MICROLAGRANGIAN MANIFOLDS AND QUASITHERMODYNAMIC FLUCTUATIONS OF NONEQUILIBRIUM STATES

ARTUR E. RUUGE

ABSTRACT. The paper deals with “quantization” and “second quantization” of phenomenological thermodynamics with respect to the Boltzmann’s constant. It is suggested to perceive the quasithermodynamic parameter (corresponding to the Boltzmann’s constant) as a mathematical analogue of the semiclassical parameter (corresponding to the Planck’s constant), and to introduce a new concept of a “thermocorpuscle” (a thermodynamic analogue of a particle where the coordinates are replaced by the nonequilibrium thermodynamic forces and the momenta are replaced by the corresponding flows). The semiclassical quantization of phenomenological thermodynamic Lagrangian manifolds yields a new system of equations for the quasithermodynamic fluctuations along a curve of evolution of a nonequilibrium physical system. This leads to a quasithermodynamic analogue of Bell’s inequalities and their violation is a new effect that can be tested experimentally. The generating function of the quasithermodynamic fluctuations (the nonequilibrium analogue of a partition function) is interpreted as an expectation value of a second quantized operator expressed via the density of “thermocorpuscles”. An analogue of the BBGKY chain of equations defines a deformation of the fluctuations by an interaction between the thermocorpuscles. In place of an interaction parameter in mechanics (the “external” Planck’s constant), one introduces the “external” Boltzmann’s constant for an asymptotic expansion of the thermodynamic collision integral.

1. INTRODUCTION

The original motivation for this paper stems from an idea to study the quasithermodynamic fluctuations in nonequilibrium statistical physics using the methods of semiclassical approximation of quantum theory. If we look at a semiclassical wave function $\psi_h(x)$, of a quantum system with $n$ degrees of freedom $x = (x_1, x_2, \ldots, x_n)$, and take a classical limit $h \to 0$ for the corresponding Wigner’s quasiprobability function $\rho_{\psi_h}(x, p)$, $p = (p_1, p_2, \ldots, p_n)$, then under some mild conditions and assumptions [6], we obtain a smooth manifold

$$\Lambda := \text{supp } \lim_{h \to 0} \rho_{\psi_h}(x, p),$$

where the limit on the right-hand side is understood in the weak sense. This manifold is a Lagrangian submanifold $\Lambda \subset \mathbb{R}_x^2 \mathbb{R}_p^2$ in the classical phase space $\mathbb{R}_x^2 \mathbb{R}_p^2$ with respect to the canonical symplectic structure $\omega = \sum_{i=1}^n dp_i \wedge dx_i$.

The Lagrangian manifolds arise in a perfectly natural way in the equilibrium thermodynamics as well. Consider, for example, a physical system consisting of $\nu$ moles of a chemical substance in a closed volume $V$. Then what happens in the...
molecular kinetic theory is that one splits $\nu$ into a product of a “very small” quantity $k_B$ (the Boltzmann’s constant) and a “very big” quantity $N$ (the number of particles in the system):

$$\nu = \frac{1}{R} k_B N,$$

where $R$ is a constant fixing the units of measurement (the universal gas constant). The basic idea is to consider a family of underlying systems with $N_\lambda = \lambda N$ particles and volume $V_\lambda = \lambda V$, where $\lambda$ is the rescaling parameter, and to derive the empirical laws from the thermodynamic limit $\lambda \to \infty$. As a model example, take the underlying system to be a system of $N_\lambda$ particles on an $n$-dimensional torus of radius $L_\lambda = \lambda^{1/n} L$, $V = L^n$, $L$ is fixed. Assume that it is described by an energy spectrum $E_0(\lambda) < E_1(\lambda) < \cdots < E_m(\lambda) < \ldots$, where each level $E_m(\lambda)$, $m \in \mathbb{Z}_{\geq 0}$, has a finite degree of degeneracy $g_m(\lambda)$. One can construct a partition function (assume that the power series converges):

$$Z^{(\lambda)}_{k_B}(\beta) := \sum_{m=0}^{\infty} g_m^{(\lambda)} \exp \left( - \frac{1}{k_B} \beta E_m^{(\lambda)} \right),$$

where $\beta = T^{-1}$ is the inverse absolute temperature of the system. The free energy $F^{(\lambda)}_{k_B}(T)$ of the system at temperature $T$ is given by

$$F^{(\lambda)}_{k_B}(T) := -k_B T \ln Z^{(\lambda)}_{k_B}(T^{-1}).$$

Take some values of $\nu$ and $V = L^n$, and assume that the free energy satisfies

$$\lambda^{-1} F^{(\lambda)}_{k_B}(T) = \nu f_{k_B}(T, v) + o(\lambda^{-1}),$$

as $\lambda \to \infty$, where $v := V/\nu$ is the volume per mole, and $f_{k_B}(T, v)$ is a smooth function. In the limit $\lambda \to \infty$ we have:

$$s = -\frac{\partial f_{k_B}(T, v)}{\partial T}, \quad p = -\frac{\partial f_{k_B}(T, v)}{\partial v},$$

where $s = S/\nu$, $S$ is the entropy of the system, and $p$ is the pressure in the system. This is precisely what one can see in the phenomenological thermodynamics. The equations (2) define a Lagrangian manifold $L \subset \mathbb{R}^4(T, v, s, p)$ with respect to symplectic structure

$$\omega := ds \wedge dT - dv \wedge dp.$$

According to [8, 9, 10], the intensive thermodynamic coordinates $(T, -p)$ can be perceived as “coordinates”, and the extensive thermodynamic coordinates $(s, v)$ can be perceived as “momenta”. More generally, one may perceive an abstract thermodynamic system with $d$ degrees of freedom $\xi = (\xi_1, \xi_2, \ldots, \xi_d)$ as a Lagrangian submanifold $\Lambda \subset \mathbb{R}^{2d}$ with respect to symplectic structure $\omega = \sum_{i=1}^{d} d\eta_i \wedge d\xi_i$, $\eta = (\eta_1, \eta_2, \ldots, \eta_d)$. The Lagrangian manifolds are a central object in the semiclassical approximation of quantum mechanics [11, 12] and it is of interest to investigate the corresponding analogy between thermodynamics and mechanics in more detail. Recently this topic has attracted some additional attention and has received some new interesting developments in [3, 13, 14, 19, 20].

In the present paper we prefer to use a little different coordinates to describe the thermodynamic Lagrangian manifolds. Denote $\varepsilon = E/\nu$ the internal energy of the
system per mole. Then at temperature $T = \beta^{-1}$ we have:

$$\varepsilon = \frac{\partial}{\partial \beta}(\beta f_B(\beta, v)).$$

The first law of thermodynamics tells us:

$$ds(\varepsilon, v) = \beta d\varepsilon + \tilde{p} dv,$$

where $\tilde{p} := p/T$, and the entropy per mole is written as a function of $\varepsilon$ and $v$, $s = s(\varepsilon, v)$. The derivatives $\beta = (\partial s/\partial \varepsilon)_v$ and $\tilde{p} = (\partial s/\partial \beta)_\varepsilon$ define a Lagrangian manifold $\Lambda_{\text{thermo}} \subset \mathbb{R}^4(\varepsilon, v, \beta, \tilde{p})$ with respect to the symplectic structure $\omega_{\text{thermo}} := d\beta \wedge d\varepsilon + d\tilde{p} \wedge dv$.

Therefore, conceptually, the entropy (per mole) of a system is an action on $\Lambda_{\text{thermo}}$. The total entropy $S_{\text{thermo}} := \nu s$ is measured in the same units as the Boltzmann constant:

$$[S_{\text{thermo}}] = [k_B].$$  \hspace{1cm} (3)

If we have an abstract mechanical system with coordinates $Q = (Q_1, Q_2, \ldots, Q_n)$ and momenta $P = (P_1, P_2, \ldots, P_n)$ and consider a Lagrangian manifold $\Lambda_{\text{mech}}$ in the phase space $(\mathbb{R}^{2n}_{Q,P}, \omega_{\text{mech}})$, where

$$\omega_{\text{mech}} := \sum_{i=1}^{n} dP_i \wedge dQ_i,$$

then we have an action $S_{\text{mech}} := \int_{\gamma} P dQ$, $\gamma$ is smooth curve on $\Lambda_{\text{mech}}$, which is measured in the same units as the Planck’s constant:

$$[S_{\text{mech}}] = [\hbar].$$  \hspace{1cm} (4)

In this context, the Heisenberg’s uncertainty principle is totally similar to the Einstein’s formula for the quasithermodynamic fluctuations:

$$\Delta P_i \Delta Q_j \sim \hbar, \quad \Delta E_l \Delta \beta_m \sim k_B,$$  \hspace{1cm} (5)

where $\Delta$ denotes the standard deviation describing a fluctuation, $i, j = 1, 2, \ldots, n$, and $l, m = 1, 2$, and we put $E_1 = E$, $E_2 = V$, $\beta_1 = \beta$, $\beta_2 = \tilde{p}$. Informally speaking, $\hbar$ is a quantum of mechanical action, and $k_B$ is a “quantum” of entropy.

The analogy between the Planck’s constant $\hbar$ and the Boltzmann’s constant $k_B$ expressed, in particular, by (3), (4), (5), is the central motive of the present paper. The general scheme is as follows. First, we generalize the idea of quantization of phenomenological thermodynamic Lagrangian manifolds to the nonequilibrium setting. This defines a generic shape of the equations describing the transfer of fluctuations along a nonequilibrium evolution curve. Under some assumptions one can assemble these fluctuations into a “quasithermodynamic wavefunction” and extract the thermodynamic Hamilton-Jacobi equation in the limit $\lambda \to \infty$. It turns out, that in nonequilibrium quasithermodynamics one can mimic Bell’s inequalities and it is natural to perceive the thermodynamic forces $X = (X_1, X_2, \ldots, X_d)$ as “coordinates”, and the corresponding flows $J = (J_1, J_2, \ldots, J_d)$ as “momenta”. We apply the second quantization to produce the “thermocorpuscles” introducing the creation and annihilation operators $a^\pm(X, J)$ in a phase space point $(X, J) \in \mathbb{R}_{X,J}^{2d}$, the symplectic structure is $\Omega := \sum_{i=1}^{d} dJ_i \wedge dX_i$. Basically, the thermodynamic analogue of the
Wigner’s equation for the single particle quasiprobability distribution is lifted to the symmetric Fock space $\mathcal{F}^\#$,

$$
\mathcal{F}^\# := \mathbb{C} \oplus L^2(\mathbb{R}^2_{X,J}) \oplus (L^2(\mathbb{R}^2_{X,J}))^{\otimes \text{symm}^2} \oplus \ldots
$$

where $\otimes \text{symm}$ denotes the symmetric tensor product. After that, it is suggested to deform the second quantized equations by an interaction between the thermocorpuscles and to mimic the basic constructions of statistical mechanics like the BBGKY chain and the collision integral. In particular, this yields a new interpretation of the nonequilibrium analogue $Z_{k_B}^{(\lambda)}(u; t)$ of the partition function in terms of the density of thermocorpuscles,

$$
Z_{k_B}^{(\lambda)}(u; t) = \int dX \exp(-uX/k_B) \int dJ \langle R^{t}_\lambda, a^+(X,J)a^-(X,J)R^{t}_\lambda \rangle,
$$

where $u = (u_1, u_2, \ldots, u_d)$ is a parameter varying in a neighbourhood of $0 = (0, 0, \ldots, 0)$, $R^{t}_\lambda \in \mathcal{F}^\#$ is a real vector depending on the rescaling parameter $\lambda$ and the time $t$, and $\langle -,-\rangle$ denotes the inner product.

2. Equilibrium Quasithermodynamics

Let us look at a phenomenological one-component thermodynamic system with two degrees of freedom, $d = 2$. To be more concrete, let the molar extensive coordinates be $\xi_1 = \varepsilon$ and $\xi_2 = v$, where $\varepsilon := E/\nu$, $v := V/\nu$, $E$ is the internal energy, $V$ is the volume of the system, $\nu$ is the number of moles. We have a phase space $\mathbb{R}^4(\xi_1, \xi_2, \eta_1, \eta_2)$ with a symplectic structure $\omega = d\eta_1 \wedge d\xi_1 + d\eta_2 \wedge d\xi_2$, where $\eta_1 = \beta$ is the inverse absolute temperature of the system, and $\eta_2 = p := \beta\rho$, $\rho$ is the pressure in the system. The system is described as a two-dimensional Lagrangian manifold $\Lambda$ in this phase space defined by equations

$$
\beta = \left(\frac{\partial s}{\partial \varepsilon}\right)_v, \quad \tilde{p} = \left(\frac{\partial s}{\partial \nu}\right)_\varepsilon,
$$

where $s = S/\nu$ is the entropy $S$ of the state of the system taken per mole. The function $s(s(\varepsilon, v))$ is a generating function of $\Lambda$ with respect to the focal coordinates $(\varepsilon, v)$. If we assume, that $\Lambda$ admits another choice of focal coordinates, say, $(\beta, \tilde{p})$, then the corresponding generating function $\tilde{\mu} = \tilde{\mu}(\beta, \tilde{p})$ is linked to $s = s(\varepsilon, v)$ via the Legendre transform,

$$
\tilde{\mu}(\beta, \tilde{p})|_{\Lambda} = [s(\varepsilon, v) - \beta \varepsilon - \tilde{p}v]|_{\Lambda}.
$$

The physical meaning of this function is just $\tilde{\mu} = -\beta \mu$, where $\mu$ is the chemical potential of the system. It is natural to consider along with $\Lambda$ a Lagrangian manifold $\Lambda^+$ of dimension $d + 1 = 3$ in the phase space $\mathbb{R}^6(x, y)$ with coordinates $x = (x_1, x_2, x_3)$, $x_1 = E$, $x_2 = V$, $x_3 = \nu$, and “momenta” $y = (y_1, y_2, y_3)$, $y_1 = \beta$, $y_2 = \tilde{p}$, $y_3 = \tilde{\mu}$. The symplectic structure $\omega^+$ is given by $\omega^+ := \sum_{i=1}^3 dy_i \wedge dx_i$, and the manifold $\Lambda^+$ can be described as

$$
\Lambda^+ := \{(x, y) \mid x_3 \neq 0, (x_1/x_3, x_2/x_3, y_1, y_2) \in \Lambda, y_3 = \tilde{\mu}(x_1/x_3, x_2/x_3)\}.
$$

One can observe, that $\Lambda^+$ admits a projectivization with respect to the extensive coordinates $x = (x_1, x_2, x_3) = (E,V,\nu)$, which can be taken as global focal coordinates on $\Lambda^+$. At the same time, all the points of $\Lambda^+$ are singular with respect to the focal coordinate plane $\mathbb{R}^3(y)$, $y = (y_1, y_2, y_3) = (\beta, \tilde{p}, \tilde{\mu})$. Our physical system can...
be identified with this $\Lambda^+$, and one should assume (for the physical reasons), that $\Lambda^+$ is connected and simply connected \[8, 14\].

It can happen, that $\Lambda^+$ admits other focal charts with a choice of coordinates different from $x = (x_1, x_2, x_3)$. Recall the notation:

$$(x_1, x_2, x_3) = (E, V, \nu), \quad (y_1, y_2, y_3) = (\beta, +\beta p, -\beta \mu).$$

In a nonsingular chart $U$ with coordinates $(x_1, x_2, x_3) = (E, V, \nu)$, the generating function $S(x_1, x_2, x_3)$ is the entropy of the system in a point $\alpha \in U \subset \Lambda^+$ corresponding to $(x_1, x_2, x_3)$, $(dS - \sum_{i=1}^3 y_i dx_i)|_{\Lambda^+} = 0$, and $S(E, V, \nu) = \nu s(\varepsilon, v), \varepsilon = E/V, v = V/\nu$. In a singular chart $U_1$ with coordinates $(y_1, x_2, x_3) = (\beta, V, \nu)$ the generating function $S_1(y_1, x_2, x_3)$ can be interpreted (up to an additive constant) as $-\beta F$, where $F$ is the free energy of the system in a point $\alpha \in U_1 \subset \Lambda^+$ corresponding to $(y_1, x_2, x_3)$. In a singular chart $U_{1,3}$ with coordinates $(y_1, y_2, y_3) = (\beta, V, -\beta \mu)$, the generating function $S_{1,3}(y_1, x_2, y_3)$ can be interpreted (up to an additive constant) as $-\beta \Omega$, where $\Omega$ is the thermodynamic potential with respect to $(T, V, \mu)$ taken in a point $\alpha \in U_{1,3} \subset \Lambda^+$ corresponding to $(y_1, x_2, y_3)$.

Let us analyse the case of the to charts $U_1$ and $U_{1,3}$ in more detail. Assume $U_1 \cap U_{1,3} \neq \emptyset$ and fix $\alpha \in U_1 \cap U_{1,3}$. In the ambient space $\mathbb{R}^6(x, y) \supset \Lambda^+$ this point acquires coordinates $x(\alpha) = (x_1(\alpha), x_2(\alpha), x_3(\alpha))$ and $y(\alpha) = (y_1(\alpha), y_2(\alpha), y_3(\alpha))$. The step from phenomenological thermodynamics to statistical thermodynamics corresponding to the first chart consists in the following. One considers a family of physical systems parametrized by $\lambda > 0$ (the rescaling parameter), $\lambda \to \infty$. Every system is placed in a thermostat and the walls of the system are fixed and are impenetrable for the particles. The inverse temperature $\beta_\lambda(\alpha) = x_1(\alpha)$ is the same for each $\lambda$, the number of particles $N_\lambda(\alpha) = \lambda x_3(\alpha) R/k_B$, together with the volume of the system $V_\lambda(\alpha) = \lambda x_2(\alpha)$, grow linearly as $\lambda \to \infty$. For simplicity, one may have in mind a system of $N$ particles moving on a $n$-dimensional torus of radius $L$ described by a Hamiltonian $\tilde{H}_N^{(L)}(g)$, where $g$ is a collection of parameters describing the interaction between the particles and the geometrical shape of the volume (the external potential). One substitutes $L = (V(\lambda(\alpha)))^{1/n}$ and $N = N_\lambda(\alpha)$, assuming that the parameters $g = g_\lambda(\alpha)$ may be adjusted as well. The free energy $\mathcal{F}_{k_B, N}(T; g)$ at absolute temperature $T$ is given by

$$\mathcal{F}_{k_B, N}(T; g) = -k_B T \ln \mathcal{Z}_{k_B, N}^{(L)}(T^{-1}; g),$$

where

$$\mathcal{Z}_{k_B, N}^{(L)}(\beta; g) := \text{Tr} \exp \left( -\frac{1}{k_B} \beta \tilde{H}_N^{(L)}(g) \right),$$

is the partition function of the canonical Gibbs distribution at absolute temperature $T = \beta^{-1}$ (we assume that the corresponding trace is finite for the required values of $\beta, L, N, and g$). The link with phenomenological thermodynamics is as follows:

$$S_1(y_1(\alpha), x_3(\alpha), x_3(\alpha)) = \lim_{\lambda \to \infty} \left[ -\lambda^{-1} \beta_\lambda(\alpha) \mathcal{F}_{k_B, N_\lambda(\alpha)}^{(V(\lambda(\alpha)))^{1/n}}(\beta_\lambda(\alpha)^{-1}; g_\lambda(\alpha)) + C(\lambda) \right], \quad (6)$$

where it is assumed that one can find functions $g_\lambda(\alpha)$ and $C(\lambda)$ so that the limit on the right-hand side exists. Denote $S^{(\lambda)}_1(\alpha)$ the expression in the square brackets on the right-hand side of (6).

Now if we take the other chart $U_{1,3}$ with coordinates $(y_1, x_2, y_3) = (\beta, V, -\beta \mu)$, then we need to consider another family of rescaled systems of another sort: the walls of the system must admit a penetration of particles, i.e. the border of the
system is described purely geometrically. One still has a rescaling parameter \( \lambda > 0 \),
the volume of the rescaled system \( V_\lambda(\alpha) = \lambda x_2(\alpha) \), but now one should look at the
partition function \( \zeta_{k_B}^{(L)}(\beta, \mu; g) \) of the grand canonical ensemble,
\[
\zeta_{k_B}^{(L)}(\beta, \mu; g) := \text{Tr} \exp \left( -\frac{1}{k_B} \beta \left[ \hat{H}^{(L)}(g) - \mu \hat{N}^{(L)}(g) \right] \right),
\]
(it is assumed that the trace is finite for the required values of \( \beta, \mu, \) and \( g \)), where \( \beta \)
and \( \mu \) are the inverse absolute temperature and the chemical potential determined
by the environment of the system, respectively, \( \hat{H}^{(L)}(g) \) is the second quantized
Hamiltonian of the multiparticle system on a \( n \)-dimensional torus of radius \( L \), \( g \) is a collection of parameters describing the interaction between the particles and the
external potential, \( \hat{N}^{(L)} \) is the second quantized operator of the number of particles
in the system \cite{2}. Denote \( \Omega_{k_B}^{(L)}(T, \mu; g) := -k_B T \ln \zeta_{k_B}^{(L)}(T^{-1}, \mu; g) \) (the potential \( \Omega \)).
One needs to substitute in place of \( \beta, \mu, \) and \( g \) some functions \( \beta_\lambda(\alpha), \mu_\lambda(\alpha), \)
and \( g_\lambda(\alpha) \), respectively, in such a way, that the average energy and the average number
of particles in the system grow linearly with \( \lambda \to \infty \). It is convenient to change the
variables \( y_1 := \beta, y_2 := -\beta \mu \), and to consider \( \tilde{\zeta}_{k_B}^{(L)}(y_1, y_2; g) := \zeta_{k_B}^{(L)}(\beta, \mu; g) \). Denote \( (\beta_\lambda(\alpha; g), -\beta_\lambda(\alpha; g) \mu_\lambda(\alpha; g)) \) the solution of the system of equations
\[
\left( \tilde{\zeta}_{k_B}^{((\lambda x_2(\alpha))^{1/n})}(y_1, y_2; g) \right)^{-1} \left( -k_B \frac{\partial}{\partial y_1} \right) \tilde{\zeta}_{k_B}^{((\lambda x_2(\alpha))^{1/n})}(y_1, y_2; g) = \lambda x_1(\alpha),
\]
\[
\left( \tilde{\zeta}_{k_B}^{((\lambda x_2(\alpha))^{1/n})}(y_1, y_2; g) \right)^{-1} \left( -k_B \frac{\partial}{\partial y_2} \right) \tilde{\zeta}_{k_B}^{((\lambda x_2(\alpha))^{1/n})}(y_1, y_2; g) = \lambda x_3(\alpha) R/k_B,
\]
with respect to \( (y_1, y_2) \), assuming it exists and is unique for the required values of \( g \).
Recall, that \( R \) denotes the universal gas constant, \( x_1(\alpha) \) corresponds to the internal
energy, \( x_2(\alpha) \) corresponds to the volume, and \( x_3(\alpha) \) corresponds to the number of
moles of the chemical substance in the phenomenological system. The link with the
phenomenological thermodynamics is as follows:
\[
S_{1,3}(y_1(\alpha), x_2(\alpha), y_3(\alpha)) = \lim_{\lambda \to \infty} \left[ -\lambda^{-1} \beta_\lambda(\alpha; g) \Omega_{k_B}^{((\lambda x_2(\alpha))^{-1})}(\beta_\lambda(\alpha; g)^{-1}, \mu_\lambda(\alpha; g); g) + \tilde{C}(\lambda) \right]_{g = \tilde{g}_\lambda(\alpha)}, \tag{7}
\]
for some functions \( \tilde{g}_\lambda(\alpha) \) and \( \tilde{C}(\lambda) \) ensuring the existence of the limit on the right-hand side. Denote \( S_{1,3}^{(\lambda)}(\alpha) \) the expression in the square brackets on the right-hand side in \( \tag{7} \).

The two functions \( S_1(y_1, x_2, x_3) \) and \( S_{1,3}(y_1, x_2, y_3) \) are related via the Legendre transform in the third argument. If we perceive them as functions on the thermo-
dynamic phase space \( \mathbb{R}^6(x, y) \), then we have
\[
S_{1,3}(y_1, x_2, y_2) \big|_{\Lambda^+(\alpha)} = \left[ S_1(y_1, x_2, x_3) - y_3 x_3 \right] \big|_{\Lambda^+(\alpha)},
\]
for any \( \alpha \in U_1 \cap U_{1,3} \). Since \( \Lambda^+ \) admits projectivization with respect to the
extensive coordinates \( x = (x_1, x_2, x_3) \), the generating functions in the focal charts are
homogeneous with respect to these coordinates:
\[
S_1(y_1, \rho x_2, \rho x_3) = \rho S_1(y_1, x_2, x_3), \quad S_{1,3}(y_1, \rho x_2, y_3) = \rho S_{1,3}(y_1, x_2, y_3),
\]
for any \( \rho > 0 \).
Consider now an abstract phenomenological thermodynamic system with extensive coordinates \( x = (x_1, x_2, \ldots, x_{d+1}) \) and intensive coordinates \( y = (y_1, y_2, \ldots, y_{d+1}) \). The corresponding thermodynamic phase space is defined as \( \mathbb{R}^{2(d+1)}(x, y) \) equipped with canonical symplectic structure \( \omega^+ = \sum_{i=1}^{d+1} dy_i \wedge dx_i \). The equilibrium states of the system are identified with a Lagrangian manifold \( \Lambda^+ \subset \mathbb{R}^{2(d+1)}(x, y) \), which is connected, simply connected, contained in the region \( x_i > 0, i = 1, 2, \ldots, d+1 \), and admits a projectivization with respect to \( x \): if \( (x(\alpha), y(\alpha)) \) are the phase space coordinates corresponding to a point \( \alpha \in \Lambda^+ \), then for any \( \rho > 0 \), the coordinates \( (\rho x(\alpha), y(\alpha)) \) correspond to another point in \( \Lambda^+ \). Let \( I \) be a subset of \( [d+1] := \{1, 2, \ldots, d+1\} \). Denote

\[
(x, y)_I := \{\{x_j\}_{j \in [d+1]\setminus I}, \{y_i\}_{i \in I}\},
\]

where the indices atop denote the order of appearance of the variables in the list read from left to the right, for example, if \( d = 2 \), \( I = \{1, 3\} \), then \( (x, y)_I = (y_1, x_2, y_3) \).

In particular, for the emptyset \( I = \emptyset \), we have \( (x, y)_\emptyset = (x_1, x_2, \ldots, x_{d+1}) \), and for \( I = [d+1] \) we have \( (x, y)_{[d+1]} = (y_1, y_2, \ldots, y_{d+1}) \).

A focal chart of type \( I \) on \( \Lambda^+ \) is an open set \( U \subset \Lambda^+ \) together with a fixed choice of local coordinates of the shape \( (x, y)_I \). For every focal chart of type \( I \) over \( U \), there exists a function \( S_{U,I}((x, y)_I) \), such that \( U \) is described by the equations:

\[
y_i = \frac{\partial S_{U,I}((x, y)_I)}{\partial x_i}, \quad x_j = -\frac{\partial S_{U,I}((x, y)_I)}{\partial y_j},
\]

where \( i \in I, j \in [d+1]\setminus I \). Observe, that for any \( \rho > 0 \),

\[
S_{U,I}((\rho x, y)_I) = \rho S_{U,I}((x, y)_I),
\]

whenever both left- and right-hand sides are defined. Observe also, that the thermodynamic Lagrangian manifold \( \Lambda^+ \) never admits a focal chart of type \([d+1]\), i.e. at least one of the local coordinates in a given focal chart must be extensive. In case \( U \) is connected, the function \( S_{U,I} \) is defined up to an additive constant, and for a fixed pair of focal charts \((U_1, I)\) and \((U_2, J)\), such that \( U_1 \cap U_2 \neq \emptyset \), it is always possible to adjust these constants in such a way that

\[
S_{U_2,J}((x, y)_J) = S_{U_1,I}((x, y)_I) - \sum_{j \in J \setminus I} y_j x_j + \sum_{i \in I \setminus J} y_i x_i,
\]

for every \( (x, y) \) corresponding to a point \( \alpha \in U_1 \cap U_2 \). This implies, that \( S_{U_1,I} \) and \( S_{U_2,J} \) are linked via the Legendre transform (denote it \( \mathcal{L}^{U_1 \cap U_2}_{I,J} \)),

\[
S_{U_2,J} = \mathcal{L}^{U_1 \cap U_2}_{I,J}(S_{U_1,I}),
\]

for every \( (x, y) \) corresponding to a point \( \alpha \in U_1 \cap U_2 \). Observe that \( \mathcal{L}^U_{I,I} = id \), for a focal chart \((U, I)\), where \( U \) is connected. If \( U \) admits focal coordinates of another type \( J \), then we have \( \mathcal{L}^U_{I,J} \circ \mathcal{L}^U_{J,I} = id \). Furthermore, if \( U \) admits focal charts of types \( I, J, \) and \( K \), then it is straightforward to check the cocyclicity condition:

\[
\mathcal{L}^U_{I,K} \circ \mathcal{L}^U_{K,J} \circ \mathcal{L}^U_{J,I} = id.
\]

Denote \( \pi_I : \Lambda^+ \to \mathbb{R}^{d+1} \) the canonical projection \( \alpha \mapsto (x(\alpha), y(\alpha))_I \), for \( I \subset [d+1] \).

Return now to the example discussed above, \( d = 3 \). We have defined the functions \( S_1^{(3)}(\alpha) \) and \( S_{1,3}^{(3)}(\alpha) \) (see the definition right after the equations (6) and (7), respectively), such that \( S_1(y_1(\alpha), x_2(\alpha), x_3(\alpha)) = \lim_{\lambda \to \infty} S_1^{(3)}(\alpha) \), for \( \alpha \in U_1 \), and
That these functions admit the following \( \lambda \) order corrections with respect to \( \lambda^{-1} \rightarrow 0 \)? In quasithermodynamics one assumes that these functions admit the following asymptotic expansions:

\[
S_1^{(\lambda)}(\alpha) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \varphi_1^{(l)}(\alpha), \quad S_{1,3}^{(\lambda)}(\alpha) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \varphi_{1,3}^{(l)}(\alpha),
\]

where \( \varphi_1^{(l)}(\alpha) \) and \( \varphi_{1,3}^{(l)}(\alpha) \) are smooth functions, \( l = 0, 1, 2, \ldots \). This can be perceived as a condition on a choice of the function \( g_\lambda(\alpha) \) present in the definitions. It is a little more convenient to work in terms of coordinates, introducing the functions \( \Phi_1^{(l)}(y_1, x_2, x_3) \) and \( \Phi_{1,3}^{(l)}(y_1, x_2, y_3) \), defined as follows: \( \Phi_1^{(l)}(y_1, x_2, x_3) = \varphi_1^{(l)}(\alpha), \alpha \in U_1 \), and \( \Phi_{1,3}^{(l)}(y_1, x_2, y_3) = \varphi_{1,3}^{(l)}(\alpha'), \alpha' \in U_{1,3} \), \( l \in \mathbb{Z}_{\geq 0} \). We know that \( \Phi_1^{(l)}(y_1, x_2, x_3) = S_1(y_1, x_2, x_3) \) and \( \Phi_{1,3}^{(l)}(y_1, x_2, y_3) = S_{1,3}(y_1, x_2, y_3) \), so the link for the leading coefficients is the Legendre transform. What is the link between the higher order coefficients corresponding to \( l \geq 1 \)?

It is more convenient to describe the link mentioned in an abstract setting. Consider a Lagrangian manifold \( \Lambda^+ \subset \mathbb{R}^{2(d+1)}(x, y) \) for an abstract phenomenological thermodynamic system as above. We assume that in every focal chart \( (U, I) \) of type \( I \subset [d+1] \), we are given a function \( S_{U,I}^{(\lambda)}(\alpha), \alpha \in U \), depending on a parameter \( \lambda > 0 \), such that there is an asymptotic expansion

\[
S_{U,I}^{(\lambda)}(\alpha) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \varphi_{U,I}^{(l)}(\alpha),
\]

as \( \lambda \to \infty \), where \( \varphi_{U,I}^{(l)}(\alpha) \) are smooth functions, \( l \in \mathbb{Z}_{\geq 0} \). Using the notation \( (8) \), define also the functions \( \Phi_{U,I}^{(l)}((x, y)_I) \) via \( \Phi_{U,I}^{(l)}((x(\alpha), y(\alpha))_I) = \varphi_{U,I}^{(l)}(\alpha), \alpha \in U \), for \( l = 0, 1, 2, \ldots \). Take another focal chart \( (W, J) \) on \( \Lambda^+ \) such that \( W \cap U \neq \emptyset \). In the leading order \( l = 0 \), in the points corresponding to \( \alpha \in W \cap U \), we have the Legendre transform:

\[
\Phi_{W,J}|_{\hat{\pi}_I(W \cap U)} = L_{J,I}^{W \cap U}(\Phi_{U,I}^{(0)}|_{\hat{\pi}_I(W \cap U)}).
\]

The formulae for the other coefficients mimic basically to the formulae of the stationary phase method. We need to assume that the Hessian

\[
\det \text{Hess} \Phi_{U,I}^{(0)}((x, y)_I) \neq 0,
\]

for every \( (x, y)_I \in \hat{\pi}_I(U) \), for every focal chart \( (U, I) \).

Let us first introduce some notation for the differential operators associated with this method. If we have a function \( f((x, y)_I) \in C_0^\infty(\hat{\pi}_I(W \cap U)) \), then we can look at an integral \( Q_{W \cap U, I}^{(\varepsilon)}[\Phi_{U,I}^{(0)}, f](x, y)_J \), \( (x, y)_J \in \hat{\pi}_J(W \cap U) \) depending on a small parameter \( \varepsilon > 0 \),

\[
Q_{W \cap U, I}^{(\varepsilon)}[\Phi_{U,I}^{(0)}, f](x, y)_J := \frac{1}{(2\pi \varepsilon)|J|^{1/2} \sqrt{\det \text{Hess} \Phi_{U,I}^{(0)}((x, y)_I)}} \int \left( \prod_{j \in J \setminus I} dx_j \right) \left( \prod_{i \in I \setminus J} dy_i \right) f((x, y)_I) \times \exp \left\{ \frac{i}{\varepsilon} \left[ \Phi_{U,I}^{(0)}((x, y)_I) - \sum_{i \in I \setminus J} y_i x_i + \sum_{j \in J \setminus I} y_j x_j \right] \right\},
\]
where $|\cdot|$ denotes the cardinality of a set. This integral admits an asymptotic expansion as $\varepsilon \to 0$ (the stationary phase method, see [5]):

$$ Q_{\varepsilon}^{(\varepsilon)}(x,y) \simeq \exp \left( -\frac{i}{\varepsilon} \Phi_{\varepsilon}(x,y) \right) Q_{\varepsilon}^{(\varepsilon)}(x,y) $$

$$ 	imes \exp \left( \frac{\pi}{4} \mu_{\varepsilon} \right) \sum_{n=0}^{\infty} (-i\varepsilon)^n \hat{\gamma}_{\varepsilon}^{(n)} \left[ \Phi_{\varepsilon}(x,y), \left( \tau_{\varepsilon}^{W,U}(x,y) \right) \right], $$

where the map $\tau_{\varepsilon}^{W,U} : \pi_I(W \cap U) \to \pi_I(W \cap U)$ is the gluing map between the focal charts $(U, I)$ and $(W, J)$ on $\Lambda^+$ defined by $\tau_{\varepsilon}^{W,U}((x(\alpha), y(\alpha))) = (x(\alpha), y(\alpha)), I$, for $\alpha \in W \cap U$, $\mu^{W,U}_{\varepsilon} \in \mathbb{Z}/4\mathbb{Z}$ is a constant (related to the Maslov index), and $\hat{\gamma}_{\varepsilon}^{(n)}[\Phi_{\varepsilon}, -]$ are linear partial differential operators, such that, for each $n$, $\hat{\gamma}_{\varepsilon}^{(n)}[\Phi_{\varepsilon}, -]$ depends only on a finite number $M_n$ of partial derivatives of $\Phi_{\varepsilon}$ in the point $(x,y) \in \pi_I(W \cap U)$. The number $M_n$ grows as $n \to \infty$, and in the leading order $n = 0$ the operator $\hat{\gamma}_{\varepsilon}^{(0)}[\Phi_{\varepsilon}, -]$ is just a multiplication over a smooth function, which does not vanish in no point of $\pi_I(W \cap U)$. Fix the choice of $\mu_{\varepsilon}^{W,U}$ in such a way that this function is positive (this is always possible).

Observe, that if we take instead of $f((x,y),I)$ a function $f^{(\varepsilon)}((x,y),I)$ depending on the small parameter $\varepsilon$, such that there is an asymptotic expansion $f^{(\varepsilon)}((x,y),I) \simeq \sum_{m=0}^{\infty} (-\varepsilon)^m f_m((x,y),I)$, where $f_m \in C^\infty_{C,0}(\pi_I(W \cap U))$, $m = 0, 1, 2, \ldots$, then, using the linearity of the operators $\hat{\gamma}_{\varepsilon}^{(n)}[\Phi_{\varepsilon}, -]$, $n = 0, 1, 2, \ldots$, we obtain:

$$ \exp \left( -\frac{i}{\varepsilon} \Phi_{\varepsilon}(x,y) \right) Q_{\varepsilon}^{(\varepsilon)}(x,y) \simeq \sum_{n=0}^{\infty} (-\varepsilon)^n \sum_{m=0}^{n} \hat{\gamma}_{\varepsilon}^{(n-m)}[\Phi_{\varepsilon}, f_m] \left( \tau_{\varepsilon}^{W,U}(x,y) \right), \quad (12) $$

for $(x,y) \in \pi_I(W \cap U)$. Now, let us assume that $f^{(\varepsilon)}$ can be written in the form $f^{(\varepsilon)} = \exp \{ g^{(\varepsilon)} \}$, where $g^{(\varepsilon)}$ is a convergent power series $g^{(\varepsilon)} = \sum_{m=0}^{\infty} (-\varepsilon)^m g_m/m!$, where $g_m \in C^\infty_{C,0}(\pi_I(W \cap U))$, $m = 0, 1, 2, \ldots$. Assume also that the left-hand side of (12) (denote it at this moment $F^{(\varepsilon)}((x,y)),I)$ can also be represented as $F^{(\varepsilon)}((x,y)),I) = \exp \{ G^{(\varepsilon)}(\tau^{W,U}_I((x,y))) \}$, where $G^{(\varepsilon)}((x,y),I)$ is a convergent power series $G^{(\varepsilon)} = \sum_{m=0}^{\infty} (-\varepsilon)^m G_m/m!$, with $G_m \in C^\infty_{C,0}(\pi_I(W \cap U))$, $m = 0, 1, 2, \ldots$ (at least this does not contradict a possibility of having an asymptotic expansion like on the right-hand side of (12)). The link between the coefficients $\{G_m\}_{m=0}^{\infty}$ and $\{g_m\}_{m=0}^{\infty}$ defines some operators, that we are going to use to describe the link between the asymptotic power series (10) corresponding to different $(U, I)$.

Take a pair of smooth functions $a(\xi)$ and $b(\xi)$, $\xi$ varies over $\mathbb{R}$, $a(\xi) = \exp(b(\xi))$. Look at the Taylor’s expansions $a(\xi) = \sum_{m=0}^{\infty} a_m \xi^m/m!$ and $b(\xi) = \sum_{n=0}^{\infty} b_n \xi^n/n!$. For the leading coefficients, $a_0 = \exp(b_0)$. If $m \geq 1$, then

$$ a_m = \left. \left\{ \left( \frac{\partial}{\partial \xi} \right)^m \exp(b(\xi)) \right\} \right|_{\xi=0} = \exp(b_0) \left\{ \left( \frac{\partial}{\partial \xi} \right)^m \sum_{p=0}^{\infty} \frac{1}{p!} [b(\xi) - b_0]^p \right\} \left|_{\xi=0} = \right.$$
\[ = e^{b_0} \sum_{p=1}^{m} \frac{1}{p!} \sum_{I_1, I_2, \ldots, I_p} b_{I_1} b_{I_2} \ldots b_{I_p}. \]

Therefore, \( a_m = A_m(b_0, b_1, b_2, \ldots, b_m) \), where
\[
A_m(b_0, b_1, b_2, \ldots, b_m) := e^{b_0} \sum_{p=1}^{m} \frac{1}{p!} \sum_{m_1, m_2, \ldots, m_p=1}^{m} m! \frac{1}{m_1! m_2! \ldots m_p!} b_{m_1} b_{m_2} \ldots b_{m_p},
\]
for every \( m = 1, 2, \ldots. \) To invert these formulae, observe that \( b_0 = \ln a_0, a_0 > 0. \)

For \( m \geq 1, \) we have
\[
b_m = \left\{ \left( \frac{\partial}{\partial \xi} \right)^m \ln(a(\xi)/a_0) \right\}_{\xi=0} = \left\{ \left( \frac{\partial}{\partial \xi} \right)^m \sum_{q=1}^{\infty} \frac{(-1)^q}{q} [a(\xi)/a_0 - 1]^q \right\}_{\xi=0} = \sum_{q=1}^{m} \frac{(-1)^q}{qa_0^q} \sum_{J_1, J_2, \ldots, J_q \subset [q], |J_1| = 0, \ldots, J_1, J_2, \ldots, J_q \neq \emptyset, |J_1| + |J_2| + \ldots + |J_q| = m} a_{J_1} a_{J_2} \ldots a_{J_q}.
\]

Therefore \( b_m = B_m(a_0, a_1, \ldots, a_m) \), where
\[
B_m(a_0, a_1, \ldots, a_m) := \sum_{q=1}^{m} \frac{(-1)^q}{qa_0^q} \sum_{m_1, m_2, \ldots, m_q=1}^{m} m! \frac{1}{m_1! m_2! \ldots m_q!} a_{m_1} a_{m_2} \ldots a_{m_q},
\]
for every \( m = 1, 2, \ldots. \) Extending naturally the notation as
\[
A_0(b_0) := \exp(b_0), \quad B_0(a_0) := \ln(a_0),
\]
we arrive at \( a_n = A_n(b_0, b_1, \ldots, b_n) \) and \( b_n = B_n(a_0, a_1, \ldots, a_n) \) for all \( n = 0, 1, 2, \ldots. \)

We can now describe how the expansions (10) corresponding to different focal charts \((U, I)\) on \( \Lambda^+ \) must be linked. Take a pair of focal charts \((U, I)\) and \((W, J)\), \( W \cap U \neq \emptyset \), and look at the expansions for \( S^{(\alpha)}_{U,I}(\alpha) \) and \( S^{(\alpha)}_{W,J}(\alpha) \) in terms of the local coordinates:
\[
\tilde{S}^{(\alpha)}_{U,I}(x, y) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \Phi^{(l)}_{U,I}(x, y), \quad \tilde{S}^{(\alpha)}_{W,J}(x, y) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \Phi^{(l)}_{W,J}(x, y),
\]
where \((x, y)\) corresponds to a point \( \alpha \) varying over \( W \cap U \), and \( S^{(\alpha)}_{U,I}(x, y) := S^{(\alpha)}_{U,I}(\tilde{\pi}^{-1}_I((x, y)_I)) \) for \((x, y)_I \in \tilde{\pi}_I(U)\), and \( S^{(\alpha)}_{W,J}(x, y) := S^{(\alpha)}_{W,J}(\tilde{\pi}^{-1}_J((x, y)_J)) \), for \((x, y)_J \in \tilde{\pi}_J(W)\). To compute the collection of coefficients \( \{ \Phi^{(l)}_{W,J}((x, y)_J) \}_{l=0}^{\infty} \), \((x, y)_J \in \tilde{\pi}_J(W \cap U)\), from a collection of coefficients \( \{ \Phi^{(l)}_{U,I}((x, y)_I) \}_{l=0}^{\infty} \), one needs to do the following:

- Compute \( \Phi^{(0)}_{W,J}((x, y)_J) \) as a Legendre transform (11) of \( \Phi^{(0)}_{U,I}((x, y)_I) \).
- Set \( b_m = \Phi^{(m+1)}_{U,I}((x, y)_I), m = 0, 1, 2, \ldots, \) and compute the corresponding \( a_n = A_n(b_0, b_1, \ldots, b_n), n = 0, 1, 2, \ldots, \) where \( A_n \) is defined in (13) and (15).
• Compute the coefficients of the asymptotic expansion on the right-hand side of (12), replacing \( \{f_m\}_{m=0}^{\infty} \) with \( \{a_m\}_{m=0}^{\infty}, a_m = a_m((x,y)_I) \):

\[
a'_n((x,y)_I) := \sum_{m=0}^{n} \hat{V}^{(n-m)}_{W \cap U,l,l,I}(\Phi^{(0)}_{U,I,a_m(\pi_{l,l}^{W \cap U}((x,y)_I))},
\]

for \( n = 0, 1, 2, \ldots \).

• Compute \( b'_n := B_n(a'_1, a'_2, \ldots, a'_n), n = 0, 1, 2, \ldots \), where \( B_n \) is defined in (13) and (15). This collection of functions \( b'_n = b'_n((x,y)_I) \) is precisely the required collection of the higher order coefficients,

\[
\Phi^{(n+1)}_{W,J}((x,y)_I) = b'_n((x,y)_I),
\]

where \( n = 0, 1, 2, \ldots \), and \((x,y)_I\) corresponds to a point in \( W \cap U \).

It can be of interest to assemble the higher order coefficients \( \Phi^{(l)}_{U,I}((x,y)_I), l = 1, 2, 3, \ldots \), into a formal power series

\[
\Phi^{(\varepsilon)}_{U,I}((x,y)_I) := \sum_{m=0}^{\infty} \varepsilon^m \Phi^{(m+1)}_{U,I}((x,y)_I),
\]

where \( \varepsilon \) is a formal variable, \( \Phi^{(\varepsilon)}_{U,I}((x,y)_I) \in C^\infty(\hat{\pi}_I(U))[[\varepsilon]] \). Note, that the coefficient \( \Phi^{(0)}_{U,I}((x,y)_I) \) is not involved in this definition. If \( U \) admits another type of focal coordinates \( J \subset [d + 1] \), then the described link between the coefficients in different focal charts yields a nonlinear map

\[
\mathcal{N}^U_{I,J} : C^\infty(\hat{\pi}_I(U))[[\varepsilon]] \rightarrow C^\infty(\hat{\pi}_J(U))[[\varepsilon]].
\]

We have \( \mathcal{N}^U_{I,I} = id \), and, in case \( U \) admits the focal coordinates of types \( I, J, \) and \( K \), one can check the cocyclicity condition:

\[
\mathcal{N}^U_{I,K} \circ \mathcal{N}^U_{K,J} \circ \mathcal{N}^U_{I,I} = id. \tag{16}
\]

Note, that about \( U \) we assume that the Hess's matrices Hess\( S_I \) and Hess\( S_J \) are non-degenerate. Intuitively, the map \( \mathcal{N}^U_{I,J} \) “extends” the Legendre transform \( \mathcal{L}^U_{I,J} \), and the cocycle condition (16) corresponds to the cocycle condition (9).

Where do the functions \( S^{(\lambda)}_{U,I}(\alpha) \) come from? Consider again a phenomenological thermodynamic system with extensive coordinates \( (x_1, x_2, x_3) = (E, V, \nu) \) (internal energy, volume, number of moles), and intensive coordinates \( (y_1, y_2, y_3) = (\beta, \beta p, -\beta \mu) \) (\( \beta \) is the inverse absolute temperature, \( p \) is pressure, \( \mu \) is chemical potential). If we know the entropy of the system as a function of extensive coordinates, \( S = S(E, V, \nu) \), then the first law of thermodynamics can be expressed as follows:

\[
dS = \beta dE + (\beta p) dV + (-\beta \mu) d\nu = \sum_{i=1}^{3} y_i dx_i, \tag{17}
\]

on the Lagrangian manifold \( \Lambda^+ \). If \( I = \{1\} \) and \( U \subset \Lambda^+ \) is fixed, then the limit \( S_I(\alpha) := \lim_{\lambda \to \infty} S^{(\lambda)}_{U,I}(\alpha) \) corresponds to the Legendre transform of \( S \) in the first argument, \( dS_I = (-Ed\beta + (\beta p) dV + (-\beta \mu) d\nu)|_U \), i.e. \( S_I = -\beta F \), where \( F \) is the
phenomenological free energy described as a function on $U \subset \Lambda^+$. If one takes the canonical Gibbs distribution

$$\tilde{w}_{k_B,N}^{(L)}(\beta; g) := \frac{1}{Z_{k_B,N}^{(L)}(\beta; g)} \exp \left( -\frac{\beta \tilde{H}_N^{(L)}(g)}{k_B} \right),$$

then one can see that the averages $\langle \tilde{H}_N^{(L)} \rangle$ and $\langle (\tilde{H}_N^{(L)} - \langle \tilde{H}_N^{(L)} \rangle)^2 \rangle$ over $\tilde{w}_{k_B,N}^{(L)}(\beta; g)$ correspond to the derivatives $(-k_B \partial / \partial \beta)$ and $(-k_B \partial / \partial \beta)^2$ of $\ln Z_{k_B,N}^{(L)}(\beta; g)$. Substituting the values $\beta = \beta_\lambda(\alpha), L = (V_\lambda(\alpha))^{1/n}, N = N_\lambda(\alpha) = \nu_\lambda(\alpha) R / k_B$, and $g = g_\lambda(\alpha)$, corresponding to the rescaled system (the rescaling coefficient $\lambda$), one extends this fact as follows: the quantities

$$C_n^{(\lambda)}(\alpha) := \left( -k_B \frac{\partial}{\partial \beta} \right)^n \frac{\lambda S_1^{(\lambda)}(\tilde{\pi}_1^{-1}(\beta, V, \nu))}{k_B} \bigg|_{(\beta, V, \nu) = (\beta_\lambda(\alpha), V_\lambda(\alpha), \nu_\lambda(\alpha))}$$

define the *cumulants* of the fluctuations of energy, $n = 1, 2, 3, \ldots$, where $\tilde{\pi}_1 : \alpha \mapsto (\beta(\alpha), V(\alpha), \nu(\alpha))$ is the canonical projection. Observe, that $C_1^{(\lambda)}(\alpha) = O(\lambda), \lambda \to \infty$, which is the *expectation value* of internal energy, and $C_2^{(\lambda)}(\alpha) = O(\lambda)$, the *variance* of the corresponding fluctuations, so the standard deviation is $O(\sqrt{\lambda})$, as it should be in quasithermodynamics. The inverse absolute temperature $\beta$ is fixed by the thermostat, and the canonically conjugate *extensive* quantity (the internal energy) fluctuates. This interpretation is naturally dualized: if an extensive quantity is fixed (for example, the volume $V$), then the canonically conjugate intensive quantity fluctuates. From the first law of thermodynamics [17], we can see, for example, that the derivatives over $V$ should define the cumulants $\tilde{C}_m^{(\lambda)}(\alpha)$ of the fluctuations of $\beta p$, where $p$ is the pressure in the system. More precisely, the derivative $\partial / \partial \beta$ should be replaced with $\lambda^{-1} \partial / \partial V$, since $V_\lambda(\alpha) = O(\lambda)$ (extensive variable), and $\beta_\lambda(\alpha) = O(1)$ (intensive variable), $\lambda \to \infty$, and we should take into account the minus sign corresponding to the Legendre transform:

$$\tilde{C}_m^{(\lambda)}(\alpha) = \left( \frac{\partial}{\partial V} \right)^m \frac{\lambda S_1^{(\lambda)}(\tilde{\pi}_1^{-1}(\beta, V, \nu))}{k_B} \bigg|_{(\beta, V, \nu) = (\beta_\lambda(\alpha), V_\lambda(\alpha), \nu_\lambda(\alpha))},$$

for $m = 1, 2, 3, \ldots$. In particular, the first two cumulants for the fluctuations of $\beta p$ (the expectation value and the variance) satisfy $C_1^{(\lambda)}(\alpha) = O(1)$ and $C_2^{(\lambda)}(\alpha) = O(\lambda^{-1}), \lambda \to \infty$, so the standard deviation corresponding to $\beta p$ is $O(1/\sqrt{\lambda})$, as it should be in quasithermodynamics.

For an abstract phenomenological thermodynamic system $\Lambda^+ \subset \mathbb{R}^{2(d+1)}(x, y)$ the interpretation of the functions $S_{U,I}^{(\lambda)}(\alpha)$ for every focal chart $(U, I)$ is as follows. We know that $[d+1] \setminus I$ cannot in thermodynamics be empty (by construction).

- The first law of thermodynamics is expressed over $(U, I)$ as follows:

$$d \varphi_{U,I}^{(0)} = \left( -\sum_{i \in I} x_i dy_i + \sum_{j \in [d+1] \setminus I} y_j dx_j \right)_{|U},$$

where $\varphi_{U,I}^{(0)}(\alpha) := \lim_{\lambda \to \infty} S_{U,I}^{(\lambda)}(\alpha), \alpha \in U \subset \Lambda^+, x = (x_1, x_2, \ldots, x_{d+1})$ are the extensive coordinates, and $y = (y_1, y_2, \ldots, y_{d+1})$ are the canonically conjugate intensive coordinates. A model example is $d = 2, (x_1, x_2, x_3) = (E, V, \nu)$
(internal energy, volume, number of moles), \((y_1, y_2, y_3) = (\beta, \beta p, -\beta \mu)\) (inverse absolute temperature \(\beta\), pressure \(p\), chemical potential \(\mu\))

- The units of measurement of \(S_{U,I}^{(\lambda)}(\alpha)\) are same as the units of measurement of the Boltzmann constant \(k_B\), \([S_{U,I}^{(\lambda)}(\alpha)] = [k_B]\). The function \(\varphi_{U,\emptyset}^{(0)}(\alpha)\) corresponding to the case \(I = \emptyset\) is the entropy of the phenomenological system in the state \(\alpha \in U \subset \Lambda^+\).

- The following collection of derivatives is interpreted as the cumulants describing the fluctuations of the vector \((x, y)_{[d+1] \setminus I}\) over a state \(\alpha \in U\):

\[
C_M^{(U,I,\lambda)}(\alpha) \mid_{\alpha = \pi^{-1}_{U,I}((x,y)_I)} := \left( \prod_{i \in I} \left( -k_B \frac{\partial}{\partial y_i} \right)^{m_i} \right) \left( \prod_{j \in [d+1] \setminus I} \left( k_B \lambda^{-1} \frac{\partial}{\partial x_j} \right)^{m_j} \right) \frac{\lambda S_{U,I}^{(\lambda)}(\pi^{-1}_{U,I}(x,y)_I)}{k_B},
\]

where \(M = (m_1, m_2, \ldots, m_{d+1}) \in \mathbb{Z}_{>0}^{d+1}\) is an integer multi-index, \(M \neq (0, 0, \ldots, 0)\), and \(\pi_{U,I} : U \ni \alpha \mapsto (x(\alpha), y(\alpha))_I\) denotes the canonical projection, \((x, y)_I \in \pi_I(U)\). The cumulants corresponding to the same \(M \in \mathbb{Z}_{>0}^{d+1}\), but computed over two different focal charts \((U, I)\) and \((W, J)\), (i.e. different families of rescaled systems) coincide asymptotically in the intersection:

\[
C_M^{(U,I,\lambda)}(\alpha) = C_M^{(W,J,\lambda)}(\alpha) + O(\lambda^{-\infty}),
\]

as \(\lambda \to \infty\), where \(\alpha \in U \cap W\).

The cocycle conditions \([12]\) basically say that one can glue a Lagrangian manifold \(\Lambda^+\). Informally speaking, phenomenological thermodynamics IS a Lagrangian manifold \(\Lambda^+\) (connected, simply connected and admitting projectivization with respect to extensive coordinates). The statistical thermodynamics IS the Gibbs distribution corresponding to a focal chart \((U, I)\) on \(\Lambda^+\), \(I \subset [d+1]\), \(I \neq [d+1]\), for example, the canonical distribution \([18]\) if \(I = \{i_0\}\) is a singleton and \(i_0\) corresponds to the inverse absolute temperature. The type \(I\) of the focal chart describes the way we “extract” the system from the outside world (fixed adiabatic walls, fixed heat-conducting walls, a non-fixed piston, walls penetrable to certain sorts of particles, etc.) This way of extraction does matter if we consider the Gibbs distribution along the families of rescaled systems by a parameter \(\lambda \to \infty\). If one is ready to sacrifice the precision modulo \(O(\lambda^{-\infty})\) describing the fluctuations of the intensive and extensive quantities, then one arrives at “quasithermodynamics”. One may say that quasithermodynamics IS a global section of the sheaf on \(\Lambda^+\) corresponding to the cocycle conditions \([16]\) (the formal variable \(\varepsilon\) corresponds to the small parameter \(\lambda^{-1}\), but should not be confused with it). This sheaf corresponds to the sheaf of \(V\)-objects in \([12]\), and the analogy with mechanics is as follows \([3]\): phenomenological thermodynamics corresponds to classical mechanics, statistical thermodynamics corresponds to quantum mechanics, and quasithermodynamics corresponds to semiclassical mechanics.

3. Nonequilibrium quasithermodynamics

We are now interested in extending the quasithermodynamics picture to a nonequilibrium setting. One may use the concept of relevant ensembles for this purpose \([23]\). Like in the previous section, to keep the story more simple, consider a multi-particle
quantum mechanical system on a $n$-dimensional torus of radius $L$. Denote $\mathcal{F}$ the Fock space of the system, and $\mathcal{H}$ the Hamiltonian of the system (a self-adjoint operator on $\mathcal{F}$). Fix a collection of observables $\hat{E}_1^{(L)}(g), \hat{E}_2^{(L)}(g), \ldots, \hat{E}_d^{(L)}(g)$ (self-adjoint operators on the Fock space of the system), depending on a collection of parameters $g = (g_1, g_2, \ldots, g_m)$. One considers the Gibbs's statistical operator

$$\hat{w}_{kB}^{(L)}(\beta; g) := \frac{1}{Z_{kB}^{(L)}(\beta; g)} \exp \left( - \frac{1}{k_B} \sum_{i=1}^d \beta_i \hat{E}_i^{(L)}(g) \right),$$

(19)

where $Z_{kB}^{(L)}(\beta; g) := \text{Tr} \left( - \sum_{i=1}^d \beta_i \hat{E}_i^{(L)}(g)/k_B \right)$ (we assume that the trace exists), $k_B$ is the Boltzmann constant, and $\beta = (\beta_1, \beta_2, \ldots, \beta_d)$. Set

$$\Phi_{kB}^{(L)}(\beta; g) := k_B \ln Z_{kB}^{(L)}(\beta; g),$$

The general idea is that the evolution of the system is well approximated by a family of operators $\{\hat{w}_{kB}^{(L)}(\beta', g')\}_t$, where $t$ is time, and $L', \beta', g'$ are certain functions (we let $L$ and $g$ change with time for more generality). One says that the state $\text{Tr}\{\hat{w}_{kB}^{(L)}(\beta'; g') - \}$ defines a relevant ensemble at the moment of time $t$.

Let us look at the collection of all possible relevant ensembles, i.e. all possible statistical operators $\hat{w}_{kB}^{(L)}(\beta; g)$. To mimic the setup of the equilibrium thermodynamics, consider a phase space $\mathbb{R}^{2(d+1)}$ with symplectic structure $\omega^+ := \sum_{i=1}^{d+1} dy_i \wedge dx_i$, $x = (x_1, x_2, \ldots, x_{d+1}), y = (y_1, y_2, \ldots, y_{d+1})$. The first $d$ coordinates $x_i$ correspond to $\hat{E}_i^{(L)}(g)$, $i \leq d$, and the last coordinate $x_{d+1}$ corresponds to the volume $V = L^n$. The first $d$ coordinates $y_i$ correspond to $\beta_i, i \leq d$, and the last coordinate $y_{d+1}$ can be perceived as a non-equilibrium analogue of the “pressure over temperature” variable in the equilibrium thermodynamics. The coordinates $x$ are termed the extensive coordinates, and the coordinates $y$ are termed the intensive coordinates. Assume that there exists a Lagrangian manifold $\Lambda^+ \subset \mathbb{R}^{2(d+1)}$, which is connected, simply connected, admits projectivization with respect to the extensive coordinates (i.e. the analogue of the Lagrangian manifold in phenomenological equilibrium thermodynamics). Assume, for simplicity, that $\Lambda^+$ admits $(y_1, y_2, \ldots, y_{d}, x_{d+1})$ as global coordinates (i.e. is covered by a single focal chart of type $[d] \subset [d+1]$). Let $\lambda \to \infty$ be a large parameter (termed the rescaling parameter), and assume that there exist functions $C(\lambda)$ and $\beta_\lambda(\alpha), g_\lambda(\alpha)$, where $\alpha$ varies over $\Lambda^+$, such that

$$\lambda^{-1} \Phi_{kB}^{((\lambda x_{d+1}(\alpha))^{1/\alpha})}(\beta_\lambda(\alpha); g_\lambda(\alpha)) - C(\lambda) \simeq \sum_{l=0}^{\infty} \lambda^{-l} \phi_{kB}^{(l)}(\alpha),$$

(20)

where $\phi_{kB}^{(l)}(\alpha)$ are smooth functions, $l = 0, 1, 2, \ldots$. This expansion should be perceived in analogy with (10), and $V_\lambda(\alpha) = \lambda x_{d+1}(\alpha)$ corresponds to the expanding volume as $\lambda \to \infty$.

Like in the equilibrium case, we should draw a distinction between phenomenological, statistical, and “quasi” nonequilibrium thermodynamics. The general idea is that the phenomenological nonequilibrium thermodynamics is described as a curve $\gamma = \{\alpha^t\}_t$ on the Lagrangian manifold $\Lambda^+$, where $t$ is time. If this is a curve of relaxation to equilibrium, then some points of $\Lambda^+$ should correspond to the equilibrium states. As a model example, one may think of a system consisting of two
ensembles. These collections of cumulants stem from the temperature). This is, of course, an approximation, depending, essentially, on the smallness of time memory effects [23], one can derive the following system of equations for $E_i(t) := x_i(\alpha^i), i = 1, 2, \ldots, d$: 

$$\frac{\partial E_i(t)}{\partial t} = J_i(\alpha^i) + \sum_{j=1}^n L_{i,j}(\alpha^i) \beta_j(t), \quad (21)$$

where $\beta_i(t) = y_i(\alpha^i), i \in [d], L_{i,j}(\alpha)$ are the Onsager coefficients corresponding to $\alpha \in \Lambda^+$ (in practice, they are often approximated as constants), and $J_i(\alpha^i), i \in [d]$, are the flows induced by $\{\beta_j(t)\}_{j \in [d]}$. These flows can be perceived, for each $i \in [d]$, as a limit

$$J_i(\alpha) := \lim_{\lambda \to \infty} \left( \lambda^{-1} \text{Tr} \left\{ \hat{\omega}_{\text{kb}}^{(L)}(\beta; g) \hat{\mathcal{J}}_i^{(L)}(g) \right\} \right|_{L=(\lambda x_{d+1}(\alpha))^{1/n}, g=\bar{g}_\lambda(\alpha)}, \quad (22)$$

where $\hat{\mathcal{J}}_i^{(L)}(g)$ are the operators of flows,

$$\hat{\mathcal{J}}_i^{(L)}(g) := \frac{i}{\hbar} \left[ \hat{\mathcal{H}}^{(L)}(g), \hat{\mathcal{E}}_i^{(L)}(g) \right],$$

the square bracket denotes a commutator, and $\hbar$ is the Planck’s constant.

Denote the left-hand side of (20) as $\mathcal{S}_\text{kb}^{(\lambda)}(\alpha)$, and let $\mathcal{S}_\text{kb}^{(\lambda)}(y_1, y_2, \ldots, y_d, x_{d+1})$ be this function expressed in terms of the focal coordinates on $\Lambda^+$ of type $[d] \subset [d+1]$, 

$$\mathcal{S}_\text{kb}^{(\lambda)}(y_1(\alpha), y_2(\alpha), \ldots, y_d(\alpha), x_{d+1}(\alpha)) = \mathcal{S}_\text{kb}^{(\lambda)}(\alpha),$$

for $\alpha \in \Lambda^+$. In analogy with the previous section, one interprets the derivatives

$$C_M^{(\lambda)}(\alpha) := \left( \prod_{i=1}^d \left(-k_B \frac{\partial}{\partial y_i}\right)^{m_i} \right) \left(k_B \lambda^{-1} \frac{\partial}{\partial x_{d+1}} \right)^{m_{d+1}} \mathcal{S}_\text{kb}^{(\lambda)}(y_1, \ldots, y_d, x_{d+1}) \right|_{\Lambda^+}, \quad (24)$$

where $\alpha \in \Lambda^+ \subset \mathbb{R}^{2(d+1)}$, $M = (m_1, \ldots, m_d, m_{d+1}) \in \mathbb{Z}_{\geq 0}^{d+1}, M \neq (0, 0, \ldots, 0)$, as the cumulants of fluctuations of $(x_1, \ldots, x_d, y_{d+1})$. The Lagrangian manifold $\Lambda^+$ is recovered from the leading term $\Phi^{(0)}_{\text{kb}}(y_1, \ldots, y_d, x_{d+1}) := \lim_{\lambda \to \infty} \mathcal{S}_\text{kb}^{(\lambda)}(y_1, \ldots, y_d, x_{d+1})$ of the asymptotic expansion in $\lambda^{-1}$ as a system of equations:

$$x_i = -\frac{\partial \Phi^{(0)}_{\text{kb}}(y_1, \ldots, y_d, x_{d+1})}{\partial y_i}, \quad y_{d+1} = \frac{1}{x_{d+1}} \Phi^{(0)}_{\text{kb}}(y_1, \ldots, y_d, x_{d+1}),$$

where $i = 1, 2, \ldots, d$.

It follows that in every point $\alpha^t$ of the curve of the evolution $\gamma = \{\alpha^t\}_t$, we have a collection of cumulants $\{C_M^{(\lambda)}(\alpha^t)\}_M$ describing the fluctuations of the measured values of the quantities associated with $\hat{\mathcal{E}}_1^{(L)}(g), \hat{\mathcal{E}}_2^{(L)}(g), \ldots, \hat{\mathcal{E}}_d^{(L)}(g)$ and $\bar{p}$, where $\bar{p}$ is the intensive quantity dual to the volume (the nonequilibrium “pressure over temperature”). This is, of course, an approximation, depending, essentially, on a successful choice of the basis variables $\hat{\mathcal{E}}_i^{(L)}(g), i \in [d]$, for the family of relevant ensembles. These collections of cumulants stem from the same Lagrangian manifold.
\(\Lambda^+\), but, in fact, one can be more general, and consider germs of different Lagrangian manifolds “attached” to each point \(\alpha^i \in \gamma \subset \Lambda^+ \subset \mathbb{R}^{2(d+1)}_x\). To see this, one needs to consider the generalized Fokker-Planck equation \([24]\). The idea of the derivation of this equation is totally similar to the one for \([21]\), except that instead of a finite collection of basis quantities \(\{\check{E}_i(L)(g)\}_{i \in [d]}\), one considers a continuous collection \(\{L^n \delta_\varepsilon(a_i - \check{E}_i(L)(g)) \}_{a_i \in \mathbb{R}^d}\), where \(\delta_\varepsilon(\cdot)\) is a smooth approximation of the Dirac’s delta, \(\delta_\varepsilon(\cdot) \to \delta(\cdot)\) as \(\varepsilon \to 0\) (in the weak sense), \(\int da \delta_\varepsilon(a) = 1\), \(a = (a_1, a_2, \ldots, a_d)\). Of course, a special care needs to be taken about treating a delta function of an operator, and also there is a technical problem related to the noncommutativity of the operators \(\check{E}_i\), \(i \in [d]\). Let us assume, for simplicity, that \(\check{E}_1(L)(g), \check{E}_2(L)(g), \ldots, \check{E}_d(L)(g)\) mutually commute (the noncommutativity can be handled as well, see, for example, \([16]\)). Then, for every \(a = (a_1, a_2, \ldots, a_d)\), we can just write

\[
\check{F}_\varepsilon(L)(a; g) := L^n \prod_{i=1}^d \delta_\varepsilon(a_i - \check{E}_i(L)(g)),
\]

without paying attention to the order of operators. Assume that the corresponding operators of flows \(\check{F}_1(L)(g), \check{F}_2(L)(g), \ldots, \check{F}_d(L)(g)\) defined in \([23]\) mutually commute as well.

In place of \(\{\check{E}_i(L)(g)\}_{i \in [d]}\), we have a collection \(\{\check{F}_\varepsilon(L)(a; g)\}_{a \in \mathbb{R}^d}\), but with an additional condition:

\[
\int da \check{F}_\varepsilon(L)(a; g) = L^n.
\]

One must also point out, that all \(\check{F}_\varepsilon(L)(a; g), a \in \mathbb{R}^d\), have the same units of measurement. In this sense, \(\{\check{F}_\varepsilon(L)(a; g)\}_{a \in \mathbb{R}^d}\) corresponds to \(\{\check{E}_i(L)(g)\}_{i \in [d]}\), where \(\check{E}_0(L)(g) := L^n - \sum_{i=1}^d \check{E}_i(L)(g)\), if we choose all \(\check{E}_i(L)(g)\), \(i \in [d]\), having the same units of measurement as the volume \(L^n\), so that we have a right to add them. The Gibbs’s statistical operator \([19]\) can be expressed as follows:

\[
\tilde{\omega}_{kB}^{(L)}(\beta; g) = \exp \left\{ - \frac{1}{k_B} \left[ \Phi_{kB}^{(L)}(\beta; g) \check{E}_0(L)(g) + \sum_{i=1}^d \left( \beta_i + \frac{\Phi_{kB}^{(L)}(\beta; g)}{L^n} \right) \check{E}_i(L)(g) \right] \right\}.
\]

Denote

\[
\check{W}_{kB}^{(L)}(\eta; g) := \exp \left( - \frac{1}{k_B} \sum_{i=0}^d \eta_i \check{E}_i(L)(g) \right),
\]

where \(\eta = (\eta_0, \eta_1, \ldots, \eta_d)\) is a collection of parameters. Observe that the trace of this operator is equal to one if we substitute \(\eta_0 = L^{-n} \Phi_{kB}^{(L)}(\beta; g)\), and \(\eta_i = \beta_i + L^{-n} \Phi_{kB}^{(L)}(\beta; g), i \in [d]\). One can also check, that

\[
\left( \sum_{i=0}^d \frac{\partial}{\partial \eta_i} \right) \ln \text{Tr}(\check{W}_{kB}^{(L)}(\eta; g)) = L^n,
\]

for any \(\eta = (\eta_0, \eta_1, \ldots, \eta_d)\). Consider a phase space with coordinates \(\mathbb{R}^{2(d+1)}_x\) with coordinates \(\eta = (\eta_0, \eta_1, \ldots, \eta_d)\), and \(\xi = (\xi_0, \xi_1, \ldots, \xi_d)\), and symplectic structure \(\check{\omega} := \sum_{i=0}^d d\eta_i \wedge d\xi_i\). The map \(\tau : \mathbb{R}^{2(d+1)}_x \to \mathbb{R}^{2(d+1)}_x, (\xi, \eta) \mapsto (x, y)\), defined by
\(x_i = \xi_i, y_i = \eta_i - \eta_0,\) for \(0 \leq i \leq d,\) and \(x_{d+1} = \xi_0 + \sum_{i=1}^{d} \xi_i,\) \(y_{d+1} = \eta_0,\) is symplectic. Therefore, in place of the Lagrangian manifold \(\Lambda^+ \subset \mathbb{R}^{(2d+1)}\) one can work in terms of the manifold \(\tau^{-1}(\Lambda^+) \subset \mathbb{R}^{2(d+1)}\), which is also Lagrangian. If one fixes the collection of parameters \(g\) and considers a Lagrangian manifold \(\Lambda^+_g\) defined by equations

\[
\begin{align*}
(x_i - \text{Tr}(\widehat{\mathcal{E}}_i^{(L)}(g) \widehat{\mathcal{W}}_{k_B}(\beta; g)))|_{\beta=(y_1, \ldots, y_d), L=x_{d+1}^{1/n}} & = 0, \\
(y_{d+1} - \frac{\partial}{\partial x_{d+1}} k_B \ln Z_{k_B}^{(x_{d+1})^{1/n}}(\beta; g))|_{\beta=(y_1, \ldots, y_d)} & = 0,
\end{align*}
\]

where \(i = 1, 2, \ldots, d\) (i.e. the generating function is \(k_B \ln Z_{k_B}^{(x_{d+1})^{1/n}}((y_1, \ldots, y_d); g)\) and the focal coordinates are \((y_1, \ldots, y_d, x_{d+1})\)), then for the manifold \(\tau^{-1}(\Lambda^+_g)\) one can claim that

\[
\left(\xi_i + k_B \frac{\partial}{\partial \eta_i} \text{Tr}\widehat{W}_{k_B}^{(L)}(\eta; g)|_{L=(\sum_{j=0}^{d} \xi_j)^{1/n}}\right)|_{\tau^{-1}(\Lambda^+_g)} = 0,
\]

for \(i = 0, 1, \ldots, d\). These \(d + 1\) equations do not characterize \(\tau^{-1}(\Lambda^+_g)\) completely, since the dimension of the manifold is \(d + 1\), while the sum of all these equations yields an identity \(0 \equiv 0\). The missing condition is the normalization:

\[
\left(\text{Tr}\widehat{W}_{k_B}^{(L)}(\eta; g)|_{L=(\sum_{j=0}^{d} \xi_j)^{1/n}}\right)|_{\tau^{-1}(\Lambda^+_g)} = 1.
\]

Now let us look at the continuous case. The analogue of the Gibbs’s statistical operator \([19]\) is as follows:

\[
\widehat{\mathcal{W}}_{k_B, \varepsilon}^{(L)}(\sigma(\cdot); g) := \exp\left(-\frac{1}{k_B} \int_{\mathbb{R}^d} da \sigma(a) \mathcal{F}_\varepsilon^{(L)}(a; g)\right),
\]

where \(\sigma(a)\) is a smooth function that we use instead of \(\{\eta_i\}_{i=0}^{d}\) in the formula \((25)\). An analogue of the phase space \(\mathbb{R}^{2(d+1)}\) is formed by pairs of functions \((F(\cdot), \sigma(\cdot))\), equipped with a canonical symplectic structure \(\bar{\omega} := \int \text{d}z (\delta \sigma)(z) \wedge (\delta F)(z)\). An analogue of the manifold \(\Lambda^+_g\), which we denote \(\widetilde{\Lambda}^+_g\), is described as follows. For every \(L > 0\), look at

\[
\mathcal{Y}_\varepsilon^{(L)}(g) := \{\sigma(\cdot) | \text{Tr}\widehat{W}_{k_B, \varepsilon}^{(L)}(\sigma(\cdot); g) = 1\}.
\]

Then \(\widetilde{\Lambda}^+_g\) is described as

\[
\widetilde{\Lambda}^+_g = \left\{(F(\cdot), \sigma(\cdot)) \mid F(a) = \left(-k_B \frac{\delta}{\delta \sigma(a)}\right) \text{Tr}\widehat{W}_{k_B, \varepsilon}^{(L)}(\sigma(\cdot); g), a \in \mathbb{R}^d, \sigma(\cdot) \in \mathcal{Y}_\varepsilon^{(L)}(g), L > 0\right\}.
\]

Since we are interested in the limit \(\varepsilon \to 0\) (i.e. the approximation \(\delta_\varepsilon(\cdot)\) becomes the Dirac’s delta), one can proceed as

\[
\text{Tr}\widehat{W}_{k_B, \varepsilon}^{(L)}(\sigma(\cdot); g) = \text{Tr} \left\{ \int da' \exp \left(-\frac{L_n}{k_B} \int da \sigma(a) \delta_\varepsilon(a - a')\right) \times \prod_{j=1}^{d} \delta_\varepsilon(a_j - \mathcal{E}_j^{(L)}(g)) \right\} + o(\varepsilon) = \int da \exp \left(-\frac{L_n}{k_B} \sigma(a)\right) \mathcal{F}_\varepsilon^{(L)}(a; g) + o(\varepsilon),
\]
where
\[
\Gamma^{(L)}_{\varepsilon}(a; g) := \operatorname{Tr}\left( \prod_{j=1}^{d} \delta_{\varepsilon}(a_j - \mathcal{E}^{(L)}_j(g)) \right)
\]
is the analogue of the statistical weight of a microcanonical distribution. Therefore, the equations describing $\tilde{\Lambda}^+_{\varepsilon,g}$ in the limit $\varepsilon \to 0$ become more simple. Computing the variational derivative, for every $a = (a_1, a_2, \ldots, a_d)$, we obtain:
\[
F(a) = L^n \exp \left( -\frac{L^n}{k_B} \sigma(a) \right) \Gamma^{(L)}_{\varepsilon}(a; g) + o(\varepsilon),
\]
where $\sigma(\cdot)$ varies over $\mathcal{Y}^{(L)}_{\varepsilon}(g)$ as $L$ varies over $\mathbb{R}_{>0}$. This motivates the following construction for the phenomenological Lagrangian manifold, which we denote just $\tilde{\Lambda}^+$. We assume, that we are given a function $\Gamma^{(L)}(a)$, $a = (a_1, a_2, \ldots, a_d)$, depending on a parameter $L > 0$. Set
\[
\mathcal{Y}^{(L)} := \{ \sigma(\cdot) \mid \int da \exp \left( -\frac{L^n}{k_B} \sigma(a) \right) \Gamma^{(L)}(a) = 1 \},
\]
for every $L > 0$. The Lagrangian manifold $\tilde{\Lambda}^+$ is described as a collection of pairs,
\[
\tilde{\Lambda}^+ := \bigcup_{L > 0} \left\{ (F(\cdot), \sigma(\cdot)) \mid \sigma(\cdot) \in \mathcal{Y}^{(L)} & \forall a : F(a) = L^n \exp \left( -\frac{L^n}{k_B} \sigma(a) \right) \Gamma^{(L)}(a) \right\} .
\]
This manifold is an analogue of $\tau^{-1}(\Lambda^+)$. Note, that the function $\Gamma^{(L)}(a)$ should be linked to the phenomenological entropy $S(a_1, \ldots, a_d; V)$ of a nonequilibrium state with volume $V = L^n$ and the values of the other extensive coordinates $(a_1, a_2, \ldots, a_d)$ via the Boltzmann’s formula:
\[
S(a_1, \ldots, a_d; V) = k_B \ln(c \Gamma^{(L)}(a)),
\]
where $k_B$ is the Boltzmann’s constant, and $c$ is an arbitrary constant having the same units of measurement as the product $a_1 a_2 \ldots a_d$ (the entropy in phenomenological thermodynamics is defined up to an additive constant).

At this point one can proceed as described above and consider the quasithermodynamic fluctuation theory over the points $(F(\cdot), \sigma(\cdot))$ in terms of the asymptotic expansions in the inverse rescaling parameter $\lambda^{-1} \to 0$. Without going into details, one can say the following. Denote the functions $(F(\cdot), \sigma(\cdot))$ corresponding to a point $\tilde{\alpha} \in \tilde{\Lambda}^+$ as $(F(\cdot; \tilde{\alpha}), \sigma(\cdot; \tilde{\alpha}))$. It follows from the construction of $\tilde{\Lambda}^+$, that to every $\tilde{\alpha} \in \tilde{\Lambda}^+$ we can associate $L(\tilde{\alpha}) := (\int da F(a; \tilde{\alpha}))^{1/n}$, and $\sigma(\cdot, \tilde{\alpha}) \in \mathcal{Y}^{(L(\tilde{\alpha}))}$. Assume that one can find a function $g_\lambda(\tilde{\alpha})$ and a function $\sigma_\lambda(\cdot; \tilde{\alpha}) \in \mathcal{Y}^{(\lambda^{1/n}L(\tilde{\alpha}))}$, such that the quantities $F(a, \tilde{\alpha}, \lambda, \varepsilon)$ defined by
\[
F(a, \tilde{\alpha}, \lambda, \varepsilon) := \left( \frac{\delta}{\delta \sigma(a)} \operatorname{Tr} \left[ W^{(\lambda^{1/n}L(\tilde{\alpha}))}_{k_B, \varepsilon} (\sigma(\cdot), g_\lambda(\tilde{\alpha})) \right] \right)_{\sigma(\cdot) = \sigma_\lambda(\cdot; \tilde{\alpha})}
\]
admit asymptotic expansions as $\lambda \to \infty$,
\[
\lambda^{-1} F(a, \tilde{\alpha}, \lambda, \varepsilon) \simeq \sum_{l=0}^\infty \lambda^{-l} F_l(a, \tilde{\alpha}, \varepsilon),
\]
where the coefficients $F_l(a, \tilde{\alpha}, \varepsilon)$, $l = 0, 1, 2, \ldots$, are smooth enough and satisfy the condition that, for every fixed $\tilde{\alpha}$ and $l$, $F_l(-, \tilde{\alpha}, \varepsilon)$ has a weak limit as $\varepsilon \to 0$, and
for the leading coefficient (corresponding to $l = 0$) this limit is $F(;\tilde{\alpha})$. Consider the operators of flows $\tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g)$ corresponding to $\tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g)$:

$$
\tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g) := \frac{i}{\hbar} \left[ \tilde{\mathcal{H}}^{(L)}(g), \tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g) \right] = -\sum_{i=1}^{d} \tilde{\mathcal{J}}_i^{(L)}(g) \frac{\partial}{\partial a_i} \tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g),
$$

where $\tilde{\mathcal{H}}^{(L)}(g)$ is the second quantized Hamiltonian of the underlying multiparticle system, and $\tilde{\mathcal{J}}_i^{(L)}(g)$ are the flow operators (23) of the observables $\tilde{\mathcal{E}}_i^{(L)}(g)$. The corresponding phenomenological flows $\mathcal{I}(a; \tilde{\alpha})$ can be perceived in analogy with (22) as follows:

$$
\mathcal{I}(a; \tilde{\alpha}) = \lim_{\varepsilon \to 0} \lim_{\lambda \to \infty} \left\{ \lambda^{-1} \text{Tr} \left[ \tilde{W}_{k_B,\varepsilon}^{(L)}(\sigma(\cdot); g) \tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g) \right] \right|_{L=\lambda^{1/n}L(\tilde{\alpha}), g=g_{\lambda}(\tilde{\alpha}), \sigma(\cdot)=\sigma(\cdot; \tilde{\alpha})} \right\}.
$$

Expanding the definitions of the quantities in the square brackets, we obtain:

$$
\text{Tr} \left[ \tilde{W}_{k_B,\varepsilon}^{(L)}(\sigma(\cdot); g) \tilde{\mathcal{F}}_\varepsilon^{(L)}(a; g) \right] = -\sum_{i=1}^{d} \frac{\partial}{\partial a_i} \text{Tr} \left\{ \exp \left( -\frac{L}{k_B} \int da' \sigma(a') \delta(a' - a) \right) \times \tilde{\mathcal{J}}_i^{(L)}(g) \right. \left. L^n \prod_{j=1}^{d} \delta_{\varepsilon}(a_j - \tilde{\mathcal{E}}_j^{(L)}(g)) \right\} + o(\varepsilon).
$$

Performing the integration in the exponent, and taking into account, that for $\tilde{\alpha} \in \tilde{\Lambda}^+$ we have $\exp(-L(\tilde{\alpha}))^n \sigma(a; \tilde{\alpha})/k_B) = F(a; \tilde{\alpha})/(L(\tilde{\alpha}))^n \Gamma(L(\tilde{\alpha}))(a)$, we obtain:

$$
\text{Tr} \left[ \tilde{W}_{k_B,\varepsilon}^{(L(\tilde{\alpha}))}(\sigma(\cdot); g) \tilde{\mathcal{F}}_\varepsilon^{(L(\tilde{\alpha}))}(a; g) \right] = -\sum_{i=1}^{d} \frac{\partial}{\partial a_i} \left( F(a; \tilde{\alpha}) \times \text{Tr} \left\{ \tilde{\mathcal{J}}_i^{(L(\tilde{\alpha}))}(g) \frac{L^n \prod_{j=1}^{d} \delta_{\varepsilon}(a_j - \tilde{\mathcal{E}}_j^{(L(\tilde{\alpha}))}(g))}{\Gamma(L(\tilde{\alpha}))(a)} \right\} \right) + o(\varepsilon).
$$

Therefore, the expression for the phenomenological flows reduces to

$$
\mathcal{I}(a; \tilde{\alpha}) = -\sum_{i=1}^{d} \frac{\partial}{\partial a_i} \left( F(a; \tilde{\alpha}) u_i(a; \tilde{\alpha}) \right),
$$

where

$$
u_i(a; \tilde{\alpha}) := \lim_{\varepsilon \to 0} \lim_{\lambda \to \infty} \left( \lambda^{-1} \text{Tr} \left\{ \tilde{\mathcal{J}}_i^{(L)}(g) \frac{L^n \prod_{j=1}^{d} \delta_{\varepsilon}(a_j - \tilde{\mathcal{E}}_j^{(L)}(g))}{\Gamma(L)(a)} \right\} \right|_{L=L(\tilde{\alpha}), g=g_{\lambda}(\tilde{\alpha})}
$$

for $i = 1, 2, \ldots, d$. Like in the beginning of this section, the evolution of the system is described as a curve $\tilde{\gamma} = \{\tilde{\alpha}^t\}_t$ on $\tilde{\Lambda}^+$. There exists an equation describing this curve, the derivation of which is totally similar to the derivation of (21), see [24] for details. Under the assumptions of the “smallness of time-memory effects” and “smallness of spatial gradients”, one arrives at the generalized Fokker-Planck equation:

$$
\frac{\partial}{\partial t} F(a; \tilde{\alpha}^t) + \sum_{i=1}^{d} \frac{\partial}{\partial a_i} \left( F(a; \tilde{\alpha}^t) u_i(a; \tilde{\alpha}^t) \right) - \sum_{i,j=1}^{d} \left( \frac{\partial}{\partial a_i} \mathcal{K}_{i,j}(a; \tilde{\alpha}^t) \frac{\partial}{\partial a_j} \right) \frac{F(a; \tilde{\alpha}^t)}{\Gamma(a^{\tilde{\alpha}^t})(a)} = 0,
$$
where $\mathcal{K}_{i,j}(a; \tilde{\alpha})$ are some coefficients (their role is similar to the Onsager coefficients $L_{i,j}(\alpha^t)$ in (21)), and $\Gamma^{(L)}(a)$ is the function defining the limiting manifold $\Lambda^+$ (i.e. the phenomenological statistical weight, see (26)). It can be more convenient to write this equation in the following form:

$$
\frac{\partial}{\partial t} F(a; \tilde{\alpha}) + \sum_{i=1}^{d} \frac{\partial}{\partial a_i} \left( F(a; \tilde{\alpha}) v_i(a; \tilde{\alpha}) \right) - \sum_{i,j=1}^{d} \left( \frac{\partial}{\partial a_i} D_{i,j}(a; \tilde{\alpha}) \frac{\partial}{\partial a_j} \right) F(a; \tilde{\alpha}) = 0,
$$

(27)

where

$$
D_{i,j}(a; \tilde{\alpha}) := \frac{\mathcal{K}_{i,j}(a; \tilde{\alpha})}{\Gamma^{(L)}(a)}
$$

are the diffusion coefficients ($i, j = 1, 2, \ldots, d$, $\tilde{\alpha} \in \tilde{\Lambda}^+$, $a \in \mathbb{R}^d$), and

$$
v_i(a; \tilde{\alpha}) := u_i(a; \tilde{\alpha}) + \sum_{j=1}^{d} \frac{\mathcal{K}_{i,j}(a; \tilde{\alpha})}{\Gamma^{(L)}(a)} \left( \frac{\partial}{\partial a_j} \ln \Gamma^{(L)}(a) \right),
$$

are the coefficients of drift ($i = 1, 2, \ldots, d$, $\tilde{\alpha} \in \tilde{\Lambda}^+$, $a \in \mathbb{R}^d$). If we denote, for every $i = 1, 2, \ldots, d$,

$$
\beta_{i}^{(L)}(a) := k_B \frac{\partial}{\partial a_i} \ln \Gamma^{(L)}(a),
$$

(these are the analogues of the inverse absolute temperature and other intensive thermodynamic quantities in the equilibrium thermodynamics), then we obtain

$$
v_i(a; \tilde{\alpha}) = u_i(a; \tilde{\alpha}) + \sum_{j=1}^{d} D_{i,j}(a; \tilde{\alpha}) \beta_j^{(L)}(a),
$$

for $i = 1, 2, \ldots, d$, $\tilde{\alpha} \in \tilde{\Lambda}^+$, $a = (a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$. The equation (27) is more familiar in the phenomenological physics, since in practice, $L(\tilde{\alpha})$ is quite often fixed along the evolution, and the coefficients depend very little on $a$ and $\tilde{\alpha}$ and can be determined empirically (recall, for example, the Fourier’s law for the heat transfer, or the Fick’s law for diffusion).

The equation (27) can be perceived as a “more precise” description of a physical system, than the one given by (21). Suppose $L(\tilde{\alpha}) = \text{const}$ along the evolution. Then it is more convenient to introduce $f(a; \tilde{\alpha}) := (L(\tilde{\alpha}))^{-n} F(a; \tilde{\alpha})$, which is normalized as $\int \text{d}a f(a; \tilde{\alpha}) = 1$. Instead of a curve $\{\alpha^t\} \subset \Lambda^+$, we now have a curve $\{\tilde{\alpha}^t\} \subset \tilde{\Lambda}^+$. This function describes the density of distribution of “fluctuations” of $E_1, E_2, \ldots, E_d$ (recall, that $\Lambda^+$ is a Lagrangian manifold embedded in a phase space with extensive coordinates $(x_1, \ldots, x_{d+1}) = (E_1, \ldots, E_d, V)$ and intensive coordinates $(y_1, \ldots, y_{d+1}) = (\beta_1, \ldots, \beta_d, \tilde{p})$, the symplectic structure is $\omega^+ = \sum_{i=1}^{d+1} \text{d}y_i \wedge \text{d}x_i$). We have a pair of consistency conditions between the two descriptions. The first one is that the entropy $S(E_1, \ldots, E_d, L^n) = \int \beta_i dE_i$, $L|_\gamma = V^{1/n} = \text{const}$, corresponding to the Lagrangian manifold $\Lambda^+$, is linked to the phenomenological statistical weight $\Gamma^{(L)}(a)$, $a = (a_1, \ldots, a_d)$ associated with $\tilde{\Lambda}^+$, via the Boltzmann’s formula:

$$
S(E_1, \ldots, E_d, L^n) = k_B \ln(\Gamma^{(L)}(E_1, \ldots, E_d)),
$$
where $k_B$ is the Boltzmann’s constant, and $c > 0$ is a constant with the same units of measurement as $(E_1E_2\ldots E_d)^{-1}$. The second is a condition on the mathematical expectation:

$$E_i(\alpha^t) = \int da \ a_i f(a; \tilde{\alpha}^t),$$

where $i = 1, 2, \ldots, d$, (note, that this equality becomes approximate, once we use the Onsager’s equations \cite{21} and the generalized Fokker-Planck equations \cite{27}). If we wish to consider \cite{27} with an initial condition $F(a, \tilde{\alpha}^t)|_{t=0} = L^nf_0(a)$, then for the rest we are not restricted in a choice of $f_0(\cdot)$ (i.e. the starting point on $\Lambda^+$ is for the rest arbitrary). In particular, consider the cumulants (assuming they exist):

$$\tilde{C}_N(t) := \left( -ik_B \frac{\partial}{\partial t} \right)^N \int da f(a, \tilde{\alpha}^t) \exp \left( \frac{i}{k_B} \sum_{i=1}^d b_i a_i \right) |_{b=0},$$

where $b = (b_1, \ldots, b_d)$, $N = (n_1, \ldots, n_d) \in \mathbb{Z}^d_0$, $\tilde{\theta} = (0, \ldots, 0) \in \mathbb{R}^d$, $N \neq \tilde{\theta}$, and we use a standard notation for multi-indices: $(-ik_B \partial/\partial b)^N := \prod_{j=1}^d (-ik_B \partial/\partial b_j)^{n_j}$. The second consistency condition mentioned is just

$$\tilde{C}_N(t) = \lim_{\lambda \to \infty} \lambda^{-1} C^{(\lambda)}(N,0)(\alpha^t),$$

if $|N| := \sum_{i=1}^d n_i = 1$, where $N = (n_1, \ldots, n_d)$, and $(N, 0) := (n_1, \ldots, n_d, 0)$. On the other hand, this equality is not required in case where $|N| \geq 2$, i.e. the cumulants $\tilde{C}_N(t)$ should be perceived as derivatives of the action function corresponding to different Lagrangian manifolds $\mathcal{M}_t$ “attached” to the points $\alpha^t \in \Lambda^+$. More precisely, we need only the germs of $\mathcal{M}_t$ of these Lagrangian manifolds (which are termed the microlagrangian manifolds). Note, that this construction is similar to the construction used in the method of canonical operator with a complex phase \cite{4} \cite{12}.

How do $\mathcal{M}_t$ change with time? More generally, we would like to consider a “deformation” of the linear fluctuation theory with respect to the rescaling parameter $\lambda \to \infty$, i.e. to introduce the functions $\tilde{C}^{(\lambda)}(N)(t)$ generalizing $\tilde{C}_N(t)$, $N \in \mathbb{Z}^d_{\neq 0}$, $N \neq \tilde{\theta}$. How does the collection $\{\tilde{C}_N^{(\lambda)}(t)\}_N$ change with time? The main idea which allows to derive these equations is as follows: let us consider the limit $\lambda^{-1} \to 0$ in analogy with the semiclassical limit $h \to 0$ of quantum mechanics (where $h$ is the small parameter of the transition corresponding to the Planck constant $\hbar$).

Suppose we have a classical mechanical system with $n$ degrees of freedom $q = (q_1, q_2, \ldots, q_n) \in \mathbb{R}^n$ described by a Hamiltonian $H(p, q)$, where $p = (p_1, p_2, \ldots, p_n)$ are the canonically conjugate momenta corresponding to $q$. Assume that $H$ is a smooth function, say, from the Schwartz space $\mathcal{S}(\mathbb{R}^{2n}_{q,p})$ of rapidly decaying functions at infinity. A semiclassical analogue of this system is described by a Hamiltonian $H_h(q, p)$ which depends on the small parameter $h \to 0$ of semiclassical approximation, $H_h(q, p) = H(q, p) + \sum_{s=1}^{r_0} ((-ih)^s/s!) H^{(s)}(q, p)$, where $r_0$ is a positive integer, and $H^{(s)} \in \mathcal{S}(\mathbb{R}^{2n}_{q,p})$, $s = 1, 2, \ldots, r_0$.

A semiclassical wave function $\psi^t_h(q)$ satisfies modulo $O(h^\infty)$ the Schrödinger equation with a small parameter $h$ in front of partial derivatives:

$$ih \frac{\partial}{\partial t} \psi^t_h(q) = H_h \left( -ih \frac{\partial}{\partial q}, q \right) \psi^t_h(q) + O(h^\infty),$$
where one assumes the Weyl quantization on the right-hand side, and one defines $O(h^\infty)$ as follows: $\psi_h(q) = O(h^\infty)$ if and only if for all $k = 1, 2, 3, \ldots$, there exists $C_k > 0$, such that $\sup_{q \in \mathbb{R}^n} |\psi_h(q)| \leq C_k h^k$. For the semiclassical wave functions there exists a limit

$$\rho'(q) := \lim_{h \to 0} |\psi_h(q)|^2,$$

(invoking the Bohr’s correspondence principle). The main idea is to try to perceive $f(a; \tilde{\alpha}^t) = L^{-n} F(a, \tilde{\alpha}^t)$ corresponding to the equation (27) in analogy with $\rho'(q)$. The small parameter $h \to 0$ should correspond to $\lambda^{-1} \to 0$. Consider first the Wigner’s \textit{quasiprobability} function associated with $\psi_h(q)$:

$$\rho_h(q, p) := \frac{1}{(2\pi h)^n} \int dq' \exp \left( -\frac{i}{h} pq \right) \bar{\psi}_h(q - \frac{q'}{2}) \psi_h(q + \frac{q'}{2}),$$

where the bar denotes the complex conjugation, and $(q, p) \in \mathbb{R}^{2n}$. It follows, that

$$\rho'(q) = \int dp \lim_{h \to 0} \rho_h(q, p),$$

where the limit under the integral on the right-hand side is taken in the weak sense.

Note, that $\rho_h(q, p)$ is real, $\rho_h(q, p) = \bar{\rho}_h(q, p)$, but, in general, does not need to be positively defined, and satisfies only the normalization condition $\int dp dq \rho_h(q, p) = 1$. The classical limit $\rho'(q, p)$, on the other hand, satisfies $\rho'(q, p) \geq 0$. Consider now $N$ semiclassical particles with an interaction between a pair of particles concentrated in $(q, p)$ and $(q', p')$ described by

$$V_h(q, p; q', p') := V(q, p; q', p') + \sum_{s=1}^{r_1} \frac{(-i h)^s}{s!} V^{(s)}(q, p; q', p'),$$

where $r_1$ is a positive integer, and $V$ and $V_s$, $s = 1, 2, 3, \ldots$, are elements of $\mathcal{S}([\mathbb{R}^{2n} \times \mathbb{R}^{2n}])$ (i.e. Swartz functions on a 2-particle phase space). To describe the \textit{kinetics} of this system, one introduces the 1-particle density function $R_{N,h}^t(q, p)$ (the analogue of $\rho_h(q, p)$) normalized on the number of particles,

$$\int_{\mathbb{R}^{2n}} dq dp R_{N,h}^t(q, p) = N,$$

and considers a limit $N \to \infty$, $h \to 0$, in such a way that

$$Nh^n = \varepsilon = \text{const} > 0.$$

Assume that we adjust the “geometry” of the configuration space of the system (this can be the radius $L$ of a $n$-dimensional torus, if the extracted system is confined on a torus), as well as the other parameters $g = (g_1, g_2, \ldots, g_m)$ describing the “nature” of the system (for instance, the radius of interaction or a parameter describing the external field) in such a way, that for the weak limit $\rho'(q, p)$ of $R_{N,h}^t(q, p)/N$ (where $N \to \infty$ and $Nh^n = \varepsilon > 0$ is fixed), one obtains an equation of the shape

$$\frac{\partial}{\partial t} \rho'(q, p) + \{H[\rho'(q, p; \varepsilon)], \rho'(q, p)\} - J[\rho'](q, p; \varepsilon) = 0,$$

where $\{-,-\}$ denotes the canonical Poisson bracket on the 1-particle phase space, with coordinates $q = (q_1, q_2, \ldots, q_n)$ and momenta $p = (p_1, p_2, \ldots, p_n)$, $\{p_i, q_j\} = \delta_{i,j}$, $i, j \in [n]$, the function $H[\rho'](q, p; \varepsilon)$ is the \textit{selfconsistent Hamiltonian}, and
$J[\rho^t](q,p;\kappa)$ is the collision integral \cite{23}. A usual way to analyse this equation is via an asymptotic expansion of the collision integral in $\kappa \to 0$ and by constructing the BBGKY chain of equations. Observe, that if we integrate \cite{28} over $p$, then this yields an expression in which the term corresponding to the Poisson bracket can be associated with the first sum in \cite{27} (the sum over one index), and the term corresponding to the collision integral can be associated with the second sum in \cite{27} (the sum over two indices).

To make the analogy more explicit, let us look at our thermodynamic system and consider along with $\mathcal{E}^{(L)}(g_1), \ldots, \mathcal{E}^{(L)}(g_d)$, the flows $\mathcal{J}^{(L)}_1(g_1), \ldots, \mathcal{J}^{(L)}_d(g_1)$ defined in \cite{23} (recall, that $L$ denotes the radius of the $n$-dimensional torus on which we consider the system, and $g = (g_1, g_2, \ldots, g_m)$ is a collection of parameters describing the interaction and the external field). Recall, that it is also assumed (for simplicity), that $\{\mathcal{E}^{(L)}_i(g)\}_{i=1}^d$ mutually commute, and that $\{\mathcal{J}^{(L)}_i(g)\}_{i=1}^d$ mutually commute. Look first at the analogue of \cite{21} (we assume that the volume $V = L^n$ is fixed along the evolution). One needs to consider the phase space $\mathbb{R}^{4d+2}$ with “coordinates” $(x_1, \ldots, x_d, x_{d+1}, \ldots, x_{2d}, V)$ and “momenta” $(y_1, \ldots, y_d, y_{d+1}, \ldots, y_{2d}, \tilde{p})$, where for $s = 1, 2, \ldots, d$, $x_s$ corresponds to $\mathcal{E}^{(L)}_s(g)$, $x_{d+s}$ corresponds to $\mathcal{J}^{(L)}_s(g)$, $y_s$ corresponds to the “inverse temperature” conjugate to $\mathcal{E}^{(L)}_s(g)$, and $y_{d+s}$ corresponds to the “inverse temperature” conjugate to $\mathcal{J}^{(L)}_s(g)$. It is convenient to denote $x_{2d+1} = V$, and $y_{2d+1} = \tilde{p}$ (the intensive quantity “pressure over absolute temperature” associated with the volume $V$). The symplectic structure is $\omega^\# := \sum_{i=1}^{2d} dy_i \wedge dx_i$, and the evolution is described as a curve $\gamma = \{\alpha^t\}_t \subset \Lambda^\#$ on a Lagrangian manifold $\Lambda^\# \subset \mathbb{R}^{4d+2}$, where $t$ is time. Denote $x_i(\alpha)$ and $y_i(\alpha)$, $i = 1, 2, \ldots, 2d+1$, the coordinates of a point $\alpha \in \Lambda^\#$ acquired in the ambient phase space. The curve satisfies the following system of equations (in analogy with \cite{21}):

$$\frac{\partial x_i(\alpha^t)}{\partial t} = J_i(\alpha^t) + \sum_{j=1}^{2d} L_{i,j}(\alpha^t) y_j(\alpha^t),$$

where $i = 1, 2, \ldots, 2d$, and $J_i(\alpha)$ and $L_{i,j}(\alpha)$, $i, j \in [2d]$, are given functions on $\Lambda^\#$ (the flows and the Onsager coefficients). These functions can be perceived as a limit $\lambda \to \infty$ of more complicated quantities associated with rescaled systems. In particular, for $J_s(\alpha), s \in [d]$, one needs to consider the flows of $\mathcal{E}^{(L)}_s(g)$ given by the operators \cite{23} and for $J_{d+s}(\alpha)$ one needs to consider the flows of the flows $\mathcal{J}^{(L)}_s(g)$,

$$\mathcal{J}^{(L)}_{d+s}(g) := \frac{i}{\hbar} \left[ \mathcal{H}^{(L)}(g), \mathcal{J}^{(L)}_s(g) \right] = \frac{i}{\hbar} \left[ \mathcal{H}^{(L)}(g), \frac{i}{\hbar} \left[ \mathcal{H}^{(L)}(g), \mathcal{E}^{(L)}_s(g) \right] \right],$$

where $s \in [d]$, (we have a family of underlying systems parametrized by $L$ and $g$). If we denote $\mathcal{E}^{(L)}_{d+s}(g) := \mathcal{J}^{(L)}_s(g), s \in [d]$, then the space of relevant ensembles is described by the Gibbs's operators

$$\mathcal{w}^{(L)}_{k_B}(\beta; g) := \frac{1}{Z^{(L)}_{k_B}(\beta; g)} \exp \left( - \frac{1}{k_B} \sum_{i=1}^{2d} \beta_i \mathcal{E}^{(L)}_i(g) \right),$$

where $\beta = (\beta_1, \ldots, \beta_{2d})$, and $Z^{(L)}_{k_B}(\beta; g)$ is determined by the normalization condition

$$\text{Tr} \mathcal{w}^{(L)}_{k_B}(\beta; g) = 1$$

(assuming the trace exists). For every $\alpha \in \Lambda^\#$ there exist functions
\( \beta_\lambda(\alpha) \) and \( g_\lambda(\alpha) \) depending on the large rescaling parameter \( \lambda \to \infty \) in such a way, that

\[
J_i(\alpha) = \lim_{\lambda \to \infty} \left\{ \lambda^{-1} \text{Tr} \left[ \mathcal{J}_i^{(L)}(g) \tilde{\omega}_k^{(L)}(\beta; g) \right] \left|_{L=(\lambda x_{d+1}(\alpha))^{1/n}, \beta=\beta_\lambda(\alpha), g=g_\lambda(\alpha)} \right. \right\},
\]

where \( i \in [2d] \). The coefficients \( L_{i,j}(\alpha) \) can be perceived in a similar way as a limit of more complicated expressions (involving, for example, the projection operators of Mori [15] and Zwanzig [25]). It is important to point out, that the system \(\{29\}\) is valid in the approximation of “small spatial gradients”. To compute the “flows of the flows”, one applies the commutator \( (i/\hbar) [\tilde{\mathcal{H}}^{(L)}(g), -] \) twice, and, therefore, we should put

\[
J_{d+s}(\alpha^t) = 0, \tag{30}
\]

where \( s \in [d] \), along the evolution curve \( \gamma = \{\alpha^t\}_t \subset \Lambda^\# \), see [24] for details. The condition \(\{30\}\) simply restricts the possible curve \( \gamma \) and implies, that the Onsager’s matrix \( \|L_{i,j}(\alpha^t)\|_{2d} = \{i,j\} \) cannot be arbitrary.

Now, if we wish to describe the “fluctuations” of the values of \( x_i(\alpha^t), y_i(\alpha^t), i \in [2d] \), corresponding to a point \( \alpha^t \in \gamma \subset \Lambda^\# \) of the evolution curve, then we should introduce a “phase space” formed by the pairs \((\tilde{F}(x), \tilde{\sigma}(x))\) \((\tilde{F} \text{ is a distribution, and } \tilde{\sigma} \text{ is a function})\), where \( x = (x_1, x_2, \ldots, x_{2d}) \in \mathbb{R}^{2d} \). For the points \( \alpha \in \Lambda^\# \), we have

\[
y_i(\alpha) = \left. \frac{\partial}{\partial x_i} \tilde{S}(x_1, \ldots, x_{2d}; V(\alpha)) \right|_{x_i = x_j(\alpha), j \in [2d]},
\]

where \( i \in [2d] \), and \( \tilde{S}(x_1, \ldots, x_{2d}; V) \) is the phenomenological nonequilibrium entropy (see [18]) (note that this is a function of extensive thermodynamic coordinates \((x_1, \ldots, x_d)\), their flows \((x_{d+1}, \ldots, x_{2d})\), and the volume \( V \)). Define the phenomenological statistical weight \( \tilde{\Gamma}^{(L)}(x_1, \ldots, x_{2d}) \) from the Boltzmann’s formula:

\[
\tilde{S}(x_1, \ldots, x_{2d}; L^n) = k_B \ln(c \tilde{\Gamma}^{(L)}(x_1, \ldots, x_{2d})),
\]

where \( c > 0 \) is a constant with the same units of measurement as \((x_1 \ldots x_{2d})^{-1}\), and construct a “huge” Lagrangian manifold \( \tilde{\Lambda}^\# \) like [26],

\[
\tilde{\Lambda}^\# := \bigcup_{L > 0} \left\{ (\tilde{F}(\cdot), \tilde{\sigma}(\cdot)) \mid \tilde{\sigma}(\cdot) \in \tilde{\mathcal{Y}}^{(L)} \& \forall x : \tilde{F}(x) = L^n \exp \left( -\frac{L^n}{k_B} \tilde{\sigma}(x) \right) \tilde{\Gamma}^{(L)}(x) \right\}, \tag{31}
\]

where \( x = (x_1, x_2, \ldots, x_{2d}) \), and \( \tilde{\mathcal{Y}}^{(L)} := \{ \tilde{\sigma} \mid \int dx \tilde{\Gamma}^{(L)}(x) \exp(-L^n\tilde{\sigma}(x)/k_B) = 1 \} \). The difference with the formula [26] is only that we have \( 2d \) arguments in the functions, in place of \( d \). The evolution of the system is a curve \( \tilde{\gamma} = \{\tilde{\alpha}^t\}_t \subset \tilde{\Lambda}^\# \) described by a system of equations similar to [27] and [29] with a condition similar to [30].

Denote \((\tilde{F}(x; \tilde{\alpha}), \tilde{\sigma}(x; \tilde{\alpha}))\) the coordinates of a point \( \tilde{\alpha} \in \tilde{\Lambda}^\# \) stemming from the phase space. Observe, that \( \tilde{\mathcal{J}}_i^{(L)}(g) = \tilde{E}_i^{(L)}(g), s \in [d] \), is present in the product of delta functions \( \prod_{i=1}^{2d} \delta(x_i - \tilde{E}_i^{(L)}(g)) \) corresponding to the analogue of the microcanonical distribution in this case. Therefore, computing \( u_i(x; \tilde{\alpha}) \) for the generalized Fokker-Planck equation (where \( i \in [2d], x = (x_1, \ldots, x_{2d}), \tilde{\alpha} \in \tilde{\Lambda}^\# \)), we obtain \( u_{s}(x; \tilde{\alpha}) = x_{d+s}, s \in [d] \). The quantities \( u_{d+s}(x; \tilde{\alpha}) \) correspond to applying
the commutator \((i/\hbar)[\hat{H}^{(L)}(g), -]\) twice (i.e., “flows of the flows”), and in the approximation of “small gradients” we should put them to zero \([25]\). In the end, this yields the following equation for \(\tilde{F}(x; \tilde{\alpha}^t)\):

\[
\frac{\partial \tilde{F}(x; \tilde{\alpha}^t)}{\partial t} + \sum_{s=1}^{d} x_{d+s} \frac{\partial \tilde{F}(x; \tilde{\alpha}^t)}{\partial x_s} + \sum_{i=1}^{2d} \frac{\partial}{\partial x_i} \left( \tilde{F}(x; \tilde{\alpha}^t) \sum_{j=1}^{2d} \tilde{D}_{i,j}(x; \tilde{\alpha}^t) \beta_j^{(L; \tilde{\alpha}^t)}(x) \right) - \sum_{i,j=1}^{2d} \left( \frac{\partial}{\partial x_i} \tilde{D}_{i,j}(x; \tilde{\alpha}^t) \frac{\partial}{\partial x_j} \right) \tilde{F}(x; \tilde{\alpha}^t) = 0,
\]

where \(\tilde{D}_{i,j}(x; \tilde{\alpha})\) are the analogue of the diffusion coefficients in \([27]\), and \(\beta_j^{(L)}(x)\) are defined as \(\tilde{\beta}_j^{(L)}(x) = (k_B \partial/\partial x_j) \ln \tilde{F}(x; \tilde{\alpha}^t), j \in [2d]\). The vector field on \(\mathbb{R}^{2d}_x\) with the components \(Y_i(x; \tilde{\alpha}) = \sum_{j=1}^{2d} \tilde{D}_{i,j}(x; \tilde{\alpha}^t) \beta_j^{(L; \tilde{\alpha}^t)}(x)\), \(i \in [2d]\), describes a contribution to the total flow associated with \(\tilde{\alpha}^{(L)}(g)\) induced by a deviation of the system from a thermodynamic equilibrium. Let us assume that the divergence of this vector field vanishes:

\[
\sum_{i=1}^{2d} \frac{\partial}{\partial x_i} Y_i(x; \tilde{\alpha}) = 0. \tag{32}
\]

Then, for every \(\tilde{\alpha} \in \tilde{\Lambda}^\#\), we can find a function \(\Phi(x; \tilde{\alpha})\) such that \(Y_i(x; \tilde{\alpha}) = \partial \Phi(x; \tilde{\alpha})/\partial x_i\), \(i \in [2d]\), \(x \in \mathbb{R}^{2d}_x\). Furthermore, let us represent this derivative as \(\partial \Phi(x; \tilde{\alpha})/\partial x_i = \sum_{j=1}^{2d} J_{i,j} \partial V(x; \tilde{\alpha})/\partial x_j\), where \(J_{i,j} = \delta_{i,j} - \delta_{i,-d,j}\), \(\delta\) is the Kronecker symbol. The matrix \(J = \|J_{i,j}\|_{i,j=1}^{2d}\) satisfies \(J^{-1} = J^T = -J\), where \((\cdot)^T\) denotes the transposed matrix. Then we have \(\partial V(x; \tilde{\alpha})/\partial x_i = \sum_{j=1}^{2d} (J^{-1})_{i,j} \partial \Phi(x)/\partial x_j\), and the function \(V(-; \tilde{\alpha})\) exists since \(\sum_{i=1}^{2d} J_{i,j} \partial \Phi(x)/\partial x_j\) is due to \(J^T = -J\). So one obtains:

\[
Y_{s}(x; \tilde{\alpha}) = \frac{\partial V(x; \tilde{\alpha})}{\partial x_{d+s}}, \quad Y_{d+s}(x; \tilde{\alpha}) = -\frac{\partial V(x; \tilde{\alpha})}{\partial x_s},
\]

for \(s \in [d]\). Substituting this into the generalized Fokker-Planck equation for \(\tilde{F}(x; \tilde{\alpha}^t)\), we obtain:

\[
\frac{\partial \tilde{F}(x; \tilde{\alpha}^t)}{\partial t} + \sum_{s=1}^{d} \left( \frac{\partial H(x; \tilde{\alpha}^t)}{\partial x_{d+s}} \frac{\partial \tilde{F}(x; \tilde{\alpha}^t)}{\partial x_s} - \frac{\partial H(x; \tilde{\alpha}^t)}{\partial x_s} \frac{\partial \tilde{F}(x; \tilde{\alpha}^t)}{\partial x_{d+s}} \right) = I(x; \tilde{\alpha}^t), \tag{33}
\]

where \(I(x, \tilde{\alpha}) := \sum_{i,j=1}^{2d} \frac{\partial}{\partial x_i} D_{i,j}(x; \tilde{\alpha}) (\partial/\partial x_j) \tilde{F}(x; \tilde{\alpha})\), and

\[
H(x; \tilde{\alpha}) := \frac{1}{2} \sum_{s=1}^{d} x_{d+s}^2 + V(x; \tilde{\alpha}),
\]

where \(x = (x_1, \ldots, x_{2d}) \in \mathbb{R}^{2d}_x\), and \(\tilde{\alpha} \in \tilde{\Lambda}^\#\). It is suggested to perceive \(H(x; \tilde{\alpha})\) as an analogue of a self-consistent Hamiltonian in mechanics (i.e., \(V(x; \tilde{\alpha})\) corresponds to the dressed potential), and the quantity \(I(x; \tilde{\alpha})\) as an analogue of the collision integral. The assumption \((32)\) about the divergence of the vector field describing
the thermodynamic flows is basically an assumption about applicability of a self-consistent picture of description, which, of course, depends on a “successful” choice of the collection of the basis quantities \( \{ \widetilde{E}_i^{(L)}(g) \}_{i=1}^d \).

The basic idea is now as follows: let us perceive \( \widetilde{F}(x; \tilde{\alpha}) \) in analogy with the weak limit of the Wigner’s quasiprobability function \( \rho_h(q,p) \) in quantum mechanics, where \((q,p) \in \mathbb{R}^{2n}_{q,p} \) and \( h \to 0 \) is the semiclassical parameter associated with the Planck constant \( h \). The weak limit \( \rho_0(q,p) = \lim_{h \to 0} \rho_h(q,p) \) is non-negatively defined, but \( \rho_h(q,p) \) itself can be negatively defined over some region (the measure of which vanishes in the limit \( h \to 0 \), see [6]). It is suggested to perceive \( \widetilde{F}(x; \tilde{\alpha}) \) as a limit

\[
\widetilde{F}(x; \tilde{\alpha}) = \lim_{\lambda \to \infty} \widetilde{F}_\lambda(x; \tilde{\alpha}),
\]

(34)

where \( \widetilde{F}_\lambda(x; \tilde{\alpha}) \) is some function (or, more generally, a distribution), depending on a large parameter \( \lambda \to \infty \), and the limit is taken in the weak sense. It is quite remarkable, that \( \widetilde{F}_\lambda(x; \tilde{\alpha}) \) does not need to be non-negatively defined. This can lead to an interesting new physical effect: the thermodynamic Bell’s inequalities. In quantum mechanics, the violation of Bell’s inequalities is related to the fact that the probability model in quantum mechanics is different from the probability model in classical mechanics: the Wigner’s quasiprobability can be negatively defined over some region of the phase space. It follows, that once we have \( \widetilde{F}_\lambda(x; \tilde{\alpha}) \) which is negatively defined over some region in \( \mathbb{R}^{2d}_x \), \( x = (x_1, \ldots, x_{2d}) \), we can mimic the Bell’s inequalities and the “entangled states” in quasithermodynamics!

Extending the analogy between mechanics and thermodynamics, one can look at \( \widetilde{F}_\lambda(x; \tilde{\alpha}) \) along the phenomenological evolution curve \( \widetilde{\gamma} = \{ \tilde{\alpha}^t \}_t \subset \tilde{\Lambda}^\# \). The general shape of the equation for \( \widetilde{F}_\lambda(x, \tilde{\alpha}^t) \) should then be as follows. Equip the affine space \( \mathbb{R}^{2d}_x \), \( x = (X, J) \), formed by the points

\[
x = (X_1, \ldots, X_d, J_1, \ldots, J_d),
\]

with a symplectic structure \( \Omega := \sum_{s=1}^d dJ_s \wedge dX_s \), where the first \( d \) coordinates \((X_1, \ldots, X_d)\) correspond to the extensive quantities (the “energies”), and the last \( d \) coordinates \((J_1, \ldots, J_d)\) correspond to the intensive quantities (the “inverse temperatures”). Recall, that the volume \( V = L^n \) is fixed (for simplicity). Consider the symmetric Fock space associated with \( L^2(\mathbb{R}^{2d}_x) \):

\[
\mathcal{F}^\# := \mathbb{C} \oplus L^2(\mathbb{R}^{2d}_x) \oplus L^2(\mathbb{R}^{2d}_x)^{\otimes \text{symm}^2} \oplus \ldots,
\]

where \( \otimes \text{symm} \) denotes the symmetric tensor power. Let \( \hat{K}_\lambda(\tilde{\alpha}) \), \( \tilde{\alpha} \in \tilde{\Lambda}^\# \), be a self-adjoint operator on \( \mathcal{F}^\# \) of the form

\[
\hat{K}_\lambda(\tilde{\alpha}) = \sum_{m=0}^M \frac{x^m}{m!} \int dx^{(0)} dx^{(1)} \ldots dx^{(m)} a^+(x^{(0)}) a^+(x^{(1)}) \ldots a^+(x^{(m)}) \times \\
\times \hat{K}_\lambda^{(m)}(\tilde{\alpha}) a^-(x^{(0)}) a^-(x^{(1)}) \ldots a^-(x^{(m)}),
\]

where \( x^{(0)}, x^{(1)}, \ldots, x^{(m)} \) vary over \( \mathbb{R}^{2d} \), \( a^\pm(x) \) are the bosonic creation and annihilation operators on \( \mathcal{F}^\# \) [2],

\[
[a^-(x), a^-(x')] = 0 = [a^+(x), a^+(x')], \quad [a^-(x), a^+(x')] = \delta(x - x'),
\]
\( x, x' \in \mathbb{R}^{2d} \), \( M \) is a fixed integer, \( \varkappa \) is a real parameter (introduced for convenience), and the operators \( \tilde{K}^{(m)}_{\lambda}(\tilde{\alpha}) \) are given by the commutators with respect to the Moyal product,

\[
\tilde{K}^{(m)}_{\lambda}(\tilde{\alpha}) = \frac{i}{\lambda - 1} \left\{ H^{(m)}_{\lambda} \left( x^{(0)} \right)^{2} + \frac{i\lambda^{-1}}{2} J \frac{\partial}{\partial x^{(0)}}, \cdots, x^{(m)} + \frac{i\lambda^{-1}}{2} J \frac{\partial}{\partial x^{(m)}}, \tilde{\alpha} \right\} - H^{(m)}_{\lambda} \left( x^{(0)} - \frac{i\lambda^{-1}}{2} J \frac{\partial}{\partial x^{(0)}}, \cdots, x^{(m)} - \frac{i\lambda^{-1}}{2} J \frac{\partial}{\partial x^{(m)}}, \tilde{\alpha} \right),
\]

where \( J = \|J_{ij}\|_{i,j=1}^{2d} \) is the canonical symplectic matrix, \( H^{(m)}_{\lambda}(x) \), \( m = 0, 1, \ldots, M \), are polynomials in \( \lambda \).

With this setup one can mimic the constructions of quantum statistical mechanics: the quasithermodynamic parameter \( \lambda^{-1} \) is similar to the semiclassical parameter \( h \), and \( \varkappa \) is similar to the interaction parameter \( g \). Observe nonetheless a rather important difference: there is a dependence of \( \tilde{K}_{\lambda}(\tilde{\alpha}) \) on the point \( \tilde{\alpha} \) of the “huge” Lagrangian manifold \( \tilde{\Lambda}^{\#} \) described as a subset of pairs \((\tilde{F}(\cdot), \tilde{\sigma}(\cdot))\) by \((31)\). Consider the second quantized analogue of the Wigner’s equation for the Weyl symbol of the square root of density matrix, and break the symmetry in time as in \((23)\):

\[
\frac{\partial R^{t}_{\lambda,\varepsilon}}{\partial t} + \tilde{K}_{\lambda}(\tilde{\alpha}) R^{t}_{\lambda,\varepsilon} = -\varepsilon (R^{t}_{\lambda,\varepsilon} - R^{(eq)}_{\lambda}),
\]

where \( R^{t}_{\lambda,\varepsilon} \in \mathcal{F}^{\#} \) is the unknown vector, and \( R^{(eq)}_{\lambda} \in \mathcal{F}^{\#} \) is fixed (the value of \( \lim_{\varepsilon \to +0} R^{t}_{\lambda,\varepsilon} \) corresponding to a state of thermodynamic equilibrium), and \( \varepsilon = +0 \).

Then we can perceive the solutions \( \tilde{F}(x; \tilde{\alpha}^{t}) \) of the generalized Fokker-Planck equation \((33)\) as a limit \((34)\), where

\[
\tilde{F}_{\lambda}(x; \tilde{\alpha}^{t}) = \lim_{\varepsilon \to +0} \left( R^{t}_{\lambda,\varepsilon}, a^{+}(x) a^{-}(x) R^{t}_{\lambda,\varepsilon} \right),
\]

and construct the analogue of BBGKY chain of equations for the higher order correlation functions. Furthermore, if there exists a generating function of fluctuations,

\[
\tilde{Z}^{(\lambda)}_{k_B}(u; \tilde{\alpha}^{t}) := \int dx \exp \left( \frac{i}{k_B} u x \right) \tilde{F}_{\lambda}(x; \tilde{\alpha}^{t}),
\]

where \( u = (u_{1}, u_{2}, \ldots, u_{2d}) \) is a parameter varying in a neighbourhood of \( \bar{0} = (0, 0, \ldots, 0) \in \mathbb{R}^{2d} \), \( u x := \sum_{i=1}^{2d} u_{i} x_{i} \), then one can compute the cumulants of fluctuations \( \tilde{C}^{(\lambda)}_{M}(t) \), \( M \in \mathbb{Z}_{\geq 0} \setminus \{0\} \), of the extensive thermodynamic quantities \( (x_{1}, \ldots, x_{d}) \) and their flows \( (x_{d+1}, \ldots, x_{2d}) \) as follows:

\[
\tilde{C}^{(\lambda)}_{M}(t) = \left( -i k_B \frac{\partial}{\partial u} \right)^{M} k_B \ln \tilde{Z}^{(\lambda)}_{k_B}(u; \tilde{\alpha}^{t}) \bigg|_{u=0}.
\]

Note that the function \( \tilde{Z}^{(\lambda)}_{k_B}(i u; \tilde{\alpha}^{t}) \) (if the corresponding analytic continuation exists) restricted to \( u_{d+1} = \cdots = u_{2d} = 0 \) yields an analogue of a partition function in nonequilibrium statistical thermodynamics. The asymptotic expansion of \( \Phi^{(\lambda)}_{k_B}(u; \tilde{\alpha}^{t}) := \lambda^{-1} k_B \ln \tilde{Z}^{(\lambda)}_{k_B}(i u; \tilde{\alpha}^{t}) \) in \( \lambda^{-1} \to 0 \) is an analogue of these data in nonequilibrium quasithermodynamics. The nonequilibrium phenomenological thermodynamics corresponds to the Lagrangian manifold \( \tilde{\Lambda}^{\#} \). The generalized Fokker-Planck
equation is a *phenomenological* equation in this terminology, and a link between the “deformed” cumulants \( \{ C_M^{(\Lambda)}(t) \}_M \) at different moments of time \( t \) is induced by (35).

4. Thermocorpuses

In the semiclassical approximation of quantum theory it is a common practice to denote the small parameter of the asymptotic expansion as \( \hbar \). In this case it is necessary to keep in mind, that the physical value of the Planck constant should not be confused with this parameter. For example, one may denote \( \hbar_{\text{phys}} = 6.6262 \times 10^{-27} \text{ erg s} \) and after that write \( \hbar \to 0 \). In quasithermodynamics, the small parameter is \( \lambda^{-1} \), where \( \lambda \) is the rescaling parameter. In can be convenient to redenote it as \( k_B \) since it stands in the same place as the Boltzmann’s constant in the exponent linking the free energy and the partition function. If we redenote the physical value of the Boltzmann constant, for example, as \( (k_B)_{\text{phys}} = 1.3807 \times 10^{-16} \text{ erg K}^{-1} \), then we may write \( k_B = \lambda^{-1} \to 0 \).

Intuitively, when the Planck’s and Boltzmann’s constants are introduced into physics, \( \hbar_{\text{phys}} \) is related to the quantization of “properties” (for example, the energy spectrum), and \( (k_B)_{\text{phys}} \) is related to the quantization of “substance”. In this sense, the formula for the quantization of energy \( E \) of a 1-dimensional harmonic oscillator is similar to the “quantization” of the number of moles \( \nu \),

\[
E - E_0 = \omega \hbar_{\text{phys}} n, \quad \nu = R^{-1} (k_B)_{\text{phys}} N,
\]

where \( n \in \mathbb{Z}_{\geq 0} \) (the number of quanta), \( N \in \mathbb{Z}_{\geq 0} \) (the number of particles), \( E_0 \) is the ground level of energy, \( \omega > 0 \) and \( R > 0 \) are parameters (the frequency of the oscillator and the universal gas constant, respectively).

In the previous section we have introduced the creation and annihilation operators \( a^\pm(x), x = (X, J) \), where \( X = (X_1, \ldots, X_d) \) are the extensive thermodynamic quantities (like the values of internal energies or the numbers of moles in different parts the system), and \( J = (J_1, \ldots, J_d) \) are the associated flows. The volume of the system \( V \) is fixed (otherwise, we need to add one more coordinate \( x_{2d+1} = V \)). It is natural to perceive these operators as the operators of creation and annihilation operators of “thermoparticles” (or, another name, could be “thermocorpuses”). We work over the space \( \mathbb{R}^d_{X, J} \) rather than \( \mathbb{R}^d_X \), or \( \mathbb{R}^d_J \), in order to avoid the discussion about the statistics of these thermodynamic particles. Are they thermo-fermions, thermo-bosons, or, perhaps, something else? Leaving this for another paper, let us look at two other effects.

**Effect no.1: The thermodynamic Bell’s inequalities.** In (33) we have \( \tilde{F}(x; \tilde{\alpha}), x = (X, J) \), which is an analogue (up to a normalization factor \( L^n \)) of the classical probability distribution (recall, that \( \gamma = \{ \tilde{\alpha} \} \subset \bar{\Lambda} \) defines a curve of evolution of the system). In particular, for \( \tilde{f}(x; \tilde{\alpha}) := L^{-n} \tilde{F}(x; \tilde{\alpha}) \), we have \( \int dx \tilde{f}(x; \tilde{\alpha}) = 1 \), and for any test function \( \varphi(x) \in C^\infty_0(\mathbb{R}^{2d}_x) \), such that \( \varphi(x) > 0 \), we have \( \int dx \varphi(x) \tilde{f}(x; \tilde{\alpha}) \geq 0 \). On the other hand, for the “deformed” distribution \( \tilde{f}_\lambda(x; \tilde{\alpha}) := L^{-n} \tilde{F}_\lambda(x; \tilde{\alpha}) \), one does not impose the latter condition, and there can exist a test function \( \varphi_\lambda \in C^\infty_0(\mathbb{R}^{2d}_x) \), such that \( \varphi_\lambda(x) > 0 \), but \( \int dx \varphi_\lambda(x) \tilde{F}_\lambda(x; \tilde{\alpha}) < 0 \). In mechanics, the fact that the Wigner’s quasiprobability function can be negatively defined over some region of the phase space is responsible for the Bell’s inequalities [1]. In particular, one
may introduce a thermodynamic wave function, considering, for example, a Cauchy problem with an initial condition stemming from an element $\psi_\lambda(X) \in L^2(\mathbb{R}_x^d)$,

$$\tilde{f}(x; \tilde{\alpha}^t)|_{t=0} = \frac{1}{(2\pi \lambda^{-1})^d} \int_{\mathbb{R}^d} dX' \exp \left(-\frac{i}{\lambda^{-1}} J X' \right) \tilde{\psi}_\lambda \left(X - \frac{X'}{2}\right) \psi_\lambda \left(X + \frac{X'}{2}\right),$$

where $X = (x_1, \ldots, x_d)$, $J = (J_1, \ldots, J_d)$. The parameter $\lambda^{-1}$ in this formula is an analogue of the semiclassical parameter $\hbar$ in quantum mechanics. If we take, for instance, a WKB-type function

$$\psi_\lambda(X) = \exp \left(\frac{i}{\lambda^{-1}} S(X)\right) \varphi_\lambda(X),$$

where $S(X)$ is real, and $\varphi_\lambda(X)$ admits an asymptotic expansion in the powers of the small parameter $\lambda^{-1}$, then we should interpret the derivatives $\partial S(X)/\partial X_i$, $i = 1, 2, \ldots, d$, as the flows $J = (J_1, J_2, \ldots, J_d)$ induced by $X = (X_1, X_2, \ldots, X_d)$. The limit $\lambda^{-1} \to 0$ corresponds to the nonequilibrium thermodynamic Hamilton-Jacobi equation. In other words, the phenomenological thermodynamic flows are the analogues of classical mechanical momenta. Intuitively, the leading coefficient

$$\lim_{\lambda \to \infty} \varphi_\lambda(X)$$

is something similar to a Gaussian exponent concentrated near a point. Observe that the quadratic function corresponding to the power of this exponent does not contain the small parameter $\lambda^{-1}$. Considering a thermodynamic system built from several similar subsystems, one may construct an entangled state and violate the Bell’s inequalities. The “observables” in this case are represented by self-adjoint operators on $L^2(\mathbb{R}_x^d)$ depending on a parameter $\tilde{\alpha} \in \tilde{\Lambda}$, and the inequalities mentioned yield a condition on the fluctuations of the measured values of $X_i$ and $J_j$, $i, j = 1, 2, \ldots, d$.

**Effect no.2: The “deformed” Boltzmann’s H-theorem.** The effect no.1 discussed above corresponds to the first quantization of phenomenological nonequilibrium thermodynamics, where the Boltzmann’s constant $k_B$ plays, in a certain sense, a role similar to the Planck’s constant $\hbar$. In statistical mechanics the particles interact with each other (for example, via an interaction potential). The parameter $g$ in front of the interaction potential is sometimes termed the external Planck constant $g = \hbar_{\text{ext}}$, since the commutation relation $[\sqrt{g}a^+(q), \sqrt{g}a^-(q')] = g\delta(q - q')$ for the bosonic creation-annihilation operators in configuration space points $q, q' \in \mathbb{R}^n$ are similar to $[b_i^+, b_j^-] = \hbar\delta_{i,j}$, where $b_i^+ = (q_j \pm \hbar\partial/\partial q_j)/\sqrt{2}$, $i, j = 1, 2, \ldots, n$. In the previous section we have seen, that it is natural to introduce an interaction between the thermocorpuscles (the collision integral). Let $\tilde{F}_\lambda(x; \tilde{\alpha}^t)$, $x = (X, J)$, be of the shape (36) corresponding in the limit $\lambda \to \infty$ to a solution of the generalized Fokker-Planck equation (33). One perceives $\tilde{F}_\lambda(x; \tilde{\alpha}^t)$ as an analogue of a one-particle kinetic function in quantum statistical mechanics. If $\tilde{F}_\lambda(x; \tilde{\alpha}^t)$ corresponds to a symbol of a $\lambda^{-1}$-pseudodifferential operator,

$$\tilde{F}_\lambda(\tilde{\alpha}^t) := \tilde{F}_\lambda \left(X, -i\lambda^{-1} \frac{\partial}{\partial X}; \tilde{\alpha}^t\right),$$

where one uses the Weyl quantization, then it is possible to consider a quantity

$$H_\lambda(\tilde{\alpha}^t) := \text{Tr}\{\tilde{F}_\lambda(\tilde{\alpha}^t)\ln\tilde{F}_\lambda(\tilde{\alpha}^t)\}. $$
In analogy with the Boltzmann’s theorem, the presence of the collision integral implies, that

\[ \frac{\partial H_{\lambda}(\tilde{\alpha}^t)}{\partial t} \leq 0, \]

along the nonequilibrium evolution curve \( \tilde{\gamma} = \{\tilde{\alpha}^t\}_t \subset \tilde{\Lambda}^\# \). This inequality should hold not just in the limit \( \lambda \to \infty \), but for all values of \( \lambda \).

As a final remark, it is of interest to point out the following. A collection of extensive thermodynamic quantities \( X = (X_1, X_2, \ldots, X_d) \) is normally a collection of local internal energies or local densities of chemical substances. If we consider the system together with its environment (i.e. the thermostat) as one big system, then we have conservation laws (for example, the total energy is fixed). It follows, that instead of the canonical symplectic structure \( \omega = \sum_{i=1}^d dJ_i \wedge dX_i \) on the whole phase space \( \mathbb{R}^{2d}_{X,J} \), one may reduce the problem to a smaller phase space separating the integrals of motion (in mechanics this step is termed a reduction of dynamics to a coisotropic submanifold). The reduced symplectic structure can be perceived as a kind of “thermodynamic gauge field” created by the environment.

5. Discussion

The main motivation of the present paper is a striking (though underappreciated) analogy between the roles played by the Planck’s and the Boltzmann’s constants in the mathematical formalism of theoretical physics. These two constants were introduced into science at more or less the same time (the beginning of the XX-th century) and in a sense capture the \( \text{Zeitgeist} \) (the spirit of the age). The concept of a thermocorpuscle, i.e. a thermodynamic particle with the thermodynamic forces \( X = (X_1, X_2, \ldots, X_d) \) in place of coordinates (internal energy, number of moles, volume, etc.), and the associated nonequilibrium thermodynamic flows \( J = (J_1, J_2, \ldots, J_d) \) in place of momenta, is, in a sense, quite natural if one investigates the analogy between the Heisenberg’s uncertainty principle and the Einstein’s quasithermodynamic fluctuation theory. Informally speaking, the Planck’s constant \( h \) and the Boltzmann’s constant \( k_B \) are very similar concepts, but corresponding to different “hierarchical” levels of organization of matter.

The symmetry between the mechanical and thermodynamic pictures is very much in line with the general philosophy of I. Prigogine [18] who tried to introduce the “irreversibility on a microscopic level of description”. A thermocorpuscle is a counterpart of a quantum particle, with \( h \) replaced by \( k_B \). It is of interest to point out, that the analogy between \( h \) and \( k_B \) is a general motivation behind the mathematical constructions in [21] and [22], where one describes a kind of “noncommutative neighbourhood” around the semiclassical and quasithermodynamic parameters. Perhaps even more complicated new theoretical structures are needed to capture the nonequilibrium statistical physics in all its aspects.

At this point it is possible to claim that the “quantization” of phenomenological thermodynamics (with \( k_B \) in place of \( h \)) and the “second quantization” of the quantized thermodynamics appear to be quite reasonable constructions. In particular, conceptually it is important to distinguish between the phenomenological thermodynamics, statistical thermodynamics, and the thermodynamics that lives between these two theories, the quasithermodynamics. This is similar to mechanics: there
is classical mechanics, quantum mechanics, and the mechanics existing between the two theories, the semiclassics [11]. If we look at the deformation quantization theory [7], then it is a standard practice to denote the “formal” parameter in the star product as $\hbar$. Now one can see, that $k_B$ is just as good:

$$f(X,J) \ast_{k_B} g(X,J) = f(X,J)g(X,J) + \sum_{s=1}^{\infty} k_B^s B_s(f,g)(X,J),$$

where $f, g \in C^\infty(\mathbb{R}^{2d}_{X,J})$, $B_s$ are bilinear differential operators, $s = 1, 2, 3, \ldots$.

In our days, due to the progress in the field of nanotechnologies, one can directly work with systems consisting of relatively small numbers of quantum particles (consider microtransistors in electronics, molecular machines in biological physics, etc.). For example, the size of a single transistor on a microchip is measured in hundreds of atoms, but certainly not in the scale of the Avogadro number. The usual empirical laws stemming from phenomenological thermodynamics, for example, the linear Ohm’s law for the electric current, break down at this scale. On the other hand, the systems are still too big to be perceived purely quantum mechanically. One needs a kind of “nanothermodynamics” in this case. If we look at mechanics and imagine that we go from a classical description to a quantum description via the semiclassics, then one can say that we first construct a Schrödinger equation with a semiclassical parameter $\hbar \rightarrow 0$ in front of partial derivatives, and then formally substitute $\hbar = 1$ (this corresponds to the “real” Schrödinger equation with $\hbar$). It looks like a natural speculation, that if one performs a similar step in quasithermodynamics (substitute $\lambda^{-1} = 1$), then one obtains the “real” quantized thermodynamics.

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Department of Mathematics and Computer Science, University of Antwerp, Middelheim Campus Building G, Middelheimlaan 1, B-2020, Antwerp, Belgium

E-mail address: artur.ruuge@ua.ac.be