Open Multi-Agent Systems with Variable Size: the Case of Gossiping

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Abstract—We consider open multi-agent systems, which are systems subject to frequent arrivals and departures of agents while the process studied takes place. We study the behavior of all-to-all pairwise gossip interactions in such open systems. Arrivals and departures of agents imply that the composition and size of the system evolve with time, and in particular prevent convergence. We describe the expected behavior of the system by showing that the evolution of scale-independent quantities can be characterized exactly by a fixed size linear dynamical system. We apply this approach to characterize the evolution of the two first moments (and thus also of the variance) for open systems of both fixed and variable size. Our approach is based on the continuous time modelling of random asynchronous events, namely gossip steps, arrivals, departures, and replacements.

Index Terms—Open multi-agent systems, Agents and autonomous systems, Cooperative control, Linear systems, Markov processes.

I. INTRODUCTION

Flexibility and scalability are among the most cited and desired features of multi-agent systems. Real-life examples include flocks of birds [1], ad-hoc networks of mobile-devices [2], the Internet, or even vehicle coordination [3], for which potential agent failures or new agent arrivals are expected to be handled by the system. However, the classical models used to study multi-agent systems assume that their composition, as complex as it can be, remains unchanged over time. This assumption then allows for characterizing asymptotic behaviors of multi-agent systems, such as convergence or synchronization.

If agents arrivals and departures are sufficiently rare as compared to the time-scale of the process that is studied by the system, this apparent contradiction may be justified, as the system is expected to be able to incorporate the effect of such event before the next one occurs. Nevertheless, this may not be the case anymore for large systems, as both the arrival or departure probability of an agent and the characteristic length of a process grow with the number of agents. Living systems with birth processes or even human societies are examples of systems whose growth is proportional to the size. Extreme environments where communication within the system may be difficult or infrequent can also lead to slow convergence rates naturally comparable to agent failure rates, and thus challenge this assumption too. Moreover, there are systems that are inherently open: think of a stretch of road shared by vehicles that keep entering and leaving it.

We consider here open multi-agent systems, which are subject to permanent arrivals and departures of agents during the execution of the process that is considered. These arrivals and departures result in new challenges for the design and analysis of such systems. An illustration is provided in Figure 1.

Figure 1. Example of dynamics of an open multi-agent system of 4 agents with random agent replacements and pairwise average gossips. Each line corresponds to the value held by an agent, and each red circle highlights a replacement event. The repeated replacements prevent convergence to consensus. See Section IV for a precise description of the system.

Firstly, every arrival and departure changes the system state dimension, whose evolution becomes challenging to analyze. The state of the system itself suffers repeated potentially important changes, that prevent it from asymptotically converging to some specific state (this is clear from Figure 1). Rather, they may approach some form of steady state behavior, which can be characterized by some relevant quantities. As in classical control in presence of perturbations, the choice of the measures is not neutral, and different descriptive quantities may behave in very different ways.

Secondly, arrivals and departures significantly impact the design of decentralized algorithms. On the one hand, departures often imply losses of information that could be needed depending on the nature of the problem. On the other hand, the desired output of the algorithm can be defined by the values held by the agents presently in the system, and thus vary over time: it can then become necessary to eliminate outdated information. Algorithms in open systems must thus be robust to arrivals and departures, and able to handle potentially
variable objectives. Such algorithms were already explored for instance for the MAX-consensus problem in [4] or the median consensus problem in [5], both subject to arrivals and departures. Moreover, algorithms designed for open systems cannot be expected to provide “0” exact results. Fundamental limitations on the performance of averaging algorithms in open systems were for instance exposed in [6], [7]. Hence, it may be preferable to maintain an approximate answer robust to perturbations rather than ensuring asymptotic convergence to an exact answer if the composition were to remain constant.

A. Contribution

We focus on the analysis of open multi-agent systems subject to random pairwise gossip interactions [8], with the goal of developing an approach applicable to general open systems. We consider all-to-all (possible) pairwise interactions, focusing on systems where departures and arrivals take place at random times, see Section II for a complete definition.

We analyze the system evolution in terms of “0” scale-independent quantities. Two such relevant quantities, that we study in this work, are given by the two first expected moments: the expected squared mean $\mathbb{E}\bar{x}^2$ and the expected mean of squares $\mathbb{E}x^2$ of the system state $x$. These quantities also provide the evolution of the expected variance $\mathbb{E}(\bar{x}^2 - \bar{x}^2)$. We show in Section III that the evolution of these quantities can be characterized exactly, and that they evolve according to an associated 2-dimensional linear system.

As a first case study, we analyze in Section IV systems subject to random replacements (i.e., a departure immediately followed by an arrival). In this simplified setting, the system size remains constant. We provide the evolution of the descriptors as a function of the rates at which replacements and gossip steps occur; those are directly related to the probability for a random event to be a replacement. We then focus on a more general case of open systems by considering systems subject to random arrivals and departures in Section V. This requires more advanced tools to monitor both the state and the size of the system at the same time.

B. Preliminary results

Preliminary results on simplified versions of the systems considered here were presented at the Allerton Conference on Communication Control and Computing (2016) [9]. Those considered deterministic arrivals and departures through the analysis of systems subject to periodic replacements, and to arrivals without departures. The analysis was performed through the study of the evolution of size-independent descriptors, namely the expected mean, the expected squared mean, and the expected variance, as a 3-dimensional linear system.

Another preliminary version was presented at the Conference on Decision and Control (2017) [10]. This one extended the results of [9] to the stochastic case, still considering fixed-size systems and growing systems where the replacements and arrivals are probabilistic events. Moreover, those results included a convergence analysis and a different choice of the moments that were studied to model the evolution of the system with a 2-dimensional linear system.

With respect to these early versions, the main new contributions in this analysis include (i) the study of completely open systems subject to independent asynchronous random arrivals and departures leading to a variable and non-monotonous system size in Section V (ii) the use of continuous time tools to model the evolution of the scale-independent quantities through asynchronous events, allowing for simpler proofs (whereas only discrete tools were considered up to now), and (iii) the derivation of bounds on the expected variance for situations where no assumption can be made on the way the agent leaving the system is selected (see Section IV.C).

C. Other works on open multi-agent systems

The possibility of agents joining or leaving the system has been recognized in computer science, and specific architectures have for example been proposed to deploy large-scale open multi-agent systems, see e.g. THOMAS project [11]. There also exist mechanisms allowing distributed computation processes to cope with the shut down of certain nodes or to take advantage of the arrival of new nodes. Typically, algorithms have been designed to maintain network connectivity into P2P networks subject to departures and arrivals [12].

Frameworks similar to open multi-agent arrivals have also been considered with VTL for autonomous cars to deal with cross-sections [13], and in the context of trust and reputation computation, motivated by the need to determine which arriving agents may be considered reliable, see e.g., the model FIRE [14]. However, the study of these algorithms’ behavior is mostly empirical.

Varying compositions were also studied in the context of self-stabilizing population protocols [1], [15], where interacting agents (typically finite-state machines) can undergo temporary or permanent failures, which can respectively represent the replacement or the departure of an agent. The objective in those works is to design algorithms that eventually stabilize on the desired answer if the system composition stops changing, i.e., once the system has become “0” closed.

Opinion dynamics models with arrivals and departures have also been empirically studied in [16], [17]. More generally, simulation-based analyses were performed in [18] for social phenomena considering arrivals and departures. Dynamic consensus in open systems has also been investigated in both in terms of stability analysis and algorithm design.

Finally, openness starts being considered for decentralized optimization, either indirectly through the use of variations of the objective function in online decentralized optimization in [20], or directly with stability analyses of decentralized gradient descent in open systems [21].

II. SYSTEM DESCRIPTION

We consider a multi-agent system whose composition evolves with time. We use integers to label the agents. We denote by $\mathcal{N}(t) \subset \mathbb{N}$ the set of agents present in the system at time $t$, and by $n(t)$ the number of agents present at time $t$, i.e., the cardinality of $\mathcal{N}(t)$. Each agent $i$ holds a value $x_i(t) \in \mathbb{R}$. We assume that the initial values of the agents at time $t = 0$ are randomly chosen according to some distribution
We consider a continuous evolution of the time $t$. In this configuration, events occur asynchronously at random times and potentially impact the state and size of the system. An event $e$ takes place according to a Poisson clock of rate $\lambda_e$ (which can depend on the size of the system $n(t)$: this will be discussed more in details in Section III). We provide the possible types of events that can occur below, where we denote by $x(t^-)$ the state of the system right before an event taking place at time $t$, and by $x(t^+)$ its state after.

(a) Gossip: Two agents $i, j \in \mathcal{N}(t^-)$ are uniformly randomly and independently selected among the $n(t^-)$ agents present in the system (with in particular the possibility of selecting twice the same agent), and they update their values $x_i, x_j$ by performing a pairwise average:

$$x_i(t^+) = x_j(t^+) = \frac{x_i(t^-) + x_j(t^-)}{2}. \quad (1)$$

(b) Arrival: One “0” new agent $i \not\in \mathcal{N}(s)$, $\forall s \leq t$ (i.e., that has never been in the system before), joins the system, so that $\mathcal{N}(t^+) = \mathcal{N}(t^-) \cup \{i\}$ and $n(t^+) = n(t^-) + 1$. The initial value $x_i(t^+) \in \mathbb{R}$ of the arriving agent is drawn independently from the same distribution $\mathcal{D}$ as for the initialization of the system.

(c) Departure: An agent $i \in \mathcal{N}(t^-)$ is selected and leaves the system, so that $\mathcal{N}(t^+) = \mathcal{N}(t^-) \setminus \{i\}$ and $n(t^+) = n(t^-) - 1$. This event may only occur if $n(t^-) > 0$. There are several ways to select the departing agent. By default, we will consider a random departure, which consists in the random uniform choice of the departing agent. We will also consider later an adversarial departure, where no assumption can be made on this choice.

Notice that the way arrivals and departures are defined imply that an agent having left the system cannot go back into it. Moreover, all the random events above are assumed independent of each other. However, we will sometimes consider for simplicity a “0” replacement event, which consists of the instantaneous combination of a departure and an arrival: an agent leaves the system and is instantaneously replaced.

A. Scale-independent quantities of interest

The aim of the study is to characterize the disagreement among agents i.e., the distance to consensus. We say that consensus is reached asymptotically when

$$\lim_{t \to \infty} \max_{(i,j) \in \mathcal{N}(t)^2} |x_i(t) - x_j(t)| = 0. \quad (2)$$

If the system dynamics does not include agent departures or arrivals, it is known that the gossip process we consider leads to consensus, see e.g., [8], [22]. The objective here is to understand how agent arrivals and departures impact the disagreement among agents. To do so, we study several quantities of interest. Because the system size may change significantly with time, we do not directly track $x(t)$, but rather focus on scale-independent quantities, i.e., quantities whose values do not scale with the size of the system. We consider in particular the empirical mean of the squares and the variance defined as

$$\bar{x}^2 = \frac{1}{n} \sum_{i \in \mathcal{N}} x_i^2,$$

$$\text{Var}(x) = \frac{1}{n} \sum_{i \in \mathcal{N}} (x_i - \bar{x})^2 = \bar{x}^2 - \bar{x}^2, \quad (3)$$

respectively, where references to time were removed to lighten the notation. Our study will focus on the evolution of $\mathbb{E}[\text{Var}(x)]$, which will also require monitoring $\mathbb{E}[\bar{x}^2]$ and $\mathbb{E}[\bar{x}^2]$. When new agents keep arriving it is impossible to achieve asymptotic consensus in the sense of [2], because the new agent’s value will with high probability be different from the value of the agents already present in the system. The study of $\mathbb{E}[\text{Var}(x)]$ will allow us to see how “0” far the system will be from consensus. The expected mean $\mathbb{E}[\bar{x}]$ could also have been monitored. It evolves following an independent one-dimensional linear system (see e.g., [9] for an analysis of $\mathbb{E}[\bar{x}]$ in a simplified setting). However, we omit this part of the study due to space limitation.

III. Preliminary tools

We first state the following preliminary lemma that provides an upper bound on the expected value of both descriptors $\mathbb{E}[\bar{x}^2]$ and $\mathbb{E}[\bar{x}^2]$, and that is proved in Appendix A. It will later allow us to bound the evolution of the variance independently of the other descriptors, and to perform further analyses.

**Lemma 1.** Consider the setting described above (where the values $x_i(t)$ are chosen in an i.i.d. way with expected value $0$ and variance $\sigma^2$), and let $t \geq 0$ be some arbitrary fixed time. Then, for any set $S$ of agents in the system at time $t$, and for any fixed or stochastic (but independent of agent values) sequence of gossips, departures and arrivals, there holds $\mathbb{E} \left( \sum_{i \in S} x_i(t) \right)^2 \leq |S| \sigma^2$. In particular,

$$\mathbb{E} \left( \bar{x}^2 | n(t) = j \right) \leq \frac{1}{j} \sigma^2; \quad \mathbb{E} \left( \bar{x}^2 | n(t) = j \right) \leq \sigma^2. \quad (4)$$

Notice that at the beginning of the process, there holds $\mathbb{E}(\bar{x}^2(0)|n(0) = j) = \frac{\sigma^2}{j}$ and $\mathbb{E}(\bar{x}^2(0)|n(0) = j) = \sigma^2$. Hence, the above lemma states that the expected values of the descriptors cannot exceed that they had at the initialization of the system. This follows that $x_j(t)$ is a combination of contributions from other agents that have potentially already left the system, and thus includes at least as much information as if no interaction ever happened.

A. Effect of the different events

We now show that the evolution of the expected moments $\mathbb{E}[\bar{x}^2]$ and $\mathbb{E}[\bar{x}^2]$ through events is governed by a 2-dimensional affine system from which we also derive the evolution of $\mathbb{E}[\text{Var}(x)]$. To lighten the notations, we denote by $\vec{x}$ the vector containing $\bar{x}^2$ and $\bar{x}^2$ so that $\text{Var}(x) = (-1, 1) \vec{x} = \bar{x}^2 - \bar{x}^2$.

We refer to $x$ (resp. $X$) as the state (resp. the corresponding descriptors vector) as it is before the event that is considered, and to $x'$ (resp. $X'$) after. The proofs of the following lemmas are provided in Appendix B for the sake of completeness.
The definition of the different types of events implies that the descriptors evolve differently depending on the system size at the moment the event takes place. Hence, we distinguish the events according to both their type and the system size at the moment they occur. The occurrence of a given event then only makes sense if the system size allows it.

**Lemma 2** (Gossip step). In a system of initially \( n \) agents subject to a gossip step event (noted \( \text{Gos}_n \)), there holds

\[
\mathbb{E}(X'|X, \text{Gos}_n) = A_{\text{Gos}_n} X,
\]

where

\[
A_{\text{Gos}_n} = \begin{pmatrix} 1 & 0 \\ \frac{1}{n} - 1 & \frac{1}{n} \end{pmatrix}.
\]

In particular,

\[
\mathbb{E}(\text{Var}(x')|\text{Var}(x), \text{Gos}_n) = \left(1 - \frac{1}{n}\right) \text{Var}(x).
\]

**Lemma 3** (Arrival). In a system of initially \( n \) agents subject to an arrival event (noted \( \text{Arr}_n \)), there holds

\[
\mathbb{E}(X'|X, \text{Arr}_n) = A_{\text{Arr}_n} X + b_{\text{Arr}_n},
\]

where

\[
A_{\text{Arr}_n} = \begin{pmatrix} \frac{n^2}{(n+1)^2} & 0 \\ 0 & \frac{1}{n+1} \end{pmatrix} \quad \text{and} \quad b_{\text{Arr}_n} = \left(\frac{1}{(n+1)^2}\right) \sigma^2.
\]

In particular,

\[
\mathbb{E}(\text{Var}(x')|\text{Var}(x), \text{Arr}_n) = \frac{n}{n+1} \left(\frac{\text{Var}(x) + \frac{\sigma^2 + \pi^2}{n+1}}{n+1}\right) \leq \frac{n}{n+1} \left(\frac{\text{Var}(x) + \frac{\sigma^2}{n}}{n+1}\right).
\]

**Lemma 4** (Departure). In a system of initially \( n \) agents subject to a (random) departure event (noted \( \text{Dep}_n \)), there holds

\[
\mathbb{E}(X'|X, \text{Dep}_n) = A_{\text{Dep}_n} X,
\]

where

\[
A_{\text{Dep}_n} = \left(1 - \frac{1}{(n-1)^2}\right) \begin{pmatrix} 1 & 0 \\ \frac{1}{n-1} & 1 \end{pmatrix}.
\]

In particular,

\[
\mathbb{E}(\text{Var}(x')|\text{Var}(x), \text{Dep}_n) = \left(1 - \frac{1}{(n-1)^2}\right) \text{Var}(x).
\]

We remark that the above result holds for random departures, where the leaving agent is randomly uniformly chosen among those in the system before the departure.

We now consider the random replacement of an agent, which consists of a random departure immediately followed by an arrival. The next result follows from a combination of Lemmas 4 and 3, the latter applied to a system of size \( n - 1 \) joined by a \( n^{th} \) agent.

**Lemma 5** (Replacement). In a system of initially \( n \) agents subject to a replacement event (noted \( \text{Rep}_n \)), there holds

\[
\mathbb{E}(X'|X, \text{Rep}_n) = A_{\text{Rep}_n} X + b_{\text{Rep}_n},
\]

where

\[
A_{\text{Rep}_n} = \left(\frac{n-2}{n} \cdot \frac{1}{n-1}\right) \quad \text{and} \quad b_{\text{Rep}_n} = \left(\frac{1}{n}\right) \sigma^2.
\]

Moreover,

\[
\mathbb{E}(\text{Var}(x')|\text{Var}(x), \text{Rep}_n) \leq \frac{n^2 - n - 1}{n^2} \text{Var}(x) + \frac{n^2 - 1}{n^2} \sigma^2.
\]

**B. General descriptors evolution**

We now provide a variation of well-known results on Markov processes. This serves as a basis to characterize the expected evolution of the descriptors conditioned by the size of the system. We show that it is linked with a flow equation governing the evolution of the system size. We begin by providing a simple illustration of application of that result.

1) Introductory example: Let us consider a Markov process \( n(t) \) that admits two states \( \alpha \) and \( \beta \). This process switches from state \( \alpha \) to \( \beta \) (resp. \( \beta \) to \( \alpha \)) according to a Poisson clock of rate \( \lambda_{\alpha \to \beta} \) (resp. \( \lambda_{\beta \to \alpha} \)), and remains unchanged in between. See Figure 2 for a graphical representation.

![Figure 2](image)

Upon this process is built a random process \( X(t) \) that evolves with \( n(t) \) through its transitions (at events \( \alpha \to \beta \) and \( \beta \to \alpha \)). In our example, let us assume

\[
\mathbb{E}(X'|X, \alpha \to \beta) = \frac{X}{2} \quad \text{and} \quad \mathbb{E}(X'|X, \beta \to \alpha) = X + \sigma^2,
\]

where \( X' \) and \( X \) respectively denote the state of \( X(t) \) after and before the event, and \( \sigma^2 \) is some positive value representing e.g. some additive noise. Then it follows that

\[
\mathbb{E}(X'|\alpha \to \beta) = \frac{1}{2} \mathbb{E}(X|\alpha \to \beta) + \frac{1}{2} \mathbb{E}(X|n = \alpha) \quad \text{and} \quad \mathbb{E}(X'|\beta \to \alpha) = \mathbb{E}(X|\beta \to \alpha) + \sigma^2 = \mathbb{E}(X|n = \beta) + \sigma^2.
\]

Our result will imply that the contributions of both states of \( n(t) \) to the evolution of \( \mathbb{E}(X(t)) \) satisfy the following flow equations

\[
\frac{d}{dt} X_{\alpha}(t) = -\lambda_{\alpha \to \beta} X_{\alpha} \pi_{\alpha} + \lambda_{\beta \to \alpha} \left(\frac{X_{\beta} + \sigma^2}{2}\right) \pi_{\beta};
\]

\[
\frac{d}{dt} X_{\beta}(t) = -\lambda_{\beta \to \alpha} X_{\beta} \pi_{\beta} + \lambda_{\alpha \to \beta} \left(\frac{X_{\alpha}}{2}\right) \pi_{\alpha},
\]

where \( X_{\alpha}(t) = \mathbb{E}(X(t)|n(t) = \alpha) \) and \( \pi_{\alpha}(t) = P(n(t) = \alpha) \) (the same holds for \( \beta \)). It thus follows from \( \mathbb{E}(X(t)) = X_{\alpha}(t) \pi_{\alpha}(t) + X_{\beta}(t) \pi_{\beta}(t) \) that

\[
\frac{d}{dt} \mathbb{E}(X) = -\lambda_{\alpha \to \beta} \left(\frac{X_{\alpha}}{2}\right) \pi_{\alpha} + \lambda_{\beta \to \alpha} \pi_{\beta} \sigma^2.
\]
2) General result: We now derive a general version of the result claimed above for a general Markov process \( n(t) \in \mathbb{N} \) and a random process \( X(t) \) that both evolve through events and remain constant in between. The evolution of \( X(t) \) through some event \( e \) is random, and its expected value follows the affine system \( \mathbb{E}(X^t|X,e) = A_e X + b_e \) (where \( X^t \) denotes the state of \( X(t) \) after the event).

Notice that for an event \( e \) to happen, the state of \( n(t) \) must allow it: in the previous example, the event \( \alpha \rightarrow \beta \) only makes sense at time \( t \) if \( n(t) = \alpha \). More generally, for an event \( e \), we denote respectively by \( s(e) \) and \( a(e) \) the source and arrival states of \( n(t) \) for that event. Namely, if event \( e \) happens at time \( t \), then it implies that \( n(t^-) = s(e) \) and \( n(t^+) = a(e) \). Hence, it follows that

\[
\mathbb{E}(X^t|e) = A_e \mathbb{E}(X|n = s(e)) + b_e. \tag{17}
\]

Moreover, we also need to introduce the following assumption.

**Assumption 1.** For any event sequence \( \xi \) and for any \( j \in \mathbb{N} \), \( \mathbb{E}(X(t)|n(t) = j, \xi) \) is nonnegative and uniformly upper bounded. Moreover, conditional to \( n(t) = j \) for some \( j \in \mathbb{N} \), the probability for two events or more to happen between times \( t - \delta t \) and \( t \) for some \( \delta t > 0 \) is in \( o(\delta t) \).

We can now state the following proposition, that is proved in Appendix C.

**Proposition 1.** In the setting described above, and under Assumption 7 there holds

\[
d \frac{d}{dt} \left( X_j \pi_j \right) = - \sum_{e: s(e) = j} \lambda_e X_j \pi_j + \sum_{e: a(e) = j} \lambda_e \left( A_e X_{s(e)} + b_e \right) \pi_{s(e)}, \tag{18}
\]

where the dependence to the time is omitted, and where \( X_j(t) = \mathbb{E}(X(t)|n(t) = j) \) and \( \pi_j(t) = \mathbb{P}(n(t) = j) \).

**Corollary 1.** If for all events \( e \) there holds \( \mathbb{E}(X^t|X,e) \leq A_e X + b_e \) (with \( A_e \) nonnegative, i.e., all elements of \( A_e \) are nonnegative), there holds

\[
d \frac{d}{dt} \left( X_j \pi_j \right) \leq - \sum_{e: s(e) = j} \lambda_e X_j \pi_j + \sum_{e: a(e) = j} \lambda_e \left( A_e X_{s(e)} + b_e \right) \pi_{s(e)}. \tag{19}
\]

Proposition 1 depicts a flow equation describing the evolution of the contribution of one given state of \( n(t) \) to that of \( \mathbb{E}X = \sum_{j \in \mathbb{N}} X_j \pi_j \). Considering the simple case where \( X(t) = 1 \), it follows that (18) reduces to simple Markov transitions between the states of \( n(t) \). The random process \( X(t) \) can be considered as some weight distribution assigned to the states of \( n(t) \) whose evolution is linked to the flow that governs that of \( n(t) \).

In the context of this paper, \( n(t) \) is the size of the system that evolves as a Markov process, and the weight distribution \( X(t) \) is some quantity that evolves accordingly (e.g., the descriptors). Proposition 1 will be used to describe how the descriptors behave accordingly with a given size of the system, and will allow us to derive their global evolution with the time.

IV. Fixed-size system

As a first case study, we assume that the number of agents in the system remains constant, and the size of the system is \( n \). This assumption only allows for two types of events among all those listed in the previous section to occur, namely gossips and replacements, respectively happening according to Poisson clocks of individual rates \( \lambda_g \) and \( \lambda_r \). This means that the frequency of events for a given agent is independent of the system size \( n \). Hence, in a system of size \( n \), it is expected that \( n \lambda_g \) and \( n \lambda_r \) gossips and replacements respectively happen per unit of time. The expected number of gossips taking place between two replacements, given by the ratio \( \rho = \lambda_g / \lambda_r \), remains constant as \( n \) grows (the same holds for the probability for a random event to be a replacement given by \( p = \frac{1}{n+1} \)).

We first provide the exact evolution of the descriptors of the system as well as its asymptotic behavior and the convergence rate of these descriptors. Then, we will consider an alternative definition of replacement events.

A. Descriptors evolution and fixed points

We first consider our default definition of replacements, namely random replacements where the replaced agent is randomly uniformly chosen among those presently in the system. The following result, based on Lemmas 2 and 3 and on Proposition 1 (where Lemma 1 ensures the validity of Assumption 1), describes the expected evolution of the system.

**Theorem 1.** In a system of fixed size subject to random replacements and gossips, there holds

\[
d \frac{d}{dt} (\mathbb{E}X(t)) = \left( -2 \lambda_r \rho - \frac{\lambda_g}{n} \right) \mathbb{E}X(t) + \left( \frac{1}{n+1} \right) \sigma^2. \tag{20}
\]

**Proof.** The proof directly follows the application of Proposition 1 combined with Lemmas 2 and 3, where \( \pi_o(t) = 1 \) and \( X_o(t) = \mathbb{E}X(t) \) for any \( t \), and where we remind that \( \lambda_{\text{Gossip}} = n \lambda_g \) and \( \lambda_{\text{Replacement}} = n \lambda_r \). This leads to the following:

\[
d \frac{d}{dt} (\mathbb{E}X(t)) = - n \left( \lambda_r - \rho \lambda_g \right) \mathbb{E}X(t) + n \lambda_r (A_{\text{Rep}} \mathbb{E}X(t) + b_{\text{Rep}}) + n \lambda_g A_{\text{Gossip}} \mathbb{E}X(t).
\]

A few algebraic steps then conclude the proof. \( \square \)

One can verify that the fixed point of (20) is

\[
\mathbb{E}X_{\text{eq}}^2 = \frac{2 + \rho}{2n(1 + \rho) - \rho} \sigma^2 \tag{21}
\]

\[
\mathbb{E}X_{\text{eq}} = \frac{2n + \rho}{2n(1 + \rho) - \rho} \sigma^2 \tag{22}
\]

leading to a variance

\[
\mathbb{E}\text{Var}(X|t)_{\text{eq}} = \frac{1 - \frac{1}{n}}{1 + \rho \left( 1 - \frac{1}{2n} \right)} \sigma^2 \sim \frac{\sigma^2}{\rho + 1}. \tag{23}
\]

where we remind \( \rho = \lambda_g / \lambda_r \) is the ratio between gossip and replacement rates.

The asymptotic values of these expressions show some interpretation. Suppose first that \( \rho \rightarrow 0 \), meaning that no gossip ever takes place. The variance is then \( \mathbb{E}\text{Var}(X|t)_{\text{eq}} = \frac{n-1}{n} \sigma^2 \), and an expected squared mean \( \mathbb{E}X_{\text{eq}}^2 = \frac{n}{n-1} \sigma^2 \). This is consistent
with a process where agents are only replaced, i.e., a system eventually consisting of agents with $n$ random i.i.d. values with mean $0$ and variance $\sigma^2$. This is also the fixed point of the affine equation in Lemma 5. For $\rho \to \infty$, the expected number of gossips steps between two replacements tends to infinity, so that a perfect averaging takes place before any replacement. We obtain in that case a variance $\mathbb{E}\text{Var}(x)|_{\text{eq}} = 0$ (this essentially corresponds to the system achieving a consensus), and an expected squared mean $\mathbb{E}x^2|_{\text{eq}} = \sigma^2 \frac{1}{2n-1}$. This latter number is lower than what would be obtained by averaging $n$ i.i.d. values. This is because it actually results from a weighted average of the values of all agents having been part of the system at some present or past time.

For large values of $n$, which we remind have no impact on the rate ratio $\rho = \lambda_g/\lambda_r$, the expected squared mean $\mathbb{E}x^2|_{\text{eq}}$ goes to $0$, while the variance $\mathbb{E}\text{Var}(x)|_{\text{eq}}$ goes to $\sigma^2/\rho$. Those results are parallel with that of [10], since the variance can be expressed as $p\sigma^2$, with $p$ the probability for a random event to be a replacement ($p = \frac{1}{1+\rho}$). More precisely, [10] relied on the average number of gossips between two replacements, given by $\rho = \frac{1}{\lambda} - 1$. Notice moreover that increasing the gossip rate makes the variance decay as more gossip steps allows the system to get closer to consensus; in opposition increasing the replacement rate makes it increase.

To illustrate Theorem 1 we consider an open system with random events (replacements or gossip steps) with $n = 50$ agents, such that on average one in twenty events is a replacement ($\rho = 19$, or equivalently with the probability for a random event to be a replacement being $p = 0.05$). The agent values are randomly drawn from a normal distribution with zero mean and constant variance $\sigma^2 = 1$. Figure 3 displays the expected evolution of the agents values $x_i(t)$ as well as the expected evolution of the descriptors with the time simulated over 10000 realizations. Those match accurately the theoretical expectations from Theorem 1 as well as with (21) and (22). The top plot exhibits that, even though convergence is obviously not achieved due to the openness of the system, the expected behavior of the estimates still presents a contracting tendency in this configuration. Besides, the bottom plots show the magnitude and convergence speed of the descriptors and allow for deducing those of the variance. In this configuration, gossip steps are frequent enough for information about agents having already left to last longer in the system: this implies a small squared mean $\bar{x}^2$ as compared to the mean of squares $\bar{x}^2$ that largely dominates in magnitude in the estimation of the variance. Nevertheless, the numerous gossip steps allow the mean of squares $\bar{x}^2$ to converge way faster than the squared mean $\bar{x}^2$ that needs arrivals to evolve. Notice that this trend could be reversed in a smaller system.

The illustration provided in the introduction (Figure 1) was obtained for an open multi-agent system of the same kind with $n = 4$ agents and $\rho = 9$ (i.e., $p = 0.1$). Notice that in such smaller and slower system, the squared mean $\bar{x}^2$ has much more impact than in the example above.

Remark 1. It is worth pointing out that a similar result can be deduced using Corollary 1 directly applied on the variance, and combined with Lemmas 2 and 5 to derive the following
upper bound on the evolution of the variance:
\[
\frac{d}{dt} \mathbb{E} \text{Var}(x(t)) \leq - \left( \lambda_g + \lambda_r + \frac{\lambda_r}{n} \right) \mathbb{E} \text{Var}(x(t)) + \frac{n^2 - 1}{n^2} \lambda_r \sigma^2. \tag{24}
\]

This result is weaker than that of Theorem 7 that allows for deriving an exact equality for the evolution of the expected variance. This highlights how following two such descriptors allows for more detailed analysis in this case, even if one were only interested in the variance. An illustration of this bound is given in Figure 4.

B. Convergence rate

We now analyze the rate at which the expected moments will eventually converge to the fixed point described above in (21) and (22). One can show that the eigenvalues of the matrix in (20) are given by
\[
\lambda = \lambda_g - 3 \lambda_r \pm \sqrt{\Delta},
\]
where \( \Delta = (\lambda_g - \lambda_r)^2 + 4 \frac{\lambda_g \lambda_r}{n}, \) so that \( \sqrt{\Delta} = |\lambda_g - \lambda_r| + o \left( \frac{1}{n} \right). \) Hence, we have
\[
\lambda_1 = -\lambda_g - \lambda_r + o \left( \frac{1}{n} \right); \quad \lambda_2 = -\lambda_g - \lambda_r - o \left( \frac{1}{n} \right). \tag{26}
\]

Notice that those are the diagonal elements of the matrix of (20). The corresponding eigenvectors are then given by
\[
v_{1,2} = \left( \frac{\rho - 1 \pm \sqrt{\rho(\rho - 1)^2 + 4 \rho/n}}{2 \rho}, \ 1 \right)^T, \tag{27}
\]
which, using the same reasoning as for the eigenvalues, leads to the following eigenvectors for large values of \( n: \)
\[
v_1 = (o(1), 1)^T; \quad v_2 = \left( \frac{\rho - 1}{\rho} + o(1), 1 \right)^T. \tag{28}
\]

Interpretation: First notice that for positive values of the gossip and replacement rates, both eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are negative, and ensure the convergence of the descriptors towards their respective fixed points.

When \( \lambda_g \gg \lambda_r, \) the impact of \( v_1 \) quickly vanishes as compared to that of \( v_2. \) Moreover, for large values of \( \lambda_g, \) \( v_2 \approx (1,1)^T. \) Both the squared mean \( \mathbb{E} \tilde{x}^2 \) and the mean of squares \( \mathbb{E} x^2 \) are thus expected to converge at the same speed, which is characterized by \( \lambda_2 \approx -2 \lambda_r. \) More precisely, the system is expected to achieve almost-consensus between each replacement, so that the descriptors converge to the same value, and thus the variance to 0. Since the system fully incorporates the effect of newly arrived agents, gossip steps end up having no impact on the convergence speed (this is observable from Lemma 2 considering that both descriptors have the same magnitude), and only the replacement rate defines the convergence speed.

When \( \lambda_g \ll \lambda_r, \) the impact of \( v_2 \) quickly vanishes. The convergence thus follows \( v_1 = (o(1), 1)^T, \) whose squared mean component \( \mathbb{E} \tilde{x}^2 \) is vanishingly small, so that the variance \( \mathbb{E} \text{Var}(x) \) becomes essentially equivalent to the mean of squares \( \mathbb{E} x^2. \) It follows from Lemmas 2 and 5 that this quantity is contracted by a factor \( \frac{n^2 - 1}{n^2} \lambda_r \sigma^2 \) at each event, no matter if it is a replacement or gossip (when neglecting the effect of \( \mathbb{E} \tilde{x}^2 \) and of external terms). The convergence speed is thus equivalently impacted by the replacement and gossip rates, which corresponds to the eigenvalue \( \lambda_1 \approx -\left( \lambda_r + \lambda_g \right). \)

Observe that there is a transition when \( \lambda_g = \lambda_r, \) where replacements and gossip steps are as frequent as each other: it follows that \( \lambda_1 = \lambda_2 \) and \( v_1 = v_2 = (o(1), 1)^T. \) Furthermore, if \( \lambda_r = 0 \) (i.e., the system closes), then it follows that \( \lambda_2 = 0. \) No more external contributions are expected and the squared mean \( \mathbb{E} \tilde{x}^2 \) remains unchanged. The mean of squares \( \mathbb{E} x^2 \) ultimately converges to the same value as \( \mathbb{E} \tilde{x}^2, \) and the variance asymptotically decays to zero following the convergence rate of \( \mathbb{E} \tilde{x}^2, \) that is conditioned by the frequency of gossip steps.

C. The case of adversarial replacements

In this section, we quickly envision an alternative way of defining the choice of the leaving agent at replacements. Up to now, it was assumed that this agent was uniformly and randomly chosen among those presently in the system. We will now consider replacements such that no assumption can be made on that choice, and refer to such events as adversarial replacements. These are a more general formulation of replacements that include those that were previously considered.

We provide an upper bound on the evolution of the expected variance in that case that is similar to that of (24). For that purpose, we first state the following lemma that characterizes the effect of adversarial departure and replacement events on the expected variance, where we remind \( x \) denotes the state of the system before the event, and \( x' \) after. The proof is provided in Appendix 3.

Lemma 6. [Adversarial departure and replacement] In a system of initially \( n \) agents (in the setting described in Section VII) subject to an adversarial departure event (noted \( \text{Dep}_n^* \)), there holds
\[
\mathbb{E} (\text{Var}(x')) | \text{Var}(x), \text{Dep}_n^* \leq \frac{n \text{Var}(x)}{n - 1}. \tag{29}
\]
Moreover, in the case of an adversarial replacement (noted \( \text{Rep}_n^* \)), there holds
\[
\mathbb{E} (\text{Var}(x')) | \text{Var}(x), \text{Rep}_n^* \leq \mathbb{E} \text{Var}(x) + \frac{\sigma^2}{n}. \tag{30}
\]

We can then provide the following theorem based on Corollary 1 where Lemma 4 ensures the validity of Assumption 1.

Theorem 2. In a system of fixed size subject to adversarial replacements and to random gossips, there holds
\[
\frac{d}{dt} (\mathbb{E} \text{Var}(x(t))) \leq -\lambda_g \mathbb{E} \text{Var}(x(t)) + \lambda_r \sigma^2. \tag{31}
\]

Proof. The proof directly follows the application of Corollary 1 where \( \pi_n(t) = 1 \) and \( X_n(t) = \mathbb{E} X(t) \), combined with Lemmas 2 and 6. \( \square \)

Figure 4 compares the evolution of the expected variance for both random and adversarial replacements through simulation, and with their respective upper bounds from (24) and from
The values of the agents are randomly drawn from a normal distribution of zero mean and with $\sigma^2 = 1$. Adversarial replacements are here defined as the arbitrary choice of the agent $j$ with minimal $|x_j(t)|$ at the time of the replacement. Notice that the upper bound \[ (31) \] is more conservative than \[ (24) \], and that the expected variance is larger for adversarial replacements than for random replacements. This directly follows from the fact that the adversarial definition of replacements includes random replacements. More generally, adversarial replacements allow arbitrarily choosing the leaving agent at replacements, so that it includes the “0” worst case departure with respect to the evolution of the variance.

V. VARYING-SIZE SYSTEM

We now consider a fully open system: agents can join and leave the system independently, and no assumption is made anymore on the size of the system at time $t$. Agents arrive in the system according to a Poisson clock with rate $\lambda_a$ that is independent of the state and size of the system. In addition, any present agent leaves the system according to another Poisson process of rate $\lambda_d$, so that the occurrence of a departure at the system scale follows a Poisson process with rate $n(t)\lambda_d$. Similarly, gossip interactions happen at the system scale according to a Poisson process with rate $n(t)\lambda_g$, so that the expected number of interactions in which a single agent is involved per unit of time does not depend on $n(t)$.

In addition, we define $n_0 = \lambda_a/\lambda_d$ the number of agents in the system such that arrivals and departures happen with equal probability (that is also interpreted as the average number of arrivals happening in the system before a given agent leaves). We also define $\bar{\gamma} = \lambda_g/\lambda_d$, the ratio between gossip and departure rates (that is proportional to the expected number of gossips experienced by an agent before leaving the system).

A. System size evolution

Provided the definitions of the events that can happen in the system, it appears that the size of the system $n(t)$ is a birth-death process where the birth rate is given by the arrival rate $\lambda_a$, and the death rate by the departure rate $\lambda_d$.

\begin{equation}
\frac{d}{dt} X_j(t) \pi_j(t) = \left( j(A_{Gos_j} - I) \lambda_g - j \lambda_d - \lambda_a \right) X_j(t) \pi_j(t) + \lambda_a \left( A_{Arr_j} X_{j-1}(t) + b_{Arr_j} \right) \pi_{j-1}(t) + (j+1) \lambda_d A_{Dep_{j+1}} X_{j+1}(t) \pi_{j+1}(t).
\end{equation}

Proposition 2. The evolution of $X_j(t)\pi_j(t)$ is given by the following:

\begin{equation}
\frac{d}{dt} X_j(t) \pi_j(t) = \left( j(A_{Gos_j} - I) \lambda_g - j \lambda_d - \lambda_a \right) X_j(t) \pi_j(t) + \lambda_a \left( A_{Arr_j} X_{j-1}(t) + b_{Arr_j} \right) \pi_{j-1}(t) + (j+1) \lambda_d A_{Dep_{j+1}} X_{j+1}(t) \pi_{j+1}(t).
\end{equation}

Proof. The proof follows the application of Proposition 1 combined with Lemmas 2, 3 and 4 with $n(t) = j$, and for which there holds $\lambda_{Gos_j} = j \lambda_g$, $\lambda_{Arr_j} = \lambda_a$ and $\lambda_{Dep_j} = j \lambda_d$. \ \hfill \square

Figure 5. Graphical representation of the birth-death process defining the evolution of the system size $n(t)$ in terms of the arrival and departure rates.

Let us denote $\pi_i(t) = P(n(t) = i)$, then the following result is known from standard properties of birth-death process.

**Lemma 7.** Assume that $n_0 = \lambda_a/\lambda_d < \infty$, then for all $i \in \mathbb{N}$ there exist steady state probabilities $\pi_i^* = \lim_{t \to \infty} \pi_i(t)$ satisfying

\begin{equation}
\pi_i^* = \lim_{t \to \infty} \pi_i(t) = \frac{n_0}{i!} e^{-n_0}, \quad (32)
\end{equation}

Proof. There holds from standard results on birth-death processes if $n(t)$ is ergodic that $\pi_i^* = \lim_{t \to \infty} \pi_i(t)$ exists and satisfies the following:

\begin{equation}
\pi_0^* = \frac{1}{1 + \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\lambda_a}{\lambda_d}} = \prod_{i=1}^{\infty} \frac{n_0}{i!} e^{-n_0},
\end{equation}

and

\begin{equation}
\pi_k^* = \pi_0^* \prod_{i=1}^{k} \frac{\lambda_a}{i \lambda_d} = \frac{n_0^k}{k!} e^{-n_0}.
\end{equation}

This concludes the proof for $n_0 = \lambda_a/\lambda_d < \infty$ which implies the ergodicity of $n(t)$. \ \hfill \square

One can verify from Lemma 7 that

\begin{equation}
\lim_{t \to \infty} \mathbb{E}n(t) = n_0, \text{ and } \lim_{t \to \infty} \mathbb{E}n(t)^2 = n_0^2 + n_0, \quad (33)
\end{equation}

and hence that the asymptotic variance of $n(t)$ is given by

\begin{equation}
\lim_{t \to \infty} \mathbb{E}\text{Var}(n(t)) = \mathbb{E}n(t)^2 - (\mathbb{E}n(t))^2 = n_0. \quad (34)
\end{equation}

Notice that these do not depend on the initial size of the system, but only on the arrival and departure rates. Moreover, it means that the asymptotic distribution of the system size is rather concentrated around $n_0$.

B. Descriptors evolution

We now study the evolution of the descriptors conditioned by the size of the system. This will allow us characterizing the evolution of these descriptors, and establishing some asymptotic behaviors. In the sequel, we refer to $X_j(t) = \mathbb{E}(X(t)|n(t) = j)$, where we remind that $X = \left(\vec{a}, \vec{a}^2\right)^T$.

**Proposition 2.** The evolution of $X_j(t)\pi_j(t)$ is given by the following:

\begin{equation}
\frac{d}{dt} X_j(t) \pi_j(t) = \left( j(A_{Gos_j} - I) \lambda_g - j \lambda_d - \lambda_a \right) X_j(t) \pi_j(t) + \lambda_a \left( A_{Arr_j} X_{j-1}(t) + b_{Arr_j} \right) \pi_{j-1}(t) + (j+1) \lambda_d A_{Dep_{j+1}} X_{j+1}(t) \pi_{j+1}(t).
\end{equation}

**Proof.** The proof follows the application of Proposition 1 combined with Lemmas 2, 3 and 4 with $n(t) = j$, and for which there holds $\lambda_{Gos_j} = j \lambda_g$, $\lambda_{Arr_j} = \lambda_a$ and $\lambda_{Dep_j} = j \lambda_d$. \ \hfill \square

Figure 4. Evolution of the expected variance of an open system of 4 agents subject to gossip steps (with rate $\lambda_g = 9$) and to replacements (with $\lambda_r = 1$); in plain blue line for random replacements, and in dash-dotted yellow line for adversarial replacements. The red dashed line provides the upper bound on the expected variance for random replacements obtained in \[ (34) \], and the purple dotted line the upper bound from Theorem 2 for adversarial replacements. The red dashed line is the fixed point of the variance for random replacements.
Corollary 2. Assume there exists $X^*_j = \lim_{t \to \infty} X_j(t)$ for all $j \in \mathbb{N}$, so that \( \frac{d}{dt} (X^*_j \pi^*_j) = 0 \), then there holds
\[
(n_0 + j + (I - A_{\text{Gos}})\gamma) X^*_j = A_{\text{Dep}_j+1} X^*_{j+1} + n_0 + j A_{\text{Arr}_j-1} X^*_{j-1} + j b_{\text{Arr}_j-1}. 
\] (36)

**Proof.** Lemma 7 allows writing $\pi^*_j = \frac{n_0^j}{j!} e^{-n_0}$, where we remind $n_0 = \lambda_\gamma / \lambda_d$. Equation (35) evaluated on $X^*_j \pi^*_j$ then becomes
\[
(\lambda_\gamma (I - A_{\text{Gos}}))j \lambda_g + j \lambda_d) X^*_j \frac{n_0^j}{j!} e^{-n_0}
= (j + 1) \lambda_d A_{\text{Dep}_j+1} + X^*_{j+1} \frac{n_0^j}{(j+1)!} e^{-n_0}
+ j \lambda_\gamma A_{\text{Arr}_j-1} X^*_{j-1} \frac{n_0^j}{(j-1)!} e^{-n_0} + \lambda_d b_{\text{Arr}_j-1} \frac{n_0^j}{(j-1)!} e^{-n_0}.
\]

Reminding that $\gamma = \lambda_\gamma / \lambda_d$, dividing by $\lambda_d$ on both sides and performing a few algebraic steps lead to the conclusion. \( \square \)

C. Bound on state variance

We characterize the evolution of the expected state variance $\mathbb{E} (\text{Var}(x(t))) = (-1, 1)\mathbb{E}X(t)$. For this purpose, we lighten the notations by defining $V_j(t) = \mathbb{E} (\text{Var}(x(t))) | n(t) = j$ and $V(t) = \mathbb{E} \text{Var}(x(t))$. We will also drop the dependence to the time of $V_j(t)$, $V(t)$, and $\pi_j(t)$ for the remainder of this section for concision matters. Moreover, we use the superscript “0” to refer to the asymptotic value of those quantities, e.g., $V^* = \lim_{t \to \infty} V(t)$.

**Proposition 3.** The evolution of $V_j \pi_j$ satisfies the following:
\[
\frac{d}{dt} (V_j \pi_j) \leq \lambda_a \left( \frac{j - 1}{j} V_{j-1} + \frac{\sigma^2}{j} \right) \pi_j - (j + 1) \lambda_d \left( 1 - \frac{1}{j^2} \right) V_{j+1} \pi_j - (\lambda_\gamma + j \lambda_d) V_j \pi_j.
\] (37)

**Proof.** The proof follows the application of Corollary 1 combined with Lemmas 2 and 3 with $n(t) = j$, and for which there holds $\lambda_{\text{Gos}_j} = j \lambda_\gamma$, $\lambda_{\text{Arr}_j} = \lambda_\gamma$, and $\lambda_{\text{Dep}_j} = j \lambda_d$. \( \square \)

**Corollary 3.** Assume there exists $V^*_j = \lim_{t \to \infty} V_j(t)$, such that \( \frac{d}{dt} (V^*_j \pi^*_j) = 0 \), then there holds
\[
(n_0 + j + \gamma) V^*_j \geq (j - 1) V^*_{j-1} + \left( 1 - \frac{1}{j^2} \right) n_0 V^*_{j+1} + \sigma^2.
\] (38)

**Proof.** Lemma 7 gives $\pi^*_j = \frac{n_0^j}{j!} e^{-n_0}$, where we remind $n_0 = \frac{n_0}{\lambda_\gamma}$. Equation (37) then yields
\[
(\lambda_a + j \lambda_\gamma + \lambda_d) V^*_j \frac{n_0^j}{j!} e^{-n_0} \leq \left( \frac{j - 1}{j} + \frac{\sigma^2}{j} \right) \lambda_\gamma \frac{n_0^{j-1}}{(j-1)!} e^{-n_0}
+ \left( 1 - \frac{1}{j^2} \right) \lambda_d \frac{n_0^{j+1}}{(j+1)!} e^{-n_0}.
\]

Reminding that $\gamma = \lambda_\gamma / \lambda_d$, dividing by $\lambda_d$ on both sides and performing a few algebraic steps lead to the conclusion. \( \square \)

We can now derive results on the asymptotic behavior of the expected variance.

**Proposition 4.** For any positive sequence of numbers $(z_j)_{j \in \mathbb{N}}$ such that $z_0 = 0$, and satisfying for all $j \geq 1$:
\[
(n_0 + j + \gamma) z_j \geq j z_{j-1} + \left( 1 - \frac{1}{(j - 1)^2} \right) n_0 z_{j+1} + \frac{n_0^j}{j!} e^{-n_0},
\] (39)
there holds
\[
V^* \leq \sum_{j=0}^{\infty} z_j \sigma^2.
\] (40)

**Proof.** Let $\tilde{V} = 0$ be the vector containing the elements $V^*_j$. There holds $V^* = \pi^T \tilde{V}$ where $\pi$ is the vector such that $[\pi]_j = \pi_j = \frac{n_0^j}{j!} e^{-n_0}$. From Corollary 3 there holds $A^T \tilde{V} \leq 1 \sigma^2$, with $[A]_{j,j} = (n_0 + j + \gamma)$, $[A]_{j,j+1} = \left( 1 - \frac{1}{j^2} \right) n_0$, and $[A]_{j+1,j} = -j$ for all $j \geq 1$.

Suppose now that there is a vector $z > 0$ such that $A^T z$ exists (i.e., $[A^T z]_j = \sum_j [A]_{j,j} [z]_j$ converges for all $i$), and satisfying $\pi \leq A^T z$, then there holds
\[
V^* = \pi^T V \leq (A^T z)^T \tilde{V} = z^T A^T z \leq z^T 1 \sigma^2,
\] which concludes the proof. \( \square \)

Note that the above proof is inspired by duality, since $z$ a feasible dual solution to $\max_{x \in \mathbb{R}} \pi^T x$ subject to $Ax \leq 1 \sigma^2$, which provides a feasible upper bound on $V^*$. In the following theorem, we provide an upper bound on the expected asymptotic variance based on the previous proposition.

**Theorem 3.** In a system subject to random arrivals and departures where agents perform gossip updates, assuming that there exists $V^*_j = \lim_{t \to \infty} V_j(t)$, there holds
\[
V^* \leq \frac{\sigma^2}{1 + \gamma} \frac{\lambda^2}{2}.
\] (41)

**Proof.** One can verify that the sequence $(z_j)_{j \in \mathbb{N}}$ with $z_0 = 0$ and $z_j = \frac{n_0^j}{j! (1 + \gamma)} e^{-n_0}$ for $j \geq 1$ satisfies (39), and it follows from Proposition 4 that
\[
V^* \leq \sum_{j=0}^{\infty} \frac{n_0^j}{j!} e^{-n_0} \frac{\sigma^2}{1 + \gamma} (e^{-n_0} - 1).
\]

This concludes the proof. \( \square \)

**Remark 2.** Interestingly, the bound (41) from Theorem 3 is similar to (25), the expected variance obtained for systems of fixed size as $n$ gets large (i.e. $\frac{\lambda^2}{2} \frac{\sigma^2}{1 + \gamma}$ with $\rho = \lambda_\gamma / \lambda_d$). Indeed, $\gamma = \lambda_\gamma / \lambda_d$ depicts the expected number of gossipers happening in the system before a departure occurs, whereas $\rho$ depicts that expected number before a replacement happens, which can be assimilated to a departure event for fixed size systems.

The bound derived in Theorem 3 is not the only one that can be obtained from Proposition 4, since any feasible $z$ leads to a bound. One can compute the “0” best possible upper bound on the expected variance in the sense of Proposition 4 by solving
\[
\min \sum_{j=0}^{\infty} \frac{1}{j!} z_j \sigma^2 \quad \text{subject to} \quad A^T z \geq \pi,
\] (42)
where the notations are those from the proof of Proposition 4. This bound corresponds, up to the duality gap, to...
max_{x \in \mathbb{R}^n} \pi^T x \text{ with } x \text{ satisfying } (38). This has been solved approximately numerically, and the derived bound is depicted in Figure 6 as well as that from Theorem 3 compared with results from an actual simulation.

It appears from the figure that the bounds are rather conservative for small values of $\gamma$, however this corresponds to very chaotic systems where agents are expected to perform very few interactions before leaving. This conservatism is partly due to Lemma 3 for which only an upper bound is provided on the expected variance. In particular, it generates a strong conservatism for small values of $n$. The additional conservatism observed for the bound of Theorem 3 follows that the sequence used to derive it is not optimal in the sense of Proposition 4 which also inherently adds conservatism by itself. Nevertheless, the bound from Theorem 3 becomes very close to the “0’best bound (42) as soon as $\gamma \approx 5$. These are both quite accurate at this point, and closely match the actual variance starting from $\gamma \approx 10$, which corresponds to agents interacting on average 10 times before leaving.

![Graph showing asymptotic expected variance](image)

**Figure 6.** Asymptotic expected variance of a system initially constituted of 5 agents with zero mean and $\sigma^2 = 1$, subject to random arrivals (with rate $\lambda_a = 1$), random departures (with rate $\lambda_d = 1$), and random gossips (with varying rate $\lambda_g$), in terms of $\gamma = \lambda_g/\lambda_d$. The simulated variance is provided in dotted blue line, whereas the upper bounds from Theorem 3 and (42) are respectively in red plain line and dashed yellow line.

VI. CONCLUSIONS

In this paper, we considered the possibility for agents to get in and out of a system, then called open. We focused on the behavior of a classical multi-agent algorithm in that context: all-to-all pairwise gossips. Whereas openness arises several challenges, including variations of the dimension of the system and the absence of usual convergence, we have shown that these systems can be characterized in terms of scale-independent quantities whose evolution is governed by fixed size linear systems. We obtained the evolution of the two first moments and of the variance for both fixed and variable size open systems through the use of continuous-time tools to model the asynchronous events. We observed that the open trait of a system may result in a significant performance drop in terms of variance reduction as compared to closed systems.

This analysis through scale-independent quantities can be extended to more general problems relying on more complex definitions of arrivals and departures, and on different types of interactions. More generally, we believe it can be a useful tool to study the behavior of open multi-agent systems in general, and a natural continuity of this work could be the analysis of interactions restricted to a graph. Another challenge left untackled consists in establishing the probability distribution of our descriptors instead of only their expected value.

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A. Proofs of Lemma 1 (Upper bound on the descriptors)

Let us fix some time $t \geq 0$ and a deterministic sequence of events (constituted of arrivals, departures and gossip steps) starting at time 0 such that $N(t) = S$.

Let $\mathcal{T}$ be the set of the labels of all the agents that have been in the system at some time between times 0 and $t$ (for simplicity those are assumed to be all different). Moreover, define $\xi(s)$ for $0 \leq s \leq t$ the $|\mathcal{T}|$-dimensional vector containing the values $x_i(s)$ of all agent $i \in \mathcal{T}$ at time $s$ in that virtual system, agents that already left the actual system at time $s$ are assumed to keep the value they held at their departure, and agents that did not join the actual system yet are assumed to hold the value with which they will be initialized at their arrival. Notice that in the virtual system, the only effect of arrivals and departures is to respectively induce or stop the modifications of the corresponding values in $\xi(s)$.

Gossip iterations result in the multiplication of $A$ by a doubly-stochastic matrix, and thus there holds $\xi(t) = A\xi(0)$ for some fixed doubly-stochastic matrix $A$. Let us compute

$$\sum_{i \in S} x_i(t) = \sum_{i \in S} \xi_i(t) = \sum_{i \in S} A_{ij} \xi_j(0) = \sum_{j \in \mathcal{T}} w_j \xi_j(0),$$

for $w_j = \sum_{i \in S} A_{ij} \geq 0$. Since the initial values held in $\xi_j(0)$ are all i.i.d. with $E[\xi_j(0)]$, there holds

$$E\left(\sum_{i \in S} x_i(t)\right)^2 = E\left(\sum_{j \in \mathcal{T}} w_j \xi_j(0)\right)^2 = \sum_{j \in \mathcal{T}} w_j^2 E(\xi_j(0))^2 + \sum_{j,k \in \mathcal{T}, j \neq k} w_j w_k E(\xi_j(0)\xi_k(0)) = \sigma^2 \sum_{j \in \mathcal{T}} w_j^2,$$

where the absence of correlation between the initial values $\xi_j(0)$ is used to nullify the crossed products in the last equality.

Since $A$ is doubly stochastic, there holds for all $j \in \mathcal{T}$ that $w_j = \sum_{i \in S} A_{ij} \leq \sum_{i \in S} A_{ij} = 1$, and thus $w_j \leq w_j$. Moreover, $\sum_{j \in \mathcal{T}} w_j = \sum_{i \in S} \sum_{j \in \mathcal{T}} A_{ij} = |S|$, and it follows

$$E\left(\sum_{i \in S} x_i(t)\right)^2 = \sigma^2 \sum_{j \in \mathcal{T}} w_j^2 \leq \sigma^2 \sum_{j \in \mathcal{T}} w_j = |S| \sigma^2,$$

which proves the result for a deterministic sequence of events. Since the events (arrivals, departures and gossip steps) are independent of the values held by the agents, one can extend the result above to stochastic event sequences by considering its expected value over all possible event sequences.

Finally, (4) follows that

$$E(\bar{x}^2|n(t) = j) = \frac{1}{j^2} E(\sum_{i \in S} x_i(t))^2;$$

$$E(\bar{x}^2|n(t) = j) = \frac{1}{j} \sum_{i \in S} E(x_i(t)) \leq \frac{1}{j} E(\sum_{i \in S} x_i(t))^2.$$
which implies the second line of (11). For the first line, taking into account $E(x_j|x) = \bar{x}$, it follows from (46) that
\[
E(\bar{x}^2|x) = \frac{1}{n-1} \left( n^2 \bar{x}^2 - 2n\bar{x}E(x_j|x) + E(x_j^2|x) \right) \\
= n^2 \left( \frac{\bar{x}^2}{n} + \frac{1}{n(n-1)} \bar{x}^2 \right).
\]
Finally, (13) follows from the direct computation of
\[
E(\operatorname{Var}(x')|\operatorname{Var}(x), \operatorname{Dep}_n) = (1, -1)E(X'|X, \operatorname{Dep}_n),
\]
which concludes the proof.

4) Proof of Lemma 3 (Replacement): The matrix equality (14) follows from a combination of Lemmas 2 and 3, the latter applied to a system of size $n-1$ joined by an $n^{th}$ agent. The inequality (16) follows
\[
E(\operatorname{Var}(x')|x, \operatorname{Rep}_n) \\
= \frac{n - 2}{n} \bar{x}^2 - \frac{\bar{x}^2}{n} \frac{\sigma^2}{n} + \frac{n - 1}{n} \bar{x}^2 + \frac{\sigma^2}{n} \\
= n \left( \frac{\bar{x}^2}{n} - \frac{n - 1}{n} \bar{x}^2 \right) + \frac{n - 1}{n^2} \sigma^2 \\
\leq \frac{n - 1}{n^2} \operatorname{Var}(x) + \left( \frac{n - 1}{n} \right)^2 \sigma^2/n,
\]
where we used Lemma 1 for the last inequality.

C. Proof of Proposition 7

Let us define the indicator random variable
\[
\chi_i(t) = \begin{cases} 1, & \text{if } n(t) = i; \\ 0, & \text{otherwise}. \end{cases}
\]
Then there holds
\[
E(\chi_i(t)X(t)) = E(\chi_i(t)X(t)|n(t) = i) \pi_i(t) \\
+ E(\chi_i(t)X(t)|n(t) \neq i) (1 - \pi_i(t)).
\]
By definition, $E(\chi_i(t)X(t)|n(t) \neq i) = 0$ and $E(\chi_i(t)X(t)|n(t) = i) = E(X(t)|n(t) = i)$, and it follows for all $i \in \mathbb{N}$
\[
E(\chi_i(t)X(t)) = X_i(t)\pi_i(t).
\]
Let us fix some $j \in \mathbb{N}$ and some time $t$, and define some small $\delta t > 0$. Moreover, define $E_0$, $E_1$, $E_{2\geq}$ the events that respectively zero, exactly one, and more than two events happened between times $t$ and $t + \delta t$ such that $n(t + \delta t) = j$. Define also $E^*$ the event that any other sequence of events happened, then
\[
X_j(t + \delta t)\pi_j(t + \delta t) = E(\chi_j(t + \delta t)X(t + \delta t)) \\
= E(\chi_j(t + \delta t)X(t + \delta t)|E_0) \cdot P(E_0) \\
+ E(\chi_j(t + \delta t)X(t + \delta t)|E_1) \cdot P(E_1) \\
+ E(\chi_j(t + \delta t)X(t + \delta t)|E_{2\geq}) \cdot P(E_{2\geq}) \\
+ E(\chi_j(t + \delta t)X(t + \delta t)|E^*) \cdot P(E^*).
\]
By definition, $E(\chi_j(t + \delta t)X(t + \delta t)|E^*) = 0$. Moreover, from Assumption 1 $P(E_{2\geq}) = o(\delta t)$. Furthermore, conditional to $E_0$ and $E_1$, $\chi_j(t + \delta t) = \chi_j(t)$, and conditional to $E_0$, there holds $X(t + \delta t) = X(t)$. Hence,
\[
E(\chi_j(t + \delta t)X(t + \delta t)|E_0) = E(X(t)|n(t) = j) = X_j(t),
\]
and from Poisson properties
\[
P(E_0) = \pi_j(t) \left( 1 - \sum_{c:i=c=j} \lambda_c \delta t + o(\delta t) \right).
\]
Finally, there holds
\[
E(\chi_j(t + \delta t)X(t + \delta t)|E_1) \cdot P(E_1) = \sum_{c:i=a=j} E(X(t + \delta t)|e) \cdot P(e).
\]
Using (17),
\[
E(X(t + \delta t)|e) = A_e \cdot E(X(t)|n(t) = s(e)) + b_c.
\]
Moreover,
\[
P(e) = \pi_{s(e)}(t) \lambda_e \delta t + o(\delta t).
\]
Hence, combining everything together gives
\[
X_j(t + \delta t)\pi_j(t + \delta t) = \pi_j(t) \left( 1 - \sum_{c:i=s(e)=j} \lambda_c \delta t \right) \cdot X_j(t) \\
+ \sum_{c:i=s(e)=j} \lambda_e (A_e X(s(e)) + b_e) \pi_{s(e)}(t) \delta t \\
+ o(\delta t).
\]

Using the fact that $E(X(t))$ is bounded provided any event sequence, and taking the limit $t \to 0$ ultimately leads to the conclusion.

D. Proof of Lemma 8 (Adversarial events)

One can show that provided $n$ values $x_j$, then
\[
\bar{x} = \arg \min \left\{ \frac{n}{\sum_{j=1}^n (x_j - y)^2} \right\}.
\]
Hence, an alternative formulation for the variance is given by
\[
\operatorname{Var}(x) = \frac{1}{n} \sum_{j=1}^n (x - x)^2 = \frac{1}{n} \min \sum_{j=1}^n (x_j - y)^2.
\]
For simplicity, assume that the agents presently in the system are labelled form 1 to $n$. Assume moreover that at a departure, the agent that leaves is labelled $n$. Denoting $\bar{x}$ the state of the system after the departure, there holds
\[
\operatorname{Var}(\bar{x}) = \frac{1}{n - 1} \min_y \sum_{j=1}^{n-1} (x_j - y)^2 \\
\leq \frac{1}{n - 1} \sum_{j=1}^{n-1} (x_j - \bar{x})^2 \\
\leq \frac{1}{n - 1} \sum_{j=1}^n (x_j - \bar{x})^2,
\]
where the nonnegativity of \((x_n - \bar{x})^2\) is used for the last inequality. Taking the expected value of the above result leads to (29), the first result of Lemma 6.

Let us now denote by \(x'\) the state of the system after an arrival that instantaneously follows the departure. Then, applying Lemma 3 to the system at state \(\tilde{x}\), it follows,

\[
E \text{Var}(x') = E x'^2 - E \bar{x}'^2
\]

\[
= \frac{n - 1}{n} E x^2 - \frac{(n - 1)^2}{n^2} E \bar{x}^2 + \left( \frac{1}{n} - \frac{1}{n^2} \right) \sigma^2
\]

\[
= \frac{n - 1}{n} E \text{Var}(\tilde{x}) + \frac{n - 1}{n^2} E \bar{x}^2 + \frac{n - 1}{n^2} \sigma^2.
\]

Applying the previous result to bound \(E \text{Var}(\tilde{x})\), and then Lemma 1 to bound \(E \bar{x}^2\), it follows

\[
E \text{Var}(x') \leq E \text{Var}(x) + \frac{1}{n^2} \sigma^2 + \frac{n - 1}{n^2} \sigma^2 = E \text{Var}(x) + \frac{1}{n} \sigma^2,
\]

which concludes the proof for the second result (30) of Lemma 6.

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