Discrete element simulations of stress distributions in silos: crossover from two to three dimensions

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The transition from two-dimensional (2D) to three-dimensional (3D) granular packings is studied using large-scale discrete element computer simulations. We focus on vertical stress profiles and examine how they change with dimensionality. We compare results for packings in 2D, quasi-2D packings between flat plates, and 3D packings. Analysis of these packings suggests that the Janssen theory does not fully describe these packings, especially at the top of the piles, where a hydrostatic-like region of vertical stress is visible in all cases. We find that the interior of the packing is far from incipient failure, while in general, the forces at the walls are close to incipient failure.

I. INTRODUCTION

The stress within a silo packed with granular material has long been of interest in the engineering and physics communities. As early as 1895, Janssen, in his pioneering work, constructed a model to describe the vertical stress in a silo. Treating the granular pack as a continuous medium where a fraction \( \kappa \) of vertical stress is converted to horizontal stress, Janssen was able to derive a simple functional form for the vertical stress. One main assumption of this model is that the forces of friction between particles and walls are at the Coulomb failure criterion: 

\[
F_t = \mu_w F_n,
\]

where \( F_t \) is the magnitude of the tangential friction force, \( F_n \) is the normal force at the wall, and \( \mu_w \) is the coefficient of friction for particle-wall contacts. This assumption is also known as incipient failure. This theory qualitatively describes the crossover to a depth-independent vertical stress, although quantitative discrepancies between the Janssen theory and experiment have been observed. The assumption of incipient failure is believed to be the source of the discrepancies, but this has never been conclusively proven due to the difficulty of experimental measurements of internal stress. Numerous improvements have been added to the theory over time, but none of these have gained wide acceptance.

Recently, well-controlled experiments have been carried out on granular packs in silos to test the suitability of Janssen’s theory in ideal conditions. Vanel and Clément constructed their initial packings by tapping a loose poured packing to increase its density. The base of the packing was then slowly moved downward to more fully mobilize the grains, and then the packing was allowed to settle. They measured the apparent mass at the bottom of the silo as a function of the filling mass of the silo. Their experiment did not follow the Janssen form, and they found the best agreement with a phenomenological model, which contains elements of Janssen’s original analysis. We describe this model in more detail in Sec III. Other more recent experimental studies in which the side walls were dragged upward to fully mobilize the packing have found agreement with the Janssen form.

A variety of discrete element and continuous finite element simulation methods have been developed to describe the stresses in a silo due to the difficulty of experimental verification. Most discrete element simulations thus far have been confined to two-dimensional (2D) systems. However, there is wide disagreement within the community on the predictive power of these models and the proper approach to take for accurate simulation. Those simulations that are carried out in three dimensions (3D) usually do so via finite-element methods that yield little information on the internal structure or forces in granular packs. Some 3D discrete element simulations have been performed, but these use periodic boundary conditions in the two directions perpendicular to gravity. Though these studies provide useful information on the internal structure of these packings, they can give no information on vertical stresses or forces induced by the walls of the container. We have recently carried out 3D discrete element simulations of stress in granular materials in cylinders. Here we compares simulations in 2D and 3D and study the crossover from two to three dimensions.

We present large-scale discrete element simulations of granular packings in 2D, quasi-2D, and 3D containers (silos). Our aim is to understand the vertical stress profiles of these granular packings, particularly the crossover from two to three dimensions, which has largely been unexplored. Both monodisperse and polydisperse particle systems are studied. We explicitly test the suitability of the Janssen theory to the vertical stress profiles produced by our

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simulations and test the validity of the Janssen theory’s assumptions. We find that the interior of the packing is far from incipient failure, while in general, the forces at the walls are close to incipient failure. In all cases, there is a region at the top of the pile where the particle-wall forces are far from incipient failure and the Janssen form is not observed.

Section II discusses the simulation technique and the method used to generate the packings. Section III presents the vertical stress profiles. We discuss their characteristics and compare our results to the classical theory of Janssen as well as two modified forms of the Janssen analysis. In Section IV we show how forces in the packings are not at incipient failure at the top and in the bulk of the packing and relate this to the failure of the Janssen analysis to fully explain our packing. Finally, we conclude and summarize the work in Section V.

II. SIMULATION METHOD

We present discrete element simulations in two and three dimensions of model systems of \( N \) mono and polydisperse spheres of fixed density \( \rho \). The mass of a given particle is \( m_i = \frac{4}{3}\pi d_i^3 \), where \( d_i \) is the diameter of a given particle. Particle diameters are uniformly distributed between \( \bar{d} \pm \Delta \). For this study, we use \( \Delta/d = 0, 0.1, \) and 0.5. \( N \) varies from 10,000 to 50,000 particles. The system is constrained by either an open rectangular box or an open cylinder with its axis along the vertical \( z \) direction. The box has length \( L \), width \( W \), and height \( H \), centered at \( x = y = 0 \) and bounded below with a flat base at \( z = 0 \). The cylinder has radius \( R \) and is centered on \( x = y = 0 \). It is bounded below with a flat base at \( z = 0 \). In some cases, a layer of randomly-arranged immobilized particles approximately 2\( \bar{d} \) high were used to provide a rough base. We vary the length \( L \) and width \( W \) of the box and the radius \( R \) of the cylinder. This work builds on previous 3D simulations of packings in a cylinder [17].

The spheres interact only on contact through a spring-dashpot interaction in the normal and tangential directions to their lines of centers. Contacting spheres \( i \) and \( j \) positioned at \( \mathbf{r}_i \) and \( \mathbf{r}_j \) experience a relative normal compression \( \delta = |\mathbf{r}_{ij} - \bar{d}| \), where \( \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j \), which results in a force

\[
\mathbf{F}_{ij} = \mathbf{F}_n + \mathbf{F}_t. 
\]

The normal and tangential contact forces are given by

\[
\mathbf{F}_n = f(\delta/d)(k_{n,t}\delta \mathbf{n}_{ij} - m_{eff} \gamma_n v_n) 
\]

\[
\mathbf{F}_t = f(\delta/d)(-k_t \Delta s_{ij} - m_{eff} \gamma_t v_t) 
\]

where \( \mathbf{n}_{ij} = \mathbf{r}_{ij}/r_{ij} \), with \( r_{ij} = |\mathbf{r}_{ij}| \). \( v_n \) and \( v_t \) are the normal and tangential components of the relative surface velocity, and \( k_{n,t} \) and \( \gamma_{n,t} \) are elastic and viscoelastic constants, respectively. \( m_{eff} = \frac{m_i m_j}{m_i + m_j} \) for interactions between particles \( i \) and \( j \), and \( m_{eff} = m_i \) for interactions between particle \( i \) and a wall, where the mass of the wall is assumed to be infinite. \( f(x) = 1 \) for Hookean (linear) contacts, while for Hertzian contacts \( f(x) = x^{D/2-1} \) — the two force models differing only in 3D. \( \Delta s_{ij} \) is the elastic tangential displacement between spheres, obtained by integrating tangential relative velocities during elastic deformation for the lifetime of the contact. The magnitude of \( \Delta s_{ij} \) is truncated as necessary to satisfy a local Coulomb yield criterion \( F_i \leq \mu F_n \), where \( F_i \equiv |\mathbf{F}_i| \) and \( F_n \equiv |\mathbf{F}_n| \) and \( \mu \) is the particle-particle friction coefficient. Frictionless spheres correspond to \( \mu = 0 \). Particle-wall interactions are treated identically, though the particle-wall friction coefficient \( \mu_w \) is set independently. A more detailed description of the model is available elsewhere [19]. We also tested an alternate form of the \( F_i \) force due to Haff and Werner [20] that does not have history effects. This model produces a hydrostatic stress profile in all cases. Without the relative transverse displacement in the force model, \( F_i = 0 \) in the quasi-static limit and the walls cannot support stress.

We present all the physical quantities in our simulations in terms of \( \bar{d} \), the average diameter; \( \bar{m} \) the mass of a particle with average diameter \( \bar{d} \); and \( g \), the force of gravity. Most of these simulations were run with \( k_n = 2 \times 10^5 \bar{m}g/\bar{d} \), \( k_t = 2\bar{m} \), and \( \gamma_n = 50\sqrt{g/\bar{d}} \). For Hookean springs we set \( \gamma_t = 0 \). In this case, the coefficient of restitution for \( \Delta = 0 \) is \( \epsilon = 0.8 \). For Hertzian springs, \( \gamma_t = \gamma_n \). For Hertzian springs in \( D = 3 \), \( \epsilon \to 0 \) as \( v \to 0 \), so that it takes much longer for energy to drain out of the pack and for the kinetic energy per particle to decrease to sufficiently low levels. For this reason, it is much more computationally expensive to use Hertzian springs to form packings. The convenient time unit is \( \tau = \sqrt{\bar{d}/\bar{g}} \), the time it takes a particle to fall its radius from rest under gravity. For this set of parameters, the time step \( \delta t = 10^{-4}\tau \). In most simulations, the particle-particle friction and particle-wall friction are the same, \( \mu = \mu_w = 0.5 \).

The two-dimensional (2D) simulations were performed by fixing \( y = 0 \) for all particles and removing any velocity or acceleration components in the \( y \) direction. Some polydispersity (\( \Delta/d > 0 \)) is required to prevent crystallization in 2D, so all 2D simulations were run with a polydispersity of \( \Delta/d = 0.1 \).
All of our results will be given in dimensionless units based on \( \bar{m}, \bar{d}, \) and \( g \). Physical experiments often use glass spheres of \( \bar{d} = 100 \mu m \) with \( \rho = 2 \times 10^3 \text{kg/m}^3 \). In this case, the physical elastic constant would be \( k_{\text{glass}} \sim 10^{10} \bar{m}g/\bar{d} \). A spring constant this high would be prohibitively computationally expensive, because the time step must have the form \( \delta t \propto k^{-\frac{1}{2}} \) for collisions to be modeled effectively. We have found that increasing \( k \) in our simulations does not appreciatively change the physical results [19].

We generate our packings by mimicking the pouring of particles at a fixed height \( Z \) into the container. For computational efficiency a group of \( M \) particles is added to the simulation on a single time step. This is done by inserting the \( M \) particles at non-overlapping positions within a thin cylindrical region of radius \( R - \bar{d} \) with height \( Z - \bar{d} < z < Z \). The \( x, y, \) and \( z \) coordinates of the particles are chosen randomly within this insertion region. The height of insertion \( z \) determines the initial \( z \)-velocity \( v_z \) of the particle — \( v_z \) is set to the value it would have after falling from a height \( Z \). After a time \( \sqrt{2}\tau \), another group of \( M \) particles is inserted. This methodology generates a steady stream of particles, as if they were poured continuously from a hopper (see Figure 1). The rate of pouring is controlled by setting \( M \) to correspond to a desired volume fraction of particles within the insertion region. For example, for an initial volume fraction of \( \phi_i = 0.13 \) in a cylinder of \( R = 10\bar{d} \), the pouring rate is \( \approx 45 \) particles/\( \tau \). This method is similar to the homogeneous “raining” methods used in experiments [21].

The simulations were run until the kinetic energy per particle was less than \( 10^{-8} \bar{m}g\bar{d} \). The resultant packing is considered quiescent and used for further analysis [16]. In 3D, Hertzian packings have difficulty reaching this criteria due to long-lived pressure waves that traverse the entire pile. In these packings, the kinetic energy per particle tends to stabilize at approximately \( 10^{-3} \bar{m}g\bar{d} \) for very long times. By adding a viscous damping term proportional to \( v \) after the pack has stabilized at \( 10^{-3} \bar{m}g\bar{d} \), we were able to reduce the kinetic energy per particle of these systems to less than \( 10^{-8} \bar{m}g\bar{d} \). A comparison of the stress profiles before and after the application of this viscous damping term showed no change. Although a change of kinetic energy of this order has a small effect on the coordination number of the packing, it does not have any effect on the vertical stress. All vertical stress profiles for Hertzian packings presented in the following sections were generated with this technique.

These simulations were performed on a parallel cluster computer built with DEC Alpha processors and Myrinet interconnects using a parallel simulation code optimized for short-range interactions [19, 22]. A typical simulation to pour 50,000 particles into a 3D cylindrical silo requires \( 5 \times 10^8 \) time steps, which requires roughly 40 CPU hours on 50 processors.

We show in Figure 1 how a typical packing is created. The figure shows three snapshots in time as particles are poured into a quasi-2D system of length \( 30\bar{d} \) and width \( 5\bar{d} \).

The strong effect of wall friction on the capacity of a silo to support stress as found experimentally is shown in Figure 2 where the particle-wall friction \( \mu_w \) is varied, while keeping the particle-particle friction constant at \( \mu = 0.5 \).
FIG. 2: Packings of \(N = 10000\) particles in U-Tubes with different \(\mu_w\). Particles were poured into the left-hand side and allowed to settle into packings. a) \(\mu_w = 0.0\) and no stress is carried by the walls. b) \(\mu_w = 0.1\) and the walls support slight stresses and there is a height difference between the left and right tubes. c) \(\mu_w = 0.5\) and there is a great difference between the left and right tubes.

The geometry of the system is a three-dimensional U-tube, with both ends open. Particles are poured into one end and the resultant packing is allowed to settle. In the case where \(\mu_w = 0\), no stress can be supported by the side walls, and the granular packing is liquid-like: the particle height is the same in both sides of the U-tube. For \(\mu_w = 0.1\), the right side is considerably lower than the left, though some particles are higher than the bend in the tube. The walls support some of the stress, but the pressure is strong enough to force some particles up the right-hand tube. For \(\mu_w = \mu = 0.5\), there are almost no particles on the right side of the U-tube, and all the weight on the left side is supported by the walls of the tube. In all these cases, the important factor is the particle-wall friction, as the particle-particle friction remains unchanged.

III. STRESSES

Of particular interest in the construction of silos is the distribution of stresses [2]. In a liquid, hydrostatic pressure increases with depth. Granular materials support shear, so the side walls of a container can support some of this shear, provided \(\mu_w > 0\). The problem of the resultant vertical stress in a silo after filling has a long history, beginning with Janssen in 1895. Janssen’s analysis [1, 22] is still in use today, although it rests on a series of assumptions which have not been fully tested.

For a container with static wall friction \(\mu_w\) and granular pack of total height \(z_0\), the Janssen analysis predicts the vertical stress \(\sigma_{zz}(z)\) at a height \(z\) is

\[
\sigma_{zz}(z) = \rho g l \left[ 1 - \exp \left( -\frac{z_0 - z}{l} \right) \right]
\]

where \(\rho\) is the volumetric density, \(l\) is the decay length, and \(z_0\) is the top of the packing. The decay length \(l\) is determined by the geometry. For a 2D container of length \(L\), \(l_{2D} = \frac{L}{2 \mu_w}\). For a quasi-2D container of length \(L\) and width \(W\), \(l_{\text{quasi-2D}} = \frac{L W}{4 L + 2 W \mu_w}\). For a 3D cylindrical container of radius \(R\), \(l_{3D} = \frac{R}{\mu_w}\) [24].

Another two-parameter fit was proposed by Vanel and Clément [4] to reconcile their experimental findings on weakly shaken packings with Janssen theory. The fit assumes a region of perfect hydrostaticity, followed by a region that conforms to the Janssen theory.

\[
z_0 - z < a : \quad \sigma_{zz}(z) = \rho g (z_0 - z)
\]

\[
z_0 - z > a : \quad \sigma_{zz}(z) = \rho g \left( a + l \left[ 1 - \exp \left( -\frac{z_0 - z - a}{l} \right) \right] \right)
\]

The two parameters are \(a\), the height of the crossover, and \(l\) the decay length, with the same values for different geometries as in the Janssen theory.
FIG. 3: Vertical stress $\sigma_{zz}$ for 2D packings. The packings in order of decreasing depth are (a) $N \approx 20000$, length $L = 30d$, (b) $N \approx 10000$, $L = 20d$, and (c) $N \approx 10000$, $L = 30d$. The particles are polydisperse with $\Delta = 0.1d$. The two fits to (a) are a fit to Janssen (Eqn. 4) (dotted line) and a fit to Vanel-Clément (Eqn. 5) (dashed line).

FIG. 4: Vertical stress for quasi-2D packings with increasing widths, shown as $(z - z_0)/d$, where $z_0$ is the top of the pile. Six separate packings are shown, with the width $W$ increasing from $1.2d$ to $5.0d$. The pressure-independent height does not increase monotonically with $W$, and it saturates at $W = 2.0d$. Results for Hookean springs and polydispersity $\Delta = 0.1d$.

We first present vertical stresses for the two-dimensional (2D) case. These are shown in Figure 3. Unlike previous studies [11], these piles are extremely deep so that the height-independent region can be easily observed. The Janssen theory does not describe the 2D stress profiles well. There is a clear linear hydrostatic-like region at the top of the packing that the Janssen theory does not account for. The Vanel-Clément form is a much closer fit to the data. In Figure 3, the deepest pile of length $30d$ is fit to both forms. The Janssen form predicts $l/d = 84.6$ and thus $\kappa = 0.354$, while the Vanel-Clement form yields $a = 35$ and $l/d = 49.6$ and thus $\kappa = 0.605$. As we will see below, there are a number of differences between 2D and 3D packings. One difference is the height of the non-saturated region, which is on the order of $5 - 7L$ in 2D and $6R$ in 3D. There is about a factor of 2 difference between the diameter or width of the container in 2D compared to 3D. In addition, the saturated pressure is also much larger in 2D than the 3D case. Both of these effects seem to arise from the necessity of all the force being carried by the side walls. A similar effect is seen at the bottom of the packing, where the slight increase in the pressure at the bottom of the pile is over a much larger region than in the 3D case as shown below. This increase occurs because geometrically the walls can not support forces from particles below this height.

To study the crossover from 2D to 3D, we studied a series of packings in rectangular containers with increasing width, ranging from $W = 1.2d$ to $5d$. We show in Figure 4 the vertical stress profiles for these quasi-2D packings. The stress profiles show the same hydrostatic region followed by saturation as the 2D profiles, though the hydrostatic region at the top of the pile is much smaller in the quasi-2D case. In addition, the saturation stress value does not increase monotonically with width. The saturation stress decreases as one goes from a width of $1.2d$ to $1.4d$, but increases again as one reaches $2d$. A width of $2d$ seems to represent a transition to a more 3D-like behavior, and the
saturation stress does not change appreciably for larger widths. This suggests that $L$ is the dominant length scale for widths greater than $2\bar{d}$, while for smaller widths, the width $W$ is the dominant length scale for determination of the vertical stress.

We also investigated quasi-2D packings with no friction on the long walls $\mu_w = 0$, while the short side walls have $\mu_w = 0.5$. This system was modeled to understand how the long side walls affect the geometry. This system, as shown in Figure 5, produces stress profiles very different from the quasi-2D case with friction on all walls. Two widths were used and both show much larger final stress values and much larger hydrostatic regions. This suggests that both sets of walls are important for determining the stress profiles for a wide range of widths. The stress profiles seen in this case resemble much more the 2D stress profiles presented in Fig. 3. These results demonstrate that experiments in thin plates meant to approximate 2D systems are more representative of 3D systems than that of pure 2D systems.

We show in Figure 6 vertical stress profiles for 3D systems in cylindrical packings. Changing the force model from a Hookean to a Hertzian interaction has a very small effect on the final packing. Changing from a monodisperse to a very highly polydisperse system ($\Delta/\bar{d} = 0.5$) also does not change the overall stress profile, although it does increase the height of the packing. This suggests that although polydispersity is required in 2D systems to avoid crystallization, it has little effect on the stress profile. It is thus reasonable to compare monodisperse 3D packings to polydisperse 2D systems.

In this case we see that the important transition is from the 2D system to the quasi-2D system. In most respects the quasi-2D system is much closer to the 3D system than the pure 2D system. The size of the hydrostatic region

FIG. 5: Vertical stress for quasi-2D packings of length $30\bar{d}$ and width $1.6\bar{d}$ (lower curve) and width $2\bar{d}$ (upper curve). The long walls of length $L = 30\bar{d}$ have $\mu_w = 0$, while the short walls have $\mu_w = 0.5$. The stress profiles are very close to those of the 2D simulations.

FIG. 6: Vertical stress $\sigma_{zz}$ for 3D cylindrical packings with radius $10\bar{d}$ and $N = 50,000$. The four different packings are (a) a monodisperse ($\Delta = 0$) packing with Hookean contacts, (b) a monodisperse packing with Hertzian contacts, (c) a polydisperse ($\Delta/\bar{d} = 0.1$) packing with Hertzian contacts, and (d) a polydisperse ($\Delta/\bar{d} = 0.5$) packing with Hertzian contacts. The packings in order of ascending height are c, a, b, d. Results for case a are averaged over 4 runs while the other three cases are for one run each. Results for $\mu = \mu_w = 0.5$. 

We also investigated quasi-2D packings with no friction on the long walls $\mu_w = 0$, while the short side walls have $\mu_w = 0.5$. This system was modeled to understand how the long side walls affect the geometry. This system, as shown in Figure 5, produces stress profiles very different from the quasi-2D case with friction on all walls. Two widths were used and both show much larger final stress values and much larger hydrostatic regions. This suggests that both sets of walls are important for determining the stress profiles for a wide range of widths. The stress profiles seen in this case resemble much more the 2D stress profiles presented in Fig. 3. These results demonstrate that experiments in thin plates meant to approximate 2D systems are more representative of 3D systems than that of pure 2D systems.

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FIG. 7: Probability distributions $P(\zeta)$ in the height-independent pressure region in the bulk of the packing (a) and at the side walls (b), each normalized by its maximum value $P(\zeta_{\text{max}})$. $\zeta = F_t/\mu F_n$ in (a) and $\zeta = F_t/\mu_w F_n$ in (b). The legend applies to both figures. Forces in the bulk are far from the Coulomb failure criterion, while those at the walls are very close to it.

changes most markedly from the pure 2D system to the quasi-2D system. The stress values in the depth of the pile are much larger for the 2D case than the quasi-2D case and are comparable to the 3D case.

IV. DISTRIBUTION OF FORCES

The previous section has demonstrated that the stress profiles of our granular packings do not conform to the Janssen theory. The historical assumption has been that the assumption of incipient failure in the Janssen analysis is responsible for the failure of the theory. We thus tested directly the Janssen assumptions of incipient failure in our simulations, i.e. whether the tangential forces at the wall are actually at the Coulomb yield criterion $F_t = \mu_w F_n$. We define $\zeta = F_t/\mu F_n$ in the bulk of the packing and $\zeta = F_t/\mu_w F_n$ for forces at the wall. If a specific force is at the Coulomb failure criterion, $\zeta = 1$. We analyzed the 3D systems shown in Figure 6. By examining the distribution of forces in the interior of our packings, we find that no particle-particle contacts are at the Coulomb criterion irrespective of method or level of polydispersity, as shown in Figure 7a. We find similar results in the interior of 2D packings. However the particle-wall forces in the height-independent stress region are much closer to the Coulomb criterion. In the four systems examined, the bulk of the particle-wall tangential forces are close to incipient failure, in contrast to the particle-particle tangential forces in the bulk, as seen in Figure 7b. The amount of polydispersity and the contact force model have essentially no effect on how close the system is to incipient failure. We have also investigated the role of $\mu_w > \mu$ in Ref. [17]. In this case, the particle-wall forces are always less than the Coulomb criterion, as the walls cannot support larger tangential forces than those supported in the bulk.

As discussed above, all of our packings exhibit a linear region in the stress profile at the top of the packing. We can also examine the Janssen assumption in this linear region. In contrast to the situation in the height-independent stress region of the packing, near the top of the pile the forces at the wall are far from the Coulomb criterion, as shown in Figure 8. This explains why the Janssen form is not obeyed in this region: the walls in this region do not support stress and thus the stress profile is hydrostatic.

V. CONCLUSIONS

We examined the differences between vertical stress in granular packings in 2D, quasi-2D, and 3D. We focused on the vertical stress of these packings and observed that 2D packings support much less vertical stress than 3D packings. In contrast, quasi-2D packings (those packings between thin walls) rapidly converge to the behavior of 3D packings as the wall thickness is increased, though their progression is not monotonic. We also show that polydispersity and different force models do not have a large effect on the vertical stress in 3D packings. In all cases, we show that the Janssen assumption of incipient failure is not justified everywhere in the packing.

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FIG. 8: Probability distributions $P(\zeta)$ at the side wall in the linear hydrostatic region at the top of the packing with $\mu = \mu_w = 0.5$, each normalized by its maximum value $P(\zeta_{\text{max}})$. $\zeta = F_t / \mu_w F_n$. In contrast to the behavior in the height-independent pressure region, the forces at the walls are far from the Coulomb failure criterion in all cases.

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