Abstract

Basic notions of continuous media mechanics are introduced for spaces with affine connections and metrics. The physical interpretation of the notion of relative acceleration is discussed. The notions of deformation acceleration, shear acceleration, rotation (vortex) acceleration, and expansion acceleration are introduced. Their corresponding notions, generated by the torsion and curvature, are considered. A classification is proposed for auto-parallel vector fields with different kinematic characteristics. Relations between the kinematic characteristics of the relative acceleration and these of the relative velocity are found. A summary of the introduced and considered notions is given. A classification is proposed related to the kinematic characteristics of the relative velocity and the kinematic characteristics related to the relative acceleration.

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Contents

1 Introduction 2

2 Relative acceleration. Deformation acceleration, shear acceleration, rotation (vortex) acceleration, and expansion acceleration 3

2.1 Physical interpretation of the notion of relative acceleration 3

2.1.1 Relative acceleration 3

2.2 Kinematic characteristics connected with the notion relative acceleration 4

2.2.1 Relative acceleration and its representation 4

2.2.2 Deformation acceleration, shear acceleration, rotation acceleration, and expansion acceleration 6

3 Classification of auto-parallel vector fields on the basis of the kinematic characteristics connected with the relative velocity and relative acceleration 8

3.1 Inertial flows. Special geodesic vector fields with vanishing kinematic characteristics, induced by the curvature, in (pseudo) Riemannian spaces 9
1 Introduction

The notion of relative acceleration is very important for the modern gravitational theories in (pseudo) Riemannian spaces or in spaces with one affine connection and metrics. The theoretical basis for construction of gravitational wave detectors is related to deviation equations describing relative accelerations induced by the curvature. For continuous media mechanics in $(\mathcal{L}_n, g)$-spaces the notion of relative acceleration and its kinematic characteristics (deformation acceleration, shear acceleration, rotation (vortex) acceleration, and expansion acceleration) could be related to the change of the volume force in a continuous medium.

In Section 2 the physical interpretation of the notion of relative acceleration is discussed. The notions of deformation acceleration, shear acceleration, rotation (vortex) acceleration, and expansion acceleration are introduced. Their corresponding notions, generated by the torsion and curvature, are considered. In Section 3 a classification is proposed for auto-parallel vector fields with different kinematic characteristics. In Section 4 relations between the kinematic characteristics of the relative acceleration and these of the relative velocity are found. In Section 5 a summary of the introduced and considered notions is given. In Section 6 a classification is proposed related to the kinematic characteristics of the relative velocity and the kinematic characteristics related to the relative acceleration.
Remark. The present paper is the third part of a larger research report on the subject with the title "Contribution to continuous media mechanics in \((L_n, g)\)-spaces" with the following contents:

I. Introduction and mathematical tools.
II. Relative velocity and deformations.
III. Relative accelerations.
IV. Stress (tension) tensor.

The parts are logically self-dependent considerations of the main topics considered in the report.

2 Relative acceleration. Deformation acceleration, shear acceleration, rotation (vortex) acceleration, and expansion acceleration

2.1 Physical interpretation of the notion of relative acceleration

Let us consider the change of the vector \(\nabla u\xi_{(a)}\perp\)

(a) along a curve \(x^i(\tau, \lambda a_0 = \text{const.})\) and

(b) along a curve \(x^i(\tau_0 = \text{const.}, \lambda a)\).

2.1.1 Relative acceleration

(a) In the first case

\[
\frac{D}{d\tau}\left(\frac{D\xi_{(a)}\perp}{d\tau}\right)(\tau_0, \lambda a_0) = \left(\nabla u\left(\nabla u\xi_{(a)}\perp\right)\right)(\tau_0, \lambda a_0) = \left(\nabla u\nabla u\xi_{(a)}\perp\right)(\tau_0, \lambda a_0) = \\
= \lim_{d\tau \to 0} \frac{\nabla u\xi_{(a)}\perp(\tau_0 + d\tau, \lambda a_0) - \nabla u\xi_{(a)}\perp(\tau_0, \lambda a_0)}{d\tau} = \\
= \lim_{d\tau \to 0} \frac{\left(\frac{D\xi_{(a)}\perp}{d\tau}\right)(\tau_0 + d\tau, \lambda a_0) - \left(\frac{D\xi_{(a)}\perp}{d\tau}\right)(\tau_0, \lambda a_0)}{d\tau}.
\]

(1)

The vector \(\nabla u\nabla u\xi_{(a)}\perp\) has two components with respect to the vector \(u\): one component collinear to \(u\) and one component orthogonal to \(u\), i.e.

\[
\nabla u\nabla u\xi_{(a)}\perp = \frac{1}{e} \cdot g(\nabla u\nabla u\xi_{(a)}\perp, u) \cdot u + \overline{g}[h_u(\nabla u\nabla u\xi_{(a)}\perp)] = \\
= \frac{\overline{t}_{(a)}}{e} \cdot u + \text{rel}(a) ,
\]

(2)

where

\[
\overline{t}_{(a)} = g(\nabla u\nabla u\xi_{(a)}\perp, u) , \quad \text{rel}(a) = \overline{g}[h_u(\nabla u\nabla u\xi_{(a)}\perp)] ,
\]

\[
\overline{t}_{(a)} = g\left(\frac{D^2\xi_{(a)}\perp}{d\tau^2}, u\right) , \quad \text{rel}(a) = \overline{g}[h_u\left(\frac{D^2\xi_{(a)}\perp}{d\tau^2}\right)] ,
\]

\[g(u, \text{rel}) = 0 .\]

The vector \(\frac{D^2\xi_{(a)}\perp}{d\tau^2}\) determines the change of the velocity \(\frac{D\xi_{(a)}\perp}{d\tau}\) at the point \((\tau_0, \lambda a_0)\) and, therefore, it describes the total acceleration of a material point at the
point \((τ₀, λ₈₀)\)

\[
totala(τ₀, λ₈₀) = a(τ₀, λ₈₀) + a(τ₀, λ₈₀),
\]

\[
a(τ₀, λ₈₀) = \left(\frac{\overline{T} \cdot u}{\epsilon}\right)(τ₀, λ₈₀),
\]

\[
a(τ₀, λ₈₀) = \nrela(τ₀, λ₈₀) = \n[u \left(\frac{D^2ξ(τ₀, λ₈₀)}{dτ^2}\right)](τ₀, λ₈₀).
\]

The total acceleration \(totala(a)\) has two parts. The first part \(a||\) is the change of the velocity \(Dξ(τ₀, λ₈₀)/dτ\) along the flow. The second part \(a⊥\) is the acceleration of the material points orthogonal to the flow. The last acceleration is exactly the relative acceleration between material points lying at the cross-section of the flow orthogonal to it. This is so, because \(a⊥\) for \(dτ \to 0\) is lying on the cross-section, orthogonal to \(u\), where this cross-section is determined by the infinitesimal vectors \(ξ(τ₀, λ₈₀)\), \(a = 1, \ldots, n - 1\), connecting the material point with coordinates \(x^a(τ₀, λ₈₀)\) with the material points with coordinates \(\{x^a(τ₀, λ₈₀ + dλ^a)\}, a = 1, \ldots, n - 1\). So, the orthogonal to \(u\) part \(rela(a)\) of the total acceleration \(totala(a)\), written as \(rela(a) = a⊥\),

\[
rela(a) = n[u \left(\frac{D^2ξ(τ₀, λ₈₀)}{dτ^2}\right)]\,
\]

has well grounds for its interpretation as relative acceleration between material points of a flow, lying on a hypersurface orthogonal to the velocity \(u\) of the flow.

The representation of the relative acceleration by means of its different parts, induced by the absolute acceleration, by the torsion, and by the curvature leads to the corresponding interpretation of the different types of relative accelerations in a flow. The different types of relative accelerations could be expressed by the different types of velocities, generated by the relative velocity. So we have a full picture of the forms of relative accelerations and of the forms of relative velocities as well as the relations between relative accelerations and relative velocities in a continuous media.

2.2 Kinematic characteristics connected with the notion relative acceleration

2.2.1 Relative acceleration and its representation

We will recall now some results already found in the considerations of the relative acceleration and its kinematic characteristics from a more general point of view \(\square\).

The notion \(relative \text{ acceleration} vector \text{ field (relative acceleration)} \ rela(a)\) can be defined (in analogous way as \(relv\)) as the orthogonal to a non-isotropic vector field \(u\) \([g(u, u) = e \neq 0]\) projection of the second covariant derivative (along the same non-isotropic vector field \(u\)) of (another) vector field \(ξ\), i.e.

\[
rela = n[u(\nabla u ξ)] = g^{ij} \cdot n[ξ \cdot (ξ^k \cdot u^l),m \cdot u^m \cdot e_i] .
\]

\(\nabla u ξ = (ξ^k \cdot u^l),m \cdot u^m \cdot e_i\) is the second covariant derivative of a vector field \(ξ\) along the vector field \(u\). It is an essential part of all types of deviation equations in \(V_n, (L_n, g)\)-, and \((T_n, g)\)-spaces \(\overline{\square} \div \overline{\square}\)

If we take into account the expression for \(\nabla u ξ\)

\[
\nabla u ξ = k[g(ξ)] - Lξu,
\]
and differentiate covariant along \( u \), then we obtain
\[
\nabla_u \nabla_u \xi = \{ \nabla_u [(k)g] \}(\xi) + (k)(g)(\nabla_u \xi) - \nabla_u (\mathcal{L}_\xi u)
\]

By means of the relations
\[
k(g)g = k, \quad \nabla_u [k(g)] = (\nabla_u k)(g) + k(\nabla_u g), \quad \{ \nabla_u [k(g)] \} g = \nabla_u k + k(\nabla_u g)g,
\]
\( \nabla_u \nabla_u \xi \) can be written in the form
\[
\nabla_u \nabla_u \xi = \frac{1}{e} \cdot H(u) + B(h_u)\xi - k(g)\mathcal{L}_\xi u - \nabla_u (\mathcal{L}_\xi u)
\]

[compare with \( \nabla_u \xi = \frac{1}{e} \cdot a + k(h_u)\xi - \mathcal{L}_\xi u \)],

where
\[
H = B(g) = (\nabla_u k)(g) + k(\nabla_u g) + k(g)k(g), \quad B = \nabla_u k + k(g)k + k(\nabla_u g)g = \nabla_u k + k(g)k - k(g)(\nabla_u g).
\]

The orthogonal to \( u \) covariant projection of \( \nabla_u \nabla_u \xi \) will have therefore the form
\[
h_u(\nabla_u \nabla_u \xi) = h_u(\frac{1}{e} \cdot H(u) - k(g)\mathcal{L}_\xi u - \nabla_u (\mathcal{L}_\xi u) + [h_u(B)h_u](\xi) .
\]

In the special case, when \( g(u, \xi) = l = 0 \) and \( \mathcal{L}_\xi u = 0 \), the above expression has the simple form
\[
h_u(\nabla_u \nabla_u \xi) = [h_u(B)h_u](\xi) = A(\xi),
\]

[compare with \( h_u(\nabla_u \xi) = [h_u(k)h_u](\xi) = d(\xi) \)].

The explicit form of \( H(u) \) follows from the explicit form of \( H \) and its action on the vector field \( u \)
\[
H(u) = (\nabla_u k)[g(u)] + k(\nabla_u g)(u) + k(g)(a) = \nabla_u [k(g)](u) = \nabla_u a.
\]

Now \( h_u(\nabla_u \nabla_u \xi) \) can be written in the form
\[
h_u(\nabla_u \nabla_u \xi) = h_u(\frac{1}{e} \cdot \nabla_u a - k(g)(\mathcal{L}_\xi u) - \nabla_u (\mathcal{L}_\xi u) + A(\xi)
\]

[compare with \( h_u(\nabla_u \xi) = h_u(\frac{1}{e} \cdot a - \mathcal{L}_\xi u) + d(\xi) \)].

The explicit form of \( A = h_u(B)h_u \) can be found in analogous way as the explicit form for \( d = h_u(k)h_u \) in the expression for \( \mathcal{L}_v \).

On the other side, the use of the relations
\[
\nabla_u (h_u(\nabla_u \xi)) = (\nabla_u h_u)(\nabla_u \xi) + h_u(\nabla_u \nabla_u \xi),
\]
\[
h_u(\nabla_u \xi) = g(\mathcal{L}_v),
\]
\[
h_u(\nabla_u \nabla_u \xi) = \nabla_u [g(\mathcal{L}_v)] - (\nabla_u h_u)(\nabla_u \xi) =
\]
\[
(\nabla_u g)(\mathcal{L}_v) + g(\nabla_u \mathcal{L}_v) - (\nabla_u h_u)(\frac{1}{e} \cdot u + \mathcal{L}_v) =
\]
\[
= (\nabla_u g)(\mathcal{L}_v) + g(\nabla_u \mathcal{L}_v) - \frac{1}{e} \cdot (\nabla_u h_u)(u) - (\nabla_u h_u)(\mathcal{L}_v) \quad (11)
\]
\[
\nabla_u h_u = \nabla_u g - [u(\frac{1}{e} \cdot g)] \cdot g(u) \otimes g(u) - \frac{1}{e} \cdot [g(a) \otimes g(u)] + g(u) \otimes g(a) +
\]
\[
+ (\nabla_u g)(u) \otimes g(u) + g(u) \otimes (\nabla_u g)(u),
\]
\[
(\nabla_u h_u)(\mathcal{L}_v) = (\nabla_u g)(\mathcal{L}_v) - \frac{1}{e} \cdot [g(a, \mathcal{L}_v) \cdot g(u) + (\nabla_u g)(u, \mathcal{L}_v) \cdot g(u)],
\]
\[
\nabla_u [g(u, \mathcal{L}_v)] = u[g(u, \mathcal{L}_v)] = 0 = (\nabla_u g)(u, \mathcal{L}_v) + g(a, \mathcal{L}_v) + g(u, \nabla_u \mathcal{L}_v),
\]
\[
(\nabla_u h_u)(\mathcal{L}_v) = (\nabla_u g)(\mathcal{L}_v) + \frac{1}{e} \cdot g(u, \nabla_u \mathcal{L}_v) \cdot g(u),
\]
\( (\nabla_u h_u)(u) = (\nabla_u g)(u) - e \cdot [u(\frac{1}{e})] \cdot g(u) - \frac{1}{e} \cdot [e \cdot g(a) + g(a, u) \cdot g(u) + e \cdot (\nabla_u g)(u) + (\nabla_u g)(u, u) \cdot g(a)] \)

we can find \( h_u(\nabla_u \nabla_u \xi) \) in the forms

\[
\begin{align*}
  h_u(\nabla_u \nabla_u \xi) &= h_u(\nabla_u (\text{rel} v)) - \frac{1}{e} \cdot [ul - g(a, \xi) - (\nabla_u g)(u, \xi)] \cdot \\
  &\cdot \left\{ \left[ \frac{ue}{e} - \frac{1}{e} \cdot g(u, a) - \frac{1}{e} \cdot (\nabla_u g)(u, u) \right] \cdot g(u) - g(a) \right\} , \quad (13)
\end{align*}
\]

\[
\begin{align*}
  \mathcal{g}[h_u(\nabla_u \nabla_u \xi)] &= \mathcal{g}[h_u(\nabla_u (\text{rel} v))] - \frac{1}{e} \cdot [ul - g(a, \xi) - (\nabla_u g)(u, \xi)] \cdot \\
  &\cdot \left\{ \left[ \frac{ue}{e} - \frac{1}{e} \cdot g(u, a) - \frac{1}{e} \cdot (\nabla_u g)(u, u) \right] \cdot u - a \right\} . \quad (14)
\end{align*}
\]

**Special case**: \( \nabla_n \)-spaces: \( \nabla_u g = 0, \ l = 0, \ e = \text{const.}, \ g(u, a) = 0. \)

\[
h_u(\nabla_u \nabla_u \xi) = h_u(\nabla_u (\text{rel} v)) - \frac{1}{e} \cdot g(a, \xi) \cdot g(a) . \quad (15)
\]

Under the additional condition \( g(a, \xi) := 0; \ h_u(\nabla_u \nabla_u \xi) = h_u(\nabla_u (\text{rel} v)). \)

Under the additional condition \( a := 0 \) (auto-parallel vector field \( u, \) inertial flow):

\[
h_u(\nabla_u \nabla_u \xi) = h_u(\nabla_u (\text{rel} v)).
\]

### 2.2.2 Deformation acceleration, shear acceleration, rotation acceleration, and expansion acceleration

The covariant tensor of second rank \( A \), named deformation acceleration tensor can be represented as a sum, containing three terms: a trace-free symmetric term, an antisymmetric term and a trace term

\[
A = sD + W + \frac{1}{n - 1} \cdot U \cdot h_u \quad (16)
\]

where

\[
D = h_u(sB) h_u \quad (17)
\]

\[
W = h_u(aB) h_u \quad (18)
\]

\[
U = \mathcal{g}[sA] = \mathcal{g}[D] \quad (19)
\]

\[
sB = \frac{1}{2} \cdot (B^{ij} + B^{ji}) \cdot e_i \wedge e_j, \quad aB = \frac{1}{2} \cdot (B^{ij} - B^{ji}) \cdot e_i \wedge e_j, \quad (20)
\]

\[
sA = \frac{1}{2} \cdot (A_{ij} + A_{ji}) \cdot e^i \wedge e^j, \quad (21)
\]

\[
sD = D - \frac{1}{n - 1} \cdot \mathcal{g}[D] \cdot h_u = D - \frac{1}{n - 1} \cdot U \cdot h_u . \quad (22)
\]

The shear-free symmetric tensor \( sD \) is the shear acceleration tensor (shear acceleration), the antisymmetric tensor \( W \) is the rotation acceleration tensor (rotation acceleration) and the invariant \( U \) is the expansion acceleration invariant (expansion acceleration). Furthermore, every one of these quantities can be divided into three parts: torsion- and curvature-free acceleration, acceleration induced by torsion and acceleration induced by curvature.

Let us now consider the representation of every acceleration quantity in its essential parts, connected with its physical interpretation.
The deformation acceleration tensor $A$ can be written in the following forms

$$A = sD + W + \frac{1}{n-1} \cdot U \cdot h_u = A_0 + G = FA_0 - \tau A_0 + G, \quad (23)$$

$$A = sD_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u + sM + N + \frac{1}{n-1} \cdot I \cdot h_u, \quad (24)$$

$$A = sFD_0 + FW_0 + \frac{1}{n-1} \cdot F U_0 \cdot h_u - \tau (sT D_0 + \tau W_0 + \frac{1}{n-1} \cdot \tau U_0 \cdot h_u) + sM + N + \frac{1}{n-1} \cdot I \cdot h_u, \quad (25)$$

where

$$A_0 = FA_0 - \tau A_0 = sD_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u, \quad (26)$$

$$FA_0 = sFD_0 + FW_0 + \frac{1}{n-1} \cdot F U_0 \cdot h_u, \quad (27)$$

$$FA_0(\xi) = h_u(\nabla_\xi u), \quad \xi_\perp = \mathcal{J}[h_u(\xi)], \quad (28)$$

$$\tau A_0 = sT D_0 + \tau W_0 + \frac{1}{n-1} \cdot \tau U_0 \cdot h_u, \quad (29)$$

$$G = sM + N + \frac{1}{n-1} \cdot I \cdot h_u = h_u(K)h_u, \quad (30)$$

$$h_u([R(u, \xi)]u) = h_u(K)h_u(\xi) \quad \text{for} \ \forall \ \xi \in T(M), \quad (31)$$

$$[R(u, \xi)]u = \nabla_u \nabla_\xi u - \nabla_\xi \nabla_u u - \nabla_{\xi, \xi} u, \quad (32)$$

$$K = K^{kl} \cdot e_k \otimes e_l, \quad K^{kl} = R^{k}_{\ mnr} \cdot g^{rl} \cdot u^m \cdot u^n, \quad (33)$$

$R^{k}_{\ mnr}$ are the components of the contravariant Riemannian curvature tensor,

$$K_a = K^{kl} \cdot e_k \wedge e_l, \quad K_a^{kl} = \frac{1}{2} \cdot (K^{kl} - K^{lk}), \quad (34)$$

$$K_s = K^{kl} \cdot e_k \cdot e_l, \quad K_s^{kl} = \frac{1}{2} \cdot (K^{kl} + K^{lk}), \quad (35)$$

$$sD = sD_0 + sM, \quad W = W_0 + N = FW_0 - \tau W_0 + N, \quad (36)$$

$$U = U_0 + I = FU_0 - \tau U_0 + I, \quad (37)$$

$$sM = M - \frac{1}{n-1} \cdot I \cdot h_u, \quad M = h_u(K_a)h_u, \quad I = \mathcal{J}[M] = g^{\alpha \beta} \cdot M_{ij}, \quad (38)$$

$$N = h_u(K_a)h_u, \quad (39)$$

$$sD_0 = sFD_0 - sT D_0 = FD_0 - \frac{1}{n-1} \cdot FU_0 \cdot h_u - (T D_0 - \frac{1}{n-1} \cdot \tau U_0 \cdot h_u), \quad (40)$$

$$sD_0 = sFD_0 - \tau D_0 - \frac{1}{n-1} \cdot (FU_0 - \tau U_0) \cdot h_u, \quad (41)$$

$$sD_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u, \quad (42)$$

$$sFD_0 = FD_0 - \frac{1}{n-1} \cdot FU_0 \cdot h_u, \quad sFD_0 = h_u(b_s)h_u, \quad (43)$$

$$b = b_s + b_a, \quad b = b^{kl} \cdot e_k \otimes e_l, \quad b^{kl} = a^{kl} \cdot g^{rl}, \quad (44)$$

$$a^k = u^k \cdot u^m, \quad b_s = b_s^{kl} \cdot e_k \cdot e_l, \quad b_s^{kl} = \frac{1}{2} \cdot (b^{kl} + b^{lk}), \quad (45)$$

$$b_a = b_a^{kl} \cdot e_k \wedge e_l, \quad b_a^{kl} = \frac{1}{2} \cdot (b^{kl} - b^{lk}), \quad (46)$$

\[ \text{Page 7} \]
\[ fU_0 = \mathcal{G}[fD_0] = g[b] - \frac{1}{e} \cdot g(u, \nabla_u a) , \quad g[b] = g_{\mathcal{G}^i}^j b^j , \quad (47) \]

\[ sT D_0 = T D_0 - \frac{1}{n-1} \cdot T U_0 \cdot h_u = sF D_0 - sD_0 , \quad T D_0 = fD_0 - D_0 , \quad (48) \]

\[ U_0 = \mathcal{G}[D_0] = fU_0 - T U_0 , \quad T U_0 = \mathcal{G}[T D_0] , \quad (49) \]

\[ fW_0 = h_u(h_u)h_u , \quad T W_0 = fW_0 - W_0 . \quad (50) \]

Under the conditions \( \xi u = 0 \) ; \( \xi = \xi_\perp = \mathcal{G}[h_u(\xi)] \), \( l = 0 \), the expression for \( h_u(\nabla_u \nabla_u \xi) \) can be written in the forms

\[ h_u(\nabla_u \nabla_u \xi_\perp) = A(\xi_\perp) = An(\xi_\perp) + G(\xi_\perp) , \quad (51) \]

\[ h_u(\nabla_u \nabla_u \xi_\perp) = fA_0(\xi_\perp) - T A_0(\xi_\perp) + G(\xi_\perp) , \quad (52) \]

\[ h_u(\nabla_u \nabla_u \xi_\perp) = (sF D_0 + fW_0 + \frac{1}{n-1} \cdot fU_0 \cdot g)(\xi_\perp) - (sT D_0 + T W_0 + \frac{1}{n-1} \cdot T U_0 \cdot g)(\xi_\perp) + (sM + N + \frac{1}{n-1} \cdot I \cdot g)(\xi_\perp) , \quad (53) \]

which enable one to find a physical interpretation of the quantities \( sD, W, U \) and of the contained in their structure quantities \( sF D_0, fW_0, fU_0, sT D_0, T W_0, T U_0, sM, N, I \). The individual designation, connected with their physical interpretation, is given in the Section 11 - Table 1. The expressions of these quantities in terms of the kinematic characteristics of the relative velocity are given in a section below.

After the above consideration the following proposition can be formulated:

**Proposition 1** The covariant vector \( g_{(rel)u} = h_u(\nabla_u \nabla_u \xi) \) can be written in the form

\[ g_{(rel)u} = h_u \left[ \frac{1}{e} \cdot \nabla_u a - \nabla_\xi u u - \nabla_u (\xi u) + T(\xi u, u) \right] + A(\xi) , \quad (54) \]

where

\[ A(\xi) = sD(\xi) + W(\xi) + \frac{1}{n-1} \cdot U \cdot h_u(\xi) . \]

For the case of affine symmetric connection \( T(w, v) = 0 \) for \( \forall \ w, v \in T(M) \), \( T_{ij} = 0, \Gamma^k_{ij} = \Gamma^k_{ji} \) and Riemannian metric \( \nabla_v g = 0 \) for \( \forall v \in T(M) \), \( g_{ij;k} = 0 \) kinematic characteristics are obtained in \( V_n \)-spaces, connected with the notion relative velocity \([?], [?], [?], [?], [?], [?] \) and relative acceleration \([?], [?], [?] \). For the case of affine non-symmetric connection \( T(w, v) \neq 0 \) for \( \forall \ w, v \in T(M) \), \( \Gamma^i_{jk} \neq \Gamma^i_{kj} \) and Riemannian metric kinematic characteristics are obtained in \( U_n \)-spaces \([?], [?] \).

### 3 Classification of auto-parallel vector fields on the basis of the kinematic characteristics connected with the relative velocity and relative acceleration

The classification of (pseudo) Riemannian spaces \( V_n \), admitting the existence of auto-parallel (in the case of \( V_n \)-spaces they are geodesic) vector fields \( (\nabla_u a = a = 0) \) with given kinematic characteristics, connected with the notion relative velocity \([?], [?], [?], [?], [?] \), can be extended to a classification of differentiable manifolds with contravariant and covariant affine connections and metrics, admitting auto-parallel vector fields with certain kinematic characteristics, connected with the relative velocity and the relative acceleration. In this way, the following two schemes for the existence of special type 1 and 2 of vector fields can be proposed (s. Table 2). Different types of combinations between the single conditions of the two schemes can also be taken under consideration.
3.1 Inertial flows. Special geodesic vector fields with vanishing kinematic characteristics, induced by the curvature, in (pseudo) Riemannian spaces

On the basis of the classification 2 the following propositions in the case of \( V_n \)-spaces can be proved:

**Proposition 2** Non-isotropic geodesic vector fields in \( V_n \)-spaces are geodesic vector fields with curvature rotation acceleration tensor \( N \) equal to zero, i.e. \( N = 0 \).

Proof: 1. From the first proposition in this subsection above, it follows that \( K_a = 0 \) and \( N = 0 \).

\[
\begin{align*}
N &= h_u(K_a)h_u = h_{ik} \cdot K_{a}^{kl} \cdot h_{ij} \cdot e^i \wedge e^j, \\
K^{kl}_a &= \frac{1}{2} \cdot (K^{kl} - K^{lk}) = \frac{1}{2} \cdot (R_{mnr}^k \cdot g^{rl} - R_{umn}^k \cdot g^{rk}) \cdot u^m \cdot u^n, \\
R_{kmnr} &= R_{nmkr} = g_{kl} \cdot R_{k}^{l \ mnr},
\end{align*}
\]

For the case of \( V_n \)-space, where

\[
R_{kmnr} = R_{nmkr} = g_{kl} \cdot R_{k}^{l \ mnr},
\]

the conditions

\[
R_{mnr}^k \cdot g^{rl} = R_{nmr}^l \cdot g^{rk}, \quad R_{mnr}^k \cdot g^{rl} - R_{umn}^k \cdot g^{rk}) \cdot u^m \cdot u^n = 0,
\]

follow and therefore

\[
K^{kl}_a = \frac{1}{2} \cdot (R_{mnr}^k \cdot g^{rl} - R_{umn}^k \cdot g^{rk}) \cdot u^m \cdot u^n = 0,
\]

\[
K_a = 0, \quad N = 0.
\]

**Proposition 3** Non-isotropic geodesic vector fields in \( V_n \)-spaces with equal to zero Ricci tensor (\( R_{ik} = R_{ik} = g_{ik}, R_{ik} = 0 \)) are geodesic vector fields with curvature rotation acceleration \( N \) and curvature expansion acceleration \( I \), both equal to zero, i.e. \( N = 0, I = 0 \).

Proof: 1. From the above proposition, it follows that \( K_a = 0 \) and \( N = 0 \).

\[
I = g[K] = g_{ij} \cdot K^{ij} = R_{mnr}^l \cdot g^{rl} \cdot g^{mk} \cdot u^m \cdot u^n = g_i^l \cdot R_{mnr}^l \cdot u^m \cdot u^n = R_{mn} \cdot u^m \cdot u^n = 0.
\]

**Proposition 4** Non-isotropic geodesic vector fields in \( V_n \)-spaces with constant curvature

\[
[R(\xi, \eta)]v = \frac{1}{n \cdot (n - 1)} \cdot R_0 \cdot [g(v, \xi) \cdot \eta - g(v, \eta) \cdot \xi], \quad \forall \xi, \eta, v \in T(M),
\]

[in index form

\[
R_{ijkl} = \frac{R_0}{n \cdot (n - 1)} \cdot (g_i^l \cdot g_{jk} - g_i^l \cdot g_{jkl}), \quad R_0 = \text{const.}
\]

are geodesic vector fields with curvature shear acceleration and curvature rotation acceleration, both equal to zero, i.e. \( sM = 0, N = 0 \).

Proof: 1. From the first proposition in this subsection above, it follows that \( N = 0 \).

\[
I = h_u(K_a)h_u = g(K_a)g,
\]

\[
sM = M - \frac{1}{n - 1} \cdot I \cdot h_u, \quad M = h_u(K_a)h_u = g(K_a)g.
\]
\[ M = g(K_s)g = g_{ik} \cdot K^{kl} \cdot g_{ij} \cdot e^i.e^j = M_{ij} \cdot e^i.e^j, \]
\[ M_{ij} = g_{ik} \cdot R^k_{\ mnr} \cdot g^{rl} \cdot g_{lj} \cdot u^m \cdot u^n = R_{imnj} \cdot u^m \cdot u^n = \frac{R_0}{n \cdot (n-1)} \cdot e \cdot h_{ij}, \quad (62) \]

\[ M = \frac{R_0 \cdot e}{n \cdot (n-1)} \cdot h_u, \quad e = g(u, u) = g_{ij} \cdot u^i \cdot u^j, \quad (63) \]
\[ I = g[K] = g^{ij} \cdot M_{ij} = \frac{1}{n} \cdot R_0 \cdot e, \quad g^{ij} \cdot h_{ij} = n - 1, \quad (64) \]

\[ sM = M - \frac{1}{n-1} \cdot I \cdot h_u = 0. \]

The projections of the curvature tensor of the type \( G = h_u(K)h_u \) (or \( R^i_{\ jkl} \cdot u^j \cdot u^k \)) along the non-isotropic vector field \( u \) acquire a natural physical meaning as quantities, connected with the kinematic characteristics curvature shear acceleration \( sM \), curvature rotation acceleration \( N \), and curvature expansion acceleration \( I \).

The projection of the Ricci tensor \( (g[K]) \), or \( R_{ik} \cdot u^i \cdot u^k \) and the Raychaudhuri identity for vector fields represent an expression of the curvature expansion acceleration, given in terms of the kinematic characteristics of the relative velocity

\[ I = g[M] = R_{ij} \cdot u^i \cdot u^j = \]
\[ = -u^j \cdot g_{lj} + g^{ij} \cdot sE_{ik} \cdot g_{lj} + g^{ij} \cdot \sigma_{ij} + g^{ij} \cdot T_{kij} + u^m - g_{mnjk} \cdot u^m - g_{mnjk} \cdot u^n - g_{mnjk} \cdot u^m + \frac{1}{2} \cdot \left( u^k \cdot (e, k \cdot u^k) + u^m - \frac{1}{4} \cdot (g_{mnjk} \cdot u^m) \cdot u^n + \frac{1}{4} \cdot (g_{mnjk} \cdot u^m \cdot u^n)^2 \right), \]
\[ \theta = \theta, \quad \theta_1 = 0 \quad \nabla_u u = a \neq 0, \quad a = 0 \quad (65) \]

In the case of \( V_n \)-spaces the kinematic characteristics, connected with the relative velocity and the relative acceleration have the forms:

a) kinematic characteristics, connected with the relative velocity

\[ d = d_0, \quad d_1 = 0 \quad k = k_0 \]
\[ \sigma = sE, \quad P = 0 \quad m = 0 \]
\[ \omega = S, \quad Q = 0 \quad q = 0 \]
\[ \theta = \theta, \quad \theta_1 = 0 \quad \nabla_u u = a \neq 0, \quad a = 0 \]

b) kinematic characteristics of a non-inertial flow, connected with the relative acceleration \( (\nabla_u u = a \neq 0) \)

\[ A = fA_0 + G \quad \tau A_0 = 0 \quad N = 0 \]
\[ G = sM + \frac{1}{n-1} \cdot I \cdot h_u \quad sT D_0 = 0 \]
\[ W = fW_0 \quad \tau W_0 = 0 \]
\[ U = fU_0 + I \quad \tau U_0 = 0 \quad \nabla_u u = a \neq 0 \]

c) kinematic characteristics of an inertial flow, connected with the relative acceleration \( (\nabla_u u = a = 0) \)

\[ A = G \quad \tau A_0 = 0 \quad N = 0 \]
\[ G = sM + \frac{1}{n-1} \cdot I \cdot h_u \quad sT D_0 = 0 \]
\[ W = 0 \quad \tau W_0 = 0 \]
\[ U = I \quad \tau U_0 = 0 \quad \nabla_u u = a = 0 \]

On the basis of the different kinematic characteristics dynamic systems and flows can be classified and considered in \( V_n \)-spaces.
3.2 Special flows with tangent vector fields over \((\mathcal{L}_n, g)\)-spaces with vanishing kinematic characteristics induced by the curvature

The explicit forms of the quantities \(G, M, N\) and \(I\), connected with accelerations induced by curvature, can be used for finding conditions for existence of special types of contravariant vector fields with vanishing characteristics induced by the curvature. \(G, M, N\) and \(I\) can be expressed in the following forms:

\[
G = h_u(K)h_u = g(K)g - \frac{1}{c} \cdot g(u) \otimes [g(u)](K)g ,
K[g(u)] = 0 ,
\]

\[
M = h_u(K_a)h_u = g(K_a)g - \frac{1}{2c} \cdot \{g(u) \otimes [g(u)](K)g + [g(u)](K)g \otimes g(u)\} =
M_{ij} = \frac{1}{2} \cdot g_{\tau \tau} \cdot g_{\tau j} + g_{\tau \tau} \cdot g_{\tau i} - \frac{1}{c} \cdot (u_i \cdot g_{\tau j} + u_j \cdot g_{\tau i}) \cdot u^{\tau \tau} R^{k} m n q \cdot u^m \cdot u^n \cdot g^{k} ,
\]

\[
I = g[K] = g[K] = R_{\rho \sigma} \cdot u^\rho \cdot u^\sigma = R_{\delta \delta} \cdot u^\delta \cdot u^\delta ,
\]

\[
N = h_u(K_a)h_u = g(K_a)g - \frac{1}{2c} \cdot \{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} =
N_{ij} = \frac{1}{2} \cdot g_{\tau \tau} \cdot g_{\tau j} - g_{\tau \tau} \cdot g_{\tau i} - \frac{1}{c} \cdot (u_i \cdot g_{\tau j} - u_j \cdot g_{\tau i}) \cdot u^{\tau \tau} R^{k} m n q \cdot u^m \cdot u^n \cdot g^{k} .
\]

By means of the above expressions conditions can be found under which some of the quantities \(M, N, I\) vanish.

3.2.1 Flows with tangent vector fields without rotation acceleration, induced by the curvature \((N = 0)\)

If the rotation acceleration \(N\), induced by the curvature vanishes, i.e. if \(N = 0\), then the following proposition can be proved:

**Proposition 5** The necessary and sufficient condition for the existence of a flow with tangent vector field \(u [g(u, u) = e \neq 0]\) without rotation acceleration, induced by the curvature (i.e. with \(N = 0\)), is the condition

\[
K_a = \frac{1}{2c} \cdot \{u \otimes [g(u)](K) - [g(u)](K) \otimes u\} .
\]

Proof: 1. Sufficiency: From the above expression it follows

\[
N = h_u(K_a)h_u =
= g(K_a)g - \frac{1}{2c} \cdot \{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} = 0 ,
\]

\[
g([g(u)](K)) = [g(u)](K)g .
\]

2. Necessity: If \(N = h_u(K_a)h_u = 0\), then

\[
g(K_a)g = \frac{1}{2c} \cdot \{g(u) \otimes [g(u)](K)g - [g(u)](K)g \otimes g(u)\} ,
K_a = \frac{1}{2c} \cdot \{u \otimes [g(u)](K) - [g(u)](K) \otimes u\} .
\]

In co-ordinate basis the necessary and sufficient condition has the forms

\[
K^{ij} = K^{ji} + \frac{1}{c} \cdot u^r \cdot (u^i \cdot K^{ij} - u^j \cdot K^{ri}) ,
\]

\[
\{R_{jn}^{mj} - R_{mj}^{jn} - \frac{1}{c} \cdot (u^r \cdot R_{mnj}^{jr} - u^r \cdot R_{mnj}^{jr}) \cdot u^l \} \cdot u^m \cdot u^n = 0 ,
\]

where

\[
R_{ijkl} = g_{mn} R_{ijkl}^n .
\]
Proposition 6 A sufficient condition for the existence of a flow with tangent vector field \( u \) \([g(u, u) = e \neq 0]\) without rotation acceleration, induced by the curvature (i.e. with \( N = 0 \)), is the condition

\[
K_a = 0 .
\] (72)

Proof: From \( K_a = 0 \) and the form for \( N, N = h_u(K_a)h_u, \) it follows \( N = 0. \)

In co-ordinate basis

\[
(R^i_{ jkl} \cdot \eta^{mi} - R^j_{ jkl} \cdot \eta^{mi}) \cdot u^k \cdot u^l = 0 ,
\]

\[
(R^i_{ jkl} - R^j_{ jik}) \cdot u^k \cdot u^l = 0 .
\] (73)

\( K_a = 0 \) can be presented also in the form

\[
[g(\xi)](\{R(u, v)\}u) - [g(v)](\{R(u, \xi)\}u) = 0 , \quad \forall \xi, v \in T(M) .
\]

In this case \( M = G = g(K)g , I = \overline{g}[G]. \)

Proposition 7 A sufficient condition for the existence of a flow with tangent vector field \( u \) \([g(u, u) = e \neq 0]\) without rotation acceleration, induced by the curvature (i.e. with \( N = 0 \)), is the condition

\[
g(\eta, [R(\xi, v)]u) = g(\xi, [R(\eta, w)]v) , \quad \forall \eta, \xi, v, w \in T(M) ,
\] (74)

or in co-ordinate basis

\[
R^i_{ jkl} = R^i_{ klij} .
\] (75)

Proof: Because of \( R(\xi, v) = -R(u, \xi) \) and for \( \eta = v \) the last expression will be identical with the sufficient condition from the above proposition.

Remark 1 If the rotation velocity \( \omega \) vanishes \( (\omega = 0) \) for an auto-parallel \( (\nabla_u u = 0) \) contravariant non-null vector field \( u, \) then the rotation acceleration tensor \( W \) will have the form \([76]\)

\[
W = \frac{1}{2} \cdot [h_u(\nabla_u \overline{g})\sigma - \sigma(\nabla_u \overline{g})h_u] .
\]

From the last expression it is obvious that under the above conditions the nonmetricity \( (\nabla_u g \neq 0) \) in a \((T_n, g)\)-space is responsible for nonvanishing the rotation acceleration \( W. \)

3.2.2 Contravariant vector fields without shear acceleration \( sM, \) induced by the curvature \((sM = 0)\)

Proposition 8 The necessary and sufficient condition for the existence of a contravariant vector field \( u \) \([g(u, u) = e \neq 0]\) without shear acceleration, induced by the curvature (i.e. with \( sM = 0 \)), is the condition

\[
M = \frac{1}{n - 1} \cdot I \cdot h_u = \frac{1}{n - 1} \cdot g[M] \cdot h_u .
\] (76)

Proof: 1. Sufficiency: From the expression for \( M \) and the definition of \( sM = M - \frac{1}{n - 1} \cdot I \cdot h_u \) it follows \( sM = 0. \)

2. Necessity: From \( sM = 0 = M - \frac{1}{n - 1} \cdot I \cdot h_u \) the form of \( M \) follows.

In co-ordinate basis the necessary and sufficient condition can be written in the form

\[
\{[g_{ij}, g_{kl}] + g_{ij} \cdot g_{kl} - \frac{1}{n} \cdot (u_i \cdot g_{lj} + u_j \cdot g_{li}) \cdot \nabla u^k - \frac{1}{n - 1} \cdot R_{mn} \cdot (g_{ij} - \frac{1}{n} \cdot u_i \cdot u_j) \} \cdot u^m \cdot u^n = 0 .
\] (77)

The condition \( sM = 0 \) is identical with the condition for \( K_s: \)

\[
K_s = \frac{1}{n - 1} \cdot I \cdot h_u + \frac{1}{2} \cdot e \cdot \{u \otimes [g(u)](K) + [g(u)](K) \otimes u \} .
\] (78)
3.2.3 Contravariant vector fields without shear and expansion acceleration, induced by the curvature ($\mathcal{M} = 0$, $I = 0$)

**Proposition 9** A sufficient condition for the existence of a contravariant vector field $u$ \([g(u,u) = e \neq 0]\) without shear and expansion acceleration, induced by the curvature (i.e. with $\mathcal{M} = 0$, $I = 0$), is the condition

\[
K_s = \frac{1}{2} \cdot \{u \otimes [g(u)](K) + [g(u)](K) \otimes u\}.
\]  

(79)

Proof: After acting on the left and on the right side of the last expression with $g$

\[
g(K_s)g = \frac{1}{2} \cdot \{g(u) \otimes [g(u)](K)g + [g(u)](K)g \otimes g(u)\} ,
\]

and comparing the result with the form for $M$, $M = h_u(K_s)h_u$ it follows that $M = 0$. Since $I = \frac{\mathcal{M}}{g}(M)$, it follows that $I = 0$ and $\mathcal{M} = 0$.

**Proposition 10** A sufficient condition for the existence of a contravariant vector field $u$ \([g(u,u) = e \neq 0]\) without shear and expansion acceleration, induced by the curvature (i.e. with $\mathcal{M} = 0$, $I = 0$), is the condition

\[
K_s = 0 .
\]

Proof: From the condition and the form of $M$, $M = h_u(K_s)h_u$, it follows that $M = 0$ and therefore $I = 0$ and $\mathcal{M} = 0$.

3.2.4 Contravariant vector fields without shear and rotation acceleration, induced by the curvature ($\mathcal{M} = 0$, $N = 0$)

**Proposition 11** A sufficient condition for the existence of a contravariant vector field $u$ \([g(u,u) = e \neq 0]\) without shear and rotation acceleration, induced by the curvature (i.e. with $\mathcal{M} = 0$ , $N = 0$), is the condition

\[
[R(u,\xi)]v = \frac{R}{n(n-1)} \cdot [g(v,u) \cdot \xi - g(v,\xi) \cdot u] ,
\]

\[
\forall v, \xi \in T(M) ,
\]

(80)

Proof: Since $v$ is an arbitrary contravariant vector field it can be chosen as $u$. Then, because of the relation

\[
h_u([R(u,\xi)]u) = h_u(K)h_u(\xi) = G(\xi) ,
\]

(81)

it follows that

\[
G = h_u(K)h_u = \frac{R}{n(n-1)} \cdot e \cdot h_u = G_s ,
\]

\[
G_a = h_u(K_a)h_u = 0 .
\]

(82)

Therefore

\[
M = G_a = \frac{R \cdot e}{n(n-1)} \cdot h_u ,
\]

\[
N = G_a = 0 ,
\]

(83)

\[
I = \frac{1}{n} \cdot R \cdot e ,
\]

\[
\mathcal{M} = 0 .
\]

(84)

In co-ordinate basis the sufficient condition can be written in the form

\[
R^i_{\quad jkl} = \frac{R}{n(n-1)} \cdot (g^i_{\quad \alpha} \cdot g^k_{\quad \beta} - g^i_{\quad \gamma} \cdot g^k_{\quad \alpha})
\]

(85)
and the following relations are fulfilled

\[ R_{jk} = R^i_{jkl} = g^i_k \cdot R^j_{ikl} = \frac{1}{n} \cdot R \cdot g_{jk}, \]
\[ R = g^{jk} \cdot R_{jk}, \]
\[ I = R_{jk} \cdot u^j \cdot u^k = \frac{1}{n} \cdot R \cdot e. \]

(86)

**Proposition 12** The necessary and sufficient conditions for the existence of \( K \) in the form

\[ K = \frac{1}{n - 1} \cdot g[K] \cdot h^u \]

are the conditions

\[ sM = 0, \quad K_a = 0. \]

Proof: 1. Sufficiency: From \( K_a = 0 \) it follows that \( K = K_s \), \( N = 0 \) and \( M = g(K_s)g = g(K)g \). Therefore, \( I = g[M] = g[K] \). From \( sM = M - \frac{1}{n-1} \cdot I \cdot h_a = 0 \), it follows that \( M = \frac{1}{n-1} \cdot g[K] \cdot h_a = g(K)g \). From the last expression, it follows the above condition for \( K \).

2. Necessity: From the condition \( K = \frac{1}{n - 1} \cdot g[K] \cdot h^u \), it follows that \( K = K_s \) and therefore \( K_a = 0 \), \( N = 0 \) and \( M = \frac{1}{n-1} \cdot g[K] \cdot h_a \), \( I = g[K] \) [because of \( h_a(h^u)h_a = h_a \), \( h_a(g)h_a = h_a \)]. From the forms of \( M \) and \( I \), it follows that \( sM = 0 \).

**Proposition 13** A sufficient condition for the existence of a contravariant vector field \( u \) \( [g(u, u) = e \neq 0] \) without shear and rotation acceleration, induced by the curvature (i.e. with \( sM = 0 \), \( N = 0 \)), is the condition

\[ K = \frac{1}{n - 1} \cdot g[K] \cdot h^u. \]

Proof: Follows immediately from the above proposition.

### 3.2.5 Contravariant vector fields without expansion acceleration, induced by the curvature \( (I = 0) \)

By means of the covariant metric \( g \) and the tensor field \( K(v, \xi) \) the notion contravariant Ricci tensor \( Ricci \) can be introduced

\[ Ricci(v, \xi) = g[K(v, \xi)], \quad \forall v, \xi \in T(M), \]

(88)

where

\[ K(v, \xi) = R^i_{jkl} \cdot g^{lm} \cdot v^j \cdot \xi^k \cdot \partial_i \otimes \partial_m = R^\alpha_{\beta \gamma \delta} \cdot g^{\gamma \delta} \cdot v^\beta \cdot \xi^\alpha \cdot e_\alpha \otimes e_\delta, \]

(89)

and the following relations are fulfilled

\[ Ricci(e_\alpha, e_\beta) = g[K(e_\alpha, e_\beta)] = R_{\alpha \beta}, \]
\[ Ricci(\partial_i, \partial_j) = g[K(\partial_i, \partial_j)] = R_{ij}, \]
\[ Ricci(u, u) = g[K(u, u)] = g[K] = I. \]

(90)

**Proposition 14** The necessary and sufficient condition for the existence of a contravariant vector field \( u \) \( [g(u, u) = e \neq 0] \) without expansion acceleration, induced by the curvature (i.e. with \( I = 0 \)), is the condition

\[ Ricci(u, u) = 0. \]
Proof: It follows immediately from the relation $\text{Ricci}(u, u) = g[K(u, u)] = g[K] = I$.

**Proposition 15** A sufficient condition for the existence of a contravariant vector field $u$ ($g(u, u) = e \neq 0$) without expansion acceleration, induced by the curvature (i.e. with $I = 0$), is the condition

$$\text{Ricci}(e_\alpha, e_\beta) = R_{\alpha\beta} = R_{\alpha\beta\gamma} = 0, \quad \text{Ricci}(\partial_i, \partial_j) = R_{ij} = R_{ijl} = 0.$$ \hspace{1cm} (91)

Proof: From $\text{Ricci}(\partial_i, \partial_j) = R_{ij} = 0$, it follows that

$$R_{ij} \cdot u^i \cdot u^j = u^i \cdot u^j \cdot \text{Ricci}(\partial_i, \partial_j) = \text{Ricci}(u, u) = I = 0.$$ 

In a non-co-ordinate basis the proof is analogous to that in a co-ordinate basis.

The existence of contravariant vector fields with vanishing characteristics, induced by the curvature, is important for mathematical models of gravitational interactions in theories over ($\mathbb{L}_n, g$)-spaces.

### 4 Kinematic characteristics connected with the relative acceleration and expressed in terms of the kinematic characteristics connected with the relative velocity

In this section the relations between the kinematic characteristics connected with the relative velocity and the kinematic characteristics connected with the relative acceleration are found. A summary of the definitions of the kinematic characteristics is also given in Table 1.

The deformation, shear, rotation and expansion accelerations can be expressed in terms of the shear, rotation and expansion velocity.

(a) Deformation acceleration tensor $A$:

$$A = \frac{1}{n} \cdot h_u(a) \otimes h_u(a) + \sigma(\overline{g})\sigma + \omega(\overline{g})\omega + \frac{2}{n-1} \cdot \theta \cdot (\sigma + \omega) + \frac{1}{n-1} \cdot (\theta^2 + \frac{\theta^2}{n-1}) \cdot h_u +$$

$$+ \sigma(\overline{g})\omega + \omega(\overline{g})\sigma + \nabla_u \sigma + \nabla_u \omega + \frac{1}{n} \cdot h_u(a) \otimes (g(u))(2 \cdot k - \nabla_u \overline{g})h_u +$$

$$+ \frac{1}{n} \cdot [\sigma(a) \otimes g(u) + g(u) \otimes \sigma(a)] + \frac{1}{n} \cdot [\omega(a) \otimes g(u) - g(u) \otimes \omega(a)] +$$

$$+ h_u(\nabla_u \overline{g})\sigma + h_u(\nabla_u \overline{g})\omega,$$ \hspace{1cm} (92)

where

$$k = \varepsilon + s - (m + q) = k_0 - (m + q), \quad k(g)L_\xi u = \nabla_{L_\xi u} u - T(L_\xi u, u).$$ \hspace{1cm} (93)
In index form

\[
A_{ij} = \frac{1}{e} \cdot \overline{g}^{li} \cdot A_{ik} \cdot a^k \cdot a^l \cdot h_{ij} + \sigma_{ik} \cdot \overline{g}^{li} \cdot \sigma_{ij} + \omega_{ik} \cdot \overline{g}^{li} \cdot \omega_{ij} + \frac{2}{n-1} \cdot \theta \cdot \sigma_{ij} + \\
+ \frac{1}{e} \cdot \overline{g}^{li} \cdot (\theta + \frac{a_i}{a^j}) \cdot h_{ij} + \\
+ \sigma_{ij} : u^k + \frac{1}{e} \cdot \overline{g}^{li} \cdot \{ \sigma_{ik} \cdot u_j + \sigma_{jk} \cdot u_i + h_{ij} + \frac{1}{2} \cdot \frac{2}{n-1} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) = \\
= D_{ij} + W_{ij} , \quad \theta = u^i \cdot \partial_i \theta = \theta_{ij} \cdot u^i ,
\]

(94)

(b) Shear acceleration tensor \( D = D - \frac{1}{n-1} \cdot U \cdot h_u:

\[
D = \frac{1}{e} \cdot h_u(a) \otimes h_u(a) + \sigma(\overline{g}) + \omega(\overline{g}) + \\
+ \frac{2}{n-1} \cdot \theta \cdot \sigma + \frac{1}{n} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \frac{1}{2} \cdot \frac{2}{n-1} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) = \\
= D_{ij} + W_{ij} , \quad \theta = u^i \cdot \partial_i \theta = \theta_{ij} \cdot u^i ,
\]

In index form

\[
D_{ij} = D_{ji} = \frac{1}{e} \cdot h_{ik} \cdot A_{ik} \cdot a^k \cdot a^l \cdot h_{ij} + \sigma_{ik} \cdot g^{kl} \cdot \sigma_{ij} + \omega_{ik} \cdot g^{kl} \cdot \omega_{ij} + \\
+ \frac{1}{2} \cdot \theta \cdot \sigma_{ij} + \frac{1}{n-1} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \frac{1}{2} \cdot \frac{2}{n-1} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{ij} + h_{jk} \cdot g^{kl} \cdot \omega_{ij}) =
\]

(95)

(c) Rotation acceleration tensor \( W \):

\[
W = \sigma(\overline{g}) \omega + \omega(\overline{g}) + \frac{2}{n-1} \cdot \theta \cdot \omega + \nabla_n \omega + \\
+ \frac{1}{2} \cdot [h_u(a) \otimes (g(u))(2 \cdot k - \nabla_u g) + h_u((g(u))(2 \cdot k - \nabla_u g)) \otimes h_u(a)] + \\
+ \frac{1}{2} \cdot [\sigma(a) \otimes g(u) + g(u) \otimes \sigma(a)] + \\
+ \frac{1}{2} \cdot [h_u(\nabla_u g \sigma + \sigma(\nabla_u g) h_u)] + \frac{1}{2} \cdot [h_u(\nabla_u g \omega - \omega(\nabla_u g) h_u)].
\]

(96)

In index form

\[
W_{ij} = -W_{ji} = \sigma_{ik} \cdot g^{kl} \cdot \omega_{lj} - \sigma_{jk} \cdot g^{kl} \cdot \omega_{li} + \frac{2}{n-1} \cdot \theta \cdot \omega_{ij} + \omega_{ij} : u^k + \\
+ \frac{1}{2} \cdot \sigma_{ik} : \{ \sigma_{ij} \cdot u_j + \sigma_{jk} \cdot u_i + h_{ij} + \frac{1}{2} \cdot \frac{2}{n-1} \cdot (\theta + \frac{a^j}{a^i}) \cdot h_{ij} + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{lj} + h_{jk} \cdot g^{kl} \cdot \omega_{li}) + \\
+ \frac{1}{2} \cdot (h_{ik} \cdot g^{kl} \cdot \omega_{lj} + h_{jk} \cdot g^{kl} \cdot \omega_{li}) =
\]

(98)

(d) Expansion acceleration \( U \):

\[
U = \frac{1}{e} \cdot g(a, a) + \overline{g}[\sigma(\overline{g}) \sigma] + \overline{g}[\omega(\overline{g}) \omega] + \theta + \frac{1}{n-1} \cdot \theta^2 + \\
+ \frac{1}{2} \cdot [2 \cdot g(u, \nabla_u u) - 2 \cdot g(u, T(a, u)) + (\nabla_u g)(a, u)] + \\
- \frac{1}{e} \cdot g(a, a) \cdot 3 \cdot g(u, a) + (\nabla_u g)(u, u)]
\]

(99)
In index form

\[ U = \frac{1}{e} \cdot \sigma_{ij} \cdot a_i \cdot a_j + g^{ij} \cdot g^{kl} \cdot \sigma_{jk} \cdot \sigma_{il} - g^{ij} \cdot g^{kl} \cdot \omega_{ik} \cdot \omega_{jl} + \]
\[ + \frac{1}{n-1} \cdot \theta^2 + \]
\[ + \frac{1}{e} \cdot g^{ij} \cdot a^k \cdot g^{ml} \cdot (e, m - g_{m}^{n} \cdot u^{n} - 2 \cdot T_{mr} \cdot u^{r} \cdot u^{mr}) - \]
\[ - u^{m} \cdot g^{ml} \cdot \omega_{m} \cdot u^{n} - \]
\[ + \frac{1}{e} \cdot \left( \partial_{k} \cdot (e, l) \cdot g_{ij} \cdot a^k \cdot u^{l} \cdot u^{j} \right) + \]
\[ + \frac{1}{e} \cdot \left( g_{ijk} \cdot a^k \cdot u^i \cdot u^j \right)^2 \]

(100)

(e) Torsion-free and curvature-free shear acceleration tensor \( s_F D_0 \)

\[ s_F D_0 = F D_0 - \frac{1}{n-1} \cdot F U_0 \cdot h_u \]

(101)

In index form

\[ (F D_0)_{ij} = (F D_0)_{ji} = \frac{1}{2} \cdot h_{ik} \cdot (a^k \cdot g^{ml} + a^l \cdot g^{mk}) \cdot h_{lj} \]

(102)

\[ F U_0 = a^k \cdot \partial_{k} - \frac{1}{e} \cdot g_{ijkl} \cdot u^k \cdot a^l \cdot u^m = \]

\[ = a^k \cdot \partial_{k} - \frac{1}{e} \cdot \left( g_{ijkl} \cdot u^k \cdot a^l \cdot u^m - g_{kl} \cdot g_{m}^{n} \cdot u^{n} \cdot u^{k} \right) \]

(103)

(f) Torsion-free and curvature-free rotation acceleration tensor \( F W_0 \)

\[ F W_0 = h_u (b_u) h_u \]

(104)

In index form

\[ (F W_0)_{ij} = -(F W_0)_{ji} = \frac{1}{2} \cdot h_{ik} \cdot (a^k \cdot g^{ml} - a^l \cdot g^{mk}) \cdot h_{lj} \]

(105)

(g) Torsion-free and curvature-free expansion acceleration \( F U_0 \) [s. (e)].

(h) Curvature-free shear acceleration tensor \( s D_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u \)

\[ D_0 = h_u (b_u) h_u - \frac{1}{e} \cdot \left( s_{P} (\sigma - \sigma) \cdot P \cdot \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \right) \]
\[ - \frac{1}{n-1} \cdot (\theta_{1} + \sigma + \theta \cdot s_{P}) - \frac{1}{n-1} \cdot (\theta_{1} + \frac{1}{n-1} \cdot \theta_{1} \cdot \theta) \cdot h_u - \nabla_{u} (s_{P}) - \]
\[ - \frac{1}{e} \cdot \left( [s_{P} (\sigma - \sigma) \cdot P + \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \right) \]
\[ - \frac{1}{e} \cdot \left( h_u (a) \otimes (g (u)) (m + q) h_u + h_u (g (u)) (m + q) \right) \cdot h_u (a) - \]
\[ - \frac{1}{e} \cdot \left( s_{P} (a) \otimes g (u) + q (u) \otimes s_{P} (a) \right) - \]
\[ - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) . \]

(106)

In index form

\[ (D_0)_{ij} = (D_0)_{ji} = \]
\[ = h_{ik} \cdot h_{jl} \cdot a^k \cdot m \cdot g^{ml} - s_{P} (\sigma + \sigma) \cdot \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \]
\[ - \frac{1}{n-1} \cdot (\theta_{1} + \sigma + \theta \cdot s_{P}) - \frac{1}{n-1} \cdot (\theta_{1} + \frac{1}{n-1} \cdot \theta_{1} \cdot \theta) \cdot h_{ij} - \]
\[ - \frac{1}{e} \cdot \left( s_{P} (\sigma + \sigma) \cdot \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \right) \]
\[ - \frac{1}{e} \cdot \left( h_u (a) \otimes (g (u)) (m + q) h_u + h_u (g (u)) (m + q) \right) \cdot h_u (a) - \]
\[ - \frac{1}{e} \cdot \left( s_{P} (a) \otimes g (u) + q (u) \otimes s_{P} (a) \right) - \]
\[ - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) . \]

(107)

(i) Curvature-free rotation acceleration tensor \( W_0 \)

\[ W_0 = h_u (b_u) h_u - \frac{1}{e} \cdot \left( s_{P} (\sigma + \sigma) \cdot \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \right) \]
\[ - \frac{1}{n-1} \cdot (\theta_{1} + \sigma + \theta \cdot s_{P}) - \frac{1}{n-1} \cdot (\theta_{1} + \frac{1}{n-1} \cdot \theta_{1} \cdot \theta) \cdot h_{ij} - \]
\[ - \frac{1}{e} \cdot \left( s_{P} (\sigma + \sigma) \cdot \frac{1}{e} \cdot (Q \sigma + \omega (\sigma) Q) - \right) \]
\[ - \frac{1}{e} \cdot \left( h_u (a) \otimes (g (u)) (m + q) h_u + h_u (g (u)) (m + q) \right) \cdot h_u (a) - \]
\[ - \frac{1}{e} \cdot \left( s_{P} (a) \otimes g (u) + q (u) \otimes s_{P} (a) \right) - \]
\[ - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) - \frac{1}{e} \cdot \left( h_u (u_{u} \sigma) + s_{P} (\nabla u \sigma) h_u \right) . \]

(108)
In index form
\[(W_0)_{ij} = -(\mathbf{W}_0)_{ji} = h_{li}^{\text{a}} \cdot h_{jl}^{\text{a}} \cdot a^k \cdot m \cdot g^m l - s P_{li} \cdot \sigma_{jl}^{\text{a}} \cdot g^k l - Q_{li}^{\text{a}} \cdot \omega_{jl}^{\text{a}} \cdot g^k l - \\
- \frac{1}{n-1} \cdot (\mathbf{\mathbf{\theta}}_1 \cdot \omega_{ij} + \mathbf{\theta} \cdot Q_{ij}) - Q_{ij;m \cdot u^m} + \\
+ s P_{lij} \cdot \omega_{jl}^{\text{a}} \cdot g^k l + Q_{lij} \cdot \sigma_{jl}^{\text{a}} \cdot g^k l - \\
- \frac{1}{c} \cdot a^k \cdot (Q_{ij} \cdot u_j - Q_{jk} \cdot u_i + h_{li}^{\text{a}} \cdot h_{jl}^{\text{a}} \cdot u^m \cdot T_{m r} \cdot u^r \cdot g^m l) + \\
+ s P_{lij} \cdot h_{jl}^{\text{a}} \cdot g^k l \cdot u^m + Q_{lij} \cdot h_{jl}^{\text{a}} \cdot g^k l \cdot u^m .
\]

(j) Curvature-free expansion acceleration \(U_0\)
\[U_0 = g[b] - \overline{\mathcal{F}}[s P(\mathcal{F})] - \overline{\mathcal{F}}(Q(\mathcal{F})\omega) - \theta_1 - \frac{1}{n-1} \cdot \mathbf{\mathbf{\theta}}_1 \cdot \mathbf{\theta} - \\
- \frac{1}{c} \cdot [g(u, T(a, u)) + g(u, \nabla u a)] .
\]

In index form
\[U_0 = a^k \cdot S_{ij} \cdot P_{si}^j \cdot \mathbf{\sigma}_{ij} - g^j \cdot Q_{ik} \cdot g^i \cdot \omega_{ij} - \theta_1 - \frac{1}{n-1} \cdot \mathbf{\mathbf{\theta}}_1 \cdot \mathbf{\theta} - \\
- \frac{1}{c} \cdot [a^k \cdot (u^m \cdot T_{m n} \cdot u^m - 2 \cdot g^i_{m n} \cdot u^i \cdot u^m - g^i \cdot a^i] + \\
+ \frac{1}{c} \cdot (e_k \cdot u^i) \cdot u^i - \frac{1}{c} \cdot (g_{m n} \cdot u^r) \cdot s \cdot u^r \cdot u^m \cdot u^m .
\]

(k) Shear acceleration tensor induced by the torsion \(s T D_0\)
\[s T D_0 = \tau D_0 - \frac{1}{n-1} \cdot \tau U_0 \cdot h_u
\]

In index form
\[(\tau D_0)_{ij} = (s T D_0)_{ij} - (D_0)_{ij} .
\]

(l) Expansion acceleration induced by the torsion \(\tau U_0\)
\[\tau U_0 = \overline{\mathcal{F}}[s P(\mathcal{F})] + \overline{\mathcal{F}}(Q(\mathcal{F})\omega) + \theta_1 + \frac{1}{n-1} \cdot \mathbf{\mathbf{\theta}}_1 \cdot \mathbf{\theta} + \frac{1}{c^3} \cdot g(u, T(a, u)) .
\]

In index form
\[\tau U_0 = \tau U_0 - U_0 .
\]

(m) Rotation acceleration tensor induced by the torsion \(\tau W_0\)
\[\tau W_0 = \frac{1}{2} \cdot [s P(\mathcal{F})] - \mathbf{\sigma}^{\text{a}} P + \frac{1}{2} \cdot [Q(\mathcal{F})\omega + \omega(\mathcal{F})Q] + \\
+ \frac{1}{c} \cdot (\mathbf{\mathbf{\theta}}_1 \cdot \sigma + \mathbf{\mathbf{\theta}} \cdot s P) + \frac{1}{c} \cdot (\mathbf{\mathbf{\theta}}_1 + \frac{1}{c} \cdot \mathbf{\mathbf{\theta}}_1 \cdot \mathbf{\theta} \cdot h_u + \nabla u(s P) + \\
+ \frac{1}{c} \cdot [s P(\mathcal{F})] + \frac{1}{c} \cdot [Q(\mathcal{F})\sigma - \sigma(\mathcal{F})Q] + \\
+ \frac{1}{c} \cdot [h_u(a) \otimes (g(u))(m + q)h_u + h_u((g(u))(m + q)) \otimes h_u(a)] + \\
+ \frac{1}{c} \cdot [s P(a) \otimes g(u) + g(u) \otimes s P(a)] + \\
+ \frac{1}{c} \cdot [h_u(\nabla u g)P + s P(\nabla u g)h_u] + \frac{1}{c} \cdot [h_u(\nabla u g)Q - Q(\nabla u g)h_u] .
\]

In index form
\[(\tau W_0)_{ij} = (s W_0)_{ij} - (W_0)_{ij} .
\]
(n) Shear acceleration tensor induced by the curvature $sM = M - \frac{1}{n-1} \cdot I \cdot h_u$

$$
M = \frac{1}{c} \cdot h_u(a) \otimes h_u(a) + \frac{1}{c} \cdot \left[ E(S) \sigma + \sigma(S) \cdot E \right] + \frac{1}{c} \cdot [S(g) \cdot \omega + \omega(g) \cdot S] + \frac{1}{n-1} \cdot (\theta_o \cdot \omega + \theta \cdot S) \cdot h_u + \nabla u \cdot (sE) +
\frac{1}{n-1} \cdot (\theta_o \cdot \theta \cdot h_u + \nabla u (sE) +
\frac{1}{n-1} \cdot [sE \cdot \omega - \omega \cdot S] + \frac{1}{2} \cdot [S(g) \cdot \sigma - \sigma(g) \cdot S] +
\frac{1}{2} \cdot [h_u (a) \otimes (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u + h_u (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u (a)] +
\frac{1}{2} \cdot [h_u (\nabla u \cdot S) \cdot E + \nabla u (\nabla u \cdot S) \cdot S \cdot \nabla u (S)] -
-h_u (b_s) h_u.
$$

In index form

$$
M_{ij} = M_{ji} = \frac{1}{c} \cdot h_{ik} \cdot a^k \cdot a^l \cdot h_{lj} + sE_{k(i} \cdot \sigma_{j)} + g^{ik} \cdot S_{k(i} \cdot \omega_{j)} + g^{ik} \cdot S_{k(i} \cdot \omega_{j)} +
\frac{1}{n-1} \cdot (\theta_o \cdot \sigma + \theta \cdot E_{ij}) + \frac{1}{n-1} \cdot (\theta_o \cdot \theta \cdot h_{ij} -
- \frac{1}{c} \cdot [E_{k(i} \cdot \omega_{j)} + \frac{g^{ik} \cdot S_{k(i} \cdot \sigma_{j)} + g^{ik} \cdot S_{k(i} \cdot \omega_{j)} +
\frac{1}{2} \cdot g_{rs} \cdot u^r \cdot \omega^s + g^{mn} \cdot u^m \cdot \omega^m - 1 \cdot g_{mn} \cdot u^m \cdot \omega^m -
\frac{1}{2} \cdot \left[ \frac{1}{2} \cdot g_{mn} \cdot u^m \cdot \omega^m \right] \right] +
\frac{1}{c} \cdot (g_{mn} \cdot \omega^m) - \frac{1}{c} \cdot \left[ \frac{1}{2} \cdot (g_{mn} \cdot \omega^m) \right] -
\frac{1}{n-1} \cdot (\theta_o \cdot \omega + \theta \cdot S) + \nabla u \cdot S + \frac{1}{c} \cdot [sE \cdot \omega - \omega \cdot S] +
\frac{1}{2} \cdot [S(g) \cdot \sigma - \sigma(g) \cdot S] +
\frac{1}{2} \cdot [h_u (a) \otimes (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u + h_u (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u (a)] +
\frac{1}{2} \cdot [h_u (\nabla u \cdot S) \cdot E + \nabla u (\nabla u \cdot S) \cdot S \cdot \nabla u (S)] -
-h_u (b_s) h_u.
$$

(o) Expansion acceleration induced by the curvature $I$

$$
I = -g[b] + \frac{1}{c} \cdot [sE \cdot \sigma + \sigma(S) \cdot \cdot E + \theta_o + \frac{1}{n-1} \cdot \theta_o \cdot \theta +
\frac{1}{2} \cdot \left[ 2 \cdot g(u, \nabla u)\cdot u - g(u, T(a, u)) + u (g(u, a))] -
- \frac{1}{2} \cdot \theta \cdot (g(u, a)) \cdot \left[ 3 \cdot g(u, a) + (\nabla u g)(u, u) \right].
$$

In index form

$$
I = R_{ij} \cdot u^i \cdot u^j = -a^i \cdot j + g^{ij} \cdot g^{ij} + g^{ij} \cdot S_{ik} \cdot \omega_{j} +
+ \frac{1}{n-1} \cdot \theta_o \cdot \theta + \frac{1}{2} \cdot \left[ a^k \cdot (e, k - u_m \cdot T_{kn} \cdot n - g_{mn} \cdot k \cdot u_m \cdot u_n -
\frac{1}{2} \cdot g_{mn} \cdot u_m \cdot u_n - \frac{1}{2} \cdot \left[ \frac{1}{2} \cdot g_{mn} \cdot u_m \cdot u_n \right] \right] +
\frac{1}{2} \cdot \left[ (e, k) \cdot u^k \right]^2 - \frac{1}{2} \cdot \left[ (e, k) \cdot u^k \right] \cdot g_{mn} \cdot r^r \cdot u^r \cdot u^m \cdot u^r + \frac{1}{2} \cdot \left[ (g_{mn} \cdot r^r \cdot u^r \cdot u^m \cdot u^r) \right].
$$

(p) Rotation acceleration tensor induced by the curvature $N$

$$
N = \frac{1}{c} \cdot [sE \cdot \sigma - \sigma(S) \cdot \cdot E + \frac{1}{c} \cdot [S(g) \cdot \omega - \omega \cdot S] +
\frac{1}{n-1} \cdot (\theta_o \cdot \omega + \theta \cdot S) + \nabla u \cdot S + \frac{1}{c} \cdot [sE \cdot \omega - \omega \cdot S] +
\frac{1}{2} \cdot (S(g) \cdot \sigma - \sigma(g) \cdot S] +
\frac{1}{2} \cdot [h_u (a) \otimes (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u + h_u (g(u)) (k_0 + k - \nabla u \cdot S) \cdot h_u (a)] +
\frac{1}{2} \cdot [h_u (\nabla u \cdot S) \cdot E + \nabla u (\nabla u \cdot S) \cdot S \cdot \nabla u (S)] -
-h_u (b_s) h_u.
$$

In index form

$$
N_{ij} = -N_{ji} = sE_{k[i} \cdot \sigma_{j]} \cdot g^{ik} + S_{k[i} \cdot \omega_{j]} \cdot g^{ik} + \frac{1}{n-1} \cdot (\theta_o \cdot \omega_{ij} + \theta \cdot S_{ij}) -
- sE_{k[i} \cdot \omega_{j]} \cdot g^{ik} - S_{k[i} \cdot \sigma_{j]} \cdot g^{ik} + S_{ij} \cdot k \cdot u^k - h_u \cdot h_{ij} \cdot a^k \cdot g^{im} +
\frac{1}{2} \cdot a^k \cdot [S_{ij} \cdot u - S_{ij} \cdot u_k + \nabla u \cdot h_{ij} \cdot g^{im} \cdot (e, m - u^m \cdot T_{mr} \cdot n \cdot r -
\frac{1}{2} \cdot u^m \cdot g^{im} + \nabla u \cdot S \cdot \nabla u (S)] -
-sE_{k[i} \cdot h_{j]} \cdot g^{ik} \cdot g^{im} \cdot a^k \cdot g^{im} \cdot u^m -
$$

(122)
Table 1. Kinematic characteristics connected with the notions relative velocity and relative acceleration. A summary of the definitions

5.1 Kinematic characteristics connected with the relative velocity

1. Relative position vector field
   (relative position vector) ............................................ ξ ⊥ = \( g( h_u(\xi) ) \)
2. Relative velocity ...................................................... \( _{rel} v = g( h_u(\nabla_u \xi) ) \)
3. Deformation velocity tensor
   (deformation velocity, deformation) .. \( d = d_0 - d_1 = \sigma + \omega + \frac{1}{\eta} \cdot \theta \cdot h_u \)
4. Torsion-free deformation velocity tensor
   (torsion-free deformation velocity, torsion-free deformation) ............................................. \( d_0 = sE + S + \frac{1}{\eta} \cdot \theta_0 \cdot h_u \)
5. Deformation velocity tensor induced by the torsion
   (torsion deformation velocity, torsion deformation) .............................................................. \( d_1 = sP + Q + \frac{1}{\eta} \cdot \theta_1 \cdot h_u \)
6. Shear velocity tensor
   (shear velocity, shear) ........................................... \( \sigma = sE - sP \)
7. Torsion-free shear velocity tensor
   (torsion shear velocity, torsion shear) ........... \( sE = E - \frac{1}{\eta} \cdot \theta_0 \cdot h_u \)
8. Shear velocity tensor induced by the torsion
   (torsion shear velocity tensor, torsion shear velocity, torsion shear) ................................................. \( sP = P - \frac{1}{\eta} \cdot \theta_1 \cdot h_u \)
9. Rotation velocity tensor
   (rotation velocity, rotation) ...................................... \( \omega = S - Q \)
10. Torsion-free rotation velocity tensor
    (torsion-free rotation velocity, torsion-free rotation) ................................................................. \( S = h_u(s)h_u \)
11. Rotation velocity tensor induced by the torsion
    (torsion rotation velocity, torsion rotation) .................. \( Q = h_u(q)h_u \)
12. Expansion velocity
    (expansion) .............................................................. \( \theta = \theta_0 - \theta_1 \)
13. Torsion-free expansion velocity
    (torsion-free expansion) .............................................. \( \theta_0 = \overline{g}[E] \)
14. Expansion velocity induced by the torsion
    (torsion expansion velocity, torsion expansion) .................. \( \theta_1 = \overline{g}[P] \)

5.2 Kinematic characteristics connected with the relative acceleration

1. Acceleration .......................................................... \( a = \nabla_u u \)
2. Relative acceleration ........................................... \( _{rel} a = g( h_u(\nabla_u \nabla_u \xi) ) \)
3. Deformation acceleration tensor
   (deformation acceleration) ........................................ \( A = sD + W + \frac{1}{\eta} \cdot U \cdot h_u \)
   .................................................................................. \( A = A_0 + G \)
4. Torsion-free and curvature-free deformation acceleration tensor
   (torsion-free and curvature-free deformation acceleration)
   ................................................................. \( fA_0 = sFD_0 + fW_0 + \frac{1}{\eta} \cdot fU_0 \cdot h_u \)
4.a. Curvature-free deformation acceleration tensor
(curvature-free deformation acceleration). \( A_0 = sD_0 + W_0 + \frac{1}{n-1} \cdot U_0 \cdot h_u \)

5. Deformation acceleration tensor induced by the torsion (torsion deformation acceleration tensor, torsion deformation acceleration)

\[ \tau A_0 = sT D_0 + \tau W_0 + \frac{1}{n-1} \cdot \tau U_0 \cdot h_u \]

5.a. Deformation acceleration tensor induced by the curvature (curvature deformation acceleration tensor, curvature deformation acceleration)

\[ A = sM + N + \frac{1}{n-1} \cdot I \cdot h_u \]

6. Shear acceleration tensor (shear acceleration)

\[ sD = D - \frac{1}{n-1} \cdot U \cdot h_u \]

\[ sD = sF D_0 - \frac{1}{n-1} \cdot F U_0 \cdot h_u \]

7. Torsion-free and curvature-free shear acceleration tensor (torsion-free and curvature-free shear acceleration)

\[ sF D_0 = F D_0 - \frac{1}{n-1} \cdot F U_0 \cdot h_u \]

7.a. Curvature-free shear acceleration tensor (curvature-free shear acceleration)

\[ sD_0 = D_0 - \frac{1}{n-1} \cdot U_0 \cdot h_u \]

8. Shear acceleration tensor induced by the torsion (torsion shear acceleration tensor, torsion shear acceleration)

\[ sT D_0 = T D_0 - \frac{1}{n-1} \cdot T U_0 \cdot h_u \]

8.a. Shear acceleration tensor induced by the curvature (curvature shear acceleration tensor, curvature shear acceleration)

\[ sD_0 = sF D_0 - sT D_0 \]

9. Rotation acceleration tensor (rotation acceleration)

\[ W = W_0 + N \]

10. Torsion-free and curvature-free rotation acceleration tensor (torsion-free and curvature-free rotation acceleration)

\[ F W_0 = h_u (b_u) h_u \]

10.a. Curvature-free rotation acceleration tensor (curvature-free rotation acceleration)

\[ W_0 = W - N \]

11. Rotation acceleration tensor induced by the torsion (torsion rotation acceleration tensor, torsion rotation acceleration)

\[ \tau W_0 = F W_0 - W_0 \]

11.a. Rotation acceleration tensor induced by the curvature (curvature rotation acceleration tensor, curvature rotation acceleration)

\[ N = h_u (K_u) h_u \]

12. Expansion acceleration (expansion acceleration)

\[ U = U_0 + I \]

\[ U = F U_0 - \tau U_0 + I \]

13. Torsion-free and curvature-free expansion acceleration (torsion-free and curvature-free expansion acceleration)

\[ F U_0 = F [F D_0] \]

13.a. Curvature-free expansion acceleration (curvature-free expansion acceleration)

\[ U_0 = F [D_0] \]

14. Expansion acceleration induced by the torsion (torsion expansion acceleration)

\[ U_0 = F [T D_0] \]

14.a. Expansion acceleration induced by the curvature (curvature expansion acceleration)

\[ I = F [M] = F [G] \]
Table 2. Classification of non-isotropic autoparallel vector fields on the basis of the kinematic characteristics connected with the relative velocity and relative acceleration

6.1 Classification on the basis of kinematic characteristics connected with the relative velocity

The following conditions, connected with the relative velocity, can characterize the vector fields over manifolds with affine connections and metrics:

1. $\sigma = 0$.
2. $\omega = 0$.
3. $\theta = 0$.
4. $\sigma = 0, \omega = 0$.
5. $\sigma = 0, \theta = 0$.
6. $\omega = 0, \theta = 0$.
7. $\sigma = 0, \omega = 0, \theta = 0$.
8. $sE = 0$.
9. $S = 0$.
10. $\theta_0 = 0$.
11. $sE = 0, S = 0$.
12. $sE = 0, \theta_0 = 0$.
13. $S = 0, \theta_0 = 0$.
14. $sE = 0, S = 0, \theta_0 = 0$.
15. $sP = 0$.
16. $Q = 0$.
17. $\theta_1 = 0$.
18. $sP = 0, Q = 0$.
19. $sP = 0, \theta_1 = 0$.
20. $Q = 0, \theta_1 = 0$.
21. $sP = 0, Q = 0, \theta_1 = 0$.

6.2 Classification on the basis of kinematic characteristics connected with the relative acceleration

The following conditions, connected with the relative acceleration, can characterize the vector fields over manifolds with affine connections and metrics:

1. $sD = 0$.
2. $W = 0$.
3. $U = 0$.
4. $sD = 0, W = 0$.
5. $sD = 0, U = 0$.
6. $W = 0, U = 0$.
7. $sD = 0, W = 0, U = 0$.
8. $sM = 0$.
9. $N = 0$.
10. $I = 0$.
11. $sM = 0, N = 0$.
12. $sM = 0, I = 0$.
13. $N = 0, I = 0$.
14. $sM = 0, N = 0, I = 0$.
15. $sT D_0 = 0$. 
16. $TW_0 = 0$.
17. $TU_0 = 0$.
18. $sTD_0 = 0$, $TW_0 = 0$.
19. $sTD_0 = 0$, $TU_0 = 0$.
20. $TW_0 = 0$, $TU_0 = 0$.
21. $sTD_0 = 0$, $TW_0 = 0$, $TU_0 = 0$.

The kinematic characteristics related to the relative velocity and the friction velocity are related to deformations and tensions. The last statement will be discussed in the next paper.

7 Conclusion

In the present paper the notion of relative acceleration is introduced and its corresponding kinematic characteristics (deformation acceleration, shear acceleration, rotation acceleration, expansion acceleration) are considered. On their basis classification of contravariant vector fields are proposed for describing flows with special properties and motions. The consideration are important for developing appropriate models of flows and continuous media in relativistic and more general (gravitational) field theories.

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