A Generic Property of Exact Magnetic Lagrangians

Mário Jorge Dias Carneiro and Alexandre Rocha.

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Abstract

We prove that for the set of Exact Magnetic Lagrangians the property “There exist finitely many static classes for every cohomology class” is generic. We also prove some dynamical consequences of this property.

1 Introduction

Let $M$ be a closed manifold equipped with an Riemannian metric $g = \langle \cdot, \cdot \rangle$. A Lagrangian $L : TM \to \mathbb{R}$ is called Exact Magnetic Lagrangian if

$$L(x, v) = \frac{\|v\|^2}{2} + \langle \eta, v \rangle$$

for some non-closed 1-form $\eta$.

This type of Lagrangian fits into Mather’s theory, as developed by R. Mañé and A. Fathi, about Tonelli Lagrangians, namely, it is fiberwise convex and superlinear. We refer the reader to the references Fathi in [6], Contreras and Iturriaga in [4] for expositions of this theory.

Let $\mathfrak{M}(L)$ be the set of action minimizing measures. Recall that $\mathfrak{M}(L)$ is the set of $\mu$ Borel probability measures in $TM$ which are invariant under the Euler-Lagrange flow $\varphi_t$ generated by $L$ and minimizes the action, that is for all invariant probability $\nu$ in $TM$:

$$\int_{TM} Ld\mu \leq \int_{TM} Ld\nu.$$ 

The set $\mathfrak{M}(L)$ is a simplex whose extremal points are the ergodic minimizing measures.

Since the Euler Lagrange flow generated by $L$ does not change by adding a closed one form $\zeta$, we also consider the action minimizing measures $\mathfrak{M}(L - \zeta)$. The minimal
action value, depends only on the cohomology class \( c = [\zeta] \in H^1(M, \mathbb{R}) \) of the closed one form, so it is denoted by \( -\alpha(c) \). It is known that \( \alpha(c) \) is the energy level that contains the Mather set for the cohomology class \( c \):

\[
\tilde{\mathcal{M}}_c = \bigcup_{\mu \in \mathcal{G}(L - \zeta)} \text{supp}(\mu).
\]

\( \tilde{\mathcal{M}}_c \) is a compact invariant set which is a graph over a compact subset \( \mathcal{M}_c \) of \( M \), the projected Mather set (see [11]). \( \mathcal{M}_c \) is laminated by curves, which are global (or time independent) minimizers. Mather also proved that the function \( c \mapsto \alpha(c) \) is convex and superlinear.

In general, \( \tilde{\mathcal{M}}_c \) is contained in another compact invariant set, which also a graph whose projection is laminated by global minimizers: the Aubry set for the cohomology class \( c \), denoted by \( \tilde{\mathcal{A}}_c \). Mañé proved that \( \tilde{\mathcal{A}}_c \) is chain recurrent and it is a challenging question to describe the dynamics of the Euler-Lagrange flow restricted to \( \tilde{\mathcal{A}}_c \). The definition of Aubry set and some its properties are given in Section 3.

Of course this question only makes sense if it is posed for generic Lagrangians, since many pathological examples can be constructed. The notion of genericity in the context of Lagrangian systems is provided by Mañé in [9]. The idea is to make special perturbations by adding a potential: \( L(x, v) + \Psi(x) \), for \( \Psi \in C^\infty(M) \).

A property is \textit{generic} in the sense of Mañé if it is valid for all Lagrangians \( L(x, v) + \Phi(x) \) with \( \Phi \) contained in a residual subset \( \mathcal{O} \).

In this setting, G. Contreras and P. Bernard proved in the work \textit{A Generic Property of Families of Lagrangian Systems} (see [1]) that generically, in the sense of Mañé, for all cohomology class \( c \) there is only a finite number of minimizing measures. This theorem is a consequence of an abstract result which is useful in different situations.

In general, when we are dealing with an specific class of Lagrangians, perturbations by adding a potential are not allowed. However, due to the abstract nature of Bernard-Contreras proof it may be addapted to the specific case like the one treated here.

The objective of this paper is to prove the genericity of finitely many minimizing measures for Exact Magnetic Lagrangians and apply it to the dynamics of the Aubry set.

Let us consider \( \Gamma^1(M) \) the set of smooth 1-forms in \( M \) endowed with the metric

\[
d(\omega_1, \omega_2) = \sum_{k \in \mathbb{N}} \arctan \left( \frac{\|\omega_1 - \omega_2\|_k}{2^k} \right),
\]
denoting by $\|\omega\|_k$ the $C^k$-norm of the 1-form $\omega$. With this metric $\Gamma^1 (M)$ is a Frechet space, it means that $\Gamma^1 (M)$ is a locally convex topological vector space whose topology is defined by a translation-invariant metric, and that $\Gamma^1 (M)$ is complete for this metric.

The main result of this paper is the following:

**Theorem 1** Let $A$ be a finite dimensional convex family of Exact Magnetic Lagrangians. Then there exists a residual subset $\mathcal{O}$ of $\Gamma^1 (M)$ such that,

$$\omega \in \mathcal{O}, L \in A \Rightarrow \dim \mathcal{M} (L + \omega) \leq \dim A.$$  

Hence there exist at most $1 + \dim A$ ergodic minimizing measures of $L + \omega$.

**Corollary 2** Let $L$ be an Exact Magnetic Lagrangian. Then there exists a residual subset $\mathcal{O}$ of $\Gamma^1 (M)$ such that for all $c \in H^1 (M, \mathbb{R})$ and for all $\omega \in \mathcal{O}$, there are at most $1 + \dim H^1 (M, \mathbb{R})$ ergodic minimizing measures of $L + \omega - c$.

The last part of this work is dedicated to prove some consequences about the dynamics. For instance, using the work of Contreras and Paterna in [5], we obtain connecting orbits between the elements of the Aubry set that contain the support of minimizing measures (the so called “static classes”).

2 Adapting the abstract setting of Bernard and Contreras

As it was pointed out previously, the proof of Theorem 1 is an application of the work of Contreras and Bernard. Here we state their result.

Assume that we are given

(i) Three topological vector spaces $E, F, G$.

(ii) A continuous linear map $\pi : F \to G$.

(iii) A bilinear map $\langle , \rangle : E \times G \to \mathbb{R}$.

(iv) Two metrizable convex compact subsets $H \subset F$ and $K \subset G$ such that $\pi (H) \subset K$.

Suppose that

1. The restriction of the map given by (iii), $\langle , \rangle |_{E \times K}$ is continuous.
2. The compact $K$ is separated by $E$. This means that, if $\mu$ and $\nu$ are two different points of $K$, then there exists a point $\omega$ in $E$ such that $\langle \omega, \mu - \nu \rangle \neq 0$.

3. $E$ is a Frechet space.

Note then that $E$ has the Baire property, that is any residual subset of $E$ is dense.

We shall denote by $H^*$ the set of affine and continuous functions defined on $H$. Given $\bar{L} \in H^*$ denote by

$$M_H (\bar{L}) = \arg \min \bar{L}$$

the set of points $\alpha \in H$ which minimizes $\bar{L}|_H$, and by $M_K (\bar{L})$ the image $\pi (M_H (\bar{L}))$. These are compacts convex subsets of $H$ and $K$.

Under these conditions we have:

**Theorem 3 (G. Contreras and P. Bernard)** For every finite dimensional affine subspace $A$ of $H^*$, there exists a residual subset $O (A) \subset E$ such that, for all $\omega \in O (A)$ and $\bar{L} \in A$, we have

$$\dim M_K (\bar{L} + \omega) \leq \dim A$$

In order to apply this theorem, we need to define the above objects in an adequate setting as follows:

Let $C$ be the set of continuous functions $f : TM \to \mathbb{R}$ with linear growth, that is

$$\|f\|_{\ell^\infty} = \sup_{\theta \in TM} \frac{|f(\theta)|}{1 + |\theta|} < +\infty$$

endowed with the norm $\|\cdot\|_{\ell^\infty}$.

We define:

- $E = \Gamma^1 (M)$ endowed with the metric $d$ defined in (1).
- $F = C^*$ is the vector space of continuous linear functionals $\mu : C \to \mathbb{R}$ provided with the weak-$\star$ topology:

$$\lim_n \mu_n = \mu \iff \lim_n \mu_n (f) = \mu (f), \forall f \in C.$$  

- $G$ is the vector space of continuous linear functionals $\mu : \Gamma^0 (M) \to \mathbb{R}$, where $\Gamma^0 (M)$ is the space of continuous 1-forms on $M$. Note that the Riemannian metric $g = \langle ., . \rangle$ allows us to represent any continuous 1-form as $\langle X, . \rangle$, for some $C^0$ vector field $X$. We endow $G$ with the weak-$\star$ topology:

$$\lim_n \mu_n = \mu \iff \lim_n \mu_n (\omega) = \mu (\omega), \forall \omega \in \Gamma^0 (M).$$
The continuous linear $\pi : F \rightarrow G$ is given by

$$\pi(\mu) = \mu|_{\Gamma^0(M)}.$$ 

For a given natural number $N$, let

$$B_N = \{(x, v) \in TM : |v| \leq N\}.$$ 

Let us denote by $M^1_N$ the set of the probability measures $\mu$ in $TM$ such that $\text{supp} \mu \subset B_N$. Define $K_N = \pi(M^1_N) \subset G$, the restriction of the probabilities in $M^1_N$ to $\Gamma^0(M)$.

**Claim 1.** $K_N$ is metrizable.

**Proof:** Since $G$ is the dual of $\Gamma^0(M)$, we define a norm in $G$ as follows

$$\|\mu\|_G = \sup_{|\omega|_{\ell^1} \leq 1} \left\{ |\mu(\omega)| \right\}.$$ 

If $\mu \in K_N$,

$$\|\mu\|_G \leq \sup_{|\omega|_{\ell^1} \leq 1} \left\{ \int_{TM} \omega d\mu \right\} \leq \sup_{|\omega|_{\ell^1} \leq 1} \left\{ \int_{TM \cap B_N} |\omega| d\mu \right\} \leq \sup_{|\omega|_{\ell^1} \leq 1} \left\{ \int_{B_N} \frac{|\omega(x, v)|}{1 + |v|} (1 + N) d\mu \right\} \leq (N + 1) \sup_{|\omega|_{\ell^1} \leq 1} \left\{ \int_{TM} |\omega|_{\ell^1} d\mu \right\} \leq N + 1.$$ 

This shows that $K_N \subset B_G$, where $B_G$ is the ball of radius $N + 1$ in $G = \Gamma^0(M)^*$. Then, by following classical theorem of Analysis, it is enough show that $\Gamma^0(M)$ is a separable vector space.

**Theorem 4** Let $E$ a Banach’s space. Then $E$ is separable if, and only if, the unit ball $B_{E^*} \subset E^*$ in the weak-$*$ topology is metrizable.

The separability of $\Gamma^0(M)$ follows from the lemma below and of the duality between 1-forms and vector fields provided by the Riemannian metric.

**Lemma 5** The space $X^0(M)$ of continuous vector fields in a compact manifold $M$ is separable.

**Proof:** By compactness of $M$, we can consider a number finite local trivializations $\hat{U}_i \subset TM \rightarrow U_i \times \mathbb{R}^n$ of the tangent bundle $TM$ and by compactness of $\overline{U}_i$, $X^0(\overline{U}_i) = C^0(\overline{U}_i, \mathbb{R}^n)$ is separable. Let $\{f^i_n\}$ be a dense subset in $X^0(\overline{U}_i)$ and $\{\alpha_i\}$ a partition
of unity subordinate to the open cover \( \{ U_i \} \). It is enough show that \( \{ \sum_i \alpha_i f_{n_i}^i \} \) is dense in \( \mathcal{X}^0(M) \). Let \( g \in \mathcal{X}^0(M) \) and consider \( g_i = \alpha_i g \). Then \( g = \sum \alpha_i g = \sum g_i \), \( \text{supp} \ g_i \subset U_i \subset \overline{U_i} \). Given \( \epsilon > 0 \) there exists \( n_i \in \mathbb{N} \) such that

\[
\left\| f_{n_i}^i - g_i \right\| < \frac{\epsilon}{2i}.
\]

Then

\[
\left\| \sum_i \alpha_i f_{n_i}^i - g \right\| = \left\| \sum_i \alpha_i f_{n_i}^i - \sum \alpha_i g \right\| \leq \sum_i \sup_{U_i} \left| f_{n_i}^i - g_i \right| < \frac{\epsilon}{2i} < \epsilon.
\]

Let us consider \((X_n)\) a dense sequence in \( \mathcal{X}^0(M) \) and \( \omega_n = \langle X_n, \cdot \rangle \in \Gamma^0(M) \). Let \( \omega = \langle X, \cdot \rangle \in \Gamma^0(M) \) and \( U_\epsilon \) be a ball in \( \Gamma^0(M) \) centered at \( \omega \), of radius \( \epsilon > 0 \). Then there exists a \( X_n \in V_\epsilon(X) \), where \( V_\epsilon(X) \) is the ball in \( \mathcal{X}^0(M) \) of radius \( \epsilon \) and center \( X \). It follows that

\[
\left\| \omega_n - \omega \right\|_{\ell_{in}} = \sup_{(x,v) \in TM} \frac{|(\omega_n - \omega)(x,v)|}{1 + |v|} \leq \sup_{x \in M} |(X_n - X)(x)| \leq \epsilon.
\]

This shows that \( \omega_n \in U_\epsilon \) and \( \Gamma^0(M) \) is separable, so \( K_N \) is metrizable. This finishes the proof of the Claim 1.

Observe that \( K_N \) is compact and convex since \( K_N = \pi(M^1_N) \), \( \pi \) is a continous map and \( M^1_N \) is a compact subset of probability measures in \( TM \).

- The bilinear mapping \( \langle \cdot, \cdot \rangle : E \times G \to \mathbb{R} \) is given by integration:

\[
\langle \omega, \mu \rangle = \int_{TM} \omega d\mu.
\]

Note that here we apply the Hahn-Banach Theorem for extends the functional \( \mu \) and that the above integral does not depend on the extension of \( \mu \) to a signed measure on \( TM \) given by Riesz representation Theorem. Moreover,

\[
\langle \cdot, \cdot \rangle : E \times K_N \to \mathbb{R}
\]

is continuous. In fact, if \( \omega_n \to \omega \) and \( \mu_n \to \mu \) with \( \omega_n \in E \) and \( \mu_n \in K_N \), then

\[
\lim_n \int_{TM} \eta d\mu_n = \int_{TM} \eta d\mu, \forall \eta \in E,
\]
and \(d(\omega_n, \omega) \to 0\) implies that given \(\epsilon > 0\), there exists \(n_0 \in \mathbb{N}\) such that
\[
\forall n \geq n_0, \|\omega_n - \omega\|_{\ell^\infty} < \frac{\epsilon}{(N + 1)}.
\]
Since \(\mu_n, \mu \in K_N\), we have
\[
\left| \int_{TM} \omega_n d\mu_n - \int_{TM} \omega d\mu \right| \leq \int_{B_N} |\omega_n - \omega| d\mu_n
\]
\[
= \int_{B_N} \frac{|\omega_n - \omega|}{1 + N} (1 + N) d\mu_n
\]
\[
\leq (1 + N) \int_{B_N} \frac{|\omega_n - \omega|}{1 + |v|} d\mu_n
\]
\[
\leq (1 + N) \int_{B_N} \|\omega_n - \omega\|_{\ell^\infty} d\mu_n < \epsilon
\]
When \(n \to \infty\),
\[
\lim_{n} \int_{TM} \omega_n d\mu_n - \int_{TM} \omega d\mu \leq \epsilon, \forall \epsilon > 0.
\]
Therefore
\[
\lim_{n} \langle \omega_n, \mu_n \rangle = \lim_{n} \int_{TM} \omega_n d\mu_n = \int_{TM} \omega d\mu = \langle \omega, \mu \rangle.
\]
\(\bullet\) \(K_N\) is separated by \(E\). This follows from the duality and approximation of continuous vector fields by smooth ones and the fact that \(K_N\) is separated by \(\Gamma^0(M)\), that is: if \(\mu, \nu \in K_N, \mu \neq \nu\), then there exists a \(\omega_0 \in \Gamma^0(M)\) such that \(\mu(\omega_0) \neq \nu(\omega_0)\) or
\[
\int_{TM} \omega_0 d\mu \neq \int_{TM} \omega_0 d\nu.
\]
The next ingredient regarding the steps followed by Bernard and Contreras is the proof of injectivity of the map \(\pi : \mathcal{M}(L) \to G\).
Recall that \(\mathcal{M}(L)\) the set of minimizing measures for \(L\) and \(\widetilde{M}_0 = \bigcup_{\mu \in \mathcal{M}(L)} \text{supp} \mu\) is the Mather set.

**Lemma 6** Let \(L\) be a Exact Magnetic Lagrangian. If \(\mu\) and \(\nu\) are two distincts minimizing measures, then there exists a 1-form \(\omega\) in \(\Gamma^0(M)\) such that
\[
\int_{TM} \omega d\mu \neq \int_{TM} \omega d\nu
\]

**Proof:** If \(\mu \neq \nu\), there exists \(A\) in the Borel sigma algebra such that \(\mu(A) \neq \nu(A)\). We can suppose \(A\) is a closed set and \(A \subset \text{supp}(\mu) \cup \text{supp}(\nu)\). The energy function for \(L\) is given by \(E(x, v) = \frac{1}{2} \|v\|^2\) and since
\[
\text{supp}(\mu) \cup \text{supp}(\nu) \subset E^{-1}(\alpha(0)) = \{(x, v) \in TM : \|v\|^2 = 2\alpha(0)\},
\]

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we have $A \subset E^{-1}(\alpha(0))$. Moreover, $A \subset \widetilde{M}_0$, where $\widetilde{M}_0$ is the Mather set. By graph property $A$ is a graph on $\pi(A)$ and we can write

$$A = \{(x, v) : x \in \pi(A) \text{ and } v = \pi^{-1}(x)\}$$

where $\pi^{-1}$ is Lipschitz on the projected Mather set. Let

$$X(x) = \begin{cases} \pi^{-1}(x), & \text{if } x \in \pi(A) \\ 0, & \text{if } x \notin \pi(A) \end{cases}$$

and consider $f_n : M \rightarrow [0,1]$ sequence of smooth bump functions

$$f_n(x) = \begin{cases} 1, & \text{if } x \in \pi(A) \\ 0, & \text{if } x \notin B_n(\pi(A)) \end{cases}$$

where $B_n(\pi(A))$ is a neighborhood of the compact $\pi(A)$:

$$B_n(\pi(A)) = \left\{ x \in M : d(x, a) < \frac{1}{n}, \text{ for some } a \in \pi(A) \right\}.$$

Let us consider $\overline{X}$ a continuous extension of $X|_{\pi(A)}$ on $M$. Then the vector field $X_n = f_n\overline{X} \in \mathfrak{X}^0(M)$, converges pointwise to $X(x)$ and

$$|\langle X_n(x), v \rangle| = |\langle f_n\overline{X}(x), v \rangle| \leq |f_n\overline{X}(x)||v| \leq |\overline{X}(x)||v|.$$

By Dominated Convergence Theorem

$$\int_{TM} \langle X_n(x), v \rangle d\mu \rightarrow \int_{TM} \langle X(x), v \rangle d\mu,$$

and

$$\int_{TM} \langle X_n(x), v \rangle d\nu \rightarrow \int_{TM} \langle X(x), v \rangle d\nu.$$

Suppose that for all $\omega \in \Gamma^0(M)$,

$$\int_{TM} \omega d\mu = \int_{TM} \omega d\nu.$$

Then we have

$$\int_{TM} \langle X_n(x), v \rangle d\mu = \int_{TM} \langle X_n(x), v \rangle d\nu.$$
Therefore
\[
\int_{TM} \langle X(x), v \rangle d\mu = \int_{TM} \langle X(x), v \rangle d\nu
\]
\[
\int_{A} \langle X(x), v \rangle d\mu = \int_{A} \langle X(x), v \rangle d\nu
\]
\[
\int_{A} \langle X(x), X(x) \rangle d\mu = \int_{A} \langle X(x), X(x) \rangle d\nu
\]
\[
\int_{A} 2\alpha(0) d\mu = \int_{A} 2\alpha(0) d\nu
\]
\[
\alpha(0) \mu(A) = \alpha(0) \nu(A),
\]
Hence \( \mu(A) = \nu(A) \) because \( \alpha(0) > 0 \) (See G. Paternain and M. Paternain in [13]). This finishes the proof.

The final step is entirely analogous to Lemma 9 of [1] and we repeat it here only for the sake of completeness. Mañé introduced a special type of probability measures, the holonomic measures which is useful to prove genericity results. A \( C^1 \) curve \( \gamma : I \subset \mathbb{R} \to M \) of period \( T > 0 \) define an element \( \mu_\gamma \in F \) by
\[
\mu_\gamma(f) = \frac{1}{T} \int_0^T f(\gamma(s), \dot{\gamma}(s)) \, ds
\]
for each \( f \in C \). Let
\[
\Theta = \{ \mu_\gamma : \gamma \in C^1(\mathbb{R}, M) \text{ periodic of integral period} \} \subset F.
\]
The set \( \mathcal{H} \) of holonomic probabilities is the closure of \( \Theta \) in \( F \). One can see \( \mathcal{H} \) is convex (see Mañé [9]). The elements \( \mu \) of \( \mathcal{H} \) satisfy \( \mu(1) = 1 \). We define the compact \( H_N \subset F \) as the set of holonomic probability measures which are supported in \( B_N \). Therefore we have \( \pi(H_N) \subset K_N \).

The each Tonelli Lagrangian \( L \) it is associated an element \( \bar{L} \in H_N^* \) as follows
\[
\mu \mapsto \int_{TM} Ld\mu, \mu \in H_N.
\]
Recalling that we have defined \( M_{H_N}(L) \) as the set of measures \( \mu \in H_N \) which minimize the action \( \int Ld\mu \) on \( H_N \).

**Lemma 7** If \( L \) is a Exact Magnetic Lagrangian then there exists \( N \in \mathbb{N} \) such that
\[
\dim M_{K_N}(L) = \dim \mathcal{M}(L).
\]
Proof: Mañé proves in [9] that $M(L) \subset H$. The Mather set $\tilde{M}_0$ is compact, therefore $M(L) \subset H_N$ for some $N \in \mathbb{N}$. Mañé also proves in [9] that minimizing measures are also all the minimizers of action functional $A_L(\mu) = \int Ld\mu$ on the set of holonomic measures, therefore $M(L) = M_{H_N}(L)$. By previous Lemma the map $\pi : M(L) \to G$ is injective, so that

$$\dim \pi (M_{H_N}(L)) = \dim \pi (M(L)) = \dim M(L)$$

Proof: (of Theorem 1) Given $n \in \mathbb{N}$ apply Theorem 3 and obtain a residual subset $O_n(A) \subset E = \Gamma^1(M)$ such that $L \in A, \omega \in O_n(A) \Rightarrow \dim M_{K_n}(L + \omega) \leq \dim A$.

Let $O(A) = \bigcap_n O_n(A)$ . By the Baire property $O(A)$ is residual. We have that $L \in A, \omega \in O(A), n \in \mathbb{N} \Rightarrow \dim M_{K_n}(L + \omega) \leq \dim A$.

Then by previous Lemma, $\dim M(L + \omega) \leq \dim A$ for all $L \in A$ and all $\omega \in O(A)$. This finishes the proof.

3 Some Dynamical Consequences

As it was pointed out in the Introduction, the Mather set $\tilde{M}_c$ associated to a cohomology class $c$ is contained in another compact invariant set called the Aubry set $\tilde{A}_c$. It is also a graph over a compact subset of the manifold $M$ and it is contained in the same energy level $\alpha(c)$ as $\tilde{M}_c$. Moreover, $\tilde{A}_c$ is chain recurrent set. All these properties are proven in [4], see also [6].

In order to state the dynamical consequences of our Theorem 1 we need to introduce the Aubry set and the concept of static classes for a general Tonelli Lagrangian.

Let us consider the action on a curve $\gamma : [0, T] \to M$ defined by

$$A_{L-c+k}(\gamma) = \int_0^T [L(\gamma, \dot{\gamma}) - \eta(\gamma)(\dot{\gamma}) + k]dt$$

where $k$ is a real number and $\eta$ is a representative of the class $c$. The energy level $\alpha(c)$, namely Mañé’s critical value of the Lagrangian $L - c$, may be characterized in several ways. $\alpha(c)$ is defined by Mañé as the infimum of the numbers $k$ such that the action $A_{L-c+k}(\gamma)$ is nonnegative for all closed curve $\gamma : [0, T] \to M$. 

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Recall that, for a given real number \( k \) the action potential \( \Phi_{L-c+k} : M \times M \to \mathbb{R} \) is defined by

\[
\Phi_{L-c+k} (x, y) = \inf_{\mathcal{A}_{L-c+k}} (\gamma)
\]

infimum taken over the curves \( \gamma \) joining \( x \) the \( y \).

Mañé proved that

\[
-\alpha(c) = \inf_{\mu \in \mathfrak{M}(L)} \int_{TM} (L - \eta) \, d\mu,
\]

where \( \eta \) is a representative of the class \( c \) and that \( \alpha(c) \) is the smallest number such that the action potential is finite, in other words, if \( k < \alpha(c) \), then \( \Phi_{L-c+k}(x, y) = -\infty \) and for \( k \geq \alpha(c) \), \( \Phi_{L-c+k}(x, y) \in \mathbb{R} \).

Observe that by Tonelli’s Theorem (See for example in [4]), for fixed \( t > 0 \), there always exists a minimizing extremal curve connecting \( x \) to \( y \) in time \( t \). The potential calculates the global (or time independent) infimum of the action. This value may not be realized by a curve.

The potential \( \Phi_{L-c+\alpha(c)} \) is not symmetric in general but

\[
\delta_M (x, y) = \Phi_{L-c+\alpha(c)} (x, y) + \Phi_{L-c+\alpha(c)} (y, x)
\]

is a pseudo-metric. A curve \( \gamma : \mathbb{R} \to M \) is called semistatic if minimizes action between any of its points:

\[
\mathcal{A}_{L-c+\alpha(c)} (\gamma|_{[a,b]}) = \Phi_{L-c+\alpha(c)} (\gamma (a) , \gamma (b)) ,
\]

and \( \gamma \) is called static if is semistatic and \( \delta_M (\gamma (a) , \gamma (b)) = 0 \) for all \( a, b \in \mathbb{R} \).

For example, the orbits contained in the Mather set \( \widetilde{\mathcal{M}}_c \) project onto static curves. The Aubry set \( \mathcal{A}_c \) is the set of the points \( (x,v) \in TM \) such that the projection \( \gamma (t) = \pi \circ \varphi_t (x, v) \) is a static curve, where \( \varphi_t \) is the Euler-Lagrange flow. We just saw that the Mather set \( \widetilde{\mathcal{M}}_c \) is contained in the Aubry set \( \widetilde{\mathcal{A}}_c \).

Denoting the projected Aubry set by \( \mathcal{A}_c \), the function \( \delta_M|_{\mathcal{A}_c \times \mathcal{A}_c} : \mathcal{A}_c \times \mathcal{A}_c \to \mathbb{R} \) is called Mather semi-distance. We define the quotient Aubry set \( (\mathcal{A}_M, \delta_M) \) to be the metric space by identifying two points \( x, y \in \mathcal{A}_c \) if their semi-distance \( \delta_M (x, y) \) vanishes. When we consider \( \delta_M \) on the quotient space \( \mathcal{A}_M \) we will call it the Mather distance and the elements of \( \mathcal{A}_M \) are called static classes for \( L - c \). Observe that the static classes are disjoint subsets of the energy level set \( \alpha(c) \) and a static curve is in the same static class.

Then we have the following corollary of the Theorem [1]:
Corollary 8 Let $L$ be a Exact Magnetic Lagrangian. Then there exists a residual subset $\mathcal{O}$ of $\Gamma^1(M)$ such that for all $c \in H^1(M, \mathbb{R})$ and for all $\omega \in \mathcal{O}$, the Lagrangian $L + \omega - c$ has at most $1 + \dim H^1(M, \mathbb{R})$ static classes.

Proof: It suffices to show that each static class supports at least one ergodic minimizing measure. In fact, let $\Lambda$ be a static class for $L + \omega - c$ and $(p, v) \in \tilde{A}_c$ with $p \in \Lambda$. For $T > 0$ we define a Borel probability measure $\mu_T$ on $TM$ by

$$\mu_T(f) = \frac{1}{T} \int_0^T f(\varphi_s(p, v)) \, ds$$

All these probability measures have their supports contained in $\tilde{A}_c$ that is a compact subset, consequently, we can extract a sequence $\mu_{T_n}$ weakly convergent to $\mu$:

$$\mu(f) = \lim_{T \to \infty} \frac{1}{T_n} \int_0^{T_n} f(\varphi_s(p, v)) \, ds,$$

which is an ergodic minimizing measure whose support is contained in $\Lambda$ (See [6] for details).

Now we present some dynamical consequences assuming that the Lagrangian $L$ has finitely many static classes. In this manner, by previous corollary, the properties presented here are generic on set of Exact Magnetic for all cohomology class.

The projected Aubry set $\mathcal{A}_c$ is chain recurrent and the static classes are connected so they are the connected components of $\mathcal{A}_c$. Moreover the static classes are the chain transitive components of $\mathcal{A}_c$ and we obtain the following cycle property: If two supports of ergodic minimizing measures are contained in a static class, then there exists a cycle consisting of static curves in the same static class connecting them.

Contreras and Paternain prove in [5] that between two static classes there exists a chain of static classes connected by heteroclinic semistatic orbits. More precisely they show

Theorem 9 Suppose that the number of static classes is finite. Then given two static classes $\Lambda_k$ and $\Lambda_l$, there exist classes $\Lambda_1 = \Lambda_k, \Lambda_2, ..., \Lambda_n = \Lambda_l$ and $\theta_1, \theta_2, ..., \theta_{n-1} \in TM$ such that for all $i = 1, ..., n - 1$ we have that $\gamma_i(t) = \pi \circ \varphi_t(\theta_i)$ are semistatic curves, $\alpha(\theta_i) \subset \Lambda_i$ and $\omega(\theta_i) \subset \Lambda_{i+1}$.

Another important property, demonstrated by P. Bernard in [2], is the semi-continuity of the Aubry set

$$H^1(M, \mathbb{R}) \ni c \mapsto \tilde{A}_c,$$

when $\mathcal{A}_M$ is finite. In order to be more precise he showed the following Theorem
Theorem 10 Let $L_k$ be a sequence of Tonelli Lagrangians converging to $L$. Then given a neighborhood $U$ of $\tilde{A}_0$ in $TM$, there exists $k_0$ such that $\tilde{A}_0(L_k) \subset U$ for each $k \geq k_0$, where $\tilde{A}_0(L_k)$ is the Aubry set for the Lagrangian $L_k$.

In fact Bernard showed that this Theorem is true with a weaker hypothesis than $\mathcal{A}_M$ be finite, namely coincidence hypothesis (See [2]).

4 Example

In this section we present an example of a Exact Magnetic Lagrangian on flat torus $\mathbb{T}^2$ whose quotient Aubry set $\mathcal{A}_M$ is a Cantor set, therefore not every Exact Magnetic Lagrangian has finitely many static classes.

Let $L : T\mathbb{T}^2 \to \mathbb{R}$ be a Exact Magnetic Lagrangian defined by

$$L(x, y, v_1, v_2) = \frac{\| (v_1, v_2) \|^2}{2} + \langle (0, f(x)), (v_1, v_2) \rangle,$$

where $f$ is a $C^2$ nonpositive and periodic function whose set of minimum points $\Gamma_{\min}$ is a Cantor set and $f|_{\Gamma_{\min}}$ is a negative constant.

In this case the system of Euler-Lagrange is given by

\[
\begin{cases}
\dot{x} = v \\
\dot{v} = -f'(x) Jv
\end{cases}
\]

where $J$ is the canonical sympletic matrix.

Lemma 11 The Mañé’s critical values of $L$ is $\alpha(0) = f(a)^2/2$, where $a \in \Gamma_{\min}$. Moreover, the closed curves $\gamma_a$ defined by $\gamma_a(t) = (a, -f(a)t)$, are static curves.

Proof: Given any curve $\beta(t) = (x(t), y(t))$ on $\mathbb{T}^2$, we have

\[
L(\beta, \dot{\beta}) = \frac{\dot{x}^2 + \dot{y}^2}{2} + f(x) \dot{y}(t) = \frac{(\dot{y} + f(x))^2 + \dot{x}^2}{2} - \frac{f(x)^2}{2} \geq -\frac{f(a)^2}{2}. \tag{3}
\]

Then

\[
\mathbb{A}_{L+f(a)^2/2}(\beta) = \int_0^T \left( L(\beta, \dot{\beta}) + \frac{f(a)^2}{2} \right) dt \geq 0,
\]

and we obtain $\alpha(0) \leq \frac{f(a)^2}{2}$. Observe that if $0 < k \leq \frac{f(a)^2}{2}$, the closed curve given by $\gamma_k(t) = \left(a, \sqrt{2kt}\right)$, where $a \in \Gamma_{\min}$, is Euler-Lagrange solution and its energy is $E = k$. Moreover,

\[
\mathbb{A}_{L+k}(\gamma_k) = \int_0^T (L(\gamma_k, \dot{\gamma_k}) + k) dt = \int_0^T \left(2k + f(a) \sqrt{2k}\right) dt.
\]

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Therefore
\[ A_{L+k}(\gamma_k) < 0 \text{ if } k < \frac{f(a)^2}{2} \quad \text{and} \quad A_{L+k}(\gamma_k) = 0 \text{ if } k = \frac{f(a)^2}{2}. \]

This shows that \( \alpha(0) = \frac{f(a)^2}{2} \) and the curve \( \gamma_a \) is semistatic, i.e., realizes the action potential. Since \( \gamma_a \) is a semistatic closed curve, it is static curve.

To complete the example, it suffices to show that the application \( \Psi : \Gamma_{\text{min}} \rightarrow \mathcal{A}_M \), given by \( \Psi (a) = [(a,0)] \), where \( [(a,0)] \) is a representative of the static class containing the curve \( \gamma_a \), is a Lipschitz bijection. In fact, since the action potential \( \Phi_{L+\alpha(0)} \) is Lipschitz, the distance \( \delta_M \) on quotient Aubry set \( \mathcal{A}_M \) also is Lipschitz.

In order to show the surjectivity of \( \Psi \) it is enough to show that the projected Aubry set \( \mathcal{A}_0 \) is exactly the union of the closed curves \( \gamma_a \) with \( a \in \Gamma_{\text{min}} \). Suppose that there exists \( p \in \mathcal{A}_0 \) such that \( \pi (p) \notin \Gamma_{\text{min}} \), where \( \pi \) is the canonical projection of \( \mathbb{T}^2 \) on \( \mathbb{R}/\mathbb{Z} \). Then there exists a neighborhood \( V_p \) of \( p \) such that \( f(x) > f(a) \) for all \( a \in \Gamma_{\text{min}} \) and \( x \in V_p \). Let \( \gamma \) be a piece, contained in \( V_p \), of the static curve passing through \( p \).

The inequality \( 3 \) implies \( A_{L+\alpha(0)}(\gamma) > 0 \). Moreover, it follows by inequality \( 3 \) which the action \( L + \alpha(0) \) of any curve is nonnegative, so \( \Phi_{L+\alpha(0)}(x,y) \geq 0 \) for all \( x,y \in \mathbb{T}^2 \). Then
\[ A_{L+\alpha(0)}(\gamma) = \Phi_{L+\alpha(0)}(\gamma(0),\gamma(T)) = -\Phi_{L+\alpha(0)}(\gamma(0),\gamma(T)) \leq 0. \]

This is a contradiction.

If \( \Psi \) is not injective there exists \( b \in \Gamma_{\text{min}}, b \neq a \) such that \( (b,0) \in [(a,0)] \). Since each static class is connected (See G. Contreras and G. Paternain in [3], Proposition 3.4) and \( b \in \pi ([(a,0)]) \) we have that \( \pi ([(a,0)]) \subset \mathbb{R}/\mathbb{Z} \) is connected so it is an interval. By total disconnectedness of \( \Gamma_{\text{min}} \), there exists \( q \in \pi ([(a,0)]) - \Gamma_{\text{min}} \). The contradiction follows of the inequality \( 3 \) by same argument above.

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