Pointwise Bounds and Blow-up for Systems of Nonlinear Fractional Parabolic Inequalities

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Abstract
We investigate nonnegative solutions \( u(x, t) \) and \( v(x, t) \) of the nonlinear system of inequalities
\[
0 \leq (\partial_t - \Delta)^\alpha u \leq v^\lambda \\
0 \leq (\partial_t - \Delta)^\beta v \leq u^\sigma
\]
in \( \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \), satisfying the initial conditions
\[
u = v = 0 \ \text{in} \ \mathbb{R}^n \times (-\infty, 0)
\]
where \( \lambda, \sigma, \alpha, \) and \( \beta \) are positive constants.

Specifically, using the definition of the fractional heat operator \( (\partial_t - \Delta)^\alpha \) given in [24], we obtain, when they exist, optimal pointwise upper bounds on \( \mathbb{R}^n \times (0, \infty) \) for nonnegative solutions \( u \) and \( v \) of this initial value problem with particular emphasis on these bounds as \( t \to 0^+ \) and as \( t \to \infty \).

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1 Introduction
In this paper we study pointwise upper bounds for nonnegative solutions \( u(x, t) \) and \( v(x, t) \) of the nonlinear system of inequalities
\[
0 \leq (\partial_t - \Delta)^\alpha u \leq v^\lambda \\
0 \leq (\partial_t - \Delta)^\beta v \leq u^\sigma
\]
in \( \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \) (1.1)
satisfying the initial conditions
\[
u = v = 0 \ \text{in} \ \mathbb{R}^n \times (-\infty, 0),
\]
where \( \lambda, \sigma, \alpha, \) and \( \beta \) are positive constants.

Our results in this paper for the system (1.1), (1.2) are an extension of our results in [24] on pointwise bounds for nonnegative solutions \( u(x, t) \) of the scalar initial value problem
\[
0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda \ \text{in} \ \mathbb{R}^n \times \mathbb{R} \\
u = 0 \ \text{in} \ \mathbb{R}^n \times (-\infty, 0),
\]
where $\lambda$ and $\alpha$ are positive constants.

As in [24], we define the fully fractional heat operator

$$(\partial_t - \Delta)^\alpha : Y^p_\alpha \to X^p$$

(1.3)

for

$$\left( p > 1 \text{ and } 0 < \alpha < \frac{n + 2}{2p} \right) \quad \text{or} \quad \left( p = 1 \text{ and } 0 < \alpha \leq \frac{n + 2}{2p} \right)$$

(1.4)

as the inverse of the operator

$$J_\alpha : X^p \to Y^p_\alpha$$

(1.5)

where

$$X^p := \bigcap_{T \in \mathbb{R}} L^p(\mathbb{R}^n \times \mathbb{R}_T), \quad \mathbb{R}_T := (-\infty, T),$$

(1.6)

$$J_\alpha f(x,t) := \int_{\mathbb{R}^n \times \mathbb{R}_t} \Phi_\alpha(x - \xi, t - \tau) f(\xi, \tau) \, d\xi \, d\tau$$

(1.7)

and

$$Y^p_\alpha := J_\alpha(X^p).$$

(1.8)

By (1.6) we mean $X^p$ is the set of all measurable functions $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R}^n \times \mathbb{R}_T)} < \infty \quad \text{for all } T \in \mathbb{R}.$$

In the definition (1.7) of $J_\alpha$, the fractional heat kernel,

$$\Phi_\alpha(x,t) := \frac{t^{\alpha - 1}}{\Gamma(\alpha)} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \chi_{(0,\infty)}(t)$$

(1.9)

is the fractional heat kernel.

When $p$ and $\alpha$ satisfy (1.4), it was shown in [24] that the operator (1.5) has among others the following properties:

(P1) it makes sense because $J_\alpha f \in L^p_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})$ for $f \in X^p$,

(P2) it is one-to-one and onto,

(P3) if $f \in X^p$ and $u = J_\alpha f$ then $f = 0$ in $\mathbb{R}^n \times (-\infty, 0)$ if and only if $u = 0$ in $\mathbb{R}^n \times (-\infty, 0)$.

By properties (P1) and (P2) we can indeed define (1.3) as the inverse of (1.5) when $p$ and $\alpha$ satisfy (1.4). Property (P3) will be needed to handle the initial conditions (1.2).

According to our results in Section 2 there are essentially only three possibilities for nonnegative solutions $u \in Y^p_\alpha$ and $v \in Y^q_\beta$ of (1.1), (1.2) depending on $n$, $\lambda$, $\sigma$, $\alpha$, $\beta$, $p$, and $q$:

(i) The only solution is $u \equiv v \equiv 0$ in $\mathbb{R}^n \times \mathbb{R}$;

(ii) There exist sharp nonzero pointwise bounds for solutions as $t \to 0^+$ and as $t \to \infty$;

(iii) There do not exist pointwise bounds for solutions as $t \to 0^+$ and as $t \to \infty$. 

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All possibilities can occur. For the precise statements of possibilities (i), (ii), and (iii) see Theorem 2.1, Theorems 2.2–2.4, and Theorems 2.5 and 2.6, respectively.

The operator (1.3) is a fully fractional heat operator as opposed to time fractional heat operators in which the fractional derivatives are only with respect to $t$, and space fractional heat operators, in which the fractional derivatives are only with respect to $x$.

Some recent results for nonlinear PDEs containing time (resp. space) fractional heat operators can be found in [2, 4, 5, 10, 14, 15, 16, 19, 23, 28, 29] (resp. [1, 3, 7, 8, 9, 11, 12, 13, 17, 20, 25, 26, 27]). Except for [24], we know of no results for nonlinear PDEs containing the fully fractional heat operator $(\partial_t - \Delta)^\alpha$. However results for linear PDEs containing this operator, including in particular

$$(\partial_t - \Delta)^\alpha u = f,$$

where $f$ is a given function, can be found in [6, 18, 21, 22].

2 Statement of Results

In this section we state our results concerning pointwise bounds for nonnegative solutions

$$u \in Y_p^\alpha \quad \text{and} \quad v \in Y_q^\beta$$

(2.1)

of the nonlinear system of inequalities

$$0 \leq (\partial_t - \Delta)^\alpha u \leq v^\lambda \quad \text{in} \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 1$$

$$0 \leq (\partial_t - \Delta)^\beta v \leq u^\sigma \quad \text{in} \ \mathbb{R}^n \times \mathbb{R}, \ n \geq 1$$

(2.2)

satisfying the initial conditions

$$u = v = 0 \quad \text{in} \ \mathbb{R}^n \times (-\infty, 0),$$

(2.3)

where

$$p, q \in [1, \infty), \ \lambda, \sigma, \alpha, \beta \in (0, \infty),$$

(2.4)

and, as in the definition in Section 1 of the operator (1.3), $p$ and $\alpha$ satisfy

$$p > 1 \quad \text{and} \quad 0 < \alpha < \frac{n + 2}{2p}$$

or

$$p = 1 \quad \text{and} \quad 0 < \alpha \leq \frac{n + 2}{2p}$$

(2.5)

and $q$ and $\beta$ satisfy

$$q > 1 \quad \text{and} \quad 0 < \beta < \frac{n + 2}{2q}$$

or

$$q = 1 \quad \text{and} \quad 0 < \beta \leq \frac{n + 2}{2q}.$$ 

(2.6)

If $p$ and $\alpha$ satisfy (2.5), $u \in Y_p^\alpha$, and $(\partial_t - \Delta)^\alpha u \geq 0$ in $\mathbb{R}^n \times \mathbb{R}$ then

$$u = J_\alpha((\partial_t - \Delta)^\alpha u) \geq 0 \quad \text{in} \ \mathbb{R}^n \times \mathbb{R}$$

by (1.7) and the nonnegativity of $\Phi_\alpha$. Thus the assumption that $u$ (and similarly $v$) is nonnegative can be omitted when studying the problem (2.1)–(2.6).

Moreover, when studying the problem (2.1)–(2.6), we can assume without loss of generality that

$$(n + 2 - 2p\alpha)q^2 \sigma \leq (n + 2 - 2q\beta)p^2 \lambda,$$

(2.7)
for otherwise switch the symbols for $u, \lambda, \alpha$, and $p$ with the symbols for $v, \sigma, \beta$, and $q$ respectively.

If (2.5) holds then either
\[ 2p\alpha < n + 2 \quad (2.8) \]
or
\[ 2p\alpha = n + 2. \quad (2.9) \]

The following Theorems 2.1–2.6 deal with solutions of (2.1)–(2.3) when (2.4)–(2.7) and (2.8) hold; the only exception being that (2.7) and (2.8) are not assumed in Theorems 2.3 and 2.4. Theorem 2.7 deals with solutions of (2.1)–(2.3) in the simpler case when (2.4)–(2.7) and (2.9) hold.

If (2.4), (2.7), and (2.8) hold then
\[ 2q\beta < n + 2, \]
\[ 0 < \sigma \leq \nu(\lambda) := \frac{(n + 2 - 2q\beta)p^2}{(n + 2 - 2p\alpha)q^2\lambda} \quad (2.10) \]

and the curves $\sigma = \nu(\lambda)$ and
\[ \sigma = \mu(\lambda) := \frac{2p\beta}{n + 2 - 2p\alpha} + \frac{n + 2}{(n + 2 - 2p\alpha)\lambda} \quad (2.11) \]

intersect at
\[ (\lambda_0, \sigma_0) = \left( \frac{(n + 2)q}{(n + 2 - 2q\beta)p}, \frac{(n + 2)p}{(n + 2 - 2p\alpha)q} \right). \quad (2.12) \]

See Figure 2.1. Thus assuming (2.4), (2.7), and (2.8) hold, the point $(\lambda, \sigma)$ belongs to one of the following five pairwise disjoint subsets of the $\lambda\sigma$-plane.

\[ A := \{ (\lambda, \sigma) : 0 < \sigma \leq \nu(\lambda) \text{ and } 1/\lambda \leq \sigma < \mu(\lambda) \} \]
\[ B := \{ (\lambda, \sigma) : 0 < \sigma \leq \nu(\lambda) \text{ and } \sigma < 1/\lambda \} \]
\[ C := \{ (\lambda, \sigma) : \mu(\lambda) < \sigma \leq \sigma_0 \text{ and } \lambda > 0 \} \]
\[ D := \{ (\lambda, \sigma) : \sigma_0 < \sigma \leq \nu(\lambda) \} \]
\[ E := \{ (\lambda, \sigma) : \sigma = \mu(\lambda) \text{ and } \lambda \geq \lambda_0 \}. \]

Note that $A, B, C,$ and $D$ are two dimensional regions in the $\lambda\sigma$-plane whereas $E$ is the curve separating $A$ and $C$. (See Figure 2.1.)

Theorems 2.1–2.6 deal with solutions of (2.1)–(2.3) when (2.4)–(2.7) and (2.8) hold and $(\lambda, \sigma)$ is in $A, B, C$ or $D$. We have no results when $(\lambda, \sigma) \in E$.

The following theorem deals with solutions of (2.1)–(2.3) when (2.4)–(2.7) and (2.8) hold and $(\lambda, \sigma) \in A$.

**Theorem 2.1.** Suppose (2.1)–(2.7) and (2.8) hold and
\[ \frac{1}{\lambda} \leq \sigma < \mu(\lambda). \]

Then
\[ u = v = 0 \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}. \]

The following theorem deals with solutions of (2.1)–(2.3) when (2.4)–(2.7) and (2.8) hold and $(\lambda, \sigma) \in B$. 

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Theorem 2.2. Suppose (2.1) – (2.7) and (2.8) hold and
\[ \sigma < \frac{1}{\lambda}. \] (2.13)

Then for all \( T > 0 \) we have
\[ \|u\|_{L^\infty(R^n \times (0,T))} \leq M_2^{\frac{\lambda}{\lambda - \lambda \sigma}} T^{\gamma_2/\sigma} \] (2.14)
and
\[ \|v\|_{L^\infty(R^n \times (0,T))} \leq M_1^{\frac{\sigma}{\lambda - \lambda \sigma}} T^{\gamma_1/\lambda} \] (2.15)
where
\[ \gamma_1 = \frac{(\beta + \alpha \sigma)\lambda}{1 - \lambda \sigma}, \quad \gamma_2 = \frac{(\alpha + \beta \lambda)\sigma}{1 - \lambda \sigma}, \] (2.16)
\[ M_1 = \frac{\Gamma(\gamma_1 + 1)\Gamma(\gamma_2 + 1)^{1/\sigma}}{\Gamma(\alpha + \gamma_1 + 1)\Gamma(\beta + \gamma_2 + 1)^{1/\sigma}}, \] (2.17)
\[ M_2 = \frac{\Gamma(\gamma_2 + 1)\Gamma(\gamma_1 + 1)^{1/\lambda}}{\Gamma(\beta + \gamma_2 + 1)\Gamma(\alpha + \gamma_1 + 1)^{1/\lambda}}, \] (2.18)
where \( \Gamma \) is the Gamma function.

By the following theorem, the bounds (2.14) and (2.15) in Theorem 2.2 are optimal.

Theorem 2.3. Suppose (2.4) – (2.6) and (2.13) hold,

\[ T > 0, \quad 0 < N_1 < M_1, \quad \text{and} \quad 0 < N_2 < M_2, \]

where \( M_1 \) and \( M_2 \) are defined in (2.17) and (2.18). Then there exist solutions
\[ u \in Y^p_{\alpha} \cap C(R^n \times R) \quad \text{and} \quad v \in Y^q_{\beta} \cap C(R^n \times R) \]
of (2.2), (2.3) such that for $0 < t < T$ we have
\[ u(0,t) \geq N_2^{1-\lambda\sigma} t^{\gamma_2/\sigma} \quad \text{and} \quad v(0,t) \geq N_1^{1-\sigma\lambda} t^{\gamma_1/\lambda} \]
where $\gamma_1$ and $\gamma_2$ are defined in (2.16).

Although the estimates (2.11) and (2.12) are optimal there still remains the question as to whether there is a single solution pair $u, v$ which has the same size as these estimates as $t \to \infty$. By the following theorem there is such a solution pair.

**Theorem 2.4.** Suppose (2.4)–(2.6) hold. Then there exist $N > 0$ and solutions $u \in Y^p_\alpha$ and $v \in Y^q_\beta$ of (2.2), (2.3) such that for $|x|^2 < t$ we have
\[ u(x,t) \geq N t^{\gamma_2/\alpha} \quad \text{and} \quad v(x,t) \geq N t^{\gamma_1/\lambda} \]
where $\gamma_1$ and $\gamma_2$ are defined in (2.16).

According to the following theorem, if $(\lambda, \sigma) \in C \cup D$ then there exist bounds as $t \to 0^+$ for solutions of (2.1)–(2.3) in neither the pointwise (i.e. $L^\infty$) sense nor in the $L^r$ sense for certain values of $r$. Moreover by Theorem 2.6 the same is true as $t \to \infty$.

**Theorem 2.5.** Suppose (2.4)–(2.7) hold,
\[ \sigma > \mu(\lambda), \quad (2.19) \]
$r \in (p, \infty]$, and $s \in (s_0, \infty]$ where
\[ s_0 = \max\{q, q\sigma_0/\sigma\} = \begin{cases} q\sigma_0/\sigma & \text{if } \sigma \leq \sigma_0 \\ q & \text{if } \sigma > \sigma_0. \end{cases} \]

Then there exist solutions $u \in Y^p_\alpha$ and $v \in Y^q_\beta$ of the initial value problem (2.2), (2.3) and a sequence $\{t_j\} \subset (0, 1)$ such that $\lim_{j \to \infty} t_j = 0$ and
\[ \|v^\lambda\|_{L^r(R_j)} = \|\partial_t - \Delta\|_{L^r(R_j)} = \infty \quad (2.21) \]
\[ \|u^\sigma\|_{L^r(R_j)} = \|\partial_t - \Delta\|_{L^r(R_j)} = \infty \quad (2.22) \]
for $j = 1, 2, \ldots$, where
\[ R_j = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : |x| < \sqrt{t_j} \text{ and } t_j < t < 2t_j\}. \quad (2.23) \]

Since (2.20) implies (2.21) and (2.22) are not true for $r = p$ and $s = q$ respectively, we see that (2.21) is optimal when $(\lambda, \sigma) \in C \cup D$ and (2.22) is optimal when $s_0 = q$ (i.e. when $(\lambda, \sigma) \in D$).

**Theorem 2.6.** Suppose (2.4), (2.7), (2.8), and (2.19) hold. Let
\[ r_0 = \frac{(n+2)(\lambda\sigma - 1)}{2(\beta + \alpha\sigma)\lambda} \quad \text{and} \quad s_0 = \frac{(n+2)(\lambda\sigma - 1)}{2(\alpha + \beta)\lambda}. \]

Then $r_0 > p, s_0 > q$, and for each $r \in [r_0, \infty]$ and $s \in [s_0, \infty]$ there exist solutions $u \in Y^p_\alpha$ and $v \in Y^q_\beta$ of the initial value problem (2.2), (2.3) and a sequence $\{t_j\} \subset (1, \infty)$ such that $\lim_{j \to \infty} t_j = \infty$ and $u$ and $v$ satisfy (2.21) and (2.22) for $j = 1, 2, \ldots$, where $R_j$ is given by (2.23).
The following theorem deals with solutions of (2.1)–(2.3) when (2.4)–(2.7) and (2.9) hold.

**Theorem 2.7.** Suppose (2.1)–(2.7) and (2.9) hold. Then the following statements are true.

(i) If $\lambda \sigma \geq 1$ then $u = v = 0$ a.e. in $\mathbb{R}^n \times \mathbb{R}$.

(ii) If $\lambda \sigma < 1$ then $u$ and $v$ satisfy (2.14) and (2.15) for all $T > 0$.

Clearly Theorem 2.7(i) is optimal and the optimality of Theorem 2.7(ii) follows from Theorems 2.3 and 2.4 because neither (2.8) nor (2.9) is assumed in those theorems.

### 3 $J_\alpha$ version of results

In order to prove our results in Section 2, we will first reformulate them in terms of the inverse $J_\alpha$ of the fractional heat operator (1.3) as follows:

If (2.4)–(2.6) hold then by properties (P1)–(P3) in Section 1 of $J_\alpha$ and the definition of the fractional heat operator (1.3), $u$ and $v$ satisfy (2.1)–(2.3) if and only if

$$f := (\partial_t - \Delta)^\alpha u \quad \text{and} \quad g := (\partial_t - \Delta)^\beta v$$

satisfy

$$f \in X^p \quad \text{and} \quad g \in X^q \quad \text{(3.1)}$$

$$0 \leq f \leq (J_\beta g)^\lambda \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R} \quad \text{(3.2)}$$

$$0 \leq g \leq (J_\alpha f)^\sigma \quad \text{(3.2)}$$

$$f = g = 0 \quad \text{in} \quad \mathbb{R}^n \times (-\infty, 0) \quad \text{(3.3)}$$

In problem (3.1)–(3.3), $f$ and $g$ are nonnegative functions in $\mathbb{R}^n \times \mathbb{R}$ and thus $J_\alpha f$ and $J_\beta g$ are well-defined nonnegative extended real valued functions in $\mathbb{R}^n \times \mathbb{R}$ without assuming (2.5) and (2.6). Hence in this section we study the problem (3.1)–(3.3) without assumptions (2.5) and (2.6).

However our results in this section for the problem (3.1)–(3.3) will only yield corresponding results for the problem (2.1)–(2.3) when (2.5) and (2.6) hold, for otherwise the fractional heat operators in (2.2) are not defined. (For a more detailed discussion of the properties of $J_\alpha$ when (2.5) and (2.6) do not hold see [24, Section 4].)

Actually in this section we will consider solutions

$$f \in X^p \quad \text{and} \quad g \in X^q \quad \text{(3.4)}$$

of the following slightly more general version of (3.2)–(3.3):

$$0 \leq f \leq K_1 (J_\beta g)^\lambda \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \ n \geq 1, \quad \text{(3.5)}$$

$$0 \leq g \leq K_2 (J_\alpha f)^\sigma \quad \text{(3.5)}$$

$$f = g = 0 \quad \text{in} \quad \mathbb{R}^n \times (-\infty, 0), \quad \text{(3.6)}$$

where

$$p, q \in [1, \infty) \quad \text{and} \quad \lambda, \sigma, \alpha, \beta, K_1, K_2 \in (0, \infty) \quad \text{(3.7)}$$

are constants.

As in Section 2 we can assume without loss of generality that

$$(n + 2 - 2p\alpha)q^2 \sigma \leq (n + 2 - 2q\beta)p^2 \lambda. \quad \text{(3.8)}$$
Under the equivalence of problems (2.1)–(2.3) and (3.1)–(3.3) discussed above, the following
Theorems 3.1–3.7 when restricted to the case that (2.5) and (2.6) hold and
\[ K_1 = K_2 = 1, \]
clearly imply Theorems 2.1–2.7 in Section 2. We will prove Theorems 3.1–3.7 in Section 5.

If (3.7) holds then either
\[ 2p\alpha < n + 2 \]
or
\[ 2p\alpha \geq n + 2. \]

**Theorem 3.1.** Suppose (3.4)–(3.8) and (3.9) hold and
\[ \frac{1}{\lambda} \leq \sigma < \mu(\lambda). \]

Then
\[ f = g = 0 \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}. \]

**Theorem 3.2.** Suppose (3.4)–(3.8) and (3.9) hold and
\[ \sigma < \frac{1}{\lambda}. \]

Then for all \( T > 0 \) we have
\[ \|f\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (K_1 K_2^{\lambda})^{\frac{1}{1-\lambda\sigma}} M_1^{\frac{\lambda}{1-\lambda\sigma}} T^{\gamma_1}, \]
\[ \|g\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (K_2 K_1^\sigma)^{\frac{1}{1-\lambda\sigma}} M_2^{\frac{\lambda}{1-\lambda\sigma}} T^{\gamma_2}, \]
\[ \|J_\alpha f\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (K_1 K_2^{\lambda})^{\frac{1}{1-\lambda\sigma}} M_2^{\frac{\lambda}{1-\lambda\sigma}} T^{\gamma_2/\sigma}, \]
and
\[ \|J_\beta g\|_{L^\infty(\mathbb{R}^n \times (0,T))} \leq (K_2 K_1^\sigma)^{\frac{1}{1-\lambda\sigma}} M_1^{\frac{\sigma}{1-\lambda\sigma}} T^{\gamma_1/\lambda}, \]

where \( \gamma_1, \gamma_2, M_1, \) and \( M_2 \) defined in (2.16)–(2.18).

**Theorem 3.3.** Suppose (3.7) and (3.11) hold,
\[ T > 0, \quad 0 < N_1 < M_1, \quad \text{and} \quad 0 < N_2 < M_2, \]

where \( M_1 \) and \( M_2 \) are defined in (2.17) and (2.18). Then there exist solutions
\[ f \in L^p(\mathbb{R}^n \times \mathbb{R}) \cap C(\mathbb{R}^n \times \mathbb{R}) \quad \text{and} \quad g \in L^q(\mathbb{R}^n \times \mathbb{R}) \cap C(\mathbb{R}^n \times \mathbb{R}) \]

of (3.5), (3.6) such that
\[ J_\alpha f, J_\beta g \in C(\mathbb{R}^n \times \mathbb{R}) \]
and, for \( 0 < t < T \),
\[ f(0,t) = (K_1 K_2^{\lambda})^{\frac{1}{1-\lambda\sigma}} N_1^{\frac{\lambda\sigma}{1-\lambda\sigma}} t^{\gamma_1}, \]
\[ g(0,t) = (K_2 K_1^\sigma)^{\frac{1}{1-\lambda\sigma}} N_2^{\frac{\lambda\sigma}{1-\lambda\sigma}} t^{\gamma_2}, \]
\[ J_\alpha f(0,t) \geq (K_1 K_2^{\lambda})^{\frac{1}{1-\lambda\sigma}} N_2^{\frac{\lambda\sigma}{1-\lambda\sigma}} t^{\gamma_2/\sigma}, \]
\[ J_\beta g(0,t) \geq (K_2 K_1^\sigma)^{\frac{1}{1-\lambda\sigma}} N_1^{\frac{\sigma}{1-\lambda\sigma}} t^{\gamma_1/\lambda}, \]

where \( \gamma_1 \) and \( \gamma_2 \) are defined in (2.16).
Theorem 3.4. Suppose (3.7) and (3.11) hold. Then there exist \( N > 0 \) and solutions \( f \in X^p \) and \( g \in X^q \) of (3.5), (3.6) such that for \( |x|^2 < t \) we have

\[
f(x, t) \geq N t^{\gamma_1}, \quad g(x, t) \geq N t^{\gamma_2},
\]

(3.23)

\[
J_\alpha f(x, t) \geq N t^{\gamma_2/\sigma}, \quad \text{and} \quad J_\beta g(x, t) \geq N t^{\gamma_1/\lambda},
\]

(3.24)

where \( \gamma_1 \) and \( \gamma_2 \) are defined in (2.16).

Theorem 3.5. Suppose (3.7), (3.8), and (3.9) hold, 

\[
\sigma > \mu(\lambda),
\]

(3.25)

\( r \in (p, \infty], \) and \( s \in (s_0, \infty], \) where 

\[
s_0 = \max\{q, q\sigma_0/\sigma\} = \begin{cases} q\sigma_0/\sigma & \text{if } \sigma < \sigma_0 \\ q & \text{if } \sigma \geq \sigma_0. \end{cases}
\]

Then there exist solutions 

\[
f \in L^p(\mathbb{R}^n \times \mathbb{R}) \quad \text{and} \quad g \in L^q(\mathbb{R}^n \times \mathbb{R})
\]

(3.26)

of the initial value problem (3.5), (3.6) and a sequence \( \{t_j\} \subset (0, 1) \) such that \( \lim_{j \to \infty} t_j = 0 \) and 

\[
\|f\|_{L^r(R_j)} = \|g\|_{L^s(R_j)} = \infty \quad \text{for } j = 1, 2, \ldots
\]

(3.27)

where 

\[
R_j = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x| < \sqrt{t_j} \text{ and } t_j < t < 2t_j\}.
\]

(3.28)

Theorem 3.6. Suppose (3.7), (3.8), (3.9) and (3.25) hold. Let 

\[
r_0 = \frac{(n + 2)(\lambda \sigma - 1)}{2(\beta + \alpha \sigma)\lambda} \quad \text{and} \quad s_0 = \frac{(n + 2)(\lambda \sigma - 1)}{2(\alpha + \lambda)\sigma}.
\]

Then \( r_0 > p, s_0 > q, \) and for each 

\[
r \in [r_0, \infty] \quad \text{and} \quad s \in [s_0, \infty]
\]

(3.29)

there exist solutions 

\[
f \in X^p \quad \text{and} \quad g \in X^q
\]

(3.30)

of the initial value problem (3.5), (3.6) and a sequence \( \{t_j\} \subset (1, \infty) \) such that \( \lim_{j \to \infty} t_j = \infty \) and 

\( f \) and \( g \) satisfy (3.27) where \( R_j \) is given by (3.28).

Theorem 3.7. Suppose (3.1)–(3.8) and (3.10) hold. Then the following statements are true.

(i) If \( \lambda \sigma \geq 1 \) then \( f = g = 0 \) a.e. in \( \mathbb{R}^n \times \mathbb{R} \).

(ii) If \( \lambda \sigma < 1 \) then \( f \) and \( g \) satisfy (3.12)–(3.15) for all \( T > 0 \).
4 Preliminarys

In the section we provide some remarks and lemmas needed for the proofs of our results in Section 3 dealing with solutions of the $J_\alpha$ problem (3.4) – (3.7).

**Remark 4.1.** If (3.7) holds and $\lambda \sigma < 1$ then the functions $F, G : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ defined in $\mathbb{R}^n \times (-\infty, 0]$ by $F = G = 0$ and defined for $(x, t) \in \mathbb{R}^n \times (0, \infty)$ by

$$F(x, t) = F(t) = M_1^{\frac{\lambda}{\lambda \sigma}} t^{\gamma_1} \quad \text{and} \quad G(x, t) = G(t) = M_2^{\frac{\lambda}{\lambda \sigma}} t^{\gamma_2},$$

where $\gamma_1, \gamma_2, M_1, \text{and } M_2$ are defined in (2.16) – (2.18), satisfy

$$F = (J_\beta G)^\lambda \quad \text{and} \quad G = (J_\alpha F)^\sigma \quad \text{in } \mathbb{R}^n \times \mathbb{R}, \quad (4.1)$$

which can be verified using the formula

$$\int_0^t \frac{(t - \tau)^{\alpha - 1} \tau^{\beta - 1}}{\Gamma(\alpha) \Gamma(\beta)} d\tau = \frac{t^{\alpha + \beta - 1}}{\Gamma(\alpha + \beta)} \quad \text{for } t, \alpha, \beta > 0. \quad (4.2)$$

Even though $F, G \notin X^p$ for all $p \geq 1$, these functions will be useful in our analysis of solutions of (3.5), (3.6) which are in $X^p$ for some $p \geq 1$.

**Remark 4.2.** It will be convenient to scale (3.5) as follows. Suppose (3.7) holds, $\lambda \sigma \neq 1, T > 0$, and $f, g, \bar{f}, \bar{g} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are nonnegative measurable functions such that $f = g = \bar{f} = \bar{g} = 0$ in $\mathbb{R}^n \times (-\infty, 0)$ and

$$f(x, t) = (K_1 K_2^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} T^{\gamma_1} \bar{f}(\bar{x}, \bar{t}) \quad \text{and} \quad g(x, t) = (K_2 K_1^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} T^{\gamma_2} \bar{g}(\bar{x}, \bar{t})$$

where $\gamma_1$ and $\gamma_2$ are defined in (2.16) and $x = T^{1/2} \bar{x}$ and $t = T \bar{t}$. Then $f$ and $g$ satisfy (3.5) if and only if $\bar{f}$ and $\bar{g}$ satisfy

$$0 \leq \bar{f} \leq \left( J_{\beta} \bar{g} \right)^\lambda \quad \text{and} \quad 0 \leq \bar{g} \leq \left( J_{\alpha} \bar{f} \right)^\sigma \quad \text{in } \mathbb{R}^n \times \mathbb{R}.$$

Moreover, for $(x, t) \in \mathbb{R}^n \times (0, \infty)$ we have

$$\frac{f(x, t)}{t^{\gamma_1}} = (K_1 K_2^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} \frac{\bar{f}(\bar{x}, \bar{t})}{\bar{t}^{\gamma_1}}, \quad \frac{g(x, t)}{t^{\gamma_2}} = (K_2 K_1^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} \frac{\bar{g}(\bar{x}, \bar{t})}{\bar{t}^{\gamma_2}},$$

$$\frac{J_\alpha f(x, t)}{t^{\gamma_2/\sigma}} = (K_1 K_2^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} J_\alpha \frac{\bar{f}(\bar{x}, \bar{t})}{\bar{t}^{\gamma_2}}, \quad \text{and} \quad \frac{J_\beta g(x, t)}{t^{\gamma_1/\lambda}} = (K_2 K_1^\frac{1}{1 - \lambda \sigma})^{\frac{1}{1 - \lambda \sigma}} J_\beta \frac{\bar{g}(\bar{x}, \bar{t})}{\bar{t}^{\gamma_1/\lambda}}.$$

**Lemma 4.1.** Suppose (3.4) – (3.7) and (3.10) hold. Then

$$f, g \in X^\infty. \quad (4.3)$$

**Proof.** Let $T > 0$ be fixed. To prove (4.3) it suffices by (3.6) to prove

$$f, g \in L^\infty(\mathbb{R}^n \times (0, T)).$$

Choose

$$p_1 > \max\{p, \sigma, \frac{(n + 2)\sigma}{2\beta}\}. \quad (4.4)$$
Define $\hat{\alpha} \in \mathbb{R}$ by
\[
\frac{2\hat{\alpha}}{n+2} + \frac{1}{2p_1} = \frac{1}{p} \leq \frac{2\alpha}{n+2}
\]
by (3.10). Then $0 < \hat{\alpha} < \alpha$ and
\[
0 < \frac{1}{p} - \frac{1}{p_1} = \frac{2\hat{\alpha}}{n+2} - \frac{1}{2p_1} < \frac{2\alpha}{n+2} < \frac{1}{p} \leq 1
\]
by (3.7)\_1. Hence by (3.6), (3.4)\_1, and Lemma A.2 we have
\[
(J_\alpha f) |_{\mathbb{R}^n \times (0, T)} \leq C(J_{\hat{\alpha}} f) |_{\mathbb{R}^n \times (0, T)} \in L^p(\mathbb{R}^n \times (0, T)).
\]
Consequently by (3.5)\_2 we have
\[
g \in L^{p_1/\sigma}(\mathbb{R}^n \times (0, T)). \tag{4.5}
\]
By (4.4) there exists $\hat{\beta} \in (0, \beta]$ such that
\[
\frac{\sigma}{p_1} < \frac{2\hat{\beta}}{n+2} < 1.
\]
Then by (3.5)\_1, (3.6), (4.5), Lemma A.2 we have
\[
f^{1/\lambda} |_{\mathbb{R}^n \times (0, T)} \leq C(J_{\beta} f) |_{\mathbb{R}^n \times (0, T)} \leq C(J_{\hat{\beta} g}) |_{\mathbb{R}^n \times (0, T)} \in L^\infty(\mathbb{R}^n \times (0, T)).
\]
Thus by (3.5)\_2, (3.6), and Lemma A.1 we have
\[
g \in L^\infty(\mathbb{R}^n \times (0, T)).
\]

**Lemma 4.2.** Suppose (3.4)\_1, (3.7) and (3.9) hold,
\[
\sigma < \mu(\lambda), \tag{4.6}
\]
\[
p_1 \in [p, \infty), \text{ and } f \in X^{p_1}. \tag{4.7}
\]
Then either
\[
f \in X^\infty \tag{4.8}
\]
or there exists a constant
\[
C_0 = C_0(n, \lambda, \sigma, \alpha, \beta, p) > 0 \tag{4.9}
\]
such that $f \in X^{p_2}$ for some $p_2 \in (p_1, \infty)$ satisfying
\[
\frac{1}{p_1} - \frac{1}{p_2} > C_0. \tag{4.10}
\]

**Proof.** If $2p_1 \alpha \geq n + 2$ then (4.8) follows from Lemma 4.1 with $p = p_1$. Hence we can assume
\[
2p_1 \alpha < n + 2. \tag{4.11}
\]
By (3.7), (3.9), and (4.6) there exists
\[
\varepsilon = \varepsilon(n, \lambda, \sigma, \alpha, \beta, p) > 0
\]
such that
\[ \alpha_\varepsilon := \alpha - \varepsilon > 0, \quad \beta_\varepsilon := \beta - \varepsilon > 0 \] (4.12)
and
\[ \sigma < \frac{2p\beta_\varepsilon}{n + 2 - 2p\alpha_\varepsilon} + \frac{n + 2}{(n + 2 - 2p\alpha_\varepsilon)\lambda}. \] (4.13)

By (4.11) and (4.12) we have
\[ n + 2 - 2p_1\alpha_\varepsilon > 2p_1\varepsilon. \] (4.14)

By (4.11), (4.12), (3.7), and (4.7) there exists \( p_3 \in (p_1, \infty) \) such that
\[ \frac{1}{p_1} - \frac{1}{p_3} = \frac{2\alpha_\varepsilon}{n + 2} < \frac{2\alpha}{n + 2} < \frac{1}{p_1} \leq 1. \] (4.15)

Hence by (3.6), (4.7), and Lemma A.2 we have
\[ J_\alpha f \in X^{p_3} \] and thus from (3.5) and (3.4) we find
that
\[ g \in X^{p_4} \] where
\[ p_4 = \max \{ q, p_3 / \sigma \} \geq 1 \] (4.16)
by (3.7). We can assume
\[ 2p_4 \beta < n + 2 \] (4.17)
for otherwise by Lemma 4.1 with \( q = p_4 \) and the roles of \((f, p, \lambda, \alpha)\) and \((g, q, \sigma, \beta)\) interchanged, we have (4.18) holds.

It follows from (4.17), (4.12), and (4.16) that there exists \( p_5 \in (p_4, \infty) \) such that
\[ \frac{1}{p_4} - \frac{1}{p_5} = \frac{2\beta_\varepsilon}{n + 2} < \frac{2\beta}{n + 2} < \frac{1}{p_4} \leq 1. \] (4.18)

Thus by (3.6), (4.16) and Lemma A.2 we have \( J_\beta g \in X^{p_5} \) and consequently by (3.5)
\[ f \in X^{p_2} \] where \( p_2 = p_5 / \lambda. \) (4.19)

Moreover, it follows from (4.19), (4.18), (4.16), (4.15), (4.14), and (4.7) that
\[
\frac{1}{p_1} - \frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{p_3} - \lambda \left( \frac{1}{p_4} - \frac{2\beta_\varepsilon}{n + 2} \right) \\
\geq \frac{1}{p_1} - \lambda \left( \frac{\sigma - \frac{2\beta_\varepsilon}{n + 2}}{p_3} \right) \\
= \frac{1}{p_1} - \lambda \left( \sigma \left( \frac{1}{p_1} - \frac{2\alpha_\varepsilon}{n + 2} \right) - \frac{2\beta_\varepsilon}{n + 2} \right) \\
= \frac{\lambda}{(n + 2)p_1} \left[ \frac{n + 2}{\lambda} - (\sigma(n + 2 - 2\alpha_\varepsilon p_1) - 2\beta_\varepsilon p_1) \right] \\
= \frac{\lambda(n + 2 - 2\alpha_\varepsilon p_1)}{(n + 2)p_1} \left[ \frac{2\beta_\varepsilon p_1}{n + 2 - 2\alpha_\varepsilon p_1} + \frac{n + 2}{(n + 2 - 2\alpha_\varepsilon p_1)\lambda} - \sigma \right] \\
\geq \frac{2\lambda p_4}{n + 2} \left[ \frac{2p_4 \beta_\varepsilon}{n + 2 - 2p_4 \alpha_\varepsilon} + \frac{n + 2}{(n + 2 - 2p_4 \alpha_\varepsilon)\lambda} - \sigma \right] \\
= C_0(n, \lambda, \sigma, \alpha, \beta, p) > 0
\] by (4.13).

\[ \square \]

**Lemma 4.3.** Suppose (3.4), (3.7), and (3.9) hold and \( \sigma < \mu(\lambda) \). Then
\[ f, g \in X^{\infty}. \] (4.20)
Proof. Starting with the assumption that \( f \) satisfies (3.14) and iterating Lemma 4.2 a finite number of times (\( m \) times is enough if \( m > 1/(pC_0) \)) we find that \( f \in X^\infty \) and hence (4.20) follows from (3.5), (3.6), and Lemma A.1. \( \square \)

Lemma 4.4. Suppose \( f, g : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) are nonnegative measurable functions satisfying (3.5) and (3.6) where

\[
\lambda, \sigma, \alpha, \beta, K_1, K_2 \in (0, \infty)
\]

and for some \( h \in \{f, g\} \) we have

\[
h \in X^\infty \quad \text{and} \quad \|h\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} \neq 0.
\]

Then \( \lambda \sigma < 1 \) and \( f \) and \( g \) satisfy (3.12)–(3.15) for all \( T > 0 \).

Proof. Suppose \( h = f \). The proof when \( h = g \) is similar and will be omitted. By (3.5) we have

\[
0 \leq f \leq K_1 K_2^\lambda (J_\beta((J_\alpha f)^\sigma))^\lambda \quad \text{in} \ \mathbb{R}^n \times \mathbb{R}
\]

and by (3.6) and (4.21) there exists \( a \geq 0 \) such that

\[
\|f\|_{L^\infty(\mathbb{R}^n \times (-\infty, a])} = 0
\]

and

\[
0 < \|f\|_{L^\infty(\mathbb{R}^n \times (-\infty, t))} < \infty \quad \text{for} \ t > a.
\]

Thus

\[
J_\alpha f = 0 \quad \text{in} \ \mathbb{R}^n \times (-\infty, a].
\]

Let \( T > a \) be momentarily fixed. Then for \((x, t) \in \mathbb{R}^n \times (a, T]\) we find from (4.23) and (4.24) that

\[
\|f\|_{L^\infty(\mathbb{R}^n \times (a, T]))} \leq \int_a^t \frac{(t - \tau)^{\alpha - 1}}{\Gamma(\alpha)} d\tau = \frac{(t - a)^\alpha}{\Gamma(\alpha + 1)}.
\]

Hence for \((x, t) \in \mathbb{R}^n \times (a, T]\) we obtain from (4.25) and (4.2) that

\[
\frac{J_\beta((J_\alpha f)^\sigma)(x, t)}{\|f\|_{L^\infty(\mathbb{R}^n \times (a, T]))} \leq \frac{1}{\Gamma(\alpha + 1)^\sigma} \int_a^t \frac{(t - \tau)^{\beta - 1}}{\Gamma(\beta)} (\tau - a)^{\alpha \sigma} d\tau = B(t - a)^{\beta + \alpha \sigma}
\]

where

\[
B = \frac{\Gamma(\alpha \sigma + 1)}{\Gamma(\alpha + 1)^\sigma \Gamma(\beta + \alpha \sigma + 1)}.
\]

Thus by (4.22) we see that

\[
\|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \leq K_1 K_2^\lambda \|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \lambda \leq K_1 K_2^\lambda B \|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \lambda \beta + \alpha \sigma
\]

for \( T > a \), which by (4.24) implies

\[
1 \leq K_1 K_2^\lambda B \|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \lambda \beta + \alpha \sigma
\]

for \( T > a \).

Thus \( \lambda \sigma < 1 \) for otherwise sending \( T \) to \( a \) in (4.26) gives a contradiction. Hence from (4.26) and (2.16) we get

\[
\|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \leq (K_1 K_2^\lambda)^\frac{1 - \lambda \sigma}{1 - \lambda \sigma} B \|f\|_{L^\infty(\mathbb{R}^n \times (a, T])] \lambda \beta + \alpha \sigma
\]

for \( T > a \).
which together with (4.28) and the nonnegativity of $a$ implies
\[ \|f\|_{L^\infty(R^n \times (0,t))} \leq (K_1K_2^\lambda) \frac{1}{1-\lambda \sigma} B^{\lambda} t^{\gamma_1} \quad \text{for } t > 0. \]  
(4.27)

Suppose for some $\delta > 0$ we have
\[ \|f\|_{L^\infty(R^n \times (0,t))} \leq \delta t^{\gamma_1} \quad \text{for } t > 0. \]  
(4.28)

Then by (4.23) and (4.2) we have for $(x,t) \in R^n \times (0, \infty)$ that
\[ (J_\alpha f)(x,t) \leq \delta \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau \gamma_1 = \delta \frac{\Gamma(\gamma_1+1)}{\Gamma(\alpha + \gamma_1 + 1)} t^{\alpha+\gamma_1} \]  
(4.29)

and hence by (4.25), (4.2), and (2.17) we find for $(x,t) \in R^n \times (0, \infty)$ that
\[ [J_\beta ((J_\alpha f)^\sigma)(x,t)]^\lambda \leq \left[ \delta^\sigma \frac{\Gamma(\gamma_1+1)^\sigma}{\Gamma(\alpha + \gamma_1 + 1)^\sigma} \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} d\tau \right]^\lambda \]  
(4.30)

because by (2.16) we have
\[ (\beta + (\alpha + \gamma_1)\sigma)\lambda = \gamma_1 \quad \text{and} \quad (\alpha + \gamma_1)\sigma = \gamma_2. \]  
(4.31)

Thus by (4.22)
\[ \|f\|_{L^\infty(R^n \times (0,t))} \leq K_1K_2^\lambda (\delta M_1)^{\lambda\sigma} t^{\gamma_1} \quad \text{for } t > 0. \]  
(4.32)

Next defining a sequence $\{ \delta_j \} \subset (0, \infty)$ by
\[ \delta_1 = K_1K_2^\lambda B^{\frac{\lambda}{1-\lambda \sigma}} \quad \text{and} \quad \delta_{j+1} = K_1K_2^\lambda (\delta_j M_1)^{\lambda\sigma} \]  
(4.33)

and using $0 < \lambda \sigma < 1$ we see that $\delta_j \to (K_1K_2^\lambda) \frac{1}{1-\lambda \sigma} M_1^{\frac{\lambda}{1-\lambda \sigma}}$ as $j \to \infty$. It therefore follows from (4.27), (4.28), and (4.31) that $f$ satisfies (3.12) for $T > 0$. Thus (4.29) holds with $\delta = (K_1K_2^\lambda) \frac{1}{1-\lambda \sigma} M_1^{\frac{\lambda}{1-\lambda \sigma}}$ and so from (3.5) we find that
\[ \|g\|_{L^\infty(R^n \times (0,t))} \leq K_2 \left( (K_1K_2^\lambda) \frac{1}{1-\lambda \sigma} M_1^{\frac{\lambda}{1-\lambda \sigma}} \frac{\Gamma(\gamma_1+1)}{\Gamma(\alpha + \gamma_1 + 1)} \right)^\sigma t^{\sigma(\alpha+\gamma_1)} \]  
(4.34)

\[ = (K_2K_2^\sigma) \frac{1}{1-\lambda \sigma} M_2^{\frac{\lambda}{1-\lambda \sigma}} t^{\gamma_2} \quad \text{for } t > 0 \]  
(4.35)

by (4.30), (2.17), and (2.18). That is $g$ satisfies (3.13) for $T > 0$. Finally, (3.14) and (3.15) follow from (3.12), (3.13), and (4.2).

5 Proofs of results for $J_\alpha$ problem.

In this section we prove our results stated in Section 3 concerning pointwise bounds for nonnegative solutions $f$ and $g$ of (3.4)–(3.7). As explained in Section 3 these results immediately imply Theorems 2.1–2.7 in Section 2.
Proof of Theorem 3.7 (resp. Theorems 3.1 and 3.2). It follows from Lemma 4.1 (resp. Lemma 4.3) that \( f, g \in X^\infty \). Hence Theorem 3.7 (resp. Theorems 3.1 and 3.2) follow(s) from Lemma 4.4.

Proof of Theorem 3.3. Let 

\[
m = \max \left\{ \left( \frac{N_2}{M_2} \right)^{\frac{\lambda^2}{1-\lambda}}, \left( \frac{N_1}{M_1} \right)^{\frac{\lambda}{1-\lambda}} \right\}
\]

and \( a_1 = (m + 1)/2 \). By (3.11) and (3.16) we have \( m \in (0, 1) \) and thus

\[
0 < m < a_1 < 1.
\]

It therefore follows from (3.11) that \( a_1^{1/\lambda} < a_1^\sigma \) and hence there exists \( a_2 \) such that

\[
0 < a_1^{1/\lambda} < a_2 < a_1^\sigma < 1
\]

which together with (5.1) and (5.2) gives

\[
\frac{a_1^{\lambda}}{a_1} > 1, \quad \frac{a_1^\sigma}{a_2} > 1,
\]

and

\[
a_1 > \left( \frac{N_1}{M_1} \right)^{\frac{\lambda}{1-\lambda}}, \quad \text{and} \quad a_2 > \left( \frac{N_2}{M_2} \right)^{\frac{\lambda}{1-\lambda}}.
\]

By Remark 4.2 we can assume \( K_1 = K_2 = T = 1 \). For \((x, t) \in \mathbb{R}^n \times \mathbb{R} \) and \( \delta \in (0, 1) \) let

\[
F_\delta(x, t) = F(t) = \psi_\delta(t)F(t) \quad \text{and} \quad G_\delta(x, t) = G(t) = \psi_\delta(t)G(t)
\]

where \( F \) and \( G \) are as in Remark 4.1 and \( \psi_\delta \in C^\infty(\mathbb{R} \to [0, 1]) \) satisfies

\[
\psi_\delta(t) = \begin{cases} 1 & \text{if } t \leq 1 \\ 0 & \text{if } t \geq 1 + \delta. \end{cases}
\]

Then for \( 1 \leq t \leq 1 + \delta \)

\[
J_\alpha F(t) - J_\alpha F_\delta(t) = \int_1^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} F(\tau)(1 - \psi_\delta(\tau)) d\tau
\]

\[
\leq \int_1^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} F(\tau) d\tau \leq F(1 + \delta) \int_1^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} d\tau
\]

\[
= F(1 + \delta) \frac{(t - 1)^\alpha}{\Gamma(\alpha + 1)} \leq F(2) \frac{\delta^\alpha}{\Gamma(\alpha + 1)}
\]

and similarly

\[
J_\beta G(t) - J_\beta G_\delta(t) \leq G(2) \frac{\delta^\beta}{\Gamma(\beta + 1)}.
\]

Thus by (4.1) we have for \( 1 \leq t \leq 1 + \delta \) that

\[
\frac{J_\alpha F_\delta(t)}{J_\alpha F(t)} = \frac{J_\alpha F(t) - (J_\alpha F(t) - J_\alpha F_\delta(t))}{G(t)^{1/\sigma}} \geq 1 - \frac{F(2)\delta^\alpha}{\Gamma(\alpha + 1)G(1)^{1/\sigma}}
\]

\[
= 1 - C\delta^\alpha
\]
and similarly for \(1 \leq t \leq 1 + \delta\) that
\[
\frac{J_\beta G_\delta(t)}{J_\beta G(t)} \geq 1 - C \delta^\beta
\]
where \(C = C(\lambda, \sigma, \alpha, \beta) > 0\). Hence choosing \(\delta \in (0,1)\) sufficiently small and using (4.1) and (5.3) we find for \(1 \leq t \leq 1 + \delta\) that
\[
G_\delta(t) \leq G(t) = (J_\alpha F(t))^\sigma \leq \sqrt{a_1^\beta a_2} (J_\alpha F_\delta(t))^\sigma
\]
and
\[
F_\delta(t) \leq F(t) = (J_\beta G(t))^\lambda \leq \sqrt{a_1^\alpha a_2} (J_\beta G_\delta(t))^\lambda
\]
which by (5.5) and (4.1) holds for all other \(t\) as well.

Next let \(\psi(x) = e^{-\psi(x)}\) where \(\psi(x) = \sqrt{1 + |x|^2} - 1\). Then for \(\varepsilon \in (0,1), \gamma > 1, \) and \(|\xi - x| < \gamma \sqrt{2}\) we have
\[
\frac{\varphi(\varepsilon \xi)}{\varphi(\varepsilon x)} = e^{-(\psi(\varepsilon \xi) - \psi(\varepsilon x))} \geq e^{-\varepsilon |\xi - x|} \geq e^{-\varepsilon \gamma \sqrt{2}}.
\]
Thus defining \(f_\varepsilon, g_\varepsilon: \mathbb{R}^n \times \mathbb{R} \to [0, \infty)\) by
\[
f_\varepsilon(x,t) = \varphi(\varepsilon x) a_1 F_\delta(t) \quad \text{and} \quad g_\varepsilon(x,t) = \varphi(\varepsilon x) a_2 G_\delta(t)
\]
we find for \(|\xi - x| < \gamma \sqrt{2}\) and \(\tau \in \mathbb{R}\) that
\[
f_\varepsilon(\xi, \tau) \geq \varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} a_1 F_\delta(\tau) \quad \text{and} \quad g_\varepsilon(\xi, \tau) \geq \varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} a_2 G_\delta(\tau).
\]
Hence for \((x, t) \in \mathbb{R}^n \times (0, 2)\) we have
\[
J_\alpha f_\varepsilon(x,t) \geq \varphi(\varepsilon x) e^{\varepsilon \gamma \sqrt{2}} a_1 \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} F_\delta(\tau) \int_{|\xi - x| < \gamma \sqrt{2}} \Phi_1(x - \xi, t - \tau) \, d\xi \, d\tau
\]
and
\[
J_\beta g_\varepsilon(x,t) \geq \varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} a_2 \int_0^t \frac{(t - \tau)^{\beta-1}}{\Gamma(\beta)} G_\delta(\tau) \int_{|\xi - x| < \gamma \sqrt{2}} \Phi_1(x - \xi, t - \tau) \, d\xi \, d\tau.
\]
But for \(x, \xi \in \mathbb{R}^n\) and \(0 < \tau < t < 2\) we find making the change of variables \(z = \frac{x - \xi}{\sqrt{4(t - \tau)}}\) that
\[
\int_{|\xi - x| < \gamma \sqrt{2}} \Phi_1(x - \xi, t - \tau) \, d\xi \geq \int_{|z| < \gamma \sqrt{4(t - \tau)}} \frac{1}{(4\pi (t - \tau))^{n/2}} e^{-\frac{|z|^2}{4(t - \tau)}} \, d\xi
\]
\[
= \frac{1}{\pi^{n/2}} \int_{|z| < \gamma /2} e^{-|z|^2} \, dz =: I(\gamma) \to 1
\]
as \(\gamma \to \infty\). Thus by (5.8), (5.9), (5.6) and (5.7) we have for \((x, t) \in \mathbb{R}^n \times (0, 1 + \delta)\) that
\[
\frac{(J_\alpha f_\varepsilon(x,t))^\sigma}{g_\varepsilon(x,t)} \geq \frac{\varphi(\varepsilon x) e^{-\varepsilon \gamma \sqrt{2}} a_1^\alpha I(\gamma)^\sigma (J_\alpha F_\delta(t))^\sigma}{\varphi(\varepsilon x)^\sigma a_2 G_\delta(t)} \geq \sqrt{a_1^\alpha a_2} I(\gamma)^\sigma e^{-\varepsilon \gamma \sqrt{2}}
\]
and

\[
\frac{(J_\beta g_\varepsilon(x,t))^\lambda}{f_\varepsilon(x,t)} \geq \frac{\varphi(\varepsilon x)^{\lambda\sigma} e^{-\lambda\sigma\varepsilon\sqrt{2}} a_2^\lambda I(\gamma) \lambda (J_\beta G_\delta(t))^\lambda}{\varphi(\varepsilon x) a_1 F_\delta(t)} \\
\geq \frac{a_2^\lambda}{a_1} I(\gamma)^\lambda e^{-\lambda\sigma\varepsilon\sqrt{2}}
\]  

(5.12)

by (3.11).

So first choosing \(\gamma\) so large that

\[
\sqrt{\frac{a_1}{a_2}} I(\gamma)^\sigma > 1 \quad \text{and} \quad \sqrt{\frac{a_2}{a_1}} I(\gamma)^\lambda > 1
\]

(we can do this by (5.3) and (5.10)) and then choosing \(\varepsilon > 0\) so small that (5.11) and (5.12) are both greater than one we see that \(f := f_\varepsilon\) and \(g = g_\varepsilon\) satisfy (3.5) in \(\mathbb{R}^n \times (0, 1 + \delta)\). Hence, since \(F_\delta\) and \(G_\delta\), and thus \(f(x, t)\) and \(g(x, t)\), are identically zero in \(\mathbb{R}^n \times ((-\infty, 0] \cup [1 + \delta, \infty))\) we have that \(f\) and \(g\) satisfy (3.5), (3.6).

From the exponential decay of \(\varphi(x)\) as \(|x| \to \infty\), we find that \(f\) and \(g\) satisfy (3.17). Also since \(f\) and \(g\) are uniformly continuous and bounded on \(\mathbb{R}^n \times \mathbb{R}\) and

\[
\int_a^b \int_{\mathbb{R}^n} \Phi_\alpha(x, t) \, dx \, dt = \frac{1}{\Gamma(\alpha + 1)} (b^\alpha - a^\alpha) \quad \text{for} \quad a < b
\]

we easily check that (3.16) holds.

Finally, from (5.4) we see for \(0 < t < 1\) that

\[
f(0, t) = a_1 F(t) \geq \left( \frac{N_1}{M_1} \right)^{\frac{\lambda\sigma}{1 - \lambda\sigma}} M_1^{\lambda\sigma} t^{\gamma_1} = N_1^{\lambda\sigma} t^{\gamma_1}
\]

and

\[
g(0, t) = a_2 G(t) \geq \left( \frac{N_2}{M_2} \right)^{\frac{\lambda\sigma}{1 - \lambda\sigma}} M_2^{\lambda\sigma} t^{\gamma_2} = N_2^{\lambda\sigma} t^{\gamma_2}
\]

and consequently we obtain (3.19) and (3.20). Hence (3.21) and (3.22) follow from (3.5), (3.6). \(\square\)

**Proof of Theorem 3.4** By Remark 4.2 with \(T = 1\) we can assume \(K_1 = K_2 = 1\). Define \(\tilde{f}, \tilde{g} : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)\) by

\[
\tilde{f}(x, t) = F(t) \chi_\Omega(x, t) \quad \text{and} \quad \tilde{g}(x, t) = G(t) \chi_\Omega(x, t)
\]  

(5.13)

where \(F\) and \(G\) are defined in Remark 4.1 and \(\Omega = \{|x|^2 < t\}\). Then using Lemma A.3 and the fact that \(\alpha + \gamma_1 = \gamma_2 / \sigma\) we obtain for \(|x|^2 < t\) that

\[
J_\alpha \tilde{f}(x, t) = \int_0^t \frac{(t - \tau)^{\alpha - 1}}{\Gamma(\alpha)} \left( \int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau) \, d\xi \right) F(\tau) \, d\tau \\
\geq C \int_{t/4}^{3t/4} (t - \tau)^{\alpha - 1} \tau^{\gamma_1} \, d\tau \\
= Ct^{\alpha + \gamma_1} = Ct^{\gamma_2 / \sigma} \\
= CG(t)^{1 / \sigma} = C \tilde{g}(x, t)^{1 / \sigma}
\]
which also holds in $\mathbb{R}^n \times \mathbb{R} \setminus \Omega$ because $\bar{g} = 0$ there. Similarly

$$J_{\beta\bar{g}}(x,t) \geq C\bar{f}(x,t)^{1/\lambda} \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}.$$ 

Thus letting

$$f = L_1\bar{f} \quad \text{and} \quad g = L_2\bar{g} \quad (5.14)$$ 

for appropriately chosen positive constants $L_1$ and $L_2$, the function $f$ and $g$ will satisfy (3.4)–(3.6).

It follows from (5.13), (5.14) and the definition of $F$ and $G$ in Remark 4.1 that there exists $N > 0$ such that (3.23) holds in $\Omega$. Thus, since $f$ and $g$ solve (3.5) we obtain (3.24), provided we decrease $N$ if necessary.

**Remark 5.1.** Suppose (3.7), (3.8), (3.9), and (3.25) hold. We will need for the proof of Theorems 3.5 and 3.6 some observations concerning the graphs of the straight lines in the $\xi\eta$-plane given by

$$\xi = (\eta - \beta)\lambda \quad \text{and} \quad \eta = (\xi - \alpha)\sigma. \quad (5.15)$$ 

These lines intersect the vertical line $\xi = \frac{n+2}{2p}$ at

$$P_2 = \left(\frac{n+2}{2p}, \eta_2\right) \quad \text{and} \quad P_3 = \left(\frac{n+2}{2p}, \eta_3\right),$$

respectively, where

$$\eta_2 = \beta + \frac{n+2}{2p\lambda} \quad \text{and} \quad \eta_3 = \left(\frac{n+2}{2p} - \alpha\right)\sigma = \frac{(n+2)\sigma}{2q\sigma_0} \quad (5.16)$$

by (2.12). Thus

$$\eta_3 < (=, >)\frac{n+2}{2q} \quad \text{if} \quad \sigma < (=, >)\sigma_0.$$ 

Moreover, it follows from (5.16) and (3.25) that

$$\eta_2 = \frac{\mu(\lambda)}{\sigma} \eta_3 < \eta_3 \quad (5.17)$$

and it follows from (3.8) and (3.25) (see Figure 2.1) that $\lambda > \lambda_0$ where $\lambda_0$ is defined in (2.12). Thus by (5.16)

$$\eta_2 < \beta + \frac{n+2}{2p\lambda_0} = \frac{n+2}{2q}. \quad (5.18)$$

The lines (5.15) are graphed in Figures 5.1a, 5.1b, and 5.1c when $\sigma < \sigma_0, \sigma = \sigma_0$, and $\sigma > \sigma_0$ respectively.

**Proof of Theorem 3.5.** Since $|R_j| < \infty$ for $j = 1, 2, \ldots$, to prove Theorem 3.5 it suffices to show for each $\varepsilon > 0$ there exist

$$r \in (p, p + \varepsilon) \quad \text{and} \quad s \in (s_0, s_0 + \varepsilon) \quad (5.19)$$

such that the conclusion of Theorem 3.5 holds.

We will use the notation and observations in Remark 5.1. Let $P_0 = (\xi_0, \eta_0)$ where

$$\xi_0 = \frac{n+2}{2p} \quad \text{and} \quad \eta_0 = \begin{cases} \eta_3 & \text{if} \quad \sigma < \sigma_0 \\ \frac{n+2}{2q} & \text{if} \quad \sigma \geq \sigma_0. \end{cases}$$
Then
\[ \eta_0 = \frac{n+2}{2s_0} \quad \text{and} \quad P_0 = \left\{ \begin{array}{ll} P_3 & \text{if } \sigma < \sigma_0 \\ \left( \frac{n+2}{2p}, \frac{n+2}{2q} \right) & \text{if } \sigma \geq \sigma_0 \end{array} \right. \]

The point \( P_0 \) is graphed in Figure 5.1. It follows from Figure 5.1 that there exist points \( P_1(x_1, \eta_1) \) in the open shaded region arbitrarily close to \( P_0 \). More precisely, fixing \( \varepsilon > 0 \), there exist \( x_1 \in (0, x_0) \) and \( \eta_1 \in (0, \eta_0) \) such that
\[ \xi_1 < (\eta_1 - \beta) \lambda \quad \text{and} \quad \eta_1 < (\xi_1 - \alpha) \sigma \] (5.20)
and
\[ \frac{n+2}{2(p + \varepsilon)} < \xi_1 < \xi_0 = \frac{n+2}{2p} \quad \text{and} \quad \frac{n+2}{2(s_0 + \varepsilon)} < \eta_1 < \eta_0 = \frac{n+2}{2s_0} \leq \frac{n+2}{2q}. \] (5.21)

Thus defining \( r \) and \( s \) by
\[ \frac{n+2}{2r} = \xi_1 \quad \text{and} \quad \frac{n+2}{2s} = \eta_1 \] (5.22)
we have \( r \) and \( s \) satisfy (5.19).

Define \( f_0, g_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[ f_0(x, t) = \left( \frac{1}{t} \right)^{\xi_1} \chi_{\Omega_0}(x, t) \quad \text{and} \quad g_0(x, t) = \left( \frac{1}{t} \right)^{\eta_1} \chi_{\Omega_0}(x, t) \] (5.23)
where
\[ \Omega_0 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t < 1 \}. \]

Then by (5.21) and Lemma A.5 we have
\[ f_0 \in L^p(\mathbb{R}^n \times \mathbb{R}) \quad \text{and} \quad g_0 \in L^q(\mathbb{R}^n \times \mathbb{R}) \] (5.24)
and for \( (x, t) \in \Omega_0 \)
\[ J_{\alpha} f_0(x, t) \geq C \left( \frac{1}{t} \right)^{\xi_1 - \alpha} \quad \text{and} \quad J_{\beta} g_0(x, t) \geq C \left( \frac{1}{t} \right)^{\eta_1 - \beta} \] (5.25)
where \( C = (n, \lambda, \sigma, \alpha, \beta, p, q, r, s) > 0. \)

Let \( \{T_j\} \subset (0, 1/2) \) be a sequence such that
\[ T_{j+1} < T_j/4, \quad j = 1, 2, \ldots \]
and define \( t_j = T_j / 2 \).

Then
\[
\Omega_j := \{ (y, s) \in \mathbb{R}^n \times \mathbb{R} : |y| < \sqrt{T_j - s} \text{ and } t_j < s < T_j \} \subset R_j \subset \Omega_0. 
\]

(5.27)

Defining \( f_j, g_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[
f_j(x, t) = \left( \frac{1}{T_j - t} \right)^{\xi_1} \chi_{\Omega_j}(x, t) \quad \text{and} \quad g_j(x, t) = \left( \frac{1}{T_j - t} \right)^{\eta} \chi_{\Omega_j}(x, t)
\]
we obtain from (5.21), (5.22), (5.27), and Lemma A.6 that
\[
\| f_j \|_{L^p} = C(n) \int_{T_j - t_j}^{T_j - t_j} \zeta^{\left( \frac{n+2}{2p} - \xi_1 \right)p - 1} d\zeta \to 0 \quad \text{as } j \to \infty,
\]
\[
\| g_j \|_{L^q} = C(n) \int_{T_j - t_j}^{T_j - t_j} \zeta^{\left( \frac{n+2}{2q} - \eta_1 \right)q - 1} d\zeta \to 0 \quad \text{as } j \to \infty,
\]
\[
\| f_j \|_{L^r(R_j)} = \| g_j \|_{L^s(R_j)} = \infty \quad \text{for } j = 1, 2, \ldots
\]
and for \( (x, t) \in \Omega_j^+ := \{ (x, t) \in \Omega_j : \frac{3T_j}{4} < t < T_j \} \) that
\[
J_\alpha f_j(x, t) \geq C \left( \frac{1}{T_j - t} \right)^{\xi_1 - \alpha} \quad \text{and} \quad J_\beta g_j(x, t) \geq C \left( \frac{1}{T_j - t} \right)^{\eta - \beta}.
\]

(5.31)

It follows from (5.23) and (5.25) that for \( (x, t) \in \Omega_0 \) we have
\[
\frac{f_0(x, t)}{(J_\beta g_0(x, t))^{\lambda}} \leq C \lambda^{(n - \lambda - \xi_1)} > 0
\]
and
\[
\frac{g_0(x, t)}{(J_\alpha f_0(x, t))^{\sigma}} \leq C \sigma^{(\xi_1 - \sigma - \eta_1)} > 0.
\]

(5.32)

Thus by (5.20) and (5.27) we find that
\[
\sup_{\Omega_0} \frac{f_0}{(J_\beta g_0)^{\lambda}} \leq C, \quad \sup_{\Omega_0} \frac{g_0}{(J_\alpha f_0)^{\sigma}} \leq C
\]
and
\[
\sup_{\Omega_j} \frac{f_0}{(J_\beta g_0)^{\lambda}} \leq 1, \quad \sup_{\Omega_j} \frac{g_0}{(J_\alpha f_0)^{\sigma}} \leq 1
\]
by taking a subsequence.

Using (5.28), (5.31), (5.20) and taking a subsequence we obtain
\[
\sup_{\Omega_j^+} \frac{f_j}{(J_\beta g_j)^{\lambda}} \leq C \sup_{(x, t) \in \Omega_j^+} \frac{(T_j - t)^{\eta - \beta - \xi_1}}{(T_j - t_j)^{\eta - \beta - \xi_1} < 1}
\]
\[
\leq C(T_j - t_j)^{\eta - \beta - \xi_1} < 1
\]
(5.34)

and similarly
\[
\sup_{\Omega_j^+} \frac{g_j}{(J_\alpha f_j)^{\sigma}} \leq C(T_j - t_j)^{\xi_1 - \sigma - \eta_1} < 1.
\]

(5.35)
It follows from (5.23), (5.28), (5.27), and (5.26) that

$$\sup_{\Omega_j} f_j = \sup_{(x,t) \in \Omega_j} \frac{(T_j - t)^{t \xi_1}}{t^{t \xi_1}} \leq 1$$  \hspace{1cm} (5.36)

and

$$\sup_{\Omega_j} g_j = \sup_{(x,t) \in \Omega_j} \frac{(T_j - t)^{t \eta_1}}{t^{t \eta_1}} \leq 1$$  \hspace{1cm} (5.37)

and letting $\Omega_j^- = \Omega_j \setminus \Omega_j^+$, using (5.25), (5.28), (5.27), (5.26) and (5.20) and taking a subsequence we obtain

$$\sup_{\Omega_j^-} f_j \leq C (2t_j)^{(\eta_1 - \beta) \lambda} \leq C \left( \frac{t_j}{2} \right)^{(\eta_1 - \beta) \lambda} \leq \frac{1}{2}$$  \hspace{1cm} (5.38)

and similarly

$$\sup_{\Omega_j^-} g_j \leq C t_j^{(\xi_1 - \alpha) \sigma} \leq \frac{1}{2}.$$  \hspace{1cm} (5.39)

Taking an appropriate subsequence of $(f_j, g_j)$ and letting

$$f = f_0 + \sum_{j=1}^{\infty} f_j \quad \text{and} \quad g = g_0 + \sum_{j=1}^{\infty} g_j$$

we see from (5.24) and (5.29) that (3.26) holds. In $\Omega_j^+$ we have by (5.33) and (5.34) that

$$f = f_0 + f_j \leq (J_\beta g_0)^{\lambda} + (J_\beta g_j)^{\lambda}$$

$$\leq C(\lambda)(J_\beta (g_0 + g_j))^\lambda \leq C(\lambda)(J_\beta g)^{\lambda}$$

and similarly by (5.33) and (5.35) that $g \leq C(\sigma)(J_\alpha f)^{\sigma}$.

In $\Omega_j^-$ we have by (5.39) and (5.38) that

$$f = f_0 + f_j \leq 2f_j \leq (J_\beta g_0)^{\lambda} \leq (J_\beta g)^{\lambda}$$

and similarly by (5.37) and (5.39) that $g \leq (J_\alpha f)^{\sigma}$. In $\Omega_0 \setminus \bigcup_{j=1}^{\infty} \Omega_j$ we have by (5.32) that

$$f = f_0 \leq C(J_\beta g_0)^{\lambda} \leq C(J_\beta g)^{\lambda}$$

and

$$g = g_0 \leq C(J_\alpha f_0)^{\sigma} \leq C(J_\alpha f)^{\sigma}.$$  \hspace{1cm}

In $\mathbb{R}^n \times \mathbb{R} \setminus \Omega_0$, $f = 0 \leq (J_\beta g)^{\lambda}$ and $g = 0 \leq (J_\alpha f)^{\sigma}$. Thus, after scaling $f$ and $g$ we see that $f$ and $g$ are solutions of (3.5) and (3.6). Also (3.27) holds by (5.30).

**Proof of Theorem 5.6.** We will use the notation and observations in Remark 5.1. Let $P_4 = (\xi_4, \eta_4)$ be the point where the lines (5.15) intersect. It follows from (5.17) and (5.18) (see Figure 5.1) that

$$0 < \xi_4 < \frac{n + 2}{2p} \quad \text{and} \quad 0 < \eta_4 < \frac{n + 2}{2q}.$$  \hspace{1cm} (5.40)
Thus, since solving the system (5.15) yields
\[ \xi_4 = \frac{n + 2}{2r_0} \quad \text{and} \quad \eta_4 = \frac{n + 2}{2s_0}, \]
we see that \( r_0 > p \) and \( s_0 > q \).

Define \( f_0, g_0 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[ f_0(x,t) = \left( \frac{1}{t} \right)^{\xi_4} \chi_{\Omega_0}(x,t) \quad \text{and} \quad g_0(x,t) = \left( \frac{1}{t} \right)^{\eta_4} \chi_{\Omega_0}(x,t) \]
where \( \Omega_0 \) is as in Lemma A.5. Then by (5.40), Lemma A.5 and the fact that \( P_4 \) satisfies (5.15) we have
\[ f_0 \in X^p, \quad g_0 \in X^q \]
and
\[ f_0 \leq C(J_\beta g_0)^\lambda \quad \text{and} \quad g_0 \leq C(J_\alpha f_0)^\sigma \quad \text{in} \ \mathbb{R}^n \times \mathbb{R} \]
where in this proof \( C \) is a positive constant depending on \( n, \lambda, \sigma, \alpha, \beta, p, q \), whose values may change from line to line.

Let \( \{ T_j \}, \{ t_j \} \subset (2, \infty) \) satisfy \( T_{j+1} > 4T_j \) and \( T_j = 2t_j \) and define \( f_j, g_j : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) by
\[ f_j(x,t) = \left( \frac{1}{T_j - t} \right)^{\xi_4} \chi_{\Omega_j}(x,t) \quad \text{and} \quad g_j(x,t) = \left( \frac{1}{T_j - t} \right)^{\eta_4} \chi_{\Omega_j}(x,t) \]
where
\[ \Omega_j = \{(x,t) \in \mathbb{R}^n \times (t_j, T_j) : |x| < \sqrt{T_j - t} \}. \]
Then
\[ \Omega_j \subset R_j \subset \Omega_0, \quad \Omega_j \cap \Omega_k = \emptyset \quad \text{for} \ j \neq k \]
\[ \inf\{ t : (x,t) \in \Omega_j \} = t_j \to \infty \quad \text{as} \ j \to \infty, \]
and by (5.44), (5.40), Lemma A.6 and the fact that \( P_4 \) satisfies (5.15) we have
\[ f_j \in L^p(\mathbb{R}^n \times \mathbb{R}), \quad g_j \in L^q(\mathbb{R}^n \times \mathbb{R}) \]
and
\[ f_j \leq C(J_\beta g_j)^\lambda \quad \text{and} \quad g_j \leq C(J_\alpha f_j)^\sigma \quad \text{in} \ \Omega_j^+ \]
where
\[ \Omega_j^+ = \{(x,t) \in \Omega_j : \frac{3T_j}{4} < t < T_j \}. \]
It follows therefore from (5.43) that
\[ f_0 + f_j \leq C((J_\beta g_0)^\lambda + (J_\beta g_j)^\lambda) \leq C(J_\beta (g_0 + g_j))^\lambda \quad \text{in} \ \Omega_j^+ \]
and similarly
\[ g_0 + g_j \leq C(J_\alpha (f_0 + f_j))^\sigma \quad \text{in} \ \Omega_j^+. \]
In \( \Omega_j^- = \Omega_j \setminus \Omega_j^+ \) we have
\[ \frac{f_j}{f_0} = \left( \frac{t}{T_j - t} \right)^{\xi_4} \leq \left( \frac{3T_j/4}{T_j/4} \right)^{\xi_4} = 3^{\xi_4} \]
and similarly \( \frac{g_j}{g_0} \leq 3^m. \)

Thus we obtain from (5.43) that
\[
f_0 + f_j \leq C f_0 \leq C (J_\beta g_0)^\lambda \leq C (J_\beta (g_0 + g_j))^\lambda \text{ in } \Omega_j^-
\]
and similarly that
\[
g_0 + g_j \leq C (J_\alpha (f_0 + f_j))^\sigma \text{ in } \Omega_j^-.
\]

Let
\[
f = f_0 + \sum_{j=1}^{\infty} f_j \quad \text{and} \quad g = g_0 + \sum_{j=1}^{\infty} g_j.
\]

Then clearly \( f \) and \( g \) satisfy (3.6) and by (5.42), (5.47), and (5.46) we see that \( f \) and \( g \) satisfy (3.30).

In \( \Omega_j \) we have by (5.45), (5.48), (5.49), (5.50), and (5.51) that
\[
f = f_0 + f_j \leq C (J_\beta (g_0 + g_j))^\lambda \leq C (J_\beta g)^\lambda
\]
and
\[
g_0 + g_j \leq C (J_\alpha (f_0 + f_j))^\sigma \leq C (J_\alpha f)^\sigma
\]
and in \((\mathbb{R}^n \times \mathbb{R}) \setminus \bigcup_{j=1}^{\infty} \Omega_j\) we have by (5.43) that
\[
f = f_0 \leq C (J_\beta g_0)^\lambda \leq C (J_\beta g)^\lambda
\]
and
\[
g = g_0 \leq C (J_\alpha f_0)^\sigma \leq C (J_\alpha f)^\sigma.
\]

Thus after scaling \( f \) and \( g \), we find that \( f \) and \( g \) satisfy (3.5).

From (5.41), (5.44), (5.45), and Lemma A.6 we find
\[
\|f\|_{L^r(\Omega)} \geq \|f_j\|_{L^r(\Omega)} = \infty \quad \text{for } j = 1, 2, \ldots
\]
and
\[
\|g\|_{L^s(\Omega)} \geq \|g_j\|_{L^s(\Omega)} = \infty \quad \text{for } j = 1, 2, \ldots.
\]

Thus, since \(|R_j| < \infty\), we have (3.27) holds for all \( r \) and \( s \) satisfying (3.29).

\[\square\]

A Auxiliary lemmas

In this appendix we provide some lemmas needed for the proofs of our results in Section 3 dealing with solutions of the \( J_\alpha \) problem (3.4)–(3.7). See [24, Section 7] for the proofs of these lemmas.

Let \( \Omega = \mathbb{R}^n \times (a, b) \) where \( n \geq 1 \) and \( a < b \). The following two lemmas give estimates for the convolution
\[
(V_{\alpha, \Omega} f)(x, t) = \int\int_{\Omega} \Phi_\alpha(x - \xi, t - \tau)f(\xi, \tau)\,d\xi\,d\tau
\]
where \( \alpha > 0 \) and \( \Phi_\alpha \) is defined in (1.9).

**Remark A.1.** Note that if \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is a nonnegative measurable function such that \( \|f\|_{L^\infty(\mathbb{R}^n \times \mathbb{R})} = 0 \) then
\[
V_{\alpha, \Omega} f = J_\alpha f \quad \text{in } \Omega := \mathbb{R}^n \times (a, b).
\]

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Lemma A.1. For $\alpha > 0$, $\Omega = \mathbb{R}^n \times (a, b)$ and $f \in L^\infty(\Omega)$ we have
\[ \|V_{\alpha, \Omega} f\|_{L^\infty(\Omega)} \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} \|f\|_{L^\infty(\Omega)}. \]

Lemma A.2. Let $p, q \in [1, \infty]$, $\alpha$, and $\delta$ satisfy
\begin{equation}
0 \leq \delta := \frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{n+2} < 1. \tag{A.2}
\end{equation}
Then $V_{\alpha, \Omega}$ maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ and for $f \in L^p(\Omega)$ we have
\[ \|V_{\alpha, \Omega} f\|_{L^q(\Omega)} \leq M \|f\|_{L^p(\Omega)} \]
where
\[ M = C(b-a)^{2\alpha-(n+2)\delta} \]
for some constant $C = C(n, \alpha, \delta)$.

Lemma A.3. Suppose $x \in \mathbb{R}^n$ and $t, \tau \in (0, \infty)$ satisfy
\begin{equation}
|x|^2 < t \quad \text{and} \quad \frac{t}{4} < \tau < \frac{3t}{4}. \tag{A.3}
\end{equation}
Then
\[ \int_{|\xi|^2 < \tau} \Phi_1(x-\xi, t-\tau) d\xi \geq C(n) > 0 \]
where $\Phi_\alpha$ is defined by (1.9).

Lemma A.4. For $\tau < t \leq T$ and $|x| \leq \sqrt{T-t}$ we have
\[ \int_{|\xi| < \sqrt{T-\tau}} \Phi_1(x-\xi, t-\tau) d\xi \geq C \]
where $C = C(n)$ is a positive constant.

Lemma A.5. Suppose $\alpha > 0$, $\gamma > 0$, $p \geq 1$, and
\[ f_0(x, t) = \left(\frac{1}{t}\right)^{\frac{\alpha+2}{2p} - \gamma} \chi_{\Omega_0}(x, t) \quad \text{where} \quad \Omega_0 = \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x|^2 < t\}. \]
Then $f_0 \in X^p$ and
\[ C_1 \left(\frac{1}{t}\right)^{\frac{\alpha+2}{2p} - \gamma - \alpha} \leq J_\alpha f_0(x, t) \leq C_2 \left(\frac{1}{t}\right)^{\frac{\alpha+2}{2p} - \gamma - \alpha} \quad \text{for} \ (x, t) \in \Omega_0 \]
where $C_1$ and $C_2$ are positive constants depending only on $n, \alpha, \gamma$, and $p$.

Lemma A.6. Suppose $\alpha > 0$, $\gamma \in \mathbb{R}$, $0 \leq t_0 < T$, $p \in [1, \infty)$, and
\[ f(x, t) = \left(\frac{1}{T-t}\right)^{\frac{\alpha+2}{2p} - \gamma} \chi_{\Omega}(x, t) \]
where
\[ \Omega = \{(x, t) \in \mathbb{R}^n \times (t_0, T) : |x| < \sqrt{T-t}\}. \]
Then
\[ J_\alpha f(x,t) \geq C \left( \frac{1}{T-t} \right)^{\frac{n+2}{2p} - \gamma - \alpha} \]
for \((x,t) \in \Omega^+ := \{(x,t) \in \Omega : \frac{T-t_0}{2} < t < T\}\) where \(C = C(n, \alpha, \gamma, p) > 0\). Moreover,
\[ f \in L^p(\mathbb{R}^n \times \mathbb{R}) \text{ if and only if } \gamma > 0 \quad (A.4) \]
and in this case
\[ \|f\|_{L^p(\mathbb{R}^n \times \mathbb{R})}^p = C(n) \int_0^{T-t_0} s^{\gamma p-1} ds. \quad (A.5) \]

References

[1] B. Abdellaoui, A. Attar, R. Bentifour, I. Peral, On fractional p-Laplacian parabolic problem with general data, Ann. Mat. Pura Appl. (4) 197 (2018) 329–356.

[2] E. Affili, E. Valdinoci, Decay estimates for evolution equations with classical and fractional time-derivatives, J. Differential Equations, https://doi.org/10.1016/j.jde.2018.09.031

[3] Boumediene Abdellaoui, Maria Medina, Ireneo Peral, Ana Primo, Optimal results for the fractional heat equation involving the Hardy potential, Nonlinear Anal. 140 (2016) 166–207.

[4] Mark Allen, A nondivergence parabolic problem with a fractional time derivative, Differential Integral Equations 31 (2018) 215–230.

[5] Mark Allen, Luis Caffarelli, Alexis Vasseur, A parabolic problem with a fractional time derivative, Arch. Ration. Mech. Anal. 221 (2016) 603–630.

[6] Ioannis Athanasopoulos, Luis Caffarelli, Emmanouil Milakis, On the regularity of the non-dynamic parabolic fractional obstacle problem, J. Differential Equations 265 (2018) 2614–2647.

[7] Matteo Bonforte, Juan Luis Vázquez, A priori estimates for fractional nonlinear degenerate diffusion equations on bounded domains, Arch. Ration. Mech. Anal. 218 (2015) 317–362.

[8] Huyuan Chen, Laurent Véron, Ying Wang, Fractional heat equations with subcritical absorption having a measure as initial data, Nonlinear Anal. 137 (2016) 306–337.

[9] Matías G. Delgado, Scott Smith, Hölder estimates for fractional parabolic equations with critical divergence free drifts, Ann. Inst. H. Poincaré Anal. Non Linéaire 35 (2018) 577–604.

[10] S. Dipierro, E. Valdinoci, V. Vespri, Decay estimates for evolutionary equations with fractional time-diffusion, J. Evol. Equ. https://doi.org/10.1007/s00028-019-00482-z

[11] Giulia Furioli, Tatsuki Kawakami, Bernhard Ruf, Elide Terraneo, Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity, J. Differential Equations 262 (2017) 145–180.

[12] Ciprian G. Gal, Mahamadi Warma, On some degenerate non-local parabolic equation associated with the fractional p-Laplacian, Dyn. Partial Differ. Equ. 14 (2017) 47–77.

[13] Mohamed Jleli, Bessem Samet, The decay of mass for a nonlinear fractional reaction-diffusion equation, Math. Methods Appl. Sci. 38 (2015) 1369–1378.
[14] Jan Kadlec, Solution of the first boundary value problem for a generalization of the heat equation in classes of functions possessing a fractional derivative with respect to the time-variable, (Russian) Czechoslovak Math. J. 16 (91) (1966) 91–113.

[15] Jukka Kemppainen, Juhana Siljander, Vicente Vergara, Rico Zacher, Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$. Math. Ann. 366 (2016) 941–979

[16] M. Mirzazadeh, Analytical study of solitons to nonlinear time fractional parabolic equations. Nonlinear Dynam. 85 (2016) 2569–2576.

[17] Luc Molinet, Slim Tayachi, Remarks on the Cauchy problem for the one-dimensional quadratic (fractional) heat equation, J. Funct. Anal. 269 (2015) 2305–2327.

[18] K. Nyström, O. Sande, Extension properties and boundary estimates for a fractional heat operator, Nonlinear Anal. 140 (2016) 29–37.

[19] Ebru Ozbilge, Ali Demir, Identification of unknown coefficient in time fractional parabolic equation with mixed boundary conditions via semigroup approach, Dynam. Systems Appl. 24 (2015) 341–348.

[20] Fabio Punzo, Enrico Valdinoci, Uniqueness in weighted Lebesgue spaces for a class of fractional parabolic and elliptic equations, J. Differential Equations 258 (2015) 555–587.

[21] Stefan G. Samko, Hypersingular integrals and their applications, Analytical Methods and Special Functions, 5, Taylor & Francis, Ltd., London, 2002.

[22] Pablo Raúl Stinga, José L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, SIAM J. Math. Anal. 49 (2017) 3893–3924.

[23] Fuqin Sun, Peihu Shi, Global existence and non-existence for a higher-order parabolic equation with time-fractional term, Nonlinear Anal. 75 (2012) 4145–4155.

[24] S. Taliabfferro, Pointwise bounds and blow-up for nonlinear fractional parabolic inequalities, J. Math. Pures Appl., in press, arXiv:1901.09964 [math.AP].

[25] Vladimir Varlamov, Long-time asymptotics for the nonlinear heat equation with a fractional Laplacian in a ball, Studia Math. 142 (2000) 71–99.

[26] Juan Luis Vázquez, Bruno Volzone, Symmetrization for linear and nonlinear fractional parabolic equations of porous medium type, J. Math. Pures Appl. (9) 101 (2014) 553–582.

[27] Juan Luis Vázquez, Arturo de Pablo, Fernando Quirós, Ana Rodríguez, Classical solutions and higher regularity for nonlinear fractional diffusion equations, J. Eur. Math. Soc. 19 (2017) 1949–1975.

[28] Vicente Vergara, Rico Zacher, Optimal decay estimates for time-fractional and other non-local subdiffusion equations via energy methods, SIAM J. Math. Anal. 47 (2015) 210–239.

[29] Quan-Guo Zhang, Hong-Rui Sun, The blow-up and global existence of solutions of Cauchy problems for a time fractional diffusion equation, Topol. Methods Nonlinear Anal. 46 (2015) 69–92.