Minimal tropical basis for Bergman fan of matroid

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Abstract

The Bergman fan of a matroid is the intersection of tropical hyperplanes defined by the circuits. A tropical basis is a subset of the circuit set that defines the Bergman fan. Yu and Yuster posed a question whether every simple regular matroid has a unique minimal tropical basis of its Bergman fan, and verified it for graphic, cographic matroids and $R_{10}$. We show every simple binary matroid has a unique minimal tropical basis. Since the regular matroid is binary, we positively answered the question.

1 Introduction

Matroids were introduced by Whitney in 1935 to try to capture abstractly the essence of dependence. A matroid is an ordered pair $(E, \mathcal{C})$ consisting of a finite set $E$ and a collection $\mathcal{C}$ of subset of $E$ having three properties:

1. $\emptyset \notin \mathcal{C}$;
2. if $C_1$ and $C_2$ are members of $\mathcal{C}$ and $C_1 \subset C_2$, then $C_1 = C_2$;
3. if $C_1$ and $C_2$ are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member $C_3$ of $\mathcal{C}$ such that $C_3 \subset (C_1 \cup C_2) \setminus \{e\}$.

$\mathcal{C}$ is called the circuit set and a member of $\mathcal{C}$ is called a circuit of $M$. Two matroids $M = (E, \mathcal{C})$ and $M' = (E', \mathcal{C}')$ are isomorphic, in which case we write $M \cong M'$, if there is a bijection $\varphi : E \rightarrow E'$ such that for every subset $X \subset E$, $X$ is a circuit of $M$ if and only if $\varphi(X)$ is a circuit of $M'$. A matroid $M = (E, \mathcal{C})$ is simple if the cardinality of every circuit of $M$ is greater than 2.
Here are some examples of matroids. Let $G$ be a finite graph, $E$ be the edge set of $G$, and $C$ be the family of edge sets of cycles in $G$. Then $(E, C)$ obeys the axioms for matroid. Any matroid that can be obtained in this way is called a cycle matroid. If graph $G$ is simple, then the cycle matroid of $G$ is simple. A matroid $M$ is said to be graphic if there is a cycle matroid $M'$ such that $M \cong M'$. Let $K$ be a field, $V$ be a vector space over $K$, $E$ be a finite subset of $V$, and $C$ be the family of minimal linearly dependent subset of $E$. Then $(E, C)$ obeys the axioms for matroid, that is a matroid. Any matroid that can be obtained in this way is called a linear matroid over $K$. A matroid $M$ is said to be representable over $K$ if there is a linear matroid $M'$ over $K$ such that $M \cong M'$. A matroid is called binary if it is representable over binary field $\mathbb{F}_2$, and regular if it is representable over every field.

The Bergman fan of a matroid is defined to be the intersection of tropical hyperplanes of tropical linear form of circuits. The Bergman of a matroid fan is a kind of tropical linear space, used as a local model of tropical manifold. A subset of the circuit set $B$ is called a tropical basis if the intersection of tropical hyperplanes of tropical linear form of circuits in $B$ is equal to the Bergman fan. A tropical basis is said to be minimal if it is minimal with respect to the inclusion relation. For an ideal $I$ of a polynomial ring over a field, if \( \{f_1, f_2, \ldots, f_m\} \) is a generator of $I$, then $V(\{f_1, f_2, \ldots, f_m\})$ is equal to $V(I)$. In the sense that the zero set of a tropical basis is equal to the Bergman fan of a matroid, a minimal tropical basis is an analogy to minimal generators of ideal. We study minimal tropical basis. In [7] Yu and Yuster showed that every graphic, cographic simple matroid and $R_{10}$ have a unique minimal tropical basis. The Seymour decomposition theorem states that every regular matroid can be decomposed into those matroids by repeated 1-, 2-, and 3-sum decompositions. Yu and Yuster posed a question whether every simple regular matroid has a unique minimal tropical basis. We proved that every simple binary matroid has a unique minimal tropical basis. Since the regular matroid is binary, we positively answered the question.

In Section 2, we recall some basic facts for matroids and Bergman fan of matroid. Then in Section 3 we gave a necessary and sufficient condition for a matroid to have a unique minimal tropical basis. We prove that a simple uniform matroid is regular if and only if it has a unique minimal tropical basis. For the Fano matroid and the so-called non-Fano matroid, we describe these minimal tropical bases. In Section 4 we show our main result that if a simple matroid is binary, then it has a unique minimal tropical basis. Finally by using Yu and Yuster’s Lemma, we give an algorithm discriminating whether
a matroid has a unique minimal tropical basis. We give examples of matroids. \( P_7 \) and \( R_6 \) are two matroids. It is known that for a filed \( K \), \( P_7 \) and \( R_6 \) are representable over \( K \) if and only if the cardinality of \( K \) is more than two. By that algorithm, we know that \( P_7 \) has a unique minimal tropical basis but \( R_6 \) does not. Therefore, for a filed \( K \), every simple matroid that is representable over \( K \) has a unique minimal tropical basis if and only if \( K \) is binary filed.

## 2 Preliminaries

We briefly recall the theory of matroid and Bergman fan of matroid. We refer the reader to [1, 2, 3] for details and further references. Let \( M = (E, \mathcal{C}) \) be a matroid and \( T \) be a subset of \( E \). We define \( \mathcal{C}_{M/T} \) as the set that consists of minimal nonempty elements of \( \{ C \setminus T \mid C \in \mathcal{C} \} \). \((E \setminus T, \mathcal{C}_{M/T})\) is a matroid. We call this matroid the contraction of \( T \) from \( M \) and write it as \( M/T \). Let \( \mathcal{C}_{M \setminus T} \) be the set \( \{ C \subset E \setminus T \mid C \in \mathcal{C} \} \). \((E \setminus T, \mathcal{C}_{M \setminus T})\) is a matroid. We call this matroid the deletion of \( T \) from \( M \) and write it as \( M \setminus T \). A minor of \( M \) is any matroid that can be obtained from \( M \) by a sequence of deletions or contractions. For \( n \) and \( d \) be natural numbers and \( n \geq d \), a uniform matroid \( U_{d,n} \) is defined as follows. The ground set is \( [n] := \{1, 2, ..., n\} \). The circuit set of \( U_{d,n} \) consists of every \((d+1)\)-subsets of \([n]\). A uniform matroid \( U_{d,n} \) is simple if and only if \( d \) is more than one. A matroid is called ternary if it is representable over \( F_3 \). A matroid \( M \) is said to be cographic if there is a graphic matroid \( M' \) such that the dual matroid of \( M' \) is isomorphic to \( M \). \( R_{10} = (E, \mathcal{C}) \) is a regular matroid. The cardinality of \( E \) is ten. \( R_{10} \) is neither graphic nor cographic.

There are characterizations for a matroid to be binary or regular.

**Theorem 1** ([3]). Let \( M \) be a matroid. The following statements are equivalent.

1. \( M \) is binary.
2. \( M \) has no minor isomorphic to \( U_{2,4} \).
3. If \( C_1 \) and \( C_2 \) are circuits, then their symmetric difference \( C_1 \triangle C_2 \) is a disjoint union of circuits.

**Theorem 2** ([2][6]). Let \( M \) be a matroid. The following statements are equivalent.

1. \( M \) is regular.
2. \( M \) is binary and ternary.
(3) $M$ can be decomposed into graphic and cographic matroids and matroids isomorphic to $R_{10}$ by repeated 1-, 2-, and 3-sum decompositions.

Let $T = (R \cup \{-\infty\}, \oplus, \odot)$ be the tropical semifield where $\oplus$ is maximum operation and $\odot$ is the usual addition. $\text{TP}^n$ is the tropical projective space. The Bergman fan or constant coefficient tropical linear space of a matroid can be described in terms of their matroid. Let $M = ([n], \mathcal{C})$ be a matroid. For a circuit $C \in \mathcal{C}$, let $V(C)$ be the set of points $x \in \text{TP}^{n-1}$ such that the maximum value in $\{x_i \mid i \in C\}$ is attained at least twice. The set $V(C) := \bigcap_{C \in \mathcal{C}} V(C)$ is a polyhedral fan called the Bergman fan or constant coefficient tropical linear space of $M$. Federico Ardila and Caroline Klivans had shown that the Bergman fan of a matroid $M$ centered at the origin is geometric realization of the order complex of the lattice of flats of $M$. The simplication of $M$ does not change lattice of flats, so we may assume that our matroid is simple.

3 Tropical basis

Let $M = ([n], \mathcal{C})$ be a matroid. A subset $B$ of $\mathcal{C}$ is defined to be a tropical basis if $V(B) := \bigcap_{C \in B} V(C)$ is equal to $V(\mathcal{C})$. A tropical basis is minimal if every its proper subset is not a tropical basis. We study minimal tropical basis and whether a matroid has a unique minimal tropical basis.

If two circuit $C_1, C_2$ have a unique element in $C_1 \cap C_2$, pasting them means taking their symmetric difference $C_1 \triangle C_2 = (C_1 \setminus C_2) \cup (C_2 \setminus C_1)$.

Proposition 1 ([7]). If $S \subset \mathcal{C}$ has the property that every other circuit of the matroid can be obtained by successively pasting circuits in $S$, then $S$ is a tropical basis.

3.1 Intersection of all tropical bases

We denote the intersection of all tropical bases of $M$ by $B_M$. We show that $M$ has a unique minimal tropical basis if and only if $B_M$ is a tropical basis.

Lemma 1. Let $C$ be a circuit. The following statements are equivalent.

(1) The circuit $C$ is in $B_M$.
(2) $\mathcal{C} \setminus \{C\}$ is not a tropical basis.
(3) There is an element $x \in \text{TP}^{n-1}$ such that $x$ is not in $V(C)$ and for every other circuit $C'$, $x$ is in $V(C')$. 

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Proof. By the definition of tropical basis, it is clear that (2) is equivalent to (3). We show (1) implies (2). If $C$ is in $B_M$, then every tropical basis contains a circuit $C$. Therefore, $C \setminus \{C\}$ is not a tropical basis. We prove (3) implies (1). Let $x$ be a point in $\text{TP}^{n-1}$ that is not in $V(C)$ but for every other circuit $C'$, $V(C')$ contains $x$. If a subset of the circuit set $B$ does not contain $C$, $x$ is not in $V(\emptyset)$, but in $V(B)$. $B$ is not a tropical basis. All tropical bases contain a circuit $C$. Therefore, $C$ is in $B_M$. \qed

Lemma 2. The following statements are equivalent.

(1) $M$ has a unique minimal tropical basis.
(2) $B_M$ is a tropical basis.

Proof. We first prove (2) implies (1). Let $B$ be the unique minimal tropical basis of $M$. By the uniqueness of minimal tropical basis and finiteness of the cardinality of the circuit set, every tropical basis of $M$ contains $B$. Therefore, $B_M = B$ is a tropical basis. We now show (1) implies (2). We show the contraposition. Let $B_1$ and $B_2$ be distinct minimal tropical bases of $M$. Then we have $B_M \subset B_1 \cap B_2 \subset B_1$. By the minimality of $B_1, B_M$ is not a tropical basis of $M$. \qed

3.2 Uniform matroid

Yu and Yuster\cite{7} found minimal tropical bases of uniform matroids. In this section we show that for a simple uniform matroid, it is regular if and only if it has a unique minimal tropical basis.

Proposition 2 (\cite{7}). For every $i \in [n]$, $B_i := \{C \mid i \in C\}$ is a minimal tropical basis of uniform matroid $U_{d,n}$.

Proposition 3. The following statements are equivalent.

(1) A simple uniform matroid $U_{d,n}$ is regular.
(2) A simple uniform matroid $U_{d,n}$ is binary.
(3) A simple uniform matroid $U_{d,n}$ has a unique minimal tropical basis.

Proof. It is clear that (1) implies (2) by definition. We show (2) implies (1). For $i \in [n]$, we have

$$U_{d,n} \setminus \{i\} \cong \begin{cases} U_{d,n-1} & d < n \\ U_{d-1,n-1} & d = n, \end{cases} \quad U_{d,n}/\{i\} \cong \begin{cases} U_{d-1,n-1} & d > 0 \\ U_{d,n-1} & d = 0. \end{cases}$$

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If \( n \) is more than \( d + 1 \), then the deletion of a \( n - d + 2 \)-elements set from \( U_{d,n} \) is isomorphic to \( U_{d,d+2} \). The contraction of a \( d - 2 \)-elements set form \( U_{d,d+2} \) is isomorphic to \( U_{2,4} \). \( U_{d,n} \) has a minor isomorphic to \( U_{2,4} \). By Theorem 1, \( U_{d,n} \) is not binary. Therefore, if a uniform matroid \( U_{d,n} \) is binary, then \( n \) is equal to \( d + 1 \) or \( d \). It is clear that \( U_{n,n} \), \( U_{n-1,n} \) are ternary. By Theorem 2, \( U_{n,n} \) and \( U_{n-1,n} \) are regular. We next show (2) implies (3). Let \( U_{d,n} \) be a simple and binary uniform matroid. Since \( U_{d,n} \) is binary, \( n \) is equal to \( d \) or \( d + 1 \). The circuit set of \( U_{n,n} \) and \( U_{n-1,n} \) are \( \emptyset \) and \( \{ [n] \} \), respectively. They have a unique minimal tropical basis. We now show (3) implies (2). We show the contraposition. If a uniform matroid \( U_{d,n} \) is not binary, then \( n \) is more than \( d + 1 \). By Proposition 2, \( B_1 \) and \( B_2 \) are minimal tropical bases of \( U_{d,n} \). There are circuits \( C_1, C_2 \) such that \( 2 \notin C_1 \) and \( 1 \notin C_2 \). Since \( C_1 \) is not in \( B_2 \) and \( C_2 \) is not in \( B_1 \), \( B_1 \) is not equal to \( B_2 \). Both are minimal tropical bases. Therefore, \( U_{d,n} \) has two minimal tropical bases.

### 3.3 Fano and non-Fano matroid

Let \( E \) be the set \( \{1, 2, \ldots, 7\} \) of points and let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be the collection of subset \( X \) of \( E \) in the left diagram and the right diagram, respectively such that \( X \) is three colinear points or four points does not contain three colinear points. \( (E, \mathcal{C}_1) \) and \( (E, \mathcal{C}_2) \) obey the axioms for matroid. \( (E, \mathcal{C}_1) \) and \( (E, \mathcal{C}_2) \) are called the Fano matroid and the non-Fano matroid, respectively. The above diagrams are called geometric representations of these matroids. A circuit \( X \) of these matroids is called a line if the cardinality of \( X \) is three. Let \( K \) be a field. The Fano matroid is representable over \( K \) if and only if the characteristic of \( K \) is two. The non-Fano matroid is representable over \( K \) if and only if the characteristic of \( K \) is not two. Therefore, the Fano matroid is not ternary but binary, and the non-Fano matroid is not binary but ternary. Yu and Yuster\[7\] showed that the Fano
matroid has a unique minimal tropical basis. The unique minimal tropical basis of the Fano matroid is 7 lines. All minimal tropical bases of the non-Fano matroid are \{6 lines, \{1,4,5,6\}\}, \{6 lines, \{2,4,5,6\}\}, \{6 lines, \{1,4,5,6\}\} and \{6 lines, \{4,5,6,7\}\}. Therefore, the Fano matroid has a unique minimal tropical basis and the non-Fano matroid does not.

4 Binary matroid

Yu and Yuster have shown that every simple graphic, cographic matroid and \(R_{10}\) have a unique minimal tropical basis. Concrete minimal tropical bases of these matroids are as follows.

**Theorem 3** (\[7\]).

1. The unique minimal tropical basis of a simple graphic matroid consists of the induced cycles.
2. The unique minimal tropical basis of a simple cographic matroid consists of the edge cuts that split the graph into two 2-edge-connected subgraphs.
3. The unique minimal tropical basis of \(R_{10}\) consists of the fifteen 4-cycles.

In this section we show that every simple binary matroid has a unique minimal tropical basis. Let \(M = ([n], \mathcal{C})\) be a matroid. It is clear that if the cardinality of \(\mathcal{C}\) is zero or one, then \(M\) has a unique minimal tropical basis. So in this section we assume that the cardinality of \(\mathcal{C}\) is more than one.

**Proposition 4.** Let \(M\) be a simple binary matroid and \(C\) be a circuit of \(M\). The following statements are equivalent.

1. The circuit \(C\) is not in \(B_M\).
2. There are circuits \(C_1, C_2\) such that \(C_1, C_2\) have a unique element in their intersection and their symmetric difference \(C_1 \Delta C_2\) is equal to \(C\).

**Proof.** By Proposition 1, it is clear that (1) implies (2). We prove that (2) implies (1). We show the contraposition. Let \(C\) be a circuit that does not satisfy condition (2). We show that for every circuit \(C'\) in \(\mathcal{C}\setminus\{C\}\), the cardinality of \(C'\setminus C\) is equal to or more than 2. By simpleness of \(M\), if \(C \cap C'\) is empty, then \(|C'\setminus C| = |C'|\) is more than 2. So we assume that \(C \cap C'\) is not empty. By axioms for
matroid (2), $|C \setminus C'|$ and $|C' \setminus C|$ are more than or equal to one. We use contradiction for proof. Assume that the cardinality of $C' \setminus C$ is equal to one. Let $e$ be a unique element of $C' \setminus C$. Since $M$ is binary, $C \triangle C'$ is a disjoint union of circuits $\biguplus_{i=1}^{k} C_i$. There is a unique circuit $C_i$ that contains the element $e$. $\biguplus_{j=1,j\neq i}^{k} C_j$ does not contain the element $e$. We have $\biguplus_{j=1,j\neq i}^{k} C_j \subset C \triangle C' = (C \setminus C') \cup (C' \setminus C) = (C \setminus (C \cap C')) \cup \{e\} \subset C \cup \{e\}$. Since $M$ is simple, the cardinality of $C_i$ is more than 2. Therefore, $\biguplus_{j=1,j\neq i}^{k} C_j$ is a proper subset of $C$. By the axioms for matroid (2), $k$ is equal to one. Hence $C \triangle C'$ is a circuit $C_1$. We have $C' \cap C_1 = C' \cap (C \triangle C') = C' \cap ((C \setminus C') \cup (C' \setminus C)) = (C' \cap (C \setminus C')) \cup (C' \cap (C \setminus C')) = \emptyset \cup \{e\} = \{e\}$. Since $C'$ contains the element $e$, $C' \cap C_1$ is equal to $\{e\}$. $C_1 \triangle C' = (C_1 \cup C') \setminus (C_1 \cap C') = (((C \setminus C') \cup (C' \setminus C)) \cup C') \setminus \{e\} = (C \cup C') \setminus \{e\} = (C \cup C') \setminus (C' \setminus C) = C$. Therefore, the circuit $C$ is the pasting of $C'$ and $C_1$, which contradicts the assumption.

$x = (x_i)_i \in \text{TP}^{n-1}$ is defined following. Assign all but one point of $C$ weight 0. Assign weight 1 to the remaining point of $C$ and to all other points of the ground set $E$. $x = (x_i)_i$ is not in $V(C)$. Since for every circuit $C'$ in $\mathcal{C} \setminus \{C\}$, $|C' \setminus C| \geq 2$, $x = (x_i)_i$ is in $V(\mathcal{C} \setminus \{C\})$. By Lemma 1, $C$ is in the intersection of all tropical bases $B_M$. □

**Lemma 3.** Let $M$ be a simple matroid and $B$ be the set consisting of circuits that can not be obtained by pasting. $B$ is a tropical basis of $M$.

**Proof.** By simpleness of $M$, if the pasting of $C_1$ and $C_2$ is a circuit $C$, then the cardinality of $C$ is more than the cardinality of $C_1, C_2$. By induction, we can check that $B$ satisfies Proposition 1’s condition. Therefore, $B$ is a tropical basis of $M$. □

**Theorem 4.** Every simple binary matroid has a unique minimal tropical basis.

**Proof.** Proposition 4 and Lemma 3 show that the intersection of all tropical bases of every simple binary matroid is a tropical basis. By Lemma 2, every simple binary matroid has a unique minimal tropical basis. □

Since the regular matroid is binary, every simple regular matroid has a unique minimal tropical basis.

5 Algorithm discriminating whether a matroid has a unique minimal tropical basis

We propose an algorithm discriminating whether a matroid has a unique minimal tropical basis. In [7] it was shown that we can determine whether $B \subset \mathcal{C}$ is a tropical basis by
0/1-points in $\text{TP}^{n-1}$. Therefore, the condition whether a subset of the circuit set is a tropical basis is computable. Let $M$ be a matroid and $\mathcal{C}$ be the circuit set and $k$ be the cardinality of $\mathcal{C}$. By Lemma 1, the intersection of of tropical bases $B_M$ is computable. We give an order $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$. Let $B_0$ be the circuit set $C_M$. For $C_i$, if $B_i \setminus \{C_i\}$ is a tropical basis, then $B_{i+1}$ is defined to be $B_i \setminus \{C_i\}$. Else $B_{i+1}$ is defined to be $B_i$. After repeating this operation for $k$ times, we get a minimal tropical basis $B_k$. By Lemma 2, if $B_k$ is equal to $B_M$, $M$ has a unique minimal tropical basis. Else $M$ has more than one minimal tropical bases.

We give examples. Let $P_7$ be the matroid that has the left above diagram as geometric representation. By that algorithm, we know $P_7$ has a unique minimal tropical basis. In [3] it was known that for a field $K$, $P_7$ is representable over $K$ if and only if the cardinality of $K$ is more than 2. So it is not binary. Therefore, the converse of Theorem 4 is not true. Let $R_6$ be the matroid that has the right above diagram as geometric representation. In [3] it was known that for a field $K$, $R_6$ is representable over $K$ if and only if the cardinality of $K$ is more than two. By that algorithm, we know $R_6$ has more than one minimal tropical bases. By Theorem 4 and examples of $P_7$ and $R_6$, we have the following theorem.

**Theorem 5.** Let $F$ be a set of fields. The following statements are equivalent.

1. If a simple matroid $M$ is representable over all fields in $F$, then $M$ has a unique minimal tropical basis.
2. The binary field $\mathbb{F}_2$ is in $F$.
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