Exact solution of the nonlinear laser passive mode locking transition

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(Dated: November 20, 2018)

We present the first statistical mechanics study of a passively mode locked laser which includes all the main physical processes, saturable absorption, Kerr nonlinearity, parabolic gain filtering and group velocity dispersion, assuming the soliton condition. We achieve an exact solution in the thermodynamic limit, where the ratio of the cavity length to the pulse width, the duty cycle, tends to infinity. The thermodynamics depends on a single dimensionless parameter $\gamma$, the ratio of the correlation length to the pulse width. The phase diagram consists of one ordered, mode-locked phase and one disordered, continuous wave phase, separated by a first order phase transition at $\gamma = 9$. The model belongs to a new class of solvable statistical mechanics models with a non-trivial phase diagram. The results are obtained with a fully controlled transfer matrix calculation, showing rigorously that passive mode locking is a thermodynamic phase transition.

PACS numbers: 42.55.Ah, 42.65.-k, 05.70.Fh

Introduction  Formation of ultrashort pulses in lasers by passive mode locking is an important branch of optics, both from the point of view of basic research and of practical applications. As such, it has been the subject of many theoretical and experimental studies over several decades, see [1] for a review. Nevertheless, until recently the central question of the threshold power needed to trigger passive mode locking was standing without a satisfactory answer.

In several recent works [2, 3, 4] considerable progress was made toward resolving this issue by including the effect of noise in a nonperturbative manner. The addition of a random element into the cavity electrodynamics turns the laser into a statistical physics system. Applying methods of equilibrium statistical mechanics, it has been shown that sufficiently strong noise destabilizes the pulses formed by a saturable absorber, and that the process of pulse formation is a first order phase transition.

However, these results were all obtained in the context of models of the laser dynamics which assumed a simple form of spectral gain filtering. Here, for the first time, we tackle the full problem of passively mode-locked laser with a fast saturable absorber, quadratic gain filtering, slow saturable gain, group velocity dispersion, and Kerr nonlinearity, assuming the soliton condition.

In the first step of the theoretical analysis we construct an exact mapping from the statistical steady state of the laser electric field to the thermal fluctuations of a string with an unstable self-interaction. We show that the thermodynamic limit is obtained when the laser cavity is much longer than one of the natural length scales, the pulse width or the correlation length. It is next shown that thermodynamics is determined solely by the dimensionless ratio $\gamma$ of the two natural length scales. We proceed to calculate exactly and explicitly the free energy, first by physical, mean field-like arguments, which are then established by a rigorous transfer matrix calculation. The free energy enables us to calculate all thermodynamic quantities, including pulse power as a function of $\gamma$, and the thermodynamic phase diagram, which consists of one ordered, mode locked phase for $\gamma > 9$, and one disordered, non-mode locked phase for $\gamma < 9$. Pulse formation is a first order phase transition. The pulsed configuration is metastable for $8 < \gamma < 9$, while the continuous wave configuration is metastable for all $\gamma > 9$.

The model closely resembles equations which have been used extensively to study kinetics of phase transitions and critical phenomena [5]. However, the opposite sign of the nonlinearity leads to markedly different phenomenology. The passive mode locking equation therefore belongs to a new class of statistical physics models.

Thermodynamics of passively mode locked lasers. Our starting point is the master equation which governs the slow dynamics of the complex envelope of the electric field $\psi(x,t)$ in a cavity of length $L$ [1, 2]

$$
\partial_z \psi = \left( \gamma_g + i \gamma_d \right) \partial^2_x \psi + \left( \gamma_s + i \gamma_k \right) |\psi|^2 \psi + g \psi + \eta .
$$

in $0 \leq z \leq L$ with periodic boundary condition, where the real constants $\gamma_g > 0$, $\gamma_d$, $\gamma_s > 0$, $\gamma_k$ are the coefficients of spectral filtering, group velocity dispersion, (fast) saturable absorption, and Kerr nonlinearity, respectively. Noise of spontaneous emission and other sources is modelled by the random term $\eta$, which is a (complex) Gaussian process with covariance $\langle \eta^*(x,t) \eta(x',t') \rangle = 2 T L \delta(x-x') \delta(t-t')$. Finally, as shown in [2], the slow saturable gain $g$, may be chosen, without significant loss of generality, such that it sets the total intracavity power $||\psi||^2 = \frac{1}{2} \int_0^L dx |\psi(x)|^2$ to a fixed value $P$. $g$ becomes then a Lagrange multiplier for the fixed power constraint.

In this Letter we consider Eq. (1) in the special but important case that $\gamma_s + i \gamma_k$ is a real multiple of $\gamma_g + i \gamma_d$, known as the soliton condition. In this case one can define a ‘Hamiltonian’ functional [2]

$$
H[\psi] = \int_0^L dx \left( - \frac{1}{2} \gamma_s |\psi(x)|^4 + \gamma_g |\psi'(x)|^2 \right) ,
$$

where $\gamma_s$, $\gamma_g$, $\gamma_d$, $\gamma_k$ are the coefficients of spectral filtering, group velocity dispersion, (fast) saturable absorption, and Kerr nonlinearity, respectively. Noise of spontaneous emission and other sources is modelled by the random term $\eta$, which is a (complex) Gaussian process with covariance $\langle \eta^*(x,t) \eta(x',t') \rangle = 2 T L \delta(x-x') \delta(t-t')$. Finally, as shown in [2], the slow saturable gain $g$, may be chosen, without significant loss of generality, such that it sets the total intracavity power $||\psi||^2 = \frac{1}{2} \int_0^L dx |\psi(x)|^2$ to a fixed value $P$. $g$ becomes then a Lagrange multiplier for the fixed power constraint.
such that the invariant measure \( \rho[\psi] \) of equation (1) is

\[
\rho[\psi] = Z^{-1} e^{-H[\psi]/(LT)} \delta(P - \|\psi\|^2),
\]

where the power constraint is enforced explicitly, and that \( \rho \) is independent of the imaginary terms in Eq. (1). The study of steady state properties of Eq. (1) is now reduced, as in [2, 3], to the that of an equilibrium statistical mechanics system with partition function \( Z \).

The Hamiltonian functional \( H \) is almost identical to the critical Ginzburg-Landau (GL) functional, the paradigm for the effective description of continuous phase transition [4]. In Eq. (2), however, in contrast with the GL functional, the coefficient of the quartic term is negative, making the null configuration unstable. The instability is countered by the power constraint, which acts nonlocally, and the thermodynamics is consequently radically different from the GL one. In particular, the one-dimensional system exhibits an ordering transition, which is impossible in the one-dimensional GL model.

The character of the steady state distribution \( \rho \) is determined by the strength of the ordering saturable absorber relative to the disordering noise. The system has two natural length scales: The pulse width \( L_p = \frac{\gamma_s}{\gamma PL} \) measures the effect of saturable absorption, while the correlation length \( L_c = \frac{\gamma_s L}{T} \) measures the effect of noise. In both cases, the effect is stronger the smaller is the associated length scale. Thus, \( L_p \) is smaller the larger is \( \gamma_s \) and is independent of \( T \), while the converse is true for \( L_c \). The third length scale in the system, the cavity length \( L \), is typically much larger than \( L_p \) and \( L_c \) in multimode lasers; we therefore study the statistical problem in the limit \( L \gg L_p, L_c \), which serves as the thermodynamic limit, neglecting small corrections of \( O(L_p/L) \) or \( O(L_c/L) \). Thermodynamic quantities, and in particular the mode locking threshold are therefore determined by the sole dimensionless parameter, the ratio \( \gamma = 4L_c/L_p = \gamma_s P^2/T \). Note that thermodynamics is independent of the spectral filtering \( \gamma_g \).

Our analysis proceeds in the textbook approach of calculating the free energy \( F = -\log Z \) of the statistical mechanics problem Eqs. (2, 3), from which other thermodynamic quantities follow. However, seeking to calculate the partition function we run into a common obstacle, namely that the functional integral in Eq. (3) is not well defined. Mathematically this is not a serious problem, since the invariant measure is well-defined [5], but in order to use \( Z \) and \( F \) we need to give a precise meaning to the functional integral, as the continuum limit of a regularized version where the integration is finite-dimensional. Given a regularization scheme with \( N \psi \) integration, we define the regularized partition function

\[
Z_N = \int \prod_{n=1}^{N} (\frac{\partial \rho_N[\psi_n] e^{-\frac{H_N[\psi_n]}{T}}}{\partial \psi_n}) \delta(P - \|\psi\|^2) \quad (5)
\]

where \( H_N \) and \( \| \cdot \|_N \) are regularized versions of \( H \) and \( \| \cdot \| \). The integration measure is multiplied by factors of \( \frac{1}{\pi} \) to make \( Z_N \) dimensionless, and by factors of \( a_N \), a regularization scheme- and \( N \)-dependent dimensionless constant, to make \( Z = \lim_{N \to \infty} Z_N \) finite. The limit is independent of the regularization scheme up to an unimportant multiplicative constant.

The rest of this Letter is devoted to the calculation of \( F \), Eq. (5), and from it the phase diagram and thermodynamic quantities, which all have simple algebraic expressions. It is shown that when \( \gamma > 9 \) the equilibrium is an ordered, mode-locked phase where the power \( P \) is divided between a single pulse and continuum fluctuations, see the top panel of Fig. 1 when \( \gamma < 9 \) the equilibrium is a disordered phase for, where the electric field consists only of spatially homogeneous fluctuations, see the bottom panel of Fig. 1.

**Mean field calculation of the free energy** As a preliminary step towards the calculation of \( F \) we examine the problem in two solvable limits. In the first, \( T \to 0 \), the evolution equation (1) approaches the noiseless dynamics without the random term \( \eta \). Then, as is well known, \( \psi \) reaches an equilibrium in the form of a soliton-like pulse \( \psi_p(z) = e^{i \phi} \sqrt{\frac{T}{2L}} \text{sech} \left( \frac{z - z_0}{2a_p} \right) \), with two initial conditions-dependent real parameters, the pulse position \( 0 \leq \phi < 2\pi \), and phase \( 0 \leq z_0 < L \) and phase \( 0 \leq \phi < 2\pi \).

In our context, \( T \to 0 \) is the validity condition for the Laplace method, otherwise known as the saddle point approximation, in Eq (4), and the solitons \( \psi_p \) take the role...
of the saddle points. It follows that $Z \sim e^{-H[\psi]/(LT)}$, so that the free energy when $L_c \gg L_p$ or $\gamma \gg 1$ is

$$F_p = \frac{H[\psi]}{LT} = -\frac{2}{\gamma^2} \frac{L^2 P^3}{4\gamma^2 T} = -\frac{LL_c}{4L_p^2}. \quad (6)$$

In the second solvable limit $\gamma_s \to 0$, $H$ becomes quadratic and, after replacing the power constraint delta function by its Fourier integral representation, the functional integral in Eq. (4) becomes gaussian. The quadratic form is then diagonalized using the Fourier representations of $\psi$, and we calculate $Z$ in a regularization scheme where $N$ is the number of $\psi$ Fourier modes kept (see Eq. (4)). The result of the gaussian integration is

$$Z_c = \int_{-\infty}^{\infty} \frac{dw}{2\pi} e^z \prod_{n=-\infty}^{\infty} \left(1 + \frac{Lw}{L_c(2\pi n)^2}\right)^{-1}. \quad (7)$$

Under the assumption that $L \gg L_c$, we can use the standard approximation methods of Euler-MacLaurin summation and saddle point integration to get the free energy in the limit $L_c \ll L_p$ or $\gamma \ll 1$

$$F_c = \frac{L}{4L_c} = \frac{L^2 T}{4\gamma P}. \quad (8)$$

Although the preceding expressions Eqs (6,8) for the free energy were obtained in limiting cases, we now argue that $F_p$, the pulse free energy, and $F_c$, the continuum free energy, may combined into an expression for $F$ valid for every $\gamma$. The argument is based on the following assumption: Configurations $\psi$ which contribute significantly to $Z$ are such that $\psi(z) = O(1)$ for most $z$, with possibly few narrow regions where $\psi(z) = O(\sqrt{L/L_p})$, whose total width is $O(L_p)$. Let $yP$, $0 \leq y \leq 1$ be the total power concentrated in regions where $\psi$ is large. For $z$ values where $\psi$ is small the nonlinear term in $H$ is negligible, and the existence of regions of large $\psi$ affects the statistics of the small $\psi$ region only in that the total available power for fluctuations is $(1 - y)P$ rather than $P$. The regions of small $\psi$ therefore contribute $F_c|_{P \to (1-y)P}$ to the total free energy. Similarly, the regions of large $\psi$ are so narrow that noise induced fluctuations make negligible contribution to the free energy in them, so that the large $\psi$ regions contribute $H[\psi]/(LT)$ to $F$. By the principle that $F$ is minimized by the Gibbs distribution the large $\psi$ regions should be such that $H[\psi]$ is minimized, that is, $\psi$ will assume a soliton-like shape with total power $yP$, and contribute $F_p|_{P \to yP}$ to $F$. We claim that since the small parameter in these arguments is $L_p/L_c$, they become exact in thermodynamic limit.

The total free energy is the sum of the pulse and continuum parts, minimized over values of $y$,

$$F = \min_y \left( -\frac{2}{\gamma^2} \frac{L^2 (yP)^3}{4\gamma^2 T} + \frac{L^2 T}{4\gamma P(1 - y)} \right)$$

$$= \min_y \frac{L}{L_p} \left( -\frac{\gamma y^3}{12} + \frac{1}{\gamma(1 - y)} \right). \quad (9)$$

$F$ can be used to easily derive other thermodynamic quantities. An order parameter which is nonzero if and only if mode locking occurs is $M = \frac{L^2}{\gamma^2T} \sqrt{\langle |\psi|^4 \rangle}$. $M$ has dimensions of power, and is proportional to the experimentally measurable RF power $\Gamma$. It follows from Eq. (9) and the definition of $F$ that $\langle |\psi|^4 \rangle = -2T \partial \gamma \Phi$. Letting $\Phi(\gamma, y)$ denote the target function in Eq. (9), and $\tilde{y}(\gamma)$ the minimizer, we calculate

$$M(\gamma) = -2T(L_p/L) \sqrt{\partial_\gamma \Phi(\gamma, \tilde{y}(\gamma))} = (\tilde{y}^3/3) \gamma P. \quad (10)$$

The most important consequence of equation (10) is that mode locking occurs whenever the minimizer $\tilde{y}(\gamma)$ is greater than zero. $\Phi(\gamma, \cdot)$ has the following behavior: For $\gamma \leq 8$ it has a single minimum at 0, while for $\gamma > 8$ there is a second minimum at $1/2(1 + \sqrt{1 - 8/\gamma})$, which becomes a global minimum when $\gamma > 9$, see Fig. 2. Therefore, as $\gamma$ is increased through $\gamma = 9$, $M$ jumps discontinuously from 0 to $\sqrt{\gamma} P$; that is, the transition is first order, the ordered phase is metastable when $8 < \gamma < 9$, and the disordered phase is metastable for all $\gamma > 9$. The pulse power in the mode locked phase is $\sqrt{\gamma} P$, as can be verified by calculating higher moment.

**Transfer matrix calculation of the free energy**

We turn to the transfer matrix calculation of $Z$ which establishes our expression (9) for the free energy with a controlled derivation. For this purpose we consider a slightly generalized partition function $Z$ with fixed values of $\psi$ at the endpoints of the interval, $\psi(0) = \psi_1$, $\psi(L) = \psi_f$, and generalized free energy $F = -\log Z$. In this calculation it is more convenient to use system parameters that do not depend on the system size, so we define

![Image of graph](image-url)
Like $Z$, $Z$ needs to be defined using a limiting procedure with a properly scaled functional measure. Let $Z_\delta$ denote the lattice regularization of $Z$ with lattice spacing $\delta$. It satisfies the identity
\begin{equation}
Z_\delta(\psi, \psi_f, \alpha, \beta, P, L) = \int \frac{\beta}{\delta^4} d\psi d\psi^* Z_\delta(\psi, \psi_f, \alpha, \beta, P - \alpha |\psi|^2, L - \delta) e^{\frac{\alpha}{2}\|\psi\|^4 - \frac{\beta}{4}|\psi_f - \psi|^2} \, (12)
\end{equation}
We next expand the right-hand-side in powers of $\delta$ and $\psi - \psi_f$ and perform the gaussian integration. This gives an equation involving $Z_\delta$ and its derivatives. Taking then the continuum limit $\delta \to 0$ gives the equation for $F$
\begin{equation}
1/\beta \frac{\partial F}{\partial \psi} + 1/2 \alpha |\psi_f|^2 + 1/4 \alpha |\psi|^4 = 0 \, , \quad (13)
\end{equation}
and from it the equation for $F$
\begin{equation}
\frac{1}{\beta} \frac{\partial F}{\partial \psi} (\partial_F)^2 - |\psi_f|^2 \frac{\partial F}{\partial P} + 1/2 \alpha |\psi_f|^4 = 0 \, , \quad (14)
\end{equation}
where the term with double $\psi_f$ derivative and the term with an $L$ derivative were dropped, since they are $O(L_p/L)$ with respect to the terms retained.

Mean field arguments, similar to those presented above to obtain $F$, can be used to calculate $F$. Like $F$, $F$ is the sum of a continuum contribution $F_c$ where the non-linear term is unimportant, and a pulse term $F_p$ where noise is unimportant. Since the continuum fluctuations are a bulk property, $F_c$ is independent of the boundary conditions to leading order, and in this order it is equal to $F_c$, which in the present parametrization reads
\begin{equation}
F_c = \frac{L^2}{4\beta P (1 - y)} \, ; \quad (15)
\end{equation}
as before, $(1 - y)$ is the relative continuum power.

Like $F_p$, $F_p$ is the negative of the maximal value of the exponent in Eq. $[11]$, subject to a given total power $yP$ and the boundary conditions. However, the fixed boundary conditions break the translation invariance, and the maximization is achieved when pulses are created near the boundaries. We use standard variational methods to obtain the result: $F_p$ is the minimum of four candidate function distinguished by the configuration of the boundary pulses labelled by the four possible sign choices in $F_p = \frac{2\sqrt{\lambda}}{\alpha} \left( 2\lambda^{\frac{1}{2}} \pm (\lambda - \frac{\alpha}{2} |\psi|^2)^{\frac{1}{2}} \mp (\lambda - \frac{\alpha}{2} |\psi_f|^2)^{\frac{1}{2}} \right) - \lambda yP \, ; \quad (16)$

In the first ambiguous sign the upper (lower) choice refers to the possibility that a pulse maximum lies at $z > 0$ ($z < 0$). The second sign choice refers similarly to the position of a pulse maximum relative to the $z = L$ boundary. $\lambda$

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