Towards ultrametric theory of turbulence

S.V. Kozyrev

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Steklov Mathematical Institute
Gubkin St. 8, 119991 Moscow, Russia

Abstract

Relation of ultrametric analysis, wavelet theory and cascade models of turbulence is discussed. We construct the explicit solutions for the nonlinear ultrametric integral equation with quadratic nonlinearity. These solutions are built by means of the recurrent hierarchical procedure which is analogous to the procedure used for the cascade models of turbulence.

1 Introduction

$p$–Adic analysis and mathematical physics attract a lot of attention, see [1], [2], [3]. $p$–Adic pseudodifferential operators (in particular, the Vladimirov operator $D^a$) and pseudodifferential equations (analogues of the equations of mathematical physics) were studied. $p$–Adic wavelets were introduced and the relation of wavelet analysis to spectral theory of $p$–adic pseudodifferential operators was described in [4]. Analysis of pseudodifferential operators and wavelets on general (locally compact) ultrametric spaces was developed in [5], [6], [7].

In the present paper we construct families of explicit solutions for the following nonlinear integro–differential equation:

\[
\frac{\partial}{\partial t} v(x, t) + \int \int F(\text{sup}(x, \xi, \eta)) v(\xi, t) (v(\eta, t) - v(x, t)) \, d\nu(\xi) \, d\nu(\eta) + \\
+ \int G(\text{sup}(x, \xi))(v(x, t) - v(\xi, t)) d\nu(\xi) = 0.
\]

Here the argument $x \in X$ belongs to the ultrametric space $X$, and $t$ (time) is a real parameter. For the investigation of this equation we use the methods of ultrametric wavelet analysis, see the Appendices 1 and 3 for the discussion.

We show that the above equation possesses solutions in the form of the product of ultrametric wavelet and exponent of time

\[ v(x, t) = e^{\omega t} \psi_{I_j}(x). \]

Moreover, we construct also more general solutions for the equation above. We construct the unique solution of Cauchy problem with the initial condition in the space $D_0(X)$ of mean zero test functions. These solutions are built by means of the recurrent hierarchical procedure. Procedures
of this kind are related to the cascade models of turbulence. Using this observation we will call the equation above the quadratic cascade equation. Ultrametric \((p\text{-adic})\) analysis was applied for the first time to description of cascade models of turbulence in \cite{8}.

The structure of the paper is as follows.

In Section 2 we construct solutions of Cauchy problem for the quadratic cascade equation.

In Section 3 (the Appendix 1) we give the exposition of ultrametric analysis.

In Section 4 (the Appendix 2) we put the proof of the crucial Lemma \ref{lem1}.

In Section 5 (the Appendix 3) we discuss the cascade models of turbulence.

\section{The quadratic cascade equation}

Consider the quadratic cascade equation

\begin{equation}
\frac{\partial}{\partial t} v(x,t) + \int \int F(\sup(x,\xi,\eta))v(\xi,t)\,(v(\eta,t) - v(x,t))\,d\nu(\xi)d\nu(\eta) + \\
+ \int G(\sup(x,\xi))(v(x,t) - v(\xi,t))d\nu(\xi) = 0.
\end{equation}

(1)

Here the complex valued functions \(F\) and \(G\) on the tree \(T(X)\) of balls in the (locally compact) ultrametric space \(X\) with the Borel measure \(\nu\) should satisfy the condition of convergence

\begin{equation}
\sum_{J>I} |F(J)|(\nu(J) - \nu(J,I)) < \infty
\end{equation}

(2)

(analogously for \(G\)), where \(\nu(J,I)\) is the measure of the maximal subball in \(J\) which contains the ball \(I\).

The third term in the above equation contains the ultrametric pseudodifferential operator, see the Appendix 1. The second term is the integral operator with quadratic nonlinearity.

We have the following important lemma.

\textbf{Lemma 1} \quad \text{Let} \ \phi, \psi \text{ be ultrametric wavelets. Denote } J \text{ (correspondingly } I \text{) the minimal ball which contains the support of } \psi \text{ (correspondingly of } \phi).\text{ Let the integration kernel } F \text{ in (2) satisfies condition (2). Then the integral}

\begin{equation}
\mathcal{I}[\phi,\psi](x) = \int \int F(\sup(x,\xi,\eta))\phi(\xi)\,(\psi(\eta) - \psi(x))\,d\nu(\xi)d\nu(\eta)
\end{equation}

\text{converges and has the form:}

\begin{equation}
\mathcal{I}[\phi,\psi](x) = \psi(x)\phi(x)\Phi_{IJ},
\end{equation}

\text{where the coefficient } \Phi_{IJ} \text{ can be non–zero only for the case when } J < I \text{ (the ball } J \text{ is a strict subset of the ball } I)\text{. In this case}

\begin{equation}
\Phi_{IJ} = \nu^2(I,J)F(I) - \nu^2(J,F(J) - \sum_{L:J<L<I} (\nu^2(L) - \nu^2(L,J))F(L).
\end{equation}

\text{ (4)}
For the proof of the above lemma see Appendix 2.

**Remark**  If we fix in (3) the order of integration in the following way

$$
\int \left[ \int F(\sup(x, \xi, \eta)) \phi(\xi) (\psi(\eta) - \psi(x)) \, d\nu(\xi) \right] \, d\nu(\eta)
$$

then the integral (3) will converge for any \( \phi, \psi \) from \( D_0(X) \) without the additional condition (2).

**Example 1** Using lemma 1 one can easily construct the particular solution of the equation (2) in the form

$$v(x, t) = e^{-\eta_I t} \psi(x)$$

where \( \psi \) is a wavelet (with the support in the ball \( I \)) and \( \eta_I \) is the corresponding eigenvalue for the pseudodifferential operator in the third term of (2):

$$\eta_I = G(I)\nu(I) + \sum_{J > I} G(J)(\nu(J) - \nu(J, I)).$$

The introduced function will be a solution of (2) because the integral \( I[\psi, \psi] \), where \( \psi \) is a wavelet, is identically zero.

**Example 2** Let us find a solution of (2) in the form

$$v(x, t) = v_1(t) \psi(x) + v_2(t) \phi(x),$$

where \( \psi \) and \( \phi \) are wavelets with the supports in the balls \( J \) and \( I \) correspondingly.

If \( I \) and \( J \) have zero intersection or \( I = J \) then \( I[\phi, \psi] = I[\psi, \phi] = 0 \) and the corresponding solution of (2) takes the form

$$v(x, t) = v_1(0)e^{-\eta_J t} \psi(x) + v_2(0)e^{-\eta_I t} \phi(x).$$

Let the balls \( I \) and \( J \) be comparable and different, for instance let \( J < I \). Then the corresponding solution of (2) takes the form

$$v(x, t) = v_1(0)e^{-\eta_J t + \phi(x)\Phi_{IJ}v_2(0)} \psi(x) + v_2(0)e^{-\eta_I t} \phi(x).$$

Here \( \Phi_{IJ} \) is given by (4) and we assume \( \eta_I \neq 0 \).

We have constructed the nontrivial solution of the nonlinear equation (2). This solution contains a double exponent (exponent of exponent) of time.

Let us consider Cauchy problem for equation (2) with \( v = v(x, t) \in D_0(X) \otimes C^1([0, \infty)) \). The space \( D_0(X) \otimes C^1([0, \infty)) \) is the inductive limit of spaces \( D_0(S) \otimes C^1([0, \infty)) \). This means that any function \( v \in D_0(X) \otimes C^1([0, \infty)) \) for any \( t \) belongs to some \( D(S) \) (where \( S \) does not depend on \( t \)) and \( v(x, t) \) is continuously differentiable with respect to \( t \) for any fixed \( x \).

We say that \( v \in D_0(X) \otimes C^1([0, \infty)) \) is the solution of Cauchy problem for (2) with the initial condition \( v_0 \in D_0(X) \), if

$$v(x, 0) = v_0(x)$$

and \( v \) satisfies (2) for \( t > 0 \).

The following theorem describes solutions of Cauchy problems for (2) with initial conditions in \( D_0(X) \).
Theorem 2  Cauchy problem with the initial condition \( v_0 \in D_0(X) \) for the integro–differential equation (2) which satisfies the condition (2) possesses the unique solution \( v \in D_0(X) \otimes C^1([0, \infty)) \).

Proof  Any function \( v \in D_0(X) \otimes C^1([0, \infty)) \) has the form of the finite linear combination of wavelets

\[
v(x, t) = \sum_{I_j} v_{Ij}(t) \psi_{Ij}(x),
\]

where \( v_{Ij} \) are in \( C^1([0, \infty)) \) and \( I \) belong to some \( S \setminus S_{\text{min}}, S \subset T(X) \). Substituting this into (2) we get

\[
\sum_{I_j} \psi_{Ij}(x) \left[ \frac{d}{dt} v_{Ij}(t) + \eta_I v_{Ij}(t) + \sum_{Jj': J > I} v_{Jj'}(t) \psi_{Jj'}(x) \Phi_{JI} \right] = 0.
\]

Since wavelets are linearly independent and wavelets of larger scale are constants on the supports of wavelets of smaller scale, the above equation is equivalent to the system of ordinary differential equations

\[
\frac{d}{dt} v_{Ij}(t) = -v_{Ij}(t) \left[ \eta_I + \sum_{Jj': J > I} v_{Jj'}(t) \psi_{Jj'}(x) \Phi_{JI} \right].
\]  (5)

Here the summation runs over the increasing sequence of balls which are larger than \( I \). This system is nonlinear (quadratic). Note that, since \( x \in I \) and the wavelet \( \psi_{Ij} \) for \( J > I \) is constant on \( I \), the coefficient \( \psi_{Jj'}(x) \) does not depend on the choice of \( x \in I \).

Let us describe the recurrent procedure of construction of the solution for the system (5). The initial condition for (2) as a function in \( D_0(X) \) has the expansion over wavelets

\[
v(x, 0) = \sum_{I_j} v_{Ij}(0) \psi_{Ij}(x).
\]  (6)

Let as choose a maximal \( I \) for which the above initial condition \( v_{Ij}(0) \) is not equal to zero (such \( I \) can be non unique). Since for a maximal \( I \) the corresponding equation in (5) is linear we obtain the corresponding exponential solution \( v_{Ij}(t) \) of the Cauchy problem:

\[
v_{Ij}(t) = v_{Ij}(0) e^{-\eta_I t}.
\]

Then for maximal subballs \( I' < I \) we substitute the obtained solution in the corresponding equations in (5) and obtain the equations for \( v_{I'j} \). These equations will be linear and will depend on the computed at the previous step function \( v_{Ij}(t) \).

Then we iterate this procedure and get solutions for all pairs \( (I, j) \):

\[
v_{Ij}(t) = v_{Ij}(0) e^{-\eta_I t - \int_0^t \sum_{Jj': J > I} v_{Jj'}(r) \psi_{Jj'}(x) \Phi_{JI} dr}.
\]  (7)

It is easy to see that if \( v_{Ij}(0) = 0 \) then \( v_{Ij}(t) = 0 \). Therefore, since the initial condition is a finite linear combination (6) of wavelets, the described recurrent procedure will give all non zero \( v_{Ij}(t) \) in a finite number of steps.

The uniqueness of the solution follows from the observation that any equation in (5) is linear as equation for \( v_{Ij}(t) \) and depends only on functions \( v_{Jj'}(t) \) related to larger scales. This finishes the proof of the theorem. \( \square \)
Remark The formula (7) gives the explicit solution of Cauchy problem for equation (2). Therefore this equation is exactly solvable, and the solution can be described with the help of the recurrent hierarchical procedure.

Remark If the initial condition \( v_0 \) for (2) belongs to some (finite dimensional) space \( D_0(S) \), \( S \subset T(X) \), then for any \( t \) the solution \( v(x,t) \) of the corresponding Cauchy problem for (2) will also belong to \( D_0(S) \). This observation is the example of the mentioned in [9] general phenomenon of existence of localized solutions for some integral equations in ultrametric analysis.

Remark The equation (2) for \( v \) in \( D_0(X) \otimes C^1([0,\infty)) \) is equivalent to the system of ordinary differential equations (5). The system (5) gives the example of the so called cascade model. Models of this kind are used for description of turbulence, see the Appendix 3. The relation of nonlinear ultrametric integral equations and cascade models of turbulence was mentioned for the first time in [8]. For discussion of cascade models of turbulence see [10].

The results of the present paper show that the cascade models are related to integrable nonlinear ultrametric integral equations and ultrametric wavelet analysis.

3 Appendix 1: Ultrametric analysis

In this Section we discuss some results on ultrametric analysis, which can be found in [5], [6], [7].

Definition 3 An ultrametric space is a set with the ultrametric \( d(x, y) \) (where \( d(x, y) \) is called the distance between \( x \) and \( y \)), i.e. a function of two variables, satisfying the properties of positivity and non degeneracy

\[
d(x, y) \geq 0, \quad d(x, y) = 0 \implies x = y;
\]

symmetricity

\[
d(x, y) = d(y, x);
\]

and the strong triangle inequality

\[
d(x, y) \leq \max(d(x, z), d(y, z)), \quad \forall x, y, z.
\]

We say that an ultrametric space \( X \) is regular, if this space satisfies the following properties:

1) The set of all the balls of nonzero diameter in \( X \) is finite or countable;

2) For any decreasing (infinite) sequence of balls \( \{I^{(k)}\} \), \( I^{(k)} \supseteq I^{(k+1)} \), the diameters of the balls tend to zero;

3) Any ball of non–zero diameter is a finite union of maximal subballs.

Ultrametric spaces are dual to directed trees. Below we describe some part of the duality construction.

For a regular ultrametric space \( X \) consider the set \( T(X) \), which contains all the balls in \( X \) of nonzero diameters, and the balls of zero diameter which are maximal subballs in balls of nonzero diameters. This set possesses a natural structure of a directed tree. Two vertices \( I \) and \( J \) in \( T(X) \)
are connected by an edge if the corresponding balls are ordered by inclusion, say \( I \supset J \) (i.e. one of the balls contain the other), and there are no intermediate balls between \( I \) and \( J \).

The partial order in \( T(X) \) is defined by inclusion of balls, this partial order is a direction. We recall that a partially ordered set is a directed set (and a partial order is a direction), if for any pair of elements there exists the unique supremum with respect to the partial order.

On the directed tree \( T(X) \) we have the natural increasing positive function which puts into correspondence to any vertex the diameter of the corresponding ball.

Assume now we have a directed tree \( T \) with the positive increasing function \( F \) on this tree. Then we define the ultrametric on the set of vertices of the tree as follows: \( d(I, J) = F(\sup(I, J)) \) where \( \sup(I, J) \) is the supremum of vertices \( I, J \) with respect to the direction.

Then we take completion of the set of vertices with respect to the defined ultrametric and eliminate from the completion all the inner points of the tree (a vertex of the tree is inner if it does not belong to the border of the tree). We denote the obtained space \( X(T) \), this space is ultrametric.

An ultrametric pseudodifferential operator is defined in the following way. Consider a \( \sigma \)-additive Borel measure \( \nu \) with countable or finite basis on a regular ultrametric space \( X \). Consider the pseudodifferential operator

\[
Tf(x) = \int T(\sup(x, y))(f(x) - f(y))d\nu(y).
\]

Here \( T(I) \) is some complex valued function on the tree \( T(X) \). The supremum \( \sup(x, y) = I \) of the points \( x, y \in X \) is the minimal ball \( I \) in \( X \), containing both points.

Let us build a basis in the space \( L^2(X, \nu) \) of complex valued functions on a regular ultrametric space \( X \) which are quadratically integrable with respect to the measure \( \nu \). We call this basis the basis of ultrametric wavelets.

Denote by \( V(I) \) the space of functions on \( X \), generated by characteristic functions of the maximal subballs in the ball \( I \) of nonzero diameter. Correspondingly, \( V_0(I) \) is the subspace of codimension 1 in \( V(I) \) of functions with zero mean with respect to the measure \( \nu \). The spaces \( V_0(I) \) for different \( I \) are orthogonal. We denote \( p_I \) the number of maximal subballs in the ball \( I \). The dimension of \( V_0(I) \) as the Euclidean space with the scalar product as in \( L^2(X, \nu) \) will be less or equal to \( p_I - 1 \).

We introduce in the space \( V_0(I) \) some orthonormal basis \( \{\psi_{IJ}\} \). If the measures of all maximal subballs in \( I \) are positive, the index \( j \) can take values \( 1, \ldots, p_I - 1 \). The next theorem shows how to construct the orthonormal basis in \( L^2(X, \nu) \), taking the union of bases \( \{\psi_{IJ}\} \) in the spaces \( V_0(I) \) over all non minimal \( I \in T(X) \) (equivalently, over all balls \( I \) of non–zero diameters).

**Theorem 4**  
1) Assume that the measure \( \nu(X) \) of the regular ultrametric space \( X \) is infinite. Then the set of functions \( \{\psi_{IJ}\} \), where \( I \) runs over all non minimal vertices of the tree \( T(X) \) is an orthonormal basis in \( L^2(X, \nu) \).

2) Let the measure \( \nu(X) \) of the regular ultrametric space \( X \) is finite and is equal to \( \nu(X) = A \). Then the set of functions \( \{\psi_{IJ}, A^{-\frac{j}{2}}\} \), where \( I \) runs over all non minimal vertices of the tree \( T(X) \) is an orthonormal basis in \( L^2(X, \nu) \).

The basis introduced in the present theorem will be called the basis of ultrametric wavelets.

The next theorem shows that the basis of ultrametric wavelets is the basis of eigenvectors for ultrametric pseudodifferential operators.
Theorem 5 Let the following series converge absolutely:
\[
\sum_{J>R} T(J)(\nu(J) - \nu(J,R))
\]  
for some ball \( R \).

Then the ultrametric pseudodifferential operator
\[
Tf(x) = \int T(\sup(x,y))(f(x) - f(y))d\nu(y)
\]
has a dense domain in \( L^2(X,\nu) \) and ultrametric wavelets from Theorem 4 are eigenfunctions of \( T \):
\[
T\psi_I(x) = \lambda_I \psi_I(x)
\]
with the eigenvalues:
\[
\lambda_I = T(I)\nu(I) + \sum_{J>I} T(J)(\nu(J) - \nu(J,I)).
\]

Here \( (J,I) \) is the maximal subball in \( J \) which contains \( I \).

A function \( f \) on an (ultrametric) space \( X \) is called locally constant, if for any arbitrary point \( x \in X \) there exists a positive number \( r \) (depending on \( x \)), such that the function \( f \) is constant on the ball with the center in \( x \) and the radius \( r \):
\[
f(x) = f(y), \quad \forall y : d(x,y) \leq r.
\]

In particular, the characteristic function \( \chi_I \) of a ball \( I \) is locally constant.

The space of test functions \( D(X) \) on the ultrametric space \( X \) is defined as the space of locally constant functions with compact support. Any test function in \( D(X) \) is a (finite) linear combination of characteristic functions of balls.

The space \( D(X) \) is the inductive limit of the finite dimensional spaces \( D(S) \), where \( S \) is a regular subtree in \( T(X) \) and \( D(S) \) is the linear span of characteristic functions of balls from \( S \). Here the finite subtree \( S \subset T(X) \) is called regular, if for any increasing edge \( IJ, I,J \in S, I < J \) the subtree \( S \) contains all edges \( KJ, K < J \).

We denote \( V_0(S) \subset D(S) \) the subspace of mean zero functions in \( D(S) \). We denote \( D_0(S) \) the factor space \( V_0(S)/V_0^\perp(S) \) over the subspace \( V_0^\perp(S) \subset V_0(S) \) of functions which can be not equal to zero only on some sets of zero measure \( \nu \). The space \( D_0(S) \) can be identified with the linear span of ultrametric wavelets \( \psi_{IJ}, I \in S\setminus S_{\min} \), where \( S_{\min} \) is the set of the minimal elements in \( S \).

We denote \( D_0(X) \) the inductive limit of the finite dimensional spaces \( D_0(S) \). The space \( D_0(X) \) is isomorphic to the linear span of ultrametric wavelets.

4 Appendix 2: Proof of Lemma 1

Consider the integral
\[
I[\phi, \psi](x) = \int \int F(\sup(x,\xi,\eta))\phi(\xi)(\psi(\eta) - \psi(x))d\nu(\xi)d\nu(\eta)
\]
for the case when \( \phi \) and \( \psi \) are ultrametric wavelets. The condition (2) guarantees the convergence of the integral. Let us denote \( I \) and \( J \) the minimal balls which contain the supports of \( \phi \) and \( \psi \) correspondingly.

Consider the following cases:

1) The balls \( J \) and \( I \) have empty intersection. Then the integral over \( \xi \) gives zero because \( \phi \) is a mean zero function and the integration kernel \( F \) is constant on \( I \).

2) Let \( I \) be the strict subset of \( J \). Then the integral over \( \xi \) can be nonzero only for \( x, \eta \in I \). In this case \( \psi(\eta) \) and \( \psi(x) \) cancel and thus the integral is equal to zero.

3) Let \( I = J \). Then the integral over \( \xi \) can be nonzero only for \( x, \eta \in I \), and, moreover, \( \text{sup}(x, \eta) < I \). In this case \( \psi(\eta) \) and \( \psi(x) \) cancel and thus the integral is equal to zero.

4) Let \( J \) be the strict subset of \( I \). Consider the following cases.

a) If \( x \notin J \) then \( \psi(x) = 0 \) and the integration kernel is constant on \( J \). Therefore the integral over \( \eta \) is equal to zero.

b) Let \( x \in J < I \). The integral takes the form

\[
\mathcal{I}[\phi, \psi](x) = \int_{\xi \in I} \int_{\eta \in \text{sup}(x, \xi)} F(\text{sup}(x, \xi, \eta)) \phi(\xi) (\psi(\eta) - \psi(x)) d\nu(\eta) d\nu(\xi) = \\
= \int_{\xi \in I} \int_{\eta \text{sup}(x, \xi, \eta) < I} F(\text{sup}(x, \xi, \eta)) \phi(\xi) (\psi(\eta) - \psi(x)) d\nu(\eta) d\nu(\xi) = \\
= \left[ \int_{\xi \text{sup}(x, \xi) = I} + \int_{\xi \text{sup}(x, \xi) < I} \right] \int_{\eta \text{sup}(x, \eta) < I} F(\text{sup}(x, \xi, \eta)) \phi(\xi) (\psi(\eta) - \psi(x)) d\nu(\eta) d\nu(\xi) = \\
= -\phi(x) \nu(I, J) F(I) \int_{\eta \text{sup}(x, \eta) < I} (\psi(\eta) - \psi(x)) d\nu(\eta) + \\
+ \phi(x) \left[ \int_{\xi \text{sup}(x, \xi) < J} + \int_{\xi \text{sup}(x, \xi) < J} \right] \int_{\eta \text{sup}(x, \eta) < I} F(\text{sup}(x, \xi, \eta)) (\psi(\eta) - \psi(x)) d\nu(\eta) d\nu(\xi) = \\
= \psi(x) \phi(x) \nu^2(I, J) F(I) + \phi(x) \nu(J) \int_{\eta \text{sup}(x, \eta) < I} F(\text{sup}(x, \eta)) (\psi(\eta) - \psi(x)) d\nu(\eta) - \\
- \psi(x) \phi(x) \int_{\xi \text{sup}(x, \xi) < I} \int_{\eta \text{sup}(x, \eta) < I} F(\text{sup}(x, \xi, \eta)) d\nu(\eta) d\nu(\xi) = \\
= \psi(x) \phi(x) \left[ \nu^2(I, J) F(I) - \nu^2(J) F(J) - \nu(J) \sum_{L:J < L < I} F(L)(\nu(L) - \nu(L, J)) - \\
- \sum_{L:J < L < I} (\nu(L) - \nu(L, J)) \left( F(L) \nu(J) + F(\text{sup}(L, K)) \sum_{K:J < K < I} (\nu(K) - \nu(K, J)) \right) \right] = \\
= \psi(x) \phi(x) \left[ \nu^2(I, J) F(I) - \nu^2(J) F(J) - 2\nu(J) \sum_{L:J < L < I} F(L)(\nu(L) - \nu(L, J)) - \\
- \sum_{L:J < L < I} (\nu(L) - \nu(L, J)) \sum_{K:J < K < I} (\nu(K) - \nu(K, J)) F(\text{sup}(L, K)) \right].
\]
Since
\[
\sum_{L:<L<I} (\nu(L) - \nu(L,J)) \sum_{K:<K<I} (\nu(K) - \nu(K,J)) F(\sup(L,K)) = \\
= \sum_{L:<L<I} (\nu(L) - \nu(L,J))^2 F(L) + 2 \sum_{L:<L<I} (\nu(L) - \nu(L,J)) F(L)(\nu(L,J) - \nu(J)),
\]
we get for the integral
\[
\mathcal{I}[\phi,\psi](x) = \psi(x)\phi(x) \left[ \nu^2(I,J)F(I) - \nu^2(J)F(J) - \\
- \sum_{L:<L<I} (\nu(L) - \nu(L,J))^2 F(L) - 2 \sum_{L:<L<I} (\nu(L) - \nu(L,J)) F(L)\nu(L,J) \right] = \\
= \psi(x)\phi(x) \left[ \nu^2(I,J)F(I) - \nu^2(J)F(J) - \sum_{L:<L<I} (\nu^2(L) - \nu^2(L,J)) F(L) \right].
\]
This finishes the proof of the lemma. □

5 Appendix 3: Hierarchical models of turbulence

In the present Section we discuss, following the book [10], the hierarchical (or cascade, or shell) models of turbulence. The construction [10] of cascade models is as follows.

1) The hierarchical basis in \(L^2(\mathbb{R}^3)\), similar to bases of wavelets, is constructed. The hierarchical basis is built using translations and dilations of some finite set of mean zero functions.

2) The hierarchical basis is applied to the investigation of the Navier–Stokes equation
\[
\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \mu \nabla^2 v + f. \tag{11}
\]
Here \(v\) is velocity, \(\rho\) is density, \(p\) is pressure, \(\mu\) is viscosity, and \(f\) is external force.

This means that we substitute in the above equation the expansion of velocity over the hierarchical basis, and investigate the obtained system of non–linear ordinary differential equations for the coefficients of the expansion:
\[
\frac{\partial v_N(t)}{\partial t} = \sum_{I,J} A_{NIJ}v_I(t)v_J(t) + \sum_B B_{NK}v_K(t) + C_N. \tag{12}
\]

3) We make the following important observation: in the obtained at the previous step system of ordinary differential equations majority of the coefficients will be small, and, moreover, will decay with the distance from the main diagonal, i.e. \(A_{NIJ}\) decays with \(|N-I| \to \infty\) or \(|N-J| \to \infty\), \(B_{NK}\) decays with \(|N-K| \to \infty\).

We define the cascade system of ordinary differential equations as follows: we put in (12) the coefficients \(A_{NIJ}, B_{NK}\), corresponding to sufficiently large \(|N-I|, |N-J|, |N-K|\) equal to zero. We obtain the system of equations with contributions corresponding to finite number of main diagonals of the tensors \(A_{NIJ}, B_{NK}\). In particular, any equation in the obtained system will
contain finite number of contributions. In this sense the cascade model is constructed by cut–off of the system [12].

The obtained in this way cascade systems of differential equations were studied numerically [10]. It was found that the behavior of some cascade models is in good agreement with the experimental results.

The hierarchical structure of the cascade models is well suited for the description of the Richardson cascade and self–similarity of turbulent flows [10]. The indices of the hierarchical basis in [12] correspond to eddies with different scales and positions. Tensor $A_{NIJ}$ describes interaction of eddies, tensor $B_{NK}$ describes dissipation of energy due to viscosity.

Let us discuss the relation of cascade models and ultrametric analysis. The natural conjecture is that the cascade system of ordinary differential equations is equivalent in some sense to the single integral equation with integration over the ultrametric argument. The first result of this kind was obtained in [8], where the $p$–adic integral equation which describes the Richardson cascade was constructed.

In the present paper we introduce the approach related to ultrametric wavelet analysis. We built the integro–differential equation (2) with the following property: the system (5) obtained by expansion of $v(x, t)$ over the basis of ultrametric wavelets is an example of the cascade system of differential equations. In this approach the eddies correspond to ultrametric wavelets, which itself are in correspondence with the elements of the hierarchical basis of [10]. The second (quadratic) term in (2) describes interaction of eddies, the third term in (2) describes dissipation. The cut–off procedure used in the construction of the cascade model in our approach takes the form of approximation of the Navier–Stokes equation (11), in the regime of developed turbulence, by the cascade equation (2). The approximation here is understood as approximation of the system (12) by the system (5).

Actually in the present paper we have described by means of ultrametric methods the simplest cascade model. We found that this simplest cascade model is integrable. This shows the relation between cascade models and theory of integrable nonlinear integral ultrametric equations.

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References

[1] V.S.Vladimirov, I.V.Volovich, Ye.I.Zelenov $p$–Adic Analysis and Mathematical Physics. Singapore: World Scientific, 1994.

[2] A.Khrennikov Non–Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models. Dordrecht: Kluwer Academic Publishers, 1997.
[3] A.N. Kochubei Pseudodifferential equations and stochastics over non–archimedean fields. New York: Marcel Dekker, 2001.

[4] S.V. Kozyrev Wavelet theory as $p$-adic spectral analysis // Izvestiya: Mathematics. V.66. N.2. P.367–376. 2002. [arXiv:math-ph/0012019]

[5] A.Yu. Khrennikov, S.V. Kozyrev Pseudodifferential operators on ultrametric spaces and ultrametric wavelets // Izvestiya: Mathematics. V.69. N.5. P.989-1003. 2005. [arXiv:math-ph/0412062]

[6] A.Yu. Khrennikov, S.V. Kozyrev Wavelets on ultrametric spaces // Applied and Computational Harmonic Analysis. V.19. P.61-76. 2005.

[7] S.V. Kozyrev Wavelets and spectral analysis of ultrametric pseudodifferential operators // Sbornik Mathematics. V.198. N.1.P.103-126. 2007. [arXiv:math-ph/0412082]

[8] S.Fischenko, E.Zelenov $p$–Adic Models of Turbulence, in: $p$–Adic Mathematical Physcis, 2–nd International Conference, Eds: A.Yu.Khrennikov, Z.Rakic, I.V.Volovich, AIP Conference Proceedings, V.286, P.174–191, AIP, Melville, New York, 2006.

[9] S.V. Kozyrev, A.Yu. Khrennikov Localization in Space for a Free Particle in Ultrametric Quantum Mechanics. Doklady Mathematics. V.74. N3. P. 906–909. 2006.

[10] P.G. Frick Turbulence: models and approaches. Lectures. (in Russian) PGTU, Perm, 1998.