Momentum Lattice for CHL String

Andrei Mikhailov*

Princeton University, Physics Department, Princeton, NJ08544.
andrei@puhep1.princeton.edu

We propose some analogue of the Narain lattice for CHL string. The symmetries of this lattice are the symmetries of the perturbative spectrum. We explain in this language the known results about the possible gauge groups in compactified theory. For the four-dimensional theory, we explicitly describe the action of S-duality on the background fields. We show that the moduli spaces of the six, seven and eight-dimensional compactifications coincide with the moduli spaces of the conjectured Type IIA, M Theory and F Theory duals. We classify the rational components of the boundary of the moduli space in seven, eight and nine dimensions.

* On leave from the Institute of Theoretical and Experimental Physics, Moscow, 117259, Russia.
1. Introduction

The nine-dimensional CHL string is an $N = 1$ supersymmetric string theory which can be obtained as an asymmetric orbifold of the ten-dimensional heterotic string. This theory may be thought of as the compactification of the $E_8 \times E_8$ heterotic string on a circle $S^1$, with the exchange of two $E_8$’s when one goes around $S^1$. It is dual to the compactification of $M$ theory on the Möbius strip [1]. The CHL string was first discovered in eight dimensions in [2]. The authors of [2] used a fermionic construction which allowed them to study the theory near some special points in the moduli space. In more invariant terms, the eight-dimensional theory can be obtained as the compactification of the $Spin(32)/\mathbb{Z}_2$ heterotic string on a torus without vector structure [3,4].

In dimensions less then nine, CHL string theory gives non-simply laced gauge groups at special subsets of the moduli space [2,5,6]. In particular, in four dimensions the set of allowed gauge groups is self-dual. This is a manifestation of the $S$-duality in four-dimensional $N = 4$ field theory.

The dual description in terms of Type IIA theory in dimensions 4, 5 and 6 was considered in [7,8]. It involves Calabi-Yau orbifolds with nontrivial RR $U(1)$ background turned on. The description of the seven-dimensional compactification in terms of $M$ theory on K3 surface with irremovable $D_4 \oplus D_4$ singularity was obtained in the recent paper [3], using the results of [9]. The eight-dimensional compactification was shown in [3] to be dual to the compactification of F Theory on K3 surface with irremovable $D_8$ singularity.

In our paper, we study the perturbative spectrum of the CHL string in nine and lower dimensions. We explicitly describe the T-duality group as the group of symmetries of certain lattice. This lattice may be thought of as an analogue of the Narain lattice for the CHL string. The moduli space of the theory is

$$O(\Gamma(D)) \setminus O(18 - D, 10 - D)/O(18 - D) \times O(10 - D)$$  \hspace{1cm}  (1.1)

where

$$\Gamma(D) = \Gamma_{9-D,9-D}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8.$$  \hspace{1cm}  (1.2)

Here $\Gamma_{n,n}(2)$ means the lattice generated by $2n$ vectors $\{e_i\}_{i=1,\ldots,n}$ and $\{f_i\}_{i=1,\ldots,n}$ with the scalar products $(e_i \cdot f_j) = 2\delta_{ij}$. In general, given the lattice $\Lambda$, we will denote $\Lambda(p)$ the lattice isomorphic to $\Lambda$ as an abelian group, but with the scalar product multiplied by $p$ (it may be thought of as “$\sqrt{p}\Lambda$”). Our sign convention for the scalar product of the
Narain lattice is in agreement with [10], and is opposite to the one usually accepted in algebraic geometry [11].

There are some useful equivalent forms for the lattice $\Gamma_{(D)}$. For example,

$$\begin{align*}
\Gamma_{(8)} &= \Gamma_{2,2} \oplus D_8 \\
\Gamma_{(7)} &= \Gamma_{3,3} \oplus D_4 \oplus D_4 \\
\Gamma_{(6)} &= \Gamma_{4,4} \oplus D_8^* (2)
\end{align*}$$

(1.3)

where $D_8^*$ is the lattice, dual to $D_8$ (the weight lattice of $D_8$).

The lattice $\Gamma_{(D)}$ is not self-dual. In section 2, we discuss the perturbative spectrum of the CHL string and the worldsheet current algebras. In section 3, we construct the momentum lattice, prove that the symmetries of this lattice are the symmetries of the perturbative spectrum, and discuss the structure of the moduli space. In section 4, we study symmetry enhancements at the special points of the moduli space, and explain how S-duality of the four-dimensional theory acts on the background fields. In section 5, we show that the moduli spaces we have found coincide with what is expected from the known F-theory, M-theory, and Type IIA duals. Some useful results from the theory of lattices, relevant to our study, are briefly reviewed in the Appendix A. Section 6 and Appendices C and D are devoted to the study of the boundary of the moduli space.

Recently we have received the preprints [12,13], where some problems discussed in our paper are studied from the different point of view.

2. Perturbative Spectrum.

To understand the structure of the perturbative spectrum, let us first consider the nine-dimensional theory. One way to construct it is to use the asymmetric orbifold of the heterotic string [5]. Let us consider the $E_8 \times E_8$ heterotic string compactified on a circle with the radius $r$ and orbifold the symmetry:

$$x^9 \rightarrow x^9 + \pi r, \quad x^I \rightarrow x^{I \pm 8}$$

(2.1)

where we have denoted $x^I$ the coordinates in the internal torus $\mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8$. In our notations, capital letters from the middle of the alphabet denote the directions along the internal torus $\mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8$, and $(I \pm 8)$ means $(I + 8)$ if $I < 9$ or $(I - 8)$ if $I \geq 9$. To understand the structure of twisted and untwisted states, let us consider string diagrams with the topology of a torus. For example, we can consider the scattering of four or more

2
gravitons with the momenta and polarizations in the uncompactified directions. We may consider a torus as a complex plane with the identifications \( z \sim z + 1 \) and \( z \sim z + \tau \).

In the \( \mathbb{Z}_2 \)-orbifold theory, we should consider the four possible boundary conditions on a torus: periodic in both directions, periodic in \( z \to z + 1 \) and periodic with the twist (2.1) in \( z \to z + \tau \), periodic in \( z \to z + \tau \) and with the twist in \( z \to z + 1 \), and with the twists in both \( z \to z + 1 \) and \( z \to z + \tau \) directions. We will denote the corresponding contributions to the amplitude as \( A_{++}(\tau) \), \( A_{+-}(\tau) \), \( A_{-+}(\tau) \) and \( A_{--}(\tau) \), respectively. The sum

\[
A(\tau) = A_{++}(\tau) + A_{+-}(\tau) + A_{-+}(\tau) + A_{--}(\tau)
\]

is modular invariant. Consider first the expression \( A_{++}(\tau) + A_{--}(\tau) \). It may be calculated as the sum of \( e^{2\pi i(\tau L_0 - \bar{\tau} L_0)} \) over those states of the heterotic string which are invariant under \((-1)^{n_0} \mathcal{P}\), where \( n_0 \) is the momentum along the ninth direction and \( \mathcal{P} \) is the operator exchanging two \( E_8 \) indices. The explicit expressions are:

\[
A_{++}(\tau) = \Phi(\{k_j\}, \{\zeta_j\}, \tau) \frac{1}{q f(q) \tau^2} (\text{Im } \tau)^{\frac{1}{2}} \sum_{m^9, n_9, I, J} q^{\frac{1}{2} p^2_L(m^9, n_9, I, J)} q^{\frac{1}{2} p^2_R(m^9, n_9, I, J)}
\]

\[
A_{--}(\tau) = \Phi(\{k_j\}, \{\zeta_j\}, \tau) \frac{1}{q f(q) \tau^2} (\text{Im } \tau)^{\frac{1}{2}} \sum_{m^9, n_9, I, J} (-1)^{n_9} q^{\frac{1}{2} p^2_L(m^9, n_9, I, J)} q^{\frac{1}{2} p^2_R(m^9, n_9, I, J)}
\]

In these formulas, the following notations are used. We have denoted \( f(q) = \prod_{k=1}^{\infty} (1 - q^k) \) and \( q = e^{2\pi i \tau} \). The momenta \( p_L \) and \( p_R \) are given by the formulae [14]:

\[
p^9_L(m, n, P) = m + \frac{g^{-1}}{2} (n + A^I (P^I - \frac{1}{2} A^I m))
\]

\[
p^9_R(m, n, P) = -m + \frac{g^{-1}}{2} (n + A^I (P^I - \frac{1}{2} A^I m))
\]

\[
p^I_L(m, n, P) = P^I - A^I m
\]

where \( g = g_{99} \) is related to the radius of compactification by the formula

\[
g = r^2 = 4 r^2 C_{HL}^2
\]

and \( A \) is the Wilson line (for the orbifold projection to make sense, the Wilson line should be diagonal). \( \Phi(\{k_j\}, \{\zeta_j\}, \tau) \) is some function of the momenta \( k_j \) and polarizations \( \zeta_j \) of the gravitons. The explicit expression for this function may be found in [15]. The only thing we will need to know about \( \Phi \) is that it is a modular function of weight \(-4\):

\[
\Phi(\{k_j\}, \{\zeta_j\}, \tau + 1) = \Phi(\{k_j\}, \{\zeta_j\}, \tau),
\]

\[
\Phi(\{k_j\}, \{\zeta_j\}, -\frac{1}{\tau}) = \tau^4 \Phi(\{k_j\}, \{\zeta_j\}, \tau)
\]
The expression $A_- + A_-$ may be written as the summation of $e^{2\pi i(\tau L_0 - \bar{\tau} L_0)}$ over the states from the twisted sector. In the twisted sector, the string is closed only modulo the symmetry (2.1). This means, that the winding number along the ninth direction, $m_9$, should be half-integer. As for the internal coordinates, they have the following expansion in terms of zero modes and oscillators:

$$X^I(\sigma + \tau) = x_0^I + p^I(\sigma + \tau) + \sum_{j \in \mathbb{Z}\setminus 0} \alpha^I_j e^{2\pi i j(\sigma + \tau)} + \sum_{j \in \mathbb{Z}} \alpha^I_{j+\frac{1}{2}} e^{2\pi i (j+\frac{1}{2})(\sigma + \tau)}$$

$$X^{I+8}(\sigma + \tau) = (x_0^I + p^I) + p^I(\sigma + \tau) + \sum_{j \in \mathbb{Z}\setminus 0} \alpha^I_j e^{2\pi i j(\sigma + \tau)} - \sum_{j \in \mathbb{Z}} \alpha^I_{j+\frac{1}{2}} e^{2\pi i (j+\frac{1}{2})(\sigma + \tau)}$$

and the eigenvalues of the operator $p^I$ belong to $\frac{1}{2}\Gamma_8$. The oscillators in the spatial direction and in the diagonal internal direction are enumerated by integers, while the oscillators in the anti-diagonal direction are enumerated by half-integers. This gives the normal ordering constant $-\frac{1}{2}$ in the twisted sector, which differs from the normal ordering constant $-1$ in the untwisted sector. The explicit expression for $A_-(\tau)$ is

$$A_-(\tau) = \frac{1}{\sqrt{q} f(q) f(\sqrt{q})} (\text{Im } \tau)^{\frac{3}{2}} \Phi(\{k_j\}, \{\zeta_j\}, \tau) \times$$

$$\times \sum_{(m,n,P) : m \in \mathbb{Z} + \frac{1}{2}, n \in \mathbb{Z}, P \in \frac{1}{2}\Gamma_8} q^{\frac{1}{2}} p_L^2(m,n,P) q^{\frac{1}{2}} p_R^2(m,n,P)$$

and $A_-(\tau) = A_-(\tau + 1)$. There are half-integer levels in the twisted sector, thus it is not true that $A_-(\tau + 1) = A_-(\tau)$. The projector

$$\frac{1}{2} \left(1 - (-1)^{n+P^2+2N'}\right)$$

where $P \in \Gamma_8 \left(\frac{1}{2}\right)$, plays the role of the orbifold projection in the twisted sector.

The expression

$$\frac{d^2 \tau}{(\text{Im } \tau)^2} (A_{++}(\tau) + A_{--}(\tau) + A_{+-}(\tau) + A_{-+}(\tau))$$

is modular invariant. Indeed, it is clearly invariant under $\tau \to \tau + 1$. Also, the expression $A_{++}(\tau)$ is invariant under $\tau \to -\frac{1}{\tau}$, since it is just a heterotic string amplitude. Let us check that

$$A_{+-}(-1/\tau) = A_{-+}(\tau)$$

Oscillators with integer labels contribute $-\frac{1}{24}$ into the normal ordering constant, while the oscillators with the half-integer labels contribute $+\frac{1}{48}$. This gives the total central charge $-16 \cdot \frac{1}{24} + 8 \cdot \frac{1}{48} = -\frac{1}{2}$ in the twisted sector.
Indeed, it follows from the Poisson resummation formula that

\[
(\text{Im } \tau)^{\frac{1}{2}} \sum_{m,n,I} (-1)^n q^{\frac{1}{2} p_L^2(m,n,I)} q^{\frac{1}{2} p_R^2(m,n,I)} = \frac{(\text{Im } (-1/\tau))^{\frac{1}{2}}}{16 \tau^4} \sum_{(m,n) \in \mathbb{Z} + \frac{1}{2}, \quad n \in \mathbb{Z}, \quad P \in \frac{1}{2} \Gamma_8} Q p_L^2(m,n,P,P) Q p_R^2(m,n,P,P)
\]

where we have denoted \( Q = e^{2\pi i(-1/\tau)} \). Also, we need the transformation law for the oscillator contributions:

\[
\frac{1}{qf^8(q)f^8(q^2)} = \frac{16 \tau^8}{Q^{1/2} f^8(Q)f^8(\sqrt{Q})}
\]

The transformation law (2.10) follows from (2.5), (2.11) and (2.12). Now, the invariance of \( A_{--} \) under \( \tau \to -\frac{1}{\tau} \) follows:

\[
A_{--}(-1/\tau + 1) = A_{+-} \left( \frac{\tau}{1-\tau} \right) = A_{+-} \left( \frac{1}{1-\tau} \right) = A_{--}(\tau - 1) = A_{--}(\tau)
\]

This proves that the measure (2.9) is modular-invariant.

Integration over the moduli space of the torus enforces the level-matching condition, which is

\[
mn + \frac{1}{2}(P^2 + Q^2) + N - \tilde{N} - 1 = 0
\]

in the untwisted sector and

\[
mn + P^2 + N - \tilde{N} - \frac{1}{2} = 0
\]

in the twisted sector (where \( P \in \Gamma_8 \left( \frac{1}{2} \right) \)).

The mass formula reads

\[
M^2 = \frac{p_L^2}{2} + \frac{p_R^2}{2} + N + \tilde{N} - a
\]

where \( a \) is 1 in the untwisted sector and \( \frac{1}{2} \) in the twisted sector.

Toroidal compactification to lower dimensions is straightforward. There is one selected direction in the torus, which we will call the ninth direction. The orbifold projection keeps those states, which are invariant under the shift by half a circle in this ninth direction with the exchange of the coordinates in two internal \( \mathbb{R}^8/\Gamma_8 \). In the twisted sector, the winding number in the ninth direction is half-integer.
There are points in the moduli space where some states become massless. The state in the twisted sector can become massless if all the oscillators are in the ground state and the momentum $K$ has $K^2 = 1$. In this case, we have an enhanced gauge symmetry in space-time and Kac-Moody algebra on the worldsheet. For the massless states in the twisted sector, some generators of the worldsheet Kac-Moody algebra act between the twisted and the untwisted sector. Indeed, we have the following set of generators:

\begin{align*}
  e_K(z) &= \sqrt{2} V_K(z), \\
  h_K(z) &= 2iK \cdot \partial X(z), \\
  f_K(z) &= \sqrt{2} V_{-K}(z)
\end{align*}

(2.16)

where $X^I(z)$ is a free left-moving boson with the operator product expansion

\begin{equation}
  \partial X^I(z) \partial X^J(0) = -\frac{\delta_{IJ}}{z^2} + \ldots
\end{equation}

(2.17)

and $V_K(z)$ is the vertex operator, creating a cut on the worldsheet. It may be defined in terms of the path integral. Insertion of such an operator means that we are integrating over those fields which have a monodromy (2.1) when we go around $z$, and

\begin{equation}
  \oint_z \partial X = 2\pi K
\end{equation}

(2.18)

The conformal dimension of such an operator is $\frac{1}{2} + \frac{1}{2}K^2$. We need $K^2 = 1$. Let us normalize $V_K(z)$ so that

\begin{equation}
  \langle V_K(z)V_{-K}(0) \rangle = \frac{1}{z^2}
\end{equation}

(2.19)

The singular part of the operator product expansion of $\partial X$ and $V_{\pm K}(z)$ is determined by (2.18) to be

\begin{equation}
  (K \cdot \partial X(z))V_{\pm K}(0) = \mp \frac{i}{z} V_{\pm K}(0) + \ldots
\end{equation}

(2.20)

The three-point function is:

\begin{equation}
  \langle V_K(w)P \cdot \partial X(z)V_{-K}(0) \rangle = \frac{i(P \cdot K)}{zw(w - z)}
\end{equation}

(2.21)

— indeed, this expression should be proportional to $\frac{1}{zw(w - z)}$, as follows from the conformal invariance, and the coefficient is determined by (2.19) and (2.20). In this expression, $P$ is an arbitrary momentum from the twisted sector. From the formula for the three-point function and the normalization condition (2.19), we get the singular terms in $V_K(w)V_{-K}(0)$:

\begin{equation}
  V_K(w)V_{-K}(0) = \frac{1}{w^2} + \frac{i}{w} K \cdot \partial X(0) + \ldots
\end{equation}

(2.22)
From the formulas (2.20), (2.22) and (2.17), we get the operator product expansions:

\[
\begin{align*}
    h_K(z)e_K(0) &= \frac{2}{z}e_K(0) + \text{regular terms} \\
    e_K(z)f_K(0) &= \frac{2}{z^2} + \frac{1}{z}h_K(0) + \text{regular terms} \\
    h_K(z)h_K(0) &= \frac{4}{z^2} + \text{regular terms}
\end{align*}
\] (2.23)

which is the operator product expansion for the generators of the Kac-Moody algebra on level two. We always get a level two algebra from the massless states in the twisted sector.

If the state in the untwisted sector becomes massless, then, as we will see below, one can get both level two and level one current algebra, although it can be level one only in \( D \leq 8 \), not in the nine-dimensional theory.

For example, consider the following state:

\[
|0, 0, 0, 0 > + |0, 0, 0, P >
\] (2.24)

where \( P \in \Gamma_8, \ P^2 = 2 \). If this state is massless, we have the following currents:

\[
\begin{align*}
    e_P(z) &= \frac{e^{iP \cdot X^{(1)}}}{z} + \frac{e^{iP \cdot X^{(2)}}}{z} \\
    h_P(z) &= iP (\partial X^{(1)} + \partial X^{(2)}) \\
    f_P(z) &= \frac{e^{-iP \cdot X^{(1)}}}{z} + \frac{e^{-iP \cdot X^{(2)}}}{z}
\end{align*}
\] (2.25)

— here we use the notation \( X^{(1)} \) and \( X^{(2)} \) for the components of \( X \) in the first and the second torus in

\[
\mathbb{R}^{16}/\Gamma_8 \oplus \Gamma_8 = (\mathbb{R}^8/\Gamma_8) \times (\mathbb{R}^8/\Gamma_8)
\]

We may compare (2.25) to the Kac-Moody generators for the level one algebra on the worldsheet of the usual heterotic string:

\[
\begin{align*}
    e_P(z) &= e^{iP \cdot X} \\
    h_P(z) &= iP \cdot \partial X \\
    f_P(z) &= e^{-iP \cdot X}
\end{align*}
\] (2.26)

The \( \hat{sl}(2) \) algebra generated by (2.25) is on the level two. If we compactify to some dimension lower than nine, we can get some level one algebras (corresponding to the massless states from the untwisted sector). For example, we may compactify the nine-dimensional CHL string on the circle of the self-dual radius. The \( sl(2) \) current algebra corresponding to the massless states with winding and momentum along this circle may be constructed exactly in the same way, as for the usual heterotic string. It is on level one.
3. The Lattice.

3.1. Momentum of a State.

We start with the nine-dimensional theory. Let us associate with each state a vector from the lattice

$$\Gamma_{1,1} \oplus \Gamma_8$$

This is done as follows. To the untwisted state with the internal momentum \((P, Q)\), winding number \(m\) and momentum \(n\) along the ninth direction, we associate the vector of the form \((l, n, R)\), where

$$R = P + Q$$
$$l = 2m$$

To the twisted state which has the internal momentum \((\frac{P}{2}, \frac{P}{2})\), and the momentum and winding number \((n, m)\) in the ninth direction, we associate the lattice vector \((l, n, P)\), with \(l = 2m\). The left and right components \(p^0_L\) and \(p^0_R\) of the momentum, as well as the diagonal part of the internal momentum \(p^0_L\), are, in the given background, functions of \((l, n, R)\) only; thus the dependence of the mass of a given state on the background fields is only through the coupling of the background to the vector \((l, n, R)\). This is one of the motivations for calling \((l, n, R)\) the momentum. The other reason is that, as we will see later, the perturbative spectrum is invariant under those transformations of the background which can be interpreted as the symmetries of the lattice generated by the momenta \((l, n, R)\).

The mass formula in both twisted and untwisted sectors, with the diagonal Wilson line \(A = (a, a)\) turned on, reads as:

$$M^2(l, n, P, Q) = \frac{1}{4}(R - la)^2 + \frac{g}{4}l^2 + \frac{1}{4g} \left( n + aR - \frac{l}{2}a^2 \right)^2 + N' + \tilde{N} - 1 \quad (3.1)$$

where the modified oscillator number \(N'\) is defined as \(N' = N + \frac{1}{2}\) in the twisted sector, and as \(N' = N + \frac{(P-Q)^2}{4}\) in the untwisted sector, \(N\) being the oscillator number. In both sectors, \(N'\) may be integer or half-integer. Our notations are summarized in the table:

| Sector | Untwisted | Twisted |
|--------|-----------|---------|
| Momentum | \((m, n, P, Q)\) | \((m + \frac{1}{2}, n, \frac{1}{2}P, \frac{1}{2}P)\) |
| \((l, n, R)\) | \((2m, n, P + Q)\) | \((2m + 1, n, P)\) |
| \(N'\) | \(N + \frac{(P-Q)^2}{4}\) | \(N + \frac{1}{2}\) |
For each vector from the lattice, we have the whole infinite tower of states associated to it. The structure of this tower is different for different vectors. We will prove that
the symmetries of the lattice are the symmetries of the perturbative spectrum. It turns out, that the vectors in the momentum lattice may be naturally grouped into three classes invariant under the symmetry group of the lattice, so that the spectrum of the massive states with given momentum depends only on which class the momentum belongs to. To prove this, let us introduce the generating function for the number of states with given momentum. Since the oscillators in the spatial direction are completely decoupled, we can without any loss of generality consider only those states for which these spatial oscillators are not excited. We define the generating function for the number of states with given momentum \((l,n,R)\) as:

\[
F(q) = \sum d(N', (l,n,R)) q^{N'}
\]  

where \(d(N')\) is the number of states with the given value of \(N'\) and given momentum \((l,n,R)\). We get the following expressions for the generating function, depending on the momentum \((l,n,R)\):

A. \(l \in 2\mathbb{Z}, R \in 2\Gamma_8\) (untwisted sector, internal momentum divisible by two):

\[
F_1(q, n \mod 2) = \frac{1}{2} \left( \frac{\Theta_8(2\tau)}{f^{24}(q)} + \frac{(-1)^n}{f^8(q)f^8(q^2)} \right)
\]  

— this expression depends on whether the momentum \(n\) along the circle is odd or even, because the orbifold projection involves \((-1)^n\).

B. \(l \in 2\mathbb{Z}, R \in \Gamma_8 \setminus 2\Gamma_8\) (untwisted sector, internal momentum cannot be divided by two):

\[
F_2(q, \bar{R}) = \frac{e^{\pi i R^2}}{2f^{24}(q)} \Theta_8(2\tau|\tau R) = \frac{1}{2f^{24}(q)} \sum_{Q \in \Gamma_8} e^{2\pi i (Q+R/2)^2} \]

— this expression depends only on the conjugacy class of \(R\) in \(\Gamma_8/2\Gamma_8\).

C. \(l \in 2\mathbb{Z} + 1, n \in \mathbb{Z}\) (twisted sector):

\[
F_3(q, \frac{1}{2}R^2 + n \mod 2) = \frac{\sqrt{q}}{2f^8(q)} \left[ \frac{1}{f^8(\sqrt{q})} - \frac{(-1)^{R^2/2+n}}{f^8(-\sqrt{q})} \right]
\]

In these formulae we have used the following notation for the \(\theta\)-function of the lattice \(\Gamma_8\):

\[
\Theta_8(\tau|z) = \sum_{S \in \Gamma_8} e^{\pi i S^2 + 2\pi i (S \cdot z)}
\]
We will explain momentarily how to derive these formulae. But first let us notice, that (3.3)—(3.5) give six different expressions for generating function, depending on whether $l$ and $n$ are even or odd, and on the conjugacy class of the momentum $R$ modulo $2\Gamma_8$. If the generating functions in all these situations were different, the spectrum would not be invariant under the symmetries of the lattice. For example, the symmetry $(l, n, R) \rightarrow (n, l, R)$ for even $l$ and odd $n$ exchanges the states from the untwisted sector with the states from the twisted sector. Since the allowed momenta and oscillator numbers are completely different in these two sectors, it is not obvious that there exists a symmetry between them. Fortunately, there is such a symmetry\footnote{There are many examples in string theory, where different sets of creation-annihilation operators give the same spectrum. Correspondence between bosons and fermions is one of them. An example of the symmetry mixing twisted and untwisted states may be found in}. In fact, as we will see later, the functions $F_1$, $F_2$ and $F_3$ satisfy certain identities, which reduce the number of different classes of vectors from six to three.

Let us explain how to derive (3.3)—(3.5). To get (3.3), we notice that those states from the untwisted sector which have $P \neq Q$ may be considered as the states of the heterotic string. The internal momenta $(P, Q)$ and $(Q, P)$ which differ only in the order of two components, give the same untwisted state after the orbifold projection (which gives a factor of $\frac{1}{2}$ on the right hand side). For $R \in 2\Gamma_8$, we may write $R = P + Q$ with $P = \frac{1}{2}R + S \in \Gamma_8$ and $Q = \frac{1}{2}R - S \in \Gamma_8$. The momentum square $\frac{1}{2}P^2 + \frac{1}{2}Q^2$ contains $\frac{S^2}{2} + \frac{S^2}{2} = S^2$ giving a contribution to $N'$ in the mass formula. After performing a summation over $S \in \Gamma_8$ and taking into account the oscillator contributions, we get the first term on the right hand side of (3.3). To get the second term, we have to take into account that for the diagonal momentum $P = Q$ (or $S = 0$) not all the excited levels of the oscillators are allowed by the orbifold projection, but only those which have $n$ plus the number of antidiagonal creation operators even.

To get (3.4), we notice that only the states with $P \neq Q$ may correspond to $R \in \Gamma_8 \setminus 2\Gamma_8$ and $n$ even. For each such state, the momentum may be written as $(Q + R, -Q)$, which gives the contribution $(Q + R/2)^2$ to $N'$.

In the formula (3.5), the first term on the right hand side comes from $A_{-+}(\tau)$, and the second term comes from $A_{--}(\tau)$. In this case, the term $N'$ in the mass formula gets contributions from the oscillators only.

It turns out that the spectrum of massive states actually depends on which of the following classes the point of the lattice belongs to:
1. \((l, n, R) \in 2(\Gamma_{1,1} \oplus \Gamma_8)\),
2. \(ln + \frac{1}{2}R^2\) is even, but \((l, n, R)\) is not in the previous class,
3. \(ln + \frac{1}{2}R^2\) is odd.

**Proof.** Begin with the class 3. The vectors from this class correspond to either case B with odd \(\frac{1}{2}R^2\), or to the case C with odd \(\frac{1}{2}R^2 + n\). Notice that if \(\frac{1}{2}R^2\) is odd, then one can always find such a vector \(Q \in 2\Gamma_8\) that \(R = P + Q\) and \(P \in \Delta\) — the root of \(E_8\). Indeed, consider the vector \(\frac{1}{2}R\). It is known [17], that the covering radius of \(\Gamma_8\) is 1. This means that there is some vector of the lattice within the distance 1 to \(\frac{1}{2}R\). Let us call it \(Q\). Then, we have:

\[
\left(\frac{1}{2}R - Q\right)^2 = \frac{1}{2}
\]

— indeed, the right hand side has to be half-integer and less or equal then one. Thus, we have to prove that

\[
F_2(q, P) = F_3(q, 1) \quad (3.7)
\]

for \(P\) a root. Let us prove a slightly more general identity:

\[
\sum_{Q \in \Gamma_8} \left( e^{2\pi i \tau (Q + R_1/2)^2} + e^{2\pi i \tau (Q + R_2/2)^2} \right) = 2\sqrt{q} \frac{f^{16}(q)}{f^8(\sqrt{q})} \quad (3.8)
\]

where \(R_1 = (1, 1, 0^{(6)})\) and \(R_2 = (1^{(4)}, 0^{(4)})\). This may be shown directly, using the known formulae for theta-series (see, for example, [18], formulas 8.180 and 8.181). We will actually need the following identities:

\[
\sum_{n \in \mathbb{Z}} e^{\pi i \tau n^2} = \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}})^2(1 - q^n),
\]

\[
\sum_{n \in \mathbb{Z}} e^{\pi i \tau (n+\frac{1}{2})^2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^{2n})(1 + q^n) \quad (3.9)
\]

Using these two identities, we get:

\[
\sum_{n_1 + \ldots + n_8 \text{ even}} \left( e^{2\pi i \tau \left( \left(\frac{1}{2}+n_1\right)^2 + \left(\frac{1}{2}+n_2\right)^2 + \sum_{j=3}^{8} n_j^2 \right)} + e^{2\pi i \tau \left( n_1^2 + n_2^2 + \sum_{j=3}^{8} (\frac{1}{2}+n_j)^2 \right)} +
\right.
\]

\[
\left. + 2e^{2\pi i \tau \left( \sum_{j=1}^{4} \left(\frac{1}{2}+n_j\right)^2 + \sum_{j=5}^{8} n_j^2 \right)} \right) =
\]

\[
= \frac{1}{2} \sum_{n_1,n_2,n_3,n_4} e^{2\pi i \tau (n_1^2+n_2^2)} e^{2\pi i \tau \left( \left(\frac{1}{2}+n_3\right)^2 + \left(\frac{1}{2}+n_4\right)^2 \right)} \left[ \sum_{n_+,n_-} e^{\pi i \tau (n_+^2+n_-^2)} \right]^2
\]

\[
= 2\sqrt{q} \frac{f^{16}(q)}{f^8(\sqrt{q})}
\]

The part of this expression odd under \(\sqrt{q} \rightarrow -\sqrt{q}\) gives the required identity (3.7).
Now let us consider the class $2$. The vectors of this class come from either type $\text{A}$ with odd $n$, or type $\text{B}$ with $\frac{1}{2}R^2$ even, or type $\text{C}$ with $\frac{1}{2}R^2 + n$ even. Notice that if $\frac{1}{2}R^2$ even but $R$ is not in $2\Gamma_8$, then there exists such a vector $S \in \Gamma_8$ that $S^2 = 4$ and $R \equiv S \mod 2\Gamma_8$. Also, all the vectors in $\Gamma_8$ with length square 4 are equivalent modulo Weyl group. Thus, we have to prove that

$$F_2(q, S) = F_3(q, 0) \quad (3.10)$$

and

$$F_1(q, 1) = F_2(q, S) \quad (3.11)$$

where $S = (2, 0^{(7)})$ (or $S = (1^{(4)}, 0^{(4)})$ — since it is related to $(2, 0^{(7)})$ via the Weyl group, we get the same $F_2(q, S)$). The identity $(3.10)$ is the part of $(3.8)$, even under $\sqrt{q} \to -\sqrt{q}$. To prove $(3.11)$, we write

$$\Theta_8(2\tau) = \frac{1}{2} \left[ \frac{f^{16}(-q)}{f^8(q^2)} + \frac{f^{16}(q)}{f^8(q^2)} \right] + 128q^2 \frac{f^{16}(q^4)}{f^8(q^2)} \quad (3.12)$$

— this expression may be derived from the explicit form of the $E_8$ lattice, using $(3.9)$, or from the fermionic description of the heterotic string $[15]$. Then, we get:

$$\Theta_8(2\tau) - \frac{f^{16}(q)}{f^8(q^2)} = \frac{1}{2} \left[ \frac{f^{16}(-q)}{f^8(q^2)} - \frac{f^{16}(q)}{f^8(q^2)} \right] + 128q^2 \frac{f^{16}(q^4)}{f^8(q^2)} \quad (3.13)$$

Notice that the first term on the right hand side is the sum of $e^{2\pi i \tau P^2}$ over $P$ in the vector conjugacy class of $so(16)$, while the second one is the sum over $P$ in the spinor conjugacy class. This is exactly how we get $f^{24}(q)F_2(q, (2, 0^{(7)}))$.

As for the vectors of class $1$, they appear only in type $\text{A}$, and it is evident that the spectrum of the massive states is the same for all of them. **This completes the proof.**

Since the definition of the classes $1, 2$ and $3$ is given in an invariant way, any automorphism of the lattice $\Gamma_{1,1} \oplus \Gamma_8$ is actually a symmetry of the perturbative spectrum.

If we further compactify CHL string on a $d$-dimensional torus, we get the lattice

$$\Gamma_{(9-d)} = \Gamma_{d,d}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8 \quad (3.14)$$

The coefficient 2 in the scalar product for $\Gamma_{d,d}(2)$ appears because the momentum $m$ along any direction in $T^d$ should be integer, which gives even $l$. The lattice consisting of the vectors of the form $(2m, n), \ m, n \in \mathbb{Z}$ with the length square $||(2m, n)||^2 = 4mn$ is $\Gamma_{1,1}(2)$. 

12
For the orbifold projection to make sense, Wilson lines in all the compactified directions should be diagonal.

As in the nine-dimensional theory, the structure of the massive states depends on the class of the vector. The sublattice of the vectors of class $1$ is now defined as

$$\Gamma_{d,d}(2) \oplus 2(\Gamma_{1,1} \oplus \Gamma_8) \subset \Gamma$$ (3.15)

This sublattice can be described in an invariant way as the sublattice of vectors of even level in $\Gamma$. The other two classes are defined in the same way as before. This classification is preserved by all the symmetries of the lattice. Thus, $O(\Gamma)$ is a group of symmetries of the perturbative spectrum.

Given the lattice $\Gamma_{(D)} = \Gamma_{d,d}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8$, consider the dual lattice:

$$(\Gamma_{d,d}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8)^* \simeq \Gamma_{d,d}(\frac{1}{2}) \oplus \Gamma_{1,1} \oplus \Gamma_8$$ (3.16)

For an arbitrary lattice $L$, we have

$$O(L) \simeq O(L^*)$$ (3.17)

(indeed, each symmetry of $L$ gives the symmetry of $L^*$, and vice versa: it follows from $(L^*)^* = L$, that the symmetry of $L^*$ corresponds to some symmetry of $L$). Since the lattice $L(2)$ may be obtained from the lattice $L$ by rescaling with the factor $\sqrt{2}$, the groups $O(L)$ and $O(L(2))$ are isomorphic. Taking this into account, we derive from (3.16) and (3.17), that

$$O(\Gamma_{d,d}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8) \simeq O(\Gamma_{d,d}(2) \oplus \Gamma_{1,1}(2) \oplus \Gamma_8(2))$$ (3.18)

Notice that $\Gamma_8(2)$ can be embedded as the diagonal sublattice into $\Gamma_8 \oplus \Gamma_8$. Then, the isomorphism (3.18) shows, in particular, that those symmetries of the $10 - d$-dimensional heterotic spectrum, which preserve the diagonal Wilson lines, are also the symmetries of the CHL spectrum. Of course, this is in agreement with what one would expect.

---

3 The level of the lattice vector is the greatest common divisor of all the scalar products of this vector with the other vectors in the lattice.
3.2. The Moduli Space.

Let us show that the moduli space of the compactified CHL string is the Grassmanian manifold of planes with signature $(0, d+9)$ in the space $\mathbb{R}^{d+1,d+9}$, modulo the symmetries of the lattice:

$$
\mathcal{M}_{9-d} = O(\Gamma(9-d)) \backslash Gr((0, d+9), \mathbb{R}^{d+1,d+9}) = O(\Gamma(9-d)) \backslash Gr((d+1,0), \mathbb{R}^{d+1,d+9}) = \\
= O(\Gamma(9-d)) \backslash O(d+1, d+9)/O(d+1) \times O(d+9)
$$

(3.19)

The choice of the background may be thought of as giving the decomposition of the momenta in the lattice into the left- and the right-moving components, according to the equation:

$$
p^L_i(l, n, R) = \frac{1}{2} \left[l^i + E^{ij}(n_j + a^I_j(R^I - \frac{1}{2}a^I_k l^k))\right] \\
p^R_i(l, n, R) = \frac{1}{2} \left[-l^i + E^{ij}(n_j + a^I_j(R^I - \frac{1}{2}a^I_k l^k))\right] \\
p^I_i = R^I - a^I_i l^i, \quad i = 1, \ldots, 8
$$

— here we have introduced, following [19], the matrix $E_{ij} = g_{ij} + b_{ij}$. For the background to have geometrical meaning, the metric $g_{ij}$ should be nonnegative definite: $g_{ij}v^iv^j \geq 0$ for any vector $v$. Then, the inverse matrix $E^{ij}$ is well defined if and only if $g_{ij}$ is nondegenerate.

Let us consider the plane in $\mathbb{R} \otimes \Gamma_D$ specified by the equation $p_R = 0$. It is clear, that different backgrounds will give us different planes. Thus, we have a map from the space of backgrounds to the space of $(0, d+9)$-planes in $\mathbb{R}^{d+1,d+9}$. This map is well defined everywhere except for those points in the moduli space, where the metric $g_{ij}$ is degenerate.

Let us consider the inverse map. Suppose that the $(0, d+9)$-plane in $\mathbb{R}^{d+1,d+9}$ is given by the equation

$$
G_{ij}l^j = n_i + (a_i \cdot R)
$$

(3.21)

Then, the corresponding background is specified as follows: $a_i$ is the Wilson line, and $G_{ij} = E_{ij} + \frac{1}{2}(a_i \cdot a_j)$. Notice that $a_i$ and $G_{ij}$ are well defined for all planes except for those which contain vectors with zero $l$- and $R$- components (and nonzero $n$-components). This means, that the corresponding direction on the plane is lightlike. Such planes are on the boundary of the moduli space.

---

4 The tensor product of the lattice with $\mathbb{R}$ is the linear space, generated by the triples $(l, n, R)$ with $l, n$ real (not necessarily integer) numbers, and $R$ arbitrary vector in $\mathbb{R}^8$ (not necessarily belonging to $\Gamma_8 \subset \mathbb{R}^8$). The system of $d+1$ equations $p_R = 0$ gives some $d+9$-dimensional plane in this vector space.
We have shown that the points of the moduli space may be parametrized by the planes in $\mathbb{R}^{d+1,d+9}$. The metric on this moduli space is read from the low-energy effective action, which is fixed by supersymmetry. The field content is that of the compactification of 10-dimensional $N = 1$ supergravity interacting with 8 vector fields. The moduli space for these theories is locally [20]:

$$\frac{SO(10 - D, 18 - D)}{SO(10 - D) \times SO(18 - D)} \times \mathbb{R}$$  (3.22)

for $D \geq 5$ (the factor $\mathbb{R}$ specifies the dilaton expectation value) and

$$\frac{SO(6, 14)}{SO(6) \times SO(14)} \times \frac{SU(1, 1)}{U(1)}$$  (3.23)

in $D = 4$ (the last factor specifies the expectation values of dilaton and axion, which is a dual of $B$ field).

Globally, the moduli space differs from (3.22) and (3.23), because we have to take into account that some apparently different backgrounds should be in fact identified because of the dualities. We certainly should identify those backgrounds which are related by series of reflections, since reflections are in fact gauge symmetries [19]. For the lattice $\Gamma_8 \oplus \Gamma_{1,1}$, it is known [17], that the whole symmetry group is in fact generated by the reflections. We do not know if the same result is true for the non-Lorentzian lattices, which arise when we study compactifications to $D < 9$. Nevertheless, we assume that we should identify those backgrounds, which may be related by the symmetry of the momentum lattice. This leads to the following form of the moduli space for $9 - d$-dimensional theory:

$$\mathcal{M}_{9-d} = O(\Gamma_{1,1} \oplus \Gamma_{d,d}(2) \oplus \Gamma_8) \backslash O(d + 1, d + 9)/O(d + 1) \times O(d + 9)$$  (3.24)

3.3. Some useful isomorphisms.

We will need some facts about the lattice

$$\Lambda_d = \Gamma_{d,d}(2) \oplus \Gamma_8$$

This lattice may be defined as the sublattice of $II_{d,s+d} = \Gamma_{d,d} \oplus \Gamma_8$, consisting of those vectors whose scalar product with any vector from a rational light-like plane of maximal
dimension is even. Indeed, let us take the lightlike plane consisting of the vectors of the form

\[
\begin{bmatrix}
0 \\
* & 0 \\
\ldots & \ldots \\
* & 0
\end{bmatrix}
\]

(3.25)

(we write elements of \( \Gamma_{d,d} \oplus \Gamma_8 \) as

\[
\begin{bmatrix}
R \\
l_1 & n_1 \\
\ldots & \ldots \\
l_d & n_d
\end{bmatrix}
\]

and the length square is \( 2l_1n_1 + \ldots + 2l_dn_d + P^2 \))

Then, those vectors which have even scalar products with the vectors from the plane (3.25), are of the form

\[
\begin{bmatrix}
2m_1 \\
2m_1 & n_1 \\
\ldots & \ldots \\
2m_d & n_d
\end{bmatrix}
\]

with integer \( m \)'s, and form a sublattice \( \Lambda_d \).

Since all the maximal rational lightlike planes are equivalent, we may have some more convenient choices. For example, to get \( \Lambda_1 \), we may use the following lightlike vector:

\[
\begin{bmatrix}
\alpha_0 + \alpha_2 + \alpha_4 + \alpha_6 \\
2 & -2
\end{bmatrix}
\]

(3.26)

Here we use the following enumeration of the roots of \( \Gamma_8 \):

---

5 There is only one such plane, modulo automorphisms of \( II_{d,d+8} \). This may be proven as follows. Given a light-like plane \( \omega \), consider some primitive vector \( v \in \omega \). Since the lattice is self-dual, one can find a primitive lightlike vector \( v^* \), such that \( (v^* \cdot v) = 1 \). Consider the sublattice orthogonal to \( v \) and \( v^* \). This sublattice is even and self-dual, thus it is isomorphic to \( II_{d-1,d-1+8} \). The intersection of \( \omega \) with this plane is a rational lightlike plane in \( II_{d-1,d-1+8} \). After applying this procedure \( d \) times, we find that \( \omega \) is the standard light-like plane in \( \Gamma_{d,d} \subset \Gamma_{d,d} \oplus \Gamma_8 \).
This is the extended Dynkin diagram; the roots are linearly dependent:

\[ \alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0 \]

The sublattice of those vectors in \( \Gamma_8 \) which have even scalar product with \( \alpha_0 + \alpha_2 + \alpha_4 + \alpha_6 \) is isomorphic to \( D_8 \) (the lattice \( D_n \) is defined as \( \{(m_1, \ldots, m_n) | \sum_j m_j \in 2\mathbb{Z} \} \)).

Thus, we get

\[ \Lambda_1 \simeq \Gamma_{1,1} \oplus D_8 \]  

(3.27)

This means, that

\[ \Gamma_{(8)} \simeq \Gamma_{2,2} \oplus D_8 \]

Let us consider the lattice

\[ \Gamma_{1,1}(2) \oplus D_8 \]  

(3.28)

This lattice may be considered as the sublattice of those vectors in \( \Gamma_{1,1} \oplus D_8 \), which have even scalar product with the vector \( v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). Let \( \{u_1, \ldots, u_8\} \) be the orthonormal basis in \( \mathbb{R}^8 \), then the lattice \( D_8 \) consists of the linear combinations \( \sum x_i u_i \) with integer coefficients \( x_i \), subject to the condition \( \sum x_i \in 2\mathbb{Z} \). Let us consider a composition of the reflection in the vector \( \begin{bmatrix} u_1 - u_3 \\ 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} u_3 - u_2 + u_5 - u_6 \\ -1 \\ 1 \end{bmatrix} \). This symmetry transforms our vector \( v \) to the vector

\[ \begin{bmatrix} -u_1 - u_2 + u_5 - u_6 + 2u_3 \\ -2 \\ 2 \end{bmatrix} \]  

(3.29)

and the vectors in \( \Gamma_{1,1} \oplus D_8 \) having even scalar product with this vector form the sublattice \( \Gamma_{1,1} \oplus D_4 \oplus D_4 \).

This gives us the following isomorphisms:

\[ \Gamma_{1,1}(2) \oplus D_8 \simeq \Gamma_{1,1} \oplus D_4 \oplus D_4, \]

\[ \Lambda_2 \simeq \Gamma_{2,2} \oplus D_4 \oplus D_4 \]  

(3.30)

Thus, the momentum lattice for the seven-dimensional compactification is

\[ \Gamma_{(7)} \simeq \Gamma_{3,3} \oplus D_4 \oplus D_4 \]  

(3.31)

We will need also the following isomorphism:

\[ \Gamma_{1,1}(2) \oplus D_4 \oplus D_4 \simeq \Gamma_{1,1} \oplus D_8^*(2) \]  

(3.32)
where $D_8^*$ is the weight lattice of $D_8$; it consists of the linear combinations $\sum x_i u_i$ with the restriction that all $x_i$ are either simultaneously integer, or simultaneously half-integer. We can prove it in the same way as we had proven (3.30). The lattice $\Gamma_{1,1} \oplus D_8^*(-2)$ is the sublattice of $\Gamma_{1,1} \oplus D_4 \oplus D_4$, consisting of those vectors which have even scalar product with the vector (3.29): that this vector actually belongs to $\Gamma_{1,1} \oplus D_4 \oplus D_4 \subset \Gamma_{1,1} \oplus D_8$, and the symmetries, relating this vector to $\left[ \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right]$, are the symmetries of $\Gamma_{1,1} \oplus D_4 \oplus D_4$. The isomorphism (3.32) implies that the momentum lattice of the six-dimensional theory may be represented as:

$$\Gamma_6 \simeq \Gamma_{4,4} \oplus D_8^*(2)$$

(3.33)

Alternative proofs of these identities using the technique of discriminant-forms are given in Appendix A.

4. Gauge groups in the compactified theory.

In this section we will show that the results of S. Chaudhuri and J. Polchinski [5] about the possible gauge groups follow from our description of the momentum lattice. We will find some gauge groups not mentioned in [5]. We also explain why the set of possible gauge groups in four dimensions turns out to be self-dual.

4.1. Gauge Groups and Roots.

As we have discussed in the previous section, the moduli space of $9-d$-dimensional compactification is locally the Grassmanian of $d+1$-dimensional spaces in $\mathbb{R}^{d+1,d+9}$. Given such a plane $\nu$, one can write a mass formula:

$$M^2(p, N', \tilde{N}) = \frac{1}{4}(\mathcal{P}_\perp p)^2 - \frac{1}{4}(\mathcal{P}_\parallel p)^2 + N' + \tilde{N} - 1 =$$

$$= \frac{1}{2}(\mathcal{P}_\perp p)^2 + 2N' - 2 = -\frac{1}{2}(\mathcal{P}_\parallel p)^2 + \tilde{N}$$

(4.1)

for the state with the momentum $p = (\{l_j\}, \{n_j\}, R)$, where the last two equalities follow from the level matching condition. We have denoted $\mathcal{P}_\parallel$ and $\mathcal{P}_\perp$ the projection on the plane and the orthogonal projection, respectively. It follows that the state is massless if and only if

$$\mathcal{P}_\parallel p = 0,$$

$$\tilde{N} = 0,$$

$$\frac{1}{4}p^2 + N' = 1$$

(4.2)

The first condition means that the plane should be orthogonal to the momentum of the massless state. Notice that for the state of the class $\mathbf{1}$, the minimal allowed value of $N'$ is
zero (the expansion of (3.3) in powers of $q$ starts with $q^0$), thus the states with $p$ of class 1 and $p^2 = 4$ will be massless. For the vectors of class 2, the minimal value of $N'$ is 1, and we the corresponding state cannot be massless. For the vectors of class 3, the minimal value of $N'$ is $\frac{1}{2}$, and it occurs with multiplicity one, as follows from (3.5) (or (3.4)). Thus, we get one massless state for each $p$ with $p^2 = 2$.

Thus, the massless states correspond to either vectors of length square 2 (such vectors can actually have only level one), or level two vectors with length square 4. This is very natural, since precisely to these two types of vectors we can associate a reflection: it is given by the formula

$$r_v(w) = w - 2 \left( \frac{v \cdot w}{(v \cdot v)} \right) v$$

— this transformation is a symmetry of the lattice if and only if half of the length square of the vector $v$ divides its level.

If at some point of the moduli space such a state becomes massless, we get an enhanced gauge symmetry. The corresponding plane in $Gr(d + 1, \mathbb{R}^{d+1,d+9})$ is invariant under the reflection in the momentum vector of the massless state. Those points in the moduli space which correspond to the non-invariant planes may be obtained by perturbing the action by the current in the Cartan subalgebra of the enhanced $SU(2)$. The T-duality corresponds to the Weyl group of the enhanced gauge group $[19]$.

We have to stress the essential difference with the description of the gauge groups for the usual heterotic compactification. Given the background specified by the plane $\omega \in \mathbb{R} \times \Gamma_{(D)}$, the roots of the gauge group belong to the sublattice $\omega^\perp \subset \Gamma_{(D)}$. For the usual heterotic string, the root system consisted of all the vectors of length square 2 in this sublattice. For the CHL compactifications, the root is either the vector with the length square 2 or the vector with the length square 4 which is on the level two in the whole lattice. Thus, given just the lattice $\omega^\perp$, one cannot yet tell what is the gauge group: one has to know the embedding $\omega^\perp \in \Gamma_{(D)}$.

It was shown in Section 3.2, that vectors of length square 2 correspond to either states from the twisted sector, or states from the untwisted sector with $P - Q \in \Delta + 2\Gamma_8$. The massless states from the level two sublattice correspond to states from untwisted sector with $P = Q$. Comparing this to the description of the world-sheet Kac-Moody algebras

---

$^6$ In the lattice $\Gamma_{1,1} \oplus \Gamma_8$, corresponding to the nine-dimensional theory, the length square of each level two vector is divisible by eight, but in lower dimensions we may have vectors of class 1 with length square four, because of the factor $\Gamma_{d,d}(2)$. 

19
in section 2, we see that those massless states whose momentum has length square 2 correspond to the level 2 $su(2)$ Kac-Moody algebra, while the length square 4 vectors correspond to the level 1 Kac-Moody algebra.

4.2. Examples of Gauge Groups.

Before proceeding with the gauge groups, let us remind here the definitions of some root systems. Unfortunately, the root systems are usually denoted by the same letters as the root lattices. To avoid confusion, we will use the bold letters for the root systems, and the usual letters for the root lattices (the lattices, generated by roots). Consider the space $\mathbb{R}^n$ with the basis $\{u_i\}_{i=1,...,n}$, $(u_i \cdot u_j) = \delta_{ij}$. We will need the following root systems:

1) $B_n$: consists of the vectors of the form $\sqrt{2}(\pm u_i \pm' u_j)$, and the vectors $\pm \sqrt{2} u_i$. Corresponds to the algebra $so(2n+1)$. The root lattice is $B_n \simeq \mathbb{Z}^n(2)$. Notice that we use the normalization of the roots of $B_n$, which differs from the normalization usually accepted in the textbooks on Lie groups [21] by the factor of $\sqrt{2}$. Our normalization agrees with the embedding of the root system in the CHL momentum lattice.

2) $C_n$: vectors of the form $\pm u_i \pm' u_j$, plus $\pm 2u_i$. Roots of the Lie algebra $sp(2n)$ (sometimes called $sp(n)$). They generate the root lattice $C_n \simeq D_n$.

3) $D_n$: $\pm u_i \pm' u_j$, $so(2n)$, the root lattice $D_n$.

4) $F_4$: roots of $C_4$ plus $\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_3 u_3 + \epsilon_4 u_4$ with all the possible signs $\epsilon_1, \ldots, \epsilon_4$. Root lattice $F_4 \simeq D_4$.

We have chosen such a normalization of the roots that the long roots have length square 4. As was explained at the end of the last subsection, this corresponds to the level one Kac-Moody algebra.

In eight dimensions we can embed the root system $D_9$ in the lattice $\Gamma_{1,1} \oplus \Gamma_8$. Indeed, consider the root system for this lattice ([17], Ch. 27). The corresponding Dynkin diagram may be obtained from the affine diagram for $E_8$ by attaching one more node to the affine root.

```
0 0
0 1
```

```
0 1
1 0
```

$\alpha_0 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8$

**Fig. 2:** Root system for $II_{1,9}$. 

20
If we remove the seventh root of $E_8$, we get the Dynkin diagram for the root system $D_9$. Thus, at the corresponding point of the moduli space we get the gauge algebra $so(18)$.

In eight dimensions, one can get the algebra $sp(20)$ in the following way. Represent the lattice as $\Gamma_{2,2} \oplus D_8$. Take the $D_8$ sublattice whose orthogonal complement is $\Gamma_{2,2}$. The root system $C_8$ can be embedded in the lattice $D_8$, so that its long roots are on the level two (indeed, the lattice generated by the root system $C_n$ is $D_n$). To get a root system $C_{10}$, we need just to add two vectors from $\Gamma_{2,2}$:

\[
\begin{bmatrix}
0 & 0 & \alpha_1 \\
0 & 1 & \\
1 & 0 & \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_8 \\
1 & 1 & 0 & -1 \\
0 & 0 & 0 & & & & & \\
\end{bmatrix}
\]

Fig. 3: Embedding the root system of $sp(20)$.

In the case $d \geq 2$ we can actually embed the root system of $sp(20)_1 \oplus so(2d - 1)_1$. Indeed, let us represent the lattice in the following form:

\[
\Gamma_{d-1,d-1}(2) \oplus \Gamma_{2,2} \oplus D_8
\]  

(4.3)

First, let us embed $C_{10}$ in $\Gamma_{2,2} \oplus D_8$, as described above. To find the root system $B_{d-1}$, let us consider the root system $A_{d-1}(2)$ (the roots of $su(d)$, rescaled by $\sqrt{2}$), embedded into $\Gamma_{d-1,d-1}(2)$ [10]. We can take the first $d - 2$ vectors of this root system to be the long roots of $B_{d-1}$, and as a short root we may choose the last root of $A_{d-1}(2)$ plus the vector

\[
\begin{bmatrix}
0 & 0 & 0 \\
1 & -1 & \\
\end{bmatrix}
\in \Gamma_{2,2} \oplus D_8
\]

which is orthogonal to all the roots of $sp(20)$. This gives the root system of $sp(20)_1 \oplus so(2d - 1)_1$.

We can also get the gauge group $sp(20 - 2n) \oplus so(2d + 2n - 1)$ for any $n \leq d + 1$. To find these gauge groups, we will need another useful isomorphism [7]. First of all, it is well known [10], that $\Gamma_{n,n}$ is isomorphic to the lattice consisting of the pairs $(w; \tilde{w})$, such that

\[
\text{The way we get the roots of } sp(20 - 2n) \oplus so(2d + 2n - 1) \text{ is essentially the translation to our language of the method suggested in [3].}
\]
\(w, \tilde{w} \in D^*_n\) and \(\tilde{w} - w \in D_n\). It turns out, that the lattice \(\Gamma_{n-1, n-1}(2) \oplus \Gamma_{1, 1}\) is isomorphic to the lattice, consisting of the vectors \(\sqrt{2}(w; \tilde{w})\), where \(w, \tilde{w} \in D^*_n\), and the difference \(w - \tilde{w}\) is either in \(D_n\) or in the vector conjugacy class of \(D^*_n\) modulo \(D_n\). This is explained in Appendix A. Notice an embedding

\[
\mathbb{Z}^n(2) \oplus \mathbb{Z}^n(-2) \subset \Gamma_{n-1, n-1}(2) \oplus \Gamma_{1, 1}
\] (4.4)

This embedding has the property that the long roots of both \(B_n\) and \(B_n(-1)\) are on the level two in the whole lattice \(\Gamma_{n-1, n-1}(2) \oplus \Gamma_{1, 1}\).

Then, for \(n > 0\) consider the embedding

\[
\mathbb{Z}^{n-1}(4) \oplus D_{10-n} \subset (\Gamma_{1, 1} \oplus D_8) \subset (\Gamma_{1, 1} \oplus D_8) \oplus \Gamma_{d-1, d-1}(2) \oplus \Gamma_{1, 1}
\] (4.5)

where all the vectors in \(\mathbb{Z}^{n-1}(4)\) are on the level two in \(\Gamma_{(D)}\). In the lattice \(\mathbb{Z}^{n-1}(4)\), let us pick \(n - 1\) vectors \(f_j\) with the scalar products

\[f_i \cdot f_j = 4 \delta_{ij}\]

Using the embedding (4.4), we can find in \(\Gamma_{d-1, d-1}(2) \oplus \Gamma_{1, 1}\) the \(2d\) vectors \(e^+_j\) and \(e^-_j\) with the scalar products

\[e^+_i \cdot e^+_j = 2 \delta_{ij}, \quad e^+_i \cdot e^-_j = 0, \quad e^-_i \cdot e^-_j = -2 \delta_{ij}\]

Then, the vectors \(e^+_j\) for \(j = 1, \ldots, d\) and \(f_j + e^-_j\) for \(j = 1, \ldots, n - 1\) may be taken as the short roots of \(so(2d + 2n - 1)\) on the level one. The long roots are expressed as sums and differences of the short roots. The root system of \(sp(20 - 2n)\) is embedded into the lattice \(D_{10-n}\) on the left hand side of (4.5).

We can find gauge groups \(F_4\) in the seven-dimensional theory. Indeed, let us use the isomorphism

\[\Gamma_{(7)} \simeq \Gamma_{3, 3} \oplus D_4 \oplus D_4\]

It follows that in the seven-dimensional compactification we can get the gauge algebra

\[so(6)_2 \oplus (F_4)_1 \oplus (F_4)_1\]

(4.6)

Indeed, we can embed the root system \(D_3\) into \(\Gamma_{3, 3}\) by embedding it into the lattice \(D_3\), which is a sublattice of \(\Gamma_{3, 3}\), as we explained above. Notice that, although the lattice \(D_3 \subset \Gamma_{3, 3}\) contains the root system \(C_3 \supset D_3\), we do not get the algebra \(sp(6)\), but only
so(6). The reason is that the long roots of sp(6) are not on the level two in the whole lattice \( \Gamma_{3,3} \). The root system \( F_4 \) is embedded into the lattice \( D_4 \).

For the six-dimensional compactification, we can embed

\[
sp(8)_1 \oplus (F_4)_1 \oplus (F_4)_1
\]

Indeed, the simple roots of the root system \( C_4 \) of \( sp(8) \) fit into \( \Gamma_{3,3} \oplus \Gamma_{1,1}(2) \) in the following way:

\[
\begin{pmatrix}
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 2
\end{pmatrix}
\]

This is in disagreement with our results: it is impossible to embed the root system \( B_4 \oplus F_4 \oplus F_4 \) into our lattice \( \Gamma_{(6)} \).

### 4.3. Topology of Gauge Groups.

We should comment on the global structure of these gauge groups. Let us first remind a few basic facts from the theory of Lie groups [21]. Suppose \( h \) is a Cartan subalgebra of the semisimple Lie algebra \( \mathfrak{g} \). Since any two elements from \( h \) commute, the corresponding adjoint operators may be simultaneously diagonalized. The eigenvalues are the linear functionals on \( h \), that is the elements of \( h^* \). They are called roots. For each root \( \alpha_i \), we have the corresponding \( sl(2) \) subalgebra \( \{ e_{\alpha_i}, f_{\alpha_i}, h_{\alpha_i} = [e_{\alpha_i}, f_{\alpha_i}] \} \). Elements \( \alpha_i^\vee = \kappa_i h_{\alpha_i} \in h \), where \( \kappa \) is the coefficient adjusted in such a way that \( [\alpha_i^\vee, e_i] = 2e_i \), are called coroots. Notice

\[
\alpha_i(\alpha_i^\vee) = 2
\]
There is an invariant bilinear form on $g$, which enables us to identify $h$ with $h^*$. We will call the corresponding isomorphism of linear spaces $\nu : h^* \rightarrow h$. We have:

$$\alpha_i^\vee = \frac{2}{(\alpha_i, \alpha_i)} \nu(\alpha_i) \quad (4.9)$$

Let us explain this formula. Consider $h \in h$ such that $[h, e_{\alpha_i}] = 0$. This means that $h \perp \nu(\alpha)$. Let us prove $h \perp \alpha^\vee$. Indeed,

$$(h \cdot \alpha_i^\vee) = \kappa^{-1}(h \cdot [e_{\alpha_i}, f_{\alpha_i}]) = \kappa^{-1}([h, e_{\alpha_i}] \cdot f_{\alpha_i}) = 0$$

Since an arbitrary element $h \in h$ orthogonal to $\nu(\alpha)$ is orthogonal also to $\alpha^\vee$, we get $\nu(\alpha) \sim \alpha^\vee$. The coefficient of proportionality may be found from (4.8).

Let us denote $Q$ the lattice generated by the roots, and $Q^\vee$ the lattice generated by the coroots. The weight lattice

$$P = (Q^\vee)^*$$

consists of those vectors which have integer scalar products with the coroots.

The global structure of the Lie group $G$ is defined by the lattice

$$X(T) = \text{Hom}(T, S^1)$$

— the lattice of characters of the maximal torus. This may be an arbitrary sublattice of $P$, containing $Q$:

$$Q \subset X(T) \subset P \quad (4.10)$$

The first homotopy group $\pi_1(G)$ of $G$ is isomorphic to $P/X(T)$.

Given the Lie algebra $g$ with the root system $\Delta$, we can define the dual Lie algebra $g^\vee$ with the dual root system $\Delta^\vee$. The topology of the dual Lie group is specified as follows:

$$X(T_{G^\vee}) = (X(T_G))^*.$$ Examples of pairs $(G, G^\vee)$ may be found in [22].

If at some point of the moduli space we have a root system $\Delta$ embedded into the sublattice orthogonal to $\omega$, then we get the corresponding group $G$ as the gauge group. Some information about the global structure of the gauge group may be obtained from the known structure of the perturbative spectrum. Consider the orthogonal projection $\pi$ on the space, generated by $\Delta$. The perturbative states are charged under the group $G$. The charges span the lattice $\pi \Gamma(D)$. This means, that at least

$$X(T) \supset \pi \Gamma(D) \quad (4.11)$$
which implies

$$\pi_1(G) \subset P/\pi \Gamma(D)$$

**Examples.** We have found the embeddings into $\Gamma(D)$ of the root systems $B_n$ for various $n$. The coroot lattice of $B_n$ is $B_n^\vee = C_n \left( \frac{1}{2} \right)$. $C_n \left( \frac{1}{2} \right)^*$ is generated by the vectors $\sqrt{2} u_j$, $j = 1, \ldots, n$, and the “spinor” $\frac{1}{\sqrt{2}} \sum_j u_j$. One can see that the “spinor” weight does actually belong to $\pi \Gamma(D)$. Indeed, the spinor weight of $B_n$ is characterized by the property that its scalar product with the short roots of $B_n$ is 1. Since the short roots of $B_n$ are on the level one in $\Gamma(D)$, there is at least one vector $v \in \Gamma(D)$ such that the scalar product of $v$ with the given short root is one. The projection $\pi v$ of this vector belongs to the spinor conjugacy class. This means that the group is in fact $Spin(2n + 1)$ — the universal covering of $SO(2n + 1)$.

The root system $sp(20)$, embedded into the eight-dimensional lattice as shown on Fig.3, corresponds to the group $Sp(20)$ (the simply connected group). Indeed, consider the vector

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$  

The scalar product of this vector with all the roots of $sp(20)$ is zero, except for the scalar product with the leftmost root, which is equal to one. This means that the corresponding state transforms in the fundamental representation of $Sp(20)$. On the other hand, for the root system $sp(20 - 2n)$ for $n > 1$, embedded in the sublattice $D_{10 - n}$ in (4.5), there are no states in the perturbative spectrum, transforming in the fundamental representation. This suggests that the corresponding group is actually $Sp(20 - 2n)/\mathbb{Z}_2$.

**4.4. S-duality in four dimensions.**

It was observed in [5], that the set of maximal gauge groups in four dimensions is self-dual: if we have the gauge group $G$ at some point in the moduli space, we must have the gauge group $G^\vee$ at some other point in the moduli space. In this subsection we will give an explanation of this fact, using our description of the lattice.

There is an involution on the moduli space, which transforms the background with the gauge symmetry $G$ to the background with the gauge symmetry $G^\vee$. This involution may be interpreted as the action of S duality on the background fields. To explain how S duality works, we have to remember that string theory has infinitely many nonperturbative states, arising from the quantization of the solitons. In this paper we have so far been concerned with the perturbative states only. In the low energy theory, we have 20 abelian gauge fields at the general point in the moduli space. Perturbative states arising from strings having nonzero momenta or winding in the compactified directions are electrically charged
under these gauge fields. From this point of view \( \Gamma \) is the lattice of electric charges. There are also nonperturbative states, carrying magnetic charges. As explained in [23], they correspond to fivebranes partially wrapping the six-torus. Given the lattice of electric charges, the allowed magnetic charges should satisfy certain quantization conditions (see [23] and references therein). These conditions require that the allowed magnetic charges belong to the lattice \( \Gamma^* \).

It was conjectured in [5], that the given four-dimensional CHL theory with the coupling constant \( \lambda \) is equivalent to the dual CHL theory with the coupling constant \( \frac{1}{\lambda} \), so that the perturbative states of the dual theory are identified with the magnetically charged nonperturbative states in the original theory. The background fields are specified by the plane \( \omega \subset \mathbb{R} \otimes \Gamma \). From the point of view of the dual theory, \( \omega \) is the plane in \( \mathbb{R} \otimes \Gamma^* \).

An important property of the four-dimensional CHL lattice is that \( \Gamma \simeq \Gamma(2) \). Indeed,

\[
\Gamma \simeq \Gamma \oplus \Gamma^* \oplus D_4 \oplus D_4 \simeq \Gamma^*(2) \tag{4.12}
\]

We have \( \Gamma(2)^* \simeq \Gamma \left( \frac{1}{2} \right) \) since \( \Gamma_{n,n} \) is self-dual, and the isomorphism \( D_4^* \simeq D_4 \left( \frac{1}{2} \right) \) is explained\(^8\) on pp.118-119 of [17]. Let us fix an isomorphism \( S : \Gamma^* \to \Gamma \left( \frac{1}{2} \right) \) (they are all conjugate by symmetries of \( \Gamma \), which will give equivalent backgrounds). Consider the background specified by the plane \( \omega \subset \mathbb{R} \otimes \Gamma \). Since \( \Gamma_4 \subset \Gamma_4^* \), we may consider this embedding as an embedding of \( \omega \) into \( \mathbb{R} \otimes \Gamma^* \). Acting on \( \omega \) by the operator \( S \in O(6,14) \), we get

\[
\omega \subset \mathbb{R} \otimes \Gamma \left( \frac{1}{2} \right) \tag{4.13}
\]

which is the same as an embedding of \( \omega \) into \( \mathbb{R} \otimes \Gamma \). This gives the required transformation of the background. This is an involution \( S^2 \omega = \omega \), commuting with the T dualities.

Suppose that we start with the background, specified by the plane \( \omega \), such that the sublattice \( \omega^\perp \cap \Gamma \) contains a root system \( \Delta \) (the long roots of this root system should be on level two in the whole lattice \( \Gamma \), not only in \( \omega^\perp \cap \Gamma \)). We want to prove that the

\(^8\) Let us explain it here for completeness. For the lattice \( D_4 = \{n_1 u_1 + \cdots + n_4 u_4 | n_1 + \cdots + n_4 \in 2\mathbb{Z} \} \), the weight lattice \( D_4^* \) consists of the vectors \( m_1 u_1 + \cdots + m_4 u_4 \), where all \( m_j \) are either simultaneously integers, or simultaneously half-integers. The simple roots \( \alpha_0, \ldots , \alpha_4 \) of \( D_4 \) (satisfying the relation \( \alpha_0 + 2 \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 0 \)) are embedded into \( D_4^*(2) \) in the following way:

\[
\alpha_0 = \frac{1}{\sqrt{2}} (u_1 - u_2 - u_3 - u_4), \quad \alpha_1 = \frac{1}{\sqrt{2}} (u_1 + u_2 + u_3 + u_4), \quad \alpha_2 = \frac{1}{\sqrt{2}} (u_2 - u_1 + u_3 - u_4), \quad \alpha_3 = \frac{1}{\sqrt{2}} (u_3 - u_1 - u_2 - u_4), \quad \alpha_4 = \frac{1}{\sqrt{2}} (u_4 - u_1 - u_2 - u_3);
\]

they generate the lattice \( D_4^*(2) \).
lattice $\omega^\perp \cap \Gamma^{(4)}$ contains the dual root system $\Delta^\vee$. To obtain the dual root system $\Delta^\vee$, we have to divide by 2 the long roots of $\Delta$, and make the overall rescaling (this rescaling is related to the coefficient $\frac{1}{2}$ in (4.13)). The long roots of $\Delta$ are on the level 2 in the whole lattice $\Gamma^{(4)}$. This implies that half of a long root belongs to $\Gamma^{*(4)}$. The short roots of $\Delta$ have integer scalar products with all the vectors from $\Gamma^{(4)}$; after rescaling of the metric by 2 they become the level two vectors in $\Gamma^{(4)}(2) = \Gamma^{(4)}$. Thus, they may be taken as the long roots of $\Delta^\vee$. This proves that $\Delta^\vee \subset (S\omega)^\perp$.

If the root system $\Delta$ was maximal in $\omega^\perp$ (that is, we could not find any other root system, containing $\Delta$ as a proper subset), then the root system $\Delta^\vee$ in $\omega^\perp \cap \Gamma^{(4)}$ is also maximal. Indeed, suppose that there is some larger root system $\hat{\Delta} \supset \Delta^\vee$ in $\omega^\perp \cap \Gamma^{*(4)}$. Then, we may use $S^2 = 1$ to prove that $\omega^\perp \cap \Gamma^{(4)}$ contains $\hat{\Delta}^\vee$. But $\hat{\Delta}^\vee$ is larger then $\Delta$, which contradicts to the assumption that $\Delta$ is maximal.

5. Comparison to Type IIA, M Theory and F Theory Duals.

5.1. Type IIA.

It was argued by Schwarz and Sen [7], that the six-dimensional compactification of the CHL string is dual to Type IIA on the singular K3 surface $Y$, with some Ramond-Ramond background fields turned on. The K3 surface involved is the quotient of a smooth K3 $X$ by certain $\mathbb{Z}_2$ involution. The corresponding involution of the cohomology lattice $H^2(X, \mathbb{Z}) = II_{3,19}$ is the exchange of the two $\Gamma_8$ sublattices. Also, the RR background field should be turned on. The origin of this RR background may be explained from the point of view of M Theory lift. Type IIA on K3 is M Theory on $K3 \times S^1$. The theory of Schwarz and Sen is the quotient by the following symmetry [7]: shift by half of the M Theory circle, and exchange of two $\Gamma_8$ sublattices in the cohomology group of K3. In this section we study the moduli space of Type IIA on K3 with involution, and show that it coincides with our answer for the moduli space of six-dimensional CHL string. This is an evidence in favour of the proposed duality.

We will consider the $\mathbb{Z}_2$ involution $\iota$ of the smooth K3 surface $X$, which acts as an identity on $H^{2,0}(X)$. Such involutions are known in algebraic geometry as Nikulin involutions [24,25]; we give an example of the Nikulin involution in Appendix B. More precisely, consider the symmetry $\iota^*$ of the lattice $L = H^2(X, \mathbb{Z}) \simeq \Gamma_{3,19}$, exchanging the two $\Gamma_8$ sublattices in $\Gamma_{3,19} \simeq \Gamma_{3,3} \oplus \Gamma_8 \oplus \Gamma_8$. The moduli space of Type IIA worldsheet conformal field theory is parametrized by the four-planes in $\mathbb{R} \otimes II_{4,20}$. If the plane is
orthogonal to the antidiagonal \((\Gamma_8)_{\text{Anti-Diag}} \subset \Gamma_8 \oplus \Gamma_8 \subset II_{4,20}\), then the symmetry \(\iota^*\) is related to the symmetry of the worldsheet conformal field theory. Indeed, the complex structure of \(X\) specifies an oriented 2-plane \(\omega\) in \(\mathbb{R} \otimes H_2(X, \mathbb{Z})\) (through the period map). This plane is a subspace of the 4-plane which specifies the background. Suppose that this plane is orthogonal to the anti-diagonal \((\Gamma_8)_{\text{Anti-Diag}} \subset \Gamma_8 \oplus \Gamma_8 \subset L\). This means that \(\iota^*\) preserves the Hodge structure of \(\mathfrak{U} \otimes H_2(X, \mathbb{Z})\), specified by the plane \(\omega\). It also preserves the cones \(V^+\) and \(V^-\) defined as \(V^+ \cup V^- = \{x \in \mathbb{R} \otimes H_2(X, \mathbb{Z}) | x^2 > 0\}\), and it leaves all the vectors with length square \(-2\) invariant. Under these conditions, the Theorem 2.7’ from [24] tells us that there exists an algebraic automorphism \(\iota\) of \(X\) (an involution), whose action in cohomology coincides with \(\iota^*\). Since the Kahler classes of all the cycles in \(H^2(K3, \mathbb{Z})\) are invariant under \(\iota^*\), the metric is invariant under \(\iota\) (since there is only one metric with the given Kahler class). The way our 4-plane parametrizes the \(B\) field and the volume of \(K3\) is explained in [11]. It follows that the \(B\)-field is \(\iota\)-invariant, if the plane is diagonal.

This means that if the point of the moduli space \(O(II_{4,20}) \setminus O(4,20)/O(4) \times O(20)\) is represented by the plane orthogonal to \((\Gamma_8)_{\text{Anti-Diag}} \subset \Gamma_{4,4} \oplus \Gamma_8 \oplus \Gamma_8\), then the symmetry of the lattice exchanging the two \(\Gamma_8\) sublattices is a symmetry of the worldsheet theory, and we may consider the corresponding orbifold. Considering just an orbifold \(K3/\iota\) does not give a new theory, because \(K3/\iota\) is birational to another \(K3\): what we would get is again \(IIA/K3\). To get the Type IIA dual of CHL string we have, as prescribed in [3], to turn on the RR flux localized on the fixed points of \(\iota\). The quotient \(X/\iota\) is singular, because \(\iota\) has eight fixed points. To study the geometry of the quotient, it is convenient first to blow up these eight singular points, and get a smooth \(K3\) surface \(Y\). Then the minimal primitive sublattice of \(H^2(Y, \mathbb{Z})\) containing the exceptional curves on \(Y\) would be isomorphic to \(D_8^*(-2)\) [24,25]. This lattice is called Nikulin lattice, and we introduce for convenience a notation:

\[ \mathcal{N} = D_8^*(-2) \]

In Appendix B, we explain, in one particular case, why the singularities generate this lattice.

The RR background field may be defined in the following way. Let \(Y'\) be \(Y\) with the eight exceptional divisors thrown away. This manifold has nontrivial first homology group [24]:

\[ H_1(Y', \mathbb{Z}) = \mathbb{Z}_2 \]
Then, the monodromy of the RR gauge field around the generator of $H_1(Y', \mathbb{Z})$ should be $-1$. It was argued in [7], that turning on this background makes the singularity irremovable.

The moduli space for $K3$ with irremovable singularities, whose vanishing cycles generate the Nikulin lattice, is parametrized by those planes in $Gr(4, \mathbb{R}^{4,20})$ which are orthogonal to the Nikulin lattice. The orthogonality condition ensures that the vanishing cycles are collapsed, and there is no $B$-field flow through them. Thus, the moduli space of our theory is locally just the locus in the moduli space of IIA/K3, corresponding to the planes orthogonal to the Nikulin lattice. Which discrete identifications should we make? We have to consider those symmetries of the lattice $II_{4,20}$, which preserve the sublattice generated by singularities. There is a unique primitive embedding of the Nikulin lattice into $II_{4,20}$ (this follows from the Theorem 2.8 in [25], we put this theorem in Appendix A for convenience), and the orthogonal sublattice is $\Gamma(6) = \Gamma_{4,4} \oplus D_8^*(-2)$. The discrete symmetries of the background are the symmetries of this lattice. Indeed, any symmetry of the lattice $II_{4,20}$ which preserves the lattice generated by singularities acts correctly on the orthogonal lattice $\Gamma(6)$. The converse is also true: given the symmetry of the sublattice $\Gamma(6)$, we may continue it to the symmetry of $II_{4,20}$. The last statement is, in fact, not completely trivial, since it is not true that $II_{4,20}$ is the direct sum of the Nikulin lattice and the lattice $\Gamma(6) = \Gamma_{4,4} \oplus D_8^*(-2)$ (such a direct sum would not be a self-dual lattice). This means, that generically speaking, we cannot just continue the symmetry of $\Gamma(6)$ as identity on the orthogonal sublattice: this would not preserve the way $\Gamma(6)$ and $\Gamma(6)\perp$ are “glued together” in $II_{4,20}$ (in other words, such a naive continuation would not be a symmetry of the lattice.) Let us explain how to find a correct continuation. To get a self-dual lattice $L$ from the sum of two non-self-dual lattices $M_1$ and $M_2$, we have to take generators of the form $px + qy$ where $x \in M_1$, $y \in M_2$, but $p, q$ are some rational numbers. The vectors in the self-dual lattice modulo those in the direct sum $M_1 \oplus M_2$ are called the glue vectors. For these vectors $px \in M_1^*$, and $qx \in M_2^*$. In fact, one can prove [17], that the map $\gamma : [px] \to [qy]$ from $M_1^*/M_1$ to $M_2^*/M_2$ is an isomorphism. We want to continue the symmetry $g_1 \in O(\Gamma(6))$ to the symmetry of the lattice $II_{4,20}$. Most of the symmetries of $\Gamma(6)$ do not act as an identity on $A_{\Gamma(6)} = \Gamma_{4,4}^*/\Gamma(6)$. Thus, we cannot continue such a symmetry $g_1$ just as an identity on $(\Gamma(6))\perp = \mathcal{N}$: what we should do instead is to find such a symmetry $g_2 \in O(\mathcal{N})$ which reduces on $\mathcal{N}^*/\mathcal{N}$ to $\gamma g_1 \gamma^{-1}$, and the proper continuation of $g_1$ will be then $g_1 \oplus g_2$. In fact, we can always find such a $g_2$, because as we prove in the Appendix A, an arbitrary symmetry $\bar{g}$ of $\mathcal{N}^*/\mathcal{N}$ may be represented by the symmetry $g$ of $\mathcal{N}$.
This gives the moduli space isomorphic to the Grassmanian modulo $O(\Gamma_{(6)})$, which coincides with our result for the momentum lattice for the six-dimensional compactification of the CHL string.

Notice that the discrete identifications corresponding to geometric symmetries of $Y$ (those which do not involve the $B$ field) may be explained in somewhat simpler way. Indeed, the geometric moduli space of $Y$ with Nikulin singularities are the same as the moduli space of those smooth K3 surfaces $X$, which admit Nikulin involution. This condition means that the corresponding plane in $\mathbb{R}^{3,19}$ should be diagonal. The subgroup of $O(II_{3,19})$ which preserves the anti-diagonal $E_8$ is $O(\Gamma_{3,3} \oplus \Gamma_8(2))$ (we have to prove that any symmetry of the sublattice $\Gamma_{3,3} \oplus \Gamma_8(2) \subset II_{3,19}$ may be extended to the symmetry of $II_{3,19}$: this may be derived from the surjectivity of the map $O(\Gamma_8(2)) \rightarrow O(\Gamma_8(2))$, which is proven in the Appendix A.) It follows from (3.18) and $\Gamma_{3,3}(2) \oplus \Gamma_8 \simeq \Gamma_{3,3} \oplus N$, that the group of geometric symmetries coincides with

$$O(\Gamma_{3,3} \oplus N)$$

which is in agreement with what we have obtained from the study of the theory on the quotient surface $Y$. This may be considered as a geometric interpretation of the isomorphism $\Gamma_{3,3}(2) \oplus \Gamma_8 \simeq \Gamma_{3,3} \oplus N$.

5.2. M Theory and F Theory Duals.

Recently, it was proposed [9,3], that the seven-dimensional CHL string is dual to the compactification of M theory on K3 surface with some irremovable $D_4 \oplus D_4$ singularity, and the eight-dimensional compactification may be described as F theory on K3 with irremovable $D_8$ singularity [3]. Let us show, that this is in agreement with our description of the moduli space of CHL string.

Consider first $D = 8$. The moduli of the F-theory compactification are parametrized by the complex structures of the elliptic K3 surfaces with a section [20], which may be thought of as the timelike planes in $\mathbb{R}^{2,18}$, modulo the action of the discrete group of symmetries $II_{2,18}$. The $D_8$ singularity means that the rational cycles, with the intersection matrix realizing the root system of $D_8$, become holomorphic. This means, that the plane in $\mathbb{R}^{2,18}$ becomes orthogonal to the roots of $D_8$. The primitive embedding of the root system of $D_8$ into $II_{2,18}$ is unique modulo the symmetries of $II_{2,18}$ (see Theorem A1 in the Appendix A.). It may be constructed in the following way:

$$D_8 \subset \Gamma_{16} \subset \Gamma_{2,2} \oplus \Gamma_{16} \simeq II_{2,18}$$

(5.1)
From this representation of the embedding, one can see that the orthogonal complement to this $D_8$ is

$$D_8^\perp = \Gamma_{2,2} \oplus D_8$$

which is our result for the Narain lattice of the eight-dimensional CHL theory. One can prove, that an arbitrary isomorphism of $D_8^\perp \subset II_{2,18}$ may be extended to the isomorphism of $II_{2,18}$. This means, that the discrete symmetries of the F Theory background are the symmetries of the lattice $D_8^\perp = \Gamma_{(8)}$. Thus, the moduli space of the CHL string in eight dimensions is isomorphic to the moduli space of its F Theory dual.

In the recent papers \cite{2,3}, it was argued that the moduli space of the eight-dimensional CHL theory is isomorphic to the moduli space of Eriques surfaces, which is the arithmetic quotient:

$$O(\Gamma) \backslash O(2,10)/O(2) \times O(10)$$

where $\Gamma = \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \oplus \Gamma_8(2)$. It follows from the isomorphism $\Gamma^* \simeq \Gamma_{(8)}(\frac{1}{2})$ and the fact that $O(L) \simeq O(L^*)$ for an arbitrary lattice $L$, that the answer (5.2) for the moduli space coincides with our answer (3.24) for the eight-dimensional theory ($d = 1$).

Consider $D = 7$. The dual theory is M Theory compactified on $K3$ with irremovable $D_4 \oplus D_4$ singularity. It follows from the Theorem A1 in the Appendix A, that there is a unique primitive embedding of $D_4 \oplus D_4$ into $II_{3,19}$. We can represent it as follows:

$$D_4 \oplus D_4 \subset E_8 \oplus E_8 \subset E_8 \oplus E_8 \oplus \Gamma_{3,3} \simeq II_{3,19}$$

Its orthogonal complement is:\footnote{Notice that in all three cases, embedding the lattice $X$ generated by singularities into the unimodular lattice, we have got the orthogonal complement to $X$ isomorphic to $\Gamma_{n,n} \oplus X$. This is not a coincidence. Indeed, it is known from the general theory of lattices (see Theorem A2 in the Appendix A), that for the sublattice in the unimodular lattice, its discriminant-form is minus the discriminant-form of the orthogonal sublattice. Since for our lattices the discriminant-forms take values in half-integers, this means that the discriminant form of the sublattice is equal to the discriminant-form of the orthogonal lattice. Since that rank and the order of $X^*/X$ for our lattice $X$ satisfies the conditions of the Theorem A1 from Appendix A, the lattice $X$ is uniquely determined by its rank, signature and the discriminant-form.}

$$(D_4 \oplus D_4)^\perp = \Gamma_{3,3} \oplus D_4 \oplus D_4$$

— this is our result for the seven-dimensional lattice. The symmetries of this lattice are the symmetries of the M Theory background. Thus, the dual theory has the same moduli space as the one we have obtained for the seven-dimensional CHL string.
6. Cusps in the Moduli Space.

One characteristic of the duality group is how many cusps we have in the moduli space of the theory. The cusps may be considered as the possible ways for going far away in the moduli space. As we have explained in the section 4, the point of the moduli space may be represented by the $10-D$-dimensional plane in $\mathbb{R}^{10-D,18-D}$, such that the scalar product is negative definite on it. The moduli space is not compact, and the infinities correspond to the possible degenerations of this scalar product. If there is at least one null-vector on the plane, then this plane is infinitely far away.

We will restrict ourselves with such degenerate planes only, that the maximal isotropic subspace is rational (the word “rational” means, that the plane goes through sufficiently many points of the lattice: one can introduce a basis, consisting of the lattice points). The rational isotropic planes of dimension $p = 1, \ldots, 10-D$ parametrize the rational components of the boundary of the moduli space: these components consist of the $10-D-p$-dimensional planes in the orthogonal complement to the given isotropic plane. (The boundary components are themselves the Grassmanians $Gr(10-D-p, \mathbb{R}^{10-D-p,18-D-p})$). There is known in mathematical literature a construction of compactifications of the quotients of Grassmanians by the discrete group (in fact, arbitrary quotients of the form $\Gamma \backslash G/K$) whose boundaries consist of the rational components only — see [27] for a recent discussion, and references therein. On the other hand, it was argued recently [28], that compactifications with irrational boundary components are physically meaningful and may be described in terms of noncommutative geometry.

Since an arbitrary isotropic plane may be embedded into some maximal isotropic plane, we will classify first the maximal rational isotropic planes in $\mathbb{R} \otimes \Gamma(D)$.

$D=9$. There is only one light-like vector in $\Gamma_{1,1} \oplus \Gamma_8$, modulo the symmetry group [17]. This may be explained in the following way. The group $O(II_{1,9})$ acts on the space $\mathbb{R}^{1,9}$, and the fundamental domain for this group may be constructed. The boundaries of the fundamental domain are the hyperplanes orthogonal to the simple roots. The Dynkin diagram for the system of simple roots is shown on Fig. 2. One can prove by direct computation that the only one light-like vector on the boundary of the fundamental domain is minus the imaginary root of $\hat{E}_8$ (the combination of roots of $\hat{E}_8$, which has zero length square).

Another proof is given in the footnote in the Section 3.3.
D=8. In the lattice corresponding to the eight-dimensional theory there are at least two rational light-like planes. One is the standard light-like plane in the lattice

\[ \Gamma_{2,2} \oplus D_8 \]  \hspace{1cm} (6.1)

and the other — the standard light-like plane in

\[ \Gamma_{1,1}(2) \oplus \Gamma_{1,1} \oplus \Gamma_8 \]  \hspace{1cm} (6.2)

We call “standard” the light-like plane, generated by the vectors of the form

\[
\begin{bmatrix}
0 \\
* \\
0 \\
* \\
\end{bmatrix}
\]  

(the notations are as explained in Section 3.3 after the formula (3.25).)

Let us prove that each light-like plane is equivalent to one of these two.

Consider a rational light-like 2-plane \( \omega \subset \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \oplus \Gamma_8 \). We will use a trick which we have learned from [11]. Consider the intersection \( \omega^{(1)} \) of \( \omega \) with the subspace \( \nu^\perp \), where \( \nu \) is a standard light-like vector in \( \Gamma_{1,1}(2) \). Then, we can apply such a symmetry of \( \Gamma_{1,1} \oplus \Gamma_8 \), that the projection of \( \omega^{(1)} \) onto \( \Gamma_{1,1} \oplus \Gamma_8 = \nu^\perp / \nu \) is the standard light-like line in \( \Gamma_{1,1} \oplus \Gamma_8 \). This means that \( \omega^{(1)} \) is now the rational light-like line in the standard light-like plane in \( \Gamma_{1,1}(2) \oplus \Gamma_{1,1} \). Using the Lemma from Appendix C, we may bring it ot one of the two standard forms: either with the orthogonal complement in \( \Gamma_{1,1}(2) \oplus \Gamma_{1,1} \) being \( \Gamma_{1,1} \), or with the orthogonal complement \( \Gamma_{1,1}(2) \). The other rational vector \( \nu' \), generating the light-like plane \( \omega \) may be chosen to belong to \( \Gamma_{1,1} \oplus \Gamma_8 \) in the first case, or to \( \Gamma_{1,1}(2) \oplus \Gamma_8 \) in the second case. In the first case, we can apply the symmetry of \( \Gamma_{1,1} \oplus \Gamma_8 \) which brings \( \nu' \) to the standard form in \( \Gamma_{1,1} \oplus \Gamma_8 \), thus we get \( \omega \) the standard light-like plane in (6.2). In the second case, \( \nu' \) may be brought to one of the two standard forms into \( \Gamma_{1,1}(2) \oplus \Gamma_8 \), as described in the Appendix C, which gives us either the case (6.1), or the case (6.2).

D=7. Now consider compactification to seven dimensions. The seven-dimensional lattice may be viewed in the three ways:

\[ \Gamma_{1,1} \oplus \Gamma_{2,2}(2) \oplus \Gamma_8 \simeq \Gamma_{2,2} \oplus \Gamma_{1,1}(2) \oplus D_8 \simeq \Gamma_{3,3} \oplus D_4 \oplus D_4 \]  \hspace{1cm} (6.3)

The standard lightlike planes in each of these three lattices give three non-equivalent cusps. Let us prove that there are no other cusps.
Consider a primitive lightlike vector \( v \in \Gamma_{1,1}(2) \). Consider the projection of \( \omega' = \omega \cap v^\perp \) to \( v^\perp / v \cong \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \oplus \Gamma_8 \). This is a light-like 2-plane, and it is equivalent to one of the two standard planes, as discussed above. Thus, we have two cases to consider:

**The first case:** The projection is equivalent to the standard light-like plane in \( \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \oplus \Gamma_8 \). Then, by the Lemma from Appendix C, the orthogonal complement to \( \omega' \) in \( \Gamma_{1,1} \oplus \Gamma_{2,2}(2) \) is either \( \Gamma_{1,1} \), or \( \Gamma_{1,1}(2) \). In case if it is \( \Gamma_{1,1} \), our plane \( \omega \) is generated by the standard light-like plane in \( \Gamma_{2,2}(2) \) and some light-like vector in \( \Gamma_{1,1} \oplus \Gamma_8 \), which by the symmetry of \( \Gamma_{1,1} \oplus \Gamma_8 \) may be brought to the unique standard form. Thus, in this case \( \omega \) is equivalent to the standard plane in the first lattice in (6.3). In the case if the orthogonal complement is \( \Gamma_{1,1}(2) \), we have \( \omega \) generated by the standard plane in \( \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \) and some lightlike vector \( v' \in \Gamma_{1,1}(2) \oplus \Gamma_8 \). Since this vector \( v' \) is equivalent to one of the two standard forms, we get either the standard plane in the first, or in the second lattice in (6.3).

**The second case:** The projection is equivalent to the standard light-like plane in \( \Gamma_{2,2} \oplus D_8 \). Then, by the Lemma from Appendix C, the orthogonal complement to this plane in \( \Gamma_{2,2} \oplus \Gamma_{1,1}(2) \) is either \( \Gamma_{1,1} \), or \( \Gamma_{1,1}(2) \). If it is \( \Gamma_{1,1} \), then the vector \( v' \) in \( \omega \) may be chosen to belong to \( \Gamma_{1,1} \oplus D_8 \). There are two standard light-like vectors in \( \Gamma_{1,1} \oplus D_8 \), thus we get the standard plane in the second or the third lattice in (6.3). If it is \( \Gamma_{1,1}(2) \), then \( v' \) belongs to \( \Gamma_{1,1}(2) \oplus D_8 \), and the Lemma from Appendix C gives two possibilities, corresponding to the standard light-like planes in the second or the third lattices in (6.3).

In the case of partial decompactification, the plane contains one or more rational light-like vectors, which then may be included into one of the standard lightlike planes. Then, the Lemma from Appendix D may be applied to bring these vectors to one of the two standard forms within the standard plane.

We should explain the physical meaning of these cusps.

Let us prove that the points far away in the moduli space correspond to the degeneration of the metric on the torus. The \( d+1 \) -dimensional plane corresponding to the point of the moduli space with the Wilson lines \( A_i = (a_i, a_i) \), the metric \( g_{ij} \) and the 2-form \( b_{ij} \) may be specified as the plane orthogonal to the \( d+9 \) -dimensional plane with the equation

\[
G_{ij}l^j = n_i + (a_i \cdot R) \tag{6.4}
\]

where

\[
G_{ij} = g_{ij} + b_{ij} + \frac{1}{2}(a_i \cdot a_j) \tag{6.5}
\]
This implies, that in the plane corresponding to our background we may pick a basis of vectors \( \{ X_p \} \), given by the following formula:

\[
X_p = \begin{bmatrix}
\delta_p^i & a_p \\
-G_{ip}
\end{bmatrix}
\tag{6.6}
\]

There is a subgroup of the duality group, which acts on the background by shifts

\[
a_i \rightarrow a_i - k_i \alpha, \quad G_{ij} \rightarrow G_{ij} - (a_i \cdot \alpha) k_j + k_i k_j
\]

where \( \alpha \) is a root of \( E_8 \) and \( \{ k_i \} \) is the set of integers. This means, that the point at infinity in the moduli space may be attained with the Wilson lines kept finite. Also, since \( b_{ij} \) is defined modulo a shift by an integer antisymmetric matrix, we may assume that \( b_{ij} \) has finite limit. The matrix of scalar products for the basis (6.6) is:

\[
(X_p \cdot X_q) = -2g_{pq}
\tag{6.7}
\]

This means that the plane generated by \( \{ X_p \} \) is at finite distance unless the metric \( g_{pq} \) becomes singular. Thus, the infinities of the moduli space correspond to the degenerations of the metric on the torus. Degeneration of the metric means that at least one eigenvalue of \( g_{ij} \) goes either to infinity, or to zero. If the corresponding eigenvector has rational coordinates, then we may interpret this limit as shrinking some cycle on a torus or considering a cycle of very large size.

If we think of the cusps as the possible ways for going to infinity in the moduli space, then different light-like planes are physically inequivalent. But if we consider the limiting theories, then some identifications occur. Indeed, notice that all the lightlike planes found have a primitive vector from some \( \Gamma_{1,1} \) sublattice in them. This means, that the corresponding limiting theories may be obtained by decompactification of the ninth dimension, giving a usual heterotic string. In the moduli space of the heterotic string on a torus, we have two cusps. They correspond to the ten-dimensional \( E_8 \times E_8 \) and \( Spin(32)/\mathbb{Z}_2 \) heterotic string. In fact, the plane with the orthogonal complement \( E_8 \) in \( \Gamma_{2,2} \oplus D_8 \) gives \( E_8 \times E_8 \) theory, while the plane with orthogonal complement \( D_8 \) gives \( Spin(32)/\mathbb{Z}_2 \) theory. In the seven-dimensional lattice \( \Gamma_{3,3} \oplus D_4 \oplus D_4 \), the plane with the orthogonal complement \( E_8 \) gives \( E_8 \times E_8 \) heterotic string in ten dimensions, and those whose complements are \( D_8 \) or \( D_4 \oplus D_4 \) give \( Spin(32)/\mathbb{Z}_2 \) heterotic string. To prove this, one has to look at the background of the heterotic string corresponding to the given cusp of the CHL string after decompactification of the ninth dimension.
Acknowledgments

I would like to thank J.H. Conway, A. Losev, D.R. Morrison, S. Sethi, E. Sharpe and especially E. Witten for many interesting and helpful discussions. This work was supported in part by RFFI Grant No. 96-02-19086 and partially by grant 96-15-96455 for support of scientific schools.

Appendix A. Discriminant-form.

There is a very convenient invariant of the lattices, called the discriminant-form \[29\]. Suppose we are given an integer lattice \(L\) (the scalar product of any two vectors is integer). Let us consider the abelian group:

\[
A_L = L^*/L
\]

Since the scalar product on the lattice is integer valued, we have a well-defined scalar product \(b\) on \(L^*/L\) taking values in \(\mathbb{Q}/\mathbb{Z}\):

\[
b : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}
\]

Also, if \(L\) is an even lattice, the quadratic form on the lattice \(L\) defines a \(\mathbb{Q}/2\mathbb{Z}\)-valued quadratic form on \(A_L\):

\[
q : A_L \to \mathbb{Q}/2\mathbb{Z}
\]

The pair \(q_L = (q, b)\) has a property

\[
q(a + a') - q(a) - q(a') \equiv 2b(a, a') \mod 2\mathbb{Z}
\]

The finite abelian group \(A_L\) together with the pair \(q_L = (q, b)\) is an invariant of the lattice, called the discriminant-form. It turns out, that in many important cases this invariant, together with the signature of the lattice, characterizes the lattice unambiguously. Namely, there is the following theorem by Kneser and Nikulin\[10\]:

**Theorem A1.** Let \(L\) be an even lattice with the the signature \((s_+, s_-), s_+ > 0, s_- > 0\), and the minimum number \(l(A_L)\) of generators of \(A_L\) satisfies an inequality:

\[
l(A_L) \leq \text{rank}(L) - 2
\]

Then there is only one even lattice \(L\) with invariants \((s_+, s_-, q_L)\).

\[10\] We use the version of this theorem, formulated in \[23\] (Theorems 2.2 and 2.8).
If $M$ is an even lattice with invariants $(t_+, t_-, q_M)$ and $L$ is an even unimodular lattice of signature $(s_+, s_-)$, subject to inequalities

$$
t_+ < s_+, \\
t_- < s_-, \\
l(A_M) \leq \text{rank}(L) - \text{rank}(M) - 2$$

then there is a unique primitive embedding of $M$ into $L$.

The reader should consult [25,29] for further details and references.

Let us compute the discriminant-forms for the CHL momentum lattices in various dimensions.

1) $L = \Gamma_{n,n}(2)/\Gamma_{n,n}(2)$: In this case $A_L = (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^n$, generated by $2n$ elements $x_i = (\frac{1}{2}, 0)$ and $y_i = (0, 1)$ in the $i$-th factor $\Gamma_{1,1}(2)^* \simeq \Gamma_{1,1}(\frac{1}{2})$. The quadratic form and scalar product are $q(x_i) \equiv q(y_i) \equiv 0$ and $b(x_i, y_j) \equiv \frac{1}{2}\delta_{ij}$.

2) $L = D_8$: $A_L = D_8^*/D_8 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, generated by $v = (1,0,\cdots)$ and $s = (\frac{1}{2}, \cdots)$. The invariants are $q(v) \equiv 1$, $q(s) \equiv 0$, $b(v,s) \equiv \frac{1}{2}$. The transformation $x = v + s, y = s$ is an isomorphism between $A$ for $D_8^*/D_8$ and $A$ for $\Gamma_{1,1}(2)^*/\Gamma_{1,1}(2)$. Indeed, we have: $q(x) \equiv q(y) \equiv 0$ and $b(x,y) \equiv \frac{1}{2}$. Thus, the theorem by Nikulin and Kneser implies:

$$\Gamma_{1,1}(2) \oplus \Gamma_8 \simeq \Gamma_{1,1} \oplus D_8$$

3) $L = D_4 \oplus D_4$: $A_L = (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^\oplus 2$, $q(v_i) \equiv q(s_i) \equiv 1$, $b(v_i, s_j) \equiv \frac{1}{2}\delta_{ij}$, $i = 1, 2$. This is isomorphic to $\Gamma_{2,2}(2)^*/\Gamma_{2,2}(2)$: $x_1 = v_1 + v_2$, $x_2 = s_1 + s_2$, $y_1 = v_1 + v_2 + s_1$, $y_2 = s_1 + s_2 + v_1$. This tells us that

$$\Gamma_{2,2}(2) \oplus \Gamma_8 \simeq \Gamma_{2,2} \oplus D_4 \oplus D_4$$

4) $L = D_8^*(2)$: $A_L$ is generated by six elements of the form $a_j = \frac{1}{\sqrt{2}}(u_j + u_{j+1})$ for $j = 1, \ldots, 6$. $q(a_j) \equiv 1$, $b(a_i, a_{i+1}) \equiv \frac{1}{2}$. An isomorphism with $A_L$ for $L = \Gamma_{3,3}(2)$ is established as follows (shown are the images of the generators of $[D_8^*(2)]^*/D_8^*(2)$ in $\Gamma_{3,3}(2)^*/\Gamma_{3,3}(2)$):

$$(a_1, a_2, a_3, a_4, a_5, a_6) = \\
\left( \begin{array}{cccccc}
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right)$$

11 Primitive means that primitive vectors of $M$ go to primitive vectors of $L$. 37
This isomorphism implies
\[ \Gamma_{3,3}(2) \oplus \Gamma_8 \simeq \Gamma_{3,3} \oplus D_8^*(2) \]

In Section 5 we needed the following property of the Nikulin lattice \( \mathcal{N} = D_8^*(-2) \): the projection map \( O(\mathcal{N}) \rightarrow O(A_{\mathcal{N}}) \) is surjective. We will prove it, using the method suggested in Section 14.1 of [29]. Let us consider the primitive embedding \( f : \mathcal{N} \rightarrow \Gamma_{16} \). We will need the following theorem

Theorem A2. (Proposition 1.6.1 in [29].) A primitive embedding of an even lattice \( S \) into an even unimodular lattice \( L \), in which the orthogonal complement is isomorphic to \( K \), is determined by an isomorphism \( \gamma : A_S \simeq A_K \) for which
\[ q_K \circ \gamma = -q_S \] (A.1)

Two such isomorphism \( \gamma \) and \( \gamma' \) determine isomorphic primitive embeddings if and only if they are conjugate via an isomorphism \( g \) of \( K \): \( \gamma = \bar{g} \gamma' \) where \( \bar{g} \) is the action of \( g \) on \( A_K \).

(Two embeddings \( i_1 : S \rightarrow L \) and \( i_2 : S \rightarrow L \) are considered as isomorphic, if there is an isomorphism \( h \) of \( L \) with the property \( i_2 = hi_1 \).)

One can prove by direct computation, that there is a unique up to isomorphism \( \Gamma_{16} \) embedding of \( \mathcal{N} \) into \( \Gamma_{16} \), whose orthogonal complement is \( \mathcal{N}^\perp \simeq \mathcal{N} \) (this embedding may be constructed as follows: \( \mathcal{N} \) is generated by \( E_1, \ldots, E_8 \) with \( E_i \cdot E_j = 2\delta_{ij} \) and \( \frac{1}{2}(E_1 + \cdots + E_8) \). Then, \( f(E_i) = u_{2i-1} + u_{2i} \). To prove that it is a unique embedding, one has to take into account that vectors \( E_i \in \mathcal{N} \) should go to the roots of \( \Gamma_{16} \), which are of the form \( \pm u_i \pm' u_j \). Then, one has to take into account that \( \frac{1}{2} \sum_{i=1}^{8} E_i \in \mathcal{N} \). Thus, for arbitrary \( h \in O(A_{\mathcal{N}}) \) there is such \( g \in O(\mathcal{N}) \) that \( h\bar{g} = \text{id} \).

In the same way, one can prove that the map \( O(\Gamma) \rightarrow O(A_{\Gamma}) \) is surjective for \( \Gamma = D_8 \) (by considering the unique primitive embedding \( D_8 \subset \Gamma_{16} \)) and \( \Gamma = D_4 \oplus D_4 \) (which may be embedded into \( \Gamma_8 \oplus \Gamma_8 \) in only one way). For the lattice \( \Gamma_8(2) \) we will prove it in a different way. We have \( A_{\Gamma_8(2)} \simeq \Gamma_8 \left( \frac{1}{2} \right) / \Gamma_8(2) \). Let us denote \( \pi : \Gamma_8 \left( \frac{1}{2} \right) \rightarrow A_{\Gamma_8(2)} \) the natural projection. The group \( \Gamma_8/2\Gamma_8 \) is generated by the projections \( \pi(\alpha_i) \) of the simple roots of \( \Gamma_8 \). Given an automorphism \( \bar{g} \) of this group which preserves the discriminant form, we consider the images \( \bar{g}(\pi(\alpha_i)) \) of the projections of the simple roots of \( \Gamma_8 \). Notice that \( q(\bar{g}(\alpha_i)) \equiv 1 \text{ mod } 2\mathbb{Z} \). As we have explained in Section 3.1, an arbitrary element of \( \Gamma_8 \left( \frac{1}{2} \right) \), whose length square is odd, is equivalent modulo \( \Gamma_8(2) \) to the element with length square one. This means, that we may choose representatives of \( \bar{g}(\pi(\alpha_i)) \) being \( \frac{1}{\sqrt{2}} \beta_i \), where \( \beta_i \) is
some root of $\Gamma_8$. Since $\bar{g}$ preserves the discriminant-form, the scalar products of the roots $\beta_j$ have the following form:

$$(\beta_i \cdot \beta_j) = \begin{cases} 
2 & \text{if } i = j \\
\pm 1 & \text{if } i \neq j \text{ and } (\alpha_i, \alpha_j) = -1 \\
0 & \text{in other cases}
\end{cases}$$

Now take into account that we have a freedom to change a sign of $\beta_j$ (because $\beta_j \equiv -\beta_j \mod 2\Gamma_8$). Since the Dynkin diagram of $\Gamma_8$ does not contain cycles, we may adjust signs of the roots $\beta_j$ in such a way that they form a system of simple roots of $\Gamma_8$. But it is known from the theory of Lie groups ([21], p.78), that any two systems of simple roots are related by some symmetry $g$ of the lattice. This symmetry is the required lift of $\bar{g}$.

As another example of how Theorem A2 works, let us prove the isomorphism which we used in Section 4:

$$\Gamma_{n,n} \simeq \{(w; \tilde{w})|w, \tilde{w} \in D_n^*, w - \tilde{w} \in D_n\} \quad (A.2)$$

There is an evident map $g : D_n(1) \to D_n(-1)$ which multiplies scalar product by $-1$. The associated map $\bar{g} : A_{D_n(1)} \to A_{D_n(-1)}$ satisfies (A.1). Thus, we may glue $D_n(1)$ and $D_n(-1)$ into an even unimodular lattice, by adding to $D_n(1) \oplus D_n(-1)$ vectors of the form $(v, gv)$ with $v \in D_n^*(1)$. In other words, we consider such vectors $(x, y) \in D_8^*(1) \oplus D_8^*(-1)$ that the conjugacy class of $x$ coincides with the conjugacy class of $y$. The lattice generated by these vectors is even and self-dual, and it coincides with $\Gamma_{n,n}$.

Also, we used in Section 5 the fact that the lattice $\Gamma_{1,1} \oplus \Gamma_{n-1,n-1}(2)$ is isomorphic to the lattice generated by $(\sqrt{2}w, \sqrt{2}\tilde{w})$ with $w$ and $\tilde{w}$ weights of $D_n$, and $w - \tilde{w}$ is either in the root lattice, or in the vector conjugacy class of $D_n$. Let us prove it. For convenience, we divide the scalar product by two. Consider the lattice $\Gamma_{1,1} \left(\frac{\Gamma}{2}\right) \oplus \Gamma_{n-1,n-1}$. This lattice may be obtained from the lattice $\Gamma_{n,n}$ by adding one half of the primitive lightlike vector (it follows from Appendix D, that there is only one primitive lightlike vector in $\Gamma_{n,n}$, modulo the symmetries). We know, that $\Gamma_{n,n}$ is isomorphic to the lattice $\{(w; \tilde{w})|w - \tilde{w} \in D_8\}$. The vector $(2s, 2\bar{s})$, where $s$ and $\bar{s}$ are spinor and conjugate spinor of $D_n$, is primitive and lightlike. Adding the generator $(s, \bar{s})$ we get the lattice consisting of $(w, \tilde{w})$ with $w - \tilde{w}$ is in scalar or vector conjugacy class of $D_n$. This is what we had to prove.
Appendix B. Example of $K3$ with Nikulin involution.

Consider the surface $X$ in the weighted projective space $\mathbb{P}_{6,4,1,1}$, specified by the following equation:

$$y^2 = x(x^2 + a(u, v)x + b(u, v)) \quad (B.1)$$

where $a(u, v)$ and $b(u, v)$ are homogeneous polynomials of degree 4 and 8. We will work in the affine coordinates with $v = 1$, and denote $a(u) = a(u, 1)$, $b(u) = b(u, 1)$. Notice that we have eight $A_1$ singularities, located at the points $x = y = 0$, $b(u) = 0$. Indeed, our equation (B.1) near such a singularity may be approximated as:

$$y^2 = ax(x - c(u - u_0)) \quad (B.2)$$

where $\frac{b}{a} = c(u - u_0) + \ldots$. This is the equation for $A_1$ singularity, located at $x = y = 0$, $u = u_0$. The existence of these eight singular points is not an accident. They may be interpreted as singularities of the quotient of another $K3$ surface by $\mathbb{Z}_2$-involution.

Let us explain it. The surface (B.1) admits an involution $x \to \tilde{x}$, $y \to \tilde{y}$ where $\tilde{x}$ and $\tilde{y}$ are given by the following equations:

$$x\tilde{x} = b(u) \quad \frac{\tilde{y}}{\tilde{x}} = -\frac{y}{x} \quad (B.3)$$

This involution preserves the holomorphic 2-form

$$\Omega = \frac{du \wedge dx}{y} \quad (B.4)$$

and has eight fixed points located at $y = 0$, $x = \frac{a}{2}$, $D = a^2 - 4b = 0$. Let us consider the quotient by this involution. The invariant functions are:

$$u, \quad Y = \frac{y}{x} \left(x - \frac{b}{x}\right), \quad X = \left(\frac{y}{x}\right)^2 = x + \frac{b}{x} + a \quad (B.5)$$

They satisfy the equation

$$Y^2 = X(X^2 - 2aX + (a^2 - 4b)) \quad (B.6)$$

This is, again, an elliptic $K3$ surface with a double section. One can see that there is a reciprocal relation between the surfaces (B.1) and (B.6): one is the $\mathbb{Z}_2$-quotient of the other, and the singularities of one surface correspond to the fixed points of $\mathbb{Z}_2$-involution of the other.
In this situation, as we have mentioned in Section 5, the lattice generated by the exceptional divisors $E_1, \ldots, E_8$ contains $\frac{1}{2} \sum_{k=1}^{8} E_k$. The appearance of the half of the sum of exceptional divisors may be explained as follows. Suppose that we have the surface $X$, with the involution $\iota$, and $Y = X/\iota$. There are many meromorphic functions on $X$, anti-invariant under $\iota$. (One can take an arbitrary meromorphic function $\psi$, not invariant under $\iota$, and consider $\phi = \psi - \iota^*\psi$ — this function will be anti-invariant.) The general anti-invariant function $\phi$ will have first order zeroes at each of the eight fixed points of $\iota$. Going to the quotient $Y = X/\iota$, we may consider this anti-invariant function as a two-valued function on $Y$: when we circle around the exceptional divisor $E_i$, this function changes its sign. Let us call this two-valued function $\phi_Y$. If the exceptional divisor is given locally by the equation $\chi(X,Y,u) = 0$, then the function $\phi_Y$ locally near the exceptional divisor may be written as $\phi_Y \sim \sqrt{\chi}$ — in other words, it has zero of order $\frac{1}{2}$ at each $E_i$. Also, the function $\phi_Y$ may have zeroes and poles of integer order at some other surfaces on $Y$. Notice that $\phi_Y^2$ is a well-defined (single-valued) function on $Y$. The corresponding principal divisor is:

$$ (\phi_Y^2) = \sum_{j=1}^{8} E_j + 2v \quad (B.7) $$

where $v$ is some divisor (combination of cycles with integer coefficients) on $Y$. This means, that the sum of the exceptional divisors is equivalent to the cycle $-2v$ in homology — in other words, the cycle $\sum_{j=1}^{8} E_j$ may be divided by two. The converse is also true. Consider the K3 manifold $Y$ with eight $A_1$ singularities, such that half of their sum belongs to the Picard lattice of $Y$. This means that the sum of exceptional divisors may be expressed as:

$$ \sum_{j=1}^{8} E_j = 2 \sum n_j c_j \quad (B.8) $$

where $c_j$ are some holomorphic curves in $Y$, $n_j \in \mathbb{Z}$. Since for the holomorphic curves on K3, the topological equivalence implies linear equivalence ([30], Ch.12), there exists a meromorphic function with first order zero at each exceptional divisor, and even order zeroes or poles at curves $c_j$. The square root $\phi_Y$ of this function is a two-valued function on $Y$, ramified at the exceptional divisors. The graph of $\phi_Y$ (surface in $Y \times \mathbb{P}^1$) is some

---

12 Linear equivalence of divisors implies topological equivalence: given $(f) = (f)_0 - (f)_\infty$, the cycle given by the equation $f = t$ interpolates between the cycle $f = 0$ and the cycle $f = \infty$, as $t$ runs from 0 to $\infty$. 41
surface $X$. When the exceptional divisors $E_j$ are blown down, this surface has a well-defined 2-form.

Let us make this reasoning more explicit for the particular example of the surface (B.1) and its $\mathbb{Z}_2$-quotient (B.6). The Picard group of the surface (B.6) is generated by the Poincare duals of the following holomorphic cycles:

- $s : (X = Y = 0)$ — the section,
- $\bar{s} : (X = Y = \infty)$ — the other section,
- $\xi : (X^2 + aX + b = 0, Y = 0)$ — the double section,
- $f : (u = \text{const})$ — the fiber,

and the eight exceptional divisors $E_i$, $i = 1, \ldots, 8$. Consider the function

$$\phi(x, y, u) = \frac{x - \frac{b}{x}}{x + \frac{b}{x} + a}$$

on the surface (B.1). This function is anti-invariant. We have:

$$\phi^2_Y(X, Y, u) = \frac{Y^2}{X^3}$$

— the function on the surface (B.6). This function has the second order zero at the double section $\xi$, and the fourth order pole at the section $s$. It also has the first order pole at each $E_i$. Indeed, the equation (B.6) in the vicinity of $E_i$ may be rewritten as

$$Y^2 \simeq -2aX(X - \alpha(u - u_0))$$

— this is the equation for the $\mathbb{Z}_2$ singularity. To blow up, we have to glue in the sphere $\mathbb{P}^1$, so that (B.11) becomes $\mathcal{O}_{\mathbb{P}^1}(-2)$. This is equivalent to introducing a function $z$ (coordinate on $\mathbb{P}^1$), such that $z^2 = \frac{X - \alpha(u - u_0)}{X}$. Then, our function $\phi^2_Y$ becomes $\frac{Y^2}{X^3} \sim \frac{z^2}{X}$ which means that it has the first order pole at the divisor. This proves that the divisor

$$2\xi - 4s - \sum_{j=1}^{8} E_j$$

is principal, thus it is topologically trivial. In other words,

$$\frac{1}{2} \sum_{j=1}^{8} E_j = -2s + \xi$$

(B.12)

Also, by looking at poles and zeroes of the function $\frac{Y}{X}$, we learn that

$$2f + s + \bar{s} - \xi = 0$$

(B.13)

Notice that the holomorphic cycles (B.9) with the relations (B.12) and (B.13) generate the Picard lattice $\Gamma_{1,1} \oplus \mathcal{N}$. 

42
Appendix C. Lightlike lines in some Lorentzian lattices.

In this section, we will classify the lightlike lines in two Lorentzian lattices: $\Gamma_{1,1} \oplus D_8$ and $\Gamma_{1,1} \oplus D_4 \oplus D_4$. We will actually give a proof for the second lattice only, since the first one can be considered in the very similar way. We may represent this lattice in two forms:

$$\Gamma_{1,1} \oplus D_4 \oplus D_4 \simeq \Gamma_{1,1}(2) \oplus D_8 \quad \text{(C.1)}$$

This gives two inequivalent examples of light-like lines: the standard light-like line in the first lattice and the one in the second lattice. They cannot be related by the automorphism of the lattice, since they have different orthogonal complements. Are there any lightlike lines, which are not equivalent to one of these two?

Consider some light-like primitive vector:

$$\begin{bmatrix} P_1 & P_2 \\ m & -n \end{bmatrix} \quad \text{(C.2)}$$

where $P_1$ and $P_2$ are some vectors from $D_4$. Our strategy will be to try to decrease $\max(P_1^2, P_2^2)$ by acting on this vector with some symmetry of the lattice. Without any loss of generality, we may suppose that $P_1^2 \geq P_2^2$ and $n \geq m > 0$. Let us consider the following symmetry:

$$\begin{bmatrix} P_1 & P_2 \\ m & -n \end{bmatrix} \rightarrow \begin{bmatrix} P_1 - m\alpha & P_2 \\ m & -m - n + (P_1 \cdot \alpha) \end{bmatrix} \quad \text{(C.3)}$$

with $\alpha^2 = 2$ and $(\alpha \cdot P_1) \geq 0$. The condition that $(P_1 - m\alpha)^2 \leq P_1^2$ may be written as:

$$(\alpha \cdot P_1) \geq m \quad \text{(C.4)}$$

From $P_1^2 + P_2^2 - 2mn = 0$ we infer that $m \leq \sqrt{P_1^2}$. (This inequality is saturated if and only if $P_1^2 = P_2^2$ and $m = n$.) Let us prove that for most of the values of $P_1$ we can find $\alpha$ which satisfies the inequality not weaker then (C.4):

$$(\alpha \cdot P_1) \geq \sqrt{P_1^2}$$

Indeed, suppose $P_1 = (p_1, p_2, p_3, p_4)$ and $p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$ (we may always bring $P_1$ to this form by some symmetry of $D_4$). Then, let us take $\alpha = (1, 1, 0, 0)$. This choice gives

$$(p_1 + p_2) \geq \sqrt{p_1^2 + p_2^2 + p_3^2 + p_4^2}$$
with equality if and only if \( p_1 = p_2 = p_3 = p_4 = p \) or \( p_2 = p_3 = p_4 = 0 \). Thus, we may continue this process until we reach \( P_1 = (p, p, p, p) \) or \( (2p, 0, 0, 0) \) and \( m = n \). At this point we should have \( P_2^2 = P_1^2 = 4p^2 \) and \( m = n = 2p \). If now \( P_2 \) is not of the form \( (p, p, p, p) \) or \( (2p, 0, 0, 0) \), then we can continue to play (C.3), but this time subtracting \( m\alpha \) from \( P_2 \), not from \( P_1 \). This process stops when we get the vector of one of the four types:

\[
\begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix},
\begin{bmatrix}
1111 & 1111 \\
2 & -2
\end{bmatrix},
\begin{bmatrix}
2000 & 1111 \\
2 & -2
\end{bmatrix},
\begin{bmatrix}
2000 & 2000 \\
2 & -2
\end{bmatrix}
\]

but the third and the fourth are related to the second via triality of \( D_4 \) (the outer automorphism). Thus, we have essentially only two possibilities. This proves that there are only two lightlike lines, given by the standard lightlike lines in (C.1) (the first one in (C.5) has orthogonal complement \( D_4 \oplus D_4 \), and the second one \( D_8 \)).

Similar (but simpler) reasoning for \( \Gamma_{1,1} \oplus D_8 \) shows that there are two and only two light-like lines there, which may be thought of the standard in two representations of the lattice:

\[
\Gamma_{1,1} \oplus D_8 \simeq \Gamma_{1,1}(2) \oplus E_8
\]

**Appendix D. Technical Lemma.**

Consider a standard light-like plane in \( \Gamma_{n,n} \oplus \Gamma_{m,m}(2) \), and a primitive vector \( v \) in it. Here we will prove that this vector may be mapped by the symmetry of the lattice to one of the two standard forms: one standard form being a standard light-like vector in the first \( \Gamma_{1,1} \) sublattice, and the other the standard light-like vector in the first \( \Gamma_{1,1}(2) \) sublattice.

To prove this, consider first the primitive vector \( w \) in the standard light-like plane in the lattice \( \Gamma_{2,2} \). It has the following form:

\[
w(x, y) = \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}
\]

(D.1)

Let us prove that we may bring it by the symmetries of \( \Gamma_{2,2} \) to the form

\[
w(\text{GCD}(x, y), 0) = \begin{bmatrix} \text{GCD}(x, y) & 0 \\ 0 & 0 \end{bmatrix}
\]

(D.2)

where \( \text{GCD}(x, y) \) is the greatest common divisor of \( x \) and \( y \). Indeed, for an arbitrary pair \((p, q)\) of integers, we may consider the following symmetries of the lattice \( \Gamma_{2,2} \):

\[
\begin{bmatrix} a & c \\ b & d \end{bmatrix} \rightarrow \begin{bmatrix} a + pb & c \\ b & d - pc \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & c \\ b & d \end{bmatrix} \rightarrow \begin{bmatrix} a & c + qd \\ b - qa & d \end{bmatrix}
\]

(D.3)
These symmetries map \( w(x, y) \) to \( w(x + py, y) \) and \( w(x, y - qx) \). But the transformations \((x, y) \to (x + y, y)\) and \((x, y) \to (x, y - x)\) on the pair of integers are the basic transformations of the Euclid algorithm for the greatest common divisor. By these transformations, one can relate \((x, y)\) to \((\text{GCD}(x, y), 0)\).

Applying these transformations to the components of \( v \) in \( \Gamma_{n,n} \) and \( \Gamma_{m,m}(2) \), we can map it to the vector in \( \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \) — the direct sum of the first \( \Gamma_{1,1} \) sublattice and the first \( \Gamma_{1,1}(2) \) sublattice. Suppose this vector has a form:

\[
v(a, b) = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}
\]  \hspace{1cm} (D.4)

Then, we can apply the transformations (D.3) of the Euclid algorithm to the pair \((a, b)\), with the only restriction that \(p\) should now be even (we realize the lattice \( \Gamma_{1,1} \oplus \Gamma_{1,1}(2) \) as consisting of the vectors of the form \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), with the condition that \(d\) is even). This allows us to bring \(v(a, b)\) to one of the vectors \(v(1, 0)\) or \(v(0, 1)\) (we have taken into account that our vector is primitive). This proves the lemma.

In the similar way, one can prove that an arbitrary \(p\)-dimensional rational subspace in the standard lightlike plane in the lattice \( \Gamma_{n,n} \oplus \Gamma_{m,m}(2) \) may be mapped to the standard lightlike plane in some \( \Gamma_{p_1,p_1} \oplus \Gamma_{p_2,p_2} \) — the direct sum of the first \(p_1\) \( \Gamma_{1,1} \) sublattices and the first \(p_2\) \( \Gamma_{1,1}(2) \) sublattices, with the condition \(p_1 + p_2 = p\).
References

[1] A. Dabholkar and J. Park, “Strings on Orientifolds”, hep-th/9604178
[2] S. Chaudhuri, G. Hockney and J. Lykken, “Maximally Supersymmetric String Theories in $D < 10$”, hep-th/9505054
[3] E. Witten, “Toroidal Compactification Without Vector Structure”, hep-th/9712028
[4] W. Lerche, C. Schweigert, R. Minasian and S. Theisen, “A Note on the Geometry of CHL Heterotic Strings”, hep-th/9711104
[5] S. Chaudhuri, J. Polchinski, “Moduli Space of CHL String “, Phys. Rev. D52 (1995) 7168, hep-th/9506048
[6] J. Park, “Orientifold and F-Theory Duals of CHL Strings”, hep-th/9611119
[7] J.H. Schwarz, A. Sen, “Type IIA Dual of the Six-Dimensional CHL Compactification”, Phys. Lett. B357 (1995) 323, hep-th/9507027
[8] S. Kachru, A. Klemm and Y. Oz, “Calabi-Yau Duals for CHL String”, hep-th/9712035
[9] K. Landsteiner, E. Lopez, “New Curves from Branes”, hep-th/9708118
[10] K.S. Narain, Phys. Lett. B169 (1985) 41
[11] P.S. Aspinwall, “K3 Surfaces and String Duality”, hep-th/9611137
[12] M. Bershadsky, T. Pantev and V. Sadov, “F-Theory and Quantized Fluxes”, hep-th/9805056
[13] P. Berglund, A. Klemm, P. Mayr and S. Theisen, “On Type IIB Vacua With Varying Coupling Constant”, hep-th/9805189
[14] K.S. Narain, M.H. Sarmadi and E. Witten, “A Note on Toroidal Compactification of Heterotic String Theory”, Nucl. Phys. B279 (1987) 369-379
[15] M. Green, J.H. Schwarz, E. Witten, “Superstring Theory”.
[16] P. Ginsparg, “Curiosities at $c = 1$”, Nucl. Phys. B295 (1988) 153-170
[17] J.H. Conway, N.J.A. Sloane, “Sphere Packings, Lattices and Groups”, Springer-Verlag 1988
[18] I.S. Gradshteyn and I.M. Ryzhik, “Table of Integrals, Series and Products”, Academic Press 1965
[19] A. Giveon, M. Porrati, E. Rabinovici, “Target Space Duality in String Theory”, hep-th/9401139
[20] A. Salam, E. Sezgin, “Supergravities in Diverse Dimensions”, Elsevier Science Publishers, B.V., and World Scientific Publishing Co., 1989
[21] A.L. Onishchik, E.B. Vinberg, “Lie Groups and Lie Algebras III”, Springer-Verlag 1994
[22] P. Goddard, J. Nyuts and D. Olive, Nucl. Phys. B125 (1977) 1
[23] J.H. Schwarz, A. Sen, “Duality Symmetries of 4D heterotic strings”, Phys. Lett. B312 (1993) 105-114

46
[24] V.V. Nikulin, “Finite Automorphism Groups of Kahler $K3$ Surfaces”, Trans. Moscow Math. Soc. 38, 71-135 (1980)
[25] D.R. Morrison, “On K3 surfaces with large Picard number”, Invent. Math. 75, 105-121 (1984)
[26] C. Vafa, “Evidence for F Theory”, Nucl. Phys. B469 (1996) 403-418
[27] W.A. Casselman, “Geometric Rationality of Satake Compactifications”, http://www.math.ubc.ca/people/faculty/cass/research.html
[28] M.R. Douglas and C. Hull, “D-branes and the Noncommutative Torus”, hep-th/9711165
[29] V.V. Nikulin, “Integral symmetric bilinear forms and some of their applications”, Math. USSR Izvestija 14, 103-167 (1980)
[30] V.A. Iskovskih, I.R. Shafarevich, “Algebraic Surfaces”, in “Algebraic Geometry - 2”, ed. I.R. Shafarevich, Springer-Verlag 1996.