Optimal structure-preserving model order reduction of second-order systems by iterative rational Krylov algorithm

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Abstract

In this paper, we focus on the efficient techniques to estimate the $\mathcal{H}_2$ optimal Structure-Preserving Model Order Reduction (SPMOR) of the second-order systems using the Iterative Rational Krylov Algorithm (IRKA). In general, the classical IRKA can be applied to the second-order system by converting it into an equivalent first-order form and get the reduced model in a first-order form. In this case, the reduced model can not preserve the structure of the second-order system which is however necessary for further manipulation. Here we develop IRKA based algorithms that enable us to generate approximate reduced second-order systems without explicitly converting the systems into a first-order form.

On the other hand, there are very challenging tasks to find reduced-order form of a large-scale system with a minimized $\mathcal{H}_2$ error norm and attain the rapid rate of convergence. To overcome these problems, this paper discusses competent techniques to determine the optimal $\mathcal{H}_2$ error norm efficiently for the second-order system. The efficiency and applicability of the proposed techniques are validated by applying them to several large-scale dynamical systems. The computation is done numerically and the achieved results are discussed in both tabular and graphical approaches.

keywords: Interpolatory projection, Krylov subspace, second-order dynamical systems, SPMOR, $\mathcal{H}_2$ error norm

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1 Introduction

We consider a second-order linear time-invariant continuous-time system of the form:

\[ M\ddot{\xi}(t) + D\dot{\xi}(t) + K\xi(t) = Hu(t), \]
\[ y(t) = L\xi(t), \]

where \( M, D, K \in \mathbb{R}^{n \times n} \) are time-invariant matrices, \( H \in \mathbb{R}^{n \times p} \) is the input matrix describing the external access to the system and \( L \in \mathbb{R}^{m \times n} \) represents the output of the measurement of the system. If \( M = I \), then the system is called a standard state-space system, or if \( M \) is invertible then the system can also be converted into a standard state-space system. The dimension of the system is \( n \) while \( \xi(t) \in \mathbb{R}^n \) is the vector of states, \( u(t) \in \mathbb{R}^p \) is the vector of control input and \( y(t) \in \mathbb{R}^m \) is the measured outputs of the system. The input and output of the system are defined in continuous-time over the interval \([0, \infty)\) and thus the system is known as a continuous-time system.

If \( p = m = 1 \), the system is called Single-Input Single-Output (SISO) system, otherwise it is called Multi-Input Multi-Output (MIMO) system. In the MIMO system we consider \( p, m \ll n \), i.e., the number of input and output of the system is much less than the number of states. We assume that the system (1) is asymptotically stable, i.e., all the finite-eigenvalues of the matrix pencil of the system lie in the left half-plane (\( \mathbb{C}^- \)). Such structured dynamical systems arise in different disciplines of science and engineering such as structural mechanics or multi-body dynamics, mechatronics, and electrical networks (see e.g., [1, 2, 3]).

Applying the Laplace-transformation in the system (1), we can find the input to output mapping which can be described by the transfer function:

\[ G(s) = L(s^2M + sD + K)^{-1}H; \quad s \in \mathbb{C}. \]  

(2)

In many engineering applications, the corresponding systems have occurred which are composed of a large number of disparate devices. Therefore, the systems become very large and complex structures. Simulation, controller design, and optimization of large-scale systems are infeasible within a reasonable computational time and the memory allocation in the computing tools. Therefore, we want to reduce the complexity of the model by applying Model Order Reduction (MOR) that well-approximates the behavior of the original model. That is the system in (1) can be replaced by the Reduced-Order Model (ROM) as

\[ \hat{M}\ddot{\hat{\xi}}(t) + \hat{D}\dot{\hat{\xi}}(t) + \hat{K}\hat{\xi}(t) = \hat{H}u(t), \]
\[ \hat{y}(t) = \hat{L}\hat{\xi}(t). \]

where \( \hat{M}, \hat{D}, \hat{K} \in \mathbb{R}^{r \times r}, \hat{H} \in \mathbb{R}^{r \times p}, \hat{L} \in \mathbb{R}^{m \times r} \) and \( r \ll n \). The transfer-function corresponding to the ROM (3) is denoted by \( \hat{G} \) and defined as

\[ \hat{G}(s) = \hat{L}(s^2\hat{M} + s\hat{D} + \hat{K})^{-1}\hat{H}; \quad s \in \mathbb{C}. \]

(4)

The ROM (3) is supposed to fulfill some of the certain approximation requirements, for instance the approximation error \( \|y(t) - \hat{y}(t)\| \) or correspondingly \( \|G(.) - \hat{G}(.)\| \) should be minimized in some suitable norm, e.g., the \( \mathcal{H}_\infty \) or \( \mathcal{H}_2 \) norms (see [4] and references therein).
The concept of projection for interpolatory model reduction was initially introduced by Skelton et al. in [5] and later by Grimme in [6] modified the approach by utilizing the rational Krylov method that introduced by Ruhe in [7]. Krylov based methods can achieve moment matching without explicitly computing moments, whereas explicit computation of moments is known to be ill-conditioned [8, 9]. These methods extremely useful for model reduction of large-scale dynamical systems. The quality of the ROM is highly dependent on the choice of the interpolation points and therefore various techniques have been developed for the selection of interpolation points [10]. Recently the issue of selecting a good set of interpolation points is linked to the problem of $\mathcal{H}_2$ optimal model reduction [11, 12]. The Iterative Rational Krylov Algorithm (IRKA) is proposed by Gugercin et al. in [13] which identifies a good choice of interpolation points that guarantees the $\mathcal{H}_2$ optimality of the interpolation points.

**Definition 1.1.** The $\mathcal{H}_2$ norm of the system (1) can be defined as

$$\|G\|_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 \, d\omega.$$  \hspace{1cm} (5)

$\mathcal{H}_2$ norm is the most appropriate way to investigate the performance of the ROMs achieved by IRKA. Estimating the $\mathcal{H}_2$ norm by the improper integral is impossible in practice.

**Definition 1.2.** The ROM (3) is called $\mathcal{H}_2$ optimal if $\hat{G}(s)$ is stable and satisfies

$$\|G(s) - \hat{G}(s)\|_{\mathcal{H}_2} = \min_{\text{dim}(G) = r} \|G(s) - \hat{G}(s)\|_{\mathcal{H}_2}.$$  \hspace{1cm} (6)

The most popular and frequently used MOR approaches for the second-order system, is first to convert the system into the equivalent first-order form and then apply any suitable technique to get a ROM. In this paper we propose the MOR techniques for the second-order system by applying the interpolatory techniques via IRKA, where the systems need not convert into the first-order form, that’s why the structure of the system remains invariant. Also, we estimate the minimized $\mathcal{H}_2$ error norm of the ROMs under certain level of tolerance, by simulating the matrix equations governed from the the error system.

## 2 IRKA for Model Order Reduction of First-Order Systems

Several MOR techniques are available for the second-order system by re-writing the system into an equivalent first-order form [14]. In this section, we briefly discuss the interpolation-based method IRKA for the first-order generalized system. Let us consider the following first-order system

$$\dot{x}(t) = Ax(t) + Bu(t),$$  \hspace{1cm} (7)

$$y(t) = Cx(t) + Du(t),$$

where $E \in \mathbb{R}^{k \times k}$ is non-singular, and $A \in \mathbb{R}^{k \times k}$, $B \in \mathbb{R}^{k \times p}$, $C \in \mathbb{R}^{m \times k}$ and $D \in \mathbb{R}^{m \times p}$. From this, to obtain the ROM

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t),$$  \hspace{1cm} (8)

$$\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}_u(t),$$
we construct the left and right transformation matrices \( W \) and \( V \), respectively, where the reduced matrices are constructed as:

\[
\begin{align*}
\hat{E} &= W^T EV, \\
\hat{A} &= W^T AV, \\
\hat{B} &= W^T B, \\
\hat{C} &= CV, \\
\hat{D}_a &= D_a.
\end{align*}
\]  

(9)

The IRKA based projection methods for MIMO systems have been provided in [15, 16], where the authors discussed that the interpolation points and the tangential directions need to be essentially updated until attain the \( H_2 \) optimality. Considering two sets of distinct interpolation points, \( \{\alpha_i\}_{i=1}^r \subset \mathbb{C} \) and \( \{\beta_i\}_{i=1}^r \subset \mathbb{C} \), the construction of the left and right transformation matrices \( W \) and \( V \), respectively, can be formed as

\[
\begin{align*}
\text{Range}(V) &= \text{span}\{(\alpha_1 E - A)^{-1} B b_1, \cdots, (\alpha_r E - A)^{-1} B b_r\}, \\
\text{Range}(W) &= \text{span}\{(\beta_1 E^T - A^T)^{-1} C^T c_1, \cdots, (\beta_r E^T - A^T)^{-1} C^T c_r\},
\end{align*}
\]

(10)

where \( b_i \in \mathbb{C}^m \) and \( c_i \in \mathbb{C}^p \) are the right and left tangential directions, respectively. With these interpolation points and tangential directions the IRKA based interpolation can be achieved.

The reduced transfer function \( \hat{G}(s) \) tangentially interpolates the original transfer function \( G(s) \) at a predefined set of interpolation points and some fixed tangential directions, such that

\[
\begin{align*}
G(\alpha_i)b_i = \hat{G}(\alpha_i)b_i, & \quad c_i^T G(\beta_i) = c_i^T \hat{G}(\beta_i), \quad \text{and} \\
c_i^T \hat{G}(\alpha_i)b_i = c_i^T G(\alpha_i)b_i & \quad \text{when } \alpha_i = \beta_i, \quad \text{for } i = 1, \cdots, r.
\end{align*}
\]

(11)

For \( j = 0, 1, \cdots, q \) the following condition will be satisfied

\[
\begin{align*}
c_i^T G^{(j)}(\alpha_i)b_i &= c_i^T \hat{G}^{(j)}(\alpha_i)b_i, \\
c_i^T C[(\alpha_i E - A)^{-1} E]^{(j)}(\alpha_i E - A)^{-1} B b_i &= c_i^T \hat{C}[(\alpha_i E - A)^{-1} E]^{(j)}(\alpha_i E - A)^{-1} B b_i,
\end{align*}
\]

(12)

where \( C[(\alpha_i E - A)^{-1} E]^{(j)}(\alpha_i E - A)^{-1} B \) is called the \( j \)-th moment of \( G(\cdot) \), and that represents the \( j \)-th derivative of \( G(\cdot) \) evaluated at the interpolation point \( \alpha_i \).

For the SISO systems, there need to consider the interpolation points but not the tangential directions. The complete procedures of IRKA for the SISO system are given in [17, 13].

The basis for the rational Krylov based model reduction requires a suitable choice of interpolation points and the use of Petrov-Galerkin conditions [19]. The summary of the first-order IRKA procedures are provided in Algorithm 1 and Algorithm 2.

### 3 IRKA for Structure-Preserving Model Order Reduction of Second-Order Systems

This section contributes the Structure-Preserving Model Order Reduction (SPMOR) of second-order system (1) via IRKA. One of the conversions of second-order system (1) into first-order form (7) is as follows

\[
\begin{align*}
x(t) &= \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \end{bmatrix}, \quad E = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}, \\
B &= \begin{bmatrix} 0 \\ H \end{bmatrix}, \quad C = \begin{bmatrix} L & 0 \end{bmatrix}, \quad \text{and } D_a = 0.
\end{align*}
\]

(13)
Algorithm 1: IRKA for First-Order MIMO Systems.

**Input:** $E, A, B, C, D_a$.

**Output:** $\hat{E}, \hat{A}, \hat{B}, \hat{C}$, $\hat{D}_a := D_a$.

1. Make the initial selection of the interpolation points $\{\alpha_i\}_{i=1}^r$ and the tangential directions $\{b_i\}_{i=1}^r$ & $\{c_i\}_{i=1}^r$.
2. $V = [(\alpha_1 E - A)^{-1} B b_1, \cdots, (\alpha_r E - A)^{-1} B b_r], \quad W = [(\alpha_1 E^T - A^T)^{-1} C^T c_1, \cdots, (\alpha_r E^T - A^T)^{-1} C^T c_r]$.
3. while (not converged) do
   4. $\hat{E} = W^T E V, \hat{A} = W^T A V, \hat{B} = W^T B, \hat{C} = C V$.
   5. for $i = 1, \cdots, r.$ do
      6. Compute $A z_i = \lambda_i \hat{E} z_i$ and $y_i^* \hat{A} = \lambda_i y_i^* \hat{E}$ for $\alpha_i \leftarrow -\lambda_i,$
         $b_i^* \leftarrow y_i^* \hat{B}$ and $c_i^* \leftarrow \hat{C} z_i^*$.
   7. end for
   8. Repeat Step-2.
   9. $i = i + 1$;
10. end while
11. Construct the reduced-order matrices by repeating Step-4.

In the conventional techniques, to obtain an efficient reduced-order model of the second-order system, at first, it is to convert into (13) essentially [20]. Then converted first-order form (7) can be implemented substantially and by applying Algorithm 1 or Algorithm 2 the equivalent first-order reduced order model (8) can be achieved.

Sometimes, that explicit formulation of (13) is prohibitive as the structure of the original model is destroyed and we can’t return back to the original system. For the large-scale second-order systems preservation of second-order structure is essential to perform the simulation, optimization, and controller design. SPMOR allows meaningful physical interpretation and provides a more accurate approximation to the full model.

In the MIMO systems, to avoid the explicit conversion and derive the SPMOR approach, the $i$-th column of $V$ and $W$ by means of the shifted linear systems need to be computed as

\[
(\alpha_i E - A) v^{(i)}_1 = B b_i, \quad \text{and} \quad (\alpha_i E^T - A^T) w^{(i)}_1 = C^T c_i, \quad i = 1, \cdots, r. \tag{14}
\]

Inserting the assumptions defined in (14) and applying matrix algebra, the system of linear equations in (14) lead to the followings

\[
\begin{bmatrix}
\alpha_i I & -I \\
K & \alpha_i M + D
\end{bmatrix}
\begin{bmatrix}
v^{(i)}_1 \\
v^{(i)}_2
\end{bmatrix} = \begin{bmatrix}
0 \\
H b_i
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
\alpha_i I & K^T \\
-I & \alpha_i M^T + D^T
\end{bmatrix}
\begin{bmatrix}
w^{(i)}_1 \\
w^{(i)}_2
\end{bmatrix} = \begin{bmatrix}
L^T c_i \\
0
\end{bmatrix}. \tag{15}
\]

Although the matrices in (15) has larger dimension $2n$, it is sparse and can be efficiently solved by suitable direct or iterative solvers [21, 22]. After the elimination and simplification of the system of linear equations governed from (14) for $v^{(i)}_1, v^{(i)}_2$,
position and velocity levels \[24\]. We can partition the projectors \(V\) where

\[
\{ \text{tangential directions} \}
\]

\[
\{ \text{interpolation points} \}
\]

\[
\text{the first-order representation} (13) \quad \text{and apply the Algorithm (2)} \quad \text{to find desired set of}
\]

\[
\text{rapid rate of convergence.}
\]

Using this idea the linear systems like (7) utilizing (13) of dimension 2 will be considered and the Algorithm (1) will be applied to find necessary set of interpolation points \(V\) respectively. Partitioning according to the techniques developed in [23], at each iteration, the first-order \(10\) end while

\[
\text{Construct the reduced-order matrices by repeating Step-4.}
\]

\[
\begin{align*}
\hat{w}_1^{(i)} & = (\alpha_i^2 M + \alpha_i D + K)^{-1} H b_i, \\
\hat{v}_2^{(i)} & = \alpha_i v_1^{(i)}, \\
\hat{w}_1^{(i)} & = (\alpha_i M^T + D^T) w_2^{(i)}, \\
\hat{w}_2^{(i)} & = (\alpha_i^2 M^T + \alpha_i D^T + K^T)^{-1} L^T c_i.
\end{align*}
\]

Using this idea the linear systems like (13) utilizing (13) of dimension 2n is replaced by an equivalent sparse linear system, which ensures the structure preservation and rapid rate of convergence.

According to the techniques developed in [23], at each iteration, the first-order representation (13) will be considered and the Algorithm (1) will be applied to find desired set of interpolation points \(\{\alpha_i\}_{i=1}^{r}\) efficiently. Also, Algorithm (1) updates the tangential directions \(\{b_i\}_{i=1}^{r}\) and \(\{c_i\}_{i=1}^{r}\) accordingly.

Due to the structure of system, we can split the desired projectors \(V\) and \(W\) as position and velocity levels [24]. We can partition the projectors \(V\) and \(W\) as

\[
V = \begin{bmatrix} V_p \\ V_v \end{bmatrix} \quad \text{&} \quad W = \begin{bmatrix} W_p \\ W_v \end{bmatrix},
\]

where \(V_p\) & \(W_p\) are the position levels and \(V_v\) & \(W_v\) are the velocity levels of \(V\) & \(W\) respectively. Partitioning \(V\) & \(W\) according to (17) and applying equation (16), we can write

\[
V_p = [v_1^{(1)}, v_2^{(1)}, \cdots, v_1^{(r)}]; \quad V_v = [v_1^{(2)}, v_2^{(2)}, \cdots, v_1^{(r)}]; \\
W_p = [w_1^{(1)}, w_2^{(1)}, \cdots, w_1^{(r)}]; \quad W_v = [w_2^{(2)}, w_2^{(2)}, \cdots, w_2^{(r)}].
\]

Again, for the SISO systems, to find the projectors \(V\) and \(W\) we need to consider the first-order representation (13) and apply the Algorithm (2) to find desired set of interpolation points \(\{\alpha_i\}_{i=1}^{r}\) only. Then for SISO systems (24) turns into the followings

\[
\begin{align*}
\hat{w}_1^{(i)} & = (\alpha_i^2 M + \alpha_i D + K)^{-1} H, \\
\hat{v}_2^{(i)} & = \alpha_i v_1^{(i)}, \\
\hat{w}_1^{(i)} & = (\alpha_i M^T + D^T) w_2^{(i)}, \\
\hat{w}_2^{(i)} & = (\alpha_i^2 M^T + \alpha_i D^T + K^T)^{-1} L^T.
\end{align*}
\]

**Algorithm 2: IRKA for First-Order SISO Systems.**

**Input**: \(E, A, B, C, D\).

**Output**: \(\hat{E}, \hat{A}, \hat{B}, \hat{C}, \hat{D} := D\).

1. Make an initial selection of the interpolation points \(\{\alpha_i\}_{i=1}^{r}\).
2. \[ V = [(\alpha_1 E - A)^{-1} B, \cdots, (\alpha_r E - A)^{-1} B], \]
3. \[ W = [(\alpha_1 E^T - A^T)^{-1} C^T, \cdots, (\alpha_r E^T - A^T)^{-1} C^T]. \]
4. while (not converged) do
5. \[ \hat{E} = W^T E V, \hat{A} = W^T A V, \hat{B} = W^T B, \hat{C} = C V. \]
6. for \(i = 1, \cdots, r\) do
7. Compute \(\hat{A} z_i = \lambda_i \hat{E} z_i\) and \(y_i^T \hat{A} = \lambda_i y_i^T \hat{E}\) for \(\alpha_i \leftarrow -\lambda_i.\)
8. end for
9. \(i = i + 1;\)
10. end while
11. Construct the reduced-order matrices by repeating Step-4.
Estimation of order reduction of the second-order system.

Algorithm 3: IRKA for Second-Order MIMO Systems.

Input: \( M, D, K, H, L \).
Output: \( M_p, D_p, K_p, H_p, \dot{L}_p, \dot{M}_v, \dot{D}_v, \dot{K}_v, \dot{H}_v, \dot{L}_v \).

1. Make the initial selection of the interpolation points \( \{c_i\}_{i=1}^r \) and the tangential directions \( \{b_i\}_{i=1}^r \).
2. Consider \( v_i^{(1)}, v_i^{(2)}, w_i^{(1)} \) and \( w_i^{(2)} \) are defined in (16) and construct
   \[
   V_p = [v_1^{(1)}, v_2^{(2)}, \cdots, v_r^{(r)}] \quad \text{and} \quad V_v = [w_1^{(1)}, w_2^{(2)}, \cdots, w_r^{(r)}],
   \]
   \[
   W_p = [w_1^{(1)}, w_2^{(2)}, \cdots, w_r^{(r)}] \quad \text{and} \quad W_v = [w_2^{(1)}, w_2^{(2)}, \cdots, w_r^{(2)}].
   \]

3. **while** (not converged) **do**

4. \( \dot{M}_p = W_p^T M V_p, \dot{D}_p = W_p^T D V_p, \dot{K}_p = W_p^T K V_p, H_p = W_p^T H, L_p = L V_p. \)
5. \( \dot{M}_v = W_v^T M V_v, \dot{D}_v = W_v^T D V_v, \dot{K}_v = W_v^T K V_v, \dot{H}_v = W_v^T H, \dot{L}_v = L V_v. \)
6. Using the first-order representation (13) in Algorithm (1) find the reduced-order matrices \( \tilde{E}, \tilde{A}, \tilde{B}, \text{and} \tilde{C} \).
7. Compute \( \tilde{A}_z = \lambda_i \tilde{E} z_i \) and \( y_i^* \tilde{A} = \lambda_i y_i^* \tilde{E} \) for \( \alpha_i \leftarrow -\lambda_i, b_i^* \leftarrow -y_i^* \tilde{B} \)
   and \( c_i^* \leftarrow \tilde{C} z_i^* \) for all \( i = 1, \cdots, r. \)
8. Repeat Step-2.
9. \( i = i + 1; \)
10. **end while**
11. Construct the reduced matrices by repeating Step-4 and Step-5.

Since there are two sets of projectors as in (18), we will get two types of SPMOR for the system (11), position level and velocity level respectively. We summarize the above ideas in Algorithm 3 and Algorithm 4 to compute the structure preserving model order reduction (3) of the second-order system (1).

4 Estimation of \( \mathcal{H}_2 \) error norm of the Reduced Order Model

The error system of the ROM (3) of the second-order system (11) using the first-order representation (13) can be written as

\[
G_{err} = G(s) - \tilde{G}(s) = C_{err}(sE_{err} - A_{err})^{-1}B_{err},
\]  

where \( G(s) \) and \( \tilde{G}(s) \) are defined in (2) and (3), respectively. In the error system (20), we have considered

\[
E_{err} = \begin{bmatrix} E & 0 \\ 0 & \tilde{E} \end{bmatrix}, \quad A_{err} = \begin{bmatrix} A & 0 \\ 0 & \tilde{A} \end{bmatrix}, \quad B_{err} = \begin{bmatrix} B \\ \tilde{B} \end{bmatrix}, \quad \text{and} \quad C_{err} = \begin{bmatrix} C & -\tilde{C} \end{bmatrix}.
\]  

Here the matrices \( E, A, B \) and \( C \) are defined in the first-order representation (13) of the second-order system (11). Also, the matrices \( \tilde{E}, \tilde{A}, \tilde{B}, \text{and} \tilde{C} \) are the reduced-order form of the first-order representation containing the reduced-order matrices defined in (3), which can be achieved by the Algorithm 3 or Algorithm 4.

Let us assume \( P_{err} \) and \( Q_{err} \) be the Gramians for the error system (20), where
Algorithm 4: IRKA for Second-Order SISO Systems.

\textbf{Input:} \( M, D, K, H, L \).
\textbf{Output:} \( \hat{M}_p, \hat{D}_p, \hat{K}_p, \hat{H}_p, \hat{L}_p, \hat{M}_v, \hat{D}_v, \hat{K}_v, \hat{H}_v, \hat{L}_v \).

1. Consider \( v_1^{(i)}, w_1^{(i)} \) and \( w_2^{(i)} \) are defined in (16) and construct
   \[ V_p = [w_1^{(1)}, w_2^{(1)}], \quad V_v = [w_1^{(2)}, w_2^{(2)}], \]
   \[ W_p = [w_1^{(1)}, w_2^{(1)}], \quad W_v = [w_1^{(2)}, w_2^{(2)}]. \]

2. Make an initial selection of the interpolation points \( \{ \alpha_i \}_{i=1}^r \).
3. Repeat Step-2.
4. The Gramians \( P_{err} \) and \( Q_{err} \) can be partitioned as
   \[ P_{err} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad Q_{err} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \] (22)

where \( P_{11}, Q_{11} \in \mathbb{R}^{n \times n}, P_{22}, Q_{22} \in \mathbb{R}^{r \times r} \) and \( P_{12}, Q_{12} \in \mathbb{R}^{n \times r} \).

The Gramians \( P_{err} \) and \( Q_{err} \) can be attained by solving the corresponding generalized Lyapunov equations
   \[ A_{err}P_{err} + P_{err}A_{err}^T + B_{err}B_{err}^T = 0, \]
   \[ A_{err}^TQ_{err} + Q_{err}A_{err} + C_{err}^TC_{err} = 0. \] (23)

Inserting (24) and (25) in (24), after simplification we have the following system of matrix equations
   \[ AP_{11}E + EP_{11}A + BB^T = 0, \]
   \[ AP_{12}E + EP_{12}A + BB^T = 0, \]
   \[ \hat{A}P_{22}E + \hat{E}P_{22}A + \hat{B}B^T = 0, \]
   \[ A^TQ_{11}E + E^TQ_{11}A + C^TC = 0, \]
   \[ A^TQ_{12}E + E^TQ_{12}A + C^TC = 0, \]
   \[ A^TQ_{22}E + E^TQ_{22}A + C^TC = 0. \] (24)

The efficient approaches to estimate the \( H_2 \) norm of the error system (20) suitably
can be written as
\[
\|G_{\text{err}}\|_{\mathcal{H}^2}^2 = \text{tr}(C_{\text{err}} P_{\text{err}} C_{\text{err}}^T) = \text{tr}(C_{\text{err}} P_{\text{err}} C_{\text{err}}^T) + 2\text{tr}(C_{\text{err}} P_{\text{err}} C_{\text{err}}^T)
\]
\[
= \|G(s)\|_{\mathcal{H}^2}^2 + \|\hat{G}(s)\|_{\mathcal{H}^2}^2 + 2\text{tr}(C_{\text{err}} P_{\text{err}} C_{\text{err}}^T).
\]  
(25)

\[
\|G_{\text{err}}\|_{\mathcal{H}^2}^2 = \text{tr}(B_{\text{err}}^T Q_{\text{err}} B_{\text{err}}) = \text{tr}(B_{\text{err}}^T Q_{\text{err}} B_{\text{err}}) + \text{tr}(B_{\text{err}}^T Q_{\text{err}} B_{\text{err}}) + 2\text{tr}(B_{\text{err}}^T Q_{\text{err}} B_{\text{err}})
\]
\[
= \|G(s)\|_{\mathcal{H}^2}^2 + \|\hat{G}(s)\|_{\mathcal{H}^2}^2 + 2\text{tr}(B_{\text{err}}^T Q_{\text{err}} B_{\text{err}}).
\]  
(26)

The $\mathcal{H}^2$ norm of the full model needs to be estimated once using the corresponding Gramian, and that is infeasible for a large-scale system by solving corresponding Lyapunov equation by the direct solvers. In this case, we can use the low-rank ADI based technique provided in Algorithm-2 of [26] or rational Krylov subspace method provided in Algorithm-1 of [27] to find the low-rank factors $Z_p$ and $Z_q$ of the Gramians $P_{11} = Z_p Z_p^T$ and $Q_{11} = Z_q Z_q^T$ defined in the first and forth equations of (24). These approaches can give us feasible ways to find the $\mathcal{H}^2$ of the full model. Then, instead using the Gramians of the full model, we can use the low-rank Gramian factors in the first part of (25) and (26) as given below.
\[
\|G(s)\|_{\mathcal{H}^2}^2 = \text{tr}(C Z_p (C Z_p)^T)
\]  
(27)

\[
\|G(s)\|_{\mathcal{H}^2}^2 = \text{tr}(B^T Q_{11} B) = \text{tr}(B^T Q_{11} B)
\]  
(28)

Since the Lyapunov equations defined by third and sixth equations in (24) corresponding to the ROMs are very small in size, they can be solved by any direct solver or MATLAB $\text{lyap}$ command in every iterations. Moreover to find the third part of (25) or (26), we can solve any of the generalized Sylvester equations given in the second and fifth equations in (24) using the sparse-dense Sylvester equation approach provided in Algorithm-4 of [11].

5 Numerical Results

To justify the accuracy and effectiveness of the proposed algorithms, they have been applied to the data generated in some large-scale models. The experiments are carried out with MATLAB® R2015a (8.5.0.197613) on a board with Intel®Core™i5 6200U CPU with a 2.30 GHz clock speed and 16 GB RAM. For the numerical experiments, the following model examples are used.

Example 5.1 ([International Space Station Model (ISSM)]). This is a structural model of the International Space Station being assembled in various stages. The aim is to model vibrations caused by docking of an incoming spaceship. The required control action is to dampen the effect of these vibrations as much as possible. The system is lightly damped and control actions will be constrained. Two models are given, which relate to different stages of completion of the Space Station [28]. The sparsity pattern of $A$ shows that it is derived from a mechanical system model. This consists of a first assembly stage (the so-called Russian service module 1R) of the International Space Station. The state dimension is $n = 270$.

Example 5.2 ([Clamped Beam Model (CBM)]). This is a structural model with $n = 348$ states. The model is obtained by spatial discretization of an appropriate partial differential equation [29]. The input represents the force applied to the structure at the free end, and the output is the resulting displacement.
Table 1: Model examples and dimensions of full models and ROMs

| Model | type   | full model ($n$) | ROM ($r$) |
|-------|--------|------------------|-----------|
| ISSM  | MIMO   | 270              | 20        |
| CBM   | SISO   | 348              | 30        |
| SOM   | SISO   | 9001             | 50        |
| BGM   | MIMO   | 17361            | 70        |

Example 5.3 ([Scalable Oscillator Model (SOM)]). This example originates in [30] with the setup that results in system (1). The triple chain oscillator model contains three chains with each of them being coupled to a fixed mounting by an additional damper on one end and fixed rigidly to a large mass that couples all three of them. The large mass is bound to a fixed amount by a single spring element. Each of the chains consists of $n_1$ equal masses and spring elements of equal stiffness. Therefore, the model parameters are the masses $m_1, m_2, m_3$ and the corresponding stiffness’s $k_1, k_2, k_3$ for the three oscillator chains, the mass $m_0$ with its spring stiffness $k_0$ for the coupling mass, the viscosity $\vartheta$ of the additional wall-mount-dampers and the length $n_1$ of each of the oscillator chains. The resulting system is of order $n_ξ = 3n_1 + 1$. The mass matrix $M = \text{diag}\{m_1I_{n_1}, m_2I_{n_1}, m_3I_{n_1}, m_0\}$. The stiffness matrix $K$ and damping matrix $D$ consist of a leading block diagonal matrix (consisting of the three stiffness matrices for the three oscillator chains) and coupling terms in the last row and column at positions $n_1$, $2n_1$ and $3n_1$ in the diagonal elements. For the numerical experiment, we consider the values of the variables as follows:

$$m_1 = 1, m_2 = 2, m_3 = 3, m_0 = 10,$$

$$k_1 = 10, k_2 = 20, k_3 = 1, k_0 = 50, \quad \vartheta = 5.$$

Input matrix $H \in \mathbb{R}^{n_ξ \times 1}$ consists of all elements with one and the output matrix $L = H^T$. The dimension of the model is 9001, i.e., $n = 9001$.

Example 5.4 ([Butterfly Gyro Model (BGM)]). The Butterfly Gyro [31] is a vibrating micro-mechanical gyro that has sufficient theoretical performance characteristics to make it a promising candidate for use in inertial navigation applications. The gyro chip consists of a three-layer silicon wafer stack, in which the middle layer contains the sensor element. The sensor consists of two wing pairs that are connected to a common frame by a set of beam elements; this is the reason the gyro is called the Butterfly. The original model consists of 17 361 degrees of freedom which results in an order $n_ξ = 17361$ second-order system. The system has a single input and 12 outputs.

Above models discussed here are available in the Model Reduction of Oberwolfach Benchmark Collection [1].

Table 1 shows the dimensions of the original models and corresponding ROMs via IRKA based SPMOR achieved by the Algorithm 3 and Algorithm 4. Figure 1a, Figure 2a, Figure 3a and Figure 4a display the comparisons of the transfer functions of the original models and the ROMs with the desired dimensions. Figure 1b, Figure 2b, Figure 3b and Figure 4b depict the absolute errors of the transfer functions in computing ROMs of the target second-order models, whereas Figure 1c, Figure 2c, Figure 3c and Figure 4c illustrate the corresponding relative errors in attaining the ROMs.

1http://www.imtek.de/simulation/benchmark/wb/35889/
Figure 1: Comparison of full model and ROM computed by Algorithm 3 of the ISSM
Figure 2: Comparison of full model and ROM computed by Algorithm 4 of the CBM
Figure 3: Comparison of full model and ROM computed by Algorithm 4 of the SOM
Figure 4: Comparison of full model and ROM computed by Algorithm 3 of the BGM
Table 2: Speed-up comparisons for ROMs

| Model  | dimension | execution time (sec) | speed-up |
|--------|-----------|----------------------|----------|
| ISSM   | full model | 270                  | $5.67 \times 10^{-4}$ | 2.78     |
|        | ROM       | 20                   | $2.04 \times 10^{-4}$ |          |
| CBM    | full model | 348                  | $3.19 \times 10^{-3}$ | 12.97    |
|        | ROM       | 30                   | $2.46 \times 10^{-4}$ |          |
| SOM    | full model | 9001                 | $1.66 \times 10^{-2}$ | 55.54    |
|        | ROM       | 50                   | $2.98 \times 10^{-4}$ |          |
| BGM    | full model | 17361                | $6.48 \times 10^{-4}$ | 609.13   |
|        | ROM       | 70                   | $1.07 \times 10^{-3}$ |          |

Table 3: $\mathcal{H}_2$ error norm of the ROM

| Model  | ROM   | $\|H - H_r\|_{\mathcal{H}_2}$ |
|--------|-------|-------------------------------|
| ISSM   | Position | $1.2 \times 10^{-6}$         |
|        | Velocity | $5.1 \times 10^{-6}$         |
| CBM    | Position | $1.1 \times 10^{-3}$         |
|        | Velocity | $3.6 \times 10^{-3}$         |
| SOM    | Position | $1.8 \times 10^{-1}$         |
|        | Velocity | $3.6 \times 10^{1}$          |
| BGM    | Position | $6.6 \times 10^{-12}$        |
|        | Velocity | $7.8 \times 10^{-12}$        |

From the provided figures, we have observed that the proposed IRKA based SP-MOR technique for finding ROMs of the second-order systems is sufficiently efficient and robust for the target models.

Table 2 represents the speed-up of the frequency responses of the target models. For the convenient comparison, we have counted the execution time for a single cycle of the frequency responses of the full models and the ROMs. It has been observed that the proposed technique is more efficient based on time for higher-dimensional models.

Table 3 depicts the $\mathcal{H}_2$ error norms of the ROMs for the target models for both position-level and velocity-level. Here, to find the $\mathcal{H}_2$ norm of the full model we have used low-rank ADI approach. It is observed that $\mathcal{H}_2$ error norm for CBM is small enough and for ISSM it is better. For BGM, the $\mathcal{H}_2$ error norm is the best, whereas, for SOM, the $\mathcal{H}_2$ error norm is less satisfactory in comparison to other target models but still not intolerable.

6 Conclusion

We have introduced the Structure-Preserving Model Order Reduction (SP-MOR) techniques for second-order systems using the Iterative Rational Krylov Algorithm (IRKA). In those techniques, the second-order systems need not convert into the first-order form explicitly which is essential for preserving some important properties of the systems. Also, SPMOR enhances the rapid rate of convergence and ensures the feasibility of the further simulations. Moreover, we have justified $\mathcal{H}_2$ error norm optimality of the achieved ROMs. We have investigated the applicability and efficiency of the proposed
techniques numerically and applied them to some practical data derived from real-world models. It has been observed that the proposed techniques provide ROMs of the target models with the desired $H_2$ error norm optimality.

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