THE GROUPS OF DIFFEOMORPHISMS AND
HOMEOMORPHISMS OF 4-MANIFOLDS WITH BOUNDARY

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ABSTRACT. We give constraints on smooth families of 4-manifolds with boundary using Manolescu’s Seiberg–Witten Floer stable homotopy type, provided that the fiberwise restrictions of the families to the boundaries are trivial families of 3-manifolds. As an application, we show that, for a simply-connected oriented compact smooth 4-manifold $X$ with boundary with an assumption on the Frøyshov invariant or the Manolescu invariants $\alpha, \beta, \gamma$ of $\partial X$, the inclusion map $\text{Diff}(X, \partial) \hookrightarrow \text{Homeo}(X, \partial)$ between the groups of diffeomorphisms and homeomorphisms which fix the boundary pointwise is not a weak homotopy equivalence. This combined with a classical result in dimension 3 implies that the inclusion map $\text{Diff}(X) \hookrightarrow \text{Homeo}(X)$ is also not a weak homotopy equivalence under the same assumption on $\partial X$. Our constraints generalize both of constraints on smooth families of closed 4-manifolds proven by Baraglia and a Donaldson-type theorem for smooth 4-manifolds with boundary originally due to Frøyshov.

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1. INTRODUCTION

The main purpose of this paper is to give constraints on smooth families of 4-manifolds with boundary using Manolescu’s Seiberg–Witten Floer stable homotopy type [34], provided that the fiberwise restrictions of the families to the boundaries are trivial families of 3-manifolds. As an application, we show that, for a
simply-connected oriented compact smooth 4-manifold $X$ with boundary with an assumption on the Frøyshov invariant of $\partial X$, the inclusion map

$$\text{Diff}(X, \partial) \hookrightarrow \text{Homeo}(X, \partial)$$

is not a weak homotopy equivalence, where $\text{Diff}(X, \partial)$ and $\text{Homeo}(X, \partial)$ denote the groups of diffeomorphisms and homeomorphisms which fix the boundary pointwise respectively. When $X$ is spin, the assumption on $\partial X$ may be replaced with a similar assumption described in terms of the Manolescu invariants $\alpha, \beta, \gamma$. This result combined with a classical theorem in dimension 3 implies that the inclusion map $\text{Diff}(X) \hookrightarrow \text{Homeo}(X)$ between the whole groups of diffeomorphisms and homeomorphisms is also not a weak homotopy equivalence under the same assumption on $\partial X$.

Our constraints on smooth families of 4-manifolds with boundary have two roots. The first is a constraint on smooth families of closed 4-manifolds proven by Baraglia [2], which can be regarded as a family version of Donaldson’s diagonalization theorem. The second is a constraint on negative-definite smooth 4-manifolds with boundary originally due to Frøyshov [15], which is a generalization of Donaldson’s diagonalization theorem to 4-manifolds with boundary. Roughly speaking, our constraints are combinations of these two.

Let us recall some background of Baraglia’s work. It is classically known that, for a smooth closed manifold of dimension $< 4$, the natural inclusion map from the group of diffeomorphisms into the group of homeomorphisms is a weak homotopy equivalence. However, in contrast, there are a large numbers of examples of manifolds of dimension $\geq 4$ for which the above inclusion map is not a weak homotopy equivalence. In dimension 4, the lowest dimension where such interesting difference happens, many authors revealed that gauge theory for families provides a strong tool to detect such phenomena. See for example [2, 3, 22, 23, 42]. In particular, Baraglia [2] recently proved that the inclusions from from the diffeomorphism groups into the homeomorphism groups are not weak homotopy equivalences for a huge class of simply-connected closed smooth 4-manifolds. This is one of important ingredients of this paper.

It is natural to try to extend Baraglia’s result to 4-manifolds with boundary. He obtained his result by giving a constraint on smooth families of closed 4-manifolds, which is a family version of Donaldson’s diagonalization theorem as mentioned above. So a natural way to extend Baraglia’s result is to obtain a constraint on smooth families of 4-manifolds with boundary. We shall carry this out based on an idea of Frøyshov [15]. Although Frøyshov used monopole Floer homology to derive his constraint, we shall use Manolescu’s Seiberg–Witten Floer stable homotopy type. This is because Baraglia’s argument is based on Furuta’s idea of finite-dimensional approximation of the Seiberg–Witten equations [18], more precisely a family version of the Bauer–Furuta invariant [4], and therefore we need to consider a family version of the relative Bauer–Furuta invariant, which lives in the Seiberg–Witten Floer stable homotopy type as far as the fiberwise restriction of a given family to the boundary is a trivial family of 3-manifolds.

To state our main theorem, let us introduce some notations. In this paper we shall consider an oriented compact smooth 4-manifold $X$ with boundary. Throughout the paper we shall assume that $b_1(X) = 0$, and that $\partial X = Y$ is a connected
oriented rational homology 3-sphere for simplicity. As the structure groups of families of $X$, we have three candidates:

$$\text{Diff}(X), \quad \text{Diff}^+(X), \quad \text{Diff}(X, \partial).$$

Here $\text{Diff}(X)$ is the whole group of diffeomorphisms, and $\text{Diff}^+(X)$ denotes the group of orientation-preserving diffeomorphisms, and $\text{Diff}(X, \partial)$ is the group of diffeomorphisms which fix the boundary pointwise. Note that any element of $\text{Diff}(X, \partial)$ preserves orientation of $X$. Note also that, if the signature of $X$ is not zero, we have $\text{Diff}(X) = \text{Diff}^+(X)$. We mainly consider $\text{Diff}(X, \partial)$ in this paper. Similarly we may define

$$\text{Homeo}(X), \quad \text{Homeo}^+(X), \quad \text{Homeo}(X, \partial).$$

as the corresponding groups of homeomorphisms. If a spin$^c$ structure or a spin structure $s$ is given on $X$, let us define groups

$$\text{Aut}(X, s), \quad \text{Aut}((X, s), \partial)$$

as follows. First $\text{Aut}(X, s)$ denote the automorphism group of the spin$^c$ or spin 4-manifold $(X, s)$. Each element of $\text{Aut}(X, s)$ is a pair $(f, \tilde{f})$ of a diffeomorphism $f$ which preserves orientation and the isomorphism class of $s$, and a lift $\tilde{f}$ of $f$ to a bundle automorphism of the principal $\text{Spin}^c(4)$ or $\text{Spin}(4)$-bundle $P$ corresponding to $s$. The group $\text{Aut}((X, s), \partial)$ is defined as the subgroup of $\text{Aut}(X, s)$ consisting of pairs $(f, \tilde{f})$ whose restrictions to $\partial X$ and $P|_{\partial X}$ are the pair of the identity maps.

Let $X \to E \to B$ be a $\text{Homeo}(X, \partial)$-bundle over a compact topological space $B$. Then we have an associated vector bundle

$$\mathbb{R}^{b^+(X)} \to H^+(E) \to B,$$

whose isomorphism class is a topological invariant of $E$. We shall explain $H^+(E)$ at the beginning of Subsection 2.3, but roughly speaking $H^+(E)$ is a bundle of maximal-dimensional positive-definite subspaces of $H^2$ of the fibers of $E$. Our constraints on smooth families will be described in term of $H^+(E)$.

For a rational homology 3-sphere $Y$ with a spin$^c$ structure $t$, we denote by $\delta(Y, t) \in \mathbb{Q}$ the Frøyshov invariant. If $Y$ is an integral homology 3-sphere, we denote by $\delta(Y)$ the Frøyshov invariant for the unique spin$^c$ structure on $Y$. The sign convention of $\delta$ in this paper is $\delta(\Sigma(2,3,5)) = 1$, which is the same as the convention of [23]. More precisely, we use $\delta$ defined by using $\mathbb{F} = \mathbb{Z}/2$-coefficient Seiberg–Witten Floer homology, which is denoted by $\delta_2$ in [36]. (The reason why we use $\mathbb{F}$-coefficient is explained in Remark [3,4].)

Now we can state the first main theorem in this paper:

**Theorem 1.1.** Let $Y$ be an oriented rational homology 3-sphere and $X$ be an oriented compact smooth 4-manifold bounded by $Y$. Assume that $b_1(X) = 0$. Let $s$ be a spin$^c$ structure on $X$ and let $t$ be the spin$^c$ structure on $Y$ defined as the restriction of $s$. Let $B$ be a compact topological space and $(X, s) \to E \to B$ a smooth $\text{Aut}((X, s), \partial)$-bundle. If $w_{b_1(X)}(H^+(E)) \neq 0$ holds, then we have

$$\frac{c_1(s)^2 - \sigma(X)}{8} \leq \delta(Y, t).$$

Theorem 1.1 is an analogue of Baraglia’s constraint [2, Theorem 1.1] for families of spin$^c$ 4-manifolds with boundary. For the case that $B = \{ \text{pt} \}$, Theorem 1.1
recovery a special case of the constraint due to Frøyshov \cite{Frøyshov2013} on the intersection form of a negative-definite 4-manifold with boundary.

For spin 4-manifolds with boundary, we have a refinement of Theorem 1.1 using the Manolescu invariants $\alpha, \beta, \gamma$ defined in \cite{Manolescu2010}, instead of $\delta$:

**Theorem 1.2.** Let $Y$ be an oriented rational homology 3-sphere and $X$ be an oriented compact smooth 4-manifold bounded by $Y$. Assume that $b_1(X) = 0$. Let $s$ be a spin structure on $X$ and let $t$ be the spin structure on $Y$ defined as the restriction of $s$. Let $B$ be a compact topological space and $(X, s) \to E \to B$ a smooth $\text{Aut}((X, s), \partial)$-bundle. Then:

- If $w_{b^+(X)}(H^+(E)) \neq 0$ holds, then we have
  \[\frac{-\sigma(X)}{8} \leq \gamma(Y, t)\]

- If $b^+(X) > 0$ and $w_{b^+(X)-1}(H^+(E)) \neq 0$ holds, then we have
  \[\frac{-\sigma(X)}{8} \leq \beta(Y, t)\]

- If $b^+(X) > 1$ and $w_{b^+(X)-2}(H^+(E)) \neq 0$ holds, then we have
  \[\frac{-\sigma(X)}{8} \leq \alpha(Y, t)\]

Theorem 1.2 is an analogue of Baraglia’s constraint \cite[Theorem 1.2]{Baraglia2017} for families of closed spin 4-manifolds with boundary. For the case that $B = \{\text{pt}\}$, F. Lin \cite[Theorem 7]{Lin2010} has proven these inequalities (for $X$ with two boundary components), which are extensions of Donaldson’s Theorems B and C to 4-manifolds with boundary.

Using Theorems 1.1 and 1.2 we may detect non-smoothable topological families of 4-manifold with boundary, stated in Theorem 1.3. As a consequence, we may detect homotopical difference between $\text{Diff}(X, \partial)$ and $\text{Homeo}(X, \partial)$ for a large class of $X$ as follows:

**Theorem 1.3.** Let $Y$ be an oriented integral homology 3-sphere. Let $X$ be a simply-connected, compact, oriented, smooth, and indefinite 4-manifold with boundary $Y$. Suppose that $\sigma(X) \leq 0$. Suppose that $X$ and $Y$ satisfy at least one of the following conditions:

- $\sigma(X) < -8$ and $\delta(Y) \leq 0$.
- $\delta(Y) < 0$.
- $X$ is spin and $-\sigma(X)/8 > \gamma(Y)$.
- $X$ is spin, $b^+(X) > 1$ and $-\sigma(X)/8 > \beta(Y)$.
- $X$ is spin, $b^+(X) > 2$ and $-\sigma(X)/8 > \alpha(Y)$.

Then the inclusion map
\[\text{Diff}(X, \partial) \hookrightarrow \text{Homeo}(X, \partial)\]
is not a weak homotopy equivalence.

As a classical fact in dimension 3, it is known that the groups of diffeomorphisms and homeomorphisms have no homotopical difference. This combined with Theorem 1.3 implies a similar result also for $\text{Diff}(X)$ and $\text{Homeo}(X)$:
Theorem 1.4. Let $X$ and $Y$ be as in Theorem 1.3. Then the inclusion map
$$\text{Diff}(X) \hookrightarrow \text{Homeo}(X)$$
is not a weak homotopy equivalence.

In Theorems 1.3 and 1.4, not just about weak homotopy equivalence, we may actually estimate the range of the degrees of homotopy groups where the difference happens for the first time: it is approximately up to $b^+(X)$. See Corollaries 1.3 and 1.5 for the precise statements.

Remark 1.5. If $X$ is spin, the assumption (3) in Theorem 1.3 is weaker than the assumption (1) there. This follows from a result by Stoffregen. For a rational homology 3-sphere $Y$ with a spin structure $t$, he showed in [44, Theorem 1.2] that
$$\alpha(Y, t) \geq \delta(Y, t) \geq \gamma(Y, t).$$
By the last inequality, if either $\sigma(X) < -8$ and $\delta(Y, t) \leq 0$, or $\delta(Y, t) < 0$ holds, we have $-\sigma(X)/8 > \gamma(Y, t)$.

It is also worth noting that we have inequalities
$$\alpha(Y, t) \geq \beta(Y, t) \geq \gamma(Y, t),$$
which follow from the definition of $\alpha, \beta, \gamma$.

Remark 1.6. There are a huge (at least infinitely many) numbers of examples of $(X, Y)$ satisfying the assumption of Theorem 1.3. For example, it is quite easy to find examples satisfying (1) of Theorem 1.3. The invariants $\alpha, \beta, \gamma, \delta$ are calculated by various authors, in particular for $\delta$. For $\alpha, \beta, \gamma$, see [36, Subsection 3.8] and [44]. Infinitely many examples of $Y$ with $\delta(Y) = 0$ are also found in [36, Subsection 3.8].

For $X$ with small $b^+$, we can compare $\pi_0(\text{Diff}(X, \partial))$ and $\pi_0(\text{Homeo}(X, \partial))$ a little more precisely, which is proven in Subsection 4.2.

Theorem 1.7. Let $Y$ be an oriented integral homology 3-sphere. Let $X$ be a simply-connected, compact, oriented, smooth, and indefinite 4-manifold with boundary $Y$. Suppose that $\sigma(X) \leq 0$. Suppose that $X$ and $Y$ satisfy at least one of the following conditions:

1. $b^+(X) = 1, -\sigma(X) < 8$ and $\delta(Y) \leq 0$.
2. $b^+(X) = 1$ and $\delta(Y) < 0$.
3. $b^+(X) = 1, X$ is spin and $-\sigma(X)/8 > \gamma(Y)$.
4. $b^+(X) = 2, X$ is spin and $-\sigma(X)/8 > \beta(Y)$.
5. $b^+(X) = 3, X$ is spin and $-\sigma(X)/8 > \alpha(Y)$.

Then the natural map
$$\pi_0(\text{Diff}(X, \partial)) \rightarrow \pi_0(\text{Homeo}(X, \partial))$$
induced from the inclusion is not a surjection. Namely, there exists a homeomorphism of $X$ fixing the boundary which is not topologically isotopic to any self-diffeomorphism of $X$ fixing the boundary.

Moreover, the map
$$\pi_0(\text{Diff}(X)) \rightarrow \pi_0(\text{Homeo}(X))$$
is also not a surjection.

Examples of $X$ satisfying the assumption of Theorem 1.7 shall be given in Examples 4.9 and 4.10, where we use $\delta$ and $\beta$ to apply Theorem 1.7 respectively.
Remark 1.8. As an obvious consequence of Theorem 1.7 in the setting of the theorem, the natural map

\[ \pi_0(\text{Diff}(X)) \to \text{Aut}(H^2(X;\mathbb{Z})) \]

is also not a surjection. Here \( \text{Aut}(H^2(X;\mathbb{Z})) \) denotes the automorphism group of the intersection form.

It is worth noting that, for a closed smooth 4-manifold \( X \), the map

\[ \pi_0(\text{Diff}(X)) \to \pi_0(\text{Homeo}(X)) \]

is often a surjection and there are only few examples of \( X \) for which (6) are known to be not surjections. See Remark 1.6 for the detail.

Lastly, we mention that there are interesting recent work on relative diffeomorphisms in dimension 4 based on techniques which are different from gauge theory. See, for example, \[7, 47, 48\].

We finish off this introduction with an outline of the contents of this paper. In Section 2 we collect necessary ingredients to prove Theorems 1.1 and 1.2. After recalling the definition of Manolescu’s Seiberg–Witten Floer stable homotopy type in Subsection 2.1, we recall some basics of the Frøyshov-type invariants \( \alpha, \beta, \gamma, \delta \), and in Subsection 2.2 we describe the families relative Bauer–Furuta invariant, from which we extract constraints on smooth families of 4-manifolds with boundary, Theorems 1.1 and 1.2. In Section 3 we prove Theorems 1.1 and 1.2 which are the main theorems of this paper. In Section 4 we consider applications of Theorems 1.1 and 1.2 mainly to the existence of non-smoothable families of 4-manifolds with boundary, stated as Theorem 4.1, and give consequences of Theorem 4.1 about comparison between various diffeomorphism groups and homeomorphism groups of 4-manifolds with boundary.

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2. Preliminaries

In this section we collect necessary ingredients to prove Theorems 1.1 and 1.2. After recalling the definition of Manolescu’s Seiberg–Witten Floer stable homotopy type \[34\] in Subsection 2.1, we recall some basics of the Frøyshov-type invariants \( \alpha, \beta, \gamma, \delta \) in Subsection 2.2. In Subsection 2.3 we describe the families relative Bauer–Furuta invariant for a family of 4-manifolds with boundary, defined if we suppose that the fiberwise restriction of the family to the boundaries is a trivial family of 3-manifolds. This is a main ingredient in the proof of Theorems 1.1 and 1.2.
2.1. Seiberg–Witten Floer stable homotopy type. In this subsection we review several notions of Manolescu’s Seiberg–Witten Floer stable homotopy type. The main references are Manolescu \cite{34} and Khandhawit \cite{23}.

Let $(Y, t)$ be an oriented $\text{spin}^c$ rational homology 3-sphere with a Riemann metric $g_Y$. Let $S$ be the spinor bundle of $t$. Fix a flat $\text{spin}^c$ reference connection $a_0$ of $(Y, s)$. For an integer $k > 2$, we define a configuration space by

$$C_k(Y, t) := (a_0 + L^2_{k-\frac{1}{2}}(i\Lambda^1_Y)) \oplus L^2_{k-\frac{1}{2}}(S).$$

The Chern–Simons–Dirac functional $CSD : C_k(Y, t) \to \mathbb{R}$ is defined by

$$CSD(a, \phi) := \frac{1}{2} \left( -\int_Y a \wedge da + \int_Y <\phi, \phi_{a_0+a}\phi > \, dv_Y \right),$$

where $\phi_{a_0+a}$ is the spin$^c$ Dirac operator for the connection $a_0 + a$. This functional is invariant under the action of the gauge group, where the gauge group $G_k(Y)$ and a subgroup $\tilde{G}_k(Y)$ of $G_k(Y)$ are defined by

$$G_k(Y) := L^2_{k+\frac{1}{2}}(Y, S^1)$$

and

$$\tilde{G}_k(Y) := \left\{ g \in G_k(Y) \middle| g = e^{i\lambda}, \int_Y f \, dv_Y = 0 \right\}.$$

The action $G_k(Y)$ on $C_k(Y, t)$ is given by the pull-back of connections and the complex mutiplication on spinors. The global slice of the action of $\tilde{G}_k(Y)$ on $C_k(Y, t)$ is given by

$$\text{Coul}_k(Y, s) = (\text{Ker} \, d^* : L^2_{k-\frac{1}{2}}(\Lambda^1_Y) \to L^2_{k-\frac{1}{2}}(\Lambda^0_Y)) \oplus L^2_{k-\frac{1}{2}}(S).$$

The $S^1$-equivariant formal gradient flow

$$v : \text{Coul}_k(Y, s) \to \text{Coul}_{k-1}(Y, s)$$

of $CSD$ with respect to the Coulomb projection of the $L^2$-metric can be written as the sum of

$$l = (\ast d, \phi_{a_0})$$

and the quadratic term

$$c(h, \psi) = (\text{pr}_{\text{Ker} \, d^*} \rho^{-1}((\psi \psi^*)_0), \rho(h)\psi - \xi(\psi)\psi),$$

where $\xi(\psi) \in \Omega^0(Y)$ is determined by the conditions

$$d\xi(\psi) = (1 - \text{pr}_{\text{Ker} \, d^*}) \circ \rho^{-1}((\psi \psi^*)_0) \quad \text{and} \quad \int_Y \xi(\psi) \, dv_Y = 0.$$

Here $\text{pr}_{\text{Ker} \, d^*}$ denotes the projection onto $\text{Ker} \, d^*$. Henceforth we just say that $t$ is spin if $t$ comes from a spin structure. Although $v$ is an $S^1$-invariant vector field in general, if $t$ is spin, we have a larger symmetry of the group $\text{Pin}(2)$, which is defined by

$$\text{Pin}(2) := S^1 \cup jS^1 \subset Sp(1).$$

When $t$ is spin, the spinor bundle has a structure of $Sp(1)$-bundle. We consider a $\text{Pin}(2)$-action on $S$ as the restriction of the natural $Sp(1)$-action on $S$. The group $\text{Pin}(2)$ acts on $\Omega^1_Y$ via the non-trivial homomorphism $\text{Pin}(2) \to O(1)$. By such actions, $\text{Pin}(2)$ acts on $\text{Coul}_k(Y, s)$ for any non-negative integer $k$. When $t$ is spin, $v$ is $\text{Pin}(2)$-equivariant. Let $\mathbb{R}$ denote the real 1-dimensional representation of $\text{Pin}(2)$ via the map $\text{Pin}(2) \to O(1)$, and $\mathbb{H}$ denote the space of quaternions, on which $\text{Pin}(2)$ acts as the restriction of the natural action of $Sp(1)$.
We have real eigenvalues
\[ \cdots < \lambda_k < \lambda_{k+1} < \lambda_{k+2} < \cdots \]
of \( l \) as an unbounded operator on \( \text{Coul}_{1/2}(Y, s) \). For \( \lambda < 0 < \mu \), we define \( V^\mu_\lambda(Y) \) as the direct sum of eigenspaces whose eigenvalues belong to \( (\lambda, \mu] \). Here we think of \( V^\mu_\lambda(Y) \) as a subspace of \( \text{Coul}_{k}(Y, s) \). We denote by
\[ p^\mu_\lambda : \text{Coul}_{k}(Y, s) \to V^\mu_\lambda(Y) \]
the \( L^2 \)-projection of \( \text{Coul}_{k}(Y, s) \) onto \( V^\mu_\lambda(Y) \). Henceforth we often abbreviate \( V^\mu_\lambda(Y) \) as \( V^\mu_\lambda \).

Since \( l \) is the sum of a real operator and a complex operator, we have the direct decomposition
\[ V^\mu_\lambda = V^\mu_\lambda(R) \oplus V^\mu_\lambda(C) \]
of a real vector space and a complex vector space. Denote by \( B(2R; V^\mu_\lambda) \) the closed ball in \( V^\mu_\lambda \) of radius \( 2R \) centered at the origin. Manolescu proved the following compactness property for the dynamical system induced by a vector field \( (V^\mu_\lambda, l + p^\mu_\lambda c) \):

**Theorem 2.1.** [34, Proposition 3] There exist sufficiently large \( R > 0 \) and \( -\lambda, \mu > 0 \) such that all trajectory \( x : \mathbb{R} \to V^\mu_\lambda \) of the flow equation
\[ \frac{\partial}{\partial t} x(t) = -(l + p^\mu_\lambda c)(x(t)) \]
which lie in \( B(2R; V^\mu_\lambda) \) actually lie in \( B(R; V^\mu_\lambda) \).

By the use of Theorem 2.1 one can see that \( B(2R; V^\mu_\lambda) \) is an isolating neighborhood of
\[ \text{Inv} B(2R; V^\mu_\lambda) := \{ x \in B(2R) \mid t \cdot x \in B(2R), \forall t \in \mathbb{R} \} \]
with respect to the flow on \( V^\mu_\lambda \) generated by \( \rho(l + p^\mu_\lambda c) \), where \( \rho \) is an \( S^1 \)-invariant bump function such that \( \rho|_{B(3R; V^\mu_\lambda)} = 0 \) and \( \rho|_{B(2R; V^\mu_\lambda)} = 1 \). Here \( t \cdot \) denotes the action of \( t \) via this flow on \( V^\mu_\lambda \). We denote by \( I^\mu_\lambda \) the \( S^1 \)-equivariant Conley index of \( \text{Inv} B(2R; V^\mu_\lambda) \) for the flow. The Seiberg–Witten Floer homotopy type is defined as
\[ \text{SWF}(Y, t) := \Sigma^{\sim n(Y, t, g_Y)} C - V^\mu_\lambda I^\mu_\lambda, \]
which makes sense in a certain suspension category. For the definition of the quantity \( n(Y, t, g_Y) \in \mathbb{Q} \) and the meaning of formal desuspensions, see [34]. (However, we shall use only \( \text{SWF}(Y, t) \) which is sufficiently suspended in that category, and so the formal desuspensions will not appear in our argument.)

When \( t \) is spin, we take \( \rho \) above to be a Pin(2)-invariant bump function, and consider Pin(2)-equivariant Conley index instead. We set
\[ \text{SWF}(Y, t) := \Sigma^{\sim n(Y, t, g_Y)} C - V^\mu_\lambda I^\mu_\lambda, \]
as a stable homotopy type of a pointed Pin(2)-space.
2.2. The Frøyshov invariant $\delta$ and the Manolescu invariants $\alpha, \beta, \gamma$. In this subsection we recall the definition of the Frøyshov invariant and the Manolescu invariants $\alpha, \beta, \gamma$. The Frøyshov invariant was originally defined in term of the monopole Floer homology \cite{Fr1}, \cite{Fr2}, but it can be interpreted also in terms of the Seiberg–Witten Floer homotopy type \cite{Mo2}, \cite{Mo3}, \cite{Mo4}. In this paper we mainly follow Manolescu’s description of the Frøyshov invariant given in \cite{Mo3}. When one considers a spin structure on a given 3-manifold, using Pin(2)-symmetry of the Seiberg–Witten equations, analogous three invariants are defined, which are the Manolescu invariants $\alpha, \beta, \gamma$ introduced in \cite{Mo3}. We also recall the definition of $\alpha, \beta, \gamma$ in this subsection. Henceforth, throughout this paper, all (co)homology will be taken with $\mathbb{F} = \mathbb{Z}/2$-coefficients. We refer the readers also to Stoffregen’s paper \cite{St} for this subsection.

Remark 2.2. The original definition of the Frøyshov invariant uses (co)homology with $\mathbb{Q}$-coefficient, not $\mathbb{F}$-coefficient. However, as noted in Remark 3.12 of \cite{Mo3}, there is no known example of 3-manifolds for which the Frøyshov invariant with $\mathbb{Q}$-coefficient and that with $\mathbb{F}$-coefficient are different.

Let $(Y, t)$ be a spin$^c$ 3-manifold and fix a Riemannian metric $g$ on $Y$ and real numbers $\lambda, \mu$ to define a finite-dimensional approximation. One can easily check that

$$(I^\mu_\lambda)^{S^1} \cong N^{S^1}/L^{S^1}$$

and $(I^\mu_\lambda)^{S^1}$ is homotopy equivalent to $V_\lambda^0(\mathbb{R})^+$. Set

$$s := \dim V_\lambda^0(\mathbb{R}).$$

Then we have

$$\tilde{H}^{s+}_S((I^\mu_\lambda)^{S^1}) \cong \tilde{H}^{s+}_S(V_\lambda^0(\mathbb{R})^+) \cong \tilde{H}^{s+}_S(S^0) \cong \mathbb{F}[U].$$

The Frøyshov invariant $\delta(Y, t)$ is defined as follows. Denote by $i : (I^\mu_\lambda)^{S^1} \hookrightarrow I^\mu_\lambda$ the inclusion. The quantity $d$ in \cite{Mo3} is defined as

(9) \hspace{1cm} d(Y, \lambda, \mu, g, t) = \min \left\{ r \equiv s \mod 2 \mid \exists x \in \tilde{H}^r_S((I^\mu_\lambda)) \text{, } U^t \cdot x \neq 0 \text{ (\forall l \geq 0)} \right\},

where an equivariant localization theorem ensures that the set in the right-hand side is not empty. This might look different from the quantity $d$ defined in Subsection 2.6 of \cite{Mo3} at first glance, but it can be seen that $d$ is just the same with Manolescu’s $d$ using an argument in the proof of Lemma 2.9 of \cite{Mo3}. (See also Definition 3.6 of \cite{St}.) Then the Frøyshov invariant $\delta(Y, t) \in \mathbb{Q}$ is defined by

(10) \hspace{1cm} \delta(Y, t) = (d(Y, \lambda, \mu, g, t) - \dim \mathbb{R} V_\lambda^0)/2 - n(Y, t, g).

It turns out that $\delta(Y, t) \in \mathbb{Q}$ depends only on $(Y, t)$. (Note that $n(Y, t, g)$ may not be an integer if $Y$ is not an integral homology sphere. If $Y$ is an integral homology sphere, then $n(Y, t, g) \in \mathbb{Z}$ and hence $\delta(Y, t) \in \mathbb{Z}$.)

Here we note an elementary observation used in the proof of one of the main theorem, Theorem 1.1.

Lemma 2.3. If $x \in \tilde{H}^r_S((I^\mu_\lambda))$ satisfies that $U^l \cdot x \neq 0$ for all $l \geq 0$, then we have $i^*x \neq 0$ in $\tilde{H}^r_S((I^\mu_\lambda)^{S^1})$. 

Proof. As well as Fact 2.5 of [14], an equivariant localization theorem implies that
\begin{equation}
\iota^* : \hat{H}_{S^1}^*(I^\mu_\lambda) \to \hat{H}_{S^1}^*((I^\mu_\lambda)^{S^1})
\end{equation}
is an isomorphism in cohomology in sufficiently high degrees. The map (11) is a
\(\hat{H}_{S^1}^*(S^0) = F[U] \)-module map. Thus we have \(\iota^* U^l \cdot x = U^l \cdot \iota^* x \) for all \(l \geq 0\).
Therefore it suffices to show that there exists \(l \geq 0\) such that \(\iota^* U^l \cdot x \neq 0\) to prove
the lemma. However, if we take \(l\) sufficiently large, \(\iota^* : \hat{H}_{S^1}^{d+r}(I^\mu_\lambda) \to \hat{H}_{S^1}^{d+r}((I^\mu_\lambda)^{S^1})\)
is an isomorphism, and we have that \(U^l \cdot x \neq 0\). Thus we obtain \(\iota^* U^l \cdot x \neq 0\) for
sufficiently large \(l\).

Lemma 2.4. Set \(d = d(Y, \lambda, \mu, g, t)\). Then there exists a cohomology class
\[ \omega \in \hat{H}_{S^1}^d(I^\mu_\lambda) \]
such that
\begin{equation}
\iota^* \omega = [V_\lambda^0(\mathbb{R})^+] \otimes U^{(d-s)/2}
\end{equation}
in
\[ \hat{H}^*(V_\lambda^0(\mathbb{R})^+) \otimes \hat{H}_{S^1}^*(S^0) \cong \hat{H}_{S^1}^*(V_\lambda^0(\mathbb{R})^+) \cong \hat{H}_{S^1}^*((I^\mu_\lambda)^{S^1}). \]
(Recall that \(d \equiv s \mod 2\), and hence \(U^{(d-s)/2}\) makes sense.)

Proof. By the definition of \(d\) given in (9) and Lemma 2.3 there exists a cohomology class \(\omega \in \hat{H}_{S^1}^d(I^\mu_\lambda)\) such that \(\iota^* \omega \neq 0\). Notice that this non-vanishing of \(\iota^* \omega\) is equivalent to (13).

Next we recall the definition of \(\alpha, \beta, \gamma\). Suppose that \(t\) comes from a spin structure. Then we have
\[ \hat{H}_{Pin(2)}^{*+s}((I^\mu_\lambda)^{S^1}) \cong \hat{H}_{Pin(2)}^{*+s}(V_\lambda^0(\mathbb{R})^+) \cong \hat{H}_{Pin(2)}^*(S^0) \cong F[q, v]/(q^4), \]
with elements \(q\) in degree 1 and \(v\) in degree 4. Let us define
\[ a(Y, \lambda, \mu, g, t) = \min \left\{ r \equiv s \mod 4 \left| \exists x \in \hat{H}_{Pin(2)}^r(I^\mu_\lambda), \ v^l \cdot x \neq 0 \ (\forall l \geq 0) \right. \right\}, \]
\[ b(Y, \lambda, \mu, g, t) = \min \left\{ r \equiv s + 1 \mod 4 \left| \exists x \in \hat{H}_{Pin(2)}^r(I^\mu_\lambda), \ v^l \cdot x \neq 0 \ (\forall l \geq 0) \right. \right\} - 1, \]
\[ c(Y, \lambda, \mu, g, t) = \min \left\{ r \equiv s + 2 \mod 4 \left| \exists x \in \hat{H}_{Pin(2)}^r(I^\mu_\lambda), \ v^l \cdot x \neq 0 \ (\forall l \geq 0) \right. \right\} - 2. \]
The definition of the invariants \(\alpha, \beta, \gamma\) valued in \(Q\) is
\begin{align}
\alpha(Y, t) &= (a(Y, \lambda, \mu, g, t) - \dim_{\mathbb{R}} V_\lambda^0)/2 - n(Y, t, g), \\
\beta(Y, t) &= (b(Y, \lambda, \mu, g, t) - \dim_{\mathbb{R}} V_\lambda^0)/2 - n(Y, t, g), \\
\gamma(Y, t) &= (c(Y, \lambda, \mu, g, t) - \dim_{\mathbb{R}} V_\lambda^0)/2 - n(Y, t, g).
\end{align}

Lemma 2.5. If \(x \in \hat{H}_{Pin(2)}^r(I^\mu_\lambda)\) satisfies that \(v^l \cdot x \neq 0\) for all \(l \geq 0\), then we have
\(\iota^* \neq 0\) in \(\hat{H}_{Pin(2)}^r((I^\mu_\lambda)^{S^1})\).

Proof. The proof is totally similar to the proof of Lemma 2.3 just use instead the fact, which is precisely Fact 2.5 of [14], that
\[ \iota^* : \hat{H}_{Pin(2)}^r(I^\mu_\lambda) \to \hat{H}_{Pin(2)}^r((I^\mu_\lambda)^{S^1}) \]
is an isomorphism in sufficiently high degrees.
Lemma 2.6. Set \( a = a(Y, \lambda, \mu, g, t), b = b(Y, \lambda, \mu, g, t), c = c(Y, \lambda, \mu, g, t) \). Then there exists a cohomology class

\[
\begin{align*}
\omega_a &\in \tilde{H}^3_{S^1}(I^a_\lambda), \\
\omega_b &\in \tilde{H}^{b+1}_{S^1}(I^a_\lambda), \\
\omega_c &\in \tilde{H}^{c+2}_{S^1}(I^a_\lambda)
\end{align*}
\]

such that

\[
\begin{align*}
t^*\omega_a &= \tau_{\text{Pin}(2)}(V^0_\lambda(\mathbb{R})) + v(a-s)/4, \\
t^*\omega_b &= \tau_{\text{Pin}(2)}(V^0_\lambda(\mathbb{R})) + qv(b-s)/4, \\
t^*\omega_c &= \tau_{\text{Pin}(2)}(V^0_\lambda(\mathbb{R})) + q^2v(c-s)/4
\end{align*}
\]

in

\[
\tilde{H}^*_{S^1}(V^0_\lambda(\mathbb{R})) \cong \tilde{H}^*_{S^1}(I^a_\lambda^{S^1}).
\]

Here \( \tau_{\text{Pin}(2)}(V^0_\lambda(\mathbb{R})) \in \tilde{H}^*_{\text{Pin}}(V^0_\lambda(\mathbb{R})) \) is the equivariant Thom class of the bundle \( V^0_\lambda(\mathbb{R}) \rightarrow \text{pt} \) over a point. (Recall that \( a, b, c \) are congruent to \( s \) mod 4, and hence \( v(a-s)/4, v(b-s)/4, v(c-s)/4 \) make sense.)

Proof. By the definition of \( a, b, c \) and Lemma 2.5, there exists cohomology classes

\[
\omega_a \in \tilde{H}^3_{S^1}(I^a_\lambda), \quad \omega_b \in \tilde{H}^{b+1}_{S^1}(I^a_\lambda), \quad \omega_c \in \tilde{H}^{c+2}_{S^1}(I^a_\lambda)
\]

whose pull-back under \( i \) do not vanish. Forgetting the degree shift by \( s \) for the moment, the non-zero cohomology classes \( i^*\omega_a, i^*\omega_b, i^*\omega_c \) are of the form \( v^l, qv^l, q^2v^m \) respectively by the degree reason:

\[
a \equiv s \mod 4, \quad b + 1 \equiv s + 1 \mod 4, \quad c + 2 \equiv s + 2 \mod 4.
\]

Recalling that the degree shift by \( s \) is occurred by multiplying the equivariant Thom class, we can determine \( l, l', l'' \) and obtain (16).

\[ \Box \]

2.3. The families relative Bauer–Furuta invariant. In this section we consider a family version of the relative Bauer–Furuta invariant.

Let \( X \) be an oriented compact smooth 4-manifold bounded by \( Y \). Assume that \( b_1(X) = 0 \) and \( Y \) is a connected rational homology 3-sphere. Let \( \mathfrak{s} \) be a spinc structure on \( X \) and let \( t \) be the spin\(^c\) structure on \( Y \) defined as the restriction of \( \mathfrak{s} \). Let \( B \) be a compact topological space. Throughout this paper, for a topological space \( F \), denote by \( \mathbb{F} \) the trivialized bundle \( B \times F \) over \( B \).

Assume that we have a Homeo\((X, \partial)\)-bundle \( X \rightarrow E \rightarrow B \). We shall define a vector bundle

\[
\mathbb{R}^{b^+(X)} \rightarrow H^+(E) \rightarrow B
\]

as follows. First, let us define the ‘maximal-positive-definite Grassmannian’

\[
\text{Gr}^+(H^2(X; \mathbb{R}))
\]

as the space of maximal-dimensional positive-definite subspace of \( H^2(X; \mathbb{R}) \) with respect to the intersection form. Since the group Homeo\((X, \partial)\) naturally acts on \( \text{Gr}^+(H^2(X; \mathbb{R})) \), we obtain a fiber bundle

\[
\text{Gr}^+(H^2(X; \mathbb{R})) \rightarrow \text{Gr}^+_E \rightarrow B
\]

from transition functions of \( E \), taking values in Homeo\((X, \partial)\). The Grassmannian \( \text{Gr}^+(H^2(X; \mathbb{R})) \) is contractible, since this is diffeomorphic to the quotient of the Lie group \( O(b^+(X), b^-(X)) \) by the maximal compact subgroup \( O(b^+(X)) \times O(b^-(X)) \).
Therefore there exists a section of $\text{Gr}_E^+ \to B$, which is unique up to isotopy. One section corresponds to a vector bundle of rank $b^+(X)$, and we denote by $H^+(E)$ the vector bundle. This vector bundle is determined uniquely by $E$ up to isomorphism, and we omit the choice of section of $\text{Gr}_E^+ \to B$ from our notation $H^+(E)$.

From here we assume that a reduction of $E$ to $\text{Aut}((X, s), \partial)$ is given. Namely, $(X, s) \to E \to B$ is a smooth fiber bundle of spin$^c$ 4-manifolds equipped with a trivialization

$$((Y, t) \to E_Y \to B) \cong ((Y, t) \to (Y, t) \times B \to B),$$

where $E_Y$ is a fiber bundle on $B$ defined to be

$$\bigsqcup_{b \in B} \partial E_b \to B.$$

Fix a fiberwise metric $g_E$ on $E \to B$ with $g_E|_{E_Y} = g_Y$, where $g_Y$ is a fixed Riemann metric on $Y$. Let $\{\tilde{A}_b\}_{b \in B}$ be a fiberwise reference spin$^c$-connection on $E$ such that $\tilde{A}_b|_{\partial E_b} = a_0$. Once we fix the data $(E, g_E)$, the following families of vector bundles over $B$

$$S^+_E, \quad S^-_E, \quad i\Lambda^*_E, \quad i\Lambda^+_E$$

are associated. The restrictions of them over $b \in B$ are the positive and negative spinor bundles with respect to $(g_{E_b}, s)$, and $i\Lambda^*_X$ and $i\Lambda^+_X$ with respect to $g_{E_b}$ respectively, where $\Lambda^*_X = \{\omega \in \Lambda^*_X \ | *_{g_{E_b}} \omega = \omega\}$. We use the notation

$$L_k^2(S^+_E), \quad L_k^2(S^-_E), \quad L_k^2(i\Lambda^*_E), \quad L_k^2(i\Lambda^+_E)$$

to denote the spaces of fiberwise $L_k^2$-sections. In order to obtain Fredholm property for a certain operator, we shall use a subspace $L_k^2(i\Lambda^1_E)_{CC}$ of $L_k^2(i\Lambda^1_E)$ defined by

$$L_k^2(i\Lambda^1_E)_{CC} := \bigsqcup_{b \in B} \{ a \in L_k^2(i\Lambda^1_{E_b}) \ | \ d^*a = 0, \ d^*ta = 0 \},$$

where $t$ is the restriction as differential forms along the inclusion $Y = \partial E_b \to E_b$. This gauge fixing condition is called double Coulomb condition and was introduced by Khandhawit [23].

**Remark 2.7.** Although Khandhawit imposed the condition $\int_Y t(*a) = 0$, we can omit this condition. This is because we have

$$\int_Y t(*a) = \int_Y t1 \wedge *na_b = \int_{E_b} (d1 \wedge *a_b) - \int_{E_b} (1 \wedge *d^*a_b) = 0$$

by the Stokes theorem for any $a_b \in L_k^2(i\Lambda^1)_{CC}$, where $n$ is the normal component. Here we used the connectivity of $Y$.

For any positive real number $\mu$, now we have the fiberwise Seiberg–Witten map over a slice

$$\mathcal{F}^\mu : L_k^2(i\Lambda^1)_{CC} \oplus L_k^2(S^+_E) \to L_{k-1}^2(i\Lambda^+) \oplus L_{k-1}^2(S^-_E) \oplus V_{-\infty}^\mu$$

defined by

$$\mathcal{F}^\mu((A_b, \Phi_b)_{b \in B}) = (\rho_b(F^+(A_b)) - (\Phi_b, \Phi_b)_0, D_{\tilde{A}_b + A_b}(\phi), p_{-\infty} \circ r_b(A_b, \Phi_b))_{b \in B},$$

where $F^+(A_b)$ is the self-dual part of the curvature of a fiberwise connection $A_b$, $\rho_b$ is the Clifford multiplication, $D_{\tilde{A}_b + A_b}$ is the fiberwise Dirac operator with respect to a connection $\tilde{A}_b + A_b$, and

$$r_b : L_k^2(i\Lambda^1)_{CC} \oplus L_k^2(S^+_E) \to \text{Coul}_k(Y, s)$$

where

$$\text{Coul}_k(Y, s) \subset H^0(\text{Gr}_E^+, S^+_E) \oplus H^0(\text{Gr}_E^+, S^-_E) \to H^0(X, s).$$
is the fiberwise restriction. We decompose \( \mathcal{F}^\mu \) as the sum of a fiberwise elliptic operator

\[
L^\mu = \{ L_b^\mu = (d^+, D_{A_b}, p_{\infty}^\mu d^*) \}_{b \in B}
\]

and a fiberwise quadratic part

\[
e^\mu = \{ e^\mu = (- (\Phi_b \otimes \Phi_b^*)_0, \rho(A_b)(\Phi_b), 0) \}_{b \in B}.
\]

We often use a decomposition of the operator \( L_b^\mu \) for each \( b \) as the sum of the real operator

\[
L_{b, \mathbb{R}}^\mu : L_k^2(iA_{E_b})_{CC} \to L_k^2(iA_{E_b}^+) \oplus V_{-\infty}^\mu(\mathbb{R})
\]

and the complex operator

\[
L_{b, \mathbb{C}}^\mu : L_k^2(S_{E_b}^+) \to L_k^2(S_{E_b}^-) \oplus V_{-\infty}^\mu(\mathbb{C})
\]

It is checked in \([23]\) that the fiberwise linear operator \( L_b^\mu \) is Fredholm on each fiber and the Fredholm index is given by

\[
2 \text{ind}_{\mathbb{C}}^{APS} D_{A_b}^+ - b^+(X) - \dim V_b^\mu,
\]

where \( 2 \text{ind}_{\mathbb{C}}^{APS} D_{A_b}^+ \) is the Fredholm index of \( L_{b, \mathbb{C}}^\mu \) as a real operator.

**Definition 2.8.** Let \( \mathcal{H}^+(E) \) be a real vector bundle defined by

\[
\mathcal{H}^+(E) := \bigsqcup_{b \in B} \text{Coker } L_{b, \mathbb{R}}^0 \to B.
\]

(\( \text{It follows from (i) of Lemma } [29]\text{ below that } \mathcal{H}^+(E) \text{ actually forms a vector bundle of rank } b^+(X). \))

Although the bundle \( \mathcal{H}^+(E) \) depends on the choice of fiberwise Riemann metric, we see that its isomorphism class is independent of choices:

**Lemma 2.9.** Under the assumption \( b_1(X) = 0 \), the operator

\[
L_{b, \mathbb{R}}^0 : L_k^2(iA_{E_b})_{CC} \to L_k^2(iA_{E_b}^+) \oplus V_{-\infty}^0(\mathbb{R})
\]

satisfies the following properties:

(i) \( L_{b, \mathbb{R}}^0 \) is an injection, and

(ii) the bundle \( \mathcal{H}^+(E) \) is isomorphic to \( H^+(E) \).

**Proof.** In order to prove this lemma, we consider the following two operators \( L_{b, \mathbb{R}}^{AHS} \) and \( L_{b, \mathbb{R}}^{AHS} \).

The first operator is the Atiyah–Hitchin–Singer operator

\[
L_{b, \mathbb{R}}^{AHS} := d^* + d^+ + \text{pr}_{H^-} \circ \tilde{r} : L_k^2(iA_{E_b}^+) \to L_k^2(iA_{E_b}^+ \oplus iA_{E_b}^-) \oplus H^-,
\]

with a boundary condition, where

- \( H^- \) is the \( L_{k-\frac{1}{2}}^2 \)-completion of the non-positive eigenspace of the operator

\[
\tilde{r} : i \text{ Im } d \oplus i \text{ Ker } d^* \oplus i\Omega^0(Y) \to i \text{ Im } d \oplus i \text{ Ker } d^* \oplus i\Omega^0(Y)
\]

defined by

\[
\tilde{r} := \begin{pmatrix}
0 & 0 & -d \\
0 & *d & 0 \\
-d^* & 0 & 0
\end{pmatrix}.
\]
Both of

\[ V := d(L_k^2(i\Lambda_{E_b}^0)) \oplus i \text{Ker}(d^*|_{L_k^2(i\Lambda^1(Y))}) \oplus L_{k-\frac{1}{2}}^2(i\Lambda^0(Y)), \]

and

\[ \text{pr}_{H^-} : V \to H^- \]

is the \( L^2 \)-projection.

Regarding the first operator \( L_b^{AHS} \), it is proved in [1] that there are fiberwise isomorphisms

\[ \text{Ker} L_b^{AHS} \cong H^1(E_b; \mathbb{R}), \quad \text{Coker} L_b^{AHS} \cong H^+(E_b; \mathbb{R}) \oplus H^0(E_b; \mathbb{R}). \]

The second is the AHS operator with a projection

\[ L_b^{AHS} := d^* + d^+ + (\text{pr}_{H^-} + \Pi) \circ \tilde{r} \]

\[ : L_k^2(i\Lambda_{E_b}^0) \to L_{k-1}^2(i\Lambda_{E_b}^0 \oplus i\Lambda_{E_b}^+) \oplus V_{-\infty}^0(\mathbb{R}) \oplus W_Y, \]

where \( W_Y = H_Y^0 \oplus d(L_{k-\frac{1}{2}}^2(i\Lambda^0(Y))) \), and the map \( \Pi \) is the \( L^2 \)-projection

\[ \Pi : V \to H_Y^0 \oplus d(L_{k-\frac{1}{2}}^2(i\Lambda^0(Y))) = W_Y. \]

Here \( H_Y^0 \) is the space of \( i\rho \)-valued constant function on \( Y \).

We are going to compare \( L_b^{AHS} \) with \( L_0 \) via \( \tilde{L}_b^{AHS} \). First let us compare \( L_b^{AHS} \) with \( \tilde{L}_b^{AHS} \):

**Lemma 2.10.** The kernels and cokernels of \( \tilde{L}_b^{AHS} \) and \( L_b^{AHS} \) are isomorphic to each other respectively, via the following isomorphism between the codomains of \( \tilde{L}_b^{AHS} \) and \( L_b^{AHS} \):

\[ \text{id} \oplus \Pi : L_{k-1}^2(i\Lambda_{E_b}^0 \oplus i\Lambda^+_{E_b}) \oplus H^- \to L_{k-1}^2(i\Lambda_{E_b}^0 \oplus i\Lambda_{E_b}^+) \oplus V^0_{-\infty}(\mathbb{R}) \oplus W_Y, \]

which is defined by

\[ \text{id} \oplus \Pi(x_1, x_2, (y_1, y_2, y_3)) := (x_1, x_2, y_2, \Pi(y_1, y_2, y_3)). \]

**Proof.** The operator \( \tilde{l} \) can be written as the sum of \(*d\) on \( \text{Ker} d^* \) and \( l \), where \( l \) is the self-adjoint operator

\[ l := \begin{pmatrix} 0 & -d^* \\ -d & 0 \end{pmatrix} : \text{Im} d \oplus i\Omega^0(Y) \to \text{Im} d \oplus i\Omega^0(Y). \]

Let us denote by \( \tilde{H}^- \) the non-positive eigenspaces of \( l \). It is checked in [23] that both of \( \tilde{H}^- \) and \( W_Y \) have \( L_{k-\frac{1}{2}}^2(i\Lambda^0(Y))_0 \) as a complement in

\[ L_{k-\frac{1}{2}}^2(i\Lambda^0(Y)) \oplus dL_{k+\frac{1}{2}}^2(i\Lambda^0(Y)), \]

where

\[ L_{k-\frac{1}{2}}^2(i\Lambda^0(Y))_0 := \left\{ a \in L_{k-\frac{1}{2}}^2(i\Lambda^0(Y)) \middle| \int_Y a \, \text{dvol} = 0 \right\}. \]

This proves \( \text{id} \oplus \Pi \) is an isomorphism. \( \square \)
Next, we compare \( \tilde{L}^{AHS}_b \) with \( L^0_b \). We have the following commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
L^2_k(i\Lambda^1)_{CC} & \rightarrow & L^2_{k-1}(i\Lambda^+_{E_b}) \\
\downarrow & & \downarrow \\
L^2_k(i\Lambda^!)_{E_b} & \rightarrow & L^2_{k-1}(i\Lambda^0_{E_b} + i\Lambda^+_{E_b}) + V^0_\infty(\mathbb{R}) + W_Y \\
\downarrow & & \downarrow \\
L^2_{k-1}(i\Lambda^0_{E_b})_0 \oplus W_Y & \rightarrow & L^2_{k-1}(i\Lambda^0_{E_b}) + W_Y \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0,
\end{array}
\]

where

\[
L^2_{k-1}(i\Lambda^0_{E_b})_0 := \left\{ a \in L^2_{k-1}(i\Lambda^0_{E_b}) \mid \int_X a \, d\text{vol} = 0 \right\}.
\]

This diagram and the snake lemma prove that there are fiberwise isomorphisms

\[
\text{Ker} \, L^0_b|_{L^2_k(i\Lambda^1)_{CC}} \cong \text{Ker} \, \tilde{L}^{AHS}_b
\]

and

\[
\text{Coker} \, L^0_b|_{L^2_k(i\Lambda^1)_{CC}} \oplus H^0(E_b; \mathbb{R}) \cong \text{Coker} \, \tilde{L}^{AHS}_b.
\]

By combining this with (18), we conclude that there are fiberwise isomorphisms

\[
\text{Ker} \, L^0_{b,R} \cong \{0\}, \quad \text{Coker} \, L^0_{b,R} \cong H^+(E_b; \mathbb{R}).
\]

This completes the proof of Lemma 2.9. \( \square \)

Next, in order to carry out finite-dimensional approximation, we take a sequence of finite-dimensional vector subbundles \( W^n_i \) of \( L^2_{k-1}(i\Lambda^+_{E}) \oplus L^2_{k-1}(S^{-}_E) \).

**Lemma 2.11.** There exists a sequence of finite-dimensional vector subbundles \( W^n_i \) of \( L^2_{k-1}(i\Lambda^+_{E}) \oplus L^2_{k-1}(S^{-}_E) \) such that

\[
\text{Coker} \, L^\mu_b \cap (L^2_{k-1}(i\Lambda^+_{E_b}) \oplus L^2_{k-1}(S^{-}_E)) \subset (W^n_i)_b
\]

for all \( b \in B \), and that the projection \( \text{pr}_{W^n_i} : L^2_{k-1}(i\Lambda^+_{E}) \oplus L^2_{k-1}(S^{-}_E) \rightarrow W^n_i \) converges to the identity map pointwise as \( n \rightarrow \infty \) for any \( b \in B \).

**Proof.** For a fixed point \( b_1 \in B \), we define \( W_{b_1} := \text{Coker} \, L^\mu_{b_1} \). By using a global trivialization of \( L^2_{k-1}(i\Lambda^+_{E}) \oplus L^2_{k-1}(S^{-}_E) \), we extend a vector space \( W_{b_1} \) to a subbundle \( \tilde{W}_{b_1} \) over \( B \). Since surjectivity is an open condition, for any element \( b \) in the small neighborhood of \( b_1 \), \( L^\mu_b \) is transversal to \( \tilde{W}_{b_1} \). Since \( B \) is compact, we can take a finite sequence of points \( b_1, \ldots, b_k \) such that \( L^\mu_b \) is transversal to \( \tilde{W}_{b_k} \) for some \( k \). Then we define

\[
W^n_i := \bigoplus \tilde{W}_{b_i}.
\]

The latter condition follows by using the trivialization of \( L^2_{k-1}(i\Lambda^+_{E}) \oplus L^2_{k-1}(S^{-}_E) \). \( \square \)
Take sequences of numbers \( \lambda_n \) and \( \mu_n \) such that \( \lambda_n \to -\infty \) and \( \mu_n \to \infty \) as \( n \to \infty \). Let us define

\[
W^n_0 := (L^\lambda_n)^{-1}(W^n_1 \oplus V^{\mu_n}).
\]

**Lemma 2.12.** There exists \( \mu_0 > 0 \) such that, for any \( \mu \) with \( \mu > \mu_0 \) and for any \( b \in B \), \( L^\mu_b \) is injective.

**Proof.** Suppose the conclusion is not true. Then we have a sequence of points \( \{\mu_n\} \) and \( \{b_n\} \subset B \) and \( x_{b_n} \in L^2_k(i\Lambda^1 \e_{E_{b_n}})_{CC} \) such that \( \mu_n \to \infty \), \( L^\mu_{b_n}(x_{b_n}) = 0 \) and \( x_{b_n} \neq 0 \). By the scalar multiplication, we assume \( \|x_{b_n}\|_{L^2_k}^2 = 1 \). Since \( B \) is compact, after taking a subsequence, we can assume that \( \{b_n\} \) converges \( b_{\infty} \in B \).

By the Fredholm property of \( L^\lambda_n \), after taking a subsequence, we can assume \( \{x_{b_n}\} \) converges to \( x_{\infty} \), which satisfies

\[
d^+(x_{\infty}) = 0, \ d^-(x_{\infty}) = 0, \ i^*x_{\infty} = 0, \ \|x_{\infty}\|_{L^2_k}^2 = 1 \text{ and } D^+_{A_{\infty}}(x_{\infty}) = 0.
\]

However this contradicts the unique continuation property for \( (d^*, d^+D^+_{A_{\infty}}) \). \( \square \)

For a sufficient large \( n \), we have an isomorphism

\[
W^n_1 \oplus V^{\mu_n}_{\lambda_n} \cong W^n_0 \oplus \text{Coker } L^{\mu_n}
\]

between the vector bundles. Moreover, since \( p^B_0(\text{pr}_{Ker\hat{d}, i^*}) : \text{Ker } L^0 \to V^\mu_0 \) is fiberwise injective, we have an identification

\[
\text{Ker } L^0 - \text{Coker } L^0 + \text{Coker } L^{\mu_n} \cong V^{\mu_n}_0
\]

as virtual vector bundles over \( B \). Thus we have

\[
W^n_1 + V^{\mu_n}_{\lambda_n} + \text{Ker } L^0 - \text{Coker } L^0 = W^n_0 \oplus V^{\mu_n}_0
\]

as virtual vector bundles on \( B \).

Applying the projection, we obtain a family of maps

\[
\text{pr}_{W^n_1} \circ \mathcal{F}^{\mu_n}|_{W^n_0} : W^n_0 \to W^n_1 \times V^{\mu_n}_{\lambda_n}
\]

whose \( S^1 \)-invariant part is given by

\[
(\text{pr}_{W^n_1} \circ \mathcal{F}^{\mu_n}|_{W^n_0})^{S^1} : W^n_0(\mathbb{R}) \to W^n_1(\mathbb{R}) \times V^{\mu_n}(\mathbb{R}).
\]

This induces a map

\[
\text{pr}_{W^n_1} \circ \mathcal{F}^{\mu_n}|_{W^n_0} : B(R, W^n_0) \to (W^n_1 \times V^{\mu_n}_{\lambda_n})^+ \,,
\]

where \( +_B \) is the fiberwise one point compactification. Here we use inner products on \( W^\pm_1 \) obtained as the restrictions of the \( L^2 \)-metrics.

For a subset \( A \) in \( V^n_\delta \), set

\[
A^+ := \{ x \in A \mid \forall t > 0, \ t \cdot x \in A \}.
\]

In order to obtain a suitable index pair used for a cohomotopy type invariant from (20), we need the following Lemma 2.13. Set

\[
K_1 := B(R, W^n_0) \cap ((\text{pr}_{W^n_1} \circ \mathcal{F}^{\mu_n})^{-1}B(\epsilon_n, W^n_1))
\]

and

\[
K_2 := S(R, W^n_0) \cap ((\text{pr}_{W^n_1} \circ \mathcal{F}^{\mu_n})^{-1}B(\epsilon_n, W^n_1)).
\]
Lemma 2.13. Suppose that the base space $B$ is compact. For a sufficiently large $R, R'$ and $n$, the compact sets

$$K_1 := p_{V_{\lambda_n}^n} \circ F_{\mu_n}(\tilde{K}_1)$$

and

$$K_2 := p_{V_{\lambda_n}^n} \circ F_{\mu_n}(\tilde{K}_2)$$

satisfy the assumption of [23] Theorem 4, [23] Lemma A.4] for $A := B(R'; V_{\lambda_n}^n)$, i.e. the following conditions hold:

(i) if $x \in K_1 \cap A^+$, then $[0, \infty) \cdot x \cap \partial A = \emptyset$ and

(ii) $K_2 \cap A^+ = \emptyset$.

Proof. The proof is essentially the same as the original proof of [23] Proposition 4.5]. We will prove by contradiction. We first prove (i). Before starting the discussion, we fix a universal constant $K$ of $\{0, \infty\}$ and $\lambda_n$. We will prove by contradiction. We first prove (i). Before starting the discussion, we fix a universal constant $K$ of [23] Corollary 4.3] for a family of metrics $g_{E_n}$. (Since $B$ is compact, we can prove the existence of such constants.) We fix constants $R$ and $R'$ with $R, R' > B_k$. We suppose there exist sequences $\{b_n\} \subset B$ with $b_n \to b_\infty$ as $n \to \infty$ and

$$\{x_n\} \subset B(R, W^n_0|_{b_n}) \cap ((p_{W^n_{1}} \circ F_{\mu_n})^{-1}B(\epsilon_n, W^n_1|_{b_n}))$$

such that there exists a sequence of approximated half trajectories $y_n : [0, \infty) \to V_{\lambda_n}^{\mu_n}$ with

$$\frac{\partial}{\partial t} y_n(t) = -(l + p_{\lambda_n}^*c)(y(t)), \quad y_n(0) = p_{\lambda_n}^{-\infty}i^*x_n$$

and

$$\|y_n(t_n)\|_{V_{\lambda_n}^{\mu_n}} = R'.$$

We need the following lemma to get a contradiction:

Lemma 2.14. Let $\{x_n\}$ be a bounded sequence in $L^2_k(i\Lambda^1)_{CC} \oplus L^2_k(S_E^n)$ such that

$$\{F_{\mu_n}(x_n)\} \cap (L^2_k(i\Lambda^1) \oplus S_E^{-}) \subset W^n, \quad \{F_{\mu_n}(x_n)\} \cap (V_{\lambda_n}^{\mu_n}) \subset V_{\lambda_n}^{\mu_n},$$

$$(L^\infty + p^n \lambda_n)(x_n) \to 0 \text{ in } L^2_k$$

and sequences $\{b_n\}$ corresponding the base points of $x_n$ in $B$ converge to $b_\infty \in B$. We also suppose that there exists a sequence of approximated half trajectories $y_n : [0, \infty) \to V_{\lambda_n}^{\mu_n}$ with

$$\frac{\partial}{\partial t} y_n(t) = -(l + p_{\lambda_n}^*c)(y(t)) \quad \text{and} \quad y_n(0) = p_{\lambda_n}^{-\infty}i^*x_n.$$  

Then, after taking a subsequence, the sequence $\{x_n\}$ converges to a solution $x_\infty$ of Seiberg–Witten equation for $E_{b_\infty}$ and the exists a Seiberg–Witten half trajectory $y_\infty$ with $y_\infty(0) = i^*x_\infty$ and $y_n(t)$ converges $y_\infty(t)$ for all $t$ in $L^2_{k+1}$.  

This is a family version of [23] Lemma 4.4]. We omit the proof since the proof of is essentially the same as that of the original one. By the use of Lemma 2.14], we have $x_\infty$ and $y_\infty$ satisfying the conclusion of Lemma 2.14]. After taking a subsequence of $\{t_n\}$, we have two cases $t_n \to t_\infty \in [0, \infty)$ and $t_n \to \infty$. This implies

$$\|y_\infty(t_\infty)\|_{L^2_{k+\frac{1}{2}}} = R' \quad \text{or} \quad \|y_\infty(\infty)\|_{L^2_{k-\frac{1}{2}}} = R'.$$

holds. However, this contradicts to the choice of $R' > B_k$. The proof of (ii) is similar. \qedsymbol
By using above lemma and [34, Theorem 4], we may take an $S^1$-invariant Conley index $(N_n, L_n)$ such that $(K_1, K_2) \subset (N_n, L_n)$.

Then we obtain an $S^1$-equivariant continuous map

\[
(23) \quad f_n := \text{pr}_{W_1^n} \circ \mathcal{F}^n |_{W^n_0} : (W^n_0(R))^{+b} \to W^n_1/B(\epsilon_n, W^n_1)^c \wedge_B I_{\lambda_n}^n,
\]

where $\wedge_B$ denotes the fiberwise smash product. We call this map (23) the families relative Bauer–Furuta invariant.

The decomposition (19) implies that this map stably can be written so that

\[
\begin{align*}
&f : \left\{ \text{ind } D^+_A \right\}_{b \in B}^{+b} \to \left( \bigoplus_{i=0}^n (Y, t, g_i) \oplus \mathbb{R}^b \right)^{+b} \wedge_B SF(Y, t), \\
&\quad \text{where } \left\{ \text{ind } D^+_A \right\}_{b \in B} \text{ denotes the virtual index bundle. Arguing exactly as in } [34],
\end{align*}
\]

one may see that this map gives rise to a topological invariant of a smooth bundle $E$ of 4-manifolds with boundary equipped with a fiberwise spin$^c$ structure, but the invariance is not necessary for our purpose in this paper.

When $s$ is spin, respecting Pin(2)-symmetry over the whole argument above, we obtain a Pin(2)-equivariant map

\[
\begin{align*}
&f : \left\{ \text{ind } D^+_A \right\}_{b \in B}^{+b} \to \left( \bigoplus_{i=0}^n (Y, t, g_i) \oplus \mathbb{R}^b \right)^{+b} \wedge_B SWF(Y, t), \\
&\text{as well.}
\end{align*}
\]

3. Proof of the main theorems

In this section we give the proofs of the main theorems, Theorems 1.1 and 1.2.

3.1. Properties of the families relative Bauer–Furuta invariant. In this subsection we summarize some properties of the relative families Bauer–Furuta invariant (23) which are deduced from Subsection 2.3. Henceforth we shall drop $n$ in (23) from our notation. Recall that the families relative Bauer–Furuta invariant for the smooth family $(X, s) \to E \to B$ is given as a fiberwise $S^1$-equivariant map between families of pointed $S^1$-spaces parametrized over $B$:

\[
(24) \quad f : W_0^{+b} \to W_1^{+b} \wedge_B I_{\lambda}^b.
\]

Here

- $I_{\lambda}^b$ is the Conley index used to define the Seiberg–Witten Floer homotopy type of $Y$, where $\mu, -\lambda$ are taken to be sufficiently large. Let $(N, L)$ be an index pair to define $I_{\lambda}^b$:

  \[
  I_{\lambda}^b = N/L.
  \]

- $W_0, W_1 \to B$ are vector bundles. Each $W_i$ is the direct sum of a real vector bundle $W_i(R)$ and a complex vector bundle $W_i(C)$ over $B$:

  \[
  W_i = W_i(R) \oplus W_i(C).
  \]

- The $S^1$-invariant part of the map (24) is obtained as the restriction of a fiberwise $S^1$-equivariant linear map between vector bundles, denoted also by the same symbol $f^{S^1}$ by an abuse of notation:

  \[
  f^{S^1} : W_0(R) \to W_1(R) \oplus V_{\lambda}^b(R) = W_1(R) \times V_{\lambda}^b(R).
  \]
Let \( p_{V^0_0(\mathbb{R})} : V_1^0(\mathbb{R}) \to V_0^0(\mathbb{R}) \) be the \( L^2 \)-projection. It follows from Lemma 2.9 that the map
\[
(\text{id}_{W_1(\mathbb{R})} \oplus p_{V_0^0(\mathbb{R})}) \circ f^{S^1} : W_0(\mathbb{R}) \to W_1(\mathbb{R}) \oplus V_0^0(\mathbb{R}) = W_1(\mathbb{R}) \times V_0^0(\mathbb{R})
\]
is a fiberwise linear injection and its fiberwise cokernel is isomorphic to the bundle \( H^+(E) \to B \).

- We have
\[
\text{rank}_C W_0(\mathbb{C}) - \text{rank}_C W_1(\mathbb{C}) = \text{ind}_C D^+_A + \dim_C V_0^0(\mathbb{C})
\]
\[
\frac{c_1(s)^2 - \sigma(X)}{8} + n(Y, t, g) + \dim_C V_0^0(\mathbb{C}).
\]
Here \( \{A_b\}_{b \in B} \) denotes a family of \( U(1) \)-connections of the family of the determinant line bundles and \( \{\text{ind} D_A^+\}_{b \in B} \) denotes the index of the families of the Dirac operators associated to \( E \).

To prove Theorem 1.1 we have to rewrite the \( S^1 \)-fixed part \((I_0\Lambda)^S\)
into the sphere \( V_0^0(\mathbb{R})^+ \)
without loss of information about the image of \( f^{S^1} \).

It is summarized as the following Lemma 3.1. Let
\[
p_1 : W_1 \times V_\Lambda^0 \to W_1,
\]
\[
p_2 : W_1 \times V_\Lambda^0 \to V_\Lambda^0
\]
be the projections.

**Lemma 3.1.** There exists a homotopy equivalence
\[
\varphi : N^{S^1}/L^{S^1} \to V_0^0(\mathbb{R})^+
\]
for which the diagram
\[
\begin{array}{ccc}
W_0(\mathbb{R})^+ & \xrightarrow{p_{V_0^0(\mathbb{R})} \circ f^{S^1}} & N^{S^1}/L^{S^1} \\
\downarrow \varphi & & \downarrow \varphi \\
V_0^0(\mathbb{R})^+ & \xrightarrow{p_2 \circ f^{S^1}} & V_0^0(\mathbb{R})^+
\end{array}
\]
commutes up to homotopy.

**Proof.** Since \( p_1 \circ f^{S^1} : W_0(\mathbb{R}) \to W_1(\mathbb{R}) \) is a fiberwise linear map,
\[
\tilde{D}(W_0(\mathbb{R})) := D(W_0(\mathbb{R})) \cap \hat{K}_1 = D(W_0(\mathbb{R})) \cap (p_1 \circ f^{S^1})^{-1}(B(\epsilon; W_1(\mathbb{R})))
\]
and
\[
\tilde{S}(W_0(\mathbb{R})) := S(W_0(\mathbb{R})) \cap \hat{K}_1 = S(W_0(\mathbb{R})) \cap (p_1 \circ f^{S^1})^{-1}(B(\epsilon; W_1(\mathbb{R})))
\]
are a disk bundle and a sphere bundle of \( W_0(\mathbb{R}) \) of some common radius respectively. Here \( \hat{K}_1, \hat{K}_2 \) are defined in (21) and (22).

Let us remark that we have
\[
(p_2 \circ f(\tilde{D}(W_0(\mathbb{R}))), p_2 \circ f(\tilde{S}(W_0(\mathbb{R})))) \subset (K_1^{S^1}, K_2^{S^1}) \subset (N^{S^1}, L^{S^1}).
\]

On the other hand, since the map (25) is a fiberwise linear injection, we have also that
\[
(p_2 \circ f(\tilde{D}(W_0(\mathbb{R}))), p_2 \circ f(\tilde{S}(W_0(\mathbb{R}))))
\subset (D(V_0^0(\mathbb{R})), D(V_0^0(\mathbb{R}))), S(V_0^0(\mathbb{R})) \times D(V_0^0(\mathbb{R})).
\]
where $D(\cdot)$ and $S(\cdot)$ are disks and spheres with appropriate radius respectively. Moreover, it is easy to check that both of the right-hand sides of (28) and (29) are index pairs for the $S^1$-invariant part of the isolated invariant set $\text{Inv}(B(2R; V^2_\lambda(\mathbb{R})))$.

It follows from this combined with an argument used to prove Proposition A.5 [23] by Khandhawit that there exists a homotopy equivalence
\[
\phi : N^S_1 / L^S_1 \to \frac{D(V^0_0(\mathbb{R})) \times D(V^\mu_\lambda(\mathbb{R}))}{S(V^0_\lambda(\mathbb{R})) \times D(V^\mu_0(\mathbb{R}))}
\]
which makes the diagram
\[
\begin{array}{ccc}
\tilde{D}(W_0(\mathbb{R}))/\tilde{S}(W_0(\mathbb{R})) & \xrightarrow{p_2 \circ f^S_1} & N^S_1 / L^S_1 \\
\downarrow & & \phi \\
D(V^0_\lambda(\mathbb{R})) \times D(V^\mu_\lambda(\mathbb{R})) & \xrightarrow{P_{V^0_\lambda(\mathbb{R})} \circ f^S_1} & D(V^0_\lambda(\mathbb{R}))/S(V^0_\lambda(\mathbb{R})).
\end{array}
\]
commutative up to homotopy. Note also an obvious commutative diagram
\[
\begin{array}{ccc}
\tilde{D}(W_0(\mathbb{R}))/\tilde{S}(W_0(\mathbb{R})) & \xrightarrow{p_2 \circ f^S_1} & D(V^0_\lambda(\mathbb{R})) \times D(V^\mu_\lambda(\mathbb{R})) \\
\downarrow & & \xrightarrow{\cong} \\
D(V^0_\lambda(\mathbb{R}))/S(V^0_\lambda(\mathbb{R})).
\end{array}
\]
Defining $\varphi$ as the composition of the vertical arrows in (30) and (31), we obtain a homotopy commutative diagram (27).

3.2. Proof of Theorem 1.1 Now we may start proving Theorem 1.1. Recall that all (co)homology are taken with $\mathbb{F} = \mathbb{Z}/2$-coefficients throughout this paper.

Proof of Theorem 1.1. Let us consider the following commutative diagram obtained by restricting the families relative Bauer–Furuta invariant onto the $S^1$-fixed-point sets:
\[
\begin{array}{ccc}
W^{+b}_0 & \xrightarrow{f} & W^{+b}_1 \wedge_B I^\mu_\lambda \\
\downarrow i_o & & \downarrow i_1 \\
W^0(\mathbb{R})^{+b} & \xrightarrow{f^S_1} & W^1(\mathbb{R})^{+b} \wedge_B (I^\mu_\lambda)_1
\end{array}
\]
Here $i_o, i_1$ denote the inclusion maps.

The following lemma can be checked in a straightforward manner, and we omit the proof.

Lemma 3.2. Let $W, W' \to B$ be vector bundles over $B$. Denote by $\text{Th}(W)$ the Thom space of $W$. Then we have:

1. The identity map $W \times I \to W \times I$ induces a well-defined map
   \[ W^{+b} \wedge_B I \to \text{Th}(W) \wedge I. \]

2. Assume that we have a fiberwise pointed map $\varphi : (W')^{+b} \to W^{+b} \wedge_B I$.
   Then $\varphi$ induces a well-defined map
   \[ \text{Th}(W') \to \text{Th}(W) \wedge I. \]
(3) For a natural number \( n \), the identity map \( W \oplus \mathbb{R}^n \to W \oplus \mathbb{R}^n \) induces a well-defined homeomorphism

\[
\text{Th}(W \oplus \mathbb{R}^n) \to \text{Th}(W) \wedge S^n.
\]

From this, it follows that the commutative diagram (32) induces the following commutative diagram:

\[
\begin{array}{ccc}
\text{Th}(W_0) & \xrightarrow{f} & \text{Th}(W_1) \wedge I^\mu_X \\
\downarrow i_0 & & \downarrow i_1 \\
\text{Th}(W_0(\mathbb{R})) & \xrightarrow{f_{S^1}} & \text{Th}(W_1(\mathbb{R})) \wedge (I^\mu_X)^{S^1}.
\end{array}
\]

Applying the functor \( \tilde{H}^*_S(\cdot ; \mathbb{F}) \), we obtain the commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^*_S(\text{Th}(W_0)) & \xrightarrow{f^*} & \tilde{H}^*_S(\text{Th}(W_1) \wedge I^\mu_X) \\
\downarrow i_0^* & & \downarrow i_1^* \\
\tilde{H}^*_S(\text{Th}(W_0(\mathbb{R}))) & \xrightarrow{(f_{S^1})^*} & \tilde{H}^*_S(\text{Th}(W_1(\mathbb{R})) \wedge (I^\mu_X)^{S^1}).
\end{array}
\]

We shall derive the divisibility of the Euler classes of some bundles using the diagram (34). To do this in our situation, we will take a cohomology class

\[\eta \in \tilde{H}^*_S(\text{Th}(W_1) \wedge I^\mu_X)\]

as follows. Henceforth, as an abbreviation, we write \( d \) for \( d(Y, \lambda, \mu, g, t) \in \mathbb{Z} \). Set

\[s = \dim V^0_1(\mathbb{R}).\]

By Lemma 2.4 there exists a cohomology class

\[\omega \in \tilde{H}^d_S(I^\mu_X)\]

satisfying the equality (12). Setting

\[d' = (d - s)/2,\]

we have

\[i^* \omega = [V^0_1(\mathbb{R})^+] \otimes U^{d'}.\]

Here recall an elementary observation used in the Künneth formula for the reduced cohomology. Let \( X_1, X_2 \) be based \( S^3 \)-spaces and \( p_1 : (X_1 \times X_2, * \times X_2) \to (X_1, *) \) and \( p_2 : (X_1 \times X_2, X_1 \times *) \to (X_2, *) \) be the projections. For cohomology classes \( \gamma_i \in \tilde{H}^*_S(X_i) \cong H^*_S(X_i, *) \), the cohomology class \( p_1^* \gamma_1 \cup p_2^* \gamma_2 \) can be thought of an element of

\[H^*_S(X_1 \times X_2, (X_1 \times *) \cup (* \times X_2)) \cong \tilde{H}^*_S(X_1 \wedge X_2).\]

Now we go back to the diagram (34) and apply the above observation to \( \text{Th}(W_1) \wedge I^\mu_X \). Let

\[p_1 : (\text{Th}(W_1) \times I^\mu_X, * \times I^\mu_X) \to (\text{Th}(W_1), *) \]

and

\[p_2 : (\text{Th}(W_1) \times I^\mu_X, \text{Th}(W_1) \times *) \to (I^\mu_X, *)\]

be the projections. Then we obtain a cohomology class

\[\eta := p_1^* \tau_S(W_1) \cup p_2^* \omega \in \tilde{H}^*_S(\text{Th}(W_1) \wedge I^\mu_X).\]
We obtain
\begin{equation}
(38) \quad i_0^* f^* \eta = (f^* S^1)^* i_1^* \eta
\end{equation}
from the commutativity of the diagram (50). Let us write down two sides of this relation (38) in detail and extract a constraint on $H^+ (E)$.

First, by Lemma 5.4, the equivariant Thom isomorphism with coefficients $\mathbb{F}$, there exists a cohomology class $\theta \in H^*_\mathbb{S}^1 (B)$ such that
\begin{equation}
(39) \quad \tau_{S^1} (W_0) \cup \pi_{W_0}^* \theta = f^* \eta,
\end{equation}
where $\pi_{W_0} : W_0 \to B$ denotes the projection. This cohomology class $\theta$ is an analogue of the cohomological mapping degree of $f$ used to extract ordinary-cohomological information from the families Bauer–Furuta invariant of a family of closed 4-manifolds.

Next, let us note the following elementary observation on Thom classes. Let $W \oplus W' \to B$ be vector bundles decomposed into a direct sum. Let $S^1$ act on a given vector bundle as the trivial action or the multiplication of complex numbers according to whether the bundle is a real or complex vector bundle. Let $i : W \hookrightarrow W \oplus W'$ be the inclusion. A basic formula used below is
\begin{equation}
(40) \quad i^* \tau_{S^1} (W \oplus W') = \tau_{S^1} (W) \cup \pi_{W_0}^* e_{S^1} (W'),
\end{equation}
which holds in $\tilde{H}_{\mathbb{S}^1} (\text{Th}(W))$.

By the previous paragraph, more precisely the formula (40), we have
\begin{equation}
(41) \quad i_0^* \tau_{S^1} (W_0) = \tau_{S^1} (W_0 (\mathbb{R})) \cup \pi_{W_0}^* e_{S^1} (W_0 (\mathbb{C})).
\end{equation}
It follows from (39) and (41) that
\begin{equation}
(42) \quad i_0^* f^* \eta = \tau_{S^1} (W_0 (\mathbb{R})) \cup \pi_{W_0}^* e_{S^1} (W_0 (\mathbb{C}) \cup \theta).
\end{equation}

Next, we calculate the right-hand side of (38). By abuse of notation we denote also by $p_1, p_2$ the projections
\begin{align*}
p_1 : & (\text{Th}(W_1 (\mathbb{R}))) \times (I^*_\lambda)^{S^1}, \ast \times (I^*_\lambda)^{S^1} \to (\text{Th}(W_1 (\mathbb{R})), \ast), \\
p_2 : & (\text{Th}(W_1 (\mathbb{R}))) \times (I^*_\lambda)^{S^1}, \text{Th}(W_1 (\mathbb{R})) \times \ast \to ((I^*_\lambda)^{S^1}, \ast)
\end{align*}
respectively. Let
\begin{equation*}
i_1 : \text{Th}(W_1 (\mathbb{R})) \hookrightarrow \text{Th}(W_1), \\
i_2 : (I^*_\lambda)^{S^1} \hookrightarrow I^*_\lambda
\end{equation*}
be the inclusions. Then, by (36) and (40), we have that
\begin{equation}
(43) \quad i_1^* \eta = p_1^* i_1^* \tau_{S^1} (W_1) \cup p_2^* i_2^* \omega = p_1^* \tau_{S^1} (W_1 (\mathbb{R})) \cup p_1^* \pi_{W_1 (\mathbb{R})}^* e_{S^1} (W_1 (\mathbb{C})) \cup p_2^* ([V_0^0 (\mathbb{R})^+] \otimes \mathbb{F}^*)
\end{equation}
in $\tilde{H}_{\mathbb{S}^1} (\text{Th}(W_1 (\mathbb{R}))) \wedge (I^*_\lambda)^{S^1})$. Let
\begin{equation*}
\Phi : \tilde{H}_{\mathbb{S}^1} (\text{Th}(W_1 (\mathbb{R}))) \wedge V_0^0 (\mathbb{R})^+ \to \tilde{H}_{\mathbb{S}^1} (\text{Th}(W_1 (\mathbb{R}))) \wedge (I^*_\lambda)^{S^1}).
\end{equation*}
be the isomorphism induced from the homotopy equivalence $\varphi : N^S^1 / L^S^1 \to V_0^0 (\mathbb{R})^+$ obtained in Lemma 5.1, where we identify $(I^*_\lambda)^{S^1}$ with $N^S^1 / L^S^1$ using
an obvious homeomorphism. Lemma 3.1 implies that we have the commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^*_{S^1}(\text{Th}(W_0(\mathbb{R}))) & \xrightarrow{(f^{S^1})^*} & \tilde{H}^*_1(\text{Th}(W_1(\mathbb{R})) \wedge (I^*_1 S^1)) \\
\downarrow & & \downarrow \\
\tilde{H}^*_1(\text{Th}(W_1(\mathbb{R})) \wedge V^0_{\eta}(\mathbb{R})^+) & \xrightarrow{\Phi} & \tilde{H}^*_1(\text{Th}(W_1(\mathbb{R})) \wedge V^0_{\eta}(\mathbb{R})^+). \\
\end{array}
\]

(44)

Note that we have an isomorphism

\[\Psi : \tilde{H}^*_1(\text{Th}(W_1(\mathbb{R})) \wedge V^0_{\eta}(\mathbb{R})^+; \mathbb{Z}) \to \tilde{H}^*_1(\text{Th}(W_1(\mathbb{R}) \oplus V^0_{\eta}(\mathbb{R})); \mathbb{Z}).\]

induced from a natural homeomorphism in Lemma 3.2 (3). Via \(\Psi\), we identify the domain and codomain of \(\Phi\). It follows from (43) that

\[\Phi^{-1} \circ i_1^* \eta = \tau_{S^1}(W_1(\mathbb{R}) \oplus V^0_{\eta}(\mathbb{R})) \cup \pi_{W_1(\mathbb{R}) \oplus V^0_{\eta}(\mathbb{R})}(e_{S^1}(W_1(\mathbb{C})) \cdot U^d),\]

where \(U^d\) denotes the action of \(U^d \in \tilde{H}^*_1(S^0; \mathbb{Z}) \cong \mathbb{F}[U]\) on \(\tilde{H}^*_1(B; \mathbb{Z})\). Recall that \(f^{S^1}\) is obtained as the restriction of a fiberwise linear map (29). Moreover, the map (29), which induces the map

\[((\text{id}_{W_1(\mathbb{R})} \oplus p_{V^0_{\eta}(\mathbb{R})}) \circ f^{S^1})^* : \tilde{H}^*_1(\text{Th}(W_1(\mathbb{R})) \wedge V^0_{\eta}(\mathbb{R})^+) \to \tilde{H}^*_1(\text{Th}(W_0(\mathbb{R})))\]

in the diagram (44), is a fiberwise linear injection and its fiberwise cokernel is isomorphic to \(H^+(E)\). It follows from this combined with (39), (41), and (45) that

\[
(f^{S^1})^* i_1^* \eta = (\text{id}_{W_1(\mathbb{R})} \oplus p_{V^0_{\eta}(\mathbb{R})}) \circ f^{S^1})^* \circ \Phi^{-1} \circ i_1^* \eta
\]

\[= \tau_{S^1}(W_0(\mathbb{R})) \cup \pi_{W_0(\mathbb{R})}(e_{S^1}(H^+(E)) \cup e_{S^1}(W_1(\mathbb{C})) \cdot U^d).\]

(46)

Since the Thom class \(\tau_{S^1}(W_0(\mathbb{R})) \in \tilde{H}^*_1(\text{Th}(W_0(\mathbb{R})))\) is a generator of \(\tilde{H}^*_1(\text{Th}(W_0(\mathbb{R})))\) as an \(H^*_1(B; \mathbb{Z})\)-module, it follows from (38), (42), and (46) that

\[e_{S^1}(W_0(\mathbb{C})) \cup \theta = e_{S^1}(H^+(E)) \cup e_{S^1}(W_1(\mathbb{C})) \cdot U^d.\]

(47)

This is an equality in \(H^*_1(B; \mathbb{Z})\), and is the desired divisibility of Euler classes.

Set \(m := \text{rank}_c W_0(\mathbb{C})\) and \(n := \text{rank}_c W_1(\mathbb{C})\). Recall that the \(S^1\)-action on \(W_1(\mathbb{C})\) is given by the scalar multiplication. Then the equivariant Euler class is written in terms of (non-equivariant) Chern classes, which is actually one of ways to define the Chern classes:

\[e_{S^1}(W_0(\mathbb{C})) = \sum_{i=0}^{m} c_i(W_0(\mathbb{C})) \otimes U^i,\]

\[e_{S^1}(W_1(\mathbb{C})) = \sum_{j=0}^{n} c_j(W_1(\mathbb{C})) \otimes U^j\]

in \(H^*_1(B; \mathbb{Z}) \cong H^*(B; \mathbb{Z}) \otimes H^*_1(pt; \mathbb{Z})\). Taking mod 2, we obtain similar equalities in \(H^*_1(B) = H^*_1(B; \mathbb{F})\). Henceforth let \(c_i(\cdot)\) denote mod 2 Chern classes. It follows from this combined with (47) that

\[
\left(\sum_{i=0}^{m} c_i(W_0(\mathbb{C})) \otimes U^i\right) \cup \theta = e_{S^1}(H^+(E)) \cup \left(\sum_{j=0}^{n} c_j(W_1(\mathbb{C})) \otimes U^j\right) \cup U^d.
\]

(48)
Here note that we have 
\[ e_s(H^+(E)) = w_{b^+}(H^+(E)) \otimes 1 \in H^*(B) \otimes H^*_S(pt) \]
since the action of $S^1$ on $H^+(E)$ is trivial. Set $k := \deg U \geq 0$. Comparing the $U$-degree highest terms in the equality (48), we obtain
\[ \theta_0 \cdot U^{m+k} = w_{b^+}(H^+(E)) \cdot U^{n+d'}, \]
where $\theta_0 \in H^*(B)$ is a non-zero cohomology class. Therefore, if $w_{b^+}(H^+(E)) \neq 0$, we have that $m + k = n + d'$, and hence $m \leq n + d'$. By (20), this inequality is equivalent to
\[ \frac{c_1(s)^2 - \sigma(X)}{8} + n(Y, t, g) + \dim_{\mathbb{C}} V^0_\lambda(C) \leq d'. \]
From the definition of the Frøyshov invariant and the definition of $d'$, which are (10) and (35) respectively, this is equivalent to the desired inequality (1). This completes the proof of Theorem 1.1. □

Remark 3.3. Baraglia [2] used local coefficient systems with fiber $\mathbb{Z}$ to derive his constraint [2, Theorem 1.1]. As a result, he obtained a constraint described in terms of the Euler class of $H^+(E)$ living in a certain cohomology with local coefficient, not $w_{b^+}(H^+(E))$. Theorem 1.1 is an analogue of the mod 2 version of his constraint. Here we explain the reason why we cannot use such local coefficients and use $\mathbb{F}$-coefficients instead in this paper. Given an $S^1$-vector bundle $W \to B$, to use the (equivariant) Thom isomorphism for $W$ with a certain local coefficient induced from a local system on the base space, we need to consider the relative cohomology $H^*_{S^1}(D(W), S(W))$, rather than $\tilde{H}^*_{S^1}(\text{Th}(W))$. This is just because there is no obvious way to define a local system on $\text{Th}(W)$ induced from a local system on the base space $B$. To use relative cohomologies, we need to have a map between pairs
\[ f : (D(W_0), S(W_0)) \to (W_1, W_1 \setminus \{0\}) \times (N, L) \]
instead of (24). But we could not figure out whether we can obtain such a map as the families relative Bauer–Furuta invariant, because it seems essential to cut the domain of $f$ by the compact sets $\tilde{K}_1, \tilde{K}_2$ in Lemma 2.13 to obtain an appropriate index pair $(N, L)$.

3.3. Proof of Theorem 1.2 The proof of Theorem 1.2 is quite similar to the proof of Theorem 1.1. Here let us summarize major difference of the settings:

- The vector bundles $W_0, W_1$ which appear in the domain and codomain of a finite-dimensional approximation $f$ of the Seiberg–Witten map are the direct sums of real vector bundles $W_i(\mathbb{R})$ and quaternionic vector bundle $W_i(\mathbb{H})$ over $B$. Here $\text{Pin}(2)$ acts on $W_i(\mathbb{R})$ as the $\pm 1$-multiplication and on $W_i(\mathbb{H})$ as the scalar multiplication of quaternions.
- We have
\[ \text{rank}_\mathbb{H} W_0(\mathbb{H}) - \text{rank}_\mathbb{H} W_1(\mathbb{H}) = \frac{-\sigma(X)}{16} + \frac{n(Y, t, g)}{2} + \dim_{\mathbb{H}} V^0_\lambda(C). \]
- $\text{Pin}(2)$-equivariant cohomology and $\text{Pin}(2)$-equivariant Thom and Euler classes are used, instead of $S^1$-equivariant cohomology, Thom classes, and Euler classes.
Proof of Theorem 1.2. Considering the restriction of a finite-dimensional approximation \( f \) to the \( S^1 \)-fixed part, we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{H}^*_\text{Pin}(2)(\text{Th}(W_0)) & \xrightarrow{f^*} & \tilde{H}^*_\text{Pin}(2)(\text{Th}(W_1) \wedge I^n_\lambda) \\
\iota_* & & \iota_* \\
\tilde{H}^*_\text{Pin}(2)(\text{Th}(W_0(\mathbb{R}))) & \xrightarrow{(f^*)^*} & \tilde{H}^*_\text{Pin}(2)(\text{Th}(W_1(\mathbb{R})) \wedge (I^n_\lambda)^{S^1}).
\end{array}
\]  

(50)

Let \( \omega \) be one of

\[
\omega_a \in \tilde{H}^2_{S^1}(I^n_\lambda), \\
\omega_b \in \tilde{H}^{b+1}_{S^1}(I^n_\lambda), \\
\omega_c \in \tilde{H}^{c+2}_{S^1}(I^n_\lambda)
\]

in Lemma 2.6, and define \( \eta \in \tilde{H}^2_{S^1}(\text{Th}(W_1) \wedge I^n_\lambda) \) using this \( \omega \) as well as \( [37] \). Repeating the proof of Theorem 1.1 using the diagram (50) and this \( \eta \), we obtain

\[
de_{\text{Pin}(2)}(W_0(\mathbb{H})) \cup \theta = e_{\text{Pin}(2)}(H^+(E)) \cup e_{\text{Pin}(2)}(W_1(\mathbb{H})) \cdot \nu^{(a-s)/4},
\]

(51)

\[
de_{\text{Pin}(2)}(W_0(\mathbb{H})) \cup \theta = e_{\text{Pin}(2)}(H^+(E)) \cup e_{\text{Pin}(2)}(W_1(\mathbb{H})) \cdot q^{(b-s)/4},
\]

(52)

\[
de_{\text{Pin}(2)}(W_0(\mathbb{H})) \cup \theta = e_{\text{Pin}(2)}(H^+(E)) \cup e_{\text{Pin}(2)}(W_1(\mathbb{H})) \cdot q^2 v^{(c-s)/4},
\]

(53)

according to the choice of \( \omega \), as well as [17]. Here \( \theta \) is an element of \( H^*_{\text{Pin}(2)}(B) \).

By an argument by Baraglia in the proof of [2] Theorem 5.1, we have that

\[
de_{\text{Pin}(2)}(W_0(\mathbb{H})) = \sum_{i=0}^{m} c_{2i}(W_0(\mathbb{H})) \otimes v^i,
\]

(54)

\[
de_{\text{Pin}(2)}(W_1(\mathbb{H})) = \sum_{j=0}^{n} c_{2j}(W_1(\mathbb{H})) \otimes v^j
\]

(55)

in \( H^*_{\text{Pin}(2)}(B) \cong H^*(B) \otimes H^*_{\text{Pin}(2)}(\text{pt}) \), where \( m = \text{rank}_{\mathbb{H}} W_0(\mathbb{H}) \) and \( n = \text{rank}_{\mathbb{H}} W_1(\mathbb{H}) \). Moreover, an argument by Baraglia in the proof of [2] Corollary 5.2, we have that

\[
e_{\text{Pin}(2)}(H^+(E)) = w_{b^+}(H^+(E))
\]

in \( H^*_{\text{Pin}(2)}(B) \).

Now we argue according to the non-vanishing of \( w_{\bullet}(H^+(E)) \) for \( \bullet = b^+, b^+ - 1, b^+ - 2 \). First let us assume that \( w_{b^+}(H^+(E)) \neq 0 \). In this case, we take \( \omega_c \) as \( \omega \).

Let us substitute (54), (55) and (56) for various Euler classes in (53). Then one may see that the right-hand side of (53) contains the term \( w_{b^+}(H^+(E)) \otimes q^2 v^{(c-s)/4} \), which is the \( v \)-degree highest non-zero term of the form \( x \otimes q^2 v^k \), where \( x \in H^*(B) \) and \( k \geq 0 \). Therefore the left-hand side of (53) should also contain a non-zero term of the form \( x \otimes q^2 v^k \). This is equivalent to the existence of a non-zero term of the form \( x \otimes q^2 v^k \) in \( \theta \). Let \( k_c \geq 0 \) be the maximum of such \( k \). Then it follows that

\[
\theta_0 \otimes q^2 v^{m+k_c} = w_{b^+}(H^+(E)) \otimes q^2 v^{n+(c-s)/4},
\]

where \( 0 \neq \theta_0 \in H^*(B) \). Thus we have \( m - n \leq (c-s)/4 \). This combined with the definition of \( \gamma \), given in [15], implies the inequality (2).
Next let us assume that $b^+(X) > 0$ and $w_{b+1}(H^+(E)) \neq 0$. In this case, we take $\omega_b$ as $\omega$. After substituting (54), (55) and (56) for the Euler classes in (52), the right-hand side of (52) contains the term $w_{b+1}(H^+(E)) \otimes q^2v^{(b-s)/4}$, which is the $v$-degree highest non-zero term of the form $x \otimes q^2v^k$. Arguing exactly as in the above paragraph, we obtain $m - n \leq (b - s)/4$, which implies the inequality (4).

Similarly, the inequality (4) is deduced from the assumption that $b^+(X) > 1$ and $w_{b+2}(H^+(E)) \neq 0$ by taking $\omega_a$ as $\omega$. This completes the proof of Theorem 1.2. □

Remark 3.4. A reader may wonder whether Pin(2)-equivariant $K$-theory can be used to extract a constraint of smooth families of spin 4-manifolds with boundary. We predict that it should be able to be established as a general statement using Manolescu’s invariant $\kappa$ introduced in [35] instead of $\alpha, \beta, \gamma$. The reason why we do not include such a study in this paper is that we could not find a potential application like Theorems 1.3 and 1.4 detected using a $K$-theoretic constraint. Theorems 1.3 and 1.4 follows from the existence of non-smoothable families (Theorem 4.1), but non-smoothability of families of that kind cannot be detected using a $K$-theoretic constraint. For the examples of non-smoothable families $E$ given in Subsection 4.1, the associated bundles $H^+(E)$ do not admit $K$-theory orientation, and the $K$-theoretic Euler class cannot make sense for them. (One way to get $K$-orientability is tensoring with $C$, but $H^+(E) \otimes C$ are trivial in those examples, and we cannot extract any information.)

4. Applications

In this section we consider applications of Theorems 1.1 and 1.2 mainly to the existence of non-smoothable families of 4-manifolds with boundary, stated as Theorem 4.1. We also describe consequences of the the existence of non-smoothable families about comparison between various diffeomorphism groups and homeomorphism groups of 4-manifolds with boundary in this section.

4.1. Non-smoothable families of 4-manifolds with boundary. In the following Theorem 4.1 non-smooth families of 4-manifolds with boundary are detected using Theorems 1.1 and 1.2. Here let us clarify the word ‘non-smooth family’ in this paper: we shall consider a continuous fiber bundle $E$ with fiber 4-manifold $X$ with boundary. If the structure group of $E$ reduces to $\text{Homeo}(X, \partial)$, but $E$ does not admit a reduction to $\text{Diff}(X, \partial)$, we say that $E$ is non-smoothable.

**Theorem 4.1.** Let $Y$ be an oriented integral homology 3-sphere. Let $X$ be a simply-connected, compact, oriented, smooth, and indefinite 4-manifold with boundary $Y$. Suppose that $\sigma(X) \leq 0$. Then:

1. If either
   (a) $\sigma(X) < -8$ and $\delta(Y) \leq 0$, or
   (b) $\delta(Y) > 0$
   holds, then there exists a non-smoothable $\text{Homeo}(X, \partial)$-bundle
   $$X \to E \to T^{b^+(X)}.$$  

2. Suppose that $X$ is spin.
   (a) If $-\sigma(X)/8 > \gamma(Y)$, there exists a non-smoothable $\text{Homeo}(X, \partial)$-bundle
   $$X \to E \to T^{b^+(X)}.$$  

(b) If $b^+(X) > 1$ and $-\sigma(X)/8 > \beta(Y)$, there exists a non-smoothable Homeo$(X, \partial)$-bundle

$$X \to E \to T^{b^+(X)-1}.$$ 

(c) If $b^+(X) > 2$ and $-\sigma(X)/8 > \alpha(Y)$, there exists a non-smoothable Homeo$(X, \partial)$-bundle

$$X \to E \to T^{b^+(X)-2}.$$ 

In [22] Kato, Nakamura and the first author introduced an idea to detect non-smoothable families of closed 4-manifold using families gauge theory and to apply them to extract difference between diffeomorphism groups and homeomorphism groups [22, Theorem 1.4, Corollary 1.5]. That was extensively generalized by Baraglia [2] soon later. Theorem 4.1 is an analogue of [2, Theorem 1.8].

To prove Theorem 4.1, we need the following results regarding topological 4-manifolds with boundaries by Freedman. Roughly speaking, these results state that Freedman’s classification result holds also for topological 4-manifolds with homology sphere boundary.

**Theorem 4.2** (See, for example, [5, 6]). Let $Y$ be an integral homology 3-sphere.

(i) The set of simply-connected compact topological 4-manifolds with boundary $Y$ having an even intersection form up to homeomorphism is determined by unimodular intersection forms up to isomorphism.

(ii) The set of simply-connected compact topological 4-manifolds with boundary $Y$ having an odd intersection form up to homeomorphism is determined by unimodular intersection forms and Kirby–Siebenmann invariant up to isomorphism.

**Theorem 4.3.** [14, 9.3C Corollary] Every integral homology 3-sphere bounds a contractible topological 4-manifold.

Now we may start the proof of Theorem 4.1. A principal idea of the construction of non-smoothable families here is based on arguments for families of closed 4-manifolds: [38, Sections 3, 4], [22, Theorem 4.1], and [2, Theorem 10.3].

**Proof of Theorem 4.1** By Remark 1.5, if $X$ is spin and satisfies the assumption in (1-a) or (1-b) of the statement of Theorem 4.1 then $X$ satisfies the assumption in (2-a). Moreover the conclusion of the case (1) is just the same as (2-a). Hence when we give a proof of the case (1), we can additionally suppose that $X$ is not spin, since the case that $X$ is spin is deduced from the case (2-a), which will be proven independently from the case (1).

Let $W$ be a contractible topological 4-manifold bounded by $Y$, whose existence is ensured by Theorem 4.3. Let $-E_8$ denote the negative-definite $E_8$-manifold.

First let us suppose that $X$ is not spin, and suppose that $\sigma(X) < -8$ and $\delta(Y) \leq 0$. Let $-\mathbb{CP}^2_{\text{fake}}$ denote the fake $\mathbb{CP}^2$, namely the closed simply-connected topological 4-manifold whose intersection form is $(-1)$ and has non-zero Kirby–Siebenmann class. Recall that a smooth 4-manifold has vanishing Kirby–Siebenmann invariant, even for the case with boundary. It follows from Theorem 4.2 that $X$ is homeomorphic to

$$b^+(X)(S^2 \times S^2)\#n(-\mathbb{CP}^2)\#(-E_8)\#(-\mathbb{CP}^2_{\text{fake}})\#W$$
for some \( n \geq 0 \). Let \( f_0 \in \text{Diff}(S^2 \times S^2) \) be an orientation-preserving self-diffeomorphism satisfying the following two properties:

- There exists an embedded 4-disk in \( S^2 \times S^2 \) such that the restriction of \( f_0 \) on the disk is the identity map.
- \( f_0 \) reverses orientation of \( H^+(S^2 \times S^2) \).

Such \( f_0 \) can be made by starting with the componentwise complex conjugation of \( \mathbb{CP}^1 \times \mathbb{CP}^1 = S^2 \times S^2 \), and deforming it around a fixed point by isotopy so that it has a fixed disk. Let \( f_1, \ldots, f_{b^+} \) be copies of \( f_0 \) on the connected sum factors of \( b^+(X)(S^2 \times S^2) \). Since \( f_i \) have fixed disks, we can extend them as homeomorphisms of \( X \) by the identity map on the other connected sum factors in \( \mathbb{CP}^1 \). Since the extended homeomorphisms obviously mutually commute, they give rise to the multiple mapping torus

\[
X \rightarrow E \rightarrow T^{b^+}.
\]

Note that the restrictions of \( f_1, \ldots, f_{b^+} \) onto the boundary are the identity maps, and hence \( E \) is a \( \text{Homeo}(X, \partial) \)-bundle. Since \( f_0 \) was taken to reverse orientation of \( H^+(S^2 \times S^2) \), it is easy to see that the associated bundle \( H^+(E) \rightarrow T^{b^+} \) satisfies \( w_{b^+}(H^+(E)) \neq 0 \).

Let \( c \in H^2(X; \mathbb{Z}) \) be a cohomology class given by

\[
c = (0, e_1, \ldots, e_n, 0, e)
\]

under the direct sum decomposition

\[
H^2(X) = H^2(b^+(X)(S^2 \times S^2)) \oplus H^2(-\mathbb{CP}^2)^{\oplus n} \oplus H^2(-E_8) \oplus H^2(-\mathbb{CP}^2_{\text{fake}}),
\]

where \( e_i \) and \( e \) are a generator of \( H^2(-\mathbb{CP}^2) \) and a generator of \( H^2(-\mathbb{CP}^2_{\text{fake}}) \) respectively. As well as an argument for families of closed 4-manifolds by Baraglia [2, Theorem 10.3], \( E \) admits a topological spin\(^c \) structure whose characteristic restricted over a fiber coincides with \( c \) above.

Now suppose that \( E \) is smoothable, namely \( E \) reduces to a \( \text{Diff}(X, \partial) \)-bundle. Then the topological spin\(^c \) structure of \( E \) above induces a smooth spin\(^c \) structure, and the restriction of the spin\(^c \) structure over a fiber, denoted by \( s \), has \( c_1(s) = c \).

Now we have \( (c_1(s)^2 - \sigma(X))/8 = 1 \), and hence Theorem [1] implies that \( 1 \leq \delta(Y) \). This contradicts the assumption that \( \delta(Y) \leq 0 \), and hence \( E \) is not smoothable.

Next, let us suppose that \( X \) is not spin, and suppose that \( \delta(Y) \leq -1 \). Most of arguments here are just the same as the arguments until previous paragraph. First, it follows from Theorem [1,2] that \( X \) is homeomorphic to

\[
b^+(X) \mathbb{CP}^2 \# b^-(X)(-\mathbb{CP}^2) \# W.
\]

Let \( f_0 \in \text{Diff}(\mathbb{CP}^2) \) be a self-diffeomorphism satisfying the following two properties:

- There exists an embedded 4-disk in \( \mathbb{CP}^2 \) such that the restriction of \( f_0 \) on the disk is the identity map.
- \( f_0 \) reverses orientation of \( H^+(\mathbb{CP}^2) \).

Such \( f_0 \) can be obtained by deforming the complex conjugation \( [z_0 : z_1 : z_2] \mapsto [\bar{z}_0 : \bar{z}_1 : \bar{z}_2] \) around a fixed point. Let \( f_1, \ldots, f_{b^+} \) be copies of \( f_0 \) on the connected sum factors of \( b^+(X)\mathbb{CP}^2 \). Extending them as homeomorphisms of \( X \), we obtain a \( \text{Homeo}(X, \partial) \)-bundle \( X \rightarrow E \rightarrow T^{b^+} \) for which \( w_{b^+}(H^+(E)) \neq 0 \). Let us take \( c \in H^2(X; \mathbb{Z}) \) defined by

\[
c = (h_1, \ldots, h_{b^+}, e_1, \ldots, e_{b^-})
\]
under
\[ H^2(X) = H^2(\mathbb{CP}^2) \oplus b^+(X) \oplus H^2(\mathbb{CP}^2) \oplus b^-(X), \]
where \( h_i \) and \( e_j \) are a generator of \( H^2(\mathbb{CP}^2) \) and a generator of \( H^2(\mathbb{CP}^2) \) respectively. Then we have \( (e^2 - \sigma(X))/8 = 0 \). Arguing exactly as in the last case, if we suppose that \( E \) is smoothable, Theorem 4.1 implies that \( 0 \leq \delta(Y) \). This contradicts the assumption that \( \delta(Y) \leq -1 \), and hence \( E \) is not smoothable.

Next, let us suppose that \( X \) is spin, and suppose that \( -\sigma(X)/8 > \gamma(Y) \). Then it follows from Theorem 4.2 that \( X \) is homeomorphic to
\[ b^+(X)(S^2 \times S^2)\#2n(-E_8)\#W \]
for some \( n \geq 0 \). Let \( f_0 \in \text{Diff}(S^2 \times S^2) \) be the orientation-preserving self-diffeomorphism taken above. As well as the non-spin case, considering copies of \( f_0 \) on the connected sum factors of \( b^+(X)(S^2 \times S^2) \) and extend them to the whole of \( X \) as homeomorphisms, we obtain a \( \text{Homeo}(X,\partial) \)-bundle \( X \to E \to T^{b^+} \) for which \( w_{b^+}(H^+(E)) \neq 0 \). As well as an argument for families of closed 4-manifolds by Baraglia [2] Theorem 10.3, \( E \) admits a topological spin structure. Arguing exactly as in the non-spin case, if we suppose that \( E \) is smoothable, Theorem 4.2 implies that \( -\sigma(X)/8 \leq \gamma(Y) \). This contradicts the assumption that \( -\sigma(X)/8 > \gamma(Y) \), and hence \( E \) is not smoothable.

The remaining cases, \( X \) is spin and \( b^+(X) > 1, -\sigma(X)/8 > \beta(Y) \), or \( b^+(X) > 2, -\sigma(X)/8 > \alpha(Y) \), are also similar. Consider copies \( f_1,\ldots, f_{b^+-1} \) or \( f_1,\ldots, f_{b^+-2} \) of \( f_0 \) on the connected sum factors of \( (b^+-1)(S^2 \times S^2) \) or \( (b^+-2)(S^2 \times S^2) \) inside \( b^+(X)(S^2 \times S^2) \), according to the assumption on \( \beta(Y) \) or \( \alpha(Y) \). Then we obtain \( X \to E \to T^{b^+-1} \) or \( X \to E \to T^{b^+-2} \) for which \( w_{b^+-1}(H^+(E)) \neq 0 \) or \( w_{b^+-2}(H^+(E)) \neq 0 \) respectively. Theorem 4.2 implies that this \( E \) is not smoothable. This completes the proof of Theorem 4.1 \( \Box \)

4.2. Comparison between Diff and Homeo. Let us extract homotopical difference between various diffeomorphism groups and homeomorphism groups obtained from Theorem 4.1. First let us start with comparison between the relative diffeomorphism and homeomorphism groups:

**Corollary 4.4.** Let \( Y \) be an oriented integral homology 3-sphere. Let \( X \) be a simply-connected, compact, oriented, smooth, and indefinite 4-manifold with boundary \( Y \). Suppose that \( \sigma(X) \leq 0 \). Suppose that \( X \) and \( Y \) satisfy at least one of the following conditions:

1. \( \sigma(X) < -8 \) and \( \delta(Y) \leq 0 \).
2. \( \delta(Y) < 0 \).
3. \( X \) is spin and \( -\sigma(X)/8 > \gamma(Y) \).
4. \( X \) is spin, \( b^+(X) > 1 \) and \( -\sigma(X)/8 > \beta(Y) \).
5. \( X \) is spin, \( b^+(X) > 2 \) and \( -\sigma(X)/8 > \alpha(Y) \).

Then the inclusion map
\[ \text{Diff}(X,\partial) \to \text{Homeo}(X,\partial) \]
is not a weak homotopy equivalence.

More precisely:

- If at least one of (1), (2), (3) is satisfied, the induced map
  \[ \pi_n(\text{Diff}(X,\partial)) \to \pi_n(\text{Homeo}(X,\partial)) \]
is not an isomorphism for some $n \in \{0, \ldots, b^+(X) - 1\}$.

- If (4) is satisfied, the induced map
  $$\pi_n(\text{Diff}(X, \partial)) \rightarrow \pi_n(\text{Homeo}(X, \partial))$$
  is not an isomorphism for some $n \in \{0, \ldots, b^+(X) - 2\}$.

- If (5) is satisfied, the induced map
  $$\pi_n(\text{Diff}(X, \partial)) \rightarrow \pi_n(\text{Homeo}(X, \partial))$$
  is not an isomorphism for some $n \in \{0, \ldots, b^+(X) - 3\}$.

**Proof.** This follows from Theorem 4.1 combined with the standard obstruction theory, as well as the proof of [2, Corollary 10.5].

**Corollary 4.5.** Let $Y$ be an oriented integral homology 3-sphere. Let $X$ be a simply-connected, compact, oriented, smooth, and indefinite 4-manifold with boundary $Y$. Suppose that $\sigma(X) \leq 0$. Suppose that $X$ and $Y$ satisfy at least one of the conditions (1)-(5) in the statement of Corollary 4.4. Then the inclusion map

$$\text{Diff}(X) \hookrightarrow \text{Homeo}(X)$$

is not a weak homotopy equivalence.

More precisely:

- If at least one of (1), (2), (3) is satisfied, the induced map
  $$\pi_n(\text{Diff}(X)) \rightarrow \pi_n(\text{Homeo}(X))$$
  is not an isomorphism for some $n \in \{0, \ldots, b^+(X)\}$.

- If (4) is satisfied, the induced map
  $$\pi_n(\text{Diff}(X)) \rightarrow \pi_n(\text{Homeo}(X))$$
  is not an isomorphism for some $n \in \{0, \ldots, b^+(X) - 1\}$.

- If (5) is satisfied, the induced map
  $$\pi_n(\text{Diff}(X)) \rightarrow \pi_n(\text{Homeo}(X))$$
  is not an isomorphism for some $n \in \{0, \ldots, b^+(X) - 2\}$.

**Proof.** Recall that, for an arbitrary orientable closed smooth 3-manifold, the inclusion map from the diffeomorphism group into the homeomorphism group is a weak homotopy equivalence. (This is a result by Cerf [8], combined with the solution to the Smale conjecture by Hatcher [20]. See [19].)

As noted by Pardon [39, Subsection 2.1], we have an exact sequence

$$1 \rightarrow \text{Diff}(X, \partial) \rightarrow \text{Diff}(X) \rightarrow \text{Diff}(Y),$$

where the image of the last map is a union of connected components. Similarly we have

$$1 \rightarrow \text{Homeo}(X, \partial) \rightarrow \text{Homeo}(X) \rightarrow \text{Homeo}(Y).$$

These exact sequences induce long exact sequences of homotopy groups, although the final maps on $\pi_0$ may not be surjections. A natural termwise inclusion from (58) to (59) gives rise to a commutative diagram between two long exact sequences. Now we can deduce from the fact in dimension 3 explained in the last paragraph and Corollary 4.4 that $\text{Diff}(X) \hookrightarrow \text{Homeo}(X)$ is not a weak homotopy equivalence, with the assistance of the five lemma.

Here we give the proof of Theorem 1.7.
Proof of Theorem 4.1. Theorem 4.1 implies that there exists a non-smoothable Homeo($X, \partial$)-bundle $X \to E \to S^1$. This implies that (5) is not a surjection. The remaining statement follows from this and the fact that Diff($Y$) $\hookrightarrow$ Homeo($Y$) is a weak homotopy equivalence again, with the assistance of the four lemma.

Remark 4.6. For a closed smooth 4-manifold $X$,
\begin{equation}
\pi_0(\Diff(X)) \to \pi_0(\Homeo(X))
\end{equation}
is often a surjection by Wall’s theorem [46] on the realizability of elements of $\text{Aut}(H^2(X; \mathbb{Z}))$ by diffeomorphisms and Quinn’s theorem [40], which shows that $\pi_0(\Homeo^+(X) \to \text{Aut}(H^2(X; \mathbb{Z}))$ is an isomorphism as far as $X$ is simply-connected. There are few known examples of closed smooth 4-manifolds $X$ for which (60) are not surjections: the first example is a $K3$ surface by Donaldson [13], and in fact so is every homotopy $K3$ surface, which one can check using a result by Morgan and Szabó [37]. It follows from Baraglia’s constraint [2, Theorem 1.1] that an Enriques surface is also an example, and so is a stabilization of an Enriques surface by the connected sum with some non-simply-connected 4-manifolds by Nakamura and the first author [24, Corollary 1.6].

4.3. Some examples. As mentioned in Remark 1.6, we can easily find a huge numbers of examples of $(X, Y)$ for which the main applications Theorem 4.1 and Corollaries 4.4 and 4.5 can apply: just find $(X, Y)$ with $\sigma(X) < -8$ and $\delta(Y) \leq 0$. This is an analogue of the assumption $|\sigma(X)| > 8$ of Baraglia’s [2, Corollary 1.9] for closed 4-manifolds. However, in our situation, we may obtain examples of $(X, Y)$ with $|\sigma(X)| \leq 8$ thanks to the assistance of the Frøyshov invariant. Let us note such an example below.

Example 4.7. We consider the Brieskorn homology 3-sphere $\Sigma(p, q, r)$ for a pairwise relatively prime triple of positive integer $(p, q, r)$. Since $\Sigma(2, 3, 5)$ admits a positive scalar curvature, one can see that
\begin{equation}
\delta(\Sigma(2, 3, 5)) = 1.
\end{equation}

On the other hand, for an odd positive integer $k$ and an odd positive integer $q$ with $q \equiv 3 \mod 4$, in [43] Savelyev constructed a family of spin bounding $W'_{q,k}$ of $-\Sigma(2, q, 2qk + 1)$ whose intersection forms are isomorphic to
\begin{equation}
\left( \begin{array}{c} q + 1 \\ 4 \end{array} \right) (-E_8) \oplus \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).
\end{equation}

Set
\begin{equation}
Y_k := (-\Sigma(2, 3, 5)) \# (-\Sigma(2, 3, 6k + 1)).
\end{equation}
Since $\delta(-\Sigma(2, 3, 6k + 1)) = 0$, we have $\delta(Y_k) = -1$. Then we consider the boundary connected sum of a simply-connected $E_8$-bouding of $-\Sigma(2, 3, 5)$ with $W'_{3,k}$, which we denote by $W_k$. Note that the intersection form of $W_k$ is isomorphic to
\begin{equation}
9 \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right),
\end{equation}
and hence $\sigma(W_k) = 0$. Moreover, $W_k$ is spin, simply-connected and

\begin{equation}
-\frac{\sigma(W_k)}{8} = 0 > -1 = \delta(Y_k)(\geq \gamma(Y_k)).
\end{equation}
This proves that \((W_k, Y_k)\) satisfies the assumption (2) (and (3)) in Corollary 4.4. Applying Corollary 4.4, we have that

\[ \pi_n(\text{Diff}(W_k, \partial)) \to \pi_n(\text{Homeo}(W_k, \partial)) \]

is not an isomorphism for some \(n \in \{0, \cdots 8\}\), and

\[ \pi_n(\text{Diff}(W_k)) \to \pi_n(\text{Homeo}(W_k)) \]

is not an isomorphism for some \(n \in \{0, \cdots 9\}\) by Corollary 4.5.

Let us give a remark on comparison between various Frøyshov-type invariants. The content of this remark is due to Ciprian Manolescu, and the authors are grateful to him.

**Remark 4.8.** The following fact is pointed out in [33, Remark 1.1]. In the work of Kutluhan, Lee, and Taubes [27], [28], [29], [30], [26], alternatively, the work of Colin, Ghiggini, and Honda [10] [11] [9] and Taubes [45], it is proved that the monopole Floer homology and the Heegaard Floer homology in coefficients \(\mathbb{Z}\) are isomorphic to each other. In particular, with \(\mathbb{F}\)-coefficients, we also have an isomorphism between the monopole Floer homology and the Heegaard Floer homology. Moreover, the \(\mathbb{Q}\)-gradings are compared in [41], [12] and [21]. This proves

\[ \frac{1}{2}d(Y, t, \mathbb{F}) = -h(Y, t, \mathbb{F}), \]

where \(d(Y, t, \mathbb{F})\) is the correction term of Heegaard Floer homology defined over \(\mathbb{F}\)-coefficient and \(h(Y, t, \mathbb{F})\) is the monopole Frøyshov invariant defined over \(\mathbb{F}\)-coefficient.

On the other hand, in [31], Lidman and Manolescu gave a grading preserving isomorphism between the \(S^1\)-equivariant cohomology of \(SWF(Y, t)\) and the monopole Floer homology over \(\mathbb{Z}\). This proves

\[ -h(Y, t, \mathbb{F}) = \delta(Y, t). \]

Summarizing the results above, we have

\[ \frac{1}{2}d(Y, t, \mathbb{F}) = \delta(Y, t). \]

The equality enables us to calculate the invariant \(\delta\) by a combinatorial way.

We also provide a family of examples satisfying the assumptions of Theorem 1.7. First let us give an example detected by the Frøyshov invariant \(\delta\).

**Example 4.9.** Let \(K\) be any knot in \(S^3\). Since the 1-surgery \(S^3_1(K)\) of \(K\) admits a positive-definite bounding \(W_1(K)\) as the trace of the surgery on \(K\), we always have

\[ \delta(S_1(K)) \leq 0. \]

We suppose that

\[ \delta(S_1(K)) < -1, \]  

(61)

where we shall give concrete examples of such \(K\) below. We also consider \(\Sigma(2, 3, 5)\) and a simply-connected \((-E_8)\)-bounding \(W'_k\) of \(\Sigma(2, 3, 5)\). We define a pair \((W_K, Y_K)\) as the boundary connected sum of \((W'_k, \Sigma(2, 3, 5))\) and \((W_1(K), S_1(K))\). Note that \(b^+(W_K) = 1\), \(W_K\) is simply-connected and the intersection form of \(W_K\) is indefinite. Therefore the pair \((W_K, Y_K)\) satisfies the assumptions of Theorem 1.7, and thus we have that

\[ \pi_0(\text{Diff}(W_K, \partial)) \to \pi_0(\text{Homeo}(W_K, \partial)) \]
and
\[ \pi_0(\text{Diff}(W_K)) \to \pi_0(\text{Homeo}(W_K)) \]
are not surjections.

In order to find a concrete family of examples of \( K \) with (61), we consider
\[ K = T(2, 2n - 1) \]
for any positive integer \( n \), where \( T(p, q) \) denotes the \((p, q)\)-torus knot. It is mentioned in [17] that
\[ -S^3_1(T(2, 2n - 1)) = \Sigma(2, 2n - 2, 4n - 3) \]
has
\[ \Gamma_{4n} = \left\{ \sum x_i e_i \left| \sum x_i \in 2\mathbb{Z}, \ 2x_i \in \mathbb{Z}, x_i - x_j \in \mathbb{Z} \right. \right\} \]
as the negative-definite intersection from of the minimal resolution \( W_{4n} \), where \( \{e_i\} \) is an orthonormal basis of \( \mathbb{R}^{4n} \). Then by using an inequality by Frøyshov [15] for \( W_{4n} \), which is the same as Theorem 1.1 for \( B = \{\text{pt}\} \), we obtain a family of estimates
\[ \left\lfloor \frac{n}{2} \right\rfloor \leq \delta(\Sigma(2, 2n - 1, 4n - 3)) \]
This proves
\[ \delta(S^3_1(T(2, 2n - 1))) \leq -\left\lfloor \frac{n}{2} \right\rfloor \]
We can see that for any positive integer \( n \geq 4 \),
\[ T(2, 2n - 1) \]
satisfies (61).

Lastly let us give an example detected by the invariant \( \beta \).

**Example 4.10.** Recall that \(-\Sigma(2, 3, 11)\) bounds an oriented compact smooth spin 4-manifold \( X' \) having the following intersection form (see, for example, [35]):
\[ 2(-E_8) \oplus 2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \]
Let \( Y = -\Sigma(2, 3, 11) \# \Sigma(2, 3, 5) \) and let \( X \) be the boundary connected sum of \( X' \) with a simply-connected \((-E_8)\)-bounding of \( \Sigma(2, 3, 5) \). Obviously \( X \) is spin, \( b^+(X) = 2 \) and \( -\sigma(X)/8 = 3 \). On the other hand, as computed by Manolescu [36 Subsection 3.8], we have
\[ \beta(-\Sigma(2, 3, 11)) = 0, \quad \alpha(\Sigma(2, 3, 5)) = 1. \]
It follows from the connected sum formula by Stoffregen [44 Theorem 1.1] that
\[ \beta(Y) \leq \beta(-\Sigma(2, 3, 11)) + \alpha(\Sigma(2, 3, 5)) = 1, \]
hence the assumption (4) of Theorem 1.7 is satisfied for \( X \) and \( Y \). Thus we have that
\[ \pi_0(\text{Diff}(X, \partial)) \to \pi_0(\text{Homeo}(X, \partial)) \]
and
\[ \pi_0(\text{Diff}(X)) \to \pi_0(\text{Homeo}(X)) \]
are not surjections.
5. Appendix

In Subsection 3.2, the proof of Theorem 1.1 we use the following version of the equivariant Thom isomorphism several times. We give equivariant Thom isomorphism theorem with local coefficients. Although we use only equivariant cohomologies in coefficients $\mathbb{F} = \mathbb{Z}/2$, Baraglia [2] made use of equivariant cohomologies in local coefficient and used the Thom isomorphism of the form Lemma 5.1.

Let $G$ be a compact Lie group. Let $B$ be a paracompact Hausdorff space and $\pi_W : W \to B$ a $G$-vector bundle over $B$. Here we regard $B$ as a $G$-space with the trivial action. Take $\rho$ be a $A$-valued local system on $B$ for a fixed Abelian group $A$.

We define the local coefficient equivariant cohomology by

$$H^*_G(B; \rho) := H^*(B \times BG; \text{pr}^* \rho),$$

where $\text{pr} : B \times BG \to B$ is the projection.

We first consider the vector bundle

$$p : W_{hG} := EG \times_G W \to (EG \times B)/G := B_{hG}.$$ 

We define the local coefficient equivariant cohomology by

$$H^*_G(D(W), S(W); \pi^*_W \rho) := H^*(D(EG \times G W), S(EG \times G S(W)); \pi^*_W \rho).$$

Lemma 5.1. We have the following isomorphisms.

(i) The multiplication of an element

$$\tau_G(W) \in \tilde{H}^*_{G}(Th(W); \mathbb{F})$$

gives an isomorphism

$$H^*_G(B; \mathbb{F}) \to \tilde{H}^{*+\text{rank} W}_{G}(Th(W); \mathbb{F}).$$

(ii) Suppose $G$ is connected. The multiplication of an element

$$\tau_G(W) \in H^*_{G}(D(W), S(W); \pi^*_W w_1(W))$$

gives an isomorphism

$$H^*_G(B; \rho) \to H^{*+\text{rank} W}_{G}(D(W), S(W); \pi^*_W \rho \otimes \pi^*_W w_1(W)),$$

where $w_1(W)$ is the orientation local system of $W$.

We give a sketch of proof of Lemma 5.1.

Proof. For the $\mathbb{F}$-coefficient, the usual Thom isomorphism theorem implies that there exists an element $\tau_G(W) \in H^*(D(EG \times G W), S(EG \times G S(W)); \mathbb{F}))$ such that

$$\cup \tau_G(W) : H^*(B \times BG; \mathbb{F}) \to H^{*+\text{rank} W}(D(EG \times G W), S(EG \times G S(W)); \mathbb{F})$$

is an isomorphism. This proves (i).

For the second statement, we use the local coefficient version of the Thom isomorphism theorem. An important point is that the orientation local system of $W$ is the same as the $\text{pr}^* w_1(W)$. Then we have an element

$$\tau_G(W) \in H^*(D(EG \times G W), S(EG \times G S(W)); \pi^*_W w_1(W))$$
such that

\[ \cup \tau_G(W) : H^*(B \times BG; pr^*\rho) \to H^{*+\text{rank} W}(D(EG \times_G W), S(EG \times_G S(W)); \pi^*_W \rho \otimes \pi^*_W w_1(W)) \]

gives an isomorphism. \qed

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