Quantum Conditional Probabilities and New Measures of Quantum Information

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Abstract

We use a novel form of quantum conditional probability to define new measures of quantum information in a dynamical context. We explore relationships between our new quantities and standard measures of quantum information, such as von Neumann entropy. These quantities allow us to find new proofs of some standard results in quantum information theory, such as the concavity of von Neumann entropy and Holevo’s theorem. The existence of an underlying probability distribution helps shed light on the conceptual underpinnings of these results.

1 Introduction

Quantum information is primarily understood in terms of von Neumann entropy and related quantities \cite{1, 2}. Due to inherently quantum phenomena such as entanglement, quantum information measures—such as conditional von Neumann entropy and mutual von Neumann information—lack well-defined underlying probability distributions. Nevertheless, despite their own somewhat unclear conceptual underpinnings, these quantities have proved useful for reframing and clarifying aspects of quantum information. Many of the relationships satisfied by classical information measures are mirrored by their quantum analogues \cite{1–3}, sometimes quite remarkably, as in the case of strong subadditivity \cite{4}.

In this paper, we define and study new forms of quantum information that complement the standard quantities. The key ingredients in our approach are conditional probability distributions, first studied in \cite{5, 6}, that provide an underlying picture for the type of information being described. In particular, we are able to provide a description of information flow in the context of open quantum systems whose dynamical evolution is well-approximated by linear, completely positive, trace-preserving (CPTP) maps, without any explicit appeal to larger Hilbert spaces or ancillary systems. We show that some standard results of quantum information theory emerge quite naturally from our perspective.

Section 2 provides some relevant background on classical and quantum information. In Section 3 we define new forms of quantum conditional entropy and quantum mutual information in terms of quantum conditional probabilities, and briefly describe a dynamical interpretation of these quantities. In Section 4 we use the results of the previous section to analyze processes under which there is growth in entropy (in the sense of Shannon) and to provide new proofs of the concavity of von Neumann entropy and quantum data processing. We demonstrate that our quantum data-processing inequality provides a natural interpretation

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of Holevo’s theorem in a dynamical context, showing that Holevo’s $\chi$ acts as an upper bound on the amount of information that can flow from a system’s initial configuration to a later one. In Section 5 we present a discussion of various ways to generalize our constructions, including to an analysis of the relationships between subsystems and the parent systems to which they belong, and to more general decompositions of density matrices than the ones that play a primary role in the paper. In Section 6 we identify connections between the constructions in this paper and previous work. We conclude in Section 7 with a brief summary of our results and interesting open questions.

2 Background

2.1 Shannon Entropy, Density Matrices, and von Neumann Entropy

Consider a classical random variable $X$ whose set of outcomes $\{x\}_x$ occur according to a probability distribution $\{p(x)\}_x$. Using this data, we can compute expectation values, standard deviations, and so on. Assuming a discrete set of outcomes, the average information encoded in the probability distribution is given by its Shannon entropy:

$$H(X) \equiv - \sum_x p(x) \log p(x).$$

In quantum theory, observables are a non-commutative generalization of random variables, with their set of eigenvalues playing the role of the set of possible outcomes. A given density matrix $\hat{\rho}$ generalizes the role of a probability distribution, allowing us to compute statistical quantities such as the expectation value of an observable $\hat{O}$:

$$\langle O \rangle = \text{Tr}[\hat{\rho}\hat{O}].$$

The simplest kind of density matrix corresponds to a pure state, and can be expressed as a projection operator of the form $|\Psi\rangle\langle\Psi|$. In this simple case, the formula (2) reduces to

$$\langle O \rangle = \text{Tr}[|\Psi\rangle\langle\Psi|\hat{O}] = \langle\Psi|\hat{O}|\Psi\rangle.$$  

In general, a density matrix has infinitely many possible decompositions over sets of projectors $\{\hat{\Pi}_\alpha\}_\alpha$,

$$\hat{\rho} = \sum_{\alpha} \lambda_\alpha \hat{\Pi}_\alpha, \quad \hat{\Pi}_\alpha = |\phi_\alpha\rangle\langle\phi_\alpha|,$$

where the set $\{\lambda_\alpha\}_\alpha$ consists of non-negative real numbers that sum to unity, and where $\{|\phi_\alpha\rangle\}_\alpha$ is not necessarily an orthonormal set of states. Each such decomposition has a corresponding Shannon entropy:

$$H(\{\lambda_\alpha\}) = - \sum_{\alpha} \lambda_\alpha \log \lambda_\alpha.$$  

The decomposition that minimizes the Shannon entropy consists of the eigenvalues and corresponding eigenprojectors of $\hat{\rho}$,

$$\hat{\rho} = \sum_i p_i \hat{P}_i, \quad \hat{P}_i = |\Psi_i\rangle\langle\Psi_i|,$$

where $\{|\Psi_i\rangle\}_i$ is the set of eigenstates of $\hat{\rho}$. The von Neumann entropy of a density matrix $\hat{\rho}$ is this minimal Shannon entropy of $\hat{\rho}$,

$$S(\hat{\rho}) \equiv - \text{Tr}[\hat{\rho} \log \hat{\rho}] = - \sum_i p_i \log p_i,$$

and therefore represents the minimum amount of average information that can be encoded in a system described by $\hat{\rho}$. 

2
2.2 Classical Conditional Entropy and its Quantum Counterpart

Classically, the conditional entropy of a random variable \( Y \) given another random variable \( X \) is defined in terms of a conditional probability distribution \( p(y|x) \) that describes correlations between possible outcomes of the two random variables \( Y \) and \( X \). Specifically, the conditional entropy of a random variable \( Y \) given that \( X \) takes the value \( x \) is defined to be

\[
H(Y|x) \equiv - \sum_y p(y|x) \log[p(y|x)].
\]

(8)

The full conditional entropy is then

\[
H(Y|X) \equiv \sum_x H(Y|x)p(x) = - \sum_{x,y} p(y|x)p(x) \log[p(y|x)],
\]

(9)

which can be thought of as the average information encoded in \( Y \) given a particular outcome of \( X \), averaged over all the possible outcomes of \( X \).

Conditional entropies satisfy the identity

\[
H(Y|X) = H(Y, X) - H(X),
\]

(10)

where \( H(Y, X) \) is the Shannon entropy of the joint distribution in \( X \) and \( Y \). The identity (10) captures the intuition that the conditional entropy measures the information about \( Y \) encoded in its correlations with \( X \) in excess of information encoded in \( X \) alone.

In the quantum case, the pair of random variables \( X \) and \( Y \) are replaced by a bipartite quantum system \( AB \), with a corresponding density matrix \( \hat{\rho}_{AB} \). The standard definition of conditional von Neumann entropy adopts the form of the classical relation (10), with \( S(\hat{\rho}_{AB}) \) in place of the classical joint entropy and \( S(\hat{\rho}_B) \) substituted for \( H(X) \), where \( \hat{\rho}_B \) is the reduced density matrix for subsystem \( B \), as defined by the partial trace over subsystem \( A \). That is, the conditional von Neumann entropy is given by

\[
S(A|B) \equiv S(\hat{\rho}_{AB}) - S(\hat{\rho}_B), \quad \hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}].
\]

(11)

Unlike classical conditional entropy, conditional von Neumann entropy defined by (11) lacks an underlying probability distribution, as can be seen from the fact that \( S(A|B) \) can be negative \(^1\) when subsystems \( A \) and \( B \) are entangled. In \(^8\), the authors introduce a conditional amplitude operator \( \hat{\rho}_{A|B} \) as one possible generalization of a conditional probability distribution, but the operator is not a density matrix, and thus lacks a clear interpretation itself. Operational approaches are quite fruitful (see \(^9\) for example), but they do not always clarify the conceptual underpinnings of such quantities.

3 Quantum Conditional Probabilities and Information

3.1 Quantum Conditional Probabilities

The type of information measures studied in this paper are built from quantum conditional probabilities first explored in the context of the minimal modal interpretation of quantum theory \(^5\) \(^6\). While the quantities we discuss here require nothing beyond standard quantum theory for their formulation, we adopt the language of the minimal modal interpretation, as it provides a useful way to describe what follows.

To start, imagine that at a given time, a quantum system is described by an ‘objective’ density matrix \( \hat{\rho}_Q \)—objective in the sense that it is empirically optimal among all possible density matrices that an external observer could assign to the system. Now suppose that from the initial time to a later time, the density

\(^1\)Specifically, by an objective density matrix, we mean a density matrix whose mixedness arises entirely from entanglement.
matrix evolves from $\hat{\rho}_Q$ to a final density matrix $\hat{\rho}_R$ according to a linear CPTP map $\mathcal{E}_{R\leftarrow Q}$:

$$\hat{\rho}_R = \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}.\quad (12)$$

The initial and final density matrices have respective spectral decompositions

$$\hat{\rho}_Q = \sum_q p_q \hat{P}_q, \quad \hat{P}_q = |\Psi_q\rangle\langle\Psi_q|,\quad (13)$$

$$\hat{\rho}_R = \sum_r p_r \hat{P}_r, \quad \hat{P}_r = |\Psi_r\rangle\langle\Psi_r|.\quad (14)$$

According to the minimal modal interpretation, every quantum system has an actual underlying state corresponding to one of the eigenstates of the system’s density matrix, but that actual underlying state is hidden from external observers unless the system’s density matrix is a projector. In our present example, the system’s actual underlying state evolves from being one of the eigenstates of $\hat{\rho}_Q$ to being one of the eigenstates of $\hat{\rho}_R$. Collectively, the eigenstates of $\hat{\rho}_Q$ represent the initial possible underlying states of the system, and the eigenstates of $\hat{\rho}_R$ represent the final possible underlying states.

The evolution of the possible underlying states of the system is defined stochastically in terms of quantum conditional probabilities. For example, the probability that the system’s later state is $|\Psi_r\rangle$ given that it was initially $|\Psi_q\rangle$ is defined to be

$$p_{E}(r|q) = \text{Tr}[\hat{P}_r \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}] = |\langle\Psi_r|\mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}|\Psi_r\rangle.\quad (15)$$

Note that throughout this paper, lower-case index labels $q, q', \ldots$ and $r, r', \ldots$ on states correspond respectively to upper-case system configuration labels $Q$ and $R$. We adopt analogous conventions for other system configuration labels.

Regardless of the interpretation of quantum theory, the quantities defined by (15) exhibit almost all of the standard properties of conditional probabilities. In particular, they are non-negative real numbers that sum to unity and satisfy the law of total probability,

$$p_r = \sum_q p_{E}(r|q)p_q.\quad (16)$$

To see this, observe that

$$p_r = \text{Tr}[\hat{P}_r \hat{\rho}_R].\quad (17)$$

Substituting (12) for $\hat{\rho}_R$ yields

$$p_r = \text{Tr}[\hat{P}_r \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}].\quad (18)$$

We now substitute the decomposition (13) of $\hat{\rho}_Q$ and use the linearity of $\mathcal{E}_{R\leftarrow Q}$ to rewrite the expression as

$$p_r = \sum_q \text{Tr}[\hat{P}_r \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}]p_q,\quad (19)$$

allowing us to arrive at (16) by identifying the trace in (19) as the quantum conditional probability (15).
The quantum conditional probabilities $p_{E}(r|q)$ can be associated with a formal density matrix

$$\hat{\rho}_{R|q}^{E} = \sum_{r} p_{E}(r|q)\hat{P}_{r},$$

(20)

which satisfies

$$\hat{\rho}_{R} = \sum_{q} p_{q}\hat{\rho}_{R|q}^{E},$$

(21)
due to (16).

A crucial difference between classical and quantum conditional probabilities is that the latter fail to satisfy Bayes’ theorem:

$$p_{E}(r|q)p_{q} \neq p_{E}(q|r)p_{r}.$$  

(22)

The failure of Bayes’ theorem reflects the non-commutativity of quantum observables, and therefore the inability to define a symmetric joint probability distribution. From a dynamical perspective, Bayes’ theorem fails due to the generic irreversibility of $E_{R\leftarrow Q}$, as is evident from the case in which $E_{R\leftarrow Q}$ represents a projective measurement.

In general, linear CPTP evolution of an eigenprojector of the initial density matrix yields a nontrivial density matrix defined by

$$\hat{\rho}_{q}^{R} \equiv E_{R\leftarrow Q}\{ \hat{P}_{q} \}.$$  

(23)

Introducing a new label $r_{q}$ to distinguish the eigenprojectors $\{ \hat{P}_{r_{q}} \}_{r_{q}}$ of this density matrix, we can write down its spectral decomposition:

$$\hat{\rho}_{q}^{R} = \sum_{r_{q}} p_{E}(r_{q}|q)\hat{P}_{r_{q}}.$$  

(24)

Note that for each fixed value of $q$, the basis of eigenprojectors $\{ \hat{P}_{r_{q}} \}_{r_{q}}$ can be different, and will generically differ from $\{ \hat{P}_{r} \}_{r}$.

Nevertheless, the set of these density matrices must combine to yield $\hat{\rho}_{R}$,

$$\hat{\rho}_{R} = \sum_{q} p_{q}\hat{\rho}_{q}^{R},$$

(25)
as a consequence of (23).

The relations (21) and (25) suggest that $\hat{\rho}_{R|q}^{E}$ and $\hat{\rho}_{q}^{R}$ are themselves related. To see how, notice that the quantum conditional probabilities $p_{E}(r|q)$ can be expressed as

$$p_{E}(r|q) = \text{Tr}[\hat{P}_{r}\hat{\rho}_{q}^{R}] = \sum_{r_{q}} p_{E}(r_{q}|q)\text{Tr}[\hat{P}_{r}\hat{P}_{r_{q}}],$$

(26)

where in passing from the first to the second line we have used the decomposition (24). The quantity inside the trace has the form of a Born probability,

$$\beta(r|r_{q}) \equiv \text{Tr}[\hat{P}_{r}\hat{P}_{r_{q}}] = |\langle \Psi_{r}|\Psi_{r_{q}} \rangle |^{2},$$

(27)

\footnote{The paper by Schack, Brun, and Caves \cite{10} is a prominent example of work that does indeed derive a quantum version of Bayes’ rule. However, these sorts of results rely on taking a large number of copies of a system’s Hilbert space to represent a large ensemble of identical systems. The conditional probabilities we define in (15) differ in essential ways from these earlier constructions, as is apparent from the fact that our conditional probabilities involve only a single instance of a system’s Hilbert space. Thus, the failure of Bayes’ theorem is compatible with these prior results.}
and therefore (26) takes the form of a law of total probability,

\[ p_E(r|q) = \sum_{r} \beta(r|q)p_E(r|q). \]  \hfill (28)

Substituting the relation (28) into the definition (20) yields

\[ \hat{\rho}_{R|q}^E = \sum_{r,r} \beta(r|q)p_E(r|q) \hat{P}_r \]
\[ = \sum_{r,r} p_E(r|q) \hat{P}_r \hat{P}_r \hat{P}_r \]
\[ = \sum_r \hat{P}_r \left( \sum_{r} p_E(r|q) \hat{P}_r \right) \hat{P}_r, \]  \hfill (29)

where in passing to the second line we have used

\[ \hat{P}_r \hat{P}_r \hat{P}_r = \beta(r|q) \hat{P}_r. \]  \hfill (30)

We thus arrive at the relation

\[ \hat{\rho}_{R|q}^E = \sum_{r,r} \hat{P}_r \hat{P}_r \hat{P}_r. \]  \hfill (31)

Note that

\[ S(\hat{\rho}_{R|q}^E) \geq S(\hat{\rho}_R^E), \]  \hfill (32)

which follows from the double stochasticity of the Born probability distribution \( \beta(r|q) \). \footnote{We discuss doubly stochastic probability distributions in the appendix, providing an explicit proof of a generalization of (32).}

So far, our description of the quantum conditional probabilities (15) has been dynamical, with \( \mathcal{E}_{R\leftarrow Q} \) thought of as an evolution map. However, the same ideas can be applied to the quantum relationships between systems and their subsystems by noting that partial traces are an example of a linear CPTP map. We provide a more detailed sketch of these ideas in Section 5. In what follows, we will continue to focus on the dynamical picture, in which a single system evolves according to \( \mathcal{E}_{R\leftarrow Q} \).

### 3.2 New Measures of Quantum Information

Combining the quantum conditional probabilities of (15) with Shannon’s entropy formula yields a new type of quantum conditional entropy. Using the initial and final density matrices defined in (13) and (14), respectively, we let

\[ J_E(R|q) \equiv -\sum_r p_E(r|q) \log[p_E(r|q)] = S(\hat{\rho}_{R|q}^E) \]  \hfill (33)

be the quantum conditional entropy of our system given that the system’s initial underlying state corresponded to the eigenstate \( |\Psi_q \rangle \) of \( \hat{\rho}_Q \). We will argue that we can interpret this quantity as the entropy added to the system during its evolution given the initial underlying state of the system. The full quantum conditional entropy is the average over all possible initial eigenstates of \( \hat{\rho}_Q \):

\[ J_E(R|Q) \equiv \sum_q J_E(R|q)p_q = -\sum_{q,r} p_E(r|q)p_q \log[p_E(r|q)]. \]  \hfill (34)
We also define a new type of quantum mutual information:

\[ I_{E}^{\mathcal{E}}(R:Q) \equiv \sum_{q,r} p_{E}(r|q)p_{q} \log \left( \frac{p_{E}(r|q)}{p_{r}} \right). \]  

(35)

The relation

\[ I_{E}^{\mathcal{E}}(R:Q) = S(\hat{\rho}_{R}) - J_{E}^{\mathcal{E}}(R|Q) \]  

(36)

follows directly from the definitions of quantum conditional entropy (34) and quantum mutual information (35), mirroring the classical identity

\[ I(Y : X) = H(Y) - H(Y|X). \]  

(37)

In a dynamical context, mutual information can be thought of as measuring the information that is shared between the initial and final system configurations.

The new forms of quantum conditional entropy and quantum mutual information defined in (33) and (35), respectively, are distinct from the traditional quantities found in the literature. As discussed in Section 2, the traditional conditional von Neumann entropy \( S(A|B) \) in equation (11) is not defined in terms of an underlying probability distribution. The traditional von Neumann mutual information \( I^{VN}(A:B) \) shared by subsystems \( A \) and \( B \) is defined as

\[ I^{VN}(A:B) \equiv S(\hat{\rho}_{A}) - S(A|B). \]  

(38)

Once again, there need not be any underlying probability distribution in these traditional definitions.

We will show that the new information measures developed in this paper satisfy inequalities that are analogous to those satisfied by (11) and (38). However, the existence of underlying quantum conditional probabilities (15) provides conceptually clearer interpretations of the sort of information measured by these new quantities.

### 3.2.1 Evolution from a Pure State

To illustrate the interpretations of the quantities (33) and (35), we examine two special cases. To start, consider a system that is initially in a known pure state \( |\Psi\rangle \). Suppose that it evolves according to a linear CPTP map \( \mathcal{E} \), so that we lose track of its initially pure state:

\[ \hat{\rho}_{R} = \mathcal{E}\{\hat{P}_{\Psi}\}, \quad \hat{P}_{\Psi} = |\Psi\rangle\langle\Psi|. \]  

(39)

In this situation, we have conditional probabilities

\[ p_{E}(r|\Psi) = p_{r}, \]  

(40)

and hence we have the quantum conditional entropy

\[ J_{E}^{\mathcal{E}}(R|Q) = J_{E}^{\mathcal{E}}(R|\Psi) = -\sum_{r} p_{r} \log p_{r} = S(\hat{\rho}_{R}). \]  

(41)

In words, the increase in the system’s entropy arises solely from the evolution of the system. We can also characterize this statement in terms of the mutual information, which vanishes,

\[ I_{E}^{\mathcal{E}}(R;\Psi) = S(\hat{\rho}_{R}) - J_{E}^{\mathcal{E}}(R|\Psi) = 0, \]  

(42)

thereby showing that no information is carried over from the system’s initial state to its final configuration.

This linear CPTP map can be thought of as modeling a process in which the system becomes more entan-
From this perspective, the quantum conditional entropy measures the growth of entanglement between a system and its environment.

### 3.2.2 Unitary Evolution

Now consider a system whose initial and final density matrices are \( \hat{\rho}_Q \) and \( \hat{\rho}_R \), as expressed in (13) and (14), respectively. Suppose that the evolution is unitary, so that for some unitary operator \( \hat{U} \), we have

\[
\hat{\rho}_R = \hat{U} \hat{\rho}_Q \hat{U}^\dagger,
\]

where \( \mathbb{I} \) is the identity. Under such evolution, the eigenvalues of \( \hat{\rho}_Q \) are unchanged and the eigenstates rotate into the set of eigenstates of \( \hat{\rho}_R \),

\[
\hat{\rho}_R = \hat{U} \hat{\rho}_Q \hat{U}^\dagger,
\]

where the upper label emphasizes that the evolution carries us from the initial configuration \( Q \) to the final configuration \( R \). In this situation, the conditional probabilities (15) are trivial,

\[
p_{E}(r|q) = \text{Tr} \left[ \hat{P}_R r \hat{P}_Q q \hat{U} \hat{U}^\dagger \right] = \delta_{rq}.
\]

The quantum conditional entropy of this process is therefore zero and the quantum mutual information is equal to the von Neumann entropy of the system, showing that the uncertainty in the state of the system before the evolution is the sole source of uncertainty in the state afterward.

### 3.3 Some Identities and Inequalities

Due to the existence of an underlying probability distribution, the quantum conditional entropy (33) and mutual information (35) satisfy various relationships familiar from classical information theory.

- Conditional entropy and mutual information are always non-negative:
  
  \[
  J_E(R|Q) \geq 0, \quad I_E(R : Q) \geq 0.
  \]

- A system’s mutual information cannot be greater than the system’s initial entropy:
  
  \[
  I_E(R : Q) \leq S(\hat{\rho}_Q).
  \]

- A system’s conditional entropy cannot be greater than the system’s final entropy:
  
  \[
  J_E(R|Q) \leq S(\hat{\rho}_R).
  \]

The inequalities (46), (47), and (48) can be proved following similar steps to those from classical information theory. We provide details in the appendix.

### 4 Entropy Growth and Data Processing

#### 4.1 Unital Evolution and Projective Measurement

A unital linear CPTP map satisfies

\[
\mathcal{E}_{R \leftarrow Q} \{ \mathbb{I} \} = \mathbb{I}.
\]

\[\text{This interpretation assumes that the map is faithful to the underlying physics, rather than capturing measurement or modeling errors.}\]
The conditional probabilities for a unital linear CPTP map are doubly stochastic:

\[
\sum_q p_E(r|q) = \text{Tr} \left[ \hat{P}_r E_{R \leftarrow Q} \left\{ \sum_q \hat{P}_q \right\} \right]
\]
\[
= \text{Tr} \left[ \hat{P}_r E_{R \leftarrow Q} \{\mathbb{1}\} \right]
\]
\[
= \text{Tr} \left[ \hat{P}_r \right]
\]
\[
= 1.
\] (50)

Thus, if the evolution of a system is unital linear CPTP, then the von Neumann entropy grows,

\[
S(\hat{\rho}_Q) \leq S(\hat{\rho}_R),
\] (51)

which follows from the law of total probability relating \( p_r \) and \( p_q \) and the double-stochasticity of \( p_E(r|q) \) in this case, as proved in the appendix.

A projective measurement without post-selection is an example of a unital process. Suppose that we measure an observable with eigenstates \( \{ |\Psi_m\rangle \}_m \). If we isolate the measurement device and refrain from learning the outcome, then the post-measurement density matrix is well-approximated by

\[
\hat{\rho}_M = \mathcal{M}\{\hat{\rho}_Q\} = \sum_m \hat{P}_m \hat{\rho}_Q \hat{P}_m,
\] (52)

which is clearly unital. As a result, we see that measurements without post-selection increase the entropy of a system.

### 4.2 Concavity of von Neumann Entropy

The quantities described earlier allow us to demonstrate certain standard properties of quantum information. Consider the concavity of von Neumann entropy,

\[
\sum_i p_i S(\hat{\rho}_i) \leq S(\hat{\rho}), \quad \hat{\rho} = \sum_i p_i \hat{\rho}_i,
\] (53)

where \( \hat{\rho} \) is an arbitrary density matrix, and the set of pairs \( \{(p_i, \hat{\rho}_i)\}_i \) is any collection of non-negative weights and density matrices that form a decomposition of \( \hat{\rho} \) with the weights summing to unity. Note that the number of elements in the set can exceed the dimension of the Hilbert space.

To prove (53), we let \( \hat{\rho} = \hat{\rho}_R \). Given a decomposition into a set of weights and density matrices \( \{(p_i, \hat{\rho}_i)\}_i \) we can define a linear CPTP map \( \mathcal{E} \) and a density matrix \( \hat{\rho}_Q \) such that \( \hat{\rho}_R = \mathcal{E}\{\hat{\rho}_Q\} \) such that the elements of the decomposition arise from \( \mathcal{E} \) applied to the eigen-decomposition of \( \hat{\rho}_Q \), with the identification of the \( i \) and \( q \) indices. From the relations (21), (25), and (31), we have,

\[
\hat{\rho}_R = \sum_q p_q \hat{\rho}_{R|q} = \sum_q p_q \hat{\rho}_q^R,
\] (54)

with

\[
\hat{\rho}_{R|q} = \sum_r \hat{P}_r \hat{\rho}_q^R \hat{P}_r.
\] (55)

Note that we have simplified the notation by suppressing some labels.

\(^5\)Note that we implicitly allow \( \mathcal{E} \) to involve a partial trace operation so that the Hilbert space dimension associated with the final density matrix \( \hat{\rho}_R \) can be smaller than that of \( \hat{\rho}_Q \).
The quantum conditional entropy (34) can be expressed as the sum
\[ J_{\mathcal{E}}(R|Q) = \sum_q p_q S(\hat{\rho}_{R|q}). \]

Thus,
\[ \sum_q p_q S(\hat{\rho}_R^q) \leq \sum_q p_q S(\hat{\rho}_{R|q}) \leq S(\hat{\rho}_R), \]
where the first inequality follows from (32), while the second is the inequality (48), demonstrating the concavity of von Neumann entropy.

### 4.3 Quantum Markovianity and Data Processing

Consider a system that evolves from \( \hat{\rho}_Q \) to \( \hat{\rho}_R \), and then to \( \hat{\rho}_S \), as described by the linear CPTP maps \( \mathcal{E}_{R\leftarrow Q} \) and \( \mathcal{E}_{S\leftarrow R} \), so that we have
\[ \hat{\rho}_R = \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\}, \quad \hat{\rho}_S = \mathcal{E}_{S\leftarrow R}\{\hat{\rho}_R\} = \mathcal{E}_{S\leftarrow R} \circ \mathcal{E}_{R\leftarrow Q}\{\hat{\rho}_Q\} = \mathcal{E}_{S\leftarrow Q}\{\hat{\rho}_Q\}. \]

Observe that
\[ \hat{\rho}_R = \mathcal{E}_{R\leftarrow Q}\{\sum_q p_q \hat{P}_q\} = \sum_q p_q \mathcal{E}_{R\leftarrow Q}\{\hat{P}_q\}, \]
with corresponding conditional probabilities
\[ p(r|q) = \text{Tr}[\hat{P}_r \mathcal{E}_{R\leftarrow Q}\{\hat{P}_q\}], \]
where we suppress the map label as the mapping will be clear from the state indices.

Similarly, we have
\[ \hat{\rho}_S = \mathcal{E}_{S\leftarrow R}\{\sum_r p_r \hat{P}_r\} = \sum_r p_r \mathcal{E}_{S\leftarrow R}\{\hat{P}_r\}, \quad p(s|r) = \text{Tr}[\hat{P}_s \mathcal{E}_{S\leftarrow R}\{\hat{P}_r\}], \]

as well as
\[ \hat{\rho}_S = \mathcal{E}_{S\leftarrow Q}\{\sum_q p_q \hat{P}_q\} = \sum_q p_q \mathcal{E}_{S\leftarrow Q}\{\hat{P}_q\}, \quad p(s|q) = \text{Tr}[\hat{P}_s \mathcal{E}_{S\leftarrow Q}\{\hat{P}_q\}]. \]

There are some subtle constraints required for the consistency of these processes. Using the law of total probability and (28), we have
\[ p_s = \sum_r p(s|r)p_r \]
\[ = \sum_{r,q} p(s|r)p(r|q)p_q. \]
\[ = \sum_{r,q,r_q} p(s|r)p(r|q)p(q|r)p_{r_q}p_{r_q}. \]

Similarly, we have
\[ p_s = \sum_q p(s|q)p_q. \]

However, recall from (23) that
\[ \mathcal{E}_{R\leftarrow Q}\{\hat{P}_q\} = \sum_{r_q} p(r_q|q)\hat{P}_{r_q}. \]
So expanding out the definition of $p(s|q)$ and using $\mathcal{E}_{S\leftarrow Q} = \mathcal{E}_{S\leftarrow R} \circ \mathcal{E}_{R\leftarrow Q}$ gives

$$
p_s = \sum_q \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ \mathcal{E}_{R\leftarrow Q} \{ \hat{P}_q \} \} \right] p_q
$$

$$
= \sum_q \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \left\{ \sum_r p(r_q|q) \hat{P}_{rq} \right\} \right] p_q
$$

$$
= \sum_q \sum_r \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ \hat{P}_{rq} \} \right] p(r_q|q) p_q
$$

$$
= \sum_q \sum_r p(s|r_q) p(r_q|q) p_q. \tag{64}
$$

Comparing (61) and (64), we find that a natural-looking consistency condition to impose would be

$$
p(s|r_q) = \sum_r p(s|r) \beta(r|r_q). \tag{65}
$$

We therefore restrict our maps $\mathcal{E}_{R\leftarrow Q}$ and $\mathcal{E}_{S\leftarrow R}$ to those satisfying (65). The existence of such maps can be demonstrated by expanding out the definitions of the conditional probabilities in (65) on both sides. On the right-hand side we have

$$
\sum_r p(s|r) \beta(r|r_q) = \sum_r \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ \hat{P}_r \} \right] \langle \Psi_r | \hat{P}_{rq} | \Psi_r \rangle
$$

$$
= \sum_r \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ | \Psi_r \rangle \langle \Psi_r | \hat{P}_{rq} | \Psi_r \rangle \} \right]
$$

$$
= \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \left\{ \sum_r \hat{P}_r \hat{P}_{rq} \hat{P}_r \right\} \right]. \tag{66}
$$

while the left-hand side of (65) is

$$
p(s|r_q) = \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ \hat{P}_{rq} \} \right]. \tag{67}
$$

We conclude that one set of maps that satisfies (65) are maps that incorporate a projective measurement along the $\{ P_r \}$ basis in their definition:

$$
\mathcal{E}_{S\leftarrow R} \left\{ \sum_r \hat{P}_r \hat{P}_{rq} \hat{P}_r \right\} = \mathcal{E}_{S\leftarrow R} \{ \hat{P}_{rq} \}. \tag{68}
$$

Conceptually, this projective measurement ensures that the intermediate composite state of the system and its environment re-factorize, thus leading to Markov-like evolution.\footnote{Note that we could have instead inserted the projective measurement step along the $\{ P_r \}$ basis into the map $\mathcal{E}_{R\leftarrow Q}$. Either way, we demonstrate the existence of a set of maps satisfying the consistency condition (65).} Putting all this together, we have

$$
p(s|q) \equiv \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow Q} \{ \hat{P}_q \} \right]
$$

$$
= \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \circ \mathcal{E}_{R\leftarrow Q} \{ \hat{P}_q \} \right]
$$

$$
= \sum_{rq} \text{Tr} \left[ \hat{P}_s \mathcal{E}_{S\leftarrow R} \{ \hat{P}_{rq} \} \right] p(r_q|q)
$$

$$
= \sum_{rq} p(s|r_q) p(r_q|q)
$$

$$
= \sum_{r,rq} p(s|r) \beta(r|r_q) p(r_q|q),
$$
which with (28) yields

\[ p(s|q) = \sum_r p(s|r)p(r|q). \]  

(69)

The mutual information shared between the initial and final configurations is

\[ I(S : Q) = \sum_{s,q} p(s|q)p_q \log \left( \frac{p(s|q)}{p_s} \right). \]  

(70)

The mutual information between the initial and intermediate configurations is

\[ I(R : Q) = \sum_{r,q} p(r|q)p_q \log \left( \frac{p(r|q)}{p_r} \right). \]  

(71)

Using (69), the difference between these two quantities can be written as

\[
I(S : Q) - I(R : Q) = \sum_{s,q,r} p(s|r)p(r|q)p_q \left( \log \left( \frac{p(s|q)}{p_s} \right) - \log \left( \frac{p(r|q)}{p_r} \right) \right)
\]

\[
= \sum_{s,q,r} p(s|r)p(r|q)p_q \log \left( \frac{p(s|q)p_r}{p_s p(r|q)} \right).
\]

Using Jensen’s inequality\(^7\), we have

\[
I(S : Q) - I(R : Q) \leq \log \left( \sum_{s,q,r} p(s|r)p(r|q)p_q \frac{p(s|q)p_r}{p_s p(r|q)} \right)
\]

\[
= \log \left( \sum_{s,q} p(s|r)p_r \frac{p(s|q)p_q}{p_s} \right)
\]

\[
= \log \left( \sum_s p_s \frac{p_s}{p_s} \right)
\]

\[
= 0.
\]

We therefore arrive at a quantum version of the data-processing inequality,

\[ I(S : Q) \leq I(R : Q), \]  

(72)

capturing the idea that the information encoded in the system’s initial configuration is increasingly diluted as the system is “processed.”

### 4.4 A Holevo-Type Bound

Let us recall the statement of Holevo’s bound. Consider a quantum system and let \( X \) be a classical random variable with possible outcomes \( \{x\}_x \) and corresponding probability distribution \( \{p_x\}_x \). Suppose that \( \{\hat{\rho}_x\}_x \) is a collection of density matrices indexed by the possible outcomes \( x \) of \( X \), and let \( \hat{\rho} \) be the correspondingly averaged density matrix:

\[ \hat{\rho} \equiv \sum_x p_x \hat{\rho}_x. \]  

(73)

\(^7\)Jensen’s inequality states that if \( f(x) \) is a convex function of its argument \( x \), then the average of \( f(x) \) provides an upper bound for the original function applied to the average of its argument. Here we apply Jensen’s inequality to \( -\log x \).
If we now measure a POVM \(\{E_y\}\) whose possible outcomes \(y\) form another classical random variable \(Y\), then Holevo’s bound states that the classical mutual information between \(X\) and \(Y\) is bounded from above by the quantity
\[
\chi \equiv S(\hat{\rho}) - \sum_x p_x S(\hat{\rho}_x) \tag{74}
\]
That is,
\[
I(X : Y) \leq \chi. \tag{75}
\]

In the two-step process described in Section 4.3, the mutual information between the initial configuration \(\hat{\rho}_Q\) and the intermediate configuration \(\hat{\rho}_R\) can be expressed as
\[
I(R : Q) = S(\hat{\rho}_R) - J(R|Q) = S(\hat{\rho}_R) - \sum_q p_q S(\hat{\rho}_R|q), \tag{76}
\]
where
\[
\hat{\rho}_R|q = \sum_r \hat{P}_r E_{R \leftarrow Q}\{\hat{P}_q\} \hat{P}_r. \tag{77}
\]

The quantity on the right-hand side of (76) is clearly an example of Holevo’s \(\chi\) quantity. We see that it emerges quite naturally as an example of our newly defined mutual information, and that Holevo’s bound (75) arises as a manifestation of our quantum data-processing inequality (72). The Holevo bound’s interpretation as a quantum version of the data-processing inequality has been discussed before (see for example [11]). Our dynamical interpretation of the bound provides another perspective that avoids any explicit embedding of the system of interest into a larger composite system. Instead, we capture the role of the broader environment through the formalism of linear CPTP maps.

5 Discussion

5.1 Systems and Subsystems

Our focus in this paper has been on a dynamical interpretation of quantum information in a system whose evolution is described by a linear CPTP map. However, as mentioned in Section 3.1, the formalism is general enough to capture structural relationships between composite quantum systems and their subsystems. To begin, consider the parent system \(AB\) formed from a pair of quantum subsystems \(A\) and \(B\) and described by the density matrix
\[
\hat{\rho}_{AB} = \sum_m p_{AB}^m \hat{P}_m^{AB}, \quad \hat{P}_m^{AB} = \{|\Psi_m^{AB}\rangle\langle\Psi_m^{AB}|\}, \tag{78}
\]
where we include the parent system’s label \(AB\) on the system’s eigenprojectors \(\hat{P}_m^{AB}\) and the corresponding probabilities \(p_{AB}^m\). The subsystem density matrices are related to \(\hat{\rho}_{AB}\) via the appropriate partial traces,
\[
\hat{\rho}_A = \text{Tr}_B[\hat{\rho}_{AB}] = \sum_a p_a^A \hat{P}_a^A, \quad \hat{\rho}_B = \text{Tr}_A[\hat{\rho}_{AB}] = \sum_b p_b^B \hat{P}_b^B, \tag{79}
\]
where the sets of eigenprojectors for subsystems \(A\) and \(B\) are \(\{\hat{P}_a^A\}_a\) and \(\{\hat{P}_b^B\}_b\), respectively.

Quantum probabilities that conditionally link subsystem eigenstates to a given eigenstate of the parent system are again defined using (15), substituting the relevant partial trace for the linear CPTP map in the formula. For instance, the conditional probability that \(|\Psi_a^A\rangle\) is the actual underlying state of subsystem \(A\)
given that the underlying state of $AB$ is $|\Psi_m^{AB}\rangle$ is

$$p(a|m) = \text{Tr}_A[\hat{P}_a A \text{Tr}_B\{\hat{P}_m^{AB}\}].$$ (80)

As in Section 3.1, the partial trace applied to system $AB$'s eigenprojector yields a density matrix

$$\hat{\rho}_m^A = \text{Tr}_B[\hat{P}_m^{AB}] = \sum_{a_m} p(a_m|m) \hat{P}_m^A.$$ (81)

We have

$$\hat{\rho}_A = \sum_m p_m \hat{\rho}_m^A = \sum_m p_m \hat{\rho}_{A|m}, \quad \hat{\rho}_{A|m} = \sum_a \hat{P}_a^A \hat{\rho}_m^A \hat{P}_a^A.$$ (82)

These relationships imply that the quantum entropy conditioned on the parent state $|\Psi_m^{AB}\rangle$ satisfies the inequality

$$S(\hat{\rho}_m^A) \leq J(A|m) = -\sum_a p(a|m) \log p(a|m) = S(\hat{\rho}_{A|m}),$$ (83)

due to the quantities $p(a|m)$ and $p(a_m|m)$ being related via the doubly stochastic distribution

$$\beta(a|a_m) = |\langle \Psi_A^A | \Psi_m^{AB} \rangle|^2.$$ (84)

It is interesting to examine the von Neumann entropy of $\hat{\rho}_m^A$,

$$S(\hat{\rho}_m^A) = -\sum_{a_m} p(a_m|m) \log p(a_m|m),$$ (85)

and to note that it is naturally interpreted as the entanglement entropy of subsystem $A$ conditioned on the parent system $AB$ actually occupying the pure state $|\Psi_m^{AB}\rangle$. Note that when the parent system is in a pure state, then $\hat{\rho}_m^A = \hat{\rho}_{A|m}$ and $J(A|m)$ is the entanglement entropy of subsystem $A$.

The full quantum conditional entropy is defined as

$$J(A|AB) = \sum_m p_m J(A|m).$$ (86)

Therefore (83) implies

$$\sum_m p_m S(\hat{\rho}_m^A) \leq J(A|AB).$$ (87)

There are also intriguing relationships between our quantum conditional entropy (33,34) and conditional von Neumann entropy (11). Observe that the inequality (47) satisfied by our version of quantum mutual information can be re-expressed as

$$S(\hat{\rho}_A) - J(A|AB) \leq S(\hat{\rho}_{AB}),$$ (88)

where the initial density matrix is taken to be $\hat{\rho}_{AB}$ and the final density matrix is $\hat{\rho}_A$. Rearranging terms and applying the definition of conditional von Neumann entropy yields

$$-S(B|A) \leq J(A|AB).$$ (89)

In the presence of entanglement, $S(B|A)$ may take on negative values, leading to a positive lower bound on $J(A|AB)$. The result naturally captures the idea that when subsystems are entangled, there is a non-zero minimal uncertainty about their states even given information about the parent system.

---

8We again adopt the language of the minimal modal interpretation, though the mathematical content involves only textbook quantum theory.
5.2 Generalizations of Quantum Conditional Probabilities

Our definition of quantum conditional probability \( \mathcal{P}_E(\rho|\kappa) \) involves the eigenprojectors of initial and final density matrices \( \hat{\Pi}_Q^\kappa \) and \( \hat{\Pi}_R^\rho \), respectively. However, as we described in Section 2, there are infinitely many decompositions of a nontrivial density matrix. Thus, we may consider quantities of the form

\[
\mathcal{P}_E(\rho|\kappa) = \text{Tr} \left[ \hat{\Pi}_R^R \hat{E}_{R\rightarrow Q} \{ \hat{\Pi}_Q^Q \} \right],
\]

(90)

where

\[
\hat{\rho}_Q = \sum_\kappa \lambda_\kappa Q \hat{\Pi}_Q^\kappa, \quad \hat{\rho}_R = \sum_\rho \lambda_\rho R \hat{\Pi}_R^\rho \]

(91)

are general convex decompositions of the system’s initial and final density matrices, respectively, with generic projection operators

\[
\hat{\Pi}_Q^Q = |\Phi_Q^Q \rangle \langle \Phi_Q^Q |, \quad \hat{\Pi}_R^R = |\Phi_R^R \rangle \langle \Phi_R^R |.
\]

(92)

Note that such sets of projectors need not be orthogonal. However, if we demand that the quantities (90) behave as probabilities, then the set \( \{ \hat{\Pi}_R^R \}_\rho \) must resolve the identity:

\[
\sum_\rho \hat{\Pi}_R^R = \mathbb{I}.
\]

(93)

Nevertheless, these quantities fail to act as fully satisfactory conditional probabilities, as they do not obey a straightforward version of the law of total probability. Instead we have

\[
\Lambda_R^R \equiv \text{Tr} \left[ \hat{\Pi}_R^R \hat{\rho}_R \right] = \text{Tr} \left[ \hat{\Pi}_R^R \hat{E}_{R\rightarrow Q} \{ \hat{\rho}_Q \} \right] = \sum_\kappa \text{Tr} \left[ \hat{\Pi}_R^R \hat{E}_{R\rightarrow Q} \{ \hat{\Pi}_Q^Q \} \right] \lambda_\kappa Q,
\]

(94)

and thus

\[
\Lambda_R^R = \sum_\kappa \mathcal{P}_E(\rho|\kappa) \lambda_\kappa Q,
\]

(95)

where we generically have \( \Lambda_R^R \neq \lambda_\rho^R \) due to the possible nonorthogonality of the projectors.

Despite their shortcomings as proper conditional probability distributions, the quantities defined in (90) may yet be of some interest for reasons we detail in Section 7.

6 Connections to Other Work

6.1 Relation to Causal Quantum Conditional States

Interest in quantum analogues of information-theoretic quantities, such as probabilities and entropies, dates back to the early work of von Neumann [12]. Conditional counterparts of these quantities have been studied in many works, often with the goal of developing operators that capture quantum conditional expectations [3] or conditional versions of density matrices [8].

Our quantum conditional probabilities are most closely related to a type of operator defined in [13] by Leifer and Spekkens. By invoking the Choi-Jamiołkowski isomorphism, Leifer and Spekkens rephrase linear CPTP evolution in terms of what they refer to as a “causal quantum conditional state” operator on a double-copy of the system’s Hilbert space. Our quantum conditional probabilities turn out to be diagonal entries in their operator. We explore these relationships in greater detail in [3].
6.2 Quantum Statistical Mechanics and Fluctuation Theorems

In [14], Esposito and Mukamel investigate definitions of work and heat, entropy production, and fluctuation theorems in the context of open quantum systems. The authors’ results rest on their construction of quantum transition matrices that can be understood in terms of the quantum conditional probabilities (15) used in this work. To see this connection, first we follow [14] and describe the evolution of an open quantum system in terms of a differential linear CPTP map

\[ \frac{d\hat{\rho}_Q}{dt} = \mathcal{K}\{\hat{\rho}_Q(t)\}. \]  

The quantum transition matrices of [14] can be expressed as

\[ w((q'|q); t) \equiv \text{Tr}\left[\hat{P}_q\mathcal{K}\{\hat{\rho}_Q\}\right]. \]  

These transition rates satisfy a differential version of the law of total probability,

\[ \frac{dp_{q'}}{dt} = \sum_q w((q'|q); t)p_q, \]  

which follows from the definition (97) and the relation

\[ \frac{dp_q}{dt} = \text{Tr}\left[\hat{P}_q\mathcal{K}\{\hat{\rho}_Q(t)\}\right], \]  

together with the orthogonality of the operators \( \hat{P}_q \) and \( d\hat{P}_q/dt \).

Using our definition (15) of quantum conditional probability, we are formally able to reproduce the constructions of [14] by considering a linear CPTP map \( \mathcal{E}_{R'\leftarrow Q} \) that we interpret as evolving the density matrix \( \hat{\rho}_Q(t) \) of a system \( Q \) at time \( t \) to the system’s density matrix \( \hat{\rho}_Q(t') \) at time \( t' = t + \delta t \), for some small time interval \( \delta t \). The map \( \mathcal{K} \) is then reproduced formally by taking

\[ \mathcal{K} = \lim_{\delta t \to 0} \frac{\mathcal{E}_{Q'}^{t'\leftarrow t} - \text{Id}}{\delta t}. \]  

Similarly, the quantum transition matrix is given by

\[ w((q'|q); t) = \lim_{\delta t \to 0} \frac{p_{q'}(q'|q) - \delta q'|q}{\delta t}, \]  

where we have introduced indices \( t' \) and \( t \) indicating the explicit time dependencies of the eigenprojectors of \( \hat{\rho}_Q(t') \) and \( \hat{\rho}_Q(t) \), respectively.

6.3 Retrodiction in Quantum Theory

In Section 3.1, we argued that our quantum conditional probabilities (15) do not generically satisfy Bayes’ theorem due to the possible irreversibility of the linear CPTP map \( \mathcal{E}_{R'\leftarrow Q} \) on which their definition depends. The implications for retrodiction—inference about past states given present conditions—are nuanced. While \( \mathcal{E}_{R'\leftarrow Q} \) may not be reversible, there may be situations in which a reverse evolution map can be defined, as explored in the context of quantum fluctuation theorems by Aw, Buscemi, and Scarini [15], which appeared while this work was in preparation. Nonetheless, the generic asymmetry inherent in the definition (15) typically precludes any retrodiction based on our formulation of quantum conditional probabilities.

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\[ \text{16} \]
7 Conclusions and Future Directions

In this work, we utilized quantum conditional probabilities (15) that were first developed in [3] to define new forms of quantum conditional entropy (34) and quantum mutual information (35). We explored how these quantities capture growth of entropy and loss of information as an open quantum system evolves according to a linear CPTP evolution map.

Thanks to the existence of an underlying conditional probability distribution, we were able to provide conceptually clear proofs of identities and inequalities satisfied by our quantum conditional entropy and mutual information, analogous to those satisfied by their classical counterparts. By contrast, the traditional von Neumann conditional entropy and mutual information generically lack any underlying conditional probabilities, rendering their definitions and relationships conceptually unclear.

One limitation of our approach is that our quantum conditional probabilities depend for their definition on the existence of a well-defined linear CPTP map. For some of the results proved in this paper including (51), this limitation is benign because the claim itself is about a sub-class of linear CPTP dynamics. For other proofs in this paper, like the concavity of von Neumann entropy (53), we were able to introduce a linear CPTP map by hand without any loss of generality.

However, our derivation of the quantum data processing inequality (72) depended on the dynamics being described by a chain of linear CPTP maps. The same is therefore true for our Holevo-type bound in (75), with \( \chi \) given by the expression on the right-hand side of (76). In general, these sorts of inequalities do appear to depend on the dynamics being at least embeddable in some linear CPTP map [2]. It would be interesting to explore whether our approach could be used to study more general forms of dynamics that can be systematically approximated as analytically or numerically controllable deviations from linear CPTP dynamics.

In light of the connections between our work and works such as [14], as described in detail in Section 6.2, it would be interesting to explore the ways our quantum conditional entropies and our other results, including our quantum data-processing inequality, may be applied in understanding open-quantum system entropy growth and fluctuation theorems.

Section 5.1 explored intriguing connections between our quantum conditional probabilities and standard quantum information-theoretic concepts that arise from the rich structure of system-subsystem relationships in quantum theory. In future work, we will continue to explore these connections, along with related concepts, such as quantum discord [18].

Despite their failure to reproduce the law of total probability, the quantities (90) do satisfy the Kolmogorov axioms for a basic probability distribution. They are also examples of more general quantities of the form

\[
F_p(\hat{A}, \hat{B}; \hat{K}) = \text{Tr} \left[ \hat{A}^p \hat{K} \hat{B}^{1-p} \hat{K}^\dagger \right],
\]

where \( \hat{A} \) and \( \hat{B} \) are positive semi-definite \( N \times N \) matrices, \( \hat{K} \) is a fixed \( N \times N \) matrix, and \( 0 \leq p \leq 1 \). Lieb proved in [19] that trace quantities of the above type are non-negative concave maps. Observe that when \( \hat{A} \) and \( \hat{B} \) are taken to be projection operators with \( p = 1/2 \), and if \( \hat{K} \) is one of the operators in a Kraus representation of \( E_{R \rightarrow Q} \), then each term in the Kraus decomposition of (90) is of the form (102). Quantities such as (102) have been central to the understanding of generalized entropies, particularly the properties of quantum relative entropy, but their implications for the existence of probability distributions in quantum theory seem worth exploring further.

The properties of (102) provide one avenue for proving the strong subadditivity of traditional von Neumann conditional entropy. As a reminder to the reader, strong subadditivity is the statement that the von Neumann conditional entropy of a system \( Q \) given systems \( R \) and \( S \) is bounded from above by the von Neumann

(See for example [17] and references therein.) Our formulation, by contrast, is built from standard elements of quantum theory, and thus time asymmetries having to do with measurement processes or other open-system dynamics are unavoidable.
conditional entropy of \( Q \) given only \( R \):

\[ S(Q|RS) \leq S(Q|R). \tag{103} \]

Strong subadditivity can then be used to prove many of the other properties satisfied by quantum entropies and related quantities. Furthermore, the surprising results of [20] can also be seen as a reflection of the strong subadditivity of von Neumann entropy. Given these wide-ranging areas, we are quite interested in exploring whether our quantum conditional probabilities and their associated quantum conditional entropy can provide some new perspectives on strong subadditivity, and hence shed some light on recent developments at the intersection of quantum information and quantum gravity.

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**Appendix: Proofs of Basic Information Inequalities**

**Properties of Doubly Stochastic Distributions**

A conditional probability distribution \( p(y|x) \) is called doubly stochastic if

\[ \sum_x p(y|x) = 1. \tag{104} \]

If \( p(y) \) and \( p(x) \) are related via a doubly stochastic distribution,

\[ p(y) = \sum_x p(y|x)p(x), \tag{105} \]

then the Shannon entropy of \( p(y) \) is greater than or equal to that of \( p(x) \). To see why, consider their difference:

\[ H(X) - H(Y) = \sum_{x,y} p(y|x)p(x) \log \left( \frac{p(y)}{p(x)} \right). \tag{106} \]

Using Jensen’s inequality, we have

\[ H(X) - H(Y) \leq \log \left( \sum_{x,y} p(y|x)p(x) \frac{p(y)}{p(x)} \right) = \log \left( \sum_{x,y} p(y|x)p(y) \right). \tag{107} \]

At this stage, we can use the double stochasticity of \( p(y|x) \) to obtain

\[ H(X) - H(Y) \leq \log \left( \sum_y p(y) \right) = 0, \tag{108} \]

and hence

\[ H(X) \leq H(Y), \tag{109} \]
as claimed. While we have explicitly proved this result using classical notation, the proof applies to von Neumann entropies linked via the quantum conditional probabilities (15) defined in Section 3.1.

Non-Negativity

The non-negativity of quantum conditional entropy follows directly from its construction from non-negative conditional probabilities that cannot be greater than one. Non-negativity of our form of quantum mutual information arises by applying Jensen inequality to the definition (35):

\[
I_E(R : Q) = - \sum_{q,r} p(r|q)p_q \log \left[ \frac{p_r}{p(r|q)} \right] \geq - \log \left[ \sum_{q,r} p(r|q)p_q \frac{p_r}{p(r|q)} \right] = - \log(1) = 0. \tag{110}
\]

These arguments thus prove (46).

Linear CPTP Evolution Cannot Increase Mutual Information

The difference between the quantum mutual information shared by the initial and final configurations, on the one hand, and the von Neumann entropy of the initial density matrix (13), on the other hand, is

\[
I_E(R : Q) - S(\hat{\rho}_Q) = - \sum_{q,r} p(r|q)p_q \log \left[ \frac{p_r}{p(r|q)} \right] + \sum_q p_q \log p_q = \sum_{q,r} p(r|q)p_q \log \left[ \frac{p(r|q)p_q}{p_r} \right]. \tag{111}
\]

The law of total probability (16) gives us

\[
p_r \geq p_E(r|q)p_q. \tag{112}
\]

Thus, the monotonicity of the logarithm implies that

\[
I_E(R : Q) - S(\hat{\rho}_Q) \leq \sum_{q,r} p(r|q)p_q \log \left[ \frac{p_r}{p_r} \right] = 0. \tag{113}
\]

We have thus proved (47).

Conditional Entropy Cannot Exceed Final Entropy

The identity (36) can be rewritten as

\[
J_E(R|Q) = S(\hat{\rho}_R) - I_E(R : Q). \tag{114}
\]

Due to the positivity of mutual information, we immediately have that conditional entropy cannot exceed the final entropy of a system after a linear CPTP process, (48).

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