I. INTRODUCTION

There is a long history of efforts to constrain dark matter properties from galactic structure (e.g. [1]). Recent numerical simulations [2] sharpen the predictions of Cold Dark Matter (CDM) structure formation models, and apparent discrepancies with the observed properties of structures from galactic to cluster scales are uncovered. The main one that has attracted a lot of attention is the cuspy halo problem, namely that CDM models predict halos that have a high density core or have an inner profile that is too steep compared to observations (see [3], but see also [4]). This has encouraged several proposals that dark matter might have properties different from those of conventional CDM (see [5] and summary therein).

On the other hand, general principles of quantum mechanics impose non-trivial constraints on some of these models. We focus here on the proposals of collisional dark matter (SADM) by [7] and of strongly self-interacting dark matter (SIDM) by [6]. Both require high level of interaction by particle physics standard: an elastic scattering cross-section of $\sigma_{el} \sim 10^{-24} (m_X/\text{GeV}) \text{cm}^2$ for the former and an annihilation cross-section of $\sigma_{ann}, v_{rel} \sim 10^{-28} (m_X/\text{GeV}) \text{cm}^2$ for the latter, where $m_X$ is the particle mass, and $v_{rel}$ is the relative velocity of approach. The proposed dark matter is therefore quite different from usual candidates such as the axion or neutralino. We show that the unitarity of the scattering matrix $S$, i.e. $S^\dagger S = 1$, which implies $(1 - S)^\dagger (1 - S) = (1 - S^\dagger) + (1 - S)$, or

$$\int d\gamma \langle \beta | 1 - S | \gamma \rangle \langle \gamma | 1 - S^\dagger | \alpha \rangle = 2 \text{Re} \langle \beta | 1 - S | \alpha \rangle \quad (1)$$

where $\alpha$ and $\beta$ represent two specified states and $\gamma$ represents a complete set of states with measure $d\gamma$. Using the definition of the scattering amplitude $A_{\beta\alpha}$

$$\langle \beta | 1 - S | \alpha \rangle \equiv -i (2\pi)^4 \delta^4(p_\beta - p_\alpha) A_{\beta\alpha} \quad (2)$$

where $p_\beta$ and $p_\alpha$ are the total four-momenta, one obtains

$$\int d\gamma (2\pi)^4 \delta^4(p_\alpha - p_\gamma) |A_{\gamma\alpha}|^2 = 2 \text{Im} A_{\alpha\alpha} \quad (3)$$

Griest and Kamionkowski [8] previously derived similar mass bounds related to the freeze-out density of thermal relics, assuming 2-body final states. In [8], we provide a general derivation for arbitrary final states using the classic optical theorem [10]. We summarize our findings in §III, discuss exceptions to our bounds, and other solutions to the cuspy halo problem.

II. DERIVING THE UNITARITY BOUNDS

Different versions of the unitarity bounds can be found in many textbooks, and can be most easily understood using non-relativistic quantum mechanics (e.g. [11]), which is probably adequate for our purpose. However, the results and derivation given here might be of wider interest e.g. for estimating thermal relic density. Here we follow closely the field theory treatment of [12].

The optical theorem [10] is a powerful consequence of the unitarity of the scattering matrix $S$, i.e. $S^\dagger S = 1$, which implies $(1 - S)^\dagger (1 - S) = (1 - S^\dagger) + (1 - S)$, or

$$\int d\gamma \langle \beta | 1 - S | \gamma \rangle \langle \gamma | 1 - S^\dagger | \alpha \rangle = 2 \text{Re} \langle \beta | 1 - S | \alpha \rangle \quad (1)$$

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if $\beta = \alpha$ in eq. (1). We are interested in the case where $\alpha$ represents a 2-body state of $X + X$ or $X + \bar{X}$ approaching each other. The final state $\gamma$, on the other hand, is completely general, and the integration over $\gamma$ covers the entire spectrum of possible final states. To be more precise, suppose $|\alpha\rangle = |k_1, s_1^1; k_2, s_2^1; n\rangle$ while $k_1$ and $k_2$ are their respective 4-momenta, and $n$ labels the particle types (e.g. mass, etc.). Recalling that $d\sigma/d\gamma \propto |A_{\alpha\gamma}|^2$, eq. (3) gives in the center of mass frame (adopted hereafter i.e. $k_1 + k_2 = 0$):

$$
\int d\gamma \frac{d\sigma}{d\gamma}(\alpha \rightarrow \gamma) = \frac{\text{Im} A_{\alpha\gamma}}{2(E_1 + E_2)|k_1|} \tag{4}
$$

This gives the total spin-averaged cross-section for scattering from $X + X$ or $X + \bar{X}$ to all possible final states.

For $X + \bar{X}$ annihilation, we exclude from the above the contribution due to elastic scattering (where type and mass of particles do not change i.e. $X + \bar{X} \rightarrow X + X$, implying $|k_1^1| = |k_1^1|$). To do so, we need the following expression for 2-body to 2-body scattering cross-section:

$$
\frac{d\sigma}{d\beta} d\beta = \frac{|A_{\beta\alpha}|^2}{4(E_1 + E_2)|k_1|}(2\pi)^4d\beta = \frac{1}{(2\pi)^4}\delta^{4}(p_\beta - p_\alpha) d\beta, \tag{7}
$$

We average over initial spin states and integrate over outgoing momenta, but focus on the elastic contribution ($n'$ in $|\beta\rangle = |k_1', s_1'; k_2', s_2'; n'\rangle$ is set to $n$ in $|\alpha\rangle$) (12):

$$
\sigma_{\text{el.}} = \frac{\pi}{k_1^2(2s_1 + 1)(2s_2 + 1)} \sum_j (2j + 1) \tag{8}
$$

The above is the total cross-section for elastic scattering (note: the same expression also describes $X + X \rightarrow X + X$ elastic scattering) that has to be subtracted from $\sigma_{\text{tot}}$ to yield the total inelastic scattering cross-section, which is relevant for annihilation into all possible final states:

$$
\sigma_{\text{inel}} = \frac{\pi}{k_1^2(2s_1 + 1)(2s_2 + 1)} \sum_j (2j + 1) \tag{9}
$$

From eq. (6) & (8), we can derive two bounds:

$$
\sigma_{\text{tot.}} \leq 4\pi|k_1|^2(2s_1 + 1)(2s_2 + 1)^{-1} \sum_j 2j + 1 \tag{10}
$$

$$
\sigma_{\text{inel.}} \leq \pi|k_1|^2(2s_1 + 1)(2s_2 + 1)^{-1} \sum_j 2j + 1 \tag{11}
$$

The first inequality uses $|\langle \ell|S|\ell\rangle|^2 \leq 1$, obtained from $\int d\gamma \langle \ell|S|\gamma\rangle\langle\gamma|S|\ell\rangle = |\langle \ell|S|\ell\rangle|^2$ and $S^\dagger S = 1$. A similar bound can be derived for $\sigma_{\text{el.}}$, as well, which coincides exactly with that for $\sigma_{\text{tot.}}$.

We pause to note that the above bounds assume only unitarity and the conservation of total energy and linear and angular momentum. No assumptions are made about the nature of the particles, whether they are composites or point-like. Nor do we assume the number of particles in the final states. To obtain useful limits from the bounds, we take the low velocity limit. Assuming the scattering amplitude $A_{\beta\alpha}$ is an analytic function of $k_1$ as $k_1 \rightarrow 0$ (exceptions will be discussed in §III), and noting that $k_1\langle k|\ell\ell\rangle$ is a polynomial function of $k$, we expect the $\ell$ partial wave contribution to $A_{\beta\alpha}$ (eq. [1]) to scale as $|k_1|^\ell$. This means in the low velocity limit, as is relevant for our purpose (typical velocity dispersion in halos range from 10 to 1000 km/s $\ll c$), the $\ell = 0$ or $s$-wave contribution dominates. Setting $\ell = 0$ in eq. (10), (11):
\[ \sigma_{\text{tot.}} \leq 16 \pi/(m_X v_{\text{rel}})^2, \quad \sigma_{\text{inel}, \text{rel}} \leq 4 \pi/(m_X^2 v_{\text{rel}}) \] (12)

where \( k_1^2 = m_X^2 |v_2 - v_1|^2 / 4 = m_X^2 v_{\text{rel}}^2 / 4 \) is used. The second inequality agrees with 3 & 4, and setting \( \ell = 0 \). Defining \((X)_{j} \equiv [(2s_1 + 1)(2s_2 + 1)]^{-1} \sum_{j, \ell , s} (2j + 1)X \), using \( S \) at the moment to denote \((\ell s n) | S | (\ell s n) \), and noting that \((1)_{j} = 1 \) for \( \ell = 0 \), it can be shown \((\pi/k_2^2) \geq (\pi/k_2^2)(1 - |S|)^2 \geq \sigma_{\text{ann.}} \geq \sigma_{\text{ann.}} \), which implies \(|S| \geq \sqrt{1 - k_2^2} \sigma_{\text{ann.}} / \pi \).

Furthermore, if \( \sigma_{\text{inel.}} \) is bounded from below, say \( \sigma_{\text{inel.}} \geq \sigma_{\text{ann.}} \), one can derive a lower bound on \( \sigma_{\text{el.}} \), using eq. \(( \text{5} ) \) & \(( \text{4} ) \), and setting \( \ell = 0 \). Two simple limiting cases: when \( m_X \) is close to the upper bound of 25 GeV, \( \sigma_{\text{el.}} \geq 4 \times 10^{-22} \text{cm}^2 / \text{GeV} / m_X \). When \( m_X \) is small, \( \sigma_{\text{el.}} \geq 5 \times 10^{-31} \text{cm}^2 / \text{GeV} / m_X \). Hence, elastic scattering is inevitable in this scenario, but can be reduced by having a sufficiently small mass.

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3. For SADM, efficient annihilation (a form of inelastic scattering) inevitably implies some elastic scattering as well. From eq. \(( \text{12} ) \), and using \( v_{\text{rel}} \sim 1000 \text{ km/s} \) as before, we have

\[ \sigma_{\text{el.}} \geq 4 \times 10^{-22} \text{cm}^2 / \text{GeV} / m_X \] (18)

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4. Recent simulations suggest that the simplest version of SIDM fails to match simultaneously the observed halo properties from dwarf galaxies to clusters 13 \& 5 (see also 13), which have \( v_{\text{rel.}} \) ranging over 3 orders of magnitude. It was suggested that an elastic scattering cross-section of \( \sigma \propto 1 / v_{\text{rel.}} \) might solve the problem. But as shown in eq. \((16) \), elastic scattering generally implies \( \sigma \propto \text{constant} \) in the small velocity limit. Hence, \( \sigma \propto v_{\text{rel.}} \) likely requires inelastic processes. As eq. \((17) \) shows, processes in which the net kinetic energy increases \((\sigma_{\text{el.}} > |k_1| \) in c.o.m. frame) can give such a velocity dependence. SADM provides an example. More generally, the net kinetic energy increase (super-elasticity) must be taken into account when considering the viability of a model with \( \sigma \propto 1 / v_{\text{rel.}} \). It may delay core collapse and make the core larger. Note, however, the general considerations in the last section does not forbid an elastic cross-section that increases as \( v_{\text{rel.}} \) decreases e.g. the \( O(v_{\text{rel.}}) \) term in eq. \((16) \) can have a negative coefficient. A \( 1 / v_{\text{rel.}} \) power-law may approximate such a cross-section, but likely only for a limited range of \( v_{\text{rel.}} \). An example is the neutron-neutron scattering cross-section, which approaches a constant for \( |k_1| \leq 10^{-2} \text{ GeV} \), and scales as \( 1 / v_{\text{rel.}} \) only for \( 10^{-2} \leq |k_1| \leq 5 \times 10^{-2} \text{GeV} \).

It is helpful to mention here possible exceptions to the above limits. Our bounds are obtained from eq. \((13) \) & \((14) \), which are the \( \ell = 0 \) (s-wave) versions of eq. \((10) \) & \((11) \). The argument for putting \( \ell = 0 \) in the small velocity limit assumes the analyticity of \( A_{\text{ann.}} \) at \( k_1 = 0 \). The latter breaks down if the interaction is long-ranged, e.g. Coulomb scattering. This is unlikely to be relevant, because there are strong constraints on dark matter with such long ranged interaction 13. Our argument for the dominance of s-wave scattering can also be invalid if there is a resonance. However, given that the scattering cross section should vary smoothly over three orders of magnitude in velocities from dwarfs to clusters, a resonance seems unlikely. Finally, the most likely situation in which the bounds break down is if the particle has a large enough size, or the interaction has a large enough effective range, \( R \), such that \( |k_1| R > 1 \) (e.g. see 16). In such
cases, higher partial waves in addition to s-wave generally contribute, and $\sigma_{\text{tot}} \lesssim 64\pi R^2$ and our arguments turn into a limit on $R \lesssim 6 \times 10^{-3}$ cm (GeV/$m_X$). The condition $|k_1|/\pi R > 1$ gives the most stringent constraint on $R$ for $v_{\text{rel}} = 10$ km/s, as appropriate for dwarf galaxies: $R \gtrsim 10^{-3}$ cm (GeV/$m_X$). One can compare this with $R$ for neutron-neutron scattering $\sim 10^{-13}$ cm.

It is intriguing that halo structure might be telling us the elementary properties, in particular the mass, of dark matter. It is interesting that several proposals to address the cuspy halo problem, such as Warm Dark Matter [18] and Fuzzy Dark Matter [19] make explicit assumptions about the mass of the particles $m_X \sim 1$ keV and $m_X \sim 10^{-22}$ eV respectively. For SIDM and SADM, astrophysical considerations generally only put constraints on the cross-section per unit mass. We have shown here that unitarity arguments imply a rather modest mass for the dark matter. It is interesting that several proposals to address the cuspy halo problem, such as Warm Dark Matter [18] and Fuzzy Dark Matter [19] can be extended to cover dark matter in the form of a Bose condensate, as has been proposed as yet another solution to the cuspy halo problem [21]. They generally require small masses as well $\lesssim 10$ eV.

A few issues are worth further investigation. Wandelt et al. [8] recently argued a version of SIDM, where the dark matter interacts strongly also with baryons, is experimentally viable, but requires $m_X \gtrsim 10^5$ GeV, or $m_X \lesssim 0.5$ GeV. Our bound here is inconsistent with the large mass region (but see exceptions above); experimental constraints on the low mass region will be very interesting ($\sigma_{\text{tot}} \lesssim 10^{-25}$ cm$^2$). It would be useful to find a micro-physics realization of the collisional scenario [21] or its variant where $\sigma$ scales appropriately with velocity to match observations. The impact of inelastic collisions on halo structures is worth exploring in more detail. It is also timely to reconsider possible astrophysical solutions to the cuspy halo problem, such as the use of mass loss mechanisms [22]. We hope to examine some of these issues in the future.

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