EXCITED RANDOM WALK IN THREE DIMENSIONS HAS POSITIVE SPEED

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1. INTRODUCTION

Excited random walk is a model of a random walk on $\mathbb{Z}^d$ which, whenever it encounters a new vertex it receives a push toward a specific direction, call it the “right”, while when it reaches a vertex it “already knows”, it performs a simple random walk. This model has been suggested in [BW] and had since got lots of attention, see [VZ]. The reason for the interest is that it is situated very naturally between two classical models: random walk in random environment and reinforced random walk. A reinforced random walk is a walk on a graph (say $\mathbb{Z}^2$) that, whenever it passes through an edge, it changes the weight of this edge, usually positively (i.e. the edge has now a greater probability to be chosen when the random walk reachers one of its end points) but possibly also negatively, with the extreme being the “bridge-burning random walk” that can never traverse the same edge twice. The problem appears naturally in brain research in connection with the evolution of neural networks. Reinforced random walk models are notoriously difficult to analyze, and even the question whether the simplest one-reinforced random walk on $\mathbb{Z}^2$ is recurrent or transient is open. See [K90, KR99, PV99, DKL02] for some known results.

Random walk in a random environment is also a model in which the environment is random, but independently of the walk. For example, one may throw a coin at every point of $\mathbb{Z}$ to decide if at this point the walk will have a push to the left or to the right, and then perform random walk on the resulting weighted graph. The independence of the walk from the environment turns out to be a powerful leverage, and many very precise results are known. See e.g. the book [H95].

Excited random walk has, seemingly, all the difficulties of reinforced random walk: the environment depends on the walk, and in a dynamic way. However, it has two significant advantages. The first is the inherent directedness: the drift of excited random walk is always in the same direction, and in particular, it can be coupled with simple random walk so that the excited is always to the right of the simple random walk. The second is the projected simple random walk of lower dimension. Thus, for example, for the excited random walk in three dimensions, its projection on the two directions orthogonal to our “right” is a simple two-dimensional random walk, up to a time change.

Thus, for example, it is clear that the three dimensional excited random walk is transient. Indeed, since a two-dimensional simple random walk visits an order of $n/\log n$ vertices, the three dimensional excited random walk must visit at least $n/\log n$ vertices. This means, roughly, that $R(n) > n/\log n - C\sqrt{n \log \log n}$ ($x_1$ denoting the first, “left-right” coordinate of $x$), and in particular that $R(n)$ drifts to the right and returns to every point only a finite number of times (in the two
dimensional case this argument does not work — see [BW] for a proof of this fact). The purpose of this note is to improve this obvious remark. We shall show that the factor $1/\log n$ is only an artifact of this argument, namely we shall prove

**Theorem 1.** Let $R(n)$ be an $\epsilon$-excited random walk. Then

$$\liminf_{n \to \infty} \frac{R(n)}{n} > 0.$$ 

The corresponding problem in two dimensions remains open. I believe that the lower limit above is in effect a limit. We will not prove it, but our techniques show great independence between different parts of an excited random walk, therefore it stands to reason that it shouldn’t be difficult.

An important element of the proof is a two dimensional result which might be of independent interest — indeed we already have another application for it, [ABK]. It reads

**Theorem 2.** Let $R_1$ and $R_2$ be two independent simple random walks on $\mathbb{Z}^2$ starting from $0$, $R_1$ of length $n$ and $R_2$ of length $m = \exp(\log^{\mu} n)$ for some $\mu \in \left[\frac{1}{2}, 1\right]$. Then

$$\mathbb{P}(\# \{v : \exists t \leq m, R_2(t) = v \text{ but } \forall s \leq n, R_1(s) \neq v\} \leq m^{3/4}) \leq C \exp(-c \log^{2\mu - 1} n).$$  

(note that $\mu$ has the elegant expression $\mu = \log \log m / \log \log n$). In particular, it shows that $R_1$ has at least $m^{3/4}$ “holes” in a $\sqrt{m}$ vicinity of zero — if $R_2$ finds them then they must exist! The theorem is sharp in the following sense: with probability $> c \exp(-C \log^{2\mu - 1} n)$, $R_1$ covers all of $B(0, m)$. The (easy) proof of sharpness will only be sketched below.

### 1.1. Around the proof.

The proof of theorem 1 is only three pages long, but its inductive nature, the number of parameters and their interdependencies make it somewhat opaque. Therefore I feel compelled to make some vague comments in preparation for the actual argument. The basic argument is a block decomposition. This approach has been tried before, but a straightforward attack does not work. If you divide your time span $[0, n]$ into blocks of length $k$ and allow yourself to “lose a factor of $1/\log k$” in each, you are left with the following obstacle: once you have one really bad block (which will happen, if $n$ is large enough), you have difficulties to say anything useful about the next block. And then about the block following it. And so on. Hence $k$ cannot be independent of $n$ — it has to be at least $\log n$ to get something. Thus the basic block approach gives (roughly) $R(n)/n > c/\log \log n$, but not a constant.

The argument here tries to work around this problem by a “restart mechanism”, namely some way to continue after encountering a bad block. This mechanism, roughly, throws away a big chunk in this case, initializes the process from two dimensional considerations, and is then forced to “pay” just a little for the bad block, and of course, they happen very rarely. The “big chunk” above, denote it by $l$, where $n \gg l \gg k$, is simply an intermediate size block. Since multiple layers are needed to get an actual constant, the easiest method to describe the structure is inductive. Thus the reader should probably keep in mind, while reading the proof, that it really describes a multi-layer structure where layer $i + 1$ is used to restart the estimates in the rare events that a block in the $i$th level failed.

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1.2. Excited random walk — notations. Let \( 0 < \epsilon \leq \frac{1}{6} \). An \( \epsilon \)-excited random walk (in three dimensions) is a random sequence \( \{R(n)\}_{n=0}^{\infty} \) of points in \( \mathbb{Z}^3 \) with the distribution defined as follows. \( R(0) = (0,0,0) \). Denote \( R(n) =: (x_1, x_2, x_3) \). Then

1. If \( R(n) = R(m) \) for some \( m < n \) \( (R(n) \) is “visited”), then \( R(n+1) \) is one of the six neighbors of \( R(n) \) in \( \mathbb{Z}^3 \) with probability \( \frac{1}{6} \) each.
2. Otherwise \( (R(n) \) is “new”), the probability is \( \frac{1}{6} + \epsilon \) for \( R(n+1) = (x_1 + 1, x_2, x_3) \) and \( \frac{1}{6} - \epsilon \) for \( R(n+1) = (x_1 - 1, x_2, x_3) \). The other neighbors have probability \( \frac{1}{6} \) each.

In both cases, the random choice is independent of the past, except for the position \( R(n) \) and whether the vertex is visited or new.

If \( \mathcal{W} \subset \mathbb{Z}^3 \) is any set and \( x \in \mathbb{Z}^3 \) is a point, then an \( \epsilon \)-excited random walk starting from \( (x, \mathcal{W}) \) is an \( \epsilon \)-excited random walk such that \( R(0) = x \), and such that if \( R(n) \in \mathcal{W} \) then rule 1 above is applies to it regardless of the past of \( R \), i.e. all vertices in \( \mathcal{W} \) are considered “visited”.

1.3. Standard notations. The notations \( C \) and \( \epsilon \) relate to absolute constants, which may be different from place to place. Sometimes we shall number them for clarity. \( C \) will usually pertain to constants which are “large enough” and \( \epsilon \) to constants which are “small enough”. The notation \( x \approx y \) is a short hand for \( cx \leq y \leq Cx \). The notations \( \ll \) and \( \gg \) have no particular additional mathematical content over \( < \) and \( > \). We only use them to stress that in a specific point the estimate is very rough, and that’s OK because it is enough for our purposes.

For a subset \( A \subset \mathbb{Z}^d \), we denote by \( \partial A \) the inner boundary, namely all vertices \( v \in A \) with at least one neighbor outside \( A \).

For a number \( x \), \( \lfloor x \rfloor \) will denote the largest integer \( \leq x \) and \( \lceil x \rceil \) will denote the smallest integer \( \geq x \).

2. Simple random walk in two dimensions — The \( e^{C \sqrt{\log n}} \) phenomenon.

Lemma 1. Let \( k \in \mathbb{N} \) and let \( r > e^{C \sqrt{k}} \). Let \( x_1, \ldots, x_k \in \partial B(0, r) \) and let \( R_1, \ldots, R_k \) be random walks starting from \( x_i \) and stopped on \( B(0, 2r) \). Then

\[
\mathbb{P}(R_i(t) \neq 0 \forall i = 1, \ldots, k, \forall t) > ce^{-C \sqrt{k}}.
\]

Proof. Let \( a \) be the harmonic potential on \( \mathbb{Z}^2 \) (see e.g. [S76]). Let \( p_i \) be the probability that \( R_i \) hits 0 before exiting \( B(0, 2r) \). Let \( \tau \) be the stopping time when \( R_i \) either hits 0 or \( \partial B(0, 2r) \). Since \( a \) is harmonic outside 0, we get

\[
a(x_i) = \mathbb{E}R_i(\tau) = p_ia(0) + (1 - p_i)\mathbb{E}(R_i(\tau) | R_i(\tau) \in \partial B(0, er)).
\]

Since \( a(x) = \frac{2}{\log 2} \log r + O(1) \) and \( a(x) = \frac{2}{\log 2} \log(2r) + O(1) \) for any \( x \in \partial B(0, er) \) (see [S76, P12.3, page 124]), and since \( a(0) = 0 \) we get

\[
p_i = \frac{\log 2}{\log 2r} + O(\log^{-2} r) = \frac{\log 2}{\log r} + O(\log^{-2} r).
\]
Taking $k$'th power we get
\[ P(R_i(t) \neq 0 \forall i = 1, \ldots, k; \forall t) = \left(1 - \frac{\log 2}{\log r} + O(\log^{-2} r)\right)^k > \left(1 - \frac{\log 2}{\sqrt{k}} + O(k^{-1})\right)^k > ce^{-(\log 2)\sqrt{k}}. \]

Lemma 2. Let $k \in \mathbb{N}$ and let $r > e^{C_1\sqrt{k}}$ for some $C_1$ sufficiently big. Let $x_{i_1}^{in}, \ldots, x_{i_k}^{in} \in \partial B(0, r)$, let $x_{j_1}^{out}, \ldots, x_{j_k}^{out} \in \partial B(0, 2r)$ and let $R_1, \ldots, R_k$ be random walks starting from $x_{i_1}^{in}$ and conditioned to hit $B(0, 2r)$ at $x_{j_1}^{out}$. Then
\[ P(R_i(t) \neq 0 \forall i = 1, \ldots, k; \forall t) > ce^{-\sqrt{k}}. \]

Proof. This follows as lemma 1 when one remembers the following fact: if $R$ is an (unconditioned) random walk starting from $x^{in} \in \partial B(0, r)$ and stopped on $\partial B(0, 2r)$, and if $E$ is any event that depends only on the portions of $R$ inside $B(0, r)$, then
\[ P(E) \approx P(E \mid R \text{ hits } B(0, 2r) \text{ in some } x^{out}). \]

This is well known. see e.g. [BK, lemma A.5] (the result there is for dimension $\geq 3$ but the same proof holds for dimension 2 with minimal changes). The constant $C_1$ comes from the constants implicit in the $\approx$ notation in (4).

Definition. Let $R$ be a random walk with some stopping time $\tau$ and let $B(x, r)$ be some ball. Define stopping times by $\tau_0^{out} = 0$ and
\[ \tau_j^{in} := \min\{t \geq \tau_{j-1}^{out} : R_i(t) \in B(x, r)\} \]
\[ \tau_j^{out} := \min\{t > \tau_j^{in} : R_i(t) \in \partial B(x, 2r)\}. \]

Let $J := \max\{j : \tau_j^{in} < \tau\}$. We call $J$ the number of visits to $B(x, r)$. In many cases we will have $k$ walks $R_i$ with stopping times $\tau_i$. In this case we define $\tau_j^{in/out}$ and $J_i$ in the same manner, and call $\sum J_i$ the total number of visits to $B(x, r)$. Note that it is possible for (some of) the random walk to start inside the ball $B(x, r)$. In this case $\tau_j^{in} = 0$ and this is considered the first visit.

Lemma 3. Let $k \in \mathbb{N}$ and let $r > e^{C_2\sqrt{k}}$ for some $C_2$ sufficiently big. Let $x_{i_1}^{in}, \ldots, x_{i_k}^{in} \in B(0, r)$, $x_{j_1}^{out}, \ldots, x_{j_k}^{out} \in B(0, 2r)$ and let $R_1, \ldots, R_k$ be random walks starting from $x_{i_1}^{in}$ and conditioned to hit $B(0, 2r)$ in $x_{j_1}^{out}$. Then
\[ P(\#\{x \in B(0, r) : R_i(t) \neq x \forall i = 1, \ldots, k; \forall t > r^{7/4}\} > c). \]

Proof. Clearly, we may assume $r$ is sufficiently large (in the sense that $r > C$) and pay only in the constant $c$ in (4); and since the probability is decreasing in $k$ that means we may also assume $k$ is sufficiently large. Let $s = e^{C_3\sqrt{k}}$ for some $C_3 < C_2$ that will be fixed, together with $C_2$, only later (however, the implicit constants $\tau_{\text{min}}$ and $k_{\text{min}}$ are assumed to be fixed after $C_2$ and $C_3$ and may depend on their values). We do need to remark in this stage that $C_2 - C_3$ is also “sufficiently large” i.e. during the proof we will only add restrictions that increase it. Let $y_1, \ldots, y_n \in B(0, r)$ satisfy that $B(y_m, 2s)$ are disjoint and that $B(y_m, 2s) \subset B(0, r) \setminus \{x_{i_1}^{in}, \ldots, x_{i_k}^{in}\}$. Clearly, we may assume $n > c(r/s)^2$. Examine the $i$'th walk (for a while, everything below will depend on this $i$ but we will not repeat this fact every time).
Define stopping times \( \tau_{i,j}^{in/out} = \tau_{i,j}^{in} \) as follows: \( \tau_0 = 0 \) and
\[
\begin{align*}
\tau_j^{in} &:= \min\{t > \tau_{j-1}^{in} : \exists m, R_i(t) \in B(x_m, s)\} \\
\tau_j^{out} &:= \min\{t > \tau_j^{in} : R_i(t) \in \partial B(x_m, 2s)\}.
\end{align*}
\]
(5)

We have left out the \( \exists m \) which might be formally needed in the definition of \( \tau_j^{out} \) since, clearly, the same \( m \) holds for both \( \tau_j^{in} \) and \( \tau_j^{out} \) \( - m \) may only change when \( j \) changes. It is easy to see that for some \( c \) sufficiently small,

\[
P(\tau_j^{out} - \tau_j^{in} < cs^2 | \mathcal{R}(0, \tau_j^{in})) < \frac{1}{4}
\]

(i.e. the estimate holds independently of the past). Notice that this uses (2) to overcome the conditioning over the past. Hence, easily,

\[
P(\tau_j^{out} < cj s^2 < \mathcal{R}(0, \tau_j^{in})) < \frac{1}{4} \quad \forall j
\]

(not necessarily the same \( c \), of course). Let \( \sigma = \sigma_i \) be the stopping time when \( R_i \) exits \( B(0, 2r) \). It is well known that \( \sigma \) is approximately \( r^2 \), with an exponentially decreasing tail, i.e.

\[
P(\sigma > \lambda r^2) < Ce^{-c\lambda}
\]

(7)

(the only difficulty is that \( R \) is conditioned, and (2) doesn’t apply. Again, we refer to a proof of a high dimensional analog result, [BK lemma A.8]).

We now return the notation \( i \). Define \( J_i := \max\{j : \tau_{i,j}^{out} < \sigma_i\} \). The same argument that gave (2) will give, with another sum,

\[
P\left(\sum_i \tau_{i,J_i} < cs^2 \sum_i J_i\right) < \frac{1}{4},
\]

(8)

while a sum over (2) would give

\[
P\left(\sum_i \sigma_i > \lambda kr^2\right) < Ce^{-c\lambda}.
\]

(9)

Picking \( \lambda \) sufficiently large such that this would be \( < \frac{1}{4} \), and combining (5) we get an estimate for \( \sum J_i \):

\[
P\left(\sum J_i \leq C_4 k(r/s^2)^2\right) \geq \frac{1}{2}.
\]

(10)

Denote this event by \( E \). Let \( \Xi \) denote the space of vectors \((\gamma_1, \ldots, \gamma_k, y_{1,1}^{in}, \ldots, y_{1,\gamma_1}^{in}, \ldots, y_{k,\gamma_k}^{in}, y_{1,1}^{out}, \ldots, y_{k,\gamma_k}^{out})\) where \( \gamma_1, \ldots, \gamma_k \) are integers and the \( y_{i,j}^{in/out} \) are points in some \( \partial B(x_m, s) \) and \( \partial B(x_m, 2s) \) respectively. For every \( \xi \in \Xi \) denote by \( \mathcal{E}_\xi \) the event that \( J_i = \gamma_i \) and that \( R_i(\tau_{i,j}^{in/out}) = y_{i,j}^{in/out} \). Since \( \xi \) clearly determines whether \( \mathcal{E} \) happened or not, define \( \Xi' \subset \Xi \) to be the collection of all \( \xi \)'s such that \( \mathcal{E}_\xi \) ensures \( \mathcal{E} \). Let \( X_m \) be the event that \( R_i(t) \neq x_m \) for all \( i \) and all \( t \) and let \( X := \sum 1\{X_m\} \).

If the total number of visits of some ball \( B(x_m, s) \) by the \( R_i \)'s is \( \leq Ck \), we may apply lemma (2) if only \( s > e^{C_1 \sqrt{Ck}} \), where \( C_1 \) is from lemma (2) i.e. if \( C_1 > C \cdot C_1 \), and we get

\[
P(X_m | \mathcal{E}_\xi) > ce^{-\sqrt{\xi}}.
\]

(11)

Notice that \( \sum J_i \) is, in effect, the sum over all balls \( B(x_m, s) \) of the total number of visits of the \( R_i \)'s. Since the number of balls is \( > c(r/s)^2 \), and since \( \mathcal{E} \) says that \( \sum J_i \leq C_4 k(r/s)^2 \), we get that for at least half of the balls the number of visits
Lemma 4. We now choose $C_3$ so that (11) is satisfied for at least half of the balls. Further, conditioning by $E_ξ$ all the balls $B(x_m, 2s)$ are independent and we get using standard estimates for independent variables, for $k$ sufficiently large,

$$\mathbb{P}(X < c(r/s)^2 e^{-\sqrt{k}} | E_ξ) < \frac{1}{2} \quad \forall ξ \in \Xi'. $$

Together with (10) we get

$$\mathbb{P}(X \geq c(r/s)^2 e^{-\sqrt{k}}) \geq \sum_{ξ \in \Xi'} \mathbb{P}(E_ξ) \mathbb{P}(X \geq c(r/s)^2 e^{-\sqrt{k}} | E_ξ) \geq \frac{1}{2} \sum_{ξ \in \Xi'} \mathbb{P}(E_ξ) = \frac{1}{2} \mathbb{P}(E) \geq \frac{1}{4}. $$

Hence we only need to check when $c(r/s)^2 e^{-\sqrt{k}} > r^{7/4}$, but this happens, for $k$ sufficiently large, when $C_2 > 4 + 8C_3$, so we may now choose $C_2$ and the lemma is proved.

**Remark.** Clearly, the same proof yields the stronger estimate

$$\mathbb{P}(|# \{x \in B(0, r): R_i(t) \neq x \forall i = 1, \ldots, k; \forall t \}| > r^{2-ε}) > c(ε).$$

However, we will not need it here.

**Lemma 4.** Let $k \in \mathbb{N}$, let $s \geq 1$ and let $r = se^{\sqrt{k}}$. Let $x_1^m, \ldots, x_k^m \in B(0, r)$ and $x_1^o, \ldots, x_k^o \in \partial B(0, 2r)$ and let $R_1, \ldots, R_k$ be random walks starting from $x_i^m$ and conditioned to hit $B(0, 2r)$ in $x_i^o$. Let $y$ satisfy $B(y, 2s) \subset B(0, r)$ and $|y - x_i^m| \geq \frac{1}{4}r$ for all $x_i^m$. Let $J$ denote the total number of visits of the $R_i$’s to $B(y, s)$. Then

$$\mathbb{P}(J > λk) \leq e^{-(cλ - C)\sqrt{k}} \quad \forall λ > 0.$$

**Proof.** The proof is a simple variation on the classic estimate for sums of independent variables. Let $J_i$ be the number of visits of $R_i$ to $B(y, s)$. The same harmonic potential estimates as in lemma 1 show that for any point in $\partial B(y, 2s)$, the probability to hit $\partial B(0, 2r)$ before hitting $B(y, s)$ is $\geq c/\log(r/s) = c/\sqrt{k}$ (here we use 2). This means that each $J_i$ has an exponential distribution with the tail decreasing faster than $(1 - c/\sqrt{k})^n$, or in other words,

$$\mathbb{E}(e^{c_1 J_i/\sqrt{k}} | J_i > 0) \leq C \quad (12)$$

for some $c_1$ sufficiently small.

We now use the condition $|y - x_i^m| > \frac{1}{4}r$: a second application of the harmonic potential argument shows that $\mathbb{P}(J_i > 0) \leq C/\sqrt{k}$ (again using 2). Plugging this into (12) gives

$$\mathbb{E}(e^{c_1 J_i/\sqrt{k}}) \leq 1 + C/\sqrt{k}$$

and since the various $J_i$’s are independent we get

$$\mathbb{E}(e^{c_1 J/\sqrt{k}}) = \prod_{i=1}^k \mathbb{E}(e^{c_1 J_i/\sqrt{k}}) \leq (1 + C/\sqrt{k})^k < e^{C\sqrt{k}}.$$

Therefore, by Chebyshev’s inequality,

$$\mathbb{P}(J > λk) = \mathbb{P}(e^{c_1 J/\sqrt{k}} > e^{c_1 λ\sqrt{k}}) \leq \frac{\mathbb{E}(e^{c_1 J/\sqrt{k}})}{e^{c_1 λ\sqrt{k}}} < e^{(C - c_1 λ)\sqrt{k}}. \quad \square$$
Remark. The value 1/4 is of course arbitrary — it can be replaced by any \( \mu > 0 \) but the constants \( C \) and \( \epsilon \) from the formulation of the lemma depend on this \( \mu \).

Proof of theorem\(^2\). As usual, we assume \( n \) is sufficiently large, as we may. The constant \( n_{\min} \) will be fixed last, at the very end of the proof. In particular we assume \( n > 1 \) so that we have no problem dividing with \( \log n \). Let \( r = \sqrt{m/\log n} \).

Clearly,

\[
\mathbb{P}(R_2[0, m] \subset B(0, r)) \leq C e^{-cn/r^2} = C e^{-c \log^2 n} \ll C \exp(-c \log^{2n^2} n) \tag{13}
\]

Denote this “bad” event by \( B_1 \).

Examine the number of visits of \( R_1 \) to \( B(0, r) \). Let \( x \in \partial B(0, 2r) \) be some point and let \( S(\tau) \) be a random walk starting from \( x \), and let \( \tau \) be the first time when \( \tau \in B(0, r) \cap \partial B(0, 2n) \). Clearly, if \( \tau(\tau) \in \partial B(0, 2n) \) then \( \tau \geq 2n - 2r > n \) and then the usual harmonic potential argument gives

\[
\mathbb{P}(\tau(S[0, n]) = 0) \geq \mathbb{P}(S(\tau) \in \partial B(0, n)) \geq 2 / \log(2n/r) > 2 / \log n.
\]

Clearly, this implies that if \( R_1(t) \in \partial B(0, 2r) \) then with probability \( > c/\log n \) this is the last visit of \( R_1 \) to \( B(0, r) \). Hence we see that the number of visits \( J \) has an exponentially decreasing tail, and in particular, for any constant \( c_2 > 0 \),

\[
\mathbb{P}(J > c_2 \log^{2\mu} n) \leq C \exp(-c \log^{2\mu} n / \log n) = C \exp(-c \log^{2\mu} n) \tag{14}
\]

where the various \( c \)’s depend on \( c_2 \). We shall fix \( c_2 \) later on. This bad event (denote it by \( B_2 \)) is the one with the largest probability, and the reason that the factor \( \exp(-C \log^{2\mu-1} n) \) appears in \( (1) \). Let

\[
k := \lfloor c_2 \log^{2\mu} n \rfloor.
\]

Define the stopping times \( x_{J_{\text{in/out}}} \) using \( (3) \) for the ball \( B(0, r) \), and from now on we shall examine \( R_1[0, \tau_{k_{\text{in}}}^\text{in/out}] \) instead of \( R_1[0, n] \). Similarly define \( \sigma \) to be the stopping time when \( R_2 \) exits \( B(0, r) \) and replace \( R_2[0, m] \) with \( R_2[0, \sigma] \). More precisely, we shall show that

\[
\mathbb{P}(\# \{ v : \exists t \leq \sigma, R_2(t) = v; \forall s \leq \tau_{k_{\text{in}}}^\text{in/out}, R_1(s) \neq v \} \leq (\sqrt{m/n})^3) \leq C \exp(-c \log^{2\mu-1} n),
\]

which will finish the theorem with \( (13) \) and \( (14) \).

Let therefore \( E_x, \xi = (x_1, \ldots, x_k, \tau_{k_{\text{in}}}^\text{in/out}, \ldots, \tau_{k_{\text{out}}}^\text{in/out}) \), be the event that \( R_1(\tau_{j_{\text{in/out}}}^\text{in/out}) = x_{j_{\text{in/out}}} \) (notice that \( x_{1_{\text{in}}}^\text{in} = 0 \)). Conditioning by \( E_x \) we get \( k \) independent walks, each one conditioned to exit \( B(0, 2r) \) at a given point. We wish to use lemma \( 4 \) with \( s := r \sqrt{2} \). First we note that \( r = \sqrt{m/\log n} \geq c(\epsilon) \exp((1/2 - \epsilon) \log^\mu n) \) for any \( \epsilon > 0 \) so

\[
s > c(\epsilon) \exp\left(\frac{1}{2} - \sqrt{c_2} - \epsilon\right) \log^\mu n \tag{17}
\]

so for \( c_2 \) sufficiently small and \( n \) sufficiently large we get \( s \geq 1 \), and we may apply lemma \( 4 \) in a meaningful way and get, for every \( B(y, s) \) satisfying \( d(y, x^\text{in}_{m}) \geq \frac{r}{4} \),

\[
\mathbb{P}(J_y > \lambda k \mid E_x) \leq e^{-(c\lambda - C)\sqrt{n}} \quad \forall \lambda > 0 \tag{18}
\]

where \( J_y \) is the number of visits to \( y \). Pick \( \lambda \) sufficiently large such that the probability above is \( \leq e^{-3\sqrt{n}} \).

Examine the set

\[
\mathcal{Y} := (B(0, 1/2) \setminus B(0, 1/4)) \cap [4s] \mathbb{Z}^2.
\]
The proof of the theorem will follow from the interactions of $R_1$ and $R_2$ with the balls $B(y, 2s)$, $y \in \mathcal{Y}$ (note that they are disjoint). As in the proof of lemma 3 we want to be able to consider the events inside each $B(y, s)$ as independent. Define therefore stopping times $\rho_{i,j}^{in/out}$ for $i = 1, 2$, similarly to (5), i.e

$$\rho_{i,j}^{in} := \min\{t \geq \rho_{i,j}^{out} : \exists y \in \mathcal{Y} \text{ s.t. } R_i(t) \in B(y, s)\}$$

and define $F_\zeta$, where $\zeta = (J_1, J_2, z_{1,1}, \ldots, z_{1,2}, z_{2,1}, \ldots, z_{2,2})$, to be the event that $R_i(\sigma_{i,j}^{in/out}) = z_{i,j}$ and that $\rho_{i,j}^{in, i+1} > \tau_k$ and $\rho_{i,j}^{out, j+1} > \sigma$, or in other words, that the total number of visits of $R_k$ to the balls $B(y, s)$, $y \in \mathcal{Y}$, is $J_i$. Denote the collection of these $\zeta$'s by $Z$. Examine first $R_1$.

1. **$R_1$ and the balls $B(y_i, 2s)$**.

Since $x^n_1 = 0$ and all other $x^n_i \in \partial B(0, r)$ we get that $|y - x_i| \geq \frac{1}{4}r$ for all $i \leq k$ and $y \in \mathcal{Y}$. Hence, since $\#\mathcal{Y} < C(r/s)^2 = C e^{2\sqrt{r}}$ with (13) and our choice of $\lambda$ we get

$$\mathbb{P}(\exists y \in \mathcal{Y} \text{ s.t. } J_y > \lambda \kappa | E_\zeta) \leq C e^{-3\sqrt{r}} < C e^{-\sqrt{r}} \leq C \exp(-c \log^2 n) \ll C \exp(-c \log^2 n)$$

Denote this event by $B_3$. Let $Z' \subset Z$ be the subset of all $\zeta$'s such that $F_\zeta$ implies $-B_3$ (clearly, if $F_\zeta$ happened then we can calculate the number of visits to every $B(y, s)$ and know whether $B_3$ happened or not). Conditioning by $F_\zeta$, $\zeta \in Z'$ we get that all balls $B(y, s)$ are independent, and we may use lemma 3 for every $B(y, s)$, if only $s > e^{C_2 \sqrt{\lambda}}$. Remembering (15) and (17), and comparing the exponents, we see that this will hold (for $n$ sufficiently large) if only

$$\frac{1}{2} - \sqrt{c_2} - \epsilon > C_2 \sqrt{\lambda c_2}$$

which, again, holds if only $c_2$ is sufficiently small. The conclusion of lemma 3 now reads

$$\mathbb{P}(\# H_y > s^{7/4} | F_\zeta) > c \quad \forall y \in \mathcal{Y}, \forall \zeta \in Z'. \quad (20)$$

where

$$H_y := \{v \in B(y, s) : \forall t < \tau_k^{out}, R_1(t) \neq v\}.$$  

2. **$R_2$ and the balls $B(y_i, 2s)$**.

The last conclusion, (20), says in effect that many balls $B(y, s)$ have “large $R_1$ holes” (the $H_y$'s) in them. Here we shall complement this with proving that $R_2$ passes through many $B(y, s)$'s and at least in one of them, through a sizable part of the $R_1$ hole.

Easily, if $B(z, 4s) \subset B(0, \frac{3}{4}r) \setminus B(0, \frac{1}{4}r)$, and if $S$ if a random walk starting from $z$ and stopped at $\partial B(z, 4s)$, then

$$\mathbb{P} \left( S \cap \bigcup_{y \in \mathcal{Y}} B(y, s) = \emptyset \right) < 1 - c. $$

Hence the probability not to intersect $\bigcup B(y, s)$ has an exponentially decreasing tail, as long as we are still within the annulus. In particular, if $B(z, C_5 \log n) \subset$
$B((0, \frac{1}{4}r) \setminus B(0, \frac{1}{4}r)$ for some $C_5$ sufficiently large, and $S$ is stopped at $\partial B(z, C_5 s \log n)$, then

$$P \left( S \cap \bigcup_{y \in Y} B(y, s) = \emptyset \right) < \frac{1}{n^2}$$

Examine now an annulus $A(a) := B((0, a + (2C_5 \log n + 4)s) \setminus B(0, a)$ where $A(a) \subset B((0, \frac{1}{4}r) \setminus B(0, \frac{1}{4}r)$. We get that with probability $> 1 - \frac{1}{n^2}$, $R_2$ intersects a ball $B(y, s) \subset A(a)$. Taking $a_i = \frac{1}{4}r + i(2C_5 \log n + 4)s$ we get a sequence of $r/s \log n$ disjoint annuli, and then

$$P(\forall y \exists y \text{ s.t. } B(y, s) \subset A(a) \land R_2 \cap B(y, s) \neq \emptyset) > 1 - \frac{r}{n^2 \log n}$$

and in particular, if $Y^*$ is the set of $y$’s such that $R_2$ intersects $B(y, s)$, then

$$P(\#Y^* < cr/s \log n) \ll C \exp(-c \log^{2\mu-1} n). \quad (21)$$

Denote this event by $B_4$. Define $Z'' \subset Z'$ to be the subset of $\zeta$’s that ensure that $B_4$ did not happen.

3. The interaction between $R_1$ and $R_2$. Next examine one $B(y, s), y \in Y^*$. For every $v \in H_y$, the harmonic potential argument shows that

$$P(\exists t \leq \sigma : R_2(t) = v) \geq c/\log s$$

hence, if $h_y := \# \{v \in H_y : \exists t \leq \sigma, R_2(t) = v \}$ then $Eh_y \geq c\#H_y/\log s$, and of course $h_y \leq \#H_y$. This shows that

$$P(h_y > 1/2 Eh_y | F_\zeta) > \frac{c}{\log s} \forall y \in Y^*, \forall \zeta \in Z.$$  

Remembering (20) and the independence of $R_1$ and $R_2$ gives

$$P(h_y > C_3 s^{7/4}/\log s | F_\zeta) > \frac{c}{\log s} \forall y \in Y^*, \forall \zeta \in Z'$$

for some $C_3$ sufficiently small. Remembering the definition of $Z''$ (see below (21)) we get

$$P(\forall y \in Y^*, h_y \geq C_3 s^{7/4}/\log s | F_\zeta) \leq \left(1 - \frac{c}{\log s}\right)^{cr/s \log n} \leq \exp(-c\log^2 n) \ll C \exp(-c \log^{2\mu-1} n) \forall \zeta \in Z''$$

(remember the definition of $k$, (15)). Throwing in (19) and (21) and summing over all $\zeta$ we get

$$P\left(\sum_{y \in Y^*} h_y \geq C_3 s^{7/4}/\log s \right) \leq C \exp(-c \log^{2\mu-1} n).$$

However, this event is what we need in (16)! Indeed, directly from the definitions,

$$\# \{v : \exists t \leq \sigma, R_2(t) = v; \forall s \leq \tau_k^{\text{out}}, R_1(s) \neq v \} \geq \sum_{y \in Y^*} h_y,$$

1This estimate is actually quite bad. The true expected value of $\#Y^*$ is $(r/s)^2/\log(r/s)$, analogous to the fact that a random walk of length $n$ passes through approximately $n/\log n$ distinct points. However, it will do for our needs.
so we need only explain why \( m^{3/4} \leq c_2 s^{7/4} / \log s \). Using (17) we see that this is equivalent to, for \( n \) sufficiently large,
\[
\frac{7}{4} \left( \frac{1}{2} - \sqrt{c_2 - \epsilon} \right) > \frac{3}{4},
\]
which holds for \( c_2 \) sufficiently small. Finally we may fix the value of \( c_2 \), get (16) and hence the theorem.

\( \square \)

**Remarks.**

(1) The only place in the proof the value \( \frac{3}{4} \) appears is in (22). Hence, as in the remark following lemma 3 (and using that remark), the theorem may be strengthened to say
\[
P(\# \{ v : \exists t \leq m, R_2(t) = v \text{ but } \forall s \leq n, R_1(s) \neq v \} \leq m^{1-\epsilon} ) \leq C(\epsilon) \exp(-c(\epsilon) \log^{2\mu-1} n).
\]

(2) As explained in the introduction, the theorem is sharp in the sense that with probability \( > c \exp(-C \log^{2\mu-1} n) \), \( R_1 \) covers all of \( B(0, m) \). Roughly, the proof is as follows: lemma 1 can be reversed to show that the probability to cover any point in \( B(0, \frac{n}{2\mu}) \) is \( > 1 - C \exp(-c\sqrt{k}) \). Similarly, the argument leading to (14) can be reversed to show that the probability to have \( \log^{2\mu} n \) visits to \( B(0, m) \) is \( > c \exp(-C \log^{2\mu-1} n) \), and these two together give the result.

3. EXCITED RANDOM WALK IN THREE DIMENSIONS

The theorem will follow very easily from the following lemma. In effect, the lemma is stronger than the theorem. The reason we need this stronger formulation is its inductive proof.

**Lemma 5.** Let \( n \in \mathbb{N} \). Let \( \mathcal{V} \subset [-\infty, -\lfloor n^{5/8} \rfloor] \times \mathbb{Z}^2 \) be any configuration of visited vertices. Let \( x \in \mathbb{Z}^3 \). Let \( R \) be an \( \epsilon \)-excited random walk starting from \( (x, \mathcal{V}) \) of length \( 2n \). Then
\[
P(R(n) > 0 \text{ and } R(2n) < R(n) + \alpha_n(\epsilon)n) \leq \exp(-c_4(\epsilon) \sqrt{\log n})
\]
where the numbers \( \alpha_n(\epsilon) \) satisfy a recursive condition ensuring that \( \alpha_n(\epsilon) \geq c(\epsilon) > 0 \).

The mystery number \( \frac{5}{8} \) is simply in the middle between the \( \frac{4}{7} \) of theorem 2 and \( \frac{1}{2} \). Since the \( \frac{1}{2} \) of theorem 2 was an arbitrary number \( < 1 \), so is this \( \frac{5}{8} \). For the impatient, the recursive condition on the \( \alpha(n) \) is (30) below where \( k \) is defined in (24) and where \( \lambda \) is some constant. It clearly ensures \( \alpha_n \geq c \).

**Proof.** All the constants during the proof will depend on \( \epsilon \), but we will not repeat this fact and only write \( C \) or \( c \) instead of \( C(\epsilon) \) and \( c(\epsilon) \). The lemma will be proved by induction, so assume the lemma holds for any \( k < n \) (we shall explain how to deal with the case \( n = 1 \), indeed with all sufficiently small \( n \), at the end). Due to this fact we need to pay special attention to the constant \( c_4 \) to ensure that it is indeed a constant and does not increase with \( n \) — hence none of the \( C \) and \( c \) below will depend implicitly on \( c_4 \).

Our first observation is that one can couple (meaning, realizing them on the same probability space) in the obvious way the excited random walk in the interval \([n, 2n]\) to a regular three dimensional random walk \( R' \) such that \( R'(i) \leq
\(R(n+i)\) for \(i \leq n\). For \(R'\) we can use a simple estimate of binomial variables to say that
\[
\mathbb{P}(\exists i \leq n : R'(i)_1 < -n^{5/8}) \leq Ce^{-cn^{1/8}} \ll C \exp(-c\sqrt{\log n}). \tag{23}
\]
Hence the same holds for \(R|n, 2n|\) and we conclude that we do not need to know anything about \(\mathcal{U}\) — with very large probability, \(R|n, 2n|\) does not intersect \(\mathcal{U}\) and we need to investigate only its intersections with \(R[0, n]\).

Let
\[
k := \left\lceil \frac{n}{\exp(\log^{1/4} n)} \right\rceil \tag{24}
\]
and let \(I_i := ]n + ik, n + (i + 1)k[\) for \(i = 0, \ldots, \left\lfloor \frac{n}{k} \right\rfloor - 1\). Let
\[
V_i := \{ v \in \mathbb{Z}^3 : \exists t \in I_i, R(t) = v \text{ and } \forall t \leq n + ik, R(t) \neq v \}.
\]

Theorem 2 allows to estimate \#\(V_i\) since if the projections of the \(V_i\)'s on the second and third coordinates are large then they themselves definitively will be. Denote the projection by \(P, P(R(0, n+ik))\) is a two dimensional random walk with the length \(\leq n + ik < 2n\). \(P(R(I_i))\) is a two dimensional random walk whose length \(m\) is a \((k, \frac{2}{3})\)-binomial variable, and in particular
\[
\mathbb{P}(m < \frac{1}{2}k) \leq Ce^{-ck}\.
\]
Assuming \(m \geq \frac{1}{2}k\) we get \(\mu = \log \log m / \log \log n = 1 - o(1)\) and hence theorem 2 says that, for \(n\) sufficiently large,
\[
\mathbb{P}(\#V_i \leq m^{3/4}) \leq C \exp(-c\log^{2\mu-1} n) \leq C \exp(-c\sqrt{\log n}) \quad \forall i.
\]

Examine the horizontal movement of \(R\) during \(I_i\) for one \(i\). There are \(\leq k\) balanced horizontal moves (meaning that they start from a visited vertex) and the number of unbalanced horizontal moves is a \((\#V_i, \frac{1}{3})\)-binomial distribution. Hence with probability \(1 - Ce^{-ck}\) it is \(\geq \frac{1}{6}\#V_i\) and if \(\#V_i > \frac{1}{2}k^{3/4}\) then we get a positive drift, namely
\[
\mathbb{P}(R(n + (i + 1)k)_1 - R(n + ik)_1 < cck^{3/4}) \leq C \exp(-c\sqrt{\log n}) \quad \forall i \tag{25}
\]
(in this formulation we no longer need to assume that \(n\) is sufficiently large).

Our purpose is to use the lemma inductively for every \(I_i, i > 0\). Let therefore
\[
a_i := \max_{i \leq n+(i-1)k} R(t)_1 + \left\lfloor k^{5/8} \right\rfloor.
\]
We translate by \(-a_i\) and use the lemma for \(k\), which will now read as
\[
\mathbb{P}(R(n+ik)_1 > a_i \text{ and } R(n+(i+1)k)_1 < R(n+ik)_1 + \alpha_k k) \leq \exp(-c_4\sqrt{\log k}). \tag{26}
\]
Denote this event by \(B_i\). The estimate (26) holds for any value of \(R[0, n+(i-1)k]\) and hence we may rewrite it as
\[
\mathbb{P}(B_i | B_0, \ldots, B_{i-2}) \leq \exp(-c_4\sqrt{\log k}).
\]
Hence the sequence \(B_{2k}\) dominates a sequence of random independent variables. Let \(\lambda = \lambda(e)\) be an integer parameter which we shall fix later. The simplest estimate now gives
\[
\mathbb{P}\left(\#\{i : B_{2i} \geq \lambda\}\right) \leq \left[\frac{n}{k}\right]^\lambda e^{-\lambda c_4\sqrt{\log k}} \leq C \exp(-c_4\lambda(1 - o(1))\sqrt{\log n}).
\]
A similar calculation holds for the odd $B_{2i+1}$, and we get
\[ \mathbb{P}(\#\{i : B_i \} \geq 2\lambda + 2) \leq C \exp(-c_4\lambda(1 - o(1))\sqrt{\log n}) \]  \tag{27}
(the +2 appears as follows: the $B_0$ for which our argument doesn’t work as is, and the very last $B_i$ where the interval $I_i$ might be cut off and we don’t want to mess with this problem).

We still need one calculation to overcome the condition $R(n + ik)_1 > a_i$ in (26). Applying the same coupling argument as in the beginning of the lemma shows that
\[ \mathbb{P}(R(n + (i - 1)k)_1 < a_i - 2n^{5/8}) \leq Ce^{-cn^{1/8}} < C \exp(-c\sqrt{\log n}). \]  \tag{28}
gives, for $n$ sufficiently large as to satisfy $2n^{5/8} < cc^{3/4}$
\[ \mathbb{P}(R(0, n + ik)_1 > a_i) > 1 - C \exp(-c\sqrt{\log n}) \text{ for all } i \]  \tag{29}
and (28) holds for any $n$ if only $C$ is sufficiently large as to make it trivial for smaller $n$’s. Summing over all $i$ we get
\[ \mathbb{P}(\exists i : R(n + ik)_1 \leq a_i) \leq \frac{\left\lfloor \frac{n}{k} \right\rfloor}{k} C \exp(-c\sqrt{\log n}) \leq C \exp(-c\sqrt{\log n}). \]  \tag{30}
Assuming that the events in (23), (27) and (29) did not happen we get that $R(2n)_1 - R(n)_1 \geq \alpha_k \left( \frac{n}{k} \right)^\lambda - 2\lambda - 2$ (note that the event of (29) implies, in particular, that for the $2\lambda + 2$ bad $i$’s, we still get that $R(n + k(i + 1)_1 - R(n + ik)_1 > 0$). Define therefore
\[ \alpha_n = \alpha_k \left( 1 - \frac{2\lambda + 2}{n/k} \right) \]  \tag{31}
and get (we now number some constants for clarity in the next part),
\[ \mathbb{P}(R(n)_1 > 0 \text{ and } R(2n)_1 < R(n)_1 + \alpha_n n) \leq \]  \tag{32}
\[ \leq C_6 \exp(-c_5\sqrt{\log n}) + C_7 \exp(-c_4\lambda(1 - o(1))\sqrt{\log n}). \]

We can now fix our parameters. Strangely enough, we start with $\lambda$, and fix it so that $\left( \frac{11}{12} \right)^\lambda < 1/2C_7$. Next we fix $N_1(\epsilon)$ sufficiently large so that for all $n > N_1(\epsilon)$ the $o(1)$ inside the second exponent is $< \frac{1}{\lambda}$, and so that $\frac{\alpha}{\lambda} > 4\lambda + 4$, so that (30) makes sense. Next we need to fix $c_4$. Let $N_2(x)$ be defined by
\[ \exp(-x\sqrt{\log N_2(x)}) = \frac{11}{12}. \]

For $c_4$ sufficiently small, we would have
\[ C_6 \exp(-c_5\sqrt{\log n}) < \frac{1}{2} \exp(-c_4\sqrt{\log n}) \text{ for all } n > N_2(c_4). \]  \tag{32}
Further, for $c_4$ sufficiently small we would have $N_2(c_4) > N_1$. Fix $c_4$ sufficiently small so as to satisfy both conditions. Hence, for all $n > N_2(c_4)$ we get
\[ C_7 \exp(-c_4\lambda(1 - o(1))\sqrt{\log n}) \leq C_7 \exp(-c_4(\lambda - 1)\sqrt{\log n}) \]  \tag{33}
\[ \leq C_7 \exp(-c_4(\lambda - 2)\sqrt{\log N_2(c_4)}) \exp(-c_4\sqrt{\log n}) \]  \tag{33}
and with (32) and (31) we get
\[ \mathbb{P}(R(n)_1 > 0 \text{ and } R(2n)_1 < R(n)_1 + \alpha_n n) \leq \exp(-c_4\sqrt{\log n}) \text{ for all } n > N_2, \]
as required. Finally, for \( n \leq N_2 \), we only need to show that
\[
\Pr(R(2n)_1 < R(n)_1 + \alpha_n n) \leq \exp(-c_4 \sqrt{\log n}) \leq \frac{11}{12}.
\]
However, setting \( \alpha(n) = \frac{1}{2n} \) we get, from the coupling of the excited random walk to the simple three-dimensional random walk, as in the beginning of the lemma, that
\[
\Pr(R(2n)_1 \leq R(n)_1) \leq \frac{5}{6}
\]
(showing this for a three-dimensional simple random walk is a straightforward calculation — but if you really don’t want to do it, replace \( \frac{11}{12} \) everywhere with \( 1 - c \) for some \( c > 0 \)) and we are done.

\[ \square \]

**Corollary.** Let \( R \) be an \( \epsilon \)-excited walk of length \( 2n \) (starting from 0). Then
\[
\Pr(R(2n)_1 \leq c(\epsilon)n) \leq \exp(-c(\epsilon)\sqrt{\log n}).
\]

**Proof.** Translate by \( n + 1 \) so that \( R \) starts from \((n + 1, 0, 0)\). Use lemma 5 with \( \mathcal{W} = \emptyset \) to get
\[
\Pr(R(n)_1 > 0 \text{ and } R(2n)_1 < R(n)_1 + \alpha_n(\epsilon)n) \leq C \exp(-c(\epsilon)\sqrt{\log n}).
\]
Since \( \alpha_n(\epsilon) \geq c(\epsilon) \), and since \( R(n)_1 > 0 \) always, we are done. \[ \square \]

**Proof of theorem** Use the last corollary for \( n = 2^i \) and get a sequence of event with probability \( \leq \exp(-c\sqrt{i}) \) hence by the Borel Cantelli lemma only a finite number of them occur, and we are done. \[ \square \]

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