Absorption of a Randomly Accelerated Particle: Gambler’s Ruin in a Different Game

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Abstract

We consider a particle which is randomly accelerated by Gaussian white noise on the line $0 < x < 1$, with absorbing boundaries at $x = 0, 1$. Denoting the initial position and velocity of the particle by $x_0$ and $v_0$ and solving a Fokker-Planck type equation, we derive the exact probabilities $q_0(x_0, v_0), q_1(x_0, v_0)$ of absorption at $x = 0, 1$, respectively. The results are in excellent agreement with computer simulations.

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A well known topic in random walk theory [1] is the problem of the “gambler’s ruin.” Initially the gambler has an amount of money \( x_0 \) and the bank the amount \( 1 - x_0 \). The gambler flips a coin repeatedly, randomly winning or losing the increment \( \epsilon \). The game ends when the gambler’s funds reach 0 or 1. The problem is to compute the probability \( q_0(x_0) \) that the gambler loses everything.

The problem is easily solved. Since \( q_0(x_0) = \frac{1}{2}[q_0(x_0 + \epsilon) + q_0(x_0 - \epsilon)]\),

\[
\frac{d^2q_0(x_0)}{dx_0^2} = 0
\]

in the limit \( \epsilon \to 0 \). From equation (1) and the boundary conditions \( q_0(0) = 1, q_0(1) = 0, \)

\[
q_0(x_0) = 1 - x_0.
\]

As the starting capital increases from 0 to 1, the probability of the gambler’s ruin decreases from 1 to 0.

Instead of the gambling scenario one could equally well imagine a particle making a random walk with infinitesimal steps \( \pm \epsilon \) on the \( x \) axis, with initial position \( 0 < x_0 < 1 \). In the course of time the particle eventually arrives at \( x = 0 \) or \( x = 1 \). The quantities \( q_0(x_0) \) in equation (2) and \( q_1(x_0) = 1 - q_0(x_0) \) represent the probabilities that the particle first reaches the edge of the interval at \( x = 0 \) and \( x = 1 \), respectively. Alternatively, we could impose absorbing boundary conditions and interprete \( q_0(x_0) \) and \( q_1(x_0) \) as the probabilities of absorption at \( x = 0 \) and at \( x = 1 \).

In this Letter we also consider a particle on the finite interval \( 0 < x < 1 \), but we assume that the changes in the velocity rather than the position of the particle are random. The particle moves according to the Langevin equation

\[
\frac{d^2x}{dt^2} = \eta(t),
\]

where the acceleration \( \eta(t) \) has the form of Gaussian white noise, with

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2\delta(t_1 - t_2).
\]
Imposing absorbing boundary conditions, we derive the probabilities $q_0(x_0, v_0)$, $q_1(x_0, v_0)$ of absorption at $x = 0$ and at $x = 1$, respectively, as functions of the initial position and velocity.

The quantity $q_0(x_0, v_0)$ can also be interpreted as the probability of a gambler’s ruin, but the game is different. The gambler has an amount of money $x(t)$ at time $t$ and the bank the amount $1 - x(t)$. Money is transferred from the bank to the gambler at a rate $v = dx/dt$, which may be positive or negative. At regular infinitesimal intervals the gambler flips a coin, randomly increasing or decreasing the rate $v$ by the increment $\Delta$. The game ends when $x$ reaches 0 or 1. The quantity $q_0(x_0, v_0)$ is the probability that a gambler with initial conditions $x_0, v_0$ loses everything.

In the case of a random walk on the $x$ axis, the probability density $P(x, x_0, t)$ at time $t$ of a particle which is initially at $x_0$ obeys the diffusion equation. For a particle which is randomly accelerated according to equations (3) and (4), the probability density $P(x, v; x_0, v_0; t)$ in the phase space $(x,v)$ satisfies the Fokker-Planck equation [2]

$$
\left( \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} - \frac{\partial^2}{\partial v^2} \right) P(x,v; x_0,v_0; t) = 0,
$$

(5)

corresponding to diffusion of the velocity, with initial condition

$$
P(x,v; x_0,v_0; 0) = \delta(x-x_0)\delta(v-v_0).
$$

(6)

In analogy with the discussion leading to differential equation (1) for $q_0(x_0)$, let us consider a discrete dynamics in which the velocity $v$ changes by $\pm\Delta$ with equal probability at time intervals $\tau$. For this dynamics

$$
P(x,v; x_0,v_0; t) = \frac{1}{2}[P(x-v\tau, v+\Delta; x_0, v_0; t-\tau) + P(x-v\tau, v-\Delta; x_0, v_0; t-\tau)],
$$

(7)

$$
q_0(x_0, v_0) = \frac{1}{2}[q_0(x_0 + v_0\tau, v_0 + \Delta) + q_0(x_0 + v_0\tau, v_0 - \Delta)].
$$

(8)

Expanding equations (7) and (8) in $\tau$ and $\Delta$, dividing by $\tau$, and taking the limit $\tau = \frac{1}{2}\Delta^2 \to 0$ gives us a “poor man’s” derivation of the Fokker-Planck equation (5) and the corresponding differential equation
\[ \left( v_0 \frac{\partial}{\partial x_0} + \frac{\partial^2}{\partial v_0^2} \right) q_0(x_0, v_0) = 0 \]  \hspace{1cm} (9)

for the probability of absorption at \( x = 0 \).

To solve equation (9) with the absorbing boundary condition

\[ q_0(0, v_0) = 1, \quad v_0 < 0, \]  \hspace{1cm} (10)

and the requirements

\[ q_0(x_0, v_0) = q_1(1 - x_0, -v_0), \]  \hspace{1cm} (11)
\[ q_0(x_0, v_0) + q_1(x_0, v_0) = 1, \]  \hspace{1cm} (12)

of reflection symmetry and total probability equal to 1, we first make the substitution

\[ \psi(x, v) = q_0(x, -v) - \frac{1}{2}. \]  \hspace{1cm} (13)

Expressed in terms of \( \psi(x, v) \), equations (9)-(12) take the form

\[ \left( v \frac{\partial}{\partial x} - \frac{\partial^2}{\partial v^2} \right) \psi(x, v) = 0, \]  \hspace{1cm} (14)
\[ \psi(0, v) = \frac{1}{2}, \quad v > 0, \]  \hspace{1cm} (15)
\[ \psi(x, v) = -\psi(1 - x, -v). \]  \hspace{1cm} (16)

Masoliver and Porrà [3] have shown how certain Fokker-Planck type equations on the finite interval \( 0 < x < 1 \) can be solved exactly. They derived an exact result for the average time \( T(x_0, v_0) \) a randomly accelerated particle with initial conditions \( x_0, v_0 \) takes to reach a boundary of the interval. The probability that the particle has not yet reached a boundary after a time \( t \) decays as \( e^{-Et} \), as discussed by Burkhardt [4]. He obtained \( E \) numerically with an approach similar to [3] and related it to the confinement free energy of a semiflexible polymer in a tube. In another application inspired by [3], Burkhardt, Franklin, and Gawronski [5] calculated the equilibrium distribution function \( P(x, v) \) of a randomly accelerated particle on the line \( 0 < x < 1 \) undergoing inelastic collisions at the boundaries [6].
The function $\psi(x, v)$ satisfies the same steady-state Fokker-Planck equation (14) as the quantity $P(x, v)$ considered in [3] and has the same Green’s function solution

$$
\psi(x, v) = \frac{v^{1/2}}{3x} \int_0^\infty du \, u^{3/2} e^{-(v^3 + u^3)/9x} \left( \frac{2v^{3/2}u^{3/2}}{9x} \right) \psi(0, u) - \frac{1}{3^{1/3} \Gamma(\frac{2}{3})} \int_0^x dy \, (x - y)^{2/3} \frac{\partial \psi(y, 0)}{\partial v}, \quad v > 0
$$

(17)
derived in [3]. Equation (17) only holds for positive $v$. For negative $v$, $\psi(x, v)$ can be obtained from equation (17) using the antisymmetry (16) under reflection.

Equation (17) determines $\psi(x, v)$ for all $x > 0$ and $v > 0$ from $\psi(0, v)$ and $\partial \psi(x, 0)/\partial v$. The first of these functions is given in equation (15). To determine the second, we set $v = 0$ in equation (17), which yields

$$
\psi(x, 0) = \frac{1}{3^{1/3} \Gamma(\frac{2}{3})} \left[ x^{-2/3} \int_0^\infty du \, u^{3/2} e^{u^3/9x} \psi(0, u) - \int_0^x dy \, (x - y)^{2/3} \frac{\partial \psi(y, 0)}{\partial v} \right].
$$

(18)

Then, substituting equation (18) in the relation $\psi(x, 0) + \psi(1 - x, 0) = 0$, which follows from (16), and using $\partial \psi(y, 0)/\partial v = \partial \psi(1 - y, 0)/\partial v$, also a consequence of (16), we obtain

$$
\int_0^1 \frac{dy}{|x - y|^{2/3}} \frac{\partial \psi(y, 0)}{\partial v} = \int_0^\infty du \, u \left[ \frac{e^{-u^3/9x}}{x^{2/3}} + \frac{e^{-u^3/9(1-x)}}{(1 - x)^{2/3}} \right] \psi(0, u).
$$

(19)

The solution to integral equation (19), derived, following [3], in Appendix B of [3], is given by

$$
\frac{\partial \psi(x, 0)}{\partial v} = \int_0^\infty du \, u \left[ R(x, u) + R(1 - x, u) \right] \psi(0, u),
$$

(20)

where

$$
R(x, u) = \frac{1}{3^{5/6} \Gamma(\frac{2}{3}) \Gamma(\frac{2}{3})} \frac{u^{1/2} e^{-u^3/9x}}{x^{7/6}(1 - x)^{1/6}} {}_1F_1 \left( -\frac{1}{6}, \frac{5}{6}, \frac{u^3(1-x)}{9x} \right),
$$

(21)

and ${}_1F_1(a; b; z)$ is the confluent hypergeometric function [8,9].

Equations (17), (20), and (21) determine $\psi(x, v)$ for all $x$ and $v$ from $\psi(0, v)$ for $v > 0$, which is known from the absorbing boundary condition (13). Substituting equations (15) and (21) in (20) leads to
\[
\frac{\partial \psi(x,0)}{\partial v} = \frac{1}{3^{1/6} \Gamma(\frac{1}{3})} [x(1-x)]^{-1/6},
\]  
(22)

and from (13), (17), and (22)

\[
\psi(x,v) = \frac{1}{2} - \frac{1}{2\pi} \int_0^x dy \frac{e^{-y^3/9(y-x)}}{(y-x)^{2/3}} [y(1-y)]^{-1/6}, 
\]  
(23)

Rewriting equation (23) in terms of \(q_0(x_0, v_0)\) using (11)-(13), we obtain our main result

\[
q_0(x_0, v_0) = 1 - q_0(1-x_0,-v_0) = \frac{1}{2\pi} \int_{x_0}^1 dy \frac{e^{-y^3/9(y-x_0)}}{(y-x_0)^{2/3}} [y(1-y)]^{-1/6}, \quad v_0 > 0, \tag{24}
\]

analogous to the solution (2) of the traditional gambler’s ruin problem.

For \(x_0 = 1\) equation (24) reproduces the expected result \(q_0(1, v_0) = 1 - q_0(0, -v_0) = 0, v_0 > 0\), corresponding to the immediate absorption of a particle that is initially at either boundary with velocity directed outward from the interval \(0 < x < 1\). For \(x_0 = 0\) and \(v_0 = 0\) the integral in equation (24) can be evaluated, yielding

\[
q_0(0, v_0) = 1 - q_0(1, -v_0) = 1 - \frac{2 \cdot 3^{2/3}}{\Gamma(\frac{1}{6})} v_0^{1/2} 1F(\frac{1}{6}; \frac{7}{6}; -\frac{1}{3} v_0^3), \quad v_0 > 0, \tag{25}
\]

\[
q_0(x_0, 0) = 1 - q_0(1-x, 0) = 1 - \frac{6 \Gamma(\frac{1}{3})}{\Gamma(\frac{1}{6})^2} x_0^{1/6} 2F(\frac{1}{6}; \frac{5}{6}; \frac{7}{6}; x_0). \tag{26}
\]

Here \(1F(a; b; z)\) and \(2F(a, b; c; z)\) are the confluent and ordinary hypergeometric functions \([8,9]\).

The probability \(q_0(x_0, v_0)\) of absorption at the origin, obtained from equation (24) by numerical integration, is shown in figure 1. The probability decreases monotonically as \(x_0\) increases with fixed \(v_0\) and as \(v_0\) increases at fixed \(x_0\), as expected. The quantity \(q_0(x_0, v_0)\) is a nonsingular function of \((x_0, v_0)\) except at the two boundary points \((0,0)\) and \((1,0)\).

The curves for \(x_0 = 0.0, 0.1, 0.3, 0.5\) become smoother near \(v_0 = 0\) as \(x_0\) increases, and for \(x_0 = 0.5, q_0(x_0, v_0) - \frac{1}{2}\) is an odd function of \(v_0\), as implied by equations (11) and (12).

The points in figure 1 show the results of computer simulations, which clearly are in excellent agreement with the analytical results. Our simulation routine is based on the exact solution \([10]\)

\[
P_{\text{free}}(x, v; x_0, v_0; t) = \frac{\sqrt{3}}{2\pi t^2} \exp \left\{ -\frac{3}{t^3} \left[ (x - x_0 - v_0 t)(x - x_0 - vt) + \frac{1}{3} (v - v_0)^2 t^2 \right] \right\} \tag{27}
\]
of the Fokker-Planck equation (3) with initial condition (6) in the absence of boundaries. Trajectories with the probability distribution $P_{\text{free}}(x_{n+1}, v_{n+1}; x_n, v_n; \Delta_{n+1})$ given by (27) are generated using the algorithm

$$x_{n+1} = x_n + v_n \Delta_{n+1} + \left(\frac{\Delta_{n+1}^3}{6}\right)^{1/2} \left(s_{n+1} + \sqrt{3} r_{n+1}\right),$$  \hspace{1cm} (28)

$$v_{n+1} = v_n + (2\Delta_{n+1})^{1/2} r_{n+1},$$  \hspace{1cm} (29)

where $x_n$ and $v_n$ are the position and velocity of the particle at time $t_n$, and $\Delta_{n+1} = t_{n+1} - t_n$. The quantities $r_n$ and $s_n$ are independent Gaussian random numbers such that

$$\langle r_n \rangle = \langle s_n \rangle = 0, \quad \langle r_n^2 \rangle = \langle s_n^2 \rangle = 1. \hspace{1cm} (30)$$

In the absence of boundaries there is no time-step error in the algorithm, i.e., the $\Delta_n$ may be chosen arbitrarily. Close to boundaries small time steps are needed.

To derive a quantitative criterion for an acceptable time step, we begin with the averages

$$\langle x(t) \rangle = x_0 + v_0 t, \quad \langle [x(t) - \langle x(t) \rangle]^2 \rangle = \frac{2}{3} t^3, \hspace{1cm} (31)$$

implied by the distribution function (27). At time $t$ the particle coordinate $x$ has a Gaussian distribution, with a maximum at $x = x_0 + v_0 t$ and the root-mean-square width $\left(\frac{2}{3} t^3\right)^{1/2}$. The effect of the boundaries on the propagation is negligible if the Gaussian peak lies almost entirely within the interval $0 < x < 1$. This is certainly the case if, say,

$$0 < x_0 + v_0 t \pm 5 t^{3/2} < 1 \hspace{1cm} (32)$$

Over the range of velocities encountered in our simulations, any $t$ which satisfies the simpler, more stringent condition

$$t < \frac{1}{10} x_0 (1 - x_0) \hspace{1cm} (33)$$

also satisfies (32).

Keeping inequality (33) in mind, we performed our simulations with the time step

$$\Delta_{n+1} = 10^{-5} + 10^{-1} x_n (1 - x_n). \hspace{1cm} (34)$$
The time step decreases as the particle approaches the boundary and has the minimum value $10^{-5}$. It is necessary to have a small nonzero minimum value. Otherwise the particle never arrives at the boundaries. Our results for the absorption probability $q_0(x_0, v_0)$ are averages based on $10^5$ trajectories for each set of initial conditions $x_0, v_0$.

Finally we note that $q_0(x_0, v_0)$ may be derived from another general Green’s function solution $\psi$ of the Fokker-Planck equation (14),

$$
\psi(x, v) = \frac{v^{1/2}}{3x} \int_0^{\infty} du \frac{u^{3/2} e^{-(v^3+u^3)/9x}}{I_{1/3} \left( \frac{2u^{3/2}}{9x} \right)} \psi(0, u) + \frac{v}{3^{2/3}\Gamma(\frac{1}{3})} \int_0^{\infty} dy \frac{e^{-v^3/9(x-y)}}{(x-y)^{4/3}} \psi(y, 0), \quad v > 0,
$$

(35)

different from (17). By substituting equation (35) in (14), one can check that the Fokker-Planck equation is indeed satisfied. On the lines $x = 0$ and $v = 0$ equation (35) reduces to the identities $\psi(0, v) = \psi(0, v)$ and $\psi(x, 0) = \psi(x, 0)$, respectively.

Equation (35) determines $\psi(x, v)$ for all $x > 0$ and $v > 0$ from $\psi(0, v), v > 0$ and $\psi(x, 0)$. The first of these functions is given in equation (15). To determine the second, we differentiate equation (35) with respect to $v$ and then set $v = 0$, which yields

$$
\frac{\partial \psi(x, 0)}{\partial v} = \frac{1}{3^{2/3}\Gamma(\frac{1}{3})} \times \left[ x^{-4/3} \int_0^{\infty} du \frac{u^2 e^{-u^3/9x}}{\psi(0, u)} - 3x^{-1/3} \psi(0, 0) - 3 \int_0^{x} \frac{dy}{(x-y)^{1/3}} \frac{\partial \psi(y, 0)}{\partial y} \right].
$$

(36)

For the absorbing boundary condition (15) the first two terms on the right-hand side of (36) cancel. Substituting equation (36) in the relation $\partial \psi(x, 0)/\partial v - \partial \psi(1-x, 0)/\partial v = 0$, which follows from (16), using the invariance of $\partial \psi(y, 0)/\partial y$ under $y \to 1-y$, and integrating with respect to $x$ yields

$$
\int_0^{1} dy \left| x - y \right|^{2/3} \frac{\partial \psi(y, 0)}{\partial y} = \text{const.}
$$

(37)

\footnote{This solution may be derived by slightly modifying the derivation in Appendix A of [5]. Setting $v = 0$ in equation (A3) of [5], solving for $W(s)$, and reinserting the result in (A3) with $v \neq 0$ yields the Laplace transform of the new solution (35).}
The function $\psi(x, 0)$ given in equations (13) and (26) satisfies equation (37). Substituting this $\psi(x, 0)$ and $\psi(0, v) = \frac{1}{2}$ into equation (35), integrating, and using (11)-(13), we obtain

$$q_0(x_0, v_0) = 1 - q_0(1 - x_0, -v_0) = \frac{2 \cdot 3^{1/3} v}{\Gamma(\frac{1}{6})^2} \int_{x_0}^{1} dy \frac{e^{-v_0^3/9(y-x_0)}}{(y - x_0)^{4/3}} (1 - y)^{1/6} \, _2F_1\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{6}; 1 - y\right), \quad v_0 > 0. \tag{38}$$

With the help of the identity

$$\int_{0}^{x} dy \frac{e^{-v^3/9(x-y)}}{(x - y)^{2/3}} f(y) = \frac{v}{3^{2/3} \Gamma(\frac{1}{3})} \int_{0}^{x} dz \frac{e^{-v^3/9(x-z)}}{(x - z)^{4/3}} \int_{0}^{z} dy \frac{f(y)}{(z - y)^{2/3}} \tag{39}$$

for arbitrary $f(y)$, one can convert expression (38) for $q_0(x_0, v_0)$ to the simpler form (24).

The second of the two Green’s function solutions (17), (35) looks simpler than the first, since no derivatives of $\psi$ appear on the right-hand side, but our main result (24) for $q_0(x_0, v_0)$ is obtained more easily from (17).

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FIG. 1. The probability $q_0(x_0, v_0)$ of absorption at the origin. The solid lines show the exact result given in equation (24). The points are the results of our computer simulations. The data points have a statistical uncertainty $\pm \delta q_0$ with $|\delta q_0| \lesssim 0.001$. 