Quantum amplitude amplification algorithm: an explanation of availability bias

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Abstract. In this article, I show that a recent family of quantum algorithms, based on the quantum amplitude amplification algorithm, can be used to describe a cognitive heuristic called availability bias. The amplitude amplification algorithm is used to define quantitatively the ease of a memory task, while the quantum amplitude estimation and the quantum counting algorithms to describe cognitive tasks such as estimating probability or approximate counting.

1 Introduction

The idea that human judgements and decision-making can evidence quantum mechanics behavior has a great deal of intuitive appeal, and it is at the basis of a recent research topic, which can be called quantum cognition. A number of authors have explored such idea, like [1] for decision making, or [2] and [3] for human judgements. The quantum-like models there proposed seem to adequately describe the experimental results: however, the potentialities of the quantum formalism have not fully explored, mainly for what concerns the quantum parallelism and a characterization of quantum algorithms in terms of human tasks.

In the present article, I propose to describe the experimental results concerning the availability bias with the quantum amplitude amplification, quantum amplitude estimation and quantum counting algorithms [5]: the first is a recent generalization of the Grover’s algorithm [6], while the other two algorithms are applications of the amplitude amplification, followed by a quantum Fourier transform. I show that these algorithms are able to model some important experimental results of cognitive science relevant to availability bias: in particular, the amplitude amplification algorithm allows to give a mathematical characterization of the ease to recall items or concepts, while the amplitude estimation/counting algorithms allow to introduce a formal connection between such ease and judgements of probability/frequency about facts. Here I do not discuss about the physical possibility for human mind to perform quantum algorithms: I only consider from a formal point of view the problem of computational complexity and the possibility to define mathematically the ease to remember. The Grover’s algorithm is an important quantum algorithm based on parallelism which allows to search in an unsorted database with a high number of items faster than any classical algorithm (quadratic speedup). One of the first attempts to use such algorithm in cognitive science (more precisely a generalization [7]) has been done by Franco [8] to describe the influence of emotions on the ease to
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remember. The same quadratic speedup is provided by the algorithms based on the amplitude amplification algorithm, based on quantum parallelism.

This article attempts to model within the quantum framework the *availability bias*, a human cognitive bias that causes people to estimate frequency or probability on the basis of how easily they can recall or imagine instances of whatever it is they are trying to estimate. The availability bias, which is at the root of many other human biases and culture-level effects, was discovered by psychologists Amos Tversky and Daniel Kahneman (2002 Nobel Prize in Economics) [4]. A simple example of the availability bias, which I discuss in the present article, is provided by a famous experiment of Tversky and Kahneman (1973) [4]: Consider the letter R: is R more likely to appear in the first position or in the third position? The most part of participants judged the first position to be more likely. However in the English language there are more words with R at the third position than at the first position. The explanation given by Tversky and Kahnemann is that people estimate the number of words based on the ease with which they can recall them, which is the availability heuristic: the first letter provides a better cue for recalling instances of words than does the third letter. It is evident from the latter example that the judgements performed by people about the words involves in theory a great number of computations. In fact, the English language contains about 500000 words, and the task previously described involves in theory computations over such set. This considerations make stronger the quantum-like point of view, since the quantum algorithms here proposed manifest a quadratic speed up, and thus are faster than any classic algorithm.

# 2 Availability bias

Availability bias is a human cognitive bias that causes people to estimate the probability or the number of particular categories of items on the basis of how easily they can recall or imagine them. In the definition of the availability bias is important to operationalize the ease with which the memory processes are performed. In particular, two different definitions have been used widely in availability experiments, giving an experimental measure of the ease of the memory task: 1) *availability-by-number*: the produced proportion of good items versus bad items in a fixed time; 2) *availability-by-speed*: the time ratio between the consumed retrieval times for the same number of good items and bad items. In general, the availability experiments involve two different groups of subjects: one which performs the memory task, and one that performs the judgements about the probability/number of items. Thus the availability experiments verify a positive correlation between the measure of ease in the memory task (by using the availability-by-number or the availability-by-speed) and the quantitative judgements performed by the subjects. I will focus the attention on the following two categories of experiments:

1) Judgements of probability: the availability effects can be due to the ease of recalling items (as in the example of [4] described in the introduction about the
likelihood of letter $R$ in the first or in the third position of English words) or to the vividness of particular events in memory: for example, in [9] subjects have been asked to estimate the probability of plane accidents (quite events rare, even if the vast majority of the population overestimates their probability).

2) Judgements of number: in [4] is described a simple recalling experiment, where subjects were presented a recorded list of names of known personalities of both sexes. After listening to the list, some subjects judged whether it contained more names of men or women, others attempted to recall the names in the list. In particular, the list included 19 names of famous personalities of one sex and 20 names of less famous personalities of the other sex. The experimental results show a positive correlation between the estimated number of persons of the more famous group and the number of recalled names of the same group.

3 Amplitude amplification algorithm

The amplitude amplification algorithm [5] is a generalization of Grover’s algorithm, and it can be used for solving the following problem: let us consider $N$ items and a boolean function $f : \{0, 1, ..., N - 1\} \rightarrow \{0, 1\}$, which partitions the items into $t$ good items (those for which $f$ is equal to 1), and $N - t$ bad items (those for which $f$ is equal to 0). It is evident that such algorithm can be used to model the retrieval tasks in cognitive science. For example, the experiment of Tversky and Kahneman [4] relevant to words with letter $R$ in first or third position can be represented as a partitioning of English words in two categories: the good items (words with $R$ at first position) and the bad items (words with $R$ at third position). Even if the mathematical details of the algorithm are described in next subsection, I now present the main features, reducing to the minimum the formalism. The intuitions here presented are similar to those preliminarily exposed in Franco [8]. The quantum amplification algorithm, like the Grover’s algorithm, is composed by three main parts:

1) The initial state, in which the $N$ items are encoded into the elements of a basis of a $N$-dimensional vector space. An important feature of the amplitude amplification algorithm, which differences it from the Grover’s algorithm, is that the items within such initial state can have different weights: in particular, the parameter $a$ is the probability to measure a good item in such initial state. In Grover’s algorithm we always have $a = t/N$. The initial state can be interpeted, in the context of cognitive processes, as a guessing state, representing the initial mental weights relevant to the items. If $a > t/N$, this means that the good items have initially more relevance than the bad items. If the guessing state is a flat distribution over all the items ($a = t/N$), this means that the subjects have no preliminar idea about good/bad items.

2) The amplification engine, which is an iterative process allowing to enhance the weights of the good items: at each step the boolean function $f$ is evaluated simultaneously over all the items, and the weights of the good items are enhanced through interference effects. Differently from the Grover’s algorithm, the efficiency of the amplification engine depends on the guessing state: the al-
The amplitude amplification algorithm allows for a further speedup when the
guessing state is such that $a > t/N$, because the number of required steps is
proportional to $1/\sqrt{a} < \sqrt{N/t}$: the initial guessing state gives higher weight to
the good items than to the bad items, making faster the retrieval process.

The interpretation of such amplification engine in the context of cognitive tasks
is in terms of subconscious processes: they allow for parallelism in the evaluation
of the boolean function over all the items, but they need a number of iterations
proportional to $1/\sqrt{a}$ to amplify the probability of good items. In other words,
the subjects are able to apply $f(x)$ on each item $x$ (thus deciding if each item is
good or bad). The algorithm suggests that such decision procedure is performed
in a parallel and subconscious way, thus faster than in a serial way.

3) A measure on the final state. The algorithm modifies the initial guess state,
producing a final state which contains almost only good states. Thus a final
measure produces one of the good items, and the recall task is finished. This
fact represents in my description the conscious act of remembering.

The amplitude amplification algorithm allows to give a simple mathematical
definition of the ease to retrieve in terms of the availability-by-speed: the time
required to find a good item is proportional to $1/\sqrt{a}$, where $a$ is the initial
guessing parameter: a high value of $a$ gives a short time to retrieve a good item.
The parameter $a$ represents how vivid are the good items in memory before
retrieving them: it can change depending on attempts of imagining instances
of the retrieved items. Analogously, the availability-by-number is the number
of good items that subjects can remember in a fixed time: it is proportional,
in our model, to $\sqrt{a}$. In the experiment on the position of letter $R$ in English
words [4], the time to produce the word is lower with $R$ as first letter than as
the third letter. Thus I assume that the guess state contains a set of $N$ items
(the most common English words), and the weight for the words beginning with
$R$ is higher than for those with $R$ at third position.

3.1 Mathematical details for the amplitude amplification algorithm

In the quantum formalism, the partition of $N$ items into good and bad items
leads to consider a $N$-dimensional Hilbert space, whose computational basis
is $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$: each vector corresponds to a particular item. Thus the
function $f$ introduces a partition of $H$ into a good subspace (spanned by the
vectors $|x\rangle$ for which $f(x) = 1$) and a bad subspace (spanned by the vectors $|x\rangle$
for which $f(x) = 0$). Thus any superposition $|s\rangle = \sum_x \psi(x)|x\rangle$ can be written
as $|s\rangle = |\psi_0\rangle + |\psi_1\rangle$, where $|\psi_1\rangle$ is the superposition of good vectors ($f(x) = 1$)
and $|\psi_0\rangle$ is the superposition of bad vectors ($f(x) = 1$).

The algorithm presents the following steps:
1) **Initial state**: prepare the vector \( A|0\rangle = |\psi_0\rangle + |\psi_1\rangle \), where \( A \) is a quantum algorithm which uses no measurement, and \( a = \langle \psi_1 | \psi_1 \rangle \) is the probability to measure a good state. If \( A \) is the quantum Fourier transform \( F_N : |x\rangle \rightarrow N^{-1/2} \sum_{y=0}^{N-1} e^{2\pi i x y} |y\rangle \), we have a uniform superposition of vector states with amplitude \( N^{-1/2} \), and \( a = t/N \) (as in standard Grover’s algorithm).

2) **Amplification engine**: apply the operator \( Q = -AS_0A^{-1}S_f \), where \( S_0 \) and \( S_f \) are conditional phase inversion operators (\( S_0 \) changes the sign of the amplitude if and only if the state is the zero state \( |0\rangle \), while \( S_f \) conditionally changes the sign of the amplitudes of the good states).

3) **Measure** the final state: obtain one of the search results, measuring the resulting state in the computational basis.

It can be shown that after \( \lceil \pi/4 \arcsin(\sqrt{a}) \rceil \) iterations (where \( x \) is the rounding of \( x \)) the measured outcome is good with probability at least \( \max(a, 1-a) \). If we have a high number of items \( N \) and \( a \ll N \), then the optimal number of iterations is proportional to \( 1/\sqrt{a} \). If \( A \) is the quantum Fourier transform the optimal number of iterations is proportional to \( \sqrt{N/t} \), which corresponds to the speedup of Grover’s algorithm. If \( a > t/N \), the number of iterations is lower than \( \sqrt{N/t} \).

4 The quantum amplitude estimation algorithm

The quantum amplitude estimation algorithm [5] allows to estimate the amplitude of a quantum state by applying at different steps the amplitude amplification algorithm. From a cognitive point of view, it allows to estimate with a good precision the probability \( a \) to find a good item (according to the partitioning introduced by function \( f \)) when the opinion state about the \( N \) items is the initial guessing state. Even if the mathematical details of the algorithm are described in next subsection, I now present its main features, reducing to the minimum the formalism. The algorithm can be decomposed in three parts:

1) **Initial state**: it is composed by the guessing state, as described before.

2) **Parallel amplifications**: different instances of the amplification engine are applied in a parallel way, with different numbers of iterations. Thus we have a double level of parallelism: in each step of the amplification engine the function \( f(x) \) is applied simultaneously to the items, and this works simultaneously for each instance of the amplification engine.

3) **Analysis** of the different amplifications: since the efficiency of each amplification engine depends on the parameter \( a \), the analysis of different instances of the amplification process with different number of iterations allow to estimate \( a \), with a few standard deviations, after a number of evaluations of \( f \) proportional to \( 1/\sqrt{a} \).

This algorithm is particularly important for the study of cognitive processes, because it allows to describe the tasks where subjects produce subjective probabilities relevant to events. In this sense, it provides the formal link between a quantum-like approach describing choices (for example, [1]) and a quantum-like approach describing subjective probabilities (for example, [2]): choices are the
effect of simple measurements on quantum states, while the subjective probabilities are the result of a quantum amplitude estimation algorithm applied on the same state. In the context of availability bias, the present algorithm can be used to describe the experiment of [4] presented in the introduction about the likelihood of letter \( R \) in the first or in the third position of English words. The retrieve process for words with \( R \) in first or third position involves two different partitioning of English words and two different amplification processes with parameters \( a \) and \( a' \). In other words, we assume that subjects’ mental state (the guess state) involves \( N \) words, and that the weight in such state relevant to words with \( R \) in first and third position is \( a \) and \( a' \) respectively. According to our model, the ease to recall words with \( R \) in first position can be described by the availability-by-number and is proportional to \( \sqrt{a} \), and the estimated probability to recall words with \( R \) in first position is near to \( a \). Thus if subjects recall more words with \( R \) in first position than in third (\( \sqrt{a} > \sqrt{a'} \)), then the estimated probability to find a word with letter \( R \) in first position is higher than the estimated probability to find words with \( R \) in third position (\( a > a' \)). The same formalism can be used to describe the experiments in [9], where subjects overestimated the probability of plane accidents, because of the vividness of such events in memory.

Like for the amplitude amplification algorithm, also in this case the produced estimated probability can be described as the result of subconscious amplification processes (with evaluations of function \( f \)) and a final analysis and measure.

### 4.1 Mathematical description of amplitude estimation algorithm

The amplitude estimation algorithm, called \( \text{Est Amp}(A, f, M) \), is able to estimate the amplitude of \( |\psi_1\rangle \) (good states superposition) in \( A|0\rangle \). It is based on the amplitude amplification algorithm. In particular:

1) **Initial state**: prepare the vector \( F_M|0\rangle A|0\rangle \), formed by two distinct registers: the first has dimension \( M \), while the second has dimension \( N \). We recall that \( F_M \) is the quantum Fourier transform \( F_M : |x\rangle \rightarrow M^{-1/2} \sum_{y=0}^{M-1} e^{2\pi ixy}|y\rangle \).

2) **Parallel amplifications**: apply the operator \( \Lambda_M(Q) \), defined by \( |j\rangle|y\rangle \rightarrow |j\rangle Q_j|y\rangle \) with \( 0 \leq j \leq M \), where \( Q = -AS_0A^{-1}S_f \) is the standard amplitude amplification engine. In other words, operator \( \Lambda_M(Q) \) applies in a parallel way different degrees of amplification, from 0 to \( M \), to the guess state \( A|0\rangle \).

3) **Find the period of the wave function**: apply \( F_M^{-1} \) to the first register and measure it, obtaining an integer \( y \). The estimated amplitude is then \( \tilde{a} = \sin^2(\pi y/M) \) with a good approximation: the accuracy of such estimate is given in Theorem 12 in [5]. In particular, to obtain a probability estimate with a few standard deviations, we have to choose \( M = \lfloor 1/\sqrt{a} \rfloor \).

### 5 The quantum counting algorithm

The quantum counting algorithm [5] allows, given a boolean function \( f \) defined on a set \( X \) of \( N \) items, to estimate the number of elements of \( X \) for which the
function $f$ is true $t = |\{x \in X \mid f(x) = 1\}|$. In other words, the algorithm allows to estimate the size of the subset of good items (those for which $f(x) = 1$). The best classical strategy is to evaluate $f$ on random elements of $X$: thus the number of evaluations in order to have a good estimate of $t$ is proportional to $N$. On the contrary, the quantum counting algorithm allows to produce good estimates for such number in approximatively $\sqrt{N}$ steps (quadratic speedup).

The quantum counting algorithm can be considered as an application of the previous amplitude estimation algorithm. In fact, if the guessing state assigns the same weight to all the items, then the estimated probability relevant to the good items is near to $t/N$: the approximate number of good items can be obtained by multiplying such estimated probability by $N$. I propose here a simple generalization of the quantum counting algorithm, which I will discuss in mathematical details in the next subsection: if the guessing state assigns non-uniform weights to the items, the probability relevant to good items is $a \neq t/N$. If for example $a > t/N$, then the estimated number of items is near to $aN > t$: we have an overestimation of the number of items, due to the guessing state in the amplification process.

Such simple generalization allows to describe the recalling experiment in [4], where subjects were presented a recorded list of names of known personalities of both sexes. After listening to the list, some subjects judged whether it contained more names of men or women, others attempted to recall the names in the list. In particular, the list included 19 names of famous personalities of one sex and 20 names of less famous personalities of the other sex. The experimental results show a positive correlation between the estimated number of persons of the more famous group and the number of recalled names of the same group. In fact the same parameter $a$ is involved both in the recalling process and in the approximate counting process: thus the ease to recall names of one group (proportional to $\sqrt{a}$) entails a higher estimated size of the same group ($aN$).

5.1 Mathematical description of quantum counting algorithm

Given a boolean function $f$ over a discrete set $X$ with $N$ elements, the quantum counting algorithm $\text{Count}(F_N, f, M)$ can be written as a special case of the amplitude estimation: $\tilde{t} = N \times \text{Est.Amp}(F_N, f, M)$. If we use, instead of the Fourier transform $F_N$, a generic operator $A$, the quantum counting algorithm $\text{Count}(A, f, M)$ does not produce a correct estimate of $t$, the number of good items. However, if $a > t/N$, the modified counting algorithm produces an estimate $\tilde{t} > t$, while if $a < t/N$, it produces an estimate $\tilde{t} < t$.

6 Conclusions

In this article I show how three important quantum algorithms can model the experimental results of availability bias. I introduce the amplitude amplification algorithm to give a mathematical characterization of the ease to recall items or
concepts. Then I present the amplitude estimation/counting algorithm, establishing a connection between the ease to retrieve and the judgements of probability/frequency about facts. The quantum description of availability bias, and in particular the use of quantum algorithms, has some advantages: 1) the economy of a quantum description, which seems to be consistent with a large number of cognitive heuristics (see for example [2], and [1]), while the classic alternatives are ad-hoc models with a very weak mathematical apparatus; 2) as noted by Manin [10], some human tasks, such as playing chess or speech generation and perception, require a great number of computations per second, as is evidenced by efficient chess playing software (based on classical algorithms). Since the characteristic time of neuronal processing is about $10^{-3}$ seconds, it seems difficult that a classical model could describe such tasks: in the experiments of words with letter $R$ of [4], the set of words on which perform the computation is in theory of 500000 elements, thus making a classic algorithm modeling the cognitive processes more difficult to apply than fast quantum algorithms.

There are some questions which need further investigations: 1) How the amplitude estimation processes can be influenced by a change in the partitioning? For example, we can choose to partition the items into two different ways. 2) Availability bias is relevant not only with estimated probabilities or approximate counting, but also with generic evaluations, like described in [11]. It should be investigated if the algorithm used in the present algorithm can be generalized also to generic evaluations (like for example course ratings).

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