Cohomology with twisted coefficients of the classifying space of a fusion system

RÉMI MOLINIER

We study the cohomology with twisted coefficients of the geometric realization of a linking system associated to a saturated fusion system $\mathcal{F}$. More precisely, we extend a result due to Broto, Levi and Oliver to twisted coefficients. We generalize the notion of $\mathcal{F}$-stable elements to $\mathcal{F}_c$-stable elements in a setting of twisted coefficient cohomology and we show that, if the coefficient module is nilpotent, then the cohomology of the geometric realization of a linking system can be computed by $\mathcal{F}_c$-stable elements. As a corollary, we show that for any coefficient module, the cohomology of the classifying space of a $p$-local finite group can be computed by these $\mathcal{F}_c$-stable elements.

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1 Introduction

Classifying spaces of groups have been well-studied since their introduction by Milnor in the 1950s. One of the first important results was the proof of the Sullivan conjecture by Lannes, Miller and Carlson in the 1980s. This made it possible to answer to some questions which had seemed impossible to solve. In particular, one discovered that the classifying space of a group $G$ has a structure more rigid and more closely linked with its structure than was thought.

The strength of this link can be highlighted with $p$-local structures. For this goal, Broto, Levi and Oliver have developed a homotopy theory of fusion systems [BLO1]. The fusion system of a finite group $G$ with Sylow $p$-subgroup $S$ is a category, $\mathcal{F}_S(G)$, introduced by Puig in the 1970s [P1], which encodes how $G$ acts on the $p$-subgroups of $S$. The objects of this category are the subgroups of $S$ and the morphisms are the group morphisms induced by conjugations by elements of $G$. From that, we can define the associated centric linking system, denoted by $\mathcal{L}_c^S(G)$ (or $\mathcal{L}$ when there is no confusion). The $p$-completion (introduced by Bousfield and Kan [BK]) of its geometric realization, $|\mathcal{L}|_p^c$, has the homotopy type of $BG^p_\mathcal{G}$ (the $p$-completion of the classifying space of $G$).
In the 1990s and for another purpose, Puig gave the definition of an abstract fusion system \( \mathcal{F} \) over a finite \( p \)-group \( S \) \([P6]\). These abstract fusion systems do not always come from a finite group. Nevertheless, for an abstract fusion system \( \mathcal{F} \), Broto, Levi and Oliver defined an abstract centric linking system \( \mathcal{L} \) associated to \( \mathcal{F} \). The \( p \)-completion of its geometric realization has all the good properties of \( p \)-completed classifying spaces of finite groups. The existence and uniqueness of a linking system associated to a saturated fusion system were shown more recently by Chermak \([Ch]\) (using the theory of partial groups). It gives rise to the theory of \( p \)-local finite groups (we refer to Aschbacher, Kessar and Oliver \([AKO]\) for more details about fusion systems in general).

A \( p \)-local finite group is a triple \((S, \mathcal{F}, \mathcal{L})\) where \( S \) is a \( p \)-group, \( \mathcal{F} \) a fusion system over \( S \) and \( \mathcal{L} \) an associated centric linking system. For a \( p \)-local finite group \((S, \mathcal{F}, \mathcal{L})\), \(|\mathcal{L}|^\wedge\) is called its classifying space. The theory of \( p \)-local finite group have been studied in details by Broto, Levi, Oliver and others (see \([BLO2]\), \([OV1]\),[5a1] and \([5a2]\)). The linking system and its geometric realization, even without \( p \)-completion, play here a fundamental and central role.

An old and well-known result due to Cartan and Eilenberg (see \([CE]\) Chap XII, Theorem 10.1) expresses the cohomology of a finite group \( G \) in a \( \mathbb{Z}((p))[G] \)-module as the submodule of “stable” elements in the cohomology of a Sylow of \( G \). This submodule of stable elements corresponds to the inverse limit over the “fusion” of the group cohomology functor. One important result in the theory of \( p \)-local finite groups is an extension of this theorem to any \( p \)-local finite group which tells us that the cohomology of the geometric realization of a linking system can be computed by \( \mathcal{F} \)-stable elements.

**Theorem 1.1** ([BLO2], Theorem B) Let \((S, \mathcal{F}, \mathcal{L})\) be a \( p \)-local finite group. The inclusion of \( BS \) in \(|\mathcal{L}|\) induces a natural isomorphism

\[
H^*(|\mathcal{L}|^\wedge, \mathbb{F}_p) \cong H^*(|\mathcal{L}|, \mathbb{F}_p) \xrightarrow{\cong} H^*(\mathcal{F}^c, \mathbb{F}_p).
\]

Here \( \mathcal{F}^c \) denotes the full subcategory of \( \mathcal{F} \) where the objects are the subgroups of \( S \) which are \( \mathcal{F} \)-centric (the analog of \( p \)-centric in the group case). From that, we can easily extend this result to any finite \( \mathbb{Z}(p) \)-module \( A \) (a proof is given in \([5a2]\) Lemma 6.12).

One question asked by Oliver in his book with Aschbacher and Kessar \([AKO]\) is the understanding of the cohomology of \(|\mathcal{L}|\) with twisted coefficients. Indeed, this cohomology appears for example in the study of extensions of \( p \)-local finite groups or, more directly, can give more information about the link between the fusion system and
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these spaces. Levi and Ragnarsson [LR] already give some tools along this line. Here we try to extend Theorem 1.1 to twisted coefficients. This cannot be expected in general. For example, if we consider \( G = A_6 \), the alternating group over 6 letters and we study its 2-local structure \((S, \mathcal{F}, \mathcal{L})\), we can show that \( \mathbb{F}_2[A_6] \) has a natural \( \mathbb{F}_2[\pi_1(|\mathcal{L}|)]\)-module and that \( H^*(S, \mathbb{F}_2[A_6]) \) is trivial in positive degree whereas \( H^* (|\mathcal{L}|, \mathbb{F}_2[A_6]) \) is not. However, with some conditions (on \( \mathcal{F} \) or on the coefficients) we can extend Theorem 1.1. Here we focus on coefficient modules with a nilpotent action and we get the following result.

**Theorem 5.3** Let \((S, \mathcal{F}, \mathcal{L})\) be a \( p \)-local finite group. If \( M \) is an abelian \( p \)-group with a nilpotent action of \( \pi_1(|\mathcal{L}|) \), then the inclusion of \( BS \) in \( |\mathcal{L}| \) induces a natural isomorphism,

\[
H^*(|\mathcal{L}|, M) \cong H^*(\mathcal{F}^c, M).
\]

Here, for \( p \) a prime, a \( p \)-group is a finite group \( P \) with cardinal \(|P|\) a power of \( p \) One important application of this result is the complete description of the cohomology of the classifying space of a fusion system in terms of \( \mathcal{F}^c \)-stable elements.

**Corollary 6.4** Let \((S, \mathcal{F}, \mathcal{L})\) be a \( p \)-local finite group. If \( M \) is an abelian \( p \)-group with an action of \( \pi_1(|\mathcal{L}|^\wedge_p) \), then there is a natural isomorphism

\[
H^*(|\mathcal{L}|^\wedge_p, M) \cong H^*(\mathcal{F}^c, M).
\]

**Organization**

After the introduction, we first give some backgrounds about \( p \)-local finite groups in the Section 2. In Section 3, we start with the notion of inclusions in a linking system in Section 3.1 and we recall the notion of \( \delta \)-functors in Section 3.2. Then, we define properly the notion of \( \mathcal{F}^c \)-stable elements in Section 3.3. In a fourth section, we first recall the notion of \( \mathcal{F}^c \)-characteristic \((S, S)\)-biset in Section 4.1. Then, we construct an idempotent of the cohomology of \( S \) using an \( \mathcal{F}^c \)-characteristic \((S, S)\)-biset in Section 4.2 and we show that the image of this idempotent is a \( \delta \)-functor in Section 4.3. We give the main results, Theorem 5.3, in the Section 5. Finally, in Section 6, we give a result about the cohomology with twisted coefficients of \( p \)-good spaces and we apply it, with Theorem 5.3, to get Corollary 6.4.
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2 Background on $p$-local finite groups

We give here a very short introduction to $p$-local finite groups. The notion of fusion systems was first introduced by Puig for modular representation purpose. Later, Broto, Levi and Oliver developed the notion of linking systems and $p$-local finite groups to study $p$-completed classifying spaces of finite groups and spaces which have closed homotopy properties. We refer the reader interested in more details to Aschbacher, Kessar and Oliver [AKO].

A fusion system over a $p$-group $S$ is a way to abstract the action of a finite group $G \geq S$ on the subgroups of $S$ by conjugation.

**Definition 2.1** Let $S$ be a finite $p$-group. A fusion system over $S$ is a small category $\mathcal{F}$, where $\text{Ob}(\mathcal{F})$ is the set of all subgroups of $S$ and which satisfies the following two properties for all $P, Q \leq S$:

(a) $\text{Hom}_S(P, Q) \subseteq \text{Mor}_\mathcal{F}(P, Q) \subseteq \text{Inj}(P, Q)$;
(b) each $\varphi \in \text{Mor}_\mathcal{F}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

The composition in a fusion system is given by composition of homomorphisms. We usually write $\text{Hom}_\mathcal{F}(P, Q) = \text{Mor}_\mathcal{F}(P, Q)$ to emphasize that the morphisms in $\mathcal{F}$ are homomorphisms.

The typical example of fusion system is the fusion system of a finite group $G$.

**Example** Let $G$ be a finite group and $S$ a $p$-subgroup of $G$. The fusion system of $G$ over $S$ is the category $\mathcal{F}_S(G)$ where $\text{Ob}(\mathcal{F}_S(G))$ is the set of all subgroups of $S$ and for all $P, Q \leq S$, $\text{Mor}_{\mathcal{F}_S(G)}(P, Q) = \text{Hom}_G(P, Q)$. The category $\mathcal{F}_S(G)$ defines a fusion system over $S$. 
In general, it is more convenient to work with fusion system when \( S \) is a Sylow \( p \)-subgroup of \( G \) or more generally, when the fusion system is saturated. For that purpose, we will try to mimic Sylow Theorems in terms of the category \( \mathcal{F}_S(G) \). There are several definitions of saturation. Here, we only give the one due to Roberts and Shpectorov and the reader interested in other equivalent definitions can find more, for example, in Aschbacher, Kessar and Oliver [AKO].

**Definition 2.2** Let \( S \) be a subgroup and \( \mathcal{F} \) a fusion system over \( S \).

(a) Two subgroups \( P, Q \leq S \) are \( \mathcal{F} \)-conjugate if they are isomorphic as objects in \( \mathcal{F} \). We denote by \( P^\mathcal{F} \) the set of all subgroups of \( S \) \( \mathcal{F} \)-conjugate to \( P \).

(b) A subgroup \( P \leq S \) is fully automatized if \( \text{Aut}_S(P) \) is a Sylow \( p \)-subgroup of \( \text{Aut}_\mathcal{F}(P) \).

(c) A subgroup \( P \leq S \) is receptive in \( \mathcal{F} \) if it has the following property: for each \( Q \leq S \) and \( \varphi \in \text{Iso}_\mathcal{F}(Q, P) \), if we set

\[
N_\varphi = \left\{ g \in N_S(Q) \mid \varphi \circ c_g \circ \varphi^{-1} \in \text{Aut}_S(P) \right\},
\]

then there is \( \overline{\varphi} \in \text{Hom}_\mathcal{F}(N_\varphi, S) \) such that \( \overline{\varphi}|_Q = \varphi \).

A fusion system \( \mathcal{F} \) over a \( p \)-group \( S \) is saturated if each subgroup of \( S \) is \( \mathcal{F} \)-conjugate to a subgroup which is fully automized and receptive.

The case of the fusion system of a finite group over one of its Sylow \( p \)-subgroup is a particular case of a saturated fusion system.

**Proposition 2.3** (Puig) Let \( G \) be a finite group. If \( S \) is a Sylow \( p \)-subgroup of \( G \), then the category \( \mathcal{F}_S(G) \) is a saturated fusion system.

For a finite group \( G \) and a subgroup \( H \leq G \), we will denote by \( \mathcal{T}_H(G) \) the transporter category of \( G \) over \( H \) which is the small category with set of objects the set of all subgroups of \( H \) and for all \( H_1, H_2 \leq G \),

\[
\text{Mor}_{\mathcal{T}_G(H)}(H_1, H_2) = \mathcal{T}_G(H_1, H_2) = \left\{ g \in G \mid gh_1g^{-1} \leq H_2 \right\}.
\]

If we just want to consider a family \( \mathcal{H} \) of subgroups of \( H \), we will denote by \( \mathcal{T}_H^\mathcal{H}(G) \) the full subcategory of \( \mathcal{T}_H(G) \) with set of objects \( \mathcal{H} \). Here, we will consider the case where \( G \) is a finite group and \( H = S \) is a Sylow \( p \)-subgroup of \( G \). We will also restrict our attention on the full subcategory \( \mathcal{T}_S^\mathcal{H}(G) = \mathcal{T}_S^\mathcal{H}(G) \) with \( \mathcal{H} = \text{Ob}(\mathcal{F}^S) \) the set of all the \( \mathcal{F} \)-centric subgroups of \( S \) (see definition 2.4).
These transporter categories are useful in the study of the $p$-local structure of a finite group $G$ and we can mention that Oliver and Ventura [OV1] extend them to the notion of transporter systems to study extension of $p$-local finite groups by a $p$-group. Nevertheless, the structure of $T_S(G)$ is too linked to $G$ (for example, we can show that $|T_S(G)| = BG$) and even if we restrict our attention on the centric subgroups, two groups $G_1$ and $G_2$ with a same Sylow $p$-subgroup $S$ and the same fusion system over $S$ can have different centric transporter categories (one can for example take $G_1$ such that $O_{p'}(G_1) \neq 0$ and $G_2 = G_1/O_{p'}(G_1)$). The good object to consider is the linking system associated to $\mathcal{F}$ introduced by Broto, Levi and Oliver [BLO2]. But first we need to define the notion of $\mathcal{F}$-centric subgroup.

**Definition 2.4** Let $\mathcal{F}$ be a saturated fusion system over a $p$-group $S$. A subgroup $P \leq S$ is $\mathcal{F}$-centric if $C_S(Q) = Z(Q)$ for every $Q \in P^\mathcal{F}$. We will denote by $\mathcal{F}^c$ the full subcategory of $\mathcal{F}$ with set of objects all the $\mathcal{F}$-centric subgroups of $S$.

If $\mathcal{F}$ is the saturated fusion system associated to a finite group $G$ with $S$ as Sylow $p$-subgroup, then a subgroup $P \leq S$ is $\mathcal{F}$-centric if, and only if, $P$ is $p$-centric, i.e. $Z(P)$ is a Sylow $p$-subgroup of $C_G(P)$. Before defining the notion of centric linking system let first recall a well-known result about saturated fusion system.

**Theorem 2.5** (Alperin’s Fusion Theorem) Let $\mathcal{F}$ be a saturated fusion system over a $p$-group $S$. Then, every morphism is a composite of restrictions of automorphisms of $\mathcal{F}$-centric subgroups.

In other words, a saturated fusion system $\mathcal{F}$ is generated by $\mathcal{F}^c$. In fact, Alperin’s Fusion Theorem is more precise and says that we just need automorphisms of $S$ and $\mathcal{F}$-essential subgroups of $S$. For more details, we refer to Section I.3 of Aschbacher, Kessar and Oliver [AKO].

**Definition 2.6** Let $\mathcal{F}$ be a fusion system over a $p$-group $S$. A centric linking system associated to $\mathcal{F}$ is a finite category $\mathcal{L}$ together with a pair of functors

$$\begin{array}{ccc} T_S^c(S) & \overset{\delta}{\longrightarrow} & \mathcal{L} & \overset{\pi}{\longrightarrow} & \mathcal{F} \\ \end{array}$$

satisfying the following conditions:

(A) $\delta$ is the identity on objects, and $\pi$ is the inclusion on objects. For each $P, Q \in \textrm{Ob}(\mathcal{L})$ such that $P$ is fully centralized in $\mathcal{F}$, $C_S(P)$ acts freely on $\text{Mor}_\mathcal{L}(P, Q)$ via $\delta_{P, P}$ and right composition, and

$$\pi_{P, Q} : \text{Mor}_\mathcal{L}(P, Q) \longrightarrow \text{Hom}_\mathcal{F}(P, Q)$$

is the orbit map for this action.
(B) For each $P, Q \in \text{Ob}(\mathcal{L})$ and each $g \in T_{\mathcal{S}}(P, Q)$, the application $\pi_{P, Q}$ sends $\delta_{P, Q}(g) \in \text{Mor}_{\mathcal{L}}(P, Q)$ to $c_g \in \text{Hom}_{\mathcal{F}}(P, Q)$.

(C) For each $P, Q \in \text{Ob}(\mathcal{L})$, all $\psi \in \text{Mor}_{\mathcal{L}}(P, Q)$ and all $g \in P$, the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\psi} & Q \\
\downarrow{\delta_{P, Q}(g)} & & \downarrow{\delta_{Q}(\psi(g))} \\
P & \xrightarrow{\psi} & Q
\end{array}
\]

commutes in $\mathcal{L}$.

The example we should have in mind is the centric linking system of a finite group, defined as follows.

**Example** Let $G$ be a finite group and $S$ a $p$-subgroup of $G$. The *centric linking system of $G$ over $S$* is the category $\mathcal{L}^c_{\mathcal{S}}(G)$ where $\text{Ob}(\mathcal{L}^c_{\mathcal{S}}(G))$ is the set of all $p$-centric subgroups of $S$ and for all $P, Q \leq S$,

\[
\text{Mor}_{\mathcal{L}^c_{\mathcal{S}}(G)}(P, Q) = T_{\mathcal{G}}(P, Q)/O^p(C_{\mathcal{G}}(P))
\]

(remark that, as $P$ is $p$-centric, $O^p(C_{\mathcal{G}}(P))$ has order prime to $p$). The category $\mathcal{L}^c_{\mathcal{S}}(G)$ with the obvious functors, $\pi$ and $\delta$, defines a centric linking system associated to $\mathcal{F}_{\mathcal{S}}(G)$.

Recently, Chermak [Ch] proved, using his theory of partial groups, that, for a given saturated fusion system, there exists a centric linking system associated to $\mathcal{F}$ which is unique (there is also an interpretation by Oliver in terms of obstruction theory in [O5]). We can generalize this definition to abstract linking system with a more flexible set of objects. This can be useful when we want to compare two $p$-local finite groups. This definition, properties and homotopy properties of linking system can be found in Aschbacher, Kessar and Oliver [AKO].

**Definition 2.7** A *$p$-local finite group* is defined to be a triple $(S, \mathcal{F}, \mathcal{L})$ where $S$ is a $p$-group, $\mathcal{F}$ a saturated fusion system over $S$, and $\mathcal{L}$ is a linking system associated to $\mathcal{F}$. The *classifying space* of $(S, \mathcal{F}, \mathcal{L})$ is then given by $|\mathcal{L}|_{p}^\wedge$.

For example, if $G$ is a finite group and $S$ a Sylow $p$-subgroup of $G$, $|\mathcal{L}^c_{\mathcal{S}}(G)|_{p}^\wedge \cong B\mathcal{G}_{\mathcal{P}}$. This was proved by Broto, Levi and Oliver in [BLO1] and was used by Oliver [O1, O2] in the proof of the Martino-Priddy conjecture.
3 Cohomology and $\mathcal{F}^c$-stable elements

3.1 Inclusion in a linking system

Before defining a cohomology functor, we need to identify how a morphism in $\mathcal{L}$ will act on a $\mathbb{Z}(p)[\pi_{\mathcal{L}}]$-module. For this, we need a better understanding of the link between $\mathcal{L}$ and its fundamental group. The notion of compatible set of inclusions was defined by Broto, Castellana, Grodal, Levi and Oliver in [5a2]. It allows us to define restriction and extension of morphisms in a linking system.

**Definition 3.1** Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. A compatible set of inclusions for $\mathcal{L}$ is a choice of morphisms $\iota^Q_P \in \text{Mor}_\mathcal{L}(P, Q)$, one for each pair of $\mathcal{F}$-centric subgroups $P \leq Q$, such that $\iota^S_P = \text{Id}_S$, and the following holds for all $P \leq Q \leq R$,

1. $\pi(\iota^Q_P)$ is the inclusion $P \leq Q$;
2. $\iota^R_Q \circ \iota^Q_P = \iota^R_P$.

We often write $\iota_P = \iota^S_P$. The existence of a compatible set of inclusions for $\mathcal{L}$ is proved in [5a2], Proposition 1.13, but an easy example is given by, for each pair of $\mathcal{F}$-centric subgroups $P \leq Q$,

$$\iota^Q_P = \delta^Q_P(1).$$

Let us fix a compatible set of inclusion $\left(\iota^Q_P\right)_{P \leq Q}$. We denote by

$$\pi_{\mathcal{L}} = \pi_1(|\mathcal{L}|, S),$$

the fundamental group of the geometric realization $|\mathcal{L}|$ with base point at the vertex $S$ and let $\mathcal{B}(\pi_{\mathcal{L}})$ be the category with a unique object, and morphism set equal to $\pi_{\mathcal{L}}$ (hence, $|\mathcal{B}(\pi_{\mathcal{L}})| = B\pi_{\mathcal{L}}$). We define a functor

$$\omega : \mathcal{L} \longrightarrow \mathcal{B}(\pi_{\mathcal{L}})$$

which sends each object to the unique one in the target and sends each morphism $\varphi \in \text{Mor}_\mathcal{L}(P, Q)$ to the class of the loop $\iota_Q \cdot \varphi \cdot \iota_P$ where $\iota_P$ is the edge $\iota_P$ followed in the other direction.

In particular, every $\mathbb{Z}(p)[\pi_{\mathcal{L}}]$-module is naturally a $\mathbb{Z}(p)[S]$-module where the action is given by the following composition:

$$\mathcal{B}(S) = \mathcal{B}(\text{Mor}_{\mathcal{T}^3(S)}(S, S)) \xrightarrow{\delta_S} \mathcal{L} \xrightarrow{\omega} \mathcal{B}(\pi_{\mathcal{L}}).$$
3.2 Background on \( \delta \)-functors

We refer the reader to [We], Chapter 2, where he can find the notion of \( \delta \)-functors, derived functors and their properties.

**Definition 3.2** Let \( A \) and \( B \) be two abelian categories.

A (contravariant) \( \delta \)-functor is a functor \( F^* : A \rightarrow B \) such that, for every short exact sequence
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]
there exist, for every \( i \), a connecting homomorphism \( \delta^i_F : F^i(A) \rightarrow F^i(C) \) such that the sequence,
\[
\cdots \rightarrow F^i(C) \rightarrow F^i(B) \rightarrow F^i(A) \xrightarrow{\delta^i_F} F^{i+1}(C) \rightarrow \cdots
\]
is exact.

If \( F^*, G^* : A \rightarrow B \) are two \( \delta \)-functors, a morphism of \( \delta \)-functors from \( F^* \) to \( G^* \) is a natural transformation \( \eta \) such that, for every short exact sequence
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0
\]
We have a commutative diagram,
\[
\begin{array}{cccccccc}
\cdots & F^i(C) & \rightarrow & F^i(B) & \rightarrow & F^i(A) & \xrightarrow{\delta^i_F} & F^{i+1}(C) & \rightarrow & \cdots \\
\downarrow{\eta_C} & \downarrow{\eta_B} & & \downarrow{\eta_A} & & \downarrow{\eta_C} & & \downarrow{\eta_C} & & \\
\cdots & G^i(C) & \rightarrow & G^i(B) & \rightarrow & G^i(A) & \xrightarrow{\delta^i_G} & G^{i+1}(C) & \rightarrow & \cdots
\end{array}
\]

With the usual composition on natural transformations, we obtain a category.

**Remark 3.3** A \( \delta \)-functor can be seen as a functor from the category \( \mathcal{S}_A \) of short exact sequences in \( A \) to \( \text{Ch}(B) \) which sends any short exact sequence to an acyclic chain complex.

If \( \eta : F \rightarrow G \) is a natural transformation then, to show that it is a morphism of \( \delta \)-functors, it is enough to prove that, for every short exact sequence
\[
0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,
\]
the following diagram is commutative for every $i$.

$$
\begin{array}{ccc}
F^i(A) & \xrightarrow{\delta^i_f} & F^{i+1}(C) \\
\downarrow{\eta_A} & & \downarrow{\eta_C} \\
G^i(A) & \xrightarrow{\delta^i_G} & G^{i+1}(C)
\end{array}
$$

One important example of a contravariant $\delta$-functor is the left derived functor of an additive and left exact functor. For a group $G$, the group cohomology functor $H^\ast(G, -) : \mathbb{Z}_p[G]\text{-Mod} \rightarrow \mathbb{Z}_p\text{-Mod}$ is the left derived functor of the fixed point functor $M \rightarrow M^G$ and then, it define a universal $\delta$-functor (by [We], Theorem 2.4.7).

**Definition 3.4** A covariant $\delta$-functor $F^\ast : A \rightarrow B$ is universal if, given another $\delta$-functor $G^\ast : A \rightarrow B$ and a natural transformation $\eta^0 : F^0 \rightarrow G^0$, there exists a unique morphism of $\delta$-functors $\eta^\ast : F^\ast \rightarrow G^\ast$ extending $\eta^0$.

In particular, if for two groups $G$ and $H$, we have a natural transformation between the functor $(-)^G$ and $(-)^H$, there is a unique morphisms of $\delta$-functors from $H^\ast(G, -)$ to $H^\ast(H, -)$ which extend this natural transformation.

For $G$ a finite group $H$ a subgroup of $G$ and $M$ a $\mathbb{Z}_p[G]$-module, restriction and conjugation by an element $g \in G$, defined in degree 0 by

$$
\begin{array}{c}
(\text{Res}^G_H)^0 : M^G \rightarrow M^H \\
x \mapsto x
\end{array}
$$

and

$$
\begin{array}{c}
c^0_g : M^{gh^{-1}} \rightarrow M^H, \\
x \mapsto g^{-1}x
\end{array}
$$

give examples of morphisms of $\delta$-functors. An other example is given by the transfer. If $G$ is a group and $H$ is a subgroup of $G$ of finite index, we can define a transfer map

$$
\text{tr}^G_H : H^\ast(H, -) \rightarrow H^\ast(G, -)
$$

as the morphism of $\delta$-functors induced by the natural transformation $(-)^H \rightarrow (-)^G$ given by, for $M$ a $\mathbb{Z}_p[G]$-module,

$$
\begin{array}{c}
M^H \rightarrow \sum_{g \in [G/H]} g^x \\
x \mapsto \sum_{g \in [G/H]} gx
\end{array}
$$
3.3 Cohomology functor

We first need to define the notion of stable elements in the setting of twisted coefficients. The idea is to define them as a projective limit of the group cohomology functor. However, when we work with twisted coefficients with an action of $\pi_L$, we cannot define our cohomology functor on all $F$. However, we can easily define a functor on $L$ and we will see that this functor factors through $F^c$.

Let $(S, F, L)$ be a $p$-local finite group and $M$ a $\mathbb{Z}(\pi_L)$-module. As we work with an action of $\pi_1(|L|)$, we can define a functor on $L$ using the bi-functoriality of group cohomology (see Weibel [We] Remark 6.7.6 for the bi-functoriality of group cohomology).

\[
\begin{align*}
H^*(-, M) : & \quad L \xrightarrow{\text{Ob}(L)} \mathbb{Z}(\pi_L)\text{-Mod} \\
& \quad P \in \text{Ob}(L) \xrightarrow{\text{H}^*(P, M)} H^*(\varphi, M) = \varphi^* \\
& \quad \varphi \in \text{Mor}_L(P, Q) \xrightarrow{\text{H}^*(\varphi, M) = \varphi^*} H^*(\pi(\varphi), \omega(\varphi)^{-1})
\end{align*}
\]

For $P, Q$ two subgroups of $S$ and $\varphi \in \text{Mor}_L(P, Q)$, $H^*(\varphi, M)$ can be also defined on the chain level as follows:

\[
\begin{align*}
\text{Hom}_{\mathbb{Z}(\pi_L)[S]}(R_\bullet, M) & \xrightarrow{\text{Hom}_{\mathbb{Z}(\pi_L)[P]}(R_\bullet, M)} \\
f & \mapsto (\omega(\varphi)^{-1} f \circ \pi(\varphi)_*)
\end{align*}
\]

where $(R_\bullet)$ is a projective resolution of the trivial $\mathbb{Z}(\pi_L)[S]$-module $\mathbb{Z}(\pi_L)$. Finally, it can also be defined as the morphism between the two derived functors of $(-)^Q$ and $(-)^P$ induced by

\[
x \in M^Q \xrightarrow{\omega(\varphi)^{-1}} x \in M^P.
\]
and it define a morphism of $\delta$-functor.

**Proposition 3.5** Let $(S, F, L)$ be a $p$-local finite group. If $P, Q \leq S$ are $F$-centric and $\varphi \in \text{Mor}_F(P, Q)$, then

\[
H^*(\varphi, -) : H^*(Q, -) \xrightarrow{\text{H}^*(P, -)} H^*(P, -)
\]

is a morphism of $\delta$-functors from $H^*(Q, -)$ to $H^*(P, -)$.

**Proof** As explained above, $H^*(\varphi, -)$ is the morphism of $\delta$-functors from the universal $\delta$-functor $H^*(Q, -)$ to $H^*(P, -)$ which is defined in degree 0 by, for $M$ a $\mathbb{Z}[\pi_L]$-module, $x \in M^Q \xrightarrow{\omega(\varphi)^{-1}} x \in M^P$. \(\square\)
By construction, this functor extend naturally the group cohomology functor defined on $\mathcal{T}_S^c(S)$.

$$
\begin{array}{ccc}
\mathcal{T}_S^c(G) & \xrightarrow{H^*(-,M)} & \mathbb{Z}_P\text{-Mod} \\
\pi & \downarrow \pi & \\
\mathcal{L} & \xrightarrow{H^*(-,M)} & \\
\end{array}
$$

In particular, for every $P \leq S$ and $g \in P$, $H^*(\delta_P(g), M) = c_g^*$.

**Proposition 3.6** Let $\varphi, \beta \in \text{Mor}_\mathcal{L}(P, Q)$ with $P, Q \in \mathcal{L}$. If $\pi(\varphi) = \pi(\beta)$ then $H^*(\varphi, M) = H^*(\beta, M)$.

**Proof** If $\pi(\varphi) = \pi(\beta)$, then there exists $u \in Z(P)$ such that $\varphi = \beta \circ \delta_P(u)$ and thus $H^*(\varphi, M) = H^*(\delta_P(u), M) \circ H^*(\beta, M)$.

However $H^*(\delta_P(u), M) = H^*(\pi(\delta_P(u), \omega(u)^{-1}) = H^*(c_u, \omega(u)^{-1}) = c_u^*$ is the automorphism of $H^*(P, M)$ induced by the conjugation by $u$, and, as $u \in Z(P) \leq P$, this is the identity (See [CE], Section XII.8, for the properties of $c_g^*$).

In particular, if $\pi(\varphi) = \text{incl}_P^Q$, then $H^*(\varphi, M) = H^*(\text{incl}_P^Q, M) = H^*(\text{incl}_P^Q, \text{Id}_M) = \text{Res}_P^Q$. Hence, we can factor our functor through $\mathcal{F}^c$ and define the $\mathcal{F}$-centric stable elements.

**Definition 3.7** An element $x \in H^*(S, M)$ is called $\mathcal{F}$-centric stable, or just $\mathcal{F}^c$-stable, if for all $P \in \text{Ob}(\mathcal{F}^c)$ and all $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$,

$$
\varphi^*(x) = \text{Res}_P^S(x).
$$

We denote by $H^*(\mathcal{F}^c, M) \subseteq H^*(S, M)$ the submodule of all $\mathcal{F}^c$-stable elements.

This submodule of $\mathcal{F}^c$-stable elements corresponds to the inverse limit of $H^*(-, M)$ on the category $\mathcal{F}^c$,

$$
H^*(\mathcal{F}^c, M) \cong \lim_{\mathcal{F}^c} H^*(-, M).
$$

Moreover, if $M$ is a $\mathbb{Z}_P$-module with a trivial action of $\pi_\mathcal{L}$, then, by Alperin’s Fusion Theorem (Theorem 2.5), $H^*(\mathcal{F}^c, M) \cong H^*(\mathcal{F}, M)$ and the notion of $\mathcal{F}^c$-stable elements extends naturally the notion $\mathcal{F}$-stable elements.

In general, we cannot expect to define a cohomology functor on all $\mathcal{F}$. Indeed, if we take a morphism $\varphi$ in $\mathcal{F}$ between two subgroups which are not $\mathcal{F}$-centric, by Alperin’s Fusion Theorem (Theorem 2.5), we can see it as a composite of restrictions of morphisms in $\mathcal{F}^c$. However, this decomposition is not unique in general and...
two different decompositions can lead to completely different morphisms in twisted cohomology. As an easy example, we can look at the trivial group \( \{ e \} \) in a given fusion system \( \mathcal{F} \): each morphism in \( \mathcal{F}^c \) restricts to the identity on \( \{ e \} \), but, if \( M \) is not a trivial \( \mathbb{Z}(p) \{ \pi \} \)-module, not every \( \varphi \in \text{Mor}(\mathcal{F}^c) \) acts trivially on \( M = M^{\{ e \}} = H^0(\{ e \}, M) \).

4 An idempotent coming from a biset

An important result in Broto, Levi and Oliver [BLO2], and a crucial tool in the proof of Theorem 1.1, is the existence of an \( \mathcal{F} \)-characteristic \( (S, S) \)-biset which leads to an idempotent of \( H^*(S, \mathbb{F}_p) \) with image \( H^*(\mathcal{F}, \mathbb{F}_p) \). Unfortunately we cannot use the same biset to get an idempotent when we consider twisted coefficients.

The first problem is that, as explained earlier, our cohomology functor cannot be defined on the whole fusion system. An \( \mathcal{F} \)-characteristic \( (S, S) \)-biset is not in general \( \mathcal{F}^c \)-generated, i.e. it cannot be decomposed in transitive \( (S, S) \)-bisets of the form \( (S \times S)/\Delta(P, \varphi) \) with \( P \in \text{Ob}(\mathcal{F}^c) \), \( \varphi \in \text{Hom}(\mathcal{F}, P, S) \) and \( \Delta(P, \varphi) = \{ (\varphi(h), h) : h \in P \} \). This problem can be bypassed working with an \( \mathcal{F}^c \)-characteristic \( (S, S) \)-biset instead of an \( \mathcal{F} \)-characteristic one.

The second problem is a bit more crucial. The composite of two left-free \( \mathcal{F}^c \)-generated \( (S, S) \)-bisets is not in general an \( \mathcal{F}^c \)-generated biset. This enables us to define in general a subcategory \( A_{\mathcal{F}^c} \) of the Burnside category with set of objects \( \text{Ob}(\mathcal{F}^c) \) and such that for each \( P, Q \in \text{Ob}(\mathcal{F}^c) \), \( A_{\mathcal{F}^c}(P, Q) = \text{Mor}_{A_{\mathcal{F}^c}}(P, Q) \) is the set of all \( \mathcal{F}^c \)-generated \( (P, Q) \)-bisets. Hence the morphism induced in cohomology by an \( \mathcal{F}^c \)-characteristic \( (S, S) \)-biset is not, in general, an idempotent.

4.1 Background on bisets and \( \mathcal{F}^c \)-characteristic bisets

Let \( G, H \) be two finite groups.

Transitive \( (G, H) \)-bisets (here, \( G \) acts on the left and \( H \) on the right) are isomorphic to bisets of the form \( (G \times H)/K \) for \( K \) a subgroup of \( G \times H \). We can then use the Goursat Lemma to describe all these subgroups. Here, we are just interested in isomorphic classes of \( (G, H) \)-bisets where the action of \( G \) is free. In this setting, the classes of transitive left-free \( (G, H) \)-bisets are given by pairs \( (K, \varphi) \), where \( K \) is a subgroup of \( H \) and \( \varphi \in \text{Hom}(K, G) \) a group homomorphism.
**Notation** For all \((K, \varphi)\), with \(K\) a subgroup of \(H\) and \(\varphi \in \text{Hom}(K, G)\) a group homomorphism, we write
\[
\Delta(K, \varphi) = \{(\varphi(k), k) ; k \in K \} \leq G \times H.
\]
For a \((G, H)\)-pair \((K, \varphi)\), the set \(\{K, \varphi\} := G \times H / \Delta(K, \varphi)\) defines a \((G, H)\)-biset and the isomorphic class of this biset is determined by the conjugacy class of \(\Delta(K, \varphi)\) and we will denote by \([K, \varphi]\) this class.

We can also define a category \(B\), often called the *Burnside category*, where the objects are the finite groups and, for all finite groups \(G, H\), \(B(G, H)\) is the set of isomorphic classes of \((G, H)\)-bisets. Composition is given by the following construction.

**Definition 4.1** Let \(G, H\) and \(K\) be finite groups, \(\Omega\) a \((G, H)\)-biset and \(\Lambda\) a \((K, G)\)-biset. We define,
\[
\Omega \circ \Lambda = \Omega \times_G \Lambda = \Omega \times \Lambda / \sim
\]
where, for all \(x \in \Omega\), \(y \in \Lambda\) and \(g \in G\), \((x, g.y) \sim (x.g, y)\).

This construction is compatible with isomorphisms, and then \(B\) with the induced composition law defines a category.

As we work with left-free bisets, we consider the subcategory \(\mathcal{A} \subseteq B\) where the objects are the same but we restrict the morphisms to isomorphic classes of left-free bisets. This gives us a category and the composition follows from the next lemma.

**Lemma 4.2** Let \(G, H\) and \(K\) be finite groups. Let \([K, \varphi] \in \mathcal{A}(G, H)\) and \([L, \psi] \in \mathcal{A}(H, K)\). Then,
\[
[K, \varphi] \circ [L, \psi] = \prod_{x \in K / H \psi(L)} [\psi^{-1}(\psi(L) \cap ^xK), \varphi \circ \psi].
\]

Let \((S, F, L)\) be a \(p\)-local finite group. The notion of \(\mathcal{F}_c\)-characteristic biset was recently introduced by Gelvin and Reeh [GRh].

**Definition 4.3** Let \(\Omega\) be a left-free \((S, S)\)-biset.

(a) We say that \(\Omega\) is \(\mathcal{F}_c\)-generated if it is the union of \((S, S)\)-bisets of the form \([P, \varphi]\) with \(P \in \text{Ob}(\mathcal{F}_c)\) and \(\varphi \in \text{Hom}_F(P, S)\).

(b) We say that \(\Omega\) is left-\(\mathcal{F}_c\)-stable if for all \(P \in \text{Ob}(\mathcal{F}_c)\) and \(\varphi \in \text{Hom}_F(P, S)\), we have \(\varphi \Omega_S \cong \rho \Omega_S\), i.e.
\[
(P[\varphi(P), \varphi^{-1}]_S) \circ [\Omega] = (P[\rho P, \text{Id}_S]) \circ [\Omega].
\]
(c) We say that $\Omega$ is right-$\mathcal{F}^c$-stable if for all $P \in \mathrm{Ob}(\mathcal{F}^c)$ and $\varphi \in \mathrm{Hom}_\mathcal{F}(P, S)$, we have $s\Omega \varphi \cong s\Omega P$, i.e. 
\[
[\Omega] \circ (s[\varphi(P), \varphi]_P) = [\Omega] \circ (s[P, \text{incl}_P^S]_P).
\]
(d) We say that $\Omega$ is non degenerate if $|\Omega|/|S| \neq 0$ modulo $p$.

If $\Omega$ satisfies all this four properties, we say that $\Omega$ is an $\mathcal{F}^c$-characteristic $(S, S)$-biset.

This definition is really similar to the one for $\mathcal{F}$-characteristic $(S, S)$-biset replacing each “$\mathcal{F}$” by “$\mathcal{F}^c$”. As explained earlier, we cannot use an $\mathcal{F}$-characteristic biset when we work with twisted coefficients. In general, an $\mathcal{F}$-characteristic biset contains transitive parts coming from morphisms between non $\mathcal{F}$-centric subgroups.

Example Let $(S, \mathcal{F}, L)$ the $p$-local of $S = D_8$ in $A_6$. The lattice of subgroups of $S$ has the following form.

The fusion is generated by the inner automorphisms of $S$, $\alpha$ an automorphism of $V$ of order 3 which permutes $Q_1, Q_2$ and $Z$, and the analog $\beta \in \text{Aut}_\mathcal{F}(V')$. The minimal $\mathcal{F}$-characteristic biset is given by the following decomposition,
\[
[D_8, \text{Id}] \sqcup [V, \alpha] \sqcup [V', \beta] \sqcup [Q_1, \beta^{-1} \circ \alpha] \sqcup [Q'_1, \alpha^{-1} \circ \beta].
\]
Here, the subgroups $Q_1$ and $Q'_1$ are not $\mathcal{F}$-centrics and every $\mathcal{F}$-characteristic biset contains this minimal $\mathcal{F}$-characteristic biset. On the other hand, an $\mathcal{F}^c$-characteristic biset is the following,
\[
[D_8, \text{Id}] \sqcup [V, \alpha] \sqcup [V', \beta].
\]

One way to obtain an $\mathcal{F}^c$-characteristic $(S, S)$-biset is by considering the set $\mathcal{G}$ of non extendable isomorphisms of $L$. If $\varphi : P \rightarrow Q$ is a non extendable morphism of $L$ and $s, s' \in S$, then $\delta^Q_s(\varphi) \circ \varphi \circ \delta^P_{s'}$ is also a non extendable isomorphism of $L$. It gives to $\mathcal{G}$ a $(S, S)$-biset structure and Gelvin and Reeh have shown the following theorem.
Theorem 4.4 ([GRh], Theorem C) The biset $\mathcal{G}$ is an $\mathcal{F}^c$-characteristic $(S,S)$-biset. Moreover, it is the unique minimal $\mathcal{F}^c$-characteristic $(S,S)$-biset and thus it is the $\mathcal{F}$-centric part of the minimal $\mathcal{F}$-characteristic $(S,S)$-biset.

4.2 An idempotent

As in [BLO2], we will use an $\mathcal{F}^c$-characteristic $(S,S)$-biset to define, for any $\mathbb{Z}_{(p)}[\pi_L]$-module $M$, an idempotent of $H^*(S,M)$.

More generally, for $P, Q, R \in \mathcal{F}^c$ with $R \leq Q$ and $\varphi \in \text{Hom}_\mathcal{F}(R, P)$, we can associate to the $(P,Q)$-pair $\{R, \varphi\} a$ morphism

$$\{R, \varphi\}_*: x \in H^*(P, M) \mapsto \text{tr}^Q_R \circ \varphi^*(x) \in H^*(Q, M).$$

If we take another $(P, Q)$-biset $\{R', \varphi'\}$ isomorphic to $\{R, \varphi\}$ (this implies that $R'$ is also $\mathcal{F}$-centric), we obtain the same morphism (by the properties of transfer and the fact that the action of $P$ on $H^*(P, M)$ is trivial). Then we can set $\{R, \varphi\}_*$ as the composite $\text{tr}^Q_R \circ \varphi^*$ and it is well-defined. We then define $\Omega_*$ for all left-free $\mathcal{F}^c$-generated $(P, Q)$-biset $\Omega$ by the sum of its transitive components.

Remark 4.5 By Proposition 3.5, for $\varphi \in \text{Mor}(\mathcal{F}^c)$, $H^*(\varphi, -)$ is a morphism of $\delta$-functor. Hence, as $\Omega_*$ is a sum of composites of transfer, restrictions and $H^*(\varphi, -)$, for $\varphi \in \text{Mor}(\mathcal{F}^c)$, which are all morphisms of $\delta$-functors it is an endomorphism of $\delta$-functors.

When we work with trivial coefficients, we can define a functor from the category $A_\mathcal{F}$, where the objects are the subgroups of $S$ and the morphisms between $P, Q \leq S$ are the isomorphic classes of $\mathcal{F}$-generated left-free $(P,Q)$-bisets, to $\mathbb{Z}_{(p)}$-$\text{Mod}$. This is possible by Lemma 4.2 and the properties of transfer.

When we work with twisted coefficients, we have to be more careful. We can only work with $\mathcal{F}^c$-generated left-free $(P,Q)$-bisets, but the set of $\mathcal{F}^c$-generated left-free $(P,Q)$-bisets for $P$ and $Q$ in $\mathcal{F}^c$ is not, in general, stable by composition! However, we still get a map from the set $A_{\mathcal{F}^c}(P, Q)$ of $\mathcal{F}^c$-generated left-free $(P,Q)$-bisets to $\text{Hom}(H^*(P,M), H^*(Q,M))$ for all $\mathbb{Z}_{(p)}[\pi_L]$-module $M$ and $P, Q \leq S$. In fact, $A_{\mathcal{F}^c}$ defines a category if, and only if, $\text{Ob}(\mathcal{F}^c)$ is a lattice, which is for example the case when $\mathcal{F}$ is constrained.

In the case of trivial coefficients, the key is the following proposition where we use explicitly that we have a functor from $A_\mathcal{F}$ to $\mathbb{Z}_{(p)}$-$\text{Mod}$. 
**Proposition 4.6** Let $(S, F, L)$ be a $p$-local finite group and $M$ be a $\mathbb{Z}_p$-module (with a trivial action of $\pi_L$). If $\Omega$ is an $F^c$-characteristic biset, then $\frac{|S|}{|H|}\Omega_\ast \in \text{End}(H^*(S, M))$ defines an idempotent with image $H^*(F^c, M)$.

**Proof** The proof is the same as the proof given by Broto, Levi and Oliver in [BLO2], Proposition 5.5, with an $F$-characteristic $(S, S)$-biset. We will show that $\text{Im} (\Omega_\ast) \subseteq H^*(F^c, M)$ and that $\Omega_\ast|_{H^*(F^c, M)} = \text{Id}_{H^*(F^c, M)}$.

Let start with the second point. If $x \in H^*(F^c, M)$, for all $P \in \text{Ob}(F^c)$ and every $\varphi \in \text{Hom}_F(P, S)$,

$$[P, \varphi]_\ast(x) = \text{tr}_{P}^S \circ \varphi^\ast(x) = \text{tr}_{P}^S \circ \text{Res}_{P}(x) = [S : P] \cdot$$

Thus, $\frac{|S|}{|H|}\Omega_\ast(x) = x$.

The first point uses the $F^c$-stability of $\Omega$ (Definition 4.3). For all $x \in H^*(S, M)$, $P \in F$ and every $\varphi \in \text{Hom}_F(P, S)$,

$$\varphi^\ast \circ \Omega_\ast(x) = (p[\varphi(P), \varphi^{-1}]S)_\ast \circ (S\Omega_\ast)_\ast(x) = (p[\varphi(P), \varphi^{-1}]S \circ S\Omega_\ast)_\ast$$

$$= (p[P, \text{incl}_P^S] \circ S\Omega_\ast)_\ast(x) = \text{Res}_{P}^S \circ \Omega_\ast(x).$$

Hence the image of $\frac{|S|}{|H|}\Omega_\ast$ is included in $H^*(F^c, M)$. $\Box$

When we work with twisted coefficients, as explained earlier, we have to be more careful. If we look at the example $(D_8, F_{D_8}(A_6), L_{D_8}(A_6))$ in degree 0 and with coefficient module $M = \mathbb{F}_p[A_6]$, we see that $\frac{|S|}{|H|}\Omega_\ast$ is not an idempotent and its image contains strictly $H^*(F_{D_8}(A_6), M)$.

In fact, only the following result still holds.

**Lemma 4.7** Let $M$ be a $\mathbb{Z}_p[\pi_L]$-module.

(a) The endomorphism $\frac{|S|}{|H|}\Omega_\ast \in \text{End}(H^*(S, M))$ restricted to $H^*(F^c, M)$ is the identity.

(b) $\text{Im} (\Omega_\ast) \subseteq H^*(S, M)^{\text{Aut}_F(S)}$.

**Proof** For (a), if $x \in H^*(F^c, M)$, for all $P \in \text{Ob}(F^c)$ and $\varphi \in \text{Hom}_F(P, S)$,

$$[P, \varphi]_\ast(x) = \text{tr}_{P}^S \circ \varphi^\ast(x) = \text{tr}_{P}^S \circ \text{Res}_{P}(x) = [S : P] \cdot$$
Hence, \( \frac{|S|}{|\Omega|} \omega_s(x) = x \). For (b), we can remark that, if \( \varphi \in \text{Aut}_\mathcal{F}(S) \),
\[
\varphi \circ \Omega_s = (S[S, \varphi]_S \circ S \Omega_s)_s = \Omega_s.
\]

We have the last equality because \( S[S, \varphi]_S \circ S \Omega_s \) is isomorphic to \( \Omega \) by \( \mathcal{F}^c \)-stability and then they induce the same morphism in cohomology. In fact, here, all the transitive \( (S, S) \)-biset which appear in the composite are \( \mathcal{F}^c \)-generated. The last equality follows by \( \mathcal{F}^c \)-stability of \( \Omega \).

In general, if the action of \( \pi_{\mathcal{L}} \) on \( M \) is not trivial, \( \frac{|S|}{|\Omega|} \omega_s \) is not an idempotent of \( H^*(S, M) \). But, if we iterate it enough times, we get one.

**Proposition 4.8** Let \( (S, \mathcal{F}, \mathcal{L}) \) be a \( p \)-local finite group and let \( M \) be an abelian \( p \)-group with an action of \( \pi_{\mathcal{L}} \). Let \( \Omega \) be an \( \mathcal{F}^c \)-characteristic \( (S, S) \)-biset. For every \( k \geq 0 \), there is a natural number \( N_{k, M} \) such that, \( \left( \frac{|S|}{|\Omega|} \Omega_k \right)^{N_{k, M}} \) defines an idempotent \( \omega_{k, M} \) of \( H^k(S, M) \) and we have
\[
H^k(\mathcal{F}^c, M) \subseteq \text{Im}(\omega_{k, M}) \subseteq H^k(S, M)^{\text{Aut}_\mathcal{F}(S)}.
\]

**Proof** To simplify the notation, we write \( \omega = \frac{|S|}{|\Omega|} \Omega_k \). For any \( k \geq 0 \), we have the following decreasing family of subgroups of \( H^k(S, M) \).
\[
H^k(\mathcal{F}^c, M) \subseteq \cdots \subseteq \text{Im}(\omega') \subseteq \text{Im}(\omega'^{-1}) \subseteq \cdots \subseteq \text{Im}(\omega^1) \subseteq \text{Im}(\omega^0) = H^k(S, M).
\]

As \( H^k(S, M) \) is a finite abelian \( p \)-group, this sequence stabilizes and then there is an \( n_0 \geq 1 \) such that for all \( n \geq n_0 \), \( \text{Im}(\omega^n) = \text{Im}(\omega^{n_0}) \). In particular, \( \omega^{n_0}|_{\text{Im}(\omega^{n_0})} \) is a permutation of the finite set \( \text{Im}(\omega^{n_0}) \) and there is an \( l \) such that \( (\omega^{n_0}|_{\text{Im}(\omega^{n_0})})^l = 1_{\text{Im}(\omega^{n_0})} \). Thus, for \( N_{k, M} = l \times n_0 \), the endomorphism \( \omega_{k, M} = \omega^{N_{k, M}} \in \text{End}(H^k(S, M)) \) is an idempotent with image \( \text{Im}(\omega^{n_0}) \supseteq H^k(\mathcal{F}^c, M) \). Finally, as \( \omega_{k, M} \) is just a finite iteration of \( \frac{|S|}{|\Omega|} \Omega_k \), \( \text{Im}(\omega_{k, M}) \subseteq H^*(S, M)^{\text{Aut}_\mathcal{F}(S)} \), by Lemma 4.7 (b).

Hence, we can define an idempotent \( \omega_s \) of \( H^*(S, M) \) as following. For every \( k \geq 0 \) and every \( x \in H^k(S, M) \),
\[
\omega_{k, M}(x) = \left( \frac{|S|}{|\Omega|} \Omega_k \right)^{\prod_{i=0}^{N_{k, M}}}(x)
\]

Besides, this definition only depends on the \( \mathcal{F}^c \)-characteristic \( (S, S) \)-biset \( \Omega \).

**Definition 4.9** For \( \Omega \) an \( \mathcal{F}^c \)-characteristic \( (S, S) \)-biset, the idempotent \( \omega_{s, -} \) obtained by this process is called the \( \mathcal{F}^c \)-characteristic idempotent associated to \( \Omega \). We will also, for any abelian \( p \)-group \( M \) with an action of \( \pi_{\mathcal{L}} \), denote by \( I^*_\Omega(M) \leq H^*(S, M) \) the image of \( \omega_{s, M} \).
**Remark 4.10** By Proposition 4.6, if the action of $\pi_L$ on $M$ is trivial, since $\omega$ is a finite iteration of $[S]|_{[T]}\Omega_*$, $\omega = [S]|_{[T]}\Omega_*$ and then $I^*_\Omega(M) = H^*(\mathcal{F}_c^\omega, M)$.

One important result of this construction is that, even if the definition does not seem natural, it defines an endomorphism of the $\delta$-functor $H^*(S, -)$ seen as a functor defined on the category of abelian $p$-groups with an action of $\pi_L$. We refer the reader to Weibel [We] section 2.1 for more details on $\delta$-functors.

**Proposition 4.11** Let $(S, \mathcal{F}, L)$ be a $p$-local finite group. If $\Omega$ is an $\mathcal{F}_c^\omega$-characteristic $(S, S)$-biset, then the $\mathcal{F}_c^\omega$-characteristic idempotent induced by $\Omega$ defines an endomorphism of the $\delta$-functor $H^*(S, -)$.

**Proof** For $M$ an abelian $p$-group with an action of $\pi_L$ and $k \geq 0$, we denote by $N_{k,M}$ a natural number as in Proposition 4.8.

We have first to show that $\omega$, the $\mathcal{F}_c^\omega$-characteristic idempotent associated to $\Omega$, defines a natural transformation from the functor $H^*(S, -)$ to itself. For every pair of abelian $p$-groups $M, N$ with an action of $\pi_L$ and every $\varphi \in \text{Hom}_{\mathcal{F}_c^\omega}(M, N)$, let consider, for $k \geq 0$ the following diagram,

\[
\begin{array}{ccccccccc}
H^k(S, M) & \xrightarrow{\text{Id}} & H^k(S, M) & \xrightarrow{\varphi_k} & H^k(S, N) & \xrightarrow{\text{Id}} & H^k(S, N) \\
\omega_{k,M} & & \omega_{k,M} & & \omega_{k,M} & & \omega_{k,M} \\
H^k(S, M) & \xrightarrow{\text{Id}} & H^k(S, M) & \xrightarrow{\varphi_k} & H^k(S, N) & \xrightarrow{\text{Id}} & H^k(S, N)
\end{array}
\]

where $\omega_{k,M,N} = \left(\frac{|S|}{|T|}\right)\prod_{i=0}^k N_{i,M} \times \prod_{i=0}^k N_{i,N}$. The middle square commutes as $\omega_{k,M,N}$ is a finite iteration of $\Omega_k$ and $\Omega_*$ is an endomorphism of $\delta$-functors by Remark 4.5. The first square commutes because, as $\omega_{k,M}$ is an idempotent of $H^k(S, M)$, $\omega_{k,M,N} = \omega_{k,M,N} = \omega_{k,M}$. Finally, the last one commutes because, as $\omega_{k,N}$ is an idempotent of $H^k(S, N)$, $\omega_{k,M,N} = \omega_{k,N} = \omega_{k,N}$. Hence, the exterior diagram commutes.

Now, to show that it defines a morphism of $\delta$-functor, let consider a short exact sequence of abelian $p$-groups with an action of $\pi_L$, $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$.

Thanks to the previous argument we just have to show that, for $k \geq 0$, the following diagram commutes,

\[
\begin{array}{cccc}
H^k(S, M) & \xrightarrow{\delta} & H^{k+1}(S, L) \\
\omega_{k,M} & & \omega_{k+1,L}
\end{array}
\]
where \( \delta \) corresponds to the connecting homomorphism. Consider then the following diagram,

\[
\begin{array}{c}
H^k(S, M) \xrightarrow{\text{Id}} H^k(S, M) \xrightarrow{\delta} H^{k+1}(S, L) \xrightarrow{\text{Id}} H^{k+1}(S, L) \\
\omega_k \downarrow \quad \omega_{k,L,M} \downarrow \quad \omega_{k+1,L,M} \downarrow \quad \omega_{k+1,L} \\
H^k(S, M) \xrightarrow{\text{Id}} H^k(S, M) \xrightarrow{\delta} H^{k+1}(S, L) \xrightarrow{\text{Id}} H^{k+1}(S, L)
\end{array}
\]

where

\[
\omega_{k,L,M} = \left( \frac{|S|}{|\Omega|} \Omega_k \right) \prod_{c=0}^{k+1} N_{i,c} \times \prod_{c=0}^{k+1} N_{i,N}
\]

and

\[
\omega_{k+1,L,M} = \left( \frac{|S|}{|\Omega|} \Omega_{k+1} \right) \prod_{c=0}^{k+1} N_{i,c} \times \prod_{c=0}^{k+1} N_{i,N}.
\]

The middle square commutes as \( \omega_{k,L,M} \) and \( \omega_{k+1,L,M} \) are finite iterations of \( \Omega_k \) and \( \Omega_{k+1} \) and \( \omega_s \) is an endomorphism of \( \delta \)-functors. The first square commutes because, as \( \omega_{k,M} \) is an idempotent of \( H^k(S, M) \), \( \omega_{k,L,M} = \omega_{k,M}^{N_{k+1,M} \times \prod_{c=0}^{k+1} N_{i,c}} = \omega_{k,M} \). The last one commutes because, as \( \omega_{L,L} \) is an idempotent of \( H^k(S, L) \), \( \omega_{k+1,L,M} = \omega_{k+1,L}^{\prod_{c=0}^{k+1} N_{i,c}} = \omega_{k+1,L} \). Thus, the exterior diagram commutes. \( \square \)

We can wonder if the image of a characteristic idempotent is always \( H^*(\mathcal{F}_c, M) \). Unfortunately not, we can still look at the 2-local finite group \((D_8, \mathcal{F}_{D_8}(A_6), \mathcal{L}_{D_8}(A_6))\) in degree 0 and with coefficient module \( M = \mathbb{F}_2[A_6] \). The image of the characteristic idempotent induced by the minimal \( \mathcal{F}_c \)-characteristic \((S, S)\)-biset contains strictly \( H^*(\mathcal{F}_{D_8}(A_6), M) \).

Still one question is open. Do two different \( \mathcal{F}_c \)-characteristic \((S, S)\)-bisets give the same idempotent? With the 2-local finite group \((D_8, \mathcal{F}_{D_8}(A_6), \mathcal{L}_{D_8}(A_6))\), we can verify by computation with GAP that two different \( \mathcal{F}_c \)-characteristic \((S, S)\)-bisets induce different morphisms in cohomology but the two induced characteristic idempotents are the same.

4.3 A \( \delta \)-functor

Let \((S, \mathcal{F}, \mathcal{L})\) be a \( p \)-local finite group, \( \Omega \) be an \( \mathcal{F}_c \)-characteristic \((S, S)\)-biset and, for \( M \) an abelian \( p \)-group with an action of \( \pi_L \), let \( \omega \in \text{End}(H^*(S, M)) \) be the associated \( \mathcal{F}_c \)-characteristic idempotent.

Let us start with the behavior of \( \delta \)-functors with idempotents.
Lemma 4.12 Let \((M_*, f_*) = \left( \cdots \xrightarrow{f_{l-2}} M_{l-1} \xrightarrow{f_{l-1}} M_l \xrightarrow{f_l} M_{l+1} \xrightarrow{f_{l+1}} \cdots \right)_{l \in \mathbb{Z}}\) be a long exact sequence in an abelian category \(A\). Let \(i_* : (M_*, f_*) \to (M_*, f_*)\) be a morphism of long exact sequences such that, for all \(l \in \mathbb{Z}, \ i_l\) is an idempotent of \(M_l\). Then the sequence

\[
\cdots \xrightarrow{f_{l-2}} \text{Im}(i_{l-1}) \xrightarrow{f_{l-1}} \text{Im}(i_l) \xrightarrow{f_l} \text{Im}(i_{l+1}) \xrightarrow{f_{l+1}} \cdots
\]

is exact.

Proof Let \(l \in \mathbb{Z}\) and \(x \in \text{Im}(i_l)\) such that \(f_l(x) = 0\). Thanks to exactness of \((M_*, f_*)\) in \(l\), there is a \(y \in M_{l-1}\) such that \(f_{l-1}(y) = x\). Thus \(x = i_l(x) = i_l \circ f_{l-1}(y) = f_{l-1} \circ i_{l-1}(y)\) and hence we obtain the exactness of \((\text{Im}(i_*), f_*)\) in degree \(l\).

Proposition 4.13 Let \(A, B\) be abelian categories and let \(F^* : A \to B\) be a \(\delta\)-functor. If \(i^* : F^* \to F^*\) is an idempotent of \(\delta\)-functor, then the functor \(\text{Im}(i^*)\) is a \(\delta\)-functor.

Proof A \(\delta\)-functor can be seen as a functor from the category \(S_A\) of short exact sequences in \(A\) to \(\text{Ch}(B)\) which sends any short exact sequence to an acyclic chain complex. A morphism of \(\delta\)-functors is then a natural transformation in this setting. With this point of view, this is just a corollary of the Lemma 4.12.

Theorem 4.14 Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group and \(\Omega\) an \(\mathcal{F}^c\)-characteristic \((S, S)\)-biset. Then, the functor \(I^*_\Omega(-)\) defines a \(\delta\)-functor from the category of finite \(\mathbb{Z}_{(p)}[\pi_\mathcal{L}]\)-modules to \(\mathbb{Z}_{(p)}\)-Mod.

Proof This is a direct corollary of Proposition 4.11 and Proposition 4.13.

5 The cohomology of the geometric realization of a linking system with nilpotent coefficients

We give here a proof of the main theorem.

Lemma 5.1 Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group. The natural inclusion \(\delta_S\) of \(B(S)\) in \(\mathcal{L}\) induces, for any \(\mathbb{Z}_{(p)}[\pi_\mathcal{L}]\)-module \(M\), a natural morphism in cohomology

\[
H^*(|\mathcal{L}|, M) \longrightarrow H^*(\mathcal{F}^c, M) \subseteq H^*(S, M).
\]
Proof For $P \in \text{Ob}(\mathcal{F}^c)$ and $\varphi \in \text{Hom}_\mathcal{F}(P, S)$, let $\tilde{\varphi} \in \text{Mor}_\mathcal{L}(P, \varphi(P))$ be such that $\pi(\tilde{\varphi}) = \varphi \in \text{Hom}_\mathcal{F}(P, \varphi(P))$. By Definition 2.6 of a linking system, for every $g \in P$, we have the following commutative diagram in $\mathcal{L}$.

Thus, we have a natural transformation between the functors $\delta_P$ and $\delta_{\varphi(P)} \circ B(\varphi)$.

Hence, the maps $|\delta_P|$ and $|\delta_{\varphi(P)} \circ B(\varphi)| = |\delta_{\varphi(P)}| \circ |B(\varphi)|$ are homotopic. In particular, the following diagram commute

and the lemma follows.

Lemma 5.2 Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group and let $\Omega$ be an $\mathcal{F}^c$-characteristic $(S, S)$-biset. Let also $0 \to L \to M \to N \to 0$ be a short exact sequence of finite $\mathbb{Z}(p)[\pi \mathcal{L}]$-modules. If $\delta_S$ induces the following isomorphisms, $H^*(|\mathcal{L}|, L) \cong I^*_\Omega(L)$ and $H^*(|\mathcal{L}|, N) \cong I^*_\Omega(N)$, then $\delta_S$ induces an isomorphism

$H^*(|\mathcal{L}|, M) \cong I^*_\Omega(M)$. 

Proof Consider the exact sequences in cohomology induced by the short exact sequence

and look at the following diagram (where $\omega_{\cdot,\cdot}$ denote the $\mathcal{F}^c$-characteristic idempotent associated to $\Omega$).
As $H^* (|\mathcal{L}|, -)$ is a $\delta$-functor and, by Theorem 4.14, $I^n_\Omega$ is also a $\delta$-functor, the two lines are exact and, as by Proposition 4.11 the $\mathcal{F}$-characteristic idempotent associated to $\Omega$ defines a morphism of $\delta$-functors, this diagram is commutative. We conclude then with the five lemma.

**Theorem 5.3** Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. If $M$ is an abelian $p$-group with a nilpotent action of $\pi_1(|\mathcal{L}|)$, then $\delta_S$ induces a natural isomorphism

$$H^* (|\mathcal{L}|, M) \cong H^* (\mathcal{F}_c, M).$$

**Proof** As the action of $\pi_\mathcal{L}$ is nilpotent, there is a sequence

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

such that, for every $1 \leq i \leq n$, the action of $\pi_\mathcal{L}$ on $M_i/M_{i-1}$ is trivial. We know, by Theorem 1.1 and Remark 4.10, that for $1 \leq i \leq n$, $\delta_S$ induces an isomorphism

$$H^* (|\mathcal{L}|, M_i/M_{i-1}) \cong H^* (\mathcal{F}_c, M_i/M_{i-1}) = I^n_\Omega (M_i/M_{i-1}).$$

By induction on $n$ and by Lemma 5.2, we get that $H^* (|\mathcal{L}|, M) \cong I^n_\Omega (M)$. Finally we also have from Lemma 5.1 that

$$\delta_S (H^* (|\mathcal{L}|, M)) \subseteq H^* (\mathcal{F}_c, M) \subseteq I^n_\Omega (M).$$

Then $H^* (|\mathcal{L}|, M) \cong H^* (\mathcal{F}_c, M) = I^n_\Omega (M)$. 

**6 The cohomology with twisted coefficients of $p$-good space**

We finish by a result on the cohomology with twisted coefficients of the $p$-completion of a $p$-good space and we apply it, with Theorem 5.3, to compare the cohomology with twisted coefficients of $|\mathcal{L}|$ and the $\mathcal{F}_c$-stable elements.

We refer the reader to Bousfield and Kan [BK] for more details about $p$-completion. There is also a brief introduction in Aschbacher, Kessar and Oliver [AKO]. Here, for $X$ a $p$-good space,

$$\lambda_X : X \to X_p^\wedge$$

will denote the structural natural transformation of the $p$-completion and we recall that it induce a isomorphism

$$H^* (X_p^\wedge, \mathbb{F}_p) \cong H^* (X, \mathbb{F}_p).$$
Lemma 6.1  Let $X$ be a space and let $0 \to L \to M \to N \to 0$ be a short exact sequence of $\mathbb{Z}(p)[\pi_1(X^\wedge_p)]$-modules. If $\lambda_X$ induces the following isomorphisms, $H^\ast(X^\wedge_p, L) \cong H^\ast(X, L)$ and $H^\ast(X^\wedge_p, N) \cong H^\ast(X, N)$, then $\lambda_X$ induces an isomorphism $H^\ast(X^\wedge_p, M) \cong H^\ast(X, M)$.

Proof  Let consider the exact sequences in cohomology induced by the short exact sequence,

$$
\cdots \to H^{n-1}(X^\wedge_p, N) \to H^n(X^\wedge_p, L) \to H^n(X^\wedge_p, M) \to H^n(X^\wedge_p, N) \to \cdots
$$

and look at the following commutative diagram,

$$
\cdots \to H^{n-1}(X^\wedge_p, N) \to H^n(X^\wedge_p, L) \to H^n(X^\wedge_p, M) \to H^n(X^\wedge_p, N) \to \cdots
$$

The two lines are exact and $\lambda_X$ gives the isomorphisms $H^\ast(X^\wedge_p, L) \cong H^\ast(X, L)$ and $H^\ast(X^\wedge_p, N) \cong H^\ast(X, N)$. We then conclude with the five lemma.

Proposition 6.2  Let $X$ be a space and $M$ be an abelian $p$-group with an action of $\pi_1(X^\wedge_p)$. If $X$ is $p$-good, then $\lambda_X$ induces a natural isomorphism $H^\ast(X^\wedge_p, M) \cong H^\ast(X, M)$.

Proof  As $X$ is $p$-good, $\lambda_X$ induces an isomorphism, $H^\ast(X^\wedge_p, \mathbb{F}_p) \cong H^\ast(X, \mathbb{F}_p)$ and $\pi_1(X^\wedge_p)$ is a $p$-group quotient of $\pi_1(X)$. In particular, the action of $\pi_1(X^\wedge_p)$ on $M$ is nilpotent: there is a sequence

$$
\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M
$$

such that, for any $1 \leq i \leq n$, $M_i/M_{i-1} \cong \mathbb{F}_p$ is the trivial module. We conclude by induction on $n$, by Lemma 6.1.

Corollary 6.3  Let $(S, \mathcal{F}, \mathcal{L})$ be a $p$-local finite group. If $M$ is an abelian $p$-group with an action of $\pi_1(|\mathcal{L}|^\wedge_p)$, $\lambda_{|\mathcal{L}|}$ induces an isomorphism $H^\ast(|\mathcal{L}|^\wedge_p, M) \cong H^\ast(|\mathcal{L}|, M)$.

Proof  As $|\mathcal{L}|$ is a $p$-good space ([AKO], Theorem III.4.17), we can apply Proposition 6.2.
Corollary 6.4 Let \((S, \mathcal{F}, \mathcal{L})\) be a \(p\)-local finite group. If \(M\) is an abelian \(p\)-group with an action of \(\pi_1(|\mathcal{L}|_p^h)\), then \(\lambda_{|\mathcal{L}|} \circ \delta_\mathcal{L}\) induces a natural isomorphism

\[ H^*(|\mathcal{L}|_p^h, M) \cong H^*(\mathcal{F}^c, M). \]

Proof As \(|\mathcal{L}|\) is \(p\)-good ([AKO], Theorem III.4.17), \(\pi_1(|\mathcal{L}|_p^h)\) is a \(p\)-group, and its action on \(M\) is nilpotent. Hence, this is just a corollary of Theorem 5.3 and Corollary 6.3.

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