Volatility Estimation of Hidden Markov Processes and Adaptive Filtration

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Abstract

The partially observed linear Gaussian system of stochastic differential equations with low noise in observations is considered. The coefficients of this system are supposed to depend on some unknown parameter. The problem of estimation of these parameters is considered and the possibility of the approximation of the filtering equations is discussed. An estimators are used for estimation of the quadratic variation of the derivative of the limit of the observed process. Then this estimator is used for nonparametric estimation of the integral of the square of volatility of unobservable component. This estimator is also used for construction of method of moments estimators in the case where the drift in observable component and the volatility of the state component depend on some unknown parameter. Then this method of moments estimator and Fisher-score device allow us to introduce the One-step MLE-process and adaptive Kalman-Bucy filter. The asymptotic efficiency of the proposed filter is discussed.

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1 Introduction

Partially observed systems and especially different Kalman filter models occupy an important place in statistics due to a large diversity of applied problems where such statistical models are used. There is extensive engineering literature devoted to identification problems for such partially observed discrete time models. Note that the continuous time systems are less studied.

This work is devoted to the construction of adaptive Kalman-Bucy filter for partially observed linear system with the small noise in observation equation. It is supposed that the coefficients of these equations depend on some unknown parameter and the adaptive filter
is constructed in several steps. First we propose a nonparametric estimator of some integral functional, then this estimator is used for construction of method of moments estimator of the unknown parameter. This estimator is used as preliminary for construction of One-step MLE-process and finally this process is used for writing the adaptive Kalman-Bucy filter. The problems of parameter estimation by observations of partially observed linear systems with small noise in observations and in state equation for more complex (multidimensional, nonlinear) models were extensively studied in filtration theory, where approximate filters were proposed and studied in different asymptotics, see, e.g., [4],[6],[7] and references there in. In particular the order of the errors of approximations are obtained for the large diversity of the statement of the problems, see [8], [14], [28], [29]. The similar problems with hidden telegraph process were studied in [3] and [15]. The statistical problems for partially observed linear and non linear systems were studied in [16], Chapter 6 and in [31]. Note that in [16] and in [31] it is supposed that the small noises are in the observation part. In particular the order of the errors of approximations are obtained for the large noise in observations only were considered recently in [21]. The construction of asymptotically optimal estimator of the parameter in volatility of unobserved component for partially observed system by discrete time observations was studied in [10],[11]. The asymptotic behavior of the filter for such partially observed nonlinear continuous time system with small noise in observations was studied in the mentioned above works [28], [29].

Let us give more detailed exposition of the presented in this work study. We consider the linear two-dimensional partially observed system

\[ dY_t = a(\vartheta, t) Y_t dt + b(\vartheta, t) dV_t, \quad Y_0 = y_0, \quad 0 \leq t \leq T, \]
\[ dX_t = f(\vartheta, t) Y_t dt + \varepsilon \sigma(t) dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \]

where the Wiener processes \( V_t, 0 \leq t \leq T \) and \( W_t, 0 \leq t \leq T \) are independent. The solution \( Y^T = (Y_t, 0 \leq t \leq T) \) of the state equation (1) can not be observed directly and we have available the observations \( X^T = (X_t, 0 \leq t \leq T) \) only. Here \( a(\cdot), b(\cdot), f(\cdot) \) and \( \sigma(\cdot) \) are some bounded functions and \( \varepsilon \in (0,1) \) is small parameter.

The conditional expectation \( m(\vartheta, t) = \mathbb{E}_\vartheta(Y_t | X_s, 0 \leq s \leq t) \) satisfies the equations of Kalman-Bucy filtration [13], [27]

\[ dm(\vartheta, t) = a(\vartheta, t) m(\vartheta, t) \, dt + \frac{\gamma(\vartheta, t) f(\vartheta, t)}{\varepsilon^2 \sigma(t)^2} [dX_t - f(\vartheta, t) m(\vartheta, t) \, dt], \]
\[ \frac{\partial \gamma(\vartheta, t)}{\partial t} = 2a(\vartheta, t) \gamma(\vartheta, t) - \frac{\gamma(\vartheta, t)^2 f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} + b(\vartheta, t)^2, \quad \gamma(\vartheta, 0) = 0, \]

with the initial value \( m(\vartheta, 0) = y_0 \). If the value of \( \vartheta \) is unknown, then to approximate \( m(\vartheta, t) \) we can use some estimator \( \tilde{\vartheta}_e \) and to use something like \( m(\tilde{\vartheta}_e, t) \), but there are some problems. For example, if \( \tilde{\vartheta}_e \) is the MLE \( \hat{\vartheta} \) constructed by observations \( X^T \), then there is a problem of definition of the Itô integral in (3)

\[ \int_0^t \frac{\gamma(\tilde{\vartheta}_e, s) f(\tilde{\vartheta}_e, s)}{\varepsilon^2 \sigma(s)^2} \, dX_s \]
because the estimator $\hat{\vartheta}_\varepsilon$ depends on the whole trajectory $X^T$. The using a series of MLEs $\hat{\vartheta}_{t,\varepsilon}$ constructed by observations $X^t = (X_s, 0 \leq s \leq t)$ for each $\varepsilon \in (0, T]$ can provide a good approximation of $m(\vartheta, t)$ but the numerical realization of such algorithm can be a difficult problem. The goal of this work is the construction of another estimator $\vartheta^*_{t,\varepsilon}$, called One-step MLE-process, which depends on the observations $X^t = (X_s, 0 \leq s \leq t)$ for each $\varepsilon \in (0, T]$. Then this estimator can be used for approximation $m(\vartheta, t)$ as follows $m(\vartheta^*_{t,\varepsilon}, t)$. We describe the error of approximation $m(\vartheta, t) - m(\vartheta^*_{t,\varepsilon}, t)$ and discuss the optimality of such approximation.

The construction of $\vartheta^*_{t,\varepsilon}$ requires a preliminary consistent estimator $\bar{\vartheta}_{\tau,\varepsilon}$, which we define with the help of nonparametric estimator of the quadratic variation of the trend coefficient of the observed process. Introduce the function

$$\Psi_\tau (\vartheta) = \int_0^\tau f (\vartheta, s)^2 b (\vartheta, s)^2 \, ds, \quad \vartheta \in \Theta, \quad \tau \in (0, T].$$

Note that we have formally

$$\left. \frac{\partial X_t}{\partial t} \right|_{\varepsilon=0} = f (\vartheta, t) Y_t.$$ 

Let us denote the trend coefficient of $X_t$ as $M_t = f (\vartheta, t) Y_t$ and put $S (\vartheta, t) = f (\vartheta, t) b (\vartheta, t)$. The process $M_t$ has the stochastic differential

$$dM_t = [f' (\vartheta, t) - a (\vartheta, t) f (\vartheta, t)] Y_t \, dt + S (\vartheta, t) \, dV_t, \quad M_0 = f (\vartheta, 0) y_0$$

and therefore $M^2_\tau$ admits the representation

$$M^2_\tau = 2 \int_0^\tau M_t \, dM_t + \int_0^\tau S (\vartheta, t)^2 \, dt.$$ 

The quadratic variation of $M_t$ can be calculated as follows

$$\Psi_\tau (\vartheta) = M^2_\tau - 2 \int_0^\tau M_s \, dM_s, \quad 0 \leq \tau \leq T.$$ 

The preliminary estimator $\bar{\vartheta}_{\tau,\varepsilon}$ is constructed in two steps. First we estimate the derivative of $X_t$, then we estimate the quadratic variation of this derivative. Introduced statistic $\Psi_{\tau,\varepsilon}$ with a fixed $\tau$ allows us to realize these two procedures simultaneously. Using this statistic and relation $\Psi_{\tau,\varepsilon} = \Psi_\tau (\bar{\vartheta}_{\tau,\varepsilon})$ we obtain the method of moments estimator $\bar{\vartheta}_{\tau,\varepsilon}$. Of course, the process $X_t$ is not differentiable and we propose, as usual in nonparametric statistics, a slow differentiation.

The asymptotic properties of the maximum likelihood estimators (MLE) and Bayesian estimator (BE) of this parameter for the model (1)-(2), were described in the work [21]. Recall that the MLE $\hat{\vartheta}_\varepsilon$ and BE $\tilde{\vartheta}_\varepsilon$ are defined by the relation

$$L(\hat{\vartheta}_\varepsilon, X^T) = \sup_{\vartheta \in \Theta} L (\vartheta, X^T), \quad \hat{\vartheta}_\varepsilon = \frac{\int_{\Theta} \hat{\vartheta} p (\vartheta) L (\vartheta, X^T) \, d\vartheta}{\int_{\Theta} p (\vartheta) L (\vartheta, X^T) \, d\vartheta}, \quad (5)$$

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where the likelihood ratio is (see [27])

\[ L(\theta, X^T) = \exp \left( \int_0^T \frac{f(\theta, t) m(\theta, t)}{\varepsilon^2 \sigma(t)^2} \, dX_t - \int_0^T \frac{f(\theta, t)^2 m(\theta, t)^2}{2 \varepsilon^2 \sigma(t)^2} \, dt \right). \] (6)

It was shown that these estimators are consistent, asymptotically normal

\[ \hat{\theta}_\varepsilon - \theta \xrightarrow{\varepsilon} N(0, I(\theta)^{-1}), \quad I(\theta) = \int_0^T \frac{\dot{S}(\theta, t)^2}{2 S(\theta, t) \sigma(t)} \, dt, \] (7)

the polynomial moments converge and the both estimators are asymptotically efficient. Here in the sequel dot means derivation w.r.t. \( \theta \) and prim means derivation w.r.t. \( t \).

The model (1)-(2) has some unusual features. For example, the empirical Fisher information

\[ I_\varepsilon(\theta) = \int_0^T \left[ \frac{\dot{f}(\theta, t) m(\theta, t) + f(\theta, t) \dot{m}(\theta, t)}{\sigma(t)} \right]^2 \, dt \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0 \]

and for all \( t \in (0, T] \) there is convergence in distribution

\[ \varepsilon^{-1/2} \left[ \frac{\dot{f}(\theta, t) m(\theta, t) + f(\theta, t) \dot{m}(\theta, t)}{\sqrt{\varepsilon \sigma(t)}} \right] \xrightarrow{\varepsilon} \frac{\dot{S}(\theta, t) \sqrt{\sigma(t)}}{\sqrt{2S(\theta, t)}} \xi_t. \]

Here \( \xi_t, t \in (0, T] \) are mutually independent standard Gaussian (\( N(0, 1) \)) random variables. This convergence provides convergence in probability

\[ \varepsilon^{-1} I_\varepsilon(\theta) = \int_0^T \left[ \frac{\dot{f}(\theta, t) m(\theta, t) + f(\theta, t) \dot{m}(\theta, t)}{\sqrt{\varepsilon \sigma(t)}} \right]^2 \, dt \longrightarrow I(\theta). \]

To calculate just one value of \( L(\theta, X^T) \) by (6) we need the solutions \( m(\theta, t), \gamma(\theta, t) \), \( 0 \leq t \leq T \) of the equations (3),(4) and therefore, as we wrote above, the numerical calculation of the MLE and BE can be difficult problem.

Here we propose a different construction of the estimator of this parameter which is computationally simpler than the MLE and which is asymptotically equivalent to the MLE [21].

2 Model of observations

Consider a non-homogeneous partially observed linear system described by the equations

\[ dX_t = f(\theta, t) Y_t \, dt + \varepsilon \sigma(t) \, dW_t, \quad X_0 = 0, \] (8)
\[ dY_t = a(\theta, t) Y_t \, dt + b(\theta, t) \, dV_t, \quad Y_0 = y_0. \] (9)
where \( f(\cdot), \sigma(\cdot), a(\cdot) \) and \( b(\cdot) \) are known, smooth functions, while \( W_t, 0 \leq t \leq T \) and \( V_t, 0 \leq t \leq T \) are two independent Wiener processes. The process \( X^T = (X_t, 0 \leq t \leq T) \) is observed and the process \( Y^T = (Y_t, 0 \leq t \leq T) \) is hidden. We are interested by the same problems: estimation of the finite-dimensional parameter \( \vartheta \in \Theta \subset \mathbb{R}^d \) and the construction of adaptive filter.

Here we study the method of moments estimators (MME) based on some nonparametric estimation of one integral, One-step MLE-process and a adaptive filter based on this estimator-process.

The limit model \((\varepsilon = 0)\) corresponds to the observations of the integral of the hidden process and the possibility of the consistent estimation is equivalent to the possibility to estimate without error the parameter of the observed Gaussian process.

### 3 Limit model

Let us now present how an estimator \( \vartheta^*_\varepsilon \) of \( \vartheta_0 \) (true value) could be obtained without error by means of the observations \( X^T = x^T = (x_t, 0 \leq t \leq T) \), when \( \varepsilon = 0 \). We suppose that we observe a Gaussian process \( x^T \) satisfying

\[
\frac{\partial x_t}{\partial t} = f(\vartheta, t) Y_t, \quad x_0 = 0, \quad 0 \leq t \leq T
\]

\[
dY_t = a(\vartheta, t) Y_t dt + b(\vartheta, t) dV_t, \quad Y_0 = y_0
\]

and aim to estimate \( \vartheta_0 \). Suppose that the function \( f(\vartheta, t) \) has a continuous derivative with respect to \( t \). Then the process \( x^T \) is described by the following stochastic differential equation

\[
dx'_t = f'(\vartheta_0, t) Y_t dt + f(\vartheta_0, t) dY_t
\]

\[
= \left[ f'(\vartheta_0, t) a(\vartheta_0, t) + f(\vartheta_0, t) \right] Y_t dt + f(\vartheta_0, t) b(\vartheta_0, t) dV_t,
\]

where \( f'(\vartheta_0, t) \) is the derivative of \( f(\vartheta_0, t) \) with respect to \( t \).

Since the volatility function depends on the unknown parameter \( \vartheta_0 \) through the product \( S(\vartheta_0, t) = f(\vartheta_0, t) b(\vartheta_0, t) \), it is clear that consistent estimation is feasible. It will be shown that the identifiability condition and the Fisher information have to be based mainly on the function \( S(\vartheta_0, t) \). For example if \( f(\vartheta, t) = \vartheta \) and \( b(\vartheta, t) = 1/\vartheta \), then consistent estimation is impossible and the (limit) Fisher information is equal to 0.

Recall that by Itô’s formula, we have the following relation for the quadratic variation

\[
x_t^2 = 2 \int_0^t x'_s dx_s + \int_0^t b(\vartheta_0, s)^2 f(\vartheta_0, s)^2 ds
\]

(10)

and the function

\[
\Psi_t = x_t^2 - 2 \int_0^t x'_s dx_s = \int_0^t b(\vartheta_0, s)^2 f(\vartheta_0, s)^2 ds \equiv K(\vartheta_0, t)
\]

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is deterministic. Under mild identifiability conditions, the observed function $\Psi_t$ defines $\vartheta_0$ for any $t \in (0, T]$ without error. For example, the estimator $\vartheta^*$ defined by the equation

$$K(\vartheta^*, t) = \Psi_t$$

is without error, i.e., $\vartheta^* = \vartheta_0$.

For example, if $f(\vartheta, t) = f(t)$, $b(\vartheta, t)^2 = h(t) + \vartheta g(t)$, where all functions and $\vartheta$ are positive, then the “estimator”

$$\vartheta_t^* = \left( \int_0^t g(s) f(s)^2 \, ds \right)^{-1} \left[ \Psi_t - \int_0^t h(s) f(s)^2 \, ds \right] = \vartheta_0.$$

Therefore the consistent estimation by observations (8) is not excluded.

Note that if we consider the limit model, then the measures that correspond to observations with different $\vartheta$ are singular. Moreover $\Psi_t$ does not depend on $a(\vartheta_0, t)$.

4 Consistent estimator

One way to estimate $\vartheta$ using observations from (8) is by estimating first the function $\Psi_t$ and then using the estimator $\Psi_{t, \varepsilon}$ to estimate $\vartheta$, i.e., to solve for some $t$ the equation

$$\Psi_{t, \varepsilon} = \int_0^t b(\tilde{\vartheta}_\varepsilon, s)^2 f(\tilde{\vartheta}_\varepsilon, s)^2 \, ds, \quad (11)$$

where $\tilde{\vartheta}_\varepsilon$ is the MME. Of course, to verify the consistency of $\tilde{\vartheta}_\varepsilon$ we need the condition of identifiability.

Another possibility is to define the estimator as

$$\tilde{\vartheta}_{t, \varepsilon} = \text{arg inf}_{\vartheta \in \Theta} \int_0^t [\Psi_{s, \varepsilon} - K(\vartheta, s)]^2 \, ds. \quad (12)$$

For example, let $f(\vartheta, t) = f(t)$ and $b(\vartheta, t) = \vartheta g(t)$. Then

$$\tilde{\vartheta}_{t, \varepsilon} = \left\{ \int_0^t \Psi_{s, \varepsilon} g(s) \, ds \big/ \int_0^t g(s)^2 \, ds \right\}^{1/2}$$

and from the consistency of $\Psi_{s, \varepsilon}$, $0 \leq s \leq t$ we obtain the consistency of $\tilde{\vartheta}_{t, \varepsilon}$.

Therefore to estimate $\vartheta$ we first estimate the function $\Psi_t$ and then using this estimator of $\Psi_t$ we estimate the parameter $\vartheta$. Below we realize this program.

5 Nonparametric estimation.

Suppose that we have the partially observed system

$$dX_t = f(t) Y_t \, dt + \varepsilon \sigma(t) \, dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (13)$$

$$dY_t = a(t) Y_t \, dt + b(t) \, dV_t, \quad Y_0 = y_0, \quad (14)$$
where $a(\cdot), b(\cdot), f(\cdot)$ and $\sigma(\cdot)$ are unknown functions.

Consider the problem of estimation of the function

$$\Psi_\tau = \int_0^\tau f(t)^2 b(t)^2 \, dt, \quad 0 < \tau \leq T$$

by observations $X^T$. Remind that this is quadratic variation of the derivative of the process $X_\tau$ at the point $\varepsilon = 0$.

Introduce the statistic

$$\Psi_{\tau, \varepsilon} = \frac{N_{\tau, \varepsilon}}{N_{\tau, \varepsilon}} \sum_{i=0}^{N_{\tau, \varepsilon}-1} \left( \frac{X_{t_{i+1}+\delta_\varepsilon} - X_{t_{i+1}} - X_{t_i+\delta_\varepsilon} - X_{t_i}}{\delta_\varepsilon} \right)^2, \quad 0 < \tau \leq T.$$

Here $t_i = i\varphi_\varepsilon, N_{\tau, \varepsilon} = \left\lfloor \frac{\tau}{\varphi_\varepsilon} \right\rfloor$, the rates $\varphi_\varepsilon \to 0, \delta_\varepsilon \to 0$ will be defined later. Just note that as the first step is derivation we wait that the rate $\delta_\varepsilon \to 0$ has to be faster than the step of discretization $\varphi_\varepsilon \to 0$.

Let us explain why this statistic can be a consistent estimator of $\Psi_\tau$. We have

$$\frac{X_{t_{i+1}+\delta_\varepsilon} - X_{t_{i+1}}}{\delta_\varepsilon} = f(t_{i+1}) Y_{t_{i+1}} + o(1), \quad \frac{X_{t_i+\delta_\varepsilon} - X_{t_i}}{\delta_\varepsilon} = f(t_i) Y_{t_i} + o(1).$$

Further, using the smoothness of the functions we can write formally (!)

$$\Psi_{\tau, \varepsilon} = \sum_{i=0}^{N_{\tau, \varepsilon}-1} \left( f(t_{i+1}) Y_{t_{i+1}} - f(t_i) Y_{t_i} \right)^2 + o(1) = \sum_{i=0}^{N_{\tau, \varepsilon}-1} f(t_i)^2 (Y_{t_{i+1}} - Y_{t_i})^2 + o(1)$$

$$= \sum_{i=0}^{N_{\tau, \varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} a(s) \, ds + \int_{t_i}^{t_{i+1}} b(s) \, dV_s \right)^2 + o(1)$$

$$= \sum_{i=0}^{N_{\tau, \varepsilon}-1} f(t_i)^2 b(t_i)^2 (V_{t_{i+1}} - V_{t_i})^2 + o(1)$$

$$= \sum_{i=0}^{N_{\tau, \varepsilon}-1} f(t_i)^2 b(t_i)^2 (t_{i+1} - t_i) + o(1)$$

$$\rightarrow \int_0^\tau f(s)^2 b(s)^2 \, ds = \Psi_\tau.$$

Therefore this statistic can be a consistent estimator of the integral $\Psi_\tau$. 

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**Proposition 1.** Let the functions \( a(\cdot), b(\cdot), f(\cdot), \sigma(\cdot) \in C^1[0,T] \) and \( \delta \varepsilon = \varepsilon, \varphi_\varepsilon = \varepsilon^{1/3} \). Then for any \( p > 0 \) there exists a constant \( C > 0 \) such that

\[
E |\Psi_{\tau,\varepsilon} - \Psi_{\tau}|^p \leq C \tau^{p/2} \varepsilon^{p/3}.
\]  

**Proof.** Now we repeat the given above expansions with description of the corresponding errors. Throughout the proof, \( C \) denote “generic” strictly positive constants, which can vary from formula to formula (and even in the same formula). We have

\[
\frac{X_{t_{i+1}+\delta}}{\delta^2} - X_{t_{i+1}} = f(t_{i+1}) Y_{t_{i+1}} + \frac{\varepsilon}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} \sigma(s) \, dW_s
\]

\[
+ \frac{1}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} [f(s) Y_s - f(t_{i+1}) Y_{t_{i+1}}] \, ds
\]

\[
= f(t_i) Y_{t_{i+1}} + [f(t_{i+1}) - f(\vartheta, t_i)] Y_{t_{i+1}} + \frac{\varepsilon}{\sqrt{\delta^2}} R_{1,\varepsilon}
\]

\[
+ \frac{f(t_{i+1})}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} [Y_s - Y_{t_{i+1}}] \, ds
\]

\[
+ \frac{1}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} [f(s) - f(t_{i+1})] Y_s \, ds.
\]

The following estimates are obtained using elementary calculations

\[
E \left( [f(t_{i+1}) - f(t_i)] Y_{t_{i+1}} \right)^2 \leq C \varphi_\varepsilon^2,
\]

\[
E R^2_{1,\varepsilon} \leq C,
\]

\[
E \left( \frac{f(t_{i+1})}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} [Y_s - Y_{t_{i+1}}] \, ds \right)^2 \leq C \delta^2,
\]

\[
E \left( \frac{1}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} [f(s) - f(t_{i+1})] Y_s \, ds \right)^2
\]

\[
\leq \frac{1}{\delta^2} \int_{t_{i+1}}^{t_{i+1}+\delta} (s - t_{i+1})^2 f'(\tilde{s})^2 \, dY_s \, ds \leq C \delta^2.
\]

Therefore

\[
E \left( \frac{X_{t_{i+1}+\delta} - X_{t_{i+1}}}{\delta^2} - \frac{X_{t_{i+1}} - X_{t_i}}{\delta^2} - f(t_i) [Y_{t_{i+1}} - Y_{t_i}] \right)^2
\]

\[
\leq C \varepsilon^2 + C \delta \varepsilon + C \varphi_\varepsilon^2.
\]

These are estimates for the asymptotic derivatives of \( X_t \) at the points \( t_i \) and \( t_{i+1} \) and of their difference.
The quadratic variation is estimated as follow

\[ Q_{\tau,\varepsilon} = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 (Y_{t_{i+1}} - Y_{t_i})^2 = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} a(s) \, ds + \int_{t_i}^{t_{i+1}} b(s) \, dV_s \right)^2 \]

\[ = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} a(s) \, ds \right)^2 + \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} b(s) \, dV_s \right)^2 \]

\[ + 2 \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \int_{t_i}^{t_{i+1}} a(s) \, ds \int_{t_i}^{t_{i+1}} b(s) \, dV_s, \]

where

\[ \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} a(s) \, ds \right)^2 \leq \varphi_\varepsilon \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \int_{t_i}^{t_{i+1}} a(s)^2 \, ds \leq C_\tau \varphi_\varepsilon, \]

and

\[ \mathbb{E} \left( 2 \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \int_{t_i}^{t_{i+1}} a(s) \, ds \int_{t_i}^{t_{i+1}} b(s) \, dV_s \right)^2 \leq C_\tau \varphi_\varepsilon^2. \]

For the stochastic integral we can write

\[ \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} b(s) \, dV_s \right)^2 \]

\[ = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( b(t_i) \left[ V_{t_{i+1}} - V_{t_i} \right] + \int_{t_i}^{t_{i+1}} [b(s) - b(t_i)] \, dV_s \right)^2. \]

Here

\[ \text{E} \left( \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \left( \int_{t_i}^{t_{i+1}} [b(s) - b(t_i)] \, dV_s \right)^2 \right) \]

\[ = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \int_{t_i}^{t_{i+1}} [b(s) - b(t_i)]^2 \, ds \]

\[ = \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 \int_{t_i}^{t_{i+1}} (s - t_i)^2 b'(\hat{s})^2 \, ds \leq C_\tau \varphi_\varepsilon^2. \]

Hence

\[ \mathbb{E} \left( Q_{\tau,\varepsilon} - \sum_{i=0}^{N_{\tau,\varepsilon}-1} f(t_i)^2 b(t_i)^2 \left[ V_{t_{i+1}} - V_{t_i} \right]^2 \right) \leq C_\tau \varphi_\varepsilon. \]
Finally

\[ K_{\tau, \epsilon} = \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i) \left( \frac{V_{t_{i+1}} - V_{t_i}}{t_{i+1} - t_i} \right) (t_{i+1} - t_i) \]

\[ \begin{align*}
&= \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i)^2 \xi_i^2 (t_{i+1} - t_i) \\
&= \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i)^2 (t_{i+1} - t_i) + \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i)^2 (\xi_i^2 - 1) (t_{i+1} - t_i). 
\end{align*} \]

Here \( \xi_i, i = 0, \ldots, N_{\tau, \epsilon} \) are independent standard Gaussian random variables, \( \xi_i \sim \mathcal{N}(0, 1) \). Therefore

\[ \mathbb{E} \left( K_{\tau, \epsilon} - \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i)^2 (t_{i+1} - t_i) \right)^2 \]

\[ = 2 \varphi_\epsilon \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^4 b(t_i)^4 (t_{i+1} - t_i) \leq C \tau \varphi_\epsilon \]
and

\[ \left| \Psi_{\tau} - \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 b(t_i)^2 (t_{i+1} - t_i) \right| \leq C \tau \varphi_\epsilon. \quad (17) \]

The obtained estimates allow us to write

\[ \mathbb{E} \left| \Psi_{\tau, \epsilon} - \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 \left[ Y_{t_{i+1}} - Y_{t_i} \right] \right|^2 \leq C \sqrt{\tau} \left[ \sqrt{\delta_\epsilon} + \varphi_\epsilon + \frac{\epsilon}{\sqrt{\delta_\epsilon \varphi_\epsilon}} \right]. \]

If we put \( \delta_\epsilon = \epsilon^q, \varphi_\epsilon = \epsilon^l \) then the equation

\[ \sqrt{\delta_\epsilon} = \varphi_\epsilon = \frac{\epsilon}{\sqrt{\delta_\epsilon \varphi_\epsilon}} \]
gives us the values \( q = 1 \) and \( l = 1/3 \), i.e., \( \delta_\epsilon = \epsilon \) and \( \varphi_\epsilon = \epsilon^{1/3} \). Hence

\[ \mathbb{E} \left| \Psi_{\tau, \epsilon} - \sum_{i=0}^{N_{\tau, \epsilon} - 1} f(t_i)^2 \left[ Y_{t_{i+1}} - Y_{t_i} \right] \right|^2 \leq C \tau^{1/2} \epsilon^{1/3}. \]

This estimate together with (16) and (17) allows us to write

\[ \mathbb{E}_\theta \left| \Psi_{\tau, \epsilon} - \Psi_{\tau} \right| \leq C \tau^{1/2} \epsilon^{1/3}. \]

The similar calculations provide the estimate of the polynomial moments too.
Remark that it is possible to prove the asymptotic normality
\[ \tau^{-1/2} \varepsilon^{-1/3} (\Psi_{\tau, \varepsilon} - \Psi_{\tau}) \Rightarrow N(0, D^2) \]
with some limit variance \( D^2 \). The estimate (15) is an intermediate result, which will be used in the next section for the construction of the preliminary estimator for One-step MLE-process.

Remark 1. The obtained rate \( \varepsilon^{1/3} \) is not optimal but is sufficient for the construction of the preliminary estimator below. An interesting result concerning estimation of the function \( \Psi_{\tau} \) by observations (13), (14) was obtained in [30]. Supposing that \( a(t) \equiv 0, \sigma(t) \equiv \sigma, f(t) \equiv 1 \) it was shown that this model of observations is asymptotically equivalent (in Le Cam’s sense) to the model of observations
\[ dX_t = \sqrt{2b(t)} \, dt + \sigma^{1/2} \varepsilon^{1/2} dW_t. \]
This equivalence shows why the rate \( \varepsilon^{1/2} \) is “more natural” for the model (13), (14).

We suppose that more detailed analysis of estimator \( \Psi_{\tau, \varepsilon} \) can show the rate \( \varepsilon^{1/3} \), but for further application of \( \Psi_{\tau, \varepsilon} \) in the construction of One-step MLE process the rate \( \varepsilon^{1/3} \) is sufficient.

6 Method of moments estimator

Let us return to the parametric partially observed linear system
\[
\begin{align*}
\text{d}X_t &= f(\vartheta, t) \, Y_t \, \text{d}t + \varepsilon \sigma(t) \, \text{d}W_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \quad (18) \\
\text{d}Y_t &= a(\vartheta, t) \, Y_t \, \text{d}t + b(\vartheta, t) \, \text{d}V_t, \quad Y_0 = y_0 \quad (19)
\end{align*}
\]
The process \( X^T = (X_t, 0 \leq t \leq T) \) is observed, the process \( Y^T = (Y_t, 0 \leq t \leq T) \) is hidden and the Wiener processes \( V^T = (V_t, 0 \leq t \leq T) \) and \( W^T = (W_t, 0 \leq t \leq T) \) as before are independent. The functions \( b(\cdot) \) and \( f(\cdot) \) are supposed to be known. The functions \( a(\cdot) \) and \( \sigma(\cdot) \) do not used in the construction of the estimator. The parameter \( \vartheta \in \Theta = (\alpha, \beta) \) is unknown and has to be estimated by observations \( X^T \).

Fix some \( \tau \in (0, T] \) and define the function
\[ \Psi_t(\vartheta) = \int_0^t f(\vartheta, s)^2 b(\vartheta, s)^2 \, ds, \quad 0 < t \leq \tau, \quad \vartheta \in \Theta \subset \mathbb{R}^d. \]
As estimator of this function we take the nonparametric estimator
\[ \Psi_{t, \varepsilon} = \sum_{i=0}^{N_{t, \varepsilon} - 1} \left( \frac{X_{t_{i+1}+\varepsilon} - X_{t_{i+1}}}{\varepsilon} - \frac{X_{t_i+\varepsilon} - X_{t_i}}{\varepsilon} \right)^2, \quad 0 < t \leq \tau. \]
Here \( t_{i+1} - t_i = \varepsilon^{1/3}, t_0 = 0 \) and \( N_{t, \varepsilon} = \lceil t \varepsilon^{-1/3} \rceil \).
The MME $\hat{\vartheta}_{\tau,\varepsilon}$ can be defined as follows. Let us fix $d$ points $0 < \tau_1 < \tau_2 < \ldots < \tau_d = \tau$ and denote the vectors $\Psi_{\tau,\varepsilon}^* = (\Psi_{\tau_1,\varepsilon}, \Psi_{\tau_2,\varepsilon}, \ldots, \Psi_{\tau_d,\varepsilon})^\top$ and $\Psi_{\tau}^*(\vartheta) = (\Psi_{\tau_1}(\vartheta), \Psi_{\tau_2}(\vartheta), \ldots, \Psi_{\tau_d}(\vartheta))^\top$. The estimator is defined as solution of the system of equations

$$
\Psi_{\tau,\varepsilon}^* = \Psi_{\tau}^*(\hat{\vartheta}_{\tau,\varepsilon}).
$$

If this equation has no solution then we put $\hat{\vartheta}_{\tau,\varepsilon} = \vartheta^o$, where $\vartheta^o$ is some value $\not\in \bar{\Theta}$.

Note that having the estimator $\Psi_{t,\varepsilon}$, $0 \leq t \leq \tau$ we can define the MDE $\tilde{\vartheta}_{\tau,\varepsilon}$ too by the equation

$$
\int_0^\tau \left[ \Psi_{t,\varepsilon} - \Psi_t(\tilde{\vartheta}_{\tau,\varepsilon}) \right]^2 \, dt = \inf_{\vartheta \in \bar{\Theta}} \int_0^\tau \left[ \Psi_{t,\varepsilon} - \Psi_t(\vartheta) \right]^2 \, dt.
$$

(21)

If these equations has more than one solution then anyone of them can be taken as the estimator of $\vartheta$.

Introduce the notation

$$
g(\vartheta_0, \nu) = \inf_{\|\vartheta - \vartheta_0\| \geq \nu} \|\Psi_{\tau}(\vartheta) - \Psi_{\tau}(\vartheta_0)\|, \quad T(\vartheta_0) = \Psi_{\tau}^*(\vartheta_0) \Psi_{\tau}^*(\vartheta_0)^\top
$$

Proposition 2. Suppose that the following conditions hold

- The functions $a(\vartheta, t), b(\vartheta, t), f(\vartheta, t), \sigma(t), (\vartheta, t) \in \bar{\Theta} \times [0, T]$ have continuous derivatives w.r.t. $t$ and two continuous derivatives w.r.t. $\vartheta$.

- The matrix $T(\vartheta_0)$ is uniformly non degenerate

$$
\inf_{\vartheta_0 \in \bar{\Theta}} \inf_{\|e\|_1, e \in \mathbb{R}^d} e^\top T(\vartheta_0) e > 0.
$$

- For any $\nu > 0$

$$
\inf_{\vartheta_0 \in \bar{\Theta}} g(\vartheta_0, \nu) > 0.
$$

Then the MME $\tilde{\vartheta}_{\tau,\varepsilon}$ is uniformly consistent and for any $p \geq 2$ there exists a constant $C = C(p) > 0$ such that

$$
\sup_{\vartheta_0 \in \bar{\Theta}} \varepsilon^{-p/3} T^{p/2} E_{\vartheta_0} \left\| \tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0 \right\|_p^p \leq C.
$$

(22)

Proof. Remark that if $\tilde{\vartheta}_{\tau,\varepsilon} \in \Theta$ then it satisfies the equation

$$
\inf_{\vartheta \in \bar{\Theta}} \left\| \Psi_{\tau,\varepsilon}^* - \Psi_{\tau}^*(\vartheta) \right\| = \left\| \Psi_{\tau,\varepsilon}^* - \Psi_{\tau}^*(\tilde{\vartheta}_{\tau,\varepsilon}) \right\| = 0.
$$
Below we use elementary inequalities and the estimate (15)
\[
\sup_{\vartheta_0 \in \Theta} P_{\vartheta_0} \left( \| \hat{\vartheta}_{\tau,\varepsilon} - \vartheta_0 \| > \nu \right)
\]
\[
= \sup_{\vartheta_0 \in \Theta} P_{\vartheta_0} \left( \inf_{\| \vartheta - \vartheta_0 \| < \nu} \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\| > \inf_{\| \vartheta - \vartheta_0 \| \geq \nu} \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\| \right)
\]
\[
\leq \sup_{\vartheta_0 \in \Theta} P_{\vartheta_0} \left( \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\| + \inf_{\| \vartheta - \vartheta_0 \| < \nu} \left\| \Psi^*_\tau \vartheta_0 - \Psi^*_\tau \vartheta_0 \right\| \right)
\]
\[
> \inf_{\| \vartheta - \vartheta_0 \| \geq \nu} \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\| - \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\|
\]
\[
= \sup_{\vartheta_0 \in \Theta} P_{\vartheta_0} \left( 2 \left\| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right\| > g (\vartheta_0, \nu) \right)
\]
\[
\leq \frac{2^{2N}}{\inf_{\vartheta_0 \in \Theta} g (\vartheta_0, \nu)^{2N}} \| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \|^{2N} \leq \frac{C T \varepsilon^{-2N/3}}{\inf_{\vartheta_0 \in \Theta} g (\vartheta_0, \nu)^{2N}} \to 0.
\]

Therefore the estimator \( \hat{\vartheta}_{\tau,\varepsilon} \) is uniformly consistent.

Recall that the MME \( \vartheta_{\tau,\varepsilon} \) is one of the solutions of the MMEq (method of moments equation)
\[
(\Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_{\tau,\varepsilon})^\top \hat{\Psi}^*_\tau (\vartheta_{\tau,\varepsilon}) = 0.
\]

Using the consistency of this estimator we can write
\[
0 = \left( \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 + \Psi^*_\tau \vartheta_0 - \Psi^*_\tau \vartheta_{\tau,\varepsilon} \right)^\top \hat{\Psi}^*_\tau (\vartheta_{\tau,\varepsilon})
\]
\[
= \left( \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 - \Psi^*_\tau \vartheta_0 + O (\varepsilon^{2/3}) \right)^\top \left( \Psi^*_\tau \vartheta_0 + O (\varepsilon^{1/3}) \right)
\]
\[
= \left( \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right)^\top \hat{\Psi}^*_\tau (\vartheta_0) - (\vartheta_{\tau,\varepsilon} - \vartheta_0) \hat{\Psi}^*_\tau (\vartheta_0)^\top \hat{\Psi}^*_\tau (\vartheta_0) + O (\varepsilon^{1/3}).
\]

Therefore
\[
\varepsilon^{-1/3}(\vartheta_{\tau,\varepsilon} - \vartheta_0) = \left( \hat{\Psi}^*_\tau (\vartheta_0)^\top \hat{\Psi}^*_\tau (\vartheta_0) \right)^{-1} \varepsilon^{-1/3} \left( \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right)^\top \hat{\Psi}^*_\tau (\vartheta_0) + O (\varepsilon^{1/3})
\]
\[
= T (\vartheta_0)^{-1} \varepsilon^{-1/3} \left( \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \right)^\top \hat{\Psi}^*_\tau (\vartheta_0) + O (\varepsilon^{1/3}).
\]

As the matrix \( T (\vartheta_0) \) is uniformly non degenerate and we have (15) these allow us to write the estimate: for any \( p \geq 2 \)
\[
\varepsilon^{-p/3} \sup_{\vartheta_0 \in \Theta} E_{\vartheta_0} \| \vartheta_{\tau,\varepsilon} - \vartheta_0 \|^p \leq C \varepsilon^{-p/3} \sup_{\vartheta_0 \in \Theta} E_{\vartheta_0} \| \Psi^*_\tau \vartheta - \Psi^*_\tau \vartheta_0 \|^p \leq C T^{p/2}
\]

\[\square\]
Example 1. Suppose that we have the model of observations (18),(19), where \( f (\vartheta, t) = \vartheta f_t(t), \vartheta \in (\alpha, \beta), \alpha > 0 \), \( b(\vartheta, t) = b(t) \) and all corresponding conditions are fulfilled. Then

\[
\Psi_{\tau}(\vartheta) = \vartheta^2 \int_0^\tau f(t)^2 b(t)^2 \, dt
\]

and the MME is

\[
\hat{\vartheta}_{\tau,\varepsilon} = \sqrt{\Psi_{\tau,\varepsilon}} \left( \int_0^\tau f(t)^2 b(t)^2 \, dt \right)^{-1/2}.
\]

This estimator is consistent and has rate of convergence \( \varepsilon^{1/3} \) (Proposition 2).

Example 2. Consider the model (18),(19), where \( f(\vartheta, t) = f(t) \) and

\[
b(\vartheta, t) = \left( h(t) + \sum_{k=1}^d \vartheta_k g_k(t) \right)^{1/2}.
\]

Here the functions \( h(\cdot) \) and \( g_k(\cdot), k = 1, \ldots, d \) are positive and all \( \vartheta > 0 \). Then

\[
\Psi_{\tau_k}(\vartheta) = \int_0^{\tau_k} f(s)^2 h(s) \, ds + \sum_{j=1}^d \vartheta_j \int_0^{\tau_k} f(s)^2 g_j(s) \, ds
\]

\[
= H_k + \sum_{j=1}^d \vartheta_j G_{jk}, \quad k = 1, \ldots, d,
\]

and we have the relation

\[
\vartheta = G^{-1}(\Psi_{\tau}(\vartheta) - H)
\]

with the obvious notations.

Hence the MME is

\[
\hat{\vartheta}_{\tau,\varepsilon} = G^{-1}(\Psi_{\tau,\varepsilon} - H)
\]

and this estimator has the properties described in the Proposition 2.

7 One-step MLE-process.

We have the same partially observed linear system

\[
\begin{align*}
dX_t &= f(\vartheta, t) Y_t \, dt + \varepsilon(t) \, dW_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
dY_t &= a(\vartheta, t) Y_t \, dt + b(\vartheta, t) \, dV_t, \quad Y_0 = 0.
\end{align*}
\]
As before the process \(X^T = (X_t, 0 \leq t \leq T)\) is observable and \(Y^T\) is hidden. All functions \(a (\cdot), b (\cdot), f (\cdot)\) and \(\sigma (\cdot)\) are supposed to be known. The parameter \(\vartheta \in \Theta \subset \mathcal{R}^d\) is unknown and has to be estimated by observations \(X^T\).

Remind that the MLE \(\hat{\vartheta}_t\) and BE \(\tilde{\vartheta}_t\) are defined by the relations (5) and their calculation requires solutions \(m (\vartheta, \cdot), \vartheta \in \Theta, \gamma (\vartheta, \cdot), \vartheta \in \Theta\) of the equations

\[
dm (\vartheta, t) = a (\vartheta, t) m (\vartheta, t) \, dt + \frac{\gamma (\vartheta, t) f (\vartheta, t)}{\varepsilon^2 \sigma (t)^2} [dX_t - f (\vartheta, t) m (\vartheta, t) \, dt], \quad m (\vartheta, 0) = y_0
\]

(24)

and

\[
\frac{\partial \gamma (\vartheta, t)}{\partial t} = 2a (\vartheta, t) \gamma (\vartheta, t) - \frac{\gamma (\vartheta, t)^2 f (\vartheta, t)^2}{\varepsilon^2 \sigma (t)^2} + b (\vartheta, t)^2, \quad \gamma (\vartheta, 0) = 0.
\]

(25)

Our goal is to construct an approximation \(\tilde{m}_t\) of the process \(m (\vartheta, t)\) (adaptive filter) by replacing \(\vartheta\) in the equations (24), (25) by some estimator-process \(\tilde{\vartheta}_{t, \varepsilon}\), \(0 \leq t \leq T\). Therefore we need a sequence of estimators \(\tilde{\vartheta}_{t, \varepsilon}, 0 < t \leq T\) which are measurable w.r.t. observations up to the time \(t\) for each \(t \in (0, T]\). To avoid the evident numerical difficulties related with the calculations MLE \(\hat{\vartheta}_{t, \varepsilon}, 0 \leq t \leq T\) we consider an approach based on the Fisher-score device and MME \(\hat{\vartheta}_{t, \varepsilon}\) as preliminary one. The proposed One-step MLE-process allows us to obtain an estimator-process, which is easy to calculate, has good asymptotic properties and can be used for the construction of adaptive filter.

The Fisher-score function for this model is

\[
\frac{\partial}{\partial \vartheta} \ln L (\vartheta, X^T) = \int_0^T \frac{\hat{M} (\vartheta, t)}{\varepsilon^2 \sigma (t)^2} [dX_t - M (\vartheta, t) \, dt],
\]

where \(M (\vartheta, t) = f (\vartheta, t) (\vartheta, t) m (\vartheta, t)\). Let us consider the construction of One-step MLE-process in the case of observations (23). Suppose that for some fixed small value \(\tau > 0\) we already have MME \(\tilde{\vartheta}_{t, \varepsilon}\) constructed by the observations \(X^\tau = (X_t, 0 \leq t \leq \tau)\). The solution of the equation (24) on the time interval \([0, \tau]\) can be written as follows

\[
m (\vartheta, t) = y_0 \Phi (\vartheta, 0, t) + \Phi (\vartheta, 0, t) \int_0^t \Phi (\vartheta, 0, s)^{-1} B (\vartheta, s) \, dX_s
\]

\[
= y_0 \Phi (\vartheta, 0, t) + \Phi (\vartheta, 0, t) \int_0^t \Phi (\vartheta, 0, s)^{-1} B (\vartheta, s) \, dX_s,
\]

where

\[
\Phi (\vartheta, 0, t) = \exp \left( \int_0^t A (\vartheta, v) \, dv \right), \quad A (\vartheta, t) = a (\vartheta, t) - \frac{\gamma (\vartheta, t) f (\vartheta, t)^2}{\varepsilon^2 \sigma (t)^2},
\]

\[
B (\vartheta, s) = \frac{\gamma (\vartheta, s) f (\vartheta, s)}{\varepsilon^2 \sigma (s)^2}, \quad H (\vartheta, s) = \Phi (\vartheta, 0, s)^{-1} \frac{\gamma (\vartheta, s) f (\vartheta, s)}{\varepsilon^2 \sigma (s)^2}.
\]
We can not put \( \tilde{\vartheta}_{\tau,\varepsilon} \) in \( m(\vartheta, t) \) because the corresponding stochastic integral Itô is not defined. Note that the function \( H(\vartheta, s) \) is continuously differentiable w.r.t. \( s \), hence the following relation

\[
\int_0^\tau H(\vartheta, s) \, dX_s = X_\tau H(\vartheta, \tau) - \int_0^\tau X_s H'(\vartheta, s) \, ds
\]

holds. Now we can put

\[
m(\tilde{\vartheta}_{\tau,\varepsilon}, t) = y_0 \Phi(\tilde{\vartheta}_{\tau,\varepsilon}, 0, \tau) + \Phi(\tilde{\vartheta}_{\tau,\varepsilon}, 0, \tau) \int_0^\tau X_s H'(\tilde{\vartheta}_{\tau,\varepsilon}, s) \, ds.
\]  

(26)

Let us define the random process \( \hat{m}(t) \), \( \tau \leq t \leq T \) by the equation

\[
d\hat{m}(t) = a(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t) \, dt + \frac{\gamma(\tilde{\vartheta}_{\tau,\varepsilon}, t) f(\tilde{\vartheta}_{\tau,\varepsilon}, t)}{\varepsilon^2 \sigma(t)^2} [dX_t - f(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t) \, dt]
\]  

(27)

with the initial value \( \hat{m}(\tau) = m(\tilde{\vartheta}_{\tau,\varepsilon}, \tau) \). The solution of this equation is

\[
\hat{m}(t) = m(\tilde{\vartheta}_{\tau,\varepsilon}, \tau) \Phi(\tilde{\vartheta}_{\tau,\varepsilon}, \tau, t) + \int_{\tau}^{t} \Phi(\tilde{\vartheta}_{\tau,\varepsilon}, s, t) \frac{\gamma(\tilde{\vartheta}_{\tau,\varepsilon}, s) f(\tilde{\vartheta}_{\tau,\varepsilon}, s)}{\varepsilon^2 \sigma(s)^2} \, dX_s.
\]

We need as well the equation for the derivative \( \hat{m}(\vartheta, t) \) on the interval \( [\tau, T] \) at the point \( \vartheta = \tilde{\vartheta}_{\tau,\varepsilon} \) (denoted \( \hat{m}(\tilde{\vartheta}_{\tau,\varepsilon}, t) = \hat{m}(t) \)), which we obtain by formal differentiation of the equation (24)

\[
d\hat{m}(t) = A(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t) \, dt + A(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t) \, dt + B(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, dX_t, \quad \tau \leq t \leq T
\]

(28)

with the initial value \( \hat{m}(\tau) = \hat{m}(\tilde{\vartheta}_{\tau,\varepsilon}, \tau) \), which can be calculated similar to the given above representation of the corresponding stochastic integral like (26).

Introduce notations:

\[
M(\tilde{\vartheta}_{\tau,\varepsilon}, t) = f(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t), \quad \bar{M}(\tilde{\vartheta}_{\tau,\varepsilon}, t) = \hat{f}(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t) + f(\tilde{\vartheta}_{\tau,\varepsilon}, t) \, \hat{m}(t),
\]

\[
I^t_\tau(\vartheta) = \int_{\tau}^{t} \frac{\dot{S}(\vartheta, s) \dot{S}(\vartheta, s)^\top}{2S(\vartheta, s) \sigma(s)} \, ds, \quad \eta_{t,\vartheta} = \frac{\tilde{\vartheta}_{\tau,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}},
\]

\[
\dot{\vartheta}_{t,\varepsilon} = \tilde{\vartheta}_{\tau,\varepsilon} + I^t_\tau(\tilde{\vartheta}_{\tau,\varepsilon})^{-1} \int_{\tau}^{t} \frac{\bar{M}(\tilde{\vartheta}_{\tau,\varepsilon}, s)}{\varepsilon \sigma(s)^2} [dX_s - M(\tilde{\vartheta}_{\tau,\varepsilon}, s)ds], \quad \tau < t \leq T;
\]

\[
\eta_{t}(\vartheta_0) = I^t_\tau(\vartheta_0)^{-1} \int_{\tau}^{t} \frac{\dot{S}(\vartheta_0, s)}{2S(\vartheta_0, s) \sigma(s)} \, dw(s), \quad \tau < t \leq T,
\]

where \( w(s), \tau \leq s \leq T \) is some Wiener process.

**Theorem 1.** Let the following assumptions hold.
1. The functions $f(\vartheta, t), b(\vartheta, t), \sigma(t), (\vartheta, t) \in \Theta \times [0, T]$ are separated from zero.

2. The conditions of Proposition 2 are fulfilled.

3. For any $t_0 \in (\tau, T]$

$$\inf_{\vartheta_0 \in \Theta} \inf_{\|e\|=1, e \in \mathbb{R}^d} e^T \mathbf{I}_T^\nu(\vartheta_0) e > 0.$$  

Then the One-step MLE-process $\vartheta_{t, \vartheta}^*, \tau < t \leq T$ has the following properties:

1. It is uniformly consistent: for any $t_0 \in (\tau, T]$ and any $\nu > 0$

$$\mathbf{P}_{\vartheta_0} \left( \sup_{t_0 \leq t \leq T} \|\vartheta_{t, \vartheta}^* - \vartheta_0\| \geq \nu \right) \longrightarrow 0,$$

2. The random process $\eta_{t, \vartheta}, t_0 \leq t \leq T$ converges in distribution in the measurable space $(\mathcal{C}[t_0, T], \mathcal{B})$ to the Gaussian process $\eta_t(\vartheta_0), t_0 \leq t \leq T$

$$\eta_{t, \vartheta} \Rightarrow \eta_t(\vartheta_0), \quad \eta_t(\vartheta_0) \sim \mathcal{N}(0, \mathbf{I}_t(\vartheta_0)^{-1}),$$

3. All polynomial moments converge: for any $p > 0$

$$\varepsilon^{-p/2} \mathbf{E}_{\vartheta_0} \|\vartheta_{t, \vartheta}^* - \vartheta_0\|^p \longrightarrow \mathbf{E}_{\vartheta_0} \|\eta_t(\vartheta_0)\|^p.$$  

(29)

Proof. The uniform consistency follows from the representation

$$\sup_{t_0 \leq t \leq T} \|\vartheta_{t, \vartheta}^* - \vartheta_0\| \leq \|\vartheta_{t, \vartheta}^* - \vartheta_0\| + C_\varepsilon \sup_{t_0 \leq t \leq T} \left\| \int_{\tau}^t \frac{\dot{M}(\vartheta_{t, \vartheta}, s)}{\sigma(s)} dW_s \right\|$$

$$+ C \int_{\tau}^T \left\| \frac{\dot{M}(\vartheta_{t, \vartheta}, s)}{\sigma(s)} \right\| \left| M(\vartheta_0, s) - M(\vartheta_{t, \vartheta}, s) \right| ds$$

and consistency of the MME $\dot{\vartheta}_{t, \vartheta}$.

We have to study the following expression

$$\frac{\vartheta_{t, \vartheta}^* - \vartheta_0}{\sqrt{\varepsilon}} = \frac{(\vartheta_{t, \vartheta}^* - \vartheta_0)}{\sqrt{\varepsilon}} + \mathbf{I}_t^\nu(\vartheta_{t, \vartheta}^*)^{-1} \int_{\tau}^t \frac{\dot{M}(\vartheta_{t, \vartheta}, s)}{\varepsilon^{3/2} \sigma(s)^2} dW_s + \int_{\tau}^t \frac{f(\vartheta_0, s) m(\vartheta_0, s) - M(\vartheta_{t, \vartheta}, s)}{\varepsilon^{3/2} \sigma(s)^2} dW_s$$

We have the following relations

$$\mathbf{I}_t^\nu(\vartheta_{t, \vartheta}^*)^{-1} = \mathbf{I}_t^\nu(\vartheta_0)^{-1} + O \left( \|\vartheta_{t, \vartheta}^* - \vartheta_0\| \right),$$

$$f(\vartheta_0, s) m(\vartheta_0, s) - M(\vartheta_{t, \vartheta}, s) = (\vartheta_0 - \vartheta_{t, \vartheta})^T \dot{M}(\vartheta_0, t) + O \left( \|\vartheta_{t, \vartheta}^* - \vartheta_0\| \right),$$

$$\int_{\tau}^t \frac{\dot{M}(\vartheta_{t, \vartheta}, s)}{\varepsilon^{3/2} \sigma(s)^2} dW_s = \int_{\tau}^t \frac{\dot{M}(\vartheta_0, s)}{\varepsilon^{3/2} \sigma(s)^2} dW_s + O \left( \|\vartheta_{t, \vartheta}^* - \vartheta_0\| \right).$$
We rewrite the equation for $M$:

$$I_t^{\varepsilon}(\tilde{\vartheta}, \varepsilon)^{-1} \int_{\tau}^{\tau} \frac{\dot{M}(\tilde{\vartheta}, \varepsilon, s)}{\varepsilon^{3/2} \sigma(s)^2} \left[ f(\vartheta_0, s) m(\vartheta_0, s) - M(\tilde{\vartheta}, \varepsilon, s) \right] ds$$

$$= I_t^{\varepsilon}(\vartheta_0)^{-1} \int_{\tau}^{\tau} \frac{\dot{M}(\tilde{\vartheta}, \varepsilon, s)}{\varepsilon^{3/2} \sigma(s)^2} \dot{\vartheta}_0 - \tilde{\vartheta}_0 + O \left( \|\tilde{\vartheta}_0 - \vartheta_0\| \right).$$

The initial value is $M(\vartheta, t) = f(\vartheta, t) m(\vartheta, t) dt$. The equation for derivative is:

$$dM(\vartheta, t) = f'(\vartheta, t) m(\vartheta, t) dt + a(\vartheta, t) M(\vartheta, t) dt - \frac{\gamma(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} M(\vartheta, t) dt$$

$$+ \frac{\gamma(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} dX_t$$

$$= -q_\varepsilon(\vartheta, t) M(\vartheta, t) dt + \frac{\gamma(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} dX_t,$$

where we denoted

$$q_\varepsilon(\vartheta, t) = \frac{\gamma_0(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2} - a(\vartheta, t) - f'(\vartheta, t).$$

The initial value is $M(\vartheta, \tau) = f(\vartheta, \tau) m(\vartheta, \tau)$. To describe the asymptotic behavior of $M(\vartheta, t)$ and $\dot{M}(\vartheta, t)$ we need some results from [21], which we recall here. The function $\gamma_* (\vartheta, t) = \varepsilon^{-1} \gamma(\vartheta, t)$ is bounded and for any $t_0 > 0$ uniformly on $t \in [t_0, T]$ converges to the function $\gamma_0(\vartheta, t) = b(\vartheta, t) \sigma(t) f(\vartheta, t)^{-1}$ ([21], Lemma 2). It was shown that we can replace the function $q_\varepsilon(\vartheta, t)$ by the function $q_*(\vartheta, t)/\varepsilon$, where

$$q_* (\vartheta, t) = \frac{\gamma_0(\vartheta, t) f(\vartheta, t)^2}{\sigma(t)^2} = \frac{b(\vartheta, t) f(\vartheta, t)}{\sigma(t)}.$$

We rewrite the equation for $M(\vartheta, t)$ without changing notation

$$dM(\vartheta, t) = - \frac{b(\vartheta, t) f(\vartheta, t)}{\varepsilon \sigma(t)^2} M(\vartheta, t) dt + \frac{b(\vartheta, t) f(\vartheta, t)}{\varepsilon \sigma(t)} dX_t$$

$$= -\frac{1}{\varepsilon} q_* (\vartheta, t) M(\vartheta, t) dt + \frac{1}{\varepsilon} q_* (\vartheta, t) dX_t$$

$$= -\frac{1}{\varepsilon} q_* (\vartheta, t) [M(\vartheta, t) - M(\vartheta_0, t)] dt + q_* (\vartheta, t) \sigma(t) d\bar{W}_t.$$

The equation for derivative is

$$d\dot{M}(\vartheta, t) = -\frac{1}{\varepsilon} q_* (\vartheta, t) \dot{M}(\vartheta, t) dt - \frac{1}{\varepsilon} \dot{q}_* (\vartheta, t) M(\vartheta, t) dt + \frac{1}{\varepsilon} \dot{q}_* (\vartheta, t) dX_t$$

$$= -\frac{1}{\varepsilon} q_* (\vartheta, t) \dot{M}(\vartheta, t) dt + \frac{1}{\varepsilon} \dot{q}_* (\vartheta, t) [M(\vartheta_0, t) - M(\vartheta, t)] dt + \dot{q}_* (\vartheta, t) \sigma(t) d\bar{W}_t.$$
As the contribution of the initial value in $\dot{M} (\vartheta, t)$ is exponentially small to simplify the exposition we will omit it below. We have

$$
\dot{M} (\vartheta, t) = \frac{1}{\varepsilon} \int_{\tau}^{t} \phi (\vartheta, s, t) \dot{q}_s (\vartheta, s) [M (\vartheta_0, s) - M (\vartheta, s)] \, ds \\
+ \int_{\tau}^{t} \phi (\vartheta, s, t) \dot{q}_s (\vartheta, s) \sigma (s) \, d\tilde{W}_s,
$$

where

$$
\phi_s (\vartheta, s, t) = \exp \left( -\varepsilon^{-1} \int_{s}^{t} q_s (\vartheta, v) \, dv \right).
$$

The asymptotics of $\dot{M} (\vartheta, t)$ is described with the help of the Lemma 1 in [21]. Note that

$$
\dot{M} (\vartheta_0, t) = \int_{\tau}^{t} \phi_s (\vartheta_0, s, t) \dot{q}_s (\vartheta_0, s) \sigma (s) \, d\tilde{W}_s,
$$

and the asymptotic of stochastic integral by the same Lemma is

$$
\int_{\tau}^{t} \phi_s (\vartheta, s, t) \dot{q}_s (\vartheta, s) \sigma (s) \, d\tilde{W}_s

= \sqrt{\varepsilon} \sqrt{b(\vartheta, t)\sigma(t)f(\vartheta, t)} \left( \frac{\dot{b}(\vartheta_0, t)}{b(\vartheta_0, t)} + \frac{\dot{f}(\vartheta, t)}{f(\vartheta, t)} \right) \xi_{t, \varepsilon} \left( 1 + O \left( \varepsilon \right) \right).
$$

Hence

$$
\varepsilon^{-1/2} \dot{M} (\vartheta_0, t) = \sqrt{b(\vartheta_0, t)\sigma(t)f(\vartheta_0, t)} \left( \frac{\dot{b}(\vartheta_0, t)}{b(\vartheta_0, t)} + \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \right) \xi_{t, \varepsilon} \left( 1 + O \left( \varepsilon \right) \right)

\implies \sqrt{b(\vartheta_0, t)\sigma(t)f(\vartheta_0, t)} \left( \frac{\dot{b}(\vartheta_0, t)}{b(\vartheta_0, t)} + \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \right) \xi_t.
$$

The first integral in (30) we write as follows

$$
\frac{1}{\varepsilon} \int_{\tau}^{t} \phi (\vartheta, s, t) \dot{q}_s (\vartheta, s) [M (\vartheta_0, s) - M (\vartheta, s)] \, ds

= \frac{(\vartheta_0 - \vartheta)}{\varepsilon} \int_{\tau}^{t} \phi (\vartheta, s, t) \dot{q}_s (\vartheta, s) \dot{M} (\vartheta, s) \, ds

= \frac{(\vartheta_0 - \vartheta)}{\varepsilon} \int_{\tau}^{t} \phi (\vartheta, s, t) \dot{q}_s (\vartheta, s) \dot{M} (\vartheta_0, s) \, ds + (\vartheta_0 - \vartheta)^2 O \left( 1 \right)

= \sqrt{\varepsilon} (\vartheta_0 - \vartheta) O \left( 1 \right) + (\vartheta_0 - \vartheta)^2 O \left( 1 \right).
$$
where $|\hat{\theta} - \theta_0| \leq |\vartheta - \hat{\vartheta}|$. We have

$$
P_{\theta_0} \left( \sup_{t_0 \leq t \leq T} |\hat{\vartheta}_{t, \varepsilon} - \theta_0| \geq \nu \right) \leq P_{\theta_0} \left( |\hat{\vartheta}_{t, \varepsilon} - \theta_0| \geq \frac{\nu}{3} \right)

+ P_{\theta_0} \left( \sup_{t_0 \leq t \leq T} \frac{1}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s) d\hat{W}_s}{\sigma(s)} \geq \frac{\nu}{3} \right)

+ P_{\theta_0} \left( \frac{|\hat{\vartheta}_{t, \varepsilon} - \theta_0|}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s) M(\hat{\vartheta}_{t, \varepsilon}, s)}{\varepsilon \sigma(s)^2} \right| ds \geq \frac{\nu}{3} \right) \). \quad (31)

Let us denote $c_0 = \inf_{\vartheta \in \Theta} \Gamma_t^{\vartheta}(\vartheta)$. Then

$$
P_{\theta_0} \left( \sup_{t_0 \leq t \leq T} \frac{1}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s) d\hat{W}_s}{\sigma(s)} \geq \frac{\nu}{3} \right)

\leq P_{\theta_0} \left( \sup_{t_0 \leq t \leq T} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s) d\hat{W}_s}{\sigma(s)} \geq \frac{\nu c_0}{3} \right)

\leq \frac{9}{\nu^2 c_0^2} E_{\theta_0} \int_{\tau}^{T} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s)^2}{\sigma(s)^2} ds \leq C \varepsilon.

The last integral in (31) is a bounded in probability function and the corresponding probability tends to zero thanks to the consistency of the estimator $\hat{\vartheta}_{t, \varepsilon}$. Therefore the estimator $\hat{\vartheta}_{t, \varepsilon}$ is uniformly on $t \in [t_0, T]$ consistent.

Further

$$
\frac{1}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s)}{\varepsilon^{3/2} \sigma(s)^2} \left[ M(\theta_0, s) - M(\hat{\vartheta}_{t, \varepsilon}, s) \right] ds

= \frac{(\vartheta_0 - \hat{\vartheta}_{t, \varepsilon})}{\Gamma_t(\hat{\vartheta}_{t, \varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t, \varepsilon}, s)^2}{\varepsilon^{3/2} \sigma(s)^2} ds (1 + O(\varepsilon^{1/3}))

= -\frac{(\hat{\vartheta}_{t, \varepsilon} - \vartheta_0)}{\sqrt{\varepsilon} \Gamma_t(\vartheta_0)} \int_{\tau}^{t} S(\vartheta_0, s) \left[ \frac{\dot{f}(\vartheta_0, s)}{\dot{f}(\vartheta_0, s)} + \frac{\dot{b}(\vartheta_0, s)}{b(\vartheta_0, s)} \right]^{2} \xi_{s, \varepsilon} ds (1 + O(\varepsilon^{1/3})) \right).$$

Recall that ([21], Lemma 4)

$$\int_{\tau}^{t} \frac{S(\vartheta_0, s)}{\sigma(s)} \left[ \frac{\dot{f}(\vartheta_0, s)}{\dot{f}(\vartheta_0, s)} + \frac{\dot{b}(\vartheta_0, s)}{b(\vartheta_0, s)} \right]^{2} \xi_{s, \varepsilon} ds

\rightarrow \int_{\tau}^{t} \frac{S(\vartheta_0, s)}{2 \sigma(s)} \left[ \frac{\dot{f}(\vartheta_0, s)}{\dot{f}(\vartheta_0, s)} + \frac{\dot{b}(\vartheta_0, s)}{b(\vartheta_0, s)} \right]^{2} ds = \Gamma_t(\vartheta_0). \quad (32)$$

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Therefore
\[
\frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{\sqrt{\varepsilon}} + \frac{1}{I_t'(\hat{\vartheta}_{t,\varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t,\varepsilon}, s)}{\varepsilon^{3/2} \sigma(s)^2} \left[ M(\vartheta_0, s) - M(\hat{\vartheta}_{t,\varepsilon}, s) \right] ds
\]
\[
= \frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{\sqrt{\varepsilon}} \left[ 1 - \frac{1}{I_t'(\hat{\vartheta}_{t,\varepsilon})} \int_{\tau}^{t} \frac{S(\hat{\vartheta}_{t,\varepsilon}, s)}{\sigma(s)} \left[ \frac{f(\hat{\vartheta}_{t,\varepsilon}, s) + \frac{\dot{b}(\hat{\vartheta}_{t,\varepsilon}, s)}{b(\vartheta_0, s)}}{f(\hat{\vartheta}_{t,\varepsilon}, s)} \right]^2 ds \right] \left( 1 + O\left( \varepsilon^{1/3} \right) \right)
\]
\[
= \frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{\sqrt{\varepsilon}} O\left( \varepsilon^{1/3} \right) = O\left( \varepsilon^{2/3} / \sqrt{\varepsilon} \right) \rightarrow 0.
\]
(33)

The convergence (32) is sufficient for the asymptotic normality of the stochastic integral
\[
\int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t,\varepsilon}, s)}{\sqrt{\varepsilon} \sigma(s)} d\bar{W}_s = \int_{\tau}^{t} \frac{M(\vartheta_0, s)}{\sqrt{\varepsilon} \sigma(s)} d\bar{W}_s \left( 1 + O\left( \varepsilon^{1/3} \right) \right) \implies \mathcal{N}(0, I_t'(\vartheta_0)).
\]

Finally we obtained the asymptotic normality of One-step MLE-process: for any \( t > \tau \)
\[
\frac{\hat{\vartheta}_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \implies \eta_t(\vartheta_0) \sim \mathcal{N}(0, I_t'(\vartheta_0)^{-1}).
\]

To prove the convergence of moments we slightly modify the calculations in (33):
\[
\frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{\sqrt{\varepsilon}} + \frac{1}{I_t'(\hat{\vartheta}_{t,\varepsilon})} \int_{\tau}^{t} \frac{M(\hat{\vartheta}_{t,\varepsilon}, s)}{\varepsilon^{3/2} \sigma(s)^2} \left[ M(\vartheta_0, s) - M(\hat{\vartheta}_{t,\varepsilon}, s) \right] ds
\]
\[
= \frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{\sqrt{\varepsilon}} \left[ 1 - \frac{1}{I_t'(\hat{\vartheta}_{t,\varepsilon})} \int_{\tau}^{t} \frac{S(\hat{\vartheta}_{t,\varepsilon}, s)}{\sigma(s)} \left[ \frac{f(\hat{\vartheta}_{t,\varepsilon}, s) + \frac{\dot{b}(\hat{\vartheta}_{t,\varepsilon}, s)}{b(\vartheta_0, s)}}{f(\hat{\vartheta}_{t,\varepsilon}, s)} \right]^2 ds \right] \left( 1 + O\left( \varepsilon^{1/3} \right) \right)
\]
\[
+ \frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)}{I_t'(\hat{\vartheta}_{t,\varepsilon})} \frac{1}{\sqrt{\varepsilon}} \int_{\tau}^{t} \frac{S(\hat{\vartheta}_{t,\varepsilon}, s)}{\sigma(s)} \left[ \frac{f(\hat{\vartheta}_{t,\varepsilon}, s) + \frac{\dot{b}(\hat{\vartheta}_{t,\varepsilon}, s)}{b(\vartheta_0, s)}}{f(\hat{\vartheta}_{t,\varepsilon}, s)} \right]^2 \left\{ \xi_{s,\varepsilon}^2 - \frac{1}{2} \right\} ds \left( 1 + O\left( \varepsilon^{1/3} \right) \right)
\]
\[
= \frac{(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)^2}{\sqrt{\varepsilon}} O(1) + (\hat{\vartheta}_{t,\varepsilon} - \vartheta_0) O(1).
\]

The terms \( \varepsilon^{-1/2}(\hat{\vartheta}_{t,\varepsilon} - \vartheta_0)^2 \), \( O(1) \) and the integral containing \( \xi_{s,\varepsilon}^2 - \frac{1}{2} \) have bounded all polynomial moments (see (23) here and the estimate (29) in [21]). Therefore for any \( p > 1 \) there exists a constant \( C > 0 \) such that
\[
E_{\vartheta_0} \left| \frac{\vartheta_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \right|^p \leq C.
\]
(34)

The presented here proof can be used for verification of the convergence of the vectors: for any \( K = 1, 2, \ldots \) and \( t_0 < t_1 < \ldots < t_K \leq T \)
\[
(\eta_{t_1,\varepsilon}, \ldots, \eta_{t_K,\varepsilon}) \implies (\eta_{t_1}(\vartheta_0), \ldots, \eta_{t_K}(\vartheta_0))\).
It is sufficient to study the convergence of the sum $\sum_{k=1}^{K} \lambda_k \eta_{t,\varepsilon}$. The estimate

$$E_{\theta_0} |\eta_{t_1,\varepsilon} - \eta_{t_2,\varepsilon}|^4 \leq C_1 |t_1 - t_2|^4 + C_2 |t_1 - t_2|^2$$

with some constants $C_1 > 0$ and $C_2 > 0$ can be verified following the same steps as it was done in [20], Theorem 1.

Having these properties of $\eta_{t,\varepsilon}$ we obtain the weak convergence of the random process $\eta_{t,\varepsilon}, t_0 \leq t \leq T$ to the random process $\eta(\vartheta_0), t_0 \leq t \leq T$.

Remark 2. Note that if $\inf_{\vartheta \in \Theta} |\dot{S}(\vartheta, \tau)| > 0$, then the condition $C_1$ is fulfilled for any $t_0 > \tau$.

Remark 3. The proposed above construction of the One-step MLE $\hat{\vartheta}_{T,\varepsilon}$ and the One-step MLE-process $\vartheta_{t,\varepsilon}, \tau < t \leq T$ are not asymptotically efficient because the limit variance of the MLE $\hat{\vartheta}$ is $I_{0}^T(\vartheta_0)^{-1}$ and the limit variance of $\vartheta_{T,\varepsilon}$ is $I_{T}^T(\vartheta_0)^{-1}$. Choosing sufficiently small $\tau$ the ratio

$$\frac{I_{0}^T(\vartheta_0)}{I_{T}^T(\vartheta_0)} \geq 1$$

can be made close to 1.

Another possibility is to consider the sequence of problems with $\tau = \tau_\varepsilon \to 0$ but slowly. The main condition on the rate of convergence of a preliminary estimator $\bar{\vartheta}_{\tau,\varepsilon}$ is

$$\frac{(\bar{\vartheta}_{\tau,\varepsilon} - \vartheta_0)^2}{\sqrt{\varepsilon}} \rightarrow 0.$$

The MME $\hat{\vartheta}_{\tau,\varepsilon}$ satisfies to this condition

$$\frac{(\hat{\vartheta}_{\tau,\varepsilon} - \vartheta_0)^2}{\sqrt{\varepsilon}} \approx \frac{\varepsilon^{2/3}}{\tau \varepsilon^{1/2}} = \frac{\varepsilon^{1/6}}{\tau} \rightarrow 0.$$

Therefore, if we take $\tau = \tau_\varepsilon = \varepsilon^{1/12}$, then

$$\frac{(\hat{\vartheta}_{\tau,\varepsilon} - \vartheta_0)^2}{\sqrt{\varepsilon}} \approx \varepsilon^{1/12} \rightarrow 0$$

and we have the asymptotic normality of the corresponding One-step estimator: for any $t > 0$

$$\frac{\vartheta_{t,\varepsilon} - \vartheta_0}{\sqrt{\varepsilon}} \Rightarrow \hat{\eta}(\vartheta_0) = \frac{1}{I_{0}^T(\vartheta_0)} \int_{0}^{t} \frac{\dot{S}(\vartheta_0, s)}{\sqrt{2S(\vartheta_0, s)\sigma(s)}} dw(s) \sim \mathcal{N}(0, I_{0}^T(\vartheta_0)^{-1}). \quad (35)$$
8 Adaptive filtration

Consider the partially observed two-dimensional linear stochastic system (23), where \( \vartheta \) is unknown parameter. The main problem is the estimation of the hidden component \( Y_t \) by observations \( X^t = (X_s, 0 \leq s \leq t) \). We can not use the equations (24), (25) directly because the true value of \( \vartheta \) is unknown. It is quite natural to replace in (24), (25) by some estimator \( \hat{\vartheta}_e \) and the corresponding solution \( m(\hat{\vartheta}_e, t) \) of (24) can be an approximation of the conditional expectation \( m(\vartheta, t) \). The properties of the estimator \( \hat{\vartheta}_e \), which we need for such approximation are: for each \( t \) it depends of the observations \( X^t \), it can be easily calculated and is asymptotically efficient. All these properties has One-step MLE-process \( \hat{\vartheta}_{t,e}, \tau_e < t \leq T \) and below we realize this construction and evaluate the error of approximation.

Let us recall the notations: \( \tau_e = \varepsilon^{1/12}, t_{i+1} = t_i + \varepsilon^{1/3}, N_{\tau_e,e} = [\varepsilon^{-1/4}], \)

\[
\Psi_{\tau_e}(\vartheta) = \int_0^{\tau_e} f(\vartheta, t)^2 b(\vartheta, t)^2 dt, \quad \hat{\Psi}_{\tau,e} = \sum_{i=0}^{N_{\tau_e,e}-1} \left( \frac{X_{t_{i+1}+\varepsilon} - X_{t_{i+1}} - X_{t_i+\varepsilon} + X_t}{\varepsilon} \right)^2, \]

\[
\dot{\hat{\vartheta}}_{\tau_e} = \Psi_{\tau_e}^{-1}(\hat{\Psi}_{\tau,e}), \quad \hat{\eta}_e(\vartheta_0) = \Gamma_0(\vartheta_0)^{-1} \int_0^{\tau_e} \frac{\dot{\hat{S}}(\vartheta_0, s)}{\sqrt{2S(\vartheta_0, s)\sigma(s)}} dW(s), \]

\[
\hat{\eta}_{\tau,e}(\vartheta_0) = \Gamma_0(\vartheta_0)^{-1} \int_0^{\tau_e} \frac{\dot{\hat{S}}(\vartheta_0, s)}{\sqrt{2S(\vartheta_0, s)\sigma(s)}} \xi_{s,e}^{(1)} dW_s, \quad \xi_{t,e}^{(1)} = \sqrt{\frac{\varepsilon}{\varepsilon}} \int_0^{\tau_e} e^{-\varepsilon} dW_{t-\frac{\varepsilon}{\varepsilon}}, \]

\[
dM(\vartheta, t) = -\frac{b(\vartheta, t)f(\vartheta, t)}{\varepsilon\sigma(t)} M(\vartheta, t) dt + \frac{b(\vartheta, t)f(\vartheta, t)}{\varepsilon\sigma(t)} dX_t, \quad (36)\]

\[
d\bar{M}(\vartheta, t) = -\frac{b(\vartheta, t)f(\vartheta, t)}{\varepsilon\sigma(t)} M(\vartheta, t) dt \]

\[
+ \frac{\dot{b}(\vartheta, t)f(\vartheta, t) + \dot{f}(\vartheta, t)b(\vartheta, t)}{\varepsilon\sigma(t)} [dX_t - M(\vartheta, t) dt], \quad (37)\]

\[
\hat{\vartheta}_{t,e} = \dot{\hat{\vartheta}}_{\tau_e} + \Gamma_{\tau_e}(\hat{\vartheta}_{\tau_e})^{-1} \int_{\tau_e}^{t} \frac{\bar{M}(\hat{\vartheta}_{\tau_e}, s)}{\varepsilon\sigma(s)^2} [dX_s - M(\hat{\vartheta}_{\tau_e}, s) ds], \quad \tau_e < t \leq T. \quad (38)\]

For the calculation \( \hat{\vartheta}_{t,e} \) we use twice the equation (36). The first time it is considered on the interval \([0, \tau_e]\) and is rewritten like (26) to provide the initial value \( M(\hat{\vartheta}_{\tau_e}, \tau_e) \) for the solution \( M(\hat{\vartheta}_{\tau_e}, t) \) of (36) on the interval \([\tau_e, T]\), i.e.,

\[
dM(\hat{\vartheta}_{\tau_e}, t) = -\frac{b(\hat{\vartheta}_{\tau_e}, t)f(\hat{\vartheta}_{\tau_e}, t)}{\varepsilon\sigma(t)} M(\hat{\vartheta}_{\tau_e}, t) dt + \frac{b(\hat{\vartheta}_{\tau_e}, t)f(\hat{\vartheta}_{\tau_e}, t)}{\varepsilon\sigma(t)} dX_t, \quad \tau_e \leq t \leq T. \]

The similar procedure we have for the solution \( \bar{M}(\hat{\vartheta}_{\tau_e}, t) \) of the equation (37). Having these two processes the estimator \( \hat{\vartheta}_{t,e} \) can be easily calculated.
The adaptive filtration for this model of observations is introduced by the equation for $\hat{m}_t$ as follows
\[
d\hat{m}_t = -\frac{b(\vartheta^*_{t,\varepsilon}, t) f(\vartheta^*_{t,\varepsilon}, t)}{\varepsilon \sigma(t)} \hat{m}_t dt + \frac{b(\vartheta^*_{t,\varepsilon}, t)}{\varepsilon \sigma(t)} dX_t, \quad \tau_{\varepsilon} \leq t \leq T
\] (39)
with initial value $\hat{m}_{\tau_{\varepsilon}} = m(\bar{\vartheta}_{\tau_{\varepsilon}}, \tau_{\varepsilon})$. Note that the calculation of $\hat{m}_t$ does not require the solution of Riccati equation.

Let us discuss the errors of approximations: $Y_t - m(\vartheta, t)$ and $Y_t - \hat{m}_t$. Recall that $Y_t$ is solution of the equation
\[
dY_t = a(\vartheta_0, t) Y_t dt + b(\vartheta_0, t) dV_t, \quad Y_0 = y_0, \quad 0 \leq t \leq T.
\]
Introduce the set
\[\mathbb{A} = \left( \omega : |\vartheta^*_{t,\varepsilon} - \vartheta_0| \leq \varepsilon^{\frac{q}{2}} \right),\]
and the random variables $\xi_{1,\varepsilon}^{(1)}, \xi_{1,\varepsilon}^{(2)}$, which are asymptotically normal
\[\xi_{1,\varepsilon}^{(1)} \Rightarrow \xi_{1,\varepsilon}^{(1)} \sim \mathcal{N}\left(0, \frac{1}{2}\right), \quad \xi_{1,\varepsilon}^{(2)} \Rightarrow \xi_{1,\varepsilon}^{(2)} \sim \mathcal{N}\left(0, \frac{1}{2}\right),\]
where $\xi_{1,\varepsilon}^{(1)}, \xi_{1,\varepsilon}^{(2)}, 0 < t \leq T$ are independent random variables. These random variables are independent of the Gaussian process $\hat{\eta}_t(\vartheta_0), 0 < t \leq T$ too.

Introduce the condition $\bar{C}_1$. The functions $f(\vartheta, t)$ and $b(\vartheta, t)$ are such that
\[\inf_{\vartheta \in \Theta} \left| \int f(\vartheta, 0) b(\vartheta, 0) + f(\vartheta, 0) b(\vartheta, 0) \right| > 0.
\]
Note that under this condition we have $\hat{\Psi}_t(\vartheta) > 0$ and $I_0(\vartheta) > 0$ for any $\tau > 0$.

**Theorem 2.** Let the conditions $\mathcal{A}, \bar{C}_1, \bar{C}_2$ be fulfilled, then,

1. If $\vartheta_0$ is known then
\[
\frac{Y_t - m(\vartheta_0, t)}{\sqrt{\varepsilon}} = \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{f(\vartheta_0, t)}} \left( \xi_{1,\varepsilon}^{(1)} - \xi_{1,\varepsilon}^{(2)} \right) \left( 1 + O(\varepsilon^{1/2}) \right).
\]

2. If $\vartheta_0$ is unknown, then on the set $\mathbb{A}$ we have the representation
\[
\frac{Y_t - \hat{m}_t}{\sqrt{\varepsilon}} = \frac{f(\vartheta_0, t)}{f(\vartheta_0, t)} \hat{\eta}_{t,\varepsilon}(\vartheta_0) Y_t + \sqrt{\frac{b(\vartheta_0, t) \sigma(t)}{f(\vartheta_0, t)}} \left( \xi_{1,\varepsilon}^{(1)} - \xi_{1,\varepsilon}^{(2)} \right) + O(\varepsilon^{q_\ast})
\]
where $q_\ast > 0$ and
\[E_{\vartheta_0} \left( \hat{\eta}_{t,\varepsilon}(\vartheta_0) Y_t \right)^2 \rightarrow E_{\vartheta_0} \hat{\eta}_t(\vartheta_0)^2 E_{\vartheta_0} Y_t^2 = I_0(\vartheta_0)^{-1} E_{\vartheta_0} Y_t^2.
\]

For any $p_\ast > 1$ there exists a constant $C_{p_\ast} > 0$ such that
\[P_{\vartheta_0}(\mathbb{A}^c) \leq C_{p_\ast} \varepsilon^{p_\ast}.
\]
Proof. Suppose that \( \vartheta_0 \) is known and we are interested by the error \( Y_t - m(\vartheta_0, t) \). Of course, we have immediately: for any \( t \in (0, T] \)

\[
E_{\vartheta_0} (Y_t - m(\vartheta_0, t))^2 = \gamma (\vartheta_0, t) = \frac{b(\vartheta_0, t) \sigma (t)}{f(\vartheta_0, t)} \varepsilon (1 + o(\varepsilon)).
\]

Further, using equation (24) and asymptotics of \( \gamma (\vartheta, t) \), we can write the equation for \( \delta (t) = Y_t - m(\vartheta_0, t) \)

\[
d\delta (t) = a(\vartheta_0, t) \delta (t) dt - \frac{\gamma (\vartheta_0, t) f(\vartheta_0, t)^2}{\varepsilon^2 \sigma (t)^2} \delta (t) dt + b(\vartheta_0, t) dV_t - \frac{\gamma (\vartheta_0, t) f(\vartheta_0, t)}{\varepsilon \sigma (t)} dW_t
\]

\[
= -q_t (\vartheta_0, t) \delta (t) dt + b(\vartheta_0, t) dV_t - \frac{\gamma (\vartheta_0, t) f(\vartheta_0, t)}{\varepsilon \sigma (t)} dW_t
\]

\[
= -\frac{1}{\varepsilon} q_t (\vartheta_0, t) \delta (t) dt + b(\vartheta_0, t) dV_t - b(\vartheta_0, t) dW_t,
\]

where we omitted the term \( O(\varepsilon) \).

Hence for any \( t > 0 \) we have the relation

\[
Y_t - m(\vartheta_0, t) = \sqrt{\frac{b(\vartheta_0, t) \sigma (t)}{f(\vartheta_0, t)}} (\xi_{t, \varepsilon}^{(1)} - \xi_{t, \varepsilon}^{(2)}) \sqrt{\varepsilon} (1 + O(\sqrt{\varepsilon})).
\]

(40)

The equation for \( \hat{m}_t \) is

\[
d\hat{m}_t = -\frac{b(\vartheta^*_{t, \varepsilon}, t) f(\vartheta^*_{t, \varepsilon}, t)}{\varepsilon \sigma (t)} \hat{m}_t dt + \frac{b(\vartheta^*_{t, \varepsilon}, t) f(\vartheta_0, t)}{\varepsilon \sigma (t)} Y_t dt + b(\vartheta^*_{t, \varepsilon}, t) dW_t
\]

Hence

\[
d(Y_t - \hat{m}_t) = a(\vartheta_0, t) Y_t dt - \frac{b(\vartheta^*_{t, \varepsilon}, t) [f(\vartheta_0, t) - f(\vartheta^*_{t, \varepsilon}, t)]]}{\varepsilon \sigma (t)} Y_t dt - b(\vartheta^*_{t, \varepsilon}, t) dW_t
\]

\[
= -\frac{b(\vartheta^*_{t, \varepsilon}, t) f(\vartheta^*_{t, \varepsilon}, t)}{\varepsilon \sigma (t)} (Y_t - \hat{m}_t) dt + b(\vartheta_0, t) dV_t.
\]

Further

\[
-\varepsilon \ln \phi_s (\vartheta^*_{t, \varepsilon}, s, t) = \int_s^t S(\vartheta^*_{v, \varepsilon}, v) \frac{\sigma(v)}{\sigma(t)} dv = (t - s) S(\vartheta_0, t) \frac{\sigma(t)}{\sigma(t)} + \int_s^t \left( \frac{S(\vartheta_0, v)}{\sigma(v)} - \frac{S(\vartheta_0, t)}{\sigma(t)} \right) dv
\]

\[
+ \int_s^t S(\vartheta^*_{v, \varepsilon}, v) - S(\vartheta_0, v) \frac{\sigma(v)}{\sigma(t)} dv
\]

\[
= (t - s) q(\vartheta_0, t) + \int_s^t (v - t) \frac{S'(\vartheta_0, \tilde{v}) \sigma(\tilde{v}) - S(\vartheta_0, \tilde{v}) \sigma'(\tilde{v})}{\sigma(\tilde{v})^2} dv
\]

\[
+ \int_s^t (\vartheta^*_{v, \varepsilon} - \vartheta_0) \frac{\hat{S}(\tilde{\vartheta}_v, v)}{\sigma(v)} dv.
\]
We have elementary estimates
\[
\left| \int_s^t (v-t) \frac{S'(\tilde{\vartheta}_0, \tilde{v}) \sigma(\tilde{v}) - S(\tilde{\vartheta}_0, \tilde{v}) \sigma'(\tilde{v})}{\sigma(v)^2} dv \right| \leq C (t-s)^2,
\]
\[
\left| \int_s^t (\tilde{\vartheta}_{v,\varepsilon}^* - \tilde{\vartheta}_0) \frac{S'(\tilde{\vartheta}_v, v)}{\sigma(v)} dv \right| \leq C (t-s) \sup_{s \leq v \leq t} |\tilde{\vartheta}_{v,\varepsilon}^* - \tilde{\vartheta}_0|.
\]

Introduce the set
\[
\mathbb{A}_\varepsilon = \left\{ \omega : \sup_{s \leq v \leq t} |\tilde{\vartheta}_{v,\varepsilon}^* - \tilde{\vartheta}_0| \leq \varepsilon^{1/2 - \nu_*} \right\},
\]
where \( \nu_* > 0 \) will be chosen later. On the set \( \mathbb{A}_\varepsilon \) we can write
\[
\left| \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) \exp \left( \frac{t-s}{\varepsilon} q(\tilde{\vartheta}_0, t) \right) - 1 \right| \leq C \frac{(t-s)^2}{\varepsilon} + C \frac{(t-s)}{\varepsilon} \varepsilon^{1/2 - \nu_*}.
\]

We have the representation
\[
Y_t - \tilde{m}_t = \frac{1}{\varepsilon} \int_{\tau_\varepsilon}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) \left[ a(\tilde{\vartheta}_0, s) \varepsilon + b(\tilde{\vartheta}_{\varepsilon}^*, s) \frac{f (\tilde{\vartheta}_{\varepsilon}^*, s) - f (\tilde{\vartheta}_0, s)}{\sigma(s)} \right] Y_s ds
\]
\[- \int_{\tau_\varepsilon}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) b(\tilde{\vartheta}_{\varepsilon}^*, s) dW_s + \int_{\tau_\varepsilon}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) b(\tilde{\vartheta}_0, s) dV_s.
\]

Denote
\[
B (\vartheta, \tilde{\vartheta}_0, s) = a(\tilde{\vartheta}_0, s) \varepsilon + \frac{b(\vartheta, s) [f (\vartheta, s) - f (\tilde{\vartheta}_0, s)]}{\sigma(s)},
\]
\[
B_* (\vartheta, \tilde{\vartheta}_0, s) = \frac{b(\vartheta, s) [f (\vartheta, s) - f (\tilde{\vartheta}_0, s)]}{\sigma(s)}, \quad \bar{\kappa} = \inf_{\vartheta, s} S (\vartheta, s) \sup_s \frac{\sigma (s)}{\sigma (s)},
\]
and fix some small \( \delta > 0 \). Then we can write the estimate
\[
\frac{1}{\varepsilon} \int_{\tau_\varepsilon}^{t-\delta} \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) |B (\tilde{\vartheta}_{s,\varepsilon}^*, \tilde{\vartheta}_0, s) Y_s| ds \leq \frac{1}{\varepsilon} \int_{\tau_\varepsilon}^{t-\delta} |B (\tilde{\vartheta}_{s,\varepsilon}^*, \tilde{\vartheta}_0, s) Y_s| ds \exp \left( - \frac{\bar{\kappa} \delta}{\varepsilon} \right).
\]

This allows us to write
\[
\frac{1}{\varepsilon} \int_{\tau_\varepsilon}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) B (\tilde{\vartheta}_{s,\varepsilon}^*, \tilde{\vartheta}_0, s) Y_s ds = \frac{1}{\varepsilon} \int_{t-\delta}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) B (\tilde{\vartheta}_{s,\varepsilon}^*, \tilde{\vartheta}_0, s) ds Y_t
\]
\[+ \frac{1}{\varepsilon} \int_{t-\delta}^t \phi_\ast (\tilde{\vartheta}_{\varepsilon}^*, s, t) B (\tilde{\vartheta}_{s,\varepsilon}^*, \tilde{\vartheta}_0, s) [Y_s - Y_t] ds + O \left( e^{-\frac{\bar{\kappa} \delta}{\varepsilon}} \right).
\]
On the set $A_{\xi}$ for the first integral in the RHS according to (41) we have

$$I_{1,\varepsilon} = \frac{1}{\varepsilon} \int_{t - \delta}^{t} \phi_{*} (\vartheta_{*}^{\varepsilon}, s, t) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \, ds$$

$$= \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) \left( 1 + O \left( \frac{\delta^{2}}{\varepsilon} \right) + O \left( \frac{\delta}{\varepsilon^{2} + \nu_{\epsilon}} \right) \right) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \, ds$$

$$= \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \, ds + O \left( \frac{\delta^{2}}{\varepsilon} \right) + O \left( \frac{\delta}{\varepsilon^{2} + \nu_{\epsilon}} \right).$$

Further

$$\frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \, ds$$

$$= \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) B (\vartheta_{0}, \vartheta_{0}, s) \, ds$$

$$+ \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) (\vartheta_{*}^{\varepsilon} - \vartheta_{0}) \hat{B} (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) ds$$

$$= \frac{\hat{B}_{*} (\vartheta_{0}, \vartheta_{0}, t) \sigma (t)}{S (\vartheta_{0}, t)} \eta_{t, \varepsilon} (\vartheta_{0}) \sqrt{\varepsilon} + O (\varepsilon) = \frac{\hat{f} (\vartheta_{0}, t)}{\sqrt{f (\vartheta_{0}, t)}} \eta_{t, \varepsilon} (\vartheta_{0}) \sqrt{\varepsilon} + O (\varepsilon).$$

Recall that $B (\vartheta_{0}, \vartheta_{0}, s) = 0$. The second integral is estimated as follows

$$\frac{1}{\varepsilon} \int_{t - \delta}^{t} \phi_{*} (\vartheta_{*}^{\varepsilon}, s, t) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) [Y_{s} - Y_{t}] \, ds$$

$$= \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \int_{s}^{t} b (\vartheta_{0}, r) dV_{r} \, ds + O (\delta)$$

$$= \frac{1}{\varepsilon} \int_{t - \delta}^{t} \exp \left( - \frac{t - s}{\varepsilon} q_{*} (\vartheta_{0}, t) \right) B (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) b (\vartheta_{0}, t) \frac{V_{t} - V_{s}}{\sqrt{\delta}} ds \sqrt{\delta} + O (\delta)$$

$$= O \left( \sqrt{\varepsilon \delta} \right) + O (\delta).$$

Here we used the relations $B (\vartheta_{0}, \vartheta_{0}, s) = (\vartheta_{*}^{\varepsilon} - \vartheta_{0}) \hat{B} (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) = \hat{B} (\vartheta_{*}^{\varepsilon}, \vartheta_{0}, s) \eta_{s, \varepsilon} (\vartheta_{0}) \sqrt{\varepsilon}$. For the stochastic integral the similar relations are

$$\int_{t - \delta}^{t - \delta} \phi_{*} (\vartheta_{*}^{\varepsilon}, s, t) b (\vartheta_{*}^{\varepsilon}, s, t) dW_{s} = O \left( \varepsilon \frac{\delta^{2}}{\varepsilon^{2}} \right)$$

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We have
\[ \int_{t-\delta}^{t} \phi_\varepsilon (\vartheta_\varepsilon, s, t) \, b (\vartheta_\varepsilon, s) \, dW_s \]
\[ = \int_{t-\delta}^{t} \exp \left( -\frac{t-s}{\varepsilon} q_\varepsilon (\vartheta_0, t) \right) \left( 1 + O \left( \frac{\delta^2}{\varepsilon} \right) + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) \right) b (\vartheta_\varepsilon, s) \, dW_s \]
\[ = \int_{t-\delta}^{t} \exp \left( -\frac{t-s}{\varepsilon} q_\varepsilon (\vartheta_0, t) \right) b (\vartheta_0, s) \, dW_s + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) \]
\[ = \sqrt{\frac{b (\vartheta_0, t) \sigma (t)}{f (\vartheta_0, t)}} \xi_{t,\varepsilon}^{(2)} + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) . \]

For the second stochastic integral we have the similar presentation
\[ \int_{t}^{t} \phi_\varepsilon (\vartheta_\varepsilon, s, t) \, b (\vartheta_\varepsilon, s) \, dV_s \]
\[ = \sqrt{\frac{b (\vartheta_0, t) \sigma (t)}{f (\vartheta_0, t)}} \xi_{t,\varepsilon}^{(1)} \sqrt{\varepsilon} + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) + O \left( \frac{\delta}{\varepsilon^{1-\nu_\varepsilon}} \right) . \]

Finally, if we set \( \delta = \varepsilon^{\frac{7}{6}} \) and \( \nu_\varepsilon = \frac{1}{6} \) then on the set \( A_\varepsilon = A \) we obtain the representation
\[ Y_\varepsilon = \frac{\tilde{\lambda}_\varepsilon - \tilde{\eta}_\varepsilon}{\sqrt{\varepsilon}} = \frac{\tilde{f} (\vartheta_0, t)}{f (\vartheta_0, t)} \eta_\varepsilon (\vartheta_0) Y_\varepsilon + \sqrt{\frac{b (\vartheta_0, t) \sigma (t)}{f (\vartheta_0, t)}} \left( \xi_{t,\varepsilon}^{(2)} - \xi_{t,\varepsilon}^{(1)} \right) + O \left( \varepsilon^{\frac{7}{6}} \right) . \]

Let us estimate the probability \( P_{\vartheta_0} (A_\varepsilon) \). Denote
\[ c_1 = \inf_{d \in \partial \Theta} V^{1-\delta} (d) \]

We have
\[ P_{\vartheta_0} \left( \sup_{t-\delta \leq s \leq t} \frac{|\vartheta_\varepsilon - \vartheta_0|}{\sqrt{\varepsilon}} \geq \varepsilon^{-\frac{1}{12}} \right) \leq P_{\vartheta_0} \left( \frac{1}{\varepsilon^{1-\delta} (\vartheta_\varepsilon)} \sup_{t-\delta \leq s \leq t} \left| \int_{t-\delta}^{t} \tilde{M} (\vartheta_\varepsilon, r) \, dW_r \right| \geq \varepsilon^{-\frac{1}{12}} \right) \]
\[ + P_{\vartheta_0} \left( \frac{|\vartheta_\varepsilon - \vartheta_0|}{\sqrt{\varepsilon^{1-\delta} (\vartheta_\varepsilon)}} \sup_{t-\delta \leq s \leq t} \left| V^{1-\delta} (\vartheta_\varepsilon) - \int_{t-\delta}^{t} \tilde{M} (\vartheta_\varepsilon, r) \tilde{M} (\vartheta_\varepsilon, r) \, dW_r \right| \geq \varepsilon^{-\frac{1}{12}} \right) . \]

To estimate the first probability we use BGD inequality [2]: for any \( p > 0 \)
\[ P_{\vartheta_0} \left( \frac{1}{\varepsilon^{1-\delta} (\vartheta_\varepsilon)} \sup_{t-\delta \leq s \leq t} \left| \int_{t-\delta}^{t} \tilde{M} (\vartheta_\varepsilon, r) \, dW_r \right| \geq \varepsilon^{-\frac{1}{12}} \right) \]
\[ \leq P_{\vartheta_0} \left( \sup_{t-\delta \leq s \leq t} \left| \int_{t-\delta}^{t} \tilde{M} (\vartheta_\varepsilon, r) \, dW_r \right| \geq c_1 \varepsilon^{-\frac{1}{12}} \right) \]
\[ \leq C \varepsilon^{\frac{p}{2}} \left( \int_{t-\delta}^{t} \frac{M (\vartheta_\varepsilon, r)}{\varepsilon^{1-\delta} (\vartheta_\varepsilon)} \, dW_r \right)^p \leq C \varepsilon^p \] (42)
Using the relations (33) and the estimate (21) in [21] we obtain the estimate for the second probability

\[
P_{\vartheta_0} \left( \frac{\vartheta_{r,e} - \vartheta_0}{\sqrt{\varepsilon}} \sup_{t-\delta \leq s \leq t} \left| I_{t_e}^{s} \left( \vartheta_{r,e} \right) - \int_{t_e}^{s} \frac{\tilde{M} \left( \vartheta_{r,e}, r \right) \tilde{M} \left( \vartheta_{r,e}, r \right)}{\varepsilon \sigma (r)^{2}} \, dr \right| \geq \frac{\varepsilon^{-\frac{1}{12}}}{2} \right) 
\]

\[
\leq P_{\vartheta_0} \left( \frac{\vartheta_{r,e} - \vartheta_0}{\sqrt{\varepsilon}} \sup_{t-\delta \leq s \leq t} \left| I_{t_e}^{s} \left( \vartheta_{r,e} \right) - \int_{t_e}^{s} \frac{\tilde{M} \left( \vartheta_{r,e}, r \right) \tilde{M} \left( \vartheta_{r,e}, r \right)}{\varepsilon \sigma (r)^{2}} \, dr \right| \geq c_1 \varepsilon^{-\frac{1}{12}} \right) 
\]

\[
\leq P_{\vartheta_0} \left( \frac{\vartheta_{r,e} - \vartheta_0}{\sqrt{\varepsilon}} \left| \int_{t_e}^{s} Q \left( \vartheta_0, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \, dr \right| \geq c_1 \varepsilon^{-\frac{1}{12}} \right) 
\]

where

\[
Q \left( \vartheta_0, r \right) = \frac{S \left( \vartheta_0, r \right)}{\sigma (r)} \left[ \frac{\tilde{f} \left( \vartheta_0, r \right) + \tilde{b} \left( \vartheta_0, r \right)}{\tilde{f} \left( \vartheta_0, r \right) + b \left( \vartheta_0, r \right)} \right]^{2}.
\]

We have

\[
\sup_{t-\delta \leq s \leq t} \left| \int_{t_e}^{s} Q \left( \vartheta_0, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \, dr \right| 
\]

\[
\leq \sup_{t-\delta \leq s \leq t} \left| \int_{t_e}^{t} Q \left( \vartheta_0, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \, dr \right| + \left| \int_{t_e}^{t} Q \left( \vartheta, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \, dr \right|
\]

\[
\leq \delta \left| Q \left( \vartheta_0, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \right| + \left| \int_{t_e}^{t} Q \left( \vartheta_0, r \right) \left( \xi_{r,e}^{2} - \frac{1}{2} \right) \, dr \right|.
\]

Remind that \( \delta = \delta_e = \varepsilon^{\frac{1}{12}} \). This relations together with (23), (42) and the estimate (21) in [21] allow us to write for any \( p > 0 \)

\[
P_{\vartheta_0} \left( \sup_{t-\delta \leq s \leq t} \frac{\left| \vartheta_{r,e}^{*} - \vartheta_0 \right|}{\sqrt{\varepsilon}} \geq \varepsilon^{-\frac{1}{12}} \right) \leq C \varepsilon^{p}.
\]

\[\square\]

**Remark 4.** The expression for \( E_{\vartheta_0} Y_t^{2} \) can be immediately obtained from the representation

\[
Y_t = y_0 \exp \left( \int_{0}^{t} a \left( \vartheta_0, v \right) \, dv \right) + \int_{0}^{t} \exp \left( \int_{s}^{t} a \left( \vartheta_0, v \right) \, dv \right) b \left( \vartheta_0, s \right) \, dV_s.
\]

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Remark 5. We have the weak convergences

\[
\frac{Y_t - m(\vartheta_0, t)}{\sqrt{\varepsilon}} \Rightarrow \sqrt{\frac{b(\vartheta_0, t)}{f(\vartheta_0, t)}} \left( \xi^{(2)}_t - \xi^{(1)}_t \right)
\]

and

\[
\frac{Y_t - \hat{m}_t}{\sqrt{\varepsilon}} \Rightarrow \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \hat{\eta}(\vartheta_0) Y_t + \sqrt{\frac{b(\vartheta_0, t)}{f(\vartheta_0, t)}} \left( \xi^{(2)}_t - \xi^{(1)}_t \right).
\]

(43) and

(44)

Recall that the Gaussian process \( \hat{\eta}(\vartheta_0), 0 < t \leq T \) and Gaussian r.v.’s \( \xi^{(1)}_t, \xi^{(2)}_t, 0 < t \leq T \) are independent.

The variances of the limit laws are

\[ \gamma_*(\vartheta_0, t) = \frac{b(\vartheta_0, t)}{f(\vartheta_0, t)} \sigma(t) \]

in the first case and

\[ \Gamma_*(\vartheta_0, t) = \frac{b(\vartheta_0, t)}{f(\vartheta_0, t)} \sigma(t) + \frac{\dot{f}(\vartheta_0, t)^2}{f(\vartheta_0, t)^2} E_{\vartheta_0} Y_t^2 = \gamma_*(\vartheta_0, t) + \frac{\dot{f}(\vartheta_0, t)^2}{f(\vartheta_0, t)^2} E_{\vartheta_0} Y_t^2 \]

in the second case.

It is interesting to note that if \( f(\vartheta, t) = f(t) \) and the function \( b(\vartheta, t) \) depends on unknown parameter \( \vartheta \) then the limit representation and the limit variance in both cases coincide. This means that the unobserved component \( Y_t \) is approximated by \( \hat{m}_t \) with the same asymptotic precision as if the value \( \vartheta \) is known.

For the integral linear and quadratic functionals we have the following limits: for any \( t_0 > 0 \)

\[
\int_{t_0}^{T} \frac{Y_t - m(\vartheta_0, t)}{\sqrt{\varepsilon}} dt \to 0,
\]

\[
\int_{t_0}^{T} \left( \frac{Y_t - m(\vartheta_0, t)}{\sqrt{\varepsilon}} \right)^2 dt \to \int_{t_0}^{T} \frac{b(\vartheta_0, t)}{f(\vartheta_0, t)} \sigma(t) dt = \int_{t_0}^{T} \gamma_*(\vartheta_0, t) dt,
\]

and

\[
\int_{t_0}^{T} \frac{Y_t - \hat{m}_t}{\sqrt{\varepsilon}} dt \Rightarrow \int_{t_0}^{T} \frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \hat{\eta}(\vartheta_0) Y_t dt,
\]

\[
\int_{t_0}^{T} \left( \frac{Y_t - \hat{m}_t}{\sqrt{\varepsilon}} \right)^2 dt \Rightarrow \int_{t_0}^{T} \left[ \frac{\dot{f}(\vartheta_0, t)^2}{f(\vartheta_0, t)^2} \hat{\eta}(\vartheta_0)^2 Y_t^2 + \gamma_*(\vartheta_0, t) \right] dt.
\]

The proofs follow from Lemma 4 in [21].
Remark 6. Consider the question of asymptotic optimality of the estimation of the conditional expectation $m(\vartheta, t)$. Remind that $m(\vartheta, t)$ itself is mean squared optimal estimator of $Y_t$.

Let us denote

$$D(\vartheta_0, t) = \frac{\dot{f}(\vartheta_0, t)^2}{f(\vartheta_0, t)^2} \mathbb{E}_{\vartheta_0}[\dot{\eta}_t(\vartheta_0) Y_t] = \frac{\dot{f}(\vartheta_0, t)^2 \mathbb{E}_{\vartheta_0} Y_t^2}{f(\vartheta_0, t)^2 \widetilde{I}_0(\vartheta_0)}.$$  

We have the lower minimax bound: for any $t \in (0, T]$ and any $\vartheta_0 \in \Theta$

$$\lim_{\nu \to \infty} \lim_{\vartheta \to \vartheta_0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \varepsilon^{-1} \mathbb{E}_{\vartheta} |\tilde{m}(t) - m(\vartheta, t)|^2 \geq D(\vartheta_0, t)^2,$$

where $\tilde{m}(t)$ is an arbitrary estimator of $m(\vartheta, t)$. We follow the proof of Theorem 1.9.1 in [12]. We have elementary estimates

$$\sup_{|\vartheta - \vartheta_0| \leq \nu} \mathbb{E}_{\vartheta} |\tilde{m}(t) - m(\vartheta, t)|^2 \geq \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} \mathbb{E}_{\vartheta} |\tilde{m}(t) - m(\vartheta, t)|^2 p(\vartheta) \, d\vartheta \geq \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} \mathbb{E}_{\vartheta} |\tilde{m}(t) - m(\vartheta, t)|^2 p(\vartheta) \, d\vartheta.$$

Here we introduced a continuous positive density $p(\vartheta)$, $\vartheta \in (\vartheta_0 - \delta, \vartheta_0 + \delta)$ and $\tilde{m}(t)$ is Bayesian estimator, which corresponds to this density $p(\cdot)$ and quadratic loss function, i.e.,

$$\tilde{m}(t) = \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} m(\vartheta, t) p(\vartheta) \, d\vartheta = \frac{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} m(\vartheta, t) p(\vartheta) L(\vartheta, X^t) \, d\vartheta}{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} p(\vartheta) L(\vartheta, X^t) \, d\vartheta}.$$

Below we change the variables $\vartheta = \vartheta_0 + \sqrt{\varepsilon} u$ and denote

$$Z_{t, \varepsilon}(u) = \frac{L(\vartheta_0 + \sqrt{\varepsilon} u, X^t)}{L(\vartheta_0, X^t)}, \quad \mathbb{U}_\varepsilon = (-\nu \varepsilon^{-1/2}, \nu \varepsilon^{-1/2}).$$

We can write

$$\frac{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} m(\vartheta, t) p(\vartheta) L(\vartheta, X^t) \, d\vartheta}{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} p(\vartheta) L(\vartheta, X^t) \, d\vartheta} = \frac{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} m(\vartheta, t) p(\vartheta) \frac{L(\vartheta, X^t)}{L(\vartheta_0, X^t)} \, d\vartheta}{\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} p(\vartheta) \frac{L(\vartheta, X^t)}{L(\vartheta_0, X^t)} \, d\vartheta} = \int_{\mathbb{U}_\varepsilon} m(\vartheta_0 + \sqrt{\varepsilon} u, t) p(\vartheta_0 + \sqrt{\varepsilon} u) Z_{t, \varepsilon}(u) \, du \left(\int_{\mathbb{U}_\varepsilon} p(\vartheta_0 + \sqrt{\varepsilon} u) Z_{t, \varepsilon}(u) \, du\right)^{-1} = m(\vartheta_0, t) + \sqrt{\varepsilon} \tilde{m}(\vartheta_0, t) \int_{\mathbb{U}_\varepsilon} p(\vartheta_0) Z_{t, \varepsilon}(u) \, du \frac{1 + o(1)}{\int_{\mathbb{U}_\varepsilon} p(\vartheta_0) Z_{t, \varepsilon}(u) \, du} = m(\vartheta_0, t) + \sqrt{\varepsilon} \frac{\tilde{f}(\vartheta_0, t)}{f(\vartheta_0, t)} m(\vartheta_0, t) \frac{\int_{\mathbb{U}_\varepsilon} u Z_{t, \varepsilon}(u) \, du}{\int_{\mathbb{U}_\varepsilon} Z_{t, \varepsilon}(u) \, du} \frac{1 + o(1)}{1 + o(1)},$$

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where we used the expansion
\[
m(\vartheta_0 + \sqrt{\varepsilon} u, t) = m(\vartheta_0, t) + u \hat{m}(\vartheta_0, t) \sqrt{\varepsilon} (1 + o(1))
\]
\[
= m(\vartheta_0, t) - u \frac{\hat{f}(\vartheta_0, t)}{f(\vartheta_0, t)} m(\vartheta_0, t) \sqrt{\varepsilon} (1 + o(1)).
\]
Therefore
\[
\frac{\hat{m}(t) - m(\vartheta_0, t)}{\sqrt{\varepsilon}} = \frac{\hat{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \int_{U_x} u Z_{t, \varepsilon}(u) \, du \int_{U_x} Z_{t, \varepsilon}(u) \, du (1 + o(1))
\]
\[
= \frac{\hat{f}(\vartheta_0, t)}{f(\vartheta_0, t)} Y_t \int_{U_x} u Z_{t, \varepsilon}(u) \, du (1 + o(1)).
\]

In the work [21], Theorem 1 it was shown that Bayes estimators constructed by observations \(X^T\) are asymptotically normal
\[
\frac{\tilde{\vartheta}_\varepsilon - \vartheta_0}{\sqrt{\varepsilon}} = \int_{U_x} u Z_{T, \varepsilon}(u) \, du \int_{U_x} Z_{T, \varepsilon}(u) \, du (1 + o(1)) \implies \tilde{\eta}_T(\vartheta_0)
\]
and all polynomial moments converge.

Hence
\[
\frac{\hat{m}(t) - m(\vartheta_0, t)}{\sqrt{\varepsilon}} \implies \frac{\hat{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \tilde{\eta}_t(\vartheta_0) Y_t
\]
and we have the convergence (\(\varepsilon \to 0\))
\[
\int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} E_\vartheta \left[ \frac{\hat{m}(t) - m(\vartheta, t)}{\varepsilon} \right]^2 p(\vartheta) \, d\vartheta \to \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} \frac{\hat{f}(\vartheta, t)^2}{f(\vartheta, t)^2} E_\vartheta \left( \tilde{\eta}_t(\vartheta) Y_t \right)^2 p(\vartheta) \, d\vartheta
\]
\[
= \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} D(\vartheta, t)^2 p(\vartheta) \, d\vartheta.
\]

Therefore we obtained the relation
\[
\lim_{\varepsilon \to 0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \varepsilon^{-1} E_\vartheta |\hat{m}(t) - m(\vartheta, t)|^2 \geq \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} D(\vartheta, t)^2 p(\vartheta) \, d\vartheta.
\]
The function \(D(\vartheta, t)^2\) is continuous on \(\vartheta\) and we have the limit
\[
\lim_{\nu \to 0} \int_{\vartheta_0 - \nu}^{\vartheta_0 + \nu} D(\vartheta, t)^2 p(\vartheta) \, d\vartheta = D(\vartheta_0, t)^2.
\]
The bound (45) is proved.
From relations (43),(44) we obtain
\[
\frac{\hat{m}_t - m(\vartheta_0, t)}{\sqrt{\varepsilon}} \implies -\frac{\dot{f}(\vartheta_0, t)}{f(\vartheta_0, t)} \hat{\eta}_t(\vartheta_0) Y_t.
\] (46)

Therefore the variance of the limit law is \(D(\vartheta_0, t)^2\). It can be shown that the convergence (46) is uniform on compacts \(K \subset \Theta\) and we have
\[
\lim_{\nu \to 0} \lim_{\varepsilon \to 0} \sup_{|\vartheta - \vartheta_0| \leq \nu} \varepsilon^{-1} E_\vartheta |\hat{m}_t - m(\vartheta, t)|^2 = D(\vartheta_0, t)^2,
\]
i.e., \(\hat{m}_t\) is asymptotically efficient (minimax) estimator of \(m(\vartheta, t)\).

**Example 3.** Consider the model of observations of Example 2
\[
\begin{align*}
\mathrm{d}X_t &= f(t) Y_t \mathrm{d}t + \varepsilon \sigma(t) \mathrm{d}W_t, \quad X_0 = 0, \quad 0 \leq t \leq T, \\
\mathrm{d}Y_t &= \alpha(t) Y_t \mathrm{d}t + \sqrt{h(t) + \vartheta g(t)} \mathrm{d}V_t, \quad Y_0 = y_0
\end{align*}
\]
where the functions \(f(\cdot), \sigma(\cdot), \alpha(\cdot), h(\cdot), g(\cdot)\) have continuous derivatives on \(t\) and are strictly positive and \(\vartheta \in (\alpha, \beta)\), where \(\alpha > 0\). It is easy to see that the conditions \(A, B, \tilde{C}_1, C_2\) in this case are fulfilled.

The preliminary estimator is:
\[
\hat{\vartheta}_{\tau_\varepsilon, \varepsilon} = \left( \Psi_{\tau_\varepsilon, \varepsilon} - \int_0^{\tau_\varepsilon} f(t)^2 h(t) \mathrm{d}t \right) \left( \int_0^{\tau_\varepsilon} f(t)^2 g(t) \mathrm{d}t \right)^{-1}
\] (47)
where \(\tau_\varepsilon = \varepsilon^{1/12}\), Fisher information
\[
I_{\tau_\varepsilon}(\vartheta) = \int_{\tau_\varepsilon}^t \frac{f(s)^2 g(s)^2}{8 [h(s) + \vartheta g(s)]^{3/2} \sigma(s)} \mathrm{d}s.
\]
The random process \(M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t)\) is solution of the equation
\[
\begin{align*}
\mathrm{d}M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t) &= \frac{f(t) \left[ h(t) + \hat{\vartheta}_{\tau_\varepsilon, \varepsilon} g(t) \right]^{1/2}}{\varepsilon \sigma(t)} \left[ \mathrm{d}X_t - M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t) \mathrm{d}t \right]
\end{align*}
\] (48)
on the interval \([\tau_\varepsilon, T]\).

For the derivative we have the following equation
\[
\begin{align*}
\mathrm{d}M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t) &= -\frac{f(t) \left[ h(t) + \hat{\vartheta}_{\tau_\varepsilon, \varepsilon} g(t) \right]^{1/2}}{\varepsilon \sigma(t)} M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t) \mathrm{d}t \\
&+ \frac{f(t) g(t)}{2 \varepsilon \sigma(t) [h(t) + \hat{\vartheta}_{\tau_\varepsilon, \varepsilon} g(t)]^{1/2}} \left[ \mathrm{d}X_t - M(\hat{\vartheta}_{\tau_\varepsilon, \varepsilon}, t) \mathrm{d}t \right].
\end{align*}
\] (49)
The initial values are defined by the relation like (26). The One-step MLE-process has the same form

\[ \vartheta^*_{t, \varepsilon} = \vartheta_{t, \varepsilon} + \frac{1}{\vartheta_{t, \varepsilon}(s) \varepsilon \sigma(s)} \int_{\tau_{t, \varepsilon}}^{t} M(\tilde{\vartheta}_{t, \varepsilon}, s) dX_s - M(\tilde{\vartheta}_{t, \varepsilon}, s) ds, \quad \tau_{\varepsilon} < t \leq T. \] (50)

The adaptive filter is

\[ d\hat{m}_t = \frac{f(t)[h(t) + \vartheta^*_{t, \varepsilon} g(t)]^{1/2}}{\varepsilon \sigma(t)} [dX_t - \hat{m}_t dt], \quad \tau_{\varepsilon} < t \leq T. \] (51)

For the initial value \( \hat{m}_{\tau_{\varepsilon}} \) we write

\[ \hat{m}_{\tau_{\varepsilon}} = y_0 \phi(\tilde{\vartheta}_{\tau_{\varepsilon}, \tau}) + \phi(\tilde{\vartheta}_{\tau_{\varepsilon}, \tau}) \left( X_{\tau} H(\tilde{\vartheta}_{\tau_{\varepsilon}, \tau}) - \int_{0}^{\tau} X_s H'(\tilde{\vartheta}_{\tau_{\varepsilon}, s}) ds \right). \] (52)

Here

\[ \phi(\vartheta, s) = \exp \left( - \int_{0}^{s} q_{\varepsilon}(\vartheta, v) dv \right), \quad q_{\varepsilon}(\vartheta, t) = -a(\vartheta, t) + \frac{\gamma(\vartheta, t) f(\vartheta, t)^2}{\varepsilon^2 \sigma(t)^2}, \]

\[ H(\vartheta, s) = \phi(\vartheta, s)^{-1} \frac{\gamma(\vartheta, s) f(\vartheta, s)}{\varepsilon^2 \sigma(s)^2}. \]

Recall that \( \hat{m}_t \) is asymptotically efficient estimator of the random process \( m(\vartheta_0, t) \).

9 Discussion

Note that the case of multidimensional parameter \( \vartheta \) can be considered following the same steps as it was done in this work.

The proposed algorithm of adaptive filtration, is numerically essentially simpler, than the algorithm based on the substitution of the MLE \( \hat{\vartheta}_{t, \vartheta} \) \( 0 < t \leq T \) because the calculation of \( \hat{\vartheta}_{t, \vartheta} \) for each \( t \) requires solutions of many equations (24)-(25), which has to be used for solution of the maximum likelihood equations. The solution of Riccati equation is needed for the calculation of the initial values only and for one value of \( \vartheta = \tilde{\vartheta}_{\tau_{\varepsilon}, \tau} \). Another advantage is the recursive structure of all equations. Of course, One-step MLE-process can be presented in a recursive form too.

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