Comment about UV regularization of basic commutators in string theories

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Abstract

Recently proposed by Hwang, Marnelius and Saltsidis zeta regularization of basic commutators in string theories is generalized to the string models with non-trivial vacuums. It is shown that implementation of this regularization implies the cancellation of dangerous terms in the commutators between Virasoro generators, which break Jacobi identity.

It is well known what the role plays the problem of regularization and renormalization of ultraviolet divergences in quantum field theory. However, the importance of the problem of the proper treatment of ultraviolet divergences arising at quantization of string theories is less understood. Perhaps, it is connected with the fact that calculating the central extensions in Virasoro algebra in string theories it is possible to escape the necessity of consideration of infinite sums due to tacitly assumed and very natural regularization providing their cancellation [1]. However, in the cases when conventional vacuum state is absent (for example in the case of tensionless conformal string [2]) explicit ultraviolet regularization of basic commutators of the theory becomes inevitable.

In recent paper by Hwang, Marnelius and Saltsidis [3] the convenient version of ultraviolet zeta regularization of basic commutators in string theories was elaborated. This scheme was applied to Hamiltonian BRST quantization [4] of string theories. The well-known results for critical dimensions obtained by the same method earlier [5] were reproduced in the new context.
In this short comment we further investigate the properties of regularization suggested in [3] constructing two simple examples. We consider traditional closed bosonic string model but with non-trivial vacuums. These vacuums are obtained by means of redefinition of creation and annihilation operators due to Bogoliubov transformation. As a result of this redefinition in the expression for central extension arise infinite sums while the traditional finite part of contribution to the central charge acquires the structures breaking Jacobi identities. The direct generalization of the regularization suggested in [3] allows calculate the finite contributions of infinite sums, which have the structure providing the total cancellation of patological terms and restoration of Jacobi identities. While our models hardly can be treated as having direct physical significance, the very fact of restoration of the correct structure of central extension looks promising. It could be interpreted as an additional argument in favour of regularization [3]. One can hope also the this scheme can be applied to more complicated theories such as quantum cosmology where the necessity of ultraviolet regularization is getting obvious [6].

The structure of the present note is the following: first, we introduce the necessary notations for closed bosonic string and write down the expression for the central charge arising in the commutator between Virasoro generators for the family of vacuums, parametrized by form of relation between energy and momentum of excitations described by the corresponding creation and annihilation operators; second, for particular choices of vacuums we show the possible sources of the break of Jacobi identities; third we write down formulae for the regularized commutators [3] and show how they modify expressions for the central extension of Virasoro algebra; finally using zeta regularization [7] we calculate renormalized expressions for central charge and manifest that they are free from the terms breaking Jacobi identitites.

To begin with let us consider closed bosonic string. One can write down the constraints in the Hamiltonian formalism in the following form:

\[ H_\perp = \frac{1}{2}p^2 + \frac{1}{2}q^2, \quad (1) \]

\[ H = pq', \quad (2) \]

where, \( H_\perp \) is the so called super-Hamiltonian constraint, \( H \) is supermomentum, \( q \) is the coordinate of string, while \( p \) is conjugate momentum (we have
omitted the indices enumerating the coordinates of d-dimensional spacetime, because the contributions of these coordinates are additive). Now let us expand \( q \) and its conjugate momentum \( p \) via creation and annihilation operators and zero-mode harmonics

\[
q = \frac{q_0}{\sqrt{2\pi}} + \sum_{k>0} \frac{1}{\sqrt{2\omega_k}\sqrt{2\pi}}(a_k e^{-ikx} + \bar{a}_k e^{ikx} + a^+_k e^{ikx} + \bar{a}^+_k e^{-ikx}), \quad (3)
\]

\[
p = \frac{p_0}{2\pi} + i \sum_{k>0} \sqrt{\frac{\omega_k}{4\pi}}(-a_k e^{-ikx} - \bar{a}_k e^{ikx} + a^+_k e^{ikx} + \bar{a}^+_k e^{-ikx}), \quad (4)
\]

where \( a_k, \bar{a}_k, a^+_k \) and \( \bar{a}^+_k \) are the annihilation and creation operators for left- and right-hand oriented excitations respectively, while \( q_0 \) and \( p_0 \) are zero modes.

Substituting Eqs. (3)-(4) into Eqs. (1)- (2) one can write down the expressions for the harmonics of super-Hamiltonian and supermomentum

\[
H_{\perp n} = ip_0 (\bar{a}^+_n - a_n) \sqrt{\frac{\omega_n}{2}}
\]

\[
-\frac{1}{2} \sum_{0<k<n} (a_{n-k} a_{n-k} + \bar{a}^+_n \bar{a}^+_{n-k}) \left( \frac{\sqrt{\omega_n \omega_{n-k}}}{2} + \frac{k(n-k)}{2\sqrt{\omega_n \omega_{n-k}}} \right)
\]

\[
+ \frac{1}{2} \sum_{k>0} (a_{n+k} \bar{a}_k + a^+_k \bar{a}^+_{n+k}) \left( \frac{k(n+k)}{\sqrt{\omega_n \omega_{n+k}}} - \frac{\sqrt{\omega_n \omega_{n-k}}}{\sqrt{\omega_n \omega_{n+k}}} \right)
\]

\[
+ \frac{1}{2} \sum_{k>0} (a_{n+k} a^+_n + \bar{a}^+_n \bar{a}^+_{n+k}) \left( \sqrt{\omega_n \omega_{n-k}} + \frac{k(n+k)}{\sqrt{\omega_n \omega_{n-k}}} \right), \quad (5)
\]

\[
H_n = -ip_0 \frac{n}{\sqrt{2\omega_n}} (a_n + \bar{a}^+_n)
\]

\[
+ \sum_{0<k<n} (\bar{a}^+_n a^+_k - a_n a_{n-k}) \frac{k}{2} \sqrt{\frac{\omega_{n-k}}{\omega_k}}
\]

\[
+ \sum_{k>0} \frac{n+k}{2} \sqrt{\frac{\omega_k}{\omega_{n+k}}} (a^+_k a_{n+k} + a^+_n a^+_k - \bar{a}^+_k a_{n+k} - \bar{a}^+_n \bar{a}^+_k)
\]

\[
+ \sum_{k>0} \frac{k}{2} \sqrt{\frac{\omega_{n+k}}{\omega_k}} (a_{n+k} \bar{a}_k + a_{n+k} \bar{a}^+_k + a^+_k \bar{a}^+_{n+k} - \bar{a}^+_n \bar{a}^+_k - \bar{a}^+_n a^+_k). \quad (6)
\]

As usual one can define Virasoro constraints as

\[
L_n = \frac{1}{2}(H_{\perp n} + H_n), \quad (7)
\]
\[ L_n = \frac{1}{2}(H_n - H_{-n}). \]  

(8)

Traditional choice of dispersiveal relation between frequency \( \omega_k \) and wave number \( k \):

\[ \omega_k = k \]

(9)

provides diagonality of Hamiltonian \( H_{\perp0} \). Besides at the choice (9) \( L \) does not depend on \( \tilde{a}, \tilde{a}^+ \) and \( \tilde{L} \) does not depend on \( a, a^+ \) respectively. Other choices of dispersiveal relations or, in other terms, other choices of creation and annihilation operators obtained by means of Bogoliubov transformations make Hamiltonian non-diagonal and the structure of Virasoro constraints becomes more involved. Nevertheless it is useful to write down the general expression for the central extension for the commutators of Virasoro generators in this case as well. It has the following form:

\[
[L_n, L_{-n}]_{\text{c.e.}} = \frac{1}{2}[H_{\perp n}, H_n]_{\text{c.e.}} \\
= \frac{1}{4} \sum_{0<k<n} \left( \frac{\omega_k(n - k)}{k} + \frac{\omega_{n-k}^2}{k} \right) \\
+ \frac{1}{4} \sum_{k>0} \left( \frac{(n + k)k}{\sqrt{\omega_k \omega_{n+k}}} - \frac{\omega_{n+k}}{\sqrt{\omega_k \omega_{n+k}}} \right) \left( k \sqrt{\frac{\omega_{n+k}}{\omega_{n-k}}} - (n + k) \sqrt{\frac{\omega_{n-k}}{\omega_{n+k}}} \right). 
\]

(10)

If we make the traditional choice of creation and annihilation operators, determined by the dispersiveal relation (9) then the second infinite sum in the expression (10) disappear, while the finite sum gives the well-known result:

\[
[L_n, L_{-n}]_{\text{c.e.}} = \frac{1}{2} \sum_{0<k<n} k(n - k) = \frac{1}{12} n(n^2 - 1). 
\]

(11)

However, the other choice of vacuum immediately implies that in the result of finite summation in the formula for central extension (10) one has terms incompatible with Jacobi identities. On the other hand infinite sum in Eq. (10) becomes divergent and needs an explicit regularization. It is interesting to check how regularization of quantum commutators suggested in ([3]) works for these cases.

First of all let us sketch briefly the algorithm suggested in ([3]). The main idea consists in the substitution instead of commutator

\[ [a_n, a_{m}^+] = \delta_{nm} \]

(12)
the regularized commutator

\[ [a_n, a_m^+] = \delta_{nm} f(n, s), \quad (13) \]

where \( f(n, s) \) is a real function which satisfies the condition

\[ f(n, 0) = 1. \quad (14) \]

This regularization should give final results when the regulator is removed \((s \to 0)\). The regulator which we choose slightly differs from that from Ref. \([3]\) and has the following form:

\[ f^{(\alpha)}(n, s) = (n + \alpha)^{-s}, \quad (15) \]

where \( \alpha \) is a positive number.

Application of this regularization to the formula \((10)\) implies that the infinite sum should be multiplied by factor

\[ f^{(\alpha)}(k, s) f^{(\alpha)}(n + k, s), \quad (16) \]

because this infinite sum arises as a sum of products of commutators

\[ [a_k, a_k^+][a_{k+n}, a_{k+n}^+] \quad (17) \]

or the corresponding commutators with \( \bar{a}, \bar{a}^+ \).

Now, let us choose for simplicity the vacuum and creation and annihilation operators defined by dispersional relation

\[ \omega_k = \omega_0, \quad (18) \]

where \( \omega_0 \) is a constant. Substituting \((18)\) into \((10)\), and multiplying the terms in an infinite sum by factor \((16)\) one has

\[ [L_n, L_{-n}]_{c.e.} = \frac{1}{4} \sum_{0 < k < n} \left( \omega_0 (n - k) + \frac{k(n - k)^2}{\omega_0} \right) \]

\[ + \sum_{k > 0} \frac{1}{4} \left( n\omega_0 - \frac{n^2 k^2}{\omega_0} \right) \frac{1}{(n + k + \alpha)^s} \frac{1}{(k + \alpha)^s}. \quad (19) \]
It is easy to calculate the finite sum in the expression (19) using summation formulae

\[
\sum_{k=1}^{n} k = \frac{n(n+1)}{2},
\]
\[
\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},
\]
\[
\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.
\]

(20)

Then one has

\[
\frac{1}{4} \sum_{0 < k < n} \left( \frac{\omega_0(n-k) + k(n-k)^2}{\omega_0} \right) = \frac{\omega_0 n(n-1)}{8} + \frac{n^2(n+1)^2}{48 \omega_0}.
\]

(21)

This expression contains terms proportional to \(n^2\) and \(n^4\), which as is well known break Jacobi identities for commutators. Now we should calculate an infinite sum in the expression (19) using \(\zeta\)-function technique [7].

Let us remind briefly the main formulae of this technique which we shall use. The definition of Riemannian \(\zeta\) - function is the following:

\[
\zeta_R(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.
\]

(22)

This function is finite at \(s > 1\) and can be analytically continued to the field \(s < 1\). In particular, we shall need the values

\[
\zeta_R(0) = -\frac{1}{2},
\]
\[
\zeta_R(-1) = -\frac{1}{12},
\]
\[
\zeta_R(-2) = 0.
\]

(23)

At \(s = 1\) Riemannian \(\zeta\) - function has the pole:

\[
\zeta_R(1 + s) = \frac{1}{s} + const + 0(s).
\]

(24)
Now, expanding factor depending on $s$ in the infinite sum in expression (19) in inverse degrees of $k$ one can get the following expression:

\[
\frac{1}{(n + k + \alpha)^s (k + \alpha)^s} k^{-2s} - 2sk^{-2s-1} \left( \alpha + \frac{n}{2} \right) \\
+s(2s + 1)k^{-2s-2} \left( \alpha + \frac{n}{2} \right)^2 - s(2s + 1)(2s + 2)k^{-2s-3} \frac{(\alpha + \frac{n}{2})^3}{3} \\
+sk^{-2s-2} \frac{n^2}{4} - s(s + 1)k^{-2s-3} \left( \alpha + \frac{n}{2} \right) \frac{n^2}{2}.
\] (25)

Contracting expression (23) with the multiplier \((n\omega_0 - \frac{n^2k}{\omega_0} - \frac{nk^2}{\omega_0})\) in the expression (19) one can get the limit $s \to 0$ using formulae (23), (24). As a result one can get renormalized contribution of the infinite sum in the expression (19)

\[
\left\{ \sum_{k>0} \frac{1}{4} \left( n\omega_0 - \frac{n^2k}{\omega_0} - \frac{nk^2}{\omega_0} \right) \right\}_{\text{renormalized}} \\
= n \left( \frac{\omega_0(1 - 2\alpha)}{8} + \frac{\alpha^3}{12\omega_0} \right) \\
+ n^2 \left( \frac{1}{48\omega_0} - \frac{\omega_0}{8} \right) - \frac{n^4}{48\omega_0}.
\] (26)

It is easy to see that dangerous terms proportional to $n^2$ and $n^4$ in the regularized expression for the infinite sum (23) exactly cancel those in the expression for the finite sum (21). Thus in such a way ultraviolet renormalization given by regulator (13), (14) and (15) provides the restoration of Jacobi identities.

One can consider also another choice of creation and annihilation operators defined by the dispersive relation

\[
\omega_k = Ak^2.
\] (27)

One can easily check that all the results of calculations for the case given by dispersion relation (27) coincide with those obtained in the case given by dispersion relation (18). (All the formulae for the former case can be obtained from the formulae for the former case by substitution $A \to 1/\omega_0$).
Thus, we have seen that the regularization of commutators suggested in Ref. (3) applied to the string model with non-standard definitions of creation and annihilation operators allows to preserve fundamental symmetric properties encoded in Jacobi identities. This could be interpreted as an additional argument in favour of this regularization. However, we should recognize that it is not true for any choice of vacuum. For example, for the vacuum defined by dispersion relation

$$\omega_k = \frac{B}{k}$$

(for this case all the calculations also could be carried out in an explicit form) we have got the term proportional to $n^5$, which cannot be cancelled by help of regularization of creation and annihilation operators. The analysis of applicability of this regularization to different types of vacuums and its application to such complicated theories as quantum cosmology is under investigation now [8].

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