\( \epsilon \)-Nash Equilibria for Major Minor LQG Mean Field Games with Partial Observations of All Agents

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Abstract

The partially observed major minor LQG and nonlinear mean field game (PO MM LQG MFG) systems where it is assumed the major agent’s state is partially observed by each minor agent, and the major agent completely observes its own state have been analysed in the literature. In this paper, PO MM LQG MFG problems with general information patterns are studied where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent’s state. The assumption of partial observations by all agents leads to a new situation involving the recursive estimation by each minor agent of the major agent’s estimate of its own state. For a general case of indefinite LQG MFG systems, the existence of \( \epsilon \)-Nash equilibria together with the individual agents’ control laws yielding the equilibria are established via the Separation Principle.

1 Introduction

Mean field game theory (MFG) studies the existence of Nash equilibria and corresponding strategies for generating them for stochastic dynamic games between large population of agents [1–3]. Basically, the theory exploits the relationship between the finite and corresponding infinite limit population

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problems to find control solutions with negligible error for the finite population problem. The key feature of this approach is the solution of both a Hamilton-Jacobi-Bellman (HJB) and a Fokker-Planck-Kolmogorov (FPK) equation, or equivalently a McKean-Vlasov stochastic differential equation, which are linked by the state distribution of the generic agent, namely the mean field of the system.

MFG theory offers a framework in which each agent interacts with the aggregate effect of all other agents. In fact, each agent optimizes its individual cost functional based on local information on its own state and information on the overall population state, i.e. the mean field. The analysis of this set of problems originated in [2,4–6], and independently in [7–9]. In [10,11], the authors analyse and solve the completely observed linear quadratic Gaussian (LQG) systems case where there is a major agent (i.e. non-asymptotically vanishing as the population size goes to infinity) together with a population of minor agents (i.e. individually asymptotically negligible). The new feature in this case is that the mean field becomes stochastic but by minor agent state extension the existence of $\epsilon$-Nash equilibria is established together with the individual agents’ control laws that yield the equilibria [11]. A convex analysis method is utilized in [12] to rederive the solutions to completely observed major minor (CO MM) LQG MFG systems, where no assumption is imposed on the evolution of the mean field in advance. A hybrid optimal control approach to CO MM LQG MFG systems with switching and stopping strategies is presented in [13,14].

In the purely minor agent case the mean field is deterministic and this obviates the need for observations on other agents’ states. This is a separate issue from that of an agent estimating its own state from partial observations on that state, see [15]. However, when a system has a major agent whose state is partially observed the standard MFG procedure for generating a Nash equilibrium needs to be extended to include estimates of the major agent’s state generated by each minor agent. In [16–18], partially observed LQG mean field games with major and minor agents (PO MM LQG MFG) have been investigated and in [19–21], a nonlinear generalization of this problem is considered. The main results in those papers are obtained with the assumptions that (i) the major agent’s state is partially observed by the minor agents and (ii) the major agent has complete observations of its own state.

The PO MM LQG MFG problems where the major agent partially observes its own state and every minor agent has complete observations on its own state and the major agent’s state are addressed in [22]. An initial investigation of the case where assumption (i) holds and the major agent has also partial observations on its own state was presented in [23]. The thorough investigation of the subject
matter together with the complete proofs and illustrative numerical experiments are reflected in the current paper (see also [24] for the case where all agents partially observe a common process). The main contributions of the current paper are summarized in the following points:

- PO MM LQG MFG problems with general information patterns are studied where (i) the major agent has partial observations of its own state, and (ii) each minor agent has partial observations of its own state and the major agent’s state.

- In the theory we present for this new, general case where (i) the major agent recursively estimates its own state, and (ii) each minor agent recursively estimates its own state, and the major agent’s estimate of its own state (in order to estimate the major agent’s feedback control input). In addition, both the major agent and minor agents generate estimates of the system’s mean field. We remark that an infinite regress does not happen here due to the asymmetric major minor (MM) feature of the MFG problem.

- MFG theory is extended to cover the general case of indefinite LQG MFG systems which alleviates positive definiteness condition of weight matrices in linear quadratic cost functionals.

- The existence of $\epsilon$-Nash equilibria together with the individual agents’ control laws yielding the equilibria is established; this is achieved in the PO MM LQG case by an application of the Separation Principle which also yields computationally tractable solutions while in the nonlinear case is far more complex (see [19–21]).

- This extension of the situation in [18], where only assumption (ii) holds, is in particular motivated by optimal execution problems in financial markets where there exist one institutional trader (interpreted as major agent) and a large population of high frequency traders (interpreted as minor agents) who attempt to maximize their own wealth. To obtain the Nash equilibrium best response trading strategy, each minor agent estimates the major agent’s inventory and trading rate based on its partial observations of market state which entails the estimation of the major agent’s self estimates. The reader is referred to the works [25–28] for more details on financial applications.
The rest of the paper is organized as follows. Section 2 introduces Partially Observed Major-Minor LQG MFG systems. The estimation and control problems for PO MM LQG MFG systems are addressed in Section 3. The simulation results and the concluding remarks are presented in Section 4 and Section 5, respectively.

2 Partially Observed Major-Minor LQG MFG Systems

A class of major-minor LQG MFG (MM LQG MFG) systems including a large population of $N$ stochastic dynamic minor agents with a stochastic dynamic major agent is considered where the agents are coupled through their cost functionals.

2.1 Dynamics

The dynamics of the major and minor agents in the class of systems under consideration are, respectively, given by

$$dx_0 = [A_0x_0 + B_0u_0]dt + D_0dw_0,$$

$$dx_i = [A(\theta_i)x_i + B(\theta_i)u_i + Gx_0]dt + Dw_i,$$

where $t \geq 0$, $1 \leq i \leq N < \infty$, $\theta_i \in \Theta$, where $\Theta$ is a parameter set. Here $x_i \in \mathbb{R}^n$, $0 \leq i \leq N$, are the states, $u_i \in \mathbb{R}^m$, $0 \leq i \leq N$, are control inputs, \{\{w_i, 0 \leq i \leq N\}\} denote $(N + 1)$ independent standard Wiener processes in $\mathbb{R}^r$ on an underlying probability space $(\Omega, \mathcal{F}, P)$ which is sufficiently large that $w$ is progressively measurable with respect to the filtration $\mathcal{F}^w = (\mathcal{F}^w_t; t \geq 0)$ on $\mathcal{F}$, and $\mathbb{E}w_iw_i^T = \Sigma$.

Assumption 1. The initial states $\{x_i(0), 0 \leq i \leq N\}$ defined on $(\Omega, \mathcal{F}, P)$ are identically distributed, mutually independent and also independent of $\mathcal{F}^w$, with $\mathbb{E}x_i(0) = 0$. Moreover, $\sup_i\mathbb{E}\|x_i(0)\|^2 \leq c < \infty$, $0 \leq i \leq N < \infty$, with $c$ independent of $N$.

The matrices $A_0, B_0, D_0, G,$ and $D$ are constant matrices of appropriate dimensions. From 2, $A(.)$ and $B(.)$ depend on the parameter $\theta$ which specifies the minor agent’s type. Minor agents are given in $K$ distinct types with $1 \leq K < \infty$. The notation $\mathcal{I}_k$ is defined as

$$\mathcal{I}_k = \{i : \theta_i = k, 1 \leq i \leq N\}, \quad 1 \leq k \leq K.$$
where the cardinality of $\mathcal{I}_k$ is denoted by $N_k = |\mathcal{I}_k|$. Then, $\pi^N = (\pi_1^N, ..., \pi_K^N)$, $\pi_k^N = \frac{N_k}{N}$, $1 \leq k \leq K$, denotes the empirical distribution of the parameters $(\theta_1, ..., \theta_N)$ sampled independently of the initial conditions and Wiener processes of the agents $A_i$, $1 \leq i \leq N$. The first assumption is as follows.

**Assumption 2.** There exists $\pi$ such that $\lim_{N \to \infty} \pi^N = \pi$ a.s.

We note that except for clarity the time argument for the stochastic and deterministic processes throughout the paper may be dropped for the purpose of notation abbreviation as in (1)-(2).

### 2.2 Cost Functionals

The individual (finite) large population infinite horizon cost functional for the major agent $A_0$ is specified by

$$J_0^N(u_0, u_{-0}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \| x_0 - \Phi(x^{(N)}) \|^2_{Q_0} + \| u_0 \|^2_{R_0} \right\} dt,$$

where $R_0 > 0$, and the individual (finite) large population infinite horizon cost functional for a minor agent $A_i$, $1 \leq i \leq N$, is given by

$$J_i^N(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \| x_i - \Psi(x^{(N)}) \|^2_{Q} + \| u_i \|^2_{R} \right\} dt,$$

where $R > 0$. We note that the major agent $A_0$ and minor agents $A_i$, $1 \leq i \leq N$, are coupled with each other through the average term $x^{(N)} = \frac{1}{N} \sum_{i=1}^{N} x_i$ in their cost functionals given by (3)-(4).

### 2.3 Observation Processes

The major agent’s partial observations $y_0$ is given by

$$dy_0 = L_0[x_0^T, (x^{(N)})^T]^T dt + R_{v_0}^{\frac{1}{2}} dv_0,$$

where $v_0$ is a standard Wiener process in $\mathbb{R}^\ell$ with $\mathbb{E}[v_0 v_0^T] = R_{v_0}$ and matrix $L_0$ is given by

$$L_0 = \begin{bmatrix} I_0 & 0_{\ell \times n} \end{bmatrix},$$

where $I_0$ is an identity matrix of appropriate size.
with $l_1^0 \in \mathbb{R}^{\ell \times n}$. The partial observations for a minor agent $A_i, 1 \leq i \leq N$, of type $k, 1 \leq k \leq K$, is given by

\[
dy_i = L_k[x_i^T, x_0^T, (x^{(N)})^T]dt + R_v^{1/2}dv_i,
\]

where $\{v_i, 1 \leq i \leq N\}$ denotes the set of $N$ independent standard Wiener processes in $\mathbb{R}^\ell$ with $\mathbb{E}[v_i v_i^T] = R_v$, and matrix $L_k$ is given by

\[
L_k = \begin{bmatrix} l_1^k & l_2^k & 0_{\ell \times n} \end{bmatrix},
\]

where $l_1^k, l_2^k \in \mathbb{R}^{\ell \times n}$.

**Control $\sigma$-Fields**

The family of partial observation information sets $\mathcal{F}^y_0$ is defined to be the increasing family of $\sigma$-fields of partial observations $\{\mathcal{F}^y_{0,t}, t \geq 0\}$ generated by the major agent $A_0$’s partial observations $(y_0(\tau), 0 \leq \tau \leq t)$ on its own state as given in (5). The set of control inputs $\mathcal{U}^{N,L}$ is defined to be the collection of linear feedback control laws adapted to $\mathcal{F}^{y_0}_{t \geq 0} = \bigvee_{i=0}^N \mathcal{F}^y_i$.

**Assumption 3. Major Agent $\sigma$-Fields and Linear Controls:** For the major agent $A_0$ the set of control inputs $\mathcal{U}^{L}_0$ is defined to be the collection of linear feedback control laws adapted to the increasing $\sigma$-fields of partial observations $\{\mathcal{F}^y_{0,t}, t \geq 0\}$.

The family of partial observation information sets $\mathcal{F}^y_i, 1 \leq i \leq N$, is defined to be the increasing $\sigma$-fields $\{\mathcal{F}^y_{i,t}, t \geq 0\}$ generated by the minor agent $A_i$’s partial observations $(y_i(\tau), 0 \leq \tau \leq t)$, on its own state and the major agent’s state, as given in (7).

**Assumption 4. Minor Agent $\sigma$-Fields and Linear Controls:** For each minor agent $A_i, 1 \leq i \leq N$, the set of control inputs $\mathcal{U}^{L}_i$ is defined to be the collection of linear feedback control laws adapted to the increasing $\sigma$-fields of partial observations $\{\mathcal{F}^y_{i,t}, t \geq 0\}$.

### 3 Estimation and Control Solutions for PO MM LQG MFG Systems

In this section we present the solution to partially observed (PO) MM LQG MFG problems where it is assumed that the major agent partially observes its own state,
and each generic minor agent partially observes its own state and the major agent’s state. The problem is first solved in the infinite population case which is far simpler to solve than the finite large population problem. Because the agents in the infinite population case are decoupled and therefore the problem reduces to the type of indefinite LQG tracking problem whose solution is given in Theorem 7. Subsequently, the $\epsilon$-Nash equilibrium property is established in Theorem 3 for the system when the infinite population control laws are applied to the finite large population PO MM LQG MFG system.

The following theorem is a restriction to the constant matrix parameter case of the general result in \cite{29}.

**Theorem 1** (Stochastic Indefinite LQ Problem \cite{29}). Let $\hat{T} > 0$ be given. For any $(\hat{s}, \hat{y}) \in [0, \hat{T}) \times \mathbb{R}^n$, consider the following linear system

$$d\hat{x} = \left[ \hat{A}\hat{x} + \hat{B}\hat{u} + \hat{b} \right] dt + \left[ \hat{C}\hat{x} + \hat{D}\hat{u} + \hat{\sigma} \right] d\hat{w}, \tag{9}$$

where $t \in [\hat{s}, \hat{T}]$, $\hat{x}(\hat{s}) = \hat{y}$ and $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$, $\hat{b}$, $\hat{\sigma}$ are matrix valued functions of suitable sizes, $\hat{w}(\cdot) \in \mathbb{R}^r$ is a standard Wiener process. Moreover, $\mathcal{F}_t = \sigma\{\hat{w}(\tau), 0 \leq \tau \leq t\}$, and $\hat{u}(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ is the set of all $\mathcal{F}_t$-adapted $\mathbb{R}^m$-valued processes such that $E \int_0^T \|u(t)\|^2 dt < \infty$.

A quadratic cost functional is given by

$$J(\hat{s}, \hat{y}, \hat{u}(\cdot)) = \mathbb{E} \left\{ \frac{1}{2} \int_0^{\hat{T}} \left[ \langle \hat{P}\hat{x}(t), \hat{x}(t) \rangle + \langle \hat{N}\hat{x}(t), \hat{u}(t) \rangle \right. \right.$$

$$\left. \left. + \langle \hat{R}\hat{u}(t), \hat{u}(t) \rangle \right] dt + \frac{1}{2} \langle \hat{P}\hat{x}(\hat{T}), \hat{x}(\hat{T}) \rangle \right\}, \tag{10}$$

with $\hat{P}$, $\hat{N}$ and $\hat{R}$ being $\mathbb{S}^n$, $\mathbb{R}^{m \times n}$ and $\mathbb{S}^m$-valued functions, respectively, and $\hat{G} \in \mathbb{S}^{n \times n}$, where $\mathbb{S}^n$ denotes symmetric matrix space of size $n$.

We also denote the set of all $\mathbb{R}^n$-valued continuous functions defined on $[s, T]$ by $C([s, T]; \mathbb{R}^n)$. Then, let $\hat{\Pi}(\cdot) \in C([\hat{s}, \hat{T}); \mathbb{S}^n)$ be the solution of the Riccati equation

$$\hat{\Pi} + \hat{\Pi}\hat{A} + \hat{A}^T\hat{\Pi} + C^T\hat{P}\hat{C} + \hat{P} - (\hat{B}^T\hat{\Pi} + \hat{N} + \hat{D}^T\hat{\Pi}\hat{D})^{-1} \times (\hat{B}^T\hat{\Pi} + \hat{N} + \hat{D}^T\hat{\Pi}\hat{D}) = 0, \quad a.e. t \in [\hat{s}, t], \quad \hat{\Pi}(\hat{T}) = \hat{P}, \tag{11}$$

where $\hat{R} + \hat{D}^T\hat{\Pi}\hat{D} > 0$, a.e. $t \in [\hat{s}, \hat{T}]$, and $\hat{s}(\cdot) \in C([\hat{s}, \hat{T}); \mathbb{R}^n)$ be the solution
of the offset equation given by
\[ \dot{s} + [\bar{A} - \bar{B}(\bar{R} + \bar{D}^T\bar{D})^{-1}(\bar{B}^T\bar{P} + \dot{s} + \bar{D}^T\bar{P}C)]^T\dot{s} + [\bar{C} - \bar{D}(\bar{R} + \bar{D}^T\bar{D})^{-1}(ar{B}^T\bar{P} + \dot{s} + \bar{D}^T\bar{P}C)]T\bar{P} = 0, \quad a.e. \ t \in [\hat{s}, \hat{T}], \ \dot{s}(\hat{T}) = 0. \]

Let us define \( \bar{\Psi} \triangleq (\bar{R} + \bar{D}^T\bar{D})^{-1}[\bar{B}^T\bar{P} + \dot{s} + \bar{D}^T\bar{P}C] \), and \( \bar{\psi} \triangleq (\bar{R} + \bar{D}^T\bar{D})^{-1}[\bar{B}^T\bar{P} + \dot{s} + \bar{D}^T\bar{P}C] \). Then the stochastic LQ problem (9)-(10) is solvable at \( \dot{s} \) with the optimal control \( \bar{u}^\circ(\cdot) \) being in the state feedback form as in
\[ \bar{u}^\circ(t) = -\bar{\Psi}(t)\dot{x}(t) - \bar{\psi}(t), \quad t \in [\hat{s}, \hat{T}]. \]

□

Henceforth we discuss the stochastic optimal control problem for the major agent, and a generic minor agent.

### 3.1 Mean Field Evolution

We introduce the empirical state average as
\[ x^{(N_k)} = \frac{1}{N_k} \sum_{j=1}^{N_k} x_j^k, \quad 1 \leq k \leq K, \]
and write \( x^{(N)} = [x^{(N_1)}, x^{(N_2)}, \ldots, x^{(N_K)}] \), where the point-wise in time \( L^2 \) limit of \( x^{(N)} \), if it exists, is called the mean field of the system and is denoted by \( \bar{x} = [\bar{x}_1, \ldots, \bar{x}_K] \). We consider for each minor agent \( A_i \) of type \( k \), \( 1 \leq k \leq K \), a uniform (with respect to \( i \) in any subpopulation \( k \), \( 1 \leq k \leq K \)) feedback control \( u_i^k \in U_{i,y} \), which is a function of

(i) minor agent’s estimate of its own state, i.e. \( \hat{x}_i|\mathcal{F}_t^y \triangleq \mathbb{E}_{i,\mathcal{F}_t^y} x_i = \mathbb{E}\{x_i|\mathcal{F}_t^y\} \),

(ii) minor agent’s estimate of the major agent’s state, i.e. \( \hat{x}_0|\mathcal{F}_t^y \triangleq \mathbb{E}_{i,\mathcal{F}_t^y} x_0 = \mathbb{E}\{x_0|\mathcal{F}_t^y\} \),

(iii) minor agent’s estimate of \( x_j \), \( 1 \leq j \leq N \), \( j \neq i \), i.e. \( \hat{x}_j|\mathcal{F}_t^y \triangleq \mathbb{E}_{i,\mathcal{F}_t^y} x_j = \mathbb{E}\{x_j|\mathcal{F}_t^y\} \),

(iv) minor agent’s estimate of the major agent’s estimate of its own state, i.e. \( (\hat{x}_0|\mathcal{F}_t^y)|\mathcal{F}_t^y \triangleq \mathbb{E}_{i,\mathcal{F}_t^y} x_0|\mathcal{F}_t^y = \mathbb{E}\{x_0|\mathcal{F}_t^y\} \),

\[ \mathbb{E}\{\hat{x}_0|\mathcal{F}_t^y\} = \mathbb{E}_{i,\mathcal{F}_t^y} x_0|\mathcal{F}_t^y = \mathbb{E}\{x_0|\mathcal{F}_t^y\} \],
(v) minor agent’s estimate of the major agent’s estimate of \( x_j \), \( 1 \leq j \leq N \), \( j \neq i \), i.e. \( (\hat{x}_j|x_0^y)|x_i^y = \mathbb{E}_i[x_j|x_0^y] = \mathbb{E}\{\hat{x}_j|x_i^y|F_i^y\} \),

(vi) bounded continuous functions of time \( m_k(.) \in C_b([0, \infty); \mathbb{R}^m) \).

Hence \( u_i^k \) is given by

\[
u_i^k = L_1^k \hat{x}_i^k + L_2^k \hat{x}_0^y + \sum_{l=1}^{K} \sum_{j=1}^{N_l} L_3^{k,l} \hat{x}_j |x_i^y + L_4^k (\hat{x}_0 |x_0^y |F_i^y) + \sum_{l=1}^{K} \sum_{j=1}^{N_l} L_5^{k,l} (\hat{x}_j |x_0^y |F_i^y) + m_k, \quad (12)\]

for matrices \( L_1^k, L_2^k, L_3^{k,l}, \) and \( L_4^k \) of appropriate dimension, which are time invariant due to the time shift invariance of the infinite horizon performance function \( (4) \) and the dynamics \( (2) \), and where \( L_3^{k,l}, L_5^{k,l} \) are assumed to depend upon \( N_l \), and satisfy \( N_l L_3^{k,l} \to \nu_3^{k,l}, N_l L_5^{k,l} \to \nu_5^{k,l} \) as \( N_l \to \infty \) for all \( k, 1 \leq k \leq K \). Substituting \( (12) \) in \( (2) \) yields

\[
dx_i = [A_k x_i + B_k L_1^k \hat{x}_i^k + B_k L_2^k \hat{x}_0^y + B_k \sum_{l=1}^{K} N_l L_3^{k,l} \hat{x}_i |x_i^y + B_k L_4^k (\hat{x}_0 |x_0^y |F_i^y) + B_k \sum_{l=1}^{K} N_l L_5^{k,l} (\hat{x}_i |x_0^y |F_i^y ) + B_k m_k + G x_0 |F_i^y] + D dw_i, \quad (13)\]

Then we take the average over the subpopulation \( k \) to obtain

\[
dx^{(N_k)} = [A_k x^{(N_k)} + B_k L_1^k \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}_i |x_i^y + B_k L_2^k \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}_0^y + B_k \sum_{l=1}^{K} N_l L_3^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} \hat{x}_i |x_i^y + B_k L_4^k \frac{1}{N_k} \sum_{i=1}^{N_k} (\hat{x}_0 |x_0^y |F_i^y) + B_k \sum_{l=1}^{K} N_l L_5^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} (\hat{x}_i |x_0^y |F_i^y ) + B_k \sum_{l=1}^{K} N_l L_5^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} \sum_{i=1}^{N_k} dw_i, \quad (14)\]
To compute the average of the estimation terms in (14), we use the state decomposition

\[
\begin{bmatrix}
\hat{x}_{i|F_i^y} \\
\hat{x}_{0|F_0^y} \\
\hat{x}_{(N_1)} \\
(\hat{x}_{0|F_0^y})_{|x_i} \\
(\hat{x}_{(N_1)}|F_0^y)
\end{bmatrix}
= \begin{bmatrix}
\hat{x}_{i|F_i^y} - x_i \\
\hat{x}_{0|F_0^y} - x_0 \\
\hat{x}_{(N_1)} - x(N_1) \\
(\hat{x}_{0|F_0^y})_{|x_i} - \hat{x}_{0|F_0^y} \\
(\hat{x}_{(N_1)}|F_0^y) - \hat{x}_{(N_1)}
\end{bmatrix} + \begin{bmatrix}
x_i - x_i \\
x_0 - x_0 \\
x(N_1) - x(N_1) \\
(\hat{x}_{0|F_0^y})_{|x_i} - \hat{x}_{0|F_0^y} \\
(\hat{x}_{(N_1)}|F_0^y) - \hat{x}_{(N_1)}
\end{bmatrix},
\tag{15}
\]

which we denote equivalently in the compact form

\[
\hat{x}_{i|F_i^y}^{k,ex} = -\hat{x}_{i|F_i^y}^{k,ex} + x_i^{k,ex},
\tag{16}
\]

for \(1 \leq i \leq N\), and \(1 \leq k \leq K\). Accordingly we rewrite (14) as

\[
dx^{(N_k)} = \left[A_kx^{(N_k)} + B_kL_i^1 \frac{1}{N_k} \sum_{i=1}^{N_k} x_i^k + B_kL_2^k x_0 + B_k \sum_{l=1}^{K} N_l L_{3,l}^{k,l} x^{(N_l)}
+ B_kL_4^k \hat{x}_{0|F_0^y} + B_k \sum_{l=1}^{K} N_l L_{5,l}^{k,l} \hat{x}_{(N_l)}
+ B_k m_k + Gx_0\right]dt
- \left[B_kL_i^k \frac{1}{N_k} \sum_{i=1}^{N_k} (x_i - \hat{x}_{i|F_i^y}) + B_kL_2^k \frac{1}{N_k} \sum_{i=1}^{N_k} (x_0 - \hat{x}_{0|F_0^y})
+ B_k \sum_{l=1}^{K} N_l L_{3,l}^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} (x_{(N_l)} - \hat{x}_{(N_l)}|F_0^y)
+ B_k \sum_{l=1}^{K} N_l L_{5,l}^{k,l} \frac{1}{N_k} \sum_{i=1}^{N_k} (\hat{x}_{(N_l)}|F_0^y) - (\hat{x}_{0|F_0^y})_{|x_i}
\right]dt
+ D \frac{1}{N_k} \sum_{i=1}^{N_k} dw_i.
\tag{17}
\]

From (17) as \(N \to \infty\) we obtain the convergence in quadratic mean to the solution to

\[
dx^k = \left[(A_k + B_kL_1^k)\bar{x}^k + (G + B_kL_2^k)x_0 + B_k \sum_{l=1}^{K} \bar{L}_{3,l}^{k,l}\bar{x}^l + B_kL_4^k \hat{x}_{0|F_0^y}
+ B_k \sum_{l=1}^{K} \bar{L}_{5,l}^{k,l} \hat{x}_{0|F_0^y} + B_k m_k\right]dt
- \left[B_kL_1^k(x_i - \hat{x}_{i|F_i^y})^k + B_kL_2^k(x_0 - \hat{x}_{0|F_0^y})^k
\right].
\]

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\[ + B_k \sum_{l=1}^{K} \bar{L}_3^{k,l} \left( \bar{x}^l - \hat{x}^l \bigg| F^y_0 \right)^k + B_k L_4^k \left( \bar{x}_{0,F_0^y} - \left( \bar{x}_{0,F_0^y} \bigg| F^y_0 \right)^k \right) \\
\] 
\[ + B_k \sum_{l=1}^{K} \bar{L}_5^{k,l} \left( \hat{x}^l \bigg| F^y_0 \right)^k \right] dt, \quad (18) \]

where the overline symbol with superscript \( k \), i.e. \( (.)^k \) denotes the infinite-population limit of the the average over subpopulation \( k \) of the corresponding terms, which are the components of \( \bar{x}^{ex} \) in (16) (see Proposition 3.1 in [18] for the convergence analysis in quadratic mean).

Subsequently, a compact representation of (18) shall be used as in

\[ d\bar{x}^k = \left( (A_k + B_k L_1^k) \bar{x}^k + (G + B_k L_2^k) x_0 + B_k \sum_{l=1}^{K} \bar{L}_3^{k,l} \bar{x}^l + B_k L_4^k \bar{x}_{0,F_0^y} \right) dt \\
\] 
\[ + B_k \sum_{l=1}^{K} \bar{L}_5^{k,l} \left( \hat{x}^l \bigg| F^y_0 \right) dt + \bar{J}_k \bar{x}^{ex} dt, \quad (19) \]

where we denote by \( \bar{x}^{ex} \) the average of the estimation errors of the minor agents of subpopulation \( k \) as \( N_k \rightarrow \infty \), and which satisfies the dynamical equation (65) in Section 3.4. Hence, the second bracket in (18) is given by \( \bar{J}_k \bar{x}^{ex} \). (Here the term \( \bar{J}_k \bar{x}^{ex} \) corrects its omission in [18].)

Therefore the mean field state vector \( \bar{x} \) satisfies

\[ d\bar{x} = \bar{A} \bar{x} dt + \bar{G} x_0 dt + \bar{H} \bar{x}_{0,F_0^y} dt + \bar{L} \bar{x} \bigg| F^y_0 \right) dt + \bar{J} \bar{x}^{ex} dt + \bar{m} dt, \quad (20) \]

where \( (\bar{x}^{ex})^T = [(\bar{x}^{1,ex})^T, \ldots, (\bar{x}^{K,ex})^T] \), and the matrices \( \bar{A}, \bar{G}, \bar{H}, \bar{L}, \bar{J}, \) and \( \bar{m} \) collect the corresponding terms in (19) and have the block matrix form

\[ \bar{A} = \begin{bmatrix} \bar{A}_1 \\ \vdots \\ \bar{A}_K \end{bmatrix}, \quad \bar{G} = \begin{bmatrix} \bar{G}_1 \\ \vdots \\ \bar{G}_K \end{bmatrix}, \quad \bar{H} = \begin{bmatrix} \bar{H}_1 \\ \vdots \\ \bar{H}_K \end{bmatrix}, \]

\[ \bar{L} = \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_K \end{bmatrix}, \quad \bar{m} = \begin{bmatrix} \bar{m}_1 \\ \vdots \\ \bar{m}_K \end{bmatrix}, \quad \bar{J} = \begin{bmatrix} \bar{J}_1 & 0 \\ \vdots & \ddots \\ 0 & \bar{J}_K \end{bmatrix}. \quad (21) \]

We note that \( \bar{A}_k, \bar{L}_k, \bar{G}_k, \bar{H}_k, \bar{m}_k, \bar{J}_k \in \mathbb{R}^{n \times nK}, \bar{J}_k \in \mathbb{R}^{n \times (3n+2nK)}, 1 \leq k \leq K \), are to be solved for using the consistency equations in
By abuse of language, the mean value of the system’s Gaussian mean field given by the state process $\bar{x} = [\bar{x}^1, ..., \bar{x}^K]$ shall also be termed the system’s mean field.

### 3.2 Major Agent: Infinite Population

The major agent’s infinite population dynamics, as the number of agents goes to infinity ($N \to \infty$), remain the same as in (1), while its infinite population individual cost functional is given by

$$J_0^\infty(u_0, u_{-0}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \|x_0 - \phi(\bar{x})\|_Q^2 + \|u_0\|_R^2 \right\} dt,$$

(22)

where $x^{(N)}$ in (3) was replaced by its $L^2$ limit, i.e. the mean field $\bar{x}$.

To solve the infinite population tracking problem for the major agent, its state is extended with the mean field process $\bar{x}$, where this is assumed to exist, i.e. $x_0^{ex} = [x_0, \bar{x}]$.

Then the Kalman filter which generates the estimates of the major agent’s state $\hat{x}_0|\mathcal{F}_0^y$ and the mean field $\hat{\bar{x}}_0|\mathcal{F}_0^y$ based on its own observations are, respectively, given by

$$d\hat{x}_0|\mathcal{F}_0^y = A_0 \hat{x}_0|\mathcal{F}_0^y dt + B_0 \hat{u}_0 dt + K_0^1 d\nu_0,$$

(25)

$$d\hat{\bar{x}}_0|\mathcal{F}_0^y = (G + H)\hat{x}_0|\mathcal{F}_0^y dt + (A + L)\hat{\bar{x}}_0|\mathcal{F}_0^y dt + \bar{m} dt + K_0^2 d\nu_0,$$

(26)

where $\hat{\bar{x}}_0|\mathcal{F}_0^y = 0$ is used (see Observation 4). Moreover, $\bar{m}$ is a deterministic process according to (19), $K_0^1$ and $K_0^2$ are the Kalman filter gains, and $\nu_0$ is the innovation process. Therefore the Kalman filter which generates the estimates of the major agent’s extended state is given by

$$\begin{bmatrix}
  d\hat{x}_0|\mathcal{F}_0^y \\
  d\hat{\bar{x}}_0|\mathcal{F}_0^y
\end{bmatrix} =
\begin{bmatrix}
  A_0 & 0_{n \times nK} \\
  G + H & A + L
\end{bmatrix}
\begin{bmatrix}
  \hat{x}_0|\mathcal{F}_0^y \\
  \hat{\bar{x}}_0|\mathcal{F}_0^y
\end{bmatrix} dt
+ \begin{bmatrix}
  B_0 \\
  0_{nK \times m}
\end{bmatrix} \hat{u}_0 dt + \begin{bmatrix}
  0_{n \times 1} \\
  \bar{m}
\end{bmatrix} dt + K_0 d\nu_0,$$

(27)

with the corresponding Kalman filter gain $K_0 = [(K_0^1)^T, (K_0^2)^T]^T$, and the innovation process $\nu_0$, respectively, given by

$$K_0 = V_0 \Pi_0^T R_{\nu_0}^{-1},$$

(28)
where $L_0 = \begin{bmatrix} \ell \times n & 0 \end{bmatrix}$, and $V_0(t)$ is the solution to the corresponding Riccati equation (31).

From (1), (20), and (27) we denote

$$\dot{A}_0 = \begin{bmatrix} A_0 \\ \dot{G} + \hat{H} \end{bmatrix}, \quad \dot{B}_0 = \begin{bmatrix} B_0 \\ 0_{nK \times n} \end{bmatrix}, \quad \dot{M}_0 = \begin{bmatrix} 0_{n \times m} \\ \cdot \end{bmatrix},$$

$$\dot{D}_0 = \begin{bmatrix} D_0 \\ 0_{nK \times r} \\ 0_{nK \times r} \end{bmatrix}, \quad \dot{J}_0 = \begin{bmatrix} 0_{n \times (3nK+2nK^2)} \\ \cdot \end{bmatrix}.$$  

(30)

Then to guarantee the convergence of the solution to the Riccati equation to a positive definite asymptotically stabilizing solution, we assume:

**Assumption 5.** $[A_0, D_0]$ is stabilizable and $[L_0, A_0]$ is detectable.

The corresponding Riccati equation is then given by

$$\dot{V}_0 = A_0 V_0 + V_0 \dot{A}_0^T - K_0 R_0 K_0^T + J_0 V_0 J_0^T + Q_{w0},$$  

(31)

where $Q_{w0} = D_0 D_0^T$, $\dot{V}(t) = \mathbb{E} \left[ \tilde{x}^{ex}(t) \tilde{x}^{ex}(t)^T \right]$ satisfies (72), and $V(0) = \mathbb{E} \left[ \left( x_0^{ex}(0) - \left( x_0^{ex}(0) \right)_{\tilde{F}_0^0} \right) \left( x_0^{ex}(0) - \left( x_0^{ex}(0) \right)_{\tilde{F}_0^0} \right)^T \right]$.

Then, utilizing the infinite horizon discounted analogy to Theorem 7, it can be shown (see Theorem 3 in Section 3.4) that the optimal control action for the major agent’s tracking problem (and hence best response MFG control input) is

$$\hat{u}_0 = -R_0^{-1} \mathbb{E}^T \left[ \Pi_0 \left( \tilde{x}_0^{ex}, \tilde{x}_0^{ex} \right)^T + s_0 \right],$$  

(32)

where $\Pi_0$ and $s_0$ are the solutions to the Riccati and offset equations given by

$$\rho \Pi_0 = \Pi_0 A_0 + A_0^T \Pi_0 - \Pi_0 B_0 R_0^{-1} B_0^T \Pi_0 + Q_0^0,$$  

(33)

$$\rho s_0 = \frac{ds_0}{dt} + (A_0 - B_0 R_0^{-1} B_0^T \Pi_0)^T s_0 + \Pi_0 M_0 \bar{n}_0 - \bar{n}_0,$$  

(34)

with $\bar{n}_0 = [I_{n \times n}, -H_0^0]^T Q_0 \bar{n}_0$ and $Q_0^0 = [I_{n \times n}, -H_0^0]^T Q_0 [I_{n \times n}, -H_0^0]$. We note $\frac{ds_0}{dt} = 0$ in (34), since $M_0$, $\bar{n}_0$ are constant.

Finally, the joint dynamics of the major agent’s closed-loop system and its Kalman filter system are given by

$$\begin{bmatrix} dx_0 \\ d\bar{x} \\ d\tilde{x}_0 | \tilde{F}_0^0 \\ d\tilde{\bar{x}} | \tilde{F}_0^0 \end{bmatrix} = A_0 \begin{bmatrix} x_0 \\ \bar{x} \\ \tilde{x}_0 | \tilde{F}_0^0 \\ \tilde{\bar{x}} | \tilde{F}_0^0 \end{bmatrix} dt + J_0 \tilde{x}^{ex} dt + M_0 dt + D_0 \begin{bmatrix} dw_0 \\ 0_{nK \times 1} \end{bmatrix},$$  

(35)
where

\[
A_0 = \begin{bmatrix} A_0 & 0_{n \times nK} \\ G & A \\ K_0 \Pi_0 & 0_{n \times nK} \end{bmatrix},
\]

\[
A_0 - K_0 \Pi_0 - B_0 R_0^{-1} B_0^T \Pi_0,
\]

\[
J_0 = \begin{bmatrix} 0_{n \times (3nK+2nK^2)} \\ 0_{(n+nK) \times (3nK+2nK^2)} \end{bmatrix},
\]

\[
M_0 = \begin{bmatrix} M_0 - B_0 R_0^{-1} B_0^T s_0 \\ M_0 - B_0 R_0^{-1} B_0^T s_0 \end{bmatrix},
\]

\[
D_0 = \begin{bmatrix} D_0 & 0_{(n+nK) \times \ell} \\ 0_{(n+nK) \times (r+nK)} & K_0 R_0^2 \end{bmatrix}.
\]

### 3.3 Minor Agent: Infinite Population

A generic minor agent’s infinite population dynamics, as the number of agents goes to infinity \((N \to \infty)\), remain the same as in (2), while its infinite population individual cost functional is given as

\[
J_\infty^i(u_i, u_{-i}) = \mathbb{E} \int_0^\infty e^{-\rho t} \left\{ \| x_i - \psi(\bar{x}) \|^2_Q + \| u_i \|^2_R \right\} dt,
\]

\[
\psi(\cdot) = H_1 x_0 + H_2^\pi \bar{x} + \eta,
\]

\[
H_2^\pi = \pi \otimes H_2 \triangleq [\pi_1 H_0, \pi_2 H_0, \ldots, \pi_K H_0].
\]

In the case where all agents have partial observations on the major agent’s state, the joint dynamics of the major agent’s closed-loop system and its Kalman filtering recursions are employed in order to solve the minor agent’s tracking problem. Before proceeding we enunciate Proposition 1, where for ease of exposition, the simple case where it is assumed that the major agent and minor agents are not coupled with the mean field (neither in their dynamics nor in their cost functional) is considered. However, each minor agent is assumed to be coupled with the major agent’s state in their cost functional. The results are extendable to the more general case described by (1)-(2) and (3)-(4), in a straightforward way.

**Proposition 2.** Estimates of Estimates Filter

Let the major agent’s dynamics be given by

\[
dx_0 = A_0 x_0 dt + B u_0 dt + dw_0,
\]
and the major agent’s observations of its own state by
\[ dy_0 = H_0 x_0 dt + dv_0, \] (40)
then the estimates of the major agent’s state based on its own observation is generated by
\[ \dot{x}_0|_{\mathcal{F}_0} = A_0 \hat{x}_0|_{\mathcal{F}_0} dt + B \hat{u}_0 dt + K_0 [dy_0 - H_0 \hat{x}_0 dt] \triangleq A_0 \hat{x}_0|_{\mathcal{F}_0} dt + B \hat{u}_0 dt + K_0 [H_0 x_0 dt + dv_0 - H_0 \hat{x}_0|_{\mathcal{F}_0} dt]. \] (41)

Next, assume the major agent’s control action is of the form
\[ u_0 = -L \hat{x}_0|_{\mathcal{F}_0}, \] (42)
then in this case the joint dynamics of the major agent’s closed-loop system and its Kalman filter system are given by
\[
\begin{bmatrix}
    dx_0 \\
    d\hat{x}_0|_{\mathcal{F}_0}
\end{bmatrix}
= \begin{bmatrix}
    A_0 \\
    K_0 H_0 & A_0 - BL - K_0 H_0
\end{bmatrix}
\begin{bmatrix}
    x_0 \\
    \hat{x}_0|_{\mathcal{F}_0}
\end{bmatrix}
\]
\[ + \begin{bmatrix}
    0 \\
    K_0 dv_0
\end{bmatrix} + \begin{bmatrix}
    dw_0 \\
    0
\end{bmatrix}. \] (43)

Finally, let the minor agent’s partial observations of the major agent’s state be given by
\[ dy_i = H_i x_0 dt + dv_i = \begin{bmatrix}
    H_i & 0
\end{bmatrix}
\begin{bmatrix}
    x_0 \\
    \hat{x}_0|_{\mathcal{F}_0}
\end{bmatrix}
\] dt + dv_i, (44)
then the process of estimates of the state of (43) based upon the observations (44) is generated by the filtering scheme
\[
\begin{bmatrix}
    d\hat{x}_0|_{\mathcal{F}_0} \\
    d(\hat{x}_0|_{\mathcal{F}_0})|_{\mathcal{F}_i}
\end{bmatrix}
= \begin{bmatrix}
    A_0 \\
    K_0 H_0 & A_0 - BL - K_0 H_0
\end{bmatrix}
\begin{bmatrix}
    \hat{x}_0|_{\mathcal{F}_0} \\
    (\hat{x}_0|_{\mathcal{F}_0})|_{\mathcal{F}_i}
\end{bmatrix}
\]
\[ + K_i \left( dy_i - \begin{bmatrix}
    H_i & 0
\end{bmatrix}
\begin{bmatrix}
    d\hat{x}_0|_{\mathcal{F}_0} \\
    d(\hat{x}_0|_{\mathcal{F}_0})|_{\mathcal{F}_i}
\end{bmatrix}
\right), \] (45)

where \( \hat{x}_0|_{\mathcal{F}_i} \triangleq \mathbb{E}_{\mathcal{F}_i}[x_0] \) denotes the minor agent A_i’s estimate of the major agent’s state, and
\[ (\hat{x}_0|_{\mathcal{F}_0})|_{\mathcal{F}_i} \triangleq \mathbb{E}_{\mathcal{F}_i}[\hat{x}_0|_{\mathcal{F}_i} = \mathbb{E}\{\hat{x}_0|_{\mathcal{F}_0} | \mathcal{F}_i\}], \]
denotes the minor agent A_i’s estimate of the major agent’s estimate of its own state. □
The proof of the Proposition 2 is straightforward and will be omitted, but we observe that the key property of the overall system which ensures its validity is that the Wiener processes \( \{w_0, v_0\} \) are independent of the noise process \( v_1 \) in (45).

Returning to the main problem, the minor agent’s state is next extended to form \( x_i^{ex} \triangleq [x_i, x_0, \bar x, \hat x_{0|x_0^y}, \hat x_{1|x_1^y}] \). Specifically this yields

\[
dx_i^{ex} = A_k x_i^{ex} dt + B_k u_i dt + J \bar x^{ex} + \mathbb{M} dt + \mathbb{D} [dw_i^T, dw_0^T, 0_{1 \times nK}, dv_0^T]^T, \tag{46}
\]

where

\[
A_k = \begin{bmatrix}
    A_k & G \\
    0_{2(n+N_K) \times n} & A_0
\end{bmatrix}, \quad
B_k = \begin{bmatrix}
    B_k \\
    0_{2(n+N_K) \times m}
\end{bmatrix},
\]

\[
J = \begin{bmatrix}
    0_{n \times (3nK+2nK^2)} \\
    J_0
\end{bmatrix}, \quad
\mathbb{M} = \begin{bmatrix}
    0_{n \times 1} \\
    M_0
\end{bmatrix},
\]

\[
\mathbb{D} = \begin{bmatrix}
    D & 0_{n \times (r+nK+\ell)} \\
    0_{2(n+N_K) \times r} & D_0
\end{bmatrix}. \tag{47}
\]

To derive the Kalman filter equations for (46), we first define \( \mathbb{L}_k = [L_k^1, L_k^2, 0_{\ell \times (n+2nK)}] \). To guarantee the convergence of the solution to the Riccati equation to a positive definite asymptotically stabilizing solution, we assume:

**Assumption 6.** The system parameter set \( \Theta = \{1, \ldots, K\} \) is such that \( [A_k, \mathbb{D}] \) is stabilizable and \( [\mathbb{L}_k, A_k] \) is detectable for all \( k, 1 \leq k \leq K \).

The Riccati equation associated with the filtering equations for (46) is then given by

\[
\dot{V}_k = A_k V_k + V_k A_k^T - K_k R_v K_k^T + \mathbb{D} \mathbb{D}^T + \mathbb{V}(t) \mathbb{V}(t)^T + Q_w, \tag{48}
\]

where \( Q_w = \mathbb{D} \mathbb{D}^T, \mathbb{V}(t) = \mathbb{E} \left[ \bar x^{ex}(t) \bar x^{ex}(t)^T \right] \) satisfies (72), and \( V_k(0) = \mathbb{E} \left[ (x_i^{ex}(0) - (x_i^{ex}(0)_{\mathcal{F}_i^y}) (x_i^{ex}(0) - (x_i^{ex}(0))_{\mathcal{F}_i^y})^T \right] \). The Kalman filter gain \( K_k \) is in turn given by

\[
K_k = V_k \mathbb{L}_k^T R_v^{-1}, \tag{49}
\]

and the innovation process \( \nu_i(t) \) is defined as in

\[
d\nu_i = dy_i - \mathbb{L}_k \left[ \hat \nu_i^{T|\mathcal{F}_i^y}, \hat \nu_0^{T|\mathcal{F}_i^y}, \hat \nu_1^{T|\mathcal{F}_i^y}, (\hat \nu_0^{0|x_0^y}_{\mathcal{F}_i^y})^{T|\mathcal{F}_i^y}, (\hat \nu_1^{0|x_0^y}_{\mathcal{F}_i^y})^{T|\mathcal{F}_i^y} \right] dt, \tag{50}
\]

where \( (\hat \nu_0^{0|x_0^y}_{\mathcal{F}_i^y})^{T|\mathcal{F}_i^y} \) and \( (\hat \nu_1^{0|x_0^y}_{\mathcal{F}_i^y})^{T|\mathcal{F}_i^y} \), respectively, denote the minor agent \( A_i \)’s estimates of the major agent’s estimates of its own state and the mean field. Then
the Kalman filter equations for a generic minor agent $A_i$, $1 \leq i \leq N$, are given as in
\begin{equation}
    d\hat{x}_{i,F}^{ex} = A_k\hat{x}_{i,F}^{ex}dt + B_k\hat{u}_idt + \mathbb{M}dt + K_kd\nu_i,
\end{equation}
where $\hat{x}_{i,F}^{ex} = 0$ (see Observation 4) is used. Clearly, (51) generates the iterated estimates $(\hat{x}_{0,F}^{0},F_i)$ and $(\hat{x}_{i,F}^{i},F_i)$ which are required to calculate $\hat{x}_{0,F}^{0}$ and $\hat{x}_{i,F}^{i}$ (see Proposition 1 in [30] for a simplified case of Estimates of Estimates Filter).

**Remark 1.** By virtue of the asymmetric information available to the major agent and a generic minor agent, an infinite regress does not occur in the process of estimating other agents’ states. In fact to calculate the best response action, the major agent only estimates its own state and hence does not estimate minor agents’ states, while each minor agent estimates its own state and the major agent’s state.

We note that by Assumption 3 the minor agent $A_i$ is able to estimate $\hat{u}_0^i$ whenever the functional dependence of the major agent’s control on it’s state is available to the minor agent through forming the conditional expectation of the major agent’s control action which by (32) is given by the following expression
\begin{equation}
    (\hat{u}_0^i)_F = \mathbb{E}\{\hat{u}_0^i|F_i\} = -R^{-1}\mathbb{B}_0 \left[ \Pi_0 \left( \left(\hat{x}_{0,F}^{0},F_i\right)^T, \left(\hat{x}_{i,F}^{i},F_i\right)^T, \left(\hat{x}_{0,F}^{0},F_i\right)^T, \left(\hat{x}_{i,F}^{i},F_i\right)^T, \left(\hat{x}_{0,F}^{0},F_i\right)^T, \left(\hat{x}_{i,F}^{i},F_i\right)^T \right)^T + s_0 \right],
\end{equation}
and which is embedded in (51). Then, utilizing the infinite horizon discounted analogy to Theorem 4 it can be shown (see Theorem 3) that the optimal control action for the minor agent $A_i$’s tracking problem (and hence best response MFG control input) is given by
\begin{equation}
    \hat{u}_i = -R^{-1}\mathbb{B}_k \left[ \Pi_k \left( \hat{x}_{i,F}^{i},F_i, \hat{x}_{i,F}^{i},F_i, \hat{x}_{i,F}^{i},F_i, \hat{x}_{i,F}^{i},F_i, \hat{x}_{i,F}^{i},F_i, \hat{x}_{i,F}^{i},F_i \right)^T + s_k \right],
\end{equation}
where the iterated estimation terms $(\hat{x}_{0,F}^{0},F_i)$ and $(\hat{x}_{i,F}^{i},F_i)$ explicitly appear, and the corresponding Riccati and offset equations are given by
\begin{equation}
    \rho \Pi_k = \Pi_k A_k + A_k^T \Pi_k - \Pi_k B_k R^{-1} B_k^T \Pi_k + Q^\pi, \forall k,
\end{equation}
\begin{equation}
    \rho s_k = \frac{ds_k}{dt} + (A_k - B_k R^{-1} B_k^T \Pi_k)^T s_k + \Pi_k \mathbb{M} - \bar{\eta}, \forall k,
\end{equation}
with $\bar{\eta} = [I_{n\times n}, -H_1, -H_2, 0_{n\times(n+1)K}]^T Q \eta$, and $Q^\pi = [I_{n\times n}, -H_1, -H_2, 0_{n\times(n+1)K}]^T Q [I_{n\times n}, -H_1, -H_2, 0_{n\times(n+1)K}]$. We note $\frac{d\theta_i}{dt} = 0$ in (55), since $\mathbb{M}, \bar{\eta}$ are constant.
3.4 Mean Field Consistency Equations

Let us denote the components of $\Pi_k$ in (54) as

$$
\Pi_k = \begin{bmatrix}
\Pi_{k,11} & \Pi_{k,12} & \Pi_{k,13} & \Pi_{k,14} & \Pi_{k,15} \\
\Pi_{k,21} & \Pi_{k,22} & \Pi_{k,23} & \Pi_{k,24} & \Pi_{k,25}
\end{bmatrix},
$$

(56)

$1 \leq k \leq K$, and where $\Pi_{k,11}, \Pi_{k,12}, \Pi_{k,14} \in \mathbb{R}^{n \times n}$, $\Pi_{k,13}, \Pi_{k,15} \in \mathbb{R}^{n \times nK}$, $\Pi_{k,21}, \Pi_{k,22}, \Pi_{k,24} \in \mathbb{R}^{2(n+nK) \times n}$, and $\Pi_{k,23}, \Pi_{k,25} \in \mathbb{R}^{2(n+nK) \times nK}$. Let us also define the block matrix $e_{k,v} = [0_{v \times v}, \ldots, 0_{v \times v}, I_v, 0_{v \times v}, \ldots, 0_{v \times v}]$ with $K$ blocks, where the $v \times v$ identity matrix $I_v$ is located at the $k$th block. Finally we define the block matrix $1_v = [I_v, \ldots, I_v]$ with $K$ blocks of identity matrix. Then we denote by

$$
\hat{e}_k = e_{k,n},
$$

(57)

$$
\tilde{e}_k = e_{k,(3n+2nK)},
$$

(58)

$$
\tilde{1} = 1_{(3n+2nK)}
$$

(59)

To obtain the mean field consistency equations, we substitute (53) in (2) to get

$$
dx_i = A_k x_i dt + G x_0 dt - B_k R^{-1}_k \mathbb{E}_k^T \left[ \Pi_k \tilde{x}_{i|\mathcal{F}_i} + s_k \right] dt + D dw_i.
$$

(60)

Then $\tilde{x}_{i|\mathcal{F}_i}$ can be written as

$$
\tilde{x}_{i|\mathcal{F}_i} = -(x_i - \tilde{x}_{i|\mathcal{F}_i}) + x_i
\tilde{x}_{i|\mathcal{F}_i} = -\tilde{x}_i + x_i,
$$

(61)

where $\tilde{x}_i$ denotes the estimation error, and the governing dynamics for $1 \leq i \leq N$, $1 \leq k \leq K$, are given by

$$
d\tilde{x}_i^k = (A_k - K_k \mathbb{E}_k) \tilde{x}_i^k + \tilde{\pi}_i^k dt - K_k R_0^k dw_i + \mathbb{D}[dw_i^T, dw_0^T, 0_{1 \times nK}, dv_0^T]^T,
$$

(62)

where $((\tilde{x}_1^k)^T, \ldots, (\tilde{x}_K^k)^T)$ satisfies (67).

Next the empirical average of (60), where (61) has been substituted, over the population of the minor agents of type $k$ is given by

$$
d\left( \frac{1}{N_k} \sum_{i=1}^{N_k} x_i^k \right) = A_k \left( \frac{1}{N_k} \sum_{i=1}^{N_k} x_i^k \right) dt + G x_0 dt
\begin{align*}
&- B_k R^{-1}_k \mathbb{E}_k^T \left[ \Pi_k \left( \frac{1}{N_k} \sum_{i=1}^{N_k} \tilde{x}_i^k + \frac{1}{N_k} \sum_{i=1}^{N_k} x_i^k \right) + s_k \right] dt + D \frac{1}{N_k} \sum_{i=1}^{N_k} dw_i.
\end{align*}
$$

(63)
As \( N_k \rightarrow \infty \), the solution to (63) converges, in quadratic mean, to the solution of

\[
d\bar{x}_k = A_k \bar{x}_k dt + Gx_0 dt - B_k R^{-1}m_k^T \left[ \Pi_k (\bar{x}_{k,ex} + \bar{x}_{k,ex}) + s_k \right] dt,
\]

where \( \bar{x}_{k,ex} = \begin{bmatrix} (\bar{x}_k^T)^T, x_0^T, \bar{x}_T^T, \hat{\xi}_{0|F_0}^T, \hat{\xi}_{T|F_0}^T \end{bmatrix}^T \), and from (62) the average of the estimation error \( \bar{x}_{i,ex} \) over subpopulation \( k, 1 \leq k \leq K \), as \( N_k \rightarrow \infty \), i.e. \( \bar{x}_{k,ex} \), is given by

\[
d\bar{x}_{k,ex} = (A_k - K_k \mathbb{I}_k) \bar{x}_{k,ex} + \mathbb{D}[0_{1 \times r}, dw_{0}^T, 0_{1 \times rK}, dv_{0}^T]^T.
\]

Note that in the derivation of (65), we use the property that \( \frac{1}{N_k} \sum_{i=1}^{N_k} w_i = w_0 \) and \( \frac{1}{N_k} \sum_{i=1}^{N_k} \nu_i = \nu_0 \), since \( w_0 \) and \( \nu_0 \) are the common processes shared between all agents of type \( k \). Moreover, the law of large numbers is used to obtain as \( N_k \rightarrow \infty \)

\[
\frac{1}{N_k} \sum_{i=1}^{N_k} K_k dw_i \xrightarrow{q.m.} 0, \quad \frac{1}{N_k} \sum_{i=1}^{N_k} dw_i \xrightarrow{q.m.} 0.
\]

Subsequently, from (65), \((\bar{x}_{ex})^T = [(\bar{x}_{1,ex})^T, \ldots, (\bar{x}_{K,ex})^T]\) satisfies

\[
d\bar{x}_{ex} = \begin{bmatrix} (A_1 - K_1 \mathbb{I}_1) \bar{e}_1 + \mathbb{J} \\ \vdots \\ (A_K - K_K \mathbb{I}_K) \bar{e}_K + \mathbb{J} \end{bmatrix} \bar{x}_{ex} dt + \begin{bmatrix} -\mathbb{D} \\ \vdots \\ -\mathbb{D} \end{bmatrix} \begin{bmatrix} 0_{r \times 1} \\ dw_0 \\ 0_{r \times K} \\ dv_0 \end{bmatrix},
\]

or equivalently in the compact form

\[
d\bar{x}_{ex} = \bar{A}_{ex} \bar{x}_{ex} dt + \bar{D}[0_{1 \times r}, dw_0^T, 0_{1 \times rK}, dv_0^T]^T.
\]

Using (66) the mean field equation (64) can be presented as

\[
d\bar{x}_k = \left[ A_k - B_k R^{-1}B_k^T \Pi_{k,11} \right] \bar{e}_k - B_k R^{-1}B_k^T \Pi_{k,13} \bar{x}_k dt
\]
\[
+ \left( G - B_k R^{-1}B_k^T \Pi_{k,12} \right) x_0 dt - B_k R^{-1}B_k^T \Pi_{k,14} \hat{f}_{0|F_0} dt
\]
\[
- B_k R^{-1}B_k^T \Pi_{k,15} \hat{\xi}_{0|F_0} dt - B_k R^{-1}B_k^T \Pi_{k,16} \bar{x}_{k,ex} dt - B_k R^{-1}B_k^T \bar{s}_k dt.
\]

Since (64) and (20) must be identical, we obtain the Consistency Equations, determining the components of \( \bar{A}, \bar{G}, \bar{H}, \bar{L}, \bar{J}, \) and \( \bar{m} \) in (20), given by the following compact set of equations

\[
\bar{A}_k = [A_k - B_k R^{-1}B_k^T \Pi_{k,11} \bar{e}_k - B_k R^{-1}B_k^T \Pi_{k,13}, \forall k],
\]
\[
\bar{G}_k = G - B_k R^{-1}B_k^T \Pi_{k,12}, \forall k,
\]

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\[ \bar{H}_k = -B_k R^{-1} B_k^T \Pi_{k,14}, \forall k, \]
\[ \bar{L}_k = -B_k R^{-1} B_k^T \Pi_{k,15}, \forall k, \]
\[ \bar{J}_k = -B_k R^{-1} B_k^T \Pi_{k}, \forall k, \]
\[ \bar{m}_k = -B_k R^{-1} B_k^T s_k, \forall k, \] \hspace{1cm} \text{(69)}

where \( \Pi_k \) and \( s_k \) satisfy (54) and (55), respectively. The set of equations (69) together with (33)-(34) and (54)-(55) form a fixed point problem which must be solved by each individual agent \( A_i, 0 \leq i \leq N \), in order to compute the matrices in the mean field dynamics (20).

Finally from (46) and (64)-(67) the Markovian dynamics of \( \bar{x}^k \) (i.e. the mean field of subpopulation \( k, 1 \leq k \leq K \)) are given by

\[
\frac{d\bar{x}^{k,ex}}{dx^{ex}} = \begin{bmatrix}
\bar{A}_k - B_k R^{-1} B_k^T \Pi_k & -B_k R^{-1} B_k^T \Pi_k \bar{e}_k \\
0 & \bar{A}_k
\end{bmatrix}
\begin{bmatrix}
\bar{x}^{k,ex} \\
\bar{e}^{ex}
\end{bmatrix}
+ \begin{bmatrix}
M - B_k R^{-1} B_k^T s_k \\
0
\end{bmatrix}
dt
+ \begin{bmatrix}
D & 0 \\
0 & \tilde{D}
\end{bmatrix}
\begin{bmatrix}
0_{r \times 1} \\
0_{rK \times 1}
\end{bmatrix}.
\] \hspace{1cm} \text{(70)}

 Remark 2. From (65) in the infinite population limit the average of the estimation errors of the minor agents of type \( k, 1 \leq k \leq K \), is driven by the major agent’s Wiener process \( w_0 \) and the measurement noise \( v_0 \) (or equivalently innovation process \( v_0 \)). In other words, it is driven by the non-zero quadratic variation processes in the dynamics of the common processes \( x_0^{ex}, \hat{x}_0^{ex} \), with which the minor agents \( A_i, 1 \leq i \leq N \), are coupled.

Subsequently, \( \bar{V}(t) = \mathbb{E}\left[\bar{e}^{ex}(t) \left( \hat{x}^{ex}(t) \right)^T\right] \) satisfies

\[
\dot{\bar{V}} = \bar{A} \bar{V} + \bar{V} \bar{A}^T + \bar{D}
\begin{bmatrix}
0_{r \times r} & I_{r \times r} \\
0_{r \times K} & 0_{r \times rK}
\end{bmatrix}
\begin{bmatrix}
0_{r \times 1} \\
0_{rK \times 1}
\end{bmatrix},
\] \hspace{1cm} \text{(71)}

and if we put \( \tilde{D} = -\bar{1}^T \bar{D} \), we obtain

\[
\dot{\bar{V}} = \bar{A} \bar{V} + \bar{V} \bar{A}^T + \tilde{Q} \tilde{Q}^T,
\] \hspace{1cm} \text{(72)}

where

\[
\tilde{Q} \tilde{Q}^T = \bar{1}^T
\begin{bmatrix}
0_{n \times n} & 0 \\
0 & Q_{w_0}
\end{bmatrix}
\begin{bmatrix}
0_{n \times K_0 R_{v_0} K_0^T}
\end{bmatrix} \bar{1}.
\] \hspace{1cm} \text{(73)}
To guarantee the convergence of the solution to the corresponding Lyapunov equation to a unique, symmetric and positive definite solution, we assume:

Assumption 7. The pair $[\tilde{A}, \tilde{Q}]$ is controllable.

Remark 3. For the case where the major agent has complete observation on its own state, and each minor agent has complete observations on their own state and the major agent’s state we have

$$\bar{x}^{k,ex}(t) = 0, \quad t \geq 0,$$

$$\mathbb{E}\{x_0|F^y_0\} = x_0, \quad (74)$$

$$\mathbb{E}\{\bar{x}|F^y_0\} = \bar{x}, \quad (75)$$

where $(76)$ holds since the major agent can compute the real value of $\bar{x}$ by observing its own state. Hence the mean field equation $(20)$ reduces to that of completely observed major minor LQG MFG systems (see [10]).

Remark 4 (Estimate of $\infty$-Population Average Estimation Error). The solution to $(67)$ is given by

$$\bar{x}^{ex}(t) = \Phi(t,0)\bar{x}^{ex}(0) + \int_0^t \Phi(t,\tau)\mathbb{D}[0_{1 \times r}, dw_0^T, 0_{1 \times rK}, dv_0^T]d\tau,$$  \hspace{1em} (77)

where $\Phi(t, \tau) = \exp (\tilde{A}(t - \tau))$. The initial estimation error of the minor agent $\mathcal{A}_i$ is given by

$$\tilde{x}_i^{k,ex}(0) = \begin{bmatrix}
\hat{x}_i|F^y_i(0) - x_i(0) \\
\hat{x}_0|F^y_0(0) - x_0(0) \\
\hat{x}|F^y_0(0) - \bar{x}(0) \\
\hat{x}|F^y_i(0) - \hat{x}_0|F^y_0(0) \\
\hat{x}|F^y_i(0) - \hat{x}_0|F^y_0(0)
\end{bmatrix} = \begin{bmatrix}
-x_i(0) \\
-x_0(0) \\
0_{nK \times 1} \\
0_{n \times 1} \\
0_{n \times 1}
\end{bmatrix}, \quad (78)$$

since the partial observation information sets $F^y_i$, $0 \leq i \leq N$, at time $t_0 = 0$ are null sets, the conditional expectations turn into total expectations which according to Assumption 7 their value is zero. Hence, the infinite-population limit of the average initial estimation error of the minor agents of subpopulation $k$ is given by

$$\bar{x}^{k,ex}(0) = [0_{1 \times n}, x_0^T(0), 0_{1 \times nK}, 0_{1 \times n}, 0_{1 \times nK}]^T, \quad (79)$$
where Assumption 7 is again used, and hence $E[\tilde{x}^{k,ex}(0)|\mathcal{F}_i^y] = 0$. Then the conditional expectation of $\tilde{x}^{ex}(t)$ with respect to $\mathcal{F}_i^y$, $0 \leq i \leq N$, i.e. $\hat{x}^{ex}_{F_i^y}(t)$, is given by

$$
\hat{x}^{ex}_{F_i^y}(t) \triangleq E[\tilde{x}^{ex}(t)|\mathcal{F}_i^y] = \Phi(t,0)E[\tilde{x}^{ex}(0)|\mathcal{F}_i^y] \\
+ E\left[\int_0^t \Phi(t,\tau)\bar{D} d\tau \bigg| \mathcal{F}_i^y\right] = 0, \quad (80)
$$

where the second term is zero due to the independence of $\{w_i, 0 \leq i \leq N\}$ and $\{v_i, 0 \leq i \leq N\}$. □

Next we define

$$
M_1 = \begin{bmatrix}
A_1 - B_1 R^{-1} B_1^T \Pi_{1,11} & 0 \\
0 & \ddots \\
B_1 R^{-1} B_1^T \Pi_{1,13} & \ddots \\
B_K R^{-1} B_K^T \Pi_{K,13}
\end{bmatrix},
$$

$$
M_2 = \begin{bmatrix}
A_K - B_K R^{-1} B_K^T \Pi_{K,11}
\end{bmatrix},
$$

$$
M_3 = \begin{bmatrix}
A_0 & 0 & 0 \\
G & \bar{A} & 0 \\
G & -M_2 & M_1
\end{bmatrix},
$$

$$
L_{0,H} = Q_0^{1/2} [I, 0, -H_0^\pi]. \quad (81)
$$

The final set of assumptions is as follows:

**Assumption 8.** The pair $(L_{0,H}, M_3)$ is observable.

**Assumption 9.** The pair $(L_a, A_0 - (\rho/2)I)$ is detectable, and for each $k, 1 \leq k \leq K$, the pair $(L_b, A_k - (\rho/2)I)$ is detectable, where $L_a = Q_0^{1/2} [I, -H_0^\pi]$ and $L_b = Q_0^{1/2} [I, -H_1, -H_2^\pi, 0_{n \times (n+nK)}]$. The pair $(A_0 - (\rho/2)I, L_b)$ is stabilizable and $(A_k - (\rho/2)I, B_k)$ is stabilizable for each $k, 1 \leq k \leq K$.

**Assumption 10.** There exists a stabilizing solution $\Pi_0, s_0, \Pi_k, s_k, \bar{A}_k, \bar{G}_k, \bar{H}_k$, $\bar{L}_k, \bar{J}_k, \bar{m}_k$ to the major-minor mean field equations (69) in the sense that the matrices

$$
A_0 - B_0 R_0^{-1} B_0^T \Pi_0 - \frac{\rho}{2} I, \\
A_k - B_k R_k^{-1} B_k^T \Pi_k - \frac{\rho}{2} I, \quad 1 \leq k \leq K,
$$

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are asymptotically stable, and
\[
\sup_{t \geq 0, 1 \leq k \leq K} e^{-\frac{\rho}{2} t} (|s_0(t)| + |s_k(t)| + |\hat{m}_k(t)|) < \infty.
\]

**Theorem 3** (\(\epsilon\)-Nash Equilibria for PO LQG MM-MFG Systems). *Subject to Assumption [1] Assumption [10] the KF-MFG state estimation scheme (27)-(31) and (48)-(51) together with the MM-MFG equation scheme (69) generate an infinite family of stochastic control laws \(U^\infty_{M_F}\), with finite sub-families \(U^N_{M_F} = \{u^o_i; 0 \leq i < N\}\), 1 \(\leq N < \infty\), given by (32) and (53), such that

(i) \(U^\infty_{M_F}\) yields a unique Nash equilibrium within the set of linear controls \(U^\infty_{y}\) such that

\[
J^\infty_i(u^o_i, u^o_{-i}) = \inf_{u_i \in U^\infty_{y}} J^\infty_i(u_i, u^o_{-i});
\]

(ii) All agent systems 0 \(\leq i \leq N\), are \(e^{-\frac{\rho}{2} t}\) discounted second order stable in the sense that

\[
\sup_{t \geq 0, 0 \leq i \leq N} e^{-\frac{\rho}{2} t} \mathbb{E} \left[\|\hat{x}_i|_{\mathcal{F}_t^y}\|^2 + \|\hat{\hat{x}}_i|_{\mathcal{F}_t^y}\|^2 + \|((\hat{x}_0|_{\mathcal{F}_0^y})|_{\mathcal{F}_t^y}\|^2 + \|(\hat{\hat{x}}_0|_{\mathcal{F}_0^y})|_{\mathcal{F}_t^y}\|^2\right] < C,
\]

with \(C\) independent of \(N\);

(iii) \(\{U^N_{M_F}; 1 \leq N < \infty\}\) yields a unique \(\epsilon\)-Nash equilibrium within the class of linear control laws \(U^N_{y}\), for all \(\epsilon\), i.e. for all \(\epsilon > 0\), there exists \(N(\epsilon)\) such that for all \(N \geq N(\epsilon)\):

\[
J^\epsilon_N(u^o_i, u^o_{-i}) - \epsilon \leq \inf_{u_i \in U^N_{y}} J^\epsilon_N(u_i, u^o_{-i}) \leq J^\epsilon_N(u^o_i, u^o_{-i}),
\]

where the major agent’s and the generic minor agent’s performance function \(J^\epsilon_N(u^o_i, u^o_{-i})\), \(u_i \in U^N_{i,y}\), 0 \(\leq i \leq N\), is given by

\[
J^\epsilon_i(u_i, u_{-i}) + \hat{E}_N,
\]

where \(J^\epsilon_i(u_i, u_{-i})\) is as in the completely observed case, \(\hat{E}_N > 0\), and when \(u_i = u^o_i\) the following limits hold:

- \(\lim_{N \to \infty} J^N_i(u^o_i, u^o_{-i}) = J^\infty_i(u^o_i, u^o_{-i})\),
- \(\lim_{N \to \infty} \hat{E}_N = \int_0^\infty e^{-\rho t} tr(Q^\pi V) dt\),

where \(V(t)\) is the solution to (31) for the major agent and the solution to (48) for a generic minor agent.
Proof. Generalizing the standard methodology in [31] and [32], we first decompose the state processes into their estimates and their estimation errors orthogonal to the corresponding estimates. Substituting the decomposed states into the performance functions and applying the smoothing property of conditional expectations with respect to the increasing filtration families $\mathcal{F}_y$ and $\mathcal{F}_0^y$ to the major and minor cost functionals respectively, we obtain the separated performance functions. This technique is applied to both finite and infinite population cases which yields the best response controls $\{\hat{u}_i^0, 0 \leq i \leq N\}$ as optimal tracking controls for the major and minor agents in the infinite population case (see [18] for the case where only the minor agent has partial observations on the major agent’s state). Specifically we form the following decompositions where the superscript ‘s’ on the resulting performance functions indicates the separation into control dependent and control independent summands.

1. Major Agent’s State Decomposition
   **Finite Population:**
   \[
   \begin{bmatrix}
   x_0 \\
   x(N)
   \end{bmatrix} =
   \begin{bmatrix}
   \hat{x}_0|\mathcal{F}_y^0 \\
   \hat{x}(N)|\mathcal{F}_0^y
   \end{bmatrix} +
   \begin{bmatrix}
   x_0 - \hat{x}_0|\mathcal{F}_y^0 \\
   x(N) - \hat{x}(N)|\mathcal{F}_0^y
   \end{bmatrix}.
   \]

   **Infinite Population:**
   \[
   \begin{bmatrix}
   x_0 \\
   \bar{x}
   \end{bmatrix} =
   \begin{bmatrix}
   \hat{x}_0|\mathcal{F}_0^y \\
   \hat{x}|\mathcal{F}_y^0
   \end{bmatrix} +
   \begin{bmatrix}
   x_0 - \hat{x}_0|\mathcal{F}_0^y \\
   \bar{x} - \hat{x}|\mathcal{F}_y^0
   \end{bmatrix}.
   \]

2. Major Agent’s Cost Functional Separation
   **Finite Population:**
   \[
   J^{s,N}_0 (u_0, u_{-0}) =
   \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left\{ \|\hat{x}_0|\mathcal{F}_0^y - H_0\hat{x}(N)|\mathcal{F}_0^y - \eta_0\|_{Q_0}^2 + \|u_0\|_{R_0}^2 \right\} dt \right]
   + \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left\{ (x_0 - \hat{x}_0|\mathcal{F}_0^y) - H_0(x(N) - \hat{x}(N)|\mathcal{F}_0^y)\right\}^2_{Q_0} dt \right].
   \tag{82}
   \]

   **Infinite Population:**
   \[
   J^{s,\infty}_0 =
   \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left\{ \|\hat{x}_0|\mathcal{F}_0^y - H_0^\pi\hat{x}_0|\mathcal{F}_0^y - \eta_0\|_{Q_0}^2 + \|u_0\|_{R_0}^2 \right\} dt \right]
   + \mathbb{E}\left[ \int_0^\infty e^{-\rho t}\left\{ (x_0 - \hat{x}_0|\mathcal{F}_0^y) - H_0^\pi(\bar{x} - \hat{x}|\mathcal{F}_0^y)\right\}^2_{Q_0} dt \right].
   \tag{83}
   \]
3. Minor Agent’s State Decomposition

**Finite Population:**

\[
\begin{bmatrix}
  x_i \\
  x_0 \\
  x(N)
\end{bmatrix} =
\begin{bmatrix}
  \hat{x}_i|F_t^y \\
  \hat{x}_0|F_t^y \\
  \hat{x}|F_t^y
\end{bmatrix}
+ \begin{bmatrix}
  x_i - \hat{x}_i|F_t^y \\
  x_0 - \hat{x}_0|F_t^y \\
  x(N) - \hat{x}(N)|F_t^y
\end{bmatrix}.
\]

**Infinite Population:**

\[
\begin{bmatrix}
  x_i \\
  x_0 \\
  \bar{x}
\end{bmatrix} =
\begin{bmatrix}
  \hat{x}_i|F_t^y \\
  \hat{x}_0|F_t^y \\
  \hat{\bar{x}}|F_t^y
\end{bmatrix}
+ \begin{bmatrix}
  x_i - \hat{x}_i|F_t^y \\
  x_0 - \hat{x}_0|F_t^y \\
  \bar{x} - \hat{\bar{x}}|F_t^y
\end{bmatrix}.
\]

4. Minor Agent’s Cost Functional Separation

**Finite Population:**

\[
J_{s,N}^i(u_i, u_{-i}) = 
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left\{ \left\| \hat{x}_i|F_t^y - H_1 \hat{x}_0|F_t^y \\
- H_2 \hat{x}(N)|F_t^y - \bar{\eta} \right\|_Q^2 + \| u_i \|_R^2 \right\} dt \right] + \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left\| (x_i - \hat{x}_i|F_t^y) \\
- H_1(x(0) - \hat{x}_0|F_t^y) - H_2(x(N) - \hat{x}(N)|F_t^y) \right\|_Q^2 dt \right]. \tag{84}
\]

**Infinite Population:**

\[
J_{s,\infty}^i = 
\mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left\{ \left\| \hat{x}_i|F_t^y - H_1 \hat{x}_0|F_t^y \\
- H_2 \hat{x}|F_t^y - \bar{\eta} \right\|_Q^2 + \| u_i \|_R^2 \right\} dt \right] + \mathbb{E} \left[ \int_0^\infty e^{-\rho t} \left\| (x_i - \hat{x}_i|F_t^y) \\
- H_1(x(0) - \hat{x}_0|F_t^y) - H_2(x - \hat{x}|F_t^y) \right\|_Q^2 dt \right]. \tag{85}
\]

As can be seen, the first integral expressions in (82), (83), (84) and (85) depend on the estimated states generated by the estimation schemes (27) and (51) for the major agent and minor agents respectively, and the second integral expressions depend only upon the respective estimation errors and on the solutions to the associated Riccati equations. The latter expressions are independent of the control.
actions and generate the additional cost \( \hat{E}_N \) in the finite population case incurred by the errors in the estimation process.

Next, the resulting infinite population tracking problems are solved for the major and minor agents in their separated forms. The control dependent summands in (83) have exactly the same structure in terms of the functional dependence on the estimated states as the infinite population cost functionals in the complete observation case have on the states. Moreover, the control dependent summands in (85) have exactly the same structure in terms of the functional dependence on the estimated states as the infinite population cost functional for the system (46) with complete observations on its own state, the major agent’s state, and the major agent’s estimates of its own state and the mean field. Hence, by the Separation Principle the infinite population Nash Certainly Equivalence equilibrium controls are given by \( \{\hat{u}_{i}, 0 \leq i \leq N\} \) in the theorem statement. Finally the infinite population control actions are applied to the finite population systems and the fact that these yield (i) \( e^{-\hat{\rho}t} \) second order system stability, and (ii) \( \epsilon \)-Nash equilibrium property, is established by the standard approximation analysis parallel to that of completely observed major-minor LQG MFG systems (see [6], [10]).

Remark 5. We note that \( (\hat{x}_0|\mathcal{F}_0^\nu) \) and \( (\hat{x}_i|\mathcal{F}_i^\nu) \) do not appear in the minor agent’s state decomposition and in its separated performance function but that they are used in the extended estimated state recursion (51) and hence appear in the control action for a minor agent.

Remark 6. The non-uniqueness of Nash equilibria which may occur in classical LQG stochastic dynamic games with specified information sets [33, 34] does not occur in this analysis. This holds since, for the specified maximal individual information sets, and subject to the hypotheses of Theorem [3] giving unique solutions to the MFG Consistency equations (as functions of the system parameters), a unique linear best response function is obtained for each agent with respect to its stochastic control problem arising from its performance function in the infinite population limit. We note that any set of controls generating a Nash equilibrium will yield the same consistency equations whose solution depends only on the system parameters.
4 Simulations

Consider a system of 100 minor agents and a single major agent. The system matrices \( \{A_k, B_k, 1 \leq k \leq 100\} \) for the minor agents are uniformly defined as

\[
A \triangleq \begin{bmatrix} -0.05 & -2 \\ 1 & 0 \end{bmatrix}, \quad B \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]

and for the major agent we have

\[
A_0 \triangleq \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B_0 \triangleq \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

The parameters used in the simulation are: \( t_{\text{final}} = 25 \text{ sec} \), \( \Delta t = 0.01 \text{ sec} \), \( \sigma_w = \sigma_{w_i} = 0.009 \), \( \sigma_v = \sigma_{v_i} = 0.0003 \), \( \rho = 0.9 \), \( \eta_0 = \eta = [0.25, 0.25]^T \), \( Q_0 = Q = I_{2 \times 2} \), \( R_0 = R = 1 \), \( H_0 = H_1 = H_2 = 0.6 \times I_{2 \times 2} \), \( G = 0_{2 \times 2} \). The true and estimated state trajectories, and the estimation errors for a single realization can be displayed for the entire population of 101 agents together, but in figures only 10 minor agents are shown for the sake of clarity.
Figure 2: 10 Minor agents’ true and estimated trajectories.

Figure 3: The mean field true and estimated trajectories.
Figure 4: The estimation errors of the major agent’s trajectory.

Figure 5: The estimation errors of the mean field trajectory.
5 Conclusions

In this paper, PO MM LQG MFG problems with general information patterns are studied where (i) the major agent has partial observations on its own state, and (ii) each minor agent has partial observations on its own state and the major agent’s state. For the general case of indefinite LQG MFG systems, the existence of $\epsilon$-Nash equilibria together with the individual agents’ control laws generating them are established via the Separation Principle. The assumption of partial observations for all agents leads to a new situation involving the recursive estimation by each minor agent of the major agent’s estimate of its own state. To the best of our knowledge, the dynamic game theoretic equilibrium which is established in this paper constitutes a rare case wherein agents explicitly generate estimates of another agent’s beliefs. Moreover, this does not give rise to an infinite regress due to the information asymmetry of the major and minor agents.

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