STABLE RATIONAL HOMOLOGY OF THE IA-AUTOMORPHISM GROUPS OF FREE GROUPS

MAI KATADA

Abstract. The rational homology of the IA-automorphism group $IA_n$ of the free group $F_n$ is still mysterious. We study the quotient of the rational homology of $IA_n$ that is obtained as the image of the map induced by the abelianization map, which we call the Albanese homology of $IA_n$. We obtain a representation-stable $GL(n, \mathbb{Q})$-subquotient of the Albanese homology of $IA_n$, which conjecturally coincides with the entire Albanese homology of $IA_n$. In particular, we obtain a lower bound of the dimension of the Albanese homology of $IA_n$ for each homological degree in a stable range. Moreover, we determine the entire third Albanese homology of $IA_n$ for $n \geq 9$. We also study the Albanese homology of an analogue of $IA_n$ to the outer automorphism group of $F_n$ and the Albanese homology of the Torelli groups of surfaces. Moreover, we study the relation between the Albanese homology of $IA_n$ and the cohomology of $Aut(F_n)$ with twisted coefficients.

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Appendix A. Properties of Albanese homology and cohomology

References

1. Introduction

The IA-automorphism group $IA_n$ of the free group $F_n$ of rank $n$ is the kernel of the canonical surjective map from the automorphism group $\text{Aut}(F_n)$ of $F_n$ to the general linear group $GL(n, \mathbb{Z})$. Magnus [29] discovered a finite set of generators for $IA_n$. Cohen-Pakianathan [8], Farb [13] and Kawazumi [20] independently determined the first homology group $H_1(IA_n, \mathbb{Z})$. They proved that the Johnson homomorphism for $\text{Aut}(F_n)$ induces an isomorphism

$$\tau : H_1(IA_n, \mathbb{Z}) \xrightarrow{\cong} \text{Hom}(H_2, \bigwedge^2 H_2),$$

where $H_2 = H_1(F_n, \mathbb{Z})$. The adjoint action of $\text{Aut}(F_n)$ on $IA_n$ induces an action of $GL(n, \mathbb{Z})$ on $H_*(IA_n, \mathbb{Z})$, and the Johnson homomorphism preserves the $GL(n, \mathbb{Z})$-action.

Krstić–McCool [24] showed that $IA_3$ is not finitely presentable and after that, Bestvina–Bux–Margalit [2] showed that $H_2(IA_3, \mathbb{Z})$ has infinite rank. However, it is still open whether or not $IA_n$ is finitely presentable for $n \geq 4$. About the second cohomology of $IA_n$, Pettet [34] determined the $GL(n, \mathbb{Z})$-subrepresentation of $H^2(IA_n, \mathbb{Q})$ that is obtained by using the Johnson homomorphism. This subrepresentation is regarded as the second Albanese cohomology of $IA_n$, which we will explain below. Day–Putman [10] obtained a finite set of generators of $H_2(IA_n, \mathbb{Z})$ as $GL(n, \mathbb{Z})$-representations. For $n = 3$, Satoh [40] detected a nontrivial irreducible subrepresentation of $H^2(IA_3, \mathbb{Q})$, which can not be detected by the Johnson homomorphism. However, the second homology of $IA_n$ has not been completely determined. It is more difficult to determine higher degree homology of $IA_n$.

For a group $G$, we consider the following quotient group of the rational homology of $G$, which is predual to what Church–Ellenberg–Farb [4] called the Albanese cohomology. The canonical surjection $\pi : G \twoheadrightarrow H_1(G, \mathbb{Z})$ induces a group homomorphism $\pi_* : H_1(G, \mathbb{Q}) \to H_1(H_1(G, \mathbb{Z}), \mathbb{Q})$ on homology. Define the Albanese homology of $G$ as

$$H^A_1(G, \mathbb{Q}) := \text{im}(\pi_* : H_1(G, \mathbb{Q}) \to H_1(H_1(G, \mathbb{Z}), \mathbb{Q})).$$

Since we have $H_1(H_1(G, \mathbb{Z}), \mathbb{Q}) \cong \bigwedge^i H_1(G, \mathbb{Q})$, where $\bigwedge^i H_1(G, \mathbb{Q})$ is the $i$-th exterior power of $H_1(G, \mathbb{Q})$, it is easier to determine the Albanese homology $H^A_i(G, \mathbb{Q})$ than the ordinary homology $H_i(G, \mathbb{Q})$.

Let $H = H_1(F_n, \mathbb{Q})$ and $U = \text{Hom}(H, \bigwedge^2 H)$. Then the Johnson homomorphism induces a $GL(n, \mathbb{Z})$-homomorphism

$$\tau_* : H_1(IA_n, \mathbb{Q}) \to H_1(U, \mathbb{Q}).$$

Therefore, we have $GL(n, \mathbb{Z})$-homomorphisms

$$H_1(IA_n, \mathbb{Q}) \to H^A_1(IA_n, \mathbb{Q}) \hookrightarrow H_1(U, \mathbb{Q}) \cong \bigwedge^i U.$$
Since $H_i(U, \mathbb{Q})$ is an algebraic $GL(n, \mathbb{Q})$-representation, the $GL(n, \mathbb{Z})$-representation structure on $H_i^A(\text{IA}_n, \mathbb{Q})$ extends to an algebraic $GL(n, \mathbb{Q})$-representation structure. In particular, since $H_i^A(\text{IA}_n, \mathbb{Q})$ is completely reducible, any subrepresentations and quotient representations of $H_i^A(\text{IA}_n, \mathbb{Q})$ can be considered as direct summands. As $GL(n, \mathbb{Q})$-representations, $H_i^A(\text{IA}_n, \mathbb{Q})$ is determined by $[1.0.1]$ and $H_2^A(\text{IA}_n, \mathbb{Q})$ is determined by Pettet [34]. To the best of our knowledge, the Albanese homology of $\text{IA}_n$ of degree greater than 2 is not known other than non-triviality [9] and upper bounds of dimensions [4], which we will explain later.

Church–Farb [6] introduced the notion of representation stability for a sequence

$$V_0 \overset{\phi_0}{\longrightarrow} V_1 \overset{\phi_1}{\longrightarrow} V_2 \rightarrow \cdots \overset{\phi_n}{\longrightarrow} V_{n+1} \rightarrow \cdots$$

of algebraic $GL(n, \mathbb{Q})$-representations, where $\phi_n : V_n \to V_{n+1}$ is a $GL(n, \mathbb{Q})$-homomorphism considering $V_{n+1}$ as a $GL(n, \mathbb{Q})$-representation via the canonical inclusion map $GL(n, \mathbb{Q}) \hookrightarrow GL(n+1, \mathbb{Q})$. Note that irreducible algebraic $GL(n, \mathbb{Q})$-representations are classified by bipartitions with at most $n$ parts, which are pairs of partitions with total length at most $n$ (see Section 2.3 for details). For a bipartition $\lambda$, let $V_{\lambda}(n)$ denote the irreducible $GL(n, \mathbb{Q})$-representation corresponding to $\lambda$. A sequence $\{V_n\}$ of algebraic $GL(n, \mathbb{Q})$-representations is representation stable if $\{V_n\}$ satisfies the following three conditions for sufficiently large $n$:

- $\phi_n : V_n \to V_{n+1}$ is injective,
- $\text{im} \phi_n$ spans $V_{n+1}$ as a $GL(n + 1, \mathbb{Q})$-representation,
- for each bipartition $\lambda$, the multiplicity of $V_{\lambda}(n)$ in $V_n$ is constant.

For example, $H_i(U, \mathbb{Q}) \cong \bigwedge^i U$ is representation stable for each $i \geq 0$.

We consider an analogue of $\text{IA}_n$ to the outer automorphism group $\text{Out}(F_n)$ of $F_n$. Let $\text{IO}_n$ denote the kernel of the canonical surjective map from $\text{Out}(F_n)$ to $GL(n, \mathbb{Z})$. Kawazumi [20] determined $H_i^A(\text{IO}_n, \mathbb{Q})$, and Pettet [34] determined $H_2^A(\text{IO}_n, \mathbb{Q})$ as $GL(n, \mathbb{Q})$-representations. However, as in the case of $\text{IA}_n$, the Albanese homology of $\text{IO}_n$ of degree greater than 2 is not known. Moreover, even non-triviality of the homology of $\text{IO}_n$ of degree greater than 2 does not seem to be known.

The aim of this paper is to determine the Albanese homology $H_i^A(\text{IA}_n, \mathbb{Q})$ and $H_i^A(\text{IO}_n, \mathbb{Q})$ as $GL(n, \mathbb{Q})$-representations. We use abelian cycles in $H_i(\text{IA}_n, \mathbb{Q})$, which are induced by $i$-tuples of mutually commuting elements of $\text{IA}_n$. Then we obtain a representation-stable $GL(n, \mathbb{Q})$-subrepresentation of $H_i^A(\text{IA}_n, \mathbb{Q})$, which conjecturally coincides with the entire $H_i^A(\text{IA}_n, \mathbb{Q})$. In particular, we obtain a lower bound of the dimension of $H_i^A(\text{IA}_n, \mathbb{Q})$ for sufficiently large $n$ with respect to $i$. We also obtain non-triviality and a lower bound of the dimension of $H_i^A(\text{IO}_n, \mathbb{Q})$ for sufficiently large $n$ with respect to $i$. Moreover, we determine $H_3^A(\text{IA}_n, \mathbb{Q})$ and $H_3^A(\text{IO}_n, \mathbb{Q})$ for $n \geq 9$. We also study the relation between the Albanese homology of $\text{IA}_n$ and the cohomology of $\text{Aut}(F_n)$ with twisted coefficients.

1.1. Non-triviality of $H_i^A(\text{IA}_n, \mathbb{Q})$. Let $i \geq 1$. First, we observe that $H_i^A(\text{IA}_n, \mathbb{Q})$ is non-trivial for $n \geq i + 1$. We detect a non-trivial representation-stable quotient of $H_i^A(\text{IA}_n, \mathbb{Q})$. 
For \( g \geq 1 \), \( \text{IA}_{2g} \) includes the Torelli group \( \mathbb{I}_{g,1} \) of a connected oriented surface of genus \( g \) with one boundary component. Church–Farb \([5]\) used an \( \text{Sp}(2g,\mathbb{Z}) \)-homomorphism

\[
\tau_i : H_i(\mathbb{I}_{g,1}, \mathbb{Q}) \to \bigwedge^{i+2} H,
\]

which was introduced by Johnson \([18]\), to detect a non-trivial representation-stable subrepresentation in the image of \( \tau_i \). The following theorem can be regarded as an analogue of this result of Church–Farb.

**Theorem 1.1** (Theorem 4.2). For \( n \geq i + 1 \), we have a surjective \( \text{GL}(n, \mathbb{Q}) \)-homomorphism

\[
H^A_i(\text{IA}_n, \mathbb{Q}) \to \text{Hom}(H, \bigwedge^{i+1} H) \cong \left( \bigwedge^{i+1} H \right) \otimes H^*,
\]

where \( H^* = \text{Hom}_\mathbb{Q}(H, \mathbb{Q}) \).

Cohen–Heap–Pettet \([9]\) obtained non-trivial subspaces of \( H^A_i(\text{IA}_n, \mathbb{Q}) \) whose dimensions are bounded above by a polynomial in \( n \) of degree \( 2i \). However, they did not consider the \( \text{GL}(n, \mathbb{Q}) \)-action on \( H^A_i(\text{IA}_n, \mathbb{Q}) \), and their non-trivial subspaces are not \( \text{GL}(n, \mathbb{Q}) \)-subrepresentations. Theorem 1.1 gives another proof of non-triviality of \( H^A_i(\text{IA}_n, \mathbb{Q}) \).

**Corollary 1.2** (Cohen–Heap–Pettet \([9]\)). For \( n \geq i + 1 \), \( H^A_i(\text{IA}_n, \mathbb{Q}) \) is non-trivial.

To observe the representation stability of the subrepresentation of \( H^A_i(\text{IA}_n, \mathbb{Q}) \) that is detected in Theorem 1.1 we write \( H \) as \( H_n \) and \( \text{Hom}_\mathbb{Q}(H, \mathbb{Q}) \) as \( H^* \). The abelian cycles and the \( \text{GL}(n, \mathbb{Q}) \)-homomorphisms that we use to prove Theorem 1.1 are compatible with the canonical inclusion maps

\[
\text{IA}_n \hookrightarrow \text{IA}_{n+1}, \quad H_n \hookrightarrow H_{n+1}, \quad H^*_n \hookrightarrow H^*_{n+1},
\]

which implies the representation stability of the subrepresentation of \( H^A_i(\text{IA}_n, \mathbb{Q}) \). Let

\[
\text{GL}(\infty, \mathbb{Q}) = \lim_{\rightarrow n} \text{GL}(n, \mathbb{Q}), \quad \text{IA}_\infty = \lim_{\rightarrow n} \text{IA}_n,
\]

and

\[
H = \lim_{\rightarrow n} H_n, \quad H^* = \lim_{\rightarrow n} (H_n)^*.
\]

We can rephrase the representation stability of the subrepresentation in the sense of \( \text{GL}(\infty, \mathbb{Q}) \)-representations as follows.

**Corollary 1.3** (Corollary 4.4). We have a surjective \( \text{GL}(\infty, \mathbb{Q}) \)-homomorphism

\[
H^A_i(\text{IA}_\infty, \mathbb{Q}) \to \left( \bigwedge^{i+1} H \right) \otimes H^*.
\]

In particular, we have \( \dim_\mathbb{Q}(H^A_i(\text{IA}_\infty, \mathbb{Q})) = \dim_\mathbb{Q}(H_i(\text{IA}_\infty, \mathbb{Q})) = \infty \).
1.2. Conjectural structure of $H^A_i(\text{IA}_n, \mathbb{Q})$. Next, we detect a subquotient representation of $H^A_i(\text{IA}_n, \mathbb{Q})$, which is conjecturally equal to the entire $H^A_i(\text{IA}_n, \mathbb{Q})$.

For $i \geq 1$, let
\[ U_i = \text{Hom}(H, \bigwedge^{i+1} H) \cong \left( \bigwedge^{i+1} H \right) \otimes H^*. \]

Note that we have $U = U_1$ and that $U_i$ vanishes for $n \leq i$. We have a direct sum decomposition
\[ U_i = U^\text{tree}_i \oplus U^\text{wheel}_i, \]
where $U^\text{tree}_i$ denotes the subrepresentation of $U_i$ that is isomorphic to $V_{i+1,1}$, and where $U^\text{wheel}_i$ denotes the other subrepresentation that is isomorphic to $V_{1,i+1}$ for $n \geq i + 1$.

For the graded $\text{GL}(n, \mathbb{Q})$-representation $U_*$, let $S^*(U_*) = \bigoplus_{i \geq 0} S^i(U_*)$, denote the graded-symmetric algebra of $U_*$. We define the traceless part $W_* = \bar{S}^*(U_*)$ of $S^*(U_*)$, which consists of elements that vanish under any contraction maps between distinct factors of $S^*(U_*)$ (see Section 2.6 for details). We can also construct $W_*$ by using an operad $\text{Com}$ of non-unital commutative algebras as we will see in Section 12.

We construct a $\text{GL}(n, \mathbb{Q})$-homomorphism
\[ F_i : H_*(U, \mathbb{Q}) \to S^*(U_*), \]
by combining two kinds of contraction maps. Then, we obtain $W_i$ as a subquotient representation of $H^A_i(\text{IA}_n, \mathbb{Q})$, which is our main result.

**Theorem 1.4** (Theorem 6.1). For $n \geq 3i$, we have
\[ F_i(H^A_i(\text{IA}_n, \mathbb{Q})) \supset W_i. \]

Theorem 1.4 implies that $H^A_i(\text{IA}_n, \mathbb{Q})$ includes a representation-stable subrepresentation which is isomorphic to $W_i$.

**Remark 1.5.** We can decompose $W_i$ into direct summands
\[ W_i = \bigoplus_{(\mu, \nu) \in P_i} W(\mu, \nu), \]
where $P_i$ denotes the set of pairs of partitions $(\mu, \nu)$ such that $\mu$ and $\nu$ are partitions of non-negative integers whose sum is $i$. Recently Lindell [27] studied the Albanese homology $H^A_i(\mathcal{I}_{g,1}, \mathbb{Q})$ of the Torelli group $\mathcal{I}_{g,1}$. Lindell’s result [27, Theorem 1.5] implies that for each pair of partitions $(\lambda, \mu)$ under some conditions, $H^A_i(\mathcal{I}_{g,1}, \mathbb{Q})$ contains an $\text{Sp}(2g, \mathbb{Q})$-subrepresentation $W_{i,g,1}^*(\lambda, \mu)$ corresponding to $(\lambda, \mu)$.

By [1.0.1], we have $H^A_1(\text{IA}_n, \mathbb{Q}) \cong W_1$ for $n \geq 3$. By [34], we have $H^A_2(\text{IA}_n, \mathbb{Q}) \cong W_2$ for $n \geq 6$. For $i = 3$, we obtain the following theorem.

**Theorem 1.6** (Theorem 11.1). For $n \geq 9$, we have a $\text{GL}(n, \mathbb{Q})$-isomorphism
\[ F_3 : H^A_3(\text{IA}_n, \mathbb{Q}) \cong W_3. \]

It seems natural to make the following conjecture.

**Conjecture 1.7** (Conjecture 6.2). For $n \geq 3i$, we have a $\text{GL}(n, \mathbb{Q})$-isomorphism
\[ F_i : H^A_i(\text{IA}_n, \mathbb{Q}) \cong W_i. \]
1.3. Coalgebra structure of $H^A_i(\text{IA}_n, \mathbb{Q})$. For any group $G$, it is well known that $H^*_i(G, \mathbb{Q})$ has a coalgebra structure. Then $H^A_i(G, \mathbb{Q})$ is a subcoalgebra of $H^*_i(H(G, \mathbb{Z}), \mathbb{Q})$. (See Section 7 for details.) We also have a coalgebra structure on $S^*(U_*)$ (see Section 2.6). Then the graded $\text{GL}(n, \mathbb{Q})$-homomorphism $F_* = \bigoplus_i F_i$ is compatible with comultiplications.

**Proposition 1.8** (Proposition 7.1). The graded $\text{GL}(n, \mathbb{Q})$-homomorphism

$$F_* = \bigoplus_i F_i : H_*(U, \mathbb{Q}) \to S^*(U_*)$$

is a coalgebra map.

For a coalgebra $A$, let Prim$(A)$ denote the primitive part of $A$. We have

$$\text{Prim}(S^*(U_*)) = U_* \subset W_*.$$  

The $\text{GL}(n, \mathbb{Q})$-homomorphism $F_*$ restricts to a graded $\text{GL}(n, \mathbb{Q})$-homomorphism

$$F_* : \text{Prim}(H^A_i(\text{IA}_n, \mathbb{Q})) \to \text{Prim}(S^*(U_*)) = U_*.$$  

Therefore, it is natural to make the following conjecture, which implies that Theorem 1.1 determines the primitive part of $H^A_i(\text{IA}_n, \mathbb{Q})$. Let $\text{Prim}(H^A_i(\text{IA}_n, \mathbb{Q}))_i$ denote the degree $i$ part of $\text{Prim}(H^A_i(\text{IA}_n, \mathbb{Q}))$.

**Conjecture 1.9** (Conjecture 7.2). For $n \geq 3i$, the $\text{GL}(n, \mathbb{Q})$-homomorphism

$$F_i : \text{Prim}(H^A_i(\text{IA}_n, \mathbb{Q}))_i \to U_i$$

is an isomorphism.

1.4. Lower bound of the dimension of $H^A_i(\text{IA}_n, \mathbb{Q})$. Church–Ellenberg–Farb [4] introduced the theory of FI-modules and studied the representation stability of $H^A_i(\text{IA}_n, \mathbb{Q})$. They obtained the following theorem about the stability and an upper bound of the dimension of $H^A_i(\text{IA}_n, \mathbb{Q})$.

**Theorem 1.10** (Church–Ellenberg–Farb [4]). For each $i \geq 0$, there exists a polynomial $P_i(T)$ of degree $\leq 3i$ such that $\dim_\mathbb{Q}(H^A_i(\text{IA}_n, \mathbb{Q})) = P_i(n)$ for sufficiently large $n$ with respect to $i$.

We obtain a lower bound of $\dim_\mathbb{Q}(H^A_i(\text{IA}_n, \mathbb{Q}))$. As a consequence of Theorem 1.10 or directly, it can be shown that the traceless part $H^*_i(U, \mathbb{Q})^{11}$ of $H^*_i(U, \mathbb{Q})$ is contained in $H^A_i(\text{IA}_n, \mathbb{Q})$, where $H^*_i(U, \mathbb{Q})^{11} \subset H^*_i(U, \mathbb{Q})$ is the subrepresentation that vanishes under any contraction maps. (See Section 5)

**Theorem 1.11** (Theorem 5.1). We have $H^*_i(U, \mathbb{Q})^{11} \subset H^A_i(\text{IA}_n, \mathbb{Q})$ for $n \geq 3i$.

We have $\dim_\mathbb{Q}(H^*_i(U, \mathbb{Q})^{11}) = P'_i(n)$ for $n \geq 3i$, where $P'_i(T)$ is a polynomial of degree $3i$. By Theorems 1.10 and 1.11, we obtain the following theorem about the dimension of $H^A_i(\text{IA}_n, \mathbb{Q})$.

**Theorem 1.12** (Theorem 5.2). We have $\dim_\mathbb{Q}(H^A_i(\text{IA}_n, \mathbb{Q})) \geq P'_i(n)$ for $n \geq 3i$. Moreover, there exists a polynomial $P_i(T)$ of degree exactly $3i$ such that we have $\dim_\mathbb{Q}(H^A_i(\text{IA}_n, \mathbb{Q})) = P_i(n)$ for sufficiently large $n$ with respect to $i$. 
1.5. Non-triviality of $H^A_i(\text{IO}_n, \mathbb{Q})$. In a way similar to $\text{IA}_n$, by using abelian cycles, we detect a non-trivial representation of $H^A_i(\text{IO}_n, \mathbb{Q})$. Note that unlike the case of $\text{IA}_n$, there is no canonical inclusion map $\text{IO}_n \hookrightarrow \text{IO}_{n+1}$, so we do not consider the representation stability for $H^A_i(\text{IO}_n, \mathbb{Q})$.

**Theorem 1.13** (Theorem 9.3). Let $i \geq 2$. For $n \geq i + 2 + \frac{1 - (-1)^i}{2}$, we have a surjective $\text{GL}(n, \mathbb{Q})$-homomorphism

$$H^A_i(\text{IO}_n, \mathbb{Q}) \twoheadrightarrow H^A_i \bigotimes H.$$  

For sufficiently large $n$ with respect to $i$, we have a surjective $\text{GL}(n, \mathbb{Q})$-homomorphism

$$H^A_i(\text{IO}_n, \mathbb{Q}) \twoheadrightarrow \text{Hom}(H, \bigotimes H^{i+1} H).$$  

By Pettet [34], we have non-triviality of $H^A_i(\text{IO}_n, \mathbb{Q})$. We obtain non-triviality of $H^A_i(\text{IO}_n, \mathbb{Q})$ for $i \geq 3$.

**Corollary 1.14.** Let $i \geq 2$. For $n \geq i + 2 + \frac{1 - (-1)^i}{2}$, $H^A_i(\text{IO}_n, \mathbb{Q})$ is non-trivial.

1.6. Lower bound of the dimension of $H^A_i(\text{IO}_n, \mathbb{Q})$. Unlike the case of $\text{IA}_n$, the stability or an upper bound of the dimension of $H^A_i(\text{IO}_n, \mathbb{Q})$ is not known. We detect a lower bound of the dimension of $H^A_i(\text{IO}_n, \mathbb{Q})$.

Let $U^O = U^\text{tree}_i$. We have an isomorphism $H^A_i(\text{IO}_n, \mathbb{Q}) \cong U^O$ by [20]. The canonical projection $\pi : \text{IA}_n \twoheadrightarrow \text{IO}_n$ induces a $\text{GL}(n, \mathbb{Q})$-homomorphism $\pi_* : H^A_i(\text{IA}_n, \mathbb{Q}) \rightarrow H^A_i(\text{IO}_n, \mathbb{Q}).$ By using Theorem 1.11, we obtain a lower bound of the dimension of $H^A_i(\text{IO}_n, \mathbb{Q})$.

**Theorem 1.15** (Theorem 9.3). For $n \geq 3i$, $H^A_i(\text{IO}_n, \mathbb{Q})$ contains a subrepresentation which is isomorphic to $H_i(U, \mathbb{Q})^{3i}$. In particular, we have $\dim_{\mathbb{Q}}(H^A_i(\text{IO}_n, \mathbb{Q})) \geq P_i^3(n)$ for $n \geq 3i$.

1.7. Conjectural structure of $H^A_i(\text{IO}_n, \mathbb{Q})$. Here, we propose a conjectural structure of $H^A_i(\text{IO}_n, \mathbb{Q})$ and the relation between $H^A_i(\text{IO}_n, \mathbb{Q})$ and $H^A_i(\text{IA}_n, \mathbb{Q})$.

Let $U^O = U^O_i$ and $U^O_i = U_i$ for $i \geq 2$. For the graded $\text{GL}(n, \mathbb{Q})$-representation $U^O = \bigoplus_{i \geq 1} U^O_i$, let $S^*(U^O_i)$ denote the graded-symmetric algebra of $U^O_i$. Let $W^O = S^*(U^O)$ denote the traceless part of $S^*(U^O)$. Then we can decompose $W^O_i$ into direct summands

$$W^O_i = \bigoplus_{(\mu, \nu) \in P^O_i} W(\mu, \nu),$$

where $P^O_i$ denotes the subset of $P_i$ consisting of pairs of partitions $(\mu, \nu)$ such that $\nu$ has no part of size 1.

By [20], we have $H^A_i(\text{IO}_n, \mathbb{Q}) \cong W^O_i$ for $n \geq 3$. By [34], we have $H^A_3(\text{IO}_n, \mathbb{Q}) \cong W^O_2$ for $n \geq 6$. For $i = 3$, we obtain the following theorem.

**Theorem 1.16** (Theorem 10.1). For $n \geq 9$, we have

$$H^A_3(\text{IO}_n, \mathbb{Q}) \cong W^O_3$$

as $\text{GL}(n, \mathbb{Q})$-representations.
It seems natural to make the following conjecture, which is an analogue of Conjecture 1.7.

**Conjecture 1.17** (Conjecture 9.7). For sufficiently large $n$ with respect to $i$, we have a $GL(n, \mathbb{Q})$-isomorphism

$$H^A_i(\text{IO}_n, \mathbb{Q}) \cong W^O_i.$$ 

On the relation between $H^A_i(\text{IO}_n, \mathbb{Q})$ and $H^A_i(\text{IA}_n, \mathbb{Q})$, the Hochschild–Serre spectral sequence for the exact sequence

$$1 \to \text{Inn}(F_n) \to \text{IA}_n \to \text{IO}_n \to 1,$$

where $\text{Inn}(F_n)$ is the inner automorphism group of $F_n$, leads to the following proposition.

**Proposition 1.18** (Proposition 9.8). For $n \geq 2$, we have a $GL(n, \mathbb{Q})$-isomorphism

$$H^A_i(\text{IA}_n, \mathbb{Q}) \cong H^A_i(\text{IO}_n, \mathbb{Q}) \oplus (H^A_{i-1}(\text{IO}_n, \mathbb{Q}) \otimes H).$$

By Theorem 1.4 and Proposition 1.18, we obtain the following proposition, which partially ensures Conjecture 1.17.

**Proposition 1.19** (Proposition 9.10). For $n \geq 3i$, we have an injective $GL(n, \mathbb{Q})$-homomorphism

$$W^O_i \oplus (W^O_{i-1} \otimes H) \to H^A_i(\text{IO}_n, \mathbb{Q}) \oplus (H^A_{i-1}(\text{IO}_n, \mathbb{Q}) \otimes H).$$

Now we have the following equivalence between conjectures about the structures of $H^A_* (\text{IA}_n, \mathbb{Q})$ and $H^A_* (\text{IO}_n, \mathbb{Q})$.

**Proposition 1.20** (Proposition 9.11). The followings are equivalent.

1. For any $i$, we have a $GL(n, \mathbb{Q})$-isomorphism $H^A_i(\text{IA}_n, \mathbb{Q}) \cong W^O_i$ for sufficiently large $n$ with respect to $i$ (cf. Conjecture 1.7).
2. Conjecture 1.17.

1.8. Relation between $H^A_* (\text{IA}_n, \mathbb{Q})$ and the cohomology of $\text{Aut}(F_n)$ with twisted coefficients. The stable cohomology of $\text{Aut}(F_n)$ with twisted coefficients has been studied by many authors. Satoh computed the first and second homology with coefficients in $H$ and $H^*$ \[38, 39\]. Djament–Vespa \[12\] and Randal-Williams \[35\] obtained the stable cohomology $H^*(\text{Aut}(F_n), H^0 \otimes p)$. Djament \[11\], Vespa \[41\] and Randal-Williams \[35\] obtained the stable cohomology $H^*(\text{Aut}(F_n), (H^*)^\otimes q)$.

Let $H^{p,q} = H^0 \otimes (H^*)^\otimes q$. Kawazumi–Vespa \[22\] studied the stable cohomology $H^*(\text{Aut}(F_n), H^{p,q})$. Their conjecture \[22\] (Conjecture 6) implies the following conjecture, where $C_{\mathcal{P}_0}$ is the wheeled PROP associated to the operadic suspension $\mathcal{P}_0$ of the operad $\text{Com}$ of non-unital commutative algebras (see Section 12 for details.)

**Conjecture 1.21** (Kawazumi–Vespa \[22\], Conjecture 6, Conjecture 12.2). For $p, q \geq 0$, we stably have an isomorphism of graded $\mathbb{Q}[G_p \times G_q]$-modules

$$H^*(\text{Aut}(F_n), H^{p,q}) = C_{\mathcal{P}_0}(p, q).$$
We make the following conjecture about the relation between the Albanese (co)homology of $\text{IA}_n$ and the cohomology of $\text{Aut}(F_n)$ with twisted coefficients, where the Albanese cohomology $H^1_A(\text{IA}_n, \mathbb{Q})$ of $\text{IA}_n$ is isomorphic to $H^1_A(\text{Aut}(F_n), \mathbb{Q})^*$ as $\text{GL}(n, \mathbb{Q})$-representations.

**Conjecture 1.22 (Conjecture 12.5).** Let $i$ be a non-negative integer and $\Lambda$ a bipartition. Then, for sufficiently large $n$, we have a linear isomorphism

$$H^i(\text{Aut}(F_n), V_{\Lambda}) \cong (H^1_A(\text{IA}_n, \mathbb{Q}) \otimes V_{\Lambda})^{\text{GL}(n, \mathbb{Z})}.$$ 

Then we have the following relation between the conjectural structure of the Albanese homology of $\text{IA}_n$ and the above two conjectures.

**Proposition 1.23 (Proposition 12.7).** Let $i$ be a non-negative integer. If two of the followings hold, then so does the third.

1. We have a $\text{GL}(n, \mathbb{Q})$-isomorphism $H^1_A(\text{IA}_n, \mathbb{Q}) \cong W_i$ for sufficiently large $n$ (cf. Conjecture 1.7).

2. Conjecture 1.21 holds for cohomological degree $i$.

3. Conjecture 1.22 holds for cohomological degree $i$.

1.9. **Conjectural structures of $H^1_A(\mathcal{I}_g, \mathbb{Q})$, $H^1_A(\mathcal{I}_{g,1}, \mathbb{Q})$ and $H^1_A(\mathcal{I}_g^1, \mathbb{Q})$.** Let $\mathcal{I}_g$ (resp. $\mathcal{I}_g^1$) denote the Torelli group of a closed surface (resp. a surface with one marked point) of genus $g$. We propose conjectural structures of the Albanese cohomology of the Torelli groups $\mathcal{I}_g$, $\mathcal{I}_{g,1}$ and $\mathcal{I}_g^1$. Note that the Albanese cohomology and homology of the Torelli groups are isomorphic since algebraic $\text{Sp}(2g, \mathbb{Q})$-representations are self-dual. In a way similar to $W_*$ we define $\mathcal{S}^*(\mathcal{X}_*)$, $\mathcal{S}^*(\mathcal{Y}_*)$ and $\mathcal{S}^*(\mathcal{Z}_*)$.

**Conjecture 1.24 (Conjecture 12.7).** We stably have $\text{Sp}(2g, \mathbb{Q})$-isomorphisms

$$H^*_A(\mathcal{I}_g, \mathbb{Q}) \cong \mathcal{S}^*(\mathcal{X}_*), \quad H^*_A(\mathcal{I}_{g,1}, \mathbb{Q}) \cong \mathcal{S}^*(\mathcal{Y}_*), \quad H^*_A(\mathcal{I}_g^1, \mathbb{Q}) \cong \mathcal{S}^*(\mathcal{Z}_*).$$

We also study the structures of the cohomology of Lie algebras associated to $\mathcal{I}_g$, $\mathcal{I}_{g,1}$ and $\mathcal{I}_g^1$.

1.10. **Outline.** The rest of the paper is organized as follows. In Section 2 we recall algebraic $\text{GL}(n, \mathbb{Q})$-representations and the notion of traceless parts. In Section 3 we recall the notion of abelian cycles. We construct an abelian cycle for $\text{IA}_n$ corresponding to each pair of partitions. In Section 4 we obtain Theorem 1.11 which induces Theorem 1.12 about the dimension of $H^1_A(\text{IA}_n, \mathbb{Q})$.

In Section 5 we study the structure of $H^1_A(\text{IA}_n, \mathbb{Q})$ and prove Theorem 1.14. In Section 6 we recall a coalgebra structure on group homology and study the coalgebra structure of $H^1_A(\text{IA}_n, \mathbb{Q})$. In Section 7 we study the algebra structure of the Albanese cohomology $H^*_A(\text{IA}_n, \mathbb{Q})$. In Section 8 we study the Albanese homology $H^1_A(\text{IO}_n, \mathbb{Q})$. We obtain Theorems 1.13 and 1.15. We also study the relation between $H^1_A(\text{IO}_n, \mathbb{Q})$ and $H^1_A(\text{IA}_n, \mathbb{Q})$. In Sections 9 and 10 we obtain $H^1_A(\text{IO}_n, \mathbb{Q})$ and $H^1_A(\text{IA}_n, \mathbb{Q})$ for $n \geq 9$. In Section 11 we study the relation between $H^*_A(\text{IA}_n, \mathbb{Q})$ and $H^*(\text{Aut}(F_n), H^{p,q})$. In Section 12 we recall algebraic $\text{Sp}(2g, \mathbb{Q})$-representations. In Section 13 we discuss the Albanese cohomology of the Torelli groups and the cohomology of Lie algebras associated to the Torelli groups. In Appendix A we give a brief summary of some properties of Albanese homology and cohomology of groups.
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2. Preliminaries

In this section, we recall representation theory of $GL(n, \mathbb{Q})$ and of $GL(\infty, \mathbb{Q})$, and introduce the notion of traceless tensor products and traceless parts of graded-symmetric algebras.

2.1. Notations and conventions. For a non-negative integer $j$, let $[j]$ denote the set $\{1, \cdots, j\}$.

Let $n \geq 1$. Let $F_n = (x_1, \cdots, x_n)$. Let $H = H(n) = H_1(F_n, \mathbb{Q}) = \bigoplus_{j=1}^{n} Qe_j$, where we fix the basis $\{e_j\}_{j=1}^{n}$ for $H$. We have $H^* = H(n)^* = H_1^1(F_n, \mathbb{Q}) = \bigoplus_{j=1}^{n} Qe^*_j$.

In what follows, we consider representations of $GL(H, \mathbb{Q}) = GL(n, \mathbb{Q})$ over $\mathbb{Q}$, which are the same as $Q[GL(n, \mathbb{Q})]$-representations. Sometimes we simply write “representations” to mean $GL(n, \mathbb{Q})$-representations.

In computations, we use the following matrices in $GL(n, \mathbb{Q})$. Let $id \in GL(n, \mathbb{Q})$ denote the identity matrix. For distinct elements $k, l \in [n]$, let $E_{k, l} \in GL(n, \mathbb{Q})$ denote the matrix that maps $e_l$ to $e_k + e_l$ and fixes $e_a$ for $a \neq l$. Then $E_{k, l}$ maps the dual basis $e^*_k$ to $e^*_k - e^*_l$ and fixes $e^*_a$ for $a \neq k$. Let $P_{k, l} \in GL(n, \mathbb{Q})$ denote the matrix that exchanges $e_k$ and $e_l$ and fixes $e_a$ for $a \neq k, l$. For $k = l$, we have $P_{k, k} = id$.

2.2. Irreducible polynomial representations of $GL(n, \mathbb{Q})$. Here we recall several notions from representation theory. See [12] for details.

Let $n \geq 1$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with at most $n$ parts is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $l(\lambda) = \max\{0 \cup \{i \mid \lambda_i > 0\}\}$ denote the length of $\lambda$ and $|\lambda| = \lambda_1 + \cdots + \lambda_l(\lambda)$ the size of $\lambda$. We write $\lambda \vdash |\lambda|$. A Young diagram for a partition $\lambda$ is a diagram with $\lambda_i$ boxes in the $i$-th row such that the rows of boxes are left-aligned. A tableau on a Young diagram is a numbering of the boxes by integers in $|\lambda|$. A tableau is standard if the numbering of each row and each column is increasing. The canonical tableau on a Young diagram is a standard tableau whose numbering starts from the first row from left to right and followed by the second row from left to right and so on.

Let
\[
a_\lambda = \sum_{\sigma \in R} \sigma \in \mathbb{Q}[S_{|\lambda|}], \quad b_\lambda = \sum_{\tau \in C} \text{sgn}(\tau) \tau \in \mathbb{Q}[S_{|\lambda|}],
\]
where $R$ (resp. $C$) is a subgroup of $S_{|\lambda|}$ preserving rows (resp. columns) of the canonical tableau on the Young diagram corresponding to $\lambda$. The Young symmetrizer $c_\lambda$ is defined by
\[
e_\lambda = b_\lambda a_\lambda \in \mathbb{Q}[S_{|\lambda|}].
\]
Let $S^\lambda = \mathbb{Q}[\mathcal{S}_\lambda]/c_\lambda$ denote the Specht module corresponding to the partition $\lambda$, which is an irreducible representation of $\mathcal{S}_{|\lambda|}$.

We call a $\text{GL}(n, \mathbb{Q})$-representation $V$ polynomial if after choosing a basis for $V$, the $(\dim V)^2$ coordinate functions of the action $\text{GL}(n, \mathbb{Q}) \to \text{GL}(V)$ are polynomial functions of the $n^2$ variables. Consider $H \cong \mathbb{Q}^n$ as the standard representation of $\text{GL}(n, \mathbb{Q})$. For a partition $\lambda$, let

$$V_\lambda = H^{|\lambda|} \otimes \mathbb{Q}[\mathcal{S}_{|\lambda|}] S^\lambda.$$  

If $\lambda$ has at most $n$ parts, then $V_\lambda$ is an irreducible polynomial $\text{GL}(n, \mathbb{Q})$-representation, and otherwise, we have $V_\lambda = 0$. For a non-negative integer $p$, by the Schur–Weyl duality, we have a decomposition of $H^\otimes p$ as $\text{GL}(n, \mathbb{Q}) \times \mathcal{S}_p$-modules

$$H^\otimes p = \bigoplus_{\lambda \vdash p \text{ with at most } n \text{ parts}} V_\lambda \otimes S^\lambda.$$ 

It is well known that irreducible polynomial representations of $\text{GL}(n, \mathbb{Q})$ are classified by partitions with at most $n$ parts, that is, any irreducible $\text{GL}(n, \mathbb{Q})$-representation is isomorphic to $V_\lambda$ for a partition $\lambda$ with at most $n$ parts.

We have the following irreducible decomposition of the tensor product of two irreducible polynomial $\text{GL}(n, \mathbb{Q})$-representations $V_\lambda$ and $V_\mu$

$$V_\lambda \otimes V_\mu = \bigoplus_{\nu} V_{\nu}^{\oplus N_{\lambda\mu}^{\nu}},$$

where $N_{\lambda\mu}^{\nu}$ is the Littlewood–Richardson coefficient.

### 2.3. Irreducible algebraic representations of $\text{GL}(n, \mathbb{Q})$.

We call a $\text{GL}(n, \mathbb{Q})$-representation $V$ algebraic if after choosing a basis for $V$, the $(\dim V)^2$ coordinate functions of the action $\text{GL}(n, \mathbb{Q}) \to \text{GL}(V)$ are rational functions of the $n^2$ variables. Here we recall irreducible algebraic representations of $\text{GL}(n, \mathbb{Q})$, which generalizes irreducible polynomial representations of $\text{GL}(n, \mathbb{Q})$. See [14, 23, 33] for details.

Let $p$ and $q$ be positive integers. For a pair $(k, l) \in [p] \times [q]$, the contraction map

$$c_{k,l} : H^\otimes p \otimes (H^*)^\otimes q \to H^\otimes p \otimes (H^*)^\otimes q - 1$$

is defined for $v_1 \otimes \cdots \otimes v_p \in H^\otimes p$ and $f_1 \otimes \cdots \otimes f_q \in (H^*)^\otimes q$ by

$$c_{k,l}((v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \otimes f_q)) = (v_k, f_l)(v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \hat{f}_l \cdots f_q),$$

where $\langle -, - \rangle : H \otimes H^* \to \mathbb{Q}$ denotes the dual pairing, and where $\hat{v}_k$ (resp. $\hat{f}_l$) denotes the omission of $v_k$ (resp. $f_l$).

For $p, q \geq 0$, let $T_{p,q}$ denote the traceless part of $H^\otimes p \otimes (H^*)^\otimes q$, which is defined by

$$T_{p,q} = \bigcap_{(k,l) \in [p] \times [q]} \ker c_{k,l} \subset H^\otimes p \otimes (H^*)^\otimes q.$$  

Note that we have $T_{0,q} = (H^*)^\otimes q$ and $T_{p,0} = H^\otimes p$.

A bipartition $\underline{\lambda} = (\lambda^+, \lambda^-)$ with at most $n$ parts is a pair of two partitions $\lambda^+$ and $\lambda^-$ such that $l(\underline{\lambda}) = l(\lambda^+) + l(\lambda^-) \leq n$. The size $|\underline{\lambda}|$ of $\underline{\lambda}$ is defined by $|\lambda^+| + |\lambda^-|$.

For a bipartition $\underline{\lambda}$ set $p = |\lambda^+|$ and $q = |\lambda^-|$. Let

$$V_{\underline{\lambda}} = T_{p,q} \otimes \mathbb{Q}[\mathcal{S}_p \times \mathcal{S}_q] (S^{\lambda^+} \otimes S^{\lambda^-}).$$
If $\Lambda$ has at most $n$ parts, then $V_{\Lambda}$ is an irreducible algebraic representation, and otherwise, we have $V_{\Lambda} = 0$. We have the following decomposition of $T_{p,q}$ as a $\text{GL}(n,\mathbb{Q}) \times (\mathfrak{S}_p \times \mathfrak{S}_q)$-modules, which generalizes the Schur–Weyl duality for $H^{\otimes p}$

\[ T_{p,q} = \bigoplus_{\Lambda \text{ bipartition with at most } n \text{ parts} \atop |\lambda^+| = p, |\lambda^-| = q} V_{\Lambda} \otimes (S^{\lambda^+} \otimes S^{\lambda^-}). \]

(See [23] Theorem 1.1.) It is well known that irreducible algebraic representations of $\text{GL}(n, \mathbb{Q})$ are classified by bipartitions with at most $n$ parts, that is, any irreducible algebraic $\text{GL}(n, \mathbb{Q})$-representation is isomorphic to $V_{\Lambda}$ for a bipartition $\Lambda$ with at most $n$ parts. (See also [33] for the correspondence between irreducible algebraic representations and bipartitions.)

For a bipartition $\Lambda = (\lambda^+, \lambda^-)$ with at most $n$ parts, we have $V_{\Lambda} = V_\mu \otimes \det^k$, where $\mu$ is a partition and $k$ is an integer satisfying

\[ (\lambda_1^+, \cdots, \lambda_i^+, 0, \cdots, 0, -\lambda_{i-1}^-, \cdots, -\lambda_1^-) = (\mu_1 + k, \cdots, \mu_n + k), \]

and where $\det$ denotes the 1-dimensional determinant representation.

The dual of $\Lambda$ is defined by $\Lambda^* = (\lambda^-, \lambda^+)$. Note that we have a $\text{GL}(n, \mathbb{Q})$-isomorphism $(V_\Lambda)^* \cong V_{\Lambda^*}$.

For two bipartitions $\lambda, \mu$ such that $n \geq l(\lambda) + l(\mu)$, we have the following irreducible decomposition of the tensor product of $V_\lambda$ and $V_\mu$,

\[ V_\lambda \otimes V_\mu = \bigoplus_{\nu \in \text{St}_{\lambda^+} \nu^\mu} V_{\nu} \otimes N_{\nu}^{\lambda^+ \mu^0}, \]

where $N_{\nu}^{\lambda^+ \mu^0} = \sum_\alpha \beta \delta (\sum_\kappa N_{\kappa \alpha}^{\lambda^+} N_{\kappa \beta}^{\mu^0}) (\sum_\kappa N_{\kappa \beta}^{\lambda^-} N_{\kappa \delta}^{\mu^0}) N_{\alpha \delta}^{\nu^+} N_{\beta \delta}^{\nu^-}$ (see [23] for details).

2.4. Generator of the traceless part $T_{p,q}$. Here we give a generator of the traceless part $T_{p,q}$ of $H^{\otimes p} \otimes (H^*)^{\otimes q}$.

We have the following explicit highest weight vectors of $T_{p,q}$.

**Lemma 2.1** (Theorems 2.7 and 2.11 of [1]). Let $p, q \geq 0$. For a bipartition $\Lambda$ with at most $n$ parts such that $|\lambda^+| = p$ and $|\lambda^-| = q$, let

\[ e(\Lambda) = (e_1^{\otimes \lambda_1^+} \otimes \cdots \otimes e_{l(\lambda^+)}^{\otimes \lambda_1^+}) \otimes ((e_n^{\otimes \lambda_n^+} \otimes \cdots \otimes e_{l(\lambda^-) + 1}^{\otimes \lambda_n^+})^{\otimes \lambda_i^-}) \in H^{\otimes p} \otimes (H^*)^{\otimes q}. \]

Let $St_{\lambda^+}$ (resp. $St_{\lambda^-}$) denote the subset of $\mathfrak{S}_p$ (resp. $\mathfrak{S}_q$) consisting of permutations which send the canonical tableau to standard tableaux on the Young diagram corresponding to $\lambda^+$ (resp. $\lambda^-$). Then the set

\[ \{(\pi b_{\lambda^+} \otimes \text{id}) \circ (\text{id} \otimes \rho_{b_{\lambda^-}}) \circ e(\Lambda) \mid |\lambda^+| = p, |\lambda^-| = q, \pi \in St_{\lambda^+}, \rho \in St_{\lambda^-}\} \]

is a basis for the space of all highest weight vectors of $T_{p,q}$.

In particular, for each $\Lambda$, $(\pi b_{\lambda^+} \otimes \text{id}) \circ (\text{id} \otimes \rho_{b_{\lambda^-}}) \circ e(\Lambda)$ generates an irreducible subrepresentation of $T_{p,q}$ which is isomorphic to $V_{\Lambda}$.

**Lemma 2.2.** Let $n \geq p + q$. Then the traceless part $T_{p,q}$ of $H^{\otimes p} \otimes (H^*)^{\otimes q}$ is generated by

\[ e_{p,q} = e_1 \otimes \cdots \otimes e_p \otimes e_{n-q+1}^* \cdots \otimes e_{n}^* \in H^{\otimes p} \otimes (H^*)^{\otimes q} \]

as a $\text{GL}(n, \mathbb{Q})$-representation.
Proof: The proof is analogous to that of [27, Lemma 2.1]. By Lemma [27.1] the traceless part $T_{p,q}$ is generated by
\[ (\pi b_+ \otimes \id) (\id \otimes \rho b_-) e(\underline{\lambda}) \mid |\lambda^+| = p, \mid \lambda^- \mid = q, \pi \in St_{\lambda^+}, \rho \in St_{\lambda^-} \].
Therefore, it suffices to show that
\[ e_J := e_{j_1} \otimes \cdots \otimes e_{j_p} \otimes e_{j_n}^* \otimes \cdots \otimes e_{j_{n-q+1}}^* \in \mathbb{Q}[\GL(n, \mathbb{Q})] e_{p,q} \]
for $j_1, \ldots, j_p \in [p], j_n, \cdots, j_{n-q+1} \in \{n, \cdots, n-q+1\}$.
Let
\[ K_p = \{ k \in [p] \mid e_{j_k} = e_{j_l} \text{ for some } l < k \} \]
and
\[ K'_q = \{ k \in [q] \mid e_{j_{n-k+l}} = e_{j_{n-l+1}}^* \text{ for some } l < k \}. \]
We can reorder the basis \( \{e_1, \cdots, e_n\} \) for $H$ by using permutation matrices in such a way that $e_{j_k} = e_k$ for $k \in [p] \setminus K_p$, that is, we may assume that $e_J$ coincides with $e_1 \otimes \cdots \otimes e_p \otimes e_{j_n}^* \otimes \cdots \otimes e_{j_{n-q+1}}^*$ except for tensor factors where $e_J$ has repeating elements. We can also assume that $e_J$ coincides with $e_{p,q}$ except for tensor factors where $e_J$ has repeating elements by reordering the basis \( \{e_1, \cdots, e_n\} \) again in such a way that $e_{j_{n-k+l}} = e_{n-k+1}$ for $k \in [q] \setminus K'_q$.

Now, we have to check that $e_J \in \mathbb{Q}[\GL(n, \mathbb{Q})] e_{p,q}$. Define
\[ E_J := \prod_{k \in K_p} (E_{j_k,k} - \id) \prod_{k \in K'_q} (id - E_{n-k+1,j_{n-k+1}}) \in \mathbb{Q}[\GL(n, \mathbb{Q})]. \]
Then we have
\[ e_J = E_J(e_{p,q}) \in \mathbb{Q}[\GL(n, \mathbb{Q})] e_{p,q}, \]
which completes the proof. \( \square \)

Corollary 2.3. Let $n \geq p+q$. Let $X$ be a $\GL(n, \mathbb{Q})$-representation with a projection $\pi : T_{p,q} \rightarrow X$. Then $X$ is generated by $\pi(e_{p,q})$ as a $\GL(n, \mathbb{Q})$-representation.

2.5. Traceless tensor products. Here we define traceless tensor products of algebraic $\GL(n, \mathbb{Q})$-representations.

By Lemma [27.1] we have a subrepresentation $V_\underline{\lambda} \subset H^{\otimes |\lambda^+|} \otimes (H^*)^{\otimes |\lambda^-|}$ which is generated by $(b_+ \otimes \id)(\id \otimes b_-) e(\underline{\lambda})$ for each bipartition $\underline{\lambda}$.

Let $\underline{\lambda}$ and $\underline{\mu}$ be two bipartitions. Let $p = |\lambda^+|, q = |\lambda^-|, r = |\mu^+|$ and $s = |\mu^-|$. Define the traceless tensor product $V_\underline{\lambda} \otimes V_\underline{\mu}$ of $V_\underline{\lambda}$ and $V_\underline{\mu}$ as
\[ V_\underline{\lambda} \otimes V_\underline{\mu} = (V_\underline{\lambda} \otimes V_\underline{\mu}) \cap T_{p+r,q+s} \subset H^{\otimes (p+r)} \otimes (H^*)^{\otimes (q+s)}. \]
In other words, $V_\underline{\lambda} \otimes V_\underline{\mu}$ is a subrepresentation of $V_\underline{\lambda} \otimes V_\underline{\mu}$ which vanishes under any contraction maps. Then we have for $n \geq l(\underline{\lambda}) + l(\underline{\mu})$,
\[ V_\underline{\lambda} \otimes V_\underline{\mu} \cong \bigoplus_{l = |\lambda^+| + |\mu^+|} V_\underline{\mu}^{\otimes N_{\underline{\mu}}} \subset \bigoplus_{\underline{\lambda}} V_\underline{\lambda}^{\otimes N_{\underline{\lambda}}} \cong V_\underline{\lambda} \otimes V_\underline{\mu}. \]

Let $M$ be an algebraic $\GL(n, \mathbb{Q})$-representation. For each bipartition $\underline{\lambda}$, define a vector space
\[ M_\underline{\lambda} = \Hom_{\GL(n, \mathbb{Q})}(V_\underline{\lambda}, M). \]
Since the category of algebraic \( \text{GL}(n, \mathbb{Q}) \)-representations is semisimple, we have a natural isomorphism

\[
\iota_M : M \overset{\cong}{\to} \bigoplus_{\Delta} V_{\Delta} \otimes M_{\Delta}.
\]

For two algebraic \( \text{GL}(n, \mathbb{Q}) \)-representations \( M \) and \( N \), we have

\[
\iota_M \otimes \iota_N : M \otimes N \overset{\cong}{\to} \left( \bigoplus_{\Delta} V_{\Delta} \otimes M_{\Delta} \right) \otimes \left( \bigoplus_{\mu} V_{\mu} \otimes N_{\mu} \right) \cong \bigoplus_{\Delta \mu} (V_{\Delta} \otimes V_{\mu}) \otimes (M_{\Delta} \otimes N_{\mu}).
\]

Define the traceless tensor product \( M \otimes^T N \) of \( M \) and \( N \) as

\[
M \otimes^T N = (\iota_M \otimes \iota_N)^{-1} \left( \bigoplus_{\Delta \mu} (V_{\Delta} \otimes V_{\mu}) \otimes (M_{\Delta} \otimes N_{\mu}) \right) \subset M \otimes N.
\]

Let \( M \otimes^0 = \mathbb{Q} \) and \( M \otimes^1 = M \). For \( n \geq 2 \), we define \( M \otimes^n = M \otimes^{n-1} \otimes M \subset M \otimes^n \) iteratively, and define the traceless part \( \widetilde{T}^* M \) of the tensor algebra \( T^* M \) as

\[
\widetilde{T}^* M = \bigoplus_{n \geq 0} M \otimes^n \subset T^* M.
\]

Let \( \Lambda^* M \) denote the exterior algebra of \( M \) and \( \text{Sym}^* M \) the symmetric algebra of \( M \). We have canonical projections \( T^* M \to \Lambda^* M \) and \( T^* M \to \text{Sym}^* M \). We define the traceless part \( \Lambda^* M \) of \( \Lambda^* M \) as the image of \( \widetilde{T}^* M \) under the above projection, and the traceless part \( \text{Sym}^* M \) of \( \text{Sym}^* M \) in a similar way.

2.6. Traceless parts of graded-symmetric algebras. Let \( M_* = \bigoplus_{i \geq 1} M_i \) be a graded algebraic \( \text{GL}(n, \mathbb{Q}) \)-representation. Let \( S^*(M_*) \) denote the graded-symmetric algebra of \( M_* \). That is, we have the following graded-commutativity

\[
y x = (-1)^{ij} x y
\]

for \( x \in M_i \) and \( y \in M_j \). Then we have \( S^k(M_i) = \text{Sym}^k(M_i) \) for even \( i \) and \( S^k(M_i) = \Lambda^k(M_i) \) for odd \( i \). We have a canonical projection \( T^* M_* \to S^*(M_*) \). Define the traceless part \( S^*(M_*) = \bigoplus_{i \geq 0} S^*(M_i)_i \) of the graded-symmetric algebra \( S^*(M_*) \) as the image of \( \widetilde{T}^* M_* \) under the projection. Then \( S^*(M_*) \) is a graded \( \text{GL}(n, \mathbb{Q}) \)-representation satisfying

\[
S^*(M_*)_i = \bigoplus_{k_1 + 2k_2 + \cdots + pk_p = i} \left( \tilde{S}^{k_1} M_1 \right) \otimes \cdots \otimes \left( \tilde{S}^{k_p} M_p \right).
\]

The graded-symmetric algebra \( S^*(M_*) \) has a coalgebra structure defined as follows. For an element \( x = x_1 \cdots x_k \in S^*(M_*) \), where \( x_j \in M_{i_j} \), and for \( \sigma \in \mathfrak{S}_k \), let \( \text{sgn}(\sigma; x) \in \{1, -1\} \) denote the sign satisfying

\[
x_{\sigma(1)} \cdots x_{\sigma(k)} = \text{sgn}(\sigma; x) x_1 \cdots x_k.
\]

Then the comultiplication \( \Delta \) is defined by

\[
\Delta(x_1 \cdots x_k) = \sum_{\sigma \in \mathfrak{S}_k} \sum_{p=0}^k \text{sgn}(\sigma; x) x_{\sigma(1)} \cdots x_{\sigma(p)} \otimes x_{\sigma(p+1)} \cdots x_{\sigma(k)}.
\]
where \( \text{Sh}(p,k-p) \subset G_k \) denotes the set of \( (p,k-p) \)-shuffles. Then we can check that the coalgebra structure of \( S^*(M_s) \) induces a subcoalgebra structure of the traceless part \( \tilde{S}^*(M_s) \). We can also check that the primitive part of \( S^*(M_s) \) is \( M_s \).

**Remark 2.4.** The traceless part \( \tilde{S}^*(M_s) \) does not inherit the algebra structure of \( S^*(M_s) \). However, we can consider an algebra structure on \( \tilde{S}^*(M_s) \) in the symmetric monoidal category that we introduce below. Let \( \text{Rep}^{\text{alg}}(GL(n,Q)) \) denote the category of algebraic \( GL(n,Q) \)-representations and \( GL(n,Q) \)-homomorphisms. The traceless tensor product \( \otimes \) and the symmetry \( \tau_{V,W} : V \otimes W \rightarrow W \otimes V \) which is the restriction of the usual symmetry \( \tau_{V,W} : V \otimes W \rightarrow W \otimes V \) form a symmetric monoidal structure \( (\text{Rep}^{\text{alg}}(GL(n,Q)), \otimes, \tau) \). Then \( \tilde{S}^*(M_s) \) is a bialgebra in \( (\text{Rep}^{\text{alg}}(GL(n,Q)), \otimes, \tau) \).

**2.7.** \( GL(\infty,Q) \)-representations. Let \( n \geq 0 \). We have an inclusion \( GL(n,Q) \hookrightarrow GL(n+1,Q) \) sending \( A \in GL(n,Q) \) to \( A \oplus 1 \in GL(n+1,Q) \). Let \( GL(\infty,Q) = \lim_{\rightarrow n} \text{GL}(n,Q) = \bigcup_{n \geq 1} \text{GL}(n,Q) \). Here we recall representation theory of \( GL(\infty,Q) \). See [37] for details.

We have a canonical inclusion \( H(n) \hookrightarrow H(n+1) \) sending the basis vector \( e_j \) to \( e_j \) for \( j \in [n] \). We also have an inclusion \( H(n)^* \hookrightarrow H(n+1)^* \) sending the dual basis vector \( e_j^* \) to \( e_j^* \) for \( j \in [n] \). Let \( H = \lim_{\rightarrow n} H(n) = \bigcup_{n \geq 1} H(n) \), and \( H^* = \lim_{\rightarrow n} (H(n)^*) = \bigcup_{n \geq 1} (H(n)^*) \).

The group \( GL(\infty,Q) \) acts on the tensor product \( H^{\otimes p} \otimes (H^*)^{\otimes q} \) for any \( p, q \geq 0 \). We call a \( GL(\infty,Q) \)-representation algebraic if it is a subquotient of a finite direct sum of tensor products \( H^{\otimes p} \otimes (H^*)^{\otimes q} \) for \( p, q \geq 0 \). Note that the category of algebraic \( GL(\infty,Q) \)-representations is not semisimple. For example, the contraction map \( H \otimes H^* \rightarrow Q \) does not split.

We consider the contraction maps \( c_{k,l} \) and the traceless part \( T_{p,q} \) for \( GL(\infty,Q) \) as in the case of \( GL(n,Q) \). For any bipartition \( \lambda \) such that \( |\lambda^+| = p \) and \( |\lambda^-| = q \), define a \( GL(\infty,Q) \)-representation \( V_{\lambda} \) as

\[
V_{\lambda} = T_{p,q} \otimes \mathbb{Q}[e_p \times e_q] \left( S^{\lambda^+} \otimes S^{\lambda^-} \right).
\]

Then as in (2.3.1), we have the following decomposition of the traceless part \( T_{p,q} \) as \( GL(\infty,Q) \)-representations

\[
T_{p,q} = \bigoplus_{\lambda=(\lambda^+,\lambda^-) \mid |\lambda^+|=p, |\lambda^-|=q} V_{\lambda} \otimes \left( S^{\lambda^+} \otimes S^{\lambda^-} \right).
\]

Moreover, as in the case of \( GL(n,Q) \), it is known that irreducible algebraic representations of \( GL(\infty,Q) \) are classified by bipartitions. (See Proposition 3.14 in [37].)

**3. Abelian cycles in \( H_i(IA_n,Q) \)**

Here we recall the notion of abelian cycles and construct abelian cycles in \( H_i(IA_n,Q) \).
3.1. Definition of abelian cycles. Let $i \geq 1$ and $\mathbb{Z}^i = \bigoplus_{k=1}^i \mathbb{Z}z_k$. Let $(\phi_1, \cdots, \phi_i)$ be an $i$-tuple of mutually commuting elements of $\text{IA}_n$. Then we have a group homomorphism $\mathbb{Z}^i \rightarrow \text{IA}_n$ mapping $z_k$ to $\phi_k$ for $1 \leq k \leq i$, which induces a group homomorphism

$$H_i(\mathbb{Z}^i, \mathbb{Q}) \rightarrow H_i(\text{IA}_n, \mathbb{Q}).$$

We have $H_i(\mathbb{Z}^i, \mathbb{Q}) \cong H_i(T^i, \mathbb{Q}) \cong \mathbb{Q}$, where $T^i$ is the $i$-dimensional torus. Let $A(\phi_1, \cdots, \phi_i) \in H_i(\text{IA}_n, \mathbb{Q})$ denote the image of the fundamental class of $H_i(T^i, \mathbb{Q})$ and call it the abelian cycle determined by $(\phi_1, \cdots, \phi_i)$.

3.2. Abelian cycles in $H_i(\text{IA}_n, \mathbb{Q})$. Magnus’s set of generators of $\text{IA}_n$ is

$$\{g_{a,b} \mid 1 \leq a, b \leq n, a \neq b\} \cup \{f_{a,b,c} \mid 1 \leq a, b, c \leq n, a < b, a \neq c \neq b\},$$

where $g_{a,b}$ and $f_{a,b,c}$ are defined by

$$g_{a,b}(x_b) = x_a x_b x_a^{-1}, \quad g_{a,b}(x_d) = x_d \text{ for } d \neq b,$$

$$f_{a,b,c}(x_c) = x_c [x_a, x_b], \quad f_{a,b,c}(x_d) = x_d \text{ for } d \neq c.$$

For $k > l$, let

$$h_{k,l} := g_{k,l} g_{k,l+1} \cdots g_{k,k-1} \in \text{IA}_n.$$

Let $\text{Aut}(F_n)(k, l)$ denote the image of the canonical injective map

$$\text{Aut}(F_{k-1}) \hookrightarrow \text{Aut}(F_n), \quad x_j \mapsto x_{j+l-1}.$$ 

Since $h_{k,l}$ fixes $x_j$ for $j \in \{1, \cdots, l-1, k, \cdots, n\}$ and maps $x_j$ to $x_k x_j x_k^{-1}$ for $j \in \{l, l+1, \cdots, k-1\}$, it follows that $h_{k,l}$ commutes with any element of $\text{Aut}(F_n)(k, l)$.

For positive integers $a$ and $r$, set

$$g_{r,a} = (g_{a+1,a} = h_{a+1,a}, h_{a+2,a}, \cdots, h_{a+r,a})$$

and

$$f_{r,a} = (f_{a,a+1,a+2}, h_{a+3,a}, h_{a+4,a}, \cdots, h_{a+r+1,a}).$$

**Lemma 3.1.** For $n \geq a + r$, $g_{r,a}$ is an $r$-tuple of mutually commuting elements. For $n \geq a + r + 1$, $f_{r,a}$ is an $r$-tuple of mutually commuting elements.

**Proof.** For $g_{r,a}$, it suffices to check that $h_{a+i,a}$ and $h_{a+j,a}$ commute for $1 \leq i < j \leq r$. Since $h_{a+j,a}$ commutes with any elements of $\text{Aut}(F_n)(a+j, a)$ and since we have $h_{a+i,a} \in \text{Aut}(F_n)(a+j, a)$, it follows that $h_{a+j,a}$ commutes with $h_{a+i,a}$.

We can also check that $f_{r,a}$ is an $r$-tuple of mutually commuting elements by using the fact that $f_{a,a+1,a+2} \in \text{Aut}(F_n)(a+j, a)$ for any $3 \leq j \leq r + 1$. \(\square\)

Let $i \geq 0$. We call $(\mu, \nu)$ a pair of partitions of total size $i$, denoted by $(\mu, \nu) \vdash i$, if $\mu$ and $\nu$ are partitions with $|\mu| + |\nu| = i$. (Note that pairs of partitions are the same as bipartitions, but we regard them as different notions.) For $n \geq i + 2l(\mu) + l(\nu)$, define an $i$-tuple $h_{(\mu,\nu)}$ of elements of $\text{IA}_n$ by

$$h_{(\mu,\nu)} = (f_{\mu_1,s(1)}, \cdots, f_{\mu_{l(\mu)},s(l(\mu))}, g_{r_1,t(1)}, \cdots, g_{r_{l(\nu)},t(l(\nu))}),$$

where

$$s(j) = 1 + \sum_{p=1}^{j-1} (\mu_p + 2), \quad t(k) = 1 + |\mu| + 2l(\mu) + \sum_{p=1}^{k-1} (\nu_p + 1).$$
Then we can check that $h_{(\mu,\nu)}$ consists of mutually commuting elements. For example, we have

\begin{align*}
    h_{(2,0)} &= f_{2,1} = (f_{1,2,3}, h_{4,1}), \\
    h_{(0,2)} &= g_{2,1} = (h_{2,1}, h_{3,1}), \\
    h_{(12,0)} &= (f_{1,1}, f_{1,4}) = (f_{1,2,3}, f_{4,5,6}), \\
    h_{(0,1^2)} &= (g_{1,1}, g_{1,3}) = (h_{2,1}, h_{4,3}), \\
    h_{(1,1)} &= (f_{1,1}, g_{1,4}) = (f_{1,2,3}, h_{5,4}).
\end{align*}

Then we obtain an abelian cycle corresponding to each pair of partitions.

**Definition 3.2.** Let $(\mu, \nu) \vdash i$ be a pair of partitions. For $n \geq i + 2l(\mu) + l(\nu)$, define $\alpha_{(\mu,\nu)} = A(h_{(\mu,\nu)}) \in H_i(IA_n, \mathbb{Q})$ as the abelian cycle corresponding to the $i$-tuple $h_{(\mu,\nu)}$ of mutually commuting elements.

**Remark 3.3.** The above construction gives an abelian cycle $\alpha_{(\mu,\nu)}$ in $H_i(IA_n, \mathbb{Z})$ for each $(\mu, \nu) \vdash i$.

### 4. Non-triviality of $H^A_i(IA_n, \mathbb{Q})$

In this section, we construct two types of contraction maps each of which detects an irreducible $GL(n, \mathbb{Q})$-quotient representation of $H^A_i(IA_n, \mathbb{Q})$.

Let

\[ U = \text{Hom}(H, \bigwedge^2 H) \cong \left( \bigwedge^2 H \right) \otimes H^*. \]

For $a, b, c \in [n]$, let

\[ e_{a,b}^c := (e_a \wedge e_b) \otimes e_c^* \in U. \]

Then we have the following basis for $U$ induced by Magnus’s set of generators of $IA_n$

\[ \{ e_{a,b}^b \mid 1 \leq a, b \leq n, a \neq b \} \cup \{ e_{a,b}^c \mid 1 \leq a, b, c \leq n, a < b, a \neq c \neq b \} \]

\[ = \{ e_{a,b}^c \mid 1 \leq a, b, c \leq n, a < b \}. \]

#### 4.1. Contraction maps

Let

\[ M_i := (H^{\otimes 2} \otimes H^*)^{\otimes i} = H^{\otimes 2i} \otimes (H^*)^{\otimes i}. \]

Let $\iota_i$ be the canonical injective map

\[ \iota_i : \bigwedge^i U \hookrightarrow M_i, \quad u_1 \wedge \cdots \wedge u_i \mapsto \sum_{\sigma \in \mathfrak{S}_i} \text{sgn}(\sigma) \varphi(u_{\sigma(1)}) \otimes \cdots \otimes \varphi(u_{\sigma(i)}), \]

where

\[ \varphi : U \hookrightarrow H^{\otimes 2} \otimes H^*, \quad (a \wedge b) \otimes d^* \mapsto (a \otimes b - b \otimes a) \otimes d^*. \]

For $i \geq 1$, let

\[ U_i = \text{Hom}(H, \bigwedge^{i+1} H) \cong \left( \bigwedge^{i+1} H \right) \otimes H^*. \]

Then we have $U = U_1$. Note that $U_i$ vanishes for $n \leq i$. We have a direct sum decomposition

\[ U_i = U_i^{\text{tree}} \oplus U_i^{\text{wheel}}, \]
where \( U_{\text{tree}} \) denotes the subrepresentation of \( U_i \) that is isomorphic to \( V_{1,i+1} \), and where \( U_{\text{wheel}}^i \) denotes the other subrepresentation that is isomorphic to \( V_{1,0} \) for \( n \geq i + 1 \). In what follows, we assume \( n \geq i + 1 \) and we identify \( U_{\text{wheel}}^i \) with \( V_{1,0} \).

Define a \( \text{GL}(n, \mathbb{Q}) \)-homomorphism
\[
c_i : M_i \to U_i \cong \left( \bigwedge^{i+1} H \right) \otimes H^*
\]
by
\[
c_i \left( \bigotimes_{j=1}^i (a_j \otimes b_j \otimes d_j^i) \right) = \left( \prod_{j=2}^i d_j^i(b_{j-1}) \right) a_1 \wedge a_2 \wedge \cdots \wedge a_i \wedge b_i \otimes d_1^i.
\]
We also define a \( \text{GL}(n, \mathbb{Q}) \)-homomorphism
\[
c_i^{\text{tree}} : M_i \to U_{\text{tree}}^i \cong V_{1,i+1}
\]
to be the composition of the contraction map \( c_i \) and the canonical projection \( U_i \to U_{\text{tree}}^i \).

Define another \( \text{GL}(n, \mathbb{Q}) \)-homomorphism
\[
c_i^{\text{wheel}} : M_i \to U_{\text{wheel}}^i \cong V_{1,0}
\]
by
\[
c_i^{\text{wheel}} \left( \bigotimes_{j=1}^i (a_j \otimes b_j \otimes d_j^i) \right) = \left( \prod_{j=1}^i d_j^i(b_{j-1}) \right) a_1 \wedge a_2 \wedge \cdots \wedge a_i,
\]
where \( b_0 \) means \( b_i \).

4.2. Non-trivial quotient representation of \( H_i^A(\text{IA}_n, \mathbb{Q}) \). Here we detect two irreducible quotient representations of \( H_i^A(\text{IA}_n, \mathbb{Q}) \) by using \( c_i^{\text{tree}} \) and \( c_i^{\text{wheel}} \).

Recall that in Definition 3.2, we defined the abelian cycle \( \alpha_{(0,i)} \in H_i(\text{IA}_n, \mathbb{Q}) \) corresponding to the pair of partition \((0,i)\) for \( n \geq i + 1 \), and that we have the following \( \text{GL}(n, \mathbb{Z}) \)-homomorphisms
\[
H_i(\text{IA}_n, \mathbb{Q}) \xrightarrow{\tau_*} H_i(U, \mathbb{Q}) = \bigwedge^i U \xrightarrow{\iota_i} M_i \xrightarrow{c_i^{\text{wheel}}} U_{\text{wheel}}^i \xrightarrow{c_i^{\text{tree}}} U_{\text{tree}}^i,
\]
where \( \tau_* \) is induced by the Johnson homomorphism.

**Lemma 4.1.** For \( n \geq i + 1 \), we have
\[
c_i^{\text{wheel}} \iota_i \tau_* (\alpha_{(0,i)}) = i! e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \in U_{\text{wheel}}^i \setminus \{0\},
\]
and
\[
c_i \iota_i \tau_* (\alpha_{(0,i)}) = (-1)^i (i+1)! e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_i^*.
\]
Therefore, for \( n \geq i + 2 \), we have
\[
(id - E_{1,n})c_i^{\text{tree}} \iota_i \tau_* (\alpha_{(0,i)}) = (-1)^i (i+1)! e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_i^* \in U_{\text{tree}}^i \setminus \{0\}.
\]
\textbf{Proof.} We have

\[ \tau_*(a_{(i,j)}) = e_{2,1}^i \wedge (e_{3,1}^i + e_{3,2}^i) \wedge \cdots \wedge (\sum_{j=1}^i c_{i+1,j}^i) \in \bigwedge^i U. \]

First, we consider the image under \( e_{i}^{\text{wheel}} \). By the definition of \( e_{i}^{\text{wheel}} \), we have

\[ c_{i}^{\text{wheel}} \iota_i \tau_*(a_{(0,i)}) = c_{i}^{\text{wheel}} \iota_i (e_{2,1}^i \wedge e_{3,1}^i \wedge \cdots \wedge e_{i+1,1}^i) \in \mathbb{Z}(e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1}). \]

Since the inclusion \( \iota_i \) gives \( i! \) copies of \( e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \), we obtain

\[ c_{i}^{\text{wheel}} \iota_i \tau_*(a_{(0,i)}) = i! \cdot e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \in V_{1,0} \setminus \{0\} \cong U^{\text{wheel}}_i \setminus \{0\}. \]

Next, we consider the image under the contraction map \( c_{i+1}^{\text{tree}} \). For \( K = (k_1, \cdots, k_{3i}) \in [n]^{3i} \), let \( e_K = \bigotimes_{j=1}^i (e_{k_j} \otimes e_{k_{2i+1-j}} \otimes e_{k_{2i+j}}) \). Then \( \{e_K \mid K \in [n]^{3i}\} \) forms a basis for \( M_i \). We write

\[ \iota_i \tau_*(a_{(0,i)}) = \sum_{K \in [n]^{3i}} a_K e_K \in M_i, \]

where \( a_K \in \mathbb{Z} \). Then we can easily check the following properties of the coefficient \( a_K \):

- for each \( K \in [n]^{3i} \), we have \( a_K \in \{0, 1, -1\} \).
- for \( K \notin [i+1]^{3i} \), we have \( a_K = 0 \).
- for \( K \in [i+1]^{3i} \) such that \( k_{2i+j} \neq \min\{k_j, k_{2i+1-j}\} \) for some \( j \in [i] \), we have \( a_K = 0 \).

By the above properties, if we have \( c_i(a_K e_K) \in (\bigwedge^{i+1} H \otimes H^*) \setminus \{0\} \), then there exists a permutation \( \sigma \in \mathfrak{S}_{i+1} \) such that for any \( j \in [i+1] \), we have \( k_j = \sigma(j) \).

\textbf{Claim 1.} For each \( \sigma \in \mathfrak{S}_{i+1} \), there uniquely exists \( K \in [i+1]^{3i} \) such that

- \( k_j = \sigma(j) \) for \( j \in [i+1] \)
- \( c_i(a_K e_K) \neq 0 \).

In particular, we have

\[ c_i(a_K e_K) = a_K (e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \cdots \wedge e_{\sigma(i+1)}) \otimes e_j^* \neq 0. \]

\textbf{Proof of Claim 1.} Define \( K \in [i+1]^{3i} \) as

\[ k_j = \begin{cases} \sigma(j) & (1 \leq j \leq i+1) \\ \min\{k_{2i+2-j}, \cdots, k_{i+1}\} & (i+2 \leq j \leq 2i) \\ \min\{k_{j-2i}, \cdots, k_{i+1}\} & (2i+1 \leq j \leq 3i). \end{cases} \]

Then we can check that

\[ c_i(a_K e_K) = a_K (e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \cdots \wedge e_{\sigma(i+1)}) \otimes e_j^*. \]

The uniqueness follows from the construction. \( \square \)

\textbf{Claim 2.} For each \( \sigma \in \mathfrak{S}_{i+1} \), let \( K(\sigma) \) denote the unique element of \([i+1]^{3i}\) that we constructed in the proof of Claim 1. Then we have \( a_{K(\sigma)} = (-1)^i \text{sgn}(\sigma) \).
Proof of Claim. We use the induction on $i$. For $i = 1$, we have
\[ a_{K(id)} = -1 = (-1)^1 \text{sgn}(id), \quad a_{K((12))} = 1 = (-1)^1 \text{sgn}((12)). \]
Assume that the statement holds for $i \geq 1$. For any $\sigma \in \mathfrak{S}_{i+1}$, define $\tau \in \mathfrak{S}_{i+1}$ as
\[ \tau(j) = \begin{cases} 
\sigma(1) & (j = 1) \\
 j - 1 & (2 \leq j \leq \sigma(1)) \\
 j & (\sigma(1) + 1 \leq j \leq i + 1). 
\end{cases} \]
Let $\tilde{\tau} = \tau^{-1}. Then $\tilde{\tau}$ maps 1 to 1, so we regard $\tilde{\tau} \in \mathfrak{S}_i$ by identifying $\{2, \ldots, i+1\}$ with $\{1, \ldots, i\}$ via the order preserving bijection. We have
\[ a_{K(\sigma)} = a_{K(\tilde{\tau}^{-1})} = \begin{cases} 
(-1)^{\sigma(1)}a_{K(\tilde{\tau})} & (\sigma(1) = 1) \\
(-1)^{\sigma(1)-2}a_{K(\tilde{\tau})} & (\sigma(1) \neq 1). 
\end{cases} \]
Therefore, in any case, we have $a_{K(\sigma)} = (-1)^{\sigma(1)}a_{K(\tilde{\tau})}$. Since we have $\text{sgn}(\tau) = (-1)^{\sigma(1)-1}$, by the hypothesis of the induction, we have
\[ (-1)^{\sigma(1)}a_{K(\tilde{\tau})} = (-1)^{\sigma(1)}(-1)^i \text{sgn}(\tilde{\tau}) = (-1)^{i+1} \text{sgn}(\tau) \text{sgn}(\tilde{\tau}) = (-1)^{i+1} \text{sgn}(\sigma), \]
which completes the proof.

By using Claims 1 and 2, we have
\[ c_i t_i \tau_* (\alpha_{(0,i)}) = \sum_{K \in [n]^{i+1}} c_i (a_K e_K) = \sum_{\sigma \in \mathfrak{S}_{i+1}} a_{K(\sigma)} (e_{\sigma(1)} \wedge e_{\sigma(2)} \wedge \cdots \wedge e_{\sigma(i+1)}) \otimes e_1^* \]
\[ = \sum_{\sigma \in \mathfrak{S}_{i+1}} a_{K(\sigma)} \text{sgn}(\sigma) (e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1}) \otimes e_1^* \]
\[ = (-1)^i (i + 1)! e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_1^*. \]
Since for $n \geq i + 2$, we have
\[ (id - E_{1,n}) c_i t_i \tau_* (\alpha_{(0,i)}) = (-1)^i (i + 1)! e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_n^* \in U_i^{\text{tree}} \setminus \{0\}, \]
the proof is complete.

Theorem 4.2. For $n \geq i + 1$, we have a surjective $\text{GL}(n, \mathbb{Q})$-homomorphism
\[ H^A_i(IA_n, \mathbb{Q}) \rightarrow \text{Hom}(H, \bigwedge^{i+1} H) \cong V_{1,0} \oplus V_{1,1+1,1}. \]
Therefore, $H^A_i(IA_n, \mathbb{Q})$ includes a $\text{GL}(n, \mathbb{Q})$-subrepresentation which is isomorphic to $\text{Hom}(H, \bigwedge^{i+1} H)$.

Proof. This directly follows from Lemma 4.1.
4.3. A representation-stable subrepresentation of $H_t^A(IA_n, \mathbb{Q})$. Here we observe that the non-trivial subrepresentation of $H_t^A(IA_n, \mathbb{Q})$ that is detected in Theorem 4.2 is representation stable.

We have a canonical inclusion map $IA_n \hookrightarrow IA_{n+1}$ sending $f \in IA_n$ to $f \in IA_{n+1}$ which is the same as $f$ on $F_n \subset F_{n+1}$ and fixes $x_{n+1}$. Let $H_t(IA_n, \mathbb{Q})$ denote the sequence of group homomorphisms $H_t(IA_n, \mathbb{Q}) \to H_t(IA_{n+1}, \mathbb{Q})$ induced by the inclusion maps. Since the inclusion map $IA_n \hookrightarrow IA_{n+1}$ sends $g_a, b \in IA_n$ to $g_a, b \in IA_{n+1}$ and $f_{a, b, c} \in IA_n$ to $f_{a, b, c} \in IA_{n+1}$, the abelian cycle $\alpha_{(0, i)} \in H_t(IA_n, \mathbb{Q})$ is sent to $\alpha_{(0, i)} \in H_t(IA_{n+1}, \mathbb{Q})$. Therefore, the element $\tau_*(\alpha_{(0, i)}) \in H_t^A(IA_n, \mathbb{Q})$ is sent to $\tau_*(\alpha_{(0, i)}) \in H_t^A(IA_{n+1}, \mathbb{Q})$.

We can check that the canonical inclusion maps $H_t(n) \hookrightarrow H_t(n+1)$ and $H_t(n)^* \hookrightarrow H_t(n+1)^*$ that we have observed in Section 2.7 send $e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_i^* \in \bigwedge^{i+1} H_t(n) \otimes H_t(n)^*$ to $e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_i^* \in \bigwedge^{i+1} (H_t(n+1) \otimes H_t(n+1))^*$. Let $(\bigwedge^{i+1} H_t(n)) \otimes H_t(n)^* \otimes (\bigwedge^{i+1} H_t(n+1)) \otimes H_t(n+1)^*$.

By Theorem 4.2 we have the following proposition.

**Proposition 4.3.** The sequence $H_t^A(IA_n, \mathbb{Q})$ includes a representation-stable sequence which is isomorphic to $(\bigwedge^{i+1} H) \otimes H^*$.

By the representation stability, we can consider an analogue of Theorem 4.2 in the category of algebraic $GL(\infty, \mathbb{Q})$-representations. Let $IA_\infty = \lim_n IA_n$ denote the direct limit. Both group homology and Albanese homology preserve direct limits (see Proposition 4.1.4), that is, we have

$$H_t(IA_\infty, \mathbb{Q}) \cong \lim_n H_t(IA_n, \mathbb{Q}), \quad H_t^A(IA_\infty, \mathbb{Q}) \cong \lim_n H_t^A(IA_n, \mathbb{Q}).$$

Since $\alpha_{(0, i)}$ is preserved under $H_t(IA_n, \mathbb{Q}) \to H_t(IA_{n+1}, \mathbb{Q})$, we obtain a cycle $\alpha_{(0, i)}$ in $H_t(IA_\infty, \mathbb{Q})$ and an element $\tau_*(\alpha_{(0, i)}) \in H_t^A(IA_\infty, \mathbb{Q})$. Therefore, by Lemma 4.1 we have a $GL(\infty, \mathbb{Q})$-homomorphism

$$c_{i, t_i} : H_t^A(IA_\infty, \mathbb{Q}) \to (\bigwedge^{i+1} H) \otimes H^*$$

such that

$$c_{i, t_i} \tau_*(\alpha_{(0, i)}) = (-1)^{(i+1)!} e_1 \wedge e_2 \wedge \cdots \wedge e_{i+1} \otimes e_i^*.$$

**Corollary 4.4.** The $GL(\infty, \mathbb{Q})$-homomorphism

$$c_{i, t_i} : H_t^A(IA_\infty, \mathbb{Q}) \to (\bigwedge^{i+1} H) \otimes H^*$$

is surjective. In particular, for each $i \geq 1$, $H_t(IA_\infty, \mathbb{Q})$ is infinite dimensional.

**Remark 4.5.** We have a non-split exact sequence

$$0 \to V_{i+1, 1} \to (\bigwedge^{i+1} H) \otimes H^* \to V_{1, 0} \to 0,$$

which generalizes the observation in Section 2.7 that $H \otimes H^* \to \mathbb{Q}$ does not split. Therefore, $H_t^A(IA_\infty, \mathbb{Q})$ is not semisimple.
5. The traceless part of $H^i_\uparrow(IA_n, \mathbb{Q})$

Let $H_i(U, \mathbb{Q})^{\uparrow\downarrow}$ denote the traceless part $\bigwedge^i U$ of $H_i(U, \mathbb{Q}) = \bigwedge^i U$. In this section, we show that the traceless part of $H_i(U, \mathbb{Q})$ is contained in $H^i_\uparrow(IA_n, \mathbb{Q})$. Note that we have $H_i(U, \mathbb{Q})^{\uparrow\downarrow} = \pi(T_{2i,i})$, where $\pi : H^{\otimes 2i} \otimes (H^*)^{\otimes i} = M_i \to \bigwedge^i U$ is defined by

$$\left( \bigotimes_{j=1}^i (a_j \otimes b_j) \right) \otimes \left( \bigotimes_{j=1}^i c_j \right) \mapsto \bigwedge_{j=1}^i ((a_j \wedge b_j) \otimes c_j) .$$

**Theorem 5.1.** Let $n \geq 3i$. We have $H_i(U, \mathbb{Q})^{\uparrow\downarrow} \subset H^i_\uparrow(IA_n, \mathbb{Q})$.

**Proof.** By Corollary 2.3, the traceless part of $H_i(U, \mathbb{Q})^{\uparrow\downarrow}$ is generated by $\pi(e_{2i,i}) = \bigwedge_{j=1}^i e_{2j-1,2j}^{n-j+1}$ as a $\text{GL}(n, \mathbb{Q})$-representation. Therefore, we have to show that $\pi(e_{2i,i}) \in H^i_\uparrow(IA_n, \mathbb{Q})$. For this purpose, we use the abelian cycle $\alpha_{(0,1^i)} \in H_i(IA_n, \mathbb{Q})$ that we defined in Definition 5.2. We have

$$\tau_*(\alpha_{(0,1^i)}) = \bigwedge_{j=1}^i e_{2j-1,2j}^{2j-1} \in H^i_\uparrow(IA_n, \mathbb{Q}).$$

We can transform $\tau_*(\alpha_{(0,1^i)})$ into $\pi(e_{2i,i})$ by an action of

$$\prod_{j=1}^i (\text{id} - E_{2j-1,n-j+1}) \in \mathbb{Q}[\text{GL}(n, \mathbb{Q})].$$

Therefore, we have $\pi(e_{2i,i}) \in H^i_\uparrow(IA_n, \mathbb{Q})$. \qed

Church–Ellenberg–Farb [4 Theorem 7.2.3] proved that for each $i \geq 0$, there exists a polynomial $P_i(T)$ of degree $\leq 3i$ such that $\text{dim}_{\mathbb{Q}}(H^i_\uparrow(IA_n, \mathbb{Q})) = P_i(n)$ for sufficiently large $n$ with respect to $i$. On the other hand, we have the following decomposition of $H_i(U, \mathbb{Q})$

$$(5.0.1) \quad H_i(U, \mathbb{Q}) = \bigwedge^i U = \bigoplus_{\lambda \vdash i} (S_\lambda(S_{12}H) \otimes S_\lambda(H^*)) ,$$

where $S_\lambda$ denotes the Schur functor, which sends a vector space $V$ to $S_\lambda(V) = V_\lambda$, and where $\lambda'$ denotes the conjugate partition $\lambda'$ to $\lambda$, which is obtained from the Young diagram corresponding to $\lambda$ by interchanging rows and columns. (See [14 Exercise 6.11].) In particular, we have $V_{12i,1} \subset H_i(U, \mathbb{Q})^{\uparrow\downarrow}$. By (5.0.1), we obtain $\text{dim}_{\mathbb{Q}}(H_i(U, \mathbb{Q})^{\uparrow\downarrow}) = P'_i(n)$ for $n \geq 3i$, where $P'_i(T)$ is a polynomial of degree $3i$. By Theorem 5.1, we obtain the following theorem.

**Theorem 5.2.** For $n \geq 3i$, we have $\text{dim}_{\mathbb{Q}}(H^i_\uparrow(IA_n, \mathbb{Q})) \geq P'_i(n)$. Moreover, there exists a polynomial $P_i(T)$ of degree $3i$ such that $\text{dim}_{\mathbb{Q}}(H^i_\uparrow(IA_n, \mathbb{Q})) = P_i(n)$ for sufficiently large $n$ with respect to $i$. 
6. The structure of \( H^4_i(IA_n, \mathbb{Q}) \)

In this section, we introduce a graded \( GL(n, \mathbb{Q}) \)-representation \( W_n \). The degree \( i \) part \( W_i \) is graded over the set of pairs of partitions of total size \( i \). We show that \( W_i \) is a subquotient of \( H^4_i(IA_n, \mathbb{Q}) \). The non-trivial quotient representations that we detected in Theorem 4.2 are the degree \((i,0)\) and \((0,i)\) parts, and the traceless part \( H_0(U, \mathbb{Q})^{11} \) is the direct sum of the degree \((1^j, 1^k)\) parts for \( j + k = i \).

6.1. Conjectural structure of \( H^4_i(IA_n, \mathbb{Q}) \).  Recall that for \( n \geq i + 1 \), we have

\[
U_i = \text{Hom}(H, \bigwedge^{i+1} H) = U_i^{\text{tree}} \oplus U_i^{\text{wheel}} \cong V_{1^{i+1},1} \oplus V_{1^i,0}.
\]

Let \( U_* = \bigoplus_{i \geq 1} U_i \), which is a graded algebraic \( GL(n, \mathbb{Q}) \)-representation. Define \( W_* = \tilde{S}^*(U_*) \) as the traceless part of the graded-symmetric algebra \( S^*(U_*) \) of \( U_* \), which we defined in Section 4.4. We can also construct \( W_* \) by using an operad \( \text{Com} \) of non-unital commutative algebras as we explain in Section 4.2.

For \( i \geq 0 \), let \((\mu, \nu) \vdash i\) be a pair of partitions. If we write

\[
\mu = (\mu_1^{k_1}, \ldots, \mu_r^{k_r}), \quad \nu = (\nu_1^{k_1'}, \ldots, \nu_s^{k_s'}),
\]

then it means that \( \mu_1 > \mu_2 > \cdots > \mu_r \) and \( \mu_j \) appears \( k_j > 0 \) times in \( \mu \) and that \( \nu_1 > \nu_2 > \cdots > \nu_s \) and \( \nu_j \) appears \( k_j' > 0 \) times in \( \nu \). Then the length of \( \mu \) (resp. \( \nu \)) is \( l(\mu) = \sum_{j=1}^r \nu_j \) (resp. \( l(\nu) = \sum_{j=1}^s \nu_j \)). Let

\[
U^{\text{tree}}_\mu = \bigotimes_{j=1}^r S^{\nu_j}(U_{\mu_j}^{\text{tree}}), \quad U^{\text{wheel}}_\nu = \bigotimes_{j=1}^s \tilde{S}^{k_j}(U_{\nu_j}^{\text{wheel}}).
\]

Then we have

\[
S^*(U_*) = \bigoplus_{i \geq 0} \bigoplus_{(\mu, \nu) \vdash i} U^{\text{tree}}_\mu \otimes U^{\text{wheel}}_\nu.
\]

Since \( W = \tilde{S}^*(U_*) \), by the definition of \( W_* \), we have

\[
W_i = \bigoplus_{(\mu, \nu) \vdash i} W(\mu, \nu) \subset S^*(U_*)_i,
\]

where

\[
W(\mu, \nu) = \bigotimes_{j=1}^r \tilde{S}^{\nu_j}(U_{\mu_j}^{\text{tree}}) \otimes \bigotimes_{j=1}^s \tilde{S}^{k_j}(U_{\nu_j}^{\text{wheel}}) \subset U^{\text{tree}}_\mu \otimes U^{\text{wheel}}_\nu.
\]

For example, we have for \( n \geq 3 \),

\[
W_1 = W(1,0) \oplus W(0,1) \cong V_{12,1} \oplus V_{1,1},
\]

and we have for \( n \geq 6 \),

\[
W_2 = W(2,0) \oplus W(0,2) \oplus W(1^2,0) \oplus W(1,1) \oplus W(0,1^2) \cong V_{13,1} \oplus V_{12,0} \oplus (V_{14,12} \oplus V_{212,2} \oplus V_{22,1}) \oplus (V_{21,1} \oplus V_{13,1}) \oplus V_{12,0}.
\]

Here, we observe that the subrepresentation \( W(\mu, \nu) \) of \( W_i \) is an image of the traceless part \( T_{i+(\mu, \mu)} \) under a projection. Let

\[
\pi_j^\mu : H^{\otimes k_j^0(1+\mu_j)} \otimes (H^*)^{\otimes k_j'} = (H^{\otimes (1+\mu_j)} \otimes H^*)^{\otimes k_j'} \rightarrow (V_{1^{1+\mu_j},1})^{\otimes k_j'} \rightarrow \tilde{S}^{k_j'}(V_{1^{1+\mu_j},1})
\]
\( \pi^{\nu_j} : H \otimes k''_{j\nu} \rightarrow (V_{i+1}) \otimes S^{k''_{j\nu}} (V_{i+1}) \)

be the composition of two canonical projections, and let

\[
\pi^{(\mu,\nu)} = \left( \bigotimes_{j=1}^{r} \pi^{\mu_j} \right) \otimes \left( \bigotimes_{j=1}^{s} \pi^{\nu_j} \right) : H \otimes \bigotimes_{j=1}^{i+1} (\mathbb{C}^{c_{\mu_{j}}}) \otimes H \rightarrow \mathbb{C}^{k''_{\mu_{j}}} \otimes \mathbb{C}^{k''_{\nu_{j}}}. 
\]

Then we have

\[
W(\mu,\nu) = \pi^{(\mu,\nu)} (T_{i+1}(\mu,\nu)).
\]

Define

\[
F_{i}(\mu,\nu) : H_i(U, \mathbb{Q}) \rightarrow \bigotimes_{i} U_{\mu}^{\text{tree}} \otimes U_{\nu}^{\text{wheel}}
\]

to be the composition of the inclusion \( i_{i} : \bigotimes_{i} U \hookrightarrow M_{i} \) and

\[
\left( \bigotimes_{j=1}^{r} \frac{1}{k''_{j}} \prod_{\mu_{j}} \right) \otimes \left( \bigotimes_{j=1}^{s} \frac{1}{k''_{j}} \prod_{\nu_{j}} \right) : M_{i} \rightarrow U_{\mu}^{\text{tree}} \otimes U_{\nu}^{\text{wheel}}.
\]

Then we have a GL\((n, \mathbb{Q})\)-homomorphism

\[
F_{i} := \bigoplus_{(\mu,\nu)=i} F_{i}(\mu,\nu) : H_i(U, \mathbb{Q}) \rightarrow \bigoplus_{(\mu,\nu)=i} U_{\mu}^{\text{tree}} \otimes U_{\nu}^{\text{wheel}}.
\]

**Theorem 6.1.** For \( n \geq 3i \), we have

\[
F_{i}(H_{i}^{A}(\mathbb{Q})) \supset W_{i}.
\]

We will prove Theorem 6.1 in the rest of this section by using the abelian cycles \( \alpha(\mu,\nu) \). In a way similar to Proposition 4.3, we can check that the Albanese homology \( H_{i}^{A}(\mathbb{Q}) \) includes a representation-stable subrepresentation which is isomorphic to \( W_{i} \).

It seems natural to make the following conjecture, which holds for \( i = 1 \) [8, 13, 20], for \( i = 2 \) [31] and for \( i = 3 \) as we will observe in Theorem 11.1.

**Conjecture 6.2.** For \( n \geq 3i \), \( F_{i} \) restricts to a GL\((n, \mathbb{Q})\)-isomorphism

\[
F_{i} : H_{i}^{A}(\mathbb{Q}) \cong W_{i}.
\]

Conjecture 6.2 implies that as a GL\((n, \mathbb{Q})\)-representation, the Albanese homology \( H_{i}^{A}(\mathbb{Q}) \) is generated by the images of abelian cycles of \( H_{i}(\mathbb{Q}) \) under \( \tau_{i} \).

We would like to realize \( W_{i} \) as a subrepresentation of \( H_{i}^{A}(\mathbb{Q}) \).

**Problem 6.3.** Construct a lift

\[
W_{i} \rightarrow H_{i}^{A}(\mathbb{Q})
\]

of the inclusion \( W_{i} \hookrightarrow F_{i}(H_{i}^{A}(\mathbb{Q})) = F_{i}(H_{i}^{A}(\mathbb{Q})) \) along \( H_{i}^{A}(\mathbb{Q}) \rightarrow F_{i}(H_{i}^{A}(\mathbb{Q})) \) as GL\((n, \mathbb{Q})\)-representations for sufficiently large \( n \) with respect to \( i \).

We would like to realize the Albanese homology \( H_{i}^{A}(\mathbb{Q}) \) as a subrepresentation of \( H_{i}(\mathbb{Q}) \).
Proof. For the abelian cycle

\[ H_i(\text{IA}_n, \mathbb{Q}) \xrightarrow{\tau_*} H_i^A(\text{IA}_n, \mathbb{Q}) \]

as $\text{GL}(n, \mathbb{Z})$-representations for sufficiently large $n$ with respect to $i$.

6.2. Computation of the contraction maps. Here we consider a condition for $(\mu, \nu)$ and $(\xi, \eta)$ that $F(\mu, \nu)(\tau_*(\alpha(\xi, \eta)))$ vanishes.

In a way similar to Lemma 4.1, we obtain the following lemma.

**Lemma 6.5.** For $n \geq i + 2$, we have

\[ e^i_{\text{wheel}_i} \tau_*(\alpha(i,0)) = 0, \]

and

\[ e^i_{\text{tree}_i} \tau_*(\alpha(i,0)) = (-1)^{i-1}(i + 1)! e_1 \wedge e_2 \wedge \cdots \wedge e_{i+2} \otimes e^*_i \in U^i_{\text{tree}} \setminus \{0\}. \]

**Proof.** For the abelian cycle $\alpha(i,0)$, we have

\[ \tau_*(\alpha(i,0)) = e_3^{i,2} \wedge \left( \sum_{j=1}^{3} e^j_{4,j} \right) \wedge \cdots \wedge \left( \sum_{j=1}^{i+1} e^j_{i+2,j} \right). \]

Let $x = e^{a_1, b_1}_{d_1} \wedge e^{a_2, b_2}_{d_2} \wedge \cdots \wedge e^{a_i, b_i}_{d_i} \in \wedge^i U$. If $d_1 \notin \{a_1, \ldots, a_i, b_i\}$, then we can check that $e^i_{\text{wheel}_i} \tau_*(x) = 0$. Therefore, we have $e^i_{\text{wheel}_i} \tau_*(\alpha(i,0)) = 0$.

The computation of the image of $\tau_*(\alpha(i,0))$ under $e^i_{\text{tree}_i} \tau_*$ is similar to the computation of the image of $\tau_*(\alpha(0,i))$ in Lemma 4.1. \qed

We call elements of the form

\[ e^{d_1}_{a_1, b_1} \wedge e^{d_2}_{a_2, b_2} \wedge \cdots \wedge e^{d_i}_{a_i, b_i} \]

the basis elements of $\wedge^i U$. Two basis elements $x = e^{d_1}_{a_1, b_1} \wedge e^{d_2}_{a_2, b_2} \wedge \cdots \wedge e^{d_k}_{a_k, b_k}$ of $\wedge^k U$ and $y = e^{r_1}_{p_1, q_1} \wedge e^{r_2}_{p_2, q_2} \wedge \cdots \wedge e^{r_i}_{p_i, q_i}$ of $\wedge^l U$ are said to be disjoint if \{a_j, b_j, d_j\}_{j=1}^{i} \cap \{p_j, q_j, r_j\}_{j=1}^{l} = \emptyset$. If basis elements $x \in \wedge^k U$ and $y \in \wedge^l U$ are disjoint, then it is easy to see that

\[ e^i_{\text{wheel}_i} \tau_*(x \wedge y) = e^i_{\text{tree}_i} \tau_*(x \wedge y) = 0. \]

Let $P_1$ denote the set of pairs of partitions of total size $i$. For $l \in \{0, \ldots, i\}$, let $P^l_1$ denote the subset of $P_1$ consisting of elements $(\mu, \nu)$ with $l(\mu) = l$. For $(\xi, \eta), (\mu, \nu) \in P^l_1$, we write $(\xi, \eta) \geq (\mu, \nu)$ if $\xi_j \geq \mu_j$ for all $j \in [l]$ and if there exist a decomposition $L_1 \sqcup \cdots \sqcup L_{l(\eta)} = \{1, \ldots, l(\eta)\}$ and $\sigma \in \mathcal{S}_l$ such that

\[ \xi_j - \mu_{\sigma(j)} = \sum_{k \in L_j} \nu_k \quad (1 \leq j \leq l), \]

\[ \eta_j = \sum_{k \in L_{l(\eta) + j}} \nu_k \quad (1 \leq j \leq l(\eta)). \]

We can check that $(P^l_1, \geq)$ is a partially ordered set with the minimum element $(1^l, 1^{i-l})$ and with the maximal elements $(\mu, 0)$ for $\mu \vdash i$ with $l(\mu) = l$. 

Lemma 6.6. For $(\xi, \eta), (\mu, \nu) \in P^d_i$, we have
\[ F_{(\mu, \nu)}(\tau_*(\alpha_{(\xi, \eta)})) = 0 \quad \text{if } (\xi, \eta) \nsucc (\mu, \nu). \]

Proof. Let $(\mu, \nu) \in P^d_i$. Suppose that we have $F_{(\mu, \nu)}(\tau_*(\alpha_{(\xi, \eta)})) \neq 0$ for $(\xi, \eta) \in P^d_i$.

We have
\[ (6.2.4) \quad \tau_*(\alpha_{(\xi, \eta)}) = \tau(\xi, 1) \land \cdots \land \tau(\xi, l) \land \tau(\eta, 1) \land \cdots \land \tau(\eta, l(\eta)) \in \bigwedge^i U, \]
where $\tau(\xi, j)$ (resp. $\tau(\eta, j)$) is obtained from $\tau_*\alpha_{(\xi, 0)}$ (resp. $\tau_*\alpha_{(0, \eta)}$) by the shift that is appeared in the definition of $\alpha_{(\xi, \eta)}$. Each $\tau(\xi, j)$ is a linear sum of basis elements $e_{a_1, b_1}^d \land e_{a_2, b_2}^b \land \cdots \land e_{a_{j-1}, b_{j-1}}^b$ of $\bigwedge^j U$ such that $d_1 \notin \{a_1, \ldots, a_{j-1}, b_1\}$. As in the proof of Lemma 6.7 for any subset $K \subset \{2, \ldots, \xi\}$, we have
\[ c_{\xi+1)^{|K|}}(e_{a_1, b_1}^d \land \bigwedge_{k \in K} e_{a_k, b_k}^b) = 0. \]

Since $e_{a_1, b_1}$ and any basis elements that appear in $\tau(\xi, p)(1 \leq p \leq l, p \neq j)$ or $\tau(\eta, q)(1 \leq q \leq l(\eta))$ are disjoint, the wedge product of $e_{a_1, b_1}$ and any other factors vanishes under $\epsilon_{\mu, \nu}^{\text{tree}}$, for any $j' \in [l]$ and $\epsilon_{\nu, j'}^{\text{wheel}}$, for any $j' \in [l(\nu)]$. Therefore, in order to satisfy $F_{(\mu, \nu)}(\tau_*(\alpha_{(\xi, \eta)})) \neq 0$, $e_{a_1, b_1}$ has to be mapped under $\epsilon_{\mu, j'}^{\text{tree}}$, for some $j' \in [l]$ as a wedge product with some other factors in $\tau(\xi, j')$. It follows that we need $\xi_j \geq \mu_j$ for all $j \in [l]$. In what follows, we restrict to the condition that for any $j \in [l]$, any element $e_{a_1, b_1}$ that appears in $\tau(\xi, j)$ is mapped under $\epsilon_{\mu, j'}^{\text{tree}}$, for some $j' \in [l]$ as a wedge product with some other factors in $\tau(\xi, j)$.

If a pair of $\sigma \in \mathcal{G}_l$ and $L_1 \cup \cdots \cup L_{l+1(\nu)} = \{1, \ldots, l(\nu)\}$ does not satisfy (6.2.2), then one of the following holds:

- there exist distinct elements $j, j' \in [l]$ such that the wedge product of some factors of a basis element which appears in $\tau(\xi, j)$ and other factors of a basis element which appears in $\tau(\xi, j')$ are mapped under $\epsilon_{\nu, j''}^{\text{wheel}}$, for some $j'' \in [l(\nu)]$,
- there exist $j \in [l]$ and $j' \in [l(\nu)]$ such that the wedge product of some factors of a basis element which appears in $\tau(\xi, j)$ and other factors of a basis element which appears in $\tau(\eta, j')$ are mapped under $\epsilon_{\nu, j''}^{\text{wheel}}$, for some $j'' \in [l(\nu)]$.

By (6.2.1), in both cases, the values under $\epsilon_{\nu, j''}^{\text{wheel}}$ are zero, which contradicts $F_{(\mu, \nu)}(\tau_*(\alpha_{(\xi, \eta)})) \neq 0$. The case where a pair of $\sigma \in \mathcal{G}_l$ and $L_1 \cup \cdots \cup L_{l+1(\nu)} = \{1, \ldots, l(\nu)\}$ does not satisfy (6.2.3) is similar. Therefore, we need such a pair, which completes the proof. \( \square \)

6.3. Proof of Theorem 6.1. To prove Theorem 6.1, we follow the previous lemma, which is an analogue of [27, Theorem 1.5].

Lemma 6.7. Let $(\mu, \nu) \vdash i$. Then for $n \geq i + 2l(\mu) + l(\nu)$, we have
\[ F_{(\mu, \nu)}(H^4_i(IA_n, Q)) \supseteq W(\mu, \nu). \]

Proof. Our proof is analogous to the proof of [27 Theorem 1.5]. By Corollary 2.3, $W(\mu, \nu) = \pi(\mu, \nu)(T_{i+1(\mu), l(\mu)})$ is generated by the element $\pi(\mu, \nu)(e_{i+1(\mu), l(\mu)})$. \( \square \)
Therefore, it suffices to show that there is an element \( x \in \mathbb{Q}[\text{GL}(n, \mathbb{Q})] \) such that \( x(F_{(\mu, \nu)}(\tau_*(\alpha_{(\mu, \nu)}))) = \pi(\mu, \nu)(e_{i+l(\mu), j(\mu)}) \).

Here, we write

\[
\mu = (\mu_1, \ldots, \mu_s(\mu)) = (\tilde{\mu}_1^1, \ldots, \tilde{\mu}_r^1), \quad \nu = (\nu_1, \ldots, \nu_{l(\nu)}) = (\tilde{\nu}_1^\nu, \ldots, \tilde{\nu}_s^\nu).
\]

As in (6.2.3), we have

\[
\tau_*(\alpha_{(\mu, \nu)}) = \tau(\mu, 1) \land \cdots \land \tau(\mu, l(\mu)) \land \tau(\nu, 1) \land \cdots \land \tau(\nu, l(\nu)) \in \bigwedge^i U.
\]

For \( 1 \leq j \leq r \) and \( 1 \leq m \leq k_j^\mu \), let \( \tau(\mu, j, m) = \tau(\mu, \phi_{\mu}(j, m)) \), where \( \phi_{\mu}(j, m) = k_1^\mu + \cdots + k_{j-1}^\mu + m \). In a similar way, we define \( \phi_{\nu}(j, m) \) and \( \tau(\nu, j, m) \). Then we have

\[
F_{(\mu, \nu)}(\tau_*(\alpha_{(\mu, \nu)}))
= \left( \bigotimes_{j=1}^r \frac{k_j^\mu}{k_j^\mu} \prod_{m=1}^{k_j^\mu} c_{\tilde{\mu}_j}^{\text{tree}}(\tilde{\mu}_j, j, m) \right) \otimes \left( \bigotimes_{j=1}^s \frac{k_j^\nu}{k_j^\nu} \prod_{m=1}^{k_j^\nu} c_{\tilde{\nu}_j}^{\text{wheel}}(\tilde{\nu}_j, j, m) \right).
\]

By Lemma 6.5, we have

\[
c_{\tilde{\mu}_j}^{\text{tree}}(\tilde{\mu}_j, j, m) = (-1)^{\tilde{\mu}_j - 1}(\tilde{\mu}_j + 1)! s(\phi_{\mu}(j, m))(e_1 \land e_2 \land e_3 \land \cdots \land e_{\tilde{\mu}_j} + 2 \otimes e_3),
\]

where \( s(j) \) is the function that we used in Section 3.2 to define the abelian cycle \( \alpha_{(\mu, \nu)} \), and where \( s(\phi_{\mu}(j, m)) \) denotes the shift homomorphism by \( s(\phi_{\mu}(j, m)) \). By Lemma 6.1, we also have

\[
ce_{\tilde{\mu}_j}^{\text{wheel}}(\tilde{\mu}_j, j, m) = (\tilde{\nu}_j)! t(\phi_{\nu}(j, m))(e_2 \land e_3 \land \cdots \land e_{\tilde{\mu}_j + 1}).
\]

Therefore, we can take an element \( x \in \mathbb{Q}[\text{GL}(n, \mathbb{Q})] \) to satisfy \( x(F_{(\mu, \nu)}(\tau_*(\alpha_{(\mu, \nu)}))) = \pi(\mu, \nu)(e_{i+l(\mu), j(\mu)}) \), which completes the proof. \( \Box \)

**Remark 6.8.** Let \( \mu \cup \nu \) denote the partition of \( i \) that is obtained from \( \mu \) and \( \nu \) by reordering the parts. We can also use the abelian cycle \( \alpha_{(0, \mu \cup \nu)} \) to prove Lemma 6.7. That is, we can also show that \( \tau_*(\alpha_{(0, \mu \cup \nu)}) \) generates \( W(\mu, \nu) \). However, we need the abelian cycle \( \alpha_{(\mu, \nu)} \) to show that \( F_{(\mu, \nu)}(H^A_4(\text{IA}_n, \mathbb{Q})) \) includes the direct sum of \( W(\mu, \nu) \).

**Proof of Theorem 6.1.** Let \( F^l_i := \bigoplus_{(\mu, \nu) \in P^l_i} F_{(\mu, \nu)} \).

For \( (\mu, \nu) \in P^l_i \) and \( (\xi, \eta) \in P^l_i \) with \( l \neq l' \), any irreducible component of \( W(\mu, \nu) \) and any irreducible component of \( W(\xi, \eta) \) are not isomorphic. Therefore, it suffices to show that we have

\[
\bigoplus_{(\mu, \nu) \in P^l_i} W(\mu, \nu) \subset F^l_i(H^A_4(\text{IA}_n, \mathbb{Q}))
\]

for any \( l \in \{0, \ldots, i \} \).

By Lemma 6.7, it suffices to show that for each \( (\xi, \eta) \in P^l_i \), the element

\[
F_{(\xi, \eta)}(\tau_*(\alpha_{(\xi, \eta)})) \in W(\xi, \eta) \subset \bigoplus_{(\mu, \nu) \in P^l_i} W(\mu, \nu)
\]

is included in \( F^l_i(H^A_4(\text{IA}_n, \mathbb{Q})) \). In what follows, we identify an element of \( W(\xi, \eta) \) with the image under the canonical inclusion \( W(\xi, \eta) \hookrightarrow \bigoplus_{(\mu, \nu) \in P^l_i} W(\mu, \nu) \).
We use the induction with respect to the partial order $\geq$ of $P_i^1$. For the minimum element $(1^i, 1^{i-1})$ of $P_i^1$, by Lemma 6.6, we have

\[ F_i^1(\tau_*(\alpha_{(1^i, 1^{i-1})})) = F_i^1(\tau_*(\alpha_{(1, 1^{i-1})})) \in F_i^1(H_i^A(IA_n, Q)). \]

For $(\xi, \eta) \in P_i^1$, suppose that for any $(\zeta, \epsilon) \leq (\xi, \eta)$, we have

\[ F_{(\xi,\eta)}(\tau_*(\alpha_{(\zeta,\epsilon)})) \in F_i^1(H_i^A(IA_n, Q)). \]

Then by Lemma 6.6 we have

\[ F_i^1(\tau_*(\alpha_{(\xi,\eta)})) = F_{(\xi,\eta)}(\tau_*(\alpha_{(\xi,\eta)})) + X(\xi, \eta) \]

where $X(\xi, \eta)$ is an element of $\bigoplus_{(\zeta, \epsilon) \leq (\xi, \eta)} W(\zeta, \epsilon)$. Therefore, by the hypothesis of the induction, we have

\[ F_{(\xi,\eta)}(\tau_*(\alpha_{(\xi,\eta)})) \in F_i^1(H_i^A(IA_n, Q)). \]

This completes the proof. $\square$

7. Coalgebra structure of $H_i^A(IA_n, Q)$

Here we recall the coalgebra structure of the rational homology of groups. We show that the map $F_* = \bigoplus_{i \geq 0} F_i : H_*(U, Q) \rightarrow S^*(U_*)$, which we constructed in Section 6, is a coalgebra map.

7.1. Coalgebra structure of $H_*(G, Q)$. Let $G$ be a group. We briefly recall the graded-cocommutative coalgebra structure $(H_*(G, Q), \Delta^G, \epsilon_*)$. (See [2] for details.)

The rational homology $H_*(G, Q)$ is defined by

\[ H_*(G, Q) = H_*(F \otimes_G Q), \]

where $F$ is a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}[G]$. Here, we take the bar resolution. The diagonal map $\Delta^G : G \rightarrow G \times G$, $g \mapsto (g, g)$ induces a homomorphism

\[ \Delta^G : H_*(G, Q) \rightarrow H_*(G, Q) \otimes H_*(G, Q), \]

which coincides with the map induced by the Alexander–Whitney map

\[ \Delta : F \rightarrow F \otimes F, \quad (g_0, \cdots, g_n) \mapsto \sum_{p=0}^n (g_0, \cdots, g_p) \otimes (g_p, \cdots, g_n). \]

(See Brown [3] Section 1 of Chapter 5.) Then the induced map $\Delta^G_*$ can be written explicitly as follows:

\[ \Delta^G_*([x_1 \otimes \cdots \otimes x_i]) = \sum_{p=0}^i [x_1 \otimes \cdots \otimes x_p] \otimes [x_{p+1} \otimes \cdots \otimes x_i] \]

for $x_1, \cdots, x_i \in G$. We also have a trivial map $\epsilon : G \rightarrow 1$, which induces

\[ \epsilon_* : H_*(G, Q) \rightarrow Q. \]

The canonical projection $\pi^G : G \rightarrow G^{ab}$ induces a coalgebra map

\[ \pi^G_* : H_*(G, Q) \rightarrow H_*(G^{ab}, Q). \]

Therefore, the coalgebra structure of $H_*(G^{ab}, Q)$ induces a subcoalgebra structure on $H_i^A(G, Q)$. 
7.2. Coalgebra structure of $H^A_\ast(IA_n, \mathbb{Q})$. As we saw in the previous subsection, we have a coalgebra structure of $H_\ast(U, \mathbb{Q})$, which is compatible with the graded $GL(n, \mathbb{Q})$-representation structure. We consider the coalgebra structure on $S^\ast(U_\ast)$ that we observed in Section 2.6. Then the two coalgebra structures are compatible in the sense of the following proposition.

**Proposition 7.1.** The graded $GL(n, \mathbb{Q})$-homomorphism

$$F_\ast = \bigoplus_{i \geq 0} F_i : H_\ast(U, \mathbb{Q}) \to S^\ast(U_\ast)$$

is a coalgebra map.

**Proof.** It suffices to show that we have

$$(F_\ast \otimes F_\ast)\Delta^U_\ast(x) = \Delta F_\ast(x)$$

for $x = x_1 \wedge \cdots \wedge x_i \in H_i(U, \mathbb{Q})$.

We have

$$F_i(x) = \sum_{(\mu, \nu) + i} F_{(\mu, \nu)}(x_1 \wedge \cdots \wedge x_i)$$

and

$$\Delta^U_\ast(x) = \sum_{p=0}^i \sum_{\sigma \in S_i \circ H(i, i-p)} \text{sgn}(\sigma)(x_{\sigma(1)} \wedge \cdots \wedge x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \wedge \cdots \wedge x_{\sigma(i)}).$$

Let $p \in \{0, \ldots, i\}$. For pairs of partitions $(\xi, \eta) \vdash p, (\zeta, \epsilon) \vdash i - p$, we write

$$\xi = (k^1_1', \ldots, k^r_1'), \quad \eta = (k^1_2', \ldots, k^r_2'), \quad \zeta = (m^1_1', \ldots, m^1_s'), \quad \epsilon = (m^1_2', \ldots, m^1_{s'}).$$

Then we can check that

$$\Delta F_i(x) = \sum_{p=0}^i \sum_{(\xi, \eta, \zeta, \epsilon) \vdash i-p} \frac{1}{(\prod_{j=1}^i k^j_1')(\prod_{j=1}^i k^j_2')!} \sum_{\tau \in S_i} \text{sgn}(\tau)$$

$$\times \left( \bigotimes_{j=1}^r \sum_{k^j_1'} k^j_1' \text{tree} \otimes \bigotimes_{j=1}^s k^j_2' \text{wheel} \right) \left( x_{\tau(1)} \otimes \cdots \otimes x_{\tau(p)} \right)$$

$$\otimes \left( \bigotimes_{j=1}^r \sum_{m^j_1'} m^j_1' \text{tree} \otimes \bigotimes_{j=1}^s m^j_2' \text{wheel} \right) \left( x_{\tau(p+1)} \otimes \cdots \otimes x_{\tau(i)} \right)$$

$$= (F_\ast \otimes F_\ast)\Delta^U_\ast(x).$$

□

By Proposition 7.1, the subcoalgebra $H^A_\ast(IA_n, \mathbb{Q}) \subset H_\ast(U, \mathbb{Q})$ is mapped to a subcoalgebra of $S^\ast(U_\ast)$, which includes $W_\ast$ as a subcoalgebra. For a coalgebra $A$, let $\text{Prim}(A)$ denote the **primitive part** of $A$. We can check that

$$\text{Prim}(S^\ast(U_\ast)) = U_\ast \subset W_\ast.$$

Since coalgebra maps preserve the primitive part, the graded $GL(n, \mathbb{Q})$-homomorphism $F_\ast$ restricts to

$$F_\ast : \text{Prim}(H^A_\ast(IA_n, \mathbb{Q})) \to \text{Prim}(S^\ast(U_\ast)) = U_\ast.$$
Conjecture 6.2 leads to the following conjecture. Let \( \text{Prim}(H^i_A(\text{IA}_n, \mathbb{Q})) \) denote the degree \( i \) part of \( \text{Prim}(H^i_A(\text{IA}_n, \mathbb{Q})) \).

**Conjecture 7.2.** For \( n \geq 3 \), the \( \text{GL}(n, \mathbb{Q}) \)-homomorphism

\[
F_i : \text{Prim}(H^i_A(\text{IA}_n, \mathbb{Q})) \to U_i
\]

is an isomorphism.

8. Albanese cohomology of \( \text{IA}_n \)

In this section, we study the subalgebra of the rational cohomology algebra \( H^*(\text{IA}_n, \mathbb{Q}) \) that Church–Ellenberg–Farb [4] called the Albanese cohomology of \( \text{IA}_n \).

### 8.1. Albanese cohomology of groups.

For a group \( G \), the **Albanese cohomology** \( H^i_A(G, \mathbb{Q}) \) of \( G \) is defined by

\[
H^i_A(G, \mathbb{Q}) = \text{im}(\pi^*: H^i_1(G, \mathbb{Z}) \to H^i(G, \mathbb{Q}))
\]

where \( \pi: G \to H^1_1(G, \mathbb{Z}) \) is the abelianization map.

We have a linear isomorphism

\[
H^i_A(G, \mathbb{Q}) \cong \text{Hom}_\mathbb{Q}(H^i_A(G, \mathbb{Q}), \mathbb{Q}).
\]

(See Lemma A.2.)

It is well known that \( H^*(G, \mathbb{Q}) \) is a graded-commutative algebra with the cup product as a multiplication, which is the dual of the comultiplication of \( H^*(G, \mathbb{Q}) \). Then \( H^i_A(G, \mathbb{Q}) \) is a subalgebra of \( H^*(G, \mathbb{Q}) \). Since we have \( H^i_1(G, \mathbb{Z}, \mathbb{Q}) \cong \bigwedge^i H^1(G, \mathbb{Q}) \), the cohomology algebra \( H^i_A(G, \mathbb{Q}) \) is generated by \( H^1_1(G, \mathbb{Q}) \) as an algebra.

### 8.2. Albanese cohomology of \( \text{IA}_n \).

As we saw in the previous subsection, the Albanese cohomology \( H^*_A(\text{IA}_n, \mathbb{Q}) \) has a graded-symmetric algebra structure and is generated by \( H^1(\text{IA}_n, \mathbb{Q}) \). Moreover, the linear isomorphism

\[
H^i_A(G, \mathbb{Q}) \cong (H^i_A(G, \mathbb{Q}))^*
\]

is a \( \text{GL}(n, \mathbb{Q}) \)-isomorphism (see Proposition A.3).

Let \( S^*(U_*)^* \) (resp. \( H_*(U, \mathbb{Q})^* \)) denote the graded dual of \( S^*(U_*) \) (resp. \( H_*(U, \mathbb{Q}) \)) and let

\[
F^*: S^*(U_*)^* \to H_*(U, \mathbb{Q})^* \to H^*_A(\text{IA}_n, \mathbb{Q})
\]

denote the composition of the dual map of \( F_* \) and the canonical surjection. Then by Proposition 7.1, \( F^* \) is an algebra map.

**Proposition 8.1.** The graded \( \text{GL}(n, \mathbb{Q}) \)-homomorphism

\[
F^*: S^*(U_*)^* \to H^*_A(\text{IA}_n, \mathbb{Q})
\]

is an algebra map.

Let \( \langle R_2 \rangle \) denote the ideal of \( H^*(U, \mathbb{Q}) \) generated by \( R_2 = \ker(\tau^*: H^2(U, \mathbb{Q}) \to H^2(\text{IA}_n, \mathbb{Q})) \). We have a surjective \( \text{GL}(n, \mathbb{Q}) \)-homomorphism

\[
H^*(U, \mathbb{Q})/\langle R_2 \rangle \to H^*_A(\text{IA}_n, \mathbb{Q}).
\]
Conjecture 8.2. The Albanese cohomology algebra $H^*_A(IA_n, \mathbb{Q})$ is stably quadratic, that is, the surjective $GL(n, \mathbb{Q})$-homomorphism

$$H^*(U, \mathbb{Q})/\langle R_2 \rangle \twoheadrightarrow H^*_A(IA_n, \mathbb{Q})$$

is an isomorphism for sufficiently large $n$ with respect to the cohomological degree.

Conjecture 8.2 holds for $*=3$. See Remark 11.8

9. Albanese homology of $IO_n$

The inner automorphism group $\text{Inn}(F_n)$ of $F_n$ is the normal subgroup of $\text{Aut}(F_n)$ consisting of $\{\sigma_x \mid x \in F_n\}$, where $\sigma_x(y) = xyx^{-1}$ for any $y \in F_n$. The outer automorphism group $\text{Out}(F_n)$ of $F_n$ is the quotient group of $\text{Aut}(F_n)$ by $\text{Inn}(F_n)$. Since we have $\text{Inn}(F_n) \subset IA_n$, we have a surjection $\text{Out}(F_n) \twoheadrightarrow GL(n, \mathbb{Z})$. Let $IO_n$ denote its kernel. That is, we have exact sequences

$$1 \to IO_n \to \text{Out}(F_n) \to GL(n, \mathbb{Z}) \to 1$$

and

$$1 \to \text{Im}(F_n) \to IA_n \xrightarrow{\pi} IO_n \to 1.$$ 

As in the case of $\text{Aut}(F_n)$, the Johnson homomorphism for $\text{Out}(F_n)$ induces an isomorphism on the first homology $[20]$

$$\tau^O : H_1(IO_n, \mathbb{Z}) \cong Hom(H, \wedge^2 H)/H \to \text{Hom}(H, \wedge^2 H)/H.$$

We can also consider $H_i(IO_n, \mathbb{Q})$ as a $GL(n, \mathbb{Z})$-representation, and the Johnson homomorphism preserves the $GL(n, \mathbb{Z})$-action. Then we have

$$H_i(IO_n, \mathbb{Q}) \cong \text{Hom}(H, \wedge^2 H)/H \cong V_{12,1}.$$ 

Let $U^O = \text{Hom}(H, \wedge^2 H)/H$. The Johnson homomorphism induces a $GL(n, \mathbb{Z})$-homomorphism on homology

$$\tau^*_O : H_i(IO_n, \mathbb{Q}) \to H_i(U^O, \mathbb{Q}).$$

In this section, we study the Albanese homology of $IO_n$ and observe some relation between $H_i^A(IO_n, \mathbb{Q})$ and $H_i^A(IA_n, \mathbb{Q})$.

9.1. A set of generators for $H_1(IO_n, \mathbb{Q})$. Here we obtain a set of generators for $H_1(IO_n, \mathbb{Q})$, which is induced by Magnus’s set of generators for $IA_n$.

The projection $\pi : IA_n \to IO_n$ induces a map

$$\pi_* : H_1(IA_n, \mathbb{Q}) \to H_1(IO_n, \mathbb{Q}).$$

Then we have the following commutative diagram of $GL(n, \mathbb{Q})$-representations:

$$
\begin{array}{ccc}
H_1(IA_n, \mathbb{Q}) & \xrightarrow{\pi_*} & H_1(IO_n, \mathbb{Q}) \\
\cong \tau & \cong \tau^O & \\
U = \text{Hom}(H, \wedge^2 H) & \xrightarrow{pr} & U^O = \text{Hom}(H, \wedge^2 H)/H,
\end{array}
$$

where the bottom map is the canonical surjection.
Magnus’s set of generators for $\text{IA}_n$ induces the following set of generators for $U^O$:
\[
\{ g_{a,b} \mid 1 \leq a, b \leq n, \ a \neq b \} \cup \{ f_{a,b,c} \mid 1 \leq a, b, c \leq n, \ a < b, \ a \neq c \neq b \},
\]
where
\[
\overline{g_{a,b}} = \text{pr} \tau(g_{a,b}) = \text{pr}(e_{a,b}^b) \in U^O, \quad \overline{f_{a,b,c}} = \text{pr} \tau(f_{a,b,c}) = \text{pr}(e_{a,b}^c) \in U^O.
\]
We can check that
\[
\tau(f_{a,b,c}) = e_{a,b}^c \in U_{1,\text{tree}} \subset U
\]
and that
\[
\tau(g_{a,b}) - \frac{1}{n-1} \left( \sum_{j=1}^n e_{a,j}^j \right) = \frac{1}{n-1} \left( \sum_{j \neq a,b} P_{j,c}^j (e_{a,b}^b - e_{a,c}^c) \right)
= \frac{1}{n-1} \left( \sum_{j \neq a,b} P_{j,c}^j (\text{id} - E_{e,b} - E_{c,b}) (e_{a,b}^c) \in U_{1,\text{tree}}^c, \right.
\]
where $c \in [n]$ is an element distinct from $a$ and $b$. Therefore, we obtain an isomorphism $U^O \cong U_{1,\text{tree}}^c$ defined by
\[
\overline{f_{a,b,c}} \mapsto e_{a,b}^c \in U_{1,\text{tree}}^c, \quad \frac{g_{a,b}}{e_{a,b}^c} \mapsto e_{a,b}^c - \frac{1}{n-1} \left( \sum_{j=1}^n e_{a,j}^j \right) \in U_{1,\text{tree}}^c,
\]
and thus we obtain the canonical injective map
\[
U^O \hookrightarrow U = U_{1,\text{tree}}^c \oplus U_{1,\text{wheel}}^c.
\]
In what follows, we consider $U^O$ as a subrepresentation of $U$.

9.2. Computation of the contraction maps. The inclusion map $\iota_i : \wedge^i U \hookrightarrow \text{M}_i$, which we defined in Section 4.1, restricts to an inclusion map
\[
\iota_i : \wedge^i U^O \hookrightarrow \text{M}_i.
\]
Then we can consider the composition of $\iota_i$ and each of the two contraction maps $c_i^\text{wheel}$ and $c_i^\text{tree}$, which we defined in Section 4.1. The abelian cycle $\alpha_{(0,i)}$ of $H_i(\text{IA}_n, \mathbb{Q})$ induces an abelian cycle $\pi_* (\alpha_{(0,i)})$ of $H_i(\text{IO}_n, \mathbb{Q})$. Here we compute the two contraction maps for $\tau_*^O \pi_* (\alpha_{(0,i)})$ as in Lemma 4.1.

**Lemma 9.1.** (1) For $i = 1$, we have $c_1^\text{wheel} \tau_*^O \pi_* (\alpha_{(0,1)}) = 0$.
(2) For $i \geq 2$, we have $c_i^\text{wheel} \iota_i \tau_*^O \pi_* (\alpha_{(0,i)}) \neq 0$ if $n \geq i + 2 + \frac{1}{i-1}$.
(3) For $i \geq 1$, we have $c_i^\text{tree} \iota_i \tau_*^O \pi_* (\alpha_{(0,i)}) \neq 0$ for sufficiently large $n$.

**Proof.** The proof is similar to that of Lemma 4.1. We have
\[
\tau_*^O \pi_* (\alpha_{(0,i)}) = \frac{y_{2,1}}{2} \wedge (y_{3,1} + y_{3,2}) \wedge \cdots \wedge \left( \sum_{j=1}^i y_{i+1,j} \right)
= \bigwedge_{k=1}^i \left( \sum_{j=1}^k \frac{n-k-1}{n-1} e_{k+1,j}^j - \frac{k}{n-1} \sum_{j=k+2}^n e_{k+1,j}^j \right).
\]
For $i = 1$, we have

$$c_1^{\text{wheel}}_{t_1} r_*^O \pi_*(\alpha_{(0, 1)}) = c_1^{\text{wheel}}_{t_1} \left( \frac{n - 2}{n - 1} e_{2, 1}^1 - \frac{1}{n - 1} \sum_{j=3}^{n} e_{2, j}^j \right)$$

$$= \frac{n - 2}{n - 1} e_2 - \frac{1}{n - 1} (n - 2) e_2 = 0,$$

which proves (1).

For $i \geq 2$, $n \geq i + 1$, we have

$$c_i^{\text{wheel}}_{t_1} r_*^O \pi_*(\alpha_{(0, 1)}) = \frac{i! (n - i - 1) \cdots (n - 2)(n - 1)^i}{(n - 1)^i} e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \in V_{i, 0}.$$ 

If $i$ is even, then we have $(n - 2) \cdots (n - i) + (-1)^i i! > 0$. If $i$ is odd, then we have $(n - 2) \cdots (n - i) + (-1)^i i! > 0$ for $n \geq i + 3$. Since we have $i! (n - i - 1) \neq 0$ for $n \geq i + 2$, we have (2).

For $i \geq 1$, $n \geq i + 1$, one can show that

$$c_i^{\text{tree}}_{t_1} r_*^O \pi_*(\alpha_{(0, 1)}) = \frac{(i + 1)! (n - i - 1) Q_i(n)}{(n - 1)^i} e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \wedge e_1 \otimes e_i^*$$

$$+ \sum_{j=i+2}^{n} \frac{R_i(n)}{(n - 1)^j} e_2 \wedge e_3 \wedge \cdots \wedge e_{i+1} \wedge e_j \otimes e_j^* \in V_{i+1, 1},$$

where $Q_i(n)$ is a monic polynomial of degree $i - 1$ and $R_i(n)$ is a polynomial of degree $i - 1$. Since $Q_i(n)$ is monic, we have (3).

Remark 9.2. We can say that to satisfy $c_i^{\text{tree}}_{t_1} r_*^O \pi_*(\alpha_{(0, 1)}) \neq 0$, the condition $n \geq 2 + \max(i, \lfloor \frac{n-1}{2} \rfloor)$ is enough. We need $n \geq i + 2$, and if $i \leq 4$, then the condition that $n \geq i + 2$ is sufficient.

Theorem 9.3. Let $i \geq 2$. For $n \geq i + 2 + \lfloor \frac{1 - (-1)^i}{2} \rfloor$, we have a surjective $\text{GL}(n, \mathbb{Q})$-homomorphism

$$H_i^A(\text{IO}_n, \mathbb{Q}) \to \bigwedge^i H \cong V_{i, 0}. $$

For sufficiently large $n$ with respect to $i$, we have a surjective $\text{GL}(n, \mathbb{Q})$-homomorphism

$$H_i^A(\text{IO}_n, \mathbb{Q}) \to \text{Hom}(H, \bigwedge^{i+1} H) \cong V_{i+1, 1} \oplus V_{i, 0}. $$

Proof. This directly follows from Lemma 9.1. 

9.3. The traceless part of $H_i^A(\text{IO}_n, \mathbb{Q})$. We have a commutative diagram

$$
\begin{array}{ccc}
H_i(\text{IA}_n, \mathbb{Q}) & \xrightarrow{\pi_*} & H_i(\text{IO}_n, \mathbb{Q}) \\
\tau_* & & \tau_* \\
&& V_{i, 0} \\
H_i(U, \mathbb{Q}) & \xrightarrow{pr_*} & H_i(U^O, \mathbb{Q}).
\end{array}
$$

Therefore, we have

$$(9.3.1) \quad \text{pr}_*(H_i^A(\text{IA}_n, \mathbb{Q})) \subset H_i^A(\text{IO}_n, \mathbb{Q}).$$
Question 9.4. Is \( \text{pr}_*: H^A_i(IA_n, \mathbb{Q}) \to H^A_i(IO_n, \mathbb{Q}) \) surjective for any \( n \)?

We have the following decomposition of \( H_i(U, \mathbb{Q}) \) as \( \text{GL}(n, \mathbb{Q}) \)-representations

\[
H_i(U, \mathbb{Q}) = \bigwedge^i U = \left( \bigwedge^i U^O \right) \oplus Y_i,
\]

where \( Y_i \) is a subrepresentation of \( H_i(U, \mathbb{Q}) \) whose irreducible decomposition does not include \( V^{\lambda} \) for any \( \lambda \) such that \( |\lambda| = 3i \). Therefore, by Theorem 5.1 we obtain the following theorem. Recall that we have \( \dim_{\mathbb{Q}}(H_i(U, \mathbb{Q})^tl) = P'_i(n) \) for \( n \geq 3i \), where \( P'_i(T) \) is a polynomial of degree \( 3i \).

**Theorem 9.5.** For \( n \geq 3i \), we have

\[
H_i(U, \mathbb{Q})^tl \subset H^A_i(IO_n, \mathbb{Q}).
\]

In particular, we have \( \dim_{\mathbb{Q}}(H^A_i(IO_n, \mathbb{Q})) \geq P'_i(n) \) for \( n \geq 3i \).

**Conjecture 9.6.** There is a polynomial \( P^O_i(T) \) of degree \( 3i \) such that we have \( \dim_{\mathbb{Q}}(H^A_i(IO_n, \mathbb{Q})) = P^O_i(n) \) for sufficiently large \( n \) with respect to \( i \).

Moreover, by Theorem 6.1, we have a direct summand \( \tilde{W}_i \) of \( H^A_i(IO_n, \mathbb{Q}) \) which is isomorphic to \( W_i \). Therefore, by (9.3.1), we have for \( n \geq 3i \),

\[
\text{pr}_*(\tilde{W}_i) \subset H^A_i(IO_n, \mathbb{Q}).
\]

9.4. **Conjectural structure of** \( H^A_i(IO_n, \mathbb{Q}) \). Here we propose a conjectural structure of \( H^A_i(IO_n, \mathbb{Q}) \).

Define \( U^O_i \) by

\[
U^O_i = \begin{cases} 
U^O \cong U^\text{tree}_1 & (i = 1) \\
U_i & (i \geq 2).
\end{cases}
\]

For the graded \( \text{GL}(n, \mathbb{Q}) \)-representation \( U^O = \bigoplus_{i \geq 1} U^O_i \), let \( S^*(U^O) \) denote the graded-symmetric algebra of \( U^O \). Let \( W^O = \tilde{S}^*(U^O) \) denote the traceless part of \( S^*(U^O) \). Let \( P^O \subset P_i \) denote the subset of \( P_i \) consisting of pairs of partitions \((\mu, \nu)\) of total size \( i \) such that \( \nu \) has no part of size 1. Then we have

\[
W^O_i = \bigoplus_{(\mu, \nu) \in P^O_i} W(\mu, \nu).
\]

We make the following conjecture, which is true for \( i = 1 \) by Kawazumi [20], for \( i = 2 \) by Pettet [34] and for \( i = 3 \) as we will observe in Theorem 10.1.

**Conjecture 9.7.** For sufficiently large \( n \) with respect to \( i \), we have a \( \text{GL}(n, \mathbb{Q}) \)-isomorphism

\[
H^A_i(IO_n, \mathbb{Q}) \cong W^O_i.
\]

Since we have a direct sum decomposition

\[
S^*(U^O_i) = \bigoplus_{(\mu, \nu) \in P^O_i} U^\text{tree}_\mu \otimes U^\text{wheel}_\nu,
\]

we have the following decomposition of \( H^A_i(IO_n, \mathbb{Q}) \) as \( \text{GL}(n, \mathbb{Q}) \)-representations

\[
H^A_i(IO_n, \mathbb{Q}) = \bigwedge^i U = \left( \bigwedge^i U^O \right) \oplus Y_i,
\]

where \( Y_i \) is a subrepresentation of \( H^A_i(IO_n, \mathbb{Q}) \) whose irreducible decomposition does not include \( V^{\lambda} \) for any \( \lambda \) such that \( |\lambda| = 3i \). Therefore, by Theorem 5.1 we obtain the following theorem. Recall that we have \( \dim_{\mathbb{Q}}(H^A_i(IO_n, \mathbb{Q})^tl) = P'_i(n) \) for \( n \geq 3i \), where \( P'_i(T) \) is a polynomial of degree \( 3i \).
there is a canonical surjective $\GL(n, \mathbb{Q})$-homomorphism

$$\Pr : S^*(U_*)_i = \bigoplus_{(\mu, \nu) \in P_i} U^\text{tree}_\mu \otimes U^\text{wheel}_\nu \to S^*(U^O_*)_i = \bigoplus_{(\mu, \nu) \in P^O_i} U^\text{tree}_\mu \otimes U^\text{wheel}_\nu.$$  

Then we have a $\GL(n, \mathbb{Q})$-homomorphism

$$G_i : H_i(U^O, \mathbb{Q}) = \bigwedge^i U^O \hookrightarrow \bigwedge^i U \xrightarrow{\Pr} S^*(U_*)_i \xrightarrow{H_i} S^*(U^O_*)_i.$$

We expect that $G_i$ restricts to a $\GL(n, \mathbb{Q})$-isomorphism $G_i : H_i^A(IO_n, \mathbb{Q}) \xrightarrow{\cong} W_i^O$, which implies that Conjecture 9.7 is true.

9.5. **Structures of $H^A_i(IO_n, \mathbb{Q})$ and $H^A_i(IA_n, \mathbb{Q})$.** Here we study the relation between the structures of $H^A_i(IO_n, \mathbb{Q})$ and $H^A_i(IA_n, \mathbb{Q})$.

Recall that we have an exact sequence of groups with $\Aut(F_n)$-actions

$$1 \to \Inn(F_n) \to IA_n \to IO_n \to 1.$$  

Since we have $\Inn(F_n) \cong F_n$ for $n \geq 2$, we identify $\Inn(F_n)$ with $F_n$. By Proposition A.8 and Remark A.7, we obtain the following proposition.

**Proposition 9.8.** For $n \geq 2$, we have a $\GL(n, \mathbb{Q})$-isomorphism

$$H^A_i(IO_n, \mathbb{Q}) \cong H^A_i(IA_n, \mathbb{Q}) \oplus (H^A_{i-1}(IO_n, \mathbb{Q}) \otimes H^1).$$  

**Proof.** The canonical injective $\Aut(F_n)$-homomorphism $F_n \hookrightarrow IA_n$ induces an injective $\Aut(F_n)$-homomorphism $(F_n)^{\text{ab}} = H_2 \to (IA_n)^{\text{ab}} = H_1(IA_n, \mathbb{Z})$. By Proposition A.8 we obtain a filtration

$$0 = F_{-1} \subset F_0 \subset \cdots \subset F_i = H^A_i(IA_n, \mathbb{Q})$$  

of $\Aut(F_n)$-modules such that there is an $\Aut(F_n)$-homomorphism

$$\iota : \bigoplus_{r=0}^i F_r/F_{r-1} \hookrightarrow \bigoplus_{p+q=i} H^A_p(IO_n, \mathbb{Q}) \otimes H^A_q(F_n, \mathbb{Q}).$$  

The $\Aut(F_n)$-actions on $H^A_i(IO_n, \mathbb{Q})$, $H^A_i(IO_n, \mathbb{Q})$, and $H^A_i(F_n, \mathbb{Q})$ induce $\GL(n, \mathbb{Z})$-actions, which extend to the structures of algebraic $\GL(n, \mathbb{Q})$-representations. Hence, the filtration $\mathcal{F}$ of $H^A_i(IO_n, \mathbb{Q})$ is a filtration of $\GL(n, \mathbb{Q})$-representations and $\iota$ is a $\GL(n, \mathbb{Q})$-homomorphism. Therefore, we have $\GL(n, \mathbb{Q})$-homomorphisms

$$H^A_i(IA_n, \mathbb{Q}) \cong \bigoplus_{r=0}^i F_r/F_{r-1} \hookrightarrow \bigoplus_{p+q=i} H^A_p(IO_n, \mathbb{Q}) \otimes H^A_q(F_n, \mathbb{Q}).$$

In order to show that $H^A_i(IO_n, \mathbb{Q}) \cong \bigoplus_{p+q=i} H^A_p(IO_n, \mathbb{Q}) \otimes H^A_q(F_n, \mathbb{Q})$, we use the cohomological Hochschild–Serre spectral sequence. We can easily check that $H^q(F_n, \mathbb{Q}) = 0$ for $q \geq 2$, and that $IO_n$ acts trivially on $H^*(F_n, \mathbb{Q})$. By Remark A.7 it suffices to show that the differential $d_2^{i, 1} : E^{0, 1}_2 \to E^{1, 0}_2$ is a zero map, which follows from the fact that $H^1(IO_n, \mathbb{Q}) \cong H^1(IO_n, \mathbb{Q}) \oplus H^1(F_n, \mathbb{Q}) = E^{1, 0}_2 \oplus E^{0, 1}_2$. This completes the proof.

To study the relation between $H^A_i(IO_n, \mathbb{Q})$ and $H^A_i(IA_n, \mathbb{Q})$, we use the following lemma.
Lemma 9.9. We have $W_i \cong W_i^O \oplus (W_{i-1}^O \otimes H)$ as $GL(n, \mathbb{Q})$-representations for $n \geq 3i$.

Proof. By the definitions of $W_i$ and $W_i^O$, it suffices to show that

$$\bigoplus_{(\mu, \nu) \in P_{i-1}^O} W(\mu, \nu) \otimes H \cong \bigoplus_{(\xi, \eta) \in \Pi_i \setminus \Pi_i^O} W(\xi, \eta). \quad (9.5.1)$$

Let $(\mu, \nu) \in P_{i-1}^O$. We write $\mu = (k_1', \cdots, k_r')$ and $\nu = (k_1'', \cdots, k_\tau'')$. Let $\mu - \mu_j = (k_1', \cdots, k_j'-1, \cdots, k_r')$ for $1 \leq j \leq r$ and $\nu + a = (k_1'', \cdots, k_\tau'', 1^a)$ for $a \geq 1$. Since we have $GL(n, \mathbb{Q})$-isomorphisms

$$W(\mu, \nu) \otimes H \cong W(\mu, \nu + 1)$$

and

$$U^{tree}_{\mu_j} \otimes H \cong (U^{tree}_{\mu_j} \otimes H) \oplus V_{1+\mu_j} \cong (U^{tree}_{\mu_j} \otimes H) \oplus \tilde{S}^{1+\mu_j} (U^\text{wheel}_1),$$

we have

$$W(\mu, \nu) \otimes H \cong (W(\mu, \nu) \otimes H) \oplus \left( \bigoplus_{j=1}^r W(\mu - \mu_j, \nu) \otimes \tilde{S}^{1+\mu_j} (U^\text{wheel}_1) \right) \quad (9.5.2)$$

$$\cong W(\mu, \nu + 1) \oplus \bigoplus_{j=1}^r W(\mu - \mu_j, \nu + 1^{1+\mu_j}).$$

Therefore, for each $(\mu, \nu) \in P_{i-1}^O$, we have

$$W(\mu, \nu) \otimes H \supset \bigoplus_{(\xi, \eta) \in \Pi_i \setminus \Pi_i^O} W(\xi, \eta).$$

Moreover, we have

$$\bigoplus_{(\mu, \nu) \in P_{i-1}^O} W(\mu, \nu) \otimes H \subset \bigoplus_{(\xi, \eta) \in \Pi_i \setminus \Pi_i^O} W(\xi, \eta) \quad (9.5.3)$$

since for distinct pairs $(\mu, \nu) \neq (\mu', \nu')$, the pairs $(\mu, \nu + 1), (\mu', \nu' + 1), (\mu - \mu_j, \nu + 1^{1+\mu_j}), (\mu' - \mu_j', \nu' + 1^{1+\mu_j'})$ for $1 \leq j \leq r$ are distinct.

Let $(\xi, \eta) \in \Pi_i \setminus \Pi_i^O$. If $\eta$ has only one part of size 1, then we have $(\mu, \nu) \in P_{i-1}^O$ such that $(\mu, \nu + 1) = (\xi, \eta)$. Otherwise, we have $(\mu, \nu) \in P_{i-1}^O$ such that $(\mu - \mu_j, \nu + 1^{1+\mu_j}) = (\xi, \eta)$. Therefore, by the decomposition (9.5.2), we obtain

$$\bigoplus_{(\mu, \nu) \in P_{i-1}^O} W(\mu, \nu) \otimes H \supset \bigoplus_{(\xi, \eta) \in \Pi_i \setminus \Pi_i^O} W(\xi, \eta).$$

Therefore, by using (9.5.3), we obtain (9.5.1). \qed

By Theorem 6.1 and Proposition 9.8, we obtain the following proposition, which partially ensures Conjecture 9.7.

Proposition 9.10. For $n \geq 3i$, we have an injective $GL(n, \mathbb{Q})$-homomorphism

$$W_i^O \oplus (W_{i-1}^O \otimes H) \hookrightarrow H_i^A (\text{IO}_n, \mathbb{Q}) \oplus (H_{i-1}^A (\text{IO}_n, \mathbb{Q}) \otimes H).$$
Proof. Let \( n \geq 3i \). By Proposition \( \ref{prop:9.8} \) we have

\[
H^i_A(\text{IA}_n, \mathbb{Q}) \cong H^A(\text{IO}_n, \mathbb{Q}) \oplus (H^A_{i-1}(\text{IO}_n, \mathbb{Q}) \otimes H).
\]

By Theorem \( \ref{thm:9.4} \) \( H^A(\text{IA}_n, \mathbb{Q}) \) contains a subrepresentation which is isomorphic to \( W_i \). Since we have

\[
W_i^O \oplus (W^O_{i-1} \otimes H) \cong W_i
\]

by Lemma \( \ref{lem:9.9} \) we have an injective \( \text{GL}(n, \mathbb{Q}) \)-homomorphism

\[
W_i^O \oplus (W^O_{i-1} \otimes H) \hookrightarrow H^A(\text{IO}_n, \mathbb{Q}) \oplus (H^A_{i-1}(\text{IO}_n, \mathbb{Q}) \otimes H).
\]

Therefore, if Conjecture \( \ref{conj:9.7} \) holds, then we stably have

\[
H^A(\text{IO}_n, \mathbb{Q}) \cong W^O_i.
\]

Recall that Conjecture \( \ref{conj:9.7} \) states that we stably have \( H^A(\text{IO}_n, \mathbb{Q}) \cong W^O_i \). Then we obtain the following relation between the conjectures about the structures of \( H^A(\text{IA}_n, \mathbb{Q}) \) and \( H^A(\text{IO}_n, \mathbb{Q}) \).

**Proposition 9.11.** The followings are equivalent.

- For any \( i \), we have a \( \text{GL}(n, \mathbb{Q}) \)-isomorphism \( H^A(\text{IA}_n, \mathbb{Q}) \cong W_i \) for sufficiently large \( n \) with respect to \( i \) (cf. Conjecture \( \ref{conj:6.2} \)).
- Conjecture \( \ref{conj:9.7} \).

Proof. If Conjecture \( \ref{conj:9.7} \) holds, then Conjecture \( \ref{conj:6.2} \) holds by Proposition \( \ref{prop:9.8} \) and Lemma \( \ref{lem:9.9} \).

Suppose that Conjecture \( \ref{conj:6.2} \) holds. We will show that Conjecture \( \ref{conj:9.7} \) also holds by induction on the homological degree \( i \). Suppose that we have \( W^O_i \cong H^A(\text{IO}_n, \mathbb{Q}) \) for sufficiently large \( n \) with respect to \( i \). For sufficiently large \( n \) with respect to \( i + 1 \), by using Lemma \( \ref{lem:9.9} \) and Proposition \( \ref{prop:9.8} \), we have

\[
W^O_{i+1} \oplus (W^O_i \otimes H) \cong W_{i+1} \cong H^A_{i+1}(\text{IA}_n, \mathbb{Q}) \cong H^A_{i+1}(\text{IO}_n, \mathbb{Q}) \oplus (H^A_{i+1}(\text{IO}_n, \mathbb{Q}) \otimes H).
\]

Therefore, since we have \( H^A_{i+1}(\text{IO}_n, \mathbb{Q}) \cong W^O_i \) by the hypothesis of the induction, we have \( H^A_{i+1}(\text{IO}_n, \mathbb{Q}) \cong W^O_{i+1} \), which completes the proof.

**Remark 9.12.** Let \( n \geq 3i \). Since we have \( W_i \cong \bigoplus_{p+q=i} W_p^O \otimes \wedge^q H \) by the definitions of \( W_i \) and \( W_i^O \), by Lemma \( \ref{lem:9.9} \) we have

\[
W_i^O \oplus (W^O_{i-1} \otimes H) \cong \bigoplus_{p+q=i} W_p^O \otimes \wedge^q H.
\]

Therefore, if Conjecture \( \ref{conj:9.7} \) holds, then we stably have

\[
H^A(\text{IO}_n, \mathbb{Q}) \otimes H^A(\text{F}_n, \mathbb{Q}) \cong H^A(\text{IO}_n, \mathbb{Q}) \otimes H^A(\text{F}_{\text{ab}}, \mathbb{Q}).
\]

10. **The third Albanese homology of \( \text{IO}_n \)**

In this section, we compute \( H^A_3(\text{IO}_n, \mathbb{Q}) \) and prove that Conjecture \( \ref{conj:9.7} \) holds for \( i = 3 \).

Hain \( \cite{Hain} \) and Sakasai \( \cite{Sakasai} \) computed the Albanese cohomology of the Torelli groups of closed surfaces of degree 2 and 3, respectively. Pettet \( \cite{Pettet} \) computed \( H^A_3(\text{IA}_n, \mathbb{Q}) \) by adapting Hain’s and Sakasai’s methods. Here we apply their method to \( H^A_3(\text{IO}_n, \mathbb{Q}) \) as follows. We have a \( \text{GL}(n, \mathbb{Q}) \)-isomorphism \( H^A_3(\text{IO}_n, \mathbb{Q}) \cong H^A_3(\text{IO}_n, \mathbb{Q})^* \).
Suppose that we have subrepresentations $S \subset H_3^A(IO_n, \mathbb{Q})$ and $T \subset \ker \tau_O^*$, where $\tau_O^* : H^3(U^O, \mathbb{Q}) \to H^3(IO_n, \mathbb{Q})$ is induced by the Johnson homomorphism $\tau^O$. Then, we have $S^* \oplus T \subset \im \tau_O^* \oplus \ker \tau_O^* \cong H^3(U^O, \mathbb{Q})$. If we can show that $H^3(U^O, \mathbb{Q}) \cong S^* \oplus T$ by counting multiplicities of irreducible components, then we obtain $H_3^A(IO_n, \mathbb{Q}) \cong S^*$ and $H_3^A(IO_n, \mathbb{Q}) \cong S$.

10.1. The third Albanese homology of $IO_n$. For $n \geq 9$, we have

$$W_3^O = W(3, 0) \oplus W(0, 3) \oplus W(1, 2) \oplus W(21, 0) \oplus W(1^3, 0)$$

= $(V_{1^4, 1}) \oplus (V_{1^3, 0}) \oplus (V_{2^2, 1} \oplus V_{21^2, 1} \oplus V_{1^4, 1})$

$\oplus (V_{221, 2} \oplus V_{21^2, 12} \oplus V_{21^3, 12} \oplus V_{1^5, 2} \oplus V_{1^5, 12})$

$\oplus (V_{32, 13} \oplus V_{321, 21} \oplus V_{31^3, 3} \oplus V_{23, 3} \oplus V_{221^2, 21} \oplus V_{221^2, 13} \oplus V_{21^4, 21} \oplus V_{1^5, 13})$

$\oplus V_{221, 2} \oplus V_{21^2, 12} \oplus V_{21^3, 12} \oplus V_{1^5, 2} \oplus V_{1^5, 12}$

$\oplus V_{2^2, 1} \oplus V_{21^2, 1} \oplus V_{21^3, 1} \oplus V_{1^5, 0}$.

10.2. Irreducible decomposition of $H_3(U^O, \mathbb{Q})$. We begin with the computation of an irreducible decomposition of $H_3(U^O, \mathbb{Q})$ as $GL(n, \mathbb{Q})$-representations. We can check the following lemma directly by hand and by using SageMath.

**Lemma 10.2.** Let $n \geq 9$. $H_3(U^O, \mathbb{Q})$ is decomposed into the following 36 irreducible $GL(n, \mathbb{Q})$-representations:

$$H_3(U^O, \mathbb{Q}) = \bigwedge^3 (V_{1^4, 1})$$

$\cong V_{32, 12} \oplus V_{321, 21} \oplus V_{31^3, 3} \oplus V_{23, 3} \oplus V_{221^2, 21} \oplus V_{221^2, 13} \oplus V_{21^4, 21} \oplus V_{1^5, 13}$

$\oplus V_{32, 12} \oplus V_{321, 21} \oplus V_{31^3, 3} \oplus V_{23, 3} \oplus V_{221^2, 21} \oplus V_{221^2, 13} \oplus V_{21^4, 21} \oplus V_{1^5, 13}$

$\oplus V_{32, 12} \oplus V_{321, 21} \oplus V_{31^3, 3} \oplus V_{23, 3} \oplus V_{221^2, 21} \oplus V_{221^2, 13} \oplus V_{21^4, 21} \oplus V_{1^5, 13}$

$\oplus V_{32, 12} \oplus V_{321, 21} \oplus V_{31^3, 3} \oplus V_{23, 3} \oplus V_{221^2, 21} \oplus V_{221^2, 13} \oplus V_{21^4, 21} \oplus V_{1^5, 13}$

10.3. Upper bound of $H_3^A(IO_n, \mathbb{Q})$. Let $\tau_O^* : H^4(U^O, \mathbb{Q}) \to H^4(IO_n, \mathbb{Q})$ denote the map induced by the Johnson homomorphism $\tau^O : IO_n \to U^O$ on cohomology. Let $R_3^O = \ker \tau_O^*$. Pettet [34] computed $R_3^O$. We will study $R_3^O$ by using $R_2^O$.

**Lemma 10.3** (Pettet [34]). Let $n \geq 3$. We have

$$R_2^O \cong V_{1, 21}.$$  

For $a, b, c \in [n]$, let

$$e_{a}^{b,c} := e_a \otimes (e_b^* \wedge e_c^*) \in U^*.$$  

**Lemma 10.4.** For $n \geq 3$, the subrepresentation $R_2^O \subset H^2(U^O, \mathbb{Q})$ is generated by

$$\beta = \sum_{j=1}^{n} e_1^{j,2} \wedge e_j^{3,1} \in H^2(U^O, \mathbb{Q}) \cong \bigwedge^2 (U^O)^*.$$
Proposition 10.6. We can check that $\beta \in H^2(U^O, \mathbb{Q}) \cong \bigwedge^2(U^O)^*$ and that $\beta$ generates an irreducible representation which is isomorphic to $V_{1,21}$. We have

$$H^2(U^O, \mathbb{Q}) \cong \bigwedge^2(V_{1,12}) \cong V_{1,22} \oplus V_{2,21} \oplus V_{1,21} \oplus V_{1,12} \oplus V_{0,12}.$$  

Since the multiplicity of $V_{1,21}$ in $H^2(U^O, \mathbb{Q})$ is 1, by Lemma 10.4, it follows that $\beta$ generates $R_2^O \subset H^2(U^O, \mathbb{Q})$.

Proof. Let $\cup : H^1(U^O, \mathbb{Q}) \otimes R_2^O \to H^3(U^O, \mathbb{Q})$

denote the restriction of the cup product. Then we have $\text{im} \cup \subset R_3^O$. In what follows, we compute $\text{im} \cup$. We can compute the tensor product $H^1(U^O, \mathbb{Q}) \otimes R_2^O$ directly by hand and by using SageMath.

Lemma 10.5. Let $n \geq 6$. We have an irreducible decomposition

$$H^1(U^O, \mathbb{Q}) \otimes R_2^O \cong V_{1,12} \otimes V_{1,21}
\cong V_{2,32} \oplus V_{1,31} \oplus V_{2,21} \oplus V_{1,21} \oplus V_{2,21} \oplus V_{1,213}
\oplus V_{1,31} \oplus V_{1,21} \oplus V_{1,14} \oplus V_{0,3} \oplus V_{0,21} \oplus V_{0,13}.$$  

In terms of $\text{GL}(n, \mathbb{Q})$-representations, we can identify the cup product map $\cup$ with

$$\land : V_{1,12} \otimes V_{1,21} \to \bigwedge^3 V_{1,12}.$$  

Proposition 10.6. For $n \geq 6$, $\text{im} \cup$ contains a $\text{GL}(n, \mathbb{Q})$-subrepresentation consisting of the following 17 irreducible representations:

$$\text{im} \cup \supset V_{2,32} \oplus V_{1,31} \oplus V_{2,21} \oplus V_{1,21} \oplus V_{2,21} \oplus V_{1,213}
\oplus V_{1,31} \oplus V_{1,21} \oplus V_{1,14} \oplus V_{0,3} \oplus V_{0,21} \oplus V_{0,13}.$$  

Proof. Let $n \geq 6$. For any distinct elements $a, b, c \in [n]$, define

$$\beta_{a,b,c}^O := \sum_{j=1}^n e_{a,b}^j \land e_{j}^{1,2} \land e_{n}^{1,1} \in \text{im} \cup$$

and

$$\beta_{a,b}^O := \sum_{j=1}^n \left( e_{b}^{a,1} - \frac{1}{n-1} \sum_{k=1}^n e_{k}^{a,k} \right) \land e_{j}^{1,2} \land e_{n}^{1,1} \in \text{im} \cup.$$  

We can use $\beta_{5,3,4}^O$ to detect 8 irreducible components of size 7 as follows. Let $\iota^* : \bigwedge^3(U^O)^* \hookrightarrow \bigwedge^3(H \otimes \bigwedge^2 H^*) \hookrightarrow (H \otimes (H^*)^\otimes 2)^\otimes 3$ denote the canonical inclusion defined in a way similar to $\iota_3$ in Section 3.1. Define a contraction map

$$\phi : (H \otimes (H^*)^\otimes 2)^\otimes 3 \to H^\otimes 2 \otimes (H^*)^\otimes 5,$$

$$\otimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_3^*(c_2)(c_1 \otimes c_3) \otimes (a_1^* \otimes b_1^* \otimes a_2^* \otimes b_2^* \otimes b_3^*).$$
Define projection maps

\[
\begin{align*}
\phi_1 : (H^*)^{\otimes 5} & \to (\bigwedge^2 H^*)^{\otimes 2} \otimes H^* , \quad \bigotimes_{i=1}^5 x_i^* \mapsto (x_1^* \wedge x_2^*) \otimes (x_3^* \wedge x_4^*) \otimes x_5^*, \\
\phi_2 : (H^*)^{\otimes 5} & \to (\bigwedge^3 H^*) \otimes (H^*)^{\otimes 2} , \quad \bigotimes_{i=1}^5 x_i^* \mapsto (x_1^* \wedge x_3^* \wedge x_4^*) \otimes (x_2^* \otimes x_5^*), \\
\phi_3 : (H^*)^{\otimes 5} & \to (\bigwedge^3 H^*) \otimes (\bigwedge^2 H^*) , \quad \bigotimes_{i=1}^5 x_i^* \mapsto (x_1^* \wedge x_2^* \wedge x_3^*) \otimes (x_4^* \wedge x_5^*), \\
\phi_4 : (H^*)^{\otimes 5} & \to (\bigwedge^4 H^*) \otimes H^* , \quad \bigotimes_{i=1}^5 x_i^* \mapsto (x_1^* \wedge x_2^* \wedge x_3^* \wedge x_4^*) \otimes x_5^*.
\end{align*}
\]

Then we have

\[
\begin{align*}
(id - P_{5,n})(id - E_{4,2})(id - E_{3,1})(id_{H^{\otimes 2} \otimes \phi_1})\phi_* (\beta_{5,3,4}^O) \\
= 4(n - 1)(e_5 \otimes e_n - e_n \otimes e_5) \otimes (e_1^* \wedge e_2^*) \otimes (e_3^* \wedge e_4^*) \otimes e_1^* \in V_{12,32}, \\
(E_{5,n} - id)(id - E_{4,2})(id - E_{3,1})(id_{H^{\otimes 2} \otimes \phi_1})\phi_* (\beta_{5,3,4}^O) \\
= 4(n - 3)(e_5 \otimes e_5 \otimes (e_1^* \wedge e_2^*) \otimes (e_3^* \wedge e_4^*) \otimes e_1^*) \in V_{2,32}, \\
(id - P_{5,n})(id - E_{4,1})(id_{H^{\otimes 2} \otimes \phi_2})\phi_* (\beta_{5,3,4}^O) \\
= 2(n + 1)(e_5 \otimes e_n - e_n \otimes e_5) \otimes (e_1^* \wedge e_2^* \wedge e_3^*) \otimes e_1^* \otimes e_1^* \in V_{12,312}, \\
(E_{5,n} - id)(id - E_{4,1})(id_{H^{\otimes 2} \otimes \phi_2})\phi_* (\beta_{5,3,4}^O) \\
= 2(n - 1)e_5 \otimes e_5 \otimes (e_1^* \wedge e_3^* \wedge e_4^*) \otimes e_1^* \otimes e_1^* \in V_{2,312}, \\
(id - P_{5,n})(id - E_{4,2})(id_{H^{\otimes 2} \otimes \phi_3})\phi_* (\beta_{5,3,4}^O) \\
= -2(n - 1)(e_5 \otimes e_n - e_n \otimes e_5) \otimes (e_1^* \wedge e_2^* \wedge e_3^*) \otimes (e_1^* \wedge e_2^*) \in V_{12,21}, \\
(E_{5,n} - id)(id - E_{4,2})(id_{H^{\otimes 2} \otimes \phi_3})\phi_* (\beta_{5,3,4}^O) \\
= -2(n + 1)e_5 \otimes e_5 \otimes (e_1^* \wedge e_2^* \wedge e_3^*) \otimes (e_1^* \wedge e_2^*) \in V_{2,21}, \\
(id - P_{5,n})(id_{H^{\otimes 2} \otimes \phi_4})\phi_* (\beta_{5,3,4}^O) \\
= 4(n - 2)(e_5 \otimes e_n - e_n \otimes e_5) \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \otimes e_1^* \in V_{12,213}, \\
(E_{5,n} - id)(id_{H^{\otimes 2} \otimes \phi_4})\phi_* (\beta_{5,3,4}^O) \\
= 4(n - 2)e_5 \otimes e_5 \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \otimes e_1^* \in V_{2,213}.
\end{align*}
\]

We can use \(\beta_{1,n,3}^O\) to detect 3 irreducible components of size 3 as follows. Define contraction maps

\[
\begin{align*}
\varphi_1 : (H \otimes (H^*)^{\otimes 2})^{\otimes 3} & \to (H^*)^{\otimes 3} , \quad \bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_2^*(c_1) a_3^*(c_2) a_1^*(c_3)b_1^* \otimes b_2^* \otimes b_3^*, \\
\varphi_2 : (H \otimes (H^*)^{\otimes 2})^{\otimes 3} & \to (H^*)^{\otimes 3} , \quad \bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto b_3^*(c_1) a_3^*(c_2) a_1^*(c_3)b_1^* \otimes a_2^* \otimes b_2^*.
\end{align*}
\]

Then we have

\[
(id - E_{2,1})(id - E_{3,1})\varphi_1 (\beta_{1,n,3}^O) = 3(n - 1)e_1^* \otimes e_1^* \otimes e_1^* \in V_{0,3}.
\]
Considering the image under the projection \((H^*)^{\otimes 3} \rightarrow \Lambda^3 H^*\), we have
\[
\varphi_1 t^*(\beta^O_{1, n, 3}) = -3(n - 1)e_1^* \wedge e_2^* \wedge e_3^* \in V_{0, 13}.
\]
Considering the image under the projection \((H^*)^{\otimes 3} \rightarrow \Lambda^2 H^* \otimes H^*\) defined by \(a^* \otimes b^* \otimes c^* \mapsto (b^* \wedge c^*) \otimes a^*\), we have
\[
(id - E_{3, 1})\varphi_{2*} t^*(\beta^O_{1, n, 3}) = 2(n - 2)(e_1^* \wedge e_2^*) \otimes e_1^* \in V_{0, 21}.
\]
We can use \(\beta^O_{1, 3, 4}\) and \(\beta^O_{1, 1}\) to detect 6 irreducible components of size 5 as follows. Define contraction maps
\[
\psi_1: (H \otimes (H^*)^{\otimes 2})^{\otimes 3} \rightarrow H \otimes (H^*)^{\otimes 4},
\]
\[
\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto \psi^*_2(c_1a_1^*(c_2)c_3 \otimes (a_1^* \otimes b_1^* \otimes b_2^* \otimes b_3^*),
\]
\[
\psi_2: (H \otimes (H^*)^{\otimes 2})^{\otimes 3} \rightarrow H \otimes (H^*)^{\otimes 4},
\]
\[
\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto \psi^*_2(c_2b_2^*(c_3) \otimes (a_1^* \otimes b_1^* \otimes a_2^* \otimes b_2^*).
\]
Considering the image under the projection \((H^*)^{\otimes 4} \rightarrow \Lambda^2 H^* \otimes (H^*)^{\otimes 2}\) defined by \(a^* \otimes b^* \otimes c^* \otimes d^* \mapsto (a^* \wedge b^*) \otimes c^* \otimes d^*\), we obtain
\[
(id - E_{3, 1})(id - E_{2, 1})\psi_1 t^*(\beta^O_{1, 3, 4}) = 2(n - 1)e_1 \otimes (e_1^* \wedge e_2^*) \otimes e_1^* \in V_{1, 31}.
\]
Considering the image under the projection \((H^*)^{\otimes 4} \rightarrow \Lambda^2 H^* \otimes H^* \otimes (H^*)^{\otimes 2}\) defined by \(a^* \otimes b^* \otimes c^* \otimes d^* \mapsto (a^* \wedge b^*) \otimes (c^* \otimes d^*)\), we have
\[
(id - E_{3, 2})(id - E_{2, 1})\psi_1 t^*(\beta^O_{3, 1}), \psi_2 t^*(\beta^O_{3, 1}))
\]
\[
= (-2(n + 1), 8(n - 1)) \cdot e_2 \otimes (e_1^* \wedge e_2^*) \otimes (e_1^* \wedge e_2^*),
\]
\[
(id - E_{3, 2})(id - E_{2, 1})\psi_1 t^*(\beta^O_{3, 1}), \psi_2 t^*(\beta^O_{3, 1}))
\]
\[
= (2n^2 - 3, -8n - 3(n - 1)) \cdot e_2 \otimes (e_1^* \wedge e_2^*) \otimes (e_1^* \wedge e_2^*).
\]
Therefore, we can see that \(\im \cup\) contains \(V_{1, 21}\) with multiplicity 2. Let \(\psi_i'\) denote the composition of \(\psi_i\) with the projection \((H^*)^{\otimes 4} \rightarrow (\Lambda^3 H^*) \otimes H^*\) defined by \(a^* \otimes b^* \otimes c^* \otimes d^* \mapsto (a^* \wedge b^* \wedge c^*) \otimes d^*\) and \(\psi_i''\) the composition with the projection defined by \(a^* \otimes b^* \otimes c^* \otimes d^* \mapsto (d^* \wedge a^* \wedge b^*) \otimes c^*\), we have
\[
(id - E_{3, 1})(\psi_i t^* (\beta^O_{3, 1}), \psi_i'' (\beta^O_{3, 1})) = (2(n - 2), 2) \cdot e_2 \otimes (e_1^* \wedge e_2^* \wedge e_3^*) \otimes e_1^*,
\]
\[
(\psi_i t^* (\beta^O_{1, 3}), \psi_i'' (\beta^O_{3, 1})) = (2n^2 - 5n - 5, 2n - 2 \cdot e_2 \otimes (e_1^* \wedge e_2^* \wedge e_3^*) \otimes e_1^*.
\]
Therefore, we can see that \(\im \cup\) contains \(V_{1, 21}\) with multiplicity 2. This completes the proof.
\[\square\]
10.4. **Proof of Theorem 10.1** Here we complete the proof of Theorem 10.1. Let $n \geq 9$. We obtained an irreducible decomposition of $H_3(U, \mathbb{Q})$ into 36 irreducibles, and 17 of them are not contained in $H_3^{A}(\text{IO}_n, \mathbb{Q})$. Therefore, it suffices to check that $H_3^{A}(\text{IO}_n, \mathbb{Q})$ contains a subrepresentation which is isomorphic to $W_3^{O}$, which is a direct sum of 19 irreducibles.

By Proposition 9.10 we have an injective $\text{GL}(n, \mathbb{Q})$-homomorphism

$$W_3^{O} \oplus (W_2^{O} \otimes H) \hookrightarrow H_3^{A}(\text{IO}_n, \mathbb{Q}) \oplus (H_2^{A}(\text{IO}_n, \mathbb{Q}) \otimes H).$$

Since we have $H_2^{A}(\text{IO}_n, \mathbb{Q}) \cong W_2^{O}$ by Pettet, we have $W_3^{O} \hookrightarrow H_3^{A}(\text{IO}_n, \mathbb{Q})$, which completes the proof.

11. The third Albanese homology of $\text{IA}_n$

In this section, we compute $H_3^{A}(\text{IA}_n, \mathbb{Q})$ and prove that Conjecture 6.2 holds for $i = 3$ by using the same method as we used to compute $H_3^{A}(\text{IO}_n, \mathbb{Q})$ in Section 10.

11.1. **The third Albanese homology of $\text{IA}_n$.** By Theorem 6.1 $H_3^{A}(\text{IA}_n, \mathbb{Q})$ contains $W_3$, which consists of the following 34 irreducible representations:

$$W(3, 0) = V_{1^3}, \quad W(0, 3) = V_{1^3},$$

$$\begin{align*}
W(21, 0) & = V_{2^2,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{1^5,1^2}, \\
W(2, 1) & = V_{2^2,1^2} \oplus V_{1^4,1^2}, \quad W(1, 2) = V_{2^2,1^2} \oplus V_{1^4,1^2}, \\
W(0, 21) & = V_{21} \oplus V_{1^3},
\end{align*}$$

$$\begin{align*}
W(1^3, 0) & = V_{3^2,1^2} \oplus V_{32,1^2} \oplus V_{32,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{1^5,1^2}, \\
W(1^2, 1) & = V_{32,1^2} \oplus V_{32,1^2} \oplus V_{22,1^2} \oplus V_{22,1^2} \oplus V_{22,1^2} \oplus V_{1^5,1^2}, \\
W(1, 1^2) & = V_{2^2,1^2} \oplus V_{2^2,1^2} \oplus V_{1^4,1^2}, \quad W(0, 1^3) = V_{1^3}.
\end{align*}$$

In the rest of this section, we prove the following theorem.

**Theorem 11.1.** Let $n \geq 9$. $H_3^{A}(\text{IA}_n, \mathbb{Q})$ is decomposed into 34 irreducible $\text{GL}(n, \mathbb{Q})$-representations:

$$H_3^{A}(\text{IA}_n, \mathbb{Q}) \cong W_3 \oplus V_{5^2,1^3} \oplus V_{32,1^2} \oplus V_{31^3,3} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{2^3,1^2} \oplus V_{1^5,1^2} \oplus V_{2^2,1^3} \oplus V_{2^2,1^3} \oplus V_{1^4,1^2} \oplus V_{21,0} \oplus V_{1^3,0}.$$
Lemma 11.2. Let \( n \geq 9 \). \( H_3(U, \mathbb{Q}) \) is decomposed into the following 61 irreducible \( \text{GL}(n, \mathbb{Q}) \)-representations:

\[
H_3(U, \mathbb{Q}) = \bigwedge^3 (V_{1,2} + V_{1,0}) \\
\cong V_{32,11} + V_{32,12} + V_{31,3,3} + V_{23,3} + V_{221,21} + V_{221,13} + V_{214,21} + V_{16,13} \\
\oplus V_{32,2} + V_{31,2} + V_{23,2} \oplus V_{212,12} + V_{212,13} + V_{213,12} + V_{213,13} \\
\oplus V_{1,2,2} + V_{1,1,2} + V_{3,1,1} + V_{2,2,1} + V_{2,1,2} + V_{1,4,1} + V_{3,4,0} + V_{4,2,0} + V_{5,3,0}.
\]

11.3. Upper bound of \( H_3^*(\text{IA}_n, \mathbb{Q}) \). Let \( \tau^* : H^i(U, \mathbb{Q}) \to H^i(\text{IA}_n, \mathbb{Q}) \) denote the map induced by the Johnson homomorphism \( \tau : \text{IA}_n \to U \) on cohomology. Let \( R_i = \ker \tau^* \). Here, we compute \( R_3 \).

By Pettet [34], we have an exact sequence of \( \text{GL}(n, \mathbb{Z}) \)-representations

\[
0 \to \text{Hom}(\text{gr}^2(\text{IA}_n), \mathbb{Q}) \xrightarrow{\iota} H^2(U, \mathbb{Q}) \xrightarrow{\tau^*} H^2(\text{IA}_n, \mathbb{Q}),
\]

where \( \text{gr}^2(\text{IA}_n) \) is the degree 2 part of the graded Lie algebra associated to the lower central series of \( \text{IA}_n \). Here \([ , ]^*\) is induced by the surjection

\[
[ , ]^* : \bigwedge^2 U \to \text{gr}^2(\text{IA}_n)
\]

which sends \( \bar{\pi} \wedge \bar{\eta} \in \bigwedge^2 U \) to \( [x, y] \in \text{gr}^2(\text{IA}_n) \). Pettet obtained an irreducible decomposition of \( R_2 \).

Lemma 11.3 (Pettet [34]). Let \( n \geq 3 \). We have

\[
R_2 \cong V_{1,21} + V_{0,12}.
\]

Since we have the following commutative diagram

\[
\begin{array}{ccc}
H^2(\text{IA}_n, \mathbb{Q}) & \xrightarrow{\tau^*} & H^2(\text{IO}_n, \mathbb{Q}) \\
\tau \downarrow & & \tau \downarrow \\
H^2(U, \mathbb{Q}) & \xrightarrow{\text{pr}^*} & H^2(U^O, \mathbb{Q}),
\end{array}
\]

we obtain

\[
(11.3.2) \quad R_2^O = \ker \tau_0^* \subset R_2 = \ker \tau^*.
\]

Therefore, we obtain a generator of \( V_{1,21} \subset R_2 \).

Lemma 11.4. The subrepresentation \( V_{1,21} \subset R_2 \) is generated by

\[
\beta = \sum_{j=1}^n e_j^{1,2} \wedge e_j^{1,1} \in H^2(U, \mathbb{Q}) \cong \bigwedge^2 U^*.
\]

We use the exact sequence (11.3.1) to find a generator of \( V_{0,12} \subset R_2 \).

Lemma 11.5. The subrepresentation \( V_{0,12} \subset R_2 \) is generated by

\[
\gamma = \sum_{j,k=1}^n e_j^{1,2} \wedge e_k^{1,k} + \sum_{j,k=1}^n e_j^{1,k} \wedge e_k^{1,2} \in H^2(U, \mathbb{Q}) \cong \bigwedge^2 U^*.
\]
Lemma 11.6. Let $n \geq 6$. We have an irreducible decomposition

$$H^1(U, \mathbb{Q}) \otimes R_2 \cong (V_{1,12} \oplus V_{0,1}) \otimes (V_{1,21} \oplus V_{0,12})$$

$$\cong V_{2,32} \oplus V_{1,22} \oplus V_{1,23} \oplus V_{2,2} \oplus V_{2,21} \oplus V_{2,21} \oplus V_{2,213} \oplus V_{1,31} \oplus V_{1,21} \oplus V_{1,212} \oplus V_{1,14} \oplus V_{0,3} \oplus V_{0,11} \oplus V_{0,13} \oplus V_{0,13} \oplus V_{0,13} \oplus V_{0,13}.$$

Proof. We use the basis for $H_2(U, \mathbb{Q}) \cong \bigwedge^2 U$ induced by the basis $\{e_{a,b}^c | 1 \leq a, b, c \leq n, a < b\}$ for $U$.

The second Johnson homomorphism gives us an inclusion map

$$\tau^{(2)} : \text{gr}^2(\Lambda_{A_n}) \hookrightarrow \text{Hom}(H, (\bigwedge^2 U \otimes H)/\bigwedge^3 H) \cong H^* \otimes (\bigwedge^2 H \otimes H)/\bigwedge^3 H.$$

(See [34, Section 2.3] for the definition of the second Johnson homomorphism.) We have $\text{gr}^2(\Lambda_{A_n}) \cong V_{21,1} \oplus V_{12,0}$, and $\text{Hom}(H, (\bigwedge^2 U \otimes H)/\bigwedge^3 H) \cong V_{21,1} \oplus V_{12,0} \oplus V_{2,0}$. In what follows, we identify elements of $\text{gr}^2(\Lambda_{A_n})$ and the images of them under $\tau^{(2)}$. The subrepresentation of $\text{gr}^2(\Lambda_{A_n})$ that is isomorphic to $V_{12,0}$ has the following basis

$$\{x_{p,q} = \sum_{j=1}^n e_j^p \otimes (e_q \wedge e_j) | 1 \leq p < q \leq n\}.$$

Let $x_{1,2}^* \in \text{Hom}(\text{gr}^2(\Lambda_{A_n}), \mathbb{Q})$ denote the dual basis vector which sends $x_{1,2}$ to 1 and the other basis vectors to 0. Then $x_{1,2}^*$ is a generator of the subrepresentation of $\text{Hom}(\text{gr}^2(\Lambda_{A_n}), \mathbb{Q})$ that is isomorphic to $V_{0,12}$.

Since we have $R_2 = \text{im}[\cdot, \cdot]^* \{1, 3, 1\}$, in order to show that $\gamma$ generates the subrepresentation of $R_2$ which is isomorphic to $V_{0,12}$, it suffices to check that

$$\gamma = (n+1)[\cdot, \cdot]^*(x_{1,2}^*).$$

We can check that for distinct elements $j, k \in [n] \setminus \{1, 2\}$,

$$x_{1,2}^*([e_{1,k}^j, e_{1,2}^k]) = 1/(n+1), \quad x_{1,2}^*([e_{1,2}^j, e_{1,2}^k]) = 1/(n+1),$$

$$x_{1,2}^*([e_{1,2}^j, e_{1,1}^k]) = 2/(n+1), \quad x_{1,2}^*([e_{1,1}^j, e_{1,1}^k]) = 2/(n+1),$$

$$x_{1,2}^*([e_{1,1}^j, e_{1,2}^k]) = 1/(n+1), \quad x_{1,2}^*([e_{1,2}^j, e_{1,1}^k]) = 1/(n+1),$$

$$x_{1,2}^*([e_{1,1}^j, e_{1,2}^k]) = 1/(n+1), \quad x_{1,2}^*([e_{1,2}^j, e_{1,1}^k]) = 3/(n+1),$$

and that

$$[\cdot, \cdot]^*(x_{1,2}^*) = 0$$

for any other basis vectors $x$ for $H_2(U, \mathbb{Q}) \cong \bigwedge^2 U$. Therefore, we have

$$[\cdot, \cdot]^*(x_{1,2}^*) = \frac{1}{n+1} \gamma.$$

Let

$$\cup : H^1(U, \mathbb{Q}) \otimes R_2 \to H^3(U, \mathbb{Q})$$

denote the restriction of the cup product. Then we have $\text{im} \cup \subset R_3$. In what follows, we compute $\text{im} \cup$. We can compute the tensor product $H^1(U, \mathbb{Q}) \otimes R_2$ directly by hand and by using SageMath.

Lemma 11.6. Let $n \geq 6$. We have an irreducible decomposition

$$H^1(U, \mathbb{Q}) \otimes R_2 \cong (V_{1,12} \oplus V_{0,1}) \otimes (V_{1,21} \oplus V_{0,12})$$

$$\cong V_{2,32} \oplus V_{1,22} \oplus V_{1,23} \oplus V_{2,2} \oplus V_{2,21} \oplus V_{2,213} \oplus V_{1,31} \oplus V_{1,21} \oplus V_{1,212} \oplus V_{1,14} \oplus V_{0,3} \oplus V_{0,21} \oplus V_{0,13} \oplus V_{0,13} \oplus V_{0,13} \oplus V_{0,13}.$$
In terms of $\text{GL}(n, \mathbb{Q})$-representations, we can identify the cup product map $\cup$ with

$$\wedge: (V_{1,12} \oplus V_{0,1}) \otimes (V_{1,21} \oplus V_{0,12}) \rightarrow \bigwedge^3 (V_{1,12} \oplus V_{0,1}).$$

**Proposition 11.7.** For $n \geq 6$, $\text{im} \cup$ contains a $\text{GL}(n, \mathbb{Q})$-subrepresentation consisting of the following 27 irreducible representations:

$$\text{im} \cup \supseteq V_{2,32} \oplus V_{1,32} \oplus V_{2,31} \oplus V_{1,31} \oplus V_{2,22} \oplus V_{1,22} \oplus V_{2,21} \oplus V_{1,21} \oplus V_{1,1} \oplus V_{0,3} \oplus V_{0,21} \oplus V_{0,11}.$$

**Proof.** By Proposition [10.6 and [11.3.2], the 8 irreducible components of size 7 and the irreducible component which is isomorphic to $V_{0,3}$ are contained in $\text{im} \cup$. By Lemmas [11.3] and [11.5] we have the following elements in $\text{im} \cup$:

$$\beta_{a,b,c} = e_a^{b,c} \wedge \beta, \quad \gamma_{a,b,c} = e_a^{b,c} \wedge \gamma.$$

Let $\iota^*: \bigwedge^3 U^* \hookrightarrow (H \otimes (H^*)^\otimes 2)^\otimes 3$ denote the canonical inclusion. We will check that $V_{0,21} \otimes V_{0,11}$ is contained in $\text{im} \cup$. We use the following contraction maps:

$$\varphi_2 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow (H^*)^\otimes 3, \quad \bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto b_3^* (c_1) a_3^* (c_2) a_1^* (c_3) b_1 \otimes a_2 \otimes b_2,$$

$$\varphi_3 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow (H^*)^\otimes 3, \quad \bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_2^* (c_1) a_3^* (c_2) a_1^* (c_3) b_1 \otimes a_2 \otimes b_2,$$

$$\varphi_4 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow (H^*)^\otimes 3, \quad \bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto b_1^* (c_1) a_3^* (c_2) a_2^* (c_3) a_1^* \otimes a_2 \otimes b_2.$$ 

Considering the image under the projection $(H^*)^\otimes 3 \rightarrow \bigwedge^3 H^*$, we have

$$(\varphi_2 \iota^* (\beta_{1,n,3}), \varphi_3 \iota^* (\beta_{1,n,3}), \varphi_4 \iota^* (\beta_{1,n,3}))$$

$$= (2(n - 1), 0, 0) \cdot e_1^* \wedge e_2^* \wedge e_3^*,$$

$$(\text{id} - E_{4,1}) (\varphi_2 \iota^* (\gamma_{1,3,4}), \varphi_3 \iota^* (\gamma_{1,3,4}), \varphi_4 \iota^* (\gamma_{1,3,4}))$$

$$= (2(n^2 - 2n - 1), 2(n^2 + n), 2(n + 1)) \cdot e_1^* \wedge e_2^* \wedge e_3^*,$$

$$(\text{id} - E_{4,3}) (\varphi_2 \iota^* (\gamma_{3,4,3}), \varphi_3 \iota^* (\gamma_{3,4,3}), \varphi_4 \iota^* (\gamma_{3,4,3}))$$

$$= (-2(n - 1), 0, 2(n^2 - 1)) \cdot e_1^* \wedge e_2^* \wedge e_3^*.$$

Therefore, we can see that $\text{im} \cup$ contains $V_{0,13}$ with multiplicity 3.

Considering the image under the projection $(H^*)^\otimes 3 \rightarrow \bigwedge^2 H^* \otimes H^*$ defined by

$$a^* \otimes b^* \otimes c^* \mapsto (a^* \wedge b^*) \otimes c^*$$

for $\varphi_3$, and defined by $a^* \otimes b^* \otimes c^* \mapsto (b^* \wedge c^*) \otimes a^*$
for $\varphi_2, \varphi_4$, we have

$$(\text{id} - E_{3,1})(\varphi_2 \iota^* (\beta_{1,n,3}), \varphi_3 \iota^* (\beta_{1,n,3}), \varphi_4 \iota^* (\beta_{1,n,3}))$$

$$= (2(n - 2), 0, 0) \cdot (e_1^* \wedge e_2^* \otimes e_1^*),$$

$$(\text{id} - E_{4,2})(\text{id} - E_{3,1}))(\varphi_2 \iota^* (\gamma_{1,3,4}), \varphi_3 \iota^* (\gamma_{1,3,4}), \varphi_4 \iota^* (\gamma_{1,3,4}))$$

$$= (2(n^2 + n - 4), 2(n^2 + n), 2(n + 1)) \cdot (e_1^* \wedge e_2^* \otimes e_1^*),$$

$$(\text{id} - E_{4,1})(\varphi_2 \iota^* (\gamma_{3,4,3}), \varphi_3 \iota^* (\gamma_{3,4,3}), \varphi_4 \iota^* (\gamma_{3,4,3}))$$

$$= (-2(n - 4), 6n, 2(n^2 - 1)) \cdot (e_1^* \wedge e_2^* \otimes e_1^*).$$

Therefore, we can see that $\text{im} \cup$ contains $V_{0,21}$ with multiplicity 3.

Lastly, we will check that 12 irreducible components of size 5 are contained in $\text{im} \cup$. We use the following contraction maps

$$\psi_1 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow H \otimes (H^*)^\otimes 4,$$

$$\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_2^* (c_1) a_3^* (c_2) c_3 \otimes (a_1^* \otimes b_1^* \otimes b_2^* \otimes b_3^*),$$

$$\psi_2 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow H \otimes (H^*)^\otimes 4,$$

$$\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_3^* (c_2) b_3^* (c_1) c_3 \otimes (a_1^* \otimes b_1^* \otimes a_2^* \otimes b_2^*),$$

$$\psi_3 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow H \otimes (H^*)^\otimes 4,$$

$$\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto a_3^* (c_2) b_3^* (c_1) c_3 \otimes (a_1^* \otimes b_1^* \otimes a_2^* \otimes b_2^*),$$

$$\psi_4 : (H \otimes (H^*)^\otimes 2)^\otimes 3 \rightarrow H \otimes (H^*)^\otimes 4,$$

$$\bigotimes_{j=1}^3 (c_j \otimes a_j^* \otimes b_j^*) \mapsto b_1^* (c_1) a_3^* (c_2) c_3 \otimes (a_1^* \otimes a_2^* \otimes b_2^* \otimes b_3^*).$$

Considering the image under the projection $(H^*)^\otimes 4 \rightarrow \bigwedge^2 H^* \otimes (H^*)^\otimes 2$ defined by $a^* \otimes b^* \otimes c^* \otimes d^* \rightarrow (a^* \wedge b^*) \otimes c^* \otimes d^*$ for $\psi_1$, and defined by $a^* \otimes b^* \otimes c^* \otimes d^* \rightarrow (b^* \wedge c^*) \otimes a^* \otimes d^*$ for $\psi_4$, we have

$$(\text{id} - E_{4,2})(\text{id} - E_{3,1}))(\psi_1 \iota^* (\beta_{1,3,4}), \psi_4 \iota^* (\beta_{1,3,4}))$$

$$= (2(n - 1), 0) \cdot e_n \otimes (e_1^* \wedge e_2^*) \otimes e_1^* \otimes e_1^*,$$

$$(\text{id} - E_{4,1})(\psi_1 \iota^* (\beta_{3,4,3}), \psi_4 \iota^* (\beta_{3,4,3}))$$

$$= (2, 2(n - 1)) \cdot e_n \otimes (e_1^* \wedge e_2^*) \otimes e_1^* \otimes e_1^*.$$

Therefore, we can see that $\text{im} \cup$ contains $V_{1,31}$ with multiplicity 2.

Considering the image under the projection $(H^*)^\otimes 4 \rightarrow \bigwedge^4 H^*$ defined by $a^* \otimes b^* \otimes c^* \otimes d^* \rightarrow a^* \wedge b^* \wedge c^* \wedge d^*$, we have

$$(\psi_1 \iota^* (\beta_{1,3,4}), \psi_4 \iota^* (\beta_{1,3,4})) = (-2(n + 1), 0) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*),$$

$$(E_{n,3} - \text{id})(\psi_1 \iota^* (\gamma_{3,4,3}), \psi_4 \iota^* (\gamma_{3,4,3})) = (8(n + 1), 4(n + 1)) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*).$$

Therefore, we can see that $\text{im} \cup$ contains $V_{1,14}$ with multiplicity 2.
Considering the image under the projection \((H^*)^4 \to (\wedge^2 H^*) \otimes (\wedge^2 H^*)\) defined by \(a^* \otimes b^* \otimes c^* \otimes d^* \mapsto (a^* \wedge b^* \wedge c^* \wedge d^*)\) for \(\psi_1, \psi_2, \psi_3,\) and defined by \((b^* \wedge c^*) \otimes (a^* \wedge d^*)\) for \(\psi_4,\) we have

\[
(id - E_{3,2})(id - E_{3,1})(\psi_1 \epsilon^* (\beta_{1,3,4}), \psi_2 \epsilon^* (\beta_{1,3,4}), \psi_3 \epsilon^* (\beta_{1,3,4}), \psi_4 \epsilon^* (\beta_{1,3,4})) = \left(\frac{1}{2}(n - 2), 0, 0, 0\right) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \wedge e_1^*.
\]

\[
(id - E_{3,2})(id - E_{3,1})(\psi_1 \epsilon^* (\beta_{1,3,4}), \psi_2 \epsilon^* (\beta_{1,3,4}), \psi_3 \epsilon^* (\beta_{1,3,4}), \psi_4 \epsilon^* (\beta_{1,3,4})) = \left(\frac{1}{2}(n - 2), 1, 0, 1\right) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \wedge e_1^*.
\]

\[
(id - E_{3,2})(id - E_{3,1})(\psi_1 \epsilon^* (\beta_{1,3,4}), \psi_2 \epsilon^* (\beta_{1,3,4}), \psi_3 \epsilon^* (\beta_{1,3,4}), \psi_4 \epsilon^* (\beta_{1,3,4})) = \left(\frac{1}{2}(n - 2), 2, 0, 0\right) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \wedge e_1^*.
\]

\[
(id - E_{3,2})(id - E_{3,1})(\psi_1 \epsilon^* (\beta_{1,3,4}), \psi_2 \epsilon^* (\beta_{1,3,4}), \psi_3 \epsilon^* (\beta_{1,3,4}), \psi_4 \epsilon^* (\beta_{1,3,4})) = \left(\frac{1}{2}(n - 2), 0, 1, 1\right) \cdot e_n \otimes (e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*) \wedge e_1^*.
\]

Therefore, we can see that \(\text{im} \cup \text{contains}\ V_{1,21^2}\) with multiplicity 4. This completes the proof.

11.4. **Proof of Theorem** [11.1] Here we complete the proof of Theorem [11.1]. By Lemma [11.2] we obtain an irreducible decomposition of \(H_3(U, \mathbb{Q})\), which consists of 61 irreducibles. By Proposition [11.7] we obtain 27 irreducible components that are not contained in \(H_3^{\text{IA}}(\text{IA}_n, \mathbb{Q})\). By Theorem [6.1] we have 34 irreducible components that are included in \(H_3^{\text{IA}}(\text{IA}_n, \mathbb{Q})\). Therefore, we obtain

\[
H_3^{\text{IA}}(\text{IA}_n, \mathbb{Q}) \cong W_3,
\]

which completes the proof of Theorem [11.1].

**Remark** 11.8. By the above proof, we have

\[
H_3(U, \mathbb{Q})/ \text{im} \cup \cong H_3^{\text{IA}}(\text{IA}_n, \mathbb{Q}).
\]

Since the image of the cup product map coincides with the degree 3 part of the ideal \((R_2) \subset H^*(U, \mathbb{Q})\), Conjecture [8.2] holds for \(*=3\).
12. Cohomology of $\text{Aut}(F_n)$ with twisted coefficients

For an algebraic $\text{GL}(n, \mathbb{Q})$-representation $V$, let $H^*_A(IA_n, V) = H^*_A(IA_n, \mathbb{Q}) \otimes V$. In this section, we study the relation between $H^*_A(IA_n, V)$ and $H^*(\text{Aut}(F_n), V)$.

12.1. Wheeled PROPs and wheeled operads. Here, we recall the notions of PROPs and operads, and wheeled versions of PROPs and operads. See [7, 22, 28, 30, 31] for precise definitions.

A PROP is a symmetric monoidal category $P = (P, \otimes, 0, S)$ with non-negative integers as objects and $m \otimes n = m + n$ for any $m, n \in \mathbb{N}$. We consider PROPs enriched over the category of graded $\mathbb{Q}$-vector spaces.

A wheeled PROP is a PROP $P$ equipped with contraction maps (or partial trace maps) $\xi^i_j : P(m, n) \to P(m - 1, n - 1)$, which can be viewed as connecting the $i$th input and the $j$th output, satisfying compatibility and unitality axioms. A non-unital wheeled PROP (or TRAP in the sense of [7]) is a wheeled PROP without identity morphisms and unitality axioms.

An operad in the category of graded $\mathbb{Q}$-vector spaces is a collection $P = \{P(n)\}_{n \geq 0}$ of graded right $\mathbb{Q}[S_n]$-modules equipped with operadic compositions, which are graded $\mathbb{Q}$-linear maps $\gamma : P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \to P(k_1 + \cdots + k_n)$, and a unit map $\eta : \mathbb{Q} \to P(1)$ satisfying associativity, equivariance and unitality axioms. A non-unital operad is an operad without unit.

For an operad $P$, a right $P$-module is a graded $\mathfrak{S}$-module $M = \{M(n)\}_{n \geq 0}$, which is a collection of graded $\mathfrak{S}_n$-modules $M(n)$, equipped with right $P$-actions, which are graded $\mathbb{Q}$-linear maps $\alpha : M(l) \otimes P(m_1) \otimes \cdots \otimes P(m_l) \to M(m_1 + \cdots + m_l)$, satisfying the operadic form of the standard axioms for a right module over an algebra. For a non-unital operad, we do not assume the unitality axioms.

A wheeled operad $P = \{P(n, m)\}_{(n, m) \in \mathbb{N} \times \{0, 1\}}$ consists of

(i) the operadic part: an operad $P_0 = \{P(n, 1)\}_{n \geq 0}$,
(ii) the wheeled part: a right $P_0$-module $P_w = \{P(n, 0)\}_{n \geq 0}$,
(iii) contraction maps $\xi_i : P_0(n) \to P_w(n - 1)$ for $1 \leq i \leq n$ satisfying compatibility with the structures $\xi$ and $\eta$.

A non-unital wheeled operad $P$ consists of a non-unital operad $P_0$, the wheeled part and the contraction maps as above.

The forgetful functor from the category of wheeled operads (resp. non-unital wheeled operads) to the category of operads (resp. non-unital operads) has a left adjoint denoted by $(\cdot)^\odot$, which is called the wheeled completion.
For any wheeled operad (resp. non-unital wheeled operad) $\mathcal{P}$, there is a wheeled PROP (resp. non-unital wheeled PROP) $\mathcal{C}_\mathcal{P}$ which is freely generated by $\mathcal{P}$. We have

\begin{align*}
\mathcal{C}_\mathcal{P}(m, n) = \bigoplus_{J \subseteq [m]} \bigoplus_{f : J \to [n]} \left( \bigotimes_{i=1}^{n} \mathcal{P}_0([f^{-1}(i)]) \right) \otimes \left( \bigotimes_{i=1}^{k} \mathcal{P}_w([|X_i|]) \right),
\end{align*}

where $P([m] \setminus J, k)$ denotes the set of partitions $X_1 \sqcup \cdots \sqcup X_k = [m] \setminus J$ such that $\min(X_1) < \cdots < \min(X_k)$, $X_1, \cdots, X_k \neq \emptyset$. (See [22, Proposition 2.3] for details.)

12.2. Stable cohomology of $\text{Aut}(F_n)$ with twisted coefficients. Here, we recall the conjectural structure of the stable cohomology of $\text{Aut}(F_n)$ with twisted coefficients given by Kawazumi–Vespa [22].

For $p, q \geq 0$, let $H^{p,q} = H^{*p} \otimes (H^*)^\otimes q$. In [22], Kawazumi and Vespa have studied the structure of the stable cohomology $H^*(\text{Aut}(F_n), H^{p,q})$ for $p, q \geq 0$. They defined a wheeled PROP $\mathcal{H}$ such that

\[ H(p, q) = \lim_{n \in \mathbb{N}} H^*(\text{Aut}(F_n), H^{p,q}). \]

They also defined a wheeled PROP $\mathcal{E}$ such that

\[ \mathcal{E}(p, q) = \bigoplus_{j \in \mathbb{N}} \text{Ext}^{+ - j}_{\mathcal{F}(\text{gr}; \mathbb{Q})}(a^\otimes q \otimes \bigwedge^j a, a^\otimes p) \]

is the direct sum of Ext-groups in the category $\mathcal{F}(\text{gr}; \mathbb{Q})$ of functors from the category $\text{gr}$ of finitely generated free groups to the category of $\mathbb{Q}$-vector spaces, where $a \in \mathcal{F}(\text{gr}; \mathbb{Q})$ is the abelianization functor.

Let $\mathcal{P}_0 = \bigoplus_{k \geq 1} \mathcal{P}_0(k)$ denote the operadic suspension of the operad $\text{Com}$ of non-unital commutative algebras. That is, $\mathcal{P}_0$ is an operad such that $\mathcal{P}_0(0) = 0$ and $\mathcal{P}_0(k) = \text{sgn}_k[k - 1]$ for $k \geq 1$, where $\text{sgn}_k[k - 1]$ is the sign representation of $\mathfrak{S}_k$ placed in cohomological dimension $k - 1$. Let $\mathcal{P}_0^\circ$ denote the wheeled completion of $\mathcal{P}_0$ and $\mathcal{C}_\mathcal{P}_0^\circ$ the wheeled PROP freely generated by the wheeled operad $\mathcal{P}_0^\circ$. Then they constructed a wheeled PROP isomorphism $\mathcal{C}_\mathcal{P}_0^\circ \xrightarrow{\simeq} \mathcal{E}$.

They constructed a morphism of wheeled PROPs $\varphi : \mathcal{H} \to \mathcal{E}$ and proposed the following conjecture.

**Conjecture 12.1** (Kawazumi–Vespa [22, Conjecture 6]). The morphism of wheeled PROP

\[ \varphi : \mathcal{H} \to \mathcal{E} \]

is an isomorphism.

Conjecture [12.1] is equivalent to the following conjecture.

**Conjecture 12.2** (Kawazumi–Vespa). Let $i, p, q$ be non-negative integers. Then, for sufficiently large $n$, we have an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

\[ H^i(\text{Aut}(F_n), H^{p,q}) = \begin{cases} 
\mathcal{C}_{\mathcal{P}_0^\circ}(p, q) & (i = p - q) \\
0 & (i \neq p - q). 
\end{cases} \]
12.3. The structure of $H^*_A(IA_n, \mathbb{Q})$. We defined the traceless part $W_*$ of the graded-symmetric algebra $S^*(U_*)$ of $U_*$ in Section 6. Here, we reconstruct $W_*$ by using the operad $\text{Com}$.

Let $O = \bigoplus_{k \geq 2} P_0(k)$ denote the non-unital suboperad of $P_0$. We have a non-unital wheeled sub-PROP $\mathcal{C}_{O^\circ}$ of $\mathcal{C}_{P_0^\circ}$ associated to the non-unital wheeled operad $O^\circ$. Then we have $\mathcal{O}^\circ_0 = \{O^\circ(n, 1)\}_{n \geq 2}$ and $\mathcal{O}^\circ_w = \{O^\circ(n, 0)\}_{n \geq 1}$. We can reconstruct $W_*$ by using the non-unital wheeled PROP $\mathcal{C}_{O^\circ}$.

**Proposition 12.3.** Let $i$ be a non-negative integer. Then, for sufficiently large $n$, we have a $\text{GL}(n, \mathbb{Q})$-isomorphism

$$W_i \cong \bigoplus_{a \sim b = i} T_{a, b} \otimes \mathbb{Q}[\mathcal{S}_a \times \mathcal{S}_b] \mathcal{C}_{O^\circ}(a, b).$$

*Here the direct sum is finite.*

**Proof.** Since $T_{b+i, b}$ is the traceless part of $H^{b+i, b}$ and we have $W_i = \tilde{S}^*(U_*)_i$, it suffices to show that

$$S^*(U_*)_i \cong \bigoplus_{b \geq 0} H^{b+i, b} \otimes \mathbb{Q}[\mathcal{S}_{b+i} \times \mathcal{S}_b] \mathcal{C}_{O^\circ}(b + i, b).$$

By [12.1], we have

$$\mathcal{C}_{O^\circ}(b + i, b) = \bigoplus_{J \subset [b+i]} \bigoplus_{f: J \to [b]} \left( \bigotimes_{j=1}^{k} \mathcal{O}^\circ_0(|f^{-1}(j)|) \right) \otimes \left( \bigotimes_{j=1}^{k} \mathcal{O}^\circ_w(|X_j|) \right).$$

Since we have $\mathcal{O}^\circ_0 = \{O^\circ(n, 1)\}_{n \geq 2}$, we have only to consider $f$ such that $|f^{-1}(j)| \geq 2$ for each $j \in [b]$. Therefore, we have

$$\mathcal{C}_{O^\circ}(b + i, b) = \bigoplus_{(\mu, \nu) = i} \text{Ind}_{\prod_j \mathcal{S}_{\mu_j+1} \times \mathcal{S}_b}^{\mathcal{S}_{b+i} \times \mathcal{S}_b} \left( \prod_j \mathcal{O}(\mu_j + 1) \otimes \mathcal{S}_{\nu_j} \right) \otimes \left( \prod_j \mathcal{O}^\circ_w(\nu_j) \otimes \mathcal{S}_{\nu_j} \right),$$

where we write $\mu = (\mu_1^{k_1}, \cdots, \mu_r^{k_r})$ and $\nu = (\nu_1^{k_1'}, \cdots, \nu_r^{k_r'})$, and where we consider $\prod_j \mathcal{S}_{\mu_j+1} \times \mathcal{S}_b$ as a subgroup of $\mathcal{S}_{b+i} \times \mathcal{S}_b$. 
Lemma 12.4.
Let $\mathcal{G}(\mu) = \prod_{j=1}^{\nu} (\mathcal{G}_{\nu_j+1} \rtimes \mathcal{G}_{k_j'})$, $\mathcal{G}(\nu) = \prod_{j=1}^{\nu} (\mathcal{G}_{\nu_j} \rtimes \mathcal{G}_{k_j'})$ and $\mathcal{G}(\mu, \nu) = \mathcal{G}(\mu) \times \mathcal{G}(\nu)$. Since we have $\mathcal{O}(\mu_j + 1) = sgn_{\mu_j + 1} [\mu_j]$ and $\mathcal{O}_w^\circ(\nu_j) = sgn_{\nu_j} [\nu_j - 1]$, we have

$$\bigoplus_{b \geq 0} H^{b+i,b} \otimes \mathbb{Q}[\mathcal{G}_{b+i} \times \mathcal{G}_{b}] \mathcal{C}_{\mathcal{O}^\circ}(b + i, b)$$

$$\bigoplus_{b \geq 0} \bigoplus_{i} \mathbb{Q}[\mathcal{G}_{b+i} \times \mathcal{G}_{b}] \text{Ind}_{\mathcal{G}_{h+i} \rtimes \mathcal{G}_{b}}^{\mathcal{G}(\mu, \nu)} \left( \bigotimes_{j=1}^{r} \mathcal{O}(\mu_j + 1)^{\otimes k_j'} \right) \otimes \left( \bigotimes_{j=1}^{s} \mathcal{O}^\circ_w(\nu_j)^{\otimes k_j''} \right)$$

$$\bigoplus_{b \geq 0} \bigoplus_{i \geq 0} \left( H^{k'_j(\mu_j+1),k'_j} \otimes \mathbb{Q}[\mathcal{G}_{b+i} \times \mathcal{G}_{k_j'}] \mathcal{O}(\mu_j + 1)^{\otimes k_j'} \right)$$

$$\bigotimes_{j=1}^{s} \left( H^{k'_j(\nu_j),0} \otimes \mathbb{Q}[\mathcal{G}_{b+i} \times \mathcal{G}_{k_j'}] \mathcal{O}_w^\circ(\nu_j)^{\otimes k_j''} \right)$$

$$\bigotimes_{b \geq 0} \bigoplus_{i \geq 0} \left( \bigotimes_{j=1}^{r} S^{b_j'} (U_{t_{j}}^\text{tree}) \bigotimes_{j=1}^{s} S^{b_j'} (U_{t_{j}}^\text{wheel}) \right)$$

$$\cong S^*(U_{s+i})$$

This completes the proof. \(\square\)

By Proposition 12.3 for sufficiently large $n$, we have a $\text{GL}(n, \mathbb{Q})$-isomorphism

$$(12.3.1) \quad W^*_i \cong \bigoplus_{a-b=i} T_{b,a} \otimes \mathbb{Q}[\mathcal{G}_a \times \mathcal{G}_b] \mathcal{C}_{\mathcal{O}^\circ}(a, b).$$

Then we have the following relation between $W^*_i$ and the wheeled PROP $\mathcal{C}_{\mathcal{P}^\circ_0}$.

**Lemma 12.4.** Let $i, p, q$ be non-negative integers. Then, for sufficiently large $n$, we have an isomorphism of $\mathbb{Q}[\mathcal{G}_p \times \mathcal{G}_q]$-modules

$$(W^*_i \otimes H^{p,q})^{\otimes \text{GL}(n, \mathbb{Z})} \cong \begin{cases} \mathcal{C}_{\mathcal{P}^\circ_0}(p, q) & (i = p - q) \\ 0 & (i \neq p - q) \end{cases}.$$
Proof. We have $H^{p,q} \cong \bigoplus_{c=0}^{\min(p,q)} T^{\oplus(\ell)}_{p-c,q-c}$. Therefore, by \[12.3.1\], we stably have isomorphisms of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$\big((W_i)^* \otimes H^{p,q}\big)_{\text{GL}(n,\mathbb{Z})} \cong \left( \bigoplus_{a,b} T_{b,a} \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \mathcal{O}_{\mathcal{O}^\circ} (a,b) \right) \otimes_{\mathbb{Q}[\mathfrak{S}_a \times \mathfrak{S}_b]} \left( \bigoplus_{c=0}^{\min(p,q)} T^{\oplus(\ell)}_{p-c,q-c} \right)_{\text{GL}(n,\mathbb{Z})}.$$ 

We can check that Conjecture 12.5 holds for $\lambda$ be a non-negative integer and $\lambda$ a bipartition. Then, for sufficiently large $n$, we have a linear isomorphism

$$H^i(\text{Aut}(F_n), V_\lambda) \cong H_A^i(\text{IA}_n, V_\lambda)_{\text{GL}(n,\mathbb{Z})}.$$ 

We can check that Conjecture 12.5 holds for $i \leq 3$ and $\lambda = (\lambda^+, 0), (0, \lambda^-)$. Conjecture 12.5 is equivalent to the following conjecture since $H^{p,q}$ is decomposed into the direct sum of $V_\lambda$'s.

Conjecture 12.6. Let $i, p, q$ be non-negative integers. Then, for sufficiently large $n$, we have an isomorphism of $\mathbb{Q}[\mathfrak{S}_p \times \mathfrak{S}_q]$-modules

$$H^i(\text{Aut}(F_n), H^{p,q}) \cong H_A^i(\text{IA}_n, H^{p,q})_{\text{GL}(n,\mathbb{Z})}.$$ 

We have the following relation between conjectures about the structure of $H_A^*(\text{IA}_n, \mathbb{Q})$ and $H^*(\text{Aut}(F_n), V)$.

Proposition 12.7. Let $i$ be a non-negative integer. If two of the follows hold, then so does the third:

1. We have a $\text{GL}(n, \mathbb{Q})$-isomorphism $H_A^*(\text{IA}_n, \mathbb{Q}) \cong W_i$ for sufficiently large $n$ (cf. Conjecture 6.3).
(2) Conjecture 12.2 holds for cohomological degree \( i \).
(3) Conjecture 12.5 holds for cohomological degree \( i \).

Proof. Since \( W_i^* \) and \( H^*_A(IA_n, \mathbb{Q}) \) are algebraic \( \text{GL}(n, \mathbb{Q}) \)-representations, (1) is equivalent to the following statement: for any \( p, q \), for sufficiently large \( n \), we have

\[
(H^i_A(IA_n, \mathbb{Q}) \otimes H^{p,q})^{\text{GL}(n,\mathbb{Z})} \cong (W_i^* \otimes H^{p,q})^{\text{GL}(n,\mathbb{Z})}.
\]

We have the following loop of possible isomorphisms for sufficiently large \( n \)

\[
H^i_A(IA_n, H^{p,q})^{\text{GL}(n,\mathbb{Z})} \cong (H^i_A(IA_n, \mathbb{Q}) \otimes H^{p,q})^{\text{GL}(n,\mathbb{Z})}
\]

\[
\cong (W_i^* \otimes H^{p,q})^{\text{GL}(n,\mathbb{Z})}
\]

\[
\cong \begin{cases} 
C^{p,q}_G(p, q) & (i = p - q) \\
0 & (i \neq p - q)
\end{cases} \text{ Lemma 12.3}
\]

\[
\cong H^i(\text{Aut}(F_n), H^{p,q}) \text{ (2)}
\]

\[
\cong H^i_A(IA_n, H^{p,q})^{\text{GL}(n,\mathbb{Z})}, \text{ (3)}
\]

which completes the proof. \( \square \)

It is well known that \( \text{Aut}(F_{2g}) \) includes the mapping class group \( \mathcal{M}_{g,1} \) of a surface of genus \( g \) with one boundary component as a subgroup. For the cohomology of \( \mathcal{M}_{g,1} \) with twisted coefficients, we stably have the following isomorphism

\[
(12.4.1) \quad H^*(\mathcal{M}_{g,1}, H^{\otimes p}) \cong (H^*(\text{Gr}_{g,1}, \mathbb{Q}) \otimes H^{\otimes p})^{\text{Sp}(2g,\mathbb{Q})}
\]

(see [17] Theorem 16.3, and see Section 14.2 for the definition of \( \text{Gr}_{g,1} \)). We also have the variants of (12.4.1) for the mapping class groups \( \mathcal{M}_g \) of a closed surface of genus \( g \) and \( \mathcal{M}_{g,1} \) of a surface of genus \( g \) with one marked point, respectively, as Garoufalidis–Getzler [13] first claimed.

A natural analogy of the isomorphism (12.4.1) to \( \text{Aut}(F_n) \) is the following.

**Conjecture 12.8.** Let \( i, p, q \) be non-negative integers. For sufficiently large \( n \), we have an isomorphism of \( \mathbb{Q}[S_p \times S_q] \)-modules

\[
H^i(\text{Aut}(F_n), H^{p,q}) \cong (H^i(\text{Gr}_{IA_n}, \mathbb{Q}) \otimes H^{p,q})^{\text{GL}(n,\mathbb{Z})},
\]

where \( \text{Gr}_{IA_n} \) is the graded Lie algebra of \( IA_n \) associated to the Andreadakis filtration of \( \text{Aut}(F_n) \).

Based on Conjectures 12.6 and 12.8 we make the following conjecture.

**Conjecture 12.9.** We stably have an isomorphism of graded \( \text{GL}(n, \mathbb{Q}) \)-representations

\[
H^*_A(IA_n, \mathbb{Q}) \cong H^*(\text{Gr}_{IA_n} \otimes \mathbb{Q}, \mathbb{Q}).
\]

If \( \text{Gr}_{IA_n} \otimes \mathbb{Q} \) is stably quadratically presented and stably Koszul, then we stably have a surjective morphism of graded \( \text{GL}(n, \mathbb{Q}) \)-representations \( H^*(\text{Gr}_{IA_n} \otimes \mathbb{Q}, \mathbb{Q}) \rightarrow H^*_A(IA_n, \mathbb{Q}) \). See Section 14.3 for the cases of the Torelli groups.
13. Algebraic Sp(2g, Q)-representations

In this section, we recall representation theory of Sp(2g, Q) and introduce the notion of traceless tensor products of Sp(2g, Q)-representations by adapting definitions in Section 2.

13.1. Irreducible Sp(2g, Q)-representations. Here, we fix a symplectic basis \{a_i, b_i \mid 1 \leq i \leq g\} for \(H_1(\Sigma_g, \mathbb{Q})\), and let 

\[ H = H_1(\Sigma_g, \mathbb{Q}), \]

where \(\Sigma_g\) is a closed surface of genus \(g\). Let \(Q : H \otimes H \rightarrow \mathbb{Q}\) denote the symplectic form such that 

\[ Q(a_i, b_i) = -Q(b_i, a_i) = 1 \quad \text{for each} \quad 1 \leq i \leq g. \]

For distinct elements \(k, l \in [p]\), the contraction map 

\[ c_{k,l}^\mathbb{Q} : H^\otimes_p \rightarrow H^\otimes_{p-2} \]

is defined in a way similar to that of GL(n, Q)-representations by using the symplectic form \(Q\). We can define the traceless part \(T_p\) of \(H^\otimes_p\) as Sp(2g, Q)-representations by 

\[ T_p = \bigcap_{1 \leq k < l \leq p} \ker c_{k,l}^\mathbb{Q} \subset H^\otimes_p. \]

For a partition \(\lambda \vdash p\), let 

\[ V_{\lambda}^\mathbb{Q} = T_p \cap V_\lambda \subset H^\otimes_p. \]

If \(\lambda\) has at most \(g\) parts, then \(V_{\lambda}^\mathbb{Q}\) is an irreducible Sp(2g, Q)-representation corresponding to \(\lambda\), and otherwise, we have \(V_{\lambda}^\mathbb{Q} = 0\). The irreducible representation \(V_{\lambda}^\mathbb{Q}\) can also be constructed as the image of the Young symmetrizer \(c_\lambda : T_p \rightarrow T_p\), and it follows that \(V_{\lambda}^\mathbb{Q}\) is generated by \((a_1 \wedge \cdots \wedge a_{\mu_1}) \otimes \cdots \otimes (a_1 \wedge \cdots \wedge a_{\mu_d})\), where \(\mu = (\mu_1, \cdots, \mu_d)\) is the conjugate of \(\lambda\). See [14, 27] for more details.

We have the following irreducible decomposition of the tensor product of two irreducible Sp(2g, Q)-representations \(V_{\lambda}^\mathbb{Q}\) and \(V_{\mu}^\mathbb{Q}\)

\[ V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q} = \bigoplus_{\nu} (V_{\nu}^\mathbb{Q})^{\otimes (N_{\nu})_{\lambda\mu}} \]

where the \(N\)'s denote the Littlewood–Richardson coefficients (see [14, Section 25.3]).

We call an Sp(2g, Q)-representation \(V\) algebraic if after choosing a basis for \(V\), the \((\dim V)^2\) coordinate functions of the action Sp(2g, Q) \(\rightarrow\) GL(V) are rational functions of the \((2g)^2\) variables.

13.2. Traceless tensor products of algebraic Sp(2g, Q)-representations. The traceless tensor products of algebraic Sp(2g, Q)-representations can be defined in a way similar to those of GL(n, Q)-representations in Section 2.

Let \(\lambda \vdash p\) and \(\mu \vdash q\). The traceless tensor product \(V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q}\) of \(V_{\lambda}^\mathbb{Q}\) and \(V_{\mu}^\mathbb{Q}\) is

\[ V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q} = (V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q}) \cap T_{p+q} \subset H^\otimes_{p+q}. \]

In other words, \(V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q}\) is a subrepresentation of \(V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q}\) which vanishes under any contraction maps. For \(g \geq l(\lambda) + l(\mu)\), we have

\[ V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q} \cong \bigoplus_{|\nu| = p+q} (V_{\nu}^\mathbb{Q})^{\otimes (N_{\nu})_{\lambda\mu}} \subset \bigoplus_{\nu} (V_{\nu}^\mathbb{Q})^{\otimes (N_{\nu})_{\lambda\mu}} \cong V_{\lambda}^\mathbb{Q} \otimes V_{\mu}^\mathbb{Q}. \]
Let $M$ be an algebraic $\text{Sp}(2g, \mathbb{Q})$-representation. For each partition $\lambda$, define a vector space

$$M_\lambda = \text{Hom}_{\text{Sp}(2g, \mathbb{Q})}(V^{\text{Sp}}_\lambda, M).$$

Since the category of algebraic $\text{Sp}(2g, \mathbb{Q})$-representations is semisimple, we have a natural isomorphism

$$\iota_M : M \cong \bigoplus \lambda V^{\text{Sp}}_\lambda \otimes M_\lambda.$$ 

For two algebraic $\text{Sp}(2g, \mathbb{Q})$-representations $M$ and $N$, we have an isomorphism

$$\iota_M \otimes \iota_N : M \otimes N \cong \left( \bigoplus \lambda V^{\text{Sp}}_\lambda \otimes M_{\lambda} \right) \otimes \left( \bigoplus \mu V^{\text{Sp}}_\mu \otimes N_{\mu} \right) \cong \bigoplus_{\lambda, \mu} (V^{\text{Sp}}_\lambda \otimes V^{\text{Sp}}_\mu) \otimes (M_\lambda \otimes N_\mu).$$

The traceless tensor product $M \tilde{\otimes} N$ of $M$ and $N$ is defined by

$$M \tilde{\otimes} N = (\iota_M \otimes \iota_N)^{-1} \left( \bigoplus_{\lambda, \mu} (V^{\text{Sp}}_\lambda \otimes V^{\text{Sp}}_\mu) \otimes (M_\lambda \otimes N_\mu) \right) \subset M \otimes N.$$ 

The traceless part $\widetilde{T}^*M$ (resp. $\widetilde{\Lambda}^* M$, $\widetilde{\text{Sym}}^* M$) of the tensor algebra $T^*M$ (resp. the exterior algebra $\Lambda^* M$, the symmetric algebra $\text{Sym}^* M$) is defined in the same way as in Section 2.

Let $M_* = \bigoplus_{i \geq 1} M_i$ be a graded algebraic $\text{Sp}(2g, \mathbb{Q})$-representation. We can also define the traceless part $\widetilde{S}^*(M_*)$ of the graded-symmetric algebra $S^*(M_*)$ as the image of $\widetilde{T}^*M_*$ under the projection $T^*M_* \rightarrow S^*(M_*)$.

We have the following relation between the traceless tensor products of $\text{Sp}(2g, \mathbb{Q})$-representations and those of $\text{GL}(2g, \mathbb{Q})$-representations.

**Lemma 13.1.** Let $M$ and $N$ be algebraic $\text{GL}(2g, \mathbb{Q})$-representations, which can be considered as algebraic $\text{Sp}(2g, \mathbb{Q})$-representations by restriction. Let $M \otimes^{\text{GL}} N$ (resp. $M \tilde{\otimes}^{\text{Sp}} N$) denote the traceless tensor product of $M$ and $N$ as $\text{GL}(2g, \mathbb{Q})$-representations (resp. as $\text{Sp}(2g, \mathbb{Q})$-representations). Then we have

$$(13.2.1) \quad M \tilde{\otimes}^{\text{Sp}} N \subset M \otimes^{\text{GL}} N.$$ 

**Proof.** It suffices to prove (13.2.1) for $M = V_\lambda \subset H^{p,q}$ and $N = V_\mu \subset H^{r,s}$, where $\lambda$ and $\mu$ are bipartitions. By the definitions of the traceless tensor products, we have

$$V_\lambda \tilde{\otimes}^{\text{GL}} V_\mu = (V_\lambda \otimes V_\mu) \cap T_{p+r,q+s}$$

and

$$V_\lambda \tilde{\otimes}^{\text{Sp}} V_\mu = (V_\lambda \otimes V_\mu) \cap \bigcap_{(k,l) \in J} \ker c_{k,l}^{\text{Sp}},$$

where $J$ consists of elements $(k, l) \in [p + q + r + s]^2$ such that $(k, l) \notin \{1, \cdots, p\}^2 \cup \{p + 1, \cdots, p + q\}^2 \cup \{p + q + 1, \cdots, p + q + r\}^2 \cup \{p + q + r + 1, \cdots, p + q + r + s\}^2$. Since we have $\bigcap_{(k,l) \in J} \ker c_{k,l}^{\text{Sp}} \subset T_{p+r,q+s}$, it follows that $V_\lambda \tilde{\otimes}^{\text{Sp}} V_\mu \subset V_\lambda \tilde{\otimes}^{\text{GL}} V_\mu$. $\square$
14. On the Albanese cohomology of the Torelli groups

Let $I_g$ (resp. $I_{g,1}$, $I_1$) denote the Torelli group of an oriented closed surface of genus $g$ (resp. with one boundary component, with one marked point). Here we adapt our arguments on $H^\bullet(I_{g,n}, \mathbb{Q})$ to $H^\bullet(I_g, \mathbb{Q})$, $H^\bullet(I_{g,1}, \mathbb{Q})$ and $H^\bullet(I_1, \mathbb{Q})$, which are algebraic $\text{Sp}(2g, \mathbb{Q})$-representations. Note that the Albanese homology of the Torelli group is isomorphic to the Albanese cohomology of the Torelli group since the algebraic $\text{Sp}(2g, \mathbb{Q})$-representations are self-dual.

The Albanese cohomology of $I_g$ of degree $\leq 3$ has been determined by Johnson [19], Hain [16], Sakasai [30] and Kupers–Randal-Williams [25].

14.1. Lindell’s result about the Albanese homology of $I_{g,1}$. Lindell [27] detected some subquotient $\text{Sp}(2g, \mathbb{Q})$-representations of the Albanese homology of $I_{g,1}$.

**Theorem 14.1** (Lindell [27] Theorem 1.5]). For a pair of partitions $(\lambda, \mu) \vdash i$ such that
\[
\lambda = (\lambda_1^0 > \cdots > \lambda_m^0 > 0), \quad \mu = (\mu_1^0 > \cdots > \mu_{m+2}^0 > 0), \quad \mu_1 = \lambda_1 + 2, \quad \text{for } g \geq i + 2l(\lambda) + l(\mu), \text{ the subrepresentation } \bigoplus_{i=1}^{\lambda_1^0} V^\text{Sp}_{\lambda^0} \text{ spanned by all irreducible subrepresentations of weight } i + 2l(\lambda) \text{ is a subquotient } \text{Sp}(2g, \mathbb{Q})\text{-representation of } H^\bullet(I_{g,1}, \mathbb{Q}).
\]

We expect that one can adapt our argument of Theorem 6.1 to the above result of Lindell to show that the direct sum $\bigoplus_{i=1}^{\lambda_1^0} W_{\lambda^0}^{I_{g,1}}(\lambda, \mu)$ is a subquotient representation of $H^\bullet(I_{g,1}, \mathbb{Q})$. In Section 14.3 we will propose a conjectural structure of $H^\bullet(I_{g,1}, \mathbb{Q})$.

14.2. Structures of the cohomology of Lie algebras associated to the Torelli groups. Let $t_g$ denote the Malcev Lie algebra of $I_g$, and $u_g$ the Lie algebra of the pronipotent radical of the relative Malcev completion of the mapping class group $M_g$ with respect to the short exact sequence $1 \to I_g \to M_g \to \text{Sp}(2g, \mathbb{Z}) \to 1$. (See [16] and [26] for details.) Let $\text{Gr} t_g$ (resp. $\text{Gr} u_g$) denote the graded Lie algebra of $t_g$ (resp. $u_g$) associated to the lower central series. The associated Lie algebras $\text{Gr} t_{g,1}$ and $\text{Gr} u_{g,1}$ of $I_{g,1}$, and $\text{Gr} t_g$ and $\text{Gr} u_g$ of $I_g$ are defined in a similar way. For $g \geq 3$, we have central extensions
\[
\begin{align}
(14.2.1) \quad & \mathbb{Q}[2] \to \text{Gr} t_g \to \text{Gr} u_g, \quad \mathbb{Q}[2] \to \text{Gr} t_{g,1} \to \text{Gr} u_{g,1}, \quad \mathbb{Q}[2] \to \text{Gr} t_g \to \text{Gr} u_g, \\
(14.2.2) \quad & \mathbb{Q}[2] \to \text{Gr} t_{g,1} \to \text{Gr} t_g, \quad \mathbb{Q}[2] \to \text{Gr} t_g \to \text{Gr} u_g,
\end{align}
\]
where $\mathbb{Q}[2]$ is the graded Lie algebra concentrated in degree 2, and an extension
\[
(14.2.3) \quad \text{Gr} p_g \to \text{Gr} u_g \to \text{Gr} u_g,
\]
where $p_g$ is the Malcev Lie algebra of the fundamental group of a closed surface of genus $g$, given by [16] (see also [26] Lemma 3.2)). Hain [16] proved that $\text{Gr} t_g$, $\text{Gr} u_g$, $\text{Gr} t_{g,1}$, $\text{Gr} u_{g,1}$, $\text{Gr} t_g$ and $\text{Gr} u_g$ are quadratically presented for $g \geq 6$. Kupers–Randal-Williams [26] proved that these graded Lie algebras are Koszul in a stable range.

In what follows, we study the structures of the cohomology of Lie algebras $\text{Gr} t_g$, $\text{Gr} u_g$, $\text{Gr} t_{g,1}$, $\text{Gr} u_{g,1}$, $\text{Gr} t_g$ and $\text{Gr} u_g$ by using computation of characters by Garoufalidis–Getzler [15].
Consider a graded algebraic $\text{Sp}(2g, \mathbb{Q})$-representation $Y_* = \bigoplus_{i \geq 1} Y_i$, $Y_1 = \Lambda^{i+2} H$. Also consider quotients $Y'_* = Y_*/(\mathbb{Q}[2])$, $X_* = Y_*/(H[1])$ and $X'_* = X_*/(\mathbb{Q}[2])$, where $\mathbb{Q}[2] \subset Y_2 = \Lambda^4 H$ is the trivial $\text{Sp}(2g, \mathbb{Q})$-subrepresentation, and where $H[1] \subset Y_1 = \Lambda^3 H$. We also consider an extension $Z_* = Y_* \oplus (\mathbb{Q}[2])$, and let $Z'_* = Z_*/(\mathbb{Q}[2]) = Y_*$. Then we obtain the following structures of the cohomology of six Lie algebras.

**Proposition 14.2.** We stably have isomorphisms of graded $\text{Sp}(2g, \mathbb{Q})$-representations

$$H^*(\text{Gr} u_g, \mathbb{Q}) \cong \tilde{S}^*(X_*), \quad H^*(\text{Gr} t_g, \mathbb{Q}) \cong \tilde{S}^*(X'_*),$$

$$H^*(\text{Gr} u_{g,1}, \mathbb{Q}) \cong \tilde{S}^*(Y_*), \quad H^*(\text{Gr} t_{g,1}, \mathbb{Q}) \cong \tilde{S}^*(Y'_*),$$

$$H^*(\text{Gr} u_g^1, \mathbb{Q}) \cong \tilde{S}^*(Z_*), \quad H^*(\text{Gr} t_g^1, \mathbb{Q}) \cong \tilde{S}^*(Z'_*),$$

where stably means that for each cohomological degree $i$, we have isomorphisms for $g \geq 3i$.

We will review the computation of Garoufalidis–Getzler [15 Theorem 1.1]. Let $A = \Lambda^*(V_{2g}^\text{Sp})/V_{2g}^\text{Sp}$ be the quadratic algebra, that is, the quotient of the exterior algebra $\Lambda^*(V_{2g}^\text{Sp})$ by the ideal generated by $V_{2g}^\text{Sp}$. Let $A^1 = \Lambda^*(V_1^\text{Sp} \oplus V_{2g}^\text{Sp})/V_{2g}^\text{Sp}$ be the quadratic algebra defined similarly. Then $A$ (resp. $A^1$) is the quadratic dual of the universal enveloping algebra of $u_g$ (resp. $u_g^1$).

Let $\Lambda$ denote the ring of symmetric functions. Let $h_n$ denote the complete symmetric function, $e_n$ the elementary symmetric function, $p_n$ the power sum symmetric function, $s_\lambda$ the Schur function, $s_{(\lambda)}$ the symplectic Schur function. Let $\omega : \Lambda \to \Lambda$, $p_n \mapsto -p_n$ denote an algebra involution, which satisfies $\omega(h_n) = (-1)^n e_n$. We have a linear automorphism of $\Lambda$

$$D : \Lambda \to \Lambda, \quad s_\lambda \mapsto s_{(\lambda)},$$

and an operation

$$\text{Exp} : \Lambda[[t]] \to \hat{\Lambda}[[t]], \quad f \mapsto \sum_{q=0}^\infty h_q \circ f,$$

where $h_q \circ f$ denotes the plethysm and $\hat{\Lambda}[[t]]$ is the completion of $\Lambda[[t]]$ with respect to the augmentation ideal. For an algebraic $\text{Sp}(2g, \mathbb{Q})$-representation $V = \bigoplus_{\lambda} (V_{\lambda}^\text{Sp})^{\otimes c(\lambda)}$, the character $\text{ch}(V)$ of $V$ is defined by $\text{ch}(V) = \sum_{\lambda} c(\lambda)s_{(\lambda)} \in \Lambda$. The character $\text{ch}_i(M_*)$ of a graded algebraic $\text{Sp}(2g, \mathbb{Q})$-representation $M_*$ is defined by $\text{ch}_i(M_*) = \sum_{q=0}^\infty (-t)^q \text{ch}(M_q) \in \Lambda[[t]]$.

Let

$$L(t) = \sum_{q=1}^\infty \left( \sum_{r=0}^{\lfloor q/2 \rfloor} h_{q+2-2r} \right) t^q \in \Lambda[[t]].$$

**Theorem 14.3** (Garoufalidis–Getzler [15 Theorem 1.1]). The characters $\text{ch}_i(A)$ of $A$ and $\text{ch}_i(A^1)$ of $A^1$ in $\Lambda[[t]]$ are

$$\text{ch}_i(A) = D\omega\text{Exp}(-h_it + L(t)), \quad \text{ch}_i(A^1) = D\omega\text{Exp}(h_0t^2 + L(t)).$$

**Remark 14.4.** The above formula for $\text{ch}_i(A^1)$ has one more copies of $h_0t^2$ in the $\text{Exp}(-)$ than the original one given in [15 Theorem 1.1]. By their proof, we have
a stable isomorphism of algebras

\[ \mathbb{C}\langle T\text{-graphs}\rangle/(\text{IH}^1) \rightarrow (A^1 \otimes T(H))^{Sp}, \]

where the left-hand side is the quotient of the algebra of trivalent graphs with ordered legs modulo the IH$^1$-relation, and where $T(H) = \bigoplus_{q=0}^{\infty} H^{\otimes q}$ is the tensor algebra. Here, the IH$^1$-relation is as follows: if a graph $G$ has a part of shape $I$ at most two of whose four endpoints are connected in $G$, then $G$ is identified with another graph which is obtained from $G$ by replacing the part of shape $I$ with a graph of shape $H$. Therefore, in degree 2, the theta-shaped graph, which they wrote as $e_2,0$, and the dumbbell-shaped graph, which has two loops connected to one edge, are not equivalent. Hence, we need one more $h_0t^2$ in the Exp($-$) of $\text{ch}_t(A^1)$.

**Remark 14.5.** Alternatively, we can compute the character $\text{ch}_t(A^1)$ by using the character $\text{ch}_t(A)$ and the Hochschild–Serre spectral sequence for the extension $\text{Gr} u_g \otimes \mathbb{Q}$ for $g \geq 3$. Since the action of $\text{Gr} u_g$ on $H^*(\text{Gr} p_g, \mathbb{Q}) = \mathbb{Q}[0] \oplus H[1] \oplus \mathbb{Q}[2]$ is trivial, we have an isomorphism of graded $\text{Sp}(2g, \mathbb{Q})$-representations

\[ H^*(\text{Gr} u_g^1, \mathbb{Q}) \cong H^*(\text{Gr} u_g, \mathbb{Q}) \otimes H^*(\text{Gr} p_g, \mathbb{Q}). \]

Therefore, we have

\[ \text{ch}_t(A^1) = \text{ch}_t(A)(1 - h_1t + t^2). \]

**Proof of Proposition 14.2.** We can check that for any graded $\text{Sp}(2g, \mathbb{Q})$-representation $M_\ast$, we have

\[ \text{ch}_t(\tilde{S}^\ast(M_\ast)) = D\text{Exp}(D^{-1}\text{ch}_t(M_\ast)). \]

The characters of $X_\ast, Y_\ast$ and $Z_\ast$ are

\[ \text{ch}_t(Y_\ast) = D \sum_{q=1}^{\infty} \left( \sum_{r=0}^{q+2/2} c_{q+2-2r} \right) (-t)^q = D\omega(L(t)), \]

\[ \text{ch}_t(X_\ast) = D\omega(-h_1t + L(t)), \]

\[ \text{ch}_t(Z_\ast) = D\omega(h_0t^2 + L(t)). \]

It follows that

\[ \text{ch}_t(\tilde{S}^\ast(X_\ast)) = D\text{Exp}(D^{-1}\text{ch}_t(X_\ast)) \]

\[ = D\text{Exp}(\omega(-h_1t + L(t))) \]

\[ = D\omega\text{Exp}(-h_1t + L(t)) \]

\[ = \text{ch}_t(A). \]

We have an isomorphism of graded $\text{Sp}(2g, \mathbb{Q})$-representations

\[ H^*(\text{Gr} u_g, \mathbb{Q}) \cong A \]

for $g \geq 3_\ast$, since $\text{Gr} u_g$ is quadratically presented for $g \geq 6$, and Koszul for $g \geq 3_\ast$. Therefore, we have $H^*(\text{Gr} u_g, \mathbb{Q}) \cong \tilde{S}^\ast(X_\ast)$. The case of $\text{Gr} u_g^1$ is similar.

For the other four graded Lie algebras, we use the Hochschild–Serre spectral sequences for extensions $\text{Gr} t_g$ of $\text{Gr} u_g$, since the action of $\text{Gr} u_g$ on $H^*(\mathbb{Q}[2], \mathbb{Q}) = H^0(\mathbb{Q}[2], \mathbb{Q}) \oplus H^1(\mathbb{Q}[2], \mathbb{Q}) = \mathbb{Q}[0] \oplus \mathbb{Q}[2]$ is trivial, we have

\[ H^*(\text{Gr} t_g, \mathbb{Q}) \cong H^*(\text{Gr} u_g, \mathbb{Q})/(H^*(\text{Gr} u_g, \mathbb{Q})[2]), \]
where $H^*(\Gr u_g, \Q)[2]$ denotes the degree-shift by 2. Therefore, we have
\[
\chi_t(H^*(\Gr t_g, \Q)) = \chi_t(H^*(\Gr u_g, \Q))(1 - t^2)
\]
\[
= (D\omega\Exp(-h_1 t + L(t))) \Exp(-t^2)
\]
\[
= D\omega\Exp(-h_1 t - h_0 t^2 + L(t))
\]
\[
= \chi_t(\bar{S}^*(X'_1)).
\]
Therefore, we have $H^*(\Gr t_g, \Q) \cong \bar{S}^*(X'_1)$, which completes the proof. \qed

14.3. Conjectural structures of the Albanese cohomology of the Torelli groups. Here we study the structures of the Albanese cohomology of $\mathcal{I}_g$, $\mathcal{I}_{g,1}$ and $\mathcal{T}_g$.

**Proposition 14.6.** We stably have surjective morphisms of graded $\Sp(2g, \Q)$-representations
\[
H^*(\Gr t_g, \Q) \to H^*_A(\mathcal{I}_g, \Q), \quad H^*(\Gr t_{g,1}, \Q) \to H^*_A(\mathcal{I}_{g,1}, \Q),
\]
\[
H^*(\Gr t_g^1, \Q) \to H^*_A(\mathcal{T}_g, \Q).
\]

**Proof.** Let $A'$ denote the quadratic algebra $A' = \bigwedge^*(V^{\Sp}_3)/(V^{Sp}_3 \oplus V^{Sp}_3)$. Since $\Gr t_g$ is quadratically presented for $g \geq 6$ \cite{16} and stably Koszul \cite{26}, in a way similar to the argument of \cite{15}, we have an isomorphism of graded $\Sp(2g, \Q)$-representations
\[
H^*(\Gr t_g, \Q) \cong A' \cong H^*(\mathcal{I}_g/\Gamma_3\mathcal{I}_g, \Q),
\]
where $\Gamma_3\mathcal{I}_g$ is the third term of the lower central series of $\mathcal{I}_g$. The natural projection $\mathcal{I}_g \twoheadrightarrow \mathcal{I}_g/\Gamma_3\mathcal{I}_g$ induces a surjective morphism of graded $\Sp(2g, \Q)$-representations
\[
H^*(\mathcal{I}_g/\Gamma_3\mathcal{I}_g, \Q) \to H^*_A(\mathcal{I}_g, \Q).
\]
Therefore, we have a surjective morphism of graded $\Sp(2g, \Q)$-representations
\[
H^*(\Gr t_g, \Q) \to H^*_A(\mathcal{I}_g, \Q).
\]
Similar arguments hold for $\mathcal{I}_{g,1}$ and $\mathcal{T}_g^1$. \qed

By Proposition 14.6, the Albanese cohomology of $\mathcal{I}_g$, $\mathcal{I}_{g,1}$ and $\mathcal{T}_g^1$ are quotient $\Sp(2g, \Q)$-representations of the cohomology of $\Gr t_g$, $\Gr t_{g,1}$ and $\Gr t_g^1$, respectively. In what follows, we propose conjectural structures of the Albanese cohomology of $\mathcal{I}_g$, $\mathcal{I}_{g,1}$ and $\mathcal{T}_g^1$. Consider the quotients $X'_1 = X_1/(\bigoplus_{i \geq 1} \Q[4i - 2])$, $Y'_1 = Y_1/(\bigoplus_{i \geq 1} \Q[4i - 2])$ and $Z'_1 = Z_1/(\bigoplus_{i \geq 1} \Q[4i - 2])$, where $\Q[4i - 2] \subset \bigwedge^4 H$ is the trivial $\Sp(2g, \Q)$-subrepresentation.

**Conjecture 14.7.** We stably have graded $\Sp(2g, \Q)$-isomorphisms
\[
H^*_A(\mathcal{I}_g, \Q) \cong \bar{S}^*(X'_1), \quad H^*_A(\mathcal{I}_{g,1}, \Q) \cong \bar{S}^*(Y'_1), \quad H^*_A(\mathcal{T}_g, \Q) \cong \bar{S}^*(Z'_1).
\]

We stably have $H^*_A(\mathcal{I}_g, \Q) \cong \bar{S}^*(X'_1)$ in degree $\leq 3$ by \cite{16} \cite{36} \cite{25}. By the definition of $\bar{S}^*(Y'_1)$, we can check that the direct sum $W_{g,1}$, which we observed in Section 14.1, is included in $\bar{S}^*(Y'_1)$.

Here, we will recall the result of Kupers–Randal-Williams \cite{25} and study the relation with Conjecture 14.7. In \cite{25} Theorem 4.1 and Sections 6.2 and 8, they
constructed an algebra homomorphism $\Phi : W \to H^*(I_g, 1, \mathbb{Q})$, where $W$ is an algebra whose character $\text{ch}_i(W)$ is
\[
\text{ch}_i(W) = D\omega \left( \prod_{\ell \geq 2} (1 - t^{4i-2}) \right) \exp \left( \frac{1}{t^2}(\sum_{q=0}^{\infty} h_q t^q) - h_0(1 + t^2) - h_1 t - h_2 t^2 \right).
\]
We can easily check that
\[
(14.3.1) \quad \text{ch}_i(W) = \text{ch}_i(\widetilde{S}^*(Y_*)')
\]
since we have
\[
\text{ch}_i(Y_*)' = D\omega \left( L(t) - \sum_{q=1}^{\infty} t^{4q-2} \right)
\]
and
\[
\text{ch}_i(W) = D\omega \exp \left( L(t) - \sum_{q=1}^{\infty} t^{4q-2} \right).
\]
In [25], they proved that for $N \geq 0$, if $H^*(I_g, 1, \mathbb{Q})$ is stably finite-dimensional for $* < N$, then $\Phi$ is an isomorphism for $* \leq N$ and is injective for $* = N + 1$. Therefore, $H^3(I_g, 1, \mathbb{Q})$ contains an $Sp(2g, \mathbb{Z})$-subrepresentation which is isomorphic to $\widetilde{S}^*(Y_*)'$, for $i \leq 3$.

In [25, Sections 7 and 8.1], they also considered the cases for $I_g$ and $I_g^1$. We can also check the variants of (14.3.1) for these two cases by using Remark 14.5.

Remark 14.8. Note that in [25], they used a different definition of $\text{ch}_i(W) = \sum_{q \geq 0} t^q \text{ch}(W_q)$ and a different involution $\omega : p_n \mapsto (-1)^n p_n$ from ours.

Remark 14.9. In [25, Remark 8.2], they claimed that $H^3_A(I_g, \mathbb{Q})$ has one fewer copies of $V_{Sp}^{2g, 3}$ than the algebraic part $H^3(I_g, \mathbb{Q})_{\text{alg}}$ of $H^3(I_g, \mathbb{Q})$, which is the union of algebraic subrepresentations of $H^3(I_g, \mathbb{Q})$. However, their computation of the characters of the third cohomology of $I_g$ and $I_g$ seems incorrect, and indeed it is possible that we have $H^3(I_g, \mathbb{Q})_{\text{alg}} = H^3_A(I_g, \mathbb{Q})$ for sufficiently large $g$.

Kawazumi–Morita [21] studied the $Sp$-invariant stable continuous cohomology $H_*^*(\lim_{g \to \infty} I_g, 1, \mathbb{Q})_{Sp}$. Their conjecture [21 Conjecture 13.8], which first appeared in [22, Conjecture 3.4], is the following.

Conjecture 14.10 (Kawazumi–Morita). We stably have isomorphisms of graded algebras
\[
H^*(I_g, 1, \mathbb{Q})_{Sp(2g, \mathbb{Z})} \cong \mathbb{Q}[e_2, e_4, \cdots],
\]
where $e_{2i}$ is the Mumford–Morita–Miller class of degree $4i$.

Furthermore, we make the following conjecture, which is closely related to, but different from, the above conjecture.

Conjecture 14.11. We stably have isomorphisms of graded algebras
\[
H^*(I_g, 1, \mathbb{Q})_{Sp(2g, \mathbb{Z})} \cong H^*_A(I_g, 1, \mathbb{Q})_{Sp(2g, \mathbb{Z})} \cong \mathbb{Q}[y_1, y_2, \cdots],
\]
\[
H^*_A(I_g^1, \mathbb{Q})_{Sp(2g, \mathbb{Z})} \cong \mathbb{Q}[z, y_1, y_2, \cdots],
\]
where $\deg y_i = 4i$ and $\deg z = 2$.

Proposition 14.12. If Conjecture [14.7] holds, then Conjecture [14.11] holds.
Proof. We have a coalgebra structure on $\widetilde{S}^*(X'_g)$ induced by the coalgebra structure of $S^*(X'_g)$ defined in Section \ref{section:coalgebra}. Since algebraic $\text{Sp}(2g,\mathbb{Q})$-representations are self-dual, we have an algebra structure on $\widetilde{S}^*(X'_g)$. Therefore, we obtain an algebra structure on $\widetilde{S}^*(X'_1)^{\text{Sp}(2g,\mathbb{Z})}$.

For a graded algebraic $\text{Sp}(2g,\mathbb{Q})$-representation $M_*$, we have
$$\widetilde{S}^*(M_*)^{\text{Sp}(2g,\mathbb{Z})} \cong S^*((M_*)^{\text{Sp}(2g,\mathbb{Z})}).$$

Since we stably have
$$(X''_g)^{\text{Sp}(2g,\mathbb{Z})} = (Y''_g)^{\text{Sp}(2g,\mathbb{Z})} = \bigoplus_{j \geq 1} \mathbb{Q}[4j], \quad (Z''_g)^{\text{Sp}(2g,\mathbb{Z})} = \mathbb{Q}[2] \oplus \bigoplus_{j \geq 1} \mathbb{Q}[4j],$$
we stably have isomorphisms of graded algebras
$$\widetilde{S}^*(X''_g)^{\text{Sp}(2g,\mathbb{Z})} \cong \widetilde{S}^*(Y''_g)^{\text{Sp}(2g,\mathbb{Z})} \cong \mathbb{Q}[y_1, y_2, \ldots], \quad \widetilde{S}^*(Z''_g)^{\text{Sp}(2g,\mathbb{Z})} \cong \mathbb{Q}[z, y_1, y_2, \ldots],$$
where $\deg y_i = 4i$, and $\deg z = 2$. Therefore, if Conjecture \ref{conjecture:14.1} holds, then Conjecture \ref{conjecture:14.1} holds. \hfill $\Box$

14.4. Relation between $H_*^A(I_g, \mathbb{Q})$, $H_*^A(I_{g,1}, \mathbb{Q})$ and $H_*^A(I_{g,1}^1, \mathbb{Q})$. Let $g \geq 2$. Let $\Sigma_g$ (resp. $\Sigma_{g,1}$, $\Sigma_{g,1}^1$) be a closed surface of genus $g$ (resp. with one boundary component, with one marked point). Let $M_{g,1}$ (resp. $M_{g,1}^1$) denote the mapping class group of $\Sigma_{g,1}$ (resp. $\Sigma_{g,1}^1$).

We have an exact sequence of groups with $M_{g,1}^1$-actions
\begin{equation}
1 \to \pi_1(\Sigma_g) \to I_{g,1}^1 \to I_g \to 1
\end{equation}
and an exact sequence of groups with $M_{g,1}$-actions
$$1 \to \pi_1(U\Sigma_g) \to I_{g,1} \to I_g \to 1,$$
where $U\Sigma_g$ denotes the unit tangent bundle of $\Sigma_g$. We have $H_*^A(\pi_1(\Sigma_g), \mathbb{Q}) \cong H_*^A(\Sigma_g, \mathbb{Q})$ and $H_1(U\Sigma_g, \mathbb{Q}) \cong H_1(\Sigma_g, \mathbb{Q})$. By using Lemma \ref{lemma:14.6} and Proposition \ref{proposition:14.8}, we obtain the following proposition.

**Proposition 14.13.** (1) We have a graded $\text{Sp}(2g, \mathbb{Q})$-isomorphism
$$H_*^A(I_{g,1}^1, \mathbb{Q}) \cong H_*^A(I_g, \mathbb{Q}) \otimes H_*^A(\pi_1(\Sigma_g), \mathbb{Q}),$$
where we have $H_*^A(\pi_1(\Sigma_g), \mathbb{Q}) \cong \mathbb{Q}[0] \oplus H[1] \oplus \mathbb{Q}[2]$.

(2) We have an injective graded $\text{Sp}(2g, \mathbb{Q})$-homomorphism
$$H_*^A(I_{g,1}, \mathbb{Q}) \to H_*^A(I_g, \mathbb{Q}) \otimes H_*^A(\pi_1(U\Sigma_g), \mathbb{Q}),$$
where we have
\begin{equation}
H_*^A(\pi_1(U\Sigma_g), \mathbb{Q}) = \mathbb{Q}[0] \oplus H[1].
\end{equation}

**Proof.** In a way similar to the proof of Proposition \ref{proposition:14.8} for each $i \geq 0$, we obtain injective $\text{Sp}(2g, \mathbb{Q})$-homomorphisms
\begin{equation}
H_*^A(I_{g,1}, \mathbb{Q}) \to \bigoplus_{p+q=i} H_*^A(I_g, \mathbb{Q}) \otimes H_*^A(\pi_1(\Sigma_g), \mathbb{Q}),
\end{equation}
\begin{equation}
H_*^i(I_{g,1}, \mathbb{Q}) \to \bigoplus_{p+q=i} H_*^p(I_g, \mathbb{Q}) \otimes H_*^q(\pi_1(U\Sigma_g), \mathbb{Q}).
\end{equation}
(1) We have $H_i^A(\pi_1(\Sigma_g), \mathbb{Q}) \cong \mathbb{Q}[0] \oplus H[1] \oplus \mathbb{Q}[2]$ since $H_*^A(\pi_1(\Sigma_g), \mathbb{Q}) \cong H_*(\Sigma_g, \mathbb{Q}) \cong \mathbb{Q}[0] \oplus H[1] \oplus \mathbb{Q}[2]$ and $H_*^A(\pi_1(\Sigma_g), \mathbb{Q}) \cong \bigwedge^* H$. Consider the cohomological Hochschild–Serre spectral sequence for the exact sequence (14.4.1). We can easily check that $\mathcal{I}_g$ acts trivially on $H^*(\pi_1(\Sigma_g), \mathbb{Q})$ and that the differentials $d_2^{0,1}$, $d_2^{0,2}$ and $d_3^{1,2}$ are zero maps. Therefore, by Lemma 14.6, the injective morphism (14.4.3) is an isomorphism.

(2) It suffices to prove (14.4.2). By using the Hochschild–Serre spectral sequence for the exact sequence

$$1 \to \mathbb{Z} \to \pi_1(U \Sigma_g) \to \pi_1(\Sigma_g) \to 1,$$

we have

$$H_i(\pi_1(U \Sigma_g), \mathbb{Q}) = \mathbb{Q}[0] \oplus H[1] \oplus H[2] \oplus \mathbb{Q}[3].$$

We also have

$$H_i(\pi_1(U \Sigma_g)^{ab}, \mathbb{Q}) = H_i(H_1(\Sigma_g), \mathbb{Q}) = \bigwedge^i H.$$

Since the Johnson homomorphism is an Sp(2g, $\mathbb{Q}$)-homomorphism, we obtain (14.4.2), which completes the proof. □

**Remark 14.14.** The composition

$$s : \Sigma_{g,1} \xrightarrow{\text{section}} U \Sigma_{g,1} \hookrightarrow U \Sigma_g$$

induces an Sp(2g, $\mathbb{Q}$)-homomorphism

$$s_* = H_i^A(\pi_1(s), \mathbb{Q}) : H_i^A(\pi_1(\Sigma_{g,1}), \mathbb{Q}) \to H_i^A(\pi_1(\Sigma_g), \mathbb{Q}),$$

which is an isomorphism by the equation (14.4.2).

**Remark 14.15.** The Albanese cohomology of the Torelli groups stably coincide with the algebraic parts of the cohomology of them in degree $\leq 2$. The case of $\mathcal{I}_g$ follows from [16] and [28]. The case of $\mathcal{I}_{g,1}$ follows from [25] and Proposition 14.13. The case of $\mathcal{I}_g$ follows from Remark 14.14. We stably have $H^3_3(\mathcal{I}_g, \mathbb{Q}) = H^3(\mathcal{I}_g, \mathbb{Q})^{alg}$ if $H^2(\mathcal{I}_g, \mathbb{Q})$ is stably finite-dimensional [30] [25]. The same holds for $\mathcal{I}_{g,1}$ by Proposition 14.13 and 25.

### 14.5. Albanese homology of $\text{IA}_{2g}$ and $\mathcal{I}_{g,1}$

We have an injective group homomorphism $\iota : \mathcal{I}_{g,1} \hookrightarrow \text{IA}_{2g}$, which induces an injective homomorphism

$$\iota_* : H_i(H_1(\mathcal{I}_{g,1}), \mathbb{Q}) \to H_i(H_1(\text{IA}_{2g}), \mathbb{Q}).$$

Let $\tilde{\iota}_*^A$ denote the restriction

$$\tilde{\iota}_*^A : H_i^A(\mathcal{I}_{g,1}, \mathbb{Q}) \to H_i^A(\text{IA}_{2g}, \mathbb{Q})$$

of $\iota_*$ to $H_i^A(\mathcal{I}_{g,1}, \mathbb{Q})$ and $H_i^A(\text{IA}_{2g}, \mathbb{Q})$, which is also injective.

**Remark 14.16.** Recall Conjectures 6.2 and 14.7 for the structures of $H_*^A(\text{IA}_{2g}, \mathbb{Q})$ and $H_*^A(\mathcal{I}_{g,1}, \mathbb{Q})$. By Lemma 14.11, we have

$$(14.5.1) \quad \bar{S}^*(Y''_*) \subset \bar{S}^*(\iota_*) = W_*$$

since we have for $i \geq 1$

$$U_i = \text{Hom}(H, \bigwedge^{i+1} H) \cong H^* \otimes \bigwedge^{i+1} H \cong H \otimes \bigwedge^{i+1} H \supset \bigwedge^{i+2} H \supset Y''_i$$
as $\text{Sp}(2g, \mathbb{Q})$-representations. The inclusion (14.5.1) conjecturally describes the injective $\text{Sp}(2g, \mathbb{Q})$-homomorphism $\iota_*^A : H^A_1(I_g, 1, \mathbb{Q}) \to H^A_1(\text{IA}_{2g}, \mathbb{Q})$, where we consider $H^A_1(\text{IA}_{2g}, \mathbb{Q})$ as an $\text{Sp}(2g, \mathbb{Q})$-representation.

We make the following conjecture about the image of $\iota_*^A$.

**Conjecture 14.17.** For $g \geq 3i$, the image $\iota_*^A(H^A_1(I_g, 1, \mathbb{Q}))$ generates $H^A_1(\text{IA}_{2g}, \mathbb{Q})$ as $\text{GL}(2g, \mathbb{Q})$-representations.

We can easily check that Conjecture 14.17 holds for $i = 1$. Conjecture 14.17 also holds for $i = 2$, which can be verified by the abelian cycles $(\rho_1)^2 \in H^A_2(I_4, 1, \mathbb{Q})$ and $\rho_2 \in H^A_2(I_3, 1, \mathbb{Q})$ given in [27] and the irreducible decomposition of $H^A_2(\text{IA}_{2g}, \mathbb{Q})$ as $\text{GL}(2g, \mathbb{Q})$-representations [34].

**Appendix A. Properties of Albanese homology and cohomology**

Here we give a brief summary of some properties about Albanese homology and cohomology.

**A.1. Albanese homology functor.** Let $\text{Gp}$ denote the category of groups and group homomorphisms and $\text{grVect}$ the category of graded $\mathbb{Q}$-vector spaces and graded linear maps. Let

$$H^A_* : \text{Gp} \to \text{grVect}$$

denote the functor which maps a group $G$ to $H^A_*(G, \mathbb{Q})$, and

$$H_* : \text{Gp} \to \text{grVect}$$

denote the functor which maps a group $G$ to $H_*(G, \mathbb{Q})$. Then we have a natural transformation

$$(\pi_*)_G := \pi_*^G : H_*(G, \mathbb{Q}) \to H^A_*(G, \mathbb{Q}),$$

where $\pi_*^G$ is the map induced by the abelianization $\pi^G : G \to G^{ab}$.

Let $\text{CCoalg}$ denote the category of graded-cocommutative coalgebras and graded coalgebra morphisms over $\mathbb{Q}$. Then we also have functors

$$H^A_* : \text{Gp} \to \text{CCoalg}, \quad H_* : \text{Gp} \to \text{CCoalg}$$

and a natural transformation

$$\pi_* : H_* \Rightarrow H^A_*.$$ 

**A.2. Filtered colimits.** A **filtered category** $I$ is a category satisfying the following two conditions:

- for any objects $i, j \in I$, there are an object $k$ and morphisms $i \to k$ and $j \to k$,
- for any parallel morphisms $f : i \to j$ and $g : i \to j$, there exists a morphism $w : j \to k$ such that $wf = wg$.

For a functor $F$ from a filtered category $I$ to another category $\mathcal{C}$, the colimit $\text{colim}_{i \in I} F_i$ of $F$ is called the **filtered colimit**.

Group homology preserves filtered colimits. We observe that the same property holds for Albanese homology.
Proposition A.1. The functor $H_*^A$ preserves filtered colimits, that is, the natural map
\[
\colim_{i \in I} H_*^A(G_i, \mathbb{Q}) \to H_*^A(\colim_{i \in I} G_i, \mathbb{Q})
\]
is an isomorphism. In particular, $H_*^A$ preserves direct limits.

Proof. Let $A, B : I \to \text{grVect}$ be two functors. For a natural transformation $\alpha_i : A_i \to B_i$, we have
\[
\colim_{i \in I} (\text{im}(A_i \xrightarrow{\alpha_i} B_i)) \cong \text{im}(\colim_{i \in I} A_i \xrightarrow{\colim_{i \in I} \alpha_i} \colim_{i \in I} B_i).
\]
Therefore, we have
\[
\colim_{i \in I} H_*^A(G_i, \mathbb{Q}) = \text{im}(\colim_{i \in I} H_*((G_i, \mathbb{Q}) \to H_*((G^\text{ab}, \mathbb{Q}))))
\cong \text{im}(H_*((\colim_{i \in I} G_i, \mathbb{Q}) \to H_*((\colim_{i \in I} G^\text{ab}, \mathbb{Q}))))
\cong \text{im}(H_*((\colim_{i \in I} G_i, \mathbb{Q}) \to H_*((\colim_{i \in I} G^\text{ab}, \mathbb{Q}))))
\cong H_*^A(\colim_{i \in I} G_i, \mathbb{Q}).
\]

□

A.3. Duality. We have the universal coefficient theorem for group homology and group cohomology. Here we observe that the universal coefficient theorem also holds for Albanese homology and cohomology.

Lemma A.2. For a group $G$, we have a linear isomorphism
\[
H_*^i(A, \mathbb{Q}) \cong (H_*^A(G, \mathbb{Q}))^*.
\]

Proof. This lemma follows from the following fact. Let $V, W$ be $\mathbb{Q}$-vector spaces and $f : V \to W$ a linear map. Define a linear map
\[
\Phi : \text{im}(f^*) \to (\text{im} f)^*
\]
by sending $\psi \in \text{im}(f^*)$ to the restriction $\phi|_{\text{im} f}$, where $\phi \in W^*$ is a linear map satisfying $\psi = \phi f$. Then we can check that $\Phi$ is a linear isomorphism. □

We have GL($n, \mathbb{Q}$)-representation structures on $H_*^i(A_n, \mathbb{Q})$ and $H_*^A(A_n, \mathbb{Q})$. Then we can check that the map $\Phi$ in the proof of Lemma A.2 is a GL($n, \mathbb{Q}$)-isomorphism. We obtain the following duality as GL($n, \mathbb{Q}$)-representations.

Proposition A.3. We have a GL($n, \mathbb{Q}$)-isomorphism
\[
H_*^i(A_n, \mathbb{Q}) \cong (H_*^A(A_n, \mathbb{Q}))^*.
\]
A.4. Albanese cohomology functor. Let
\[ H^*_A : \text{Gp}^{\text{op}} \to \text{grVect} \]
denote the functor which maps a group \( G \) to \( H^*_A(G, \mathbb{Q}) \), and
\[ H^* : \text{Gp}^{\text{op}} \to \text{grVect} \]
the functor which maps a group \( G \) to \( H^*(G, \mathbb{Q}) \). Then we have a natural transformation
\[ \iota_G : H^*_A(G, \mathbb{Q}) \hookrightarrow H^*(G, \mathbb{Q}). \]

For any group \( G \), we have a graded-commutative algebra structure on \( H^*(G, \mathbb{Q}) \) with the cup product as the multiplication. Let \( \text{CAlg} \) denote the category of graded-commutative \( \mathbb{Q} \)-algebras and graded algebra morphisms. Then we also have functors
\[ H^*_A : \text{Gp}^{\text{op}} \to \text{CAlg}, \quad H^* : \text{Gp}^{\text{op}} \to \text{CAlg} \]
and a natural transformation
\[ \iota : H^*_A \Rightarrow H^*. \]

The following property holds for Albanese cohomology as in the case of group cohomology.

**Proposition A.4.** The functor \( H^*_A \) preserves filtered limits, that is, the natural map
\[ H^*_A(\text{colim}_{i \in I} G_i, \mathbb{Q}) \to \lim_{i \in I} H^*_A(G_i, \mathbb{Q}) \]
is an isomorphism.

**Proof.** By Proposition A.1 and Lemma A.2, it follows that
\[
H^*_A(\text{colim}_{i \in I} G_i, \mathbb{Q}) \cong (H^*_A(\text{colim}_{i \in I} G_i, \mathbb{Q}))^* \cong (\colim_{i \in I} H^*_{A}(G_i, \mathbb{Q}))^* \cong \lim_{i \in I} H^*_{A}(G_i, \mathbb{Q}).
\]

\[ \square \]

A.5. Hochschild–Serre spectral sequence. Here we study Albanese homology of groups by using the Hochschild–Serre spectral sequences for exact sequences of groups. See [42, Chapter 5] for details of spectral sequences and their convergence.

For an exact sequence of groups
\[ 1 \to N \xrightarrow{\iota} G \to Q \to 1, \]
we have the following commutative diagram whose rows are exact
\[
\begin{array}{ccccccccc}
1 & \to & N & \xrightarrow{\iota} & G & \xrightarrow{\pi} & Q & \to & 1 \\
\downarrow{\pi_N} & & \downarrow{\pi_G} & & \downarrow{\pi_Q} & & & & \\
N^{\text{ab}} & \xrightarrow{\iota_*} & G^{\text{ab}} & \xrightarrow{\pi_*} & Q^{\text{ab}} & \to & 1.
\end{array}
\]

We have the Hochschild–Serre spectral sequence which converges to \( H_{p+q}(G, \mathbb{Q}) \):
\[ E^2_{p,q} = H_p(Q, H_q(N, \mathbb{Q})) \Rightarrow H_{p+q}(G, \mathbb{Q}). \]
Suppose that \( \iota_* \) is injective. Then we also have the Hochschild–Serre spectral sequence which converges to \( H_{p+q}(G^{ab}, Q) \):

\[
\tilde{E}^2_{p,q} = H_p(Q^{ab}, H_q(N^{ab}, Q)) \Rightarrow H_{p+q}(G^{ab}, Q).
\]

Since \( Q^{ab} \) acts trivially on \( H_q(N^{ab}, Q) \), we have

\[
H_p(Q^{ab}, H_q(N^{ab}, Q)) \cong H_p(Q^{ab}, Q) \otimes H_q(N^{ab}, Q),
\]

and thus we have \( \tilde{E}^2_{p,q} = \tilde{E}^\infty_{p,q} \).

A morphism between short exact sequences of groups induces a morphism of spectral sequences. Thus, the abelianization induces a morphism of spectral sequences

\[
f^*_p : E^*_{p,q} \to \tilde{E}^*_{p,q}.
\]

By using the above two spectral sequences, we obtain the following proposition.

**Proposition A.5.** Suppose that \( \iota_* : N^{ab} \to G^{ab} \) is injective. Then we have an injective graded \( Q \)-linear map

\[
H^A_*(G, Q) \to H^A_*(Q, Q) \otimes H^A_*(N, Q).
\]

**Proof.** We can check that the image of

\[
f^2_{p,q} : H_p(Q, H_q(N, Q)) \to H_p(Q^{ab}, H_q(N^{ab}, Q)) = H_p(Q^{ab}, Q) \otimes H_q(N^{ab}, Q)
\]

is \( H^A_p(Q, Q) \otimes H^A_q(N, Q) \). Since \( \tilde{E}^\infty_{p,q} = \tilde{E}^2_{p,q} \), we have

\[
\text{im} f^\infty_{p,q} \subset \text{im} f^2_{p,q} = H^A_p(Q, Q) \otimes H^A_q(N, Q).
\]

Since the map \( f^\infty_{p,q} \) is compatible with

\[
\pi_* : H_{p+q}(G, Q) \to H^A_{p+q}(G, Q) \hookrightarrow H_{p+q}(G^{ab}, Q),
\]

we have a filtration of \( Q \)-vector spaces

\[
0 = F_{-1} \subset F_0 \subset \cdots \subset F_{n-1} \subset F_n = H^A_n(G, Q)
\]

satisfying \( F_r/F_{r-1} = \text{im} f^\infty_{r,n-r} \) for \( 0 \leq r \leq n \). Therefore, we have

\[
H^A_n(G, Q) \cong \bigoplus_{p+q=n} \text{im} f^\infty_{p,q} \subset \bigoplus_{p+q=n} H^A_p(Q, Q) \otimes H^A_q(N, Q).
\]

\[\Box\]

If we consider the cohomological Hochschild–Serre spectral sequences, then we have

\[
E^p_{2,q} = H^p(Q, H^q(N, Q)) \Rightarrow H^{p+q}(G, Q).
\]

Suppose that \( \iota_* : N^{ab} \to G^{ab} \) is injective. Then we also have

\[
\tilde{E}^p_{2,q} = H^p(Q^{ab}, H^q(N^{ab}, Q)) \Rightarrow H^{p+q}(G^{ab}, Q).
\]

**Lemma A.6.** Suppose that \( \iota_* : N^{ab} \to G^{ab} \) is injective. Suppose also that \( Q \) acts trivially on \( H^*(N, Q) \), that \( E^p_{2,q} = 0 \) for any \( q \geq 3 \), and that the differentials \( d_2^{0,1}, d_2^{0,2}, d_3^{0} \) are zero maps. Then we have a graded \( Q \)-linear isomorphism

\[
H^A_*(G, Q) \cong H^A_*(Q, Q) \otimes H^A_*(N, Q).
\]
Proof. Since $Q$ acts trivially on $H^\ast(N, \mathbb{Q})$, we have $E^{p,q}_2 = E^{p,0}_2 \otimes E^{0,q}_2$ for any $p, q$. Since $E^{p,q}_2 = 0$ for any $q \geq 3$ and the differentials $d^{0,1}_2, d^{2,0}_2$ are zero maps, by the multiplicative structure of the cohomological Hochschild–Serre spectral sequence, we have $E^{p,q}_2 = E^{p,q}_\infty$ and $E^{p,q}_2 = \bar{E}^{p,q}_\infty$ for any $p, q$. Therefore, we have $\text{im} f^{p,q}_2 = \text{im} f^{p,q}_2 = H^n_A(Q, \mathbb{Q}) \otimes H^n_A(N, \mathbb{Q})$ for any $p, q$. It follows that $H^n_A(G, \mathbb{Q}) \cong H^n_A(Q, \mathbb{Q}) \otimes H^n_A(N, \mathbb{Q}).$

Remark A.7. Suppose that $\iota_* : N^{ab} \to G^{ab}$ is injective, that $Q$ acts trivially on $H^\ast(N, \mathbb{Q})$, that $E^{2,0}_2 = 0$ for any $q \geq 2$, and that the differential $d^{2,1}_2 : E^{2,1}_2 \to E^{2,0}_2$ is a zero map. Then by Lemma A.6, we have $H^n_A(G, \mathbb{Q}) \cong H^n_A(Q, \mathbb{Q}) \otimes H^n_A(N, \mathbb{Q})$.

In what follows, we study a group action on spectral sequences. Let $K$ be a group. A $K$-group $G$ is a group $G$ with a left $K$-action satisfying $k \cdot (gg') = (k \cdot g)(k \cdot g')$ for $k \in K$ and $g, g' \in G$. A morphism of $K$-groups is a group homomorphism compatible with the $K$-action. Let $K \text{Gp}$ denote the category of $K$-groups and $K$-group homomorphisms. Let $K \text{Mod}$ denote the category of left $\mathbb{Q}[K]$-modules and $\mathbb{Q}[K]$-homomorphisms. Let

$$1 \to N \xrightarrow{\iota} G \to Q \to 1$$

be an exact sequence in $K \text{Gp}$. Then the Hochschild–Serre spectral sequence associated to this exact sequence is defined in $K \text{Mod}$. Proposition A.5 extends to the following proposition. Since the category $K \text{Mod}$ is not necessarily semisimple, we do not have a direct sum decomposition of $H^n_A(G, \mathbb{Q})$ as $\mathbb{Q}[K]$-modules.

Proposition A.8. Let

$$1 \to N \xrightarrow{\iota} G \to Q \to 1$$

be an exact sequence in $K \text{Gp}$. Suppose that $\iota_* : N^{ab} \to G^{ab}$ is injective. Then $H^n_A(G, \mathbb{Q})$ has a filtration in $K \text{Mod}$

$$0 = F^{-1} \subset F_0 \subset \cdots \subset F_{n-1} \subset F_n = H^n_A(G, \mathbb{Q})$$

such that there is an injective $\mathbb{Q}[K]$-homomorphism

$$\bigoplus_{r=0}^n F_r / F_{r-1} \hookrightarrow \bigoplus_{p+q=n} H^p_A(Q, \mathbb{Q}) \otimes H^q_A(H, \mathbb{Q}).$$

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