Bulk and edge excitations of a $\nu = 1$ Hall ferromagnet

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Abstract

In this article, we shall focus on the collective dynamics of the fermions in a $\nu = 1$ quantum Hall droplet. Specifically, we propose to look at the quantum Hall ferromagnet. In this system, the electron spins are ordered in the ground state due to the exchange part of the Coulomb interaction and the Pauli exclusion principle. The low energy excitations are ferromagnetic magnons. In order to obtain an effective Lagrangian for these magnons, we shall introduce bosonic collective coordinates in the Hilbert space of many-fermion systems. These collective coordinates describe a part of the fermionic Hilbert space. Using this technique, we shall interpret the magnons as bosonic collective excitations in the Hilbert space of the many-electron Hall system. Furthermore, by considering a Hall droplet of finite extent, we shall also obtain the effective Lagrangian governing the spin collective excitations at the edge of the sample.

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I. Introduction

Quantum Hall ferromagnets have generated a considerable amount of interest in recent years. The fact that Hall states should exhibit ferromagnetism is somewhat surprising. In conventional Hall systems, the electron spins are aligned by the strong external magnetic field. Thus, in the absence of spontaneous alignment, these Hall states do not qualify as ferromagnetic states. However, particular samples, where the coupling of the electronic spins to the external magnetic field is negligible for a variety of reasons, do exhibit a spontaneous alignment of the spins. These are the Hall ferromagnets [1],[2],[3].

Traditionally, the integer and the fractional Hall effects are distinguished by the origin of the gap in the single particle spectrum. In the integer effects, the gap which is equal to the cyclotron gap, is produced by the energy difference between the successive Landau levels. On the other hand, in the fractional effects, the gap is produced due to inter-electronic interactions in the strongly correlated Hall fluid.

There is, however, a scenario, where despite integer filling, the gap in the single particle spectrum is due to inter-electron interactions. For instance, in GaAs, the effective mass in the conduction band, which appears in the expression for the cyclotron gap, is much smaller than the actual mass of the electron, which appears in the expression for the Zeeman gap, through the Bohr magneton $\mu_B$. Thus the cyclotron gap increases effectively by a factor of $\sim 14$. Spin-orbit scattering reduces the effective $g$ factor by a factor of $\sim 5$. Thus the Zeeman energy is vanishingly small compared to the cyclotron energy (a factor of $\sim 70$ smaller)[1].

Imagine that, starting from the value of 2, the $g$ factor in the sample can be gradually brought down to zero. Initially, for $g = 2$, the $\nu = 1$ state described a state with a uniform density of $\frac{B}{2\pi}$ electrons per unit area, with each electron in the spin “up” state, say. With the reduction of the Zeeman gap, one would expect this state to be rendered unstable as the two spin orientations gradually become degenerate. Experimentally, however, a gap is
observed, indicating the $\nu = 1$ state remains stable [4].

A rather heuristic argument may be adduced to explain this phenomenon. In the lowest Landau level, the kinetic energy of the electrons has a fixed value. Thus, to minimise the Coulomb energy, the spatial part of the many-electron wave function should be totally antisymmetric. This in turn will require, by the Pauli principle, that the spin part of the wave function be completely symmetric. Thus, the Coulomb interaction causes the spins to align and is instrumental in producing the observed gap which stabilises the $\nu = 1$ state in the absence of the Zeeman coupling. The state is genuinely ferromagnetic as the spins are spontaneously aligned. In this situation, the conventional distinction between the integer and the fractional effects becomes somewhat blurred.

Since the ground state exhibits the spontaneous breaking of the original global spin $SU(2)$ symmetry (for $g = 0$) down to a $u(1)$ symmetry, the excitation spectrum must contain gapless Goldstone bosons, namely, the ferromagnetic magnons. The effective Lagrangian governing the dynamics of the magnons has been obtained previously in a variety of ways [1],[2],[5].

It is well known that the excitations located at the edges of finite Hall samples play a crucial role in the physics of these samples. The so-called edge states have been studied in considerable depth for the integer and the fractional effects [6]. In contrast, despite some seminal work [7] on the edge excitations of ferromagnet Hall samples, a systematic derivation of the effective Lagrangian governing these excitations seems to be in order.

In this article, we propose to use a recently developed technique [8] for introducing bosonic collective variables as coordinates in many-fermion Hilbert spaces, in studying the bulk and the edge excitations of the $\nu = 1$ Hall ferromagnet.

I. Bosonic Effective Lagrangian for Fermions

In this section, we shall review a method, originally developed in [8], of introducing bosonic collective coordinates in the Hilbert space of many-fermion systems. We shall
further obtain, starting from the second-quantised fermionic Lagrangian, an effective Lagrangian governing the bosonic collective coordinates.

Let us consider a system of \( N \) fermions, free to reside at \( K \) sites, with \( K \geq N \). Let us denote the sites by \( \alpha (\alpha = 1, 2, \ldots, K) \). An \( N \)-body fermionic Fock state is given by

\[
|\alpha_1, \alpha_2, \ldots, \alpha_N \rangle = \psi_{\alpha_1}^\dagger \psi_{\alpha_2}^\dagger \cdots \psi_{\alpha_N}^\dagger |0\rangle. \tag{1.1}
\]

where \( \psi_\alpha^\dagger \) are the fermionic creation operators satisfying the standard anti-commutation relation \( \{\psi_\alpha, \psi_\beta^\dagger\} = \delta_{\alpha\beta} \) and where \( |0\rangle \) is the fermionic vacuum state. The number of independent Fock states describing this system is given by

\[
D_f = \frac{K!}{(K-N)!N!} \tag{1.2}.
\]

Let \( A \) denote a set of \( N \) indices \( \alpha_1, \alpha_2, \ldots, \alpha_N \). A general state vector \( |\Psi\rangle \) belonging to the \( N \)-particle Fock space is given by

\[
|\Psi\rangle = \sum_{\alpha_1,\alpha_2,\ldots,\alpha_N} \frac{1}{\sqrt{N!}} |\alpha_1, \alpha_2, \ldots, \alpha_N\rangle \psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}
\equiv \sum_A |A\rangle \Psi_A \tag{1.3}
\]

where \( \psi_{\alpha_1,\alpha_2,\ldots,\alpha_N} \) is a totally anti-symmetric complex field with \( N \) indices. We normalise it to \( \langle \Psi | \Psi \rangle = 1 \). Thus,

\[
\sum_{\alpha_1,\alpha_2,\ldots,\alpha_N} \frac{1}{N!} |\psi_{\alpha_1,\alpha_2,\ldots,\alpha_N}|^2 = \sum_A |\Psi_A|^2 = 1. \tag{1.4}
\]

Based on (1.3), we may look upon \( \Psi_A \) as the coordinates of the fermionic Hilbert space.

From simple group theoretic considerations, we also know that the completely antisymmetric irreducible representation of the group \( SU(K) \), in terms of antisymmetric tensors with \( N \) indices, has a dimensionality equal to \( D_f \). Thus, the group \( SU(K) \) acts naturally on the \( \psi_{\alpha_1,\alpha_2,\ldots,\alpha_N} \). Namely,

\[
\psi_{\alpha_1\alpha_2\cdots\alpha_N} = \frac{1}{\sqrt{N!}} \Phi_{\beta_1\beta_2\cdots\beta_N} u_{\alpha_1\beta_1} u_{\alpha_2\beta_2} \cdots u_{\alpha_N\beta_N} \tag{1.5}
\]
where the indices $\alpha, \beta$ run from 1 to $K$. The transformation (1.5) is equivalently written as

$$\Psi_A = \sum_B D_{AB}(u) \Phi_B,$$

where $D(u)$ is the totally anti-symmetric irreducible representation of $SU(K)$, of dimensionality $D_f$.

Using (1.5), we perform a change of coordinates from $\Psi_A$ to $u, \Phi_B$. The $u$ are the parameters of $SU(K)$ and we call them collective coordinates. If we restrict the range of $B$ to less than $D_f - \frac{K^2}{2}$, the new coordinates do not cover the entire Hilbert space. Nonetheless, if we choose the restricted $\Phi_B$ judiciously, we could hope to include the important collective states in the truncated Hilbert space. In this paper, we restrict $B$ to one single value, namely, a set of $1, 2, \cdots N$. We set

$$\Phi_{\beta_1 \beta_2 \cdots \beta_N} = \epsilon_{\beta_1 \beta_2 \cdots \beta_N},$$

where $\epsilon_{\beta_1 \beta_2 \cdots \beta_N}$ is the Levi-Civita symbol for the set $1, 2, \cdots N$. Substituting (1.7) into (1.5) and thence into (1.3), we obtain $|\Psi\rangle \equiv |u\rangle$ where

$$|u\rangle = \prod_{\alpha=1}^{N} \tilde{\psi}_\alpha^\dagger |0\rangle$$

with

$$\tilde{\psi}_\alpha^\dagger = (\psi^\dagger u)_\alpha, \{\tilde{\psi}_\alpha, \tilde{\psi}_\beta^\dagger\} = \delta_{\alpha\beta}.$$

We get $\langle u|u \rangle = 1$. The state $|u\rangle$ is obviously constructed by filling up $N$ sites sequentially with fermions created by $\tilde{\psi}_\alpha^\dagger$.

Let us define an operator $\mathcal{P}$ as

$$\mathcal{P} \equiv \int du \ |u\rangle \langle u|.$$

We can show (see Appendix A) that $\mathcal{P}$ has the property of a projection operator $\mathcal{P}^2 = \mathcal{P}$, if we use the appropriate Haar measure for $SU(K)$ in $du$. 

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The partition function of the fermionic system in the subspace defined by $\mathcal{P}$ may be expressed as a path integral, which is obtained in the usual manner:

$$Z = \int \mathcal{D}u e^{i \int dt \langle u(t) | (i\partial_t - H) | u(t) \rangle},$$

(1.11)

where $\mathcal{D}u$ is the appropriate Haar measure. The effective Lagrangian in the subspace defined by $\mathcal{P}$ is thus obtained as

$$L_{\text{eff}} = \langle u(t) | (i\partial_t - H) | u(t) \rangle.$$  

(1.12)

Recalling the definition of $|u\rangle$ from (1.8), we see that

$$\langle u(t) | i\partial_t | u(t) \rangle = \text{tr}(\rho_0 u^\dagger i\partial_t u)$$

(1.13)

where

$$(\rho_0)_{\alpha\beta} \equiv \langle u | \tilde{\psi}_\beta \psi_\alpha | u \rangle.$$  

(1.14)

Thus,

$$(\rho_0)_{\alpha\beta} = \delta_{\alpha\beta}, \alpha, \beta \leq N$$

$$= 0 \quad \text{otherwise.}$$ 

(1.15)

Let us consider a many body Hamiltonian of the form

$$H = H^{(1)} + H^{(2)},$$

with

$$H^{(1)} = \psi_\alpha^\dagger h^{(1)}_{\alpha\beta} \psi_\beta$$  

(1.16)

and

$$H^{(2)} = \psi_{\alpha_1}^\dagger \psi_{\alpha_2}^\dagger h^{(2)}_{\beta_1\beta_2} \psi_{\beta_1} \psi_{\beta_2}. $$

(1.17)

The corresponding effective Hamiltonian is then given by

$$H_{\text{eff}} \equiv \langle u | H | u \rangle = \text{tr}(\rho_0 u^\dagger h^{(1)} u + (u\rho_0 u^\dagger)_{\beta_1\alpha_1} (u\rho_0 u^\dagger)_{\beta_2\alpha_2} [h^{(2)}_{\beta_1\beta_2} - h^{(2)}_{\beta_2\beta_1}])$$

(1.18)
and the effective Lagrangian is, from (1.13) and (1.18),

\[
L_{\text{eff}} = \text{tr} \left( \rho_0 u^\dagger (i \partial_t - h^{(1)} u) - \sum_{\alpha_1, \alpha_2, \beta_1, \beta_2} \left( (u \rho_0 u^\dagger)^{\beta_1 \alpha_1}(u \rho_0 u^\dagger)^{\beta_2 \alpha_2} \left[ h^{(2)}_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} - h^{(2)}_{\beta_2 \beta_1}^{\alpha_1 \alpha_2} \right] \right) \right). \tag{1.19}
\]

The third term in (1.19) is the so called Direct term and the last term, the Exchange term.

We have thus obtained the effective Lagrangian governing the dynamics of the bosonic collective coordinate \(u\) starting from the fermionic second quantised action.

**II. Planar Fermions in the Lowest Landau Level**

Before introducing the bosonic collective coordinates, for the particular case of the Hall ferromagnet, let us briefly recapitulate the basic physics of planar fermions subjected to a strong magnetic field orthogonal to the plane. We assume from the onset that the Zeeman coupling of the electron spins to this magnetic field is zero, due to the vanishingly small value of the effective g-factor.

The single particle Hamiltonian is the celebrated Landau Hamiltonian. If \(\vec{A}\) is the gauge potential (we choose the symmetric gauge for convenience) giving rise to the external magnetic field, we have

\[
h_0 = \frac{1}{2m} (\vec{p} - \vec{A})^2 \equiv \frac{1}{2m} \vec{\Pi}^2 \tag{2.1}
\]

where \(\Pi^x = -i \partial_x - \frac{B}{2} y\), \(\Pi^y = -i \partial_y + \frac{B}{2} x\). Defining \(\pi \equiv \frac{1}{\sqrt{2B}} (\Pi^x - i \Pi^y)\) and \(\pi^\dagger\) as its complex conjugate, we have,

\[
[\pi, \pi^\dagger] = 1. \tag{2.2}
\]

The large degeneracy \((\frac{B}{2\pi})^2\) states per unit area) of the single particle spectrum is expressed through the introduction of the guiding centre coordinates:

\[
\hat{X} \equiv \dot{x} - \frac{1}{B} \hat{\Pi}^y, \quad \hat{Y} \equiv \dot{y} + \frac{1}{B} \hat{\Pi}^x, \quad [\hat{X}, \hat{Y}] = \frac{i}{B}. \tag{2.3}
\]

The holomorphic combination and its complex conjugate,

\[
\hat{a} \equiv \sqrt{\frac{B}{2}} (\hat{X} + i \hat{Y}), \quad \hat{a}^\dagger \equiv \sqrt{\frac{B}{2}} (\hat{X} - i \hat{Y}), \tag{2.4}
\]
satisfy
\[ [\hat{\pi}, \hat{\pi}^\dagger] = 1, \quad [\hat{a}, \hat{a}^\dagger] = 1, \quad \hat{a} \hat{\pi} = \hat{\pi} \hat{a} = 0. \quad (2.5) \]

These are related to the coordinate operators by
\[ \sqrt{\frac{B}{2}} (\hat{x} + i\hat{y}) \equiv \hat{z} = \hat{a} - i\hat{\pi}^\dagger, \quad \sqrt{\frac{B}{2}} (\hat{x} - i\hat{y}) \equiv \hat{\bar{z}} = \hat{a}^\dagger + i\hat{\pi}. \quad (2.6) \]

Of course \([\hat{z}, \hat{\bar{z}}] = 0\).

The eigenbasis of \(h_0\) may be taken to be \(|n, l\rangle\), where \(n, l = 0, 1, 2 \cdots \infty\). The index \(n\) is called the Landau level index. \(n = 0\) corresponds to the lowest Landau level (L.L.L.). For a given \(n, l\) measures the degeneracy of the energy eigenstate. We have
\[ \hat{\pi}|n, l\rangle = \sqrt{n}|n-1, l\rangle, \quad \hat{\pi}^\dagger|n, l\rangle = \sqrt{n+1}|n+1, l\rangle \quad (2.7) \]

and
\[ \hat{a}|n, l\rangle = \sqrt{l}|n, l-1\rangle, \quad \hat{a}^\dagger|n, l\rangle = \sqrt{l+1}|n, l+1\rangle. \quad (2.8) \]

Thus,
\[ \hat{\pi}^\dagger \hat{\pi}|n, l\rangle = n|n, l\rangle, \quad \hat{a}^\dagger \hat{a}|n, l\rangle = l|n, l\rangle. \quad (2.9) \]

We can also define a coherent state basis for \(\hat{a}\). If
\[ |\xi\rangle \equiv e^{\xi \hat{a}^\dagger}|0\rangle \quad (2.10) \]

we can easily check that \(\hat{a}|\xi\rangle = \xi|\xi\rangle\). The inner product of two coherent states is given by \(\langle \eta | \xi \rangle = e^{\bar{\eta} \xi} \) and the resolution of the identity is \(\int d^2 \xi e^{-|\xi|^2} |\xi\rangle \langle \xi | = I\), where \(d^2 \xi \equiv \frac{d \text{Re} \xi d \text{Im} \xi}{\pi}\). The coherent state basis is related to the \(|l\rangle\) basis through \(\langle l | \xi \rangle = \frac{\xi^l}{\sqrt{l!}}\). The L.L.L. wave function is given by
\[ \langle \vec{x} | 0, l \rangle = \sqrt{\frac{B}{2\pi}} \frac{1}{\sqrt{l}} e^{-\frac{1}{2} |z|^2} \bar{z}^l \quad (2.11) \]
in the \(|l\rangle\) basis and
\[ \langle \vec{x} | 0, \xi \rangle = \sqrt{\frac{B}{2\pi}} e^{-\frac{1}{2} |z|^2 + \bar{z} \xi} \quad (2.12) \]
in the coherent state basis.

A smooth function of \( \hat{z}, \hat{\bar{z}} \), may be expanded in the following manner:

\[
A(\hat{z}, \hat{\bar{z}}) = A(\hat{a} - i\hat{\pi}^\dagger, \hat{a}^\dagger + i\hat{\pi}) = \sum_{p,q} \frac{(\hat{\pi}^\dagger)^p(\hat{\pi})^q}{p! q!} \partial_z^p \partial_{\bar{z}}^q A(z, \bar{z})|_{z=\hat{a}, \bar{z}=\hat{a}^\dagger}. \tag{2.13}
\]

Here, the symbol \( \dagger \cdot \cdot \cdot \dagger \) indicates that since the \( \hat{\pi}, \hat{\pi}^\dagger \) have been normal ordered, the \( \hat{a}, \hat{a}^\dagger \) are automatically anti-normal ordered. Now when such a function is projected onto the L.L.L., only the term with \( p = 0, q = 0 \) survives. Thus, in the L.L.L., the function \( A(\hat{z}, \hat{\bar{z}}) \rightarrow \dagger A(\hat{a}, \hat{a}^\dagger) \dagger \). It is instructive to express this anti-normal ordered operator in the coherent state basis. Let

\[
\dagger A(\hat{a}, \hat{a}^\dagger) \dagger \equiv \sum_{p,q} \frac{1}{p! q!} (\hat{a}^\dagger)^p (\hat{a})^q A_{pq} \tag{2.14}
\]

We now insert the resolution of the identity in the coherent state basis between the \( \hat{a} \)s and the \( \hat{a}^\dagger \)s in (2.14). We then obtain

\[
\dagger A(\hat{a}, \hat{a}^\dagger) \dagger = \int d^2\xi e^{-|\xi|^2} A(\xi, \bar{\xi}) \langle \bar{\xi}| \tag{2.15}
\]

where \( A(\xi, \bar{\xi}) \equiv \sum_{p,q} \frac{1}{p! q!} A_{pq}(\xi)^p(\bar{\xi})^q \). Thus the product of two individually anti-normal ordered operators is given by

\[
\dagger A(\hat{a}, \hat{a}^\dagger) \dagger \dagger B(\hat{a}, \hat{a}^\dagger) \dagger = \int d^2\xi e^{-|\xi|^2} A(\xi, \bar{\xi}) \ast B(\xi, \bar{\xi}) \langle \bar{\xi}| \tag{2.16}
\]

where the \( \ast \) product is given by

\[
A(\xi, \bar{\xi}) \ast B(\xi, \bar{\xi}) \equiv \sum_{n=0}^\infty \frac{(-1)^n}{n!} \partial_{\xi}^n A \partial_{\bar{\xi}}^n B. \tag{2.17}
\]

The star product is associative in that \( (A \ast B) \ast C = A \ast (B \ast C) \). This concept of the star product should be very familiar to the aficionados of non-commutative field theories. In fact, the field theory of fermions in the L.L.L. is an instance of such a field theory, where the non-commutativity is restricted to the spatial coordinates.
Upto this point, we have considered only the single particle Landau Hamiltonian. However, there are other contributions to the many particle Hamiltonian, which we shall now discuss.

The interaction between the electrons is the Coulomb interaction, whose contribution, as indicated in (1.18), splits naturally into a direct part and an exchange part. It is the exchange part which is instrumental in producing ferromagnetic behaviour in the Hall droplet.

Projected on to the L.L.L., the Coulomb term is written as:

$$H_c = \frac{1}{2} \int d^2z_1 \int d^2z_2 \ e^{-|z_1|^2-|z_2|^2} \ psi_{\alpha}(z_1)psi_{\beta}(z_2)V(\sqrt{\frac{2}{B}}|z_1-z_2|)psi_{\beta}(\bar{z}_2)psi_{\alpha}(\bar{z}_1)$$  \hspace{1cm} (2.18)

where $V$ is the Coulomb interaction and $psi_{\alpha}$ are the second quantised electron operators with spin $\alpha$, projected onto the L.L.L.. Let $|psi_{\alpha}\rangle$ be the abstract notation for the second quantised electron operator, with $\langle \vec{x}|psi_{\alpha}\rangle \equiv psi_{\alpha}(\vec{x})$ being the corresponding field operator. Then projection to the L.L.L. entails:

$$\langle \vec{x}|psi_{\alpha}\rangle \equiv psi_{\alpha}(\vec{x}) \rightarrow \langle \vec{x}| \sum_{l=0}^{\infty} |0, l\rangle \langle 0, l|psi_{\alpha}\rangle.$$  \hspace{1cm} (2.19)

We define $c_{\alpha}(l) \equiv \langle 0, l|psi\rangle$. This is the operator that destroys an electron, in the L.L.L., with index $l$ and spin $\alpha$. Then upon using (2.11), we have

$$psi_{\alpha}(\vec{x}) \rightarrow \sqrt{\frac{B}{2\pi}} \sum_{l=0}^{\infty} \frac{1}{\sqrt{l}} e^{-\frac{1}{2}|\vec{x}|^2} \vec{z}^l c_{\alpha}(l) \equiv e^{-\frac{1}{2}|\vec{z}|^2} psi_{\alpha}(\vec{z}).$$  \hspace{1cm} (2.20)

This is the $psi(\vec{z})_{\alpha}$ that appears in (2.18).

Apart from the Coulomb interaction, we require that the electrons be confined to a finite portion of the plane. This requires the introduction of a suitable confining potential.

Let us introduce a radially symmetric confining potential $v(r)$ that confines $N$ electrons in a droplet of radius $R$.

$$v_c(\vec{z}, \tilde{z}) = \gamma|\tilde{z}|^2,$$  \hspace{1cm} (2.21)

where $\gamma$ is the strength of the confining potential.
In view of the fact that the coordinate operators do not commute when projected to the L.L.L., the confining potential is really the Hamiltonian of a one dimensional harmonic oscillator, with the coordinate operators acting as canonically conjugate variables. The eigenenergies of $v$ are therefore given by

$$\epsilon_n = \gamma (n + \frac{1}{2}), \quad n = 0, 1, 2, \cdots \infty. \quad (2.22)$$

If we fill the available single particle states with $N$ particles to form a droplet with the lowest possible energy, the energy of this droplet would be

$$E_{tot} \equiv \sum_{n=0}^{N-1} \epsilon_n = \gamma \frac{N^2}{2}. \quad (2.23)$$

We know that the degeneracy in the L.L.L. is given by $\frac{B}{2\pi}$ particles per unit area. If the radius of the droplet is $R$, we have the relation

$$N = \left(\frac{B}{2\pi}\right)(\pi R^2) \Rightarrow N = \frac{BR^2}{2}. \quad (2.24)$$

This tells us that if the strength of the magnetic field is held fixed the area of the droplet scales with the number of electrons. *

The ground state of the many electron system is formed by filling up the single particle states of the confining potential sequentially with electrons. Furthermore, as is well known, the ground state of the system is ferromagnetic for large $N$. Thus the electrons in the ground state all have say, spin “up”. Let us now try to relate all this to the formulation

* From (2.23), it is clear that $E_{tot} \sim N^2$ as this is done. Thus, from (2.24), it seems that the energy increases quadratically with the size of the sample. However, for the energy to be a properly extensive quantity, we require that it should be directly proportional to the area (i.e. to $N$). The way to resolve this is to consider a confining potential whose strength is of order one (in $N$) in the bulk but is of order $N$ at the boundary. This form of the confining potential shall be seen to have a crucial significance in determining the leading order terms in the bosonic effective Lagrangian that we shall compute further on.
of the collective theory given in section I. The available sites ($K$ in number) correspond to the single particle eigenstates of the confining potential (labelled by integers). Furthermore, each state has two values of the spin index associated with it. Thus, in line with what we have said in section I, the bosonic collective fields are now expressed in terms of unitary operators belonging to the fundamental representation of $SU(2K)$, with $K \to \infty$.

Similarly, the operator $\hat{\rho}_0$ of section I is given in the present context by

$$\hat{\rho}_0 = \sum_{n=0}^{N-1} |n\rangle \langle n| \otimes \Omega$$

(2.25)

where $\Omega$ is a $2 \times 2$ Hermitean matrix incorporating the information about the spin of the many body ground state. Since the ground state of the system is ferromagnetic, $\Omega = P_+ \equiv \frac{1}{2}(I + \sigma_3)$. $P_+$ is the projector onto the spin “up” state.

We can express the operator $\hat{\rho}_0$ in the coherent state basis:

$$\rho_0(|z|^2) \equiv e^{-|z|^2} \langle \bar{z}| \hat{\rho}_0(\hat{a}, \hat{a}^\dagger)|z\rangle.$$  

(2.26)

For a large number of particles ($N \to \infty$),

$$\rho_0(|z|^2) = \theta(N - |z|^2) P_+.$$  

(2.27)

This was obtained in [9]. We have reproduced the derivation, for completeness, in Appendix B.

(2.27) immediately tells us that in order to convert this into a statement about the spatial extent of the system, we should require that $|z|^2 = \frac{N}{R^2}r^2$, whence the theta function would look like $\theta(N - \frac{N}{R^2}r^2) = \theta(R^2 - r^2)$. This in turn tells us that $z \sim \sqrt{N}$.

III. Effective Lagrangian for the Collective Excitations of the Quantum Hall Ferromagnet

In this section, we shall present a somewhat detailed derivation of the bosonic effective Lagrangian given in (1.19), for the case of the quantum Hall ferromagnet. In the sequel, we shall take all the operators to have been projected onto the L.L.L..
Since the operator $\hat{u}$ is unitary, we can write it as $\hat{u} \equiv e^{i\hat{A}}$, where $\hat{A}$ is a Hermitian operator. Let us also define the function $g(z, \bar{z}) \equiv e^{iA(z, \bar{z})}$. As we have noted in the previous section, $z \sim \sqrt{N}$. Thus a derivative with respect to $z$, acting on $u(z, \bar{z})$ will carry with it a factor of $N^{-\frac{1}{2}}$. For a large value of $N$, we thus have a natural parameter to expand $u$ in. In fact,

$$u(z, \bar{z}) = g(z, \bar{z}) - g_{\bar{z}, z} + O\left(\frac{1}{N^2}\right)$$  \hspace{1cm} (3.1)

where $g_{\bar{z}, z} \equiv i \int_0^1 d\alpha \ e^{-i\alpha A} \partial_{\bar{z}} \partial_z e^{i\alpha A}$.

Now, using the coherent state basis and the star product defined earlier, we have

$$A \equiv \text{tr} \ \rho_0 \ \hat{u}^\dagger i\partial_t \hat{u} = i \ \text{tr} \ \int d^2z \ \rho_0(|z|^2) \ u(z, \bar{z}, t)^\dagger \ * \ \partial_t u(z, \bar{z}, t)$$
$$= i \ \text{tr} \ \int d^2z \ \theta(N - |z|^2) \ P_+ \ u(z, \bar{z}, t)^\dagger \ * \ \partial_t u(z, \bar{z}, t),$$  \hspace{1cm} (3.2)

where the trace on the r.h.s. is over the spin indices. Using (3.1), we obtain,

$$A = i \int d^2z \ \theta(N - |z|^2) \ \text{tr} \ P_+ \left[ g^\dagger \partial_t g - \partial_z g^\dagger \partial_z g - \partial_t g_{\bar{z}, z} - (g_{\bar{z}, z})^\dagger \partial_t g + \cdots \right].$$  \hspace{1cm} (3.3)

After some rather straightforward manipulation of the above equation, using the relation $g^\dagger g_{\bar{z}, z} + \text{c.c.} = g^\dagger \partial_z g \ g^\dagger \partial_z g$, we get

$$A = A_{\text{bulk}} + A_{\text{edge}},$$  \hspace{1cm} (3.4)

where

$$A_{\text{bulk}} \equiv i \int d^2z \ \theta(N - |z|^2) \ \text{tr} \ P_+ \ g^\dagger \partial_t g$$  \hspace{1cm} (3.5)

and

$$A_{\text{edge}} \equiv \frac{i}{2} \int d^2z \ \delta(N - |z|^2) \ \text{tr} \ P_+ \left( g^\dagger (\partial_{\bar{z}} - z \partial_z) g \ g^\dagger \partial_t g + [g^\dagger z \partial_z g, g^\dagger \partial_t g] \right).$$  \hspace{1cm} (3.6)

Equation (3.5) is a leading contribution to the effective Lagrangian for the bulk and (3.6), a leading contribution to the edge dynamics.
The second term in (1.19) is the contribution of the confining potential $v_c$ to the effective Lagrangian. In the coherent state representation, it is written as:

$$B \equiv -\text{tr} \hat{\rho}_0 \hat{u}^\dagger v_c \hat{u} = - \text{tr} \int d^2z \ \theta(N - |z|^2) \ P_+ \ (u^\dagger \ast v_c \ast u). \quad (3.7)$$

Using (3.1) and the definition of the star product, we get, after a simple calculation,

$$B = \int d^2z \ \theta(N - |z|^2)[-v_c + \text{tr} P_+(\partial_z v_c \hat{g}^\dagger \partial_z g - \partial_z v_c \hat{g}^\dagger \partial_z g)]$$

$$+ \int d^2z \ \delta(N - |z|^2) v_c \text{tr} P_+ g^\dagger \partial_z g \ g^\dagger \partial_z g. \quad (3.8)$$

The first term in (3.8) is the bulk term and the second, the edge contribution.

Let us now look at the contribution of the exchange part of the Coulomb interaction to the effective Lagrangian. In equation (2.18), we have already given the form of the Coulomb interaction, projected to the L.L.L.. We shall start from this expression and obtain the exchange part from it. Before doing so, let us establish some notation. Let $k \equiv \frac{1}{\sqrt{2B}}(k_x + ik_y)$ and $\bar{k}$ be its complex conjugate. Let $V(k, \bar{k})$ be the Fourier transform of the Coulomb interaction. In this notation,

$$V(|\vec{x}_1 - \vec{x}_2|) = \int d\vec{k} \ V(k, \bar{k}) \ e^{-i(k \vec{z}_1 + \bar{k} \vec{z}_1)} \ e^{i(k \vec{z}_2 + \bar{k} \vec{z}_2)}. \quad (3.9)$$

Further, adopting the notation introduced in equation (2.19), we obtain

$$H_C = \frac{1}{2} \int d\vec{k} \ V(k, \bar{k}) \ \left< \psi_{\alpha} \ e^{-i\vec{\chi}} \ |\psi_{\alpha}\right> \left< \psi_{\beta} \ e^{i\vec{\chi}} \ |\psi_{\beta}\right> \quad (3.10)$$

where $e^{i\vec{\chi}} \equiv e^{i\vec{k} \hat{a}} e^{i\bar{k} \hat{a}^\dagger}$ and $e^{-i\vec{\chi}}$ is its Hermitian conjugate. Let us further denote $e^{i\chi(1)} \equiv e^{i\bar{k} \vec{z}_1} e^{i\vec{k} \vec{z}_1}$ and similarly for $e^{i\chi(2)}$.

In the notation of (1.19),

$$h^{(2)\beta_1 \beta_2}_{\alpha_1 \alpha_2} = \frac{1}{2} \int d\vec{k} \ V(k, \bar{k}) \ (e^{-i\vec{\chi}})^{\alpha_1 \beta_1} (e^{i\vec{\chi}})^{\alpha_2 \beta_2}. \quad (3.11)$$

Hence, the expectation value of the Coulomb Hamiltonian in the state $|u\rangle$ is

$$\langle H_C \rangle = \frac{1}{2} \int d\vec{k} \ V(k, \bar{k}) \ (\hat{u} \hat{\rho}_0 \hat{u}^\dagger)^{\beta_1 \alpha_1} (\hat{u} \hat{\rho}_0 \hat{u}^\dagger)^{\beta_2 \alpha_2} \left[ (e^{-i\vec{\chi}})^{\alpha_1 \beta_1} (e^{i\vec{\chi}})^{\alpha_2 \beta_2} - (e^{-i\vec{\chi}})^{\alpha_1 \beta_2} (e^{i\vec{\chi}})^{\alpha_2 \beta_1} \right]. \quad (3.12)$$
The first term is the direct contribution of the Coulomb term and the second, the exchange contribution.

Let us first focus on the exchange contribution. From (3.12), it is given by

\[ C \equiv L_{eff}^{(ex)} = \frac{1}{2} \int d\vec{k} \, V(k, \bar{k}) \, \text{tr} \left( \hat{\rho}_0 \hat{u}^\dagger e^{-i\chi} \hat{u} \hat{\rho}_0 \hat{u}^\dagger e^{i\chi} \hat{u} \right). \]  

(3.13)

Then, introducing the coherent state basis and the star product, we write the contribution to the effective bosonic Lagrangian, of the exchange part as

\[ C = \frac{1}{2} \int d\vec{k} \, V(k, \bar{k}) \int d^2 z_1 \, d^2 z_2 e^{-|z_1|^2} e^{-|z_2|^2} \text{tr} \left[ \hat{\rho}_0 |z_1\rangle (u^\dagger \ast e^{-i\chi(1)} \ast u)(z_1, z_1) \langle z_1| \hat{\rho}_0 |z_2\rangle (u^\dagger \ast e^{i\chi(2)} \ast u)(z_2, z_2) \langle z_2| \right]. \]  

(3.14)

Thus,

\[ C = \frac{1}{2} \int d\vec{k} \, V(k, \bar{k}) \int d^2 z_1 \, d^2 z_2 e^{-|z_1|^2} e^{-|z_2|^2} \langle z_1| \hat{\rho}_0 |z_2\rangle \text{tr} P_+ (u^\dagger \ast e^{-i\chi(1)} \ast u)(z_1, z_1) P_+ (u^\dagger \ast e^{i\chi(2)} \ast u)(z_2, z_2). \]  

(3.15)

The spin projector, \( P_+ \), has been explicitly written in (3.15).

After a somewhat lengthy calculation, presented in detail in Appendix C and D, we obtain

\[ C = -\frac{1}{2l} \sqrt{\frac{\pi}{2}} \int d^2 z \, \theta(N - |z|^2) [\text{tr} \, P_+ g^\dagger \partial_z g^\dagger \partial_z g - \text{tr} \, P_+ g^\dagger \partial_z g \text{tr} \, P_+ g^\dagger \partial_z g] + \frac{1}{2l} \sqrt{\frac{\pi}{2}} \int d^2 z \, \delta(N - |z|^2) \text{tr} \, P_+ g^\dagger (z \partial_z - z \partial_z) g. \]  

(3.16)

Here \( l = \frac{1}{\sqrt{B}} \) is the magnetic length. The first term is the bulk contribution and the second, that from the edge.

The contribution of the direct term is given by (see (3.12)),

\[ D \equiv L_{eff}^{(dir)} = -\frac{1}{2} \int d^2 k \, V(k, \bar{k}) \, \text{tr} (\hat{\rho}_0 \hat{u}^\dagger e^{-i\chi} \hat{u}) \, \text{tr} (\hat{\rho}_0 \hat{u}^\dagger e^{i\chi} \hat{u}). \]  

(3.17)
Upon introducing the coherent states, we obtain

\[ D = -\frac{1}{2} \int d^2k \, V(k, \bar{k}) \int d^2z_1 d^2z_2 \, \theta(N - |z_1|^2) \, \theta(N - |z_2|^2) \]

\[ \text{tr}P_+(u^\dagger * e^{-i\chi(1)} * u) \, \text{tr}P_+(u^\dagger * e^{i\chi(2)} * u). \]  

(3.18)

Now, by using the definition of the star product and the expansion of \( u \) in terms of \( g \), we may, upon suitable integration by parts, obtain

\[ \int d^2z_1 \, \theta(N - |z_1|^2) \, \text{tr}P_+(u^\dagger * e^{-i\chi(1)} * u) \approx \int d^2z_1 \, \theta(N - |z_1|^2) \, \text{tr}P_+(1 - [g^\dagger \partial z g, g^\dagger \partial_z g]) e^{-i\chi(1)} \]

\[ + \int d^2z_1 \, \delta(N - |z_1|^2) \, \text{tr}P_+(g^\dagger (\bar{z} \partial \bar{z} - z \partial z) g) e^{-i\chi(1)}. \]  

(3.19)

We drop the term that does not involve \( g \) from (3.20) as it is the Coulomb contribution to the fermionic ground state and not to the collective excitations. Further, we can easily convince ourselves that \( \delta \rho(z, \bar{z}) \), which is the deviation of the mean local density from its constant ground state value, is given by

\[ \delta \rho(z, \bar{z}) = -\theta(N - |z|^2) \, \text{tr}P_+[g^\dagger \partial z g, g^\dagger \partial_z g] + \delta(N - |z_1|^2) \, \text{tr}P_+(g^\dagger (\bar{z} \partial \bar{z} - z \partial z) g). \]  

(3.20)

Thus, using (3.19) and (3.20), we can rewrite (3.18) as

\[ D = -\frac{1}{2} \int d\bar{x}_1 d\bar{x}_2 \, V(|\bar{x}_1 - \bar{x}_2|) \, \delta \rho(\bar{x}_1) \, \delta \rho(\bar{x}_1). \]  

(3.21)

Interestingly enough, the deviation from the mean ground state density picks up contributions both in the bulk and at the edge.

**IV. Scaling of the terms in the bosonic effective Lagrangian with \( N \)**

In the preceding section, we have obtained the leading contributions to the bosonic effective Lagrangian that emerge from the underlying microscopic fermionic action. We have considered the number of particles in the system, \( N \gg 1 \) and have developed a systematic derivative expansion scheme, with \( 1/\sqrt{N} \) as the small parameter, to identify
the leading contributions. In this section, we shall study how the various terms scale with 
$N$ and rewrite the various terms as integrals over real spatial coordinates. In equation
(3.5), we note that in order to write the theta function in terms of spatial coordinates,
we have to take $z = \frac{\sqrt{N}}{R} \eta$, where $R$ is the radius of the droplet given by $R = \sqrt{\frac{2N}{B}}$
and $\eta \equiv x + iy$. Then we get $\theta(N - |z|^2) = \theta(R^2 - r^2)$. This just means that the bulk
contribution has support inside of the droplet. The measure $d^2z$ then becomes $\frac{B}{2\pi} dx \, dy$.
Thus, we have, from (3.5),
\[ A_{\text{bulk}} = i \frac{B}{2\pi} \int d\vec{x} \, \theta(R^2 - r^2) \, \text{tr} P_+ \, g^\dagger \, \partial_t g. \quad (4.1) \]
This term is proportional to the area of the droplet and hence is proportional to $N$. Again,
from (3.6), we have,
\[ A_{\text{edge}} = \frac{1}{8\pi} \int_0^{2\pi} d\theta \, \text{tr} P_+ \left( \{g^\dagger i \partial_\theta g, g^\dagger i \partial_t g\} + [g^\dagger r \partial_r g, g^\dagger i \partial_\theta g] \right)_{r=R}. \quad (4.2) \]
This term is of $O(1)$ in $N$ as expected, as it is a boundary term and as such should be
independent of the number of particles in the droplet.

The contributions due to the confining potential are given in (3.8). We have argued
previously that for a proper thermodynamic limit to exist, that is, for the energy of the
droplet to scale as $N$, the confining potential should be of order one (in $N$) in the bulk.
Alternatively, $\partial_z v \sim 1/\sqrt{N}$ in the bulk. Thus, we see that the second term in $B$ is an order
one contribution to the bulk effective Lagrangian. Thus it is a subleading contribution
(compared to $O(N)$ contributions) and may be dropped. Again,
\[ B_{\text{edge}} = -\frac{\omega}{4\pi} \int_0^{2\pi} d\theta \, \text{tr} P_+ \left( - (g^\dagger r \partial_r g)^2 + [g^\dagger r \partial_r g, g^\dagger i \partial_\theta g] + (g^\dagger i \partial_\theta g)^2 \right)_{r=R} \quad (4.3) \]
where $\omega \equiv \frac{\sqrt{\langle r=R \rangle}}{2N}$, and is of order one in $N$.

Let us now turn our attention to the contributions of the Coulomb term.
\[ C_{\text{bulk}} = - \frac{1}{2l} \frac{1}{\sqrt{2\pi}} \int d\vec{x} \, \theta(R^2 - r^2) \left[ \text{tr} P_+ \, g^\dagger \partial_\eta g \, g^\dagger \partial_\eta g - \text{tr} P_+ \, g^\dagger \partial_\eta g \, \text{tr} P_+ \, g^\dagger \partial_\eta g \right]. \quad (4.4) \]
Similarly, using \( \eta \partial_\eta - \eta \partial_\eta = i \partial_\theta \), where \( \theta \) is the plane polar angle, we get

\[
C_{\text{edge}} = \frac{1}{8l} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} d\theta \; \text{tr}\; P_+ g^\dagger i \partial_\theta g |_{r=R}. 
\]

(4.5)

The contribution in (4.4) is proportional to the area of the droplet and is thus of order \( N \), whilst that in (4.5) is of order one in \( N \).

Similar arguments can be provided for the contributions from the direct part of the Coulomb interaction.

The leading contributions to the effective Lagrangian, governing the collective excitations in the bulk, are given by

\[
L_{\text{eff}}^{(\text{bulk})} = \int d\vec{x} \left[ \frac{B}{2\pi} \text{tr}\; P_+ g^\dagger i \partial_\theta g - \frac{1}{2l} \frac{1}{\sqrt{2\pi}} \left( \text{tr}\; P_+ g^\dagger \partial_\eta g \; g^\dagger \partial_\eta g - \text{tr}\; P_+ g^\dagger \partial_\eta g \; \text{tr}\; P_+ g^\dagger \partial_\eta g \right) \right] 
\]

\[+ \frac{1}{2\pi^2} \int_D d\vec{x}_1 \int_D d\vec{x}_2 \; V(|\vec{x}_1 - \vec{x}_2|) \left( \text{tr}\; P_+ [g^\dagger \partial_\eta g, g^\dagger \partial_\eta g] \right) \left( \text{tr}\; P_+ [g^\dagger \partial_\eta g, g^\dagger \partial_\eta g] \right). \]

(4.6)

The subscript \( D \) indicates that the integral is over all \( r < R \).

Similarly, we obtain the effective Lagrangian governing the collective dynamics at the edge of the droplet

\[
L_{\text{eff}}^{(\text{edge})} = \int_0^{2\pi} d\theta \left[ \frac{1}{8\pi} \text{tr}\; P_+ \left\{ \{ g^\dagger i \partial_\theta g, g^\dagger i \partial_\theta g \} + [ g^\dagger r \partial_\eta g, g^\dagger i \partial_\theta g ] \right\} |_{r=R} 
\]

\[+ \frac{\omega}{4\pi} \text{tr}\; P_+ \left( - (g^\dagger r \partial_\eta g)^2 + [ g^\dagger r \partial_\eta g, g^\dagger i \partial_\theta g ] + (g^\dagger i \partial_\theta g)^2 \right) |_{r=R} 
\]

\[- \frac{1}{2l} \frac{1}{\sqrt{2\pi}} \text{tr}\; P_+ g^\dagger i \partial_\theta g |_{r=R} \right] 
\]

\[- \frac{1}{8\pi} \int_0^{2\pi} d\theta_1 d\theta_2 \; V(|\vec{x}_1 - \vec{x}_2|) \left( \text{tr}\; P_+ g^\dagger i \partial_\theta_1 g \right) \left( \text{tr}\; P_+ g^\dagger i \partial_\theta_2 g \right) |_{r_1, r_2=R}. \]

(4.7)

At this point, we note that if \( \hat{\rho}_0 = \sum_{n=0}^{\infty} |n\rangle \langle n| \; P_+ \), all the available single particle states in the L.L.L. would have been filled and the effective Lagrangian would have been entirely a bulk effective Lagrangian (as the droplet would in this case fill the entire plane).

In fact, in this case,

\[
\rho_0(|z|^2) \equiv e^{-|z|^2} \langle \bar{z} | \hat{\rho}_0 | z \rangle = P_. \]

(4.8)
Thus, the effective Lagrangian would be given by (4.6) with the support of the integral over \(x, y\) extending over the entire droplet. Namely,

\[
L_{\text{eff}}^{(\text{bulk})} = \int d\vec{x} \left[ \frac{B}{2\pi} \text{tr} P_g g^\dagger i\partial_t g - \frac{1}{2l} \frac{1}{\sqrt{2\pi}} (\text{tr} P_+ g^\dagger \partial_\eta g \ g^\dagger \partial_\eta g - \text{tr} P_+ g^\dagger \partial_\eta g \ \text{tr} P_+ g^\dagger \partial_\eta g) \right] \\
+ \frac{1}{2\pi^2} \int_D d\vec{x}_1 \int_D d\vec{x}_2 \ v_c(|\vec{x}_1 - \vec{x}_2|) \ (\text{tr} P_+ [g^\dagger \partial_\eta_1 g, g^\dagger \partial_\eta_1 g]) \ (\text{tr} P_+ [g^\dagger \partial_\eta_2 g, g^\dagger \partial_\eta_2 g])
\]

(4.9)

This is the effective Lagrangian that has been discussed extensively in the literature [1],[2],[5], in the context of ferromagnetic magnons in the Hall ferromagnet.

**V. Simplifying the effective Lagrangian**

In this section, we shall look closely at the various contributions to the bulk and the edge effective Lagrangians and comment on their relative importance. We shall look separately at the the bulk and the edge contributions.

In terms of the familiar Euler angles, we can parametrise \(g\) as

\[
g \equiv e^{-i \frac{\phi}{2} \sigma_3} \ e^{-i \frac{\theta}{2} \sigma_2} \ e^{-i \frac{\chi}{2} \sigma_3}
\]

(5.1)

with

\[
g \sigma_3 g^\dagger \equiv \hat{m} \cdot \vec{\sigma} , \ \hat{m}^2 = 1
\]

(5.2)

where

\[
\hat{m} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).
\]

Alternatively, we may parametrise \(g\) in a more conventional manner as

\[
g \equiv e^{i \vec{A} \cdot \vec{\sigma}}
\]

\[
g^\dagger i \partial_\mu g = -\vec{\sigma} \cdot (\partial_\mu \vec{A} + \vec{A} \times \partial_\mu \vec{A}).
\]

(5.3)

where we have retained upto the quadratic in \(\vec{A}\). Furthermore, we shall assume the following ansatz for \(\vec{A}\) [2],

\[
\vec{A} = \frac{1}{2}(m_2, -m_1, 0)
\]

(5.4)
and consider $m_3 \simeq 1$, with $|\vec{m}_T| \ll 1$, where $\vec{m}_T \equiv (m_1, m_2, 0)$.

Then the various contributions to the bulk effective Lagrangian may be written as:

$$A_{\text{bulk}} = -\frac{1}{2} B \int_{D} d\vec{x} (1 - \cos \theta) \partial_t \phi$$

$$= \frac{1}{2} B \int_{D} d\vec{x} \int_{0}^{1} d\lambda \, \hat{m}_\lambda(\vec{x}, t) \cdot [\partial_t \hat{m}_\lambda(\vec{x}, t) \times \partial_\lambda \hat{m}_\lambda(\vec{x}, t)]$$  \hspace{1cm} (5.5)

where $\hat{m}_\lambda \equiv \hat{m}(\lambda \theta, \phi)$. We require $\hat{m}_{\lambda=0} = \hat{e}_3$ and $\hat{m}_{\lambda=1} = \hat{m}$. Explicitly, we may choose

$$\hat{m}_\lambda(\theta, \phi) = (\sin \lambda \theta \sin \phi, \sin \lambda \theta \cos \phi, \cos \lambda \theta).$$

The second line in (5.5) is the well-known geometric phase generic to quantum ferromagnets and is proportional to the area of the droplet. The suffix $D$ indicates that the integral has support over the entire area of the droplet.

As we have argued before, we may drop $B_{\text{bulk}}$.

$$C_{\text{bulk}} = -\frac{1}{32l \sqrt{2\pi}} \int_{D} d\vec{x} (\partial_\alpha \hat{m})^2 - \frac{1}{4l} \sqrt{\frac{\pi}{2}} \int_{D} d\vec{x} \rho_p(\vec{x}, t)$$  \hspace{1cm} (5.6)

where $\rho_p \equiv -\frac{1}{8\pi} \epsilon_{\alpha \beta} \hat{m} \cdot (\partial_\alpha \hat{m} \times \partial_\beta \hat{m})$ is the well known Pontryagin index density. The integral of $\rho_p$ over all space gives an integer, the Pontryagin index, which is a topological quantity.

$$D_{\text{bulk}} = -\frac{1}{2} \int_{D} d\vec{x}_1 d\vec{x}_2 \rho_p(\vec{x}_1, t) V(|\vec{x}_1 - \vec{x}_2|) \rho_p(\vec{x}_2, t)$$  \hspace{1cm} (5.7)

Thus, the bulk effective Lagrangian is given by:

$$L^{(\text{bulk})}_{\text{eff}} = \int_{D} d\vec{x} [\frac{B}{4\pi} \int_{0}^{1} d\lambda \hat{m}_\lambda \cdot [\partial_t \hat{m}_\lambda \times \partial_\lambda \hat{m}_\lambda] - \frac{1}{32l \sqrt{2\pi}} (\partial_\alpha \hat{m})^2 - \frac{1}{4l} \sqrt{\frac{\pi}{2}} \rho_p]$$

$$- \frac{1}{2} \int_{D} d\vec{x}_1 d\vec{x}_2 \rho_p(\vec{x}_1) V(|\vec{x}_1 - \vec{x}_2|) \rho_p(\vec{x}_2).$$  \hspace{1cm} (5.8)

This Lagrangian has been obtained in a variety of ways in the literature. It governs the dynamics of the ferromagnetic magnons $[1],[2],[5]$. 

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Let us now look at the effective Lagrangian governing the edge excitations. To simplify matters, we shall focus on those excitations that satisfy the condition \( \partial_r \hat{m}(r, \theta, t)|_{r=R} = 0 \).

Then, using (5.3), (5.4),

\[
A_{\text{edge}} \simeq \frac{1}{16\pi} \int_0^{2\pi} d\theta [\partial_t \hat{m} \cdot \partial_\theta \hat{m}]_{r=R}.
\]

Further,

\[
B_{\text{edge}} \simeq -\frac{\omega}{16\pi} \int_0^{2\pi} d\theta [(\partial_\theta \hat{m})^2]_{r=R}
\]

where \( \omega \equiv \frac{\nu \omega_{(r=R)}}{2N} \).

Interestingly enough, to the leading order, the contributions of the Coulomb interaction to the Lagrangian for the edge, are zero.

Thus,

\[
\mathcal{L}^{(\text{edge})}_{\text{eff}} = \frac{1}{16\pi} \int_0^{2\pi} d\theta [\partial_t \hat{m} \cdot \partial_\theta \hat{m} - \omega (\partial_\theta \hat{m})^2]_{r=R}.
\]

These edge excitations are obviously chiral in nature.

We see that even though the bulk modes (ferromagnetic magnons) and the edge modes (chiral excitations) are both gapless, they owe their dynamics to completely different sources. For the bulk modes, which are the Goldstone modes corresponding to a spontaneous breaking of the global spin symmetry \( SU(2) \rightarrow u(1) \), the exchange part of the Coulomb interaction is not merely crucial to their dynamics, but is truly their \textit{raison d’\'etre}.

On the other hand, the chiral edge excitations are somewhat generic to confined Hall fluids. Let us consider a specific instance. We shall consider a conventional \( \nu = 1 \) H all droplet where the spin degrees of freedom are completely frozen by the Zeeman term. In this case, \( \rho(|z|^2) = \theta(N - |z|^2) \) and \( g \) is simply a \( u(1) \) phase. The edge contribution to the effective Lagrangian can be easily shown to be

\[
\mathcal{L}^{(\text{edge})}_{\text{eff}} = \frac{1}{2\pi} \int_0^{2\pi} d\theta [\partial_\theta \phi \partial_t \phi - \omega (\partial_\theta \phi)^2],
\]

where \( g = e^{i\phi} \).
This is the familiar Lagrangian obtained in the literature [10] for a case where the Coulomb interaction is known to be unimportant. The corresponding bulk contribution trivially vanishes.

This should convince us that the edge excitations are extremely ubiquitous and as such do not owe their existence to the Coulomb interaction. Thus, to the leading order, the edge excitations of a quantum Hall ferromagnet are qualitatively similar to those for the standard $\nu = 1$ integer Hall droplet.

VI. Conclusions

In this article, we have utilised the bosonisation technique introduced in [8] to discuss the collective excitations of a finite-sized Hall sample where the Landé $g$ factor is vanishingly small. For a large but finite sample, $(\text{Area} \gg \frac{1}{B})$, we know that these excitations are ferromagnetic magnons whose dynamics is governed by the exchange part of the Coulomb interaction between the electrons.

We have shown that the effective Lagrangian for the collective excitations splits naturally into two pieces, one having support in the bulk of the droplet and the other at the edge.

The bulk effective Lagrangian coincides with that computed for an infinite sample [1],[2],[5]. The edge excitations, which are chiral in nature, are to the leading order, unaffected by the Coulomb interaction between the electrons. The mere fact that the electrons are confined to a droplet suffices to produce chiral edge excitations, which are qualitatively similar to those for the conventional $\nu = 1$ Hall droplet.

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Let $\mathcal{D}(u)$ denote the totally antisymmetric irreducible representation of $SU(K)$, of dimensionality $\binom{K}{N}$. Then

$$D_{A0} = \frac{1}{\sqrt{N!}} \varepsilon_{\beta_1 \cdots \beta_N} u_{\alpha_1 \beta_1} \cdots u_{\alpha_N \beta_N} \quad (A.1)$$

where the suffix 0 refers to the sequence 1, 1, $\cdots$ N and A runs over $\binom{K}{N}$ values. Further, let

$$|A\rangle \equiv |\alpha_1, \alpha_2 \cdots \alpha_N\rangle \quad (A.2)$$

where $|\alpha_1, \alpha_2 \cdots \alpha_N\rangle$ has been given in (1.3). Thus, from (1.3), (1.6), (A.1) and (A.2), we get

$$|u\rangle = \sum_A |A\rangle D_{A0}(u) \quad (A.3)$$

and

$$\langle u| = \sum_A \langle A| D^*_{A0}(u). \quad (A.4)$$

We further know that

$$\int du \, D_{A_1B_1}(u) \, D^*_{A_2B_2}(u) = \delta_{A_1A_2} \, \delta_{B_1B_2}. \quad (A.5)$$

Again,

$$\langle u_1|u_2\rangle = \sum_{A_1A_2} \langle A_1|A_2\rangle \, D^*_{A_10}(u_1) \, D_{A_20}(u_2)$$

$$= \sum_A D^*_{A0}(u_1) \, D_{A0}(u_2)$$

$$= D_{00}(u_1^{-1}u_2). \quad (A.6)$$

Thus,

$$P^2 = \int du_1du_2 \, |u_1\rangle \langle u_1|u_2\rangle \langle u_2|$$

$$= \int du_1du_2 \, |u_1\rangle \, D_{00}(u_1^{-1}u_2) \, \langle u_2|. \quad (A.7)$$
Let
\[ u_3 \equiv u_1^{-1} u_2. \]  
(A.8)

Then, as \( du_1 = du_3 \), (Haar Measure) and \( \langle u_2 \rangle = \langle u_1 u_3 \rangle \), we get
\[
P^2 = \int du_1 du_3 \langle u_1 \rangle D_{00}(u_3) \langle u_1 u_3 \rangle
= \int du_1 du_3 \langle u_1 \rangle D_{00}(u_3) \sum_A \langle A \rangle D^*_{A0}(u_1 u_3)
= \int du_1 du_3 \langle u_1 \rangle D_{00}(u_3) \sum_{A_1 A_2} \langle A_1 \rangle D^*_{A_1A_2}(u_1) D_{A_20}(u_3)
= \int du_1 \langle u_1 \rangle \sum_A \langle A \rangle D^*_{A0}(u_1)
= P. \]  
(A.9)

Appendix B

Let
\[ \hat{\rho}_0 \equiv \sum_{n=0}^{N-1} |n\rangle \langle n| \]  
(B.1)

Further, we define
\[ e^{-|z|^2} \langle \bar{z} | \hat{\rho}_0 | z \rangle \equiv \rho_0(|z|^2). \]  
(B.2)

Thus,
\[ \rho_0(|z|^2) = e^{-|z|^2} \sum_{n=0}^{N-1} \langle \bar{z} | n \rangle \langle n | z \rangle
= e^{-|z|^2} \sum_{n=0}^{N-1} \frac{(|z|^2)^n}{n!}. \]  
(B.3)

Now, let
\[ F(x, N) \equiv \sum_{n=0}^{N-1} \frac{x^n}{n!} e^{-x}. \]  
(B.4)

Then,
\[ \partial_x F(x, N) = -e^{-x} \frac{x^{N-1}}{(N-1)!} \simeq -e^{-x} \frac{x^N}{N!}. \]  
(B.5)
for large \( N \). Using Striling’s formula

\[
N! \approx \sqrt{2\pi} \ e^{-N} \ (N)^{N+\frac{1}{2}} \tag{B.6}
\]

and the saddle-point method

\[
e^{-x} \ x^N = e^{-(x-N \ln x)} \simeq e^{-(N-N \ln N)} \ e^{-\frac{(x-N)^2}{2N}} \tag{B.7}
\]

we get

\[
-\partial_x F(x, N) \simeq \frac{1}{\sqrt{2\pi N}} \ e^{-\frac{(x-N)^2}{2N}}. \tag{B.8}
\]

Therefore,

\[
-\partial_{|z|^2} \rho_0(|z|^2) \simeq \frac{1}{\sqrt{2\pi N}} \ e^{-\frac{|z|^2-N}{2N}}
= \frac{1}{\sqrt{2\pi N}} \ e^{-N(|z|^2-N)}
\]

\[
\simeq \frac{1}{N} \ \delta\left(\frac{|z|^2}{N} - 1\right)
= \delta\left(|z|^2 - N\right) \tag{B.9}
\]

where \( \sigma \equiv \frac{|z|^2}{N} \). Thus,

\[
\rho_0(|z|^2) \approx \theta(N - |z|^2). \tag{B.10}
\]

**Appendix C**

From (3.15), the exchange part of the Coulomb interaction is

\[
C = \frac{1}{2} \int d\vec{k} V(|\vec{k}|) \int d^2z_1 \ d^2z_2 e^{-|z_1|^2-|z_2|^2} |\langle \bar{z}_1 | \bar{\rho} | z_2 \rangle|^2 \times
\]

\[
\times \ \text{tr} \left[ P_+ \left( u^\dagger e^{-i\chi(1)} * u \right) (z_1, \bar{z}_1) \ P_+ \left( u^\dagger e^{i\chi(2)} * u \right) (z_2, \bar{z}_2) \right]. \tag{C.1}
\]

But,

\[
(u^\dagger e^{-i\chi(1)} * u)_{\beta\gamma} (z_1, \bar{z}_1) = \delta_{\beta\gamma} e^{-i\chi(1)} - (\partial_{\bar{z}_1} u^\dagger u)_{\beta\gamma} (\partial_{z_1} e^{-i\chi(1)}) - (u^\dagger \partial_{z_1} u)_{\beta\gamma} (\partial_{\bar{z}_1} e^{-i\chi(1)})
+ \frac{1}{2} (\partial^2_{z_1} u^\dagger u)_{\beta\gamma} (\partial^2_{z_1} e^{-i\chi(1)}) + (\partial_{\bar{z}_1} u^\dagger \partial_{z_1} u)_{\beta\gamma} (\partial_{\bar{z}_1} \partial_{z_1} e^{-i\chi(1)}) + \frac{1}{2} (u^\dagger \partial^2_{z_1} u) (\partial^2_{\bar{z}_1} e^{-i\chi(1)}) \tag{C.2}
\]
Similarly for \((u^\dagger e^{ix(2)} * u)(z_2, \bar{z}_2)\).

Thus,

\[
C = \frac{1}{2} \int d^2z_1 \ d^2z_2 e^{-|z_1|^2 - |z_2|^2} |\langle \bar{z}_1 | \hat{\rho} | z_2 \rangle|^2 \left[ \text{tr} P_+ V - \text{tr} P_+ (u^\dagger \partial_{z_1} u) \partial_{\zbar_1} V - h.c. \right. \\
- \text{tr} P_+ (u^\dagger \partial_{z_2} u) \partial_{\zbar_2} V - h.c. + \text{tr} P_+ (u^\dagger \partial_{z_1} u) P_+ (u^\dagger \partial_{z_2} u) + h.c. \right.
\\
\left. + (u^\dagger \partial_{z_1} u) P_+ (\partial_{\zbar_2} u^\dagger u) \partial_{\zbar_1} \partial_{\zbar_2} V + h.c. + \text{tr} P_+ (\partial_{\zbar_1} u^\dagger u) \partial_{\zbar_1} \partial_{\zbar_2} V + 1 \to 2 + \cdots \right](C.3)
\]

Let

\[
z_1 \equiv \sqrt{N} z + \frac{1}{2} \bar{z}; \ z_2 \equiv \sqrt{N} \bar{z} - \frac{1}{2} \bar{z}
\]

and

\[
u(\sqrt{N} z, \sqrt{N} \bar{z}) \equiv g(Z, \bar{Z}). \quad (C.4)
\]

Then,

\[
u(z_1, \bar{z}_1) = g(Z, \bar{Z}) + \frac{1}{2\sqrt{N}} \zbar \partial_Z g(Z, \bar{Z}) + \frac{1}{2\sqrt{N}} \bar{z} \partial_{\bar{Z}} g(Z, \bar{Z}) + \frac{1}{4N} |z|^2 \partial_Z \partial_{\bar{Z}} g(Z, \bar{Z}) + \cdots . \quad (C.6)
\]

Again,

\[
\partial_{z_1} u(z_1, \bar{z}_1) = \frac{1}{\sqrt{N}} \partial_Z \left[ g(Z, \bar{Z}) + \frac{1}{2\sqrt{N}} \bar{z} \partial_Z g(Z, \bar{Z}) + \frac{1}{2\sqrt{N}} \bar{z} \partial_{\bar{Z}} g(Z, \bar{Z}) + \cdots \right] \quad (C.7)
\]

\[
u(z_2, \bar{z}_2) = g(Z, \bar{Z}) - \frac{1}{2\sqrt{N}} \zbar \partial_Z g(Z, \bar{Z}) - \frac{1}{2\sqrt{N}} \bar{z} \partial_{\bar{Z}} g(Z, \bar{Z}) + \frac{1}{4N} |z|^2 \partial_Z \partial_{\bar{Z}} g(Z, \bar{Z}) + \cdots \quad (C.8)
\]

\[
\partial_{z_2} u(z_2, \bar{z}_2) = \frac{1}{\sqrt{N}} \partial_Z \left[ g(Z, \bar{Z}) - \frac{1}{2\sqrt{N}} \bar{z} \partial_Z g(Z, \bar{Z}) - \frac{1}{2\sqrt{N}} \bar{z} \partial_{\bar{Z}} g(Z, \bar{Z}) + \cdots \right] \quad (C.9)
\]

\[
\partial_{z_1}^2 u(z_1, \bar{z}_1) = \partial_{z_2}^2 u(z_2, \bar{z}_2) = \frac{1}{N} \partial_{\bar{Z}}^2 g(Z, \bar{Z}) \cdots . \quad (C.10)
\]

Further,

\[
\partial_{z_1} V = -\partial_{z_2} V = \partial_2 V \quad (C.11)
\]

and thus,

\[
z \partial_{z} V = \bar{z} \partial_{\zbar} V = -\frac{1}{2} V. \quad (C.12)
\]
Again, from Appendix D, we get
\[ e^{-|z_1|^2-|z_2|^2}|\langle \bar{z}_1 | \hat{\rho} | z_2 \rangle|^2 = e^{-|z|^2} \left[ e^{\frac{1}{2\sqrt{N}}(z \bar{\partial}_z - \bar{z} \partial_z)} \theta(1 - |z|^2) \right] \left[ e^{-\frac{1}{2\sqrt{N}}(z \bar{\partial}_z - \bar{z} \partial_z)} \theta(1 - |z|^2) \right]. \]

(D.13)

Furthermore, \( d^2 z_1 \ d^2 z_2 = N d^2 Z \ d^2 z. \)

Therefore, using the above results, we obtain
\[ C = \frac{N}{2} \int d^2 z e^{-|z|^2} \int d^2 Z \left[ e^{\frac{1}{2\sqrt{N}}(z \bar{\partial}_z - \bar{z} \partial_z)} \theta(1 - |z|^2) \right] \left[ e^{-\frac{1}{2\sqrt{N}}(z \bar{\partial}_z - \bar{z} \partial_z)} \theta(1 - |z|^2) \right] \times \]
\[ \left[ V + \frac{1}{2N} V \text{tr} P_+ g^\dagger \partial Z \partial Z g + h.c. + \frac{1}{N} V \text{tr} P_+ \partial Z g^\dagger \partial Z g \right. \]
\[ - \frac{2}{N} \bar{z} \partial_z V \text{tr} P_+ (g^\dagger \partial Z g g^\dagger \partial Z g - g^\dagger \partial Z g P_+ g^\dagger \partial Z g) \].

(C.14)

The first term contains no excitations. Furthermore, the terms arising from the derivatives of the theta functions are subleading in \( N \). Hence we drop these terms from further consideration.

Thus,
\[ C = \frac{1}{8} \int d^2 z e^{-|z|^2} V \int d^2 Z \theta(1 - |Z|^2) \left( 2 \text{tr} P_+ \partial Z (\partial Z g^\dagger g) + h.c. \right) \]
\[ - \int d^2 z e^{-|z|^2} \bar{z} \partial_z V \int d^2 Z \theta(1 - |Z|^2) \text{tr} P_+ (g^\dagger \partial Z g g^\dagger \partial Z g - g^\dagger \partial Z g P_+ g^\dagger \partial Z g). \] (C.15)

For \( V = \frac{1}{r} \),
\[ \int d^2 z e^{-|z|^2} V = \sqrt{\frac{\pi}{2}} \frac{1}{l}, \int d^2 z e^{-|z|^2} \bar{z} \partial_z V = -\frac{1}{2} \sqrt{\frac{\pi}{2}} \frac{1}{l} \] (C.16)

where \( l \equiv \frac{1}{\sqrt{B}} \) is the magnetic length.

After performing the integral over \( z, \bar{z} \), let us set \( Z \to \frac{1}{\sqrt{N}} z \).

Then,
\[ C = -\frac{1}{2l} \sqrt{\frac{\pi}{2}} \int d^2 z \theta(N - |z|^2) \text{tr} P_+ (g^\dagger \partial_z g g^\dagger \partial_z g - g^\dagger \partial_z g P_+ g^\dagger \partial_z g) \]
\[ + \frac{1}{4l} \sqrt{\frac{\pi}{2}} \int d^2 z \delta(N - |z|^2) \text{tr} P_+ g^\dagger (\bar{z} \partial_z - z \partial_z) g. \] (C.17)
This is the expression quoted in (3.17)

**Appendix D**

We wish to prove:

\[
\langle \tilde{z}_2 | \hat{f} | z_1 \rangle = e^{\tilde{z}_2 z_1} \int d^2 \omega d^2 z e^{-|z|^2} e^{\tilde{\omega}(z-z_1) - \omega(\tilde{z}-\tilde{z}_2)} \langle \tilde{z} | \hat{f} | z \rangle. \tag{D.1}
\]

The *R.H.S.* can be reorganised as

\[
R.H.S. = \int d^2 \omega e^{\tilde{z}_2 z_1} e^{-\tilde{\omega} z_1 + \omega \tilde{z}_2} \int d^2 z e^{-|z|^2 + \tilde{\omega} z - \omega \tilde{z}} f(z, \tilde{z}). \tag{D.2}
\]

But,

\[
\int d^2 z e^{-|z|^2 + \tilde{\omega} z - \omega \tilde{z}} f(z, \tilde{z}) = e^{-|\omega|^2} e^{-\partial_\omega \partial_{\tilde{\omega}}} f(-\omega, \tilde{\omega}). \tag{D.3}
\]

Thus,

\[
R.H.S. = e^{\tilde{z}_2 z_1} \int d^2 \omega e^{-|\omega|^2} e^{-\tilde{\omega} z_1 + \omega \tilde{z}_2} e^{-\partial_\omega \partial_{\tilde{\omega}}} f(-\omega, \tilde{\omega}). \tag{D.4}
\]

Let

\[
\xi \equiv \omega + z_1; \bar{\xi} \equiv \tilde{\omega} - \tilde{z}_2. \tag{D.5}
\]

Then,

\[
R.H.S. = \int d^2 \xi e^{-|\xi|^2} e^{-\partial_\xi \partial_{\bar{\xi}}} f(-\xi + z_1, \bar{\xi} + \tilde{z}_2)
= e^{\partial_{z_1} \partial_{\tilde{z}_2}} \int d^2 \xi e^{-|\xi|^2} e^{-\xi \partial_{z_1} + \bar{\xi} \partial_{\tilde{z}_2}} f(z_1, \tilde{z}_2)
= e^{\partial_{z_1} \partial_{\tilde{z}_2}} e^{-\partial_{z_1} \partial_{\tilde{z}_2}} f(z_1, \tilde{z}_2)
= \langle \tilde{z}_2 | \hat{f} | z_1 \rangle
= L.H.S. \tag{D.6}
\]

We further wish to show that:

\[
\int d^2 \eta f(\eta, \bar{\eta}) \int d^2 \omega e^{\tilde{\omega}(\eta-z) - \omega(\bar{\eta}-\bar{z})} = f(z, \tilde{z}). \tag{D.7}
\]
Let
\[ \chi \equiv \eta - z. \quad (D.8) \]

Then,
\[ \bar{\omega}(\eta - z) - \omega(\bar{\eta} - \bar{z}) = 2i(\omega_x \chi_y - \omega_y \chi_x). \quad (D.9) \]

Therefore, from (D.7),
\[
L.H.S. = \int d^2 \eta f(\eta, \bar{\eta}) \frac{1}{\pi} \int d\bar{\omega} e^{2i(\omega_x \chi_y - \omega_y \chi_x)}
= \int d\bar{\eta} f(\bar{\eta}) \delta(\bar{\eta} - \bar{z})
= f(z, \bar{z}). \quad (D.10)
\]