SUMMATION METHOD IN OPTIMAL CONTROL PROBLEM WITH DELAY

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To N.K. Nikolski on occasion of his 80th birthday

1. Introduction

We consider the control problem for differential equation with distributed delay

\begin{equation}
\dot{x}(t) = x(t - 1) + \int_{-1}^{0} x(t + \tau) \varphi(\tau) d\tau + u(t), \quad t > 0
\end{equation}

with the initial conditions

\begin{equation}
x(t) = x_0(t), \quad t \in (-1, 0); \quad x(0) = \xi, \quad \xi \in \mathbb{C}, \quad x_0 \in L^2(-1, 0).
\end{equation}

The distribution of delay is defined by the function \(\varphi \in L^2(-1, 0)\) and the control function \(u\) is to be chosen to steer the initial condition \((\xi, x_0(t))\) to the desired state.

Let \(\mathcal{M} = \mathbb{C} \times L^2(-1, 0)\) be the space of all initial data \(\mathcal{M} = \{\bar{x} = (\xi, x(t)), \xi \in \mathbb{C}, \ x_0 \in L^2(-1, 0)\}\).

The equation (1.1) always has a solution \(x(t) = x(t; \bar{x}, u)\) and we deal with null controllability problem: given \(T > 1\) and \(\bar{x} \in \mathcal{M}\), find the control function \(u\) which steers the initial data to zero, i.e. \(x(t; \bar{x}, u) = 0, \ t \in (T - 1, T)\).

It is known (see e.g. [1]) that such control function always exists, and can be given by a bounded operator. Actually there exist many such functions. We consider the problem of finding the optimal control. In other we look for the problem

\begin{equation}
\|u\|_{L^2(0,T)} \rightarrow \min, \quad u \in U(x_0, \xi; T),
\end{equation}

here \(U(x_0, \xi; T)\) stays for all control functions which steers the initial data to zero at the moment \(T\).
Such kind of problems were investigated in various settings by many authors. We refer the reader to (8, 7, 10, 5) to mention a few. We also refer the reader to 7, 5 for comprehensive survey and description of the current stay of art.

We combine the semigroup techniques from 6 with the projection techniques, see e.g. 2 and 9. These techniques lead one to the study of the eigenfunction expansions for the corresponding non-selfadjoint operator. Since the corresponding systems of eigenfunctions do not form a Riesz basis, the expansions diverge, generally speaking. In 2 the authors prove the point-wise convergence on the inner intervals of the segment [T - 1, T], yet the problem of construction the optimal control on the whole interval remains open. In this article we consider a simplest model of time delay equation and suggest the reconstruction procedure which converges in the whole $L^2(0, T)$. We believe the same procedure can be applied in much more general settings.

We adjust for this case the techniques in 4, this techniques cannot be applied directly because the generating function does not satisfy the Muckenhoupt condition. Also it is worth to mention that when constructing the optimal control for a single eigenvector we arrived to already familiar problem which appeared in study of the hereditary completeness of sets of exponential functions: given a set of exponents $\Lambda \in \mathbb{C}$ such that, the corresponding exponential functions $\{\exp(i\lambda t)\}$ are incomplete in $L^2(0, T)$ we need to construct a function which is in the span of $\{\exp(i\lambda t)\}$ and is orthogonal to all but one of these exponential functions. This problem has been considered in 3 in a very general setting, however the explicit construction of such functions is a complicated problem. In the present article we restrict ourselves to the simplest model case when $\varphi = 0$.

2. THE HOMOGENEOUS PROBLEM

The corresponding homogeneous problem has the form

$$\dot{x}(t) = x(t - 1) + \int_{-1}^{0} x(t + \tau) \varphi(\tau) d\tau, \quad t > 0$$

with the initial conditions

$$\begin{pmatrix} x(0) \\ x(1) \end{pmatrix} = \vec{x}_0 \in \mathcal{M}$$

This problem admits the unique solution $x(t) = x(t, \vec{x}_0)$ and also generates the natural semigroup

$$S_t = e^{At} : \mathcal{M} \to \mathcal{M}, \quad S_t \vec{x}_0 = \begin{pmatrix} x(t) \\ x(t+1) \end{pmatrix}.$$
The infinitesimal operator of the group is defined by the relation
\[ A : \vec{x}_0 \mapsto \left( x_0(-1) + \int_{-1}^0 x_0(\tau) \varphi(\tau) d\tau \right). \]
Its domain
\[ \mathcal{D}_A = \{ \left( \frac{\xi}{x(t)} \right), \ x \in W^2_1(-1,0), \ \xi = x(0) \}. \]
Let
\[ D(z) = -iz + e^{-iz} + \int_{-1}^0 e^{i \tau z} \varphi(\tau) d\tau \]
be the characteristic function of the homogeneous problem. Its indicator diagram is \([0, i]\).
We denote by \( \Lambda = \{ \lambda_n \} \) the set of all zeros of \( D \). It can be easily seen that
\[ \lambda_n = -\pi/2 + 2\pi n + i \log |n| + o(1), \ n \in \mathbb{Z}. \]
Here and in what follows we refer the reader to [11] for facts about entire functions. We denote by \( \text{For simplicity we assume } \Lambda \subset \mathbb{C}_+ \) and that \( D \) has no multiple zeroes, this may be not true for a finite number of zeros only.

We also have
\[ |D(z)| \asymp \begin{cases} |z|, & \text{for } z = x + iy \in \mathbb{C}_+, \ y < \log x, \ \text{dist}(z, \Lambda) > \epsilon \\ e^y, & \text{for } z = x + iy \in \mathbb{C}_+, \ y > \log x, \ \text{dist}(z, \Lambda) > \epsilon. \end{cases} \]

The propositions below are straightforward (again we refer to [11] for the proof of completeness and minimality).

**Proposition 2.1.** The spectra of \( A \) is simple and coincides with \( i\Lambda \). The corresponding eigenvectors are of the form \( \vec{e}_\lambda = \left( \exp(\frac{1}{i\lambda \tau}) \right), \lambda \in \Lambda. \)

Let \( \tilde{\mathcal{E}}_\Lambda = \{ \vec{e}_\lambda \}_{\lambda \in \Lambda} \) be the set of all eigenfunctions.

**Proposition 2.2.** The system \( \tilde{\mathcal{E}}_\Lambda \) is complete and minimal in \( \mathcal{M} \).

By \( \mathcal{X}_\Lambda = \{ \vec{x}_\lambda \}_{\lambda \in \Lambda} \subset \mathcal{M}, \vec{x}_\lambda = \left( \frac{\xi_{\lambda}}{x_{\lambda}(\tau)} \right) \), we denote the system biorthogonal to \( \tilde{\mathcal{E}}_\Lambda \).

We will use the relations
\[ e^{A\tau} \vec{e}_\lambda = e^{i\lambda \tau} \vec{e}_\lambda, \ e^{A^*\tau} \vec{x}_\lambda = e^{-i\lambda \tau} \vec{x}_\lambda, \ \lambda \in \Lambda. \]

**Proposition 2.3.** The following relations take place
\[ (2.3) \quad \xi_{\lambda} = \frac{-i}{D'(\lambda)}. \]
(2.4) \[ \int_{-1}^{0} e^{itz} x(\lambda(t)) dt = \frac{1}{D(\lambda)} e^{-iz} + \int_{-1}^{0} e^{itz} \varphi(\tau) d\tau + i\lambda. \]

Since \( D(\lambda) = 0 \), one can rewrite (2.4) as

(2.5) \[ \int_{-1}^{0} e^{itz} x(\lambda(t)) dt = \frac{1}{D'(\lambda)} \left[ e^{-izt} - e^{-i\lambda t} \right] + \int_{-1}^{0} e^{-izt} - e^{-i\lambda t} \varphi(t) dt. \]

In what follows we use both (2.4) and (2.5).

Even after being normalised, the system \( \tilde{E}_\Lambda \) does not form a Riesz basis in \( \mathcal{M} \). However for each \( \bar{x} \in \mathcal{M} \) we can consider it formal Fourier series

(2.6) \[ \bar{x} \sim \sum_{\lambda \in \Lambda} \langle \bar{x}, \bar{e}_\lambda \rangle \bar{e}_\lambda. \]

3. Structure of the optimal solution

Denote the operator \( \mathcal{B} : \mathbb{C} \rightarrow \mathcal{M} \) as \( \mathcal{B}a = (0, a) \), \( a \in \mathbb{C} \). The problem (1.1), (1.2) can now be written as

(3.1) \[
\begin{cases}
\dot{\bar{x}}(t) = A\bar{x}(t) + \mathcal{B}u(t), & t > 0, \\
\bar{x}(0) = \bar{x}_0.
\end{cases}
\]

Its solution has the form

\[ \bar{x}(t) = e^{At} \bar{x}_0 + \int_{0}^{t} e^{A(t-\tau)} \mathcal{B}u(\tau) d\tau. \]

Fix now \( T > 1 \). The admissibility condition \( u \in U(T; \bar{x}_0) \) reads as

\[ e^{AT} \bar{x}_0 = -\int_{0}^{T} e^{A(T-\tau)} \mathcal{B}u(\tau) d\tau. \]

The set of exponential functions \( \mathcal{E}_\Lambda = \{e^{i\lambda t}\} \) is incomplete in \( L^2(0, T) \). Let \( \mathcal{E}_\Lambda(T) \) be the closure of its span in \( L^2(0, T) \) and \( \mathcal{E}_\Lambda(T)^\perp = L^2(0, T) \ominus \mathcal{E}_\Lambda(T) \).

We follow the result from [2], see also [9], Ch.7.4.

**Theorem 3.1.** Let \( u_\lambda \in L^2(0, T) \) solve the problem (1.3) for \( \bar{x} = \bar{e}_\lambda \) and \( v_\lambda(t) = -u_\lambda(T-t) \). Then

\[ v_\lambda \in \mathcal{E}_\Lambda(T) \]

and also

\[ \int_{0}^{T} e^{i\mu t} v_\lambda(t) dt = \delta_{\lambda, \mu}, \; \mu \in \Lambda. \]
Lemma 3.2. Let \( v \in L^2(0,T) \) be such that

\[
0 = \int_0^T e^{At} \mathcal{B}v(t) dt.
\]

Then \( v \in \mathcal{E}_\Lambda(T) \). 

Proof. Given \( \lambda \in \Lambda \) relation (3.2) yields

\[
0 = \int_0^T \langle e^{At} \mathcal{B}v(t), \bar{x}_\lambda \rangle dt = \int_0^T \langle \mathcal{B}v(t), e^{A^*t} \bar{x}_\lambda \rangle dt = \int_0^T \langle \mathcal{B}v(t), e^{-i\lambda t} \bar{x}_\lambda \rangle dt.
\]

It remains to mention that \( \mathcal{B}v(t) = \left( \begin{array}{c} v(t) \\ 0 \end{array} \right), \bar{x}_\lambda = \left( \begin{array}{c} x_\lambda(t) \\ \xi_\lambda \end{array} \right), \) therefore \( \langle \mathcal{B}v(t), e^{-i\lambda t} \bar{x}_\lambda \rangle = \bar{x}_\lambda v(t) e^{-i\lambda t} \) and \( \xi_\lambda \neq 0 \). \( \square \)

Lemma 3.3. Let \( \lambda \in \Lambda \) and \( v \in L^2(0,T) \) be such that

\[
e^{AT} \bar{x}_\lambda = \int_0^T e^{At} \mathcal{B}v(t) dt.
\]

Then

\[
\int_0^T v(t)e^{-i\mu t} dt = \bar{y}_\lambda^{-1} e^{i\lambda t} \delta_{\lambda,\mu}, \mu \in \Lambda.
\]

Proof. is similar to that of the previous lemma. We write

\[
e^{i\lambda t} \delta_{\lambda,\mu} = \langle e^{AT} \bar{x}_\lambda, \bar{x}_\mu \rangle = \int_0^T \langle e^{At} \mathcal{B}v(t), \bar{x}_\mu \rangle dt
\]

and then apply similar arguments. \( \square \)

Corollary 3.4. The optimal control linearly depends on initial data: if \( u_1 \) and \( u_2 \) are solutions of the problem (1.3) for \( \bar{x} = \bar{x}_1, \bar{x}_2 \) respectively, then \( u_1 + u_2 \) solves this problem for \( \bar{x} = \bar{x}_1 + \bar{x}_2 \).

Now in order to solve the general problem we need to reconstruct an arbitrary initial data \( \bar{x} \in \mathcal{M} \) from its expansion (2.6).

4. Summation method

4.1. Formulation of the result. In this section we describe linear summation method for the series (2.6), i.e we describe the matrices \((w_n(\lambda))_{n \in \mathbb{N}, \lambda \in \Lambda}\) with the following properties:

(4.1) \( w_n(\lambda) \to 1 \) as \( n \to \infty \), for each \( \lambda \in \Lambda \);

(4.2) \#\{n; w_n(\lambda) \neq 0\} < \infty \), for each \( n \in \mathbb{N} \);

(4.3) \( S_n \bar{x} \to \bar{x} \) as \( n \to \infty \), here \( S_n \bar{x} = \sum_{\Lambda \in \Lambda} w_n(\lambda) \langle \bar{x}, \bar{x}_\lambda \rangle \bar{x}_\lambda \).
Given the sequences \(\{l_n\}, \{R_n\}\), satisfying the conditions

\[
\begin{align*}
(4.4) & \quad l_n, R_n \to \infty \text{ as } n \to \infty, \\
(4.5) & \quad l_n^2/n \to 0, \ n/R_n \to 0 \text{ as } n \to \infty, \\
(4.6) & \quad e^{-\pi l_n^2/2 R_n} \to 0 \text{ as } n \to \infty.
\end{align*}
\]

**Remark 4.1.** It suffices, for example, to take \(l_n = |n|^{1/4}, R_n = n^4\).

Consider the function

\[
W_n(z) = e^{-l_n \frac{\pi}{2} \log \frac{z}{z+1}}, \quad z \in \mathbb{C}_+.
\]

**Theorem 4.2.** Let, for \(\lambda \in \Lambda\)

\[
w_n(\lambda) = \begin{cases} W_n(\lambda), & |\lambda| < R_n, \\ 0, & \text{otherwise} \end{cases}
\]

and the operator \(S_n\) be defined by

\[
S_n \vec{x} = \sum_{\lambda \in \Lambda} w_n(\lambda) \langle \vec{x}, \vec{e}_\lambda \rangle \vec{e}_\lambda.
\]

Then \(S_n \vec{x} \to \vec{x}\), as \(n \to \infty\).

### 4.2. Preliminary estimates.

**Lemma 4.3.**

\[
W_n(z) \xrightarrow{\text{z}} 1, \text{ as } n \to \infty \text{ on each compact set } K \subset \mathbb{C}_+.
\]

**Lemma 4.4.**

\[
|W_n(z)| < e^{-\pi l_n^2/2}, \quad z \in \mathbb{C}_+, |z| = R_n.
\]

These lemmas are straightforward.

On the real line we have

\[
W_n(t) = \begin{cases} e^{i l_n \log \frac{t_n}{t+n}}, & |t| < n, \\ e^{i l_n \log \frac{t_n}{t-n}}, & |t| > n \end{cases}
\]

**Lemma 4.5.**

\[
\sup_n \left| \int_{1<|t|<n} W_n(t) \frac{dt}{t} \right| < \infty
\]

**Lemma 4.6.**

\[
\sup_n \left| \int_{|t|>n} W_n(t) \frac{dt}{t} \right| < \infty
\]
Proof of Lemma 4.5
We have
\[ I_n := \int_{1<|t|<n} W_n(t) \frac{dt}{t} = \left( \int_{1<|t|<n/l_n} + \int_{n/l_n<|t|<n} \right) = I_{n,1} + I_{n,2} \]

The estimate of \( I_{n,1} \) is straightforward. We have for \( |t| < n/l_n \)
\[ l_n \log \left| \frac{t-n}{t+n} \right| < \frac{l_n}{n} |t| < < 1, \]
Respectively
\[ I_{n,1} = \int_{1<|t|<n/l_n} \left[ 1 + O\left( \frac{l_n}{n} |t| \right) \right] \frac{dt}{t} = O(1). \]

When estimating \( I_{2,n} \) we consider each interval \( \pm[n/l_n, n] \) separately. Consider the integral
\[ I'_{n,2} = \int_{n/l_n<t<n} e^{il_n \log \left| \frac{t-n}{t+n} \right|} \frac{dt}{t} \]
By changing of variables
\[ u = \frac{n-t}{n+t}; \ \tau = 1-u; \ v = \log(1-\tau), \]
and extracting the main term on each step, we finally arrive to the integral \( \int^\infty_{1/n} e^{il_n v} \frac{dt}{v} \),
which is bounded uniformly in \( n \).

Proof of Lemma 4.6 goes in the same vien. We write
\[ \int_{|t|>n} W_n(t) \frac{dt}{t} = \left( \int_{|t|>nl_n} + \int_{n<|t|<nl_n} \right) W_n(t) \frac{dt}{t} \]
and use the similar reasonings.

4.3. Proof of Theorem 4.2. a. Let operator \( S_n \) be defined by (4.3). It follows from (4.9) that \( S_n\bar{x} \to \bar{x} \) as \( n \to \infty \) on the dense set of finite linear combinations of \( \bar{e}_\lambda \). So it suffices to prove the uniform boundedness of all operators \( S_n \). Consider instead the adjoint operator
\[ S^*\bar{y} = \sum_\lambda w_n(\lambda) \langle \bar{y}, \bar{e}_\lambda \rangle \bar{x}_\lambda. \]
Actually it is more convenient to deal with the conjugate objects:
\[ \bar{S^*}\bar{y} = \sum_\lambda w_n(\lambda) \langle \bar{e}_\lambda, \bar{y} \rangle \bar{x}_\lambda =: \bar{s}_n. \]
Let \( \vec{s}_n = (s_n(t)) \) and \( \vec{y} = (y(t)) \). Since \( \vec{x}_\lambda = (x_\lambda(t)) \) we have

\[
\sigma_n = \sum_\lambda \lambda \eta \bar{\xi}_\lambda + \sum_\lambda \lambda \bar{\eta} \xi_\lambda \int_{-1}^{0} e^{i\lambda t} \bar{y}(t) dt = \sigma_n^{(1)} + \sigma_n^{(2)},
\]

(4.11)

\[
s_n(t) = \sum_\lambda \lambda \eta \bar{x}_\lambda(t) + \sum_\lambda \lambda \bar{\eta} \xi_\lambda \int_{-1}^{0} e^{i\lambda \tau} \bar{y}(\tau) d\tau \bar{x}_\lambda(t) = s_n^{(1)}(t) + s_n^{(2)}(t).
\]

(4.12)

Each of the four summands in the right of (4.11), (4.12) should be estimates separately.

Consider the contour \( \Gamma_n = L_n \cup C_n \). Here \( L_n = [-R_n, R_n] \), \( C_n = \{ R_n e^{i\theta}; \theta \in [0, \pi] \} \). The arcs \( C_n \) uniformly in \( n \) satisfy the Carleson condition, hence, for any function \( h \) from the Hardy space \( H(\mathbb{C}_+) \),

\[
\left( \int_{C_n} |h(\zeta)|^2 |d\zeta| \right)^{1/2} \leq \text{Const} \|h\|.
\]

We also assume that \( \text{dist}(\Lambda, \Gamma_n) > 0 \) uniformly in \( n \).

b. Estimate of \( \sigma_n^{(1)} \). We use (2.3). By the residues

\[
2i\pi \sigma_n^{(1)} = \int_{\Gamma_n} \frac{W_n(\zeta)}{D(\zeta)} d\zeta = \left( \int_{C_n} + \int_{L_n} \right) \frac{W_n(\zeta)}{D(\zeta)} d\zeta = I_n + J_n.
\]

(4.13)

That \( I_n \) is bounded (and even tends to zero as \( n \to 0 \)) is very clear, this follows from (4.6), (4.10), and (2.2). In order to see the boundedness of \( J_n \) it suffices to mention that, according Lemma 4.6 the integral

\[
\int_{L_n} \frac{W_n(t)}{it + 1} dt
\]

is bounded uniformly in \( n \). Therefore it suffices to establish the boundedness of

\[
\int_{L_n} W_n(t) \left\{ \frac{1}{D(t)} - \frac{1}{it + 1} \right\} dt.
\]

The later is evident since the expression in parenthesis uniformly \( O(t^{-2}) \) as \( t \to \infty \).

c. Estimate of \( \sigma_n^{(2)} \) Denote

\[
Y(z) = \int_{-1}^{0} e^{iz\tau} y(\tau) d\tau.
\]

(4.14)

As before we have

\[
2i\pi \sigma_n^{(2)} = \int_{\Gamma_n} W_n(\zeta) \frac{Y(\zeta)}{D(\zeta)} d\zeta = \left( \int_{C_n} + \int_{L_n} \right) W_n(\zeta) \frac{Y(\zeta)}{D(\zeta)} d\zeta = I_n + J_n.
\]

The estimate of \( J_n \) is now straightforward because \( Y \in L^2(\mathbb{R}) \) uniformly in \( y \) in the unit ball of \( L^2(-1,0) \) and \( |D(t)| \asymp |t| \) as \( t \to \pm \infty \).
In order to estimate \( I_n \) we observe that the function \( \Psi(\zeta) = e^{izY(\zeta)} \) belongs to the Hardy space \( H^2(\mathbb{C}_+) \) and has the same \( L^2(\mathbb{R}) \) norm as \( Y \). Also, it follows from (2.2) that
\[
\left| \frac{Y(\zeta)}{D(\zeta)} \right| < |\Psi(\zeta)|, \, \zeta \in C_n.
\]

Now we use that all \( C_n \)'s are uniformly Carleson curves in \( \mathbb{C}_+ \) and also (4.10). We obtain
\[
|J_n| \lesssim e^{-ln\pi/2} \int_{C_n} |\Psi(\zeta)||d\zeta| \leq e^{-ln\pi/2}|C_n|^{1/2} \left( \int_{C_n} |\Psi(\zeta)|^2|d\zeta| \right)^{1/2} \lesssim R_n^{1/2}e^{-ln\pi/2} \|\Psi\|.
\]

It remains to refer to (4.6).

d. Estimate of \( \|s_n^{(1)}\| \). We apply the Fourier transform and use representation (2.5):
\[
(s_n^{(1)})^*(x) = \sum_{\lambda} w_n(\lambda) \frac{1}{D'(\lambda)} \left( e^{-izt} - e^{-i\lambda t} \right) \frac{1}{z - \lambda} + \int_{-1}^{0} \frac{e^{-izt} - e^{-i\lambda t}}{z - \lambda} \varphi(t) dt
\]

Therefore it suffices to obtain uniform in \( t \) estimates of \( \|\Phi_{n,t}\|_{L^2(\mathbb{R})} \) where
\[
\Phi_{n,t}(x) = \sum_{\lambda} w_n(\lambda) \frac{1}{D'(\lambda)} e^{-izt} - e^{-i\lambda t} \frac{1}{x - \lambda}
\]

Since \( \Lambda \subset \mathbb{C}_+ \) this function belongs to the Hardy space \( \mathcal{H}^2(\mathbb{C}_-) \) and
\[
\|\Phi_{n,t}\|_{L^2(\mathbb{R})} = \sup\left\{ \left| \int_{-\infty}^{\infty} \Phi_{n,t}(x)h(x)dx \right| : h \in \mathcal{H}^2(\mathbb{C}_+), \|h\| = 1 \right\}.
\]

We obtain two terms to be estimated:
\[
\sum_{\lambda} \frac{w_n(\lambda)}{D'(\lambda)} e^{i\lambda t} \int_{-\infty}^{\infty} \frac{h(x)}{x - \lambda} dx = 2i\pi \sum_{\lambda} \frac{w_n(\lambda)}{D'(\lambda)} e^{-i\lambda t} h(\lambda) =: a_{1,n},
\]

and
\[
\sum_{\lambda} \frac{w_n(\lambda)}{D'(\lambda)} \int_{-\infty}^{\infty} \frac{e^{i\lambda t} h(x)}{x - \lambda} dx =: a_{2,n}.
\]

Once again we have
\[
a_{1,n} = \left( \int_{L_n} + \int_{C_n} \right) W_n(\zeta) \frac{e^{i\zeta t}}{D(\zeta)} h(\zeta) d\zeta.
\]

The estimate of the first integral follows blantly from the Schwartz inequality. In order to estimate the second one we observe that \( |e^{it\zeta}/D(\zeta)| \lesssim 1, \zeta \in C_n \) uniformly in \( t \) and \( n \) and then again use the fact that all \( C_n \)'s are uniformly Carlesonian together with (4.6):
\[
\int_{C_n} W_n(\zeta) \frac{e^{i\zeta t}}{D(\zeta)} h(\zeta) d\zeta \lesssim \int_{C_n} |W_n(\zeta)||h(\zeta)||d\zeta| \lesssim e^{-\pi n/2} R_n^{1/2} \|h\| \lesssim \|h\|.
\]
In order to estimate $a_{n,2}$ we represent $h$ as

$$h(x) = \int_{0}^{\infty} e^{i\xi x} \omega(\xi) d\xi$$

for some $\omega \in L^2(0, \infty)$.

For $t \in [-1, 0]$ we have

$$h(x) = \int_{t}^{0} e^{i\xi x} \omega(\xi - t) d\xi + \int_{0}^{\infty} e^{i\xi x} \omega(\xi) d\xi = h_t^- (x) + h_t^+ (x).$$

Then $h_t^\pm \in \mathcal{H}(\mathbb{C}_\pm)$, $\|h_t^\pm\| \leq \|h\|$ and

$$a_{2,n} = \sum_{\lambda} w_n(\lambda) \frac{h_t^+(\lambda)}{D(\lambda)}.$$

The rest of the estimate is similar to that of $a_{n,1}$.

e. Estimate of $\|s_n^{(2)}\|$. Again we apply the Fourier transform but now use representation (2.4).\footnote{2.4}.

$$\widehat{(s_n^{(2)})}(x) = \sum_{\lambda} w_n(\lambda) Y(\lambda) \frac{e^{-ix} + \int_{-1}^{0} e^{ix\varphi(\tau)} d\tau + i\lambda}{x - \lambda},$$

here $Y(\lambda)$ is defined in (4.14).

Once again we use the duality reasons

$$\|\widehat{s_n^{(2)}}\| = \sup \{ \left| \int_{-\infty}^{\infty} \widehat{s_n^{(2)}}(x) h(x) dx \right| : h \in \mathcal{H}^2(\mathbb{C}_+), \|h\| \leq 1 \}.$$

Let the functions $h_t$ be defined as in (4.16). Consider the function

$$f(x) = h_{-1} + \int_{-1}^{0} h_t(x) \varphi(t) dt.$$

We have $f \in \mathcal{H}^2(\mathbb{C}_+)$, $\|f\| \prec 1$ and

$$\int_{-\infty}^{\infty} \widehat{s_n^{(2)}}(x) h(x) dx = \sum_{\lambda} w_n(\lambda) \frac{1 + i\lambda}{D(\lambda)} Y(\lambda) f(\lambda).$$

This expression can be estimated in a similar way as when estimating $\|s_{n,1}\|$. \hfill \Box

We have proved the following statement.

**Theorem 4.7.** Given the control problem (1.1), (1.2), let $u_0(t)$ be the optimal solution defined by (1.3). Let, further, $\Lambda$ and $\tilde{E}_\Lambda$ be the corresponding eigenvalues and eigenvectors and also

$$\tilde{x}_0 \sim \sum_{\lambda \in \Lambda} \langle \tilde{x}_0, \tilde{x}_\lambda \rangle \tilde{e}_\lambda$$
be the formal Fourier expansion of the initial data $\bar{x}_0$.

Then

$$u_0(t) = \lim_{n \to \infty} \sum_{\lambda} w_n(\lambda) \langle \bar{x}_0, \bar{x}_\lambda \rangle u_\lambda(t),$$

here $w_n(\lambda)$ are defined in (4.7) and the functions $u_\lambda \in L^2(0, T)$ are defined by the relations

$$u_\lambda(t) = -v_\lambda(T - t); \quad v_\lambda \in \mathcal{E}(\Lambda), \quad \int_0^T e^{i\mu t} v_\lambda(t) dt = \delta_{\lambda, \mu}, \quad \mu \in \Lambda.$$ 

The limit exists in $L^2(0, T)$ sense.

5. A MODEL CASE

a. In this section we consider the simplest model case

$$\dot{x}(t) = x(t - 1) + u(t), \quad t > 0; \quad (5.1)$$

$$x(t) = x_0(t), \quad t \in (-1, 0), \quad x(0) = \xi_0. \quad (5.2)$$

For this case we give an explicit construction of the corresponding functions $v_\lambda$ for $T \in (1, 2)$. We denote $\delta = T - 1$. The characteristic function has the form

$$D(z) = -iz + e^{-iz}. \quad (5.3)$$

b. Description of $\mathcal{E}(\Lambda)^\perp$. Let $g \in L^2(0, T), g \in \mathcal{E}(\Lambda)^\perp$. Then the entire function

$$G(z) = \int_0^T e^{itz} g(t) dt$$

vanishes on $\Lambda$ and the standard reasonings on comparing its growth with that of $D(z)$ yield

$$G(z) = e^{it} D(z) \Omega(z), \quad (5.4)$$

where $\Omega$ runs through the set of all functions of the form

$$\Omega(z) = \int_0^{\delta} e^{itz} \omega(t) dt, \quad x \Omega(x) \in L^2(\mathbb{R}) \quad (5.5)$$

The later yields

$$\omega \in W^2_1(0, \delta), \quad \omega(0) = \omega(\delta) = 0 \quad (5.6)$$

and, by (5.3) - (5.4), we obtain

$$g(t) = \begin{cases} 
    \omega'(t - 1), & t \in (1, 1 + \delta), \\
    \omega(t), & t \in (0, \delta), \\
    0, & \text{otherwise}
\end{cases}$$
This is the description of $E(\Lambda)^\perp$.

c. Description of the biorthogonal system. Fix $\lambda \in \Lambda$ and let a function $v_\lambda \in L^2(0, T)$ satisfy
\[
\int_0^T e^{i\mu t}v_\lambda(t)dt = \delta_{\lambda, \mu}, \quad \mu \in \Lambda.
\]
Denote
\[
V_\lambda(z) = \int_0^T e^{izt}v_\lambda(t)dt.
\]
As before we have
\[
V_\lambda(z) = e^{i\lambda D'(\lambda)}e^{iz}D(z),
\]
where the function
\[
P_\lambda(z) = \int_0^\delta e^{itz}p_\lambda(t)dt, \quad p_\lambda \in L^2(0, \delta)
\]
Satisfies
\[
(5.7) \quad P_\lambda(\lambda) = \int_0^\delta e^{ixp_\lambda(t)}dt = 1.
\]
We remind that $D(\lambda) = 0$. Therefore
\[
\frac{e^{ix}D(x)}{x - \lambda} = -i e^{ix} - i e^{i(x-\lambda)} - 1 = -i \left[ e^{ix} + \int_0^1 e^{i\tau}e^{-i\lambda \tau}d\tau \right]
\]
and, denoting $a_\lambda = 1/(e^{i\lambda}D'(\lambda))$ we obtain
\[
v_\lambda(t) = -ia_\lambda \left[ p_\lambda(t-1) + \int_{t-1}^t e^{-i\lambda(t-u)}p_\lambda(u)du \right]
\]
When $p_\lambda$ runs through $L^2(0, \delta)$ the function $v_\lambda$ runs through all functions in $L^2(0, T)$ orthogonal to $E_\Lambda \setminus \{e^{i\lambda t}\}$. The condition $v_\lambda \in E_\Lambda^\perp$ leads one to condition on $p_\lambda$.

d. We assume that the functions $\omega, p_\lambda$ are defined on the whole $\mathbb{R}$ and vanish outside of $(0, \delta)$. The condition $v_\lambda \in E_\Lambda$ takes the form
\[
(5.8) \quad 0 = \int_0^T [\omega(t) + \omega'(t-1)] \left[ p_\lambda(t-1) + e^{-i\lambda t} \int_{t-1}^t e^{-i\lambda \tau}p_\lambda(\tau)d\tau \right] dt,
\]
for all $\omega \in W_1^2(0, \delta)$, $\omega(0) = \omega(\delta) = 0$.

When opening the parenthesis we obtain four summands $A_i, i = 1, 2, 3, 4$. Consider each of them separately.
\[
A_1 = \int_0^T \omega(t)p_\lambda(t-1)dt = 0,
\]
Therefore, (5.9) implies the equation
\[
A_2 = \int_0^T \omega(t) \int_{t-1}^t \overline{p_\lambda} \tau e^{-i\lambda(t-\tau)} d\tau = \int_0^\delta \omega(t)(\overline{p_\lambda} * \varepsilon_\lambda)(t) dt;
\]
\[
A_3 = \int_0^T \omega'(t-1) p_\lambda(t-1) dt = -\int_0^T \omega(t-1) p_\lambda'(t-1) dt.
\]
Here we assume that \( p_\lambda' \) exists. With this assumption we will arrive to equation with respect to \( p_\lambda' \) which has smooth solution. This suffices since we already know that there is only one \( p_\lambda \), satisfying all conditions.

When considering \( A_4 \) we use (5.7). We have
\[
A_4 = \int_0^T \omega'(t-1) e^{-i\lambda t} \int_{t-1}^t \overline{p_\lambda} \tau e^{i\lambda \tau} d\tau dt = \int_0^T \omega'(t-1) e^{-i\lambda t} \int_{t-1}^t \overline{p_\lambda} \tau e^{i\lambda \tau} d\tau dt =
\]
\[
e^{-i\lambda t} \int_0^\delta \omega(t) e^{-i\lambda t} \left( 1 - \int_{t-1}^t \overline{p_\lambda} \tau e^{i\lambda \tau} d\tau \right) dt = e^{-i\lambda t} \left[ i\lambda \int_0^\delta \omega(t) e^{-i\lambda t} dt - \int_0^\delta \omega(t)(\overline{p_\lambda} * \varepsilon_\lambda)'(t) dt \right].
\]

We collecting these relations together and remark that \( \omega \) can be any function in \( W^2_1(0, \delta) \), satisfying (5.6). We then arrive to the equation
\[
(5.9) \quad -\overline{p_\lambda}'(t) - e^{-i\lambda}(\overline{p_\lambda} * \varepsilon_\lambda)'(t) + (\overline{p_\lambda} * \varepsilon_\lambda)(t) - e^{-i\lambda} \varepsilon_\lambda'(t) = 0, \quad 0 < t < \delta.
\]
The left-hand side of this equation is well-defined for all \( t > 0 \). Denote it by \( \alpha(t) \). Let \( \mathcal{P}_\lambda(s) \) and \( e^{-sA}(s) \) be the Laplace transform of \( \overline{p_\lambda} \) and \( \alpha \) respectively. Taking once again into account that \( D(\lambda) = 0 \), we transform (5.9) into equation for \( \mathcal{P}_\lambda(s) \):
\[
\mathcal{P}_\lambda(s)(1 - s^2) = (\lambda^2 - i\lambda \overline{p_\lambda}(0)) + sp_\lambda(0) + e^{-sA}(s).
\]
The last summand in the right-hand side does not influence the values of \( \overline{p_\lambda} \) on \( (0, \delta) \). Therefore
\[
\overline{p_\lambda}(t) = [\lambda^2 - i\lambda \overline{p_\lambda}(0)] \sinh t + \overline{p_\lambda}(0) \cosh t, \quad t \in (0, \delta).
\]
The value \( \overline{p_\lambda}(0) \) should now be chosen to meet (5.7). We omit the corresponding calculation.

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