Stability and Critical Phenomena of Black Holes and Black Rings

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Abstract

We revisit the general topic of thermodynamical stability and critical phenomena in black hole physics, analyzing in detail the phase diagram of the five dimensional rotating black hole and the black rings discovered by Emparan and Reall. First we address the issue of microcanonical stability of these spacetimes and its relation to thermodynamics by using the so-called Poincaré (or “turning point”) method, which we review in detail. We are able to prove that one of the black ring branches is always locally unstable, showing that there is a change of stability at the point where the two black ring branches meet. Next we study divergence of fluctuations, the geometry of the thermodynamic state space (Ruppeiner geometry) and compute the appropriate critical exponents and verify the scaling laws familiar from RG theory in statistical mechanics. We find that, at extremality, the behaviour of the system is formally very similar to a second order phase transition.

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1 Introduction

Topology change of event horizons constitutes a fundamental issue in gravity. From a broad perspective, one of its most important implications comes from its relation to the problem of resolution of singularities. Instabilities leading to the destruction of an event horizon can in principle produce naked singularities, hence violating the Cosmic Censorship Conjecture. The latter constitutes one of the major unsolved issues of the classical theory, and a complete understanding of this problem must imply deep consequences at the quantum level. Somehow, the Cosmic Censorship can be thought as the classical mechanism for resolution of singularities, which ultimately are expected to be cured by any correct quantum completion of gravity.

In a String Theory context, singularity resolution involving a (manifold) topology change was first emphasized in [1], where the inclusion of nonperturbative string states (namely, black hole states) was shown to provide a mechanism for the resolution of the
conifold singularity. Gravitational phase transitions also play an important role in the \( \text{AdS/CFT} \) correspondence, starting from the original work of Witten \cite{2} on nucleation of black holes in Anti-de-Sitter space (the Hawking-Page phase transition \cite{3}) and its relation to deconfinement at finite temperature in gauge theories. This is of interest both for the study of the dual phenomena in the corresponding gauge theory (recent works include e.g. \cite{4}) and, also, because it seems to provide the possibility of investigating gravitational singularities from the field theory point of view \cite{5}.

Interestingly, higher dimensional General Relativity provides indications of topology change of event horizons, the best known example being the Gregory-Laflamme instability \cite{6}. Recently this phenomenon has been studied in the presence of compact dimensions by a number of authors \cite{7} (for a recent review see \cite{8}). In general, the question of topology change of event horizons makes sense whenever different phases of black objects (black holes, black strings, black \( p \)-branes or others) can exist, a general feature of higher dimensional General Relativity and low-energy String Theory.

Of course, and at least at classical level, the key issue to be investigated about possible transitions between different black phases concerns their dynamical stability, i.e. stability of the solution under field perturbations, or classical stability under Lorentzian time evolution. However, not much is known about the classical stability of solutions of GR above four dimensions. Apart from the well known instabilities of the Gregory-Laflamme type, very little is known about many other solutions of interest, like e.g. those with angular momentum. For those, some partial or indirect checks of dynamical stability have been put forward \cite{9} but, in most cases, an exhaustive and rigorous proof of (in)stability is still lacking.

A way to address the problem of stability of event horizons comes from black hole thermodynamics. In ordinary thermodynamics of extensive systems, local thermodynamical stability (defined as equivalent to the condition that the Hessian of the entropy has no positive eigenvalues) is related to the dynamical stability of the system. However, such a parallel cannot be inferred for black holes, which constitute non-extensive, non-additive thermodynamical systems. The reasons for this depart from standard thermodynamics can be phrased in different ways: the long-range behaviour of gravitational forces, the well known scaling of the BH entropy with the area (rather than the volume), or the fact that BHs constitute elementary entities that cannot be subdivided into separate systems, not even in any idealized manner.

Critical thermodynamic behaviour of black holes was first studied in \cite{10} and, since then, by a number of authors (see e.g. \cite{11} and references therein). However, the interpretation of the divergences in the specific heats and generalized susceptibilities (in particular, their interpretation as a true phase transition and their relation to the stability of the system) is not clear and has been subject of debate \cite{12,13,14,15,16,17,18}. In a String Theory context some partial attempts to relate dynamical stability and local thermodynamical stability have been put forward as well \cite{19}, but currently the present status of this subject is not clear \cite{20} (see also the third reference listed in \cite{4}). We will comment on the relation between the Gubser-Mitra conjecture and the “Poincaré method” that we
In this paper we will explore these issues in the context of five dimensional gravity. This scenario seems interesting to us due to the discovery made by Emparan and Reall of a new BH phase: an asymptotically flat, rotating black hole solution with horizon topology $S^1 \times S^2$ and carrying angular momentum along the $S^1$ — what they called a “black ring” \[21\]. Such black ring solutions exist in a region in parameter space (i.e. mass and angular momentum) which overlaps with that of the five dimensional spherical rotating black hole with rotation in just one plane studied in \[22\]. This fact provides the first known example of BH non-uniqueness in asymptotically flat space, and it raises questions on stability and on the possibility of transitions between the different phases.

Let us summarize the main properties of these spacetimes. The five dimensional rotating black hole \[22\] has an upper “Kerr bound” in its angular momentum per unit mass, the extremal configuration that saturates the bound being a naked singularity. In what follows, we will be using the dimensionless quantity

$$x^2 \sim \frac{J^2}{M^3}$$

(where $J$ and $M$ are respectively the total angular momentum and mass) as the “control parameter”\[1\] of the problem — analogous, say, to the temperature in a liquid-gas system when studied in the canonical or grand canonical ensembles. The same parameter has been already used in investigations of black rings in \[23\]. It is normalized in such a way that the five dimensional rotating black hole solution exists in the range

Black Hole : $0 \leq x < 1$,

$x = 1$ being the extremal (singular) limit. On the other hand, one can have two different black ring spacetimes with horizon topology $S^1 \times S^2$ when the total angular momentum per unit mass exceeds a certain minimal value $x_{\text{min}}$ given by:

$$x_{\text{min}} = \sqrt{\frac{27}{32}} \approx 0.919.$$ 

One of them, that we will call the “large” black ring, can have angular momentum per unit mass unbounded from above. The other one, that we call “small” black ring, cannot\[2\]. One has:

$$\begin{cases} 
\text{Large Black Ring :} & x_{\text{min}} \leq x < \infty, \\
\text{Small Black Ring :} & x_{\text{min}} \leq x < 1. 
\end{cases}$$

At $x_{\text{min}}$ both solutions coincide. For a comparison of the respective entropies see Fig. \[1\]. One can see that near extremality the large black ring is entropically favoured (its entropy equals that of the rotating BH at $x = 2\sqrt{2}/3 \approx 0.943$). This fact was taken in

\[1\]A proper order parameter will be defined in Section \[5\].

\[2\]The labels “large” and “small” for the two different black rings have already been introduced in \[23\].
Ref. [21] as a first indication of a phase transition from the black hole to a black ring when the former rotates fast enough, i.e. as we vary $x$. Even if such a topology changing process is not believed to be possible classically [24], one might expect to find some evidence of it by exploring issues like stability and the thermodynamics of these spacetimes. The study of the general properties of this phase diagram is the main purpose of this paper.

There are two main concrete questions that we will address here. The first one is that of the classical stability of these solutions and its relation to thermodynamics. We will deal with this issue in Section 3 and our main tool will be the so-called “Poincaré” or “turning point” method.

The second question is that of to which extent we are allowed to call all these configurations “phases” in any thermodynamical or statistical mechanical sense. This is a nontrivial question which has to do with the properties of the classical “configuration space” (i.e. the space of metrics): namely, if all these solutions correspond or not to extrema of some common underlying thermodynamic potential. If so, the general theory of critical phenomena tells us that, near a critical point, the generalized susceptibilities and fluctuations have to diverge in a very specific way. We will address these issues in Sections 4 and 5. Our main tools will be the study of the geometry of the thermodynamic space state and the general theory of second order phase transitions.

**Summary of Results**

After reviewing in Section 2 the main properties of rotating black holes and black rings, we focus in Section 3 on the issue of stability. We revise the usual arguments relating dynamical and local thermodynamical stability, and recall why these cannot be applied to non-extensive systems such as BHs. Instead, we will use the so-called “Poincaré method of stability” [25] as it has been already applied to BH physics in [15, 16, 17, 26]. This technique is of remarkable simple applicability, and it essentially consists on plotting the appropriate phase diagrams. We will review it in detail and comment on the suitability of this method for the study of BHs. Contrary to the standard analysis based on the sign of the specific heat, the Poincaré method does not predict any instability in e.g. the Schwarzschild or Kerr spacetimes. We will use it to study the BH/BR system in the microcanonical ensemble, finding that the only point at which a change of stability can occur is at $x_{\text{min}}$. One of our main results is in fact that the small black ring is not only globally, but also locally unstable. The large black ring turns out to be a “more stable” phase in a sense that we will explain. On the other hand, we do not find any change of stability along the whole BH branch. In particular, we will see that nothing special happens at the point in the BH branch where the specific heat changes sign (that is, the five dimensional analogue of the “critical point” found by Davies in [10] and discussed since then by many other authors).

In Section 4 we discuss divergence of fluctuations and critical behaviour. We will see that quadratic fluctuations can be evaluated at $x_{\text{min}}$ and $x = 1$, due to the fact that at those points the relevant degrees of freedom of the system are just a few and under control.
We will see that at both points fluctuations diverge, but that the origin of this divergent behaviour is very different at $x_{\text{min}}$ and at extremality. Next we will use the geometry of the thermodynamic state space as introduced by Ruppeiner [27], and explain how this technique should be interpreted and applied to the BH case. In particular, the divergences of the “thermodynamic curvature” are interpreted in Ruppeiner theory as indicators of critical phenomena. We will compute this quantity for the BH/BR system, finding that it diverges both at $x_{\text{min}}$ and $x = 1$.

In Section 5 we compute the various critical exponents at the points where fluctuations diverge. We check the usual scaling laws familiar from Renormalization Group theory in statistical mechanics, finding that these are obeyed at extremality, but not at $x_{\text{min}}$. We thus conclude that $x = 1$ might be identified, in a certain sense that we will explain, with the critical point of a second order phase transition. The phenomenon at $x_{\text{min}}$ seems, on the other hand, very different: there fluctuations diverge because, as a consequence of the change of stability, the thermodynamic potential becomes locally flat. We discuss our results and prospects for future work in Section 6.

2 The Black Hole and Black Ring Spacetimes

2.1 The $d = 5$ Rotating Black Hole

In [22] Myers and Perry found black hole solutions of higher dimensional GR with arbitrary angular momenta. In particular, the $d = 5$ BH solution with rotation in a single plane is found to be given by the line element:

$$ds^2 = -dt^2 + \Delta (dt - a \sin^2 \theta \, d\psi)^2 + \sin^2 \theta (r^2 + a^2) d\psi^2 + \Psi \, dr^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\phi^2,$$

with $0 \leq \psi \leq 2\pi$, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi/2$ and the functions $\Delta$, $\Psi$ and $\rho$ being:

$$\Delta \equiv \frac{\mu}{r^2 + a^2 \cos^2 \theta}, \quad \Psi \equiv \frac{r^2 + a^2 \cos^2 \theta}{r^2 + a^2 - \mu}, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta. \quad (2.2)$$

The parameters $\mu$ and $a$ set the physical ADM mass and angular momentum via the relations:

$$M = \frac{3\pi}{8G} \mu, \quad J = \frac{2}{3} Ma. \quad (2.3)$$

This solution has a “ring singularity” at $\rho = 0$ and a event horizon (of spherical topology) at $r_H^2 = \mu - a^2$, whose existence thus requires the bound

$$a^2 < \mu \quad \iff \quad \frac{J^2}{M^3} < \frac{32G}{27\pi}. \quad (2.4)$$

The extremal case ($\mu = a^2$) is a naked singularity.
Thermodynamic Quantities

Let us introduce the following dimensionless quantity (a suitably normalized angular momentum “per unit mass”):

\[ x \equiv \left( \frac{27\pi}{32G} \right)^{1/2} \frac{J}{M^{3/2}} = \frac{a}{\sqrt{\mu}}. \]  

(2.5)

In terms of it and the mass parameter \( \mu \), the area, temperature and angular velocity of the horizon are given by:

\[ A_{BH} = 2\pi \mu^{3/2} \sqrt{1 - x^2}, \quad T_{BH} = \frac{1}{2\pi \sqrt{\mu}} \sqrt{1 - x^2}, \quad \Omega_{BH} = \frac{x}{\sqrt{\mu}}. \]  

(2.6)

In particular, one can see that the area decreases when \( a^2 \to \mu \) at fixed mass (i.e. \( x \to 1 \)), and it vanishes precisely in the extremal limit. This is because, as we increase the angular momentum of the BH, the event horizon gets “flattened”, becoming a (singular) disc of zero area in the extremal limit (for an explicit analysis of this geometry see e.g. [9]).

2.2 The Black Ring Spacetime

Let us review now the solution found by Emparan and Reall in [21]³. This solution describes a five dimensional, asymptotically flat black hole whose horizon has \( S^1 \times S^2 \) topology. The solution is regular on and outside the event horizon provided that it has angular momentum along the \( S^1 \) direction. Both the black ring and the rotating black hole solution are described by the metric:

\[
\begin{align*}
ds^2 &= -\frac{F(x)}{F(y)} \left( \frac{dt + R\sqrt{\lambda \nu} (1 + y) d\psi}{d\psi} \right)^2 + \\
&\quad + \frac{R^2}{(x - y)^2} \left[ -F(x) \left( G(y) d\psi^2 + \frac{F(y)}{G(y)} dy^2 \right) + F^2(y) \left( \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right) \right],
\end{align*}
\]  

(2.7)

where the functions introduced above are:

\[ F(\xi) = 1 - \lambda \xi, \quad G(\xi) = (1 - \xi^2)(1 - \nu \xi). \]  

(2.8)

\( R \) and \( \nu \) are parameters whose appropriate combinations give the total ADM mass and angular momentum, while \( \lambda \) has to take specific values in order to avoid conical singularities. In particular, there are two possibilities:

\[
\begin{align*}
\lambda &= 1, \quad \text{(black hole)}, \\
\lambda &= \frac{2\nu}{1 + \nu^2}, \quad \text{(black ring)}.
\end{align*}
\]  

(2.9)

³Note however that here we will be using the conventions of [28].
Once $\lambda$ is fixed one has therefore two parameters: $R$ (with dimensions of length and setting the “radius” of the BH or the black ring) and $\nu$ (dimensionless and varying from 0 to 1). In the BH case $\nu = 0$ represents a non rotating black hole, while $\nu = 1$ yields the extremal case. In the BR case, and as explicitly shown below, as $\nu$ takes values from 0 to 1 we get two different black ring configurations, which coincide only at $\nu = 1/2$. At $\nu = 0$ we have an infinitely thin and large black ring with infinite angular momentum per unit mass. The angular momentum per unit mass of this object takes its minimal value at $\nu = 1/2$. From $\nu = 1/2$ to $\nu = 1$ the solution describes a second black ring of different geometry, which approaches the singular BH spacetime in the extremal limit $\nu \to 1$. At $\nu = 1/2$ both solutions coincide, yielding the black ring spacetime with minimum possible angular momentum per unit mass. These properties can be visualized e.g. from the entropy diagram of Fig. [11].

In [28] the physical parameters of both the black ring and black hole solutions are found to be:

$$
A = 8\pi^2 R^3 \frac{\sqrt{\lambda(1+\lambda)(\lambda-\nu)^{3/2}}}{(1+\nu)^2(1-\nu)}, \quad M = \frac{3\pi R^2 \lambda(1+1)}{4G \nu + 1}, \quad \nu = 1/2.
$$

We recover the black hole or black ring values by using (2.9). Finally, let us recall that the physical parameters in (2.10) satisfy the Smarr relation given by

$$
M = \frac{3}{2} \left( \frac{T A}{4G} + \Omega J \right) \quad (2.11)
$$

for any value of $\lambda$, which in particular means that both the black hole and the black ring spacetimes satisfy the same Smarr relation.

**Thermodynamic Quantities**

In order to compare entropies and eventually carry out a thermodynamic analysis, we will need a fundamental relation $A = A(M, J)$ and the corresponding “equations of state” for the black ring spacetime too. Defining the parameters $\mu$ and $x$ in terms of the physical mass and angular momentum as in (2.3) and (2.5), one finds:

$$
A_{BR} = \sqrt{2\pi^2} \mu^{3/2} \sqrt{\nu(1-\nu)}, \quad T_{BR} = \frac{1}{2\sqrt{2\pi} \sqrt{\mu}} \sqrt{\frac{1-\nu}{\nu}}, \quad \Omega_{BR} = \frac{\sqrt{2}}{\sqrt{\mu}} \sqrt{\frac{\nu}{1+\nu}} \quad (2.12)
$$

where $\nu = \nu(J^2/M^3)$ is given by any of the following two values:

$$
\begin{align*}
\nu_S &= \sqrt{\frac{8}{x}} \cos \Theta, \\
\nu_L &= \sqrt{\frac{8}{x}} \sin \Theta
\end{align*}
$$

(2.13)
the angle $\Theta$ being

$$\Theta = \frac{1}{3} \arctan \sqrt{\frac{32}{27} x^2 - 1}.$$  

(2.14)

These explicit expressions are obtained by inverting the parametrization from (2.10) and taking into account the conditions to avoid naked singularities\(^4\). The fact that the angular momentum at fixed mass is bounded from below at the precise value $x_{\text{min}} = \sqrt{27/32} \approx 0.919$ [21] is explicit in the formula above.

In (2.13) $\nu_S$ and $\nu_L$ correspond to the two possible black rings found in [21]: $\nu_S$ corresponds to what we will call “small” ring (i.e. the one whose angular momentum is also bounded from above at $x = 1$), while $\nu_L$ corresponds to the “large” black ring (the one with unbounded angular momentum). As explained, in the limit $x \to \infty$ the horizon of this large black ring takes the shape of an infinitely large (along the $S^1$) and thin ring, hence the name. The fact that the angular momentum of the small BR is also bounded from above, while the one of the large BR remains unbounded, can be deduced by inspection of (2.13) and recalling that $\nu$ is constrained to take values in $[0, 1)$ [28].

2.3 Entropy Diagrams

The expressions (2.6) and (2.12) allow for an explicit comparison of the respective entropies. We recover the plot already found in [21], which is shown in Fig. 1.

![Figure 1: Plot of $A/(2\pi^2 \mu^{3/2})$ against $x = \sqrt{27\pi/32J/M^{3/2}}$ for the Myers-Perry BH (dashed line) and the two black rings (see [21]). At $x_{\text{min}} = \sqrt{27/32} \approx 0.919$ black ring formation becomes possible. At $x = 2\sqrt{2}/3 \approx 0.943$ the entropy of the large BR exceeds that of the black hole.](image)

This figure is the first indication of the possibility of a phase transition when we vary the angular momentum at fixed mass. As we see, beyond $x = 2\sqrt{2}/3 \approx 0.942$ the entropy of the large black ring exceeds the one of the black hole, which rapidly decreases to zero at $x = 1$. Emparan and Reall pointed out in their original work that this fact may be taken as an indication of a BH/BR phase transition.

\(^4\)This fundamental relation has already been given in parametric form in [23].
Motivated by this fact, we will study the stability properties and the behaviour of fluctuations of these spacetimes. However, our results will lead us to the conclusion that, from the point of view of stability, the relevant points in this phase diagram are $x_{\text{min}}$ and $x = 1$, and that nothing special seems to be happening at $x = 2\sqrt{2}/3$. We will comment on the implications of this result in the Conclusions.

### 3 Dynamical vs. Thermodynamical Stability

To argue in favour or against a possible phase transition, the first issue we should address is that of the stability of the classical solutions. To face this problem, one may think that a way to circumvent the study of linear perturbations of the metric might come from thermodynamics. In ordinary thermodynamics of extensive systems, local thermodynamical stability (defined as the Hessian of the entropy having no positive eigenvalues) is linked to the dynamical stability of the system: a positive mode in the Hessian means that at least some kind of small fluctuations within the system are entropically favoured, implying that the system is unstable against these. However, the simple argument leading to this conclusion [29] relies on the additivity of the entropy — a property which does not hold for black holes. The best known example of this failure is Schwarzschild black hole: a stable configuration with negative specific heat (i.e. a positive Hessian).

In order to make clear the differences with the case of BH thermodynamics, let us first review the standard criteria of local thermodynamical stability. To fix ideas, consider an extensive thermodynamic system in the microcanonical ensemble with a fundamental relation $S = S(M, J)$ (where $M$ and $J$ stand for the internal energy and some other extensive parameter). Divide it into two identical subsystems and consider a spontaneous fluctuation that transfers some amount of energy $dM$ from one subsystem into the other. Then, if the system is to be stable, one requires the total entropy after this energy transfer to be smaller (or equal) than the previous total entropy:

$$S(M + dM, J) + S(M - dM, J) \leq 2S(M, J),$$

since, otherwise, energy transfers within the system will take place spontaneously, therefore meaning that the system would be unstable against redistributions of mass from homogeneity (this would be the dynamical unstable mode). The differential form of the above condition is:

$$H_{MM} \equiv \frac{\partial^2 S}{\partial M^2} \leq 0.$$  \hspace{1cm} (3.2)

So (3.2) is a necessary condition for a system to be stable. Similar considerations lead to

$$H_{JJ} \equiv \frac{\partial^2 S}{\partial J^2} \leq 0$$

for transfers of $J$ within the system. For “cooperative” transfers of both $M$ and $J$ we have
to require the Hessian $H$ of $S(M, J)$ to not to be positive definite. Hence

$$\det H = \frac{\partial^2 S}{\partial M^2} \frac{\partial^2 S}{\partial J^2} - \left( \frac{\partial^2 S}{\partial M \partial J} \right)^2 \geq 0.$$  \hspace{1cm} (3.4)

So conditions (3.2)-(3.4) have to be satisfied if the system is to be stable. Only two of them are necessary, the “invariant” condition being a restriction on the sign of the two eigenvalues of the Hessian matrix (for example, (3.2) and (3.4) imply (3.3)).

In standard thermodynamics the violation of any of the above inequalities is taken as an indication of instability and a phase transition. They are equivalent to the more familiar criteria of positivity of the various specific heats and susceptibilities (for example, (3.2) implies positivity of the specific heat at constant $J$, etc.)

### 3.1 Stability of Black Holes and Non-Extensive Systems

Note that the above argument relies on the additivity property of the entropy when merging subsystems and, more fundamentally, on the very possibility of describing a given system as made up of constituent subsystems. Standard thermodynamical stability criteria fail in general when applied to black holes because their entropy is non-additive (the BH entropy is never an homogeneous first-order function of its variables), and because in GR a local definition of mass and other familiar “should-be-extensive” quantities is not possible. Therefore, BHs cannot be thought as made up of any constituent subsystems each of them endowed with its own thermodynamics, and as such they are non-extensive objects. Let us recall that extensivity of the entropy is a postulate in ordinary thermodynamics [29], and many general results in thermodynamic theory just fail if this property is not assumed.

The study of non-extensive thermodynamics has been worked out to some extent, though (see e.g. [30] and references therein), and a concrete proposal to avoid the problems just mentioned in the BH case has been put forward in a series of articles by several authors [15, 16, 17]. These works are based on the so-called “Poincaré method of stability” [25]. This method was previously applied to gravitating systems in [31], where it was shown to work in the task of showing up the existence of unstable modes of an equilibrium solely from the properties of the equilibrium sequence, without having to solve any eigenvalue equation. Formal proofs of the method include those given by Katz [31] and Sorkin [32, 13]. A somewhat alternative and more intuitive interpretation has been given in [26]. Let us review these stability analyses before applying them to the BH/BR system.

**Setup**

As emphasized in [26], one has to consider a single black hole as a whole system. In this way, any discussion related to the possibility of dividing it into subsystems or to the additivity property of the entropy simply does not take place. Second, one has to put the black hole in a given thermodynamic ensemble. The values of the fixed thermodynamic quantities that specify the ensemble will be the usual global quantities that one can define in GR, thus avoiding any problem related to the non-locality of their definitions. This is
just what we will be doing if, for example, the chosen ensemble is the microcanonical and our starting point is the fundamental relations (2.6) and (2.12).

Considering the BH as a whole system means that, for instance, negative specific heat at constant $J$ only indicates an instability of the canonical ensemble: small interchanges of energy between the BH and an external thermal bath render the system unstable. However, we do not expect such considerations to be related to the issue of classical stability of the gravitational solution under small perturbations.

**Equilibrium States, Entropy Maximum Postulate and Fundamental Relations**

The next step is to make a stability analysis based only on maximization of entropy. In particular, we do not want to assume additivity of the entropy function. Before proceeding to such an analysis, let us first make some definitions and set some notations (we follow closely Sorkin [32, 13]).

- Let $\mathcal{M}$ be the space of possible states or configurations of a given system. A given configuration is specified by a point $X$ in $\mathcal{M}$. Let $\{\mu^i\}$ denote the set of independent thermodynamic variables that specify a given ensemble, and $S$ be the corresponding Massieu function. Both $S$ and the set of $\{\mu^i\}$ are functions on $\mathcal{M}$ which we assume to be smooth (for our purposes, having at least continuous second derivatives) except maybe in some boundary of $\mathcal{M}$.

- **Equilibrium states** (stable or not) occur at points in $\mathcal{M}$ which are extrema of $S$ under displacements $dX$ for which $d\mu^i = 0$. At any equilibrium state one can define *conjugate thermodynamic variables* $\{\beta_i\}$ such that, for all displacements $dX$,

$$dS = \beta_i d\mu^i,$$

reflecting the fact that, *at equilibrium*, the Massieu function only depends on the values of the thermodynamic variables $\{\mu^i\}$. Unless the relations $d\mu^i = 0$ are not linearly independent, Eq. (3.5) defines the quantities $\{\beta_i\}$ uniquely. The set of equilibria are thus a submanifold $\mathcal{M}_{eq}$ of the configuration space. Curves in this submanifold are called “equilibrium sequences”, and points in $\mathcal{M}_{eq}$ can be labeled by the corresponding values of the $\{\mu^i\}$, which are referred to as “control parameters”. The points in $\mathcal{M} \setminus \mathcal{M}_{eq}$ are called *off-equilibrium states*.

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5In the microcanonical ensemble the $\{\mu^i\}$ are variables like the energy and the Massieu function is the entropy. In other ensembles, the Massieu function is given by the Legendre transform of the entropy.

6In practice this will mean “extremality” (see below).

7These names come from catastrophe theory, which is the mathematical framework for the study of the phenomena will be interested in. See e.g. [33].

8More properly, one should speak about *constrained equilibria*, since in principle an entropy function is only guaranteed to exist at equilibrium. It is of course difficult to say what “constrained equilibria” means in a gravitational setting. In this respect, we will have to assume that thermodynamic functions are somehow well defined for off-equilibrium states (see e.g. [34, 35]), at least in the vicinity of $\mathcal{M}_{eq}$. 

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The entropy maximum postulate states that unconstrained locally stable equilibrium states take place at points in $M_{eq}$ where $S$ is a local maximum w.r.t. arbitrary variations $dX$ in $M$ that preserve $d\mu^i = 0$.

A fundamental relation for the equilibrium states is an explicit expression for the function

$$S_{eq} = S(\mu^i),$$

i.e. the integral of (3.6). Given a fundamental relation, the variation of the Massieu function as given by Eq. (3.5) serves to calculate explicit expressions for the conjugate thermodynamic variables $\{\beta_i\}$ at equilibrium:

$$\beta_i(\mu^j) = \frac{\partial S_{eq}}{\partial \mu^i}.$$

The above equations constitute the equations of state of the system.

The entropy maximum postulate is therefore a statement about the behaviour of $S$ along off-equilibrium curves in $M$, not about its variations along $M_{eq}$. In general, the properties of $S$ along $M_{eq}$ do not tell us anything about the local stability of an equilibrium, and expressions like (3.2) where $S$ is the equilibrium entropy do not provide information on local stability if one cannot invoke additivity of the entropy over constituent subsystems. However, the equilibrium entropy as given by (3.6) is often all we know. In particular, it is all we know when considering BHs, because expressions such as (2.6) or (2.12) only provide values of the entropy function restricted to a very particular region of configuration space: namely, the series of (equilibrium) black hole solutions.

Therefore, questions about local stability based on maximization of entropy (or Massieu function) are not questions about the properties of the restricted function $S_{eq}$. What one needs is an extended Massieu function, henceforth denoted by $\hat{S}$, that reflects the behaviour of the quantity $S$ off the black hole states, at least in their vicinity. However, the form of $\hat{S}$ is not known in general. To have it, one would need a parametric family of spacetimes describing deformations of the black hole series, and then to compute all the relevant thermodynamic quantities from them.

The “Turning Point Method”

The so-called “Poincaré” or “turning point” method allows, under certain assumptions, to infer some information about the behaviour of $\hat{S}$ and on stability by using only the equilibrium equations of state. In particular, it does not require any knowledge of the explicit form of the extended Massieu function. We sketch here the proof originally given by Katz in [31]. It has been reformulated for systems like BHs with several control parameters in [35]. Other proofs have been given by Sorkin [32, 13].

Assume that an expression for the extended Massieu function is given, at least in the neighbourhood of $M_{eq}$. We can parametrize such a function as:

$$\hat{S} = \hat{S}(X^\rho; \mu^i),$$

(3.8)
where the \( \{ X^\rho \} \) denote a set of off-equilibrium variables. Equilibria take place at the values \( X_{eq}^\rho = X_{eq}^\rho (\mu^i) \) which are solutions of:

\[
\partial_\rho \hat{S} \equiv \frac{\partial \hat{S}}{\partial X^\rho} = 0 .
\]  

(3.9)

The equilibrium Massieu function \( S_{eq}(\mu^i) = \hat{S}(X_{eq}^\rho(\mu^i); \mu^i) \). Define off-equilibrium conjugate variables \( \hat{\beta}_i \),

\[
\hat{\beta}_i(X^\rho; \mu^j) \equiv \partial_i \hat{S} \equiv \frac{\partial \hat{S}}{\partial \mu^i} .
\]  

(3.10)

These give the equilibrium values of Eq. (3.7) when evaluated at equilibrium, that is, \( \beta_i(\mu^j) = \hat{\beta}_i(X_{eq}^\rho(\mu^i); \mu^j) \). The point \( X_{eq}^\rho \) represents a stable equilibrium if and only if it is a local maximum of \( \hat{S} \) at fixed \( \{ \mu^i \} \). Therefore, one has to consider the eigenvalues of the matrix

\[
\partial_{\rho \sigma} \hat{S}(X^\tau; \mu^i) \bigg|_{X^\tau = X_{eq}^\tau(\mu^i)} .
\]  

(3.11)

A change of stability along \( X_{eq}^\rho(\mu^i) \) takes place when one of these eigenvalues becomes zero and changes sign. Assume now that the variables \( \{ X^\rho \} \) can be chosen in such a way that the above matrix is diagonal. The eigenvalues we are interested in are then:

\[
\lambda_\rho(\mu^i) \equiv \partial^2_\rho \hat{S}(X^\sigma; \mu^i) \bigg|_{X^\sigma = X_{eq}^\sigma(\mu^i)} .
\]  

(3.12)

These are called “Poincaré coefficients of stability”. An instability appears whenever there is some \( \lambda_\rho > 0 \). What we will show now is how it is possible to get information about the sign of an eigenvalue near a point where it vanishes without having to compute the spectrum of \( (\partial_{\rho \sigma} \hat{S})_{eq} \).

Consider the identities provided by equations (3.9) when the l.h.s. is evaluated at \( X_{eq}^\rho \). Choose some \( \mu^a \) (for fixed \( a \)) and take the partial derivative with respect to it at both sides of (3.9). Because of the choice of normal coordinates, one gets the set of relations:

\[
(\partial_\rho \hat{\beta}_a)_{eq} + \lambda_\rho \partial_a X_{eq}^\rho = 0 ,
\]  

(3.13)

where there is no sum over \( \rho \) in the second term. Now consider Eq. (3.10) for \( \hat{\beta}_a \) (with the same \( a \) as before) also evaluated at \( X_{eq}^\rho \). Taking again the derivative w.r.t. \( \mu^a \) gives:

\[
\frac{\partial \hat{\beta}_a}{\partial \mu^a} = (\partial_a^2 \hat{S})_{eq} + \sum_\rho (\partial_\rho \hat{\beta}_a)_{eq} \partial_a X_{eq}^\rho
\]  

(3.14)

(no sum over \( a \)). The important thing is that the l.h.s. contains only the derivative of the equilibrium equation of state. Using (3.13) we get:

\[
\frac{\partial \hat{\beta}_a}{\partial \mu^a} = (\partial_a^2 \hat{S})_{eq} - \sum_\rho \frac{(\partial_\rho \hat{\beta}_a)_{eq}^2}{\lambda_\rho} .
\]  

(3.15)
Assume for a moment that at a generic equilibrium \( (\partial^2 \tilde{S})_{eq} \) is finite and that the coefficients \( (\partial \hat{\beta}_a)_{eq} \) are finite and nonzero (the failure of these assumptions will be discussed below). In such a case, the last equation shows that when a change of stability occurs (i.e. when one or several eigenvalues \( \lambda_\rho \) become zero and change sign) the plot of the conjugacy diagram \( \beta_a(\mu^a) \) along the equilibrium series has a vertical tangent, since \( \partial \beta_a / \partial \mu^a \) has to diverge there — this is a “turning point” (see Fig. 2). The first consequence of this is that:

(i) If one single point of an equilibrium sequence is shown to be fully stable, then all equilibria in the sequence are fully stable until the first turning point is reached.

Let us consider the general case in which one can have degeneracy of zeromodes at a given equilibrium state, and let \( \hat{\rho} = 1, \ldots, N \) label the eigenvalues becoming zero. Near the turning point one has:

\[
\frac{\partial \beta_a}{\partial \mu^a} \approx - \sum_{\hat{\rho}} \frac{(\partial \hat{\beta}_a)_{eq}^2}{\lambda_{\hat{\rho}}} + \cdots \text{ (finite terms)}.
\]

(3.16)

This shows that:

(ii) At a turning point in the diagram \( \beta_a(\mu^a) \), the branch with negative slope near the turning point is always unstable, since a negative slope near the turning point means that at least one term in the r.h.s. of (3.16) has to be negative.

In a generic case, however, we cannot conclude that the branch with positive slope near the turning point is stable: for example, there can always be positive eigenvalues that do not change sign and, in such a case, they will never show up in a conjugacy diagram. So the positive slope branch is not guaranteed to be fully stable unless we do know that at least one point in the branch is in fact fully stable. On the other hand:

(iii) If the spectrum of zeromodes is nondegenerate, the degree of stability changes by one at a turning point. The branch with positive slope near the turning point is always more stable than the one with negative slope.

Here “more stable” means that it has exactly one negative mode less than the negative slope branch, since a nondegenerate spectrum means that no more than one eigenvalue is changing sign at a time. The positive slope branch will be stable against the particular eigenvalue that changes sign at the turning point and, in that sense, it is “more stable”.

In any case, what one can always see with this method is the existence of instabilities: the existence of a turning point from which a negative slope branch emerges necessarily implies the existence of at least one unstable mode. Point (ii) above asserts that this conclusion remains true even without the assumption of a nondegenerate spectrum of zeromodes and without any knowledge of stability of any particular point in the sequence \( \text{32} \; \text{13} \).
Bifurcations

However, it is still possible to have an eigenvalue $\lambda_\tilde{\rho}$ that changes sign but whose behaviour does not show up as a turning point in the conjugacy diagrams $\beta_a(\mu^n)$. This can only happen if, at the point of change of stability, the coefficient $(\partial_\tilde{\rho}\beta_a)_{eq}$ also vanishes. Then we will not see in general any vertical tangent. It can be shown that this can only happen at a bifurcation, i.e. when there is another sequence of equilibria that crosses the considered one, the intersection point being the bifurcation. One can indeed prove that: \[ A \]

(iv) Changes of stability can only occur at turning points or bifurcations.

The general result is therefore that, in the absence of bifurcations, the stability character of a sequence cannot change until the equilibrium curve meets a turning point. In particular, stability does not change when the slope of $\beta_a(\mu^n)$ changes sign when passing through zero, thus ruling out the criteria based on the sign of the specific heats (see e.g. Eq. (3.2)) for nonextensive systems. This is because, in general, the sign of the slope of the diagram $\beta_a(\mu^n)$ does not say anything about the stability character of any mode. It is only in the vicinity of a turning point where both things can be related via Eq. (3.16).

Vertical Asymptotes

One might have wondered why, from the appearance of a zero eigenvalue in (3.15), we infer the existence of a vertical tangent, rather than a vertical asymptote. The reason is that at a vertical asymptote there are no equilibrium states at all, and therefore they signal the (asymptotic) endpoint or “boundary” of the equilibrium sequence. In particular, one cannot speak about any “change of stability” there, since if equilibria exist beyond a vertical asymptote they belong to a different sequence.

On the other hand, it seems more reasonable to expect the divergence at an asymptote to be a consequence of the non-differentiability of $\hat{S}$ at the boundary of the sequence, rather than to an eigenvalue approaching zero. The 4 dimensional Kerr BH discussed below is an example where indeed one can prove that the divergence at a vertical asymptote has to be due to some non-differentiability of $\hat{S}$. We will consider in detail vertical asymptotes in the cases studied below, and we will see that there are important differences w.r.t. the behaviour at turning points.

In the following we will admit non-smoothness of $\hat{S}$ only at such points. If one does not have at least continuous second order derivatives then thermodynamic coefficients do not exist, and the Gaussian approximation to the entropy breaks down.

Examples

A typical conjugacy diagram is shown in Fig. 2. The point $P$ is a turning point. The upper branch is unstable, while the lower branch can be stable or more stable. The sign of the slope also changes at the horizontal tangent at $Q$, but this has no relation with a change of stability. Also, if somehow one knows that there are no bifurcations, then one
can conclude that the stability of such a curve can change only at $P$. This means that the lower branch would preserve its stability character beyond $Q$, even if the slope changes sign there.

Let us note that here is a simpler reason to discard the point $Q$ as a point of stability change along the equilibrium series. When crossing $Q$, the control parameter $\mu^a$ does not remain constant, so stability considerations based on maximization of $\hat{S}$ simply do not apply there, since $\hat{S}$ is maximum at constant $\mu^i$ (note, on the other hand, that $d\mu^a = 0$ at $P$). This was essentially the reason invoked in [26] to disregard horizontal tangents in conjugacy diagrams.

![Figure 2: Generic plot of a conjugate variable $\beta_a$ against $\mu^a$ along an equilibrium sequence. The point $P$ is a turning point, where a change of stability can take place. No changes of stability occur at $Q$, even if the slope changes sign there.](image)

As a concrete example to make the previous arguments explicit, consider the four-dimensional Kerr black hole in the microcanonical ensemble [15]. The Massieu function is the entropy $S$ and the control parameters are $\{\mu^i\} = \{M, J\}$. The conjugate thermodynamic coordinates are $\{\beta_i\} = \{\beta, \omega\}$, $\beta$ being the inverse temperature and $\omega = -\Omega/T$. For the study of stability we need the plots of $\beta(M)$ at constant $J$ and $\omega(J)$ at constant $M$ along the equilibrium series. Such plots are given in Fig. 3.

The minimum at $M_D$ in the plot of $\beta(M)$ is exactly the point discussed by Davies [10]. There the specific heat at constant $J$ changes sign through an infinite discontinuity, a fact that has been interpreted in the literature as a sign of a phase transition. However, this point is a horizontal tangent, and therefore no changes of stability take place at $M_D$. Actually, these plots show that there are no turning points at all. Therefore, according to this analysis, no changes of stability occur in all the way down from the Schwarzschild limit to extremality. There could be bifurcations, but in this case these are ruled out due to the uniqueness of time independent BH solutions of the Einstein equations in four dimensions. Since we know that one point of the curve (namely, the Schwarzschild limit) is stable [38], we can conclude that stability is preserved along the whole curve, at least
Figure 3: Conjugacy diagrams for the four dimensional Kerr black hole [15].

(a) $\beta(M)$ at fixed $J$. $M_{\text{ext}}$ stands for the minimal mass at fixed $J$ (the Kerr bound). $M \to \infty$ is the Schwarzschild limit. $M_D$ is the “Davies’ point” $J_D^2 = (2\sqrt{3} - 3)M_D^4$ [10].

(b) $\omega(J)$ at fixed mass. $J_{\text{ext}}$ stands for the Kerr bound. As in the plot of $\beta(M)$, one finds a vertical asymptote there.

against the dynamical modes accounted for in this analysis (see the discussion below). The non-appearance of instabilities in these diagrams is a result which clearly agrees nicely with the fact that Kerr black holes are actually stable [39].

One should consider the origin of the divergence at the vertical asymptote at the Kerr bound in view of of Eq. (3.15). If the divergence at extremality was only due to some $\lambda_\rho$ approaching zero, then that eigenvalue had to be positive (i.e. unstable), since the slope is negative. The absence of turning points in these curves would imply that this unstable mode never changes sign along the whole series, thus meaning that every point is unstable. This is impossible due to the fact that one point of the curve, the Schwarzschild limit, is stable. So the divergence at extremality (or, at least, the leading contribution to it) has to be due to the non-differentiability of $\hat{S}$ at the Kerr bound. This makes sense, since the extremal black hole has zero temperature and $\beta$ and $\omega$ themselves diverge there. So even without knowing about the behaviour of $\hat{S}$, one should expect a divergence due to the “zero temperature contribution”.

In this case one can say even more about the behaviour of $\hat{S}$ near the vertical asymptote. In principle, both coefficients $(\partial^2_a \hat{S})_{\text{eq}}$ and $(\partial_\rho \beta_a)_{\text{eq}}$ in (3.15) can diverge. However, given the fact that we know that all the eigenvalues $\lambda_\rho$ have to stay negative near extremality, the fact that $\partial_\beta / \partial \mu^a$ is negative there implies that, near extremality:

$$\frac{\partial \beta_a}{\partial \mu^a} \approx (\partial^2_a \hat{S})_{\text{eq}} + \cdots \text{ (subleading terms)}.$$  \hspace{1cm} (3.17)

Note that the above approximation means that the behaviour of the phase diagrams $\beta_a(\mu^a)$ is controlled by completely different physics at a vertical asymptote and at a turning point.
(compare to \(\text{3.16}\)). This will be important in Section 4 when discussing fluctuations and trying to isolate the relevant degrees of freedom both at turning points and at vertical asymptotes.

### 3.2 Relation to Dynamical Stability

One should ask about a precise relation between the study of stability based on the Poincaré method and the dynamical stability under linear perturbations of the black hole solution. There are two issues one has to consider at this point: the choice of the ensemble and the number and nature of fluctuations.

#### Choice of the Ensemble

We will be interested in the stability of solutions like (2.1) and (2.7). These describe isolated BHs, for which the mass and angular momentum are conserved quantities that remain fixed under Lorentzian time evolution. The appropriate ensemble is thus the microcanonical, and the relevant potential is the entropy. Inequivalence of the stability limits of different ensembles in self-gravitating systems has been discussed in \([15, 16, 17, 26, 40]\) and, from a more general point of view, in \([41]\). This becomes clear from the precedent discussion on the existence of turning points: a different ensemble means a different choice of the Massieu function and, typically, a reciprocal choice of the conjugate variables, which may in general interchange horizontal tangents ("Davies’ points") by vertical ones (turning points). This is what would happen if, for example, we decide to put the above discussed \(d = 4\) Kerr black hole in the canonical ensemble. In that case stability would be linked to the maximization of the "free energy" \(\phi = S - \beta M\). The conjugacy diagram of interest would be \(M = M(\beta)\), and such a plot has a vertical tangent where \(\beta = \beta(M)\) happens to have a horizontal one. Therefore we see, as mentioned at the beginning of this Section, that the point \(M_D\) only means a change of stability of the black hole when put in contact with a thermal bath, but not of an isolated BH.

#### Nature of Fluctuations

As discussed above, the appearance of a single unstable mode at a turning point is enough to ensure instability of a black hole solution. However, we are still faced with the question of which dynamical (metric) fluctuations are accounted for in a conjugacy diagram. Put differently, we would like to know which deformations of the metric carry physically the off-equilibrium fluctuations denoted above by \(\{X^\rho\}\). In general one will have some identification of the kind

\[
X^\rho \sim \delta g^{(\rho)}_{\mu\nu},
\]

where the \(\{\delta g^{(\rho)}_{\mu\nu}\}\) stand for a suitable set of metric perturbations. This has to be so because there are no other possibilities for coordinates in the configuration space\(^9\). The problem is

\(^9\)For an explicit derivation in specific cases of these kind of identifications from a gravitational path-integral point of view see e.g. \([40, 35]\) and references therein.
to identify which particular perturbations are encoded in our analysis.

This question is important because, in principle, we do not have enough arguments to conclude that full Poincaré stability (if achieved) is enough to guarantee full dynamical stability. First it seems difficult that in a general case the conjugate thermodynamic variables could encode all possible physical metric perturbations. In addition, note that applicability of the turning point method requires some assumptions on the spectrum of \((\partial_{\rho\sigma}\hat{S})_{\text{eq}}\); namely, a discrete spectrum and the possibility of choosing normal coordinates for the off-equilibrium variables. Without any knowledge about the off-equilibrium function \(\hat{S}\), and in the lack of a rigorous proof, one has to admit the possibility that these requirements may be, in a generic case, too strong to be satisfied by the whole spectrum of \((\partial_{\rho\sigma}\hat{S})_{\text{eq}}\) under arbitrary linearized perturbations. There could be some perturbations that do not satisfy the required conditions for them to be “scanned” with this method, and some of those could be unstable modes.

Perturbations satisfying the above criteria are probably those keeping some degree of symmetry\(^{10}\). In [15, 16] it has been argued that, for rotating BHs, Poincaré stability ensures stability against axisymmetric perturbations\(^{11}\). It would be nice to have an explicit proof of this.

Therefore we see that the turning point method seems reliable when detecting the appearance of instabilities. Moreover, one of its main advantages is its extreme simplicity and, most importantly, the fact that it can be applied to non-extensive systems such as BHs. The drawbacks of this method are, however, the following: 1) Bifurcations do not show up in the conjugacy diagrams. 2) Branches with a positive slope at a turning point are not ensured to be fully stable unless one can prove stability of at least one point in the branch. 3) If it cannot be proven that the spectrum of zeromodes is nondegenerate, then one has no control on the relative number of negative modes between two branches meeting at a turning point. 4) In principle, there might be dynamical modes that this treatment of fluctuations does not encode.

Before ending this Section, let us mention that this stability analysis only requires from the considered system an evolution dictated by the extremization of some potential. In our case, such a potential is the BH area, which we know that obeys (classically) an extremum law due to Hawking’s area theorem. In particular, application of this method does not require interpretation of the black hole area or surface gravity as a thermodynamic entropy or temperature.

\(^{10}\)On physical grounds, it is also unlikely that asymmetric perturbations will extremize the entropy (see e.g. [12]). In such a case, the most natural possibility is that the system meets another branch of less-symmetric equilibria (i.e. a bifurcation).

\(^{11}\)See also [13] for another example concerning the symmetry characterizing the relevant perturbations.
3.3 Stability of Black Holes and Black Rings

Let us now apply these considerations on stability to the five dimensional black hole/black ring system. Both for the sake of comparison and for later use, let us turn first to the study of the Hessian of the entropy.

3.3.1 Standard Thermodynamical Stability

We denote the elements of the Hessian by $H_{ij} = \partial^2 S/\partial \mu^i \partial \mu^j$, where the variables \{\mu^i\} are \{M, J\}. For the rotating BH the fundamental relation that we need is given by (2.6). In terms of the variable $x$ defined in (2.5) and the mass parameter $\mu$ one finds that (from now on we set $G = 1$):

\[
\begin{align*}
H_{MM} &= \frac{2^3}{3\sqrt{\mu}} \frac{1-4x^2}{(1-x^2)^{3/2}} \\
H_{MJ} &= \frac{2^3}{\mu} \frac{x}{(1-x^2)^{3/2}} \\
H_{JJ} &= -\frac{2^3}{\mu^{3/2}} \frac{1}{(1-x^2)^{3/2}} \\
det H &= -\frac{2^6}{3\mu^2} \frac{1}{(1-x^2)^2}
\end{align*}
\]

where in the r.h.s. we have quoted the divergent part in the expansion around $\epsilon \equiv 1-x$, which we shall need later. The plots of the diagonal elements and the determinant at fixed mass are given in Fig. 4.

![Figure 4: $H_{MM}(x)$, $H_{JJ}(x)$ and det $H(x)$ for the rotating BH. The curve changing sign at $x = 1/2$ is that of the element $H_{MM}$. $H_{JJ}$ and det $H$ are always negative.](image)

The element $H_{MM}$ is proportional to minus the inverse specific heat $c_J$ at constant $J$, so one can see that $c_J$ changes sign from negative to positive values through an infinite discontinuity at $x = 0.5$, exactly as in the four dimensional case. Therefore the same
To perform the Poincaré analysis and investigate the existence of turning points we need the plots of $\nu$ where one only has to replace $x$ with $\nu$ and $\beta$ now. Since this solution approaches the BH solution in the extremal limit (see e.g. Fig. 1). Notice that $\nu = 1/2$ gives the ring with lowest angular momentum, $\nu = 0$ represents the infinite angular momentum limit of the large black ring, and $\nu = 1$ is the $x = 1$ limit of the small black ring. For later use, let us write down the divergent part of the expansions around $\delta \equiv 2\nu - 1$ and $\epsilon \equiv 1 - \nu$:

\[
\begin{align*}
H_{MM} & \sim -\frac{2^3 \sqrt{2}}{3 \mu} \frac{1}{\delta} + \mathcal{O}(1), \\
H_{MJ} & \sim +\frac{2^5}{\mu} \frac{\nu}{\sqrt{1 + \nu (2\nu - 1) (1 - \nu)^3/2}}, \\
H_{JJ} & \sim -\frac{2^6 \sqrt{2}}{\mu^{3/2}} \frac{\nu^{3/2}}{(1 + \nu)^2 (2\nu - 1) (1 - \nu)^3/2}, \\
det H & \sim -\frac{2^8}{3 \mu^2} \frac{1}{\delta} + \mathcal{O}(1),
\end{align*}
\]

where one only has to replace $\nu$ by the appropriate value $\nu_S(x)$ or $\nu_L(x)$ in (2.13). Recall that $\nu = 1/2$ gives the ring with lowest angular momentum, $\nu = 0$ represents the infinite angular momentum limit of the large black ring, and $\nu = 1$ is the $x = 1$ limit of the small black ring. For later use, let us write down the divergent part of the expansions around $\delta \equiv 2\nu - 1$ and $\epsilon \equiv 1 - \nu$:

\[
\begin{align*}
H_{MM} & \sim -\frac{2^3 \sqrt{2}}{\sqrt{\mu}} \frac{1}{\delta} + \mathcal{O}(1), \\
H_{MJ} & \sim +\frac{2^5}{\mu} \frac{\nu}{\epsilon^{3/2}} + \frac{2^4 \sqrt{2}}{3 \sqrt{\mu}} \frac{1}{\epsilon^{1/2}} + \mathcal{O}(\epsilon^{1/2}), \\
H_{JJ} & \sim -\frac{2^6 \sqrt{2}}{\mu^{3/2}} \frac{1}{\epsilon^{1/2}} - \frac{2^3 \sqrt{2}}{3 \mu^{3/2}} \frac{1}{\epsilon^{1/2}} + \mathcal{O}(\epsilon^{1/2}), \\
det H & \sim -\frac{2^8}{3 \mu^2} \frac{1}{\epsilon^2} - \frac{2^8}{3 \mu^2} \frac{1}{\epsilon} + \mathcal{O}(1).
\end{align*}
\]

Fig. 5 is a plot of the different quantities at fixed mass for both the small and the large black ring. Again, we can see that for both configurations there is always a positive eigenvalue of the Hessian: the SBR has one positive eigenvalue, while the LBR has two positive eigenvalues for all $x$. The divergence of the small black ring at $x = 1$ was expected, since this solution approaches the BH solution in the extremal limit (see e.g. Fig. 1). Notice the infinite discontinuity at $x_{\text{min}}$ between the small and large black ring cases, even if both solutions coincide and are nonsingular at $x_{\text{min}}$.

3.3.2 Turning Points and Stability Analysis

To perform the Poincaré analysis and investigate the existence of turning points we need now the plots of $\beta(M)$ at fixed angular momentum and $\omega(J)$ ($\omega \equiv -\Omega/T$) at fixed mass.

\[\text{Section 3.1} \]
Figure 5: Plots of $H_{MM}$, $H_{JJ}$ and $\det H$ as a function of $x$ for both black rings.

(a) $H_{MM}(x)$, $H_{JJ}(x)$ and $\det H(x)$ for the small black ring. All quantities diverge both at the lower bound $x_{\text{min}}$ and at the upper one $x = 1$.

(b) $H_{MM}(x)$, $H_{JJ}(x)$ and $\det H(x)$ for the large black ring. All quantities diverge at the lower bound $x_{\text{min}}$ and tend asymptotically to zero at $x \to \infty$.

For the black hole and black rings, the conjugate variables are respectively given by:

$$
\begin{align*}
\beta_{\text{BH}} &= \frac{2\pi \sqrt{\mu}}{\sqrt{1 - x^2}}, \\
\omega_{\text{BH}} &= -\frac{2\pi x}{\sqrt{1 - x^2}}.
\end{align*}
$$

\hspace{2cm}

$$
\begin{align*}
\beta_{\text{BR}} &= 2\pi \sqrt{2} \sqrt{\mu} \sqrt{\frac{\nu}{1 - \nu}}, \\
\omega_{\text{BR}} &= -\frac{4\pi \nu}{\sqrt{1 - \nu^2}}.
\end{align*}
$$

(3.21)

All these curves, for the BH and BR phases, are represented in Fig. 6.

Focusing first on the BH curves (dashed lines) we see that there are no turning points, and therefore no changes of stability are shown along the black hole series. Given the fact that the five dimensional Schwarzschild solution is stable \[44\], we can conclude that the five dimensional Myers-Perry BH with just one angular momentum is stable, unless:

a) there are bifurcations, which would imply non-uniqueness of the BH solution other than black rings, or

b) one admits the possibility of the existence of dynamical modes that the turning point method does not “see”.

With respect to point a) above, let us point out that the point $x \approx 0.943$ in Fig. 4 is not a bifurcation, since the BH and the LBR are different spacetimes. One can check that the function $\beta_{\text{BH}} = \beta_{\text{BH}}(M)$ has a minimum (horizontal tangent) at $M = 27\pi J^2 / 8$ (i.e. $x = 1/2$) which is not shown in the figure\[12\]. This is the “Davies’ point” where the specific

\[12\] The whole BH curves are analogous to those of Fig. 9.
Figure 6: The conjugacy diagrams $\beta(M)$ and $\omega(J)$ for all phases of the BH/BR system along their corresponding equilibrium series. The dashed lines represent the BH-curves. We see the existence of a turning point in both diagrams at the point where both BR branches meet. The SBR and BH curves have a vertical asymptote at extremality. $\beta_{BH}(M)$ has a minimum (not shown in the plot) at the value of $M$ that corresponds to $x = 1/2$.

(a) $\beta(M)$ at fixed $J$. $M_{ext}$ stands for the minimal allowed mass at fixed angular momentum in the BH and SBR cases (the “Kerr” or extremal bound $x = 1$). $M_{min}$ is the mass at $x_{min}$.

(b) $\omega(J)$ at fixed mass.

heat changes sign but, as already discussed, no changes of stability occur there. On the other hand, the extremal limit is a vertical asymptote, again like in the four dimensional case. The five dimensional case is completely analogous in all respects, and therefore we can equally conclude that the divergence at $x = 1$ comes from the non-differentiability of $\hat{S}$, and that expressions like (3.17) also apply here.

Before proceeding to the analysis of the BR curves, let us derive some general properties which, assuming no bifurcations, will be of use when analyzing the degeneracy of the spectrum of zeromodes. The general expressions of the kind of (3.15) are, in this case:

$$
\begin{align*}
\partial_M \beta &= (\partial_M \hat{S})_{eq} - \sum_{\sigma} \frac{(\partial_{\sigma} \hat{\beta})_{eq}^2}{\lambda_{\sigma}} - \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \hat{\beta})_{eq}^2}{\lambda_{\tilde{\rho}}}, \\
\partial_J \omega &= (\partial_J \hat{S})_{eq} - \sum_{\sigma} \frac{(\partial_{\sigma} \hat{\omega})_{eq}^2}{\lambda_{\sigma}} - \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \hat{\omega})_{eq}^2}{\lambda_{\tilde{\rho}}}, \\
\partial_M J S &= (\partial_M \hat{S})_{eq} - \sum_{\sigma} \frac{(\partial_{\sigma} \hat{\beta})_{eq}(\partial_{\sigma} \hat{\omega})_{eq}}{\lambda_{\sigma}} - \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \hat{\beta})_{eq}(\partial_{\tilde{\rho}} \hat{\omega})_{eq}}{\lambda_{\tilde{\rho}}}. 
\end{align*}
$$

These are the elements of the equilibrium Hessian. The off-diagonal term (= $\partial_M \omega = \partial_J \beta$) can be computed in a similar fashion as we did in Section (3.1) to compute the diagonal
ones. In the equations above, sum over $\sigma$ denotes sum over never-zero eigenvalues, while sum over $\tilde{\rho}$ denotes the sum over all eigenvalues becoming zero at some point. Note first that, near such a point, the diagonal elements of the Hessian have to diverge all with the same power:

$$H_{MM} \sim H_{JJ} \sim \frac{1}{\lambda_1} + \cdots \quad \text{(subleading terms)},$$

$(3.23)$

$\lambda_1$ denoting the eigenvalue(s) contributing with the highest power to the sum over $\tilde{\rho}$. Notice that this need not be the case if the divergence was not due to the appearance of a zeromode. On the other hand, the leading contribution to the determinant of the Hessian is given by:

$$\det H \sim \left( \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \tilde{\beta})_{\text{eq}}^2}{\lambda_{\tilde{\rho}}} \right) \left( \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \tilde{\omega})_{\text{eq}}^2}{\lambda_{\tilde{\rho}}} \right) - \left( \sum_{\tilde{\rho}} \frac{(\partial_{\tilde{\rho}} \tilde{\beta}_{\text{eq}})(\partial_{\tilde{\rho}} \tilde{\omega}_{\text{eq}})}{\lambda_{\tilde{\rho}}} \right)^2 + \cdots \quad \text{(s.t.)},$$

$(3.24)$

thus meaning that, if there are $N > 1$ zeromodes $\tilde{\rho}$, then the leading contribution to the determinant near a turning point will in general go like:

$$\det H \sim \frac{1}{\lambda_1 \lambda_2} + \cdots \quad \text{(s.t.)},$$

$(3.25)$

where $\lambda_2$ represents the eigenvalue with the highest subleading contribution to the divergence of $H_{ij}$ in $(3.23)$. If, on the contrary, there is no degeneracy of zeromodes, then one will always have the nontrivial property:

$$H_{MM} \sim H_{JJ} \sim H_{MJ} \sim \det H \sim \frac{1}{\lambda_1} + \cdots \quad \text{(s.t.)},$$

$(3.26)$

since the leading divergence in $(3.24)$ will exactly vanish, and the contribution of order $1/\lambda_1$ to the off-diagonal term cannot be cancelled. Moreover, in view of $(3.22)$, the numerical coefficients $h^{(1)}_{ij}$ of the leading contributions to the divergence of the elements $H_{ij}$ have to satisfy the exact relation:

$$\left( h^{(1)}_{MJ} \right)^2 = h^{(1)}_{MM} h^{(1)}_{JJ}. $$

$(3.27)$

This remains true order by order in degree of divergence also in the case of a degenerate spectrum in which all zeromodes contribute with a different degree of divergence.

Let us turn now to the analysis of stability of black rings. From the conjugacy diagrams we see that there are two special points: $x_{\text{min}}$ and $x = 1$. At $x_{\text{min}}$ we find a turning point. Taking into account the general results of the preceding Section, we therefore conclude that the small black ring is always unstable. From the entropy diagram in Fig. 1 we knew that the small black ring was globally unstable, since it is the phase with lower entropy. Now we see that it is unstable also locally.

$^{13}$In the case where $\lambda_1$ is degenerate, the divergence will go like $\sim 1/\lambda_1^2$. 

25
What about the stability of the large black ring? First we see, from the behaviour of the Hessian near the turning point (3.20), that there is only one divergent piece and, in addition, that the necessary conditions (3.26) and (3.27) for a nondegenerate spectrum are satisfied. Therefore this provides strong evidence that the large black ring is more stable (in degree one) than the small ring. However, as discussed above, we cannot infer from this fact its full dynamical stability. Actually, one expects on physical grounds that at least the limit of large $x$ may suffer from a Gregory-Laflamme instability: the large angular momentum limit describes a very long and thin ring, and so it may be considered as a “black string limit” of the configuration [23, 45]. If there are GL instabilities in the LBR, there are three possibilities to explain the absence of turning points in the conjugacy diagrams.

a) The first one is that these diagrams are “blind” against GL modes. This would be the case if, as hinted at in [15, 16], the dynamical modes accounted for in this analysis are only the ones preserving axisymmetry. Notice that instabilities of the Gregory-Laflamme type tend to distort the shape of the string along its axis. In our case, this means deformations along the $S^1$ within the plane of rotation, and therefore they are not axisymmetric perturbations.

b) The second possibility is that the GL instability is present for all values of $x$. In this case the LBR would be always unstable but there would be no turning points.

c) The last possibility is that the emergence of GL modes actually appears as a bifurcation. This would imply the existence of a new branch of solutions beyond some angular momentum per unit mass, most probably describing non-uniform black rings [24, 46]. We consider this latter possibility an interesting one which deserves further investigation.

A further heuristic argument, due to Emparan and Reall, in favour of another instability of the large and small black rings is the following. As discussed in [45], one expects the limit $x = x_{\text{min}}$ to be unstable: one simply has to throw matter with no angular momentum into the ring with minimal $x$. This would decrease the value of the angular momentum per unit mass of the final state and, given the absence of black ring states at $x < x_{\text{min}}$, such a process would destabilize the system. This could happen if possibilities analogous to a) or b) above apply to such an instability.

Let us comment now on the origin of the divergence at the vertical asymptote that the SBR curves exhibit at $x = 1$. As discussed for the black hole case, it seems most likely that such a divergence appears due to the fact that $\beta$ and $\omega$ themselves diverge there because of the zero temperature at $x = 1$. In the SBR case, however, one has to admit the possibility that the divergence can be due to an unstable mode that approaches zero at extremality since, contrary to the BH, the SBR has negative modes for all $x$. We want to argue here that this does not seem the case. The BH and SBR solutions approach each other as they get close to $x = 1$, so one should expect the physical origin of this divergence to be the same
in both cases. In fact, by looking at the leading contributions \( \sim \epsilon^{-3/2} \) to the divergence in the expansion of the SBR Hessian (3.20) at \( x = 1 \), we see that these exactly coincide with those of the BH (3.18) up to an overall constant factor. So we find evidence that the divergence at \( x = 1 \), or at least the leading contribution \( \sim \epsilon^{-3/2} \), is due to the fact that the temperature vanishes there. Therefore, as in the four dimensional case, we will write:

\[
H_{ij} \approx (\partial_{ij}\tilde{S})_{eq} + \cdots \quad \text{(subleading terms)},
\]

both for the BH and the SBR near extremality (we are assuming that this expression also holds for the off diagonal term of (3.22)). This will be of importance in the next Section, when we will consider the behaviour of fluctuations.

Summarizing, our analysis proves local instability of the SBR and presents strong evidence that the LBR is a “more stable” phase, even though it does not guarantee its full dynamical stability. In fact, one expects the LBR to suffer from instabilities which do not show up in the conjugacy diagrams, the possible reasons for this being the ones discussed above. On the other hand, these conjugacy diagrams do not show any special property neither at \( x = 1/2 \) (“Davies’ point”) nor at \( x = 2\sqrt{2}/3 \) (where the LBR becomes entropically favoured, see Fig. 1).

4 Fluctuations, Critical Behaviour and Thermodynamic Geometry

When an unstable point is reached one expects fluctuations to diverge, as it also happens for instance at the critical point of a second order phase transition. However, both phenomena are not the same. Let us next elaborate on this issue in some detail.

Before proceeding we want to make clear what we mean by “fluctuations” in the context of the off-equilibrium formalism introduced above. What we want to consider here are general motions in configuration space, i.e. motions parametrized by \( \{\Delta X^\rho, \Delta \mu^i\} \). Note that for the study of stability we needed to work at fixed \( \{\mu^i\} \), as required by entropy maximization. Here, however, we will not be interested in stability considerations, but rather in the changes induced in the system as we move in all possible directions of configuration space. This is why now we will consider variations like \( \{\Delta \mu^i\} \) as well.

The physical significance of the variations in the \( \{X^\rho\} \)- and in the \( \{\mu^i\} \)-directions is, however, different. The off-equilibrium variables \( \{X^\rho\} \) should be considered as “dynamical”, in the sense that they obey a first order equilibrium equation \( (\partial_{\rho}\tilde{S} = 0) \). Therefore, the variations \( \{\Delta X^\rho\} \) can be thought as spontaneous fluctuations of the system. On the other hand, the set of variables \( \{\mu^i\} \) are our control parameters (in the microcanonical ensemble, the mass and the angular momentum) and, as such, they do not fluctuate spontaneously. In any case, we can always consider the changes induced in the system as we change these control parameters (“adiabatically”, say), in an attempt to map out the behaviour of the system (or better, the thermodynamic potential) in configuration space at
the vicinity of equilibrium\textsuperscript{14}. At this point, let us recall that all our considerations are purely classical, and therefore we are ignoring any quantum effects like Hawking radiation (in such a case, spontaneous variations in mass and angular momentum would be of course physical).

### 4.1 Fluctuations at Turning Points and Vertical Asymptotes

Let us first consider fluctuations around equilibria near a turning point, following in part the discussions in \[40, 36, 35\]. If the eigenvalue $\lambda_1$ becoming zero there is nondegenerate, then the fluctuations will be basically driven by fluctuations in the corresponding mode $X^1$, since this is the direction along which the Massieu potential becomes flat, the restoring forces are small and then $\Delta X^1$ can be large. This will induce off-equilibrium fluctuations in the conjugate variables $\hat{\beta}_i$, which will be well approximated by:

$$\delta \hat{\beta}_i \approx (\partial_1 \hat{\beta}_i)_{eq} \Delta X^1$$ \hspace{1cm} (4.1)

(unless $(\partial_1 \hat{\beta}_i)_{eq}$ is vanishingly small, which can only happen at a bifurcation — see Section 3.11. This approximation remains valid also if there are other zeromodes, but still the one approaching faster to zero is nondegenerate.

The probability for the system to be at $X = X_{eq} + \Delta X^1$ is proportional to the exponential of the Massieu function evaluated at $X$ \[29, 47\], which will be well approximated by its second order variation along that direction:

$$P(X_{eq} + \Delta X^1) \sim \exp \left[ \hat{S}_{eq} + \frac{1}{2} \lambda_1 (\Delta X^1)^2 \right].$$ \hspace{1cm} (4.2)

The last two expressions together with Eq. (3.16) show that, near a turning point which is not a bifurcation and where $\lambda_1$ is nondegenerate, the probability density for a fluctuation $\delta \hat{\beta}_i$ is given by

$$P(\delta \hat{\beta}_i) \sim \exp \left[ -\frac{1}{2} \frac{(\delta \hat{\beta}_i)^2}{\partial \hat{\beta}_i / \partial \mu^i} \right].$$ \hspace{1cm} (4.3)

Calculating the normalization factor and evaluating the mean square deviation (from the side with $\lambda_1 < 0$) is straightforward, and gives the result \[30, 39, 35\]:

$$\langle (\delta \hat{\beta}_i)^2 \rangle \approx \frac{\partial \hat{\beta}_i}{\partial \mu^i}.$$ \hspace{1cm} (4.4)

The same result was obtained in \[26, 17\] via somewhat different considerations. There the $\delta \hat{\beta}_i$ were taken from the start as good coordinates to parametrize off-equilibrium states, thus casting all the discussion on off-equilibrium fluctuations directly in terms of them (without passing through the \{X$^\rho$\} coordinates used here). In such a case, the above

\footnote{Note that variations $\Delta \mu^i$ will not in general run along the equilibrium series unless they are correlated with the $\Delta X^\rho$ as dictated by the equilibrium equation $X^\rho_{eq} = X^\rho(\mu^i)$.}
formula, which here is obtained only as an approximation near a turning point, turns out to be valid everywhere\textsuperscript{15}. However, in general one will need a different and more complete parametrization to account for all off-equilibrium states. In such a case the above equality is ensured to hold only near a turning point.

So we see that, when the system approaches a turning point from the stable side, the quadratic fluctuations of the conjugate variables diverge. In the BH/BR system this happens at $x_{\text{min}}$, where we also expect the condition of $\lambda^i$ being nondegenerate to be satisfied (see last Section). In such a case we can write, near $x_{\text{min}}$ (see (3.22)):

$$H_{MM} \approx -\frac{(\partial_1 \hat{\beta})^2}{\lambda^i}, \quad H_{JJ} \approx -\frac{(\partial_1 \hat{\omega})^2}{\lambda^i}, \quad H_{MJ} \approx -\frac{(\partial_1 \hat{\beta})(\partial_1 \hat{\omega})}{\lambda^i}.$$ (4.5)

Let us recall that the dominant term in the expansion of the BR Hessian (3.20) exactly obeys (3.27), i.e. the numerical coefficients are in perfect agreement with the approximations above. Calculating the ratio $\delta \hat{\beta}/\delta \hat{\omega}$ from (4.1) allows to write, from (4.4) and $H_{MJ}$ above, the off-diagonal correlations $\langle \delta \hat{\beta}_i \delta \hat{\beta}_j \rangle$. We thus obtain the general expression:

$$\langle \delta \hat{\beta}_i \delta \hat{\beta}_j \rangle \approx +H_{ij}$$ (4.6)

for correlations near a turning point. The expression above is then telling us how fluctuations of the LBR diverge as the system approaches $x_{\text{min}}$.

Let us turn now to the fluctuations in the BH and the SBR cases near the vertical asymptote at $x = 1$. Here the phenomenon is very different, since to start with is not related to a change of stability. We saw in the last Section that the leading divergence of the elements of the Hessian can be explained by the divergence of the elements $(\partial_{ij} \hat{S})_{eq}$. Eq. (3.28) implies that these coefficients are the ones that dominate also the series expansion of the fluctuations $\delta \hat{\beta}$ and $\delta \hat{\omega}$ (since it implies that $(\partial_{ij} \hat{S})_{eq} \gg (\partial_1 \beta)_{eq} (\partial_1 \beta)_{eq}$) and, in addition, that they are well approximated by the equilibrium Hessian. Therefore we can write:

$$\delta \hat{\beta}_i \approx H_{ij} \Delta \mu^j.$$ (4.7)

So we see that, contrary to the previous case (see Eq. (4.1)), the variations $\delta \hat{\beta}_i$ are now dominated by the variations in the variables $\{\mu^i\}$, which are small but still induce the main contribution to $\delta \hat{\beta}_i$ because of the large coefficient. Variations in the $\{\mu^i\}$-directions will be governed by a probability density which, using (3.28), can be written as approximately proportional to

$$P(\mu^i_{eq} + \Delta \mu^i) \sim \exp \left[ \hat{S}_{eq} + \beta_i \Delta \mu^i + \frac{1}{2} H_{ij} \Delta \mu^i \Delta \mu^j \right].$$ (4.8)

\textsuperscript{15}This can be seen from the general formalism by substituting in (4.2) (which in this case holds for all modes, which are precisely the $\delta \hat{\beta}_i$) the exact value of $(\partial^2 \hat{S})_{eq}$ as obtained from Eq. (3.15) and taking each $\hat{\beta}_i$ as independent from the $\{\mu^i\}$ and the rest of the $X^\rho$ ($\sim \hat{\beta}_j$, $j \neq i$).
Expressions (4.7) and (4.8) formally coincide with those used in the usual treatments of fluctuations in thermodynamic fluctuation theory\textsuperscript{16} \cite{17}. The computation of correlations gives the result:

$$\langle \delta \hat{\beta}_i \delta \hat{\beta}_j \rangle \approx -H_{ij} + \beta_i \beta_j \approx -H_{ij}$$  \hspace{1cm} (4.9)

(note the minus in the final expression sign w.r.t. (4.6)). The computation of these fluctuations requires some care, however, due to the presence of a positive eigenvalue in the Hessian matrix. Its derivation is outlined in the Appendix. The first equality is the exact result as computed from the (approximate) probability distribution (4.8). The additional contribution to the Hessian is due to the presence of the linear term in the probability distribution. However, such a contribution can be neglected near extremality, since it is subleading as $\epsilon \to 0$ (see Eqns. (3.18), (3.20) and (3.21)).

Summarizing, we see that both near extremality and near the turning point fluctuations diverge, and that at those points correlations can be computed from the equilibrium Hessian.

Another situation where fluctuations typically diverge is at a critical point. However, as pointed out in \cite{26, 18}, this is in general a different phenomenon from a change of stability. A critical point is the endpoint of a “coexistence curve”, characterized by the existence of two competing local maxima of the corresponding Massieu function, describing for example the liquid and gas phases of a fluid. At the critical point both maxima coalesce, which entails a locally flat potential and an absence of restoring forces. This is what makes fluctuations become large near the critical point. On the other hand, the divergence of fluctuations related to a change of stability is in general due to a different mechanism: the flattening of the potential because the system enters into a series of local minima (unstable equilibria) from a series of local maxima (stable equilibria).

Next we would like to investigate these phenomena in the BH/BR system. To this aim, further information on the critical behaviour of a system can be obtained from the geometry of the thermodynamic state space, which is what we shall introduce next.

Before ending this Section the following comment is in order. Note that, at least in the SBR case (and probably in the LBR also), we are considering fluctuations of an \textit{unstable} system (i.e. with some $\lambda_\rho > 0$) . Therefore, to make sense of all approximations above, we have been forced to suppress by hand all variations along the unstable $\{X^\rho\}$-directions, which will be of course the dynamically relevant ones. All our considerations are thus purely formal, in the sense that we are restricting ourselves to study these configurations along the directions in configuration space in which no unstable modes are present. It is in this sense in which this analysis on fluctuations (and the subsequent discussion on a phase transition) has to be understood.

\textsuperscript{16} Up to the linear term in the probability distribution, whose effect just amounts to a shift of the mean values $\langle \delta \hat{\beta}_i \rangle$ (see Appendix). As shown below, the contribution of this term to the mean square deviations is subleading. The linear term is nonvanishing here because the $\{\mu^\rho\}$ do not obey any “equilibrium equation” (i.e. the vanishing of a first order variation) — only the $\{X^\rho\}$ variables do.
4.2 Critical Phenomena and the Geometry of the Thermodynamic State Space

It has been observed by several authors in different contexts that the entropy function induces a natural metric on the thermodynamic state space, and that the geometric invariants constructed out of it may provide interesting information about the phases of the model under consideration. This formalism was pioneered by Ruppeiner, and we refer the reader to [27] for a very detailed review and various examples. Ruppeiner formalism relies on the usual thermodynamic properties of extensive systems. Here we want to argue that, modulo some careful reinterpretations, it can be applied also to BHs near turning points and vertical asymptotes. Previous works where Ruppeiner and related thermodynamic geometries have been applied to black hole physics include [48], but the issue of the non-extensivity of BHs has not been discussed there.

Let us first review how this formalism can be motivated in ordinary (extensive) thermodynamics [27]. Let us consider the microcanonical ensemble, in which one starts with a homogeneous, isolated system in the thermodynamic limit. The purpose is to investigate thermodynamic fluctuations in a given open subsystem (to be thought as a “small” subsystem) of fixed volume \( V \). The microcanonical variables \( \{\mu^i\} \) are fixed within the total system, but the ones specifying the state of the subsystem \( \{\mu^i_V\} \) can fluctuate around their equilibrium values \( \{\mu^i_{V,eq}\} \), which will be set by those of the environment. All off-equilibrium states of the subsystem are assumed to be uniquely specified by the values of the \( \{\mu^i_V\} \).

According to the basic assumption of thermodynamic fluctuation theory, the probability density \( P_V \) of finding the subsystem in a state characterized by values \( \{\mu^i_V\} \) of the remaining variables of state (all except the volume, which is fixed) is given by:

\[
P_V(\mu^i_V) \sim \exp \left[ S_{tot}(\mu^i_V; \mu^i_{eq}) \right],
\]

where the \( S_{tot} \) is the entropy of the total system and \( \mu^i_{eq} \) are the average (equilibrium) values of the environment. The function \( S_{tot}(\mu^i_V; \mu^i_{eq}) \) has, as a function of \( \{\mu^i_V\} \), a local maximum when the densities of the subsystem equal those of the environment, i.e. at \( \mu^i_V = \mu^i_{V,eq} \approx V \mu^i_{eq}/V_{tot} \). The above formula describes fluctuations of the subsystem around this equilibrium value. If one now assumes extensivity, i.e. the fact that the entropy of the total system \( S_{tot} \) is the sum of that of the subsystem \( S_V \) plus that of the environment, and expands around the equilibrium point up to second order, one gets [17] [27]:

\[
P_V(\mu^i_V) = \sqrt{\frac{\det g(\mu_{eq})}{(2\pi)^{n-1}}} \exp \left[ -\frac{1}{2} g_{ij}(\mu_{eq}) \Delta \mu^i_V \Delta \mu^j_V \right],
\]

where \( (n - 1) \) is the number of independent fluctuating variables once the volume has been fixed. In the expression above \( g_{ij} \) is defined as:

\[
g_{ij}(\mu_V) \equiv -\frac{\partial^2 S_V}{\partial \mu^i_V \partial \mu^j_V},
\]
and $\Delta \mu_i^V = \mu_i^V - \mu_{i\text{eq}}^V$. Sum over repeated indices ($i, j = 1, 2, \ldots, n - 1$) is understood in the argument of the exponential. In standard treatments the formula above is written in terms of the entropy and other quantities per volume (i.e. $S_V = V S$, etc). Here we prefer to leave it in terms of the absolute quantities of the subsystem itself to make closer contact with the black hole case.

So far this is standard thermodynamic fluctuation theory [47]. Ruppeiner proposed to interpret the Hessian of the entropy $g_{ij}$ as a metric in the thermodynamic state space. This metric is normally known in the statistical mechanics literature as the “Ruppeiner metric”, and it is supposed to govern the fluctuations of the system. The meaning of distance as defined by this metric is the following: the less the probability of a fluctuation between two states, the further apart they are w.r.t. the geodesic distance defined by this metric. In other words, the metric is an indicator of the “closeness” of probability distributions. A straightforward computation based on the Gaussian distribution shows that the average values of fluctuations vanish:

$$\langle \Delta \mu_i^V \rangle = 0,$$

and that the main square deviations are given by the contravariant metric tensor:

$$\langle \Delta \mu_i^V \Delta \mu_j^V \rangle = g^{ij}(\mu_{\text{eq}}),$$

which becomes an alternative definition of the thermodynamic metric.

Ruppeiner theory is covariant, in the sense that the line element built from the metric defined in (4.12) is taken to be an invariant quantity. This prescription serves to tell us what to do if the chosen coordinates to describe fluctuations of the subsystem are not the usual extensive quantities. If for example we choose to work with the conjugate variables in the entropy representation

$$\beta_{V_i} = \frac{\partial S_V}{\partial \mu_i^V},$$

the line element is written as:

$$ds^2 = g_{ij}(\mu_V) d\mu_i^V d\mu_j^V = -d\beta_{V_i} d\mu_i^V = \frac{\partial^2 \phi(\beta_V)}{\partial \beta_{V_i} \partial \beta_{V_j}} d\beta_{V_i} d\beta_{V_j},$$

where the new Massieu function $\phi$ is the Legendre transform of the entropy w.r.t. the variables $\{\beta_{V_i}\}$:

$$\phi(\beta_V) = S_V - \mu_i^V \beta_{V_i}.$$  

Equation (4.16) gives the expression for the metric in the new coordinates:

$$\tilde{g}_{ij}(\beta_V) = \frac{\partial^2 \phi(\beta_V)}{\partial \beta_{V_i} \partial \beta_{V_j}}.$$  

Expressions like (4.13) and (4.14) regarding fluctuations hold in any chosen coordination of the state space.

$^{17}$Our use of upper and lower indices could be confusing here. Notice however that a change of variables to the $\{\beta_i\}$ makes them become the new genuine contravariant coordinates. Expression (4.17) gives therefore the covariant components of the metric in the new coordinates.
**Thermodynamic Curvature**

A very special quantity in Ruppeiner theory is the curvature computed from the thermodynamic metric. For a two dimensional system \((n-1=2)\), as will be later our case of interest, the curvature scalar contains all nontrivial information about the metric itself (in higher dimensional cases one may need other components of the curvature tensor).

We will not review here the derivations leading to the meaning of the thermodynamic curvature scalar. We will simply recall its main properties, referring the interested reader to the Ref. [27]. First, computation of the scalar curvature for known statistical models provides indication that it is a measure of interactions. For example, it is zero for the ideal gas, but the thermodynamic metric turns out to be curved in the Van der Waals case. But, most importantly for our purposes, the curvature diverges at a critical point, and it does so in the same way as the correlation volume \(\xi^d\), i.e.

\[
R_{\text{crit}} \sim \xi^d, \quad (4.19)
\]

where \(\xi\) is the correlation length and \(d\) the effective spatial dimension of the system. The reason for this to be so can be put a bit more precisely: calculations show [27] that the scalar curvature measures the smallest volume where one can describe the considered subsystem as surrounded by a uniform environment. At criticality, one expects such a volume to scale as \(\xi^d\).

**Applicability to Black Holes**

So far we have sketched the main features of Ruppeiner formalism as they can be motivated from ordinary, extensive thermodynamics. We deduced the notion of a metric in the thermodynamic state space by considering a subsystem of a much bigger extensive system in the microcanonical ensemble. Such a total system effectively acts as a reservoir for the subsystem, of fixed volume \(V\). A careful derivation of the theory just presented requires extensivity of the entropy function. We have in fact invoked extensivity to arrive to the first place where the thermodynamic metric appears, Eq. (4.11). How can all this be conceivably applied to non-extensive systems such as black holes?

We will try to argue now that this is in fact the case. All considerations exposed so far come from the Gaussian expansion (4.11). In statistical mechanics, extensivity is required to infer the fundamental relation \(S_V\) from the knowledge of \(S_{\text{tot}}\), which is what is usually known. It is also needed to “split” the degrees of freedom of the subsystem from those of the reservoir. But note that, in practice, this splitting only serves to fix the equilibrium values of the subsystem. On the other hand, since we are fixing one scale, the function \(S_V\) will not be an homogeneous function of its variables. In practice, what this means is that we will be working with an entropy function \(S_V\) which is not extensive within the subsystem. The latter and its fluctuations are the object of physical interest, and the formalism described above does not meet in principle any difficulty as long as: 1) the (non-homogeneous) function \(S_V\) is known, independently of its relation to \(S_{\text{tot}}\); and 2) we have an expression for the relevant off-equilibrium variations of the subsystem around its (known)
equilibrium state. This is precisely the situation we found for black hole near turning points and vertical asymptotes, which therefore have to be identified with “Ruppeiner’s subsystem”.

In Ruppeiner theory, general off-equilibrium states of the subsystem are described exactly in Eq. (4.11) by the values of the \( \{ \mu_i^v \} \). That is, all possible off-equilibrium states of the subsystem are supposed to be well described by the thermodynamic limit. In the the BH case, however, we do not have in general any control on off-equilibrium fluctuations, since they are given by the completely unknown function \( \hat{S}(X^P; \mu^i) \). The analog of the Ruppeiner metric (i.e. the coefficients of the quadratic expansion of the entropy) will be given, in a general case, by the many-dimensional metric with elements:

\[
g_{ij} = -\left( \partial_{ij} \hat{S} \right)_{eq}, \quad g_{i\rho} = -\left( \partial_{i\rho} \hat{S} \right)_{eq}, \quad g_{\rho\rho} = -\lambda_\rho. \tag{4.20}
\]

However, near a turning point or a vertical asymptote, the expressions (4.2) and (4.8) provide a good description of off-equilibrium fluctuations. Moreover, the quadratic approximation to the entropy is computable, and the “effective” elements of the Ruppeiner metric are just a few and can be computed from the equilibrium Hessian. In fact, one only has to use the prescription (4.14) for the inverse metric as read from equations (4.6) and (4.9).

### 4.3 Thermodynamic Curvature for the BH and BR Spacetimes

As explained above, the elements of the Ruppeiner metric can be computed from the expressions for quadratic fluctuations. Near \( x_{\text{min}} \) and \( x = 1 \) we will write, according to (4.6) and (4.9)\(^18\):

\[
g^{ij} = \langle \delta \hat{S}_i \delta \hat{S}_j \rangle = \pm H_{ij}, \tag{4.21}
\]

where the plus sign stands for the LBR at \( x_{\text{min}} \) and the minus sign for the SBR and BH at \( x = 1 \). Let us compute now the curvature scalar from these metric elements. For convenience, we give below the expressions as computed from \( g^{ij} = +H_{ij} \), and we will provide their values for all \( x \). The physically meaningful limits are however those at \( x_{\text{min}} \) and \( x = 1 \) with the appropriate sign. The results for the rotating black hole and black rings are, respectively:

\[
\begin{align*}
R_{\text{BH}} &= \frac{2}{\pi^2 \mu^{3/2}} \frac{1}{\sqrt{1 - x^2}}, \\
R_{\text{BR}} &= \frac{2\sqrt{2}}{\pi^2 \mu^{3/2}} \frac{\sqrt{\nu(\nu^2 + 2\nu - 2)}}{(2\nu - 1)^2 \sqrt{1 - \nu}}.
\end{align*}
\tag{4.22}
\]

These quantities are plotted in Fig. 7. We see that divergences of the curvature (i.e. indicators of critical behaviour according to Ruppeiner theory) are found at \( x_{\text{min}} \) and \( x = 1 \), in agreement with our previous analysis on stability and fluctuations.

\(^{18}\)For simplicity, we are neglecting the subleading contribution \( \beta_i \beta_j \) in (4.9). It can be checked that this does not change the power behaviour of the divergence of the thermodynamic curvature at extremality, which is what we will be interested in.
5 Scaling Laws in the BH/BR System

Ruppeiner analysis confirms that the points $x_{\text{min}}$ and $x = 1$ are candidates of critical points. However, divergences in the thermodynamic curvature have been found to happen also when the system enters into an unphysical region. In [49], for instance, computation of the thermodynamic scalar for the Van der Waals model showed that it diverges also at the so-called spinoidal curve. In our case, both at $x_{\text{min}}$ and at $x = 1$ the BH/BR system enters in an unphysical region (namely, naked singularities). One can also argue that the origin of these divergences could just be the divergent nature of fluctuations at those points (which, as discussed in Section 4.1, do not necessarily mean critical behaviour). In order to clarify this issue, we examine next the scaling relations between the different critical exponents both at $x_{\text{min}}$ and $x = 1$.

5.1 Critical Exponents and Order Parameter

We shall first define the appropriate susceptibilities and critical exponents that are suited to the microcanonical ensemble. Following [18] we define them as follows:

$$
\chi_J \equiv \frac{\partial^2 S}{\partial M^2} \sim \epsilon_{M}^{-\alpha}, \epsilon_{J}^{-\phi}. \\
\chi_M \equiv \frac{\partial^2 S}{\partial J^2} \sim \epsilon_{M}^{-\gamma}, \epsilon_{J}^{-\sigma}. 
$$

These are the natural generalizations of the expressions familiar from other ensembles (for example, in the grand canonical ensemble the thermodynamic potential is the Gibbs free energy, and the specific heat is given by its second derivative with respect to the temperature, etc). Also, in the expressions above the $\epsilon$ parameters are the suitable generalizations of what e.g. would be the “reduced temperature” $(T - T_{\text{crit}})/T_{\text{crit}}$ in the usual treatments (i.e. (grand)canonical ensembles) of more familiar examples like liquid/gas transitions, etc.
We want to study the scaling relations at the two points of the system where fluctuations diverge, i.e. at \( x_{\min} \) and at extremality, \( x = 1 \). At each of these, we define the parameters \( \epsilon_M \) and \( \epsilon_J \) to be:

At \( x_{\min} \):
\[
\begin{align*}
\epsilon_M &= \frac{M_{\min} - M}{M_{\min}}, \\
\epsilon_J &= \frac{J - J_{\min}}{J_{\min}},
\end{align*}
\]

At \( x = 1 \):
\[
\begin{align*}
\epsilon_M &= \frac{M - M_{\text{ext}}}{M_{\text{ext}}}, \\
\epsilon_J &= \frac{J_{\text{ext}} - J}{J_{\text{ext}}},
\end{align*}
\]

(5.2)

where
\[
\begin{align*}
M_{\min} &= \left( \frac{\pi J^2}{G} \right)^{1/3}, \\
J_{\min} &= \sqrt{\frac{GM^3}{\pi}}, \\
M_{\text{ext}} &= \left( \frac{27\pi J^2}{32G} \right)^{1/3}, \\
J_{\text{ext}} &= \sqrt{\frac{32GM^3}{27\pi}}.
\end{align*}
\]

(5.3)

Other critical exponents tell us about the behavior of the order parameter of the transition considered. In general, there is no obvious choice for an order parameter. In [18] it was proposed to choose, as an order parameter for BH phase transitions, the difference of the “intensive” (conjugate) variables between the two phases. In our case this means:

\[
\begin{align*}
\text{At } x = x_{\min}: \quad \eta &\equiv \omega_{\text{SBR}} - \omega_{\text{LBR}}, \\
\text{At } x = 1: \quad \eta &\equiv \omega_{\text{SBR}} - \omega_{\text{BH}}.
\end{align*}
\]

(5.4)

These parameters go to zero at the (presumed) transition point, as they should. For these order parameters, the natural generalizations of the critical exponents \( \beta \) and \( \delta \) familiar from statistical mechanics is given by:

\[
\eta \sim \epsilon_M^\beta \sim \epsilon_J^{1/\delta}.
\]

(5.5)

With these definitions, from equations (3.18) and (3.19) the critical exponents are readily computed to be:

At \( x = x_{\min} \):
\[
\begin{align*}
\text{Small Black Ring}: \quad \alpha &= \varphi = \gamma = \sigma = 1/2, \\
\text{Large Black Ring}: \quad \alpha &= \varphi = \gamma = \sigma = 1/2, \\
\beta &= 1/\delta = 1/2.
\end{align*}
\]

(5.6)

At \( x = 1 \):
\[
\begin{align*}
\text{Black Hole}: \quad \alpha &= \varphi = \gamma = \sigma = 3/2, \\
\text{Small Black Ring}: \quad \alpha &= \varphi = \gamma = \sigma = 3/2, \\
\beta &= 1/\delta = -1/2.
\end{align*}
\]

Incidentally, let us notice that these critical exponents at extremality precisely coincide with those found in [17, 18], where four dimensional black holes were considered. In
that case, the two proposed phases were the inner and outer horizons, the black hole solution being thought then as a “coexistence curve”. In the five dimensional case we are considering, our candidates for the two different phases are instead the black ring and the black hole.

5.2 Scaling Laws and Correlation Length in the BH/BR System

With these critical exponents at our disposal, let us check now if the scaling laws are obeyed. First of all, note that a nontrivial result is the equality of the critical exponents $\alpha, \varphi, \gamma$ and $\sigma$ at both phases of each transition point considered (something which, in principle, need not be so). As for the scaling relations involving them and the other critical exponents defined so far, we quote here their expressions. They are given by \[47\]:

\[
\begin{align*}
\alpha + 2\beta + \gamma &= 2, \\
\beta(\delta - 1) &= \gamma, \\
\varphi(\beta + \gamma) &= \alpha.
\end{align*}
\] (5.7)

We can verify that these scalings are obeyed by the BH/BR system, both at $x = x_{\text{min}}$ and at $x = 1$.

However, according to the general theory of critical phenomena, further scaling laws involving additional critical exponents have to be satisfied near a critical point \[47\]. The remaining critical exponents are those related to the behaviour of the two-point correlation function and the correlation length. The two-point correlation function $G(r)$ introduces a new critical exponent $\zeta$ defined by:

\[
G(r) \sim \frac{\exp(-r/\xi)}{r^{d-2-\zeta}}
\] (5.8)

($d$ is again the spatial dimension of the system). On the other hand, the divergence of the correlation length $\xi$ at the critical point is expressed in terms of the critical exponents $\mu$ and $\nu$ as:

\[
\xi \sim \epsilon_M^{\nu} \sim \epsilon_J^\mu.
\] (5.9)

At a critical point these exponents obey the scaling laws given by \[47\]:

\[
\begin{align*}
\nu(2 - \zeta) &= \gamma, \\
2 - \alpha &= \nu d, \\
\mu(\beta + \gamma) &= \nu.
\end{align*}
\] (5.10)

To verify these relations we need to compute the correlation length and the two-point correlator. However, there is no obvious way (not even an obvious definition) to compute these quantities for a black hole geometry. In any case, Eq. \[41.19\] provides us with an
explicit algorithm to compute the quantities $\nu d$ and $\mu d$ if applicability of Ruppeiner theory is assumed. One obtains:

\[
\begin{align*}
\text{At } x_{\text{min}}: & \quad \nu d = \mu d = 1, \\
\text{At } x = 1: & \quad \nu d = \mu d = 1/2.
\end{align*}
\]

(5.11)

Relations (5.10) are obeyed at extremality for any effective dimension $d$, but are not satisfied at $x_{\text{min}}$.

We therefore see that $x = 1$ satisfies additional scaling laws not obeyed at $x_{\text{min}}$. In this respect, we can conclude that the behaviour of the BH/SBR system near extremality enjoys properties which, formally, are analogous to those of a critical point of a second order phase transition in which the SBR is always the “disfavoured” or metastable phase. As explained at the end of Section 4.1, this statement has to be understood as describing the properties of the system only along the stable directions in configuration space.

On the other hand, at $x_{\text{min}}$ we do not find an analogous behaviour of the large and small ring phases. In spite of the divergent behaviour of fluctuations, we see that the scalings related to the critical exponents of the correlation length are not satisfied there. This is in agreement with our expectations about the origin of divergent fluctuations at $x_{\text{min}}$ where, as we saw, what we have is a change of stability. As discussed in Section 4.1, this phenomenon is very different from that of a critical point.

### 6 Conclusions and Outlook

We have addressed the issues of stability, divergence of fluctuations and critical phenomena in the BH/BR system. Our results give a consistent and unified picture of the properties of the phase diagram of black holes and black rings. The two main tools that we have used are the Poincaré method of stability and the geometry of the thermodynamic state space. From a quite general point of view, we have put special emphasis on the consequences of the non-extensivity of BH thermodynamics, and we argued on the suitability of the methods used here for the study of BHs. The main difference between the approach followed in this paper and others in the BH literature is that ours is based crucially on the study of the off-equilibrium function $\tilde{S}(X^p; \mu^i)$, and not on the properties of the equilibrium entropy $S_{\text{eq}}(\mu^i)$. The main advantage of the Poincaré method is precisely that, just by plotting the right phase diagrams of an equilibrium sequence, it can be used to infer some relevant properties about $\tilde{S}(X^p; \mu^i)$, even if such a function is completely unknown.

Let us he summarize the main results concerning the BH/BR system. First, the small black ring has been shown to be locally unstable. Concerning the large black ring, we found strong evidence that this is a more stable configuration, i.e. one with less unstable modes. It is nevertheless true that this does not imply its full dynamical stability and, in fact, one expects such a configuration to be unstable. The only place in the phase diagram of Fig 1 where we see a change of stability is at $x_{\text{min}}$, where both BR branches meet.
We are able to evaluate thermodynamic fluctuations and compute the second moments of correlations near $x_{\text{min}}$ and $x = 1$. In this respect, we have discussed in detail the differences between turning points and vertical asymptotes. We saw in Section 4.1 that, if one considers fluctuations only along the stable directions, near a turning point the fluctuations of the conjugate variables are dominated by the less stable mode $X^1$, i.e.

$$\delta \beta \sim (\partial_1 \beta_i)_{\text{eq}} \Delta X^1,$$

due to the fact that $\Delta X^1$ is large. On the contrary, near a vertical asymptote the fluctuations $\delta \beta_i$ are dominated by those in the usual thermodynamic variables, i.e.

$$\delta \beta \sim (\partial_j \beta_i)_{\text{eq}} \Delta \mu^j,$$

due to the fact that $(\partial_j \beta_i)$ is divergent. This is likely to be a more general phenomenon.

Finally, the computation of the various critical exponents allows to formally interpret $x = 1$ as a critical point along the stable directions in configuration space, while at $x_{\text{min}}$ the divergence of fluctuations seems to be related only to a change of stability. This fits with our expectations from the stability diagrams in Fig. 6 since at a turning point we only expect a change in stability. Evidence of a critical point at a vertical asymptote has also been found in [17, 18].

The extremal configurations at $x = 1$ are naked singularities. However, the general properties of a system at criticality concern its behaviour near the critical point, and our configurations are nonsingular there. A critical point indicates the endpoint of a “coexistence curve”, along which a first order phase transition between the two competing phases can take place. It is true, however, that in the BH/SBR system such a first order phase transition would never be physically relevant in the dynamics of these spacetimes: not only because the SBR is locally unstable, but also because this phase always has lower entropy than the BH. Also, we do not know about any phases beyond the “critical point”. In any case, the fact that the scaling relations are satisfied looks nontrivial. It would be interesting to understand better the physical relevance of this property.

On the other hand, the absence of any other points of change of stability (in addition to the general arguments to rule out stability considerations based on the sign of the specific heat) makes it very unlikely to interpret the change of sign in the specific heat in the BH branch as a phase transition. Moreover, at $x = 1/2$ it is not clear at all between which two phases there could be a transition.

Finally let us comment on the other special point in the phase diagram of Fig. 11, namely, $x = 2\sqrt{2}/3$. Our results do not indicate so far the existence of any special properties there, neither in the BH nor in the LBR branch. Given this fact, we therefore conclude that if both branches are stable around this point, the only possibility we are left with is that of a first order phase transition between both. We find no indications at all to interpret $x = 2\sqrt{2}/3$ as a critical point of a second order phase transition between the BH and the LBR.
From the point of view of the applicability of the techniques used in this paper, it
would be nice to understand in a more precise way the relation between the off-equilibrium
fluctuations \( \{X^\rho \} \) considered here and the dynamical modes \( \{\delta g_{\mu\nu} \} \) of metric perturbations.
Given the simplicity of the turning point method, it would be very interesting to find an
explicit and precise connection, and see which dynamical modes these conjugacy diagrams
are actually probing. Even if, in general, one cannot expect to recover full information on
dynamical stability from the Poincaré method, it seems clear that this method looks reliable
at least to detect instabilities. By this we mean that Poincaré instability (a negative slope
near a turning point) necessarily implies the existence of at least one unstable dynamical
mode (while the opposite is not necessarily true).

In this respect, a natural question to ask is about the relation (if any) of the Poincaré
method used here and the Gubser-Mitra conjecture \(^{19}\). The latter relates, for black
brane solutions, standard local thermodynamical instability to the presence of dynamical
instabilities in the same sense as discussed here\(^{19}\). We have argued against the applicability
to BHs of the criterion of stability based on the sign of the various specific heats. Note,
however, that the GM conjecture does not include compact horizons, just black branes
with some noncompact translational symmetry. This means that, along the noncompact
directions, one has extensivity of the mass per unit length and additivity of the entropy
per unit length. Therefore standard thermodynamic criteria should apply in these cases,
at least for perturbations generating lumps along the uniform directions. Note that these
kind of GL instabilities are in perfect qualitative agreement with the simple example that
we discussed in the introduction of Section \(^3\) (a negative Hessian means that system is
unstable against redistribution of mass and tends to become non-uniform). It would be
extremely interesting to explore this issue in more detail and see, in particular, if the
appearance of this kind of instabilities is related to bifurcations of equilibria.

Concerning the use of the thermodynamic metric and its relation to BH physics, it
would be desirable to have a precise definition and understanding of the correlation length
for a BH geometry. What we have used here is the divergence of the thermodynamic
curvature. This is probably measuring the typical size of fluctuations in the BH geometry
as compared to the size of the BH itself. It would be nice to work this out in an explicit way.

The techniques we have been using are completely general, and obvious extensions
of this work include their application to other gravitational phase transitions of interest.
Work on the directions proposed here is in progress.

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\(^{19}\)That is, the existence of a classical instability under Lorentzian time evolution.
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A Fluctuations Near Extremality

Let us sketch here the derivation of equation (4.9) for fluctuations near extremaility. In ordinary systems with positive specific heat, the probability distribution just reduces to an ordinary Gaussian integration due to the fact that the eigenvalues of the Hessian matrix are all negative. Here, however, we always have one positive eigenvalue, which makes the normalization constant of the probability distribution strictly infinite if one is to integrate fluctuations over the whole real line. Therefore, in evaluating the expressions (4.9), we need to impose a cutoff in phase space, i.e.

\[
\begin{align*}
\langle \delta \hat{\beta}_i \delta \hat{\beta}_j \rangle (\Lambda_M, \Lambda_J) &= \frac{1}{N(\Lambda_M, \Lambda_J)} \int_{-\Lambda_M}^{\Lambda_M} d(\Delta M) \int_{-\Lambda_J}^{\Lambda_J} d(\Delta J) \delta \hat{\beta}_i \delta \hat{\beta}_j \exp(\hat{S}), \\
N(\Lambda_M, \Lambda_J) &= \int_{-\Lambda_M}^{\Lambda_M} d(\Delta M) \int_{-\Lambda_J}^{\Lambda_J} d(\Delta J) \exp(\hat{S}),
\end{align*}
\]

where \(\hat{S}\) stands for the quadratic approximation to the entropy near the vertical asymptote, i.e. Eq. (4.8), and the variations \(\delta \hat{\beta}_i\) are given by Eq. (4.7).

Before proceeding, let us note that a cutoff is always necessary: if one is to make sense of any truncated series expansion in \(\Delta \mu^i\) of the entropy or \(\delta \hat{\beta}_i\), then the variations \(\Delta \mu^i\) cannot be arbitrarily large. This is also the case when one has convergent Gaussian integrals. However one typically has highly peaked distributions, and thus any physical cutoff is naturally larger than the width of the Gaussian. Therefore a good approximation which yields a simple analytic result is given by integrating over the whole real line, which is what we always do. In our case, the variations in \(\Delta M\) and \(\Delta J\) have to be smaller than a certain quantity \(\Lambda_{M,J}\) such that: a) they are much smaller than the total mass and angular momentum, b) they are small enough to be considered independent (since for large values they would be related the extremality bound \(x < 1\)), and c) the truncated series expansions (4.7) and (4.8) remain good approximations. The latter condition can be used to give a simple estimation of the cutoff near extremality, namely:

\[
\Lambda_{M,J} \sim \mathcal{O}(\epsilon),
\]

for \(\epsilon = 1 - x\). This is obtained by computing the third derivatives of the equilibrium entropy, which behave as \(\sim \epsilon^{-5/2}\) near the vertical asymptote. These give a simple estimation of
the order of magnitude of the next contribution in the expansion of both $\hat{S}$ and $\delta \hat{\beta}$. Making them negligible w.r.t. the second order contribution (see Eqs. (3.18) and (3.20)) yields (A.2).

In our case we always have one positive and one negative eigenvalue. Now, depending on which quantity we want to compute, we can always arrange the integrand in such a way that the contribution arising from the positive (“divergent”) mode cancels out in the final result for any choice of the cutoff $\Lambda_{M,J}$. As a matter of fact, one can check that:

$$\langle \delta \hat{\beta} \delta \hat{\beta} \rangle (\Lambda_M = \infty, \Lambda_J) = -H_{MM} + \beta^2, \quad (A.3)$$

$$\langle \delta \hat{\omega} \delta \hat{\omega} \rangle (\Lambda_M = \infty, \Lambda_J = \infty) = -H_{JJ} + \omega^2.$$  

This results are exact and do not require any explicit form of the coefficients in the expansion of the entropy, only that $H_{MM} < 0$, $H_{JJ} < 0$ and $\det H < 0$, as in our case. The final dependence in the cutoff that we retain finite in each case (which would give a divergence if taken to be infinite) exactly cancels in the numerator and the denominator of the general expression (A.1). To get this result we are approximating, in each case, the integral over the negative (“convergent”) mode with a finite cutoff with the integral over the whole real line. To justify such an approximation, let us note that the width $\sigma_i$ of the Gaussian integrals is given by ($i = M, J$)

$$\sigma_i \sim 1/\sqrt{-H_{ii}} \sim O(\epsilon^{3/4}) \quad (A.4)$$

near extremality (see Eqs. (3.18) and (3.20)). This is always smaller than the cutoff (A.2).

Finally, the easiest way to find a simple approximation for the mixed correlations $\langle \delta \hat{\beta} \delta \hat{\omega} \rangle$ is the following. One can check that:

$$\langle (\beta + \delta \hat{\beta}) \delta \hat{\omega} \rangle (\Lambda_M = \infty, \Lambda_J) = -H_{MJ}, \quad (A.5)$$

while, on the other hand:

$$\langle \delta \hat{\omega} \rangle (\Lambda_M = \infty, \Lambda_J) = -\omega. \quad (A.6)$$

Since $\langle (\beta + \delta \hat{\beta}) \delta \hat{\omega} \rangle = \beta \langle \delta \hat{\omega} \rangle + \langle \delta \hat{\beta} \delta \hat{\omega} \rangle$, one gets:

$$\langle \delta \hat{\beta} \delta \hat{\omega} \rangle = -H_{MJ} + \beta \omega. \quad (A.7)$$

Equations (A.3) and (A.7) complete therefore the result (4.9) anticipated in Section 4.

Of course, the calculation above can also be carried out by using the symmetric expressions:

$$\langle (\omega + \delta \hat{\omega}) \delta \hat{\beta} \rangle (\Lambda_M = \infty, \Lambda_J) = -H_{MJ}, \quad \langle \delta \hat{\beta} \rangle (\Lambda_M = \infty, \Lambda_J) = -\beta. \quad (A.8)$$

Note the shifts in the mean fluctuations, $\langle \delta \hat{\beta} \rangle = -\beta$ and $\langle \delta \hat{\omega} \rangle = -\omega$, which are due to the nonvanishing linear terms in the expansion of the entropy.
References

[1] A. Strominger, “Massless black holes and conifolds in string theory,” Nucl. Phys. B 451 (1995) 96 [arXiv:hep-th/9504090].

[2] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [arXiv:hep-th/9803131].

[3] S. W. Hawking and D. N. Page, “Thermodynamics Of Black Holes In Anti-De Sitter Space,” Commun. Math. Phys. 87, 577 (1983).

[4] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, “The Hagedorn / deconfinement phase transition in weakly coupled large N gauge theories,” arXiv:hep-th/0310285.
  O. Aharony, J. Marsano, S. Minwalla and T. Wiseman, “Black hole - black string phase transitions in thermal 1+1 dimensional supersymmetric Yang-Mills theory on a circle,” arXiv:hep-th/0406210.
  T. Harmark and N. A. Obers, “New phases of near-extremal branes on a circle,” JHEP 0409 (2004) 022 [arXiv:hep-th/0407094].

[5] P. Kraus, H. Ooguri and S. Shenker, “Inside the horizon with AdS/CFT,” Phys. Rev. D 67, 124022 (2003) [arXiv:hep-th/0212277].
  L. Fidkowski, V. Hubeny, M. Kleban and S. Shenker, “The black hole singularity in AdS/CFT,” JHEP 0402, 014 (2004) [arXiv:hep-th/0306170].

[6] R. Gregory and R. Laflamme, “Black Strings And P-Branes Are Unstable,” Phys. Rev. Lett. 70 (1993) 2837 [arXiv:hep-th/9301052].

[7] B. Kol, “Topology change in general relativity and the black-hole black-string transition,” [arXiv:hep-th/0206220]
  T. Wiseman, “From black strings to black holes,” Class. Quant. Grav. 20 (2003) 1177 [arXiv:hep-th/0211028].
  T. Harmark and N. A. Obers, “Phase structure of black holes and strings on cylinders,” Nucl. Phys. B 684 (2004) 183 [arXiv:hep-th/0309230].
  E. Sorkin, “A critical dimension in the black-string phase transition,” Phys. Rev. Lett. 93, 031601 (2004) [arXiv:hep-th/0402216].
  M. I. Park, “The final state of black strings and p-branes, and the Gregory-Laflamme instability,” [arXiv:hep-th/0405045].
  B. Kol and E. Sorkin, “On black-brane instability in an arbitrary dimension,” [arXiv:gr-qc/0407058].

[8] B. Kol, “The phase transition between caged black holes and black strings: A review,” [arXiv:hep-th/0411240].

[9] R. Emparan and R. C. Myers, “Instability of ultra-spinning black holes,” JHEP 0309 (2003) 025 [arXiv:hep-th/0308056].
V. Cardoso and J. P. S. Lemos, “New instability for rotating black branes and strings,” arXiv:hep-th/0412078.

V. Cardoso, G. Siopsis and S. Yoshida, “Scalar perturbations of higher dimensional rotating and ultra-spinning black holes,” Phys. Rev. D 71 (2005) 024019 arXiv:hep-th/0412138.

[10] P. C. W. Davies, “Thermodynamics Of Black Holes,” Proc. Roy. Soc. Lond. A 353 (1977) 499.

[11] C. O. Lousto, “Some thermodynamic aspects of black holes and singularities,” Int. J. Mod. Phys. D6 (1997) 575. arXiv:gr-qc/9601006.

[12] A. Curir, “Rotating black holes as dissipative spin-thermodynamical systems,” Gen. Rel. Grav. 13 (1981) 417.
A. Curir, “Black hole emissions and phase transitions,” Gen. Rel. Grav. 13 (1981) 1177.

[13] R. D. Sorkin, “A Stability Criterion For Many Parameter Equilibrium Families,” Astrophys. J. 257 (1982) 847.

[14] D. Pavon and J. M. Rubi, “Nonequilibrium Thermodynamic Fluctuations Of Black Holes,” Phys. Rev. D37 (1988) 2052.
D. Pavon, “Phase transition in Reissner-Nordstrom black holes,” Phys. Rev. D43 (1991) 2495.

[15] O. Kaburaki, I. Okamoto and J. Katz, “Thermodynamic stability of Kerr black holes”, Phys. Rev. D47 (1993) 2234.

[16] J. Katz, I. Okamoto and O. Kaburaki, “Thermodynamic stability of pure black holes,” Class. Quant. Grav. 10, 1323 (1993).

[17] O. Kaburaki, “Critical Behavior of Extremal Kerr-Newman Black Holes,” Gen. Rel. Grav. 28 (1996) 843.

[18] O. Kaburaki, “Scaling laws at the critical point on black hole equilibrium series,” Phys. Lett. A217 (1996) 315.

[19] S. S. Gubser and I. Mitra, “The evolution of unstable black holes in anti-de Sitter space,” JHEP 0108 (2001) 018 arXiv:hep-th/0011127.
H. S. Reall, “Classical and thermodynamic stability of black branes,” Phys. Rev. D 64 (2001) 044005 arXiv:hep-th/0104071.

[20] D. Marolf and B. C. Palmer, “Gyrating strings: A new instability of black strings?,” arXiv:hep-th/0404139.
P. Bostock and S. F. Ross, “Smeared branes and the Gubser-Mitra conjecture,” arXiv:hep-th/0405026.
[21] R. Emparan and H. S. Reall, “A rotating black ring in five dimensions,” Phys. Rev. Lett. 88 (2002) 101101 [arXiv:hep-th/0110260].

[22] R. C. Myers and M. J. Perry, “Black Holes In Higher Dimensional Space-Times,” Annals Phys. 172 (1986) 304.

[23] R. Emparan, “Rotating circular strings, and infinite non-uniqueness of black rings,” JHEP 0403 (2004) 064. arXiv:hep-th/0402149.

[24] G. T. Horowitz and K. Maeda, “Fate of the black string instability,” Phys. Rev. Lett. 87 (2001) 131301. arXiv:hep-th/0105111.

[25] H. Poincaré, 1885. Acta. Math. 7, 259.

[26] O. Kaburaki, “Should the entropy be concave?,” Phys. Lett. A185 (1994) 21.

[27] G. Ruppeiner, “Riemannian geometry in thermodynamic fluctuation theory”, Rev. Mod. Phys. 67 (1995) 605.

[28] H. Elvang and R. Emparan, “Black rings, supertubes, and a stringy resolution of black hole non-uniqueness,” JHEP 0311 (2003) 035. arXiv:hep-th/0310008.

[29] H. B. Callen, “Thermodynamics and an introduction to thermostatics”, 2nd. edition, John Wiley & Sons, 1985.

[30] P. T.Landsberg, “Thermodynamics and statistical mechanis,” Dover, New York, 1990.

[31] J. Katz, “On the number of unstable modes of an equilibrium,” Mon. Not. R. Astr. Soc. 183 (1978) 765. J. Katz, “On the number of unstable modes of an equilibrium — II,” Mon. Not. R. Astr. Soc. 189 (1979) 817.

[32] R. Sorkin, “A Criterion For The Onset Of Instability At A Turning Point,” Astrophys. J. 249 (1981) 254.

[33] V. I. Arnold, “Catastrophe Theory,” Springer-Verlag, 1986.

[34] G. Horwitz and J. Katz, “Steepest-descent technique and stellar equilibrium statistical mechanics I. Newtonian clusters in a box,” Astrophys. J. 211 (1977) 226.

[35] J. Katz and I. Okamoto, “Fluctuations in Isothermal Spheres,” Mon. Not. Roy. Astron. Soc. 317 (2000) 163. arXiv:astro-ph/0004179.

[36] I. Okamoto, J. Katz and R. Parentani, “A Comment on fluctuations and stability limits with application to 'superheated' black holes,” Class. Quant. Grav. 12 (1995) 443 arXiv:gr-qc/9412038.
[37] G. Iooss and D. Joseph, “Elementary Stability and Bifurcation Theory,” 2nd. edition, Springer-Verlag, 1990.

[38] B. S. Kay and R. M. Wald, “Linear Stability Of Schwarzschild Under Perturbations Which Are Nonvanishing On The Bifurcation Two Sphere,” Class. Quant. Grav. 4 (1987) 893.

[39] S. A. Teukolsky, “Perturbations Of A Rotating Black Hole 1. Fundamental Equations For Gravitational Electromagnetic, And Neutrino Field Perturbations,” Astrophys. J. 185 (1973) 635.
S. A. Teukolsky, “Perturbations Of A Rotating Black Hole 2. Dynamical Stability Of The Kerr Metric.” Astrophys. J. 185 (1973) 635.

[40] R. Parentani, J. Katz and I. Okamoto, “Thermodynamics of a black hole in a cavity,” Class. Quant. Grav. 12, 1663 (1995) [arXiv:gr-qc/9410015].

[41] R. Parentani, “The inequivalence of thermodynamic ensembles,” [arXiv:gr-qc/9410017]

[42] R.D. Sorkin, R. M. Wald and Z. J. Zhang, “Entropy of self-gravitating radiation”, Gen. Rel. Grav. 13 (1981) 1127.

[43] J. L. Friedman, J. R. Ipser and R. D. Sorkin, “Turning Point Method For Axisymmetric Stability Of Rotating Relativistic Stars,” Astrophys. J. 325 (1988) 722.

[44] V. Cardoso, O. J. C. Dias and J. P. S. Lemos, “Gravitational radiation in D-dimensional spacetimes,” Phys. Rev. D 67 (2003) 064026 [arXiv:hep-th/0212168].
A. Ishibashi and H. Kodama, “Stability of higher-dimensional Schwarzschild black holes,” Prog. Theor. Phys. 110 (2003) 901 [arXiv:hep-th/0305185].

[45] R. Emparan and H. Reall, “The End of Black Hole Uniqueness,” Gen. Rel. Grav. 34 (2002) 2057.

[46] S. S. Gubser, “On non-uniform black branes,” Class. Quant. Grav. 19 (2002) 4825. [arXiv:hep-th/0110193].

[47] L. D. Landau and E. M. Lifshitz, “Statistical physics,” Pergamon, 1980.

[48] S. Ferrara, G. W. Gibbons and R. Kallosh, “Black holes and critical points in moduli space,” Nucl. Phys. B500 (1997) 75. [arXiv:hep-th/9702103]. J. Aman, I. Bengtsson and N. Pidokrajt, “Geometry of black hole thermodynamics,” Gen. Rel. Grav. 35 (2003) 1733. [arXiv:gr-qc/0304015].

[49] D. C. Brody and A. Ritz, “Geometric Phase Transitions,” [arXiv:cond-mat/9903168]