Multi-particle entanglement

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1 Introduction

Multi-particle entanglement is genuinely different from entanglement in quantum systems consisting of two parts. The prefix multi may refer here to quantum systems composed of a macroscopic number of subsystems, such as the parts of an interacting many-body system, or it may mean merely “three”. To fathom what is so different consider, say, a quantum system that is composed of three qubits. Each of the qubits is thought to be held by one of the paradigmatic distantly separated parties. It may come as quite a surprise that states of such composite quantum systems may contain tri-partite entanglement, while at the same time showing no bi-partite entanglement at all. Such quantum states can only be generated when all parties come together and prepare the state using local physical devices. Whenever any two parties group together, the state becomes separable, and hence contains no bi-partite entanglement at all.

In this chapter, we aim at fleshing out in what ways this multi-partite setting is different from the situation that we encountered earlier in this book. It is still true that entanglement can be conceived as that property of states that can be exploited to overcome constraints of locality. Yet, locality refers here to the several distinct subsystems, and we indeed already encounter a much richer situation when asking questions of what states are equivalent up to a mere local change of basis. In sharp contrast to the bi-partite setting, there is no longer a natural “unit” of entanglement, the role that was taken by the maximally entangled state of a system of two qubits. Quite strikingly, the very concept of being maximally entangled becomes void. Instead, we will see that in two ways there are “inequivalent kinds of entanglement”. We will explore some of the ramifications of these inequivalent kinds of entanglement.

Space limitations do not allow for a treatment of this subject matter in full detail, yet, we aim at “setting the coordinates” and guiding through the extensive literature in this field. Our coordinate system chosen for this chapter has the axes labeled pure and mixed states on the one hand, entanglement in single specimens and the asymptotic setting on the other hand. We very briefly mention ways to detect multi-particle entanglement, and introduce the concept of
stabilizer and graph states. Finally, we stress that multi-particle entanglement does not only have applications in information processing as such, but also in metrology, for example in the context of precision frequency standards using trapped ions. This chapter is emphasising the theory of multi-particle entanglement in finite-dimensional quantum systems, however, we will mention key experimental achievements whenever possible, notably using ion traps and purely optical systems.

2 Pure states

We will first fix one dimension in our coordinate system, and consider multi-particle entanglement of pure quantum states. This is the study of state vectors in a Hilbert space

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes ... \otimes \mathcal{H}_N$$

of a quantum mechanical system of \(N\) constituents. We will first take a closer look at entanglement in single specimens of multi-partite systems, that is of single “copies”. We will then turn to the asymptotic regime, where one asks questions of inconvertability when one has many identically prepared systems at hand.

2.1 Classifying entanglement of single specimens

A theory of entanglement should not discriminate states that differ only by a local operation. Here, “local operation” can mean merely a change of local bases (LU operations) or, else, general local quantum operations assisted by classical communication, that are either required to be succesful at each instance (LOCC) or just stochastically (SLOCC). For each notion of locality, the questions that have to be addressed are how many equivalence classes exist, how are they parameterized and how can one decide whether two given states belong to the same class? We will briefly touch upon these problems, limiting our attention to equivalence under LU operations and SLOCC in turn.

For the case of LU-equivalence of bi-partite qubit states, the Schmidt normal form

$$\sin \theta |0, 0\rangle + \cos \theta |1, 1\rangle.$$ 

(2)

gives a concise answer to all the above questions. Two quantum states are LU-equivalent if and only if their respective Schmidt normal forms coincide. All classes are parameterized by only one real parameter: the angle \(\theta\).

Some simple parameter counting arguments show that in the case of \(N\) qubit systems the situation must be vastly more complex. Indeed, disregarding a global phase, it takes \(2^{N+1} - 2\) real parameters to fix a normalized quantum state in \(\mathcal{H} = (\mathbb{C}^2)^\otimes N\). The group of local unitary transformations \(\text{SU}(2) \times \cdots \times \text{SU}(2)\) on the other hand has \(3N\) real parameters. Because the set of state vectors that are LU equivalent to a given \(|\psi\rangle\) is the same as the image of \(|\psi\rangle\) under all local unitaries, the dimension of an equivalence class cannot exceed \(3N\) (it can be less if \(|\psi\rangle\) is stabilized by a continuous subset of the local unitaries). Therefore, one needs at least \(2^{N+1} - 3N - 2\) real numbers to parameterize the sets of inequivalent pure quantum states [1]. Perhaps surprisingly – considering the rough nature of the argument – this lower bound
turns out to be tight \[2\]. It is a striking result that the ratio of non-local to local parameters grows exponentially in the number of systems. In particular, the finding rules out all hopes of a naive generalization of the Schmidt normal form. A general pure tri-partite qubit state, say, cannot be cast into the form

\[
\sin \theta |0, 0, 0\rangle + \cos \theta |1, 1, 1\rangle
\]

by the action of local unitaries \[3\].

Considerable effort has been undertaken to describe the structure of LU-equivalence classes by the use of invariants or normal forms \[1, 2, 4, 5, 6\]. By now, even for a general multi-particle system a normal form is known which is a generalization of the Schmidt form in the sense that it uses a minimal number of product vectors from a factorisable orthonormal basis to express a given state \[2\]. To give the reader an impression of how such generalized forms look like, we will briefly sketch the derivation of the simplest case, being defined on three qubits \[6\]. We start with a general state vector

\[
|\psi\rangle = \sum_{i,j,k} \alpha_{i,j,k} |i,j,k\rangle.
\]

(4)

Define two matrices \(T_0, T_1\) by

\[
(T_i)_{j,k} = \alpha_{i,j,k}.
\]

If we apply a unitary operator \(U_1\) with matrix elements \(u_{i,j}\) to the first qubit, then the matrix \(T_0\) transforms according to

\[
T_0' = u_{0,0} T_0 + u_{0,1} T_1.
\]

The algebraic constraint \(\det(T_0') = 0\) amounts to a quadratic equation in \(u_{0,0}/u_{0,1}\) and can thus always be fulfilled. We go on to diagonalize \(T_0'\) by applying two unitaries \(U_2, U_3\) to the second and third system such that

\[
U_2 T_0' U_3 = \begin{pmatrix}
\lambda_0 & 0 \\
0 & 0
\end{pmatrix}
\]

(5)

By absorbing phases into the definition of the basis states \(|0\rangle_1, |1\rangle_1, |1\rangle_2, |1\rangle_3\), we arrive at

\[
(U_1 \otimes U_2 \otimes U_3)|\psi\rangle = \lambda_0 |0, 0, 0\rangle + \lambda_1 e^{i\phi} |1, 0, 0\rangle + \lambda_2 |1, 1, 0\rangle + \lambda_3 |1, 1, 1\rangle
\]

(6)

with real coefficients \(\lambda_i\). Normalization requires \(\sum_i \lambda_i^2 = 1\). It is shown in Ref. \[6\] that \(0 \leq \phi \leq \pi\) can always be achieved and further, that for a generic state vector the form \[6\] is unique. In accordance with the formula we derived earlier, the normal form depends on five independent parameters.

How does the situation look like if one allows the local operations to be SLOCC? An SLOCC protocol that maps a state vector \(|\psi\rangle\) to \(|\phi\rangle\) with some probability of success consists of several rounds in each of which the parties perform operations on their respective systems, possibly depending on previous measurement results. One can think of the protocol as splitting into different branches with each measurement. It should be clear that \(|\psi\rangle \rightarrow |\phi\rangle\) is possible if and only if at least one of these branches does the job. The effect of each single branch on the state vector can be described by one Kraus operator \(A_i\) per system:

\[
|\psi\rangle \rightarrow (A_1 \otimes \cdots \otimes A_N)|\psi\rangle.
\]

(7)

In this context generic means for all state vectors but a set of measure zero.
If $|\psi\rangle$ and $|\phi\rangle$ are equivalent under SLOCC, then the operators $A_i$ can be chosen to be invertible \[9, 7, 8\]. Note that the term filtering operation is used synonymously with SLOCC.

Having thus established a framework for dealing with SLOCC operations, we can repeat the parameter counting argument from the LU case. By simply substituting the local unitary group by $SL(\mathbb{C}^2) \times \cdots \times SL(\mathbb{C}^2)$ one finds a lower bound of $2^N+1 - 6N - 2$ parameters that are necessary to label SLOCC equivalence classes of qubit systems. The inevitable next step would be to adopt the classification of equivalence classes by invariants and normal forms from the LU case to the SLOCC one. While this has indeed been done \[7\], we are instead going to focus on a particularly interesting special case. Indeed, for the case of three qubits, the above estimation formula does not give a positive lower bound for the number of parameters and therefore one might expect that there is only a discrete set of inequivalent classes.

Five SLOCC-inequivalent subsets of three-qubit pure states can be identified by inspection \[9\]. Product vectors

$$|\psi\rangle_1|\phi\rangle_2|\pi\rangle_3$$

(8)
certainly form a class of their own because local operations can never create entanglement between previously unentangled systems. For the same reason vectors of the form

$$|\psi\rangle_1|\Phi\rangle_2,3$$

(9)

with some non-factoring state vector $|\Phi\rangle_2,3$ constitute an SLOCC-equivalence class, the class of bi-partite entangled states that factor with respect to the split 1-23 of the set of systems. There are three such splits (1-23, 12-3, 13-2) giving rise to three bi-partite classes. Calling these sets equivalence classes is justified, because any two entangled bi-partite pure states are equivalent under SLOCC for qubit-systems. Finally, we are left with the set of fully entangled vectors that admit no representation as tensor products. Do they form a single equivalence class? It turns out that this is not the case.

To understand why this happens, we will employ an invertible SLOCC invariant \[9, 10\]. Any pure state can be written in the form

$$|\psi\rangle = \sum_{i=1}^{R} \alpha_i |\psi_i\rangle_1 \otimes \cdots \otimes |\psi_i\rangle_N.$$  

(10)

Now let $R_{\text{min}}$ be the minimal number of product terms needed to express $|\psi\rangle$. A moment of thought shows that this number is constant under the action of invertible filtering operations (we will re-visit this invariant in Section 4 where its logarithm is called the Schmidt measure). Now consider the states vectors

$$|\text{GHZ}\rangle = (|0,0,0\rangle + |1,1,1\rangle)/\sqrt{2},$$

$$|\text{W}\rangle = (|0,0,1\rangle + |0,1,0\rangle + |1,0,0\rangle)/\sqrt{3}.$$  

(11, 12)

It takes a few lines \[9\] to show that there is no way of expressing $|\text{W}\rangle$ using only two product terms and hence the two states cannot be converted into each other by SLOCC. In this sense, there are two “inequivalent forms” of pure tri-partite entanglement of three qubits. Notably, neither form can be transformed into the other with any probability of success (however, see Section 3.1). Three-qubit W-states and GHZ-states have already been experimentally
realized, both purely optically using postselection \cite{11,12} and in ion traps \cite{13}. This picture is complete: any fully entangled state is SLOCC equivalent to either $|\text{GHZ}\rangle$ or $|\text{W}\rangle$ \cite{9}. We conclude that the three qubits pure states are partitioned into a total of six SLOCC equivalence classes.

We take the occasion to exemplify some concepts in multi-particle entanglement theory by studying the properties of the GHZ and the W state. A simple calculation shows that after a measurement of the observable corresponding to the Pauli matrix $X_1$ on the first qubit, both $|\text{GHZ}\rangle$ and $|\text{W}\rangle$ collapse into a Bell state on the systems labeled 2 and 3 \textit{regardless of the measurement outcome}. Because either state is invariant under system permutations, we can project – with certainty – any pair of systems into a Bell state by performing a suitable measurement on the remaining qubit. This property can immediately be generalized to states on more than three systems and is known as \textit{maximal connectedness} \cite{14}. The maximum degree of entanglement of the state into which a pair can be projected by suitable local measurements on the other parts is the \textit{localizable entanglement} \cite{15}.

The two states behave differently, however, if a system is traced out. Specifically, tracing out the first qubit of the GHZ state will leave the remaining systems in a complete mixture. Yet, for $|\text{W}\rangle$ we have

$$
\text{tr}_1[|\text{W}\rangle\langle\text{W}|] = \frac{1}{3}|0,0\rangle\langle0,0| + \frac{2}{3}|\Psi^+\rangle\langle\Psi^+|
$$

where $|\Psi^+\rangle = (|0,1\rangle + |1,0\rangle)/\sqrt{2}$. The operator in Eq. (13) is a mixed \textit{entangled bi-partite} state. It is in that sense, that the entanglement of $|\text{W}\rangle$ is more \textit{robust} under particle loss than the one of $|\text{GHZ}\rangle$ \cite{9}. Can a “super-robust” fully entangled three-qubit state be conceived that leaves any pair of systems in a Bell state if the third particle is lost? Unfortunately not, because if $|\psi\rangle$ does not factor, $\text{tr}_1[|\psi\rangle\langle\psi|]$ is mixed – and in particular not fully entangled \cite{15}. This phenomenon has been dubbed the \textit{monogamous nature} of entanglement.

### 2.2 Asymptotic manipulation of multi-particle quantum states

Needless to say, instead of manipulating quantum systems at the level of single specimens, entanglement manipulation is meaningful in the asymptotic limit. Here, one assumes that one has many identically prepared systems at hand, in a state $\rho^\otimes n$, and aims at transforming them into many other identical states $\sigma^\otimes m$, for large $n$ and $m$, involving collective operations. This is the asymptotic setting that we have previously encountered in this book. As mentioned before, one does not require that the target state is reached exactly, but only with an error that is asymptotically negligible. This setting is notably different from the one of the previous subsection, where transformations were considered for single specimens of multi-partite quantum systems.

It is instructive again to briefly reconsider the situation when only two subsystems are present. In that case, the task of classifying different “kinds” of entanglement is void. Any bi-partite entanglement of pure states is essentially equivalent to that of an EPR-pair. One can asymptotically transform any bi-partite state into a number of maximally entangled qubits pairs and back, the achievable optimal rate in this transformation being given by the entropy of the reduction \cite{17}. In this sense, one can say that there is only one ingredient to asymptotic
bi-partite entanglement: this is the maximally entangled qubit pair. Pure states can be characterized by the content of this essential ingredient, giving rise to the \textit{entropy of entanglement}, which is the unique measure of entanglement. How is this for multi-particle systems?

Again, it turns out that the situation is much more complex than before. Before stating how the situation is like in the multi-particle setting, let us first make the concept of \textit{asymptotic reversibility} more precise. If $\rho \otimes n$ can be transformed under LOCC into $\sigma \otimes m$ to arbitrary fidelity, there is no reason why $n/m$ should be an integer. So to simplify notation, one typically also takes non-integer yields into account. One says that $|\psi \rangle \otimes x$ is \textit{asymptotically reducible to} $|\phi \rangle \otimes y$ under LOCC, if for all $\delta, \varepsilon > 0$ there exist natural $n, m$ such that

$$\left| \frac{n}{m} - \frac{x}{y} \right| < \delta, \| \Psi(|\psi \rangle \langle \psi| \otimes n),|\phi \rangle \langle \phi| \otimes m \|_1 > 1 - \varepsilon. \quad (14)$$

Here, $\|A\|_1 = \text{tr}|A| = \text{tr}((A^\dagger A)^{1/2})$ denotes the trace-norm of an operator $A$ as a distance measure, and $\Psi$ is quantum operation which is LOCC. If both $|\psi \rangle \otimes x$ can be transformed into $|\phi \rangle \otimes y$ as well as $|\phi \rangle \otimes y$ into $|\psi \rangle \otimes x$, the transformation is \textit{asymptotically reversible}. In the bipartite case, it is always true that any $|\psi \rangle$ can be transformed into

$$|\psi^+ \rangle \otimes E(|\psi \rangle \langle \psi|), \quad |\psi^+ \rangle = (|0, 0 \rangle + |1, 1 \rangle)/\sqrt{2} \quad (15)$$

where $E(|\psi \rangle \langle \psi|) = S(\text{tr}_2(|\psi \rangle \langle \psi|))$ is the entropy of entanglement, and this transformation is asymptotically reversible [17, 18, 19]. Such a maximally entangled qubit pair can hence be conceived as the only essential ingredient in bi-partite entanglement of pure states. This holds true not only for qubit systems, but for systems of any finite dimension, and with small technicalities even for infinite-dimensional systems [20].

In the multi-partite setting, there is no longer a single essential ingredient, but many different ones. For pure states on $H = H_1 \otimes \ldots \otimes H_N$, given a set of state vectors

$$S = \{|\psi_1 \rangle, \ldots, |\psi_k \rangle\} \quad (16)$$

for some $k$, one may consider their \textit{entanglement span} as the set of pure states that can reversibly be generated using $S$ under asymptotic LOCC [13]. In the bi-partite setting, the entanglement span is always given by the set of all pure states (not taking the trivial case into account where $S$ contains only product states). In the multi-particle case, however, it is meaningful to introduce the concept of a \textit{minimal reversible entanglement generating set} (MREGS). An MREGS $S$ is a set of pure states such that any other state can be generated from $S$ by means of reversible asymptotic LOCC. It must be minimal in the sense that no set of smaller cardinality possesses the same property [18, 21, 22].

After this preparation, what is now the MREGS for, say, a tri-partite quantum system? The irony is that even in this relatively simple case, no conclusive answer is known. Only a few states have been identified that must be contained in any MREGS. At first one might be tempted to think that three different maximally entangled qubit pairs, shared by two systems each, already form an MREGS. This natural conjecture is not immediately ruled out by what we have seen in the previous subsection: after all, we do not aim at transforming quantum states of single specimens, but rather allow for asymptotic state manipulation. Yet, it can be shown that merely to consider maximally entangled qubit pairs is not sufficient to construct
What is more, even
\[ S = \{ |\psi^+\rangle_{1,2}, |\psi^+\rangle_{1,3}, |\psi^+\rangle_{2,3}, |\text{GHZ}\rangle \} \] (17)
does not suffice. All these pure states are *inequivalent with respect to asymptotic reducibility*, but there are pure states that cannot be reversibly generated from these ones alone [23]. So again, we see that there are *inequivalent kinds of entanglement*. Because we allowed for asymptotic manipulations, the present inequivalence is even more severe than the one encountered in the last section.

To find general means for constructing MREGS constitutes one of the challenging open problems of the field: as long as this question is generally unresolved, the development of a “theory of multi-particle entanglement” in the same way as in the bi-partite setting seems unfeasible. Whereas in the latter case the “unit” of entanglement is entirely unambiguous – it is the EPR-pair – there is no substitute for it in sight for multi-partite systems. This motivates after all to consider more pragmatic approaches to grasp multi-particle entanglement.

## 3 Mixed states

### 3.1 Classifying mixed state entanglement

The program pursued in Section 2 – parameterizing all equivalence classes of states under various types of local operations – can in principle also be applied to mixed states [24, 25]. However, we will content ourselves with sketching a more rough classification scheme based on *separability properties* [25].

At the lowest level there is the class of states that can be prepared using LOCC alone. Its members are called *fully separable* and can be written in the form
\[ \rho = \sum_i p_i (\rho_i^1 \otimes \cdots \otimes \rho_i^N). \] (18)

Evidently, states of this kind do not contain entanglement. Now arrange the \(N\) parts of the multi-particle system in \(k \leq N\) groups. We can conceive the groups as the constituents of a \(k\)-partite quantum system. This *coarse graining* procedure is called forming a \(k\)-*partite split* of the system and indeed, we have less explicitly encountered that concept before in Section 2. Having set up this terminology, it is meaningful to ask with respect to which splits a given quantum state is fully separable. Two states belong to the same *separability class* if they are separable with respect to the same splits. Clearly, being in the same class in this sense is a necessary condition for being equivalent under any type of local operations.

A state is referred to as \(k\)-*separable*, if it is fully separable considered as a state on some \(k\)-partite split. By the use of this terminology, the separability classes can be brought into a hierarchy, where \(k\)-separable classes are considered to be more entangled than \(l\)-separable ones for \(k < l\). States that are not separable with respect to any non-trivial split are *fully inseparable*.

The number of all splits of a composite system grows exorbitantly fast with the number \(N\) of its constituents. One is naturally tempted to reduce the complexity by identifying *redundancies* in this classification. After all, once it is established that a state is fully separable,
there is no need to consider any further splits. While such redundancies certainly exist, pinpointing them turns out to be subtle and indeed gives rise to one of the more peculiar results in quantum information theory, as will be exemplified by means of our standard example, the three-qubit system.

The five possible splits of three systems (1-2-3, 12-3, 1-23, 13-2, 123) have already been identified in Section 2. It is a counter-intuitive fact that there are mixed states that are separable with respect to any bi-partite split but are not fully separable [28]. An analogous phenomenon does not exist for pure states. The following sub-classes of the set of bi-separable states are all non-empty [25].

- **1-qubit bi-separable states** with respect to the first system are separable for the split 1-23 but not for 12-3 or 13-2.
- **2-qubit bi-separable states** with respect to the first and second system are separable for the split 1-23 and 2-13, but not for 12-3.
- **3-qubit bi-separable states** are separable with respect to any bi-partite split but are not fully separable.

Together with the inseparable states and the fully separable ones, the above sets constitute a complete classification of mixed three qubit states modulo system permutations [26].

We end this subsection with a refinement of the class of inseparable states that will play a role in the following subsection [27]. In this paragraph, the fully separable states are denoted by $S$, the bi-separable ones by $B$, and lastly, the set of all mixed states including the fully inseparable ones by $F$. Clearly, $S \subset B \subset F$ is a hierarchy of convex sets. Now, recall the two different classes of genuine three-qubit pure state entanglement that were identified in Section 2. We define $W$ to be the set of states that can be decomposed as a convex combination of bi-separable ones and projections onto $W$-type vectors and finally rename the set of all states $GHZ$. This leaves us with a finer partitioning $S \subset B \subset W \subset GHZ$ of the state space in terms of convex sets. The definition suggests that the $GHZ$-type vectors are in some way more entangled than the $W$-states – which until now we had no reason to suspect.

In order to justify the construction, we need to employ another tool from Section 2: the generalized Schmidt normal form. A pure three-particle state is SLOCC-equivalent to the $W$-state if and only if its Schmidt normal form reads

$$
\lambda_0|0,0,0\rangle + \lambda_1|1,0,0\rangle + \lambda_2|1,0,1\rangle + \lambda_3|1,1,0\rangle,
$$

that is, if $\lambda_4 = \phi = 0$. Comparing (19) to the general form (6) shows that $|W\rangle + \varepsilon|1,1,1\rangle$ must be of $GHZ$-type for any $\varepsilon > 0$. Physically, that means that even though one cannot turn a $GHZ$-type vector into $|W\rangle$ using SLOCC, one can approximate it as closely as desired. Therefore, we can transform states (at least approximately) from the outer to the inner classes: $GHZ \rightarrow W \rightarrow B \rightarrow S$ by means of non-invertible local filtering operations. Note that invertible local operations leave the classification of a state invariant. As a last remark, formula (19) and the parameter counting considerations in Section 2 show that both the product vectors and the $W$-type vectors form a subset of measure zero among all pure states.

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2Bi-separable means 2-separable.
3The higher Schmidt measure of the W-state even suggest the contrary.
Notwithstanding, it can be shown \[27\] that \( S \) as well as \( W \setminus B \) are of finite volume in the set of mixed states.

### 3.2 Methods of detection

One way of experimentally detecting multi-particle entanglement is to perform a complete quantum state tomography, and to see whether the resulting estimated state is consistent with an entangled state.\(^4\) Depending on the context, this can be a costly procedure. It may be desirable to detect entanglement without the need of acquiring full knowledge of the quantum state. Such an approach can be advantageous when certain types of measurements are more accessible than others, and when one intends to detect entanglement based on data from these restricted types of measurements, as such insufficient to fully reconstruct the state. This is where entanglement witnesses come into play.

An entanglement witness \( A \) is an observable that is guaranteed to have a positive expectation value on the set \( S \) of all separable states. So whenever the measurement of \( A \) on some quantum state \( \rho \) gives a negative result, one can be certain that \( \rho \) contains some entanglement. It is, however, important to keep in mind that witnesses deliver only sufficient conditions. In addition to \( S \), there might be other, non-separable states that have a positive expectation value with respect to \( A \).

We are now going to take a more systematic look at this technique and, at the same time, generalize it from \( S \) to any compact convex set \( C \) in state space. To that end, we note that the set of quantum states \( \sigma \) that satisfy the equation \( \text{tr}[\sigma A] = 0 \) for some observable \( A \) form a hyperplane which partitions the set of states into two half-spaces. If \( C \) is contained in one of these half-spaces, the plane is called a supporting hyperplane of \( C \). Each half-space is characterized by the fact that for all its respective elements \( \sigma \) the sign of \( \text{tr}[\sigma A] \) is fixed. Now, if \( \rho \) is a state contained in the half-space “opposing” \( C \), we have, for all \( \sigma \in C \),

\[
\text{tr}[A\rho] < 0, \quad \text{tr}[A\sigma] \geq 0.
\]  

(20)

But \( \text{tr}[A\rho] \) is nothing but the expectation value of \( A \) and a negative result suffices to assert that \( \rho \not\in C \). In this way, entanglement witnesses witness entanglement.

Witnesses can be constructed for all the convex sets that appeared in the classification of the previous subsection. For example, a GHZ witness is an operator that detects states that are not of W-type. It is not difficult to see that

\[
A_{\text{GHZ}} = \frac{3}{4} - |\text{GHZ}\rangle\langle\text{GHZ}|
\]  

(21)

\(^4\)The question that we will only touch here is the one of how to computationally determine whether a known quantum state is in one of the mentioned separability classes. It turns out that already in the bi-partite case, deciding separability is an NP-hard problem \[29\]. One can nevertheless construct hierarchies of efficiently decidable sufficient criteria for a state being, say, fully inseparable. This is possible in a way insuring that every fully entangled state is necessarily detected in some step of the hierarchy \[29,31\]. One route towards finding such criteria is to cast the problem into a polynomially constrained optimization problem, involving polynomials of degree three only. This is feasible due to the fact that any Hermitian matrix for which \( \text{tr}[M^2] = 1, \text{tr}[M^3] = 1 \) is a matrix that satisfies \( \text{tr}[M] = 1, M = M^2, M \geq 0 \), so is one that corresponds to a pure quantum state \[30,32\]. This can be used to parameterize the separable states from some separability class in terms of polynomial expressions. Relaxing the problem to hierarchies of efficiently decidable semi-definite programs then amount to a two-way test of being fully inseparable \[30\]. For alternative algorithms for deciding multi-particle entanglement, see Refs. \[31,32,33\].
is a GHZ witness: We have that $\langle \text{GHZ}|\rho|\text{GHZ} \rangle \leq 3/4$ for any W-type state, and hence $\text{tr}[A_{\text{GHZ}}\rho] \geq 0$ for any W-type state. More generally, such witnesses can be constructed as $A_{\text{GHZ}} = Q - \varepsilon I$ with an appropriate $\varepsilon > 0$, where $Q \geq 0$ is a matrix that does not have any W-type state in its kernel. In turn, $\rho = |\text{GHZ}\rangle\langle \text{GHZ}|$ is a state that will evidently be detected as not being of W-type. Similarly, a W witness is given by

$$A_{\text{W}} = \frac{2}{3} I - |\text{W}\rangle\langle \text{W}|,$$

(22)

 Needless to say, such witnesses are an especially convenient tool in the multi-partite setting, when they are evaluated using only local measurements. Just in the same way as one can choose a basis consisting of product matrices when performing a tomographic measurement, expectation values of witness operators can be obtained with appropriate local measurements, using local decompositions [37]. The detection of multi-particle entanglement using witness operators has already been experimentally realized [38]. Indeed, one of the estimated witness operators in the experiment was of the form given in Eq. (22).

4 Quantifying multi-particle entanglement

Entanglement measures give an answer to the question to what degree a quantum state is entangled. Their values are typically related to the usefulness of the state for quantum information applications. Entanglement measures can, for example, be related to teleportation fidelities or rates at which a secure key can be extracted. As has been pointed out in case of the bi-partite setting, there are two approaches to quantify entanglement. Firstly, in the axiomatic approach, one specifies certain criteria that any meaningful entanglement measure must satisfy, and identifies functions that fulfil all these criteria. Secondly, one may quantify a state’s entanglement directly in terms of rates of a certain protocol that can optimally be achieved using that state.

In the case considered in this chapter, the route taken in the bi-partite setting is not accessible: in particular, one cannot evaluate asymptotic rates at which one can distill the elements of an MREGS from a given state. This would be the analogue of the distillable entanglement. In turn, the cost would correspond to the rates at which one can prepare a state asymptotically starting from MREGS elements. This route is inaccessible; one of the reasons being that the MREGS are unknown.

More pragmatically, one can still quantify multi-particle entanglement in terms of meaningful functions that are multi-particle entanglement monotones, that is, positive functions vanishing on separable states that do not increase under LOCC, equipped with some physical interpretation.

- The Schmidt measure $E_S$ [10, 39] is the logarithm of the minimal number of products in a product decomposition $E_S = \log_2(R_{\text{min}})$ (see Eq. (17)). It provides a classification of multi-particle entangled states and is an entanglement monotone. In the bi-partite case, this measure reduces to the Schmidt rank, i.e., the rank of the reduction. This measure is particularly suitable to quantify entanglement in graph states with many constituents.
Another candidate is the global entanglement $E_{\text{Global}}$ of Ref. [40]. This is a measure of entanglement for an $N$-qubit system, equipped with a Hilbert space $\mathcal{H} = (\mathbb{C}^2)^\otimes N$. 5

The geometric measure of entanglement [41] makes use of a geometric distance to the set of product states:

$$E_{\text{Geometric}} = \min ||\psi\rangle\langle\psi| - \sigma||_2,$$

where $||.||_2$ is the Hilbert-Schmidt norm, and the minimum is taken over all product states.

The tangle [16] is a measure of entanglement suitable for systems consisting of three qubits. This measure of entanglement is based on the entanglement of formation, or rather on the concurrence, as

$$\tau(\rho) = C^2(\rho_{123}) - C^2(\rho_{12}) - C^2(\rho_{13}).$$

Here, $C(\rho_{ij})$ is the concurrence of the reduction with respect to systems labeled $i,j$, and $C(\rho_{123})$ is the concurrence of $\rho$ in the split 1 - 23. The concurrence, in turn, is given by $C(\rho) = \max\{\lambda_3^{1/2} - \lambda_2^{1/2} - \lambda_3^{1/2} - \lambda_1^{1/2}\}$, where $\lambda_1, \ldots, \lambda_4$ are the singular values of $\rho \hat{\rho}$, non-increasingly ordered, and $\hat{\rho} = 1 \otimes 1 - \rho_1 \otimes 1 - 1 \otimes \rho_2 + \rho$. As is by no means obvious, this quantity is invariant under permutation of the three systems and is in fact an entanglement monotone for three-qubit systems. It can be efficiently computed and applied to mixed states without the need for taking convex hulls.

The relative entropy of entanglement in the multi-partite setting is defined as the minimal distance of a given state to the set of fully separable states, quantified in terms of the quantum relative entropy [42].

5This measure of entanglement is defined as follows: Starting point is a map $f_j$

$$f_j(b) |b_1, \ldots, b_N\rangle = \delta_{b_j}|b_1, \ldots, b_{j-1}, b_{j+1}, \ldots, b_N\rangle,$$

where the vectors $|b_1, \ldots, b_N\rangle$ with $b_i \in \{0, 1\}$ span the Hilbert space $\mathcal{H}$. The right hand side of Eq. 23 is hence either zero, or the entry $b_j$ is omitted. This map can be extended to a map $O^2 \otimes (\mathbb{C}^2)^\otimes N \rightarrow (\mathbb{C}^2)^\otimes N-1$ by linearity. In turn, for two vectors $x, y \in (\mathbb{C}^2)^\otimes N-1$ one may write $x = \sum_i x_i |i\rangle$ and $y = \sum_i y_i |i\rangle$ with $0 \leq i \leq 2^{N-1}$. For a state vector $|\psi\rangle$ the quantity

$$E_{\text{Global}} = \frac{4}{N} \sum_{j=1}^{N} d(f_j(0)|\psi\rangle, f_j(1)|\psi\rangle),$$

where $d(x, y) = \sum_{i<j} |x_i y_j - x_j y_i|^2$, is indeed an monotone on pure states. Convex hulls of pure-state entanglement monotones deliver then convex monotones for mixed quantum states.

5Stabilizer states and graph states

We now turn to a specific class of multi-particle entangled states which provides a very useful theoretical “laboratory”: The stabilizer formalism provides a powerful picture for grasping a wide class of states and operations. Stabilizer states are multi-qubit quantum states that play a crucial role in quantum information science, in particular in the field of quantum error correction. It is beyond the scope of the present chapter to give an introduction to the rich...
literature on the stabilizer formalism. Instead, we will very briefly introduce the very concept of a stabilizer state and a graph state.

Stabilizer states form a set of quantum states that allow for an efficient description in terms of the operators they are eigenstates of. Let us exemplify this using the familiar GHZ state on three qubits. Recall that $X, Y$ and $Z$ denote the well-known Pauli operators. It is not difficult to see that the state vector $|\text{GHZ}\rangle = (|0,0,0\rangle + |1,1,1\rangle)/\sqrt{2}$ is an eigenstate of $Z_1 \otimes Z_2 \otimes \mathbb{I}_3$, $\mathbb{I}_1 \otimes Z_2 \otimes Z_3$, and $X_1 \otimes X_2 \otimes X_3$ to the eigenvalue $+1$. This alone should not be too surprising. But then, the state vector of the GHZ state is the only state vector that has this property, up to a global phase. So instead of explicitly writing down the GHZ state vector, we could have specified it by saying that it is an eigenstate of $Z_1 Z_2$, $Z_2 Z_3$, and $X_1 X_2 X_3$.

This idea can be pushed much further – and this is when the advantage of such a formalism becomes apparent. The central ingredient is the Pauli group. For a single system, it is given by

$$G = \{ \pm \mathbb{I}, \pm i \mathbb{I}, \pm X, \pm i X, \pm Y, \pm i Y, \pm Z, \pm i Z \}.$$ (27)

The phases ensure that the group is actually closed under multiplication. The Pauli group on $N$ qubits, $G_N$, in turn consists of $N$-fold tensor products of elements of $G$. It is a basic fact from linear algebra, that a set of $N$ operators $\{P_1, \cdots, P_N\}$ from $G_N$ allow for a basis of common eigenvectors if they commute mutually. The key insight lies in the observation that this basis always contains a unique element which is a common eigenvector $|\psi\rangle$ of all $P_i$ to the eigenvalue $+1$ [44]. In other words, the operators $P_i$ stabilize $|\psi\rangle$. Clearly then, $|\psi\rangle$ is also stabilized by any product of elements of $\{P_i\}_{i=1 \cdots N}$. The set of all such products forms an abelian group, the stabilizer group which is said to be generated by the $P_i$. The vector $|\psi\rangle$ is the associated stabilizer state, which is, again, uniquely defined by the requirement

$$P_i |\psi\rangle = |\psi\rangle.$$ (28)

We have yet to make the claim precise that the stabilizer formalism offers an efficient description. State vectors are usually specified by their expansion coefficients with respect to some product basis in Hilbert space. By computing lower bounds of the Schmidt measure (cf. Section 4), e.g., it can be established that there are stabilizer states that require in the order of $2^N$ non-vanishing terms when described in any product basis [39, 14]. Their stabilizer group, on the other hand, is determined by only $N$ generators.

There is an even more compact description of stabilizer states, based on the familiar concept of a graph $G(V, E)$ which is specified by a set of vertices $V$ and an edge set $E$ [39, 43, 45]. To each graph on $N$ vertices, a stabilizer group is associated by the following construction. We label the vertices with numbers 1 to $N$ and denote by $N_a$ the neighbors of the $a$-th vertex, that is, the set of vertices directly connected to $a$. Now, to any vertex $a$, we associate an element $K_a$ of the Pauli group via

$$K_a = X_a \prod_{b \in N_a} Z_b.$$ (29)

Using the fact that the relation of “being a neighbor” is symmetric, one can show that the $K_a$ commute mutually and therefore specify a unique stabilized $|G\rangle$, the graph state vector of $G$. As an example, consider a linear graph on four vertices. It gives rise to the generators
\{X_1 Z_2, Z_1 X_2 Z_3, Z_2 X_3 Z_4, Z_3 X_4\} and that the following vector is stabilized by each of them

\[
|\text{Cluster}\rangle = \frac{1}{4} \bigotimes_{a=1}^{4} (|0\rangle Z_{a+1} + |1\rangle),
\]

where we have set \(Z_5 = 1\). It is the \textit{four-qubit cluster state} [14], an instance of a family of states which form the central resource for measurement based quantum computing. The four-qubit cluster state has recently been prepared in an optical experiment [46].

Any stabilizer state can be brought into the form of a graph state using only local unitaries [45]. In particular, this means that all multi-particle entanglement properties of stabilizer states can be described entirely in terms of properties of graphs. The same holds true for the effects of local Pauli measurements [39, 45, 47] and \textit{Clifford} operations [39, 48] on graph states. Multi-particle entanglement, for example in terms of the Schmidt measure, can be assessed for graph states [39]. Stabilizer circuits can be simulated computationally more cheaply when expressed in terms of graph states [49] using the rules of Ref. [39]. They also form a convenient and physically motivated testbed to assess the question how robust multi-particle entangled states may be under decoherence processes [50, 51].

### 6 Applications of multi-particle entangled states

Any protocol of quantum information science making use of quantum systems with more than two constituents may be conceived as an application of multi-particle entanglement. To pinpoint the specifics of multi-particle entanglement that make a certain task possible is yet less straightforward. Multi-particle entanglement is certainly crucial for quantum error correction, where the idea is to encode logical qubits into a larger number of qubits in a multi-particle entangled state, as a protection against the entanglement with an environment beyond actual control. This, in John Preskill’s words, to “fight entanglement with entanglement”. In quantum key distribution, we will encounter several applications of multi-particle entanglement. In quantum computing, multi-particle entanglement plays a key role. In measurement based computing, as we will see later, multi-particle entangled states forms the resource. The use of multi-particle entanglement can then even be “monitored” in the course of the computation [39]. It is, however, not yet entirely understood what exact criteria concerning their entanglement the involved states have to fulfil to render an efficient classical simulation impossible.

Yet, multi-particle entanglement does not only facilitate processing or transmission of information, but also allow for applications in \textit{metrology} [52, 53, 54, 55]. We will shortly sketch an idea to enhance the accuracy of the \textit{estimation of frequencies} using multi-particle entangled states. This applies in particular to frequency standards based on laser-cooled ions, which can achieve very high accuracies [55]. Starting point is to prepare \(N\) ions that are loaded in a trap in some internal state with state vector \(|0\rangle\). One may then drive an atomic transition with natural frequency \(\omega_0\) to a level \(|1\rangle\) by applying an appropriate Ramsey pulse with frequency \(\omega\), such that the ions are in an equal superposition of \(|0\rangle\) and \(|1\rangle\). After a free

\[\text{In the context of quantum information theory, a Clifford operation is a unitary operator that maps elements of the Pauli group to elements of the Pauli group under conjugation.}\]
evolution for a time $t$, the probability to find the ions in level $|1\rangle$ is given by
\[
p = \frac{1 + \cos((\omega - \omega_0)t)}{2}.
\]
(31)

Given such a preparation, one finds that if one estimates the frequency $\omega_0$ with such a scheme, the uncertainty in the estimated value is given by
\[
\delta \omega_0 = \frac{(NT/t) - 1}{2}.
\]
(32)

This theoretical limit, the shot-noise limit, can in principle be overcome when entangling the ions initially. This idea has been first explored in Ref. [54], where it was suggested to prepare the ions in a $N$-particle GHZ state with state vector $|\text{GHZ}\rangle = (|0, 0, ..., 0\rangle + |1, 1, ..., 1\rangle)/\sqrt{2}$.

With such a preparation, and neglecting decoherence effects, one finds an enhanced precision,
\[
\delta \omega_0 = \frac{(T/t) - 1}{2}/N,
\]
(33)
beating the above limit by a factor of $1/\sqrt{N}$. Unfortunately, while the GHZ-state provides some increase in precision in an ideal case, it is at the same time subject to decoherence processes. A more careful analysis shows that under realistic decoherence models this enhancement actually disappears for the GHZ state. Notwithstanding these problems, the general idea of exploiting multi-particle entanglement to enhance frequency-measurements can be made use of: For example, for $N = 4$ the partially entangled preparation
\[
|\psi\rangle = \lambda_0(|0, 0, 0, 0\rangle + |1, 1, 1, 1\rangle) + \lambda_1(|0, 0, 0, 1\rangle + |0, 0, 1, 0\rangle + |0, 1, 0, 0\rangle)
+ |1, 0, 0, 0\rangle + |1, 1, 1, 0\rangle + |1, 1, 1, 1\rangle + |1, 0, 1, 1\rangle + |0, 1, 1, 1\rangle
+ \lambda_2(|0, 0, 1, 1\rangle + |0, 1, 0, 1\rangle + |1, 0, 0, 0\rangle + |1, 1, 0, 0\rangle + |1, 0, 1, 0\rangle + |0, 1, 1, 0\rangle),
\]
(34)
can lead to an improvement of more than 6%, when the probability distribution $\lambda_0, \lambda_1, \lambda_2$ is appropriately chosen and appropriate measurements are performed [55]. For four ions, exciting experiments have been performed in the meantime [56], and applied for two ions to precision spectroscopy [57], indeed showing that the shot noise limit can be beaten with the proper use of entanglement.

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