THE WEYL LAW FOR ALGEBRAIC TORI

IAN PETROW

Abstract. We give an asymptotic evaluation for the number of automorphic characters of an algebraic torus $T$ with bounded analytic conductor. The analytic conductor which we use is defined via the local Langlands correspondence for tori by choosing a finite dimensional complex algebraic representation of the $L$-group of $T$. Our results therefore fit into a general framework of counting automorphic representations on reductive groups by analytic conductor.

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1. Introduction

A basic question in the analytic theory of automorphic forms is the following:

Question. Given a connected reductive algebraic group $G$ over a number field $k$, how many irreducible cuspidal automorphic representations of $G$ are there?  

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To make sense of the Question, one needs to choose a positive real-valued invariant by which to order the representations of $G$. Sarnak, Shin and Templier [SST16] have proposed using the analytic conductor.

On the groups $GL_m$, the analytic conductor has a standard definition [IS00], but over more general reductive groups it is less well understood. The most canonical (but not necessarily the most practical) definition is through the local Langlands conjectures. Let $r : \mathbb{C}G \to \mathbb{C}GL_m(C)$ be a finite dimensional algebraic representation of the complex $L$-group of $G$. The local Langlands conjectures predict the existence of maps

$$r_{\pi,v} : \mathcal{A}_v(G) \to \mathcal{A}_v(GL_m)$$

at every place $v$ of $k$ from the local unitary dual $\mathcal{A}_v$ of $G$ to that of $GL_m$. One then defines the analytic conductor $c(\pi,r)$ of an irreducible automorphic representation $\pi$ with respect to $r$ by

$$c(\pi,r) = \prod_v c_v(r_{\pi,v} \pi_v),$$

where the conductors on the right hand side are the “classical” local analytic conductors on $GL_m$.

The universal counting Question seems to be quite difficult at the level of generality in which we have stated it. Only very recently has there been progress in a few special cases. Over an arbitrary number field, the cases $G = GL_1$, $GL_2$, and under additional assumptions, $GL_m$ with $m \geq 3$ have been resolved in a preprint of Brumley and Milićević [BM18]. The case that $G$ is a one-dimensional torus over $\mathbb{Q}$ splitting over an imaginary quadratic extension was treated in work of Brooks and the author [BP18], and Lesesvre has studied the case that $G$ is the units group of a quaternion algebra in his Paris 13 PhD thesis [Les18].

In this paper, we present an answer to the Question for $G = T$ a torus defined over a number field $k$ and $r$ an arbitrary complex algebraic representation of its $L$-group. Even though the groups we are dealing with are abelian, our results are not easy, as we work with a very general notion of conductor. Indeed, the difficulties involved are already evident in the intricacy of the statement of the final result. As its reward, working with such a general notion of conductor reveals some of the richness that any general answer to the Question must exhibit. For example, the power of $X$ in the asymptotic count of automorphic characters (see [L4]) need not be an integer, but rather is a positive rational with denominator at most $m$. Further, we find that arbitrary integer powers of $\log X$ are possible in the asymptotic count (see example [L8]). Another interesting aspect of our results is the resemblance of the automorphic counting question to the Manin conjecture, which we present in subsection [L2].

We make some precise definitions in order to give the statement of our result. Let $T$ be an algebraic torus defined over a number field $k$. Let $A(T)$ denote the group of continuous unitary characters of $T(k)\backslash T(A)$, where $A$ is the adèle ring of $k$. We call elements of $A(T)$ automorphic characters; they are the basic objects of study in this paper. Let $K/k$ be the minimal Galois extension over which $T$ splits, and let $G = \text{Gal}(K/k)$. Let $X^*(T)$ and $X_*(T)$ be the character and cocharacter lattices of $T$, and $\hat{T} = \text{Hom}(X_*(T), \mathbb{C}^\times)$ the complex dual torus. We make the identification $X_*(T) = X^*(\hat{T})$. Each of these objects admits a natural action of $G$. Let $\hat{L}T = \hat{T} \times G$ be the $L$-group of $T$, and pick $r : \hat{L}T \to \mathbb{C}GL_m(C)$ an algebraic representation of $\hat{L}T$. Generally, we will write $n = \dim T$ and $m = \dim r$. Pick $\nu$ a Haar measure on the locally compact group $A(T).

The main goal of this paper is to give an asymptotic formula for $\nu(\{\chi \in A(T) : c(\chi,r) \leq X\})$, where $c(\chi,r)$ is the analytic conductor (defined in [2]), as $X$ tends to infinity. The statement of the result requires a few more constructions. The restriction of $r$ to $\hat{T}$ breaks up as a direct sum

$$r|_{\hat{T}} = \bigoplus_{\mu \in X_*(T)} V_\mu.$$
Let $M$ be the set of co-weights $\mu$ appearing in this decomposition, counted with multiplicity. Let $S \subseteq M$ denote a subset of the coweights with multiplicity and $S^c$ its complement. For such an $S$, we define the complex diagonalizable group
\begin{equation}
D(S) = \bigcap_{\mu \in S^c} \ker \mu \subseteq \hat{T}.
\end{equation}
The restriction $r|_{\hat{T}}$ is faithful if and only if $D(\emptyset) = \{1\}$, and in that case we let
\begin{equation}
A = A(T, r) = \max\left\{\frac{\dim D(S) + 1}{|S|} : S \subseteq M, \quad D(S) \neq \{1\}\right\}.
\end{equation}

**Theorem 1.1.** Suppose that $r|_{\hat{T}}$ is faithful. Then there exists a non-zero polynomial $P = P_{\nu,r,T}$ and $\delta = \delta_{r,T} > 0$ such that
\[\nu(\{\chi \in A(T) : c(\chi, r) \leq X\}) = X^A P(\log X) + O_{\nu,r,T}(X^{A-\delta}).\]
If $r|_{\hat{T}}$ is not faithful, then the left hand side is infinite for some finite $X$.

The dependence of $P$ and the implicit constant on $\nu$ is linear, since Haar measure is unique up to scaling. Here is a simple corollary of Theorem 1.1.

**Corollary 1.2.** Let $T$ be a torus of dimension $n$, $r$ an $m$-dimensional complex representation of its $L$-group, and $\nu$ a Haar measure on $A(T)$. We have
\[\nu(\{\chi \in A(T) : c(\chi, r) \leq X\}) \gg_{T,r,\nu} X^{\frac{n+1}{m}}.\]
If $r|_{\hat{T}}$ is faithful, then for all $\varepsilon > 0$ we have
\[\nu(\{\chi \in A(T) : c(\chi, r) \leq X\}) \ll_{\varepsilon,T,r,\nu} X^{2+\varepsilon}.\]

**Proof.** By Theorem 1.1 it suffices to give uniform lower and upper bounds on $A$. For the lower bound, note that $D(M) = \hat{T}$ which gives $A \geq \frac{n+1}{m}$. For the upper bound, observe that for any $S \subseteq M$ we have $\dim D(S) \leq |S|$, since $\dim D(\emptyset) = 0$ and $\text{codim} \ker \mu \leq 1$ for any $\mu \in M$. Therefore $A \leq \max\{|S| + 1 : S \neq \emptyset\} \leq 2$. \hfill $\square$

We can give an expression for the degree of the polynomial $P$, but this requires a few more definitions. Since $M$ was formed from the restriction of a representation of $L^T$, the group $G$ acts on $M$, and also on the power set $2^M = \{S : S \subseteq M\}$. This action preserves $|S|$ as well as $\dim D(S)$, so $G$ also acts on the set
\begin{equation}
\Sigma = \{S \neq \emptyset : \frac{\dim D(S) + 1}{|S|} = A\}.
\end{equation}
Let
\begin{equation}
\lambda = \text{lcm}_{S \subseteq M} |\pi_0(D(S))|,
\end{equation}
where $\pi_0(D(S))$ denotes the group of connected components of $D(S)$. The group $(\mathbb{Z}/\lambda\mathbb{Z})^\times$ acts on $\pi_0(D(S))$ for each $S \subseteq M$ by $\ell.y = y^\ell$, $y \in \pi_0(D(S))$, $\ell \in (\mathbb{Z}/\lambda\mathbb{Z})^\times$. Let $\zeta_\lambda$ be a primitive $\lambda$th root of unity, and let $\hat{K} = K(\zeta_\lambda)$ and $\tilde{G} = \text{Gal}(\hat{K}/k)$. The enlarged group $\tilde{G}$ acts on the fibered set with base $\Sigma$ given by
\begin{equation}
\tilde{\Sigma} = \{(S, y) : S \in \Sigma, \quad y \in \pi_0(D(S))\}.
\end{equation}
Indeed, we have inclusions
\begin{equation}
\tilde{G} \hookrightarrow G \times \text{Gal}(k(\zeta_\lambda)/k) \hookrightarrow G \times (\mathbb{Z}/\lambda\mathbb{Z})^\times
\end{equation}
given by restricting automorphisms to $K$ and to $\mathbb{Q}(\zeta)$. If $g \in \tilde{G}$ restricts to $(\mathfrak{g}, g^*) \in G \times (\mathbb{Z}/\lambda \mathbb{Z})^\times$, then $g$ acts on $\tilde{\Sigma}$ by
\[ g.(S, y) = (\mathfrak{g}S, \mathfrak{g}^*g^*). \]

Finally, let
\[ \tilde{\Sigma}_0 = \tilde{\Sigma} - \{(S, 1) : \dim D(S) = 0\}. \]

Since the deleted set is preserved by the action of $\tilde{G}$, we also have that $\tilde{G}$ acts on $\tilde{\Sigma}_0$.

**Theorem 1.3.** The polynomial $P$ appearing in Theorem 1.1 satisfies
\[ \deg P = |\tilde{G}\backslash\tilde{\Sigma}_0| - 1. \]

Theorems 1.1 and 1.3 settle a problem of Sarnak, Shin and Templier [SST16, (4)] for the universal family of automorphic characters on a torus in the greatest possible generality.

Remarks:

1. If one is willing to assume the Lindelöf hypothesis for Hecke characters, then any $\delta < (2m^2)^{-1}$ is admissible in the statement of Theorem 1.1.
2. If one assumes that $r$ is irreducible, then $\lambda = 1$. The assumption that $r$ is irreducible simplifies many examples, since it implies that $\tilde{G} = G$ and $\tilde{\Sigma} = \Sigma$.
3. One is also very interested in the form of the leading constant in the asymptotic in Theorem 1.1 especially if it admits an interpretation in terms of the geometry or arithmetic of $T$. While in principle our method yields an expression for the leading constant, it is not so easy to write down in explicit form. One reason is that at primes $p$ that ramify in $K/k$, we can only show that the number of characters of $T(k_p)$ is not so large as to affect the power of $X$ or $\log X$ in the final answer. Another reason is that we cannot exclude the possibility that terms corresponding to non-identity global units of $T$ may contribute to the leading term of the polynomial $P$ (see example 1.7).
4. The invariant $A$ in Theorem 1.1 and the power of $\log X$ in Theorem 1.3 are sufficiently complicated as to suggest that any general answer to the Question at the beginning of this paper would be quite onerous to state in full generality.
5. To resolve the counting problem at archimedean places for general $r$, one seems to require the use of a Brascamp-Lieb inequality due to Barthe [Bar98]. This is striking, as one does not typically expect such a deep analytic input to be necessary to resolve such a question in analytic number theory. The use of the Brascamp-Lieb inequality suggests that the counting problem for a general reductive group is difficult indeed, as already in the case of tori one needs to go much beyond an explicit understanding of the local Langlands correspondence.
6. Another interpretation of the families of automorphic characters studied in this paper is the following. Let $T$, $K/k$ be as above, $T^\vee = \text{Hom}(X_s(T), G_m)$ be the algebraic dual torus, and $S = \text{Res}_{K/k} G_m$. Given a faithful irreducible algebraic representation $r$ of $L^1T$, one obtains an injective morphism $i : T^\vee \to S$ by restriction of $r$. Such an injective map $i$ gives rise to an $L$-homomorphism $L^1T \to L^1S$, and so Langlands predicts that there exists a transfer of automorphic characters $i_* : \mathcal{A}(T/k) \to \mathcal{A}(GL_1/K)$. Conversely, given $i : T^\vee \to S$, there exists a faithful irreducible algebraic representation $r$ of $L^1T$ extending $i$ such that $L(s, \chi, r) = L(s, i_* \chi)$ for all $\chi \in \mathcal{A}(T)$, where the left hand side is the Langlands $L$-function and the right hand side is the Hecke $L$-function.
7. The automorphic counting problem outlined at the beginning of this paper has applications to the Ramanujan conjecture on general reductive groups (see the surveys [Sar05] and [Sha04]). Outside the case $G = GL_n$, the Ramanujan conjecture is known to be false, but all automorphic forms for which it fails are expected to arise as functorial transfers from lower rank groups. For analytic applications, one would like to show that the Ramanujan
conjecture cannot fail “too often” in a quantitative sense in terms of analytic conductor. At the very least, to do so one would need to estimate the sizes of subfamilies of $A(G)$ coming from functorial transfers of automorphic characters of tori, and the so present paper paves the way for putting the above program into action.

1.1. Examples.

**Example 1.4.** Let $T = GL_1 = G_m$. Then $\hat{T} = C^\times$ and $G$ is trivial. We choose $r = id = z : C^\times \to C^\times$ as representation of the $L$-group. Then $A(T)$ is the set of primitive Hecke characters over $k$, and $c(\chi, r)$ is the standard notion of analytic conductor of a Hecke character, which we denote by $C(\chi)$ in all of the examples that follow. The multiset of co-weights is the singleton $M = \{z\}$, and $2^{\{z\}} = \{\emptyset, \{z\}\}$. We have $D(\emptyset) = \{1\}$ and $D(\{z\}) = C^\times$, so we have $A = 2$, and $\deg P = 0$. Therefore there are $\sim c_k X^2$ primitive Hecke characters of analytic conductor bounded by $X$ for some constant $c_k > 0$ with a power saving error term. The same result has also recently been announced by Brunley and Miličević.

**Example 1.5.** Let $T = GL_1 = G_m$ as above, but take as representation the 1001-dimensional representation $r = z^{1001} : C^\times \to GL_{1001}(\mathbb{C})$. The set $A(T)$ consists of Hecke characters $\chi$ as above, whereas $r$ assigns to $\chi$ the conductor $C(\chi)^{1001}$. The multiset of co-weights is $\{z, \ldots, z\}$, where $z$ is repeated 1001 times, and only the full set has $D(S) \neq \{1\}$. Thus, $A = 2/1001$, and one recovers that there are $\sim c_k X^{2/1001}$ Hecke characters of $r$-conductor less than $X$. This shows that the power of $X$ in Theorem 1.1 can be arbitrarily small.

**Example 1.6.** Keep $T = GL_1 = G_m$ as above, but take as representation $r = z^2 \oplus z^3 : C^\times \to GL_2(\mathbb{C})$. This is a 2-dimensional faithful representation of the $L$-group but it is not irreducible. The set $A(T)$ is as in the previous two examples, but now $r$ assigns to $\chi$ the conductor $C(\chi^2)C(\chi^3)$. The set of co-weights is $\{z^2, z^3\}$, and the subsets $S$ and groups $D(S)$ are

$$S = \emptyset, \{z^2\}, \{z^3\}, \{z^2, z^3\},$$

$$D(\emptyset) = \{1\}, D(\{z^2\}) = \mu_3, D(\{z^3\}) = \pm 1, D(\{z^2, z^3\}) = C^\times.$$

Therefore $A = 1$, and the maximum is attained on all $S \neq \emptyset$. We have the Galois group $G = 1$, but the enlarged Galois group is $\hat{G} \simeq (\mathbb{Z}/6\mathbb{Z})^\times$. We have

$$\hat{\Sigma} = \{(z^2, 1), (z^2, z_3), (z^2, \overline{z}_3), (z^3, 1), (z^3, -1), (z^2, z^3, 1)\},$$

$$\hat{\Sigma}_0 = \{(z^2, z_3), (z^2, \overline{z}_3), (z^3, -1), (z^2, z^3, 1)\}.$$

The group $\hat{G}$ acts on $\hat{\Sigma}_0$ by swapping $(z^2, z_3)$ and $(z^2, \overline{z}_3)$ and fixing $(z^3, -1)$ and $(z^2, z^3, 1)$. Therefore there exists a constant $c_{23}$ so that

$$\nu(\{\chi \in A(GL_1) : C(\chi^2)C(\chi^3) \leq X\}) \sim c_{23} X(\log X)^2.$$

**Example 1.7.** Let $T = G_m \times G_m$. This torus has $L$-group equal to $C^\times \times C^\times$. There is no faithful irreducible representation of this $L$-group. Take the faithful 2-dimensional representation $z_1 \oplus z_2$. Classically, this corresponds to counting pairs of primitive Hecke characters $(\chi_1, \chi_2)$ with the conductor $C(\chi_1)C(\chi_2)$. There are $\sim c_{12} X^2 \log X$ pairs of Hecke characters of conductor bounded by $X$, for some $c_{12} > 0$.

For a general torus $T$, we will see later that there is a term which potentially contributes to the main term of $\nu(\{\chi \in A(T) : c(\chi, r) \leq X\})$ as $X \to \infty$ for each global unit of $T$. In the example $T = G_m \times G_m$ here, there are four global units: $(1, 1), (1, -1), (-1, 1)$ and $(-1, -1)$. It is interesting to note that the contribution from $(1, 1)$ is of size $X^2 \log X$, each of $(1, -1)$ and $(-1, 1)$ make a contribution of size $X^2$, and $(-1, -1)$ contributes a smaller a power of $X$. 5
Example 1.8. Let $K/k$ be any degree $n$ Galois extension with group $G$. Let $T = (\text{Res}_{K/k} \mathbf{G}_m)/\mathbf{G}_m$, where $\mathbf{G}_m$ is embedded diagonally. One can express the dual torus as

$$\hat{T} = \{(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n : z_1 \cdots z_n = 1\}.$$

Define $r : L \to \text{GL}_n(\mathbb{C})$ by setting $r((1, \cdots, 1) \times \sigma)$ to be the permutation matrix in $\text{GL}_n(\mathbb{C})$ defined by $\sigma \in G \subseteq S_n$, and

$$r((z_1, \ldots, z_n) \times 1) = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix}.$$ 

The set of coweights is $\{z_i\}_{i=1, \ldots, n}$, where $z_i$ represents projection onto the $i$-th coordinate. For any $\varnothing \neq S \in 2^{\{z_i\}}$ we have $\dim D(S) = |S| - 1$, so $A = 1$, and the maximum in (1.4) is attained for all $S$ with $|S| \geq 2$, and so $\tilde{\Sigma}_0 = \{S : |S| \geq 2\}$. Since $r$ is irreducible we have $\hat{G} = G$.

Suppose now that $G \simeq S_n$. Then

$$\hat{\Gamma} \setminus \tilde{\Sigma}_0 = \{\{S : |S| = 2\}, \ldots, \{S : |S| = n\}\}$$

and so we have

$$v(\chi \in \mathcal{A}(T) : c_r(\chi) \leq X) \sim c_r X \log X^{n-2}.$$ 

This is an example of logarithms arising in the asymptotic formula for “natural” reasons, giving a negative answer to a question of Sarnak [Sar08]. Also note that if e.g. $[K : k] = 4$ and $G \neq S_4$ the asymptotic is $\sim c_G X \log X)^3$ for some constants $c_G$, whereas if $G \simeq S_4$ it is $\sim c_{S_4} X \log X)^2$. This shows that the power of $\log X$ in the asymptotic formula is sensitive to the arithmetic of the torus.

1.2. Relation to the Manin conjecture. The automorphic counting question introduced at the outset of this paper is reminiscent of the Manin conjecture on the number of rational points of bounded height on a Fano variety. We briefly review the latter to point out a few of its features.

Let $V$ be a Fano variety over $k$, and $\mathcal{L}$ a very ample line bundle. Let $s_0, \ldots, s_m$ be global sections of $\mathcal{L}$ with no common zeros, and $\phi = \phi_{s_0, \ldots, s_m} : V \to \mathbb{P}^m$ be the natural morphism associated to these data. Let $H(x)$ be the absolute exponential Weil height on $\mathbb{P}^m(k)$. Then $h_{s_0}(x) = h(\phi(x))$ is a height function on $V(k)$ relative to $\mathcal{L}$, $s_0, \ldots, s_m$. If $s'_0, \ldots, s'_m$ is another choice of global sections for the same $\mathcal{L}$ with $\phi' = \phi_{s'_0, \ldots, s'_m} : V \to \mathbb{P}^m$, then $h_{s'_0}(x) = h_{s'_0}(x) + O(1)$ as $x \in V(k)$ varies [Sil86] Thm. 3.1]. Following Batyrev-Manin [BM90], for $U \subseteq V$ a Zariski open let

$$N_U(\mathcal{L}, X) = \# \{x \in U(k) : h_{s_0}(x) \leq X\}.$$ 

Let $N^{\text{eff}}_U(\mathcal{L}, X) = \# \{x \in U(k) : h_{s_0}(x) \leq X\}$.

Conjecture 1.9 (Batyrev-Manin Conj. C'). Let $V$ be a Fano variety with canonical bundle $\omega_V$ not effective. If $U$ is sufficiently small, we have

$$N_U(\mathcal{L}, X) \sim c X^{\alpha(\mathcal{L})} \log X^{t(\mathcal{L}) - 1}$$

as $X \to \infty$ for some positive constant $c$. Here,

$$\alpha(\mathcal{L}) = \inf \{\lambda \in \mathbb{R} : \lambda[\mathcal{L}] + [\omega_V] \in N^{\text{eff}}_U\},$$

and $t(\mathcal{L})$ is the codimension of the minimal face of $\partial N^{\text{eff}}_U$ containing $\alpha(\mathcal{L})[\mathcal{L}] + [\omega_V]$.

The analogy between the automorphic counting question and the Manin conjecture is as follows, and should be viewed as an expression of the deep conjectures of Langlands. The role of the ambient space is played by $\mathbb{P}^m(k) \leftrightarrow \mathcal{A}(\text{GL}_m)$, into which $V(k) \leftrightarrow \mathcal{A}(G)$ embeds. The embedding is given by the data $\mathcal{L}, s_0, \ldots, s_m$ on the Manin side, and (conjecturally) on the automorphic side by $r : L \to \text{GL}_m(\mathbb{C})$. Indeed, $\mathcal{L}, s_0, \ldots, s_m$ determine a morphism $V \to \mathbb{P}^m$ whereas the representation $r$ (conjecturally) determines $r_a : \mathcal{A}(G) \to \mathcal{A}(\text{GL}_m)$. The absolute exponential Weil height $H(x)$
for $x \in P^n(k)$ on the Manin side corresponds to the analytic conductor $c(\pi)$, $\pi \in A(\text{GL}_m)$ on the automorphic side. The height function $h_\phi(x)$ relative to $\phi$ corresponds to the analytic conductor $c(\pi, r)$ relative to $r$ as in (1.1).

The invariant $\alpha(\mathcal{L})$ appearing in the Manin conjecture and the invariant $A$ appearing in Theorem 1.1 both are expressible in terms of combinatorial geometry problems, see the computations with matroids in section 6 of this paper.

At least in the special case of tori, Theorems 1.1 and 1.3 suggest that $t(\mathcal{L})$ on the Manin side corresponds to the set of orbits $\tilde{G}_s^0 \tilde{\Sigma}_0$. In both cases, the power of log comes from the possible embeddings of $V$ or $A(T)$ in ambient space that are “extremal” in the combinatorial geometry problem defining $\alpha(\mathcal{L})$ or $A$.

The leading constant in Manin’s conjecture has been given a conjectural interpretation in terms of adelic volumes by Peyre [Pey95] and Chambert-Loir and Tschinkel [CLT10]. For a discussion of the significance of the leading constant in the automorphic counting problem, see [BM18 §1.5].

While the analogy presented here is striking, it only goes so far. In Manin’s conjecture, there is a canonical choice of $\mathcal{L}$, that is, one takes $\mathcal{L} = -\omega_V$, the anti-canonical bundle. In the automorphic setting, there is apparently no canonical choice of complex representation $r$ of the $L$-group of $G$. Moreover, in the setting of Manin’s conjecture the set of possible height functions corresponds to the ample cone of $V$, whereas in the automorphic setting, the set of possible height functions on $A(T)$ is a much larger object. Lastly, we remark that the invariant $t(\mathcal{L})$ in Manin’s conjecture is an essentially global invariant of $V$. On the other hand, $\tilde{G}_s^0 \tilde{\Sigma}_0$ has a somewhat more local nature, as we shall see in section 6 of this paper.

1.3. Outline of the proof. To prove Theorems 1.1 and 1.3 it suffices to study the generating series

$$Z(s) = \int_{A(T)} \frac{1}{c(\chi, r)^s} d\nu(\chi).$$

In this paper we show that $Z(s)$ converges absolutely and uniformly on compacta in the right half-plane $\text{Re}(s) > A$. Then we prove that $Z(s)$ continues to a meromorphic function polynomially bounded in vertical strips in the right half-plane $\text{Re}(s) > A - (2m^2)^{-1}$, and show that the only pole of $Z(s)$ in this domain occurs at $s = A$ and that it is of order $|\tilde{G}_s^0 \tilde{\Sigma}_0|$. Theorems 1.1 and 1.3 then follow from a standard application of Perron’s formula.

In section 2 we study the local analytic conductors in detail, taking particular care to deal with places ramified in $K/k$. We define “abelian” local analytic conductors $\tilde{c}_\pi(\chi, v)$ for each non-archimedean place $v$ of $k$ that control the standard local analytic conductors, but which are accessible to the counting techniques later in the paper. We also restrict the local Langlands correspondence to a distinguished compact subgroup of $T(k_v)$ at each finite place $v$ of $k$.

The main goal of section 3 is to decompose $Z(s)$ into a product of local generating series at each place $v$ of $k$. This is not immediately possible due to the presence of the global units of $T$. What is possible is to express $Z(s)$ as a sum over the global units of series attached to each unit, which themselves factor over places of $k$. We also prove a key global finiteness lemma related to the units of $T$ in section 6 and the second assertion of Theorem 1.1.

Sections 4, 5 and 6 are devoted to the computation of the local generating series at unramified, ramified and archimedean places, respectively. The heart of the paper is in section 4 where an expression for the Dirichlet series coefficients at unramified primes is given in terms of the dimensions of the groups $D(S)$, and the number of fixed points of local Galois group actions, already reminiscent of those in the statement of Theorem 1.3. In section 5 we cannot give such an explicit evaluation of the local generating series at ramified primes, but we can show that these series converge absolutely in a right half-plane containing $s = A$. Since there are only finitely many ramified primes, this is sufficient for our purposes but obscures the leading constant of the
polynomial $P$ in Theorem 1.1 In section 6 we study the archimedean local generating series. To show that these converge absolutely in a right half-plane containing $s = A$, we make use of a Brascamp-Lieb inequality and the theory of matroids.

Finally, in section 7 we pull together the local results of sections 4, 5 and 6 to show that the global generating series attached to each global unit admits an analytic continuation by comparison with a product of Hecke $L$-functions. The global finiteness lemma of section 3 shows that as we vary the global unit, only finitely many different such $L$-series can occur. The sum over global units then converges absolutely, establishing the analytic properties of $Z(s)$.

1.4. Index of notation.

| Notation | Definition | Location |
|----------|------------|----------|
| $k, \mathbf{A}$ | a number field and its ring of adèles | 1.1 |
| $T, n$ | an algebraic torus of dimension $n$ defined over $k$ | 1.1 |
| $\mathcal{A}(T)$ | the Pontryagin dual of $T(k) \backslash T(\mathbf{A})$ | 1.1 |
| $\mathcal{K}, G$ | the minimal extension of $k$ splitting $T$, $\text{Gal}(K/k)$ | 1.1 |
| $\widehat{T}$ | the complex dual torus: $\text{Hom}(X_+(T), \mathbb{C}^\times)$ | 1.1 |
| $L^T$ | the $L$ group of $T$: $\widehat{T} \rtimes G$ | 1.1 |
| $r, m$ | an $m$-dimensional complex algebraic representation of $L^T$ | 1.1 |
| $\nu$ | a Haar measure on $\mathcal{A}(T)$ | 1.1 |
| $\varepsilon(\chi, r)$ | the (local or global) analytic conductor with respect to $r$ | 2.2 |
| $M, \mu$ | the set of coweights $\mu$ of $r$, counted with multiplicity | 1.2, 2.4 |
| $M$ | a matrix with entries in $\mathbb{Z}$ encoding the coweights of $r$ | 2.4 |
| $S, 2^M$ | a subset of $M$, the set of all subsets of $M$ | 1.1 |
| $D(S)$ | a complex diagonalizable subgroup of $\widehat{T}$ | 1.3 |
| $A$ | the power of $X$ in the main theorem | 1.4 |
| $\Sigma$ | set of nonzero $S \subseteq M$ which attain $A$ | 1.5 |
| $\lambda$ | $\text{lcm}_{S \subseteq M} |\pi_0(D(S))|$ | 1.6 |
| $\mathcal{K}, \mathcal{G}$ | $K$ adjoin the $\lambda$th roots of unity, $\text{Gal}(\mathcal{K}/k)$ | 1.1 |
| $\mathcal{S}$ | a fibered set with base $\Sigma$ and fiber $\pi_0(D(S))$ | 1.7 |
| $\mathcal{S}_0$ | $\mathcal{S}$ with the subset $\{(S, 1): \dim D(S) = 0\}$ deleted | 1.9 |
| $C(\chi)$ | the analytic conductor of a Hecke character $\chi$ | 1.1 |
| $\text{Res}_{K/k}$ | restriction of scalars, i.e. “Weil restriction” | 1.1 |
| $F$ | $\text{a local field over which } T \text{ is defined}$ | 2.2 |
| $L, G$ | the minimal extension of $F$ splitting $T$, $\text{Gal}(L/F)$ | 2.2 |
| $\tilde{\nu}(\chi, r)$ | “abelian” local analytic conductor | Def 2.6 |
| $v, w$ | a valuation of $k$ and a unique valuation of $K$ extending it | 2.2 |
| $W_F, W_{L/F}$ | Weil group and relative Weil group of a local field | 2.2 |
| $\sigma$ | the canonical map $W_{L/F} \to \text{Gal}(L/F)$ | 2.2 |
| $\mathcal{A}^\wedge$ | the group of continuous unitary characters of $A$ | 2.2 |
| $(\varrho, V)$ | a complex Galois representation | 2.2 |
| $q_F$ | the cardinality of the residue field of $F$ | 2.1, 2.2, 2.5 |
| $G_F$ | the absolute Galois group of $F$ | 2.1 |
| $\xi, \varphi$ | a cohomology class and a Langlands parameter $\varphi = \xi \times \sigma$ | 2.5 |
| $c(\varrho), \tilde{c}(\varrho)$ | the Artin conductor and abelian conductor of $(\varrho, V)$ | Def 2.2, 2.5 |
| $\mathcal{N}$ | the norm map, i.e. the product of Galois conjugates | 2.7 |
| $G^v, W^v_{L/F}$ | higher ramification groups with upper numbering | 2.1 |
| $c_\mathcal{F}(\varrho)$ | conductor of $(\varrho, V)$ with respect to a filtration $\mathcal{F}$ | Def 2.3 |
| $\mathcal{M}, \mathcal{O}_L^\wedge$ | the standard filtration on $\mathcal{O}_L^\wedge$ | 2.1 |
### Notation

| Notation | Definition | Location |
|----------|------------|----------|
| π_F, π_L, π, Ψ | uniformizers of F, L, and the maximal ideals they generate | [2.2] |
|  \hat{H}^n | Tate cohomology groups | [2.2] |
| R | the restriction to O_L^\times map out of H^1(W_L/F, \hat{T}) | [2.10] |
| T, n_1, n_2, n_3 | the set of geometric components of a variety | [3.1] |
| T^\wedge | a typical element of T^\wedge | [2.21] |
| ((w, e), α, (w', α')) | an element of X_s(G_m)^{n_1} \times (S^1)^{n_2} \times (C^\times)^{n_3}, F local non-arch | [2.2] |
| A_i, C_i, B_i, B_3^\wedge | matrices formed from explicit parameterization of coweights | [2.4] |
| U_N(T) | global norm-units of T | [3.1] |
| Cl_N(T) | norm-class group of T | [3.2] |
| S_\infty | set of archimedean places of k | [3.1] |
| V(T) | \{(\chi_\infty, \chi_f) : \chi_\infty(x)\chi_f(x) = 1 \text{ for all } x \in U_N(T)\} | [3.1] |
| V_\infty | \{\chi_\infty : \chi_\infty(x) = 1 \text{ for all } x \in U_N(T)\} | [3.6] |
| V_\fin | \{\chi_f = \prod_{v|\infty} NT(\mathcal{O}_w)^\wedge \} | [3.1] |
| U_v(s, x) | local generating series at unramified places | [3.8] |
| R_v(s, x), A_v(s, x) | similar, at ramified and archimedean places | [3.1] |
| B | set of places of k with (q_k, \lambda) \neq 1 \text{ or } v \text{ ramified} | [3.1] |
| T_S | an auxiliary torus attached to S \in 2M | [3.1] |
| α^{-1}(x) | fibers of a canonical map α : T_S \rightarrow T | [3.1] |
| K', Γ | a field s.t. all components of α^{-1}(x) are defined, Gal(K'/k) | [3.2] |
| p(V) | the set of geometric components of a variety V | [3.2] |
| N | the non-negative integers | [3.1] |
| P_{\leq}(c) | a finite subgroup of Hom_G(\mathcal{O}_L^\times, \hat{T}) | [1.2] |
| P_{\geq}(c) | a "sharp" subset of P_{\leq}(c) | [1.3] |
| Π_{\leq}(c, x), Π_{\geq}(c, x) | character sums over P_{\leq}(c) and P_{\geq}(c) | [1.4] |
| D_\kappa(c) | a generalization of D(S) | [1.7] |
| α, λ | the residue fields of F, L, their characteristic | [4] |
| a(S, x) | the number of Frobenius-fixed components of α^{-1}(x) | [1.1] |
| a(S) | a(S, 1); the number of Frobenius-fixed points of π_0(D(S)) | [1.12] |
| S_{red} | the maximal Galois-stable subset of S | [1.11] |
| H_M | a polytope in \mathbb{R}_{\geq 0}^n given by a matrix M | [6.5] |
| B_\kappa | \inf\{\|x\|_\kappa : x \in H_M\} | [6.7] |
| (N, J), r(S), | a matroid, its rank function | Def 6.6, 6.8 |
| P_1, P_2, P_3 | the matroid polytope and matroid base polytope | Def 6.9 |
| \Psi', D_\Psi, D_\Psi' | a prime of K' above \Psi, decomposition groups of \Psi, \Psi' | [7.1] |
| Θ, | equal up to an absolutely convergent Euler product | Def 7.2 |
| C | a conjugacy class of Γ | [7.1] |
| Σ_\Psi | subset of Σ fixed by D_\Psi | [7.8] |
| a_C(S, x) | the number of C-fixed components of α^{-1}(x) | [7.10] |
| Σ_{a,b} | set of S \in 2^M such that \dim D(S) = a and |S| = b | [7.12] |
| Σ_{a,b} | similar to Σ_0, but with respect to Σ_{a,b} | [7.14] |
| V_{a,b}, ψ_{a,b} | permutation representation of Γ on Σ_{a,b}, its character | [7.1] |

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2. Conductors

Let \( v \) be a place of \( k \) and denote by \( k_v \) the completion of \( k \) at \( v \). An automorphic character \( \chi \in \mathcal{A}(T) \) admits a factorization

\[
\chi = \bigotimes_v \chi_v
\]

in which all but finitely many of the \( \chi_v \) are trivial on the maximal compact subgroup of \( T(k_v) \). Likewise, the global \( L \)-group \( \mathcal{L}_T \) gives rise to local \( L \)-groups at each place of \( T \). Indeed, the torus \( T \) is defined over \( k_v \) and for each \( v \) there exists a unique minimal Galois extension \( K \) of \( k_v \) splitting \( T \). We have \( \text{Gal}(K/k) \hookrightarrow \text{Gal}(K/k_v) \), and local \( L \)-groups \( \mathcal{L}_{T_v} \hookrightarrow \mathcal{L}_T \), and so \( r \) determines a finite dimensional complex algebraic representation of the local \( L \)-group \( \mathcal{L}_{T_v} \) for each \( v \) by restriction.

The analytic conductor \( c(\chi, r) \) is a local invariant of \( \mathcal{A}(T) \) defined by

\[
c(\chi, r) = \prod_v c_v(\chi_v, r|_{\mathcal{L}_{T_v}}),
\]

where \( c_v(\chi_v, r|_{\mathcal{L}_{T_v}}) \) are local analytic conductors, all but finitely many of which are equal to 1.

Thus, it suffices to restrict our attention to local analytic conductors, which are the subject of the rest of this section of the paper. For the remainder of section 2, we let \( T \) denote a torus defined over a local field \( F \), splitting over a minimal Galois extension \( L/F \). We now write \( G = \text{Gal}(L/F) \) for the local Galois group and \( \mathcal{L} = \hat{T} \times G \) for the local \( L \)-group, dropping the subscript \( v \). Finally, we suppose we are given a finite dimensional complex algebraic representation \( r : \mathcal{L} \to \mathbb{G}_a \), and write \( c(\chi, r) \) for the local analytic conductor (again dropping \( v \) from the notation).

Following [Bor79], we define \( c(\chi, r) \) by passing through the local Langlands correspondence and taking the conductors from the Galois representation associated to \( \chi \) and \( r \). To that end, recall the Weil group \( W_F \) of a local field (see e.g. [Ta77 §1]), and the relative Weil group

\[
W_{L/F} = \frac{W_F}{[W_L, W_L]} \hookrightarrow W_L^\text{ab}.
\]

The relative Weil group can be thought of as a group extension of \( G \) by \( L^\times \), i.e. the sequence

\[
1 \longrightarrow L^\times \longrightarrow W_{L/F} \overset{\sigma}{\longrightarrow} G \longrightarrow 1
\]

is exact, where the first map is the Artin map of local class field theory.

For a topological abelian group \( A \), \( \text{Hom}(A, C^\times) \) denotes the group of continuous complex characters of \( A \), and \( A^\times \) denotes the subgroup of unitary characters, that is to say the Pontryagin dual. For \( M \) a \( G \)-module, \( H^1(G, M) \) denotes the first group cohomology group defined using continuous cocycles. The local Langlands correspondence for tori [Lan97] asserts that there is a canonical isomorphism

\[
\text{Hom}(T(F), C^\times) \simeq H^1(W_{L/F}, \hat{T}),
\]

where \( W_{L/F} \) acts on \( \hat{T} \) via the map \( \sigma \) (see (2.2)). Langlands also shows that (2.3) restricts to

\[
T(F)^\times \simeq H^1(W_{L/F}, \hat{T}_u),
\]
where $\hat{T}_u = X_u(T) = \text{Hom}(X_u(T), S^1)$. We will use (2.4) when $F$ is an archimedean local field.

If $\chi \in \text{Hom}(T(F), C^\times)$ corresponds to $\xi \in H^1(W_{L/F}, \hat{T})$ across (2.3) and $\sigma$ is as in (2.2), then

$$\varphi = \xi \times \sigma : W_{L/F} \to L^T$$

is a conjugacy class of homomorphisms called the Langlands parameter of $\chi$. The composition

$$r \circ \varphi : W_{L/F} \to \text{GL}(V)$$

is the complex Galois representation associated to $\chi$ and $r$. One defines local $L$ and $\varepsilon$-factors associated to finite dimensional complex Galois representations as in [Tat79, §3]. If $F$ is non-archimedean, we define local analytic conductor $c(\chi, r)$ in terms of the $\varepsilon$-factor of $r \circ \varphi$, and if $F$ is archimedean we define $c(\chi, r)$ in terms of the $L$-factor of $r \circ \varphi$.

We now summarize the contents of the rest of section 2. In subsections 2.1-2.3 we restrict to the non-archimedean case. In subsection 2.1 the $\varepsilon$-factor of a complex Galois representation $(\varrho, V)$ is given in terms of the Artin conductor $c(\varrho)$ of $(\varrho, V)$ (see (2.6)). If $L/F$ is ramified we are unable to perform the counting arguments in section 4 with respect to the Artin conductor. To get around this obstacle we introduce the abelian conductor $\hat{c}$ (see Definition 2.3), which is more amenable to counting (see section 5) and which controls the Artin conductor (see Lemma 2.7).

The goal of subsections 2.2 and 2.3 is to restrict the Langlands isomorphism (2.3) to the ring of integers of $F$. It turns out that it is more natural to restrict to the image of the norm map $N : T(O_L) \to T(O_F)$. In subsection 2.2 we prove the key lemma that the image of the norm map has finite index. In subsection 2.3 we accomplish the restriction.

In subsection 2.4 we study the archimedean case.

### 2.1. The Artin conductor

We now restrict our attention to non-archimedean local fields. Let $dx$ be a Haar measure on $F$, $\psi$ a non-trivial additive character of $F$, and $dx'$ the dual Haar measure relative to $\psi$. Given a finite dimensional complex representation $(\varrho, V)$ of the Weil group $W_F$, Tate [Tat79, §3] defines the $\varepsilon$-factor $\varepsilon(V, \psi, dx) = \varepsilon(\varrho, \psi, dx)$ attached to these data.

**Definition 2.1.** If $\xi$ corresponds to $\chi \in \text{Hom}(T(F), C^\times)$ under the Langlands isomorphism (2.3), $\sigma$ is as in (2.2), and $\varphi = \xi \times \sigma$, then the quantity

$$c(\chi, r) = |\varepsilon(r \circ \varphi, \psi, dx)|^2$$

is called the local analytic conductor of $\chi$ with respect to $r$.

Tate [Tat79, §3.4.2] shows that $\varepsilon(V, \psi, dx)$ is additive, and in particular only depends on the isomorphism class of $V$. If $(\varrho, V)$ is unitary we have (see [Tat79, §3.4.7]) that

$$|\varepsilon(V, \psi, dx)|^2 = q_F^{c(\varrho)}(\delta(\psi)dx/dx')^{\dim(V)}.$$

In particular, since $\varepsilon(V, \psi, dx)$ only depends on the isomorphism class of $(\varrho, V)$, it suffices for (2.6) to hold that $(\varrho, V)$ be unitarizable. Here $q_F$ is the cardinality of the residue field of $F$ and $c(\varrho)$ is the Artin conductor of the representation $(\varrho, V)$. The factor $(\delta(\psi)dx/dx')^{\dim(V)}$ is explained in Tate and we do not need to elaborate on it here.

Next, we review the definition of the Artin conductor $c(\varrho)$ of a finite-dimensional complex representation $\varrho : W_F \to \text{GL}(V)$ of the Weil group of a non-archimedean local field. The classical Artin conductor is an invariant of a finite dimensional complex representation of a finite Galois group $\text{Gal}(E/F)$. For more discussion of the classical Artin conductor see [Ser79, Ch.VI] or [Ulm16, §4]. We give a slightly nonstandard definition of the Artin conductor of a finite-dimensional complex representation of $W_{L/F}$ following [Ulm16] (this goes back at least to [DDT97]).

Let $G_F = \text{Gal}(\overline{F}/F)$ and let $W_F \to G_F$ be the inclusion given as part of the data of a Weil group (see [Tat79, §1.4.1]). For any $v \in [-1, \infty)$, let $W^v_F$ be the inverse image of the (upper numbering) higher ramification group $G^v_F$ by this inclusion (see Serre [Ser79] for definitions, especially Ch. IV,
Let $W_{L/F}^v$ be the image of $W_F^v$ by the canonical projection $W_F \to W_{L/F}$. We have therefore a decreasing filtration of groups:
\[
\cdots \subseteq W_{L/F}^v \subseteq \cdots \subseteq W_{L/F}^2 \subseteq W_{L/F}^1 \subseteq W_{L/F}^0 \subseteq W_{L/F}.
\]

Proposition/Definition 2.2. For a finite dimensional complex representation $\varrho : W_{L/F} \to \text{GL}(V)$, the number
\[
c(\varrho) = \int_{-1}^{\infty} \text{codim}(V^{\varrho(W_{L/F}^v)}) \, dv
\]
is called the Artin conductor of $(\varrho, V)$. The value of $c(\varrho)$ only depends on $\varrho|_{W_{L/F}^0}$, and extends the notion of Artin conductor for complex representations of finite Galois groups.

Proof. Since there are no breaks in the upper-numbering filtration between $-1$ and $0$, and the upper numbering is left-continuous (see [Ser79, Ch.4 §3]), it follows that the Artin conductor $c(\varrho)$ only depends on the restriction of $\varrho$ to $W_{L/F}^0$. Since $W_{L/F}^0$ is compact and profinite, and $\text{GL}(V)$ has no small subgroups, it follows that $H = \ker \varrho|_{W_{L/F}^0}$ is a finite index open subgroup of $W_{L/F}^0$.

We have that $\varrho|_{W_{L/F}^0}$ then factors through the finite quotient $W_{L/F}^0/H$. Then the inverse image of $H$ in $W_{L/F}^0$ is also finite index, and contains $[W_L, W_L]$, thus (see Tate [Tat79 §1.4.5]) we have that $H = W_{L/E}^0$ for some finite extension $L^{ab}/E/F$. By Serre [Ser79, Ch. IV, Prop. 14] we have that
\[
\frac{W_{L/F}^0}{H} = \frac{W_{L/F}^0}{W_{L/F}^0 H} = \left( \frac{W_{L/F}}{W_{L/E}} \right)^0 \approx \text{Gal}(E/F)^0.
\]
and indeed, for all $v \in (-1, \infty)$ that
\[
\frac{W_{L/F}^v}{H \cap W_{L/F}^v} \approx \frac{W_{L/F}^v H}{H} = \left( \frac{W_{L/F}}{W_{L/E}} \right)^v \approx \text{Gal}(E/F)^v.
\]

Therefore to $\varrho|_{W_{L/F}^0}$ there is associated a finite extension $E/F$ with $E \subseteq L^{ab}$, and $\varrho$ factors through the representation $\varrho' : \text{Gal}(E/F) \to \text{GL}(V)$ given by composing with the isomorphisms above. It is shown in [Ulm16 §4] for finite dimensional complex representations of finite Galois groups that the standard definition of the Artin conductor matches the one given in the Proposition/Definition with the higher ramification groups $G^v(E/F)$ in place of the Weil group and $\varrho'$ in place of $\varrho$. \qed

The Artin conductor is a special case of the following more general notion of conductor.

Definition 2.3. For $G$ a group endowed with a filtration $\mathcal{F} = (G^v)$ and $\varrho : G \to \text{GL}(V)$ a finite dimensional complex representation, we call
\[
c_\mathcal{F}(\varrho) = \int_{-1}^{\infty} \text{codim} V^{\varrho(G^v)} \, dv
\]
the conductor of $(\varrho, V)$ with respect to $\mathcal{F}$.

With $G = W_{L/F}$ and $\mathcal{F}$ given by the upper numbered filtration, $c_\mathcal{F}(\varrho)$ the Artin conductor of $(\varrho, V)$.

Next, we introduce an “abelian” conductor $\tilde{c}(\varrho)$. The Artin conductor of a representation $(\varrho, V)$ is controlled by the abelian conductor, and in the case that the representation factors through $W_{L/F}$ for $L/F$ an unramified extension, the abelian conductor is identical to the Artin conductor.

We denote for any $v \in (-1, \infty)$ the groups $\mathcal{O}_L^{(v)} = 1 + \pi_L^{[v]} \mathcal{O}_L$ for $v > 0$ and $\mathcal{O}_L^{(v)} = \mathcal{O}_L^{\infty}$ for $0 \geq v > -1$. In particular the function $(-1, \infty) \to \{\text{subgroups of } L^{\times}\}$ given by $v \mapsto \mathcal{O}_L^{(v)}$ is locally constant on $(-1, \infty) - \mathbb{Z}$, and left-continuous (with respect to the discrete topology on $\{\text{subgroups of } L^{\times}\}$) at each integer.
**Lemma 2.4.** For any real number \( v > -1 \), the following diagram commutes and the horizontal rows are short exact:

\[
\begin{array}{cccccc}
1 & \rightarrow & \mathcal{O}_L^{(v)} & \rightarrow & W_{L/F}^v & \rightarrow & G^v & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 1 \\
1 & \rightarrow & L^\times & \rightarrow & W_{L/F} & \rightarrow & G & \rightarrow & 1
\end{array}
\]

**Proof.** The map

\[
L^\times \rightarrow \frac{W_L}{[W_L, W_L]} \rightarrow \frac{W_F}{[W_F, W_F]} = W_{L/F}
\]

is defined via the Artin map and is clearly an inclusion. Under the Artin map, we have that \( \mathcal{O}_L^{(v)} \subseteq L^\times \) maps onto the \( v \)th higher ramification group of \( W_{L/F}^{ab} \) in the upper numbering (see Serre [Ser79, Ch. XV, Thm. 2]), and we have

\[
\mathcal{O}_L^{(v)} = \mathcal{O}_L^{([v])} \subseteq W_{L/F}^{[v]} \subseteq W_{L/F}^v
\]

for all real \( v > -1 \). Thus, the left half of the above diagram commutes and all four maps are injections.

By [Ser79, Ch. IV, Prop. 2], the cokernel of each of the left horizontal maps is

\[
\frac{W_{L/F}^v}{W_{L/F}^v \cap W_{L}^{ab}} \simeq \frac{W_{L/F}^v W_{L}^{ab}}{W_{L}^{ab}}.
\]

We apply [Ser79, Ch. IV, Prop. 14] with \( G = W_{L/F} \) and \( H = W_{L}^{ab} \) to see that

\[
\frac{W_{L/F}^v W_{L}^{ab}}{W_{L}^{ab}} \simeq \left( W_{L/F}/W_{L}^{ab} \right)^v.
\]

Finally, by the third group isomorphism theorem and [Tat79, §1.1] we conclude that

\[
\left( W_{L/F}/W_{L}^{ab} \right)^v \simeq G^v,
\]

since the canonical inclusion \( \varphi : W_F \rightarrow G_F \) has dense image. \( \square \)

We are finally ready to give a definition of the abelian conductor \( \tilde{c}(\varphi) \).

**Definition 2.5.** Let \((\varphi, V)\) be a finite-dimensional complex representation of \( W_{L/F}, W_{L/F}^0, L^\times \), or \( \mathcal{O}_L^\times \), where the latter two groups are viewed as subgroups of \( W_{L/F} \) as in Lemma 2.4. Let \( \mathcal{U} \) be the filtration given by \( (\mathcal{O}_L^{(v)}) \). Then

\[
\tilde{c}(\varphi) = c_{\mathcal{U}}(\varphi)
\]

is called the abelian conductor of \((\varphi, V)\).

**Remark:** The abelian conductor \( \tilde{c} \) is additive in the sense that if \( \varphi = \varphi_1 \oplus \varphi_2 \), then \( \tilde{c}(\varphi) = \tilde{c}(\varphi_1) + \tilde{c}(\varphi_2) \).

**Definition 2.6.** The abelian local analytic conductor \( \tilde{c}(\chi, r) \) attached to \( \chi, r, \psi, dx \) is the complex number

\[
\tilde{c}(\chi, r) = q_F^{C(\chi \times \sigma)}(\delta(\psi)dx/dx')^{\dim(C(\chi \times \sigma))},
\]

where \( \xi \in H^1(W_{L/F}, \hat{\mathcal{F}}) \) corresponds to \( \chi \in \text{Hom}(T(F), \mathcal{C}) \) by (2.3).

Finally, we note that the abelian conductor controls the Artin conductor and vice-versa. Let \( v_0 = \inf\{v : G^v(L/F) = \{1\}\} \). For example, if \( L/F \) is unramified then \( v_0 = -1 \).
Lemma 2.7. We have
\[
\tilde{c}(\varphi) \leq c(\varphi) \leq \tilde{c}(\varphi) + (v_0 + 1) \dim V.
\]
In particular, if \( L/F \) is unramified, then \( \tilde{c}(\varphi) = c(\varphi) \).

Proof. For the first inequality, simply observe that
\[
\text{codim}(V^{\varphi(\mathcal{O}_L)}) \leq \text{codim}(V^{\varphi(W_{L/F})}).
\]
For the second, we use the fact that if \( v > v_0 \) then \( \text{codim}(V^{\varphi(\mathcal{O}_L)}) = \text{codim}(V^{\varphi(W_{L/F})}) \). We have
\[
c(\varphi) = \int_{-1}^{v_0} \text{codim}(V^{\varphi(W_{L/F})}) dv + \int_{v_0}^{\infty} \text{codim}(V^{\varphi(W_{L/F})}) dv
\leq (v_0 + 1) \text{codim}(V^{\varphi(W_{L/F})}) + \int_{v_0}^{\infty} \text{codim}(V^{\varphi(\mathcal{O}_L)}) dv
\leq (v_0 + 1) \text{codim}(V^{\varphi(W_{L/F})}) + \int_{-1}^{\infty} \text{codim}(V^{\varphi(\mathcal{O}_L)}) dv
\leq (v_0 + 1) \dim V + \tilde{c}(\varphi).
\]

Corollary 2.8. We have for \( r, \chi, \xi, \psi, dx \) as above that
\[
\tilde{c}(\chi, r) | c(\chi, r) | q_F^{(v_0 + 1)m} \tilde{c}(\chi, r).
\]
In particular, if \( L/F \) is unramified then \( \tilde{c}(\chi, r) = c(\chi, r) \).

The reason we prefer the abelian conductor is the following.

Proposition 2.9. Let \( r \) be a finite-dimensional complex algebraic representation of \( \hat{L}T \) of dimension \( m \). Let \( M \) be the set of co-weights of \( r \) with multiplicity, as in (1.2). Suppose that \( \chi \leftrightarrow \xi \) under the local Langlands correspondence (2.3). Then
\[
\tilde{c}(\chi, r) = \sum_{\mu \in M} c(\mu \xi^{\mathcal{O}_{\hat{L}}}) \left( \delta(\psi) dx / dx' \right)^m,
\]
where \( c \) is the standard notion of conductor of a quasicharacter of \( \hat{L}^\times \).

Proof. Let \( \varphi = \xi \times \sigma : W_{L/F} \to \hat{L}T \) be the Langlands parameter associated to \( \chi \in \text{Hom}(T(F), C^\times) \) by the local Langlands correspondence. By definition, the abelian local analytic conductor only depends on
\[
\varphi|_{\mathcal{O}_{\hat{L}}} = \xi|_{\mathcal{O}_{\hat{L}}} \times 1.
\]
We have that
\[
r \circ \varphi|_{\mathcal{O}_{\hat{L}}} = r|_{\hat{F}} \circ \xi|_{\mathcal{O}_{\hat{L}}} = \bigoplus_{\mu \in M} \mu \circ \xi|_{\mathcal{O}_{\hat{L}}},
\]
so that by the additivity of the abelian conductor
\[
\tilde{c}(r \circ \varphi) = \sum_{\mu \in M} c(\mu \circ \xi).
\]
2.2. The norm map. Because of the demands of our local to global decomposition in section 3, we must restrict the local Langlands correspondence (2.3) to a compact subgroup of \( T(F) \) which is finite index in the maximal compact subgroup \( T(O_F) \). As we will see in the following subsection 2.3, the natural choice is to restrict the Langlands correspondence to the image of the norm map (2.7) 
\[
N : T(O_L) \to T(O_F)
\]
defined by the product of Galois conjugates. In this subsection prove an important lemma describing the image of \( N \), which we write \( NT(O_L) \).

We begin with a preliminary but crucial result.

**Lemma 2.10.** Suppose \( L/F \) is unramified. The map \( N : T(O_L) \to T(O_F) \) is surjective.

**Proof.** See [Ama69, Cor. of Thm. 1].

More generally, we have the following result.

**Lemma 2.11.** We have
\[
\frac{|T(O_F)|}{|NT(O_L)|} \leq e_{L/F}^d \frac{|T(F)|}{|NT(L)|},
\]
where \( e_{L/F} \) is the ramification index of \( L/F \).

**Proof.** We have an exact sequence
\[
1 \to O_L^\times \to L^\times \to Z \to 0,
\]
and since \( X^*(T) \) is a free abelian group this leads to
\[
1 \to T(O_L) \to T(L) \to \text{Hom}(X^*(T), Z) \to 0.
\]
Recall that (see [Ono61, §2.1]) \( T(O_F) \simeq \text{Hom}_G(X^*(T), O_L^\times) \), and that this is the unique maximal compact subgroup of \( T(F) \simeq \text{Hom}_G(X^*(T), L^\times) \). Taking \( G \)-invariants we have
\[
1 \to T(O_F) \to T(F) \to \text{Hom}_G(X^*(T), Z) \to H^1(G, T(O_L)) \to \cdots
\]
and a commutative diagram
\[
\begin{array}{ccc}
0 & \to & T(O_L) \to T(L) \to \text{Hom}(X^*(T), Z) \to 0 \\
\downarrow N & & \downarrow N & & \downarrow N \\
0 & \to & T(O_F) \to T(F) \to \text{Hom}_G(X^*(T), Z).
\end{array}
\]
The rightmost map \( N \) above is given by
\[
N : \text{Hom}(X^*(T), Z) \to \text{Hom}(X^*(T), Z)
\]
\[
\ell \mapsto \sum_{\sigma \in G} \ell^\sigma
\]
where \( \ell^\sigma(\chi) = \ell(\chi^{\sigma^{-1}}) \). Let
\[
\ker(N : \text{Hom}(X^*(T), Z) \to \text{Hom}(X^*(T), Z)) = \{ \ell \in \text{Hom}(X^*(T), Z) : \sum_{\sigma \in G} \ell^\sigma = 0 \}.
\]
Now
\[
T(L) = \text{Hom}(X^*(T), L^\times)
\]
and
\[
T(F) = \text{Hom}_G(X^*(T), L^\times)
\]
and the middle map \( N \) is given by

\[
N : \text{Hom}(X^*(T), L^x) \to \text{Hom}_G(X^*(T), L^x)
\]

\[
\psi \mapsto \prod_{\sigma \in G} \psi^\sigma,
\]

where \( \psi^\sigma \) is given by \( \psi^\sigma(\chi) = \psi(\chi^{\sigma^{-1}})^\sigma \). Let

\[
T_1 = \ker(N : T(L) \to T(F)) = \{ \psi \in \text{Hom}(X^*(T), L^x) : \prod_{\sigma \in G} \psi^\sigma = 1 \}.
\]

Define the valuation map

\[
v : T_1 \to \mathfrak{R}
\]

\[
\psi \mapsto v(\psi)
\]

where \( v(\psi)(\chi) = v(\psi(\chi)) \). The snake lemma gives us an exact sequence

\[
\begin{array}{cccc}
T_1 & \overset{\nu}{\longrightarrow} & \mathfrak{R} & \overset{\delta}{\longrightarrow} \frac{T(\mathcal{O}_F)}{NT(\mathcal{O}_L)} \overset{T(F)}{\longrightarrow} \frac{T(F)}{NT(L)}.
\end{array}
\]

We claim that

\[
\left| \frac{\mathfrak{R}}{v(T_1)} \right| \leq \epsilon_{L/F}^{\dim T}.
\]

Indeed, let \( \ell \in \mathfrak{R} \) be arbitrary. We claim that \( \epsilon_{L/F} \ell \in v(T_1) \). The first claim follows from this second claim on letting \( \ell \) run through a \( \mathbb{Z} \)-basis for \( \mathfrak{R} \), so it suffices to show this. Now we show the second claim. Choose \( \pi_F \) a uniformizer for \( F \). For \( \ell \in \mathfrak{R} \) let

\[
\psi_\ell \in \text{Hom}(X^*(T), L^x)
\]

be defined by

\[
\psi_\ell(\chi) = \pi_F^{\ell(\chi)}.
\]

Note that

\[
\prod_{\sigma \in G} \psi_\ell^\sigma(\chi) = \prod_{\sigma \in G} \psi_\ell(\chi^{\sigma^{-1}})^\sigma = \prod_{\sigma \in G} \pi_F^{\ell(\chi^{\sigma^{-1}})} = \prod_{\sigma \in G} \pi_F^{\ell(\chi)^{\sigma^{-1}}} = \pi_F^0 = 1,
\]

since \( \ell \in \mathfrak{R} \). Therefore \( \psi_\ell \in T_1 \). Note also that

\[
v(\psi_\ell)(\chi) = v(\psi(\chi)) = v(\pi_F^{\ell(\chi)}) = \epsilon_{L/F} \cdot \ell(\chi).
\]

Thus we have shown that for all \( \ell \in \mathfrak{R} \) we have \( \epsilon_{L/F} \ell \in v(T_1) \), as claimed.

By the exact sequence (2.8) we have that

\[
\left| \frac{T(\mathcal{O}_F)}{NT(\mathcal{O}_L)} \right| \leq \left| \frac{\mathfrak{R}}{v(T_1)} \right| \cdot \left| \frac{T(F)}{NT(L)} \right| \leq \epsilon_{L/F}^{\dim T} \cdot \left| \frac{T(F)}{NT(L)} \right|.
\]

\[\square\]

Recall the definition of the Tate cohomology groups \( \hat{H}^n \) from e.g. [Ser79, Ch.VIII]. We have

\[
\frac{T(F)}{NT(L)} = \hat{H}^0(G, T),
\]

and by the Nakayama-Tate theorem (see e.g. [PR94, Thm. 6.2])

\[
\hat{H}^0(G, T) \simeq \hat{H}^2(G, X^*(T)).
\]

Since \( X^*(T) \) is a finitely generated abelian group, we have by e.g. [AW67, §6 Cor. 2] that

\[
|\hat{H}^2(G, X^*(T))| < \infty,
\]

and so it follows that \( \left| \frac{T(F)}{NT(L)} \right| \) is finite.
2.3. **The Langlands pairing.** The goal of this section is to restrict \[2.3\] to the compact subgroup \(NT(0_L)\) of \(T(F)\). To do this, we re-formulate the Langlands correspondence \[\text{[La97]}\] as a perfect pairing

\[(2.9)\quad T(F) \otimes H^1(W_{L/F}, \hat{T}) \to \mathbb{C}^x,\]

which we call the Langlands pairing. We write \(G^0\) for the inertia subgroup of \(G\). The following is the main result of this section.

**Proposition 2.12.** Write \(R(H^1)\) for the image of the restriction to \(O_L^n\) map

\[(2.10)\quad R: H^1(W_{L/F}, \hat{T}) \to \text{Hom}_G(O_L^n, \hat{T}).\]

The subgroup \(R(H^1)\) of \(\text{Hom}_G(O_L^n, \hat{T})\) is of index at most \(\leq |H^1(G^0, \hat{T})| \cdot |H^2(G^0, \hat{T})|\). The Langlands pairing restricts to a perfect pairing

\[NT(0_L) \otimes R(H^1) \to \mathbb{C}^x.\]

In particular, if \(L/F\) is unramified, Langlands pairing restricts to a perfect pairing

\[(2.11)\quad T(0_F) \otimes \text{Hom}_G(O_L^n, \hat{T}) \to \mathbb{C}^x.\]

**Corollary 2.13.** The abelian local analytic conductor \(\tilde{c}(\chi, r)\) only depends on \(\chi|_{NT(0_L)}\). Of course, we would like the group \(R(H^1)\) to be of finite index, and indeed this is true.

**Lemma 2.14.** We have that \(H^i(G^0, \hat{T})\) is a finite group for all \(i \geq 1\).

**Proof.** See \[\text{[CE56]}\] chapter XII, theorems 4.1 and 6.4. \(\square\)

**Proof of Lemma 2.14** In Theorem 2.15 we take \(G = G^0, A = \hat{T}, \) and \(C = \mathbb{C}^x\) to obtain that

\[(2.12)\quad H^i(G^0, \hat{T}) \otimes H_i(G^0, X_*(T)) \to \mathbb{C}^x\]

is a perfect pairing. Now, by \[\text{[AW67]}\] \([6\ Cor. 1]\) we have that \(H^i(G^0, \hat{T})\) is a group of finite exponent. Furthermore, \(X_*(T)\) is a finitely-generated \(G^0\)-module, so by \[\text{[AW67]}\] \([6\ Cor. 2]\) we have that \(H_i(G^0, X_*(T))\) is a finite group. The result now follows from the duality theorem. \(\square\)

To prepare for the proof of Proposition 2.12, we review the proof of the Langlands correspondence \[\text{[2.3]}\]. To do so, we recall the following explicit descriptions of group cohomology and homology (the same exposition appeared in the appendix of \[\text{[BP18]}\]).

For this paragraph, let \(G\) be a group and \(M\) a left \(G\)-module. Computing via the inhomogeneous resolution gives the usual description of group cohomology

\[H^1(G, M) = \left\{ \text{maps } \xi: G \to M \text{ satisfying } \xi(gh) = \xi(g) + g \xi(h) \right\}/\left\{ \text{maps such that there exists } m \in M, \xi(g) = gm - m \right\}.\]

If \(\bigoplus S N\) is a direct sum of copies of an abelian group \(N\) indexed by a set \(S\), let \(\delta_s(n) \in \bigoplus S N\) be the element which is \(n\) in the spot indexed by \(s\) and \(0\) elsewhere. Computing via the inhomogeneous resolution then gives the following description of group homology

\[H_1(G, M) = \left\{ \left( m_g \right)_{g \in G} \left| \sum_g (g^{-1}m_g - m_g) = 0 \right\} \right\}/d(\bigoplus G \times G M),\]
where \( d(\delta_{g,h}(m)) = \delta_h(g^{-1}m) - \delta_{gh}(m) \). If \( G' < G \) is a finite-index normal subgroup, there is an action of \( G/G' \) on \( H_1(G', M) \) by the rule \( g \ast \delta_g(m) = \delta_{g'g^{-1}}(gm) \). There also exists a natural map

\[
\text{Trace} : H_1(G, M) \to H_1(G', M)^{G/G'},
\]

which may be computed as follows: pick coset representatives \( g_1, g_2, \ldots, g_n \) for \( G/G' \). Then any \( g \in G \) determines a permutation \( \tau \in S_n \) by the rule \( g_g = \tau g' \) (where \( g' \in G' \)), and

\[
\text{Trace}(\delta_g(m)) = \sum_{i} \delta_{g_i g g_{r(i)}}(g_i m).
\]

Now we return to our review of the Langlands correspondence (2.3). In particular, \( G = \text{Gal}(L/F) \) again. Let us begin by recalling two easy steps. First, we have the following standard isomorphisms

(2.13) \[ T(L) \cong \text{Hom}(X^*(T), L^\times) \cong L^\times \otimes X_*(T) \cong H_1(L^\times, X_*(T)), \]

and taking \( G \)-invariants

(2.14) \[ T(F) \cong \text{Hom}(X^*(T), L^\times)^G \cong (L^\times \otimes X_*(T))^G \cong H_1(L^\times, X_*(T))^G. \]

Second, Langlands proves the following mild extension of Theorem 2.15. Let \( \alpha \in W_{L/F}, \chi \in X_*(T) \) such that \( \delta_\alpha(\chi) \) is a cycle representing a class in \( H_1(W_{L/F}, X_*(T)) \). Let \( \xi \) be a cocycle representing a class in \( H^1(W_{L/F}, \hat{T}) \). Langlands [Lan97] shows that the pairing

(2.15) \[ \cup : H_1(W_{L/F}, X_*(T)) \otimes H^1(W_{L/F}, \hat{T}) \to \mathbb{C}^\times \]

defined by

\[ \cup : \delta_\alpha(\chi) \otimes \xi \mapsto \chi(\xi(\alpha)) \]

is a perfect pairing.

The difficult part of Langlands’s proof of (2.3) is that the map

(2.16) \[ \text{Trace} : H_1(W_{L/F}, X_*(T)) \to H_1(L^\times, X_*(T))^G \]

is an isomorphism. Combining (2.14), (2.15) and (2.16), we obtain the Langlands pairing (2.9).

We now discuss what the Langlands pairing has to do with the norm map. The Artin map (see (2.22)) induces a map

\[ H_1(L^\times, X_*(T)) \to H_1(W_{L/F}, X_*(T)) \]

so that the triangle

\[
\begin{array}{ccc}
H_1(L^\times, X_*(T)) & \xrightarrow{N} & H_1(L^\times, X_*(T))^G \\
\downarrow & & \Downarrow \cong \\
H_1(W_{L/F}, X_*(T))
\end{array}
\]

commutes. Here \( N \) is the norm map defined as a product of Galois conjugates, and the vertical map is the trace map (2.16). Composing with the isomorphisms (2.13) and (2.14), we have that the norm map \( N : T(L) \to T(F) \) factors through the homology group \( H_1(W_{L/F}, X_*(T)) \):

(2.17) \[ T(L) \xrightarrow{N} T(F) \]

\[
\begin{array}{ccc}
T(L) & \xrightarrow{N} & T(F) \\
\downarrow & & \Downarrow \cong \\
H_1(W_{L/F}, X_*(T)).
\end{array}
\]
Proof of Proposition 2.12. Let

\[ \text{Ann}(NT(\mathcal{O}_L)) \subseteq H^1(W_{L/F}, \hat{T}) \]

be the annihilator of \( NT(\mathcal{O}_L) \) with respect to the Langlands pairing \((2.9)\). To prove the proposition, it suffices to compute \( \text{Ann}(NT(\mathcal{O}_L)) \), and show that

\[ \frac{H^1(W_{L/F}, \hat{T})}{\text{Ann}(NT(\mathcal{O}_L))} \cong R(H^1), \]

is as described in the statement of the proposition.

**Lemma 2.16.** Let

\( (2.18) \quad R : H^1(W_{L/F}, \hat{T}) \to \text{Hom}_G(\mathcal{O}_L^\chi, \hat{T}) \)

be the restriction to \( \mathcal{O}_L^\chi \) map (note \( \mathcal{O}_L^\chi \) acts trivially on \( \hat{T} \)). Then we have

\[ \text{Ann}(NT(\mathcal{O}_L)) = \ker(R). \]

**Proof.** Let

\[ t : H_1(W_{L/F}, X_*(T)) \to T(F) \]

be the isomorphism obtained by composing the trace map \((2.16)\) with the series of isomorphisms \((2.14)\). The first step is to give an explicit description for the inverse image \( t^{-1}(NT(\mathcal{O}_L)) \subseteq H_1(W_{L/F}, X_*(T)) \). Later, we use the explicit description for the cup product pairing \((2.15)\) to compute \( \text{Ann}(NT(\mathcal{O}_L)) \).

The main trick to compute \( t^{-1}(NT(\mathcal{O}_L)) \) is to use the commuting triangle \((2.17)\), as the trace map is difficult to work with directly. Restricting \((2.17)\) to the maximal compact of \( T(L) \) we obtain

\[ T(\mathcal{O}_L) \xrightarrow{N} NT(\mathcal{O}_L) \cong t^{-1}(NT(\mathcal{O}_L)), \]

where all arrows are surjective. Since we understand the diagonal arrow much better than the vertical one, this yields a description for \( t^{-1}(NT(\mathcal{O}_L)) \). In the above explicit description for group homology, it is the subgroup of \( H_1(W_{L/F}, X_*(T)) \) generated by sums of all possible homology classes \( \delta_\alpha(\chi) \) as \( \alpha \) runs over \( \mathcal{O}_L^\chi \subset W_{L/F} \).

We now use the description \( t^{-1}(NT(\mathcal{O}_L)) = \langle \delta_\alpha(\chi) \rangle_{\alpha \in \mathcal{O}_L^\chi} \) and compute the annihilator

\[ \text{Ann}(NT(\mathcal{O}_L)) = \text{Ann}(t^{-1}(NT(\mathcal{O}_L))) \subseteq H^1(W_{L/F}, \hat{T}) \]

across \((2.15)\).

First we prove \( \text{Ann}(NT(\mathcal{O}_L)) \supseteq \ker(R) \). Let \( \xi \) represent a class in \( \ker(R) \). Then \( \xi \) vanishes on \( \mathcal{O}_L^\chi \) by definition, and we have \( \xi \cup \delta_\alpha(\chi) = 1 \) for any \( \delta_\alpha(\chi) \in t^{-1}(NT(\mathcal{O}_L)) \) by our explicit description in the previous paragraph, and the definition of \((2.15)\). Therefore \( \ker(R) \subseteq \text{Ann}(t^{-1}(NT(\mathcal{O}_L))) \).

Now we prove \( \text{Ann}(NT(\mathcal{O}_L)) \subseteq \ker(R) \). Suppose \( \xi \in H^1(W_{L/F}, \hat{T}) \) does not represent any class in \( \ker(R) \). Then there exists a \( \beta \in \mathcal{O}_L^\chi \) for which \( \xi(\beta) \neq 1 \). Since \( \xi(\beta) \neq 1 \) there exists \( \chi \in X_*(T) \) not vanishing on \( \xi(\beta) \in \hat{T} \). Since \( \mathcal{O}_L^\chi \) acts trivially on \( X_*(T) \), we have that \( \chi_\beta \) is a cycle, and thus represents a homology class. Thus \( \delta_\beta(\chi) \in t^{-1}(NT(\mathcal{O}_L)) \) and \( \xi \cup \delta_\beta(\chi) \neq 1 \), so \( \xi \notin \text{Ann}(t^{-1}(NT(\mathcal{O}_L))) \). Therefore \( \ker(R)^c \subseteq \text{Ann}(t^{-1}(NT(\mathcal{O}_L)))^c \), so we have \( \ker(R) = \text{Ann}(t^{-1}(NT(\mathcal{O}_L))) \). \(\square\)
By Lemma 2.16 we have shown that
\[ NT(\mathcal{O}_L) \otimes \frac{H^1(W_{L/F}, \hat{T})}{\ker(R)} \to \mathbb{C}^\times \]
is a perfect pairing. It now suffices to show that
\[ \frac{H^1(W_{L/F}, \hat{T})}{\ker(R)} \simeq R(H^1) \leq \text{Hom}_G(\mathcal{O}_L^\times, \hat{T}) \]
is of index at most \( |H^1(G^0, \hat{T})| \cdot |H^2(G^0, \hat{T})| \), as in the statement of Proposition 2.12.

Consider the inertia group \( W_{L/F}^0 \) acting on \( \hat{T} \), and the exact sequence
\[ 1 \to \mathcal{O}_L^\times \to W_{L/F}^0 \to G^0 \to 1 \]
as in Lemma 2.4. We take the inflation-restriction exact sequence attached to these data
\[ (2.19) \quad 1 \to H^1(G^0, \hat{T}) \to H^1(W_{L/F}^0, \hat{T}) \to \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \to H^2(G^0, \hat{T}). \]
These give
\[ 1 \to H^1(G^0, \hat{T}) \to H^1(W_{L/F}^0, \hat{T}) \to \ker(g) \to 1, \]
where \( \ker(g) \) is a subgroup of \( \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \) of index at most \( |H^2(G^0, \hat{T})| \). We take Frobenius invariants of this to obtain a sequence
\[ 1 \to H^1(G^0, \hat{T}) \to H^1(W_{L/F}^0, \hat{T}) \to \ker(g)^1 \to H^1(\mathbf{Z}, H^1(G^0, \hat{T})), \]
where \( 1 \in \mathbf{Z} \) acts by arithmetic Frobenius on \( \hat{T} \). Since a cocycle is determined by its value on a generator, we have
\[ |H^1(\mathbf{Z}, H^1(G^0, \hat{T}))| \leq |H^1(G^0, \hat{T})|. \]
Therefore \( r'(H^1(W_{L/F}^0, \hat{T})^Z) \) has index at most \( |H^1(G^0, \hat{T})| \) in \( \ker(g)^Z \), and \( \ker(g) \) has index at most \( |H^2(G^0, \hat{T})| \) in \( \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \), so \( \ker(g)^Z \) has index at most \( |H^2(G^0, \hat{T})| \) in \( \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \).
Thus \( r'(H^1(W_{L/F}^0, \hat{T})^Z) \) has index at most \( |H^1(G^0, \hat{T})| \cdot |H^2(G^0, \hat{T})| \) in \( \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \).

Consider again \( W_{L/F}^0 \) acting on \( \hat{T} \), and take the exact sequence
\[ 1 \to W_{L/F}^0 \to W_{L/F} \to \mathbf{Z} \to 1. \]
Taking the inflation-restriction exact sequence associated to these we have
\[ 1 \to H^1(\mathbf{Z}, \hat{T}^G^0) \to H^1(W_{L/F}^0, \hat{T}) \to H^1(W_{L/F}^0, \hat{T})^Z \to 1. \]

Here the term \( H^2(\mathbf{Z}, \hat{T}^G^0) \) vanishes because the cohomological dimension of \( \mathbf{Z} \) is one (see [Bro94, Ch. VIII, §2]). We have that \( R(H^1) = (r' \circ r'')(H^1(W_{L/F}^0, \hat{T})) \), and by the above remarks we conclude that \( R(H^1) \) has index at most \( |H^1(G^0, \hat{T})| \cdot |H^2(G^0, \hat{T})| \) in \( \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \), as was to be shown.

In the case that \( L/F \) is an unramified extension, we have \( G^0 = \{ 1 \} \), so that the first part of Proposition 2.12 gives us that \( R(H^1) = \text{Hom}_{G^0}(\mathcal{O}_L^\times, \hat{T}) \). We have \( NT(\mathcal{O}_L) = T(\mathcal{O}_F) \) by Lemma 2.10 so that the Langlands pairing restricts to the perfect pairing (2.11). \( \square \)
2.4. Archimedean local fields. We assume in this subsection that $F, L$ are archimedean local fields, with $T$ defined over $F$ and splitting over $L$. Let $\Gamma_R(s) = \pi^{-s/2} \Gamma(s/2)$ and $\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s)$. If $(\varrho, V)$ is a complex Galois representation of the Weil group $W_F$, then the $L$ factor of $(\varrho, V)$ is of the form

$$L(s, V) = \prod_i \Gamma_{F_i}(s + \kappa_{\varrho, i}),$$

where each $F_i = R$, or $C$, $\dim V = \sum_i \deg F_i$, and $\kappa_{\varrho, i} \in C$. Let $dx$ be a Haar measure on $F$, $\psi$ a non-trivial additive character of $F$, and $dx'$ the dual Haar measure relative to $\psi$.

**Definition 2.17.** Suppose $F$ is an archimedean local field. If $\xi$ corresponds to $\chi \in \text{Hom}(T(F), C^\times)$ under the Langlands isomorphism (2.3), $\sigma$ is as in (2.2), and $\varphi = \xi \times \sigma$, then the quantity

$$c(\chi, r) = \prod_i (|\kappa_{\varrho \circ \varphi, i}| + 1)^{\deg F_i} (\delta(\psi) dx/dx')^{\dim r}$$

is called the archimedean local analytic conductor of $\chi$ with respect to $r$.

Note that our definition differs slightly from the standard definition in that the $+1$ above is typically replaced by a $+2$ or $+3$. The reason some authors prefer $+2$ or $+3$ is to ensure that as $\dim V$ varies, there are only a finite number of representations of bounded conductor. Since we will always consider $\dim r$ to be fixed in this paper, we prefer $+1$ as it makes some of the computations in section 3 more elegant.

The definition of the $L$-factor $L(s, V)$ for archimedean places is given in [Tat79] sections 3.1.1, 3.1.2 and 3.3.1 in terms of the classification of the finite dimensional irreducible representations of $W_F$ given in section 2.2.2. Therefore, we must make explicit parameterizations of the possible Langlands parameters $\varphi = \xi \times \sigma : W_{L/F} \to L^T$, as well as the possible representations $r : L^T \to \text{GL}(V)$, in order to be able to find their compositions among the classification [Tat79] §2.2.2.

For the rest of this subsection we assume $T$ is defined over an archimedean local field $F \cong R$. The case that $F \cong C$ is strictly easier, and follows closely the case that $T \cong (\text{Res}_{C/R} G_m)^n$, which we elaborate on below.

A torus defined over $F$ splits over a quadratic extension $L \cong C$, and so $G = \text{Gal}(L/F)$ is a group of order two, whose elements we write $\{1, \tau\}$. Let us recall the explicit description of the Weil groups for archimedean local fields. We have

$$W_F = L^\times \cup L^\times j,$$

and

$$W_L = L^\times,$$

where we write elements of $W_F$ as words in $z$ and $j$ and we have the rules $jzz^{-1} = \tau z$ and $j^2 = -1$. We also have

$$\pi : W_{F/F} = W_F^{ab} \cong F^\times$$

where the isomorphism $\pi$ is given by

$$\pi(z) = |z|^2, \quad \text{and} \quad \pi(j) = -1.$$  

Also note that

$$W_{L/F} = W_F.$$

In this subsection, we take $L^T = \hat{T} \times G$ to be the local archimedean $L$-group, and $L^T_u = \hat{T}_u \times G$, where $\hat{T}_u = \text{Hom}(X_s(T), S^1)$. The reason to study $L^T_u$ instead of $L^T$ is that the unitary characters of $T(F)$ correspond under the local Langlands correspondence (2.3) to $H^1(W_{L/F}, \hat{T}_u)$ (see [Lan97 Thm. 1]).

Now we choose isomorphisms $L \cong C$ and

$$T(F) \cong T = (R^\times)^{n_1} \times (S^1)^{n_2} \times (C^\times)^{n_3}$$

(2.20)
so that \( \dim T = n = n_1 + n_2 + 2n_3 \). By computing with the inflation-restriction exact sequence and using facts about the group cohomology of finite cyclic groups, we have an explicit parameterization of the \( L \)-equivalence classes of Langlands parameters \( \varphi : W_{L/F} \to L T_u \). They are given by

\[
(2.21) \quad \varphi(z) = \left( \left| z \right|^{w_1}, \ldots, \left| z \right|^{w_n}, \left( \left| z \right| \right)^{\alpha_1}, \ldots, \left( \left| z \right| \right)^{\alpha_{n_2}} \right), \]

\[
\left( \left( \left| z \right| \right)^{\alpha_1'}, \left| z \right|^{w_1'-\alpha_1'}, \left( \left| z \right| \right)^{\alpha_2'}, \ldots, \left( \left| z \right| \right)^{\alpha_{n_3}'}, \ldots, \left( \left| z \right| \right)^{-\alpha_{n_3}'}, \left| z \right|^{w_{n_3}'-\alpha_{n_3}'} \right) \times 1
\]

and

\[
\varphi(j) = \left( (-1)^{\epsilon_1}, \ldots, (-1)^{\epsilon_{n_1}}, 1, \ldots, 1, (1, (-1)^{\alpha_1'}), \ldots, (1, (-1)^{\alpha_{n_3}'}) \right) \times \tau.
\]

Here, \( w_i \in i \mathbb{R}, \epsilon_i \in \{0, 1\}, \alpha_i \in \mathbb{Z}, \alpha_i' \in \mathbb{Z}, \) and \( w_i' \in \mathbb{C} \) such that \( w_i' - \alpha_i' \in i \mathbb{R} \). We write

\[
(2.22) \quad T^\wedge = (iR^{n_1} \times (\mathbb{Z}/2\mathbb{Z})^{n_1}) \times \mathbb{Z}^{n_2} \times (iR^{n_3} \times \mathbb{Z}^{n_3}),
\]

so that Langlands parameters may be parameterized by \((\{w, \epsilon\}, \alpha, (w', \alpha')) \in T^\wedge \). We also need an explicit description of the representation \( r \). The representation \( r \) decomposes into irreducible representations, and we can parameterize all irreducible representations of \( L T \) by the set of orbits \( G, X^*(T) \) using Mackey theory (see [Ser77 §8.2]). We now study this parameterization explicitly. Corresponding to \( (2.21) \) we have an isomorphism of \( G \)-modules

\[
(2.23) \quad X^*(T) \cong X = X_*(G_m)^{n_1} \times X_*(S^1)^{n_2} \times X_*(\text{Res}_{\mathbb{C}/\mathbb{R}} G_m)^{n_3}
\]

\[
\mu_x \leftrightarrow x,
\]

where \( X_*(G_m) = \mathbb{Z} \) with \( G \) acting trivially, \( X_*(S^1) = \mathbb{Z} \) with \( \tau \in G \) acting by sending \(-1\) to \( 1 \), and \( X_*(\text{Res}_{\mathbb{C}/\mathbb{R}} G_m) = \mathbb{Z}^2 \) with \( \tau \in G \) acting by swapping the two factors. Each \( x \in X \) is contained in a \( G \)-orbit of size 1 or 2. We have the following three genres of isomorphism classes of irreducible representations of \( L T \):

1a) If \( x \) is fixed by \( G \), i.e. is in an orbit of size one, then \( \mu_x \) is an irreducible representation of \( L T \).

1b) If \( x \) is fixed by \( G \), i.e. is in an orbit of size one, then \( \mu_x \otimes (\text{sign}) \) is an irreducible representation of \( L T \).

2) If \( x \) is not fixed by \( G \), i.e. is in an orbit of size two, then \( V_x = \text{Ind}_{T}^{L T} \mu_x \) is an irreducible representation of \( L T \). It only depends on the orbit of \( x \). That is, \( V_x \cong V_{x'} \) and this representation is not isomorphic to any other \( V_{x''} \), \( x'' \neq x, \tau x \).

Therefore we get a decomposition

\[
r|_{L T} = \bigoplus_{i=1}^{m_1} \mu_{x_i} \oplus \bigoplus_{i=1}^{m_2} \left( \mu_{x_i'} \otimes (\text{sign}) \right) \oplus \bigoplus_{i=1}^{m_3} V_{x''_i},
\]

for some \( x_i, x_i' \) which are fixed by \( G \) and some \( x''_i \) which are not fixed by \( G \), and where \( m_1 + m_2 + 2m_3 = m = \dim r \).

To work out the archimedean \( L \)-factor for each Langlands parameter \( \varphi \) (as in \( (2.21) \)) and each irreducible representation of \( L T \) (as in (1a),(1b),(2), above), we must compute these representations of \( W_F \) explicitly enough to be able to recognize them in the classification of irreducible representations given in [Laf79 §2.2.2].

1a) Suppose \( x \) is fixed by \( G \). Then we have

\[
(2.24) \quad x = (a_1, \ldots, a_{n_1}, 0, \ldots, 0, (b_1, b_1), \ldots, (b_{n_3}, b_{n_3})).
\]
The associated representations of $W_{L/F}$ are

$$(\mu_x \circ \varphi)(z) = \prod_{i=1}^{n_1} |z|^{a_{i,w_i}} \prod_{i=1}^{n_3} |z|^{2b_i(w_i'-\alpha_i')} = (\pi(z))^{\frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i')}$$

$$(\mu_x \circ \varphi)(j) = (-1)^{\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i} = (\pi(j))^{\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i}.$$  

As a character of $F^\times$ this is

$$\left( \frac{|y|}{y} \right)^{\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i} \left| \frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i') \right|.$$  

Following [Tat79 §3.1.1], the $L$-function of this character of $W_{L/F}$ is

$$L(s, \mu_x \circ \varphi) = \Gamma_R \left( s + \frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i') + \left( \sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i \pmod{2} \right) \right).$$  

Here and below, by $(n \pmod{2})$ we mean the integer 0 or 1 according to the value of $n$ modulo 2.

1b) Suppose $x$ is fixed by $G$. Then $x$ is as in [222], and we have the characters of $W_{L/F}$

$$(\mu_x \otimes \text{(sign)} \circ \varphi)(z) = \prod_{i=1}^{n_1} |z|^{a_{i,w_i}} \prod_{i=1}^{n_3} |z|^{2b_i(w_i'-\alpha_i')} = (\pi(z))^{\frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i')}$$

$$(\mu_x \otimes \text{(sign)} \circ \varphi)(j) = (-1)^{\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i} = (\pi(j))^{\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i}.$$  

As a character of $F^\times$ this is

$$\left( \frac{|y|}{y} \right)^{1+\sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i} \left| \frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i') \right|.$$  

Following [Tat79 §3.1.1], the $L$-function of this character of $W_{L/F}$ is

$$L(s, \mu_x \otimes \text{(sign)} \circ \varphi) = \Gamma_R \left( s + \frac{1}{2} \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} b_i (w_i'-\alpha_i') + \left( 1 + \sum_{i=1}^{n_1} a_i \varepsilon_i + \sum_{i=1}^{n_3} \alpha_i' b_i \pmod{2} \right) \right).$$  

2) Suppose $x$ is not fixed by $G$. Then we have

$$x = (a_1, \ldots, a_{n_1}, c_1, \ldots, c_n, b_1, b_1'), \ldots, (b_{n_3}, b_{n_3}'),$$

where at least one of the $c_i \neq 0$ or one of the $b_i \neq b_i'$. Then we have

$$(V_x \circ \varphi)(z) = \left( \begin{array}{cc} (\mu_x \circ \varphi)(z) & 0 \\ 0 & (\mu_{tx} \circ \varphi)(z) \end{array} \right)$$

$$(V_x \circ \varphi)(j) = \left( \begin{array}{cc} (\mu_x \circ \varphi)(j) & 0 \\ 0 & (\mu_{tx} \circ \varphi)(j) \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right).$$  

In order to find this representation in the classification of [Tat79 §2.2.2], we must recognize it as the induction of some character. We have

$$(\mu_x \circ \varphi)(z) = \prod_{i=1}^{n_1} |z|^{a_{i,w_i}} \prod_{i=1}^{n_2} z^{-\alpha_i c_i} |z|^{\alpha_i c_i} \prod_{i=1}^{n_3} z^{-\alpha_i'(b_i+b_i')-2b_i\alpha_i'}$$

and

$$(\mu_{tx} \circ \varphi)(z) = \prod_{i=1}^{n_1} |z|^{a_{i,w_i}} \prod_{i=1}^{n_2} z^{\alpha_i c_i} |z|^{-\alpha_i c_i} \prod_{i=1}^{n_3} z^{\alpha_i'(b_i+b_i')-2b_i\alpha_i'}.$$
The power of \( z \) in \((\mu_x \circ \varphi)(z)\) is
\[
- \sum_{i=1}^{n_2} \alpha_i c_i - \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i)
\]
and the power of \( z \) in \((\mu_{rx} \circ \varphi)(z)\) is
\[
\sum_{i=1}^{n_1} \alpha_i c_i + \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i).
\]
Exactly one of these two is negative. Following Tate, the rule for recognizing which character this representation is induced from is: choose \((\mu_x \circ \varphi)(z)\) or \((\mu_{rx} \circ \varphi)(z)\) according to which has a negative power of \( z \). Then the representation is induced off that character.

The power of \(|z|\) in \((\mu_x \circ \varphi)(z)\) is
\[
\sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_2} \alpha_i c_i + \sum_{i=1}^{n_3} w'_i (b_i + b'_i) - 2 \sum_{i=2}^{n_3} b'_i \alpha'_i
\]
and the power of \(|z|\) in \((\mu_{rx} \circ \varphi)(z)\) is
\[
\sum_{i=1}^{n_1} a_i w_i - \sum_{i=1}^{n_2} \alpha_i c_i + \sum_{i=1}^{n_3} w'_i (b_i + b'_i) - 2 \sum_{i=2}^{n_3} b_i \alpha'_i.
\]
Note that
\[
\sum \alpha_i c_i - 2 \sum b'_i \alpha'_i = \left( \sum \alpha_i c_i + \sum \alpha'_i (b_i - b'_i) \right) - \sum \alpha'_i b_i - \sum \alpha'_i b'_i
\]
and
\[
- \sum \alpha_i c_i - 2 \sum b_i \alpha'_i = \left( - \sum \alpha_i c_i - \sum \alpha'_i (b_i - b'_i) \right) - \sum \alpha'_i b_i - \sum \alpha'_i b'_i.
\]
Therefore if the power of \( z \) in \((\mu_x \circ \varphi)(z)\) is negative we have that the power of \(|z|\) in it is
\[
\sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} w'_i (b_i + b'_i) + \left| \sum_{i=1}^{n_3} \alpha_i c_i + \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i) \right| - \sum_{i=1}^{n_3} \alpha'_i (b_i + b'_i),
\]
and if the power of \( z \) in \((\mu_{rx} \circ \varphi)(z)\) is negative we get that the power of \(|z|\) in it is given by exactly the same formula. So we find the representation \( V_x \circ \varphi \) of \( W_{L/F} \) is induced from the character of \( W_L \) given by
\[
\left| \sum_{i=1}^{n_1} a_i c_i + \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i) \right| \left| \sum_{i=1}^{n_3} \alpha_i c_i + \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i) \right| - \sum_{i=1}^{n_3} \alpha'_i (b_i + b'_i).
\]
Then by \cite{Tat79} §3.3.1 we can conclude that
\[
L(s, \varphi, V_x) = \Gamma_C \left( s + \sum_{i=1}^{n_1} a_i w_i + \sum_{i=1}^{n_3} (w'_i - \alpha'_i)(b_i + b'_i) + \left| \sum_{i=1}^{n_2} \alpha_i c_i + \sum_{i=1}^{n_3} \alpha'_i (b_i - b'_i) \right| - \sum_{i=1}^{n_3} \alpha'_i (b_i + b'_i) \right).
\]

We now collect the above results in a more compact form. Given a representation \( \rho \) we determine a \( 3 \times 3 \) block matrix \( M = M(\rho) \) as follows. Take a decomposition
\[(2.25)\]
\[
\rho|_{L/T} = \bigoplus_{i=1}^{m_1} \mu_{x_i} + \bigoplus_{i=1}^{m_2} \left( \mu_{x'_i} \otimes \langle \text{sign} \rangle \right) + \bigoplus_{i=1}^{m_3} V_{x''_i},
\]
where each \( x_i, x'_i \in X \) are fixed by the action of \( G \) and each \( x''_i \in X \) is in a \( G \)-orbit of cardinality 2. We may write explicitly
\[(2.26)\]
\[
x_i = (a_{i1}, a_{i2}, \ldots, a_{in_1}, 0, \ldots, 0, (b_{i1}, b_{i1}), \ldots, (b_{in_3}, b_{in_3})),
\]

and similarly for $x'_i$. Likewise we may write
\begin{equation}
(2.27) \quad x''_i = ([a_{i1}, a_{i2}, \ldots, a_{in_1}, c_{i1}, \ldots, c_{in_2}, (b_{i1}, b'_{i1}), \ldots, (b_{in_3}, b'_{in_3})]),
\end{equation}
with at least one $c_{ij} \neq 0$ or $b_{ij} \neq b'_{ij}$. Now define the matrix
\begin{equation}
(2.28) \quad M = M(r) = \begin{pmatrix} A_1 & 0 & B_1 \\ A_2 & 0 & B_2 \\ A_3 & C & B_3 \end{pmatrix}
\end{equation}
where: $A_1 \in M_{m_1 \times n_1}(\mathbb{Z})$ is given by $A_1 = ((a_{ij}))$ with $a_{ij}$ as in (2.27), $B_1 \in M_{m_1 \times n_3}(\mathbb{Z})$ is given by $B_1 = ((b_{ij}))$ where $b_{ij}$ is also as in (2.27). Next, $A_2$ and $B_2$ are defined similarly to $A_1$ and $B_1$ but using the coordinates for $x'_i$ instead of those of $x_i$ as in (2.27). Finally, $A_3 \in M_{m_3 \times n_1}(\mathbb{Z})$ is given by $(a_{ij})$ where $a_{ij}$ are taken from (2.27), $C \in M_{m_3 \times n_2}(\mathbb{Z})$ is given by $C = (c_{ij})$ where $c_{ij}$ are taken from (2.27), and $B_3 \in M_{m_3 \times n_3}(\mathbb{Z} \times \mathbb{Z})$ is given by $B_3 = ((b_{ij}, b'_{ij}))$, where $(b_{ij}, b'_{ij})$ is also taken from (2.27).

We can write the elements $\varphi$ as length $n_1 + n_2 + n_3$ block column vectors, i.e. as
\begin{equation}
\begin{pmatrix} \langle w, \epsilon \rangle \\ \alpha \\ \langle w', \alpha' \rangle \end{pmatrix} \in (\mathbb{C}^{n_1} \times (\mathbb{Z}/2)^{n_1}) \times \mathbb{Z}^{n_2} \times (\mathbb{C}^{n_3} \times \mathbb{Z}^{n_3}).
\end{equation}

We define block-matrix multiplication as follows. Matrices of the form $A$ multiplied on an element $(w, \epsilon) \in (i\mathbb{R})^{n_1} \times (\mathbb{Z}/2)^{n_1}$ are defined to be
\begin{align*}
A_1(w, \epsilon) &= \frac{1}{2} A_1 w + (A_1 \epsilon \mod 2), \\
A_2(w, \epsilon) &= \frac{1}{2} A_2 w + (A_2 \epsilon + 1 \mod 2), \\
A_3(w, \epsilon) &= A_3 w,
\end{align*}
where the products on the right hand sides are the usual matrix multiplication. A matrix of the form $C$ multiplied on an element $\alpha \in \mathbb{Z}^{n_2}$ is defined to by the standard matrix multiplication $C \alpha$. We define also for elements $\langle w', \alpha' \rangle \in \mathbb{C}^{n_3} \times \mathbb{Z}^{n_3}$ the multiplication $B_1(\langle w', \alpha' \rangle) = B_1(\langle w' - \alpha' \rangle)$ where on the right hand side we have usual matrix multiplication. We also define the multiplication of $B_2$ in exactly the same way. So, in summary,
\begin{equation}
(2.29) \quad \langle A_1 | 0 | B_1 \rangle \begin{pmatrix} \langle w, \epsilon \rangle \\ \alpha \\ \langle w', \alpha' \rangle \end{pmatrix} = \frac{1}{2} A_1 w + B_1(\langle w' - \alpha' \rangle) + (A_1 \epsilon + B_1 \alpha' \mod 2) \in (i\mathbb{R} \times \{0, 1\})^{m_1}
\end{equation}
and
\begin{equation}
(2.30) \quad \langle A_2 | 0 | B_2 \rangle \begin{pmatrix} \langle w, \epsilon \rangle \\ \alpha \\ \langle w', \alpha' \rangle \end{pmatrix} = \frac{1}{2} A_2 w + B_2(\langle w' - \alpha' \rangle) + (A_2 \epsilon + B_2 \alpha' + 1 \mod 2) \in (i\mathbb{R} \times \{0, 1\})^{m_2}.
\end{equation}

Finally, we define the multiplication of $\langle A_3 | C | B_3 \rangle$ on $\langle (w, \epsilon), \alpha, (w', \alpha') \rangle$ as follows. Let $B_3^+ \in M_{m_3 \times n_3}(\mathbb{Z})$ be the matrix $((b_{ij} + b'_{ij}))$ formed from the entries of $B_3$ and $B_3^- \in M_{m_3 \times n_3}(\mathbb{Z})$ the matrix with entries $((b_{ij} - b'_{ij}))$. Then we define
\begin{equation}
(2.31) \quad \langle A_3 | C | B_3 \rangle \begin{pmatrix} \langle w, \epsilon \rangle \\ \alpha \\ \langle w', \alpha' \rangle \end{pmatrix} = A_3 w + B_3^+(\langle w' - \alpha' \rangle) + [C \alpha + B_3^-(\alpha')] \in (i\mathbb{R} \times \mathbb{N})^{m_3},
\end{equation}
where the absolute values means take the absolute value of each entry. This defines a multiplication

\[(2.32) \quad M\varphi = \begin{pmatrix} A_1 & 0 & B_1 \\ A_2 & 0 & B_2 \\ A_3 & C & B_3 \end{pmatrix} \begin{pmatrix} (w, \epsilon) \\ \alpha(w', \alpha') \end{pmatrix} \in (i\mathbb{R} \times \{0,1\})^{m_1+m_2} \times (i\mathbb{R} \times \mathbb{N})^{m_3}.\]

If \( M = M(r) \) is as above and \( \varphi \) is a Langlands parameter \((2.21)\), we have

\[(2.33) \quad L(s, r \circ \varphi) = \prod_{i=1}^{m_1+m_2} \Gamma_{\mathbb{R}}(s + (M\varphi)_i) \prod_{i=m_1+m_2+1}^{m_1+m_2+m_3} \Gamma_{\mathbb{C}}(s + (M\varphi)_i),\]

where \((M\varphi)_i\) means the \( i \)th entry of \( M\varphi \in (i\mathbb{R} \times \{0,1\})^{m_1+m_2} \times (i\mathbb{R} \times \mathbb{N})^{m_3}\).

Let \( x \in T(F) \), which according to our chosen isomorphism \((2.20)\) we can express as

\[x \mapsto (x_1, x_2, \ldots, x_j, \ldots, x_{n_1}, x_{n_2}, \ldots) \in (\mathbb{R}^\times)^{n_1} \times (S^1)^{n_2} \times (\mathbb{C}^\times)^{n_3},\]

where \( x_j \in \mathbb{R}^\times \) for \( j = 1, \ldots, n_1 \), \( x_j' \in S^1 \) for \( j = 1, \ldots, n_2 \), and \( x_j'' \in \mathbb{C}^\times \) for \( j = 1, \ldots, n_3 \). Then if \( \chi \in T(F)^\times \) corresponds to the Langlands parameter \( \varphi \) with parameterization \((2.21)\) across the Langlands correspondence \((2.4)\), we have that \( \chi \) is given explicitly by

\[(3.1) \quad \chi(x) = (\text{sgn } x_1)^{\alpha_1} \cdots (\text{sgn } x_{n_1})^{\alpha_{n_1}} |x_1|^{\alpha_{n_1}} \cdots |x_{n_2}|^{\alpha_{n_2}} \cdots |x_{n_3}|^{\alpha_{n_3}} = \prod_{j=1}^{n_3} |x_j''|^{|\alpha_{n_1}'| - \alpha_{n_2}'},\]

3. **Global preliminaries**

3.1. **Local to global.** The main goal of this subsection is the following proposition, which reduces the global counting problem to a local one. Recall for each place \( v \) of \( k \) the extension \( K_v \) of \( k_v \) and corresponding valuation \( w \) of \( K \). Let \( \mathcal{O}_w \) and \( \mathcal{O}_v \) be the rings of integers in \( K_w \) and \( k_v \), respectively, and \( NT(\mathcal{O}_w) \) be the image of the norm map \( N : T(\mathcal{O}_w) \to T(\mathcal{O}_v) \). Let \( S_x \) be the set of archimedean places of \( k \) and let

\[(3.1) \quad U_N(T) = T(k) \cap \prod_{v \in S_x} T(k_v) \times \prod_{v \notin S_x} NT(\mathcal{O}_w) \subset T(A)\]

be the global norm-units of the torus \( T \). Recall that for all unramified non-archimedean places of \( k \) we have \( NT(\mathcal{O}_w) = T(\mathcal{O}_v) \) by Lemma \(2.10\) and for all ramified non-archimedean places we have that \( NT(\mathcal{O}_w) \) is a finite-index subgroup of \( T(\mathcal{O}_v) \) by Lemma \(2.11\). Thus, \( U_N(T) \) is a finite index subgroup of the global units of the torus \( U(T) \), which by the Dirichlet units theorem for tori \([Shy77]\) is a finitely-generated abelian group.

**Proposition 3.1.** There exists \( c \in \mathbb{R}_{>0} \), depending only on \( T \) and \( \nu \) so that

\[Z(s) = c \sum_{\tau \in \text{Cl}_N(T)^\times} \sum_{x \in U_N(T)} \prod_{v \in S_x} \int_{T(k_v)^\times} \chi_v(x) \frac{d\chi_v}{\mathcal{C}_v(\chi_v, r)^s} \sum_{\nu \in S_x} \frac{\chi_v(x)}{\mathcal{C}_v(\chi_v, \nu, r)^s} \]

for any \( s \) for which the right hand side converges absolutely.

**Proof.** We have an exact sequence of locally compact commutative groups

\[(3.2) \quad 1 \longrightarrow T(k) \prod_{v \in S_x} T(k_v) \prod_{v \notin S_x} NT(\mathcal{O}_w) \longrightarrow T(A) \longrightarrow \text{Cl}_N(T) \longrightarrow 1,\]

where \( \text{Cl}_N(T) \) is a finite group, since the standard class number of \( T \) is finite (see \([Ono61\] Thm. 3.1)) and since \( \prod NT(\mathcal{O}_w) \) has finite index in \( \prod T(\mathcal{O}_v) \) by Lemma \(2.11\). By Pontryagin duality and taking the quotient by \( T(k) \), we have a dual exact sequence

\[(3.3) \quad 1 \longrightarrow \text{Cl}_N(T)^\times \longrightarrow A(T) \longrightarrow V(T) \longrightarrow 1,\]

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Proof. See [Bou67, Ch.II §7 Prop. 8].

We have an exact sequence

\[ 1 \longrightarrow U_N(T) \longrightarrow \prod_{v \in S_\infty} T(k_v) \longrightarrow V_\infty^\wedge \longrightarrow 1 \]

where

\[ (3.4) \quad V(T) = \left( \prod_{v \in S_\infty} T(k_v) \prod_{v \notin S_\infty} NT(O_v)/U_N(T) \right)^\wedge. \]

That is, \( V(T) \) is the set of pairs \((\chi_\infty, \chi_f)\), where \( \chi_\infty = \prod_{v | \infty} \chi_v \) and \( \chi_f = \prod_{v | \infty} \chi_v \) such that \( \chi_\infty(x)\chi_f(x) = 1 \) for all \( x \in U_N(T) \). We give

\[ \left( \prod_{v \in S_\infty} T(k_v) \prod_{v \notin S_\infty} NT(O_v) \right)^\wedge \approx \prod_{v \in S_\infty} T(k_v)^\wedge \prod_{v \notin S_\infty} NT(O_v)^\wedge \]

the Haar measure such that each \( NT(O_v)^\wedge \) has counting measure (it is discrete, being the dual of a compact group), and \( T(k_v)^\wedge \) has the Haar measure given by products of counting and the standard Lebesgue measure (for more details, see subsection [2.4]). Then by [Bou04, VII.44] there is a constant \( c' \) such that for any integrable function \( f \) on \( A(T) \)

\[ \int_{A(T)} f(\chi) \, d\nu(\chi) = c' \sum_{\theta \in Cl_N(T)^\wedge} \int_{V(T)} f((\chi_\infty, \chi_f)\theta) \, d\mu((\chi_\infty, \chi_f)). \]

Let \( f(\chi) = c(\chi, r)^{-s} \) for \( \text{Re}(s) \) sufficiently large. We write each \( \theta = \theta_x \theta_f = \prod_v \theta_v \). Since \( c(\chi, r) \) is defined as a product of local factors, we have for each \( \theta \) that

\[ \int_{V(T)} \frac{1}{c((\chi_\infty, \chi_f)\theta, r)^s} d\mu((\chi_\infty, \chi_f)) = \sum_{\chi_f \in V_{\text{fin}}} \frac{1}{c_{\text{fin}}((\chi_f, r, \theta_f)^s \int_{\chi_f^{-1}V_\infty} \frac{1}{c_{\text{fin}}((\chi_\infty, \theta_x, r)^s} d\chi_\infty \]

\[ = \sum_{\chi_f \in V_{\text{fin}}} \frac{1}{c_{\text{fin}}((\chi_f, r, \theta_f)^s \int_{V_\infty} \frac{1}{c_{\text{fin}}((\chi_\infty, \theta_x, r)^s} d\chi_\infty, \]

where

\[ V_\infty = \left\{ \chi_\infty \in \prod_{v \in S_\infty} T(k_v)^\wedge : \chi_\infty(x) = 1, \text{ for all } x \in U_N(T) \right\}, \]

and

\[ (3.7) \quad V_{\text{fin}} = \prod_{v \notin S_\infty} NT(O_v)^\wedge. \]

Within the integral over \( V_\infty \) above we mean by \( \chi_f^{-1} \) any element of \( \prod_{v \in S_\infty} T(k_v)^\wedge \) taking the same values as \( \chi_f(x)^{-1} \) on all \( x \in U_N(T) \).

Lemma 3.2 (Poisson Summation). Let \( H \leq G \) be locally compact commutative groups such that the quotient \( G/H \) is compact. Let \( f \in L^1(G) \) and suppose that its Fourier transform

\[ \hat{f}(\psi) = \int_G f(g)\overline{\psi}(g) \, dg \]

belongs to \( L^1(G^\wedge) \). Then for almost every \( x \in G \) we have

\[ \int_H f(xh) \, dh = \frac{1}{\text{Vol}(G/H)} \sum_{\psi \in (G/H)^\wedge} \hat{f}(\psi)\psi(x). \]

Proof. See [Bou67, Ch.II §1 Prop. 8].
and by Pontryagin duality

\[ 1 \longrightarrow V_\infty \longrightarrow \prod_{v \in S_{\infty}} T(k_v)^{\vee} \longrightarrow U_N(T)^{\vee} \longrightarrow 1. \]

Since \( U_N(T) \) is discrete, we have that \( U_N(T)^{\vee} \) is compact. Therefore, Lemma 3.2 applies with \( G = \prod_{v \in S_{\infty}} T(k_v)^{\vee} \) and \( H = V_\infty \), and we derive from (3.5) the formula claimed in the statement of the proposition.

In light of Proposition 3.1, there are three distinct cases to consider:

1. **Unramified places.** In this case, \( NT(\mathcal{O}_v)^{\vee} = T(\mathcal{O}_v)^{\vee} \) by Lemma 2.10 and \( c(\chi_v \theta_v, r) = \overline{c}(\chi_v, r) \) by Theorem 2.8 and Corollary 2.13. Therefore we have

\[
U_v(s, x) = \sum_{\chi_v \in NT(\mathcal{O}_v)^{\vee}} \frac{\chi_v(x)}{c(\chi_v \theta_v, r)^s} = \sum_{\chi_v \in T(\mathcal{O}_v)^{\vee}} \frac{\chi_v(x)}{\overline{c}(\chi_v, r)^s}. \tag{3.8}
\]

2. **Ramified places.** In this case we do not have any of the advantages listed in the above unramified case. However, there are only finitely many ramified \( v \), so we need less. We will show in section 5 that

\[
R_v(s, x) = \sum_{\chi_v \in NT(\mathcal{O}_v)^{\vee}} \frac{\chi_v(x)}{c(\chi_v \theta_v, r)^s}
\]

converges absolutely in some right half plane with \( s = A \) in its interior, where \( A \) is the invariant defined in (1.1). We also treat as ramified places any finite place \( v \) for which \( (\lambda, q_{k_v}) \neq 1 \), where \( q_{k_v} \) is the cardinality of the residue field of non-archimedean local field associated to \( v \), and \( \lambda \) is the natural number introduced in the introduction. We call the set of places for which either \( v \) is ramified or \( (\lambda, q_{k_v}) \neq 1 \) the “bad” set of places, and denote that set by \( B \).

3. **Archimedean places.** If \( v \) is archimedean then we have

\[
A_v(s, x) = \int_{T(k_v)^{\vee}} \frac{\chi_v(x)}{\overline{c}(\chi_v, r)^s} d\chi_v. \tag{3.10}
\]

An expression for this integral via the local Langlands correspondence can be derived from Definition 2.17 and equations (2.33) and (2.34).

In terms of the notation just introduced in (3.8), (3.9), and (3.10), the result of Proposition 3.1 reads

\[
Z(s) = c \sum_{x \in U_N(T)} \prod_{v \in S_{\infty}} A_v(s, x) \prod_{v \notin S_{\infty} \cup B} U_v(s, x) \sum_{\theta \in Cl_N(T)^{\vee}} \prod_{v \in B} R_v(s, x). \tag{3.11}
\]

### 3.2. A global finiteness lemma

Now that we have defined the global norm-units \( U_N(T) \) play in the proof of Theorem 1.1 is apparent, we prove an important global finiteness lemma related to them. As in the introduction, let \( G = \text{Gal}(K/k) \) the global Galois group, and \( S \subset M \) be a subset of the co-weights of \( r \). Let \( G_S = \text{Stab}_G S \subseteq G \). Then \( G_S \) acts on \( S \), and also on its complement \( S^c \). Let \( K_S \) be the intermediate field in the extension \( K/k \) corresponding to \( G_S \) under the Galois correspondence. Then there is a torus \( T_S \) defined over \( K_S \) by taking its cocharacter lattice to be \( \mathbb{Z}^{[S^c]} \), where the coordinates are indexed by \( \mu \in S^c \), and \( G_S \) acts by permuting these. We have then a map of group schemes \( \alpha : T_S \rightarrow T \) over \( K_S \) given by the map of cocharacter lattices

\[
\alpha_* : \mathbb{Z}^{[S^c]} \rightarrow X_*(T)
\]

\[
(0, \ldots, 1, \ldots, 0) \mapsto \mu,
\]

where the 1 is in the \( \mu \)-slot and the other coordinates are all 0. Now, given a \( K_S \)-point \( x \) of \( T \), we define the (not necessarily irreducible) variety \( \alpha^{-1}(x) \) to be the fiber of \( \alpha \) over \( x \). Since \( \alpha^{-1}(x) \) is
noetherian, it has finitely many irreducible components. Let \( p(\alpha^{-1}(x)) \) denote the set of irreducible components. There is a continuous action of \( G_{K_S} \) on \( p(\alpha^{-1}(x)) \), and therefore there exists a finite extension \( K'_e \) of \( K_S \) such that all geometric components of \( \alpha^{-1}(x) \) are defined over \( K'_e \). Without loss of generality, we may assume that \( \tilde{K} \subseteq K'_e \). It will be shown later that if \( x = 1 \), then all of the components of \( \alpha^{-1}(1) \) are in fact defined over \( \tilde{K} \). On the other hand, the next lemma shows that we can pick a single \( K'/k \) over which all geometric components of \( \alpha^{-1}(x) \) for all \( x, S \) are defined.

**Lemma 3.3.** Let \( \alpha : T_1 \to T_2 \) be a map of tori defined over a number field \( k \). Every component of \( \alpha^{-1}(x) \) for all \( x \in U_N(T_2) \) is defined over a single finite extension of \( k \). The number of geometric components of \( \alpha^{-1}(x) \) is uniformly bounded as a function of \( x \).

**Proof.** Given \( \alpha^{-1}(x), \alpha^{-1}(y) \) two fibers with fields of definition of their irreducible components \( K_1 \) and \( K_2 \), we show that all of the components of \( \alpha^{-1}(xy) \) are defined over the compositum \( K_1 K_2 \).

Suppose more generally that \( X_1 \) and \( X_2 \) are varieties over \( k \) which have all of their irreducible components defined over \( K_1 \) and \( K_2 \), respectively. Then, all of the components of \( X_1 \times X_2 \) are defined over \( K_1 K_2 \). Indeed, if \( C \) is a irreducible component of \( X_1 \times X_2 \), then the projection maps \( \pi_1 \) and \( \pi_2 \) send \( \pi_1 : C \to C_1 \) and \( \pi_2 : C \to C_2 \), where \( C_1 \) and \( C_2 \) are irreducible components of \( X_1 \) and \( X_2 \). By the universal property, \( C \to C_1 \times C_2 \), and \( C_1 \times C_2 \) is defined over \( K_1 K_2 \), so \( C \) is as well.

We have an isomorphism

\[
\alpha^{-1}(x) \times \alpha^{-1}(y) \cong \alpha^{-1}(xy) \times \alpha^{-1}(y^{-1})
\]

\[(u, v) \mapsto (uv, v^{-1}),\]

which thus induces a bijection on components. Let \( K_{12} \) be a field of definition of all the irreducible components of \( \alpha^{-1}(xy) \). Then we have that any component is defined over \( K_1 K_2 \) and also over \( K_{12} K_2 \), so \( K_1 K_2 \) is an extension of \( K_{12} \), and thus all components of \( \alpha^{-1}(xy) \) are defined over \( K_{12} K_2 \).

The norm-units \( U_N(T) \) form a finitely generated abelian group by [Shy77], Lemmas 2.10 and 2.11. Let \( \zeta, \epsilon_1, \ldots, \epsilon_s \) be generators, and let \( K_0, \ldots, K_s \) be fields over which all irreducible components of \( \alpha^{-1}(\zeta), \alpha^{-1}(\epsilon_1), \ldots, \alpha^{-1}(\epsilon_s) \) are defined, respectively. Then we have that all components of all \( \alpha^{-1}(x) \), as \( x \) varies over \( U_N(T) \) are defined over \( K_0, \ldots, K_s \), a finite extension of \( k \).

The second assertion follows from e.g. [Sta18, Tag 055A] and quasicompactness. \( \square \)

By Lemma 3.3 we may take \( \tilde{K} \subseteq K' \) to be a finite Galois extension of \( K \) over which all irreducible components of all \( \alpha^{-1}(x) \) for all \( S, x \) are defined. Let \( \Gamma = \text{Gal}(K'/k) \). The finite group \( \Gamma \) acts on the set of irreducible components \( p(\alpha^{-1}(x)) \) for any \( x \). We will show later that if \( x = 1 \), the action of \( \Gamma \) on \( \alpha^{-1}(1) \) factors through \( \Gamma \to \hat{G} \).

Now suppose \( p \) is an unramified prime of \( k \). The valuation \( w \) extending \( v \) from the first paragraph of this section determines a unique prime \( \mathfrak{P} \) of \( K \) lying over \( p \). We also choose for each \( \mathfrak{P} \), a prime \( \mathfrak{P}' \) of \( K' \) lying over \( \mathfrak{P} \). Under the restriction map \( \Gamma \to G \), \( Fr_{\mathfrak{P}'} \) maps to \( Fr_{\mathfrak{P}} \).

If \( Fr_{\mathfrak{P}} \in G \) fixes \( S \), i.e. \( Fr_{\mathfrak{P}} \in G_S \), then \( T_S \) is defined over \( k_p \), the completion of \( k \) at \( p \). There exists a model for \( T \) over the ring of integers \( \mathcal{O}_k \) of \( k \) called the canonical model [Vos98, Ch. IV, §11.2], which we denote by \( \mathcal{T} \). The \( \mathcal{O}_k \)-scheme \( \mathcal{T} \) has the property that for each prime \( p \) of \( \mathcal{O}_k \) and each completion \( k_p \) of \( k \) with ring of integers \( \mathcal{O}_p \) we have that \( \mathcal{T}(\mathcal{O}_p) \) is the unique (see [Ono61, §2.1]) maximal compact subgroup of \( T(k_p) \). Therefore, each of \( T, T_S \), and \( \alpha^{-1}(x) \) is a variety defined over \( \mathcal{O}_p \) for each prime \( p \) of \( k \) for which \( Fr_{\mathfrak{P}} \) fixes \( S \), and so each fiber \( \alpha^{-1}(x) \) is also defined over the residue field \( f_p \) of \( k_p \) and has all of its geometric components defined over the residue field of \( K_{\mathfrak{P}'} \).

### 3.3. Proof of second assertion of Theorem 1.1

**Theorem 3.4.** If \( r \mid \mathcal{T} \) is not faithful, then \( \nu(\{ x \in A(T) : c(x, r) \leq X \}) = \infty \) for some finite \( X \).
Lemma 3.5. There exist constants $K, \varepsilon > 0$ and a subset $X(\chi_f) \subseteq \chi_f^{-1}V_{\infty}$ for each $\chi_f \in V_{\text{fin}}$ such that $\nu(X(\chi_f)) > \varepsilon$ and

$$\sup \{ \prod_{v \notin S_{\infty}} c_v(\chi_v, r) : \chi_{\infty} \in X(\chi_f) \} \leq K$$

for all $\chi_f \in V_{\text{fin}}$.

Proof. We give an explicit description of the sets $\chi_f^{-1}V_{\infty}$ in terms of the corresponding Langlands parameters across \([2.3]\). Let $x_1, \ldots, x_s$ be generators for $U_N(T)$. Recall $\mathbf{T}$ from \([2.20]\). For $v \in S_{\infty}$ let $\sigma_v : U_N(T) \hookrightarrow T(k_v) \simeq \mathbf{T}$ be the corresponding embedding. As in subsection \([2.4]\) we write

$$\sigma_v x_i = (\ldots, x_{vi}, \ldots, x'_{vi}, \ldots, x''_{vi}, \ldots),$$

where $x_{vi} \in R_{\infty}$, $x'_{vi} \in S_{\infty}$, and $x''_{vi} \in C_{\infty}$.

Consider the image of $\chi_f^{-1}V_{\infty}$ across the map

$$\prod_{v \in S_{\infty}} T(k_v)^{\wedge} \rightarrow \prod_{v \in S_{\infty}} \mathbf{T}^{\wedge},$$

where $\mathbf{T}^{\wedge}$ was given in \([2.22]\). We write elements of $\prod_{v \in S_{\infty}} \mathbf{T}^{\wedge}$ as $((w_v, \epsilon_v), (\alpha_v, (w'_v, \alpha'_v)))_{v \in S_{\infty}}$. Then the image of $\chi_f^{-1}V_{\infty}$ across \([3.12]\) is an affine hyperplane in $\prod_{v \in S_{\infty}} \mathbf{T}^{\wedge}$ cut out by

$$\prod_{v \in S_{\infty}} \prod_{j=1}^{n_1} (\text{sgn } x_{vi})^{\epsilon_j} |x_{vi}|^{w_j} \prod_{j=1}^{n_2} x''_{vi}^{\alpha_j} \prod_{j=1}^{n_3} x''_{vi}^{\alpha_j} |x''_{vi}|^{w'_j - \alpha'_j} = \chi_f(x_i)^{-1}.$$
for all generators $x_i, i = 1, \ldots, s$ of $U_N(T)$. Since $\chi_f(x_i) \in S^1$ for all $\chi_f$ and $x_i$, the affine hyperplane in $\prod_{v \in S_x} T^\wedge$ described by (3.13) intersects a fixed (independent of $\chi_f$) compact set around the origin, say $U_0$, in a set of positive measure bounded below independently of $\chi_f$.

For each $v \in S_x$, the set $\{\chi \in T(k_v)^\wedge : c_v(\chi, r) \leq X\}$ is in bijection under the local Langlands correspondence with

$$(3.14) \quad H_v = \{\varphi : \prod_{i=1}^{m_1+m_2} ((M\varphi)_i - 1) \prod_{i=m_1+m_2+1}^{m_1+m_2+m_3} ((M\varphi)_i - 1)^2 \leq X\}$$

by the results of section 2.4. Let

$$L_v(X) = \{\varphi : \sum_{i=1}^{m_1+m_2} ((M\varphi)_i - 1) \sum_{i=m_1+m_2+1}^{m_1+m_2+m_3} ((M\varphi)_i - 1)^2 \leq (m_1 + m_2 + m_3)X^{1/(m_1+m_2+m_3)}\}.$$ 

By the am-gm inequality, we have $L_v(X) \subseteq H_v(X)$. If $X$ is sufficiently large, then $L_v(X)$ contains any fixed compact set in $T^\wedge$, and in particular $U_0 \subseteq \prod_{v \in S_x} L_v(X)$. Taking $K = X^{1/S_x}$, the lemma is proved. \hfill \Box

The fibered set $V_{\text{fin},0} \times X(\chi_f) \subseteq V(T)$ is our candidate for a set of infinite measure and bounded analytic conductor. By Lemma 3.5 and additivity of measure we have

$$\nu(V_{\text{fin},0} \times X(\chi_f)) = \nu \left( \bigcup_{\chi_f} \{(\chi_x, \chi_f) : \chi_x \in X(\chi_f)\} \right) = \sum_{\chi_f} \nu(\{(\chi_x, \chi_f) : \chi_x \in X(\chi_f)\}) \geq \infty,$$

yet $c(\chi, r) \leq K$ for any $\chi = (\chi_x, \chi_f) \in V_{\text{fin},0} \times X(\chi_f)$. \hfill \Box

4. Unramified Computations

In this section $T$ is a torus defined over a non-archimedean local field $F$, splitting over an unramified extension $L/F$ with group $G = \text{Gal}(L/F)$. Recall from Proposition 2.12 the isomorphism

$$\text{Hom}_\mathbb{C}(\mathcal{O}_L^\wedge, \hat{T}) \simeq T(\mathcal{O}_F)^\wedge \quad \xi \mapsto \chi_{\xi}.$$ 

By Proposition 2.9 we have

$$(4.1) \quad U_v(s, x) = \sum_{\chi \in T(\mathcal{O}_F)^\wedge} \frac{\chi(x)}{\hat{c}(\chi, r)^s} = (\delta(\psi)dx/dx')^m \sum_{\xi \in \hat{\text{Hom}}_\mathbb{C}(\mathcal{O}_L^\wedge, \hat{T})} \chi_{\xi}(x)q_F^{-s} \sum_{\mu \in \mathbb{C}} c(\mu, \xi).$$

Recall from the introduction the set $M$ of co-weights of $r$. Let $\mathbb{N}$ denote the set of non-negative integers, and let us index the coordinates of $\mathbb{N}^M$ by $\mu \in M$. For each $c = (c_\mu)_{\mu \in M} \in \mathbb{N}^M$, consider the following sets of Langlands parameters (restricted to $\mathcal{O}_L^\wedge$):

$$(4.2) \quad P_{\leq}(c) = \{\xi \in \text{Hom}_\mathbb{C}(\mathcal{O}_L^\wedge, \hat{T}) : c(\mu, \xi) \leq c_\mu, \text{ for all } \mu \in M\}$$

and

$$(4.3) \quad P_\geq(c) = \{\xi \in \text{Hom}_\mathbb{C}(\mathcal{O}_L^\wedge, \hat{T}) : c(\mu, \xi) = c_\mu, \text{ for all } \mu \in M\}.$$ 

If $r|_\hat{T}$ is faithful, then the sets $P_{\leq}(c)$ and $P_\geq(c)$ are finite, and in that case we consider the sums

$$(4.4) \quad \Pi_{\leq}(c, x) = \sum_{\xi \in P_{\leq}(c)} \chi_{\xi}(x) \quad \text{and} \quad \Pi_{\geq}(c, x) = \sum_{\xi \in P_{\geq}(c)} \chi_{\xi}(x).$$

For example, $\Pi_{\leq}(c, 1) = |P_{\leq}(c)|$ and $\Pi_{\geq}(c, 1) = |P_{\geq}(c)|$. 

Thus the product in the statement of the proposition is finite, running up to
for all
coordinates and let
admits an action of
the latter set. Since
the coefficients of
Proposition 4.1. Suppose \( r|_{\mathcal{F}} \) is faithful, \( c \in \mathbb{N}^M \) is \( G \)-fixed, \( L/F \) is unramified, and \( (q_F, \lambda) = 1 \). If \( \chi_{\xi}(x) = 1 \) for all \( \xi \in P_{\xi}(c) \), then
and if there exists \( \xi \in P_{\xi}(c) \) such that \( \chi_{\xi}(x) \neq 1 \) then \( P_{\xi}(c, x) = 0 \).
Proof. Suppose that \( \chi_{\xi}(x) = 1 \) for all \( \xi \in P_{\xi}(c) \). Then \( P_{\xi}(c, x) = |P_{\xi}(c)| \) and it suffices to count the latter set. Since \( r|_{\mathcal{F}} \) is faithful, then \( D_k(c) = \{1\} \) for sufficiently large \( k \in \mathbb{N} \), and so there exists
Thus the product in the statement of the proposition is finite, running up to \( k_0 - 1 \).
A parameter \( \xi \) is in \( P_{\xi}(c) \) if and only if
for all \( k \in \mathbb{N} \). In particular, every \( \xi \in P_{\xi}(c) \) is trivial on \( 1 + \mathfrak{P}^{k_0} \). We inductively construct all of the \( \xi \in P_{\xi}(c) \) by extending the trivial homomorphism \( 1 + \mathfrak{P}^{k_0} \to \mathcal{T} \) backwards along the standard filtration.
Consider two base cases: \( k_0 = 0 \) and \( k_0 = 1 \). If \( c \) is such that \( k_0 = 0 \) then \( D_k(c) = \{1\} \) for all \( k \in \mathbb{N} \) and \( P_{\xi}(c) = \{1\} \), so the formula in the statement of the proposition holds. If \( c \) is such that \( k_0 = 1 \) then \( \xi(1 + \mathfrak{P}) = \{1\} \) for all \( \xi \in P_{\xi}(c) \), and the possible extensions of \( \xi \) to \( \mathcal{O}_L^\times \) are parametrized by
So the formula in the statement of the proposition holds.
Now suppose as the induction hypothesis that
\[
|P_{\xi}(c)| = |\text{Hom}_G(\ell^\times, D_0(c))| \prod_{k=1}^{k_0-1} |\text{Hom}_G(\ell, D_k(c))|
\]
for all $c$ such that $k_0 \leq K$. Consider $c$ such that $k_0 = K + 1$. Then all $\xi \in P_\leq(c)$ satisfy $\xi(1 + \mathfrak{P}^{K+1}) = \{1\}$, and the possible extensions the trivial map $1 + \mathfrak{P}^K \to \hat{T}$ to elements of $\text{Hom}_G((1 + \mathfrak{P}^K)/\mathfrak{P}^{K+1}, D_K(c))$ are parameterized by

$$\text{Hom}_G((1 + \mathfrak{P}^K)/\mathfrak{P}^{K+1}, D_K(c)) \simeq \text{Hom}_G(\ell, D_K(c)),$$

since $L/F$ is unramified. Therefore (4.8) holds for $c$ such that $k_0 = K + 1$. By induction, (4.8) holds for all $c \in \mathbb{N}^M$.

By the normal basis theorem, there exists $\alpha \in \ell$ such that

$$\{\alpha, \alpha q_{\ell}, \alpha q_{\ell}^2, \ldots, \alpha q_{\ell}^{\dim D_k(c)}\}$$

is a basis for $\ell$ over the residue field of $F$. A $G$-equivariant homomorphism in $\text{Hom}_G(\ell, D_k(c))$ is determined by its value on $\alpha$, which is of additive order $\text{char}(\ell)$ in $\ell$. Since $(q_{\ell}, \lambda) = 1$, the element $\alpha$ cannot map non-trivially into the component group of any $D_k(c)$. There are

$$\text{char}(\ell)^{\dim D_k(c)}$$

elements of order dividing $\text{char}(\ell)$ in the connected component of the identity of $D_k(c)$. Hence

$$|\text{Hom}_G(\ell, D_k(c))| = \text{char}(\ell)^{\dim D_k(c)},$$

and we have shown the first part of the Proposition.

If there exists $\xi \in P_\leq(c)$ such that $\chi_\xi(x) \neq 1$, then it immediately follows from orthogonality of characters that $\Pi_\leq(c, x) = 0$, hence the second part of the proposition.

Proposition 4.1 is only valid for $G$-fixed $c \in \mathbb{N}^M$ (since otherwise $D_k(c)$ is not a $G$-module, and $G$-equivariant homomorphisms into $D_k(c)$ do not make any sense). However, we can always reduce to the case that $c$ is $G$-fixed by the following Lemma.

Lemma 4.2. If $c \in \mathbb{N}^M$ is not $G$-fixed, then $\Pi_\leq(c, x) = 0$.

Proof. Suppose $c$ is not fixed by $G$, so that $|M| \geq 2$. Without loss of generality suppose there exists $\sigma \in G$ such that $\mu^\sigma = \mu'$ but that $c_{\mu'} > c_{\mu}$. Suppose for a contradiction that there exists $\xi \in \text{Hom}_G(\mathfrak{o}_L^x, \hat{T})$ such that $c(\mu \circ \xi) = c_{\mu}$ and $c(\mu' \circ \xi) = c_{\mu'}$. If $z \in \mathfrak{o}_L^x$ then the Galois equivariance of $\xi$ says

$$\mu \circ \xi(\sigma z) = \mu' \circ \xi(z).$$

If $z \in 1 + \mathfrak{P}^{c_{\mu}}$, then we also have $\sigma z \in 1 + \mathfrak{P}^{c_{\mu}}$. But then $c(\mu \circ \xi) \leq c_{\mu}$ implies that $c(\mu \circ \xi \circ \sigma) \leq c_{\mu}$, and $c(\mu' \circ \xi) = c(\mu \circ \xi \circ \sigma)$, so $c_{\mu'} = c(\mu' \circ \xi) \leq c_{\mu}$, contradiction.

Later, in Lemma 4.3, we shall see that only the $c \in \mathbb{N}^M$ all of whose entries are 0 or 1 will matter for the location and order of the rightmost pole of $Z(s)$. Therefore we spend the rest of the section devoting particular attention to this case.

If all of the entries of $c$ are 0 or 1, then $D_k(c) = \{1\}$ for all $k \geq 1$. Therefore we restrict our attention to the case $k = 0$. We make a change of variables, and instead consider subsets $S \subseteq M$ as in the introduction. The change of variables is given by the $G$-equivariant bijection

(4.9) \{0, 1\}^M \simeq 2^M

(4.10) $c \leftrightarrow \{\mu : c_{\mu} = 1\},$

with $G$ acting on $2^M$ as in the introduction. Define the quantities $\Pi_\leq(S, x)$ and $\Pi_\geq(S, x)$ via the above bijection $c \leftrightarrow S$ in terms of $\Pi_\leq(c, x)$ and $\Pi_\geq(c, x)$, and define $D(S) = D_0(c)$ as in the introduction.

Let $\text{Fr} \in G$ denote the Frobenius element. By Lemma 4.2, it is no loss of generality to suppose that $\text{Fr} S = S$. Recall from section 3 the map of tori $\alpha : T_S \to T$. We saw in that section that the fiber $\alpha^{-1}(x)$ is a variety defined over the residue field $f$ of $F$, and that there is a finite Galois extension.
Suppose \( p(\alpha^{-1}(x)) \) is the finite set of geometric components. There is a continuous action of \( \text{Gal}(L'/F) \) on \( p(\alpha^{-1}(x)) \). Let \( \text{Fr}' \in \text{Gal}(L'/F) \) denote a Frobenius automorphism, and write

\[
\tag{4.11}
a(S, x) = \#\{y \in p(\alpha^{-1}(x)) : \text{Fr}' y = y\}.
\]

The number \( a(S, x) \) does not depend on the choice of \( \text{Fr}' \), since the inertia subgroup of \( \text{Gal}(L'/F) \) acts trivially on \( \alpha^{-1}(x) \).

**Lemma 4.3.** Suppose \( \text{Fr}S = S \), and that \( L/F \) is unramified. Then

\[
\Pi_{\xi}(S, x) = \begin{cases} 
(a(S, x) + O_{T,x}(q_F^{-1/2})) q_F^{\dim D(S)} & \text{if } \dim D(S) \geq 1 \\
a(S, x) & \text{if } D(S) \text{ is finite}.
\end{cases}
\]

**Proof.** Let \( f, \ell \) be the residue fields of \( F, L \). By definition of \( T \) and \( T_S \), we have exact sequences

\[
1 \longrightarrow D(S) \longrightarrow \hat{T} \longrightarrow \hat{T}_S,
\]

and

\[
1 \longrightarrow \text{Hom}_G(\ell^\times, D(S)) \longrightarrow \text{Hom}_G(\ell^\times, \hat{T}) \longrightarrow \text{Hom}_G(\ell^\times, \hat{T}_S).
\]

Since \( L/F \) is unramified and \( \text{Fr} \) fixes \( S \), we have \( T_S \) and \( T \) are defined over \( f \). By Pontryagin duality we have an exact sequence

\[
1 \longrightarrow (T(f)/\alpha(T_S(f)))^\wedge \longrightarrow T(f)^\wedge \longrightarrow T_S(f)^\wedge.
\]

By Proposition 2.12 we have that the local Langlands correspondence for tori restricts to

\[
T(f)^\wedge \simeq \text{Hom}_G(\ell^\times, \hat{T}),
\]

and likewise for \( T_S \), since they both split over \( L/F \), which is unramified. Therefore we have

\[
1 \longrightarrow \text{Hom}_G(\ell^\times, D(S)) \longrightarrow \text{Hom}_G(\ell^\times, \hat{T}) \longrightarrow \text{Hom}_G(\ell^\times, \hat{T}_S)
\]

\[
\text{Hom}_G(\ell^\times, D(S)) \simeq (T(f)/\alpha(T_S(f)))^\wedge.
\]

By the five lemma of homological algebra, we conclude that

\[
\text{Hom}_G(\ell^\times, D(S)) \simeq (T(f)/\alpha(T_S(f)))^\wedge.
\]

By orthogonality of characters we have

\[
\Pi_{\xi}(S, x) = \sum_{\xi \in \text{Hom}_G(\ell^\times, D(S))} \chi_{\xi}(x) = \begin{cases} 
|\text{Hom}_G(\ell^\times, D(S))| & \text{if } x \in \alpha(T_S(f)) \subset T(f) \\
0 & \text{if } x \notin \alpha(T_S(f)).
\end{cases}
\]

If it is the case that \( x \in \alpha(T_S(f)) \), then

\[
\Pi_{\xi}(S, x) = |\text{Hom}_G(\ell^\times, D(S))| = \frac{|T(f)|}{|\alpha(T_S(f))|} = \frac{|T(f)|}{|T_S(f)|} |\ker(\alpha : T_S(f) \to T(f))|.
\]

In either case of \( x \in \alpha(T_S(f)) \) or not, we have that

\[
\Pi_{\xi}(S, x) = \frac{|T(f)|}{|T_S(f)|} |\alpha^{-1}(x)|.
\]

We use the Lang-Weil theorem \cite{LW54} to count the number of points on the \( \alpha^{-1}(x) \) over finite fields. We quote the following version of the Lang-Weil theorem from T.Tao’s blog “The Lang-Weil
bound”, 31 Aug, 2012. Define the “complexity” of a variety $V$ as follows. Let $\overline{F}$ denote an algebraic closure of a finite field $f$. If $V$ is defined over $\overline{F}$ by

$$V = \{ x \in \overline{F}^d : P_1(x) = \ldots = P_m(x) = 0 \},$$

then the complexity $C$ is defined to be the maximum of $d, m$, and the degrees of the $P_i$.

**Theorem 4.4** (Lang-Weil). Let $V$ be a variety of complexity at most $C$ defined over $\overline{F}$. Then one has

$$|V(f)| = (p(V) + O_C(|f|^{1/2}))|f|^{\dim(V)}$$

where $p(V)$ is the number of geometrically irreducible components of $V$ of dimension $\dim(V)$ that are invariant with respect to the Frobenius endomorphism $x \mapsto x^{|f|}$ associated to $f$.

If in fact the dimension of $V$ is zero, it is not hard to see that $|V(f)| = p(V)$.

We will apply the Lang-Weil theorem to the varieties $X$ and $\overline{F}$, and by e.g. [Mil17, Rem. 5.42] that

$$\dim \alpha^{-1}(x) + \dim \alpha(T_S) = \dim T_S,$$

and by [Mil17] A.73 that

$$\dim D(S) = \dim T - \dim \alpha(T_S).$$

Combining these, we have

$$\dim T - \dim T_S + \dim \alpha^{-1}(x) = \dim D(S).$$

From the Lang-Weil theorem, we conclude the lemma. 

In the special case that the global unit $x = 1$ we state the leading constant in Lemma 4.3 in a more convenient fashion. Let $\Fr \in G$ and

$$a(S) = |\{ y \in \pi_0(D(S)) : \Fr y^{q^F} = y \}|.$$

**Lemma 4.5.** Suppose $\Fr S = S$, and that $L/F$ is unramified. Then $a(S, 1) = a(S)$.

**Proof.** We give an alternate computation of $|\Hom_G(\ell^\times, \hat{F})|$. Let $x$ denote a generator for the cyclic group $\ell^\times$. Then $\Hom_G(\ell^\times, D(S))$ is in bijection with the set $\{ z \in D(S) : \Fr z = z^{q^F} \}$ of possible images of $x$ in $D(S)$. This set is equal to the kernel $K$ of the $G$-equivariant homomorphism

$$D(S) \to D(S)$$

given by

$$z \mapsto \frac{z^{q^F}}{\Fr z}.$$

We have an exact sequence of $G$-modules

$$X^*(D(S)) \xrightarrow{\varphi} X^*(D(S)) \longrightarrow X^*(K) \longrightarrow 1.$$  

The map $\varphi$ is given by $\varphi(\chi) = q^F \chi - \chi^{\Fr}$, where we have written $X^*(D(S))$ in additive notation. Our goal is to compute the cardinality of $X^*(K)$, which equals the cardinality of $K$ itself.

Write $X = X^*(D(S))$, $X_t$ for the torsion subgroup, and $X_f = X/X_t$. The map $\varphi : X \to X$ induces maps $X_t \to X_t$ and $X_f \to X_f$, both of which we also denote $\varphi$. We write $Q = X^*(K)$.
for the cokernel of \( \varphi : X \to X \), \( Q_t \) for the cokernel of \( \varphi : X_t \to X_t \), and \( Q_f \) for the cokernel of \( \varphi : X_f \to X_f \). In summary, we have a commutative diagram

\[
\begin{array}{c}
1 \quad X_t \quad X \quad X_f \quad 1 \\
\downarrow \varphi \quad \downarrow \varphi \quad \downarrow \varphi \\
1 \quad X_t \quad X \quad X_f \quad 1 \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
Q_t \quad Q \quad Q_f \\
\end{array}
\]

The lower right horizontal arrow is surjective since both arrows on the other side of the square it forms are both surjective.

We show that the top right \( \varphi \) is injective. Indeed, let \( \chi \in X_f \) satisfy \( \varphi(\chi) = 0 \). Since \( \varphi \) is \( G \)-equivariant, we also have \( \varphi(\chi^{F_i}) = 0 \), for all \( i \). Since \( \varphi(\chi) = 0 \) we have \( q_F \chi = \chi^{F_i} \), and so \( \chi^{F_i} \equiv 0 \pmod{q_F} \). But similarly, since \( \varphi(\chi^{F_i}) = 0 \) we have that \( \chi^{F_i} \equiv 0 \pmod{q_F^2} \). Therefore \( \chi \equiv 0 \pmod{q_F^2} \). Repeating this process ad infinitum, we conclude that \( \chi = 0 \in X_f \), so the top right \( \varphi \) is injective.

Then by the snake lemma we have that \( Q_t \to Q \), and so the bottom row of the diagram forms an exact sequence of finitely-generated abelian groups. We have that \( Q \) is finite if both \( Q_t \) and \( Q_f \) are, and in this case \( |Q| = |Q_t||Q_f| \).

Let us begin with \( Q_f \). The map \( \varphi \) on \( X_f \) is given in matrices by \( q_F I - A \), where \( A \) is some matrix of integers for which \( A[G] = I \). Putting \( q_F I - A \) in Smith normal form \( q_F I - A = UDV \) with \( U, V \in \text{GL}_{\dim D(S)}(\mathbb{Z}) \), we have

\[
Q_f \cong \frac{\mathbb{Z}}{d_1 \mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{d_{\dim D(S)} \mathbb{Z}},
\]

with each \( d_i \) is equal to \( q_F \pm 1 \), and so \( |Q_f| = q_F^{\dim D(S)}(1 + O(q_F^{-1})) \).

Now we compute \( |Q_t| \). For any endomorphism of a finite abelian group \( f : A \to A \), we have that \( |\ker f| = |\operatorname{coker} f| \). Let \( K_t \) be the kernel of \( \varphi : X_t \to X_t \), which therefore has the same cardinality as \( Q_t \). But the cardinality of \( K_t \) is exactly the quantity \( a(S) \) defined above the statement of the Lemma.

We have shown that \( |Q_t| = a(S) \) and \( |Q_f| = q_F^{\dim D(S)}(1 + O(q_F^{-1})) \), and since \( |Q| = |Q_t||Q_f| \), we conclude the lemma.

Finally, we apply the foregoing results on \( \Pi_\varepsilon(S, x) \) to derive the final results for \( \Pi_\varepsilon(S, x) \). Let

\[
\mu(i \in T) = \begin{cases} 
1 & \text{if } i \notin T \\
-1 & \text{if } i \in T.
\end{cases}
\]

The main tool is (4.6), which we re-state for the sets \( S \) as

\[
(4.13) \quad \Pi_\varepsilon(S, x) = \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) \Pi_\varepsilon(S - T, x).
\]

For a set \( S \) we denote

\[
(4.14) \quad S_{\text{red}} = \{ \mu \in S : \sigma \mu \in S \text{ for all } \sigma \in G \}.
\]

The set \( S_{\text{red}} \) is now \( G \)-fixed, and by (4.13) and Lemma 4.2 we have

\[
(4.15) \quad \Pi_\varepsilon(S, x) = \Pi_\varepsilon(S_{\text{red}}, x).
\]

Recall from (3.4) the positive rational \( A \).
Lemma 4.6. For any $\emptyset \neq S \subseteq M$ such that $\text{Fr} S = S$, and

$$\frac{\dim D(S) + 1}{|S|} \geq A,$$

we have

$$\Pi_\geq(S, x) = \begin{cases} \left( a(S, x) + O(q_F^{-1/2}) \right) q_F^{\dim D(S)} & \text{if } \dim D(S) \geq 1 \\ a(S, x) - 1 & \text{if } \dim D(S) = 0 \text{ and } D(S) \neq \{1\} \\ 0 & \text{if } D(S) = \{1\}. \end{cases}$$

Proof. Suppose first that $\dim D(S) = 0$ and $a(S, x) \neq 0$. Then for any $T \subseteq S$ we also have $\dim D(T) = 0$ since $D(T) \subseteq D(S)$. By (4.13) and (4.15) we have

$$\Pi_\geq(S, x) = \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) \Pi_{\leq((S - T)_{\text{red}}, x)}.$$

Since $a(S, x) \neq 0$, we have by Lemma 4.3 and Proposition 4.1 that $\chi_\ell(x) = 1$ for all $\xi \in P_\ell(S)$. Then for any $S' \subseteq S$ we also have $\chi_\ell(x) = 1$ for all $\xi \in P_\ell(S')$. Using Proposition 4.1 and Lemma 4.3 again, we have

$$\Pi_\geq(S, x) = a(S, x) + \sum_{\emptyset \neq T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) a((S - T)_{\text{red}}, x).$$

For any $\emptyset \neq T \subseteq S$ we have

$$\frac{\dim D(T) + 1}{|T|} = \frac{1}{|T|} \geq \frac{1}{|S|} = \frac{\dim D(S) + 1}{|S|} \geq A,$$

thus $D(T) = \{1\}$ by definition of $A$. For all $T \neq \emptyset$, we have $(S - T)_{\text{red}} \subseteq S$. Thus, if $(S - T)_{\text{red}} \neq \emptyset$, then we have $D((S - T)_{\text{red}}) = \{1\}$. On the other hand, if $(S - T)_{\text{red}} = \emptyset$, then we also have $D((S - T)_{\text{red}}) = \{1\}$ by the fidelity of $r|_{\tilde{F}}$. Therefore

$$\Pi_\geq(S, x) = a(S, x) + \sum_{\emptyset \neq T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T)$$

$$= a(S, x) - 1 + \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T)$$

$$= a(S, x) - 1,$$

since we assumed $S \neq \emptyset$.

Now suppose that $\dim D(S) \geq 1$ and $a(S, x) = 0$. As above, we have by (4.13), (4.15), Proposition 4.1 and Lemma 4.3 that

$$\Pi_\geq(S, x) = \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) \Pi_{\leq((S - T)_{\text{red}}, x)}$$

$$= q_F^{\dim D(S)} \left( 1 + O(q_F^{-1/2}) \right) \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) a((S - T)_{\text{red}}, x),$$

as $q_F \to \infty$. Suppose that $T$ is such that $\dim D((S - T)_{\text{red}}) = \dim D(S)$. If $(S - T)_{\text{red}} = \emptyset$, then

$$1 \leq \dim D(S) = \dim D((S - T)_{\text{red}}) = 0,$$

which is a contradiction with the fidelity of $r|_{\tilde{F}}$. Therefore we may assume that $(S - T)_{\text{red}} \neq \emptyset$. If $T \neq \emptyset$ then

$$\frac{\dim D((S - T)_{\text{red}}) + 1}{|(S - T)_{\text{red}}|} = \frac{\dim D(S) + 1}{|(S - T)_{\text{red}}|} > \frac{\dim D(S) + 1}{|S|} \geq A.$$
Therefore $D((S - T)_{\text{red}}) = \{1\}$, and this is a contradiction with $\dim D((S - T)_{\text{red}}) = \dim D(S)$. Thus, the only $T \subseteq S$ which satisfies $\dim D((S - T)_{\text{red}}) = \dim D(S)$ is $T = \emptyset$, from which we conclude the statement in the lemma.

Now suppose $a(S, x) = 0$. If $\dim D(S) \geq 1$ then by (4.13), Lemma 4.3 and the triangle inequality, the statement of the lemma holds.

To finish the proof of lemma, it remains to consider the case that $\dim D(S) = 0$. If $D(S) = \{1\}$, then we must have $a(S, x) \neq 0$, so suppose that $\dim D(S) = 0$ and $D(S) \neq \{1\}$. Suppose $|S| \geq 2$ and $S$ is maximal such that $\dim D(S) = 0$ and $D(S) \neq \{1\}$. There exists $\mu \notin S$, since $D(M) = \hat{T}$. We claim that $D(S \cup \mu) \geq 1$. Indeed, by maximality, either $\dim D(S \cup \mu) \geq 1$ or $D(S \cup \mu) = \{1\}$. But the second of these can’t happen since $D(S) \neq \{1\}$ already, and $D(\cdot)$ is monotonic. So $\dim D(S \cup \mu) \geq 1$. But then
\[
\frac{\dim D(S \cup \mu) + 1}{|S \cup \mu|} \geq \frac{2}{|S| + 1} > \frac{1}{|S|} \geq A,
\]
since $|S| \geq 2$. This is a contradiction with the definition of $A$. Hence, $|S| = 1$. Then we have
\[
\Pi_{\xi}(S, x) = \sum_{T \subseteq S} \mu(1 \in T) \cdots \mu(m \in T) \Pi_{\xi}((S - T)_{\text{red}}, x)
\]
\[
= \Pi_{\xi}(S, x) - \Pi_{\xi}(\emptyset, x)
\]
\[
= -1.
\]
Here, $\Pi_{\xi}(S, x) = 0$ by the assumption that there exists $\xi \in P_{\xi}(S)$ with $\chi_{\xi}(x) \neq 1$, and $\Pi_{\xi}(\emptyset, x) = 1$ since only the trivial character appears in the definition of $\Pi_{\xi}(\emptyset, x)$. $\square$

5. Ramified computations

In this section, $T$ is a torus defined over a non-archimedean local field $F$ and splitting over a finite Galois extension $L$. Let $G = \text{Gal}(L/F)$ be the corresponding group. Let $p, \mathfrak{P}$ be the maximal ideals of $F, L$, respectively.

By Theorem 2.8, Corollary 2.13 and Proposition 2.9 we have
\[
|R_{s}(s, x)| \leq \sum_{\chi_{\nu} \in NT(\mathfrak{O}_{\nu})} \frac{1}{|\hat{\xi}(\chi_{\nu}, r)|^{\text{Re}(s)}}
\]
\[
= \sum_{\chi_{\nu} \in NT(\mathfrak{O}_{\nu})} \frac{1}{|\hat{\xi}(\chi_{\nu}, r)|^{\text{Re}(s)}}
\]
\[
= (\delta(\psi)dx/dx')^{m} \sum_{\xi \in \text{Hom}_{C}(\mathfrak{O}_{L}^{\times}, \hat{T})} q_{F}^{-\text{Re}(s)} \sum_{\mu \in \mathfrak{P}} \chi_{\xi}(\mu)^{\nu}.
\]
(5.1)

The conductor $c$ appearing in the last line of (5.1) is in fact $c = c_{\mathfrak{U}}$ as in Definition 2.23 where $\mathfrak{U}$ is the standard filtration on $\mathfrak{O}_{L}^{\times}$ (see Definition 2.23). Next, we construct yet another filtration and compare it to $\mathfrak{U}$.

By the normal basis theorem, there exists an element $\alpha \in L$ such that $\{\alpha^{g} : g \in G\}$ is a basis for $L/F$. The $\{\alpha^{g}\}$ all have the same valuation (e.g. [Ser79, Ch.2 Cor 3]), so by clearing numerators or denominators, there exists $\beta \in \mathfrak{O}_{L}^{\times}$ such that $\{\beta^{g} : g \in G\}$ is a basis for $L$. We define an injective map of $\mathfrak{O}_{F}[G]$-modules
\[
f : \mathfrak{O}_{F}[G] \rightarrow \mathfrak{O}_{L}
\]
by $f(1) = \beta$. Its image is finite-index in $\mathfrak{O}_{L}$ since $\{\beta^{g}\}$ span $L$.

Let $\nu \geq 1$ be sufficiently large so that the $p$-adic exponential function
\[
\exp : \mathfrak{P}^{\nu} \rightarrow 1 + \mathfrak{P}^{\nu}
\]
is well-defined and an isomorphism. Then let \( g : \mathcal{O}_F[G] \hookrightarrow \mathcal{O}_L^\times \) be defined as the composition of the following sequence of injective maps
\[
g : \mathcal{O}_F[G] \xrightarrow{f} \mathcal{O}_L \xrightarrow{\mathfrak{n}} \mathfrak{P}^\nu \xrightarrow{\exp} 1 + \mathfrak{P}^\nu \xrightarrow{\tau} \mathcal{O}_L^\times.
\]
The homomorphism \( g \) has finite cokernel. Let \( V^n = g(p^n\mathcal{O}_F[G]) \) and \( V = (V^n) \) be the corresponding filtration of \( \mathcal{O}_L^\times \). We have for all \( n \geq 0 \) that
\[
V^n \subseteq \mathcal{O}_L^{(\nu + e_{L/F}n)},
\]
where \( e_{L/F} \) is the ramification index of \( L/F \) and \( \nu \) is as above. Indeed, if \( x \in p^n\mathcal{O}_F[G] \) then we write \( x \) as
\[
x = \sum_{g \in G} a_g g
\]
with \( a_g \in p^n \) for all \( g \in G \). So we have
\[
f(x) = \sum_{g \in G} a_g \beta^g \in p^n = \mathfrak{P}^{e_{L/F}n},
\]
so that \( f(p^n\mathcal{O}_F[G]) \subseteq \mathfrak{P}^{e_{L/F}n} \).

Now let us consider the conductor \( c_V \) defined with respect to the filtration \( V \) and compare \( c_U \) and \( c_V \). Let \( \chi : \mathcal{O}_L^\times \to \mathbf{C}^\times \) be a character. If \( \chi|_{\mathcal{O}_L^\times} = 1 \) then \( c_U(\chi) \leq n \), and if \( \chi|_{\mathcal{O}_L^\times} \neq 1 \) then \( c_U(\chi) \geq n + 1 \). Similarly, if \( \chi|_{V^n} = 1 \) then \( c_V(\chi) \leq n \), and if \( \chi|_{V^n} \neq 1 \) then \( c_V(\chi) \geq n + 1 \). Therefore by (3.7.2) we have
\[
c_U(\chi) \geq e_{L/F}c_V(\chi) + \nu - e_{L/F} + 1.
\]

Then we have
\[
\sum_{\xi \in \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T})} q_F^{-\Re(s)\sum_{\mu} c_U(\mu\xi)} \leq \sum_{\xi \in \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T})} q_F^{-\Re(s)e_{L/F}\sum_{\mu} c_V(\mu\xi)}.
\]

But now the summand only depends on the restriction of \( \xi \) to \( V^0 \). We have an exact sequence
\[
1 \longrightarrow \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T}) \longrightarrow \text{Hom}_G(\mathcal{O}_L^\times, \hat{T}) \longrightarrow \text{Hom}_G(V^0, \hat{T}) \longrightarrow \cdots
\]
The kernel is a finite group, since \( \mathcal{O}_L^\times/V^0 \) is a finite group and \( \hat{T} \) only has finitely many points of order dividing the cardinality of this group. Thus we have
\[
\sum_{\xi \in \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T})} q_F^{-\Re(s)\sum_{\mu} e_{L/F}c_V(\mu\xi)} \leq \left| \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T}) \right| \sum_{\xi \in \text{Hom}_G(V^0, \hat{T})} q_F^{-\Re(s)e_{L/F}\sum_{\mu} c_V(\mu\xi)}.
\]
But also
\[
\text{Hom}_G(V^0, \hat{T}) \cong \text{Hom}_G(\mathcal{O}_F[G], \hat{T}) \cong \text{Hom}(\mathcal{O}_F, \hat{T}).
\]
If \( \xi \leftrightarrow \tau \in \text{Hom}(\mathcal{O}_F, \hat{T}) \) across this isomorphism, then \( c_V(\mu \circ \xi) = c_W(\mu \circ \tau) \), where the latter is the conductor with respect to the filtration \( W = (p^n) \) of the additive group \( \mathcal{O}_F \).

Therefore
\[
|R_v(s, x)| \leq (\delta(\psi)dx/dx')^n \left| \text{Hom}_G(\mathcal{O}_L^\times/V^0, \hat{T}) \right| \sum_{\tau \in \text{Hom}(\mathcal{O}_F, \hat{T})} q_F^{-\Re(s)e_{L/F}\sum_{\mu} c_W(\mu\tau)}.
\]

Then one computes similarly to section (3). Let
\[
\Pi_{\xi}(c) = \left| \{ \tau \in \text{Hom}(\mathcal{O}_F, \hat{T}) : c_W(\mu \circ \tau) \leq c_\mu \text{ for all } \mu \in M \} \right|.
\]
If \( r \mid_{\hat{T}} \) is faithful and \( c \in \mathbb{N}^M \) is \( G \)-fixed then

\[
\Pi_{\leq}(c) = \prod_{k=0}^{\infty} \left| \text{Hom}(p^{k-1}/p^k, D_k(c)) \right| = \prod_{k=0}^{\infty} q_F^\dim D_k(c).
\]

By the fidelity of \( r \mid_{\hat{T}} \), the product is actually a finite product.

We have therefore that

\[
\frac{|R_v(s,x)|}{(\delta(\psi)dx/dx')^m} \leq q_F^{-\Re(s)(\nu-e_{L/F}+1)} \left| \text{Hom}_G(O_L^*/V^0, T) \right| (1 - q_F^{-\Re(s)}\sum_{c \in \mathbb{N}^M} \prod_{k=0}^{\infty} q_F^{\dim D_k(c)}).
\]

Therefore, \( R_v(s,x) \) converges absolutely and uniformly on compacts for all

\[
\Re(s) > \frac{1}{e_{L/F}} \limsup_{c \in \mathbb{N}^M} |c|^{-1} \sum_{k \geq 0} \dim D_k(c) \leq \max\left\{ \frac{\dim D(S)}{|S|} : D(S) \neq \{1\} \right\}.
\]

6. Archimedean computations

Recall (3.10) for \( v \) an archimedean place of \( k, x \in T(k_v) \) that

\[
A_v(s,x) = \int_{T(k_v)} \frac{\chi(x)}{c(\chi,r)^s} d\chi.
\]

Assume that \( k_v \simeq \mathbb{R} \), and recall (2.10) that we have chosen an isomorphism

\[
T(k_v) \simeq T = (\mathbb{R}^x)^{n_1} \times (S^1)^{n_2} \times (\mathbb{C}^x)^{n_3}
\]

\[
x \mapsto (\ldots, x_j, \ldots, x_j', \ldots, x_j'', \ldots),
\]

with \( x_j \in \mathbb{R}^x, x_j' \in S^1, \) and \( x_j'' \in \mathbb{C}^x \). The case that \( k_v \simeq \mathbb{C} \) is very similar to the situation that \( n_1 = n_2 = 0 \) above, so we ignore it and assume that \( k_v \simeq \mathbb{R} \) for the remainder of this section.

**Theorem 6.1.**

1. The integral \( A_v(s,x) \) converges absolutely and uniformly on compacts in the domain

\[
\Re(s) > \sigma_0 = \max\left\{ \frac{\dim D(S)}{|S|} : \dim D(S) \geq 1 \right\}.
\]

2. For \( s \) in the above region of absolute convergence, we have

\[
A_v(s,x) \ll_{T,r,s} \prod_{1 \leq j \leq n_1} \frac{1}{1 + |\log |x_j||} \prod_{1 \leq j \leq n_3} \frac{1}{1 + |\log |x_j''||},
\]

with at most polynomial growth in \( s \) in a vertical strip.

3. For any real \( \sigma_0 < \sigma \leq 2 \), we have that \( A_v(\sigma,x) \) is non-negative real.

The proof of Theorem 6.1 will occupy the remainder of section 6 of this paper. In subsection 6.1 we reduce assertion (1) to a problem in combinatorial geometry (see Proposition 6.3). The main input in the proof of assertion (1) is a Brascamp-Lieb inequality, which is a heavy tool from pure analysis. In subsection 6.2 we give some background information on matroids and polymatroids, and in subsection 6.3 we solve the combinatorial geometry problem. Assertion (2) follows immediately. Finally, in subsection 6.4 we prove assertion (3) of Theorem 6.1.
Recall the definition \(2.28\) of the matrix \(M = M(r)\), which gives by \(2.32\) a map \(M : T^\alpha \rightarrow (iR \times \{0,1\})^{m_1 + m_2} \times (iR \times N)^{m_3}\). By Definition \(2.17\), \(2.33\), and \(2.34\) we have for some constant \(a\) depending only on the choice of \(\nu\) that

\[
A_v(s, x) = a \sum_{\alpha, \alpha' \in \mathbb{Z}^{m_2}} \sum_{\in (0,1)^{n_1}} \int_{w \in R} \int_{w' \in R} \chi_{w, \epsilon, \alpha, w', \alpha'}(x) \prod_{i=1}^{m_1 + m_2} ((|M \varphi_i| + 1)^s \prod_{i=m_1 + m_2 + 1}((|M \varphi_i| + 1)^{2s} dw \, dw',
\]

where \(\chi_{w, \epsilon, \alpha, w', \alpha'}\) is a unitary character of \(T = (R^\times)^n_1 \times (S^1)^{n_2} \times (C^\times)^{n_3}\), which was given explicitly in terms of \(w, \epsilon, \alpha, w', \alpha'\) in \(2.34\).

6.1. Convergence. We apply the triangle inequality \(A_v(s, x)\). For \(i = 1, \ldots, m_1 + m_2\), the \((M \varphi)_i \in iR\) or \(iR + 1\) by inspection of \(2.29\), \(2.30\). For such \(i\) we apply the inequality

\[
\frac{1}{\sqrt{x^2 + 1} + 1} \leq \frac{1}{|x| + 1}.
\]

Then we make the change of variables \(w_j \mapsto iw_j\) and \(w'_j \mapsto \alpha_j \mapsto iw'_j\), so that \(\chi \in T(k_v)^\alpha \) unitary implies that \(w_j, w'_j \in R\).

We introduce some notation to record the result (see \(6.3\)) of the aforementioned manipulations of \(A_v(s, x)\). Let \(M_{re}\) denote the \((m_1 + m_2 + m_3) \times (n_1 + n_3)\) matrix

\[
M_{re} = \begin{pmatrix}
A_1 & B_1 \\
A_2 & B_2 \\
A_3 & B_3^+
\end{pmatrix},
\]

were \(A_1, A_2, A_3, B_1, B_2, B_3^+\) were defined in subsection \(2.3\). Such a matrix acts on \(\bar{w} = (w, w') \in R^{n_1 + n_3}\) by the usual multiplication of matrices. Let also

\[
M_{int} = \begin{pmatrix}
C & B_3\end{pmatrix},
\]

where \(C\) and \(B_3^+\) were also defined in subsection \(2.3\). The integral \((m_3 \times (n_2 + n_3))\) matrix \(M_{int}\) acts on \(\bar{\alpha} = (\alpha, \alpha') \in \mathbb{Z}^{m_2 + n_3}\) by the usual multiplication of matrices. The result of our inequalities and changes of variable is

\[
(6.3) \quad A_v(s, x) \ll \sum_{\alpha, \alpha' \in \mathbb{Z}^{m_2 + n_3}} \int_{m_1 + m_2} \int_{i=1}^{m_1 + m_2} \frac{1}{(|(M_{re} \bar{w})| + 1)^{Re(s)}} \prod_{i=m_1 + m_2 + 1}^{m_1 + m_2 + m_3} \frac{1}{(|(M_{re} \bar{w})| + 1)^{Re(s)}} \prod_{i=m_1 + m_2 + 1} d\bar{w},
\]

or, rearranging,

\[
(6.4) \quad A_v(s, x) \ll \int_{m_1 + m_2} \int_{i=1}^{m_1 + m_2} \frac{1}{(|(M_{re} \bar{w})| + 1)^{Re(s)}} \prod_{i=m_1 + m_2 + 1}^{m_1 + m_2 + m_3} \frac{1}{(|(M_{re} \bar{w})| + 1)^{Re(s)}} \prod_{i=m_1 + m_2 + 1} d\bar{w}.
\]

Before proceeding with the estimation of \((6.3)\) or \((6.4)\), we first describe a result in combinatorial geometry. Let \(M \in M_{m \times n}(R)\) be an \(m \times n\) matrix with real entries.

**Definition 6.2.** For any \(\alpha \geq \beta \geq 1\) we say that \(M\) is \((\alpha, \beta)\)-biased if there exist \(\alpha\) rows of \(M\) such that any basis of \(R^n\) formed from rows of \(M\) contains at least \(\beta\) of the distinguished \(\alpha\) rows.
For example, note that any full-rank \( m \times n \) matrix is \((m,n)\)-biased.

Let us now write \( a_i, i = 1, \ldots, m \) for the rows of \( M \), and consider the convex polytope \( H_M \) cut out by the the following inequalities on \( \mathbb{R}_r \):

\[
\sum_{i=1}^{m} x_i = n
\]

and

\[
\sum_{i \in S} x_i \leq \dim(\text{span}(\{a_i : i \in S\}))
\]

for every subset \( S \subseteq \{1, \ldots, m\} \). Write \( \| \cdot \|_{\infty} \) for the \( L^{\infty} \)-norm on \( \mathbb{R}^m \), i.e. for \( x \in \mathbb{R}^m \) we set

\[
\|x\|_{\infty} = \max(|x_1|, \ldots, |x_m|).
\]

The norm \( \| \cdot \|_{\infty} \) is a convex and piecewise-linear function on \( \mathbb{R}^m \). Let

\[
B_\infty = B_\infty(M) = \inf \{\|x\|_{\infty} : x \in H_M\}.
\]

**Proposition 6.3.** Let \( M \) be any full-rank \( m \times n \) matrix with real entries. We have that

\[
B_\infty = \max \{\frac{\beta}{\alpha} : M \text{ is } (\alpha, \beta)\text{-biased}\}.
\]

The proof of Proposition 6.3 will be given in subsection 6.3. We now give the proof of assertion (1) of Theorem 6.1, assuming Proposition 6.3.

We estimate \( A_v(s, x) \) starting with the integral over \( \bar{w} \) in the interior of (6.3). The main tool to bound such an integral is a Brascamp-Lieb inequality. The following necessary and sufficient conditions were first proven in [Bar98], and then re-stated in the form below by [CLL04, §4].

**Theorem 6.4 (Brascamp-Lieb Inequality).** Let \( a_1, \ldots, a_m \) be non-zero vectors in \( \mathbb{R}^n \) which span \( \mathbb{R}^n \), and let \( M \) be the \( m \times n \) matrix whose rows are \( a_i \). Let \( \bar{p} = (p_1^{-1}, \ldots, p_m^{-1}) \in \mathbb{R}_r^m \). Let \( \bar{f} = (f_i)_{i=1, \ldots, m} \) be an \( m \)-tuple of non-negative measurable functions \( f_i : \mathbb{R} \to \mathbb{R}_r \). Then

\[
\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(\langle a_i, x \rangle) \, dx \ll_{m,n,M,\bar{p}} \prod_{i=1}^{m} \|f_i\|_{L^{p_i}(\mathbb{R})}
\]

if and only if \( \bar{p} \in H_M \subset \mathbb{R}_r^m \). Here the implied constant depends on \( m, n, M, \bar{p} \), but not on \( \bar{f} \).

Note that \( H_{M_{\text{re}}} \) is compact, so the infimum in (6.7) is attained, say by \( \bar{B} = (B_1, \ldots, B_m) \in H_{M_{\text{re}}} \). Note that \( r|\tilde{\alpha} \) faithful implies that \( M_{\text{re}} \) is full-rank. We apply Theorem 6.3 to the interior integral over \( \bar{w} \) of (6.3) with \( M = M_{\text{re}}, \bar{p} = \bar{B} \),

\[
f_1(x) = \cdots = f_{m_1+m_2}(x) = \frac{1}{(|x| + 1)^{\text{Re}(s)}},
\]

and

\[
f_i(x) = \frac{1}{(x^2 + (M_{\text{int}} \alpha_i^{-2})^{1/2} + 1)^{2\text{Re}(s)}}, \text{ for } i = m_1 + m_2 + 1, \ldots, m_1 + m_2 + m_3.
\]

We obtain

\[
A_v(s, x) \ll_{r,T}
\]

\[
\prod_{i=1}^{m_1+m_2} \|(|x| + 1)^{-\text{Re}(s)}\|_{L^{B_i^{-1}}(\mathbb{R})} \sum_{\pi \in \mathcal{Z}^{m_2+m_3}} \prod_{i=m_1+m_2+1}^{m_1+m_2+m_3} \|\left(x^2 + (M_{\text{int}} \alpha_i^{-2})^{1/2} + 1\right)^{-2\text{Re}(s)}\|_{L^{B_i^{-1}}(\mathbb{R})}.
\]

Now we need to bound the sum over \( \bar{w} \) in (6.8). Luckily, a version of the Brascamp-Lieb inequality on \( \mathbb{Z}^n \) has been given by [BCCT10, Thm. 2.4].
Theorem 6.5 (Bennett, Carbery, Christ and Tao). Let $G$ and $\{G_i : 1 \leq i \leq m\}$ be finitely generated Abelian groups. Let $\varphi_i : G \to G_i$ be homomorphisms. Let $p_i \in [1, \infty]$. Then

$$\text{rank}(H) \leq \sum_i p_i^{-1} \text{rank}(\varphi_i(H))$$

for every subgroup $H$ of $G$ if and only if there exists a constant $C < \infty$ such that

$$\sum_{y \in G} \prod_{i=1}^m (f_i \circ \varphi_i)(y) \leq C \prod_i \|f_i\|_{\ell^1(G_i)}$$

for all $f_i : G_i \to [0, \infty]$.

Let $a_i$, $i = 1, \ldots, m_3$ denote the rows of $M_{\int}$. We have that $x \in H_{\int}$ if and only if

$$\sum_{i=1}^{m_3} x_i = n$$

and

$$\sum_{i \in S} x_i \leq \text{rank}(\text{span}_\mathbb{Z}\{a_i : i \in S\})$$

for every subset $S \subseteq \{1, \ldots, m\}$, by tensoring with $\mathbb{R}$. The discussion on [BCCT10, p. 649] shows that (6.11) and (6.12) imply that

$$\text{rank}(H) \leq \sum_{i=1}^{m_3} x_i \text{rank}(\varphi_i(H))$$

for every subgroup $H$ of $\mathbb{Z}^{n_2+n_3}$, where $\varphi_i : \mathbb{Z}^{n_2+n_3} \to \mathbb{Z}$ is given by $x \mapsto \langle a_i, x \rangle$. The infimum in (6.7) is attained, say by $B' = (B'_1, \ldots, B'_m) \in H_{\int}$. We apply Theorem 6.5 with $G = \mathbb{Z}^{n_2+n_3}$, $G_i = \mathbb{Z}$, $\varphi_i$ as above, $p_i = B'_i^{-1}$, and

$$f_i(x) = \frac{1}{(\sqrt{x^2 + \alpha^2} + 1)^2 \text{Re}(s)} \|L_{B_i^{-1}}(\mathbb{R})\|_{\ell^{B_i^{-1}}(\mathbb{R})}$$

for $i = 1, \ldots, m_3$,

to obtain that

$$A_v(s, x) \ll_{r, T} \prod_{i=1}^{m_1+m_2} \|(|x| + 1)^{-\text{Re}(s)}\|_{L_{B_i^{-1}}(\mathbb{R})} \prod_{i=m_1+m_2+1}^{m_1+m_2+m_3} \|(|x| + 1)^{-2\text{Re}(s)}\|_{L_{B'_i^{-1}}(\mathbb{R})} \ell_{B'_i^{-1}}(\mathbb{Z}).$$

The right hand side converges as soon as

$$\text{Re}(s) > \max\left\{\frac{\max \{B_i, B'_i/2 : B_i \geq B'_i\}}{B_i} \right\} \max_{i=1, \ldots, m_1+m_2} \{B_i, B'_i/2 : B_i \geq B'_i\}.$$

We can also bound $A_v(s, x)$ starting with (6.4) instead of (6.3). We apply first Theorem 6.5 then Theorem 6.3 i.e. in the reverse order from above, to show that $A_v(x, s)$ converges absolutely whenever

$$\text{Re}(s) > \max\left\{\frac{\max \{B_i, B'_i/2 : B_i \geq B'_i\}}{B_i} \right\} \max_{i=1, \ldots, m_1+m_2} \{B_i, B'_i/2 : B_i \geq B'_i\}.$$

Putting together (6.15) and (6.16) we have that $A_v(s, x)$ converges absolutely when

$$\text{Re}(s) > \max \{\max\{B_1, \ldots, B_{m_1+m_2}, \frac{B_{m_1+m_2+1}}{2}, \ldots, \frac{B_{m_1+m_2+m_3}}{2}\}, \max \{\frac{B_i}{2} : i = 1, \ldots, m_3\}\}.$$
where in the first max \( \alpha = \alpha_1 + \alpha_2 \), and the distinguished set (in the definition of \((\alpha, \beta)\)-bias) contains \( \alpha_1 \) of the first \( m_1 + m_2 \) rows and \( \alpha_2 \) of the last \( m_3 \) rows. Let

\[
M' = \begin{pmatrix}
A_1 & 0 & B_1 & B_1 \\
A_2 & 0 & B_2 & B_2 \\
A_3 & C & B_3^+ + B_3^- & B_3^+ - B_3^- \\
A_3 & -C & B_3^+ - B_3^- & B_3^+ + B_3^-
\end{pmatrix}.
\]

We claim that

\[
\max\{\frac{\beta}{\alpha_1 + 2\alpha_2} : M_{\text{re}} \text{ is } (\alpha, \beta)\text{-biased}\} \leq B_x(M').
\]

Indeed, suppose the first maximum on the left hand side is larger. Then there are \( \alpha = \alpha_1 + \alpha_2 \) distinguished rows of \( M_{\text{re}}, \alpha_1 \) of which are among the first \( m_1 + m_2 \) rows, and \( \alpha_2 \) are among the last \( m_3 \) rows. Choose the corresponding \( \alpha_1 \) rows of \( M' \) among the first \( m_1 + m_2 \) rows, and the corresponding \( 2\alpha_2 \) rows, i.e. \( \alpha_2 \) pairs of rows of \( M' \) from among the last \( 2m_3 \) rows. This set of \( \alpha_1 + 2\alpha_2 \) rows of \( M' \) shows that \( M' \) is \((\alpha_1 + 2\alpha_2, \beta)\)-biased. So

\[
\max\{\frac{\beta}{\alpha_1 + 2\alpha_2} : M_{\text{re}} \text{ is } (\alpha, \beta)\text{-biased}\} \leq B_x(M').
\]

Similarly, suppose the second maximum on the left hand side of (6.18) is larger. Then there are \( \alpha \) distinguished rows of \( M_{\text{int}} \), and we choose the corresponding \( 2\alpha \) rows, i.e. \( \alpha \) pairs of rows from among the last \( 2m_3 \) rows of \( M' \). This distinguished set of \( 2\alpha \) rows of \( M' \) shows that \( M' \) is \((2\alpha, \beta)\)-biased. So

\[
\frac{1}{2} \max\{\frac{\beta}{\alpha} : M_{\text{int}} \text{ is } (\alpha, \beta)\text{-biased}\} \leq B_x(M'),
\]

which finishes the proof of (6.18).

Finally, there is a bijection between the rows of \( M' \) and the coweights of \( r \) via the isomorphism (2.28). Under this isomorphism, sets of rows of \( M' \) correspond bijectively to subsets \( S \subseteq M \) of coweights of \( r \), and \( |S| = \alpha \) and \( \dim D(S) = \beta \). This concludes the proof of assertion (1) of Theorem 6.1.

Assertion (2) of Theorem 6.1 follows immediately. Indeed, returning to equation (6.2), we integrate by parts once in each variable \( w_i, w'_i \), and apply part (1) of the theorem.

6.2. **Background on matroids and polymatroids.** The key observation in the proof of Proposition (6.3) is that the definition of \((\alpha, \beta)\)-bias makes sense more generally for matroids, and \( H_M \) is exactly the matroid base polytope associated to the matroid \( M \). To this end, we next recall some background on matroids and polymatroids. The following exposition was communicated to the author by R. Zenklusen.

**Definition 6.6 (Matroid).** A **matroid** is a pair \((N, \mathcal{I})\) where \( N \) is a finite set and \( \mathcal{I} \subset 2^N \) is a family of “independent” subsets of \( N \) satisfying the following axioms.

1. \( \mathcal{I} \neq \emptyset \)
2. If \( I \in \mathcal{I} \) and \( J \subset I \) then \( J \in \mathcal{I} \).
3. If \( I, J \in \mathcal{I} \) and \( |J| > |I| \) then there exists \( e \in J \setminus I \) such that \( I \cup \{e\} \in \mathcal{I} \).

**Example 6.7 (Linear Matroid).** If \( N \) is a set of vectors spanning a vector space, and \( \mathcal{J} \) is the set of linearly independent subsets of \( N \), then \((N, \mathcal{J})\) is a matroid. One calls such a matroid a **linear matroid**.

If \((N, \mathcal{J})\) is a matroid, then the set of bases \( \mathcal{B} \subset \mathcal{J} \) is the set of maximal subsets of \( \mathcal{J} \), ordered by inclusion. If \((N, \mathcal{J})\) is a linear matroid, then \( \mathcal{B} \) consists of subsets of vectors which form a basis.
Definition 6.8. The rank function of a matroid is the function \( r : 2^N \to \mathbb{Z}_{\geq 0} \) given by
\[
r(S) = \max\{|I| : I \in \mathcal{I}, I \subseteq S\}.
\]

If \((N, \mathcal{I})\) is a linear matroid, then \( r(S) \) is the dimension of the space spanned by the vectors in \( S \). By the definitions of \( \mathcal{B} \) and \( r \) we have
\[
\mathcal{B} = \{ I \in \mathcal{I} : r(I) = r(N) \}.
\]
Let \((N, \mathcal{I})\) be a matroid and let \( 1_I \in \{0, 1\}^N \) be the indicator function of \( I \). If \( S \) is a finite set of points in \( \mathbb{R}^N \), then we write \( \text{conv}(S) \) for the convex hull formed from those points.

Definition 6.9. The matroid polytope of \((N, \mathcal{I})\) is
\[
P_I = \text{conv}(\{1_I : I \in \mathcal{I}\}) \subset \mathbb{R}^N
\]
and the matroid base polytope is
\[
P_B = \text{conv}(\{1_B : B \in \mathcal{B}\}) \subset \mathbb{R}^N.
\]

Lemma 6.10. The rank function of a matroid \((N, \mathcal{I})\) satisfies the following properties.

- \( r : 2^N \to \mathbb{Z} \)
- \( r \) is submodular:
  \[
r(A) + r(B) \geq r(A \cup B) + r(A \cap B)
  \]
- \( r \) is monotone: \( r(A) \geq r(B) \) for all \( B \subseteq A \subseteq N \)
- \( r \) is non-negative: \( r(A) \geq 0 \) for all \( A \subseteq N \)
- \( r \) satisfies \( r(A \cup \{e\}) \leq r(A) + 1 \) for all \( A \subseteq N \) and \( e \in N \).

If \( r \) is any function enjoying these 5 properties, then there exists a unique matroid whose rank function is \( r \).

Proof. See [Sch03, §39.7]. \( \square \)

We can also express the matroid polytope and matroid base polytope in terms of the rank function as follows. Let \( x \in \mathbb{R}_{\geq 0}^N, e \in N \) and \( x_e \) be the \( e \)-th component of \( x \). For a subset \( S \subseteq N \) we set \( x(S) = \sum_{e \in S} x_e \). In terms of \( x(S) \), we have (see [Sch03, Cor. 40.2b])
\[
P_I = \{ x \in \mathbb{R}_{\geq 0}^N : x(S) \leq r(S) \text{ for all } S \subseteq N \}.
\]
Then \( P_B \) is one face of the matroid polytope \( P_I \) given by a supporting hyperplane, i.e. we have (see [Sch03, Cor. 40.2d])
\[
P_B = P_I \cap \{ x \in \mathbb{R}^N : x(N) = r(N) \}.
\]

Theorem 6.11 (Matroid Intersection). Let \((N, \mathcal{I}_1)\) and \((N, \mathcal{I}_2)\) be two matroids on the same ground set. Then we have
\[
\max\{|I| : I \in \mathcal{I}_1 \cap \mathcal{I}_2\} = \min_{A \subseteq N} \{ r_1(A) + r_2(N \setminus A) \}.
\]

Proof. See [Sch03, Thm. 41.1]. \( \square \)

Definition 6.12 (Polymatroid). A polymatroid on \( N \) is a polytope
\[
P_f = \{ x \in \mathbb{R}_{\geq 0}^N : x(S) \leq f(S) \text{ for all } S \subseteq N \}
\]
where \( f : 2^N \to \mathbb{R}_{\geq 0} \) is a submodular and monotone function.

Theorem 6.13 (Polymatroid intersection). Let \( f_1, f_2 : 2^N \to \mathbb{R}_{\geq 0} \) be two submodular and monotone functions. We have
\[
\sup\{ x(N) : x \in P_{f_1} \cap P_{f_2} \} = \min_{A \subseteq N} \{ f_1(A) + f_2(N \setminus A) \}.
\]
Proof. See [Sch03, Cor. 46.1b]. □

The following definition generalizes Definition 6.2 from linear matroids to matroids.

**Definition 6.14 (\((\alpha, \beta)\)-bias for matroids).** We say a matroid \((N, J)\) is \((\alpha, \beta)\)-biased if there exists \(S \subseteq N\) with \(|S| = \alpha\) and such that

\[|B \cap S| \geq \beta\]

for all bases \(B \subseteq N\).

**Lemma 6.15.** A subset \(S \subseteq N\) satisfies \(r(N) - r(N\setminus S) \geq \beta\) if and only if for any basis \(B \subseteq N\) we have \(|B \cap S| \geq \beta\).

Proof. “Only if”: Let \(B \subseteq N\) be any basis. By sub-modularity of the rank function we have

\[r(B \cap S) + r(N\setminus S) \geq r(N),\]

but \(B \cap S\) is independent, so we have

\[|B \cap S| = r(B \cap S) \geq r(N) - r(N\setminus S) \geq \beta.\]

“If”: Suppose that \(B \in \mathbb{B}\) is such that \(|B \cap S|\) is minimal as we range over all bases. Equivalently, \(B\) is such that \(|B\setminus(B \cap S)|\) is maximal. We claim that \(B\setminus(B \cap S)\) is maximal by inclusion among independent sets which are disjoint from \(S\). From this claim it follows by definition of the rank function that

\[|B\setminus(B \cap S)| = r(N\setminus S),\]

and so \(r(N) - r(N\setminus S) = |B \cap S| \geq \beta\).

If the claim were false, then there would exist \(e \notin S\) and \(\notin B\setminus(B \cap S)\) such that

\[(B\setminus(B \cap S)) \cup \{e\} = (B \cup \{e\})\setminus(B \cap S)\]

is an independent set, by matroid axiom (3). Since the set in (6.20) is independent and \(r(B \cup \{e\}) = r(N)\), we can complete it to a basis \(\tilde{B} \subseteq B \cup \{e\}\). But then we have

\[(B \cup \{e\})\setminus(B \cap S) = \tilde{B}\setminus(\tilde{B} \cap S),\]

from which it follows that

\[|\tilde{B}\setminus(\tilde{B} \cap S)| = |(B \cup \{e\})\setminus(B \cap S)| = |B\setminus(B \cap S)| + 1.\]

This contradicts the minimality of \(|B \cap S|\) among all bases \(B \in \mathbb{B}\). Therefore the claim is true. □

**Corollary 6.16.** A matroid \((N, J)\) is \((\alpha, \beta)\)-biased if and only if there exists \(S \subseteq N\) with \(|S| = \alpha\) and \(r(N) - r(N\setminus S) \geq \beta\).

6.3. **Proof of Proposition 6.3.** Considering the level sets of the \(L^\infty\) norm, the \(B_{\infty}\) defined in (6.7) becomes

\[B_{\infty} = \inf\{\lambda \geq 0 : P_2 \cap [0, \lambda]^N \neq \emptyset\}.

(Aside: compare this and (6.19) to the discussion of the Manin conjecture in subsection 1.2.) From the description (6.19) of the matroid basis polytope in terms of the rank function, we have

\[B_{\infty} = \inf\{\lambda \geq 0 : \sup\{x(N) : x \in P_2 \cap [0, \lambda]^N\} = r(N)\}.

Now we re-interpret \([0, \lambda]^N\) as the polymatroid defined by the function

\[f : 2^N \rightarrow \mathbb{R}_{\geq 0},\]

\[f(S) = \lambda |S|\]

Then, by the polymatroid intersection theorem (Theorem 6.13), we have

\[\sup\{x(N) : x \in P_2 \cap [0, \lambda]^N\} = \min_{A \subseteq N} \{r(A) + f(N\setminus A)\} = \min_{A \subseteq N} \{r(A) + \lambda |N\setminus A|\}.\]
Since this last min is always $\leq r(N)$, to characterize $B_{\infty}$ it suffices to find the smallest $\lambda \geq 0$ such that for all $A \subseteq N$

$$(6.21) \quad r(A) + \lambda |N\setminus A| \geq r(N),$$

that is to say

$$B_{\infty} = \inf \{ \lambda \geq 0 : r(A) + \lambda |N\setminus A| \geq r(N) \text{ for all } A \subseteq N \}.$$ 

It changes nothing to swap $A$ with $N\setminus A$, so

$$B_{\infty} = \inf \{ \lambda \geq 0 : r(N\setminus A) + \lambda |A| \geq r(N) \text{ for all } A \subseteq N \}.$$ 

If $A = \emptyset$ then the inequality is satisfied for all $\lambda$, so suppose not. We then have by Corollary 6.16

$$B_{\infty} = \min_{A \subseteq N} \left\{ \frac{r(N) - r(N\setminus A)}{|A|} : A \neq \emptyset \right\} = \max_{\alpha, \beta} \left\{ \frac{\beta}{\alpha} : (N, J) \text{ is } (\alpha, \beta)-biased \right\}.$$ 

### 6.4. Positivity.

To prove assertion (3) of Theorem 6.1 we first establish one-variable versions of the result.

**Lemma 6.17.** For all real $0 < \sigma \leq 2$ the Fourier transforms of the following functions are positive or $\pm \infty$:

$$f(x) = \frac{1}{(\sqrt{x^2 + 1} + 1)^\sigma}, \quad g(x) = \frac{1}{(|x| + 1)^\sigma} - \frac{1}{(\sqrt{x^2 + 1} + 1)^\sigma}.$$ 

**Proof.** When $\xi > 0$ we have by contour shifting

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e(-\xi x)}{(\sqrt{x^2 + 1} + 1)^\sigma} \, dx = i \int_{1}^{\infty} e^{-2\pi \xi x} \left( \frac{1}{(i\sqrt{x^2 - 1} + 1)^\sigma} - \frac{1}{(i\sqrt{x^2 - 1} + 1)^\sigma} \right) \, dx$$

$$= i \int_{1}^{\infty} \frac{e^{-2\pi \xi x}}{x^{\sigma}} \left( (-i\sqrt{x^2 - 1} + 1)^\sigma - (i\sqrt{x^2 - 1} + 1)^\sigma \right) \, dx$$

$$= 2 \int_{1}^{\infty} \frac{\sin(\sigma \arctan \sqrt{x^2 - 1})}{x^{\sigma}} e^{-2\pi \xi x} \, dx > 0.$$ 

The value $\hat{f}(0)$ is clearly positive if it converges, and if $\xi < 0$ we follow the same steps as above, shifting the contour up instead of down.

Now we show $\hat{g}(\xi)$ is positive or $\pm \infty$. For a real parameter $0 \leq \beta \leq 1$ define

$$\hat{g}_\beta(\xi) = \int_{-\infty}^{\infty} \frac{e(-\xi x)}{(\sqrt{x^2 + \beta^2} + 1)^\sigma} \, dx,$$ 

so that $\hat{g}(\xi) = \hat{g}_0(\xi) - \hat{g}_1(\xi)$. We have

$$\hat{g}(\xi) = \int_{1}^{0} \frac{d}{d\beta} \hat{g}_\beta(\xi) \, d\beta = \sigma \int_{-\infty}^{\infty} \frac{e^{-\xi x}}{\sqrt{x^2 + \beta^2} + 1} \, dx d\beta.$$ 

(6.23)
Suppose that $\xi > 0$. The interior integral has a branch cut from $-i\beta$ to $-i\infty$. To evaluate the integral, we shift the contour around this branch. We have

\[
\int_{-\infty}^{\infty} \frac{e(-\xi x)}{\sqrt{x^2 + \beta^2(\sqrt{x^2 + \beta^2 + 1})^{\sigma+1}}} \, dx = \int_{\beta}^{\infty} e^{-2\pi\xi x} \left( \frac{1}{\sqrt{x^2 - \beta^2(i\sqrt{x^2 - \beta^2 + 1})^{\sigma+1}}} + \frac{1}{\sqrt{x^2 - \beta^2(-i\sqrt{x^2 - \beta^2 + 1})^{\sigma+1}}} \right) \, dx
\]

\[
= 2 \int_{\beta}^{\infty} \frac{e^{-2\pi\xi x}}{\sqrt{x^2 - \beta^2(x^2 - \beta^2 + 1)}} \cos((\sigma + 1) \arctan(\sqrt{x^2 - \beta^2})) \, dx
\]

\[
= 2 \int_{0}^{\pi/2} e^{-2\pi\xi \sqrt{\tan^2 \theta + \beta^2}} (\cos \theta)^{\sigma-1} \cos((\sigma + 1)\theta) \, d\theta.
\]

Now we return to (6.23), change order of integration, and change variables $\sqrt{\tan^2 \theta + \beta^2} \to y$ to find

\[
\hat{g}(\xi) = 2\sigma \int_{0}^{\pi/2} \int_{\tan \theta}^{\sec \theta} e^{-2\pi\xi y} \, dy (\cos \theta)^{\sigma-1} \cos((\sigma + 1)\theta) \, d\theta.
\]

Let $h_\sigma(\theta)$ be the anti-derivative of $(\cos \theta)^{\sigma-1} \cos((\sigma + 1)\theta)$. By integrating by parts we have

\[
\hat{g}(\xi) = -2\sigma \int_{0}^{\pi/2} \frac{1}{\cos \theta} (\sin \theta) e^{-2\pi\xi \sec \theta} - e^{-2\pi\xi \tan \theta} h_\sigma(\theta) \, d\theta.
\]

Observe that $(\sin \theta) e^{-2\pi\xi \sec \theta} - e^{-2\pi\xi \tan \theta} \leq 0$, while $h_\sigma(\theta) \geq 0$ for all $0 < \sigma \leq 2$ and $0 \leq \theta \leq \pi/2$. Thus $\hat{g}(\xi) \geq 0$ for $\xi > 0$. The case $\xi = 0$ is obvious and $\xi < 0$ follows by a similar calculation. □

It follows from Lemma [6.17] that the Fourier transforms of

\[
\frac{1}{(|x| + 1)^\sigma}, \quad \frac{1}{(|x| + 1)^\sigma} + \frac{1}{(\sqrt{x^2 + \beta^2 + 1})^\sigma}
\]

are also everywhere positive or $+\infty$.

**Lemma 6.18.** For all real $0 < \sigma \leq 2$ and $\xi \in \mathbb{R}/\mathbb{Z}$, the Fourier series

\[
\sum_{\beta \in \mathbb{Z}} \frac{e(\beta \xi)}{1 + |\beta|^{\sigma}}
\]

is positive or $+\infty$.

**Proof.** Recall the Dirichlet and Fejér kernels

\[
D_u(x) = \sum_{|n| \leq u} e(n x) \quad \text{and} \quad F_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} D_k(x).
\]

By summing by parts twice we have

\[
\sum_{\beta \in \mathbb{Z}} \frac{e(\beta \xi)}{1 + |\beta|^{\sigma}} = \sum_{n=0}^{\infty} \left( \frac{1}{(n + 3)^\sigma} - \frac{2}{(n + 2)^\sigma} + \frac{1}{(n + 1)^\sigma} \right) (n + 1) F_{n+1}(\xi).
\]

The factor inside the parentheses above is positive by the mean value theorem. The Fejér kernel is also positive, and therefore the series is positive wherever it converges. □
Lemma 6.19. For all real $0 < \sigma \leq 2$ and $\xi \in \mathbb{C}^\times$, the function

$$\sum_{\beta \in \mathbb{Z}} \left( \frac{\beta}{\xi} \right)^\beta \int_{\mathbb{R}} \frac{e(-|\xi|x)}{(\sqrt{x^2 + \beta^2 + 1})^\sigma} \, dx$$

is positive or $+\infty$.

Proof. By summation by parts twice, it suffices to show that the second forward difference in $\beta$ of

$$f(\beta, \xi) = \int_{\mathbb{R}} \frac{e(-|\xi|x)}{(\sqrt{x^2 + \beta^2 + 1})^\sigma} \, dx$$

is positive. Recall the definition of $\hat{g}_\beta(\xi)$ from [6,22], in terms of which we have

$$f(\beta + 2, \xi) - 2f(\beta + 1, \xi) + f(\beta, \xi) = \int_\beta^{\beta+1} \frac{d}{d\gamma} \int_\gamma^{\gamma+1} \frac{d}{d\alpha} \hat{g}_\alpha(|\xi|) \, d\alpha \, d\gamma.$$

As in the proof of Lemma 6.17, we have

$$f(\beta + 2, \xi) - 2f(\beta + 1, \xi) + f(\beta, \xi) = -\sigma \int_\beta^{\beta+1} \frac{d}{d\gamma} \int_\gamma^{\gamma+1} \frac{e(-|\xi|x)}{(\sqrt{x^2 + \beta^2 + 1})^\sigma} \, dx \, d\gamma.$$

By summation by parts twice, it suffices to show that the second forward difference in $\beta$ of

$$f(\beta + 2, \xi) - 2f(\beta + 1, \xi) + f(\beta, \xi) = \int_\beta^{\beta+1} \frac{d}{d\gamma} \int_\gamma^{\gamma+1} \frac{d}{d\alpha} \hat{g}_\alpha(|\xi|) \, d\alpha \, d\gamma.$$

Recall $h_\sigma(\theta) \geq 0$ is defined to be the anti-derivative of $(\cos \theta)^\sigma(\cos((\sigma + 1)\theta))$. Let also

$$H(\beta, \theta, \xi) = \int_\beta^{\beta+1} \frac{d}{d\theta} \left( \frac{(\gamma + 1)e^{-2\pi|\xi|\sqrt{\tan^2 \theta+(\gamma+1)^2}}}{\sqrt{\tan^2 \theta+(\gamma+1)^2}} - \frac{\gamma e^{-2\pi|\xi|\sqrt{\tan^2 \theta+\gamma^2}}}{\sqrt{\tan^2 \theta+\gamma^2}} \right) \, d\gamma.$$

Then, by integrating by parts we have

$$f(\beta + 2, \xi) - 2f(\beta + 1, \xi) + f(\beta, \xi) = 2\sigma \int_0^{\pi/2} H(\beta, \theta, \xi) h_\sigma(\theta) \, d\theta.$$

It suffices to show that $H(\beta, \theta, \xi)$ is non-negative for all $\beta \in \mathbb{N}$, $\xi \in \mathbb{C}^\times$, and $0 \leq \theta \leq \pi/2$. Set

$$N(\gamma, \theta, \xi) = \gamma e^{-2\pi|\xi|\sqrt{\tan^2 \theta+(\gamma+1)^2}} \tan \theta \sec^2 \theta \left( \frac{2\pi|\xi|\sqrt{\tan^2 \theta+\gamma^2}}{\sqrt{\tan^2 \theta+\gamma^2}} + 1 \right)$$

so that

$$H(\beta, \theta, \xi) = \int_\beta^{\beta+1} (N(\gamma, \theta, \xi) - N(\gamma + 1, \theta, \xi)) \, d\gamma$$

$$= \tan \theta \sec^2 \theta \left( \int \frac{\sqrt{\tan^2 \theta+(\beta+1)^2}}{\sqrt{\tan^2 \theta+\beta^2}} - \int \frac{\sqrt{\tan^2 \theta+(\beta+2)^2}}{\sqrt{\tan^2 \theta+(\beta+1)^2}} \right) e^{-2\pi|\xi|y} \left( \frac{2\pi|\xi||y| + 1}{y^2} \right) \, dy.$$

The integrand here is decreasing for all $y > 0$, thus $H(\beta, \theta, \xi) \geq 0$. □

Consider the Fourier dual pair

$$\mathbf{T} = (\mathbb{R}^\times)^{n_1} \times (S^1)^{n_2} \times (\mathbb{C}^\times)^{n_3} \leftrightarrow (i\mathbb{R}^{n_1} \times (\mathbb{Z}/2\mathbb{Z})^{n_1}) \times \mathbb{Z}^{n_2} \times (i\mathbb{R}^{n_3} \times \mathbb{Z}^{n_3}) = \mathbf{T}^\wedge.$$
and the spaces of tempered distributions $S'(\mathbf{T})$ and $S'(\mathbf{T}^\vee)$. Recall $M$ and $M \varphi$ from subsection 2.4. Let $f_i \in S'(\mathbf{T}^\vee)$ be given for $i = 1, \ldots, m_1 + m_2$ by the function
\[ f_i(w, \epsilon, \alpha, w', \alpha') = \frac{1}{((M \varphi)_i + 1)^\sigma}. \]
and for $i = m_1 + m_2 + 1, \ldots, m_1 + m_2 + m_3$ by the function
\[ f_i(w, \epsilon, \alpha, w', \alpha') = \frac{1}{((M \varphi)_i + 1)^{2\sigma}}. \]
We have that the Fourier series/transform of each of these is also a tempered distribution: $\mathcal{F}(f_i) \in S'(\mathbf{T})$. By Lemmas (6.17), (6.18), and (6.19), each of $\mathcal{F}(f_i)$ is a positive distribution for $\sigma_0 < \sigma \leq 2$. By assertion (I) of Theorem 6.1 and (6.2), the $(m_1 + m_2 + m_3)$-fold convolution of the distributions $\mathcal{F}(f_i)$ is defined and we have
\[ A_\psi(\sigma, x) = \prod_{i=1}^{m_1 + m_2 + m_3} \mathcal{F}(f_i)(x), \]
for all $x \in \mathbf{T}$. Since the convolution of positive distributions is positive, we have that $A_\psi(\sigma, x)$ takes positive values for $\sigma_0 < \sigma \leq 2$ and all $x \in T(k_v)$, as was to be shown.

7. Counting

In this section we prove the analytic continuation of the generating series $Z(s)$. Following Tate [Tat79 §3.5] we take $\psi$ to be a non-trivial additive character of $\mathbf{A}/k$ and $dx$ the Haar measure on $\mathbf{A}$ such that $\int_{\mathbf{A}/k} dx = 1$. Let $\psi_v$ be the local component of $\psi$ at a place $v$, $dx = \prod_v dx_v$ be any factorization of $dx$ into a product of local measures such that the ring of integers $\mathcal{O}_v$ at all but finitely many $v$ gets measure 1, and $\delta(\psi_v)$ be the function defined in [Tat79 §3.4.5]. Recall the notation of (3.8), (3.9), and (3.10), and set
\[ U(s, x) = \prod_{v \notin B \cup S_\mathbf{X}} \left( \delta(\psi_v) dx_v / dx'_v \right)^{sm} U_v(s, x), \quad R(s, x) = \sum_{\theta \in \text{Cl}_N(T)^\vee} \prod_v \left( \delta(\psi_v) dx_v / dx'_v \right)^{sm} R_v(s, x), \]
and
\[ A(s, x) = \prod_{v \in S_\mathbf{X}} \left( \delta(\psi_v) dx_v / dx'_v \right)^{sm} A_v(s, x). \]
Proposition 3.1 states that for some positive real number $c$ we have
\[ Z(s) = \int_{\mathbf{A}(T)} \frac{1}{c(\chi, r)^s} dv(\chi) = c \left( \prod_v \left( \delta(\psi_v) dx_v / dx'_v \right)^{-sm} \right) \sum_{x \in U_N(T)} A(s, x) U(s, x) R(s, x). \]
By [Tat79 §3.5] the product
\[ D_k = \prod_v \left( \delta(\psi_v) dx_v / dx'_v \right) \]
do not depend on the factorization of the global $dx$, nor on the choice of global additive character $\psi$, but only on $k$.

To determine the analytic properties of $Z(s)$, it suffices to determine the analytic properties of $A(s, x)$, $U(s, x)$, and $R(s, x)$, and to show that sum over $x \in U_N(T)$ in (7.1) converges absolutely.
7.1. Unramified places.

**Theorem 7.1.**

1. The series $U(s, x)$ converges absolutely and uniformly on compacta in the right half-plane $\Re(s) > A$, and admits a meromorphic continuation to the right half-plane $\Re(s) > A - (2m^2)^{-1}$ with at most a pole at $s = A$.

2. The series $U(s, 1)$ has a pole at $s = A$ of order $|\hat{G}\setminus\hat{\Sigma}_0|$ with positive leading constant in its Laurent series expansion.

3. The possible pole of $U(s, x)$ at $s = A$ is of order $\leq |\hat{G}\setminus\hat{\Sigma}_0|$, has positive leading coefficient, and Laurent series expansion bounded by that of $U(s, 1)$ at $s = A$.

4. Away from a small neighborhood $N$ surrounding $s = A$, we have the bound

   $$U(s, x) \ll_{N, \Re(s)} (1 + |s|)^K,$$

   uniformly in $x$, for some constant $K$ depending only on $T$ and $r$.

We devote the rest of this subsection to proving Theorem 7.1.

Since our computations in this section are global in nature, we switch to the notation of prime ideals. To each place $v \notin S_x \cup B$ there is associated a prime ideal $\mathfrak{p}$ of $k$, and we have $q_{\mathfrak{p}} = N(\mathfrak{p})$, the absolute ideal norm of $\mathfrak{p}$. Recall from section 3 the (global) Galois extensions $k \subseteq K \subseteq K'$ with groups $\Gamma = \mathrm{Gal}(K'/k)$ and $G = \mathrm{Gal}(K/k)$. Recall from section 3 we have chosen for each prime $\mathfrak{p}$ of $k$ a prime $\mathfrak{p}'$ of $K'$ lying over $\mathfrak{p}$. Let $D_{\mathfrak{p}} \subseteq G$ and $D_{\mathfrak{p}'} \subseteq \Gamma$ be decomposition groups at $\mathfrak{p}$. The results of section 4 apply with $D_{\mathfrak{p}}$ and $D_{\mathfrak{p}'}$ playing the role of the local Galois groups.

We have the restriction map $\Gamma \to G$ under which $\mathrm{Fr}_{\mathfrak{p}'} \mapsto \mathrm{Fr}_{\mathfrak{p}}$. When $x = 1 \in U_N(T)$ we will also use the map

$$\hat{G} \hookrightarrow G \times \mathrm{Gal}(k(\zeta_\lambda)/k)$$

$$\mathrm{Fr}_{\mathfrak{p}'} \mapsto (\mathrm{Fr}_{\mathfrak{p}}, N(\mathfrak{p})),$$

where $N(\mathfrak{p}) \in \mathrm{Gal}(k(\zeta_\lambda)/k)$ denotes the automorphism of $k(\zeta_\lambda)$ sending a primitive $\lambda$th root of unity $\zeta_\lambda$ to $\zeta_\lambda^{N(\mathfrak{p})}$ (see (1.8)). In particular, the value of $N(\mathfrak{p})$ modulo $\lambda$ is determined by $\mathrm{Fr}_{\mathfrak{p}'} \in \hat{G}$.

By (1.1) and (1.5) we have

$$U(s, x) = \prod_{\mathfrak{p} \notin B, c \in N^M} \sum_{\mathfrak{m} \subseteq \mathfrak{n}^c} \frac{\Pi_{=}(c, x)}{N(\mathfrak{p})^{\sigma_0(c)}}.$$

**Definition 7.2.** If $F_i(s)$ and $F_2(s)$ are meromorphic functions defined in $\Re(s) > \sigma_i$, $i = 1, 2$ and there exists an analytic function $G(s)$ given by an absolutely and uniformly convergent Euler product in $\Re(s) > \sigma_0$ such that $F_1(s) = G(s)F_2(s)$, then we say that $F_1$ equals $F_2$ up to an absolutely convergent Euler product in $\Re(s) > \sigma_0$ and write $F_1 \approx F_2$ in $\Re(s) > \sigma_0$.

**Lemma 7.3.** Suppose $F_1$ and $F_2$ are as in Definition 7.2.

1. If $F_1 \approx F_2$ in $\Re(s) > \sigma_0$, then $F_1$ admits a meromorphic continuation to $\Re(s) > \max(\sigma_0, \min(\sigma_1, \sigma_2))$.

2. The relation $\approx$ defines an equivalence relation on meromorphic functions on $\Re(s) > \max(\sigma_0, \min(\sigma_1, \sigma_2))$.

3. If $F_1 \approx F_2$ in $\Re(s) > \sigma_0$, then $F_1$ and $F_2$ have the same poles to the same orders in the domain $\Re(s) > \max(\sigma_0, \min(\sigma_1, \sigma_2))$.

The proofs are easy exercises, so we omit them. Since we do not give an expression for the leading constant in Theorem 7.1 it suffices to study $U(s, x)$ up to the $\approx$ equivalence.

Recall the change of variables $c \leftrightarrow S$ given by (4.9), (4.10), and let

$$U_0(s, x) = \prod_{\mathfrak{p} \notin B, S \subseteq M} \sum_{S \subseteq M} \frac{\Pi_{=}(S, x)}{N(\mathfrak{p})^{\sigma_0(S)}}.$$
Lemma 7.4. We have $U(s, x) \approx U_0(s, x)$ in $\text{Re}(s) > A - (2m^2)^{-1}$. 

Proof. It suffices to show that the series

$$(7.2) \prod_{p \neq B} \frac{\sum_{c \in \mathbb{N}^M} \Pi_\varepsilon(c, x)N(p)^{-\varepsilon|c|}}{\sum_{c \in \{0, 1\}^M} \Pi_\varepsilon(c, x)N(p)^{-\varepsilon|c|}}$$

converges absolutely and uniformly on compacta in $\text{Re}(s) > A - \frac{1}{2m^2}$. 

Consider $p \neq B$ with $N(p)$ sufficiently large. The factor of the product (7.2) at $p$ is

$$(7.3) 1 + \sum_{c \in \mathbb{N}^M} \frac{\Pi_\varepsilon(c, x)}{N(p)^{\varepsilon|c|}} - \left( \sum_{c \in \mathbb{N}^M} \frac{\Pi_\varepsilon(c, x)}{N(p)^{\varepsilon|c|}} \right) \left( \sum_{c \in \{0, 1\}^M} \frac{\Pi_\varepsilon(c, x)}{N(p)^{\varepsilon|c|}} \right) + \cdots$$

Consider the indices $c \in \mathbb{N}^M$ appearing in the first sum on the right hand side. We may assume that $c$ is such that $\Pi_\varepsilon(c, x) \neq 0$. In particular, it is $D_q$-fixed by Lemma 4.2.

Lemma 7.5. If $c = (c_i) \in \mathbb{N}^M$ is such that $\max c_i \geq 2$ and $\Pi_\varepsilon(c, x) \neq 0$ then $\dim D_k(c) \geq 1$ for all $k = 0, \ldots, \max c_i - 1$.

Proof. Let us choose an ordering of the $\mu \in M$, say $\mu_1, \ldots, \mu_m$, and write $c_i = c_{\mu_i}$. We choose the ordering such that $c_1$ is maximal among $c_1, \ldots, c_m$, thus $c_1 \geq 2$. By (1.6)

$$\Pi_\varepsilon(c, x) = \sum_{d_2 \leq c_2} \cdots \sum_{d_m \leq c_m} \mu(2^{d_2}) \cdots \mu(2^{d_m}) (\Pi_\varepsilon(c - (0, d_2, \ldots, d_m), x) - \Pi_\varepsilon(c - (1, d_2, \ldots, d_m), x)).$$

Since $\Pi_\varepsilon(c, x) \neq 0$ there exists $d_2, \ldots, d_m \in \{0, 1\}$ such that

$$(7.4) \Pi_\varepsilon(c - (0, d_2, \ldots, d_m), x) \neq \Pi_\varepsilon(c - (1, d_2, \ldots, d_m), x).$$

We have by Proposition 4.1 that

$$\Pi_\varepsilon(c, x) = \left| \text{Hom}_{D_p} \left( (\mathcal{O}_L/\mathfrak{q})^\times, D_0(c) \right) \right| \prod_{k=1}^\infty \text{char} / \mathfrak{p}^{\dim D_k(c)}.$$ 

Thus, since $c_1 \geq 2$ we have

$$(7.5) \frac{\Pi_\varepsilon(c - (0, d_2, \ldots, d_m))}{\Pi_\varepsilon(c - (1, d_2, \ldots, d_m))} = \text{char} / \mathfrak{p}^{\dim D_{c_1-1}(c - (0, d_2, \ldots, d_m)) - \dim D_{c_1-1}(c - (1, d_2, \ldots, d_m))}.$$

By (7.4) the quantity in (7.5) is $\neq 1$. Note that $D_k(c)$ is monotonic in $c$, i.e. if $c' \leq c$ coordinate-wise then for all $k \geq 0$ we have

$$D_k(c') \subseteq D_k(c).$$

Thus

$$\text{char} / \mathfrak{p}^{\dim D_{c_1-1}(c - (0, d_2, \ldots, d_m)) - \dim D_{c_1-1}(c - (1, d_2, \ldots, d_m)) > 1},$$

from which we conclude

$$1 \leq \dim D_{c_1-1}(c - (0, d_2, \ldots, d_m)) - \dim D_{c_1-1}(c - (1, d_2, \ldots, d_m))$$

$$\leq \dim D_{c_1-1}(c - (0, d_2, \ldots, d_m)) \leq \dim D_{c_1-1}(c) \leq \dim D_k(c)$$

for all $1 \leq k \leq c_1 - 1$. 

We have the trivial bounds $|\Pi_\varepsilon(c, x)| \leq \Pi_\varepsilon(c, 1) \leq \Pi_\varepsilon(c, 1)$ and by Lemmas 4.1 and 4.3 the bound

$$\Pi_\varepsilon(c, 1) \ll_{T, r} \prod_{k=0}^{\infty} N(p)^{\dim D_k(c)}.$$
Note that

\[ |c| = \sum_{k=0}^{\infty} \left| \{ \mu \in M : c_\mu > k \} \right|, \]

so

\[ \frac{\prod_{c \in \mathbb{C}} (c, x)}{N(p)^{|c|}} \ll \prod_{k=0}^{\infty} N(p)^{\dim D_k(c) - s|\{ \mu : c_\mu > k \}|}. \]

Therefore we have e.g. for the first sum in (7.3) that

\[ \sum_{c \in \mathbb{N}} \frac{\prod_{c \in \mathbb{C}} (c, x)}{N(p)^{|c|}} \ll \sum_{c : \max c_\mu \geq 2} \prod_{k=0}^{\infty} N(p)^{\dim D_k(c) - s|\{ \mu : c_\mu > k \}|}. \]

By Lemma 7.5 we have the product here has at least two non-one factors for each \( c \) in the outer sum. Now we take the product of (7.3) over \( p \notin B \), and find that (7.2) converges absolutely and uniformly on compacta in the region

\[ (7.6) \sup_{i \geq 2} \max_{1 \leq j \leq i} \left\{ \frac{\dim D(S_1) + \cdots + \dim D(S_i) + 1}{|S_1| + \cdots + |S_i|} : D(S_j) \neq \{1\} \right\}. \]

**Lemma 7.6.** For any integers \( a, a' \) and any \( 1 \leq c, c' \leq m \) one has

\[ \frac{a + a' + 1}{c + c'} \leq \max \left\{ \frac{a + 1}{c}, \frac{a' + 1}{c'} \right\} - (2m^2)^{-1}. \]

**Proof.** Elementary. \( \square \)

By Lemma 7.6 we conclude that (7.2) converges absolutely and uniformly in the region \( \Re(s) > A - (2m^2)^{-1} \).

We arrange the product \( U_0(s, x) \) over (finitely many) conjugacy classes \( C \subseteq \Gamma \):

\[ (7.7) U_0(s, x) = \prod_{C \subseteq \Gamma} \prod_{p \in B} \sum_{p \in C} \prod_{S \subseteq M} \Pi_=(S, x) N(p)^{-s|S|}. \]

Consider the interior product of (7.7).

**Lemma 7.7.** Let

\[ (7.8) \Sigma^p = \{ S \subseteq M : S \neq \emptyset, \ Fr_p S = S, \ \frac{\dim D(S) + 1}{|S|} \geq A \}. \]

We have

\[ U_0(s, x) \approx \prod_{C \subseteq \Gamma} \prod_{p \notin B} \left( 1 + \sum_{S \in \Sigma^p} \Pi_=(S, x) N(p)^{-s|S|} \right) \]

in \( \Re(s) > A - (2m^2)^{-1} \).

**Proof.** By Lemma 4.2 we have if \( Fr_p S \neq S \) then \( \Pi_=(S, x) = 0 \). Consider the sets \( \emptyset \neq S \subseteq M \) which satisfy \( Fr_p S = S \), and take

\[ (2^M \setminus \{\emptyset\})^{D_p} = \Sigma^p \cup \Sigma^p^c. \]

Let

\[ F_1(s) = \prod_{C \subseteq \Gamma} \prod_{p \notin B} \left( 1 + \sum_{S \in \Sigma^p} \Pi_=(S, x) N(p)^{-s|S|} \right), \]
and

\[ F_2(s) = \prod_{C \subseteq \Gamma} \prod_{\mathfrak{p} \not\in B \text{ Fr}_\mathfrak{p} \in C} \left( 1 + \sum_{S \in \Sigma^p} \Pi_{\Sigma}(S, x) N(p)^{-s|S|} \right). \]

By Lemma 7.6 we have

\[ \prod_{\mathfrak{p} \not\in B} \sum_{S \in M \text{ Fr}_\mathfrak{p} \in C} \Pi_{\Sigma}(S, x) N(p)^{-s|S|} \approx F_1(s) F_2(s) \]

in Re(s) > A - (2m^2)^{-1}. Note that for any \( S \in \Sigma^p \) we have

\[ A - \frac{\dim D(S) + 1}{|S|} \geq m^{-2}, \]

since \( A \) and \( \frac{\dim D(S) + 1}{|S|} \) are two elements of the Farey sequence of order \( m \). We have by the estimate \( |\Pi_{\Sigma}(S, x)| \leq \Pi_{\Sigma}(S, 1) \), Lemma 4.3 and (7.9) that \( F_2(s) \) converges absolutely and uniformly in the region Re(s) \( \geq A - (2m^2)^{-1} \). Thus, \( U_0(s, x) \approx F_1(s) \) in Re(s) > A - (2m^2)^{-1}. \( \square \)

Recall the quantity \( a_C(S, x) \) from (4.11). Similarly, let

\[ a_C(S, x) = \#\{y \in p(\alpha^{-1}(x)) : \mathfrak{F}_\mathfrak{p} \ y = y\}, \]

which only depends on the conjugacy class \( C \subseteq \Gamma \) of \( \mathfrak{F}_\mathfrak{p} \). We have by Lemmas 7.4, 7.7 and 4.6

\[ U(s, x) \approx \prod_{C \subseteq \Gamma} \prod_{\mathfrak{p} \not\in B \text{ Fr}_\mathfrak{p} \in C} \left( 1 + \sum_{S \in \Sigma^p} N(p)^{\dim D(S) - s|S|} \begin{cases} a_C(S, x) & \text{if } \dim D(S) \geq 1 \\ a_C(S, x) - 1 & \text{if } \dim D(S) = 0 \end{cases} \right) \]

in Re(s) > A - (2m^2)^{-1}. Note that if \( \frac{\dim D(S) + 1}{|S|} > A \) then \( D(S) = \{1\} \) by the definition of \( A \), and so \( a_C(S, x) = 1 \) and the term corresponding to \( S \) above vanishes.

We split up the sum in (7.11) over the possible values of \( \dim D(S), |S| \). The parameter space is

\[ P = \{(a, b) : a \geq 0, b \geq 1, \frac{a + 1}{b} = A\}. \]

Since the case \( a = 0 \) is different in (7.11), so we also introduce

\[ P_0 := \{(a, b) : a \geq 1, b \geq 1, \frac{a + 1}{b} = A\}. \]

If \( A = 1 \), then \( P = P_0 \cup \{(0, 1)\} \), and otherwise \( P = P_0 \). Let also

\[ \Sigma_{a, b} = \{S \in \Sigma : \dim D(S) = a, \ |S| = b\}, \]

so that

\[ \Sigma = \bigcup_{(a, b) \in P} \Sigma_{a, b}. \]

Note that \( \Sigma_{0, 1/A} \) is non-empty only if \( A = 1 \). With this notation, we have that

\[ U(s, x) \approx \prod_{C \subseteq \Gamma} \prod_{\mathfrak{p} \not\in B \text{ Fr}_\mathfrak{p} \in C} \sum_{S \in \Sigma_{a, b}} (1 + N(p)^{-s}) a_C(S, x)^{-1} \prod_{(a, b) \in P_0} \sum_{S \in \Sigma_{a, b}} (1 + N(p)^{a - bs}) a_C(S, x) \]

in Re(s) > A - (2m^2)^{-1}. Let

\[ \tilde{\Sigma}_{a, b} = \begin{cases} \{(S, y) : S \in \Sigma_{a, b}, y \in p(\alpha^{-1}(x))\} & \text{if } (a, b) \neq (0, 1) \\ \{(S, y) : S \in \Sigma_{0, 1}, y \in p(\alpha^{-1}(x)), y \neq 1\} & \text{if } (a, b) = (0, 1) \end{cases} \]
The group \( \Gamma \) acts on each \( \hat{\Sigma}_{a,b} \) through \( G \) on \( S \) and through its Galois action on \( p(x) \) with \( y \in \pi(D(S)) \) (see Lemma 4.5) is \( (\overline{\mathfrak{g}}, g^*) \). Let \( V_{a,b} \) be the permutation representation of \( \Gamma \) acting on \( \hat{\Sigma}_{a,b} \). Let \( \psi_{a,b} \) be its character, and \( C \) a conjugacy class of \( \Gamma \). Then \( \psi_{a,b}(C) \) is the number of \( \text{Fr}_{p'} \)-fixed points on \( \hat{\Sigma}_{a,b} \). In these terms, we have

\[
U(s, x) \approx \prod_{C \subseteq \Gamma} \prod_{\mathfrak{p} \nmid B} \prod_{\text{Fr}_{p'} \in C} \prod_{(a,b) \in P} \left( 1 + N(p)^{a-bs} \right)^{\psi_{a,b}(C)}.
\]

Now we decompose \( V_{a,b} \) into irreducible representations \( V_i \) of \( \Gamma \)

\[
V_{a,b} = \bigoplus_i V_i
\]

with corresponding irreducible characters \( \psi_i \), so that

\[
\psi_{a,b} = \sum_i m_{a,b,i} \psi_i
\]

for some \( m_{a,b,i} \in \mathbb{N} \). Then we have

\[
U(s, x) \approx \prod_{C \subseteq \Gamma} \prod_{\mathfrak{p} \nmid B} \prod_{(a,b) \in P} \prod_i \left( 1 + \psi_i(C)N(p)^{a-bs} \right)^{m_{a,b,i}}
\]

in \( \text{Re}(s) > A - (2m)^{-1} \).

Moving the products over \( C \) and \( \mathfrak{p} \) to the inside, we have

\[
(7.15) \quad U(s, x) \approx \prod_{(a,b) \in P} \prod_i L^{(B)}(bs - a, V_i)^{m_{a,b,i}}
\]

in \( \text{Re}(s) > A - (2m)^{-1} \), where \( L^{(B)}(bs - a, V_i) \) denotes the Artin \( L \)-function attached to \( V_i \) with archimedean primes and primes in \( B \) omitted. Since the \( V_i \) are permutation representations, we have by e.g. [CR06, ex. 43.1], that there exists a subgroup \( H \) of \( \Gamma \) and a character \( \chi_i \) of \( H \) so that

\[
V_i = \text{Ind}^\Gamma_H \chi_i.
\]

Then by the Artin formalism

\[
L^{(B)}(bs - a, V_i) = L^{(B)}(bs - a, \chi_i),
\]

and thus all \( L \)-functions appearing in (7.15) are Hecke \( L \)-functions for Hecke characters \( \chi_i \) of some fields intermediate to \( K'/k \). By the analytic continuation of Hecke \( L \)-series, this establishes part (1) of Theorem 7.1.

Since \( L(1, \chi_i) \neq 0 \) for any Hecke characters \( \chi_i \), the number of poles at \( s = A \) appearing in (7.15) is equal to the number of trivial characters appearing among the Hecke characters \( \chi_i \), counted with multiplicity. By e.g. Serre [Ser77, ex. 2.6], the number of trivial characters is equal to the number of orbits

\[
\sum_{a,b \in P} |\Gamma \backslash \hat{\Sigma}_{a,b}|.
\]

If \( x = 1 \) then this matches \( |\hat{G}\backslash \hat{\Sigma}_0| \) as defined in the introduction. Thus we have established part (2) of Theorem 7.1.

For \( \sigma > A \) we have that \( |U(s, x)| \leq U(\sigma, 1) \), where \( \sigma = \text{Re}(s) \). Therefore

\[
\sum_{a,b \in P} |\Gamma \backslash \hat{\Sigma}_{a,b}| \leq |\hat{G}\backslash \hat{\Sigma}_0|
\]
for any \( x \in U_N(T) \), and the Laurent series expansion of \( U(s, x) \) around \( s = A \) is bounded in absolute value by that of \( U(s, 1) \). By e.g. (7.13), the leading constant in the Laurent expansion is positive real. This establishes part (4) of Theorem (7).

By Lemma 3.3, the group \( \Gamma \) is finite and does not depend on \( x \in U_N(T) \). Therefore there are only finitely many irreducible representations \( V_i \) of \( \Gamma \), and so the product over \( i \) in (7.15) has finitely many factors, bounded uniformly in terms of \( x \). By the second part of Lemma 3.3, the dimension of \( V_{a,b} \) is uniformly bounded in terms of \( x \). Therefore, the multiplicities \( m_{a,b,i} \) remain uniformly bounded as \( x \in U_N(T) \) varies. Therefore, the degree and conductor of the (partial) \( L \)-function on the right hand side of (7.15) remain uniformly bounded as \( x \) varies over \( U_N(T) \) (but depend on \( r \) and \( T \), of course). By a standard convexity bound on the Hecke \( L \)-functions in (7.15) inside the critical strip, there exists \( K = K(r, T) > 0 \) independent of \( x \) so that when \( \sigma > A - (2m^2)^{-1} \) and \( s \) avoids a small neighborhood \( N \) around \( s = A \), we have

\[
U(s, x) \ll_{N, \sigma} (1 + |s|)^K.
\]

7.2. Ramified places.

**Lemma 7.8.** The function \( R(s, x) \) converges absolutely and uniformly on compacta in the region \( \text{Re}(s) > A - (2m^2)^{-1} \). It takes a positive value at the point \( s = A \).

**Proof.** Recall that

\[
R(s, x) = \sum_{\theta \in \text{Cl}_N(T)} \prod_{\nu \in B} \left( \delta(\psi_\nu)dx_\nu/dx'_\nu \right)^{sm} R_\nu(s, x).
\]

We saw in section 5 for each \( \nu \in B \) that \( \left( \delta(\psi_\nu)dx_\nu/dx'_\nu \right)^{sm} R_\nu(s, x) \) converges absolutely and uniformly on compacta in

\[
\text{Re}(s) > \max\left\{ \frac{\dim D(S)}{|S|} : D(S) \neq \{1\} \right\} \geq A - m^{-2},
\]

a region which includes the point \( s = A \). Since the sum over class group characters in (7.17) is finite and \( B \) consists of finitely many places, the function \( R(s, x) \) converges absolutely, absolutely and uniformly on compacta in the region \( \text{Re}(s) > A - (2m^2)^{-1} \). The first statement of (7.8) holds.

We now show the second assertion of Lemma 7.8. The product over \( \nu \in B \) is a finite product. Let us enumerate the places appearing as \( v_1, \ldots, v_s \). Let us denote by

\[
c \mid |B|^\infty
\]

the set of all positive integers \( c \) of the form \( q_{k_1}^{n_1} \cdots q_{k_s}^{n_s} \), for \( n_1, \ldots, n_s \in \mathbb{N} \). In this paragraph, we write \( \mu(d) \) for the \( \text{m"obius} \) function defined with respect to numbers \( d \mid |B|^\infty \), that is, where \( q_{k_{v_i}} \) plays the role of the primes. For \( c \mid |B|^\infty \) we let

\[
H_c^s = \{ (\theta, \chi_{v_1}, \ldots, \chi_{v_s}) \in \text{Cl}_N(T)^s \times \prod_{\nu \in B} \text{NT}(\mathcal{O}_w) : \prod_{i=1}^s q_{k_{v_i}}^{-c_{v_i}(\chi_{v_i}, \theta_{v_i})} = c \},
\]

where \( c_{v_i} \) is the Artin conductor associated to \( \chi_{v_i} \theta_{v_i} \) via \( r \). We also define

\[
H_c = \{ (\theta, \chi_{v_1}, \ldots, \chi_{v_s}) \in \text{Cl}_N(T)^s \times \prod_{\nu \in B} \text{NT}(\mathcal{O}_w) : \prod_{i=1}^s q_{k_{v_i}}^{-c_{v_i}(\chi_{v_i}, \theta_{v_i})} \mid c \}.
\]

The set \( H_c \) is a group. We have the relations

\[
H_c = \bigcup_{d 
mid c} H_d^s, \quad \text{and} \quad H_c^s = \bigcup_{d 
mid c} \mu(d)H_{c/d},
\]

where the unions are over integers \( d \mid |B|^\infty \) which also divide \( c \), and \( \mu(d) \) is defined as before.
In terms of these definitions, we have

\[ R(s, x) = \sum_{c \mid |B|} \frac{1}{c^s} \sum_{(\theta, \chi_{v_{1}}, \ldots, \chi_{v_{k}}) \in H_{c}^k} \chi_{v_{1}}(x) \cdots \chi_{v_{k}}(x) \]

\[ = \sum_{c \mid |B|} \frac{1}{c^s} \sum_{d \mid c} \mu(d) \sum_{(\theta, \chi_{v_{1}}, \ldots, \chi_{v_{k}}) \in H_{c/d}} \chi_{v_{1}}(x) \cdots \chi_{v_{k}}(x) \]

\[ = \left( \sum_{c \mid |B|} \frac{1}{c^s} \right)^{-1} \sum_{c \mid |B|} \frac{1}{c^s} \sum_{(\theta, \chi_{v_{1}}, \ldots, \chi_{v_{k}}) \in H_{c}} \chi_{v_{1}}(x) \cdots \chi_{v_{k}}(x). \]

The first factor is evidently positive real for all \( s > 0 \), since it is a product over finitely many “primes”. By orthogonality of characters, the second is a Dirichlet series with non-negative integer coefficients, hence it takes a positive real value wherever it converges absolutely.

\[ \square \]

7.3. Conclusion. We now prove that \( Z(s) \) continues to a meromorphic function polynomially bounded in vertical strips in the right half-plane \( \text{Re}(s) > A - (2m^2)^{-1} \), with at most a pole at \( s = A \) of order \( |G \setminus \Sigma_0| \). To do so, we show that (7.1) converges absolutely and uniformly on compacta in \( \text{Re}(s) > A - (2m^2)^{-1} \).

For each \( v \in S_{\infty} \), let

\[ \sigma_v : T(k) \hookrightarrow \mathbf{T} = \left\{ \begin{array}{ll} (\mathbf{R}^X)^{n_1} \times (S^1)^{n_2} \times (\mathbf{C}^X)^{n_3} & \text{if } v \text{ real} \\ (\mathbf{C}^X)^n & \text{if } v \text{ complex} \end{array} \right \} \]

be the corresponding embedding. By (7.1), Theorems 6.1, 7.1, and Lemma 7.8 we have for any \( s \neq A \) with \( \text{Re}(s) > A - (2m^2)^{-1} \) that

\[ (7.18) \quad Z(s) \ll_{r, T, \nu} \sum_{x \in U_N(T)} |U(s, x)||R(s, x)| \prod_{v \in S_{1, \infty}} \prod_{j=1}^{n_1} \frac{1}{1 + |\log |(\sigma_v x)_j||} \prod_{j=1}^{n_3} \frac{1}{1 + |\log |(\sigma_v x)_{j'}||} \]

where \( S_{1, \infty} \) and \( S_{2, \infty} \) are the set of real and complex archimedean places, respectively. Since \( R(s, x) \) is a finite sum, we have the bound \( |R(s, x)| \ll R(\sigma, 1) \), where \( \sigma = \text{Re}(s) \). By Theorem 7.1 part (4), we also have \( U(s, x) \ll_{\sigma} (1 + |s|)^K \), uniformly in \( x \), and for \( s \) avoiding a neighborhood around \( s = A \). Thus

\[ (7.19) \quad Z(s) \ll_{r, T, \nu, \sigma} R(\sigma, 1)(1 + |s|)^K \sum_{x \in U_N(T)} \prod_{v \in S_{1, \infty}} \prod_{j=1}^{n_1} \frac{1}{1 + |\log |(\sigma_v x)_j||} \prod_{j=1}^{n_3} \frac{1}{1 + |\log |(\sigma_v x)_{j'}||} \]

If \( T \) is anisotropic, then \( U_N(T) \) is finite and so we have shown that \( Z(s) \) continues to a meromorphic function polynomially bounded in vertical strips in the right half-plane \( \text{Re}(s) > A - (2m^2)^{-1} \).

Suppose then that \( T \) is isotropic. By [Shy77], and Lemmas 2.10 and 2.11 we have

\[ \text{rank} U_N(T) = \sum_{v \in S} \text{rank} X^*(T)^{G_{kv}} - \text{rank} X^*(T)^G. \]

If \( v \) is a real place we have \( \text{rank} X^*(T)^{G_{kv}} = n_1 + n_3 \), and if \( v \) is a complex place we have \( \text{rank} X^*(T)^{G_{kv}} = n \). Since \( T \) is isotropic we have \( \text{rank} X^*(T)^G \geq 1 \). There are \(|S_{1, \infty}|(n_1 +
shown that \(Z\) continues to a meromorphic function polynomially bounded in vertical strips in the right half-plane \(\text{Re}(s) > A - (2n^2)^{-1}\).

By the absolute convergence of (7.18) and assertion (3) of Theorem 7.1, the sum of Laurent series at \(s = A\) of \(U(s, x)R(s, x)A(s, x)\) also converges absolutely. Therefore \(Z(s)\) has a pole of order \(|G \setminus \Sigma_0|\) at \(s = A\) with positive leading term in its Laurent series expansion.

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ETH ZÜRIC̈H - DEPARTEMENT MATHEMATIK, HG G 66.4, RÂMISTRASSE 101, 8092 ZÜRIC̈H, SWITZERLAND
E-mail address: ian.petrow@math.ethz.ch