Composite control with observer for switched stochastic systems subject to multiple disturbances and input saturation

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Abstract
This paper investigates the composite hierarchical anti-disturbance control problem for switched systems with saturating inputs, in which the multiple disturbances including normal-bounded disturbances, white noises and nonlinear uncertainties are considered. According to whether the states of the system be available or not, two types of disturbance observers and a state observer are designed, respectively. With the values of the observers, the observer-based composite controllers are constructed accordingly, under which an uniform form of the closed-loop system is presented. By utilising the minimum dwell-time technique and stochastic control theory, a new criterion of the mean-square exponential stability is presented for the closed-loop system with a weighted $H_{\infty}$ performance level. The design conditions of the observers for the disturbances generated by exosystem and states of the system are proposed by propositions, respectively. Based on the gains of the observers, the solvable synthesis conditions of the composite controllers are presented for the cases of the available and unavailable states of the system. The simulations of GE-90 aero-engine model and a numerical example are employed to verify the applicability of the obtained results.

1 INTRODUCTION

Disturbances exist widely in all actual industrial processes and can degrade controlled performance or destroy even the stability of the systems. With the increasing demands of high control precision, it attracts considerable attention to anti-disturbance control techniques [1]. There exist many advanced control schemes to achieve the disturbance attenuation and rejection, such as robust $H_{\infty}$ control [2, 3], sliding mode control [4], disturbance compensation control [6, 7], and so on. The above-mentioned control strategies can be applied effectively to deal with one type of disturbances or a single equivalent disturbance. However, in practical engineering, the control system is usually subject to different types of multiple disturbances, such as bounded noises, stochastic noises, deterministic disturbances etc. Therefore, it could not achieve the required control accuracy for the system with multiple disturbances by each of above control methods separately. For fulfilling the need of high-precision control, a composite hierarchical anti-disturbance control (CHADC) scheme has been put forward, which mainly contains two parts, that is, classifying the disturbances based on the disturbance characteristics and constructing the targeted composite anti-disturbance controller [8]. When the exosystem generated disturbance and $H_{2}$-norm bounded disturbance are considered simultaneously, a composite control strategy has been proposed for various control systems by integrating the disturbance-observer-based control (DOBC) strategy and $H_{\infty}$ control strategy [9–13]. In [14, 15], the above control scheme has been improved by introducing the technique of the fuzzy control or adaptive control for the system with the nonlinear uncertainty as well as the above two types of disturbances. Recently, the CHADC has been developed in the stochastic field when both the deterministic and stochastic disturbances are considered [16–18].
2 | PROBLEM STATEMENT AND PRELIMINARIES

2.1 | System description

Consider the following switched stochastic system:

\[
\begin{align*}
\dot{x}(t) &= A_{\phi(t)}x(t) + B_{\phi(t)}\text{sat}[u(t)] + F_{\phi(t)}f(x(t), t) \\
&\quad + E_{1,\phi(t)}d_1(t) + D_{\phi(t)}x(t)\delta(t), \\
y(t) &= Cx(t),
\end{align*}
\]  
(1)

where \(x(t) \in \mathbb{R}^n\) are system states; \(u(t) \in \mathbb{R}^m\) are control inputs; \(y(t) \in \mathbb{R}^q\) are system outputs; \(f(x(t), t) \in \mathbb{R}^{q}\) is nonlinear vector function which describes the system uncertainty and satisfies:

Assumption 1.

1. \(f(0, t) = 0\),
2. for a prescribed constant matrix \(U\), \(\|f(x_1(t), t) - f(x_2(t), t)\| \leq \|U(x_1(t) - x_2(t))\|\) holds.

\(d_i(t) \in \mathbb{R}^p\) are the disturbance inputs in \(L_2[0, \infty]\); the multiplicative noise \(\delta(t)\) is a white noise; the disturbances \(d_i(t)\) are matched with control inputs and generated by the exogenous system described as:

\[
\begin{align*}
\dot{u}(t) &= W_{\phi(t)}u(t) + E_{2,\phi(t)}d_2(t), \\
d_0(t) &= V_{\phi(t)}u(t),
\end{align*}
\]
(2)

where \(u(t) \in \mathbb{R}^r\) are disturbance system states; \(d_2(t)\) is an additional noise in \(L_2[0, \infty]\). The piecewise constant function \(\phi(t)\) is called as switching rule taking the value in finite set \(S = \{1, 2, \ldots, N\}\). For any switching time instant \(t_k, \phi(t) = \phi(t_k) = i_k \in S\) when \(t \in [t_k, t_{k+1})\). The notation \([T^*]\) stands for a set of switching signals in which the successive switching times satisfy \(t_{k+1} - t_k \geq T^*,\) where \(T^*\) is usually called as minimum dwell time. The saturation function \(\text{sat} : \mathbb{R}^n \rightarrow \mathbb{R}^n\) is defined as

\[
\text{sat}(u) = [\text{sat}(u_1) \; \text{sat}(u_2) \; \ldots \; \text{sat}(u_m)]^T,
\]

where \(\text{sat}(u_i) = \text{sgn}(u_i)\min(1, |u_i|)\) with a saturation level \(\rho\).

Here, we define a complete probability space as \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\) with a filtration \(\mathcal{F}_t\) satisfying the usual conditions. From [34], the white noise \(\delta(t)\) can be described as \(\frac{d\omega(t)}{dt}\), where \(\omega(t)\) is standard Wiener process in \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\). Therefore, for \(i \in S\), system (1) and exogenous system (2) can be rewritten as

\[
\begin{align*}
\dot{x}(t) &= [A_i x(t) + B_i \text{sat}[u(t)] + V_i u(t)] + F_i f(x(t), t) \\
&\quad + E_{1,i}d_1(t) + D_i x(t)\delta(t), \\
\dot{u}(t) &= [W_i u(t) + E_{2,i}d_2(t)]dt.
\end{align*}
\]
(3)
For system (3), we make an assumption as:

**Assumption 2.** For \( \forall i \in S, (A_i, B_i) \) is controllable, \((W_i, B_i V_i)\) is observable.

In what follows, the composite controllers will be constructed for the cases of available and unavailable states of system (1), respectively.

### 2.2 The case of available states of the system

Firstly, all of the states of system (1) are assumed to be available. To estimate the disturbance \( \hat{d}_i(t) \), a disturbance observer is designed as follows:

\[
\begin{aligned}
\hat{d}_i(t) &= V_j \hat{d}(t), \\
\hat{d}(t) &= \xi(t) - L \chi(t), \\
d\xi(t) &= [(W_i + L B_i V_i) \hat{d}(t) + L(A_i \chi(t) + B_i u(t) + F_i f(\chi(t), t))]dt.
\end{aligned}
\]

(4)

Based on the disturbance estimation \( \hat{d}_i(t) \), a composite controller is constructed for system (1) as

\[ u(t) = K_i x(t) - \hat{d}_i(t), \]

(5)

where \( K_i, i \in S \) are gains of the controller.

### 2.3 The case of unavailable states of the system

Next, under the assumption of the unavailable states of system (1), a state observer should be constructed as

\[
\begin{aligned}
d\hat{x}(t) &= [A_i \hat{x}(t) + B_i u(t) + \hat{d}_i(t)] + F_i f(\hat{x}(t), t) \\
&- L_{d2} y(t) + \hat{y}(t)]dt, \\
\hat{y}(t) &= C \hat{x}(t),
\end{aligned}
\]

(6)

where \( \hat{x}(t) \in \mathbb{R}^n \) is the estimate of \( x(t) \), \( \hat{y}(t) \in \mathbb{R}^p \) is the observer output, \( L_{d2} \in \mathbb{R}^{mxq} \) are the gains of the observer and \( \hat{d}_i(t) \) can be observed by the following observer

\[
\begin{aligned}
\hat{d}_i(t) &= V_j \hat{d}(t), \\
\hat{d}(t) &= \xi(t) - L_{d2} y(t), \\
d\xi(t) &= [(W_i + L A_i C \hat{x}(t) + B_i u(t) + F_i f(\hat{x}(t), t))]dt.
\end{aligned}
\]

(7)

Based on the observers (6) and (7), a composite controller is designed for system (1) as

\[ u(t) = K_i \hat{x}(t) - \hat{d}_i(t), \]

(8)

where \( K_i, i \in S \) are gains of the controller.

**Remark 1.** In this paper, the disturbance observer (4) and composite controller (5) have been designed based on the information of states of the system (1). On the other hand, for the case of unavailable states of the system, we first proposed the composite control scheme (8) by designing a state observer (6) and disturbance observer (7) based on the outputs of the system. Therefore, a systematic anti-disturbance control problem would be studied for the switched system with multiple disturbances under the available or unavailable states of the system.

**Remark 2.** Comparing the existing results of anti-disturbance control of the switched system [30, 31], more types of disturbances and input saturation are considered in this paper. For the cases of available and unavailable states of the system, the corresponding composite controllers has been designed based on the observer technique.

### 2.4 Closed-loop system description

Denoting a function as \( \varphi(\bullet) = \bullet - \text{sat}(\bullet) \), the augmented closed-loop system can be presented as

\[
\begin{aligned}
d\chi(t) &= [\bar{A}_i \chi(t) + F_i f(\chi(t), t) + B_i \varphi(\bar{K}_i, \chi(t))] \\
&+ E_i d(t)] dt + D_i \chi(t) d\omega(t),
\end{aligned}
\]

(9)

where \( \bar{K}_i \) is the control output, and the matrices in the closed-loop system (9) are described, respectively, under controller (5) as

\[
\begin{array}{c}
\chi^T(t) = [x^T(t) \ e^T(t)], \\
e_0(t) = u(t) - \hat{d}(t), \\
f(\chi(t), t) = f(\chi(t), t), \\
\bar{C}_i = \text{diag}(C_{1i}, C_{2i}), \\
\bar{A}_i = \begin{bmatrix} A_i + B_i K_i & B_i V_i \\ 0 & W_i + L B_i V_i \end{bmatrix}, \\
B_i = \begin{bmatrix} -B_i \\ -L B_i \end{bmatrix}, \\
\bar{B}_i = \begin{bmatrix} E_{1i} \\ L E_{1i} \end{bmatrix}, \\
\bar{E}_i = \begin{bmatrix} F_i \\ 0 \end{bmatrix}, \\
D_i = \begin{bmatrix} D_i & 0 \\ LD_i & 0 \end{bmatrix}, \\
d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}.
\end{array}
\]

and under controller (8) as

\[
\begin{array}{c}
\chi^T(t) = [x^T(t) \ e^T(t) \ \epsilon^T(t)], \\
e_0(t) = u(t) - \hat{d}(t), \\
\bar{C}_i = \text{diag}(C_{1i}, C_{2i}, C_{3i}), \\
\bar{A}_i = \begin{bmatrix} A_i + B_i K_i & B_i K_i & B_i V_i \\ 0 & A_i + L A_i C_i & B_i V_i \\ 0 & L_2 C A_i & W_i + L_2 C B_i V_i \end{bmatrix}, \\
\bar{B}_i = \begin{bmatrix} -B_i \\ -L B_i \end{bmatrix}, \\
\bar{B}_i = \begin{bmatrix} E_{1i} \\ L E_{1i} \end{bmatrix}, \\
\bar{E}_i = \begin{bmatrix} F_i \\ 0 \end{bmatrix}, \\
D_i = \begin{bmatrix} D_i & 0 \\ LD_i & 0 \end{bmatrix}, \\
d(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}.
\end{array}
\]
\begin{align*}
\dot{y}_i &= \begin{bmatrix}
-B_i \\
0 \\
-L_2CB_i
\end{bmatrix}, \quad \dot{E}_i = \begin{bmatrix}
E_{i1} & 0 \\
E_{i1} & 0 \\
L_2E_{i1} & E_{2i}
\end{bmatrix}, \\
\dot{P}_i &= \begin{bmatrix}
F_i & 0 \\
0 & F_i
\end{bmatrix}, \quad \dot{D}_i = \begin{bmatrix}
D_i & 0 \\
D_i & 0 \\
0 & 0
\end{bmatrix},
\end{align*}

where \( z(t) \) is control output.

**Lemma 1** [35]. For the above defined function \( \varphi(v) = v - \text{sat}(v) \), there exists a real number \( \varepsilon \in (0, 1) \) such that

\[ \varphi'(v) \varphi \leq \varepsilon v^T v. \]  

**Lemma 2** [36]. Given any constant \( \lambda > 0 \) and any matrices \( M, \Gamma, U \) of compatible dimensions, then

\[ 2x^T M \Gamma U x \leq \lambda x^T M M^T x + \lambda^{-1} x^T U^T U x, \]  

for all \( x \in \mathbb{R}^n \), where \( \Gamma \) is an uncertain matrix satisfying \( \Gamma^T \Gamma \leq I \).

### 3 SYSTEM PERFORMANCE ANALYSIS

In this part, we will develop a new criterion of the mean-square exponential stability with the weighted \( H_\infty \) performance for the closed-loop system in (9) under the minimum dwell-time switching rule.

**Theorem 1.** Given scalars \( \kappa_i > 0, \gamma > 0 \), if there exist scalars \( \delta > 1 \), \( \alpha_i > 0 \), matrices \( \Psi_i > 0 \), such that for \( i \in S \),

\[ \begin{bmatrix}
\Theta_i + \tilde{C}_i^T \tilde{C}_i \oplus \kappa_i \tilde{P}_i \tilde{E}_i \oplus \gamma^2 I
\end{bmatrix} < 0, \]

where

\[ \Theta_i = \text{sym}\{P_i^T A_i\} + D_i^T P_i D_i + \alpha_i P_i F_i T_i P_i + \alpha_i^{-1} U_i^T U_i + P_i B_i \tilde{B}_i^T P_i + \varepsilon K_i^T K_i, \]

then system (9) is mean-square exponentially stable with a weighted \( H_\infty \) performance level \( \gamma \) for any \( \varphi \in S(T^*) \) with \( T^* \geq \max_{i \in S} \{ t_{\text{min}} \} \), where \( \nu \geq 1 \) satisfies

\[ P_i \leq \nu P_j, \quad \forall i, j \in S. \]

**Proof.** Please see the Appendix A. \( \square \)

**Remark 4.** In [38], the research has been made on anti-disturbance control of the nonlinear systems with exosystem generated disturbance and input saturation. A locally sector bounded nonlinearity and a locally convex polynomial approximation are chosen to treat the saturation function, which require the states of the system stay in a bounded set. However, it might not fulfill this requirements for the switched stochastic system due to effect of the system switching and stochastic factors. In this paper, by using a global condition in Lemma 1 and matrix inequality technique, the input saturation is effectively handled.

**Remark 5.** Here, a minimum dwell-time approach is chosen to design the switching rule, which might be conservative than the average dwell-time approach from a theoretical point of view. That is because the switching rule satisfying the minimum dwell-time requirement might be easy to be applied in practice, when the asymptotic stability or stabilisation is considered. On the other hand, the result in Theorem 1 can be generalised to the case under the average dwell-time based on the result in [39].

### 4 COMPOSITE CONTROLLER DESIGN

Now, by using convex linearisation method, we will give some solvable design conditions of the controller (5) and (8) based on the result in Theorem 1.
4.1 The design method of controller (5)

**Proposition 1.** Given scalars $\kappa_1 > 0$, the disturbance error system $\dot{e}_0(t) = (W_\varphi(t) + L B_\varphi(t)) \dot{v}_0(t)$ is exponentially stable under arbitrary switching signal, if there exist matrices $G$ and $P > 0$ such that

$$\text{sym}(PW_i + GB_i V_i) + \kappa_i P < 0, \quad \forall i \in S. \quad (16)$$

Moreover, the gain of disturbance observer (4) can be chosen by $L = P^{-1}G_i$.

**Proof.** Setting $G = LP$, (16) is equivalent to

$$\text{sym}(PW_i + LB_i V_i) + \kappa_i P < 0, \quad \forall i \in S. \quad (17)$$

A common Lyapunov function is chosen as

$$V(e_0(t), \varphi(t)) = e_0(t)^T P e_0(t), \quad (18)$$

where $P > 0$.

The time derivative along the state trajectory of the disturbance error system is computed as

$$\dot{V}(e_0(t), \varphi(t)) = e_0(t)^T \dot{P} e_0(t). \quad (19)$$

For any $i \in S$, combing (17) with (19) yields that

$$\dot{V}(e_0(t), i) = -\kappa_i V(e_0(t), i). \quad (20)$$

From [37], it is obtained that (20) guarantees the disturbance error system to be exponentially stable under arbitrary switching signal. \hfill \square

**Theorem 2.** Given scalars $\kappa_i > 0$, $\gamma > 0$ and matrix $L$ satisfying Proposition 1, if there exist scalars $\delta > 1$, $\alpha_i > 0$, matrices $X_{i_1} > 0$, $X_{i_2} > 0$, $M_i$ such that for $i \in S$,

$$\begin{bmatrix} \Omega_{i_1} & \Omega_{i_2} \\ \ast & \Omega_{i_2} \end{bmatrix} < 0, \quad (21)$$

where

$$\Omega_{i_1} = \begin{bmatrix} \pi_{i_1} & B_i V_i X_{i_1} + B_i B_i^T L_i^T \\ & \pi_{i_2} \end{bmatrix},$$

$$\Omega_{i_2} = \begin{bmatrix} F_i \pi_{i_2} & X_i D_i^T U^T X_i C_i^T \\ & \pi_{i_2} \end{bmatrix},$$

$$\Omega_{i_2} = \text{diag}(-\gamma^2 I, -\delta I, -X_i, -\alpha_i I, -I),$$

then the argument system in (9) is mean-square exponentially stable with a weighted $H_{\infty}$ performance level $\gamma$ for any $\varphi \in \mathcal{S}[T^*]$ with $T^* \geq \max_{i \in S} \frac{1}{\kappa_i}$, where $\nu \geq 1$ satisfies

$$X_i \leq \nu X_i, \quad \forall i, j \in S. \quad (22)$$

Moreover, the gains of the composite controller can be chosen as

$$K_i = M_i X_i^{-1}, \quad i \in S. \quad (23)$$

**Proof.** Setting $P_i = X_i^{-1}$, applying Schur complement lemma and pre- and post-multiplying the inequality (21) by $\text{diag}(P_i, P_i)$, one has that (21) is equivalent to (14). Noticing $X_i = \text{diag}(X_{i_1}, X_{i_2})$, it is easy to get that (22) can guarantee (15). Next, the proof can follow Theorem 1. \hfill \square

4.2 The design method of controller (8)

**Proposition 2.** Given scalars $\kappa_i > 0$, the disturbance error system $\dot{e}_0(t) = (A_\varphi(t) + L_{A_\varphi(t)}) \dot{v}_0(t)$ is exponentially stable under arbitrary switching signal, if there exist matrices $G_{i_1}$ and $P > 0$ such that

$$\text{sym}(PW_i + G_i C_i) + \kappa_i P < 0, \quad \forall i \in S. \quad (24)$$

Moreover, the gains of state observer (7) can be chosen by $L_{A_1} = P^{-1}G_{i_1}$.

**Proposition 3.** Given scalars $\kappa_i > 0$, the disturbance error system $\dot{e}_0(t) = (W_\varphi(t) + L_2 C_\varphi(t)) \dot{v}_0(t)$ is exponentially stable under arbitrary switching signal, if there exist matrices $G$ and $P > 0$ such that

$$\text{sym}(PW_i + G_i C_i) + \kappa_i P < 0, \quad \forall i \in S. \quad (25)$$

Moreover, the gain of disturbance observer (6) can be chosen by $L_2 = P^{-1}G_{i_2}$.

**Theorem 3.** Given scalars $\kappa_i > 0$, $\gamma > 0$, matrices $A_{i_1}$ and $L_2$ satisfying Proposition 2 and Proposition 3, respectively, if there exist scalars $\delta > 1$, $\alpha_i > 0$, matrices $X_{i_1} > 0$, $X_2 > 0$, $M_i$ such that for $i \in S$,

$$\begin{bmatrix} \tilde{\Omega}_{i_1} & \tilde{\Omega}_{i_2} & \tilde{\Omega}_{i_3} \\ \ast & \ast & \ast \end{bmatrix} \begin{bmatrix} \tilde{E}_i & \pi_{i_2} & \tilde{X}_i D_i^T U^T X_i C_i^T \\ & \ast & \ast \end{bmatrix} \begin{bmatrix} \tilde{E}_i & \pi_{i_2} & \tilde{X}_i D_i^T U^T X_i C_i^T \\ & \ast & \ast \end{bmatrix}^T < 0, \quad (26)$$

where

$$\tilde{\Omega}_{i_1} = \begin{bmatrix} \pi_{i_1} & B_i M_i & \pi_{i_2} \\ * & * & * \\ * & * & * \end{bmatrix},$$

$$\tilde{\Omega}_{i_2} = \begin{bmatrix} E_i & \pi_{i_2} & X_i D_i^T U^T X_i C_i^T \\ & \ast & \ast \end{bmatrix},$$

$$\tilde{\Omega}_{i_3} = \text{diag}(-\gamma^2 I, -\delta I, -X_i, -\alpha_i I, -I),$$

$$\tilde{\Omega}_{i_4} = \begin{bmatrix} \pi_{i_1} & B_i M_i & \pi_{i_2} \\ * & * & * \\ * & * & * \end{bmatrix},$$

$$\tilde{\Omega}_{i_5} = \begin{bmatrix} E_i & \pi_{i_2} & X_i D_i^T U^T X_i C_i^T \\ & \ast & \ast \end{bmatrix},$$

$$\tilde{\Omega}_{i_6} = \text{diag}(-\gamma^2 I, -\delta I, -X_i, -\alpha_i I, -I).$$
with

\[
\dot{\bar{X}}_i = \text{sym}(A\bar{X}_i + B_i\bar{u}_i) + \alpha_iF_iF_i^T + B_i\bar{B}_i^T + \kappa_iX_i,
\]

\[
\dot{\bar{X}}_2 = B_1VX_2 + B_3B_3^T C_2^T,
\]

\[
\dot{\bar{X}}_{22} = \text{sym}(A_2\bar{X}_{22} + L_1C\bar{X}_1) + \alpha_iF_iF_i^T + \kappa_iX_i,
\]

\[
\dot{\bar{X}}_{23} = B_1VX_2 + X_1^T A_1C^T L_2^T + \alpha_iF_iF_i^T ,
\]

\[
\dot{\bar{X}}_{33} = \text{sym}(W_iX_2 + L_2CB_3V_iX_2) + \kappa_iX_i
\]

\[\dot{\bar{X}}_i = \begin{bmatrix} M_i^T \\ -M_i^T \\ X_2 \end{bmatrix}, \quad \bar{X}_i = \begin{bmatrix} X_{i1} \\ 0 \\ 0 
\end{bmatrix}, \quad U_i = \begin{bmatrix} U_{i1}^T \\ 0 \\ 0 \end{bmatrix},
\]

the argument system in (7) is mean-square exponentially stable with a weighted \( H_{\infty} \) performance level \( \gamma \) for any \( \varphi \in \mathcal{S}(T^+) \) with \( T^+ \geq \max_{\varphi \in \mathcal{S}(T^+)} \{ J_{\varphi} \} \), where \( \nu \geq 1 \) satisfies

\[
X_i \leq \nu X_j, \quad \forall i, j \in S. \tag{27}
\]

Moreover, the gains of the composite controller can be chosen as

\[
K_i = M_iX_i^{-1}, \quad i \in S. \tag{28}
\]

Proof. Given constant matrices \( U_1 \) and \( U_2 \), by Assumption 1, we can have the following inequality

\[
\| f(\bar{X}(t), t) \| \leq \left\| \begin{bmatrix} f(x(t), t) \\ f_x(t) \end{bmatrix} \right\| \leq \begin{bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{bmatrix} \begin{bmatrix} x(t) \\ 0 \\ 0 \end{bmatrix} = \| \bar{U} \dot{x}(t) \|, \tag{29}
\]

where

\[
\bar{U} = \begin{bmatrix} U_1 \\ 0 \\ U_2 \\ 0 \end{bmatrix}.
\]

Denoting \( P_i = X_i^{-1}, r = 1, 2 \), pre-and post-multiplying the inequality (40) by \( \bar{P}_i = \text{diag}(P_1, P_1, P_2) \), and using Schur complement lemma yield that

\[
\begin{bmatrix} \bar{\Theta}_i + \bar{C}_i^T \bar{C}_i + \kappa_i \bar{P}_i & \bar{P}_i \bar{E}_i^T \\ * & -\gamma^2 I \end{bmatrix} < 0. \tag{30}
\]

It is obvious that (30) has the same form with (14). On the other hand, one has (22) can guarantee (15). The proof can follow Theorem 1.

Remark 6. In this paper, we choose two steps strategy to obtain the solvable design conditions of the composite controllers, that is, solving the gains of the observer to guarantee the error system be exponentially stable and proposing the design conditions for the controller based on the observers satisfying Theorem 1, which is inspired by [17].

5 | SIMULATED EXAMPLES

In this section, the simulations of two examples are shown to illustrate the effectiveness of the proposed composite control schemes for the cases of available and unavailable states of system (1), respectively.

5.1 | Example 1

Consider a GE-90 aero-engines model borrowed in [31] with the fan speed increment \( \bar{z}_1(t) \), the core speed increment \( \bar{z}_2(t) \), the fuel flow increment \( \bar{u}(t) \) and the engine pressure ratio \( \bar{z}_3(t) \), the states of which can be available and the parameters of which are given as follows

\[
A_1 = \begin{bmatrix} -3.33 & 1.45 \\ 0.85 & -4.28 \end{bmatrix}, A_2 = \begin{bmatrix} -2.65 & 1.31 \\ 0.52 & -3.24 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 262.22 \\ 773.84 \end{bmatrix}, B_2 = \begin{bmatrix} 274.64 \\ 814.37 \end{bmatrix}, F_1 = \begin{bmatrix} 0.2 \\ 0.4 \end{bmatrix}, \tag{31}
\]

\[
E_2 = \begin{bmatrix} 0.2 \\ 0.6 \end{bmatrix}, D_1 = \begin{bmatrix} -0.3 & 0.8 \\ 0.8 & -0.4 \end{bmatrix}, E_{11} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix},
\]

\[
D_2 = \begin{bmatrix} 1 & 0.3 \\ 0.9 & 1 \end{bmatrix}, E_{12} = \begin{bmatrix} 0.5 \\ 0.7 \end{bmatrix}, C_{11}^T = \begin{bmatrix} 0.1 \end{bmatrix}^T, \tag{32}
\]

\[
C_{12}^T = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix}^T, C_{21}^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T, C_{22}^T = \begin{bmatrix} 0 \end{bmatrix}^T.
\]

Here, the disturbance \( d_i(t) \) is generated by system (2) with the following parameters

\[
W_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},
\]

\[
V_1 = \begin{bmatrix} 0.8 & 1 \end{bmatrix}, V_2 = \begin{bmatrix} 0.8 & 1 \end{bmatrix}, \tag{32}
\]

\[
E_{21} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}, E_{22} = \begin{bmatrix} 0.9 \\ 0.6 \end{bmatrix}.
\]

In what follows, we set \( \kappa_1 = \kappa_2 = 0.5, \gamma = 2, \nu = 1.1 \) and \( U = [0.1 \ 0.2] \). By Proposition 1, we can obtain the gain of the
disturbance observer (4) as follows

\[ L = \begin{bmatrix} -0.0343 & 0.0110 \\ 0.3425 & -0.1183 \end{bmatrix}. \]  

(33)

Next, the minimum dwell time and the feedback control gain \( K_i \) can be solved by Theorem 2 with (33) as

\[ T^* = 0.1906, \]

\[ K_1 = \begin{bmatrix} -0.8786 & 1.3371 \end{bmatrix}, \]  

(34)

\[ K_2 = \begin{bmatrix} -0.8433 & -0.8194 \end{bmatrix}. \]

This simulation is made under \( f(x(t), t) = 0.1x_2 e^{-0.1t} \), \( d_1(t) = 5e^{-t} \), \( d_2(t) = e^{-t} \) and the switching signal in Figure 1 satisfying the minimum dwell time \( T \geq T^* \). Figure 2 shows the states of system (1) under controller (5) with parameters (31), (33) and (34). In Figure 3, we plot not only the disturbance \( d_0(t) \) and its estimation \( \hat{d}_0(t) \) but also the error between them, which makes clear that the observer (4) can estimate the disturbance \( d_0 \) instantly. The signals of the control output are shown in Figure 4. From Figures 2-4, it is verified that the GE-90 aero-engines system with (31) under the composite controller (5) based on the disturbance observer (4) and minimum switching rule is mean-square exponentially stable with a prescribed \( H_\infty \) performance level.
Remark 7. In [31], a composite control was investigated for the switched system with the norm-bounded disturbance and exosystem generated disturbance, in which the disturbance observer based controller is designed under the assumption of available states of the system. In this example, we consider not only the above two types of disturbances but also the uncertainties of the system, the stochastic disturbance and input saturation. The control scheme might be ineffective for the switched system subject to above mentioned multiple disturbances and input saturation.

5.2 | Example 2

The system (1) with two subsystems is presented to illustrating the effectiveness of the proposed control design scheme for the case of the unavailable systems states, the parameters of which are shown as

\[
A_1 = \begin{bmatrix} -0.9 & 0.2 \\ -0.3 & -1.3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.3 & -0.2 \\ -0.1 & 0.1 \end{bmatrix},
\]

\[
B_1 = \begin{bmatrix} 0.3 \\ 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.5 & 0.2 \\ 0.1 & 1 \end{bmatrix},
\]

\[
F_1 = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.6 & 0.1 \end{bmatrix},
\]

\[
D_2 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, \quad E_{11} = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}.
\]
The parameters of the disturbance $d_0(t)$ are given as follows

$$W_1 = \begin{bmatrix} -0.2 \\ 0.3 \end{bmatrix}, \quad W_2 = \begin{bmatrix} -0.2 \\ -0.4 \end{bmatrix},$$

$$V_1 = [0.1 \ 0.1], \quad V_2 = [0.1 \ 0.1],$$

$$E_{21} = [0.2 \ 0.1], \quad E_{22} = [0.1 \ 0.2].$$

First, the following gains of the observers (6) and (7) can be obtained by solving Propositions 2 and 3, respectively

$$L_{11} = \begin{bmatrix} -0.9500 \\ 1.6500 \end{bmatrix}, \quad L_{12} = \begin{bmatrix} -0.3500 \\ 1.0000 \end{bmatrix}, \quad L_2 = \begin{bmatrix} -0.2299 \\ 0.7592 \end{bmatrix}.$$  

By setting the scalars as $\kappa_1 = \kappa_2 = 0.1$, $\gamma = 3$, and $\nu = 1.1$, $U_1 = U_2 = [1 \ 2]$, we can obtain the parameters of the controller.
The simulation results are shown in Figures 5–9. In Figure 5, a switching signal satisfying the minimum dwell time $T^* \geq 0.1906$ is given. From Figures 7–8, the states $x(t)$ and the disturbance $d_0(t)$ can be estimated effectively by observers (6) and (7) with gains (37). From Figure 6, it can be verified the system (1) with the assumption of unavailable state information can be stochastically exponentially stabilised in mean-square under the composite control scheme (8) with gains (38). Meanwhile, Figure 6 and Figure 9 show that the disturbance attenuation and rejection has been achieved.

6 | CONCLUSION

The composite anti-disturbance control problem of switched system with nonlinear uncertainties, multiple disturbances and input saturation has been studied by combining strategies of the observer based control, robust stochastic control and $H_\infty$ control. For the cases of the available or unavailable states of the system, the proposed composite control schemes can guarantee the closed-loop system to be mean-square exponentially stable with a weighted $H_\infty$ performance level under a minimum
switching rule, which are based on the disturbance and state observers. The simulations of GE-90 aero-engine model and a numerical example have successfully demonstrated the effectiveness of the proposed composite control schemes.

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APPENDIX A

Proof. For any switching signals $\varphi \in \mathcal{S}[T^+]$, the corresponding switching sequence is supposed as $\{(l_0, i_0); (t_1, i_1); \ldots; (l_k, i_k), \ldots; (l_0 = 0, i_0 \in \mathcal{S}, k = 0, 1, 2, \ldots\}$. For $t \in [t_k, t_{k+1})$, one has $\varphi(t) = i_k$. Choose a Lyapunov function for the mode $i_k$ as follows

$$V_{i_k}(\chi(t)) \triangleq \chi^T(t)P_{i_k}\chi(t). \tag{A.1}$$

It is easy to get that

$$a_{i_k}\|\chi(t)\|^2 \leq V_{i_k}(\chi(t)) \leq b_{i_k}\|\chi(t)\|^2, \tag{A.2}$$

where $a_{i_k} = \lambda_{\min}(P_{i_k})$ and $b_{i_k} = \lambda_{\max}(P_{i_k})$.

For a fixed $i_k$ and by It\'o formula, the stochastic differential can be computed along the solution of argument system (9) as

$$dV_{i_k}(\chi(t)) = \mathcal{L}V_{i_k}(\chi(t)) + 2\chi^T(t)P_{i_k}\bar{D}_{i_k}\chi(t) + \chi^T(t)\bar{D}_{i_k}P_{i_k}\chi(t). \tag{A.3}$$

where

$$\mathcal{L}V_{i_k}(\chi(t)) = 2\chi^T(t)P_{i_k}\chi(t) = \chi^T(t)(\text{sym}[P_{i_k}A_{i_k}] + D_{i_k}^TP_{i_k}\bar{D}_{i_k})\chi(t)$$

$$+ 2\chi^T(t)P_{i_k}\bar{F}_{i_k}\chi(t) + 2\chi^T(t)P_{i_k}E_{i_k}d(t) + 2\chi^T(t)P_{i_k}\bar{B}_{i_k}\varphi(K_{i_k}\chi(t)).$$

Under Assumption 1 with a given matrix $U$, using lemmas (1) and (2), there exist positive scalar $\alpha_{i_k}$ and real number $\varepsilon \in (0, 1)$ such that

$$dV_{i_k}(\chi(t)) \leq \chi^T(t)\Theta_{i_k}\chi(t) + 2\chi^T(t)P_{i_k}\bar{D}_{i_k}\chi(t)d\omega(t), \tag{A.4}$$

where

$$\Theta_{i_k} = \text{sym}(P_{i_k}A_{i_k}) + D_{i_k}^TP_{i_k}\bar{D}_{i_k} + \alpha_{i_k}P_{i_k}\bar{F}_{i_k}\bar{F}_{i_k}^TP_{i_k}$$

$$+ \alpha_{i_k}^{-1}U^TU + P_{i_k}\bar{B}_{i_k}\bar{B}_{i_k}^TP_{i_k} + \varepsilon K_{i_k}^TK_{i_k}.$$

When $d(t) \equiv 0$, (14) implies

$$\mathcal{L}V_{i_k}(\chi(t)) \leq -\kappa_{i_k}\chi^T(t)P_{i_k}\chi(t) = -\kappa_{i_k}V_{i_k}(\chi(t)). \tag{A.5}$$

It is easy to further get that

$$d[f_{i_k}\chi(t)] \leq 2\chi^T(t)P_{i_k}\bar{D}_{i_k}\chi(t)d\omega(t). \tag{A.6}$$

Under the switching signal $\varphi \in \mathcal{S}[T^+]$ and the corresponding switching sequence, for $t \in [t_k, t_{k+1})$, integrating both sides of (A.6) from $t_k > 0$ to $t$ and taking expectation yield that

$$\mathbb{E}[V_{i_k}(\chi(t))] \leq e^{-\kappa_{i_k}(t-t_k)}\mathbb{E}[V_{i_k}(\chi(t_k))]. \tag{A.7}$$

Noting that $P_i = X_i^{-1}$, the inequality (15) is equivalent to $P_i \prec \nu P_i, \forall i, j \in \mathcal{S}$. At the switching time instant $t_k$, one can obtain that

$$\mathbb{E}[V_{i_k}(\chi(t_k))] \leq \nu \mathbb{E}[V_{i_{k-1}}(\chi(t_{k-1}))]. \tag{A.8}$$

From (A.7) and (A.8), the following inequality can hold

$$\mathbb{E}\{V_{i_k}(\chi(t))\} \leq e^{-\kappa_{i_k}(t-t_k)}\mathbb{E}\{V_{i_k}(\chi(t_k))\} \leq \nu e^{-\kappa_{i_k}(t-t_k)}\mathbb{E}\{V_{i_k}(\chi(t_{k-1}))\} \leq \cdots \leq \nu^k e^{-\kappa_{i_k}(t-t_k)}\mathbb{E}\{V_{i_k}(\chi(t_{k-1}))\} \leq \nu^k e^{-\kappa_{i_k}(t-t_k)}\mathbb{E}\{V_{i_k}(\chi(t_0))\}. \tag{A.9}$$
Denoting $\kappa = \min_{\in S} \{\kappa_i\}$ and noting that $t_0 = 0$, (A.9) can be written as

$$E[V_{i_0}(\chi(t))] \leq e^{\mu t} E[V_{i_0}(\chi(0))].$$

(10.10)

For switching sequence of $q \in S[T^*]$, we have $t_k - t_{k-1} \geq T^*$, which yields that $t \leq kT^*$. For $\nu \geq 1$ and $T^* \geq \max_{i \in S} \frac{\ln \nu}{\kappa_i}$, we can obtain the following inequality from (A.10)

$$E[V_{i_k}(\chi(t))] \leq e^{-\nu \mu} E[V_{i_0}(\chi(0))],$$

(11.11)

where $\mu = \kappa - \frac{\ln \nu}{T^*} > 0$. From (A.2) and (11.11), we obtain

$$E[\|\chi(t)\|^2] \leq \eta e^{-\nu \mu} \|\chi(0)\|^2,$$

(12.12)

where $\eta = \frac{b}{a}$ with $a = \min \{a_i\}$ and $b = \max \{b_i\}$ for $i \in S$.

By definition 1, (12.12) implies system (9) with $d(t) \equiv 0$ is mean-square exponentially stable.

When system (9) is considered to be with disturbances $d(t)$, the following performance index with positive constant $\gamma$ is introduced as

$$J(t) = \mathcal{L} V_{i_k}(\chi(t)) + \kappa_{i_k} V_{i_k}(\chi(t)) + \tilde{\zeta}^T(t) \tilde{\zeta}(t) - \gamma^2 d^T(t) d(t) , \quad t \in [t_k, t_{k+1}).$$

(13.13)

Combining (14), (A.3), (A.4) with (13.13) yields that $J(t) < 0$, which implies

$$\mathcal{L} V_{i_k}(\chi(t)) + \kappa_{i_k} V_{i_k}(\chi(t)) < -(\tilde{\zeta}^T(t) \tilde{\zeta}(t) - \gamma^2 d^T(t) d(t)).$$

(14.14)

From (A.3) and (14.14), it is easy to see that

$$dV_{i_k}(\chi(t)) < 2 \tilde{\chi}^T(t) P_{i_k} D_i \chi(t) d\omega(t) - \kappa_{i_k} V_{i_k}(\chi(t))$$

$$- \tilde{\zeta}^T(t) \tilde{\zeta}(t) + \gamma^2 d^T(t) d(t).$$

(15.15)

As the above process, for $t \in [t_k, t_{k+1})$, one has

$$E[V_{i_k}(\chi(t))]$$

$$< e^{\kappa_{i_k}(t-t_0)} E[V_{i_0}(\chi(0))]$$

$$- \gamma^2 E \left[ \int_0^t e^{-\kappa_{i_k}(t-s)} (\tilde{\zeta}^T(s) \tilde{\zeta}(s) - \gamma^2 d^T(s) d(s)) ds \right]$$

$$- \cdots$$

$$- \gamma^0 E \left[ \int_0^t e^{-\kappa_{i_k}(t-s)} (\tilde{\zeta}^T(s) \tilde{\zeta}(s) - \gamma^2 d^T(s) d(s)) ds \right].$$

(16.16)