Online Actuator Selection and Controller Design for Linear Quadratic Regulation over a Finite Horizon

Lintao Ye* Ming Chi* Zhi-Wei Liu* Vijay Gupta†

January 26, 2022

Abstract

We study the simultaneous actuator selection and controller design problem for linear quadratic regulation over a finite horizon, when the system matrices are unknown a priori. We propose an online actuator selection algorithm to solve the problem which specifies both a set of actuators to be utilized and the control policy corresponding to the set of selected actuators. Specifically, our algorithm is a model based learning algorithm which maintains an estimate of the system matrices using the system trajectories. The algorithm then leverages an algorithm for the multiarmed bandit problem to determine the set of actuators under an actuator selection budget constraint and also identifies the corresponding control policy that minimizes a quadratic cost based on the estimated system matrices. We show that the proposed online actuator selection algorithm yields a sublinear regret.

1 Introduction

In large-scale control system design, the number of actuators (or sensors) that can be installed is often limited by budget or complexity constraints. The problem of selecting a subset of all the candidate actuators (or sensors), in order to optimize a system objective while satisfying a budget constraint is a classic problem referred to as actuator (or sensor) selection (e.g., [36, 19, 28, 31, 37, 29, 35, 38]). However, most of the existing work on this problem assumes the knowledge of the system model when designing the actuator (or sensor) selection algorithms. In this work, we are interested in the situation when the system model is not known a priori (e.g., [22]). In such a case, the existing algorithms for the actuator selection problem and the corresponding analysis tools do not apply.

Specifically, we study the simultaneous actuator selection and controller design problem for a finite-horizon Linear Quadratic Regulation (LQR) setting (e.g., [2]). The goal is to select a subset of actuators under a cardinality constraint, while minimizing the quadratic cost function over the finite horizon. We assume that the system model is not known a priori to the designer. Since the system model is not known, an online setting of the problem is natural, where we aim to solve the actuator selection and controller design problem over multiple rounds and there is a cardinality constraint on the set of selected actuators in each round of the problem. After the completion of each round, the LQR cost incurred by the selected actuators and the designed controller for that round is revealed. This setting corresponds to the episodic setting in reinforcement learning (e.g., [32, 24]). In order to solve the problem, we provide an online algorithm and characterize the regret of the algorithm. The notion of regret is a typical metric to characterize the performance of online optimization algorithms in an unknown environment (e.g., [8, 20]).

A major challenge in our problem is that in order to obtain a solution to the problem, the online actuator selection algorithm needs to specify both the set of selected actuators and the corresponding control policy. Since the system matrices are not known, it is not possible to directly use the well-known formula for the optimal LQR control policy (e.g., [6]), given a set of selected actuators. In order to tackle this challenge, the online actuator selection algorithm that we propose and the corresponding analysis combines ideas from

*School of Artificial Intelligence and Automation, Huazhong University of Science and Technology, Wuhan, China; {yelintao93, chiming, zwliu}@hust.edu.cn.
†Department of Electrical Engineering, the University of Notre Dame, IN, USA; vgupta2@nd.edu.
model based learning algorithms for LQR (e.g., [12, 9]) and online algorithms for the multiarmed bandit problem (e.g., [5]). Specifically, during certain rounds of the actuator selection (and controller design) problem, the proposed online algorithm focuses on estimating the system matrices. In other rounds of the problem, the online algorithm first uses an algorithm for the multiarmed bandit problem as a subroutine to select the set of actuators and then determines the corresponding control policy based on the estimated system matrices. By finding an appropriate frequency of these rounds, we show that the online actuator selection algorithm achieves a regret of $\tilde{O}(\sqrt{T})$, where $T$ is the number of rounds in the actuator selection problem and $\tilde{O}(\cdot)$ hides logarithmic factors in $T$.

Related Work

Actuator selection and sensor selection in control systems have been studied in the literature extensively. Due to its combinatorial nature, the problem has been shown to be NP-hard in general (e.g., [38]). Therefore, much work in the literature provides approximation algorithms to solve the problem with performance guarantees (e.g., [28, 34, 18]), often leveraging results from combinatorial optimization (e.g., [27, 7]). However, most of the previous work assumes the system matrices to be known, which is a departure from our work. Exceptions are [16, 30], where the authors studied an online sensor selection problem, however, for the estimation of a static random variable. The goal there is to minimize the estimation error of the static random variable using sensor measurements, where the objective function associated with the estimation error is not specified a priori. The authors considered a similar episodic setting to the one that we described above and proposed an online sensor selection algorithm with regret analysis. Another related work is [15], where the authors considered an unknown continuous-time linear time-invariant system without stochastic input and studied the problem of selecting a subset of actuators under a cardinality constraint such that a metric associated with the controllability of the system is optimized. The authors proposed an online actuator selection algorithm and showed that the algorithm will select the optimal set of actuators after a finite number of time steps.

The LQR problem with unknown system matrices (without the actuator or selection selection component) has been widely studied recently as a benchmark for reinforcement learning (e.g., [1, 11, 14, 33, 17, 26]). The setting in this direction closest to ours is the so called model based learning, where the algorithms estimate the system matrices using the system trajectories and design the control policy based on the estimated system matrices. This way of designing the control policy is also known as the certainty equivalence approach (e.g., [4]). Specifically, the authors in [10, 25] provided an online algorithm for the LQR problem with unknown system matrices and showed that the regret of the algorithm is $\tilde{O}(\sqrt{N})$, where $N$ is the number of time steps in the LQR problem and $\tilde{O}(\cdot)$ hides logarithmic factors in $N$. Note that the authors in [1, 11, 10, 25] considered the infinite horizon LQR setting. We extend the analyses and results in [25] to the finite horizon LQR setting when solving the problem considered in this paper.

Contributions

We formulate an online actuator selection and controller design problem for LQR over a finite horizon, when the system matrices are not known a priori. In order to solve the problem, we propose an online actuator selection algorithm which specifies both the set of selected actuators and the corresponding control policy. Specifically, the online actuator selection algorithm maintains estimates of the system matrices using the system trajectories. Moreover, the online actuator selection algorithm uses an algorithm for the multiarmed bandit problem as a subroutine to select the set of selected actuators and then determines the corresponding control policy based on the estimated system matrices. We analyze the regret of the proposed algorithm and show that the regret is $\tilde{O}(\sqrt{T})$ with high probability, where $T$ is the number of rounds in the problem and $\tilde{O}(\cdot)$ hides logarithmic factors in $T$. When analyzing the regret of our algorithm, we also extend the certainty equivalence approach proposed in [25] for learning LQR over an infinite horizon to the finite-horizon setting.

Notation and terminology

The sets of integers and real numbers are denoted as $\mathbb{Z}$ and $\mathbb{R}$, respectively. The set of integers (resp., real numbers) that are greater than or equal to $a \in \mathbb{R}$ is denoted as $\mathbb{Z}_{\geq a}$ (resp., $\mathbb{R}_{\geq a}$). For a real number $a$, let $[a]$ be the smallest integer that is greater than or equal to $a$. The space of $n$-dimensional real vectors is
Consider a discrete-time linear time-invariant system
\[ x_{k+1} = Ax_k + Bu_k + w_k, \]  
where \( A \in \mathbb{R}^{n \times n} \) is the system dynamics matrix, \( x_k \in \mathbb{R}^n \) is the state vector, \( B \in \mathbb{R}^{n \times m} \) is the input matrix, \( u_k \in \mathbb{R}^m \) is the input vector (i.e., control policy), and \( w_k \) is a zero-mean white Gaussian noise process with covariance \( W \) for all \( k \in \mathbb{Z}_{\geq 0} \).

### 2.1 Linear Quadratic Regulation

Given the system in Eq. (1) and a length \( N \in \mathbb{Z}_{\geq 1} \) of the time horizon, the goal of the LQR problem over a finite horizon is to find control policies \( u_0, u_1, \ldots, u_{N-1} \) that solve
\[
\min_{u_0, \ldots, u_{N-1}} \mathbb{E} \left[ \sum_{k=0}^{N-1} x_k^\top Q_k x_k + u_k^\top R_k u_k + x_N^\top Q_f x_N \right],
\]  
where the expectation is taken with respective to \( w_0, \ldots, w_{N-1} \). We assume that \( Q_k \in S_n^+ \) and \( R_k \in S_{m+}^+ \) for all \( k \in \{0, 1, \ldots, N-1\} \), and \( Q_f \in S_n^+ \). It is well-known that the optimal solution to Problem (2) has the following form (e.g., [6, Chapter 3]):
\[
\hat{u}_k = K_k x_k \quad \forall k \in \{0, 1, \ldots, N-1\},
\]  
where the gain matrix \( K_k \in \mathbb{R}^{m \times n} \) is given by
\[
K_k = -(B^\top P_{k+1} B + R_k)^{-1} B^\top P_{k+1} A,
\]  
and where \( P_k \in S_n^+ \) in Eq. (4) is given by the following recursion:
\[
P_k = A^\top P_{k+1} A - A^\top P_{k+1} B (B^\top P_{k+1} B + R_k)^{-1} B^\top P_{k+1} A + Q_k
\]  
initialized with \( P_N = Q_f \). Thus, when the system matrices are known, one can obtain the optimal solution to the LQR problem (i.e., Problem (2)) using Eqs. (3)-(5). Also note that the optimal control policy given by Eq. (3) is a linear state-feedback policy. Given an initial condition \( x_0 \), the corresponding minimum cost of Problem (2) can be obtained as
\[
J(x_0) = x_0^\top P_0 x_0 + \sum_{k=0}^{N-1} \text{Tr}(P_{k+1} W).
\]  

Suppose that the system matrices \( A \in \mathbb{R}^{n \times n} \) and \( B^{n \times m} \) are unknown to the system designer. As we described in Section 1, we consider a standard episodic setting in reinforcement learning (e.g., [32, 24]), corresponding to the LQR problem defined in Eq. (2), and we aim to solve the LQR problem for a number of \( T \in \mathbb{Z}_{\geq 1} \) rounds (i.e., episodes) in an online manner.
2.2 Online Actuator Selection and Controller Design

Let $\mathcal{G} \triangleq [q]$ be the set that contains all the candidate actuators. Denote $B = [B_1 \cdots B_q]$, where $B_i \in \mathbb{R}^{n \times m_i}$ for all $i \in \mathcal{G}$ with $\sum_{i \in \mathcal{G}} m_i = m$. For any $i \in [q]$, $B_i$ corresponds to a candidate actuator for the system given by Eq. (1) that can be potentially selected and installed. In each round $t \in [T]$ of the LQR problem defined in Eq. (2), we consider the scenario where only a subset of actuators out of all the candidate actuators is selected to provide inputs to the system given by Eq. (1), due to, e.g., budget constraints. After the completion of each round $t \in [T]$, the system is reset to an initial state.

Now, consider any round $t \in [T]$, and suppose that a set of actuators $\mathcal{S} \subseteq \mathcal{G}$ is selected in round $t$, where $\mathcal{S} = \{i_1, \ldots, i_H\}$ with $|\mathcal{S}| = H$, and where $H \in \mathbb{Z}_{\geq 1}$ is a cardinality constraint on the set of actuators that can be chosen in round $t$. We denote by $B_{\mathcal{S}} \triangleq [B_{i_1} \cdots B_{i_H}]$ the input matrix associated with the actuators in $\mathcal{S}$, and denote by $u_{k,i}^{(t)} = [u_{k,i_1}^{(t)} \cdots u_{k,i_H}^{(t)}]^{\top}$ the control policies of the actuators in $\mathcal{S}$ at time step $k \in \{0, 1, \ldots, N-1\}$ in round $t \in [T]$, where $u_{k,i} \in \mathbb{R}^{m_i}$ is the control input provided by actuator $i \in \mathcal{S}$. Based on the set $\mathcal{S} \subseteq \mathcal{G}$ of selected actuators, the system dynamics in round $t \in [T]$ can then be written as

$$x_{k+1}^{(t)} = Ax_k^{(t)} + Bu_{k,S}^{(t)} + w_k^{(t)},$$

for $k \in \{0, 1, \ldots, N-1\}$, where $x_k^{(t)}$ and $u_k^{(t)}$ are the state and noise at time step $k$ in round $t$, respectively. Similarly, we assume that $w_k^{(t)}$ is a zero-mean white Gaussian noise process with $\mathbb{E}[w_k^{(t)}(w_k^{(t)})^{\top}] = W$ for all $k \in \{0, 1, \ldots, N-1\}$. As we mentioned above, the system state is reset to a starting point after the completion of each round. For simplicity, we assume throughout this paper that $x_0^{(t)} = 0$ for all $t \in [T].$ \footnote{Note that our analysis can be extended to the case when $x_0^{(t)} \sim \mathcal{N}(0, \Sigma_0)$, as one may view $x_0^{(t)}$ as $u_0^{(t)}$ in the analysis.}

Next, for any round $t \in [T]$ and any $\mathcal{S} \subseteq \mathcal{G}$ (with $|\mathcal{S}| = H$), we denote $u_{\mathcal{S}}^{(t)} = (u_{0,\mathcal{S}}^{(t)}, u_{1,\mathcal{S}}^{(t)}, \ldots, u_{N-1,\mathcal{S}}^{(t)})$, and define the following (quadratic) cost (associated with round $t$ when the set of actuators $\mathcal{S}$ is chosen to provide $u_{\mathcal{S}}^{(t)}$):

$$J_t(\mathcal{S}, u_{\mathcal{S}}^{(t)}) = \left( \sum_{k=0}^{N-1} x_k^{(t)} \cdot Q^{(t)} x_k^{(t)} + u_k^{(t)} \cdot Q_{\mathcal{S}} u_k^{(t)} \right) + x_N^{(t)} \cdot Q_{f} x_N^{(t)},$$

where $Q^{(t)}, Q_{\mathcal{S}} \in \mathbb{S}_{++}^n$ and $R^{(t)} \in \mathbb{S}_{++}^{m_{\mathcal{S}}}$ are the cost matrices during round $t$, and $R_{\mathcal{S}}^{(t)} \in \mathbb{S}_{++}^{m_{\mathcal{S}}}$ (with $m_{\mathcal{S}} = \sum_{i \in \mathcal{S}} m_i$) is a submatrix of $R^{(t)}$ that corresponds to the set $\mathcal{S}$. \footnote{That is, the matrix $R_{\mathcal{S}}^{(t)}$ is obtained by deleting the rows and columns of $R^{(t)}$ indexed by the elements in the set $\mathcal{S} \setminus \mathcal{S}$.} Here, we focus on the scenario where the cost matrices $Q^{(t)}$ and $R^{(t)}$ are time-invariant within any round $t \in [T]$, but can be different across different rounds. Throughout this paper, we assume that $Q^{(t)}, Q_{\mathcal{S}}^{(t)}$ and $R^{(t)}$ are known for all $t \in [T]$. Note from Eq. (8) that the cost function of round $t \in [T]$, i.e., $J_t(\cdot)$, depends on both the actuators selected in round $t$ and the inputs provided by the selected actuators in round $t$.

2.3 Regret of Online Algorithm

We now aim to design an online algorithm such that at the beginning of each round $t \in [T]$, the algorithm decides a set of selected actuators $\mathcal{S}_t \subseteq \mathcal{G}$ (with $|\mathcal{S}_t| = H$) for round $t$ and a corresponding control policy $u_{\mathcal{S}_t}^{(t)} = (u_{0,\mathcal{S}_t}^{(t)}, u_{1,\mathcal{S}_t}^{(t)}, \ldots, u_{N-1,\mathcal{S}_t}^{(t)})$ provided by the actuators in $\mathcal{S}_t$. Letting $x^{(t)} = (x_0^{(t)}, x_1^{(t)}, \ldots, x_N^{(t)})$ for all $t \in [T]$, the decision of the online algorithm is made based on: (a) the system state trajectories $x^{(1)}, \ldots, x^{(t-1)}$, and (b) all the previous decisions made by the algorithm, i.e., $(\mathcal{S}_1, \ldots, \mathcal{S}_{t-1})$ and $(u_{\mathcal{S}_1}^{(1)}, \ldots, u_{\mathcal{S}_{t-1}}^{(t-1)}).$ \footnote{Since we have assumed that $Q^{(t)}, Q_{\mathcal{S}}^{(t)}$ and $R^{(t)}$ are known, we see from Eq. (8) that $J_t(\mathcal{S}_1, u_{\mathcal{S}_1}^{(1)}), \ldots, J_{t-1}(\mathcal{S}_{t-1}, u_{\mathcal{S}_{t-1}}^{(t-1)})$ are also available to the online algorithm.}

One can show, using the arguments in Section 2.1, that for a given set of selected actuators $\mathcal{S} \subseteq \mathcal{G}$ in round $t \in [T]$, the optimal control control policy that minimizes $\mathbb{E}[J_t(\mathcal{S}, u_{\mathcal{S}}^{(t)})]$ is given by

$$u_{k,S}^{(t)} = K_{k,S}^{(t)} x_k^{(t)} \quad \forall k \in \{0, 1, \ldots, N - 1\},$$

where $K_{k,S}^{(t)}$ is the Kalman filter gain for $k \in \{0, 1, \ldots, N - 1\}$.
where $K_{k,S}^{(t)} \in \mathbb{R}^{m_S \times n}$ is the gain matrix given by
\begin{equation}
K_{k,S}^{(t)} = -(B_S^T P_{k+1,S}^{(t)} B_S + R_S^{(t)})^{-1} B_S^T P_{k+1,S}^{(t)} A, \tag{10}
\end{equation}
and $P_{k,S}^{(t)} \in \mathbb{S}^n_{+}$ satisfies the following recursion:
\begin{equation}
P_{k,S}^{(t)} = Q^{(t)} + A^T P_{k+1,S}^{(t)} A - A^T P_{k+1,S}^{(t)} B_S (B_S^T P_{k+1,S}^{(t)} B_S + R_S^{(t)})^{-1} B_S^T P_{k+1,S}^{(t)} A, \tag{11}
\end{equation}
initialized with $P_{N,S}^{(t)} = Q_N^{(t)}$. Recalling the assumption that $x_0^{(t)} = 0$, we have from Eqs. (6) and (8) that
\begin{equation}
J_t(S) \triangleq \min_{u_{S,T}^{(t)}} \mathbb{E}[J_t(S, u_{S,T}^{(t)})] = \sum_{k=0}^{N-1} \text{Tr}(P_{k+1,S}^{(t)} W). \tag{12}
\end{equation}
Note that when the system matrices $A$ and $B$ are unknown, one cannot directly use Eq. (10) to obtain the optimal control policy $\tilde{u}_{k,S}^{(t)}$ for a given $S \subseteq \mathcal{G}$ in round $t \in [T]$.

Based on the above arguments, we now introduce a metric to characterize the performance of the online algorithm that we propose to solve the actuator selection and controller design problem defined in Section 2.2. Specifically, we use $A$ to denote a general online algorithm for the problem. In order to characterize the performance of algorithm $A$, we first define $J^*$ to be the optimal cost of the following optimization problem:
\begin{equation}
\min_{S_1,\ldots,S_T} \min_{u_{S_1}^{(t)},\ldots,u_{S_T}^{(t)}} \mathbb{E} \left[ \sum_{t=1}^{T} J_t(S_t, u_{S_t}^{(t)}) \right] \tag{13}
\end{equation}
s.t. $S_t \subseteq \mathcal{G}$, $|S_t| = H ~ \forall t \in [T]$,

where $J_t(S_t, u_{S_t}^{(t)})$ is defined in Eq. (8). Note that $J_*$ is the minimum expected accumulative costs that one will incur under the cardinality constraint on the set of actuators that can be chosen in each round. Using (12), we can rewrite (13) as
\begin{equation}
\min_{S_1,\ldots,S_T} \sum_{t=1}^{T} \sum_{k=0}^{N-1} \text{Tr}(P_{k+1,S_t}^{(t)} W) \tag{14}
\end{equation}
s.t. $S_t \subseteq \mathcal{G}$, $|S_t| = H ~ \forall t \in [T]$,

where $P_{k+1,S_t}^{(t)}$ is given by Eq. (11). We then define the following performance metric of Algorithm $A$:
\begin{equation}
R_A = \mathbb{E}_A \left[ \sum_{t=1}^{T} J_t(S_t, u_{S_t}^{(t)}) \right] - J^*, \tag{14}
\end{equation}
where $(S_1,\ldots,S_T)$ is the sequence of the sets of actuators chosen by Algorithm $A$, $(u_{S_1}^{(1)},\ldots,u_{S_T}^{(T)})$ is the corresponding sequence of control policies chosen by Algorithm $A$, and $\mathbb{E}_A[\cdot]$ denotes the expectation with respect to the potential randomness of algorithm $A$, which we shall discuss in detail later. Note from Eq. (14) that $R_A$ is obtained by comparing the actual accumulative cost incurred by algorithm $A$ after $T$ rounds against the minimum expected accumulative cost (e.g., $[1,10]$). Also note that $R_A$ defined in Eq. (14) is known as the regret of Algorithm $A$, and is a typical performance metric for online algorithms (e.g., $[8,1]$).

## 3 Controller Design Using Certainty Equivalence Approach

As we described in Section 2.3, an online algorithm for the actuator selection and controller design problem needs to decide a set of selected actuators $S_t \subseteq \mathcal{G}$ (with $|S_t| = H$) for any round $t \in [T]$ and a corresponding control policy $u_{S_t}^{(t)} = (u_{0,S_t}^{(t)},u_{1,S_t}^{(t)},\ldots,u_{N-1,S_t}^{(t)})$ provided by the actuators in $S_t$. In Section 3, we focus on the controller design part of the online algorithm that we will propose, i.e., how the online algorithm identifies
the control policy \( u^{(t)}_S = (u^{(t)}_{0,S}, u^{(t)}_{1,S}, \ldots, u^{(t)}_{N-1,S}) \) given that a set of actuators \( S \subseteq \mathcal{G} \) is selected. Later in Section 4, we will present the overall online algorithm.

For the controller design, we leverage the certainty equivalence approach, which has been studied for the LQR problem over an infinite horizon with unknown system model (e.g., \([11, 25]\)). Specifically, in the certainty equivalence approach, we design a control policy based on estimated system matrices, denoted as \( \hat{A} \) and \( \hat{B} \). Naturally, the performance of the resulting certainty equivalent controller depends on the estimation errors \( \| \hat{A} - A \| \) and \( \| \hat{B} - B \| \). In the following, we extend the analysis in \([25]\) for the certainty equivalence approach for LQR over an infinite horizon to the finite-horizon setting of interest to us.

Suppose that a set of actuators \( S \subseteq \mathcal{G} \) (with \(|S| = H\)) is chosen to provide control inputs in round \( t \in [T] \). Recall that the optimal control policy is given by \( \hat{u}^{(t)}_S = K^{(t)}_{k,S}x^{(t)}_k \), where \( K^{(t)}_{k,S} \) given in Eq. (10) depends on the system matrices \( A \) and \( B \). Since \( A \) and \( B \) are unknown a priori, we first leverage the system trajectories, including the state and input history, to obtain the estimated system matrices \( \hat{A} \) and \( \hat{B} \). The certainty equivalent controller for the finite-horizon LQR problem is then given by

\[
\hat{u}^{(t)}_k = \hat{K}^{(t)}_{k,S}x^{(t)}_k \quad \forall k \in \{0, 1, \ldots, N-1\},
\]

where

\[
\hat{K}^{(t)}_{k,S} = -(\hat{B}^T_{S}P^{(t)}_{k+1,S}\hat{B} + R^{(t)}_S)^{-1}\hat{B}^T_{S}P^{(t)}_{k+1,S}\hat{A},
\]

and \( \hat{P}^{(t)}_{k,S} \in \mathbb{S}^n_+ \) satisfies the following recursion:

\[
\hat{P}^{(t)}_{k,S} = Q^{(t)} + \hat{A}^T\hat{P}^{(t)}_{k+1,S}\hat{A} - \hat{A}^T\hat{P}^{(t)}_{k+1,S}\hat{B}(\hat{B}^T\hat{P}^{(t)}_{k+1,S}\hat{B} + R^{(t)}_S)^{-1}\hat{B}^T\hat{P}^{(t)}_{k+1,S}\hat{A},
\]

initialized with \( \hat{P}^{(0)}_{N,S} = Q^{(0)}_I \). In other words, we obtain \( \hat{K}^{(t)}_{k,S} \) based on the estimated system matrices \( \hat{A} \) and \( \hat{B} \), and apply the resulting control policy \( \hat{u}^{(t)}_S = \hat{K}^{(t)}_{k,S}x^{(t)}_k \) to the true system corresponding to the matrices \( A \) and \( B \). Moreover, we denote the cost associated with \( S \) and \( \hat{u}^{(t)}_S = \hat{K}^{(t)}_{k,S}x^{(t)}_k \) as

\[
\hat{J}_t(S) = E[J_t(S, \hat{u}^{(t)}_S)],
\]

where \( J_t(S, \hat{u}^{(t)}_S) \) is defined in Eq. (8). Using similar arguments to those in, e.g., \([6, \text{Chapter 3}]\), one can show that

\[
\hat{J}_t(S) = \sum_{k=0}^{N-1} \text{Tr}(\hat{P}^{(t)}_{k+1,S}W),
\]

where \( \hat{P}^{(t)}_{k,S} \) satisfies the following recursion:

\[
\hat{P}^{(t)}_{k,S} = Q^{(t)} + \hat{K}^{(t)}_{k,S}^T R^{(t)} S \hat{K}^{(t)}_{k,S} + (A + BS\hat{K}^{(t)}_{k,S})^T \hat{P}^{(t)}_{k+1,S} (A + BS\hat{K}^{(t)}_{k,S}),
\]

initialized with \( \hat{P}^{(0)}_{N,S} = Q^{(0)}_I \).

Next, we aim to characterize the performance of the certainty equivalent controller given in (15). To this end, we will provide an upper bound on \( \hat{J}_t(S) - J_t(S) \) in terms of the estimation error corresponding to \( A \) and \( B \), where \( J_t(S) \) is defined in Eq. (12).

### 3.1 Perturbation Bounds on Relevant Matrices

Suppose that \( \| A - \hat{A} \| \leq \varepsilon \) and \( \| B - \hat{B} \| \leq \varepsilon \) with \( \varepsilon \in \mathbb{R}_{>0} \). In this subsection we provide upper bounds on \( \| K^{(t)}_{k,S} - \hat{K}^{(t)}_{k,S} \| \) and \( \| P^{(t)}_{k,S} - \hat{P}^{(t)}_{k,S} \| \), where \( K^{(t)}_{k,S} \) (resp., \( \hat{K}^{(t)}_{k,S} \)) is given by Eq. (10) (resp., Eq. (16)), and \( P^{(t)}_{k,S} \) (resp., \( \hat{P}^{(t)}_{k,S} \)) is given by Eq. (11) (resp., Eq. (17)). Recall that we have assumed that the matrices \( Q^{(t)}, Q^{(t)}_I \in \mathbb{S}^n_+ \), and \( R^{(t)} \in \mathbb{S}^m_+ \) are known for all \( t \in [T] \). We will make the following mild assumptions on these matrices.

---

4We will provide more details on how we obtain \( \hat{A} \) and \( \hat{B} \) in Section 4.
Assumption 1. We assume that (a) $Q_f^{(t)} \in \mathbb{S}^n_{++}$ for all $t \in [T]$; and (b) $\sigma_n(Q_f^{(t)}) \geq 1$ and $\sigma_m(R^{(t)}) \geq 1$ for all $t \in [T]$.

Note that assuming $\sigma_n(Q_f^{(t)}) \geq 1$ and $\sigma_m(R^{(t)}) \geq 1$ is not more restrictive than assuming $Q_f^{(t)} \in \mathbb{S}^n_{++}$ and $R^{(t)} \in \mathbb{S}^m_{++}$. This is because multiplying both sides of Eq. (8) by a positive constant does not change $K^{(t)}_{k,S}$ given by Eq. (10). In order to simplify the notations in the sequel, for all $S \subseteq \mathcal{G}$, we denote

$$
\Gamma_S = \max_{t \in [T]} \max_{k \in [N]} \Gamma_{k,S}^{(t)},
$$

and

$$
\tilde{\Gamma}_S = 1 + \Gamma_S,
$$

where

$$
\Gamma_{k,S}^{(t)} = \max \{ \|A\|, \|B\|, \|P_{k,S}^{(t)}\|, \|K_{k-1,S}^{(t)}\| \}.
$$

Moreover, we denote

$$
\sigma_Q = \max_{t \in [T]} \max_{k \in [N]} \sigma_1(Q_f^{(t)}),
$$

$$
\sigma_R = \max_{t \in [T]} \sigma_1(R^{(t)}).
$$

We have the following result whose proof is provided in Appendix A.

Lemma 1. Consider any $S \subseteq \mathcal{G}$, any $t \in [T]$ and any $k \in [N]$. Let $\varepsilon \in \mathbb{R}_{\geq 0}$ and $D \in \mathbb{R}_{\geq 1}$ be such that $D \varepsilon \leq 1/6$. Suppose that $\|A - \hat{A}\| \leq \varepsilon$, $\|B_S - \hat{B}_S\| \leq \varepsilon$, and $\|P_{k,S}^{(t)} - \hat{P}_{k,S}^{(t)}\| \leq D \varepsilon$, and that Assumption 1 holds. Then,

$$
\|K_{k-1,S} - \hat{K}_{k-1,S}^{(t)}\| \leq 3 \tilde{\Gamma}_S \varepsilon, \tag{24}
$$

and

$$
\|P_{k-1,S}^{(t)} - \hat{P}_{k-1,S}^{(t)}\| \leq 20 \tilde{\Gamma}_S \sigma_R D \varepsilon, \tag{25}
$$

where $\tilde{\Gamma}_S$ is defined in Eq. (22), and $\sigma_R$ is defined in (23).

To proceed, we introduce the following assumption on the controllability of the pair $(A,B)$ of the system in Eq. (1); similar assumptions can be found in, e.g., [10, 25].

Assumption 2. For any $S \subseteq \mathcal{G}$ with $|S| = H$, we assume that the pair $(A,B_S)$ in the system given by Eq. (1) satisfies that $\sigma_1(C_{\ell,S}) \geq \nu$, where $\ell \in [n-1]$, $\nu \in \mathbb{R}_{>0}$ and

$$
C_{\ell,S} \triangleq [B_S \ A B_S \ \cdots \ \ A^{\ell-1} B_S].
$$

If Assumption 2 is satisfied, we say that the pair $(A,B_S)$ is $(\ell,\nu)$-controllable (e.g., [25]). Note that if $(A,B_S)$ is controllable, $(A,B_S)$ can be $(\ell,\nu)$-controllable for some $\ell \in [n-1]$ that is much smaller than $n$. For example, supposing that rank($B_S$) = $n$, then $(A,B_S)$ is (1, $\nu$)-controllable. Furthermore, one can check that a sufficient condition for Assumption 2 to hold is that for any $S \subseteq \mathcal{G}$, the pair $(A,B_s)$ is $(\ell,\nu)$-controllable. Denoting

$$
\hat{C}_{\ell,S} = [\hat{B}_S \ \hat{A} \hat{B}_S \ \cdots \ \hat{A}^{\ell-1} \hat{B}_S] \forall S \subseteq \mathcal{G},
$$

we have the following lower bound on $\sigma_n(\hat{C}_{\ell,S})$ from [25].

Lemma 2. [25, Lemma 6] Consider any $S \subseteq \mathcal{G}$. Suppose that $\|A - \hat{A}\| \leq \varepsilon$ and $\|B_S - \hat{B}_S\| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}_{\geq 0}$. Under Assumption 2,

$$
\sigma_n(\hat{C}_{\ell,S}) \geq \nu - \varepsilon \ell^{3/2} \beta^{-1} (\|B_S\| + 1),
$$

where $\beta \triangleq \max\{1, \varepsilon + \|A\|\}$.

We see from Lemma 2 that if $\varepsilon$ is small enough, then $\sigma_n(\hat{C}_{\ell,S}) > 0$, i.e., rank($\hat{C}_{\ell,S}$) = $n$ and the pair $(\hat{A},\hat{B}_S)$ is controllable. We then have the following result whose proof is included in Appendix A.
Lemma 3. Consider any $S \subseteq G$ with $|S| = H$ and any $t \in [T]$. Suppose that Assumptions 1-2 hold, and that $\|A - \hat{A}\| \leq \varepsilon$ and $\|B_S - \hat{B}_S\| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}_{\geq 0}$. Then, for all $k \in \{N - \gamma \ell : \gamma \in \mathbb{Z}_{\geq 0}, \gamma \ell \leq N\}$, it holds that

$$\|P_{k,S}^{(t)} - \hat{P}_{k,S}^{(t)}\| \leq \mu_{k,S}^{(t)} \varepsilon,$$  \hspace{1cm} (26)

under the assumption that $\varepsilon$ is small enough such that $\mu_{k,S}^{(t)} \varepsilon \leq 1$ with

$$\mu_{k,S}^{(t)} \triangleq 32\ell^2 \beta^{2(\ell - 1)}(1 + \nu^{-1})(1 + \|B_S\|)^2\|P_{k,S}^{(t)}\| \max\{\sigma_Q, \sigma_R\};$$  \hspace{1cm} (27)

where $\beta \triangleq \max\{1, \varepsilon + \|A\|\}$, and $\sigma_Q, \sigma_R$ are defined in (23).

Let us further denote

$$\mu_S \equiv 32\ell^2 \beta^{2(\ell - 1)}(1 + \nu^{-1})\Gamma_S^3 \max\{\sigma_Q, \sigma_R\}.$$  \hspace{1cm} (28)

Now, combining Lemma 1 and Lemma 3 yields the following result, which provides upper bounds on $\|K_{k,S}^{(t)} - \hat{K}_{k,S}^{(t)}\|$ and $\|P_{k,S}^{(t)} - \hat{P}_{k,S}^{(t)}\|$ for all $k$. The proof of the following result is included in Appendix A.

**Proposition 1.** Consider any $S \subseteq G$ with $|S| = H$. Suppose that Assumptions 1-2 hold, and that $\|A - \hat{A}\| \leq \varepsilon$, $\|B_S - \hat{B}_S\| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}_{\geq 0}$. Then, for all $k \in \{0, 1, \ldots, N\}$, it holds that

$$\|P_{k,S}^{(t)} - \hat{P}_{k,S}^{(t)}\| \leq (20\Gamma_S^3 \sigma_R)^{f-1} \mu_S \varepsilon,$$  \hspace{1cm} (29)

under the assumption that $\varepsilon$ is small enough such that the right hand side of (29) is smaller than or equal to $1/6$, where $\Gamma_S$ is defined in Eq. (22), $\sigma_R$ is defined in Eq. (23), and $\mu_S$ is defined in Eq. (28). Moreover, for all $k \in \{0, 1, \ldots, N - 1\}$, it holds that

$$\|K_{k,S}^{(t)} - \hat{K}_{k,S}^{(t)}\| \leq 3\Gamma_S^3(20\Gamma_S^3 \sigma_R)^{f-1} \mu_S \varepsilon,$$  \hspace{1cm} (30)

under the same assumption on $\varepsilon$.

### 3.2 Perturbation Bound on Cost

Supposing that $\|A - \hat{A}\| \leq \varepsilon$ and $\|B - \hat{B}\| \leq \varepsilon$ with $\varepsilon \in \mathbb{R}_{\geq 0}$, in this subsection we provide an upper bound on $\hat{J}(S) - J_i(S)$, where $J_i(S)$ and $\hat{J}_i(S)$ are defined in Eqs. (12) and (18), respectively. We begin with the following result; the proof is included in Appendix A.

**Lemma 4.** Consider any $S \subseteq G$ and any $t \in [T]$. Let $x_k^{(t)}$ be the state corresponding to the certainty equivalence control $u_k^{(t)} = \hat{K}_{k,S}^{(t)}x_k$, i.e., $x_k^{(t)} = (A + B_S\hat{K}_{k,S}^{(t)})x_k + w_k^{(t)}$, where $w_k^{(t)}$ is the zero-mean white Gaussian noise process with covariance $W$ for all $k$. Then,

$$\hat{J}_i(S) - J_i(S) = \sum_{k=0}^{N-1} \mathbb{E} \left[ x_k^{(t)\top} \Delta K_{k,S}^{(t)\top} (R^{(t)} + B_S^\top P_{k,S} B_S) \Delta \hat{K}_{k,S} x_k^{(t)} \right],$$  \hspace{1cm} (31)

where $\Delta K_{k,S}^{(t)} \triangleq \hat{K}_{k,S}^{(t)} - K_{k,S}^{(t)}$. To proceed, consider any $S \subseteq G$ and any $t \in [T]$. For all $k_1, k_2 \in \{0, 1, \ldots, N\}$ with $k_2 \geq k_1$, we use $\Psi_{k_2,k_1}^{(t)}(S)$ to denote the transition matrix corresponding to $A + B_S K_{k_2,\cdot}^{(t)}$, i.e.,

$$\Psi_{k_2,k_1}^{(t)}(S) = (A + B_S K_{k_2-1,\cdot}^{(t)})(A + B_S K_{k_2-2,\cdot}^{(t)}) \cdots (A + B_S K_{k_1,\cdot}^{(t)}),$$  \hspace{1cm} (32)

and $\Psi_{k_2,k_1}^{(t)}(S) \triangleq I$ if $k_1 = k_2$, where $K_{k,S}^{(t)}$ is given by Eq. (10). Similarly, we denote

$$\hat{\Psi}_{k_2,k_1}^{(t)}(S) = (A + B_S \hat{K}_{k_2-1,\cdot}^{(t)})(A + B_S \hat{K}_{k_2-2,\cdot}^{(t)}) \cdots (A + B_S \hat{K}_{k_1,\cdot}^{(t)}),$$  \hspace{1cm} (33)

and $\hat{\Psi}_{k_2,k_1}^{(t)}(S) \triangleq I$ if $k_1 = k_2$, where $\hat{K}_{k,S}^{(t)}$ is given by Eq. (16). We then have the following results; the proofs are included in Appendix A.
Lemma 5. Consider any $S \subseteq G$ with $|B| = H$ and any $t \in [T]$. Suppose that Assumptions 1-2 hold. Then, there exist finite constants $\zeta_5 \in \mathbb{R}_{\geq 1}$ and $\eta_5 \in \mathbb{R}$ with $0 < \eta_5 < 1$ such that $\|\Phi_{k_2,k_1}^{(t)}(S)\| \leq \zeta_5 \eta_5^{k_2-k_1}$ for all $k_1, k_2 \in \{0, 1, \ldots, N\}$ with $k_2 \geq k_1$.

Lemma 6. Consider any $S \subseteq G$ with $|S| = H$ and any $t \in [T]$. Suppose that Assumptions 1-2 hold, and that $\|K_{k,S}^{(t)} - \hat{K}_{k,S}^{(t)}\| \leq \varepsilon$ for all $k \in \{0, 1, \ldots, N - 1\}$, where $\varepsilon \in \mathbb{R}_{>0}$. Let $\zeta_6 \in \mathbb{R}_{\geq 1}$ and $\eta_6 \in \mathbb{R}$ with $0 < \eta_6 < 1$ be such that $\|\Phi_{k_2,k_1}^{(t)}(S)\| \leq \zeta_6 \eta_6^{k_2-k_1}$ for all $k_1, k_2 \in \{0, 1, \ldots, N\}$ with $k_2 \geq k_1$. Then, for all $k_1, k_2 \in \{0, 1, \ldots, N\}$ with $k_2 \geq k_1$, it holds that

$$\|\hat{\Psi}_{k_2,k_1}^{(t)}(S)\| \leq \zeta_6 \frac{1 + \eta_6}{2}^{k_2-k_1},$$

under the assumption that $\varepsilon \leq \frac{1 - \eta_6}{2H \|B\|_{\infty}}$.

Now, combining Lemmas 4 and 6, and recalling Proposition 1, we obtain the following result.

Proposition 2. Consider any $S \subseteq G$ with $|S| = H$ and any $t \in [T]$. Suppose that Assumptions 1-2 hold, and that $\|A - \hat{A}\| \leq \varepsilon$ and $\|B - \hat{B}\| \leq \varepsilon$, where $\varepsilon \in \mathbb{R}_{>0}$. Let $\zeta_7 \in \mathbb{R}_{\geq 1}$ and $\eta_7 \in \mathbb{R}$ with $0 < \eta_7 < 1$ be such that $\|\Phi_{k_2,k_1}^{(t)}(S)\| \leq \zeta_7 \eta_7^{k_2-k_1}$ for all $k_1, k_2 \in \{0, 1, \ldots, N\}$ with $k_2 \geq k_1$. Then, it holds that

$$\hat{J}_i(S) - J_i(S) \leq \frac{4\min\{n, m_5\}N\zeta_7^2}{1 - \eta_7^2} \sigma_1(W) (\sigma_R + \Gamma_S^{(t)}(2\Gamma_S^9\sigma_R)^{-1} \mu_S)^2 \varepsilon^2,$$

under the assumption that

$$\varepsilon \leq \frac{1 - \eta_7}{6\|B\|_{\infty} \zeta_7^2 \Gamma_S^{-3} (2\Gamma_S^9 \sigma_R)^{1 - \ell} \mu_S^{-1}},$$

where $J_i(S)$ and $\hat{J}_i(S)$ are defined in Eqs. (12) and (18), respectively, $m_5 = \sum_{i \in S} m_i$, $\sigma_R$ is defined in (23), and $\Gamma_S$ is defined in Eq. (21) with $\Gamma_S = \Gamma_S + 1$.\n
Proof. First, we provide an upper bound on $\Sigma_k^{(t)} = E[x_k^{(t)} x_k^{(t)\top}]$ for all $k \in \{0, 1, \ldots, N - 1\}$. Recalling that we have assumed that $x_0^{(t)} = 0$, it follows that $\Sigma_0^{(t)} = 0$. Considering any $k \in \{N - 1\}$, one can show via Eq. (7) that

$$\Sigma_k^{(t)} = (A + B_k^{(t)}) \Sigma_{k-1}^{(t)} (A + B_k^{(t)})^\top + W,$$

which implies that

$$\Sigma_k^{(t)} = \sum_{i=0}^{k-1} \hat{\Psi}_{k-1,i}^{(t)}(S) i W \hat{\Psi}_{k-1,i}^{(t)}(S)^\top,$$

where $\hat{\Psi}_{k-1,i}^{(t)}(S)$ is defined in Eq. (33). Now, under the assumption on $\varepsilon$ given in (36), we can apply the upper bound on $\|\hat{\Psi}_{k,i}^{(t)}(S)\|$ in Lemma 6 and obtain

$$\|\Sigma_k^{(t)}\| \leq \sigma_1(W) \zeta_7^2 \sum_{i=1}^{k} \left(\frac{1 + \eta_7}{2}\right)^{2(k-i)} \leq \frac{\sigma_1(W) \zeta_7^2}{1 - \frac{1 + \eta_7}{2}^2} \leq \frac{4\sigma_1(W) \zeta_7^2}{1 - \eta_7^2}.$$

One can also show via Eq. (31) in Lemma 4 that

$$\hat{J}_i(S) - J_i(S) \leq \sum_{k=0}^{N-1} \|\Sigma_k^{(t)}\| \|R^{(t)} + B_k^{\top} P_k^{(t)} B_k\| \|\Delta \hat{K}_k^{(t)}(S)\|^2_F.$$

Moreover, under the assumption on $\varepsilon$ given in (36), we can also apply the upper bound on $\|\Delta \hat{K}_k^{(t)}(S)\|$ in Proposition 1, where $\Delta \hat{K}_k^{(t)}(S) \in \mathbb{R}^{m_5 \times n}$. Also noting that $\|\Delta \hat{K}_k^{(t)}(S)\|^2_F \leq \min\{n, m_5\} \|\Delta \hat{K}_k^{(t)}(S)\|^2$, and recalling the definition of $\Gamma_S$, we can then combine the above arguments together and obtain (35).
4 Algorithm Design

In this section, we formally describe the online algorithm that we propose for the actuator selection (and controller design) problem. Recall from our discussions in Section 2.3 that an online algorithm for the actuator selection problem needs to decide the set of selected actuators for each round \( t \in [T] \) and the control policy corresponding to the set of selected actuators. In Section 3, we have introduced the certainty equivalence approach to choosing the control policy when a set of actuators is selected for a given round \( t \in [T] \). In order to select the set of actuators for each round \( t \in [T] \), we will leverage an online algorithm for the multiarmed bandit problem introduced in [5, Section 8] (i.e., the \textbf{Exp3.S} algorithm). To better present our results, we briefly review the \textbf{Exp3.S} algorithm for the multiarmed bandit problem and the corresponding regret analysis from [5].

4.1 \textbf{Exp3.S} Algorithm for Multiarmed Bandit

Consider the multiarmed bandit problem in which at the beginning of each round \( t \in [T] \), we need to choose an action from a finite set \( Q \) of possible actions. Choosing \( i_t \in Q \) for round \( t \in [T] \) incurs a cost, denoted as \( y_t(i) \), which is revealed at the end of round \( t \). An instance of the multiarmed bandit problem is then specified by a number of rounds \( T \), a finite set \( Q \) of possible actions, and costs of actions \( y(1), \ldots, y(T) \) with \( y_t = (y_t(1), \ldots, y_t(|Q|)) \) for all \( t \in [T] \), where \( y_t(i) \in [0, y_b] \) (with \( y_b \in \mathbb{R}_{>0} \)) denotes the cost of choosing action \( i \) in round \( t \), for all \( t \in [T] \) and for all \( i \in Q \). An online algorithm \( A_M \) for the multiarmed bandit problem now needs to decide an action \( i_t \in Q \) for each round \( t \in [T] \), based on all the available information so far, i.e., \( i_{t'} \) and \( y_{t'}(t') \) for all \( t' \in \{1, \ldots, t-1\} \). In particular, it was shown in [5, Corollary 8.2] that for any sequence of actions \( i^T \equiv (i_1, \ldots, i_T) \), the \textbf{Exp3.S} algorithm yields the following regret bound:

\[
R_M(i^T) \equiv E_M\left[\sum_{t=1}^{T} y_t(i_t)\right] - \sum_{t=1}^{T} y_t(i_t) \\
\leq y_b h(i^T) \sqrt{|Q| T \log(|Q| T)} + 2y_b e \sqrt{\frac{|Q| T}{\log(|Q| T)}},
\]

(37)

where \( E_M[\cdot] \) denotes the expectation with respective to the randomness of the \textbf{Exp3.S} algorithm,\(^5\) and

\[
h(i^T) \equiv 1 + |\{1 \leq \ell < T : i_\ell \neq i_{\ell+1}\}|.
\]

(38)

See [5, Section 8] for more details about the \textbf{Exp3.S} algorithm.

Remark 1. As shown in [5], the regret bound in (37) holds under the assumption that for any \( i \in Q \) and any \( t \in [T] \), \( y_t(i_t) \) does not depend on the previous actions chosen by the online algorithm, i.e., \( i_1, \ldots, i_{t-1} \). Other than this assumption, \( y_t(i_t) \) can be any real number in \([0, y_b]\), and there is no statistical assumption on \( y_t(i_t) \).

We will call the \textbf{Exp3.S} algorithm as a subroutine in our online algorithm for the actuator selection problem described in Section 2.2. Specifically, we let the set of possible actions \( Q \) corresponding to the \textbf{Exp3.S} algorithm contain \( \binom{|G|}{H} \) actions, where recall that \( G \) is the set of candidate actuators and \( H \) is the cardinality constraint on the set of selected actuators in each round \( t \in [T] \). Each action in \( Q \) now corresponds to a set \( S \subseteq G \) with \(|S| = H\). Suppose that the set \( S_t \subseteq G \) with \(|S_t| = H\) is selected by \textbf{Exp3.S} in round \( t \in [T] \), and that a control policy \( u_{S_t}^{(t)} = (u_{0,S_t}^{(t)}, u_{1,S_t}^{(t)}, \ldots, u_{N-1,S_t}^{(t)}) \) is chosen for the actuators in \( S_t \). We then feedback the cost \( J_t(S_t, u_{S_t}^{(t)}) \) defined in Eq. (8) as the cost that \textbf{Exp3.S} would incur by choosing (the action corresponding to) \( S_t \).

4.2 Online Actuator Selection Algorithm

We are now ready to formally introduce the online algorithm (Algorithm 1) to solve the actuator selection problem defined in Section 2.2. We will make the following assumption on the noise process \( w_k \) in the system given by Eq. (7).

\(^5\)Specifically, the \textbf{Exp3.S} algorithm chooses the action \( i_t \) for any round \( t \in [T] \) in a random manner [5].
Assumption 3. We assume that (a) for all distinct \(k_1, k_2 \in \{0, 1, \ldots, N - 1\}\) and for all distinct \(t_1, t_2 \in [T]\), the noise terms \(w_{k_1}^{(t_1)}\) and \(w_{k_2}^{(t_2)}\) are independent; and (b) the covariance of \(w_k^{(t)}\) satisfies \(W = \sigma^2 I_n\), where \(\sigma \in \mathbb{R}_{\geq 0}\).

Assumption 3(b) is made here for simplicity. Our analysis in the remaining of this paper can be extended to \(w_k\) with general covariance matrix \(W\), by considering \(\sigma_1(W)\) and \(\sigma_n(W)\) in the analysis. Recall that \(G = [g]\) and that at most \(H \in \mathbb{Z}_{\geq 1}\) actuators can be selected in each round \(t \in [T]\). We also rely on the following assumption; similar assumptions can be found in, e.g., [11, 25, 10, 9].

Assumption 4. We assume that there exists a partition \(G = \bigcup_{i \in [p]} G_i\) such that \(|G_i| \leq H\) and there is a known stabilizable \(K_{G_i} \in \mathbb{R}^{m_{G_i} \times n}\) with \(\|A + B_{G_i} K_{G_i}\|^k \leq \zeta_0 \eta_0^k\) for any \(k \in \mathbb{R}_{>0}\) and any \(i \in [p]\), where \(m_{G_i} = \sum_{j \in G_i} m_j\), \(\zeta_0 \in \mathbb{R}_{>1}\) and \(\eta_0 \in \mathbb{R}_{>0}\) with \(0 < \eta_0 < 1\).

Note that under Assumption 2, such a stabilizable \(K_{G_i}\) is guaranteed to exist for any \(i \in [p]\). Moreover, similarly to our arguments in the proof of Lemma 5, the stability of the matrix \(A + B_{G_i} K_{G_i}\) ensures via the Gelfand formula (e.g., [21]) that such finite constants \(\zeta_0 \geq 1\) and \(0 < \eta_0 < 1\) exist. Now, for all \(t \in [T]\), all \(k \in \{0, 1, \ldots, N - 1\}\) and all \(S \subseteq G\), we denote

\[
\zeta_{k, S}^{(t)} = \left[ x_k^{(t)\top} u_{k, S}^{(t)\top} \right] \top,
\]

where \(x_k^{(t)}\) is the state of the system at time step \(k\) in round \(t\), and \(u_{k, S}\) is the input provided by the set \(S\) of actuators at time step \(k\) in round \(t\). Also recall that we denote \(u_{S}^{(t)} = (u_0^{(t), u_1^{(t)}, \ldots, u_{N-1}^{(t)}\top})\). The online algorithm for the actuator selection problem is then given in Algorithm 1, which uses the \texttt{Exp3.S} algorithm and Algorithm 2 as subroutines, where

\[
k \triangleq \max \left\{ \max_{j \in [p]} \|K_{G_j}\|, 1 \right\}.
\]

Algorithm 1 Online actuator selection algorithm

**Input:** Parameters \(\tau_1, \lambda, T\), and \(K_{G_i}\) for all \(i \in [p]\) from Assumption 4.

1. Set \(n_e = \lceil \sqrt{T}/(\tau_1 p) \rceil\), \(\tau_2 = (T - n_e \tau_1 p)/n_e\), and \(T_{1,1} = 1\).
2. for \(i = 1\) to \(n_e\) do
3. \hspace{1em} for \(j = 1\) to \(p\) do
4. \hspace{2em} Set \(T_{i,j+1} = T_{i,j} + \tau_1\).
5. \hspace{1em} Set \(T_{i,p+1} = T_{i,1} + \tau_2\).
6. for \(i = 1\) to \(n_e\) do
7. \hspace{1em} for \(j = 1\) to \(p\) do
8. \hspace{2em} for \(t = T_{i,j}\) to \(T_{i,j+1} - 1\) do
9. \hspace{3em} Set \(S_t = G_j\).
10. \hspace{3em} Play \(u_{G_j}^{(t)}\), where \(u_{G_j}^{(t)} \overset{i.i.d.}\sim N(K_{G_j}x_k^{(t)}, 2\sigma^2 \kappa^2 I) \forall k \in \{0, \ldots, N - 1\}\).
11. \hspace{2em} Obtain \(\hat{A}(i)\) and \(\hat{B}(i)\) from \texttt{LSE}(i).
12. \hspace{1em} for \(t = T_{i,p+1}\) to \(T_{i+1,1} - 1\) do
13. \hspace{2em} Select \(S_t\) with \(|S_t| = H\) using \texttt{Exp3.S}.
14. \hspace{1em} for \(k = 0\) to \(N - 1\) do
15. \hspace{2em} Obtain \(K_{k, S_t}\) using \(\hat{A}(i)\) and \(\hat{B}(S_t, i)\) via Eq. (16).
16. \hspace{2em} Play \(u_{k, S_t}^{(t)} = K_{k, S_t} x_k^{(t)}\).
17. Feedback \(J_t(S_t, u_{S_t}^{(t)})\) defined in Eq. (8) as the cost that \texttt{Exp3.S} would incur by choosing \(S_t\).

In words, Algorithm 1 divides the \(T\) rounds in the actuator selection problem into \(n_e\) epochs. During each epoch \(i \in [n_e]\), Algorithm 1 contains two phases, i.e., the estimation phase and the control phase, where the estimation phase contains \(\tau_1 p\) rounds and the control phase contains \(\tau_2\) rounds.\footnote{Note that we assume for simplicity that \(n_e (\tau_1 p + \tau_2) = T\) in Algorithm 1; otherwise one can change the number of rounds in the last epoch accordingly.} Specifically,
in lines 7-11, Algorithm 1 obtains estimates of $A$ and $B$, i.e., $\hat{A}(i)$ and $\hat{B}(i)$. This is achieved by playing the known stabilizable $K_{G_j}$ from Assumption 4 for $T_1$ rounds and for all $j \in [p]$ and then using the least squares estimation algorithm given in Algorithm 2. After the estimation phase, Algorithm 1 leverages the \texttt{Exp3.S} algorithm in lines 12-17 to select a set of actuators $S_t$ with $|S_t| = H$ for round $t$. The control policy corresponding to $S_t$ is then designed based on $\hat{A}(i)$ and $\hat{B}_{S_t}(i)$ using Eq. (16), where $\hat{B}_{S_t}(i)$ denotes a submatrix of $\hat{B}(i)$ that contains columns from $\hat{B}(i)$ corresponding to $S_t$.

5 Regret Analysis

In this section, we provide an upper bound on the regret of Algorithm 1 defined in Eq. (14), which holds with high probability. First, we analyze the estimation error of the least squares estimation given in Algorithm 2. To proceed, consider any epoch $i \in [n_e]$ in Algorithm 1. For any $G_j$ (with $j \in [p]$) described in Assumption 4, we denote

$$V_{G_j}(i) = \lambda I + \sum_{i=1}^{T_1} \sum_{t=1}^{N-1} \sum_{k=0}^{l,j} z_{k,G_j}^{(t)} z_{k,G_j}^{(t)\top},$$

where $\lambda \in \mathbb{R}_{>0}$, $T_1, T_1, T_1+1$ are given in Algorithm 1, and $z_{k,G_j}^{(t)}$ is given in Eq. (39). We then have the following result, which characterizes the estimation error of $\hat{G}_j(i)$ given in line 2 of Algorithm 1; the proof is similar to that of [10, Lemma 6] and is omitted here for conciseness.

**Lemma 7.** Consider any $G_j$ from Assumption 4, where $j \in [p]$. Let $\Delta_{G_j}(i) = \Theta_{G_j} - \hat{\Theta}_{G_j}(i)$, where $\Theta_{G_j} = [A \quad B_{G_j}]$. Suppose that Assumption 3 holds. Then, for any $\delta \in \mathbb{R}$ with $0 < \delta < 1$, the following holds with probability at least $1 - \delta$:

$$\text{Tr}(\Delta_{G_j}(i)\top V_{G_j}(i)\Delta_{G_j}(i)) \leq 4\sigma^2 n \log \left( \frac{n \det(V_{G_j}(i))}{\delta \det(A)} \right) + 2\lambda \|\Theta_{G_j}\|_F^2 \quad \forall i \in [n_e],$$

where $n_e$ is defined in Algorithm 1.

For notational simplicity in the sequel, let us denote

$$\vartheta = \max\{|A|, |B|\},$$

$$\varepsilon_0 = \min_{S \subseteq G_j, |S| = H} \frac{1 - \eta_S}{6\|B_S\|_F} \hat{\Gamma}_S^{-3} (20\hat{\Gamma}_S \sigma R)^{1-\epsilon} \mu_S^{-1},$$

$$\zeta = \max_{S \subseteq G_j, |S| = H} \zeta_S, \quad \eta = \max_{S \subseteq G_j, |S| = H} \eta_S,$$

$$\hat{\kappa} = \max_{S \subseteq G_j, |S| = H} \left( \frac{\hat{\Gamma}_S + \frac{1 - \eta_S}{2\|B_S\|_F}}{\zeta_S} \right),$$

where $\hat{\Gamma}_S$ (resp., $\Gamma_S$) is defined in Eq. (22) (resp., (21)), $\zeta_S$ and $\eta_S$ are provided in Lemma 5, and $\mu_S$ is defined in Eq. (28). We have the following result, which characterizes the regret of Algorithm 1 defined in Eq. (14).
Theorem 1. Suppose that Assumptions 1-4 hold. Consider any $\delta \in \mathbb{R}_{>0}$ with $0 < \delta < 1$. In Algorithm 1, let
\[
\tau_1 = \frac{160n\left(\frac{\lambda d^2_n}{\sigma^2} + 2(n + m)\log \left(\frac{2n}{\tau_0 \frac{\lambda d^2_n}{\sigma^2}}\right)\right)}{(N - 1)^2 \delta^2},
\]
where
\[
z_b = \frac{20c_0^2(1 + \kappa)^2\sigma^2}{(1 - \eta_0)^2} (2\eta_0^2 + \kappa^2m + n) \log \frac{8NT}{\delta}.
\]
Then, for any $T > \tau_1 p$ with $T > 2$,
\[
R_A = \tilde{O}(\sqrt{T})
\]
holds with probability at least $1 - \delta$, where $\tilde{O}(\cdot)$ hides polynomial factors in $\log(T/\delta)$.

Remark 2. Note that the expression for $\tau_1$ in Eq. (43) requires the knowledge of the parameters given in (42). Such a requirement is typical in learning based algorithms for LQR with unknown system models (e.g., \cite{14, 11, 10, 25, 9}). In other words, some knowledge of the (unknown) system is needed when designing the algorithms. In fact, if we only set $\tau_1$ to be greater than or equal to the right hand side of Eq. (43) (while keeping other conditions to be the same in Theorem 1), our analysis for Theorem 1 directly yields that the regret of Algorithm 1 is also $\tilde{O}(\sqrt{T})$ with high probability. Thus, Theorem 1 provides a sufficient condition on $\tau_1$ such that Algorithm 1 yields a sublinear regret in $T$.

5.1 Proof of Theorem 1

First, recalling lines 8-10 in Algorithm 1, let us consider any $i \in [n]$, any $j \in [p]$ and any $T_{i,j} \leq t \leq T_{i,j+1} - 1$. One can show that the state of the system in Eq. (7) satisfies that
\[
x_k^{(t)} = (A + B_{G_j} K_{G_j}) x_k^{(t)} + B_{G_j} \tilde{w}_k^{(t)} + w_k^{(t)},
\]
for all $k \in \{0, \ldots, N - 1\}$, where $x_0^{(t)} = 0$ as we assumed, $K_{G_j}$ comes from Assumption 4, and $\tilde{w}_k^{(t)} \sim \mathcal{N}(0, 2\sigma^2 \kappa^2 I)$ with $\kappa$ defined in Eq. (40). In other words, $u_k^{(t)}$ in line 10 of Algorithm 1 satisfies that $u_k^{(t)} = K_{G_j} x_k^{(t)} + \tilde{w}_k^{(t)}$ for all $k \in \{0, \ldots, N - 1\}$. Note that $w_k^{(t)}$ is independent of $w_k^{(t)}$ for all $k, k' \in \{0, \ldots, N - 1\}$. Denote
\[
\bar{T} = \{t : T_{i,j} \leq t \leq T_{i,j+1} - 1, i \in [n], j \in [p]\} \cap [T],
\]
\[
T = [T] \setminus \bar{T}.
\]
Let $(S_1, \ldots, S_T)$ be the sets of actuators selected by Algorithm 1 and let $(S_1^*, \ldots, S_T^*)$ be an optimal solution to Problem (13). Recalling Eq. (14), the regret of Algorithm 1 (denoted as $A$) can then be written as
\[
R_A = \mathbb{E}_A \left[ \sum_{t \in \bar{T}} J_t(S_t, u_{S_t}^{(t)}) \right] + \mathbb{E}_A \left[ \sum_{t \in T} J_t(S_t, u_{S_t}^{(t)}) \right] - \sum_{t=1}^T J_t(S_t^*),
\]
where $J_t(S_t, u_{S_t}^{(t)})$ is defined in Eq. (8) with $u_{S_t}^{(t)}$ given by Algorithm 1, and $J_t(S_t^*)$ is given by Eq. (12). Denoting
\[
R_1 = \mathbb{E}_A \left[ \sum_{t \in \bar{T}} J_t(S_t, u_{S_t}^{(t)}) \right] - \sum_{t \in \bar{T}} J_t(S_t^*),
\]
\[
R_2 = \mathbb{E}_A \left[ \sum_{t \in T} J_t(S_t, u_{S_t}^{(t)}) \right] - \sum_{t \in T} J_t(S_t^*, u_{S_t}^{(t)}),
\]
\[
R_3 = \sum_{t \in \bar{T}} (J_t(S_t^*, u_{S_t}^{(t)}) - \hat{J}_t(S_t^*)),
\]
\[
R_4 = \sum_{t \in \bar{T}} (\hat{J}_t(S_t^*) - J_t(S_t^*)).
\]
where \( \hat{J}_i(S^*_t) \) is given by Eq. (18). Now, one can show that \( R_A \) can be further written as

\[
R_A = R_1 + R_2 + R_3 + R_4. \tag{47}
\]

In order to prove the (high probability) upper bound on \( R_A \), we will provide upper bounds on \( R_1, R_2, R_3, \) and \( R_4 \) in the sequel. To this end, for any \( 0 < \delta < 1 \), we first define the following probabilistic events:

\[
E_w = \left\{ \|w_k^{(t)}\| \leq \sigma \sqrt{5n \log \frac{8NT}{\delta}}, \forall t \in [T], \forall k \in \{0, \ldots, N-1\} \right\},
\]

\[
E_w = \left\{ \|\hat{w}_k^{(t)}\| \leq \kappa \sigma \sqrt{10m \log \frac{8NT}{\delta}}, \forall t \in \tilde{T}, \forall k \in \{0, \ldots, N-1\} \right\},
\]

\[
E_\Theta = \left\{ \text{Tr}(\Delta \mathcal{G}_j(i)^T V_{\mathcal{G}_j}(i) \Delta \mathcal{G}_j(i)) \leq 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\lambda I)} \right) + 2\lambda \|\Theta_{\mathcal{G}_j}\|_F^2, \forall i \in [n_e], \forall j \in [p] \right\},
\]

\[
E_z = \left\{ \sum_{l=1}^{T_{i,j+1}-1} \sum_{t=T_{i,j}}^{N-1} \sum_{k=0}^{z_k(i)} z_k(i)^T \leq \frac{(N-1)\tau_{i,j} \sigma^2}{80} I, \forall i \in [n_e], \forall j \in [p] \right\}.
\]

Letting

\[
E = E_w \cap E_{\delta} \cap E_\Theta \cap E_z, \tag{48}
\]

we have the following result; the proof is included in Appendix B.

**Lemma 8.** For any \( 0 < \delta < 1 \), the event \( E \) defined in Eq. (48) satisfies \( \mathbb{P}(E) \geq 1 - \delta/2 \).

Next, supposing that \( E \) holds, we characterize the estimation error associated with \( \hat{\Theta}_{\mathcal{G}_j}(i) \) given by Algorithm 2, for all \( j \in [p] \) and all \( i \in [n_e] \). The proof of the following result is included in Appendix B.

**Lemma 9.** Consider any \( 0 < \delta < 1 \), and suppose that the event \( E \) defined in Eq. (48) holds. For any \( i \in [n_e] \) and any \( j \in [p] \), it holds that \( \|\Theta_{\mathcal{G}_j}(i) - \Theta_{\mathcal{G}_j}\| \leq \sqrt{\varepsilon_0^2 / t} \), where \( \Theta_{\mathcal{G}_j} = [\mathcal{A} \quad \mathcal{B}_{\mathcal{G}_j}] \) and \( \varepsilon_0 \) is defined in (42).

### 5.1.1 Upper bound on \( R_1 \)

First, from the definition of Algorithm 1, we know that \( R_1 \) satisfies that

\[
R_1 = \sum_{t \in \tilde{T}} J_i(S_t, u_{S_t}^{(t)}) - \sum_{t \in \tilde{T}} J_i(S^*_t)
\]

\[
\leq \sum_{t \in \tilde{T}} J_i(S_t, u_{S_t}^{(t)}).
\]

Considering any \( t \in \tilde{T} \) and noting lines 8-10 in Algorithm 1, we have from Eqs. (8) and (45) that

\[
J_i(S_t, u_{S_t}^{(t)}) = \left( \sum_{k=0}^{N-1} x_k^{(t)\top} Q^{(i)} x_k^{(t)} + u_{k,S_t}^{(t)\top} R_{S_t}^{(i)} u_{k,S_t}^{(t)} \right) + x_N^{(t)\top} Q_{N\top}^{(i)} x_N^{(t)}
\]

where \( K_{\mathcal{G}_j} \) is provided by Assumption 4. Thus, we have

\[
R_1 \leq \max\{\sigma_Q, \sigma_R\} \left( \sum_{t \in \tilde{T}} \sum_{k=0}^{N-1} x_k^{(t)\top} x_k^{(t)} + u_{k,S_t}^{(t)\top} u_{k,S_t}^{(t)} \right) + x_N^{(t)\top} x_N^{(t)}
\]

where \( \sigma_Q, \sigma_R \in \mathbb{R}_{\geq 1} \) are defined in (23). Recall that \( u_{k,\mathcal{G}_j}^{(t)} = K_{\mathcal{G}_j} x_k^{(t)} + \hat{w}_k^{(t)} \) for all \( k \in \{0, \ldots, N-1\} \), where \( \hat{w}_k^{(t)} \) i.i.d. \( \mathcal{N}(0, 2\sigma^2 \kappa^2 I) \) with \( \kappa \) defined in Eq. (40). It follows that under the event \( E \) defined in Eq. (48),

\[
\|u_{k,\mathcal{G}_j}^{(t)}\| \leq \kappa \|x_k^{(t)}\| + \kappa \sigma \sqrt{10m \log \frac{8NT}{\delta}}.
\]
which implies that
\[ \|x_k^{(t)}\|^2 \leq 2\kappa^2\|x_k^{(t)}\|^2 + 20\kappa^2\sigma^2m \log \frac{8NT}{\delta}. \]

Moreover, we see from (46) and the definition of Algorithm 1 that \( |\bar{T}| = \eta_e\tau_1p \leq \sqrt{T} + \tau_1p \). Now, leveraging the upper bound on \( \|x_k^{(t)}\| \) for all \( t \in \mathcal{T} \) and all \( k \in \{0, \ldots, N\} \) given by (74) in the proof of Lemma 9, one can show via the above arguments that under the event \( \mathcal{E} \),
\[ R_1 \leq \max\{\sigma_Q, \sigma_R\} \frac{(\sqrt{T} + \tau_1p)(2\kappa^2 + 1)\epsilon^2_0}{(1 - \eta_0)^2} \frac{(20(\bar{v}^2 + 1)\kappa^2\sigma^2m + 10\sigma^2n) \log \frac{8NT}{\delta}}{\delta}, \]
where \( \eta_0, \zeta_0 \) are provided in Assumption 4, and \( \bar{v} \) is defined in (42).

5.1.2 Upper bound on \( R_2 \)

Consider any \( t \in \mathcal{T} \) and any \( S \subseteq \mathcal{G} \) with \( |S| = H \). Noting lines 14-16 in Algorithm 1, one can show that the state of the system in Eq. (7) corresponding to \( S \) satisfies that
\[ x_k^{(t)} = \sum_{i=0}^{k} \hat{\psi}_{k,i}^{(t)}(S)w_i^{(t)}, \]
where \( x_0^{(t)} = 0 \), and \( \hat{\psi}_{k,i}^{(t)}(S) \) is defined in Eq. (33). Moreover, supposing that the event \( \mathcal{E} \) holds, we know from Lemma 9 that \( \|\hat{\Theta}_{G,i} - \Theta_{G,i}\| \leq \sqrt{\frac{2}{\eta_0}} \epsilon_0 \) for all \( i \in [n_e] \) and all \( j \in [p] \). It follows that \( \hat{A}(i) \) and \( \hat{B}(i) \) obtained in line 11 of Algorithm 1 satisfy that \( \|\hat{A}(i) - A\| \leq \epsilon_0 \) and \( \|\hat{B}(i) - B\| \leq \epsilon_0 \) for all \( i \in [n_e] \), which also implies that \( \|\hat{B}_S(i) - B_S\| \leq \epsilon_0 \) for all \( i \in [n_e] \), where \( B_S(i) \) contains the columns of \( B(i) \) that correspond to \( S \). Now, one can obtain from the choice of \( \epsilon_0 \) in (42) and Proposition 1 that
\[ \|\hat{K}_{k,S}^{(t)} - K_{k,S}^{(t)}\| \leq \frac{1 - \eta S}{2\|B_S\|\zeta S} \forall k \in \{0, \ldots, N - 1\}, \]
which also implies that
\[ \|\hat{K}_{k,S}^{(t)}\| \leq \hat{k} \forall k \in \{0, \ldots, N - 1\}, \]
where \( \hat{k} \) is defined in (42), and \( \hat{K}_{k,S}^{(t)} \) and \( K_{k,S}^{(t)} \) are given by Eqs. (16) and (10), respectively. We then have from Lemma 6 that
\[ \|\hat{\psi}_{k_2,k_1}^{(t)}(S)\| \leq \zeta_S \frac{1 + \eta S}{2} k_2 - k_1, \]
for all \( k_1, k_2 \in \{0, \ldots, N - 1\} \) with \( k_2 \geq k_1 \), where \( \zeta_S, \eta_S \) are provided in Lemma 5 with \( 0 < \frac{1 + \eta S}{2} < 1 \). One can now use similar arguments to those for [9, Lemma 38] and show that
\[ \|x_k^{(t)}\| \leq \frac{2\zeta_S}{1 - \eta S} \max_{i \in \mathcal{T}} \max_{k \in \{0, \ldots, N - 1\}} u_k^{(t)}. \]
Thus, under the event \( \mathcal{E} \) defined in Eq. (48), we have that
\[ \|x_k^{(t)}\| \leq \frac{2\zeta\sigma}{1 - \eta} \sqrt{5m \log \frac{8NT}{\delta}}, \]
for all \( t \in \mathcal{T} \) and all \( k \in \{0, \ldots, N\} \), where \( \zeta, \eta \) are defined in (42). Furthermore, we recall from Eq. (8) that
\[ J_k(S, u_S^{(t)}) = \left( \sum_{k=0}^{N-1} x_k^{(t)}Q^{(t)}x_k^{(t)} + u_{k,S}^{(t)}R_S^{(t)}u_{k,S}^{(t)} \right) + x_N^{(t)}Q_f^{(t)}x_N^{(t)} \]
\[ = \left( \sum_{k=0}^{N-1} x_k^{(t)^T} (Q^{(t)} + \hat{K}_{k,S}^{(t)^T}R_S^{(t)}\hat{K}_{k,S}^{(t)})x_k^{(t)} \right) + x_N^{(t)^T}Q_f^{(t)}x_N^{(t)}, \]
where we use the fact that $u^{(t)}_{k,s} = \hat{K}^{(t)}_{k,s} x_k^{(t)}$ in line 16 in Algorithm 1. It then follows from our above arguments that under the event $\mathcal{E}$,

$J_t(S, u^{(t)}_S) \leq N(\sigma_Q + \hat{\kappa}^2 \sigma_R R) \frac{4C^2 \sigma^2}{(1 - \eta)^2} 5n \log \frac{8NT}{\delta}$.  

(51)

To proceed, recall that we use the Exp3.S algorithm in Algorithm 1 to select $S_t$ for all $t \in T$. As we argued in Section 4.1, each action in the Exp3.S algorithm corresponds to a set $S_t \subseteq G$ with $|G| = H$, and the set of possible actions $Q$ in the Exp3.S algorithm contain $(\binom{H}{T})$ actions (i.e., $|Q| = \binom{H}{T}$). Moreover, the cost of the action corresponding to $S_t$ in round $t \in [T]$ is given by $J_t(S, u^{(t)}_S)$. Thus, we can replace $y_h$ in (37) with the upper bound in (51) and obtain that under the event $\mathcal{E}$,

$R_2 = \mathbb{E}_A \left[ \sum_{t \in T} J_t(S_t, u^{(t)}_S) - \sum_{t \in T} J_t(S_t^*, u^{(t)}_{S_t^*}) \right] \leq N(\sigma_Q + \hat{\kappa}^2 \sigma_R R) \frac{4C^2 \sigma^2}{(1 - \eta)^2} 5n \log \frac{8NT}{\delta} \left( h(S^*) \sqrt{|Q| T \log(|Q|T)} + 2e \sqrt{\frac{|Q|T}{\log(|Q|T)}} \right)$, 

(52)

with

$h(S^*) = 1 + \{1 \leq t < T : S_t^* \neq S_{t+1}^*\}$, 

(53)

where recall that $(S_1^*, \ldots, S_T^*)$ is an optimal solution to Problem (13).

5.1.3 Upper bound on $R_3$

First, consider any $t \in T$. Similarly to our arguments in Section 5.1.2, we see that $J_t(S_t^*, u^{(t)}_{S_t^*})$ is given by

$J_t(S_t^*, u^{(t)}_{S_t^*}) = \left( \sum_{k=0}^{N-1} x_k^{(t)T} (Q^{(t)} + \hat{K}^{(t)}_{k,s} R^{(t)}_S(x_k^{(t)} - u_k^{(t)})) + x_N^{(t)T} Q_f x_N \right) = \left( \sum_{k=0}^{N-1} x_k^{(t)T} \tilde{P}^{(t)}_{k,s} x_k^{(t)} - x_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} x_{k+1}^{(t)} + 2w_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} (A + B_S) x_k^{(t)} \right)$

$+ \left( \sum_{k=0}^{N-1} \tilde{P}^{(t)}_{k+1,s} w_k^{(t)T} x_N \right) + x_N^{(t)T} Q_f x_N$, 

where we note that $\tilde{P}^{(t)}_{N,s} = Q_f^{(t)}$. Recalling the definition of $R_3$, we see that

$R_3 = \left( \sum_{t \in T} \left( \sum_{k=0}^{N-1} 2w_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} (A + B_S) x_k^{(t)} + w_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} w_k^{(t)} - J_t(S_t^*) \right) \right)$

$= \left( \sum_{t \in T} \left( \sum_{k=0}^{N-1} 2w_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} (A + B_S) x_k^{(t)} + w_k^{(t)T} \tilde{P}^{(t)}_{k+1,s} w_k^{(t)} - \sigma^2 \text{Tr}(\tilde{P}^{(t)}_{k+1,s})) \right) \right)$.  

(54)

Now, for any $t \in T$, one can apply Eq. (20) recursively to show that

$\tilde{P}^{(t)}_{k,s} = \left( \sum_{i=k}^{N-1} \tilde{\Psi}^{(t)}_{i,k} (S_i^*) (Q^{(t)} + \hat{K}^{(t)}_{i,s} R^{(t)}_S (x_i^{(t)} - u_i^{(t)})) + \tilde{\Psi}^{(t)}_{N,k} (S_N^*) \tilde{P}^{(t)}_{N,s} (S_N^*) \right) + \tilde{\Psi}^{(t)}_{N,k} (S_N^*) \tilde{P}^{(t)}_{N,s} (S_N^*)$, 

16
for all $k \in \{0, \ldots, N - 1\}$, where $\hat{\Psi}^{(t)}_{l,k} (S^t_l)$ is defined in Eq. (33). Next, suppose that the event $\mathcal{E}$ defined in Eq. (48) holds. Similarly to our arguments in Section 5.1.2, we know that with the choice of $\varepsilon_0$ in (42),

$$
\| \hat{K}^{(t)}_{k,S^t_l} \| \leq \hat{\kappa} \quad \forall k \in \{0, \ldots, N - 1\},
$$

where $\hat{\kappa}$ is defined in (42), and $\hat{K}^{(t)}_{k,S}$ is given by Eq. (16). We also have from Lemma 6 that

$$
\| \hat{\Psi}^{(t)}_{k_2,k_1} (S^t_l) \| \leq \zeta S^t_l \left( \frac{1 + \eta S^t_l}{2} \right)^{k_2 - k_1},
$$

for all $k_1, k_2 \in \{0, \ldots, N - 1\}$ with $k_2 \geq k_1$, where $\zeta S^t_l, \eta S^t_l$ are provided in Lemma 5 with $0 < \frac{1 + \eta S^t_l}{2}$ < 1. For any $k \in \{0, \ldots, N - 1\}$, one can then show that

$$
\| \hat{P}^{(t)}_{k,S^t_l} \| \leq (\sigma_Q + \sigma R \hat{\kappa})^2 \zeta S^t_l \sum_{i=0}^{N-k-1} \left( \frac{1 + \eta S^t_l}{2} \right)^{2i}
\leq (\sigma_Q + \sigma R \hat{\kappa})^2 \zeta S^t_l \frac{4 \zeta^2}{1 - \eta S^t_l}
\leq (\sigma_Q + \sigma R \hat{\kappa})^2 \frac{4 \zeta^2}{1 - \eta^2},
$$

where $\sigma_Q, \sigma_R$ are defined in (23), and $\zeta, \eta$ are defined in (42). Furthermore, we recall from our arguments in Section 5.1.2 that under the event $\mathcal{E}$ defined in Eq. (48),

$$
\| x_k \| \leq \frac{2 \zeta \sigma}{1 - \eta} \sqrt{\frac{8NT}{5n \log \frac{8NT}{\delta}}},
$$

for all $k \in \{0, \ldots, N - 1\}$.

To proceed, let us denote

$$
V^{(t)}_{k,S^t_l} = \hat{P}^{(t)}_{k,k+1,S^t_l} (A + B S^t_l) x_k^{(t)}.
$$

From our arguments above, we see that under $\mathcal{E}$,

$$
\| V^{(t)}_{k,S^t_l} \| \leq \| \hat{P}^{(t)}_{k+1,S^t_l} \| \| A + B S^t_l \| \| x_k^{(t)} \|
\leq \frac{16 \sigma Q(R \hat{\kappa})^2 \theta^2 \zeta^3}{(1 - \eta^2)(1 - \eta)} \sqrt{\frac{5n \log \frac{8NT}{\delta}}{5n \log \frac{8NT}{\delta}}},
$$

for all $k \in \{0, \ldots, N - 1\}$ and all $t \in T$, where $\theta$ is defined in (42), which implies that

$$
\sum_{t \in T} \sum_{k=0}^{N-1} \| V^{(t)}_{k,S^t_l} \|^2 \leq NT \frac{256 \sigma^2 Q(R \hat{\kappa})^2 \theta^2 \zeta^6}{(1 - \eta^2)^2 (1 - \eta)^2} \sqrt{5n \log \frac{8NT}{\delta}}.
$$

Noting from Assumption 3 that $w_k^{(t)} \overset{i.i.d.}{\sim} N(0, \sigma^2 I)$ for all $k \in \{0, \ldots, N - 1\}$ and for all $t \in [T]$, one can now apply [10, Lemma 30] and obtain that under the event $\mathcal{E}$, the following holds with probability at least $1 - \delta/4$:

$$
2 \sum_{t \in T} \sum_{k=0}^{N-1} w_k^{(t)} V^{(t)}_{k,S^t_l} \leq 64 \sqrt{NT} \frac{2 \sigma^2 Q(R \hat{\kappa})^2 \theta^2 \zeta^6}{(1 - \eta^2)(1 - \eta)} \sqrt{5n \log \frac{8NT}{\delta}}. \tag{55}
$$

Moreover, based on our arguments above, one can apply [10, Lemma 31] and obtain that under the event $\mathcal{E}$, the following holds with probability at least $1 - \delta/4$:

$$
\sum_{t \in T} \left( \sum_{k=0}^{N-1} w_k^{(t)} V^{(t)}_{k,S^t_l} \right)^2 \leq 8 \sigma Q(R \hat{\kappa})^2 \frac{4 \zeta^2}{1 - \eta^2} \sigma^2 \sqrt{NT \log \frac{16NT}{\delta}}. \tag{56}
$$

Recalling the decomposition of $R_3$ in (54), we can apply (55)-(56) together with a union bound and obtain an upper bound on $R_3$ that holds with probability at least $1 - \delta/2$ under the event $\mathcal{E}$. 

17
5.1.4 Upper bound on $R_4$

In order to provide an upper bound on $R_4$, we use the result in Proposition 2. Specifically, as we argued in Section 5.1.1, under the event $\mathcal{E}$ defined in Eq. (48), $A(i)$ and $B(i)$ obtained in line 11 of Algorithm 1 satisfy that $\|A(i) - A\| \leq \varepsilon_0/\sqrt{i}$ and $\|B(i) - B\| \leq \varepsilon_0/\sqrt{i}$ for all $i \in [n_e]$, which also implies that $\|B_S(i) - B_S\| \leq \varepsilon_0/\sqrt{i}$ for all $i \in [n_e]$ and all $S \subseteq G$ with $|G| = H$, where $B_S(i)$ contains the columns of $B(i)$ that correspond to $S$. Supposing that $\mathcal{E}$ holds, for any $t \in T$, we then have from the choice of $\varepsilon_0$ in (42) and Proposition 2 that

$$\hat{J}_t(S^*_i) - J_t(S^*_i) \leq \hat{D}_S \varepsilon_0^2 i,$$

where $\hat{D}_S$ encapsulates the factors on the right hand side of (35) before $\varepsilon$. Recalling that we have assumed that $n_e(\tau_1 p + \tau_2) = T$ (see footnote 6) in Algorithm 1, we see from (46) that $|T| = T - n_e \tau_1 p = n_e \tau_2$. Thus, from the definition of Algorithm 1, one can show that under the event $\mathcal{E}$,

$$R_4 = \sum_{t \in T} (\hat{J}_t(S^*_i) - J_t(S^*_i)),
\leq \sum_{j=1}^{n_e} \hat{D}_j \varepsilon_0^2 = \tilde{D}_2 \varepsilon_0^2 \sum_{j=1}^{n_e} \frac{1}{j},$$

where $\tilde{D} \triangleq \max_{S \subseteq G, |S| = H} \hat{D}_S$. Since $\sum_{j=1}^{n_e} 1/j \leq 1 + \log n_e$, we obtain that under the event $\mathcal{E}$,

$$R_4 \leq \tilde{D}_2 \varepsilon_0^2 (1 + \log n_e). \tag{57}$$

5.1.5 Upper bound on $R_A$

Finally, we combine the upper bounds on $R_1, R_2, R_3$, and $R_4$ together. Specifically, we have provided upper bounds on $R_1, R_2$, and $R_4$ that hold under the event $\mathcal{E}$ defined in Eq. (48). Moreover, we have provided an upper bound on $R_3$ that hold with probability at least $1 - \delta/2$, under the event $\mathcal{E}$. Since $\mathbb{P}(\mathcal{E}) \geq 1 - \delta/2$ from Lemma 8, we can further apply a union bound and obtain an upper bound on $R_A$ that holds with probability at least $1 - \delta$. Now, from the choice of the parameter $\tau_i$ given by Eq. (43) and the definitions of $n_e, \tau_2$ in Algorithm 1, we see that $\tau_1 = \mathcal{O}(\log T/\delta)$, $n_e = \mathcal{O}(\sqrt{T})$, and $\tau_2 = \mathcal{O}(\sqrt{T})$, where $\mathcal{O}(\cdot)$ hides factors that are polynomial in $\log(T/\delta)$. It then follows from (49), (52), (55)-(56), and (57) that $R_A = \mathcal{O}(\sqrt{T})$ holds with probability at least $1 - \delta$.

5.2 Discussions

Several remarks pertaining to $R_A$ given by Theorem 1 are in order. First, we know from the proof of Theorem 1 that $R_A$ also contains the factors $|Q| = (\mathcal{H})^H$ and $h(S^*)$, where $h(S^*)$ is defined in Eq. (53). In general, the factor $|Q|$ comes from the combinatorial nature of the actuator selection problem. Noting that $|Q| \leq |G|^H$, we see that $|Q|$ will be polynomial in $|G|$ if $H$ is a fixed constant. In other words, $|Q|$ will not be a bottleneck in $R_A$ when the number of actuators that is allowed for each round $t \in [T]$ is small compared to the number of all the candidate actuators. Moreover, supposing that $Q^{(t)} = Q$, $Q^{(t)} = Q_f$, and $R^{(t)} = R$ for all $t \in [T]$, we see from (13) that $S^*_1 = \cdots = S^*_e$, which implies via Eq. (38) that $h(S^*) = 1$.

Second, we note from the proof of Theorem 1 that $R_A$ contains factors that are polynomial in the problem parameters, including $n, m, d, \kappa, \kappa, \sigma, \sigma_R, \sigma_Q, \Gamma, S, N$. Moreover, one can show that the requirement on $T$, i.e., $T > \tau_1 p$, is equivalent to that $T$ is greater than a factor that is polynomial in the problem parameters. Such polynomial factors are scalable in the sense that when the problem parameter, e.g., the system dimension $n$ or the horizon length $N$, scales large, the factors do not grow exponentially in the problem parameter. (e.g., [23]).

Finally, note that the regret bound $R_A = \mathcal{O}(\sqrt{T})$ is sublinear in $T$. When other problem parameters are fixed, we have that $R_A / T \to 0$ as $T \to \infty$. In other words, the regret per round of Algorithm 1 tends to 0, as the number of rounds increases.
5.3 Simulation Results

In this subsection, we provide simulation results to validate the theoretical results in Theorem 1. Specifically, we generate random matrices $A$ and $B$ that satisfy Assumption 2, and we set the cost matrices to be $Q^{(t)} = I$, $Q_f^{(t)} = 2I$ and $R^{(t)} = I$ for all $t \in [T]$. We set the covariance matrix of the disturbance $w_k^{(t)}$ to be $W = I$ for all $k \in \{0, \ldots, N-1\}$ and all $t \in [T]$, and set the number of time steps in each round $t \in [T]$ to be $N = 5$. Now, we obtain the regret per round of Algorithm 1, i.e., $R_A/T$, for different values of $T$, where we set $\tau_1$ in Algorithm 1 as $\tau_1 = O(\log T)$. Note that for a given value of $T$, we obtain the averaged $R_A/T$ over 10 experiments. We see from Fig. 1 that $R_A/T$ decreases and tends to 0 as $T$ increases, which matches with our discussions in Section 5.2.

![Figure 1: Regret per round of Algorithm 1 vs. $T$](image)

6 Conclusion

We formulated an online actuator selection and controller design problem for linear quadratic regulation over a finite horizon, when the system matrices are unknown a priori. We proposed an online actuator selection algorithm to solve the problem which specifies the set of selected actuators under a budget constraint and determines the control policy corresponding to the set of selected actuators. The proposed algorithm is a model based learning algorithm which maintains estimates of the system matrices obtained from the system trajectories. The algorithm leverages an algorithm for the multiarmed bandit problem to select the set of selected actuators and determines the corresponding control policy based on estimated system matrices. We showed that the proposed online actuator selection algorithm yields a sublinear regret.

References

[1] Y. Abbasi-Yadkori and C. Szepesvári. Regret bounds for the adaptive control of linear quadratic systems. In Proc. Conference on Learning Theory, pages 1–26, 2011.

[2] B. D. Anderson and J. B. Moore. *Optimal control: linear quadratic methods*. Courier Corporation, 2007.

[3] B. D. Anderson and J. B. Moore. *Optimal filtering*. Courier Corporation, 2012.

[4] K. J. Åström and B. Wittenmark. *Adaptive control*. Courier Corporation, 2013.

[5] P. Auer, N. Cesa-Bianchi, Y. Freund, and R. E. Schapire. The nonstochastic multiarmed bandit problem. SIAM Journal on Computing, 32(1):48–77, 2002.

[6] D. P. Bertsekas. *Dynamic programming and optimal control: Vol. 1 4th Edition*. Athena Scientific, 2017.

[7] A. A. Bian, J. M. Buhmann, A. Krause, and S. Tschatschek. Guarantees for greedy maximization of non-submodular functions with applications. In Proc. International Conference on Machine Learning, pages 498–507, 2017.

[8] S. Bubeck. Introduction to online optimization. *Lecture Notes*, 2011.
[9] A. Cassel, A. Cohen, and T. Koren. Logarithmic regret for learning linear quadratic regulators efficiently. In Proc. International Conference on Machine Learning, pages 1328–1337, 2020.

[10] A. Cohen, T. Koren, and Y. Mansour. Learning linear-quadratic regulators efficiently with only $\sqrt{T}$ regret. In Proc. International Conference on Machine Learning, pages 1300–1309, 2019.

[11] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu. Regret bounds for robust adaptive control of the linear quadratic regulator. arXiv preprint arXiv:1805.09388, 2018.

[12] S. Dean, H. Mania, N. Matni, B. Recht, and S. Tu. On the sample complexity of the linear quadratic regulator. Foundations of Computational Mathematics, 20(4):633–679, 2020.

[13] L. El Ghaoui. Inversion error, condition number, and approximate inverses of uncertain matrices. Linear algebra and its applications, 343:171–193, 2002.

[14] M. Fazel, R. Ge, S. Kakade, and M. Mesbahi. Global convergence of policy gradient methods for the linear quadratic regulator. In Proc. International Conference on Machine Learning, pages 1467–1476, 2018.

[15] F. Fotiadis and K. G. Vamvoudakis. Learning-based actuator placement for uncertain systems. In Proc. IEEE Conference on Decision and Control, pages 90–95, 2021.

[16] D. Golovin, M. Faulkner, and A. Krause. Online distributed sensor selection. In Proc. ACM/IEEE International Conference on Information Processing in Sensor Networks, pages 220–231, 2010.

[17] B. Gravell, P. M. Esfahani, and T. Summers. Learning optimal controllers for linear systems with multiplicative noise via policy gradient. IEEE Transactions on Automatic Control, 66(11):5283–5298, 2021.

[18] B. Guo, O. Karaca, T. Summers, and M. Kamgarpour. Actuator placement under structural controllability using forward and reverse greedy algorithms. IEEE Transactions on Automatic Control, 66(12):5845–5860, 2021.

[19] V. Gupta, T. H. Chung, B. Hassibi, and R. M. Murray. On a stochastic sensor selection algorithm with applications in sensor scheduling and sensor coverage. Automatica, 42(2):251–260, 2006.

[20] E. Hazan. Introduction to online convex optimization. Foundations and Trends in Optimization, 2(3-4):157–325, 2016.

[21] R. A. Horn and C. R. Johnson. Matrix analysis. Cambridge University Press, 2012.

[22] Z.-S. Hou and Z. Wang. From model-based control to data-driven control: Survey, classification and perspective. Information Sciences, 235:3–35, 2013.

[23] M. Ibrahimi, A. Javanmard, and B. V. Roy. Efficient reinforcement learning for high dimensional linear quadratic systems. In Proc. International Conference on Neural Information Processing Systems, pages 2636–2644, 2012.

[24] C. Jin, Z. Yang, Z. Wang, and M. I. Jordan. Provably efficient reinforcement learning with linear function approximation. In Proc. Conference on Learning Theory, pages 2137–2143, 2020.

[25] H. Mania, S. Tu, and B. Recht. Certainty equivalence is efficient for linear quadratic control. arXiv preprint arXiv:1902.07826, 2019.

[26] H. Mohammadi, A. Zare, M. Soltanolkotabi, and M. R. Jovanovic. Convergence and sample complexity of gradient methods for the model-free linear quadratic regulator problem. IEEE Transactions on Automatic Control, 2021.

[27] G. L. Nemhauser, L. A. Wolsey, and M. L. Fisher. An analysis of approximations for maximizing submodular set functions. Mathematical Programming, 14(1):265–294, 1978.
A Proofs pertaining to the certainty equivalence approach

A.1 Proof of Lemma 1

In this proof, we drop the dependency of various terms on $S$ and $t$ for notational simplicity, while the proof holds for any $S \subseteq \mathcal{G}$ and any $t \in [T]$. For example, we write $P_{k,S}^{(t)}$ as $P_k$, and write $B_S$ as $B$. To prove (24), we first note that

$$
\|B^T P_k B - \hat{B}^T \hat{P}_k \hat{B}\| \leq \|B^T P_k B - B^T P_k \hat{B}\| + \|B^T P_k \hat{B} - B^T \hat{P}_k \hat{B}\| + \|B^T \hat{P}_k \hat{B} - \hat{B}^T \hat{P}_k \hat{B}\|,$$

which implies that

$$
\|B^T P_k B - \hat{B}^T \hat{P}_k \hat{B}\|
\leq \|\hat{B}\|\|P_k\|\|\hat{B}\| + \|\hat{B}\|\|\hat{B}\|\|D\|\|\hat{P}_k\|\|\hat{B}\|
\leq (\Gamma + \epsilon)(\Gamma + D\epsilon)\epsilon + (\Gamma + \epsilon)\Gamma D\epsilon + \Gamma^2 \epsilon
\leq \hat{\Gamma}^2 \epsilon + \hat{\Gamma}^2 D\epsilon + \Gamma^2 \epsilon \leq 3\hat{\Gamma}^2 D\epsilon,
$$

(58)

where $\Gamma$ (i.e., $\Gamma_S$) is defined in Eq. (21). The first inequality in (58) uses the fact that $\epsilon \leq D\epsilon \leq 1$, and the second inequality in (58) uses the facts that $\hat{\Gamma} \geq 1$ and $D \geq 1$. Note that $\sigma_n(R^{(t)}) \geq 1$ from Assumption 1. Also recalling the definitions of $K_{k-1,S}^{(t)}$ and $\hat{K}_{k-1,S}^{(t)}$ in Eqs. (10) and (16), respectively, the rest of the proof for (24) now follows from similar arguments to those for [25, Lemma 2].
To prove Eq. (25), one can first use Eq. (16) to rewrite Eq. (17) as
\[
\hat{P}_{k-1} = Q + K_{k-1}^T R_k k_{k-1} + (\hat{A} + \hat{B}K_{k-1})^T \hat{P}_k (\hat{A} + \hat{B}K_{k-1}).
\]
Similarly, one can obtain from Eqs. (10)-(11) the following:
\[
P_{k-1} = Q + K_{k-1}^T R_k k_{k-1} + (A + BK_{k-1})^T P_k (A + BK_{k-1}).
\]
Now, using similar arguments to those above for (58), one can show via (24) that
\[
\|A + BK_{k-1} - \hat{A} - \hat{B}K_{k-1}\| \\
\leq \|A - \hat{A}\| + \|BK_{k-1} - \hat{B}K_{k-1}\| + \|B\hat{K}_{k-1} - \hat{B}K_{k-1}\| \\
\leq \varepsilon + \Gamma \varepsilon + (\Gamma + \varepsilon)3\Gamma^3 D\varepsilon \leq 4\Gamma^4 D\varepsilon.
\]
(59)
Denoting \(\hat{L}_{k-1} \triangleq \hat{A} + \hat{B}K_{k-1}\) and \(L_{k-1} = A + BK_{k-1}\), we have that
\[
\|P_{k-1} - \hat{P}_{k-1}\| \leq \|K_{k-1}^T R_k k_{k-1} - \hat{K}_{k-1}^T R\hat{K}_{k-1}\| + \|L_{k-1}^T P_k L_{k-1} - \hat{L}_{k-1}^T \hat{P}_k \hat{L}_{k-1}\|.
\]
Similarly, one can show that
\[
\|K_{k-1}^T R_k k_{k-1} - \hat{K}_{k-1}^T R\hat{K}_{k-1}\| \\
\leq (\Gamma + 3\Gamma^3 D\varepsilon)\sigma_1(R)3\Gamma^3 D\varepsilon + 3\Gamma^3 D\varepsilon \sigma_1(R)\Gamma \varepsilon \\
\leq 3\Gamma^3 \sigma_1(R)\varepsilon (2\Gamma + 3\Gamma^3 L) \\
\leq 6\Gamma^4 \sigma_1(R)D\varepsilon + 9\Gamma^6 \sigma_1(R)D^2 \varepsilon^2.
\]
(60)
Let us also denote \(\Delta L_{k-1} \triangleq L_{k-1} - \hat{L}_{k-1}\). Noting that \(\|A + BK_{k-1}\| \leq \hat{\Gamma}^2\) and recalling (59), one can show that
\[
\|L_{k-1}^T P_k L_{k-1} - \hat{L}_{k-1}^T \hat{P}_k \hat{L}_{k-1}\| \\
\leq (\|\Delta L_{k-1}\| + \hat{\Gamma}^2)\varepsilon (\varepsilon + \varepsilon^2) + (\|\Delta L_{k-1}\| + \hat{\Gamma}^2)\varepsilon + \|\Delta L_{k-1}\| \varepsilon \hat{\Gamma}^2 \\
\leq 16\hat{\Gamma}^6 D^2 \varepsilon^2 + 4\hat{\Gamma}^7 D\varepsilon + 4\hat{\Gamma}^6 D\varepsilon^2 + 4\hat{\Gamma}^4 D\varepsilon + 4\hat{\Gamma}^6 D\varepsilon,
\]
(61)
where we use the fact that \(D\varepsilon \leq 1\). The inequality in (25) now follows from combining (60) and (61), and noting the facts that \(D\varepsilon \leq 1/6\) and \(\sigma_1(R^{(t)}) \leq \sigma_R\).

A.2 Proof of Lemma 3

Our proof is based on a similar idea to that for the proof of [25, Proposition 3]. To simplify the notations in the proof, we assume that \(N = \varphi \ell\) for some \(\varphi \in \mathbb{Z}_{\geq 1}\); otherwise we only need to focus on the time steps from \(N - \varphi \ell\) to \(N\) of the LQR problem given in Eq. (2), where \(\varphi\) is the maximum positive integer such that \(N - \varphi \ell \geq 0\). Under the assumption that \(N = \varphi \ell\), we need to show that (26) holds for \(k \in \{0, \ell, \ldots, \varphi \ell\}\). Note that (26) holds for \(k = N\), since \(P_{N,S}^{(t)} = \hat{P}_{N,S}^{(t)} = Q_f^{(t)}\). In the rest of this proof, we again drop the dependency of various terms on \(S\) and \(t\) for notational simplicity, while the proof works for any \(S \subseteq \mathcal{G}\) (with \(|S| = H\)) and any \(t \in [T]\). First, for any \(\gamma \in \mathbb{Z}_{\geq 1}\) (with \(\gamma \ell \leq N\)), let us consider the noiseless LQR problem for the system given in Eq. (1), i.e., \(x_{k+1} = Ax_k + Bu_k\), from time step \(\gamma \ell\) to \(N\). Let the initial state \(x_{\gamma \ell}\) be any vector in \(\mathbb{R}^n\) with \(\|x_{\gamma \ell}\| = 1\). Similarly to Eq. (8), we define the following cost:
\[
\hat{J}(A, B, u_{\gamma \ell: N-1}) \triangleq \left(\sum_{j=\gamma \ell}^{\gamma \ell+\ell-1} \sum_{k=0}^{\ell} x_{j+1+k}^T Q x_{j+1+k} + u_{j+1+k}^T R u_{j+1+k}\right) + x_N^T Q_f x_N,
\]
where \(u_{\gamma \ell: N-1} = (u_{\gamma \ell}, \ldots, u_{N-1})\). Again, we know from, e.g., [6], that the minimum value of \(\hat{J}(A, B, u_{\gamma \ell: N-1})\) (over all control policies \(u_{\gamma \ell: N-1}\)) is achieved by \(\hat{u}_K = K_{k,S} x_k\) for all \(k \in \{\gamma \ell, \gamma \ell + 1, \ldots, N - 1\}\), where \(K_{k,S}\)
is given by Eq. (10). Moreover, we know that \( \hat{J}(A, B, \bar{u}_{\gamma,N-1}) = x_{\gamma_{\ell}}^T P_{\gamma_{\ell}} x_{\gamma_{\ell}} \), where \( P_{\gamma_{\ell}} \) can be obtained from Eq. (11) with \( P_N = Q_f \).

Next, consider another LTI system given by \( \hat{x}_{k+1} = \hat{A}_k \hat{x}_k + \hat{B}_k \bar{u}_k \) over the same time horizon and starting from the same initial state \( \hat{x}_{\gamma_{\ell}} = x_{\gamma_{\ell}} \) as we described above. Define the corresponding cost as

\[
J(\hat{A}, \hat{B}, \bar{u}_{\gamma,N-1}) = \left( \sum_{j=0}^{L-1} \sum_{k=0}^{t-1} \hat{x}_{j+k}^T Q \hat{x}_{j+k} + \bar{u}_{j+k}^T R \bar{u}_{j+k} \right) + \hat{x}_N^T Q_f \hat{x}_N,
\]

where \( \bar{u}_{\gamma,N-1} = (\bar{u}_{\gamma}, \ldots, \bar{u}_{N-1}) \). Similarly, the minimum value of \( J(\hat{A}, \hat{B}, \bar{u}_{\gamma,N-1}) \) (over all control policies \( \bar{u}_{\gamma,N-1} \)) is achieved by \( u_k^\ast = \hat{K}_k(s) \hat{x}_k \) for all \( k \in \{ \gamma, \gamma + 1, \ldots, N - 1 \} \), where \( \hat{K}_k(s) \) is given in Eq. (16).

The minimum cost is given by \( J(\hat{A}, \hat{B}, u'_{\gamma,N-1}) = x_{\gamma_{\ell}}^T \hat{P}_{\gamma_{\ell}} x_{\gamma_{\ell}} \), where \( \hat{P}_{\gamma_{\ell}} \) can be obtained from Eq. (17) with \( \hat{P}_N = Q_f \). Moreover, we note that

\[
J(\hat{A}, \hat{B}, u'_{\gamma,N-1}) \leq J(\hat{A}, \hat{B}, \bar{u}_{\gamma,N-1}),
\]

where \( \bar{u}_{\gamma,N-1} \) is an arbitrary control policy and the inequality follows from the optimality of \( u'_{\gamma,N-1} \). Recalling that \( \varepsilon \) is assumed to be small enough such that the right-hand side of (26) is smaller than or equal to 1, one can obtain from Lemma 2 that \( \sigma_n(\hat{C}, \bar{s}) \geq \frac{\sigma}{2} > 0 \), which implies that the pair \((\hat{A}, \hat{B})\) is controllable. Now, one can follow similar arguments to those for the proof of [25, Proposition 3] and show that \( \hat{u}_{\gamma,N-1} \) can be chosen such that \( \hat{x}_{\gamma_{\ell}} = \bar{x}_{\gamma_{\ell}} \) for all \( \gamma' \in \{ \gamma, \gamma + 1, \ldots, \phi \} \). It then follows from the above arguments that

\[
x_{\gamma_{\ell}}^T \hat{P}_{\gamma_{\ell}} x_{\gamma_{\ell}} \leq \left( \sum_{j=0}^{L-1} \sum_{k=0}^{t-1} \hat{x}_{j+k}^T Q \hat{x}_{j+k} + \bar{u}_{j+k}^T R \bar{u}_{j+k} \right) + \hat{x}_N^T Q_f \hat{x}_N \leq J(\hat{A}, \hat{B}, \bar{u}_{\gamma,N-1}),
\]

One can further follow similar arguments to those for the proof of [25, Proposition 3] and show that \( \bar{x}_{\gamma,N-1} \) in Eq. (62) can be chosen such that the following holds:

\[
x_{\gamma_{\ell}}^T \hat{P}_{\gamma_{\ell}} x_{\gamma_{\ell}} - x_{\gamma_{\ell}}^T P_{\gamma_{\ell}} x_{\gamma_{\ell}} \leq \frac{1}{2} \mu_{\gamma_{\ell}} \varepsilon,
\]

under the assumption that \( \frac{1}{2} \mu_{\gamma_{\ell}} \varepsilon \leq 1 \), where \( \mu_{\gamma_{\ell}} \) (i.e., \( \mu_{\gamma_{\ell}, S}^{(t)} \)) is defined in Eq. (27). Now, reversing the roles of \((A, B)\) and \((\hat{A}, \hat{B})\) in the arguments above, one can also obtain that

\[
x_{\gamma_{\ell}}^T P_{\gamma_{\ell}} x_{\gamma_{\ell}} - x_{\gamma_{\ell}}^T \hat{P}_{\gamma_{\ell}} x_{\gamma_{\ell}} \leq \frac{1}{2} \mu_{\gamma_{\ell}} \varepsilon,
\]

under the assumption that \( \frac{1}{2} \mu_{\gamma_{\ell}} \| \hat{P}_{\gamma_{\ell}} \| \varepsilon \leq 1 \).\(^7\) Note from Eq. (11) and Assumption 1 that \( P_{\gamma_{\ell}} \geq Q \geq I_n \), and note that (63) and (64) hold for any \( x_{\gamma_{\ell}} \in \mathbb{R}^n \) with \( \| x_{\gamma_{\ell}} \| \leq 1 \) as we discussed above. It then follows from (63) that \( \lambda_1(\hat{P}_{\gamma_{\ell}}) \leq \lambda_1(P_{\gamma_{\ell}}) + 1 \), i.e., \( \| \hat{P}_{\gamma_{\ell}} \| \leq \| P_{\gamma_{\ell}} \| + 1 \leq 2 \| P_{\gamma_{\ell}} \| \). Hence, we have from (63) and (64) that \( \lambda_1(\hat{P}_{\gamma_{\ell}} - P_{\gamma_{\ell}}) \leq \mu_{\gamma_{\ell}} \varepsilon \) and \( \lambda_1(P_{\gamma_{\ell}} - \hat{P}_{\gamma_{\ell}}) \leq \mu_{\gamma_{\ell}} \varepsilon \), which further implies (26).

---

\(^7\)Note that the proof technique in [25] is for the infinite-horizon (noiseless) LQR problem, which can be adapted to the finite-horizon setting studied here. The details of such an adaption are omitted for conciseness.
inequality again follows from the assumption on $\varepsilon$. Repeatedly applying (25) in Lemma 1, we obtain that 
\[ \| P^{(t)}_{\gamma_{t-j},S} - \hat{P}^{(t)}_{\gamma_{t-j},S} \| \leq (20\delta^3 \sigma^2) \| \mu_S \| \varepsilon \] for all $j \in [\ell - 1]$. Thus, we have shown that (29) also holds for all 
\[ k \in \{ \gamma_{t-j} \colon \gamma \in [\varepsilon], j \in [\ell - 1] \}. \]
Combining the above arguments together completes the proof of (29) for all 
\[ k \in \{ 0, \ldots, N \}. \]
The proof of (30) now follows from similar arguments to those for (24) in the proof of 
Lemma 1.

A.4 Proof of Lemma 4

For notational simplicity, we again drop the dependency of various terms on $t$ and $S$ in this proof. First, we let $J(x_k)$ be the cost of using the optimal control gain $K_k$ given in Eq. (10), starting from the state $x_k$, 
where $x_k$ (i.e., $x_k^{(t)}$) is the state at time step $k$ when the certainty equivalence control $u_{k'} = \hat{K}_{k'} x_{k'}$ is used 
for all $k' \in \{ 0, \ldots, k - 1 \}$. Therefore, we have from our discussions in Sections 2.1 that 
\[ J(x_k) = x_k^T P_k x_k + \sum_{i=k}^{N-1} P_{k+1} W, \] (65)
where $P_k$ (i.e., $P_{k,S}$) is given by Eq. (11) with $P_N = Q_f$. Since $x_0 = 0$ as we assumed, we have from Eqs. (12) and 
(65) that $J(x_0) = J_t(S)$. Denoting $c_k = x_k^T Q x_k + u_k^T R u_k$, where $u_k = \hat{K}_k x_k$, we can rewrite $\hat{J}_t(S)$ defined in 
Eq. (18) as 
\[ \hat{J}_t(S) = \mathbb{E} \left[ \sum_{k=0}^{N-1} c_k + x_N^T Q f x_N \right]. \]
It now follows that 
\[ \hat{J}_t(S) - J_t(S) = \mathbb{E} \left[ \sum_{k=0}^{N-1} c_k + J(x_k) - J(x_k) \right] + x_N^T Q f x_N - J(x_0) \]
\[ = \mathbb{E} \left[ \sum_{k=0}^{N-1} c_k + J(x_{k+1}) - J(x_k) \right]. \] (66)
To obtain Eq. (66), we use the telescope sum and note from Eq. (65) that $J(x_N) = x_N^T Q f x_N$. Next, considering a single term in the summation on the right hand side of Eq. (66), we have 
\[ \mathbb{E} \left[ c_k + J(x_{k+1}) - J(x_k) \right] = \mathbb{E} \left[ x_k^T (Q + \hat{K}_k^T R K_k) x_k + x_{k+1}^T P_{k+1} x_{k+1} - x_k^T P_k x_k - P_{k+1} W \right]. \]
Noting that $x_{k+1} = (A + B \hat{K}_k) x_k + w_k$ and recalling that $w_k$ is a zero-mean white Gaussian noise process, we then obtain that 
\[ \mathbb{E} \left[ c_k + J(x_{k+1}) - J(x_k) \right] = \mathbb{E} \left[ x_k^T (Q + \hat{K}_k^T R K_k) x_k + x_k^T ((A + B \hat{K}_k)^T P_{k+1} (A + B \hat{K}_k) - P_k) x_k \right]. \] (67)
Since $P_k$ satisfies the recursion in Eq. (11), one can use Eq. (10) and obtain 
\[ P_k = Q + K_k^T R K_k + (A + B K_k)^T P_{k+1} (A + B K_k). \]
Using similar arguments to those in the proof of [14, Lemma 10], one can now show that 
\[ \mathbb{E} \left[ c_k + J(x_{k+1}) - J(x_k) \right] = \mathbb{E} \left[ x_k^T \Delta K_k^T (R + B^T P_k B^T) \Delta K_k x_k \right], \] (68)
where $\Delta K_k = \hat{K}_k - K_k$. It then follows from Eqs. (66) and (68) that Eq. (31) holds, completing the proof. ■
A.5 Proof of Lemma 5

First, under Assumptions 1-2, we know from, e.g., [3, 6], that $K_{0,S}^{(t)} \to K_{S}^{(t)}$ as $N \to \infty$, where $K_{0,S}^{(t)}$ is given by Eq. (10) and $K_{S}^{(t)}$ is given by

$$K_{S}^{(t)} = -(B_{S}^{\top}P_{S}^{(t)}B_{S} + R_{S}^{(t)})^{-1}B_{S}^{\top}P_{S}^{(t)}A,$$

where $P_{S}^{(t)} \in \mathbb{S}_{++}$ satisfies the following Ricatti equation:

$$P_{S}^{(t)} = Q^{(t)} + A^{\top}P_{S}^{(t)}A - A^{\top}P_{S}^{(t)}B(SB_{S}^{\top}P_{S}^{(t)}B_{S} + R_{S}^{(t)})^{-1}B_{S}^{\top}P_{S}^{(t)}A.$$  \hspace{1cm} (70)

We also know from, e.g., [3, 6], that $\rho(A + B_{S}K_{S}^{(t)}) < 1$.

Thus, we see that given any $\delta \in \mathbb{R}_{>0}$, there exists a finite constant $N(\delta) \in \mathbb{Z}_{\geq 1}$ such that $\|K_{k,S}^{(t)} - K_{S}^{(t)}\| \leq \delta$ for all $0 \leq k \leq N - N(\delta)$. Denoting $\Delta K_{k,S}^{(t)} = K_{k,S}^{(t)} - K_{S}^{(t)}$, we have

$$A + B_{S}K_{k}^{(t)}(S) = L_{S} + B_{S}\Delta K_{k}^{(t)}(S),$$

where $L_{S} \triangleq A + B_{S}K_{S}^{(t)}$. Since $\rho(L_{S}) < 1$ as we argued above, we have from the Gelfand formula (e.g., [21]) that there exist finite constants $\zeta_{t,S} \geq 1$ and $0 < \eta_{t,S} < 1$ such that $\|(A + B_{S}K_{S}^{(t)})^{k}\|_{} \leq \zeta_{t,S}\eta_{t,S}^{k}$ for all $k \geq 0$.

Now, choosing $\delta = \frac{1-m_{S}}{2\|B_{S}\|_{\psi_{t,S}}}$, one can then show via our arguments above and Lemma 10 in Appendix C that

$$\|\Psi_{k_{2},k_{1}}^{(t)}(S)\| \leq \zeta_{t,S}(\frac{1+\eta_{t,S}}{2})^{k_{2}-k_{1}},$$

for all $k_{2} \leq N - N(\delta)$ and $k_{1} \leq N - N(\delta)$ with $k_{2} \geq k_{1}$. Next, let us define

$$\tilde{\zeta}_{t,S} = \max\left\{ \frac{\|\Psi_{j,i}^{(t)}(S)\|}{(1+\eta_{t,S})^{j-i}} : N - N(\delta) \leq i \leq j \leq N \right\}.$$

We then have that

$$\|\Psi_{k_{2},k_{1}}^{(t)}(S)\| \leq \max\{\tilde{\zeta}_{t,S}, \zeta_{t,S}\}(\frac{1+\eta_{t,S}}{2})^{k_{2}-k_{1}},$$

for all $k_{1}, k_{2} \in \{0, 1, \ldots, N\}$ with $k_{2} \geq k_{1}$. Note from Lemma 11 in Appendix C that $\|K_{k,S}^{(t)} - K_{S}^{(t)}\| \leq \tilde{\psi}_{S}\eta_{S}^{N-k}$ with $\tilde{\psi}_{S} \triangleq \psi_{S}\zeta_{S}(\Gamma_{S} + \|P_{S}^{(t)}\|\Gamma_{S}(1 + 2\|P_{S}^{(t)}\|\Gamma_{S}^{\top})$, where $\psi_{S}, \zeta_{S}, \eta_{S}$ are finite constants with $0 < \eta_{S} < 1$ and $\Gamma_{S}$ is defined in Eq. (21). One can then show that $N(\delta) = \frac{\log(\delta/\tilde{\psi}_{S})}{\log\eta_{S}}$, where $\delta = \frac{1-m_{S}}{2\|B_{S}\|_{\psi_{t,S}}}$. Since $\|A + B_{S}K_{S}^{(t)}\| \leq \tilde{\Gamma}_{S}^{2}$, where $\tilde{\Gamma}_{S}$ is defined in Eq. (22), we obtain that

$$\tilde{\zeta}_{t,S} \leq (\frac{2\tilde{\Gamma}_{S}^{2}}{1+\eta_{t,S}})^{\frac{\log(1/\tilde{\psi}_{S})}{\log\eta_{S}}},$$

which implies that $\tilde{\zeta}_{t,S}$ is a finite constant (that does not depend on $k_{1}, k_{2}$). Setting $\zeta_{S} = \max_{t \in [T]} \max\{\tilde{\zeta}_{t,S}, \zeta_{t,S}\}$ and $\eta_{S} = \max_{t \in [T]} \frac{1+\eta_{t,S}}{2}$, we complete the proof of the lemma.

A.6 Proof of Lemma 6

Recall the definition of $\hat{\Psi}_{k_{2},k_{1}}^{(t)}(S)$ (resp., $\hat{\Psi}_{k_{2},k_{1}}^{(t)}(S)$) in Eq. (32) (resp., Eq. (33)), and note that

$$A + B_{S}\hat{K}_{k,S}^{(t)} = A + B_{S}K_{k,S}^{(t)} + B_{S}(\hat{K}_{k,S}^{(t)} - K_{k,S}^{(t)}),$$

for all $k \in \{0, 1, \ldots, N - 1\}$. The proof now follows from Lemma 10 in Appendix C. \hfill \blacksquare
B Proofs pertaining to the regret bound

B.1 Proof of Lemma 8

First, we know from Assumption 3 that $w_k(t) \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2 I)$ for all $k \in \{0, \ldots, N - 1\}$ and for all $t \in [T]$. One can then apply [9, Lemma 34] and obtain that $\mathbb{P}(\mathcal{E}_w) \geq 1 - \delta/8$. Similarly, recalling that $w_k(t) \overset{i.i.d.}{\sim} \mathcal{N}(0, 2\sigma^2 \kappa^2 I)$ for all $k \in \{0, \ldots, N - 1\}$ and for all $t \in [T]$, one can apply [9, Lemma 34] and obtain that $\mathbb{P}(\mathcal{E}_w) \geq 1 - \delta/8$.

Next, for any $j \in [p]$, we have from Lemma 7 that with probability at least $1 - \delta/(8p)$,

$$\text{Tr}(\Delta_{\mathcal{G}_j}(i)\top V_{\mathcal{G}_j}(i)\Delta_{\mathcal{G}_j}(i)) \leq 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\Lambda)} \right) + 2\lambda \|\Theta_{\mathcal{G}_j}\|_F^2 \quad \forall i \in [n_e].$$

Applying a union bound over all $j \in [p]$, we obtain that $\mathbb{P}(\mathcal{E}_{\mathcal{G}_j}) \geq 1 - \delta/8$.

Finally, recalling Eq. (45), for any $i \in [n_e]$, any $j \in [p]$ and any $T_i, j \leq t \leq T_i, j + 1 - 1$, we denote a sigma field $\mathcal{F}_{k, j}^{(t)} = \sigma(x_0^{(t)}, u_0^{(t)}, \ldots, x_k^{(t)}, u_k^{(t)}_{\mathcal{G}_j})$ for all $k \in \{0, \ldots, N - 1\}$, where $u_k^{(t)}_{\mathcal{G}_j} = K_{\mathcal{G}_j} x_k^{(t)} + \tilde{w}_k^{(t)}$ with $\tilde{w}_k^{(t)} \overset{i.i.d.}{\sim} \mathcal{N}(0, 2\sigma^2 \kappa^2 I)$. Note that for any $k \in [N - 1]$, $z_k^{(t)} = [x_k^{(t)\top} u_k^{(t)\top}]\top$ is conditional Gaussian given $\mathcal{F}_{k-1, j}^{(t)}$. One can then use similar arguments to those for [10, Lemma 34] and show that

$$\mathbb{E}[z_k^{(t)}\top z_k^{(t)}\top | \mathcal{F}_{k-1, j}^{(t)}] \succeq \frac{\sigma^2}{2} I,$$

for all $k \in [N - 1]$. Now, noting from the choice of $\tau_1$ in Eq. (43) that $\tau_1 \geq 200(m + n) \log^{\frac{96n \log p}{\delta}}$, one can apply [9, Lemma 36] to show that for any $i \in [n_e]$ and any $j \in [p]$, the following holds with probability at least $1 - \delta/(8n_e p)$:

$$\sum_{l=1}^{T_i, j+1-1} \sum_{t=T_i, j}^{N-1} z_k^{(t)} z_k^{(t)\top} \geq \frac{(N - 1)\tau_1 i \sigma^2}{80} I.$$

Applying a union bound over all $i \in [n_e]$ and all $j \in [p]$ yields that $\mathbb{P}(\mathcal{E}_z) \geq 1 - \delta/8$.

Combining the above arguments together and applying a union bound over $\mathcal{E}_w$, $\mathcal{E}_{\mathcal{G}_j}$, $\mathcal{E}_{\Theta}$, and $\mathcal{E}_z$, we complete the proof of the lemma.

B.2 Proof of Lemma 9

Consider any $i \in [n_e]$ and any $j \in [p]$. First, under $\mathcal{E}_{\Theta}$, we have

$$\text{Tr}(\Delta_{\mathcal{G}_j}(i)\top V_{\mathcal{G}_j}(i)\Delta_{\mathcal{G}_j}(i)) \leq 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\Lambda)} \right) + 2\lambda \|\Theta_{\mathcal{G}_j}\|_F^2$$

$$\leq 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\Lambda)} \right) + 2\lambda \min\{n, n + m_{\mathcal{G}_j}\} \|\Theta_{\mathcal{G}_j}\|_F^2$$

$$\leq 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\Lambda)} \right) + 2\lambda n \theta^2,$$

where $\Delta_{\mathcal{G}_j}(i) = \hat{\Theta}_{\mathcal{G}_j}(i) - \Theta_{\mathcal{G}_j}$, $m_{\mathcal{G}_j} = \sum_{i \in \mathcal{G}_j} m_i$ and $\theta$ is defined in (42). Next, under $\mathcal{E}_z$, we have

$$V_{\mathcal{G}_j}(i) = \Lambda + \sum_{l=1}^{T_i, j+1-1} \sum_{t=T_i, j}^{N-1} z_k^{(t)} z_k^{(t)\top}\mathcal{G}_j$$

$$\geq \frac{(N - 1)\tau_1 i \sigma^2}{80} I.$$

Combining (71) and (72) together and rearranging terms, we obtain

$$\|\Delta_{\mathcal{G}_j}(i)\|_F^2 \leq \|\Delta_{\mathcal{G}_j}(i)\|_F^2 \leq \frac{80}{\tau_1 i \sigma^2 (N - 1)} \left( 4\sigma^2 n \log \left( \frac{8np \det(V_{\mathcal{G}_j}(i))}{\delta \det(\Lambda)} \right) + 2\lambda n \theta^2 \right).$$
Next, we aim to provide an upper bound on $\|V_{\tilde{G}}(i)\|$. We see that
\[
\|V_{\tilde{G}}(i)\| \leq \lambda + \sum_{l=1}^{i} \sum_{t=\tau_{\tilde{G}}}^{\tau_{\tilde{G},i-1}} \sum_{k=0}^{N-1} \|z_{k,\tilde{G}}^{(t)}\|^2, \tag{73}
\]
where $z_{k,\tilde{G}}^{(t)} = [x_{k}^{(t)\top} \ u_{k,\tilde{G}}^{(t)\top}]^{\top}$ with $u_{k,\tilde{G}}^{(t)} = K_{\tilde{G}} \ x_{k}^{(t)} + \tilde{w}_{k}^{(t)}$ and $\tilde{w}_{k}^{(t)} \sim N(0, 2\delta^2 \epsilon I)$, where $\kappa$ is defined in Eq. (40). Noting Eq. (45) and recalling from Assumption 4 that $\|(A + B_{\tilde{G},K_{\tilde{G}}})^k\| \leq \zeta \eta^k$ for all $k \in \mathbb{R}_{\geq 0}$, where $0 < \eta < 1$, one can now show that (e.g., [9, Lemma 38])
\[
\|x_{k}^{(t)}\| \leq \frac{\zeta \eta}{1 - \eta^k} \max_{t \in \tilde{T}} \max_{k \in \{0, \ldots, N-1\}} (B_{\tilde{G}, \tilde{G}} \tilde{w}_{k}^{(t)} + \tilde{w}_{k}^{(t)}),
\]
for all $t \in \tilde{T}$ and all $k \in \{0, \ldots, N\}$, where $\tilde{T}$ is defined in (46). Thus, under $\mathcal{E}$, we have
\[
\|x_{k}^{(t)}\| \leq \frac{\zeta \eta}{1 - \eta^k} \sqrt{20\delta^2 \epsilon^2 \kappa^2 \sigma^2 + 10\sigma^2 n} \sqrt{\log \frac{8NT}{\delta}}, \tag{74}
\]
Since $\|z_{k,\tilde{G}}^{(t)}\| \leq \|x_{k}^{(t)}\| + \|u_{k,\tilde{G}}^{(t)}\| \leq (1 + \kappa)\|x_{k}^{(t)}\| + \|\tilde{w}_{k}^{(t)}\|$, one can combine the above arguments and show that under $\mathcal{E}$,
\[
\|z_{k,\tilde{G}}^{(t)}\| \leq \frac{\zeta \eta(1 + \kappa)}{1 - \eta^k} \sqrt{20\delta^2 \epsilon^2 \kappa^2 \sigma^2 + 10\sigma^2 n} \sqrt{\log \frac{8NT}{\delta}} + \kappa \sigma \sqrt{10m \log \frac{8NT}{\delta}}, \tag{75}
\]
where $z_b$ is given in (42). Plugging (75) into (73), we obtain
\[
\|V_{\tilde{G}}(i)\| \leq \lambda + i\tau_1 N z_b,
\]
which implies that
\[
\log \frac{8np \det(V_{\tilde{G}}(i))}{\delta \det(\lambda I)} \leq \log \left( \frac{8np(\lambda + i\tau_1 N z_b)}{\delta \lambda} \right)^{m \varphi_j + n} = (m \varphi_j + n) \log \left( \frac{8np}{\delta} + \frac{8np i\tau_1 N z_b}{\lambda} \right) \leq (m + n) \log \left( \frac{8np}{\delta} + \frac{8np i\tau_1 N z_b}{\lambda} \right),
\]
where the second inequality follows from the fact that $pi\tau_1 \leq pn_\tau \tau_1 \leq T$. Combining the above arguments together, one can show via the choice of $\tau_1$ in Eq. (43) and algebraic manipulations that $\|\Delta_{\tilde{G}}(i)\|^2 \leq \epsilon^2 / i$, which completes the proof of the lemma.

C Technical Lemmas

Lemma 10. Consider a sequence of matrices $M_0, M_1, \ldots$, where $M_k \in \mathbb{R}^{n \times n}$ for all $k \geq 0$, and a sequence of matrices $\Delta_0, \Delta_1, \ldots$, where $\Delta_k \in \mathbb{R}^{n \times n}$ and $\|\Delta_k\| \leq \varepsilon$ for all $k \geq 0$. Suppose that there exist $\zeta \in \mathbb{R}_{>0}$ and $\eta \in \mathbb{R}_{>0}$ such that
\[
\|M_{k_2-1}M_{k_2-2} \cdots M_{k_1}\| \leq \zeta \eta^{k_2-k_1}, \tag{76}
\]
for all $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ with $k_2 > k_1$. Then, the following holds:
\[
(M_{k_2-1} + \Delta_{k_2-1})(M_{k_2-2} + \Delta_{k_2-2}) \cdots (M_{k_1} + \Delta_{k_1}) \leq \zeta (\varepsilon + \eta)^{k_2-k_1}, \tag{77}
\]
for all $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ with $k_2 > k_1$. 

\[\]
where

which implies via Eq. (80) that

Under Assumptions 1-2, one can use similar arguments to those in [3, Chapter 4.4] and prove Eq. (78) and (82) of [25, Lemma 5].

First, one can expand the left hand side of (77) into $2^{k_2-k_1}$ terms. For all $r \in \{0, 1, \ldots, k_2 - k_1\}$ and for all $s \in \{1, 2, \ldots, (k_2-k_1)\}$, let $G_{r,s}$ denote a term in the expansion whose degree of $\Delta_i$ is $r$ and whose degree of $M_i$ is $k - r$, where $(k_r^{k_1})$ is the number of terms in the expansion with the degree of $\Delta_i$ to be $r$. For instance, the term $M_{k_2-1}\Delta_{k_2-2}M_{k_2-3}\Delta_{k_2-4}M_{k_2-5} \cdots M_{k_1}$ may be denoted as $G_{2,s}$ for $s \in \{1, 2, \ldots, (k_2-k_1)\}$. Under the above notation, the left hand side of (77) can be written as $\sum_{r=0}^{k_r^{k_1}} \sum_{s=1}^{(k_r^{k_1})} G_{r,s}$. From (76), we also note that $\|G_{r,s}\| \leq \zeta^{r+1}\eta^{k_2-k_2-r} \varepsilon^r$. This is because the $\Delta_i$s in the term $G_{r,s}$ split the $M_i$'s in $G_{r,s}$ into at most $r + 1$ disjoint groups, and $\|\Delta_i\| \leq \varepsilon$ for all $i \in \mathbb{Z}$.

For instance, we see that $M_{k_2-1}\Delta_{k_2-2}M_{k_2-3}\Delta_{k_2-4}M_{k_2-5} \cdots M_{k_1} \leq \zeta^3\eta^{k_2-k_2-2}\varepsilon^2$. The rest of the proof then follows from the proof of [25, Lemma 5].

**Lemma 11.** Consider any $S \subseteq B$ with $|S| = H$, any $t \in \{T\}$ and any $k \in \{N\}$. Denote $\Delta P_{k,S}^{(i)} = \mathbf{P}_{k,S}^{(i)} - \mathcal{F}_S$, where $\mathbf{P}_{k,S}^{(i)}$ and $\mathcal{F}_S$ are given by Eqs. (11) and (70), respectively. Suppose that Assumptions 1-2 hold. Then,

$$
\Delta P_{k,S}^{(i)} = (L_S^k)^{N-k}(F_{N,S}^{(i)} - F_S^{(i)})\psi_{N,k}(\mathcal{S}),
$$

where $L_S \triangleq A + B_SK_S^{(i)}$, and $\|\psi_{N,k}(\mathcal{S})\| \leq \psi_S$ for all $k \in \{N\}$, where $\psi_S \in \mathbb{R}_{>0}$ is a finite constant. Moreover, suppose that $\|B_S^T\Delta P_{k,S}B_S\| \leq 1/2$. Then,

$$
\|K_{k-1} - K\| = \|(R + B^TP_kB)^{-1}B^TP_kA - (R + B^TPB)^{-1}B^TPA\|,
$$

which implies that $\|K_{k-1} - K\| = \|\Delta K_{k-1} + \Delta \tilde{K}_{k-1}\|$, where

$$
\Delta \tilde{K}_{k-1} = (R + B^TP_kB)^{-1}B^TP_k(P_k - P),
$$

and

$$
\Delta K_{k-1} = (R + B^TP_kB)^{-1} - (R + B^TPB)^{-1}B^TP.
$$

Recalling from Assumption 1 that $\sigma_n(R) \geq 1$, one can then show that

$$
\|\Delta \tilde{K}_{k-1}\| \leq \|B\||\Delta P_k||. \quad (81)
$$

Next, note that $R + B^TP_kB = R + B^TPB + B^TP\Delta P_kB$, where $\|B^T\Delta P_kB\| \leq 1/2$ and $\sigma_n(R + B^TPB) \geq 1$. One can now apply the results in [13, Section 7] and obtain that

$$
\|(R + B^TP_kB)^{-1} - (R + B^TPB)^{-1}\| \leq \frac{\|B^T\Delta P_kB\|}{\sigma_n(R + B^TPB)(\sigma_n(R + B^TPB) - \|B^T\Delta P_kB\|)}.
$$

It follows that

$$
\|(R + B^TP_kB)^{-1} - (R + B^TPB)^{-1}\| \leq 2\|B^T\Delta P_kB\|,
$$

which implies via Eq. (80) that

$$
\|\Delta \tilde{K}_{k-1}\| \leq 2\|PB\||B^T\Delta P_kB||. \quad (82)
$$

Combining (81) and (82) yields

$$
\|K_{k-1}^{(i)} - K_{k-1}^{(i)}\| \leq (\|B_S^k\| + 2\|B_S^T\||B_S||^3)\|\Delta P_{k,S}^{(i)}||. \quad (83)
$$

Recalling that $\rho(L_S) < 1$ (e.g., [3, 6]), we know from the Gelfand formula (e.g., [21]) that there are finite constants $\zeta' \geq 1$ and $0 < \zeta'' < 1$ such that $\|L_S^{(k')}\| \leq \zeta''\eta_S^{(k')}k'$ for all $k' \geq 0$. Thus, plugging (78) into (83) and recalling the definition of $\Gamma_S$ in Eq. (21), we obtain (79). □