THE PICARD GROUP OF THE CATEGORY OF $C_n$-EQUIVARIANT STABLE HOMOTOPY THEORY

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Abstract. For a finite group $G$, there is a map $RO(G) \to \text{Pic}(Sp^G)$ from the real representation ring of $G$ to the Picard group of $G$-spectra. This map is not known to be surjective in general, but we prove that when $G$ is cyclic this map is indeed surjective and in that case we describe $\text{Pic}(Sp^G)$ explicitly.

We also show that for an arbitrary finite group $G$ homology and cohomology with coefficients in a cohomological Mackey functor do not see the part of $\text{Pic}(Sp^G)$ coming from the Picard group of the Burnside ring. Hence these homology and cohomology calculations can be graded on a smaller group.

1. Introduction

The Picard group of the category of $G$-spectra is the group of isomorphism classes of invertible $G$-spectra (in the homotopy category) under the smash product. It has been studied by tom Dieck, Petrie, Fausk, Lewis, and May among others. See for example \cite{8, 9, 10, 11, 12, 3}.

There is a natural map $RO(G) \to \text{Pic}(Sp^G)$ from the real representation ring of $G$ to the Picard group of $G$-spectra, sending a virtual representation to the isomorphism class of the corresponding virtual representation sphere. We prove the following result:

**Theorem A.** Let $C_n$ denote the cyclic group of order $n$. The map

$$RO(C_n) \to \text{Pic}(Sp^{C_n})$$

is surjective, and

$$\text{Pic}(Sp^{C_n}) \cong \prod_{d|n} (\mathbb{Z}/d)^{\times}/\{\pm 1\} \times \prod_{d|n} \mathbb{Z}.$$ 

Here the first factor comes from Pic of the Burnside ring of $C_n$ and the second factor is related to the dimension of the virtual representation.

The Picard group provides a natural home for grading equivariant stable homotopy groups. Making the notion of Pic($Sp^G$)-graded homotopy groups of a $G$-spectrum precise depends on some choices, but Dugger \cite{2} proves that it is possible to make coherent choices and obtain well defined homotopy groups.

This means that rather than grading $C_n$-equivariant stable homotopy groups on $RO(C_n)$ we can grade them on the much smaller group $\text{Pic}(Sp^{C_n})$. When considering homology or cohomology with coefficients in a cohomological Mackey functor, further simplifications are possible.

**Theorem B.** Let $G$ be a finite group, suppose $[X]$ is in the kernel of $d : \text{Pic}(Sp^G) \to C(G)$, and suppose $M$ is a cohomological Mackey functor. Then

$$X \wedge H M \simeq H M.$$
Here $C(G)$ is the abelian group of functions from the set of conjugacy classes of subgroups of $G$ to $\mathbb{Z}$ under pointwise addition. By [3, Theorem 0.1] we have an exact sequence

$$0 \to \text{Pic}(A(G)) \to \text{Pic}(Sp^G) \xrightarrow{d} C(G),$$

where $A(G)$ is the Burnside ring of $G$. We can interpret Theorem B as saying that homology and cohomology with coefficients in a cohomological Mackey functor do not see the part of $\text{Pic}(Sp^G)$ coming from $\text{Pic}(A(G))$.

**Example 1.1.** Consider the cyclic group $C_p$ for an odd prime $p$. Then $RO(C_p) \cong \mathbb{Z}^{(p-1)/2}$. Theorem A says that instead of grading $C_p$-equivariant stable homotopy groups on $\mathbb{Z}^{(p-1)/2}$ we can grade them on $(\mathbb{Z}/p)^\times/\{\pm 1\} \times \mathbb{Z}^2$. If $p = 3$ there is no difference, but for large $p$ the group $\text{Pic}(Sp^{C_p})$ is much smaller.

Moreover, Theorem B says that homology or cohomology groups with coefficients in a cohomological Mackey functor can be graded on just $\mathbb{Z}^2$ regardless of the prime.

**Remark 1.2.** In the existing literature the reduction from grading on $RO(G)$ to grading on a smaller group is usually done after localizing at $p$. See for example [5, Prop. 2.1] (and the sentence just before that, which emphasizes the $p$-local context).

1.1. **Conventions.** We assume that the reader is familiar with the theory of Mackey functors and with equivariant stable homotopy theory. Any category of “genuine” $G$-equivariant spectra with the usual homotopy category will do. For concreteness, we will work in the category of orthogonal $G$-spectra as described in [4].

We will use notation like $M$ for a Mackey functor, and write $M(G/H)$ for the value of $M$ at the subgroup $H$. Given subgroups $K, L \leq H \leq G$ we have a transfer map $\text{tr}_K^H : M(G/K) \to M(G/H)$ and a restriction map $\text{R}_L^H : M(G/H) \to M(G/L)$. The composite $\text{R}_L^H \circ \text{tr}_K^H$ can be computed using a “double coset formula”. Each $M(G/H)$ also has an action of $N_H(G)/H$, and there are conjugation maps between $M(G/H)$ and $M(G/Hg^{-1})$, but these will play no role in the current paper. See for example [13] for details.

1.2. **Organization.** In Section 2 we compute $\text{Pic}(A(C_n))$ and in Section 3 we complete the proof of Theorem A.

In Section 4 we describe explicit representatives of $\text{Pic}(A(C_n))$, and in Section 5 we describe explicit $C_n$-spectra representing $\text{Pic}(Sp^{C_n})$.

In Section 6 we explain how our choices from the previous two sections give a $\text{Pic}(Sp^{C_n})$-trivialisation of $Sp^{C_n}$ in the sense of Dugger [2], and we explain in what sense $\text{Pic}(Sp^{C_n})$-graded stable homotopy groups of a homotopy commutative $G$-spectrum are “graded commutative”.

Finally, in Section 7 we prove Theorem B. This turns out to boil down to a calculation of the Picard group of $H\mathbb{Z}$.

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2. **The Picard group of the Burnside ring**

Much of the material in this section can be found in [9] and [12], using slightly different language. But we have been unable to find the exact calculations we need in the literature. Let $G$ be a finite group. Suppose $G$ has $r$ conjugacy classes of
subgroups represented by $H_1, \ldots, H_r$. If we need to be explicit about it we will let $H_1 = G$ and $H_r = e$.

We let $C(G) \cong \mathbb{Z}^r$ denote the abelian group of functions from the set of conjugacy classes of subgroups to $\mathbb{Z}$ under pointwise addition. In fact it will be convenient to consider $C(G)$ as a ring under pointwise addition and multiplication.

Let $A(G)$ denote the Burnside ring of $G$, which as an abelian group is free on $[G/H_1], \ldots, [G/H_r]$. The multiplication is given by decomposing $G/H_i \times G/H_j$ into orbits, and $[G/H_1] = [G/G]$ is the multiplicative identity. Let

$$w : A(G) \to C(G)$$

be the Burnside ghost map, given on a finite $G$-set $X$ by $X \mapsto (H_i \mapsto |X^{H_i}|)$, or $X \mapsto (|X^{H_1}|, \ldots, |X^{H_r}|)$. This map is injective, and the ring structure on $A(G)$ is determined by requiring that $w$ is a ring map.

**Remark 2.1.** Recall from [1] Corollary 2) that the Burnside ring $A(G)$ is isomorphic to the ring $W_G(\mathbb{Z})$ of $G$-Witt vectors of $\mathbb{Z}$. If we represent an element in $A(G)$ by $(a_1, \ldots, a_r)$ where $a_i$ denotes the coefficient of $[G/H_i]$ then the Burnside ghost map is given by $(a_1, \ldots, a_r) \mapsto \langle x_1, \ldots, x_r \rangle$ with

$$x_i = \sum |(G/H_j)^{H_i}| a_j,$$

with the sum being over all $j$ so that $H_i$ is subconjugate to $H_j$. This can be compared to the Witt vector ghost map, which is given by the same formula except with

$$x_i = \sum |(G/H_j)^{H_i}| a_j^{[H_i:H_j]}.$$

(Here $[H_i : H_j]$ denotes the index of the appropriate conjugate of $H_i$ in $H_j$.)

Let $X = \text{Spec}(C(G))$, let $Y = \text{Spec}(A(G))$, and let $f : X \to Y$ be Spec of the Burnside ghost map. If we complete at a prime $p$ which does not divide $|G|$, $f$ is an isomorphism. Now we can consider the following short exact sequence of sheaves of abelian groups on $Y$, with $Q$ defined as the quotient:

$$0 \to \mathcal{O}_Y^\times \to f_*\mathcal{O}_X^\times \to Q \to 0.$$

The sheaf $Q$ is concentrated at those $p$ which divide $|G|$. Because $f$ is finite, each $Q_p^\wedge$ is finite and it follows that the global sections of $Q$ is a finite abelian group.

**Theorem 2.2** (tom Dieck [9] Section 4). For any finite group $G$, the Picard group of $A(G)$ sits in a 4-term exact sequence

$$0 \to A(G)^\wedge \to (\mathbb{Z}^r)^\wedge \to \Gamma(Y, Q) \to \text{Pic}(A(G)) \to 0.$$

**Corollary 2.3.** For any finite group $G$ the Picard group $\text{Pic}(A(G))$ is finite.

tom Dieck goes on to discuss a procedure for computing $\Gamma(Y, Q)$ when $G$ is abelian, but we can actually compute $\Gamma(Y, Q)$ in general. Because $Q$ is concentrated at a finite set of primes, $\Gamma(Y, Q) \cong \prod_p Q_p$ is a product of stalks. And the stalk $Q_p$ can be computed as the cokernel

$$(A(G) \otimes \mathbb{Z}_p)^\wedge \overset{w}{\to} (\mathbb{Z}_p^r)^\wedge \to Q_p.$$

Now we put this together with the “Cauchy-Frobenius-Burnside” relations, which say that $x$ is in the image of the Burnside ghost map if and only if, for each $i$, the
Proof. We have one subgroup $d$ image of this group of units under the Burnside ghost map is the diagonal from Theorem 2.4.

Theorem 2.4. The Picard group of the Burnside ring $A(C_n)$ is given by

$$\text{Pic}(A(C_n)) \cong \prod_{d \mid n, d \neq 1, 2} \left( \mathbb{Z}/(C_n/C_n/d) \right)^\times \cong \prod_{d \mid n} \left( \mathbb{Z}/d \right)^\times.$$ 

Proof. We have one subgroup $C_n/d \leq C_n$ for each $d \mid n$, and we get

$$\Gamma(Y, Q) \cong \prod_{d \mid n} \left( \mathbb{Z}/(C_n/C_n/d) \right)^\times \cong \prod_{d \mid n} \left( \mathbb{Z}/d \right)^\times.$$ 

We can omit the $d = 1$ and $d = 2$ factors, as these are trivial.

If $n$ is odd we have $A(C_n)^{\times} \cong \{ \pm 1 \}$ with the $-1$ represented by $-\gamma C_n/C_n$. The image of this group of units under the Burnside ghost map is the diagonal $\{ \pm 1 \}$, so if $n$ has $r$ divisors the map $A(C_n)^{\times} \to (\mathbb{Z}^r)^\times$ is the diagonal map $\{ \pm 1 \} \to \{ \pm 1 \}^r$. The cokernel of this map is isomorphic to $\{ \pm 1 \}^r$, and $\text{Pic}(A(C_n)) \cong \Gamma(Y, Q)/\{ \pm 1 \}^r$ with each $\{ \pm 1 \}$ acting on a single $(\mathbb{Z}/d)^\times$ for $d \neq 1$. This gives the result for $n$ odd.

If $n$ is even we have $A(C_n)^{\times} \cong \{ \pm 1 \}^2$ with generators represented by $-1 = -\gamma C_n/C_n$ and $\tau = [C_n/C_n/2] - [C_n/C_n]$. The image of $-1$ under the Burnside ghost map is $C_d \mapsto -1$ for all $d \mid n$, and the image of $\tau$ is $C_d \mapsto \begin{cases} 1 & \text{if } d \mid n/2 \\ -1 & \text{if } d \nmid n/2 \end{cases}$

The cokernel of this map is isomorphic to $\{ \pm 1 \}^{r-2}$ with each $\{ \pm 1 \}$ acting on a single $(\mathbb{Z}/d)^\times$ for $d \neq 1, 2$. This gives the result for $n$ even. 

This completes the description of the factor $\prod_{d \mid n} \left( \mathbb{Z}/d \right)^\times/\{ \pm 1 \}$ in the expression for $\text{Pic}(Sp^{C_n})$ in Theorem A.

One can, with some difficulty, give an explicit description of an invertible $A(C_n)$-module representing each element of $\text{Pic}(A(C_n))$. But we will wait until Section 3 to do so, as the perspective of Mackey functors makes the description more intuitive. In Section 5 we find an explicit $C_n$-spectrum representing each element of $\text{Pic}(A(C_n))$.

3. The image of $d : \text{Pic}(Sp^{C}) \to C(G)$

Recall from III.5.1 that $f \in C(G)$ is a Borel-Smith function if it satisfies a certain list of congruences.
For any invertible \( X \in \text{Sp}^G \), \( d(X) \) is a Borel-Smith function, and by [11, Theorem III.5.4] the map \( d \circ j \cdot \text{RO}(G) \to \text{C}(G) \) surjects onto the subgroup of Borel-Smith functions when \( G \) is nilpotent. In particular this holds when \( G \) is abelian.

Because the image of \( d \) is a subgroup of a free abelian group, it is itself free and after choosing a basis we get a short exact sequence

\[
0 \to \text{Pic}(A(G)) \to \text{Pic}(\text{Sp}^G) \to \mathbb{Z}^r \to 0
\]

for some \( r \). Here \( r \) could be smaller than the number of conjugacy classes of subgroups of \( G \). When \( G \) is nilpotent \( r \) is equal to the number of conjugacy classes of cyclic subgroups of \( G \).

This can be completed to the following diagram with exact rows:

\[
\begin{array}{ccc}
0 & \to & \text{RO}_0(G) \\
\downarrow & & \downarrow \\
0 & \to & \text{Pic}(A(G))
\end{array}
\to
\begin{array}{ccc}
\text{RO}(G) & \to & \text{RO}(G) \\
\downarrow & & \downarrow \approx \\
\text{Pic}(\text{Sp}^G) & \to & \mathbb{Z}^r \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

If we choose a splitting we get

\[
\text{Pic}(\text{Sp}^G) \cong \text{Pic}(A(G)) \times \mathbb{Z}^r,
\]

and every element in the second factor (or more precisely every element \((1, b)\) where \(1 \in \text{Pic}(A(G))\) is the identity element) is represented by a virtual representation sphere.

We once again specialise to \( G = C_n \). Recall that the irreducible real representations of \( C_n \) are the trivial representation \( 1 \), the sign representation \( \sigma \) if \( n \) is even, and the 2-dimensional representations \( \lambda(i) \) for \( i = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor \) with \( z \in C_n \subset \mathbb{C} \) acting by multiplication by \( z^i \).

First, \( C(C_n) \) is a free abelian group of rank equal to the number of divisors of \( n \). The subgroup of Borel-Smith functions is the finite index subgroup generated by the functions \( \{ f_d \}_{d \mid n} \) defined as follows: We have \( f_1(H) = 1 \) for all \( h \), and if \( 2 \mid n \) we have \( f_2(H) = 1 \) for \( H \leq C_{n/2} \) and zero otherwise. For \( d \neq 1, 2 \) we have \( f_d(H) = 2 \) for \( H \leq C_{n/d} \) and zero otherwise. We will write \( \prod_{d \mid n} \mathbb{Z} \) for the subgroup of Borel-Smith functions, and observe that these are exactly the dimension functions of the irreducible \( C_n \)-representations. (But note that multiple irreducible representations can have the same dimension function.) We then have a short exact sequence

\[
0 \to \text{Pic}(A(C_n)) \to \text{Pic}(\text{Sp}^C_{C_n}) \to \prod_{d \mid n} \mathbb{Z} \to 0.
\]

Now we go ahead and choose the following splitting \( \prod_{d \mid n} \mathbb{Z} \to \text{Pic}(\text{Sp}^C_{C_n}) \): For \( b = (b_d) \in \prod_{d \mid n} \mathbb{Z} \) we let

\[
Y_b = S^{b_1} \wedge S^{b_2} \sigma \wedge \bigwedge_{d \mid n, d \neq 1, 2} S^{b_d} \lambda(n/d).
\]

Here we omit the smash factor \( S^{b_2} \sigma \) if \( n \) is odd.

This completes the description of the factor \( \prod_{d \mid n} \mathbb{Z} \) in the expression for \( \text{Pic}(\text{Sp}^C_{C_n}) \) in Theorem 5 and together with Section 2 this completes the proof of Theorem 5.
4. The Picard group of Mackey functors

There is another natural notion of a Picard group in $G$-equivariant stable homotopy theory. Recall, e.g. from [13], that the category of $G$-Mackey functors is a symmetric monoidal category under the box product $\boxtimes$ with unit the Burnside Mackey functor $\mathbb{A}_G$. We can then consider the group $\text{Pic}(\text{Mack}_G)$ of isomorphism classes of invertible $G$-Mackey functors under $\boxtimes$.

**Theorem 4.1.** For any finite group $G$ we have a natural isomorphism $\text{Pic}(\mathbb{A}(G)) \cong \text{Pic}(\text{Mack}_G)$.

**Proof.** We have an adjunction $F : \text{Mod}_{\mathbb{A}(G)} \rightleftarrows \text{Mack}_G : U$.

Here $F$ is defined by $F(M)(G/H) = A(H) \otimes_{A(G)} M$, with restriction and transfer maps induced by the restriction and transfer maps between Burnsides rings. The functor $U$ is defined by $U(N) = N(G/G)$, with $A(G)$-module structure given by $[G/H] \cdot x = \text{tr}_H^G(R_H^K(x))$.

The composite $U \circ F$ is naturally isomorphic to the identity, while $F \circ U(N)$ picks out the sub-Mackey functor of $N$ with $F \circ U(N)(G/H)$ generated by $\text{tr}_K^H(R_K^K(x))$ for $x \in N(G/G)$ and $K \leq H$.

Both functors map invertible objects to invertible objects, and $F \circ U$ is naturally isomorphic to the identity when restricted to invertible Mackey functors, so $F$ and $U$ induce inverse isomorphisms on Picard groups.

Next we describe a certain family of $C_n$-Mackey functors, starting with the Burnside Mackey functor $\mathbb{A} = \mathbb{A}_{C_n}$. As abelian groups we have $\mathbb{A}(C_n/C_m) = \mathbb{A}(C_m) = \mathbb{Z}\{x_d^n\}_{d|m}$ with $x_d^n$ representing the finite $C_m$-set $C_m/C_m/d$ or cardinality $d$. The transfer maps are given by $\text{tr}^k : A(C_m) \to A(C_{km})$ $\text{tr}^k(x_d^m) = x_{km}^m$, and the restriction maps are determined by $R^k(x_1^n) = x_1^{n/k}$ for all $k \mid n$. The action of $C_n/C_m$ on $\mathbb{A}(C_n/C_m)$ is trivial.

**Definition 4.2.** Suppose $a = (a_d) \in \prod_{d \mid n} \mathbb{Z}$ satisfies $a_e \mid a_d$ for $e \mid d$, and $a_d \neq 0$ for each $d$. Let $\mathbb{A}_a$ denote the Mackey functor defined in the same way as $\mathbb{A}$, except with restriction maps determined by $R^k(x_1^n) = a_k x_1^{n/k}$.

**Remark 4.3.** The remaining restriction maps are determined as follows: For a generator $x_1^{n/k}$ we use that $R^{kt} = R^t \circ R^k$ to conclude that $R^t(x_1^{n/k}) = \frac{a_{n/k}}{a_1} x_1^{n/kt}$. Any other generator $x_d^m$ for $d > 1$ is in the image of some transfer map, and in this case $R^k(x_d^m)$ is determined by how restriction and transfer maps commute.

In general the box product of Mackey functors is difficult to compute, but for this particular family of Mackey functors we have the following:

**Lemma 4.4.** Given Mackey functors $\mathbb{A}_a$ and $\mathbb{A}_b$ as defined above we have $\mathbb{A}_a \boxtimes \mathbb{A}_b \cong ab \mathbb{A}$.
where \((ab)_d = a_d b_d\).

**Proof.** We have

\[
\text{A}^a \circ \text{A}^b(C_n/C_m) = \text{A}^a(C_n/C_m) \otimes \text{A}^b(C_n/C_m) \oplus \text{Im}(\text{tr}) / \sim,
\]

where the equivalence relation is given by Frobenius reciprocity. If we denote the basis for \(\text{A}^a(C_n/C_m)\) by \(\{x^m\}\) and the basis for \(\text{A}^b(C_n/C_m)\) by \(\{y^m\}\), Frobenius reciprocity tells us that

\[
x^m_q \otimes y^m_r \sim \text{tr}^q(x^m/q \otimes R^q(y^m)) \quad \text{for } q > 1
\]

\[
x^m_q \otimes y^m_r \sim \text{tr}^r(R^q(x^m_q) \otimes y^m_{rq}) \quad \text{for } r > 1
\]

Hence \(\text{A}^a \circ \text{A}^b(C_n/C_m) \cong \mathbb{Z}\{x^m_1 \otimes y^m_1\} \oplus \text{Im}(\text{tr})\), and because \(R^k(x^m_1 \otimes y^m_1) \cong R^k(x^m_1) \otimes R^k(y^m_1)\) the result follows. \(\square\)

We would like to understand when \(\text{A}^a\) is invertible and when \(\text{A}^a \cong \text{A}^b\). To do this we can study what happens when we do a change of basis, for example by replacing \(x^n_1\) by \(x^n_1 + \sum_{d > 1} c_d x^n_d\) for certain \(c_d \in \mathbb{Z}\).

But before we do that, we restrict our attention to those \(a\) for which \(a_r | a_s\) for all relatively prime \(r\) and \(s\).

**Definition 4.5.** Let \(a = (a_d) \in \prod_{d | n} \mathbb{Z}\) with each \(a_d \neq 0\). Let \(\text{A}^a\) denote the Mackey functor defined in the same way as \(\text{A}\), except with restriction maps determined by

\[
R^k(x^n_1) = \left( \prod_{d | k} a_d \right) x^{n/k}_1.
\]

Any Mackey functor \(\text{A}^a\) is of the form \(\text{A}^b\) for \(b_k = \prod_{d | k} a_d\), but the reverse is not true. Note that we still have

\[
\text{A}^a \circ \text{A}^b \cong \text{A}^{ab}.
\]

**Proposition 4.6.** Given tuples \(a = (a_d)\) and \(b = (b_d)\) we have the following:

1. Suppose \(a_d \equiv b_d \mod d\) for each \(d\). Then \(\text{A}^a \cong \text{A}^b\).
2. Suppose \(a_d = \pm b_d\) for each \(d\). Then \(\text{A}^a \cong \text{A}^b\).
3. Suppose \(\text{A}^a \cong \text{A}^b\). Then \(a_d \equiv \pm b_d \mod d\) for each \(d\).

It follows that \(\text{A}^a\) is invertible if and only if each \(a_d\) is invertible in \(\mathbb{Z}/d\), and in that case the isomorphism class of \(\text{A}^a\) is determined by the equivalence class of \(a_d\) in \((\mathbb{Z}/d)^\times / \{ \pm 1 \}\) for each \(d\). By Theorem 1 and Theorem 4.1 we know how many isomorphism classes of invertible \(C_n\)-Mackey functors we are supposed to have, and because the above result describes the same number of invertible \(C_n\)-Mackey functors it follows that these are all of them.

**Proof.** We perform an appropriate change of basis. If we start with \(\text{A}^a\) and we want to add \(k\) to \(a_k\), we replace \(x^n_1\) by

\[
y^n_1 = x^n_1 + \left( \prod_{d | k, d \neq k} a_d \right) x^n_k
\]
By the equivariant Hurewicz theorem [6, Theorem 2.1] it then suffices to show that 
\( \pi_1(X) \) is in the kernel of \( \tau \) for appropriate \( m \).

For the last part, note that an isomorphism can be interpreted as a change of basis, and that any change of basis from \( \{ x_q^m \} \) to \( \{ y_q^m \} \) with the two properties that the description of the transfer maps remains intact and that \( R^k(y_q^m) \) is a multiple of \( y_1^{n/k} \) must be a combination of the changes of basis described above.

**Remark 4.7.** For any \( n \) there is one change of basis that does not change \( a = (a_d) \), namely replacing each \( x_q^m \) by \( -x_q^m \). This corresponds to the unit \( -1 = [C_n/C_n] \) in \( \text{A}(C_n) \).

If \( n \) is even there is one other change of basis that does not change \( a = (a_d) \), given by \( y_q^1 = -x_1^1 + x_2^1 \). This corresponds to the unit \( \tau = [C_n/C_{n/2}] - [C_n/C_n] \) in \( \text{A}(C_n) \).

Any element of \( \text{Pic}(\text{Mack}_n) \) is represented by \( A^a \) with \( 1 \leq a_d < \frac{d}{2} \) in a unique way. Given \( a = (a_d) \in \prod_{d|n, d \neq 1,2} \mathbb{Z} \) with each \( a_d \) relatively prime to \( d \), let \( \hat{a} \) denote the representative of the equivalence class of \( a \) in \( \prod_{d|n, d \neq 1,2} (\mathbb{Z}/d)\times/\{\pm 1\} \) with \( 1 \leq a_d < \frac{d}{2} \).

Given \( a \), we have multiple isomorphisms \( A^a \to \hat{A}^\hat{a} \). The set of such isomorphisms is in bijection with \( \text{Aut}(A) \). As observed in the proof of Theorem 2.4, for \( n \) odd we have \( \text{Aut}(A) \cong \mathbb{Z}/2 \) and for \( n \) even we have \( \text{Aut}(A) \cong \mathbb{Z}/2 \times \mathbb{Z}/2 \).

We can pick out a preferred isomorphism as follows. Let us denote the above basis for \( A^a \) by \( \{ x_q^m \} \) and the basis for \( \hat{A}^\hat{a} \) by \( \{ y_q^m \} \). If \( n \) is odd we insist that \( x_1^1 \to y_1^1 \) (as opposed to \( -y_1^1 \)) and this uniquely determines the isomorphism. If \( n \) is even we also insist that the isomorphism sends \( x_1^1 \) to \( y_1^1 + \sum_{d \geq 2} c_d y_d^1 \) (as opposed to \( -y_1^1 \)).

We will get back to this in Section 5.

5. A \( \mathcal{A}_0 \) Calculation

In this section we find explicit representatives for the \( \text{Pic}(\text{A}(C_n)) \) factor of \( \text{Pic}(\text{Sp}^C) \) from Theorem [A]

**Theorem 5.1.** Let \( X = S^{\lambda(n/d) - \lambda(an/d)} \) for some \( a \) relatively prime to \( d \). Then \( \mathcal{A}_n X = 0 \) for any integer \( n \neq 0 \) and \( \mathcal{A}_n X \cong \hat{A}^{\hat{a}} \) for \( \hat{a} = (a_e) \) with \( a_e = \begin{cases} a & e = d \\ 1 & \text{otherwise} \end{cases} \).

**Proof.** Because \( X \) is invertible and \( [X] \) is in the kernel of \( d : \text{Pic}(\text{Sp}^C) \to C(C_n) \), we already know that \( \mathcal{A}_n X \) is an invertible \( A \)-module and that \( \mathcal{A}_n X = 0 \) for \( n \neq 0 \). By the equivariant Hurewicz theorem [6, Theorem 2.1] it then suffices to show that \( H_0(X; \hat{A}) \cong \hat{A}^{\hat{a}} \).
For ease of notation, let $d' = n/d$. We give $S^{\lambda(d')}$ a $C_n$-CW structure with a 0-cell $C_n/C_n$, a 1-cell $C_n/C_{d'}$, and a 2-cell $C_n/C_{d'}$. Similarly, we give $S^{-\lambda(a,d')}$ a $C_n$-CW structure with a 0-cell $C_n/C_n$, a $(1)$-cell $C_n/C_{d'}$, and a $(2)$-cell $C_n/C_{d'}$.

We then get a $C_n$-CW structure on the smash product by taking a product of the two $C_n$-CW structures, and we get a chain complex which computes $H_*(X;A)(C_n/C_n)$ by taking the total complex of the following double complex:

\[ A(C_n/C_n) \leftarrow A(C_n/C_{d'}) \leftarrow A(C_n/C_{d'}) \]

\[ A(C_n/C_{d'}) \leftarrow A(\prod C_n/C_{d'}) \leftarrow A(\prod C_n/C_{d'}) \]

\[ A(C_n/C_{d'}) \leftarrow A(\prod C_n/C_{d'}) \leftarrow A(\prod C_n/C_{d'}) \]

Here we have identified $C_n/C_{d'} \times C_n/C_{d'}$ with $\prod C_n/C_{d'}$. If we want to compute $H_*(X;A)(C_n/C_n)$ we can take the product of each of these $C_n$-sets with $C_n/C_n$ before evaluating.

We know what every entry is, and we can compute all the maps. Evaluated on $C_n/C_n$ we get the following:

\[ \mathbb{Z}\{x^n_q\} \xrightarrow{tr^d} \mathbb{Z}\{x^{d'}_q\} \xrightarrow{0} \mathbb{Z}\{x^d_q\} \]

\[ \Delta \]

\[ \nabla \]

\[ \mathbb{Z}\{x^{d'}_q\} \xrightarrow{\nabla} \mathbb{Z}\{x^d_q(j)\}_{0 \leq j < d} \xrightarrow{1-s} \mathbb{Z}\{x^d_q(j)\}_{0 \leq j < d} \]

\[ \mathbb{Z}\{x^d_q\} \xrightarrow{0} \mathbb{Z}\{x^d_q(j)\}_{0 \leq j < d} \xrightarrow{1-s} \mathbb{Z}\{x^d_q(j)\}_{0 \leq j < d} \]

Here we consider all $q$ which divide $n$ in the upper left corner and all $q$ which divide $d'$ in the rest of the diagram. The maps labelled $\nabla$ are fold maps, the maps labelled $\Delta$ are diagonal maps, $s$ shifts the index $j$ by 1 (modulo $d$) and $sh_a$ shifts the index $j$ by $a$ (modulo $d$).

We can compute the homology of the total complex using a small spectral sequence. If we first take homology with respect to the horizontal maps and then with respect to the vertical maps we find that $H_0(X;A) \cong \prod_{q \mid n} \mathbb{Z}$ as expected, with two types of generators:

**Type 1:** If $d \not\mid q$, the generator $x^n_q \in A(C_n/C_n)$ from the upper left corner survives the spectral sequence.

**Type 2:** If $d \mid q$, the diagonal entry $\sum_{0 \leq j < d} x^d_q(j)$ for $q' = q/d$ from the lower right corner survives the spectral sequence.

The type 2 generators are cycles in the double complex, but the type 1 generators are not because $R^{d'}$ applied to one of the type 1 generators is nonzero.
It suffices to consider the generator \( x^n_1 \) in the upper left corner. We can find an actual generator represented by \( x^n_1 \) by considering the following zig-zag:

\[
\begin{array}{c}
\vdots \\
\downarrow \\
\bullet - \bullet (1) \\
\downarrow \\
\sum_{0 \leq j < a} x^d_1(j) \\
\end{array}
\]

In other words, the generator represented by \( x^n_1 \) in the upper left hand corner in the spectral sequence is the sum of \( x^n_1 \) in the upper left, \( x^d_1(1) \) in the middle, and \( \sum_{0 \leq j < a} x^d_1(j) \) in the lower right hand corner.

Now we need to compute \( \text{Re} \) of this generator. If \( d \nmid e \) then \( \text{Re}(x^n_1) = x^{n/e}_1 \) in the upper left corner still survives the spectral sequence. So in this case \( \text{Re} \) does not pick up any additional factors.

If \( d \mid e \) then \( C_n/C_{d'} \times C_n/C_e \cong \prod_d C_n/C_e \) and the upper left transfer map becomes a fold map. Hence \( \text{Re}(x^n_1) = x^{n/e}_1 \) in the upper left corner is in the image of the horizontal transfer map, and we are left with \( \text{Re}(\sum_{0 \leq j < a} x^d_1(j)) \) in the lower right hand corner. Because we have \( a \) terms we now pick up a factor of \( a \). This completes the calculation, and hence the proof.

Given \( a = (a_d) \) we can then consider the smash product \( X_{\eta} = \bigwedge_{d \mid n} S^{\lambda(n/d) - \lambda(a_d n/d)} \) and we find that \( \pi_\bullet X \cong \mathcal{A}^a \).

### 6. Coherence

In \[2\], Dugger explains how it is possible to make coherent choices and obtain well defined \( \text{Pic}(Sp^G) \)-graded homotopy groups of a \( G \)-spectrum as well as an associative and “graded commutative” multiplication on \( \Sigma_\infty(S^0) \).

Here we put “graded commutative” in quotation marks because we have to keep track of units in \( A(G) \) rather than just \( \pm 1 \).

Let us mix additive and multiplicative notation in the Picard group by writing each \((\mathbb{Z}/d)^\times / \{\pm 1\}\) multiplicatively and each \( \mathbb{Z} \) additively.

For each \((a, b) \in \text{Pic}(Sp^G) \cong \prod_{d \mid n} (\mathbb{Z}/d)^\times / \{\pm 1\} \times \prod_{d \mid n} \mathbb{Z}\)

we define \( \pi \in \prod_{d \mid n} \mathbb{Z} \) and \( X_\pi \) as in Section \[3\] We also define \( Y_b \) as in Section \[3\]

As explained in \[2\] Section 7 we then need to define coherent maps

\[
X_{\pi a} \wedge Y_{b+b'} \rightarrow (X_\pi \wedge Y_b) \wedge (X_\pi \wedge Y_{b'})
\]
The coherence maps for the $Y_b$ are easiest: As in [2] Section 6], they are determined by the canonical maps $S^0 \to S^{-V} \wedge S^V$ for $V$ one of 1, $\sigma$, and $\lambda(n/d)$ for $d \mid n$ and $d \neq 1, 2$.

The coherence maps for the $X_\pi$ are slightly more complicated, but we already dealt with that in Section[3]. We specified a particular isomorphism $\Delta^a \otimes \Delta^{a'} \to \Delta^{a+a'}$ for each pair $(a, a')$ and a particular isomorphism $\Delta^a \to \Delta^b$ for each $a$. These then specify particular isomorphisms $X_a \wedge X_{a'} \to X_{a+a'}$ and $X_a \to X_\pi$.

With these choices, we then get particular maps

$$X_\pi \wedge X_{a'} \to X_{\pi \wedge a'} \to X_{\pi a'}.$$ 

For coherence we have to check that the two maps

$$X_\pi \wedge X_{a'} \wedge X_{a''} \to X_{\pi \wedge a' \wedge a''}$$

agree. But this is clear, because on $\mathbb{P}_n$ both send $x_1^1 \otimes y_1^1 \otimes z_1^1$ to $w_1^1$ and, if $n$ is even, both send $x_1^n \otimes y_1^n \otimes z_1^n$ to $w_1^n + \sum_{d \geq 2} c_d w_d^n$.

In general, the $G$-equivariant stable homotopy groups of spheres, or of a homotopy commutative $G$-ring spectrum, are graded commutative with units in the Burnside ring. The same is true for $\text{Pic}(Sp^G)$-graded homotopy groups:

**Theorem 6.1.** With notation as above, the $\text{Pic}(Sp^G)$-graded stable homotopy groups of spheres are graded commutative in the sense that if $x \in \pi^G_{a,b}(S)$ and $y \in \pi^G_{a',b'}(S)$ then

$$xy = (-1)^{b_1b_1'} yx$$

if $n$ is odd and

$$xy = (-1)^{b_1b_1'} (-\tau)^{b_2b_2'}$$

for $n$ even, where $\tau = [C_n/C_n/2] - [C_n/C_n]$.

**Proof.** This is folklore for $RO(C_n)$-graded homotopy groups. See e.g. [2] Prop. 1.18 for a related result. In the $RO(C_n)$-equivariant context we can compute the “signs” in the units of $A(C_n)$ by computing the trace of the identity element on each $S^V$ for $V$ irreducible. For $S^1$ with a trivial action we pick up the usual $-1$.

If $V = \lambda(i)$ for some $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ then the composite

$$S^0 \xrightarrow{\eta} S^{-V} \wedge S^V \xrightarrow{\lambda} S^V \wedge S^{-V} \xrightarrow{\eta} S^0$$

has degree 1 in the underlying non-equivariant category. This shows that $x_1^1 \mapsto x_1^1$ on $\mathbb{P}_n$. If $n$ is odd this suffices. If $n$ is even, we can also determine whether $x_1^n \mapsto x_1^n + \ldots$ or $x_1^n \mapsto -x_1^n + \ldots$ by taking geometric $C_n$-fixed points. But after taking geometric fixed points the above composite becomes

$$S^0 \to S^0 \wedge S^0 \xrightarrow{\lambda} S^0 \wedge S^0 \to S^0,$$

which again has degree 1 and not $-1$.

For $S^2$, the trace has degree $-1$ in the underlying non-equivariant category and degree 1 after taking geometric $C_n$-fixed points. This shows that $x_1^1 \mapsto -x_1^1$ and $x_1^n \mapsto x_1^n + \ldots$, which is exactly what multiplication by $-1$ does.

To complete the proof, it suffices to observe that none of the isomorphisms $X_a \to X_\pi$ do anything unexpected on $\mathbb{P}_n$. □
7. Cohomological Mackey functors and the Picard group

Recall that a Mackey functor $M$ is cohomological if $\text{tr}_K^H \circ R_K^H$ is multiplication by the index $[H : K]$ for all subgroups $K \leq H \leq G$.

When restricted to the subcategory of cohomological $G$-Mackey functors, $\square$ is a symmetric monoidal product with unit $\mathbb{Z}$ (rather than $\mathbb{A}_G$, which is not cohomological).

Proof of Theorem 7.4. Suppose $[X]$ is in the kernel of $d : \text{Pic}(Sp^G) \to C(G)$ and let $\underline{M}$ be a cohomological Mackey functor.

To show that $X \wedge H \underline{M} \simeq H \underline{M}$ it suffices to show that $\underline{\pi}_n(X \wedge H \underline{M}) \cong \underline{\pi}_n(X) \square \underline{M}$ is isomorphic to $\underline{M}$. Because $\underline{M}$ is cohomological we have $\underline{\pi}_n(X) \square \underline{M} \cong \underline{\pi}_n(X) \square H \underline{M}$, so it suffices to show that $\underline{\pi}_n(X) \square H \underline{M}$ is isomorphic to $\underline{Z}$.

Choose $Y$ with $X \wedge \simeq S_0^G$. Then $\underline{M}_1 = \underline{\pi}_n(X) \square \underline{Z}$ is invertible with inverse $\underline{M}_1 = \underline{\pi}_n(Y) \square \underline{Z}$. So it suffices to show that the only invertible $\underline{Z}$-module is $\underline{Z}$ itself. In other words, it suffices to show that the Picard group of the category of $\underline{Z}$-modules is trivial.

First observe that if $\underline{M}_1$ is an invertible $\underline{Z}$-module then $\underline{M}_1(G/H) \cong \underline{Z}$ for all $H \leq G$, and $\underline{M}_1$ is determined by the restriction maps $R_K^H : \underline{M}_1(G/H) \cong \underline{Z} \to \underline{M}_1(G/K) \cong \underline{Z}$ for $K \leq H \leq G$. These must satisfy $R_K^H | [H : K]$, plus a compatibility condition coming from the formula for $R_K^H \circ \text{tr}_K^H$ for each pair $K, L \leq H$.

We first consider the case when $G$ is a $p$-group. Let $P_1, \ldots, P_n$ denote the index $p$ subgroups. Denote the restriction map $\text{Mack}_G \to \text{Mack}_H$ by $\downarrow_{P_i}^G$. By induction we can assume $\downarrow_{P_i}^G \underline{M}_1 \cong \underline{Z}$. Then $R_{P_i}^G \downarrow_{P_i}^G$ is always multiplication by $p$, so either all the $R_{P_i}^G$ are $\pm 1$ and $\underline{M}_1 \cong \underline{Z}$ or all the $R_{P_i}^G$ are $\pm p$. But if all the $R_{P_i}^G$ are $\pm p$ then that will remain true in $\underline{M}_1 \square \underline{M}_1$, so $\underline{M}_1$ is not invertible.

Finally we consider an arbitrary finite group $G$ with Sylow subgroups $P_1, \ldots, P_n$ of order $p_1^{i_1}, \ldots, p_n^{i_n}$. Again we can assume by induction that $\downarrow_{P_i}^G \underline{M}_1 \cong \underline{Z}$ for each $i$. But then $R_{P_i}^G$ is multiplication by the same integer $N$ for each $i = 1, \ldots, n$. Now $N | p_i^{i_i}$ for each $i$, so $N = \pm 1$ and again we conclude that $\underline{M}_1 \cong \underline{Z}$. \hfill $\square$

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