A CHARACTERIZATION OF SIMPLICIAL LOCALIZATION
FUNCTORS

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Abstract. We characterize simplicial localization functors among relative
functors from relative categories to simplicial categories as any choice of ho-
motopy inverse to the delocalization functor of Dwyer and the second author.

1. AN OVERVIEW

We start with some preliminaries.

1.1. Relative categories. As in [BK] we denote by RelCat the category of
(small) relative categories and relative functors between them, where by a relative
category we mean a pair \((C, W)\) consisting of a category \(C\) and a subcategory
\(W \subset C\) which contains all the objects of \(C\) and their identity maps and of which
the maps will be referred to as weak equivalences and where by a relative func-
tor between two such relative categories we mean a weak equivalence preserving
functor.

1.2. Homotopy equivalences between relative categories. A relative functor
\(f : X \rightarrow Y\) between two relative categories (1.1) is called a homotopy equiva-
lence if there exists a relative functor \(g : Y \rightarrow X\) (called a homotopy inverse
of \(f\)) such that the compositions \(gf\) and \(fg\) are naturally weakly equivalent (i.e.
can be connected by a finite zigzag of natural weak equivalences) to the identity
functors of \(X\) and \(Y\) respectively.

1.3. DK-equivalences. A map in the category SCat of simplicial categories (i.e.
categories enriched over simplicial sets) is [Be] called a DK-equivalence if it in-
duces weak equivalences between the simplicial sets involved and an equivalence
of categories between their homotopy categories, i.e. the categories obtained from
them by replacing each simplicial set by the set of its components.

Furthermore a map in RelCat will similarly be called a DKequivalence if its
image in SCat is so under the hammock localization functor [DK2]

\[ L^H : \text{RelCat} \rightarrow \text{SCat} \]

(or of course the naturally DK-equivalent functors \(\text{RelCat} \rightarrow \text{SCat}\) considered in
[DK] and [DHKS, 35.6]).

We will denote by both

\[ \text{DK} \subset \text{SCat} \quad \text{and} \quad \text{DK} \subset \text{RelCat} \]

the subcategories consisting of these DK-equivalences.

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Next we define what we mean by

1.4. Simplicial localization functors. In defining DK-equivalences in \( \text{RelCat} \) we used the hammock localization functor and not one of the other DK-equivalent functors mentioned because, for our purposes here it seemed to be the more convenient one. However in other situations the others are more convenient and it therefore makes sense to define in general a simplicial localization functor as any functor \( \text{RelCat} \to \text{SCat} \) which is naturally DK-equivalent to the functors mentioned above \((1.3)\).

We also need

1.5. The relativization functor. In contrast with the situation mentioned in 1.4 there is a preferred choice for a relativization functor

\[ R: \text{SCat} \to \text{RelCat} \]

which is a kind of inverse of the simplicial localization functor, namely the delocalization mentioned in \([\text{DK3, 2.5}]\) which assigns to an object \( A \in \text{SCat} \) its relative flattening which is the relative category which consists of

(i) a category which is the Grothendieck construction on \( A \), where \( A \) is considered as a simplicial diagram of categories, and

(ii) its subcategory obtained by applying the same construction to the subobject of \( A \) which consists of its objects only.

Our main result then is

1.6. Theorem. A relative functor

\[ (\text{RelCat}, \text{DK}) \to (\text{SCat}, \text{DK}) \]

is a simplicial localization functor \((1.5)\) iff it is a homotopy inverse \((1.2)\) of the realization functor \((1.5)\)

\[ \text{Rel}: (\text{SCat}, \text{DK}) \to (\text{RelCat}, \text{DK}) \].

We end with some

1.7. Comments on the proof of 1.6. The proof of theorem 1.6 heavily involves some of the results of \([\text{DK}]\) and \([\text{DK3, 2.5}]\) and we therefore first (in \(2\)) review some of the results of these papers.

In \(3\) we then actually prove theorem 1.6. It turns out however that in addition to the results mentioned in \(2\) we need a property of the hammock localization of which we will give two proofs. The first is a very short one based on a remark of Toen and Vezzosi \([\text{TV, 2.2.1}]\) involving the homotopy category of \( \text{SCat} \). The other, which is due to Bill Dwyer, relies heavily on \([\text{DK}]\) and \([\text{DK2}]\) and is longer, but has the “advantage” of taking place in the model category itself.

2. Preliminaries

In preparation for the proof (in \(3\)) of theorem 1.6 we review here some of the results of \([\text{DK}], [\text{DK2}]\) and \([\text{DK3, 2.3}]\) which will be needed.
2.1. The hammock localization. In the proof of [1.6] we will make extensive use of the hammock localization $L^H$ of [DK2] because, unlike the other simplicial localization functors, it has the property that every relative category $(C, W)$ comes with a natural embedding $C \to L^H(C, W)$.

2.2. The category RelSCat. This will be the category which has as its objects the pairs $(A, U)$ where $A \in \text{SCat}$ and $U \subset A$ is a subobject which contains all the objects of $A$.

One then can consider RelCat as a full subcategory of RelSCat and [DK2, 2.5] extend the functor $L^H: \text{RelCat} \to \text{SCat}$ to a functor $L^H: \text{RelSCat} \to \text{SCat}$ by sending an object of RelSCat to the diagonal of the bisimplicial set obtained from it by dimensionwise application of the hammock localization.

To deal with [DK] and [DK3, 2.5] it will be convenient to introduce a notion of

2.3. Neglectable categories. Given an object $(A, U) \in \text{RelSCat}$ (2.2) we will say that $U$ is neglectable in $A$ if every map of $U$ goes to an isomorphism in $\pi_0 A$.

2.4. Some results from [DK]. [DK, 3.4 and 5.1] imply

(i) Let $A$ be a category, let $U$ and $V \subset A$ be subcategories which contain all the objects of $A$ and let $U \cup V \subset A$ denote the subcategory spanned by $U$ and $V$ and assume that $V$ is neglectable in $L^H(A, U)$. Then the induced map

$$L^H(A, U) \to L^H(A, U \cup V) \in \text{SCat}$$

is a DK-equivalence.

Similarly [DK, 6.4] implies

(ii) Let $(B, V) \in \text{RelSCat}$ be such that $V$ is neglectable in $B$. Then the induced map (2.1 and 2.2)

$$B \to L^H(B, V) \in \text{SCat}$$

is a DK-equivalence.

We end with a brief review of

2.5. The relativization functor [DK3, 2.5]. The relativization functor is the functor

$$\text{Rel}: \text{SCat} \to \text{RelCat}$$

which sends an object $A \in \text{SCat}$ to the object $(bA, \text{id}) \in \text{RelCat}$, where $bA$ is the flattening of $A$, i.e. the category which has as objects the pairs $(A, n)$, where $A$ is an object of $A$ and $n$ is an integer $\geq 0$ and which has as maps $(A_1, n_1) \to (A_2, n_2)$ the pairs $(a, q)$ where $a$ is a map $A_1 \to A_2 \in A_{n_2}$ and $q$ is a simplicial operator from dimension $n_1$ to dimension $n_2$ and $\text{id} \in bA$ is the subcategory consisting of the maps $(a, q)$ for which $a$ is an identity map.

It then was noted in [DK3, 2.5] that, for every object $A \in \text{SCat}$, there exists an object $\overline{A} \in \text{SCat}$ with the same object set as $bA$ with the following properties:

(i) There is a natural monomorphism $A \to \overline{A}$ which is a DK-equivalence.
(ii) There is a natural embedding $bA \to \overline{A}$ with the property that (if the image of $\text{bid} \in bA$ in $\overline{A}$ is also denoted by $\text{bid}$) the induced map

$$L^H(bA, \text{bid}) \to L^H(\overline{A}, \text{bid}) \in \text{SCat}$$

is a DK-equivalence.

(iii) $\text{bid}$ is neglectable in $\overline{A}$ (2.3) and hence the embedding (2.1 and 2.2)

$$\overline{A} \to L^H(\overline{A}, \text{bid}) \in \text{SCat}$$

is a DK-equivalence.

It follows that

(iv) $A$ and $L^H \text{Rel } A$ can be connected by the natural zigzag of DK-equivalences

$$A \to \overline{A} \to L^H(\overline{A}, \text{bid}) \leftarrow L^H(bA, \text{bid}) = L^H \text{Rel } A$$

which in turn implies that

(v) Rel is a relative functor (1.3)

$$\text{Rel}: (\text{SCat, DK}) \to (\text{RelCat, DK}) .$$

3. A proof of theorem 1.6

To prove theorem 1.6 is suffices, in view of 2.5(iv) and (v), to prove

3.1. Proposition. Every object $(C, W) \in \text{RelCat}$ is naturally DK-equivalent to $\text{Rel } L^H(C, W)$.

Proof. Consider the commutative diagram in $\text{RelSCat}$ (2.2)

$$
\begin{array}{cccc}
(C, W) & \xrightarrow{a} & (bL^H(C, W), \text{bid}) = RL^H(C, W) \\
\downarrow{c} & & \downarrow{c} \\
(bL^H(C, W), \text{bid} \cup W) & \xleftarrow{b} & (L^H(C, W), W) \\
\downarrow{d} & & \downarrow{g} \\
(L^H(C, W), \text{bid} \cup W) & \xrightarrow{f} & (L^H(C, W), \text{bid})
\end{array}
$$

in which

- $c$ is as in 2.1, $f$ is as in 2.5(i), $d$ and $e$ are as in 2.5(ii) and $a$ is the unique map such that $da = fc$, and
- the symbol $\cup$ is as in 2.4(i) and, in the formulas which involve two $W$’s, the second $W$ is the image of the $W$ in the upper left $(C, W)$.

Then it suffices to show that $a$ and $b$ are DK-equivalences in $\text{RelCat}$ or equivalently that $L^H a$ and $L^H b$ are DK-equivalences in $\text{SCat}$.

This is done as follows:

The map $f$ admits a factorization

$$(L^H(C, W), W) \xrightarrow{x} (L^H(C, W), W) \xrightarrow{y} (L^H(C, W), \text{bid} \cup W)$$
in which clearly $W$ is neglectable (2.3) in $L^H(C, W)$ and hence, in view of 2.3(ii) and 2.4(i), $L^H x$ is a DK-equivalence and $W$ is neglectable in $L^H(C, W)$. It follows that (2.4(iii)) $L^H y$ is a DK-equivalence and hence (2.4(ii)) so is $L^H g$.

Furthermore, in view of 2.3(ii), $L^H e$ is a DK-equivalence and consequently $L^H d$ is a DK-equivalence and $W$ is neglectable in $L^H(b L^H(C, W), bid)$ which implies that $L^H b$ is a DK-equivalence.

It thus remains to prove that $L^H a$ is a DK-equivalence, but this now follows from

3.2. Proposition. $L^H c: L^H(C, W) \rightarrow L^H(L^H(C, W), W)$ is a DK-equivalence.

We will give two proofs of this proposition.

The first is short and is based on a remark by Toen and Vezzosi involving the homotopy category $Ho SCat$ of $SCat$.

The other, due to Bill Dwyer, is longer but takes place inside the model category $S O-Cat$ of the simplicial categories with a fixed object set $O$ (in this case the object set of $C$).

They both involve the commutative diagram $C \rightarrow L^H(C, W) \rightarrow L^H(L^H(C, W), W)$

(i) in which the unmarked maps are as in 2.1, and

(ii) in which, in view of 2.4(ii) the right hand map is a DK-equivalence.

3.3. The short proof. In view of [TV, 2.2.1] the maps at the right and the bottom in (i) have the same image in $Ho SCat$ and as (2.2(ii)) the one on the right is a DK-equivalence, so is the one at the bottom.

3.4. The longer proof. We start with a brief discussion of

(i) Homotopy pushouts

Given a model category together with a choice of cofibrant replacement functor and a choice of functorial factorization of maps into a cofibration followed by a trivial fibration, associate with every zigzag $Y \leftarrow X \rightarrow Z$ a commutative diagram $Y^c \leftarrow X^c \rightarrow Z^c$

as follows. The pentagon is obtained by applying the cofibrant approximation functor and the two triangles by means of the functorial factorization. Consequently the maps indicated ~ are weak equivalences.

Then the pushout $Y^c \amalg_{X^c} Z^c$ is a homotopy pushout of the zigzag $Y \leftarrow X \rightarrow Z$. 
Clearly this construction is functorial in the sense that every diagram of the form

\[
\begin{array}{c}
Y_0 & \xleftarrow{y} & X_0 & \xrightarrow{x} & Z_0 \\
\downarrow & & \downarrow & & \downarrow \\
Y_1 & \xleftarrow{\bar{y}} & X_1 & \xrightarrow{\bar{x}} & Z_1
\end{array}
\]

induces a map

\[
Y_0 \amalg X_0 \Rightarrow Z_0 \Rightarrow Y_1 \amalg X_1 \Rightarrow Z_1
\]

which is a weak equivalence whenever \(y, x\) and \(z\) are.

Next we discuss

(ii) **The original simplicial localization functor \(L\)** \([\text{DK}]\)

We will work in the model category \(S_{O-Cat}\) \([\text{DK}]\) of the simplicial categories with a fixed object set \(O\) (which will be the object set of \(C\)). The weak equivalences in the model structure are the DK-equivalences.

As \([\text{DK} 4.1]\) \(L(C, W)\) is the pushout of the zigzag

\[
F_*C \leftarrow F_*W \rightarrow F_*W[F_*W^{-1}]
\]

and the map \(F_*W \rightarrow F_*C\) is a cofibration, it follows from \([\text{DK} 8.1]\) that this pushout is also a homotopy pushout. Consequently

\((*)\) \(L(C, W)\) is naturally DK-equivalent to

\[
(F_*C)^{\mathbb{c}} \amalg (F_*W)^{\mathbb{c}} \cong (F_*W[F_*W^{-1}])^{\mathbb{c}}
\]

Now we turn to

(iii) **The hammock localization \(L^H\)** \([\text{DK2}]\)

In view of \([\text{DK2} 2.5]\) the functors \(L\) and \(L^H\) are naturally DK-equivalent and there exists a diagram of the form

\[
\begin{array}{c}
F_*C & \xleftarrow{F_*W} & F_*W[F_*W^{-1}] \equiv L(W, W) \\
\cong & \cong & \\
F_*C & \xleftarrow{F_*W} & L^H(F_*W, F_*W) \\
\downarrow & & \downarrow \\
C & \xleftarrow{W} & L^H(W, W)
\end{array}
\]

in which the vertical maps are DK-equivalences. Hence

\((*)\) \(L^H(C, W)\) is naturally DK-equivalent to

\[
C^{\mathbb{c}} \amalg W^{\mathbb{c}} \cong L^H(W, W)^{\mathbb{c}}
\]

and \(L^H(L^H(C, W), W)\) is naturally DK-equivalent to

\[
C^{\mathbb{c}} \amalg W^{\mathbb{c}} \cong L^H(W, W)^{\mathbb{c}} \amalg W^{\mathbb{c}} \cong L^H(W, W)^{\mathbb{c}}
\]
in which the two middle maps \( W^c \to L^H(W, W)^c \) are the same and which therefore is the same as

\[
Q = \colim \begin{array}{ccc}
\mathcal{C}^c & \overset{W^c}{\leftarrow} & L^H(W, W)^c \\
\downarrow & & \downarrow \\
\mathcal{L}^H(W, W)^c & \overset{L^H(W, W)^c}{\leftarrow}
\end{array}
\]

It follows that diagram 3.2(i) is DK-equivalent to the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C}^c & \overset{\mathcal{C}^c \amalg W^c}{\to} & L^H(W, W)^c \\
\downarrow & & \downarrow \\
\mathcal{C}^c \amalg W^c & \overset{\mathcal{C}^c \amalg L^H(W, W)^c}{\to} & Q
\end{array}
\]

in which \( u \) is obtained by mapping the zigzag \( \mathcal{C}^c \leftarrow W^c \to L^H(W, W)^c \) to the upper zigzag in the diagram whose colimit is \( Q \) and \( v \) is obtained by mapping it to the lower zigzag.

In view of 3.2(ii) the map \( u \) is a DK-equivalence and as \( v = Tu \) where \( T : Q \to Q \) denotes the automorphism which switches the two copies of \( L^H(W, W)^c \), so is the map \( v \) and therefore also the desired map

\[ L_c : L^H(C, W) \to L^H(L^H(C, W), W) \]

References

[BK] C. Barwick and D. M Kan, Relative categories; another model for the homotopy theory of homotopy theories, To appear.

[Be] J. E. Bergner, A model category structure on the category of simplicial categories, Trans. Amer. Math. Soc. 359 (2007), no. 5, 2043–2058.

[DHKS] W. G. Dwyer, P. S. Hirschhorn, D. M. Kan, and J. H. Smith, Homotopy limit functors on model categories and homotopical categories, Mathematical Surveys and Monographs, vol. 113, American Mathematical Society, Providence, RI, 2004.

[DK1] W. G. Dwyer and D. M. Kan, Simplicial localizations of categories, J. Pure Appl. Algebra 17 (1980), no. 3, 267–284.

[DK2] , Calculating simplicial localizations, J. Pure Appl. Algebra 18 (1980), no. 1, 17–35.

[DK3] , Equivalences between homotopy theories of diagrams, Algebraic topology and algebraic K-theory (Princeton, N.J., 1983), Ann. of Math. Stud., vol. 113, Princeton Univ. Press, Princeton, NJ, 1987, pp. 180–205.

[TV] B. Toën and G. Vezzosi, Homotopical algebraic geometry. I. Topos theory, Adv. Math. 193 (2005), no. 2, 257–372.

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