Note on integrability of certain homogeneous Hamiltonian systems

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Abstract

In this paper we investigate a class of natural Hamiltonian systems with two degrees of freedom. The kinetic energy depends on coordinates but the system is homogeneous. Thanks to this property it admits, in a general case, a particular solution. Using this solution we derive necessary conditions for the integrability of such systems investigating differential Galois group of variational equations.

Key words: integrability obstructions; Liouville integrability; differential Galois theory; systems in polar coordinates; systems in curved spaces

1 Introduction

It seems that the most effective methods of proving non-integrability are based on application of the differential Galois theory. For Hamiltonian systems necessary conditions for the integrability in the Liouville sense are given by the Morales-Ramis theorem.

**Theorem 1.1** (Morales-Ruiz and Ramis). Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neighbourhood of a phase curve $\Gamma$ corresponding to a particular solution. Then, the identity component $G^0$ of the differential Galois group $G$ of variational equations along $\Gamma$ is Abelian.
For a detailed exposition and a proof see e.g. [4, 5].

The above theorem has found a very effective application for natural systems given by the following Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + V(q), \quad (1.1)$$

where \(V(q)\) is a homogeneous function of degree \(k \in \mathbb{Z}\), and \(q = (q_1, \ldots, q_n)\) and \(p = (p_1, \ldots, p_n)\) are the generalised coordinates and momenta, respectively. Let us note that for application of Theorem 1.1 we have to know a particular solution of the considered system. In general it is a difficult problem how to find such a solution. However for systems given by (1.1) with a homogeneous potential \(V(q)\) it is well known that if \(d \in \mathbb{C}^n\) is a non-zero solution of nonlinear system \(V'(d) = d\), then functions

$$q(t) = \varphi(t) d, \quad p(t) = \varphi(t) d, \quad \ddot{\varphi} = -\varphi^{k-1}, \quad (1.2)$$

determine a particular solution of Hamilton’s equations. The variational equations along this solution split into a direct product of second order equations of the form

$$\ddot{x} = -\lambda \varphi(t)^{k-2} x, \quad (1.3)$$

where \(\lambda\) is an eigenvalue of Hessian \(V''(d)\). The necessary conditions for the integrability have the form of arithmetic restrictions on \(\lambda\), see e.g. [4, 5]. The crucial role in derivation of these conditions plays the Yoshida change of independent variable which transforms equation (1.3) into the Gauss hypergeometric equation [8].

Hamiltonian (1.1) describes a particle moving under influence of potential forces in flat Euclidean space \(\mathbb{R}^n\). It is a natural to ask what is an analog of homogeneous systems in curved spaces. There is no obvious answer to this question. We have to take into account the form of metric of the configuration space as well as the form of the potential. We leave a general discussion of this problem to a separate paper and here we consider systems with two degrees of freedom given by the following Hamiltonian

$$H = T + V, \quad T = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right), \quad V = r^m U(\varphi), \quad (1.4)$$

where \(m\) and \(k\) are integers, and \(k \neq 0\). If we consider \((r, \varphi)\) as the polar coordinates, then the kinetic energy corresponds to a singular metric on a plane or a sphere. We assume that \(U(\varphi)\) is a complex meromorphic function of variable \(\varphi \in \mathbb{C}\), and we do not require that \(U(\varphi)\) is periodic.

The main result of this paper is the following theorem which gives necessary conditions for the integrability of Hamiltonian systems given by (1.4). For its formu-
lation we need to define the following sets

\begin{align*}
J_0(k, m) &:= \left\{ \frac{1}{k} (mp + 1) (2mp + k) \mid p \in \mathbb{Z} \right\}, \quad (1.5) \\
J_1(k, m) &:= \left\{ \frac{1}{2k} (mp - 2) (mp - k) \mid p = 2r + 1, r \in \mathbb{Z} \right\}, \quad (1.6) \\
J_2(k, m) &:= \left\{ \frac{1}{8k} \left[ 4m^2 \left( p + \frac{1}{2} \right)^2 - (k - 2)^2 \right] \mid p \in \mathbb{Z} \right\}, \quad (1.7) \\
J_3(k, m) &:= \left\{ \frac{1}{8k} \left[ 4m^2 \left( p + \frac{1}{3} \right)^2 - (k - 2)^2 \right] \mid p \in \mathbb{Z} \right\}, \quad (1.8) \\
J_4(k, m) &:= \left\{ \frac{1}{8k} \left[ 4m^2 \left( p + \frac{1}{4} \right)^2 - (k - 2)^2 \right] \mid p \in \mathbb{Z} \right\}, \quad (1.9) \\
J_5(k, m) &:= \left\{ \frac{1}{8k} \left[ 4m^2 \left( p + \frac{1}{5} \right)^2 - (k - 2)^2 \right] \mid p \in \mathbb{Z} \right\}, \quad (1.10) \\
J_6(k, m) &:= \left\{ \frac{1}{8k} \left[ 4m^2 \left( p + \frac{2}{5} \right)^2 - (k - 2)^2 \right] \mid p \in \mathbb{Z} \right\}, \quad (1.11)
\end{align*}

and we put

\[ J_a(k, m) := J_0(k, m) \cup J_1(k, m) \cup J_2(k, m). \quad (1.12) \]

**Theorem 1.2.** Assume that \( U(\varphi) \) is a complex meromorphic function and there exists \( \varphi_0 \in \mathbb{C} \) such that \( U'(\varphi_0) = 0 \) and \( U(\varphi_0) \neq 0 \). If the Hamiltonian system defined by Hamiltonian (1.4) is integrable in the Liouville sense, then number

\[ \lambda := 1 + \frac{U''(\varphi_0)}{kU'(\varphi_0)}, \quad (1.13) \]

belongs to set \( J(k, m) \), which is defined by the following table

| No. | \( k \) | \( m \) | \( J(k, m) \) |
|-----|--------|--------|--------------|
| 1   | \( k = -2(mp + 1) \) | \( m \in \mathbb{C} \) | \( J_0(k, m) \cup J_1(k, m) \cup J_2(k, m) \) |
| 2   | \( k \in \mathbb{Z} \setminus \{0\} \) | \( m \) | \( J_a(k, m) \cup J_3(k, m) \cup J_4(k, m) \) |
| 3   | \( k = 2(mp - 1) \pm \frac{1}{3}m \) | \( m \in \mathbb{Z} \setminus \{0\} \) | \( J_a(k, m) \cup J_3(k, m) \cup J_4(k, m) \) |
| 4   | \( k = 2(mp - 1) \pm \frac{1}{2}m \) | \( m \) | \( J_3(k, m) \cup J_4(k, m) \) |
| 5   | \( k = 2(mp - 1) \pm \frac{3}{5}m \) | \( m \in \mathbb{Z} \setminus \{0\} \) | \( J_a(k, m) \cup J_3(k, m) \cup J_4(k, m) \) |
| 6   | \( k = 2(mp - 1) \pm \frac{1}{5}m \) | \( m \) | \( J_3(k, m) \cup J_4(k, m) \) |

Table 1: Integrability table. Here \( k, m, p, q \in \mathbb{Z} \) and \( k \neq 0 \).
The above theorem tells us that if \( k = -2(mp + 1) \), then the Morales-Ramis Theorem 1.1 does not give any obstruction for the integrability of the considered systems. Let us notice that this is an infinite family of systems. For systems (1.1) with homogeneous potentials only two cases of this type are such distinguished, namely \( k = \pm 2 \) [4, 5].

For each pair \((k, m)\) of integers which do not satisfy relation \( k = -2(mp + 1) \), \( p \in \mathbb{Z} \), Theorem 1.2 restricts admissible values \( \lambda \) to the set \( J_a(k, m) \). If \( m \) is not a multiple of 2, 3, and 5 these are the only restrictions. Otherwise, if \( m \) is a multiple of \( q \in \{2, 3, 5\} \), and \( k \) takes appropriate value, then the set of admissible values of \( \lambda \) contains additional elements. These are Cases 3–6 in Table 1.

Let us note that the above theorem remains valid for rational \( k \) and \( m \). In such extended version we require that \( k \) is a non-zero rational number, and the restriction contained in the third column of Table 1 can be ignored. For the proof of this extended version one has to apply a reasoning similar to that one used in [1].

Let us remark that there is also other possibility to generalise systems given by (1.1) with homogeneous potentials in such a way that they will admit a straight line particular solution and the variational equations can be reduced to a direct product of hypergeometric equations. In [6] the authors consider system with Hamiltonian

\[
H = T(p) + V(q),
\]

where \( T \) and \( V \) are homogeneous functions of integer degrees. To find a straight line particular solution one must solve overdetermined system of nonlinear equations

\[
\dot{T}(c) = c, \quad \dot{V}(c) = c,
\]

that has a solution only in special cases. Moreover, this generalisation does not have a form of a natural Hamiltonian system. In other words, except the case when \( \text{deg} \, T = 2 \), it cannot be considered as a Hamiltonian function of a point in a curved space.

2 Proof of Theorem 1.2

Equations of motion corresponding to Hamiltonian (1.4) have the form

\[
\begin{align*}
\dot{r} &= \frac{\partial H}{\partial p_r} = r^{m-k}p_r, \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = r^{m-k-3}p^2 - \frac{1}{2}(m-k)r^{m-k-1}\left(\frac{p^2 + p^2}{r^2}\right) - mr^{m-1}U(\varphi), \\
\dot{\varphi} &= \frac{\partial H}{\partial p_\varphi} = r^{m-k-2}p_\varphi, \\
\dot{p}_\varphi &= -\frac{\partial H}{\partial \varphi} = -r^mU'(\varphi).
\end{align*}
\]

If \( U'(\varphi_0) = 0 \) for a certain \( \varphi_0 \in \mathbb{C} \), then system (2.1) has two dimensional invariant manifold

\[
N = \{(r, p_r, \varphi, p_\varphi) \in \mathbb{C}^4 \mid \varphi = \varphi_0, \ p_\varphi = 0\}.
\]

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Indeed, equations (2.1) restricted to $\mathcal{N}$ read

\[
\dot{r} = r^{m-k}p_r, \quad p_r = -\frac{1}{2}(m-k)r^{m-k-1}p^2_r - mr^{m-1}U(\phi_0).
\] (2.3)

Hence, $\mathcal{N}$ is foliated by phase curves parametrised by energy $E$

\[
E = \frac{1}{2}r^{m-k}p_r^2 + r^mU(\phi_0).
\] (2.4)

Taking into account that $\dot{r} = r^{m-k}p_r$ we can rewrite equation (2.4) in the form

\[
\dot{r}^2 = 2r^{m-k}\{E - r^mU(\phi_0)\}.
\] (2.5)

Let $[R, P_R, \Phi, P_\Phi]^T$ denote the variations of $[r, p_r, \phi, p_\phi]^T$. Then, the variational equations along a particular solution lying on $\mathcal{N}$ take the form

\[
\frac{d}{dt} \begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix} = C \begin{bmatrix} R \\ P_R \\ \Phi \\ P_\Phi \end{bmatrix},
\] (2.6)

with

\[
C = \begin{bmatrix}
-\frac{1}{2}(l-1)lr^{l-2}p_r^2 - (m-1)mr^{m-2}U(\phi_0) & lr^{l-1}p_r & 0 & 0 \\
0 & -lr^{l-1}p_r & 0 & 0 \\
0 & 0 & -r^{m-2}U''(\phi_0) & 0 \\
0 & 0 & 0 & r^{l-2}
\end{bmatrix},
\]

where we introduced auxiliary parameter $l = m - k$. Equations for $\Phi$ and $P_\Phi$ form a closed subsystem which is called normal variational equations. This system can be rewritten as a one second-order differential equation

\[
\ddot{\Phi} + P\dot{\Phi} + Q\Phi = 0, \quad P = (k - m + 2)r^{m-k-1}p_r, \quad Q = r^{2m-k-2}U''(\phi_0).
\] (2.7)

In order to rationalise it we make the transformation

\[
t \rightarrow z = \frac{U(\phi_0)}{E}r^m(t),
\] (2.8)

for $E \neq 0$, that gives immediately

\[
z^2 = -2Em^2r^{m-k-2}z^2(z - 1), \quad \dot{z} = Emr^{m-k-2}z [(k - 4m + 2)z + 3m - k - 2].
\]

Equation (2.7) after such a change of independent variable takes the form

\[
z(z - 1)\Phi''(z) + \left[\frac{2m + k + 2}{2m}z - \frac{k + m + 2}{2m}\right]\Phi'(z) + \frac{k(1 - \lambda)}{2m^2}\Phi(z) = 0,
\] (2.9)

where prime denotes derivative with respect to $z$ and

\[
\lambda = 1 + \frac{U''(\phi_0)}{kU(\phi_0)}.
\]
Equation (2.9) is a special case of the Gauss hypergeometric differential equation whose general form is the following

\[ z(z-1)\Phi''(z) + [(\alpha + \beta + 1)z - \gamma] \Phi'(z) + \alpha \beta \Phi(z) = 0, \]  

(2.10)

and \( \alpha, \beta \) and \( \gamma \) are parameters, see e.g. [7, 3]. In our case the parameters take the forms

\[ \alpha = \frac{k + 2 - \Delta}{4m}, \quad \beta = \frac{k + 2 + \Delta}{4m}, \quad \gamma = \frac{k + 2 + m}{2m}, \]  

(2.11)

where

\[ \Delta = \sqrt{(k-2)^2 + 8k\lambda}. \]

The differences of exponents at singularities \( z = 0, z = 1 \) and at \( z = \infty \) are given by

\[ \rho = 1 - \gamma, \quad \sigma = \gamma - \alpha - \beta, \quad \tau = \beta - \alpha, \]

respectively, so in our case they are

\[ \rho = \frac{m - k - 2}{2m}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{\Delta}{2m}. \]  

(2.12)

If Hamilton equations (2.1) are integrable in the Liouville sense, then by Theorem 1.1 the identity component of the differential Galois group of variational equations (2.6) as well as normal variational equations (2.9) is Abelian, so in particular it is solvable. Necessary and sufficient conditions for solvability of the identity component of the differential Galois group for the Riemann \( P \) equation as well as its special form: the hypergeometric equation are well known thanks to the Kimura theorem which we recall in Appendix A.

The proof of Theorem 1.2 consists in a direct application of Theorem A.1 to our Gauss hypergeometric equation (2.9).

The condition A of Theorem A.1 is fulfilled if at least one of the following numbers

\[ \rho + \sigma + \tau = \frac{2m - k - 2 + \Delta}{2m}, \]
\[ -\rho + \sigma + \tau = \frac{k + 2 + \Delta}{2m}, \]
\[ \rho - \sigma + \tau = \frac{-k - 2 + \Delta}{2m}, \]
\[ \rho + \sigma - \tau = \frac{2m - k - 2 - \Delta}{2m} \]

is an odd integer. If it is the first one, then \( \lambda \in \mathcal{J}_0(k, m) \), and if it is the second one, then \( \lambda \in \mathcal{J}_1(k, m) \). It is easy to check that if the third or fourth of the above numbers is an odd integer, then \( \lambda \in \mathcal{J}_0(k, m) \cup \mathcal{J}_1(k, m) \). This exhaust all the possibilities in Case A of Theorem A.1.

Now, we pass to Case B of Theorem A.1. In this case the quantities \( \rho \) or \( -\rho \), \( \sigma \) or \( -\sigma \) and \( \tau \) or \( -\tau \) must belong to Table 2 called Schwarz’s table. As \( \sigma = \frac{1}{2} \) only items 1, 2, 4, 6, 9, or 14 of the Table 2 are allowed. We analyse them case by case.
Case 1.

- $\pm \rho = 1/2 + s$, for a certain $s \in \mathbb{Z}$, then $k = -2(mp + 1)$ for a certain $p \in \mathbb{Z}$. In this case $\tau$ is an arbitrary number, so $\lambda$ is arbitrary.

- $\pm \tau = 1/2 + p$, for a certain $p \in \mathbb{Z}$, then $\lambda \in J_2(k,m)$. In this case $\rho$-arbitrary, and thus $k$ can be arbitrary.

Case 2. In this case $\pm \tau = 1/3 + p$, for a certain $p \in \mathbb{Z}$, and $\pm \rho = 1/3 + s$, for a certain $s \in \mathbb{Z}$. The first condition implies that $\lambda \in J_3(k,m)$. If the second condition is fulfilled, then

$$k = 2(mp - 1) \pm \frac{1}{3}m. \quad (2.13)$$

Case 4. We have two possibilities:

- if $\pm \rho = 1/3 + s$, and $\pm \tau = 1/4 + p$ for certain $s, p \in \mathbb{Z}$, then $k$ is given by (2.13), and $\lambda \in J_4(k,m)$.

- if $\pm \rho = 1/4 + s$, and $\pm \tau = 1/3 + p$, for certain $p, s \in \mathbb{Z}$, then

$$k = 2(mp - 1) \pm \frac{1}{2}m, \quad (2.14)$$

and $\lambda \in J_3(k,m)$.

Case 6.

- If $\pm \rho = 1/3 + s$ and $\pm \tau = 1/5 + p$, for some $s, p \in \mathbb{Z}$, then $k$ is given by (2.13) and $\lambda \in J_5(k,m)$.

- If $\pm \rho = 1/5 + s$ and $\pm \tau = 1/3 + p$, for some $s, p \in \mathbb{Z}$, then

$$k = 2(mp - 1) \pm \frac{3}{5}m, \quad (2.15)$$

and $\lambda \in J_3(k,m)$.

Case 9.

- If $\pm \rho = 2/5 + s$, and $\pm \tau = 1/5 + p$, for some $s, p \in \mathbb{Z}$, then

$$k = 2(mp - 1) \pm \frac{1}{5}m, \quad (2.16)$$

and $\lambda \in J_5(k,m)$.

- If $\pm \rho = 1/5 + s$, and $\tau = 2/5 + p$, for some $s, p \in \mathbb{Z}$, then $k$ is given by (2.15) and $\lambda \in J_6(k,m)$.
Case 14.

- If \( \pm \rho = 2/5 + s \), and \( \tau = 1/3 + p \), for some \( s, p \in \mathbb{Z} \), then \( k \) is given by (2.16) and \( \lambda \in J_3(k, m) \).

- If \( \pm \rho = 1/3 + s \), and \( \pm \tau = 2/5 + p \), for some \( s, p \in \mathbb{Z} \), then \( k \) is given by (2.13) and \( \lambda \in J_6(k, m) \).

Now, collecting all items with the same form of \( k \) and taking into account that it has to be an integer we obtain Table 1.

### 3 Application of Theorem 1.2

Here we consider two examples of Hamilton functions of the form (1.4).

**Example 1.** First of all we check separable cases for Hamiltonian (1.4). Its Hamilton-Jacobi equation takes the form

\[
\frac{1}{2} r^{m-k} \left[ \left( \frac{\partial S}{\partial r} \right)^2 + r^{-2} \left( \frac{\partial S}{\partial \phi} \right)^2 \right] + r^m U(\phi) = E, \tag{3.1}
\]

where \( S = S(r, \phi) \) is Hamilton’s characteristic function. We look for \( S \) postulating its additive form

\[ S = S_r(r) + S_\phi(\phi). \]

Then we can rewrite (3.1) in the following way

\[
r^{-k} \left( \frac{dS_r}{dr} \right)^2 + r^{-(k+2)} \left( \frac{dS_\phi}{d\phi} \right)^2 + 2U(\phi) = 2r^{-m}E. \tag{3.2}
\]

This equation separates when \( k = -2 \) and then we obtain

\[
r^2 \left( \frac{dS_r}{dr} \right)^2 - 2r^{-m}E = \alpha, \quad \left( \frac{dS_\phi}{d\phi} \right)^2 + 2U(\phi) = -\alpha, \tag{3.3}
\]

where \( \alpha \) is a separation constant. So, in this case the Hamiltonian of the system has the form

\[
H = \frac{1}{2} r^{m+2} \left( p_r^2 + \frac{p_\phi^2}{r^2} \right) + r^m U(\phi), \tag{3.4}
\]

and it is integrable with the following additional first integral

\[
G = \frac{p_\phi^2}{2} + U(\phi).
\]

Let us note that the case \( k = -2 \) is contained in the first item of Table 1.
Example 2. Now we consider the Hamilton function

\[ H = \frac{1}{2} r^{m-k} \left( p^2 + \frac{p^2 \phi}{r^2} \right) - r^m \cos \phi. \]  

(3.5)

It has the form (1.4) with \( U(\phi) = -\cos \phi \). As \( U'(\phi) = \sin \phi \), we take \( \phi_0 = 0 \). Since \( U''(0)/U(0) = -1 \), we have \( \lambda = (k - 1)/k \). Comparing this value with forms of \( \lambda \) in sets \( I_j(k, m) \) for \( j = 0, \ldots, 6 \) we obtain the following conditions:

- if \( \lambda \in I_0(k, m) \), then \( 2m^2 p^2 + (k + 2)m p + 1 = 0 \), and this implies that
  \[ [4mp + k + 2]^2 = k^2 + 4k - 4, \]  
  (3.6a)

- if \( \lambda \in I_1(k, m) \), then \( m^2 p^2 - (k + 2)m p + 2 = 0 \), and this implies that
  \[ [2(mp - 1) - k]^2 = k^2 + 4k - 4, \]  
  (3.6b)

- if \( \lambda \in I_2(k, m) \), then
  \[ [m(2p + 1)]^2 = k^2 + 4k - 4, \]  
  (3.6c)

- if \( \lambda \in I_3(k, m) \), then
  \[ [2m(3p + 1)]^2 = 9(k^2 + 4k - 4), \]  
  (3.6d)

- if \( \lambda \in I_4(k, m) \), then
  \[ [m(4p + 1)]^2 = 4(k^2 + 4k - 4), \]  
  (3.6e)

- if \( \lambda \in I_5(k, m) \), then
  \[ [2m(5p + 1)]^2 = 25(k^2 + 4k - 4), \]  
  (3.6f)

- if \( \lambda \in I_6(k, m) \), then
  \[ [2m(5p + 2)]^2 = 25(k^2 + 4k - 4). \]  
  (3.6g)

It is easy to see that if one of the above conditions is fulfilled, then

\[ k^2 + 4k - 4 = q^2, \]  

(3.7)

for a certain \( q \in \mathbb{Z} \). Rewriting this equality in the form

\[ (k + 2 + q)(k + 2 - q) = 8 \]

and considering all decompositions of \( 8 = (\pm 1) \cdot (\pm 8) = (\pm 2) \cdot (\pm 4) = (\pm 4) \cdot (\pm 2) = (\pm 8) \cdot (\pm 1) \), we obtain that \( k \in \{-5, 1\} \). With these values of \( k \) one can easily
find that $\lambda = (k - 1) / k \in J_0(k, m)$ iff $m \in \{-1, 1\}$. Hence, we have the following four cases with the following $m, k$ and $l = m - k$:

1. $m = 1$, $k = -5$, $l = 6$,
2. $m = -1$, $k = 1$, $l = -2$,
3. $m = 1$, $k = 1$, $l = 0$,
4. $m = -1$, $k = -5$, $l = 4$,

(3.8)

Similarly, if $\lambda \in (k - 1) / k \in J_1$ with $k \in \{-5, 1\}$, then $m \in \{-2, -1, 1, 2\}$. Thus, besides cases (3.8) we have additionally the following ones

5. $m = 2$, $k = 1$, $l = 1$,
6. $m = -2$, $k = 1$, $l = -3$,
7. $m = 2$, $k = -5$, $l = 7$,
8. $m = -2$, $k = -5$, $l = 3$.

(3.9)

Now, we show that there are no other cases when the necessary conditions for the integrability given by Theorem 1.2 in items 2-6 of Table 1 are satisfied. In fact, for both values $k \in \{-5, 1\}$ we have $k^2 + 4k - 4 = 1$, thus the right-hand sides of conditions (3.6) are given explicitly. As the left- and the right-hand sides of equations (3.6d), (3.6f) and (3.6g) have different parities these equalities cannot hold. Equalities (3.6c) and (3.6e) are fulfilled only for $m = \pm 1$ or $m = \pm 2$ and $p = 0$, respectively, but these values are already given in the above listed cases.

Surprisingly all cases (3.8) are integrable and in fact superintegrable.

**Case 1.** In this case we have the Hamiltonian of the following form

$$H = \frac{1}{2} r^6 \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r \cos \varphi. \quad (3.10)$$

This system has two additional, functionally independent first integrals of the second order in momenta

$$F_1 := r^2 p_\varphi^2 \cos(2\varphi) - r^3 p_r p_\varphi \sin(2\varphi) + r^{-1} \sin \varphi \sin(2\varphi),$$
$$F_2 := r^2 p_\varphi^2 \sin(2\varphi) + r^3 p_r p_\varphi \cos(2\varphi) - r^{-1} \sin \varphi \cos(2\varphi). \quad (3.11)$$

**Case 2.** We have the following Hamiltonian

$$H = \frac{1}{2} r^{-2} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-1} \cos \varphi, \quad (3.12)$$

and two additional functionally independent first integrals take the form

$$F_1 := -r^{-2} p_\varphi^2 \cos(2\varphi) + r^{-1} p_r p_\varphi \sin(2\varphi) + r \sin \varphi \sin(2\varphi),$$
$$F_2 := -r^{-2} p_\varphi^2 \sin(2\varphi) + r^{-1} p_r p_\varphi \cos(2\varphi) + r \sin \varphi \cos(2\varphi). \quad (3.13)$$
Case 3. Hamilton function and additional first integrals are the following

\[
H = \frac{1}{2} \left( p_r^2 + \frac{p^2}{r^2} \right) - r \cos \phi,
\]

\[
F_1 := r^{-1} p^2 \phi \cos \phi + p_r p_\phi \sin \phi + \frac{1}{2} r^2 \sin^2 \phi,
\]

\[
F_2 := \left( p_r^2 - r^{-2} p_\phi^2 \right) \cos \phi \sin \phi + r^{-1} p_r p_\phi \cos(2 \phi) - r \sin \phi.
\]

(3.14)

Case 4. In this case we have respectively:

\[
H = \frac{1}{2} r^4 \left( p_r^2 + \frac{p^2}{r^2} \right) - r^{-1} \cos \phi,
\]

\[
F_1 := r p^2 \phi \cos \phi - r^2 p_r p_\phi \sin \phi + \frac{1}{2} r^{-2} \sin^2 \phi,
\]

\[
F_2 := r^4 \left( p_r^2 - r^{-2} p_\phi^2 \right) \cos \phi \sin \phi - r^3 p_r p_\phi \cos(2 \phi) - r^{-1} \sin \phi.
\]

(3.15)

(a) section plane \( r = 1 \) with coordinates \((\phi, p_\phi)\)  
(b) magnification of region around unstable periodic solution

Figure 1: Poincaré cross sections on energy level \( E = -0.5 \) for Hamiltonian system given by (3.5) with \( m = -2, k = 1 \) corresponding to Case 6

In cases with parameters given in (3.9) we have integrable as well as non-integrable systems. Namely cases 5 and 8 are integrable.

Case 5.

\[
H = \frac{1}{2} r \left( p_r^2 + \frac{p^2}{r^2} \right) - r^2 \cos \phi,
\]

\[
F := r^{-1} (p_\phi^2 - r^2 p_r^2) \cos \phi + r^2 (1 + \cos^2 \phi) + 2 p_r p_\phi \sin \phi.
\]

(3.16)
$r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) magnification of region around unstable periodic solution

Figure 2: Poincaré cross sections on energy level $E = -0.3$ for Hamiltonian system given by (3.5) with $m = 2, k = -5$ corresponding to Case 7

(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) section plane $\varphi = 0$ with coordinates $(r, p_r)$

Figure 3: Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (3.5) with $m = -2, k = 2$

Case 8.

\[ H = \frac{1}{2} r^3 \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-2} \cos \varphi, \]
\[ F := r (p_\varphi^2 - r^2 p_r^2) \cos \varphi + r^{-2} (1 + \cos^2 \varphi) - 2 r^2 p_r p_\varphi \sin \varphi. \]

However the Poincaré sections for Hamiltonian system (3.5) with parameters given in Cases 6 and 7 in (3.9) show chaotic area, see Figures 1-2. Chaos appears just in vicinities of unstable periodic solutions visible on magnifications 1(b) and 2(b).

Cases from the first item of Table 1 are generically non-integrable, see Poincaré sections on Figures 3-5 showing large chaotic regions. However in order to prove...
(a) section plane \( r = 1 \) with coordinates \((\varphi, p_\varphi)\) (b) section plane \( \varphi = 0 \) with coordinates \((r, p_r)\)

Figure 4: Poincaré cross sections on energy level \( E = -0.6 \) for Hamiltonian system given by (3.5) with \( m = -1, k = 8 \)

(a) section plane \( r = 1 \) with coordinates \((\varphi, p_\varphi)\) (b) section plane \( \varphi = 0 \) with coordinates \((r, p_r)\)

Figure 5: Poincaré cross sections on energy level \( E = -0.5 \) for Hamiltonian system given by (3.5) with \( m = 1, k = -6 \)

the non-integrability of Hamiltonians with \( m \) and \( k \) given by this item higher order variational equations must be used.

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A Gauss hypergeometric equation

The Riemann $P$ equation is the most general second order differential equation with three regular singularities [7, 3]. If we place using homography these singularities at $z \in \{0, 1, \infty\}$, then it has the form

$$\frac{d^2\eta}{dz^2} + \left(\frac{1 - \alpha - \alpha'}{z} + \frac{1 - \gamma - \gamma'}{z - 1}\right) \frac{d\eta}{dz} + \left(\frac{\alpha\alpha'}{z^2} + \frac{\gamma\gamma'}{(z - 1)^2} + \frac{\beta\beta' - \alpha\alpha' - \gamma\gamma'}{z(z - 1)}\right) \eta = 0,$$

(A.1)

where $(\alpha, \alpha'), (\gamma, \gamma')$ and $(\beta, \beta')$ are the exponents at the respective singular points. These exponents satisfy the Fuchs relation

$$\alpha + \alpha' + \gamma + \gamma' + \beta + \beta' = 1.$$

We denote the differences of exponents by

$$\rho = \alpha - \alpha', \quad \sigma = \beta - \beta', \quad \tau = \gamma - \gamma'.$$

In particular the Gauss hypergeometric equation

$$\frac{d^2\eta}{dz^2} + \left(\frac{(\alpha + \beta + 1)z - \gamma}{z(z - 1)}\right) \frac{d\eta}{dz} + \frac{\alpha\beta}{z(z - 1)} \eta = 0,$$

(A.2)

where $\alpha, \beta, \gamma$ are parameters with the exponent differences given by

$$\rho = 1 - \gamma, \quad \sigma = \gamma - \alpha - \beta, \quad \tau = \beta - \alpha.$$

is a special form of Riemann $P$ equation. Necessary and sufficient conditions for solvability of the identity component of the differential Galois group of (A.1) and (A.2) are given by the following theorem due to Kimura [2].

**Theorem A.1.** The identity component of the differential Galois group of the Riemann $P$ equation (A.1) is solvable iff

A. at least one of the four numbers $\rho + \sigma + \tau$, $-\rho + \sigma + \tau$, $\rho + \sigma - \tau$, $\rho - \sigma + \tau$ is an odd integer, or

B. the numbers $\rho$ or $-\rho$ and $\sigma$ or $-\sigma$ and $\tau$ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz’s Table 2.

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Table 2: The Schwarz’s table. Here $r, s, p \in \mathbb{Z}$

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 1 | $1/2 + r$ | $1/2 + s$ | arbitrary complex number |
| 2 | $1/2 + r$ | $1/3 + s$ | $1/3 + p$ |
| 3 | $2/3 + r$ | $1/3 + s$ | $1/3 + p$ | $r + s + p$ even |
| 4 | $1/2 + r$ | $1/3 + s$ | $1/4 + p$ |
| 5 | $2/3 + r$ | $1/4 + s$ | $1/4 + p$ | $r + s + p$ even |
| 6 | $1/2 + r$ | $1/3 + s$ | $1/5 + p$ |
| 7 | $2/5 + r$ | $1/3 + s$ | $1/3 + p$ | $r + s + p$ even |
| 8 | $2/3 + r$ | $1/5 + s$ | $1/5 + p$ | $r + s + p$ even |
| 9 | $1/2 + r$ | $2/5 + s$ | $1/5 + p$ |
| 10 | $3/5 + r$ | $1/3 + s$ | $1/5 + p$ | $r + s + p$ even |
| 11 | $2/5 + r$ | $2/5 + s$ | $2/5 + p$ | $r + s + p$ even |
| 12 | $2/3 + r$ | $1/3 + s$ | $1/5 + p$ | $r + s + p$ even |
| 13 | $4/5 + r$ | $1/5 + s$ | $1/5 + q$ | $r + s + p$ even |
| 14 | $1/2 + r$ | $2/5 + s$ | $1/3 + p$ |
| 15 | $3/5 + r$ | $2/5 + s$ | $1/3 + p$ | $r + s + p$ even |

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