Binding of antikaons and $\Lambda(1405)$ clusters in light kaonic nuclei

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The energy spectra of light-mass kaonic nuclei were investigated using the theoretical framework of the 0s-orbital model with zero-range $KN$ and $KK$ interactions of effective single-channel real potentials. The energies of the $KN$, $KNN$, $KNNN$, $KKN$, and $KKN$ systems were calculated in the cases of weak- and deep-binding of the $KN$ interaction, which was adjusted to fit the $\Lambda(1405)$ mass with the energy of the $KN$ bound state. The results qualitatively reproduced the energy systematics of kaonic nuclei calculated via other theoretical approaches. In the energy spectra of the $KNN$ and $KKNN$ systems, the lowest states $KNN(J^P, T = 0^+, 1/2^+)$ and $KKNN(0^+, 0)$ were found to have binding energies approximately twice and four times as large as that of the $KN(1/2^-, 0)$ state, respectively. Higher ($J^P, T$) states including $KNN(1^-, 1/2^+)$, $KKNN(0^+, 1)$, and $KKN(1^+, 1)$ were predicted at energies of 9–25 MeV below the antikaon-decay threshold. The effective $\Lambda(1405)$-$\Lambda(1405)$ interaction in the $KKNN$ system was also investigated via a $KN + KN$-cluster model. Strong and weak $\Lambda(1405)$-$\Lambda(1405)$ attractions were obtained in the $S^\pi = 0^+ + S^\pi = 1^+$ channels, respectively. The $\Lambda(1405)$-$\Lambda(1405)$ interaction in the $KKNN$ system was compared with the effective $d$-$d$ interaction in the $NNNN$ system, and the properties of dimer-dimer interactions in hadron and nuclear systems were discussed.

I. INTRODUCTION

Kaonic nuclei have recently become a hot topic in hadron physics. In particular, light-mass kaonic nuclei have been intensively investigated to understand the structures of exotic hadrons that are interpreted as multi-hadron systems called hadron molecular states. One candidate for the simplest such system is the $\Lambda(1405)$ state (denoted as $\Lambda^\ast$), which is the lowest negative-parity $\Lambda$ state with $(J^P, T) = (1/2^-, 0)$. The $\Lambda(1405)$ state is a narrow resonance observed in the $\pi\Sigma$ spectra at an energy slightly below the $\bar{K}N$ threshold, and is considered to be a $\bar{K}N$ quasibound state produced by a strong $\bar{K}N$ interaction. This picture of the $\bar{K}N$ molecular state for the $\Lambda(1405)$ resonance led to the concept of kaonic states with an antikaon deeply bound via the $\bar{K}N$ interaction in light-mass nuclei such as $K^-pp$ and $K^-ppn$ [1-3], for which various few-body calculations have been developed [4-19]. Several experiments have been performed in searching of the $\bar{K}N$ state [20-23], but the evidence has not yet been confirmed [24-25]. A similar challenging is investigating double-kaonic nuclei with two antikaons [26-28].

For kaonic nuclei of mass number $A = 2$, intensive studies of the $\bar{K}N$ system have been performed by many groups. To clarify the properties of the strangeness dibaryon, the $\bar{K}KN$ system is also a key issue. It has also attracted interest in properties of the $\bar{K}K$ interaction concerning the kaon condensation in dense nuclear matter. To experimentally search for the quasibound $\bar{K}KNN$ state, a formation mechanism via a $\Lambda^\ast + \Lambda^\ast$ doorway was proposed [29]. Furthermore, the effective $\Lambda^\ast\Lambda^\ast$ interaction in the $\bar{K}KNN$ system may draw general interest in dimer-dimer interactions in hadron systems. For such a system consisting of isospin SU(2) bosons and fermions, the question of what role is played by the nucleon Fermi statistics and antikaon Bose statistics in the effective dimer-dimer interaction between two $\Lambda^\ast$ particles arises.

The original idea for kaonic nuclei was based on the phenomenological $\bar{K}N$ interaction called the Aihashi-Yamazaki (AY) interaction [1-3]. The AY interaction is characterized by an extremely strong $\bar{K}N$ attraction in the $T = 0$ channel, which reproduces an $\Lambda(1405)$ mass at an energy 27 MeV below the $\bar{K}N$ threshold reported in Particle Data Group (PDG) table [30]. On the other hand, a weaker $\bar{K}N$ interaction was proposed in detailed analyses of $\pi\Sigma$ scattering based on the chiral SU(3) effective field theory, from which the $\Lambda^\ast$ resonance-pole position was obtained at only 8–12 MeV below the $\bar{K}N$ threshold [31]. In three-body calculations of the $\bar{K}NN$, the deep-type AY interaction predicted a deeply bound $\bar{K}NN$ state, whereas the weak-type chiral interaction obtained a smaller binding energy (B.E.) for the $\bar{K}NN$ state. Moreover, for other kaonic nuclei, theoretical predictions of the binding energies are spread over a range because of the uncertainty of $\bar{K}N$ interactions and model ambiguities in theoretical treatments, such as the energy dependence of the interaction as well as channel coupling.

In this paper, my aim is to investigate the energy spectra of single- and double-kaonic nuclei, particularly, the $\bar{K}KNN$ system. I do not intend to predict precise values of the energy spectra, which may depend upon the details of hadron-hadron interactions as well as model treatments. Instead, I investigate the energy systematics of kaonic nuclei to understand their global features and to extract universal properties independently from the uncertainty in hadron-hadron interactions. In this paper, I apply a simple model of the 0s-orbital configuration to kaonic nuclei and calculate their energy spectra by assuming zero-range $\bar{K}N$ and $\bar{K}K$ interactions of effective single-channel real potentials. The $\bar{K}N$ interaction is tuned to fit the $\Lambda^\ast$ mass with the $\bar{K}N$ bound state in two cases of weak- and deep-binding. As for the $NN$ interaction, I adopt a finite-range effective $NN$ interaction adjusted to reproduce $S$-wave $NN$-scattering lengths. I
discuss the important roles of isospin symmetry in the energy spectra of kaonic nuclei. I also investigate the effective $\Lambda^*-\Lambda^*$ interaction with a $KN+KN$-cluster model. For comparison with the effective $d-d$ interaction in the $N{NNN}$ system, I discuss the properties of dimer-dimer interactions and binding mechanisms in hadron and nuclear systems.

This paper is organized as follows. In Sec. [II] the theoretical frameworks for the $0s$-orbital and $KN+KN$ cluster models are explained. In Sec. [III] the results of the $0s$-orbital model for kaonic nuclei are presented. In Sec. [IV] the $KNN$ system is investigated via the $KN+KN$-cluster model, and the effective $\Lambda^*\Lambda^*$ interactions are discussed. Finally, a summary is given in Sec. [V].

II. FORMULATION OF SINGLE- AND DOUBLE-KAONIC NUCLEI

A. $0s$-orbital model for single- and double-kaonic nuclei

For single and double-kaonic nuclei with $AK$ antikaons and $A$ nucleons, I assume the $0s$-orbital configuration with one-range Gaussian wave functions and express the wave functions for the $(J^\pi, T)$ states with angular momentum $(J)$, parity $(\pi)$, and isospin $(T)$ as

$$ \Psi^{(J^\pi,T)}_{KNN} (r_1, \ldots, r_A) = \phi^K_{0} (r_1) \cdots \phi^K_{0} (r_A) \phi^N_{0} (r_1) \cdots \phi^N_{0} (r_A) \otimes [s_1 \cdots s_A] S \otimes [t_1 \cdots t_A] T, $$

$$ \phi^K_{0} (r) = \left( \frac{2\mu_K}{\pi} \right)^{3/4} e^{-\mu_K r^2}, $$

$$ \phi^N_{0} (r) = \left( \frac{2\mu_N}{\pi} \right)^{3/4} e^{-\mu_N r^2} $$

with $J = S$ and $\pi = (-1)^AK$. Here, nucleon spins $s_i$ are coupled to the total nuclear spin $S$, and antikaon isospins $t_i$ and nucleon isospins $t_i$ are coupled to the total isospin $T$. The spatial configuration of identical particles in the $0s$-orbit is symmetric. The nucleon Fermi statistics are taken into account in the nucleon-spin and -isospin configuration. For double-kaonic nuclei, the isospins of two antikaons in the $0s$-orbit are coupled to form an isovector ($\tau_K = 1$) as $[t_1 t_2]_{\tau_K = 1}$ to satisfy the Bose statistics.

The Gaussian width parameter $\nu_K$ for $\phi^K_{0}$ is chosen to be

$$ \nu_K \equiv \frac{m_K}{m_N} \nu_N $$

with a ratio $m_K/m_N \approx 1/2$ of the antikaon mass $m_K$ to the nucleon mass $m_N$, such that the center-of-mass (cm) motion can be exactly removed from the total wave function. Hence, the antikaon orbit $\phi^K_{0}$ has a broader distribution than the nucleon orbit $\phi^N_{0}$. The internal wave functions of the $NN$, $\bar{K}K$, and $\bar{K}N$ pairs can be written as

$$ \Phi^{\bar{K}K} (r_{ij}'; r_{ij}) = \left( \frac{\nu_K}{\pi} \right)^{3/4} e^{-\frac{\nu_K}{2} r_{ij}'^2}, $$

$$ \Phi^{NN} (r_{ij}) = \left( \frac{\nu_N}{\pi} \right)^{3/4} e^{-\frac{\nu_N}{2} r_{ij}^2}, $$

$$ \Phi^{KN} (r_{ij}) = \left( \frac{\lambda}{\pi} \right)^{3/4} e^{-\frac{\lambda}{2} r_{ij}^2} $$

where $r_{kl} = r_k - r_k$. The root-mean-square (rms) distances $\sqrt{\langle r_{kl}^2 \rangle}$ of the $NN$, $\bar{K}K$, and $\bar{K}N$ pairs are given as $R_{NN} = 1/\sqrt{2\nu_N}$, $R_{KK} = 1/\sqrt{2\nu_K}$, and $R_{KN} = 1/\sqrt{2\lambda}$, respectively.

1. Wave functions of single-kaonic nuclei

The $\bar{K}N$ bound state with $(J^\pi, T) = (1^-, 0)$ corresponding to $\Lambda^*$ is expressed as

$$ \Phi^{(1^-)}_{KNN} (1', 1) = \phi^K_0 (r_1') \phi^N_0 (r_1) \otimes s_1 \otimes [t_1, t_1]_{T=0}, $$

where the nucleon spin $s_1 = \{ \uparrow, \downarrow \}$ specifies the intrinsic spin of the $\Lambda^*$ system.

The $KNN$ states with $(J^\pi, T) = (0^-, \frac{1}{2})$, $(0^-, \frac{3}{2})$, and $(1^-, \frac{1}{2})$ are written as

$$ \Psi^{(0^-)}_{KNNN} (1', 1, 2) = \phi^K_0 (r_1') \phi^N_0 (r_1) \phi^N_0 (r_2) \otimes s_1 s_2 s_3 S = 0 \otimes [t_1, t_1, t_2]_{\tau_K = 1}, $$

$$ \Psi^{(1^-)}_{KNNN} (1', 1, 2) = \phi^K_0 (r_1') \phi^N_0 (r_1) \phi^N_0 (r_2) \otimes s_1 s_2 s_3 S = 1 \otimes [t_1, t_1, t_2]_{\tau_K = 0}, $$

where $\tau_K$ indicates the total nucleon isospin. The $J = 0$ states contain an isovector $NN$ pair, and the $J = 1$ state contains a deuteron-like isoscalar $NN$ pair because of the nucleon Fermi statistics for the total nucleon spin, $S = J$. The $(0^-, \frac{3}{2})$ state is the lowest $KNN$ state, which has been intensively studied by three-body calculations, whereas the $(1^-, \frac{1}{2})$ state was predicted to be a higher $KNN$ state [15, 16].

For kaonic nuclei with mass numbers $A = 3$ and $A = 4$, I consider the $(J^\pi, T) = (1^-, 0)$ and $(0^-, \frac{1}{2})$ states with $^3$H and $^4$He cores, respectively, as

$$ \Psi^{(1^-)}_{KNNNN} (1', 1, 2, 3) = \phi^K_0 (r_{1'}) \phi^N_0 (r_1) \phi^N_0 (r_2) \phi^N_0 (r_3) \otimes s_1 s_2 s_3 S = \frac{1}{2} \otimes [t_1, t_2, t_3]_{\tau_K = \frac{1}{2}}, $$

$$ \Psi^{(0^-)}_{KNNNN} (1', 1, 2, 3, 4) = \phi^K_0 (r_{1'}) \phi^N_0 (r_1) \phi^N_0 (r_2) \phi^N_0 (r_3) \phi^N_0 (r_4) \otimes s_1 s_2 s_3 s_4 S = 0 \otimes [t_1, t_2, t_3, t_4]_{\tau_K = 0}, $$

where $[t_1, t_2, t_3, t_4]_{\tau_K = 0}$.
2. Wave functions of double-kaonic nuclei

The $0s$-orbital states of double-kaonic nuclei contain an isovector $KK$ pair because of the Bose statistics. The $KKN$ system with $(J^\pi, T) = (0^+, 0)$ is given by

$$\Psi_{KKN}^{(0^+, 0)}(1', 2', 1, 2) = \phi^K(r_1')\phi^K_0(r_2')\phi^K(r_1)\phi^K_0(r_2)$$

$$\otimes s_1 \otimes \left[ t_{11}t_{22}^*_{\tau_K = 1} t_{11}^* \right]_{T = \frac{1}{2}}, \quad (14)$$

For the $KKNN$ system with the 0s-orbital configuration, the $(J^\pi, T) = (0^+, T)$ states consist of isovector $NN$ and $KK$ pairs, which are coupled to the total isospins $T = 0, 1, and 2$, and the $(J^\pi, T) = (1^+, T)$ state contains an isoscalar $NN$ pair and an isovector $KK$ pair as

$$\Psi_{KKNN}^{(1^+, 0)}(1', 2', 1, 2) = \phi^K(r_1')\phi^K_0(r_2')\phi^K_0(r_1)\phi^K(r_2)$$

$$\otimes s_1 s_2 \otimes \left[ t_{11}t_{22}^*_{\tau_K = 1} t_{11}^* \right]_{T = 1}, \quad (15)$$

$$\Psi_{KKNN}^{(1^+, 0)}(1', 2', 1, 2) = \phi^K(r_1')\phi^K_0(r_2')\phi^K_0(r_1)\phi^K(r_2)$$

$$\otimes s_1 s_2 s_3 \otimes \left[ t_{11}t_{22}^*_{\tau_K = 1} t_{11}^* \right]_{T = 1}, \quad (16)$$

B. $KN + KN$ cluster model for $KKNN$

To discuss the effective $\Lambda^* - \Lambda^*$ interaction, I apply a $KN + KN$-cluster model to the $KKNN$ system. I consider two $(KN)$-clusters with the 0s-orbital configuration located at $-\mathbf{R}/2$ and $\mathbf{R}/2$ with a distance of $R = |\mathbf{R}|$ as

$$\Psi_{KN + KN}^{(S^0, T)}(\mathbf{R}; 1', 2', 1, 2) = n_0 A_{12} S_{12'} \left( \phi^K_{1/2}(r_1')\phi^K_{1/2}(r_2')\phi^K(r_1)\phi^K(r_2) \right)$$

$$\otimes s_1 s_2 \otimes \left[ t_{11}^* t_{22}^*_{\tau_K = 1} t_{11}^* \right]_{T = 0}, \quad (17)$$

$$\phi^K(r) = \left( \frac{2\nu_K}{\pi} \right)^{3/4} e^{-\nu_K(r - x)^2}, \quad (18)$$

$$\phi^K_N(r) = \left( \frac{2\nu_N}{\pi} \right)^{3/4} e^{-\nu_N(r - x)^2}, \quad (19)$$

where $n_0$ is the normalization factor. The operators $A_{12}$ and $S_{12'}$ are the antisymmetrized and symmetrized operators for nucleons and kaons, respectively,

$$A_{12} = \frac{1 - P_{12}}{\sqrt{2}}, \quad S_{12'} = \frac{1 + P_{12}}{\sqrt{2}}, \quad (20)$$

which are equivalent to the internal-parity-projection operators of the $NN$ and $KK$ pairs.

Hereafter, I consider the $T = 0$ states with $\tau_K = \tau_N \equiv \tau$ leading to the asymptotic $\Lambda^* + \Lambda^*$ state at $R \to \infty$, and take the isospin $\tau = 0$ and $\tau = 1$ components, which I denoted as $\Psi_{KN + KN}^{(S^0, 0)}(\mathbf{R}, \tau; 1', 2', 1, 2)$, into account. The wave function with configuration mixing $(\tau$-mixing) of $\tau = 0$ and $\tau = 1$ is given by

$$\Psi_{KN + KN}^{(S^0, 0)}(\mathbf{R}; 1', 2', 1, 2) = \sum_{\tau = 0, 1} c_\tau \Psi_{KN + KN}^{(S^0, 0)}(\mathbf{R}, \tau; 1', 2', 1, 2), \quad (21)$$

where the coefficients $c_\tau$ are determined by diagonalization of the norm and Hamiltonian matrices for $\tau = \{1, 0\}$ at each distance $R$. Note that the parities $\pi$ of the $KN + KN$ system is related to the total nucleon spin $S$ as $S^0 = 0^+$ and $1^-$ because of the nucleon Fermi and antikaon Bose statistics. These correspond to the selection rule of $S^0 = 0^+$ and $1^-$ for two $\Lambda^*$ particles in Fermi statistics.

For the $KN + KN$-cluster state in the $S^0 = 0^+$ channel, the $\tau = 1$ component has a spatial-even $KK$ pair and a singlet-odd $NN$ pair, while the $\tau = 0$ component consists of a spatial-odd $KK$ pair and a singlet-odd $NN$ pair. The former, the $\tau = 1$ component, becomes equivalent to the lowest 0s-orbital $KKNN$ state with $(J^\pi, T) = (0^+, 0)$ at $R = 0$. The latter is forbidden in the 0s-orbital model space and therefore goes to an excited configuration with two $0p$-orbital particles in the $R \to 0$ limit.

The $KN + KN$-cluster state in the $S^0 = 1^-$ channel leads to a spin-aligned $\Lambda^* + \Lambda^*$ state having negative parity. The $\tau = 1$ component of the $S^0 = 1^-$ state has a spatial-even $KK$ pair and a triplet-odd $NN$ pair, whereas the $\tau = 0$ component is composed of a spatial-odd $KK$ pair and a triplet-even $NN$ pair. Neither component is not allowed in the 0s-orbital configuration. Instead, the $\tau = 1$ component becomes one-antikaon excitation and the $\tau = 0$ component goes to a one-nucleon excitation in the $R \to 0$ limit.

It is worth noting that the spatial part of the $KKNN$-cluster wave functions before $A_{12}$ and $S_{12'}$ can be rewritten in the separable form of the cm, inter-cluster, and $KN$-cluster internal wave functions with Jacobi coordinates as

$$\phi^K_{1/2}(r_1')\phi^K_{1/2}(r_1)\phi^K(r_2)\phi^K(r_2')$$

$$= \phi_{cm}(r_{cm}) \otimes \phi_{rel}(\mathbf{R}, r_{rel}) \otimes \phi^K(r_{1'}) \otimes \phi^K(r_{2'}), \quad (22)$$

$$\phi_{cm}(r_{cm}) = \left( \frac{2 \cdot 4\gamma}{\pi} \right)^{3/4} e^{-4\gamma r_{cm}^2}, \quad (23)$$

$$\phi_{rel}(\mathbf{R}, r_{rel}) = \left( \frac{2\gamma}{\pi} \right)^{3/4} e^{-\gamma(r_{rel} - \mathbf{R})^2}, \quad (24)$$

$$\gamma = \frac{m_1 + m_2}{2m_1}, \quad (25)$$

$$r_{cm} = \frac{1}{2} \left\{ \frac{m_N r_{1'} + m_K r_1}{m_N + m_K} + \frac{m_N r_{2'} + m_K r_2}{m_N + m_K} \right\}, \quad (26)$$

$$r_{rel} = \frac{m_N r_{2'} + m_K r_2}{m_N + m_K} - \frac{m_N r_{1'} + m_K r_1}{m_N + m_K}. \quad (27)$$
C. Hamiltonian and two-body interactions

In the present calculation, I omit the charge-symmetry breaking and the Coulomb force. The Hamiltonian for single-kaonic nuclei \( \bar{K}N^A \) with mass number \( A \) is given by

\[
H_{\bar{K}N^A} = t^\text{kin}_N^A + \sum_i A_i t^\text{kin}_i - T^\text{cm}
+ \sum_{T=0,1} \sum_i A_{T,i} v^T_{\bar{K}N}(1', i) + \sum_{i<j} v^{(ST)}_{NN}(i, j),
\]

(28)

where \( t^\text{kin} \) is the single-particle kinetic energy, \( T^\text{cm} \) is the cm kinetic energy, \( v^T_{\bar{K}N} \) is the \( T \) component of the \( \bar{K}N \) interaction, and \( v^{(ST)}_{NN} \) with \( (ST) = (10), (01), (11), \) and \( (00) \) indicate the triplet-even, singlet-even, triplet-odd, and singlet-odd components of the \( NN \) interaction, respectively. Similarly, the Hamiltonian for the double-kaonic nuclei \( \bar{K}\bar{K}N^A \) is given by

\[
H_{\bar{K}\bar{K}N^A} = t^\text{kin}_{\bar{K}} + t^\text{kin}_{N^A} + \sum_i A_i t^\text{kin}_i - T^\text{cm}
+ \sum_{T=0,1} \sum_i A_{T,i} v^T_{\bar{K}\bar{K}}(1', i) + \sum_{i<j} v^{(ST)}_{NN}(i, j) + \sum_{T=0,1} v^{(ST)}_{\bar{K}N}(1', 2').
\]

(29)

In the present 0s-orbital model, the single-particle kinetic energies of antikaons and nucleons have the same value as

\[
\langle \phi_0^N | t^\text{kin} | \phi_0^N \rangle = \langle \phi_0^{\bar{K}} | t^\text{kin} | \phi_0^{\bar{K}} \rangle = \frac{3h\omega}{4} \equiv T_0^\text{kin},
\]

(30)

where \( h\omega = 2\hbar^2
\nu_N/m_N \). The cm kinetic energy term is also the same value \( T_{\text{cm}}^\text{kin} = T_0^\text{kin} \). Thus, the total kinetic energy is given by \( T_{\text{kin}}^\text{kin} = (A_{\text{tot}} \times 1) - T_0^\text{kin} \), where \( A_{\text{tot}} \) is the total particle number \( A_{\text{tot}} = A + A_{\bar{K}} \).

For the \( NN \) interaction, I adopt a finite-range effective central interaction of the Volkov \( NN \) force [22], which is often used with cluster models for nuclear systems. In the present calculation, the \( NN \) spin-orbit and tensor interactions are omitted. The Volkov central \( NN \) force is given in two-range Gaussian form as

\[
v^{(ST)}_{NN}(i, j) = u^{(ST)}_{NN}(r_{ij}) P_{ij}^{(ST)},
\]

(31)

\[
u^{(ST)}_{NN}(r) = f^{(ST)}_{NN}(r) \sum_{k=1,2} V_k e^{-r^2/2a^2_k},
\]

(32)

where \( P_{ij}^{(ST)} \) is the projection operator to the \( (ST) \) state of the \( NN \) pair. The range parameters \( \eta_k \) and global-strength parameters \( V_k \) are given in the Volkov parameterization, whereas the strength ratios \( f^{(ST)}_{NN} \) of four components \( (ST) = (10), (01), (11), \) and \( (00) \) are adjustable parameters, which I tune to fit the \( S \)-wave \( NN \)-scattering lengths and the \( \alpha + \sigma \)-scattering phase shifts. For the spatial parts of the expectation values of the \( NN \) interaction for the 0s-orbital \( NN \) pair, I use the notation

\[
V^{(ST)}_{NN} \equiv \langle \phi_0^N \phi_0^N | u^{(ST)}_{NN}(r) | \phi_0^N \phi_0^N \rangle.
\]

(33)

For the \( \bar{K}N \) and \( \bar{K}\bar{K} \) interactions, I consider the \( S \)-wave interactions of the effective single-channel real potentials and assume zero-range (delta function) forces for simplicity. The imaginary part of the \( \bar{K}N \) interaction, which corresponds to the \( \pi \Sigma \) decays via the \( \Sigma \cdot \bar{K}N \) coupling, is omitted. The \( \bar{K}N \) interaction in the \( T = 0 \) and \( T = 1 \) channels is written as

\[
v^{T=0}_{\bar{K}N}(i', j) = u^{T=0}_{\bar{K}N}(r_{ij}) P_{ij}^{T=0},
\]

(34)

\[
v^{T=1}_{\bar{K}N}(i', j) = u^{T=1}_{\bar{K}N}(r_{ij}) P_{ij}^{T=1},
\]

(35)

with the delta function \( u^T_{\bar{K}N}(r) = U^T_{\bar{K}N} \delta(r) \). Here, the isospin-projection operators can be expressed as \( P_{kl}^{T=0} = \frac{1-\tau^z_{\bar{K}}}{2} \) and \( P_{kl}^{T=1} = \frac{1+\tau^z_{\bar{K}}}{2} \). For the \( \bar{K}\bar{K} \) interaction, the spatial-even term exists only in the \( T = 1 \) channel and is given by

\[
v^{T=1}_{\bar{K}\bar{K}}(i', j') = u^{T=1}_{\bar{K}\bar{K}}(r_{ij}) P_{ij}^{T=1},
\]

(36)

\[
v^{T=0}_{\bar{K}\bar{K}}(i', j') = u^{T=0}_{\bar{K}\bar{K}}(r_{ij}) P_{ij}^{T=0},
\]

(37)

The strengths \( U^{T=0}_{\bar{K}N} \), \( U^{T=0}_{\bar{K}K} \), and \( U^{T=1}_{\bar{K}N} \) of the interactions are tuned as follows. I first adjust the strength \( U^{T=0}_{\bar{K}N} \) of the \( \bar{K}N \) interaction in the \( T = 0 \) channel to make the \( \Lambda^+ \) energy fit with the energy \( T^{\text{kin}}_0 + V^{T=0}_{\bar{K}N} \) of the \( \bar{K}N \) state. The strengths \( U^{T=1}_{\bar{K}N} \) and \( U^{T=1}_{\bar{K}K} \) are adjusted to reproduce the strength ratios \( F^{T=1}_{\bar{K}N} \equiv v^{T=1}_{\bar{K}N}(r)/u^{T=0}_{\bar{K}N}(r) \) and \( F^{T=1}_{\bar{K}K} \equiv u^{T=1}_{\bar{K}K}(r)/u^{T=0}_{\bar{K}K}(r) \) of the \( \bar{K}N \) and \( \bar{K}\bar{K} \) interactions used in other theoretical works with kaonic nuclei. The adopted values of these parameters are explained later.

D. Parameter settings

For the \( NN \) interaction, I use the values of \( V_1 = -60.65 \text{ MeV} \), \( V_2 = 61.14 \text{ MeV} \), \( \eta_1 = 1.80 \text{ fm} \), and \( \eta_2 = 1.01 \text{ fm} \) of the Volkov No.2 parametrization [22]. I tune the ratio parameters \( f^{(ST)}_{NN} \) to fit the experimental data of the \( S \)-wave \( NN \)-scattering lengths in the spin-triplet and -singlet channels and the \( \alpha + \sigma \)-scattering phase shifts and set values of \( f^{(10)}_{NN} = 1.3 \), \( f^{(01)}_{NN} = 0.7 \), \( f^{(11)}_{NN} = -0.2 \), and \( f^{(00)}_{NN} = -0.2 \). This parametrization describes a stronger triplet-even \( NN \) interaction to form
a bound deuteron state and a weaker singlet-even $NN$ interaction describing an unbound $nn$ state. The odd-channel $NN$ interactions are weak repulsions.

In the present calculation, I adopt two sets of parameters of the $0s$-orbit width ($\nu_N$) and the strengths of the $KN$ and $\bar{K}\bar{K}$ interactions. One is the set-I parametrization for the weak-binding case, and the other is the set-II parametrization for the deep-binding case. In each parametrization, I use fixed $\nu_N$ and $\nu_K$ values consistently for all kaonic and normal nuclei. In the set-I (weak-binding) case, I use $\nu_N = 0.16$ fm$^{-2}$, which was optimized for the deuteron energy in the $0s$-orbital model with the tuned $NN$ interaction. In the set-II (deep-binding) case, I choose $\nu_N = 0.25$ fm$^{-2}$ which reproduces the binding energy and nuclear size of the $^4$He system. To determine the strengths $U_{KN}^{T=0}$, $U_{KN}^{T=1}$, and $U_{K\bar{K}}^{T=1}$ of the $KN$ and $K\bar{K}$ interactions, I adopt the $\Lambda^*$ energy ($\epsilon_{\Lambda^*}$) and strength ratios $F_{KN}^{T=1}$ and $F_{K\bar{K}}^{T=1}$ that are given by a weak-type chiral interaction for the set-I (weak-binding) case, and those that are given by the deep-type phenomenological $\Lambda Y$ interaction for the set-II (deep-binding) case.

For the set-I (weak binding) case with $\nu_N = 0.16$ fm$^{-2}$, I adjust $U_{KN}^{T=0}$ to fit $\epsilon_{\Lambda^*} = -10$ MeV, which corresponds to the $\Lambda^*$ resonance-pole position of the chiral SU(3) analysis [31]. For the strength $U_{KN}^{T=1}$, I adopt the value $F_{KN}^{T=1} = 0.457$ of the effective single-channel $\bar{K}N$ potentials derived from the chiral SU(3) coupled-channel analysis. The original $KN$ potential in Ref. [31] is energy-dependent and contains imaginary terms, but I omit the energy dependence and use only the real part of the interaction at the $\Lambda^*$ resonance-pole position (1.421 MeV of the $\Lambda^*$ mass). For the strength $U_{K\bar{K}}^{T=1}$, I take the value $F_{K\bar{K}}^{T=1} = -0.345$ of a $\bar{K}K$ interaction from Ref. [28].

For the set-II (deep-binding) case with $\nu_N = 0.25$ fm$^{-2}$, $U_{KN}^{T=0}$ is adjusted to fit $\epsilon_{\Lambda^*} = -27$ MeV from the PDG value of $\Lambda^*$ [30]. To determine $U_{KN}^{T=1}$ and $U_{K\bar{K}}^{T=1}$, I employ the value $F_{KN}^{T=1} = 0.294$ of the deep-type single-channel $\Lambda Y$ interaction [1] [3], and the value $F_{K\bar{K}}^{T=1} = -0.175$ from Ref. [28].

The expectation values of the single-particle kinetic energy and the spatial parts of the $NN$, $\bar{K}N$, and $\bar{K}\bar{K}$ interaction terms of the two-particle pairs are listed in Table I. In both the set-I and set-II cases, the $\bar{K}N$ attraction is stronger in the $T = 0$ channel than in the $T = 1$ channel with a factor of 2-3, and the $T = 1$ $\bar{K}\bar{K}$ interaction is the weak repulsion. Note that the $\bar{K}N$ and $K\bar{K}$ interactions adopted here are delta forces renormalized to reproduce energy expectation values in the present $0s$-orbital model space with a given $\nu_N$ value. Such renormalized delta forces cannot be applied to variational calculations beyond the assumed model setting.

### III. RESULTS OF KAONIC NUCLEI WITH THE $0s$-ORBITAL MODEL

#### A. Energy counting

With the present $0s$-orbital model, I calculate the energies $E_{KNN}^{(J^P,T)}$ and $E_{K\bar{K}NN}^{(J^P,T)}$ of the $(J^P,T)$ states of the kaonic nuclei, $KNN^A$ and $K\bar{K}NN^A$. These are obtained by counting the spin and isospin components of the $NN$, $\bar{K}N$, and $K\bar{K}$ pairs and can be expressed simply with the expectation value terms, $T_0^{\text{kin}}$, $V_{NN}^{(ST)}$, $V_{KN}^{T=0}$, and $V_{K\bar{K}}^{T=1}$. Hence, I obtain the lowest $(J^P,T)$ states of each system of the $KNN(1/2^-,0)$, $KNN(0^-,1/2)$, $K\bar{K}NN(1/2^+,1/2)$, $K\bar{K}NN(0^+,0)$, $K\bar{K}NN(1/2^-,0)$, and $K\bar{K}NN(0^-,1/2)$.

The energy of $KNN(1/2^-,0)$ corresponding to the $\Lambda^*$ state is

$$E_{KNN}^{(1/2^-,0)} = T_0^{\text{kin}} + V_{KN}^{T=0} = \epsilon_{\Lambda^*},$$

(38)

which is used as an input to determine the interaction strengths in the present framework. For $K\bar{K}NN(0^-,1/2)$, $K\bar{K}NN(1/2^+,1/2)$, $K\bar{K}NN(0^+,0)$, the energies are given by

$$E_{K\bar{K}NN}^{(0^-,0)} = 3T_0^{\text{kin}} + V_{KN}^{T=0},$$

(40)

where $\epsilon_{\Lambda^*} = T_0^{\text{kin}} + V_{KN}^{T=0}$ is the energy of a two-neutron state with the $0s$-orbital configuration and has a positive value. The energies of $K\bar{K}NN(1/2^-,0)$ and $K\bar{K}NN(0^-,1/2)$ are written as

$$E_{KNN}^{(0^-,1/2)} = 4T_0^{\text{kin}} + 6\left(\frac{1}{2}V_{NN}^{T=0}\right) + 6\left(\frac{1}{2}V_{NN}^{T=1}\right) + 4\left(\frac{1}{4}V_{KN}^{T=1}\right) + 4\left(\frac{3}{4}V_{KN}^{T=1}\right) = \epsilon_{\Lambda^*} + \epsilon_{\Lambda^*} = 3V_{KN}^{T=1}.$$

(44)

where $\epsilon_{\Lambda^*} = T_0^{\text{kin}} + 3V_{NN}^{(10)} + 3V_{NN}^{(10)}$ and $\epsilon_{\Lambda^*} = 3T_0^{\text{kin}} + 3V_{NN}^{(10)} + 3V_{NN}^{(10)}$ are the energies of the $^3$H and $^4$He nuclei with the $0s$-orbital configuration, respectively.
TABLE I: Expectation values and factors for energies of kaonic and normal nuclei in the 0s-orbital model. Upper: the single-particle kinetic energy and the spatial part of the expectation values of the interaction terms for the NN, \( \bar{K}N \), and \( \bar{K}\bar{K} \) pairs for set-I and set-II parametrization in units of MeV. Lower: factors of each term as \( A_{\text{tot}} \) for the kinetic term, and the product of the number of pairs and the isoscalar component per pair for the interaction terms.

| nuclei \((J^\pi, T)\) | set-I \((\nu_N = 0.16 \text{ fm}^{-2})\) | set-II \((\nu_N = 0.25 \text{ fm}^{-2})\) |
|--------------------------|-------------------------------|-------------------------------|
| \( \bar{K}N(1/2^-, 0) \) | \( T^{\text{kin}}_0 \) | 9.95 | 15.55 |
| \( \bar{K}NN(0^-, 1/2) \) | \( V^{(10)}_{NN} \) | -11.55 | -16.32 |
| \( \bar{K}NN(0^-, 3/2) \) | \( V^{(01)}_{NN} \) | -6.22 | -8.79 |
| \( \bar{K}NN(1^-, 1/2) \) | \( V^{(01)}_{\bar{K}N} \) | -19.95 | -42.55 |
| \( \bar{K}NN(1^-, 1/2) \) | \( V^{(10)}_{\bar{K}N} \) | -9.11 | -12.51 |
| \( \bar{K}NN(1^-, 1/2) \) | \( V^{(11)}_{\bar{K}K} \) | 4.59 | 4.97 |

The energy counting for the lowest and other \((J^\pi, T)\) states is summarized in Table I. The factor for each interaction term is given by the product of the number of pairs and the spin-isospin component per pair. The strong \( KN \) interaction in the \( T = 0 \) channel generally induces isoscalar \( KN \) correlation. On the other hand, for \( NN \) pairs, an isoscalar \( NN \) correlation is favored because the triplet-even \((ST) = (10)\) term is stronger than the singlet-even \((ST) = (01)\) term in the effective \( NN \) interaction. Furthermore, nuclear systems in the 0s-orbit favor spin and/or isospin saturation as in the \(^4\text{He}\) system because of the Pauli principle of nucleons. In kaonic nuclei, the isoscalar \( \bar{K}N \) and \( NN \) correlations compete against each other. In \( A = 2 \) kaonic nuclei, the \( \bar{K}NN(0^-, 1/2) \) and \( K\bar{K}NN(0^-, 0) \) states containing an isovector \((ST) = (01)\) \( NN \) pair are energetically favored over the \( \bar{K}NN(1^-, 1/2) \) and \( K\bar{K}NN(1^+, 1) \) states with a dueteron-like \((ST) = (10)\) \( NN \) pair, indicating that isoscalar \( KN \) correlation is superior to isoscalar \((ST) = (10)\) \( NN \) correlation. In kaonic nuclei with \( A \geq 3 \), the isospin saturation occurs in the nuclear pair; consequently, the isoscalar \( KN \) correlation gradually decreases with the increase of \( A \), as can be seen in the reduction of the \( T = 0 \) component of the \( KN \) pairs. The fraction of the \( T = 0 \) component is 1 in the \( \bar{K}N(1/2^-, 0) \) state, \( 3/4 \) in the \( K\bar{K}N(0^-, 1/2) \), \( \bar{K}KN(1/2^+, 1/2) \), and \( \bar{K}KN(0^+, 0) \) states, \( 1/3 \) in the \( \bar{K}NN(1/2^-, 0) \) state, and \( 1/4 \) in the \( K\bar{K}NN(0^-, 1/2) \) state.

B. Energy spectra of kaonic nuclei

The calculated energies obtained using set-I (weak-binding) and set-II (deep-binding) are listed in Tables II and III respectively. For kaonic nuclei, the total energies \((E)\), \( \bar{K} \)-separation energies \((S_{\bar{K}})\), and \( \Lambda^* \)-separation energies \((S_{\Lambda^*})\) are shown. For normal nuclei, the total energies, nucleon-separation energies \((S_N)\), and deuteron-separation energies \((S_d)\) are shown. Moreover, the contributions of the kinetic energy and \( NN \), \( KN \), and \( K\bar{K} \)-interaction terms are listed in the table.

In the lowest states, i.e., \( \bar{K}N(1/2^-, 0) \), \( K\bar{K}NN(0^-, 1/2) \), \( K\bar{K}NN(1/2^-, 0) \), \( K\bar{K}NN(0^-, 1/2) \), and \( K\bar{K}NN(0^+, 0) \), an antikaon and a \( \Lambda^* \) are deeply bound due to the remarkable contribution of the \( KN \) interaction with \( S_{\Lambda^*} \gtrsim 10 \) MeV in the set-I result and \( S_{\Lambda^*} \gtrsim 20 \) MeV in the set-II result. An exception is the \( K\bar{K}NN(1/2^+, 1/2) \), which is almost bound close to the \( K + \Lambda^* \)-threshold energy in the set-I result and is weakly bound with \( S_{\Lambda^*} \sim 7.0 \) MeV in the set-II result. The other \((J^\pi, T)\) states of the kaonic nuclei are...
TABLE II: The energies of the \((J^p, T)\) states of kaonic and normal nuclei calculated by the \(0s\)-orbital model with the set-I (weak-binding) parametrization. The total energy \(E = - \text{B.E.}\) and the contributions of kinetic \((T^{\text{kin}})\), \(NN (v_{NN})\), \(\bar{K}N (v_{\bar{K}N})\), and \(\bar{K} \bar{K} (v_{\bar{K}\bar{K}})\) interactions are listed. Separation energies \(S_{\bar{K}}\) and \(S_{\Lambda^*}\) for kaonic nuclei and \(S_N\) and \(S_d\) for normal nuclei are also shown. Energies are in units of MeV.

| set-I \((\epsilon_{\Lambda^*} = -10 \text{ MeV}; \nu_N = 0.16 \text{ fm}^{-2})\) | \(T^{\text{kin}}\) \(v_{NN}\) \(v_{\bar{K}N}\) \(v_{\bar{K}\bar{K}}\) | \(E\) | \(S_{\bar{K}}\) | \(S_{\Lambda^*}\) |
|---|---|---|---|---|
| \(KN(1/2^-, 0)\) | 10.0 | 0.0 | -20.0 | 0.0 | -10.0 | 10.0 | - |
| \(\bar{K}NN(0^-, 1/2)\) | 19.9 | -6.2 | -34.5 | 0.0 | -20.8 | (20.8) | 10.8 |
| \(\bar{K}NN(0^-, 3/2)\) | 19.9 | -6.2 | -18.2 | 0.0 | -4.5 | (4.5) | - |
| \(\bar{K}NN(1^-, 1/2)\) | 19.9 | -11.6 | -23.6 | 0.0 | -15.3 | 13.7 | 5.3 |
| \(\bar{K}\bar{K}N(1/2^+, 1/2)\) | 19.9 | 0.0 | -34.5 | 4.6 | -10.0 | -0.01 | -0.01 |
| \(\bar{K}\bar{K}N(0^+, 0)\) | 29.9 | -6.2 | -69.0 | 4.6 | -40.7 | 19.9 | 20.7 |
| \(\bar{K}\bar{K}N(0^+, 1)\) | 29.9 | -6.2 | -58.1 | 4.6 | -29.9 | 9.1 | - |
| \(\bar{K}\bar{K}N(0^+, 2)\) | 29.9 | -6.2 | -36.5 | 4.6 | -8.2 | 3.7 | - |
| \(\bar{K}\bar{K}N(1^+, 1)\) | 29.9 | -11.6 | -47.3 | 4.6 | -24.4 | 9.1 | - |
| \(\bar{K}NN(1/2^-, 0)\) | 29.9 | -26.7 | -43.6 | 0.0 | -40.4 | 33.6 | 38.8 |
| \(\bar{K}NNN(0^-, 1/2)\) | 39.8 | -53.3 | -47.3 | 0.0 | -60.8 | 37.3 | 54.0 |

| set-II \((\epsilon_{\Lambda^*} = -27 \text{ MeV}; \nu_N = 0.25 \text{ fm}^{-2})\) | \(T^{\text{kin}}\) \(v_{NN}\) \(v_{\bar{K}N}\) \(v_{\bar{K}\bar{K}}\) | \(E\) | \(S_{\bar{K}}\) | \(S_{\Lambda^*}\) |
|---|---|---|---|---|
| \(KN(1/2^-, 0)\) | 15.6 | 0.0 | -42.6 | 0.0 | -27.0 | 27.0 | - |
| \(\bar{K}NN(0^+, 1/2)\) | 31.1 | -8.8 | -70.1 | 0.0 | -47.8 | (47.8) | 20.8 |
| \(\bar{K}NN(0^-, 3/2)\) | 31.1 | -8.8 | -25.0 | 0.0 | -2.7 | (2.7) | - |
| \(KN(1^-, 1/2)\) | 31.1 | -16.3 | -40.0 | 0.0 | -25.3 | 24.5 | -1.7 |
| \(\bar{K}\bar{K}N(1/2^+, 1/2)\) | 31.1 | 0.0 | -70.1 | 5.0 | -34.0 | 7.0 | 7.0 |
| \(\bar{K}\bar{K}N(0^+, 0)\) | 46.7 | -8.8 | -140.2 | 5.0 | -97.3 | 49.6 | 43.3 |
| \(\bar{K}\bar{K}N(0^+, 1)\) | 46.7 | -8.8 | -110.1 | 5.0 | -67.3 | 19.5 | - |
| \(\bar{K}\bar{K}N(0^+, 2)\) | 46.7 | -8.8 | -50.1 | 5.0 | -7.2 | 4.5 | - |
| \(\bar{K}\bar{K}N(1^+, 1)\) | 46.7 | -16.3 | -80.1 | 5.0 | -44.8 | 19.5 | - |
| \(\bar{K}NNN(1/2^-, 0)\) | 46.7 | -37.7 | -82.6 | 0.0 | -73.6 | 67.0 | 72.8 |
| \(\bar{K}NNN(0^-, 1/2)\) | 62.2 | -75.3 | -80.1 | 0.0 | -93.2 | 64.5 | 86.7 |

TABLE III: Same as Table II but results are calculated with the set-II (deep-binding) parametrization.

| set-II \((\epsilon_{\Lambda^*} = -27 \text{ MeV}; \nu_N = 0.25 \text{ fm}^{-2})\) | \(T^{\text{kin}}\) \(v_{NN}\) \(v_{\bar{K}N}\) | \(E\) | \(S_{\bar{K}}\) | \(S_{\Lambda^*}\) | \(S_N\) | \(S_d\) |
|---|---|---|---|---|---|---|
| \(NN(0^+, 1)\) | 15.6 | -8.8 | 6.8 | - | - |
| \(NN(1^+, 0)\) | 15.6 | -16.3 | -0.8 | 0.8 | - |
| \(NNN(1/2^+, 1/2)\) | 31.1 | -37.7 | -6.6 | 5.8 | - |
| \(NNN(0^+, 0)\) | 46.7 | -75.3 | -28.7 | 22.1 | 27.1 |

relatively unfavored because they have weaker isoscalar \(KN\) correlations than the lowest states.

I compare the present results with other theoretical results in Table IV. The binding energies of the lowest states are compared with the theoretical values from Refs. [17–19] of the few-body calculations using weak-
type and deep-type $\bar{K}N$ interactions. The energy spectra of the set-I result agrees reasonably well with the results of other calculations with weak-type chiral interactions, and the set-II result corresponds well with other theoretical results with deep-type $\bar{A}Y$ interactions. In Table IV I also compare the present results for rms distances $R_{NN}$, $R_{KN}$, and $R_{\bar{K}\bar{K}}$ for $NN$, $KN$, and $\bar{K}\bar{K}$ pairs in kaonic nuclei with other theoretical results. The rms distances are constant for a fixed $\nu_N$ value in the present 0s-orbital model, whereas they are dependent on the system in other calculations with few-body approaches that include dynamical effects. Nevertheless, the present calculations using set-I and set-II yield reasonable results for $R_{NN}$, $R_{KN}$, and $R_{\bar{K}\bar{K}}$ in kaonic nuclei that are comparable to other calculations of weak-type and deep-type interactions, respectively. Hence, the present choices of $\nu_N$ adopted for sets-I and II are reasonable for global descriptions of the system sizes of kaonic nuclei.

In Fig. 1 the energy spectra of kaonic nuclei are shown together with other theoretical results. Figure 1(a) shows the set-I (weak-binding) result in comparison with other theoretical results for weak-type chiral interactions from Refs. 12 15 19. Figure 1(b) shows the set-II (deep-binding) result compared with other calculations for the deep-type $\bar{A}Y$ interaction. In each group of weak- and deep-type calculations, the present calculation describes the energy systematics of other theoretical results. This means that the binding energies of kaonic nuclei are not very sensitive to the details of the $\bar{K}N$ interaction but essentially depend upon the energy of the $\bar{K}N$ bound state corresponding to $\Lambda^*$. Moreover, the leading part of the binding energies may be understood by simple energy counting in the present 0s-orbital model, in which the spin and isospin symmetries play essential roles in the binding mechanism of light-mass kaonic nuclei.

C. $\bar{K}NN$ system

In the $\bar{K}NN$ system, the $(J^\pi, T) = (0^+, 0)$ state is the lowest and has been investigated by many groups as a deeply bound $K^-pp$ system. For this state, I obtain a binding energy that approximately two times larger than the $\Lambda^*$-binding energy (see Table II and Fig. 1(a) for the set-I result, and Table III and Fig. 1(b) for the set-II result). This energy relation $E_{\bar{K}NN}^{(0^+,0)} \approx 2\epsilon_{\Lambda^*}$ is naively understood by the energies for two $\bar{K}N$ pairs in the $\bar{K}NN(0^-, 1/2)$ state. Quantitatively, it can be described by the present model with energy counting as

$$E_{\bar{K}NN}^{(0^+,0)} = 2\epsilon_{\Lambda^*} + \frac{1}{2}(V_{KN}^{T=1} - V_{KN}^{T=0}) + V_{NN}^{(01)},$$

meaning that the $NN$ attraction compensates for the energy loss by reducing the isoscalar $\bar{K}N$ correlation in $\bar{K}NN(0^-, 1/2)$. It is interesting to compare the energies of $\bar{K}NN(0^-, 1/2)$ and $\bar{K}\bar{K}N(1/2^+, 1/2)$. The former is deeply bound and the latter is weakly (or almost) bound, even though the two systems have the same degree of $\bar{K}N$ interaction. As shown in Eqs. (39) and (40), the energy difference is just the last term, $V_{NN}^{(01)}$ of the $NN$ attraction in $\bar{K}NN(0^-, 1/2)$ and the $V_{\bar{K}K}^{T=1}$ term of the $\bar{K}\bar{K}$ repulsion in $\bar{K}\bar{K}N(1/2^+, 1/2)$. This brings about a significant difference in the $\Lambda^*$-separation energies of two systems indicating the important role of the singlet-even $NN$ attraction in the binding mechanism of the $\bar{K}NN(0^-, 1/2)$ state.

The $\bar{K}NN(1^-, 1/2)$ state with a deuteron-like $(ST) = (10)$ $NN$ pair has a higher energy than the lowest $\bar{K}NN(0^-, 1/2)$ state having a $(ST) = (01)$ $NN$ pair, despite the triplet-even $NN$ interaction being stronger than the singlet-even $NN$ interaction. This is because there is no isoscalar $\bar{K}N$ correlation in $\bar{K}NN(1^-, 1/2)$ with a fraction 1/4 of the $T = 0$ component. In the set-I result, the $\bar{K}NN(1^-, 1/2)$ state is weakly bound with $S_{\Lambda^*} = 5.3$ MeV. This result is consistent with the prediction of $S_{\Lambda^*} = 9$ MeV for a coupled-channel Faddeev calculation 16. In the set-II result, I obtained $E_{\bar{K}NN}^{(1^-,1/2)} = -25.3$ MeV, which is slightly higher than the $\Lambda^*$-decay threshold at $-27.0$ MeV.

D. $\bar{K}KNN$ system

The $(J^\pi, T) = (0^+, 0)$ state is the lowest of the $\bar{K}KNN$ system, and is deeply bound because of the strong isoscalar $\bar{K}N$ correlation with a fraction of 3/4 for the $T = 0$ component. In both the set-I and II results, the energy of $\bar{K}KNN(0^+, 0)$ is approximately twice that of $\bar{K}NN(0^-, 1/2)$, meaning that the $K$-separation energy is almost constant between the two systems. This energy relation,

$$E_{\bar{K}KNN}^{(0^+,0)} \approx 2E_{\bar{K}NN}^{(0^-,1/2)},$$

is also roughly satisfied in other theoretical results for Refs. 17 18. In the present 0s-orbital model, it is easy to see the energy relation from Eqs. (39) and (41) as

$$E_{\bar{K}KNN}^{(0^+,0)} = 2E_{\bar{K}NN}^{(0^-,1/2)} - \frac{1}{2}\epsilon_{nn} + V_{\bar{K}K}^{T=1},$$

where the last two terms yield minor contributions as $-\frac{1}{2}\epsilon_{nn} = -1.9$ MeV and $V_{\bar{K}K}^{T=1} = 4.6$ MeV for the set-I case and $-\frac{3}{4}\epsilon_{nn} = -3.4$ MeV and $V_{\bar{K}K}^{T=1} = 5.0$ MeV for the set-II case and cancel each other.

Let me discuss the binding mechanism of a singlet-even $NN$ pair in the $\bar{K}NN$ and $\bar{K}KNN$ systems in a Born-Oppenheimer picture of light-mass antikaons around heavy-mass nucleons. Two nucleons in the singlet-even channel are unbound without antikaons, but they are deeply bound by a surrounding antikaon in the $\bar{K}NN$ system and further deeply bound by two antikaons in the $\bar{K}KNN$ system. The mechanism for binding the two nucleons by an antikaon in the $\bar{K}^-pp$ system was originally interpreted as a super-strong nuclear force caused by a migrating $\bar{K}$ meson by Yamazaki and Akaishi 3 44. In
the perturbative picture, the constant $S_R$ in the $\bar{K}NN$ and $\bar{K}\bar{K}NN$ systems can be described by the condensation of two antikaons in the same orbit around two nucleons. if the $\bar{K}K$ interaction is minor. It should be noted that, when three antikaons around two nucleons are considered, the additional(third) antikaon no longer exhibits isoscalar $\bar{K}N$ correlation because the isospin is already saturated in the $\bar{K}\bar{K}NN$ system.

The higher states $\bar{K}\bar{K}NN(0^+, 1)$, $\bar{K}\bar{K}NN(0^+, 2)$, and $\bar{K}\bar{K}NN(1^+, 1)$, exhibit weaker isoscalar $\bar{K}N$ correlations because these states have lower symmetry in the isospin coupling between antikaons and nucleons than $\bar{K}NN(0^+, 0)$ does. In particular, the $\bar{K}\bar{K}NN(0^+, 1)$ and $\bar{K}\bar{K}NN(0^+, 2)$ states are composed of isovector $NN$ and $\bar{K}N$ pairs coupled to $T = 1$ and $T = 2$, respectively, and the $\bar{K}\bar{K}NN(1^+, 1)$ state contains an isoscalar $NN$ pair.

Comparing $\bar{K}\bar{K}NN(0^+, 1)$ and $\bar{K}\bar{K}NN(1^+, 1)$, one can see the competition between the isoscalar $\bar{K}N$ and $NN$ correlations. $\bar{K}\bar{K}NN(0^+, 1)$ has a moderate isoscalar $\bar{K}N$ component with a fraction of 1/2 but no isoscalar $NN$ component, whereas the $\bar{K}\bar{K}NN(1^+, 1)$ state contains a pure isoscalar $NN$ component but no isoscalar $\bar{K}N$ correlation with a fraction 1/4 of the $T = 0$ component. For these two states, $S_R$ is constant at 9.1 MeV for the set-I case and 19.5 MeV for the set-II case. The energy difference between $\bar{K}\bar{K}NN(0^+, 1)$ and $\bar{K}\bar{K}NN(1^+, 1)$ is the same value as the energy difference between $\bar{K}NN(0^+, 1/2)$ and $\bar{K}NN(1^-, 1/2)$, i.e., approximately 5 MeV in the set-I result and $\approx 20$ MeV in the set-II result.

In future experimental searches for double-kaonic nuclei, the $\bar{K}\bar{K}NN$ states might be observed as the quasibound resonances in the invariant mass spectra of such modes as the $\Lambda\Lambda$, $\Lambda\Sigma^{\pm}\pi^\mp$, and $\Xi^-\pi$ decays. The $\Lambda\Lambda$ mode for the $T = 0$ spectrum shows the $\bar{K}\bar{K}NN(0^+, 0)$

| $\nu_N$ (fm$^{-2}$) | $\bar{K}N(1/2^-, 0)$ | $\bar{K}NN(0^-, 1/2)$ | $\bar{K}\bar{K}NN; (0^+, 0)$ | $\bar{K}NNN(1/2^-, 0)$ | $\bar{K}NNN(0^-, 1/2)$ | $NN(1^+, 0)$ | $NNN(1/2^+, 1/2)$ | $NNN(0^+, 0)$ |
|---------------------|----------------------|-----------------------|-------------------------|------------------------|-----------------------|-------------|----------------|-------------|
| Present             | 0.16                 | 10                    | 20.8                    | 40.7                   | 40.4                  | 1.6         | 6.7             | 23.5        |
| Maeda [18]          | 0.25                 | 27                    | 47.8                    | 97.3                   | 73.6                  | 0.8         | 6.6             | 28.7        |
| Ohnishi [19]        |                      |                       | 23.8                    | 43                     | 42                    |             |                 |             |
| Barnea [17]         |                      |                       | 19                      | 93                     | 42                    | 1.6         |                 |             |
| Chiral              |                      |                       | 1.69                    | 193                    | 1.8                   | 22          |                 |             |
| Chiral              |                      |                       | 2.25                    | 1.41                   | 2.17                  | 2.50        |                 | 2.31        |
| Chiral              |                      |                       | 1.25                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.41                    | 1.41                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.69                    | 1.89                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.41                    | 1.9                    | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.89                    | 1.75                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.9                     | 1.8                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.75                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
|                     |                      |                       | 1.73                    | 1.73                   | 1.77                  | 1.77        |                 | 1.77        |
E. Antiknock binding in single-kaonic nuclei

In Fig. 2(a), I plot the $\bar{K}$-separation energies ($S_{\bar{K}}$) for the lowest states of single-kaonic nuclei calculated with set-I and II. For comparison, I also show theoretical results from Refs. [18, 19]. In each group of weak- and deep-type calculations, the $A$ dependence of $S_{\bar{K}}$ exhibits a similar trend, in that $S_{\bar{K}}$ increases gradually up to $A = 3$ and becomes saturated at $A = 4$ because the isoscalar $\bar{K}NN$ correlation vanishes in the nuclear-isospin-saturated system. In Fig. 2(b), I compare the separation energies, $S_{\bar{K}}$ for single-kaonic nuclei and $S_{\Lambda}$ for normal nuclei of set-I, which are plotted as functions of $A_{\text{tot}} - 1$. In the $A = 2$ and 3 systems, a nucleon in a normal nucleus is rather weakly bound because of the relatively weak $NN$ interaction compared with an antikaon that is deeply bound by the strong $\bar{K}N$ attraction in a kaonic nucleus. However, at $A = 4$ for $^{4}\text{He}$, $S_{\Lambda}$ increases drastically. This is in contrast with the gradual change of $S_{\bar{K}}$ with the increase of $A$ in kaonic nuclei.

IV. RESULTS OF $\bar{K}N + \bar{K}N$-CLUSTER MODEL

A. Effective $\Lambda^{*-}\Lambda^{*}$ interaction

To investigate the effective $\Lambda^{*-}\Lambda^{*}$ interaction, I apply the $\bar{K}N + \bar{K}N$-cluster model to the $\bar{K}NN$ system with total isospin $T = 0$. As described in Sec. II B, I assume $0s$-orbital configuration for each $\bar{K}N$ cluster and consider the two-cluster wave function with a distance $R$. In the cluster limit at a large distance $R$, each $\bar{K}N$ cluster forms an isoscalar $\bar{K}N$ bound state that corresponds to the $\Lambda^{*}$ state. In this asymptotic $\Lambda^{*} + \Lambda^{*}$ state, two $S^{\pi} = 0^{+}$ and $1^{-}$ channels are allowed because of Fermi statistics of $\Lambda^{*}$ particles, and their energies are degenerate at $R \to \infty$. 

FIG. 1: Energies $E = -\text{B.E.}$ of the kaonic nuclei. (a) Upper: the set-I (weak-binding) results are shown together with other theoretical results obtained by few-body approaches using weak-type chiral interactions by Doté et al. (ORB type-I case) [12], Barnea et al. (BGL case) [17], Maeda et al. (weak-chiral-regime case) [18], Ohnishi et al. (Kyoto type-I case) [19], Bayar et al. (normal-radius case) [15], and Oset et al. (normal-radius case) [16]. (b) Lower: the set-II (deep-binding) results are shown together with other theoretical results obtained by few-body approaches using weak-type chiral interactions by Doté et al. [12] and Ohnishi et al. [19].
The mixing of $\tau = 0$ and $\tau = 1$ components with respect to the isospin $\tau_K = \tau_N = \tau$ of $NN$ and $\bar{K}K$ pairs, which are coupled to total isospin $T = 0$, is taken into account in each $S^\pi$ channel.

The $KN + \bar{K}N$-cluster wave function with $\tau$ mixing can smoothly connect two limits; the shell model state at $R \to \infty$ and the asymptotic $\Lambda^*+\Lambda^*$ state at $R \to \infty$. As the two $\Lambda^*$-clusters approach each other, the isospin rearrangement occurs through the isospin exchange between two clusters via the $KN$ and $NN$ interactions.

Because of the Bose and Fermi statistics, there are selection rules in the spatial symmetry of the $\bar{K}K$ and $NN$ pairs as follows. In the $(S^\pi T) = (0^+0)$ channel, the $KN + \bar{K}N$-cluster system is described by a linear combination of two isospin components; the $\tau = 1$ component with spatial-even $NN$ and $\bar{K}K$ pairs, and the $\tau = 0$ component with spatial-odd $NN$ and $\bar{K}K$ pairs.

In the shell-model limit at $R \to \infty$, the former component goes to the $0s$-orbital $\bar{K}KNN(0^+,0)$ state, which is the lowest state in the $0s$-orbital model. The latter $\tau = 0$ component is forbidden in the $0s$-orbital model space and instead goes to a $(0s)^2(0p)^2$ configuration with an antikaon and a nucleon excited into $0p$-orbits. On the other hand, in the $(S^\pi T) = (1^-0)$ channel with negative parity, either one of $NN$ and $\bar{K}K$ pairs is a spatial-odd state. The $\tau = 0$ component contains a spatial-odd $KK$ pair, whereas the $\tau = 1$ component has a spatial-odd $NN$ pair. In the shell-model limit at $R \to 0$, they become excited $(0s)^3(0p)$ states. In the $\tau = 0$ component, there’s an antikaon excitation and in the $\tau = 1$ component, there’s a nucleon excitation.

Such selection rules for $\bar{K}K$ and $NN$ pairs play important roles in the effective $\Lambda^*\Lambda^*$ interaction, particularly at short distances. In the asymptotic $\Lambda^*+\Lambda^*$ state at a large $R$, each channel of $S^\pi = 0^+$ and $1^-$ contains $\tau = 1$ and $\tau = 0$ components with a ratio of 3:1. On the other hand, in the shell-model limit at $R \to \infty$, the $\tau$-mixing in the $S^\pi = 0^+$ channel is equivalent to the mixing of the $(0s)^4$ and $(0s)^2(0p)^2$ configurations, and that in the $S^\pi = 1^-$ channel corresponds to the configuration mixing of the antikaon and nucleon excitations in the $(0s)^3(0p)$ configuration.

I calculate the energy of the $KN + \bar{K}N$-cluster state with and without $\tau$-mixing at each distance $R$. In Fig. 3 I show the $R$ dependence of the total energy of the $KN + \bar{K}N$ state in the $S^\pi = 0^+$ and $1^-$ channels. Note that the asymptotic $\Lambda^*+\Lambda^*$ state at a large distance $R$ contains an additional energy cost $T^\text{kin}_0$ for localization of the relative motion, and the energy calculated with $\tau$-mixing equals $2\epsilon_{\Lambda^*} + T^\text{kin}_0$ at sufficiently large $R$. In both the $S^\pi = 0^+$ and $1^-$ channels, I obtain the energy minimum at $R \to 0$ in the energy curve with $\tau$-mixing, which indicates an attractive $\Lambda^*\Lambda^*$ interaction. The attraction of the $\Lambda^*$-$\Lambda^*$ interaction in the $S^\pi = 0^+$ channel is strong enough to form a deeply bound $\bar{K}KNN(0^+,0)$ state in the shell-model limit, whereas that in the $S^\pi = 1^-$ channel is weaker.

In Table 1 I list values at $R \to 0$ for energy contributions and probability $P(\tau = 1)$ for the $\tau = 1$ component. I also show the $\bar{K}$- and $\Lambda^*$-separation energies of the $\bar{K}KNN$ states, which are evaluated by the energy in the shell-model limit, as measured from the corresponding decay-threshold energies. For the $S^\pi = 0^+$ state in the shell-model limit, the $\tau = 1$ configuration is dominant and the $\tau = 0$ mixing effect is negligibly small, meaning that the $0s$-orbital model used in the present work well approximates the deeply bound $\Lambda^*+\Lambda^*$ state in the $S^\pi = 0^+$ channel.

For the $S^\pi = 1^-$ channel, I obtain a value of $\delta_{\Lambda^*} = 2.1$ MeV in the set-II result under $\tau$-mixing (see Table 1), indicating that the $\Lambda^*\Lambda^*$ attraction forms a (quasi) bound $S^\pi = 1^-$ state at a slightly lower energy than the two-cluster-threshold energy, $2\epsilon_{\Lambda^*}$. In the set-I result with $\tau$-mixing, a negative value $\delta_{\Lambda^*} = -2.8$ MeV.
is obtained for the $\Lambda^*$-separation energy. This suggests that the $\Lambda^*$-$\Lambda^*$ attraction in the $S^\pi = 1^-$ channel is insufficient to form a bound state, but may produce a $L^\pi = 1^-$ resonance near the $2\Lambda^*$-threshold energy. The weaker $\Lambda^*$-$\Lambda^*$ attraction in the $S^\pi = 1^-$ channel than in the $S^\pi = 0^+$ channel is described by a much weaker $KK$ attraction and a somewhat weaker $NN$ attraction as well as a larger kinetic-energy loss for the $0p$-orbit excitation.

Let me discuss the $S^\pi = 1^-$ state in more detail. Comparing the energies of the $\tau = 1$ and $\tau = 0$ configurations without $\tau$-mixing, the $\tau = 1$ component is favored at all $R$ because of the stronger isoscalar $KN$ correlation than the $\tau = 0$ component. In the result with $\tau$-mixing, the $\tau = 1$ configuration dominates the $S^\pi = 1^-$ state in the shell-model limit with probability $P(\tau = 1) = 0.84$ for set-I and $P(\tau = 1) = 0.92$ for set-II. In Table [VI], I show the energy contributions of the $KN + KN$ state at $R \to 0$, measured from twice of the internal-energy contributions of the $\Lambda^*$ cluster. From the table, one can see that the $\tau = 1$ component with a single-nucleon excitation gains energy in the $KN$ attraction but loses energy due to $KK$ repulsion, whereas the $\tau = 0$ component with a single-antikaon excitation gains energy through $NN$ attraction but somewhat loses energy through the $KN$ attraction.

At the same time, it is observed in the shell-model limit of $KN + KN$ with $S^\pi = 1^-$, the $\tau = 1$ and $\tau = 0$ configurations can compete and the mixing ratio depends upon details of the interactions. In the present calculation, the nucleon excitation in the $\tau = 1$ component is favored over the antikaon excitation in the $\tau = 0$ component; this can be explained by the nucleon feeling a broader $KN$ mean-field, allowing it to more easily excite into the $0p$-orbit than an antikaon, because light-mass antikaons have broader density distributions than nucleons in the present model. The $\tau$-mixing ratio may change if the antikaon mass is heavier than the physical antikaon mass. For example, if I assume equal kaon and nucleon masses $m_K = m_N$ and keep the other parameters unchanged, I obtain a lower energy for the $\tau = 0$ component with an antikaon excitation than for the $\tau = 1$ component with a nucleon excitation in the small-$R$ region, resulting in significant $\tau$-mixing in the $KN + KN$ state with $S^\pi = 1^-$. 

### B. Comparison of $\Lambda^*$-$\Lambda^*$ and $d$-$d$ Interactions

The $\Lambda^*$-$\Lambda^*$ interaction previously discussed is regarded as an effective dimer-dimer interaction in the kaonic nuclei. I here compare its properties with those of the $d$-$d$ interaction in nuclear systems. A deuteron is a weakly bound $(ST) = (10)$ $NN$ state. I describe the $NN + NN$ system in the $S^\pi = 0^+, 1^-$, and $2^+$ channels with a $d$-$d$ cluster model called the Brink-Bloch model [33] as done in Ref. [35]. The detailed properties of the effective $d$-$d$ interaction have been investigated in the previous paper [35]. In the present paper, I show the energy of the $NN + NN$ system at $R \to 0$ for the set-I parametrization, and discuss the roles of kinetic- and potential-energy contributions in the effective dimer-dimer interactions of the two systems.

I can classify the $S^\pi$ states of two systems based on the number of spatial-odd $KK$ and $NN$ pairs. Because of the nucleon Fermi statistics, the $S^\pi = 0^+$, $S^\pi = 1^-$, and $S^\pi = 2^+$ states of the $NN + NN$ system contain zero, one, and two spatial-odd $NN$ pairs, respectively. Similarly, in the $KN + KN$ system, the $\tau = 1$ and $\tau = 0$ components of the $S^\pi = 0^+$ state contain zero and two spatial-odd pairs, respectively, while the $S^\pi = 1^-$ state has one spatial-odd pair. In the shell-model limit, these states having no, one, and two spatial-odd pairs correspond to the $(0s)^4$, $(0s)^3(0p)$, $(0s)^2(0p)^2$ configurations, which have kinetic energy of $3T^\text{kin}_0$, $(3 + \frac{2}{3})T^\text{kin}_0$, and $(3 + \frac{4}{3})T^\text{kin}_0$, respectively.

In Fig. [IV] and Table [VI], I show the results of the $KN + KN$ and $NN + NN$ systems in the shell-model limit. The energy contributions measured from twice of the internal energies of a single cluster are shown. From the energy spectra of Fig. [IV], the two clusters in the lowest $S^\pi = 0^+$ channel for the $(0s)^4$ configuration are deeply bound in both systems. The binding energy for the two $\Lambda^*$ clusters from the threshold is approximately 20 MeV, coinciding with that for two deuteron clusters. However, the detailed contributions of the four pairs between the two clusters differ. According to the energy counting in the present model, the $\Lambda^*$-$\Lambda^*$ and $d$-$d$ binding energies are given as

$$\Delta E_{KKNN}^{(0^+,0)} = E_{KKNN}^{(0^+,0)} - 2\epsilon_{\Lambda^*},$$
$$\Delta E_{NNNN}^{(0^+,0)} = E_{NNNN}^{(0^+,0)} - 2\epsilon_d,$$

where

$$\Delta E_{KKNN}^{(0^+,0)} = T^\text{kin}_0 + V_{NN}^{(10)} + V_{KN}^{T=0} + V_{KK}^{T=1}.$$

In the $NNNN$ system, the potential energy contribution is always attractive for all four $NN$ pairs between the two clusters, while in the $KKNN$ system, the strong attraction in the isoscalar $KN$ pair compensates for the repulsion in the isovector $KK$ pair.

In the $S^\pi = 1^-$ channel of the $NNNN$ and $KKNN$ systems at $R \to 0$, two clusters gain some amount of potential energy but lose kinetic energy for one $0p$-orbit excitation in the $(0s)^3(0p)$ configuration. In the $KKNN$ system, the $\Lambda^*$-$\Lambda^*$ interaction in the $S^\pi = 1^-$ channel is the weak attraction and almost forms a bound state at an energy close to the $2\Lambda^*$ threshold. To gain the $KN$ attraction efficiently from the $\Lambda^* + \Lambda^*$ state to the shell-model-limit state, isospin rearrangement plays an essential role. This is a unique characteristic of kaonic nuclei, but cannot be seen in the $NN + NN$ system because such isospin rearrangement is not allowed in the $(S^\pi T) = (1^- 0)$ state. Hence, there is no attraction of the $d$-$d$ interaction in the $S^\pi = 1^-$ channel.

The $NNNN(2^+0)$ state and the $\tau = 0$ component of the $KKNN(0^+0)$ state correspond to the $(0s)^4(0p)^2$
configuration and have much higher energy than the two-cluster threshold. A comparison of the two systems shows that the kinetic-energy loss is the same, but the total energy differs significantly because of the difference in the potential-energy contributions (see Fig. 4). As shown in Table VI, the $NNN(2^+0)$ state gains potential energy because of the attractive $NN$ interaction, whereas the $\tau = 0$ component of the $KKNN(0^+0)$ state containing isoscalar $KK$ and $NN$ pairs loses the potential energy of the $KN$ and $KK$ interactions.

V. SUMMARY

I investigated the energy systematics of single- and double-kaonic nuclei in the mass number $A \leq 4$ region with the 0s-orbital model using zero-range $KN$ and $KK$ interactions. The $KN$ interaction was tuned to fit the $\Lambda(1405)$ mass with the energy of the $KN$ bound state. For the $NN$ interaction, I adopted the Volkov finite-range central interaction with a tuned parametrization adjusted to reproduce the $S$-wave $NN$-scattering lengths. I calculated the energy spectra of the $\bar{K}NN$, $\bar{K}NNN$, $KKNN$, $KKN$, and $KKNN$ systems in the cases of weak- and deep-binding and compared the results with other theoretical calculations with weak-type chiral and deep-type $\Lambda Y$ interactions. The present results qualitatively reproduce the energy systematics of kaonic nuclei calculated via other theoretical approaches. In the present 0s-orbital model, the energy spectra of kaonic nuclei were given by simple energy counting of isospin components of $NN$, $KN$, $KK$ pairs. The approximate energy relations for the lowest states of the $\bar{K}N$, $\bar{K}NN$, and $\bar{K}NNN$ systems were obtained as $E_{\bar{K}N}^{(0^-,1/2)} \approx 2\epsilon_{\Lambda^*}$ and $E_{\bar{K}NN}^{(0^+,0)} \approx 2E_{\bar{K}NN}^{(0^-,1/2)}$, which are universal features that are independent of the $\Lambda(1405)$ mass.

For the $\bar{K}NN$ and $\bar{K}NNN$ systems, I discussed the important roles of the isospin symmetry in the energy spectra of the $(J^\pi,T)$ states. In addition to the lowest $\bar{K}NN(0^-,1/2)$ and $\bar{K}NNN(0^+,0)$ states containing the isovector $(ST) = (01)$ $NN$ pair, I also obtain the $\bar{K}NNN(0^+,1)$, $\bar{K}NN(1^-,1/2)$, and $\bar{K}NNN(1^+,1)$ states. The latter two have isoscalar $(ST) = (10)$ $NN$ pairs like deuterons. The predicted $K$-separation energies for these states are $S_K=9-25$ MeV. In future experimental searches for $\bar{K}NNN$ states, the $\bar{K}NNN(0^+,1)$ and $\bar{K}NNN(1^+,1)$ states may contribute to the $T = 1$ components of invariant mass spectra.

I also investigated the effective $\Lambda^*-\Lambda^*$ interaction with the $\bar{K}N + KN$-cluster model and obtained a strong attraction in the $S^\pi = 0^+$ channel and a weak attraction in the $S^\pi = 1^-$ channel. In comparing the $\Lambda^*-\Lambda^*$ interaction in the $\bar{K}KN$ system with the $d$-$d$ interaction in the $NNNN$ system, I discussed the properties of dimer-dimer interactions in hadron and nuclear systems.

In the present calculation, zero-range real potentials were used for the $\bar{K}N$ and $KK$ interactions. Moreover, kaonic nuclei were simply described using the 0s-orbital and cluster models. Despite such simple theoretical treatments of interactions and wave functions, the present results succeeded in globally describing the energy systematics of kaonic nuclei obtained with precise few-body calculations. The energy-counting rule in the present model is useful for understanding the leading properties of energy spectra in kaonic nuclei; it also enables one to ex-

![FIG. 3: Energies of the $\bar{K}N + KN$-cluster system with inter-cluster distances $R$ for (a) the $S^\pi = 0^+$ and (b) $S^\pi = 1^-$ states of the set-I result. (c) Those for the $S^\pi = 1^-$ state of the set-II result. The energies calculated with and without $\tau$-mixing are shown. Arrows show the $\Lambda^* + \Lambda^*$ threshold energy, which is $T_0$ below the asymptotic energy at $R \to \infty$ obtained with $\tau$-mixing.](image-url)
The set-I and II results are shown in the upper and lower parts, respectively.

**TABLE V**: Energies of the $\bar{K}N + \bar{K}N(S^*T)$ states in the shell-model ($R \rightarrow 0$) limit, as calculated with and without $\tau$-mixing. The set-I and II results are shown in the upper and lower parts, respectively.

| $\tau$-mixing | $\tau = 1$ | $\tau = 0$ | $\bar{K}N + \bar{K}N$ $(S^*T) = (0^+0)$ |
|----------------|-----------|-----------|----------------------------------|
| $E$            | -41.1     | 20.3      | 21.1                             |
| $S_K$          | 0.99      | 1         |                                 |
| $S_{A^*}$      |           |           |                                  |
| $P(\tau = 1)$  |           |           |                                  |

**TABLE VI**: Energies of the $\bar{K}N + \bar{K}N(S^*T)$ and $NN + NN(S^*T)$ states in the shell-model limit, as measured from the two-cluster threshold energies. The set-I result of the total energy ($\Delta E$), kinetic ($\Delta T^{\text{kin}}$), $NN$ ($\Delta v_{NN}$), $\bar{K}N$ ($\Delta v_{\bar{K}N}$), and $\bar{K}\bar{K}$ ($\Delta v_{\bar{K}\bar{K}}$) interaction-energy contributions measured from twice of the internal energies of a single cluster are listed. For the $\bar{K}N + \bar{K}N$ states, the results obtained with and without $\tau$-mixing are shown. All energies are in units of MeV.

| $\tau$-mixing | $\tau = 1$ | $\tau = 0$ | $\bar{K}N + \bar{K}N$ $(S^*T) = (1^−0)$ |
|----------------|-----------|-----------|----------------------------------|
| $E$            | -56.1     | 30.9      | 2.1                              |
| $S_K$          | 0.92      | 0         |                                 |
| $S_{A^*}$      |           |           |                                  |
| $P(\tau = 1)$  |           |           |                                  |

$KNN$ configuration

| $\Delta T^{\text{kin}}$ | $\Delta v_{NN}$ | $\Delta v_{\bar{K}N}$ | $\Delta v_{\bar{K}\bar{K}}$ | $\Delta E$ |
|-------------------------|-----------------|------------------------|-----------------------------|-----------|
| $\bar{K}N + \bar{K}N$ $(0^+0)$ |                  |                        |                              |           |
| $\tau = 1$              | 10.0            | -6.2                   | -29.6                       | 4.6       |
| $\tau = 0$              | 0               | 0.7                    | 8.2                         | 0.0       |
| $\bar{K}N + \bar{K}N$ $(1^-0)$ |                  |                        |                              |           |
| $\tau = 1$              | 16.6            | -1.2                   | -16.4                       | 3.9       |
| $\tau = 0$              | 16.6            | 0.7                    | -17.2                       | 4.6       |

$NN + NN$ configuration

| $\Delta T^{\text{kin}}$ | $\Delta v_{NN}$ | $\Delta E$ |
|-------------------------|-----------------|-----------|
| $NN + NN$ $(0^+0)$      |                  |           |
| $\tau = 1$              | 10.0            | -30.2     |
| $NN + NN$ $(1^-0)$      |                  |           |
| $\tau = 1$              | 16.6            | -8.3      |
| $NN + NN$ $(2^+0)$      |                  |           |
| $\tau = 0$              | 23.2            | -5.4      |

tract universal features independently from the details of the hadron-hadron interactions. For precise predictions
FIG. 4: Energy spectra of the $K\bar{K}NN$ and $NNNN$ systems calculated with the $KN+KN$ and $NN+NN$-cluster models in the shell-model limit. The energies are measured from the two-cluster threshold energies, $2\epsilon_{\Delta}$ for the $K\bar{K}NN$ system and $2\epsilon_{d}$ for the $NNNN$ system. For the $K\bar{K}NN$ system, the energies without $\tau$-mixing and the $S^+ = 1^-$ energy with $\tau$-mixing are shown. Energy levels above the threshold are not shown. The configurations are plotted.

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