STABILITY OF SYZYGY BUNDLES ON AN ALGEBRAIC SURFACE

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Introduction

The purpose of this note is to prove the stability of the syzygy bundle associated to any sufficiently positive embedding of an algebraic surface.

Let $X$ be a smooth projective algebraic variety over an algebraically closed field $k$, and let $L$ be a very ample line bundle on $X$. The syzygy (or kernel) bundle $M_L$ associated to $L$ is by definition the kernel of the evaluation map

$$\text{eval}_L : H^0(L) \otimes_k \mathcal{O}_X \rightarrow L.$$

Thus $M_L$ sits in an exact sequence

$$0 \rightarrow M_L \rightarrow H^0(L) \otimes_k \mathcal{O}_X \rightarrow L \rightarrow 0.$$

These vector bundles (and some analogues) arise in a variety of geometric and algebraic problems, ranging from the syzygies of $X$ to questions of tight closure. Consequently there has been considerable interest in trying to establish the stability of $M_L$ in various settings. When $X$ is a smooth curve of genus $g \geq 1$, the situation is well-understood thanks to the work of several authors ([13], [3], [8], [1], [4], [12]): in particular $M_L$ is stable as soon as $\deg L \geq 2g+1$. When $X = \mathbb{P}^n$ and $L = \mathcal{O}_{\mathbb{P}^n}(d)$, the stability of $M_L$ was established by Flenner [9, Cor. 2.2] in characteristic 0 and by Trivedi [14] in characteristic $> 0$ for many $d$. A more general statement, due to Coandă [6], treats the bundles associated to possibly incomplete linear subseries of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Motivated by questions of tight closure, the stability of syzygy bundles on $\mathbb{P}^n$ arising from a somewhat more general construction has been studied by Brenner [2] and by Costa, Marques and Miró-Roig [7], [11]. In dimension 2, Camere [5] recently proved that kernel bundles on $K3$ and abelian surfaces are stable.

We show here that if $L$ is a sufficiently positive divisor on any smooth projective surface $X$, then $M_L$ is stable with respect to a suitable hyperplane section of $X$. Specifically, fix an ample divisor $A$ and an arbitrary divisor $P$ on $X$. Given a large integer $d$, set

$$L_d = dA + P,$$

and write $M_d = M_{L_d}$. Our main result is

**Theorem A.** If $d$ is sufficiently large, then $M_d$ is slope stable with respect to $A$.

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Recall that the conclusion means that if $F \subseteq M_d$ is a subsheaf with $0 < \operatorname{rank}(F) < \operatorname{rank}(M_d)$, then

$$\frac{c_1(F) \cdot A}{\operatorname{rank} F} < \frac{c_1(M_d) \cdot A}{\operatorname{rank} M_d}.$$ 

Since a slope-stable bundle is also Gieseker stable, it follows that $M_d$ is parametrized for $d \gg 0$ by a point on the moduli space of bundles on $X$ with suitable numerical invariants. On the other hand, working over $\mathbb{C}$, Camere [5, Proposition 2] shows that if $H^1(X, \mathcal{O}_X) = 0$, and if the natural map

$$H^0(X, K_X) \otimes H^0(X, L) \rightarrow H^0(X, K_X + L)$$

is surjective for some very ample line bundle $L$, then $M_L$ is rigid. But this surjectivity is automatic if $K_X$ is globally generated and $L$ is sufficiently positive. Hence we deduce

**Corollary B.** Let $X$ be a complex projective surface with vanishing irregularity $q(X) = 0$, and assume that $K_X$ is globally generated. Then $M_d$ corresponds to an isolated point of the moduli space of stable vector bundles on $X$ when $d \gg 0$.

It is natural to suppose that the analogue of Theorem A holds also for varieties of dimension $\geq 3$, but unfortunately our proof does not work in this setting. However if Pic($X) \cong \mathbb{Z}$, then the argument of Coandă [6] goes through with little change to establish:

**Proposition C.** Assume that $\dim X \geq 2$ and that Pic($X) = \mathbb{Z} \cdot [A]$ for some ample divisor $A$. Write $L_d = dA$. Then $M_d = \text{def} M_{L_d}$ is $A$-stable for $d \gg 0$.

As in [5] the strategy for Theorem A is to reduce the question to the stability of syzygy bundles on curves, but we avoid the detailed calculations appearing in that paper. In order to explain the idea, we sketch a quick proof of Camere’s result [5, Theorem 1] that if $L$ is a globally generated ample line bundle on a K3 surface $X$, then $M_L$ is $L$-stable. Supposing to the contrary, let $F \subseteq M_L$ be a saturated destabilizing subsheaf, and fix a general point $x \in X$. Consider now a general curve $C \in |L \otimes m_x|$: we may suppose that $F$ sits as a sub-bundle of $M_L$ along $C$. Restriction to $C$ yields a diagram:

$$(*) \quad \begin{array}{ccc}
F|C & \hookrightarrow & \mathcal{O}_C \\
\downarrow & & \downarrow \\
0 & \longrightarrow & M_L|C \longrightarrow \overline{M_{\Omega_C}} \longrightarrow 0,
\end{array}$$

where $\overline{M_{\Omega_C}}$ is the syzygy bundle on $C$ associated to $\Omega_C = L|C$. But $\overline{M_{\Omega_C}}$ is semi-stable by [13], while

$$\mu(F|C) \geq \mu(M_L|C) > \mu(\overline{M_{\Omega_C}}).$$

It follows that $F|C$ cannot inject into $\overline{M_{\Omega_C}}$, and hence the two sub-bundles $F|C$ and $\mathcal{O}_C$ of $M_L|C$ have a non-trivial intersection, which in turn implies that $\mathcal{O}_C$ is contained in $F|C$. On the other hand, consider the fibres at $x$ of the various bundles in play. The vertical map in $(*)$ corresponds to a fixed subspace $F(x) \subseteq H^0(X, L \otimes m_x)$. So we would be asserting that the equation defining a general curve $C \in |L \otimes m_x|$ lies in this subspace,
and this is certainly not the case. The proof of Theorem A in general proceeds in an analogous manner, the main complication being that we have to deal with a trivial vector bundle of higher rank appearing in the bottom row of (*).

Concerning the organization of the paper, Section 1 is devoted to the proof of Theorem A. Proposition C appears in Section 2, where we also propose some open problems.

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1. PROOF OF MAIN THEOREM

This section is devoted to the proof of Theorem A from the Introduction.

We start by fixing notation and set-up. As in the Introduction, $X$ is a smooth projective surface, and $L_d = dA + P$ where $A$ is an ample divisor, and $P$ is an arbitrary divisor on $X$. For the duration of the argument we fix an integer $b \gg 0$ such that $(bA + P - K_X)$ is very ample, and so that $H^1(X, L_b) = 0$, and put

$$B = L_b = bA + P.$$

Observe that $b$ and $B$ are independent of $d$. We also assume henceforth that $d$ is sufficiently large so that $L_d$ and $L_{d-b}$ are very ample.

Fixing such an integer $d$, assume now that $M_d = M_{L_d}$ is not $A$-stable. Recall that this means that there exists a non-trivial subsheaf

$$F_d \subseteq M_d$$

such that

$$\frac{c_1(F_d) \cdot A}{\text{rank } F_d} \geq \frac{c_1(M_d) \cdot A}{\text{rank } M_d}.$$

Without loss of generality we assume that $F_d \subseteq M_d$ is saturated, and we fix a point $x = x_d \in X$ at which $F_d$ is locally free. Writing $m_x$ for the ideal sheaf of $x$, we suppose finally that $d$ is always large enough so that the natural mapping

$$(1.1) \quad H^0(X, (d - b)A \otimes m_x) \otimes H^0(X, B) \longrightarrow H^0(X, L_d \otimes m_x)$$

is surjective.

The plan is to use the stability of syzygy bundles on curves to show that if $d \gg 0$, then no such $F_d$ can actually exist. To this end, consider a general curve

$$C_d \in |(d - b)A| = |L_d - B|$$

passing through the fixed point $x \in X$. We may assume that $C_d$ is smooth and irreducible, and that $M_d/F_d$ is locally free along $C_d$. Observe also that for any torsion free sheaf $\mathcal{F}$ on $X$ that is locally free along $C_d$, one has

$$\mu_A(\mathcal{F}) = \frac{1}{(d - b)} \cdot \mu(\mathcal{F}|C_d).$$

In particular, if $\mathcal{F}$ is $A$-unstable as a sheaf on $X$, then $\mathcal{F}|C_d$ is unstable on $C_d$. 
We now consider the restriction of $M_d$ and $F_d$ to $C_d$. Writing $\overline{M}_d = M_{L_d}$ for the syzygy bundle on $C_d$ of the restriction $\overline{L}_d = L_d|C_d$, we have an exact sequence

$$0 \longrightarrow H^0(B)_{C_d} \longrightarrow M_d|C_d \longrightarrow \overline{M}_d \longrightarrow 0,$$

where the term on the left is the trivial bundle on $C_d$ with fibre $H^0(X, B)$. We complete this to a diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_d \\
& \downarrow & \downarrow \\
& F_d|C_d & \longrightarrow \overline{N}_d & \longrightarrow 0 \\
& \downarrow & \downarrow \\
& H^0(B)_{C_d} & \longrightarrow M_d|C_d & \longrightarrow \overline{M}_d & \longrightarrow 0 \\
\end{array}
\]

of vector bundles on $C_d$, where $\overline{N}_d$ denotes the image of $F_d|C_d$ in $\overline{M}_d$, and $K_d$ is the kernel of the resulting map $F_d|C_d \longrightarrow \overline{N}_d$.

Observe now that $L_d|C_d \equiv \text{lin} (C_d + B)|C_d \equiv \text{lin} (K_X + C_d + Q)|C_d$ for some very ample divisor $Q$ on $X$. In particular, $\deg(L_d|C_d) > 2g(C_d) + 1$, and hence $\overline{M}_d$ is stable on $C_d$ thanks to [8]. On the other hand, it follows from the bottom row of (1.2) that $\mu(M_d|C_d) > \mu(\overline{M}_d)$, and hence

\[
\mu(F_d|C_d) \geq \mu(M_d|C_d) > \mu(\overline{M}_d).
\]

Therefore $F_d|C_d$ cannot be a subsheaf of $\overline{M}_d$, and hence $K_d \neq 0$.

The following two lemmas constitute the heart of the proof. The first asserts that the destabilizing subsheaf $F_d$ must have large rank.

**Lemma 1.1.** One has

$$\text{rank}(F_d) \geq h^0((L_d - B) \otimes m_x) = h^0(L_d - B) - 1.$$ 

The second lemma shows that if $d$ is sufficiently large, then the vertical inclusion on the left of (1.2) is the identity.

**Lemma 1.2.** If $d \gg 0$, then $K_d = H^0(B)_{C_d}$.

Granting these assertions for now, we give the proof of Theorem A. We need to show that if $d \gg 0$ then the picture introduced above cannot occur. To this end, we consider the fibres at the fixed point $x \in X$ of the vector bundles appearing in the left hand square of (1.2). Recalling that the fibre $M_d(x)$ of $M_d$ at $x$ is canonically identified with $H^0(X, L_d \otimes m_x)$, these take the form

\[
\begin{array}{ccc}
F_d(x) & \longrightarrow & \overline{N}_d \\
& \downarrow & \downarrow \\
H^0(X, B) & \longrightarrow & H^0(X, L_d \otimes m_x). \\
\end{array}
\]
Here the bottom map is the natural inclusion determined by a local equation for \( C_d \in |(L_d - B) \otimes m_x | \). It follows from Lemma 1.2 that \( H^0(X, B) \) maps into the subspace \( F_d(x) \subsetneq H^0(X, L_d \otimes m_x) \). So for the required contradiction, it is enough to show that as \( C_d \) varies over an open subset of \( |(L_d - B) \otimes m_x | \), the images of the corresponding embeddings of \( H^0(X, B) \) span all of \( H^0(X, L_d \otimes m_x) \). But this follows from the surjectivity of the map (1.1).

Proof of Lemma 1.1. We continue to work with the diagram (1.4), and we write \( P_{\text{sub}}(W) \) for the projective space of one-dimensional subspaces of a vector space \( W \). Multiplication of sections gives rise to a finite morphism:

\[
\mu_d : P_{\text{sub}}(H^0(X, B)) \times P_{\text{sub}}(H^0(X, (L_d - B) \otimes m_x)) \longrightarrow P_{\text{sub}}(H^0(X, L_d \otimes m_x)).
\]

Set

\[
Z = \mu_d^{-1}(P_{\text{sub}}(F(x))).
\]

Then

\[
(\ast) \quad \dim P_{\text{sub}}(F(x)) \geq \dim Z
\]

thanks to the finiteness of \( \mu_d \). On the other hand, for general \( C_d \in |(L_d - B) \otimes m_x | \), the image of the corresponding inclusion

\[
H^0(X, B) \subseteq H^0(X, L_d \otimes m_x)
\]

must meet the subspace \( F(x) \subseteq H^0(X, L_d \otimes m_x) \) non-trivially: indeed, this follows from (1.2) and the fact that \( K_d(x) \neq 0 \). But this means that the projection

\[
(\ast\ast) \quad \text{pr}_2 : Z \longrightarrow P_{\text{sub}}(H^0(X, (L_d - B) \otimes m_x)))
\]

is dominant. The Lemma follows upon combining (\ast) and (\ast\ast).

Proof of Lemma 1.2. Since \( M_d/F_d \) is locally free along \( C_d \), it follows from (1.2) that \( K_d \) is a saturated subsheaf of \( H^0(B)_{C_d} \), so it suffices to show that \( \text{rank} K_d = h^0(B) \). The argument is numerical. First, note from (1.2) and the stability of \( M_d \) that

\[
\mu(F_d|C_d) = \frac{\text{deg } K_d + \text{deg } N_d}{\text{rank } F_d} \leq \frac{\text{deg } N_d}{\text{rank } F_d}
\]

\[
= \mu(N_d) \cdot \left( \frac{\text{rank } N_d}{\text{rank } F_d} \right)
\]

\[
< \mu(M_d) \cdot \left( 1 - \frac{\text{rank } K_d}{\text{rank } F_d} \right).
\]

Now \( \text{deg}(M_d|C_d) = \text{deg}(M_d) \), and since

\[
\mu(M_d|C_d) \leq \mu(F_d|C_d),
\]

equation (1.5) yields:

\[
\frac{\text{deg}(M_d|C_d)}{\text{rank } M_d + h^0(B)} < \frac{\text{deg}(M_d|C_d)}{\text{rank } M_d} \cdot \left( 1 - \frac{\text{rank } K_d}{\text{rank } F_d} \right).
\]
Observing that \( \deg(M_d | C_d) < 0 \), this is equivalent to the inequality

\[
\frac{1}{\rank M_d + h^0(B)} > \frac{1}{\rank M_d} \cdot \left(1 - \frac{\rank K_d}{\rank F_d}\right),
\]

i.e.:

\[
\frac{\rank M_d}{\rank M_d + h^0(B)} > \left(1 - \frac{\rank K_d}{\rank F_d}\right).
\]

Thus

\[
\frac{\rank K_d}{\rank F_d} > 1 - \frac{\rank M_d}{\rank M_d + h^0(B)} = \frac{h^0(B)}{\rank M_d},
\]

and hence

\[
(*) \quad \frac{\rank K_d}{\rank F_d} > \frac{h^0(B)}{\rank M_d} \cdot \frac{\rank F_d}{\rank M_d}.
\]

But by the previous lemma, \( \rank F_d \geq h^0(X, L_d - B) - 1 \). Furthermore, \( B \) is independent of \( d \), and \( \rank M_d = h^0(X, L_d) - 1 \). Thus as \( d \) grows, the fraction on the right in (*) becomes arbitrarily close to \( 1 \). It follows that

\[
\frac{\rank K_d}{\rank F_d} > h^0(B) - 1
\]

provided that \( d \gg 0 \), and hence \( \rank K_d = h^0(B) \), as required.

\[\square\]

2. Complements

In this section we first of all prove Proposition C by adapting the method of proof of Theorem 1.1 in \cite{6}. Then we propose some open problems.

2.1. Coandă’s Argument. We begin by stating (without proof) two preliminary results on which the method rests; the first of these is a cohomological characterization of stability, and the second is a vanishing theorem of Green.

**Lemma 2.1.** Let \( E \) be a vector bundle on \( X \). If for every \( r \) with \( 0 < r < \rk(E) \) and for every line bundle \( N \) on \( X \) with \( \mu_L(N^r E \otimes N) \leq 0 \) one has \( H^0(N^r E \otimes N) = 0 \), then \( E \) is \( L \)-stable. \[\square\]

**Lemma 2.2.** (\cite{10} 3.a.1) Let \( N, N' \) be line bundles on \( X \) and assume \( N \) is very ample. Then for \( r \geq h^0(N') \), we have \( H^0(N^r M_N \otimes N') = 0 \). \[\square\]

**Proof of Proposition C** Let \( X \) be a smooth projective variety of dimension \( n \geq 2 \) for which \( \Pic(X) \cong Z \cdot [A] \) for an ample line bundle \( A \). Consider the function \( q : \mathbb{N} \to \mathbb{Q} \) defined by \( q(t) = \frac{h^0(tA) - 1}{t} \). Since \( q(t) = \frac{A^n}{n!} t^{n-1} + O(t^{n-2}) \) for \( t \gg 0 \) by asymptotic Riemann-Roch, there exists a positive integer \( d_0 \) satisfying the following properties:

(1) For all integers \( a \) satisfying \( 1 \leq a \leq d_0 - 1 \), we have \( q(a) < q(d_0) \).

(2) For all integers \( d \geq d_0 \), we have that \( dA \) is very ample and \( q(d) < q(d + 1) \).

\[\text{In fact, } (h^0(L_d) - h^0(L_d - B)) = O(d), \text{ whereas } h^0(L_d) \text{ grows quadratically in } d.\]
An immediate consequence is that \( q(a) < q(d) \) whenever \( d \geq d_0 \) and \( 1 \leq a \leq d - 1 \). For the rest of the proof we fix an integer \( d \geq d_0 \).

Recalling that \( \text{Pic}(X) = \mathbb{Z} \cdot [A] \) by assumption, it suffices by Lemmas \( 2.1 \) and \( 2.2 \) to show that given integers \( a \) and \( 0 < r < h^0(dA) - 1 \), one has the implication

\[
\mu_A(\Lambda^r M_d \otimes O_X(aA)) \leq 0 \iff r \geq h^0(aA),
\]

where as before \( M_d = M_{dA} \). This is automatic for \( a \leq 0 \), so we assume \( a \geq 1 \) throughout.

We have that

\[
\mu_A(\Lambda^r M_d \otimes O_X(aA)) = r \cdot \mu_A(M_d) + a \cdot (A^n) = (A^n) \cdot \left( a - \frac{dr}{h^0(dA) - 1} \right)
\]

Our assumption that \( \mu_A(\Lambda^r M_d \otimes O_X(aA)) \leq 0 \) then implies that \( a \leq \frac{dr}{h^0(dA) - 1} \), or

\[
r \geq a \cdot \left( \frac{h^0(dA) - 1}{d} \right).
\]

In particular, \( a < d \), so \( 1 \leq a \leq d - 1 \). We will be done once we verify that

\[
a \cdot \left( \frac{h^0(dA) - 1}{d} \right) > h^0(aA) - 1.
\]

for \( 1 \leq a \leq d - 1 \). But \( 2.4 \) is equivalent to \( q(a) < q(d) \), so this follows from our assumption on \( d \).

\[
\square
\]

**Remark 2.3** (Rigidity of \( M_L \)). Let \( L \) be a very ample line bundle on a smooth complex projective variety \( X \) of dimension \( \geq 3 \) with \( H^1(X, O_X) = 0 \). Then arguing as in the proof of [5, Proposition 1], one sees that \( M_L \) is rigid, i.e. \( \text{Ext}^1(M_L, M_L) = 0 \). Consequently, in the situation of Proposition \( \square \) \( M_d \) again represents an isolated point of the moduli space of bundles when \( \text{dim}_K X \geq 3 \) and \( d \gg 0 \).

**2.2. Some Open Problems.** Recall that if \( X \) is a smooth curve of genus \( g \), then \( M_L \) is stable as soon as \( \text{deg} L \geq 2g + 1 \). This suggests

**Problem 2.4.** Find an effective version of Theorem A.

Presumably one would want to work with divisors of the sort \( L = K_X + B + N \) with \( B \) satisfying a suitable positivity hypothesis, and \( N \) nef.

It is also interesting to ask whether \( M_d \) satisfies some stronger stability properties:

**Problem 2.5.** As before, let \( L_d = dA + P \), and put \( M_d = M_{dA} \). Is \( M_d \) slope stable with respect to any polarization on \( X \) when \( d \gg 0 \)? In characteristic \( p > 0 \), is it strongly stable?

Finally, we conjecture that our main result extends to all dimensions.

**Conjecture 2.6.** Let \( X \) be a smooth projective variety of dimension \( n \), and define \( M_d \) as above. Then \( M_d \) is \( A \)-stable for every \( d \gg 0 \).
References

[1] A. Beauville, *Some stable vector bundles with reducible theta divisor*, Manuscripta Math. **110** (2003), p. 343-349

[2] H. Brenner, *Looking out for stable syzygy bundles*, with an appendix by Georg Hein, Adv. Math. **219** (2008), no. 2, p. 401-427

[3] D. Butler, *Normal generation of vector bundles over a curve*, J. Differential Geom. **39** (1994), p. 1-34

[4] C. Camere, *About the stability of the tangent bundle of \( \mathbb{P}^n \) restricted to a curve*, C. R. Acad. Sci. Paris, Ser. I **346** (2008), p. 421-426

[5] C. Camere, *About the stability of the tangent bundle of \( \mathbb{P}^n \) restricted to a surface*, Math. Zeit. **271** (2012), no. 1-2, p. 499-507

[6] I. Coandă, *On the stability of syzygy bundles*, Internat. J. Math. **22** (2011), no. 4, p. 515-534.

[7] L. Costa, P. Macias Marques and R. M. Miró-Roig, *Stability and unobstructedness of syzygy bundles*, J. Pure Appl. Algebra **214** (2010), no. 7, p. 1241-1262

[8] L. Ein and R. Lazarsfeld, *Stability and restrictions of Picard bundles, with an application to the normal bundles of elliptic curves*, in *Complex Projective Geometry*, London Math. Soc. Lecture Note Ser., **179** (1992), p. 149-156

[9] H. Flenner, *Restrictions of semistable bundles on projective varieties*, Comment. Math. Helv. **59** (1984), p. 635-650

[10] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Diff. Geom. **19** (1984), p. 125-171.

[11] P. Macias Marques and R. M. Miró-Roig, *Stability of syzygy bundles*, Proc. Amer. Math. Soc. **139** (2011), p. 3155-3170

[12] E. Mistretta, *Stability of line bundle transforms on curves with respect to low codimensional subspaces*, J. London Math. Soc. (2) **78** (2008), p.172-182

[13] K. Paranjape and S. Ramanan, *On the canonical ring of a curve*, in Algebraic geometry and commutative algebra, vol. II, Kinokuniya, Tokyo (1988) p. 503-516

[14] V. Trivedi, *Semistability of syzygy bundles on projective spaces in positive characteristics*, Internat. J. Math. **21** (2010), no. 11, p. 1475-1504

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