The Application of the Newman-Janis Algorithm in Obtaining Interior Solutions of the Kerr Metric.

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(May 25, 2021)

In this paper we present a class of metrics to be considered as new possible sources for the Kerr metric. These new solutions are generated by applying the Newman-Janis algorithm (NJA) to any static spherically symmetric (SSS) “seed” metric. The continuity conditions for joining any two of these new metrics is presented. A specific analysis of the joining of interior solutions to the Kerr exterior is made. The boundary conditions used are those first developed by Dormois and Israel. We find that the NJA can be used to generate new physically allowable interior solutions. These new solutions can be matched smoothly to the Kerr metric. We present a general method for finding such solutions with oblate spheroidal boundary surfaces. Finally a trial solution is found and presented.

PACS numbers: 04.20.-g, 04.20.Jb, 97.10.Kc

I. INTRODUCTION

Since the discovery of the Kerr metric [1] many attempts have been made to find a physically reasonable interior matter distribution that may be considered as its source. For a review of some of these approaches the reader is referred to the introductions of [2] and [3]. Though much progress has been made results have been generally disappointing. As far as we can tell nobody has obtained a physically satisfactory interior solution. This seems surprising given the success of matching internal spherical symmetric solutions to the Schwarzschild metric. The problem is not simply that the loss of one degree of symmetry makes the derivation of analytic results that much more difficult. Severe restrictions are placed on an interior metric by maintaining that it must be joined smoothly to the Kerr metric. Further restrictions are placed on interior solutions to ensure that they correspond to physical objects. Furthermore since the Kerr metric has no radiation field associated with it its source metric must also be non-radiating. This places even further constraints on the structure of the interior metric [4]. Given the strenuous nature of these limiting conditions is not surprising to learn that as yet nobody obtained a truly satisfactory solution to the problem of finding sources for the Kerr metric. In general the failure is due to an internal structure that has unphysical properties, or a failure to satisfactorily match the boundary conditions.

This paper tries to overcome some these problems by examining a broad class of interior metrics which, by construction, are matched smoothly to the Kerr metric. Concurrently all the sources that we will be examining are stationary and axially symmetric (SAS) and as a consequence they are not generators of gravitational radiation. The boundary conditions used, those first discussed by Dormois [5] and developed further by Israel [6], are explained in more detail later. At this stage we simply point out that in order to evaluate the Dormois-Israel conditions a knowledge of the surface separating the two space-times (interior and exterior) is required. Unlike many previous attempts, however, we do not specify the surface prior to evaluating the Dormois-Israel conditions. In fact we take the reverse approach, that is the surface between the two metrics is determined by imposing the Dormois-Israel criteria. This then allows us to explore all possible surfaces on which the two metrics can be matched continuously. To the best of the authors knowledge this approach has not been previously examined.

After the original discovery of the Kerr metric Newman and Janis showed that this result could be “derived” by making an elementary complex transformation to the Schwarzschild solution [7]. This same method was then used to obtain a new SAS solution to Einstein’s field equations now known as the Kerr-Newman metric [8]. The Kerr-Newman space-time is that associated with the exterior geometry of a rotating massive charged black-hole. For a modern review of the Newman-Janis algorithm (NJA) for obtaining both the Kerr and Kerr-Newman metrics see [9].

At the time of publication there was no valid reason as to why this method worked. Many physicist considered this ad hoc procedure to be a “fluke” and not worthy of further investigation. However by means of a very elegant mathematical method, Schiffer et al [10] gave a rigorous proof as to why the Kerr metric can be considered as a complex transformation of the Schwarzschild space-time. We will not go into the details of this paper, but simply

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state that it tells us nothing about the success of such a complex transformation on a spherically symmetric interior solution to give an axially symmetric one. The reason for this is that the proof they gave relies on two assumptions. The first assumption is that the metric belongs to the same algebraic class as the Kerr-Newman solution, namely the Kerr-Schild class (KS) [1]. The second is that the metric corresponds to a empty solution of Einstein’s field equations. In the case we study these assumptions are not made and hence the proof is not applicable. It is clear, by the generation of the Kerr-Newman metric, that all the components of the stress-energy tensor need not be zero for the NJA to be successful. In fact Gürses and Gürsey in 1975 [12] showed that if a metric can be written in KS form then a complex transformation “is allowed in general relativity.”

Since it is not possible to represent a perfect fluid (with the exception of the cosmological fluid) by a KS space-time [4], there is no proof as to the validity of complex transformations in generating physically reasonable stationary axially symmetric (SAS) interior solutions from spherically symmetric ones. This, of course, does not imply that the complex transformation method is valid only on KS space-times. It simply means we have no conception as to whether or not it will yield sensible results. Recently KS geometries have been considered in looking for elastic-solid bodies as sources for the Kerr metric [13] [14]. This approach seems promising but preliminary inquiries show that the sources are either surrounded by a shell-like distribution of stress energy or else exhibit a ring singularity. For these reasons we feel that it is important to investigate the nature of complex coordinate transformations in non KS space-times.

There are two main themes that run through this paper. The first is the application of the NJA to any static spherically symmetric (SSS) seed metric. The second is the joining of two SAS metrics on a static axially symmetric boundary surface. These two themes are linked together with the aim of finding a physically reasonable source for the Kerr metric. As far as the authors are aware this combined approach is entirely new.

The rest of the outline of the paper is as follows. In the second section [II] we briefly review the Newman-Janis algorithm NJA for obtaining SAS metrics from SSS ones. The resulting metric is written in terms of two arbitrary functions. These functions give the physical properties of the internal structure. Furthermore we make use of a coordinate transformation so that this new metric is written in Boyer-Lindquist type coordinates. This makes the physical interpretation much clearer and decreases the amount of algebra required in calculating various metric properties. The third section [III] outlines the boundary conditions used in the Dormois-Israel formalism. In this section we develop a set of boundary conditions for the joining of any two SAS metrics on a arbitrary static axially symmetric boundary surface. The term boundary surface is used according to Israel’s original definition. This means that the surfaces separating the two geometries has a vanishing surface stress-energy tensor, i.e. no thin shells. In this section we place particular emphasis on those SAS metrics that can be generated from SSS metrics via the NJA. Following on directly from this section [IV] examines explicitly the case when the exterior metric corresponds to the Kerr metric. In this section the boundary conditions for matching of an interior generated by the NJA to the Kerr metric are given. Also in this section we discuss in what sense we use the term “physically reasonable source of the Kerr metric”. This places even further constraints on the interior metric form. Having a established a rigorous formalism in the previous sections in the next section [V] we look for solutions to which the NJA may be successfully applied. We examine one such solution which we have called the “trial solution”. We find that it has a physically reasonable seed metric associated with it and matches smoothly to the Kerr metric on an oblate spheroid. The last section [VI] is the conclusion and sums up all the new results that have been obtained in this paper.

II. THE NEWMAN-JANIS ALGORITHM.

We do not claim to be the first authors to examine SAS interior solutions generated by the NJA. In fact a paper by Herrera and Jiménez [15] obtains the same general metric form after the NJA is applied as we do. However the similarity ends there. In our results the metric is cast in a more tangible form, we use a different set of boundary conditions and we examine different specific metric functions.

Despite the work by Newman and Janis, the work by Herrera and Jiménez and some further papers on the subject the NJA is not a well known area of general relativity. For this reason we believe it is necessary to give a detailed outline as to what it actually is. We start by reviewing this algorithm as applied to a static spherically symmetric “seed” metric,

$$ds^2 = e^{2\phi(r)}dt^2 - e^{2\lambda(r)}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\Phi^2),$$  \hspace{1cm} (1)

with the signature chosen to be consistent with Newman and Janis' original paper. Following Newman and Janis, equation (1) is written in advanced Eddington–Finkelstein coordinates, i.e. the $g_{rr}$ component is eliminated by a change of coordinates and a cross term is introduced. Specifically this is done by advancing the time coordinate so that $dt = du + fdr$ and setting $f = \pm e^{\lambda(r) - \phi(r)}$, we choose the positive case, again to be consistent with the Newman-Janis formulation. Once this is done the metric in these new coordinates is,
\[ ds^2 = e^{2\phi(r)} du^2 + 2e^{\lambda(r) + \phi(r)} du dr - r^2 (d\theta^2 + \sin^2 \theta d\Phi^2). \] 

Written in contravariant form this is
\[
g^{\mu\nu} = \begin{pmatrix} 0 & e^{-\lambda(r) - \phi(r)} & 0 & 0 \\ e^{-\lambda(r) - \phi(r)} & -e^{-2\lambda(r)} & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2 \sin^2 \theta) \end{pmatrix}. \] 

This is done so that the above metric may be written in the terms of its null tetrad vectors,
\[
g^{\mu\nu} = l^\mu n^\nu + l'^\mu n'^\nu - m^\mu \bar{m}^\nu - m'^\mu \bar{m}'^\nu, \tag{4} \]

where
\[
l^\mu = \delta^\mu_1 \tag{5} \\
n^\mu = -\frac{1}{2} e^{-2\lambda(r)} \delta^\mu_1 + e^{-\lambda(r) - \phi(r)} \delta^\mu_0 \tag{6} \\
m^\mu = \frac{1}{\sqrt{2r}} \left( \delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3 \right). \tag{7} \]

The bar indicates a complex conjugate. This complex null tetrad system forms the starting point for the “derivation” of Kerr-Newman space-times. As has been already stated this procedure is known to be a valid method for KS geometries but its extension into non KS type metrics is still to be thoroughly examined. To be consistent exactly the same transformations as those originally performed by Newman and Janis are made. That is coordinates are advanced by the following complex increments:
\[
u \to \nu' = \nu - ia \cos \theta, \quad r \to r' = r + ia \cos \theta, \quad \theta \to \theta', \quad \Phi \to \Phi'. \tag{8} \]

By keeping \( r' \) and \( \nu' \) real (that is considering the transformations as a complex rotation of the \( \theta, \Phi \) planes) one obtains the following tetrad.
\[
l^\mu = \delta^\mu_1 \tag{9} \\
n^\mu = -\frac{1}{2} e^{-2\lambda(r,\theta)} \delta^\mu_1 + e^{-\lambda(r,\theta) - \phi(r,\theta)} \delta^\mu_0 \tag{10} \\
m^\mu = \frac{1}{\sqrt{2r + ia \cos \theta}} \left( ia \sin \theta (\delta^\mu_1 - \delta^\mu_2) + \delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3 \right). \tag{11} \]

All primes are now dropped for convenience of notation but one must recall that the new functions \( e^{\lambda(r,\theta)} \) and \( e^{\phi(r,\theta)} \) are not the same as the old ones. In fact, the new functions depend on both \( r \) and \( \theta \) whereas the old ones had only an \( r \) dependence.

The metric formed from the above null vectors using (4) is,
\[
g^{\mu\nu} = \begin{pmatrix} -\frac{a^2 \sin^2 \theta}{\Sigma} & e^{-\lambda(r,\theta) - \phi(r,\theta)} + \frac{a^2 \sin^2 \theta}{\Sigma} & 0 & -\frac{a}{\Sigma} \\ . & -e^{-2\lambda(r,\theta)} - \frac{a^2 \sin^2 \theta}{\Sigma} & 0 & \frac{a}{\Sigma} \\ . & . & -1 & 0 \\ . & . & . & -\frac{1}{\Sigma \sin^2 \theta} \end{pmatrix}. \tag{12} \]

where \( \Sigma = r^2 + a^2 \cos^2 \theta \). In the covariant form this is
\[
g_{\mu\nu} = \begin{pmatrix} e^{2\phi(r,\theta)} & e^{\lambda(r,\theta) + \phi(r,\theta)} & 0 & \frac{\sin^2 \theta e^{\phi(r,\theta)} (e^{\lambda(r,\theta)} - e^{\phi(r,\theta)})}{a^2} \\ . & 0 & 0 & -ae^{\phi(r,\theta) + \lambda(r,\theta)} \sin^2 \theta \\ . & . & -\Sigma & 0 \\ . & . & . & -\sin^2 \theta (\Sigma + a^2 \sin^2 \theta e^{\phi(r,\theta)} (2e^{\lambda(r,\theta)} - e^{\phi(r,\theta)})) \end{pmatrix}. \tag{13} \]

As the metric is symmetric the “\( a \)” is used to indicate \( g^{\mu\nu} \neq g^{\nu\mu} \). The form of this metric gives the general result of the NJA to any SSS seed metric.

The metric given in equation (13), though relatively simple, is still hard to work with. To eradicate this problem one can make a gauge transformation so that the only off-diagonal component is \( g_{\Phi\Theta} \). This makes it easier to compare
with the more usual Boyer-Lindquist form of the Kerr metric and to interpret physical properties such as frame dragging. It also aids in the calculation and evaluation of the Einstein tensor. To do this, the coordinates $u$ and $\Phi$ are redefined in such a way that the metric in the new coordinate system has the properties described above. More explicitly, if we advance the coordinates in the following way, $du = dt + g(r)dr$ and $d\Phi = d\Phi' + h(r)dr$, with the following functional forms of $f$ and $g$:

$$g(r) = -\frac{e^{\lambda(r,\theta)}(\Sigma + a^2 \sin^2 \theta e^{\lambda(r,\theta)} + \phi(r,\theta))}{e^{\phi(r,\theta)}(\Sigma + a^2 \sin^2 \theta e^{2\lambda(r,\theta)})} \quad (14)$$

$$h(r) = -\frac{ae^{2\lambda(r,\theta)}}{\Sigma + a^2 \sin^2 \theta e^{2\lambda(r,\theta)}} \quad (15)$$

then after some algebraic manipulations one finds that in this coordinate system the metric is,

$$g_{\mu\nu} = \begin{pmatrix}
0 & 0 & a \sin^2 \theta e^{\phi(r,\theta)}(e^{\lambda(r,\theta)} - e^{\phi(r,\theta)}) \\
0 & -\Sigma/(\Sigma e^{-2\lambda(r,\theta)} + a^2 \sin^2 \theta) & 0 \\
0 & 0 & -\Sigma \\
-\Sigma & 0 & -\sin^2 \theta(\Sigma + a^2 \sin^2 \theta e^{\phi(r,\theta)} (2e^{\lambda(r,\theta)} - e^{\phi(r,\theta)})) & 0
\end{pmatrix}. \quad (16)$$

We state that this metric represents the complete family of metrics that may be obtained by performing the NJA on any static spherical symmetric seed metric, written in Boyer-Lindquist type coordinates. The validity of these transformations requires that $R + a^2 \sin^2 \theta e^{2\lambda r} \not= 0$, which is always the case since $e^{2\lambda r} > 0$. We note here that the choice of $e^{\phi(r,\theta)} = e^{-2\lambda(r,\theta)} = 1 + (Q^2 - 2mr)/\Sigma$ corresponds to the Kerr-Newman solution, where $Q$ and $m$ are the charge and mass of the body respectively.

### III. THE MATCHING OF TWO SPACE-TIMES ON A COMMON BOUNDARY SURFACE

The problem of matching two separate space-times on a common surface is a well explored area of general relativity. A large number of papers exist on the subject, and now an algebraic programme is available to speed up calculations. However Israel's original paper still remains a definitive work on the subject. Although the formalism was originally developed to examine the motion of expanding bubbles in the universe it has applications that range far wider. We will not go into a detailed discussion of the Dormois-Israel formalism, we will simply use it as it was originally developed.

When matching to two different metrics on a common surface the main problem one faces is that of choosing an appropriate coordinate system. One needs to be convinced that if any discontinuities do occur they are due solely to the topology and not the coordinate systems. To eliminate this problem a common coordinate system must be chosen. It is argued that in this new coordinate system the continuous properties should be the metric topology and not the coordinate systems. To eliminate this problem a common coordinate system must be chosen.

In our case an attempt is made to join an interior solution, which acts as a source, to the Kerr metric. It seems reasonable then to assume that the surface dividing the two space-times is both static and axially symmetric. With this in mind we examine how the Dormois-Israel formalism can be simplified and what restrictions it places on both the surface and the internal metric.

The approach taken here is as follows; one starts by specifying the coordinates on which the surface is defined. Let us suppose that both the interior and exterior metrics are defined in the coordinate system $x^\alpha = \{t, r, \theta, \Phi\}$. Let us further suppose that the boundary surface separating these two space-times is a function connecting the two variables $r$ and $\theta$. This simply states that the separating surface is both static and axially symmetric. If this is true then the two coordinates $r$ and $\theta$ may be eliminated by a single parameter $\tau$. As a consequence of this, the surface coordinates may be expressed as $\zeta^1 = \{t, \tau, \Phi\}$. These are the types of boundary surfaces that we study. If in general, by the definition of a boundary surface, two coordinates can be replaced by a single one at the surface, then the four metric $g_{\alpha\beta}$ may be replaced by the three metric $\gamma_{ij}$. Note the we are using the convention that Greek indices run from 0 to 3 and Latin from 0 to 2. The four metric is then projected onto the surface three metric by the following relationship,

$$\gamma_{ij} = \frac{\partial x^\alpha}{\partial \zeta^i} \frac{\partial x^\beta}{\partial \zeta^j} g_{\alpha\beta}. \quad (17)$$

It is handy at this stage to introduce the following notation; let $M^+$ correspond to the space-time geometry exterior to the object we are examining. In the case we are most interested in, $M^+$ corresponds to Kerr geometry. The interior geometry is given the symbol $M^-$. For any metric-dependent quantity $X$ one must specify the region of space-time
in which it is to be calculated. The notation \( X^+ \) means that the quantity \( X \) is calculated in the exterior space-time geometry \( M^+ \). The notation \( X^+ \vert_s \) signifies that the value \( X \) is calculated in \( M^+ \) and evaluated at the surface. Finally we use the notation that \( [X] \equiv X^+ \vert_s - X^- \vert_s \), which measures the jump discontinuity in the value of \( X \) as calculated by the two metrics and evaluated at the surface.

Israel’s first boundary condition demands the continuity of the first fundamental form, that is \( [^3 g_{ij}] = 0 \). If the two metrics to be joined may both be written in a form in which the only off-diagonal term is \( g_{t \Phi} \) then the above results reduce to

\[
[3 g_{tt}] = [4 g_{tt}] = 0 \tag{18} \\
[3 g_{t \Phi}] = [4 g_{t \Phi}] = 0 \tag{19} \\
[3 g_{\Phi \Phi}] = [4 g_{\Phi \Phi}] = 0 \tag{20} \\
[3 g_{r r}] = [4 g_{r r}] + \left( \frac{\partial r}{\partial \tau} \right)^2 4 g_{r r} + \left( \frac{\partial \theta}{\partial \tau} \right)^2 4 g_{\theta \theta} = 0. \tag{21}
\]

All these four equations when used in conjunction with the metric (16) are simplified to the two conditions

\[
\left[ e^{2 \lambda(r, \theta)} \right] = \left[ e^{2 \phi(r, \theta)} \right] = 0. \tag{22}
\]

The above equations give the continuity of metric coefficients at the boundary surface. The next of Israel’s boundary conditions involves the properties of the extrinsic curvature. Most work done with the matching of extrinsic curvature has been based on the thin shell formalism (for examples see [18] to [21]), however the general technique may be used in a much wider range of scenarios. The primary result that we are interested in is that for boundary surfaces (i.e. surfaces that have a surface energy-momentum tensor which is identically zero) all components of the extrinsic curvature are continuous at the surface.

The extrinsic curvature measures the rate of change of the normal vector as it moves along the boundary surface. It is given explicitly by the expression

\[
K_{ij} = n_{ij} = -n^\gamma \left( \frac{\partial^2 x^\gamma}{\partial \zeta^i \partial \zeta^j} + \Gamma^\gamma_{\alpha \beta} \frac{\partial x^\alpha}{\partial \zeta^i} \frac{\partial x^\beta}{\partial \zeta^j} \right), \tag{23}
\]

where \( n^\gamma \) is the unit normal to the surface and \( \Gamma^\gamma_{\alpha \beta} \) are the Christoffel symbols associated with a given metric. The components of the normal \( n^\gamma \) clearly depend on the hyper-surface separating the two space-time regions \( M^+ \) and \( M^- \). In general if the surface is specified by the equation

\[
F(x^\alpha) = 0 \tag{25}
\]

then the components of the normal vector are

\[
n^\gamma = \pm \frac{\partial x^\gamma}{\sqrt{\partial \beta F \partial \beta F}}. \tag{26}
\]

The \( \pm \) determines whether the normal is a space-like or time-like vector. In the case of static axially symmetric surfaces then

\[
F(r, \theta) = 0. \tag{27}
\]

\( R(\theta) \) is some unknown function of \( \theta \) which specifies the boundary surface. In this case the unit normal, which we consider to be a time-like vector, is given by

\[
n^\gamma = -\frac{\delta^\gamma_1 - \partial_\theta R(\theta) \delta^2_2}{\sqrt{g^{rr} - g^{\theta \theta} (\partial_\theta R(\theta))^2}}. \tag{28}
\]

Using the above equation along with (24) and (18) to (21) one finds that the matching of extrinsic curvature reduces to the following set of constraints on the metric components
\[ [K_{\mu}] = g^{\tau \tau} [g_{\tau \tau}] + g^{\theta \phi} \partial_\theta R(\theta) [g_{\mu \phi}] = 0 \]  
\[ [K_{\ell \phi}] = g^{\tau \tau} [g_{\phi \tau}] + g^{\theta \phi} \partial_\theta R(\theta) [g_{\phi \phi}] = 0 \]  
\[ [K_{\phi \phi}] = g^{\tau \tau} [g_{\phi \tau}] + g^{\theta \phi} \partial_\theta R(\theta) [g_{\phi \phi}] = 0 \]  
\[ [K_{\tau \tau}] = \frac{1}{2} \left( \frac{\partial \tau}{\partial \tau} \right)^2 \left( g^{\tau \tau} [g_{\tau \tau}] + \partial_\theta R(\theta) g^{\theta \theta} [g_{\theta \theta}] \right) \]
\[ + \frac{\partial \tau}{\partial \tau} \frac{\partial \theta}{\partial \theta} \left( g^{\tau \tau} [g_{\theta \theta}] - \partial_\theta R(\theta) g^{\theta \theta} [g_{\theta \theta}] \right) \]
\[ - \frac{1}{2} \left( \frac{\partial \theta}{\partial \tau} \right)^2 \left( g^{\tau \tau} [g_{\theta \theta}] + \frac{\partial R(\theta)}{\partial \theta} g^{\theta \theta} [g_{\theta \theta}] \right) = 0. \]

In the above equations all metric components refer to the four metrics \( \{ g_{\alpha \beta} \} \) and are evaluated at the surface. The notation \( X, \alpha \equiv \partial_\alpha X = \frac{\partial X}{\partial \alpha} \) is used. Note also that since all components of the extrinsic curvature are zero common factors have been removed. If the two metrics to be joined can be generated by the NJA and hence may be written in the form \( \{ \text{eqns} \} \) then equations \( \{ 29 \} \) to \( \{ 33 \} \) may be greatly simplified. Assuming \( a \neq 0 \) we find that above equations are satisfied by

\[ g^{\tau \tau} \left[ e^{2\phi(r, \theta)} \right] - \partial_\theta R(\theta) g^{\theta \theta} \left[ e^{2\phi(r, \theta)} \right] = 0 \]  
\[ g^{\tau \tau} \left[ e^{2\lambda(r, \theta)} \right] - \partial_\theta R(\theta) g^{\theta \theta} \left[ e^{2\lambda(r, \theta)} \right] = 0. \]

The common factors mentioned above involve the rotation parameter \( a \). When this is zero the only surviving component of the extrinsic curvature is \( K_{\ell \tau} \). If this is the case the boundary conditions are expressed simply by \( \{ 33 \} \) as expected. If \( \partial_\theta R(\theta) \neq 0 \) then equation \( \{ 32 \} \) is simplified using the relations \( \{ 16 \} \) and \( \{ 27 \} \) to read

\[ \partial_\theta R(\theta) \left( g^{\tau \tau} [g_{\theta \theta}] + g^{\theta \theta} [g_{\theta \theta}] \right) + 2 g^{\tau \tau} [g_{\theta \theta}] = 0. \]

However if \( \partial_\theta R(\theta) = 0 \) then \( [K_{\tau \tau}] \equiv 0 \).

The above equations \( \{ 22 \} \) and \( \{ 33 \} \) to \( \{ 34 \} \) form a complete set of boundary conditions for the joining of any two stationary axially symmetric metrics generated by the NJA when applied to any SSS seed metric on an axially symmetric boundary surface. In the next section we examine some of the properties of these continuity conditions when the exterior metric is that due to Kerr.

**IV. POSSIBLE SOURCES FOR THE KERR METRIC**

The ultimate goal of this paper is find new static axially symmetric solutions to Einstein’s field equations which may be considered as sources for the Kerr metric. With this in mind we need to examine the properties of \( \{ 22 \}, \{ 33 \} \) and \( \{ 34 \} \) when the exterior metric is Kerr. In this case the boundary conditions become,

\[ e^{2\phi(r, \theta)}|_s = 1 - \frac{2MR(\theta)}{\Sigma_s} \]  
\[ e^{2\lambda(r, \theta)}|_s = \frac{\Sigma_s}{\Sigma_s - 2MR(\theta)} \]  
\[ \frac{2M}{\Sigma_s} \left( R(\theta)^2 - a^2 \cos^2 \theta \right) - e^{2\phi(r, \theta)}|_s = \]  
\[ - \frac{\partial_\theta R(\theta)}{\Delta_s} \left( \frac{4a^2MR(\theta) \cos \theta \sin \theta}{\Sigma_s^2} + e^{2\phi(r, \theta)}|_s \right), \]
\[ \frac{2M}{\Delta_s^2} \left( a^2 \cos^2 \theta - R^2(\theta) \right) - e^{2\lambda(r, \theta)}|_s = \]  
\[ \frac{\partial_\theta R(\theta)}{\Delta_s} \left( \frac{4a^2MR(\theta) \cos \theta \sin \theta}{\Delta_s^2} - e^{2\lambda(r, \theta)}|_s \right), \]
\[ \frac{2M}{\Sigma_s} \left( R(\theta)^2 - a^2 \cos^2 \theta \right) - e^{-2\lambda(r, \theta)}|_s = \]  
\[ - \frac{\partial_\theta R(\theta)}{\Delta_s} + \frac{2}{\Sigma_s} \left( \frac{4a^2MR(\theta) \cos \theta \sin \theta}{\Sigma_s^2} + e^{-2\lambda(r, \theta)}|_s \right), \]
\[ \Delta_s = R(\theta)^2 - 2MR(\theta) + a^2 \]  
\[ \Sigma_s = R(\theta)^2 + a^2 \cos^2 \theta. \] 

These results have been obtained from a purely geometric point of view. So far the only physical constraint invoked is that the surface stress energy tensor be identically zero.

When it comes to looking for sources of the Kerr metric then the interior metric must correspond to some physically sensible matter distribution. This obviously places further constraints on the interior metric. One of the most fundamental properties we expect from an interior Kerr metric is that as \( a \to 0 \) the seed metric is recovered and is an interior Schwarzschild solution. The recovery of the seed metric for \( a = 0 \) is inherent in the properties of (16) and the transformations (8). However what is not necessarily true, and has not even been discussed until now, is that the metric (10) corresponds to a physical reasonable solution to Einstein’s field equations.

Before proceeding any further it should be made clear by what is meant by the term “physically reasonable”. Their are a number of criteria that one could use. The ones that we are using, as we consider them to be the most fundamental, are the following:

- The strong and weak energy conditions are obeyed, i.e. the density \( \rho \) is always positive and the density is always greater than the pressure \( P \): \( \rho \geq 0; \rho \geq p \).
- \( P \) and \( \rho \) are monotonically decreasing as we move out from the center.
- \( P \) and \( \rho \) are related by a sensible equation of state.
- The interior is matched smoothly to the exterior.

With this in mind, we need to provide a seed metric for the NJA which satisfies the boundary conditions along with the above physical constraints. The technique frequently used for obtaining such interior seed solutions is to combine Einstein’s field equations along with the conservation laws for a given stellar model, such as a perfect fluid with a given equation of state. In the case of SSS perfect fluids this lends to the Oppenheimer-Volkov equation [23]. Even with these simplifications analytic solutions to the Oppenheimer-Volkov equation are difficult to obtain, see [4] for some examples. It is noted in [4] that the few known analytic solutions generally fail on physical grounds. This is a fairly major hurdle in the application of NJA to interior solutions since exact analytic expressions are required in order to be successfully.

**V. A TRIAL SOLUTION**

One of the great difficulties we face in looking for solutions sources of the Kerr metric is the matching of boundary conditions on appropriate surfaces. The approach we are going to take is not the usual one of guessing the solution and then seeing if the boundary conditions match. In fact we are going to take the reverse approach. We will start with interior structures that join smoothly to the Kerr metric. We will then examine the physical properties of these structures via the generation of the Einstein tensor. If these structures are “physically reasonable” we will then claim to have found an interior solution that may be considered as a source for the Kerr metric.

Although the boundary conditions (36) to (39) come in a relatively simple form, they are still rather difficult to work with. However this situation is rectifiable by considering surfaces described by \( \partial R(\theta)/\partial \theta = 0 \). The justification for choosing such simplified boundary surfaces goes beyond just making the equations more easily solvable. To understand this the first thing we must realise is that the Kerr metric written in Boyer-Lindquist coordinates leads to a confusing interpretation of the variable \( r \). We wish to emphasise the point again here that the four degrees of gauge freedom in relativity invoke an ambiguity of coordinate definitions. Consider for example the Kerr metric as first written by Kerr in Cartesian coordinates;

\[ ds^2 = d\bar{t}^2 - dx^2 - dy^2 - dz^2 - \frac{2m\bar{\rho}^4}{\bar{\rho}^4 + a^2 z^2} \left[ \frac{\bar{\rho}(x dx + y dy) - a(x dy - y dx)}{\bar{\rho}^2 + a^2} + \frac{z}{\bar{\rho}} dz + d\bar{t} \right]^2, \] 

where \( \bar{\rho} \) is determined implicitly, up to a sign, by

\[ \bar{\rho}^4 - (x^2 + y^2 + z^2 - a^2)\bar{\rho}^2 - a^2 z^2 = 0. \]
The coordinates $x, y, z$ in the Cartesian form are related to $r, \theta, \phi$ in the Boyer-Lindquist form in the following way:

$$x = r \sin \theta \cos \phi + a \sin \theta \sin \phi; \quad y = r \sin \theta \sin \phi - a \sin \theta \cos \phi; \quad z = r \cos \theta. \quad (45)$$

If we take the magnitude of the radial vector in spherical coordinates ($radius$) to have its usual definition,

$$radius^2 = x^2 + y^2 + z^2, \quad (46)$$

then by substitution of (13) into (46) we obtain that

$$radius^2 = r^2 + a^2 \sin^2 \theta. \quad (47)$$

Since a surface is described by $r = R(\theta)$ it is apparent why $\partial R(\theta)/\partial \theta = 0$, i.e. $R(\theta) = R = constant$, is a sensible choice of boundary surface. The surface defined by

$$radius^2 - a^2 \sin^2 \theta = R^2 \quad (48)$$

is and oblate spheroid. The oblate spheroid is a surface of revolution swept out by an ellipse rotating about its minor axis. An oblate spheroid defined by eq.(13) transforms to a sphere in the limit $a \to 0$. Clearly these are the types of surfaces one would expect for a uniformly rotating star. If, as we have argued, the boundary surface separating the interior and exterior solutions is an oblate spheroid then equations (36) to (40) simplify to

$$e^{2\phi(r,\theta)}|_s = 1 - \frac{2MR}{\Sigma s} \quad (49)$$

$$e^{2\lambda(r,\theta)}|_s = \frac{\Sigma s}{\Sigma s - 2MR} \quad (50)$$

$$\frac{2M(R^2 - a^2 \cos^2 \theta)}{\Sigma s^2} = e^{2\phi(r,\theta)} - _s \quad (51)$$

$$\frac{2M(a^2 \cos^2 \theta - R^2)}{\Delta s^2} = e^{2\lambda(r,\theta)} - _s. \quad (52)$$

The above equations give the boundary conditions for the joining of a SAS metric generated by the NJA to the Kerr metric on a static oblate spheroidal boundary surface. With these constraint equations it is possible to look for new sources of the Kerr metric.

The first solution of this kind we examine, and the one which is perhaps the simplest, is

$$e^{2\lambda(r,\theta)} = \frac{\Sigma s^2 - 2M(R^2r + a^2 \cos \theta(2R - r))}{\Delta s^2}. \quad (53)$$

Using this guess it should be possible to obtain, a density and pressure profile as well as finding an exact solution for $e^{2\phi(r,\theta)}$ by the use of Einstein’s equations. It is shown by the extremely low number of exact solutions Einstein’s equations that such a method, even for SSS metrics, is extremely difficult to work with. The use of algebraic programmes such as Cartan and grtensor cut down most of the algebra however they do not reduce the problem to a solvable state.

Although it is a complex task to examine the nature of stationary axially symmetric metrics the examination of the non-rotating case is much simpler. It is always useful to examine the slow or zero rotation limit of such solutions since one would expect that these limits must also represent physical objects. By examining the $a \to 0$ limit of such metrics we are examine the properties of the seed metric. Recall that in the previous section we showed that the NJA could not be successfully applied to it any arbitrary seed metric.

We know that our trial solution for $e^{2\lambda(r,\theta)}$ given by (53) obeys the boundary conditions on an oblate spheroid surface. One method for obtaining the interior structure might be to examine various functions of $e^{2\phi(r,\theta)}$ that obey the boundary conditions (14) and (51). Once this has been done these functions along with (53) could be fed into the general metric (10), from this the stress-energy tensor may be obtained by equating it to the Einstein tensor. However the authors are opposed to such a method as it relies on a great deal of guess work. As such it does not guarantee anything about the physical nature of the objects we are examining.

A more sensible approach, we believe, is to examine the properties of the seed metric first. If the seed metric corresponds to a physically sensible object then the hope is that the new metric generated by the NJA will also represent a physical object. To begin this process the $a \to 0$ limit of taken of the metric (10) so that is (11) and (53) becomes

$$e^{2\lambda(r,\theta)} = \frac{\Sigma s^2 - 2M(2R - r)}{\Delta s^2}. \quad (54)$$
\[ e^{2\lambda(r_*)} = \frac{1 - xr_*}{(1 - x)^2} \]  

where \( x = 2M/R \) and \( r_* = r/R \).

The theory of radially symmetric distributions of matter is a well explored field. One of the classic papers on this subject is by Wyman [28]. For an isotropic fluid sphere described by (1) the pressure \( P \) and density \( \rho \) satisfy the relations

\[
\begin{align*}
P_* &= e^{-2\lambda}(2\phi'/r_* + 1/r_*^2) - 1/r_*^2 \\
\rho_* &= e^{-2\lambda}(2\phi''/r_* - \lambda'\phi' + (\phi' - \lambda')/r_* + \phi^2) \\
P'_* &= -(P_* + \rho_*)\phi'.
\end{align*}
\]

The notation used is that \( \lambda \) and \( \phi \) are understood to be only functions of \( r_* \). \( P_* = PR^2, \rho_* = \rho R^2 \) and the prime denotes derivatives with respect to \( r_* \). The equality of equations (55) and (56) imply (58). From (57) and (54) the density profile is

\[
\rho_* = \frac{1}{r_*^2} - \frac{(1 - x)^2}{r_*^2(r_*x - 1)^2}.
\]

This density profile corresponds to that of a physically sensible stellar object as it as always positive and monotonically decreasing for all values of \( x \) such that \( 0 < x < 1 \). For this model the density vanishes at the surface for all allowed values of \( x \).

Given that the function \( e^{-2\lambda} \) is known then the equality of (55) and (54) results in a Riccati equation of first order in \( \phi' \). There are various methods for finding the solutions such equations [24] and once this is done the pressure profile may be obtained. For the example given the resulting Riccati equation is

\[
\phi'' = \frac{1}{r_*^2} + \frac{r_*x - 1}{r_*^2(x - 1)^2} + \frac{x}{2r_*(1 - r_*x)} + \left[ \frac{1}{r_*} + \frac{x}{2r_*x - 1} \right] \phi' - \phi'^2.
\]

Since this is a second order differential equation the solution involves two constants of integration. These two constants are determined by making sure that at the surface the pressure is zero and that the metric is Schwarzschild. In general the solutions are defined by functions more complicated than the elementary transcendental functions. Thus far we have tried without success to obtain analytic solutions. As a result we have decided to perform numerical integration to examine the solutions quantitatively.

The integration routine we have chosen to use is the ubiquitous fourth-order adaptive step-size Runge-Kutta routine. Using this routine we are able to see how \( \phi' \) varies as a function of \( r_* \). This along with the the known relation \( e^{-2\lambda} \) enables us to determine the pressure profile of the stellar object. The pressure and density profiles are shown in figures (1) to (4) for \( x = 0.3 \) as typical for neutron stars.

VI. CONCLUSION

To make it clear what new discoveries and investigations have been made in this paper it is important to make a brief summary. Recall that in the first section [1] we gave a history to the discovery of the Kerr metric and of the Newman-Janis algorithm (NJA). We discussed the fact that the formalism has been well developed in the context of metrics that can be written in Kerr-Schild (KS) form. Furthermore we pointed out that interior solutions, with the exception of the pure radiation solutions, can not been written in the KS form. Essential this means that the NJA as applied to interior solutions has not been appropriately explored. With this in mind we set about the task of rectifying this situation. The second section [2] commenced by generalising the Newman-Janis algorithm for any static spherically symmetric (SSS) “seed” metric. We pointed out that although this has previously been investigated by others its lack of success may be attributed to an inconvenient choice of coordinates and to boundary conditions which are too stringent. To eliminate this problem a coordinate transformation was made after applying the NJA so that this new metric could be written in Boyer-Lindquist type form. This result to the best of the authors knowledge is completely new. The metric formed by this algorithm belongs to a special class of metrics which are both stationary and axially symmetric (SAS). The Kerr metric belongs to this class.

In the third section [3] we established the constraints placed by matching smoothly two such metrics on any static axially symmetric surface. The boundary conditions we used were those first developed Dormois and Israel. We found that for metrics generated by the NJA the boundaries conditions are a relatively simple set of constraint equations.
In looking for interior solutions to rotating massive bodies one would expect that the exterior to be described by the Kerr metric. Section IV therefore was devoted to examining the properties of an interior metric that matches smoothly to the Kerr metric. Also in this section we elaborated on what is meant by the term “physically reasonably” in describing the properties of stellar objects.

Finally in the last section before the conclusion we determined what sort of interior metrics could be matched smoothly to the Kerr metric on oblate spheroidal surfaces. We found one such solution we have termed the trial solution in which the metric coefficient $e^\lambda(r,\theta)$ was determined explicitly. We argued that although it would be just as simple to find functions of $e^\phi(r,\theta)$ that matched the boundary conditions on such surfaces, one has no a priori reason to believe that these will correspond to a physically reasonable stellar model. To work around this problem we decided that to begin with a physically reasonable seed metric. To make sure this was the case we took the zero rotation limit ($a = 0$) of $e^\lambda(r,\theta)$. We found that the density profile resulting from this was physical sensible. The density $\rho$ as a function of the radius $r$ was always positive, decreased monotonically out from the center and although it was infinite at the center the total mass was still finite.

The pressure profile on the other hand was a slightly more difficult to explicate. We started by looking at seed metrics which described perfect fluids with isotropic pressure. Although this simplified calculations greatly it meant that in order to find exact solutions of $e^\phi$ we needed to integrate a first order non-liner equation known as the Riccati equation. The task of finding such solutions is non trivial as solutions generally can not expressed as simple transcendental functions. However such equations can be integrated numerically without too much effort.

Once this was done it was possible to determine the pressure profiles for various mass to surface radius ratios ($x = 2M/R$). An examination of one such profile for a typical value $x$ for compact objects namely $x = 0.3$ showed extremely encouraging results. The pressure $P_*$ profile as a function of normalised radius $r_*$ satisfied all our specified criteria for being physically reasonable. The pressure was a monotonically decreasing, the strong energy condition was obeyed and although the pressure diverged at $r = 0$ the amount of energy within a given volume was still finite.

From the original investigations made in this paper we can confidently state that the NJA may be applied to find new sources of the Kerr metric. We have shown that there exists physically sensible seed metrics to which when the NJA is applied new SAS metrics are generated. These metrics are considered as sources of the Kerr metric. The new solutions have continuous boundary conditions on oblate spheroidal surfaces. At present the physically properties of these new metrics has not been fully investigated. The sole reason for this being that it is a somewhat arduous task. However we are currently embarking on this project. The authors feel confident that further work in this area will complete the task of obtaining the long sort after sources for the Kerr metric.

VII. ACKNOWLEDGEMENTS

SPD would like to thank the Australian Postgraduate Programme for support during the completion of this work. SPD would also like to thank the University of Padova for its hospitality during his stay there in which most of this work was done. SPD and RT would also like to thank (in alphabetical order) N. J. Cornish, F. De Felice, N. E. Frankel, G. Magli and P. Szekeres for many helpful discussion. We would also like to thank C. P. Dettmann for providing the skeleton programme for the fourth-order Runge-Kutta routine.

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FIG. 1. The density profile of the trial solution with $a = 0$ and $x = 0.3$

FIG. 2. The pressure profile of the trial solution with $a = 0$ and $x = 0.3$

FIG. 3. Combined density and pressure profiles of the trial solution with $a = 0$ and $x = 0.3$

FIG. 4. Pressure versus density plot for the trial solution with $a = 0$ and $x = 0.3$
