ON THE STABLE CENTER CONJECTURE

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Abstract. Let $G$ be a connected reductive group over a local non-archimedean field $F$. The stable center conjecture provides an intrinsic decomposition of the set of smooth irreducible representations of $G(F)$, which is only slightly coarser than the conjectural decomposition into $L$-packets. In this note we propose a way to verify this conjecture for depth zero representations.

To Gérard Laumon on his 60th birthday

Contents

1. Introduction 1
2. Geometric projector to the depth zero spectrum 5
3. Application to the classical (stable) Bernstein center 10
4. The case of the unit element 12
5. Geometric construction and stability of $z_0$ 17
6. Proof of Conjecture 5.7 for $G = SL_2$. 20
References 25

1. Introduction

1.1. Notation. (a) Let $F$ be a local non-archimedean field, let $W_F$ be the Weil group of $F$, and let $W'_F$ be the Weil-Deligne group.

(b) Let $G$ be a connected reductive group over $F$, which we for simplicity always assume to be split, and let $\hat{G} = {}^L G^0$ be the connected Langlands dual group (over $\mathbb{C}$).

(c) Let $R(G)$ be the category of smooth representations of $G(F)$, and let $\text{Irr}(G)$ be the set of equivalence classes of irreducible objects in $R(G)$.

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1.2. **The local Langlands conjecture.**

(a) Recall that the local Langlands conjecture asserts the existence of a decomposition $\operatorname{Irr}(G) = \biguplus \lambda \Pi_\lambda$, where $\lambda$ runs over a set of $\hat{G}$-conjugacy classes of Frobenius semi-simple Langlands parameters $W'_F \to \hat{G}$, and $\Pi_\lambda$ is a finite set, called the $L$-packet, corresponding to $\lambda$.

(b) The local Langlands conjecture predicts the existence of a coarser decomposition $\operatorname{Irr}(G) = \biguplus \lambda \tilde{\Pi}_\lambda$, where $\lambda$ runs over a set of $\hat{G}$-conjugacy classes of Frobenius semi-simple continuous homomorphisms $W'_F \to \hat{G}$, and $\tilde{\Pi}_\lambda := \bigcup \lambda |_{W'_F = \lambda} \Pi_\lambda$ is a finite union of $L$-packets, which we will call an extended $L$-packet.

1.3. **The Bernstein center.**

(a) Let $Z_G = Z(R(G))$ be the Bernstein center of $G(F)$, that is, the algebra of endomorphisms of the identity functor $\text{Id}_{R(G)}$ (see [Ber]). It is well-known that $Z_G$ is a commutative algebra.

(b) By the Schur lemma, each $z \in Z_G$ defines a function $f_z : \operatorname{Irr}(G) \to \mathbb{C}$ such that $z[\pi] = f_z(\pi) \cdot \text{Id}_\pi$ for all $\pi \in \operatorname{Irr}(G)$. Moreover, Bernstein showed that the map $z \mapsto f_z$ is injective.

(c) By the definition, each $z \in Z_G$ defines an endomorphism $z_{\text{reg}}$ of the regular representation $C_c^\infty(G(F))$ of $G(F) \times G(F)$, hence an invariant distribution $\mu_z$ on $G(F)$ such that $\mu_z(\phi) = z_{\text{reg}}(\phi)(1)$ for all $\phi \in C_c^\infty(G(F))$.

(d) Similarly, each $z \in Z_G$ defines an endomorphism $z_{\mathcal{H}}$ of the Hecke algebra $\mathcal{H}(G(F))$. For every smooth representation $(\pi, V)$ of $G$, $v \in V$ and $\nu \in \mathcal{H}(G(F))$, we have an equality $z(\nu(v)) = (z_{\mathcal{H}}(\nu))(v)$.

1.4. **The stable Bernstein center.** We define the stable center of $G$ to be the linear subspace $Z^s_G \subset Z_G$ of $z \in Z_G$ such that the invariant distribution $\mu_z$ on $G(F)$ is stable.

1.5. **The stable center conjecture.**

(a) The subspace $Z^s_G \subset Z_G$ is a subalgebra.

(b) For every algebra homomorphism $\chi : Z^s_G \to \mathbb{C}$, the set of $\pi \in \operatorname{Irr}(G)$ such that $z(\pi) = \chi(z)$ for all $z \in Z^s_G$ is an extended $L$-packet (see [1.2(b)]).

**Remark.** The Lie algebra analogue of statement (a) follows from the result of Waldspurger [W] which says that the space of stable distribution on the Lie algebra of $G$ is invariant under Fourier transform.

1.6. **Decomposition of $R(G)$.** (a) Denote by $R(G)_0 \subset R(G)$ (resp. $R(G)_{>0} \subset R(G)$) the subcategory of representations $\pi \in R(G)$, all of whose irreducible subquotients have depth zero (resp. positive depth) (see [MP1, MP2]). Set $Z^0_G := Z(R(G)_0)$ and $Z^s_G := Z(R(G)_{>0})$. 
(b) It follows from results of Bernstein [Ber] and Moy-Prasad [MP1 MP2] that
the category $R(G)$ decomposes as a direct sum $R(G) = R(G)_0 \oplus R(G)_{>0}$. Therefore
the Bernstein center $Z_G$ decomposes as a direct sum $Z_G = Z_G^0 \oplus Z_G^{>0}$.

Explicitly, $Z_G^0 \subset Z_G$ consists of all $z \in Z_G$ such that $z|_\pi = 0$ for all $\pi \in R(G)$ of
positive depth, hence the unit $z^0 \in Z_G^0$ is the projector to the depth zero spectrum.
We also set $Z_G^{st,0} := Z_G^0 \cap Z_G^{st}$.

(c) It follows from results of Bernstein [Ber], Moy-Prasad [MP1, MP2] and Deligne–
Lusztig [DL] that the category $R(G)_0$ further decomposes as a direct sum $R(G)_0 = \bigoplus_\theta R(G)_{0,\theta}$, indexed by the set of conjugacy classes of semisimple $\theta \in \hat{G}$ such that $\theta^q$ is
conjugate to $\theta$.

(d) The decomposition of (c) implies decomposition $Z_G^0 = \bigoplus_\theta Z_G^0_{\theta}$ of the Bernstein
center. Each $Z_G^0_{\theta}$ is a unital subalgebra, and we set $Z_G^{st,\theta} := Z_G^0_{\theta} \cap Z_G^{st}$.

(e) The decomposition from (c) induces a decomposition $\text{Irr}(G)_0 = \sqcup_\theta \text{Irr}(G)_{0,\theta}$ of the
set of equivalence classes $\text{Irr}(G)_0$ of smooth irreducible representations of depth
zero. In particular, for every $\pi \in \text{Irr}(G)_0$ one can associate a semisimple conjugacy
class $\theta(\pi) \in \hat{G}$ such that $\theta(\pi)^q$ is conjugate to $\theta(\pi)$.

1.7. Depth zero representations.  (a) One expects that the local Langlands
correspondence $\pi \mapsto \lambda(\pi)$ preserves depth. In particular, a representation $\pi \in \text{Irr}(G)$
is of depth zero if an only if $\lambda(\pi) : W_F' \to \hat{G}$ is tamely ramified, that is, trivial on
the wild inertia subgroup of $W_F$.

(b) We choose a lift $\text{Fr} \in W_F$ of the Frobenius element. This choice defines a
bijection between the set of tamely ramified Langlands parameters $\lambda : W_F' \to \hat{G}$ and
the set of pairs $s, u \in \hat{G}$ such that $s \in \hat{G}$ is semisimple and $sus^{-1} = u^q$.

(c) Similarly, Frobenius-semisimple continuous homomorphisms $\overline{\lambda} : W_F \to \hat{G}$
are in bijection pairs $s, \theta \in \hat{G}$ such that $s, \theta \in \hat{G}$ are semisimple and $s\theta s^{-1} = \theta^q$. Moreover, if a Langlands parameter $\lambda$ corresponds to the pair $(s, u)$, then its
restriction $\lambda|_{W_F}$ corresponds to the pair $(s, u_{ss})$.

(d) We see that the local Langlands conjecture for depth zero representations predicts the existence of the decomposition $\text{Irr}(G)_0 = \sqcup_{(s, \theta)} \text{Irr}_{s, \theta}$ of $\text{Irr}(G)_0$ into extended
$L$-packets. In other words, for every $\pi \in \text{Irr}(G)_0$, one should be able to associate
a pair $(s, \theta) = (s(\pi), \theta(\pi))$ such that $\pi \in \text{Irr}_{s, \theta}$ and one expects that the conjugacy
class $\theta(\pi)$ coincides with the one from [Lu1] (e).

1.8. Known cases. In the case when the group $G$ is adjoint, Lusztig [Lu1, Lu2]
parametrized the set $\text{Irr}(G)_0^1$ of irreducible depth zero unipotent representations, thus verifying a (more refined version of) Langlands conjecture in this case. In
particular, for every $\pi \in \text{Irr}(G)_0^1$, Lusztig associated a semisimple conjugacy class
$s(\pi) \in \hat{G}$. 
Moreover, taking into account recent works of Lusztig, it looks like the existence of the decomposition (1.7) of all depth zero representation is within reach. Therefore we assume from now on that \( s(\pi) \in \hat{G} \) is defined for every \( \pi \in \operatorname{Irr}(G)_0 \).

Now we can restate the stable center conjecture for depth zero representation in a more precise form.

1.9. The depth zero stable center conjecture.

(a) The subspace \( Z_{st,0} \subseteq Z^0_G \) is a unital subalgebra. In particular, the projector to the depth zero spectrum is stable.

(b) Element \( z \in Z^0_G \) belongs to \( Z_{st,0} \) if and only if \( f_z(\pi') = f_z(\pi'') \) for all \( \pi', \pi'' \in \operatorname{Irr}(G)_0 \) such that \( s(\pi') = s(\pi'') \) and \( \theta(\pi') = \theta(\pi'') \).

1.10. Plan of the paper. The goal of this note is to outline an approach to a proof the depth zero stable center conjecture.

Our paper is organized as follows. In Sections 2 and 3 we outline our strategy. Then in Section 4 and 5 we sketch a way to implement this strategy for the unit element (see [BVK2] for the details). Namely, we provide a geometric construction of the projector to the depth zero spectrum \( z^0 \in Z_G \), write a conjectural formula for the restriction of \( \mu_{z_0} \) to the regular semi-simple locus \( G_{rss}(F) \) of \( G(F) \), and deduce the stability of \( z^0 \in Z_G \). Finally, in Section 6 we prove our formula for \( \mu_{z_0} \) in the simplest case \( G = SL_2 \).

1.11. Our conventions on categories. (a) Our approach is based on "categorification". Since our construction involves homotopy limits, the derived categories are not suitable for our purposes. Instead, we use the language of stable \( \infty \)-categories (see [Lur]) for the details. On the other hand, because of the expositional nature of this paper, we use \( \infty \)-categories only as a "black box".

(b) More precisely, for every scheme or algebraic stack \( X \) of finite type over an algebraically closed field, we assume the existence of the corresponding stable \( \infty \)-category \( D(X) \), whose homotopy category is the bounded derived category of constructible sheaves \( D(X) = D^b_c(X, \mathbb{Q}_l) \). In addition, we assume that the six functors are lifted to \( \infty \)-categories (see, for example, [LZ]).

(c) We say that two objects of the \( \infty \)-category are equivalent and write \( A \sim B \), if they become isomorphic in the homotopy category. In particular, by writing \( F \sim G \) for two objects of the \( \infty \)-category \( D(X) \), we indicate that they are isomorphic in the derived category \( D(X) \).

(d) To simplify the notation, we will call all \( \infty \)-categories simply "categories". The same applies to monoidal categories.

1.12. Related works. Our work was influenced by the beautiful paper of Vogan on local Langlands conjectures ([VO]). In the process of writing this note we have
learned that versions of the stable Bernstein center and the stable center conjecture were also considered by Haines [Ha] and Scholze–Shin [SS].

Another approach to harmonic analysis on $p$-adic groups via $l$-adic sheaves was proposed in recent papers by Lusztig [Lu4], [Lu5]. It is different from ours: for example, loc. cit. deals with characters of irreducible representations rather than elements of Bernstein center. However, we expect the two constructions to be related.

2. Geometric projector to the depth zero spectrum

In this section we describe the construction of the geometric projector to the depth zero spectrum, which is central for our approach. Our first goal is to ”categorify” the Hecke algebra and the Bernstein center.

2.1. Notation. (a) Let $k$ be an algebraically closed field. We set $E := k((t))$, and fix a prime number $l$ different from the characteristic of $k$. Let $G$ be a semi-simple and simply connected reductive group over $k$, and let $G_E$ be an ind-group scheme over $k$ such that $G_E(k) = G(E)$.

(b) Let $I \subset G_E$ be an Iwahori subgroup, let $Fl$ be the affine flag variety $G_E/I$, and let $\widetilde{W}$ be the affine Weyl group of $G$. For every $w \in \widetilde{W}$, we denote by $Fl^w \subset Fl$ be the closure of the $I$-orbit $Iw \subset Fl$, and by $G^w \subset G_E$ the pre-image of $Fl^w$.

(c) Let $I^+ \subset I$ be the pro-unipotent radical of $I$. For each $n \in \mathbb{N}$ we denote by $I^+_n \subset I^+$ the $n$-th congruence subgroup of $I^+$.

2.2. Categorification of the Hecke algebra.

(a) For every $n \in \mathbb{N}$ and $w \in \widetilde{W}$, the quotient $I^+_n \backslash G^w$ is a scheme of finite type, therefore we can consider the category $\mathcal{D}(I^+_n \backslash G^w)$.

(b) For every $w \leq w' \in \widetilde{W}$ and $n \in \mathbb{N}$, we have a closed embedding $\iota_{w,w'} : I^+_n \backslash G^w \hookrightarrow I^+_n \backslash G^{w'}$, which gives rise to a fully faithful functor

$$(\iota_{w,w'})_* : \mathcal{D}(I^+_n \backslash G^w) \rightarrow \mathcal{D}(I^+_n \backslash G^{w'}) .$$

Similarly, for every $n' > n$ and $w \in \widetilde{W}$ we have a smooth morphism

$\pi_{n',n} : I^+_n \backslash G^w \rightarrow I^+_n \backslash G^{w'}$, which gives rise to the fully faithful functor

$\pi_{n',n}^* : \mathcal{D}(I^+_n \backslash G^w) \rightarrow \mathcal{D}(I^+_n \backslash G^{w'})$. This defines an inductive system $\{\mathcal{D}(I^+_n \backslash G^w)\}_{n,w}$.

(c) We set $\mathcal{D}(G_E) := \text{colim}_n \text{colim}_w \mathcal{D}(I^+_n \backslash G^w)$. This is a categorical counterpart of the Hecke algebra. Also for every $n$, we set $\mathcal{D}(I^+_n \backslash G) := \text{colim}_w \mathcal{D}(I^+_n \backslash G^w)$.

2.3. Convolution. (a) The multiplication map $m : G_E \times G_E \rightarrow G_E$ defines the convolution $m : \mathcal{D}(G_E) \times \mathcal{D}(G_E) \rightarrow \mathcal{D}(G_E)$, which we denote by $\ast$.

Explicitly, for every $\mathcal{F}, \mathcal{G} \in \mathcal{D}(G_E)$, we choose $n, r \in \mathbb{N}$ and $u, w \in \widetilde{W}$ such that $\mathcal{F} \in \mathcal{D}(I^+_n \backslash G^u)$ and $\mathcal{G} \in \mathcal{D}(I^+_r \backslash G^w)$. Increasing $r$ we can assume that $I^+_r$ acts
trivially on $I_n^+ \backslash G_E^w$. Then $m$ induces a map $\overline{m} : (I_n^+ \backslash G_E^w) \times (I_n^+ \backslash G_E^w) \to (I_n^+ \backslash G_E)$, and we set

$$\mathcal{F} \star \mathcal{G} := \overline{m}(\mathcal{F} \boxtimes \mathcal{G}) \in \mathcal{D}(I_n^+ \backslash G_E) \subset \mathcal{D}(G_E).$$

One can check that $\mathcal{F} \star \mathcal{G}$ is independent on the choice of $n$ and $r$. Under the convolution $\star$, $\mathcal{D}(G_E)$ has a structure of a monoidal category without unit.

(b) Let $\delta_{I_n^+} \in \mathcal{D}(I_n^+ \backslash G_E) \subset \mathcal{D}(G_E)$ be the constant sheaf $\mathcal{Q}_I$ supported at $I_n^+ \subset I_n^+ \backslash G_E$. Then it follows by definition that $\delta_{I_n^+} \star \delta_{I_n^+} = \delta_{I_n^+}$ for all $n$. In particular, we have $\delta_{I^+} \star \delta_{I^+} = \delta_{I^+}$.

2.4. The categorical Bernstein center. (a) Denote by $\mathcal{D}(G_E)^2$ the category $\mathcal{D}(G_E \times G_E)$. The standard action $(g, h)(x) := gxh^{-1}$ of $G_E \times G_E$ on $G_E$ induces an action of the monoidal category $\mathcal{D}(G_E)^2$ on $\mathcal{D}(G_E)$ by the two-sided convolution.

(b) Denote by $\mathcal{Z}(G_E)$ the monoidal category $\text{End}_{\mathcal{D}(G_E)^2}(\mathcal{D}(G_E))$ of endomorphisms of the module category $\mathcal{D}(G_E)$ over the monoidal category $\mathcal{D}(G_E)^2$. The category $\mathcal{Z}(G_E)$ is a geometric counterpart of the Bernstein center of $G_E$.

(c) Denote by $\text{End} \mathcal{D}(G_E) = \text{End}_{\mathcal{D}(G_E)^{op}} \mathcal{D}(G_E)$ the monoidal category of endomorphisms of the module category $\mathcal{D}(G_E)$ over the monoidal category $\mathcal{D}(G_E)$, where the monoidal category $\mathcal{D}(G_E)^{op}$ acts on $\mathcal{D}(G_E)$ by the right convolution.

We denote by $\omega : \mathcal{Z}(G_E) \to \text{End} \mathcal{D}(G_E)$ the forgetful functor.

2.5. Parahoric subgroups. (a) Denote by $\text{Par}$ the category, corresponding to the partially ordered set of standard parahorics $\mathcal{P} \supset I$ of $G_E$ with order, opposite to the inclusion.

(b) For every $\mathcal{P} \in \text{Par}$, we denote by $\text{End}^\mathcal{P} \mathcal{D}(G_E)$ the category of $\text{Ad} (\mathcal{P})$-equivariant elements of $\text{End} \mathcal{D}(G_E)$. For every pair $\mathcal{P} \subset \mathcal{Q} \in \text{Par}$, we have a forgetful functor $\text{End}^\mathcal{Q} \mathcal{D}(G_E) \to \text{End}^\mathcal{P} \mathcal{D}(G_E)$.

(c) Notice that the forgetful functor $\omega$ from 2.4 (c) has a natural monoidal lift $\omega_{\mathcal{P}} : \mathcal{Z}(G_E) \to \text{End}^\mathcal{P} \mathcal{D}(G_E)$ for each $\mathcal{P} \in \text{Par}$, hence a natural lift to the homotopy limit $\tilde{\omega} : \mathcal{Z}(G_E) \to \lim_{\mathcal{P}} \text{End}^\mathcal{P} \mathcal{D}(G_E)$. Notice that the ind-group scheme $G_E$ is generated by standard parahoric subgroups $\mathcal{P}$, we expect that $\tilde{\omega}$ is an equivalence.

2.6. The center of the categorical Iwahori-Hecke algebra.

(a) Consider monoidal categories $\mathcal{D}_{I^+}(G_E) := \mathcal{D}(I^+ \backslash G_E / I^+) \subset \mathcal{D}(G_E)$ and $\mathcal{D}_{I^+}^T(G_E) := \mathcal{D}(\frac{I^+ \backslash G_E / I^+}{T})$, where $T$ acts by conjugation, with unit $\delta_{I^+}$. We have a forgetful functor $\mathcal{D}_{I^+}^T(G_E) \to \mathcal{D}_{I^+}(G_E)$.

(b) We denote by $\mathcal{D}_{I^+}(G_E)^2$ a category $\mathcal{D}((I^+ \backslash G_E / I^+)^2)$. It is a monoidal category, acting on $\mathcal{D}_{I^+}(G_E)$ by the two-sided convolution. We define $\mathcal{Z}_{I^+}(G_E)$ as $\text{End}_{\mathcal{D}_{I^+}(G_E)^2}(\mathcal{D}_{I^+}(G_E))$ and call it the center of the categorical Iwahori-Hecke algebra.
(c) Note that $D_I^+(G_E)$ is a full subcategory of $D(G_E)$ and that the restriction map $\Phi \mapsto \Phi|_{D_I^+(G_E)}$ gives rise to a monoidal functor $R : Z(G_E) \to Z_I^+(G_E)$.

(d) Denote by $\text{ev}_I^+ : Z_I^+(G_E) \to D_I^+(G_E)$ the evaluation functor $\Phi \mapsto \omega(\Phi)(\delta^+_I)$. The functor $\text{ev}_I^+$ has a natural lift a monoidal functor $\text{ev}_I^{+T} : Z_I^+(G_E) \to D_I^{+T}(G_E)$.

The following conjecture is central for what follows.

\textbf{Conjecture 2.7.} The restriction functor $R : Z(G_E) \to Z_I^+(G_E)$ has a right adjoint $\tilde{A} : Z_I^+(G_E) \to Z(G_E)$. Moreover, $\tilde{A}$ is monoidal, by which we mean that $\tilde{A}$ preserves convolution but does not necessary map unit to unit.

\section*{2.8. A finite-dimensional analog.}

(a) As before, $G$ be a connected reductive group over $k$, $B \subset G$ a Borel subgroup, $U \subset B$ the unipotent radical of $B$, and $T \subset B$ a maximal torus. Let $\mathcal{Z}_U(G)$ center of the categorical Hecke algebra, constructed similarly to its affine analog $\mathcal{Z}_I^+(G_E)$. The action of $G$ on $G/U$ gives rise to the monoidal functor $a_1 : D(G) \to \mathcal{Z}_U(G)$.

It is proven in [BKVI] that $a_1$ induces an equivalence of the derived categories. Similar results in the $D$-module settings were also shown earlier by Ben Zvi-Nadler [BN] and Bezrukavnikov-Finkelberg-Ostrik [BFO].

(b) Our proof is based on an explicit construction of the inverse of $a_1$. Consider functor $\tilde{b}_t : \mathcal{Z}_U(G) \to D^G(G)$, defined as a composition of the evaluation map $\text{ev}_U^T : \mathcal{Z}_U(G) \to D_U^T(G) = D(U_{G/U}^T)$, the $\ast$-pullback $D_U^T(G) \to D^B(G)$, and the averaging functor $\text{Av}_{G/B} : D^B(G) \to D^G(G)$.

Note that the composition $\tilde{b}_t \circ a_1 : D^G(G) \to D^G(G)$ is a convolution with the Springer sheaf $\mathcal{S}$. We show that the composition $a_1 \circ \tilde{b}_t : \mathcal{Z}_U(G) \to \mathcal{Z}_U(G)$ is a convolution with $a_1(\mathcal{S}) \in \mathcal{Z}_U(G)$, deduce from this that the natural $W$-action on $\mathcal{S}$ gives rise to the $W$-action on $\tilde{b}_t$, and that $b_t := \tilde{b}_t^{\text{W-inv}}$ is the inverse of $a_t$.

\section*{2.9. Our strategy.}

Notice that the restriction functor $R$ from 2.6 is an affine analog of the functor $a_1$ from 2.8. Therefore we expect that its adjoint $\tilde{A}$ can be constructed in a way analogous to the construction of $b_t$, described in 2.8 (b).

In other words, we expect that $\tilde{A}$ is the composition of the averaging functor with respect to $Fl = G_E/I$ and the functor of ”derived $\hat{W}$-skew-invariants”. Since $Fl$ is an ind-scheme, and the group $\hat{W}$ is infinite, our first task is to give precise meaning to this composition.

Our approach is to replace the averaging functor with respect to $Fl$ by the limit of the averaging functors with respect to a family of closed $I$-invariant subschemes $Y \subset Fl$. Also, we replace ”derived $\hat{W}$-invariants” by the ”homotopy limit” of $W_T$-invariants, where $W_T$ runs the set of all finite parabolic Weyl subgroups $W_T \subset \hat{W}$.
Namely, for every closed $I$-invariant subscheme $Y \subset FL$, and every parabolic finite Weyl subgroups $\tilde{W}_p \subset \tilde{W}$, we consider the composition $\tilde{A}^Y_p : Z_{I^+}(G_E) \to \mathcal{D}(G_E)$ of the averaging functor with respect to $Y$ and the functor of derived $W_p$-skew-invariants. Let $A^Y : Z_{I^+}(G_E) \to \mathcal{D}(G_E)$ be the homotopy limit $\text{holim}_P \tilde{A}^Y_p$ of $\tilde{A}^Y_p$.

We show that for every $F \in Z$ we have a natural action of the finite Weyl group $W$ on $\mathcal{D}(G_E)$. Thus projective system $\{A^Y\}_Y$ stabilizes, thus projective system $\{A^Y\}_Y$ defines a functor $A : Z_{I^+}(G_E) \to \text{End} \mathcal{D}(G_E)$. Finally, we show that $A$ lifts to a functor $\tilde{A}$, which is monoidal and right adjoint to $R$.

Now we describe our strategy in detail.

2.10. Averaging functors. (a) Let $Y \subset FL$ be a locally closed subscheme, let $\tilde{Y} \subset G_E$ the preimage by $Y$, and let $\tilde{Y} \times_I G_E$ be the quotient of $\tilde{Y} \times G_E$ by $I$ given by the action $(y, g)h := (yh, h^{-1}gh)$.

(b) Every $Y$ as in (a) gives rise to a diagram

\[
\begin{array}{ccc}
I^+G_E/I^+ & \xrightarrow{a_1} & \tilde{Y} \times_I G_E \xrightarrow{a_2} G_E,
\end{array}
\]

where $a_1([y, g]) := [g]$ and $a_2([y, g]) := [gyg^{-1}]$, and we define the "averaging" functor

\[Av^Y = (a_2)_! \circ a^*_1 : \mathcal{D}_{I+}(G_E) \to \mathcal{D}(G_E).\]

Using 2.6 (d) also we set

\[Av^Y := Av^Y \circ \text{ev}_{I+} : \mathcal{Z}_{I^+}(G_E) \to \mathcal{D}(G_E).\]

(c) For every $Y$ as in (a) and $P \in \text{Par}$ we consider a locally closed subscheme $YP := (\tilde{Y}P)/I \subset FL$, and functors $Av^Y_P := Av^YP$ and $A^Y_P := A^YP$.

(d) Notice that for every $Y$ as in (a), and every closed subscheme $Y' \subset Y \subset FL$, the restriction maps induce morphisms $Av^Y \to Av^{Y'}$ and $A^Y \to A^{Y'}$ of functors.

2.11. Notation. Denote by $\mathcal{D}(G_E)^{\text{Par}}$ the category of functors $\text{Fun}(\text{Par}, \mathcal{D}(G_E))$. For every $P \in \text{Par}$ we denote by $\text{ev}(P) : \mathcal{D}(G_E)^{\text{Par}} \to \mathcal{D}(G_E)$ the evaluation at $P$. We also denote by $\text{lim}_{\text{Par}} : \mathcal{D}(G_E)^{\text{Par}} \to \mathcal{D}(G_E)$ the functor, which associates to each $\alpha \in \text{Fun}(\text{Par}, \mathcal{D}(G_E))$ its limit in $\mathcal{D}(G_E)$.

Conjecture 2.12. (a) For every $Y$ as in 2.10 (a) and every $P \in \text{Par}$, the functor $Av^Y_P$ has a natural action of the finite Weyl group $W_P \subset \tilde{W}$.

(b) Denote by $A^Y_P := (Av^Y)_W^{\text{par}} : \mathcal{Z}_{I^+}(G_E) \to \mathcal{D}(G_E)$ the composition of $A^Y_P$ with a functor of skew-invariants. Then for every $P \subset Q \in \text{Par}$, we have a natural morphism of functors $f_{Q, P} : A^Y_P \to A^Y_Q$ compatible with compositions.

(c) There exists a functor $A^Y : \mathcal{Z}_{I^+}(G_E) \to \mathcal{D}(G_E)^{\text{Par}}$ such that the composition $\text{ev}(P) \circ A^Y_P$ is equivalent to $A^Y_P$ for every $P \in \text{Par}$. 
Conjecture 2.15. For every $A$, functors $2.14$. Remark. Since derived categories do not have limits, in order to define $X \in D$ and $i$, the composition

$$A^Y := \lim_{Par} \circ A^Y_{\gamma,n} : \mathcal{Z}_{I^+}(G_E) \to D(G_E)^{Par} \to D(G_E).$$

(b) Notice that by 2.10 (d), the functors $\{A^Y\}_Y$, indexed by closed $I$-invariant subschemes $Y \subset Fl$, form a projective system. In particular, for every $\mathcal{F} \in \mathcal{Z}_{I^+}(G_E)$ and $\mathcal{X} \in D(G_E)$, we get a projective system $\{A^Y(\mathcal{F}) \ast \mathcal{X}\}_Y$ in $D(G_E)$.

2.14. Remark. Since derived categories do not have limits, in order to define functors $A^Y$ we are forced to work with $\infty$-categories.

Conjecture 2.15. For every $\mathcal{F} \in \mathcal{Z}_{I^+}(G_E)$ and $\mathcal{X} \in D(G_E)$, the projective system $\{A^Y(\mathcal{F}) \ast \mathcal{X}\}_Y$ stabilizes. Moreover, $\lim_Y(A^Y(\mathcal{F}) \ast \delta_{I^+}) \in D(G_E)$ is equivalent to $ev_{I^+}(\mathcal{F})$.

2.16. Functor $A : \mathcal{Z}_{I^+}(G_E) \to \text{End} D(G_E)$.

(a) We denote by $\iota : G_E \to G_E$ the map $g \mapsto g^{-1}$ and by $\iota^* : D(G_E) \to D(G_E)$ the corresponding pullback map.

(b) Since $D(G_E)$ is a monoidal category we have a natural monoidal functor $\alpha : D(G_E) \to \text{End}(D(G_E))$ such that $\alpha(\mathcal{X})(\mathcal{Y}) = \iota^*(\mathcal{X}) \ast \mathcal{Y}$.

(c) By Conjecture 2.15 there exists a limit

$$A = \lim_Y(\alpha \circ A^Y) : \mathcal{Z}_{I^+}(G_E) \to \text{End} D(G_E).$$

Explicitly, we have $A(\mathcal{F}) \sim \lim_Y(\iota^*(A^Y(\mathcal{F})) \ast \mathcal{X})$ for every $\mathcal{F} \in \mathcal{Z}_{I^+}(G_E)$ and $\mathcal{X} \in D(G_E)$.

(d) For every $Y$, we denote by $\text{pr}_Y : A \to \alpha \circ A^Y$ the natural projection.

2.17. Locally constant measures on the regular semi-simple locus.

(a) For every $\gamma \in G(E)$ and $n \in \mathbb{N}$, we denote by $i_{\gamma,n}$ the inclusion $\gamma I^+_n \hookrightarrow G_E$ and by $\tilde{i}_{\gamma,n} : D(G_E) \to D(\gamma I^+_n)$ and $(i_{\gamma,n})! : D(\gamma I^+_n) \to D(G_E)$ the corresponding functors of restriction and extension of zero.

We denote by $r_{\gamma,n} : \text{End} D(G_E) \to \text{Hom}(D(\gamma I^+_n), D(I^+_n))$ the functor $r_{\gamma,n}(\phi) := \tilde{i}_{\gamma,n} \circ \phi \circ (i_{\gamma,n})!$.

(b) We say that the restriction of $\mathcal{F} \in \text{End} D(G_E)$ to the regular semi-simple locus $G^{ss}_E$ of $G_E$ is locally constant, if for every $\gamma \in G^{ss}(E)$, there exist $\mathcal{X} \in D(G_E)$, $n \in \mathbb{N}$ and a morphism $\phi : \mathcal{F} \to \alpha(\mathcal{X})$ in $\text{End} D(G_E)$ such that restriction $r_{\gamma,n}(\phi) : r_{\gamma,n}(\mathcal{F}) \to r_{\gamma,n}(\alpha(\mathcal{X}))$ is an equivalence.

In this case we say that $\mathcal{F}$ equals $\mathcal{X}$ in a neighborhood of $\gamma$.

Conjecture 2.18. (a) Monoidal functor $A : \mathcal{Z}_{I^+}(G_E) \to \text{End} D(G_E)$ has a natural monoidal lift $\tilde{A} : \mathcal{Z}_{I^+}(G_E) \to \mathcal{Z}(G_E)$. Moreover, $\tilde{A}$ is right adjoint to $R$. 
(b) For every $\mathcal{F} \in \mathcal{Z}_{I^+}(G_E)$, the restriction of $\mathcal{A}(\mathcal{F})$ to $G^\text{rss}$ is locally constant. Moreover, the projection $\text{pr}_Y(\mathcal{F}) : \mathcal{A}(\mathcal{F}) \to \alpha(\mathcal{A}^Y(\mathcal{F}))$ from 2.10 (d) has the property that for every $\gamma \in G^\text{rss}(E)$ there exists $n \in \mathbb{N}$ and such that the restriction $r_{\gamma,n}(\text{pr}_Y(\mathcal{F}))$ is an equivalence for all sufficiently large $Y$.

3. Application to the classical (stable) Bernstein center

Assume now that $k$ is an algebraic closure of the finite field $\mathbb{F}_q$, and that $G$ has an $\mathbb{F}_q$-structure. Then all the objects defined in the previous section have $\mathbb{F}_q$-structures, and we denote by $\mathcal{D}^{\text{Fr}}(G_E)$ and $\mathcal{Z}^{\text{Fr}}(G_E)$ the corresponding categories of Weil (Frobenius equivariant) objects.

3.1. The "sheaf-function correspondence". We set $F := \mathbb{F}_q((t))$, and choose a field isomorphism $\mathbb{F}_q \cong \mathbb{F}_q$. Denote by $\mu_n$ the Haar measure on $G(F)$ such that $\int_{I^+} \mu_n = 1$.

(a) To every object $\mathcal{X} \in \mathcal{D}^{\text{Fr}}(G_E)$ we associate an element $[\mathcal{X}]$ of the Hecke algebra $\mathcal{H}(G(F))$. To define $[\mathcal{X}]$, we choose an $n \in \mathbb{N}$ such that $\mathcal{X} \in \mathcal{D}^{\text{Fr}}(I^+_n \backslash G_E)$. Then the "trace of Frobenius" map associates to $\mathcal{X}$ a function $[\mathcal{X}]_n \in C^\infty(I^+_n \backslash G(F)) \subset C^\infty_c(G(F))$, and we set $[\mathcal{X}] := [\mathcal{X}]_n \mu_n$. It is easy to check that $[\mathcal{X}]$ does not depend on a choice of $n$.

(b) For every $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^{\text{Fr}}(G_E)$ we have $[\mathcal{X} \ast \mathcal{Y}] = [\mathcal{X}] \ast [\mathcal{Y}]$. Also for every $n \in \mathbb{N}$ element $[\delta_{I^+_n}] \in \mathcal{H}(G(F))$ is the unit of Hecke algebra $\mathcal{H}(G(F), I^+_n)$.

(c) We claim that every $\mathcal{F} \in \text{End}^{\text{Fr}}(\mathcal{D}(G_E))$ defines a unique element $[\mathcal{F}]$ of $\text{End}_{\mathcal{H}(G(F))^{\text{op}}} \mathcal{H}(G(F))$, that is, $[\mathcal{F}]$ is an endomorphism of $\mathcal{H}(G(F))$ commuting with right convolutions, such that $[\mathcal{F}][[\mathcal{X}]] = [\mathcal{F}(\mathcal{X})]$ for every $\mathcal{X} \in \mathcal{D}^{\text{Fr}}(G_E)$.

The uniqueness follows from the fact that the image of the map $[\cdot] : \mathcal{D}^{\text{Fr}}(G_E) \to \mathcal{H}(G(F))$ spans all of $\mathcal{H}(G(F))$. To show the existence we have to show that for every tuple $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathcal{D}^{\text{Fr}}(G_E)$ and $a_1, \ldots, a_n \in \mathbb{Q}_l$ such that $\sum_i a_i [\mathcal{X}_i] = 0$, then $\sum_i a_i [\mathcal{F}(\mathcal{X}_i)] = 0$. Choose $n \in \mathbb{N}$ such that $\mathcal{X}_i \in \mathcal{D}^{\text{Fr}}(I^+_n \backslash G_E)$ for all $i$. Then $\mathcal{X}_i = \delta_{I^+_n} \ast \mathcal{X}_i$, hence $\mathcal{F}(\mathcal{X}_i) = \mathcal{F}(\delta_{I^+_n}) \ast \mathcal{X}_i$, thus $\sum_i a_i [\mathcal{F}(\mathcal{X}_i)] = [\mathcal{F}(\delta_{I^+_n})] \ast (\sum_i a_i [\mathcal{X}_i]) = 0$.

(d) Let $\mathcal{F} \in \mathcal{Z}^{\text{Fr}}(G_E)$, and let $\tilde{\mathcal{F}} \in \text{End}^{\text{Fr}}(\mathcal{D}(G_E))$ be the image of $\mathcal{F}$. Then the element $[\tilde{\mathcal{F}}] \in \text{End}_{\mathcal{H}(G(F))^{\text{op}}} \mathcal{H}(G(F))$ from (c) also commutes with left convolutions. Thus $\mathcal{F}$ defines an element $[\mathcal{F}]$ of the Bernstein center $\mathcal{Z}_{G(F)}$ of $G(F)$.

(e) Note that functor $\tilde{\mathcal{A}}$ from Conjecture 2.7 is Frobenius-equivariant, hence it defines a monoidal functor $\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^{\text{Fr}} : \mathcal{Z}^{\text{Fr}}(G_E) \to \mathcal{Z}^{\text{Fr}}(G_E)$. Similar observation apply to the functor $\mathcal{A}$ from 2.16.

3.2. Locally constant function on $G^{\text{rss}}(F)$. It follows from Conjecture 2.18 (a),(b) that for each $\mathcal{F} \in \mathcal{Z}^{\text{Fr}}_{I^+}(G_E)$, the restriction of $\mathcal{A} = \tilde{\omega} \circ \tilde{\mathcal{A}}(\mathcal{F}) \in \text{End} \mathcal{D}(G_E)$
to $G_E^{rs}$ is locally constant. Then the invariant distribution $\mu_{[A(F)]}(G_E^{rs}(F))$ on $G_E^{rs}(F)$ has the form $\phi_{[A(F)]}\mu_0$ for some $\phi_{[A(F)]} \in C^\infty_c(G_E^{rs}(F))$.

Our next goal is to describe this function explicitly.

**3.3. Notation.** Let $G_E$ be the quotient of $G_E$ by the adjoint action of $I$. For every $\gamma \in G_E^{rs}(F)$, we denote by $pr_\gamma : Fl \to \frac{I^+G_E/I^+}{F}$ the composition of the averaging morphism $G_E/I \to \mathcal{O}_F$, given by the formula $g \mapsto g^{-1}\gamma g$, and the projection $\mathcal{O}_F \to \frac{I^+G_E/I^+}{F}$. In particular, $pr_\gamma$ defines functor $pr_{\gamma}^* : \mathcal{D}^{T,Fr}_I(G_E) \to \mathcal{D}^{Fr}(Fl)$ (see 2.6). Let $ev_{I^+}^T : \mathcal{Z}_{I^+}^{Fr}(G_E) \to \mathcal{D}^{T,Fr}_I(G_E)$ be the functor from 2.6 (d).

**Conjecture 3.4.** Let $F \in \mathcal{Z}_{I^+}^{Fr}(G_E)$ and $\gamma \in G_E^{rs}(F)$. Then

(a) each homology group $H_i(Fl, pr_{\gamma}^*(ev_{I^+}^T(F)))$ has a natural structure of a finitely generated $\mathbb{Q}[\hat{W}]$-module.

(b) We have an equality

$$\phi_{[A(F)]}(\gamma) = \text{Tr}(Fr, H_*(\hat{W}, H_i(Fl, pr_{\gamma}^*(ev_{I^+}^T(F))))^{sgn}),$$

where for every representation $V$ of $\hat{W}$, we denote by $V^{sgn}$ its twist by the sign character.

**3.5. Gaitsgory’s central sheaves.** (a) For a tame rank one local system $\theta$ on $T$, we denote by $\mathcal{D}_{I,\theta}(G_E) \subset \mathcal{D}_{I^+}(G_E)$ the full subcategory of complexes which are $T$-monodromic with respect to the left and right actions with generalized eigenvalues of monodromy being $\theta$. Similarly, we define $\mathcal{D}_{I,\theta}(G_E)^2$ and $\mathcal{Z}_{I,\theta}(G_E)$. Then $\mathcal{Z}_{I,\theta}(G_E)$ is a full subcategory in $\mathcal{Z}_{I^+}(G_E)$.

(b) A generalization of construction by Gaitsgory [Ga] gives a monoidal functor $\Psi_{\theta} : \text{Rep} \hat{G} \to \mathcal{Z}_{I,\theta}(G_E)$ for each $\theta$ (see [Be] for the corresponding assertion for derived categories). If $Fr^*(\theta) = w_*(\theta)$ for some $w \in W$, then $\Psi_{\theta}$ lifts to a monoidal functor to $\Psi_{Fr}^* : \text{Rep} \hat{G} \to \mathcal{Z}_{I^+}^{Fr}(G_E)$. Therefore by Conjecture 2.7, for each $\theta$ we have a monoidal functor $\Phi_{\theta} := \mathcal{A} \circ \Psi_{\theta}^* : \text{Rep} \hat{G} \to \mathcal{Z}_{I^+}^{Fr}(G_E)$, hence an algebra homomorphism $[\Phi_{\theta}] := [\cdot] \circ \Phi_{\theta} : K^0(\text{Rep} \hat{G}) \to \mathcal{Z}_{G(E)}.$

(c) For each dominant weight $\mu$ of $\hat{G}$, we denote by $V_{\mu}$ the irreducible representation of $\hat{G}$ of highest weight $\mu$, and set $Z_{\mu}^\theta := [\Phi_{\theta}](V_{\mu}) \in \mathcal{Z}_{G(E)}$.

Note that $\theta$ defines a semi-simple conjugacy class in $\hat{G}$ such that $\theta^\theta \sim \theta$.

**Conjecture 3.6.** (a) For every $\theta$ and $\mu$, the element $Z_{\mu}^\theta \in \mathcal{Z}_{G(E)}$ belongs to $\mathcal{Z}_{G(E)}^{st,\theta}$.

In other words, $Z_{\mu}^\theta$ is stable, $Z_{\mu}^\theta(\pi) = 0$ for each $\pi \in \text{Irr}(G)_{>0}$, and $Z_{\mu}^\theta(\pi) = 0$ for each $\pi \in \text{Irr}(G)_{0}$ with $\theta' \neq \theta$.

(b) For each $\pi \in \text{Irr}(G)_{0}$, we have an equality $Z_{\mu}^\theta(\pi) = \text{Tr}(s(\pi), V_{\mu})$. 


Conjecture 3.7. The linear subspace $Z_{G(F)}^{st,0} \subset Z_{G(F)}^0$ is the span of all $\{Z_{\mu,1,\mu,\theta}\}$, taken over all tame local systems $\theta$ and all dominant weights $\mu$, while the linear subspace $Z_{G(F)}^{st,0} \subset Z_{G(F)}^0$ is the span of all $\{Z_{\mu}^0\}_\mu$.

3.8. Remark. Note that since $[\Phi_\theta] : K^0(\operatorname{Rep} \hat{G}) \to Z_{G(F)}$ is an algebra homomorphism, conjectures 3.6 (a) and 3.7 imply the depth zero stable center conjecture.

4. The case of the unit element

In this section we prove Conjectures 2.12, 2.15 and 2.18 for the unit element $\operatorname{Id} \in Z_{I^+}(G_E)$. Recall that $ev_{I^+}^\flat(\operatorname{Id}) = \delta_{I^+} \in D_{I^+}^+(G_E)$, thus $A^\nu_Y(\operatorname{Id}) = Av_Y^\nu(\delta_{I^+})$.

The following result is Conjecture 2.12 for the unit element.

**Theorem 4.1.** (a) For every $\mathcal{P} \in \operatorname{Par}$, element $Av_Y^\nu(\delta_{I^+}) \in D(G_E)$ is equipped with an action of the finite Weyl group $W_\mathcal{P} \subset \tilde{W}$. We set $A^\nu_Y := Av_Y^\nu(\delta_{I^+})^{W_\mathcal{P},\operatorname{sgn}}$.

(b) For every pair $\mathcal{P} \subset \mathcal{Q} \in \operatorname{Par}$, we have a natural morphisms $f_{\mathcal{Q},\mathcal{P}} : A^\nu_Y \to A^\nu_Y$ compatible with compositions.

(c) There exists a functor $A^\nu_Y : \operatorname{Par} \to D(G_E)$, which maps every $\mathcal{P} \in \operatorname{Par}$ to $A_Y^\nu$ and every inclusion $\mathcal{P} \hookrightarrow \mathcal{Q}$ to $f_{\mathcal{Q},\mathcal{P}}$.

**Sketch of a proof.** In order to avoid technical discussion involving $\infty$-categories, we only discuss parts (a) and (b) in the framework of derived categories. We are going to deduce both assertions from the classical Springer theory.

(a) By definition, $Av_Y^\nu(\delta_{I^+}) \in D(G_E)$ coincides with the pushforward $\nu_! \delta_{I^+}$, where $\nu := \nu_! : \tilde{\mathcal{P}} \times_I I \to G_E$ is given by $[y,g] \mapsto ygy^{-1}$. Thus $\nu_! \mathcal{P}$ can be written as a composition

$$\tilde{\mathcal{P}} \times_I I \xrightarrow{\nu_!'} \tilde{\mathcal{P}} \times \mathcal{P} \xrightarrow{\nu_!} G_E,$$

where $\tilde{\mathcal{P}} \times \mathcal{P}$ is the quotient of $\tilde{\mathcal{P}} \times \mathcal{P}$ by the action of $\mathcal{P}$ given by $(y,g)h := (yh,h^{-1}gh)$, the morphism $\nu_!' = \nu_!'$ is induced by the inclusion $I \hookrightarrow \mathcal{P}$, and $\nu_!''([y,g]) := ygy^{-1}$. We claim that already $\nu_!(\delta_{I^+})$ is equipped with an action of $W_\mathcal{P}$.

Let $L_\mathcal{P} := \mathcal{P}/\mathcal{P}^+$ be the “Levi subgroup” of $\mathcal{P}$, let $B_\mathcal{P} := I/\mathcal{P}^+ \subset L_\mathcal{P}$ be the corresponding Borel subgroup, and let $U_\mathcal{P} := I^+/\mathcal{P}^+ \subset B_\mathcal{P}$ be the unipotent radical. Denote by $B_\mathcal{P} \setminus B_\mathcal{P}$ and $L_\mathcal{P} \setminus L_\mathcal{P}$ are quotient stacks with respect to the adjoint actions. Then we have a Cartesian diagram

$$\begin{array}{ccc}
\tilde{\mathcal{P}} \times_I I & \longrightarrow & B_\mathcal{P} \setminus B_\mathcal{P} \\
\downarrow_{\nu_!} & & \downarrow_{\operatorname{pr}} \\
\tilde{\mathcal{P}} \times \mathcal{P} & \longrightarrow & L_\mathcal{P} \setminus L_\mathcal{P},
\end{array}$$
where \( \text{red}_I \) and \( \text{red}_P \) are defined by the rule \([y, g] \mapsto [g] \). Denote by \( \delta_{U_P} \) the constant sheaf supported on \( U_P \). Then by the proper base change theorem we have an isomorphism \( \nu'_I(\delta_{I^+}) \cong \text{red}_P^*(\text{pr}_1(\delta_{U_P})) \).

Since \( \text{pr}_1(\delta_{U_P}) \) is the Springer sheaf on \( L_P \setminus L_P \), it is equipped with an action of the Weyl group \( W_P \), implying assertion (a).

(b) First we describe \( A_Y^V \) explicitly. Denote by \( i : \{1\} \hookrightarrow U_P \) the inclusion, and set \( \delta_1 := \iota_U(\delta_{U_P}) \). For every \( P \in \mathcal{P} \) we denote by \( i_P : \mathcal{P}^+ \rightarrow I^+ \) the inclusion, and set \( \delta_{P^+} := (i_P)_* \iota^{\delta}_{P^+}(\delta_{I^+}) \). Then \( \delta_{P^+} \cong \text{red}_P^*(\delta_1) \).

Since \( \delta_1 \) is naturally isomorphic to the skew-invariants \( \text{pr}_1(\delta_{U_P})^{\mathcal{P}^+} \) of the Springer sheaf, we conclude that \( A_{P^+}^V := \text{Av}_{P'}^V(\delta_{I^+})^{\mathcal{P}^+} \) is naturally isomorphic to \( (\nu'_P)_!(\delta_{P^+}) \).

For every two standard parahoric subgroups \( P \subset Q \), we set \( Q^+ \subset \mathcal{P}^+ \), so the embedding \( i_Q : Q^+ \hookrightarrow I^+ \) decomposes as \( Q^+ \xrightarrow{i} \mathcal{P}^+ \xrightarrow{i_P} I^+ \). In particular, we have a canonical morphism \( \delta_{Q^+} = i^* \delta_{P^+} \rightarrow \delta_{P^+} \).

Recall that \( \nu'_Q : \tilde{Y} \mathcal{P} \times \mathcal{P} \mathcal{P} \rightarrow G_E \) decomposes as a composition of the projection \( \pi : \tilde{Y} \mathcal{P} \times \mathcal{P} \mathcal{P} \rightarrow \tilde{Y} \mathcal{P} \times \mathcal{P} \mathcal{Q} \) and \( \nu'_\mathcal{Q} : \tilde{Y} \mathcal{Q} \times \mathcal{Q} \mathcal{Q} \rightarrow G_E \). We define \( f_{\mathcal{Q}, \mathcal{P}} : A_{\mathcal{Q}}^V \rightarrow A_P^V \) as the composition

\[
(\nu'_\mathcal{Q})_!(\delta_{\mathcal{Q}^+}) \rightarrow (\nu'_\mathcal{P})_!(\delta_{\mathcal{P}^+}) \rightarrow (\nu'_P)_!(\delta_{P^+}),
\]

where the first map is induced by the pullback \( \pi^* \) of the proper map \( \pi \), while the second one is induced by the morphism \( \delta_{\mathcal{Q}^+} \rightarrow \delta_{P^+} \), defined above. \( \square \)

As in [2.13] we set \( A^V := \lim_{\mathcal{P} \rightarrow \mathcal{P}_I}(A_Y^V) \in \mathcal{D}(G_E) \). The following notation will be used later.

4.2. Notation. (a) We denote by \( \tilde{\Delta} \) the set of simple affine roots of \( G \).

(b) For every proper subset \( J \subset \tilde{\Delta} \), we denote by \( \mathcal{P}(J) \in \mathcal{P} \) the parahoric subgroup such that \( J \) is the set of simple roots of \( L_P \).

(c) For every \( w \in \tilde{W} \), we set \( Y_w := I_w \subset Fl \), and denote by \( J_w \) the set of \( \alpha \in \tilde{\Delta} \) such that \( w(\alpha) > 0 \).

(d) For each \( n > 0 \), let \( \tilde{W}(n) \subset \tilde{W} \) be the set of all \( w \in \tilde{W} \) such that \( w(\alpha) < n \), that is, \( w(\alpha) - n < 0 \), for every \( \alpha \in \tilde{\Delta} \). We also set \( \tilde{W}(0) = \{1\} \).

The following result is the first part of Conjecture [2.15] for the case of the unit element.

Theorem 4.3. For every \( \mathcal{X} \in \mathcal{D}(G_E) \), the projective system \( \{A^V \ast \mathcal{X}\}_Y \) stabilizes.

Sketch of a proof. By the definition of \( \mathcal{D}(G_E) \), for each \( \mathcal{X} \in \mathcal{D}(G_E) \) there exists \( n \in \mathbb{N} \) such that \( \delta_{I^+_n} \ast \mathcal{X} = \mathcal{X} \). Thus it is enough to show that the system \( \{A^V \ast \delta_{I^+_n}\}_Y \) stabilizes for each \( n \).

Since \( \sum_{\alpha \in \Delta} \alpha = 1 \), each set \( \tilde{W}(n) \) is finite. We set \( Y(n) := \bigcup_{w \in \tilde{W}(n)} Fl^{\leq w} \). The following result is a more precise version of Theorem 4.3 for \( \mathcal{X} = \delta_{I^+_n} \).
Claim 4.4. For every closed $I$-invariant subscheme $Y \supset Y(n)$ of $\text{Fl}$ the induced morphism $A^Y \ast \delta_{I_n^+} \to A^Y(n) \ast \delta_{I_n^+}$ is an equivalence.

The proof of Claim 4.4 is based on the following result.

Claim 4.5. Let $w \in \hat{W}$, and let $J \subset J'$ be two proper subsets of $\hat{\Delta}$ such that $w(\alpha) > 0$ for every $\alpha \in J'$ and $w(\alpha) - n > 0$ for every $\alpha \in J' \setminus J$.

Then the morphism $f_{P(J') P(J)} : A^w_{P(J')} \ast \delta_{I_n^+} \to A^w_{P(J)} \ast \delta_{I_n^+}$ is an equivalence.

First we derive Claim 4.4 from Claim 4.5. By induction on the number of orbit-insets in $Y \setminus Y(n)$, it is enough to show that for every closed $I$-invariant subscheme $Y \subset \text{Fl}$ and every $w \in \hat{W} \setminus \hat{W}(n)$ such that $Y_w \subset Y$ is open, the induced map $A^Y \ast \delta_{I_n^+} \to A^{Y \setminus Y_w} \ast \delta_{I_n^+}$ is an equivalence. Set $Y' := Y \setminus Y_w$.

In order to avoid technical discussion involving homotopy limits, we only show an equality $\langle A^Y \ast \delta_{I_n^+} \rangle = \langle A^Y \ast \delta_{I_n^+} \rangle$ in the Grothendieck group $K^0(\mathcal{D}(G_E))$.

Consider element $\langle A^w \rangle := \sum_{J \subset J_w} (-1)^{|\hat{\Delta}| - |J|} \langle A^w_{P(J)} \rangle \in K^0(\mathcal{D}(G_E))$. We claim that $\langle A^Y \rangle = \langle A^Y \rangle + \langle A^w \rangle$. Indeed, let $J \subset \hat{\Delta}$ be a proper subset. Since $Y_w \subset Y$ is open, we have $\hat{Y} P(J) = \hat{Y} P(J) \cup \hat{Y} P(J)$, if $J \subset J_w$ and $\hat{Y} P(J) = \hat{Y} P(J)$, otherwise. This implies the equality $\langle A^Y \rangle = \langle A^Y \rangle + \langle A^w \rangle$ in $K^0(\mathcal{D}(G_E))$. Therefore it is enough to show that $\langle A^w \rangle \ast \langle \delta_{I_n^+} \rangle = 0$ for every $w \in \hat{W} \setminus \hat{W}(n)$.

By the definition of $\hat{W}(n)$, for each $w \in \hat{W} \setminus \hat{W}(n)$ there exists $\alpha \in \hat{\Delta}$ such that $w(\alpha) - n > 0$. In particular, $\alpha \in J_w$. It now follows from Claim 4.5 that for every $J \subset J_w \setminus \alpha$ the map $A^w_{P(J \cup \alpha)} \ast \delta_{I_n^+} \to A^w_{P(J)} \ast \delta_{I_n^+}$ is an equivalence. Hence we have an equality $\langle A^w_{P(J)} \rangle \ast \langle \delta_{I_n^+} \rangle = \langle A^w_{P(J \cup \alpha)} \rangle \ast \langle \delta_{I_n^+} \rangle$. Since $\langle A^w \rangle \ast \langle \delta_{I_n^+} \rangle$ equals

$$\sum_{J \subset J_w \setminus \alpha} (-1)^{|\hat{\Delta}| - |J|} (\langle A^w_{P(J)} \rangle \ast \langle \delta_{I_n^+} \rangle - \langle A^w_{P(J \cup \alpha)} \rangle \ast \langle \delta_{I_n^+} \rangle),$$

we see that $\langle A^w \rangle \ast \langle \delta_{I_n^+} \rangle = 0$. \hfill \square

It remains to show Claim 4.5.

4.6. Sketch of a proof of Claim 4.5. First we show the equality $P(J)^+ = P(J')^+ \cdot (P(J)^+ \cap w^{-1} I_n^+ w)$. For this it suffices to show that $U_{w(\alpha)} = wU_{\alpha}w^{-1}$ is contained in $I_n^+$, or equivalently, that $w(\alpha) - n > 0$ for every affine root $\alpha$ which satisfy $U_{\alpha} \subset P(J)^+$ and $U_{\alpha} \not\subset P(J')^+$.

Notice that inclusion $U_{\alpha} \subset P(J)^+$ holds if and only if $\alpha = \sum_{\alpha_i \in \Delta} n_i \alpha_i$ such that $n_i \geq 0$ for all $\alpha_i$, and $n_i > 0$ for some $\alpha_i \notin J$. Thus we have to show that $w(\alpha) - n > 0$ for every $\alpha = \sum_{\alpha_i \in J} n_i \alpha_i$ such that $n_i \geq 0$ for every $\alpha_i \in J'$ and
Let $η$ be an isomorphism (type over $k$) of fibrations of affine spaces. Then the morphism of functors $\eta$ induces an equivalence.

By definition, element $A_{P(J)}^{Y_w} \ast \delta_{I_n^+} \in D(G_E)$ equals $(\eta_J) ! \delta_J$, where $\eta_J$ is the map $I_n^+ \times P(J)^+ \times I_n^+ \to G_E$ given by the rule $(g, y, h) \mapsto gwzw^{-1}g^{-1}h$, and $\delta_J$ is the sheaf $1_{I_n^+} \otimes \delta_P(J)^+ \otimes \delta_{I_n^+}$, where $1_{I_n^+} \in D(I_n^+)$ is the constant sheaf. Moreover, the morphism $f_{P(J), P(J)'} : A_{P(J)}^{Y_w} \ast \delta_{I_n^+} \to A_{P(J)}^{Y_w} \ast \delta_{I_n^+}$ is induced by the morphism $\delta_P(J)^+ \to \delta_P(J)^+$ defined in the proof of Theorem 4.1 (b).

We set $R_J := (P(J)^+ \cap w^{-1}I_n^+ w)$ and consider the action of $R_J$ on $I_n^+ \times P(J)^+ \times I_n^+$ given by $(g, y, h)z := (g, yz, (gwzw^{-1}g^{-1}h))$. Then $\eta_J$ can be written as a composition $I_n^+ \times P(J)^+ \times I_n^+ \xrightarrow{pr_{J'}} (I_n^+ \times P(J)^+ \times I_n^+)/R_J \xrightarrow{\eta_J} G_E$, thus $A_{P(J)}^{Y_w} \ast \delta_{I_n^+} = (\eta_J) ! (pr_{J'}) ! \delta_J$.

Using the equality $P(J)^+ = \mathcal{P}(J')^+ \cdot R_J$, shown at the beginning of the proof, we see that the inclusion $\mathcal{P}(J')^+ \hookrightarrow \mathcal{P}(J)^+$ induces an isomorphism $(I_n^+ \times \mathcal{P}(J')^+ \times I_n^+)/R_{J'} \to (I_n^+ \times \mathcal{P}(J)^+ \times I_n^+)/R_J$ and identifies $\eta_J$ with $\eta_{J'}$.

To finish the proof of the claim, it suffices to show that the adjoint map $\delta_{\mathcal{P}(J)^+} = i_{J'}^+ \delta_{\mathcal{P}(J)^+} \to \delta_{\mathcal{P}(J)^+}$ induces an equivalence $(pr_{J'}) ! \delta_J \to (pr_{J'}) ! \delta_J$. Since $\delta_J \sim pr_{J'} (pr_{J'}) ! \delta_J$, this follows from the following lemma.

**Lemma 4.7.** Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes of finite type over $k$, such that $g$ and $g \circ f$ are smooth morphisms, all of whose fibers are fibrations of affine spaces. Then the morphism of functors $\text{adj} : g_! f_! g' ! \to g_! g'$ is an equivalence.

**Proof.** By our assumption of $g$ and $g \circ f$, both adjoint maps $g_! g' ! \to \text{Id}$ and $g_! f_! g' ! = (g \circ f)_!(g \circ f)^! \to \text{Id}$ are equivalences.

**4.8.** As in 2.16, we deduce from Theorem 4.3 the existence of the limit $A := \lim_Y \alpha(A^Y)$. Explicitly, for every $X \in \mathcal{D}(G_E)$ we have $A(X) \sim \lim_Y (\iota^*(A^Y) \ast X)$. We denote by $pr_Y : A \to \alpha(A^Y)$ the canonical projection.

The following result is Conjecture 2.18 for the case of the unit element.

**Theorem 4.9.** (a) Endomorphism $A \in \text{End} \mathcal{D}(G_E)$ has a natural lift to an element $\tilde{A} \in \mathcal{Z}(G_E)$.

(b) The restriction of $A$ to $G_E^{\text{ess}}$ is locally constant. Moreover, the projection $pr_Y : A \to \alpha(A^Y)$ has the property that for every $\gamma \in G^{\text{ess}}(E)$ there exists $n \in \mathbb{N}$ and such that the restriction $r_{\gamma,n}(pr_Y)$ is an equivalence for all sufficiently large $Y$. 

$n_i > 0$ for some $\alpha_i \in J' \setminus J$. But this follows from the fact that $w(\alpha) > 0$ for every $\alpha \in J$, and $w(\alpha) - n > 0$ for every $\alpha \in J' \setminus J$.

Consider $I_n^+ := I^+ \cap w/I^w_1 I$. Then the map $g \mapsto gw$ defines an isomorphism between $I_n^+$ and $Y_w = IwI/I$. Moreover, since $J \subset J' \subset J_w$ the maps $IwI/I \to IwP(J)/P(J) \to IwP(J')/P(J')$ are isomorphisms.


Sketch of a proof. (a) In order to show that $A$ has a natural lift to an element of $\mathcal{Z}(G_E)$, it is enough to show that $A$ has a compatible system of natural lifts to an element of $\text{End}^P D(G_E)$ for every $P \in \text{Par}$. To see this note that $A$ can be defined as a limit $\alpha(A^Y)$ taken over all closed subschemes of $Fl$, which are $P$-invariant. Then each $A^Y$ is $\text{Ad}(P)$-equivariant, thus $A$ is $\text{Ad}P$-equivariant as well.

(b) Fix $\gamma \in G^{rss}(E)$. First we show that there exists $n \in \mathbb{N}$ such that the restriction $i_{\gamma,n}^*(A^Y) \in D(\gamma I_n^+)$ is constant for all $Y$.

Observe that there exists $m \in \mathbb{N}$ such that $\gamma I_m^+ \subset G_E^{rss}$ and we have equality $Fl_{\gamma'} = Fl_{\gamma}$ of affine Springer fibers for every $\gamma' \in \gamma I_m^+ \cap (G_{\gamma})_E$. This immediately implies that the restriction of $A^Y$ to $\gamma I_m^+ \cap (G_{\gamma})_E$ is constant. Next we note that since each $Y$ is $I$-invariant, each element $A^Y$ is $\text{Ad}I$-equivariant. Since $\text{Ad}I(\gamma I_m^+ \cap (G_{\gamma})_E)$ contains $\gamma I_n^+$ for some $n$, we conclude that the restriction $i_{\gamma,n}^*(A^Y) \in D(\gamma I_n^+)$ is constant.

Now we deduce Theorem 4.9 from Theorem 4.3. Since $A$ is the limit of $\{\alpha(A^Y)\}_Y$, it will suffice to show that for some $n \in \mathbb{N}$ the projective system $\{r_{\gamma,n}(\alpha(A^Y))\}_Y$ stabilizes. Notice that $r_{\gamma,n}(\alpha(A^Y))$ only depends on the restriction $i_{\gamma,n}^*(A^Y) \in D(\gamma I_n^+)$, thus it remains to show that the projective system $\{i_{\gamma,n}^*(A^Y)\}_Y$ stabilizes.

As we have seen in the beginning of the proof, each $i_{\gamma,n}^*(A^Y) \in D(\gamma I_n^+)$ is constant, therefore we have a natural equivalence between $i_{\gamma,n}^*(A^Y)$ and $i_{\gamma,n}^*(A^Y * \delta_{I_n^+})$. Thus it will suffice to show that the projective system $\{A^Y * \delta_{I_n^+}\}_Y$ stabilizes. But this follows from Theorem 4.3.

The following lemma will be used in the next section.

Lemma 4.10. For every $P \in \text{Par}$ we have a natural equivalence $A(\delta_{P^+}) \sim \delta_{P^+}$.

Sketch of a proof. First we show the assertion for $P = I$. Claim 4.4 for $n = 0$ implies that for every closed $I$-invariant subscheme $Y \supset Y(0) \subset Fl$ the map $A^Y * \delta_{I^+} \to A^Y(0) * \delta_{I^+}$ is an equivalence. Thus $A(\delta_{I^+}) \sim A^Y(0) * \delta_{I^+}$. Since $Y(0) = Y_1$, it follows from Claim 4.5 for $w = 1$ and $n = 0$ that the map $A^Y_{P^+} * \delta_{I^+} \to A^Y_{I^+} * \delta_{I^+}$ is an equivalence, therefore $A^Y_{P^+} * \delta_{I^+} \sim A^Y_{I^+} * \delta_{I^+}$. Since $Y_1$ is a point, we get that $A^Y_{I^+} = A^Y_{I^+}(\delta_{I^+}) = \delta_{I^+}$, hence $A(\delta_{I^+}) \sim \delta_{I^+} \sim \delta_{I^+}$.

Now let $P \in \text{Par}$ be arbitrary. It will suffice to construct an isomorphism $A(\delta_{P^+}) \cong \delta_{P^+}$ in the derived category. Using classical Springer theory we have an isomorphism $\delta_{P^+} \cong Av_{P/I}(\delta_{I^+})^{W_{P,sgn}}$. Therefore we have an isomorphism

$$A(\delta_{P^+}) \cong A(Av_{P/I}(\delta_{I^+})^{W_{P,sgn}}) \cong (A(Av_{P/I}(\delta_{I^+})))^{W_{P,sgn}}.$$ 

Since $A$ is $\text{Ad}(P)$-equivariant (by Theorem 4.9 (a)), we have a natural isomorphism of functors $A \circ Av_{P/I} \cong Av_{P/I} \circ A$. Therefore isomorphism $A(\delta_{I^+}) \cong \delta_{I^+}$ implies...
that
\[ A(\delta_{P^+}) \cong (Av_{P/1}(A(\delta_{I^+})))^{W_{P,sgn}} \cong Av_{P/1}(\delta_{I^+})^{W_{P,sgn}} \cong \delta_{P^+}. \]

\[ \square \]

5. Geometric construction and stability of \( z_0 \)

From now on we assume that we are in the situation of Section 3. Note that element \( \tilde{A} \in \mathcal{Z}(G_F) \) from Theorem 4.9 (a) is Frobenius equivariant, and therefore defines an element \( [\tilde{A}] \in \mathcal{Z}_{G(F)} \) of the Bernstein center (see 3.1(d)). By construction, for every \( X \in D^{Fr}(G_E) \), we have an equality \( [\tilde{A}](\delta_{\mathcal{X}}) = [\tilde{A}(\mathcal{X})] \in \mathcal{H}(G(F)). \)

**Theorem 5.1.** Element \( [\tilde{A}] \in \mathcal{Z}_{G(F)} \) equals the projector \( z^0 \in \mathcal{Z}_{G(F)} \) to the depth zero spectrum.

**Sketch of a proof.** By the definition of \( z^0 \), we have to show that \( [\tilde{A}] \) acts as an identity on each irreducible representations \( (\pi, V) \) of \( G(F) \) of depth zero, and acts by zero on each irreducible representations of positive depth.

Assume first that an irreducible representation \( (\pi, V) \) of \( G(F) \) is of depth zero. Then there exists \( P \in Par \) such that \( V^P \neq 0 \), thus it remains to show that \( [\tilde{A}](v) = v \) for all \( v \in V^P \). By Lemma 4.10 and observations of 3.1 we conclude that \( [\tilde{A}][\delta_{P^+}] = [A(\delta_{P^+})] = [\delta_{P^+}] \).

Note that \( [\delta_{P^+}] \in \mathcal{H}(G(F)) \) is the unit of the Hecke algebra \( \mathcal{H}(G(F), P^+) \), hence for each \( v \in V^P \) we have \( [\delta_{P^+}](v) = v \). Therefore by the observation of 1.3(d), we conclude that
\[
[\tilde{A}](v) = [\tilde{A}][\delta_{P^+}](v)] = ([\tilde{A}][\delta_{P^+}]) (v) = [\delta_{P^+}](v) = v.
\]

Assume now that an irreducible representation \( (\pi, V) \) of \( G(F) \) is of depth \( r > 0 \). Then the result of Moy-Prasad [MP1, MP2], there exists a parahoric subgroup \( Q \in Par \) and a non-degenerate character \( \xi : Q_r/Q_{r,+} \rightarrow \mathbb{C}^* \) such that the \( \xi \)-isotypical component \( V_\xi := \text{Hom}_{Q_r}(\xi, V) \) is non-zero. It remains to show that \( [\tilde{A}](v) = 0 \) for every \( v \in V_\xi \). Observe that \( \xi \) defines an element \( h_\xi := [\xi Q_r] \in \mathcal{H}(G(F)) \). Then \( h_\xi(v) = v \) for every \( v \in V_\xi \), hence arguing as in the depth zero case, it is enough to show that \( [\tilde{A}](h_\xi) = 0 \).

Recall that by definition (see 4.8), \( [\tilde{A}](h_\xi) \) equals \( [\xi(A^Y)] * h_\xi = [A^Y] * h_{\xi-1} \) for each sufficiently large \( Y \) and also \( A^Y = \sum_{P \in Par} (-1)^{rk G - rk P} [A^Y_P] \).

It suffices to show that \( [A^Y_P] * h_\xi = 0 \) for every locally closed subscheme \( Y \subset F^l \) and every \( P \in Par \). Since \( A^Y_P = Av^Y_P(\delta_{P^+}) \), it suffices to check that for every \( g \in G(F) \) the restriction of \( \xi \) to \( Q_r \cap gP^+g^{-1} \) is non-trivial. The latter assertion follows from the fact that \( r > 0 \) and \( \xi \) is non-degenerate (see [MP2, Prop. 6.4]). \[ \square \]
Our next goal is to propose an explicit formula for $\mu_{z^0}$ and to deduce its stability. For this we need certain results on the homology of affine Springer fibers.

5.2. Homology of affine Springer fibers. (a) For each $\gamma \in G^{rss}(F)$, we denote by $F_l \gamma \subset F_l$ the corresponding affine Springer fiber, and by $H_i(F_l \gamma) = H_i(F_l \gamma, \mathbb{Q}_l)$ the corresponding homology group. For each subscheme $Y \subset F_l$ we denote by $Y \cap F_l \gamma \subset F_l$.

(b) As it was observed by Lusztig ([Lu2]), each $H_i(F_l \gamma)$ is equipped with an action of the affine Weyl group $\tilde{W}$ of $G$. More precisely, mimicking the proof of Theorem 4.1 (a), we show that for each subscheme $Y \subset F_l$ and each $P \in Par$, the homology group $H_i((Y P)_\gamma)$ is equipped with an action of the finite Weyl group $W_P$. Then $H_i(F_l \gamma) = \colim Y H_i((Y P)_\gamma)$ is equipped with $W_P$-action for each $P \in Par$, which together define a $\tilde{W}$-action.

(c) The natural action of the group $G(E)$ on $F_l$ induces an action of the centralizer $G_\gamma(E)$ on the affine Springer fiber $F_l \gamma$, hence on the homology groups $H_i(F_l \gamma)$. Moreover, the actions of $\tilde{W}$ and $G_\gamma(E)$ on $H_i(F_l \gamma)$ commute.

(d) Consider finitely generated group $\Lambda_\gamma := X_*(G_\gamma)_r$. Using Kottwitz lemma, [Ko Lem 2.2], $\Lambda_\gamma$ can be identified with the group of connected components of the ind-group scheme $(G_\gamma)_E$. Therefore there exists a canonical surjective homomorphism $G_\gamma(E) \to \Lambda_\gamma$ such that the action of $G_\gamma(E)$ on $H_i(F_l \gamma)$ is induced by the action of $\Lambda_\gamma$.

(e) By (b)-(d), the homology groups $H_i(F_l \gamma)$ are equipped with commuting actions of $\tilde{W}$ and $\Lambda_\gamma$.

5.3. The canonical map. (a) Recall that $\tilde{W} = \Lambda \rtimes W$ of the lattice $\Lambda := X_*(T)$ is the group cocharacters of the abstract Cartan $T$ of $G$.

(b) Since $G_\gamma \subset G$ is a maximal torus, we have natural isomorphism $\varphi : T \to G_\gamma$ defined up to $W$-conjugacy. Any such $\varphi$ induces a homomorphism $\pi_\varphi : \Lambda = X_*(T) \to X_*(G_\gamma) \to \Lambda_\gamma$.

hence an algebra homomorphism

$$\text{pr}_\gamma : \mathbb{Q}_l[\Lambda]^W \to \mathbb{Q}_l[\Lambda] \to \mathbb{Q}_l[\Lambda_\gamma].$$

Notice that since $\varphi$ is defined up to $W$-conjugacy, the homomorphism $\pi_\varphi$ is unique up to a $W$-conjugacy therefore $\text{pr}_\gamma$ is independent of $\varphi$.

The following result asserts that the actions of $\tilde{W}$ and $\Lambda_\gamma$ from 5.2 are "compatible". Assume that the characteristic of $k$ is sufficiently large.
Theorem 5.4. There exists a finite $\tilde{W} \times \Lambda_\gamma$-equivariant filtration $\{F_j H_i(FL_\gamma)\}_j$ of $H_i(FL_\gamma)$ such that the action of $\mathcal{O}_I[\Lambda]^W \subset \mathcal{O}_I[\Lambda]$ on each graded piece $\text{gr}^j H_i(FL_\gamma)$ is induced from the action of $\mathcal{O}_I[\Lambda_\gamma]$ via homomorphism $\text{pr}_\gamma$.

Sketch of a proof. The main ingredient of the proof is an analogous result of Yun ([Yun, Thm 1.3]) for Lie algebras. The argument of Yun is very involved. First he treats the case when $x \in \text{Lie} I \cap \text{Lie} \text{G}^{\text{rss}}(F)$ is regular semi-simple whose reduction $\overline{x} \in \text{Lie} I/\text{Lie} I^+$ is regular semi-simple. In this case, the affine Springer fiber $FL_x$ is discrete, Lusztig’s action of $\tilde{W}$ on $H_i(FL_x)$ comes from an action of $\tilde{W}$ on $FL_x$. Moreover, the restriction of this action to $\Lambda \subset \tilde{W}$ coincides with the geometric action of $\Lambda = \Lambda_\gamma$.

To show the result in general, Yun uses global method, extending some of the results of Bao Chau Ngo ([Ngo]). To deduce the assertion for groups, we use quasi-logarithms and the topological Jordan decomposition (see [BV2]).

Corollary 5.5. For each $\gamma \in \text{G}^{\text{rss}}(F)$, each homology group $H_i(FL_\gamma)$ is a finitely generated $Q_l[\tilde{W}]$-module.

Sketch of a proof. Each $H_i(FL_\gamma)$ is a finitely generated $G_\gamma(E)$-module, therefore the assertion follows from Theorem 5.4 (compare [BV2]).

Remark. This statement appears also as Conjecture 3.6 in [Lu4]. It is also mentioned in loc. cit. that the statement should follow from the result of [Yun].

5.6. Note that it follows from Theorem 5.4 and Theorem 4.9 (b) that the restriction of $\mu_0 = \mu_{[\lambda]}$ to $G^{\text{rss}}(F)$ has the form $\phi_{z_0} \mu_0$ for some $\phi_{z_0} \in C^\infty(G^{\text{rss}}(F))$. Again $\mu_0$ is the Haar measure on $G(F)$ normalized by condition that $\int_{I^+} \mu_0 = 1$.

Taking into account Theorem 5.1, the following conjecture is Conjecture 3.4 (b) for the unit element. In section 6 we prove it for $G = SL_2$.

Conjecture 5.7. For each topologically unipotent $\gamma \in G^{\text{rss}}(F)$, we have an equality

\[
\phi_{z_0}(\gamma) = \text{Tr}(\text{Fr}_s, H_*(\tilde{W}, H_*(FL_\gamma)^{\text{sgn}})),
\]

where by $^{\text{sgn}}$ we denote twist by the sign character.

The following result is a particular case of Conjecture 3.6 (a).

Theorem 5.8. Assume that Conjecture 6.6 holds. Then $z^0 \in \mathcal{Z}_{G(F)}$ is stable.

Sketch of a proof. Since the characteristic of $k$ is sufficiently large, if follows from a theorem of Harish-Chandra that it is enough to show the stability of the restriction
of \(\mu_\varphi\) to \(G^{rss}(F)\). In other words, we have to check that \(\phi_\varphi(\gamma) = \phi_\varphi(\gamma')\) for every stably conjugate \(\gamma, \gamma' \in G^{rss}(F)\).

Since \(\Av^V(\delta_{\mathcal{P}^+}) \in \mathcal{D}(G_E)\) is supported on topologically unipotent elements of \(G_E\), the measure \(\mu_{[\mathbf{\Lambda}]} = \mu_\varphi\) and hence also function \(\phi_\varphi\) is supported on the topologically unipotent elements of \(G^{rss}(F)\).

Thus we can assume that \(\gamma\) is topologically unipotent. In this case, by formula (\(\Pi\)) implies that it is enough to show that \(G_\gamma(E)\) acts unipotently on each \(H_i(\tilde{W}, H_j(Fl_\gamma)^{sgn})\). Since \(H_i(\tilde{W}, H_j(Fl_\gamma)^{sgn}) = H_i(\Lambda, H_j(Fl_\gamma))^{W, sgn}\), it is enough to show that \(G_\gamma(E)\) acts unipotently on each \(H_i(\Lambda, H_j(Fl_\gamma))\).

By definition, the group \(\Lambda\) hence the algebra \(\oplus_{W}[\Lambda]^W\) acts trivially on \(H_i(\Lambda, H_j(Fl_\gamma))\). Therefore assertion follows from the theorem of Yun (Theorem 5.4) asserting that two actions are compatible. \(\square\)

6. Proof of Conjecture 5.7 for \(G = SL_2\).

Though many of the arguments can be carried out in general, in this section we will always assume that \(G = SL_2\).

6.1. Notation. (a) Let \(L\) be a field of characteristic zero. The group algebra \(A := L[\tilde{W}]\) is equipped with a natural filtration \(\{A_n\}_n\), where \(A_n\) is the span of all \(w \in \tilde{W}\) such that \(l(w) \leq n\).

(b) Let \(V\) be an \(A\)-module. Recall that an increasing filtration \(V = \{V_n\}_n\) of \(V\) by \(L\)-vector spaces is called a good filtration if \(A_1(V_n) \subset V_{n+1}\) for every \(n \in \mathbb{N}\) and \(A_1(V_n) = V_{n+1}\), if \(n\) is sufficiently large.

(c) We denote by \(\overline{A} := \oplus A_i/A_{i-1}\) the graded algebra of \(A\). It is known that \(\overline{A}\) is Noetherian. For every \(w \in \tilde{W}\), we denote by \(\overline{w} \in \overline{V}_{l(w)}\) the image of \(w\).

(d) Let \(\tilde{S} = \{s_0, s_1\} \subset \tilde{W}\) be the set of simple affine reflections. For each \(s \in \tilde{S}\), we denote by \(\langle s \rangle \subset \tilde{W}\) the cyclic group generated by \(s\).

The following purely algebraic result is crucial for our argument.

**Theorem 6.2.** Let \(V\) be a finitely generated \(A\)-module. Then for every good filtration \(\{V_n\}_n\) of \(V\) and every sufficiently large \(n\), we have an equality

\[
\sum_i (-1)^i \dim H_i(\tilde{W}, V) = \dim(V_n + s_0V_n)^{(s_0)} + \dim(V_n + s_1V_n)^{(s_1)} - \dim V_n.
\]

The proof Theorem 6.2 is based on the following lemma.

**Lemma 6.3.** Let \(\overline{V} = \oplus_{n \geq 0} \overline{V}_n\) be a finitely generated graded \(\overline{A}\)-module. Then for each sufficiently large \(n\) we have

(a) \(\overline{s}_0(\overline{V}_n) + \overline{s}_1(\overline{V}_n) = \overline{V}_{n+1}\) and \(\overline{s}_0(\overline{V}_n) \cap \overline{s}_1(\overline{V}_n) = 0\).
(b) \( \ker \overline{s} \cap V_n = \im \overline{s} \cap V_n \) for every \( s \in \overline{S} \).

**Proof.** (a) We have to show that \( L \)-vector spaces \( \overline{V}/(\im s_0 + \im s_1) \) and \( \im s_0 \cap \im s_1 \subset \overline{V} \) are finite-dimensional. Since \( \overline{s}_0 \) and \( \overline{s}_1 \) generate the algebra \( \overline{A} \) over \( L \), while \( \overline{V} \) is finitely generated over \( \overline{A} \), the quotient \( \overline{V}/(\im s_0 + \im s_1) \) is a finitely generated \( \overline{A}/(\overline{s}_0, \overline{s}_1) \)-module, thus a finite dimensional \( L \)-vector space.

Since \( \overline{s}_0^2 = \overline{s}_1^2 = 0 \), the intersection \( \im \overline{s}_0 \cap \im \overline{s}_1 \) is contained in \( \overline{U} := \ker \overline{s}_0 \cap \ker \overline{s}_1 \), so it will suffice to show that \( \overline{U} \) is finite dimensional. Since \( \overline{s}_0 \) and \( \overline{s}_1 \) generate \( \overline{A} \), we conclude that \( \overline{U} \subset \overline{V} \) is an \( \overline{A} \)-submodule. Since \( \overline{A} \) is Noetherian, \( \overline{U} \) is a finitely generated \( \overline{A} \)-module. But each \( \overline{s}_i \) acts on \( \overline{U} \) by zero, therefore \( \overline{U} \) is a finitely generated \( \overline{A}/(\overline{s}_0, \overline{s}_1) \)-module, thus a finite dimensional \( L \)-vector space.

(b) Since \( \overline{s}_0^2 = 0 \), we have an inclusion \( \im \overline{s} \subset \ker \overline{s} \), hence it is enough to show that the quotient \( \ker \overline{s}/\im \overline{s} \) is finite dimensional. Using the induction on the number of generators of \( \overline{V} \), we can assume that \( \overline{V} \) is generated by one element \( v \in \overline{V}_k \). We claim that in this case we have an inclusion \( \ker \overline{s} \subset \im \overline{s} + (\ker \overline{s}_0 \cap \ker \overline{s}_1) + Lv \), which implies the assertion, since by (a), \( \ker \overline{s}_0 \cap \ker \overline{s}_1 \) is finite-dimensional.

Every \( x \in \ker \overline{s} \) can be written as a finite linear combination \( x = \sum_{w \in \overline{W}} a_w \overline{w}(v) \), where \( a_w \in L \) for \( w \in \overline{W} \). We set \( x_+ := \sum_{w \neq 1 \in \overline{W}, sw > w} a_w \overline{w}(v) \) and \( x_- := \sum_{w \in \overline{W}, sw < w} a_w \overline{w}(v) \). Then \( x = x_+ + x_- + a_1 v \). Since for every \( w \) with \( sw < w \) we have \( \overline{w} = \overline{s} \overline{w} \overline{s} \overline{w} \), we have \( x_- \in \im \overline{s} \) it will suffice to show that \( x_+ \in \cap_{\ell \in \overline{S}} \ker \overline{s} \).

Assume first that \( t \neq s \). Notice that for every \( w \neq 1 \in \overline{W} \) satisfying \( sw > w \) also satisfies \( tw < w \). Therefore arguing as above we conclude that \( x_+ \in \cap_{\ell \in \overline{S}} \ker \overline{s} \).

Thus it remains to show that \( x_+ \in \ker \overline{s} \).

Since \( x \in \ker \overline{s} \), we have \( \overline{s}(x) = \overline{s}(x_+) + \overline{s}(x_-) + a_1 \overline{s}(v) = 0 \). Since \( x_- \) belongs to \( \im \overline{s} \subset \ker \overline{s} \), this implies that \( \overline{s}(x_+) = -a_1 \overline{s}(v) \). Using the fact that \( a_1 \overline{s}(v) \in \overline{V}_{k+1} \), while \( \overline{s}(x_+) = \bigoplus_{i > k+1} \overline{V}_i \), we conclude that \( \overline{s}(x_+) = -a_1 \overline{s}(v) = 0 \), thus \( x_+ \in \ker \overline{s} \).

\( \square \)

**Corollary 6.4.** Let \( V \) be a finitely generated \( A \)-module with a good filtration \( \{V_n\}_n \). Then every \( s \in \overline{S} \) and sufficiently large \( n \), we have the following:

(a) For each \( s \in \overline{S} \), we have an equality \( V_{n+1} \cap sV_{n+1} = V_n + sV_n \).

(b) For each \( s \in \overline{S} \) we have an isomorphism

\[
V_{n+1}/(V_n + sV_n) \cong (V_{n+1} + sV_{n+1})^{(s)}/(V_n + sV_n)^{(s)}.
\]

(c) The canonical map \( V_{n+1}/V_n \to V_{n+1}/(V_n + s_0 V_n) \oplus V_{n+1}/(V_n + s_1 V_n) \) is an isomorphism.

**Proof.** Since \( \{V_n\}_n \) is a good filtration, the graded module \( \overline{V} := \oplus_n (V_n/V_{n-1}) \) is a finitely generated \( \overline{A} \)-module, so Lemma 6.3 applies.
(a) The inclusion $V_n + sV_n \subset V_{n+1} \cap sV_{n+1}$ holds for all $n$. Conversely, let $x \in V_{n+1} \cap sV_{n+1}$. Then the image $\overline{x} \in \overline{V}_{n+1} = V_{n+1}/V_n$ of $x$ belongs to $\text{Ker} \overline{x} \cap \overline{V}_{n+1}$. Then by Lemma 6.3 (b) for each sufficiently large $n$, we have $\overline{x} \in \text{Im} \overline{x} \cap \overline{V}_{n+1}$, thus $x \in V_n + sV_n$.

(b) Consider the composition

$$\text{pr}_s : V_{n+1} \rightarrow (V_{n+1} + sV_{n+1})^{(s)} \rightarrow (V_{n+1} + sV_{n+1})^{(s)}/(V_n + sV_n)^{(s)},$$

where the first map is $x \mapsto x + sx$, and the second one is the projection. Then $\text{pr}_s$ is surjective, so it remains to show that $\text{Ker} \text{pr}_s = V_n + sV_n$. The inclusion $V_n + sV_n \subset \text{Ker} \text{pr}_s$ is clear. To show the opposite inclusion, notice that if $x \in \text{Ker} \text{pr}_s$, then $x + sx \in V_n + sV_n$, hence $sx \in V_{n+1}$, thus $\overline{x} \in \overline{V}_{n+1}$ belongs to $\text{Ker} \overline{x}$. Now by Lemma 6.3 (b) we have $\overline{x} \in \text{Im} \overline{x} \cap \overline{V}_{n+1}$, thus $x \in V_n + sV_n$.

(c) We claim that for a sufficiently large $n$, the natural morphism

$$V_{n+1}/V_n \rightarrow V_{n+1}/(V_n + sV_n) \oplus V_{n+1}/(V_n + sV_n)$$

is an isomorphism. For that we have to show that $(V_n + sV_n) + (V_n + sV_n) = V_{n+1}$ and $(V_n + sV_n) \cap (V_n + sV_n) = V_n$. But this follows immediately from Lemma 6.3 (a).

Now we are ready to prove Theorem 6.2.

Proof of Theorem 6.2. First we claim that the right hand side of (2) stabilizes. For this it is sufficient to show that for each sufficiently large $n$, we have an equality

$$\dim V_{n+1}/V_n = \sum_{s \in S} \dim(V_{n+1} + sV_{n+1})^{(s)}/(V_n + sV_n)^{(s)}.$$

But this is a consequence of Corollary 6.4 (b) and (c).

Denote by $\text{RHS}_V$ the limit of the expression of the right hand side of (2). By a standard argument, $\text{RHS}_V$ is independent of the good filtration $V$, thus we will denote it simply by $\text{RHS}_V$.

Denote by $K_0(A)$ the Grothendieck group of the category of finitely generated $A$-modules, and set $K_0(A)_{Q} := K_0(A) \otimes_{Z} Q$. Note that the map $V \mapsto \text{RHS}_V$ extends to the homomorphism $\text{RHS} : K_0(A)_{Q} \rightarrow Q$ of $Q$-vector spaces. Since the map $V \mapsto \dim(H_s(W, V))$ also extends to the homomorphism $\text{LHS} : K_0(A)_{Q} \rightarrow Q$ of $Q$-vector spaces, it is easy to see that $\text{LHS}(\{V\}) = \text{RHS}(\{V\})$ for elements $\{V\}$ belonging to some basis of $K_0(A)_{Q}$. Let $\hat{T} := \text{Spec} L[A]$ be the dual torus of $T$. Then $\hat{T}$ is equipped with an action of the Weyl group $W$, and $K_0(A)_{Q}$ is canonically identified with the equivariant $K$-theory $K^W_0(\hat{T})_{Q}$. As is well-known (see, for example, [VI]) $K^W_0(\hat{T})_{Q}$ is generated over $Q$ by the structure sheaf $\mathcal{O}_{\hat{T}}$ and coherent sheaves, supported on the finite
subscheme \{-1, 1\} \subset \widehat{T}. Passing back to \(A\)-modules, we conclude that it is enough to check (1) in the case when \(V = A\) and in the case when \(V\) is finite dimensional.

Assume first that \(V = A\). In this case, \(H_i(\widehat{W}, V) = \text{Tor}_i^A(L, A) = 0\) if \(i > 0\) and \(H_0(\widehat{W}, V) = L \otimes_A A = L\), thus the left hand side of (1) is one. Consider filtration \(V_n := A_n\). Then \((V_n + sV_n)^{(s)}\) has a basis consisting of elements \(w + sw\), where \(sw > w\) and \(l(w) \leq n\). Thus \(\dim(V_n + sV_n)^{(s)} = n + 1\), while \(\dim V_n = 2n + 1\), thus the right hand side of (1) is also one.

We now consider the case where \(V\) is finite-dimensional. In this case, it follows from the main result of [BV1], that for each \(n\), and every sufficiently large \(l\), there is a unique \(\tilde{W}\)-invariant. Since \(V\) is \([\sigma]\)-locally finite, we can assume that \(\sigma\) acts on \(V\) by a character of \(\langle \sigma \rangle\), thus \(V\) is of finite type, thus \(\dim(V_n + sV_n)^{(s)} = n + 1\), while \(\dim V_n = 2n + 1\), thus the right hand side of (1) is also one.

We now consider the case where \(V\) is finite-dimensional. In this case, we have to show the equality \(\dim H_*(\widehat{W}, V) = \dim V_{(s_0)} + \dim V_{(s_1)} - \dim V\). Since \(0 \to L[\widehat{W}] \to L[\widehat{W}/(s_0)] \oplus L[\widehat{W}/(s_1)] \to L \to 0\) is a projective resolution of the \(A\)-module \(L\), the equality follows.

6.5. Notation. Let \(\langle \sigma \rangle\) be a cyclic group with generator \(\sigma\), and let \(V\) be an \(L[\langle \sigma \rangle]\)-module. We say that \(V\) is \(\langle \sigma \rangle\)-locally finite, if \(V\) is a union of finite dimensional submodules.

Corollary 6.6. Let \(V\) be an \(L[\langle \sigma \rangle] \times \langle \sigma \rangle\)-module, which is finitely generated \(L[\langle \sigma \rangle] \)-module and \(\langle \sigma \rangle\)-locally finite \(L[\langle \sigma \rangle]\)-module. Then for every good filtration \(\{V_n\}_n\) on \(V\) and every sufficiently large \(n\) we have an equality

\[
\text{Tr}(\sigma, H_*(\widehat{W}, V)) = \text{Tr}(\sigma, (V_n + s_0V_n)^{(s_0)}) + \text{Tr}(\sigma, (V_n + s_1V_n)^{(s_1)}) - \text{Tr}(\sigma, V_n)
\]

Proof. Since \(V\) is \(L[\langle \sigma \rangle]\)-locally finite, \(V\) decomposes as a sum of generalized \(\sigma\)-eigen spaces \(V = \bigoplus_{(a)} V_{(a)}\). Since action of \(\sigma\) on \(V\) commutes with \(\widehat{W}\), each \(V_{(a)}\) is automatically \(\tilde{W}\)-invariant. Since \(V\) is finitely generated over \(\tilde{W}\), we conclude that this decomposition is finite, so we can assume that \(\sigma\) has a unique generalized eigen value \(a\).

In this case, \(V = \bigcup_n \text{Ker}(\sigma - aI)^n\). Since each \(\text{Ker}(\sigma - aI)^n\) is \(\tilde{W}\)-invariant, we conclude from the finite generation of \(V\) that \(V = \text{Ker}(\sigma - aI)^n\) for some \(n\). Next, we can replace \(V\) by its semi-simplification, thus assuming that \(\sigma\) acts on \(V\) by a scalar \(a\). Twisting \(V\) by a character of \(\langle \sigma \rangle\), we can assume that \(\sigma\) acts on \(V\) trivially. In this case the assertion follows from Theorem [6.2].

6.7. Application to affine Springer fibers.

(a) For each \(n \in \mathbb{N}\), we denote by \(Fl_{\leq n} \subset Fl\) be the union \(\bigcup_w Fl_{\leq w}\) taken over all \(w \in \tilde{W}\) such that \(l(w) \leq n\). Also for every \(s \in \tilde{S}\) we set \(Fl_{\leq n,s} := Fl_{\leq n}P(s)\).

(b) Note that since \(G = SL_2\) element \(\gamma \in G^{rss}(F)\) is either elliptic or split. Moreover, if \(\gamma \in G^{rss}(F)\) is elliptic, then \(Fl_{\gamma}\) is of finite type, thus \(Fl_{\gamma} = Fl_{\leq n}\), if \(n\) is sufficiently large.

(c) Assume now that \(\gamma \in G^{rss}(F)\) is split, and that \(G_\gamma(O_F)\) is contained in \(I\). In this case, it follows from the main result of [BV1], that for each \(n\) sufficiently large
maps $H_i(Fl_{\gamma}^n) \to H_i(Fl_{\gamma})$ and $H_i(Fl_{\gamma}^{n,s}) \to H_i(Fl_{\gamma})$ are injective. Moreover, it follows from Theorem 5.4 (see [BV2]) that $\{H_i(Fl_{\gamma}^n)\}_n$ is a good filtration of $H_i(Fl_{\gamma})$.

**Lemma 6.8.** Assume that either $\gamma \in G^{rss}(F)$ is elliptic or that $\gamma \in G^{rss}(F)$ is split, and that $G_\gamma(O_F)$ is contained in $I$. Then for each $i \in \mathbb{N}$, $s \in S$ and a sufficiently large $n$, we have an equality

\[(4) \quad H_i(Fl_{\gamma}^n) + sH_i(Fl_{\gamma}^n) = H_i(Fl_{\gamma}^{n,s}) \subset H_i(Fl_{\gamma}).\]

**Proof.** As we mentioned in 6.7, for all sufficiently large $n$, we get inclusions

\[H_i(Fl_{\gamma}^n) \subset H_i(Fl_{\gamma}^{n,s}) \subset H_i(Fl_{\gamma}^{n+1}) \subset H_i(Fl_{\gamma}).\]

Recall that the subspace $H_i(Fl_{\gamma}^{n,s}) \subset H_i(Fl_{\gamma})$ is $s$-invariant (see 5.2 (b)). Therefore we have inclusions

\[(5) \quad H_i(Fl_{\gamma}^n) + sH_i(Fl_{\gamma}^n) \subset H_i(Fl_{\gamma}^{n+1}) \subset H_i(Fl_{\gamma}) \cap sH_i(Fl_{\gamma}^{n+1}).\]

Since $\{H_i(Fl_{\gamma}^n)\}_n$ is a good filtration of $H_i(Fl_{\gamma})$, it follows from Corollary 6.4 (a) that the right hand side of (5) is contained in the left hand side when $n$ is sufficiently large.

Now we are ready to prove Conjecture 5.7 for $G = SL_2$

**6.9. Proof of Conjecture 5.7 for $G = SL_2$.** It follows from Theorem 5.4 that the left hand side of (1) equals $(\mu_2/\mu_0)(\gamma) = (\mu_{\gamma_0}/\mu_0)(\gamma)$. Then by Theorem 4.9 (b), it is equal to $([A^{FlF\leq n}]_{\mu_0}(\gamma)$ for each sufficiently large $n$. We see that the left hand side of (1) equals

\[(6) \quad \sum_{s \in S} \dim H_i(Fl_{\gamma}^{n,s})_{\langle v \rangle} - \dim H_i(Fl_{\gamma}^n)\]

for each sufficiently large $n \in \mathbb{N}$.

Consider the \(\overline{Q}_l\)-representation $V := H_i(Fl_{\gamma})^{sgn}$ of $\overline{W} \times Fr$ over $\overline{Q}_l$. By Corollary 5.5, $V$ is a finitely generated $\overline{Q}_l[\overline{W}]$-module. Since $H_i(Fl_{\gamma}) = \text{colim}_n H_i(Fl_{\gamma}^n)$ and each $H_i(Fl_{\gamma}^n)$ is finite dimensional and Fr-equivariant, $V$ is a locally finite $\overline{Q}_l[Fr]$-module.

Note that both sides of (1) do not change if we replace $\gamma$ by its $G(F)$-conjugate. Therefore we can assume that we are in the situation of 6.7, that is, either $\gamma$ is elliptic or $\gamma$ is split and $G_\gamma(O_F) \subset I$. In both cases, $\{H_i(Fl_{\gamma}^n)\}_n$ is a good filtration of $V$, so it follows from a combination of Corollary 6.6 and Lemma 6.8 that the expression of (6) equals the right hand side of (1). \qed
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