Exactly solvable model of dissipative vortex tunneling

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I consider the problem of vortex tunneling in a two-dimensional superconductor. The vortex dynamics is governed by the Magnus force and the Ohmic friction force. Under-barrier motion in the vicinity of the saddle point of the pinning potential leads to a model with quadratic Hamiltonian which can be analytically diagonalized. I find the dependence of the tunneling probability on the normal state quasiparticle relaxation time $\tau$ with a minimum at $\omega_0\tau \sim 1$, where $\omega_0$ is the level spacing of the quasiparticle bound states inside the vortex core. The results agree qualitatively with the available experimental data.

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I. INTRODUCTION

The dynamics of vortices in Type-II superconductors has been one of the most attractive areas of research in recent years because of its importance from scientific as well as technological point of view. To science, vortices present an example of extended topologically stable objects with extremely rich properties. On the technological side, the motion of vortices is the source of supercurrent dissipation - the circumstance that limits high magnetic fields application of high-$T_c$ materials. In view of this fact, the exploration of the limits of strong pinning of vortices has been actively pursued both in theory and experiment. It has long been known that the thermally-activated depinning rate strongly decreases as the temperature is lowered and eventually saturates at values believed to be set by quantum tunneling. The present article explores this quantum regime.

The difficulty of the problem is in the fact that there are two major forces that govern the dynamics of the vortex: Hall (Magnus) force and friction force. While the first one is conservative and can be treated through a single vortex description, the second one is not: the energy dissipates into the environment. Therefore, one needs to consider the combined vortex-plus-environment system. In the pioneering paper by Feigel’man et al., which took into account both forces, the environmental degrees of freedom were integrated out with the use of path-integral techniques to produce an effective vortex action description. The time non-locality of the resulting action, however, limited the analysis to qualitative conclusions. Later, the same approach was independently taken by Morais Smith et al., but they also failed to go beyond the scaling analysis of the effective action.

This paper is devoted to the study of an exactly solvable model of two-dimensional dissipative vortex tunneling. It has a quadratic Hamiltonian of the vortex coupled to the environment and is solvable by an analytic diagonalization. The present model is a generalization of the quadratic model of one-dimensional dissipative particle tunneling that was solved by Ford et al. The results of the solution show the dependence of the tunneling rate on the ratio of the Magnus and dissipative forces. The tunneling rate has a minimum near the point where these two forces become equal.

This paper is organized as follows: I first start with a non-dissipative case in Sec. 2; the dissipative model is introduced in Sec. 3, where some of its general features are analyzed; in Sec. 4 I describe the solution of the dissipative tunneling problem and compare the results with available experimental data; conclusions follow in Sec. 5. Units with $\hbar = 1$ are used throughout the paper.

II. VORTEX TUNNELING: NON-DISSIPATIVE CASE

In order to analyze the motion of a vortex one needs first to identify the forces that govern its dynamics. Both in classical and in quantum fluids there is an intrinsic Magnus force that acts on a vortex when it moves relative to a (super)fluid. This force is normal both to the vortex tangent and to the vortex velocity, relative to the fluid, and is linear with in the latter. In this respect, it is very similar to Lorentz force that acts on charged particles moving in magnetic fields. Since Magnus force does not produce any work, it is possible to formulate vortex dynamics using Hamiltonian formalism. This formalism was successfully applied to the problem of quantum-mechanical vortex nucleation by Volovik. However, it was not until much later that Haldane and Wilczek realized that, just like Lorentz force is a manifestation of a geometric (Aharonov-Bohm) phase associated with a motion of a particle, Magnus force should arise as a consequence of a geometric phase associated with vortex motion. Later, Ao and Thouless extended this idea to vortices in superconductors. To clarify the physics involved, I will sketch the derivation of the geometric phase done by Ao and Thouless.
Consider the ground state many-body wave function of a two-dimensional superconductor: \( \Psi_0(\mathbf{r}_1, \ldots, \mathbf{r}_N) \), where \( \mathbf{r}_j = (x_j, y_j) \) are the coordinates of electrons. A vortex at \( \mathbf{r} = (x, y) \) can be created by “spinning up” the system:

\[
\Psi_0 = \exp \left\{ \frac{i}{2} \sum_{j=1}^{N} \theta(\mathbf{r}_j - \mathbf{r}) \right\} \Psi_0'(\mathbf{r}_1, \ldots, \mathbf{r}_N; \mathbf{r}).
\]

Here it is assumed that \( \Psi_0 \) is “close” to \( \Psi_0 \), i.e., it is obtained from the latter by a continuous deformation; \( \theta(\mathbf{r}_j - \mathbf{r}) = \arctan \left( \frac{y - y_j}{x - x_j} \right) \). Note that the factor \( \frac{i}{2} \) in the exponent comes from the fact that the condensate is made of Cooper pairs. Now one can adiabatically move \( \mathbf{r} \) around a closed contour keeping the coordinates of electrons fixed. The phase accumulated at the end will come from the exponent and will be given by \( \pi \) times the number of electrons encircled by the path of the vortex. This phase can be described by the inclusion of a geometric phase term in the action:

\[
S = \int dt \alpha x(t)y(t).
\]

Here \( \alpha = \pi n_s \) is the Hall coefficient and \( n_s \) is the density of electrons in the condensate. One sees that the length scale, associated with this geometric phase term is \( 1/\sqrt{n_s} \), which is one of the smallest in the problem. One expects then that the dynamics will be dominated by the Magnus force.

Next I include the vortex potential energy in the action:

\[
S = \int dt \left[ \alpha \dot{x} y - V(x, y) \right].
\]

To clarify the picture I go over to the Hamiltonian formulation. First of all, it is easy to identify canonically conjugate variables. Indeed, from the above action one sees that the variable, conjugate to \( x \) is \( \alpha y \). The Hamiltonian is then equal to the potential energy:

\[
H = V(x, y),
\]

\[
[x, y] = \frac{i}{\alpha}.
\]

For an arbitrary \( V(x, y) \) there is a problem of operator ordering. This problem has been addressed by Girvin and Jach.\(^\text{[6]}\) In the model studied below, however, this difficulty does not appear.

A comment should be made about the relevance of vortex mass which is not included in the model. Using the analogy between a superconducting vortex and a particle in magnetic field (i.e., similar origins of the corresponding geometric phases) this omission of the vortex mass amounts to the effective Lowest Landau Level approximation. Such an approximation is warranted if the transitions to higher Landau levels can be neglected. This can be checked by comparing matrix elements \( |V_{nm}| \) of the potential energy \( V(x, y) \) between different Landau levels with the spacing \( \omega_c \) between Landau levels. The ratio \( |V_{nm}|/\omega_c \) depends on the particular potential under consideration and can be estimated in our problem of vortex tunneling to be \( \approx 2\pi/S \), where \( S \) is the action involved in tunneling. The value of this action can be extracted from experimental data (to be discussed in Sec. 4) and is \( \approx 50 \). Thus, the ratio \( |V_{nm}|/\omega_c \approx 0.1 \) and the omission of the vortex mass is indeed warranted. However, it is possible that in some other situations the mass of a vortex is relevant and should be included. This can be easily done since the mass term in the Hamiltonian is quadratic and thus does not lead to any substantial complications.

Consider now the tunneling problem in which an impurity potential, which pins the vortex, is tilted by a Magnus force due to an applied depinning supercurrent (see Fig. 1). For a vortex to tunnel out of the pinning site it has to overcome a potential barrier. This leads to a thermally-activated depinning rate at high temperatures. Here I am considering the problem of quantum tunneling which gives dominant contribution to the depinning rate at low temperatures. One can argue then that for sufficiently strong depinning supercurrents the tunneling exponent is dominated by the under-barrier motion in the vicinity of the saddle point of the potential (see Fig. 1). This leads one to consider the problem of tunneling in the following model potential:

\[
V = \frac{\kappa_y}{2} y^2 - \frac{\kappa_x}{2} x^2, \quad \kappa_x, \kappa_y > 0.
\]

This becomes the problem of tunneling across an inverted parabolic barrier \(-\kappa_x x^2\) in 1D quantum mechanics if one identifies the momentum \( p = \alpha y \), and the mass \( m = \alpha^2/\kappa_y \) of the 1D particle. The transmission coefficient for this problem can be calculated exactly and is given by

\[
\frac{\kappa_y}{2} y^2 - \frac{\kappa_x}{2} x^2. \]
\[ D_0(E) = \frac{1}{1 + \exp \left( \frac{2\pi E}{\sqrt{m}} \right)} \]  
\[ \Omega_0 = \sqrt{\frac{n_x}{m}} = \frac{\sqrt{n_x k_B}}{\alpha}. \]

Here \(-E\) is the energy of the vortex as measured from the value at the saddle point. \(E\) equals the activation energy in the high temperature thermally-assisted tunneling regime.

In comparison, the previous works on the dissipative vortex tunneling problem[15] considered the model with an added cubic term: \( V = \frac{\kappa_3}{3} y^3 - \frac{\kappa_5}{5} x^5 - \frac{\kappa_7}{7} x^7 \) which has a minimum at \( x = -\frac{2\kappa_7}{\kappa_5} \). I argue that the role of the cubic term is mainly to set the position of the localized state and its energy and that it is not essential for the tunneling itself. Meanwhile, setting \( \kappa_3 = 0 \) produces a quadratic Hamiltonian which allows for an exact diagonalization when the dissipation is added. Finally, let me mention that the model Eq. (5) (without dissipation) has been considered before by Fertig and Halperin[14] in a different context: the tunneling of electrons in two dimensions in the presence of a strong magnetic field. It later became the basis for the Chalker-Coddington model of the Integer Quantum Hall transition[13].

The result Eqs. (6,8) shows an interesting feature of the vortex tunneling problem. If \( \kappa_y \) is decreased while both \( \kappa_x \) and the height of the barrier \( E \) are kept constant the transmission coefficient \( D_0 \) decreases implying stronger pinning. In particular, for \( \kappa_y = 0 \) the vortex no longer tunnels across the barrier. This result can be understood by observing that for \( \kappa_y = 0 \) the Hamiltonian becomes invariant with respect to the \( y \) coordinate of the vortex. This leads to the conservation of the conjugate momentum which is \( \alpha x \). Thus, the tunneling process in which \( x \) changes is forbidden by a new conservation law. This means that the pinning potentials which are translationally invariant along the direction of the biasing supercurrent are good candidates for strong vortex pinning.

III. DISSIPATION

So far we have considered a vortex in a pure superconductor. In reality, the presence of impurities that sets the normal state resistance at low temperatures introduces a finite quasiparticle scattering time \( \tau \). This fact has to be taken into account and the adiabatic arguments presented above need to be modified. These modifications have been worked out by Kopnin and Kravstov[16] and by Kopnin and Salomaa[11] (for a simple new look at the problem see also[14]). They showed that a finite relaxation time breaks the adiabaticity in the spectral flow of the quasiparticle vortex core bound states and leads to two main effects: the Hall coefficient \( \alpha \) is reduced from its pure value \( \pi n_s \), and there appears a friction force acting on the vortex, \( F = -\eta v \). The coefficients \( \alpha \) and \( \eta \) are given by:

\[ \alpha = \pi n_s \frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2}, \]
\[ \eta = \pi n_s \frac{\omega_0 \tau}{1 + (\omega_0 \tau)^2}. \]

Here, \( \omega_0 \) is the level spacing of quasiparticle bound states inside the vortex core. Thus, the parameter \( \omega_0 \) sets relative importance of the Magnus and friction forces. This can be quantified by defining the Hall angle: \( \tan \Theta_H = \alpha/\eta = \omega_0 \tau \). Thus, only in the “supercon by limit” \( \omega_0 \tau \gg 1 \) can one neglect the dissipative force and consider a pure quantum problem.

It should be mentioned that there is an alternative point of view expressed by Ao et al[17] according to which the Hall coefficient is a topological number and thus is not renormalized. It turns out that the analysis of experimental results of van Dalen et al[20] presented in Sec. 4 suggests that there is indeed a renormalization of the Hall coefficient \( \alpha \). Meanwhile, I take \( \alpha \) and \( \eta \) as two phenomenological parameters.

In the following I use the approach of Ford et al[20] to the problem of dissipation in quantum mechanics. I model the friction force by a linear coupling to a bath of oscillators (to simplify notation I use \( x_i, \alpha = \{1, 2\} \) for \( x \) and \( y \) coordinates of the vortex):

\[ H = V(x_\alpha) + \sum_{aj} \left[ \frac{p_{aj}^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 (q_{aj} - x_\alpha)^2 \right] \]
\[ + \frac{\alpha}{\eta} \sum_j \frac{1}{\alpha} \left[ q_{aj} \right] - \frac{1}{\alpha} \left[ p_{aj} \right] + \eta \delta_{ab} \delta_{ij} \]

Vortex coordinates \( x_1 \) and \( x_2 \) are coupled to two independent identical sets of oscillators in a simple way: they shift their equilibrium positions. The difference between this model the one considered by Ford et al[20] is that, due to a different physical context, here both \( x_1 \) and \( x_2 \) need to be coupled to the oscillator baths in order to account for friction in both directions.

The Heisenberg equations of motion corresponding to Eq. (10) are:

\[ \alpha \epsilon_{ab} \dot{x}_b = -\frac{\partial V}{\partial x_a} + \sum_j m_j \omega_j^2 (q_{aj} - x_\alpha), \]
\[ \dot{q}_{aj} + \omega_j^2 q_{aj} = \omega_j^2 x_\alpha. \]

The solution of Eq. (13) is:

\[ q_{aj}(t) = q^h_{aj}(t) + x_\alpha(t) - \int_{-\infty}^{t} dt' \cos[\omega_j(t - t')] \dot{x}_\alpha(t'), \]

where \( q^h_{aj}(t) \) is the solution of the corresponding homogeneous equation. Substituting Eq. (14) into Eq. (12) one obtains:
\[
\alpha_{ab} \ddot{x}_b = -\frac{\partial V}{\partial x_a} - \int_{-\infty}^{t} dt' \mu(t-t') \dot{x}_a(t') + N_a(t), \tag{15}
\]

\[
\mu(t) = \theta(t) \sum_j m_j \omega_j^2 \cos(\omega_j t),
\]

\[
N_a(t) = \sum_j m_j \omega_j^2 a^h_j(t).
\]

Here \(\mu(t)\) is the so-called memory kernel, \(N_a(t)\) is the noise force and \(\theta(t)\) is the Heaviside step function.

Next I consider the so-called Ohmic case when the friction force acting on the vortex is strictly linear in the vortex velocity. This is realized by taking the following distribution of oscillator strengths:

\[
\eta(\omega) = \frac{\pi}{2} \sum_j m_j \omega_j^2 \left[ \delta(\omega - \omega_j) + \delta(\omega + \omega_j) \right] = \text{const.} \tag{18}
\]

With this choice Eq. (15) becomes:

\[
\alpha_{ab} \ddot{x}_b = -\frac{\partial V}{\partial x_a} - \eta \dot{x}_a + N_a(t). \tag{19}
\]

Naturally, this can be called the force balance equation: it balances Magnus, potential, friction and noise forces. This equation shows that the model Eq. (10) indeed describes friction force linear in the velocity. A more general version of this equation is known as the quantum Langevin equation.

**IV. DISSIPATIVE VORTEX TUNNELING**

Returning to our problem of tunneling in the vortex depinning by an external supercurrent, one now has to consider Eq. (10) with \(V\) given by Eq. (11). The crucial observation due to Ford et al. (made for a similar problem) is that the resulting system is a set of coupled oscillators with one of them having a negative spring constant \((-\kappa_x)\). This leads to a spectrum which contains an isolated mode with purely imaginary frequency \(\Omega^*\). After the diagonalization of the Hamiltonian one obtains an oscillator with a (shifted) negative spring constant that is decoupled from the rest of the oscillators (all with positive spring constants). It is this new negative spring constant that determines the tunneling amplitude. The transmission coefficient is then given by Eq. (11) with a substitution of \(\Omega_0 \to \Omega^*\):

\[
D(E) = \frac{1}{1 + \exp \left\{ \frac{2\pi E}{\hbar \Omega^*} \right\}}. \tag{20}
\]

The value of the imaginary eigenmode frequency \(i\Omega^*\) can be found by diagonalizing the Hamiltonian; a much easier way is to observe that this eigenmode should satisfy the force balance equation, Eq. (19), without the noise term \(i.e., the corresponding equation for the expectation values; it should be stressed that noise term can be separated due to the linearity of the equations). This leads to the following equation on \(\Omega^*\):

\[
\det \begin{pmatrix}
-\kappa_x + \eta \Omega^* & \alpha \Omega^* \\
-\alpha \Omega^* & -\kappa_y + \eta \Omega^*
\end{pmatrix} = 0. \tag{21}
\]

It is easy to understand this equation qualitatively using the following simple arguments: roughly, all oscillators in the bath (see Eq. (10)) can be divided into two types, those with frequencies smaller and bigger than \(\Omega^*\). Oscillators with small frequencies do not follow the motion of the vortex and each one effectively increases the value of \(\kappa_y\) and \(-\kappa_x\) by \(m_j \omega_j^2\). Those with high frequencies adjust to the vortex motion and thus do not contribute to the increase of \(k_x\)'s. Then a simple estimate gives:

\[
\kappa_y \to \kappa_y + \sum_{\omega_j \leq \Omega^*} m_j \omega_j^2 \approx \kappa_y + C \eta \Omega^*,
\]

\[
-\kappa_x \to -\kappa_x + \sum_{\omega_j \leq \Omega^*} m_j \omega_j^2 \approx -\kappa_x + C \eta \Omega^*, \tag{22}
\]

with \(C \sim 1\). This indeed agrees qualitatively with the correct Eq. (21).

Eqs. (20,21) constitute the solution of the problem. In the rest of this section the implications of this solution for the existing experimental data of van Dalen et al. will be discussed. In these experiments quantum dynamical relaxation rate of magnetization \(Q\) was measured in oxygen depleted thin films of \(\text{YBa}_2\text{Cu}_3\text{O}_x\) as a function of the oxygen content \(x\). Changing \(x\) changes the normal state quasiparticle relaxation time \(\tau\) which, in turn, is expected to change in the Hall and friction coefficients. I use expressions in Eq. (14) to compare the results of the solution of the present model with the experiment.

The dynamical zero temperature magnetization relaxation rate is \(Q = 1/S\) where \(S\) is the action involved in a tunneling event. It determines the transmission coefficient of the pinning barrier as \(D \sim \exp(-S)\). In reality \(Q\) is small: \(Q_\infty \equiv Q(\omega_0 \tau \to \infty) \approx 0.02\), therefore Eq. (21) implies: \(Q = \Omega^*/2\pi E\). As the oxygen content, and thus
is varied \( \Omega^* \) (given by the solution of Eqs. (21, 31)) changes while \( E \) (the height of the pinning barrier) stays constant. In Fig. 3 the calculated ratio \( Q/Q_\infty \) is plotted as a function of \( \omega_0 \tau \) for several values of \( \delta = \kappa_x / \kappa_y \). A simple calculation gives that \( \delta \sim \sqrt{1 - J/J_c} \), where \( J_c \) is the critical biasing current density at which the minimum of the pinning potential disappears.

From Eq. 3 one can see that the Hall angle is \( \tan \Theta_H = \alpha/\eta = \omega_0 \tau \), therefore in the “superclean limit” \( \omega_0 \tau \to \infty \) the dynamics is dominated by the Magnus force, while in the opposite “dirty limit” \( \omega_0 \tau \to 0 \) it is dominated by the force of friction. Fig. 2 shows that in the superclean limit \( Q \) saturates while in the dirty limit it varies as \( 1/\omega_0 \tau \). Qualitatively, both of these behaviors were known before. What was not known, however, was the behavior of \( Q \) in the crossover region \( \omega_0 \tau \sim 1 \).

In the experiments of van Dalen et al.\(^{21}\) only the regime \( \omega_0 \tau \leq 1 \) was realized. There \( Q \) shows roughly a \( 1/(\omega_0 \tau) \) dependence. In the region \( \omega_0 \tau \sim 1 \) the onset of a crossover in \( Q \) is visible, but the minimum of \( Q \) can hardly be observed since no data are available in the superclean limit \( \omega_0 \tau \gg 1 \). In view of this, experiments on cleaner films would be desirable to check the existence of the minimum in \( Q \) - the main qualitative predictions of the above analysis.

Finally, it should be emphasized that the \( 1/(\omega_0 \tau) \) behavior of \( Q \) in the dirty limit that was observed in the experiment is a clear evidence in favor of the correctness of Eq. (3). Indeed, if the Hall coefficient \( \alpha \) were not renormalized by disorder it would imply that \( Q \) in the dirty limit should always be smaller than its superclean limit value \( Q_\infty \) due to the general tendency of dissipation to suppress tunneling.\(^{22}\) The experimental results of van Dalen et al.\(^{23}\) show that this is not the case.

V. CONCLUSION

In this article, I have considered a model of dissipative vortex tunneling. In this model the system-plus-environment Hamiltonian is quadratic and can be analytically diagonalized. The results were obtained for the dynamical magnetization relaxation rate \( Q \) as a function of the Hall angle \( \tan \Theta_H = \alpha/\eta = \omega_0 \tau \), where \( \omega_0 \) is the level spacing of the quasiparticle vortex core bound states. The results show a \( 1/(\omega_0 \tau) \) dependence of \( Q \) in the “dirty” limit \( \omega_0 \tau \ll 1 \), saturation in the “superclean” limit \( \omega_0 \tau \gg 1 \) and a minimum at \( \omega_0 \tau \sim 1 \). These predictions were compared with the available experimental data of van Dalen et al.\(^{21}\) Results agree qualitatively in the “dirty” regime \( \omega_0 \tau \ll 1 \). On the other hand, the predicted minimum cannot be found since no data are available in the “superclean” regime \( \omega_0 \tau \gg 1 \).

Although I have only considered here the case of a vortex in a two-dimensional film, the results can be carried over to the three-dimensional case as well. Following the arguments of Brandt\(^{23}\) one can show that the effect of the extra elastic term in the energy of a vortex line is mainly to set a length scale along the vortex - the size of the tunneling nucleus. Meanwhile, the effect of dissipation on vortex tunneling should be qualitatively the same.

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