SOME RESULTS ON SEMISIMPLE SYMMETRIC SPACES AND INVARIANT DIFFERENTIAL OPERATORS

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Abstract: Let $G$ be a connected real semisimple Lie group with finite a center and $\sigma$ be an involutive automorphism of $G$. Suppose that $H$ is a closed subgroup of $G$ with $G^\sigma_e \subset H \subset G^\sigma$, where $G^\sigma$ is the fixed points group of $\sigma$, and $G^\sigma_e$ denotes its identity component. The coset space $X = G/H$ is then a semisimple symmetric space. Our purpose is to construct a compact real analytic manifold $\hat{X}$ in which the semisimple symmetric space $X = G/H$ is realized as an open subset, and that $G$ acts analytically on it. Using the Cartan decomposition $G = KA_0$, we must compactify the vectorial part $A_0$.

In [6], using the action of the Weyl group, we constructed a compact real analytic manifold in which the semisimple symmetric space $G/H$ is realized as an open subset, and that $G$ acts analytically on it. Our construction is a motivation of the Oshima’s construction, and it is similar to those in Shimeno and Sekiguchi for semisimple symmetric spaces. In this note, first we illustrate the construction via the case of $SL(n, IR)/SO_e(1, n-1)$ and then show that the system of invariant differential operators on $X = G/H$ extends analytically on the compactification $\hat{X}$.

Keywords: symmetric spaces, Weyl group, Cartan decomposition, compactification

1 Introduction

Let $G$ be a connected real semisimple Lie group with a finite center, $\sigma$ be an involutive automorphism of $G$ and $X = G/H$ be the corresponding semisimple symmetric space. Here $H$ is a closed subgroup of $G$ with $G^\sigma_e \subset H \subset G^\sigma$, where $G^\sigma$ is the fixed points group of $\sigma$, and $G^\sigma_e$ denotes its identity component.

Denote $\theta$ for the Cartan involution which commutes with $\sigma$ and $K$ the fixed points of $\theta$. Then, $K$ is a $\sigma$-stable maximal compact subgroup of $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$. The involutions of $\mathfrak{g}$ induced by $\sigma$ and $\theta$ are denoted by the same letters, respectively.

Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{a} = \mathfrak{t} \oplus \mathfrak{p}$ is the decompositions of $\mathfrak{g}$ into $+1$ and $-1$ eigenspaces for $\sigma$ and $\theta$, respectively, where $\mathfrak{h}$ (resp. $\mathfrak{t}$) is the Lie algebra of $H$ (resp. $K$). Fix a maximal abelian subspace $a$ in $\mathfrak{p} \cap \mathfrak{q}$ and let $a'$ denote the dual space of $a$. The corresponding analytic subgroup $A$ of $a$ in $G$ is then called the vectorial part of $X$. For a $\alpha \in a'$, put

$$\mathfrak{g}_\alpha = \{ Y \in \mathfrak{g} \mid [H, Y] = \alpha(H)Y, \forall H \in a \}.$$

Then the set $\Sigma = \{ \alpha \in a^* \mid \mathfrak{g}_\alpha \neq \{0\}, \alpha \neq 0 \}$ defines a root system with the inner product induced by the Killing form $<,>$ of $\mathfrak{g}$. Moreover, the Weyl group $W$ of $\Sigma$ is defined with the normalizer $N_K(a)$ of $a$ in $K$, modulo the centralizer $M = Z_K(a)$ of $a$ in $K$. It acts naturally on $a$ and coincides via this action with the reflection group of the root system $\Sigma$.

Let $\Delta = \{ \alpha_1, \ldots, \alpha_l \}$ be a fundamental system of $\Sigma$, where the number $l$ which equals dim $a$ is called the split rank of the symmetric space $X$, and denote $\Sigma^+$ for the corresponding set of all $\alpha$.

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positive roots in $\Sigma$, and $W_{K\cap H}$ for the normalizer $N_{K\cap H}(a)$ of $a$ in $K$ modulo the centralizer $Z_{K\cap H}(a)$ of $a$ in $K$. We see that $W_{K\cap H}$ is a subgroup of $W$. For each element $\omega$ of $W$ we fix a representative $\omega$ in $N_K(a)$ so that $\omega \in N_{K\cap H}(a)$ if $\omega \in W_{K\cap H}$.

In [6], using the action of the Weyl group, we constructed a compact real analytic manifold in which the semisimple symmetric space $G/H$ is realized as an open subset, and that $G$ acts analytically on it. By this way, we first, construct an imbedding of $A_{IR}$ into a compact real analytic manifold $\hat{A}_{IR}$ which is called a compactification of $A_{IR}$. Then, we construct the compact manifold $\tilde{X}$ based on the action of Weyl group on $\hat{A}_{IR}$ and consider the real analytic structure of $\tilde{X}$ induced from the real analytic structure of $\hat{A}_{IR}$. Our construction is a motivation of the Oshima’s construction, and it is similar to those in Shimeno and Sekiguchi for semisimple symmetric spaces.

In this note, first, we recall some notation and results concerning the compactification of semisimple symmetric spaces constructed in [6] and illustrate the construction via the case of $SL(n,IR)/SO_e(1,n-1)$. Then, we show that the system of invariant differential operators on $X = G/H$ extends analytically on the compactification $\tilde{X}$.

## 2 A compact imbedding of symmetric spaces

In this section, we recall some notation and results concerning the compactification of semisimple symmetric spaces constructed in [6].

Denote $g_\mathbb{C}$ for the complexification of $g$ and $G_\mathbb{C}$ for the corresponding analytic group. For simplicity, we assume that $G$ is the real form of the complex Lie group $G_\mathbb{C}$. Let $a_\mathbb{C}$ be the complexification of $a$ and $A_\mathbb{C}$ be the analytic subgroup of $a_\mathbb{C}$ in $G_\mathbb{C}$. For each $a \in A_\mathbb{C}$ and $\alpha \in \Sigma$ we define $a^\alpha = e^{a,\log a} \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$ and consider the subset

$$A_{IR} = \{ a \in A_\mathbb{C} | a^\alpha \in \mathbb{R}, \quad \forall \alpha \in \Sigma \}.$$

Let $(\mathbb{C}^*)^\Sigma$ be the set of complexes $z = (z_\beta)_{\beta \in \Sigma}$, where $z_\beta \in \mathbb{C}^*$, and $\text{CIP}^1$ be the 1-dimensional complex projective space. Consider the map $\varphi : A_\mathbb{C} \to (\mathbb{C}^*)^\Sigma$ defined by $\varphi(a) = (a^\alpha)_{\alpha \in \Sigma}, \forall a \in A_\mathbb{C}$. Then, for every $z = (z_\alpha)_{\alpha \in \Sigma} \in \varphi(A_\mathbb{C})$, we have

$$z_{-\alpha} = (z_\alpha)^{-1}, \forall \alpha \in \Sigma$$

$$z_\alpha = \prod_{\gamma \in \Delta} (z_\gamma)^{k(\alpha,\gamma)}, \forall \alpha \in \Sigma^+, \alpha = \sum_{\gamma \in \Delta} k(\alpha,\gamma) \cdot \gamma. \quad (2.2)$$

Using the natural imbedding of $(\mathbb{C}^*)^\Sigma$ into $(\text{CIP}^1)^\Sigma$, we get an imbedding map of $A_\mathbb{C}$ into $(\text{CIP}^1)^\Sigma$ denoted also by $\varphi$.

Let $M = \{ z \in (\text{IRIP}^1)^\Sigma | z_{-\alpha} = z_{\alpha}^{-1}, \forall \alpha \in \Sigma \}$, where $\text{IRIP}^1$ is the 1-dimensional real projective space. By definition, $M$ is compact. Moreover, the subset

$$U_{\Sigma^+} = \{ m = (m_\alpha, m_{-\alpha}) \in M | m_\alpha \in \mathbb{R}, m_{-\alpha} \in \mathbb{R}^+ \cup \{\infty\}, \forall \alpha \in \Sigma^+ \}$$

is an open subset in $(\text{IRIP}^1)^{\Sigma^+}$, and we get a homeomorphism $\chi_{\Sigma^+} : U_{\Sigma^+} \to \mathbb{R}^{\Sigma^+}$ defined by $\chi_{\Sigma^+}(m) = (m_\alpha)_{\alpha \in \Sigma^+}, \forall m \in U_{\Sigma^+}$. 

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Recall that $W$ acts on $M$ by $(w, z) = z_{w^{-1}a}$, $\forall a \in \Sigma$, $w \in W$, $z \in M$. So, we have $U_{w(z^+)} = w \cdot U_{z^+}$, $\forall w \in W$. By a similar way as in [5, Lemma 1.2], we see that the pair \{ $U_{z^+}, \chi_{z^+}$ \} is a chart on $M$, and $\{ U_{w(z^+)} \chi_{w(z^+)} \}_{w \in W}$ defines an atlas of charts on $M$ such that $M$ becomes a real analytic submanifold.

By definition, $\phi(A_{IR})$ is a subset of $(IR)^2$. Denote $\hat{A}_{IR}$ for the closure of $\phi(A_{IR})$ in $(IR)^2$. It follows from (2.1) and (2.2) that $\hat{A}_{IR}$ is a compact subset in $M$. Now, we define an atlas of charts on $\hat{A}_{IR}$ induced from the atlas on $M$.

Let $u_\Delta$ be the subset of $u_\Sigma^+$ consisting of elements $m = (m_\alpha, m_{-\alpha})$ such that $m_\alpha = \prod_{y \in \Delta} (m_y)^{k(\alpha, y)}$, $\forall \alpha \in \Sigma^+$, $\Delta = \sum_{y \in \Delta} k(\alpha, y) \gamma$. Then, $u_\Delta$ is an open subset in $\hat{A}_{IR}$. It follows that $\chi_{\Sigma^+}(u_\Delta) = \{ x \in \IR^\Sigma^+ | x_\alpha = \prod_{y \in \Delta} (x_y)^{k(\alpha, y)} \}$, and we get a homeomorphism $\chi_\Delta : u_\Delta \rightarrow \IR^\Delta$ defined by $\chi_\Delta(m) = (m_y)_{y \in \Delta}$, $\forall m \in u_\Delta$.

**Theorem 2.1** $\hat{A}_{IR}$ is a compact real analytic manifold that is called a compactification of $A_{IR}$. The set of charts $\{ u_{w(\Delta)} \chi_{w(\Delta)} \}_{w \in W}$ defines an atlas of charts on $\hat{A}_{IR}$ so that the manifold $\hat{A}_{IR}$ is covered by $|W|$-many charts.

Now, consider the subset $\hat{A}_{IR} = \{ \tilde{a} \in \hat{A}_{IR} | (\tilde{a})^a \in [-1, 1] \}$ and recall that the Weyl group $W$ acts on $\hat{A}_{IR}$ as follows $(w, \tilde{a}) = (\tilde{a})_{w^{-1}(\alpha)}$, $\forall w \in W$, $\forall \tilde{a} \in \hat{A}_{IR}$. Then, (see [5, Lemma 2.1]) we have $W \cdot \hat{A}_{IR} = \hat{A}_{IR}$. Moreover, for each element $\tilde{a} \in \hat{A}_{IR}$, we have a unique decomposition $\tilde{a} = a_{fin} \cdot e(\tilde{a})$, where $a_{fin} \in A_{IR}$ and $e(\tilde{a}) \in \hat{A}_{IR}$ such that $e(\tilde{a})^\gamma \in \{ -1, 0, 1, \infty \}$, $\forall \gamma \in \Delta$.

Note that $e(\tilde{a}) \in \{ -1, 0, 1, \infty \} \Delta^+$ and for all $\alpha = \sum_{y \in \Delta} k(\alpha, \gamma) \gamma$, $\gamma \in \Sigma$, we have $e(\tilde{a})^\alpha = \prod_{\gamma \in \Delta} (e(\tilde{a})^\gamma)^{|k(\alpha, \gamma)|}$.

The motivation of the Oshima's definition, $e(\tilde{a})$ is called an extended signature of the element $\tilde{a}$. Now, we define parabolic subalgebras with respect to extended signatures $e(\tilde{a})$, for all $\tilde{a} \in \hat{A}_{IR}$.

First, we consider $\tilde{a}_e \in \hat{A}_{IR}$, and let $F_e = \{ \gamma \ | \ e_\gamma = e(\tilde{a}_e)^\gamma \neq 0 \}$ be a subset of the simple root system $\Delta$ with respect to the extended signature $e = e(\tilde{a}_e)$. Denote $\Sigma_e = (\Sigma_{\gamma \in F_e} IR) \cap \Sigma$ and suppose that $W_e$ is the subgroup of $W$ generated by reflections with respect to $\gamma$ in $F_e$. Let $P_e$ be the parabolic subgroup of $G$ with the corresponding Langlands decomposition $P_e = M_e A_e N_e$ so that $M_e A_e$ is the centralizer of $a$ in $G$, and the Lie algebra $N_e$ of $N_e$ equals $\sum_{\alpha \in \Sigma^+} g_{\alpha}$. Then, (see [11]) we define a parabolic subalgebra $p_e = m_e + a_e + \sum_{\alpha \in \Sigma^+} g_{\alpha}$ of $\mathfrak{g}$ and its Langlands decomposition $p_e = m_e + a_e + n_e$ so that $a_e \subset a_e$.

Let $P_e$ denote the corresponding parabolic subgroup of $p_e$ in $G$. It follows that $P_e = M_e A_e N_e$ is the corresponding Langlands decomposition of $P_e$, and we define a closed subgroup $P(\tilde{e})$ of $\mathfrak{g}$ by $P(\tilde{e}) = (M_e \cap \omega^{-1} H(\omega) A_e N_e)$, where $\omega$ is a representative of $\omega \in W$ in $N_e(a)$ so that $\omega \in N_{K \cap H}(a)$ if $\omega \in W_{K \cap H}$.
In general, for each \( \eta = \eta(\bar{a}) \) with \( \bar{a} = w. \bar{a}_t \in \hat{A}_{\text{IR}} \), we firstly, consider the parabolic subalgebra \( P_\eta \) with respect to the element \( \epsilon = \epsilon(\bar{a}_t) \). Then, we define a parabolic subalgebra \( P_\eta = w. P_e w^{-1} \) based on the action of the Weyl group \( W \) on the parabolic subalgebra \( P_e \) (see [3]).

Now, we define an equivalence relation on the product manifold \( G \times \hat{A}_{\text{IR}} \). Let \( x = (g, \bar{a}) \) be an element of \( G \times \hat{A}_{\text{IR}} \), where \( \bar{a} = w. \bar{a}_t \) and \( t = (t_\gamma)_{\gamma \in \Delta} \). Then, we put \( sgn \ x = \epsilon(\bar{a}_t) = sgn \ t \), which is an element of \( \{-1,0,1\}^\Delta \). Here, \( sgn \ t = (sgn \ t_\gamma)_{\gamma \in \Delta} \) and for an \( s \) in IR, we define \( sgn \ s = 1 \) (resp. \( 0 \)) if \( s > 0 \) (resp. \( s = 0 \) or \( s < 0 \)).

Using \( F_x, \Sigma_x \) and \( W_x \) instead of \( F_{x'}, \Sigma_x' \) and \( W_x' \), respectively, we define a parabolic subalgebra

\[
p_x = m_x + a_x + \sum_{a \in \Sigma_x} g_a + \sum_{a \in \Sigma_x' \setminus \Sigma_x} g_a
\]

of \( g \) and its Langlands decomposition \( p_x = m_x + a_x + n_x \) so that \( a_e \subset a_g \).

Moreover, denote \( P_x \) for the corresponding subgroup of \( p_x \) in \( G \), we get the Langlands decomposition \( P_x = M_x A_x N_x \), and \( P(x) = (M_x \cap \omega^{-1} H \omega) A_x N_x \) is a closed subgroup of \( G \). Let \( \{H_1, H_2, \ldots, H_l\} \) denote the dual basis of \( \Delta = \{\alpha_1, \ldots, \alpha_i\} \), that is, \( H_j \in a \) and \( a(H_j) = \delta_{ij}, \forall i, j = 1, 2, \ldots, l \) and put \( a(x) = \exp(-\sum_{\gamma \in \Sigma_x} \log |t_\gamma| \ H_\gamma) \), where \( H_\gamma \in \{H_1, H_2, \ldots, H_l\} \) with respect to \( \gamma \).

Note that for all elements \( x = (g, \omega \bar{a}_t) \) and \( x' = (g', \omega' \bar{a}_t') \) of \( G \times \hat{A}_{\text{IR}} \) such that \( sgn \ x = sgn \ x' \), we have \( W_x = W_{x'} \).

**Definition 2.2** We say that two elements \( x = (g, \omega \bar{a}_t) \) and \( x' = (g', \omega' \bar{a}_t') \) of \( G \times \hat{A}_{\text{IR}} \) are equivalent if and only if the following conditions hold:

(i) \( sgn \ x = sgn \ x' \)

(ii) \( W_{K\cap H} \omega W_x = W_{K\cap H} \omega' W_x \)

(iii) \( ga(x) P(x) = g'a(x') (M_x \cap \omega^{-1} H \omega) A_x N_x \).

Then, Definition 2.2 really gives an equivalence relation, which we write \( x \sim x' \). The quotient space of \( G \times \hat{A}_{\text{IR}} \) by this equivalence relation becomes a topological space with the quotient topology and denoted by \( \hat{X} \).

Let \( \pi : G \times \hat{A}_{\text{IR}} \to \hat{X} \) be the natural projection. As the action of \( G \) on \( G \times \hat{A}_{\text{IR}} \) is compatible with the equivalence relation, we can define an action of \( G \) on \( \hat{X} \) by

\[
g_1 \pi(g, \bar{a}) = \pi(g_1 g, \bar{a}), \quad \forall g, g_1 \in G, \bar{a} \in A_{\text{IR}}.
\]

2.3

For each \( \epsilon \in \{-1,0,1\}^\Delta \), we put \( \hat{A}_{\text{IR}, \epsilon} = \{\bar{a} \in \hat{A}_{\text{IR}} \mid \bar{a} = \omega \bar{a}_t, \epsilon(\bar{a}_t) = \epsilon\} \) and denote \( X_\epsilon = \pi(G \times \hat{A}_{\text{IR}, \epsilon}) \). Then, we have

**Proposition 2.3** (i) \( \hat{X} \) is a compact connected \( G \)-space, and \( \hat{X} = \cup_{\epsilon \in \{-1,0,1\}^\Delta} X_\epsilon \) gives the orbital decomposition of \( \hat{X} \) for the action of \( G \) on it.

(ii) Each \( X_\epsilon \) is homeomorphic to \( G/P(\epsilon) \). There are just \( 2|\Delta| \) open orbits that are isomorphic to \( G/H \), and the number of compact orbits in \( \hat{X} \) equals that of the elements of the coset \( W_{K\cap H} \setminus W \).
3 The case of $SL(n, \mathbb{IR})/SO_e(1, n - 1)$

Consider the real semi-simple Lie group $G = SL(n, \mathbb{IR})$ and denote $g = sl(n, \mathbb{IR})$ for the corresponding Lie algebra of $G$. Suppose that $\theta$ is the Cartan involution defined by $\theta(X) = (^\top X)^{-1}$, $\forall X \in G$, and $K = SO(n, \mathbb{IR})$ is the maximal compact subgroup in $G$ with respect to $\theta$. Then, $g = \mathfrak{t} \oplus \mathfrak{p}$ is the Cartan decomposition of $g$ with respect to $\theta$, where $\mathfrak{t} = so(n, \mathbb{IR})$ is the Lie algebra of $K$. Moreover, we have that $G/K = SL(n, \mathbb{IR})/SO(n, \mathbb{IR})$ is a Riemannian symmetric space of non-compact type.

Let $\sigma$ be the involution of $G$ defined by

$$\sigma(X) = f\theta(X)j, \forall X \in G, \text{where } j = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. $$

Then, the fixed points group $G^\sigma$ of $\sigma$ is $SO(1, n - 1)$, and $g = \mathfrak{h} \oplus \mathfrak{a}$ is the decomposition of $g$ with respect to $\sigma$, where $\mathfrak{h} = so(1, n - 1)$ is the Lie algebra of the corresponding identity component $H = G^\sigma_e = SO_e(1, n - 1)$.

It follows that $X = G/H = SL(n, \mathbb{IR})/SO_e(1, n - 1)$ is a semisimple symmetric space of rank $n - 1$, and we get a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ defined by

$$\mathfrak{a} = \{ \begin{pmatrix} t_1 & 0 & \cdots & 0 \\ 0 & t_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & t_n \end{pmatrix} \mid t_1 + t_2 + \ldots + t_n = 0 \}. $$

Note that $\mathfrak{a}$ is at the same time maximal abelian in $g$. By definition, the root system $\Sigma$ of $\mathfrak{a}$ in $\mathfrak{g}$ is $\Sigma = \{ e_i - e_j \mid 1 \leq i \neq j \leq n \}$, and the Weyl group $W$ is isomorphic to $S_n$, where $S_n$ is the symmetric group of order $n$. Moreover, the corresponding analytic subgroup in $G$ of $\mathfrak{a}$ defined by

$$A = \{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \mid a_1a_2\ldots a_n = 1, \quad a_i > 0 \} \cong (0, \infty)^{n-1}. $$

Then, we get

$$A_{\mathbb{IR}} = \{ \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix} \mid a_1a_2\ldots a_n = 1 \} \cong (\mathbb{IR^*})^{n-1}. $$

By definition, we have

$$\mathcal{M} = \{ z \in (\mathbb{IR^1}^\Sigma) \mid z_{-\alpha} = z_{\alpha}^{-1}, \forall \alpha \in \Sigma \} = \{ (z_\gamma, z_{-\gamma}) \mid z_\gamma \in \mathbb{IR^1}(\mathbb{IR}), \gamma \in \Sigma^+ \} \cong (\mathbb{IR^1}^\Sigma)^{\Sigma^+}. $$

Moreover, $\mathcal{U}_{\Sigma^+} = \{ m = (m_\alpha, m_{-\alpha}) \in \mathcal{M} \mid m_\alpha \in \mathbb{IR}, \forall \alpha \in \Sigma^+ \} \cong \mathbb{IR^+}^n$, where $|\Sigma^+| = \frac{n(n-1)}{2}$ and the corresponding homeomorphism $\chi_{\Sigma^+}: \mathcal{U}_{\Sigma^+} \to \mathbb{IR^+}^n$ defined by $\chi_{\Sigma^+}(m) = (m_\alpha)_{\alpha \in \Sigma^+}$, $\forall m \in \mathcal{U}_{\Sigma^+}$. 
It follows that the pair \( \{ \mathcal{U}_\Sigma, \chi_\Sigma^+ \} \) is a chart on \( \mathcal{M}_\Sigma \), and \( \{ \mathcal{U}_w(\Sigma^+), \chi_w(\Sigma^+) \}_{w \in \mathcal{W}} \) defines an atlas of charts on \( \mathcal{M} \) such that the manifold \( \mathcal{M} \) is covered by \( n! \)-many charts. By definition, we see that
\[
\mathcal{U}_\Delta = \{ \ m \in \mathcal{M} \mid m_\alpha = \prod_{\gamma \in \Delta} (m_{\gamma})^{k(\alpha, \gamma)}, \ a = \sum_{\gamma \in \Delta} k(\alpha, \gamma), \ \forall \alpha \in \Sigma^+ \}
\]
\[
\chi_\Sigma^+(\mathcal{U}_\Delta) = \{ \ x \in \mathbb{R}^{\Sigma^+} \mid x_\alpha = \prod_{\gamma \in \Delta} (x_{\gamma})^{k(\alpha, \gamma)} \} \cong \mathbb{R}^\Delta \cong \mathbb{R}^{n-1}.
\]
Then, we get a homeomorphism \( \chi_\Delta : \mathcal{U}_\Delta \to \mathbb{R}^{n-1} \) defined by \( \chi_\Delta(m) = (m_{\gamma})_{\gamma \in \Delta} \ \forall m \in \mathcal{U}_\Delta. \)

Hence, \( \mathcal{A}_{\mathbb{R}} \cong \mathbb{R}^{n-1} \cup \{ \infty \} \cong S^{n-1} \) is a compact real analytic manifold, and the set of charts \( \{ \mathcal{U}_w(\Delta), \chi_w(\Delta) \}_{w \in \mathcal{W}} \) defines an atlas of charts on \( \mathcal{A}_{\mathbb{R}} \) so that the manifold \( \mathcal{A}_{\mathbb{R}} \) is covered by \( n! \)-many charts.

It follows from Proposition 2.3 that the compactification \( \hat{\mathcal{X}} \) of the symmetric space \( \mathcal{X} = \text{SL}(n, \mathbb{R})/\text{SO}_e(1, n - 1) \) is a compact connected \( \text{SL}(n, \mathbb{R}) \)-space, and there are just \( 2^{n-1} \) open orbits that are isomorphic to \( \text{SL}(n, \mathbb{R})/\text{SO}_e(1, n - 1) \). Moreover, the number of compact orbits in \( \hat{\mathcal{X}} \) equals \( n \), the number of elements of the coset \( W_{\text{rel}} \setminus W \).

## 4 Invariant differential operators

In this section, we shall show that the system of invariant differential operators on \( \mathcal{X} = G/H \) extends analytically on the compact \( G \)-space \( \hat{\mathcal{X}} \). First, we recall after [9] on the structure of the algebra of invariant differential operators on \( G/H \).

For a real or complex Lie subalgebra \( \mathfrak{u} \) of \( \mathfrak{g}_C \), we denote \( \mathcal{U}(\mathfrak{u}) \) for the universal enveloping algebra of \( \mathfrak{u}' \), where \( \mathfrak{u}' \) is the complex Lie subalgebra of \( \mathfrak{g} \) generated by \( \mathfrak{u} \). Now, we retain the notation in section 2. First, the complex linear extensions of the involution \( \sigma \) and \( \theta \) on \( \mathfrak{g}_C \) are also denoted by the same letters. Let \( \mathfrak{b} \) be a maximal abelian subspace of \( \mathfrak{a} \) containing \( \mathfrak{a} \). Denote \( \Sigma(\mathfrak{b}) \) for the root system for the pair \( (\mathfrak{g}_C, \mathfrak{b}_C) \) and \( \Sigma(\mathfrak{b})^+ \) for the set of positive roots with respect to the compatible orders for \( \Sigma(\mathfrak{b}) \) and \( \Sigma \). Put \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma} \alpha \). Denote \( \mathfrak{n}_C \) for the nilpotent subalgebra of \( \mathfrak{g}_C \) corresponding to \( \Sigma(\mathfrak{b})^+ \) and \( \mathfrak{P} = \sigma(\mathfrak{n}_C) \). From the Iwasawa decomposition \( \mathfrak{g}_C = \mathfrak{P} \oplus \mathfrak{b}_C \oplus \mathfrak{h}_C \) and the Poincaré-Birkhoff-Witt theorem, it follows that
\[
\mathcal{U}(\mathfrak{g}) = \mathfrak{P} \oplus \mathcal{U}(\mathfrak{P}) \oplus \mathcal{U}(\mathfrak{b}_C) \oplus \mathcal{U}(\mathfrak{h}_C).
\]

Let \( \delta \) be the projection of \( \mathcal{U}(\mathfrak{g}) \) to \( \mathcal{U}(\mathfrak{b}) \) with respect to this decomposition and \( \eta \) be the algebra automorphism of \( \mathcal{U}(\mathfrak{b}) \) defined by \( \eta(Y) = Y - \rho(Y) \) for \( Y \in \mathfrak{b} \). Then, the map \( \gamma = \eta \circ \delta \) induces the Harish-Chandra isomorphism:
\[
\gamma : \mathcal{U}(\mathfrak{g})^{\mathfrak{b}}/((\mathcal{U}(\mathfrak{g})^{\mathfrak{b}} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}) \to \mathcal{U}(\mathfrak{b})^{W(\mathfrak{b})},
\]
where \( \mathcal{U}(\mathfrak{g})^{\mathfrak{b}} \) is the centralizer of \( \mathfrak{b} \) in \( \mathcal{U}(\mathfrak{g}) \), and \( \mathcal{U}(\mathfrak{b})^{W(\mathfrak{b})} \) is the set of the elements in \( \mathcal{U}(\mathfrak{b}) \) that are invariant under the Weyl group \( W(\mathfrak{b}) \).

Moreover, by a similar way with the proof of Lemma 2.1 in [9], we have:

**Lemma 4.1** For \( h \in H \), \( \text{Ad}(h) \) acts trivially on the algebra \( \mathcal{U}(\mathfrak{g})^{\mathfrak{b}}/((\mathcal{U}(\mathfrak{g})^{\mathfrak{b}} \cap \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}) \).

**Denote** \( \mathbf{D}(G/H) \) for the algebra of invariant differential operators on \( G/H \) and \( \mathbf{D}(G/G_0) \) for the algebra of invariant differential operators on \( G/G_0 \). Then, we see that \( \mathbf{D}(G/G_0) \) is
naturally isomorphic to the algebra $\mathcal{U}(\mathfrak{g})^b/(\mathcal{U}(\mathfrak{g})^b \cap \mathcal{U}(\mathfrak{g})b)$, and it follows from Lemma 4.1 that $\mathbf{D}(G/H)$ is also isomorphic to this algebra. Hence, we get the algebra isomorphism

$$\overline{\gamma}: \mathbf{D}(G/H) \rightarrow \mathcal{U}(b)^W(b)$$

4.2

by identifying algebras $\mathbf{D}(G/H)$ and $\mathcal{U}(\mathfrak{g})^b/(\mathcal{U}(\mathfrak{g})^b \cap \mathcal{U}(\mathfrak{g})b)$. Moreover, we have the natural projection

$$\pi: \mathcal{U}(\mathfrak{g})^b \rightarrow \mathbf{D}(G/H)$$

which satisfies $\gamma = \overline{\gamma} \circ \pi$.

Now, we will study the $G$-invariant differential operators on the $G$-manifold $\hat{X}$ constructed in Section 2 based on the invariant differential operators on the manifold $X = G/H$.

Indeed, let $\omega_1, \ldots, \omega_s$ be a complete set of representatives of the coset $W_{KH} \setminus W$, where

$$s = [W: W_{KH}]$$

Then, by applying Corollary 7.10 in [10] and using the decomposition

$$\mathcal{U}(\mathfrak{g}) = \pi_C \mathcal{U}(\pi_c \oplus b_c) \oplus \mathcal{U}(b) \oplus \mathcal{U}(\mathfrak{g}) Ad(\omega_j^{-1}) b_c$$

in place of (4.1), we can define an isomorphism

$$\overline{\gamma}_j: \mathbf{D}(G/\omega_j H \omega_j^{-1}) \rightarrow \mathcal{U}(b)^W(b)$$

4.3

in the same way as $\overline{\gamma}$.

For each $\omega \in W$ and $\varepsilon \in \{-1, 1\}$, put $\hat{A}_{IR, \varepsilon} = \{ \tilde{a} \in \hat{A}_{IR} \mid \tilde{a} = \omega. \tilde{a}, \varepsilon(\tilde{a}) = \varepsilon \}$ and $X_\varepsilon^\omega = \pi(G \times \hat{A}_{IR, \varepsilon})$. Then, by Proposition 2.3 in [6], we know that the orbit $X_\varepsilon^\omega = \pi(G \times \hat{A}_{IR, \varepsilon})$ is isomorphic to $G/H$. Moreover, we can define the isomorphism

$$\lambda_\varepsilon^\omega: G/\omega H \omega^{-1} \rightarrow X_\varepsilon^\omega$$

4.4

defined by $\lambda_\varepsilon^\omega(g \omega H \omega^{-1}) = \pi(g, \omega, \tilde{a})$, for all $g \in G$, where $\text{sgn} \ t = \varepsilon$.

As $Ad(\omega^{-1})$ defines an isomorphism of $\mathcal{U}(\mathfrak{g})^b$ onto $\mathcal{U}(\mathfrak{g})^b Ad(\omega^{-1})b$, it induces an isomorphism of $\mathbf{D}(G/H)$ onto $\mathbf{D}(G/\omega H \omega^{-1})$, which is also denoted by $Ad(\omega^{-1})$. Combining this with (4.2) and (4.3), we have (see [9, Lemma 2.4])

$$(\overline{\gamma}_j \circ Ad(\omega^{-1}))(D) = \overline{\gamma}(D),$$

4.5

for any $D \in \mathbf{D}(G/H)$.

Denote $\mathbf{D}(\hat{X})$ for the algebra of $G$-invariant differential operators on the manifold $\hat{X}$ whose coefficients are real analytic functions. Then, we have

**Theorem 4.2** There exists a surjective algebra isomorphism $\lambda: \mathbf{D}(\hat{X}) \rightarrow \mathcal{U}(b)^W(b)$

that is given by $\lambda(D) = \overline{\gamma}_j \circ (\lambda_\varepsilon^\omega)^{-1}(D|X_\varepsilon^\omega)$, which does not depend on the choice of $\varepsilon \in \{-1, 1\}$ and $j = 1, 2, \ldots, s$.

**Proof.** As $X_\varepsilon^\omega$ is open in the connected manifold $\hat{X}$ and $\overline{\gamma}_j$ is an isomorphism, we see that $\lambda$ is surjective. Now, fix an element $u \in \mathcal{U}(b)^W(b)$. Then, the differential operator $D_\varepsilon^j = \lambda_\varepsilon^\omega \circ (\overline{\gamma}_j)^{-1}(u)$ on $X_\varepsilon^\omega$ is $G$-invariant and by (4.5), we have
\[
D_{\varepsilon}^j = \lambda_{\varepsilon}^{\omega_j} \circ (\overline{y})^{-1}(u) = \lambda_{\varepsilon}^{\omega_j} \circ (\overline{y} \circ \overline{y}^{-1})(u) = \lambda_{\varepsilon}^{\omega_j} \circ \text{Ad}(\omega^{-1})(\overline{y}^{-1}(u)).
\]

Combining this with (4.4) and the definition of \( \overline{X} \), we see that \( D_{\varepsilon}^j \) does not depend on \( j \). Then, we define a \( G \)-invariant differential operator \( D_U \) on \( U = \bigcup_{\varepsilon \in \{-1,1\}} X_{\varepsilon}^{\omega} \) by

\[
D_U|X_{\varepsilon}^{\omega} = \lambda_{\varepsilon}^{\omega_j} \circ (\overline{y})^{-1}(u)
\]

for \( \varepsilon \in \{-1,1\} \) and \( j = 1, 2, \ldots, s \).

Since \( U \) is open dense in \( \overline{X} \), in order to get the theorem we have only to show that \( D_U \) has an analytic extension on \( \overline{X} \). From Theorem 3.6 in [6], we get that

\[
\overline{X} = \bigcup_{g \in G, 1 \leq j \leq s} \Omega_{g}^{\omega_j}.
\]

Based on this, we see that \( D_U \) has an analytic extension on \( \overline{X} \) if \( D_U|U \cap \Omega_{g}^{\omega_j} \) has an analytic extension on \( \Omega_{g}^{\omega_j} \). By a similar way with the proof of Theorem 2.5 in [9], we can prove this assertion. Accordingly, the theorem is proved.

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