On color-critical \((P_5, \overline{P}_5)\)-free graphs

Harjinder S. Dhaliwal\(^1\)  Angèle M. Hamel\(^1\)  Chinh T. Hoang\(^1\)  Frédéric Maffray\(^2\)  Tyler J. D. McConnell\(^1\)  Stefan A. Panait\(^1\)

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\(^1\) Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario, Canada
\(^2\) CNRS, Laboratoire G-SCOP, UJF, INPG, Grenoble, France

Abstract

A graph is \(k\)-critical if it is \(k\)-chromatic but each of its proper induced subgraphs is \((k-1)\)-colorable. It is known that the number of 4-critical \(P_5\)-free graphs is finite, but there is an infinite number of \(k\)-critical \(P_5\)-free graphs for each \(k \geq 5\). We show that the number of \(k\)-critical \((P_5, \overline{P}_5)\)-free graphs is finite for every fixed \(k\). Our result implies the existence of a certifying algorithm for \(k\)-coloring \((P_5, \overline{P}_5)\)-free graphs.

Keywords: Graph coloring, \(P_5\)-free graphs

1 Introduction

Graph coloring is a well-studied problem in computer science and discrete mathematics. Determining the chromatic number of a graph is a NP-hard problem. But for many classes of graphs, such as perfect graphs, the problem can be solved in polynomial time. Recently, much research has been done on coloring \(P_5\)-free graphs. Finding the chromatic number of a \(P_5\)-free graph is NP-hard \([10]\), but for every fixed \(k\), the problem of coloring a graph with \(k\) colors admits a polynomial-time algorithm \([6, 7]\). Research has also been done on \((P_5, \overline{P}_5)\)-free graphs (graphs without \(P_5\) and its complement \(\overline{P}_5\)). In \([4]\), a polynomial-time algorithm is found for finding an approximate weighted coloring of a \((P_5, \overline{P}_5)\)-free graph (definitions not given here will be given later). Recently, \([8]\) gave a polynomial time algorithm for finding a minimum weighted coloring of a \((P_5, \overline{P}_5)\)-free graph.

The algorithms in \([6, 7, 8]\) produce a \(k\)-coloring if the input graph is \(k\)-colorable. However, when the graph is not, the algorithms do not produce an easily verified certificate for a “NO” answer. The point of view in this article is motivated by the
idea of a “certifying algorithm”. An algorithm is certifying if it returns with each output a simple and easily verifiable certificate that the particular output is correct. For example, a certifying algorithm for the bipartite graph recognition would return either a 2-coloring of the input graph, thus proving that it is bipartite, or an odd cycle, thus proving it is not bipartite. A certifying algorithm for planarity would return either an embedding of the graph in a plane, or one of the two Kuratowski subgraphs proving the input graph is not planar.

A graph is $k$-critical if it is $k$-chromatic but each of its proper induced subgraphs is $(k - 1)$-colorable. In [1] and also [11], a certifying algorithm for 3-colorability of $P_5$-free graphs is provided by showing that the number of 4-critical $P_5$-free graphs is finite. Given this result, one may ask whether the same statement holds for $k$-critical $P_5$-free graphs for any $k \geq 4$. However, [3] shows the number of $k$-critical $P_5$-free graphs is infinite for $k \geq 5$. Here we prove that the number of $k$-critical $(P_5, \overline{P}_5)$-free graphs is finite for every fixed $k$, thereby establishing a certifying algorithm for $k$-colorability of $(P_5, \overline{P}_5)$-free graphs. In section 2 we give definitions and background on our problem. In section 3, we give the proofs of our main results. In section 4, we give the exact number of $k$-critical $(P_5, \overline{P}_5)$-free graphs for $k \leq 8$; in particular, we will construct a list of all 5-critical $(P_5, \overline{P}_5)$-free graphs.

2 Definitions and background

A $k$-coloring of a graph $G = (V, E)$ is a mapping $f : V \rightarrow \{1, \ldots, k\}$ such that $f(u) \neq f(v)$ whenever $uv \in E$. Given a coloring, a color class is the set of all vertices of the same color. The chromatic number $\chi(G)$ of a graph $G$ is the smallest $k$ such that $G$ is $k$-colorable. $G$ is $k$-chromatic if $\chi(G) = k$. A graph $G$ is $k$-critical if it is $k$-chromatic and none of its proper induced subgraphs is $k$-chromatic (that is, all of its proper induced subgraphs are $(k - 1)$-colorable). We say that a graph is critical if it is $k$-critical for some $k$. Let $N(v)$ be the set of neighbors of $v$. A set $X$ of vertices of a graph $G = (V, E)$ is a module if for all $v \not\in X$, $X \subseteq N(v)$ or $N(v) \cap X = \emptyset$. Module $X$ is trivial if $|X| = 1$ or $X = |V|$. Unless otherwise stated, a module in this paper is non-trivial. A vertex of $G$ is universal if it is adjacent to every other vertex of $G$. Vertices $u, v$ are comparable if $N(u) \subseteq N(v)$, or vice versa. If $X$ is a set of vertices of $G$, then $G[X]$ denotes the subgraph if $G$ induced by $X$. A set $A$ of vertices is complete to a set $B$ of vertices if there are all edges between $A$ and $B$. Given two graphs $G$ and $H$, the graph $F$ is the join of $G$ and $H$ if $F$ is obtained by taking $G$ and $H$ and joining every vertex in $G$ to every vertex in $H$ by an edge. As usual, $K_t$ denotes the clique on $t$ vertices; and $C_t$ denotes the induced cycle on $t$ vertices. The complement of $G$ is denoted by $\overline{G}$.

Let $G$ be a graph with a module $M$, where $M$ needs not be non-trivial. Consider the graph $G'$ obtained from $G$ by removing $M$, adding another graph $H$, and adding all edges between a vertex $x \in G - M$ and all vertices of $H$ whenever $x$ has a neighbor
in $M$ in the graph $G$. We say that $G'$ is obtained from $G$ by substituting $H$ for $M$. A buoy is a graph $B$ obtained from a $C_5$ by, for each vertex $x$ of the $C_5$, substituting a graph $B_x$ for $x$. The graph $B_x$ is a bag of the buoy. Thus, the buoy will have 5 bags, and we label these $B_1, B_2, B_3, B_4, B_5$ in the cyclic order. Without loss of generality, we will start with $B_1$ when discussing bags. Note each $B_i$ is a module of $B$. The buoy $B$ is full if $B$ contains all vertices of $G$; $B$ is universal if every vertex in $G - B$ is universal; $B$ is a join buoy if every vertex of $B$ is adjacent to every vertex of $G - B$. A pseudo-buoy is defined as a buoy except that any of the sets $B_1, \ldots, B_5$ may be empty. The following two lemmas are folklore and are easy to establish.

**Lemma 2.1** In a critical graph there do not exist two comparable vertices. \(\square\)

**Lemma 2.2** A critical graph is connected. \(\square\)

The following lemma is also well known and we will rely on it for our proofs.

**Lemma 2.3** A critical graph that is not a clique does not contain a clique cutset.

**Proof of Lemma 2.3** Let $G$ be a $k$-critical graph with a clique cutset $C$. Thus, $G - C$ can be partitioned into two sets $A, B$ such that there are no edges between $A$ and $B$. Let $G_A$ (resp., $G_B$) be the subgraph of $G$ induced by $C \cup A$ (resp., $C \cup B$). Consider a coloring of $G_A$ (resp., $G_B$) with $\chi(G_A)$ (resp., $\chi(G_B)$) colors. Since $G$ is critical, we have $\chi(G_A) < k$ and $\chi(G_B) < k$. Without loss of generality, we may assume $\chi(G_A) \geq \chi(G_B)$. Then a $\chi(G_A)$-coloring of $G$ can be obtained by identifying the colors of $G_A$ and $G_B$ on $C$. But this implies $\chi(G) = \chi(G_A) < k$, a contradiction. \(\square\)

**Lemma 2.4** \([3]\) Given a $(P_5, \overline{P}_5)$-free graph $G$, every $C_5$ of $G$ is contained in a buoy which is either full or a module of $G$. \(\square\)

A graph $G$ is perfect if for each induced subgraph $H$ of $G$, the chromatic number of $H$ equals the number of vertices in a largest clique of $H$. It follows that if $G$ is perfect and $k$-critical, then $G$ is the graph $K_k$, the clique on $k$ vertices.

**Lemma 2.5** \([2]\) $(P_5, \overline{P}_5, C_5)$-free graphs are perfect. \(\square\)

### 3 The structure of $k$-critical $(P_5, \overline{P}_5)$-free graphs

In this section, we study properties of maximal buoys in $(P_5, \overline{P}_5)$-free graphs and we show every critical $(P_5, \overline{P}_5)$-free graph $G$ can be constructed by two simple operations: (i) $G$ is the join of two smaller critical graphs, or (ii) $G$ is obtained from the $C_5$ by substituting some smaller critical graphs for each vertex on the $C_5$. First, we need to establish a number of preliminary results.
Lemma 3.1 If $G$ is $k$-critical with a non-trivial module $M$, then $M$ is $\ell$-critical for some $\ell < k$.

Proof of Lemma 3.1 As $M$ is a module, we can partition the vertices of $G$ into three sets: $M; N$, the set of vertices in $G - M$ adjacent to the vertices of $M$; and $R$, the set of vertices in $G - M$ having no neighbors in $M$.

Let $\chi(M) = \ell$ for some $\ell \leq k$. If $\ell = k$, then $N$ must be empty since any color assigned to a vertex in $N$ must be different from the colors assigned to vertices in $M$ (and so $G$ would require more than $k$ colors). This would imply that $G$ is not connected, a contradiction to Lemma 2.2. So, we know $\ell < k$.

Now, we need to to show $M$ is $\ell$-critical; that is, we must show that $M - y$ is $(\ell - 1)$-colorable for any vertex $y \in M$.

Suppose on the contrary that there is vertex $x \in M$ with $\chi(M - x) = \ell$. Now consider an assignment $\gamma$ of colors to $G$ which uses the same colors for $R$ and $N$ as in $\beta$ but uses the coloring of $\theta$ for $M$, that is, for a vertex $x$, if $x \in R \cup N$, then $\gamma(x) = \beta(x)$; and if $x \in M$, then $\gamma(x) = \theta(x)$. Then $\gamma$ is a valid coloring because for any two adjacent vertices $x, y$, we have $\gamma(x) \neq \gamma(y)$; in particular, if $x \in M$ and $y \in N$ then $\gamma(x) \in C_M$ and $\gamma(y) \in C_N$ and so $\gamma(x) \neq \gamma(y)$. So, $\gamma$ uses the same number of colors as $\beta$; that is, $G$ is $(\ell - 1)$-colorable, a contradiction. □

Lemma 3.2 Let $G = (V, E)$ be any graph. Suppose that $V$ admits a partition into two non-empty sets $V_1$ and $V_2$ such that $V_1$ is complete to $V_2$. Then $G$ is critical if and only if the two graphs $G[V_1]$ and $G[V_2]$ are critical.

Proof. Let $k = \chi(G)$. Write $G_i = G[V_i]$ for each $i \in \{1, 2\}$.

First suppose that $G$ is $k$-critical. If $V_i$ is a single vertex then it is 1-critical; otherwise, $V_i$ is module of $G$ and so by Lemma 3.1 $G[V_i]$ is critical.

Now suppose that each of $G_1$ and $G_2$ is critical. Let $k_1 = \chi(G_1)$ and $k_2 = \chi(G_2)$. So $k = \chi(G) = \chi(G_1) + \chi(G_2) = k_1 + k_2$. Pick any $x \in V_1$. Since $G_1$ is critical, $G_1 - x$ admits a $(k_1 - 1)$-coloring. We can combine this coloring with any $k_2$-coloring of $G_2$, using a disjoint set of colors, to obtain a $(k - 1)$-coloring of $G$. The same holds if $x \in V_2$. So $G$ is critical. □

Lemma 3.3 Let $G = (V, E)$ be a $k$-critical graph for some $k$. Let $M$ be a module of $G$ with $\chi(M) = \ell$ for some $\ell$. Let $G'$ be the graph obtained from $G$ by substituting a clique $K$ on $\ell$ vertices for $M$. Then $G'$ is also $k$-critical.
Lemma 3.4 It is easy to see that $\chi(G') = \chi(G) = k$. We need to prove every proper induced subgraph of $G'$ is $r$-colorable for some $r < k$. Let $H'$ be a proper induced subgraph of $G'$. Let $N$ be the set of vertices of $G$ with some neighbor in $M$ and let $R = V - N$. With respect to $G'$, the sets $N$ and $R$ remain unchanged (vertices in $N$ would have neighbors in $K$).

Suppose some vertex $x$ in $K$ does not belong to $H'$. Let $t$ be the number of vertices of $K$ that are in $H'$ with $t < \ell$. In $M$, consider an induced subgraph $M_t$ with chromatic number $t$. Such graph exists since $t < \ell = \chi(M)$. Consider the subgraph of $G$ induced by $N \cup R \cup M_t$. It admits an $r$-coloring $\alpha$ for some $r < k$. In this coloring, at least $t$ colors appear in $M$. From $\alpha$, we can construct an $r$-coloring of $H'$ by (i) for $v \in H' \cap (N \cup R)$, giving $v$ the color $\alpha(v)$, (ii) giving each of the $t$ vertices of $K \cap H'$ a distinct color used by $\alpha$ on $M$.

Thus, we may assume all vertices of $K$ belong to $H'$. It follows that some vertex $x \in N \cup R$ is not in $H'$. Let $H$ be the proper induced subgraph of $G$ obtained from $H'$ by substituting $M$ for $K$. Since $H$ admits an $r$-coloring $\alpha$ for some $r < k$, we may obtain an $r$-coloring of $H'$ from $\alpha$ by (i) for $v \in H' \cap (N \cup R)$, giving $v$ the color $\alpha(v)$, (ii) giving each of the vertices of $K$ a distinct color that $\alpha$ uses on $M$. \[\square\]

**Lemma 3.4** Let $G$ be a critical $(P_5, \overline{P_5})$-free graph. Then a maximal buoy of $G$ is full or a join buoy.

Here, “Maximal” is meant with respect to set-inclusion. In particular, a maximal buoy may not be a largest buoy.

**Proof of Lemma 3.4** By induction on the number of vertices. Let $G = (V, E)$ a $k$-critical $(P_5, \overline{P_5})$-free graph. By the induction hypothesis, we may assume the Lemma holds for any critical $(P_5, \overline{P_5})$-free graph with fewer vertices than $|V|$. Let $B$ be a maximal buoy of $G$. We may assume $B$ is not full, for otherwise we are done. By Theorem 2.1, we know $B$ is a module of $G$. Let $N$ be the set of vertices of $G$ with some neighbor in $B$ and let $R = V - N - B$. We know $R \neq \emptyset$, for otherwise $B$ is a join buoy and we are done. Let $G'$ be the graph obtained from $G$ by substituting a clique $K$ on $\chi(B)$ vertex for $B$. By Lemma 3.3, we know $G'$ is also $k$-critical. With respect to $G'$, the sets $N$ and $R$ remain unchanged (vertices in $N$ would have neighbors in $K$). The substitution operation does not create a new $P_5$ or $\overline{P_5}$, so $G'$ is $(P_5, \overline{P_5})$-free. Graph $G'$ is not a clique because there is a non-edge between $R$ and $K$. By Lemma 2.5 $G'$ must contain a $C_5$ and we consider a maximal buoy $B'$ of $G'$. Now, note that $G'$ has fewer vertices than $G$ ($B$ has a non-edge but $K$ is a clique). The induction hypothesis implies $B'$ is a full or a join buoy of $G'$.

Suppose $B'$ is a full buoy. Since $K$ is a module of $G'$, $K$ lies entirely in one bag of $B'$. We can obtain a full buoy of $G$ from $B'$ by substituting $B$ for $K$, and we are done. Now, we may assume $B'$ is a join buoy of $G'$, that is, $G'$ can be partitioned into two set $F_1 = B'$ and $F_2$ such that there are all edges between $F_1$ and $F_2$. Since there
are no edges between $R$ and $K$, $R \cup K$ must lie completely in one $F_i$. By substituting $B$ for $K$, we see that $B'$ is a join buoy of $G$. □

**Lemma 3.5** For integer $h \geq 0$, let $H$ be any graph that is a pseudo-buoy with bags $B_1, \ldots, B_5$ such that, for each $i \mod 5$, $B_i$ is $k_i$-colorable, where $k_1, \ldots, k_5$ are integers that satisfy $k_i + k_{i+1} \leq h$ for each $i \mod 5$. Then:

(i) If $\sum_{i=1}^{5} k_i \leq 2h$, then $\chi(H) \leq h$.

(ii) If $\sum_{i=1}^{5} k_i > 2h$ and each $B_i$ is $k_i$-chromatic, then $\chi(H) > h$.

**Proof of Lemma 3.5** Proof of (i). We establish property (i) by induction on $h$. Suppose $\sum_{i=1}^{5} k_i \leq 2h$. If $h = 0$, the property holds trivially. Now assume that $h \geq 1$. Say that a pair $(i, i+1)$ is tight if $k_i + k_{i+1} = h$. Say that a pair $(j, j+2)$ is good if every tight pair contains an element from $\{j, j+2\}$. It is a routine matter to see that either (a) there is a good pair, or (b) all five pairs $(i, i+1) (i = 1, \ldots, 5)$ are tight. In case (b), we have $5h = \sum_{i=1}^{5} (k_i + k_{i+1}) = 2 \sum_{i=1}^{5} k_i \leq 4h$, which is impossible. So there is a good pair. Up to relabelling, assume that $(1, 3)$ is a good pair. Pick any $k_1$-coloring of $B_1$, and let $S'$ be a color class in that coloring; if $k_1 = 0$, i.e., $B_1 = \emptyset$, then let $S' = \emptyset$. Likewise, pick any $k_3$-coloring of $B_3$, and let $S''$ be a color class in that coloring; if $k_3 = 0$, let $S'' = \emptyset$. Let $S = S' \cup S''$ and $H^* = H \setminus S$. Thus $H^*$ is a pseudo-buoy with bags $B_1 \setminus S', B_2, B_3 \setminus S'', B_4, B_5$. The fact that $(1, 3)$ is a good pair implies that $H^*$ satisfies the induction hypothesis for the integer $h-1$. So $H^*$ is $(h-1)$-colorable, and consequently $H$ is $h$-colorable, just use the $h$-th color on the vertices of $S$.

Proof of (ii). Suppose on the contrary that $\chi(H) \leq h$. So the vertices of $H$ can be partitioned into $h$ stable sets $S_1, \ldots, S_h$. For each $i \in \{1, \ldots, 5\}$, let $\sigma_i$ be the number of sets among $S_1, \ldots, S_h$ that have non-empty intersection with $B_i$. The definition of a pseudo-buoy implies that each stable set $S_j$ has non-empty intersection with at most two of $B_1, \ldots, B_5$. It follows that $\sum_{i=1}^{5} \sigma_i \leq 2h$. So $\sum_{i=1}^{5} \sigma_i < \sum_{i=1}^{5} k_i$, which implies that there is an integer $i \in \{1, \ldots, 5\}$ such that $\sigma_i < k_i$. Hence the non-empty intersections of $S_1, \ldots, S_h$ in $B_i$ form a coloring of $B_i$ with strictly fewer colors than $k_i$, a contradiction. □

Let $C_k$ be the family of $k$-critical $(P_5, \overline{P}_5)$-free graphs. Clearly, $C_1 = \{ K_1 \}$ and $C_2 = \{ K_2 \}$. In general we have $K_k \in C_k$. We are now in a position to prove the main theorem of this section.

**Theorem 3.6** For any $k \geq 2$, a graph $G$ is in $C_k$ if and only if it can be obtained by any of the following two constructions:

— **Construction 1:** $G$ is the join of a member of $C_{k_1}$ and a member of $C_{k_2}$ for positive integers $k_1$ and $k_2$ such that $k_1 + k_2 = k$.

— **Construction 2:** $G$ is a buoy with bags $B_1, \ldots, B_5$ such that, for each $i \mod 5$,
Proof of Theorem 3.6. Let \( k \geq 2 \).

(I) We prove that any graph obtained by Construction 1 or 2 is in \( C_k \).

First suppose that \( G \) is obtained by Construction 1, i.e., \( G \) is the join of a member \( G_1 \) of \( C_{k_1} \) and a member \( G_2 \) of \( C_{k_2} \) for positive integers \( k_1 \) and \( k_2 \) such that \( k_1 + k_2 = k \). By Lemma 3.2, \( G \) is \( k \)-critical.

Now suppose that \( G \) is obtained by Construction 2, with the same notation as in the theorem. By Lemma 3.5 (ii) (with \( h = k - 1 \)), we have \( \chi(G) \geq k \). Pick any \( x \in B_1 \). We know that \( B_1 - x \) is \((k_1 - 1)\)-colorable. Moreover, we have \( (k_1 - 1) + \sum_{i=2}^{5} k_i = 2(k - 1) \). So the graph \( G - x \), which is the buoy with bags \( B_1 - x, B_2, B_3, B_4, B_5 \), satisfies the hypothesis of Lemma 3.5 (i) with \( h = k - 1 \), and consequently it is \((k - 1)\)-colorable. The same holds for every vertex \( x \) in \( G \). This implies that \( \chi(G) = k \) and \( G \) is \( k \)-critical.

(II) Now we prove the converse part of the theorem. Let \( G = (V, E) \) be a member of \( C_k \), i.e., \( G \) is \((P_5, \overline{P}_5)\)-free and \( k \)-critical. If \( \overline{G} \) is not connected, then we can apply Lemma 3.2 and it follows that \( G \) is obtained by Construction 1. Therefore assume that \( \overline{G} \) is connected. Thus \( G \) is not a clique and by Lemma 3.5 \( G \) contains a \( C_5 \), which is a buoy. Consider a maximal buoy \( B \) of \( G \). By Lemma 3.4 \( B \) is a full buoy or a join buoy.

Suppose \( B \) is a join buoy of \( G \). Consider the set \( A = V - B \). Since \( A \) is complete to \( B \), by Lemma 3.2 both \( A \) and \( B \) are critical, and so \( G \) is obtained by Construction 2.

So, we may assume \( B \) is a full buoy, that is, it contains all vertices of \( G \). Let the bags of \( B \) be \( B_1, \ldots, B_5 \). Let \( k_i = \chi(B_i) \) for each \( i \in \{1, \ldots, 5\} \). For each \( i \) we must have \( k_i + k_{i+1} \leq k - 1 \), for otherwise \( \chi(G[B_i \cup B_{i+1}]) = k \), which contradicts the fact that \( G \) is \( k \)-critical. Also we must have \( \sum_{i=1}^{5} k_i \geq 2k - 1 \), for otherwise, by Lemma 3.5 (i), \( G \) is \((k - 1)\)-colorable. Each \( B_i \) (with at least two vertices) is a module of \( G \), and so by Lemma 3.1 \( B_i \) is \( k_i \)-critical. Suppose that \( \sum_{i=1}^{5} k_i \geq 2k - 1 \). Pick any \( x \in V \). Then \( G - x \) is a pseudo-buoy that satisfies Lemma 3.5 (ii), so \( \chi(G - x) \geq k \), which contradicts the fact that \( G \) is \( k \)-critical. So we have \( \sum_{i=1}^{5} k_i = 2k - 1 \). This shows that \( G \) can be obtained by Construction 2. \( \square \)

**Theorem 3.7** For every \( k \), \( C_k \) is a finite set.

*Proof of Theorem 3.7* By induction on \( k \). When \( k = 3 \), it is easy to see \( C_3 \) contains two graphs: the \( C_3 \) and \( C_5 \). Let \( J_k \) be the set of graphs of \( C_k \) constructed by Construction 1. Let \( B_k \) be the set of graphs of \( C_k \) constructed by Construction 2. We have \( C_k = J_k \cup B_k \) by Theorem 3.6. Let \( f(k) \) (respectively, \( j(k), b(k) \)) be the cardinality of \( C_k \) (respectively, \( J_k, B_k \)). Each graph in \( J_k \) is constructed by taking the join of a graph in \( C_i \) and a graph in \( C_{k-i} \) for \( i = 1, 2, \ldots, k - 1 \). It follows that \( j(k) \leq \sum_{i=1}^{k-1} f(i)f(k-i) \).
Consider a graph $G$ in $B_k$ which is a full buoy. Graph $G$ has five bags, each of which is a graph of $C_i$ with $i \leq k - 2$. It follows that $b(k) \leq (f(k - 2))^5$. Since $f(k) = j(k) + b(k)$, we have $f(k) \leq \sum_{i=1}^{k-1} f(i)f(k-i) + (f(k-2))^5$. So $f(k)$ is a function in $k$ and the result follows.

For completeness, we will give a bound for $f(k)$ using Knuth’s up-arrow notation. Define a single up-arrow operation to be $a \uparrow b = a^b$. Next define a double arrow operation to be $a \uparrow\uparrow b = a \uparrow (a \uparrow \ldots \uparrow a)$, that is, a tower of $b$ copies of $a$. It is easy to prove by induction that $f(k) \leq 5 \uparrow\uparrow k$. □

4 $k$-critical $(P_5, \overline{P}_5)$-free graphs for small $k$

By Theorem 3.7 the number of $k$-critical $(P_5, \overline{P}_5)$-free graphs is finite for every $k$. In this section, we will refine the argument to establish sharp bounds on $|C_k|$ for small values of $k$. In particular, we will construct all 5-critical $(P_5, \overline{P}_5)$-free graphs.

Let classes $B_k$ and $J_k$ be defined as in Theorem 3.7. Recall that $C_1 = \{K_1\}$, $C_2 = \{K_2\}$ and $C_3 = \{K_3, C_5\}$, so $B_1 = B_2 = \emptyset$ and $B_3 = \{C_5\}$.

When $A$ and $B$ are two sets of graphs, let $A \otimes B$ be the set of graphs that are the join of a member of $A$ and a member of $B$. We know that $J_k = \bigcup_{k_1+k_2=k} C_{k_1} \otimes C_{k_2}$. This means that each member $G$ of $J_k$ is either the join of several buoys or the join of $K_p$ and a member of $C_{k-p}$ for some positive $p$; in the latter case $G$ is also the join of $K_1$ and a member of $C_{k-1}$. It follows that we can write

$$J_k = (C_1 \otimes C_{k-1}) \cup \bigcup_{k_1} B_{k_1} \otimes \cdots \otimes B_{k_p}$$

where the union is over all vectors $(k_1, \ldots, k_p)$ such that $k_1 + \cdots + k_p = k$ and $k_i \geq 3$.

When $G$ is a member of $B_k$ we associate with $G$ its pattern, which is the numerical vector $(k_1, \ldots, k_5)$ (with the same notation as in Theorem 3.6) that satisfies the constraints $k_i > 0$, $k_i + k_{i+1} \leq k - 1$ for each $i$ mod 5 and $\sum_{i=1}^{5} k_i = 2k - 1$. Several non-isomorphic members of $B_k$ can have the same pattern. Conversely, given a pattern $(k_1, \ldots, k_5)$ that satisfies the constraints, one can construct a member of $B_k$ as a buoy $(B_1, \ldots, B_5)$ where $B_i$ is chosen from $C_{k_i}$ for each $i$, and Theorem 3.6 says that every member of $B_k$ is constructed that way (different choices may yield the same member of $B_k$ due to isomorphism). We illustrate this for the values $k = 4$, $k = 5$ and $k = 6$.

For $k = 4$, the only pattern that satisfies the constraints $k_i > 0$, $k_i + k_{i+1} \leq 3$ for each $i$ mod 5 and $\sum_{i=1}^{5} k_i = 7$ is, up to circular permutation, $(1,2,1,2,1)$. The corresponding member of $B_4$ is graph $T_3$ in Figure 1. By (1), we have $J_4 = C_1 \otimes C_3$, so the members of $J_4$ are the graphs $T_1$ and $T_2$ in Figure 1. Hence $|C_4| = 3$.

For $k = 5$, the patterns that satisfy the constraints $k_i > 0$, $k_i + k_{i+1} \leq 4$ for each $i$ mod 5 and $\sum_{i=1}^{5} k_i = 9$ are, up to circular permutation, $(2,2,2,2,1), (2,1,3,1,2)$ and $(1,3,1,3,1)$. Pattern $(2,2,2,2,1)$ yields the graph $F_4$ on Figure 2. Since $C_3 =$
Figure 1: All 4-critical \((P_5, \overline{P}_5)\)-free graphs

Figure 2: All 5-critical \((P_5, \overline{P}_5)\)-free graphs. In the graphs \(F_4, \ldots, F_9\), the ovals represent the bags and the double line denotes all edges between the two bags.
\{K_3, C_5\}, pattern (2, 1, 3, 1, 2) yields graphs \(F_5\) and \(F_6\), and pattern (1, 3, 1, 3, 1) yields graphs \(F_7\), \(F_8\) and \(F_9\). So \(|B_5| = 6\). By (1), we have \(J_5 = C_1 \otimes C_4\), so \(J_5\) consists of graphs \(F_1\), \(F_2\), \(F_3\) on Figure 2. Hence \(|C_5| = 9\).

For \(k = 6\), the patterns that satisfy the constraints \(k_i > 0, k_i + k_{i+1} \leq 5\) for each \(i \mod 5\) and \(\sum_{i=1}^{5} k_i = 11\) are, up to circular permutation, \((2, 2, 2, 2, 3), (2, 3, 2, 3, 1), (2, 3, 1, 3, 2), (1, 4, 1, 3, 2)\) and \((1, 4, 1, 4, 1)\). Since \(|C_1| = 1, |C_2| = 1, |C_3| = 2\) and \(|C_4| = 3, \) we see that \((2, 2, 2, 2, 3)\) yields two graphs, \((2, 3, 2, 3, 1)\) yields four graphs, \((2, 3, 1, 3, 2)\), yields three graphs, \((1, 4, 1, 3, 2)\) yields six graphs, and \((1, 4, 1, 4, 1)\) yields, up to symmetry, six graphs. So \(|B_6| = 21\). By (1), we have \(J_6 = (C_1 \otimes C_5) \cup (B_3 \otimes B_3)\), so \(|J_6| = 9 + 1 = 10\). Hence \(|C_6| = 31\).

Similar computations lead to \(|C_7| = 185\) and \(|C_8| = 1487\).

### 5 Conclusion

The algorithm in [8] accepts as input any graph \(G\) that is \((P_5, \overline{P_5})\)-free and determines whether \(G\) is \(k\)-colorable in time \(O(n^3)\), where \(n\) is the number of vertices of \(G\). However, when \(G\) is not \(k\)-colorable, the algorithm does not produce an easily verifiable certificate for a “NO” answer.

Now, given a \((P_5, \overline{P_5})\)-free graph \(G\), we have a simple polynomial-time algorithm for finding a \(k\)-critical induced subgraph \(H\) of \(G\), as follows. Consider a vertex \(x\), and determine if \(G \setminus x\) is \(k\)-colorable; if it is not \(k\)-colorable, then we remove \(x\) from consideration; if it is \(k\)-colorable, then \(x\) is in \(H\). Repeat the process for all other vertices. The vertices that have not been removed from consideration form the desired graph \(H\). Now it is easy to check whether \(H\) satisfies the properties given in Theorem 3.6 because, by Theorem 3.7, there is only a finite number of \(k\)-critical graphs. So it takes constant time to check whether \(H\) belongs to the family \(C_k\) of \(k\)-critical graphs.

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