HYPERPLANE ARRANGEMENTS AND COMPACTIFICATIONS OF VECTOR GROUPS

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ABSTRACT. Matroid Schubert varieties have recently played an essential role in the proof of the Dowling-Wilson conjecture and in Kazhdan-Lusztig theory for matroids. We study these varieties as equivariant compactifications of affine spaces, and give necessary and sufficient conditions to characterize them. We also generalize the theory to include partial compactifications and morphisms between them. Our results resemble the correspondence between toric varieties and polyhedral fans.

1. INTRODUCTION

A matroid Schubert variety is a singular algebraic variety constructed from a hyperplane arrangement, which is not a Schubert variety, but which plays the role of a Schubert variety in the role it plays in Kazhdan-Lusztig theory for matroids [Bra+20]. Let $H_1, \ldots, H_n$ be a collection of linear hyperplanes in a finite dimensional complex vector space $V$. Assume that the arrangement is essential, meaning that the common intersection of the hyperplanes is zero. The associated matroid Schubert variety is the closure of $V$ in $(\mathbb{P}^1)^n$ via the embedding $V \subseteq V/H_1 \times \ldots \times V/H_n \subseteq \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$.

Matroid Schubert varieties were first studied by [AB16], where the authors showed that the combinatorics of the matroid associated to the arrangement determined much of the geometry of the variety. The intersection cohomology of matroid Schubert varieties was used in [HW17] to prove Dowling and Wilson’s Top Heavy conjecture for matroids in the realizable case.

The affine reciprocal plane of an essential hyperplane arrangement is the intersection of the matroid Schubert variety with the affine chart $(\mathbb{C}^\times \cup \{\infty\})^n \subseteq (\mathbb{P}^1)^n$. Reciprocal planes are studied in [Ter02, PS06, EPW16, KV19], where the authors also observe a two way street between the combinatorics of arrangements and the geometry of their reciprocal planes. The intersection cohomology of the projectivized reciprocal plane was used in [EPW16] to prove that the coefficients of the Kazhdan-Lusztig polynomial of a realizable matroid are non-negative.

In this paper we study the geometry of matroid Schubert varieties through the lens of equivariant compactifications. If $G$ is an algebraic group, then an equivariant compactification of $G$ is a proper variety $X$ containing $G$ as a dense open set, and an action $G \times X \to X$ extending the group law $G \times G \to G$. With the word “proper” omitted, we call $X$ an equivariant partial compactification. For example, a toric variety is by definition an equivariant partial compactification of the algebraic torus $T = (\mathbb{C}^\times)^d$. One of the main theorems in toric geometry states that all normal toric varieties arise from polyhedral fans. A matroid Schubert variety is an equivariant compactification of the additive group $V = \mathbb{C}^d$, which we will call a vector group.
To see the equivariant structure, note that \( \mathbb{C} \subseteq \mathbb{P}^1 \) is an equivariant compactification of the additive group \( \mathbb{C} \), and so \( \mathbb{C}^n \subseteq (\mathbb{P}^1)^n \) is an equivariant compactification of \( \mathbb{C}^n \). Therefore the closure of any subgroup \( V \subseteq \mathbb{C}^n \) in \((\mathbb{P}^1)^n\) is an equivariant compactification of \( V \), because the action of \( V \) on itself preserves its closure.

The main purpose of this paper is to prove the following characterization of which equivariant compactifications of vector groups arise as matroid Schubert varieties.

**Theorem 1.1.** An equivariant compactification \( Y \) of the vector group \( V = \mathbb{C}^d \) is isomorphic to a matroid Schubert variety if and only if \( Y \) is normal as a variety, \( Y \) has only finitely many orbits, and each orbit contains a point which can be reached by a limit \( \lim_{t \to \infty} tv \), for \( v \in V \).

The limit condition in the above theorem is analogous to the fact that any orbit in a normal toric variety can be reached by a one-parameter subgroup of the torus. Because the fan corresponding to a normal toric variety is constructed by considering the limits of one-parameter subgroups, it is natural to look for an analogous correspondence only for equivariant compactifications of \( V \) where every orbit is reached by a one-variable limit.

In the course of proving the above theorem, we give another characterization in which the limit condition is replaced by the stronger condition that each orbit admits a normal slice satisfying certain properties. The second characterization resembles a key geometric property of matroid Schubert varieties: for each flat in a hyperplane arrangement, the matroid Schubert variety of the restriction is a normally nonsingular slice through the corresponding orbit [Bra+20].

Finally, we prove an equivalence of categories which generalizes both characterizations to include partial compactifications as well as morphisms between them. The objects in the first category are equivariant partial compactifications of \( V \) satisfying the conditions of the above theorem, or the stronger formulation involving normal slices. The objects in the second category we call *partial hyperplane arrangements*, which include all essential hyperplane arrangements as examples.

1.1. **Equivariant compactifications.** We assume all varieties are irreducible and separated over \( \mathbb{C} \). Suppose that \( G \) is a commutative linear algebraic group, acting on a variety \( X \). Given a point \( x \in X \), we write \( G \cdot x \subseteq X \) for the orbit and \( G_x \subseteq G \) for the stabilizer.

The main tool we will use throughout the paper is the following notion of a slice (Definition 2.4), which is standard for actions of Lie groups and algebraic groups [Gle50, Mos57, MY57, Pal61]. A (Zariski) slice through \( x \in X \) is a \( G_x \)-stable subvariety \( Z_x \subseteq X \) containing \( x \), such that \( G \cdot Z_x \subseteq X \) is an open set, and

\[
G \cdot Z_x \cong G \times Z_x / \sim, \; \text{where} \; (gh, z) \sim (g, hz) \; \text{for all} \; g \in G, h \in G_x, z \in Z_x.
\]

Geometrically, \( Z_x \) is a normal slice through the orbit \( G \cdot x \), and \( G \cdot Z_x \) is neighborhood of \( G \cdot x \) that admits a product structure, similar to a tubular neighborhood. We use the words “Zariski slice” to emphasize the difference between the above notion and that of an étale slice.

In order to state our results, we make the following abbreviations. Suppose now that \( X \) is an equivariant partial compactification of \( G \). We say \( X \) satisfies

- **FO** (**F**inite **O**rbits) if \( X \) has finitely many \( G \)-orbits,
- **OP** (**O**ne-parameter subgroups) if for every \( G \)-orbit \( G \cdot x \subseteq X \), there is a one-dimensional algebraic subgroup of \( G \) whose closure in \( X \) intersects \( G \cdot x \), and
• SL (Slices) if there exists a Zariski slice through every point of $X$.

The following is our main result on matroid Schubert varieties, which implies Theorem 1.1.

**Theorem A.** Suppose that $Y$ is an equivariant compactification of the vector group $V = \mathbb{C}^d$. Then the following are equivalent.

(i) $Y$ is equivariantly isomorphic to a matroid Schubert variety.

(ii) $Y$ is normal and satisfies FO and SL.

(iii) $Y$ is normal and satisfies FO and OP.

The original aim of this project was to prove that the third statement implies the first, however we have found that it is most natural to prove that the third statement implies second, and then prove that the second implies the first. For this reason, we view the existence of slices as a more fundamental property of matroid Schubert varieties.

The study of equivariant compactifications of vector groups was initiated by [HT99], and we recommend [AZ20] for a survey. We will see in Section 1.3 that matroid Schubert varieties show many parallels to toric varieties, however the study of general equivariant compactifications of vector groups has little in common with toric geometry [AZ20]. In particular, toric varieties satisfy FO, OP, and SL, whereas these conditions need not hold for equivariant compactifications of vector groups, as the following examples show.

**Example 1.2.** Consider the action of a vector group $V$ of dimension at least two on its projective closure $\mathbb{P}(V \oplus \mathbb{C})$, where the action on the boundary is trivial. This compactification satisfies SL and OP, but not FO.

**Example 1.3 ([HT99]).** Consider the action of the two-dimensional vector group $\mathbb{C}^2$ on $\mathbb{P}^2$ where $(a_1, a_2)$ acts as

$$
\exp \begin{pmatrix}
0 & a_1 & a_2 \\
0 & 0 & a_1 \\
0 & 0 & 0
\end{pmatrix}
= \begin{pmatrix}
1 & a_1 & a_2 + \frac{1}{2}a_1^2 \\
0 & 1 & a_1 \\
0 & 0 & 1
\end{pmatrix}.
$$

This action has one two-dimensional orbit (with which we can identify $\mathbb{C}^2$), one one-dimensional orbit, and one zero-dimensional orbit, so FO holds. However SL and OP fail for the one dimensional orbit.

### 1.2. Equivariant partial compactifications

We now describe how Theorem A extends to equivariant partial compactifications of vector groups, as well as morphisms between them. We define a morphism of equivariant compactifications of vector groups to be a map of varieties, which restricts to a linear map from the first vector group to the second.

Let us first review some hyperplane arrangement terminology. We will work only with arrangements of linear hyperplanes in a finite dimensional complex vector space. We do not consider arrangements of affine hyperplanes. We say that a hyperplane arrangement is *essential* if the common intersection of the hyperplanes is zero. A *flat* of a hyperplane arrangement is a linear subspace of the ambient vector space which can be written as the intersection of several hyperplanes. We consider the ambient vector space to be a flat, because it arises from the empty intersection of hyperplanes.
Remark 1.4. Following the standard convention, we equip the collection of flats with the partial order given by reverse inclusion, writing $F \leq G$ if $F$ and $G$ are flats such that $G \subseteq F$. When the arrangement is essential, this partial order gives the collection of flats the structure of a finite geometric lattice, or equivalently, a simple matroid. For our purposes, the partial order will only be used in Example 1.6 and in the fact that will refer to the flats of an essential hyperplane arrangement as the “lattice of flats.”

Viewing an essential hyperplane arrangement as its lattice of flats, we make the following generalization:

**Definition 1.5.** A partial hyperplane arrangement in $V = \mathbb{C}^d$ is a finite collection $\mathcal{L}$ of vector subspaces of $V$, such that

(i) $\{0\} \in \mathcal{L}$,
(ii) if $F, F' \in \mathcal{L}$ then $F \cap F' \in \mathcal{L}$,
(iii) for each $F \in \mathcal{L}$, the set $\{F' \in \mathcal{L} : F' \subseteq F\}$ is the lattice of flats of an essential hyperplane arrangement in the vector space $F$.

**Example 1.6.** Suppose that $\mathcal{L}$ is the lattice of flats of an essential hyperplane arrangement, and $\mathcal{P} \subseteq \mathcal{L}$ is an order filter (i.e. an upward closed set under the partial order of reverse inclusion.) Then $\mathcal{P}$ is a partial hyperplane arrangement.

**Example 1.7.** Here we give an example of a partial hyperplane arrangement in $\mathbb{C}^4$. Let $\mathcal{L}$ consist of the zero subspace of $\mathbb{C}^4$, together with the affine cones of the points, lines, and planes in $\mathbb{P}^3$ listed in Fig. 1.

![Figure 1. The projectivization of a partial hyperplane arrangement in $\mathbb{C}^4$.](image)

**Example 1.8.** Here we give an example of a partial hyperplane arrangement which cannot be realized as an order filter in the lattice of flats of a hyperplane arrangement. Consider the partial hyperplane arrangement $\mathcal{L}$ in $\mathbb{C}^3$ consisting of the proper coordinate subspaces, and a general line. Any hyperplane passing through the general line will intersect one of the coordinate hyperplanes in a non-coordinate line. Therefore if there is hyperplane arrangement whose lattice of flats contains $\mathcal{L}$, then $\mathcal{L}$ cannot be upward closed.

**Definition 1.9.** Suppose that $\mathcal{L}_i$ is a partial hyperplane arrangement in a vector group $V_i$ for $i = 1, 2$, and $T : V_1 \to V_2$ is a linear map. Then we say $T$ is a morphism of partial hyperplane arrangements if
(i) for each $F_1 \in \mathcal{M}_1$ there exists $F_2 \in \mathcal{M}_2$ such that $T(F_1) \subseteq F_2$,
(ii) for each $F_1 \in \mathcal{M}_1$ and $F_2 \in \mathcal{M}_2$, $T^{-1}(F_2) \cap F_1 \in \mathcal{M}_1$.

*Example* 1.10. In the case where $\mathcal{M}_i$ is the lattice of flats of a hyperplane arrangement $\mathcal{A}_i$, then $T$ is a morphism of partial hyperplane arrangements if and only if the preimage of each hyperplane in $\mathcal{A}_2$ is either a hyperplane in $\mathcal{A}_1$ or is $V_1$.

The following is our main result, in maximal generality.

**Theorem B.** There is a fully faithful embedding of categories from partial hyperplane arrangements to equivariant partial compactifications of vector groups, such that the following are equivalent for an equivariant partial compactification $Y$.

(i) $Y$ arises from a partial hyperplane arrangement.
(ii) $Y$ is normal and satisfies FO and SL.
(iii) $Y$ is normal and satisfies FO and OP.

Given an equivariant partial compactification $Y$ of $V$ arising from a partial hyperplane arrangement $\mathcal{L}$, then $\mathcal{L}$ can be recovered as the collection of all stabilizers of points in $Y$. See Proposition 3.3.

1.3. **Analogy with toric varieties.** For the remainder of the introduction, we say that an equivariant partial compactification of $V$ satisfying any of the equivalent conditions of Theorem B is a *linear V-variety*. In this section, we explain the analogy between Theorem B and the correspondence between normal toric varieties and polyhedral fans. From now on, we assume all toric varieties are normal.

Toric varieties satisfies FO, SL, and OP, and once we impose these conditions onto an equivariant partial compactification of a vector group, there is dictionary (Fig. 2) which is similar to the dictionary between toric varieties and fans. In both cases the idea is to cover the variety with “simple” open sets. The main difference is that these open sets are affine in the torus case and non-affine in the vector group case.

1.3.1. **One-parameter subgroups.** Let $T$ be an algebraic torus. A one-parameter subgroup of $T$ is an algebraic group homomorphism from $\mathbb{C}^\times$ to $T$. The one-parameter subgroups of $T$ form a finitely generated free abelian group $N$, and write $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$ for the corresponding real vector space. Let $V$ be a vector group. By a one-parameter subgroup of $V$, we mean an algebraic group homomorphism from $\mathbb{C}$ to $V$. The one-parameter subgroups of $V$ naturally correspond to the elements of $V$, so $V$ will play the role of both $T$ and $N$.

1.3.2. **Full dimensional cones and essential hyperplane arrangements.** The toric varieties arising from full dimensional cones are exactly the affine toric varieties that have no torus factors. If $\sigma \subseteq N_\mathbb{R}$ is full dimensional (strictly convex rational polyhedral) cone, then there is a canonical embedding of tori $T \subseteq \prod_{u \in \mathcal{H}} \mathbb{C}^\times$ where $\mathcal{H}$ is the unique minimal basis of the dual semigroup. Note that this embedding is only canonical when $\sigma$ is full dimensional. The corresponding toric variety is the closure of $T$ in $\prod_{u \in \mathcal{H}} (\mathbb{C}^\times \cup \{0\})$. If $\mathcal{A}$ is an essential hyperplane arrangement in $V$, then there is a canonical embedding of vector groups $V \subseteq \prod_{H \in \mathcal{A}} V/H$. The corresponding matroid Schubert variety is the closure of $V$ in $\prod_{H \in \mathcal{A}} (V/H \cup \{\infty\})$. 

| Sec. | Combinatorics | Geometry | Combinatorics | Geometry |
|------|---------------|----------|---------------|----------|
| 1.3.1 | N T           | V        | V             |
| 1.3.2 | Full          | Essential| Matroid       |
|       | dimensional   | hyperplane| Schubert      |
|       | cones in N_R  | arrangements| varieties |
| 1.3.3 | Cones in N_R  | Essential| Simple linear |
|       | Affine toric  | hyperplane| V-variety     |
|       | varieties with| arrangements|            |
|       | no torus factors | in F ⊆ V |            |
| 1.3.4 | Fans in N_R   | Partial  | Linear        |
|       | Toric varieties| hyperplane| V-variety     |
|       |                | arrangements|            |
|       |                | in F ⊆ V  |            |
| 1.3.5 | Cones in a fan| Orbits,  | Orbits,       |
|       | Σ              | distinguished| distinguished |
|       |                | points, and| points, and   |
|       |                | invariant affine| invariant   |
|       |                | opens in X_Σ | opens in     |
|       |                |            | Y_L          |

Figure 2. Correspondences for toric varieties versus correspondences for linear V-varieties.

1.3.3. Simple partial compactifications. Suppose an algebraic group G acts on a variety X with finitely many orbits. We say that X is simple if there is a unique closed orbit. Since the orbits form a finite stratification, X can be covered with simple G-stable open sets. Simple toric varieties are exactly affine toric varieties by [Sum74, Corollary 2], and every affine toric variety arises from a sublattice \( N' \subseteq N \) and a full dimensional cone \( \sigma \subseteq N' \otimes \mathbb{R} \). Simple linear V-variety varieties are not affine, however by Proposition 2.15 and Theorem A every simple linear V-variety arises from a vector subspace \( F \subseteq V \) and an essential hyperplane arrangement in \( F \).

1.3.4. Partial compactifications. Toric varieties are constructed from affine toric varieties by gluing according to the fan, and likewise linear V-variety varieties are constructed from simple linear V-variety varieties by gluing according to the partial hyperplane arrangement. See Section 5.3.7 for more details.

1.3.5. Orbit correspondences. Let Σ be a fan in \( N_\mathbb{R} \) corresponding to a toric variety \( X_\Sigma \). Let \( \sigma^o \) denote the relative interior of a cone \( \sigma \in \Sigma \). That is, \( \sigma^o = \sigma \setminus \bigcup \tau \) where the union is over \( \tau \in \Sigma \) such that \( \tau \subseteq \sigma \). The cones \( \sigma \in \Sigma \) are in one-to-one correspondence with distinguished points \( x_\sigma \in X_\Sigma \), given by

\[
N \cap \sigma^o = \{ \lambda \in N : \lim_{t \to 0} \lambda(t) = x_\sigma \}.
\]

Let \( \mathcal{L} \) be a partial hyperplane arrangement corresponding to a linear V-variety \( Y_\mathcal{L} \). Given \( F \in \mathcal{L} \), write \( F^o = F \setminus \bigcup F' \) where the union is over \( F' \in \mathcal{L} \) such
that $F' \subseteq F$. The flats $F \in \mathcal{L}$ are in one-to-one correspondence with distinguished points $y_F \in Y_{\mathcal{L}}$, given by

$$F^o = \{ v \in V : \lim_{t \to \infty} tv = y_F \}.$$ 

In both cases, each orbit contains exactly one distinguished point. Therefore cones (resp. flats) also correspond to orbits, and to simple invariant open sets in $X_{\mathcal{L}}$ (resp. $Y_{\mathcal{L}}$). See Section 3 for more details.

1.4. Structure of the paper. In Section 2 we prove some consequences of $\text{FO}$ and $\text{SL}$ which will be used throughout the paper, and we prove the equivalence of Item ii and Item iii in Theorem B. In Section 3 we prove the analog of the orbit-cone correspondence for linear $V$-varieties. In Section 4 we prove Theorem A, and in Section 5 we prove Theorem B. The appendix describes matroid Schubert varieties in coordinates.

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2. Slices

2.1. Slices of group actions. We begin by reviewing the definition of homogeneous fiber spaces, following [VP94, Chapter II.4.8]. Suppose that $G$ is an algebraic group, $H$ is an algebraic subgroup, and $Z$ is a quasiprojective variety on which $H$ acts. Then there exists a variety $G * H \mathcal{Z}$ called the homogeneous fiber space, which parameterizes equivalence classes in $G \times Z$, where

$$(gh, z) \sim (g, hz), \quad \text{for all } g \in G, h \in H, z \in Z.$$ 

There is a canonical map $G \times Z \to G * H \mathcal{Z}$ sending a point $(g, z)$ to its equivalence class, which we write as $[g, z]$. The universal property which characterizes $G * H \mathcal{Z}$ is the following: if $\pi : G \times Z \to X$ is a map of varieties such that $\pi(gh, z) = \pi(g, hz)$ for all $g \in G, h \in H, z \in Z$, then there is a unique factorization as follows.

$$\begin{array}{ccc}
G \times Z & \xrightarrow{\pi} & X \\
& \searrow & \\
& G * H \mathcal{Z} & \xrightarrow{= \tau} \\
& & \xrightarrow{\text{pr}_1} \\
& & G/H
\end{array}$$

From the universal property, we see that there is a canonical map

$$\tau : G * H \mathcal{Z} \to G/H, \quad [g, z] \mapsto gH,$$

with each fiber isomorphic to $Z$. We call $\tau$ the canonical fibration. If $H \subseteq G$ is normal and has a splitting $s : G \to H$, then there is a $G$-equivariant isomorphism

$$G * H \mathcal{Z} \cong G/H \times Z, \quad [g, z] \mapsto (gH, s(g) \cdot z),$$

which makes the following diagram commute, where $\text{pr}_1$ is the projection.

$$\begin{array}{ccc}
G * H \mathcal{Z} & \xrightarrow{=} & G/H \times Z \\
& \searrow & \downarrow \text{pr}_1 \\
& G/H & \xrightarrow{\tau}
\end{array}$$
Remark 2.1. For the remainder of the paper we will take $G$ to be commutative. Therefore it is possible for us to avoid defining $G *_H Z$ by choosing splittings and working with $G/H \times Z$ instead. While $G/H \times Z$ is a simpler construction, we have found that thinking in terms of the more canonical construction $G *_H Z$ shortens and clarifies the rest of the paper enough to make it worthwhile.

We need the following lemmas, which are formal consequence of the definitions.

**Lemma 2.2** (Associativity of $*$). Suppose that $H' \subseteq H \subseteq G$ are closed subgroups and $H'$ acts on a variety $Z'$. Then there is a natural isomorphism

$$G *_H (H *_{H'} Z') \cong G *_{H'} Z', \quad [g, [h, z]] \mapsto [gh, z].$$

**Lemma 2.3** (Orbits and stabilizers of $G *_H Z$).

(i) There is a one-to-one correspondence between $G$-orbits in $G *_H Z$ and $H$-orbits in $Z$ which sends $G \cdot [v, z]$ to $H \cdot z$.

(ii) Suppose that $G$ is commutative, and $x = [v, z] \in G *_H Z$. Then the stabilizers $G_x$ and $H_z$ coincide as subgroups of $G$.

**Definition 2.4.** Suppose that $X$ is an algebraic variety with a $G$-action. If $x \in X$ is a point with stabilizer $G_x$, we say that a $G_x$-stable locally closed subvariety $Z_x \subseteq X$ containing $x$ is a **(Zariski) slice** at $x$ if the natural map

$$G *_{G_x} Z_x \rightarrow X, \quad [g, z] \mapsto g \cdot z$$

is a $G$-equivariant Zariski open embedding.

The point $x$ is in the image of $G *_{G_x} Z_x$, so we have that $G *_{G_x} Z_x$ is identified with a $G$-stable neighborhood of the orbit $G \cdot x$.

We will often use the following criterion for open embeddings to prove the existence of slices.

**Theorem 2.5** (Zariski’s main theorem). Suppose that $\pi : X \rightarrow Y$ is a morphism of varieties which is birational and injective on closed point, and that $Y$ is normal. Then $\pi$ is an open embedding.

For the above formulation, we refer to [Vak](#) Exercise 29.6.D) and the surrounding discussion. For our purposes, checking injectivity on closed points can be rephrased as follows.

**Lemma 2.6.** Suppose that $x \in X$ and $Z_x \subseteq X$ is a $G_x$-stable subvariety containing $x$. Then $G *_{G_x} Z_x \rightarrow X$ is injective on closed points if $g_1 \cdot z_1 = g_2 \cdot z_2$ implies $g_2^{-1}g_1 \in G_x$ for all $g_1, g_2 \in G$ and $z_1, z_2 \in Z_x$.

2.2. **Partial compactifications with slices.** For this subsection let $G$ be a commutative linear algebraic group, and $X$ a normal equivariant partial compactification of $G$ such that FO and SL hold. We will first collect some basic consequences.

**Lemma 2.7.** If $x \in X$ has a slice $Z_x$, then $Z_x \cap G$ is a coset of $G_x$.

**Proof.** We have that $G$ is contained in any invariant open neighborhood of $X$, and the natural map

$$G *_{G_x} Z_x \rightarrow X$$

is a $G$-equivariant open embedding, so there must be $[v, z] \in G *_{G_x} Z_x$ mapping to $G$. Therefore $Z_x \cap G \neq \emptyset$. We also have that $G *_{G_x} Z_x \cong G/G_x \times Z_x$ embeds as a Zariski open set in the variety $X$, so $Z_x$ and thus $Z_x \cap G$ is irreducible of
dimension \( \dim G - \dim G/G_x = \dim G_x \). Finally we note that the only irreducible 
\( G_x \)-invariant closed subsets of \( G \) of dimension \( \dim G_x \) are cosets.

There is a special point in each orbit corresponding to the trivial coset:

**Definition 2.8.** We say that a point \( x \in X \) is distinguished if it has a slice 
containing the identity of \( G \).

It follows from Lemma 2.7 that every orbit contains a distinguished point, and 
we will see in Lemma 2.11 that every orbit contains at most one distinguished point.

The orbits of \( X \) form a finite stratification, so each orbit \( G \cdot x \) has a unique 
smallest \( G \)-invariant open neighborhood defined as follows.

**Definition 2.9.** The minimal \( G \)-invariant neighborhood \( U_x \) of \( x \in X \) is given 
by the union of all orbits \( G \cdot y \) such that \( G \cdot x \subseteq G \cdot y \).

There exists a unique slice through \( x \in X \) contained in \( U_x \), defined as follows:

**Definition 2.10.** The minimal slice through \( x \in X \) is \( Z_x \cap U_x \), where 
\( Z_x \) is any slice of \( x \).

It follows by [VP94, Proposition II.4.21] that the minimal slice through \( x \) is 
indeed a slice, and uniqueness of the minimal slice for distinguished points (the 
case of an arbitrary point follows easily) comes from the following observation:

**Lemma 2.11.** The minimal slice \( Z_x \) at a distinguished point \( x \in X \) is the closure 
of \( G \cdot x \) in \( U_x \).

**Proof.** Because \( x \) is distinguished, \( Z_x \) contains the identity of \( G \). Since \( G \cap Z_x \) 
is a coset of \( G_x \) by Lemma 2.7, \( G \cap Z_x = G_x \). Thus the closure of \( G_x \) in \( U_x \) is 
contained in \( Z_x \). Because \( G \cdot G_x Z_x \cong G/G_x \times Z_x \) embeds as an open set in \( X \), we 
get that \( Z_x \) is irreducible of dimension \( \dim G - \dim G/G_x = \dim G_x \), and closed in 
\( U_x \). Therefore the closure of \( G_x \) in \( U_x \) equals \( Z_x \), since they are both irreducible 
closed subvarieties of \( U_x \) of the same dimension.

**Example 2.12 (Distinguished points and minimal slices of \( \mathbb{P}^1 \)).**

(i) Consider \( \mathbb{P}^1 = \mathbb{C} \cup \{0\} \cup \{\infty\} \) as an equivariant compactification of \( \mathbb{C}^\times \).
The distinguished points are 1, 0, and \( \infty \) with minimal invariant open neighborhoods \( \mathbb{C} \cdot 1, \mathbb{C} \cdot 0 \), and \( \mathbb{C} \cdot \infty \) respectively. The minimal slices are 
\( \{1,\mathbb{C} \cdot 0 \}, \mathbb{C} \cdot \{0\} \), and \( \mathbb{C} \cdot \{\infty\} \) respectively. Note that \( \mathbb{P}^1 \) is a non minimal 
slice through both 0 and \( \infty \).

(ii) Consider \( \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \) as an equivariant compactification of \( \mathbb{C} \). The distin-
guished points are 0 and \( \infty \) with minimal invariant open neighborhoods \( \mathbb{C} \) and \( \mathbb{P}^1 \) respectively. The minimal slices are \( \{0\} \) and \( \mathbb{P}^1 \) respectively.

In Appendix A, we demonstrate the notions developed in this section for matroid 
Schubert varieties.

We now prove that the class of varieties we are working with is closed under 
taking slices:

**Lemma 2.13.** If \( x \in X \) is a distinguished point, then the minimal slice \( Z_x \) is a 
normal partial compactification of \( G_x \) satisfying \( FO \) and \( SL \).

**Proof.** By Lemma 2.11, \( Z_x \) is an equivariant partial compactification of \( G_x \), and 
\( U_x \) is normal by [VP94, Proposition II.4.22]. By Lemma 2.3, the \( G_x \)-orbits of \( Z_x \)
correspond to the $G$-orbits of an open set in $X$, so $Z_x$ has finitely many $G_x$-orbits. Finally we check that $Z_x$ has slices. For a point $y \in Z_x$, simply take the slice $Z_y$ through $y$ in $X$. Since $y \in Z_x \subseteq U_x$, $G \cdot x \subseteq G \cdot y$ by definition of $U_x$. Therefore $G_y \subseteq G_x$, so $Z_y \subseteq Z_x$ by Lemma 2.11. Now consider the diagram

$$
G \ast G_y Z_y \longrightarrow X \\
G_x \ast G_y Z_y \longrightarrow Z_x
$$

where the horizontal maps are the open embeddings $[v, z] \mapsto v \cdot z$, and the left vertical map is given by $[v, z] \mapsto [v, z]$. We wish to show that the bottom arrow is an open embedding, for which we use Theorem 2.5. The bottom arrow restricts to an isomorphism $G_x \ast G_y G_y \cong G_x$, so it is birational. All three arrows except the bottom one are already known to be injective on closed points, so the bottom arrow is injective on closed points. □

**Remark 2.14.** It follows from [VP94, Proposition II.4.21] that every orbit closure satisfies $\text{SL}$ for the group $G/G_x$. So modulo normality, the class of varieties studied in this section is also closed under taking orbit closures.

### 2.3. Topology of orbit stratification.

In the previous section we studied partial compactifications of tori and vector groups simultaneously. In this section, we will use properties of vector groups which fail for tori.

**Proposition 2.15.** If $X$ is an equivariant partial compactification of a vector group $V$ satisfying $\text{FO}$ and $\text{SL}$, and $x \in X$ is a distinguished point, then the minimal slice $Z_x$ is proper and has $x$ as the unique $V_x$-fixed point.

The proof follows from a general topological observation about varieties stratified into affine spaces. We say that an algebraic cell decomposition of a variety $X$ is a partition $X = \sqcup \alpha S_\alpha$ into finitely many locally closed subvarieties $S_\alpha$ called cells, such that each cell is isomorphic to an affine space and the closure of a cell is a union of cells.

**Lemma 2.16.** Suppose that $Z$ is a connected variety with an algebraic cell decomposition that has at least one zero dimensional cell. Then $Z$ is proper and has exactly one zero dimensional cell.

**Proof.** Consider the singular cohomology with compact support $H^i_c(Z; \mathbb{Q})$. Since $Z$ is connected, we have that $H^0_c(Z; \mathbb{Q})$ is zero if $Z$ is not proper and one dimensional if $Z$ is proper. The lemma follows from the well known fact that $$\dim H^i_c(Z; \mathbb{Q}) = \#\{i\text{-dimensional cells in } Z\}.$$ One way to prove the above equation is by inducting on the number of cells as follows. Suppose $S$ is a cell of lowest dimension in $Z$ and $U$ is its open complement. Since $S \cong \mathbb{A}^r$ for some $r$, $H^i_2(S; \mathbb{Q}) = \mathbb{Q}$ and $H^i_c(S; \mathbb{Q}) = 0$ for $i \neq 2r$. Then the above equation follows from induction using long exact sequence $$\ldots \rightarrow H^i_2(U; \mathbb{Q}) \rightarrow H^i_c(Z; \mathbb{Q}) \rightarrow H^i_c(S; \mathbb{Q}) \rightarrow H^{i+1}_c(U; \mathbb{Q}) \rightarrow \ldots.$$ □

**Proof of Proposition 2.17.** We have by Lemma 2.13 that $Z_x$ has finitely many $V_x$ orbits. Each orbit of $V_x$ is isomorphic to an affine space, and as is true of any algebraic group action, orbits are locally closed and the closure of an orbit is a union
of orbits. Thus the $V_x$-orbits of $Z_x$ form an algebraic cell decomposition. Since $x \in Z_x$ is a zero dimensional cell, the proposition follows from Lemma 2.16. □

**Remark 2.17.** In the notation of Proposition 2.15, it follows that the minimal slice through $x$ is the unique slice through $x$, as opposed to the torus case. See Example 2.12.

**Remark 2.18.** In the case where $X$ is a toric variety with torus $T$, the minimal slice $Z_x$ through a point $x \in X$ is not proper but rather affine. However $Z_x$ still has $x$ as the unique $T_x$-fixed point for a different reason. This is due to the fact that disjoint $T$-invariant closed sets in an affine $T$-variety can be separated by an invariant function [Dol03, Lemma 6.1]. Since $Z_x$ has a dense $T_x$-orbit, all invariant functions are constant. Therefore all invariant closed sets intersect, so $x$ is the only $T_x$-fixed point.

### 2.4. Slices and one-parameter subgroups

In this section we prove that Item ii and Item iii in Theorem B are equivalent. We break the proof into two lemmas.

**Definition 2.19.** Suppose that $X$ is an equivariant partial compactification of a vector group $V$, and $x \in X$. Define

$$V^0_x = V_x \setminus \bigcup V_y$$

where the union is over $y \in X$ such that $V_y \subseteq V_x$.

**Lemma 2.20.** Suppose $X$ is a normal equivariant partial compactification of a vector group $V$, satisfying FO and SL. Let $x \in X$ be a distinguished point, and let $v \in V$. Then $\lim_{t \to \infty} tv = x$ if and only if $v \in V^0_x$. In particular, $X$ satisfies OP.

**Proof.** Suppose that $v \in V^0_x$. By Proposition 2.15, $Z_x$ is proper, so $\lim_{t \to \infty} tv$ must converge to a boundary point of $Z_x \supseteq V_x$. In addition, $v$ lies in the stabilizer of $\lim_{t \to \infty} tv$, so $\lim_{t \to \infty} tv$ be a $V_x$-fixed point. By Proposition 2.15, $x$ is the unique $V_x$-fixed point in $Z_x$, so $\lim_{t \to \infty} tv = x$. To prove the other direction, we note that $V$ is partitioned into sets of the form $V^0_y$ for $y \in X$, so if $v \notin V^0_x$ then $\lim_{t \to \infty} tv = y$ for some $y \neq x$. □

**Lemma 2.21.** Suppose that $X$ is a normal equivariant partial compactification of $V$ satisfying FO and OP. Then $X$ satisfies SL.

**Proof.** We wish to construct a slice through a point $x \in X$. We first explain why it is enough to show that the quotient map

$$V \to V/V_x$$

extends to a $V$-equivariant map

$$\tau : U_x \to V/V_x,$$

where $U_x$ is the minimal invariant open neighborhood of Definition 2.9. We can assume without loss of generality that $\tau(x) = 0$, since the translation of a slice is a slice. Setting $Z_x := \tau^{-1}(0)$, we have that $V_x$ acts on $Z_x$ and $x \in Z_x$. To show that $Z_x$ is a slice, we must check that the natural map

$$V *_{V_x} Z_x \to X, \quad [v, z] \mapsto v \cdot z$$
is an open embedding. For this we use Theorem 2.5. As before we have that $V \ast_{\nu_x} Z_x \to X$ restricts to an isomorphism $V \ast_{\nu_x} V_x \cong V$, so it is birational. By Lemma 2.6 we must show that if $v_1, v_2 \in V$ and $z_1, z_2 \in Z_x$ such that

$$v_1 \cdot z_2 = v_1 \cdot z_2,$$

then $v_1 - v_2 \in V_x$. We check this by applying $\tau$:

$$\tau(v_1) \cdot \tau(z_2) = \tau(v_2) \cdot \tau(z_2) \quad \text{by equivariance of } \tau,$$

$$\tau(v_1) = \tau(v_2) \quad \text{because } z_1, z_2 \in Z_x = \tau^{-1}(0),$$

$$v_1 - v_2 \in V_x \quad \text{because } \tau \text{ extends } V \to V/V_x.$$

Next we show how to construct $\tau$. We wish to construct a $V$-equivariant map

$$\text{Sym}(V/V_x)^{\vee} \to H^0(U_x, \mathcal{O}_X),$$

so it is enough to show that if $f \in V'$ vanishes on $V_x$, then $f$ can extend to $U_x$. Since $U_x$ is normal, it suffices to show that $f$ does not have a pole along any codimension one orbit. Let $L \subseteq V$ be a one dimensional vector subspace of $V$, and let $y$ be the boundary point of $L$ in $X$. Assume that $y \in U_x$. By our assumption that $X$ satisfies OP, it is enough to show that $f$ does not have a pole along $V \cdot y$. Since the action of $L$ fixes the boundary of $L$, $L \subseteq V_y$. We also have $V_y \subseteq V_x$ by definition of $U_x$. Therefore $f$ vanishes on $L$. Now let $L'$ denote the translation of $L$ by a generic vector, and $y'$ the boundary point of $L'$. Since $f$ is linear, $f$ is constant and nonzero on $L'$. Thus $f^{-1}$ is constant and nonzero on $L'$. If $f^{-1}$ is undefined at $y'$, then since $y'$ is generic in $V \cdot y$, $f$ has a zero along $V \cdot y$. If on the other hand $f^{-1}$ is defined at $y'$, then $f^{-1}$ cannot vanish at $y'$ by continuity, so $f$ does not have a pole along $V \cdot y$.

This completes the proof that the statements Item [3] and Item [3] in Theorem [3] are equivalent.

3. THE ORBIT-FAT CORRESPONDENCE

Now that we have proved the equivalence of the statements Item [3] and Item [3] in Theorem [3] we will refer to an equivariant partial compactification of a vector group $V$ which satisfies either of these conditions as a linear $V$-variety. In Lemma 2.20 we showed that if $X$ is a linear $V$-variety and $x \in X$ is a distinguished point, then $v \in V_x^{\circ}$ (Definition 2.19) if and only if $\lim_{t \to \infty} tv = x$. As a consequence, we have:

**Corollary 3.1.** If $X$ is a linear $V$-variety, then there is a canonical bijection between any two of the following sets.

- Orbits of $X$
- Distinguished points of $X$
- Stabilizers of points of $X$

Moreover, each of the above sets is functorial on the category of normal equivariant partial compactifications of vector groups, and the bijections between them are natural.

**Proof.** The correspondence between orbits and distinguished points is automatic, and we have by Lemma 2.20 that any distinguished point $x \in X$ can be recovered from its stabilizer $V_x$ by taking the limit $\lim_{t \to \infty} tv$ for $v \in V_x^{\circ}$. Thus all three sets are in correspondence.
Suppose that $T$ is a morphism of linear vector group varieties. It is automatic that orbits are mapped inside of orbits and stabilizers are mapped inside of stabilizers. Since distinguished points are the set of points that arise as limits of one-parameter subgroups, distinguished points are mapped to distinguished points. Naturality of these correspondences follows formally.

**Definition 3.2.** Let $X$ be a linear $V$-variety. The **partial hyperplane arrangement** $\mathcal{L}(X)$ associated to $X$ is the collection of stabilizers of points in $X$.

To justify the definition of $\mathcal{L}(X)$, we will prove:

**Proposition 3.3.**

(i) If $X$ is a proper linear $V$-variety, then $\mathcal{L}(X)$ is the collection of flats of an essential hyperplane arrangement in $V$.

(ii) If $X$ is a linear $V$-variety then $\mathcal{L}(X)$ is a partial hyperplane arrangement in $V$.

By combining Corollary 3.1 and Proposition 3.3 we have a natural one-to-one correspondence between the orbits of $X$ and the flats of its relative hyperplane arrangement $\mathcal{L}(X)$, as described in Section 1.3.5.

**Lemma 3.4.** If $X$ is a linear $V$-variety, then $\mathcal{L}(X)$ is closed under intersections.

**Proof.** Suppose that $x, y \in X$ are distinguished points, and consider the action of $V_x \cap V_y$ on the closure $\overline{V_x \cap V_y}$ in $X$. Let $Z_x$ be the minimal slice through $x$. Then $V_x \subseteq Z_x$ by Lemma 2.11, so $\overline{V_x \cap V_y}$ is a closed subvariety of $Z_x$. Then by Proposition 2.14, $Z_x$ is proper, so $\overline{V_x \cap V_y}$ is proper. By the Borel fixed point theorem [Hum75, Chapter 21.2], there exists a $(V_x \cap V_y)$-fixed point $z \in \overline{V_x \cap V_y}$. Thus $V_x \cap V_y \subseteq V_z$. To show the opposite inclusion, note that $z \in \overline{V_x \cap V_y} \subseteq Z_x \cap Z_y \subseteq U_x \cap U_y$, where $U_x$ is the minimal invariant neighborhood. Therefore by definition of $U_x$, we have $x, y \in V \cdot z$, and thus $V_x \subseteq \overline{V_x \cap V_y}$.

To prove Proposition 3.3 (i), it now suffices to prove that any stabilizer $V_x \subseteq V$ is the intersection of the codimension one stabilizers containing it. For this we need the following lemmas.

**Lemma 3.5.** Suppose that $G$ is a linear algebraic group acting on a variety $X$, and $Z_x$ is a slice through $x \in X$. Then any regular function on $G \cdot x$ extends to the neighborhood $G *_{G_x} Z_x \subseteq X$.

**Proof.** This follows from the universal property of $G *_{G_x} Z_x$, applied to the map $G \times Z_x \to \mathbb{C}$ given by $(g, z) \mapsto f(g \cdot x)$ where $f$ is a regular function on $G \cdot x$.

**Lemma 3.6.** If $U$ is a connected algebraic variety which has nonconstant global regular functions, and $U \subseteq K$ is a compactification, then the boundary $K \setminus U$ has an irreducible component of codimension one in $K$.

**Proof.** Assume for a contradiction that every component of $K \setminus U$ has codimension at least two. Consider the inclusion of the normalizations $\tilde{U} \subseteq \tilde{K}$. Since $U$ has nonconstant global regular functions, then so must $\tilde{U}$. The normalization map is finite and therefore preserves the codimension of the boundary, so $\tilde{K} \setminus \tilde{U}$ is a closed set of codimension at least two. Thus any regular function on $\tilde{U}$ extends to $\tilde{K}$, so...
we get that the proper variety  \( \tilde{K} \) has nonconstant global regular functions, which is a contradiction.

**Proof of Proposition 3.3 (i)** Suppose that \( x \in X \) is a distinguished point with stabilizer \( V_x \) of codimension at least two in \( V \). We wish to show that \( V_x \) is the intersection of the codimension one stabilizers containing it, so by induction it is enough to find \( y, z \in X \) such that

\[
V_x = V_y \cap V_z, \quad \dim V_y = \dim V_z = \dim V_x + 1.
\]

By Corollary 3.3 \( V \cdot y \neq V \cdot z \) implies \( V \cdot y \neq V \cdot z \). Therefore it suffices to show that the orbit closure \( \overline{V \cdot x} \) contains two distinct orbits of codimension one in \( \overline{V \cdot x} \). Suppose that \( V \cdot y \subseteq \overline{V \cdot x} \) is an orbit of codimension one in \( \overline{V \cdot x} \). Since \( V_x \subseteq V \) is codimension at least two, \( \dim V \cdot y > 0 \), and so we can choose a nonconstant regular function \( f \) on \( V \cdot y \). By Lemma 3.5 \( f \) extends to a regular function on the minimal invariant neighborhood \( U_y \supseteq V \cdot x \cup V \cdot y \), which is nonconstant when restricted to \( V \cdot x \cup V \cdot y \). Therefore by Lemma 3.6 with \( U = V \cdot x \cup V \cdot y \) and \( K = \overline{V \cdot x} \), there is another orbit \( V \cdot z \subseteq \overline{V \cdot x} \) of codimension one. \( \square \)

**Proof of Proposition 3.3 (ii)** We will apply Proposition 3.3 (i) to the slices of \( X \). We have that \( \{0\} \in \mathcal{L}(X) \), and by Lemma 3.4 \( \mathcal{L}(X) \) is closed under intersections. It remains to show that for \( F \in \mathcal{L}(X) \), \( \{G \in \mathcal{L}(X) : G \subseteq F\} \) is the collection of flats of a partial hyperplane arrangement in \( F \). Suppose that \( F \) is the stabilizer of the distinguished point \( x \in X \). Then the slice \( Z_x \) is a proper linear \( F \)-variety by Lemma 2.13 and Proposition 2.15 so the set of stabilizer \( \mathcal{L}(Z_x) \) is the collection of flats of an essential hyperplane arrangement in \( F \) by Proposition 3.3 (i). Then by Lemma 2.3 \( \mathcal{L}(Z_x) = \{G \in \mathcal{L}(X) : G \subseteq F\} \). \( \square \)

4. PROOF OF THEOREM A

Let \( X \) be an equivariant compactification of \( V \), such that \( X \) is normal and satisfies FO and SL. We have shown in Lemma 2.20 and Lemma 2.21 that this is equivalent to assuming \( X \) is normal and satisfies FO and OP. In Corollary A.2 we show that matroid Schubert varieties satisfy FO and SL, and normality of matroid Schubert varieties follows from [Bri03, Theorem 1] together with [AB16, Theorem 1.3(c)]. Thus it only remains to show that \( X \) is equivariantly isomorphic to a matroid Schubert variety associated to a hyperplane arrangement in \( V \).

By Proposition 3.3 there exists an essential hyperplane arrangement \( \mathcal{A} = \{H_1, \ldots, H_n\} \) in \( V \) whose lattice of flats is the collection of stabilizers of \( X \). We write

\[
\Phi_{\mathcal{A}} : V \to V/H_1 \times \cdots \times V/H_n,
\]

for the induced linear embedding, and \( Y_{\mathcal{A}} \) for the Schubert variety of \( \mathcal{A} \). Our goal is to show that there exists an isomorphism

\[
T : X \to Y_{\mathcal{A}(X)}
\]

extending the isomorphism

\[
\Phi_{\mathcal{A}} : V \to \Phi_{\mathcal{A}(V)}.
\]

For each hyperplane \( H_i \), we denote by \( x_i, Z_i \), and \( V \ast_{H_i} Z_i \subseteq X \) the corresponding distinguished point, slice, and minimal \( V \)-invariant open neighborhood, respectively. Explicitly, \( Z_i \) is the closure of \( H_i \) in \( X \), \( x_i \) is the \( H_i \)-fixed point in \( Z_i \), and \( V \ast_{H_i} Z_i \) is embedded in \( X \) as the union of all \( V \)-orbits in \( X \) which intersect \( Z_i \) (see
Section \([2.2]\). Because \(Z_i\) is proper (Proposition \([2.15]\)) and \(X\) is separated, \(Z_i \subseteq X\) is closed. Recall from Section \([2.1]\) that \(Z_i\) is the fiber of the trivial \(V\)-equivariant fibration

\[
\tau : V \ast_{H_i} Z_i \to V/H_i.
\]

Therefore \(Z_i\) is a prime Cartier divisor in \(X\), so there is an associated line bundle \(\mathcal{O}_X(Z_i)\). Fix a linearization of \(\mathcal{O}_X(Z_i)\), which exists by [Dol03, Theorem 7.2]. We then have a linear action

\[
V \cong H^0(X, \mathcal{O}_X(Z_i)).
\]

Because \(\tau\) is \(V\)-equivariant, the translations of \(Z_i\) under the action of \(V\) are the fibers of \(\tau\), so letting \(Z'_i \neq Z_i\) be any such translation, we have that \(Z_i\) and \(Z'_i\) are linearly equivalent and disjoint. Thus \(\mathcal{O}_X(Z_i)\) is globally generated, since the sections (up to scaling) corresponding to \(Z_i\) and \(Z'_i\) have no common zeros. So far, we have that \(Z_i\) defines a \(V\)-equivariant morphism

\[
T_i : X \to \mathbb{P}(H^0(X, \mathcal{O}_X(Z_i)))^\vee.
\]

Finally, we have that the target of \(T_i\) is \(\mathbb{P}^1\) from a general observation:

**Lemma 4.1.** Suppose \(X\) is a proper normal variety, and \(Z\) and \(Z'\) are prime Cartier divisors which are linearly equivalent and such that \(Z \cap Z' = \emptyset\). Then the space of global sections of \(\mathcal{O}_X(Z)\) is two dimensional.

**Proof.** Let \(i : Z \to X\) denote the inclusion, and consider the short exact sequence

\[
0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(Z) \longrightarrow \mathcal{O}_X(Z) \otimes i_* \mathcal{O}_Z \longrightarrow 0.
\]

By the projection formula, the sheaf on the right is isomorphic to \(i_*(\mathcal{O}_X(Z) \otimes \mathcal{O}_Z)\). However the restriction of \(\mathcal{O}_X(Z)\) to \(Z\) is trivial because \(Z\) can be moved to the disjoint divisor \(Z'\). Thus \(\mathcal{O}_X(Z) \otimes i_* \mathcal{O}_Z \cong i_* \mathcal{O}_Z\). Now take the long exact sequence in cohomology.

\[
0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(Z)) \longrightarrow H^0(X, i_* \mathcal{O}_Z) \longrightarrow \ldots
\]

Since \(X\) and \(Z\) are proper and irreducible, \(\dim H^0(X, \mathcal{O}_X) = \dim H^0(X, i_* \mathcal{O}_Z) = 1\). Therefore \(\dim H^0(X, \mathcal{O}_X(Z)) \leq 2\). We also have that the sections (up to scaling) corresponding to \(Z\) and \(Z'\) are independent, so \(\dim H^0(X, \mathcal{O}_X(Z)) = 2\). \(\Box\)

Let us choose coordinates on the target of \(T_i\):

\[
s_0, s_1 \in H^0(X, \mathcal{O}_X(Z_i)), \quad \text{div}(s_0) = Z_i, \quad s_1 \neq 0 \text{ is } V\text{-fixed}.
\]

The section \(s_1\) exists because \(V\) is unipotent. For any isomorphism between \(\mathcal{O}_X(Z_i)|_V\) and \(\mathcal{O}_V\), we have that \(s_0|_V\) is sent to a linear form vanishing on \(H_i\), and \(s_1|_V\) is sent to a constant since \(s_0\) is \(V\)-fixed. Thus there is a commutative square

\[
\begin{array}{ccc}
X & \xrightarrow{T_i} & \mathbb{P}(V/H_i \oplus \mathbb{C}) \\
\downarrow & & \downarrow \\
V & \longrightarrow & V/H_i
\end{array}
\]

where the right vertical arrow is the embedding

\[
V/H_i \to \mathbb{P}^1, \quad v \mapsto [s_0(v) : s_1(v)].
\]
From this it follows that the product map \( X \to \prod_{i=1}^{n} \mathbb{P}(V/H_i \oplus \mathbb{C}) \) extends \( \Phi_{\mathcal{A}} \), and thus we can define a morphism
\[
T : X \to Y_{\mathcal{A}}(X), \quad T := (T_1, \ldots, T_n).
\]

Since \( T \) is birational, by Theorem 2.5 we can show that \( T \) is an isomorphism by showing that it is bijective on closed points. Since \( T \) extends \( \Phi_{\mathcal{A}} \), it is a morphism of linear \( V \)-varieties. The set of stabilizers of \( X \) is the lattice of flats of \( \mathcal{A} \) by assumption, and one can prove in coordinates that the set of stabilizers of \( Y_{\mathcal{A}} \) is the image under \( \Phi_{\mathcal{A}} \) of the lattice of flats of \( \mathcal{A} \) (see Corollary A.2 (iii)). Thus \( T \) carries the set of stabilizers of \( X \) bijectively onto the set of stabilizers of \( Y_{\mathcal{A}} \). Furthermore, \( T \) carries the distinguished points of \( X \) bijectively onto the distinguished points of \( Y_{\mathcal{A}} \) by Corollary 3.1. Let \( x \in X \) be a distinguished point with stabilizer \( V_x \), and let \( T(x) \in Y_{\mathcal{A}} \) be the corresponding distinguished point with stabilizer \( \Phi_{\mathcal{A}}(V_x) \). We have the following commutative square relating the orbit \( V \cdot x \) in \( X \) and the corresponding orbit \( \Phi_{\mathcal{A}}(V) \cdot T(x) \) in \( Y_{\mathcal{A}} \).

\[
\begin{array}{ccc}
V \cdot x & \xrightarrow{T} & \Phi_{\mathcal{A}}(V) \cdot T(x) \\
\downarrow & & \downarrow \\
V/V_x & \xrightarrow{\Phi_{\mathcal{A}}(V)/\Phi_{\mathcal{A}}(V_x)} & 
\end{array}
\]

Since the bottom arrow is an isomorphism, it follows that the top arrow is an isomorphism, and we take the disjoint union over all distinguished points to obtain that \( T \) is a bijection on closed points.

This completes the proof of Theorem A.

5. Proof of Theorem B

5.1. Overview. In this section \( V = \mathbb{C}^d \) will denote a vector group, and we will use the term linear \( V \)-variety to mean a normal equivariant partial compactification \( V \subseteq X \), which has finitely many orbits and a slice through every point. We have proved in Section 2.4 that Item ii and Item iii in Theorem B are equivalent. To prove Theorem B it remains to show the following.

**Theorem 5.1.** Given a linear \( V \)-variety \( X \), let \( \mathcal{L}(X) \) denote the associated partial hyperplane arrangement of Definition 3.2.

(i) **Functoriality:** If \( X_i \) is a linear \( V_i \)-variety for \( i = 1, 2 \) and \( T : X_1 \to X_2 \) is a morphism, then the restricted linear map \( T : V_1 \to V_2 \) is a morphism of partial hyperplane arrangements \( \mathcal{L}(X_1) \to \mathcal{L}(X_2) \).

(ii) **Full faithfulness:** If \( X_i \) is a linear \( V_i \)-variety for \( i = 1, 2 \) and \( T : V_1 \to V_2 \) is a morphism of partial hyperplane arrangements \( \mathcal{L}(X_1) \to \mathcal{L}(X_2) \), then \( T \) extends uniquely to a morphism of linear \( V \)-varieties \( X_1 \to X_2 \).

(iii) **Essential surjectivity:** If \( \mathcal{L} \) is an essential hyperplane arrangement in \( V \), then there exists a linear \( V \)-variety \( X \) such that \( \mathcal{L}(X) \cong \mathcal{L} \).

The proof of essential surjectivity describes how to construct the linear \( V \)-variety associated to a partial hyperplane arrangement.

5.2. Morphisms. In this section we prove functoriality and full faithfulness. To prove functoriality, we generalize the proof of Lemma 3.4, appealing again to the Borel fixed point theorem.
Proof of Theorem 5.1 (i). Since the stabilizer of \( x \in X_1 \) is mapped into the stabilizer of \( T(x) \in X_2 \), it follows that for each flat of \( \mathcal{L}(X_1) \) is mapped into a flat of \( \mathcal{L}(X_2) \), as required in Item 1 of Definition 1.9. If \( F_1 \in \mathcal{L}(X_1) \) and \( F_2 \in \mathcal{L}(X_2) \), it remains to show that \( T^{-1}(F_2) \cap F_1 \in \mathcal{L}(X_1) \). Write \( Z_1 \) for the minimal slices through the distinguished point \( x_1 \in X_1 \) corresponding to \( F_1 \). Then \( Z_1 \) is proper by Proposition 2.13 so \( T^{-1}(F_2) \cap Z_1 \) is proper. By the Borel fixed point theorem, there exists \( z \in T^{-1}(x_2) \cap Z_1 \) such that
\[
T^{-1}(F_2) \cap F_1 \subseteq (V_1)_z.
\]
We now show the opposite inclusion. By Proposition 2.13 there is a unique \( F_1 \)-fixed point in \( Z_1 \), and \( F_1 \cdot z \) contains a \( F_1 \)-fixed point, so \( x_1 \in F_1 \cdot z \). Therefore \( (V_1)_z \subseteq F_1 \). On the other hand, \( z \in T^{-1}(x_2) \), so \( (V_1)_z \subseteq T^{-1}(F_2) \).

Let us start by proving full faithfulness in the compact case.

Lemma 5.2. Suppose that \( \mathcal{A}_i \) is an essential hyperplane arrangement in the vector group \( V_i \) for \( i = 1, 2 \), and \( T : V_1 \rightarrow V_2 \) is a linear map such that the preimage of each hyperplane in \( \mathcal{A}_2 \) is either a hyperplane in \( \mathcal{A}_1 \) or is \( V_1 \). Then \( T \) extends to a morphism between matroid Schubert varieties \( Y_{\mathcal{A}_1} \rightarrow Y_{\mathcal{A}_2} \).

Proof. Since \( Y_{\mathcal{A}_1} \) is the closure of \( V_1 \subseteq \prod_{H \in \mathcal{A}_1} P(V_1/H \oplus \mathbb{C}) \), we are reduced to showing that \( T \) extends to a map
\[
\prod_{H_1 \in \mathcal{A}_1} P(V_1/H_1 \oplus \mathbb{C}) \rightarrow \prod_{H_2 \in \mathcal{A}_2} P(V_2/H_2 \oplus \mathbb{C}).
\]
Fix \( H_2 \in \mathcal{A}_2 \). If \( T^{-1}(H_2) = V_1 \), then the \( H_2 \) component of the displayed function can be defined to be constant with value 0 in \( V_2/H_2 \). Otherwise \( T^{-1}(H_2) \in \mathcal{A}_1 \), in which case the \( H_2 \) component of the displayed function can be defined as projection onto \( P(V_1/T^{-1}(H_2) \oplus \mathbb{C}) \) followed by the map
\[
[v + T^{-1}(H_2) : z] \mapsto [T(v) + H_2 : z].
\]
To prove full faithfulness in general, we apply Lemma 5.2 to the slices in a linear \( V \)-variety, which are matroid Schubert varieties by Proposition 2.13 and Theorem A.

Proof of Theorem 5.1 (ii). Since \( V_1 \subseteq X_1 \) is dense, uniqueness follows immediately. We now argue that an extension \( T : X_1 \rightarrow X_2 \) exists.

Suppose that \( F_i \in \mathcal{L}(X_i) \) for \( i = 1, 2 \) such that \( T(F_1) \subseteq F_2 \), and denote by \( Z_i \subseteq X_i \) the slices through the corresponding distinguished points. By Proposition 2.13, Lemma 2.13, and Theorem A, \( Z_i \) is the matroid Schubert variety associated to a hyperplane arrangement in \( F_i \). By Lemma 2.4, the hyperplane arrangement corresponding to \( Z_i \) is given by those flats of \( \mathcal{L}(X_i) \) which are contained in \( F_i \). Therefore, since \( T \) is a morphism of partial hyperplane arrangements, the hypotheses of Lemma 5.2 are satisfied for the restriction \( T|_{F_i} : F_1 \rightarrow F_2 \). Thus \( T|_{F_i} \) extends to a morphism \( T|_{F_i} : Z_1 \rightarrow Z_2 \).

We can now extend \( T \) to the open set \( V_1 \ast_{F_1} Z_1 \subseteq X_1 \) by sending \([v, z] \) to \([T(v), T|_{F_1}(z)] \) in \( V_2 \ast_{F_2} Z_2 \). As \( F_1 \) and \( F_2 \) vary, the open sets \( V_1 \ast_{F_1} Z_1 \) cover \( X_1 \) (here we are using Item 1 in Definition 1.9), and the extensions of \( T \) to \( V_1 \ast_{F_1} Z_1 \) are unique and thus compatible on intersections, so they define a global extension \( X_1 \rightarrow X_2 \).
5.3. Construction of linear $V$-varieties. Now we turn to essential surjectivity. Let $\mathcal{L}$ be a partial hyperplane arrangement in $V$.

5.3.1. A diagram of hyperplane arrangements. From Definition 1.3 we have that for every $F \in \mathcal{L}$, there is an essential hyperplane arrangement $A_F$ in the vector space $F$ whose lattice of flats is $\{ F' \in \mathcal{L} : F' \subseteq F \}$. It follows immediately that $A_F$ is unique. If $F' \subseteq F$ are elements of $\mathcal{L}$, then $A_{F'}$ is the restriction of $A_F$ to the flat $F'$.

5.3.2. A diagram of matroid Schubert varieties. For each $F \in \mathcal{L}$, we have the matroid Schubert variety $Y_{A_F}$ associated to the hyperplane arrangement $A_F$. If $F' \in \mathcal{L}$ is contained in $F$, then $F'$ is a flat of $A_F$. Therefore $Y_{A_{F'}}$ is the slice through a distinguished point $x' \in Y_{A_F}$ by the coordinate given in Corollary A.2 (vi). So $\mathcal{L}$ indexes a diagram of matroid Schubert varieties, where each arrow is the inclusion of a slice.

5.3.3. A diagram of open embeddings. Given $F' \subseteq F$ elements of $\mathcal{L}$, we have an open embedding $F \ast_F Y_{A_{F'}} \subseteq Y_{A_F}$ because $Y_{A_{F'}}$ is a slice through $x_{F'}$. It is straightforward to check that $V \ast_F$ preserves open embeddings, so by the associativity property of Lemma 2.2 we have an open embedding

$$V \ast_G Y_{A_{F'}} \cong V \ast_F (F \ast_G Y_{A_{F}}) \subseteq V \ast_F Y_{A_{F'}}.$$ 

Therefore $\mathcal{L}$ indexes a diagram of open embeddings between the varieties $V \ast_F Y_{A_{F'}}$ for $F \in \mathcal{L}$. By Lemma 2.3 $V \ast_F Y_{A_{F'}}$ has finitely many $V$-orbits, and again by the associativity property of Lemma 2.2 each point $[v, y] \in V \ast_F Y_{A_{F'}}$ has a slice $V \ast_F Z_y$, where $Z_y$ is a slice through $y \in Y_{A_{F'}}$. Thus $V \ast_F Y_{A_{F'}}$ is a linear $V$-variety.

5.3.4. Cocycle condition. To verify that the $V \ast_F Y_{A_{F'}}$ glue together, we must check the cocycle condition [Har77, Exercise II.2.12]. This reduces to the following lemma.

**Lemma 5.3.** If $F', F'' \subseteq F$ are elements of $\mathcal{L}$, then

$$V \ast_{F'} Y_{A_{F'}} \cap V \ast_{F''} Y_{A_{F''}} = V \ast_{F' \cap F''} Y_{A_{F' \cap F''}}$$

considered as open sets in $V \ast_F Y_{A_{F'}}$.

**Proof.** A point $[v, z] \in V \ast_{F'} Y_{A_{F'}}$ lies in $V \ast_{F'} Y_{A_{F'}}$ if and only if $z \in Y_{A_{F''}}$, so we are reduced to showing that $Y_{A_{F'}} \cap Y_{A_{F''}} = Y_{A_{F' \cap F''}}$, considered as closed sets in $Y_{A_{F'}}$. To check this one can use the set theoretic formula of Proposition A.1. □

5.3.5. Separation. We now prove that the variety $Y_{\mathcal{L}}$ glued from the $V \ast_F Y_{A_{F'}}$ is separated. By [Har77, Corollary II.4.2], checking that the diagonal morphism $Y_{\mathcal{L}} \to Y_{\mathcal{L}} \times Y_{\mathcal{L}}$ is a closed embedding reduces to the following lemma.

**Lemma 5.4.** Suppose that $F = F' \cap F''$ are elements of $\mathcal{L}$, and write

$$i : V \ast_{F'} Y_{A_{F'}} \to V \ast_{F''} Y_{A_{F''}}, \quad [v, y] \mapsto [v, y] \times [v, y].$$

Then $i$ has a closed image.

**Proof.** Consider the canonical fibration defined in Section 2.1

$$\tau_F : V \ast_{F'} Y_{A_{F'}} \to V/F, \quad [v, y] \mapsto v + F.$$ 

We get that the following square is commutative.
\[
\begin{array}{ccc}
V \ast F^\ast Y_{df} & \stackrel{i}{\longrightarrow} & V \ast F^\ast Y_{df'}, Y_{df''} \\
V/F \stackrel{j}{\longrightarrow} V/F' \times V/F''
\end{array}
\]

\[\tau_F \quad \tau_{F'} \times \tau_{F''}\]

Notice that \(j\) is a closed embedding, since \(F = F' \cap F''\). We also have that \(\tau_F\) is proper (it’s conjugate to the projection \(V/F \times Y_{df} \to V/F\), so \(f\) is proper. Then \(i\) is proper by \([\text{Har77}, \text{Corollary II.4.8(e)}]\), and thus closed.

5.3.6. **Linearity.** Since the action of \(V\) extends to each open set in a cover of \(Y_{\mathcal{L}}\) and is compatible on intersections, the action extends to \(Y_{\mathcal{L}}\). From the fact that \(Y_{\mathcal{L}}\) is normal, that it has finitely many \(V\)-orbits, and that every point has a slice. Thus \(Y_{\mathcal{L}}\) is a linear \(V\)-variety.

**Proof of Theorem 5.1 (iii).** We have constructed a linear \(V\)-variety \(Y_{\mathcal{L}}\), and we wish to show that the collection of stabilizers \(\mathcal{L}(Y_{\mathcal{L}})\) coincides with \(\mathcal{L}\).

First we check that each flat of \(\mathcal{L}\) is the stabilizer of a point in \(Y_{\mathcal{L}}\). Suppose that \(F \in \mathcal{L}\). Then consider the point \([0, x_F] \in V \ast F^\ast Y_{df} \subseteq Y_{\mathcal{L}}\), where \(x_F\) is the unique fixed point of \(Y_{df}\). By Lemma 2.3 \(F\) is the stabilizer of \([0, x_F] \in Y_{\mathcal{L}}\).

Now we check that the stabilizer of each point of \(Y_{\mathcal{L}}\) is a flat of \(\mathcal{L}\). From the construction of \(Y_{\mathcal{L}}\), we have that every point is contained in an open set isomorphic to \(V \ast F^\ast Y_{df}\) for \(F \in \mathcal{L}\). Suppose that \([v, y] \in V \ast F^\ast Y_{df}\). Then the stabilizer of \([v, y]\) with respect to the action of \(V\) is equal the stabilizer of \(y \in Y_{df}\) with respect to the action of \(F\) by Lemma 2.3 and is therefore equal to a flat in \(\mathcal{L}\) by Corollary A.2 (iii) \(\Box\)

This completes the proof of Theorem B.

5.3.7. **Comparison with toric varieties.** We conclude this section by explaining how the construction of a toric variety from a polyhedral fan can be made to look like the construction above. In order to be consistent with \([\text{CLS11}]\), we will use \(n\) for the dimension of a toric variety rather than \(d\) as we have been doing so far. We will use \(d\) for the dimension of a cone. Let us fix the following notation.

\[\begin{align*}
\text{• } N & \cong \mathbb{Z}^n, \text{ a lattice of dimension } n, \\
\text{• } T & = \text{Spec } \mathbb{C}[N^\vee], \text{ the corresponding } n\text{-dimensional torus,} \\
\text{• } \Sigma & , \text{ a fan of strictly convex rational polyhedral cones in } N \otimes_{\mathbb{Z}} \mathbb{R}, \\
\text{• } \sigma & \in \Sigma, \text{ a cone of dimension } d, \\
\text{• } U_{\sigma,N} & , \text{ the toric variety of dimension } n \text{ corresponding to } \sigma \text{ as a cone in } N \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{([CLS11], Theorem 1.2.18),} \\
\text{• } x_\sigma & \in U_{\sigma,N}, \text{ the distinguished point in the minimal } T\text{-orbit} \quad \text{([CLS11], Chapter 3.2),} \\
\text{• } N_\sigma & = \text{span}_\mathbb{Z}(\sigma \cap N), \text{ the sublattice of dimension } d \text{ generated by } \sigma, \\
\text{• } T_\sigma & = \text{Spec } \mathbb{C}[N_\sigma^\vee], \text{ the } d\text{-dimensional torus corresponding to } N_\sigma, \\
\text{• } U_{\sigma,N_\sigma} & , \text{ the toric variety of dimension } d \text{ corresponding to } \sigma \text{ considered as a cone in } N_\sigma \otimes_{\mathbb{Z}} \mathbb{R}, \\
\text{• } \mathcal{H}_\sigma & , \text{ the unique minimal basis (see ([CLS11], Proposition 1.2.23)) for the semigroup} \\
& S_{\sigma,N_\sigma} = \{ u \in \text{Hom}_\mathbb{Z}(N_\sigma, \mathbb{Z}) : u \text{ is nonnegative on } \sigma \}. \\
\text{The fan } \Sigma \text{ indexes a commutative diagram of inclusions among its cones, as in Section 5.3.1.}\end{align*}\]

The cone \(\sigma\) defines an embedding of the torus \(T_\sigma\) in the larger torus \((\mathbb{C}^\times)^{\mathcal{H}_\sigma}\) (see \([\text{CLS11}], \text{Definition 1.1.7}\)), and the closure of \(T_\sigma \subseteq (\mathbb{A}^1)^{\mathcal{H}_\sigma}\) is \(U_{\sigma,N_\sigma}\).
Matroid flats:
Let Corollary A.2. Proposition A.1 set of indices corresponding to non-infinite entries of $x$ of the linear subspace $V$ corresponding to non-infinite entries of $x$, if and only if $F \in \mathcal{F}$ and $\pi_F(x) \in \pi_F(V)$. Equivalently, $Y_{\mathcal{F}} = \bigcup_{F \in \mathcal{F}} \pi_F(V) \times \{\infty\}^{E \setminus F} \subseteq (\mathbb{P}^1)^E$.

Let us now demonstrate in explicit coordinates the objects defined in Section 2.2.

Corollary A.2. Let $x, y \in Y_{\mathcal{F}}$, and write $F, F' \subseteq E$ for the set of indices corresponding to non-infinite entries of $x$ and $y$ respectively.

(i) The $V$-orbit of $x$ is $V \cdot x = \pi_F(V) \times \{\infty\}^{E \setminus F}$.
(ii) The distinguished point in the V-orbit of x is \( x_F = \{0\}^F \times \{\infty\}^{E^\vee F} \).

(iii) The stabilizer of x is \( V_x = V \cap (\{0\}^F \times \mathbb{C}^{E^\vee F}) \).

(iv) The minimal V-invariant open neighborhood \( U_x \) of x contains y if and only if \( F \subseteq F' \). Equivalently,

\[
U_x = \bigcup_{F' \in \mathcal{F}, F \subseteq F'} \pi_{F'}(V) \times \{\infty\}^{E^\vee F'}.
\]

(v) The minimal slice \( Z_x \) through x contains y if and only if \( F \subseteq F' \) and \( \pi_{F'}(x) = \pi_{F}(y) \). Equivalently

\[
Z_x = \bigcup_{F' \in \mathcal{F}, F \subseteq F'} \left( \pi_{F'}(V) \cap \left( \pi_{F}(x) \times \mathbb{C}^{E^\vee F} \right) \right) \times \{\infty\}^{E^\vee F'}.
\]

(vi) Set \( Y_{af} \) equal to the closure of \( V \cap \mathbb{C}^{E^\vee F} \) in \( (\mathbb{P}^1)^{E^\vee F} \). The minimal slice through the distinguished point \( x_F = \{0\}^F \times \{\infty\}^{E^\vee F'} \) of \( Y_{af} \) is \( Z_{x_F} = \{0\}^F \times Y_{af} \).

**Proof.** We begin with Corollary A.2 (i). Since \( x_i = \infty \) for \( i \notin F \), we have that \( v \in V \) acts on x via \((v \cdot x)_i = v_i + x_i\) for \( i \in F \) and \((v \cdot x)_i = \infty \) for \( i \notin F \). Thus \( V \cdot x \subseteq \pi_F(V) \times \{\infty\}^{E^\vee F} \). For the reverse inclusion, let \( y \in \pi_F(V) \times \{\infty\}^{E^\vee F} \) and choose \( v \in V \) such that \( \pi_F(v) = \pi_F(y) - \pi_F(x) \). Then \( v \cdot x = y \).

We have that \( v \cdot x = x \) if and only \( v_i + x_i = x_i \), for all \( i \in F \), so Corollary A.2 (iii) follows.

To prove Corollary A.2 (iv), let \( y \in Y_{af} \) and write \( F' = \{i \in E : y_i \neq \infty\} \). We wish to show that the set of \( y \) for which \( F \subseteq F' \) is equal to the minimal open neighborhood \( U_x \). We first note that the set of \( y \) such that \( F \subseteq F' \) is a V-stable open set of \( V \cdot x \), so it contains the minimal one. To show the reverse inclusion we must check that if \( F \subseteq F' \), then \( \pi_F(V) \times \{\infty\}^{E^\vee F} \subseteq \pi_{F'}(V) \times \{\infty\}^{E^\vee F'} \). Since \( F \in \mathcal{F} \) is a flat, we may choose a vector \( v \in V \) such that \( v_i = 0 \) for \( i \in F \) and \( v_i \neq 0 \) for \( i \notin F \). Then for each value of \( t \in \mathbb{C} \), \( \pi_{F'}(tv) \times \{\infty\}^{E^\vee F'} \in \pi_{F'}(V) \times \{\infty\}^{E^\vee F'} \), but as \( t \to \infty \), the limit lies in \( \pi_F(V) \times \{\infty\}^{E^\vee F} \).

We now turn to Corollary A.2 (v). We have that

\[
Z_x := \{y \in Y_{af} : F \subseteq F', \pi_F(x) = \pi_F(y)\}
\]

is contained in \( U_x \) by Corollary A.2 (iv) and so we just need to check that it is a slice. We have that \( V_x \) acts on \( Z_x \), so we use Lemma 2.4 to show that the induced map \( V_x \times Z_x \to Y_{af} \) is injective on closed points. Suppose \( v \cdot z = v' \cdot z' \) for \( v, v' \in V \) and \( z, z' \in Z_x \). Then \( \pi_F(v) + \pi_F(z) = \pi_F(v') + \pi_F(z') \), however \( \pi_F(z) = \pi_F(z') = \pi_F(x) \), so \( \pi_F(v - v') = 0 \) as required. Since \( V \cap Z_x \) is a coset of \( V_z \), we have that \( V \times Z_x \to Y_{af} \) is birational, and thus an open embedding by Theorem 2.7.

Now Corollary A.2 (ii) and Corollary A.2 (vi) follow from knowing the slice through \( x_F \). \( \Box \)

**References**

[AB16] Federico Ardila and Adam Boocher. “The closure of a linear space in a product of lines”. In: *J. Algebraic Combin.* 43.1 (2016), pp. 199–235. DOI: 10.1007/s10801-015-0634-8 (10 10 14 20).

[AZ20] Ivan Arzhantsev and Yulia Zaitseva. “Equivariant completions of affine spaces”. In: *arXiv e-prints*, arXiv:2008.09828 (Aug. 2020) (19 3).
REFERENCES

[Bra+20] Tom Braden, June Huh, Jacob P. Matherne, Nicholas Proudfoot, and Botong Wang. “Singular Hodge theory for combinatorial geometries”. In: arXiv e-prints, arXiv:2010.06088 (Oct. 2020), arXiv:2010.06088. arXiv:2010.06088 [math.CO] (↑1 [2]).

[Bri03] Michel Brion. “Multiplicity-free subvarieties of flag varieties”. In: Com-mutative algebra (Grenoble/Lyon, 2001). Vol. 331. Contemp. Math. Amer. Math. Soc., Providence, RI, 2003, pp. 13–23. DOI:10.1090/conm/331/05900 (↑14).

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. Toric varieties. Vol. 124. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2011, pp. xxiv+841. DOI:10.1090/gsm/124 (↑19, 20).

[Dol03] Igor Dolgachev. Lectures on invariant theory. Vol. 296. London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2003, pp. xvi+220. DOI:10.1017/CBO9780511615436 (↑11, 15).

[EPW16] Ben Elias, Nicholas Proudfoot, and Max Wakefield. “The Kazhdan-Lusztig polynomial of a matroid”. In: Adv. Math. 299 (2016), pp. 36–70. DOI:10.1016/j.aim.2016.05.005 (↑1).

[Gle50] A. M. Gleason. “Spaces with a compact Lie group of transformations”. In: Proc. Amer. Math. Soc. 1 (1950), pp. 35–43. DOI:10.2307/2032430 (↑2).

[Har77] Robin Hartshorne. Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496 (↑18, 19).

[HT99] Brendan Hassett and Yuri Tschinkel. “Geometry of equivariant compactifications of $G^a_n$”. In: Internat. Math. Res. Notices 22 (1999), pp. 1211–1230. DOI:10.1155/S1073792899000665 (↑3).

[Hum75] James E. Humphreys. Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975, pp. xiv+247 (↑13).

[HW17] June Huh and Botong Wang. “Enumeration of points, lines, planes, etc”. In: Acta Math. 218.2 (2017), pp. 297–317. DOI:10.4310/ACTA.2017.v218.n2.a2 (↑1).

[KV19] Mario Kummer and Cynthia Vinzant. “The Chow form of a reciprocal linear space”. In: Michigan Math. J. 68.4 (2019), pp. 831–858. DOI:10.1307/mmj/1571731287 (↑1).

[Mos57] G. D. Mostow. “Equivariant embeddings in Euclidean space”. In: Ann. of Math. (2) 65 (1957), pp. 432–446. DOI:10.2307/1970058 (↑2).

[MY57] D. Montgomery and C. T. Yang. “The existence of a slice”. In: Ann. of Math. (2) 65 (1957), pp. 108–116. DOI:10.2307/1969667 (↑3).

[Pal61] Richard S. Palais. “On the existence of slices for actions of non-compact Lie groups”. In: Ann. of Math. (2) 73 (1961), pp. 295–323. DOI:10.2307/1970335 (↑2).

[PS06] Nicholas Proudfoot and David Speyer. “A broken circuit ring”. In: Beiträge Algebra Geom. 47.1 (2006), pp. 161–166 (↑1).

[PXY18] Nicholas Proudfoot, Yuan Xu, and Ben Young. “The Z-polynomial of a matroid”. In: Electron. J. Combin. 25.1 (2018), Paper No. 1.26, 21 (↑20).
[Sun74] Hideyasu Sumihiro. “Equivariant completion”. In: J. Math. Kyoto Univ. 14 (1974), pp. 1–28. DOI: 10.1215/kjm/1250523277 (↑6).

[Ter02] Hiroaki Terao. “Algebras generated by reciprocals of linear forms”. In: J. Algebra 250.2 (2002), pp. 549–558. DOI: 10.1006/jabr.2001.9121 (↑1).

[Vak] Ravi Vakil. The Rising Sea: Foundations of Algebraic Geometry. November 18, 2017 version. URL: http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf (↑8).

[VP94] E. B. Vinberg and V. L. Popov. Invariant theory. Ed. and trans. by I. R. Shafarevich. Vol. 55. Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin, 1994, pp. vi+284. DOI: 10.1007/978-3-662-03073-8 (↑8 9 10).

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