Squeezing-enhanced quantum sensing with quadratic optomechanics

Sheng-Dian Zhang,† Jie Wang,† Qian Zhang,† Ya-Feng Jiao, Yun-Lan Zuo, Şahin K. Özdemir, Cheng-Wei Qu, Franco Nori, and Hui Jing

1 Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China
2 Department of Engineering Science and Mechanics, and Materials Research Institute, Pennsylvania State University, University Park, State College, Pennsylvania 16802, USA
3 Department of Electrical and Computer Engineering, National University of Singapore, Singapore 117583, Singapore
4 Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan
5 Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA
† The authors contributed equally to this work
† Jinghui73@foxmail.com

1. INTRODUCTION
The development of quantum-enhanced sensors aimed at the sensitive measurement of time, temperature, pressure, or electromagnetic fields has witnessed considerable progress in recent years [1, 2], with a broad spectrum of approaches including the use of elementary particles [3–6], superconducting circuits [7–9], optical systems [10–12], and solid-state mechanical devices [13, 14]. In particular, cavity optomechanical (COM) [15–17] and electromechanical sensors [18–20] are remarkably well suited for the measurement of weak forces or very small displacements [21]. Importantly, their standard quantum limit (SQL), which results from the combined effects of backaction noise and photon shot noise, can be broken by the use of optical fields with appropriate quantum correlations (see e.g. [22, 23]). For example, in an impressive recent experiment, the sub-SQL displacement measurement in a COM system with a macroscopic 40 kg mirror was achieved by injecting squeezed light in the otherwise empty port of the system [24].

COM displacement sensors typically rely on the linear coupling between the displacement of the mechanical element and the electromagnetic field. However, such a coupling is not appropriate for energy or phonon number measurements, which require instead an optomechanical coupling that is quadratic in the mechanical displacement [25]. This coupling also allows for applications such as two-phonon cooling [26], and a variety of quantum non-demolition (QND) measurements [27–32]. Quadratic COM systems (where the cavity detuning is proportional to the square of the mechanical displacement, i.e., $\omega_{\text{cav}}(x) \propto x^2$ [33]) have been demonstrated using, e.g., levitated nanospheres [34], membrane-in-the-middle cavities [33, 35–37], photonic crystals [38, 39], and atomic gases [40]. Also, selective linear or quadratic COM coupling was achieved via homodyne measurements and utilized to create non-Gaussian mechanical states [41, 42]. However, one known issue of quadratic coupling is the linear dissipative coupling typically associated with it and there has been significant interest in exploiting quantum noise interference to cancel the residual linear backaction in the bad-cavity limit, allowing one to make QND measurements of mechanical energy using a quadratic COM system [43]. A recent publication proposed a novel geometry that significantly solves this problem and results in a dramatic reduction of backaction noise [44].

In this paper we expand on the study of quadratic optomechanical sensors [28, 43, 44] and demonstrate theoretically that the inclusion of intracavity optical squeezing [48–50] can result in a remarkable improvement in their sensitivity. Our proposed scheme, which is compatible with other available techniques of fabricating and engineering advanced COM sensors, provides a way to further enhance the power of quadratic COM sensors for applications ranging from quantum metrology to tests of fundamental laws of physics.

2. SQUEEZED QUADRATIC OPTOMECHANICS
We consider an ideal membrane-in-the-middle (MIM) Fabry-Pérot cavity with a thin dielectric membrane located either at a node or antinode of the standing wave mode and coupled quadratically to the field [33, 51], allowing for quantum non-demolition readout of the membrane’s phonon numbers [35]. An additional nonlinear $\chi^{(2)}$
medium, coupled quadratically to the cavity field, induces intracavity squeezing, integrated with an intracavity squeezing. It is driven by a pump field of frequency $\omega_p$ at twice the signal frequency $\omega_s$ [52], see Fig. 1(a). We limit our considerations to the case where the membrane has a low enough reflection that it will not split the cavity into two sub-cavities [53, 54].

The intracavity second-order nonlinear optical process is described by the Hamiltonian [55]

$$\hat{H}_{(2)} = \hbar \Delta_c \hat{a}^\dagger \hat{a} + \hbar \Delta_p \hat{a}_p^\dagger \hat{a}_p + i \hbar \chi^{(2)} \left( \hat{a}_p^2 e^{i \theta} - \hat{a}_p^\dagger \hat{a}_p e^{-i \theta} \right),$$

(1)

where $\hat{a}_p$ and $\hat{a}$ are the boson operators of pump and signal modes, of frequencies $\omega_p = 2\omega_s$; $\Delta_p$ is the detuning between the the pump drive and the nearest cavity mode frequencies; $\Delta_c$ is the detuning between the signal and the nearest cavity mode frequencies, and $\hat{a}$ is the associated phase of $\chi^{(2)}$.

We assume that the pump field is strong enough that it can be treated classically, and characterized by a large mean ‘photon number’ $n_p$. Eliminating the associated optomechanical interaction adiabatically and including the driving $E_c$ of the signal mode, we obtain the effective model Hamiltonian at its simplest level [33, 45]:

$$\hat{H} = \hbar \Delta_c \hat{a}^\dagger \hat{a} + \frac{\hbar}{2} \Omega_m \left( \hat{q}_m^2 + \hat{p}_m^2 \right) - \hbar g_0 \hat{a}_p^\dagger \hat{a}_p \hat{q}_m^2 + i \hbar G \left( \hat{a}_p^2 e^{i \theta} - \hat{a}_p^\dagger e^{-i \theta} \right) + i \hbar \left( E_c \hat{a}^\dagger - E_c^* \hat{a} \right),$$

(2)

where $\hat{q}_m$ and $\hat{p}_m$ are the position and momentum operators of mechanical mode at frequency $\Omega_m$; $g_0$ represents single-photon COM coupling strength, which quantifies the interaction between a single phonon and a single photon; $G = \chi^{(2)}/\sqrt{n_p}$ is the nonlinear gain coefficient, and $E_c$ is the driving amplitude. We stress that both the quadratic COM coupling and the squeezing-enhanced COM systems were already well-established in experiments. For examples, a high-finesse MIM system was utilized for direct measurements of the membrane’s displacement [33] and, by tuning the suitable position of the membrane, the quadratic coupling strength can be greatly enhanced for 3 orders of magnitude, indeed reaching a purely quadratic COM system [35, 56]. Such a quadratic COM system was also experimentally demonstrated by levitating a nanosphere in a suitable potential [34]. We also note that in a recent experiment, by using an intracavity parametric amplifier, phase-sensitive manipulations of an input squeezed vacuum was demonstrated [57]. Similarly, loss suppressions and thus giant enhancement of sensitivities were also demonstrated in experiments by inserting such optical amplifiers into interferometers [58, 59]. Indeed, the merits of quantum squeezing in enhancing linear COM sensors have already been confirmed in experiments and the main purpose of our present work is to confirm that such a merit also exists for a quadratic COM system. Hence it is reasonable to expect that even for a hybrid COM system with both linear and quadratic couplings, the positive effects of quantum squeezing will still exist, which we plan to further study in our future work (we note that in a very recent work, the linear coupling was confirmed to be not detrimental for quantum entanglement emerging in such a hybrid COM system [60]).

Here we use the experimentally feasible parameter values, i.e., the cavity quality factor $Q = 1 \times 10^7$ [61], the total optical decay rate $\kappa/2\pi = 3$ MHz [61], including both the decay rate $\kappa_{ex}$ at the input mirror and the intra-cavity decay rate $\kappa_0$, with ‘efficiency’ $\eta_e = \kappa_{ex}/(\kappa_0 + \kappa_{ex})$, and the mechanical quality factor $Q_m = 5 \times 10^8$, with the mechanical frequency $\Omega_m/2\pi = 1$ MHz [61], the effective mass $m_{eff} = 1$ ng [61], and the associated decay rate $\Gamma_m$ [61]. We note that a second-order nonlinearity of $\chi^{(2)}/2\pi = 80$ kHz was realized [52], confirming the feasibility of $G = 0.246$. Very recently, a new optomechanical experiment using an optical crystal with third-order nonlinearity has demonstrated that with this nonlinearity-assisted system, optical spring effect can be enhanced [62]. Figure 1(b) shows that the nonlinear gain coefficient $G$ increases with the pump laser power and the second-order nonlinearity, indicating the required parametric gain occurs at large pump detunings [45].

Neglecting the higher-order nonlinear terms [63] in the quantum fluctuations results in coupled linear equations

$$\dot{\hat{a}} = - \left( i \Delta - \frac{\chi}{2} \right) \hat{a} + 2g \delta \hat{q}_m + 2G e^{i \theta} \hat{a}^\dagger,$$

$$\dot{\hat{q}}_m = \Omega_m \hat{q} + \sqrt{\eta_e \kappa_{ex}} \hat{f}_{m,\text{in}} + \sqrt{1 - \eta_e} \kappa \hat{f}_{a,\text{in}},$$

$$\dot{\hat{p}}_m = - \Gamma_m \hat{p}_m + 2g \delta \hat{q}_m + \sqrt{2 \Gamma_m \Omega_m} \hat{f}_{m,\text{in}},$$

(3)

where $\hat{f}_{a,\text{in}}$ and $\hat{f}_{m,\text{in}}$ are the noise operators associated with the input cavity mirror and the internal losses, and $\Delta = \Delta_c - g_0 \hat{p}_m$ is the
effective optical detuning. The flowchart of Fig. 2(b) illustrates the various couplings involved in Eq. (3). A variable on the right-hand side of an equation of motion is connected to a variable on the left-hand side by arrows, showing that δp_{out} is independent of δq_{in}, a consequence of the cancellation of the associated coefficient, i.e., $-\Omega_m + 2\delta q_0 n_c = 0$, where $n_c = \Omega_m/(2\delta q_0)$.

Direct measurements of intracavity fields are typically challenging, and one often measures the field that escapes the resonator instead. The relationship between the input field and the output field is given by the input-output relation $\delta_{out} = \sqrt{n_c} \delta_{in} - \delta_{in}$ [63]. As illustrated in Fig. 2(a-b), the parameters used in our work are indeed in the optimally sensitive regime at the border between the stable and unstable regions.

Figure 2(b) shows that in the quadratic COM system under consideration, the flow of signal and noise between $\delta_{in}$ and $\delta_{out}$ is unidirectional, in contrast to the situation for linear COM systems. This causes the mechanical susceptibility of the quadratic COM sensor to differ from the expression $\Omega_m/(\Omega_m^2 - \Omega^2 - i\Omega^2$ of those systems [63].

One way to measure the frequency-dependent force noise is homodyne detection [64], whereby the output signal is mixed at a 50:50 beam splitter with a local oscillator, with a phase $\phi$ between the signal and the reference field. The photocurrent $I_{\phi}$ at the output of the balanced detector is then proportional to a rotated field quadrature

$$\delta q_{\phi}^\theta(\Omega) = \delta q_{\phi} \cos \phi + \delta p_{\phi} \sin \phi.$$  \hspace{1cm} (4)

Introducing the correlation functions [65]

$$\langle \delta q_{\phi} | \omega \rangle \delta q_{\phi} (\Omega) \rangle = \langle \delta q_{\phi} | \omega \rangle \delta q_{\phi} (\Omega) \rangle = \frac{1}{2} \delta (\omega + \Omega),$$

$$\langle \delta q_{\phi} | \omega \rangle \delta p_{\phi} (\Omega) \rangle = - \langle \delta p_{\phi} | \omega \rangle \delta q_{\phi} (\Omega) \rangle = \frac{1}{2} \delta (\omega + \Omega),$$

$$\langle \delta p_{\phi} | \omega \rangle \delta p_{\phi} (\Omega) \rangle = \bar{n}_m \delta (\omega + \Omega),$$

where $\omega = 0$ and $\bar{n}_m = [\exp(\hbar \Omega_m/k_B T) - 1]^{-1}$ denotes the thermal phonon occupancy. The output amplitude and phase quadrature spectrum can be expressed as [64]

$$S_{qq}^{out}(\Omega) = \frac{1}{2} \langle \delta q_{out}(\Omega), \delta q_{out}(-\Omega) \rangle = \frac{1}{2} \kappa_- |\Omega| + \bar{n}_m |N_-| |\Omega|^2,$$

$$S_{pp}^{out}(\Omega) = \frac{1}{2} \langle \delta p_{out}(\Omega), \delta p_{out}(-\Omega) \rangle = \frac{1}{2} \kappa_+ |\Omega| + \bar{n}_m |N_+| |\Omega|^2.$$  \hspace{1cm} (6)

in the above two equations, we introduced the following definitions

$$\kappa_- = |A_-|^2 + |B_-|^2 + |C_-|^2 + |D_-|^2,$$

$$\kappa_+ = |A_+|^2 + |B_+|^2 + |C_+|^2 + |D_+|^2.$$  \hspace{1cm} (7)

The parameters $A_\pm$, $B_\pm$, $C_\pm$, $D_\pm$, and $N_\pm$ can be derived through straightforward algebraic calculations (see Supplement 1 [46] for their lengthy expressions). $\kappa_\pm$ denotes the contributions of shot noise and backaction noise to the output amplitude or phase quadrature spectrum $S_{qq}^{out}$, while $N_\pm$ is from the noise imprinted by mechanical motion. The symmetrized cross-correlation spectrum is then written as

$$S_{pq}^{out}(\Omega) = \frac{1}{2} \langle \langle \delta q_{out}(\Omega), \delta p_{out}(-\Omega) \rangle \rangle = \text{Re} \left\{ \frac{1}{2} \kappa_{co} |\Omega| + \bar{n}_m N |\Omega|^2 \right\},$$  \hspace{1cm} (8)

with $\kappa_{cr} = B_- A_-^* A_- B_-^* + D_- C_-^* C_- D_-^*$, and $\kappa_{si} = A_- A_-^* + B_- B_-^* + C_- C_-^* + D_- D_-^*$. Here $\kappa_{co} = \kappa_{cr} + i\kappa_{si}$, which contains the squeezed-dependent correlations between shot noise and backaction noise, and $N = N_+^2 N_-$. [66]. The output spectrum thus contains amplitude or phase vacuum noise, thermal occupations, and
The added noise is
\[ \bar{n}_{\text{add}}[\Omega] = C + \frac{1}{16\eta_c\Gamma_m^2|\chi_m|^2} \] (13)
where the multi-photon cooperativity is defined as
\[ C \equiv C_{\text{SQL}} = \frac{1}{4\sqrt{\eta_c\Gamma_m|\chi_m|^2}}. \] (14)
It is minimized to
\[ \bar{n}_{\text{add}}[\Omega] = \frac{1}{2\sqrt{\eta_c\Gamma_m|\chi_m|^2}}. \] (15)
for
\[ C \equiv C_{\text{SQL}} = \frac{1}{4\eta_c\Gamma_m|\chi_m|^2}. \] (16)
where \( \chi_m = -\Omega_m/(\Omega^2 + i\Omega_m) \) is the mechanical susceptibility of the system, which quantifies the response of the oscillator to external forces. So that in the absence of squeezing the minimum output force noise is given by
\[ S_{\text{FF}}[\Omega] = 2\hbar \eta_c \Gamma_m \quad \text{with} \quad \bar{n}_{\text{add}}[\Omega]. \] (17)
It is clear from Fig. 2(b) that a direct way to counter the effect of the backaction noise is to introduce another path from \( \delta q \) to \( \delta p \) using intracavity squeezing [47]. Then, without standard phase detection, the imprecision and backaction noises can be correlated by tuning the parametric phase of \( \chi^{(2)} \) medium. Thus, a decreased parametric phase corresponds to a lower detection sensitivity in the stable region because of the narrower range for the multi-photon cooperativity [Fig. 3(a)].

To simultaneously achieve quantum noise suppression and force signal amplification, the values of the scaled cooperativity \( (C/C_{\text{SQL}}) \) and the squeezed parameters should be chosen within the stable region [Fig. 3(b)]. For the parameters of our numerical examples it yields the quantum noise that is 3.5 decibels below the SQL [see Fig. 3(b)]. We note that in a very recent experiment, using a linear COM system assisted by quantum correlations [24], a joint quantum uncertainty that is 3 decibels below the SQL was shown after subtracting thermal noises. Here, we define the degree of the squeezing as
\[ \sigma = \lg \left( \frac{S_{\text{FF}}}{S_{\text{SQL}}} \right). \] (18)
Quantum-enhanced force measurement can be simply characterized by the enhancement factor due to the squeezing
\[ \zeta = \frac{\min \{ S_{\text{FF}}(G = 0, \theta = 0) \}}{\min \{ S_{\text{FF}}(G \neq 0, \theta \neq 0) \}}. \] (19)
When the thermal noise of the system has been significantly reduced, for instance by utilizing dilution refrigeration or precooling, further enhancement could be further improved by injecting squeezed vacuum into the optical cavity [48, 69].

Notably, the mechanical susceptibility can transduce force into the displacement of the membrane and quantify the response of the mechanical resonator to the detected force [63]. In the quadratic COM system, the mechanical response [derived from Eq. (10)] to the detected force is significantly enhanced [Fig. 4(a)] due to the larger mechanical susceptibility, enabling a remarkable amplification of the force signal and corresponding to a low quantum noise given by Eq. (11). Hence, the enhanced mechanical response is important to achieve better measurement sensitivity. As shown in Fig. 4(a), the optimal mechanical response derived from Eq. (10) for the quadratic COM sensor can be further enhanced by introducing intracavity squeezing. Therefore, from the analyses made above, according to Eq. (19), combined with the additional merit of quantum squeezing, the quadratic COM systems can be more beneficial by incorporating the additional merit of quantum squeezing [Fig. 4(b)].

The advantage of the quadratic COM system is mainly manifested in quantum-noise-dominated situations, which becomes marginal with increasing thermal noises. The high sensitivity is predicted close to the boundary between the stable and unstable regimes [70], as shown in Fig. 2(a). The sensitivity of force measurements is mainly limited by the thermal Langevin force, with the PSD given by [71]

$$S_{\Omega m, F} = 2m_{\text{eff}}k_B T \frac{\Omega_m}{Q_m},$$

(20)

where $k_B$ is Boltzmann’s constant, and $T$ is the bath temperature. In practice, thermal noise can lower the measurement sensitivity. Nevertheless, we estimate that under realistic conditions, the force sensitivity can still reach ($10.2 \text{ aN}^2$/Hz) even at room temperature (which can be optimized as ($0.26 \text{ aN}^2$/Hz) at cryogenic temperatures), approaching the level of the state-of-the-art sensors with force noises in the range 10–100 aN Hz$^{-1/2}$ at room temperature (or less than 1 aN Hz$^{-1/2}$ at cryogenic temperatures) [72]. We estimate that by using the state-of-the-art membrane [73], the force noise even can be reduced to ($9.9 \text{ aN}^2$/Hz) at the temperature of 0.2 K.

For a highly reflective membrane, another practical concern is the backaction arising from the underlying linearity of hybridized modes [28, 43, 44]. However, this technical challenge has not prevented the advances in quadratic COM systems [28, 43, 44]. In fact, linear backaction can be effectively suppressed in practice through structural design or active feedback [42, 44], or by using highly tunable COM systems such as levitated particles, photonic crystals, electromechanical devices, and cold atoms [34, 39, 40, 74, 75]. Indeed, the merits of quantum squeezing in enhancing linear COM sensors have already been confirmed in experiments and the main purpose of our present work is to confirm that such a merit also exists for a quadratic COM system. Hence it is reasonable to expect that even for a hybrid COM system with both linear and quadratic couplings, the positive effects of quantum squeezing will still exist—specific topic we plan to further calculate and verify in our next work.

4. CONCLUSION

In summary, we have shown that the performance of quadratic COM sensors can be significantly enhanced by intracavity squeezing. We find that the mechanical response to weak force signals can be significantly amplified with considerably reduced quantum noise in these systems, promising sub-SQL force measurements with experimentally accessible parameters. We expect that by combining it with other existing techniques of fabricating and operating COM-based sensors, such as those involving feedback control [15, 76] or advanced materials with much higher mechanical Q factors [73, 77], it is possible to further improve its performance in practice. Such an improved COM sensor can be useful for a wide range of applications requiring ultrahigh sensitivity [78–83]. It is our hope that these results will stimulate further efforts toward building and utilizing quantum-squeezing-enhanced sensors, such as those based on levitated spheres [34], cold atoms [40], dissipative or near-field COM systems [42].

Funding. National Key Research and Development Program of China (2022YFE0102400); National Natural Science Foundation of China (12147156); National Natural Science Foundation of China (11774086, 11935006); Multidisciplinary University Research Initiative (FA9550-21-1-0202); the Foundational Questions Institute Fund (FQXi) (FQXi-IAF19-06); the Asian Office of Aerospace Research and Development (AOARD) (FA2386-20-1-0469); the Japan Society for the Promotion of Science (JSPS) and the Ministry of Education, Science, Sport, and Technology, Japan (S-005143-01); the Science and Technology Innovation Program.
Acknowledgments. We thank Pierre Meystre for helpful discussions and good suggestions.

Disclosures. The authors declare no conflicts of interest.

Data availability. Data underlying the results presented in this paper may be obtained from the authors upon reasonable request.

Supplemental document. See Supplement 1 for supporting content.

REFERENCES

1. A. A. Clerk, M. H. Devoret, S. M. Girvin, et al., Rev. Mod. Phys. 82, 1155 (2010).
2. C. L. Degen, F. Reinhard, and P. Cappellaro, Rev. Mod. Phys. 89, 035002 (2017).
3. E. Pedrozo-Peñaflé et al., Nat. (London) 588, 414 (2020).
4. K. C. McCormick, J. Keller, S. C. Burd, et al., Nature 572, 86 (2019).
5. S. Wu, G. Bao, J. Guo, et al., Sci. Adv. 9, eaad1760 (2023).
6. T. Chalopin, C. Bouazza, A. Evrard, et al., Nat. Commun. 9, 4955 (2018).
7. M. A. Castellanos-Beltran, K. D. Irwin, G. C. Hilton, et al., Nat. Phys. 4, 929 (2008).
8. K. M. Backes et al., Nat. (London) 590, 238 (2021).
9. K. Xu, Y.-R. Zhang, Z.-H. Sun, et al., Phys. Rev. Lett. 128, 150501 (2022).
10. R. Schnabel, Phys. Rep. 684, 1 (2017).
11. B. J. Lawrie, P. D. Lett, A. M. Marino, and R. C. Pooser, ACS Photon. 6, 1307 (2019).
12. Y. Wang, H.-L. Zhang, J.-L. Wu, et al., Sci. China Phys. Mech. Astron. 66, 103131 (2023).
13. X.-D. Chen, E.-H. Wang, L.-K. Shan, et al., Nat. Commun. 14, 1288 (2023).
14. K. A. Gilmore, M. Affolter, R. J. Lewis-Swan, et al., Science 373, 673 (2021).
15. E. Gavartin, P. Verlot, and T. J. Kippenberg, Nat. Nanotechnol. 7, 509 (2012).
16. M. Tse et al., Phys. Rev. Lett. 123, 231107 (2019).
17. M. J. Yap et al., Nat. Photonics 14, 19 (2020).
18. J. B. Clark, F. Lecoq, R. W. Simmonds, et al., Nat. Phys. 12, 683 (2016).
19. E. E. Wollman, C. U. Lei, A. J. Weinstein, et al., Science 349, 952 (2015).
20. J.-M. Pirkkalainen, E. Damskägg, M. Brandt, et al., Phys. Rev. Lett. 115, 243601 (2015).
21. B.-B. Li, L. Ou, Y. Lei, and Y.-C. Liu, Nanophotonics 10, 2799 (2021).
22. N. S. Kampel, R. W. Peterson, R. Fischer, et al., Phys. Rev. X 7, 021008 (2017).
23. D. Mason, J. Chen, M. Rossi, et al., Nat. Phys. 15, 745 (2019).
24. H. Yu et al., Nat. (London) 583, 43 (2020).
25. M. Bhattacharya, H. Uys, and P. Meystre, Phys. Rev. A 77, 033819 (2008).
26. A. Nunnenkamp, K. Bankje, J. G. E. Harris, and S. M. Girvin, Phys. Rev. A 82, 021806 (2010).
27. A. M. Jayich, J. C. Sankey, B. M. Zwickl, et al., New J. Phys. 10, 095008 (2008).
28. H. Miao, S. Danilishin, T. Corbitt, and Y. Chen, Phys. Rev. Lett. 103, 100402 (2009).
29. L. Dellantonio, O. Kyrilenko, F. Marquardt, and A. S. Sørensen, Nat. Commun. 9, 3621 (2018).
30. B. D. Hauer, A. Metelmann, and J. P. Davis, Phys. Rev. A 98, 043804 (2018).
31. M. Ludwig, A. H. Safavi-Naeini, O. Painter, and F. Marquardt, Phys. Rev. Lett. 109, 063601 (2012).
32. A. A. Clerk, F. Marquardt, and J. G. E. Harris, Phys. Rev. Lett. 104, 213603 (2010).
72. D. Hälg et al., Phys. Rev. Appl. 15, L021001 (2021).
73. M. J. Bereyhi, A. Beccari, R. Groth, et al., Nat. Commun. 13, 3097 (2022).
74. X. Ma, J. J. Viennot, S. Kotler, et al., Nat. Phys. 17, 322 (2021).
75. R. Burgwal and E. Verhagen, Nat. Commun. 14, 1526 (2023).
76. G. I. Harris, D. L. McAuslan, T. M. Stace, et al., Phys. Rev. Lett. 111, 103603 (2013).
77. A. Beccari, D. A. Visani, S. A. Fedorov, et al., Nat. Phys. 18, 436 (2022).
78. Y. Zheng, L.-M. Zhou, Y. Dong, et al., Phys. Rev. Lett. 124, 223603 (2020).
79. C. F. Ockeloen-Korppi, E. Damskägg, J.-M. Pirkkalainen, et al., Phys. Rev. Lett. 118, 103601 (2017).
80. D. Gao, W. Ding, M. Nieto-Vesperinas, et al., Light Sci. Appl. 6, e17039 (2017).
81. F. Lecocq, J. B. Clark, R. W. Simmonds, et al., Phys. Rev. X 5, 041037 (2015).
82. A. H. Safavi-Naeini, S. Gröblacher, J. T. Hill, et al., Nat. (London) 500, 185 (2013).
83. I. D. Stoev, B. Seelbinder, E. Erben, et al., eLight 1, 1 (2021).
Supplemental Material for
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Sheng-Dian Zhang,1,* Jie Wang,1,* Qian Zhang,1,* Ya-Feng Jiao,1 Yun-Lan Zhu,1 
Şahin K. Özdemir,2 Cheng-Wei Qiu,3 Franco Nori,4,5 and Hui Jing1,†

1Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, 
Department of Physics and Synergetic Innovation Center for Quantum Effects 
and Applications, Hunan Normal University, Changsha 410081, China 
2Department of Engineering Science and Mechanics, and Materials Research Institute, 
Pennsylvania State University, University Park, State College, Pennsylvania 16802, USA 
3Department of Electrical and Computer Engineering, 
National University of Singapore, Singapore 117583, Singapore 
4Theoretical Quantum Physics Laboratory, RIKEN Cluster for Pioneering Research, Wako-shi, Saitama 351-0198, Japan 
5Physics Department, The University of Michigan, Ann Arbor, Michigan 48109-1040, USA 
* These authors contribute equally to this work 
† To whom correspondence should be addressed; E-mail: jinghui73@foxmail.com 
(Dated: August 6, 2024)

Here, we present more technical details on quantum-squeezing-enhanced quadratic optomechanical sensing, including: (1) detailed derivations of the linearized Hamiltonian; (2) the output noise spectrum; (3) degenerate optical parametric oscillations; (4) discussions on stability conditions; (5) signal-to-noise ratio and the optimal variance of the rotated field quadrature; (6) extended applications to the state-of-the-art quantum sensors.

S1. DERIVATION OF THE LINEARIZED HAMILTONIAN

In our cavity optomechanical (COM) system, the pump laser with frequency \( \omega_p \) has twice the frequency of the signal laser (\( \omega_s \)). Each laser tone (pump and signal) is quasi-resonant with a particular optical normal mode of the a Fabry-Pérot cavity, thus we refer to these optical modes as pump and signal mode, respectively [S1]. The flexible dielectric membrane is placed at a location of \( q_0 = j \lambda_p / 4 = k \lambda_s / 4 \) (j, k integers) [S2], i.e., the common node (or antinode) of the intracavity standing waves [S3], where \( \lambda_p \) and \( \lambda_s \) are the resonant wavelengths for the pump and signal modes, respectively. We then form a realistic description incorporating intrinsic losses and the coupling of the mechanical resonator to the optical modes, which yields the total Hamiltonian in a rotating frame [S4, S5]:

\[
\hat{H} = \hbar \Delta_p \hat{a}^{\dagger} \hat{a} + \hbar \Delta_p \hat{a}^{\dagger}_p \hat{a}_p + \frac{\hbar}{2} \Omega_m \left( \hat{q}_m^2 + \hat{p}_m^2 \right) - \hbar \eta_m \left( \hat{g}_0 \hat{a}^{\dagger} \hat{a} + \hat{g}_p \hat{a}^{\dagger}_p \hat{a}_p \right) \\
+ i \hbar \chi^{(2)} \left( \hat{a}^{\dagger} \hat{a}_p e^{i \Phi} - \hat{a}^{\dagger}_p e^{-i \Phi} \right) + i \hbar \left( \hat{E}_c \hat{a}^{\dagger} + \hat{E}_p \hat{a}^{\dagger}_p - \text{H.C.} \right),
\]

(S1)

where we wrote in a frame where the pump and signal modes phase space rotate at frequency \( \omega_p \) and \( \omega_s \), respectively, and the driving amplitudes are \( |\hat{E}_c| = \sqrt{\kappa_p \hat{P}_c / (\hbar \omega_c)} \), \( |\hat{E}_p| = \sqrt{\kappa_p \hat{P}_p / (\hbar \omega_p)} \). The detunings of the optical modes are \( \Delta_c = \Omega_c - \omega_s \), \( \Delta_p = \Omega_p - \omega_p \), with \( g_0 \) and \( g_p \) the COM coupling strength of the signal and pump modes, respectively.

Thus, the equations of motion can be given by

\[
\begin{align*}
\dot{\hat{a}} &= - \left( i \Delta_p + \frac{\kappa}{2} \right) \hat{a} + i g_0 \hat{a}^2 \hat{a} + 2 \chi^{(2)} \hat{a}^{\dagger} \hat{a}_p e^{i \Phi} + \hat{E}_c, \\
\dot{\hat{a}}_p &= - \left( i \Delta_p + \frac{\kappa}{2} \right) \hat{a}_p + i g_p \hat{a} \hat{a}_p^2 - \chi^{(2)} \hat{a}^2 e^{-i \Phi} + \hat{E}_p, \\
\dot{\hat{q}}_m &= \Omega_m \hat{p}_m, \\
\dot{\hat{p}}_m &= -\Omega_m \hat{q}_m - \Gamma_m \hat{p}_m + 2 g_0 \left( \hat{g}_0 \hat{a}^{\dagger} \hat{a} + \hat{g}_p \hat{a}^{\dagger}_p \hat{a}_p \right). 
\end{align*}
\]

(S2)

To proceed, we derive the classical equations for the steady-state values under the condition of strong optical driving

\[
\begin{align*}
- \left( i \Delta_p + \frac{\kappa}{2} \right) a + 2 \chi^{(2)} e^{i \Phi} a + |\hat{E}_c| e^{i \Phi} &= 0, \\
- \left( i \Delta_p + \frac{\kappa + \kappa_p}{2} \right) a_p - \chi^{(2)} e^{-i \Phi} a^2 + |\hat{E}_p| e^{i \Phi} &= 0, \\
-\Omega_m + 2 g_0 a^2 + 2 g_p a^2 p &= 0.
\end{align*}
\]

(S3)
where \( \Phi (\varphi) \) is the phase of the pump (signal) laser. Herein, we choose \( \xi = \frac{\kappa}{2} = \frac{\kappa}{2}, \delta_0 = \delta_0 = \frac{\delta_0}{2} \), then the steady-state solutions are

\[
2g_0\alpha_c \alpha_c = \frac{\Omega_m - 2g_0\alpha_p \alpha_p}{2g_0} = \frac{\Omega_m - 8g_0\alpha_p \alpha_p}{2g_0} = \frac{\Omega_m - 4\alpha_p \alpha_p}{2g_0} = \frac{\Omega_m - 4\alpha_p \alpha_p}{2g_0}.
\]

Thus,

\[
\alpha_p = \frac{\sqrt{\Delta^2 + \frac{k^2}{4}} - \sqrt{\Delta^2 + \frac{k^2}{4} - 16\chi^2(2) \left( |\varepsilon_p| - \frac{\chi^2(2)\Omega_m}{2g_0} \right)} / 8\chi(2),}
\]

\[
= \frac{\sqrt{\Delta^2 + \frac{k^2}{4}} - \sqrt{\Delta^2 + \frac{k^2}{4} - 16\chi^2(2) \left( |\varepsilon_p| - \frac{\chi^2(2)\Omega_m}{2g_0} \right)} / 8\chi(2),}
\]

\[
\frac{1}{8\chi(2)} \sqrt{\frac{k^2 + 4\Delta^2}{4}} - \frac{1}{8\chi(2)} \sqrt{\frac{k^2 + 4\Delta^2}{4} - 16\chi^2(2) \left( |\varepsilon_p| - \frac{\chi^2(2)\Omega_m}{2g_0} \right)} / 8\chi(2),}
\]

\[
= \frac{1}{16\chi(2)} \sqrt{k^2 + 4\Delta^2} - \frac{1}{16\chi(2)} \sqrt{\frac{k^2 + 4\Delta^2}{16}} - \frac{1}{16\chi(2)} \left( |\varepsilon_p| - \frac{\chi^2(2)\Omega_m}{2g_0} \right).}
\]
For the resonance case without intracavity squeezing, the additional term $\Sigma[Q]$ is negligible. The effective susceptibility can be written at the simplest level as

$$\chi_{\text{eff}}^{(2)}[Q] = -\frac{1}{m_{\text{eff}}(\Omega^2 + i\Gamma_m)}.$$  \hfill (S9)

Here, we consider the effect of the fluctuations of the pump mode. For a strong pump field, this mode can be eliminated adiabatically, which yields the shifts of the cavity linewidth, the COM coupling rate, and the mechanical resonance frequency:

$$\kappa_s^{\text{eff}} = \kappa_s + \frac{16\nu_s^2 n_s}{\kappa_p + 2\Delta_m}, \quad \Omega_m^{\text{eff}} = \frac{16G_0^2 \Delta}{\kappa_p + 4\Delta_m^2}, \quad G_s = G_s \pm \frac{4\nu_\alpha G_p e^{\pm i\theta}}{\kappa_p + 2i\Delta_m},$$  \hfill (S10)

where $\Delta = \Delta_c - g_0 q_m^2$, $\Delta' = \Delta_p - g_p q_m^2$, $G_s = g_0 q_m \sqrt{2\Gamma_m}$, and $G_p = g_p q_m \sqrt{2\Gamma_p}$. Thus, the optical losses are slightly modified by the pump mode due to the photon up-conversion [S1]. The additional COM coupling and the mechanical eigenfrequency indicate the contributions of the photon-phonon coupling for the pump mode [S1]. Then, the fluctuations of the pump mode can be neglected under a large detuning and a small second-order nonlinearity [S1].

### S2. THE OUTPUT QUADRATURES

After introducing phenomenologically the various dissipation mechanisms and associated input noise, the Hamiltonian yields readily the quantum Langevin equations

$$\dot{a} = -\left(\frac{i\Delta_c + \kappa}{2}\right) a + i g_0 \theta a^2 m + 2G e^{i\Phi} a^+, $$

$$+ \epsilon_c + \sqrt{\eta\kappa} f_{a,\text{in}}, + \sqrt{\nu_\alpha \kappa} f_{a,0}, $$

$$\dot{q}_m = \Omega_m \hat{p}_m, $$

$$\dot{p}_m = -\Omega_m \hat{q}_m - \Gamma_m \hat{p}_m + 2g_0 q_m a^+ a + \sqrt{2\Gamma_m} \hat{f}_{\text{in}},$$  \hfill (S11)

where $f_{a,\text{in}}$ and $f_{a,0}$ are the noise operators associated with the input cavity mirror and internal losses, respectively. The forces acting on the mechanical membrane are

$$\hat{F}_{\text{in}} = \hat{F}_{\text{th}} + \hat{F}_{\text{sig}},$$  \hfill (S12)

where $\hat{F}_{\text{th}}$ and $\hat{F}_{\text{sig}}$ are the scaled thermal force and the force signal to be detected, respectively, with dimension Hz$^{1/2}$, respectively. All noise operators have zero mean values

$$\langle f_{a,\text{in}} \rangle = \langle f_{a,0} \rangle = \langle \hat{F}_{\text{th}} \rangle = \langle \hat{F}_{\text{sig}} \rangle = 0.$$  \hfill (S13)

Because of the nonlinear COM interaction, Eqs. (S11) do not form a closed set of operator equations. We proceed by considering the situation of a strong driving, and expand each operator as the sum of its classical mean value and a small quantum fluctuation, i.e. $\hat{a} = a + \delta a$, $\hat{q}_m = q_m + \delta q_m$, and $\hat{p}_m = p_m + \delta p_m$, with $\langle \delta a \rangle = \langle \delta q_m \rangle = \langle \delta p_m \rangle = 0$. This yields the classical mean value equations of motion

$$\dot{a} = -\left(\frac{i\Delta_c + \kappa}{2}\right) a + 2G e^{i\Phi}, $$

$$\dot{q}_m = -\Omega_m \theta \dot{q}_m - \Gamma_m \dot{p}_m + 2g_0 q_m a^2, $$

$$\dot{p}_m = p_m,$$  \hfill (S14)

where the effective optical detuning is $\Delta = \Delta_c - g_0 q_m^2$, and $\Phi$ describes the phase of the driving field. Here we take intracavity field as the phase reference, i.e. $a \in \mathbb{R} > 0$, in which case the steady-state mean values become: $|a|^2 = \Omega_m/2g_0$, $p_m = 0$, and

$$|\delta q_m|^2 = \frac{1}{g_0} \left( \Delta_c - \frac{\epsilon_c}{a} \sin \Phi - 2G \sin \theta \right).$$  \hfill (S15)

We now introduce the ‘position’ and ‘momentum’-like operators of the optical field,

$$\hat{q} = \frac{1}{\sqrt{2}} \left( a^+ + a \right), $$

$$\hat{p} = \frac{i}{\sqrt{2}} \left( a^+ - a \right),$$  \hfill (S16)

and the associated optical noise operators

$$\hat{f}_{q,\text{in}} = \frac{1}{\sqrt{2}} \left( f_{a,\text{in}}^+ + f_{a,\text{in}} \right), \quad \hat{f}_{p,\text{in}} = \frac{i}{\sqrt{2}} \left( f_{a,\text{in}}^+ - f_{a,\text{in}} \right);$$

$$\hat{f}_{q,0} = \frac{1}{\sqrt{2}} \left( f_{a,0}^+ + f_{a,0} \right), \quad \hat{f}_{p,0} = \frac{i}{\sqrt{2}} \left( f_{a,0}^+ - f_{a,0} \right).$$  \hfill (S17)
In the Fourier domain expressions for the output quadratures:

\[ \dot{q}_{\text{out}} [\Omega] = \left( \kappa/2 + 2G \cos \theta - i\Omega \right)^{-1} \left[ \sqrt{\eta_c} \left( \Delta_c - 2G \sin \theta - G_0^2 \rho_0 \right) \right. \dot{f} + \left( \eta_c \kappa - 1 \right) \dot{f}_{q,\text{in}} + \kappa \sqrt{(1 - \eta_c)} \eta_c \dot{f}_{q,0} \] ,

\[ \dot{p}_{\text{out}} [\Omega] = \left( \kappa/2 + 2G \cos \theta - i\Omega \right)^{-1} \left[ -\sqrt{\eta_c} \left( \Delta_c - 2G \sin \theta - G_0^2 \rho_0 \right) \right. \dot{q} + \left( \eta_c \kappa - 1 \right) \dot{f}_{p,\text{in}} + \kappa \sqrt{(1 - \eta_c)} \eta_c \dot{f}_{p,0} \] .

(SI8)

The essential step in quantum sensing is to observe the output fluctuations of physical quantities to be measured in the Fourier domain, i.e.,

\[ \left( \begin{array}{c} \delta q_{\text{out}} \\ \delta p_{\text{out}} \end{array} \right) = \left( \begin{array}{cccc} A_- & B_- & C_- & D_- \\ -B_+ & A_+ & D_+ & C_+ \end{array} \right) \left( \begin{array}{c} \dot{f}_{q,\text{in}} \\ \dot{f}_{p,\text{in}} \\ \dot{f}_{q,0} \\ \dot{f}_{p,0} \end{array} \right) + \tilde{F}_{\text{in}} \right). 

(SI9)

where

\[ A_\pm [\Omega] = \rho \kappa (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} \eta_c - 1, \]
\[ B_\pm [\Omega] = \rho \kappa (\Delta - 2G \sin \theta - 4G^2 \chi_m)(\kappa/2 - 2G \cos \theta - i\Omega)^{-1} (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} \eta_c, \]
\[ C_\pm [\Omega] = \rho \kappa (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} \sqrt{(1 - \eta_c)} \eta_c, \]
\[ D_\pm [\Omega] = \rho \kappa (\Delta - 2G \sin \theta - 4G^2 \chi_m)(\kappa/2 - 2G \cos \theta - i\Omega)^{-1} (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} \sqrt{(1 - \eta_c)} \eta_c, \]
\[ N_\pm [\Omega] = 2\rho \kappa (\Delta - 2G \sin \theta)(\kappa/2 - 2G \cos \theta - i\Omega)^{-1} (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} \sqrt{2\kappa \eta_c \Gamma_m}, \]
\[ \rho = \left[ 1 + (\kappa/2 - 2G \cos \theta - i\Omega)^{-1} (\kappa/2 + 2G \cos \theta - i\Omega)^{-1} (\Delta + 2G \sin \theta) \right]^{-1}. \]

(SI20)

and \( \chi_m = -\Omega_m / (\Omega^2 + \Omega_m^2) \) is the mechanical susceptibility of the system, which quantifies the response of the oscillator to external forces.

For the case without intracavity squeezing (\( G = 0, \theta = 0 \)), the above coefficients related to the quadratic coupling are

\[ A_\pm [\Omega] = \rho \kappa (\kappa/2 - i\Omega)^{-1} \eta_c - 1, \]
\[ B_\pm [\Omega] = \rho \kappa (\Delta - 2G \sin \theta - 4G^2 \chi_m)(\kappa/2 - 2G \cos \theta - i\Omega)^{-2} \eta_c, \]
\[ C_\pm [\Omega] = \rho \kappa (\kappa/2 - 2G \cos \theta - i\Omega)^{-2} \eta_c, \]
\[ D_\pm [\Omega] = \rho \kappa (\Delta - 2G \sin \theta)(\kappa/2 - 2G \cos \theta - i\Omega)^{-2} \sqrt{(1 - \eta_c)} \eta_c, \]
\[ N_\pm [\Omega] = 2\rho \kappa (\Delta - 2G \sin \theta)(\kappa/2 - 2G \cos \theta - i\Omega)^{-2} \sqrt{\kappa \eta_c \Gamma_m}, \]
\[ \rho = \left[ 1 + (\kappa/2 - 2G \cos \theta - i\Omega)^{-2} (\Delta - 2G \sin \theta) \right]^2. \]

(SI21)

**S3. SECOND-ORDER NONLINEAR PROCESSES**

In the case of strong optical drives, the nonlinear gain coefficient is derived from the steady-state equations:

\[ G = \chi^{(2)} \left( \tau - \sqrt{\tau^2 + \Omega_m^2 / 8\rho_0} - \left| \xi_p \right| \right) / 4\chi^{(2)}, \]

(SI22)

where \( \tau = \sqrt{k^2 + 4\Delta^2} / (16\chi^2) \), \( \left| \xi_p \right| = \sqrt{\rho \eta_c P_p / (\kappa \omega_p)}, \) \( P_p \) quantifies the pump power for the \( \chi^{(2)} \) crystal. The nonlinear gain coefficient is enhanced with the increase of the power of the pump laser and the second-order nonlinearity [Fig. S1(a), left panel], following the characteristic optical parametric oscillation (OPO) power curves [S7]. However, the photons circulating in the cavity is reduced when increasing the detuning of the pump field, which results in the suppression of the nonlinear gain coefficient [Fig. S1(a), right panel]. Figure S1(b) schematically illustrates the \( \chi^{(2)} \) nonlinear process, where the OPO model can be treated as two coupled cavities with spontaneous parametric down-conversion [S7]. The visible pump laser at frequency \( \omega_p \), drives the \( \chi^{(2)} \) crystal, producing a pair of infrared signal and idler lights at frequencies \( \omega_s \) and \( \omega_i \), which satisfies the energy-matching condition \( \omega_p = \omega_s + \omega_i \). For degenerate OPOs (\( \omega_s = \omega_i = \omega_p / 2 \)), a single parametric oscillation is realized at half the frequency of the pump laser. Whereas for non-degenerate cases (\( \omega_s \neq \omega_i \)), the OPO process is operated at two distinct resonances centered about the pump.
S4. STABILITY CONDITIONS

The stability or instability of the system is determined by the signs of the real parts of the eigenvalues of the dynamical evolution matrix \( M \). To find the eigenvalues \( \lambda \), it is necessary to solve the characteristic equation \( \det(M - \lambda I) = 0 \), which is reduced to an algebraic equation of the 4th degree: 

\[
\lambda^4 + M_3\lambda^3 + M_2\lambda^2 + M_1\lambda + M_0 = 0.
\]

Applying the Routh-Hurwitz method, we obtain the necessary and sufficient conditions for the system stability:

\[
0 < M_3, \quad 0 < M_3M_2 - M_1, \\
0 < M_0, \quad 0 < M_3M_2M_1 - (M_1^2 + M_2^2M_0).
\]  
(S23)

These conditions allow to determine whether all the roots in the characteristic equation have negative real parts. Thus, we can use them to justify the system stability without solving the characteristic equation itself. Herein, we focus on the resonance case \( (\Delta \approx \Delta_c = 0) \), thereby the first three inequalities in Eq. (S23) yield the first two stability conditions: \( G/\kappa < 0.25, -\pi < \theta < 0 \). To proceed, by exploiting the last inequality in Eq. (S23), we formulate the stability criterion functions \( \Theta \) as

\[
\Theta = M_3M_2M_1 - (M_1^2 + M_2^2M_0).
\]  
(S24)

Then, the signs of \( \Theta \) provide the remaining stability requirements:

\[
\text{sgn}(\Theta) = \begin{cases} 1, & \text{implies stability,} \\ 0, & \text{otherwise, implies instability.} \end{cases}
\]  
(S25)

As shown in main text, the parameters used in our numerical calculations are chosen truly in the stable region. In particular, the required signal power can be derived from Eq. (S15):

\[
P_c = \frac{\hbar\Omega}{4\kappa\kappa_c} \left[ (\kappa - 4G)^2 + 8G\kappa(1 - \cos \theta) \right],
\]  
(S26)

which is tens of microwatts and can be attained with accessible experimental conditions [S8]. In principle, a system tends to be sensitive to external perturbations in the unstable region. Then, the sensitive region in the stable realm locates near the dividing line between stability and instability.

S5. SIGNAL-TO-NOISE RATIO AND THE OPTIMAL VARIANCE

The performance of the state-of-the-art sensors is commonly quantified by the signal-to-noise ratio (SNR). In our system, the spectral density and the SNR of the signal force are respectively estimated by [S10]

\[
S_{FF}^{\text{est}, \phi}[\Omega] = S_{FF}^{\text{sig}} + S_{FF}^{\phi}, \\
\text{SNR}[\Omega] = \frac{S_{FF}^{\text{est}, \phi}}{S_{FF}^{\text{est}} - S_{FF}^{\phi}}.
\]  
(S27)
The spectral density of the apparent force experienced by the oscillator is described as $\vec{S}_\text{app}$. As shown in Fig. S2(a), the SNR can reach 1.14 at the temperature of 0.2 K.

Figure S2(b)-(c) characterizes the variance of the generalized rotated field quadrature, which is given by [S11]

$$
V_{\phi} \{ \Omega \} = \int_{-\infty}^{\infty} \text{Re} \left\{ \frac{\tilde{S}_{\phi,\text{out}}}{2\pi} \right\} d\Omega
$$

(S28)

where the optical output spectrum is expressed as [S10]

$$
|S_{\phi,\text{out}}| = |S_{\phi}| \cos^2 \phi + |S_{\theta}| \sin^2 \phi + \text{Re} \left\{ \frac{\tilde{S}_{\phi,\text{out}}}{2\pi} \right\} \sin (2\phi).
$$

(S29)

Such a variance reaches its lowest value when choosing proper cooperativity and homodyne angle.

**S6. EXTENDED APPLICATIONS TO PRECISION MEASUREMENTS**

Table S1 provides a comparison of performance metrics for recently reported COM sensors including the force sensor described in this work.

| Sensors           | Experiment (Y/N) | Temperature (K) | Mean phonon occupations | Reported sensitivity | Equivalent force sensitivity | References |
|-------------------|------------------|-----------------|--------------------------|----------------------|-----------------------------|------------|
| Magnetometer      | Y                | 300             | $1.1 \times 10^6$        | $(400 \text{nT})^2\text{Hz}$ | $(2.4 \text{pN})^2\text{Hz}$ | [S12]      |
| Magnetometer      | Y                | 300             | $1.2 \times 10^6$        | $(200 \text{pT})^2\text{Hz}$ | $(1.2 \text{nN})^2\text{Hz}$ | [S13]      |
| Magnetometer      | Y                | 300             | $1.0 \times 10^6$        | $(5 \text{nT})^2\text{Hz}$ | $(0.75 \text{nN})^2\text{Hz}$ | [S14]      |
| Torque sensor     | Y                | $\sim 1 \text{mK}$ | $2.8$                    | $(1.3 \text{zN \text{m}})^2\text{Hz}$ | $(0.43 \text{fN})^2\text{Hz}$ | [S15]      |
| Ultrasound sensor | Y                | 300             | $1.3 \times 10^8$        | $(8 \text{\muPa})^2\text{Hz}$ | $(370 \text{fN})^2\text{Hz}$ | [S16]      |
| This work         |                  | 300             | $6.2 \times 10^6$        | $(10.2 \text{aN})^2\text{Hz}$ |                             |            |
|                   |                  | 10              | $2.1 \times 10^5$        | $(1.86 \text{aN})^2\text{Hz}$ |                             |            |
|                   |                  | 0.2             | $4.2 \times 10^3$        | $(0.26 \text{aN})^2\text{Hz}$ |                             |            |

**Table S1.** Extended applications to the state-of-the-art COM sensors. The resolution of the accelerometers is quantified by noise-equivalent acceleration in units of $g^2 \text{Hz}$, where $1 \text{g} = 9.81 \text{m/s}^2$.

**REFERENCES**

[S1] V. Peano, H. G. L. Schwefel, C. Marquardt, and F. Marquardt, “Intracavity Squeezing Can Enhance Quantum-Limited Optomechanical Position Detection through Deamplification,” Phys. Rev. Lett. 115, 243603 (2015).

[S2] M. Bhattacharya, H. Uys, and P. Meystre, “Optomechanical trapping and cooling of partially reflective mirrors,” Phys. Rev. A 77, 033819 (2008).

[S3] J. C. Sankey, C. Yang, B. M. Zwickl, A. M. Jayich, and J. G. E. Harris, “Strong and tunable nonlinear optomechanical coupling in a low-loss system,” Nat. Phys. 6, 707–712 (2010).

[S4] J.-Q. Liao and F. Nori, “Single-photon quadratic optomechanics,” Sci. Rep. 4, 6302 (2014).
[S5] W. Qin, A. Miranowicz, H. Jing, and F. Nori, “Generating Long-Lived Macroscopically Distinct Superposition States in Atomic Ensembles,” Phys. Rev. Lett. 127, 093602 (2021).

[S6] T. K. Paraíso, M. Kalaei, L. Zang, H. Pfeifer, F. Marquardt, and O. Painter, “Position-Squared Coupling in a Tunable Photonic Crystal Optomechanical Cavity,” Phys. Rev. X 5, 041024 (2015).

[S7] A. W. Bruch, X. Liu, J. B. Surya, C.-L. Zou, and H. X. Tang, “On-chip $\chi^{(2)}$ microring optical parametric oscillator,” Optica 6, 1361–1366 (2019).

[S8] X. Zhang, Q.-T. Cao, Z. Wang, Y.-x. Liu, C.-W. Qiu, L. Yang, Q. Gong, and Y.-F. Xiao, “Symmetry-breaking-induced nonlinear optics at a microcavity surface,” Nat. Photonics 13, 21–24 (2018).

[S9] David Hälg, Thomas Gisler, Yeghishe Tsaturyan, Letizia Catalini, Urs Grob, Marc-Dominik Krass, Martin Héritier, Hinrich Mattiat, Ann-Katrin Thamm, Romana Schirhagl, Eric C. Langman, Albert Schliesser, Christian L. Degen, and Alexander Eichler, “Membrane-based scanning force microscopy,” Phys. Rev. Appl. 15, L021001 (2021).

[S10] V. Sudhir, R. Schilling, S. A. Fedorov, H. Schutz, D. J. Wilson, and T. J. Kippenberg, “Quantum Correlations of Light from a Room-Temperature Mechanical Oscillator,” Phys. Rev. X 7, 031055 (2017).

[S11] C. Meng, G. A. Brawley, J. S. Bennett, M. R. Vanner, and W. P. Bowen, “Mechanical squeezing via fast continuous measurement,” Phys. Rev. Lett. 125, 042604 (2020).

[S12] S. Forstner, S. Prams, J. Knittel, E. D. Van Ooijen, J. D. Swaim, G. I. Harris, A. Szorkovszky, W. P. Bowen, and H. Rubinsztein-Dunlop, “Cavity optomechanical magnetometer,” Phys. Rev. Lett. 108, 120801 (2012).

[S13] S. Forstner, E. Sheridan, J. Knittel, C. L. Humphreys, G. A. Brawley, H. Rubinsztein-Dunlop, and W. P. Bowen, “Ultrasensitive optomechanical magnetometry,” Adv. Mater. 26, 6348–6353 (2014).

[S14] B.-B. Li, J. Bílek, U. B. Hoff, L. S. Madsen, S. Forstner, V. Prakash, C. Schäfermeier, T. Gehring, W. P. Bowen, and U. L. Andersen, “Quantum enhanced optomechanical magnetometry,” Optica 5, 850–856 (2018).

[S15] M. Wu, A. C. Hryciw, C. Healey, D. P. Lake, H. Jayakumar, M. R. Freeman, J. P. Davis, and P. E. Barclay, “Dissipative and Dispersive Optomechanics in a Nanocavity Torque Sensor,” Phys. Rev. X 4, 021052 (2014).

[S16] S. Basiri-Esfahani, A. Armin, S. Forstner, and W. P. Bowen, “Precision ultrasound sensing on a chip,” Nat. Commun. 10, 132 (2019)