TWO-TRACK CATEGORIES

DAVID BLANC AND SIMONA PAOLI

Abstract. We describe a 2-dimensional analogue of track categories, called two-track categories, and show that it can be used to model categories enriched in 2-type mapping spaces. We also define a Baues-Wirsching type cohomology theory for track categories, and explain how it can be used to classify two-track extensions of a track category $D$ by a module over $D$.

Introduction

In [29], Dwyer and Kan showed that any model category $\mathcal{E}$ can be provided with simplicial function complexes, in such a way that the resulting simplicially enriched category $X_*$ encodes the homotopy theory of $\mathcal{E}$. As in the case of individual topological spaces, it is often useful to approximate $X_*$ by its Postnikov sections $P^nX_*$ for $n \geq 0$, obtained by applying the $n$-th Postnikov functor to each mapping space of $X_*$. Hence $P^nX_*$ is a category “enriched in $n$-types” – that is, in simplicial sets whose homotopy groups vanish in dimensions $> n$. Consecutive Postnikov sections are related as usual via their $k$-invariants, which take value in certain $(\mathcal{S}, \mathcal{O})$-cohomology groups (see [30]).

It is convenient to have algebraic models for $P^nX_*$, in particular if they allow for an explicit description of the (systems of) homotopy groups (see §5.9), Postnikov towers, and $k$-invariants for $X_*$. For instance, in the case $n = 1$ the fundamental groupoid $\hat{\pi}_1Y$ of an individual simplicial set (or topological space) $Y$ provides an algebraic model the 1-type of $Y$ – that is, the homotopy type of $P^1Y$. If we use Kan’s version of the Postnikov section, so that both functors $\hat{\pi}_1$ and $P^1$ strictly commute with products, they extend to simplicially enriched categories. Moreover, the nerve functor from groupoids to simplicial sets lands in 1-types, and is also monoidal, so it extends to the enriched setting, providing an inverse up to homotopy to $\hat{\pi}_1$. As a result, categories enriched in groupoids, called track categories, provide an algebraic model for $P^1X_*$, up to homotopy.

Ideally, such an algebraic description would allow one to better understand the homotopy theory of the original model category $\mathcal{E}$ as a whole – e.g., by providing an explicit calculus of higher homotopy (or cohomology) operations. For example, let $\mathcal{E}_Y$ consist of a single space $Y$ together with all maps from $Y$ into finite products of mod-$p$ Eilenberg-Mac Lane spaces, and maps between them. Its homotopy category $\text{ho} \mathcal{E}_Y$ thus encodes the mod $p$ cohomology of $Y$ as an algebra over the Steenrod algebra. If $X_*$ is the corresponding simplicially enriched category, its track category $\hat{\pi}_1X_*$ records in addition all secondary mod-$p$ cohomology operations on $Y$. Moreover, in this case $\hat{\pi}_1X_*$ can be described as a “linear track extension” of $\text{ho} \mathcal{E}_Y$ by a certain natural system $\mathcal{K}$ on $\text{ho} \mathcal{C}$.
DAVID BLANC AND SIMONA PAOLI

(see §5 below), and this extension is classified by a class in the third Baues-Wirsching cohomology group \( H^3_{BW}(\text{ho} C; \mathcal{E}_Y) \). This class may be identified in turn with the 0-th \( k \)-invariant for \( X_\bullet \) (see [4, Theorem 6.5]). Baues and his collaborators have shown how this description can be used in practice to elucidate the secondary structure of \( H^*(Y; \mathbb{F}_p) \) and make computations in the Adams spectral sequence (see [2, 8]).

0.1. The 2-dimensional case. In this paper we describe an approach to the 2-dimensional case, which both provides a setting for studying the “tertiary Steenrod algebra” in the spirit of Baue’s work, and indicates how one might be able to proceed to higher dimensions.

This approach involves two main steps:

(a) We first construct a functorial model of 2-types, which we call two-typical double groupoids.

Many algebraic structures have been shown to model 2-types of topological spaces, beginning (in the connected case) with the crossed modules of [17] and the double groupoids with connections of [22]. More general models include the homotopy double groupoids of [21], the homotopy bigroupoids of [37], the strict 2-groupoids of [48], and the weak 2-groupoids of [56].

The advantage of the two-typical double groupoids is the explicit description of the model associated to a Kan complex, its homotopy groups, and its Postnikov tower.

(b) More directly relevant to our present purpose is the fact that the resulting functor \( Q_t : \mathcal{S} \to \text{DbGpd}_t \) preserves products. Thus by applying \( Q_t \) to each mapping space of a simplicially enriched category \( X_\bullet \), we obtain a convenient model for “categories enriched in 2-types” – more precisely, for \((P^2 \mathcal{S}, \mathcal{O})\)-categories, which are categories \( \mathcal{Y}_\bullet \) enriched in simplicial spaces for which \( \text{map}_{\mathcal{Y}_\bullet}(a, b) \cong P^2 \text{map}_{\mathcal{Y}_\bullet}(a, b) \) for each \( a, b \in \mathcal{O} = \text{Obj } Y_\bullet \).

In the second part, we define a Baues-Wirsching type cohomology \( H^*_{BW}(\mathcal{D}; \mathcal{M}) \) for an (ordinary) track category \( \mathcal{D} \), with coefficients in an appropriate notion of a natural system \( \mathcal{M} \) on \( \mathcal{D} \). We then show how one can associate an “underlying homotopy track category” \( \mathcal{D} = \text{ho} \mathcal{G} \) to any two-track category \( \mathcal{G} \), as well as a natural system \( \mathcal{M} = \Pi_2 \mathcal{G} \) on \( \mathcal{D} \), and explain how \( \mathcal{G} \), thought of as an extension of \( \mathcal{D} \) by \( \mathcal{M} \), is classified by a naturally defined class \( \chi_\mathcal{G} \in H^4_{BW}(\mathcal{D}; \mathcal{M}) \). Finally, we show that this cohomology theory is naturally isomorphic (with a shift in dimension) to the \((\mathcal{S}, \mathcal{O})\)-cohomology of Dwyer and Kan for the corresponding simplicially enriched category \( X_\bullet = \mathcal{N} \mathcal{D} \), and that \( \chi_\mathcal{G} \) corresponds to the first \( k \)-invariant of \( X_\bullet \).

As with most new constructions of a known cohomology theory, one should view the cohomology theory \( H^*_{BW} \) mainly as an alternative approach to the computation of the \((\mathcal{S}, \mathcal{O})\)-cohomology of a \((P^2 \mathcal{S}, \mathcal{O})\)-category. Since it is very difficult to use the original definition of Dwyer and Kan to carry out explicit calculations, it is hoped that our definition will make it more accessible – in particular, to Baues-Jibladze type computations of the Adams spectral sequence (see [7, 8] and [5]).

See [3] for a different approach to the 2 (and higher \( n \))-dimensional cases.

0.2. Notation and conventions. For any category \( \mathcal{C} \), \( s\mathcal{C} := \mathcal{C}^{\Delta^{op}} \) is the category of simplicial objects over \( \mathcal{C} \). We abbreviate \( s\text{Set} \) to \( \mathcal{S} \). If \( \mathcal{C} \) is concrete, the \( n \)-skeleton
sk_nX_\bullet \in sC \ of any \ X_\bullet \in sC \ is \ generated \ under \ the \ degeneracy \ maps \ by \ X_0, \ldots, X_n.

The \ n\text{-}coskeleton \ functor \ csk_n : sC \to sC \ is \ left \ adjoint \ to \ sk_n.

We \ denote \ by \ S_{cf} \ the \ full \ subcategory \ of \ S \ consisting \ of \ Kan \ complexes \ – \ i.e., \ fibrant \ (and \ cofibrant) \ simplicial \ sets. \ For \ X \in S_{cf}, \ we \ can \ use \ csk_{n+1}X \ as \ a \ model \ for \ the \ n\text{-}th \ Postnikov \ section \ P^nX. \ For \ each \ n \geq 0, \ let \ P^nS \ denote \ the \ full \ subcategory \ of \ S \ consisting \ of \ simplicial \ sets \ X \ for \ which \ the \ natural \ map \ X \to P^nX \ is \ a \ weak \ equivalence \ (\text{i.e.,} \ \pi_i(X,x) = 0 \ for \ all \ x \in X \ and \ i > n). \ An \ n\text{-}type \ is \ (the \ homotopy \ equivalence \ class \ of) \ an \ object \ in \ P^nS.

For \ a \ bisimplicial \ set \ W_{\bullet,\bullet} \in sS = ssSet, \ we \ think \ of \ the \ first \ index \ as \ the \ \text{horizontal} \ direction \ and \ the \ second \ index \ as \ the \ \text{vertical} \ direction, \ and \ for \ each \ (fixed) \ n \geq 0 \ we \ write \ W_n^h \in S \ for \ the \ simplicial \ set \ with \ (W_n^h)_i := W_{n,i} \ for \ i \geq 0. \ Similarly, \ if \ f : W_{\bullet,\bullet} \to V_{\bullet,\bullet} \ is \ a \ map \ in \ sS, \ we \ write \ f_n^h \ for \ its \ restriction \ to \ W_n^h \in S.

For \ any \ n \geq 0, \ a \ map \ f : X_\bullet \to Y_\bullet \ in \ S \ is \ called \ an \ n\text{-}equivalence \ if \ it \ induces \ isomorphisms \ f_* : \pi_0X_\bullet \to \pi_0Y_\bullet \ \text{(of sets), \ and} \ f^# : \pi_i(X_\bullet, x) \to \pi_i(Y_\bullet, f(x)) \ \text{for every} \ 1 \leq i \leq n \ \text{and} \ x \in X_0.

Let \ Cat \ denote \ the \ category \ of \ small \ categories, \ and \ S_{pd} \ the \ full \ subcategory \ of \ groupoids. \ If \ \langle V, \otimes \rangle \ is \ a \ monoidal \ category, \ we \ denote \ by \ V\text{-}Cat \ the \ collection \ of \ all \ (not \ necessarily \ small) \ categories \ enriched \ over \ V \ (see [18, §6.2]). \ For \ any \ set \ \mathcal{O}, \ denote \ by \ \mathcal{O}\text{-}Cat \ the \ category \ of \ all \ small \ categories \ \mathcal{D} \ having \ Obj \ \mathcal{D} = \mathcal{O}, \ with \ functors \ which \ are \ the \ identity \ on \ objects \ as \ morphisms. \ A \ \langle V, \mathcal{O} \rangle\text{-}category \ is \ a \ category \ \mathcal{D} \in \mathcal{O}\text{-}Cat \ enriched \ over \ \mathcal{V}, \ with \ mapping \ objects \ map^\mathcal{D}_\mathcal{V}(-, -) \in \mathcal{V}. \ The \ category \ of \ all \ \langle V, \mathcal{O} \rangle\text{-}categories \ will \ be \ denoted \ by \ \langle V, \mathcal{O} \rangle\text{-}Cat.

The \ main \ examples \ of \ \langle V, \otimes \rangle \ to \ keep \ in \ mind \ are \ \langle Set, \times \rangle, \ \langle S_{pd}, \times \rangle, \ \langle \mathcal{S}, \otimes \rangle, \ \langle \mathcal{C}, \otimes \rangle, \ \text{and} \ \langle \mathcal{S}, \times \rangle, \ \text{and} \ \langle \mathcal{C}, \otimes \rangle, \ \text{where} \ \mathcal{C} \ is \ the \ category \ of \ cubical \ sets \ (see §7.5 \ below).

We \ obtain \ further \ variants \ by \ applying \ any \ (strictly) \ monoidal \ functor \ P : \langle V, \otimes \rangle \to \langle V', \otimes' \rangle \ to \ a \ \langle V, \mathcal{O} \rangle\text{-}category \ \mathcal{C}, \ \text{where} \ \text{by} \ \text{a} \ \text{slight} \ \text{abuse} \ \text{of} \ \text{notation} \ \text{we} \ \text{use} \ \text{the} \ \text{same} \ \text{name} \ \text{for} \ \text{the} \ \text{prolonged} \ \text{functor}. \ \text{For} \ \text{example}, \ \text{given} \ \text{an} \ \langle \mathcal{S}, \mathcal{O} \rangle\text{-}category \ \mathcal{X}_\bullet, \ \text{for} \ \text{each} \ n \geq 1 \ \text{we} \ \text{have} \ \text{a} \ \langle P^n\mathcal{S}, \mathcal{O} \rangle\text{-}category \ \mathcal{Y}_\bullet := P^n\mathcal{X}_\bullet, \ \text{in} \ \text{which} \ \text{each} \ \text{mapping} \ \text{space} \ \mathcal{Y}_\bullet(a,b) \ \text{is} \ \text{the} \ \text{n\text{-}th} \ \text{Postnikov} \ \text{section} \ P^n\mathcal{X}_\bullet(a,b). \ \text{More} \ \text{precisely}, \ \text{we} \ \text{use} \ \text{a} \ \text{functorial} \ \text{product}

0.3. Organization. Section[1] provides a review of groupoids and track categories. Section[2] discusses double groupoids, and in particular those which are two-typical (Definition[2.21]). In Section[3] we show that two-typical double groupoids model 2-types (Theorem[3.6]), and in Section[4] we use this notion to define two-track categories, and show their equivalence with \langle P^2\mathcal{S}, \mathcal{O} \rangle\text{-}categories, up to weak equivalence (Corollary[4.3]). An alternative model using Gray categories is given by Proposition[4.8].
Coefficient systems on track categories are defined in Section 5 and are used in Section 6 to define the cohomology of track categories (extending the Baues-Wirsching cohomology of small categories), and show its equivalence with the \((\mathcal{S}, \mathcal{O})\)-cohomology of Dwyer and Kan (Theorem 7.21). Finally, in Section 8 we discuss two-track extensions of track categories and show how they are classified by a suitable Baues-Wirsching type cohomology class, which may be identified with the first \(k\)-invariant of the corresponding \((P^2\mathcal{S}, \mathcal{O})\)-category (Theorem 8.10).

0.4. Acknowledgements. We would like to thank Hans Baues and the referee for many useful comments and corrections. This research was supported by BSF grant 2006039.

1. Groupoids and track categories

We first recall some standard definitions and facts about groupoids and track categories.

1.1. Definition. Recall that a groupoid is a small category \(G\) in which all morphisms are isomorphisms. As for any category, it can be described by a diagram of sets:

\[
\begin{array}{ccc}
G_1 \times_{G_0} G_1 & \overset{c}{\longrightarrow} & G_1 \\
\downarrow{\scriptscriptstyle d_0} & & \downarrow{\scriptscriptstyle t} \\
G_1 & \overset{i}{\longrightarrow} & G_0
\end{array}
\]

where \(G_0\) is the set of objects of \(G\) and \(G_1\) the set of arrows. Here \(s\) and \(t\) are the source and target functions, \(i\) associates to an object its identity map, \(d_0\) and \(d_2\) are the respective projections, with “inverses” \(s_0\) and \(s_1\), and \(c\) is the composition, satisfying the appropriate identities.

We can think of (1.2) as the 2-skeleton of a simplicial set (with \(G_2 := G_1 \times_{G_0} G_1\), \(d_1 = c : G_2 \to G_1\), and so on). The nerve functor \(N : \mathcal{Gpd} \to \mathcal{S}\) (cf. [53]) assigns to \(G\) the corresponding 2-coskeletal simplicial set \(NG\), so

\[
(NG)_n := \underbrace{G_1 \times_{G_0} G_1 \times_{G_0} \ldots \times_{G_0} G_1}_{n}
\]

for all \(n \geq 2\), with face maps determined by the associativity of the composition \(c\).

Given a groupoid \(G\) as above, taking the coequalizer of \(s\) and \(t\) in (1.2) yields the set \(\pi_0 G\) (which may be identified with the usual set \(\pi_0 NG\) of path components of the nerve – that is, the coequalizer of \(d_0, d_1 : NG_0 \to NG_1\)).

Note that the usual cartesian product yields a monoidal structure on the category \(\mathcal{Gpd}\) of groupoids (with the trivial groupoid on one object as the unit).

1.4. Definition. A track category is a (small) category enriched in groupoids, and a \((\mathcal{Gpd}, \mathcal{O})\)-category is a track category with object set \(\mathcal{O}\). We may identify the category \((\mathcal{Gpd}, \mathcal{O})\)-Cat of all such track categories with \(\mathcal{Gpd}(\mathcal{O}\text{-Cat})\), the category of internal groupoids in \(\mathcal{O}\text{-Cat}\).

We use the notation \(\mathcal{D} = (\mathcal{D}_1 \rightarrow \mathcal{D}_0)\) to indicate that the track category \(\mathcal{D}\) has \(\mathcal{D}_0\) as its category of 0-cells (objects \(a, b \in \mathcal{O}\)) and 1-cells (maps \(f : a \to b\)), and \(\mathcal{D}_1\) as its category of 0-cells and 2-cells (or tracks \(\xi : f \Rightarrow g\)). For fixed \(a, b \in \mathcal{O} := \text{Obj} \mathcal{D}\), we denote by \(\xi \oplus \zeta\) the vertical (internal) composition of tracks \(f \overset{\xi}{\Rightarrow} g \overset{\zeta}{\Rightarrow} h\) in the groupoid \(\mathcal{D}(a, b)\). The homotopy category of \(\mathcal{D}\) in \(\mathcal{O}\text{-Cat}\), denoted by \(\Pi_0 \mathcal{D}\) or \(\text{ho} \mathcal{D}\),
is obtained by applying \( \pi_0 \) to each groupoid \( D(a,b) \). This has equivalence classes of 1-cells (with respect to the 2-cells, which are all invertible) as morphisms.

For a groupoid \( G \), we let \( G^\delta \) denote the semi-discrete groupoid with the same objects as \( G \), with \( G^\delta(a,a) = G(a,a) \) for each \( a \in \text{Obj} \ G \), and \( G^\delta(a,b) = \emptyset \) for \( a \neq b \) (i.e., a disjoint union of groups). This notation extends to track categories. Given a category \( \mathcal{E} \in \mathcal{O}\text{-Cat} \), a \((\mathcal{S}pd, \mathcal{E})\)-category is a track category \( D \) with \( \mathcal{D}_0 = \mathcal{E} \).

The nerve functor \( N : \mathcal{S}pd \to \mathcal{S} \) has a left adjoint, which coincides with the fundamental groupoid functor \( \hat{\pi}_1 : \mathcal{S} \to \mathcal{S}pd \) (cf. [38, Chapter 2]) when applied to Kan complexes. Moreover, \( \hat{\pi}_1 \) and \( N \) induce a one-to-one correspondence between 1-types (i.e., isomorphism classes in \( \text{ho} \mathcal{S} \leq 1 \)) and equivalence classes of groupoids.

Since \( N \) commutes with products, it extends to a functor \( \mathcal{N} : (\mathcal{S}pd, \mathcal{O})\text{-Cat} \to (\mathcal{S}, \mathcal{O})\text{-Cat} \) defined by taking the nerve of each groupoid \( D(a,b) \) for \( a, b \in \mathcal{O} \). \( \mathcal{N} \) has a left adjoint \( \mathcal{P} : (\mathcal{S}, \mathcal{O})\text{-Cat} \to (\mathcal{S}pd, \mathcal{O})\text{-Cat} \), which is defined for fibrant \((\mathcal{S}, \mathcal{O})\text{-Cat}\) (see §7.1 below) by applying \( \hat{\pi}_1 \) to each mapping space — again, because \( \hat{\pi}_1 \) commutes with products. The fibrancy is needed here because when \( X \in \mathcal{S} \) is not fibrant, the usual construction of \( \hat{\pi}_1 X \) involves first replacing it by a Kan complex (cf. [35, I, §8]). Moreover, since the nerve of a groupoid is 2-coskeletal, \( \mathcal{N} \) in fact lands in the category \((P^1 \mathcal{S}, \mathcal{O})\text{-Cat} \) of “categories enriched in 1-types” (see (0.2)), so we have functors:

\[
\begin{align*}
(P^1 \mathcal{S}, \mathcal{O})\text{-Cat} & \xrightarrow{\mathcal{P}} (\mathcal{S}pd, \mathcal{O})\text{-Cat} \\
\mathcal{N} & \xrightarrow{} (\mathcal{S}pd, \mathcal{O})\text{-Cat}
\end{align*}
\]

Under the identifications \((\mathcal{S}pd, \mathcal{O})\text{-Cat} \cong \mathcal{S}pd(\mathcal{O}\text{-Cat}) \) and \((\mathcal{S}, \mathcal{O})\text{-Cat} \cong \mathcal{sO}\text{-Cat} \), the adjoint pair \((1.5) \) corresponds to the adjunction \( \mathcal{S}pd(\mathcal{O}\text{-Cat}) \rightleftarrows \mathcal{sO}\text{-Cat} \) between the nerve functor on internal groupoids and its left adjoint.

We say that a morphism \( f \) in \((\mathcal{S}pd, \mathcal{O})\text{-Cat} \) is a weak equivalence if \( \mathcal{N}f \) is a weak equivalence in the standard model structure on \((\mathcal{S}, \mathcal{O})\text{-Cat} \) (see §7.1 below). If \((\mathcal{S}pd, \mathcal{O})\text{-Cat}/\sim\) denotes the localization of \((\mathcal{S}pd, \mathcal{O})\text{-Cat} \) with respect to weak equivalences, and similarly for \((\mathcal{S}, \mathcal{O})\text{-Cat} \) or \((P^n \mathcal{S}, \mathcal{O})\text{-Cat} \), then \( \mathcal{P} \) and \( \mathcal{N} \) induce equivalences between \((P^1 \mathcal{S}, \mathcal{O})\text{-Cat}/\sim\) and \((\mathcal{S}pd, \mathcal{O})\text{-Cat}/\sim\).

\[\text{2. Two-Typical Double Groupoids}\]

As noted in the Introduction, there are several algebraic structures which model 2-types. We now describe a certain kind of double groupoid which can be used as such models. These are equipped with a pair of adjoint functors, which enables us to pass back and forth between \( P^2 \)-simplicial sets and such double groupoids.

\[\text{2.1. Definition.} \ A \text{ double groupoid} \ (\text{cf. [33]}) \text{ is a groupoid internal to } \mathcal{S}pd: \text{ in other words, a diagram of the form } (1.2) \text{, with } G_0 \text{ and } G_1 \text{ in } \mathcal{S}pd \text{, rather than } \text{Set}. \text{ A double groupoid } G_{**} \text{ may thus be described explicitly by a diagram (of sets) of the}\]
identities. We think of \( G \) are source and target maps for one of the four groupoids, and the curved maps are the identities. We think of \( G_{*0} \) and \( G_{*1} \) as the horizontal groupoids of \( G_{**} \), and \( G_{0*} \) and \( G_{1*} \) as the vertical ones. The category of double groupoids is denoted by \( DbGpd \).

2.3. **Definition.** By applying the nerve functor \( N : Spd \to S \) horizontally to a double groupoid \( G_{**} \in DbGpd \), we obtain a simplicial groupoid \( N^hG_{**} \in sSpd \). If we then apply \( N \) again vertically, we obtain a bisimplicial set \( N^vN^hG_{**} \in sS \), called the double nerve of \( G_{**} \). Taking its diagonal yields the diagonal nerve of \( G_{**} \), so that the functor \( \mathcal{N}_d : DbGpd \to S \) is the composite of:

\[
DbGpd \xrightarrow{N^h} sSpd \xrightarrow{N^v} sS \xrightarrow{\text{diag}} S.
\]

From general categorical considerations it is clear that \( \mathcal{N}_d : DbGpd \to S \) must have a left adjoint; however, it is hard to describe this adjoint explicitly in a useful way. We now define a functor \( Q_d : S \to DbGpd \), equivalent up to homotopy to this adjoint, which has a particularly simple form when \( X \in S \) is fibrant, and which takes values in a convenient subcategory of \( DbGpd \). We shall use this construction of \( Q_dX \) as our canonical double groupoid model for \( X \). In fact, \( Q_d \) will be the composite of two functors, so we start with the following:

2.4. **Definition.** Let the functor \( \text{or}^* : S \to sS \) be induced by the ordinal sum \( \text{or} : \Delta^2 \to \Delta \) (where the category \( \Delta \) of finite ordinals is the indexing category for simplicial objects). Given \( X \in S \), the bisimplicial set \( \text{or}^*X \) can be described as “total \( \text{Dec} \)” of \( [10] \) (see also \( [28] \)), as follows:

Let \( \text{Aug}S \) denote the category of augmented simplicial sets. There is a functor \( \text{Dec} : S \to \text{Aug}S \) which forgets the last face operator. This has a right adjoint \( + : \text{Aug}S \to S \), which forgets the augmentation. The adjoint pair \( (\text{Dec},+) \) gives rise to a comonad, and the resulting comonad resolution of \( X \) is \( \text{or}^*X \).

Explicitly, this has the form:

\[
\begin{align*}
\cdots & \xrightarrow{X_5} \xrightarrow{X_4} \xrightarrow{X_3} \\
\cdots & \xrightarrow{X_4} \xrightarrow{X_3} \xrightarrow{X_2} \\
\cdots & \xrightarrow{X_3} \xrightarrow{X_2} \xrightarrow{X_1}
\end{align*}
\]

The following fact is straightforward:
2.5. Lemma. Let $X$ be a fibrant simplicial set. Then $\text{Dec } X$ is also fibrant and the map $\partial : \text{Dec } X \to X$ is a fibration.

2.6. Definition. The functor $Q_d : S \to \mathcal{D}bGpd$ is defined to be the composite $Q_d := \hat{Q} \circ \text{or}^*$, where $\hat{Q}$ is the left adjoint to the double nerve functor $N^v N^h : \mathcal{D}bGpd \to \mathcal{S}s\mathcal{S}et$.

2.7. Remark. The internal case (in groups) recovers the fundamental $\mathcal{C}at^2$-group functor of a simplicial group (see [23]).

In general, it is hard to describe the left adjoint $\hat{Q} : \mathcal{S}s\mathcal{S}et \to \mathcal{D}bGpd$ in a useful way. In order to get a simple description in certain cases, we need the following:

2.8. Definition. We say that a simplicial set $X$ is $\text{csk}_2$-fibrant if $\text{csk}_2 X$ is fibrant. This is equivalent to saying that $\Lambda^k[n] \to X$ has a filler $\Delta[n] \to X$ for each $0 < n \leq 2$. Similarly, we say that $f : X \to Y$ is a $\text{csk}_2$-fibration if $\text{csk}_2 f$ is a fibration.

2.9. Remark. Recall that when $X \in S$ is fibrant, its fundamental groupoid $\hat{\pi}_1 X$ has a particularly simple description: its set of objects is $X_0$, and for $x, x' \in X_0$, the morphism set $(\hat{\pi}_1 X)_1(x, x')$ is $\{\tau \in X_1 : d_0 \tau = x, d_1 \tau = x'\}/\sim$, where $\sim$ is determined by the 2-simplices of $X$. We write $(\hat{\pi}_1 X)_1$ for $X_1/\sim$.

Note that $\hat{\pi}_1 : S_{\text{cf}} \to \mathcal{S}pd$ factors through $\text{csk}_2$, so this description is valid for any $\text{csk}_2$-fibrant $X$. Note also that if $X_\bullet$ is a simplicial groupoid, $\pi : X_{n1} \to X_{n0}$ is the target map of the groupoid $X_n$ for each $n \geq 0$.

2.10. Proposition. Let $X_\bullet \in \mathcal{S}pd$ be a (horizontal) simplicial groupoid, for which the simplicial sets $X_0$ and $X_1$ are $\text{csk}_2$-fibrant, and the morphism $d_1^0 : X_1 \to X_0$ is a $\text{csk}_2$-fibration. Then the left adjoint $Q^h : \mathcal{S}pd \to \mathcal{D}bGpd$ to the nerve $N^h : \mathcal{D}bGpd \to \mathcal{S}pd$, applied to $X_\bullet$, is $\hat{\pi}_1^h X_\bullet$.

2.11. Proposition. Let $X_\bullet \in \mathcal{S}pd$ be such that $X_0$ and $X_1$ are $\text{csk}_2$-fibrant for each $i \geq 0$, and $d^0_1 : X_1 \to X_0$ and $d^v_1 : X_1 \to X_0$ are $\text{csk}_2$-fibrations. Then

(i) $(N^v_1 \hat{\pi}_1 X_\bullet)_i$ is fibrant for all $i \geq 0$.
(ii) $(N^v_1 \hat{\pi}_1 X_\bullet)_1$ is $\text{csk}_2$-fibrant.
(iii) $\hat{d}^0_1 : (N^v_1 \hat{\pi}_1 X_\bullet)_1 \to (N^h_1 \hat{\pi}_1 X_\bullet)_0$ is a $\text{csk}_2$-fibration.
(iv) $\hat{d}^v_0 : (N^v_1 \hat{\pi}_1 X_\bullet)_i \to (N^v \hat{\pi}_1 X_\bullet)_0$ is a fibration.

For the proofs of these Propositions, see Appendix A.

2.12. Corollary. If $X_\bullet \in \mathcal{S}pd$ satisfies the hypotheses of Proposition 2.11, then $\hat{Q}_d X_\bullet = \hat{\pi}_1^h \hat{\pi}_1 X_\bullet$, and thus for a Kan complex $X \in S_{\text{cf}}$ we have $\hat{Q}_d X = \hat{\pi}_1^h \hat{\pi}_1 X_\bullet$.

2.13. Remark. The functor $Q_d$ is not actually the left adjoint of $N_d$. However, if we replace $\text{diag} : \mathcal{S}s\mathcal{S}et \to S$ by the homotopy equivalent functor $\overline{W} : \mathcal{S}s\mathcal{S}et \to S$ of [1, §III], which is right adjoint to $\text{or}^*$, then $Q_d$ will be left adjoint to $\overline{W} N^v N^h$ (see [24]).

2.14. Two-typical double groupoids.

We now have a natural choice for modeling a space $Y$ by a double groupoid: choose a Kan complex $X$ weakly equivalent to $Y$, and take $Q_d X \in \mathcal{D}bGpd$. It turns out that the double groupoids obtained in this way have several convenient properties. First of all, it is not hard to see that $Q_d X$ is symmetric, in the following sense:
2.15. **Definition.** A double groupoid $G_{**}$ is called **symmetric** if $G_{0*} \cong G_{*0}$ and $G_{1*} \cong G_{*1}$ are isomorphic groupoids. In this case $G_{**}$ may be described more succinctly by a diagram

$$
\begin{align*}
\xymatrix{ G[1] \ar[rr]^{d_0} & & G[0] \ar[ll]^{d_1} \\
& G[0] \ar[rr]_{d_0} & & G[1] \ar[ll]_{d_1} }
\end{align*}
$$

in which

$$
\begin{align*}
G[1] = (G_2 \xrightarrow{d^{[1]}_0} G_0) \quad \text{and} \quad G[0] = (G_1 \xrightarrow{d^{[0]}_0} G_0)
\end{align*}
$$

are groupoids (isomorphic to $G_{1*}$ and $G_{0*}$, respectively). To make this precise, one should expand (2.17) to a diagram of the form (2.2), satisfying suitable axioms.

Applying $\pi_0$ to $G_{[1]}$ and $G_{[0]}$ yields the following diagram of sets:

$$
\pi_0 G_{[1]} \xrightarrow{\pi_0} \pi_0 G_{[0]}
$$

where the two maps are induced by $d_0, d_1 : G_{[1]} \to G_{[0]}$. This is equivalent to applying the functor $\pi_0$ either horizontally or vertically to $G_{**}$ in the diagram (2.2) (or equivalently, to the symmetric bisimplicial set $N_v N_h G_{**}$).

In addition, Propositions 2.10 and 2.11 imply that $Q_t X$ satisfies certain fibrancy conditions:

2.19. **Definition.** For any groupoid $G$, let $G^d$ denote the discrete groupoid on the set $\pi_0 G$. This comes equipped with a map of groupoids $\gamma : G \to G^d$.

A double groupoid $G_{**}$ is called **weakly globular** if

(a) The map $\gamma : G_{0*} \to G_{0*}^d$ is a weak equivalence.

(b) Both $N^v d^0_0$ and $N^v d^1_1$ are fibrations of simplicial sets, where $d^0_0, d^1_1 : G_{1*} \to G_{0*}$ are maps of (vertical) groupoids.

2.20. **Remark.** Note that a strict 2-groupoid (i.e., a groupoid enriched in groupoids — cf. [17, §7.7]) is an example of a weakly globular double groupoid, since a map of simplicial sets with discrete target is a fibration. The internal version of this concept (in groups) is the notion of weakly globular cat$^2$-group of [50] (in this case the fibrancy conditions are automatically satisfied).

2.21. **Definition.** A double groupoid is called **two-typical** if it is symmetric and weakly globular. The full subcategory of $\mathcal{D}bGpd$ whose objects are two-typical double groupoids will be denoted by $\mathcal{D}bGpd_{tt}$.

2.22. **Lemma.** Let $p : G \to G'$ be a map of groupoids. Suppose that the diagram:

$$
\begin{align*}
\xymatrix{ \Lambda^k[n] \ar[r]^f \ar[d]_i & N G \ar[d]^{Np} \\
\Delta[n] \ar[r]_h & N G' }
\end{align*}
$$

commutes.
has a lift \( \Delta[n] \to N\mathcal{G} \) when \( n = 1 \). Then \( Np \) is a fibration.

**Proof.** This follows because the nerve of a groupoid is 2-coskeletal.

We now show that the functor \( Q_t : \mathcal{S} \to \mathcal{DbGpd} \) of Definition 2.6 indeed takes values in \( \mathcal{DbGpd}_t \) (when applied to a Kan complex):

2.23. **Proposition.** If \( X \) is a fibrant simplicial set, then \( Q_tX \) is a two-typical double groupoid.

**Proof.** Let \( X \) be a fibrant simplicial set. Evidently \( Q_tX \) is symmetric (that is, \( Q_tX_{i,j} \cong Q_tX_{j,i} \) for all \( i, j \)), since \( \ast \) \( X \) is. Recall that for \( i > 0 \):

\[
\begin{align*}
(\ast \ast X)_{0*} &= \text{Dec} \ X = (\ast \ast X)_{00} \\
(\ast \ast X)_{1*} &= \text{Dec} (\ast \ast X)_{11} \\
(\ast \ast X)_{2*} &= \text{Dec} (\ast \ast X)_{22}.
\end{align*}
\]

By Lemma 2.5 it follows that \( \ast \ast X \) satisfies the hypotheses of Proposition 2.11. Hence, by Proposition 2.11 we see that \( \pi_1^h \ast \ast X \in s\mathcal{Gpd} \) satisfies the hypotheses of Proposition 2.10. Thus if \( Q^h \) is the left adjoint to \( N^h \), we see \( Q^h \pi_1^v \ast \ast X \) is computed by applying \( \pi_1^h \) levelwise in the horizontal direction in the bisimplicial set \( N^v \pi_1^v \ast \ast X \). That is:

\[
(2.24) \quad \hat{Q} \ast \ast X = Q^h \pi_1^v \ast \ast X = \pi_1^h \pi_1^v \ast \ast X.
\]

Furthermore, by Proposition 2.11 the bisimplicial set \( N^v \pi_1^v \ast \ast X \) itself satisfies the hypotheses of Proposition 2.11 therefore we conclude by Proposition 2.11(3) that \( N d_0^v \) is a fibration, for \( d_0^v : (\hat{Q} \ast \ast X)_{s1} \to (\hat{Q} \ast \ast X)_{s0} \).

To show that \( \hat{Q} \ast \ast X \) is weakly globular, it remains to show that the vertical groupoid \( (\hat{Q} \ast \ast X)_{0s} = \hat{\pi}_1 \text{Dec} \ X \) is equivalent to a discrete groupoid. Recall that if we let \( c(X_0) \) denote the constant simplicial set on the set \( X_0 \), there are maps \( c(X_0) \xrightarrow{s} \text{Dec} \ X \xrightarrow{v} c(X_0) \) with \( v \circ s = \text{Id} \), and there is a simplicial homotopy equivalence \( v \circ s \simeq \text{Id} \). Thus the simplicial sets \( c(X_0) \) and \( \text{Dec} \ X \) are weakly equivalent and therefore have equivalent fundamental groupoids. There are thus equivalences of groupoids \( r : (\hat{Q} \ast \ast X)_{0s} \to \hat{\pi}_1 c(X_0) \) and \( t : \hat{\pi}_1 c(X_0) \to (\hat{Q} \ast \ast X)_{0s} \), with \( r \circ t = \text{Id} \), and \( \hat{\pi}_1 c(X_0) \) is the discrete groupoid on the set \( X_0 \).

3. **Two-typical double groupoids and 2-types**

We now show that a two-typical double groupoid \( G_{**} \) is a 2-type, in the sense that its diagonal nerve, the simplicial set \( \mathcal{N}_dG_{**} \), is in \( P^2\mathcal{S} \) (cf. \{0.2\}). Actually, we prove a little more:

3.1. **Proposition.** For any weakly globular \( G_{**} \in \mathcal{DbGpd} \), the realization \( X := \mathcal{N}_dG_{**} \in \mathcal{S} \) is a 2-type.

**Proof.** Since \( \mathcal{N}_dG_{**} \) is the diagonal of the bisimplicial set \( N^vN^hG_{**} \), it suffices to find a simplicial groupoid \( Y_\bullet \) such that \( \mathcal{N}_dY_\bullet := \text{diag} N^vY_\bullet \) is a 2-type, together with a
map of simplicial groupoids \( N^hG_{ss} \to Y_* \) which is a weak equivalence in each simplicial dimension, so that the induced map

\[
N_dG_{ss} = \text{diag } N^vN^hG_{ss} \xrightarrow{\gamma} \tilde{N}_dY_*
\]

is a weak equivalence, too.

Recall that a \textit{Tamsamani weak 2-groupoid} is a simplicial groupoid \( Y_* \in s\mathcal{Spd} \) such that:

(a) \( Y_0 \) is discrete;

(b) The Segal maps

\[
(3.2) \quad \eta_n : Y_n \to \underbrace{Y_1 \times_{Y_0} Y_1 \times_{Y_0} \ldots \times_{Y_0} Y_1}_{n}
\]

are equivalences of groupoids (that is, \( N\eta_n \) is a weak equivalence). The full subcategory of such objects in \( s\mathcal{Spd} \) is denoted by \( \text{Tam}_2 \).

(c) The simplicial set \( \pi_0 Y_* \) obtained by applying \( \pi_0 \) in each simplicial dimension, is the nerve of a groupoid.

In this case, the simplicial set \( \tilde{N}_dY_* \) is in \( P^2\mathcal{S} \). Moreover, simplicial sets of the form \( \tilde{N}_dY_* \) model all 2-types of simplicial sets, up to homotopy (see [56]).

The idea of the construction of \( Y_* \) is to make \( Z_* \) into a globular object by replacing the groupoid \( Z_0 = G_{0*} \) with the discrete groupoid \( G^{d}_{0*} \), in a way that does not change the homotopy type of \( Z_* \). By pushing out the unique degeneracy map \( s_{(n)} : Z_0 \to Z_n \) along the discretization map \( \gamma : G_{0*} \to G^{d}_{0*} \) one obtains a simplicial groupoid \( Y_* \); since \( \gamma \) is an equivalence, it is easily seen that the simplicial map \( Z_* \to Y_* \) is a levelwise equivalence. This, together with the fact that the Segal maps in \( Z_* \) are isomorphisms and the maps \( N\eta_0, N\eta_1 : NZ_1 \to NZ_0 \) are fibrations also implies that the Segal maps of \( Y_* \) are equivalences. In conclusion, \( Y_* \) will be a Tamsamani weak 2-groupoid with the same homotopy type as \( Z_* \). The details are as follows:

Let \( Z_* = N^hG_{ss} \) and \( Y_0 := G^{d}_{0*} \) (see Definition 2.19). Note for each \( n \geq 0 \), there is a unique morphism \( s_{(n)} : [0] \to [n] \) in \( \Delta^{op} \). We use the same notation \( s_{(n)} : W_0 \to W_n \) for any simplicial object \( W_* \). For each \( n > 0 \), \( Y_n \) is defined to be the pushout in \( \mathcal{E}cat \):

\[
\begin{array}{ccc}
G_{0*} =: Z_0 & \xrightarrow{s_{(n)}} & Z_n \\
\gamma =: f_0 & & \downarrow f_n \\
G^{d}_{0*} =: Y_0 & \xrightarrow{\sigma_{(n)}} & Y_n
\end{array}
\]

We claim that \( Y_n \) is a groupoid and \( f_n \) is a weak equivalence. In fact, since \( G_{ss} \) is weakly globular, \( \gamma \) is an equivalence of groupoids, and thus also a categorical equivalence. The map \( s_{(n)} \) is injective on objects. But the pushout in \( \mathcal{E}cat \) of a categorical equivalence by a map which is injective on objects is a categorical equivalence (see [44]); hence \( f_n \) is a categorical equivalence, and thus a weak equivalence. Since a category equivalent to a groupoid is itself a groupoid, \( Y_n \) is a groupoid. This proves the claim.

Let \( \phi : [n] \to [m] \) be any morphism in \( \Delta^{op} \). Then \( \phi s_{(n)} = s_{(m)} \), by the uniqueness, so that \( f_n \phi s_{(n)} = f_m s_{(m)} = \sigma_{(m)} f_0 : Z_0 \to Y_m \).

From the universal property of pushouts there is thus a unique map \( \hat{\phi} : Y_n \to Y_m \) such that \( \hat{\phi} f_n = f_m \phi \) and \( \hat{\phi} \sigma_{(n)} = \sigma_{(m)} \).
In particular we have maps $\hat{\partial}_i : Y_n \to Y_{n-1}$ for $0 \leq i \leq n$ and $\hat{\sigma}_i : Y_{n-1} \to Y_n$ for $0 \leq i < n$.

The maps $\hat{\partial}_i$ and $\hat{\sigma}_i$ $(0 \leq i \leq n < \infty)$ satisfy the simplicial identities, so that these make $Y_* = (Y_n)_{n=0}^\infty$ into a simplicial groupoid. To see this, let $\phi : [n] \to [m]$, and $\psi : [m] \to [k]$ be any morphisms in $\Delta^\op$ with $\xi := \phi \circ \psi$. Then:

$$\hat{\xi} \sigma_{(n)} = \sigma_k = \hat{\phi}_1 \sigma_{(m)} = \hat{\phi}_1 \hat{\phi} \sigma_{(n)}$$

and $\hat{\xi} f_n = f_k \hat{\phi} = f_k \psi \phi = \hat{\phi} f_m \phi = \hat{\psi} \hat{\phi} f_n$.

It follows by universality of pushouts that $\hat{\xi} = \hat{\psi} \hat{\phi}$. In particular, since the simplicial identities are satisfied by the maps $\partial_i$ and $\sigma_i$, they are satisfied by $\hat{\partial}_i$ and $\hat{\sigma}_i$.

We now prove that $Y_* \in s\mathcal{Gpd}$ is a Tamsamani weak 2-groupoid. By construction, $Y_0 = N^vG_{s0}^d$ is discrete. We need to prove that, for each $n \geq 2$, the map $\pi_n f : \pi_n Y_* \to \pi_n Y_{n+1}$ is an equivalence of groupoids.

Consider the case $n = 2$. There is a commutative diagram in $\mathcal{S}$:

$$\begin{array}{ccc}
NZ_2 & \xrightarrow{f_2} & NZ_1 \times_{NZ_0} NZ_1 \\
\downarrow f_1 & & \downarrow f_1 \times f_1 \\
NY_2 & \xrightarrow{\eta_2} & NY_1 \times_{NY_0} NY_1
\end{array}$$

We claim that $f_1 \times f_1$ is a weak equivalence. In fact, there is a commutative diagram in $\mathcal{S}$:

$$\begin{array}{ccc}
NZ_1 & \xrightarrow{Nd_0} & NZ_0 & \xrightarrow{Nd_1} & NZ_1 \\
\downarrow Nf_1 & & \downarrow Nf_0 & & \downarrow Nf_1 \\
NY_1 & \xrightarrow{Nd'_0} & NY_0 & \xrightarrow{Nd'_1} & NY_1
\end{array}$$

In this diagram, the map in each column is a weak equivalence; the map $Nd_0$ is a fibration since $G_{\ast\ast}$ is weakly globular. Since $NY_0$ is constant, the map $Nd'_0$ satisfies the hypotheses of Lemma 2.22 and is therefore a fibration. Since the standard model structure on $\mathcal{S}$ is right proper, we can apply [33 Proposition 13.3.9] to conclude that the induced map of pullbacks $f_1 \times f_1$ is a weak equivalence, as claimed.

From above, we know that $f_2$ is also a weak equivalence. The commutativity of (3.4) therefore implies that $\eta_2$ is a weak equivalence. Similarly one shows that $\eta_m$ is a weak equivalence for each $n > 2$. To show that $Y_\ast$ is a Tamsamani weak 2-groupoid, it remains to check that $\pi_0 Y_\ast$ is the nerve of a groupoid. For this, notice that there is a commutative diagram in $\mathcal{S}$:

$$\begin{array}{ccc}
NZ_1 & \xrightarrow{Nd_0} & NZ_0 & \xrightarrow{Nd_1} & NZ_1 \\
\downarrow Nf_0 & & \downarrow Nf_0 & & \downarrow Nf_0 \\
NZ_1 & \xrightarrow{Ndd_0} & NY_0 & \xleftarrow{Ndd_0} & NY_1
\end{array}$$

Again each vertical map is a weak equivalence and each horizontal map is a fibration. Thus we conclude that the induced map of pullbacks

$$N(Z_1 \times Z_0 Z_1) = NZ_1 \times_{NZ_0} NZ_1 \to NZ_1 \times_{NY_0} NZ_1 = N(Z_1 \times Y_0 Z_1)$$

is a weak equivalence. Thus the map of groupoids $Z_1 \times_{Z_0} Z_1 \to Z_1 \times_{Y_0} Z_1$ is an equivalence of categories. Since the functor $\pi_0 : \mathcal{Gpd} \to \mathcal{Set}$ preserves fibre products
over discrete objects, we have an isomorphism
\[(\pi_0Z_\bullet)_n \cong \pi_0(\underbrace{Z_1 \times Z_0 \ldots \times Z_0}_n Z_1) \cong (\pi_0Z_1 \times \pi_0Z_0 \ldots \times \pi_0Z_0 \pi_0Z_1)_{\times n}
\cong (\pi_0Z_\bullet)_1 \times (\pi_0Z_\bullet)_0 \ldots \times (\pi_0Z_\bullet)_0 (\pi_0Z_\bullet)_1^n.
\]
This shows that the simplicial set \(\pi_0Z_\bullet\) has all Segal maps isomorphisms, and is therefore the nerve of a category. In fact, since \(G_{**}\) is a double groupoid, \(\pi_0Z_\bullet\) is the nerve of a groupoid.

Since for each \(n \geq 0\), \(f_n : Z_n \to Y_n\) is a weak equivalence, \((\pi_0Y_\bullet)_n = (\pi_0Z_\bullet)_n\).

Hence \(\pi_0Y_\bullet \cong \pi_0Z_\bullet\), so that from above \(\pi_0Y_\bullet\) is also the nerve of a groupoid, as required. This actually defines a functor \(T : DbGpd_\bullet \to \mathcal{S}\) with \((T(G_{**})) = Y_\bullet\).

This concludes the proof that \(Y_\bullet\) is a Tamsamani weak 2-groupoid. From \([56]\), we know that \(\tilde{N}_d Y_\bullet\) is a 2-type. Since the map \(f : N^hG_{**} = Z_\bullet \to Y_\bullet\) is a levelwise weak equivalence, the map \(N^h(f)\) is a levelwise weak equivalence in \(s\mathcal{S}\), and therefore \(N_d f\) is a weak equivalence. It follows by (3.2) that \(N_d G_{**} \simeq \tilde{N}_d Y_\bullet\), so that \(N_d G_{**}\) is a 2-type.

3.5. Remark. We actually can read off more information from a two-typical double groupoid \(G_{**}\), than just its 2-type. Namely, we can describe algebraically its Postnikov decomposition. This will be useful later in defining the homotopy track category of a 2-track category (see [4.4] and the notion of 2-track extension (see 3.1). First, observe that in the proof of Proposition 3.1 we have shown that if \(G_{**}\) is any weakly globular double groupoid, then the diagram (2.18), obtained by applying the coequalizer \(\pi_0\) (horizontally or vertically) to \(G_{**}\) itself has the structure of a groupoid, which we call the fundamental groupoid of \(G_{**}\), and denote by \(\pi_1G_{**}\).

Also, there is a simplicial map \(Z_\bullet = N^hG_{**} \to c\pi_1G_{**}\), where \(c\pi_1G_{**}\) denotes the constant simplicial groupoid on \(\pi_1G_{**}\). Likewise, applying \(\pi_0\) in each simplicial dimension to the simplicial groupoid \(Y_\bullet\) yields the nerve of a groupoid \(\pi_0Y_\bullet\). By the proof of 3.1, the map \(Z_\bullet \to Y_\bullet\) induces an isomorphism \(\pi_1G_{**} \cong \pi_1Y_\bullet\). Further, by [56], \(\pi_0\pi_1Y_\bullet = \pi_0B Y_\bullet\) and \(\pi_1\pi_1\pi_0(Id_\bullet) = \pi_1(\pi_0B Y_\bullet, \ast)\), where \(B = \text{diag} \circ \tilde{N}\) is the realization functor.

Since \(Z_\bullet\) and \(Y_\bullet\) have the same homotopy types, we conclude that the map \(Z_\bullet \to c\pi_1G_{**}\) induces isomorphisms of homotopy groups in dimension 0 and 1. Hence this map gives the last stage of the Postnikov decomposition of \(G_{**}\). In Theorem 3.6 we will show that, given \(X \in P^2\mathcal{S}\), \(Q_t X\) represents the 2-type of \(X\). The algebraic description of the Postnikov decomposition of \(Q_t X\) given above translates into the fact that, for each \(g \in G_0\), the fundamental group of the groupoid \(G_{[1]} = G_2 \sqsupseteq G_1\) based at \(Id_g\) (in the notation of [2.15]) is isomorphic to the local system \(\pi_2(X, [g])\). Thus we can actually recover the Postnikov system of a 2-type \(X\) algebraically from its two-typical model \(Q_t X\).

Proposition 3.1 shows that the functor \(N_d : DbGpd_\bullet \to \mathcal{S}\) takes values in \(P^2\mathcal{S}\) (see 0.2) — i.e., the realization of a two-typical double groupoid is a 2-type. As before, we say that a map \(f\) in \(DbGpd\) is a weak equivalence if \(N_d f\) is a weak equivalence in \(\mathcal{S}\).

We now show that we have a one-to-one correspondences of weak equivalence classes of objects of \(DbGpd\) and \(P^2\mathcal{S}\):
3.6. Theorem. The functors $Q_i : P^2 S \to \mathcal{D}bGpd_i$ and $N_d : \mathcal{D}bGpd_i \to P^2 S$ induce equivalences of categories after localization:

$$P^2 S \sim \frac{\text{diag}}{Q_i} \to \mathcal{D}bGpd_i \sim \frac{\text{diag}}{N_d}.$$ 

To show this, we shall need the following concept:

3.8. Definition. A map $f : W_{\bullet \bullet} \to V_{\bullet \bullet}$ of bisimplicial sets is called a diagonal $n$-equivalence if $f^h_k : W^h_k \to V^h_k$ is an $(n-k)$-equivalence for each $k \leq n$.

3.9. Proposition. If $f : W_{\bullet \bullet} \to V_{\bullet \bullet}$ is a diagonal $n$-equivalence, then the induced map $\text{diag} f : \text{diag} W_{\bullet \bullet} \to \text{diag} V_{\bullet \bullet}$ is an $n$-equivalence.

For the proof, see Appendix B.

Proof. By [51, I, §1] we must show that for any Kan complex $X \in P^2 S$ there is a weak equivalence $X \simeq N_1 Q_1 X$, where $Q_1 X = \hat{\pi}_1^h \pi_1^v$ or $^*X$ (cf. [2.6]), and that the natural map $\text{or}^* X \to N^v \hat{\pi}_1^v$ or $^*X$ is a weak equivalence in each (horizontal) simplicial dimension. Moreover, since $(\text{or}^* X)_{\bullet \bullet}$ is homotopically trivial for each $i \geq 0$, we have $(\text{or}^* X)_{i \bullet} \simeq N^v \hat{\pi}_1^v(\text{or}^* X)_{i \bullet}$, so

$$\text{diag} \circ^* X \simeq \text{diag} N^v \hat{\pi}_1^v \circ^* X.$$ 

Furthermore, if $c(X)_{i \bullet}$ is the constant simplicial set on $X_i$, the augmentation map of bisimplicial sets $\circ^* X \to c(X)$ is a weak equivalence in each (horizontal) simplicial dimension, so it induces a weak equivalence

$$\text{diag} \circ^* X \simeq \text{diag} c(X) = X.$$ 

Now consider the unit map

$$\hat{\pi}_1^v \circ^* X \to N^h \hat{\pi}_1^h \hat{\pi}_1^v \circ^* X = N^h Q_1 X,$$

for the comonad $N_1 : S \to S$.

This map is a diagonal 2-equivalence. In fact, $(N^v N^h Q_1 X)_0^h = (N^h Q_1 X)_0^h = N^h \pi_1 \text{Dec} X$ and $(N^v \hat{\pi}_1^v \circ^* X)_0^h = \text{Dec} X$. Moreover, the map $\text{Dec} X \to N^h \hat{\pi}_1 \text{Dec} X$ is a weak equivalence, hence in particular a 2-weak equivalence, since the Kan complex $\text{Dec} X$ is simplicially homotopy equivalent to a constant simplicial set (cf. [35 I, §6]). Further, for each $i > 0$, the map:

$$(N^v \hat{\pi}_1^v \circ^* X)^h_i \to (N^h Q_1 X)^h_i = (N^h \hat{\pi}_1^h \hat{\pi}_1^v \circ^* X)^h_i$$

is a 1-equivalence, proving the claim.

It follows from Proposition 3.9 that there is a 2-equivalence

$$\text{diag} N^v \hat{\pi}_1^v \circ^* X \simeq N_d Q_1 X.$$ 

Thus (3.10), (3.11), and (3.12) imply that there is a 2-equivalence

$$X \simeq N_d Q_1 X.$$ 

Since, by hypothesis, $X$ is in $P^2 S$, and by Proposition 3.1, $N_d \hat{Q} \circ^* X$ is in $P^2 S$, it follows that (3.13) is a weak equivalence.
Finally, given \( G_{**} \in DbGpd \), by (3.12) there is a weak equivalence
\[
\mathcal{N}_d G \simeq \mathcal{N}_d \mathcal{Q}_d \mathcal{N}_d G_{**}
\]
This shows that \( G_{**} \) and \( \mathcal{Q}_d \mathcal{N}_d G_{**} \) are weakly equivalent. This completes the proof that the functors in (3.7) are equivalences of categories. \( \square \)

4. TWO-TRACK CATEGORIES

We are now in a position to define the main subject of this paper:

4.1. Definition. A two-track category with object set \( \mathcal{O} \) is a \((DbGpd_1, \mathcal{O})\)-category, where the enrichment is with respect to the cartesian monoidal structure. We denote the category of all two-track categories by 2-Track.

4.2. Proposition. The functor \( \mathcal{Q}_t : (S, \mathcal{O})\)-Cat \( \to \) \((DbGpd_1, \mathcal{O})\)-Cat associates to a fibrant \((S, \mathcal{O})\)-category \( X \) a two-track category \( \mathcal{G} = \mathcal{Q}_t X \) with object set \( \mathcal{O} \).

Proof. From its definition, it is straightforward that \( \mathcal{Q}_t \) preserves products. Since the fundamental groupoid functor preserves products, and products in functor categories are computed pointwise, it follows from (2.24) that \( \mathcal{Q}_t = \mathcal{Q} \circ \mathcal{O} : S \to DbGpd \) preserves products. It thus extends to \((S, \mathcal{O})\)-Cat (applied to each mapping space). \( \square \)

Similarly, the functor \( \mathcal{N}_d = \text{diag } N^v N^h : DbGpd \to S \) preserves products and therefore induces a functor on two-track categories which we also denote by \( \mathcal{N}_d : (DbGpd_1, \mathcal{O})\)-Cat \( \to \) \((P^2 S, \mathcal{O})\)-Cat. We say that a morphism \( f \) in \((DbGpd_1, \mathcal{O})\)-Cat is a weak equivalence if \( \mathcal{N}_d f \) is a weak equivalence of \((S, \mathcal{O})\)-categories, and we denote by \((DbGpd_1, \mathcal{O})\)-Cat/\(\sim\) and \((P^2 S, \mathcal{O})\)-Cat/\(\sim\), respectively the localizations with respect to weak equivalences (so that \((P^2 S, \mathcal{O})\)-Cat/\(\sim\) is actually the full subcategory \(\text{ho } (P^2 S, \mathcal{O})\)-Cat of \(\text{ho } (S, \mathcal{O})\)-Cat with respect to the model category structure described in §7.1). We then deduce from Theorem 3.6.

4.3. Corollary. The functors \( \mathcal{Q}_t : (P^2 S, \mathcal{O})\)-Cat \( \to \) \((DbGpd_1, \mathcal{O})\)-Cat and \( \mathcal{N}_d : (DbGpd_1, \mathcal{O})\)-Cat \( \to \) \((P^2 S, \mathcal{O})\)-Cat induce equivalence of categories between \((P^2 S, \mathcal{O})\)-Cat/\(\sim\) and \((DbGpd_1, \mathcal{O})\)-Cat/\(\sim\).

4.4. The homotopy track category of a two-track category. Because a two-track category \( \mathcal{G} \) is enriched in two-typical double groupoids, which are in particular symmetric, we can apply (2.16) to each mapping object to obtain a concise description of \( \mathcal{G} \) in the form

\[
\begin{array}{c}
\mathcal{G}_{[1]} \\
\downarrow s \\
\mathcal{G}_{[0]}
\end{array}
\]

in which \( \mathcal{G}_{[1]} = (\mathcal{G}_2 \circ \mathcal{G}_1) \) and \( \mathcal{G}_{[0]} = (\mathcal{G}_1 \circ \mathcal{G}_0) \) are track categories, as in (2.17). We call a morphism \( \xi \) of \( \mathcal{G}_1 \) a 1-track, and write \( \xi : f \Rightarrow g \), where \( f = d_0 \xi \) and \( g = d_1 \xi \). Similarly, a morphism \( \alpha \) of \( \mathcal{G}_2 \) is called a 2-track, and we write \( \alpha : d_0 \alpha \Rightarrow d_1 \alpha \).

The additional data mentioned in Remark 3.5 is available also here: thus the homotopy track category of a two-track category \( \mathcal{G} \), written \( \tilde{\Pi}_1 \mathcal{G} \) or \( \text{ho } \mathcal{G} \), is obtained by taking the coequalizer of the maps of track categories \( \mathcal{G}_{[1]} \circ \mathcal{G}_{[0]} \), or equivalently, by applying the fundamental groupoid functor \( \tilde{\pi}_1 \) to each two-typical mapping object \( \mathcal{G}(a, b) \).
(a, b ∈ G_0). This track category \( D := \hat{\Pi}_1 \mathcal{G} \) itself has an associated homotopy category \( \text{ho} \ D = \Pi_0 \mathcal{D} \) (see [4.4], which we denote simply by \( \Pi_0 \mathcal{G} \).

Moreover, we have certain abelian track category \( \Pi_2 \mathcal{G} \) over \( \hat{\Pi}_1 \mathcal{G} \), which will be described in the next section (see [5.9]), which together with \( \hat{\Pi}_1 \mathcal{G} \) (and \( \Pi_0 \mathcal{G} \)) can be thought of as the Postnikov system for \( \mathcal{G} \). We summarize this situation by the following diagram:

\[
\begin{array}{c}
\Pi_2 \mathcal{G} \\
\downarrow \downarrow \quad \downarrow \downarrow \\
\mathcal{G}[1] \quad \mathcal{G}[0] \quad \hat{\Pi}_1 \mathcal{G}
\end{array}
\]

\( d_0 \)

\( d_1 \)

In Section 8 we will further analyze the structure of (4.6) as a “two-track extension” of \( \hat{\Pi}_1 \mathcal{G} \) by \( \Pi_2 \mathcal{G} \).

4.7. A Gray category model of two-track categories. We wish to point out some connections with other higher-categorical structures. These are not needed for the subsequent sections.

There is a functor \( L : \text{DbGpd}_d \to \text{Tam}_2 \) defined as follows: in \( N^h_{G_{ss}} \), replace the groupoid \( G_{0s} \) by the discrete groupoid \( G_{0s}^d \), with \( d_i : G_{1s} \to G_{0s} \) \((i = 0, 1)\) by \( \gamma d_i \), and \( s_0 : G_{0s} \to G_{1s} \) by \( s_0 \sigma \sigma \) where \( \sigma : G_{0s} \to G_{0s} \) is the natural section. The verification that the resulting simplicial groupoid \( L(G_{ss}) \) is a Tamsamani weak 2-groupoid is as in the proof of Proposition 3.1. In addition, there is a natural weak equivalence \( L(G_{ss}) \to T(G_{ss}) \) (see proof of Proposition 3.1), so \( L(G_{ss}) \) is weakly equivalent to \( N^h(G_{ss}) \). This functor \( L \) clearly preserves products.

Recall from [5.6] that there is a product-preserving functor \( M : \text{Tam}_2 \to \text{BiGpd} \) to the category of bigroupoids. Furthermore, there is a strictification functor \( \text{st} : \text{BiGpd} \to \text{2-Gpd} \) to the category of strict 2-groupoids which is monoidal with respect to the cartesian product in \( \text{BiGpd} \) and the Gray tensor product in \( \text{2-Gpd} \) (see [36]).

All three functors \( L, M, \) and \( \text{st} \) preserve homotopy types (of the classifying spaces). The composite \( P^2 \mathcal{S} \circ M \circ L : \text{DbGpd}_d \to \text{2-Gpd} \) is also monoidal, and thus extends to a functor \( K : (\text{DbGpd}_d, O)\text{-Cat} \to (\text{2-Gpd}, O)\text{-Cat} \), where the target is the full subcategory of Gray categories whose 2- and 3-cells are invertible.

The Street nerve functor \( \mathcal{N} : \text{2-Gpd} \to \mathcal{S} \) (cf. [55], and see §6.1 below) is monoidal with respect to the Gray tensor product in \( \text{2-Gpd} \) and the cartesian product in \( \mathcal{S} \) (see [57]), and is weakly equivalent to the diagonal nerve. Therefore, it extends to a functor \( \mathcal{N} : (\text{2-Gpd}, O)\text{-Cat} \to (\mathcal{S}, O)\text{-Cat} \), and we have:

4.8. Proposition. The functors \( P^2 \mathcal{S} \circ M \circ L : (P^2 \mathcal{S}, O)\text{-Cat} \to (\text{2-Gpd}, O)\text{-Cat} \) and \( \mathcal{N} : (\text{2-Gpd}, O)\text{-Cat} \to (P^2 \mathcal{S}, O)\text{-Cat} \) induce equivalence of categories between \( (P^2 \mathcal{S}, O)\text{-Cat}/\sim \) and \( (\text{2-Gpd}, O)\text{-Cat}/\sim \).

Here weak equivalences in \( (2\text{-Gpd}, O)\text{-Cat} \) are defined by means of \( \mathcal{N} \).

5. Coefficient systems on track categories

In order to define a cohomology theory for track categories, we first have to describe the possible coefficient systems. These have already been identified, in Quillen’s approach, as the abelian group objects in the appropriate over-category (cf. [52, §2]):
5.1. Definition. A module over a track category $\mathcal{D} = (\mathcal{D}_1\to\mathcal{D}_0)$ is an abelian group object $\mathcal{M} = (\mathcal{M}_1\to\mathcal{M}_0)$ in the over-category $(\mathcal{Spd},\mathcal{D}_0)\text{-Cat}/\mathcal{D}$.

In order to make this more explicit, note that by definition a module $\mathcal{M}$ over $\mathcal{D}$ comes equipped with a map of track categories (in fact, of $(\mathcal{Spd},\mathcal{D}_0)$-categories) $p : \mathcal{M} \to \mathcal{D}$, which has a section $s : \mathcal{D} \to \mathcal{M}$ (with $p \circ s = \text{Id}$).

Applying the “semi-discretization” functor $(-)^\delta$ of [14] yields a map of semi-discrete track categories $p^\delta : \mathcal{M}^\delta \to \mathcal{D}^\delta$, and we denote by $\mathcal{K} := \text{Ker}(p^\delta)$ the kernel of this map (as a disjoint union of homomorphisms of groups), so that $\mathcal{K} \in (\mathcal{Spd}^\delta, \mathcal{D}_0)\text{-Cat}$. In fact, since $\mathcal{M} \in ((\mathcal{Spd}, \mathcal{D}_0)\text{-Cat}/\mathcal{D})_{ab}$, we see that $\mathcal{K} \in ((\mathcal{Spd}^\delta, \mathcal{D}_0)\text{-Cat})_{ab}$.

5.2. Remark. Note that in any track category $\mathcal{D}$, the categorical (horizontal) composition $\circ_h : \mathcal{D}_i(a,b) \times \mathcal{D}_i(b,c) \to \mathcal{D}_i(a,c)$ ($i = 0, 1$) is determined by the interchange law:

$$\xi \circ_h \zeta = [(f_0)_*\xi] \oplus [g_\ast^\delta \xi] = [g_\ast \xi] \oplus [(f_1)_*\zeta]$$

for tracks $\zeta : g_0 \Rightarrow f_1 : a \to b$ and $\xi : f_0 \Rightarrow f_1 : b \to c$. Here $(f_0)_*\zeta = s_0(f_0) \circ_h \zeta$, where the degeneracy $s_0 : \mathcal{D}_0 \to \mathcal{D}_1$ embeds $\mathcal{D}_0$ in $\mathcal{D}_1$ by mapping $f : a \to b$ to the identity track $\text{Id} : f \Rightarrow f$ in $\mathcal{D}_1(a,b)$.

5.4. Definition. A natural system (in abelian groups) on a track category $\mathcal{D} = (\mathcal{D}_1\to\mathcal{D}_0)$ is a collection $\mathcal{K}$ of abelian groups $K_f$ indexed by $f \in \mathcal{D}_0(a,b)$, equipped with:

(a) For every composable sequence $a \overset{g}{\Rightarrow} b \overset{f}{\Rightarrow} c$ in $\mathcal{D}_0$, pre- and post-composition homomorphisms $f_* : K_g \to K_{fg}$ and $g^* : K_f \to K_{fg}$, with $(fg)_* = g_\ast f_\ast$ and $(fg)^* = g^\ast f^\ast$.

(b) A homomorphism $\xi_* : K_{f_0} \to K_{f_1}$ for every $\xi \in \mathcal{D}_1(f_0, f_1)$, such that for any $\zeta \in \mathcal{D}_1(f_1, f_2)$ we have $\langle \xi \oplus \zeta \rangle_* = \zeta_* \xi_*$. In particular, this reduces to an action of the group $\mathcal{D}_1(f, f)$ on $K_f$ when $f = f_0 = f_1$.

(c) The two structures commute – that is, if we use (5.3) to define $\circ_h : K_f \times K_g \to K_{fg}$ for any $a \overset{g}{\Rightarrow} b \overset{f}{\Rightarrow} c$ in $\mathcal{D}_0$, then:

$$\langle \xi \circ_h \alpha \rangle \circ_h \langle \zeta \circ_h \beta \rangle = \langle \xi \circ_h \alpha \circ_h \beta \rangle$$

for $\zeta : g_0 \Rightarrow g_1$ in $\mathcal{D}_1(a,b)$, $\xi : f_0 \Rightarrow f_1$ in $\mathcal{D}_1(b,c)$, $\alpha \in K_{f_0}$, and $\beta \in K_{g_0}$.

5.6. Proposition. For a fixed track category $\mathcal{D} = (\mathcal{D}_1\to\mathcal{D}_0)$, there is a bijective correspondence between modules $\mathcal{M}$ over $\mathcal{D}$ and natural systems $\mathcal{K}$ on $\mathcal{D}$ (up to isomorphism), defined by $\mathcal{M} \mapsto \text{Ker}(p^\delta)$, where $p^\delta$ is as in 5.1.

Proof. Since $\mathcal{K} := \text{Ker}(p^\delta)$ is an abelian group object in $(\mathcal{Spd}^\delta, \mathcal{D}_0)\text{-Cat}$, for every $a, b \in \mathcal{O} := \text{Obj}(\mathcal{D}_0)$, $\mathcal{K}(a,b)$ is a disjoint union of abelian groups $K_f$, one for each $f \in \mathcal{D}_0(a,b)$. Moreover, the maps of semi-discrete track categories $K \to \mathcal{M}^\delta \overset{p^\delta}{\to} \mathcal{D}^\delta$ and $s^\delta$ induce a split exact sequence of groups

$$\begin{array}{cccccc}
0 & \longrightarrow & K_f & \longrightarrow & \mathcal{M}_1(f, f) & \overset{p_f}{\longrightarrow} & \mathcal{D}_1(f, f) & \longrightarrow & 1
\end{array}$$

for each $f \in \mathcal{D}_1(a,b)$. Thus if we write $M_f := \mathcal{M}_1(f, f)$ and $D_f := \mathcal{D}_1(f, f)$ for the two automorphism groups, we have a semi-direct product of groups: $M_f \cong K_f \ltimes D_f$. 

The fundamental groupoid functor to each simplicial mapping space $X$ of $\xi$-morphism $\psi$ induces an isomorphism $\psi^* : G(f_0, f_1) \cong G(f_0, f_1)$ for every $f_0, f_1 \in D_0(a, b) = M_0(a, b)$. In particular, this holds for $\psi = s(\xi)$, for any $\xi \in D_1(f_0, f_1)$.

Combining this with (5.7), we see that

$$\sigma : \prod_{D_1(f_0, f_1)} K_{f_0} \cong G(f_0, f_1)$$

(as sets), where the isomorphism $\sigma$ maps $(\xi, \alpha)$ in $D_1(f_0, f_1) \times K_{f_0}$ to $s(\xi) \cdot (i_{f_0}(\alpha))$ in $M_1(f_0, f_1)$, which we shall denote simply by $\xi \cdot \alpha$.

Because $p|G$ is a map of groupoids, the vertical composition in $G$ respects the splitting (5.8) in the sense that

$$(\xi, \alpha) \circ (\xi', \alpha') = (\xi \circ \xi', \alpha + \xi^{-1} \alpha'(\alpha'))$$

for $f_0 \xrightarrow{\xi} f_1 \xrightarrow{\xi'} f_2$ in $D_1(a, b)$, $\alpha \in K_{f_0}$, and $\alpha' \in K_{f_1}$.

Finally, because $i : K \to M$ and $s : D \to M$ are maps of track categories (in $(\mathbb{S}pd, D_0)$-Cat), we see that $\sigma$ of (5.8) respects the horizontal compositions $\circ_h$, so (5.5) holds. Moreover, it is clear from this description that one can reconstruct the module $\mathcal{M}$ from the natural system $\mathcal{K}$. $\square$

5.9. Example. The main example of a natural system is obtained as follows:

Let $X_\bullet$ be an $(\mathcal{S}, \mathcal{O})$-category – or equivalently, a simplicial category in $s\mathcal{C}at$, with fixed object set $\mathcal{O}$, and let $\mathcal{D} := \mathcal{P} \mathcal{X}_\bullet$ the track category obtained by applying the fundamental groupoid functor to each simplicial mapping space $X_\bullet(a, b)$ (for $(a, b) \in \mathcal{O}$) of $X_\bullet$ (assuming these are Kan complexes), so $\mathcal{D}_0 = X_0$. Fix some $n \geq 2$.

For each $f \in \mathcal{D}_0(a, b)$, let $K_f := \pi_n(X_\bullet(a, b); f)$. Since both $\mathcal{P} \mathcal{X}$ and $\pi_n$ commute with products in $\mathcal{S}$, the composition structure maps $\circ : X_\bullet(a, b) \times X_\bullet(b, c) \to X_\bullet(a, c)$ induce $\circ_h : K_f \times K_g \to K_{gf}$, as well as

$$\pi_1(X_\bullet(a, b); f_0, f_1) \times \pi_1(X_\bullet(b, c); g_0, g_1) \to \pi_1(X_\bullet(a, c); g_0 \circ f_0, g_1 \circ f_1).$$

The action of the fundamental groupoid on the higher homotopy groups defines the isomorphism $\xi^* : K_{f_0} \to K_{f_1}$ for each $\xi \in \pi_1(X_\bullet(a, b); f_0, f_1)$, and satisfies (5.5) by naturality of this action (see [14 §1.13]).

Yet a third way of looking at modules over track categories is the following, in the spirit of [10 §1]:

5.10. Definition. For any track category $\mathcal{D}$, $\text{Fac} \mathcal{D}$, the category of factorizations of $\mathcal{D}$ having as objects the maps of $\mathcal{D}_0$, and as morphisms $(h, k, \xi) : f \to g$ “homotopy commuting” squares:

$$\begin{array}{ccc} a & \xrightarrow{f} & c \\ \downarrow{h} & & \downarrow{k} \\ b & \xrightarrow{g} & d \end{array}$$

(5.11)
so that $\xi : \ell \circ f \circ k \Rightarrow g$. The composition defined by concatenation of squares:

$$
\begin{array}{ccc}
a & \overset{f}{\longrightarrow} & a' \\
\downarrow^{k} & \downarrow^{\xi} & \downarrow^{\ell} \\
b & \overset{g}{\longrightarrow} & b' \\
\downarrow^{m} & \downarrow^{\zeta} & \downarrow^{n} \\
c & \overset{h}{\longrightarrow} & c'
\end{array}
$$

so $(k, \ell, \xi) : f \rightarrow g$ and $(m, n, \zeta) : g \rightarrow h$ compose to $(km, n\ell, m^*n_\zeta \oplus \xi : f \rightarrow h)$. Note that \text{Fac} \mathcal{D} is the Grothendieck construction on the functor $\mathcal{D}_0^{op} \times \mathcal{D}_0 \rightarrow \text{Cat}$ which sends $(a, b)$ to the groupoid $\mathcal{D}(a, b)$. A natural system on $\mathcal{D}$ is then just a functor $K : \text{Fac} \mathcal{D} \rightarrow \text{AbSp}$. More generally, if $\mathcal{C}$ is any category, a natural system in $\mathcal{C}$ on $\mathcal{D}$ is a functor $K : \text{Fac} \mathcal{D} \rightarrow \mathcal{C}$. Such a $K$ assigns to each $f : a \rightarrow b$ in $\mathcal{D}_0$ an object $\mathcal{K}_f \in \mathcal{C}$, and to each diagram (5.11) a morphism $\star_{(k,h,\xi)} : K_f \rightarrow K_g$ in $\mathcal{C}$, where:

$$(5.13) \quad \star_{(k,h,\xi)}(\alpha) = \xi_s(k, h^*(\alpha)) = \xi_s[\alpha \circ_h s_0(k)]$$

(cf. (5.3)). In other words, the operation $\star_{(k,h,\xi)}$ is composed of three simpler operations: $\xi_s$, pre- and post-composition (as we can see by letting $k$, $n$, and either $\xi$ or $\zeta$ in (5.12) be identity maps or tracks).

The category of natural systems in $\mathcal{C}$ on $\mathcal{D}$ (with natural transformations as morphisms) will be denoted by $\mathcal{NS}_\mathcal{D}(\mathcal{C}) = \mathcal{C}^{\text{Fac} \mathcal{D}}$.

5.14. **Free natural systems.** Note that there is a forgetful functor $U : \mathcal{NS}_\mathcal{D}(\text{Set}) \rightarrow \text{Obj}(\text{Set})^{\text{Arr} \mathcal{D}_0}$, which assigns to a natural system $\mathcal{K} : \text{Fac} \mathcal{D} \rightarrow \text{Set}$ the corresponding set function on objects $\text{Obj}(\mathcal{K}) : \text{Obj}(\text{Fac} \mathcal{D}) = \text{Arr} \mathcal{D}_0 \rightarrow \text{Obj}(\text{Set})$. This has a left adjoint $\mathcal{F} : \text{Obj}(\text{Set})^{\text{Arr} \mathcal{D}_0} \rightarrow \mathcal{NS}_\mathcal{D}(\text{Set})$, constructed as follows: given a function $K : \text{Arr} \mathcal{D}_0 \rightarrow \text{Obj}(\text{Set})$, the free natural system $\mathcal{F}K : \text{Fac} \mathcal{D} \rightarrow \text{Set}$ is defined for any $g \in \text{Arr} \mathcal{D}_0$ by:

$$(5.15) \quad (\mathcal{F}K)_g := K(f) \times \text{Hom}_{\text{Fac} \mathcal{D}}(f, g),$$

where we identify $(x, (h, k, \xi))$, with $\xi_s(k, h^*(x))$, where $x \in K(f)$ and $(h, k, \xi) : f \rightarrow g$ are as in (5.11). The description of $\mathcal{F}K$ on morphisms is given by (5.12). Note that $K(g)$ embeds in $(\mathcal{F}K)_g$ by $x \mapsto (x, \text{Id})$.

6. **Cohomology of track categories**

In order to define a Baues-Wirsching type cohomology theory for track categories, we first consider:

6.1. **Nerves for track categories.** Several concepts of nerves for higher categories have appeared in the literature (see [11, 16, 43]). We shall need the following version, due in this form to Steet in [55] (but see also [27]):

If $\mathcal{D} = (\mathcal{D}_1 \supset \mathcal{D}_0)$ is a track category, its nerve $\mathcal{N} \mathcal{D}$ is the 3-coskeletal simplicial set $\mathcal{N}_\bullet$ with:

(a) $\text{sk}_1 \mathcal{N}_* = \mathcal{N} \mathcal{D}_0$ (the usual nerve of the category $\mathcal{D}_0$);
(b) \( N_2 \) has a 2-simplex for every \( \xi \in \mathcal{D}_1(g \circ h, f) \), with faces:

\[
\begin{align*}
(6.2) & & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \\
& & \
\end{align*}
\]

(c) \( N_3 \) has a 3-simplex \( \tau \) for every identity in \( \mathcal{D}_1 \) of the form

\[
(6.3) f \eta \oplus \zeta = h \theta \oplus \xi
\]

for a diagram:

\[
(6.4)
\]

so that the (flattened) 3-simplex \( \tau \) is:

\[
(6.5)
\]

(with outer edges identified pairwise).

By setting \( m = gh \), \( k = fg \), and \( \ell = fgh \) (with identity tracks) we obtain \( \zeta = f \eta \) and \( \xi = h \theta \). Similarly, for \( h = \text{Id} \), \( m = g \), and \( \eta = \text{Id} : g \Rightarrow g \), we have \( \zeta \oplus \theta = \xi \). Finally, since the (horizontal) composition of tracks \( \eta : f \Rightarrow f' \) and \( \theta : g \Rightarrow g' \) satisfies

\[
\theta \circ_h \eta = g \eta \oplus (f')^* \theta = f^* \theta \oplus f' \eta,
\]

one can recover all the structure of \( \mathcal{D} \) from \( N_\bullet \).

6.5. **Nerves and natural systems.** For any simplicial set \( N_\bullet \) define \( \partial_{\text{max}} : N_n \to N_1 \) by \( \partial_{\text{max}}(\sigma) := d_1 d_2 \cdots d_{n-1} \sigma \in N_1 \) (the edge between the initial and final vertex of \( \sigma \)). In particular, when \( N_\bullet = \mathcal{N}_\mathcal{D} \) we may define \( N_n[f] := \{ \sigma \in N_n \mathcal{D} : \partial_{\text{max}}(\sigma) = f \} \) (for any arrow \( f \) in \( \mathcal{D}_0 \) and \( n \geq 1 \)). For \( n = 0 \) we set \( N_n[\text{Id}_a] = \{ a \} \) and \( N_n[f] = \emptyset \) otherwise. This defines a function \( \text{Arr} \mathcal{D}_0 \to \text{Obj}(\mathsf{Set}) \), and thus for each \( n \geq 0 \) we have a free natural system in sets on \( \mathcal{D} \) denoted by \( \tilde{N}_n \mathcal{D} \) (see §5.14).
6.6. Remark. Observe that for $n = 3$, the collection $N_3\mathcal{D}$ of all 3-simplices in the nerve of a track category $\mathcal{D}$ itself constitutes a natural system (in sets) on $\mathcal{D}$, with $f \mapsto N_3[f]$ on objects of $\text{Fac} \mathcal{D}$. We define the operation $\ast$ of (5.13) on a 3-simplex $\tau$ indexed by a diagram (6.4) as follows:

Given a diagram (5.11) of the following special form:

\[
\begin{array}{ccc}
0 & \rightarrow & 3 \\
\downarrow \lambda & & \downarrow e \\
1 & \rightarrow & 4
\end{array}
\]

we define $\lambda_\ast e_\ast \tau$ to be the 3-simplex indexed by

\[
\begin{array}{ccc}
0 & \rightarrow & 3 \\
\downarrow \lambda & & \downarrow e \\
1 & \rightarrow & 4 \\
\downarrow \lambda & & \downarrow e \\
2 & \rightarrow & 4
\end{array}
\]

The precomposition $\mu_\ast i_\ast \tau$, when (5.11) has the form:

\[
\begin{array}{ccc}
0 & \rightarrow & 3 \\
\downarrow \lambda & & \downarrow e \\
1 & \rightarrow & 3 \\
\downarrow \lambda & & \downarrow e \\
2 & \rightarrow & 3
\end{array}
\]

is defined analogously.

This allows us to think of a 3-simplex $\tau$ indexed by the diagram (6.4) as encoding a morphism in $\text{Fac} \mathcal{D}$ from $g$ to $\ell$ (compare §5.10), with $\tau$ thus representing the operation

\[
\tau_\ast := \eta_\ast h_\ast \zeta_\ast f_\ast
\]

in any natural system $\mathcal{K} : \text{Fac} \mathcal{D} \rightarrow \mathcal{C}$.

In particular, any 2-simplex $\rho$ as in (6.2) can be thought of as a degenerate 3-simplex, and thus yields two operations for a natural system $\mathcal{K} : \text{Fac} \mathcal{D} \rightarrow \mathcal{C}$, namely:

\[
\rho_\ast := \xi_\ast g_\ast : \mathcal{K}_h \rightarrow \mathcal{K}_f \quad \text{and} \quad \rho_\ast := \xi_\ast h_\ast : \mathcal{K}_g \rightarrow \mathcal{K}_f .
\]

6.10. Baues-Wirsching type cohomology. If $\mathcal{D}$ is a track category, the face and degeneracy maps of the nerve $N_\ast = N \mathcal{D}$ induce maps of natural systems as follows:

(a) If $\phi = d_i : N_n \mathcal{D} \rightarrow N_{n-1} \mathcal{D}$ \quad (0 < i < n) or $\phi = s_j : N_n \mathcal{D} \rightarrow N_{n+1} \mathcal{D}$ \quad (0 \leq j \leq n), we define $\tilde{\phi} : \tilde{N}_n \mathcal{D} \rightarrow \tilde{N}_{n\pm1} \mathcal{D}$ to be $\mathcal{F}\phi$.

(b) Given $\sigma \in N_n \mathcal{D}$, consider the 2-simplex $\rho_0 = d_2 \cdots d_{n-1} \sigma$ of $\mathcal{N} \mathcal{D}$, and define the map of natural systems $\tilde{d}_0 : \tilde{N}_n \mathcal{D} \rightarrow \tilde{N}_{n-1} \mathcal{D}$ by setting $\tilde{d}_0(\iota(\sigma)) := \rho_0^\ast(d_0 \sigma)$, (see (6.9)). This extends to all of $\tilde{N}_n \mathcal{D}$ by the adjointness of $\mathcal{F}$.

(c) Similarly define $\tilde{d}_n : \tilde{N}_n \mathcal{D} \rightarrow \tilde{N}_{n-1} \mathcal{D}$ by setting $\tilde{d}_n(\iota(\sigma)) := (d_1 \cdots d_{n-2} \sigma)_\ast(d_n \sigma)$.
This makes \( \tilde{N}_n := (\tilde{N}_n^D)_{n=0}^\infty \) into a simplicial object in the category \( \text{NS}_D(\text{Set}) \) of natural systems in \( \text{Set} \) on \( D \). Compare this to the description of the usual Baues-Wirsching complex for a category \( C \) in terms of a two-sided bar construction \( B_\bullet(G) \) in \([3, \S 3]\).

Now let \( K \) be a natural system (in \( \text{AbSp} \)) on a track category \( D \). We define a cosimplicial set \( C^\bullet(D; K) \) by setting \( C^n(D; K) := \text{Hom}_{\text{NS}_D(\text{Set})}(\tilde{N}_n^D, UK) \), where \( U : \text{AbSp} \to \text{Set} \) is the forgetful functor. Since \( UK \) is an abelian group object in \( \text{NS}_D(\text{Set}) \), we see that \( C^\bullet(D; K) \) is actually a cosimplicial abelian group (or equivalently, a cochain complex). Its cohomotopy (i.e., the cohomology of the corresponding cochain complex) is defined to be the \textit{Baues-Wirsching cohomology} of \( D \) with coefficients in \( K \), written \( H^n_{\text{BW}}(D; K) := \pi^n(C^\bullet(D; K)) \) (compare \([10, \S 1] \) and \([4, \S 2]\)).

6.11. \textit{Remark.} Note that the embedding \( \text{Set} \hookrightarrow \text{Spd} \) defined by taking the discrete groupoid on a set extends to a full and faithful functor \( \mathcal{O}\text{-Cat} \hookrightarrow (\text{Spd}, \mathcal{O})\text{-Cat} \), so we can think of any small category \( E \) as a (discrete) track category \( D_E \). Since \( \text{Fac} D_E \), a natural system \( K \) on \( D_E \) is just a natural system on \( E \) (see \([10, \S 1]\)), and thus \( H^n_{\text{BW}}(D_E; K) \), the Baues-Wirsching cohomology of \( E \).

Furthermore, if the track category \( D \) is actually an internal category in groups (equivalently, a crossed module), \( H^n_{\text{BW}}(D; K) \) may be identified with the cohomology of the classifying space \( BD \) (see \([34]\) and \([49]\)).

7. \textit{Comparison with \((\mathcal{S}, \mathcal{O})\text{-cohomology}\)}

Recall from \([1, 2]\) that an \((\mathcal{S}, \mathcal{O})\text{-category} \ X_\bullet \) is a category enriched in simplicial sets (cf. \([1, 2]\) – or equivalently, a simplicial object in \( \mathcal{O}\text{-Cat} \) – with constant object set \( \mathcal{O} \).

7.1. \((\mathcal{S}, \mathcal{O})\text{-categories}.\) In \([30, \S 1]\), Dwyer and Kan define a model category structure on \((\mathcal{S}, \mathcal{O})\text{-Cat} \), in which the fibrations and weak equivalence are defined objectwise (that is, on each mapping space \( X_\bullet(a, b) \)). As noted in \([3, 4]\) if \( X_\bullet \) is fibrant, then \( \tilde{\pi}_1 X_\bullet \) is a track category, and \( \pi_n X_\bullet \) is a module over \( \tilde{\pi}_1 X_\bullet \) for each \( n \geq 2 \). Moreover, a map \( \Phi : M_\bullet \to N_\bullet \) in \((\mathcal{S}, \mathcal{O})\text{-Cat} \) is a weak equivalence if and only if it induces an isomorphism in \( \pi_n \) for all \( n \geq 1 \).

Note that homotopy functors for simplicial sets which strictly preserve products extend to \((\mathcal{S}, \mathcal{O})\text{-Cat} \), with the usual properties. For example, given an \((\mathcal{S}, \mathcal{O})\text{-category} \ X_\bullet \), for each \( n \geq 1 \) we have a \((P^n\mathcal{S}, \mathcal{O})\text{-category} \ Y_\bullet := P^n X_\bullet \) in which each mapping space \( Y_\bullet(a, b) \) is the \( n \)-the Postnikov section \( P^n X_\bullet(a, b) \).

Similarly if \( M \) is a module over a track category \( D \) \([5, 1]\), applying the twisted Eilenberg-Mac Lane functor \( E_D(-, n) := K(-, n) \times N D \) objectwise to \( M \) (see \([13, \S 5]\)) yields the \textit{(twisted) Eilenberg-Mac Lane} \((\mathcal{S}, \mathcal{O})\text{-category} \ E_D(M, n) \) in \((\mathcal{S}, \mathcal{O})\text{-Cat}/N D \), using the natural map \( q : E_D(M, n) \to P^1 E_D(M, n) \simeq N D \). This map \( q \) is equipped with a section \( s : ND \to E_D(M, n) \), making \( E_D(M, n) \) into an abelian group object in \((\mathcal{S}, \mathcal{O})\text{-Cat}/N D \) (see \([30]\) or \([4, \S 3]\)).
Finally, for each $n \geq 0$ there is an $n$-th $k$-invariant square functor, which assigns to each $(S, O)$-category $Y_*$ a homotopy pull-back square:

\[
\begin{align*}
P^{n+1}Y_* & \xrightarrow{p^{(n+1)}} P^nY_* \\
BD & \xrightarrow{k_n} E_D(M, n+2)
\end{align*}
\]

(over $BD$), where $D := \pi_1 Y_*$ and $M$ is the module $\pi_{n+1} Y_*$ over $D$. The map $k_n : P^n Y \to E_D(M, n+2)$ is called the $n$-th (functorial) $k$-invariant for $Y_*$. Compare [13 Proposition 6.4].

7.3. Remark. There is a forgetful functor $U : \mathcal{C}at \to \mathcal{D}i\mathcal{S}$ to the category of directed graphs, whose left adjoint $F : \mathcal{D}i\mathcal{S} \to \mathcal{C}at$ is the free category functor (both $U$ and $F$ are the identity on objects). This pair of adjoint functors defines a comonad generated by the inclusions $d_i$ of $K$. $\mathcal{D}$ is a $\mathcal{C}at$-category, and $\mathcal{O}$ is a $\mathcal{D}i\mathcal{S}$-category whose left adjoint $D$ is the module $\pi_{n+1} Y_*$ over $\mathcal{D}$. The map $k_n : P^n Y \to E_D(M, n+2)$ is called the $n$-th (functorial) $k$-invariant for $Y_*$. Compare [13 Proposition 6.4].

7.4. Definition. Given a track category $\mathcal{D}$, a module $\mathcal{M}$, and an $(S, O)$-category $X_*$ equipped with a map $X_* \to \mathcal{N}D$, Dwyer, Kan, and Smith define the $n$-th $(S, O)$-cohomology group of $X_*$ with coefficients in $\mathcal{M}$ to be

\[
H^n_{(S, O)}(X_*/\mathcal{N}D; \mathcal{M}) := [X_* , E_D(M,n)]_{(S, O)-\mathcal{C}at/\mathcal{N}D} = \pi_0 \text{map}_{\mathcal{N}D}(X_* , E_D(M,n)) .
\]

When $X_* \to \mathcal{N}D$ is a weak equivalence, we abbreviate $H^n_{(S, O)}(X_*/\mathcal{N}D; \mathcal{M})$ to $H^n_{(S, O)}(\mathcal{D}; \mathcal{M})$.

Another model for topologically enriched categories is provided by $(\mathcal{C}, O)-\mathcal{C}at$, where $\mathcal{C}$ denotes the category of cubical sets:

7.5. Definition. Let $\Box$ denote the Box category, whose objects are the abstract cubes $\{I^n\}_{n=0}^\infty$ (where $I := \{0,1\}$ and $I^0$ is a single point). The morphisms of $\Box$ are generated by the inclusions $d_i : I^{n-1} \to I^n$ and projections $s_i : I^n \to I^{n-1}$ for $1 \leq i \leq n$ and $\varepsilon \in \{0,1\}$. A contravariant functor $K : \Box^{op} \to \text{Set}$ is called a cubical set.

We write $K_n$ for the set $K(I^n)$ of $n$-cubes in $K$. The collection of all these, for $n \geq 0$, form a category $\mathcal{C}_K$ (with inclusions as morphisms). The two maps $d_i : K_n \to K_{n-1}$ ($\varepsilon \in \{0,1\}$) induce the $i$-th face maps of $K$, and the map $s_j : K_n \to K_{n+1}$ (induced by $s^j$) is called the $j$-th degeneracy. The sub-cubical set of $K$ generated by $K_0, K_1, \ldots, K_n$ under the degeneracies is called the $n$-th cubical skeleton of $K$, written $\text{sk}_n K$.

7.6. Definition. If $K$ and $L$ are two cubical sets, their cubical tensor $K \otimes L \in \mathcal{C}$ is defined:

\[
K \otimes L := \text{colim}_{I^j \in \mathcal{C}_K, I^k \in \mathcal{C}_L} I^{j+k} .
\]

This defines a symmetric monoidal structure on cubical sets.

Cubical sets are related to simplicial sets by a pair of adjoint functors

\[
\mathcal{C}_s := \mathcal{S}_\text{cub}^T ,
\]
where the triangulation functor $T$ is defined by: $TK := \text{colim}_{r \in C_K} \Delta[1]^n$, and the cubical singular functor $S_{\text{cub}}$ is defined by $(S_{\text{cub}} X)(I^n) := \text{Hom}_{\Delta}(TI^n, X)$. These induce equivalences of the corresponding homotopy categories. For further details, see [26], or the surveys in [41, 42]. The adjoint pair (7.7) prolong to functors between $(\mathcal{C}, \mathcal{O})\text{-Cat}$ and $(\mathcal{S}, \mathcal{O})\text{-Cat}$, which also induce equivalences of homotopy categories (cf. [15]).

7.8. Models for $(P^1\mathcal{C}, \mathcal{O})\text{-categories}$. Any track category $\mathcal{D}$ is equivalent to one in which $\mathcal{D}_0$ is free on its homotopy category $\Pi_0\mathcal{D}$ (see [31 (A.2)]). We use this to give an explicit description of a cofibrant and fibrant 2-coskeletal $(\mathcal{C}, \mathcal{O})\text{-category}$ $W$ weakly equivalent to $S_{\text{cub}}N\mathcal{D}$ directly in terms of $\mathcal{D}$, in the spirit of the Boardman-Vogt “W-construction” (see [15] §3):

(a) The 0-cubes of the cubical mapping space $W(a, b)$ correspond to maps in the free category $\mathcal{D}_0 = F\text{ ho }\mathcal{D}$, which in turn are described by composable sequences

$$f_* = (a = a_{n+1} \xrightarrow{f_{n+1}} a_n \xrightarrow{f_n} \cdots a_i \xrightarrow{f_{i+1}} a_i \xrightarrow{f_i} a_{i-1} \xrightarrow{f_{i-1}} \cdots a_1 \xrightarrow{f_1} a_0 = b)$$

in $\text{ho }\mathcal{D}$. The corresponding vertex is denoted by $f_* \otimes := f_1 \otimes \cdots \otimes f_n \otimes f_{n+1}$, using the monoidal structure in $\mathcal{C}$.

(b) For any track

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$a$};
  \node (b) at (1,1) {$b$};
  \node (c) at (2,0) {$c$};
  \node (f) at (1,0) {$f$};
  \node (h) at (1,-1) {$h$};
  \node (g) at (0,1) {$g$};
  \draw[->] (a) to (f);
  \draw[->] (a) to (h);
  \draw[->] (f) to (c);
  \draw[->] (g) to (h);
  \draw[->] (g) to (f);
  \draw[->] (h) to (c);
\end{tikzpicture}
\end{center}

in $\mathcal{D}$, we have a 1-cube $\xi : f \otimes g \Rightarrow h$ in $W(a, b)$ (where either $f$ or $g$ could be $\text{Id}$).

(c) For any diagram in $\mathcal{D}$ of the form (6.4), satisfying (6.3), we have a 2-cube in $W$

$$f \otimes g \otimes h \xrightarrow{f \otimes n} f \otimes m \xrightarrow{\otimes h} k \otimes h \xrightarrow{\xi} \ell \, .$$

(d) In general, an $n$-cube $I^f_{\bullet}$ of $W(a, b)$ is determined by the following data:

- A composable sequence $f_*$ in $\mathcal{D}_0$ of length $n + 1$ as in (7.9).
- For every $1 \leq i \leq j \leq n$, a map $h^{[i,j]} : a_j \to a_i$ in $\mathcal{D}_0$, where $h^{[i,i+1]} := f_{i+1}$ and $h^{[i,i]} := \text{Id}_{a_i}$.
- For each partition of the form:

$$\alpha = (0 = i_0 < i_1 < \cdots < i_{k-1} < i_k = n + 1) \, ,$$

we thus obtain a composable sequence

$$h_\bullet[\alpha] = (a_{n+1} = a_{i_k} \xrightarrow{h^{[i_k-1,i_k]}} a_{i_{k-1}} \xrightarrow{h^{[i_{k-2},i_{k-1}]}} a_{i_{k-2}} \cdots a_i \xrightarrow{h^{[0,i_1]}} a_0 = a_0) \, .$$

In $\mathcal{D}_0$ of length $k$. The vertices of $I^f_{\bullet}$ are indexed by $\alpha \in A$, with the corresponding vertex denoted by $g_\bullet[\alpha] \otimes$. In particular, the vertex indexed by $f_* \otimes := d_1^0 d_2^0 \cdots d_n^0 I^f_{\bullet}$ is called the initial vertex of $I^f_{\bullet}$, and the vertex indexed by the single map $h^{[0,n+1]} = d_1^0 d_2^0 \cdots d_n^0 I^f_{\bullet}$ is called the terminal vertex of $I^f_{\bullet}$. By analogy with (6.6) we denote $h^{[0,n+1]}$ by $\partial_{\text{max}} I^f_{\bullet}$. 
• For every $1 \leq i \leq j \leq k \leq n$, a track $\xi^{[i,j,k]} : h^{[i,j]} \circ h^{[j,k]} \Rightarrow h^{[i,k]}$ in $D$, which is the identity track if $i = j$ or $j = k$ (or both).

• These tracks are compatible in that for every $1 \leq i \leq j \leq k \leq \ell \leq n$, the diagram

$$
\begin{array}{ccc}
h^{[i,j]} \circ h^{[j,k]} \circ h^{[k,\ell]} & \Rightarrow & h^{[i,j]} \circ h^{[j,\ell]} \\
(\xi)^{[i,j,k]} & \downarrow & (\xi)^{[i,j,\ell]}
\end{array}
$$

(7.13)

commutes in $D$.

• Given $\alpha$ as above and $0 < j < k$, let $(\alpha, \hat{j})$ be the partition obtained from $\alpha$ by omitting $i_j$. There is an edge in $\mathcal{I}^\bullet$ from $\hat{g}_* [\alpha]^{\otimes}$ to $\hat{g}_* [\alpha, \hat{j}]^{\otimes}$, indexed by

$$h^{[i_0,i_1]} \otimes \ldots \otimes h^{[i_{j-1},i_j-1]} \otimes \xi^{[i_{j-1},i_j,i_j+1]} \otimes h^{[i_{j+1},i_{j+2}]} \otimes \ldots \otimes h^{[i_{k-1},i_k]} .$$

• The 2-faces in $\mathcal{I}^\bullet$ are similarly determined by $\alpha$ as above and a choice of two indices $0 < j' < j'' < k$ to be omitted, and so on.

7.14. Remark. Let $S^n \in \mathcal{C}_n$ be a pointed cubical sphere (e.g., $I^n / \partial I^n$), which corepresents $\pi_n$ for pointed cubical sets. Given a fibrant cubical set $K$, with $V := P^1 K$ a fibrant cubical model for $N \hat{\pi}_1 K$, the twisted sphere $S^n \otimes V$ corepresents the $\hat{\pi}_1 K$-module $\{\pi_n(K,k)\}_{k \in K_0}$ in $\mathcal{C} / V$.

More generally, if $X \in (\mathcal{C}, \mathcal{O})$-Cat is fibrant and $W$ is a cofibrant model for $P^1 X$, then for each $a, b \in \mathcal{O}$ the twisted cubical $n$-sphere $S^n \otimes W(a,b)$ generates under pre- and post-composition a $(\mathcal{C}, \mathcal{O})$-category $S^n_{(a,b)} \otimes W$ in $(\mathcal{C}, \mathcal{O})$-Cat/W, with $sk_{n-1}(S^n_{(a,b)} \otimes W) = sk_{n-1} W$. This again corepresents $\pi_n X(a,b)$, so the various choices of $a, b \in \mathcal{O}$ (for fixed $n$) together corepresent the natural system $\pi_n X$ of Example 5.9.

7.15. Cubical Eilenberg-Mac Lane categories. Let $D$ be a track category and $\mathcal{K}$ be a natural system on $D$, with $\mathcal{M}$ the corresponding module over $D$ (cf. Proposition 5.6), and let $W$ be the $(\mathcal{C}, \mathcal{O})$-model for $D$ constructed above. For each $n \geq 2$, we can use it to construct an explicit fibrant $(\mathcal{C}, \mathcal{O})$-model $E$ for $E_D(\mathcal{M}, n)$, as follows:

We start with $sk_{n-1} E := sk_{n-1} W$, and let $W_n[f] := \{I \in W_n : \partial_{\text{max}} I = f\}$ for each arrow $f \in D_0$ (compare (6.6)). The degenerate $n$-cubes of $E$ are those of $W$, and the $n$-cubes of $E$ are defined by setting $E_n := \{(I, \alpha) : I \in W_n, \alpha \in \mathcal{K}_{\partial_{\text{max}} I}\}$, so $E_n[f] = \mathcal{K}_f \times W_n[f]$ for each $f \in D_0$. The face maps are zero if $\alpha \neq 0$, $d_j^e(I, 0) := d_j^e I \in W_{n-1}$, and the degeneracies are formal (i.e., we add symbols $s_j(I, \alpha) \in E_{n+1}$ for each $1 \leq j \leq n$). Note that $(I, \alpha)$ is never degenerate for $\alpha \neq 0$, even though $I$ itself may be degenerate (i.e., some or all of the factors $f_i$ of the sequence $f_\bullet$ indexing $I$ may be identity maps) or decomposable.

For each $J \in W_{n+1}$ indexed by $f_\bullet = (a_{n+2} \xrightarrow{f_{n+2}} a_{n+1} \ldots a_1 \xrightarrow{f_1} a_0)$, consider a collection $\bar{a} = (\alpha_i^0, \alpha_i^1)^{n+1}_{i=1}$ of elements in $\mathcal{K}$, with

$$\alpha_i^e \in \mathcal{K}_{\partial_{\text{max}}(d_i^e I)}$$

and

$$\alpha_i^1 = 0 \text{ for } 2 \leq i \leq n .$$
If $\bar{a}$ satisfies the cycle condition

$$
(7.18) \quad \xi^{[0,n+1,n+2]}_\ast f_{n+2}^\ast \alpha_1^1 + \sum_{i=2}^{n} (-1)^i (\alpha_i^1 - \alpha_i^0) + (-1)^{n+1} \xi^{[0,1,n+2]}_\ast (f_0)_\ast \alpha_{n+1}^1 = 0,
$$

then we have a unique $(n+1)$-cube $(J, \bar{a})$ in $E_{n+1}$, with $d_i^0 (J, \bar{a}) = (d_i^0 J, \alpha_i^0)$. We can think of (7.18) as a “matching face condition” for the collection $\bar{a}$, as in a Kan complex (cf. [33] I, §3).

Finally, $E = E_D(M, n)$ is $(n + 1)$-coskeletal. Using Remark 7.14, we see that $\pi_n E \cong M$ as $D$-modules, and $\pi_n E = 0$ for $2 \leq i \neq n$.

7.19. Example. Note that (7.18) succinctly encodes the condition that $\pi_n E \cong M$ as a module over $D$ (see §5.9): by making suitable choices of $\bar{a}$ (with only two nonzero entries $a_i^\xi$), we can ensure that all of the identities of §5.4 are satisfied. For example, let $(I, \alpha) \in E_n$, with $I$ indexed by $f_\ast = (a_{n+1} \rightarrow f_{n+1} \rightarrow \cdots \rightarrow f_1 \rightarrow a_0)$:

(a) Post-composing $(I, \alpha)$ with the $0$-cube of $W \subseteq E$ indexed by $g : a_0 \rightarrow b$ yields a “formal” composite $I' := g \otimes (I, \alpha) \in E_n$, with $\partial_{\text{max}} I' = g \otimes \partial_{\text{max}} I = g \cdot h^{[0,n+1]}$, because the (free) category $D_0$ is identified with $W_0$ by construction. In order for $I'$ to be identified up to homotopy with $(I', g_{*} \alpha)$, we have an $(n + 1)$-cube $(J, \bar{a}) \in E_{n+1}$ such that $d_i^0 J = I'$, $d_i^1 J = (g \otimes I, g_{*} \alpha)$, and $d_i^2 J = (s_i d_i^1 I, 0)$ is degenerate for all $2 \leq i \leq n + 1$ and $\varepsilon = 0, 1$.

(b) One can similarly construct an $(n + 1)$-cube identifying $(I \otimes h, h^* \alpha)$ with $h \otimes (I, \alpha)$.

(c) If $\psi : h^{[0,n+1]} \rightarrow k$ is a track in $D$, let $J'' \in W_n$ be the $n$-cube obtained from $I$ by replacing its terminal vertex $\partial_{\text{max}}I = h^{[0,n+1]}$ by $k$, and post-composing all edges ending in $h^{[0,n+1]}$ with $\psi$. Again we have an $(n + 1)$-cube $(J, \bar{a}) \in E_{n+1}$ with $d_i^0 (J, \bar{a}) = (I, \alpha)$ and $d_i^1 (J, \bar{a}) = (J'', \psi_{*} \alpha)$. All other faces are determined by the requirement that all edges are degenerate (indexed by $1d$), except for $d_i^0 d_i^1 \cdots d_i^{n+1} (J, \bar{a}) = \psi$.

For instance, if $n = 2$, $I$ is as in (7.10), and $\psi : \ell \rightarrow k$ is a track in $D$, $(J, \bar{a})$ will be:

$$
(7.20) \quad \begin{array}{c}
\text{Id} & \xrightarrow{\theta \otimes h} & k \otimes h & \xrightarrow{\xi} & \ell \\
\downarrow & & \downarrow & & \\
\downarrow & & \downarrow & & \\
\text{Id} & \xrightarrow{\text{Id}} & k & \xrightarrow{\psi_{*} \alpha} & k
\end{array}
$$

with the top square being $(\alpha, I)$, and the bottom $(\psi_{*} \alpha, I'')$.

(d) Finally, a suitable choice of $(J, \bar{a})$ ensures that $\alpha_1 + \alpha_2 = \alpha_3$ in $\pi_{n} E$.

We are now in a position to prove the following analogue of [4] Theorem 3.9:
7.21. Theorem. Let \( \mathcal{D} \) be a track category, with \( X_\bullet := \mathcal{N} \mathcal{D} \) the corresponding \((\mathcal{P}^1 \mathcal{S}, \mathcal{O})\)-category, and let \( \mathcal{K} \) be a natural system on \( \mathcal{D} \), with \( \mathcal{M} \) the corresponding module over \( \mathcal{D} \). For each \( n \geq 1 \), the \( n \)-th Baues-Wirsching cohomology group \( H^n_{\mathcal{BW}}(\mathcal{D}; \mathcal{K}) \) is then naturally isomorphic to the \((n - 1)\)-st \((\mathcal{S}, \mathcal{O})\)-cohomology group \( H^n_{(\mathcal{S}, \mathcal{O})}(\mathcal{N} \mathcal{D}; \mathcal{M}) \).

Proof. First note that the adjoint pair \((\mathcal{J}, \tilde{\mathcal{I}})\) induce a natural isomorphism between the \((\mathcal{S}, \mathcal{O})\)-cohomology group \( H^n_{(\mathcal{S}, \mathcal{O})}(\mathcal{N} \mathcal{D}; \mathcal{M}) \) and the \((\mathcal{C}, \mathcal{O})\)-cohomology group

\[
H^n_{(\mathcal{C}, \mathcal{O})}(W/\mathcal{D}; \mathcal{M}) := [W, E_\mathcal{D}(\mathcal{M}, n)]_{(\mathcal{C}, \mathcal{O})-cat/W},
\]

where \( W \) is a cofibrant model for \( S_{\text{cub}} \mathcal{N} \mathcal{D} \).

As in [7.8] we may assume that \( \mathcal{D}_0 \) is free, and identify \( f \otimes g \) in \( W_0 \) with the composite \( fg \) in \( \mathcal{D}_0 \). For each arrow \( f \in \mathcal{D}_0 \) set \( W_n[f] := \{ I \in W_n : \partial_{\max} I = f \} \) (compare [6.6]). Given \( \bullet \overset{g}{\to} \bullet \overset{f}{\to} \bullet \overset{h}{\to} \bullet \) in \( \mathcal{D}_0 \), we may use the cubical pre- or post-composition with 0-vertices of \( W \) to define operations \( g^* : W_n[f] \to W_n[fg] \) and \( h_* : W_n[f] \to W_n[hf] \), respectively. Furthermore, for each track \( \psi : f \circ g \Rightarrow h \) and \( I = I' \otimes I'' \in W_n[f \otimes g] \) with \( I' \in W_n[f] \) and \( I'' \in W_n[g] \), we obtain a new \( n \)-cube \( \psi I \) by post-composing all tracks indexing edges into the terminal vertex \( f \otimes g \) of \( I \) with \( \psi \), as in Example 7.19 – see bottom square in Figure (7.20). Under these operations, \( W_n \) constitutes a natural system in sets on \( \mathcal{D}_0 \).

Note that a map \( \phi : W \to E_\mathcal{D}(\mathcal{M}, n) \) of \((\mathcal{C}, \mathcal{O})\)-categories over \( W \) is determined by its values on the indecomposable \( n \)-cubes in \( W \), and \( \phi \) must take \( I \in W_n \), with \( f := \partial_{\max} I \) to an \( n \)-cube \( (I, \phi(I)) \in E_n \) with \( \phi(I) \in \mathcal{K}_f \). Moreover, any \((n + 1)\)-cube \( J \in W_{n+1} \) must map to \((J, \tilde{a})\) satisfying (7.18), so that the map \( \tilde{\phi} : W_n \to \mathcal{K} \) described above is in fact a map of natural systems over \( \mathcal{D} \). Conversely, any such map \( \tilde{\phi} \) in \( NS_\mathcal{D} \) satisfies the cocycle condition, and so uniquely extends to \( W_{n+1} \); it thus determines a map \( \phi : W \to E_\mathcal{D}(\mathcal{M}, n) \) over \( W \), since both \( W \) and \( E_\mathcal{D}(\mathcal{M}, n) \) are \((n + 1)\)-coskeletal. Therefore, we can compute \([W, E_\mathcal{D}(\mathcal{M}, n)]_{(\mathcal{C}, \mathcal{O})-cat/W}\) as the cohomology of the cochain complex associated to the cubical abelian group \( C^* := \text{Hom}_{NS_\mathcal{D}}(W, \mathcal{K}) \). – see [4, §3] and compare [12] §4.

Finally, the \( n \)-cubes in \( W_n[f] \), indexed by composable sequences \( f_* \) of length \( n + 1 \) in \( \mathcal{D}_0 \), with the additional track data described in [7.8] are clearly in one-to-one correspondence with the \((n + 1)\)-simplices of \( \mathcal{N} \mathcal{D} \). By (7.17), the face maps \( d_i \) \((1 < i < n + 1)\) are not relevant, so we can actually identify the cochain complex associated to the cubical abelian group \( C^* := \text{Hom}_{NS_\mathcal{D}}(W, \mathcal{K}) \) with that associated to the cosimplicial abelian group \( C^* (\mathcal{D}; \mathcal{K}) := \text{Hom}_{NS_\mathcal{D}}(\mathcal{N}(\mathcal{D}), \mathcal{K}) \) used to define \( H^n_{\mathcal{BW}}(\mathcal{D}; \mathcal{K}) \) (with a dimension shift). \( \square \)

8. Two-track extensions of track categories

Now let \( \mathcal{G} \) be a two-track category. As noted in [1.4] we can associate to \( \mathcal{G} \) its homotopy track category \( \Pi_0 \mathcal{G} = \text{ho} \mathcal{G} \) by taking the coequalizer of the maps of track categories \( \mathcal{G}_1 \to \mathcal{G}_0 \) (in the notation of (1.5)). At the same time, one can associate with \( \mathcal{G} \) a natural system \( \mathcal{K} := \Pi_2 \mathcal{G} \) on \( \mathcal{D} \), which may be identified with \( \pi_2 \mathcal{N} \mathcal{G} \) described in Example 5.9.

8.1. Definition. A two-track extension of track category \( \mathcal{D} \) by a natural system \( \mathcal{K} \) on \( \mathcal{D} \) is two-track category \( \mathcal{G} = (\mathcal{G}_{[1]} \to \mathcal{G}_{[0]}) \) such that \( \text{ho} \mathcal{G} \) is weakly equivalent to \( \mathcal{D} \), and \( \mathcal{K} \).
is isomorphic to $\Pi_2\mathcal{G}$ under this equivalence. We use the following diagram to describe this situation:

\[(8.2)\]
\[\begin{array}{c}
\mathcal{K} \\
\downarrow d_0
\end{array}
\begin{array}{c}
\mathcal{G}[1]
\downarrow d_1
\end{array}
\begin{array}{c}
\mathcal{G}[0]
\downarrow \mathcal{D}
\end{array}\]

(compare \((4.0)\)).

As an immediate consequence of Theorem \((7.21)\) we deduce (compare \([6, (4.6)]\)):

8.3. **Corollary.** Equivalence classes of two-track extensions of a track category $\mathcal{D}$ by a natural system $\mathcal{K}$ are in one-to-one correspondence with elements of $H^4_{\mathrm{BW}}(\mathcal{D}; \mathcal{K})$.

**Proof.** Let $Y_\bullet := \mathcal{N}_a\mathcal{G}$ be the $(P^2\mathcal{S}, \mathcal{O})$-category corresponding to a two-track extension $\mathcal{G}$ of $\mathcal{D}$ by $\mathcal{K}$, and let $\mathcal{M} = \pi_2 Y_\bullet$ be the module over $\pi_1 Y_\bullet$ corresponding to the natural system $\mathcal{K}$ under the identification of $\pi_1 Y_\bullet$ with $\mathcal{D}$. By \([32, \S 3]\), a $(P^2\mathcal{S}, \mathcal{O})$-category $Y_\bullet$ with first Postnikov section $X_\bullet := P_1 Y_\bullet$ is determined up to weak equivalence by its first $k$-invariant $k_1$. By Corollary \((4.3)\), $\mathcal{G}$ is determined up to weak equivalence by $Y_\bullet$, and thus by $k_1 \in H^3(\mathcal{N}^3\mathcal{D}; \mathcal{M})$. By Theorem \((7.21)\) this cohomology group may be identified with $H^4_{\mathrm{BW}}(\mathcal{D}; \mathcal{M})$.

\[\square\]

8.4. **The Baues-Wirsching type class.** In \([10]\) an explicit cohomology class in $H^3_{\mathrm{BW}}(\mathcal{E}; \mathcal{M})$ was constructed classifying all linear track extensions of the category $\mathcal{E}$ by a natural system $\mathcal{M}$ on $\mathcal{E}$ -- that is, track categories $\mathcal{D}$ such that $\mathrm{ho} \mathcal{D} \cong \mathcal{E}$ and $\pi_1 \mathcal{D} \cong \mathcal{M}$. (Not all track categories have this form, since in general we can only expect $\pi_1 \mathcal{D}$ to be a natural system in groups.) We now do the same for all two-track extensions:

Let $\mathcal{G}$ be a two-track category with homotopy track category $\mathcal{D} = \Pi_2\mathcal{G}$, and natural system $\mathcal{K} := \Pi_2\mathcal{G}$ on $\mathcal{D}$. We choose a section $s : \mathcal{D} \to \mathcal{G}_[1]$ for $\gamma$ as above -- that is, for each $a, b \in \mathcal{O}$, we choose once and for all a map $s(f) \in \mathcal{G}_0(a, b)$ representing each homotopy class $f \in \mathcal{D}_0(a, b)$, and a 2-cell $s(\xi) : s(f) \to s(f')$ in $\mathcal{G}_1(a, b)$ for each track $\xi : f \Rightarrow f'$ in $\mathcal{G}_1(a, b)$ (with no compatibility assumptions).

A 4-cocycle in $C^*(\mathcal{D}; \mathcal{K})$ is given by a map of natural systems $\tilde{\phi} : \mathcal{N}^4(\mathcal{D}) \to \mathcal{K}$, which is determined in turn by assigning to each 4-simplex $\sigma \in \mathcal{N}^4\mathcal{D}$ an element $\phi(\alpha) \in K_{\partial_{\max} \sigma}$. Now any 3-simplex $\tau \in \mathcal{N}^3_3 \mathcal{D}$ is determined by a diagram \((6.4)\) in $\mathcal{D}$ satisfying \((6.3)\), which means that its lift

\[(8.5)\]

has a 2-track $s(\tau) \in \mathcal{G}_2$ (see \([4.4]\)) with

\[(8.6)\]

$s(\tau) : s(f)_* s(\eta) \oplus s(\xi) \Rightarrow s(h)_* s(\theta) \oplus s(\xi)$

which represents an element $\psi_s(\tau) \in K_\ell$, by definition of $\mathcal{K} = \Pi_2\mathcal{G}$. This extends to a map of natural systems $\psi : \mathcal{N}^3(\mathcal{D}) \to \mathcal{K}$. 

Finally, we define \( \tilde{\phi} : \tilde{N}_4(D) \to K \) by setting
\[
\phi(\sigma) := \sum_{i=0}^{4} (-1)^i \tilde{\psi}_a(d_i \sigma) \quad \text{for} \quad \sigma \in N_4D.
\]

8.8. Lemma. \( \tilde{\phi} \) is a 4-cocycle in \( C^4(D; K) \).

Proof. Given a 5-simplex \( \alpha \in N_5D \), we see
\[
(\delta \tilde{\phi})(\alpha) := \sum_{j=0}^{5} (-1)^j (d^j \tilde{\phi})(\alpha) = \sum_{j=0}^{5} (-1)^j \tilde{\phi}(d_j \alpha) = \sum_{j=0}^{5} (-1)^j \left( \sum_{i=0}^{4} (-1)^i \tilde{\psi}_a(d_i d_j \alpha) \right)
\]
which vanishes by the usual simplicial identities. \( \square \)

8.9. Definition. We denote the cocycle defined above by \( \tilde{\phi}_G \in C^4(\hat{\pi}_1 G; \Pi_2 G) \), and the corresponding cohomology class, called the Baues-Wirsching class for \( G \), by \( \chi_G \in H^4(\hat{\pi}_1 G; \Pi_2 G) \).

8.10. Theorem. If \( V_* \) is a \( (P^2\mathcal{S}, \mathcal{O}) \)-category and \( \mathcal{G} \) is the associated two-track category, the first \( k \)-invariant for \( V_* \) corresponds to the cohomology class \( \chi_G \) defined above under the natural isomorphism of Theorem 7.21.

Proof. Let \( X = S_{\text{cub}} V_* \) be the \( (P^2\mathcal{C}, \mathcal{O}) \)-category corresponding to \( V_* \), and consider the following square of the form (7.2) in \((\mathcal{C}, \mathcal{O})\text{-Cat}:
\[
\begin{array}{ccc}
X = P^2X & \xrightarrow{i^{(2)}} & Y \\
\downarrow{p^{(2)}} & & \downarrow{q} \\
BD = P^1X & \xrightarrow{\text{PO}} & Z
\end{array}
\]

where \( D = \hat{\pi}_1 X \cong \hat{\pi}_1 V_* \) is the track category associated to \( X \).

By [13], Proposition 6.4], the homotopy pushout \( Z \) in \( (\mathcal{C}, \mathcal{O})\text{-Cat} \) satisfies \( P^3Z \cong E_D(\pi_2 X, 3) \). Thus if \( r^{(3)} : Z \to P^3Z \) is the structure map of the Postnikov tower, the first \( k \)-invariant for \( V_* \) (or equivalently, for \( X \)) is represented by the map \( k_1 := r^{(3)} \circ q \sim r^{(3)} \circ j \) in \([W, E_D(\pi_2 X, 3)]_{(\mathcal{C}, \mathcal{O})\text{-Cat}/W} \).

We use the cubical version of Kan’s original model for the Postnikov system, so that \( (P^k X)_n \) consists of \( \sim_k \)-equivalence classes of \( n \)-cubes in \( X \), where \( I \sim_k J \Leftrightarrow \text{sk}_k^e I = \text{sk}_k^e J \) (cf. [35], VI, §2]. We assume that \( X \) is fibrant (so each mapping space \( X(u,v) \) is a cubical Kan complex).

Factor the structure map \( p^{(2)} : P^2X \to P^1X \) as a cofibration \( i^{(2)} : P_2X \to Y \) followed by a weak equivalence, so that the pushout above is in fact a homotopy pushout, as required. Thus \( Y_i = X_i \) for \( i \leq 2 \), while \( Y_3 = (X_3/\sim_2) \) II \( \hat{Y} \), where \( \hat{Y} \) consists of a 3-cube \( J = J_3 \) for every collection \( T := (I_{0}^{e}, I_{1}^{e}, \ldots, I_{0}^{e}, I_{3}^{e}) \) of 2-cubes in \( X_2 \) with matching faces (such that \( d_i^e J = I_i^e \) for \( 1 \leq i \leq 3 \) and \( e \in \{0,1\} \)). The pushout \( Z \) is the reduction modulo \( \sim_2 \) of the image of \( i^{(2)} \) (without changing \( \hat{Y} \)). The 3-cubes \( J_{(t,0,\ldots,0)} \) for non-null homotopic \( I \) represent \( D = \hat{\pi}_1 X \) in \( P^3Z \cong E_D(\hat{\pi}_1 X, 2) \).

Let \( W \sim \mathcal{N}D \) be the cofibrant replacement for \( P^1X \) constructed as in [7.8]. The weak equivalence of \((\mathcal{C}, \mathcal{O})\text{-categories} \varphi : W \to Y \) is then defined as follows:
TWO-TRACK CATEGORIES

(a) For every indecomposable 0-cube \( (f) \in \mathcal{D}_0 = \mathcal{F} \text{ho} \mathcal{D} \), corresponding to a homotopy class \( f \in [X u, X v]_{\text{ho} C} \), \( \varphi(f) \) is a choice of a representative \( s(f) \) in \( (P_0 X)_0 = X(u, v)_0 \).

(b) For a (non-composite) 1-cube \( I \) corresponding to a track \( \xi : f \otimes g \Rightarrow h \) in \( W_1 \), the 1-cube \( \varphi(I) \) is a choice of a homotopy \( s(\xi) : s(f) \circ s(g) \Rightarrow s(h) \) representing \( \xi \).

(c) For a (non-composite) 2-cube \( J \) as in \( (7, 10) \), indexed by a diagram in \( \mathcal{D} \) of the form \( (6.4) \) satisfying \( (6.3) \), the 2-cube \( \varphi(J) \) is a choice of a 2-track \( \alpha : f \circ \eta \oplus \zeta \Rightarrow h^* \theta \oplus \xi \) (cf. \( 4.4) \).

(d) Finally, the faces of any 3-cube \( K \) form a set of matching 2-cubes, whose image under \( \varphi \) has a canonical fill-in \( T \in Y \), and we set \( \varphi(K) := T \).

The map \( W \simeq P^1 X \to P^3 Z \) represents \( k_1 \), and (as in the proof of Theorem \( 7.21 \)) it is determined by its image on the 3-cubes of \( W \). Using the description in \( 7.8 \), we see that the 3-cube:

\[
\begin{align*}
  f_1 \otimes f_2 \otimes f_3 \otimes f_4 & \quad \xrightarrow{\xi^{[0,1,2]} \otimes f_4} \quad f_1 \otimes f_2 \otimes f_4 \otimes f_4 \quad \xrightarrow{f_1 \otimes f_2 \otimes \xi^{[2,4]}} \quad f_1 \otimes f_2 \otimes h^{[2,4]} \quad \xrightarrow{f_1 \otimes \xi^{[0,2,4]}} \quad f_1 \otimes h^{[1,4]} \\
  h^{[0,2]} \otimes f_3 \otimes f_4 & \quad \xrightarrow{\xi^{[0,2,3]} \otimes f_4} \quad h^{[0,3]} \otimes f_4 \quad \xrightarrow{f_1 \otimes h^{[1,3]}} \quad h^{[0,3]} \otimes h^{[2,4]} \quad \xrightarrow{\xi^{[0,1,3]} \otimes f_4} \quad h^{[0,3]} \otimes f_4 \\

  h^{[0,3]} \otimes f_4 & \quad \xrightarrow{\xi^{[0,1,3]} \otimes f_4} \quad h^{[0,2]} \otimes h^{[2,4]} \quad \xrightarrow{\xi^{[0,2,4]}} \quad h^{[0,3]} \otimes f_4 \\

  h^{[0,2]} \otimes f_4 & \quad \xrightarrow{\xi^{[0,2,3]} \otimes f_4} \quad h^{[0,3]} \otimes h^{[2,4]} \\

  h^{[0,4]} & \quad \xrightarrow{f_1 \otimes h^{[1,4]}} \quad h^{[0,4]} 
\end{align*}
\]

is sent under \( k_1 \) to the cube in \( Y \) described by replacing \( f_1 \otimes f_2 \otimes f_3 \otimes f_4 \) by \( s(f_1) \otimes s(f_2) \otimes s(f_3) \otimes s(f_4) \), and so on. This is just what the cocycle of \( (8.7) \) does to the corresponding 4-simplex of \( N \mathcal{D} \), under the isomorphism of Theorem \( 7.21 \). \( \square \)

APPENDIX A: FIBRANCY CONDITIONS ON DOUBLE GROUPOIDS

In this appendix we prove some technical facts about double groupoids:

Proposition \( (2.10) \). Let \( X_\bullet \in s\mathcal{Spd} \) be a simplicial groupoid, for which the simplicial sets \( X_0 \) and \( X_1 \) are csk2-fibrant, and the morphism \( d_1^v : X_1 \to X_0 \) is a csk2-fibration. Then the left adjoint \( Q^h : s\mathcal{Spd} \to \mathcal{DbGpd} \) to the nerve \( N^h : \mathcal{DbGpd} \to s\mathcal{Spd} \), applied to \( X_\bullet \), is \( \tilde{\pi}_1^h X_\bullet \).

Proof. Let \( X_{11}/\sim := (\tilde{\pi}_1^h X_\bullet)_1 \) and \( X_{10}/\sim := (\tilde{\pi}_1^h X_\bullet)_0 \) be the fundamental groupoids of the csk2-fibrant (horizontal) simplicial sets \( X_{10} \) and \( X_{11} \), respectively, and let \( [\alpha] \in X_{11}/\sim \) denote the equivalence class of \( \alpha \in X_{11} \).

We first show that there is a well defined composition map:

\[
(8.12) \quad X_{11}/\sim \times_{X_{10}/\sim} X_{11}/\sim \longrightarrow X_{11}/\sim
\]

where \( \overline{d}_i^v : X_{11}/\sim \to X_{10}/\sim \) is induced by \( d_i^v : X_{11} \to X_{10} \) for \( i = 0, 1 \).

For any \( ([\alpha], [\beta]) \in X_{11}/\sim \times_{X_{10}/\sim} X_{11}/\sim \), we have \( d_i^v([\alpha]) = \overline{d}_0^v([\beta]) \) – that is, \( \overline{d}_i^v([\alpha]) = [\overline{d}_0^v([\beta])] \).
in $X_{10}/\sim$. This means that there is a $\tau \in X_{20}$ such that:

$$d_0^h \tau = d_1^v \alpha \quad d_1^v \tau = s_0^h d_1^v \alpha \quad \text{and} \quad d_2^h \tau = d_0^v \beta .$$

On the other hand, since $d_1^v : X_{11} \rightarrow X_{02}$ is a $\text{csk}_2$-fibration, we have a lifting

$$\begin{array}{ccc}
\Lambda^1(2) & \xrightarrow{(\alpha,s_0^h d_1^v \alpha)} & X_{11} \\
\downarrow & \swarrow & \downarrow \quad \xi \\
\Delta[2] & \xrightarrow{\tau} & X_{22}
\end{array}$$

and $d_1^v \xi = \tau$.

The following picture summarizes the situation:

The prism at the top of the picture represents the filler $\xi \in X_{21}$ for the horn $(\alpha, s_0^h d_1^v \alpha)$. If we let $\hat{\alpha} := d_2^h \xi$ (the front face of the prism), then we have:

$$d_0^h \xi = \alpha, \quad d_1^h = s_0^h d_1^v \alpha, \quad \text{and} \quad d_2^h \xi = \hat{\alpha} .$$

This means that $\alpha \sim \hat{\alpha}$ in the equivalence relation determined by the 2-simplices of $X_{11}$ - that is, $[\alpha] = [\hat{\alpha}]$ in $X_{11}/\sim$.

Furthermore, since $d_1^v \hat{\alpha} = d_2^h \tau = d_0^v \beta$, we see that $\hat{\alpha}$ and $\beta$ are composable in $X_1$. We define the composition $m$ of (8.12) by $m([\alpha],[\beta]) = [\hat{\alpha} \circ \beta]$, where $\circ$ denotes the vertical composition in $X_1$.

To see that $m$ is independent of the choice of representatives for $[\alpha]$ and $[\beta]$ and the lift $\xi$, suppose $\alpha \sim \alpha'$. Then $\hat{\alpha} \sim \hat{\alpha}'$, so there is a $\gamma \in X_{21}$ with

$$d_0^h \gamma = \hat{\alpha}, \quad d_1^h \gamma = s_0^h d_1^v \hat{\alpha}, \quad \text{and} \quad d_2^h \gamma = \hat{\alpha}' .$$

Let $\delta = \gamma \circ s_0^h \beta$. Then

$$d_0^h \delta = \hat{\alpha} \circ \beta , \quad d_1^h \delta = (s_0^h d_1^v \hat{\alpha}) \circ \beta, \quad \text{and} \quad d_2^h \delta = \hat{\alpha}' \circ \beta .$$

This shows that $\hat{\alpha} \circ \beta \sim \hat{\alpha}' \circ \beta$, and therefore $m([\alpha],[\beta]) = m([\alpha'],[\beta'])$.

Given $\alpha \in X_{11}$, consider two different choices of lifts $\xi$ yielding $\hat{\alpha}$ and $\hat{\alpha}'$. Then $\hat{\alpha} \sim \hat{\alpha}'$. Arguing as above, this implies that $\hat{\alpha} \circ \beta \sim \hat{\alpha}' \circ \beta$. We conclude that $m$ is well defined.

For all $([\alpha],[\beta],[\gamma]) \in X_{11}/\sim \times X_{10}/\sim \times X_{11}/\sim \times X_{10}/\sim \times X_{11}/\sim$ we have:

$$m(m([\alpha],[\beta],[\gamma])) = m([\alpha],[\beta \circ \gamma]) = [\hat{\alpha} \circ (\hat{\beta} \circ \gamma)] = [((\hat{\alpha} \circ \hat{\beta}) \circ \gamma)] ,$$

and since $\hat{\alpha} \circ \hat{\beta} \sim \hat{\alpha} \circ \beta \sim \hat{\alpha} \circ \hat{\beta}$, the two lines of (8.13) are equal, so $m$ is associative.
Since the vertical elements in $X_{\bullet 1}$ are invertible with respect to $\circ$, which induces $m$, we have a groupoid

$$X_{11}/\sim \times_{X_{10}} X_{11}/\sim \xrightarrow{m} X_{11} \xrightarrow{d_0} X_{10}/\sim$$

(in the notation of (1.2)), which we denote by $X_{\bullet}/\sim$.

Notice that, from the above, there is an isomorphism for all $n \geq 2$ :

$$\left( (X_{11} \times_{X_{10}} \cdots \times_{X_{10}} X_{11}) /_{\sim} \right) \Rightarrow \left( (X_{11}/\sim \times_{X_{10}/\sim} \cdots \times_{X_{10}/\sim} X_{11}/\sim) \right).$$

Now if $N^h : \mathcal{D}bGpd \to sGpd$ is the (horizontal) nerve functor, from the above we have a double groupoid $Q^h X$ with

$$(N^h Q^h X)_n = \hat{\pi}_1(X_{\bullet n}).$$

The groupoid of objects and vertical morphisms of $Q^h X$ is $X_{0\bullet}$; the groupoid of objects and horizontal morphisms is $\hat{\pi}_1(X_{0\bullet})$; the groupoid of objects and horizontal morphisms and squares is $X_{1\bullet}/\sim$; and the groupoid of vertical morphism and squares is $\hat{\pi}_1(X_{1\bullet})$.

We now show that $Q^h$ is left adjoint to the nerve functor $N^h$. Since $N^h$ is fully faithful, a morphism $f : Q^h X_\bullet \to G_{\bullet \bullet}$ in $\mathcal{D}bGpd$ corresponds uniquely to a morphism $N^h f : N^h Q^h X_\bullet \to N^h G_{\bullet \bullet}$ in $sGpd$. The latter amounts to morphisms $(N^h f)_n : (N^h Q^h X_\bullet)_n \to (N^h G_{\bullet \bullet})_n$ in $sGpd$ commuting with face and degeneracy operators. But $(N^h Q^h X_\bullet)_n = N^h \hat{\pi}_1(X_{\bullet n})$ and $(N^h G_{\bullet \bullet})_n = N^h G_{\bullet n}$. Thus each $(N^h f)_n$ corresponds uniquely to a morphism $\hat{\pi}_1 X_{\bullet n} \to G_{\bullet n}$ and thus, by adjunction, to a morphism $\overline{f}_n : X_{\bullet n} \to N^h G_{\bullet n}$. Using the fact that $Q^h N^h = \text{Id}$, it is straightforward to check that all the $\overline{f}_n$ commute with face and degeneracy operators, hence correspond uniquely to a morphism $X_{\bullet} \to N^h G_{\bullet \bullet}$ in $sGpd$.

\[ \square \]

**Proposition (2.11).** Let $X_{\bullet \bullet} \in sS$ be such that $X_{i\bullet}$ and $X_{\bullet i}$ are csk$_2$-fibrant for each $i \geq 0$, and $d^h_0 : X_{1\bullet} \to X_{0\bullet}$ and $d^v_0 : X_{1\bullet} \to X_{0\bullet}$ are csk$_2$-fibrations. Then

(i) $(N^v \hat{\pi}_1 X_{\bullet \bullet})_\bullet$ is fibrant for all $i \geq 0$.

(ii) $(N^v \hat{\pi}_1 X_{\bullet \bullet})_{1\bullet}$ is csk$_2$-fibrant.

(iii) $\overline{f}^h_0 : (N^v \hat{\pi}_1 X_{\bullet \bullet})_{1\bullet} \to (N^v \hat{\pi}_1 X_{\bullet \bullet})_{0\bullet}$ is a csk$_2$-fibration.

(iv) $\overline{f}^v_0 : (N^v \hat{\pi}_1 X_{\bullet \bullet})_{1\bullet} \to (N^v \hat{\pi}_1 X_{\bullet \bullet})_{0\bullet}$ is a fibration.

**Proof.** (i) $(N^h \hat{\pi}_1 X_{\bullet \bullet})_\bullet$ is fibrant since it is the nerve of a groupoid.

(ii) This follows as in the proof of Proposition A, with vertical and horizontal directions switched.
(iii) Consider the diagram

\[
\begin{array}{c}
\Delta[0] \\
\approx \\
\Lambda^k[1] \\
\Delta[1]
\end{array} \xymatrix{
\ar[r]^\alpha & X_{1/\sim} & \ar[l]_{d_0} \\
\ar[r]^q & X_{1/\sim} \\
\ar[r]_{d_0} & X_{0/\sim}
}
\]

Since \(d_0^v\) is a \(\text{csk}_2\)-fibration, there is a lift \(\xi: \Delta[1] \to X_{1/\sim}\). Then \(q\xi: \Delta[1] \to X_{1/\sim}\) is the required factorization.

It remains to show that there is a lift for

\[
\begin{array}{c}
\Lambda^k[2] \\
\Delta[2]
\end{array} \xymatrix{
\ar[r]^{([\alpha],[\beta])} & X_{1/\sim} & \ar[l]_{d_0} \\
\ar[r]^q & X_{1/\sim} \\
\ar[r]_{d_0} & X_{0/\sim}
}
\]

By (i) we know that \(([\alpha],[\beta])\) factors as

\[
\Lambda^k[2] \xymatrix{
\ar[r]^{(\hat{\alpha},\beta)} & X_{1/\sim} \\
\ar[r]^q & X_{1/\sim} \\
\ar[r]_{d_0} & X_{0/\sim}
}
\]

Since \(d_0^v\) is a \(\text{csk}_2\)-fibration, it has a lift \(\xi: \Delta[2] \to X_{1/\sim}\). Thus \(q\xi\) is the required factorization.

(iv) By Lemma 2.22 it is enough to show that there is a lift in the diagram

\[
\begin{array}{c}
\Lambda^k[1] \\
\Delta[1]
\end{array} \xymatrix{
\ar[r] & X_{1/\sim} & \ar[l] \\
\ar[r] & X_{0/\sim}
}
\]

Notice that this factor as follows

\[
\begin{array}{c}
\Lambda^k[1] \\
\Delta[1]
\end{array} \xymatrix{
\ar[r] & X_{1/\sim} & \ar[l]_{d_0} \\
\ar[r]^q & X_{1/\sim} \\
\ar[r]_{d_0} & X_{0/\sim}
}
\]

Since \(d_0^h\) is a \(\text{csk}_2\)-fibration, there is a lift \(\xi: \Delta[1] \to X_{1/\sim}\). Thus \(q\xi\) is the required lift in (8.14).

\[\square\]

Appendix B: \(n\)-diagonals of bisimplicial sets

In this appendix we prove an elementary fact about bisimplicial sets, which is presumably known, but which we have not found in the literature. It follows immediately from the Bousfield-Friedlander spectral sequence (cf. [19 Theorem B.5]) in the cases where it converges, but is actually true for any bisimplicial set. In this paper we only need the result for \(n = 2\), in which case the proof can be much simplified; but we prove the general case for future reference.
Proposition (3.9). If \( f : W_{\bullet} \to V_{\bullet} \) is an diagonal n-equivalence (Definition 3.8), then the induced map \( \text{diag} f : \text{diag} W_{\bullet} \to \text{diag} V_{\bullet} \) is an n-equivalence.

Proof. Step I. First note that for any \( W_{\bullet} \in s\mathcal{S} \) we have \( \text{diag} W_{\bullet} \simeq \text{hocolim}_k V^v_k \) as a diagram in “vertical” simplicial sets (see [20] XII, §3.4), and since (7.2) is a Quillen equivalence, it suffices to prove the result for

\[
\text{hocolim}_k |f^v| : \text{hocolim}_k |W^v_k| \to \text{hocolim}_k |V^v_k|
\]

(See [39] §18.9.8).

For any \( V_{\bullet} \in s\mathcal{C} \), we denote by \( \tilde{V}_{\bullet} \) the corresponding restricted simplicial object (also called ss-object – see [15]), in which we forget the degeneracies: in other words, \( \tilde{V}_{\bullet} \in \mathcal{C}^\Delta_{\text{res}} \), where \( \Delta_{\text{res}} \) is the category of finite ordered sets and order-preserving monomorphisms.

By [54] Prop. A.1(iv)), the homotopy colimit of the diagram \( \{ |W^v_k| : \Delta \to \mathcal{J}op \}_{k=0}^{\infty} \) is weakly equivalent to the homotopy colimit of \( \{ |\tilde{W}^v_{\bullet}|_k : \Delta_{\text{res}} \to \mathcal{J}op \}_{k=0}^{\infty} \), which we denote by \( \tilde{W}_{\bullet} : \Delta_{\text{res}} \to \mathcal{J}op. \)

Finally, recall that \( \text{hocolim} \tilde{X}_{\bullet} \) can be described explicitly as

\[
(8.15) \quad \prod_{n=0}^{\infty} X_n \times \Delta[n] / \sim
\]

where \((d_i x, \sigma) \sim (x, d_i^\sigma)\) for each \( x \in X_n \) and \( \sigma \in \Delta[n-1] \), where \( d_i : \Delta[n-1] \to \Delta[n] \) is the inclusion of the i-th face (cf. [53] §1)).

Step II. Note that, for any \( Y \in \mathcal{S} \), \( |Y| \) is a CW complex, and realization commutes with skeleta: \( \text{sk}_n |Y| = |\text{sk}_n Y| \).

For any restricted simplicial object \( \tilde{X}_{\bullet} \in \mathcal{C}^\Delta_{\text{res}} \) (where \( \mathcal{C} \) is either \( \mathcal{S} \) or \( \mathcal{J}op \)), we define its diagonal n-skeleton to be the restricted simplicial object \( \text{dsk}_n \tilde{X}_{\bullet} \) with \( (\text{dsk}_n \tilde{X}_{\bullet})_k = \text{sk}_{n-k} \tilde{X}_k \) (so \( (\text{dsk}_n \tilde{X}_{\bullet})_k = \emptyset \) for \( k > n \)). Let \( j_n : \text{dsk}_n \tilde{X}_{\bullet} \hookrightarrow \tilde{X}_{\bullet} \) be the inclusion (a diagonal \((n-1)\)-equivalence).

For any CW complex \( W \), all cells in the relative CW complex \( (W \times \Delta[k], W \times \partial \Delta[k]) \) in dimensions \( \leq n \) are products of standard i-simplices with cells of \( \text{sk}_{n-k} W \). Thus we see from (8.15) that when \( \mathcal{C} = \mathcal{J}op \), \( j_n \) induces a cellular isomorphism of the n-skeleton of the homotopy colimit of the diagram \( \text{dsk}_n \tilde{X}_{\bullet} : \Delta_{\text{res}} \to \mathcal{J}op \) with that of \( \tilde{X}_{\bullet} \), since they have the same cells in dimensions \( \leq n \). In particular, \( \text{hocolim}(j_n) : \text{hocolim} \text{dsk}_n \tilde{X}_{\bullet} \to \text{hocolim} \tilde{X}_{\bullet} \) is an \((n-1)\)-equivalence. Thus if \( \tilde{Z}_{\bullet} = |\tilde{Z}|^v \) for some \( \tilde{Z}_{\bullet} : \Delta_{\text{res}} \to \mathcal{S} \), then also \( \text{hocolim}(j_n) : \text{hocolim} \text{dsk}_n \tilde{Z}_{\bullet} \to \text{hocolim} \tilde{Z}_{\bullet} \) is an \((n-1)\)-equivalence.

Step III. Recall that if \( K \in \mathcal{S} \) is a Kan complex, its n-th Postnikov section \( P^n K \) may be identified with the \((n+1)\)-st coskeleton functor \( \text{csk}_{n+1} K \), which factors through its left adjoint, the \((n+1)\)-st skeleton (see [31] §1.3) Moreover, one can use any Postnikov tower functor on \( \mathcal{S} \) or \( \mathcal{J}op \) to obtain functorial \( k \)-invariants, as in [13] §5-6). What we shall in fact be doing in each case is to start with \( \text{csk}_{k+1} K \) as the bottom section of our Postnikov tower, for the appropriate \( k \), and then use the pullbacks along the functorial \( k \)-invariants to define the higher sections. We may replace \( K \) by the limit of the resulting tower of fibrations.
Given a diagonal $n$-equivalence $f : W_{\bullet} \to V_{\bullet}$, let $g : \bar{X}_{\bullet} \to \bar{Y}_{\bullet}$ denote the associated map $\bar{f}^v : \bar{W}^v \to \bar{V}^v$ of restricted simplicial objects in $\mathcal{S}$. We may assume that $\bar{X}_{\bullet}$ and $\bar{Y}_{\bullet}$ are Reedy fibrant, and $g$ is a Reedy fibration (see [39, §15]). In particular, for each $n \geq 0$, $\bar{X}_n = W_n^v$ and $\bar{Y}_n = V_n^v$ are Kan complexes, and the maps $\delta_n^X : \bar{X}_n \to M_n\bar{X}_{\bullet}$ and $\delta_n^Y : \bar{Y}_n \to M_n\bar{Y}_{\bullet}$ are fibrations, where the matching object of any (restricted) simplicial object $\bar{U}_{\bullet}$ is defined

\begin{equation} M_n\bar{U}_{\bullet} = \{(u_0, \ldots, u_n) \in (\bar{U}_{n-1})^{n+1} \mid d_iu_j = d_ju_i \text{ for all } 0 \leq i < j \leq n\} \end{equation}

(see [20, X, §4.5]). Note that all face maps $d_i : \bar{U}_n \to \bar{U}_{n-1}$ are defined by post-composing $\delta_i^U : \bar{U}_n \to M_n\bar{U}_{\bullet}$ with projection onto the $i$-th factor $\bar{U}_{n-1}$.

Finally, let $g := f \circ j_{n+1} : \bar{U}_{\bullet} \to \bar{Y}_{\bullet}$, where $\bar{U}_{\bullet} := \text{dsk}_{n+1}\bar{X}_{\bullet}$ is the diagonal $(n+1)$-skeleton.

**Step IV.** We shall now construct a factorization

\begin{equation} \bar{U}_{\bullet} \xrightarrow{\psi} \bar{Z}_{\bullet} \xrightarrow{h} \bar{Y}_{\bullet} \end{equation}

of $g$, with $h$ a vertical weak equivalence and $\text{dsk}_{n+1}\psi$ an isomorphism. We do so by induction on the (horizontal) simplicial dimension $m \geq 0$:

(a) Assume that we have defined $\bar{Z}_{\bullet}$ and constructed the factorization (8.17) through (horizontal) simplicial dimension $m - 1$, and let $\ell := n - m$.

First, use $P^\ell = \text{csk}_{\ell+1}$ and the functorial $k$-invariants as above to define a commutative diagram in $\mathcal{S}$ as follows:

\begin{equation} \begin{array}{ccc} \bar{U}_m & \xrightarrow{p^{(\ell)}} & P^\ell\bar{U}_m \xrightarrow{(P^\ell g_m)^*k_{\ell}} K(\pi_{\ell+1}\bar{Y}_m, \ell + 2) \\ \downarrow{g_m} & \simeq & \downarrow{q^{(\ell)}_g} \\ \bar{Y}_m & \xrightarrow{q^{(\ell)}} & P^\ell\bar{Y}_m \xrightarrow{k_{\ell}} K(\pi_{\ell+1}\bar{Y}_m, \ell + 2) \end{array} \end{equation}

(8.18)

where $p^{(\ell)}$ and $q^{(\ell)}$ are the structure maps of the Postnikov tower. We define $\bar{Z}_m \in \mathcal{S}$ by means of its Postnikov tower, starting with $P^\ell(\bar{Z}_m) := P^\ell\bar{U}_m$ with $P^\ell h_m = \text{Id}$. Let $P^\ell+1(\bar{Z}_m)$ be the homotopy fiber of $(P^\ell g_m)^*k_{\ell}$.

Using the following (vertical) map of the two horizontal fibration sequences on the right, we obtain a dotted map as indicated in:

\begin{equation} \begin{array}{ccc} \bar{U}_m & \xrightarrow{p^{\ell+1}} & P^{\ell+1}\bar{Z}_m \xrightarrow{g_*k_{\ell}} K(\pi_{\ell+1}\bar{Y}_m, \ell + 2) \\ \downarrow{g_m} & & \downarrow{p^{\ell+1}g_m} \\ \bar{Y}_m & \xrightarrow{q^{\ell+1}} & P^{\ell+1}\bar{Y}_m \xrightarrow{k_{\ell}} K(\pi_{\ell+1}\bar{Y}_m, \ell + 2) \end{array} \end{equation}

(8.19)

using the indicated pullback. This yields factorization:

\begin{equation} P^{\ell+1}\bar{U}_m \xrightarrow{P^{\ell+1}\psi_m} P^{\ell+1}\bar{Z}_m \xrightarrow{P^{\ell+1}h_m} P^{\ell+1}\bar{Y}_m \end{equation}

(8.20)

of $P^{\ell+1}g_m$. 

Since $g$ was a diagonal $n$-equivalence, $g_m$ is an $\ell$-equivalence, so $P^\ell g_m$ is a weak equivalence. Thus $P^{\ell+1}h_m$ is a weak equivalence (since we pulled back the $k$-invariants for $\tilde{Y}_m$ along a weak equivalence).

Note that $P^\ell \psi_m$ is an isomorphism of simplicial sets, so that $\sk_{\ell+1} \psi_m$ is an isomorphism, too (since $P^\ell = \csk_{\ell+1}$).

(b) Proceeding as in (b) we pull back the rest of the Postnikov system for $\tilde{Y}_m$ along weak equivalences, obtaining a weak equivalence $h_m: \tilde{Z}_m \to \tilde{Y}_m$ and the factorization as in (8.20).

(c) To complete the construction we must define the (horizontal) face maps of $\tilde{Z}_\bullet$, where we assume by induction that they have been defined through (horizontal) simplicial dimension $m-1$. We also assume that the resulting $(m-1)$-truncated restricted simplicial object in $\mathcal{S}$, $\sk^{h}_{m-1} \tilde{Z}_\bullet$, is Reedy fibrant, and that $\sk^{h}_{m-1} h$ is a Reedy fibration. Since $\tilde{X}_\bullet$ (and thus $\sk^{h}_{m-1} \tilde{X}_\bullet$) was already Reedy fibrant, and $\sk^{v}_{n-1-i} \tilde{Z}_i = \sk^{v}_{n-1-i} \tilde{U}_i = \sk^{v}_{n-1-i} \tilde{X}_i$, making $\tilde{Z}_\bullet$ fibrant does not require any changes in (vertical) simplicial dimension $\leq n-i$. Similarly, when changing $\sk^{h}_{m-1} h$ into a fibration, we need make no changes in (vertical) simplicial dimension $\leq n-i$.

Note that since $\sk^{h}_{m-1} \tilde{Z}_\bullet$ is Reedy fibrant, the limit defining the matching object $M_m \tilde{Z}_\bullet$, in (8.10) is actually a homotopy limit (by the dual of [25 Cor. 19.5(2)]), and similarly for $M_m \tilde{Y}_\bullet$. Since $\sk^{h}_{m-1} h$ is a weak equivalence, so is $M_m h: M_m \tilde{Z}_\bullet \to M_m \tilde{Y}_\bullet$. Moreover, since limits preserve fibrations, $M_m h$ is also a fibration. But then we have a lifting:

$$
\begin{array}{ccc}
U_m & \xrightarrow{\sk^h_m \delta^U_m} & M_m \tilde{Z}_\bullet \\
\psi_m \downarrow & & \downarrow \delta^Z_m \\
Z_m & \xrightarrow{\delta^Z_m \circ h_m} & M_m \tilde{Y}_\bullet,
\end{array}
$$

and $\delta^Z_m: \tilde{Z}_m \to M_m \tilde{Z}_\bullet$ determines the (horizontal) face maps $d^h_i: \tilde{Z}_m \to \tilde{Z}_{m-1}$ ($0 \leq i \leq m$), making $\sk^{h}_{m} \tilde{Z}_\bullet$ into an (m-truncated) restricted simplicial object, $\sk^{h}_{m} h: \sk^{h}_{m} \tilde{Z}_\bullet \to \sk^{h}_{m} \tilde{Y}_\bullet$ into a weak equivalence, and $\sk^{h} \psi: \sk^{h}_{m} \tilde{U}_\bullet \to \sk^{h}_{m} \tilde{Z}_\bullet$ into a map between such objects. This completes the induction.

**Step V.** To complete the proof, note that $\tilde{Z}_\bullet$ and $\tilde{Y}_\bullet$ are weakly equivalent, so hocolim $\tilde{Z}_\bullet \simeq$ hocolim $\tilde{Y}_\bullet$. But by construction $\text{dsk}_{n+1} \tilde{Z}_\bullet = \tilde{U}_\bullet = \text{dsk}_{n+1} \tilde{X}_\bullet$ (isomorphic as restricted simplicial objects in $\mathcal{S}$), so by Step II, hocolim $\tilde{X}_\bullet$, and hocolim $\tilde{Z}_\bullet$ are $n$-equivalent. Thus by Step I, diag $W_{\bullet \bullet}$ and diag $V_{\bullet \bullet}$ are $n$-equivalent, and a diagram chase shows that diag $f$ is an $n$-equivalence, as claimed. □

**References**

[1] M. Artin & B. Mazur, “On the van Kampen theorem”, Topology 5 (1966), pp. 178-189.
[2] H.-J. Baues, The algebra of secondary cohomology operations, Birkhäuser Prog. in Math. 247, Basel, 2006.
[3] H.-J. Baues, “Higher order track categories and the algebra of higher order cohomology operations”, preprint, 2009, [arXiv:0903.2878](http://arxiv.org/abs/0903.2878).
[4] H.-J. Baues & D. Blanc “Comparing cohomology obstructions”, preprint, 2010.
[5] H.-J. Baues & D. Blanc “Stems and spectral sequences”, preprint, 2010.
[6] H.-J. Baues & W. Dreckman, “The cohomology of homotopy categories and the general linear group”, *K-Theory* 3 (1989), pp. 307-338.
[7] H.-J. Baues & M.A. Jibladze, “Computation of the $E_3$-term of the Adams spectral sequence”, preprint, 2004.
[8] H.-J. Baues & M.A. Jibladze, “Secondary derived functors and the Adams spectral sequence”, *Topology* 45 (2006), pp. 295-324.
[9] H.-J. Baues, M.A. Jibladze, & A.P. Tonks, “Cohomology of monoids in monoidal categories”, in J.-L. Loday, J.D. Stasheff, & A.A. Voronov, eds., *Operads: Proceedings of Renaissance Conference (Hartford,CT/Luminy, 1995)* Contemp. Math. 202, AMS, Providence, RI 1997, pp. 137-165.
[10] H.J. Baues & G. Wirsching, “The cohomology of small categories”, *J. Pure & Appl. Alg.* 38 (1985), pp. 187-211.
[11] C. Berger, “A cellular nerve for higher categories”, *Adv. in Math.* 169 (2002), pp. 118-175.
[12] D. Blanc, “Generalized André-Quillen Cohomology”, *J. Homotopy & Rel. Structures* 3 (2008), pp. 161-191.
[13] D. Blanc, W.G. Dywer, & P.G. Goerss, “The realization space of a II-algebra: a moduli problem in algebraic topology”, *Topology* 43 (2004), pp. 857-892.
[14] D. Blanc, M.W. Johnson, & J.M. Turner, “On Realizing Diagrams of II-algebras”, *Alg. & Geom. Top.* 6 (2006), pp. 763-807.
[15] D. Blanc, M.W. Johnson, & J.M. Turner, “Higher homotopy operations and Cohomology”, to appear in *J. K-Theory*.
[16] V. Blanco, M. Bullejos Lorenzo, & E. Faro, “A Full and faithful Nerve for 2-categories”, *Appl. Cat. Str.* 13 (2005), pp. 223-233.
[17] F. Borceux, *Handbook of Categorical Algebra, Vol. 1: Basic Category Theory*, Encyc. Math. & its Appl. 50, Cambridge U. Press, Cambridge, UK, 1994.
[18] F. Borceux, *Handbook of Categorical Algebra, Vol. 2: Categories and Structures*, Encyc. Math. & its Appl. 51, Cambridge U. Press, Cambridge, UK, 1994.
[19] A.K. Bousfield & E.M. Friedlander, “Homotopy theory of Γ-spaces, spectra, and bisimplicial sets”, in M.G. Barratt and M.E. Mahowald, eds., *Geometric Applications of Homotopy Theory, II*, Lec. Notes Math. 658, Springer, Berlin-New York, 1978, pp. 80-130.
[20] A.K. Bousfield & D.M. Kan, *Homotopy limits, Completions, and Localizations*, Lec. Notes Math. 304, Springer, Berlin-New York, 1972.
[21] R. Brown, K.A. Hardie, K.H. Kamps, & T. Porter, “A homotopy double groupoid of a Hausdorff space”, *Theory Appl. Categ.* 10 (2002), pp. 71-93.
[22] R. Brown & C.B. Spencer, “Double groupoids and crossed modules”, *Cahiers Top. Géom. Diff. Cat.* 17 (1976), pp. 343-362.
[23] M. Bullejos, A.M. Cegarra, & J.W. Duskin, “On cat$^n$-groups and homotopy types”, *J. Pure & Appl. Alg.* 86 (1993), pp. 135-154.
[24] A.M. Cegarra & J. Remedios, “The relationship between the diagonal and the bar constructions on a bisimplicial set”, *Top. & Appl.* 153 (2005), pp. 21-51.
[25] W. Chachólski & J. Scherer, *Homotopy theory of diagrams*, Memoirs AMS 736, American Mathematical Society, Providence, RI, 2002.
[26] D.-C. Cisinski, “Les préfaisceaux comme modèles des types d’homotopie”, *Astérisque* 308, Soc. Math. France, Paris, 2006.
[27] J.-M. Cordier, “Sur la notion de diagramme homotopiquement cohérent”, *Cahiers Top. Géom. Cat.* 23 (1982), pp. 93-112.
[28] J.W. Duskin, *Simplicial methods and the interpretation of “triple” cohomology*, Memoirs AMS 3 (No. 163), AMS, Providence, RI, 1975.
[29] W.G. Dwyer & D.M. Kan, “Function complexes in homotopical algebra”, *Topology* 19 (1980), pp. 427-440.
[30] W.G. Dwyer & D.M. Kan, “Simplicial localizations of categories”, *J. Pure & Appl. Alg.* 17 (1980), No. 3, pp. 267-284.
[31] W.G. Dwyer & D.M. Kan, “An obstruction theory for diagrams of simplicial sets”, *Proc. Kon. Ned. Akad. Wet. - Ind. Math.* 46 (1984) pp. 139-146.
[32] W.G. Dwyer, D.M. Kan, & J.H. Smith, “An obstruction theory for simplicial categories”, Proc. Kon. Ned. Akad. Wet. - Ind. Math. 89 (1986), pp. 153-161.

[33] C. Ehresmann, “Catégories doubles et catégories structurées”, Comptes Rend. Acad. Sci., Paris 256 (1963), pp. 1198-1201.

[34] G.J. Ellis, “Homology of 2-types”, J. Lond. Math. Soc. (2) 46 (1992), pp. 1-27.

[35] P.G. Goerss & J.F. Jardine, Simplicial Homotopy Theory, Progress in Mathematics 179, Birkhäuser, Basel-Boston, 1999.

[36] R. Gordon, A.J. Power, & R.H. Street, Coherence of tricategories, Memoirs AMS 117, AMS, Providence, RI, 1995.

[37] K.A. Hardie, K.H. Kamps, & R.W. Kieboom, “A homotopy bigroupoid of a topological space”, Appl. Cat. Struct. 9 (2001), pp. 311-327.

[38] P.J. Higgins, Notes on Categories and Groupoids, Van Nostrand Reinhold Mathematical Studies 32, Van Nostrand Reinhold Co., London, 1971.

[39] P.S. Hirschhorn, Model Categories and their Localizations, Math. Surveys & Monographs 99, AMS, Providence, RI, 2002.

[40] L. Illusie, Complexe cotangent et déformations. II, Lec. Notes Math. 283, Springer, Berlin-New York, 1972.

[41] J.F. Jardine, “Cubical homotopy theory: a beginning”, preprint, 2002.

[42] J.F. Jardine, “Categorical homotopy theory”, Homology, Homotopy & Applic. 8 (2006), pp. 71-144.

[43] M. Johnson & R.F.C. Walters, “On the nerve of an n-category”, Cahiers Top. Géom. Diff. Cat. 28 (1987), pp. 257-282.

[44] A. Joyal & R. Street, “Pullbacks equivalent to pseudo pullbacks”, Cahiers de Topologie et Géom. Diff. Catég. 34 (1993), pp. 153-156.

[45] D.M. Kan, “Is an ss complex a css complex?”, Adv. in Math. 4 (1970), pp. 170-171.

[46] S. Lack & S. Paoli, “2-Nerves for bicategories”, K-Theory 38 (2008), pp. 153-175.

[47] S. Mac Lane & J.H.C. Whitehead, “On the 3-type of a complex”, Proc. Nat. Acad. Sci. USA 36 (1950), pp. 41-48.

[48] I. Moerdijk & J.-A. Svensson, “Algebraic classification of equivariant homotopy 2-types, I”, J. Pure Appl. Alg. 89 (1993), pp. 187-216.

[49] S. Paoli, “(Co)homology of crossed modules with coefficients in a π1-module”, Homol. Homotopy Applic. 5 (2003), pp. 261-296.

[50] S. Paoli, “Weakly globular catn-groups and Tamsamani’s model”, Adv. in Math. 222 (2009), pp. 621-727.

[51] D.G. Quillen, Homotopical Algebra, Springer-Verlag Lec. Notes Math. 20, Berlin-New York, 1963.

[52] D.G. Quillen, “On the (co-)homology of commutative rings”, Applications of Categorical Algebra, Proc. Symp. Pure Math. 17, AMS, Providence, RI, 1970, pp. 65-87.

[53] G.B. Segal, “Classifying spaces and spectral sequences”, Pub. Math. Inst. Hautes Et. Sci. 34 (1968), pp. 105-112.

[54] G.B. Segal, “Categories and cohomology theories”, Topology 13 (1974), pp. 293-312.

[55] R.H. Street, “The algebra of oriented simplices”, J. Pure Appl. Alg. 49 (1987), 283-335.

[56] Z. Tamsamani, “Sur des notions de n-catégorie et n-groupeoide non-strictes via des ensembles multi-simpliciaux”, K-theory 16 (1999), pp. 51-99.

[57] D. Verity, Complicial sets characterising the simplicial nerves of strict ω-categories, Memoirs AMS 193, American Mathematical Society, Providence, RI, 2008.

Department of Mathematics, University of Haifa, 31905 Haifa, Israel

E-mail address: blanc@math.haifa.ac.il
E-mail address: paoli@math.haifa.ac.il