Monodromy operators and symmetric correlators

D Chicherin\(^1\) and R Kirschner\(^2\)

\(^1\)St. Petersburg Department of Steklov Mathematical Institute of Russian Academy of Sciences, Fontanka 27, 191023 St. Petersburg, Russia and Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, St. Petersburg, 199178 Russia

\(^2\)Institut für Theoretische Physik, Universität Leipzig, PF 100 920, D-04009 Leipzig, Germany.

E-mail: Roland.Kirschner@itp.uni-leipzig.de

Abstract. Yangian symmetric correlators are defined as eigenfunctions of monodromy operators. Examples and relations are given and an evaluation method of the symmetry condition is described.

1. Yangian symmetry

We consider \(N\)-point correlators in \(n\) dimensions, i.e. functions of \(nN\) variables interpreted as \(N\) points \(x_i\) with coordinates \((x_{a,i})^{n}_{a=1} = (x_{1,i}, x_{2,i}, \cdots, x_{n,i})\). We shall study a symmetry condition imposed on the correlators in terms of the monodromy matrix of a \(N\)-site spin chain with Jordan-Schwinger-type \([1, 2, 3, 4, 5, 6, 7, 8, 9, 10]\) representations of \(g_{\ell}n\). Correlators with this Yangian symmetry are known to appear as kernels of Yang-Baxter operators, as evolution kernels describing high-energy asymptotics of QCD and composite operator renormalisation and as scattering amplitudes \([11, 12, 13, 14]\).

The notion of Yangian algebra was introduced by Drinfeld \([18]\) in general form. The case related to \(g_{\ell}n\) was worked out earlier by L. Faddeev and collaborators \([19, 20, 21]\) in the QISM formulation. The relation of the latter formulation to the one in algebra generators by Drinfeld is well explained in \([22]\).

In this contribution we recall the concept and shortly review some results of our paper \([23]\). Beyond this, we describe in detail a method of evaluating the symmetry condition and work out non-trivial examples.

Associated with the coordinates we shall work with \(nN\) Heisenberg canonical pairs \(p_i = (p_{a,i})^{n}_{a=1} = (p_{1,i}, p_{2,i}, \cdots, p_{n,i})\) and \(x_i = (x_{a,i})^{n}_{a=1} = (x_{1,i}, x_{2,i}, \cdots, x_{n,i})\), \([p_{a,i}, x_{b,j}] = \delta_{a,b}\delta_{i,j}\). We consider the partition of the index set \(N = \{1, 2, \cdots, N\}\) labeling these points in two non-overlapping subsets \(I\) and \(J\), \(I \cup J = N, I \cap J = \emptyset\). The partition of the set of \(N\) sites into the subsets \(I, J\) can also be denoted as signature, i.e. by a sequence of symbols "\(^+\)" and "\(^-\)" where the symbol "\(^+\)" is put for a site in \(I\) and "\(^-\)" for a site in \(J\).

We define the action of \(g_{\ell}n\) on the points in dependence of their signature as

\[
\delta x_{c,i} = \left[ L^+_i \right]_{a,b} x_{c,i}, \quad i \in I, \quad \delta x_{c,j} = \left[ L^-_j \right]_{a,b} x_{c,j}, \quad j \in J
\]

\[
\left[ L^+_i \right]_{a,b} = p_{a,i} x_{b,i}, \quad \left[ L^-_j \right]_{a,b} = -x_{a,j} p_{b,j}.
\] (1)
We denote the inner product by $(\cdot, \cdot)$,

$$(ji) \equiv (ij) \equiv (x_i \cdot x_j) \equiv x_{a,i}x_{a,j}. \quad (2)$$

and notice that for $i \in I, j \in J$ it is $gl_n$ invariant in the sense $[L_i^+ + L_j^-, (x_i \cdot x_j)] = 0$. Monomials of the form

$$\Phi_{I,J} = \prod_{i \in I, j \in J} (x_i \cdot x_j)^{\lambda_{ij}} \quad (3)$$

are $gl_n$ invariant and general $gl_n$ invariant correlators are superpositions of such monomials with varying exponents which can take complex values. In general the coordinates are complex valued.

The $L$ matrices are defined in terms these generator matrices by adding a spectral parameter being a complex number,

$$L_k^+(u_k) = u_k + p_kx_k, \quad L_k^-(u_k) = u_k - x_kp_k, \quad (4)$$

or in component form

$$[L_k^+(u_k)]_{a,b} = u_k\delta_{a,b} + p_{a,k}x_{b,k}, \quad [L_k^-(u_k)]_{a,b} = u_k\delta_{a,b} - x_{a,k}p_{b,k}. \quad (5)$$

The $L$ matrices are well known as the basic tool of treating the spin chain by the Quantum Inverse Scattering Method (QISM) [19, 20, 24, 25].

We are going to study symmetry conditions on correlators going beyond the mentioned $gl_n$ invariance to be formulated in terms of the monodromy matrix

$$T^{\alpha_1 \cdots \alpha_N}_{1 \cdots N}(u_1, \cdots, u_N) = L_{1}^{\alpha_1}(u_1)L_{2}^{\alpha_2}(u_2) \cdots L_{N}^{\alpha_N}(u_N), \quad (6)$$

where $\alpha_1 \alpha_2 \cdots \alpha_N$ denotes the signature, i.e. $\alpha_k = \pm$ at $k = 1, \cdots, N$. The lower indices refer to the chain site with the representation (the quantum space) where the operators act nontrivially.

A Yangian symmetric correlator is defined as a $gl_n$ invariant correlator of definite dilatation weights being a solution of the monodromy eigenvalue relation

$$T(u_1, \cdots, u_N) \cdot \Phi(u_1, \cdots, u_N) = E(u_1, \cdots, u_N) \Phi(u_1, \cdots, u_N). \quad (7)$$

Here the matrix $T$ with operator elements acts on the correlator function resulting in the r.h.s. proportional to the unit matrix.

2. Similarity transformations and eigenfunctions

The study of similarity transformations of the monodromy matrix by monomial expressions (3) allows to obtain simple symmetric correlators. The action of the similarity transformation on a single $L$-operator depends on whether its label falls into the sets $i \in I$ or $j \in J$,

$$(x_i \cdot x_j)^{-\lambda_{ij}}L_i^+(u_i)(x_i \cdot x_j)^{\lambda_{ij}} = L_i^+(u_i) + \lambda_{ij}l_{ij}$$
$$\quad (x_i \cdot x_j)^{-\lambda_{ij}}L_i^-(u_j)(x_i \cdot x_j)^{\lambda_{ij}} = L_i^-(u_j) - \lambda_{ij}l_{ij} \quad (8)$$

where we use the abbreviation $l_{ij} \equiv \frac{x_i \cdot x_j}{(x_i \cdot x_j)^{\lambda_{ij}}}$. In the following we also adopt the notation $\Phi \circ T = \Phi T \Phi^{-1}$.

It is easy to perform the similarity transformation of the monodromy matrix of the two-site chain, $T^{+\cdots}(u_1, u_2)$ (6), using (8)

$$(x_1 \cdot x_2)^{\lambda_{12}} \circ L_i^+(u_1)L_i^-(u_2) = (L_i^+(u_1 + \lambda_{12}) + \lambda_{12}(l_{21} - 1))(L_i^-(u_2 - \lambda_{12}) + \lambda_{12}(1 - l_{21})).$$
Then taking into account the relations $x_1 l_{21} = x_1$, $l_{21} x_2 = x_2$, $l_{21} (l_{21} - 1) = 0$ we obtain

$$(x_1 \cdot x_2)^{\lambda_{12}} \circ T^{+-}(u_1, u_2) = T^{+-}(u_1 + \lambda_{12}, u_2 - \lambda_{12}) + \lambda_{12} (u_1 - u_2 + \lambda_{12}) (1 - l_{21}). \quad (9)$$

At the special value $\lambda_{12} = u_2 - u_1$ the remainder in the previous formula vanishes

$$(x_1 \cdot x_2)^{u_2-u_1} \circ T^{+-}(u_1, u_2) = T^{+-}(u_2, u_1). \quad (10)$$

Such similarity transformation leads to the permutation of the two spectral parameters $u_1 \leftrightarrow u_2$. Then applying both sides of (9) to the basic state $1$ one obtains the eigenvalue relation for the monodromy matrix $T$ of the latter equation on the basic state $1$ we find the eigenvalue relation does not turn to zero for any nonzero $\lambda$

$$u \leftrightarrow u$$ implies parameter permutation while in the second the parameters are untouched.

2 transpositions reducing the number of chain sites by one and apply the second similarity transformation by

$$x 

Next we turn to the configuration $++$ considering the similarity transformation of the monodromy matrix $T^{-+}(u_1, u_2)$. Similar to the previous calculation one obtains easily

$$(x_1 \cdot x_2)^{\lambda_{21}} \circ T^{-+}(u_1, u_2) = T^{++}(u_1, u_2) + \lambda_{21} l_{21} (u_1 - u_2 - \lambda_{21} - (p_1 \cdot x_1) - (x_2 \cdot p_2)). \quad (12)$$

Contrary to (10) the spectral parameters stay untouched and the operator remainder term does not turn to zero for any nonzero $\lambda_{21}$. Taking $\lambda_{21} = u_1 - u_2 - n$ and acting with both sides of the latter equation on the basic state $1$ we find the eigenvalue relation

$$T^{-+}(u_1, u_2) \cdot \Phi^{+-} = u_1 (u_2 + 1) \Phi^{-+}, \quad \Phi^{-+} = (x_1 \cdot x_2)^{u_1-u_2-n}. \quad (13)$$

We shall see further that the relations (10) and (12) are crucial in our discussion. The first one implies parameter permutation while in the second the parameters are untouched.

Let us consider the three point correlator with signature $++-$. The similarity transformation of the monodromy matrix $T^{+++}(u_1, u_2, u_3)$ by $\Phi^{++-} = (x_1 \cdot x_3)^{\lambda_{12}} (x_2 \cdot x_3)^{\lambda_{23}}$ is performed in two steps. At first we permute $u_2 \leftrightarrow u_3 (9)$,

$$(x_2 \cdot x_3)^{\lambda_{23}} \circ T^{+-+}(u_1, u_2, u_3) = T^{++-}(u_1, u_3, u_2), \quad \lambda_{23} = u_3 - u_2.$$

Then we act on a constant function in second space $l_2$,

$$T^{+++}(u_1, u_3, u_2) \cdot l_2 = (u_3 + 1) T^{+++}_{l_3} (u_1, u_2)$$

reducing the number of chain sites by one and apply the second similarity transformation by means of (9) obtaining

$$E^{++-} = u_1 (u_2 + 1) (u_3 + 1), \quad \Phi^{++-} = (13)^{u_2-u_1} (23)^{u_3-u_2}. \quad (14)$$

Now we consider the 4-point correlator with signature $+-+$. The pattern to implement the similarity transformation of $T^{++-}$ by

$$\Phi^{++-} = (x_1 \cdot x_2)^{\lambda_{12}} (x_1 \cdot x_4)^{\lambda_{14}} (x_2 \cdot x_3)^{\lambda_{23}} (x_3 \cdot x_4)^{\lambda_{34}} \quad (15)$$

is analogous to the above calculations of the three-point correlation functions. At first the transpositions $u_1 \leftrightarrow u_2$ and $u_3 \leftrightarrow u_4$ are performed due to (9)

$$(x_1 \cdot x_2)^{\lambda_{12}} (x_3 \cdot x_4)^{\lambda_{34}} \circ T^{++-}(u_1, u_2, u_3, u_4) = T^{++-}(u_2, u_1, u_4, u_3)$$

at $\lambda_{12} = u_2 - u_1$, $\lambda_{34} = u_4 - u_3$. Then we apply (12)

$$(x_2 \cdot x_3)^{\lambda_{23}} \circ T^{++-}(u_2, u_1, u_4, u_3) = T^{++-}(u_2, u_1, u_4, u_3) + \hat{r}$$
and notice that at $\lambda_{32} = u_1 - u_4 - n$ the remainder $\tilde{r}$ vanishes after acting on a basic state in the second and the third spaces $1, 2, 3$

\[ T^{+-+}(u_2, u_1, u_4, u_3) \cdot 1213 = u_1(u_4 + 1)T_{14}^{+-}(u_2, u_3). \]

Finally we permute $u_2 \leftrightarrow u_3$ (9),

\[ (x_1 \cdot x_4)^{\lambda_{14}} \circ T_{14}^{+-}(u_2, u_3) = T_{14}^{+-}(u_3, u_2) \]

at $\lambda_{14} = u_3 - u_2$. Thus we have shown that (7) holds with

\[ E^{+-+} = u_1u_2(u_3 + 1)(u_4 + 1), \quad \Phi^{+-+} = (12)^{u_2-u_1}(14)^{u_3-u_2}(23)^{u_1-u_4-n}(34)^{u_4-u_3}. \]

3. A general method

We consider the symmetry condition (7) for $N$-point correlators with the signature $++---$

\[ L_1^+(u_1)L_2^+(u_2)L_3^-(u_3)\ldots L_N^-(u_N)\Phi = E\Phi. \]

We use the inversion relations

\[ \begin{align*}
L^+(u) & L^+(-u - 1 - (x \cdot p)) = u(-u - 1 - (x \cdot p)), \\
L^-(u) & L^-(-u - 1 + (p \cdot x)) = u(-u - 1 + (p \cdot x))
\end{align*} \]

(17)

to rewrite the condition in the form

\[ \frac{E}{u_1u_2(u_1 + 1 + (x_1 \cdot p_1))(u_2 + 1 + (x_2 \cdot p_2))}L_2^+(-u_2 - 1 - (x_2 \cdot p_2))L_3^+(-u_1 - 1 - (x_1 \cdot p_1)) \Phi = 0 \]

We look for the solution in the link-integral form

\[ \Phi_{f,J} = \int \phi(c) \exp \left(-\sum c_{ij}(x_i \cdot x_j)\right) \prod dc_{ij}. \]

(18)

In particular, using the integral formula for the Gamma function

\[ y^\lambda = \frac{\Gamma(1 + \lambda)}{2\pi i} \int_\mathcal{C} dc (-c)^{-\lambda-1}e^{-cy} \]

(19)

where the contour $\mathcal{C}$ encircles clockwise the positive real semi-axis starting at $+\infty - i\epsilon$, surrounding 0, and ending at $+\infty + i\epsilon$, the monomial ansatz (3) acquires the link form with

\[ \phi(c) = \prod c_{ij}^{\lambda_{ij}}. \]

We need the more general ansatz

\[ \phi(c_{ij}) = \prod c_{ij}^{-\lambda_{ij}} f(z_{i_1i_2}) \]

(20)

involving a function of the cross ratios

\[ z_{i_1i_2}^{j_1j_2} = \frac{c_{i_1j_1}c_{i_2j_2}}{c_{i_1j_2}c_{i_2j_1}}, \quad i_1, i_2 \in I, j_1, j_2 \in J \]

In the considered case we have $i_1 = 1, i_2 = 2$ and the upper labels can be omitted. We choose as a complete set of independent cross ratios $z_j = z_{jj+1}, j = 3, \ldots, N - 1$. 

\[ \begin{align*}
E^{+-+} & = u_1u_2(u_3 + 1)(u_4 + 1), \\
\Phi^{+-+} & = (12)^{u_2-u_1}(14)^{u_3-u_2}(23)^{u_1-u_4-n}(34)^{u_4-u_3}. \end{align*} \]
Applying the similarity relations (8) and

\[ L^+ (u) \cdot 1 = (u + 1)I, \quad L^- (u) \cdot 1 = uI \]

the symmetry condition with the ansatz (20) can be written as

\[
\int \exp \left( -\sum c_{ij} (x_i \cdot x_j) \right) \psi(c, x) \prod dc_{ij} = 0,
\]

\[
\psi(c, x) = \left( \prod_{j=3}^N \left( 1 + \frac{1}{u_j} D_{1j} l_{j1} + \frac{1}{u_j} D_{2j} l_{j2} \right) - \tilde{E} \left( 1 + \frac{1}{\tilde{u}_2} \sum_{j=3}^N D_{2j} l_{j2} \right) \left( 1 + \frac{1}{\tilde{u}_1} \sum_{j=3}^N D_{1j} l_{j1} \right) \right) \phi,
\]

where

\[
\tilde{u}_i = u_i + \sum_{j=3}^N \lambda_{ij}, \quad i = 1, 2, \quad \tilde{E} = \frac{E \tilde{u}_1 \tilde{u}_2}{\prod_{j=1}^N u_j (\tilde{u}_1 + 1)(\tilde{u}_2 + 1)}, \quad D_{ij} = \frac{\partial}{\partial c_{ij}} c_{ij}.
\]

The generalisation to other split signature configurations is evident. The dependence on \( x \) of \( \psi \) is in the matrices \( l_{ji} \). The products of the latter are expressed again in those matrices multiplied by cross ratios now in \( (x_i \cdot x_j) \),

\[
X_{j1, j2} = \frac{(x_1 \cdot x_{j1})(x_2 \cdot x_{j2})}{(x_1 \cdot x_1)(x_2 \cdot x_2)}.
\]

For example, we have

\[ l_{j22} l_{j1} = X_{j2j1} l_{j1} \cdot l_{j2} - X_{j1j2} l_{j1} \cdot l_{j2}.
\]

In the integrand of the link integral a factor of the coordinate cross ratio \( X_{j2j1} \) can be transformed by partial integration to a factor of a cross ratio in the link variables resulting in the rule

\[
D_{1j1} D_{2j2} X_{j2j1} \rightarrow D_{1j1} D_{2j1} z_{j1j2}.
\]

We transform also the operators \( D_{1j1}, D_{2j2} \) by expressing the differentiation with respect to the link variables \( c_{1j}, c_{2j} \) in terms of differentiation with respect to the cross ratios,

\[
D_{1j1} \phi(c) = \prod c_{ij}^{-1-\lambda_{ij}} (-\lambda_{1j} + z_j \partial_{z_j} - z_{j-1} \partial_{z_{j-1}}) f(z).
\]

Note that \( f(z) \) involves just \( z_3, ..., z_{N-1} \) and the differentiations with index values \( j = 2, j = N \) appearing formally here have to be substituted by zero. For the condition to hold it is sufficient that the transformed integrand of the link integral vanishes, i.e.

\[
\begin{align*}
&\left(1 + \sum_{j=3}^N \frac{1}{u_j} (\tilde{l}_{j1} + \tilde{l}_{j2}) + \sum_{k=2}^{N-2} \sum_{3 \leq j_1 < j_2 < ... < j_k \leq N} \prod_{s=2}^k \frac{1}{u_{j_s}} \cdot \right. \\
&\left.\left[ (-\lambda_{1j_s} - \lambda_{2j_s} z_{j_s,j_{s-1}} + (d_{j_s} - d_{j_{s-1}})(1 - z_{j_s,j_{s-1}})) \tilde{l}_{j1} + \tilde{l}_{j2} z_{j1j_{s1}} \right) f(z) \right.
\end{align*}
\]

\[
= \tilde{E} \left( 1 + \frac{1}{\tilde{u}_1} \sum_{j=3}^N \tilde{l}_{j1} + \frac{1}{\tilde{u}_2} \sum_{j=3}^N \tilde{l}_{j2} + \frac{1}{\tilde{u}_1 \tilde{u}_2} \sum_{j=3}^N (-\lambda_{2j} + d_{j-1} - d_j) \frac{1}{z_{3j}} \sum_{j=3}^N \tilde{l}_{j1} z_{3j} \right) f(z).
\]

We use the abbreviations

\[
\tilde{l}_{ji} = (-\lambda_{ij} - 1 + D_{ij}) l_{ji}, \quad d_j = z_j \partial_{z_j}, \quad d_2 = 0, d_N = 0.
\]
Since the coordinate dependent matrices $l_{ij}$ and the unit matrix are linearly independent the matrix condition implies a sequence of conditions. In the non-degenerate case, where none of the expressions $(-1 - \lambda_{ij} + D_{ij}) f(z), i = 1, 2, j = 3, \ldots N$ vanishes identically, also $l_{ij}$ are linearly independent. The inherent Yangian structure amounts in a hierarchy of the conditions owing to the maximal inverse power of spectral parameters involved (not regarding the $u$-factors in $E$), and the less involved lower order conditions fix the solution essentially. The higher order conditions then appear as consistency relations, which can be fulfilled by adjusting the involved parameters, the exponents $\lambda_{ij}$ and the spectral parameters $u_s$.

Let us show some explicit calculation steps in the case $N = 5$. The condition of order zero is fulfilled if $E = 1$. The condition arising as coefficient at $\tilde{l}_{23}$ is of first order and implies just $u_5 = \tilde{u}_2$. The conditions coming along with $l_{52}, l_{41}, l_{42}$ are of second order and read in this ordering

$$
(\tilde{u}_1 - \tilde{u}_2 + \lambda_{25} + \lambda_{24} z_4 + \lambda_{23} z_3 z_4 - d_3 (1 - z_3) z_4 - d_4 (1 - z_4)) f(z_3, z_4) = 0,
$$

$$
(u_5 - u_4 - \lambda_{25} - \lambda_{15} z_4 + d_4 (1 - z_4)) f(z_3, z_4) = 0,
$$

$$
((\tilde{u}_1 - u_4) \lambda_{25} + (u_4 u_5 - \tilde{u}_1 u_5 + \tilde{u}_1 \lambda_{15} u_4 \lambda_{24}) z_4 - u_4 \lambda_{23} z_3 z_4 -
$$

$$
-(\tilde{u}_1 - u_4) d_4 (1 - z_4) + u_4 d_3 (1 - z_3) z_4) f(z_3, z_4) = 0.
$$

In the last condition the derivative terms can be substituted by using the first ones resulting in the conditions on the parameters $(u_4 - \tilde{u}_1) u_5 = 0, \ (u_4 - \tilde{u}_1) (u_4 - u_3) = 0$. Let us continue with the case $\tilde{u}_1 = u_4$. Then the condition coming with $l_{52}, l_{41}$ decouple and result in the solution

$$
f(z_3, z_4) = z_3^{\lambda_{24} - \lambda_{15} (1 - z_3)^{-1} - \lambda_{23} - \lambda_{24} + \lambda_{15}} z_4^{u_4 - u_5 + \lambda_{25} (1 - z_4)^{-1} - \lambda_{15} - \lambda_{25} + u_5 - u_4}.
$$

We quote the results for the 3 and 4 point correlators with signature of the considered form. In the case $N = 4$ we have the above form (20) with

$$
f(z) = z^{u_3 - u_4 + \lambda_{24} (1 - z)^{-1} - \lambda_{23} - \lambda_{24} - u_3 + u_4}, \quad \lambda_{14} = \lambda_{23}, \quad \tilde{u}_1 = u_3, \quad \tilde{u}_2 = u_4, \quad z = z_{34}.
$$

In the case $N = 3$

$$
f = 1, u_3 = \tilde{u}_2, \lambda_{23} = \tilde{u}_2 - \tilde{u}_1.
$$

The latter reproduces the above result (14) for the symmetric 3-point correlator. The 4-point correlator obtained here and the one obtained in the previous section (16) are related by the signature transposition to be described below.

4. Relations between symmetric correlators

There are several ways to generate symmetric correlators from given ones.

One source of such relations is the structure of the $L$ operators. In particular they obey the inversion relations (17) and the relation of matrix transposition (see (5))

$$
[L^\pm (u)]_{ba} = - [L^\mp (-u - 1)]_{ab}.
$$

These relations give rise to relations for the monodromy matrices and to transformations of the symmetry condition. This implies in particular that a symmetric correlator $\Phi^{\alpha_1 \cdots \alpha_N}$ is proportional to the one with opposite signature $-\alpha_1 \cdots - \alpha_N$ and transformed spectral parameters and also to the one with the reversed order of points. Further interesting relations are the ones involving cyclic shifts of the points reminiscent to crossing relations [15, 16, 17]. For
example, if the first point is attributed sign $+$ the relation for the cyclic shift moving this point to the last position reads

$$T_{2-N_1}^{\alpha_2\cdots\alpha_N}(u_2,\cdots, u_N, u_1-n) \cdot \Phi = E_0 \cdot \Phi, \quad E_0 = \frac{(u_1 + 1 - n)(u'_1 + 1)}{u_1 u'_1} E(u_1,\cdots, u_N), \quad (25)$$

where $u'_1 = -u_1 - 1 - \sum_j \lambda_{ij}$.

The operation of elementary canonical transformation $C_i$

$$C_i^{-1} x_i , c_i = -p_i , \quad C_i^{-1} p_i , c_i = x_i. \quad (26)$$

acting at site $i$ interchanges $L_i^+(u)$ with $L_i^+(u)$ and $\Phi(\cdots, x_i, \cdots) \cdot 1$ by $\Phi(\cdots, p_i, \cdots) \cdot \delta(x_i)$. In the discussion of this operation we restrict the related coordinate variables to real values. Applying this operation at one site $i$ leads from a regular to a singular correlator. Let us apply the operation to the sites $i,j$ of different signature. The symmetric correlator with $+$ at $i$ and $-$ at $j$ is transformed into a symmetric correlator with the transposed signature, $-$ at $i$ and $+$ at $j$. The result is a regular correlator again, a singular one appears only intermediately.

To perform the transposition we write the original correlator in link integral form and $\delta(x_i)\delta(x_j)$ in Fourier integrals.

$$\Phi \rightarrow C_{ij} \Phi = \int dc dp_i dp_j \phi(c) e^{-Q} = \int dc \phi(c) e^{-Q'},$$

$$Q \equiv c_{i_1j_1}(x_{i_1} \cdot x_{j_1}) - c_{ij}(p_i \cdot p_j) + ic_{ij}(p_i \cdot x_{j_1}) + ic_{ij}(x_{i_1} \cdot p_j) - i(p_i \cdot x_i) - i(p_j \cdot x_j).$$

Here $i_1,j_1$ is summed over the ranges $I,J$ respectively avoiding the fixed values $i,j$. Performing the integrals over $p_i,p_j$ we obtain a link integral of the above form with the quadratic form in the exponential replaced by

$$Q' = c'_{i'j'}(x_{i'} \cdot x_{j'}) , \quad c'_{i_1j_1} = c_{i_1j_1} - \frac{c_{i_1j}c_{i_1j'}}{c_{ij}}, \quad c'_{ij} = \frac{c_{ij}}{c_{ij}}, \quad c'_{jj'} = -\frac{1}{c_{ij}}.$$

The last step of the transposition operation is the change of integration variables $c_{ij} \rightarrow c'_{ij}$, where $i'$ run over $I$ and $j'$ run over $J$. In particular one can check that the 4-point correlators with the signatures $+--$ and $++-$ are related in this way.

Correlation functions can be regarded as kernels of integral operators provided the integration can be defined.

$$[\hat{R} \cdot \psi](x_{K+1},\cdots , x_N) = \int dx_1 \cdots dx_K R(x_1,\cdots , x_K, x_{K+1},\cdots , x_N) \psi(x_1,\cdots , x_K). \quad (27)$$

The subsequent action of these operators is then also defined. This means that the convolution of kernels by the given integration prescription results in further correlation functions. We assume that the integration allows for a simple integration by parts defining the transposition rule,

$$\int dx \phi(x)L^\pm(u)\psi(x) = \int dx \psi(x)L^\pm(u)\phi(x), \quad \quad (28)$$

$$[L^{\pm T}(u)]_{ab} = - [L^{\pm}(-u - 1)]_{ab}. \quad (29)$$

We impose the condition that the dilatation weight of an integrated point plus the weight $n$ of the measure adds up to zero. By this restriction on the dilatation weights the integration becomes compatible with the symmetry, i.e. the reduction to the projective space carrying the $sl_n$ irreducible representations should be always defined. With the transposition relation (29)
this results in the compatibility of the symmetry conditions with the convolution; the result of this convolution of symmetric correlators is a symmetric correlator.

This procedure of connecting solutions has reminiscence to the BCFW [26] prescription for super YM amplitudes in the formulation by integral [12].

Yangian symmetric correlators are the solutions of the eigenvalue relations of monodromy operators restricted to definite dilatation weights at the points. To each of the \( n \)-dimensional points a dilatation weight, a spectral parameter and a signature are associated. The individual spectral parameters are useful ingredients; their role is not restricted to the one of expansion parameter in the monodromy operators as being generating functions of the Yangian algebra generators.

The Jordan-Schwinger type representations of \( g_{\ell n} \) is a simple basic case of higher rank \((A_{n-1})\) symmetry. Generic irreducible representations of this type are characterized by just one parameter (related to the dilatation weight). Such representations are the building units by which general representations can be constructed iteratively. Extensions of the described construction of symmetric correlators to general types of \( g_{\ell n} \) representations or to quantum deformation are straightforward.

The simplicity of the Jordan-Schwinger case is manifest in the simple form of the L-matrices, leading by elementary steps to relations for similarity transformations, inversion, matrix transposition and operator conjugation. These transformations result in relations for monodromy operators and consequently for symmetric correlators.

The \( n \) degrees of freedom associated with any chain site can be related to a \( n \) dimensional position space. In this way the applications of the corresponding integrable dynamical system are not necessarily restricted to one or two (discrete) dimensions. The one-dimensional structure of the associated spin chain is reflected in the cyclicity property of the correlators.

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