SPECTRAL BOOTSTRAP CONFIDENCE BANDS FOR LÉVY-DRIVEN
MOVING AVERAGE PROCESSES

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Abstract. In this paper we study the problem of constructing bootstrap confidence intervals for the Lévy density of the driving Lévy process based on high-frequency observations of a Lévy-driven moving average processes. Using a spectral estimator of the Lévy density, we propose a novel implementations of multiplier and empirical bootstraps to construct confidence bands on a compact set away from the origin. We also provide conditions under which the confidence bands are asymptotically valid.

1. INTRODUCTION

The continuous-time Lévy-driven moving average processes are defined as

\[ Z_t = \int_{-\infty}^{\infty} K(t-s) \, dL_s \]  

(1.1)

where \( K \) is a deterministic kernel and \( L = (L_t)_{t \in \mathbb{R}} \) is a two-sided Lévy process with a Lévy triplet \((\gamma, \sigma, \nu)\). The conditions which guarantee that this integral is well-defined are given in the pioneering work by Rajput and Rosinski [13]. For instance, if \( \int x^2 \nu(dx) < \infty \), it is sufficient to assume that \( K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \).

Continuous-time Lévy-driven moving average processes (and slightly modified versions of them) are widely used for the construction of many popular models such as Lévy-driven Ornstein-Uhlenbeck processes, fractional Lévy processes, CARMA processes, Lévy semistationary processes and ambit fields, cf. Barndorff-Nielsen, Benth and Veraart [1], Podolskij [12]. Most of these models can be applied to financial and physical problems. For instance, the choice \( K(t) = t^\alpha e^{-\lambda t} \mathbb{1}_{[0,\infty)}(t) \) with \( \lambda > 0 \) and \( \alpha > -1/2 \) (known as Gamma-kernel) is used for modeling volatility and turbulence, see e.g. Barndorff-Nielsen and Schmiegel [2]. Otherwise, the choice \( K(t) = e^{-\lambda t} (t) \) (known as well-balanced Ornstein-Uhlenbeck process) can be used for the analysis of the SAP high-frequency data, see Schnurr and Woerner [15].

This paper is devoted to statistical inference for continuous-time Lévy-driven moving average processes. Assuming that the high-frequency equidistant observations of the process \((Z_t)\) are given, we aim to estimate the characteristic triplet of the process \((L_t)\). Recently, Belomestny, Panov and Woerner [4] considered the statistical estimation of the Lévy measure \( \nu \) from the low-frequency observations.

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of the process \((Z_t)\). The approach presented in [4] is rather general - in particular, it works well under various choices of \(K\). Nevertheless, this approach is based on the superposition of the Mellin and Fourier transforms of the Lévy measure, and therefore its practical implementation can meet some computational difficulties. In [3], another method was presented, which essentially uses the following theoretical observation. For any kernel \(K\), the characteristic function \(\Phi(u) := E[e^{iuZ}]\) of the process \((Z_t)\) and the characteristic exponent \(\psi(u)\) of the process \((L_t)\) are connected via the formula:

\[
\Phi(u) = \exp \left( \int_{\mathbb{R}} \psi(uK(s)) \, ds \right).
\]

It was noted in [3] that under the choice \(K(x) = (1 - \alpha |x|)^\frac{1}{\alpha}\) this formula can be inverted without use of an additional integral transformations, that is, the function \(\psi\) can be represented via \(\Phi\) and its derivatives. Therefore, the characteristic exponent can be estimated from the observations of the process \((Z_t)\), and further application of the Fourier techniques leads to a consistent estimator of the Lévy triplet.

The current paper is devoted to the estimation of the Lévy measure \(\nu\) in the same model as in [3] but based on high-frequency observations of the process \((Z_t)\). Moreover, we are interested in uniform bootstrap confidence bands for \(\nu\). We propose a novel implementation of the multiplier and empirical bootstrap procedures to construct confidence bands on a compact set away from the origin. We also provide conditions under which the confidence bands are asymptotically valid. Our approach can be viewed as an extension of the recent work [10] where bootstrap confidence bands are constructed for the case of high-frequency observations of the Lévy process \((L_t)_{t \geq 0}\) itself.

The paper is organised as follows. In Section 2 we formulate our main statistical problem and propose an estimator for the underlying Lévy density \(\nu\). We also discuss how to construct confidence bands for \(\nu\). Section 3 contains a detailed description of the bootstrap procedure and results on the validity of the bootstrap confidence bands. Some numerical results on simulated data are shown in Section 4. Finally, in Section 5 all proofs are collected.

2. SET-UP

We shall consider continuous-time Lévy-driven moving average processes \((Z_t)_{t \geq 0}\) of the form:

\[
Z_t = \int_{-\infty}^{\infty} K(t - s) \, dL_s,
\]

where \(K\) is a symmetric kernel given by

\[
K_\alpha(x) = \begin{cases} 
(1 - \alpha |x|)^\frac{1}{\alpha}, & |x| \leq \alpha^{-1}, \\
0, & \text{else}
\end{cases}
\]

for some \(\alpha \in (0, 1), L = (L_t)_{t \in \mathbb{R}}\) is a two-sided Lévy process with the Lévy triplet \((\gamma, \sigma, \nu)\). Note that as a limiting case for \(\alpha \to 0\) we get the exponential kernel \(K_0(x) = \exp(-x)\). It follows from [13] that the process \((Z_t)\) is well-defined and
ininitely divisible with the characteristic function:

$$E[e^{iuZ_t}] = \exp \left\{ iu\gamma_Z(t) - \frac{1}{2}u^2\sigma_Z^2(t) + \int_{\mathbb{R}} \left[ e^{iux} - 1 - iux1_{\{|x|\leq 1\}} \right] \nu_Z(t, dx) \right\},$$

where

$$\gamma_Z(t) = \gamma \int K(t-s) \, ds + \int \int xK(t-s) \left[ 1_{\{|xK(t-s)|\leq 1\}} - 1_{\{|x|\leq 1\}} \right] \nu(dx) \, ds,$$

$$\sigma_Z^2(t) = \sigma^2 \int K^2(t-s) \, ds,$$

and

$$\nu_Z(t, dx) = \int \int 1_{B(xK(t-s))} \nu(dx) \, ds, \quad B \in B(\mathbb{R}).$$

Furthermore, under our choice of the kernel function $K$, we can represent the characteristic exponent $\psi$ of the Lévy-process $(L_t)$ via the characteristic function $\Phi_{\Delta}$ of the increments $Z_{t+\Delta} - Z_t$. We explicitly derive for $\Psi_{\Delta}(u) := \log(\Phi_{\Delta}(u))$ (see Lemma 3),

$$\Psi_{\Delta}(u) = \int_{-\infty}^{\infty} \psi(u(K_{\alpha}(x + \Delta) - K_{\alpha}(x))) \, dx = (L_{\alpha}\psi)(\Delta u) + S_{\Delta}(u),$$

where the operator $L_{\alpha}$ is defined as

$$(L_{\alpha}f)(x) := \frac{2}{1-\alpha} x^{-\frac{\alpha}{1-\alpha}} \int_0^x f(z) z^{\frac{2\alpha-1}{1-\alpha}} \, dz$$

for any locally bounded function $f$ and

$$\psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} \left( e^{iux} - 1 - iux1_{\{|x|\leq 1\}} \right) \nu(dx).$$

Moreover, if

$$\int |x|^p \, \nu(x) \, dx < \infty$$

for some natural $p > 1$ then the function $S_{\Delta}$ satisfies (see Lemma 4)

$$\lim_{\Delta \to 0} \Delta^{-(l+\alpha)} \mathcal{K}_\Delta^{(l)}(u/\Delta) = 0, \quad l = 0, \ldots, p,$$

and as a result we have convergence

$$\Psi_{\Delta}(u/\Delta) \to \Psi(u) := L_{\alpha}\psi(u), \quad u \in \mathbb{R}$$

for $\Delta \to 0$. Furthermore by inverting the operator $L_{\alpha}$, we get from (2.3)

$$\psi''(u) = \frac{1}{\Delta^2} (L_{\alpha}^{-1}\Psi_{\Delta})''(u/\Delta) - \frac{1}{\Delta^2} (L_{\alpha}^{-1}S_{\Delta})''(u/\Delta)$$

with

$$(L_{\alpha}^{-1}f)(x) := \frac{\alpha}{2} f(x) + \frac{1-\alpha}{2} x f'(x).$$

On the other hand, under the condition (2.5) with $p = 2$, we obtain from (2.4),

$$\psi''(u) = -\sigma^2 - \int_{\mathbb{R}} e^{iux} \rho(x) \, dx,$$
where \( \rho(x) := x^2 \nu(x) \). Therefore, we can apply the inverse Fourier transform to get

\[
\rho(x) = -\frac{1}{2\pi \Delta^2} \int_\mathbb{R} e^{-iux} [(\mathcal{L}_\alpha^{-1} \Psi_\Delta)'(u/\Delta) + \Delta^2 \sigma^2] \, du - R_\Delta(u),
\]

where

\[
R_\Delta(u) := \int_\mathbb{R} e^{-iux} r_\Delta(u) \, du, \quad r_\Delta(u) := (1/\Delta)^2 (\mathcal{L}_\alpha^{-1} S_\Delta)'(u/\Delta).
\]

In view of (2.6) and (2.8), the term \( R_\Delta \) is of smaller order in \( \Delta \) than the first term in (2.9) and we can consider the limiting case (2.7) in (2.9).

In this work we assume that we observe a discretised (high-frequency) trajectory of the limiting Lévy process \( X_0, X_\Delta, \ldots, X_{n\Delta} \) with characteristic function \( \Phi(u) := \mathbb{E}[\exp(iuX_1)] = \exp(\Psi(u)) = \exp(\mathcal{L}_\alpha \psi(u)) \). This assumption is mainly done to simplify analysis and avoid difficulties related to the time dependence structure of the process \( (Z_t) \). Still the main features of the underlying inverse problem (e.g. the structure of the inverse operator \( \mathcal{L}_\alpha^{-1} \)) remains reflected in our statistical analysis. An extension to the case where one directly observes the process \( (Z_t)_{t \geq 0} \) will also be discussed.

Let us now describe our estimation procedure. Let \( W \) be an integrable kernel function such that

\[
\int_\mathbb{R} W(x) \, dx = 1, \quad \int_\mathbb{R} |x|^{p+1} |W(x)| \, dx < \infty, \quad \int_\mathbb{R} x^l W(x) \, dx = 0, \quad l = 1, \ldots, p,
\]

and suppose that the Fourier transform \( \varphi_W \) of \( W \) is supported in \([-1, 1]\). Motivated by (2.9), we propose to estimate \( \rho \) via the estimator:

\[
\hat{\rho}_n(x) := -\frac{1}{2\pi} \int_\mathbb{R} e^{-iux} \left[ (\mathcal{L}_\alpha^{-1} \hat{\Psi})''(u) + \hat{\sigma}_n^2 \right] \varphi_W(uh_n) \, du,
\]

where \( \hat{\Psi} := \Delta^{-1} \log(\hat{\Phi}_{\Delta X}) \) with

\[
\hat{\Phi}_{\Delta X}(u) := -\frac{1}{n} \sum_{j=1}^n e^{iu(\Delta X)_j}, \quad u \in \mathbb{R},
\]

\((\Delta X)_j := X_{\Delta j} - X_{\Delta (j-1)}\), \( h_n \) is a sequence of positive numbers (bandwidths) such that \( h_n \to 0 \) as \( n \to \infty \), and \( \hat{\sigma}_n^2 \) is an estimator of \( \sigma^2 \). Our aim is to construct confidence bands for the transformed Lévy density \( \rho \) on a compact set \( I \) in \( \mathbb{R} \setminus \{0\} \) and to prove validity of the proposed confidence bands. To this end, we shall use the Gaussian multiplier (or wild) bootstrap.

3. Main results

3.1. Construction of confidence bands. Using the equations (2.9) and (2.10), the difference \( \hat{\rho}_n(x) - \rho(x) \) can be represented as

\[
\hat{\rho}_n(x) - \rho(x) = \underbrace{(\hat{\rho}_n(x) - \bar{\rho}(x))}_{R_n(x)} + \underbrace{(\bar{\rho}(x) - \rho(x))}_{I_{\alpha, n}(x) + I_{\sigma, n}(x)},
\]
where
\[
\tilde{\rho}(x) := -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \left[ (\mathcal{L}_\alpha^{-1}(\Psi'))''(u) + \Delta \sigma^2 \right] \varphi_W(uh_n) \, du,
\] (3.2)

\[
I_{\sigma_n^2}(x) := -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} (\tilde{\sigma}_n^2 - \sigma^2) \varphi_W(uh_n) \, du,
\]

\[
I_{\rho_n}(x) := \left[ \rho + (h_n^{-1}W(\cdot/h_n)) \right](x) - \rho(x).
\]

Later we show that under suitable assumptions (Assumption 1), the terms \(I_{\rho_n}\) and \(I_{\sigma_n^2}\) are asymptotically (as \(n \to \infty\) and \(\Delta \to 0\)) smaller than \(R_n\) and hence can be neglected when constructing the confidence interval for the transformed Lévy density \(\rho\). Further note that

\[
(\mathcal{L}_\alpha^{-1}(\Psi'))''(u) - (\mathcal{L}_\alpha^{-1}(\Psi'))''(u) = Q_0(u)\mathbb{D}(u) + Q_1(u)\mathbb{D}'(u) + Q_2(u)\mathbb{D}''(u) + Q_3(u)\mathbb{D}'''(u),
\]

where \(\mathbb{D}(u) := \hat{\Phi}_{\Delta X}(u) - \Phi_{\Delta X}(u), \Phi_{\Delta X}(u) := \exp(\Delta \mathcal{L}_\alpha \psi(u))\) and

\[
Q_0(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( \frac{2 - \alpha}{2} \left( \frac{\Phi_{\Delta X}'(u)}{\Phi_{\Delta X}(u)} \right)^2 - \frac{\Phi_{\Delta X}''(u)}{\Phi_{\Delta X}(u)} \right) + \frac{1 - \alpha}{2} u \left( \frac{\Phi_{\Delta X}''(u)}{\Phi_{\Delta X}(u)} + 6 \frac{\Phi_{\Delta X}'(u)\Phi_{\Delta X}'(u)}{(\Phi_{\Delta X}(u))^2} - 4 \left( \frac{\Phi_{\Delta X}'(u)}{\Phi_{\Delta X}(u)} \right)^3 \right)
\]

\[
= \frac{\Delta}{\Phi_{\Delta X}(u)} \left( \frac{2 - \alpha}{2} \left( \Delta(\Psi'(u))^2 - \Psi''(u) \right) - u \frac{1 - \alpha}{2} (\Psi''(u) - 3\Delta \Psi''(u)\Psi'(u) + \Delta^2(\Psi'(u))^3) \right),
\]

(3.4)

\[
Q_1(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( 3u \left( 1 - \alpha \right) \left( \frac{\Phi_{\Delta X}'(u)}{\Phi_{\Delta X}(u)} \right)^2 - 3u \frac{\Phi_{\Delta X}''(u) - 1 - \alpha}{2} \Phi_{\Delta X}(u) \right)
\]

\[
= \frac{\Delta}{\Phi_{\Delta X}(u)} \left( 3u \frac{1 - \alpha}{2} \left( \Delta(\Psi'(u))^2 - \Psi''(u) \right) - \left( 2 - \alpha \right) \Psi'(u) \right),
\]

(3.5)

\[
Q_2(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( \frac{2 - \alpha}{2} - 3u \frac{\Phi_{\Delta X}'(u) - 1 - \alpha}{2} \right)
\]

\[
= \frac{1}{\Phi_{\Delta X}(u)} \left( \frac{2 - \alpha}{2} - 3u \Delta \Psi'(u) \frac{1 - \alpha}{2} \right),
\]

(3.6)

\[
Q_3(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( \frac{1 - \alpha}{2} \right).
\]

With the above notations \(R_n(x)\) becomes

\[
R_n(x) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixu} \left[ Q_0(u)\mathbb{D}(u) + Q_1(u)\mathbb{D}'(u) + Q_2(u)\mathbb{D}''(u) + Q_3(u)\mathbb{D}'''(u) \right] \varphi_W(uh_n) \, du
\]

(3.7)
The representation (3.8) is crucial for our analysis. Consider now the process
\[ T_n(x) := \frac{\sqrt{n}}{s(x)} R_n(x) \]
(3.9)
where \( s^2(x) \) is given by
\[ s^2(x) := \text{Var} \left[ \sum_{m=0}^{3} i^m (\Delta X)^m K_{m,n}(x - (\Delta X)_1) \right] \]
(3.10)
Under some conditions, we shall show that there exists a tight \( \ell^\infty(1) \)-sequence of Gaussian random variables \( T_n^G \) with zero mean and the same covariance function as one of \( T_n \), and such that the distribution of \( \|T_n^G\|_I := \sup_{x \in I} |T_n^G(x)| \) asymptotically approximates the distribution of \( \|T_n\|_I \) in the sense that
\[ \sup_{z \in \mathbb{R}} |P \{ \|T_n\|_I \leq z \} - P \{ \|T_n^G\|_I \leq z \} | \to 0, \quad n \to \infty. \]
Accordingly, the construction of confidence bands reduces to estimating the quantiles of the r.v. \( T_n^G \). To this end we shall use bootstrap. Define
\[ c_n^G(1-\tau) := \inf \{ z \in \mathbb{R} : P \{ \|T_n^G\|_I \leq z \} \geq 1 - \tau \} \]
for \( \tau \in (0,1) \), then the \( 1-\tau \)-confidence band for \( \rho \) is of the form:
\[ \widehat{C}_{1-\tau}(x) = \left[ \tilde{\rho}_n(x) - \frac{s(x)}{\sqrt{n}\Delta} c_n^G(1-\tau), \tilde{\rho}_n(x) + \frac{s(x)}{\sqrt{n}\Delta} c_n^G(1-\tau) \right], \quad x \in I. \]
Since \( \rho(x) \in \widehat{C}_{1-\tau}(x) \) for all \( x \in I \) means that
\[ \left\| \frac{\sqrt{n}\Delta(\tilde{\rho}_n(\cdot) - \rho(\cdot))}{s(\cdot)} \right\|_I \leq c_n^G(1-\tau), \]
we can show that
\[ P \left\{ \rho(x) \in \widehat{C}_{1-\tau}(x), \quad \forall x \in I \right\} = P \{ \|T_n^G\|_I \leq c_n^G(1-\tau) \} + o(1) \]
as \( n \to \infty \). Hence \( \widehat{C}_{1-\tau}(x) \) is a valid confidence band for \( \rho \) on \( I \) with an approximate level \( 1-\tau \). However, we still need to estimate the quantile \( c_n^G(1-\tau) \). In what follows we consider the Gaussian multiplier (or wild) bootstrap to estimate the quantile \( c_n^G(1-\tau) \).
Gaussian multiplier bootstrap. The main idea of the Gaussian multiplier bootstrap consists in reweighting estimated influence functions using mean zero and unit variance pseudo-random variables, see, e.g. [8] for more details. On the one hand, the advantage of this method compared to the conventional bootstrap is that it can avoid recomputing the estimator in each bootstrap repetition, and as a result we reduce the calculation time. On the other hand, one of the disadvantages of the Gaussian multiplier bootstrap is that it is necessary to obtain an analytical expression for the corresponding influence function. In our case, this method will be used as follows. First we simulate $N$ independent centred Gaussian random variables $\omega_1, \ldots, \omega_n \sim N(0, 1)$, independent of the data $D_n = \{(\Delta X)_j\}_{j=0}^n$ and construct the multiplier process $\hat{T}_{MB}^n(x)$ of the form:

$$\hat{T}_{MB}^n(x) := \frac{1}{s_n(x)\sqrt{n}} \left( \sum_{m=0}^3 \left( \sum_{j=1}^n \omega_j \{i^m(\Delta X)_{ij}^{m} \hat{K}_{m,n}(x - (\Delta X)_j) - n^{-1} \sum_{j=1}^n i^m(\Delta X)_{ij}^{m} \hat{K}_{m,n}(x - (\Delta X)_j) \} \right) \right),$$

where

$$\hat{K}_{m,n}(z) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuz} \hat{Q}_m(u) \varphi_W(uh_n) \, du$$

and $\hat{Q}_m(u)$ is based on a bootstrapped version of the empirical characteristic function $\hat{\Phi}_{\Delta X}$:

$$\hat{\Phi}_{\Delta X}(u) := \frac{1}{n} \sum_{j=1}^n \omega_j e^{iu(\Delta X)_j}, \quad u \in \mathbb{R}.$$ 

Furthermore, we estimate $c_n^G(1 - \tau)$ using quantile $c_{MB}^{\hat{\omega}}(1 - \tau)$ of the distribution of $\|\hat{T}_{MB}^n\|_1$, conditional on the data $D_n$. The latter quantity can be computed via simulations. As a result, the confidence band takes the form

$$\hat{C}_{\hat{\omega}}^{\hat{\omega}}(x) := \left[ \hat{\rho}_n(x) - \frac{\hat{s}_n(x)}{\sqrt{n\Delta}} c^{\hat{\omega}}_n(1 - \tau), \hat{\rho}_n(x) + \frac{\hat{s}_n(x)}{\sqrt{n\Delta}} c^{\hat{\omega}}_n(1 - \tau) \right], \quad x \in I.$$

3.2. Validity of bootstrap confidence bands. In this section, we will present the main result, which proves the validity of the confidence band $\hat{C}_{\hat{\omega}}^{\hat{\omega}}(x)$.

Assumption 1. We assume that the following conditions are fulfilled.

(i) $\int_{\mathbb{R}} |x|^{6+\varsigma} \nu(x) \, dx < \infty$ for some $\varsigma \in [0, 1].$

(ii) Let $r > 0$ and let $p$ be an integer such that $p < p \leq p + 1$. The function $\rho$ is $p$-times differentiable, and $(\rho)^p$ is $(r - p)$-Hölder continuous.

(iii) It holds $h_n^3 \gtrsim \Delta, \ h_n^{r+1} \Delta^{1/2} n^{1/2}(\log h_n^{-1})^{-1} \to 0$ and $(n\Delta h_n^3)^{-1/2}(\log n)^{1/2} \to 0$.

\footnote{The function $f : \mathbb{R} \to \mathbb{R}$ is called $\alpha$-Hölder continuous for $\alpha \in (0, 1]$, if
$$\sup_{x,y \in \mathbb{R}, x \neq y} \frac{|f(y) - f(x)|}{|y - x|^\alpha} < \infty.$$}
(iv) The estimator $\hat{\sigma}^2$ satisfies
\[ |\sigma^2 - \hat{\sigma}^2| \cdot \|h_n^{-1}W(\cdot/h_n)\|_I = o_P(\Delta^{-1/2}h_n^{-1}n^{-1/2}\log h_n^{-1}). \]

**Discussion.** Condition (i) is a moment condition and is equivalent to finiteness of \((6+)\)-th moment of the increments process \(X_{i+\Delta} - X_i\) (see Lemma 8 for more details). And finally, Condition (iv) guarantees that the term \(|\sigma^2 - \hat{\sigma}^2| \cdot \|h_n^{-1}W(\cdot/h_n)\|_I\) is of smaller order as compared to the order of the leading term in \(\hat{\rho}_n(x) - \rho(x)\).

Now we formulate the main theorem of this section, which shows the convergence of the proposed Gaussian approximation.

**Theorem 1.** (Gaussian approximation)
Under our assumptions, for sufficiently large \(n\), there exists a tight Gaussian random variable \(T_n^G\) in \(\ell^\infty(I)\) with zero mean and covariance function of the form
\[ W(x, y) := I_\Delta[B_1(x, \cdot)B_1(y, \cdot)] - I_\Delta[B_1(x, \cdot)]I_\Delta[B_1(y, \cdot)] \]
\[ + I_\Delta[B_2(x, \cdot)B_2(y, \cdot)] - I_\Delta[B_2(x, \cdot)]I_\Delta[B_2(y, \cdot)] \]
\[ + i[I_\Delta[B_1(y, \cdot)B_2(x, \cdot)] - I_\Delta[B_1(x, \cdot)B_2(y, \cdot)]] \]
\[ + i[I_\Delta[B_1(x, \cdot)]I_\Delta[B_2(y, \cdot)] - I_\Delta[B_1(y, \cdot)]I_\Delta[B_2(x, \cdot)]] \]
the integral operator \(I_\Delta\) is defined as \(I_\Delta[f] := \int f(v) P_\Delta(dv)\), \(s(x) = \sqrt{s^2(x)}\) has the form (5.22) and
\[ B_1(x, v) := K_{0,n}(x - v) + \sigma^2K_{2,n}(x - v), \]
\[ B_2(x, v) := \sigma K_{1,n}(x - v) - \sigma^3K_{3,n}(x - v). \]
Moreover it holds
\[ \left| \left| T_n \right| \right|_I - \left| \left| T_n^G \right| \right|_I = o_P\left(h_n^{1/2}\log h_n^{-1}\right), \quad n \to \infty. \]

Building on Theorem 2, the following result formally establishes the asymptotic validity of the multiplier bootstrap confidence band \(\hat{C}^{MB}_{1-\tau}(x)\).

**Theorem 2.** (Validity of bootstrap confidence bands). Under Assumption 1 we have that
\[ P\{\rho(x) \in \hat{C}^{MB}_{1-\tau}(x), \quad \forall x \in I\} \to 1 - \tau, \]
as \(n \to \infty\). Moreover the supremum width of the confidence band of \(\hat{C}^{MB}_{1-\tau}(x)\) is of order \(O_p\left((n\Delta h_n^3)^{-1/2}\log n\right)\).

**Discussion on choosing for \(\Delta, n\) and \(h_n\).** From the lemma 12 applies \(\inf_{x \in I} s^2(x) \gtrsim \Delta h_n^{-3}\), which leads to the first assumption 1 (iii), namely \(h_n^3 \gtrsim \Delta\). According to the representation 3.1 applies
\[ \hat{\rho}_n(x) - \rho(x) = \left(\hat{\rho}_n(x) - \bar{\rho}(x)\right) + \left(\bar{\rho}(x) - \rho(x)\right). \]
Under the condition of the dominance of the convergence rate of the first term \(\|R_n(x)\|_R = O((\Delta^{-1/2}h_n^{-1}n^{-1/2}\log h_n^{-1}))\) follows assumption 1 (iv), namely \(|\sigma^2 - \hat{\sigma}^2| \cdot \|h_n^{-1}W(\cdot/h_n)\|_I = o_P(\Delta^{-1/2}h_n^{-1}n^{-1/2}\log h_n^{-1})\) and assumption 1 (iii), namely
From the proof of the theorem 1 it follows that \( h_{n}^{r+1} \Delta^{1/2} n^{1/2} (\log h_{n}^{-1})^{-1} \rightarrow 0 \). If the two terms are supposed to be significant, the last condition is represented in the form \( h_{n}^{r+1} \Delta^{1/2} n^{1/2} (\log h_{n}^{-1})^{-1} \geq 1 \), which leads to a relationship between \( n \) and \( h_{n} \). Let \( \log h_{n}^{-1} \lesssim \log n \lesssim n^{\epsilon} \), where \( \epsilon \rightarrow 0 \) then applies

\[
\begin{align*}
    h_{n}^{r} & \lesssim \Delta^{-1/2} n^{-1/2} \log h_{n}^{-1} \\
    h_{n}^{r} & \lesssim h_{n}^{-5/2} n^{-1/2} \log h_{n}^{-1} \\
    h_{n}^{r+5/2} & \lesssim n^{-1/2 + \epsilon} \\
    h_{n} & \lesssim n^{-1/2 + \epsilon} \\
    n & \lesssim h_{n}^{-2+\epsilon}.
\end{align*}
\]

From the proof of the theorem 1 it follows that

\[
\frac{\sqrt{n} \Delta (\hat{p}_{n}(x) - p(x))}{s(x)} = T_{n}(x) + O_{P}\left(h_{n}^{1/2} \log h_{n}^{-1}\right)
\]

and

\[
\left\| \|G_{n}\|_{\mathcal{F}} - V_{n}\right\| = O_{P}\left\{ \frac{\log n^{1+1/q}}{n^{1/2 - 1/q} \sqrt{\Delta h_{n}^{-1}}} + \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \right\} = O_{P}\left(\frac{\log n}{(\Delta h_{n}^{-1})^{1/6}}\right)
\]

further follows

\[
\begin{align*}
    h_{n}^{1/2} \log h_{n}^{-1} & \gg \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \\
    h_{n}^{1/2} \log h_{n}^{-1} & \gg \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \\
    h_{n}^{5/6} \log h_{n}^{-1} & \gtrsim n^{-1/6} \log n \\
    h_{n} & \gtrsim n^{-1/5}.
\end{align*}
\]

We also find the relationship between \( n \) and \( h_{n} \) so that the error of the Gaussian approximation \( \left\| \|G_{n}\|_{\mathcal{F}} - V_{n}\right\| \) is comparable to the approximation error \( \left\| T_{n}(x) \right\|_{\mathcal{F}} \). Let \( \log h_{n}^{-1} \lesssim \log n \lesssim n^{\epsilon} \), where \( \epsilon \rightarrow 0 \), then applies

\[
\begin{align*}
    h_{n}^{1/2} \log h_{n}^{-1} & \gtrsim \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \\
    h_{n}^{1/2} \log h_{n}^{-1} & \gtrsim \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \\
    h_{n}^{5/6} \log h_{n}^{-1} & \gtrsim n^{-1/6} \log n \\
    h_{n} & \gtrsim n^{-1/5}.
\end{align*}
\]

Furthermore, it should be noted that the bootstrap approximation \( \left\| T_{n}^{MB}(x) \right\|_{\mathcal{F}} \) of a Gaussian process \( \left\| T_{n}^{G}(x) \right\|_{\mathcal{F}} \) according to the theorem 2 has the order

\[
\left\| \|G_{n}\|_{\mathcal{F}} - V_{n}^{G}\right\| = O_{P}\left\{ \frac{\log n^{2+1/q}}{n^{1/2 - 1/9} \sqrt{\Delta h_{n}^{-1}}} + \frac{\log n^{7/4 + 1/q}}{(\Delta h_{n}^{-1})^{1/4}} \right\} = O_{P}\left(\frac{\log n^{7/4 + 1/q}}{(\Delta h_{n}^{-1})^{1/4}}\right).
\]

Therefore the rate of convergence of this approximation is faster than the one mentioned above in the theorem 1 order, namely applies

\[
\begin{align*}
    h_{n}^{1/2} \log h_{n}^{-1} & \gg \frac{\log n}{(\Delta h_{n}^{-1})^{1/6}} \gg \frac{\log n^{7/4 + 1/q}}{(\Delta h_{n}^{-1})^{1/4}}.
\end{align*}
\]
It is also important to note that according to the theorem 2, the the supremum width of the confidence band should also converge to 0:
\[
(n\Delta h_n^3)^{-1/2} \sqrt{\log n} \rightarrow 0.
\]
Since \( h_n^{-3/2} (\log n)^3 (\log h_n^{-1})^{-3} \ll h^{-2} \sqrt{\log n} \) then applies
\[
\begin{align*}
h_n^{-1/2} (n\Delta h_n^{-1})^{-1/6} \log n (\log h_n^{-1})^{-1} & \rightarrow 0, \\
h_n^{1/2} \log h_n^{-1} (n\Delta h_n^{-1})^{1/6} (\log n)^{-1} & \rightarrow \infty
\end{align*}
\]
if assumption \((n\Delta h_n^3)^{-1/2} \sqrt{\log n} \rightarrow 0\) satisfies. Since the expression \((n\Delta h_n^3)^{-1/2} \sqrt{\log n} \rightarrow 0\) has a slower order of convergence than the expression \(h_n^{-1/2} (n\Delta h_n^{-1})^{-1/6} \log n (\log h_n^{-1})^{-1} \rightarrow 0\), then the expression \(h_n \gtrsim n^{-1/5}\) should be specified:
\[
\begin{align*}
(n\Delta h_n^3)^{-1/2} \sqrt{\log n} & \rightarrow 0 \\
(nh_n^6)^{-1/2} \sqrt{\log n} & \rightarrow 0 \\
h_n^{-3} & \lesssim n^{1/2(1-\varepsilon)} \\
h_n^{-1/6(1-\varepsilon)} & \gtrsim n^{-1/6(1-\varepsilon)} \\
& \gtrsim n^{-6/(1-\varepsilon)},
\end{align*}
\]
where \(\log n \ll n^\varepsilon, \varepsilon \rightarrow 0\). The supremum width of the confidence band is minimal if \(n \approx h_n^{-\frac{2+\alpha}{2+\alpha}}\), da \((n\Delta h_n^3)^{-1/2} \sqrt{\log n} \rightarrow 0 \iff h^{-6} \log n/n \rightarrow \min\) applies. Then the following applies to the relationship between \(n\) and \(h_n\):
\[
\begin{align*}
h_n^{-6/(1-\varepsilon)} & \leq n \leq h_n^{-\frac{2+\alpha}{2+\alpha}} \\
n^{-1/6(1-\varepsilon)} & \leq h_n \leq n^{-\frac{1-2\alpha}{5-\alpha}}.
\end{align*}
\]

4. NUMERICAL RESULTS

Consider the integral (2.1) with the kernel \(K_\alpha\) from the class (2.2) for some \(\alpha \in (0, 1)\), and the Lévy process \((L_t)\) defined by
\[
L_t = \gamma t + \sigma W_t + CPP_t^{(1)} \cdot 1\{t \geq 0\} + CPP_t^{(2)} \cdot 1\{t < 0\},
\]
(4.1)
\[
CPP_t^{(k)} := \sum_{j=1}^{N_t^{(k)}} Y_j^{(k)}, \quad k = 1, 2,
\]
where \(\gamma \in \mathbb{R}\) is a drift, \(\sigma \geq 0\), \(W_t\) is a Brownian motion, \(N_t^{(1)}, N_t^{(2)}\), are two Poisson processes with intensity \(\lambda, Y_1^{(1)}, Y_2^{(1)}\), ..., and \(Y_1^{(2)}, Y_2^{(2)}\), are i.i.d. rv’s with an absolutely continuous distribution, and all \(Y_t, N_t^{(1)}, N_t^{(2)}, W_t\) are jointly independent. For simulation study, we take \(\gamma = 5, \lambda = 1\) and \(\sigma = 0\), and aim to estimate the corresponding Lévy density of \((L_t)\) under different choices of the parameter \(\alpha\), namely \(\alpha = 0.5, 0.8\) and \(0.9\).

Simulation. Recall that the Lévy-driven moving average process \(Z_t\) satisfying 2.1 is observed at \(n\) discrete instants \(t_j = j\Delta, j = 1, \ldots, n\), with regular sampling interval and our estimation procedure is based on the random variables \((\Delta Z)_{1}, \ldots, (\Delta Z)_{n}\) which are independent, identically distributed, with common characteristic function \(\Phi\). We assume that, as \(n\) tends to infinity, \(\Delta = \Delta_n\) tends to 0 and \(n\Delta\) tends to infinity.
For $k = 1, 2$, denote the jump times of $L_t^{(k)}$ by $s_1^{(k)}, s_2^{(k)}, \ldots$, corresponding to the jump sizes $Y_1^{(k)}, Y_2^{(k)}, \ldots$. $Y_1^{(k)}$ and $Y_2^{(k)}$ are independent r.v’s with standard exponential distribution with parameter $\lambda$. Note that

$$Z_t = \begin{cases} 
\frac{2\gamma}{1+\alpha} + \sum_{j \in J^{(1)}} \left(1 - \alpha |t - s_1^{(j)}| \right)^{1/\alpha} Y_1^{(j)}, & \text{if } t \geq \frac{1}{\alpha} \\
\frac{2\gamma}{1+\alpha} + \sum_{j \in J^{(2)}} \left(1 - \alpha |t - s_1^{(j)}| \right)^{1/\alpha} Y_1^{(j)} + \sum_{j \in J^{(3)}} \left(1 - \alpha |t + s_2^{(j)}| \right)^{1/\alpha} Y_2^{(j)}, & \text{if } t < \frac{1}{\alpha},
\end{cases}$$

where

$$J^{(1)} := \left\{ j : t - \frac{1}{\alpha} \leq s_1^{(j)} \leq t + \frac{1}{\alpha} \right\},$$

$$J^{(2)} := \left\{ j : 0 \leq s_1^{(j)} \leq t + \frac{1}{\alpha} \right\},$$

$$J^{(3)} := \left\{ j : 0 \leq s_2^{(j)} \leq \frac{1}{\alpha} - t \right\}.$$

Finally, the limiting Lévy process is defined by

$$X_j := \frac{(Z_j \Delta - Z_{(j-1)\Delta})}{\Delta}, j = 1, \ldots, n.$$

Typical trajectory of the of the limiting Lévy process $X_t := (\Delta Z)_t/\Delta$ is presented in Figure 4.1.

**Figure 4.1.** Typical trajectory of the limiting Lévy process $X_t := (\Delta Z)_t/\Delta$ with the value of the parameter $\alpha = 0.5$.

**Estimation.** Following the ideas from Section 2, we estimate the transformed Lévy measure by Equation (2.10) under different choices of $\alpha$. To show the convergence properties of the considered estimates, we provide simulations with different values of $n$. Figure 4.2 shows an estimate of the real part of the characteristic exponent $\psi$ of the Lévy process $(L_t)$ through discrete observations of the limit Lévy process $(X_t)$. It is important to note that a good estimate of the characteristic exponent $\psi(u)$ is obtained when $u \in (0, 2)$. Figure 4.3 shows the estimator of the
transformed Lévy density $\rho$ through discrete observations of the Limit-Lévy process $(X_t)$. The estimation of the Lévy densities based on 25 simulation runs are presented in Figures 4.4.

On the one hand, a priori choice for the parameter $h_n$ can be found using the interval for $u$ where the characteristic function of the process $\Delta X$ can be approximated by empirical characteristic function. On the other hand, a priori choice of the parameter $h_n$ has to consider the assumption 1 (iii). Note that the parameter $h_n$ is chosen by numerical optimization. Namely, for each choice of $\alpha$, we first estimate the Lévy densities for each $h_n$ from an equidistant grid (from 0.05 to 0.5 with step 0.05), and then analyze the quality of estimation in terms of the minimal mean square error. Because the best results are obtained for $h_n$ from 0.1 to 0.2, we
reproduce the estimation procedure for $h_n$ from another grid (from 0.08 to 0.25 with step 0.01). After several iterations, we stop the procedure. It is important to note that in the real-life examples, the aforementioned strategy for choosing $h_n$ should be changed, because the comparison with respect to the mean square error is not possible. One should rather use adaptive methods. The simulation results illustrated in the figure 4.4 show that the convergence rates significantly depend on the parameter $\alpha$. More precisely, it turns out that the quality of estimation increases with growing $\alpha$, and the best rates correspond to the case when $\alpha$ is close to 1. This can be explained by the fact that observations become less dependent as $\alpha$ increases. Let us remark that in Figure 4.4 we show the real parts of the estimate $\hat{\nu}_n(x)$. The imaginary part of the considered estimate is quite small (of order $10^{-8}$) and is shown in the Figure 4.5. Finally, following the ideas from Section 2, we construct the confidence interval for the transformed Lévy density $\rho$ via the Gaussian multiplier bootstrap method with parameters $\alpha = 0.8$, $n = 10^5$ and the confidence level 0.9. The dashed line in Figure 4.6 represents the estimator $\hat{\rho}_n$ of the transformed Lévy density $\rho$ (red line).

5. Proofs

For a symmetric kernel $K_\alpha$ of the form (2.2) we first show 2.3.

Lemma 3. We have

$$\Psi_\Delta(u) = \int_{-\infty}^{\infty} \psi(u (K_\alpha(x + \Delta) - K_\alpha(x))) \, dx = \mathcal{L}_\alpha \psi(\Delta u) + S_\Delta(u),$$
where the operator $L_\alpha$ is defined as

$$L_\alpha f(x) := \frac{2}{1 - \alpha} x^{-\frac{\alpha}{\alpha-1}} \int_0^x f(z) z^{\frac{2\alpha-1}{\alpha-1}} \,dz$$

for any locally bounded function $f$ and

$$\psi(u) = i \gamma u - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux 1_{\{|x| \leq 1\}}) \nu(dx).$$
Furthermore, $S_\Delta(u)$ has the form

$$S_\Delta(u) = S_1(u) + S_2(u)$$

(5.1)

$$= \int_{-1/\alpha}^{1/\alpha} \left[ \psi(u(K_\alpha(x + \Delta) - K_\alpha(x))) - \psi(uK_\alpha'(x)) \right] dx$$

$$+ \int_{-1/\alpha}^{-1/\alpha-\Delta} \psi(uK_\alpha(x + \Delta)) dx.$$

**Proof.** In the previously described scenario, the characteristic function $\Phi_\Delta$ of the increment process $Z_{t+\Delta} - Z_t$ has the form

$$\Phi_\Delta(u) := E[\exp(iu(Z_{t+\Delta} - Z_t))] = \exp(\Psi_\Delta(u)),$$

where

$$\Psi_\Delta(u) := \int_{-\infty}^{\infty} \psi(u(K_\alpha(x + \Delta) - K_\alpha(x))) dx$$

The last expression can be obtained using Lemma 5.5 in Sato [14] and taking into account the fact that

$$u(Z_{t+\Delta} - Z_t) = \int_{t}^{t+\Delta} u(K_\alpha(x) - K_\alpha(x)) dL_s.$$ 

We also calculate the first two derivatives of $K_\alpha$,

$$K_\alpha'(x) = - (1 - \alpha x)^{1-\alpha} = -K_\alpha^{1-\alpha}(x), \quad \forall x > 0$$

$$K_\alpha''(x) = (1 - \alpha)(1 - \alpha x)^{1-2\alpha} = (1 - \alpha)K_\alpha^{1-2\alpha}(x), \quad \forall x > 0.$$ 

Then the characteristic function $\Phi_\Delta$ of the increment process $Z_{t+\Delta} - Z_t$ has the form:

$$\Phi_\Delta(u) = \exp \left[ \int_{-\infty}^{\infty} \psi(u(K_\alpha(x + \Delta) - K_\alpha(x))) dx \right]$$

$$= \exp \left[ 2 \int_{0}^{1/\alpha} \psi(uK_\alpha'(x)) dx + S_\Delta(u) \right]$$

(5.2)

$$= \exp \left[ 2 \int_{0}^{1/\alpha} \frac{\psi(uK_\alpha'(x))}{K_\alpha''(x)} dK_\alpha'(x) + S_\Delta(u) \right]$$

$$= \exp \left[ \frac{2}{1 - \alpha} \int_{0}^{1} \psi(u\Delta y)^{2\alpha-1} dy + S_\Delta(u) \right]$$

$$= \exp \left[ \frac{2}{1 - \alpha} (u\Delta)^{-\frac{\alpha}{1-\alpha}} \int_{0}^{u\Delta} \psi(z)^{\frac{2\alpha-1}{1-\alpha}} dz + S_\Delta(u) \right]$$

$$= \exp \left[ L_\alpha u(\Delta) + S_\Delta(u) \right],$$

where

$$S_\Delta(u) = S_1(u) + S_2(u)$$

(5.3)

$$= \int_{-1/\alpha}^{1/\alpha} \left[ \psi(u(K_\alpha(x + \Delta) - K_\alpha(x))) - \psi(uK_\alpha'(x)) \right] dx$$

$$+ \int_{-1/\alpha-\Delta}^{-1/\alpha} \psi(uK_\alpha(x + \Delta)) dx.$$ 

□
Note that the distribution of $Z_{t+\Delta} - Z_t$ is infinitely divisible. Next, we prove (2.6).

**Lemma 4.** Let
$$\int |x|^p \nu(x) \, dx < \infty$$
for some $p \in \mathbb{N}$. Then applies

$$\lim_{\Delta \to 0} \Delta^{-(p+\alpha)} S_\Delta^{(p)}(u/\Delta) = 0,$$
where $S_\Delta \in C^p(\mathbb{R})$ is defined by 5.3.

**Proof.** We have
$$S_1^{(p)}(u) = \int_{-1/\alpha}^{1/\alpha} (K_\alpha(x + \Delta) - K_\alpha(x))^p \psi^{(p)}(u(K_\alpha(x + \Delta) - K_\alpha(x))) \, dx$$
$$- \int_{-1/\alpha}^{1/\alpha} (\Delta K_\alpha'(x))^p \psi^{(p)}(u\Delta K_\alpha'(x)) \, dx$$
$$= \int_{-1/\alpha}^{1/\alpha} \left[ (K_\alpha(x + \Delta) - K_\alpha(x))^p - (\Delta K_\alpha'(x))^p \right]$$
$$\times \psi^{(p)}(u(K_\alpha(x + \Delta) - K_\alpha(x))) \, dx$$
$$+ \int_{-1/\alpha}^{1/\alpha} (\Delta K_\alpha'(x))^p \left[ \psi^{(p)}(u(K_\alpha(x + \Delta) - K_\alpha(x))) - \psi^{(p)}(u\Delta K_\alpha'(x)) \right] \, dx$$
$$:= I_1 + I_2.$$

Using the fact that $||K_\alpha'||_\infty < \infty$, we derive the integral form of the remainder term in the Taylor’s formula:

$$(5.5) \quad \left| (K_\alpha(x + \Delta) - K_\alpha(x))^p - (\Delta K_\alpha'(x))^p \right| \leq B_\alpha \Delta^p \int_{x}^{x+\Delta} (x + \Delta - t)|K_\alpha''(t)| \, dt$$
$$= B_\alpha \Delta^p \int_0^{\Delta} (\Delta - t)|K_\alpha''(t + x)| \, dt.$$

If $\int_{\mathbb{R}} |K_\alpha''(t)| \, dt < \infty$, then we have

$$|I_1| \lesssim \Delta^{p+1}, \quad |I_2| \lesssim \Delta^{2p}, \quad \Delta \to 0.$$

Similarly
$$S_2^{(p)}(u) = \int_{-1/\alpha}^{1/\alpha} \frac{(K_\alpha(x))^p \psi^{(p)}(u(K_\alpha(x)))}{K_\alpha'(x)} \, dx$$
$$= \int_{-1/\alpha}^{1/\alpha} \left( \frac{K_\alpha(x)}{K_\alpha'(x)} \right)^p \psi^{(p)}(u(K_\alpha(x))) dK_\alpha(x)$$
$$= \int_0^{(\alpha\Delta)^{1/\alpha}} y^{p+\alpha-1} \psi^{(p)}(uy) \, dy$$
$$= u^{-p-\alpha} \int_0^{u(\alpha\Delta)^{1/\alpha}} z^{p+\alpha-1} \psi^{(p)}(z) \, dz.$$
Further
\[ \psi^{(p)}(z) = i^p \int_{\mathbb{R}} x^p e^{izu} \nu(x) \, dx \lesssim \int |x|^p \nu(x) \, dx \]
and hence
\[
S_2^{(p)}(\frac{u}{\Delta}) = \left( \frac{u}{\Delta} \right)^{-p-\alpha} \int_0^{u(\Delta)^{-1/\alpha} \Delta^{-1}} z^{p+\alpha-1} \psi^{(p)}(z) \, dz 
\lesssim \left( \frac{u}{\Delta} \right)^{-p-\alpha} \int_0^{u(\Delta)^{-1/\alpha} \Delta^{-1}} z^{p+\alpha-1} \left( \int |x|^p \nu(x) \, dx \right) \, dz 
\lesssim \Delta^{(p+\alpha)/\alpha}.
\]
(5.7)
Combining 5.6 and 5.7, we get
\[
S_2^{(p)}(\frac{u}{\Delta}) \lesssim \Delta^{p+1}.
\]
Since \(0 < \alpha < 1\), it follows that
\[
\lim_{\Delta \to 0} \Delta^{-(p+\alpha)} S_2^{(p)}(\frac{u}{\Delta}) = 0.
\]
□

Next, we formulate and prove some auxiliary lemmas that we need to prove the main results. In the sequel we assume that \(h_n, \Delta \to 0\) as \(n \to \infty\) and \(\lesssim\) stands for inequality up to a constant not depending on \(h_n, \Delta\) and \(n\).

**Lemma 5.** We have for any \(h_n > 0\) with \(h_n^3 \gtrsim \Delta\),
\[
\inf_{|u| \leq h_n^{-1}} |\Phi_{\Delta X}(u)| \gtrsim 1.
\]

**Proof.** Recall that
\[
\Phi_{\Delta X}(u) = \exp \left[ \frac{2\Delta}{1-\alpha} u^{-\frac{\alpha}{1-\alpha}} \int_0^u \psi(z) z^{\frac{2n-1}{1-\alpha}} \, dz \right].
\]
By using the Taylor expansion we obtain for any \(u, x \in \mathbb{R}\),
\[
|e^{iu} - 1 - iux||_{|x| \leq 1} \leq \frac{|x^2u^2|}{2},
\]
so that
\[
|\psi(u)| \leq \frac{\sigma^2u^2}{2} + |\gamma u| + \frac{u^2}{2} \int_{\mathbb{R}} x^2 \nu(x) \, dx \lesssim h_n^{-2}.
\]
Then the infimum of \(\Phi_{\Delta X}(u)\) can under the condition \(h_n^3 \gtrsim \Delta\) be estimated by
\[
\inf_{|u| \leq h_n^{-1}} |\Phi_{\Delta X}(u)| \geq \exp \left( -\Delta \sup_{|u| \leq h_n^{-1}} \left| \frac{2}{1-\alpha} u^{-\frac{\alpha}{1-\alpha}} \int_0^u \psi(z) z^{\frac{2n-1}{1-\alpha}} \, dz \right| \right)
\geq \exp \left( -\Delta \sup_{|u| \leq h_n^{-1}} \left| \frac{\sigma^2u^2}{2} + \gamma u \right| \right)
\gtrsim e^{-\Delta h_n^{-2}} \gtrsim e^{-h_n} \gtrsim 1.
\]
This completes the proof. □

**Lemma 6.** Define \(\rho(x) = x^2 \nu(x)\), then \(|\rho * (h_n^{-1}W(\cdot/h_n)) - \rho||_{\mathbb{R}} \lesssim h_n^r\), where \(*\) denotes the convolution.
Proof. Using the change of variables, we may rewrite the term as
\[
\rho \left( h_n^{-1}W \left( \cdot / h_n \right) \right) - \rho(x) = \int_{\mathbb{R}} \{ \rho(x - yh_n) - \rho(x) \} W(y) \, dy.
\]
By the Tailor’s expansion we obtain for any \( x, y \in \mathbb{R} \),
\[
\rho(x - yh_n) - \rho(x) = \sum_{i=1}^{p-1} \frac{\rho^{(i)}(x)}{i!} (-yh_n)^i + \frac{\rho^{(p)}(x - \theta yh_n)}{p!} (-yh_n)^p,
\]
for some \( \theta \in (0,1) \). Since \( \rho^{(p)}(r-p) \) \( \alpha \)-Hölder continuous, we can assert that
\[
H := \sup_{x,y \in \mathbb{R}, x \neq y} \left| \frac{\rho^{(p)}(x) - \rho^{(p)}(y)}{|x - y|^{r-p}} \right| < \infty.
\]
Furthermore, since \( \int_{\mathbb{R}} y^l W(y) \, dy = 0 \), for \( l = 1, \ldots, p \), we conclude that for any \( x \in \mathbb{R} \),
\[
\left| \int_{\mathbb{R}} \{ \rho(x - yh_n) - \rho(x) \} W(y) \, dy \right| = \left| \int_{\mathbb{R}} \left[ \rho(x - yh_n) - \rho(x) - \sum_{i=1}^{p} \frac{\rho^{(i)}(x)}{i!} (-yh_n)^i \right] W(y) \, dy \right|
\]
\[
= \left| \int_{\mathbb{R}} \left[ \frac{\rho^{(p)}(x - \theta yh_n)}{p!} (-yh_n)^p \right] W(y) \, dy \right|
\]
\[
= \left| \int_{\mathbb{R}} \frac{\rho^{(p)}(x - \theta yh_n)}{p!} (-yh_n)^p \, dy \right| \leq \frac{Hh_n^r}{p!} \int_{\mathbb{R}} |y|^r |W(y)| \, dy.
\]
This completes the proof. \( \square \)

Lemma 7. Suppose that \( \int_{\mathbb{R}} |x|^p \nu(x) \, dx < \infty \) for some \( p \geq 1 \) and let
\[
\Psi(u) := \frac{2}{1 - \alpha} \int_0^1 \psi(uy) y^{\frac{2\alpha-1}{\alpha}} \, dy,
\]
then we have\(^{18}\)
\[
(i) \quad \| \Psi' \|_{L^1} \lesssim \frac{1}{h_n},
\]
\[
(ii) \quad \| \Psi^{(k)} \|_{L^1} \lesssim 1, \quad k \geq 2.
\]
Proof. Under the assumption we have
\[
\Psi'(u) = \frac{2}{1 - \alpha} \int_0^1 y \psi'(uy) y^{\frac{2\alpha-1}{\alpha}} \, dy
\]
\[
= \frac{2}{1 - \alpha} \int_0^1 \left( -\sigma^2 uy^2 + iy(\gamma + \int_{\mathbb{R}} x(e^{iux} - \mathbb{1}_{|x| \leq 1}) \nu(dx)) \right) y^{\frac{2\alpha-1}{\alpha}} \, dy,
\]
\[
\Psi''(u) = \frac{2}{1 - \alpha} \int_0^1 y^2 \psi''(uy) y^{\frac{2\alpha-1}{\alpha}} \, dy
\]
\[
= \int_0^1 \left( -\sigma^2 y^2 - y^2 \int_{\mathbb{R}} x^2 e^{iux} \nu(dx) \right) y^{\frac{2\alpha-1}{\alpha}} \, dy,
\]
\[
\Psi^{(k)}(u) = \frac{2}{1 - \alpha} \int_0^1 y^k \psi^{(k)}(uy) y^{\frac{2\alpha-1}{\alpha}} \, dy
\]
\[
= \int_0^1 y^k \left[ \int_{\mathbb{R}} x^k e^{iux} \nu(dx) \right] y^{\frac{2\alpha-1}{\alpha}} \, dy, \quad k \geq 2,
\]
and the assertion follows. \( \square \)

\(^{18}\)The supremum is found on the interval \( I_1 \) so that \( \varphi_W(uh_n) \) supported in \([-1,1] \)
Lemma 8. For \( k \in \mathbb{N} \) we have
\[
\mathbb{E} [ |\Delta X|^{2k} ] \lesssim \Delta,
\]
where \((\Delta X)_t\) the increment process of the limiting Lévy process \((X_t)\).

Proof. Since \( \Phi_{\Delta X}(u) = \exp[\Delta \Psi(u)] \), we have
\[
\mathbb{E} [ \Delta X^{2k} ] = (-1)^{2k} \Phi_{\Delta X}^{(2k)}(0) = (-1)^{k} \frac{d^{2k}}{du^{2k}} \exp[\Delta \Psi(u)] \big|_{u=0}.
\]
Note that
\[
\Phi_{\Delta X}^{(2k)}(u) = \Delta \Phi_{\Delta X}(u)(\Psi^{(2k)}(u) + a_1 \Delta \Psi^{(2k-1)}(u) \Psi'(u) + \ldots + \Delta^{2k-1}(\Psi'(u))^{2k})
\]
where \( a_i \in \mathbb{R} \), for \( i = 1, \ldots, 2k - 1 \). This observation completes the proof. \( \square \)

Lemma 9. For \( k = 0, 1, 2, 3 \) we have
\[
\left\| \mathbb{D}^{(k)}(u) \right\|_{L_n} = \mathcal{O}_P(n^{-1/2} \Delta^{(k \wedge 1)/2} \log h_n^{-1}),
\]
where \( \mathbb{D}(u) = \hat{\Phi}_{\Delta X}(u) - \Phi_{\Delta X}(u) \).

Proof. To prove this statement we use Lemma 8 together with proof of the Theorems 1 (see [14]), which shows that for the weight function \( \omega(u) = (\log(e + |u|))^{-1} \) under Assumption 1, we obtain
\[
C_k := \sup_n \mathbb{E} \left[ \| \sqrt{n} \Delta^{-(k \wedge 1)/2} \mathbb{D}^{(k)}(u) \omega(u) \|_{\mathbb{R}} \right] < \infty
\]
for \( k = 0, 1, 2, 3 \). Furthermore,
\[
\| \sqrt{n} \Delta^{-(k \wedge 1)/2} \mathbb{D}^{(k)}(u) \omega(u) \|_{\mathbb{R}} \geq \sqrt{n} \Delta^{-(k \wedge 1)/2} \| \mathbb{D}^{(k)}(u) \|_{L_n} \inf_{|u| \leq h_n^{-1}} \omega(u)
\]
Since
\[
\inf_{|u| \leq h_n^{-1}} \omega(u) = \inf_{|u| \leq h_n^{-1}} (\log(e + |u|))^{-1} = \left( \sup_{|u| \leq h_n^{-1}} \log(e + |u|) \right)^{-1} = \log(1 + h_n^{-1})^{-1}
\]
we conclude that
\[
\mathbb{E} \left[ \| \mathbb{D}^{(k)}(u) \|_{L_n} \right] \leq \frac{C_k \Delta^{(k \wedge 1)/2}}{\sqrt{n} \inf_{|u| \leq h_n^{-1}} \omega(u)} \lesssim n^{-1/2} \Delta^{(k \wedge 1)/2} \log h_n^{-1}.
\]
This completes the proof. \( \square \)

Note that the Lemma 5 and Lemma 9 imply
\[
\inf_{|u| \leq h_n^{-1}} \left| \hat{\Phi}_{\Delta X}(u) \right| \geq \inf_{|u| \leq h_n^{-1}} \left| \Phi_{\Delta X}(u) \right| - o_p(1) \gtrsim 1 - o_p(1).
\]

Lemma 10. Let \( R_n(x) \) be the form
\[
R_n(x) = -\frac{1}{2\pi \Delta} \int_{\mathbb{R}} e^{-iux} [Q_0(u) \mathbb{D}(u) + Q_1(u) \mathbb{D}'(u) + Q_2(u) \mathbb{D}''(u) + Q_3(u) \mathbb{D}'''(u)] \varphi_W(uh_n) du,
\]
Let us first consider the integral

\[ Q_0(u) = \frac{\Delta}{\Phi_{\Delta X}(u)} \left( \frac{2 - \alpha}{2} (\Delta (\Psi'(u))^2 - \Psi''(u)) - \frac{1 - \alpha}{2} (\Psi'''(u) - 3\Delta \Psi''(u) \Psi'(u) + \Delta^2 (\Psi'(u))^3) \right), \]

Then applies

\[ \|R_n(x)\|_R = O(\Delta^{-1/2} h^{-1}_n n^{-1/2} \log h^{-1}_n). \]

Proof. Let us first consider the integral

\[ I_0(x) = \int_{\mathbb{R}} e^{-iu x} Q_0(u) \mathbb{D}(u) \varphi_W(u h_n) \, du. \]

According to the lemmas 9, 7 and 5 we get

\[ \|I_0(x)\|_R \lesssim \Delta \|\mathbb{D}(u)\|_{L_h}(\Delta h^{-2}_n + 1 + h^{-1}_n (1 + \Delta h^{-1}_n + \Delta^2 h^{-3}_n)) \]

\[ \lesssim \Delta h^{-1}_n n^{-1/2} \log h^{-1}_n. \]

Analogous to \( I_0(x)\|_R \) applies to the integrals

\[ I_i(x) = \int_{\mathbb{R}} e^{-iu x} Q_i(u) \mathbb{D}^{(i)}(u) \varphi_W(u h_n) \, du \]

for \( i = 1, 2, 3 \):

\[ \|I_1(x)\|_R \lesssim \Delta \|\mathbb{D}'(u)\|_{L_h}(h^{-1}_n (\Delta h^{-2}_n + 1)) \]

\[ \lesssim \Delta^{3/2} h^{-1}_n n^{-1/2} \log h^{-1}_n, \]

\[ \|I_2(x)\|_R \lesssim \|\mathbb{D}''(u)\|_{L_h}(1 + \Delta h^{-2}_n) \]

\[ \lesssim \Delta^{1/2} n^{-1/2} \log h^{-1}_n, \]

\[ \|I_3(x)\|_R \lesssim \|\mathbb{D}'''(u)\|_{L_h} h^{-1}_n \]

\[ \lesssim \Delta^{1/2} h^{-1}_n n^{-1/2} \log h^{-1}_n. \]

Thus the claim of the lemma holds such that

\[ \|R_n(x)\|_R = O(\Delta^{-1/2} h^{-1}_n n^{-1/2} \log h^{-1}_n). \]

Lemma 11. Let \( P_\Delta \) denote the distribution of the r.v. \( X_\Delta - X_0 \). The measures \( g^{2m} P_\Delta(dy) \) for \( m = 1, 2, 3 \) have Lebesgue densities \( g^{2m}_\Delta \). For any compact set in \( \mathbb{R} \setminus \{0\} \) and any \( \varepsilon_0 > 0 \), let \( I^\varepsilon_0 = \{ x \in \mathbb{R} : d(x, I) \leq \varepsilon_0 \} \), where \( d(x, I) = \inf_{y \in I} |x - y| \).

We have

(i) \[ \inf_{y \in I^\varepsilon_0} g^{2m}_\Delta(y) = \inf_{y \in I^\varepsilon_0} (g^{2m} P_\Delta)(y) \gtrsim \Delta, \] for \( m = 1, 2, 3 \) \( \Delta \to 0 \).
\[(ii)\]
\[\|g_\Delta^2\|_\mathbb{R} \lesssim \Delta^{1/2} \text{ and } \|g_\Delta^{2m}\|_\mathbb{R} \lesssim \Delta, \quad m = 2, 3 \quad \Delta \to 0,\]
for some sufficiently small \(\varepsilon_0 > 0\) such that \(0 \notin I^{\varepsilon_0}\).

**Proof.** Note that
\[
\psi'(uy) = -u\sigma^2 y^2 + iy(\gamma + \int \mathbb{E}(e^{iyx} - \mathbb{I}_{[-1,1]}(x))\nu(dx))
\]
\[
\psi''(uy) = -\sigma^2 y^2 - \int e^{iyx}(yx)^2 \nu(x) dx
\]
\[
= - \int y^{-1}e^{iux}t^2\nu(t/y) dt
\]
\[
\psi^{(k)}(uy) = \int e^{iyx}(iyx)^k \nu(x) dx = \int y^{-1}e^{iux}t^k\nu(t/y) dt,
\]
where \(\nu_\sigma = \sigma^2 y^2\delta_0 + t^2 \nu, \ t = yx\). It also follows that

\[
\Psi'(u) = \frac{2}{1-\alpha} \int_0^1 \left(-\sigma^2 y^2 + iy(\gamma + \int \mathbb{E}(e^{iyx} - \mathbb{I}_{[-1,1]}(x))\nu(dx))\right) y^{\frac{2\alpha-1}{\alpha}} dy
\]
\[
= \frac{2}{1-\alpha} \int_0^1 \left(\int e^{iut}(-\sigma^2 \delta_1 y^2 + \delta_2 i \gamma y + tv(t/y)) dt\right) y^{\frac{3\alpha-2}{\alpha}} dy
\]
\[
= \frac{2}{1-\alpha} \int e^{iut} \left(\int_0^1 (-\sigma^2 \delta_1 y^2 + \delta_2 i \gamma y + tv(t/y)) y^{\frac{3\alpha-2}{\alpha}} dt\right) dt,
\]
\[
\Psi''(u) = \frac{2}{1-\alpha} \int_0^1 \psi''(uy) y^{\frac{2\alpha-1}{\alpha}} dy = \frac{2}{1-\alpha} \int_0^1 \left(\int e^{iut}t^2\nu_\sigma(t/y) dt\right) y^{\frac{3\alpha-2}{\alpha}} dy
\]
\[
= \frac{2}{1-\alpha} \int e^{iut} t^2 \int_0^1 y^{\frac{3\alpha-2}{\alpha}} \nu_\sigma(t/y) dy dt,
\]
\[
\Psi^{(k)}(u) = \frac{2}{1-\alpha} \int_0^1 \psi^{(k)}(uy) y^{\frac{2\alpha-1}{\alpha}} dy = \frac{2}{1-\alpha} \int e^{iut}t^k \int_0^1 y^{\frac{3\alpha-2}{\alpha}} \nu(t/y) dy dt.
\]

From infinite divisibility of the process \((\Delta X)_t\), it follows that \(\Phi_{\Delta X}(u) = \left(\Phi_{\Delta/2}(u)\right)^2\).
Then
\[
\Phi''_{\Delta X}(u) = \Phi_{\Delta X}(u)\left(\Delta \Psi''(u) + (\Delta \Psi'(u))^2\right)
\]
\[
= \Delta \Psi''(u)\Phi_{\Delta X}(u) + 4\left(\Delta/2 \Psi'(u)\Phi_{\Delta/2}(u)\right)^2
\]
\[
= \Delta \Psi''(u)\Phi_{\Delta X}(u) + 4\left(\Phi_{\Delta/2}(u)\right)^2.
\]

Furthermore
\[
\int e^{iux} x^2 P_\Delta(dx) = \frac{2}{1-\alpha} \Delta \int e^{iut} t^2 \int_0^1 y^{\frac{3\alpha-2}{\alpha}} \nu_\sigma(t/y) dy dt \int e^{iut} P_\Delta(dt) + 4 \int e^{iut} t P_{\Delta/2}(dt)^2.
\]
Then the first claim follows from (5.9), we have

\[ P \]

Hence

\[ x^2 P_\Delta = \frac{2}{1 - \alpha} \Delta \left( t^2 \int_0^1 y^{\frac{\alpha}{\alpha - 2}} \nu_{\sigma}(t/y) \, dy \right) \ast P_\Delta + 4(x P_{\Delta/2}) \ast (x P_{\Delta/2}), \]

\[ x^3 P_\Delta = \frac{2}{1 - \alpha} \Delta \left( \left( t^3 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast P_\Delta + \right. \]

\[ \left. + \left(t^2 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu_{\sigma}(t/y) \, dy \right) \ast (x P_\Delta) \right) + 8(x P_{\Delta/2}) \ast (x^2 P_{\Delta/2}), \]

\[ x^4 P_\Delta = \frac{2}{1 - \alpha} \Delta \left( t^4 \int_0^1 y^{\frac{4 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast P_\Delta \]

\[ + 2 \left( t^3 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast (x P_\Delta) + \left( t^2 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu_{\sigma}(t/y) \, dy \right) \ast (x^2 P_\Delta) \]

\[ + 8(x^2 P_{\Delta/2}) \ast (x^2 P_{\Delta/2}) + 8(x^3 P_{\Delta/2}) \ast (x P_{\Delta/2}), \]

\[ x^5 P_\Delta = \frac{2}{1 - \alpha} \Delta \left( t^5 \int_0^1 y^{\frac{5 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast P_\Delta + 4 \left( t^3 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast (x P_\Delta) \]

\[ + 6 \left( t^4 \int_0^1 y^{\frac{4 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast (x^2 P_\Delta) + 4 \left( t^3 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast (x^3 P_\Delta) \]

\[ + \left( t^2 \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu_{\sigma}(t/y) \, dy \right) \ast (x^4 P_\Delta) + 32(x^3 P_{\Delta/2}) \ast (x^2 P_{\Delta/2}) \]

\[ + 24(x^3 P_{\Delta/2}) \ast (x^3 P_{\Delta/2}) + 8(x^5 P_{\Delta/2}) \ast (x P_{\Delta/2}). \]

For any \( \varepsilon > 0 \), it holds due to the Markov inequality

\[ P_\Delta([-\varepsilon, \varepsilon]) = P(|\Delta X| > \varepsilon) \leq \varepsilon^{-2}E[\Delta X^2] \lesssim \Delta. \]

Then it follows \( P_\Delta([-\varepsilon, \varepsilon]) = 1 - O(\Delta) \) and since \( \inf_{t \in I_1} \left( t^k \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \gtrsim 1 \) we have

\[ \inf_{t \in I_1} \left( t^k \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \ast P_\Delta \]

\[ \gtrsim P_\Delta[-\varepsilon_1/2, \varepsilon_1/2] \inf_{t \in I_1} \left( t^k \int_0^1 y^{\frac{3 \alpha - 2}{\alpha - 2}} \nu(t/y) \, dy \right) \gtrsim 1. \]

Then the first claim follows from (5.9),

\[ \inf_{y \in I_0} y^{2m}_{\Delta}(y) = \inf_{y \in I_0} \left( y^{2m} P_\Delta \right)(y) \gtrsim \Delta, \quad m = 1, 2, 3. \]

Furthermore due (5.9) and

\[ \| y P_{\Delta/2} \|_{L_1} = E[X_{\Delta/2}] \leq \sqrt{E[X_{\Delta/2}^2]} \lesssim \Delta^{1/2}, \]
Lemma 12. For the variance $s^2(x)$ defined by (5.22), we have the following estimate:

$$\inf_{x \in \mathcal{I}} s^2(x) \gtrsim \Delta h_n^{-3},$$

for sufficiently large $n$.

Proof. We have

$$s^2(x) = \sum_{m=0}^{3} \text{Var}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)]$$

$$= \sum_{m=0}^{3} \left( \mathbb{E}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)] - \mathbb{E}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)]^2 \right).$$

In order to determine the infimum of the variance, we compute the supremum of the expected value $\mathbb{E}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)]$ and the infimum of $\mathbb{E}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)]$ for $m = 0, 1, 2, 3$. Note that

$$\mathbb{E}[(\Delta X)^m K_{m,n}(x - (\Delta X)_1)] = \int_{\mathbb{R}} y^m K_{m,n}(x - y) P_\Delta(dy).$$
Further we get

\[
\int_{\mathbb{R}} K_{0,n}(x-y)P_\Delta(dy) = \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{-iu(x-y)}Q_0(u)\varphi_W(uh_n) \, du \right] P_\Delta(dy)
\]
\[
= \int_{\mathbb{R}} \left[ \int_{\mathbb{R}} e^{-iu(x-y)}P_\Delta(dy) \right] Q_0(u)\varphi_W(uh_n) \, du
\]
\[
= \int_{\mathbb{R}} e^{-iu x}q_{\Delta X}(u)Q_0(u)\varphi_W(uh_n) \, du
\]
\[
= \Delta \int_{-1/h_n}^{1/h_n} e^{-iu x} \left( \frac{2-\alpha}{2} \left( (\Psi'(u))^2 - \Psi''(u) \right) 
- u \frac{1-\alpha}{2} \left( (\Psi'(u))^2 - 3\Delta \Psi''(u)\Psi'(u) - \Delta^2 (\Psi'(u))^3 \right) \right) du.
\]

Furthermore

\[
\int_{-1/h_n}^{1/h_n} e^{-iu x} u\Psi'''(u) \, du \leq \int_{-1/h_n}^{1/h_n} e^{-iu x} u \, du = -2i \int_0^{1/h_n} u \sin(u x) \, du
\]
\[
= \frac{2i}{xh_n} \cos(x/h_n) + \frac{1}{x^2} \sin(x/h_n) \lesssim h_n^{-1}
\]

for \( x \in I \). Analogously we get

\[
\int_{-1/h_n}^{1/h_n} e^{-iu x} u\Psi''(u) \Psi'(u) \, du \lesssim h_n^{-1} \int_{-1/h_n}^{1/h_n} e^{-iu x} u \, du \lesssim h_n^{-2},
\]
\[
\int_{-1/h_n}^{1/h_n} e^{-iu x} u(\Psi'(u))^3 \, du \leq h_n^{-3} \int_{-1/h_n}^{1/h_n} e^{-iu x} u \, du \lesssim h_n^{-4},
\]
\[
\int_{-1/h_n}^{1/h_n} e^{-iu x} (\Psi'(u))^2 \, du \leq h_n^{-2} \int_{-1/h_n}^{1/h_n} e^{-iu x} \, du \lesssim h_n^{-2},
\]
\[
\int_{-1/h_n}^{1/h_n} e^{-iu x} \Psi''(u) \, du \leq \int_{-1/h_n}^{1/h_n} e^{-iu x} \, du \lesssim 1.
\]

Then due to (5.10) and (5.11) we get

\[
\int_{\mathbb{R}} K_{0,n}(x-y)P_\Delta(dy) \lesssim \Delta(h_n^{-2} + 1 + h_n^{-1} + \Delta^2 h_n^{-4}) \lesssim h_n^{-1}.
\]
Analogously

\begin{equation}
\int_{\mathbb{R}} yK_{1,n}(x-y)P_{\Delta}(dy) = \int_{\mathbb{R}} y^{3} \left[ \int e^{-iu(x-y)}Q_{1}(u)\varphi_{W}(uh_{n}) \, du \right] P_{\Delta}(dy) \\
= \int_{\mathbb{R}} \left[ \int ye^{-iu(x-y)}P_{\Delta}(dy) \right] Q_{1}(u)\varphi_{W}(uh_{n}) \, du \\
= \int_{\mathbb{R}} e^{-iu} \Phi_{\Delta X}(u)Q_{1}(u)\varphi_{W}(uh_{n}) \, du \\
= \Delta^{2} \int_{-1/h_{n}}^{1/h_{n}} e^{-iu} \left( \Psi''(u) + \Delta(\Psi'(u))^{2} - (2 - \alpha)\Psi'(u) \right) \, du \\
\lesssim \Delta^{2}(\Delta h_{n}^{-4} + h_{n}^{-2} + h_{n}^{-4}) \lesssim \Delta,
\end{equation}

\begin{equation}
\int_{\mathbb{R}} y^{2}K_{2,n}(x-y)P_{\Delta}(dy) = \int_{\mathbb{R}} y^{2} \left[ \int e^{-iu(x-y)}Q_{2}(u)\varphi_{W}(uh_{n}) \, du \right] P_{\Delta}(dy) \\
= \int_{\mathbb{R}} \left[ \int y^{2}e^{-iu(x-y)}P_{\Delta}(dy) \right] Q_{2}(u)\varphi_{W}(uh_{n}) \, du \\
= \int_{\mathbb{R}} e^{-iu} \Phi_{\Delta X}(u)Q_{2}(u)\varphi_{W}(uh_{n}) \, du \\
= \Delta^{2} \int_{-1/h_{n}}^{1/h_{n}} e^{-iu} \left( \Psi''(u) + \Delta(\Psi'(u))^{2} \right) \\
\times \left( \frac{2 - \alpha}{2} - 3u\Delta\Psi'(u)\frac{1 - \alpha}{2} \right) \, du \\
\lesssim \Delta (\Delta h_{n}^{-2} + h_{n}^{-2} + \Delta^{2}h_{n}^{-4}) \lesssim \Delta,
\end{equation}

\begin{equation}
\int_{\mathbb{R}} y^{3}K_{3,n}(x-y)P_{\Delta}(dy) = \int_{\mathbb{R}} y^{3} \left[ \int e^{-iu(x-y)}Q_{3}(u)\varphi_{W}(uh_{n}) \, du \right] P_{\Delta}(dy) \\
= \int_{\mathbb{R}} \left[ \int y^{3}e^{-iu(x-y)}P_{\Delta}(dy) \right] Q_{3}(u)\varphi_{W}(uh_{n}) \, du \\
= \int_{\mathbb{R}} e^{-iu} \Phi_{\Delta X}(u)Q_{3}(u)\varphi_{W}(uh_{n}) \, du \\
= \Delta^{2} \int_{-1/h_{n}}^{1/h_{n}} e^{-iu} \left( u^{1 - \alpha/2} (\Psi'''(u) + 3\Delta\Psi''(u)\Psi'(u) \\
+ \Delta^{2}(\Psi'(u))^{3}) \right) \, du, \\
\lesssim \Delta (h_{n}^{-1} + \Delta h_{n}^{-2} + \Delta^{2}h_{n}^{-4}) \lesssim \Delta h_{n}^{-1}.
\end{equation}
To estimate the infimum \( E[(\Delta X)^{2m}K_{m,n}^2(x-(\Delta X)_1)] \) for \( m = 0, 1, 2, 3 \), we consider

\[
E[(\Delta X)^{2m}K_{m,n}^2(x-(\Delta X)_1)] = \int_{\mathbb{R}} y^{2m}K_{m,n}^2(x-y)P_\Delta(dy) \\
= \int_{\mathbb{R}} K_{m,n}^2(x-y)g_\Delta^{2m}(y)dy \\
= \int_{\mathbb{R}} K_{m,n}^2(y)g_\Delta^{2m}(x-y)dy.
\]

Since \( K_{m,n}(z) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iz\Phi}Q_m(u)\varphi_W(uh_n)du \), we have according to the Plancherel’s theorem:

\[
\int_{\mathbb{R}} K_{m,n}^2(y)dy = -\frac{1}{2\pi} \int_{\mathbb{R}} |Q_m(u)\varphi_W(uh_n)|^2du,
\]

where

\[
Q_0(u) = \frac{\Delta}{\Phi_{\Delta X}(u)} \left( \frac{2-\alpha}{2} (\Delta(\Psi'(u))^2 - \Psi''(u)) \right. \\
- u^2 \frac{1-\alpha}{2} \left( \Psi'''(u) - 3\Delta \Psi'(u)\Psi''(u) - \Delta^2(\Psi'(u))^3 \right),
\]

\[
Q_1(u) = \frac{\Delta}{\Phi_{\Delta X}(u)} \left( 3u^2 \frac{1-\alpha}{2} (\Delta(\Psi'(u))^2 + \Psi''(u)) - (2-\alpha)\Psi'(u) \right),
\]

\[
Q_2(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( 2 - \alpha \frac{1}{2} - 3u^2 \Delta \Psi'(u) \frac{1}{2} \right),
\]

\[
Q_3(u) = \frac{1}{\Phi_{\Delta X}(u)} \left( u - \alpha \frac{1}{2} \right).
\]

Furthermore applies

\[
\inf_{x \in I} \int_{\mathbb{R}} K_{m,n}^2(y)g_\Delta^{2m}(x-y)dy \geq \inf_{x \in I} g_\Delta^{2m}(x-y) \int_{\mathbb{R}} K_{m,n}^2(y)dy \\
\geq \frac{1}{2\pi} \inf_{x \in I} g_\Delta^{2m}(x-y) \int_{\mathbb{R}} |Q_m(u)\varphi_W(uh_n)|^2du \\
= \frac{1}{2\pi} \inf_{x \in I} g_\Delta^{2m}(x-y) \int_{-1/h_n}^{1/h_n} |Q_m(u)|^2du.
\]

Since \( |\Phi_{\Delta X}(u)| \leq 1 \) for all \( u \), it follows for \( m = 3 \):

\[
(5.16) \quad \int_{-1/h_n}^{1/h_n} |Q_3(u)|^2du = \int_{-1/h_n}^{1/h_n} \frac{1}{\Phi_{\Delta X}^2(u)} \left( u - \alpha \frac{1}{2} \right)^2 du \geq \int_{-1/h_n}^{1/h_n} \left( u - \alpha \frac{1}{2} \right)^2 du \\
= \left( \frac{1-\alpha}{2} \right)^2 \int_{-1/h_n}^{1/h_n} u^2du = \frac{(1-\alpha)^2}{2} \frac{1}{3} \left( \frac{1}{h_n} \right)^3 = O(1/h_n^3)
\]
The same argument applies to $m = 2$:

$$
\int_{-1/h_n}^{1/h_n} |Q_2(u)|^2 du = \int_{-1/h_n}^{1/h_n} \frac{1}{\Phi^2_{\Delta x}(u)} \left( \frac{2 - \alpha}{2} - 3u \Delta \Psi'(u) \frac{1 - \alpha}{2} \right)^2 du \\
\geq \left( \frac{2 - \alpha}{2} \right)^2 \int_{-1/h_n}^{1/h_n} du - \frac{3}{2}(2 - \alpha)(1 - \alpha) \int_{-1/h_n}^{1/h_n} u \Psi'(u) du \\
+ 9 \Delta^2 \left( \frac{1 - \alpha}{2} \right)^2 \int_{-1/h_n}^{1/h_n} (u \Psi'(u))^2 du.
$$

Furthermore

$$
\int_{-1/h_n}^{1/h_n} u \Psi'(u) du = \frac{2}{1 - \alpha} \left[ \int_{-1/h_n}^{1/h_n} \left( \int_0^1 u(-\sigma^2 uy + i\gamma) y^{\alpha/2} dy \right) du \\
+ \int_{-1/h_n}^{1/h_n} \left( \int_0^1 \int_\mathbb{R} iux(e^{iyx} - \mathbb{I}_{|x| \leq 1}) \nu(dx) y^{\alpha/2} dy \right) \right] \\
= \mathcal{O}(h_n^{-3} + h_n^{-2}) = \mathcal{O}(h_n^{-3}).
$$

Similarly $\int_{-1/h_n}^{1/h_n} (u \Psi'(u))^2 du = \mathcal{O}(h_n^{-5})$ and we have

$$
\int_{-1/h_n}^{1/h_n} |Q_2(u)|^2 du = \mathcal{O}(h_n^{-1} + \Delta h_n^{-3} + \Delta^2 h_n^{-5}) = \mathcal{O}(h_n^{-1}), \tag{5.17}
$$

$$
\int_{-1/h_n}^{1/h_n} |Q_1(u)|^2 du \geq \Delta^2 \int_{-1/h_n}^{1/h_n} \left( 3u \frac{1 - \alpha}{2} (\Delta \Psi'(u))^2 + \Psi''(u) - 2 \alpha \Psi'(u) \right)^2 du \\
= \mathcal{O}(\Delta^2(\Delta^2 h_n^{-7} + \Delta h_n^{-5} + h_n^{-3})) = \mathcal{O}(\Delta^2 h_n^{-3}), \tag{5.18}
$$

$$
\int_{-1/h_n}^{1/h_n} |Q_0(u)|^2 du = \Delta^2 \int_{-1/h_n}^{1/h_n} \left( \frac{2 - \alpha}{2} (\Delta \Psi'(u))^2 - \Psi''(u) \right) \\
- u \frac{1 - \alpha}{2} (\Psi''(u) - 3 \Delta \Psi''(u) \Psi'(u) - \Delta^2 \Psi'(u))^2 du \\
= \mathcal{O}(\Delta^2(h_n^{-3} + \Delta h_n^{-3} + h_n^{-1} + \Delta^2 h_n^{-5} + \Delta^3 h_n^{-7} + \Delta^4 h_n^{-9})) \\
= \mathcal{O}(\Delta^2 h_n^{-3}), \tag{5.19}
$$

Taking into account Lemma 11, we get

$$
\inf_{x \in I} \int_R K_{0,n}^2(x - y) P_\Delta(dy) \gtrsim \Delta^2 h_n^{-3} \\
\inf_{x \in I} \int_R y^2 K_{1,n}^2(x - y) P_\Delta(dy) \gtrsim \Delta^3 h_n^{-3} \\
\inf_{x \in I} \int_R y^4 K_{2,n}^2(x - y) P_\Delta(dy) \gtrsim \Delta h_n^{-1} \\
\inf_{x \in I} \int_R y^6 K_{3,n}^2(x - y) P_\Delta(dy) \gtrsim \Delta h_n^{-3} \tag{5.20}
$$

Finally, by combining (5.20) with (5.12)-(5.15), we prove the claim. $\Box$
5.1. Proof of Theorem 1. Using the equations (3.1) and (3.2), the difference $\hat{\rho}_n(x) - \rho(x)$ can be represented as

$$\hat{\rho}_n(x) - \rho(x) = \left(\frac{\hat{\rho}_n(x) - \tilde{\rho}(x)}{R_n(x)}\right) + \left(\frac{\tilde{\rho}(x) - \rho(x)}{I_{\sigma^2}(x) + I_{\rho_n}(x)}\right),$$

where

$$\tilde{\rho}(x) := -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu^2} \left(\mathcal{L}_n^{-1}(\Psi)''(u) + \Delta^2 \sigma^2\right) \varphi_W(uh) du,$$

$$I_{\sigma^2}(x) := -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu^2}(\sigma_n^2 - \sigma^2) \varphi_W(uh) du,$$

$$I_{\rho_n}(x) := \left[\rho * (h^{-1}W(\cdot/h))\right](x) - \rho(x).$$

Under assumptions 1 (iv) and lemma 6, the terms $I_{\rho_n}$ and $I_{\sigma^2}$ are asymptotically (as $n \to \infty$ and $\Delta \to 0$) smaller than $R_n$ and hence can be neglected when constructing the confidence interval for the transformed Lévy density $\rho$. With the notations 3.8 for $R_n(x)$, namely

$$R_n(x) = \frac{1}{n\Delta} \sum_{m=0}^{\infty} \left(\sum_{j=1}^{3} \{i^m(\Delta X)^m J_{m,n}(x - (\Delta X)_j) \right)$$

$$- \mathbb{E}\left[\left\{i^m(\Delta X)^m K_{m,n}(x - (\Delta X)_1)\right\}\right],$$

where the kernel functions $K_{m,n}(z)$, $m = 0, 1, 2, 3$, are defined as

$$K_{m,n}(z) := -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-iu^2} Q_m(u) \varphi_W(uh_n) du$$

consider the process

$$(5.21) T_n(x) := \frac{\sqrt{n}\Delta}{s(x)} R_n(x),$$

where $s^2(x)$ is given by

$$(5.22) s^2(x) := \mathbb{V}a\left[\sum_{m=0}^{\infty} i^m(\Delta X)^m K_{m,n}(x - (\Delta X)_1)\right].$$

Further we show that exists a tight $\ell^\infty(I)$-sequence of Gaussian random variables $T_n^G$ with zero mean and the same covariance function as one of $T_n$, and such that the distribution of $\|T_n^G\|_I := \sup_{x \in I} |T_n^G(x)|$ asymptotically approximates the distribution of $\|T_n\|_I$ in the sense that

$$\sup_{z \in \mathbb{R}} \mathbb{P}\left\{\|T_n\|_I \leq z\right\} - \mathbb{P}\left\{\|T_n^G\|_I \leq z\right\} \to 0, \quad n \to \infty.$$

In what follows, we always assume Assumption 1. The proofs rely on modern empirical process theory. For a probability measure $Q$ on a measurable space $(\mathcal{S}, \mathcal{S})$ and a class of measurable functions $\mathcal{F}$ on $\mathcal{S}$ such that $\mathcal{F} \in L^2(Q)$, let $N(\mathcal{F}, \|\cdot\|_Q; \varepsilon)$ denote the $\varepsilon$-covering number for $\mathcal{F}$ with respect to the $L^2(Q)$-seminorm $\|\cdot\|_Q$. See Section 2.1 in [16] for details. Let $\overset{d}{=} \overset{d}{=}$ denote the equality in distribution. Consider the function class

$$\mathcal{F}_{i,n} = \left\{y \mapsto \frac{(iy)^i}{s(x)} K_{i,n}(x - y) : x \in I\right\}.$$
According to Lemma 12
\[ \inf_{x \in I} s^2(x) \gtrsim \Delta h_n^{-3} \]
and we have
\begin{equation}
(5.23) \quad \frac{\sqrt{n} \Delta(\hat{\rho}_n(x) - \rho(x))}{s(x)} = T_n(x) + o_P \left( h_n^{1/2} \log h_n^{-1} \right)
\end{equation}
uniformly in \( x \in I \). Under condition (iii) of Assumption 1, the expression \( h_n^{1/2} \log h_n^{-1} \) converges to 0. Further we approximate \( \| \) uniformly in \( x \)...

Note that according lemma 11 the increment process \( \Delta X \sim X_0 \) has the distribution \( \Delta X \sim \Delta X_1 \), so that \( y^{2m} P_\Delta(dy) = g_2^{2m}(y)dy \), with \( \|g_2\|_\infty \lesssim \Delta^{1/2} \) and \( \|g_2^{2m}\|_\infty \lesssim \Delta \) for \( m = 2, 3 \). Hence
\[ E[(\Delta X)^{2m} K_m^2(x - (\Delta X)_1)] = \int_R y^{2m} K_m^2(x - y) P_\Delta(dy) = \int_R K_m^2(x - y) g_2^{2m}(y) dy \leq \|g_2\|_\infty \int_R K_m^2(x - y) dy \]

Since \( K_m(z) = -\frac{1}{2\pi} \int_R e^{-iuz} Q_m(u) \varphi_W(uh_n) du \), we have according to the Plancherel’s theorem,
\[ \int_R K_m^2(x) dy = -\frac{1}{2\pi} \int_R |Q_m(u) \varphi_W(uh_n)|^2 du. \]

Using (5.16), (5.17), (5.18) and (5.19), we get that
\begin{align}
E[K_3^2(x - (\Delta X)_1)] & \lesssim \Delta^2 h_n^{-3}, \\
E[(\Delta X)^2 K_1^2(x - (\Delta X)_1)] & \lesssim \Delta^{5/2} h_n^{-3}, \\
E[(\Delta X)^4 K_2^2(x - (\Delta X)_1)] & \lesssim \Delta h_n^{-1}, \\
E[(\Delta X)^6 K_3^2(x - (\Delta X)_1)] & \lesssim \Delta h_n^{-3},
\end{align}

holds and it also follows that
\[ \sup_{f_{i,n},n \in F_i,n} E \left[ \sum_{i=1}^3 f_{i,n}^2((\Delta X)_1) \right] \lesssim \frac{\Delta h_n^{-3}}{\Delta h_n^{-1}} \lesssim 1. \]
Furthermore
\[
\|K_{0,n}(\cdot - (\Delta X)_1)\|_\mathbb{R} \lesssim \Delta(h_n^{-2} + 1 + h_n^{-1} + \Delta^2 h_n^{-4}) \lesssim h_n^{-1},
\]
(5.25)
\[
\|K_{1,n}(\cdot - (\Delta X)_1)\|_\mathbb{R} \lesssim \Delta(h_n^{-3} + h_n^{-1}) \lesssim h_n^{-1},
\]
\[
\|K_{2,n}(\cdot - (\Delta X)_1)\|_\mathbb{R} \lesssim 1 + \Delta h_n^{-2} \lesssim 1,
\]
\[
\|K_{3,n}(\cdot - (\Delta X)_1)\|_\mathbb{R} \lesssim h_n^{-1}.
\]
Hence
\[
\sup_{f_{i,n} \in \mathcal{F}_{i,n}} \left\| \sum_{i=0}^{3} f_{i,n}((\Delta X)_1) \right\|_\mathbb{R} \lesssim \frac{h_n^{-1}}{\sqrt{h_n^{-2}}} \lesssim \sqrt{h_n}/\sqrt{\Delta}
\]
and we have
(5.26)
\[
\sup_{f_{i,n} \in \mathcal{F}_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}((\Delta X)_1) \right] \lesssim \sup_{f_{i,n} \in \mathcal{F}_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}^2((\Delta X)_1) \right] \sqrt{h_n}/\sqrt{\Delta} \lesssim \sqrt{h_n}/\sqrt{\Delta},
\]
\[
\sup_{f_{i,n} \in \mathcal{F}_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}^4((\Delta X)_1) \right] \lesssim \sup_{f_{i,n} \in \mathcal{F}_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}^2((\Delta X)_1) \right] h_n/\Delta \lesssim h_n/\Delta.
\]

Due Theorem 2.1 in [6] with \( B(f) \equiv 0, A \lesssim 1, v \lesssim 1, \sigma \sim 1, b \lesssim \frac{\sqrt{\log n}}{\sqrt{\Delta}}, \gamma \lesssim \frac{1}{\log n} \) and \( q \) sufficiently large, we derive that there exist a random variable \( V_n \) with the same distribution as \( \|U_n\|_{\mathcal{F}_{i,n}} \) such that
(5.27)
\[
\left\| G_n \right\|_{\mathcal{F}_{i,n}} - V_n \right\| = \mathcal{O}_P \left( \frac{\log n}{n^{1/2 - 1/q} \sqrt{\Delta h_n^{-1}}} + \frac{1}{n^{1/6}} \right) = \mathcal{O}_P \left( \frac{\log n}{(n\Delta h_n^{-1})^{1/6}} \right).
\]

Taking into account Assumption 1 (iii), the expression (5.23) converges to zero slower than (5.27). Then the statement 3.14 follows. In addition, for
\[
(f_{j,n}(y) = (iy)^j K_{j,n}(x - y)/s(x)
\]
we define \( T_n^G(x) = U_n \left( \sum_{i=0}^{3} f_{i,n}(x) \right), x \in I, \) and we observe, that there exists a tight Gaussian random variable \( T_n^G(x) \) in \( \mathcal{C}^\infty(I) \) with expected value zero and the same covariance function as for \( T_n \). The following concentration inequality holds (see Theorem 2.1 in [9] for any \( \varepsilon > 0, \)
(5.28)
\[
\sup_{z \in \mathbb{R}} P \left\{ \left| \|T_n^G\|_I - z \right| \leq \varepsilon \right\} \leq 4\varepsilon (1 + E \left[ \|T_n^G\|_I \right]).
\]

According to the Corollary 2.1 in [9], Theorem 3 in [7] and the representation 5.27 for \( \|G_n\|_{\mathcal{F}_{i,n}} - V_n \), we can claim that there exists a sequence \( \varepsilon_n \to 0 \) such
\[
P \left\{ \left| \|G_n\|_{\mathcal{F}_{i,n}} - V_n \right| \geq \varepsilon_n \left( h_n^{1/2} \log h_n^{-1} \right) \right\} \leq \varepsilon_n.
\]

Since \( \|G_n\|_{\mathcal{F}_{i,n}} = \|T_n\|_I \) and \( V_n \overset{d}{=} \|T_n^G\|_I \), we have that
\[
P \left\{ \|T_n\|_I \leq z \right\} \leq P \left\{ \|T_n^G\|_I \leq z + \varepsilon_n \left( h_n^{1/2} \log h_n^{-1} \right) \right\} + \varepsilon_n \leq \]
\[
P \left\{ \|T_n^G\|_I \leq z \right\} + 4\varepsilon_n \left( h_n^{1/2} \log h_n^{-1} \right) (1 + E \left[ \|T_n^G\|_I \right]) + \varepsilon_n,
\]
for all \( z \in \mathbb{R} \). In this way we have
\[
P \left\{ \|T_n\|_I \leq z \right\} \geq P \left\{ \|T_n^G\|_I \leq z \right\} - 4\varepsilon_n \left( h_n^{1/2} \log h_n^{-1} \right) (1 + E \left[ \|T_n^G\|_I \right]) - \varepsilon_n,
\]
for all $z \in \mathbb{R}$.

It follows from Corollary 2.2.8 (see [16]) together with $\text{Var}\left[\sum_{i=0}^{3} f_{i,n}(\Delta X)_{1}\right] = 1$, that

$$
E[\|T_{n}^{G}\|_{I}] = E[\|U_{n}\|_{f_{i,n}}] \lesssim \int_{0}^{1} \sqrt{1 + \log(1/\varepsilon \sqrt{\Delta h_{n}^{-1}})} \, d\varepsilon \lesssim (\log n)^{1/2}.
$$

The proof is completed.

5.2. **Proof of Theorem 2.** The proof scheme of the validity of bootstrap confidence bands was introduced by Kato and Kurisu [10] and can be represented as follows.

Step 1: Conditional distribution of the supremum of the multiplier process $\|\hat{T}_{n}^{MB}\|_{I}$ consistently estimates the distribution of the Gaussian supremum $\|T_{n}^{G}\|_{I}$ in the sense that

$$
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left\{ \|\hat{T}_{n}^{MB}\|_{I} \leq z \mid D_{n}\right\} - \mathbb{P}\left\{ \|T_{n}^{G}\|_{I} \leq z \right\} \right| = o_{P}(1)
$$

Step 2: In addition together with Theorem 1 we have that

$$
\sup_{z \in \mathbb{R}} \left| \mathbb{P}\left\{ \Delta \sqrt{n}(\hat{\rho}_{n}(\cdot) - \rho(\cdot)) \mid I \leq z \right\} - \mathbb{P}\left\{ \|T_{n}^{G}\|_{I} \leq z \right\} \right| \rightarrow 0.
$$

Step 3: Combining steps 1 and 2 leads to the conclusion of Theorem 2. For a precise proof of Theorem 2 we need the following technical lemma.

**Lemma 13.** We have

$$
\|s_{n}^{2}(\cdot)/s^{2}(\cdot) - 1\|_{I} = o_{P}\left\{ (n\Delta h_{n}^{-1})^{-1} \log n \right\}^{1/2}.
$$

This Lemma can be proved using the technique in [10] (see Lemma 8.10) together with Corollary 5.1 and A.1 in [5].

**Proof.** First using Lemma 9 note that

$$
\left\| \frac{\mathbb{D}}{\Phi_{\Delta X} \Phi_{\Delta X}} \right\|_{\mathbb{R}} \lesssim n^{-1/2} \log h_{n}^{-1},
$$

$$
\left\| \frac{\mathbb{D}}{\Phi_{\Delta X} \Phi_{\Delta X}} \right\|_{\mathbb{R}} \lesssim \frac{\mathbb{D}'}{\Phi_{\Delta X} \Phi_{\Delta X}} + \frac{\Delta \mathbb{D} \Psi'}{\Phi_{\Delta X} \Phi_{\Delta X}} \lesssim n^{-1/2} \log h_{n}^{-1}(\Delta^{1/2} + \Delta h_{n}^{-1}) \lesssim n^{-1/2} \log h_{n}^{-1} \Delta^{1/2},
$$

$$
\left\| \frac{\mathbb{D}}{\Phi_{\Delta X} \Phi_{\Delta X}} \right\|_{\mathbb{R}} \lesssim n^{-1/2} \log h_{n}^{-1}(\Delta^{1/2} + \Delta^{3/2} h_{n}^{-1} + \Delta^{2} h_{n}^{-2}) \lesssim n^{-1/2} \log h_{n}^{-1} \Delta^{1/2},
$$

$$
\left\| \frac{\mathbb{D}}{\Phi_{\Delta X} \Phi_{\Delta X}} \right\|_{\mathbb{R}} \lesssim n^{-1/2} \log h_{n}^{-1}(\Delta^{1/2} + \Delta^{3/2} h_{n}^{-1} + \Delta^{2} h_{n}^{-2} + \Delta^{3} h_{n}^{-3}) \lesssim n^{-1/2} \log h_{n}^{-1} \Delta^{1/2}.
$$
Furthermore
\[ \hat{K}_{3,n}(z) - K_{3,n}(z) = -\frac{1}{2\pi} \int_{\mathbb{R}} e^{-izu} \left( \hat{Q}_{3}(u) - Q_{3}(u) \right) \varphi_{W}(uh_n) \, du \]
\[ = -\frac{1 - \alpha}{4\pi} \int_{\mathbb{R}} e^{-izu} u \left( \frac{-\mathbb{D}(u)}{\Phi_{\Delta X}(u)\Phi_{\Delta X}(u)} \right) \varphi_{W}(uh_n) \, du \]
and
\[ \| \hat{K}_{3,n} - K_{3,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \log h_n^{-1} \int_{-1/h_n}^{1/h_n} e^{-iux} u \, du \lesssim n^{-1/2} h_n^{-1} \log h_n^{-1}. \]

Analogously
\[ \| \hat{K}_{2,n} - K_{2,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \log h_n^{-1} \left( h_n^{-1} + \Delta^{1/2} \int_{-1/h_n}^{1/h_n} e^{-iux} u \, du \right) \lesssim n^{-1/2} h_n^{-1} \log h_n^{-1}, \]
\[ \| \hat{K}_{1,n} - K_{1,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \log h_n^{-1} \Delta^{1/2}(h_n^{-1} + \Delta^{1/2}h_n^{-1}) \lesssim n^{-1/2} \Delta^{1/2} h_n^{-1} \log h_n^{-1}, \]
\[ \| \hat{K}_{0,n} - K_{0,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \log h_n^{-1} \Delta^{1/2}(h_n^{-1} + \Delta^{1/2}h_n^{-1}) \lesssim n^{-1/2} \Delta^{1/2} h_n^{-1} \log h_n^{-1}. \]
Since \( \| \hat{K}_{i,n} - K_{i,n} \|_{\mathbb{R}} \leq \| \hat{K}_{i,n} - K_{i,n} \|_{\mathbb{R}} + \| \hat{K}_{i,n} + K_{i,n} \|_{\mathbb{R}} \), we have according to (5.25),
\[ \| \hat{K}_{0,n} - K_{0,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \Delta^{3/2} h_n^{-2} \log h_n^{-1}, \]
\[ \| \hat{K}_{1,n} - K_{1,n} \|_{\mathbb{R}} \lesssim n^{-1/2} \Delta^{3/2} h_n^{-2} \log h_n^{-1}, \]
\[ \| \hat{K}_{2,n} - K_{2,n} \|_{\mathbb{R}} \lesssim n^{-1/2} h_n^{-1} \log h_n^{-1}, \]
\[ \| \hat{K}_{3,n} - K_{3,n} \|_{\mathbb{R}} \lesssim n^{-1/2} h_n^{-1} \log h_n^{-1}. \]

Next we have for \( i = 0, 1, 2, 3 \)
\[ \frac{1}{n} \sum_{j=1}^{n} (\Delta X)^{ij}_{j} \left\{ \hat{K}_{i,n} \left( \cdot - (\Delta X)_{j} \right) - K_{i,n} \left( \cdot - (\Delta X)_{j} \right) \right\}_{f} = O_{p}(n^{-1/2} \Delta^{(i+1/2)} h_n^{-1} \log h_n^{-1}). \]

and analogously
\[ \frac{1}{n} \sum_{j=1}^{n} (\Delta X)^{ij}_{j} \left\{ \hat{K}_{i,n} \left( \cdot - (\Delta X)_{j} \right) - K_{i,n} \left( \cdot - (\Delta X)_{j} \right) \right\}_{f} = O_{p}(n^{-1/2} \Delta^{5/2} h_n^{-2} \log h_n^{-1}), \]
\[ \frac{1}{n} \sum_{j=1}^{n} (\Delta X)^{ij}_{j} \left\{ \hat{K}_{i,n} \left( \cdot - (\Delta X)_{j} \right) - K_{i,n} \left( \cdot - (\Delta X)_{j} \right) \right\}_{f} = O_{p}(n^{-1/2} h_n^{-1} \log h_n^{-1}), \]
\[ \frac{1}{n} \sum_{j=1}^{n} (\Delta X)^{ij}_{j} \left\{ \hat{K}_{i,n} \left( \cdot - (\Delta X)_{j} \right) - K_{i,n} \left( \cdot - (\Delta X)_{j} \right) \right\}_{f} = O_{p}(n^{-1/2} h_n^{-2} \log h_n^{-1}). \]

By the previous statement, we conclude that
\[ \tilde{s}_{n}^{2}(x) = \bar{s}_{n}^{2}(x) + O_{p}(n^{-1/2} \Delta^{2/4} h_n^{-2} \log h_n^{-1}) \]
uniformly in $x \in I$, where
\[
\tilde{s}_n^2(x) := \sum_{m=0}^{3} \left( \frac{1}{n} \sum_{j=0}^{n} \left[ (\Delta X)^{2m}_j K_{m,n} (x - (\Delta X)_j) \right] 
- \left( \frac{1}{n} \sum_{j=0}^{n} \left[ (\Delta X)^{2m}_j K_{m,n} (x - (\Delta X)_j) \right] \right)^2 \right).
\]

Moreover, since $\inf_{x \in I} s^2(x) \gtrsim \Delta h_n^{-3}$ we obtain that
\[
\frac{n^{-1/2} \Delta h_n^{-2} \log h_n^{-1}}{\Delta h_n^{-3}} = \frac{n^{-1/2} \log h_n^{-1}}{\log n} \ll \frac{1}{n \Delta h_n^{-1}}.
\]

Finally let us prove that $\|s_n^2(\cdot)/s^2(\cdot) - 1\|_I = o_P \{(n \Delta h_n^{-1} \log n)^{1/2}\}$. It follows from (5.12), (5.13), (5.14) und (5.15),
\[
\begin{align*}
\|E[K_{0,n}(\cdot - (\Delta X)_1)/s(\cdot)]\|_I & \lesssim \Delta h_n^{-1} / \sqrt{\Delta h_n^{-3}} \lesssim (\Delta h_n)^{1/2} \\
\|E[(\Delta X)_1 K_{1,n}(\cdot - (\Delta X)_1)/s(\cdot)]\|_I & \lesssim \Delta / \sqrt{\Delta h_n^{-3}} \lesssim \Delta^{1/2} h_n^{3/2} \\
\|E[(\Delta X)_1^2 K_{2,n}(\cdot - (\Delta X)_1)/s(\cdot)]\|_I & \lesssim \Delta / \sqrt{\Delta h_n^{-3}} \lesssim \Delta^{1/2} h_n^{3/2} \\
\|E[(\Delta X)_1^2 K_{3,n}(\cdot - (\Delta X)_1)/s(\cdot)]\|_I & \lesssim \Delta h_n^{-1} / \sqrt{\Delta h_n^{-3}} \lesssim (\Delta h_n)^{1/2}.
\end{align*}
\]

Note that
\[
\sup_{f_{i,n} \in F_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}^2((\Delta X)_1) \right] \lesssim \sup_{f_{i,n} \in F_{i,n}} \mathbb{E} \left[ \sum_{i=0}^{3} f_{i,n}^4((\Delta X)_1) \right] \lesssim h_n / \Delta
\]

and
\[
\sup_{f_{i,n} \in F_{i,n}} \left\| \sum_{i=0}^{3} f_{i,n} \right\|_{\mathcal{R}} \lesssim \sup_{f_{i,n} \in F_{i,n}} \left\| \sum_{i=0}^{3} f_{i,n}^2 \right\|_{\mathcal{R}} \lesssim h_n / \Delta.
\]

Together with Corollary 5.1 in [5] and Theorem 2.14.1 in [16] we get
\[
\begin{align*}
\left\| \frac{1}{n} \sum_{i=0}^{3} \left( \sum_{j=1}^{n} (f_{i,n}^2((\Delta X)_j) - E[f_{i,n}^2((\Delta X)_j)]) \right) \right\|_{F_{i,n}} & \lesssim \sqrt{\frac{\log n}{n \Delta h_n^{-1}}} + \frac{\log n}{n \Delta h_n^{-1}} \lesssim \sqrt{\frac{\log n}{n \Delta h_n^{-1}}}, \\
\left\| \frac{1}{n} \sum_{i=0}^{3} \left( \sum_{j=1}^{n} (f_{i,n}((\Delta Z)_j) - E[f_{i,n}((\Delta Z)_j)]) \right) \right\|_{F_{i,n}} & \lesssim \sqrt{\frac{1}{n \Delta h_n^{-1}}} \lesssim \sqrt{\frac{\log n}{n \Delta h_n^{-1}}}.
\end{align*}
\]

Finally we have $\|s_n^2(\cdot)/s^2(\cdot) - 1\|_I = O_P((n \Delta h_n^{-1} \log n)^{1/2})$. This completes the proof. \hfill \Box

We get from (5.29)-(5.30),
\[
\left\| \left( \sum_{j=1}^{n} \omega_j \right) \left\{ \frac{1}{n} \sum_{j=1}^{n} (\Delta X)^{2m}_j \{ \hat{K}_{m,n}(\cdot - (\Delta X)_j) - K_{m,n}(\cdot - (\Delta X)_j) \} \right\} \right\|_f = O_p(\Delta^{1/2} h_n^{-1} \log h_n^{-1}).
\]
Hence
\[
\left\| \sum_{j=1}^{n} \omega_j \{ (\Delta X)^m_j \{ \hat{K}_{m,n} (\cdot - (\Delta X)) - K_{m,n} (\cdot - (\Delta X)) \} \} \right\|_I \\
\leq \mathcal{O}_p(n^{-1/2}h_n^{-1} \log h_n^{-1}) E \left[ \sqrt{\sum_{j=1}^{n} (\Delta X)^{2m}_j} \right] = \mathcal{O}_p(\Delta^{1/2}h_n^{-1} \log h_n^{-1}).
\]

Since \(1/\tilde{s}_n(x)_I = \mathcal{O}_p(1/\sqrt{\Delta h_n^{-3}})\), using Lemma 13 we obtain
\[
(5.31) \quad \hat{T}^\text{MB}_n(x) = \left[ 1 + o_p \left( \sqrt{\frac{\log n}{n\Delta h_n^{-1}}} \right) \right] \left( T^\text{MB}_n(x) + \mathcal{O}_p \left( h_n^{1/2} \log h_n^{-1} \right) \right)
\]

Applying Theorem 2.2 in [6] with \(B(f) = 0, A \lesssim 1, v \lesssim 1, \sigma \sim 1, b \lesssim \frac{\sqrt{\log n}}{\sqrt{\Delta}}, \gamma \lesssim \frac{1}{\log n}\) and sufficiently large \(q\), we conclude that there exists a random variable \(V_n^\xi\) whose conditional distribution given \(D_n\) is identical to the distribution of \(\|U_n\|_{\mathcal{F}_n,I}\), that is, \(P\left( \|V_n^\xi\|_I \leq z \ | \ D_n \right) = P\left( \|T^G_n\|_I \leq z \right)\) for all \(z \in \mathbb{R}\) almost surely, and such that
\[
(5.32) \quad \mathbb{E}\left[ \|G_n^\xi\|_{\mathcal{F}_n,I} - V_n^\xi \right] = \mathcal{O}_p \left( \frac{\log n}{n^{1/2-1/4}} \right) + \mathcal{O}_p \left( \frac{\log n}{n\Delta h_n^{-1}} \right) = \mathcal{O}_p \left( \frac{\log n}{n\Delta h_n^{-1}} \right)
\]

This in turn implies that there exists a sequence of constants \(\varepsilon_n \to 0\) such that
\[
P\left( \|G_n^\xi\|_{\mathcal{F}_n,I} - V_n^\xi \geq \varepsilon_n \frac{\log n}{n\Delta h_n^{-1/4}} \ | \ D_n \right) \to 0.
\]

The condition (iii) of Assumption 1 guarantees that the expression (5.32) converges to 0 and with speed faster than one of the expression (3.14). Since \(\|G_n^\xi\|_{\mathcal{F}_n,I} = \|T^\text{MB}_n\|_I\), we get together with the bound \(\mathbb{E}\left[ \|T^G_n\|_I \right] \lesssim (\log n)^{1/2}\) and the anti-concentration inequality (5.28),
\[
P\left( \|T^\text{MB}_n\|_I \leq z \ | \ D_n \right) \leq P\left( \|V_n^\xi\|_I \leq z + \varepsilon_n h_n^{-1/2} \log h_n^{-1} \ | \ D_n \right) + o_p(1)
\]
\[
= P\left( \|T^G_n\|_I \leq z + \varepsilon_n h_n^{-1/2} \log h_n^{-1} \right) + o_p(1) \leq P\left( \|T^G_n\|_I \leq z \right) + o_p(1)
\]

For the same reason, we conclude that
\[
P\left( \|T^\text{MB}_n\|_I \leq z \ | \ D_n \right) \geq P\left( \|T^G_n\|_I \leq z \right) - o_p(1).
\]

This argument shows together with (5.31) that
\[
(5.33) \quad \sup_{x \in \mathbb{R}} P\left( \|\hat{T}^\text{MB}_n(\cdot)\|_I \leq z \ | \ D_n \right) - P\left( \|T^G_n\|_I \leq z \right) \to 0.
\]

To conclude the proof, it remains to show that
\[
(5.34) \quad P\left( \nu(x) \in \hat{C}^\text{MB}_{1-\tau} (x) \ \forall x \in I \right) \to 1 - \tau.
\]

Let us recall that it follows from Theorem 1 together with the bound \(\mathbb{E}\left[ \|T^G_n\|_I \right] \lesssim (\log n)^{1/2}\) that
\[
\rho(x) \in \hat{C}^\text{MB}_{1-\tau} (x) \ \forall x \in I, \text{ if and only if } \left\| \frac{\sqrt{n} \Delta (\hat{\rho}_n(\cdot) - \rho(\cdot))}{\tilde{s}_n(\cdot)} \right\|_I \leq \hat{c}^\text{MB}_n (1 - \tau)
\]
and we have \( \| T_n \|_I = O_P\{\log n\}^{1/2} \). Let us remark that
\[
\frac{\sqrt{n} \Delta (\hat{\rho}_n(x) - \rho(x))}{\hat{s}_n(x)} = \frac{s(x) \sqrt{n} \Delta (\hat{\rho}_n(x) - \rho(x))}{\hat{s}_n(x)}
= (1 + o_P\{n^{-1/2} \log h_n^{-1}\}) \left[ T_n(x) + o_P(h_n^{1/2} \log h_n^{-1}) \right]
= T_n(x) + o_P(h_n^{1/2} \log h_n^{-1}).
\]

Now if we recall the conclusion of Theorem 1 and the anti-concentration inequality (5.28), we get
\[
\sup_{z \in \mathbb{R}} P\left\{ \left\| \frac{\sqrt{n} \Delta (\hat{\rho}_n(\cdot) - \rho(\cdot))}{\hat{s}_n(\cdot)} \right\|_I \leq z \right\} - P\left\{ \| T_n^G \|_I \leq z \right\} \to 0.
\]

Note that due to (5.33) together with argument similar to Step 3 in the proof of Theorem 2 in [11], we can find a sequence of constants \( \varepsilon'_n \to 0 \) such that
\[
e_n^G (1 - \tau - \varepsilon'_n) \leq \hat{e}_n^M (1 - \tau) \leq e_n^G (1 - \tau + \varepsilon'_n)
\]
with probability approaching one. This implies that
\[
P\left\{ \left\| \frac{\sqrt{n} \Delta (\hat{\rho}_n(\cdot) - \rho(\cdot))}{\hat{s}_n(\cdot)} \right\|_I \leq e_n^M (1 - \tau + \varepsilon'_n) \right\}
\leq P\left\{ \left\| \frac{\sqrt{n} \Delta (\hat{\rho}_n(\cdot) - \rho(\cdot))}{\hat{s}_n(\cdot)} \right\|_I \leq e_n^G (1 - \tau + \varepsilon'_n) \right\} + o(1)
\leq P\left\{ \| T_n^G \|_I \leq e_n^G (1 - \tau + \varepsilon'_n) \right\} + o(1) = 1 - \tau + o(1)
\]
For the same reason, we have upper bound for the probability, which has the form \( 1 - \tau - o(1) \). Due the Borell-Sudakov-Tsirelson inequality (see Lemma A.2.2 in [16] for more details) we have
\[
e_n^G (1 - \tau + \varepsilon'_n) \lesssim E[\| T_n^G \|_I] + \sqrt{1 + \log(1/(\tau - \varepsilon'))} \lesssim (\log n)^{1/2}.
\]
If we combine this with (5.36), we get \( \hat{e}_n^M (1 - \tau) = O_P(\sqrt{\log n}) \) with the supremum width of the confidence band \( \hat{C}_{1-\tau}^M \) bounded as
\[
2 \sup_{x \in I} \hat{s}_n(x) \hat{e}_n^M (1 - \tau) \lesssim (1 + o_P(1)) \sup_{x \in I} s(x) \hat{e}_n^M (1 - \tau)
= O_P((\Delta h_n^{3/2})^{-1/2} \sqrt{\log n})
\]
This observation completes the proof of Theorem 2 for the multiplier bootstrap case.

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