Lazy beats crazy:
a spurious yet mathematical justification for laissez-faire

Stéphane Le Roux

May 11, 2014

Abstract

In perfect-information games in extensive form, common knowledge of rationality triggers backward induction, which yields a Nash equilibrium. That result assumes much about the players’ knowledge, while it holds only for a subclass of the games in extensive form. Alternatively, this article defines a non-deterministic evolutionary process, by myopic and lazy improvements, that settles exactly at Nash equilibrium (in extensive form). Importantly, the strategical changes that the process allows depend only on the structure of the game tree, they are independent from the actual preferences. Nonetheless, the process terminates if the players have acyclic preferences; and even if some preferences are cyclic, the players with acyclic preferences stop improving eventually! This result is then generalised in games played on DAGs or infinite trees, and it is also refined by assigning probabilities to the process and perturbing it.

Keywords: games in extensive form, improvement, evolution, termination, Nash equilibrium.

1 Introduction

Game theory is the theory of competitive interactions between decision makers having different interests, and its primary purpose is to further understand such real-world interactions through mathematical modelling. Apart from some earlier related works, the field of game theory is usually said to be born in the first part of the 20th century, especially thanks to von Neumann [12], but also Borel [3] and some others. Since then, it has been applied to many concrete areas such as economics, political science, evolutionary biology, computer science, etc.

The remainder of Section 1 recalls the definitions of games and equilibrium, and motivates the evolutionary perspective on games in extensive form; Section 2 states and proves the main result of the paper; and Section 3 generalises the main result in several directions.

1.1 Games in normal form

In most areas of application the models, i.e. the games involve many players and many outcomes, which are meant to describe faithfully the many shades of situations involving many stakeholders. In these multi-outcome games an outcome may be, e.g., a function
assigning a real number, also called a payoff, to every player. In this case it is implicitly assumed that each player prefers to receive greater numbers. More generally, the outcomes may also be abstract objects that each player may compare via a binary relation called the preference of the player. It is important to allow for arbitrary preferences (as opposed to transitive, reflexive, total, etc.) since, for example, the conclusion of [14] shows that linearly ordered preferences do not account for partially ordered preferences. Abstract games in normal form are defined below.

**Definition 1 (Games in normal form)** They are tuples \( \langle A, (S_a)_{a \in A}, O, v, (\prec_a)_{a \in A} \rangle \) satisfying the following:

- \( A \) is a non-empty set (of players, or agents),
- \( \prod_{a \in A} S_a \) is a non-empty Cartesian product (whose elements are the strategy profiles and where \( S_a \) represents the strategies available to Player \( a \)),
- \( O \) is a non-empty set (of possible outcomes),
- \( v : \prod_{a \in A} S_a \to O \) (the outcome function that values the strategy profiles),
- Each \( \prec_a \) is a binary relation over \( O \) (modelling the preference of Player \( a \)).

Informally, in such a game each player is not trying to win against one single opponent by making her/him lose, but is instead trying to maximise the outcome of the play w.r.t. her own preference. So far and in general, such games do not have a natural notion of recommendation, i.e. of best strategy to play, but the famous notion of Nash equilibrium provides valuable information about games, albeit incomplete. Nash equilibrium is usually defined as a strategy profile, i.e. a combination of one strategy per player, yielding an outcome that no player can increase along her own preference by unilaterally changing her individual strategy. The traditional notion of Nash equilibrium is rephrased below in the abstract setting with a subtle semantic change (but remains the same in extension): each binary relation \( \prec_a \), which I call preference, is the complement of the inverse of what is traditionally called preference.

**Definition 2 (Nash equilibrium)** Let \( \langle A, (S_a)_{a \in A}, O, v, (\prec_a)_{a \in A} \rangle \) be a game in normal form. A strategy profile (profile for short) \( s \) in \( S := \prod_{a \in A} S_a \) is a Nash equilibrium if it makes every Player \( a \) stable, i.e. \( v(s) \not\prec_a v(s') \) for all \( s' \in S \) that differ from \( s \) at most at the \( a \)-component.

\[
NE(s) := \forall a \in A, \forall s' \in S, \neg(v(s) \prec_a v(s')) \land \forall b \in A - \{a\}, s_b = s'_b
\]

Three games in normal form are represented below as arrays. They all involve two players, say \( a \) and \( b \), two strategies for \( a \) (resp. \( b \) ), namely \( a_l \) and \( a_r \) (resp. \( b_l \) and \( b_r \) ), and outcomes/payoff functions in \( \mathbb{R}^2 \). Recall that, informally, Player \( a \) (resp. \( b \) ) prefers payoff functions with greater first (resp. second) component. In the first game, if Player \( a \) picks the strategy \( a_l \) and Player \( b \) picks \( b_l \), the strategy profile \( (a_l, b_l) \) then yields payoff 1 for \( a \) and 0 for \( b \). This profile is not a Nash equilibrium (NE for short) because, by
changing strategies, Player \( a \) can convert the profile \((a_l, b_l)\) into \((a_r, b_l)\) and obtain payoff 2, which is greater than 1. The game has two NE, namely profiles \((a_r, b_l)\) and \((a_l, b_r)\). The second game has no NE and the third game, which enjoys some symmetry, has two Nash equilibria. Inasmuch as existence of NE is desirable, the second example is problematic. For finite games with real-valued payoffs as below, however, Nash [10] proved that allowing randomised/mixed strategy instead of pure ones yields a new game that is guaranteed to have a NE. The only mixed NE of the second game below assigns probability \( \frac{1}{2} \) to each of the strategies, and the corresponding expected payoffs are \( \frac{3}{2} \).

Nash’s theorem is one of the most celebrated results in game theory, although it raises a few well-known issues as a modelling tool.

The first issue is the implicit assumption that the players are able to randomise their strategies, moreover independently, which is not always the case in the real world.

Second issue, the players compare the profiles only via the expected values of the payoffs. But, if asked to choose between receiving one heap of gold for sure and three heaps of gold with probability one half, I would choose one heap for sure.

Third issue, there may be several NE in one game like the third game above and no universal way of deciding which to play, whether mixed strategies and communications between players are allowed or not. (In the third game above, the profile assigning probability \( \frac{1}{2} \) to each of the strategies is a mixed NE with expected payoffs \( \frac{3}{4} \) and could be seen as a recommendation, albeit non-optimal, but some more complex games seem to lack even such a recommendation.)

Fourth issue, even if starting from a given profile, little is known on how players could incrementally (but non-cooperatively) modify the profile and reach an NE, even though Nash’s both proofs of existence of NE consider equilibria as fixed points of a synchronous profile improvement. In [10] the players change their individual strategies simultaneously into their respective best responses against the current opponent strategies, and then Nash invokes Kakutani Fixed Point Theorem [6]; in [11] the players simultaneously change strategies towards (but not fully into) their best responses, and then Nash invokes Brouwer Fixed Point Theorem (see [4] and [5]), which is a special case of [6]. The second game above shows that the improvement from [10] does not converge to NE. Indeed, starting at profile \((a_l, b_l)\) such a dynamic system would enter a cycle visiting the four pure, \textit{i.e.}, non-mixed profiles. The left-hand game below shows that the improvement from [11] does not converge to NE either. Indeed, starting at the profile that assigns probability \( \frac{3}{4} \) to \( a_l \) and \( b_l \), such a dynamic system would visit the profile that assigns probability \( \frac{1}{4} \) to \( a_l \) and \( b_l \) and then come back to the original profile.

\[
\begin{array}{ccc}
    & b_l & b_r \\
 a_l & 1, 0 & 5, 0 \\
a_r & 2, 4 & 5, 3 \\
\end{array}
\quad
\begin{array}{ccc}
    & b_l & b_r \\
 a_l & 0, 3 & 3, 0 \\
a_r & 3, 0 & 0, 3 \\
\end{array}
\quad
\begin{array}{ccc}
    & b_l & b_r \\
 a_l & 2, 1 & 0, 0 \\
a_r & 0, 0 & 1, 2 \\
\end{array}
\]

\[
\begin{array}{ccc}
    & b_l & b_m & b_r \\
 a_l & 1, 0 & 0, 1 & 0, 0 \\
a_m & 0, 1 & 1, 0 & 0, 0 \\
a_r & 0, 0 & 0, 0 & 2, 2 \\
\end{array}
\]
Since none of the two synchronous improvements that were designed by Nash converges towards an NE, let us now consider the asynchronous improvement below, as an alternative improvement that might enjoy convergence properties.

**Definition 3 (Convertibility, induced preference over profiles, and improvement)**

- Let \( \langle A, \langle S_a \rangle_{a \in A}, O, v, (\prec_a)_{a \in A} \rangle \) be a game in normal form. For \( s \) and \( s' \) profiles in \( \prod_{a \in A} S_a \), let \( c \rightarrow_a s' := \forall b \in A - \{a\}, s_b = s'_b \).
- Given a game \( \langle A, \langle S_a \rangle_{a \in A}, O, v, (\prec_a)_{a \in A} \rangle \), let \( s \prec_a s' \) denote \( v(s) \prec_a v(s') \). So in this article \( \prec_a \) may also refer to the induced preference over the profiles.
- Let \( \rightarrow_a := \prec_a \cap c \rightarrow_a \) be the individual improvement reductions of the players and let \( \rightarrow := \cup_{a \in A} \rightarrow_a \) be the collective improvement reduction.

**Observation 4** The Nash equilibria of a game are exactly the sinks, i.e., the terminal profiles of the collective improvement \( \rightarrow \).

However, in the right-hand game above, starting from profile \((a_l, b_l)\) and following \( \rightarrow \) yields a cycle that misses the only NE, namely \((a_r, b_r)\).

### 1.2 Games in extensive form

The games in extensive form are another well-studied class of games. (Their and the related formal, inductive definitions can be found in [8] and [9].) A finite real-valued game in extensive form is an object built on a finite rooted tree; each internal node is owned by exactly one player, and each leaf encloses a real-valued payoff function, i.e. a tuple assigning one real number to each player. The leftmost object below is such a game and, intuitively, it is played as follows: Player \( b \) at the root chooses left or right. Right yields payoff 1 for \( a \) and 0 for \( b \) (and the game ends), left requires a choice from Player \( a \), and so on until a leaf is reached.

|       |       |       |       |       |
|-------|-------|-------|-------|-------|
|       |       |       |       |       |
|       |       |       |       |       |
|       |       |       |       |       |
|       |       |       |       |       |
|       |       |       |       |       |
|       |       |       |       |       |

A strategy of a player is an object that specifies, for each node of the game that the player owns, the (unique) choice that she would make if the play ever reached this node.
Calling such an object a strategy leads to a natural embedding of the games in extensive form into the games in normal form. It is exemplified by the last figure above, where $bb_r$ means that $b$ chooses left at the root and right at the other node. The second, third and fourth objects from the left above represent three of the eight possible profiles for the game, where the double lines represent the strategical choices. Such a profile induces one unique payoff function, by following the unique choices from the root to the leaves. This embedding provides games in extensive form with an inherited notion of NE. For instance, the second object above is not an NE, which one may equivalently read off the array. Indeed, at least one of the player, namely $b$, can improve upon her payoff by changing her strategy from "right-right" to "left-left" and obtain 3 instead of 0. The third and fourth objects above are the two NE of the game.

Kuhn [7] proved that every such game has an NE, and Osborn and Rubinstein [13] proved that every such game with abstract outcomes (instead of payoff functions at the leaves) has an NE provided that the preferences are strict weak orders. As recalled below, a strict weak order is a strict partial order whose negation is transitive.

**Definition 5 (Strict weak order)** The axioms are as follows:

\[
\begin{align*}
\forall x, \neg (x \prec x) & \quad \text{irreflexivity} \\
\forall x, y, z, & \quad x \prec y \land y \prec z \Rightarrow x \prec z \quad \text{transitivity} \\
\forall x, y, z, & \quad \neg (x \prec y) \land \neg (y \prec z) \Rightarrow \neg (x \prec z) \quad \text{transitivity of the complement}
\end{align*}
\]

Kuhn’s proof and its structure-preserved abstraction by [13] rely on the notion of backward induction (BI for short), which may be seen as a recursive (multi-)function. Informally, if the game is a leaf, it returns the unique profile, else it performs BI on the children/subtrees and let the owner of the root choose one of the newly computed subprofiles whose induced outcome is maximal according to her own preference. This procedure computes an NE when the preferences are derived from the usual implicit order over real-valued payoffs and more generally whenever they are strict-weak orders over abstract outcomes. Performed on the game above, BI computes the fourth object above. When the preferences are strict-weak orders, BI actually always computes a subgame perfect equilibrium (SPE), where an SPE is an NE whose substrategies profiles, rooted at the children of the NE’s root, are also SPE. The concept of SPE was introduced by Selten only in 1965 although it coincides with the outputs of backward induction used by Kuhn [7] in 1953 and by [13]. Actually the strict weak order condition in [13] is the weakest condition guaranteeing that backward induction yields an NE/SPE, as characterised by the following equivalence.

**Observation 6** Consider all the games in extensive form involving some given players and preferences over some given outcomes; the three propositions below are equivalent.

1. The preference of each player is a strict weak order.
2. Backward induction always yields a Nash equilibrium.
3. Backward induction always yields a subgame perfect equilibrium.
Proof 1 ⇒ 3 by BI, following [13], 2 ⇒ 1 by definition, and 2 ⇒ 1 by contraposition. If one Player $a$ has a preference that is not a strict weak order, let us make a case distinction. First case, there exists an outcome $x$ such that $x \prec_a x$, so a leaf game enclosing $x$ has no NE. Second case, there exist outcomes $x$, $y$ and $z$ such that $x \prec y$ and $y \prec z$ but $\neg(x \prec z)$, so that the leftmost profile below is a valid output of the backward induction on the leftmost game, but it is not even an NE since Player $a$ can change her two choices at once and obtain outcome $z$, as in the second profile. Third case, there exist outcomes $x$, $y$ and $z$ such that $\neg(x \prec y)$ and $\neg(y \prec z)$ but $x \prec z$. The third profile below is a valid output of the backward induction on the second game, but it is not even an NE since Player $a$ can change her two choices at once and obtain outcome $z$, as in the rightmost profile.

Nonetheless, Observation 6 above is not a definitive answer to existence of SPE/NE. The sufficient, strict-weak-order condition in [13] was indeed generalised into a necessary and sufficient condition in [8] and [9]. By Observation 6 above, BI alone cannot do the job in the general case, although it is still the main ingredient of the proof below.

Observation 7 Consider all the games in extensive form involving some given players and preferences over some given outcomes; the three propositions below are equivalent.

1. The preference of each player is acyclic.
2. Every sequential game has a Nash equilibrium.
3. Every sequential game has a subgame perfect equilibrium.

Proof 1 ⇒ 3 in three steps. First, consider linear extensions of the preferences. Second, perform a usual BI for the extended preferences. Third, note that an NE for the extended preferences is also an NE for the original preferences.

This work, started by Kuhn, therefore solves for the class of abstract games in extensive form the first two issues raised in Section 1.1 by the concept of Nash equilibrium for real-valued games in normal form: probabilities are no longer needed.

Furthermore, the concept of SPE is not just a technical tool meant to exhibit some NE. Indeed, Aumann wrote in [2] "if common knowledge of rationality obtains in a game of perfect information, then the backward induction outcome is reached." Let us exemplify this informally through the game below: Player $a$ considers the node that she owns just above the leaves $(4, 0)$ and $(3, 4)$ (resp. $(2, 1)$ and $(1, 2)$) and decides to play left to maximise her payoff, should a play of the game ever reach this node. Player $b$ knows that $a$ is rational and would play left, so $b$ decides to play right to maximise her payoff, should a play of the game ever reach this node. Player $a$ knows that $b$ is rational and that $b$ knows that $a$ is rational, so $a$ knows that $b$ would play right, so $a$ decides to play left at the root.
By giving this nice interpretation of BI, Aumann promotes subgame perfect equilibrium as recommendation concept (for the stakeholders) and as prediction (for the outside observers), while NE does not enjoy these properties in general, as mentioned above. Together with Kuhn’s algorithm to compute SPE, it solves partly the last two issues raised in Section 1.1 by the concept of Nash equilibrium for real-valued games in normal form. (Note that Aumann’s interpretation is questioned, e.g., in [15], due to counter-factual reasoning.)

Unfortunately, to make it work, Aumann assumes "that the payoffs to each player at different leaves of the game tree are different", which dramatically reduces the class of games that may benefit from this epistemic insight. Indeed in the left-hand game below, if only common knowledge of rationality is assumed, a has no clue what b would play, and therefore cannot commit to any decision. Basically, Aumann’s interpretation requires that each player use a total ordering not only on the payoffs but also on the payoff functions, so that the players may be predictable enough, which holds automatically if every payoff occurs at most once, as assumed by Aumann.

More generally, allowing partially-ordered outcomes to occur at several leaves of the game, which is utterly desirable for the sake of generality, raises the following two related issue. First, the same issue that made Aumann exclude multiple occurrence of payoffs. (In the right-hand game below, if x and y are incomparable for b although a prefers x to z and z to y, the same issue arises as with the real-valued payoffs.) Second, the issue that lead to Observation 6. The algorithmic side of both issues may be solved at once by linear extensions of the preferences, as done in the proof of Observation 7, but this generates two game-theoretic issues: first, in connection with Aumann’s epistemic interpretation, how should a player know which linear extensions prevail for the other players if only common knowledge of rationality holds? second, such an extension may distort the meaning of the original partial order, which is the only actual and exact account of the player’s preference.

This article introduces (for 1 ⇒ 2 of Observation 7) an alternative proof that solves all the above-mentioned issues. The alternative proof relies on Observation 4, which holds also for games in extensive form, albeit the collective improvement → may fail to terminate even for very simple games in extensive form, as shown by the cycle below.
What seems to make things go wrong in the cycle above, is the unnecessary changes of choices by Player $b$ when improving upon the current profile. This motivates a restriction of the convertibility relations $\cdot \to \cdot$ for games in extensive form. Informally, the restriction requires some minimalism (or laziness!) from each player who changes strategies to induce a new play: only choice modifications that actually contribute to the modification of the induced play are performed. Section 2 will show that rewriting Definition 3 using the lazy convertibility instead of the plain convertibility yields a collective lazy improvement that terminates if the preferences are acyclic and that it still satisfies (the lazy version of) Observation 4.

Like the proof of Observation 7, the above-mentioned alternative proof shows more than mere existence of NE. The alternative proof yields an algorithm, just like the original proof, but the significant differences are as follows:

1. The concept of SPE is not needed, all NE are possible outputs of the new algorithm.
2. The proof allows for fully arbitrary preferences and does not modify them, thus preserving their original meaning.
3. The game-theoretic interpretation of the alternative proof does not rely on the subtle concept of common knowledge of rationality; instead, it has an evolutionary, distributed flavour and it makes sense even (or especially) for short-sighted players oblivious of part of the game, including their opponents.
4. Informally, instead of figuring out "the right strategies" directly, the players will repeatedly, asynchronously, and lazily modify their strategies to improve upon the current induced outcome. Each player whose preference is acyclic stabilises quickly although the players with cyclic preferences (the crazy!) may loop forever.
5. When only non-crazy players are involved, the collective improvement stops after at most $(h - 1) \cdot (l - 1)$ steps where $h$ is the cardinality of the longest preference chains and $l$ is the number of leaves in the game. Observation 20 shows that the bound is tight and it also happens to be an example where backward induction would be faster than the new algorithm. (But high speed is not the main point here.)

Before Section 2 which introduces the aforementioned alternative proof, and then Section 3 which proves, disproves, or only hints at possible generalisations or variants, let us briefly discuss how to represent finite games in extensive form in a rigorous manner.

1.3 How to represent finite games in extensive form

To write formal proofs, especially proofs that can be checked automatically by a computer, one should choose a formalism carefully to represent the objects that we have in mind. It makes it easier not only to convince ourselves that the represented objects are the ones that we have in mind, but also to prove properties about these objects. For instance, if the natural numbers are defined inductively ($0$ is a natural and for every natural $n$, the object $n + 1$ is also a natural), one may prove a predicate $P$ on the naturals by induction along
the definition: it suffices to show that \( P(0) \) holds and that for all natural \( n \), if \( P(n) \) holds then \( P(n + 1) \) also holds. Likewise, finite game structures in extensive form, i.e. more simply, finite rooted tree can be defined inductively, which provides a convenient induction proof principle for free. The definition is more complex than for natural numbers, though: it says that a single node is a tree, and that a non-empty list of trees (meant to be the children) together with a fresh node (meant to be the root) is also a tree. This is a definition by mutual induction, since it interweaves the definition of trees and of lists of trees, see [8] or [9] for more details. To save some space and time in this article I will not be this formal; I will rather perform semi-formal inductions on trees by ignoring the fact that lists are inductive objects.

2 Lazy improvement in games in extensive form

After a definition and some lemmas, Theorem 12 shows that in any sequence of lazy improvements, every non-crazy player, i.e. having acyclic preference, stabilises eventually regardless of what the (crazy) opponents do. Using a very different proof, Theorem 18 shows the same, plus a tight bound for stabilisation.

In this section, let \( A \) be a non-empty set of players, let \( O \) be a non-empty set of outcomes, and for each Player \( a \) in \( A \) let \( \prec_a \) be an arbitrary preference over \( O \) and \( S_a \) be a non-empty strategy set.

Definition 8 below defines lazy convertibility (inductively) and lazy improvement.

**Definition 8 (Lazy convertibility and lazy improvement)**

- If \( s \) is a leaf profile, \( s \prec_a s \) for all \( a \in A \).
- Let two profiles \( s \) and \( s' \) have the profiles \( s_0, \ldots, s_n \) and \( s'_0, \ldots, s'_n \) as respective children. Let Player \( a \) choose \( s_i \) at the root of \( s \) and \( s'_k \) at the root of \( s' \) and assume that \( s_j = s'_j \) for \( j \neq k \). If \( s_k \prec_b s'_k \) and if \( b = a \) or \( i = k \), define \( s \prec_b s' \).

Let \( \leadsto_a := \prec_a \cap \prec_a \) be the lazy improvement reduction of Player \( a \) and let \( \leadsto := \cup_{a \in A} \leadsto_a \) be the collective lazy improvement.

The three observations below can be proved by induction on the definition of \( \prec_a \) (or \( \leadsto_a \)) and \( s \) (or \( s' \)), respectively.

**Observation 9**

1. \( \prec_a \subseteq \leadsto_a \) for all \( a \).
2. \( s \prec_a s' \) iff the profiles \( s \) and \( s' \) differ at most at nodes owned by Player \( a \) on the play induced by \( s' \).
3. \( s \prec_a s' \) iff first, \( s \prec_a s' \) and second, if \( s' \) differs from \( s \) at \( n \) nodes and if \( s'' \) induces the same leaf as \( s' \) then \( s'' \) differs from \( s \) at \( n \) nodes or more.
To exemplify Definition 8, Player \( a \) can lazily convert the leftmost strategy profile below into each of the profiles below, but not into any other profile. Player \( a \) is written bold face at nodes where changes occur.

\[
\begin{array}{ccccccc}
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\end{array}
\]

Contrary to the convertibility relations \( \xrightarrow{c} \), which are equivalence relations, the lazy convertibility relations \( \xleftarrow{c} \) are certainly reflexive but in general neither symmetric nor transitive. For instance, Player \( a \) cannot lazily convert the rightmost profile above back into the leftmost one. In the additional example below, Player \( a \) can convert the leftmost profile to the middle profile but not to the rightmost profile.

\[
\begin{array}{ccccccc}
\text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\
\end{array}
\]

The lazy convertibility enjoys a useful property nonetheless, that the usual convertibility does not: if a player dismisses a play during a sequence of lazy conversion, only the very same player is later able to make the last step to induce the same play again, possibly induced by a different profile. This phenomenon is more formally stated by Lemma 10.

**Lemma 10** If \( s \xrightarrow{c} s_0 \xrightarrow{c} \ldots \xrightarrow{c} s_n \xleftarrow{c} t \), where \( s \) and \( s' \) induce the same play, and if this play is different from the plays that are induced by the \( s_i \), then \( a = b \).

**Proof** Let us prove the claim by induction on the underlying game. Since the play induced by \( s_0 \) is different from the play induced by \( s \), these profiles are not just leaves, but proper trees instead. During the assumed \( \xleftarrow{c} \) reduction of \( s \), its subprofile that is chosen by the root owner in \( s \) undergoes a \( \xleftarrow{c} \) reduction too, say \( t \xleftarrow{c} t_0 \xleftarrow{c} \ldots \xleftarrow{c} t_n \xleftarrow{c} t' \), where \( t \) and \( t' \) induce the same play (and the root owner chooses \( t' \) in \( s' \)). If all these subprofiles are equal, Player \( a \) must be the root owner (of \( s \)), since \( s \) and \( s_0 \) induce different plays by assumption, and \( b \) is also the root owner since \( s_n \) and \( s' \) induce different play, so \( a = b \).

Now let \( t_j \) be the first subprofile different from \( t \), so \( t_j \) induces a play different from \( t \) and \( t' \). If \( t_k \) and \( t' \) induce different plays but \( t_k+1 \) and \( t' \) induce the same play for some \( k \geq j \), then \( s_k \) and \( s' \) induce the same play by definition of \( \xleftarrow{c} \), contradiction with the assumptions of the lemma, so all \( t_j, \ldots t_n \) induce plays different from that of \( t' \). If \( t_1 \neq t \), then \( a = b \) by the induction hypothesis, else \( a \) must be the root owner and does not choose \( t_1 \) in \( s_1 \). The first time that \( a \) chooses some \( t_i \) again must be in \( s_j \); indeed if it were before, \( s \) and \( s_i \) would induce the same play, and if it were after, \( a \) could not change \( t_{j-1} \) into \( t_j \). Therefore \( t_{j-1} \xleftarrow{c} t_j \xleftarrow{c} \ldots \xleftarrow{c} t_n \xleftarrow{c} b t' \) and \( a = b \) by the induction hypothesis.

Despite the restrictive property from Lemma 10, the lazy convertibility is in some sense as effective as the usual convertibility, as suggested by Proposition 11.

**Proposition 11** The Nash equilibria of a game are exactly the terminal profiles of the lazy improvement \( \rightarrow \).
Proof The Nash equilibria are terminal profiles of \( \rightarrow \) due to the inclusion \( \rightarrow \subseteq \rightarrow \) and Observation 4. Conversely, note that if \( s \rightarrow_a s' \) then there exists \( s'' \) such that \( s \rightarrow_a s'' \) and \( s' \) and \( s'' \) induce the same outcome. (It suffice to prove by induction that if \( s \rightarrow_a s' \) then there exists \( s'' \) such that \( s \rightarrow_a s'' \) and \( s' \) and \( s'' \) induce the same play.) \( \square \)

Theorem 12 The players who perform infinitely many lazy-improvement steps in a given sequence of lazy improvements have cyclic preferences.

Proof Assume that Player \( a \) has an acyclic preference but makes infinitely many lazy improvements in the sequence \( (s_n)_{n \in \mathbb{N}} \) where \( s_n \rightarrow s_{n+1} \) for all \( n \). By finiteness of the set of strategy profiles, let \( s_l \) occurs infinitely often in the sequence and let \( k > 0 \) such that \( s_l = s_{l+k} \) and such that Player \( a \) performs a lazy improvement in the sequence between \( s_l \) and \( s_{l+k} \). Up to index renaming, let us assume that \( l = 0 \), that the first step is \( s_0 \rightarrow_a s_1 \), and that the outcome induced by \( s_0 \) is \( \prec_a \)-minimal (existence by acyclicity!) among the outcomes upon which Player \( a \) lazy-improves during the cycle. Note that \( s_0 \) and \( s_1 \) induce different plays since \( s_0 \prec_a s_1 \), and let \( s_{i+1} \) (where \( 0 < i \)) be the first profile along the cycle that induces the same play as \( s_0 \). Lemma 10 on the sequence \( s_0 \rightarrow_a s_1 \rightarrow \cdots \rightarrow s_i \rightarrow s_{i+1} \) shows that \( s_i \rightarrow_a s_{i+1} \), so \( s_i \prec_a s_{i+1} \), whence \( s_i \prec_a s_0 \), which contradicts \( \prec_a \)-minimality of \( s_0 \).

Together with Proposition 11 the following corollary shows the equivalence between all preferences being acyclic and universal existence of NE.

Corollary 13 The collective lazy improvement reduction terminates for all games iff all preferences are acyclic.

Proof The difficult implication of the equivalence is a corollary of Theorem 12. For the other implication, note that if \( x_0 \prec_a x_1 \prec_a \cdots \prec_a x_n \prec_a x_0 \), then \( \rightarrow_a \) does not terminate on the profile below.

\[
x_0 \rightarrow_a x_1 \rightarrow_a \cdots \rightarrow_a x_n
\]

The proof of Theorem 12 by contradiction, gives a quick argument but no deep insight on how and how fast the reduction terminates. The remainder of this section gives an alternative proof with complexity bounds and deeper insight. Definition 14 below assigns a function to every game and a function to every profile. More specifically, given a game and a player, the first function counts the number of children (or edges) that the player is bound to ignore when choosing any strategy for this game, and the second function is a refinement of the first function: given a profile, a player, and an outcome, it counts the number of nodes in the profile where the player ignores a child-profile inducing this outcome. (For convenience and since it does not lead to any ambiguity here, implicit currying and the like are widely used below, e.g., by using the types \( A \times B \rightarrow C \) and \( A \rightarrow B \rightarrow C \) one for another.)

Definition 14 (Dismissed outcomes of a game and of a profile) The dismissed outcomes of a game \( g \) is a function \( \Delta(g) \) of type \( A \rightarrow \mathbb{N} \), and it is defined inductively below.

- \( \Delta(g,a) := 0 \) if \( g \) is a leaf game.
• If Player $a$ owns the root of a game $g$ whose children are $g_0, \ldots, g_n$ then
  
  $\Delta(g, b) := \sum_{j=0}^{n} \Delta(g_j, b)$ for all $b \neq a$.

  $\Delta(g, a) := (\sum_{j=0}^{n} \Delta(g_j, a)) + n$

The dismissed outcomes of a profile $s$ is a function $\delta(s)$ of type $A \to O \to \mathbb{N}$, or equivalently in this case, of type $A \times O \to \mathbb{N}$, and it is defined inductively below.

• $\delta(s, a, o) := 0$ if $s$ is a leaf profile.

• If Player $a$ owns the root of a profile $s$ and chooses the subprofile $s_i$ among $s_0, \ldots, s_n$ then

  $\delta(s, b, o) := \sum_{j=0}^{n} \delta(s_j, b, o)$ for all $b \neq a$.

  $\delta(s, a, o) := (\sum_{j=0}^{n} \delta(s_j, a, o)) + |\left\{ j \in \{0, \ldots, n\} - \{i\} \mid v(s_j) = o \right\}|$

The smaller array below describes the function $\Delta(g)$, where $g$ is the underlying game of the left-hand profile $s$ below, and the right-hand array describes the function $\delta(s)$. For instance $\delta(s, b, y) = 2$ because Player $b$ ignores/dismisses the outcome $y$ twice: once at the left-most internal node, after two leftward moves, when choosing outcome $x$ rather than $y$, and also once after one rightward move, also when choosing $x$ rather than $y$. Note that the only leaf that is not accounted for by the function of the dismissed outcome of a profile/game is the leaf that is induced by the profile.

\[\begin{array}{ccc}
  & \Delta(g, \cdot) & \\
  a \mapsto 5 & a \mapsto 1 & 0 \mapsto 3 \\
  b \mapsto 3 & b \mapsto 0 & 2 \mapsto 1 & 0
\end{array}\]

Observation 15 below relates the two functions from Definition 14. It refers to $s2g$, a function that returns the underlying game of a given profile, see \[8\] or \[9\] for a proper definition.

**Observation 15**

1. Let $s$ be a profile and $a$ be a player, then $\Delta(s2g(s), a) = \sum_{o \in O} \delta(s, a, o)$.

2. Let $g$ be a game, then $1 + \sum_{a \in A} \Delta(g, a)$ equals the number of leaves of $g$.

**Proof**

1. By induction on $s$. If $s$ is a leaf profile, the claim holds since $\Delta(s2g(s), a) = 0 = \delta(s, a, o)$ by definition, so now let $s$ be a profile where the root owner $a$ chooses $s_i$ among subprofiles $s_0, \ldots, s_n$. For $b \neq a$ a Definition 14 and the induction hypothesis yield $\Delta(s2g(s), b) = \sum_{j=0}^{n} \Delta(s2g(s_j), b) \overset{I.H.}{=} \sum_{j=0}^{n} \sum_{o \in O} \delta(s_j, b, o) = \sum_{o \in O} \sum_{j=0}^{n} \delta(s_j, b, o) = \sum_{o \in O} \delta(s, b, o)$. Similarly, $\Delta(s2g(s), a) = \sum_{j=0}^{n} \Delta(s2g(s_j), a) + n \overset{I.H.}{=} \sum_{j=0}^{n} \sum_{o \in O} \delta(s_j, a, o) + |\left\{ j \in \{0, \ldots, n\} - \{i\} \mid v(s_j) \in O \right\}| = \sum_{o \in O} \left( \sum_{j=0}^{n} \delta(s_j, a, o) + |\left\{ j \in \{0, \ldots, n\} - \{i\} \mid v(s_j) = o \right\}| \right) = \sum_{o \in O} \delta(s, a, o)$.
2. By induction on \( g \). This holds for every leaf game \( g \) since \( \Delta(g, a) = 0 \) by definition. Let \( g \) be a game whose root is owned by Player \( a \) and whose subgames are \( g_0, \ldots, g_n \). The number of leaves in \( g \) is the sum of the numbers of leaves in the \( g_j \), that is, \( \sum_{j=0}^{n} (1 + \sum_{b \in A} \Delta(g_j, b)) \) by induction hypothesis. This, equals \( 1 + \sum_{j=0}^{n} \sum_{b \in A \setminus \{a\}} \Delta(g_j, b) + n + \sum_{j=0}^{n} \Delta(g_j, a) \), which, in turn, equals \( 1 + \sum_{b \in A \setminus \{a\}} \Delta(g, b) + \Delta(g, a) \) by definition.

\[ \square \]

Lemma 16 below states conservation of the outcomes that are dismissed by a player in a profile during a lazy conversion of another player. Intuitively, it is because a lazy conversion of a player cannot modify the subtreasures that are ignored by the other players, even though she owns node therein.

**Lemma 16** \( s \xrightarrow{\alpha} s' \land b \neq a \Rightarrow \delta(s, b) = \delta(s', b) \)

**Proof** By induction on the profile. It holds for leaves, so let \( s \xrightarrow{\alpha} s' \) with subprofiles \( s_0, \ldots, s_n \) and \( s'_0, \ldots, s'_n \), respectively. By definition of \( \xrightarrow{\alpha} \) we have \( s_j \xrightarrow{\alpha} s'_j \) for all \( j \), and therefore \( \delta(s_j, b, o) = \delta(s'_j, b, o) \) by induction hypothesis. If the root owner is different from \( b \), then \( \delta(s, b, o) = \sum_{j=0}^{n} \delta(s_j, b, o) = \sum_{j=0}^{n} \delta(s'_j, b, o) = \delta(s', b, o) \) by definition of \( \delta \). If \( b \) is the root owner, she chooses the \( i \)-th subprofile in both \( s \) and \( s' \) since \( b \neq a \), and moreover \( s'_j = s_j \) for all \( j \) distinct from \( i \). So \( \delta(s, b, o) = \sum_{j=0}^{n} \delta(s_j, b, o) + |\{ j \in \{0, \ldots, n\} \setminus \{i\} \cup \{v(s_j) = o\} | = \sum_{j=0}^{n} \delta(s'_j, b, o) + |\{ j \in \{0, \ldots, n\} \setminus \{i\} \in \{v(s'_j) = o\} | = \delta(s', b, o) \).

However, the conservation does not fully hold for the player who converts the profile, unless the induced outcomes are the same for both profiles. In Lemma 17 below, \( eq \) is just a boolean representation of equality: \( eq(x, x) := 1 \) and \( eq(x, y) := 0 \) for \( x \neq y \).

**Lemma 17** \( s \xrightarrow{\alpha} s' \Rightarrow \delta(s, a) + eq(v(s)) = \delta(s', a) + eq(v(s')) \)

**Proof** By induction on the profile \( s \). It holds for leaves, so let \( s \xrightarrow{\alpha} s' \) with subprofiles \( s_0, \ldots, s_n \) and \( s'_0, \ldots, s'_n \), respectively. If the root owner is distinct from \( a \), she chooses the same \( i \)-th subprofile in both \( s \) and \( s' \), so \( \delta(s, a, o) + eq(v(s), o) = \sum_{0 \leq j \leq n \land j \neq i} \delta(s_j, a, o) + \delta(s_i, a, o) + eq(v(s_i), o) = \sum_{0 \leq j \leq n \land j \neq i} \delta(s'_j, a, o) + \delta(s'_i, a, o) + eq(v(s'_i), o) = \delta(s', a, o) + eq(v(s'), o) \) by definition of \( \delta \), since \( s_j = s'_j \) for \( j \neq i \), and by induction hypothesis.

If \( a \) is the root owner, let \( a \) choose the \( i \)-th and \( k \)-th subprofiles in \( s \) and \( s' \), respectively. Let \( N := \delta(s, a, o) + eq(v(s), o) \), so \( N = \sum_{0 \leq j \leq n \land j \neq k} \delta(s'_j, a, o) + |\{ j \in \{0, \ldots, n\} \setminus \{i\} \cup \{v(s_j) = o\} | + eq(v(s_k), o) = |\{ j \in \{0, \ldots, n\} \setminus \{v(s_j) = o\} | \}

first with \( x := i \) and then with \( x := k \) yields \( N = \sum_{0 \leq j \leq n \land j \neq k} \delta(s'_j, a, o) + |\{ j \in \{0, \ldots, n\} \setminus \{v(s_j) = o\} | + eq(v(s_k), o) = |\{ j \in \{0, \ldots, n\} \setminus \{v(s_j) = o\} | + \delta(s_k, a, o) + eq(v(s_k), o) \). Since \( s_k \xrightarrow{\alpha} s'_k \) by definition of lazy convertibility, and by the induction hypothesis, let us further rewrite \( \delta(s_k, a, o) + eq(v(s_k)) \) with \( \delta(s'_k, a, o) + eq(v(s'_k), o) \) in \( N \). Folding Definition 14 yields \( N = \delta(s', a, o) + eq(v(s'), o) \).  

The two lemmas above suggest that whenever a player lazily converts a profile to obtain a better outcome, some measure decreases a bit with respect to her preference, but does not change for the other players. The collective lazy improvement should therefore terminate, and even quite quickly, as proved below.

13
Theorem 18  Consider a game \( g \) where Player \( a \) has an acyclic preference whose chains have at most \( h \) elements. Then in any sequence of collective lazy improvements, the number of steps performed by Player \( a \) is bounded by \((h - 1) \cdot \Delta(g, a)\).

Proof  For every outcome \( o \) let \( h(a, o) \) be the maximal cardinality of the \( \prec_a \)-chains whose \( \prec_a \)-maximum is \( o \), and note that \( o \prec_a o' \) implies \( h(a, o) < h(a, o') \). For every profile \( s \) let \( M(s, a) := \sum_{o \in O} (h(a, o) - 1) \Delta(s, a, o) \) and note that \( 0 \leq M(s, a) \leq (h - 1) \cdot \Delta(s2g(s), a) \) by Observation [15]. Let \( s \rightarrow_a s' \) be a lazy improvement step, so \( s \prec_a s' \) and \( v(s) \prec_a v(s') \). By definition, then \( M(s, a) - M(s', a) = \sum_{o \in O} (h(a, o) - 1) \cdot (\delta(s, a, o) - \delta(s', a, o)) = h(a, v(s')) - h(a, v(s)) > 0 \) by Lemma [17]. Let \( s \rightarrow_b s' \) be a lazy improvement step where \( b \neq a \), then \( M(s, a) = M(s', a) \) by Lemma [16].

Corollary 19 below rephrases Corollary 13 and adds a complexity result that follows from Theorem 18 and Observation 15.1.

Corollary 19  The collective lazy improvement terminates for all games iff all preferences are acyclic, in which case the number of sequential lazy improvement steps is at most \((h - 1) \cdot (l - 1)\) where \( h \) bounds the cardinality of the preference chains and \( l \) is the number of leaves.

Observation 20  
1. The maximal length of lazy improvement sequences is bounded in a quadratic manner in the size of the game in general and linearly when \( h \) from Corollary 19 is fixed.

2. The quadratic bound may be attained.

Proof  (of 20.2) For \( n \in \mathbb{N} \), consider the underlying game of the following strategy profile, where \( y \prec_a x_0 \prec_a x_1 \prec_a \cdots \prec_a x_n \) and \( x_1 \prec_b y \) for all \( i \).

```
        a
       / \  \
      b   b
     / \  / \
    x_0 y x_1 y x_n y
```

Let us prove by induction on \( n \) the existence of a sequence of \( \frac{(n+2)(n+3)}{2} - 2 \) lazy improvement steps when starting from the strategy profile above. For the base case \( n = 0 \), there are \( 1 = \frac{(0+2)(0+3)}{2} - 2 \) lazy improvement steps. For the inductive case, let Player \( a \) make \( n \) lazy improvements in a row, by choosing \( x_1 \), then \( x_2 \), and so on until \( x_n \). At that point, let Player \( b \) improve from \( x_n \) to \( y \) and then let Player \( a \) come back to \( x_0 \). So far, \( n + 2 \) lazy improvement steps have been performed. Now let us ignore the substrategy profile involving \( x_n \) (and \( y \)). By induction hypothesis, \( \frac{(n+1)(n+2)}{2} - 2 \) additional lazy improvement steps can be performed in a row. Since \( (n + 2) + \frac{(n+1)(n+2)}{2} - 2 = \frac{(n+2)(n+3)}{2} - 2 \), we are done.

In [2] Aumann gave a very interesting meaning to subgame perfect equilibria and backward induction by invoking common knowledge of rationality, as mentioned in Section 1.2. Aumann’s interpretation thus suggests that SPE are more relevant and meaningful than common NE, but in the present article, Corollaries 13 and 19 grant the common NE the first-class citizenship. A possible interpretation reads as follows: At first, each player plays according to a given strategy, and then the same game is repeated, \( i.e. \), played
(infinitely) many rounds. If the initial profile is not an NE, some player will eventually notice a possible improvement for herself after finitely many rounds, and change strategies accordingly. The strategy improvement is performed in a lazy way because strategy changes are costly, and greedily by picking any better response (instead of best responses only, because a bird in the hand is worth two in the bush). The process will stabilise quickly according to Corollary [19] and the NE of the game are the sink/terminal states of this distributed, evolutionary process. Such an interpretation particularly suits players with low level of awareness, since a player need not know the preferences or the existence of the other players; at first a player need not even know all of her options since the lazy improvement does not require any maximality (but requiring it would preserve quick termination nonetheless). In this respect, Corollaries [13] and [19] and their above interpretation are a complement to Kuhn’s backward induction [7] and its interpretation by Aumann. Unfortunately, Aumann’s interpretation requires ”that the payoffs to each player at different leaves of the game tree are different”, whereas the present article does not assume anything about the outcomes, not even that they be real-valued, or about their distribution onto the tree leaves.

Theorems [12] and [18] reach even further: they do not assume anything about the preferences, which are arbitrary binary relations, but they prove that the collective lazy improvement reaches quickly a semi Nash equilibrium and then possibly switches between different such semi-NE only. In such a case the crazy players, those with cyclic preferences, may not only model somewhat irrational decision makers, but also nature and its apparently arbitrary changes, whether they are cyclic like the seasons or not.

Besides foreseeable applications of Corollary [19] in computer science, more surprising areas of application are worth investigating. Let us consider a very simple example in evolutionary biology: The players are the species and their interactions, e.g., prey-predator or competition for similar resources are (arguably) modelled by a repeated game in extensive form, where one generation of every species lasts for one round only. The strategy of one player corresponds to the genotype of this species, and the set of possible plays compatible with the strategy corresponds to the phenotype of this species, so that different genotypes may lead to the same phenotype. Changing one choice at one node of the game corresponds to an elementary, atomic mutation. Overtime, unlikely mutations will occur, possibly several at once but only for one species at a time, although this restriction may be problematic from a biological point of view. The laziness of the bursts of mutations may be justified by the low probability of mutation, so that a non-lazy mutation is far more unlikely to occur than the lazy mutation inducing the same new play. The mutants fully replace the original breed for the next round iff the mutants are fitter, i.e. obtain a better outcome according to their preferences. This simplistic selection process may be problematic from a biological point of view, but note that arbitrary preferences allow for subtle definitions of fitness. Especially, impossibility or non-direct possibility of some mutations can be modelled easily by non-linear preferences, e.g. non-transitive. Finally, let us forgive the various inaccuracies of this highly non-quantitative model and conclude by Theorem [18] that species with acyclic preferences stop evolving rather quickly!

A successful use of quantitative game theory for evolutionary biology was introduced in 1973 by Maynard Smith and Price [16], which lead to the concept of evolutionary stable
strategy (ESS). (For a detailed technical introduction see, e.g., [17].) Put simply, the notion of ESS is relevant in two-player symmetric games in normal form; the ESS are defined as the probabilistic NE that are robust against some specific small perturbations (the mutations), and are characterised as the strategies that survive a selection process against the other strategies.

Some drawbacks of Theorem 18 as a model for evolutionary biology have already been mentioned above, now let us list some of its possible advantages in comparison to the concept of ESS: it can model interactions between many different species; mutation is an internal, detailed process; mutation and selection are involved together in the same evolutionary process; (pure) NE always exist in games in extensive form and are the results of a quick distributed process. There is an additional, major issue raised by this model, though: game-theoretic interactions occur between different species only, through the game, whereas selection occurs within each species independently. It is yet unclear whether this model can be refined and become acceptable from a biological point of view, but I believe that it is worth mentioning.

Let us consider another very short example in sport competitions, where an overall match may be (arguably) modelled by a repeated game in extensive form, where each round corresponds to a new phase of the match. The teams start the match with predefined strategies given by their respective coaches who adjust and improve their teams’ strategies against their opponents dynamically, during the match. It is reasonable to consider lazy improvement in this case because coaches do not have time to think or even talk about counter-factual, short-term-irrelevant issues. Between the time scale of evolution and that of a sport match, there should be a wide spectrum of possible applications of Theorem 18.

In the real-world, however, not exactly the same game may be played over and over again, because the rules of the game may change due to new technologies, increasing knowledge, shortage of resources, or climate changes that were not foreseen when designing the original game. So one may wonder whether stability is likely to be reached, and enjoyed long enough, before the game changes again. Here the (relatively low) algorithmic complexity of the lazy improvement may serve as a threshold and answer the question. Note that the issue whether complex systems, such as economies, could actually reach their promised equilibria or not was already mentioned in [1].

3 Generalisations and variants

Section 3.1 shows that the lazy improvement should remain asynchronous to enjoy termination properties; Section 3.2 shows that the notion of strategy should keep including counter-factual choices for the lazy improvement to preserve its termination properties; Section 3.3 characterises the families of linear preferences that guarantee termination of (an extension of) lazy improvement for all games that are built on finite directed acyclic graphs (DAG), as opposed to trees only; Section 3.4 shows that restricting the lazy convertibility in DAG games in three different meaningful ways is not sufficient to ensure termination of these very lazy improvements for all acyclic preferences; Section 3.5 shows that lazy does not beat crazy in DAG games, even for two-player win-lose games; Section 3.6 shows that the lazy improvement finds approximate Nash equilibria in infinite
games with continuous payoff functions; and Section 3.7 refines the lazy improvement into an evolutionary model with perturbations.

3.1 Synchronous lazy improvement

If players were allowed to lazily improve simultaneously, as opposed to asynchronously only, the following cycle could occur. (It is a cycle up to symmetry only, but a proper cycle of length four may be easily derived from it.)

\[
\begin{array}{cccc}
 & a & b & a \\
\hline
a & 1,1 & 1,1 & a \\
\hline
3,2 & 2,0 & 2,0 & 3,2
\end{array}
\]

3.2 Representation of strategies

The technical notion of strategy that is used in this article to represent the intuitive concept of a strategy (in games in extensive form) is not the only possible notion. An alternative notion does not require choices from a player at every node that she owns, but only at nodes that are not ruled out by the strategy of the same player. The three objects below are such minimalist, alternative strategy profiles, where double lines still represent choices. Up to symmetry, they constitute from left to right a cycle of improvements that could be intuitively described as lazy, so an actual cycle of length eight can easily be inferred from the short pseudo cycle. This may happen because, although the improvements may look lazy, Player \(a\) forgets about her choices in a subgame (of the root) when leaving it, and may settle for different choices when coming back to the subgame. This suggests that even counter-factual choices are sometimes relevant.

\[
\begin{array}{cccc}
 & a & b & a \\
\hline
a & \text{W} & \text{L} & a \\
\hline
W&L&L&W&L&W&W&W
\end{array}
\]

3.3 Finite games and lazy improvement on directed acyclic graphs

The concept of game in extensive form and the related concepts of strategy, Nash equilibrium, and subgame perfect equilibrium can be easily generalised in the setting of directed acyclic graphs, as opposed to rooted trees; it is done below.

**Definition 21 (Dag games, NE, SPE)**

- Given a non-empty set of players and a non-empty set of outcomes, a \(DAG\) game is a family of binary relations (the preferences, indexed by the player set) together with a finite directed acyclic graph (\(DAG\)) satisfying the following:
  1. exactly one node has no predecessor, and it is called the root.
2. The sinks of the DAG are labelled with one outcome each; they are the leaves.
3. The non-sink nodes are labelled with (i.e. are owned by) one player each.

- A strategy profile for a DAG game is a DAG game where exactly one of the outgoing edges of each non-leaf node is chosen, i.e., tagged by the owner.

- A strategy profile induces one unique leaf, by starting from the root and following the unique choices. The preferences over outcomes are thus lifted to preferences over profiles, i.e. a player compares two profiles by comparing the two induced outcomes. A player can convert a profile into another profile when the choices are the same for both profiles at every node that is owned by any other player. The notions of (individual) strategy, Nash equilibrium and subgame perfect equilibrium follow naturally.

Similarly to Observation 7, linear extension of preferences and backward induction lead to Observation 22 below.

**Observation 22** Every DAG game has an SPE iff every player has an acyclic preference.

Let us extend the notion of lazy convertibility to DAG games, not following Definition 8 but Observation 9.2 instead, partly because the inductive definition of DAGs is not as convenient as that of rooted trees. (An extension of the lazy improvement following Observation 9.3 is considered in Section 3.4.)

**Definition 23 (Lazy convertibility and improvement in DAGs)**

- $s \xleftarrow{a} s'$ iff the profiles $s$ and $s'$ differ at most at nodes owned by Player $a$ on the play induced by $s'$.

- Let $\rightarrow_{a} := \prec_{a} \cap \xleftarrow{a}$ be the lazy improvement reduction of Player $a$ and let $\rightarrow := \bigcup_{a \in A} \rightarrow_{a}$ be the collective lazy improvement.

Similarly to Proposition 11, Proposition 24 below suggests that laziness is not too restrictive, and a proof would be similar to that of Proposition 11.

**Proposition 24** The Nash equilibria of a DAG game are exactly the terminal profiles of the lazy improvement $\rightarrow$.

**Lemma 25** Let $A$ be a non-empty set and for all $a \in A$ let $\prec_{a}$ be a strict linear order over some non-empty set $O$. The following assertions are equivalent.

1. $\forall a, b \in A, \forall x, y, z \in O, \neg(z \prec_{a} y \prec_{a} x \land x \prec_{b} z \prec_{b} y)$.
2. There exists a partition $\{O_{i}\}_{i \in I}$ of $O$ and a linear order $<$ over $I$ such that:
(a) \( i < j \) implies \( x <_a y \) for all \( a \in A \) and \( x \in O_i \) and \( y \in O_j \).

(b) \( <_b|_{O_i} = <_a|_{O_i} \) or \( <_b|_{O_i} = <_a|_{O_i}^{-1} \) for all \( a, b \in A \).

If the assertions hold, one may also assume that \( <_b|_{O_i} = <_a|_{O_i}^{-1} \) is witnessed for all \( i \in I \), moreover \( O_i \) is always a \( <_a \)-interval for all \( i \in I \) and \( a \in A \), as implied by \( 2a \).

Proof: \( 2 \Rightarrow 1 \) is straightforward, so let us assume \( 1 \). Let \( x \sim y \) stand for \( \exists a, b \in A, x \leq_a y \leq_b x \), which defines a reflexive and symmetric relation, and note that due to the forbidden pattern \( z <_a y <_a x \land x <_b z <_b y \) the following holds: if \( x <_a y \) and \( y <_b x \), then \( x <_a z <_a y \) iff \( y <_b z <_b x \). To show that \( \sim \) is transitive too, let us assume that \( x \sim y \sim z \). If \( x, y, z \) are not pairwise distinct, \( x \sim z \) follows directly, so let us assume that they are pairwise distinct, so by assumption there exist \( a, b, c, d \in A \) such that \( y <_a x <_b y <_c z <_d y \). To show that \( x \sim z \) there are three cases depending on where \( y \) lies with respect to \( y <_a x \), all cases invoking the (above-mentioned) forbidden-pattern argument: if \( y <_a z <_a x \) then \( x <_b z <_b y \), and \( x \sim z \) follows; if \( y <_a x <_a z \) then \( z <_d x <_a y \) and subsequently \( y <_c x <_c z \), by invoking twice the forbidden-pattern argument, and \( x \sim z \) follows; third case, let us assume that \( z <_a y <_a x \). If \( x <_b z \) then \( x \sim z \) follows, and if \( z <_c x \) then \( z <_a x <_b y \), so \( y <_c x <_c z \), and \( x \sim z \) follows. Therefore \( \sim \) is an equivalence relation; let \( \{O_i\}_{i \in I} \) be the corresponding partition of \( O \).

Now let us show that the \( \sim \)-classes are \( \leq_a \)-intervals for all \( a \), so let \( x \sim y \) and \( x <_a z <_a y \). By definition of \( \sim \), there exists \( b \) such that \( y <_b x \), in which case \( y <_b z <_b x \) by the forbidden-pattern argument, so \( x \sim z \) by definition.

Let \( x \in O_i \) and \( y \in O_j \) be such that \( x \leq_a y \). Since \( O_i \) and \( O_j \) are intervals, \( x' <_a y' \) for all \( x' \in O_i \) and \( y' \in O_j \). Since \( \neg(x' \sim y') \) by assumption, \( x' <_b y' \) for all \( b \in A \), by definition of \( \sim \). In this case defining \( i < j \) meets the requirements.

Before proving \( 2b \) let us prove that if \( x <_a y \) and \( y <_b x \) and \( z \sim y \), then \( z <_a y \) iff \( y <_b z \); this is trivial if \( z \) equals \( x \) or \( y \), so let us assume that \( x \neq z \neq y \), and also that \( z <_a y \). If \( z <_b y \), then \( y <_c z \) for some \( c \) since \( z \sim y \), and wherever \( x \) may lie with respect to \( y <_c z \), it always yields a forbidden pattern using \( <_a \) or \( <_b \), so \( y <_b z \). The converse is similar, it follows actually from the application of this partial result using \( <_b^{-1} \) and \( <_a^{-1} \) instead of \( <_a \) and \( <_b \).

Now assume that \( <_b|_{O_i} \neq <_a|_{O_i} \) for some \( O_i \), so \( x <_a y \) and \( y <_b x \) for some \( x, y \in O_i \). Let \( z, t \in O_i \). By the claim just above \( z <_a y \) iff \( y <_b z \), so by the same claim again \( z <_a t \) iff \( t <_b z \), which shows that \( <_b|_{O_i} = <_a|_{O_i}^{-1} \). This proves the equivalence.

Finally, let us assume that the assertions hold. By definition of \( \sim \), if \( O_i \) is not a singleton, \( x <_a y \) and \( y <_b x \) for some \( a, b \in A \) and \( x, y \in O_i \), so \( <_b|_{O_i} = <_a|_{O_i}^{-1} \) is witnessed. □

Theorem 26 Let \( A \) be a non-empty set of players and \( O \) be a non-empty set of outcomes and assume that every Player \( a \) in \( A \) has a strict linear preference \( <_a \) over \( O \). The following assertions are equivalent.

1. The collective \( <_a \)-lazy improvement terminates for every DAG game that is built using \( A \) and \( O \).

2. For all Players \( a \) and \( b \), and for all outcomes \( x, y, z \) such that \( z <_a y <_a x \), the patterns \( x <_b y <_b z \) and \( x <_b z <_b y \) are ruled out.
3. There exist a partition \( \{ O_i \}_{i \in I} \) of the set of outcomes and a strict linear order \(<\) over \( I \) such that every \( O_i \) contains one or two outcomes and \((i < j \land x \in O_j \land y \in O_i) \Rightarrow x <_a y \) for all Players \( a \).

**Proof** \( \emptyset \Rightarrow \exists \) By the implication \( \exists \Rightarrow \emptyset \) of Lemma 23 there exist a partition \( \{ O_i \}_{i \in I} \) and a strict linear order \(<\) over \( I \) satisfying \((i < j \land x \in O_j \land y \in O_i) \Rightarrow x <_a y \) for all Players \( a \). Since one may also assume that \( <_b|_{O_x} = <_a|_{O_x} \) is witnessed for all \( i \in I \), again by Lemma 23 and since the pattern \((z <_a y <_a x) \land (x <_b y <_b z) \) is forbidden, each \( O_i \) contains at most two outcomes.

\( \emptyset \Rightarrow \exists \) Let us first prove the implication for every two-player win-lose game with \( n \) non-leaf nodes. For this purpose, let us number the non-leaf nodes from 1 to \( n \) such that the root is numbered 1 and the numbers decrease (possibly by more than one) when moving along the directed edges. Note that this numbering is possible by topological sorting, i.e. linear extension. To every profile \( s \), let us associate a boolean \( n \)-tuple \( b(s) = (b(s)_1, \ldots, b(s)_n) \) such that \( b(s)_i = 1 \) if the owner of the \( i \)-th node wins according to the leaf induced by the subprofile rooted at the \( i \)-th node. Now it suffices to show that \( s 
lessdot \ s' \Rightarrow b(s) <_{lex} b(s') \) where \( <_{lex} \) is the usual lexicographic order: consider the node of lowest number \( k \) among the nodes (such as the root) that induce different outcomes in \( s \) and \( s' \), i.e. making \( a \) lose in \( s \) and win in \( s' \). The choices in \( s \) and \( s' \) must also differ at node \( k \) which is therefore owned by Player \( a \), so \( b(s)_k = 0 \) and \( b(s')_k = 1 \) and \( b(s)_i = b(s')_i \) for all \( i < k \).

Let us now prove the full statement together with a complexity bound. Let \( l(n) \) be the maximal length of the sequences of collective lazy improvements in win-lose DAG games with \( n \) non-leaf nodes. (These DAG games are finitely many up to isomorphism w.r.t. lazy improvement, so \( l(n) \) is well-defined.) Since DAG games are finite objects, one may assume without loss of generality that \( I \) (from assumption \( \exists \)) equals \( \{ 1, \ldots, m \} \). Let us prove by induction on \( m \) that every sequence of collective lazy improvement in a DAG game with \( n \) non-leaf nodes has length at most \( l(n) \cdot m \). First case, \( m = 1 \): if \( O_1 \) contains one outcome only, every profile is a Nash equilibrium; else let \( \{ x, y \} := O_1 \). The only difference with the two-player win-lose case above is that many players may be involved. Let \( A_x \) (resp. \( A_y \)) be the set of players who prefer \( x \) to \( y \) (resp. \( y \) to \( x \)) and let us derive a two-player game by relabelling the non-leaf nodes, i.e. by replacing the players in \( A_x \) with the meta-player \( A_x \) (resp. in \( A_y \) with \( A_y \)). Since lazy improvement in the original game can be faithfully simulated by lazy improvement in the derived game, this settles the case \( m = 1 \).

If the sequence starts at a profile inducing an outcome in \( O_1 \), after at most \( l(n) \) steps it must either stop or lead to a profile inducing an outcome in \( O_2 \cup \cdots \cup O_m \). To see this let us derive the "minor" DAG game that involves only the nodes of the original DAG game (and the edges in between) from where outcomes in \( O_1 \) may be reach. The game has at most \( n \) non-leaf nodes so, by the case \( m = 1 \), its collective lazy improvement sequences have length at most \( l(n) \). If the sequence does not start at a profile inducing an outcome in \( O_1 \), it will never reach such a profile since \( x <_a y \) for all \( a \in A \) and \( x \in O_1 \) and \( y \in \bigcup_{2 \leq i} O_i \), so let us now derive the "minor" DAG game that involves only the nodes of the original DAG game (and the edges in between) from where outcomes in \( \bigcup_{2 \leq i} O_i \) may be reach. By induction hypothesis, the sequence has length at most \( l(n) \cdot (m - 1) \), so the induction is proved by (appropriate) concatenation of sequences.
assume that $z <_a y <_a x$ and $x <_b y <_b z$. The picture below represents a cycle of length 11 of the collective lazy improvement in a "linear" DAG game. Each of the 11 parts of the picture represents a profile and should be understood as follows: the root of the game is owned by Player $a$, at the top; apart from the eight drawn, non-terminal nodes, there are three (non-drawn) terminal nodes enclosing outcomes $x$, $y$ and $z$; there is a (non-drawn) directed edge from each node to each other node that lies graphically beneath; there is a directed edge from each non-terminal node to each terminal node; the only edges that are actually drawn represent strategic choices; and Player $a$ (resp. $b$) choosing a terminal node enclosing $x$, $y$, $z$ is denoted $ax$, $ay$, $az$ (resp. $bx$, $by$, $bz$).

Likewise, the picture below represents a cycle of length eight when $x <_b z <_b y$.

The conditions on the preferences may be phrased this way: the players may disagree at the microscopic level but they must agree at the macroscopic level.

Finally, note that the bounds of algorithmic complexity that are used for $\leq$ in the proof above are not likely to be optimal: the bound $l(n) \cdot m$ might be (slightly) refined, and it would not be surprising if $l(n)$ were equal to $2^n$ instead of $2^n$, which comes from the lexicographic order.

### 3.4 Very lazy improvement

In Section 3.3 above, Definition 23 generalises the lazy convertibility, which was originally defined for games in extensive form only. That generalisation uses the characterisation from Observation 9.2 of lazy convertibility, whereas using the characterisation from Ob-
servation [9,3] leads to a stricter, i.e. set-theoretically smaller, very lazy convertibility. The difference between the two lazy-convertibility relations is as follows: the first one requires to minimise the choice changes when seeking to induce a given play, i.e. a given path from the root, whereas the second one requires to minimise the number of choice changes when seeking to induce a given terminal node (which may be induced by several distinct plays). For DAG games let us be even more restrictive and require minimality of choice changes when seeking to induce a given outcome.

It is actually possible to define meaningful, alternative very lazy convertibility relations by restricting \( \mathcal{C} \), e.g. by requiring to minimise the distance between the original play and the modified play when seeking to induce a given outcome (where the distance relates to the product topology on words), or by requiring to minimise the length of the modified play when seeking to induce a given outcome. One may hope that at least one of these three very lazy convertibility relations yields a terminating very lazy improvement when the preferences are acyclic, but the following picture dashes this hope.

The following picture is a (substantial) modification of the second cycle from Section 3.3. Four families of outcomes \( x, y, z, t \) are involved and the transitive preferences are defined by \( z_i \prec_a y_j \prec_a t_k \prec_a x_l \) and \( x_i \prec_b z_j \prec_b t_k \prec_b y_l \) whatever the subscripts may be; only the nodes and directed edges that are used (and displayed) at some point in the cycle actually exist in the underlying DAG game; and it is irrelevant to know which player owns the two bullet-point nodes.

The first cycle from Section 3.3 could also be modified to construct a similar counter-example, but it would probably require five outcomes and a substantially longer cycle.

Finally, note that it is still unclear whether an interesting version of Theorem 26 holds for any of these very lazy improvements.

### 3.5 Lazy does not beat crazy in win-lose DAG games

Theorems [12] and [18] not only show termination of the collective lazy improvement when all preferences are acyclic, they also show eventual stabilisation of players with acyclic preferences regardless of the preferences of their opponents. Likewise, Theorem 26 shows termination of the collective lazy improvement in a class of DAG games including win-lose two-player DAG games, but the picture below represents a cycle (up to symmetry) of lazy
improvement in a "win-lose" two-player DAG game where Player a prefers 1 over 0 and b has a cyclic preference. (Note that using 0, 0', 1, and 1' instead of 0 and 1 only would make the cycle below also valid for any of the three very lazy improvements mentioned in Section 3.4.)

The lazy-beats-crazy property, which holds in games in extensive form, is closely related to the existence of a measure that is decreasing for the lazily-improving player but left unchanged for the other players. The cycle above shows that such a measure does not exist for DAG games, which makes more challenging to improve the bound of algorithmic complexity of $l(n)$ from $2^n$ to, e.g., $2n$, as mentioned in the end of Section 3.3.

3.6 Lazy improvement in infinite games in extensive form

This section considers games that are built on infinite trees with finite branching; the outcomes are real-valued payoff functions and are assigned continuously to the plays, i.e. to the infinite paths starting at the root, where the underlying topology is the product topology when finite choice sets at each node are endowed with the discrete topology. An inductive definition of lazy convertibility seems to make little sense for infinite games, so let us use the alternative characterisations from Observations 9.2. Observation 27 below refers to the well-known definition of $\epsilon$-NE for some non-negative real number $\epsilon$: an $\epsilon$-NE is a strategy profile such that no player can improve upon her current payoff by more than $\epsilon$, so that the 0-NE are the NE.

Observation 27 Let us consider the specific infinite games that are described just above.

1. Lemma 10 still holds.

2. Theorem 12 still holds (even for infinitely many players) when considering lazy $\epsilon$-improvement and $\epsilon$-Nash equilibrium instead of lazy improvement and NE.

3. It is possible to give an algorithmic bound (following Theorems 18) that depends on $\epsilon$ and on the bounds and modulus of continuity of the payoff functions.

Proof (of 2) It is sufficient to consider one "rational" Player a (preferring greater payoffs) and one "crazy" Player b (preferring everything) who could represent all the opponents of a. The payoff function $f$ of Player a is continuous on a compact set, so its range is bounded; wlog it is included in $[0,1]$. Let $A_k := \left[\frac{\epsilon(k-1)}{2}, \frac{\epsilon(k+1)}{2}\right]$ for every natural number $k \leq \frac{2}{\epsilon}$, so that the sets $f^{-1}(A_k)$ constitute an open cover of the set of possible plays. Since the clopen balls form a basis of the product topology, each $f^{-1}(A_k)$ is a union of clopen balls, so by compactness there exists a finite clopen-ball cover $B_1, \ldots, B_m$ of the set of possible plays such that each $B_l$ is included in some $f^{-1}(A_{k_l})$. (When there are two candidates $k$ and $k+1$ for $k_l$, let us pick $k$.) Moreover, since two overlapping
clopen balls must relate by inclusion, let us choose the $B_1, \ldots, B_m$ pairwise disjoint. Since $|f(p) - \frac{c_k}{2}| < \frac{\epsilon}{2}$ whenever $p \in B_i$ by construction, let us approximate the payoff function $f$ by a piecewise constant function $f'$ such that $f'(p) := \frac{c_k}{2}$ whenever $p \in B_1$; this $f'$ amounts to a finite game in extensive form. This way, every lazy $\epsilon$-improvement $s \xrightarrow{\epsilon a} t$ in the infinite game defined by $f$ is simulated by a lazy improvement $s' \xrightarrow{a} t'$ in the finite game defined by $f'$, where $s'$ and $t'$ are the truncated (or partially defined) versions of $s$ and $t$ for the finite game: indeed, first note that $s \xrightarrow{\epsilon a} t$ implies $s' \xrightarrow{a} t'$, and then note that if $f \circ p(s) + \epsilon < f \circ p(t)$, where $p(s) \in B_1$ and $p(t) \in B_{1'}$ are the plays induced by the profiles $s$ and $t$, then $\frac{c_k}{2} < f \circ p(s) + \frac{\epsilon}{2} < f \circ p(t) - \frac{\epsilon}{2} < \frac{c_k}{2}'$, thus implying $f' \circ p(s') < f' \circ p(t')$. As for crazy $b$, each lazy improvement $s \xrightarrow{b} t$ in the infinite game corresponds to $s' \xrightarrow{b} t'$ in the finite game, possibly with $s' = t'$ although $s \neq t$. This shows that any sequence of lazy improvements in $f$ where $a$ occurs infinitely often would be simulated by a similar sequence in $f'$, which is ruled out by Theorem 12.

3.7 Evolution with perturbations

The end of Section 2 has introduced a toy application of Theorem 18 in evolutionary biology. Despite its many drawbacks, one selling point of this alternative model is its internalised mutation process. This process is not very subtle, though, so let us refine it now by adding small perturbations to the collective lazy improvement, after transforming the lazy improvement into a quantitative, i.e., stochastic process. The perturbation probabilities are meant to be extremely small in comparison to those of the lazy improvement, and the perturbations are restricted to occur only at NE through any atomic strategic change, i.e., at one single node of the game, that does not modified the induced outcome. In some sense, this is the smallest possible refinement of the model for which a new and interesting phenomenon arises, as exemplified in this section; but first, the notion of Markov chain of a game in extensive form is defined below.

**Definition 28 (Markov chain of a game in extensive form)** Given a game with $l$ leaves, let us define a Markov chain where the states are the possible strategy profiles for the game. Let $0 < p, \epsilon < \frac{1}{24}$ be to real numbers and let us define three types of transitions between states: a transition with probability $p$ that corresponds to lazy improvement; a transition with probability $\epsilon$ between each NE and each profile that disagrees with the NE at exactly one node and that induce the same outcome; and at each profile a self-loop transition whose probability is set by the other transitions.

**Observation 29** Given a game chain from Definition 28 consider the behaviour of its stationary distribution(s) as $\epsilon$ approaches zero. This behaviour is (qualitatively) independent from $p$, and a continuity-like argument shows that it converges towards a distribution that assigns zero to every non-NE of the game. Actually, the set of profiles being assigned a non-zero weight in this limit distribution would remain the same if each transition probability $p$ and $\epsilon$ were replaced with probabilities between $p$ and $\alpha p$ and between $\epsilon$ and $\alpha \epsilon$, respectively, for some $0 < \alpha$. (Under the condition, e.g., that $0 < (1 + \alpha)p, (1 + \alpha)\epsilon < \frac{1}{24}$)

Observation 29 leads to the following definition of evolutionary stable profile, which are all NE.
Definition 30 (Stable profiles of games in extensive form) A profile is stable if it is assigned a non-zero weight in some limit distribution of the Markov chain of the game.

The picture on the right-hand side below represents the Markov chain of the game on the left-hand side. The plain arrows represent the $p$-transitions and the dashed arrows the $\epsilon$-transitions. It is easy to see that the limit distribution assigns full weight to the only SPE of the game, upper-left profile, while the other NE is not stable, lower-right profile. By exhibiting a non-stable NE, this example shows that Definition 30 is non-trivial.

The example above raises the question whether the stable profiles are exactly the SPE, but the following counter-example says no. In the picture below the profiles are named from A to H. The only SPE of the game is A and there are two other NE, namely F and G. Again, the plain arrows represent the $p$-transitions and the dashed arrows the $\epsilon$-transitions.

\[
\begin{array}{cccc}
A & B & C & D \\
\begin{array}{c}
a \\
b \begin{array}{c}
\leftarrow \end{array}
\end{array} & \begin{array}{c}
a \\
b \begin{array}{c}
\leftarrow \end{array}
\end{array} & \begin{array}{c}
a \\
b \begin{array}{c}
\leftarrow \end{array}
\end{array} & \begin{array}{c}
a \\
b \begin{array}{c}
\leftarrow \end{array}
\end{array} \\
\begin{array}{c}
3,1 \\
3,1 \\
3,1 \\
3,1
\end{array} & \begin{array}{c}
0,0 \\
0,0 \\
0,0 \\
0,0
\end{array} & \begin{array}{c}
0,0 \\
0,0 \\
0,0 \\
0,0
\end{array} & \begin{array}{c}
0,0 \\
0,0 \\
0,0 \\
0,0
\end{array} \\
\end{array}
\]
The transition matrix $M$ of the Markov chain of the game is displayed below, where the letters from A to H still refer to the states/profiles and where the non-displayed values are zeros. Note that $M_{ij}$ represents the probability of transition from state $i$ to state $j$, whereas it is usually the other way round among the Markov-chain community.

$$
\begin{pmatrix}
    A & B & C & D & E & F & G & H \\
    A & 1 - \epsilon & p & p & & & & \\
    B & & 1 - 2p & p & & & & \\
    C & & & 1 - 2p & p & & & \\
    D & & & & 1 - p & \epsilon & p & \\
    E & & & & & 1 - p & \epsilon & \\
    F & & & & & & 1 - 2\epsilon & \epsilon & \\
    G & & & & & & & 1 - 2\epsilon & \\
    H & & & & & & & & 1 - p
\end{pmatrix}
$$

One can check that the vector $(7p, 3\epsilon, 6\epsilon, 12\epsilon, 4\epsilon, 4p, 5p, 5\epsilon)$ is an eigenvector of the matrix (actually the only one up to scalar multiplication). Therefore the limit distribution assigns weights $\frac{7}{16}$ to the SPE and $\frac{1}{4}$ and $\frac{3}{16}$ to the other NE. Moreover, since the two NE that are not SPE are linked by $\epsilon$-transitions both ways, it is reasonable to consider them together and to notice that the sum of their weights is greater than the weight of the SPE.

This refinement of the evolutionary model from Section 2 may still not be realistic from the point of view of evolutionary biology, which is difficult to assess now, but at least it lead to an interesting notion of stability for NE in games in extensive form. Furthermore, this notion may be relevant to generic evolution, albeit non-biological.

**Acknowledgements** I thank Volker Betz for explanations on Markov chains, Arno Pauly for inspiring discussions, *e.g.*, that lead to Section 3.6, and Victor Poupet and Martin Ziegler for discussions and useful corrections on drafts of the paper.

### 4 Conclusion

The deterministic backward induction had given some special Nash equilibria (the subgame perfect equilibria) in some special games in extensive form (where real-valued payoffs occur at most once) a very interesting meaning: they are the results of a complex reasoning that is based on the common-knowledge of rationality. Alternatively, the non-deterministic, lazy improvement now gives all Nash equilibria in all games in extensive form a new characterisation and a new meaning: they are the sinks of myopic and lazy evolution! Furthermore, should crazy players (or Mother Nature) evolve lazily along cyclic trends, this would by no means prevent the so-called rational players to stop evolving eventually, whence lazy beats crazy.

Under a necessary and sufficient condition on preferences, the termination of the lazy improvement at Nash equilibrium still holds for games that are built on directed acyclic graphs instead of mere tree, but the lazy-beats-crazy property does no longer hold even for very simple DAG games.
References

[1] Mario Amendola and Jean-Luc Gaffard. Out of Equilibrium. Oxford University Press, 1998.

[2] Robert J. Aumann. Backward induction and common knowledge of rationality. Games and Economic Behavior, 8:6–19, 1995.

[3] Emile Borel. La théorie du jeu et les équations intégrales à noyau symétrique. Comptes Rendus Hebdomadaires des séances de l’académie des Sciences, 173:1304–1308, 1921.

[4] L.E.J. Brouwer. über Abbildung von Mannigfaltigkeiten. Mathematische Annalen, 71:97–115, 1911.

[5] Jacques Hadamard. Note sur quelques applications de l’indice de kronecker. Jules Tannery: Introduction la théorie des fonctions d’une variable, 2:437477, 1910.

[6] Shizuo Kakutani. A generalization of Brouwer’s fixed point theorem. Duke Math. J., 8:457–459, 1941.

[7] Harold W. Kuhn. Extensive games and the problem of information. Contributions to the Theory of Games II, 1953.

[8] Stéphane Le Roux. Generalisation and formalisation in game theory. Ph.D. thesis, Ecole Normale Superiéure de Lyon, January 2008. Under supervision of Pierre Lescanne.

[9] Stéphane Le Roux. Acyclic preferences and existence of sequential Nash equilibria: a formal and constructive equivalence. In TPHOLs, International Conference on Theorem Proving in Higher Order Logics, Lecture Notes in Computer Science, pages 293–309. Springer, August 2009.

[10] John Nash. Equilibrium points in n-person games. Proceedings of the National Academy of Sciences, 36:48–49, 1950.

[11] John Nash. Non-cooperative games. Annals of Mathematics, 54, 1951.

[12] John von Neumann and Oskar Morgenstern. Theory of Games and Economic Behavior. Princeton Univ. Press, Princeton, 1944.

[13] Martin J. Osborne and Ariel Rubinstein. A Course in Game Theory. The MIT Press, 1994.

[14] Stéphane Le Roux. From winning strategy to Nash equilibrium. arXiv:1203.1866v3.

[15] Dov Samet. Hypothetical knowledge and games with perfect information. Games and Economic Behavior, 17:230–251, 1996.

[16] J. Maynard Smith and G.R. Price. The logic of animal conflicts. Nature, 246:15–18, 1973.

[17] J.W. Weibull. Evolutionary game theory. The MIT Press, 1997.