Relic Gravitational Waves from Cosmic Strings: Updated Constraints and Opportunities for Detection

R.R. Caldwell\(^1\), R.A. Battye\(^{1,2}\), and E.P.S. Shellard\(^1\)

\(^1\) University of Cambridge, D.A.M.T.P.
Silver Street, Cambridge CB3 9EW, U.K.

\(^2\) Theoretical Physics Group, Blackett Laboratory, Imperial College
Prince Consort Road, London SW7 2BZ, U.K.

Abstract

We examine the spectrum of gravitational radiation emitted by a network of cosmic strings, with emphasis on the observational constraints and the opportunities for detection. The analysis improves over past work, as we use a phenomenological model for the radiation spectrum emitted by a cosmic string loop. This model attempts to include the effect of the gravitational back-reaction on the radiation emission by an individual loop with a high frequency cut-off in the spectrum. Comparison of the total spectrum due to a network of strings with the recently improved bound on the amplitude of a stochastic gravitational wave background, due to measurements of noise in pulsar signal arrival times, allows us to exclude a range of values of $\mu$, the cosmic string linear mass density, for certain values of cosmic string and cosmological parameters. We find the conservative bound $G\mu/c^2 < 5.4(\pm 1.1) \times 10^{-6}$ which is consistent with all other limits. We consider variations of the standard cosmological scenario, finding that an under dense, $\Omega_0 < 1$ universe has little effect on the spectrum, whereas the portion of the spectrum probed by gravitational wave detectors is strongly sensitive to the thermal history of the cosmological fluid. We discuss the opportunity for the observation of this stochastic background by resonant mass and laser interferometer gravitational wave detectors.

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I. INTRODUCTION

Cosmic strings are line-like topological defects which may have formed during a phase transition in the early universe [1,2]. Strings which formed with a mass-per-unit-length \( \mu \) such that \( G\mu/c^2 \sim 10^{-6} \) may be responsible for the formation of the large-scale structure and cosmic microwave background anisotropy observed in the universe today. In order to test the validity of the cosmic string scenario, it is necessary to compare observations of our own universe with the predictions of the cosmic string model. In this work we present updated results of detailed computations of the spectrum of gravitational radiation emitted by a network of cosmic strings [3]. We use these results to obtain limits on the cosmic string mass-per-unit-length and to predict whether the spectrum is within the sensitivity of forthcoming gravitational wave detectors.

The spectrum of gravitational radiation due to cosmic strings has been previously considered in detail in ref. [3] (and references therein), where the methods used in the present paper were described. Here we benefit from recent work, ref. [4], which suggests that the effect of the radiative back-reaction on a string loop is to damp out the higher oscillation modes. Hence, the current work represents an improvement due to the introduction of a phenomenological frequency cut-off, and due to the updated observational constraints.

The most recent analysis of pulsar signal arrival times gives the limit on the spectral density of gravitational radiation

\[
\Omega_{\text{gr}}(f) \equiv \frac{f}{\rho_{\text{crit}}} \frac{\rho_{\text{gr}}}{df} \bigg|_{f_{\text{obs}}} < 9.3 \times 10^{-8} h^{-2} \quad (95\% \text{ CL})
\]

in a logarithmic frequency interval at \( f_{\text{obs}} = (8 \text{ yrs})^{-1} \). This analysis corrects errors in previous work [6,7], and uses an improved method for testing the hypothesis that the timing noise is due to a stochastic gravitational wave background. For these reasons we use the latest bound, equation (1.1), to obtain a constraint on \( G\mu/c^2 \) for given values of the cosmic string and cosmological parameters.

The paper proceeds as follows. In section II we present the model for the cosmic string loop radiation spectrum. This model improves over past computations in that the effect of the gravitational back-reaction is included. In section III we give an analytic estimate of the spectrum of radiation emitted by a network of strings, allowing us to study the model dependencies of the spectrum. In section IV we present the results of our numerical computation of the radiation spectrum. Here we obtain limits on \( G\mu/c^2 \) for given values of the cosmic string and cosmological parameters. In section V we discuss the opportunities for the observation of the stochastic gravitational wave background by the forthcoming generation of gravitational wave detectors. We conclude in section VI.

II. COSMIC STRING LOOP RADIATION SPECTRUM

The spectrum of gravitational radiation emitted by a network of cosmic string loops is obtained using the product of a background \( \Omega = 1 \) FRW cosmological model, an extended one-scale model for the evolution of a network of cosmic strings, and a model of the emission of gravitational radiation by cosmic string loops. The procedure by which the spectrum is
computed has been presented in detail in ref. [3]. In this section we discuss the model for radiation by an individual loop.

The model of the emission of gravitational radiation by cosmic string loops is composed of the following three elements.

1. A loop radiates with power $P = \Gamma G \mu^2 c$. The dimensionless radiation efficiency, $\Gamma$, depends only on the loop configuration, rather than overall size. Recent studies of realistic loops indicate that the distribution of values of the efficiency has a mean value $\langle \Gamma \rangle \approx 60$ [8].

2. The frequency of radiation emitted by a loop of invariant length $L$ is $f_n = 2n/L$ where $n = 1, 2, 3, \ldots$ labels the oscillation mode.

3. The fraction of the total power emitted in each mode of oscillation $n$ at frequency $f_n$ is given by the coefficient $P_n$ where

$$P = \left( \sum_{n=1}^{\infty} P_n \right) G \mu^2 c = \Gamma G \mu^2 c. \quad (2.1)$$

Analytic and numerical studies suggest that the radiation efficiency coefficients behave as $P_n \propto n^{-q}$ where $q$ is the spectral index.

This model has several shortcomings. First, the spectral index $q$ has not been well determined by the numerical simulations. Numerical work suggests $q = 4/3$ [9] as occurs with cuspy loops, loops along which points momentarily reach the velocity of light, based on simulations of a network of cosmic strings. However, these simulations have limited resolution of the important small scale features of the long strings and loops. Hence, the evidence for $q = 4/3$ is not compelling. Analytic work suggests that $q = 2$ [10], characteristic of kinky loops, loops along which the tangent vector changes discontinuously as a result of intercommutation, may be more realistic. Second, the effect of back-reaction on the motion of the string has been ignored. In this model, a loop radiates at all times with a fixed efficiency, in all modes, until the loop vanishes. As we shall next argue, the back-reaction will result in an effective high frequency cut-off in the oscillation mode number. Thus, the loop will only radiate in a finite number of modes, and hence in a finite range of frequencies. The resolution of these issues may have strong consequences for the entire spectrum produced by a network of strings.

Recent advances in the understanding of radiation back-reaction on global strings suggest various modifications to the simplified model of emission by cosmic string loops. There are remarkable similarities between gravitational radiation and Goldstone boson radiation from strings [11], which we believe allow us to make strong inferences as to the nature of gravitational radiation back-reaction. For example, the same, simple model for the emission of gravitational radiation by cosmic strings may be transferred over to global strings: in the absence of Goldstone back-reaction, global string loops radiate at a constant rate, at wavelengths given by even sub-multiples of the loop length, with an efficiency as described by an equation similar to (2.1). Hence, our argument proceeds as follows. Fully relativistic field theory simulations of global strings have been carried out [11], where it was observed that the power in high oscillation modes is damped by the Goldstone back-reaction on periodic global
strings. An analytic model of Goldstone back-reaction [4], as a modification of the classical Nambu-Goto equations of motion for string, was developed which successfully reproduces the behaviour observed in field theory simulations. That is, high frequency modes are damped rapidly, whereas low frequency modes are not. Thus, we are motivated to rewrite equation (2.1) for global strings, and by analogy for cosmic strings, as

$$ P = \left( \sum_{n=1}^{n_*} P_n \right) G \mu^2 c = \Gamma G \mu^2 $$  \hspace{1cm} (2.2)

where \( n_* \) is a cut-off introduced to incorporate the effects of back-reaction. By comparing the back-reaction length-scale to the loop size, we estimate that such a cut-off should be no larger than \( \sim (\Gamma G \mu/c^2)^{-1} \). The ongoing investigations of global and cosmic string back-reaction [12] have not yet reached the level of precision where a firm value of \( n_* \) may be given. As we demonstrate later, the effect on the radiation spectrum is significant only for certain values of the cut-off.

### III. ANALYTIC ESTIMATE OF THE RADIATION SPECTRUM

Analytic expressions for the spectrum of gravitational radiation emitted by a network of cosmic strings have been derived in ref. [3]. While these analytic expressions are simplified for convenience, they offer the opportunity to examine the various dependencies of the spectrum on cosmic string and cosmological parameters.

The spectrum of gravitational radiation produced by a network of cosmic strings has two main features. First is the ‘red noise’ portion of the spectrum with nearly equal gravitational radiation energy density per logarithmic frequency interval, spanning the frequency range \( 10^{-8} \text{Hz} \lesssim f \lesssim 10^{10} \text{Hz} \). This spectrum corresponds to gravitational waves emitted during the radiation-dominated expansion era. This feature of the spectrum may be accessible to the forthcoming generation of gravitational wave detectors. Second is the peak in the spectrum near \( f \sim 10^{-12} \text{Hz} \). The amplitude and slope of the spectrum from the peak down to the flat portion of the spectrum is tightly constrained by the observed limits on pulsar timing noise.

#### A. Red Noise Portion of the Spectrum

An analytic expression for the ‘red noise’ portion of the gravitational wave spectrum is given as follows:

$$ \frac{f}{\rho_{\text{ crit}}} \frac{d\rho_{\text{ gr}}}{df} = \frac{8\pi}{9} A \frac{\Gamma(G\mu)^2}{\alpha c^4} \frac{\left(1 - \frac{\langle v^2 \rangle}{c^2}\right)(\beta - 3/2 - 1)}{(z_{\text{eq}} + 1)} 10^{-8} \text{Hz} \lesssim f \lesssim 10^{10} \text{Hz} $$  \hspace{1cm} (3.1)

$$ A \equiv \rho_\infty d_H^2(t)c^2/\mu \hspace{1cm} \beta \equiv \left[1 + f_r \alpha d_H(t)c/(\Gamma G \mu t)\right]^{-1} $$  \hspace{1cm} (3.2)

In the above expressions, \( \rho_\infty \) is the energy density in ‘infinite’ or long cosmic strings, \( \alpha \) is the invariant length of a loop as a fraction of the physical horizon length \( d_H(t) \) at the time of formation, \( \langle v^2 \rangle \) is the rms velocity of the long strings, and \( f_r \approx 0.7 \) is a correction for
the damping of the relativistic center-of-mass velocity of newly formed string loops. All quantities are evaluated in the radiation era; $d_H(t) = 2ct$, $A = 52 \pm 10$, and $\langle v^2 \rangle / c^2 = 0.43 \pm 0.02$ \[12\].

The above expression for the spectrum has been obtained assuming no change in the number of relativistic degrees of freedom, $g$, of the background radiation-dominated fluid. However, the annihilation of massive particle species as the cosmological fluid cools leads to a decrease in the number of degrees of freedom, and a redshifting of all relativistic particles not thermally coupled to the fluid. This has the effect of modifying the amplitude of the spectral density \[14\], equation (3.1), by a factor $(g(T_f)/g(T_i))^{1/3}$ where $g(T_i,f)$ is the number of degrees of freedom at temperatures before and after the annihilations. Using a minimal GUT particle physics model as the basis of the standard thermal history, we see that the red noise spectrum steps downwards with growing frequency.

$$\frac{f}{\rho_{\text{crit}}} \frac{d\rho_{\text{gr}}}{df} = \frac{8\pi}{9} A \frac{\Gamma(G\mu)^2}{\alpha c^4} [1 - \langle v^2 \rangle / c^2] (\beta^{-3/2} - 1) (z_{eq} + 1)$$

$$\times \begin{cases} 1 & 10^{-8} \text{ Hz} \lesssim f \lesssim 10^{-10} \alpha^{-1} \text{ Hz} \\ (3.36/10.75)^{1/3} = 0.68 & 10^{-10} \alpha^{-1} \text{ Hz} \lesssim f \lesssim 10^{-4} \alpha^{-1} \text{ Hz} \\ (3.36/106.75)^{1/3} = 0.32 & 10^{-4} \alpha^{-1} \text{ Hz} \lesssim f \lesssim 10^{8} \text{ Hz} \end{cases}$$

Hence, the red noise spectrum is sensitive to the thermal history of the cosmological fluid. The locations of the steps in the spectrum are determined by the number of relativistic degrees of freedom as a function of temperature, $g(T)$. As an example, we present the effect of a non-standard thermal history on the spectrum in Figure [12]. In this sample model, the number of degrees of freedom $g(T)$ decreases by a factor of 10 at the temperatures $T = 10^9$, $10^5$, $1 \text{ GeV}$. The effect on the spectrum is a series of steps down in amplitude with increasing frequency; detection of such a shift would provide unique insight into the particle physics content of the early universe at temperatures much higher than may be achieved by terrestrial particle accelerators. In the case of a cosmological model with a thermal history such that $g(T_i) \gg g(T_f)$ for $T_i > T_f$, all radiation emitted before the cosmological fluid cools to $T_i$ will be redshifted away by the time the fluid reaches $T_f$. As we discuss later, such a sensitivity of the spectrum to the thermal history affects the nucleosynthesis bound on the total energy in gravitational radiation, and the opportunity to detect high frequency gravitational waves.

**B. Peaked Portion of the Spectrum**

We now turn our attention to the peaked portion of the gravitational wave spectrum. The shape of this portion depends on the model for the emission by a loop, presented in section [13]. The dominant behaviour of the peaked portion of the spectrum is given by
\[
\frac{f}{\rho_{\text{crit}}} \frac{d\rho_{\text{gr}}}{df} \approx \begin{cases} 
C_1/f^{(q-1)} & 1 < q < 2 \\
C_2/f & q \geq 2
\end{cases}
\]
for \(10^{-12} \text{ Hz} \lesssim f \lesssim 10^{-8} \text{ Hz}. \quad (3.4)

Here \(C_{1,2}\) are dimensionful quantities which depend on \(G\mu/c^2\), \(\alpha\), \(\Gamma\), \(A\), \(q\) and \(n_*\). A lengthy expression displaying the full dependence of the spectrum on these parameters is not particularly enlightening. However, the qualitative behaviour, described in more detail in ref. [11], is as follows. The overall height of the spectrum depends linearly on \(G\mu/c^2\), while the frequency at which the peaked spectrum gives way to the red noise spectrum depends inversely on \(\alpha\). The important result is that for values of the mode cut-off \(n_* \lesssim 10^2\), the spectrum drops off as \(1/f\) for any value \(q \geq 4/3\). As a demonstration, sample spectra with various values of \(n_*\) are displayed in Figure 2. Hence, the introduction of a sufficiently low mode cut-off eliminates the dependence of the spectrum on \(q\), the loop spectral index.

We have also examined the spectrum of gravitational radiation produced by the cosmic string network in an open FRW space-time, with \(0.1 < \Omega_0 < 1\). For this range of values of the cosmological density parameter, the portion of the spectrum produced at a time \(t\) is shifted downward by a factor \(\Omega(t)\). Sample spectra for various values of \(\Omega_0\) are displayed in Figures 3-4. In the case that the spectrum drops off slower than \(1/f\), the spectral density at frequencies as high as \(f \sim 10^{-5} \text{ Hz}\) is diluted for \(\Omega_0 < 1\). In the case that the spectrum drops off as \(1/f\), only at lower frequencies, \(f < \sim 10^{-10} \text{ Hz}\), is the spectral density affected.

### IV. OBSERVATIONAL BOUNDS ON THE RADIATION SPECTRUM

In this section we determine the observational constraint on the cosmic string mass-per-unit-length \(G\mu/c^2\). To begin, we discuss the recent analyses of the pulsar timing data, after which we apply the newly obtained bounds to the cosmic string gravitational wave background.

The observations used to place a limit on the amplitude of a stochastic gravitational wave background consist of pulse arrival times for PSR B1937+21 and PSR B1855+09 [6]. Although there has been some recent controversy regarding the analysis of the pulsar timing data [7], the work by McHugh et al [5] best assesses the likelihood that the timing residuals are due to gravitational radiation. We note that all analyses to date have assumed a flat, red noise spectrum for the gravitational wave spectral density. Such an assumption is only justified for a restricted range of frequencies in the case of a background due to cosmic strings, as we have demonstrated in the preceding section. Hence, a statistical analysis which uses a realistic model of the cosmic string spectrum may obtain a different limit on the amplitude of the spectral density.

We now present values of the parameter \(G\mu/c^2\) for values of \(\alpha\) which satisfy the pulsar timing constraint on the gravitational radiation spectrum. Contours of constant \(\Omega_{\text{gr}}\) in the logarithmic frequency bin \(f = (8 \text{ yrs})^{-1}\), given by (1.1), in \((\alpha, G\mu/c^2)\) parameter space, are shown in Figure 3. We have used cosmological parameters \(\Omega_0 = 1\) and \(h \in [0.5, 0.75]\) with the cosmic string loop radiation efficiency \(\Gamma = 60\). We find

\[
G\mu/c^2 < \begin{cases} 
2.0(\pm 0.4) \times 10^{-6} (2h)^{-8/3} & q = 4/3 \\
5.4(\pm 1.1) \times 10^{-6} & q \geq 2 \text{ or } n_* \lesssim 10^2
\end{cases}
\quad (4.1)
\]
These constraints correspond to the maximum value of $G\mu/c^2$ along the contour of constant $\Omega_{gr}$. In the case $q = 4/3$, this maximum occurs near $\alpha = \Gamma G\mu/c^2$, the expected size of newly formed loops based on considerations of the gravitational back-reaction, while for the $q \geq 2$ or $n_* \lesssim 10^2$ case, the maximum occurs at a slightly smaller value of $\alpha$. For both larger and smaller values of $\alpha$ the bounds become more stringent, as described in [3]. The unusual dependence on $h$ is due to the contribution from high mode number waves emitted in the matter era, for which the amplitude depends on both the slope of the spectrum and the time of radiation-matter equality. The quoted errors are due to uncertainties in the cosmic string model parameters measured by the numerical simulations [13]. For the case of an open universe, there is no change in the $q \geq 2$ or $n_* \lesssim 10^2$ bound. However, the $q = 4/3$ bound is weakened by a factor $\sim 1/\Omega_0$.

We now comment on the validity of the model which we have used to generate the gravitational radiation spectra. We have shown that the observational bounds on the total spectrum are sensitive to the value of the loop spectral index $q$, unless there is a back-reaction cut-off $n_* \lesssim 10^2$. Furthermore, we have noted that there is uncertainty in the characteristic value of the loop spectral index, $q$. Hence, we feel that it is more reasonable to take the conservative bound of $(4.1)$ at the present. Next, consider the extended one-scale model, described in refs. [2,3], for the evolution of the string network. This model assumes that the long string energy density scales relative to the background energy density, with the dominant energy loss mechanism due to the formation of loops of a characteristic scale. A more sophisticated model, by Austin et al [15], attempts to include the effect of the gravitational back-reaction on the long-term evolution of the string network; results suggest that an effect of the back-reaction may be to lower the scaling density in long strings at late times, beyond the reach of numerical simulations. Hence, there is some uncertainty as to how accurately the extended one-scale model describes the evolution of the string network. However, we do not believe that these considerations could result in a decrease in the amplitude of the gravitational wave background by more than $\sim 50\%$. Thus, we quote $G\mu/c^2 < 5.4(\pm 1.1) \times 10^{-6}$ as a conservative bound on the cosmic string mass-per-unit-length.

The bounds computed in ref. [3] due to the constraint on the total energy density in gravitational waves at the time of nucleosynthesis remain valid. For a limit on the effective number of neutrino species $N_\nu < 3.1, 3.3, 3.6$, the bound on the cosmic string mass-per-unit-length is $G\mu/c^2 < 2, 6, 10 \times 10^{-6}$ respectively, evaluated at $\alpha = \Gamma G\mu/c^2$. The big-bang nucleosynthesis limit on the number of effective neutrino species is a conservative $N_\nu < 4$, owing to uncertainties in the systematic errors in the observations of light element abundances [16]. Hence, until the observations are refined, the nucleosynthesis bound is weaker than the pulsar timing bound. Furthermore, the translation of the limit on $N_\nu$ into the bound on $G\mu/c^2$ is sensitive to the thermal history of the cosmological fluid [14]. The bound on the string mass-per-unit-length may be considerably weakened if the cosmological fluid possessed many more relativistic degrees of freedom in the early universe beyond those given by a minimal GUT model.

Comparing detailed computations of the large angular scale cosmic microwave background temperature anisotropies induced by cosmic strings [17] with observations, the cosmic string mass-per-unit-length has been normalized to

$$G\mu/c^2 = 1.05^{+0.35}_{-0.20} \times 10^{-6}. \quad (4.2)$$
Therefore, given the uncertainties in the extended one-scale model, we find the gravitational radiation spectrum to be compatible with observations.

V. DETECTION OF THE RADIATION SPECTRUM

We would like to determine whether the stochastic gravitational wave spectrum emitted by cosmic strings may be observed by current and planned detectors. Because all ground-based detectors operate at frequencies \( f > \sim 10^{-3} \) Hz, we need only consider the ‘red noise’ portion of the gravitational wave spectrum (5.3). Noting that the spectral density, \( \Omega_{\text{gr}}(f) \), has a minimum value when \( \alpha \to 0 \) (this has been pointed out by Allen in [18]) the predicted spectrum is bounded from below by

\[
\Omega_{\text{gr}}(f) \geq \frac{24\pi}{9} A f_{r} \left(1 - \frac{\langle v^2 \rangle/c^2}{1 + z_{\text{eq}}}(G\mu/c^2) \left(g(T_0)/g(T_{\text{GUT}})\right)^{1/3}\right)
\]

\[
\geq 1.4 \times 10^{-9} \text{ for } 10^{-8} \text{ Hz} \lesssim f \lesssim 10^{10} \text{ Hz}. \quad (5.1)
\]

Here we have used the normalization in (4.2) for \( G\mu/c^2 \), Hubble parameter \( h = 0.75 \), and assumed a minimal GUT thermal history. Hence, this lower bound is valid up to frequencies \( f \sim 10^{-3}\alpha^{-1} \) Hz \( \sim 10 \) Hz based on our knowledge of the number of relativistic degrees of freedom, \( g \), of the primordial fluid up to temperatures \( T \sim 10^{3} \) GeV. Notice that a measurement of the spectral density due to cosmic strings at higher frequencies would sample \( g \) at higher temperatures.

We may in turn place a lower bound on the amplitude of the dimensionless strain predicted for the gravitational wave emitted by cosmic strings:

\[
h_{c} = 1.3 \times 10^{-18} h \sqrt{\Omega_{\text{gr}}(f)} \left(\frac{f}{1 \text{ Hz}}\right)^{-1}
\]

\[
\geq 3.6 \times 10^{-23} \left(\frac{f}{1 \text{ Hz}}\right)^{-1} \text{ for } 10^{-6} \text{ Hz} \lesssim f \lesssim 10^{8} \text{ Hz}. \quad (5.2)
\]

The expressions (5.1–5.2) are useful for comparison with the planned sensitivities of the forthcoming generation of gravitational wave detectors [19].

The most promising opportunity to probe for a stochastic gravitational wave background due to cosmic strings is through a cross-correlation of the observations of the advanced LIGO, VIRGO and LISA interferometers. It is estimated that the advanced LIGO detectors will have the sensitivity

\[
h_{3/yr} = 5.2 \times 10^{-25} \left(\frac{f}{1 \text{ kHz}}\right)^{1/2}
\]

for stochastic waves (equation 125c of [20]), sufficient to measure the minimum predicted strain (5.2) near \( f \sim 100 \) Hz in a 1/3-year integration time. More recent calculations [21] confirm that the orientation of the advanced LIGO interferometers will be sufficient in order to detect the cosmic string gravitational wave background. For the LISA project [22], comparison of the projected strain sensitivity \( h_c \sim 10^{-20} \) at the frequency \( f \sim 10^{-3} \) Hz [22] with (5.2) indicates that the space-based interferometer will be capable of detecting a gravitational radiation background produced by a network of cosmic strings.
Other ground-based interferometric gravitational wave detectors are in development or under construction. The GEO600 and TAMA300 detectors, operating near frequencies $f \sim 10^3$ Hz, may also be capable of measuring a cosmic string generated background.

A network of resonant mass antennae, such as bar and TIGA detectors may probe for a stochastic background. Successful detection by these antennae will require improved sensitivity and longer integration time. However, cross-correlation between a narrow-band bar and a wide-band interferometric detector may improve the opportunities. Estimates of the sensitivity of such a system, assuming optimum detector alignment [23], indicate that

$$\sqrt{\frac{h_{\text{int}} h_b}{h_b}} \gtrsim 2.6 \times 10^{-19} \sqrt{\Omega_{\text{gr}}(f)} \left( \frac{f}{1 \text{ kHz}} \right)^{-3/2} \left( \frac{t_{\text{obs}}}{10^7 \text{s}} \right)^{1/2}$$

is necessary to detect a background $\Omega_{\text{gr}}$. Hence, for a 1/3-year observation time, the bar and interferometer strain sensitivities at 1 kHz must be better than $\sim 10^{-23}$ in order to detect the cosmic string background.

We stress that the amplitude of the cosmic string gravitational wave background for frequencies $f \gtrsim 10$ Hz is sensitive to the number of degrees of freedom of the cosmological fluid at temperatures $T \gtrsim 10^3$ GeV. The amplitude of the cosmic string background at LISA-frequencies, near $f \sim 10^{-3}$ Hz, is firm, since the cosmological fluid near the temperature $T \sim 10$ MeV is well understood. However, at the higher frequencies probed by ground-based detectors, our uncertainty in the number of degrees of freedom of the cosmological fluid, as determined by the correct model of particle physics at that energy scale, may reduce the predicted amplitude (5.1) of gravitational radiation.

**VI. CONCLUSION**

In this paper, we presented improved calculations of the spectrum of relic gravitational waves emitted by cosmic strings. We demonstrated that the effect of a gravitational back-reaction on the radiation spectrum of cosmic string loops, for which there is an effective mode cut-off $n* \lesssim 10^2$, may serve to weaken the pulsar timing bound on the cosmic string mass-per-unit-length. Arguing for a model of radiation by loops, for which either the spectral index is $q \geq 2$ or there is an emission mode cut-off $n* \lesssim 10^2$, we obtain the conservative bound $G\mu/e^2 < 5.4(\pm1.1) \times 10^{-6}$ due to observations of pulsar timing residuals. We believe this bound to be robust, as the spectrum depends weakly on the precise value of the mode cut-off, up to $n* \sim 10^2$. We have noted the interesting result that the flat, red noise portion of the gravitational wave spectrum is sensitive to the thermal history of the cosmological fluid, revealing features of the particle physics content at early times. Finally, we have pointed out that the generation of advanced LIGO, VIRGO and LISA interferometers should be capable of detecting the predicted stochastic gravitational wave background due to cosmic strings.

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FIGURES

FIG. 1. The effect of a non-standard thermal history of the cosmological fluid on the amplitude of the red noise portion of the gravitational wave spectrum is shown. The solid curve displays the spectrum produced using a minimal GUT with a maximum $g = 10^6.75$. The dashed curve shows the spectrum produced allowing for a hypothetical, non-standard evolution of $g(T)$, as might occur if there were a series of phase transitions, or a number of massive particle annihilations as the universe cooled. For temperatures $T > 10^{9}$ GeV, the number of degrees of freedom is $g = 10^4$. For $10^5$ GeV $< T < 10^9$ GeV, $g = 10^3$. For $T < 10^5$ GeV, the standard thermal scenario is resumed.

FIG. 2. The effect of a cut-off in the radiation mode number on the spectrum of gravitational radiation is shown. Curves for the loop radiation spectral index $q = 2, 4/3$ for various values of $n_*$ are shown. The vertical line shows the location of the frequency bin probed by pulsar timing measurements. For $n_*\lesssim 10^2$ the shape of the spectrum is insensitive to the value of $q$ for purposes of pulsar timing measurements. For increasing $n_*$, more radiation due to late-time cosmic string loops is emitted in the pulsar timing frequency band.

FIG. 3. The effect of a low density, $\Omega_0 < 1$ universe on the peaked portion of the gravitational wave spectrum. The solid, long- and short-dashed curves represent spectra for $\Omega_0 = 1, 0.6, 0.2$. The vertical line shows the location of the frequency bin probed by pulsar timing measurements. For the loop spectral index $q = 2$, a low density universe dilutes only the lowest frequency waves, corresponding the radiation emitted by loops still present today.

FIG. 4. The effect of a low density, $\Omega_0 < 1$ universe on the peaked portion of the gravitational wave spectrum. The solid, long- and short-dashed curves represent spectra for $\Omega_0 = 1, 0.6, 0.2$. The vertical line shows the location of the frequency bin probed by pulsar timing measurements. For the loop spectral index $q = 4/3$, a low density universe leads to a dilution of gravitational waves with wavelengths up to $f \sim 10^{-5}$ Hz.

FIG. 5. Curves of constant $\Omega_{gr}$ in $(\alpha, G\mu/c^2)$ parameter space are shown. For a given value of $\alpha$, these figures give the observational bound on $G\mu/c^2$ in the case $h = 0.5, 0.75$. In each figure, the constraining curves for $q = 10, 2, 4/3$ are given by the solid, long-, and short-dashed curves. The light dashed lines show $\alpha = \Gamma G\mu/c^2$. The most conservative constraint is $G\mu/c^2 < 5.4 \times 10^{-6}$. 
REFERENCES

[1] T. W. B. Kibble, J. Phys. A 9, 1387 (1976); Ya. B. Zel’dovich, Mon. Not. Roy. Astron. Soc. 192, 663 (1980); A. Vilenkin, Phys. Rev. Lett. 46, 1169 (1981);
[2] A. Vilenkin and E.P.S. Shellard, Cosmic strings and other topological defects (Cambridge University Press, Cambridge, 1994).
[3] R.R. Caldwell and Bruce Allen, Phys. Rev. D 45, 3447 (1992).
[4] R.A. Battye and E.P.S. Shellard, Nuc. Phys. B 423, 260 (1994); R.A. Battye and E.P.S. Shellard, Phys. Rev. Lett. 75, 4354 (1995); R.A. Battye and E.P.S. Shellard, Phys. Rev. D 53, 1811 (1996).
[5] M.P. McHugh, G. Zalamansky, F. Vernotte, and E. Lantz, “Pulsar timing and the upper limits on a gravitational wave background: a Bayesian approach”, preprint (1996).
[6] V.M. Kaspi, J.H. Taylor, and M.F. Ryba, Astrophys. J. 428, 713 (1994).
[7] S.E. Thorsett and R.J. Dewey, Phys. Rev. D 53, 3468 (1996).
[8] Paul Casper, private communication (1996); Paul Casper and Bruce Allen, Phys. Rev. D 52, 4337 (1995).
[9] Bruce Allen and E.P.S. Shellard, Phys. Rev. D 45, 1898 (1992).
[10] D. Garfinkle and T. Vachaspati, Phys. Rev. D 36, 2229 (1987).
[11] R.A. Battye, PhD Thesis (Cambridge University, 1995).
[12] R.A. Battye and E.P.S. Shellard, work in progress.
[13] D. Bennett and F. Bouchet, Phys. Rev. D 41, 2408 (1990); B. Allen and E. P. S. Shellard, Phys. Rev. Lett. 64, 119 (1990).
[14] D. Bennett, Phys. Rev. D 34, 3592 (1986); D. Bennett, Phys. Rev. D 34, 3932(E) (1986).
[15] E. Copeland, T.W.B. Kibble, and Daren Austin, Phys. Rev. D 45, 1000 (1992); Daren Austin, E.J. Copeland, and T.W.B. Kibble, Phys. Rev. D 48, 5594 (1993); Daren Austin, E.J. Copeland, and T.W.B. Kibble, Phys. Rev. D 51, 2499 (1995).
[16] Craig J. Copi, David N. Schramm, and Michael S. Turner, “The Big-Bang Nucleosynthesis Limit to the Number of Neutrino Species”, FERMILAB-Pub-96/122-A, astro-ph/9606059 (1996).
[17] B. Allen, R. R. Caldwell, E. P. S. Shellard, A. Stebbins and S. Veeraraghavan, “Large Angular Scale CMB Anisotropy Induced by Cosmic Strings”, submitted to Phys. Rev. Lett. (May, 1996).
[18] B. Allen, “The Stochastic Gravity-Wave Background: Sources and Detection”, in Proceedings of the Les Houches School on Astrophysical Sources of Gravitational Waves, ed. J.-A. Marck and P. Lasota (Springer-Verlag, 1996).
[19] For a survey of planned and present experiments, see Gravitational Wave Experiments, ed. E. Coccia, G. Pizzella, and F. Ronga (World Scientific, Singapore, 1995).
[20] Kip Thorne, in Three Hundred Years of Gravitation, ed. S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987).
[21] E.E. Flanagan, Phys. Rev. D 48, 2389 (1993).
[22] Kip Thorne, “Gravitational Waves”, in Proceedings of the 1994 Snowmass Summer Study: Particle and Nuclear Astrophysics and Cosmology in the Next Millenium, ed. E.W. Kolb and R.D. Peccei (World Scientific, Singapore, 1995); K. Danzmann et al, “Laser Interferometer Space Antenna: Pre-Phase A Report”, February, 1996.
[23] Pia Astone, J. Alberto Lobo, and Bernard Schutz, Class. Quantum Grav. 11, 2093 (1994).
Figure 5

Graphs showing the relationship between $\log_{10}(G\mu/c^2)$ and $\log_{10}(\alpha)$ for $h=0.50$ and $h=0.75$. The graphs display two curves, one solid and one dashed, indicating different values or conditions.