Manipulating quantum coherence of charge states in interacting double-dot Aharonov–Bohm interferometers

Jinshuang Jin, Shikuan Wang, Jiahuan Zhou, Wei-Min Zhang, and YiJing Yan

1 Department of Physics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, People’s Republic of China
2 Department of Physics, Hangzhou Dianzi University, Hangzhou 310018, People’s Republic of China
3 Department of Physics and Center for Quantum Information Science, National Cheng Kung University, Tainan 70101, Taiwan
4 Hefei National Laboratory for Physical Sciences at the Microscale and Collaborative Innovation Center of Chemistry for Energy Materials (iChEM), University of Science and Technology of China, Hefei 230026, People’s Republic of China
5 Authors to whom any correspondence should be addressed.

E-mail: jsjin@hznu.edu.cn, wzhang@mail.ncku.edu.tw and yanyj@ustc.edu.cn

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Abstract

We investigate the dynamics of charge-state coherence in a degenerate double-dot Aharonov–Bohm interferometer with finite inter-dot Coulomb interactions. The quantum coherence of the charge states is found to be sensitive to the transport setup configurations, involving both the single-electron impurity channels and the Coulomb-assisted ones. We numerically demonstrate the emergence of a complete coherence between the two charge states, with the relative phase being continuously controllable through the magnetic flux. Interestingly, a fully coherent charge qubit arises at the double-dots electron pair tunneling resonance condition, where the chemical potential of one electrode is tuned at the center between a single-electron impurity channel and the related Coulomb-assisted channel. This pure quantum state of charge qubit could be experimentally realized at the current–voltage characteristic turnover position, where differential conductance sign changes. We further elaborate the underlying mechanism for both the real-time and the stationary charge-states coherence in the double-dot systems of study.

1. Introduction

The investigations of semiconductor quantum dots have long aroused a great deal of attention. There are the systems of a great controllability, in terms of their geometric shape and size, and internal energy levels and couplings. These features make quantum dots particularly useful in the study of mesoscopic physics, as well as the potential applications for nanotechnology and quantum information processing. Quantum dots are also promising candidates for realizations of scalable quantum computer, implemented with the electron charge and/or spin qubits, even with Majorana fermions [1–8].

In this paper, we will be interested in the generation and manipulation of the charge-state coherence in interacting double-dot Aharonov–Bohm (AB) interferometer systems. Double-dot AB interferometers have been extensively studied both experimentally [9–11] and theoretically [10–24] for the carrier electron interference in nonequilibrium quantum transport. It is manifested in the conductance oscillations tuned by the externally applied magnetic flux [9–11]. Many investigations into interacting double-dot AB interferometer systems have focused on the intra-dot Coulomb interaction which has significant effects on the Fano resonance [18], the tunnel magnetoresistance [14], the full counting statistics [17], and the Fano–Kondo interplay [21]. The effect of inter-dot Coulomb interactions has also been studied [16, 20, 23], in relating Dicke-like, Fano-like, and orbital–Kondo resonance characteristics [16, 20, 23]. On the other hand, without including intra-dot or inter-dot Coulomb interactions, transient transport properties with respect to various geometric setups and symmetries in double-dot AB systems have also been investigated in detail recently [19, 24].
We focus in this work on the coherent properties of charge qubit, as created in interacting double-dot AB interferometers. We solve the coherent dynamics of charge states in interacting double-dot AB interferometers using the hierarchical equations-of-motion (HEOM) approach [25] with numerically accurate evaluations [26–28]. In the double-dot AB interferometer transport model involving inter-dot Coulomb interactions, the electron tunneling processes can be fully coherent at a low temperature [29]. Approaches, such as the random matrix method [29, 30] and Born–Markovian master equation [31–33], break down here, as they overlook the non-Markovian and virtual tunneling processes [34]. The equation of motion (EOM)-based rate equation approach [35, 36] and the exact master equation approach [37, 38] also do not capture the physics of the double-dot AB interferometers when finite inter-dot Coulomb interaction is considered. Mathematically, HEOM is developed based on the path integral influence functional theory [39–41] but operationally and numerically makes the practical calculations to interacting open systems more tractable. The underlying hierarchical construction resolves the combined effects of the dissipative coupling strength, memory time, Coulomb interactions, and higher-order electron cotunneling processes. HEOM is essentially a nonperturbative approach with the following features [25–28]: (1) for noninteracting fermionic systems, the hierarchical dynamics truncated at the second-tier level is formally exact for the reduced density matrix and transport current; (2) for interacting systems, HEOM terminates formally at a finite tier level, whereas the practical HEOM evaluations are performed with a rapid numerical convergence. The HEOM method has been used extensively in accurate characterizations on strongly correlated quantum transport systems. The applications include the studies of transient electronic transport [42–45], thermopower [46], and the spectral density of local impurity system in the Kondo regime [27, 47].

Meanwhile, much progress has been made in the investigation of charge qubit coherence in the serial double-dot systems, in which the property of the charge qubit has been manipulated with voltage pulses [4, 48–51] and/or resonant microwave driving [52, 53]. Coherence properties of the charge qubit in noninteracting double-dot AB interferometers has revealed an interesting phenomenon of phase localization [54, 55]. Coherent control of charge qubit states through large asymmetric couplings to the leads has been explored both experimentally and theoretically [11, 56, 57]. Decoherence dynamics in double-dot AB systems has also been extensively studied [58–60]. With inter-dot Coulomb interaction being taken into account, and using the HEOM method [25], we show in this paper that the charge-state coherence can be optimally achieved at the so-called double-dots electron pair tunneling resonance condition [61], and then the coherence phase can be manipulated by the applied AB flux. The electron pair tunneling resonance was originally found [61] in the single impurity Anderson model system, with the characteristic peak showing in the $d^2I/dV^2 \sim V$ curve. We find that the same pair tunneling resonance mechanism plays important roles in the generation of the charge-state coherence in interacting double-dot AB interferometer systems. A finite inter-dot Coulomb interaction is necessary in establishing the inter-dot electron pair tunneling resonance. This leads to the fully coherent charge states, with tunable phase by the AB magnetic flux, and both the vacuum and double-occupation states are effectively suppressed. This scenario differs remarkably from the previous studies with either noninteracting [19, 57] and weak-intermediate inter-dot Coulomb-interaction [55] or large Coulomb-interaction limit [55, 58].

It should also be pointed out that compared to a single QD with Zeeman-splitting performing the electron spin qubit proposed in the pioneering work of Loss and DiVincenzo [2], the advantages of the charge qubit in terms of the interacting double-dot AB interferometers is that the AB flux control of charge qubit coherence in double-dot AB interferometers is much easier. This is because the single QD with Zeeman-splitting to perform the qubit requires a precise control of local magnetic field within a single QD. Furthermore, in the present systems, the characteristic peak (dip) shows rather in the $I \sim V$ curve. Therefore, the coherent charge qubit as generated could be easily identified in experiments.

The rest of paper is organized as follows. In section 2, after a brief introduction of the model for double-dot AB interferometers, we apply the HEOM approach [25–28] to the coherence dynamics of the charge states of the double-dot AB systems. In section 3, the dynamics of the quantum coherence of the charge qubit is solved from the converged stationary results of the HEOM in different tunneling regimes in the presence of finite inter-dot Coulomb interaction. We then study the real-time dynamics and elaborate the underlying mechanism of the observed nonequilibrium charge-states coherence from weak to strong inter-dot Coulomb interactions in section 4. Finally, a summary is given in section 5.

2. Methodology

Consider an interacting double-dot embedded in an AB interferometer, its total Hamiltonian, $H_T = H_b + H_b + H_{AB}$, consists of three parts. The central parallel double-dot system is modeled by
Here, \( a^\dagger_\alpha (a_\alpha) \) denotes the annihilation (creation) operator of the electron in the dot-\( \alpha \) orbital state with the degenerate energy \( \varepsilon \), and \( U \) is the inter-dot Coulomb interaction strength. In equation (1), the spin degeneracy on each dot is lifted. This can be achieved with a local spin-splitting micromagnet [62], unless the spin–orbit coupling dominates the electron tunneling between the dots and the leads in special cases [63–65]. Together with the properly applied bias potential, each dot would only contribute one transport channel to the charge qubit generation.

The electrodes are modeled as noninteracting electrons reservoirs bath, i.e.,

\[
H_{\text{b}} = \sum_{\alpha k} (\varepsilon_{\alpha k} \mathbf{c}_\alpha^\dagger \mathbf{c}_\alpha + U n_{\alpha k} \mathbf{n}_{\alpha k}) \tag{2}
\]

with \( \alpha = L, R \), under the applied bias voltage potential \( eV = \mu_L - \mu_R \). Here, \( \mathbf{c}_\alpha^\dagger (\mathbf{c}_\alpha) \) denotes the creation (annihilation) operator of the electron with momentum \( k \) in the specified \( \alpha \)-reservoir. The electrons tunneling between the dots and the reservoirs is described by the tunneling Hamiltonian,

\[
H_{\text{SH}} = \sum_{\alpha k} (\varepsilon_{\alpha k} - \mu_{\alpha k} \mathbf{c}_\alpha^\dagger \mathbf{c}_\alpha + \text{h.c.}), \tag{3}
\]

with the AB flux \( \Phi \)-induced phase factors satisfying

\[
\phi_{11} - \phi_{12} + \phi_{22} - \phi_{21} = \phi \equiv 2\pi \Phi / \Phi_0. \tag{4}
\]

Here, \( \Phi_0 \) denotes the flux quantum. Without loss of generality, we adopt (due to the gauge invariant) [54, 66],

\[
\phi_{11} = -\phi_{12} = \phi_{22} = -\phi_{21} = \phi / 4. \tag{5}
\]

The hybridization spectral function assumes Lorentzian,

\[
I_{\text{hw}} (\omega) \equiv \pi e^{i\delta \omega - \phi_{\alpha k}} \sum_k \mathbf{e}_{\alpha k}^* \mathbf{e}_{\alpha k} \delta (\omega - \varepsilon_{\alpha k}) = \frac{\Gamma_{\text{hw}} W^2}{\omega^2 + W^2}, \tag{6}
\]

with the equal coupling strengths,

\[
\Gamma_{\alpha 11} = \Gamma_{\alpha 22} = \Gamma / 2, \\
\Gamma_{\alpha 12} = \Gamma_{\alpha 21} = \Gamma_{\alpha 21} = \Gamma_{\alpha 12} = e^{i\delta/2} \Gamma / 2. \tag{7}
\]

Throughout this work, we take the unit of \( e = \hbar = 1 \), for the electron charge and the Planck constant. In numerical calculations we set \( \mu_L = -\mu_R = V / 2 \) and fix the bandwidth at \( W = 10 \text{ meV} \) for electrodes.

The involved charge states in the double dots are \([00], [10], [20], [01], [11], [21] \), i.e., the empty, the dot-1 occupied, the dot-2 occupied, and double-dots-occupancy states, respectively. The charge qubit takes single-electron states \([10], [20] \) as its physical basis. The quantum coherence properties of all the double-dots states are described by the reduced system density matrix, \( \rho (t) \equiv \text{tr}_{\text{b1}} \rho_{\text{tot}} (t) \), i.e., the partial trace of the total density operator \( \rho_{\text{tot}} \) over the electrode bath degrees of freedom.

We implement the HEOM formalism [25],

\[
\dot{\rho}_{j}^{(n)}(t) = -i [\mathcal{L}_S + \sum_{r=1}^{n} \gamma_{j,r}] \rho_{j}^{(n)}(t) - i \sum_{j} \mathcal{A}_{j} \rho_{j}^{(n+1)}(t) - i \sum_{r=1}^{n} \sum_{\mu} (-)^{\lambda-\mu} \mathcal{C}_{j}^{\mu} \dot{\rho}_{j}^{(n-1)}(t), \tag{8}
\]

to study the real-time dynamics of the reduced system density matrix, \( \rho^{(0)}(t) \equiv \rho (t) \), whereas \( \rho_{j}^{(n)}(t) \equiv \rho_{j}^{(n)}(t) \), with \( \rho^{(n<0)}(t) \equiv 0 \). In equation (8), \( \mathcal{L}_S = \{ H_{\text{S}}, \cdot \} \) defines the reduced system Liouvillian; \( \mu \equiv (z, \alpha, u, \kappa) \) and \( \nu \equiv (z, \alpha, u, \kappa) \) denote the specified collective indexes for the Grassmannian superoperators, \( \mathcal{A}_{j} \equiv \mathcal{A}_{\text{ suc}} = \mathcal{A}_{j}^\alpha = \mathcal{A}_{j}^u = \mathcal{A}_{j}^\kappa = \mathcal{A}_{j}^z \), and \( \mathcal{C}_{j} \equiv \mathcal{C}_{\text{ suc}} = \mathcal{C}_{j}^\alpha = \mathcal{C}_{j}^u = \mathcal{C}_{j}^\kappa = \mathcal{C}_{j}^z \).

\[
\mathcal{A}_{j}^{z} \hat{O}_{\kappa} = \hat{a}_{\alpha}^{z} \hat{O}_{\kappa} \pm \hat{O}_{\kappa} \hat{a}_{\alpha}^{z}, \quad \mathcal{A}_{j}^{\alpha} \hat{O}_{\kappa} = \mathcal{A}_{j}^{\alpha} \hat{O}_{\kappa} \pm \hat{O}_{\kappa} \mathcal{A}_{j}^{\alpha},
\]

\[
\mathcal{C}_{j}^{z} \hat{O}_{\kappa} = \sum_{\nu} (\mathcal{C}_{j}^{\alpha} \hat{a}_{\nu}^{z} \hat{O}_{\kappa} \pm \hat{O}_{\kappa} \mathcal{C}_{j}^{\alpha} \hat{a}_{\nu}^{z}).
\]

Here, \( \hat{O}_{\kappa} \) is an arbitrary operator, with even (+) or odd (−) fermionic parity, such as \( \rho^{(2\mu)}(t) \) or \( \rho^{(2\mu+1)}(t), \) respectively. The superindex \( \kappa = \pm, -(\pm \text{is its opposite sign}) \) is used to redefine the fermion creation and annihilation operators, \( \hat{a}_{\alpha}^{\dagger} \equiv \hat{a}_{\alpha}^{z} \) and \( \hat{a}_{\alpha} \equiv \hat{a}_{\alpha}^{z} \), in equation (9); \( \kappa \) arises from the nonequilibrium interacting reservoirs bath correlation functions [25–28, 67, 68], in an exponent expansion form of

\[
\mathcal{C}_{j}^{z} \hat{O}_{\kappa} (t) = \sum_{\nu=1}^{K} \mathcal{C}_{j}^{\alpha} \hat{a}_{\nu}^{z} e^{-\gamma_{\nu} t}. \tag{10}
\]

The stationary solutions to equation (8) can be obtained by using the conditions, \( \rho_{j}^{(n=0,\kappa)} = 0; \forall n \). These together with the normalization constraint, \( \text{tr}\rho^{(0)} = 1 \), lead to equation (8) a set of coupled linear equations for solving \( \rho_{j}^{(n=0,\kappa)} = 0 \) of a hierarchy truncated at \( L \)-tier. In practical calculations, an iterative quasiminimal
residual algorithm [69, 70] is employed for solving the large-sized coupled linear equations [28]. Equation (8) can also be called the dissipation equation of motion (DEOM) [71–73]. The latter is a quasi-particle theory, which identifies the physical meaning of individual \( \rho^{(n)}_j(t) = \rho^{(n)}_{k-1,j}(t) \). Besides equation (8), the DEOM theory includes also the underlying dissipator algebra, especially the generalized Wick’s theorem [71–73]. This extends the real-time dynamics further to the interacting bath subspace. Not only the transient transport current [42, 43, 47], but also the nonequilibrium current–current correlation functions can then be evaluated [72].

The fermionic HEOM formalism is formally exact when \( n_{\text{max}} = 2N_c N_d \), where \( N_d \) denotes the number of spin–orbital states, and \( N_c = 2 \) for the two signs of \( z = + \) and \( - \). The above property arises from the fact that \( A_z^+ \) (see equation (9)) assumes either the commutator or anticommutator actions of \( a_z^+ \) that is a fermionic operator. On the other hand, the total number of the dynamical variables in the HEOM formalism, equation (8), increases combinatorially with the exponential terms of the reservoir bath correlation functions, equation (10). Nevertheless, the full evaluations of the formally exact equation (8), up to the above theoretical maximum tier, \( n_{\text{max}} \), is practically often unnecessary. As an efficient and universal numerical method [25–28, 44, 74], HEOM converges rapidly and uniformly with increasing the truncated tier level, \( L = n_{\text{trun}} \), by setting all \( \rho^{(n>L)}_j = 0 \), at a sufficiently large \( L \). As long as the numerical convergence is reached, the results are guaranteed to be quantitatively accurate. The minimal truncation tier \( L \) required to achieve convergence is closely dependent on the configurations of system as well as bath. Energetic properties such as the strength of electron correlation, the system–reservoir coupling, and the temperature will all have influence on \( L \). It is difficult to have an a priori estimation for the required minimal \( L \). In practice, the convergence with respect to \( L \) is tested case by case. For the AB double–dot system exemplified in this work, the HEOM evaluations effectively converge at \( L = 3 \) tier level.

It is also worth pointing out that HEOM is a quasi-particle theory, for not just the central impurity system, but also the itinerary electrons (or dissipators) from reservoirs [71–73]. While it is exact at the L-dissipators level, regardless the strength of Coulomb interaction in the impurity, the truncated HEOM evaluation treats also the higher-order effects in a mean-field manner [25–28, 71–73]. These unique features distinct the HEOM approach from the conventional EOM scheme of decoupling higher-orders in calculating Green’s functions (GFs). As pointed out in the introduction, the HEOM technique has been used extensively in accurate characterizations on strongly correlated quantum transport systems, in particular it reproduces Kondo peaks in the spectral density of local impurity systems [27, 47]. In the present work, the HEOM evaluations effectively converged at \( L = 3 \) tier level indicates that the HEOM is more efficient than the standard EOM scheme in decoupling higher-order GFs [75, 76] for finite Coulomb interaction. The more details of the HEOM method and its benchmark comparisons with other techniques can be found in [25–28].

3. Coherence of charge qubit

3.1. Coherence control with bias voltage

We focus on the quantum coherence of the two charge states, \( \{1, 2\} \), which constitute a charge qubit with the single-electron occupation. The interested charge–qubit density operator \( \rho_c(t) \) is the \( 2 \times 2 \) sub-matrix of the reduced system \( \rho(t) \). The latter and also other \( \rho^{(n)}_j(t) \) spans over all the four Fock states, \( \{0\}, \{1\}, \{2\}, \{\downarrow\} \). The probability of single electron occupation is given by \( p_c = \text{tr} \rho_c = \rho_{11} + \rho_{22} \leq 1 \), and the charge qubit phase \( \theta = \text{arctan}[\text{Im}(\rho_{12})/ \text{Re}(\rho_{12})] \). The nonzero probabilities of the empty and double-occupation states, \( \rho_{00} \) and \( \rho_{dd} \), are the leakage effects [54, 77]. Denote \( \delta = \rho_{11} - \rho_{22} \) for the probability difference between the two charge states. The charge qubit entropy is given by \( S_c = -\text{tr}(\rho_c \ln \rho_c) \), with \( \rho_c = \rho_c / \text{tr}(\rho_c) \) and \( S^\text{max}_c = \ln 2 \) for two-level systems. Thus \( \chi_c = 1 - S_c / \ln 2 \), satisfying \( 0 \leq \chi_c \leq 1 \), can be used as a purity measure, with \( \chi_c = 1 \) indicating a truly pure state of the charge qubit. In noninteracting double-dot AB interferometers, a phenomenon of phase localization, namely the charge qubit phase does not change as the change of the AB flux, was found [54, 55]. The main finding in this work is to reveal the fact that the charge qubit phase will become controllable through the AB flux when inter-dot Coulomb interaction is considered.

Figure 1 shows the nonequilibrium steady-state results, as functions of the applied voltage, on the charge-qubit state properties given in (a) and (b), the leakage effects in (c), and the transport current in (d), respectively, where the Coulomb interaction of \( U = 0.5 \) meV and the AB phase of \( \phi = \pi/2 \) are used, see the description in the figure caption for the details. The observed results can be understood in terms of the interplay between different tunneling channels involved in individual transport regimes. First of all, there are two types of tunneling channels: the single-electron impurity channels with the degenerate energy levels at \( \varepsilon_1 = \varepsilon_2 = \varepsilon \), and the Coulomb-assisted channels at \( \varepsilon + U \). Concerning further their positions in relation to the applied voltage window, we identify three transport regimes, indicated in figure 1 in terms of I, II, and III, respectively. Let us start with the double-dots state versus voltage, the \( \rho - V \) characteristics, reported in figure 1(a)–(c).
Regime I: $\varepsilon + U > \varepsilon > \mu_{\text{rel}}$. This is the cotunneling regime, with the bias window containing no tunneling channels. The resulted single-electron occupation ($\rho_e = \rho_{11} + \rho_{22}$) and double occupation ($\rho_{dd}$) are both negligible. The full probability of empty state ($\rho_{00}$) emerges.

Regime II: $\varepsilon + U > \mu_{\text{L(R)}} > \varepsilon > \mu_{\text{rel}}$. This is the Coulomb-blockade (CB) regime, and is of particular interest in the present work. The most striking scenario occurs at the bias voltage of $|V| = 2\varepsilon + U$. There emerges a nearly pure charge qubit state, with the single-electron occupation, $p_e = \rho_{11} + \rho_{22}$, and the purity parameter, $\chi_e$, both close to their maximum values of 1. These results are specified with the arrows on the solid curves in figure 1(a), where $\Gamma = 0.02$ meV and $k_B \Gamma = 0.02$ meV are adopted for demonstration. At $|V| = 2\varepsilon + U$, while $\rho_{00} + \rho_{dd} = 1 - p_e \approx 0$, we have also $\delta = \rho_{11} - \rho_{22} = 0$ and therefore $|\rho_{12}| \approx 0.5$, as seen in figures 1(b) and (c). The dashes curves in figure 1(a) go with the increased lead coupling strength, $\Gamma = 0.1$ meV. Apparently, increasing temperature also decreases the purity of the charge qubit state. Note that in the present symmetric bias setup, $|V| = 2\varepsilon + U$ amounts actually to the pair tunneling resonance condition [61], which would also occur in the Coulomb participated regime (II'), where $\mu_L > \varepsilon + U > \mu_R > \varepsilon$; see section 3.2 for the details.

Regime III: $\mu_{\text{L(R)}} > \varepsilon, \varepsilon + U > \mu_{\text{rel}}$. This is the sequential-dominated regime, as both the single-electron impurity and Coulomb-assisted tunneling channels fall inside the bias window. The results here are similar to those of $U = 0$ and weak $U$ obtained in [54] and [55], respectively. Indeed, as reported there before, the phase localization of charge qubit appears at the value of $\theta = -\pi/2$ or $\pi/2$ [54, 55]. Either of these two values corresponds to $\text{Re} \rho_{12} = 0$ (see figure 1(b)). The fact that $\text{Re} \rho_{12}$ vanishes in the sequential-dominated regime (III) is rather robust against the flux; see the discussion for figure 2 later.
proximity to their maximum values of 1, could be experimentally located at the aforementioned I–V characteristic turnover position, where the differential conductance sign changes.

3.2. Charge qubit phase at pair transfer resonance versus AB magnetic flux

Examine now the charge qubit phase, \( q = \text{arctan}(\text{Im}(\rho_{12}) / \text{Re}(\rho_{12})) \), as functions of the AB flux. We focus on the case of \( V = |2\varepsilon + U| \), at which the charge qubit is in close proximity to a pure state; see figure 1(a). It is noticed that in the present symmetric bias setup, \( V = |2\varepsilon + U| \) satisfies the pair tunneling resonance condition [61],

\[
\mu_\alpha - \varepsilon = \varepsilon + U - \mu_\alpha,
\]

(11)

Beside the CB regime (II) described earlier, the pair tunneling occurs also in another transport setup configuration:

- Regime II': \( \mu_L > \varepsilon + U > \mu_R > \varepsilon \), the Coulomb participation (CP) regime. In this regime, the transport would be primarily driven by the electron tunneling from dots to R-lead. The relevant pair tunneling resonance is therefore equation (11) with \( \alpha = R \). On the other hand, in the CB regime (II), where \( \varepsilon + U > \mu_L > \varepsilon > \mu_R \), the resonance follows equation (11) with \( \alpha = L \), as the transport would now be primarily driven by the electron tunneling from L-lead to dots.

Figure 2. The accurate results for the stationary charge state varies with the magnetic flux for different Coulomb interaction at the bias voltage \( V = 0.7 \text{ meV} \). (a) The characteristic coherence is denoted by the relative phase \( \theta \), and (b) and (c) the real and imaginary parts of \( \rho_{12} \). The probabilities of the states for single-electron occupation, empty, and double occupation are illustrated in (d), (e), and (f), respectively. Here, \( U = 0.1 \text{ meV} \) is for sequential-dominated regime (III), and \( U = 0.5 \text{ meV} \) is for CB regime (II) (black-curves, where \( \varepsilon = 0.1 \text{ meV} \)) and CP regime (II') (green-curves, where \( \varepsilon = -0.6 \text{ meV} \)) with satisfying \( V = |2\varepsilon + U| \). The other parameters are the same as in figure 1.
Figure 2 reports the nonequilibrium steady-state results, as functions of the applied AB flux $\phi/\pi \equiv 2\psi/\Phi_0$. Both the CB (black-curves) and CP (green-curves) cases are operated at the pair tunneling resonance voltage, $V = |2\varepsilon + U| = 0.7$ meV, with the same $U = 0.5$ meV, but different values of $\varepsilon = 0.1$ meV and $-0.6$ meV, respectively. Included for comparison are also the sequential-dominated regime (III) counterparts (red-dashed-curves), exemplified with $U = 0.1$ meV and $\varepsilon = 0.1$ meV at $V = 0.7$ meV. As that in figure 1, where $\phi = \pi/2$, the sequential-dominated regime displays the phase localization at $\theta = -\pi/2$ or $\pi/2$, for $V < 0$ and $V > 0$, respectively [54]. This result is independent of the flux at $\phi = 2m\pi$, with $m$ being an integer, as shown by the red-dashed horizontal parts in figure 2(a), where $\Re \rho_{12} = 0$, see figure 2(b). The single-electron occupation, $p_\nu$, and the leakage effects, $\rho_{00}$ and $\rho_{dd}$, as shown by the red-dashed curves in figures 2(d)–(f), also agree with those noninteracting ($U = 0$) results reported in [54].

On the other hand, in both the CB (II) and CP (II’) regimes, while $\Im \rho_{12}$ is rather independent of the inter-dot Coulomb coupling, $\Re \rho_{12}$ significantly deviates from the zero-value behavior in the $U = 0$ case. The single-electron occupation is remarkably enhanced, and meanwhile the leakage effects are greatly suppressed. Especially, at $|V| = |2\varepsilon + U|$ that satisfies the pair tunneling resonance condition [61], the charge qubit state, as inferred from figures 1(a) and (b) for its $\chi_\nu \approx 1$ and $\theta \equiv \rho_{11} - \rho_{22} = 0$, corresponds to a pure-state proximity of

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + e^{-i\theta}|2\rangle).$$

(12)

The AB flux-tuned phase, as shown in figure 2(a) for $V > 0$, is given by

$$\theta = \begin{cases} 
\pi - \phi/2; & \text{CB regime (II)} \\
\phi/2; & \text{CP regime (II’)} 
\end{cases}$$

(13)

For the bias $V < 0$, the above relations hold with exchange of $\phi$ to $-\phi$ (not shown in figure 2), due to the phase-lead symmetry relations underlying equation (5).

Therefore, the manipulation of the charge state coherence is performed at the electrons pair tunneling resonance, equation (11). The arbitrary initially double-dot state would evolve into the fully coherent charge state (equation (12)), with the relative phase precisely tuned by the AB flux (equation (13)). Remarkably, as shown in figure 2, while the singularity occurs in the CB regime (II) at $\phi = 2m\pi$, the CP counterparts are completely free of the singularity. The nearly pure charge qubit state, with both the purity parameter, $\chi_\nu$, and occupation number, $p_\nu$, in close proximity to their maximum values of 1. It is interesting to notice that in the present CP regime (II’) setup, the double-occupation level locates below the transport window; i.e., $\epsilon_d = 2\varepsilon + U = -0.7$ meV < $\mu_R = -0.35$ meV in study. However, its occupation number $\rho_{dd}$ remains very small, under the pair tunneling resonance voltage; see figure 2(f). Involved here is also the interference resonance that overcomes the leakage from the desired charge qubit state.

4. Coherence dynamics analysis

4.1. Charge qubit coherence dynamics

To further explore the underlying mechanism of the full coherence realization of the charge qubit states in the AB interferometers, we study the evolution of $\rho_{12}(t)$, the transient charge-qubit coherence, in both the CB regime (II) (left-panels) and the CP regime (II’) (right-panels), with the initial empty state in the double dots ($\rho_{00}(0) = 1$). The results are presented in figure 3. It shows that the short-time ($t \lesssim 1/\Gamma$) dynamics in both the regime II and II’ are quite similar as that in the sequential-dominated regime (III) reported previously in [54] and [55]. The short-time dynamics of $\rho_{12}(t)$ is dominated by the electron tunneling through the single-electron impurity channels ($\varepsilon$), with little contributions from the Coulomb-assisted channel ($\varepsilon + U$). Denote $\rho_{12}(t) = |\rho_{12}(t)|e^{i\theta(t)}$, the relative phase between the two charge states, $|1\rangle$ and $|2\rangle$, is found to be $\theta(t) = \phi/2$ when $t \lesssim 1/\Gamma$. For $t > 1/\Gamma$, the charge-qubit coherence becomes sensitive to the specific tunneling regimes. The relative locations of the single-occupation ($\varepsilon$) and the double-occupation ($\varepsilon + U$) transport channels with respect to the bias window take the crucial role when $t > 1/\Gamma$. Especially the nonequilibrium charge qubit operated in either the CB regime (II) or the CP regime (II’) remains a proximity to a full coherence dynamics, as shown by equation (12) in the long-time limit. The characteristics of $\Im \rho_{12}(t)$ are similar in these two transport regimes, without changing signs as the time evolves, while $\Re \rho_{12}(t)$ behaviors differ remarkably. In the CB regime (II) (see figure 3(a)), $\Re \rho_{12}(t)$ will switch the sign (unless $\phi = 2m\pi$ that is associated with $\theta = \pi/2$ for $V > 0$), which leads to the relative phase $\theta = \pi - \phi/2$. On the other hand, in the CP regime (II’), as shown in figure 3(b), $\Re \rho_{12}(t)$ has no such sign change and the relative phase remains as the case of $\theta(t) = \phi/2$. Note that the above results in the CB regime and the CP regime could be changed significantly by weakening the Coulomb interaction or increasing the bias voltage or temperature. For example, the full coherence of charge qubit
dynamics would be immediately broken down and the charge qubit state is reduced to the phase localization, a phenomenon revealed in the noninteracting ($U = 0$) limit \cite{54, 55}.

It is worth to point out here that the existence of energy states higher than the aforementioned transport windows would not change the general dynamic behavior, either short- or long-time regions. As described earlier, the short-time dynamics is dominated by the electron tunneling through the single-electron impurity channels ($\varepsilon$). In the long-time regime, the electron pair tunneling resonance takes full effects and suppresses the occupations in the high-energy states.

4.2. Mechanistic analysis

The above nonequilibrium features on the charge-states coherence dynamics, including the short-time, long-time and stationary state behaviors, can be understood as follows. Taking the following transformation on the electron operators in the dots \cite{78},

$$\hat{a}_1 = (a_1 + a_2)/\sqrt{2} \quad \text{and} \quad \hat{a}_2 = (a_1 - a_2)/\sqrt{2},$$

(14)

the system Hamiltonian described in equation (1) are invariant under this transformation,

$$\hat{H}_S = \sum_{\alpha = 1, 2} \varepsilon \hat{a}^\dagger_{\alpha} \hat{a}_{\alpha} + U \hat{n}_1 \hat{n}_2, \quad \text{with} \quad \hat{n}_{\alpha} = \hat{a}^\dagger_{\alpha} \hat{a}_{\alpha}. \quad (15)$$

The tunneling Hamiltonian, equation (3), becomes

$$\hat{H}_{SB} = \sum_{\alpha \beta \kappa \ell} \ell_{\alpha \beta \kappa \ell} \hat{a}^\dagger_{\alpha \kappa} \hat{c}_{\beta \ell} + \text{H.c.,} \quad (16)$$

with $\ell_{\alpha \beta \kappa \ell} = \sqrt{2} t_{\alpha \beta \kappa \ell} \cos(\phi/4)$ and $\ell_{\beta \alpha \ell}^* = \ell_{\ell \kappa \beta \alpha}^* = i \sqrt{2} t_{\beta \alpha \ell} \sin(\phi/4)$. For clarity, we plot the electrons tunneling through the degenerate double-dot AB interferometers described by the tunneling Hamiltonian equations (3) and (16) which are schematically depicted in figures 4(a) and (b), respectively. As it is well known, figure 4(a) is a commonly schematic plot of double-dot AB interferometers used in many papers on quantum dot AB interferometers \cite{17, 54, 57, 78}. An extension of such basis change has also been applied to double-dot system

\begin{figure*}
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The coherence evolution of $\rho_{12}(t)$ for CB regime (II) ($\varepsilon = 0.1$ meV) and the CP regime (II') ($\varepsilon = -0.6$ meV) setups in the left ((a) and (b)) and right ((c) and (d)) panels, respectively, with the strong inter-dot Coulomb interaction $U = 0.5$ meV. The other parameters are the same as in figure 2.}
\end{figure*}
coupled to Majorana fermions [8, 79]. Figure 4(b) is the schematic view for the detail mechanistic analysis of the charge coherence in terms of the new basis after the unitary transformation of equation (14).

The transformed hybridization spectral function, with the equal coupling strengths of equation (7), remain the Lorentzian form of equation (6), but having

$$
\tilde{\Gamma}_{111} = \Gamma \cos^2(\phi/4), \quad \tilde{\Gamma}_{222} = \Gamma \sin^2(\phi/4),
$$

$$
\tilde{\Gamma}_{121} = \tilde{\Gamma}_{12}^* = \tilde{\Gamma}_{212} = \tilde{\Gamma}_{21}^* = i \Gamma \sin(\phi/2)/2.
$$

From $|u\rangle = a_u^\dagger |0\rangle$ and $|\bar{u}\rangle = \bar{a}_u^\dagger |0\rangle$ with equation (14), we have

$$
\rho_{11} = (\rho_{11} + \rho_{22} + 2 \Re \rho_{12})/2,
$$

$$
\rho_{22} = (\rho_{11} + \rho_{22} - 2 \Re \rho_{12})/2,
$$

$$
\rho_{12} = (\rho_{11} - \rho_{22}) - 2i \Im \rho_{12}/2.
$$

Equations (17) and (18) are used in the following analysis, with the focus on the flux $\phi$-dependent charge qubit phase, $\theta = \arctan[\Im(\rho_{11})/\Re(\rho_{12})]$.

Let us start with the two special scenarios, where the inter-dot Coulomb interaction does not play the role, and the charge qubit phases are always localized.

(i) At $\phi = 2m\pi$, we have $\theta = 0$ or $\pi$ (see the black-lines in figure 3). As inferred from equation (17), this scenario has either $\tilde{\Gamma}_{111} = 0$ or $\tilde{\Gamma}_{222} = 0$, with odd or even $m$, respectively, and also $\tilde{\Gamma}_{121} = 0$. The electron tunnels through only one of the transformed single-occupation channels, either $|1\rangle$ or $|\bar{1}\rangle$, see figure 4(b). Consequently, $\rho_{12} = 0$, since there is no interference between these two states. In this case, $\rho_{12}(t)$ is always real, as inferred from equation (18), and the charge qubit phase is localized at $\theta = 0$ or $\pi$. Physically the above scenario amounts to the transport setup involving only one single spinless electronic level. The inter-dot Coulomb interaction does not play any roles in this scenario, and the double-occupation is always $\rho_{12}(t) = 0$. The long-time probabilities of the empty and the single-electron occupied states are equal, i.e., $p_0 = 0.5$ and $p_e = 0.5$. The latter is via either $\rho_{11}$ or $\rho_{22}$, exclusively.

(ii) At $\phi = (2m + 1)\pi$, we have $\theta = \pm \pi/2$ (see the green-lines in figure 3). As inferred from equation (17), this scenario goes by $\tilde{\Gamma}_{111} = \tilde{\Gamma}_{222} = |\tilde{\Gamma}_{121}| = \Gamma/2$, resulting in $\rho_{11}(t) = \rho_{22}(t) = |\rho_{12}(t)|$. This is the case of a full interference with equal probability. In this case, $\rho_{12}(t)$ is always pure imaginary, as inferred from equation (18), and the charge qubit phase is localized at $\theta = \pm \pi/2$. Physically, a full interference with equal probability is an interference resonance. It leads to the maximum value of $p_e = \rho_{11} + \rho_{22} \approx 1$ in the long-time region. Both the vacancy and double occupations are suppressed. This interference resonance behavior is independent of inter-dot Coulomb interaction.
Turn to the situations of \( \phi = n\pi \), away from the above two special scenarios, and the inter-dot Coulomb interaction will play the roles. The general remarks on the nonspecial situations are as follows. (a) In the short-time \((t \lesssim 1/\Gamma)\) region, electrons tunnel mainly through two single-electron impurity channels, with \( \varepsilon_1 = \varepsilon_2 = \varepsilon \). According to the flux-dependent tunneling rate of equation (17), one of them could be called the fast channel and the other be the slow one [80]. More precisely, \([1] \) is the fast channel when \( \phi < \pi/4 \), whereas it is the slow one when \( \phi > \pi/4 \). The fast channel dominates in short time. The above analysis also dictates the sign of \( \text{Re}[\rho_{12}(t)] = [\rho_{11}(t) - \rho_{22}(t)]/2 \) (see equation (18)) in the short-time region. As time evolves, the slow channel occupation gradually accumulates. The sign of \( \text{Re}[\rho_{12}(t)] \) would change if the population inversion could occur. For example, the individual curve in figure 3(a) changes sign, while that in figure 3(c) does not. We will elaborate these observations later; (b) When \( t > 1/\Gamma \), Coulomb-assisted \((\varepsilon + U)\)-channels play roles. These are the transfer channels, rather than the double-occupation state of energy \( 2\varepsilon + U \).

Focus hereafter the long-time behavior for \( \phi = n\pi \), which depends on both single-electron \( \varepsilon \)-channel and Coulomb-assisted \((\varepsilon + U)\)-channel. Apparently, the nonequilibrium property manifests the interplay between these two transfer channels and their relative locations with respect to the bias window. The cotunneling regime (I) is not the interest of this work, since it generates no significant population in the charge qubit state. On the other hand, the sequential-dominant regime (III), where \( \mu_L > \varepsilon, \varepsilon + U > \mu_R \), the interested transfer channels both fall inside the bias window. This is similar to the well-studied Coulomb-free scenario [54, 55], with the results being summarized as follows. In the wide-band-reservoirs limit, the probabilities of electrons tunneling through \([1] \) and \([2] \) would be equal, i.e., \( \rho_{11} = \rho_{22} \) (unless \( \phi = 2n\pi \), the special scenario-(i) described earlier, with \( \theta = 0 \) or \( \pi \)). Again, as inferred from equation (18), the resultant \( \rho_{12} \) is pure imaginary. Phase localization occurs at \( \theta = \pm \pi/2 \), the same value of the special scenario-(ii), but without the aforementioned full interference resonance condition. The leakage effect cannot be neglected; see the regime-III parts of figure 1(c).

From the above detailed analysis, it shows that different behaviors of charge qubit coherence, as depicted in figures 2 and 3 and also equation (13), are rooted at the facts that in the CB regime it is the single-electron \( \varepsilon \)-channel inside the bias window, whereas in the CP regime it is the Coulomb-assisted \((\varepsilon + U)\)-channel. Actually, the aforementioned fast versus slow \( \varepsilon \)-channels, discussed in relation to the short-time region properties, are physically concerned with the CB regime. Involving there the dynamical Coulomb blockade processes [80], leads to electron accumulation in the slow channel, and further the population inversion along evolution. Consequently, \( \text{Re}[\rho_{12}(t)] \) experiences the sign change, as depicted in figure 3(a). This also leads to the charge qubit phase transition at \( \phi = 0 \); see the black-curve in figure 2(a). The CP regime is just the opposite to the CB regime. Now it is the Coulomb-assisted \( \varepsilon + U \)-channels inside the bias window, whereas the single-electron ones are outside. There are no dynamical Coulomb blockage effects; neither the slow channel accumulation nor the population inversion. The resulted relative phase follows \( \theta = \phi/2 \) without jump; see the green-line in figure 2(a).

5. Conclusion

In summary, we have demonstrated that inter-dot Coulomb interactions would play a crucial role in operating degenerate double-dots as a charge qubit. Finite Coulomb interaction could result in dynamical Coulomb-assisted transport channels. Together with single-electron ones they comprise electron tunneling interference pairs, whenever \( \varepsilon + U > \mu_L > \varepsilon > \mu_R \) (Coulomb blockage regime) or \( \mu_L > \varepsilon + U > \mu_R > \varepsilon \) (CP without blockage). The pair tunneling interference is responsible for the coherence control of a charge qubit, including its relative phase, via the applied bias voltage and magnetic flux. A fully coherent charge qubit emerges at the double-dots electron pair tunneling resonance, \( \varepsilon + U - \mu_o = \mu_{\alpha} - \varepsilon \) (see equation (11)). This amounts to \( |V| = |2\varepsilon + U| \), provided \( \mu_L = -\mu_R = V/2 \) that the bias voltage is applied symmetrically to two leads. Interestingly, the pair tunneling resonance can be located at the \( I-V \) characteristic turnover position, as specified by the arrows in figure 1(d). Therefore, the information on a fully coherent charge qubit would be experimentally extracted from where the differential conductance sign changes.

Moreover, the charge qubit phase, operated especially in the CP regime, can be smoothly manipulated via the applied magnetic flux (see equation (13)). This is different from the Coulomb blockage scenario, where the Coulomb-assisted \((\varepsilon + U)\)-channel is above the transport window. The underlying dynamical blockage induces population inversion in the long-time region, and consequently the relative phase change, as inferred from figures 3(a) and (b). In contrast, in the CP regime, the \((\varepsilon + U)\)-channel is within the transport window and does not have the aforementioned blockage effect, as seen from figures 3(c) and (d). All these observations are elaborated via the real-time dynamical and stationary properties of the charge qubit coherence, and also on the basis of a canonical transformation; see section 4.
In conclusion, we present an experimentally viable approach to the preparation and manipulation of charge qubit coherence in double-dots AB interferometers. The predictions of this work and the underlying principles are closely related to the field of quantum information/computation in general.

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