One-dimensional Disordered Density Waves and Superfluids: The Role of Quantum Phase Slips and Thermal Fluctuations

Andreas Glatz and Thomas Nattermann

Institut für Theoretische Physik, Universität zu Köln
Zülpicher Str. 77, D-50937 Köln, Germany

(Dated: March 22, 2022)

PACS numbers: 71.10.Pm, 72.15.Rn, 73.20.Mp, Jc

The low temperature phase diagram of 1D disordered quantum systems like charge or spin density waves, superfluids and related systems is considered by a full finite T renormalization group approach, presented here for the first time. At zero temperature the consideration of quantum phase slips leads to a new scenario for the unpinning (delocalization) transition. At finite T a rich cross-over diagram is found which reflects the zero temperature quantum critical behavior.

The results obtained for the CDWs/SDWs have a number of further applications on disordered quantum systems: they relate e.g. to the localization transition of Luttinger liquids [3, 4], tunnel junction chains [13], superfluids [2]. Josephson coupled chains of these systems if the coupling is treated in mean-field theory [3]. In most parts of this paper, we will use the terminology of CDWs.

Well below the mean field condensation temperature $T_{MF}$ of the CDW, the electron density $\rho(x)$ can be written in the form [1]

$$\rho(x) = \rho_0(1 + Q^{-1}\partial_x\varphi) + \rho_1 \cos[p(\varphi + Qx)] + \ldots$$

(1)

where $Q = 2k_F$, $k_F$ is the Fermi–momentum, $\rho_0$ the mean electron density and $\rho_1$ is proportional to the amplitude of the complex (mean field) order parameter $\Delta e^{i\varphi} \sim \langle b_Q + b_{-Q}^\dagger \rangle$, $b_Q$, $b_{-Q}^\dagger$ denotes the phonon creation and annihilation operator, respectively. $\varphi(x)$ is a slowly varying phase variable. Neglecting fluctuations in $\Delta$, the Hamiltonian of the CDW is given by

$$\hat{H} = \int_0^L \left\{ \frac{c}{2} \left[ \left(\frac{\partial}{\partial x}\right)^2 + (\partial_x\varphi)^2 \right] + \sum_i U_i \rho(x) \delta(x-x_i) \right. + W \cos \left( \frac{q\pi}{h} \int_0^x dy \hat{P}(y) \right) \right\} dx$$

(2)

where $[\hat{P}(x),\hat{\varphi}(x')] = \frac{h}{c} \delta(x-x')$, $c = \frac{h v_F}{\pi}$ denotes the elastic constant, $v_F$ the Fermi–velocity, $v$ the effective velocity of the phonon excitations and $f(T)$ the condensate density [4]. Note that $f(T)$ and $\Delta(T)$ vanish at $T_{MF}$ whereas $v$ remains finite. The third term results from the effect of impurities of random potential strenght $U_i = \pm U_{imp}$ and position $x_i$ and includes a forward and a backward scattering term proportional to $\rho_0$ and $\rho_1$, respectively. We will assume, that the mean impurity distance $l_{imp}$ is large compared with the wave length of the CDW and that the disorder is weak, i.e. $1 \ll l_{imp} \ll c/(U_{imp}\rho_1)$. In this case the Fukuyama–Lee length $L_{FL} = (c/(U_{imp}^2))^{2/3}$ is large compared with the impurity distance, here $U = U_{imp}\rho_1/\sqrt{l_{imp}}$. The
fourth term in (2) describes the influence of quantum phase slips by \( \varphi = \pm \pi \tau \) and will be further discussed below (2). The model (2) includes the four dimensionless parameters \( t = T/\pi \lambda_c, \quad K = h v/\pi c, \quad u^2 = U^2/\Lambda^2 \pi c^2 \) and \( w = W/\pi \Lambda^2 \), which measure the strength of the thermal, quantum, and disorder fluctuations and the probability of phase slips, respectively. \( \Lambda = \pi/a \) is a momentum cut-off. Although for CDWs and SDWs \( K \)-values of the order \( 10^{-1} \) and 1, respectively, have been discussed at \( T = 0 \) [3], the expressions relating \( K \) and \( t \) to the microscopic theory lead to the conclusion that both diverge by approaching \( T_{MP} \) whereas the ratio \( K/t \) remains finite. The classical region of the model given by \( K \ll t \) which can be rewritten as the condition, that the thermal de Broglie wave length \( \lambda_T = K/(t \Lambda) \) of the phason excitations is small compared with the lattice spacing \( a \).

In order to determine the phase diagram we adopt a standard Wilson-type renormalization group calculation, which starts from a path integral formulation of the partition function corresponding to Hamiltonian (2) with \( u, w \ll 1 \). We begin with the renormalization of the disorder term and put \( w = 0 \) for the moment. The system is transformed into a translationally invariant problem using the replica-trick. Going over to dimensionless spatial and imaginary time variables, \( \Lambda x \to x \) and \( \Lambda \nu \tau \to \tau \), the replicated action is given by \( \sigma = (u \rho_0 \Lambda/\rho_1(Q))^2 \)

\[
\frac{S^{(n)}}{\hbar} = \frac{1}{2 \pi K} \sum_{\alpha, \beta} \int_0^{L \Lambda} \int_0^{K/t} dx \int_0^\infty d\tau \left\{ [\partial_x \varphi_\alpha]^2 + [\partial_\tau \varphi_\alpha]^2 \right\} \delta_{\alpha \beta} \\
- \frac{1}{2K} \int_0^{K/t} d\tau' \left[ u^2 \cos \beta \left( \varphi_\alpha(x, \tau) - \varphi_\beta(x, \tau') \right) \right. \\
+ \sigma \partial_x \varphi_\alpha(x, \tau) \partial_x \varphi_\beta(x, \tau') \right\}.
\]

(3)

Integrating over the high momentum modes of \( \varphi(x, \tau) \) in a momentum shell of infinitesimal width \( 2\pi/b \leq |q| \leq 2\pi \) but arbitrary frequencies and rescaling \( x \to x' = x/b, \tau \to \tau' = \tau/b \) we obtain the following renormalization group flow equations (up to one loop):

\[
\frac{dK}{dl} = -\frac{1}{2} p^2 u^2 K B_0(p^2 K, K/2t) \coth \frac{K}{2t}, 
\]

(4)

\[
\frac{du^2}{dl} = \left[ 3 - \frac{p^2 K}{2t} \coth \frac{K}{2t} \right] u^2, \\
\frac{dt}{dl} = t,
\]

(5)

\[
B_1(\nu, y) = \int_0^\infty d\tau \int_0^\infty dx g_0(\tau, x) \cos \left( y - \tau \right) (\cos \nu - y)^{-1} \left[ 1 + \left( \frac{\nu}{y} \right)^2 \left( \cosh \frac{\nu}{y} - \cos \frac{\nu}{y} \right) \right]^{1/2},
\]

(6)

where \( l = \ln b \) and \( g_0(\tau, x) = \delta(x) \tau^2 \). Note that \( B_0(p^2 K, K/2t) \to 0 \) for \( K \to 0 \).

The equation for the flow of \( \sigma \) is more involved and will not be discussed here since it does not feed back into the other flow equations. Indeed, we can get rid of the forward scattering term by rewriting \( \hat{\varphi}(x) = \varphi_0(x) + \varphi_f(x) \) with \( \varphi_f(x) = \int_0^\infty dy g(y, x) \), \( \langle g(x) \rangle = 0 \) and \( \langle g(x)g(x') \rangle = \frac{2\sigma}{\nu} \delta(x - x') \). The phase correlation function \( C(x, \tau) = \langle \varphi(x, \tau) - \varphi(0, 0) \rangle^2 \rangle = C(x, \tau) + C_f(x) \) has therefore always a contribution \( C_f(x) \sim |x|/\xi_T \) with \( \xi_T^{-1} \sim (\sigma \ell) = |x| \). Since all further remarks about phase correlations refer to \( C_f(x, \tau) \) we will drop the subscript \( b \).

There is no renormalization of \( t \) (i.e. \( c \)) because of a statistical tilt symmetry [4]. The special case \( t = 0 \) was previously considered in [3] (with \( p = \sqrt{2} \)). The flow equation for \( K \) obtained in (3) for \( w = 0 \) deviates slightly from [3], which can be traced back to the other RG-procedures. The critical behavior is however the same: there is a Kosterlitz-Thouless (KT) transition \( K^* \) at \( K_u \) between a disorder dominated pinned and a free unpinned phase which terminates in the fixed point \( K_u^* = 6/p^2 \). \( u_0 \) denotes the bare value of the disorder and \( K_u \) is given by \( u^2 = g_K^*(K_u - K_u^*)^{-2} \log \Lambda \). \( \eta = B_0(p^2 K_u^*, \infty) \). In the pinned phase the parameters \( K, u \) flow into the classical, strong disorder region: \( K \to 0, u \to \infty \). The correlation function \( C(x, 0) \sim |x|/\xi_u \) increases linearly with \( |x| \). Integration of the flow equations gives for small initial disorder and \( K \ll K_u \) an effective correlation length \( \xi_u \approx \Lambda^{-1}(AL_{FL})^{-1} \) at which \( u \) becomes of the order unity. Close to the transition line \( K_u \) shows KT behavior. For \( K > K_u \) \( \xi_u \) diverges and \( C(x, \tau) \sim K(l = \log |z|) \log |z| \) where \( |z| = \sqrt{x^2 + \tau^2} \). Note that \( K(l) \) saturates on large scales at a value \( K_{eff}(u_0) \).

For large values of \( u \) our flow equations break down, but we can find the asymptotic behavior in this phase by solving the initial model in their strong pinning limit \( U_{imp} \gg 1, K = 0 \) exactly. A straightforward but somewhat clumsy calculation yields for the pair correlation function \( C(x, \tau) = \frac{2x}{pQx}(1 - \frac{\alpha}{\sinh \nu})(Qx) \) where \( \alpha = \pi/(pQ\lambda_0) \). The connection to the weak pinning model follows by choosing \( \lambda_0 \approx L_{FL} \).

At finite temperatures thermal fluctuations destroy the quantum interference effects which lead to the pinning of the CDW at \( t = 0 \). The RG flow of \( u \) in the region \( K < K_u \) first increases and then decreases. \( \xi \) can be found approximately by integrating the flow equations until the maximum of \( u(l) \) and \( t(l)/(1 + K(l)) \) is of order one. This can be done in full generality only numerically (see Fig. 1). It is however possible to discuss several special cases analytically. The zero temperature correlation length can still be observed as long as this is smaller than \( \lambda_T \) which rewrites for \( K \) not too close to \( K_u^* \) as \( t < t_K \approx K_u^*(1 - K_u/K_u^*)^{-1} \), \( t_u \approx (AL_{FL})^{-1} \). We call this domain the quantum disordered region. For \( K > K_u \) the correlation length \( \xi \) is given by \( \lambda_T \).
is larger than given by purely thermal fluctuations. For scales smaller than \( \lambda_T \), \( C(x, \tau) \) still increases as \( \sim \log |z| \) with a continuously varying coefficient \( K_{\text{core}}(u_0) \). In this sense one observes quantum critical behavior in that region, despite of the fact, that the correlation length is now finite for all values of \( K \). In the classical disordered region \( t_K < t < t_u \) the correlation length is roughly given by \( L_{\text{FL}} \) as follows from previous studies. In the remaining region \( t_u \lesssim t \) we adopt an alternative method by mapping the (classical) one-dimensional problem onto the Burgers equation with noise. In this case the RG-procedure applied to this equation becomes trivial since there is only a contribution from a single momentum shell and one finds for the correlation length \( \xi^{-1} \approx \frac{\pi}{2} f(T)(1 + (2\pi/p)^2(t_u/t)\Lambda) \). The phase diagram depicted in Fig. 1 is the result of the numerical integration of our flow equations and shows indeed the various cross-overs discussed before.

So far the phase field was considered to be single valued. Taking into account also amplitude fluctuations of the order parameter the phase may change by multiples of \( 2\pi \) by orbiting (in space and imaginary time) a zero of the amplitude. Such vortices correspond to quantum phase slips described by the last term in \([8]\) (with \( q = 2 \)), which we discuss here under equilibrium conditions. This operator superposes two translations of \( \varphi \) by \( \pm q\pi \) left from \( x \), i.e. it changes coherently the phase by \( \pm q\pi \) in a macroscopic region. For vanishing disorder the model can be mapped on the sine-Gordon Hamiltonian for the \( \theta \)-field (with \( K \) replaced by \( K^{-1} \)) by using the canonical transformation \( \tilde{P} = -\frac{\hbar}{2} \partial_x \tilde{\theta} \) and \( -\frac{\hbar}{2} \partial_x \tilde{\varphi} = \tilde{\Pi} \). To see the connection to space-time vortices one rewrites the action of interacting vortices as a classical 2D Coulomb gas which is subsequently mapped to the sine-Gordon model [17]. The initial value \( w_0 \) of \( w \) is proportional to the fugacity \( \omega_0 \) of the space-time vortices which may be non-negligible close to \( T_{MF} \), where the action \( S_{\text{core}} \approx h/(\pi K) \) of the vortex core is small. Performing an analogous calculation as before (but with \( u = 0 \)) the RG-flow equations read:

\[
\frac{dK}{dt} = -\pi \frac{q^4 w^2}{2 K^3} B_2 \left( \frac{q^2}{K} \right) \coth \frac{K}{2t},
\]

\[
\frac{dt}{dt} = \left[ 1 - \pi \frac{q^4 w^2}{2 K^3} B_1 \left( \frac{q^2}{K} \right) \coth \frac{K}{2t} \right] t,
\]

\[
\frac{dw}{dt} = \left[ 2 - \frac{q^2}{4K} \coth \frac{K}{2t} \right] w,
\]

where \( B_{1,2} \) are given in \([9]\) with \( g_1 = 2 \tau^2 \cos x \) and \( g_2 = (x^2 + \tau^2) \cos x \). From \([8]\) - \([11]\) we find, that for \( t = u = 0 \) quantum phase slips become relevant (i.e. \( w \) grows) for \( K > K_w \) with \( K_w = q^2 / 8 \) (\( q = 2 \) for CDWs). In this region vortices destroy the quasi long range order of the CDW, \( C(x, \tau) \sim |z|/\xi_w \). The transition is of KT type with a correlation length \( \xi_w \) \((w(\log \xi_w) \approx 1)\) diverging at \( K_w + 0 \) \([12]\). At finite temperatures \( w \) first increases, but then decreases and flows into the region of large \( t \) and small \( w \). Thus quantum phase slips become irrelevant at finite temperatures. This can be understood as follows: at finite \( t \) the 1D quantum sine-Gordon model can be mapped on the Coulomb gas on a torus of perimeter \( K/t \) since periodic boundary conditions apply now in the \( \tau \)-direction. Whereas the entropy of two opposite charges increases for separation \( L \gg K/t \) as \( \log(LK/t) \), their action increases linearly with \( L \). Thus, the charges remain bound. The one-dimensional Coulomb gas has indeed only an insulating phase \([8]\).

![FIG. 1: The low temperature cross-over diagram of a one-dimensional CDW. The amount of disorder corresponds to a reduced temperature \( t_u \approx 0.1 \). In the classical and quantum disordered region, respectively, essentially the \( t \equiv 0 \) behavior is seen. The straight line separating them corresponds to \( \lambda_T \approx a \). In the quantum critical region the correlation length is given by \( \lambda_T \). Pinning (localization) occurs only for \( t = 0, K < K_u^* \).](image)

![FIG. 2: T = 0 phase diagram for a CDW with quantum phase slips. If \( qp < 4\sqrt{3} \) there is a single transition between a low-K pinned and a high-K unpinned phase. In both phases the correlation length is finite. If \( qp > 4\sqrt{3} \) these two phases are separated by a third phase in which phase slips are suppressed and \( C(x, \tau) \sim \log |z| \). Both transitions disappear at finite \( t \).](image)

It is now interesting to consider the combined influence of disorder and phase slips. In doing this we write an approximate expression for the action of a single vortex in a region of linear extension \( L \) as

\[
S_{\text{vortex}} - S_{\text{core}} = \left( \frac{q^2}{4K} - 2 \right) \log L - \frac{u(L)}{K}.
\]

For very low \( K(< K_u, K_w) \) where \( u(L) \approx u_0 L^{3/2} \) the disorder always favours vortices on the scale of the effective Fujimura-Lee length \( \xi_w \). These vortices will be pinned in space by disorder. On the other hand, for very large values of \( K(> K_u, K_w) \) phase vortices are not influenced by disorder since \( u(L) \) is renormalized...
to zero. In the remaining region we have to distinguish the cases $K_u \gtrsim K_w$. For $K_w < K < K_u$ (i.e. $q_p < 4 \sqrt{3}$) and $u_0 = 0$ the phase correlations are lost on the scale of the KT correlation length $\xi_w$ of the vortex unbinding transition. Not too close to this transition $\xi_w \Lambda \approx e^{(S_{\text{core}}/2)(1-K_u/K)}^{-1}$. Switching on the disorder, $u$ will be renormalized by strong phase fluctuations which lead to an exponential decay of $u \sim u_0 e^{-\text{const}L/\xi_w}$ such that disorder is irrelevant for the vortex gas as long as $\xi_w \lesssim \xi_u$. We expect that the relation $\xi_u \approx \xi_u$ determines the position of the phase boundary between a pinned low-$K$ phase where vortices are favoured by the disorder and an unpinned high-$K$ phase where vortices are induced by quantum fluctuations. This line terminates in $K_w$ for $u_0 \to 0$ (see Fig. 2). If $S_{\text{core}}$ is large, $\xi_w$ will be large as well and $\xi_u \approx \xi_u$ will be reached only for $K \approx K_u$. For moderate values of $S_{\text{core}}$, the unpinning transition may be lowered considerably by quantum phase slips. In the opposite case $K_u < K < K_w$ (i.e. $q_p > 4 \sqrt{3}$) phase fluctuations renormalize weak disorder to zero such that vortices are still suppressed until $K$ reaches $K_w$ where vortex unbinding occurs. In this case two sharp phase transitions have to be expected.

Our flow equations describe also the effect of a commensurate lattice potential on the CDW: if the wave length $\pi/k_F$ of the CDW modulation is commensurate with the period $a$ of the underlying lattice such that $\pi/k_F = n/(qa)$ with $n,q$ integer, an Umklapp term $w \cos q\varphi$ appears in the Hamiltonian [3]. We obtain the results in this case from (3) - (4) (and the conclusions derived from them) if we use the replacements $K \to K^{-1}$, $t \to t/K^2$ and $w \to w/K^2$. Thus the lattice potential is relevant for $K < K_w$ with $K_w = 8/q^2$.

Next we consider the application of the results obtained so far to a one-dimensional Bose fluid. Its density operator is given by eq. (1) if we identify $Q/\pi = \rho_0 = \rho_1$ ($p = 2$), $\partial_t \varphi$ is conjugate to the phase $\theta$ of the Bose field [9]. With the replacements $K \to K^{-1}$, $t \to t/K^2$ and $w = 0$, (3) describes the action of the 1D-superfluid in a random potential. $v$ denotes the phase velocity of the sound waves with $v/(\pi K) = \rho_0/m$ and $\pi v K = \kappa/(\pi^2 \rho_0^2)$ where $\kappa$ is the compressibility. The transition between the superfluid and the localized phase occurs for $K_v^* = 2/3$ [3]. Thermal fluctuations again suppress the disorder and destroy the superfluid localization transition in 1D. In contrast to CDWs here the $\theta$-field may have vortex-like singularities in space–time and the flow equations (3) - (4) apply again. The vortex unbinding transition appears at $K_w$ with $K_w = 1/2$ ($q = 2$). If both $w$ and $u$ are non-zero we can use the canonical transformation to rewrite the vortex contribution in the form $w \cos(q\varphi)$. For $K < K_u$, $K_w$ both perturbations are irrelevant and the system is superfluid. For $K_w < K < K_u$ the decay of $u$ is stopped due to the suppression of the $\varphi$ fluctuations due to $w$. An Imry-Ma-argument shows further, that the $q\varphi = \pi(n+1/2)$ state is destroyed on the scale $\xi \approx \xi_u/w^2(\log \xi)$ by arbitrary weak disorder, i.e. vortices become irrelevant above this scale. On larger scales one can expects that quantum fluctuations wash out the disorder, the system is still superfluid. Finally, at $K > K_u$, $K_w$ both perturbations are relevant and superfluidity is destroyed.

To conclude we have shown, that in 1D CDWs/SDWs and superfluids disorder driven zero temperature phase transitions are destroyed by thermal fluctuations leaving behind a rich cross-over behavior. Quantum phase slips in CDWs and superfluids lead to additional phase transitions and shift the unpinning transition in CDWs to smaller $K$-values. Coulomb hardening and dissipative quantum effects will be discussed in a forthcoming publication [21].

Acknowledgement: The authors thank A. Altland, S. Bravovski, T. Emig, L. Glazman, S. Korshunov, B. Rosenow and S. Scheidl for useful discussions.

[1] G. Grüner, Rev. Mod. Phys. 60, 1128 (1988); “DENSITY WAVES IN SOLIDS”, Addison-Wesley (1994)
[2] Proceedings of the ECRYS-Workshop9 , S. Brazovski and P. Monceau P. (Editors), J. de Physique IV, Vol. 9, (1999).
[3] G. Blatter, M.V. Feigel’man, V. Geshkenbein, A.I. Larkin and V.M. Vinokur, Rev.Mod.Phys. 66, 1125 (1994).
[4] T. Nattermann, S. Scheidl, Adv. Phys. 49, 607 (2000).
[5] J.L. Cardy and S. Ostlund, Phys. Rev. B 25, 6899 (1982).
[6] J. Villain and J. F. Fernandez, Z. Phys. B 54, 139 (1984).
[7] M. Feigel’man, Sov. Phys. JETP 52, 555 (1980).
[8] H. Fukuyama, Lecture Notes in Physics 217, 387 (1984).
[9] M. Giamarchi and H.J. Schulz, Europhys. Lett. 3, 1287 (1987). Phys. Rev. B 37, 325 (1988).
[10] S. Chakravarty, B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. 60, 1057 (1988), Phys. Rev. B 39, 2344 (1989).
[11] S.E. Korshunov, Sov. Phys. JETP 68, 609 (1989).
[12] see e.g. by R.M. Bradley and S. Doniach, Phys. Rev. B30, 1138 (1984), M.P.A. Fisher and G. Grinstein, Phys. Rev. Lett. 60, 208 (1988), S.R. Renn and J.M. Duan, Phys. Rev. Lett. 76, 3400 (1996), A. D. Zaikin et al. Phys. Rev. Lett 78 1552 (1997).
[13] K. Maki, Phys. Lett. A 202, 313 (1995).
[14] U. Schultz, J. Villain, E. Brezin and H. Orland, J. Stat. Phys. 51, 1 (1988).
[15] J.M.Kosterlitz, D.J.Thouless, J. Phys. C6, 1181 (1974).
[16] A. D. Huse, C. L. Henley and D. S. Fisher, Phys. Rev. Lett. 55, 2924 (1985).
[17] J. V. Jose, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, Phys. Rev B 16, 1217 (1977).
[18] A. Lenard, J. Math. Phys. 2, 682 (1961).
[19] F. D. M. Haldane, Phys. Rev. Lett. 47, 1840 (1981); note that in this paper the meaning of $\varphi$ and $\theta$ is interchanged.
[20] A. Glatz and T. Nattermann, to be published.