AN ESTIMATE OF THE ROOT MEAN SQUARE ERROR INCURRED WHEN APPROXIMATING AN \( f \in L^2(\mathbb{R}) \) BY A PARTIAL SUM OF ITS HERMITE SERIES

MEI LING HUANG, RON KERMAN, SUSANNA SPEKTOR

Abstract. Let \( f \) be a band-limited function in \( L^2(\mathbb{R}) \). Fix \( T > 0 \) and suppose \( f' \) exists and is integrable on \([-T,T]\). This paper gives a concrete estimate of the error incurred when approximating \( f \) in the root mean square by a partial sum of its Hermite series.

Specifically, we show, for \( K = 2n \), \( n \in \mathbb{Z}_+ \),

\[
\left[ \frac{1}{2T} \int_{-T}^{T} |f(t) - (S_K f)(t)|^2 dt \right]^{1/2} \leq \left( 1 + \frac{1}{K} \right) \left[ \frac{1}{2T} \int_{|t|>T} f(t)^2 dt \right]^{1/2} + \frac{1}{2T} \int_{|\omega|>N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} \right)
\]

in which \( S_K f \) is the \( K \)-th partial sum of the Hermite series of \( f \), \( \hat{f} \) is the Fourier transform of \( f \), \( N = \frac{\sqrt{2K+1} + \sqrt{2K+3}}{2} \) and \( f_N = (\hat{f} \chi_{(-N,N)})^\wedge(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(N(t-s))}{t-s} f(s) ds \). An explicit upper bound is obtained for \( S_a(K,T) \).

1. Introduction

We recall that the \( k \)-th Hermite function, \( h_k \), is given at \( t \in \mathbb{R} \) by

\[ h_k(t) = (-1)^k \gamma_k e^{\frac{1}{2}t^2} \frac{d^k e^{-t^2}}{dt^k}, \quad k = 0, 1, \ldots, \]

where \( \gamma_k = \pi^{-1/2} 2^{-k/2} (k!)^{-1/2} \). Given \( f \in L^2(\mathbb{R}) \), its Hermite series is

\[ \sum_{k=0}^{\infty} c_k h_k, \]

in which

\[ c_k = \int_{\mathbb{R}} f(t) h_k(t) dt, \quad k = 0, 1, 2, \ldots. \]

We seek an estimate in the root mean square of how well the \( K \)-th partial sum of the Hermite series of \( f \), namely,

\[ (S_K f)(t) = \sum_{k=0}^{K} c_k h_k(t), \]

approximates it. Our principal result is

Theorem 1. Consider a band-limited function \( f \in L^2(\mathbb{R}) \). Fix \( T > 0 \) and suppose \( f' \) exists and is integrable on \( I_T = [-T,T] \).

Then, with \( K = 2n \), \( n \in \mathbb{Z}_+ \), and \( S_K f \) the \( K \)-th partial sum of the Hermite series of \( f \), one has

\[ N = \frac{\sqrt{2K+1} + \sqrt{2K+3}}{2} \]

and \( S_K f \) the \( K \)-th partial sum of the Hermite series of \( f \), one has

\[ \left( 1 + \frac{1}{K} \right) \left[ \frac{1}{2T} \int_{|t|>T} f(t)^2 dt \right]^{1/2} \right)
\]

in which \( f_N = (\hat{f} \chi_{(-N,N)})^\wedge \).
The goal now is to find, for an appropriate $T$, the smallest $K$ to ensure the right hand side of \([1]\) satisfies a given bound. To do this we need explicit bounds for $S_K(K,T)$. We will make careful use of the estimates of the kernel of an integral representation of $S_Kf$ due to Sansone; see [S]. These estimates show that the core of the partial sum operator is the Dirichlet operator, $F_N$, defined at $f$ on $I_T$ by

$$
(F_Nf)(t) = \frac{1}{\pi} \int_{-T}^{T} \sin(N(t-s)) f(s) ds, \quad t \in I_T.
$$

A key fact, used repeatedly in the derivation at our estimates, is that the Hilbert transform, $H$, given, for suitable $f$ at almost all $x \in \mathbb{R}$ by

$$(Hf)(x) = \frac{1}{\pi} (P) \int_{\mathbb{R}} \frac{f(y)}{x-y} dy = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|x-y| > \epsilon} \frac{f(y)}{x-y} dy,$$

is a unitary operator on $L^2(\mathbb{R})$. Also important will be certain identities of Bedrosian, valid for band-limited functions $f \in L^2(\mathbb{R})$, namely,

$$H(f \sin(a \cdot))(t) = f(t) \cos at$$

and

$$H(f \cos(b \cdot))(t) = f(t) \sin bt, \quad t \in \mathbb{R},$$

for fixed $a, b \in \mathbb{R}$. See [B].

The error involved in approximating $f$ by $F_N f$ is established in Lemma 2 in the next section. The Sansone estimates are intensively studied in Section 3. These enable the proof of Theorem 1 in the following section, where $S_K(n,T)$ is defined. An explicit estimate of $S_K(n,T)$ is described in an appendix.

The estimate of the root mean square error in \([1]\) is both more specific and more easily calculated than the one in the paper [KHB] of the first two authors and M. Brannan. In the final section we revisit the trimodal distribution studied in that paper.

2. Approximation using the Dirichlet operator

In this section we prove

**Lemma 2.** Given $f \in L^2(\mathbb{R})$, $N, T \in \mathbb{R}_+$, and $f_T = f \chi_{(-T,T)}$, set

$$(F_N f_T)(t) = \frac{1}{\pi} \int_{-T}^{T} \frac{\sin(N(t-s))}{t-s} f(s) ds, \quad t \in I_T = [-T,T].$$

Then,

$$\left[ \int_{-1}^{1} |f(t) - (F_N f_T)(t)|^2 dt \right]^{1/2} \leq \left[ \int_{|t| > T} |f(t)|^2 dt \right]^{1/2} + \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} \tag{1}$$

**Proof.** Using the standard notation $\text{sinc} t = \sin t / t$, we have

$$F_N f_T = \sqrt{\frac{2}{\pi}} N(\text{sinc} t) * f_T \tag{2} \quad t \in \mathbb{R},$$

whence

$$\widehat{F_N f_T} = \chi_{(-N,N)} \hat{f}_T,$$

\[\text{(1)}\] We define the Fourier transform, $\hat{f}$, of $f$ by

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t} dt, \quad \omega \in \mathbb{R}.$$

\[\text{(2)}\] The convolution, $g * h$, of $g$ and $h$ in $L^2(\mathbb{R})$ is here defined by $(g * h)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(t-s) h(s) ds, \quad t \in \mathbb{R}$. One has $(g \ast h)(\omega) = \hat{g}(\omega) \hat{h}(\omega), \omega \in \mathbb{R}$. See [W].
Thus,
\[
\left[ \int_{-T}^{T} |f(t) - (F_N f_T)(t)|^2 dt \right]^{1/2} \leq \left[ \int_{\mathbb{R}} |f(t) - (F_N f_T)(t)|^2 dt \right]^{1/2} = \left[ \int_{\mathbb{R}_+} |\hat{f}(\omega) - \hat{F_N f_T}(\omega)|^2 d\omega \right]^{1/2}
\]
\[
\leq \left[ \int_{|\omega|>N} \left| \frac{1}{2\pi} \int_{\mathbb{R}} \chi_T(y) f(y) e^{-i\omega y} dy \right|^2 d\omega \right]^{1/2} = \int_{|\omega|>N} \left( \chi_T f(\omega) \right)^2 d\omega \right]^{1/2}
\]
\[
\leq \left[ \int_{|\omega|>N} \left| f(\omega) \right|^2 d\omega \right]^{1/2} + \left[ \int_{|\omega|>N} \left| \chi_T f(\omega) \right|^2 d\omega \right]^{1/2} \leq \left[ \int_{|\omega|>N} \left| f(\omega) \right|^2 d\omega \right]^{1/2} + \left[ \int_{|\omega|>N} \left| \chi_T f(\omega) \right|^2 d\omega \right]^{1/2}
\]
\[
\leq \left[ \int_{|\omega|>N} \left| f(\omega) \right|^2 d\omega \right]^{1/2} + \left[ \int_{|t|>T} \left| f(t) \right|^2 dt \right]^{1/2}
\]

\[\square\]

3. The Sansone estimates

To begin, we describe Sansone’s analysis of the usual expression for \((\sum_K f_T)(t) = \sum_{k=0}^{K} C_k h_k(t)\), when \(K\) is even, say \(K = 2n\). For ease of reference to [S] we work with the variables \(x\) and \(\alpha\) rather than \(t\) and \(s\).

Now, according to [S, p. 372, (4) and (5)],

\[(S_{2n}, f)(x) = \sqrt{\frac{2n+1}{2}} \int_{-T}^{T} k_{2n}(x, \alpha) f(\alpha) d\alpha,\]

where

\[k_{2n}(x, \alpha) = \frac{h_{2n+1}(x) h_2(\alpha) - h_{2n+1}(\alpha) h_2(x)}{x - \alpha}\]

and \(f_T = f_T \chi_T\). Further, by (7) and (8) on p. 373 and the first two estimates on p. 374, together with (15.1) and (15.2) on p. 362, one has

\[(4) \quad \sqrt{\frac{2k+1}{2}} k_{2n}(x, \alpha)(x - \alpha) = -C^{(n)}[A(x)B(\alpha) - A(\alpha)B(x)],\]

in which

\[-C^{(n)} = \frac{1}{\pi} \left(1 + \frac{c}{12n}\right), \quad |c| < 3,\]

\[A(y) = \sin(\sqrt{4n+3}y) - \frac{y^3}{6} \cos(\sqrt{4n+3}y) + \frac{T(2n+1, y)}{h_{2n+1}(0) \sqrt{4n+3}}\]

and

\[B(y) = \frac{y^3}{6} \sin(\sqrt{4n+1}y) + \frac{T(2n, \alpha)}{h_{2n}(0) \sqrt{4n+1}}.\]

The functions \(T(2n, y)\) and \(T(2n+1, y)\) are defined through the equations

\[h_{2n}(x) = h_{2n}(0) \cos(\sqrt{4n+1}x) + \frac{h_{2n}(0)}{4n+1} \frac{y^3}{\sqrt{4n+1}} \sin(\sqrt{4n+1}y) + \frac{T(2n, y)}{4n+1}\]

and

\[h_{2n+1}(y) = h_{2n+1}(0) \sin(\sqrt{4n+3}y) - \frac{h_{2n+1}(0)}{4n+3} \frac{y^3}{6} \cos(\sqrt{4n+3}y) + \frac{T(2n+1, y)}{4n+3}.\]

Here,

\[|a_n| = \frac{1}{|h_{2n+1}(0)|} \frac{1}{\sqrt{4n+1}} < \frac{4}{\pi^{1/2}} \sqrt{\frac{3}{2}} \frac{1}{n^{3/4}},\]

\[|b_n| = \frac{1}{|h_{2n}(0)|} \frac{1}{\sqrt{4n+1}} < \frac{\pi^{1/2}}{4} \sqrt{\frac{3}{2}} \frac{1}{n^{3/4}},\]

\[|T(2n+1, y)| < \frac{y^2}{\pi^{1/2} n^{1/4}} \left( \frac{y^4}{18} + 1 \right) + \frac{4}{187} \frac{y^{17/2}}{\sqrt{4n+1}},\]

and

\[|T(2n, y)| < \frac{y^2}{\pi^{1/2} n^{1/4}} \left( \frac{y^4}{18} + 1 \right) + \frac{4}{187} \frac{y^{17/2}}{\sqrt{4n+3}}.\]
Expanding the products in (3) yields

\[
\sqrt{\frac{2n+1}{2}} k_{2n}(x, \alpha)(x-\alpha) = -\frac{1}{\pi} \left(1 + \frac{\varepsilon}{6K}\right) \left(\sin(N(x-\alpha)) + \sum_{k=1}^{5} M_k^{(n)}(x, \alpha)\right), \ |\varepsilon| < 3,
\]

where, firstly,

\[
M_1^{(n)}(x, \alpha) = \cos(N(x+\alpha)) \sin((x-\alpha)/2N) - 2\sin^2((x+\alpha)/4N) \sin(N(x-\alpha)),
\]

as shown on pp. 375 of [S]. Again, on p. 376 we find

\[
\sqrt{4n+1}M_2^{(n)}(x, \alpha) = \frac{-x^3}{6} \sin(\sqrt{4n+1}x) \sin(\sqrt{4n+3}\alpha) + \frac{\alpha^3}{6} \sin(\sqrt{4n+1}\alpha) \sin(\sqrt{4n+3}x)
\]

\[
= \frac{\alpha^3 - x^3}{6} \sin(\sqrt{4n+1}x) \sin(\sqrt{4n+3}\alpha) + \frac{\alpha^3}{6} \sin((x-\alpha)/2N) \sin(N(x+\alpha)) - \sin((x+\alpha)/2N) \sin(N(x-\alpha))
\]

An argument similar to the one for \(\sqrt{4n+1}M_2^{(n)}(x, \alpha)\) gives

\[
\sqrt{4n+3}M_3^{(n)}(x, \alpha) = \frac{\alpha^3}{6} \cos(\sqrt{4n+1}x) \cos(\sqrt{4n+3}\alpha) - \frac{x^3}{6} \cos(\sqrt{4n+1}\alpha) \cos(\sqrt{4n+3}x)
\]

\[
= \frac{\alpha^3 - x^3}{6} \cos(\sqrt{4n+1}x) \cos(\sqrt{4n+3}\alpha) - \frac{x^3}{6} \sin(N(x+\alpha)) \sin((x-\alpha)/2N) + \sin((x+\alpha)/2N) \sin(N(x-\alpha))
\]

Next,

\[
\sqrt{(4n+1)(4n+3)}M_4^{(n)}(x, \alpha)
\]

\[
= \frac{\alpha^3 x^3}{36} \left[-\cos(\sqrt{4n+3}x) \sin(\sqrt{4n+1}+\alpha) + \cos(\sqrt{4n+3}\alpha) \sin(\sqrt{4n+1}x)\right]^{(3)}
\]

\[
= \frac{\alpha^3 x^3}{72} \left[\sin(\sqrt{4n+3}x - \sqrt{4n+1}\alpha) - \sin(\sqrt{4n+3}x + \sqrt{4n+1}\alpha)\right]
\]

\[
+ \sin(\sqrt{4n+1}x - \sqrt{4n+3}\alpha) + \sin(\sqrt{4n+1}x + \sqrt{4n+3}\alpha)\right]^{(3)}
\]

\[
= \frac{\alpha^3 x^3}{36} \sin(N(x-\alpha)) \cos((x+\alpha)/2N) - \sin((x-\alpha)/2N) \sin(N(x+\alpha))\right]^{(3)}
\]

Finally,

\[
M_5^{(n)}(x, \alpha) = a_n \left[T(2n+1, x) \cos(\sqrt{4n+1}\alpha) - T(2n+1, x) \cos(\sqrt{4n+3}x) + T(2n+1, x) \frac{\alpha^3 \sin(\sqrt{4n+1}\alpha)}{\sqrt{4n+1} + \alpha^3 \sin(\sqrt{4n+3}x)}\right]
\]

\[
+ b_n \left[T(2n, x) \sin(\sqrt{4n+1}\alpha) - T(2n, x) \sin(\sqrt{4n+3}x) + T(2n, x) \frac{\alpha^3 \cos(\sqrt{4n+1}\alpha)}{\sqrt{4n+1} + \alpha^3 \cos(\sqrt{4n+3}x)}\right]
\]

\[
+ a_n b_n \left[T(2n+1, x) T(2n, x) - T(2n+1, x) T(2n, x)\right].
\]

To prove Theorem 1 we will require the following estimates of integrals involving terms on the right hand side of (3).

3.1

\[
\left|\int_{-T}^{T} M_k^{(n)}(x, \alpha) \frac{f(\alpha)}{x-\alpha} d\alpha\right| \leq \frac{1}{2N} \left|\int_{-T}^{T} \cos(N(x+\alpha)) \sin(x-\alpha/2N) f(\alpha) d\alpha\right|
\]

\[
+ 2 \left|\int_{-T}^{T} \sin^2((x+\alpha)/4N) \sin(N(x-\alpha)) \frac{f(\alpha)}{x-\alpha} d\alpha\right| = I(x) + II(x).
\]

\((3)\) The expression for \(\sqrt{(4n+1)(4n+3)}M_4^{(n)}(x, \alpha)\) on p. 345–372 of [S] is incorrect.
To begin,
\[ I(x) \leq \frac{1}{2N} \left[ \left| \int_{-T}^{T} \cos(N(x + \alpha)) \left[ \sin((x - \alpha)/2N) - 1 \right] f(\alpha) \, d\alpha \right| + \left| \int_{-T}^{T} \cos(N(x + \alpha)) f(\alpha) \, d\alpha \right| \right] \]
\[ \leq \frac{1}{2N} \left[ \int_{-T}^{T} \frac{1}{6} \left( x - \alpha \right)^2 N d\alpha + \left| \frac{f(-T) + |f(T)|}{N} \right| + \frac{1}{N} \int_{-T}^{T} |f^{(1)}(\alpha)| \, d\alpha \right] \]
\[ \leq \frac{1}{48N^3} \left[ x^2 \int_{-T}^{T} |f(\alpha)| \, d\alpha + 2x \int_{-T}^{T} |f(\alpha)\alpha| \, d\alpha + \int_{-T}^{T} |f(\alpha)\alpha^2| \, d\alpha \right] + \frac{1}{2N^2} \left[ |f(-T)| + |f(T)| + \int_{-T}^{T} |f^{(1)}(\alpha)| \, d\alpha \right]. \]

Thus,
\[ \left[ \frac{1}{2T} \int_{-T}^{T} I(x)^2 \, dx \right]^{1/2} \leq \frac{1}{48N^3} \left[ \left[ \frac{1}{2T} \int_{-T}^{T} x^2 \, dx \right]^{1/2} \int_{-T}^{T} |f(\alpha)| \, d\alpha \right. \]
\[ + 2 \left[ \frac{1}{2T} \int_{-T}^{T} x^2 \, dx \right]^{1/2} \int_{-T}^{T} |f(\alpha)\alpha| \, d\alpha + \frac{T}{\sqrt{3}} \int_{-T}^{T} |f(\alpha)\alpha^2| \, d\alpha \right. \]
\[ + \frac{1}{2N^2} \left[ |f(-T)| + |f(T)| + \int_{-T}^{T} |f^{(1)}(\alpha)| \, d\alpha \right]. \]

Again,
\[ II(x) = \left[ \int_{-T}^{T} (1 - \cos((x + \alpha)/2N)) \sin(N(x - \alpha)) f(\alpha) \frac{x}{x - \alpha} \, d\alpha \right] \leq \left[ \int_{-T}^{T} \frac{(x + \alpha)/2N)^2}{2} \sin(N(x - \alpha)) f(\alpha) \frac{x}{x - \alpha} \, d\alpha \right] \]
\[ \leq \frac{1}{8N^2} \left[ x^2 \int_{-T}^{T} \sin(N(x - \alpha)) \frac{x}{x - \alpha} \, d\alpha \right] + 2|x| \left[ \int_{-T}^{T} \sin(N(x - \alpha)) \frac{f(\alpha)\alpha}{(x - \alpha)} \, d\alpha \right] \]
\[ + \left[ \int_{-T}^{T} \sin(N(x - \alpha)) \frac{f(\alpha)\alpha^2}{x - \alpha} \, d\alpha \right] = \frac{1}{8N^2} [III(x) + IV(x) + V(x)]. \]

Now,
\[ IV(x) \leq 2T \left| \sin(Nx) \right| \int_{-T}^{T} \frac{f(\alpha)\cos(N\alpha)}{x - \alpha} \, d\alpha - \cos(Nx) \int_{-T}^{T} \frac{f(\alpha)\sin(N\alpha)}{x - \alpha} \, d\alpha \right|, \]
so,
\[ \left[ \frac{1}{2T} \int_{-T}^{T} IV(x)^2 \, dx \right]^{1/2} \leq 2\pi T^{1/2} \left[ \int_{-T}^{T} |f(\alpha)|^2 \, d\alpha \right]^{1/2}. \]

Similarly,
\[ \left[ \frac{1}{2T} \int_{-T}^{T} V(x)^2 \, dx \right]^{1/2} \leq \sqrt{2\pi} T^{-1/2} \left[ \int_{-T}^{T} |f(\alpha)|^2 \, d\alpha \right]^{1/2}. \]

Next,
\[ \left| \int_{-T}^{T} \sin(N(x - \alpha)) \frac{f(\alpha)}{x - \alpha} \, d\alpha \right| \leq \int_{-T}^{T} \frac{f(\alpha)\sin(N\alpha)}{x - \alpha} \, d\alpha + \int_{-T}^{T} \frac{f(\alpha)\cos(N\alpha)}{x - \alpha} \, d\alpha \]
and, by Bedrosian’s identity,
\[ \int_{-T}^{T} \frac{f(\alpha)\sin(N\alpha)}{x - \alpha} \, dx = \int_{-\infty}^{\infty} \frac{f(\alpha)\sin(N\alpha)}{x - \alpha} \, d\alpha - \int_{-\infty}^{\infty} \frac{f(\alpha)\sin(N\alpha)}{x - \alpha} \chi_{|\alpha|>T} \, d\alpha \]
\[ = \int_{-\infty}^{\infty} f(N\alpha) \sin(N\alpha) \, d\alpha + \int_{-\infty}^{\infty} \frac{f(\alpha) - f(N\alpha)}{x - \alpha} \sin(N\alpha) \, d\alpha - \int_{-\infty}^{\infty} \frac{f(\alpha) - f(N\alpha)}{x - \alpha} \sin(N\alpha) \chi_{|\alpha|>T} \, d\alpha, \]
where \(f_N = (f\chi_{(-N,N)})^\vee.\)

A similar result holds with \(\sin(N\alpha)\) replaced by \(\sin(N\alpha).\)

Thus,
\[ \left[ \frac{1}{2T} \int_{-T}^{T} III^2 \, dx \right]^{1/2} \leq \sqrt{2\pi} T^{3/2} \left[ \left[ \int_{-T}^{T} f_N(\alpha)^2 \, d\alpha \right]^{1/2} + \left[ \int_{|\alpha|>T} f(\alpha)^2 \, d\alpha \right]^{1/2} + \left[ \int_{|\omega|>N} |\hat{f}(\omega)|^2 \, d\omega \right]^{1/2} \right]. \]
since
\[
\int_{-\infty}^{\infty} |f(\alpha) - f_N(\alpha)|^2d\alpha = \int_{-\infty}^{\infty} |\tilde{f}(\omega) - f_N(\omega)|^2d\omega = \int_{|\omega|>N} |\tilde{f}(\omega)|^2d\omega.
\]
In sum,
\[
\left[\frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} M_1^{(n)}(x, \alpha) \frac{f(\alpha)}{x - \alpha} d\alpha \right|^2 dx \right]^{1/2} \leq \frac{1}{48N^4} \left[ T^2 \int_{-T}^{T} |f(\alpha)|^2 d\alpha + \frac{T}{3} \int_{-T}^{T} |f(\alpha)|^2 d\alpha + \int_{-T}^{T} |f(\alpha)|^2 d\alpha \right]
\]
\[+ \frac{1}{2N^2} \left[ |f(-T)| + |f(T)| + \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha \right] + \frac{1}{8N^2} \left[ 2\pi T^{1/2} \left[ \int_{-T}^{T} |f(\alpha)|^2 d\alpha \right]^{1/2} + \sqrt{2\pi T}^{1/2} \left[ \int_{-T}^{T} |f(\alpha)|^2 d\alpha \right]^{1/2} \right]
\]
\[+ \sqrt{2\pi T}^{3/2} \left( \left[ \int_{-T}^{T} f_N(\alpha)^2 d\alpha \right]^{1/2} + \left[ \int_{|\alpha|>T} |f(\alpha)|^2 d\alpha \right]^{1/2} + \left[ \int_{|\omega|>N} |\tilde{f}(\omega)|^2 d\omega \right]^{1/2} \right) \]

3.2
\[
\int_{-T}^{T} M_2^{(n)}(x, \alpha) f(x) d\alpha \leq \frac{1}{\sqrt{4n+1}} \int_{-T}^{T} \alpha^3 - x^3 \sin(\sqrt{4n+3}\alpha)f(\alpha) d\alpha + \frac{1}{\sqrt{4n+1}} \int_{-T}^{T} \sin((x-\alpha)/2N)\sin(N(x+\alpha)) f(\alpha) (x+\alpha) \alpha^3 d\alpha.
\]
Now,
\[
I(x) \leq \frac{1}{6\sqrt{4n+1}} \left[ \int_{-T}^{T} \sin(\sqrt{4n+3}\alpha)f(\alpha)\alpha^2 d\alpha \right] + x \left[ \int_{-T}^{T} \sin(\sqrt{4n+3}\alpha)f(\alpha) d\alpha \right] + x^2 \left[ \int_{-T}^{T} \sin(\sqrt{4n+3}\alpha)f(\alpha) d\alpha \right]
\]
\[
\leq \frac{1}{6\sqrt{(4n+1)(4n+3)}} \left[ T^2(\sin(\sqrt{4n+3}\alpha)f(\alpha)\alpha^2 d\alpha) + \left[ T^2(\sin(\sqrt{4n+3}\alpha)f(\alpha) \alpha^2 d\alpha) \right] + \left[ (x+T)^2 f(\alpha) \alpha^2 d\alpha \right] + 2 \left[ x^2 (x+T)^2 f(\alpha) \alpha^2 d\alpha \right]
\]
\[
+ |x| \left( \int_{-T}^{T} f^{(1)}(\alpha)\alpha^2 d\alpha + \int_{-T}^{T} |f(\alpha)|^2 d\alpha \right) + x^2 \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha
\]
Thus,
\[
\left[ \frac{1}{2T} \int_{-T}^{T} x^2 d\alpha \right]^{1/2} \leq \frac{1}{6\sqrt{(4n+1)(4n+3)}} \left[ 1 + \frac{1}{\sqrt{3}} + \frac{3}{\sqrt{3}} \right] \left( \int_{-T}^{T} |f(-T)| + |f(T)| \right)
\]
\[+ \frac{T^2}{\sqrt{5}} \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha + \frac{2}{\sqrt{3}} \left( \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha + \int_{-T}^{T} |f(\alpha)| d\alpha \right) + \int_{-T}^{T} |f^{(1)}(\alpha)^2| d\alpha + 2 \int_{-T}^{T} |f(\alpha)| d\alpha
\]
Next,
\[
II(x) \leq \frac{1}{12N\sqrt{4n+1}} \left[ \int_{-T}^{T} \sin((x-\alpha)/2N) - 1 |f(\alpha)\alpha^3| d\alpha + \int_{-T}^{T} \sin((N(x+\alpha)) f(\alpha)\alpha^3 d\alpha \right]
\]
\[
\leq \frac{1}{12N\sqrt{4n+1}} \left[ \int_{-T}^{T} \left( \frac{x-\alpha}{2N} \right)^3 f(\alpha)\alpha^3 d\alpha + \frac{2}{N} \int_{-T}^{T} \left| \frac{d}{d\alpha} f(\alpha) \alpha^3 \right| d\alpha \right]
\]
\[
\leq \frac{1}{288N^3\sqrt{4n+1}} \left[ \int_{-T}^{T} f(\alpha)\alpha^3 d\alpha + 2 |x| \int_{-T}^{T} f(\alpha)\alpha^3 d\alpha + \int_{-T}^{T} f(\alpha)\alpha^5 d\alpha \right]
\]
\[+ \frac{1}{6N^2\sqrt{4n+1}} \left[ T^3 (|f(-T)| + |f(T)|) + \int_{-T}^{T} f^{(1)}(\alpha)\alpha^3 d\alpha + 3 \int_{-T}^{T} f(\alpha)\alpha^2 d\alpha \right].
\]
Hence,
\[
\left[ \frac{1}{2T} \int_{-T}^{T} \Pi(x)^2 dx \right]^{1/2} \leq \frac{1}{288N^3 \sqrt{4n+1}} \left[ \frac{T^2}{\sqrt{5}} \int_{-T}^{T} |f(\alpha)|^3 d\alpha + \frac{2T}{\sqrt{5}} \int_{-T}^{T} |f(\alpha)|^4 d\alpha + \int_{-T}^{T} |f(\alpha)|^5 d\alpha \right] + \frac{1}{6N^2 \sqrt{4n+1}} \left[ T^3(|f(-T)| + |f(T)|) + \int_{-T}^{T} |f^{(1)}(\alpha)|^3 d\alpha + 3 \int_{-T}^{T} |f^{(1)}(\alpha)|^2 d\alpha \right].
\]

Finally,
\[
III(x) \leq \frac{1}{48N^2 \sqrt{4n+1}} \left[ |x|^3 \int_{-T}^{T} |f(\alpha)|^3 d\alpha + 3x^2 \int_{-T}^{T} |f(\alpha)|^4 d\alpha + \int_{-T}^{T} \text{sinc}(N(x-\alpha)) f(\alpha)^4 d\alpha \right]
\leq \frac{1}{12 \sqrt{4n+1}} \left[ |x|^3 \int_{-T}^{T} ((x + \alpha)/2N)^2 |f(\alpha)|^3 d\alpha + \int_{-T}^{T} ((x + \alpha)/2N)^2 |f(\alpha)|^4 d\alpha \right]
+ \frac{1}{12N \sqrt{4n+1}} \left[ T \left( \int_{-T}^{T} \frac{\sin(N\alpha) f(\alpha)^3}{x-\alpha} d\alpha \right) \right.
+ \left. \int_{-T}^{T} \frac{\cos(N\alpha) f(\alpha)^4}{x-\alpha} d\alpha \right].
\]

Altogether, then,
\[
\left[ \frac{1}{2T} \int_{-T}^{T} III(x)^2 dx \right]^{1/2} \leq \frac{1}{48N^2 \sqrt{4n+1}} \left[ \frac{T^3}{\sqrt{7}} \int_{-T}^{T} |f(\alpha)|^3 d\alpha + \frac{3T^2}{\sqrt{5}} \int_{-T}^{T} |f(\alpha)|^4 d\alpha + \frac{3T}{\sqrt{3}} \int_{-T}^{T} |f(\alpha)|^5 d\alpha \right]
+ \frac{1}{6N \sqrt{4n+1}} \left[ T \left( \frac{\int_{-T}^{T} |f(\alpha)|^3 d\alpha}{6 |x|} \right)^{1/2} \right. \left. + \left( \frac{\int_{-T}^{T} |f(\alpha)|^2 d\alpha}{|x|} \right)^{1/2} \right].
\]

3.3

\[
\left| \int_{-T}^{T} M_2^{(n)}(x, \alpha) f(\alpha) d\alpha \right| \leq \frac{1}{\sqrt{4n+3}} \left| \int_{-T}^{T} \frac{\alpha^3 - x^3}{6(x-\alpha)} \cos(\sqrt{4n+3} \alpha) f(\alpha) d\alpha \right|
+ \frac{|x|^3}{\sqrt{4n+3}} \left| \int_{-T}^{T} \frac{\sin((x-\alpha)/2N) \sin(N(x+\alpha)) f(\alpha)}{12 N^2} d\alpha \right|
+ \frac{|x|^3}{\sqrt{4n+3}} \left| \int_{-T}^{T} \frac{\sin((x+\alpha)/2N) \sin(N(x-\alpha)) f(\alpha)}{12 N^2} d\alpha \right|
= I(x) + II(x) + III(x).
\]

Now, I(x) here is, essentially, the same as the I(x) involved in \( M_2^{(n)}(x, \alpha) \), we find
\[
I(x) \leq \frac{1}{6 \sqrt{4n+3}} \left[ \int_{-T}^{T} \frac{\cos(\sqrt{4n+3} \alpha) f(\alpha)}{x-\alpha} d\alpha \right] \left[ 2 |x| \int_{-T}^{T} \frac{\cos(\sqrt{4n+3} \alpha) f(\alpha)}{x-\alpha} d\alpha \right] + x^2 \right| \int_{-T}^{T} \frac{\sin(\sqrt{4n+3} \alpha) f(\alpha)}{x-\alpha} d\alpha \right|
\leq \frac{1}{6(4n+3)} \left[ T^2 |f(-T)| + |f(T)| \right] + \int_{-T}^{T} |f^{(1)}(\alpha) f^{(2)}(\alpha)| d\alpha + \int_{-T}^{T} \frac{|f(\alpha)|}{\sqrt{4n+3}} d\alpha
+ \left| \frac{|f(-T)| + |f(T)|}{\sqrt{4n+3}} + \int_{-T}^{T} \frac{|f^{(1)}(\alpha)|}{\sqrt{4n+3}} d\alpha \right].
\]

Then,
\[
\left[ \frac{1}{2T} \int_{-T}^{T} I(x)^2 dx \right]^{1/2} \leq \frac{1}{6(4n+3)} \left[ (1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}}) I^2 + 2 \int_{-T}^{T} |f(\alpha)| d\alpha + \frac{T}{\sqrt{3}} \int_{-T}^{T} |f(\alpha)| d\alpha
\right. + \frac{T^2}{\sqrt{5}} \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha + \frac{T}{\sqrt{5}} \int_{-T}^{T} |f^{(1)}(\alpha)| d\alpha + \int_{-T}^{T} |f^{(1)}(\alpha)|^2 d\alpha.
\]
Next,
\[
\begin{align*}
\text{H}(x) & \leq \frac{|x|^3}{12N\sqrt{4n} + 3} \left[ \int_{-T}^{T} |\text{sinc}(x - \alpha)/2N - 1||f(\alpha)|d\alpha + \left| \int_{-T}^{T} \sin(N(x + \alpha)f(\alpha))d\alpha \right| \right] \\
& \leq \frac{|x|^3}{12N\sqrt{4n} + 3} \left[ \int_{-T}^{T} \frac{1}{6} (\frac{x - \alpha}{2N})^2 |f(\alpha)|d\alpha + \left[ \int_{-T}^{T} \cos(N\alpha)f(\alpha)d\alpha + \int_{-T}^{T} \sin(N\alpha)f(\alpha)d\alpha \right] \right] \\
& \leq \frac{|x|^3}{12N\sqrt{4n} + 3} \left[ \int_{-T}^{T} \frac{1}{24N^2} (x - \alpha)^2 |f(\alpha)|d\alpha + \frac{2}{N} |f(-T)| + |f(T)| + \frac{2}{N} \int_{-T}^{T} |f^{(1)}(\alpha)|d\alpha \right] \\
& \quad + \frac{28N^4\sqrt{4n} + 3}{6N^2\sqrt{4n} + 3} \left[ x^2 \int_{-T}^{T} |f(\alpha)|d\alpha + 2|x| \int_{-T}^{T} |f(\alpha)|d\alpha + \int_{-T}^{T} |f(\alpha)|d\alpha \right] \\
& \quad + \frac{|x|^3}{6N^2\sqrt{4n} + 3} |f(-T)| + |f(T)| + \int_{-T}^{T} |f^{(1)}(\alpha)|d\alpha.
\end{align*}
\]

Hence,
\[
\begin{align*}
\left[ \frac{1}{2T} \int_{-T}^{T} \text{H}(x)^2 dx \right]^{1/2} & \leq \frac{T^3}{6\sqrt{7}N^2\sqrt{4n} + 3} \left[ |f(-T)| + |f(T)| + \int_{-T}^{T} |f^{(1)}(\alpha)|d\alpha \right] \\
& \quad + \frac{T^3}{288N^4\sqrt{4n} + 3} \left[ \frac{1}{\sqrt{7}} \int_{-T}^{T} |f(\alpha)|\alpha^2d\alpha + \frac{2T^{3/2}}{\sqrt{9}} \int_{-T}^{T} |f(\alpha)|\alpha d\alpha + \frac{T^2}{\sqrt{11}} \int_{-T}^{T} |f(\alpha)|d\alpha \right].
\end{align*}
\]

Finally, with \(T_4(t) = 1 - \frac{t^2}{2} + \frac{t^4}{120}\),
\[
\begin{align*}
\text{H}(x) & \leq \frac{|x|^3}{12\sqrt{4n} + 3} \left[ \int_{-T}^{T} |\text{sinc}((x + \alpha)/2N) - T_4((x + \alpha)/2N)|f(\alpha)||x + |\alpha||d\alpha \right. \\
& \quad \left. + \int_{-T}^{T} \text{sinc}(N(x - \alpha))T_4((x + \alpha)/2N)f(\alpha)(x + \alpha)d\alpha \right] \\
& \leq \frac{|x|^3}{12\sqrt{4n} + 3} \left[ \int_{-T}^{T} \frac{1}{5040} \left( \frac{x + \alpha}{2N} \right)^6 |f(\alpha)||x + |\alpha||d\alpha \\
& \quad + \int_{-T}^{T} \text{sinc}(N(x - \alpha)) \left( 1 - \frac{(x + \alpha)^2}{24N^2} + \frac{(x + \alpha)^4}{1920N^4} \right) f(\alpha)(x + \alpha)d\alpha \right] \\
& \leq \frac{1}{387020N^6\sqrt{4n} + 3} \left[ \int_{-T}^{T} |f(\alpha)|d\alpha + 7|x|^9 \int_{-T}^{T} |f(\alpha)|d\alpha + 21x^8 \int_{-T}^{T} |f(\alpha)|^2d\alpha \\
& \quad + 35|x|^7 \int_{-T}^{T} |f(\alpha)|^3d\alpha + 35x^6 \int_{-T}^{T} |f(\alpha)|^4d\alpha + 21|x|^5 \int_{-T}^{T} |f(\alpha)|^5d\alpha \\
& \quad + 7x^4 \int_{-T}^{T} |f(\alpha)|^6d\alpha + |x|^3 \int_{-T}^{T} |f(\alpha)^7d\alpha \\
& \quad + \frac{1}{12N\sqrt{4n} + 3} \left[ |x|^4 \left( \int_{-T}^{T} \left| \frac{\sin(N\alpha)f(\alpha)}{x - \alpha} \right| d\alpha + \int_{-T}^{T} \left| \frac{\cos(N\alpha)f(\alpha)}{x - \alpha} \right| d\alpha \right) \\
& \quad + |x|^3 \left( \int_{-T}^{T} \left| \frac{\sin(N\alpha)f(\alpha)}{x - \alpha} \right| d\alpha + \int_{-T}^{T} \left| \frac{\cos(N\alpha)f(\alpha)}{x - \alpha} \right| d\alpha \right) \right] \\
& \quad + \frac{1}{288N^2\sqrt{4n} + 3} \left[ x^3 \int_{-T}^{T} \left( \left( |x| + |\alpha| \right)^3 + \frac{(|x| + |\alpha|)^5}{80N^2} \right) |f(\alpha)|d\alpha \right] \\
& \quad \left. + \frac{1}{387020N^6\sqrt{4n} + 3} \left( IV(x) + \frac{1}{12N\sqrt{4n} + 3} IV(x) + \frac{1}{288N^2\sqrt{4n} + 3} (VI(x) \right. \right] \\
& \quad \left. + \frac{1}{2T} \int_{-T}^{T} |f(\alpha)|d\alpha + \frac{1}{12N\sqrt{4n} + 3} \int_{-T}^{T} |f(\alpha)|d\alpha + \frac{1}{288N^2\sqrt{4n} + 3} \int_{-T}^{T} |f(\alpha)|^2d\alpha \right]^{1/2}.
\end{align*}
\]

Altogether, then,
\[
\begin{align*}
\left[ \frac{1}{2T} \int_{-T}^{T} \text{H}(x)^2 dx \right]^{1/2} & \leq \frac{1}{387020N^6\sqrt{4n} + 3} \left[ \frac{1}{2T} \int_{-T}^{T} (IV(x))^2 dx \right]^{1/2} + \frac{1}{12N\sqrt{4n} + 3} \left[ \int_{-T}^{T} \frac{1}{2T} \int_{-T}^{T} V(x)^2 dx \right]^{1/2} \\
& \quad + \frac{1}{288N^2\sqrt{4n} + 3} \left[ \frac{1}{2T} \int_{-T}^{T} (VI(x))^2 dx \right]^{1/2},
\end{align*}
\]
where

\[
\frac{1}{2T} \int_{-T}^{T} (IV(x)^2) dx \leq \frac{T^{10}}{\sqrt{21}} \int_{-T}^{T} |f(\alpha)| d\alpha + \frac{T^9}{\sqrt{19}} \int_{-T}^{T} |f(\alpha)\alpha| d\alpha \\
+ \frac{21T^8}{\sqrt{17}} \int_{-T}^{T} |f(\alpha)\alpha^2| d\alpha + \frac{35T^7}{\sqrt{15}} \int_{-T}^{T} |f(\alpha)\alpha^3| d\alpha + \frac{35T^6}{\sqrt{13}} \int_{-T}^{T} |f(\alpha)\alpha^4| d\alpha \\
+ \frac{21T^5}{\sqrt{11}} \int_{-T}^{T} |f(\alpha)\alpha^5| d\alpha + \frac{7T^4}{\sqrt{9}} \int_{-T}^{T} |f(\alpha)\alpha^6| d\alpha + \frac{T^3}{\sqrt{7}} \int_{-T}^{T} |f(\alpha)\alpha^7| d\alpha .
\]

Observe that, by Bedrosian's identity,

\[
\int_{-T}^{T} \frac{\sin(N\alpha)f(\alpha)}{x - \alpha} d\alpha = \int_{-\infty}^{\infty} \frac{\sin(N\alpha)f(\alpha)}{x - \alpha} d\alpha - \int_{-\infty}^{\infty} \sin(N\alpha)f(\alpha)T_{\alpha > T}(\alpha) d\alpha \\
= \int_{-\infty}^{\infty} \frac{\sin(N\alpha)f_N(\alpha)}{x - \alpha} d\alpha - \int_{-\infty}^{\infty} \sin(N\alpha)[f(\alpha) - f_N(\alpha)] x - \alpha d\alpha - \int_{-\infty}^{\infty} \sin(N\alpha)f(\alpha)T_{\alpha > T}(\alpha) d\alpha \\
= \pi \cos(Nx)f_N(x) + \int_{-\infty}^{\infty} \frac{\sin(N\alpha)[f(\alpha) - f_N(\alpha)]}{x - \alpha} d\alpha - \int_{-\infty}^{\infty} \sin(N\alpha)f(\alpha)T_{\alpha > T}(\alpha) d\alpha,
\]

where \( f_N = (\hat{f})_{(-N,N)}^N \). A similar result holds with \( \sin(N\alpha) \) replaced by \( \cos(N\alpha) \). Thus,

\[
\left[ \frac{1}{2T} \int_{-T}^{T} V(x)^2 dx \right]^{1/2} \leq \frac{1}{\sqrt{2T}} \left[ 2\pi \int_{-T}^{T} |f_N(\alpha)\alpha^4| \right]^{1/2} \leq \pi T^{3/2} \int_{-\infty}^{\infty} \frac{|f(\alpha) - f_N(\alpha)|^2}{x - \alpha} d\alpha \\
+ 2\pi T^3 \left[ \int_{|\alpha| > T} |f_N(\alpha)\alpha^4| dx \right] + \pi T^7/2 \left[ \int_{|\omega| > N} (f(\omega))^2 d\omega \right]^{1/2},
\]

since

\[
\int_{-\infty}^{\infty} |f(\alpha) - f_N(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |\hat{f}(\omega) - \hat{f}_N(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |\hat{f}(\omega) - \hat{f}(\omega)\xi_{(-N,N)}(\omega)|^2 d\omega = \int_{|\omega| > N} (f(\omega))^2 d\omega.
\]

Again,

\[
\left[ \frac{1}{2T} \int_{-T}^{T} IV(x)^2 dx \right]^{1/2} \leq \left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} |x|^3 \left( 8(|x|^3 + |\alpha|^3) + 2(|x|^5 + |\alpha|^5) \right) |f(\alpha)| d\alpha \right|^2 dx \right]^{1/2}.
\]

Expanding the integrand on the right hand side of the last inequality, we find

\[
\left[ \frac{1}{2T} \int_{-T}^{T} IV(x)^2 dx \right]^{1/2} \leq \frac{1}{\sqrt{2T}} \left[ \int_{-T}^{T} x^3 dx \right]^{1/2} \left[ \int_{-T}^{T} |f(\alpha)\alpha^4| d\alpha \right] + \frac{2}{\sqrt{5N^2}} \left[ \int_{-T}^{T} x^6 dx \right]^{1/2} \left[ \int_{-T}^{T} |f(\alpha)\alpha^3| d\alpha \right] \\
+ \frac{8}{\sqrt{13}} T^6 \left[ \int_{-T}^{T} |f(\alpha)| d\alpha \right] + \frac{8}{\sqrt{5}} T^3 \left[ \int_{-T}^{T} |f(\alpha)\alpha^3| d\alpha \right]
\]

\[
= \frac{8}{\sqrt{13}} T^6 \left[ \int_{-T}^{T} |f(\alpha)| d\alpha \right] + \frac{8}{\sqrt{5}} T^3 \left[ \int_{-T}^{T} |f(\alpha)\alpha^3| d\alpha \right] + \frac{2}{\sqrt{5N^2}} T^3 \left[ \int_{-T}^{T} |f(\alpha)\alpha^5| d\alpha \right].
\]

3.4

\[
\int_{-T}^{T} M_4(\alpha)(x,\alpha) f(\alpha) d\alpha \leq \frac{|x|^3}{36\sqrt{(4n+1)(4n+3)}} \left[ \int_{-T}^{T} \sin(N(x-\alpha)) \cos((x+\alpha)/2N) f(\alpha) \alpha^3 x - \alpha \right]
\]

\[
+ \frac{T^3}{18\sqrt{(4n+1)(4n+3)}} \left[ \int_{-T}^{T} \cos(N\alpha) \sin(\alpha/2N) f(\alpha) \alpha^3 x - \alpha \right]
\]

\[
+ \frac{T^3}{18\sqrt{(4n+1)(4n+3)}} \left[ \int_{-T}^{T} \cos(N\alpha) \sin(\alpha/2N) f(\alpha) \alpha^3 x - \alpha \right].
\]
Hence,
\[
\left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} M_1^{(n)}(x, \alpha) f(\alpha) \, d\alpha \right|^2 \, dx \right]^{1/2} \leq \frac{\sqrt{2} T^{5/2}}{9(4n + 1)(4n + 3)} \left[ \int_{-T}^{T} |f(\alpha)\alpha^2| \, d\alpha \right]^{1/2}.
\]

3.5

The expression
\[
\left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} M_1^{(n)}(x, \alpha) f(\alpha) \, d\alpha \right|^2 \, dx \right]^{1/2}
\]

is dominated by the sum of five terms, which we now consider in turn.

(i) The term
\[
\left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} a_n[T(2n + 1, x) \cos(\sqrt{4n + 1} \alpha) - T(2n + 1, \alpha) \cos(\sqrt{4n + 1} x)] \frac{f(\alpha)}{x - \alpha} \right|^2 \, dx \right]^{1/2}
\]
is no bigger than
\[
\frac{|a_n|}{\sqrt{2T}} \left[ \int_{-T}^{T} \left| \int_{-T}^{T} \frac{\cos(\sqrt{4n + 1} \alpha) f(\alpha)}{x - \alpha} \, d\alpha \right|^2 \, dx \right]^{1/2} + \frac{|a_n|}{\sqrt{2T}} \left[ \int_{-T}^{T} \left| \int_{-T}^{T} \frac{T(2n + 1, x) \cos(\sqrt{4n + 1} \alpha) f(\alpha)}{x - \alpha} \, d\alpha \right|^2 \, dx \right]^{1/2}
\]
\[
\leq \frac{|a_n|}{\sqrt{2T}} \left[ 2 \int_{-T}^{T} \left| T(2n + 1, x) f_N(x) \right|^2 \, dx \right]^{1/2} + \sup_{x \in [-T, T]} \left| T(2n + 1, x) \right| \left( \left( \int_{|\alpha| > T} |f(\alpha)|^2 \, d\alpha \right)^{1/2} + \left( \int_{|\omega| > N} |\tilde{f}(\omega)|^2 \, d\omega \right)^{1/2} \right)
\]
\[
\leq 4 \frac{\pi}{2T} \sqrt{\frac{3}{2}} \frac{1}{n} \left[ 2 \left( \int_{-T}^{T} |f_N(\alpha)\omega(\alpha)|^2 \, d\alpha \right)^{1/2} + \omega(T) \left[ \left( \int_{|\alpha| > T} |f(\alpha)|^2 \, d\alpha \right)^{1/2} + \left( \int_{|\omega| > N} |f(\omega)|^2 \, d\omega \right)^{1/2} \right], \right]
\]
in which
\[
\omega(\alpha) = \alpha^2 \left( a^4 + 1 \right) \frac{1}{\pi^{1/2}} + \frac{2}{187} \frac{|\alpha|^{17/2}}{n^{1/4}}.
\]

(ii) Arguing as in (i) we have
\[
\left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} a_n \left[ T(2n + 1, x) \frac{\alpha^3 \sin(\sqrt{4n + 1} \alpha)}{6 \sqrt{4n + 1}} - T(2n + 1, \alpha) \frac{\alpha^3 \sin(\sqrt{4n + 1} x)}{6 \sqrt{4n + 1}} \right] \frac{f(\alpha)}{x - \alpha} \right|^2 \, dx \right]^{1/2}
\]
\[
\leq \frac{|a_n|}{6\sqrt{4n + 1}} \left[ \int_{-T}^{T} \left| \int_{-T}^{T} \frac{\sin(\sqrt{4n + 1} \alpha) f(\alpha)\alpha^3}{x - \alpha} \, d\alpha \right|^2 \, dx \right]^{1/2} + \pi T^3 \left[ \int_{-T}^{T} |T(2n + 1, \alpha) f(\alpha)|^2 \, d\alpha \right]^{1/2}
\]
\[
\leq \frac{|a_n|}{6\sqrt{4n + 1}} \left[ \int_{-T}^{T} \left| \int_{-T}^{T} \frac{\sin(\sqrt{4n + 1} \alpha) f(\alpha)\alpha^3}{x - \alpha} \, d\alpha \right|^2 \, dx \right]^{1/2} + \pi T^3 \left[ \int_{-T}^{T} |T(2n + 1, \alpha) f(\alpha)|^2 \, d\alpha \right]^{1/2}
\]
\[
\leq 2 \frac{\pi}{3T} \sqrt{\frac{3}{2}} \frac{1}{n \sqrt{4n + 1}} \left[ \omega(T) \left[ \int_{-T}^{T} |f(\alpha)\alpha^3|^2 \, d\alpha \right]^{1/2} + T^3 \left[ \int_{-T}^{T} |f(\alpha)\omega(\alpha)|^2 \, d\alpha \right]^{1/2} \right].
\]

(iii) The mean square on $I_T$ of
\[
\int_{-T}^{T} b_n \left[ T(2n, x) \sin(\sqrt{4n + 3} \alpha) - T(2n, \alpha) \sin(\sqrt{4n + 3} x) \right] \frac{f(\alpha)}{x - \alpha}
\]
is dominated by

$$\frac{|b_n|}{\sqrt{2T}} \left[ \int_{-T}^{T} \left| T(2n, x) \int_{-T}^{T} \frac{\sin(\sqrt{4n+3}\alpha) f(\alpha)}{x - \alpha} \, d\alpha \right|^2 \right]^{1/2}$$

$$\leq \frac{|b_n|}{\sqrt{2T}} \left[ 2 \left( \int_{-T}^{T} |T(2n, x)f_N(x)|^2 \right)^{1/2} + \sup_{|x| \leq T} |T(2n, x)| \left( \int_{|\alpha| > T} f(\alpha)^2 d\alpha \right)^{1/2} + \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2}$$

$$\leq \frac{1}{24} \sqrt{\frac{\pi}{2T}} \sqrt{\frac{3}{2n}} \frac{1}{n^{1/2}} \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} + \omega(T) \left( \int_{|\alpha| > T} f(\alpha)^2 d\alpha \right)^{1/2} + \left( \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right)^{1/2}$$.

(iv) The method of (ii) applied to the estimation of the square mean on $I_T$, of

$$\int_{-T}^{T} b_n \left[ -T(2n, x) \frac{x^3 \cos(\sqrt{4n+3}\alpha)}{\sqrt{4n+3}} + T(2n, x) \frac{\alpha^3 \cos(\sqrt{4n+3}\alpha)}{\sqrt{4n+3}} \right] f(\alpha) \, dx$$

leads to the upper bound

$$\frac{1}{\sqrt{2T}} \frac{|b_n|}{6\sqrt{4n+3}} \frac{1}{n} \left[ \omega(T) \left( \int_{-T}^{T} |f(\alpha)\alpha^3|^2 d\alpha \right)^2 \right]^{1/2} + T^3 \left[ \int_{-T}^{T} |f(\omega)\alpha|^2 d\alpha \right]^{1/2}$$

$$\leq \frac{1}{24} \sqrt{\frac{\pi}{2T}} \sqrt{\frac{3}{2n^{1/2}}} \frac{1}{n^{1/2}} \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} \leq \sqrt{\frac{3}{T^2 n^{1/2}}} \left[ \int_{-T}^{T} |f(\omega)\alpha|^2 d\omega \right]^{1/2}$$.

(v) The square mean, on $I_T$, of

$$\int_{-T}^{T} b_n \left[ T(2n, x) T(2n, x) - T(2n+1, x) T(2n, x) \right] f(\alpha) d\alpha$$

is, by a now familiar argument,

$$\leq \frac{|a_n| |b_n|}{\sqrt{2T}} \left[ \int_{-T}^{T} T(2n+1, x) \int_{-T}^{T} T(2n, x) f(\alpha) \, dx \right]^{1/2} + \left[ \int_{-T}^{T} T(2n, x) \int_{-T}^{T} T(2n, x) f(\alpha) \, dx \right]^{1/2}$$

$$\leq \frac{1}{\sqrt{2T}} \frac{|a_n| |b_n|}{6} \omega(T) \left[ \int_{-T}^{T} |f(\alpha)\alpha|^2 d\alpha \right]^{1/2} \leq \sqrt{\frac{3}{T^2 n^{1/2}}} \left[ \int_{-T}^{T} |f(\omega)\alpha|^2 d\omega \right]^{1/2}$$.

4. The proof of the Theorem 1

By (2), (3), and (4) one has

$$(S_K f)(x) = -C^{(n)} \left( F_N f_T(x) + \sum_{k=1}^{5} \int_{-T}^{T} M_k^{(n)}(x, \alpha) \frac{f_T(\alpha)}{x - \alpha} d\alpha \right),$$

in which

$$-C^{(n)} = \frac{1}{\pi} \left( 1 + \frac{\varepsilon}{12n} \right), \quad |\varepsilon| < 3.$$

Thus, by Lemma 2 and (4),

$$\left[ \frac{1}{2T} \int_{-T}^{T} |f(x) - (S_K f)(x)|^2 dx \right]^{1/2} \leq \frac{1}{2T} \left[ \int_{-T}^{T} |f(x) - F_N(\alpha)|^2 dx \right]^{1/2} + \frac{1}{2} \left[ \int_{-T}^{T} \frac{\sin(N(x - \alpha))}{x - \alpha} f(\alpha) d\alpha \right]^{1/2}$$

$$\leq \frac{1}{2T} \left[ \int_{|\alpha| > T} f(x)^2 dx \right]^{1/2} + \frac{1}{2T} \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} + \frac{1}{K \sqrt{2T}} \left[ \int_{|\alpha| < T} |F_N(\alpha)|^2 d\alpha \right]^{1/2} + \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2}$$

$$+ \frac{1}{\pi} \left( 1 + \frac{1}{2K} \right) S_n(K, T) \leq \left( 1 + \frac{1}{K} \right) \left[ \frac{1}{2T} \left[ \int_{|\alpha| > T} f(\alpha)^2 d\alpha \right]^{1/2} + \frac{1}{2T} \left[ \int_{|\omega| > N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} \right]^{1/2} + \frac{1}{K} \left[ \frac{1}{2T} \int_{|\alpha| < T} |F_N(\alpha)|^2 d\alpha \right]^{1/2} + \frac{1}{\pi} \left( 1 + \frac{1}{2K} \right) S_n(K, T).$$
where, once again,

\[
S_a(K, T) = \sum_{k=1}^{5} \left[ \frac{1}{2T} \int_{-T}^{T} \left| \int_{-T}^{T} \frac{M_k^{(n)}(x, \alpha)}{x - \alpha} f(\alpha) \, d\alpha \right|^2 \, dx \right]^{1/2}.
\]

An explicit estimate of \( S_a(K, T) \) is described in the appendix using the ones involving \( M_k^{(n)}(x, \alpha) \), \( k = 1, \ldots, 5 \) in Section 3.

5. An Example

Example 1 of [KHB] involved the Hermite series approximation of the trimodal density function

\[
f(t) = 0.5\phi(t) + 3\phi(10(t - 0.8)) + 2\phi(10(t - 1.2)),
\]

in which

\[
\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \quad t \in \mathbb{R},
\]

is the standard normal density. Figure 1 above shows \( f \) is essentially supported in \([-3, 3]\). Again, from the graph of \( |\hat{f}| \) in Figure 4(c) of [KHB] we see it effectively lives in \([-8, 8]\).

Taking \( T = 3 \) and \( n = 250 \) (so \( N = 31.6544 \) \( K = 500 \)), we obtain

\[
\frac{1}{6} \int_{|t|<3} |f(t) - (S_{500}f)(t)|^2 dt < 0.02361.
\]

One always has

\[
\left[ \frac{1}{2T} \int_{|t|<T} |g(t) - (S_Kg)(t)|^2 dt \right]^{1/2} \leq \sup_{|t|<T} |g(t) - (S_Kg)(t)|,
\]

so, if the supremum norm is rather large, the smaller root mean square norm gives a better measure of the average size of \( |g(t) - (S_Kg)(t)| \). In our case

\[
\sup_{|t|<3} |f(t) - (S_{500}f)(t)| < 0.0025.
\]

Therefore, the supremum norm is here the better measure. Nevertheless, it is the computable estimates giving (6) that lead us to Figure 2 and hence to (7).

We observe that the graph in Figure 2 is of the error function \( f - S_{500}f \) approximated by \( f - S_{40}f - \sum_{k=41}^{500} \langle f, d_k \rangle d_k \), where \( d_k \) is the Dominici approximation to \( b_k \) given in Theorem 1.1 of [KHB].

The term involving \( S_a(500, 3) \) in (6) makes the biggest contribution to the upper bound in (1). Thus,

\[
1.002 \left[ \frac{1}{6} \int_{|t|>3} f(t)^2 dt \right]^{1/2} < 0.00051,
\]

\[
1.002 \left[ \frac{1}{6} \int_{|\omega|>31.6544} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} < 0.00088.
\]
and 
\[ \frac{1}{500} \left( \frac{1}{6} \int_{|t|<3} f(t)^2 dt \right)^{1/2} < 0.00062, \]

while 
\[ \frac{1}{\pi} \left[ 1 + \frac{1}{1000} \right] S_a(500, 3) < 0.02161. \]

For the convenience of the reader we have gathered together in an appendix the terms that make up \( S_a(n, T) \).

**Appendix**

Take the indicated multiples of the terms \( \int_{-T}^{T} |f(\alpha)| d\alpha \) etc. and then add them to get an estimate of the Sansone sum, \( S_a(K, T) \) in formula (6).

\[
\begin{align*}
\int_{-T}^{T} |f(\alpha)| d\alpha & : \frac{T^2}{48\sqrt{5}N^3} + \frac{T^3}{384\sqrt{5}N^4} + \frac{1}{8N^3} + \frac{T}{6\sqrt{3}(4n+1)(4n+3)} + \frac{T}{6\sqrt{3}(4n+3)} \\
& + \frac{288\sqrt{11}N^3\sqrt{4n+3}}{3870720\sqrt{21}N^4 + 3N^6} + \frac{36\sqrt{13}N^2\sqrt{4n+3}}{720\sqrt{17}N^4\sqrt{4n+3}}; \\
\int_{-T}^{T} |f(\alpha)\alpha| d\alpha & : \frac{T}{48\sqrt{3}N^3} + \frac{3T^2}{128\sqrt{3}N^4} + \frac{1}{5(4n+3)} + \frac{1}{3\sqrt{3}(4n+1)(4n+3)} \\
& + \frac{T^4}{144\sqrt{9}N^3\sqrt{4n+3}} + \frac{7T^6}{3870720\sqrt{19}N^4 + 3N^6}; \\
\int_{-T}^{T} |f(\alpha)\alpha^2| d\alpha & : \frac{1}{48N^4} + \frac{T}{128\sqrt{3}N^4} + \frac{1}{2N^2\sqrt{4n+1}} + \frac{T^3}{288\sqrt{7}(4n+1)N^3} \\
& + \frac{21T^8}{3870720\sqrt{15}N^4 + 3N^6}; \\
\int_{-T}^{T} |f(\alpha)\alpha^3| d\alpha & : \frac{1}{384N^4} + \frac{T^3}{48\sqrt{7}N^2\sqrt{4n+1}} + \frac{1}{2N^2\sqrt{4n+1}} + \frac{T^2}{288\sqrt{5}(4n+1)N^3} \\
& + \frac{T^5}{35T^7} + \frac{3870720\sqrt{15}N^4 + 3N^6}{36\sqrt{7}N^2\sqrt{4n+3}}.
\end{align*}
\]
\[
\int_{-T}^{T} \left| f(\alpha)^5 \right| d\alpha : \frac{1}{288N^3\sqrt{4n+1}} + \frac{T}{16\sqrt{3}N^2\sqrt{4n+1}} + \frac{21T^5}{3870720\sqrt{13}N^6\sqrt{(4n+3)}} + \frac{T^3}{720\sqrt{7}(4n+3)N^4};
\]
\[
\int_{-T}^{T} \left| f(\alpha)^6 \right| d\alpha : \frac{1}{48N^2\sqrt{4n+1}} + \frac{T}{3870720\sqrt{5}\sqrt{4n+3}N^6};
\]
\[
\int_{-T}^{T} \left| f(\alpha)^7 \right| d\alpha : \frac{T^3}{3870720\sqrt{7}\sqrt{4n+3}N^6};
\]
\[
\int_{-T}^{T} \left| f^{(1)}(\alpha) \right| d\alpha : \frac{1}{2N^2} + \frac{T}{6\sqrt{3}(4n+1)(4n+3)} + \frac{T^2}{6\sqrt{3}(4n+3)} + \frac{T^3}{6\sqrt{7}N^2\sqrt{4n+3}};
\]
\[
\int_{-T}^{T} \left| f^{(1)}(\alpha)^2 \right| d\alpha : \frac{1}{6\sqrt{(4n+1)(4n+3)}} + \frac{1}{6(4n+3)};
\]
\[
\int_{-T}^{T} \left| f^{(1)}(\alpha)^3 \right| d\alpha : \frac{1}{6\sqrt{(4n+1)}N^2};
\]
\[
\omega(\alpha) = \alpha^2 \left( \frac{\alpha^4}{18} + 1 \right) \frac{1}{\pi^{1/2}} + \frac{|\alpha|^{17/2}}{187n^{1/4}};
\]
\[
\left[ \int_{-T}^{T} \left| f(\alpha)^2 \right| d\alpha \right]^{1/2} : \frac{\pi T^{5/2}}{6\sqrt{2N\sqrt{4n+3}}};
\]
\[
\left[ \int_{-T}^{T} \left| f(\alpha)^3 \right| d\alpha \right]^{1/2} : \frac{\pi T^{1/2}}{8N^2T^{-1/2}};
\]
\[
\left[ \int_{-T}^{T} f_N(\alpha)^2 \right]^{1/2} : \frac{\sqrt{2\pi}T^{3/2}}{8N^2};
\]
\[
\left[ \int_{-T}^{T} f_N(\alpha)^4 \right]^{1/2} : \frac{\sqrt{2\pi}T^{-1/2}}{12N\sqrt{4n+3}};
\]

where \( f_N = (f_{\xi(-N,N)})^\vee = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin N(t-s)}{t-s} f(s) ds; \)
\[
\left[ \int_{-T}^{T} |f(\alpha)\omega(\alpha)|^2 d\alpha \right]^{1/2} : \frac{\sqrt{2}}{3} \pi^{1/2} T^{5/2} \left( \frac{3}{2} \right)^{1/4} \frac{1}{n\sqrt{4n+1}} + \frac{1}{24\sqrt{2}} \pi^{3/2} T^{5/2} \left( \frac{3}{2} \right)^{1/4} \frac{1}{n\sqrt{4n+3}} + \sqrt{2\pi}T^{-1/2} \frac{1}{n^2}\omega(T);
\]
\[
\left[ \int_{-T}^{T} |f_N(\alpha)\omega(\alpha)|^2 d\alpha \right]^{1/2} : 4\sqrt{2}\pi^{1/2} T^{-1/2} \left( \frac{3}{2} \right)^{1/4} \frac{1}{n} + \frac{1}{12\sqrt{2}} \pi^{3/2} T^{5/2} \left( \frac{3}{2} \right)^{1/4} \frac{1}{n};
\]
\[
\left| f(-T) \right| + \left| f(T) \right| : \frac{1}{2N^2} + \frac{T}{8N^3} \left( 1 + \frac{1}{\sqrt{3}} \right) + \frac{T^2}{6\sqrt{(4n+1)(4n+3)}} \left( 1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} \right) + \frac{T^3}{6\sqrt{N^2}\sqrt{4n+1}} + \frac{T^2}{6(4n+3)} \left( 1 + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{5}} \right) + \frac{T^3}{6\sqrt{7}N^2\sqrt{4n+3}};
\]
\[
\left[ \int_{|\alpha|>T} |f(\alpha)|^2 d\alpha \right]^{1/2} 
= 2\sqrt{2\pi^{1/2}} T^{-1/2} \left( \frac{3}{2} \right)^{1/4} \omega(T) \frac{1}{n} + \frac{\sqrt{2\pi T^{7/2}}}{12 N \sqrt{4n+3}} 
+ \frac{1}{24\sqrt{2}} \pi^{3/2} T^{-1/2} \left( \frac{3}{2} \right)^{1/4} \omega(T) \frac{1}{n},
\]

\[
\left[ \int_{|\omega|>N} |\hat{f}(\omega)|^2 d\omega \right]^{1/2} 
= 2\sqrt{2\pi^{1/2}} T^{-1/2} \left( \frac{3}{2} \right)^{1/4} \omega(T) \frac{1}{n} + \frac{\sqrt{2\pi T^{7/2}}}{12 N \sqrt{4n+3}} 
+ \frac{1}{24\sqrt{2}} \pi^{3/2} T^{-1/2} \left( \frac{3}{2} \right)^{1/4} \omega(T) \frac{1}{n}.
\]

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Mei Ling Huang Address: Department of Mathematics and Statistics, Brock University, St. Catharines, Canada
E-mail address: mhuang@brocku.ca

Ron Kerman Address: Department of Mathematics and Statistics, Brock University, St. Catharines, Canada
E-mail address: rkerman@brocku.ca

Susanna Spektor Address: Department of Mathematics and Statistics, Brock University, St. Catharines, Canada
E-mail address: sanaspek@gmail.com