Analytical calculation formulas for capacities of classical and classical-quantum channels

Masahito Hayashi  Fellow, IEEE

Abstract

We derive an analytical calculation formula for the channel capacity of a classical channel without any iteration while its existing algorithms require iterations and the number of iteration depends on the required precision level. Hence, our formula is its first analytical formula without any iteration. We apply the obtained formula to examples and see how the obtained formula works in these examples. Then, we extend it to the channel capacity of a classical-quantum (cq-) channel. Many existing studies proposed algorithms for a cq-channel and all of them require iterations. Our extended analytical algorithm have also no iteration and output the exactly optimum values.

Index Terms

mutual information, maximization, channel capacity, classical-quantum channel, analytical algorithm

I. INTRODUCTION

One of key problems in classical and quantum information theory is the maximization of information quantities. However, it is not so easy to perform such a maximization analytically because all of existing methods require a certain iterations, whose number depends on the required precision level. The most common maximization problem is the channel capacity, which is given as the maximization of mutual information [5], and its calculation has been studied by Arimoto [1], Blahut [2], and their related studies [6], [7], [8]. However, these are iterative approximation algorithms to calculate the maximum of the mutual information. In addition, the reference [3] calculated only its upper bound and the references [23], [9] developed other type of method to approximately calculate it. Hence, they cannot calculate the exact value for the channel capacity. As variants, the references [13], [24] extended the above method to the wire-tap capacity [11], [12] when the wire-tap channel is degraded. Also, the references [14], [15], [16], [17], [18] extended it to the quantum setting, so called the capacity of classical-quantum channel. However, these results are also iterative approximation algorithms.

This paper proposes an algorithm to analytically calculate the channel capacity of the classical channel without iteration. The proposed algorithm is composed of solving simultaneous linear equations and calculation of logarithm and exponential because it employs an information-geometrical structure. However, the proposed method works under certain conditions. Since our method is analytical, we can derive several analytical formula for the capacity when these conditions are satisfied. Then, to see this possibility, we apply our algorithm to several examples, and derives analytical expressions of the capacities in these examples. Further, we extend our analytical algorithm to the calculations of the capacity of classical-quantum channel.

The remaining part of this paper is organized as follows. First, Section II derives our algorithm for the capacity of a classical channel. Section III applies the obtained result to several examples. Next, Section IV extends this method to the capacity of classical-quantum channel. Finally, Section V discusses the merit and the demerit of our method over existing methods.

Masahito Hayashi is with Shenzhen Institute for Quantum Science and Engineering, Southern University of Science and Technology, Nanshan District, Shenzhen, 518055, China, International Quantum Academy (SIQA), Futian District, Shenzhen 518048, China, Guangdong Provincial Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Nanshan District, Shenzhen 518055, China, and Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan. (e-mail: hayashi@sustech.edu.cn, masahito@math.nagoya-u.ac.jp)
II. Capacity of classical channel

We consider the input and output alphabets \( \mathcal{X} := \{1, \ldots, n_1\} \) and \( \mathcal{Y} := \{1, \ldots, n_2\} \) that are finite sets. We denote the sets of probability distributions on \( \mathcal{X} \) and \( \mathcal{Y} \) by \( \mathcal{P}_\mathcal{X} \) and \( \mathcal{P}_\mathcal{Y} \), respectively. For distributions \( P, Q \in \mathcal{P}_\mathcal{X} \), the entropy \( H(P) \) and the divergence \( D(P||Q) \) are defined as

\[
H(P) := -\sum_{x \in \mathcal{X}} P(x) \log P(x),
\]

\[
D(P||Q) := \sum_{x \in \mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}.
\]

A channel from \( \mathcal{X} \) to \( \mathcal{Y} \) is given as conditional distribution on \( \mathcal{Y} \) conditioned with \( \mathcal{X} \). That is, using the notation \( W_x(y) := W(y|x) \), it can be considered as a map \( W : \mathcal{X} \to \mathcal{P}_\mathcal{Y} \). For \( Q_X \in \mathcal{P}_\mathcal{X} \) and \( Q_Y \in \mathcal{P}_\mathcal{Y} \), \( W \cdot Q_X \in \mathcal{P}_\mathcal{Y} \), \( W \times Q_X \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}} \), and \( Q_X \times Q_Y \in \mathcal{P}_{\mathcal{X} \times \mathcal{Y}} \) are defined by \( (W \cdot Q_X)(x, y) := \sum_{x \in \mathcal{X}} W(y|x)Q_X(x) \), \( (W \times Q_X)(x, y) := W(y|x)Q_X(x) \), and \( (Q_X \times Q_Y)(x, y) := Q_X(x)Q_Y(y) \) respectively.

The channel capacity of a channel \( W \) is given by [5], [26] p.124

\[
C(W) := \max_{Q_X \in \mathcal{P}_\mathcal{X}} \sum_{x \in \mathcal{X}} Q_X(x) D(W_x||W \cdot Q_X)
\]

\[
= \min_{Q_Y \in \mathcal{P}_\mathcal{Y}} \max_{x \in \mathcal{X}} D(W_x||Q_Y)
\]

\[
= \min_{Q_X \in \mathcal{P}_\mathcal{X}} \max_{x \in \mathcal{X}} D(W_x||W \cdot Q_X). \tag{3}
\]

To discuss \( C(W) \), we assume the following conditions.

\( (A) \quad W_1, \ldots, W_{n_1} \) are linearly independent.

Then, we have the following;

**Lemma 1:** Assume Condition (A). When a distribution \( Q_Y = W \cdot Q_X \) realizes the maximum in (3), it satisfies the following condition; \( D(W_x||Q_Y) \) does not depend on \( x \in \text{supp}(Q_X) \).

Lemma [1] is shown in Appendix [B]. We define the set \( \mathcal{M}_0 \) as

\[
\mathcal{M}_0 := \{ Q_Y \in \mathcal{P}_\mathcal{Y} \mid Q_Y = \sum_{x \in \mathcal{X}} c(x)W_x, \sum_{x \in \mathcal{X} \setminus \mathcal{X}_0} c(x) = 1 \}. \tag{4}
\]

Due to Condition (A), only one distribution \( Q_Y \in \mathcal{M}_0 \) satisfies the following condition;

\( (B) \quad D(W_x||Q_Y) \) does not depend on \( x \in \mathcal{X} \).

In the following, we denote the element of \( \mathcal{M}_0 \) to satisfy the condition (B) by \( Q_{Y,*} \). Since \( Q_{Y,*} \) belongs to \( \mathcal{M}_0 \), there exists a function \( \hat{Q}_{X,*} \) on \( \mathcal{X} \) as the solution of the following equation;

\[
\sum_{x \in \mathcal{X}} W(y|x)\hat{Q}_{X,*}(x) = Q_{Y,*}(y). \tag{5}
\]

Condition (A) guarantees the uniqueness of \( \hat{Q}_{X,*} \). We have the following theorem;

**Theorem 1:** Assume Condition (A). The following conditions are equivalent

(i) The relation \( D(W_x||Q_{Y,*}) = C(W) \) holds.

(ii) The function \( \hat{Q}_{X,*} \) satisfies the condition

\[
\hat{Q}_{X,*}(x) \geq 0 \text{ for } x \in \mathcal{X} \tag{6}
\]

Theorem [1] is shown in Appendix [C].

Hence, under the condition (ii), the capacity \( C(W) \) is given by \( D(W_x||Q_{Y,*}) \). To consider the case that the condition (ii) does not hold, we prepare the following theorem. For any function \( f \) on \( \mathcal{X} \), we define \( \mathcal{N}(f) := \{ x \in \mathcal{X} \mid f(x) < 0 \} \) and \( \mathcal{N}^c(f) := \{ x \in \mathcal{X} \mid f(x) \geq 0 \} \).
Theorem 2: Assume Condition (A). Then, we have

$$C(W) = \max_{Q_X \in \mathcal{P}_{X \to W}(Q_{X*,+})} \sum_{x \in \mathcal{N}^c(Q_{X*,+})} Q_X(x) D(W_x \| W \cdot Q_X).$$

(7)

Theorem 2 is shown in Appendix D. Therefore, the capacity $C(W)$ is obtained only with the input set $\mathcal{N}^c(Q_{X*,+})$. That is, the function $\hat{Q}_{X*}$ gives an important information for computing $C(W)$.

We choose $n_2 - 1$ linearly independent functions $f_1, \ldots, f_{n_2 - 1}$ on $\mathcal{Y}$ such that

$$\sum_{y \in \mathcal{Y}} W_{n_2} (y) f_j (y) = 0$$

(8)

for $j = 1, \ldots, n_2 - 1$. We define the matrix $(h_{i,j})$

$$h_{i,j} := \sum_{y \in \mathcal{Y}} W_i (y) f_j (y).$$

(9)

Given an $n_2 - 1$-dimensional parameter $\theta = (\theta^1, \ldots, \theta^{n_2 - 1})$, we define the distribution $P_{\theta,Y}$ as

$$P_{\theta,Y}(y) = e^{\sum_{j=1}^{n_2 - 1} f_j (y) \theta^j - \phi(\theta)},$$

(10)

where

$$\phi(\theta) := \log \left( \sum_{y \in \mathcal{Y}} e^{\sum_{j=1}^{n_2 - 1} f_j (y) \theta^j} \right).$$

(11)

The parameterization (10) is called the natural parameter [4].

We have the following theorem.

Theorem 3: Assume that the parameters $\theta^1, \ldots, \theta^{n_2 - 1}$ satisfies the condition

$$\sum_{j=1}^{n_2 - 1} h_{i,j} \theta^j = -H(W_i) + H(W_{n_2}).$$

(12)

Then, we have

$$D(W_x \| P_{\theta,Y}) = \phi(\theta) - H(W_{n_2}).$$

(13)

for $x \in \mathcal{X}$.

Proof: The condition (12) implies that

$$\sum_{y \in \mathcal{Y}} W_i (y) \sum_{j=1}^{n_2 - 1} f_j (y) \theta^j = \sum_{j=1}^{n_2 - 1} h_{i,j} \theta^j$$

$$= -H(W_i) + H(W_{n_2}).$$

(14)

For $x(\neq n_2) \in \mathcal{X}$, we have

$$D(W_x \| P_{\theta,Y}) = \sum_{y \in \mathcal{Y}} W_x (y) \left( \log W_x (y) - \log P_{\theta,Y} (y) \right)$$

$$= -H(W_x) - \sum_{y \in \mathcal{Y}} W_x (y) \left( \sum_{j=1}^{n_2 - 1} f_j (y) \theta^j - \phi(\theta) \right)$$

$$= -H(W_x) - \left( -H(W_x) + H(W_{n_2}) - \phi(\theta) \right)$$

$$= \phi(\theta) - H(W_{n_2}).$$

(15)
Also, we have

\[
D(W_n||P_{\theta,Y}) = \sum_{y \in \mathcal{Y}} W_{n_2}(y) \left( \log W_{n_2}(y) - \log P_{\theta,Y}(y) \right)
\]

\[
= -H(W_{n_2}) - \sum_{y \in \mathcal{Y}} W_x(y) \left( \sum_{j=1}^{n_2-1} f_j(y) \theta^j - \delta(\theta) \right)
\]

\[
= -H(W_{n_2}) - \left( -\delta(\theta) \right) = \delta(\theta) - H(W_{n_2}).
\]

(16)

We define the set \( \mathcal{E}_0 \) as

\[
\mathcal{E}_0 := \{ P_{\theta,Y} \mid \text{The condition (12) holds.} \}
\]

(17)

**Lemma 2**: The set \( \mathcal{M}_0 \cap \mathcal{E}_0 \) is composed of one element \( P_{\theta,Y} \).

**Lemma 3** is shown in Appendix E. Therefore, \( P_{\theta,Y} \) equals \( Q_{Y,s} \).

Now, we assume the following condition (Condition (C)).

(C) \( n_1 = n_2 \) and \( W_1, \ldots, W_{n_1} \) are linearly independent.

Then, only one set of parameters \( \theta^1, \ldots, \theta^{n_2-1} \) satisfies the condition (12). Due to Theorem 3, solving the equation (12), we find \( Q_{Y,s} \) as \( P_{\theta,Y} \). To construct our algorithm, we add the \( n_2 \)-th function \( f_{n_2} \) on \( \mathcal{Y} \) and define \( h_{i,j} \) by (9) for \( i, j = 1, \ldots, n_2 \). We rewrite the equation (5) as

\[
\sum_{x \in \mathcal{X}} \hat{Q}_{X,s}(x) h_{x,j} = \sum_{x \in \mathcal{X}} \hat{Q}_{X,s}(x) \sum_{y \in \mathcal{Y}} W_x(y) f_j(y)
\]

\[
= \sum_{y \in \mathcal{Y}} P_{\theta,Y}(y) f_j(y).
\]

(18)

We obtain the function \( \hat{Q}_{X,s} \) on \( \mathcal{X} \) as the solution of (18), and \( W \cdot \hat{Q}_{X,s} = Q_{Y,s} \) satisfies the condition (B). When the function \( \hat{Q}_{X,s} \) satisfies the condition (6), the value \( D(W_x||P_{\theta,Y}) \) is the capacity of the channel \( W \) due to Theorem 3. Therefore, we have Algorithm 1 to compute \( C(W) \) under Condition (C).

In fact, \( (W_i(j))_{i,j} \) and \( (f_j(i))_{i,j} \) form \( n_2 \times n_2 \) matrices. When \( (f_j(i))_{i,j} \) is the inverse matrix of \( (W_i(j))_{i,j} \), \( (h_{i,j})_{i,j} \) is the identity matrix. Due to Theorem 1, Theorem 3 does not necessarily work for calculating \( C(W) \). Hence, based on Theorems 1 and 3, we propose Algorithm 1 to check the condition in Theorem 1 and output compute \( C(W) \) under this condition.

In Algorithm 1, Step 1 has calculation complexity \( O(n_2^3) \). Steps 2 and 3 have calculation complexity \( O(n_2^3) \) because \( h_{i,j} \) is an upper triangle matrix. Step 5 has calculation complexity \( O(n_2^3) \). Hence, the total calculation complexity is \( O(n_2^3) \).

**Algorithm 1** Exact algorithm for classical channel capacity

Step 1: Choose \( f_1, \ldots, f_{n_2} \) such that \( (f_j(i))_{i,j} \) is the inverse matrix of \( (W_i(j))_{i,j} \). Hence, \( h_{i,j} = \delta_{i,j} \).

Step 2: Set the parameter \( \theta^i = -H(W_i) + H(W_{n_2}) \) for \( i = 1, \ldots, n_2 - 1 \), which is the solution of (12).

Step 3: Calculate \( \delta(\theta) \) by using (11).

Step 4: Calculate \( \hat{Q}_{X,s}(x) \) := \( \sum_{y \in \mathcal{Y}} P_{\theta,Y}(y) f_x(y) \), where \( P_{\theta,Y}(y) \) is calculated by (10). This step follows from (18).

Step 5: If the condition (6) holds, we consider that the condition in Theorem 1 holds and output \( \delta(\theta) - H(W_n) \) as the capacity. Otherwise, we consider that the condition in Theorem 1 does not hold and output “the capacity cannot be computed.”

Next, instead of Condition (C), we consider the following condition.
(C') The relation $n_1 \geq n_2$ holds. Any $n_2$ elements among $W_1, \ldots, W_{n_1}$ are linearly independent. Under this condition, we choose $n_2$ elements $x_1, \ldots, x_{n_2}$ in $\mathcal{X}$ so that we can apply Algorithm [1] if the capacity is calculated, it is denoted by $C(W; x_1, \ldots, x_{n_2})$.

In this case, we need to try $\binom{n_1}{n_2}$ combinations, which requires too large calculation amount. However, it is possible to avoid such repetition as follows. First, we apply the conventional iterative algorithm by [11], [2] or the improved iterative algorithm by [8]. Then, we obtain an approximately optimal input distribution. If the distribution has the majority of the probability in $n_2$ elements of $\mathcal{X}$, we apply Algorithm [1] to the case when $\mathcal{X}$ is the above $n_2$ elements. In this case, we do not need $\binom{n_1}{n_2}$ repetitions. That is, the above hybrid method works for exact calculation.

However, if $C(W; x_1, \ldots, x_{n_2})$ depends on the choice of $n_2$ elements $x_1, \ldots, x_{n_2}$, and the difference is very small, this idea does not work. In this case, it is expected that the approximately optimal input distribution the majority of the probability in more than $n_2$ elements of $\mathcal{X}$. Hence, the above method does not work.

Also, even under Condition (C'), there is the case that the support of the optimal input distribution is composed of a smaller element than $n_2$. In this case, even when we apply Algorithm [1] for $\binom{n_1}{n_2}$ combinations, we cannot obtain the capacity.

When only Condition (A) holds, $Q_{Y,\ast}$ can be characterized as follows.

**Theorem 4:** Assume that $h_{i,j} = 0$ for $j = n_1 + \ldots, n_2 - 1$ and the parameters $\theta^1, \ldots, \theta^{n_2-1}$ satisfies the condition [9]. When the parameters $\theta^{n_1}, \ldots, \theta^{n_2-1}$ are given as

$$
(\theta^{n_1}, \ldots, \theta^{n_2-1}) = \arg\min_{\eta^{n_1}, \ldots, \eta^{n_2-1}} \phi(\theta^1, \ldots, \theta^{n_1-1}, \eta^{n_1}, \ldots, \eta^{n_2-1}),
$$

we have $P_{\theta,Y} = Q_{Y,\ast}$.

**Proof:** Since the objective function in [19], the parameters $\theta^{n_1}, \ldots, \theta^{n_2-1}$ achieves the minimum [19] if and only if

$$
\frac{\partial \phi(\theta^1, \ldots, \theta^{n_2-1})}{\partial \theta^j} = 0 \quad \text{for} \quad j = n_1, \ldots, n_2 - 1.
$$

The above condition is equivalent to

$$
\sum_{y \in \mathcal{Y}} f_j(y) P_{\theta,Y}(y) = 0 \quad \text{for} \quad j = n_1, \ldots, n_2 - 1.
$$

Due to [113], when $\theta^{n_1}, \ldots, \theta^{n_2-1}$ are given by [19], $P_{\theta,Y}$ belongs to $\mathcal{M}_0$. Due to the uniqueness by Lemma [2] we obtain the desired statement.

Theorem 4 guarantees that $Q_{Y,\ast}$ is given as the solution of the minimization [19], which is a convex minimization. While the analytical solution of [19] is difficult in general, it is possible in the following case. We impose the following condition for the functions $f_{n_1}, \ldots, f_{n_2-1}$; (i) $f_j(y)$ takes non-zero value only with two elements $y_j, y'_j \in \mathcal{Y}$ for $j = n_1 + \ldots, n_2 - 1$. (ii) The sets $\{y_j, y'_j\}$ for $j = n_1 + \ldots, n_2 - 1$ are disjoint with each other.

In this case, the relation [19], i.e., [20], can be simplified as

$$
f_j(y_j) e^{f_j(y_j)\theta^j + \sum_{i=1}^{n_1-1} f_i(y_i)\theta^i} + f_j(y'_j) e^{f_j(y'_j)\theta^j + \sum_{i=1}^{n_1-1} f_i(y'_i)\theta^i} = 0
$$

for $j = n_1 + \ldots, n_2 - 1$. The equation [22] is solved as

$$
\theta^j = \frac{1}{f_j(y_j) - f_j(y'_j)} \left( \sum_{i=1}^{n_1-1} (f_i(y'_i) - f_i(y_j))\theta^i + \log \frac{-f_j(y'_j)}{f_j(y_j)} \right)
$$

for $j = n_1 + \ldots, n_2 - 1$. Therefore, we can analytically calculate $P_{\theta,Y} = Q_{Y,\ast}$ under the conditions (i) and (ii).
III. Example

A. Output system with two elements

First, we consider the case with \( Y = \{1, 2\} \). When \( \mathcal{X} \) and the channel \( W \) satisfies Condition (C') in this case, the method described after Condition (C') works well as follows. For any two elements \( x_1 \neq x_2 \in \mathcal{X} \), the channel only with two inputs \( x_1, x_2 \) always satisfies the condition (6) because \( Q_{Y,x} \) is located between \( W_{x_1} \) and \( W_{x_2} \). Hence, the condition in Theorem 1 holds. Therefore, it is sufficient to derive a general formula for the capacity when two elements in \( \mathcal{X} \) are fixed.

Therefore, in the following, we consider the case with \( \mathcal{X} = \{1, 2\} \) and \( Y = \{1, 2\} \). We define the distributions \( W_x \) for \( x \in \mathcal{X} \) by the following vector form:

\[
W_1 := \begin{pmatrix} 1 - p \\ p \end{pmatrix}, \quad W_2 := \begin{pmatrix} 1 - q \\ q \end{pmatrix}.
\]

For simplicity, we assume that \( q > p \). We define the \( 2 \times 2 \) matrix \( V \) as \( V := (W_1, W_1) \). The inverse matrix is

\[
V^{-1} = \frac{1}{q - p} \begin{pmatrix} q & q - 1 \\ -p & 1 - p \end{pmatrix}
\]

In this case, the parameter \( \theta \) is one-dimensional and is solved to \( h(q) - h(p) \), where \( h(p) \) is the binary entropy. Then, \( \phi(\theta) \) is calculated as

\[
\phi(\theta) = \log(e^{\frac{q(h(q) - h(p))}{q-p}} + e^{\frac{-p(h(q) - h(p))}{q-p}})
= \log(e^{\frac{q(h(q) - h(p))}{q-p}(1 + e^{-\frac{(q+p)(h(q) - h(p))}{q-p}}))
= \frac{q(h(q) - h(p))}{q - p} + \log(1 + e^{-\frac{(q+p)(h(q) - h(p))}{q-p}}).
\]

The capacity is calculated as

\[
C(W) = \phi(\theta) - h(p)
= \frac{p(h(q) - qh(p))}{q - p} + \log(1 + e^{-\frac{(q+p)(h(q) - h(p))}{q-p}}),
\]

which is a general capacity formula with \( \mathcal{X} = \{1, 2\} \) and \( Y = \{1, 2\} \). Then,

\[
P_{\theta,Y} = \begin{pmatrix} e^{\frac{q(h(q) - h(p))}{q-p}} - \phi(\theta) \\ e^{-\frac{p(h(q) - h(p))}{q-p}} - \phi(\theta) \end{pmatrix}.
\]

Hence, the optimal input distribution is

\[
\hat{Q}_{x,*} = \begin{pmatrix} \frac{1}{q - p} \left( q - e^{-\frac{p(h(q) - h(p))}{q-p}} - \phi(\theta) \right) \\ \frac{1}{q - p} \left( -p + e^{-\frac{p(h(q) - h(p))}{q-p}} - \phi(\theta) \right) \end{pmatrix}.
\]

B. Output system with three elements

1) General problem description: Next, we consider the case with \( Y = \{1, 2, 3\} \). In this case, Algorithm 1 does not necessarily work even under the condition (C). Moreover, the method described after Condition (C') does not necessarily work even under the condition (C'). To see such a case, we consider the following
example with $\mathcal{X} = \{1, 2, 3, 4\}$ and $\mathcal{Y} = \{1, 2, 3\}$ with $\epsilon \in [0, 1/2]$. We define the distributions $W_x$ for $x \in \mathcal{X}$ by the following vector form:

$$W_1 := \begin{pmatrix} 1 - \epsilon \\ 0 \\ \epsilon \end{pmatrix}, \quad W_2 := \begin{pmatrix} 0 \\ 1 - \epsilon \\ \epsilon \end{pmatrix}$$

$$W_3 := \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}, \quad W_4 := \begin{pmatrix} \frac{1}{2} - \epsilon \\ \frac{1}{2} - \epsilon \\ 2 \epsilon \end{pmatrix}.$$  \hspace{1cm} (31)

We define $3 \times 3$ matrix $V_j$ for $j \in \mathcal{X}$ as $V_1 := (W_2, W_3, W_4)$, $V_2 := (W_1, W_3, W_4)$, $V_3 := (W_1, W_2, W_4)$, $V_4 := (W_1, W_2, W_3)$. Their inverse matrices are

$$V_1^{-1} = \begin{pmatrix} \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & 0 \\ \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2} \\ \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{1}{2} \end{pmatrix}$$

$$V_2^{-1} = \begin{pmatrix} \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & 0 \\ \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2} \\ \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{1}{2} \end{pmatrix}$$

$$V_3^{-1} = \begin{pmatrix} \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & 0 \\ \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2} \\ \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{1}{2} \end{pmatrix}$$

$$V_4^{-1} = \begin{pmatrix} \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & 0 \\ \frac{1}{2(1-\epsilon)} & \frac{1}{2(1-\epsilon)} & \frac{1}{2} \\ \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{2(1-\epsilon)}{2-2\epsilon} & \frac{1}{2} \end{pmatrix}.$$  \hspace{1cm} (35)

Also, we have

$$H(W_1) = H(W_2) = h(\epsilon)$$

$$H(W_3) = \log 2, \quad H(W_4) = h(2\epsilon) + (1 - 2\epsilon) \log 2.$$  \hspace{1cm} (36), \hspace{1cm} (37)

When we apply Algorithm 1 to the three components in $V_j$, we denote $\theta$, $\phi(\theta)$, $P_{\theta,Y}$, $\hat{Q}_{X,*}$, and $\phi(\theta) - H(W_n)$ by $\theta_j$, $\phi_j(\theta_j)$, $P_{j,Y}$ and $\hat{Q}_{j,X}$, and $C_j$, respectively. In the following, we discuss our model dependently of the value of $j$.

2) Case that $j = 4$: First, we consider the case that $j = 4$, i.e., the channel is composed of three inputs $\{1, 2, 3\}$. Then, we have

$$\theta_4 = \begin{pmatrix} h_{4,\epsilon} \\ h_{4,\epsilon} \end{pmatrix}$$

with $h_{4,\epsilon} := \log 2 - h(\epsilon)$ and

$$\phi_4(\theta_4) = \log(2 + e^{h_{4,\epsilon}}).$$

Thus,

$$C_4 = \phi_4(\theta_4) - \log 2 = \log(1 + \frac{e^{h_{4,\epsilon}}}{2})$$

$$P_{4,Y} = \begin{pmatrix} e^{-\phi_4(\theta_4)} \\ e^{-\phi_4(\theta_4)} \\ e^{h_{4,\epsilon}-\phi_4(\theta_4)} \end{pmatrix}.$$  \hspace{1cm} (41)
Therefore,

$$
\hat{Q}_{4,X} = V_4^{-1} P_{4,Y} = \begin{pmatrix}
\frac{1}{2} e^{h_{4,x} \phi_4(\theta_4)} \\
\frac{1}{2e} e^{h_{4,x} \phi_4(\theta_4)} \\
2e^{-\phi_4(\theta_4)} - \frac{1-\epsilon}{\epsilon} e^{h_{4,x} \phi_4(\theta_4)}
\end{pmatrix}.
$$

(Eq. 42)

While the first and second components of $\hat{Q}_{4,X}$ are always positive value, the third component has a possibility to have a negative value. The non-negativity of the first component is equivalent to the following condition;

$$
1 \geq g_1(\epsilon),
$$

(Eq. 43)

where $g_1(\epsilon) := \frac{1-\epsilon}{2e} e^{h_{4,x}}$. In (43), the first inequality corresponds to the non-negativity of the third component and the second inequality corresponds to the non-negativity of the first component. Fig. 1 numerically plots the function $g_1(\epsilon)$. It shows that $\hat{Q}_{4,X}$ is a probability distribution when $0.3588 \leq \epsilon$. That is, $C_4$ is achievable for $0.3588 \leq \epsilon$. When $\epsilon < 0.3588$, the third component of $\hat{Q}_{4,X}$ is negative. Hence, $C_4$ is not achievable. Due to Theorem 2, the optimal input distribution in this case has the support in $\{1, 2\}$. In this case, due to the symmetry, the uniform distribution on $\{1, 2\}$ is optimal. That is, the capacity with the input set $\{1, 2, 3\}$ is

$$
C_* := -(1-\epsilon) \log \frac{1-\epsilon}{2} - \epsilon \log \epsilon - h(\epsilon) = (1-\epsilon) \log 2.
$$

(Eq. 44)

Fig. 1. Graphs of functions $g_1$, $g_2$, and $g_3$ with logarithmic scale. Red dashed curve expresses $g_1$. Blue dashed curve expresses $g_2$. Purple solid curve expresses $g_3$. Green solid line expresses 1. Red dashed curve $g_1$ and Blue dashed curve $g_2$ across Green solid line at 0.3588 and 0.4286, respectively.
3) Case that \( j = 3 \): We consider the case that \( j = 3 \), i.e., the channel is composed of three inputs \( \{1, 2, 4\} \). Then, we have

\[
\theta_3 = \begin{pmatrix} h_{3,\epsilon} \\ h_{3,\epsilon} \end{pmatrix}
\]  

(45)

with \( h_{3,\epsilon} := h(2\epsilon) + (1-2\epsilon) \log 2 - h(\epsilon) \) and

\[
\phi_3(\theta_3) = \log(2e^{2h_{3,\epsilon}} + e^{-\frac{(1-2\epsilon)h_{3,\epsilon}}{\epsilon}}) = \log(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}) + 2h_{3,\epsilon}.
\]  

(46)

Thus,

\[
C_3 = \phi_3(\theta_3) - h(2\epsilon) - (1-2\epsilon) \log 2
\]

\[
= \log(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}) + 2h_{3,\epsilon} - h(2\epsilon) - (1-2\epsilon) \log 2
\]

\[
= \log(2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}) + h(2\epsilon) + (1-2\epsilon) \log 2 - 2h(\epsilon)
\]  

(47)

Hence,

\[
P_{3,Y} = \frac{1}{2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}} \begin{pmatrix} 1 & 1 \\ 1 & e^{-\frac{h_{3,\epsilon}}{\epsilon}} \end{pmatrix}.
\]  

(48)

Therefore,

\[
\hat{Q}_{3,X} = V_3^{-1} P_{3,Y}
\]

\[
= \frac{1}{2 + e^{-\frac{h_{3,\epsilon}}{\epsilon}}} \begin{pmatrix} 1 - \frac{1-2\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \\ 1 - \frac{1-2\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \\ -2 + \frac{1}{\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \end{pmatrix}.
\]  

(49)

While the first and second components of \( \hat{Q}_{3,X} \) are always positive value, the third component has a possibility to have a negative value. The non-negativity of the first component is equivalent to the following condition;

\[
g_2(\epsilon) \geq 1 \geq g_3(\epsilon),
\]  

(50)

where \( g_2(\epsilon) := \frac{1-\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \) and \( g_3(\epsilon) := \frac{1-2\epsilon}{2\epsilon} e^{-\frac{h_{3,\epsilon}}{\epsilon}} \). In (50), the first inequality corresponds to the non-negativity of the third component and the second inequality corresponds to the non-negativity of the first component. However, as numerically plotted in Fig. 1, \( g_3(\epsilon) \leq 1 \) for \( \epsilon < \frac{1}{2} \) and \( g_2(\epsilon) < 1 \) for \( \epsilon < 0.4286 \). Hence, when \( \epsilon \geq 0.4286 \), \( C_3 \) is achievable, i.e., it gives the capacity under the case \( j = 3 \).

When \( \epsilon < 0.4286 \), the third component of \( \hat{Q}_{3,X} \) is negative. In this case, due to Theorem 2, the optimal distribution has support \( \{1, 2\} \).

4) Case that \( j = 1 \): Next, we consider the case that \( j = 1 \), i.e., the channel is composed of three inputs \( \{2, 3, 4\} \). Then, we have

\[
\theta_1 = \begin{pmatrix} h_{3,\epsilon} \\ h_{1,\epsilon} \end{pmatrix}
\]  

(51)

with \( h_{1,\epsilon} := h(2\epsilon) - 2\epsilon \log 2 \) and

\[
\phi_1(\theta_1) = \log\left( e^{-\frac{1}{1-\epsilon} h_{3,\epsilon} - \frac{1}{2(1-\epsilon)} h_{1,\epsilon}} + e^{\frac{1}{1-\epsilon} h_{3,\epsilon} - \frac{1}{2(1-\epsilon)} h_{1,\epsilon}} + e^{\frac{1}{2} h_{1,\epsilon}} \right)
\]

\[
= \log\left( \frac{1}{1-\epsilon} \left( \frac{1-2\epsilon}{4} \right)^{\frac{1}{1-\epsilon}} + (1-\epsilon) \left( \frac{1-2\epsilon}{4} \right)^{-\frac{1}{1-\epsilon}} + 4\epsilon(1-2\epsilon) \right) + h_{1,\epsilon}
\]  

(52)
Thus,
\[
C_1 = \phi_1(\theta_1) - h(2\epsilon) - (1 - 2\epsilon) \log 2
\]
\[
= \log \left( \frac{1}{1 - \epsilon} \left( \frac{1 - 2\epsilon}{4} \right)^\frac{1 - 2\epsilon}{2 - 2\epsilon} + (1 - \epsilon) \left( \frac{1 - 2\epsilon}{4} \right)^\frac{1 - 2\epsilon}{2 - 2\epsilon} + 4\epsilon(1 - 2\epsilon)^\frac{1 - 2\epsilon}{2 - 2\epsilon} \right) - \log 2
\]  
(53)

Since
\[
e^{-\frac{1}{2\epsilon} h_1,\epsilon} = 4\epsilon(1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}},
\]  
(54)
we have
\[
P_{1,Y} = \begin{pmatrix}
\frac{-1}{1 - \epsilon^2} & \frac{1 - 2\epsilon}{2 - 2\epsilon} + \frac{1 - 2\epsilon}{4} - \frac{1 - 2\epsilon}{2 - 2\epsilon} \\
\frac{1 - 2\epsilon}{2 - 2\epsilon} & \frac{1 - 2\epsilon}{2 - 2\epsilon} - \frac{1 - 2\epsilon}{4} + 2(1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}}
\end{pmatrix}.
\]
(55)

Therefore,
\[
\hat{Q}_{1,X} = V_{1}^{-1} P_{1,Y}
\]
\[
= e^{h_{1,\epsilon} - \phi_1(\theta_1)} \begin{pmatrix}
\frac{3 - 2\epsilon}{2(1 - \epsilon)} \frac{1 - 2\epsilon}{2 - 2\epsilon} + \frac{1 - 2\epsilon}{4} - \frac{1 - 2\epsilon}{2 - 2\epsilon} \\
\frac{1 - 2\epsilon}{2(1 - \epsilon)} \frac{1 - 2\epsilon}{2 - 2\epsilon} - \frac{1 - 2\epsilon}{4} + 2(1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2 - 2\epsilon}}
\end{pmatrix}.
\]
(56)

Taking the limit \(\epsilon \to 0\), we have
\[
\lim_{\epsilon \to 0} \hat{Q}_{1,X} = \begin{pmatrix}
\frac{7}{10} & \frac{3}{5} \\
\frac{3}{5} & \frac{7}{10} - \frac{4}{9} \frac{10}{9}
\end{pmatrix}.
\]
(57)

Fig. 2 shows numerical plots of \(\hat{Q}_{1,X}(2), \hat{Q}_{1,X}(3), \text{ and } \hat{Q}_{1,X}(4)\). Although \(\hat{Q}_{1,X}(3)\) and \(\hat{Q}_{1,X}(4)\) are always positive, \(\hat{Q}_{1,X}(2)\) is positive only for \(\epsilon \geq 0.3972\). Hence, when \(\epsilon < 0.3972\), due to Theorem 2, the capacity of case \(j = 1\) equals the capacity of the channel with inputs 3 and 4. In this case, we cannot use Algorithm 1 because the number of input system is smaller than the number of the output system. Assume that \(P_X(3) = 1 - p\) and \(P_X(4) = p\). The mutual information between \(X\) and \(Y\) is
\[
(1 - 2\epsilon p) \log 2 + h(2\epsilon p) - (1 - p) \log 2 - p(h(2\epsilon) + (1 - 2\epsilon) \log 2)
\]
\[
= h(2\epsilon p) - ph(2\epsilon).
\]
(58)

Then, the maximum mutual information is achieved when \(p = \frac{1}{2\epsilon(1 + e^{\frac{1 - 2\epsilon}{2\epsilon}})} = \frac{1}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}} = \frac{1}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}} \). The capacity of this case is
\[
C_{1*} := h(\frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}}) - h(\frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}})
\]
\[
= \frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}} \log(2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}})
\]
\[
- (1 - \frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}}) \log(1 - \frac{2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}}) - \frac{1 - 2\epsilon}{2\epsilon + (1 - 2\epsilon)^{\frac{1 - 2\epsilon}{2\epsilon}}} \log(1 - 2\epsilon)
\]  
(59)

Due to the symmetry, we can discuss the case with \(j = 2\).
5) Derivation of $C(W)$: Based on the above discussion, we discuss the capacity of the channel $W$ with input system $\mathcal{X} = \{1, 2, 3, 4\}$. When $0 \leq \epsilon \leq 0.3588$, $C_1$ is the capacity for the case $j = 1$, and $C_*$ is the capacity for the cases $j = 3, 4$. Since $C_* \geq C_1$ in this case, $C_*$ is the capacity of the channel $W$.

When $0.3588 < \epsilon \leq 0.3972$, $C_1$ is the capacity for the case $j = 1$, $C_4$ is the capacity for the cases $j = 3$, and $C_4$ is the capacity for the cases $j = 4$. Since $C_4 \geq C_1, C_1$ in this case, $C_4$ is the capacity of the channel $W$.

In fact, as seen in Fig. 2, 4 curves $C_1, C_3, C_4$, and $C_*$ intersect at 0.3972. For $0.3972 < \epsilon \leq 1/2$, $C_*$ is the capacity for the case $j = 1$, $C_3$ or $C_*$ is the capacity for the cases $j = 3$, and $C_4$ is the capacity for the cases $j = 4$. Since $C_* \geq C_3, C_4$ in this case, $C_*$ is the capacity of the channel $W$. Overall, the capacity $C(W)$ of the channel $W$ is calculated as follows.

$$C(W) = \begin{cases} 
C_* & \text{when } 0 \leq \epsilon \leq 0.3588 \\
C_4 & \text{when } 0.3588 < \epsilon \leq 0.3972 \\
C_* & \text{when } 0.3972 < \epsilon \leq 1/2 
\end{cases} \quad (60)$$

IV. Capacity of Classical-Quantum Channel

Next, we discuss a classical-quantum channel from the classical system $\mathcal{X} := \{1, \ldots, n_1\}$ to the quantum system $\mathcal{H}$ with dimension $n_2$, which is given as a set of density matrices $\{W_j\}_{j=1}^{n_1}$. We denote the set of density matrices on $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$. For density matrices $\rho, \sigma \in \mathcal{S}(\mathcal{H})$, the entropy $H(\rho)$ and the divergence $D(\rho||\sigma)$ are defined as

$$H(\rho) := -\text{Tr} \rho \log \rho, \quad D(\rho||\sigma) := \text{Tr} \rho (\log \rho - \log \sigma). \quad (61)$$
Fig. 3. Graphs of functions $C_1, C_3, C_4, C_*$ and $C_{**}$. Black solid curve expresses $C_1$, Blue dashed curve expresses $C_3$. Red dashed curve expresses $C_4$. Green solid line expresses $C_*$. Purple solid curve expresses $C_{**}$. Its enlarged view is given as Fig. 4.

Fig. 4. Enlarged view of graphs of functions $C_1, C_3, C_4, C_*$ and $C_{**}$. The explanations for 5 curves are the same as Fig. 3. 4 curves $C_1, C_3, C_4$, and $C_*$ intersect at 0.3972. In particular, $C_1$ touches $C_{**}$ at 0.3972 $C_3$ and $C_4$ touch $C_*$ at 0.3588 and 0.4286, respectively. That is, the inequalities $C_1 \geq C_{**}$ and $C_3, C_4 \geq C_*$ hold always.
Under this classical-quantum channel, given an input probability distribution \( P \in P_\mathcal{X} \) on the classical system \( \mathcal{X} \), the capacity of classical-quantum channel \( W = \{W_j\}_{j=1}^{n_1} \) is defined as \([27], [28], [29], [30]\)

\[
C_q(W) := \max_{P \in P_\mathcal{X}} \sum_{x \in \mathcal{X}} P(x) D\left( W_x \parallel \sum_{x' \in \mathcal{X}} P(x') W_{x'} \right).
\] (62)

The capacity of classical-quantum channel has the following form \([31], [32]\)

\[
C_q(W) = \min_{\sigma \in S(\mathcal{H})} \max_{x \in \mathcal{X}} D(W_x \parallel \sigma),
\] (63)

Statements similar to statements in section II can be shown in this case of cq-channel by using quantum information geometry based on Kubo-Mori-Bogoliubov Fisher information, which is directly linked to quantum relative entropy \(61) \[4], [38, Chapter 7\]. Here, for the calculation of \( C_q(W) \), we consider only the algorithm corresponding to Algorithm 1. Hence, we consider the case under the following condition similar to Condition (C).

(D) \( n_1 = n_2^2 \) and \( W_1, \ldots, W_{n_2^2} \) are linearly independent.

Then, we consider the following condition;

(E) \( D(W_x \parallel \sigma) \) does not depend on \( x \in \mathcal{X} \).

Since the dimension of the convex full of \( \{W_x\}_{x \in \mathcal{X}} \) equals the dimension of \( S(\mathcal{H}) \), due to Condition (D), only one density matrix \( \sigma \) on \( \mathcal{H} \) satisfies the condition (E). We denote such a density matrix by \( \sigma_* \). The relation \(63) \) guarantees that

\[
C_q(W) \leq D(W_x \parallel \sigma_*) \tag{64}
\]

for \( x \in \mathcal{X} \).

We choose \( n_2^2 - 1 \) linearly independent Hermitian matrices \( A_1, \ldots, A_{n_2^2-1} \) on \( \mathcal{H} \) such that

\[
\text{Tr} W_{n_2^2} A_j = 0
\] (65)

for \( j = 1, \ldots, n_2^2 - 1 \). We define the matrix \((h_{i,j})\)

\[
h_{i,j} := \text{Tr} W_i A_j.
\] (66)

Given an \( n_2^2 - 1 \)-dimensional parameter \( \theta = (\theta^1, \ldots, \theta^{n_2^2-1}) \), we define the density matrix \( \rho_\theta \) as

\[
\rho_\theta = \exp \left( \sum_{j=1}^{n_2^2-1} A_j^i \theta^j - \phi(\theta) \right),
\] (67)

where

\[
\phi(\theta) := \log \text{Tr} \exp \left( \sum_{j=1}^{n_2^2-1} A_j^i \theta^j \right).
\] (68)

We have the following theorem.

\textbf{Theorem 5:} Assume that the parameters \( \theta^1, \ldots, \theta^{n_2^2-1} \) satisfies the condition

\[
\sum_{j=1}^{n_2^2-1} h_{i,j} \theta^j = -H(W_i) + H(W_{n_2^2}). \tag{69}
\]

Then, we have

\[
D(W_x \parallel \rho_\theta) = \phi(\theta) - H(W_{n_2^2}) = C_q(W), \tag{70}
\]

for \( x \in \mathcal{A} \).
Hence, when we have

\[ \text{Tr} W_i \sum_{j=1}^{n_2-1} A_j \theta_j^i = \sum_{j=1}^{n_2-1} h_{i,j} \theta_j^i = -H(W_i) + H(W_{n_2}^*). \]  

(71)

For \( x(\neq n_2^2) \in \mathcal{X} \), we have

\[ D(W_x \| \rho_{\theta}) = \text{Tr} W_x (\log W_x - \log \rho_{\theta}) \]
\[ = -H(W_x) - \text{Tr} W_x \left( \sum_{j=1}^{n_2-1} A_j \theta_j^i - \phi(\theta) \right) \]
\[ = -H(W_x) - (-H(W_x) + H(W_{n_2}^*) - \phi(\theta)) \]
\[ = \phi(\theta) - H(W_{n_2}^*). \]  

(72)

Also, we have

\[ D(W_{n_2}^* \| \rho_{\theta}) = \text{Tr} W_{n_2} (\log W_{n_2}^* - \log \rho_{\theta}) \]
\[ = -H(W_{n_2}^*) - \text{Tr} W_x \left( \sum_{j=1}^{n_2-1} A_j \theta_j^i - \phi(\theta) \right) \]
\[ = -H(W_{n_2}^*) - (-\phi(\theta)) = \phi(\theta) - H(W_{n_2}^*). \]  

(73)

Hence, when \( D(W_x \| \rho_{\theta}) \) does not depend on \( x \in \mathcal{X} \), it realizes the capacity.

Proof: The condition (12) implies that

\[ \text{Tr} W_i \sum_{j=1}^{n_2-1} A_j \theta_j^i = \sum_{j=1}^{n_2-1} h_{i,j} \theta_j^i = -H(W_i) + H(W_{n_2}^*). \]  

(71)

Due to Theorem 5, when \( \theta \) satisfies the condition (69), the density matrix \( \rho_{\theta} \) equals \( \sigma_* \). To construct our algorithm, we add the \( n_2^2 \)-th Hermitian matrix \( A_{n_2} \) and define \( h_{i,j} \) by (66) for \( i, j = 1, \ldots, n_2^2 \). To find the input distribution \( \hat{Q}_{X,*} \) to achieve the maximum (63), we consider the equation \( \sum_x W(y|x) \hat{Q}_{X,*}(x) = \rho_{\theta} \), which can be rewritten as

\[ \sum_{x \in \mathcal{X}} \hat{Q}_{X,*}(x) h_{x,j} \left( = \sum_{x \in \mathcal{X}} \hat{Q}_{X,*}(x) \text{Tr} W_x A_j \right) = \text{Tr} \rho_{\theta} A_j. \]  

(74)

If we have

\[ \hat{Q}_{X,*}(x) \geq 0 \text{ for } x \in \mathcal{X}, \]  

(75)

since \( \rho_{\theta} = \sigma_* \) (64) guarantees that

\[ D(W_x \| \rho_{\theta}) = C_q(W), \]  

(76)

i.e., the solution gives the capacity.

Therefore, in the same way as Algorithm 1, we propose Algorithm 2 based on Theorem 5. Now, we describe two Hermitian matrices \( X, Y \) on \( \mathcal{H} \) by two \( n^2 \)-dimensional vectors \( x = (x_j)_{j=1}^{n_2^2} \) and \( y = (y_j)_{j=1}^{n_2^2} \) as follows.

\[ X = \sum_{j=1}^{n_2^2} x_j |j \rangle \langle j| \]
\[ + \sum_{j=1}^{n_2^2} \sum_{j'=1}^{n_2^2} \frac{x_{n_2^2+j}(j-1)/2+j'}{\sqrt{2}} (|j \rangle \langle j'| + |j' \rangle \langle j|) \]
\[ + \sum_{j=1}^{n_2^2} \sum_{j'=1}^{n_2^2} \frac{x_{n_2^2+j-1}(j-1)/2+j'}{\sqrt{2}} (i|j \rangle \langle j'| - i|j' \rangle \langle j|). \]  

(77)
Then, we have
\[ \text{Tr } XY = \sum_{j=1}^{n_2^2} x_j y_j. \] (78)

In this sense, \((W_1, \ldots, W_{n_2^2})\) and \((A_1, \ldots, A_{n_2^2})\) can be considered as \(n_2^2 \times n_2^2\) matrices. Then, Step 1 of Algorithm 2 can be done by calculating the inverse matrix of the matrix corresponding to \((W_1, \ldots, W_{n_2^2})\). Hence, Step 1 has calculation complexity \(O(n_2^4)\). In Step 2, the calculation of all of \(H(W_i)\) needs calculation complexity \(O(n_1 n_2^3)\). Hence, Step 2 has calculation complexity \(O(n_2^5)\) in total. In Step 3, the calculation of \(\sum_{j=1}^{n_2^2-1} A_j \theta^j\) has calculation complexity \(O(n_2^6)\). and the calculation of \(\exp\left(\sum_{j=1}^{n_2^2-1} A_j \theta^j\right)\) and its trace has calculation complexity \(O(n_2^3)\). Step 3 has calculation complexity \(O(n_2^4)\) in total. In Step 5, the calculation of all of \(\text{Tr } \rho_\theta A_j\) has calculation complexity \(O(n_1 n_2^3)\) since exponentiation \(\exp\left(\sum_{j=1}^{n_2^2-1} A_j \theta^j\right)\) and its trace are already calculated. Hence, the total calculation complexity is \(O(n_2^6)\).

Algorithm 2: Exact algorithm for classical channel capacity

Step 1: Choose \(A_1, \ldots, A_{n_2^2}\) such that \(h_{i,j}\) is the identity matrix.
Step 2: Set the parameter \(\theta^j = -H(W_i) + H(W_{n_2^2})\) for \(i = 1, \ldots, n_2^2 - 1\), which is the solution of (69).
Step 3: Calculate \(\phi(\theta)\) by using (68).
Step 4: Calculate \(Q_{X,*}(x) := \text{Tr } \rho_\theta A_j\), where \(\rho_\theta\) is calculated by (67).
Step 5: If the condition (75) holds, we consider that (76) holds and output \(\phi(\theta) - H(W_n)\) as the capacity. Otherwise, we output “the capacity cannot be computed.”

V. COMPARISON

In the calculation of the capacity of classical channel, when an error \(\epsilon\) is allowed, the conventional method \([11, 2]\) has calculation amount \(O\left(n_1 n_2 \log \frac{n_1}{\epsilon}\right)\) because each iteration has calculation amount \(n_1 n_2\) and the number of iteration is \(O\left(\frac{\log n_1}{\epsilon}\right)\). While it is smaller than our method (Algorithm 1) when \(n_1 = n_2\), our method derives the exact value of the maximum without iteration.

When only Condition (A) holds, we can consider to solve the minimization (19) due to Theorem 4. However, it is difficult to analytically solve (19) in general. Since this method needs larger calculation amount to obtain \(\theta^1, \ldots, \theta^{n_1-1}\), the algorithm based on Theorem 4 does not have advantage over the conventional method \([11, 2]\) except for the case that the minimization (19) is analytically solved.

Next, we compare Algorithm 2 with existing algorithms for the capacity of a classical-quantum channel. The algorithm by \([14, 18]\) has calculation complexity \(O\left(\max(n_1, n_2) n_2^3 \log n_1 + n_1 n_2^3\right)\). The algorithm by \([16]\) has calculation complexity \(O\left(\frac{\max(n_1, n_2) n_2^3 \log n_1}{\epsilon}\right)\). Unfortunately, these existing algorithms are smaller than our method, Algorithm 2 when \(n_1 = \frac{n_2}{3}(n_2 - 1) + 1\) or \(n_1 = n_2^2\). However, our method derives the exact value of the maximum without iteration, which is an advantage over existing methods.

VI. CONCLUSION AND FUTURE STUDY

We have proposed an exact algorithm to calculate the channel capacities of classical and classical-quantum channels. However, we have various conditions to apply our formulas. Therefore, it is a future problem to remove conditions. Indeed, Toyota \([10]\) studied information geometrical structure \([4]\) for Arimoto-Blahut algorithm for the capacity of a classical channel. Hence, it is an interesting topic to derive an information theoretical characterization of our method.

Further, it is a challenging problem to extend our method to the maximization of Gallager’s function, i.e., Rényi mutual information, including classical-quantum setting, which is related to the exponential
decreasing rate [33], [34] of the decoding error probability and the strong converse exponent [35], [36], [37]. As another future study, we can consider an extension of our formulas to wire-tap channel capacity [11], [12], [13], [24].

ACKNOWLEDGMENTS

The author was supported in part by the National Natural Science Foundation of China (Grant No. 62171212) and Guangdong Provincial Key Laboratory (Grant No. 2019B1212030002). The author is very grateful to Mr. Shoji Toyota for helpful discussions.

APPENDIX A

SUMMARY FOR INFORMATION GEOMETRY

To show Theorems [1] and [2] we summarize basic knowledge for information geometry, which was established in the reference [4]. The following contents are used in Appendices [C] and [D]. Given a finite probability space \( X \) and a distribution \( P_X \) on \( X \), we define an exponential family as follows. Consider \( l \) linearly independent random variables \( f_1, \ldots, f_l \) on \( X \). We define the distribution \( P_{\theta,X} \) as

\[
P_{\theta,X}(x) := P_X(x)e^{\sum_{j=1}^{l} \theta_j f_j(x) - \phi(\theta)},
\]

where \( \phi(\theta) := \log \sum_{x \in X} P_X(x)e^{\sum_{j=1}^{l} \theta_j f_j(x)} \). The set \( \mathcal{E} := \{ P_{\theta,X} | \theta \in \mathbb{R}^l \} \subseteq \mathcal{P}_X \) is called an exponential family generated by random variables \( f_1, \ldots, f_l \). Also, the set

\[
\mathcal{M} := \{ Q_X \in \mathcal{P}_X | Q_X \text{ satisfies } (81) \}
\]

is called the mixture family generated by the constraint

\[
\sum_{x \in X} f_j(x)Q_X(x) = a_j.
\]

The following is a typical example of a mixture family. For a subset \( X_0 \subseteq X \), we define the mixture family \( \mathcal{M}_{X_0} \) as

\[
\mathcal{M}_{X_0} := \{ Q_Y \in \mathcal{P}_Y | Q_Y(x) = \sum_{x \in X \setminus X_0} c(x)W_x, \sum_{x \in X \setminus X_0} c(x) = 1 \}.
\]

When \( X_0 \) is the empty set, \( \mathcal{M}_{X_0} \) coincides with \( \mathcal{M} \). Also, we simplify \( \mathcal{M}_{\{x\}} \) to \( \mathcal{M}_x \).

Theorem 6: There uniquely exists an element \( P_{X,*} \in \mathcal{E} \cap \mathcal{M} \). Any elements \( P_{X,1} \in \mathcal{M} \) and \( P_{X,2} \in \mathcal{E} \) satisfy

\[
D(P_{X,1} || P_{X,2}) = D(P_{X,1} || P_{X,*}) + D(P_{X,*} || P_{X,2}).
\]

Using this theorem, we can show the following corollaries.

Corollary 1: Given a distribution \( Q_X \) on \( X \), there uniquely exists an element \( Q_{X,*} \in \mathcal{M} \) such that

\[
D(P_{X,1} || Q_X) = D(P_{X,1} || Q_{X,*}) + D(Q_{X,*} || Q_X)
\]

for any element \( P_{X,1} \in \mathcal{M} \). \( Q_{X,*} \) is called the projection of \( Q_X \) to \( \mathcal{M} \), and is denoted by \( \Gamma_{\mathcal{M}}^{(m)}(Q_X) \).

Corollary 2: Given a distribution \( Q_X \) on \( X \), there uniquely exists an element \( Q_{X,*} \in \mathcal{E} \) such that

\[
D(Q_X || P_{X,2}) = D(Q_X || Q_{X,*}) + D(Q_{X,*} || P_{X,2})
\]

for any element \( P_{X,2} \in \mathcal{E} \). \( Q_{X,*} \) is called the projection of \( Q_X \) to \( \mathcal{E} \), and is denoted by \( \Gamma_{\mathcal{E}}^{(e)}(Q_X) \).

Now, we consider a one-parameter exponential family \( \{ P_t \} \).

Lemma 3: For \( t_1 \leq t_2 \leq t_3 \), we have

\[
D(P_{t_1} || P_{t_2}) + D(P_{t_2} || P_{t_3}) \leq D(P_{t_1} || P_{t_3}).
\]

Proof: Let \( J_t \) be the Fisher information in the one-parameter exponential family \( \{ P_t \} \). Then, we have

\[
D(P_t || P_{t'}) = \int_{t'}^{s} J_s(s-t')ds.
\]

The expression (87) implies (86).
APPENDIX B
PROOF OF LEMMA

This lemma can be shown by contradiction. If \( D(W_x || Q_Y) \) depends on \( x \in \text{supp}(Q_X) \), we define \( X_0 := \{ x_0 \in X | D(W_{x_0} || Q_Y) < \max_{x \in X} D(W_x || Q_Y) \} \). With a small \( \epsilon > 0 \), we choose \( Q_{X,\epsilon} \) as

\[
Q_{X,\epsilon}(x_0) := Q_X(x_0) + \frac{\epsilon}{|X_0|}
\]

\[
Q_{X,\epsilon}(x) := Q_X(x) - \frac{\epsilon}{|X \setminus X_0|}
\]

for \( x_0 \in X_0 \) and \( x \in X \setminus X_0 \). Then, we have \( D(W_{x_0} || W \cdot Q_{X,\epsilon}) < D(W_{x_0} || W \cdot Q_X) \) and \( D(W_x || W \cdot Q_{X,\epsilon}) > D(W_x || W \cdot Q_X) \). Hence, with sufficiently small \( \epsilon > 0 \), we have

\[
\max_{x \in X} D(W_x || W \cdot Q_{X,\epsilon}) < \max_{x \in X} D(W_x || W \cdot Q_X),
\]

which contradicts the assumption of contradiction.

APPENDIX C
PROOF OF THEOREM

Assume the condition (ii). For \( Q_X \neq \hat{Q}_{X,*} \in \mathcal{P}_X \), we have \( \max_{x \in X} D(W_x || W \cdot Q_X) > \max_{x \in X} D(W_x || W \cdot \hat{Q}_{X,*}) \). Hence, \( Q_{Y,*} \) achieves \( C(W) \), which implies Condition (i).

Assume the condition (i). There exists \( Q_X \in \mathcal{P}_X \) such that \( D(W_x || Q_{Y,*}) = \sum_{x \in X} Q_X(x) D(W_x || W \cdot Q_X) \). For \( x \in \text{supp}(Q_X) \), we have

\[
D(W_x || W \cdot Q_X) = D(W_x || Q_{Y,*}).
\]

Then, the distribution \( \hat{Q}_{Y,*} := \Gamma_{\hat{Q}_{X,*}}^{(m)}(Q_{Y,*}) \) satisfies \( D(W_x || Q_{Y,*}) = D(W_x || \hat{Q}_{Y,*}) + D(\hat{Q}_{Y,*} || Q_{Y,*}) \). Hence, \( D(W_x || Q_{Y,*}) \geq D(W_x || \hat{Q}_{Y,*}) \). Since \( \min_{Q_Y \in \mathcal{P}_Y} \max_{x \in X} D(W_x || Q_Y) = \max_{x \in X} D(W_x || W \cdot Q_X) \), we have \( D(W_x || W \cdot Q_X) = D(W_x || Q_{Y,*}) = D(W_x || \hat{Q}_{Y,*}) \) for \( x \in \text{supp}(Q_X) \). Hence, \( D(\hat{Q}_{Y,*} || Q_{Y,*}) = 0 \), i.e., \( \hat{Q}_{Y,*} = Q_{Y,*} \). That is, \( Q_{Y,*} \) belongs to \( \mathcal{M}_0 \). Due to Condition (A), the condition \( \hat{Q}_{X,*} \) uniquely determines \( Q_{Y,*} \). Hence, \( W \cdot \hat{Q}_{X,*} = Q_{Y,*} = W \cdot Q_X \). Due to Condition (A), \( \hat{Q}_{X,*} = Q_X \), which implies the condition (ii).

APPENDIX D
PROOF OF THEOREM

Due to Condition (A), there uniquely exists a distribution \( Q_{X,*} \in \mathcal{P}_X \) to achieve the capacity \( C(W) \). It is sufficient to show that \( Q_{X,*}(x_0) = 0 \) for any element \( x_0 \in \text{supp}(\hat{Q}_{X,*}) \). For this aim, we fix an arbitrary element \( x_0 \in \text{supp}(\hat{Q}_{X,*}) \).

**Step 1:** We show that there exists a distribution \( Q_{Y,0} \in \mathcal{M}_{x_0} \) such that

\[
\max_{x \in X} D(W_x || Q_{Y,*}) \geq \max_{x \in X} D(W_x || Q_{Y,0}) = D(W_{x'} || Q_{Y,0})
\]

for \( x' \in X \setminus \{ x_0 \} \).

We choose a function \( f_{x_0} \) on \( X \) such that

\[
\sum_{y \in Y} f_{x_0}(y) W_{x_0}(y) = 1,
\]

\[
\sum_{y \in Y} f_{x_0}(y) W_x(y) = 0
\]
for \( x(\neq x_0) \in \mathcal{X} \). We denote \(-\hat{Q}_{X,*}(x_0) > 0\) by \( a \). Then, we have
\[
\frac{1}{1 + a} Q_{Y,*} + \frac{a}{1 + a} W_{x_0} \in \mathcal{M}_{x_0}.
\] (95)

The combination of (94) and (95) implies that
\[
\sum_{y \in \mathcal{Y}} f_{x_0}(y) \left( \frac{1}{1 + a} Q_{Y,*}(y) + \frac{a}{1 + a} W_{x_0}(y) \right) = 0.
\] (96)

Then, the combination of (93) and (96) yields that
\[
\sum_{y \in \mathcal{Y}} f_{x_0}(y) Q_{Y,*}(y) = -a.
\] (97)

The distribution \( Q_{Y,0} := \Gamma_{\mathcal{M}_{x_0}}^{(m)}(Q_{Y,*}) \in \mathcal{M}_{x_0} \) satisfies
\[
D(Q_Y || Q_{Y,*}) = D(Q_Y || Q_{Y,0}) + D(Q_{Y,0} || Q_{Y,*})
\] (98)
for any \( Q_Y \in \mathcal{M}_{x_0} \). We define the exponential family \( \mathcal{E}_1 := \{ Q_{Y,t} \}_{t \in \mathbb{R}} \) as
\[
Q_{Y,t}(y) := Q_{Y,0}(y) e^{t f_{x_0}(y) - \varphi(y)},
\] (99)
where
\[
\varphi(y) := \log \sum_{y \in \mathcal{Y}} Q_{Y,0}(y) e^{t f_{x_0}(y)}.
\] (100)

Hence, \( Q_{Y,0} \) coincides with the case with \( t = 0 \). We choose \( t_* \) such that \( Q_{Y,t_*} = Q_{Y,*} \). The relation (97) guarantees that \( t_* < 0 \). Also, we choose \( t_0 \) as \( Q_{Y,t_0} = \Gamma_{\mathcal{E}_1}^{(e)}(W_{x_0}) \). Then, we have
\[
D(W_{x_0} || Q_{Y,t}) = D(W_{x_0} || Q_{Y,t_0}) + D(Q_{Y,t_0} || Q_{Y,t}).
\] (101)
for any \( t_0 \in \mathbb{R} \). The relation (97) guarantees that \( t_0 > 0 \). Since \( t_* < 0 \) and \( t_0 > 0 \), Lemma 3 yields that
\[
D(Q_{Y,t_0} || Q_{Y,0}) \leq D(Q_{Y,t_0} || Q_{Y,t_*}) - D(Q_{Y,0} || Q_{Y,t_*}).
\] (102)

The combination of (101) and (102) guarantees that
\[
D(W_{x_0} || Q_{Y,0}) \overset{(a)}{=} D(W_{x_0} || Q_{Y,t_0}) + D(Q_{Y,t_0} || Q_{Y,0})
\]
\[
\overset{(b)}{\leq} D(W_{x_0} || Q_{Y,t_0}) + D(Q_{Y,t_0} || Q_{Y,t_*}) - D(Q_{Y,0} || Q_{Y,t_*}) \overset{(c)}{=} D(W_{x_0} || Q_{Y,t_*}) - D(Q_{Y,0} || Q_{Y,t_*})
\]
\[
\overset{(d)}{=} D(W_{x_0} || Q_{Y,*}) - D(Q_{Y,0} || Q_{Y,*}) \overset{(e)}{=} D(W_{x'} || Q_{Y,*}) \overset{(f)}{=} D(W_{x'} || Q_{Y,0})
\] (103)
for \( x' \in \mathcal{X} \setminus \{ x_0 \} \). Each step is shown in the following way. Steps (a) and (c) follow from (101). Step (b) follows from (102). Step (d) follows from \( Q_{Y,t_*} = Q_{Y,*} \). Step (e) follows from \( D(W_{x_0} || Q_{Y,*}) = D(W_{x'} || Q_{Y,*}) \). Step (f) follows from (98). Then, we have
\[
\max_{x \in \mathcal{X}} D(W_x || Q_{Y,*}) \geq D(W_{x'} || Q_{Y,*}) \overset{(a)}{=} D(W_{x'} || Q_{Y,0}) \overset{(b)}{=} \max_{x \in \mathcal{X}} D(W_x || Q_{Y,0}),
\] (104)
where each step is shown as follows. Step (a) follows from (98). Step (b) follows from (103). Hence, we obtain (92).

**Step 2:** We choose a function \( \hat{Q}_{X,1} \) on \( \mathcal{X} \setminus \{ x_0 \} \) such that
\[
\sum_{x \in \mathcal{X} \setminus \{ x_0 \}} \hat{Q}_{X,1}(x) W_x = Q_{Y,1},
\] (105)
Fig. 5. Relation for $\mathcal{E}_1$, $\mathcal{M}_{x_0}$, $Q_{Y,*} = Q_{Y,t_*}$, $Q_{Y,0}$, and $Q_{Y,t_0}$.

where $\hat{Q}_{X,1}$ uniquely exists because $Q_{Y,1} \in \mathcal{M}_{x_0}$. We show the desired statement $Q_{X,*}(x_0) = 0$ when $\hat{Q}_{X,1}(x) \leq 0$ for $x \in \mathcal{X} \setminus \{x_0\}$.

In this case, it is sufficient to show that $Q_{X,*}(x_0) = \hat{Q}_{X,1}$, i.e., $\hat{Q}_{X,1}$ achieves the capacity $C(W)$. We have

$$
\sum_{x \in \mathcal{X} \setminus \{x_0\}} Q_{X,1}(x)D(W_x\|Q_{Y,1}) = \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|Q_{Y,1}) \overset{(a)}{=} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|Q_{Y,1}) \overset{(b)}{=} \max_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \sum_{x \in \mathcal{X} \setminus \{x_0\}} Q_X(x)D(W_x\|W \cdot Q_X) \overset{(c)}{=} \min_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|W \cdot Q_X) \overset{(d)}{=} \min_{Q_Y \in \mathcal{M}_{x_0}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|Q_Y) \overset{(e)}{=} \min_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|W \cdot Q_X) \overset{(f)}{=} \min_{Q_X \in \mathcal{P}_{\mathcal{X} \setminus \{x_0\}}} \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|W \cdot Q_X) \overset{(g)}{=} \min_{x \in \mathcal{X}} D(W_x\|W \cdot Q_{X,1}).
$$

Each step is shown in the following way. Step (a) follows from the second equation in (92). Step (b) follows from Theorem 1. Step (c) follows from (3). Step (d) is shown as follows. Since $Q_Y \mapsto D(W_x\|Q_Y)$ is convex, $Q_Y \mapsto \max_{x \in \mathcal{X} \setminus \{x_0\}} D(W_x\|Q_Y)$ is also convex. Since $Q_{Y,1}$ achieves a local minimum, it also achieve the global minimum in $\mathcal{M}_{x_0}$.

Step (e) is shown as follows. For $Q_X \in \mathcal{P}_X$, the distribution $Q_Y := \Gamma_{\mathcal{M}_{x_0}}^{(m)}(W \cdot Q_X)$ satisfies

$$
D(W_x\|W \cdot Q_X) = D(W_x\|Q_Y) + D(Q_Y\|W \cdot Q_X) \geq D(W_x\|Q_Y') \text{ for } x \in \mathcal{X} \setminus \{x_0\},
$$

(107)
which shows (e).

Hence, we have

$$C(W) = \sum_{x \in X \setminus \{x_0\}} Q_{X,1}(x) D(W_x || W \cdot Q_{X,1}).$$  \hspace{1cm} (108)$$

**Step 3:** We show the desired statement $Q_{X,*}(x_0) = 0$ when there exists $x_1 \in X \setminus \{x_0\}$ such that $Q_{X,1}(x_1) < 0$. Applying the same discussion as Step 1, we find that there exists a distribution $Q_{Y,2} \in \mathcal{M}_{\{x_0, x_1\}}$ such that

$$\max_{x \in X} D(W_x || Q_{Y,1}) \geq \max_{x \in X} D(W_x || Q_{Y,2}) = D(W_x || Q_{Y,2})$$  \hspace{1cm} (109)$$

for $x' \in X \setminus \{x_0, x_1\}$. Then, we choose $\hat{Q}_{X,2}$ in the same way as (105). If $\hat{Q}_{X,2}(x) \geq 0$ for $x \in X \setminus \{x_0, x_1\}$, we find that $Q_{X,*}(x_0) = 0$ in the same way as Step 2. Otherwise, we repeat the above procedure up to $i$ times until we have $\hat{Q}_{X,i}(x) \geq 0$ for $x \in X \setminus \{x_0, x_1, \ldots, x_{i-1}\}$. Once we obtain the above condition, we find $Q_{X,*}(x_0) = 0$ in the same way as Step 2.

**APPENDIX E**

**PROOF OF LEMMA**

To show Lemma 2, we prepare functions $\tilde{f}_1, \ldots, \tilde{f}_{n_2-1}$ to satisfy the condition in Theorem 4. We denote the distribution defined in (10) based on these functions $f_1, \ldots, f_{n_2-1}$ by $P_{\theta, Y}$. Such functions are given as linear combination of the original functions $f_1, \ldots, f_{n_2-1}$ by using coefficient $\alpha_j^i$, as

$$\sum_j f_j \alpha_j^i = \tilde{f}_j^i.$$  \hspace{1cm} (110)$$

Hence, we have

$$\sum_{j'=1}^{n_2-1} \tilde{f}_j^i(y) \theta^{j'} = \sum_{j=1}^{n_2-1} f_j(y) \left( \sum_{j'=1}^{n_2-1} \alpha_j^i \theta^{j'} \right).$$  \hspace{1cm} (111)$$

Using this relation, we find that $\bar{P}_{\theta, Y} = \bar{P}_{\bar{\theta}, Y}$, where $\bar{\theta}^i = \sum_{j'=1}^{n_2-1} \alpha_j^i \bar{\theta}^{j'}$. Thus, the set $\mathcal{E}_0$ can be characterized with the new functions $f_1, \ldots, f_{n_2-1}$. Therefore, without loss of generality, we can assume that the functions $f_1, \ldots, f_{n_2-1}$ satisfies the condition in Theorem 4.

We choose $\theta^1, \ldots, \theta^{n_2-1}$ satisfies the condition (9). Then, we have

$$\mathcal{E}_0 = \{ P_{\theta^1, \ldots, \theta^{n_2-1}, \eta^{n_1-1, n_2-1, Y}}(\eta^{n_1, \ldots, n_2-1}) \in \mathbb{R}^{n_2-n_1} \}.$$  \hspace{1cm} (112)$$

Hence, $\mathcal{E}_0$ is an exponential family generated by $f_{n_1}, \ldots, f_{n_2-1}$.

Since $f_{i,j} = 0$ for $j = n_1 + \ldots, n_2 - 1$, $\mathcal{M}_0$ can be written as

$$\mathcal{M}_0 = \{ Q_Y \in \mathcal{P}_Y | \sum_{y \in Y} f_j(y) Q_Y(y) = 0 \text{ for } j = n_1, \ldots, n_2 - 1 \}.$$  \hspace{1cm} (113)$$

Hence, $\mathcal{M}_0$ is a mixture family generated by the same functions $f_{n_1}, \ldots, f_{n_2-1}$. Therefore, Theorem 6 implies Lemma 2.
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