The Shift-Dimension of Multipersistence Modules
joint work with Wojciech Chachólski and René Corbet
arXiv:2112.06509

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January 13, 2022
Topological Data Analysis

Studying the shape of data...
...with tools from (algebraic) topology

Behind the scenes

- commutative algebra
- multigraded algebra
- algebraic geometry

New invariants of multigraded modules...
...arising from TDA
Applications

- medical and life sciences
- Blue Brain Project: study of brain networks
- PersLay: a neural network layer for graph classification
- distinguishing point processes on a unit square

...whenever data arise!

AILoGeLiS

- consortium on algebra, topology, and geometry in the life sciences
- revealing structures of data with algebra, topology, and geometry
- institutes: KTH, MPI MiS, Oxford, MIT, and EPFL
- website: www.altogelis.com
Barcoding

Main tool: persistent homology

Associating barcodes to data

**Input:** point cloud \( \{p_i\} \subseteq \mathbb{R}^N \)

1. \( X_\varepsilon := \bigcup_{p_i} B_\varepsilon(p_i) \)

2. Increase \( \varepsilon \rightarrow \text{nerve} \) filtered simplicial complex

3. For all \( n \): \( n \)-th homology with coefficients in \( K \) naturally is a finitely generated \( \mathbb{N} \)-graded \( K[t] \)-module \( P_n \)

4. Structure theorem for finitely generated modules over \( \text{PIDs} \):

\[
P_n \cong \bigoplus_{i} K[t]t^{\alpha_i} \oplus \bigoplus_{j} K[t]t^{\beta_j} / K[t]t^{\beta_j+\gamma_j}
\]

**Output:** barcode \( \{[\alpha_i, \infty), [\beta_j, \beta_j + \gamma_j]\} \)

**Fact:** This invariant is discrete, complete, and stable!
An example from (Bauer, 2018)

**Figure:** Thickening a point cloud

**Figure:** The associated simplicial complex
Barcode of the first homology group
Birth and death of a 1-cycle
Multiparameter persistence

Study of **multifiltered** simplicial complexes (Carlsson–Zomorodian, 2009)

**Algebraic counterpart**

\[ \mathbb{N}^r \text{-graded } K[x_1, \ldots, x_r]\text{-modules } M = \bigoplus_{a \in \mathbb{N}^r} M_a \]

**Challenges**

- no higher-dimensional analogue of barcodes
- lack of **stable, algorithmic** invariants

**Multipersistence modules as functors**

Let \( G \in \{\mathbb{N}^r, \mathbb{R}^r_{\geq 0}\} \) (more general monoids in (Corbet–Kerber, 2018))

\[
\begin{align*}
\text{Fun}((G, \leq), \text{Vect}_K) & \quad \text{isom. of cats} \quad G\text{-graded } K[G]\text{-modules} \\
\bigcup \\
\text{Tame}((G, \leq), \text{Vect}_K) & \quad \cong \quad \text{finitely presented } G\text{-graded } K[G]\text{-modules}
\end{align*}
\]
Turning discrete into stable invariants

\( T \) a set
\( f \) a discrete invariant \( f : T \to \mathbb{N} \)
\( d \) an extended pseudometric \( d : T \times T \to \mathbb{R}_{\geq 0} \cup \{\infty\} \)
\( \mathcal{M} \) measurable functions \([0, \infty) \to [0, \infty)\) endowed with interleaving distance

Definition & Theorem (Gäfvert–Chachólski, 2017)

The **hierarchical stabilization** of \( f \) at \( x \in T \), denoted \( \hat{f}(x) \in \mathcal{M} \), is

\[
\hat{f}(x)(\tau) := \min \{ f(y) \mid y \in T : d(x, y) \leq \tau \}.
\]

For any choice of \( d \), \( \hat{f} : T \to \mathcal{M} \) is 1-Lipschitz.

Measuring distances between tame functors

How to construct metrics—in best case in a way that is suitable for learning tasks?
The poset obtained from adding one element $\infty$ to $\mathbb{R}^r_{\geq 0}$

**Definition**

A **persistence contour** is a functor $C: \mathbb{R}^r_{\infty} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^r_{\infty}$ such that for every $x \in \mathbb{R}^r_{\infty}, \tau, \varepsilon \in \mathbb{R}_{\geq 0}$:

1. $x \leq C(x, \varepsilon)$ and
2. $C(C(x, \varepsilon), \tau) \leq C(x, \varepsilon + \tau)$.

**Example (standard contour)**

$C(x, \varepsilon) = x + \varepsilon \cdot v$, where $v = (v_1, \ldots, v_r)$

**$\varepsilon$-neighborhoods of 0**

For $\varepsilon \in \mathbb{R}_{\geq 0}$ define

$$D_\varepsilon := \{ G \in \text{Tame}(\mathbb{R}^r_{\geq 0}, \text{Vect}_K) \mid C(x, \varepsilon) \neq \infty \Rightarrow G(x \leq C(x, \varepsilon)) = 0 \}.$$

Multigraded Betti numbers

Hilbert’s syzygy theorem

Every f.g. \(\mathbb{N}^r\)-graded \(\mathbb{K}[x_1, \ldots, x_r]\)-module \(M\) has a minimal free resolution \(F_\bullet\) of length at most \(r\), i.e., there exists an exact sequence of \(\mathbb{N}^r\)-graded modules

\[
F_\bullet: \quad F_r \xrightarrow{\delta_r} \cdots \xrightarrow{} F_0 \xrightarrow{\delta_0} M \xrightarrow{} 0,
\]

where the ranks of the \(F_i\) are simultaneously minimized.

Definition

The rank of \(F_i\) in a minimal free resolution of \(M\) as above is called the **i-th total multigraded Betti number** of \(M\) and is denoted by \(\beta_i(M)\).
Figure: Example realizations of point processes on the unit square

Figure 3 in (Chachólski–Riihimäki, 2020)
Distinguishing point processes via contours

Figure: Mean $\hat{\beta}_0$ of $H_1$ of 200 simulations of point processes with respect to the standard contour (left), distance contour (middle), and shift contour (right)$^3$

$^3$Figure 5 in (Chachólski–Riihimäki, 2020)
A new invariant of multigraded modules

$M$ a finitely generated $\mathbb{N}^r$-graded $\mathbb{K}[x_1, \ldots, x_r]$-module

Theorem & Definition (Chachólski–Corbet–S., 2021)

The stabilization of $\beta_0$ w.r.t. the metric “arising from the standard contour” in the direction of $\nu = (\nu_1, \ldots, \nu_r) \in \mathbb{N}^r$ gives rise to

$$\dim_{\nu}(M) = \min \{ \ell \mid \exists m_1, \ldots, m_\ell \in M : x_1^{\nu_1} \cdots x_r^{\nu_r} \cdot M \subseteq \langle m_1, \ldots, m_\ell \rangle \},$$

the **shift-dimension** of $M$. Such $\{m_1, \ldots, m_\ell\}$ $\nu$-generate $M$ and are a $\nu$-basis of $M$ for $\ell = \dim_{\nu}(M)$.

Computing $\widehat{\beta}_0$ . . .

- . . . is NP-hard in general (Gäfvert–Chachólski, 2017)
- linear-time algorithm for quotients of monomial ideals (Chachólski–Corbet–S.)
Examples of the shift-dimension

0-dimension

\[
\dim_0(M) = \beta_0(M), \text{ the minimal number of (homogeneous) generators of } M
\]

Free multigraded modules

\[
F = \mathbb{K}[x_1, \ldots, x_r](-a_1, \ldots, -a_r) \cong \mathbb{K}[x_1, \ldots, x_r] \cdot x_1^{a_1} \cdots x_r^{a_r}, \ a_1, \ldots, a_r \in \mathbb{N}, \text{ is } \nu\text{-generated by } x_1^{a_1+\nu_1} \cdots x_r^{a_r+\nu_r}. \text{ Hence } (\dim_{\nu}(F))_{n \in \mathbb{N}} = 1, 1, 1, \ldots
\]

Quotient of homogeneous monomial ideals

Let \( M = \langle x_1^3 x_2, x_1 x_2^3 \rangle / \langle x_1^4 x_2^4 \rangle, \ \nu = \mathbb{1} \in \mathbb{N}^2. \) Then \( M, x_1 x_2 M \subseteq \langle x_1^3 x_2, x_1 x_2^3 \rangle, \ x_1^2 x_2^2 M \subseteq \langle x_1^3 x_2^3 \rangle, \) and \( x_1^3 x_2^3 M = 0. \) Hence \((\dim_{\nu}(M))_{n \in \mathbb{N}} = 2, 2, 1, 0, 0, \ldots\)
$$M = \langle x_1^3x_2, x_1x_2^3 \rangle / \langle x_1^4x_2^4 \rangle \in \text{Mod}(\mathbb{K}[x_1, x_2])$$

**Figure:** Visualization of $M$, $x_1x_2M$, and $x_1^2x_2^2M$

One reads:

- $M, x_1x_2M \subseteq \langle x_1^3x_2, x_1x_2^3 \rangle$, $x_1^2x_2^2M \subseteq \langle x_1^3x_2^3 \rangle$, and $x_1^3x_2^3M = 0$.
- $\dim_{(0,0)}(M) = \dim_{(1,1)}(M) = 2$, $\dim_{(2,2)}(M) = 1$, and $\dim_{(3,3)}(M) = 0$. 


Algebraic properties of the shift-dimension

\[ M = I \triangleleft \mathbb{K}[x_1, \ldots, x_r] \]

\[ \dim_v(I : (x_1^{v_1} \cdots x_r^{v_r})) \leq \beta_0(I) \text{ for all } v \in \mathbb{N}^r \]

Epimorphisms

If \( \varphi : M \to N \), then \( \dim_v(M) \geq \dim_v(N) \).

Proof: The image of a \( v \)-basis of \( M \) is a set of \( v \)-generators of \( N \).

Successively killing non-\( v \)-basis-elements

\( m_1 \in M \) not in any \( v \)-basis of \( M \), \([m_2]\) not in any \( v \)-basis of \( M/\langle m_1 \rangle \), \ldots

\[ M \to M/\langle m_1 \rangle \to \cdots \to M/\langle m_1, \ldots, m_\ell \rangle =: M_\ell. \]

Iterating this process stabilizes after a finite number \( \ell \) of iterations. In \( M_\ell \), every element is contained in some \( v \)-basis of \( M_\ell \).
To be, or not to be in a \(\nu\)-basis, that is the question.

Lemma

An element \(m \in M\) can be extended to a \(\nu\)-basis of \(M\) if and only if

\[
\dim_{\nu}(M/\langle m \rangle) = \dim_{\nu}(M) - 1.
\]

Proof.

\(\Rightarrow\) If \(\{m, m_2, \ldots, m_{\dim_{\nu}(M)}\}\) is a \(\nu\)-basis of \(M\), then \(\{m_2, \ldots, m_{\dim_{\nu}(M)}\}\) is a \(\nu\)-basis of \(M/\langle m \rangle\).

\(\Leftarrow\) If \(\{[m_2], \ldots, [m_{\dim_{\nu}(M)}]\}\) is a \(\nu\)-basis of \(M/\langle m \rangle\), then the elements \(m, m_2, \ldots, m_{\dim_{\nu}(M)}\) \(\nu\)-generate \(M\).
Algebraic properties of the shift-dimension

Non-additivity

Let $M = \langle x_1 \rangle / \langle x_1 x_2^2 \rangle$, $N = \langle x_2 \rangle / \langle x_1^2 x_2 \rangle$. Then $\dim_{(1,1)}(M) = \dim_{(1,1)}(N) = 1$. Since $x_1(x_1 x_2, x_1 x_2) = (x_1^2 x_2, 0) = x_1 x_2(x_1, 0)$, $x_2(x_1 x_2, x_1 x_2) = (0, x_1 x_2^2) = x_1 x_2(0, x_2)$ in $M \oplus N$,

$$x_1 x_2 (M \oplus N) \subseteq \langle (x_1 x_2, x_1 x_2) \rangle.$$

Hence $\dim_{(1,1)}(M \oplus N) = 1 \neq 2 = \dim_{(1,1)}(M) + \dim_{(1,1)}(N)$.

Additivity for some cases

For $M, N$ as in one of the following three cases (extension in progress)

1. $M$ and $N$ free multigraded modules
2. $\dim_v(M) \leq 1$ and $N$ free
3. $M$ a monomial homogeneous ideal, $N$ free of rank 1

the $v$-dimension is additive, i.e.,

$$\dim_v(M \oplus N) = \dim_v(M) + \dim_v(N).$$
Proposition

Let $0 \xrightarrow{} M \xrightarrow{\varphi} L \xrightarrow{\psi} N \xrightarrow{} 0$ be a short exact sequence of persistence modules. Then for all $v, w \in \mathbb{N}^r$, the following two inequalities hold:

1. $\dim_{v+w}(L) \leq \dim_v(M) + \dim_w(N)$, and
2. $\dim_v(L) \leq \dim_v(N) + \beta_0(M)$. 
References

[Bau18] Ulrich Bauer.
Persistent homology and the stability theorem.
Talk given at TAGS – Linking Topology to Algebraic Geometry and Statistics in Leipzig on February 19, 2018. Slides of the talk available at https://ulrich-bauer.org/persistence-talk-leipzig.pdf, 2018.

[CCS21] René Corbet, Wojciech Chachólski, and Anna-Laura Sattelberger.
The shift-dimension of multipersistence modules.
Preprint arXiv:2112.06509, 2021.

[CK18] René Corbet and Michael Kerber.
The representation theorem of persistence revisited and generalized.
J. Appl. and Comput. Topology, 2:1–31, 2018.

[CR20] Wojciech Chachólski and Henri Riihimäki.
Metrics and stabilization in one parameter persistence.
SIAM J. Appl. Algebra Geom., 4(1):69–98, 2020.

[CZ09] Gunnar Carlsson and Afra Zomorodian.
The Theory of Multidimensional Persistence.
Disc. Comp. Geom., 42:71–93, 2009.

[GC17] Oliver Gäfvert and Wojciech Chachólski.
Stable invariants for multidimensional persistence.
arXiv:1703.03632, 2017.

[Ghr08] Robert Ghrist.
Barcodes: the persistent topology of data.
Bull. Amer. Math. Soc., 45:61–75, 2008.

[MS05] Ezra Miller and Bernd Sturmfels.
Combinatorial Commutative Algebra, volume 227 of Graduate Texts in Mathematics.
Springer-Verlag, New York, 2005.

[SCL+17] Martina Scolamiero, Wojciech Chachólski, Anders Lundman, Ryan Ramanujam, and Sebastian Öberg.
Multidimensional Persistence and Noise.
Found. Comput. Math, 17(6):1367–1406, 2017.

[ZC05] Afra Zomorodian and Gunnar Carlsson.
Computing persistent homology.
Discrete Comput. Geom., 33(2):249–274, 2005.
Tusen takk for oppmerksamheten!