Certain new bounds considering the weighted Simpson-like type inequality and applications

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Abstract

We investigate a weighted Simpson-type identity and obtain new estimation-type results related to the weighted Simpson-like type inequality for the first-order differentiable mappings. We also present some applications to $f$-divergence measures and to higher moments of continuous random variables.

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1 Introduction and preliminaries

The following inequality is named the Simpson integral inequality:

$$\left| \frac{1}{6} \left[ f(r_1) + 4f\left( \frac{r_1 + r_2}{2} \right) + f(r_2) \right] - \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} f(x) \, dx \right| \leq \frac{1}{2880} \| f^{(4)} \|_{\infty} (r_2 - r_1)^4,$$

where $f : [r_1, r_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on $(r_1, r_2)$, and $\| f^{(4)} \|_{\infty} = \sup_{t \in (r_1, r_2)} |f^{(4)}(t)| < \infty$.

To see more recent results and the related generalizations with respect to (1.1), we refer the readers to [1–3, 5–18, 22–26, 29–34] and the references therein.

Let us recall that Miheşan [20] presented a class of mappings, called $(\alpha, m)$-convex functions, as follows: A mapping $f : [0, b^*] \to \mathbb{R}$, $b^* > 0$, is said to be $(\alpha, m)$-convex if

$$f(\lambda x + m(1 - \lambda)y) \leq \lambda^\alpha f(x) + m(1 - \lambda^\alpha)f(y)$$

for all $x, y \in [0, b^*]$ and $\lambda \in [0, 1]$ with some fixed $(\alpha, m) \in (0, 1] \times (0, 1]$. Shuang et al. [28] proved the following result for such mappings.

**Theorem 1.1** Let $f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R}$ be a differentiable function on $\mathbb{R}_0$, let $r_1, r_2 \in \mathbb{R}_0$, $r_1 < r_2$, and let $f' \in L^1[r_1, r_2]$. If $|f'|^q$ is $(\alpha, m)$-convex on $[0, \frac{r_2}{m}]$ for $(\alpha, m) \in (0, 1] \times (0, 1]$ and

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If \( q > 1 \), then
\[
\left| \frac{1}{8} f(r_1) + 6f\left(\frac{r_1 + r_2}{2}\right) + f(r_2) \right| - \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} f(x) \, dx \\
\leq \frac{r_2 - r_1}{4} \left( \frac{(q - 1)(3(2q - 1)(q^2 - 1) + 1)}{(2q - 1)^2} \right)^{1 - \frac{1}{q}} \\
\times \left\{ \left[ \frac{1}{1 + \alpha} \left| f'(r_1) \right|^q + \left( \frac{m\alpha}{1 + \alpha} \right) \left| f'\left(\frac{r_1 + r_2}{2}\right) \right|^\frac{q}{2} \right]^{\frac{1}{q}} \\
+ \left[ \frac{1}{1 + \alpha} \left| f'\left(\frac{r_1 + r_2}{2}\right) \right|^q + \left( \frac{m\alpha}{1 + \alpha} \right) \left| f'\left(\frac{r_2}{m}\right) \right|^\frac{q}{2} \right]^{\frac{1}{q}} \right\}.
\]

Noor et al. [21], introduced the class of \((\alpha, m, h)\)-convex functions that unifies several new and known classes of convex functions as follows.

**Definition 1.1** ([21]) Let \( h : J \subseteq \mathbb{R} \to \mathbb{R} \). A function \( f : I \subseteq \mathbb{R} \to (0, \infty) \) is said to be \((\alpha, m, h)\)-convex function if
\[
f(tx + m(1 - t)y) \leq h(t^\alpha)f(x) + mh(1 - t^\alpha)f(y)
\]
(1.2)
for all \( x, y \in I \) and \( t \in [0, 1] \) with some fixed \((\alpha, m) \in (0, 1) \times (0, 1)\).

Note that in [21] the authors have forgotten to write the second \( m \) in (1.2) in their original definition.

Let us discuss several particular cases of Definition 1.1.

I. If \( h(t) = t^s \) for \( s \in (0, 1) \), then Definition 1.1 reduces to the definition of \((\alpha, m, s)\)-convexity.

II. If \( h(t) = t^s \) for \( s \in (0, 1) \) and \( \alpha = 1 \), then Definition 1.1 reduces to the definition of \((s, m)\)-convexity.

III. If \( h(t) = t \), then Definition 1.1 reduces to the definition of \((\alpha, m)\)-convexity.

IV. If \( h(t) = 1 \), then Definition 1.1 reduces to the definition of \((m, P)\)-convexity.

V. If \( h(t) = t(1 - t) \) and \( \alpha = 1 \), then Definition 1.1 reduces to the definition of \((m, tgs)\)-convexity.

VI. If \( h(t) = \sqrt{t\sqrt{t}} \) and \( \alpha = 1 \), then Definition 1.1 reduces to the definition of \( m-MT \)-convexity.

Also, the following theorem was proved in [19]. It obtains an estimation-type result associated with the weighted Simpson-type inequality for \( h \)-convex mappings using Hölder’s inequality.

**Theorem 1.2** Let \( f : [r_1, r_2] \to \mathbb{R} \) be a differentiable function on \((r_1, r_2)\) such that \( f' \in L^1[r_1, r_2] \), and let \( w : [r_1, r_2] \to \mathbb{R} \) be continuous and symmetric with respect to \( \frac{r_1 + r_2}{2} \). If \( |f'|^q \)
is $h$-convex on $[r_1, r_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then

\[
\left| \frac{1}{6(r_2 - r_1)} \left[ f(r_1) + 4f \left( \frac{r_1 + r_2}{2} \right) + f(r_2) \right] \int_{r_1}^{r_2} w(x) \, dx - \frac{1}{r_2 - r_1} \int_{r_1}^{r_2} w(x)f(x) \, dx \right|
\]

\[
\leq \frac{r_2 - r_1}{12} \|w\|_{[r_1, r_2], \infty} \cdot \left( \frac{1 + 2^{p+1}}{3(p + 1)} \right)^{\frac{1}{p}} \cdot 2^{\frac{1}{q}}
\]

\[
\times \left\{ \left[ |f'(r_1)|^q \int_0^{\frac{1}{2}} h(t) \, dt + |f'(r_2)|^q \int_{\frac{1}{2}}^1 h(t) \, dt \right]^{\frac{1}{q}}
\]

\[
+ \left[ |f'(r_1)|^q \int_{\frac{1}{2}}^1 h(t) \, dt + |f'(r_2)|^q \int_0^{\frac{1}{2}} h(t) \, dt \right]^{\frac{1}{q}} \right\}.
\]

Different from [19] and [28], our purpose in this paper is to give some new bounds related to the weighted Simpson-like type inequality for the first-order differentiable mappings.

To obtain the principal results, we presume that the absolute value of the derivative of the considered mapping is $(\alpha, m, h)$-convex. Next, we substitute this hypothesis with the boundedness of the derivative and with a Lipschitz condition for the derivative of the considered mapping to establish integral inequalities with new estimation-type results. Also, we provide some applications to $f$-divergence measures and to higher moments of continuous random variables.

**2 Main results**

To obtain our main results, we need the following lemma.

**Lemma 2.1** Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on $I^o$, $a, b \in I^o$ with $a < b$, and let $w : [a, b] \to \mathbb{R}$ be symmetric with respect to $\frac{a + b}{2}$. If $f', w \in L^1[a, b]$, then

\[
\frac{1}{8(b - a)} \left[ f(a) + 6f \left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b - a} \int_a^b w(x)f(x) \, dx
\]

\[
= \frac{b - a}{4} \left\{ \int_0^1 p_1(t)f \left( ta + (1 - t) \frac{a + b}{2} \right) \, dt
\]

\[
+ \int_0^1 p_2(t)f \left( \frac{a + b}{2} + (1 - t)b \right) \, dt \right\},
\]

where

\[
p_1(t) = \frac{3}{4} \int_0^1 w \left( sa + (1 - s) \frac{a + b}{2} \right) \, ds - \int_0^t w \left( sa + (1 - s) \frac{a + b}{2} \right) \, ds
\]

and

\[
p_2(t) = \frac{1}{4} \int_0^1 w \left( s \frac{a + b}{2} + (1 - s)b \right) \, ds - \int_0^t w \left( s \frac{a + b}{2} + (1 - s)b \right) \, ds.
\]
Proof Integrating by parts and changing the variables, we have

\[
\mathcal{I}_1 = \int_0^1 p_1(t)f\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
= \int_0^1 \left[ \frac{3}{4} \int_0^t w\left(sa + (1-s)\frac{a+b}{2}\right) ds \right. \\
- \int_0^t w\left(sa + (1-s)\frac{a+b}{2}\right) ds \left. \right] f\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
= -\frac{2}{b-a} \left[ \frac{3}{4} \int_0^1 w\left(sa + (1-s)\frac{a+b}{2}\right) ds \right. \\
- \int_0^t w\left(sa + (1-s)\frac{a+b}{2}\right) ds \left. \right] f\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
= -\frac{2}{b-a} \left[ -\frac{1}{4} f(a) + \frac{3}{4} f\left(\frac{a+b}{2}\right) \right] \int_0^1 w\left(sa + (1-s)\frac{a+b}{2}\right) ds \\
- \frac{2}{b-a} \int_0^1 w\left(ta + (1-t)\frac{a+b}{2}\right)f\left(ta + (1-t)\frac{a+b}{2}\right) dt \\
= -\frac{1}{(b-a)^2} \left[ f(a) + 3f\left(\frac{a+b}{2}\right) \right] \int_a^{\frac{a+b}{2}} w(x) dx - \frac{4}{(b-a)^2} \int_a^{\frac{a+b}{2}} w(x)f(x) dx.
\]

Similarly, we get

\[
\mathcal{I}_2 = \int_0^1 p_2(t)f\left(ta + (1-t)b\right) dt \\
= -\frac{2}{b-a} \left[ \frac{1}{3} \int_0^1 w\left(s\frac{a+b}{2} + (1-s)b\right) ds \right. \\
- \int_0^t w\left(s\frac{a+b}{2} + (1-s)b\right) ds \left. \right] f\left(ta + (1-t)b\right) dt \\
= -\frac{2}{b-a} \left[ -\frac{1}{3} f(a) + \frac{3}{4} f\left(\frac{a+b}{2}\right) \right] \int_0^1 w\left(s\frac{a+b}{2} + (1-s)b\right) ds \\
- \frac{2}{b-a} \int_0^1 w\left(t\frac{a+b}{2} + (1-t)\frac{a+b}{2}\right)f\left(t\frac{a+b}{2} + (1-t)\frac{a+b}{2}\right) dt \\
= -\frac{1}{(b-a)^2} \left[ 3f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^{\frac{a+b}{2}} w(x) dx - \frac{4}{(b-a)^2} \int_a^{\frac{a+b}{2}} w(x)f(x) dx.
\]

Since \(w(x)\) is symmetric with respect to \(\frac{a+b}{2}\), we have

\[
\int_a^{\frac{a+b}{2}} w(x) dx = \int_a^{\frac{b}{a}} w(x) dx = \frac{1}{2} \int_a^b w(x) dx.
\]

Thus we have

\[
\frac{b-a}{4}(\mathcal{I}_1 + \mathcal{I}_2) \\
= \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) dx - \frac{1}{b-a} \int_a^b w(x)f(x) dx,
\]

which completes the proof.
Throughout the work, we write \( \|w\|_{[a,b],\infty} = \sup_{x \in [a,b]} |w(x)| \) for a continuous mapping \( w : [a, b] \to \mathbb{R} \). Next, we derive our main results.

**Theorem 2.1** Let \( f : \mathbb{R}_0 = [0, \infty) \to \mathbb{R} \) be a differentiable function on \( \mathbb{R}_0, a, b \in \mathbb{R}_0, a < b \), let \( f' \in L^1([a,b], \mathbb{R}) \), and let \( w : [a, b] \to \mathbb{R} \) be continuous and symmetric with respect to \( \frac{a+b}{2} \). If \( |f'|^q \) for \( q \geq 1 \) is \((a, m, h)-\)convex on \([0, \frac{5}{16}]\) with some fixed \((a, m) \in (0, 1] \times (0, 1] \) then

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \|w\|_{\mathbb{R}_0, \infty} \left( 5 \pi \right)^{-\frac{1}{q}} \\
\times \left\{ \left[ \int_0^1 \left( \frac{3}{4} - t \right) \left| h(t') |f'(a)|^q + h(1-t') m \left| f' \left( \frac{a+b}{2m} \right) \right|^q \right| \, dt \right\}^{\frac{1}{q}} \\
+ \left[ \int_0^1 \frac{1}{4} \left( h(t') |f' \left( \frac{a+b}{2} \right)\right|^q + h(1-t') m \left| f' \left( \frac{b}{m} \right) \right|^q \right] \, dt \}^{\frac{1}{q}}. \tag{2.2}
\]

**Proof** Applying Lemma 2.1 and using the fact that \( \|w\|_{[a, \frac{a+b}{2}], \infty} \|w\|_{\frac{a+b}{2}, \infty} \leq \|w\|_{[a,b], \infty} \), we have

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \left\{ \int_0^1 \frac{3}{4} \int_0^1 w \left( sa + (1-s) \frac{a+b}{2} \right) \, ds \\
- \int_0^t w \left( sa + (1-s) \frac{a+b}{2} \right) \left\| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right\| \, dt \\
+ \int_0^1 \frac{1}{4} \int_0^1 w \left( sa + (1-s) \frac{b}{2} \right) \, ds \\
- \int_0^t w \left( sa + (1-s) \frac{b}{2} \right) \left\| f' \left( ta + (1-t) b \right) \right\| \, dt \right\}^{\frac{1}{q}} \\
\leq \frac{b-a}{4} \|w\|_{[a,b], \infty} \left\{ \int_0^1 \frac{3}{4} \int_0^1 ds - \int_0^t ds \left\| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right\| \, dt \\
+ \int_0^1 \frac{1}{4} \int_0^1 ds - \int_0^t ds \left\| f' \left( ta + (1-t) \frac{b}{2} \right) \right\| \, dt \\
= \frac{b-a}{4} \|w\|_{[a,b], \infty} \left\{ \int_0^1 \frac{3}{4} - t \left\| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right\| \, dt \\
+ \int_0^1 \frac{1}{4} - t \left\| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right\| \, dt \} \right\}. \tag{2.3}
\]

Using the power mean inequality, we have

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \|w\|_{[a,b], \infty}.
\]
\[ \begin{align*}
\times \left\{ \left( \int_0^1 |\frac{3}{4} - t| \, dt \right)^{1 - \frac{1}{q}} \left[ \int_0^1 |\frac{3}{4} - t| \left| f' \left( ta + (1 - t) \frac{a + b}{2} \right) \right|^q \, dt \right]^{\frac{1}{q}} \\
+ \left( \int_0^1 |\frac{1}{4} - t| \, dt \right)^{1 - \frac{1}{q}} \left[ \int_0^1 |\frac{1}{4} - t| \left| f' \left( \frac{a + b}{2} + (1 - t)b \right) \right|^q \, dt \right]^{\frac{1}{q}} \right\}. \quad (2.4)
\end{align*} \]

From (2.3) and (2.4) we get the inquired inequality in (2.2), since

\[ \int_0^1 |\frac{1}{4} - t| \, dt = \int_0^1 |\frac{3}{4} - t| \, dt = \frac{5}{16}, \quad (2.5) \]

and using the \((\alpha, m, h)\)-convexity of \(|f'|^q\) on \([0, \frac{b}{a}]\), we have

\[ \left| f' \left( ta + (1 - t) \frac{a + b}{2} \right) \right|^q \leq h(t^\alpha) |f'(a)|^q + h(1 - t^\alpha) m \left| f' \left( \frac{a + b}{2m} \right) \right|^q \quad (2.6) \]

and

\[ \left| f' \left( \frac{a + b}{2} + (1 - t)b \right) \right|^q \leq h(t^\alpha) \left| f' \left( \frac{a + b}{2} \right) \right|^q + h(1 - t^\alpha) m \left| f' \left( \frac{b}{m} \right) \right|^q. \quad (2.7) \]

Direct computation provides the following cases. □

**Corollary 2.1** If we take \( q = 1 \) in Theorem 2.1, then we have the following inequality for \((\alpha, m, h)\)-convex functions:

\[ \left| \frac{1}{8(b - a)} \left[ f(a) + 6f \left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b - a} \int_a^b w(x)f(x) \, dx \right| \]

\[ \leq \frac{b - a}{4} \| w \|_{[\alpha, b], \infty} \left\{ \int_0^1 |\frac{3}{4} - t| \left( h(t^\alpha) |f'(a)| + h(1 - t^\alpha) m \left| f' \left( \frac{a + b}{2m} \right) \right| \right) \, dt \right. \]

\[ + \left. \int_0^1 |\frac{1}{4} - t| \left( h(t^\alpha) \left| f' \left( \frac{a + b}{2} \right) \right| + h(1 - t^\alpha) m \left| f' \left( \frac{b}{m} \right) \right| \right) \, dt \right\}. \]

**Remark 2.1** Consider Corollary 2.1.

(i) Putting \( h(t) = 1 \), we have the following inequality for \((m, P)\)-convex functions:

\[ \left| \frac{1}{8(b - a)} \left[ f(a) + 6f \left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b - a} \int_a^b w(x)f(x) \, dx \right| \]

\[ \leq \frac{5(b - a)}{64} \| w \|_{[\alpha, b], \infty} \]

\[ \times \left\{ \left[ |f'(a)| + \left| f' \left( \frac{a + b}{2} \right) \right| \right] + m \left[ \left| f' \left( \frac{a + b}{2m} \right) \right| + \left| f' \left( \frac{b}{m} \right) \right| \right] \right\}. \]
(ii) Putting \( h(t) = t(1-t) \) and \( \alpha = 1 \), we have the following inequality for \((m,tgs)-\text{convex functions:}\)

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|
\]

\[
\leq \frac{71(b-a)}{3 \cdot 2^{11}} \|w\|_{[a,b],\infty} \Bigg\{ \left| f'(a) \right| + \left| f'(\frac{a+b}{2}) \right| \Bigg\}
\]

\[
+ m \left[ \left| f' \left( \frac{a+b}{2m} \right) \right| + \left| f' \left( \frac{b}{m} \right) \right| \right].
\]

(iii) Putting \( h(t) = t^s \) and using the inequality \((1-t^s)^s \leq 2^{1-s} - t^m \) for \( t \in [0,1] \) with some fixed \( \alpha \in (0,1], s \in (0,1) \), we have the following inequality for \((\alpha,m,s)-\text{convex functions:}\)

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|
\]

\[
\leq \frac{b-a}{4} \|w\|_{[a,b],\infty} \left\{ \left| \Delta_1 f'(a) \right| + \left| \left( \frac{5 \cdot 2^{1-s}}{16} - \Delta_1 \right)m \left| f' \left( \frac{a+b}{2m} \right) \right| \right\}
\]

\[
+ \left\{ \Delta_2 \left| f' \left( \frac{a+b}{2} \right) \right| + \left| \left( \frac{5 \cdot 2^{1-s}}{16} - \Delta_2 \right)m \left| f' \left( \frac{b}{m} \right) \right| \right\},
\]

where

\[
\Delta_1 = \frac{3^{2\alpha+2} + 2^{2\alpha+1} \alpha - 2^{2\alpha+2}}{2^{2\alpha+1}(\alpha+1)(2\alpha+2)},
\]

\[
\Delta_2 = \frac{1 + 2^{2\alpha+1}(3\alpha + 2)}{2^{2\alpha+1}(\alpha+1)(2\alpha+2)}.
\]

**Theorem 2.2** Suppose that all assumptions of Theorem 2.1 are satisfied. Then

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|
\]

\[
\leq \frac{b-a}{4} \|w\|_{[a,b],\infty} \left( \frac{5}{16} \right)^{1-\frac{1}{q}}
\]

\[
\times \left\{ \left[ \int_0^1 \left( h \left( 1 - \left( \frac{1-t}{2} \right)^a \right) m \left| f' \left( \frac{a}{m} \right) \right|^q + h \left( \frac{1-t}{2} \right)^a \right) \left| f'(b) \right|^q \right] \, dt \right\}^{\frac{1}{q}}
\]

\[
+ \left\{ \int_0^1 \left( h \left( \left( \frac{t}{2} \right)^a \right) \left| f'(a) \right|^q + h \left( 1 - \left( \frac{t}{2} \right)^a \right) m \left| f' \left( \frac{b}{m} \right) \right|^q \right) \, dt \right\}^{\frac{1}{q}}. \tag{2.8}
\]

**Proof** Noting that \( ta + (1-t) \frac{a+b}{2} = \frac{t^2}{4}a + \frac{1-t^2}{4}b \) and using the \((\alpha,m,h)-\text{convexity of } |f'|^q \) on \([0,\frac{1}{m}]\), for any \( t \in [0,1] \), we have the inequality

\[
|f' \left( ta + (1-t) \frac{a+b}{2} \right)|^q
\]

\[
\leq h \left( 1 - \left( \frac{1-t}{2} \right)^a \right) m \left| f' \left( \frac{a}{m} \right) \right|^q + h \left( \frac{1-t}{2} \right)^a \left| f'(b) \right|^q. \tag{2.9}
\]
and, similarly,

$$\left| f'(\frac{t^a + b}{2} + (1-t)b) \right|^q \leq h\left(\frac{t^a}{2}\right)^q |f'(a)|^q + h\left(1-\frac{t^a}{2}\right)^q m\left|f'(\frac{b}{m})\right|^q. \quad (2.10)$$

Continuing from inequality (2.4) in the proof of Theorem 2.1 and using (2.9) and (2.10) with (2.5), we obtain the desired result in (2.8). This completes the proof. \( \square \)

**Corollary 2.2** If we take \( q = 1 \) in Theorem 2.2, then the following inequality for \((\alpha, m, h)\)-convex functions holds:

$$\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a + b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|$$

$$\leq b - a \|w\|_{[a,b],\infty} \left\{ \int_0^1 \left[ h\left(1-\frac{t^a}{2}\right)^a \left| f'(\frac{a}{m})\right| + h\left(\frac{t^a}{2}\right)^a \left| f'(\frac{b}{m})\right| \right] \, dt \right\}.$$ 

**Remark 2.2** Consider Corollary 2.2.

(i) Putting \( h(t) = t^s \) for \( s \in (0, 1] \) and using the inequality \((1-t^a)^s \leq 2^{-s} - t^{as} \) for \( t \in [0, 1] \) with some fixed \( s \in (0, 1] \) again, we have the following inequality for \((\alpha, m, s)\)-convex functions:

$$\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a + b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|$$

$$\leq b - a \|w\|_{[a,b],\infty} \left\{ \left(2^{1-s} - \left(\frac{2^{-as}}{1 + as}\right)\right) m \left[ \left| f'(\frac{a}{m})\right| + \left| f'(\frac{b}{m})\right| \right] \right\}.$$ 

(ii) Putting \( h(t) = 1 \), we have the following inequality for \((m, P)\)-convex functions:

$$\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a + b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|$$

$$\leq b - a \|w\|_{[a,b],\infty} \left\{ m \left[ \left| f'(\frac{a}{m})\right| + \left| f'(\frac{b}{m})\right| \right] \right\}.$$ 

(iii) Putting \( h(t) = t(1-t) \) and \( \alpha = 1 \), we have the following inequality for \((m, tgs)\)-convex functions:

$$\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a + b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right|$$

$$\leq b - a \|w\|_{[a,b],\infty} \left\{ m \left[ \left| f'(\frac{a}{m})\right| + \left| f'(\frac{b}{m})\right| \right] \right\}.$$ 

The next result deals with the case where \( |f'|^q \) for \( q > 1 \) is \((\alpha, m, h)\)-convex.
Theorem 2.3 Let \( f : \mathbb{R}_0 \rightarrow \mathbb{R} \) be a differentiable function on \( \mathbb{R}_0, a, b \in \mathbb{R}_0, a < b, \) let \( f' \in L^1[a, b], \) and let \( w : [a, b] \rightarrow \mathbb{R} \) be continuous and symmetric with respect to \( \frac{a+b}{2}. \) If \( |f'|^q \) for \( q > 1 \) is \((\alpha, m, h)\)-convex on \([0, \frac{a}{m}]\) with some fixed \((\alpha, m) \in (0, 1) \times (0, 1],\) then

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \| w \|_{[a,b]} Q^{1-\frac{1}{q}} \left\{ \left[ \int_0^1 \left( h(t^s) \left| f'(a) \right|^q + h(1-t^s) m \left| f' \left( \frac{a+b}{2m} \right) \right|^q \right] \, dt \right\}^{\frac{1}{q}} \\
+ \left[ \int_0^1 \left( h(t^s) \left| f' \left( \frac{a+b}{2} \right) \right|^q + h(1-t^s) m \left| f' \left( \frac{b}{m} \right) \right|^q \right] \, dt \right\}^{\frac{1}{q}},
\]

(2.11)

where

\[
Q = \frac{(q-1)(3^{2q-1}/(q-1) + 1)}{(2q-1)2^{2q-1}/(q-1)}.
\]

Proof Using the Hölder inequality for (2.3), we have

\[
\int_0^1 \left| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right|^q \, dt + \int_0^1 \left| f' \left( t \frac{a+b}{2} + (1-t)b \right) \right|^q \, dt \\
\leq \left\{ \left( \int_0^1 \left| \frac{a+b}{2} \right|^q \, dt \right)^{\frac{1}{q}} \left[ \int_0^1 \left| f' \left( ta + (1-t) \frac{a+b}{2} \right) \right|^q \, dt \right]^{\frac{1}{q}} \\
+ \left( \int_0^1 \left| \frac{a+b}{2} \right|^q \, dt \right)^{\frac{1}{q}} \left[ \int_0^1 \left| f' \left( t \frac{a+b}{2} + (1-t)b \right) \right|^q \, dt \right]^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]

(2.12)

From (2.6), (2.7), and (2.12) we get the desired inequality in (2.11), since

\[
\int_0^1 \left| \frac{a+b}{2} \right|^q \, dt = \int_0^1 \left| \frac{a+b}{2} \right|^q \, dt = \frac{(q-1)(3^{2q-1}/(q-1) + 1)}{(2q-1)2^{2q-1}/(q-1)} = Q.
\]

Now, we state some particular cases of Theorem 2.3.

Corollary 2.3 In Theorem 2.3, putting \( h(t) = t^s \) and using the inequality \( (1-t^s)^\alpha \leq 2^{1-s} - t^{s\alpha} \) for \( t \in [0, 1] \) with some fixed \( \alpha \in (0, 1], s \in (0, 1] \) again, we have the following inequality for \((\alpha, m, s)\)-convex functions:

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \| w \|_{[a,b]} Q^{1-\frac{1}{q}} \left\{ \left[ \frac{1}{1+s\alpha} \left| f'(a) \right|^q + \frac{2^{1-s} - 1}{1+s\alpha} m \left| f' \left( \frac{a+b}{2m} \right) \right|^q \right]^{\frac{1}{q}} \\
+ \left[ \frac{1}{1+s\alpha} \left| f' \left( \frac{a+b}{2} \right) \right|^q + \frac{2^{1-s} - 1}{1+s\alpha} m \left| f' \left( \frac{b}{m} \right) \right|^q \right]^{\frac{1}{q}} \right\}^{\frac{1}{q}}.
\]
Remark 2.3 In Corollary 2.3, if \( f(a) = f\left(\frac{a+b}{2}\right) = f(b) \) with \( m = 1 = \alpha \), then the following inequality for \( s \)-convex functions holds:

\[
\left| f\left(\frac{a + b}{2}\right) \int_a^b w(x) \, dx - \int_a^b w(x) f(x) \, dx \right|
\leq \frac{(b-a)^2}{4} \|w\|_{[a,b],\infty} Q^{1-\frac{s}{2}} \left\{ \frac{1}{1+s} |f'(a)|^q + \left(2^{1-s} - \frac{1}{1+s}\right) |f\left(\frac{a+b}{2}\right)|^q \right\}^{\frac{1}{2}}
\]

\[
= \left[ \frac{1}{1+s} \left| f\left(\frac{a+b}{2}\right) \right|^q + \left(2^{1-s} - \frac{1}{1+s}\right) |f(b)|^q \right]^{\frac{1}{2}}
\]

Corollary 2.4 Consider Theorem 2.3.

(i) If we take \( h(t) = 1 \), then the following inequality for \((m, P)\)-convex functions holds:

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x) f(x) \, dx \right|
\leq \frac{b-a}{4} \|w\|_{[a,b],\infty} Q^{1-\frac{s}{2}} \left\{ \frac{1}{6} \left[ |f'(a)|^q + m \left| f\left(\frac{a+b}{2}\right) \right|^q \right] \right\}^{\frac{1}{2}}
\]

\[
+ \left[ \left| f\left(\frac{a+b}{2}\right) \right|^q + m \left| f\left(\frac{b}{m}\right) \right|^q \right]^{\frac{1}{2}}
\]

(ii) If we take \( h(t) = t(1-t) \) and \( \alpha = 1 \), then the following inequality for \((m, tgs)\)-convex functions holds:

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x) f(x) \, dx \right|
\leq \frac{b-a}{4} \|w\|_{[a,b],\infty} Q^{1-\frac{s}{2}} \left\{ \frac{1}{6} \left[ |f'(a)|^q + m \left| f\left(\frac{a+b}{2m}\right) \right|^q \right] \right\}^{\frac{1}{2}}
\]

\[
+ \left[ \left| f\left(\frac{a+b}{2}\right) \right|^q + m \left| f\left(\frac{b}{m}\right) \right|^q \right]^{\frac{1}{2}}
\]

(iii) If we take \( h(t) = \frac{\sqrt{t}}{2\sqrt{t}} \) and \( \alpha = 1 \), then the following inequality for \( m\)-MT-convex functions holds:

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x) f(x) \, dx \right|
\leq \frac{b-a}{4} \|w\|_{[a,b],\infty} Q^{1-\frac{s}{2}} \left\{ \frac{1}{6} \left[ |f'(a)|^q + m \left| f\left(\frac{a+b}{2m}\right) \right|^q \right] \right\}^{\frac{1}{2}}
\]

\[
+ \left[ \left| f\left(\frac{a+b}{2}\right) \right|^q + m \left| f\left(\frac{b}{m}\right) \right|^q \right]^{\frac{1}{2}}
\]

A similar result may be stated.
**Theorem 2.4** Suppose that all assumptions of Theorem 2.3 are satisfied. Then

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \left\| w \right\|_{[a,b],\infty} Q^{1-\frac{1}{p}} \times \left\{ \left[ \int_0^1 \left( h \left( 1 - \frac{t}{2} \right)^\alpha m f' \left( \frac{a}{m} \right)^q + h \left( \frac{1-t}{2} \right)^\alpha \right) \, dt \right]^{\frac{q}{q+1}} + \left[ \int_0^1 \left( h \left( \frac{t}{2} \right)^\alpha \right) f' \left( \frac{b}{m} \right)^q + h \left( 1 - \frac{t}{2} \right)^\alpha \right) m f' \left( \frac{b}{m} \right)^q \, dt \right]^{\frac{q}{q+1}} \right\}.
\]

(2.13)

**Proof** The proof of Theorem 2.4 is analogous to that of Theorem 2.3 by using \( ta + (1-t) \frac{ab}{2} = \frac{1+t}{2}a + \frac{1-t}{2}b \) and \( \frac{ab}{2} t + (1-t)b = \frac{t}{2}a + (1-\frac{t}{2})b \).

The following result holds for \((a,m,s)-\text{convexity}.

**Theorem 2.5** Let \( f : \mathbb{R}_0 \to \mathbb{R} \) be a differentiable function on \( \mathbb{R}_0 \), \( a, b \in \mathbb{R}_0 \), \( a < b \), let \( f' \in L^1[a,b] \), and let \( w : [a,b] \to \mathbb{R} \) be continuous and symmetric with respect to \( \frac{ab}{2} \). If \( |f'|^q \) is \((a,m,s)-\text{convex} on [0, \frac{ab}{2}] for some fixed \((a,m) \in (0,1) \times (0,1), p+1 + q = 1 \) and \( q > 1 \), then

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \left\| w \right\|_{[a,b],\infty} \left( \frac{1 + 3p+1}{(p+1)4q+1} \right)^{\frac{1}{p}} \times \left\{ \left[ \int_0^1 \left( t^\alpha \left| f' \left( \frac{a+b}{2m} \right) \right|^q \right) \, dt \right]^{\frac{1}{p}} + \left[ \int_0^1 \left( \frac{a+b}{2} - (1-t)b \right)^\alpha \, dt \right]^{\frac{1}{p}} \right\}.
\]

(2.14)

**Proof** Since \(|f'|^q\) is \((a,m,s)-\text{convex} on [0, \frac{ab}{2}]\), using the Hölder inequality for (2.3), we have

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \right| \\
\leq \frac{b-a}{4} \left\| w \right\|_{[a,b],\infty} \left( \frac{1 + 3p+1}{(p+1)4q+1} \right)^{\frac{1}{p}} \times \left\{ \left[ \int_0^1 \left( t^{\alpha} + \frac{a+b}{2m} \right)^q \, dt \right]^{\frac{1}{q}} + \left[ \int_0^1 \left( \frac{a+b}{2} - (1-t)b \right)^\alpha \, dt \right]^{\frac{1}{q}} \right\}.
\]

(2.15)
From (2.15) we get the desired inequality in (2.14), since

\[ \int_0^1 \! t^{\alpha} \, dt = \frac{1}{1 + \alpha}, \]

and using the inequality \((1 - t^\alpha)^s \leq 2^{1-s} - t^{\alpha s}\) for \(t \in [0,1]\) with some fixed \(\alpha \in (0,1]\), \(s \in [0,1]\), we have

\[ \int_0^1 \! (1 - t^\alpha)^s \, dt \leq \int_0^1 \! (2^{1-s} - t^{\alpha s}) \, dt = 2^{1-s} - \frac{1}{1 + \alpha}. \]

□

Now, we point out a particular case of Theorem 2.5.

**Corollary 2.5** If we take \(s = 1\) and \(m = 1 = \alpha\) in Theorem 2.5, then the following inequality for convex functions holds:

\[
\left| \frac{1}{8(b-a)} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x) f(x) \, dx \right| \\
\leq \frac{b-a}{2^{2+\frac{1}{p}}} \left\| w \right\|_{[a,b],\infty} \left(1 + \frac{3^{p+1}}{(p+1)4^{p+1}}\right)^{\frac{1}{2}} \left\{ \left[ \left| f'(a) \right|^q + \left| f'(\frac{a+b}{2}) \right|^q \right]^{\frac{1}{q}} \right. \\
\left. + \left[ \left| f'\left(\frac{a+b}{2}\right) \right|^q + \left| f'(b) \right|^q \right]^{\frac{1}{q}} \right\}.
\]

Next, we would like to point out some published results that are particular cases of the obtained main results.

**Remark 2.4** In Lemma 2.1, if we take \(w(x) = 1\), then identity (2.1) becomes the following equation proved by Shuang et al. [28]:

\[
\frac{1}{8} \left[ f(a) + 6f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \\
= \frac{b-a}{4} \int_0^1 \left[ \left( \frac{3}{4} - t \right) f'\left( ta + (1-t)\frac{a+b}{2} \right) + \left( \frac{1}{4} - t \right) f'\left( \frac{a+b}{2} + (1-t)b \right) \right] \, dt.
\]

**Remark 2.5** If we take \(h(t) = t\) and \(w(x) = 1\) in Theorems 2.1 and 2.3, then we obtain Theorems 3.1 and 3.5 established by Shuang et al. [28], respectively.

### 3 Further estimation results

If the considered function \(f'\) is bounded from below and above, then we have the following result.

**Theorem 3.1** Let \(f : I \subseteq \mathbb{R} \to \mathbb{R}\) be a differentiable function on \(I', a, b \in I', a < b, and let w : [a, b] \to \mathbb{R}\) be continuous and symmetric with respect to \(\frac{a+b}{2}\). Assume that \(f'\) is integrable on \([a, b]\) and there exist constants \(m < M\) such that \(-\infty < m \leq f'(x) \leq M < +\infty\) for all
where \( p_1(t) \) and \( p_2(t) \) are defined in Lemma 2.1.

**Proof** From Lemma 2.1 we have

\[
\begin{align*}
\frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx &- \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \\
&- \frac{(b-a)(m+M)}{8} \left[ \int_0^1 p_1(t) \, dt + \int_0^1 p_2(t) \, dt \right] \\
&\leq \frac{5(b-a)(M-m)}{64} \|w\|_{[a,b],\infty},
\end{align*}
\]

(3.1)

So

\[
\mathcal{T} := \frac{1}{8(b-a)} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \\
- \frac{(b-a)(m+M)}{8} \left[ \int_0^1 p_1(t) \, dt + \int_0^1 p_2(t) \, dt \right] \\
= \frac{b-a}{4} \left\{ \int_0^1 p_1(t) \left[ f'(ta+(1-t)\frac{a+b}{2}) - \frac{m+M}{2} \right] \, dt \\
+ \int_0^1 p_2(t) \left[ f'(ta+(1-t)\frac{a+b}{2}) - \frac{m+M}{2} \right] \, dt \right\} \\
+ \frac{(b-a)(m+M)}{8} \left\{ \int_0^1 p_1(t) \, dt + \int_0^1 p_2(t) \, dt \right\}.
\]

Therefore

\[
|\mathcal{T}| \leq \frac{b-a}{4} \left\{ \int_0^1 |p_1(t)| \left| f'(ta+(1-t)\frac{a+b}{2}) - \frac{m+M}{2} \right| \, dt \\
+ \int_0^1 |p_2(t)| \left| f'(ta+(1-t)\frac{a+b}{2}) - \frac{m+M}{2} \right| \, dt \right\} \\
\leq \frac{(b-a)(M-m)}{8} \left\{ \int_0^1 |p_1(t)| \, dt + \int_0^1 |p_2(t)| \, dt \right\}.
\]
Since \( f' \) satisfies \(-\infty < m \leq f'(x) \leq M < +\infty \), we have

\[
m - \frac{m + M}{2} \leq f'(x) - \frac{m + M}{2} \leq M - \frac{m + M}{2},
\]

which implies that

\[
\left| f'(x) - \frac{m + M}{2} \right| \leq \frac{M - m}{2}.
\]

Also, since \( w \) is symmetric with respect to \( \frac{a + b}{2} \), we get

\[
|T| \leq \frac{(b - a)(M - m)}{8} \left\{ \int_0^1 \frac{3}{4} t \int_0^1 w \left( sa + \left(1 - s \right) \frac{a + b}{2} \right) ds \right. \\
+ \left. \int_0^1 \frac{1}{4} t \int_0^1 w \left( t \frac{a + b}{2} + \left(1 - s \right) b \right) ds \right\} \\
\leq \frac{(b - a)(M - m)}{8} \||w||_{[a,b],\infty} \\
\times \left\{ \int_0^1 \frac{3}{4} \int_0^1 ds \int_0^t \left| ds \right| dt + \int_0^1 \frac{1}{4} \int_0^1 ds \int_0^t \left| ds \right| dt \right\} \\
\leq \frac{5(b - a)(M - m)}{64} \||w||_{[a,b],\infty}.
\]

This ends the proof. \( \square \)

**Corollary 3.1** In Theorem 3.2, if \( w(x) = 1 \), then we get

\[
\left| \frac{1}{8} f(a) + 6f \left( \frac{a + b}{2} \right) + f(b) \right| - \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{(b - a)(9M - m)}{64}. \tag{3.2}
\]

**Proof** If we take \( w(x) = 1 \), then the relation \( ||w||_{[a,b],\infty} = 1 \) implies that

\[
\left| \frac{1}{8} f(a) + 6f \left( \frac{a + b}{2} \right) + f(b) \right| - \frac{1}{b - a} \int_a^b f(x) \, dx \\
\leq \frac{(b - a)(m + M)}{8} \left| \int_0^1 p_1(t) \, dt + \int_0^1 p_2(t) \, dt \right| + \frac{5(b - a)(M - m)}{64} \\
\leq \frac{(b - a)(m + M)}{8} \left[ \left| \int_0^1 p_1(t) \, dt \right| + \left| \int_0^1 p_2(t) \, dt \right| \right] + \frac{5(b - a)(M - m)}{64} \\
\leq \frac{(b - a)(m + M)}{16} + \frac{5(b - a)(M - m)}{64} \\
= \frac{(b - a)(9M - m)}{64}. \quad \square
\]

Our next goal is an estimation-type result with respect to the weighted Simpson-like type inequality when the derivative of the considered function \( f' \) satisfies a Lipschitz condition.
Theorem 3.2 Let \( f : I \subseteq \mathbb{R} \to \mathbb{R} \) be a differentiable function on \( I^0 \), \( a, b \in I^0 \), \( a < b \), and let \( w : [a, b] \to \mathbb{R} \) be continuous and symmetric with respect to \( \frac{a+b}{2} \). Assume that \( f' \) is integrable on \([a, b]\) and satisfies a Lipschitz condition for some \( L > 0 \). Then

\[
\left\| \frac{1}{8(b-a)} \left[ f(a) + 6f\left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx \right\| \leq \frac{41(b-a)^2L}{3 \cdot 2^8} \|w\|_{[a,b],\infty},
\]

(3.3)

where \( p_1(t) \) and \( p_2(t) \) are defined in Lemma 2.1.

Proof From Lemma 2.1 we have

\[
\begin{align*}
\frac{1}{8(b-a)} & \left[ f(a) + 6f\left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \\
= & \frac{b-a}{4} \left\{ \int_0^1 p_1(t) \left[ f'\left( ta + (1-t)\frac{a+b}{2} \right) - f'(a) + f'(b) \right] \, dt \right\} \\
& + \int_0^1 p_2(t) \left[ f'\left( \frac{a+b}{2} + (1-t)b \right) - f'(b) \right] \, dt \\
= & \frac{b-a}{4} \left\{ \int_0^1 p_1(t) \left[ f'\left( ta + (1-t)\frac{a+b}{2} \right) - f'(a) \right] \, dt \right\} \\
& + \int_0^1 p_2(t) \left[ f'\left( \frac{a+b}{2} + (1-t)b \right) - f'(b) \right] \, dt \\
& + \frac{b-a}{4} \left\{ \int_0^1 p_1(t)f'(a) \, dt + \int_0^1 p_2(t)f'(b) \, dt \right\}.
\end{align*}
\]

Then

\[
R = \frac{1}{8(b-a)} \left[ f(a) + 6f\left( \frac{a + b}{2} \right) + f(b) \right] \int_a^b w(x) \, dx - \frac{1}{b-a} \int_a^b w(x)f(x) \, dx \\
- \frac{b-a}{4} \left\{ \int_0^1 p_1(t)f'(a) \, dt + \int_0^1 p_2(t)f'(b) \, dt \right\}
\]

\[
= \frac{b-a}{4} \left\{ \int_0^1 p_1(t) \left[ f'\left( ta + (1-t)\frac{a+b}{2} \right) - f'(a) \right] \, dt \right\} \\
& + \int_0^1 p_2(t) \left[ f'\left( \frac{a+b}{2} + (1-t)b \right) - f'(b) \right] \, dt.
\]

Since \( f' \) satisfies Lipschitz conditions for some \( L > 0 \), we have

\[
\left| f'\left( ta + (1-t)\frac{a+b}{2} \right) - f'(a) \right| \leq L \left| ta + (1-t)\frac{a+b}{2} - a \right| = L|1-t|\left( \frac{b-a}{2} \right)
\]

and

\[
\left| f'\left( \frac{a+b}{2} + (1-t)b \right) - f'(b) \right| \leq L \left| \frac{a+b}{2} + (1-t)b - b \right| = L|t|\left( \frac{b-a}{2} \right).
\]
Hence

\[ |\mathcal{R}| \leq \frac{(b-a)^2L}{8} \left\{ \int_0^1 (1-t)|p_1(t)| \, dt + \int_0^1 t|p_2(t)| \, dt \right\}. \]

Also, since \( w \) is symmetric with respect to \( \frac{a+b}{2} \), we get

\[ |\mathcal{R}| \leq \frac{(b-a)^2L}{8} \|w\|_{[a,b],\infty} \left\{ \int_0^1 (1-t) \left| \frac{3}{4} - t \right| \, dt + \int_0^1 t \left| \frac{1}{4} - t \right| \, dt \right\} \leq \frac{41(b-a)^2L}{3 \cdot 2^8} \|w\|_{[a,b],\infty}. \]

This completes the proof. \( \square \)

**Corollary 3.2** In Theorem 3.2, if \( w(x) = 1 \), then we get

\[
\left| \frac{1}{8} \left[ f(a) + 6f \left( \frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\leq \frac{41(b-a)^2L}{3 \cdot 2^8} + \frac{b-a}{16} \left[ f'(a) + f'(b) \right]. \quad (3.4)
\]

### 4 Applications

#### 4.1 \( f \)-divergence measures

In various applications of probability theory, one of the primary themes is discovering a proper measure of distance between any two probability distributions. Let a set \( \psi \) and a \( \sigma \)-finite measure \( \mu \) be given and consider the set of all probability densities on \( \Omega := \{ \rho : \psi \to \mathbb{R}, \rho(x) > 0, \int_\psi \rho(x) \, d\mu(x) = 1 \} \).

Let \( f : (0, \infty) \to \mathbb{R} \) be a given mapping and consider \( D_f(\rho, \tau) \) defined by

\[
D_f(\rho, \tau) := \int_\psi \rho(x)f \left[ \frac{\tau(x)}{\rho(x)} \right] \, d\mu(x), \quad \rho, \tau \in \Omega. \quad (4.1)
\]

If \( f \) is convex, then (4.1) is known as the Csiszár \( f \)-divergence [4].

Shioya and Da-te [27] presented the Hermite–Hadamard (HH) divergence

\[
D_{HH}^f(\rho, \tau) = \int_\psi \rho(x) f \left( \frac{\tau(x)}{\rho(x)} \right) \frac{\tau(x)}{\rho(x)} \, d\mu(x), \quad \rho, \tau \in \Omega, \quad (4.2)
\]

where \( f \) is convex on \((0, \infty)\) with \( f(1) = 0 \). In the same paper [27], they also gave the property of HH divergence that \( D_{HH}^f(\rho; \tau) \geq 0 \) with equality if and only if \( \rho = \tau \).
Proposition 4.1 Let all assumptions of Theorem 2.5 hold with \( f(1) = 0 \). If \( \rho, \tau \in \Omega \), then

\[
\left| \frac{1}{8} \left[ D_{f}(\rho, \tau) + 6 \int_{\Psi} \rho(x)f\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) d\mu(x) \right] - D_{f_{\text{HH}}}(\rho, \tau) \right| \\
\leq \left( \frac{7}{3 \cdot 2^q} \right)^{\frac{1}{2}} \left\{ \left[ f'(1) \right]^2 \int_{\Psi} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} d\mu(x) \right. \\
+ \int_{\Psi} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) \right]^2 d\mu(x) \right. \\
+ \left[ \int_{\Psi} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) \right]^2 d\mu(x) \right]^{\frac{1}{2}} \\
\left. + \int_{\Psi} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\tau(x)}{\rho(x)} \right) \right]^2 d\mu(x) \right\}.
\] (4.3)

**Proof** Let \( \Psi_1 = \{ x \in \psi : \tau(x) > \rho(x) \} \), \( \Psi_2 = \{ x \in \psi : \tau(x) < \rho(x) \} \), and \( \Psi_3 = \{ x \in \psi : \tau(x) = \rho(x) \} \).

Obviously, if \( x \in \Psi_3 \), then equality holds in (4.3). Now if we take \( w(x) = 1 \) and \( q = 2 \) in Corollary 2.5, then for \( a = 1, b = \frac{\tau(x)}{\rho(x)} \) and \( x \in \Psi_1 \), multiplying both sides of the obtained results by \( \rho(x) \) and then integrating on \( \Psi_1 \), we get

\[
\left| \frac{1}{8} \int_{\Psi_1} \rho(x)f\left( \frac{\tau(x)}{\rho(x)} \right) d\mu(x) + 6 \int_{\Psi_1} \rho(x)f\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) d\mu(x) \right|
\]

\[
- \int_{\Psi_1} \rho(x) \int_{\frac{\tau(x)}{\rho(x)}}^{\frac{\tau(x)}{\rho(x)}} f(t) \, dt \, d\mu(x) \right| \\
\leq \left( \frac{7}{3 \cdot 2^q} \right)^{\frac{1}{2}} \left\{ \left[ f'(1) \right]^2 \int_{\Psi_1} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} d\mu(x) \right. \\
+ \int_{\Psi_1} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) \right]^2 d\mu(x) \right. \\
+ \left[ \int_{\Psi_1} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) \right]^2 d\mu(x) \right]^{\frac{1}{2}} \\
\left. + \int_{\Psi_1} \frac{\left( \tau(x) - \rho(x) \right)^2}{\rho(x)} \left[ f'\left( \frac{\tau(x)}{\rho(x)} \right) \right]^2 d\mu(x) \right\}.
\] (4.4)

Similarly, if \( x \in \Psi_2 \), then using Corollary 2.5 for \( a = \frac{\tau(x)}{\rho(x)} \), \( b = 1 \), multiplying both sides of the obtained results by \( \rho(x) \), and integrating on \( \Psi_2 \), we get

\[
\left| \frac{1}{8} \int_{\Psi_2} \rho(x)f\left( \frac{\tau(x)}{\rho(x)} \right) d\mu(x) + 6 \int_{\Psi_2} \rho(x)f\left( \frac{\rho(x) + \tau(x)}{2\rho(x)} \right) d\mu(x) \right|
\]

\[
- \int_{\Psi_2} \rho(x) \int_{\frac{\tau(x)}{\rho(x)}}^{\frac{\tau(x)}{\rho(x)}} f(t) \, dt \, d\mu(x) \right|
\[
\leq \left( \frac{\frac{7}{3} \cdot 2^n}{2^n} \right)^{\frac{1}{2}} \left\{ \left[ \left| f'(1) \right|^2 \int_{\varphi_2} \frac{(\tau(x) - \rho(x))^2}{\rho(x)} \, d\mu(x) \right. \right.
\]
\[
+ \int_{\varphi_2} \frac{(\tau(x) - \rho(x))^2}{\rho(x)} \left| f'(\frac{\tau(x) + \rho(x)}{2\rho(x)}) \right|^2 \, d\mu(x) \right\}^{\frac{1}{2}}
\]
\[
+ \left[ \int_{\varphi_2} \frac{(\tau(x) - \rho(x))^2}{\rho(x)} \left| f'(\frac{\tau(x) + \rho(x)}{2\rho(x)}) \right|^2 \, d\mu(x) \right]^{\frac{1}{2}} \right\}.
\]

Adding inequalities (4.4) and (4.5) and then using the triangle inequality, we get the desired result. \qed

4.2 Random variable

Suppose that for \(0 < a < b\), \(w : [a, b] \rightarrow [0, +\infty)\) is a continuous probability density of a continuous random variable \(X\) that is symmetric about \(\frac{a+b}{2}\). Also, for \(r \in \mathbb{R}\), suppose that the \(r\)th moment

\[
E_r(X) = \int_a^b x^r w(x) \, dx
\]

(4.6)
is finite.

Since \(w\) is symmetric and \(\int_a^b w(x) \, dx = 1\), we have

\[
E(X) = \int_a^b xw(x) \, dx = \frac{a + b}{2},
\]

(4.7)

which follows from

\[
\int_a^b xw(x) \, dx = \int_a^b (b + a - x)w(b + a - x) \, dx
\]

\[
= \int_a^b (b + a - x)w(x) \, dx.
\]

Based on the above-mentioned derivations, we obtain the following estimates of the \(r\)th moment.

(a) If we consider \(f(x) = x^r\) on \([a, b]\) for \(r \geq 2\), then the function \(|f'(x)|^q = r^q x^{(r-1)}\) with \(q > 1\) is a convex function. Therefore, using this function in Remark 2.3 with \(s = 1\) and in Corollary 2.5, respectively, we have

\[
\frac{|(E(X))^r - E_r(X)|}{(E(X))^r} \leq \left( \frac{r(b-a)^2}{2^{\frac{1}{2}} q^{\frac{1}{2}}} \parallel w \parallel_{[a, b], \infty} \right)^{1 - \frac{1}{q}} + \left( \left( \frac{a + b}{2} \right)^{q(r-1)} + \left( \frac{a + b}{2} \right)^{r-1} \right)^{\frac{1}{q}}
\]

\[
+ \left( \left( \frac{a + b}{2} \right)^{q(r-1)} + \left( \frac{a + b}{2} \right)^{r-1} \right)^{\frac{1}{q}}
\]

\[
\times \left\{ \left[ q^{(r-1)} + \left( \frac{a + b}{2} \right)^{q(r-1)} \right]^{\frac{1}{q}} + \left[ (a + b)^{q(r-1)} + b^{q(r-1)} \right]^{\frac{1}{q}} \right\}
\]

\[
\leq \left( \frac{r(b-a)^2}{2^{\frac{1}{2}} q^{\frac{1}{2}}} \parallel w \parallel_{[a, b], \infty} \right)^{1 - \frac{1}{q}} + \left( \left( \frac{a + b}{2} \right)^{q(r-1)} + \left( \frac{a + b}{2} \right)^{r-1} \right)^{\frac{1}{q}}
\]

\[
+ \left( \left( \frac{a + b}{2} \right)^{q(r-1)} + \left( \frac{a + b}{2} \right)^{r-1} \right)^{\frac{1}{q}}
\]

\[
\times \left\{ \left[ q^{(r-1)} + \left( \frac{a + b}{2} \right)^{q(r-1)} \right]^{\frac{1}{q}} + \left[ (a + b)^{q(r-1)} + b^{q(r-1)} \right]^{\frac{1}{q}} \right\}
\]
and
\[
\left| \frac{a^r + b^r}{8} + \frac{3}{4} (E(X))^r - E_r(X) \right| 
\leq \frac{r(b-a)^2}{2^{p+1/2}} \|w\|_{[a,b],\infty} \left( \frac{1 + 3p+1}{(p+1)4^{p+1}} \right)^{1/2} 
\times \left\{ \left[ a^{(r-1)} + \left( \frac{a+b}{2} \right)^{q(r-1)} \right]^{1/4} + \left[ \left( \frac{a+b}{2} \right)^{q(r-1)} + b^{q(r-1)} \right]^{1/4} \right\}.
\]

(b) If we consider \( f(x) = x^r \) on \([a, b]\) for \( r \in \mathbb{R} \), then
\[ m = r a^{r-1} \leq f'(x) = r x^{r-1} \leq r b^{r-1} = M, \] and so from (3.2) in Corollary 3.1 we have
\[
\left| \frac{a^r + b^r}{8} + \frac{3}{4} (E(X))^r - E_r(X) \right| \leq \frac{r(b-a)^{-1}}{64}.
\]

(c) If we consider \( f(x) = x^r \) on \([a, b]\) for \( r \in \mathbb{R} \), then the Lipschitz constant \( L = \sup_{x \in [a, b]} |f'(x)| = \sup_{x \in [a, b]} r x^{r-1} \) is equivalent to
\[
L = \begin{cases} 
rb^{r-1}, & r \geq 1, \\
ra^{r-1}, & r < 1.
\end{cases}
\]
So from (3.4) in Corollary 3.2 we have
\[
\left| \frac{a^r + b^r}{8} + \frac{3}{4} (E(X))^r - E_r(X) \right| = \begin{cases} 
\frac{r(b-a)^{-1}}{16} [a^{r-1} + (1 + 41(b-a)/48)b^{r-1}], & r \geq 1, \\
\frac{r(b-a)^{-1}}{16} [(1 + 41(b-a)/48)a^{r-1} + b^{r-1}], & r < 1.
\end{cases}
\]

Remark 4.1 Applications based on the obtained results to special means can be given, and we omit the details.

5 Conclusions
Based on a new weighted Simpson-like type integral identity, we obtained certain estimation-type results with respect to the weighted Simpson-like type inequality for the first-order differentiable mappings. Some particular cases are considered, which can be derived from the main results in the present paper. It is an interesting topic to apply these estimations to \( f \)-divergence measures and to higher moments of continuous random variables.

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