Multi-Scale Perturbation Analysis in Hydrodynamics of the Superfluid Turbulence. Derivation of the Dresner Equation.

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Abstract

The Hydrodynamics of Superfluid Turbulence (HST) describes the flows (or counterflows) of HeII in the presence of a chaotic set of vortex filaments, so called superfluid turbulence. The HST equations govern both a slow variation of the hydrodynamic variables due to dissipation related to the vortex tangle and fast processes of the first and second sound propagation. This circumstance prevents an effective numerical simulations of the problems of unsteady heat transfer in HeII. By virtue of a pertinent multi-scale perturbation analysis we show how one can eliminate the fast processes to derive the evolution equation for the slow processes only. We then demonstrate that the long-term evolution of a transient heat load of moderate intensity obeys the nonlinear heat conductivity equation, often referred to as the Dresner equation. We also compare our approach against the Dresner phenomenological derivation and establish a range of validity of the latter.

Keywords: HeII, Superfluid Turbulence, Unsteady Heat Transfer

Introduction and scientific background

As it is known, upon exceeding some critical value of the counterflow velocity in case of the motion induced by the heat load applied to one end of a channel, a chaotic set (or tangle) of quantized vortex filaments emerges in superfluid helium. The presence of the vortex tangle dramatically changes all the hydrodynamic properties of HeII in comparison with those prescribed by usual two-fluid hydrodynamics (See for details book by Donnelly and review article by Tough). Due to a small value of the critical velocity such situation takes place almost in any experiment on studying hydrodynamic and heat transfer phenomena in HeII. Consequently, any flow or counterflow of HeII should be investigated with the help of the set of hydrodynamics equations coupled with Vinen equation for the vortex line density dynamics (see ). This set
of equations known as the Hydrodynamics of Superfluid Turbulence (HST) have been obtained in papers of Nemirovskii and Lebedev [4], Geurst [5], and Yamada et. al. [6] within the scope of different approaches. The Hydrodynamics of Superfluid Turbulence as well as the methods for its investigation are outlined in the review paper by Nemirovskii and Fiszdon [7]. The set of HST equations is extremely cumbersome, therefore, there is no wonder that to achieve quantitative results in virtually any case of practical interest one is bound to turn to numerical methods. Yet, numerical simulation of nonstationary flows of HeII faces one serious obstacle. The point here is that the set of HST equations that unifies the hydrodynamics equations of superfluid helium and the Vinen equation for the vortex line density evolution is initially of hyperbolic type. As a result, a slow variation of hydrodynamic variables due to dissipation is accompanied by the fast processes related to propagation (and possible reflections from the boundaries of the channel) of the first and second sounds. If one is interested in the slow evolution of the temperature, velocities and the vortex line density only, particular details of sound propagation and manifold reflections come out completely irrelevant, yet requiring rather extensive numerical resources. Indeed, a typical time of transient processes is of order of seconds, whereas a typical mesh size in numerical investigations of acoustic phenomena should be of order of at least microseconds to account for the forming of the shock fronts. Thus, it seems attractive to try to get rid of the fast modes by a pure analytical procedure. In the present paper we realize effective separation of the slow from the fast modes using multi-scale (in the case multi-time) asymptotic perturbation techniques (See, for example, book by Nayfeh [8]. The main idea here is to introduce several time scales, each of which responsible for the description of a corresponding process. As an example we recall, perhaps, the simplest possible case of the overdamped harmonic oscillator. Given a certain relation between the frequency and the damping decrement, sinusoidal oscillations die away shortly, whereas slow exponential relaxation may take quite a while to die out. Therefore, the long-term dynamic behavior changes drastically. The same is true for our case: dimensional analysis of the HST equations shows that there is a dimensionless criterion, Strouhal number $Sh$ that is the ratio of the counterflow decrement to inverse time that takes the heat pulse to cross the channel. Depending on the input parameters of the problem the number $Sh$ can be either large or small. The latter circumstance allows us to come up with an effective procedure for separation of the fast from the slow modes. The paper is organized as follows. The second section is devoted to dimensionless analysis of the set of HST equations. In the third section we carry out the multi-scale asymptotic perturbation procedure for the case of heat load of moderate intensity, of order of a few $W/cm^2$ i.e. for $Sh \gg 1$. We then establish that the slow processes are governed by the nonlinear heat conductivity equation similar to that derived phenomenologically by Dresner [9]. We also discuss the criteria of validity of the Dresner equation and the range of its applicability. In conclusion we resume the obtained results.

**Dimensionless equations of HST**

As it was noted above the full set of the HST equations is very complicated and in general, does not permit any analytical investigation at all. Having in mind to study processes related to heat transfer in HeII, we propose the following simplifications.

(i) First, we restrict ourselves to the case of counterflow of HeII which occurs whenever one applies a heat load to a reservoir filled with superfluid helium.

(ii) Second, we study only quasi-one-dimensional cases, i.e. either pure one-dimensional ($d=1$),
or cylindrical \((d=2)\), or spherical \((d=3)\) geometries.

(iii) Next, we neglect by the nonlinear convective terms on the left-hand sides of the equations of motion. Usually these terms come in play in the cases of very large heat fluxes (when no separation of the "fast" and "slow" mode is possible at all) and are responsible for the forming of the shock fronts.

(iv) Finally, we drop the dissipation function which is very small in the cases to be studied. The above assumptions correspond to the situations brought about in the overwhelming majority of experiments on transient heat transfer. To describe the counterflow of HeII it is convenient to choose the following set of variables: dimensionless velocity of the normal component \(V'_n\), dimensionless temperature \(T'\) and dimensionless square root of the vortex line density (VLD) \(G\) \((G = \sqrt{L/L_\infty}\), where \(L_\infty\) is equilibrium with respect to the normal velocity value of VLD). Under the assumptions listed above the set of equation of HST (see for details e.g. review article of Nemirovskii and Fiszdon [7]) is reduced to the following form:

\[
\begin{align*}
\frac{\partial V'_n}{\partial t'} + \frac{\partial T'}{\partial x'} &= - Sh G^2 V'_n \\
\frac{\partial T'}{\partial t'} + \frac{1}{x'^{d-1}} \frac{\partial}{\partial x'} x'^{d-1} V'_n &= 0,
\end{align*}
\]

\[
\frac{\partial G}{\partial t'} + M_L \frac{\partial G}{\partial x'} = - \frac{V_i Sh}{2} (G^2 V'_n - G^3),
\]

with the following dimensionless variables:

\[
t = \frac{L}{U} t', \quad x = L x', \quad V_n = V_{n0} V'_n, \quad T = \frac{\sigma V_{n0} T'}{\sigma T U'}, \quad \mathcal{L} = \mathcal{L}_\infty G^2.
\]

Here \(L\) is a length of the channel, \(U\) is the velocity of the second sound, \(V_{n0}\) is a characteristic value of the normal velocity at the boundary \(V_{n0} = q/\rho \sigma T\) (here \(q\) is a heat flux, which is actually a function of time, \(\sigma\) is entropy per unit mass and \(\sigma_T\) is a derivative of entropy with respect to the temperature).

At this stage it is worth making a few remarks concerning the set of equations \((1)-(3)\). If one used dimensional variables \(V_n, T\) and \(\mathcal{L}\) instead of their dimensionless counterparts, one would easily recognize the equations widely used in studying the dynamics of intense heat pulses. Nevertheless a somewhat awkward use of variable \(G\) in lieu of the vortex line density is necessary to develop the multi-scale perturbation procedure.

It is easy to see from set \((1)-(3)\) that the dynamics of the hydrodynamic variables is specified by several dimensionless criteria, namely the Vinen parameter \(V_i\), the Strouhal parameter \(Sh\) and the Mach number \(M_L\). They are defined as:

\[
Sh = \frac{L}{U \tau_d}, \quad M_L = \frac{V_i}{U}, \quad V_i = \frac{\alpha}{A(T) \rho_s \rho_n} \left(\frac{\alpha}{\beta}\right)
\]

Here we have introduced the following notations: \(\alpha\) and \(\beta\), the parameters of the Vinen equation; \(A(T)\) the Gorter-Mellink constant; \(V_L\) is a drift velocity of the vortex tangle; \(\tau_d\), a characteristic time of attenuation of the second sound due to the vortex tangle introduced by Vinen (see [3]):

\[
\tau_d = \frac{\rho_s^2}{A \rho^3 V_{n0}^2}.
\]
Numerical estimation of parameters shows that within the temperature interval 1.4 – 2.1 K parameter changes from 0.2 to 1.5. It does not depend on applied heat flux (or on quantity \( V_{a0} \)). The Mach number, although dependent on the heat flux, remains very small. As far as parameter \( Sh \) is concerned, it strongly depends on the heat flux and can be either large or small compared to unity:

\[
Sh << 1, \text{ for small heat flux} \tag{7}
\]

\[
Sh >> 1, \text{ for large heat flux} \tag{8}
\]

Thus, the main parameter, drastically affecting the behavior of the system, is the Strouhal parameter \( Sh \) which is nothing else than the ratio of the counterflow damping decrement due to an interaction with the vortex tangle to the inverse time of the flight of the heat pulse. Conditions (7)-(8) can be used for effective separation of the processes with characteristic time to an interaction with the vortex tangle to the inverse time of the flight of the heat pulse. Conditions (7)-(8) can be used for effective separation of the processes with characteristic time to an interaction with the vortex tangle to the inverse time of the flight of the heat pulse. Conditions (7)-(8) can be used for effective separation of the processes with characteristic time to an interaction with the vortex tangle to the inverse time of the flight of the heat pulse.

Having in mind to investigate the cases of practical interest, we estimate \( Sh \) numbers for heat load of moderate intensity \( q = 1 \div 10 \) W/cm\(^2\) and channel sizes \( L = 10 \div 10^2 \) cm. Using the definitions of the Strouhal number as well as the dynamical parameters we obtain that in the temperature region \( T = 1.4 \div 2.1 \) K the Strouhal number is of order of \( 10^2 – 10^5 \), i.e. much greater than unity. Therefore, we can restrict ourselves to studying the cases of large \( Sh \) numbers.

**Multi-time scale method**

In this section we carry out elimination of the fast processes. As mentioned above we will concentrate on the case of large Strouhal parameter \( Sh >> 1 \). Put in another way, attenuation of the second sound is assumed so strong that the convective regime of heat transfer quickly gives way to another regime, reminiscent of nonlinear heat conduction in usual newtonian fluids. Formally this leads to the degeneration of the initially hyperbolic equations into a parabolic equation describing the slow evolution of the velocity and temperature fields. To demonstrate this transition explicitly we invoke a pertinent multi-scale asymptotic perturbation theory (see e.g.Nayfeh). Following this method, we introduce different time scales.

\[
t_0 = t'; \quad t_1' = \epsilon t'; \quad t_2' = \epsilon^2 t' \quad \ldots \ldots,
\]

where \( \epsilon = 1/Sh << 1 \). We look for a solution to the set of the HST equations in the form of an asymptotic series

\[
V' = V_0'(x', t_0', t_1', t_2') + \epsilon V_1'(x', t_0', t_1', t_2') + \epsilon^2 V_2'(x', t_0', t_1', t_2') + \ldots
\]

\[
T' = T_0'(x', t_0', t_1', t_2') + \epsilon T_1'(x', t_0', t_1', t_2') + \epsilon^2 T_2'(x', t_0', t_1', t_2') + \ldots
\]

\[
G = G_0(x', t_0', t_1', t_2') + \epsilon G_1(x', t_0', t_1', t_2') + \epsilon^2 G_2(x', t_0', t_1', t_2') + \ldots
\]

In accordance with multi-scale perturbation analysis (see Nayfeh) the coefficients in the series for dimensionless normal velocity \( V' \), dimensionless temperature \( T' \) and quantity \( G \) are supposed to have an order of smallness \( O(\epsilon^3) \), i.e. to be of order of unity. The simple chain rule follows from relations:

\[
\frac{\partial}{\partial t'} = \frac{\partial}{\partial t_0'} + \epsilon \frac{\partial}{\partial t_1'} + \epsilon^2 \frac{\partial}{\partial t_2'} \tag{11}
\]
The next step in study of the slow evolution of the heat pulse consists in substituting the multi-time scale series (9)-(10) into equations of HST (1)-(3). Gathering terms of the same order of magnitude with respect to $\varepsilon$ we come up with a chain of equations leading to divergent (secular) solutions. Canceling step by step these secularities we then obtain a hierarchy of equations of different orders in parameter $\varepsilon$, governing different stages of the evolution of the fields of temperature, counterflow velocity and the square root of the vortex line density.

Let us develop the outlined scheme in detail. Excluding the temperature variable $T'$ from the set of equations (1)-(3) we rewrite it in the as:

\begin{align}
G^2 V_n' - G^3 &= -\frac{2}{V_t} \varepsilon \left( \frac{\partial G}{\partial t'} + M_L \frac{\partial G}{\partial x'} \right), \\
\frac{\partial}{\partial t'} (G^2 V_n') &= \varepsilon \left[ \frac{\partial}{\partial x'} \left( \frac{1}{x'^d-1} \frac{\partial}{\partial x'} x'^d - 1 V_n' \right) - \frac{\partial^2 V_n'}{\partial t'^2} \right].
\end{align}

Implementation of the above procedure depends on a particular statement of the problem under consideration. To be specific, we assume that at the moment a heater is switched on the thermal front starts out propagating into undisturbed bulk of helium which is, in fact, the case in most experiments. Such statement, however, brings up some apparent paradox. Indeed, substituting series (10) for $G$ and $V$ and using rule (11) we find out that to zero order in $\varepsilon$, quantity $G$ is equal to $V_0(x', t_0', t_1', t_2)$. Put in another way, it follows that VLD $\mathcal{L}(t)$ immediately takes its equilibrium value with respect to the normal velocity. But that is wrong as there actually is some finite time $\tau_V$ of development of the VLD (see [3]). Hence, we seem to end up in quite a contradictory situation. To find a way out, let us compare the characteristic time of the vortex tangle development $\tau_V$ against the characteristic time of propagation of heat pulse $L/U$. As it is well known (see [3]), the former is determined by relation:

$$
\tau_V = a(T) q^{-3/2}.
$$

Taking into account a scatter of data it follows

$$
q^{3/2} L \sim 40 \div 140 \left[ \frac{W^{3/2}}{cm^2} \right].
$$

It is seen from (15) that for heat pulses of moderate amplitude $1 [W/cm^2] \leq q \leq 10 [W/cm^2]$ propagating in channels of length $L \sim 10 \div 10^2 cm$, which are typical of experiments on transient heat transfer in HeII, the quantities $\tau_V$ and $L/U$ are of the same order of magnitude. It implies that for the time of order $(L/U)$, typical of the slow processes, the vortex line density has enough time to adjust to the velocity field, i.e. to take its equilibrium (with respect to the relative velocity) value $\mathcal{L}_\infty = \frac{\rho V^2}{\rho_s (\alpha)^2}$. In other words, if we start our scheme not at $t = 0$ when there are no vortices in the bulk of helium but at time $t = \tau_V$ when the VLD has adjusted to its equilibrium value, we can then proceed without any discrepancies. At times smaller than $\tau_V$ when $G$ is small relations (12)-(13) can be rewritten as

\begin{align}
\left[ \frac{\partial}{\partial x'} \frac{1}{x'^d-1} \frac{\partial}{\partial x'} x'^d - \frac{\partial^2}{\partial t'^2} \right] V_n' &= 0
\end{align}
\[
G^2 v' = \frac{\partial G_0}{\partial t'} + M_L \frac{\partial G}{\partial x'}
\]  

(17)

The meaning of relations (16) and (17) is obvious. For times smaller than \( L/U \) the heat pulse propagates according to wave-like equation because of the absence of vortices (excluding extremely small background value). The vortex line density, being very small in comparison with its equilibrium value \( \mathcal{L}_\infty \), changes mainly by first generating term of the Vinen equation\(^1\).

Let us now proceed with the general scheme. The following step in studying the slow evolution of the heat pulse consists in substituting multi-scale variables (9)-(10) into equations (13)-(12). Gathering terms of the same order, we wind up with the chain of equations:

To order \( \epsilon^0 \):

\[
V_0' = V_0'(x', t_1', t_2') \quad G_0 = V_0'.
\]

(18)

To order \( \epsilon^1 \):

\[
V_1' - G_1 = \frac{2M_L}{V'_{1}} \frac{1}{V_0} \frac{\partial V_1'}{\partial x'}, \quad V_0'^2 \frac{\partial}{\partial t} (V_1' + 2G_1) = -\frac{\partial}{\partial t} G_0^3 + \frac{\partial}{\partial x'} (V_0'(x') d - \frac{\partial}{\partial x'} (V_0'(x') - 1 V_0')).
\]

(19)

Using (18) the second equation of set (19) can be integrated over \( t_0' \) to yield:

\[
3V_0'^2 V_1' = f(x', t_1', t_2') + \frac{4M_L}{V'_{1}} \frac{1}{V_0} \frac{\partial V_1'}{\partial x'} + t_0' \left[ \frac{\partial}{\partial x'} (x' d - 1 V_0') - \frac{\partial}{\partial t} G_0^3 \right].
\]

(20)

It is seen that for \( t_0' \rightarrow \infty \) the quantity \( V_1' \rightarrow \infty \) as well. In order to cancel this divergence we have to require the quantity inside the brackets to be equal to zero. This procedure is called the cancellation of divergencies in secular terms in an asymptotic series. Having accomplished this procedure, we figure out that to the first order in \( \epsilon^1 \) the evolution of the heat pulse obeys the set of equations below:

\[
V_1' = V_1'(x', t_1', t_2'),
\]

(21)

\[
V_1' - G_1 = \frac{2M_L}{V'_{1}} \frac{1}{V_0'^2} \frac{\partial V_1'}{\partial x'},
\]

(22)

\[
\frac{\partial}{\partial t_1'} V_0^3 = \frac{\partial}{\partial x'} \frac{1}{x'^d - 1} \frac{\partial}{\partial x'} (x'^d - 1 V_0'),
\]

(23)

Thus we have derived a set of equations which describes slow variations of hydrodynamic variables (with the values of parameters discussed above). It can then be inferred from (18) and (23) that the leading part of the normal velocity depends only on the slow time and is governed by the parabolic-type evolution equation. This is the main result of the paper. It holds true provided that all the necessary criteria for application of the outlined perturbation procedure are met (see discussion above). Later we shall establish the range of validity of our

\(^1\)To see what would exactly happen at times smaller than \( \tau_\nu \) one had to resolve a fairly difficult problem of propagation of the heat pulse which creates the vortices on its way and interacts with the very vortices. The corresponding investigations are described in the review article [7].
approach. At this point, however, we complete the calculation. To this end let us study the next, $\epsilon^2$ order. We follow exactly the same procedure as above, cancelling divergences one step at a time. Leaving out simple but somewhat tedious calculations we write down the final result:

To order $\epsilon^2$.

\begin{align*}
V_0' &= V_0'(x', t_1') \\
V_1' &= V_1'(x', t_1') \\
G_1 &= G_1(x', t_1') \\
G_0 &= V_0' \\
V_1' - G_1 &= 2\frac{M}{V_0'} \frac{1}{V_0'} \frac{\partial V_0'}{\partial x'} \\
3 \frac{\partial}{\partial t_1'} (V_0'^2 V_1') &= \frac{\partial}{\partial x'} \frac{1}{V_0'} \frac{\partial}{\partial x'} (x'^{d-1} V_1') + 4\frac{M}{V_1'} \frac{1}{V_0'} \frac{\partial V_0'}{\partial x'} .
\end{align*}

(24)

It is plain that the last equation accounts for a small correction to the velocity field and hence is of little interest to us.

The Dresner nonlinear heat conductivity equation

Let us consider the first $\epsilon^1$ order in greater detail. With the goal to compare our results to the well-known Dresner heat conductivity equation (see e.g. [9]) we consider here only the simplest geometry ($d = 1$). Combining equation (21) with the results of the previous section we arrive at the set of dimensionless equations for the normal velocity, and the temperature:

\begin{align*}
\frac{\partial V_0'^3}{\partial t_1'} &= \frac{\partial^2 V_0'}{\partial x'^2}, \\
\frac{\partial T'}{\partial t_1'} + \frac{\partial V_0'}{\partial x'} &= 0.
\end{align*}

(25)

Here $V_0', T'$, are zero terms in series (10) for the dimensional normal velocity and the temperature. We recall that to this order in $\epsilon$ the vortex line density takes its equilibrium value with respect to normal velocity, e.i. $L = V_0'^2$. Next, excluding quantity $V_0'$ from the set of equations written above we obtain the relation:

\begin{align*}
\frac{\partial T'}{\partial t_1'} &= \epsilon^{-1/3} \frac{\partial}{\partial x'} \left( \frac{\partial T'}{\partial x'} \right)^{1/3} \\
\frac{\partial T'}{\partial x'} &= -Sh V_0'^3.
\end{align*}

(27)

Relation (27) formally coincides with the widely used nonlinear heat-conductivity equation derived by Dresner [9]. In this connection it is worth discussing the method used by Dresner. He started with the Gorter-Mellink relation which in our notation transforms to:

\begin{align*}
\frac{\partial T'}{\partial x'} &= -Sh V_0'^3 \\
\frac{\partial}{\partial x'} &= q
\end{align*}

Recalling that heat flux $q$ is related to normal velocity $V_n$ via $q = STV_n$, he concluded that the heat flux was proportional to the cube root of the temperature gradient.

\begin{align*}
q \propto \left( \frac{\partial T'}{\partial x'} \right)^{1/3}.
\end{align*}
Furthermore, using the energy conservation law,
\[ \frac{\partial S}{\partial t} + \frac{\partial S V_n}{\partial x} = 0. \]

Dresner derived an equation similar to equation (27). His approach, however, is not self-consistent from the point of view of the full set of the HST equations. Indeed, the Gorter-Mellink relation (28) corresponds to a steady-state and its direct use for a nonstationary case is not valid. The correct equation should be obtained from equations (1), in which quantity \( G \) is replaced by its equilibrium value \( V_n' \).

\[ \frac{\partial V_n'}{\partial t'} + \frac{\partial T'}{\partial x'} = -Sh V_n'^3 \]

Throwing out this term implies that \( V_n' = V_n'(x) \) which, in turn, means that the temperature is also a function of \( x \) only and does not change in time. Accordingly, \( T' \) connected to \( V_n' \) via Gorter-Mellink relation does not change in time.

In our approach \( \frac{\partial V_n'/\partial t'}{\partial t'} \neq 0 \) and there is no contradiction. Our careful consideration explicitly demonstrates that the regime of the nonlinear heat-conductivity equation takes place only for "slow" time (\( \sim L/U \)) and validity of the overall procedure requires meeting the criteria dwelt upon above. The proposed allows also to determine a range of validity of the Dresner equation (27). One restriction is that slow regime takes place at a time scale of order \( (L/U) \) therefore a parabolic type heat-transfer equation works at times greater than \( \tau_d \). The latter then provides the lower limit of the applicability of the evolution equation derived above, equation (23).

On the other hand, the obvious upper limit is furnished by the boiling time calculated in \[10\].

To conclude this chapter we would like to point out the region of fulfillment of condition (8) in terms of the heat flux and the size of the channel. Using the definition of the Strouhal number \[13\] and thermodynamical parameters, we obtain that in the temperature region \( T = 1.4 \div 2.1K \) condition (8) is equivalent to the following one:

\[ q^2 L \gg 0.2 \div 0.6\left[\frac{W^2}{cm^4}\right]. \]

(29)

Thus for heat load of moderate intensity \( q = 1 \div 10 \ W/cm^2 \) and for sizes of the setup \( L = 10 \div 10^2 cm \) relation (8) and consequently (29) are valid to good accuracy.

**Conclusion**

We have outlined the procedure of separation of the fast from the slow processes in the equations of Hydrodynamics of Superfluid Turbulence. As an illustration of the developed procedure, we studied the slow stage of the evolution of the transient heat load of moderate intensity. It has been shown how the wave-like behaviour of the heat pulse described by initially hyperbolic equations gives way to the parabolic-type non-linear heat conductivity equation. The latter formally coincides with the one derived earlier by Dresner on the phenomenological grounds. Unlike the approach advocated by Dressner, our consideration is not only self-consistent but also allows to determine a range of validity of the elimination procedure.
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