AN ALTERNATIVE DEFINITION OF THE COMPLETION OF
METRIC SPACES

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Abstract. In this article, the author proposes another way to define the com-
pletion of a metric space, which is different from the classical one via the dense
property, and prove the equivalence between two definitions. This definition
is based on considerations from category theory, and can be generalized to
arbitrary categories.

1. Introduction

In Definition 1.2.4 of [1], Zhang and Lin gave out the notion of a completion of
a metric space as follows.

Definition 1. A metric space $Y$ is called a completion space of a given metric space
$X$, if it is the smallest complete metric space containing $X$. Here the smallest means
that every metric space containing $X$ contains $Y$ as its subspace.

Remark. A metric space $(X_1, \rho_1)$ is said to be (isometrically) embedded in a metric
space $(X_2, \rho_2)$, if $(X_1, \rho_1)$ is isometrically isomorphic to a subspace, which is en-
dowed with the induced metric, of $(X_2, \rho_2)$. Under this interpretation, we regard
$(X_1, \rho_1)$ itself a subspace of $(X_2, \rho_2)$; see [1, p. 10].

However, the completion space under this definition is not unique, i.e., there
exists two spaces that are both completion spaces of a given space but are not
isometrically isomorphic. Nevertheless, it is well known that if we alter the re-
quirement of $Y$ being the “smallest” by demanding $X$ to be dense in $Y$, then the
completion space is unique in the sense of isometric isomorphism. Thus, we have
the following definition; see [2, p. 74].

Definition 1*. A metric space $Y$ is called a completion space of a metric space
$X$, if $Y$ is complete and $X$ is dense in $Y$.

In this article, we amend Definition 1 to be well-defined and equivalent to Defi-
nition 1*, and show that it fits itself well in the language of categories.

2. Preliminaries in Category Theory

In mathematics, category theory [3] deals in an abstract way with mathematical
structures and relationships between them.

Definition 2. A category consists of

(i) a class of objects, and
(ii) for any two objects $A$ and $B$, a set $\text{Hom}(A, B)$ of morphisms from $A$ to $B$ such that the following axioms hold:

- **Composition:** if $f : A \to B$, $g : B \to C$ are two morphisms, then the composite morphism $g \circ f : A \to C$ is defined;
- **Associativity:** if $f : A \to B$, $g : B \to C$, $h : C \to D$ are morphisms, then $h \circ (g \circ f) = (h \circ g) \circ f$;
- **Identity:** for every object $X$, there exists a morphism $1_X : X \to X$ called the identity morphism for $X$, such that for every morphism $f : A \to B$, we have $1_B \circ f = f = f \circ 1_A$.

There are abundant examples of categories, ranging from algebra to geometry, and also to analysis. As illustrations, all groups (rings, fields) as objects and group (ring, field) homomorphisms as morphisms form the category of groups (rings, fields); all vector spaces over a fixed field $K$ as objects and $K$-linear transformations as morphisms form the category of $K$-vector spaces; and all differentiable manifolds (topological spaces) being objects, and smooth maps (continuous maps) between them the morphisms, form the category of differentiable manifolds (topological spaces).

It is easy to verify that all metric spaces together with isometries between them form a category, where metric spaces serves as objects and isometries as morphisms. This category is called the category of metric spaces. An isometry here means a map $\varphi : (X, \rho_X) \to (Y, \rho_Y)$, such that for any $x, x' \in X$, $\rho_X(x, x') = \rho_Y(\varphi(x), \varphi(x'))$ holds. Notice that a metric space $X$ is a subspace of $Y$ if and only if there exists a morphism from $X$ to $Y$.

3. The Definition of Completion and its Generalization

Now, we can define the completion of a metric space according to the algebraic structures – morphisms between objects – on the category of metric spaces:

**Definition 1**. A complete metric space $(Y, \rho_Y; \varphi_Y)$ containing a given metric space $(X, \rho_X)$, where $\varphi_Y$ is the morphism from $X$ to $Y$, is called a completion space of $(X, \rho_X)$, if for every complete metric space $(Z, \rho_Z; \varphi_Z)$ containing $(X, \rho_X)$, $\varphi_Z$ being the morphism from $X$ to $Z$, there exists a morphism $\varphi$ from $(Y, \rho_Y)$ to $(Z, \rho_Z)$ such that the diagram

\[
\begin{array}{ccc}
(Y, \rho_Y) & \xrightarrow{\varphi} & (Z, \rho_Z) \\
\downarrow{\varphi_Y} & & \downarrow{\varphi_Z} \\
(X, \rho_X) & \xleftarrow{\varphi} &
\end{array}
\]

commutes, i.e. for all $x \in X$, $\varphi_Z(x) = \varphi \circ \varphi_Y(x)$ holds.

To see that Definition 1 is well-defined and coincides with Definition 1*, we have the following theorem, which is also the main result of this paper.

**Theorem 1.** Let $(X, \rho_X)$ be a metric space. If $(Y, \rho_Y; \varphi_Y)$ is a complete metric space containing $(X, \rho_X)$ as its subspace, where $\varphi_Y$ is the embedding map, then the following statements are equivalent:

1. $(X, \rho_X)$ is isometric to a dense subspace of $(Y, \rho_Y)$, or more precisely, $\varphi_Y(X)$ is dense in $(Y, \rho_Y)$;
(2) For every complete metric space \((Z, \rho_Z; \varphi_Z)\) containing \((X, \rho_X)\), \(\varphi_Z\) the embedding map, there exists an embedding \(\varphi\) from \((Y, \rho_Y)\) to \((Z, \rho_Z)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
(Y, \rho_Y) & \xrightarrow{\varphi} & (Z, \rho_Z) \\
\downarrow{\varphi_Y} & & \downarrow{\varphi_Z} \\
(X, \rho_X) & \xrightarrow{\varphi} & (Z, \rho_Z)
\end{array}
\]

Next section will be devoted to the proof of this theorem.

From this theorem, we see that the completion space of a metric space in Definition 1* satisfies the required properties of that in Definition 1** and vice versa, which shows that the two definitions are equivalent. The advantage of Definition 1** over Definition 1* is that Definition 1** can be generalized to arbitrary categories, using commutative diagrams: Let \(C\) be a category, \(P\) a property, and \(S = \{X \in C \mid X\text{ has property } P\}\) be the subset containing all elements satisfying \(P\). If for any object \(A\) in \(S\), the only 1-1 morphism from \(A\) to itself is the identity map, then we can define the \(P\)-tion of any object in \(C\) as follows.

**Definition 3.** Let \(X\) be an object in \(C\), \(Y\) an object containing \(X\), i.e. there exists a 1-1 morphism \(f_Y\) from \(X\) to \(Y\). \(Y\) is called a \(P\)-tion of \(X\), if \(Y \in S\) and for any \(Z \in S\) containing \(X\), \(f_Z\) a 1-1 morphism from \(X\) to \(Z\), there exists a 1-1 morphism \(f\) from \(Y\) to \(Z\), such that the diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{f_Z} & Z \\
\downarrow{f_Y} & & \downarrow{f} \\
X & \xrightarrow{f_Z} & Z
\end{array}
\]

commutes.

**Remark.** In the category of metric spaces, since all morphisms are isometric embeddings, there are automatically 1-1 by the property of isometries.

### 4. Proof of Theorem 1

**Proof of Theorem 1.** (1) \(\Rightarrow\) (2): Suppose that \((Z, \rho_Z)\) is a complete metric space containing \((X, \rho_X)\) and \(\varphi_Z\) denotes the embedding map from \((X, \rho_X)\) to \((Z, \rho_Z)\).

For every \(y \in (Y, \rho_Y)\), since \(\varphi_Y(X)\) is dense in \((Y, \rho_Y)\), there exists a sequence \(\{x_n\}_{n=1}^{\infty} \subset (X, \rho_X)\) such that \(\varphi_Y(x_n) \to y\) in \((Y, \rho_Y)\) as \(n \to \infty\). By the fact that both \(\varphi_Y\) and \(\varphi_Z\) are embedding, i.e. they preserve metrics, then both \(\{x_n\}_{n=1}^{\infty}\) and \(\{\varphi_Z(x_n)\}_{n=1}^{\infty}\) are Cauchy sequences in corresponding spaces. By assumption, \((Z, \rho_Z)\) is complete, then there exists a \(z \in (Z, \rho_Z)\) such that

\[
\lim_{n \to \infty} \varphi_Z(x_n) = z.
\]

Define \(\varphi : (Y, \rho_Y) \to (Z, \rho_Z), \varphi(y) = z\), where \(z = \lim_{n \to \infty} \varphi_Z(x_n)\) is determined as above. Now we verify that \(\varphi\) is well-defined (i.e. independent of the choice of \(\{x_n\}_{n=1}^{\infty}\)) and satisfies the desired demand.
(i) Suppose that \( \{x_n\}_{n=1}^{\infty} \) and \( \{x'_n\}_{n=1}^{\infty} \) are two sequences in \((X, \rho_X)\), both \(\{\varphi_Y(x_n)\}_{n=1}^{\infty}\) and \(\{\varphi_Z(x'_n)\}_{n=1}^{\infty}\) tend to \(y \in Y\). Then

\[
\lim_{n \to \infty} \rho_X(x_n, x'_n) = \lim_{n \to \infty} \rho_Y(\varphi_Y(x_n), \varphi_Y(x'_n)) = \rho_Y(y, y) = 0,
\]

and

\[
\lim_{n \to \infty} \rho_Z(\varphi_Z(x_n), \varphi_Z(x'_n)) = \lim_{n \to \infty} \rho_X(x_n, x'_n) = 0,
\]

which implies

\[
\lim_{n \to \infty} \varphi_Z(x_n) = \lim_{n \to \infty} \varphi_Z(x'_n).
\]

Therefore, \(\varphi\) is well-defined.

(ii) \(\varphi\) is an isometry. For any \(y, y' \in (Y, \rho_Y)\), there exist sequences \(\{x_n\}_{n=1}^{\infty}\) and \(\{x'_n\}_{n=1}^{\infty}\) in \((X, \rho_X)\) such that \(y = \lim_{n \to \infty} \varphi_Y(x_n)\) and \(y' = \lim_{n \to \infty} \varphi_Y(x'_n)\). By definition, \(\varphi(y) = \lim_{n \to \infty} \varphi_Z(x_n)\) and \(\varphi(y') = \lim_{n \to \infty} \varphi_Z(x'_n)\). Since \(\varphi_Y\) and \(\varphi_Z\) are both isometries, and the limit operation commutes with distance functions, so we have

\[
\rho_Z(\varphi(y), \varphi(y'))
= \rho_Z\left(\lim_{n \to \infty} \varphi_Z(x_n), \lim_{n \to \infty} \varphi_Z(x'_n)\right) = \lim_{n \to \infty} \rho_Z(\varphi_Z(x_n), \varphi_Z(x'_n))
= \lim_{n \to \infty} \rho_X(x_n, x'_n) = \lim_{n \to \infty} \rho_Y(\varphi_Y(x_n), \varphi_Y(x'_n))
= \rho_Y\left(\lim_{n \to \infty} \varphi_Y(x_n), \lim_{n \to \infty} \varphi_Y(x'_n)\right)
= \rho_Y(y, y').
\]

Therefore, \(\varphi\) is an isometry.

(iii) For every \(y = \varphi_Y(x) \in \varphi_Y(X)\), let \(x_n \equiv x\) for all natural numbers \(n\). Then

\[
\varphi(y) = \lim_{n \to \infty} \varphi_Z(x_n) = \lim_{n \to \infty} \varphi_Z(x) = \varphi_Z(x).
\]

Hence, for all \(x \in X\), \(\varphi(\varphi_Y(x)) = \varphi_Z(x)\), and then, the diagram in (2) commutes.

(2) \(\Rightarrow\) (1): Consider \(Z \equiv \overline{\varphi_Y(X)}\), the closure of \(\varphi_Y(X)\) in \((Y, \rho_Y)\). Since \((Y, \rho_Y)\) is complete and \(Z\) is closed in \((Y, \rho_Y)\), we obtain that \((Z, \rho_Z)\) is also complete, where \(\rho_Z\) is the metric induced from \(\rho_Y\). By assumption, there exists an embedding \(\varphi\) from \((Y, \rho_Y)\) to \((Z = \varphi_Y(X), \rho_Z)\) such that the diagram

\[
\begin{array}{ccc}
(Y, \rho_Y) & \xrightarrow{\varphi} & (Z = \varphi_Y(X), \rho_Z) \\
| \quad \varphi_Y \quad | & \downarrow \quad \varphi \quad | & \downarrow \quad \varphi \quad | \\
(X, \rho_X) & \xrightarrow{\varphi_Y} & (Z = \varphi_Y(X), \rho_Z)
\end{array}
\]

commutes, i.e., \(\varphi|_{\varphi_Y(X)} = id\). For every \(y \in Y\), \(\varphi(y)\) belongs to \(\varphi_Y(X)\). Thus, for any \(\varepsilon > 0\), there exists a \(x' \in X\) such that

\[
\rho_Z(\varphi(y), \varphi_Y(x')) < \varepsilon.
\]

Since both \(\varphi\) and \(\varphi_Y\) preserve metric, and \(\varphi|_{\varphi_Y(X)} = id\), we have

\[
\rho_Y(y, \varphi_Y(x')) = \rho_Z(\varphi(y), \varphi(\varphi_Y(x'))) = \rho_Z(\varphi(x), \varphi(x')) < \varepsilon.
\]
Thus, $\varphi_Y(X)$ is dense in $Y$, which completes the proof of Theorem 1.

\section*{References}
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