GLOBAL WEAK SOLUTION AND SMOOTH SOLUTION OF THE PERIODIC INITIAL VALUE PROBLEM FOR THE GENERALIZED LANDAU-LIFSHITZ-BLOCH EQUATION IN HIGH DIMENSIONS

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Abstract. In this paper, by using the Galerkin method and energy estimates, the global weak solution and the smooth solution to the generalized Landau-Lifshitz-Bloch (GLLB) equation in high dimensions are obtained.

1. Introduction. As we all know, the Landau–Lifshitz equation well describes the magnetization dynamics of ferromagnets at low temperature and many important results have been obtained, see [5]. The Landau–Lifshitz-Gilbert equation is described as follows

\[ Z_t = Z \times \Delta Z - \lambda Z \times (Z \times \Delta Z), \]

where \( Z(x, t) = (Z_1(x, t), Z_2(x, t), Z_3(x, t)) \) is a magnetization functional vector. \( \lambda > 0 \) is a Gilbert constant. “×” denotes the vector outer product. In order to describe the dynamics of the magnetization vector \( Z \) in a ferromagnetic body for a wide range of temperatures, Garanin [4, 3] derived the Landau–Lifshitz–Bloch (LLB) equation from statistical mechanics with the mean field approximation in 1990. Additionally, at high temperature (\( \theta \geq \theta_c \), \( \theta_c \)-Curie value), LLB model is also satisfactory. In [1], within the continuum thermodynamic framework, Berti and Giorgi derived the evolution equation for the magnetization vector in a ferromagnetic body and they pointed out that the procedure led to the generalized Landau-Lifshitz model for magnetically saturated bodies.

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The GLLB equation is given as follows

$$Z_t = -\gamma Z \times H^{\text{eff}} + \frac{L_1}{|Z|^2} (Z \cdot H^{\text{eff}}) Z - \frac{L_2}{|Z|^2} Z \times (Z \times H^{\text{eff}}),$$  \hspace{0.5cm} (2)$$

where $|\cdot|$ is the Euclidean norm, $\gamma > 0$ is the gyromagnetic ratio, and $L_1$, $L_2$ are the longitudinal and transverse damping parameters, respectively, $H^{\text{eff}}$ is the effective field. Let $\gamma a_\| = L_1$, $\gamma a_\perp = L_2$, here $a_\|$ and $a_\perp$ are dimensionless damping parameters which are defined in [2]

$$a_\|(\theta) = \frac{2\theta}{3\theta_c}, \quad a_\perp(\theta) = \left\{ \begin{array}{ll} \lambda \left(1 - \frac{\theta}{3\theta_c}\right), & \text{if } \theta < \theta_c \\ a_\|(\theta), & \text{if } \theta \geq \theta_c, \end{array} \right.$$  

where $\lambda > 0$ is a constant. In [6], the author asserted that if $L_1 = L_2$, (2) can be reduced as follows

$$Z_t = \Delta Z + Z \times \Delta Z - k(1 + \mu|Z|^2)Z, \quad (k, \mu > 0)$$  \hspace{0.5cm} (3)$$

and the existence of the weak solution for the equation (3) has been obtained.

The purpose of this paper is to discuss the following initial value problem for the GLLB in high dimensions.

$$Z_t = \Delta Z + Z \times \Delta Z + f(x, t, Z)$$  \hspace{0.5cm} (4)$$

$$Z(x, 0) = Z_0(x),$$  \hspace{0.5cm} (6)$$

where $x + 2De_i = (x_1, x_2, \cdots, x_{i-1}, x_i + 2D, x_{i+1}, \cdots, x_n)$, $i = 1, 2, \cdots, d$, $D > 0$, $\Omega \subset \mathbb{R}^d$ represents the d-dimensional cube with width 2D along each direction, i.e. $\Omega = \{x = (x_1, x_2, \cdots, x_d), |x_i| \leq D, i = 1, 2, \cdots, d\}$, $Q_T = \{(x, t), x \in \Omega, 0 \leq t \leq T\}$, and $Z(x, t) = (z_1(x, t), z_2(x, t), z_3(x, t))$ is a three-dimensional vector, $f(x, t, Z)$ is a known three dimensional vector of variables $x \in \mathbb{R}^d$, $t \in \mathbb{R}^+$, $Z \in \mathbb{R}^3$, $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$.

The structure of the paper is as follows. In section 2, we obtain the approximate solution to the periodic initial value problem (4)-(6), and the estimation of the approximate solution is derived in section 3. In section 4, we get the weak solution of (4)-(6). Finally, we prove the existence and uniqueness of the smooth solution for the problem (4)-(6) in high dimensions.

2. Approximate solution to the periodic initial value problem (4)-(6). In order to prove the existence of global weak solution to the problem (4)-(6), we use the Galèrkin method. For this, we construct the Galèrkin approximate solution.

Let $w_j(x)$, $(j = 1, 2, \cdots)$ be unit eigenfunctions satisfying equation $\Delta w_j + \lambda_j w_j = 0$ with periodicity, namely, $w_j(x - De_i) = w_j(x + De_i)$ $(i = 1, 2, \cdots, d)$. $\lambda_j$ $(j = 1, 2, \cdots)$ are the corresponding eigenvalues different from with each other, and \{w_j(x)\} consists of the orthogonal base in $L^2$.

Denote the approximate solution of the problem (4)-(6) in the following form

$$Z_N(x, t) = \sum_{s=1}^{N} \alpha_s N(t) w_s(x),$$  \hspace{0.5cm} (7)$$
where $\alpha_s(t) \ (t \in \mathbb{R}^+ \ (s = 1, 2, \cdots, N; \ N = 1, 2, \cdots)$ are the three dimensional vector-valued functions satisfying the following system of ordinary differential equations of first order.

\[
\int_{\Omega} Z_{Nt}(x,t)w_s(x)dx = \int_{\Omega} (Z_N(x,t) \times \Delta Z_N(x,t))w_s(x)dx \tag{8}
\]

\[
+ \int_{\Omega} \Delta Z_Nw_s(x)dx + \int_{\Omega} f(x,t,Z_N)w_s(x)dx, \ s = 1, 2, \cdots, N,
\]

with the initial condition

\[
\int_{\Omega} Z_N(x,0)w_s(x)dx = \int_{\Omega} Z_0w_s(x)dx, \ s = 1, 2, \cdots, N. \tag{9}
\]

It is obvious that

\[
\int_{\Omega} Z_{Nt}(x,t)w_s(x)dx = \alpha_{sN}^t(t), \quad \int_{\Omega} Z_N(t,0)w_s(x)dx = \alpha_sN(0), \tag{10}
\]

with

\[
Z_s = \int_{\Omega} Z_0(x)w_sdx, \ s = 1, 2, \cdots, N \tag{11}
\]

are the coefficients of expansion $Z_0(x) = \sum_{s=1}^{N} Z_s w_s(x)$.

3. Estimation for the approximate solution. In order to prove the existence of the global weak solution, we assume that:

(i) The $3 \times 3$ Jacobi matrix $f_Z(x,t,Z)$ is semi-bounded, i.e., there is a constant $b \geq 0$ such that for any $\xi \in \mathbb{R}^3$

\[
\xi \cdot f_Z \cdot \xi \leq b|\xi|^2, \tag{12}
\]

where “$\cdot$” denotes the three-dimensional inner product.

(ii) (4) is homogeneous, i.e., $f(x,t,0) \equiv 0$. Moreover, it is assumed that for some constants $A$ and $B$:

\[
\begin{cases}
|f(x,t,Z)| \leq A|Z| + B, \quad 2 \leq l \leq 2 + \frac{d}{2}, \quad d \geq 2, \\
|\nabla_x f(x,t,Z)| \leq A|Z|^{1+\frac{d}{2}} + B.
\end{cases} \tag{13}
\]

(iii) Vector function $Z_0(x) \in H^1_{\text{per}}(\Omega)$.

**Lemma 3.1.** Let condition (i) hold. $Z_0(x) \in L^2(\Omega)$, $f(x,t,0) \in L^2(Q_T)$. Then the approximate solution $Z_N(x,t)$ meets the estimation

\[
\sup_{0 \leq t \leq T} \|Z_N(\cdot,t)\|_{L^2(\Omega)} \leq K_1, \quad \int_0^t \|\nabla Z_N(\cdot,t)\|^2_{L^2(\Omega)} dt \leq K_1, \tag{14}
\]

where $K_1$ is independent of $N$.

**Proof.** Multiplying (8) by $\alpha_{sN}(t)$ and summing from $s = 1$ to $s = N$, we have

\[
\int_{\Omega} Z_{Nt} \cdot Z_N dx = \int_{\Omega} \Delta Z_N \cdot Z_N dx + \int_{\Omega} (Z_N \times \Delta Z_N) \cdot Z_N dx + \int_{\Omega} f(x,t,Z_N) \cdot Z_N dx,
\]

or

\[
\int_{\Omega} Z_{Nt} \cdot Z_N dx = \int_{\Omega} \Delta Z_N \cdot Z_N dx + \int_{\Omega} (Z_N \times \Delta Z_N) \cdot Z_N dx + \int_{\Omega} f(Z_N) \cdot Z_N dx. \tag{16}
\]
In addition,
\[
\int_{\Omega} Z_N t \cdot Z_N \, dx = - \frac{1}{2} \frac{d}{dt} \| Z_N(\cdot, t) \|_{L^2(\Omega)}^2,
\]
\[
\int_{\Omega} (Z_N \times \Delta Z_N) \cdot Z_N \, dx = 0,
\]
\[
\int_{\Omega} \Delta Z_N \cdot Z_N \, dx = - \int_{\Omega} \nabla Z_N \ast \nabla Z_N \, dx,
\]
where \( \nabla Z_N \) is a \( d \times 3 \) tensor and \( \ast \) is the inner product in a \( d \)-dimensional space.

It follows from the homogeneous condition that
\[
\int_{\Omega} \Delta Z_N \cdot Z_N \, dx = - \| \nabla Z_N \|_{L^2(\Omega)}^2.
\]

For the last term on the right hand side of (16), we have
\[
\int_{\Omega} f(Z_N) \cdot Z_N \, dx = \int_0^1 dt \int_{\Omega} \frac{\partial f(x, t, Z_N(x, t))}{\partial z} \cdot Z_N(x, t) \cdot Z_N(x, t) \, dx
\]
\[
+ \int_{\Omega} f(x, t, 0) \cdot Z_N(x, t) \, dx,
\]
and hence
\[
\int_{\Omega} f(Z_N) \cdot Z_N \, dx \leq (b + 1) \| Z_N(\cdot, t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| f(x, t, 0) \|_{L^2(\Omega)}^2.
\]

This proves the lemma.

**Lemma 3.2.** Let conditions (i)-(iii) hold. For the approximate solution \( Z_N(x, t) \),
\[
\sup_{0 \leq t \leq T} \| \nabla Z_N \|_{L^2(\Omega)}^2 \leq K_2, \int_0^T \| \Delta Z_N \|_{L^2(\Omega)}^2 \, dt \leq K_2.
\]

(18)

hold, where \( K_2 \) is independent of \( N \).

**Proof.** Multiplying (8) by \(-\lambda_s \alpha_s N(t)\) and summing from \( s = 1 \) to \( s = N \), we have
\[
\int_{\Omega} Z_N t \cdot \Delta Z_N \, dx = \int_{\Omega} (Z_N \times \Delta Z_N) \cdot \Delta Z_N \, dx + \int_{\Omega} \Delta Z_N \cdot \Delta Z_N \, dx + \int_{\Omega} f(Z_N) \cdot \Delta Z_N \, dx.
\]

(19)

The first term in the above equality is
\[
\int_{\Omega} Z_N t \cdot \Delta Z_N \, dx = - \frac{1}{2} \frac{d}{dt} \| \nabla Z_N \|_{L^2(\Omega)}^2.
\]

(20)

The last term on the right hand side of (19) is
\[
\int_{\Omega} f(Z_N) \cdot \Delta Z_N \, dx = - \int_{\Omega} Df(Z_N) \ast \nabla Z_N \, dx
\]
\[
= - \int_{\Omega} \nabla x f \cdot \nabla Z_N \, dx - \int_{\Omega} (\frac{\partial f}{\partial z}) \cdot \nabla Z_N \ast \nabla Z_N \, dx.
\]

(21)
Since $\frac{\partial f}{\partial z}$ is semi-bounded, one has
\[ \int_\Omega \left( \frac{\partial f}{\partial z} \cdot \nabla Z_N \right) \cdot \nabla Z_N \, dx \leq b \| \nabla Z_N (\cdot, t) \|_{L^2(\Omega)}^2. \]
The first term on the right hand side of (21) can be estimated by
\[ \left| \int_\Omega \nabla f \cdot \nabla Z_N \, dx \right| \leq \frac{1}{2} \| \nabla Z_N (\cdot, j) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \nabla f (\cdot, t, Z_N (\cdot, t)) \|_{L^2(\Omega)}^2. \]
It follows from (13) that
\[ \| \nabla f (\cdot, t, Z_N (\cdot, t)) \|_{L^2(\Omega)} \leq C_1 \int_\Omega |Z_N (\cdot, t)|^{2+4/d} \, dx + C_2, \]
where $C_1, C_2$ are constants. From the Sobolev inequality, we have
\[ \int_\Omega |Z_N (\cdot, t)|^{2+4/d} \, dx \leq C_3 \| Z_N (\cdot, t) \|_{L^2(\Omega)}^{4/d} \| \nabla Z_N (\cdot, t) \|_{L^2(\Omega)}^2 + C_4 \| Z_N (\cdot, t) \|_{L^2(\Omega)}^{2+4/d}, \]
where $C_3, C_4$ are constants.
Finally, by (19), we have
\[ \frac{d}{dt} \| \nabla Z_N (\cdot, t) \|_{L^2(\Omega)}^2 + 2 \| \Delta Z_N (\cdot, t) \|_{L^2(\Omega)}^2 \leq (2b + C_5) \| \nabla Z_N (\cdot, t) \|_{L^2(\Omega)}^2 + C_6, \tag{22} \]
where $C_5, C_6$ are constants which are dependent only on $\sup_{0 \leq t \leq T} \| Z_N (\cdot, t) \|_{L^2(\Omega)}$, but independent of $N$. Applying the Gronwall inequality to the last inequality, from (22), we can set $\int_0^T \| \Delta Z_N (\cdot, t) \|_{L^2(\Omega)}^2 \, dt \leq K_4$, which completes the proof. \( \square \)

**Lemma 3.3.** Let (i)-(iii) hold. For the above approximate solutions,
\[ \sup_{0 \leq t \leq T} \| Z_{Nt} (\cdot, t) \|_{H^{-(2+4/d/2)}(\Omega)} \leq K_4 \tag{23} \]
holds with $K_4$ independent of $N$.

**Proof.** Let $\varphi (x) \in H_0^{2+4/d/2}(\Omega)$ be a test function.
\[ \varphi (x) = \varphi_N (x) + \overline{\varphi}_N (x), \]
\[ \varphi_N (x) = \sum_{s=1}^N \beta_s w_s (x), \]
\[ \overline{\varphi}_N (x) = \sum_{s=N+1}^\infty \beta_s w_s (x), \]
where $\beta_s = \int_\Omega \varphi (x) w_s (x) \, dx$, $s = 1, 2, \ldots$.
If $s \geq N + 1$, then
\[ \int_\Omega Z_{Nt} w_s (x) \, dx = 0. \]
Else if $1 \leq s \leq N$, one has
\[ \int_\Omega Z_{Nt} w_s (x) \, dx = - \int_\Omega (Z_N \times \nabla Z_N) \cdot \nabla w_s \, dx \tag{24} \]
\[ - \int_\Omega \nabla Z_N \cdot \nabla w_s \, dx + \int_\Omega f (x, t, Z_N (x, t)) w_s \, dx. \]
Now, we estimate each term on the right hand side of (24). For the first term, we have
\[
\left| \int_{\Omega} (Z_N \times \nabla Z_N) \ast \nabla w_s \, dx \right| \leq C_7 \| \nabla w_s \|_{L^{\infty}(\Omega)} \| Z_N(\cdot, t) \|_{L^2(\Omega)} \| \nabla Z_N(\cdot, t) \|_{L^2(\Omega)}.
\]
(25)

It yields from the Sobolev inequality
\[
\| \nabla w_s \|_{L^{\infty}(\Omega)} \leq C_8 \| w_s \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)}.
\]
(26)

Therefore, we obtain
\[
\left| \int_{\Omega} (Z_N \times \nabla Z_N) \ast \nabla w_s \, dx \right| \leq C_9 \| w_s \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)},
\]
(27)

where \(C_9\) depends only on \(\sup_{0 \leq t \leq T} \| Z_N(\cdot, t) \|_{H^1(\Omega)}\) but is independent of \(N\).

For the second term on the right hand side of (24), we get
\[
\left| \int_{\Omega} \nabla Z_N \ast \nabla w_s \, dx \right| \leq C_{10} \| \nabla Z_N(\cdot, t) \|_{L^2(\Omega)} \| \nabla w_s \|_{L^2(\Omega)}.
\]
(28)

For the last term on the right hand side of (24), we have
\[
\left| \int_{\Omega} f(x, t, Z_N(x, t)) w_s \, dx \right| \leq \| w_s \|_{L^{\infty}(\Omega)} \int_{\Omega} |f(x, t, Z_N(x, t))| \, dx
\]
\[
\leq \| w_s \|_{L^{\infty}(\Omega)} (A \int_{\Omega} |Z_N(x, t)|^l \, dx + B)
\]
\[
\leq C_{11} \| w_s \|_{L^{\infty}(\Omega)}
\]
\[
\leq C_{12} \| w_s \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)},
\]
(29)

where
\[
\int_{\Omega} |Z_N(x, t)|^l \, dx \leq C_{13} \| Z_N(\cdot, t) \|_{L^2(\Omega)}^{(1-\alpha)l} \| \nabla Z_N(\cdot, t) \|_{L^2(\Omega)}^\alpha.
\]
(30)

with \(\alpha = \frac{d}{2} - \frac{d}{l}, 2 \leq l \leq \frac{2d}{d-2}\).

Henceforth, we have
\[
\left| \int_{\Omega} Z_N t w_s \, dx \right| \leq C_{14} \| w_s \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)},
\]
(31)

where \(C_{14}\) depends on \(\sup_{0 \leq t \leq T} \| Z_N(\cdot, t) \|_{H^1(\Omega)}\) but is independent of \(0 \leq t \leq T\) and \(N\).

Similarly, setting
\[
\left| \int_{\Omega} Z_N t \phi_N \, dx \right| \leq C_{14} \| \phi_N \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)}
\]
(32)

and hence
\[
\left| \int_{\Omega} Z_N t \phi \, dx \right| \leq C_{14} \| \phi \|_{H^{2+\lfloor d/2 \rfloor}(\Omega)}, \quad \forall \phi \in H^{2+\lfloor d/2 \rfloor}_0(\Omega).
\]
(33)

The proof is completed. \(\square\)

Applying the fixed point theorem and the priori estimates of the approximate solutions, we can prove the existence of global solution \(\alpha_{sN}(t) (0 \leq t \leq T, s = 1, 2, \ldots, N)\).

**Lemma 3.4.** Let (i)-(iii) hold. Then the ordinary differential system (8)-(9) admits at least one global continuously differentiable solution \(\alpha_{sN}(t) (0 \leq t \leq T, s = 1, 2, \ldots, N)\).
Lemma 3.5. Let (i)-(iii) hold. Then for the approximate solution
\[ \|Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t)\|_{L^2(\Omega)} \leq K_4(\Delta t)^{\frac{1}{2 + 4/3}} \]  
holds, where \( K_4 \) is independent of \( N \), and \( 0 \leq t, t + \Delta t \leq T \).

Proof. It follows from the Sobolev inequality with negative index that
\[ \|Z_N(\cdot, t)\|_{L^2(\Omega)} \leq C_{15}\|Z_N(\cdot, t)\|_{H^{-1}(\Omega)}^{\frac{1}{2}}\|Z_N(\cdot, t)\|_{H^1(\Omega)}^{\frac{1}{2}} \]  
Applying this inequality to \( Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t) \), we get
\[ \|Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t)\|_{L^2(\Omega)} \leq C_{15}\|Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t)\|_{H^{-1}(\Omega)}^{\frac{1}{2}}\|Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t)\|_{H^1(\Omega)}^{\frac{1}{2}} \]  
where \( 0 \leq t, t + \Delta t \leq T \).

Thus,
\[ \|Z_N(\cdot, t + \Delta t) - Z_N(\cdot, t)\|_{L^2(\Omega)} \leq C_{15}(\Delta t)^{\frac{1}{4 + 4/3}}\sup_{0 \leq t \leq T}\|Z_N(\cdot, t)\|_{H^{-1}(\Omega)}^{\frac{1}{2}}\sup_{0 \leq t \leq T}\|Z_N(\cdot, t)\|_{H^1(\Omega)}^{\frac{1}{2}} \]  
which completes the lemma. \( \square \)

4. Weak solution of (4)-(6).

4.1. Convergence of approximate solutions. In order to get a weak solution to (4)-(6), we set \( N \to \infty \).

Definition 4.1. A vector-valued function \( Z(x, t) \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)) \) is called a weak solution to system (4)-(6), if for any test function \( \varphi(x, t) \in C^1(Q_T), \varphi(x, T) = 0 \), then
\[ \int_0^T \int_\Omega [\varphi_t Z(x, t) - \nabla \varphi(x, t) \ast (Z(x, t) \times \nabla Z(x, t))] + \varphi(x, t) f(x, t, Z(x, t))]dxdt \]
\[ + \int_\Omega \varphi(x, 0) Z_0(x)dx = 0 \]  
holds.

The Galerkin approximate function \( \varphi(x, t) \) is
\[ \varphi_N(x, t) = \sum_{s=1}^N \beta_s(t) w_s(x) \]
which uniformly converges to \( \varphi(x, t) \) in \( C^1(Q_T) \) as \( N \to \infty \), where
\[ \beta_s N(t) = \int_\Omega \varphi(x, t) w_s(x) dx, \quad s = 1, 2, \ldots, N. \]

It follows from former estimates that \( Z_N(x, t) \) the following integral equality holds
\[ \int_0^T \int_\Omega [\varphi N_t Z_N(x, t) - \nabla \varphi N(x, t) \ast (Z_N(x, t) \times \nabla Z_N(x, t))] \]
\[ + \varphi_N(x, t) f(x, t, Z_N(x, t))]dxdt + \int_\Omega \varphi(x, 0) Z_0(x)dx = 0. \]
4.2. Uniform boundedness and convergence. It follows from former estimates that \( Z_N(x,t) \) is uniformly bounded in the space

\[
G = L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap W^{(1)}_{\infty}(0,T; H^{2+|d/2|}(\Omega)).
\]  

(40)

Then we may choose a subsequence \( Z_{N_i}(x,t) \) from \( Z_N(x,t) \) such that \( Z_{N_i}(x,t) \) converges in \( L^p(0,T; L^2(\Omega)) \) to a vector \( Z(x,t) \in L^p(0,T; H^1(\Omega)) \) with \( 1 < p < \infty \) and \( \nabla Z_{N_i}(x,t) \) converges in \( L^p(0,T; L^2(\Omega)) \) to a vector \( \nabla Z(x,t) \). On the other hand, since \( Z_{N_i}(x,t) \) is bounded in \( L^p(0,T; H^{-2+|d/2|}(\Omega)) \) \( (1 < p < \infty) \), we may assume that \( Z_{N_i}(x,t) \) weakly converges to a vector \( w(x,t) \in L^p(0,T; H^{-2+|d/2|}(\Omega)) \) \( (1 < p < \infty) \).

Now, we claim that \( w(x,t) = Z_i(x,t) \). In fact,

\[
\int_0^T \int_\Omega Z_{N_i}(x,t) dx dt = -\int_0^T \int_\Omega Z_{N_i}(x,t) \phi_i(x,t) dx dt,
\]

where \( \phi(x,t) \) is any test function with a compact support set in \( Q_T \). Let \( N_i \to \infty \), we get

\[
\int_0^T \int_\Omega w(x,t) dx dt = -\int_0^T \int_\Omega Z(\varphi_i(x,t) dx dt.
\]

The claim follows.

This means that

\[
Z(x,t) \in G_p = L^p(0,T; H^1(\Omega)) \cap W^{(1)}_{\infty}(0,T; H^{-2+|d/2|}(\Omega)), \quad 1 < p < \infty
\]

(43)

and the norm of \( Z(x,t) \) in \( G_p \) is uniformly bounded in \( p \). Then \( Z(x,t) \in G_\infty \). Since

\[
\int_0^T \| \Delta Z_{N_i}(x,t) \|^2_{L^2(\Omega)} dt \leq K, \quad \int_0^T \| \Delta Z(x,t) \|^2_{L^2(\Omega)} dt \leq K, \quad N_i \to \infty.
\]

Thus \( Z(x,t) \in G \).

Then, we have the following lemma.

**Lemma 4.2.** Suppose that (i)-(iii) hold. Then for the limit function \( Z(x,t) \) of the approximate \( Z_N(x,t) \),

\[
\sup_{0 \leq t \leq T} \| Z(\cdot,t) \|_{H^1(\Omega)} \leq K_5,
\]

(44)

\[
\sup_{0 \leq t \leq T} \| Z_i(\cdot,t) \|_{H^{-2+|d/2|}(\Omega)} \leq K_6,
\]

(45)

\[
\| Z_i(\cdot,t + \Delta t) - Z(\cdot,t) \|_{L^2(\Omega)} \leq K_7(\Delta t)^{\frac{\delta}{4+|d/2|}},
\]

(46)

hold, where \( K_5, K_6, K_7 \) are independent of \( 0 \leq t, t + \Delta t \leq T \).

Choose a suitable subsequence from \( \{ Z_{N_i}(x,t) \} \) (for simplicity, we still denote it by \( \{ Z_{N_i}(x,t) \} \) such that for some vector-valued function \( Z(x,t) \), \( Z_{N_i}(x,t) \to Z(x,t) \) strongly in \( L^2(\Omega) \) for each \( 0 \leq t \leq T \), and \( \| Z(\cdot,t) \|_{L^2(\Omega)} \to \| Z(\cdot,t) \|_{L^2(\Omega)} \) uniformly in \( 0 \leq t \leq T \). More precisely, \( Z_{N_i}(x,t) \) converges to \( Z(x,t) \) in \( C^{(0,1/(3+|d/2|+\delta))}(0,T; L^2(\Omega)), \delta > 0 \).

Now we consider the limit of integral (39) as \( N_i \to \infty \). Since as \( N_i \to \infty \), \( \{ \varphi_{N_i}(x,0) \} \) uniformly converges in \( \Omega \) to \( \phi(x,0) \); \( \{ Z_{N_i}(x,0) \} \) converge in \( L^2(\Omega) \) to...
Global weak solution and smooth solution for the GLLB ...

Let (i)-(iii) hold and \( T \) be a sequence such that \( T_k \to \infty \) as \( k \to \infty \). Let \( \{Z_{N_k,i}(x,t)\} \) (\( k = 0, 1, 2, \ldots; i = 1, 2, \ldots \)) be the \( i \)-th subsequence of \( \{Z_N(x,t)\} \) such that

(a) \( Z_{N_{k+1},i}(x,t) \subset Z_{N_k}(x,t) \);

(b) \( Z_{N_k,i}(x,t) \) is a weak solution of problem (8)(9). Then the problem (8)(9) admits at least one global weak solution

\[
Z(x,t) \in L^\infty(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \cap C^{0,\frac{1}{2m-1}}(0,T;L^2(\Omega)).
\]
Theorem 4.4. Suppose 
and meets the integral relation (38).

Hence we may take the diagonal method to choose a subsequence \( \{Z_{k}\} \)
which weakly converges to \( Z \) in \( G \). Hence \( Z(x,t) \) is in
\begin{equation}
G_{\infty} = L_{loc}^{1}(0, \infty; H^{1}(\Omega)) \cap H_{loc}^{1}(0, \infty; H^{2}(\Omega)) \subset \cap L_{loc}^{1}(0, \infty; H^{1}(\Omega)) \cap C^{0, \frac{1}{d+2}(\Omega)}, k = 1, 2, 3, ...
\end{equation}
and meets the integral relation (38).

Theorem 4.5. Suppose
(I) the 3 \times 3 Jacobi matrix \( f_{x}(x,t,z) \) is semi-bounded, i.e., for \( (x,t) \in Q_{\infty} \) and \( z \in \mathbb{R}^{3} \), (34) holds.
(II) for any \( 0 < T < \infty \), \( (x,t) \in Q_{T} \) and \( z \in \mathbb{R}^{3} \)
\begin{equation}
\begin{cases}
|f(x,t,z)| \leq A(T)|z| + B(T), \\
|\nabla f(x,t,z)| \leq A(T)|z|^{l+2/m} + B(T), \\
f(x,t,0) = 0
\end{cases}
\end{equation}
hold, where \( A(T), B(T) \) are positive constants depending on \( T \), and
\[ 2 \leq l \leq 2 + 4/(d-2), d \geq 2. \]

(III) \( Z_{0}(x) \in H^{1}(\Omega). \)
Then there is a global weak solution of (8)-(9)
\[ Z(x,t) \in L_{loc}^{1}(0, \infty; H^{1}(\Omega)) \cap C^{0, \frac{1}{d+2}(\Omega)}, 0, \infty; L^{2}(\Omega)). \]
If \( b < 0 \), it follows from (17) that
\[ \frac{d}{dt} \| Z(\cdot, t) \|_{L^{2}(\Omega)}^{2} \leq -2b \| Z(\cdot, t) \|_{L^{2}(\Omega)}^{2} \]
\text{and } \( f(x,t,0) \equiv 0 \), then
\[ \| Z(\cdot, t) \|_{L^{2}(\Omega)} \leq \| Z_{0}(x) \|_{L^{2}(\Omega)} \exp\{-|b|t\}, \quad 0 \leq t < \infty. \]

We conclude that the similar inequality also holds for \( \lim \) \( Z(x,t) \), that is, we have

Theorem 4.5. Let (I)-(III) hold. Then for the weak solution \( Z(x,t) \) to the problem
(8)-(9) in \( Q_{\infty} \),
\[ \lim_{t \to \infty} \| Z(\cdot, t) \|_{L^{2}(\Omega)} = 0. \]

4.4. Uniqueness of smooth solution.

Theorem 4.6. Assume that \( f(x,t,Z) \) is twice continuously differentiable with respect to \( x \) and \( Z \), and the Jacobi matrix \( f_{Z}(x,t,Z) \) is semi bounded. Then the classical solution to (8)-(9) is unique.

Proof. Let \( u(x,t), Z(x,t) \) be the two solutions of (8)-(9). Setting \( w(x,t) = u(x,t) - Z(x,t) \), we have
\[ \frac{d}{dt} \| w(\cdot, t) \|_{L^{2}(\Omega)}^{2} + \| \nabla w(\cdot, t) \|_{L^{2}(\Omega)}^{2} = -2 \int_{\Omega} \nabla w \star (w \times \nabla u) + 2 \int_{\Omega} w \frac{\partial f}{\partial Z} \cdot w, \]
\[
\frac{d}{dt} \| \nabla w(\cdot, t) \|_{L^2(\Omega)}^2 + \| \Delta w(\cdot, t) \|_{L^2(\Omega)}^2 \\
= -2 \int_{\Omega} \nabla w \ast (w \times \nabla Z) - 2 \int_{\Omega} \nabla w \ast \frac{\partial \tilde{f}}{\partial Z} \cdot \nabla w \\
- 2 \int_{\Omega} \nabla w \ast \left( \frac{\partial^2 \tilde{f}}{\partial Z^2} w \right) \cdot \nabla u - 2 \int_{\Omega} \nabla w \ast \frac{\partial f(x, t, Z)}{\partial Z} \cdot \nabla w,
\]

where
\[
\frac{\partial \tilde{f}}{\partial Z} = \int_0^1 \frac{\partial f(x, t, \tau u + (1 - \tau)Z)}{\partial Z} dZ,
\]
\[
\frac{\partial^2 \tilde{f}}{\partial Z^2} = \int_0^1 \frac{\partial^2 f(x, t, \tau u + (1 - \tau)Z)}{\partial Z^2} dZ,
\]
\[
\nabla \frac{\partial \tilde{f}}{\partial Z} = \int_0^1 \nabla \frac{\partial f(x, t, \tau u + (1 - \tau)Z)}{\partial Z} dZ.
\]

Hence \( w(x, t) \in C^{(3,1)}(Q_T) \) satisfies the homogeneous equation and initial conditions. Thus
\[
\frac{d}{dt} \| w(\cdot, t) \|_{H^1(\Omega)}^2 \leq C_{18} \| w(\cdot, t) \|_{H^1(\Omega)}^2,
\]
which completes the proof. \( \square \)

Finally, we consider the “blow up” problem for the weak solution \( Z(x, t) \) of the following initial value problem for Landau-Lifshitz equation:
\[
Z_t = Z \times \Delta Z + f(x, t, Z),
\]
\[
Z|_{t=0} = Z_0(x).
\]
And we have the following results.

**Theorem 4.7.** If
\[
Z \cdot f(x, t, Z) \geq C_0 |Z|^{2+\delta}, \quad (x, t) \in Q_T, \quad Z \in \mathbb{R}^3,
\]
where \( C_0 > 0, \delta > 0, \) and \( \| Z_0(x) \|_{L^2(\Omega)} > 0, \) then the weak solution \( Z(x, t) \in W_2^{(2,1)}(Q_T) \) to (8) blows up at finite time, i.e., for some finite \( t_0 > 0, \)
\[
\lim_{t \to t_0^-} \| Z(\cdot, t) \|_{L^2(\Omega)} = \infty.
\]

**Proof.** Multiplying (54) by \( Z \) and integrating over \( \Omega \) with respect to \( x, \) we have
\[
\frac{1}{2} \frac{d}{dt} \| Z(\cdot, t) \|_{L^2(\Omega)}^2 = -2 \int_{\Omega} Z(\cdot, t) \cdot f(x, t, Z) dx \geq C_1 \int_{\Omega} |Z|^{2+\delta} dx
\]
\[
\geq C_1 (\text{mes} \Omega)^{-\frac{\delta}{2}} \| Z \|_{L^{2+\delta}(\Omega)}^{2+\delta} + C_0 \| Z \|_{L^2(\Omega)}^2 \geq C_1 (\text{mes} \Omega)^{-\frac{\delta}{2}} \| Z \|_{L^{2+\delta}(\Omega)}^{\frac{1}{2+\delta}}
\]
and then
\[
\| Z \|_{L^2(\Omega)} \geq (\| Z_0 \|_{L^{2+\delta}(\Omega)}^{\frac{1}{2+\delta}} + C_0 \delta (\text{mes} \Omega)^{-\frac{\delta}{2}})^{-\frac{1}{\delta}}.
\]
This proves the theorem. \( \square \)

In the following, we consider a more general blow up result for Landau-Lifshitz equation
\[
u_t = \nabla \ast A(x, t, u) \cdot \nabla u + f(x, t, u),
\]
where \( u(x, t) = (u_1(x, t), ..., u_n(x, t)) \) is a multi-dimensional vector valued unknown defined in \( Q_T = \{ x \in \Omega \subset \mathbb{R}^m, 0 \leq t \leq T \} \) and “\( \ast \)” and “\( \ast \)\( \ast \)” denote the linear
product in $\mathbb{R}^n$ and $\mathbb{R}^d$, respectively. $A(x,t,u)$ is a nonsingular and zero-definite matrix and $f(x,t,u)$ is an $n$ dimensional vector-valued function satisfying
\[ u \cdot f(x,t,u) \geq C_0 |u|^{2+\delta} \] (58)
for $(x,t) \in Q_T$, $u \in \mathbb{R}^n$, $C_0 > 0$ and $\delta > 0$.

Multiplying (57) by $u$ and integrating over $\Omega$, we get
\[ \frac{1}{2} \frac{d}{dt} \|u(\cdot,t)\|^2_{L^2(\Omega)} = \int_{\partial \Omega} (u \cdot A \cdot \nabla u) \ast \nu_m \, dx - \int_{\Omega} (\nabla u \ast A \cdot \nabla u) \, dx + \int_{\Omega} u \cdot f \, dx. \] (59)
Consider the first boundary value problem
\[ u(x,t) = 0, \quad x \in \partial \Omega, \quad 0 \leq t \leq T, \]
\[ u(x,0) = \varphi(x), \quad x \in \Omega; \] (60)
or the second boundary value problem\[ A(x,t,u(x,t)) \cdot \nabla u(\cdot,t) \ast \nu_m = 0, \quad x \in \partial \Omega, \quad 0 \leq t \leq T. \]
\[ u(x,0) = \varphi(x), \quad x \in \Omega. \] (62)
We have
\section*{Theorem 4.8.}
Under the above conditions on $A$ and $f$, for the solution $u(x,t)$ of (57), (60)-(61), we have
\[ \|u(\cdot,t)\|_{L^2(\Omega)} \to 0, \quad t \to t_1 - 0, \] (64)
where $t_1 > 0$ is finite and $\|\varphi\|_{L^2(\Omega)} > 0$.

5. Existence and uniqueness of the smooth solution for the problem (4)-(6) in high dimensions. By using the fixed point theorem, it is easy to get

\section*{Theorem 5.1 (Local smooth solution).}
Let (i)-(iii) hold, and $f(x,t,z) \subset C^n(Q_T)$. Then there is a constant $T_0 > 0$ such that a unique smooth solution $z(x,t) \subset C^n(Q_{T_0})$ for the problem (4)-(6) exists, where $Q_{T_0} = \{0 \leq t \leq T_0, x \in \Omega\}$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$.

In order to get the classical global solution for problem (4)-(6), we need to give higher order a priori estimates.

\section*{Lemma 5.2.}
If
\begin{enumerate}
\item[(1)] $Z_0(x) \in H^2(\Omega)$, $\Omega \subset \mathbb{R}^2$;
\item[(2)] $f(x,t,Z) \cdot Z \leq 0$
\end{enumerate}
hold. Then we have
\[ \sup_{0 \leq t \leq T} \|Z(\cdot,t)\|_{L^\infty(\Omega)} \leq \|Z_0(x)\|_{L^2(\Omega)}. \] (65)

\section*{Proof.}
Since
\[ \frac{1}{p} \frac{d}{dt} \|Z(\cdot,t)\|_{L^p(\Omega)} = \frac{1}{p} \int_{\Omega} |Z|^{p-2} Z \cdot Z_t \, dx \]
\[ = \frac{1}{p} \int_{\Omega} |Z|^{p-2} Z \cdot (\Delta Z + Z \times \Delta Z + f(x,t,Z)) \, dx \]
\[ \leq \frac{1}{p} \int_{\Omega} |Z|^{p-2} \nabla Z^T \nabla Z \, dx \leq 0, \]
thus
\[ \|Z(\cdot,t)\|_p \leq \|Z(\cdot,0)\|_p \leq \|Z_0(x)\|_{H^2}. \]
Let $p \to \infty$, then the proof of the lemma is completed. \hfill \Box

**Lemma 5.3.** Let $d = 2, 3$ and the assumptions of Lemma 5.2 hold. If $f(x, t, Z) \in C'(Q_T)$ and

$$|\partial_x f(x, t, Z)| + |f_Z(x, t, Z)| \leq K_1, \|Z_0\|_{H^2(\Omega)} \leq M.$$ 

Then for the smooth solution of the problem (4)-(6), we have

$$\|Z(\cdot, t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\nabla Z(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq \|Z_0\|_{H^2(\Omega)} \forall \; T > 0, \; t \in [0, T],$$

and

$$\|\nabla Z(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \|\Delta Z(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq K_2, \forall \; T > 0, \; t \in [0, T],$$

where $K_1$ and $K_2$ depend on $\|Z_0\|_{H^2(\Omega)}$.

**Proof.** Taking the Scalar product of 3-dimensional space with $Z$ in both sides of (4), and integrating the result over $\Omega \times [0, t]$, we obtain

$$\|Z(x, t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\nabla Z(\cdot, t)\|_{L^2(\Omega)}^2 ds + 2\int_0^t \int_\Omega f(x, t, Z) \cdot Z dx ds \leq \|Z_0(x)\|_{L^2(\Omega)}^2.$$ 

Thus we get

$$\|Z(\cdot, t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\nabla Z\|_{L^2(\Omega)}^2 ds \leq \|Z_0(x)\|_{L^2(\Omega)}^2.$$ 

Next taking the scalar product with $\Delta Z$ in (4), and integrating the result over $\mathbb{R}^d \times [0, t]$, we have

$$\|\nabla Z(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \|\Delta Z(\cdot, s)\|_{L^2(\Omega)}^2 ds = \int_0^t \int_\Omega \Delta \cdot f(x, t, Z) dx + \|\nabla Z_0\|_{L^2(\Omega)}^2 \leq \int_0^t \int_\Omega \nabla Z \cdot \nabla f dx + \|\nabla Z_0\|_{L^2(\Omega)}^2 \leq \int_0^t \int_\Omega \nabla Z \cdot \partial_x f + f_Z \cdot \nabla Z dx + \|\nabla Z_0\|_{L^2(\Omega)}^2 \leq C \int_0^t \|\nabla Z\|_{L^2(\Omega)}^2 ds + C.$$ 

By using Gronwall's inequality, we get (67). \hfill \Box

**Lemma 5.4.** Let $d = 2$ and the initial data $Z_0(x) \in H^m(m \geq 2)$. Additionally, assume the conditions of Lemma 5.3 hold and

$$|f(x, t, Z)| \leq A|Z|^l + B, \; A, \; B > 0, \; l \leq 2.$$ 

Then, for the smooth solution of the problem (4)-(6), we have

$$\|\Delta Z(\cdot, t)\|_{L^2(\Omega)}^2 + 2\int_0^t \|\Delta Z(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq C(T, \|Z_0\|_{H^2(\Omega)}), \forall \; T > 0, \; t \in [0, T],$$

and

$$\|Z_t(\cdot, t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla Z_t(\cdot, s)\|_{L^2(\Omega)}^2 ds \leq C(T, \|Z_0\|_{H^2(\Omega)}), \forall \; T > 0, \; t \in [0, T].$$
Moreover, if \( m \geq 3 \), then we get

\[
\| \Delta \nabla Z(\cdot, t) \|^2_{L^2(\Omega)} + 2 \int_0^t \| \Delta^2 Z(\cdot, s) \|^2_{L^2(\Omega)} ds \leq C(T, \| Z_0 \|_{H^3(\Omega)}), \forall T > 0, \ t \in [0, T],
\]

and

\[
\| \nabla Z_\ell(\cdot, t) \|^2_{L^2(\Omega)} + \int_0^t \| \Delta Z_\ell(\cdot, s) \|^2_{L^2(\Omega)} ds \leq C(T, \| Z_0 \|_{H^3(\Omega)}), \forall T > 0, \ t \in [0, T].
\]

**Proof.** Applying Laplace operator to both sides of equation (4), we have

\[
\int_\Omega \Delta Z_t \cdot \Delta Z dx = \int_\Omega \Delta^2 Z \cdot \Delta Z dx + 2 \sum_{j=1}^2 \int_\Omega \partial_{x_j} Z \times \Delta \partial_{x_j} Z \cdot \Delta Z dx
\]

\[
+ \int_\Omega Z \times \Delta^2 Z \cdot \Delta Z dx + \int_\Omega \Delta f(x, t, Z) \cdot \Delta Z dx.
\]

Integrating the above equation by parts, we get

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta Z(x, t)|^2 dx + \int_\Omega |\nabla \Delta Z(x, t)|^2 dx \]

\[
= \sum_{j=1}^2 \int_\Omega \partial_{x_j} Z \times \Delta \partial_{x_j} Z \cdot \Delta Z dx + \int_\Omega \Delta f \cdot \Delta Z dx.
\]

With the help of Hölder inequality, it follows that

\[
\sum_{j=1}^2 \int_\Omega \partial_{x_j} Z \times \Delta \partial_{x_j} Z \cdot \Delta Z dx \leq 2 \| \nabla Z \|_{L^4(\Omega)} \| \Delta Z \|_{L^4(\Omega)} \| \Delta \nabla Z \|_{L^2(\Omega)}.
\]

By Gagliardo-Nirenberg’s inequality, we conclude that

\[
\| \nabla Z \|_{L^4} \leq C \| \nabla Z \|_{H^2}^{1/2} \| \nabla Z \|_{L^2}^{3/4},
\]

\[
\| \Delta Z \|_{L^4} \leq C \| \Delta Z \|_{H^2}^{1/2} \| \Delta Z \|_{L^2}^{1/2}.
\]

Furthermore,

\[
\| \nabla Z \|_{H^2} \leq C(\| \Delta \nabla Z \|_{L^2} + \| \nabla Z \|_{L^2}),
\]

\[
\| \Delta Z \|_{H^1} \leq C(\| \Delta \nabla Z \|_{L^2} + \| \Delta Z \|_{L^2}).
\]

Hence, we get

\[
\left| \sum_{j=1}^2 \int_\Omega \partial_{x_j} Z \times \Delta \partial_{x_j} Z \cdot \Delta Z dx \right| \leq C \| \Delta Z \|_{L^4}^{1/2} \| \Delta \nabla Z \|_{L^2}^{7/4} + C,
\]

where \( C \) depends on \( \| \nabla Z_0 \|_{L^2(\Omega)} \). From Young’s inequality, it asserts that the right hand side of the above inequality is smaller than

\[
\frac{1}{4} \| \Delta \nabla Z \|^2_{L^2(\Omega)} + C(1 + \| \Delta Z \|^4_{L^2(\Omega)}).
\]

The last term of the left hand side of (72) is

\[
\int_\Omega \Delta f(x, t, Z) \cdot \Delta Z dx \leq C \| \Delta Z \|_{L^4}^{1/2} \| \nabla Z \|_{L^2}^2 + C \]

\[
\leq \frac{1}{4} \| \Delta \nabla Z \|^2_{L^2(\Omega)} + C.
\]
Back to (72), we have
\[ \frac{d}{dt} \int_{\Omega} |\Delta Z(\cdot,t)|^2 dt \leq C\|\Delta Z\|_{L^2(\Omega)}^4 + C. \]

The generalized Gronwall's inequality say that if \( f' \leq (f \cdot g) + C \) then \( f \leq C \exp(\int_0^t gds) + C \), so if we replace \( f \) and \( g \) by \( \|\Delta Z\|_{L^2(\Omega)}^2 \) and (67) implies the boundedness of \( \int_0^t gds \), then
\[ \|\Delta Z\|_{L^2(\Omega)}^2 \leq C. \]

Hence (68) holds.

By using induction arguments, we can obtain the following lemma.

**Lemma 5.5.** Let \( d = 2, \nabla Z_0 \in H^k (k \geq 2) \), \( f(x,t,Z) \in C^k(Q_T) \) and the conditions of Lemma 5.2, Lemma 5.4 hold. Then the smooth solution of problem (4)-(6) satisfies the following a priori estimate
\[ \sup_{0 \leq t \leq T} \|D^{m+1}Z(\cdot,t)\|_{L^2(\Omega)}^2 + \int_0^T \|D^{m+2}Z(\cdot,s)\|_{L^2(\Omega)}^2 ds \leq C, \quad 2 \leq m \leq k, \quad (73) \]
where \( C \) depends on \( T \) and \( \|\nabla Z_0(x)\|_{H^k} \).

We have the following Theorems.

**Theorem 5.6.** Let \( d = 2, Z_0 \in H^m (m \geq 2) \). Assume that \( f(x,t,Z) \) satisfy the following conditions
1. \( f(x,t,Z) \in C^m(Q_T) \), \( f(x,t,Z) \cdot Z \leq 0 \);
2. \( |f(x,t,Z)| \leq A|Z|^l + B \),
where \( A, B \) are positive constants, and \( l \geq 2 \). Then for any \( T > 0 \), there exists a unique solution \( Z(x,t) \) of the problem (4)-(6) satisfying
\[ \partial_t^j \partial_x^\alpha Z \in L^\infty(0,T;L^2(\Omega)), \]
\[ \partial_t^h \partial_x^\beta Z \in L^\infty(0,T;L^2(\Omega)) \]
with \( 2j + |\alpha| \leq m \) and \( 2h + |\beta| \leq m + 1 \).

**Theorem 5.7.** Let \( d \geq 3, Z_0 \in H^m (m \geq 2) \) and \( \|Z_0\|_{H^2} \) is sufficiently small. Under the conditions of Theorem 5.1, for any \( T > 0 \), there exists a unique solution \( Z(x,t) \) of the problem (4)-(6) satisfying
\[ \partial_t^j \partial_x^\alpha Z \in L^\infty(0,T;L^2(\Omega)), \]
\[ \partial_t^h \partial_x^\beta Z \in L^\infty(0,T;L^2(\Omega)) \]
with \( 2j + |\alpha| \leq m \) and \( 2h + |\beta| \leq m + 1 \).

**Proof.** Along the same line of the proof of 5.6, it asserts that
\[ \sum_{j=1}^3 \int_\Omega \partial_x^j Z \times \Delta \partial_x^j Z \Delta Z dx \leq C\|\nabla Z\|_{L^6(\Omega)} \|\Delta Z\|_{L^3(\Omega)} \|\Delta \nabla Z\|_{L^2(\Omega)} \]
\[ \leq C\|Z_0\|_{H^2} \|\Delta Z\|_{H^2} \|\Delta \nabla Z\|_{L^2(\Omega)} \]
\[ \leq \frac{1}{3}\|\Delta Z\|_{L^2(\Omega)}, \]
where the hypothesis \( \|Z_0(x)\|_{H^2} \leq 1 \) has been used. We also can obtain the result from the argument of Theorem 5.7.
Theorem 5.8. For any $d$ and any smooth initial data, if the solution $Z(x,t)$ of the problem (4)-(6) satisfies $\|\nabla Z(\cdot,t)\|_{L^\infty} \leq C$ in $Q_T = \Omega \times [0,T]$, $\Omega \subset \mathbb{R}^d$, then $Z(x,t)$ is smooth in $Q_T$.

By using the standard method, we can get the following results.

Theorem 5.9. (Uniqueness of the solution) Let $u$ and $V$ be two smooth solution of problem (4)-(6), with the same initial data $u_0 = v_0 \in H^\infty(\Omega)$, $\Omega \subset \mathbb{R}^d$, then, for any $d$, we have $u \equiv v$.

Theorem 5.10. (Initial value problem) For the initial value problem (4)-(6), Theorem 5.6-Theorem 5.9 also hold for $x \in \mathbb{R}^d$.

Proof. The estimates for the problem (4)-(6) are independent of $D$ and let $D \to \infty$. Then the theorem is proved.

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