A CURVATURE FLOW APPROACH TO $L_p$-CHRISTOFFEL-MINKOWSKI PROBLEM FOR $p > 1$

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Abstract. We study the motion of smooth, closed, strictly convex hypersurfaces in $\mathbb{R}^{n+1}$ expanding in the direction of their normal vector field with speed depending on the $k$-th elementary symmetric polynomial of the principal radii of curvature $\sigma_k$. As an application, we give a unified flow approach to $L_p$-Christoffel-Minkowski problem for $p > 1$.

Keywords: curvature flows, Christoffel-Minkowski problem, convexity.
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1. Introduction

Let $\mathcal{M}$ be an oriented, closed $n$-dimensional manifold. We embed $\mathcal{M}$ in the Euclidean $(n+1)$-space by

$$\tilde{X}_0 : \mathcal{M} \to \mathbb{R}^{n+1}$$

and denote its image by $\tilde{\mathcal{M}}_0 = \tilde{X}_0(\mathcal{M})$. We assume that $\mathcal{M}_0$ is strictly convex. Then we consider a family of maps $\tilde{X} : \mathcal{M} \times [0, \tilde{T}) \to \mathbb{R}^{n+1}$, with $\tilde{X}_\tau = \tilde{X}(\cdot, \tau) : \mathcal{M} \to \mathbb{R}^{n+1}$ satisfying the initial value problem

$$\begin{cases}
\frac{\partial}{\partial \tau} \tilde{X}(x, \tau) = f(\nu) \left( \frac{E_{n-k}}{E_n} (\tilde{h}_{ij}) \right)^\alpha \nu(x, \tau) & \text{in } \mathcal{M} \times [0, \tilde{T}),
\tilde{X}(\cdot, 0) = \tilde{X}_0 & \text{in } \mathcal{M},
\end{cases}$$

where

- $\nu$ and $\tilde{h}_{ij}$ denote the outward normal vector and the second fundamental form of the evolving hypersurface $\tilde{\mathcal{M}}_\tau = \tilde{X}(\mathcal{M})$ respectively.
- $\alpha$ is a positive constant and $f$ is a positive smooth function defined on the unit sphere $\mathbb{S}^n$.
- $E_i = \sum_{j_1 < j_2 < \cdots < j_i} \kappa_{j_1} \kappa_{j_2} \cdots \kappa_{j_i}$ is the $i$-th elementary symmetric polynomial of principal curvatures for $0 \leq i \leq n$. ($E_0 = 1$)

We consider a normalization of the flow (1.1) given by

$$X(x, t) = e^{-t} \tilde{X}(x, \tau(t)),$$

where

$$t = t(\tau) = \frac{1}{p} \log \int_{\mathbb{S}^n} \tilde{u}^p(x, \tau) d\mu - \frac{1}{p} \log \int_{\mathbb{S}^n} \tilde{u}_0^p(x, 0) d\mu$$

and

$$d\mu = f^{-\frac{1}{p}}(x) dx, \quad p = 1 + \frac{1}{\alpha}.$$

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It can be seen later on that $X$ satisfies the following normalised flow

$$
\begin{align*}
\frac{\partial}{\partial t} X(x,t) &= \phi(t) f(\nu) \left( \frac{E_n - k}{E_n} (h_{ij}) \right)^\alpha \nu(x,t) - X \quad \text{in } M \times [0,T), \\
X(\cdot,0) &= X_0 \quad \text{in } M,
\end{align*}
$$

where

$$
\phi(t) = \int_{S^n} u^p(x,t) d\mu \bigg/ \int_{S^n} u(x,t)^{p-1} \sigma_k^\alpha (u_{ij} + u \delta_{ij}) f(x) d\mu
$$

Let $\tilde{X}(\cdot, \tau)$ be a smooth solution to the curvature curvature flow (1.1) and let $\tilde{u}(\cdot, \tau)$ be its support function. The flow (1.1) can be reduced to the initial value problem for support function $\tilde{u}$:

$$
\begin{align*}
\frac{\partial \tilde{u}}{\partial \tau}(x, \tau) &= f(x) \sigma_k^\alpha (\tilde{u}_{ij} + \tilde{u} \delta_{ij}) \\
\tilde{u}(\cdot, 0) &= \tilde{u}_0.
\end{align*}
$$

Similarly, if $X(\cdot, t)$ be a smooth solution to the curvature curvature flow (1.5) and let $u(\cdot, t)$ be its support function. The flow (1.5) can be reduced to the initial value problem for support function $u$:

$$
\begin{align*}
\frac{\partial u}{\partial t}(x, t) &= \phi(t) f(x) \sigma_k^\alpha (u_{ij} + u \delta_{ij}) - u \\
u(\cdot, 0) &= u_0.
\end{align*}
$$

Our motivation to study the flow (1.5) is due to the significance of its solitons in convex geometry. A positive homothetic self-similar solution of (1.5), when exists, is a solution to

$$
\hat{f}(x) u^{1-p} \sigma_k (u_{ij} + u \delta_{ij}) = c
$$

for some $c > 0$ and where $\hat{f}(x) = \int_{S^n} f(x) \, d\sigma_n(x)$. One would like to find necessary and sufficient conditions on a function $f$ such that a positive strictly convex solution exists. Here the strict convexity of a solution, $u$, is understood as the strict convexity of the associated closed hypersurface. The pairs $(p = 1, k = 1)$, $(p = 1, k = n)$, $(p \neq 1, k = n)$ of this equation are known in order as the Christoffel problem, the Minkowski problem and the $L_p$-Minkowski problem. In general, this equation is known as the $L_p$-Christoffel-Minkowski problem.

We mainly get the following result.

**Theorem 1.1.** Assume that $\hat{f} \in C^\infty(S^n)$ is a positive function such that

$$(\hat{f}^\frac{k+1}{p} \sigma_{k+1}^{p+1})_{ij} + \hat{f}^\frac{k+1-p}{p} \delta_{ij}$$

is positive definite and $M_0 \subset \mathbb{R}^{n+1}$ is a strictly convex, closed hypersurface which contains the origin in its interior.

(i) If $k \alpha \leq 1$, then the normalised flow (1.5) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. For each $t \in [0, \infty)$, $M_t = X(S^n, t)$ is a closed, smooth and strictly convex hypersurface and the support function $u(x,t)$ of $M_t = X(S^n, t)$ converges smoothly, as $t \to \infty$, to the unique positive, smooth and strictly convex solution of the equation (1.8) with $f$ replaced by $\lambda_0 \hat{f}$ for some $\lambda_0 > 0$.

(ii) If $f$ is in addition even function and the initial hypersurface $M_0$ is origin-symmetric, then the normalised flow (1.5) has a unique smooth solution, which exists for any time $t \in [0, \infty)$. For each $t \in [0, \infty)$, $M_t = X(S^n, t)$ is a closed, smooth, strictly convex and origin-symmetric hypersurface and the support function $u(x,t)$ of $M_t = X(S^n, t)$ converges smoothly, as $t \to \infty$,
to the unique positive, smooth, strictly convex and even solution of the equation \((1.8)\) with \(\hat{f}\) replaced by \(\lambda_0 \hat{f}\) for some \(\lambda_0 > 0\).

(iii) If \(f \equiv 1\), then the normalised flow \((1.5)\) has a unique smooth solution, which exists for any time \(t \in [0, \infty)\). For each \(t \in [0, \infty)\), \(M_t = X(S^n, t)\) is a closed, smooth and strictly convex hypersurface and the support function \(u(x, t)\) of \(M_t = X(S^n, t)\) converges smoothly, as \(t \to \infty\), to a sphere.

**Remark 1.1.** Recently, \([19]\) and \([24]\) use a similar flow to give a new proof to the well-known \(L_p\) Christoffel-Minkowski problem for the case \(p \geq k + 1\) \([18]\) without using the constant rank theorem. Our proof gives a unified flow approach to \(L_p\)-Christoffel-Minkowski problem for \(p \geq k + 1\) \([18]\) and for \(1 < p < k + 1\) and even \(\hat{f}\) \([17]\) without using the constant rank theorem. Our results give partial answers to Question 1 and 2 in \([17]\), also Question 1 and 2 in \([19]\).

The equation \((1.8)\) arises naturally in the \(L_p\) Brunn-Minkowski theory, see \([20, 25]\). The \(L_p\)-Minkowski problem is also well-understood (except the case \(p \leq -n - 1\)) and we refer the reader to the essential papers \([3, 7, 20, 22, 21]\) for motivation and the most comprehensive list of results, see also \([25]\). If \(p = 1, k < n\), much less is known and more restrictions need to be imposed on \(\hat{f}\). In \([19]\), Guan-Ma proved a deformation lemma which allowed them to establish if a function is \(\hat{f} \in C^\infty(S^n)\) is \(k\)-convex, e.g. \(D_iD_j\hat{f} + \hat{f} \sigma_{ij}\) is non-negative definite, then the equation \((1.8)\) for \(p = 1, k < n\) has a strictly convex solution. Later in \([18]\), using the deformation lemma, Hu, Ma and Shen proved that if \(p \geq k + 1, k < n\) and \(\hat{f} \in C^\infty(S^n)\) is \((p + k - 1)\)-convex, then \((1.8)\) admits a positive strictly convex solution. Recently, for \(1 < p < k + 1\) and even prescribed data, under the \((p + k - 1)\)-convexity of \(\hat{f}\), an existence result was proved by Guan and Xia in \([17]\) using a refined gradient estimate and the constant rank theorem.

2. Preliminaries

2.1. Setting and General facts.

For later convenience, we first state our conventions on Riemann Curvature tensor and derivative notation. Let \(M\) be a smooth manifold and \(g\) be a Riemannian metric on \(M\) with Levi-Civita connection \(D\). For a \((s, r)\) tensor field \(D\) on \(M\), its covariant derivative \(D\alpha\) is a \((s, r + 1)\) tensor field given by

\[
DD(Y^1,..,Y^n,X_1,...,X_r,X) = D_X D(Y^1,..,Y^n,X_1,...,X_r) \equiv X(D(Y^1,..,Y^n,X_1,...,X_r)) - D(D_X Y^1,..,Y^n,X_1,...,X_r)
\]

\[- \ldots - \alpha(Y^1,..,Y^n,X_1,...,D_X X_r),\]

the coordinate expression of which is denoted by

\[
DD = (\alpha_{k_1,...,k_r;k_{r+1}}^{l_1,...,l_s}).
\]

We can continue to define the second covariant derivative of \(\alpha\) as follows:

\[
D^2\alpha(Y^1,..,Y^n,X_1,...,X_r,X,Y) = (D_Y (D\alpha))(Y^1,..,Y^n,X_1,...,X_r,X),
\]

the coordinate expression of which is denoted by

\[
D^2\alpha = (\alpha_{k_1,...,k_r;k_{r+1}k_{r+2}}^{l_1,...,l_s}).
\]

In particular, for a function \(u : M \to \mathbb{R}\), we have the following important identity

\[
D^2 f(X,Y) = XY(f) - (DY X)f.
\]
Similarly, we can also define the higher order covariant derivative of $\alpha$:
\[ D^3 \alpha = D(D^2 \alpha), \ldots, \]
and so on. For simplicity, the coordinate expression of the covariant differentiation will be denoted by indices without semicolons, e.g.
\[ u_i, \quad u_{ij} \quad \text{or} \quad u_{ijk} \]
for a function $u : M \to \mathbb{R}$.

Our convention for the Riemannian curvature $(3,1)$-tensor $R_m$ is defined by
\[ R_m(X, Y)Z = -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]}Z. \]

By picking a local coordinate chart $\{x^i\}_{i=1}^n$ of $M$, the component of the $(3,1)$-tensor $R_m$ is defined by
\[ R_m\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = R^{l}_{ijk}\frac{\partial}{\partial x^l}, \]
and $R_{ijkl} = g_{lm} R_{ijk}^m$. Then, we have the standard commutation formulas (Ricci identities):
\[ (2.1) \quad \alpha_{k_1\ldots k_r;\ ji} - \alpha_{k_1\ldots k_r;\ ij} = \sum_{a=1}^{r} R_{ijk_a} m_a \alpha_{k_1\ldots k_{a-1}k_{a+1}\ldots k_r} - \sum_{b=1}^{s} R_{ijm_b} \alpha_{k_1\ldots k_r} m_b \alpha_{k_1\ldots k_r}. \]

We list some facts which will be used frequently. For the standard sphere $S^n$ with the sectional curvature 1 and the standard metric $\sigma$,
\[ R_{ijkl} = \sigma_{ik}\sigma_{jl} - \sigma_{il}\sigma_{jk}. \]
A special case of Ricci identity for a function $u : M \to \mathbb{R}$ will be:
\[ u_{kji} - u_{kij} = R_{ijk}^m u_m. \]
In particular, for a function $u : S^n \to \mathbb{R}$,
\[ (2.2) \quad u_{kji} - u_{kij} = \sigma_{ik} u_j - \sigma_{ij} u_k. \]

Let $(M, g)$ be an immersed hypersurface in $\mathbb{R}^{n+1}$ and $\nu$ be a given unit outward normal.

The second fundamental form $h_{ij}$ of the hypersurface $M$ with respect to $\nu$ is defined by
\[ h_{ij} = -\left\langle \frac{\partial^2 X}{\partial x^i \partial x^j}, \nu \right\rangle_{\mathbb{R}^{n+1}}. \]

2.2. Basic properties of convex hypersurfaces.

We first recall some basic properties of convex hypersurfaces. Let $M$ be a smooth, closed, uniformly convex hypersurface in $\mathbb{R}^{n+1}$. Assume that $M$ is parametrized by the inverse Gauss map
\[ X : S^n \to M. \]
The support function $u : S^n \to \mathbb{R}$ of $M$ is defined by
\[ u(x) = \sup\{\langle x, y \rangle : y \in M\}. \]
The supremum is attained at a point $y$ such that $x$ is the outer normal of $M$ at $y$. It is easy to check that
\[ (2.3) \quad y = u(x)x + Du(x), \]
where $D$ is the covariant derivative with respect to the standard metric $\delta_{ij}$ of the sphere $S^n$.

Hence
\[ (2.4) \quad r = |y| = \sqrt{u^2 + |Du|^2}. \]
The second fundamental form of $\mathcal{M}$ is given by, see e.g. [1, 27],
\[(2.5)\quad h_{ij} = u_{ij} + \delta_{ij},\]
where $u_{ij} = D_j D_i u$ denotes the second order covariant derivative of $u$ with respect the spherical metric $\delta_{ij}$. By Weingarten’s formula,
\[(2.6)\quad \delta_{ij} = \langle \partial_{\nu} \partial_{x^i}, \partial_{\nu} \partial_{x^j} \rangle = h_{ik} g^{kl} h_{jl},\]
where $g_{ij}$ is the metric of $\mathcal{M}$ and $g^{ij}$ is its inverse. It follows from (2.5) and (2.6) that the principal radii of curvature of $\mathcal{M}$, under a smooth local orthonormal frame on $\mathbb{S}^n$, are the eigenvalues of the matrix
\[b_{ij} = u_{ij} + u \delta_{ij}.\]
In particular, write $\sigma_k$ for the $k$-th elementary symmetric polynomial of the principal radii of curvature, we have
\[
\frac{E_n}{E_{n-k}}(h_{ij}) = \sigma_k(u_{ij} + u \delta_{ij}).
\]

**Lemma 2.1.** Let $\tilde{X}(:, \tau)$ be a smooth solution to the curvature curvature flow (1.1) with $\tau \in [0, \tilde{T})$ and for each $\tau \geq 0$, $\tilde{\mathcal{M}}_\tau = \tilde{X}(\mathbb{S}^n, \tau)$ be a smooth, closed and strictly convex hypersurface. Then, $X$ given by (1.2) satisfies the normalised flow (1.5).

**Proof.** By virtue of the relation (1.3), we have
\[
\frac{dt}{d\tau} = \int_{\mathbb{S}^n} \hat{u}^{p-1}(x, t) \frac{\partial \hat{u}}{\partial \tau} d\mu / \int_{\mathbb{S}^n} \hat{u}^p(x, \tau) d\mu
\]
Thus,
\[(2.7)\quad \frac{d\tau}{dt} = e^{(1-k\alpha)t} \phi(t)
\]
Recalling that
\[X(x, t) = e^{-t} \tilde{X}(x, \tau(t)).\]
Thus, the rescaled $E_k$
\[E_k(h_{ij}(x, s)) = E_k(\tilde{h}_{ij}(x, \tau(t))) e^{kt}.
\]
In view of (2.7) and (2.2), one infers that
\[
\frac{\partial}{\partial t} X(x, t) = e^{-t} \frac{\partial}{\partial \tau} X \frac{d\tau}{dt} - X
\]
\[= e^{-k\alpha t} \phi(t) f(\nu) \left( \frac{E_{n-k}}{E_n} \tilde{h}_{ij} \right)^\alpha \nu(x, t) - X
\]
\[= \phi(t) f(\nu) \left( \frac{E_{n-k}}{E_n} h_{ij} \right)^\alpha \nu(x, t) - X,
\]
thus completing the proof. □
2.3. The entropy of the flow.

Lemma 2.2. Let $X(\cdot, t)$ be a smooth solution to the curvature flow \((1.5)\) with $t \in [0, T)$ and for each $t \geq 0$, $M_t = X(S^n, t)$ be a smooth, closed and strictly convex hypersurface. Then,

$$\int_{S^n} u^p(x, t) d\mu$$

is preserved under the flow \((1.5)\), where

$$d\mu = f^{-\frac{1}{\alpha}}(x) dx.$$

Proof. \(\square\)

Define the entropy

$$J_{p,k}(X(\cdot, t)) = \int_{S^n} u_k dx$$

Lemma 2.3. Let $X(\cdot, t)$ be a smooth solution to the curvature flow \((1.5)\) with $t \in [0, T)$ and for each $t \geq 0$, $M_t = X(S^n, t)$ be a smooth, closed and strictly convex hypersurface. Then, $J_{p,k}(u)$ is nondecreasing under the flow \((1.5)\) and the inequality holds if and only if $u$ is a solution to \((1.8)\).

Proof. A direct computation implies that by virtue of the divergence structure and the equation \((1.7)\)

$$\frac{1}{k + 1} \frac{d}{dt} J_{p,k}(X(\cdot, t)) = \int_{S^n} u_t \sigma_k dx$$

$$= (\int_{S^n} d\mu)^{-1} \left( \int_{S^n} u^{-p} \sigma_k^{1+\alpha} f(x)^{1+\frac{1}{\alpha}} d\mu \int_{S^n} d\mu \right. - \int_{S^n} u^{-1} \sigma_k f(x) d\mu \int_{S^n} u^{-p} \sigma_k f(x)^{\frac{1}{\alpha}} d\mu \right)$$

$$= (\int_{S^n} d\mu)^{-1} \left( \int_{S^n} v^{1+\alpha} d\mu \int_{S^n} d\mu - \int_{S^n} v^\alpha d\mu \int_{S^n} v d\mu \right),$$

where

$$v = u^{-p} \sigma_k f(x)^{\frac{1}{\alpha}} \quad \text{and} \quad d\mu = u^p f(x)^{-\frac{1}{\alpha}} dx.$$

It follows by Holder inequality that

$$\frac{d}{ds} J_{p,k}(u) \geq 0$$

with the equality if and only if

$$v(x, s) = h(s).$$

\(\square\)

We introduce some properties for convex bodies which will be used in the sequel. First, we recall the following Lemma, see Lemma 2.6 in [23] for the proof.
Lemma 2.4. Let \( \Omega \) be a convex body containing the origin in its interior and \( u \) and \( r \) be the support function and radial function of \( \Omega \), and \( x_{\max} \) be the point such that \( u(x_{\max}) = \max_{S^n} u \). Then,

\[
u(x) \geq x \cdot x_{\max} u(x_{\max}), \quad x \in S^n.
\]

Furthermore, we need to some properties of \( S(b_{ij}) = \sigma^1_k(b_{ij}) \) due to its inverse concavity.

Lemma 2.5.

\[
\sum_{i} S^{ii} \geq (C_{k}^{n})^\frac{1}{k}.
\]

(2.8)

\[
S^{ij} h^{lm} b_{ij} b_{jm} \geq S^2.
\]

(2.9)

\[
S^{ij} h^{lm} \xi_{ij} \xi_{jm} + 2 e^{ijkl} \xi_{ij} \xi_{lm} \geq 2 S^{-1}(S^{ij} \xi_{ij})^2.
\]

(2.10)

Proof. For the proof, see [2] the first two inequality and [27] for the last inequality. \( \square \)

3. Prior estimates

3.1. Gradient estimates.

Lemma 3.1. Let \( u(x,t) \in C^\infty(S^n \times [0,T]) \) be a strictly convex solution to the flow (1.7), then

\[
\max_{S^n} |D \log u| \leq C
\]

for \( k \alpha \leq 1 \).

Proof. Since \( D \log u = D \log \tilde{u} \), it is sufficient to estiamte \( D \log \tilde{u} \). Let \( \varphi = \log \tilde{u} \), then

\[
\frac{\partial \varphi}{\partial \tau}(x,\tau) = f(x)e^{k\alpha - 1} \sigma^1_k(\varphi_{ij} + \varphi_i \varphi_j + \delta_{ij}) = Q(x,\varphi, D\varphi, D^2 \varphi)
\]

Set \( \psi = \frac{|D\varphi|^2}{2} \). By differentiating the \( \psi \),we have

\[
\frac{\partial \psi}{\partial \tau} = \frac{\partial}{\partial \tau} D_m \varphi D^m \varphi = D_m \varphi D^m \varphi = D_m Q D^m \varphi.
\]

Then,

\[
\frac{\partial \psi}{\partial \tau} = Q^{ij} D_i D_j D_m \varphi D^m \varphi + Q^k D_k \varphi D^m \varphi + (k \alpha - 1)|D \varphi|^2 + D_m \log f D^m \varphi Q.
\]

Interchanging the covariant derivatives, we have

\[
D_{ij} \psi = D_j (D_{mi} \varphi D^m \varphi) = D_{mi} \varphi D^m \varphi + D_{mi} \varphi D^m \varphi = (D_{ijm} \varphi + R^l_{imj}) D^m \varphi + D_m D^m_D \varphi.
\]

Therefore, we can express \( D_{ijm} \varphi D^m \varphi \) as

\[
D_{ijm} \varphi D^m \varphi = D_{ij} \psi - R^l_{imj} D_l \varphi D^m \varphi - D_{mi} \varphi D^m \varphi.
\]
Then, in view of the fact $R_{(imj)} = \sigma_{im} \sigma_{ij} - \sigma_{ij} \sigma_{im}$ on $S^n$ we have

\begin{equation}
\frac{\partial \psi}{\partial \tau} = Q^{ij} D_i D_j \psi + Q^k D_k \psi - Q^{ij} (\sigma_{ij} |D\varphi|^2 - D_i \varphi D_j \varphi)
\end{equation}

\begin{equation*}
- Q^{ij} D_{mi} \varphi D_m^i \varphi + ((k\alpha - 1)|D\varphi| + C)Q|D\varphi|.
\end{equation*}

Since the matrix $Q^{ij}$ is positive definite, the forth and fifth terms in the right of (3.2) are non-positive. And noticing that the sixth term in the right of (3.2) is also non-positive if $k\alpha < 1$ and $|D\varphi| \geq \frac{C}{1-k\alpha}$. So we got the equation about $\psi$ as follows:

\begin{equation}
\frac{\partial \psi}{\partial \tau} \leq Q^{ij} D_i D_j \psi + Q^k D_k \psi \quad \text{in} \quad \Omega \times (0, \infty),
\end{equation}

\begin{equation}
\psi(x, 0) = \frac{|D\varphi(x, 0)|^2}{2} \quad \text{in} \quad \Omega.
\end{equation}

Using the maximum principle, we get the gradient estimates of $\varphi$. For $k\alpha = 1$, we have

\begin{equation}
\frac{\partial \varphi}{\partial \tau}(x, \tau) = f(x)\sigma_k^\alpha (\varphi_{ij} + \varphi_i \varphi_j + \delta_{ij}) = Q(x, D\varphi, D^2\varphi)
\end{equation}

Thus,

\begin{equation}
\frac{\partial}{\partial \tau} \left( \frac{\partial \varphi}{\partial \tau} \right) = Q^{ij} D_i D_j \left( \frac{\partial \varphi}{\partial \tau} \right) + Q^k D_k \left( \frac{\partial \varphi}{\partial \tau} \right).
\end{equation}

Therefore,

\begin{equation}
\frac{1}{C} \leq \frac{\partial \varphi}{\partial \tau} \leq C,
\end{equation}

which implies

\begin{equation}
\frac{1}{C} \leq \sigma_k \leq C.
\end{equation}

Assume that $\psi$ attains its maximum at $(x_0, t_0)$. Since $D\psi = 0$ at $x_0$, we can choose $\{e_1, \ldots, e_n\}$ at $x_0$ such that

\begin{equation}
e_1 = \frac{D\varphi}{|D\varphi|}
\end{equation}

and

\begin{equation}
a_{ij} = \varphi_{ij} + \varphi_i \varphi_j + \delta_{ij} = \text{diag}\{1 + \varphi_1^2, 1 + \varphi_2^2, \ldots, 1 + \varphi_n^2\}.
\end{equation}

We can have at $(x_0, t_0)$ from (3.2)

\begin{equation}
0 \leq -Q^{ij} (\sigma_{ij} |D\varphi|^2 - D_i \varphi D_j \varphi) + CQ|D\varphi|.
\end{equation}

Thus,

\begin{equation}
\sum_i \frac{\sigma_{ii}^k}{\sigma_k} - \frac{\sigma_{k1}^1}{\sigma_k} \leq \frac{1}{\alpha |D\varphi|}.
\end{equation}

Since $\frac{\sigma_{ii}^1}{\sigma_k} \leq \frac{1}{1 + |D\varphi|^2}$, thus

\begin{equation}
\sum_i \frac{\sigma_{ii}^l}{\sigma_k} \leq \frac{1}{\alpha |D\varphi|} + \frac{1}{1 + |D\varphi|^2}.
\end{equation}

Noticing that $\sum_i \frac{\sigma_{ii}^l}{\sigma_k} \geq C(n, k)\frac{1}{\sigma_k}$ by (2.3), we obtain

\begin{equation}
\psi \leq C.
\end{equation}

Therefore, we prove our claim.
3.2. $C^0$-estimates and $\phi(t)$ estimates.

**Lemma 3.2.** Let $u(x,t) \in C^\infty(S^n \times [0,T))$ be a strictly convex solution to the flow (1.7), then we have

\[
\frac{1}{C} \leq u(x,t) \leq C
\]

and

\[
|Du(x,t)| \leq C
\]

provided either (i) $p \geq k + 1$; or (ii) $f$ and $u_0$ are in addition even functions.

**Proof.** Since

\[
\int_{S^n} u^p(x,t) d\mu
\]

is constant along the normalized flow, we have by Lemma 2.4

\[
\int_{S^n} u^p(x,0) d\mu = \int_{S^n} u^p(x,t) d\mu \geq \int_{\{x \cdot x, t > 0\}} (x \cdot x_t u)^p(x_t, t) d\mu \geq \max_{S^n} u^p(\cdot, t)/C.
\]

where $x_t$ is a point at where $u(\cdot, t)$ attains its spatial maximum. This yields the second inequality in (3.3). By (2.3),

\[
\max_{S^n} |Du(\cdot, t)| \leq 2 \max_{S^n} u(\cdot, t).
\]

This yields the inequality in (3.4). Now we left the first inequality in (3.3) to prove.

Case (i): $p \geq k + 1$.

Clearly,

\[
\max_{S^n} u^p(\cdot, t) \geq \int_{S^n} u^p(x,t) d\mu = \int_{S^n} u^p(x,0) d\mu / C.
\]

By virtue of the Gradient estimate (3.1), we obtain

\[
\max_{S^n} \log u(x,t) - \min_{S^n} \log u(x,t) \leq C \max_{S^n} |D \log u(x,t)| \leq C.
\]

This together with (3.7) shows the positive lower bound of $u$.

Case (ii): $f$ and $u_0$ are even.

Here we use the idea in [6]. Assume $u(x,t)$ is not uniformly bounded away from 0. Since $f$ and $u_0$ are even, $u(x,t)$ is even. Thus, $K_t$ is origin-symmetric body, where $K_t$ is the convex body containing the origin and $\partial K_t = M_t$. Thus, $K_t$ converges to a convex body contained in a lower-dimensional subspace. This means that

\[
u(x,t) \to 0
\]

as $t \to \infty$ almost everywhere with respect to the spherical Lebesgue measure. Combined with bounded convergence theorem, we conclude

\[
\int_{S^n} u^p(x,t) d\mu \to 0
\]

as $t \to \infty$ which is a contraction to Lemma 2.2. \qed
Lemma 3.3. Let \( u(x,t) \in C^\infty(S^n \times [0,T)) \) be a strictly convex solution to the flow (1.7), if we have
\[
1/C \leq u(x,t) \leq C,
\]
then we obtain
\[
1/C \leq \phi(t) \leq C.
\] (3.8)

Proof. Since
\[
C^{-1} \min_{S^n} u^{p+k\alpha-1}(\cdot,t) \leq \int_{S^n} u(x,s)^{p-1} \sigma_k^\alpha (u_{ij} + u\delta_{ij}) f d\mu \leq C \max_{S^n} u^{p+k\alpha-1}(\cdot,t),
\]
thus if we have lower and upper estimates for \( u(x,t) \), then we have lower and upper bounds on \( \phi(t) \). \( \square \)

3.3. \( \sigma_k \) estimates.

Set
\[
F = \phi(t)f(x)\sigma_k^\alpha
\]
and
\[
F^{ij} = \alpha \phi(t)f(x)\sigma_k^{\alpha-1}\sigma_k^{ij}
\]
Clearly,
\[
F^{ij}b_{ij} = k\alpha F.
\]

Lemma 3.4. Under the normalized flow (1.7), we have the following evolution equations
\[
\frac{\partial}{\partial t}(f(x)\sigma_k^\alpha) - F^{ij}(f(x)\sigma_k^\alpha)_{ij} = F^{ij}\delta_{ij}(f(x)\sigma_k^\alpha) - k\alpha F, \tag{3.9}
\]
\[
\frac{\partial}{\partial t}u - F^{ij}u_{ij} = (1-k\alpha)F - u + uF^{ij}\delta_{ij}. \tag{3.10}
\]
\[
\frac{\partial}{\partial t} \frac{1}{2}u^2 - F^{lm}(\frac{1}{2}r^2)_{lm} = (k\alpha+1)uF - F^{lm}b_{il}b_{lm} + F(D\log f, Du) - r^2. \tag{3.11}
\]

Proof. First, by virtue of (1.7), we have
\[
\frac{\partial}{\partial t}(f(x)\sigma_k^\alpha) = f(x)\frac{\partial}{\partial t}(\sigma_k^\alpha) = \alpha f(x)\sigma_k^{\alpha-1}\sigma_k^{ij}(u_{ij} + u\delta_{ij}) = F^{ij}(f(x)\sigma_k^\alpha)_{ij} + F^{ij}\delta_{ij}(f(x)\sigma_k^\alpha) - k\alpha \phi(t)(f(x)\sigma_k^\alpha),
\]
which verifies (3.3). Secondly, using (1.7) again, we can get (3.10) through the following computations
\[
\frac{\partial}{\partial t}u - F^{ij}u_{ij} = F - u - F^{ij}(b_{ij} - u\delta_{ij}) = F - u - k\alpha F + uF^{ij}\delta_{ij} = (1-k\alpha)F - u + uF^{ij}\delta_{ij}.
\]
Rewritten the equation above as follows,
\[
\frac{\partial}{\partial t}(\frac{1}{2}u^2) - F^{ij}(\frac{1}{2}u^2)_{ij} = uu_t - F^{ij}(u_{ij}u + u_iu_j)
\]
Moreover, we have by (1.7) and the Ricci identity (2.2)

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |Du|^2 \right) - F^{lm} \left( \frac{1}{2} |Du|^2 \right)_{lm} = u_i u_{it} - F^{lm} (u_{ilm} u_i + u_{ti} u_{lm}) \\
= u_i F^{lm} (u_{imi} + u_i \delta_{im}) - F^{lm} (u_{ilm} u_i + u_{il} u_{im}) + F \langle D \log f, Du \rangle - |Du|^2 \\
= u_i F^{lm} (u_{imi} - u_{ilm}) + F^{lm} \delta_{im} |Du|^2 - F^{lm} u_{il} u_{im} + F \langle D \log f, Du \rangle - |Du|^2 \\
= F^{lm} (u_{il} u_{im} - \delta_{im} |Du|^2) + F^{lm} \delta_{im} |Du|^2 - F^{lm} u_{il} u_{im} + F \langle D \log f, Du \rangle - |Du|^2 \\
= F^{lm} u_{il} u_{im} - F^{lm} u_{im} u_{il} + F \langle D \log f, Du \rangle - |Du|^2 \\
= F^{lm} u_{il} u_{im} + 2u F^{lm} b_{il} - u^2 F^{lm} \delta_{im} - F^{lm} b_{il} b_{im} + F \langle D \log f, Du \rangle - |Du|^2 \\
= F^{lm} u_{il} u_{im} + 2k \alpha u F - u^2 F^{lm} \delta_{im} - F^{lm} b_{il} b_{im} + F \langle D \log f, Du \rangle - |Du|^2.
\]

in view of

\[
F^{lm} b_{il} b_{im} = F^{lm} (u_{il} + u \delta_{il})(u_{im} + u \delta_{im}) \\
= F^{lm} u_{il} u_{im} + F^{lm} \delta_{im} u^2 + 2u F^{lm} u_{lm} \\
= F^{lm} u_{il} u_{im} + 2u F^{lm} b_{lm} - u^2 F^{lm} \delta_{lm}.
\]

Thus,

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} |Du|^2 \right) - F^{lm} \left( \frac{1}{2} |Du|^2 \right)_{lm} = (k \alpha + 1)u F - F^{lm} b_{il} b_{im} + F \langle D \log f, Du \rangle - r^2.
\]

\[\blacksquare\]

Lemma 3.5. Let \( u(x,t) \in C^\infty(S^n \times [0,T)) \) be a strictly convex solution to the flow (1.7), if we have

\[ \frac{1}{C} \leq u(x,t) \leq C, \]

\[ |Du| \leq C, \]

and

\[ \frac{1}{C} \leq \phi(t) \leq C, \]

then we obtain

\[ \sigma_k (u_{ij} + u \delta_{ij}) \leq C. \]

Proof. We consider

\[ P(x,t) = \log (f(x) \sigma_k^2) - \log u - \log (1 - \frac{r^2}{2}), \]

where \( \frac{1}{t} > \max_{S^n} \frac{r^2}{2} \). Fixed an arbitrary \( T' \in (0,T) \) and assume that \( P \) attains its maximum on \( \Sigma^n \times [0,T] \) at \( (x_0, t_0) \) with \( t_0 > 0 \) (otherwise we are done). Thus, we have at \( (x_0, t_0) \)

\[ D P(x,t) = D \log (f(x) \sigma_k^2) - D \log u - D \log (1 - \frac{r^2}{2}) = 0. \]

(3.12)
We have by the evolution equations (3.9), (3.10) and (3.11)

\[
\frac{\partial}{\partial t}\log(f(x)\sigma_k^\alpha) - F^{ij}(\log(f(x)\sigma_k^\alpha))_{ij} \\
\leq F^{ij}(\log(f(x)\sigma_k^\alpha))_i(\log(f(x)\sigma_k^\alpha))_j + F^{ij}\delta_{ij},
\]

\[
\frac{\partial}{\partial t}\log(1 - \epsilon r^2) - F^{im}(\log(1 - \epsilon r^2))_{ij} \\
\geq F^{ij}(\log(1 - \epsilon r^2))_i(\log(1 - \epsilon r^2))_j - (k\alpha + 1)\frac{\epsilon uF}{1 - \epsilon r^2} \\
+ \frac{\epsilon}{1 - \epsilon r^2}F^{lm}b_{il}b_{im} - \frac{\epsilon F}{1 - \epsilon r^2}\langle D\log f, Du \rangle.
\]

Combing the three equations above, we can obtain at the point \((x_0, t_0)\)

\[
\frac{\partial}{\partial t}P - F^{ij}P_{ij} \leq 2F^{ij}\log(1 - \epsilon r^2)_i(\log u)_j \\
- (1 - k\alpha)\frac{F}{u} + 1 + [(k\alpha + 1)u + k\alpha]\frac{\epsilon uF}{1 - \epsilon r^2} \\
- \frac{\epsilon}{1 - \epsilon r^2} F^{lm}b_{il}b_{im} + \frac{\epsilon F}{1 - \epsilon r^2}\langle D\log f, Du \rangle.
\]

(3.13)

Noticing that

\[
F^{ij}\log(1 - \epsilon r^2)_i(\log u)_j = -F^{ij}\frac{\epsilon r_i u_j}{1 - \epsilon r^2}u = -F^{ij}\frac{\epsilon r_i \sum b_{il}u_l u_j}{1 - \epsilon r^2}u \leq 0,
\]

here we use that \(r_i = \sum b_{il}u_l\). Thus, we arrive at \((x_0, t_0)\) from (3.13).

By (2.9), we have

\[
\sigma_k^{ij}b_{il}b_{il} \geq k\sigma_k^{1+\frac{1}{k}}.
\]

Thus, at the point \((x_0, t_0)\)

\[
0 \leq \frac{\partial}{\partial t}P - F^{ij}P_{ij} \leq C_1 F - C_2 F^{lm}b_{il}b_{im}.
\]

By (2.9), we have

\[
\sigma_k^{ij}b_{il}b_{il} \geq k\sigma_k^{1+\frac{1}{k}}.
\]

Thus, at the point \((x_0, t_0)\)

\[
0 \leq \frac{\partial}{\partial t}P - F^{ij}P_{ij} \leq C_1 F - C_2 F\sigma_k^{\frac{1}{k}}.
\]

So, we obtain at the point \((x_0, t_0)\)

\[
\sigma_k \leq C,
\]

which implies our result.
Lemma 3.6. Let \( u(x,t) \in C^\infty(S^n \times [0,T]) \) be a strictly convex solution to the flow (1.7), if we have
\[
\frac{1}{C} \leq u(x,t) \leq C, \\
|Du| \leq C,
\]
and
\[
\frac{1}{C} \leq \phi(t) \leq C,
\]
then we obtain
\[
C \leq \sigma_k(u_{ij} + u\delta_{ij}).
\]

Proof. We consider
\[
P(x,t) = \log(f(x)\sigma_k^\alpha) - \frac{A}{2}r^2,
\]
where \( A \) will be choose large later. We have by (3.9) and (3.11)
\[
\frac{\partial}{\partial t} \log(f(x)\sigma_k^\alpha) - F^{ij}(\log(f(x)\sigma_k^\alpha))_{ij} \geq -k\alpha\phi(t)
\]
and
\[
\frac{\partial}{\partial t} r^2 - F^{lm}(r^2)_{lm} \leq (k\alpha + 1)uF + F(D\log f, Du) - r^2.
\]
Therefore, we have by choosing \( A \geq \max \frac{k\alpha\phi(t)}{r^2} \)
\[
\frac{\partial}{\partial t} P - F^{ij}P_{ij} \geq -A(k\alpha + 1)uF - AF(D\log f, Du) + Ar^2 - k\alpha\phi(t)
\]
\[
\geq A(C_1 - C_2F).
\]
So, we conclude our solution by the Maximum principle. \( \square \)

3.4. \( C^2 \)-estimates.

Lemma 3.7. Let \( X(x,t) \in C^\infty(S^n \times [0,T]) \) be a strictly convex solution to the flow (1.5), if we have
\[
\frac{1}{C} \leq u(x,t) \leq C, \\
\frac{1}{C} \leq \phi(t) \leq C
\]
and
\[
C \leq \sigma_k(u_{ij} + u\delta_{ij}),
\]
then the principle curvatures of \( \mathcal{M}_t = X(\cdot, t) \) satisfy
\[
\kappa_i \leq C.
\]
Proof. We consider the function

\[ W(x, t) = \log \Lambda(x, t) - \log u(x, t), \]

where \( A \) is a large constant to be determined later and

\[ \Lambda(x, t) = \max \{ \eta^{ij} \xi_i \xi_j : |\xi| = 1 \}. \]

Fixed an arbitrary \( T' \in (0, T) \) and assume that \( W \) attains its maximum on \( S^0 \times [0, T] \) at \((x_0, t_0)\) with \( t_0 > 0 \) (otherwise we are done). Choose Riemannian normal coordinates at \((\xi_0, t_0)\) such that at this point we have

\[ \sigma_{ij} = \delta_{ij}, \quad h_{ij} = \kappa_i \delta_{ij}, \quad \kappa_i \leq \ldots \leq \kappa_2 \leq \kappa_1. \]

Let

\[ w(x, t) = \log h^{11}(x, t) - \log u(x, t). \]

Then, \( w \) attains its maximum at \((x_0, t_0)\). So we obtain at \((x_0, t_0)\)

\begin{equation}
0 \leq \frac{\partial}{\partial t} w(x, t) = b_{11} \frac{\partial}{\partial t} h^{11}(x, t) - \frac{u_t}{u} = -h^{11} b_{11;1} - \frac{u_t}{u},
\end{equation}

(3.14)

\begin{equation}
0 = D_i w(x, t) = b_{11} D_i h^{11} - \frac{u_i}{u} = -h^{11} b_{11;i} - \frac{u_i}{u}.
\end{equation}

(3.15)

and

\begin{equation}
0 \geq D^2_{ij} w(x, t) = \frac{1}{h^{11}} \frac{1}{h^{11};j} - \frac{1}{(h^{11})^2} h^{11};h^{11} j - \frac{u_{ij}}{u} + \frac{u_i u_j}{u^2}
\end{equation}

\begin{equation}
= -h^{11} b_{11;ij} + 2h^{11} \sum_l h^{li} b_{il;1} b_{jl;1} - (h^{11})^2 b_{11;i} b_{11;j} - \frac{u_{ij}}{u} + \frac{u_i u_j}{u^2}
\end{equation}

(3.16)

here we use the equation (3.15) to get the last equality. Rewriting the evolution equation (1.7) as

\[ \log(u_t + u) = \alpha \log \sigma_k + \log f + \log \phi(t). \]

Differentiating with \( x^1 \) gives

\[ \frac{u_{t1} + u_1}{u_t + u} = \frac{\alpha}{\sigma_k} \sigma^{ij}_k b_{ij;1} + (\log f)_1. \]

and differentiating again with \( x^1 \) shows

\[ \frac{u_{t11} + u_{11}}{u_t + u} = \frac{(u_{t1} + u_1)^2}{(u_t + u)^2} + \frac{\alpha}{\sigma_k} (\sigma^{ij}_k b_{ij;1} + (\sigma^{ij}_{k} b_{ij;1}) + \frac{\alpha}{(\sigma_k)^2} (\sigma^{ij}_k b_{ij;1})^2 + (\log f)_1. \]

which implies together with (3.14)

\[ 0 \leq -h^{11} \left( \frac{u_{t11} + u_{11}}{u_t + u} + \frac{-u_{11} - u}{u_t + u} + \frac{u + u_t}{u_t + u} \right) - \frac{u_t}{u(u_t + u)} \]

\[ = -h^{11} \frac{u_{t11} + u_{11}}{u_t + u} + \frac{2}{u_t + u} - h^{11} \frac{1}{u}, \]
Thus,

\[
0 \leq -\alpha h_k^{11} \frac{(\sigma_k^{ij} b_{ij;11} + \sigma_k^{ij;lm} b_{ij;1b_{lm;1}})}{\sigma_k^2} + \frac{\alpha h_k^{11}}{\sigma_k} (\sigma_k^{ij} b_{ij;1})^2 - \frac{h_k^{11}}{\sigma_k} \frac{(u_t + u)^2}{(u_t + u)^2} - h_k^{11} (\log f)_{11} - h_k^{11} \frac{1}{u} + \frac{2}{u_t + u}.
\]

(3.17)

According to the Ricci identity, we have

\[ b_{ij;11} = b_{11;ij} - \delta_{ij} b_{11} - \delta_{ij} b_{11} + \delta_{11} b_{ij} \]

Plugging the identity above into (3.17) and employing (3.16), we can obtain

\[
0 \leq -\frac{\alpha h_k^{11}}{\sigma_k} \left( \sigma_k^{ij} (b_{11;ij} - \delta_{ij} b_{11} + b_{ij}) + \sigma_k^{ij;lm} b_{ij;1b_{lm;1}} \right)
+ \frac{\alpha h_k^{11}}{\sigma_k^2} (\sigma_k^{ij} b_{ij;1})^2 - \frac{h_k^{11}}{\sigma_k} \frac{(u_t + u)^2}{(u_t + u)^2} - h_k^{11} (\log f)_{11} - h_k^{11} \frac{1}{u} + \frac{2}{u_t + u}
\]

\[
\leq \frac{\alpha h_k^{11}}{\sigma_k} \left( -2\sigma_k^{ij} \sum_l h_k^{ll} b_{il;1} b_{jl;1} - \sigma_k^{ij;lm} b_{ij;1b_{lm;1}} + \frac{1}{\sigma_k} (\sigma_k^{ij} b_{ij;1})^2 \right)
+ \frac{\alpha h_k^{11}}{\sigma_k} \sigma_k^{ij} (\delta_{ij} b_{11} - b_{ij}) + \frac{\alpha}{\sigma_k} \frac{\sigma_k^{ij} b_{ij}}{u} - \frac{\delta_{ij}}{\sigma_k}
- \frac{h_k^{11}}{\sigma_k} \frac{(u_t + u)^2}{(u_t + u)^2} - h_k^{11} (\log f)_{11} - h_k^{11} \frac{1}{u} + \frac{2}{u_t + u}
\]

\[
\leq \frac{\alpha h_k^{11}}{\sigma_k} \left( -2\sigma_k^{ij} \sum_l h_k^{ll} b_{il;1} b_{jl;1} - \sigma_k^{ij;lm} b_{ij;1b_{lm;1}} + \frac{1}{\sigma_k} (\sigma_k^{ij} b_{ij;1})^2 \right)
- \frac{h_k^{11}}{\sigma_k} \frac{(u_t + u)^2}{(u_t + u)^2} - h_k^{11} (\log f)_{11} - (k\alpha + 1)h_k^{11} + \frac{k\alpha - 1}{u} + \frac{2}{u_t + u}.
\]

To proceed, we need to use the following inequality which clearly is implied by (2.10)

\[ 2\sigma_k^{ij} h_k^{ll} b_{il;1} b_{jl;1} + \frac{\sigma_k^{ij;kl} b_{ij;1b_{kl;1}}}{\sigma_k} \geq \frac{k + 1}{k} \frac{(\sigma_k^{ij} b_{ij;1})^2}{\sigma_k}. \]

Thus,

\[
0 \leq -\frac{(k\alpha + 1)\alpha}{k} \frac{h_k^{11}}{\sigma_k} (\sigma_k^{ij} b_{ij;1})^2 - \frac{2\alpha h_k^{11}}{\sigma_k} \sigma_k^{ij} b_{ij;1} (\log f)_1
- h_k^{11} \frac{(\log f)_{11}}{2} - h_k^{11} (\log f)_{11} - (k\alpha + 1) h_k^{11} + C.
\]

here we use the following Cauchy inequality to get the last inequality

\[
\frac{(k\alpha + 1)\alpha}{k} \frac{1}{(\sigma_k^2)} (\sigma_k^{ij} b_{ij;1})^2 + 2\frac{\alpha}{\sigma_k} \sigma_k^{ij} b_{ij;1} (\log f)_1 + \frac{k\alpha}{(k\alpha + 1)} ((\log f)_1)^2 \geq 0.
\]

Thus,

\[
0 \leq -\frac{h_k^{11}}{k\alpha + 1} ((\log f)_1)^2 - h_k^{11} (\log f)_{11} - (k\alpha + 1) h_k^{11} + C.
\]

Since

\[
(\hat{f}^{\frac{1}{p+1-k}})_{ij} + \hat{f}^{\frac{1}{p+1-k}} \delta_{ij}
\]

is positive definite, we have

\[
(\hat{f}^{\frac{1}{p+1-k}})_{11} + \hat{f}^{\frac{1}{p+1-k}} > 0.
\]
By a simple computation,

\[ \hat{f}(\hat{f})_{11} - \frac{p + k - 2}{p + k - 1}((\hat{f})_{1})^2 + (p + k - 1)\hat{f}^2 > 0. \]

So,

\[ (\log \hat{f})_{11} + \frac{1}{p + k - 1}((\log \hat{f})_{1})^2 + p + k - 1 > 0. \]

Noticing that \( \hat{f} = f^{1/\alpha} \), then

\[ (\log f)_{11} + \frac{p - 1}{p + k - 1}((\log f)_{1})^2 + k\alpha + 1 > 0. \]

Thus,

\[ h^{11} \leq C, \]

which completes our proof.

4. The convergence of the normalised flow

4.1. The first two cases.

With the help of the prior estimates in the section above, we show the long-time existence and asymptotic behaviour of the normalised flow (1.5). The parts (i) and (ii) in Theorem 1.1 is a consequence of the following Proposition 4.1.

Proposition 4.1. Assume that \( \hat{f} \in C^\infty(S^n) \) is a positive function such that

\[ (\hat{f})_{\frac{1}{p+k-1}} + \hat{f}^{\frac{1}{p+k-1}}\delta_{ij} \]

is positive definite and \( u_0 \in C^\infty(S^n) \) is positive and strictly convex. If either (i) \( k\alpha \leq 1 \), or (ii) \( f \) and \( u_0 \) are even functions, then there is a unique positive, smooth and strictly convex solution \( u \) to (1.7), which exists for any time \( t \in [0, \infty) \), and \( u(x, t) \) converges for a sequence of times to a positive, smooth and strictly convex solution of the equation (1.8) with \( \hat{f} \) replaced by \( \lambda_0 \hat{f} \) for some \( \lambda_0 > 0 \).

Proof. Since the equation (1.7) is parabolic, we have the short time existence. Let \( T \) be the maximal time such that \( u(\cdot, t) \) is a positive, smooth and strictly convex solution to (1.7) for all \( t \in [0, T) \).

Lemmas 3.2, 3.3 and 3.6 enable us to apply Lemma 3.7 to the equation (1.7) and thus we can deduce a uniformly lower estimate for the smallest eigenvalue of \( \{u_{ij} + u\delta_{ij}\}(x, t) \}. This together with Lemma 3.5 implies

\[ C^{-1}I \leq (u_{ij} + u\delta_{ij})(x, t) \leq CI, \quad \forall (x, t) \in S^n \times [0, T), \]

where \( C > 0 \) depends only on \( n, \alpha, f \) and \( u_0 \). This shows that the equation (1.7) is uniformly parabolic. Using Evans-Krylov estiamte and Schauder estimate, we obtain

\[ |u|_{C^{1,m}([0,T])} \leq Ct,m \]

for some \( C_{l,m} \) independent of \( T \). Hence \( T = \infty \). The uniqueness of the smooth solution \( u(\cdot, t) \) follows by the parabolic comparison principle.

By the monotonicity of \( J_{k,p} \) (See Lemma 2.3), and noticing that

\[ |J_{k,p}(\cdot, t)| \leq C, \quad \forall t \in [0, \infty), \]

we conclude that

\[ \int_0^\infty |J_{k,p}(X(\cdot,t))| \leq C. \]

Hence, there is a sequence \( t_i \to \infty \) such that

\[ \frac{d}{dt} J_{k,p}(X(\cdot,t_i)) \to 0. \]

In view of Lemma 2.3, we see that \( u(\cdot,t_i) \) converges smoothly to a positive, smooth and strictly convex \( u_\infty \) solving (1.8) with \( f \) replaced by \( \lambda_0 \hat{f} \) with \( \lambda_0 = \lim_{t_i \to 0} (\phi(t_i))^{\frac{1}{\alpha}}. \)

\[ \blacksquare \]

4.2. The remain case \( f \equiv 1. \)

Now, we only leave the part (iii) in Theorem 1.1 to prove.

**Lemma 4.2.** Assume \( \alpha > 0, f \equiv 1 \) and \( \tilde{u}_0 \in C^\infty(S^n) \) is positive and strictly convex. Let \( \tilde{u}(\cdot,\tau) \) be a positive, smooth and strictly convex solution to \( (1.6) \), and let \( \tilde{T} \in (0,\infty) \) be the time such that \( \max_{S^n} \tilde{u}(\cdot,\tau) < \infty \) for all \( \tau < \tilde{T} \), while

\[ \lim_{\tau \to \tilde{T}} \max_{S^n} \tilde{u}(\cdot,\tau) = \infty. \]

Then, there is a constant \( C \) depending only on \( n,k,\alpha \) and \( u_0 \) such that

\[ \lim_{\tau \to \tilde{T}} \max_{S^n} \tilde{u}(\cdot,\tau) = 1 \]

and

\[ \sup_{\tau \in [0,\tilde{T}]} \min_{S^n} \tilde{u}(\cdot,\tau) \leq C. \]

**Proof.** Let

\[ \Theta(t) = \begin{cases} 
[(1 - k\alpha)\tau + a^{1-k\alpha}]^{\frac{1}{1-k\alpha}}, & \text{if } k\alpha < 1; \\
ae^\tau, & \text{if } k\alpha = 1; \\
[a^{-(k\alpha-1)} - (k\alpha - 1)\tau]^{-\frac{1}{k\alpha-1}}, & \text{if } k\alpha > 1; 
\end{cases} \]

where \( \min_{S^n} u_0 \leq a \leq \max_{S^n} u_0 \) for \( k\alpha \leq 1 \) and \( a = [(k\alpha - 1)\tilde{T}]^{-\frac{1}{k\alpha-1}} \) for \( k\alpha > 1 \). From (15), we know that for any \( (x,\tau) \in S^n \times [0,\tilde{T}) \)

\[ \Theta(t) - C \leq \tilde{u}(x,\tau) \leq \Theta(t) + C. \]

Thus,

\[ 1 \leq \lim_{\tau \to \tilde{T}} \max_{S^n} \tilde{u}(\cdot,\tau) \leq \lim_{\tau \to \tilde{T}} \Theta(t) + C = 1. \]

\[ \blacksquare \]

The part (iii) in Theorem 1.1 is a consequence of the following Proposition 4.3.

**Proposition 4.3.** Assume \( \alpha > 0, f \equiv 1 \) and \( u_0 \in C^\infty(S^n) \) is positive and strictly convex. Then there is a unique positive, smooth and strictly convex solution to (1.7) for all the time \( t \geq 0 \).

When \( t \to \infty, u(\cdot,t) \) converges smoothly to the constant \( \lambda_0 = (\frac{1}{|S^n|} \int_{S^n} u_0^p dx)^{\frac{1}{p}} \), where \( p = 1 + \frac{1}{\alpha} \) as in (14).
Proof. Since the equation (1.7) is parabolic, we have the short time existence. Let $T$ be the maximum time such that $u(\cdot,t)$ is a positive, smooth and strictly convex solution to (1.7). By (3.5) and (3.6), we obtain
\[
\max_{\mathbb{S}^n} (u + |Du|) \leq C.
\]
Noticing that $\tilde{u}(\cdot,\tau)$ and $u(\cdot,t(\tau))$ only differ a multiplier, we deduce
\[
\max_{\mathbb{S}^n} u(\cdot,t) \leq C \min_{\mathbb{S}^n} u(\cdot,t), \quad \forall t \in [0,T).
\]
This together with the fact $\int_{\mathbb{S}^n} u(x,t)dx = \int_{\mathbb{S}^n} u_0(x)dx$ implies that
\[
\min_{\mathbb{S}^n} u(\cdot,t) \geq C^{-1}, \quad \forall t \in [0,T).
\]
Using Lemmas 3.3, 3.6, 3.5, and 3.7 we can have
\[
C^{-1} \leq \phi(t) \leq C, \quad \forall t \in [0,T),
\]
and
\[
C^{-1} I \leq (u_{ij} + u\delta_{ij})(x,t) \leq CI, \quad \forall (x,t) \in \mathbb{S}^n \times [0,T),
\]
where $C > 0$ depends only on $n, \alpha, f$ and $u_0$. This shows that the equation (1.7) is uniformly parabolic. Using Evans-Krylov estimate and Schauder estimate, we obtain
\[
|u|_{C^{l,m}(\mathbb{S}^n \times [0,T])} \leq C_{l,m}
\]
for some $C_{l,m}$ independent of $T$. Hence $T = \infty$. By (1.1),
\[
\max_{\mathbb{S}^n} |u(x,t) - \lambda_0| \to 0, \quad \text{as} \quad t \to \infty
\]
for some constant $\lambda_0$. Since $\int_{\mathbb{S}^n} u(x,t)dx = \int_{\mathbb{S}^n} u_0(x)dx$, we necessarily have $\lambda_0 = (\frac{1}{|\mathbb{S}^n|} \int_{\mathbb{S}^n} u_0^p dx)^{\frac{1}{p}}$, which completes the proof. \qed

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