \tag{5.2}
\]

Proof. By the fundamental theorem of calculus, we have
\[
v^p(x, t)^p - u^p(x, t) = \int_0^1 \frac{d}{ds}(sv + (1 - s)u)^p(x, t) \, ds
\]
\[
= p(v(x, t) - u(x, t))\Theta^{-1},
\]
where $\Theta$ is a number between $v(x, t)$ and $u(x, t)$. The claim is trivial if $v(x, t) \leq u(x, t)$. If $v(x, t) \geq u(x, t)$ then
\[
\Theta \geq u(x, t) \geq \{\vartheta |x|^\sigma(1 - p)t\}^q
\]
by Theorem 2. Hence (5.2) is also true and the claim follows. \qed

We perform an alternative estimate for $w(x, t)$. By the claim,
\[
(v - u)(x, t) \leq \int_0^t G(t-s)B\{V(v^p - u^p)_+\}(x, s) \, ds
\]
\[
\leq pq \int_0^t s^{-1} G(t-s)B\{V(\vartheta |\cdot|^\sigma)^{-1}w\}(x, s) \, ds
\]
\[
= pq \int_0^t s^{-1} G(t-s)B\{w\}(x, s) \, ds.
\]

As above, if $0 < t \leq T$, by taking the norm, we get
\[
\|w(\cdot, t)\|_{R,a} \leq pq\theta^{-1} \int_0^t s^{-1}\beta(t-s)\|w(\cdot, s)\|_{R,a} \, ds,
\]
\[
\leq pq\theta^{-1} \beta(T) \int_0^t s^{-1}\|w(\cdot, s)\|_{R,a} \, ds. \tag{5.3}
\]

We introduce the function
\[
H(t) = pq\theta^{-1}\beta(T) \int_0^t s^{-1}\|w(\cdot, s)\|_{R,a} \, ds.
\]
Then (5.3) becomes
\[ H'(t) \leq pq\theta^{-1}\beta(T)t^{-1}H(t). \]
Let \( 0 < \varepsilon < T \). By integration, we have for all \( \varepsilon \leq t < T \) that
\[ H(t) \leq H(\varepsilon)e^{-pq\theta^{-1}\beta(T)t}H(\varepsilon). \]
Using (5.1) and the definition of \( h \), we get that
\[ H(\varepsilon) = pq\theta^{-1}\beta(T)\int_0^\varepsilon s^{-1}\|w(\cdot, s)\|_{R,\alpha}ds \]
\[ \leq pq\theta^{-1}\beta(T)(1+q)(1-p)^q\int_0^\varepsilon s^{-1+q}ds \]
\[ = pq\theta^{-1}\beta(T)(1+q)(1-p)^q\varepsilon^q \]
\[ = C\varepsilon^q. \]
Thus we obtain
\[ H(t) \leq C\varepsilon^q e^{-pq\theta^{-1}\beta(T)t}H(\varepsilon). \]
We choose \( \theta \in (p, 1) \) so that \( p\theta^{-1} \in (0, 1) \). Since \( \beta(T) = e^{(1+\varepsilon)T} \rightarrow 1 \) as \( T \rightarrow 0 \), we can choose \( T > 0 \) sufficiently small such that \( q - pq\theta^{-1}\beta(T) > 0 \). Observe that \( T \) depends only on \( p \). Therefore passing \( \varepsilon \rightarrow 0 \), we obtain \( H(t) = 0 \) for all \( 0 < t < T \). This implies in particular that
\[ v(x, t) \leq u(x, t) \quad \text{on } \mathbb{R}^n \times [0, T/2]. \]
Finally, we employ the semigroup argument. By the preceding result, we have \( u(T/2) \geq v(T/2) \geq 0 \) with \( u(T/2) \neq 0 \), hence \( Mu(T/2) \geq Mv(T/2) \geq 0 \) with \( Mu(T/2) \neq 0 \) on \( \mathbb{R}^n \).
The translations \( u_{T/2}, v_{T/2} \) satisfy
\[ u_{T/2}(x, t) \geq G(t)Mu(T/2) + \int_0^t G(t-s)B\left\{ V^p u_{T/2}^p \right\}(x, s)ds \]
\[ v_{T/2}(x, t) \geq G(t)Mv(T/2) + \int_0^t G(t-s)B\left\{ V^p v_{T/2}^p \right\}(x, s)ds. \]
Since the choice of \( T \) depends only on \( p \), we obtain as above that \( u_{T/2} \geq v_{T/2} \) for \( t \in [0, T/2] \). Thus \( u \geq v \) on \( Q_T \). Similarly, we obtain that \( u \geq v \) on \( Q_{3T/2}, Q_{2T}, \ldots \) and so forth. Therefore \( u \geq v \) on \( \mathbb{R}^n \times [0, \infty) \). \( \square \)

**Remark 6** It is not difficult to modify the proof of the above theorem to get the following generalization. Assume that
\[ \vartheta_0|x|^\sigma \leq V(x, t) \leq \vartheta_1|x|^\sigma \quad (\sigma \geq 0). \]
Then for \( 0 < p < \frac{\sigma}{\sigma+1} \), the preceding theorem is true with \( \vartheta_0|x|^\sigma \) replaced by \( V(x, t) \).

We conclude this section with the following existence and uniqueness result. The proof follows immediately from the comparison theorem above.

**Theorem 4** Let \( 0 < p < 1, \sigma \geq 0, \vartheta_0 > 0, \) and \( a \leq -\frac{\sigma}{1-p} \). Then for each \( u_0 \in BC_a(\mathbb{R}^n) \) such that \( u_0 \geq 0 \) and \( u_0 \neq 0 \), there exists a unique global mild solution \( u \in C([0, \infty); BC_a(\mathbb{R}^n)) \), \( u \geq 0 \) to the Cauchy problem
\[
\begin{aligned}
\left\{ \begin{array}{ll}
\partial_t u - \Delta \vartheta u = \vartheta_0|x|^\sigma u^p & x \in \mathbb{R}^n, t > 0, \\
u(x, 0) = u_0(x) & x \in \mathbb{R}^n.
\end{array} \right.
\end{aligned}
\]
6 Non-uniqueness of solutions; classification

We study the problem \((1.1)\) when \(V = \vartheta |x|^\sigma \) (\(\sigma \geq 0\)) and \(u_0 = 0\), that is, we consider the Cauchy problem

\[
\begin{cases}
\partial_t u - \Delta \partial_t u = \Delta u + \vartheta |x|^\sigma u^p & \text{in } \mathbb{R}^n \times [0, \infty), \\
u|_{t=0} = 0 & \text{in } \mathbb{R}^n.
\end{cases}
\] (6.1)

According to Theorem 1, this problem admits a global mild solution \(0 \leq u \in Z^a_{\infty}\) where \(a \leq -\frac{\sigma}{1-p}\).

In the case \(\sigma = 0\), by a recent work of the first author, the problem (6.1) admits infinitely many solutions. In fact, every non-trivial solution to the problem in this case has the form

\[u(x,t) = U_0(x, (t-\tau)_+)\text{ for some } \tau \geq 0,\]

where

\[U_0 = (\vartheta_0 (1-p) t)^\frac{1}{1-p}\]

is the maximal solution to the zero-initial condition problem.

For the general case \(\sigma \geq 0\), we prove in this section that the same conclusion is also true. For this, let us give a definition.

**Definition 3** By a maximal solution of the problem (6.1), we mean a function \(U \in Z^a_{\infty}\), where \(a \leq -\frac{\sigma}{1-p}\), such that \(U\) satisfies \(u = M_{u_0=0} U\) on \(Q_{\infty}\) and

\[u \leq U \quad \text{on } Q_{\infty}, \text{ for any other solutions } u \text{ of (6.1)}.\]

We have the following uniqueness of maximal solution.

**Theorem 5** The problem (6.1) admits a unique maximal solution.

*Proof.* The uniqueness is obvious. For the existence, let \(u_m \in Z^a_{\infty}\), for each \(m\), be the unique solution of the the problem \(u_m = M_{u_0=0} U_m\), that is \(u_m\) satisfies

\[u_m(x,t) = m^{-1} + \int_0^t G(t-s)B \{\vartheta_0 |x|^\sigma u_m^p\} (x,s) ds.\]

By the comparison principle and the lower grow-up rate, we have

\[\{\vartheta_0 |x|^\sigma (1-p)t\}^{1-p} \leq u_\ell(x,t) \leq u_m(x,t) \quad \text{for all } \ell > m.\]

Thus \(u_m\) converges pointwise to a function \(U\) with

\[U(x,t) \geq \{\vartheta_0 |x|^\sigma (1-p)t\}^{1-p}.\]

Applying the monotone convergence theorem, it follows that \(U\) satisfies \(U = M_0 U\). Employing the same argument as in Theorem 1, we obtain \(U \in Z^a_{\infty}\).

To show that \(U\) is a maximal solution, we assume that \(u \in Z^a_{\infty}\) is a mild solution of (6.1). By Theorem 3 and the choice of \(u_m\), then we have

\[u(x,t) \leq u_m(x,t) \quad \text{on } Q_{\infty} \text{ for all } m.\]

Since \(u_m \rightarrow U\) as \(m \rightarrow \infty\), it follows that \(u \leq U\) on \(Q_{\infty}\). Therefore \(U\) is a maximal solution of (6.1). \(\square\)
Remark 7 We will denote the maximal solution of (6.1) by $U_\sigma$.

Lemma 10 If $0 \leq u \in \mathcal{Z}_\infty^a$ is a mild solution of (6.1) and $t_0 > 0$, then $u(\cdot, (t-t_0)_+)$ is a mild solution of (6.1).

Proof. The proof is immediate from the semigroup property (Lemma 5).

Proposition 4 If $0 \leq u \in \mathcal{Z}_\infty^a$ is a mild solution of (6.1) and there is a sequence $\{(x_m, t_m)\}$ in $\mathbb{R}^n \times (0, \infty)$ such that

$$u(x_m, t_m) > 0 \quad \text{for all } m, \quad \text{and } t_m \to 0^+,$$

then $u = U_\sigma$.

Proof. It follows immediately that $u \leq U_\sigma$. By Lemma 8, we have $u(x, t) > 0$ for all $x \in \mathbb{R}^n$ and $t > 0$. Let $\tau > 0$. By the preceding lemma, $U_\sigma(\cdot, (t-\tau)_+)$ is a mild solution of (6.1) having zero values for $t \in [0, \tau]$. Hence using the comparison principle (Theorem 3), we find that

$$U_\sigma(x, (t-\tau)_+) \leq u(x, t) \quad \text{on } Q_\infty.$$

This is true for all $\tau > 0$, hence we have by continuity that $U_\sigma \leq u$. Therefore $u \equiv U_\sigma$. □

Now we can characterize any nontrivial solutions of (6.1).

Theorem 6 Let $U_\sigma$ be the maximal solution of (6.1). If $0 \leq u \in \mathcal{Z}_\infty^a$ is a nontrivial mild solution of (6.1), then

$$u(x, t) = U_\sigma(x, (t-\tau)_+),$$

for some $\tau \geq 0$.

Proof. Let $0 \leq u \in \mathcal{Z}_\infty^a$ be a nontrivial mild solution of (6.1). Define $\tau = \inf\{t > 0 : u(x, t) > 0 \text{ for some } x \in \mathbb{R}^n\}$. If $\tau = 0$, it follows by the preceding proposition that $u \equiv U_\sigma$. So we assume that $\tau > 0$. In this case, the function $\tilde{u}(x, t) := u(x, (t-\tau)_+)$ is a mild solution of (6.1) according to the preceding lemma. Introducing a new time variable $\xi = t - \tau$, then $\tilde{u}$ satisfies

$$\tilde{u}(x, \xi) = \int_0^\xi \mathcal{G}(t-s) \mathcal{B} \left\{ \vartheta_0 | \cdot |^{\sigma} \tilde{u}^p \right\} (x, s) ds$$

and $\tilde{u}(x, \xi) > 0$ for all $\xi > 0$. We thus obtained by the preceding proposition that

$$\tilde{u}(x, \xi) = U_\sigma(x, \xi) \quad \text{for all } x \in \mathbb{R}^n, \xi > 0.$$

This is equivalent to that $u(x, t) = U_\sigma(x, t-\tau)$ for all $t > \tau$. Since $u(\cdot, t) \equiv 0$ for $0 \leq t < \tau$, we can conclude that $u(x, t) = U_\sigma(x, (t-\tau)_+)$. □

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