In this paper, we establish new integral representations for the remainder term of the known asymptotic expansion of the logarithm of the Barnes $G$-function. Using these representations, we obtain explicit and numerically computable error bounds for the asymptotic series, which are much simpler than those obtained earlier by other authors. We find that along the imaginary axis, suddenly infinitely many exponentially small terms appear in the asymptotic expansion of the Barnes $G$-function. Employing one of our representations for the remainder term, we derive an exponentially improved asymptotic expansion for the logarithm of the Barnes $G$-function, which shows that the appearance of these exponentially small terms is in fact smooth, thereby proving the Berry transition property of the asymptotic series of the $G$-function.

1. Introduction and main results

In a series of papers published between 1899 and 1904 [1–4], Barnes introduced and studied certain generalizations of the classical gamma function $\Gamma(z)$, called the multiple gamma functions $\Gamma_k(z)$. Among them, the case $G(z) = 1/\Gamma_2(z)$, being the so-called double gamma function or the Barnes $G$-function, is of special interest. This function has several important applications in many different areas of mathematics, such as the theory of $p$-adic $L$-functions [5] or the study of determinants of Laplacians [6–10].

It is known that the $G$-function is an entire function and has the Weierstrass canonical product

$$G(z + 1) = (2\pi)^{z/2} e^{-[z(z+1)+\gamma z^2]/2}$$

$$\times \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^n e^{-z^2/(2n)}.$$
where \( \gamma = 0.57721566 \ldots \) is the Euler–Mascheroni constant. It also satisfies the functional equation \( G(z + 1) = \Gamma(z)G(z). \)

In January 2000, Richard Askey proposed, in the panel discussion of the San Diego symposium on asymptotics and applied analysis, a deeper study of this function. In particular, the derivation of an asymptotic expansion for \( G(z) \) is necessary for approximating this function for large values of \( z \).

Motivated by the function’s increasing interest and Askey’s proposition, Ferreira & López [11] established the following expansion in terms of the Bernoulli numbers

\[
\log G(z + 1) = \frac{1}{4} z^2 + z \log \Gamma(z + 1) - \left( \frac{1}{2} \zeta(z + 1) + \frac{1}{12} \right) \log z - \log A \\
+ \sum_{n=1}^{N-1} \frac{B_{2n+2}}{2n(2n + 1)(2n + 2)z^{2n}} + R_N(z),
\]

(1.1)

for \( |\arg z| < \pi/2 \), where

\[
R_N(z) = \int_0^{+\infty} \left( \frac{1}{2} \coth \left( \frac{t}{2} \right) - \sum_{k=0}^{N} \frac{B_{2k}}{(2k)!} \right) \frac{e^{-zt}}{t^3} \, dt.
\]

(1.2)

The constant \( A \) is called the Glaisher–Kinkelin constant, and can be expressed as

\[
\log A = \frac{\gamma + \log(2\pi)}{12} - \frac{\zeta'(2)}{2\pi^2} = \frac{1}{12} - \zeta'(-1) = 0.24875447 \ldots,
\]

where \( \zeta \) denotes Riemann’s zeta function. They also showed that this expansion can be extended to the wider sector \( |\arg z| < \pi \), by rotating the path of integration in (1.2). By estimating the error term \( R_N(z) \), they deduced the following asymptotic expansion for the Barnes \( G \)-function:

\[
\log G(z + 1) \sim \frac{1}{4} z^2 + z \log \Gamma(z + 1) - \left( \frac{1}{2} \zeta(z + 1) + \frac{1}{12} \right) \log z - \log A \\
+ \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n + 1)(2n + 2)z^{2n}},
\]

(1.3)

as \( z \to \infty \) in the sector \( |\arg z| \leq \pi - \delta < \pi \) with any fixed \( 0 < \delta < \pi \).

In this paper, we use (1.1) as the definition of \( R_N(z) \) for \( |\arg z| < \pi \) and \( N \geq 1 \). Throughout the paper, empty sums are taken to be zero.

The aim of this paper is to establish new integral representations for the remainder \( R_N(z) \). Using these representations, we obtain several new properties of the asymptotic expansion (1.3), including simple, explicit and numerically computable bounds for \( R_N(z) \), exponentially improved asymptotic expansion, and the smooth transition of the Stokes discontinuities.

In theorem 1.1., we give two integral representations for the remainder \( R_N(z) \) of the asymptotic series (1.3). In formula (1.6), \( \text{Li}_2(w) \) denotes the Dilogarithm, defined by

\[
\text{Li}_2(w) = \sum_{n=1}^{\infty} \frac{w^n}{n^2}
\]

(1.4)

for \( |w| \leq 1 \).

**Theorem 1.1.** For any \( N \geq 1 \), the remainder \( R_N(z) \) in (1.1) has the representations

\[
R_N(z) = \frac{1}{z^{2N}} \left( -1 \right)^{N+1} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1+st/z^2} \, ds \right) t^{2N} \log(1 - e^{-2\pi t}) \, dt,
\]

(1.5)

\[
= \frac{1}{z^{2N}} \left( -1 \right)^N \int_0^{+\infty} \frac{t^{2N-1}}{1+(t/z)^2} \text{Li}_2(e^{-2\pi t}) \, dt
\]

(1.6)
for $|\arg z| < \pi/2$, and

$$R_N(z) = -\frac{1}{2N(2N+1)} \int_0^{+\infty} \frac{B_{2N+1}(t-\lfloor t \rfloor)}{(t+z)^{2N}} dt,$$

$$|R_N(z)| \leq \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \min \left\{ \left( \csc(2\theta), \sqrt{\frac{e}{4} (2N + \frac{5}{2})} \right) \right\} \quad \text{if} \quad \frac{\pi}{4} < |\theta| \leq \frac{\pi}{2} \quad (1.7)$$

for $|\arg z| < \pi$, where $B_n(z)$ denotes the $n$th Bernoulli polynomial.

We note that (1.6) can also be written in terms of the Bose–Einstein integral $G_p(w)$ [12, pp. 611–612], because $Li_2(e^{-2\pi t}) = G_1(-2\pi i)$. The second theorem provides two bounds for the error term $R_N(z)$, one being valid in the closed sector $|\arg z| \leq \pi/2$ the other being valid in the larger one $|\arg z| < \pi$.

**Theorem 1.2.** Let $\theta = \arg z$. For any $N \geq 1$, the remainder $R_N(z)$ in (1.1) satisfies

$$|R_N(z)| \leq \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \left\{ \min \left( \csc(2\theta), \sqrt{\frac{e}{4} (2N + \frac{5}{2})} \right) \right\} \quad \text{if} \quad \frac{\pi}{4} < |\theta| \leq \frac{\pi}{2} \quad (1.9)$$

and

$$|R_N(z)| \leq \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \sec^{2N+1} \left( \frac{\theta}{2} \right) \quad (1.10)$$

for $|\arg z| < \pi$. Moreover, if $z > 0$, then $R_N(z)$ has the sign of the first neglected term and its absolute value is smaller than that term.

In theorem 1.3., we give alternative bounds for the sectors $\pi/4 < |\arg z| < \pi$.

**Theorem 1.3.** Let $\theta = \arg z$. For any $N \geq 1$, the remainder $R_N(z)$ in (1.1) satisfies

$$|R_N(z)| \leq \frac{\csc(2(\theta - \phi^*))}{\cos^{2N+1} \phi^*} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \quad (1.11)$$

for $\pi/4 < |z| < \pi$, where $0 < \phi^* < \pi/2$ is the unique solution of the implicit equation

$$(2N + 3) \cos(3\phi^* - 2\theta) = (2N - 1) \cos(\phi^* - 2\theta), \quad (1.12)$$

that satisfies $-\pi/2 + \theta < \phi^* < \pi/2$ if $3\pi/4 < \theta < \pi$; $-\pi/2 + \theta < \phi^* < -\pi/4 + \theta$ if $\pi/2 \leq \theta < 3\pi/4$; and $0 < \phi^* < -\pi/4 + \theta$ if $\pi/4 < \theta < \pi/2$. Similarly, we have

$$|R_N(z)| \leq \frac{\csc(2(\theta - \phi^*))}{\cos^{2N+1} \phi^*} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \quad (1.11)$$

for $-\pi < \arg z < -\pi/4$, where $-\pi/2 < \phi^* < 0$ is the unique solution of the equation (1.12) that satisfies $-\pi/2 < \phi^* < \pi/2 + \theta$ if $-\pi < \theta < -3\pi/4$; $\pi/4 + \theta < \phi^* < \pi/2 + \theta$ if $-3\pi/4 < \theta \leq -\pi/2$ and $\pi/4 + \theta < \phi^* < 0$ if $-\pi/2 < \theta < -\pi/4$.

Note that all these error bounds involve the absolute value of the first omitted term of the asymptotic expansion (1.3), multiplied by a factor which depends only on $\arg z$ and the index $N$ of the first neglected term. The bounds obtained originally by Ferreira & López [11] have more complicated forms. Later, employing a result of Pedersen [13], Ferreira [14] established the following simpler bound

$$|R_N(z)| \leq \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \sec^{2N} \theta, \quad (1.11)$$

for $|\arg z| < \pi/2$ and $N \geq 1$. It is easy to see that this bound is weaker than (1.9).

In theorem 1.3., we give a new convergent series expansion for the logarithm of the Barnes $G$-function. This expansion involves two infinite series. The terms of the first infinite series are finite sums whose length depend on an arbitrary sequence of non-negative integers $N_1, N_2, N_3, \ldots$. This infinite series is in fact a certain decomposition of the truncated version of the asymptotic...
series (1.3) (see the remark after the theorem). The terms of the second infinite series involve
the terminant function \( \hat{T}_p(\omega) \), whose definition and basic properties are given in §4. In §4a,
we show that, as a manifestation of the Stokes phenomenon, suddenly infinitely many exponentially
small terms appear in the asymptotic expansion (1.3) as \( \arg z \) passes through the values \( \pm \pi/2 \).
An optimal choice of the indices \( N_1, N_2, N_3, \ldots \) and the asymptotic behaviour of the terminant
functions show that the appearance of these exponentially small terms is in fact smooth, proving
the Berry transition property of the asymptotic series (1.3) (see §4b). Moreover, with the optimal
choice of the indices \( N_1, N_2, N_3, \ldots \), (1.13) is an exponentially improved asymptotic expansion for
\( \log G(z + 1) \). We remark that a very similar behaviour was found for the asymptotic series of the
gamma function [15–17, pp. 279–288] and the Hurwitz zeta function [18].

**Theorem 1.4.** For an arbitrary sequence of non-negative integers \( N_1, N_2, N_3, \ldots \), the logarithm of the
Barnes G-function has the exact expansion

\[
\log G(z + 1) = \frac{1}{4} z^2 + z \log \Gamma(z + 1) - \left( \frac{1}{2} z(z + 1) + \frac{1}{12} \right) \log z - \log A
\]

\[
- \sum_{k=1}^{\infty} \frac{1}{(2\pi k)^2} \sum_{n=0}^{N_k-1} (-1)^n \frac{2(2n+1)!}{(2\pi k z)^{2n+2}}
\]

\[
- \sum_{k=1}^{\infty} \left( \hat{T}_{2N_k+1}(2\pi k z) \frac{2\pi k z}{2\pi k^2} e^{2\pi k z} + \hat{T}_{2N_k+1}(-2\pi k z) \frac{2\pi k^3}{2\pi k^2} e^{-2\pi k z} \right),
\]

(1.13)

provided that \( |\arg z| < \pi \).

The special case \( N_k = N - 1 \geq 0 \) \((k \geq 1)\) yields the following convergent series for the remainder
\( R_N(z) \) of the asymptotic series (1.3):

\[
R_N(z) = - \sum_{k=1}^{\infty} \left( \hat{T}_{2N_k-1}(2\pi k z) \frac{2\pi k z}{2\pi k^2} e^{2\pi k z} + \hat{T}_{2N_k-1}(-2\pi k z) \frac{2\pi k^3}{2\pi k^2} e^{-2\pi k z} \right),
\]

for \( |\arg z| < \pi \). In deriving this expansion, we used the well-known representation of the Bernoulli
numbers in terms of the Riemann zeta function [12, equation 25.6.2] and the definition of \( R_N(z) \).

The rest of the paper is organized as follows. In §2, we prove the formulæ for the remainder
term stated in theorem 1.1. In §3, we prove the error bounds given in theorems 1.2 and 1.3. In
§4, we prove the expansion presented in theorem 1.4, and provide a detailed discussion of the
Stokes phenomenon related to the asymptotic expansion (1.3). In §5, we present some numerical
examples to demonstrate the accuracy of our error bounds given in theorems 1.2 and 1.3. The
paper concludes with a short discussion in §6.

2. Proof of the representations for the remainder term

Here, we prove theorem 1.1. In the first part, we prove (1.5) and (1.6), whereas in the second part,
we derive (1.7) and (1.8).

(a) Proof of formulæ (1.5) and (1.6)

Our starting point is the representation

\[
\log G(z + 1) = \frac{1}{2} z \log(2\pi) - \frac{1}{2} z(z + 1) + z \log \Gamma(z + 1) - \int_0^z \log \Gamma(s + 1) \, ds
\]

for \( |\arg z| < \pi \), with the logarithm taking its principal value [12, equation 5.17.4]. As for the path
of integration, we take a straight line that connects 0 to \( z \). For the logarithm of the gamma function
\[ \log \Gamma(s + 1) \text{ under the integral, we introduce the expansion} \]
\[ \log \Gamma(s + 1) = \left( s + \frac{1}{2} \right) \log s - \frac{1}{2} \log(2\pi) + \mu(s), \]
where \( \mu(s) \) is the Binet function and \( |\arg s| < \pi \) [19, pp. 56–58]. Integrating term by term, we deduce
\[ \log G(z + 1) = \frac{1}{4} z^2 + z \log \Gamma(z + 1) - \left( \frac{1}{2} z(z + 1) + \frac{1}{12} \right) \log z - \log A + R_1(z) \]
with
\[ R_1(z) = \log A + \frac{1}{12} \log z - \int \mu(s) \, ds. \]

Now suppose that \( |\arg z| < \pi/2 \) and \( |\arg s| < \pi/2 \). If we substitute the following formula
\[ \mu(s) = -\frac{1}{\pi} \int_0^{+\infty} \frac{s}{s^2 + t^2} \log(1 - e^{-2\pi t}) \, dt, \]
valid for \( |\arg s| < \pi/2 \) [19, p. 66], we find
\[ R_1(z) = \log A + \frac{1}{12} \log z + \frac{1}{2\pi} \int_0^{+\infty} \log \left( 1 + \left( \frac{z}{t} \right)^2 \right) \log(1 - e^{-2\pi t}) \, dt \]
\[ = \log A - \frac{1}{\pi} \int_0^{+\infty} \log \left( 1 + \left( \frac{t}{2} \right)^2 \right) \log(1 - e^{-2\pi t}) \, dt, \]
The first integral can be calculated using integration by parts
\[ -\frac{1}{\pi} \int_0^{+\infty} \log t \log(1 - e^{-2\pi t}) \, dt = 2 \int_0^{+\infty} \frac{t \log t}{e^{2\pi t} - 1} \, dt - 2 \int_0^{+\infty} \frac{t}{e^{2\pi t} - 1} \, dt \]
\[ = -\frac{\gamma + \log(2\pi)}{12} + \frac{\zeta'(2)}{2\pi^2} = -\log A, \]
leading to the result
\[ R_1(z) = \frac{1}{2\pi} \int_0^{+\infty} \log \left( 1 + \left( \frac{t}{2} \right)^2 \right) \log(1 - e^{-2\pi t}) \, dt, \]
which is valid for \( |\arg z| < \pi/2 \). Next, we substitute the expansion
\[ \log \left( 1 + \left( \frac{t}{2} \right)^2 \right) = \sum_{n=1}^{N-1} (-1)^{n+1} t^{2n} \frac{1}{n!} + (-1)^N t^{2N} \frac{1}{2^{2N}} \int_0^1 \frac{2z^{2N-1}}{1 + (st/z)^2} \, ds \]
with \( N \geq 1 \), to deduce that
\[ R_1(z) = \sum_{n=1}^{N-1} \frac{(-1)^{n+1}}{2n^{2n}} \frac{1}{\pi} \int_0^{+\infty} t^{2n} \log(1 - e^{-2\pi t}) \, dt \]
\[ + \frac{1}{2^{2N}} \frac{(-1)^N}{\pi} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/z)^2} \, ds \right) t^{2N} \log(1 - e^{-2\pi t}) \, dt \]
\[ = \sum_{n=1}^{N-1} \frac{B_{2n+2}}{2n(2n+1)(2n+2)} \frac{(-1)^{n+1}}{\pi} \int_0^{+\infty} t^{2n} \log(1 - e^{-2\pi t}) \, dt \]
\[ + \frac{1}{2^{2N}} \frac{(-1)^N}{\pi} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/z)^2} \, ds \right) t^{2N} \log(1 - e^{-2\pi t}) \, dt, \]
where we have used the known representation of the Bernoulli numbers [12, equation 24.7.5]
\[ \frac{B_{2n+2}}{(2n+1)(2n+2)} = \frac{(-1)^{n+1}}{\pi} \int_0^{+\infty} t^{2n} \log(1 - e^{-2\pi t}) \, dt. \]
Finally, by the definition of \(R_N(z)\), we conclude that
\[
R_N(z) = \frac{1}{z^{2N}} \frac{(-1)^{N+1}}{2} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/z)^2} \, ds \right) t^{2N} \log(1 - e^{-2\pi t}) \, dt
\]
for \(|\arg z| < \pi/2\) and \(N \geq 1\). To prove (1.6), we perform integration by parts in (1.5) and employ the integral representation
\[
\text{Li}_2(w) = -\int_0^w \frac{\log(1 - t)}{t} \, dt,
\]
which is valid for \(|\arg (w - 1)| < \pi\) [12, equation 25.12.2].

(b) Proof of formulae (1.7) and (1.8)

Our starting point is the functional relationship with the Hurwitz zeta function \(\zeta(s, z)\), due to Vardi [10]
\[
\log G(z + 1) = z \log \Gamma(z + 1) - \zeta'(-1, z + 1) + \frac{1}{12} - \log A
\]
\[
= z \log \Gamma(z + 1) - \zeta'(-1, z) - z \log z + \frac{1}{12} - \log A,
\]
(2.2)
for \(\Re(z) > 0\), with \(\zeta'(s, z) = \partial \zeta(s, z)/\partial s\). Applying formula [12, equation 25.11.7] with \(a = z - 1\), \(n = 1\), and the functional equation [12, equation 25.11.3] for the Hurwitz zeta function, yields
\[
\zeta(s, z) = \frac{z^{-s}}{2} + \frac{z^{1-s} \Gamma(s + 3)}{12} - \frac{\Gamma(s + 3)}{6 \Gamma(s)} \int_0^{+\infty} \frac{B_3(t - \lfloor t \rfloor)}{(t + z)^{s+3}} \, dt,
\]
for \(z > 1\) and \(\Re(s) > -2\) with \(s \neq 1\). Differentiation with respect to \(s\) and the substitution \(s = -1\) gives
\[
\zeta'(-1, z) = -\frac{1}{4} z^2 + \left( \frac{z(z - 1)}{2} + \frac{1}{12} \right) \log z + \frac{1}{12} + \frac{1}{6} \int_0^{+\infty} \frac{B_3(t - \lfloor t \rfloor)}{(t + z)^2} \, dt.
\]
Substituting this expression into (2.2), using the definition of \(R_1(z)\) and analytic continuation in \(z\), we deduce
\[
R_1(z) = \frac{1}{6} \int_0^{+\infty} \frac{B_3(t - \lfloor t \rfloor)}{(t + z)^2} \, dt
\]
(2.3)
for \(|\arg z| < \pi\). Using the identity
\[
B_n'(x) = nB_{n-1}(x)
\]
and integration by parts, induction on \(N\) yields the formulae
\[
R_N(z) = -\frac{1}{(2N - 1)2N} \int_0^{+\infty} \frac{B_{2N}(t - \lfloor t \rfloor)}{(t + z)^{2N+1}} \, dt
\]
(2.4)
\[
= -\frac{1}{2N(2n + 1)} \int_0^{+\infty} \frac{B_{2N+1}(t - \lfloor t \rfloor)}{(t + z)^{2N}} \, dt
\]
for \(N \geq 2\). Thus, the representation (1.7) is proved. To prove (1.8), we use the first formula in (2.4)
\[
R_N(z) = \frac{B_{2N+2}}{2N(2N + 1)(2N + 2)z^{2N}} + R_{N+1}(z)
\]
\[
= -\frac{1}{(2N + 1)(2N + 2)} \int_0^{+\infty} \frac{B_{2N+2}}{(t + z)^{2N+1}} \, dt - \frac{1}{(2N + 1)(2N + 2)} \int_0^{+\infty} \frac{B_{2N+2}(t - \lfloor t \rfloor)}{(t + z)^{2N+1}} \, dt
\]
\[
= -\frac{1}{(2N + 1)(2N + 2)} \int_0^{+\infty} \frac{B_{2N+2}(t - \lfloor t \rfloor)}{(t + z)^{2N+1}} \, dt.
\]
3. Proof of the error bounds

Here, we prove theorems 1.2 and 1.3. To make the subsequent formulae simpler, we introduce the notation

\[
\ell(\theta) = \begin{cases} 
|\csc(2\theta)| & \text{if } \frac{\pi}{4} < |\theta| < \frac{\pi}{2} \\
1 & \text{if } |\theta| \leq \frac{\pi}{4}.
\end{cases}
\] (3.1)

It is easy to show that for any \( r > 0 \),

\[
\frac{1}{1 + r e^{-2\theta}} \leq \ell(\theta).
\] (3.2)

From now on, we write \( \theta = \arg z \). Applying this inequality and the formula (2.1) for the Bernoulli numbers to the representation (1.5), we obtain

\[
|R_N(z)| \leq \frac{1}{|z|^{2N}} \frac{1}{\pi} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/|z|)^2} e^{-2i\theta} \right) \left| t^{2N} \log(1 - e^{-2\pi t}) \right| dt,
\]

\[
\leq \ell(\theta) \frac{1}{|z|^{2N}} \frac{1}{\pi} \int_0^{+\infty} \left( \int_0^1 s^{2N-1} \right) \left| t^{2N} \log(1 - e^{-2\pi t}) \right| dt,
\]

\[
= \ell(\theta) \frac{1}{2N|z|^{2N}} \frac{-1}{\pi} \int_0^{+\infty} t^{2N} \log(1 - e^{-2\pi t}) dt,
\]

\[
= \frac{|B_{2N+2}|}{2N(2N + 1)(2N + 2)|z|^{2N}} \ell(\theta).
\]

This partially proves (1.9). To prove the statement for the case \( z > 0 \), we note that \( 0 < 1/(1 + (st/|z|)^2) < 1 \) for any positive \( s, t \) and \( z \), and therefore from (1.5), we obtain

\[
0 < (-1)^N R_N(z) \leq \frac{1}{z^{2N}} \frac{-1}{\pi} \int_0^{+\infty} \left( \int_0^1 s^{2N-1} \right) \left| t^{2N} \log(1 - e^{-2\pi t}) \right| dt,
\]

\[
= (-1)^N \frac{B_{2N+2}}{2N(2N + 1)(2N + 2)z^{2N}}.
\]

Now, let \( 0 < \varphi < \pi/2 \) be an acute angle that may depend on \( N \). First, suppose that \( \pi/4 + \varphi < \arg z \), then the path of integration in (1.5) through the angle \( \varphi \). Doing so, we find

\[
R_N(z) = \frac{1}{z^{2N}} \frac{(-1)^{N+1}}{\pi} \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/|z|)^2} e^{i\varphi} \right) \left| t^{2N} \log(1 - e^{-2\pi t}) \right| dt
\]

\[
= \frac{1}{z^{2N}} \frac{(-1)^{N+1}}{\pi} \left( e^{i\varphi} \cos \varphi \right)^{2N+1}
\]

\[
\times \int_0^{+\infty} \left( \int_0^1 \frac{s^{2N-1}}{1 + (st/|z| \cos \varphi)^2} e^{-2i(t-\varphi)} \right) \left| t^{2N} \log(1 - e^{-2\pi(t e^{i\varphi}/\cos \varphi)}) \right| dt,
\]

for \( \pi/4 + \varphi < \arg z < \pi/2 + \varphi \) and \( N \geq 1 \). To obtain a simple estimation for \( R_N(z) \), we first note that

\[
|\log(1 - e^{-2\pi(t e^{i\varphi}/\cos \varphi)})| \leq - \log(1 - |e^{-2\pi(t e^{i\varphi}/\cos \varphi})|) = - \log(1 - e^{-2\pi t})
\]
for any $t > 0$ and $0 < \varphi < \pi/2$. Employing this inequality, the bound (3.2) and the formula (2.1) for the Bernoulli numbers, we deduce

$$|R_N(z)| \leq \frac{\csc(2(\theta - \varphi))}{\cos^{2N+1}\varphi} \frac{1}{|z|^{2N}} \left|\frac{1}{\pi} \int_0^{\infty} \left( \int_0^{1} s^{2N-1} ds \right) t^{2N} \log(1 - e^{-2\pi t}) dt\right|,$$

$$= \frac{\csc(2(\theta - \varphi))}{\cos^{2N+1}\varphi} \frac{1}{2N|z|^{2N}} \left|\frac{1}{\pi} \int_0^{\infty} t^{2N} \log(1 - e^{-2\pi t}) dt\right|,$$

$$= \frac{\csc(2(\theta - \varphi))}{\cos^{2N+1}\varphi} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}}$$

for $\pi/4 + \varphi < \arg z < \pi/2 + \varphi$ and $N \geq 1$. The minimization of the factor

$$\frac{\csc(2(\theta - \varphi))}{\cos^{2N+1}\varphi}, \quad (3.3)$$

can be done by applying a lemma of Meijer’s [20, p. 956]. In our case, Meijer’s lemma gives that the minimizing value $\varphi = \varphi^*$ in (3.3) is the unique solution of the implicit equation

$$(2N + 3) \cos(3\varphi^* - 2\theta) = (2N - 1) \cos(\varphi^* - 2\theta), \quad (3.4)$$

that satisfies $-\pi/2 + \theta < \varphi^* < \pi/2$ if $3\pi/4 \leq \theta < \pi$; $-\pi/2 + \theta < \varphi^* < -\pi/4 + \theta$ if $\pi/2 \leq \theta < 3\pi/4$; and $0 < \varphi^* < -\pi/4 + \theta$ if $\pi/4 < \theta < \pi/2$. With this choice of $\varphi$, we have

$$|R_N(z)| \leq \frac{\csc(2(\theta - \varphi^*))}{\cos^{2N+1}\varphi^*} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}} \quad (3.5)$$

for $\pi/4 < \arg z < \pi$. By the reflection principle, we have $|R_N(\bar{z})| = |R_N(\bar{\bar{z}})| = |R_N(z)|$, which, in combination with (3.5), gives

$$|R_N(z)| \leq \frac{\csc(2(\theta - \varphi^*))}{\cos^{2N+1}\varphi^*} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}}$$

for $-\pi < \arg z < -\pi/4$. Here, $-\pi/2 < \varphi^* < 0$ is the unique solution of the equation (3.4), that satisfies $-\pi/2 < \varphi^* < \pi/2 + \theta$ if $-\pi < \theta \leq -3\pi/4$; $\pi/4 + \theta < \varphi^* < \pi/2 + \theta$ if $-3\pi/4 < \theta \leq -\pi/2$ and $\pi/4 + \theta < \varphi^* < 0$ if $-\pi/2 < \theta < -\pi/4$. The proof of theorem 1.3 is complete.

To complete the proof of (1.9), note that when $\arg z = \theta = \pi/2$, the minimizing value $\varphi^*$ in (3.4) is given explicitly by

$$\varphi^* = \arctan\left(\frac{1}{\sqrt{2N + 2}}\right),$$

and therefore we have

$$\frac{\csc(2(\theta - \varphi^*))}{\cos^{2N+1}\varphi^*} \leq \frac{\csc(2(\pi/2 - \varphi^*))}{\cos^{2N+1}\varphi^*} = \frac{1}{2} \left(1 + \frac{1}{2N+2}\right)^N \sqrt{2N + 2} \leq \frac{1}{2} \sqrt{e\left(2N + \frac{5}{2}\right)},$$

as long as $\pi/4 + \varphi^* < \theta \leq \pi/2$ and $N \geq 1$. We also have

$$\frac{1}{2} \sqrt{e\left(2N + \frac{5}{2}\right)} \geq \frac{1}{2} \sqrt{e\left(2 + \frac{5}{2}\right)} \geq \csc(\theta),$$

for $\pi/4 < \theta \leq \pi/4 + \varphi^* \leq \pi/4 + \arctan\left(\frac{1}{2}\right)$, therefore

$$|R_N(z)| \leq \frac{1}{2} \sqrt{e\left(2N + \frac{5}{2}\right)} \frac{|B_{2N+2}|}{2N(2N+1)(2N+2)|z|^{2N}},$$

provided that $\pi/4 < \theta \leq \pi/2$ and $N \geq 1$. For the conjugate sector $-\pi/2 < \theta \leq -\pi/4$, we can simply use $|R_N(\bar{z})| = |R_N(\bar{\bar{z}})| = |R_N(z)|$. This finishes the proof of (1.9).
Finally, we prove (1.10) with the help of the expression (1.8). First, we show that the quantity $B_{2N+2} + B_{2N+2}(t - |t|)$ does not change sign for any $t \geq 0$ with a fixed $N \geq 1$. Indeed, by the Fourier series of the Bernoulli polynomials [12, equation 24.8.1], we have

$\left(-1\right)^N \left(B_{2N+2} + B_{2N+2}(t - |t|)\right) = \frac{2(2N + 2)!}{(2\pi)^{2N+2}} \sum_{k=1}^{\infty} \frac{1 + \cos(2\pi kt)}{k^{2N+2}} \geq 0$.

It is easy to show that

$$|t + z| \geq (t + |z|) \cos \left(\frac{\theta}{2}\right)$$

for $|\arg z| < \pi$ and $t \geq 0$. Applying this inequality for (1.8), we obtain the desired result

$$|R_N(z)| \leq \frac{1}{(2N + 1)(2N + 2)} \int_0^{+\infty} \frac{|B_{2N+2} + B_{2N+2}(t - |t|)|}{|t + z|^{2N+1}} \, dt,$$

for $|\arg z| < \pi$ and $t \geq 0$. Applying this inequality for (1.8), we obtain the desired result

$$|R_N(z)| \leq \frac{1}{(2N + 1)(2N + 2)} \int_0^{+\infty} \frac{|B_{2N+2} + B_{2N+2}(t - |t|)|}{|t + z|^{2N+1}} \, dt.$$ 

In the final step, we have made use of (1.9).

4. Exponentially improved asymptotic expansion

We shall find it convenient to express our exponentially improved expansion in terms of the (scaled) terminant function, which is defined in terms of the incomplete gamma function as

$$\hat{T}_p(w) = \frac{e^{\pi i p} \Gamma(p)}{\Gamma(1 - p, w)} = \frac{e^{\pi i p} w^{1-p} e^{-w}}{2\pi i} \int_0^{+\infty} \frac{t^{p-1} e^{-t}}{w + t} \, dt \quad \text{for} \quad p > 0 \quad \text{and} \quad |\arg w| < \pi,$$

and by analytic continuation elsewhere. Olver [21] showed that when $p \sim |w|$ and $w \to \infty$, we have

$$\hat{T}_p(w) = \frac{1}{2} + \frac{1}{2} \text{erf} \left(\frac{1}{2} |w|\right) + O \left(\frac{e^{-(1/2)|w|^2}(\varphi)}{|w|^{1/2}}\right)$$

for $-\pi + \delta \leq \arg w \leq 3\pi - \delta$, $0 < \delta \leq 2\pi$; and

$$e^{-2\pi i p} \hat{T}_p(w) = -\frac{1}{2} + \frac{1}{2} \text{erf} \left(-\frac{1}{2} |w|\right) + O \left(\frac{e^{-(1/2)|w|^2}(\varphi)}{|w|^{1/2}}\right)$$

for $-3\pi + \delta \leq \arg w \leq \pi - \delta$, $0 < \delta \leq 2\pi$. Here, $\varphi = \arg w$ and erf denotes the error function. The quantity $c(\varphi)$ is defined implicitly by the equation

$$\frac{1}{2} c^2(\varphi) = 1 + i(\varphi - \pi) - e^{i(\varphi - \pi)},$$

and corresponds to the branch of $c(\varphi)$ which has the following expansion in the neighbourhood of $\varphi = \pi$:

$$c(\varphi) = (\varphi - \pi) + \frac{i}{6} (\varphi - \pi)^2 - \frac{1}{36} (\varphi - \pi)^3 - \frac{i}{270} (\varphi - \pi)^4 + \cdots.$$

For complete asymptotic expansions, see Olver [21]. We remark that Olver uses the different notation $F_p(w) = e^{-\pi i p} \hat{T}_p(w)$ for the terminant function and the other branch of the function $c(\varphi)$. For further properties of the terminant function, see, for example, Paris & Kaminski [17, ch. 6].
(a) The discontinuous treatment of the Stokes multipliers

The asymptotic expansion of \( \log G(z + 1) \) in the sector \( |\arg z| < \pi/2 \) is given by (1.3). We shall obtain an expansion in the sector \( \pi/2 < |\arg z| < \pi \). We expand the dilogarithm in (1.6) using its series (1.4) and apply partial fraction decomposition to deduce

\[
R_1(z) = \sum_{k=1}^{\infty} \frac{1}{2\pi k^2} \left( \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-2\pi k t}}{it - z} \, dt + \frac{1}{2\pi i} \int_0^{+\infty} \frac{e^{-2\pi k t}}{it + z} \, dt \right). \tag{4.4}
\]

To find the analytic continuation of \( R_1(z) \) to the sector \( \pi/2 < \arg z < \pi \), we rotate the path of integration in the first integral to obtain

\[
R_1(z) = -\sum_{k=1}^{\infty} \frac{e^{2\pi i k z}}{2\pi i k^2} + R_1(z e^{-\pi i}), \tag{4.5}
\]

for \( \pi/2 < \arg z < \pi \). By the connection formula \( R_1(z) = \overline{R_1(z)} \), we have, similarly,

\[
R_1(z) = \sum_{k=1}^{\infty} \frac{e^{-2\pi i k z}}{2\pi i k^2} + R_1(z e^{\pi i}), \tag{4.6}
\]

for \( -\pi < \arg z < -\pi/2 \). Using the asymptotic expansion (1.3) for \( R_1(z e^\pm \pi i) \), we finally obtain

\[
R_1(z) \sim \mp \sum_{k=1}^{\infty} \frac{e^{\pm 2\pi i k z}}{2\pi i k^2} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}} \tag{4.7}
\]

as \( z \to \infty \) in the sectors \( \pi/2 < \pm \arg z < \pi \).

The case \( |\arg z| = \pi/2 \) must be considered separately. On the positive imaginary axis, we put \( z = iy \), \( y > 0 \). Integrating once by parts in (2.3) and using the Fourier series of the Bernoulli polynomial \( B_2(t - \lfloor t \rfloor) \) [12, equation 24.8.1], we find

\[
R_1(iy) = -\frac{1}{2} \int_0^{+\infty} \frac{B_2(t - \lfloor t \rfloor)}{t + iy} \, dt = \frac{iy}{2} \int_0^{+\infty} \frac{B_2(t - \lfloor t \rfloor)}{t^2 + y^2} \, dt - \frac{1}{2} \int_0^{+\infty} \frac{t B_2(t - \lfloor t \rfloor)}{t^2 + y^2} \, dt,
\]

\[
= \frac{2iy}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{+\infty} \frac{\cos(2\pi nt)}{t^2 + y^2} \, dt - \frac{2}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{+\infty} \frac{t \cos(2\pi nt)}{t^2 + y^2} \, dt.
\]

The first integral can be evaluated using Fourier transforms, and we find

\[
\frac{2iy}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{+\infty} \frac{\cos(2\pi nt)}{t^2 + y^2} \, dt = -\sum_{n=1}^{\infty} \frac{1}{2} \frac{e^{-2\pi ny}}{2\pi in^2}.
\]

The second integral can be written in terms of the exponential integral [22, equation 3.723/5.], hence we arrive at

\[
-\frac{2}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{+\infty} \frac{t \cos(2\pi nt)}{t^2 + y^2} \, dt = \frac{1}{(2\pi)^2} \sum_{k=-\infty}^{\infty} \frac{1}{k^2} e^{-2\pi ky} \text{Ei}(2\pi ky). \tag{4.8}
\]

From Watson’s lemma [23, pp. 71–72; 24, pp. 20–21], we obtain that

\[
e^{\pm 2\pi ky} \text{Ei}(\pm 2\pi ky) = \pm PV \int_0^{+\infty} \frac{e^{-2\pi ky t}}{1 + t} \, dt \sim \sum_{n=1}^{\infty} \frac{(n - 1)!}{(2\pi ky)^n}, \quad k > 0,
\]

as \( y \to +\infty \), where \( \text{PV} \) denotes the Cauchy principal value. Substituting this expansion into (4.8), changing the order of summation and using the representation of the Bernoulli numbers in terms
of the Riemann zeta function [12, equation 25.6.2], we find
\[- \frac{2}{(2\pi)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{0}^{\infty} \frac{t \cos(2\pi nt)}{t^2 + y^2} dt \sim \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)(iy)^{2n}},\]
as \(y \to +\infty\). Combining all the partial results, we arrive at
\[R_1(z) \sim - \sum_{k=1}^{\infty} \frac{1}{2} \frac{e^{2\pi ikz}}{2\pi ik^2} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}},\]
as \(z \to \infty\) with \(\arg z = \pi/2\). By the connection formula \(R_1(\zeta) = R_1(z)\), we have, similarly,
\[R_1(z) \sim \sum_{k=1}^{\infty} \frac{1}{2} \frac{e^{-2\pi ikz}}{2\pi ik^2} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}}\]
as \(z \to \infty\) with \(\arg z = -\pi/2\). We can unify the series (1.3), (4.7), (4.9) and (4.10) as follows
\[\log G(z+1) \sim \frac{1}{4} z^2 + z \log \Gamma(z+1) - \left( \frac{1}{2} z(z+1) + \frac{1}{12} \right) \log z - \log A + \sum_{n=1}^{\infty} S_k(\theta) e^{\pm 2\pi ikz} + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n+1)(2n+2)z^{2n}},\]
where
\[S_k(\theta) = \begin{cases} 0 & \text{if } |\theta| < \frac{\pi}{2} \\ \mp \frac{1}{2} \frac{1}{2\pi ik^2} \quad & \text{if } \theta = \pm \frac{\pi}{2} \\ \mp \frac{1}{2\pi ik^2} & \text{if } \frac{\pi}{2} < |\theta| < \pi, \end{cases}\]
and \(\theta = \arg z\). The upper or lower sign is taken in (4.11) and (4.12) according as \(z\) is in the upper or lower half-plane. It seems that there is a discontinuous change in the coefficients of the exponential terms when \(\arg z\) changes continuously across \(\arg z = \pm \pi/2\). We have encountered a Stokes phenomenon with Stokes lines \(\arg z = \pm \pi/2\). The function \(S_k(\theta)\) is called the Stokes multiplier of the subdominant exponential \(e^{\pm 2\pi ikz}\). The forms of these multipliers are in agreement with Dingle’s non-rigorous ‘final main rule’ in his theory of terminants [25, p. 414], namely that half the discontinuity occurs on reaching the Stokes line, and half on leaving it the other side.

In the important papers [26,27], Berry provided a new interpretation of the Stokes phenomenon; he found that assuming optimal truncation, the transition between compound asymptotic expansions is of error function type, thus yielding a smooth, although very rapid, transition as a Stokes line is crossed.

Using the expansion given in theorem 1.4, we shall show in §4b that the asymptotic expansion of the Barnes G-function exhibits the Berry transition between the two asymptotic series across the Stokes lines \(\arg z = \pm \pi/2\). More precisely, we shall find that the subdominant exponentials \(e^{\pm 2\pi ikz}\) ‘switch on’ in a rapid and smooth way as \(\arg z\) pass through the values \(\pm \pi/2\).

(b) Smoothing the Stokes discontinuities

We begin with the proof of theorem 1.4. We employ the well-known expression for non-negative integer \(N_k\)
\[\frac{1}{1-x} = \sum_{n=0}^{2N_k-1} x^n + \frac{x^{2N_k}}{1-x}, \quad x \neq 1,\]
to expand the fractions under the integrals in (4.4) in powers of $\pm i t / z$, leading to the result

$$R_1(z) = - \sum_{k=1}^{\infty} \frac{1}{(2 \pi k)^2} \sum_{n=0}^{N_k-1} (-1)^n \frac{2(2n + 1)!}{(2\pi k z)^{2n+2}} + \sum_{k=1}^{\infty} \frac{1}{2 \pi k^2} \left( -\frac{1}{z^{2N_k}} \int_0^{+\infty} \frac{t^{2N_k} e^{-2\pi k t}}{z - it} \, dt + \frac{1}{z^{2N_k}} \int_0^{+\infty} \frac{t^{2N_k} e^{-2\pi k t}}{z + it} \, dt \right),$$

for $|\arg z| < \pi / 2$. The integrals can be written in terms of the terminant function, and by analytic continuation, we arrive at

$$R_1(z) = - \sum_{k=1}^{\infty} \frac{1}{(2 \pi k)^2} \sum_{n=0}^{N_k-1} (-1)^n \frac{2(2n + 1)!}{(2\pi k z)^{2n+2}} - \sum_{k=1}^{\infty} \left( \hat{T}_{2N_k+1}(2\pi k z) \frac{e^{2\pi k z}}{2\pi i k^2} + \hat{T}_{2N_k+1}(-2\pi k z) \frac{e^{-2\pi k z}}{2\pi i k^2} \right),$$

for $|\arg z| < \pi$. The absolute convergence of the infinite series involving the terminant functions is justified by the large $w$ behaviour of $\hat{T}_p(w)$ [17, p. 260]. The proof of theorem 1.4 is complete.

Now, we show that the asymptotic expansion of the Barnes $G$-function has the Berry transition property. We choose $N_k \sim \pi k |z|$ for each positive $k$. With this choice, in the upper half-plane the terms involving $\hat{T}_{2N_k+1}(2\pi k z)$ are exponentially small, the dominant contribution comes from the terms involving $\hat{T}_{2N_k+1}(-2\pi k z)$. From (4.1) and (4.3), the terminant functions have the asymptotic behaviour

$$\hat{T}_{2N_k+1}(2\pi k z) \sim \frac{1}{2} + \frac{1}{2} \text{erf} \left( \left( \theta - \frac{\pi}{2} \right) \sqrt{\pi k |z|} \right)$$

provided that $\arg z = \theta$ is close to $\pi / 2$ and $z$ is large. Similarly, in the lower half-plane, the dominant contribution is controlled by the terms involving $\hat{T}_{2N_k+1}(-2\pi k z)$. From (4.2) and (4.3), we have

$$\hat{T}_{2N_k+1}(-2\pi k z) \sim -\frac{1}{2} + \frac{1}{2} \text{erf} \left( \left( \theta + \frac{\pi}{2} \right) \sqrt{\pi k |z|} \right)$$

under the assumptions that $\arg z = \theta$ is close to $-\pi / 2$ and $z$ is large. Therefore, the approximate functional form of the Stokes multipliers near $\arg z = \pm \pi / 2$ is

$$\pm \frac{1}{2\pi i k^2} \left( \frac{1}{2} \pm \frac{1}{2} \text{erf} \left( \left( \theta \mp \frac{\pi}{2} \right) \sqrt{\pi k |z|} \right) \right).$$

For example, in the upper half-plane, the Stokes multipliers change from approximately 0 when $\theta < \pi / 2$ to approximately $-1 / 2\pi i k^2$ when $\theta > \pi / 2$. On the Stokes line $\theta = \pi / 2$, they take the value $-\frac{1}{2} (1 / 2\pi i k^2)$ up to an exponentially small error. Hence, we have proved that the transition of the Stokes multipliers across the Stokes lines $\arg z = \pm \pi / 2$ is smooth.

5. Numerical examples

In table 1, we present some numerical results that demonstrate the accuracy of our error bounds given by theorems 1.2 and 1.3. It is seen from the table that if $|\arg z| \leq \pi / 2$, then our bounds are quite accurate. However, they become worse and worse as $|\arg z|$ approaches $\pi$. Nevertheless, for $\pi / 2 < \pm \arg z < \pi$, we can use the connection formula

$$R_N(z) = \sum_{k=1}^{\infty} \frac{e^{\pm 2\pi k z}}{2\pi i k^2} + R_N(z e^{\mp \pi i}) = \mp \frac{\text{Li}_2(e^{\pm 2\pi k z})}{2\pi i} + R_N(z e^{\mp \pi i}),$$

which follows from (4.5) and (4.6).
6. Discussion

In this paper, we have established new integral representations for the remainder term of the asymptotic expansion of the logarithm of the Barnes G-function. Using these representations, we have derived new simple, explicit and numerically computable error bounds for this asymptotic series. We have found that along the imaginary axis, suddenly infinitely many exponentially small terms appear in the asymptotic expansion of the Barnes G-function. By employing one of the representations for the remainder term, we have obtained an exponentially improved asymptotic expansion for the logarithm of the Barnes G-function, which shows that the appearance of these exponentially small terms is in fact smooth, showing the Berry transition property of the asymptotic series of the G-function.

If we substitute into (1.3) the well-known asymptotic expansion of \( \log \Gamma(z + 1) \), we obtain

\[
\log G(z + 1) \sim -\frac{3}{4} z^{-2} + \frac{z}{2} \log(2\pi) + \left( \frac{1}{12} - \frac{1}{12} \right) \log z + \frac{1}{12} - \log A + \sum_{n=1}^{\infty} \frac{B_{2n+2}}{2n(2n + 2)z^{2n}},
\]

as \( z \to \infty \) in the sector \( |\arg z| < \pi - \delta < \pi \) with any fixed \( 0 < \delta \leq \pi \). This is exactly the asymptotic series that was given originally by Barnes [2]. Using the known error bounds [28] and the exponentially improved version [17, pp. 279–288] of the asymptotic series of \( \log \Gamma(z + 1) \), together with the results of this paper, similar properties of Barnes’ series (6.1) can be obtained.

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