Real Functions and its Differentiation
in Alternative Set Theory

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Abstract
In the previous paper (Kiri Sakahara and Takashi Sato. Basic Topological Concepts and a Construction of Real Numbers in Alternative Set Theory. arXiv e-prints, arXiv:2005.04388, May 2020), the authors displayed basic topological concepts and a construction of a system of real number in alternative set theory (AST). The present paper is a continuation of that research providing additional treatments of real functions. The basic properties of differentiation in AST are preserved as in the conventional calculus.

1 Introduction
The authors displayed basic topological concepts and a construction of a system of real number in alternative set theory (AST, for short) in Sakahara and Sato [2]. The present paper is a continuation of that research providing additional treatments of real functions.

Almost all concepts, formulations and statements displayed here are due to Tsujishita [3]. The only exception is Theorem 7 which asserts that derivatives of real functions given here coincide with that defined in the traditional manner. Proofs of the statements are also virtually identical. However, the paper does not skip many of them since there remain essential differences with regard to the way sets are formulated, so too the way proofs proceed.

The readers can find relatively compact explanation of AST, such as an axiomatic system of AST, in Vopěnka and Trlifajová [5] and Sakahara and Sato [1, 2].

2 A system of numbers
Let us start with constructing a number system in accordance with Vopěnka [4]. The class of natural numbers $N$ is defined as:

$$N = \left\{ x : \forall y \in x \ (y \subseteq x) \land (\forall y, z \in x \ (y \subseteq z \lor y = z \lor z \subseteq y)) \right\},$$

while the class of finite natural numbers $FN$ consists of the numbers represented by finite sets

$$FN = \{ x \in N : \text{Fin}(x) \}$$

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in which $\text{Fin}(x)$ means that each subclass of $x$ is a set. The class of all integers $\mathbb{Z}$ and that of all rational numbers are defined respectively as:

$$Z = N \cup \{-a ; a \in N \land a \neq 0\} \quad \text{and} \quad Q = \left\{ \frac{x}{y} ; \ x, y \in Z \land y \neq 0 \right\}.$$

$BQ \subseteq Q$ denotes the class of bounded rational numbers and $FQ \subseteq BQ$ the class of finite rational numbers, i.e.,

$$BQ = \{ q \in Q ; (\exists i \in FN) (|q| \leq i) \} \quad \text{and} \quad FQ = \left\{ q \in Q ; (\exists x, y \in FN) \left( q = \frac{x}{y} \lor q = -\frac{x}{y} \right) \right\}.$$

Real numbers are defined in AST as an equivalence class of bounded rational numbers. The reason behind this construction lies in the human’s inability to distinguish two mutually close rational numbers. This idea is grasped by the indiscernibility equivalence $\equiv$, on the class $Q$ of all rational numbers. One of the definitions of the indiscernibility equivalence $\equiv$ is given as:

$$p \equiv q \equiv \left( (\exists k) (\forall i > 0) (|p| < k \land |p - q| < \frac{k}{i}) \lor (\forall k) ((p > k \land q > k) \lor (p < -k \land q < -k)) \right)$$

in which the letters $i, j, k$ denote finite natural numbers, i.e., $i, j, k \in FN$ for notational ease, hereafter. For each $q \in Q$ the notation $\text{mon} (q) = \{ s \in Q ; s \equiv q \}$, called the monad of $q$, represents the class of all rational numbers which are indiscernible from $q$. A real number is denoted as a monad $\text{mon} (q)$ of some rational number $q$. Two limiting cases are denoted as:

$$\infty = \{ q \in Q ; (\forall i) (q > i) \} \quad \text{and} \quad -\infty = \{ q \in Q ; (\forall i) (q < -i) \}.$$

The class of all real numbers $R$ is defined as:

$$R \equiv \{ \text{mon} (x) ; x \in BQ \} = BQ / \equiv.$$

Let us denote a real continuum as $\mathcal{R} = (Q, \equiv)$, where a continuum is a pair of classes $\mathcal{C} = (C, \equiv_C)$, in which a set-theoretically definable class $C$ is called as a support of $\mathcal{C}$.

Finite arithmetic operation of real numbers is same as usual. The countable sum of nonnegative real number $r_i$ is defined by the following. Let $b_i \in r_i$. Since the numbers $r_i$ are nonnegative, prolonging the sequence $(b_i)_{i \in FN}$ onto a set $(b_i)_{i \in \alpha}$, there exists an infinite natural number $\delta \leq \alpha$, $\delta \notin FN$ which satisfies for any infinite natural number $\gamma \leq \delta$, $\gamma \notin FN$ the following indiscernibility equivalence

$$\sum_{n=0}^{\gamma} b_n \equiv \sum_{n=0}^{\delta} b_n.$$

We put $\sum_{i \in FN} r_i = r$, where $r \in R \cup \{ \pm \infty \}$ satisfies $\sum_{n=0}^{\delta} b_n \in r$. 

\footnote{This concept is due to Tsujishita [3].}
3 Morphisms

Let $C_i (i = 1, 2)$ be continua, then a function $F : C_1 \rightarrow C_2$ is continuous if $x \doteq y$ implies $F(x) \doteq F(y)$. Two continuous functions $F$ and $G$ are indiscernible, denoted simply as $F \doteq G$, if for all $x \in C_1$, $F(x) \doteq G(x)$ follows.

The morphism $\mathcal{F}$ between two continua is defined as follows.

(1) A morphism from $C_1$ to $C_2$ is a monad mon $(F)$ denoted simply as $[F]$ for some continuous function $F$ from $C_1$ to $C_2$.

(2) If $C_i (i = 1, 2)$ are continua, the notation $\mathcal{F} : C_1 \rightarrow C_2$ means that $\mathcal{F}$ is a morphism from $C_1$ to $C_2$.

(3) If $\mathcal{F}$ is a morphism, then the expression $G \in \mathcal{F}$ means $G \doteq F$ in which $\mathcal{F} = [F]$. If $G \in \mathcal{F}$, we say that the morphism $\mathcal{F}$ is represented by $G$ and $G$ represents $\mathcal{F}$.

(4) If $\mathcal{F}$ and $\mathcal{G}$ are morphisms represented respectively by $F$ and $G$, then the expression $\mathcal{F} = \mathcal{G}$ means $F \doteq G$.

It is essential that morphisms are defined as the monads of continuous functions. When a set-theoretically definable function $F : C_1 \rightarrow C_2$ has an indiscernible gap at $x \in C_1$, that is, $\neg(F(x) \doteq F(y))$ for some $y \doteq x$, its value at mon $(x)$ cannot be determined uniquely since $x \doteq c_1$ y but $\neg(F(y) \doteq c_2 F(x))$, thus, $\text{mon}_{\mathcal{F}}(F(x)) \cap \text{mon}_{\mathcal{G}}(F(y)) = \emptyset$.

Contrary to the framework of Tsujihita, in which the morphisms are not guaranteed to be classes, they are in AST.

The identity morphism $\text{id}_{C}$ is represented by the identity function $\text{id}_C$. Given $\mathcal{F}_1 : C_1 \rightarrow C_2$ and $\mathcal{F}_2 : C_2 \rightarrow C_3$, the composition $\mathcal{F}_2 \circ \mathcal{F}_1 : C_1 \rightarrow C_3$ is given as the morphism $[F_2 \circ F_1]$ represented by the composition $F_2 \circ F_1$ of $F_1 : C_1 \rightarrow C_2$ and $F_2 : C_2 \rightarrow C_3$.

A morphism $\mathcal{F} : C_1 \rightarrow C_2$ is injective and surjective if it is represented by a continuous function $F : C_1 \rightarrow C_2$ satisfying respectively

$$(\forall x, y \in C_1) (F(x) \doteq F(y) \text{ implies } x \doteq y).$$

and

$$(\forall x_2 \in C_2) (\exists x_1 \in C_1) (F(x_1) \doteq x_2).$$

A morphism $\mathcal{F} : C_1 \rightarrow C_2$ is an equivalence if there is a uniquely determined morphism $\mathcal{F}^{-1}$, which is the inverse of $\mathcal{F}$, satisfying

$$\mathcal{F}^{-1} \circ \mathcal{F} = \text{id}_{C_1} \text{ and } \mathcal{F} \circ \mathcal{F}^{-1} = \text{id}_{C_2}.$$
Every pair of representations of $\mathcal{F}$ and $\mathcal{F}^{-1}$, say $F : C_1 \to C_2$ and $F^{-1} : C_2 \to C_1$, satisfies

$$F^{-1} \circ F \cong id_{C_1} \quad \text{and} \quad F \circ F^{-1} \cong id_{C_2}.$$ 

$F^{-1}$ is said to be almost inverse of $F$.

If there is an equivalence $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$, the continuum $\mathcal{C}_1$ is said to be equivalent to $\mathcal{C}_2$ denoted as $\mathcal{C}_1 \simeq \mathcal{C}_2$.

Let $\mathcal{C}$ be a continuum and $\mathcal{C}_i \subset \mathcal{C}$ ($i = 1, 2$) be subcontinua. An equivalence $\mathcal{A} : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be a quasi-identity if its representation satisfies $\alpha(x) = x$ for all $x \in C_1$. A quasi-identity, by definition, is uniquely determined if it exists.

**Proposition 1** (Proposition 2.4.2 of Tsujishita [3]). Let $r$ be a nonzero nonfinite rational number, in which there exists $\tau \in \mathbb{N} \setminus FN$ and $r = \frac{1}{\tau}$. The inclusion

$$\iota_r : rZ \to Q$$

represents a quasi-identity, for which the function $\kappa_r : Q \to rZ$ defined by

$$\kappa_r(s) \equiv \left[ \frac{s}{r} \right] r$$

in which $[x]$ denotes the integer part of $x$, gives an almost inverse of $\iota_r$.

**Proof.** It is obvious that $\kappa(\iota(nr)) = nr$ for $n \in \mathbb{N}$. Since $x < [x+1] \leq x + 1$, the following inequations follow

$$s = \left( \frac{s}{r} \right) r < \left[ \frac{s}{r} + 1 \right] r \leq \left( \frac{s}{r} + 1 \right) r = s + r,$$

thus, $\left[ \frac{s}{r} \right] r \cong s$ follows since $r \neq 0$. Finally, $\iota \circ \kappa \cong id$, follows. \hfill \Box

## 4 Continua of functions

Let $\mathcal{C}_i$ ($i = 1, 2$) be continua in which $C_1$ is set-theoretically definable\(^3\). The *continuum of functions* are given as

$$\mathcal{F}(\mathcal{C}_1, \mathcal{C}_2) = (\text{Fun}(C_1, C_2), \simeq)$$

Subcontinua of $\mathcal{F}(\mathcal{C}_1, \mathcal{C}_2)$ consisting of continuous functions are continua of morphisms from $\mathcal{C}_1$ to $\mathcal{C}_2$, denoted as $\mathcal{F}_C(\mathcal{C}_1, \mathcal{C}_2)$.

While $\mathcal{F}(\mathcal{C}_1, \mathcal{C}_2)$ is not necessarily equivalent to $\mathcal{F}(\mathcal{C}_1', \mathcal{C}_2')$ even if $\mathcal{C}_1 \simeq \mathcal{C}_1'$, continua of morphism $\mathcal{F}_C(\mathcal{C}_1, \mathcal{C}_2)$ always do as is shown in the next proposition.

**Proposition 2** (Proposition 5.1.1 of Tsujishita [3]). Let $\mathcal{C}_i$ and $\mathcal{C}_i'$ ($i = 1, 2$) be continua and $G_i : C_i \to C_i'$ be representations of equivalences with almost inverse $G_i^{-1} : C_i' \to C_i$. Define functions from Fun $(C_1, C_2)$ to Fun $(C_1', C_2')$ as

$$\alpha(F) \equiv G_2 \circ F \circ G_1^{-1},$$

$\alpha(F) \equiv G_2 \circ F \circ G_1^{-1}$,

$\alpha(F)$

\(^3\) A class $\{ x : \varphi(x) \}$ is said to be set-theoretically definable if $\varphi(x)$ is a set formula.

\(^4\) Since $C_1$ is not always a set, Fun $(-, -)$, which is a class of all functions from left $-$ to right $-$, remains a codable class. In regard to codable class, see 1.5 of Vopěnka [3].
and from \( \text{Fun}(C'_1, C'_2) \) to \( \text{Fun}(C_1, C_2) \) as

\[
\beta(F') \equiv G_2^{-1} \circ F' \circ G_1.
\]

Then, \( \alpha \) represents an equivalence \( \mathcal{A} : \mathcal{F}_C(C_1, C_2) \to \mathcal{F}_C(C'_1, C'_2) \) with an almost inverse \( \mathcal{B} \), one of whose representations is \( \beta \).

**Proof.** Let \( F, H \in \mathcal{F}_C(C_1, C_2) \) which satisfies \( F \equiv H \) and \( x \in C'_1 \). Then \( F(G_1^{-1}(x)) \equiv H(G_1^{-1}(x)) \) holds. Consequently the next equations hold

\[
\alpha(F)(x) = G_2(F(G_1^{-1}(x))) \equiv G_2(H(G_1^{-1}(x))) = \alpha(H)(x)
\]

and, thus, \( \alpha \) is continuous. Continuity of \( \beta \) is verified in a similar way.

Next, we show that \( \beta \) is an almost inverse of \( \alpha \). Since \( G_i^{-1} \) are almost inverse of \( G_i \), \( G_i^{-1}(G_i(x)) \equiv x \) holds for every \( x \in C_i \) and \( i = 1, 2 \). Then \( F(G_i^{-1}(G_i(x))) \equiv F(x) \) follows by the continuity of \( F \). Consequently,

\[
\beta(\alpha(F))(x) = G_2^{-1}(G_2(F(G_1^{-1}(G_1(x))))) \equiv F(x)
\]

and the equation below holds

\[
\beta(\alpha(F)) \equiv F.
\]

Similarly, \( \alpha(\beta(F')) \equiv F' \) follows. \( \Box \)

### 5 Real functions

Let \( C \) be a continuum. A **real function on** \( C \) is a morphism from \( C \) to \( R \).

**Example 1 (Polynomial functions).** The polynomial function \( \mathcal{F} : R \to R \) is given simply as

\[
\mathcal{F}(x) \equiv \sum_{i=0}^{n} p_i \cdot x^i.
\]

in which \( p_i \in R \).

**Example 2 (Exponential).** For \( \tau \in N \setminus FN \) and \( q \in Q \), define the approximation of the \( \tau \)th power of the Napier’s constant by

\[
\exp(q, \tau) \equiv \sum_{i=0}^{\tau} \frac{q^i}{i!}.
\]

Uniqueness of the value of the exponential is verified by Proposition 7.2.1 of Tsujishita [3]. Then the **exponential function** is given as

\[
e^x = \exp(x) \equiv \text{mon} (\exp(q, \tau)) \quad \text{where } q \in x.
\]
in which the choices of $\tau \in N \setminus FN$ and $q \in x$ are arbitrary. Uniqueness of its value of the function also verified by Proposition 7.2.5 and fundamental properties are verified by Propositions 7.2.6 and 7.2.10 of Tsujishita [3].

It is worth mentioning that one of the usual ways to define exponential function is given as:

$$\exp(x) \equiv \sum_{t=0}^{\infty} \frac{x^t}{t!}.$$  

As it is defined in the last paragraph of Section 2, countable sum of a given sequence of real numbers $\left(\frac{q^t}{t!}\right)_{t \in \tau}$ is given by that of prolonged sequence of their representative rational numbers, say, $\left(\frac{q^t}{t!}\right)_{t \in \tau}$ in which $q \in x$.

**Example 3 (Logarithm).** For $\tau \in N \setminus FN$ and $q \in \{x \in \mathbb{Q}; x > 0\}$, define the approximation of the natural logarithm of $q$ by

$$\log(q, \tau) \equiv \max \left\{k \in \mathbb{Z}; \left(\exp \left(\frac{k}{\tau}, \tau\right) \leq q \right) \wedge (|k| \leq \tau^2)\right\}.$$  

Then the natural logarithm function can be given as

$$\log(x) \equiv \text{mon} \left(\log(q, \tau)\right) \quad \text{where} \quad q \in x, \quad \text{for} \quad x \in (0, \infty),$$

in which the choices of $\tau \in N \setminus FN$ and $q \in x$ are arbitrary. Uniqueness of its value and fundamental properties are verified in Proposition 7.2.12 and 7.2.13 of Tsujishita [3].

As it is seen in the last example, the definition of the function coincides with one of the usual ways to define it. Not only these two cases, every traditional real function can be dealt exactly the same manner.

In the remainder of the section, let us investigate behaviors of real functions divided by indiscernible intervals to probe into features regarding derivatives of real functions, which we take up in the next section.

**Proposition 3 (Proposition 7.5.1 of Tsujishita [3]).** Let $F, G$ be rational-valued functions on $C$ and $a, b \in C$ which are mutually indiscernible, that is, $F \equiv G$ and $a \equiv_C b$, then the following two conditions are equivalent: for any given finite $n \in FN$

$$\left( (\exists \varepsilon_a \in \text{mon} (0)) (d(x, a) > |\varepsilon_a|) \wedge (x \equiv_C a) \Rightarrow \left( \frac{|F(x)|}{d(x, a)^n} \leq 0 \right) \right) \quad (1)$$

and

$$\left( (\exists \varepsilon_b \in \text{mon} (0)) (d(x, b) > |\varepsilon_b|) \wedge (x \equiv_C b) \Rightarrow \left( \frac{|G(x)|}{d(x, b)^n} \leq 0 \right) \right). \quad (2)$$

**Proof.** Suppose that the property $\text{(1)}$ holds. Choose $x \in C$ which satisfies $d(x, a) > |\varepsilon_a|$ and $x \equiv_C a$. Put $\varepsilon_b$ as:

$$\varepsilon_b = \max \{|\varepsilon_a| + d(a, b), 3 \cdot d(a, b)| \leq 0.$$
Then for all \( x \in C \) which satisfies \( 0 \leq d(x, b) > \varepsilon_b \), the following inequations follow by the triangle inequality:

\[
d(x, a) \geq d(x, b) - d(a, b) > |\varepsilon_a|,
\]
and

\[
d(x, a) \geq d(x, b) - d(a, b) > 3 \cdot d(a, b) - d(a, b) = 2 \cdot d(a, b).
\]

The last inequality implies that

\[
d(x, b) \geq d(x, a) - d(a, b) > \frac{1}{2} \cdot d(x, a),
\]
and, therefore, the indiscernibility equivalence below follows:

\[
|G(x)| \leq 2^n \frac{|F(x)|}{d(x, a)^n} = 0.
\]

The converse is true by the similar argument. \( \square \)

**Proposition 4** (Proposition 7.5.3 of Tsujishita [3]). Let \( \varepsilon, \eta \in \left\{ \frac{1}{\gamma}; \gamma \in N \setminus FN \right\} \), \( \alpha : [0, 1]^n \rightarrow [0, 1]^n \), where \( [0, 1]_\varepsilon \equiv \{ x\varepsilon; x \in [0, 1] \} \), represents an equivalence, and \( F \) be a continuous rational-valued function on \( [0, 1]^n \). Then for \( n \in FN \) and \( a \in [0, 1]^n \),

\[
\left( (\exists \delta_\eta \in \text{mon } (0)) (d(y, \alpha(a)) > |\delta_\eta|) \land (y \equiv \alpha(a)) \Rightarrow \left( \frac{|F(y)|}{d(y, \alpha(a))^n} \equiv 0 \right) \right) \tag{3}
\]

if and only if

\[
\left( (\exists \delta_\varepsilon \in \text{mon } (0)) (d(x, a) > |\delta_\varepsilon|) \land (x \equiv a) \Rightarrow \left( \frac{|F(\alpha(x))|}{d(x, a)^n} \equiv 0 \right) \right). \tag{4}
\]

**Proof.** Suppose the property (3) holds. Let \( \beta \) be an almost inverse of \( \alpha \). Since \( \alpha \) represents an equivalence there exists \( \delta \in \text{mon } (0) \) which satisfies

\[
d(x, \alpha(x)) < |\delta|
\]
for all \( x \in [0, 1]_\varepsilon \). By the triangle inequality, the following inequalities are fulfilled for all \( a, x \in [0, 1]_\varepsilon \):

\[
d(\alpha(x), \alpha(a)) \leq d(x, \alpha(x)) + d(x, a) + d(a, \alpha(a)) < d(x, a) + 2|\delta|.
\]

Let us choose \( x \in [0, 1]_\varepsilon \) which satisfies \( d(x, a) > 4|\delta| \) and \( x \equiv a \). Then, the following inequality is drawn from the last one.

\[
d(\alpha(x), \alpha(a)) < \frac{3}{2} d(x, a).
\]

It implies that the following indiscernible equality.

\[
\frac{|F(\alpha(x))|}{d(x, a)^n} \leq \left( \frac{3}{2} \right)^n \cdot \frac{|F(\alpha(x))|}{d(\alpha(x), \alpha(a))^n} \equiv 0.
\]

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The converse case can also be shown by almost the same argument. Suppose the property (4) holds. Put \( b = \alpha(a) \). Since \( a \neq \beta(b) \), the next property is met

\[
\left( (\exists \delta_b \in \text{mon}(0)) (d(x, \beta(b)) > |\delta_b|) \land (x \neq \beta(b)) \right) \Rightarrow \left( \frac{|F(\alpha(x))|}{d(x, \beta(b))^n} = 0 \right).
\]

The following inequality is also satisfied since \( \beta \) is an almost inverse of \( \alpha \)

\[d(y, \beta(y)) < |\gamma|\]

for all \( y \in [0, 1]_{\eta} \). By the same argument as the previous case, the inequality below follows:

\[d(\beta(y), \beta(b)) \leq d(y, \beta(y)) + d(x, b) + d(b, \beta(b)) < d(y, b) + 2|\gamma|.
\]

Let us choose \( y \in [0, 1]_{\eta} \) which satisfies \( d(y, b) > 4|\gamma| \) and \( y \neq b \). Then, the following inequality is drawn from the last one.

\[d(\beta(y), \beta(b)) < \frac{3}{2}d(y, b) = \frac{3}{2}d(y, \alpha(a)).\]

It implies that the following inequality:

\[
\frac{|F(y)|}{d(y, \alpha(a))^n} = \frac{|F(\alpha(\beta(y)))|}{d(y, \alpha(a))^n} \leq \left( \frac{3}{2} \right)^n \cdot \frac{|F(\alpha(\beta(y)))|}{d(\beta(y), \beta(b))^n} \leq 0.
\]

\( \square \)

6 Differentiation

Let \( \varepsilon > 0 \) be a nonfinite rational number \( \varepsilon = \frac{1}{\tau} \) for \( \tau \in N \setminus \{F\} \), and \( ([0, 1]_{x}, \sim) \) be a continuum representing a real interval \([0, 1]\), in which \([0, 1]_{x} \equiv \left\{ \frac{n}{\tau} : 0 \leq n \leq \tau \right\} \land (n \in N) \} \).

Let \((f, [0, 1]_{x})\) denote a pair of a continuous rational-valued function \( f \) on \([0, 1]_{x} \) and a \([0, 1]_{x}\)-valued function \( \kappa_{\varepsilon} : [0, 1] \rightarrow [0, 1]_{x} \) given in Proposition 4 as

\[
\kappa_{\varepsilon}(s) = \left[ \frac{s}{\varepsilon} \right] \varepsilon.
\]

It is said that a real function \( \mathcal{F} \) is represented by \((f, [0, 1]_{x})\), or simply by \( f \) \( \mathcal{F} \) iff \( f \circ \kappa_{\varepsilon} \) represents \( \mathcal{F} \).

Let us also denote \([0, 1]_{x} \setminus \{1\} \) as \([0, 1]_{x}^{-} \), and \( x^{+} \equiv x + \Delta x \) in which \( \Delta x \equiv \varepsilon \). Then a difference quotient of a rational-valued function \( f \) on \([0, 1]_{x} \) at \( x \in [0, 1]_{x}^{-} \) is given as

\[
\frac{\Delta f(x)}{\Delta x} = \frac{f(x^{+}) - f(x)}{\Delta x}.
\]

\( \Delta f(x) \equiv f(x^{+}) - f(x) \) is a difference of \( f \) at \( x \), and \( \frac{\Delta f}{\Delta x} \) is a difference quotient function of \( f \).

At a first glance, it may seem that the difference quotient function depends on the choice of \( \varepsilon \), but it is not. It is verified by the next proposition.

\( \text{The existence of such } f \text{ is guaranteed by Proposition 4 and 2.} \)
Proposition 5 (Proposition 8.2.1 of Tsuji[3]). Let \( \mathcal{F} \) be a real function on \([0,1]\). Let \( (f_i, [0,1]_\varepsilon) \ (i = 1, 2) \) be representations of \( \mathcal{F} \) such that the difference quotients of \( f_i \) are continuous. Then the real functions on \([0,1]\) represented by the difference quotients \( \left( \frac{\Delta f_i}{\Delta x}, [0,1]_\varepsilon \right) \ (i = 1, 2) \) coincide.

To prove the proposition, let us first confirm that difference quotients coincide with derivatives.

Theorem 6 (Theorem 8.3.1 of Tsuji[3]). If \( f \) is a function on \([0,1]_\varepsilon\) with continuous difference quotients, then for \( a \in [0,1]_\varepsilon \),

\[
\left( (\exists \eta \in \text{mon}(0)) \ (x - a > |\eta|) \wedge (x \equiv a) \Rightarrow \left( \frac{f(x) - f(a)}{x - a} - \frac{\Delta f}{\Delta x}(a) = 0 \right) \right).
\]

Proof. Pick \( x \in [0,1]_\varepsilon \) which satisfies \( x \equiv a \) and \( a < x \), then

\[
f(x) - f(a) = \sum_{a \leq u < x} \frac{\Delta f}{\Delta x}(u)(u^+ - u)
= \sum_{a \leq u < x} \frac{\Delta f}{\Delta x}(a)(u^+ - u) + \sum_{a \leq u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right) (u^+ - u)
= \frac{\Delta f}{\Delta x}(a)(x - a) + \sum_{a \leq u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right) (u^+ - u).
\]

Then, the following equation is met:

\[
f(x) - f(a) - \frac{\Delta f}{\Delta x}(a)(x - a) = \sum_{a \leq u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right) (u^+ - u).
\]

Since \( \frac{\Delta f}{\Delta x} \) is continuous and \( a \equiv x \), \( \left| \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right| < |q| \) for all \( q \in FQ \). Thus, for any \( q \in FQ \), the next inequations are met.

\[
\left| \sum_{a \leq u < x} \left( \frac{\Delta f}{\Delta x}(u) - \frac{\Delta f}{\Delta x}(a) \right) (u^+ - u) \right| < \frac{x - a}{\varepsilon} \cdot |q| \cdot \varepsilon = (x - a) \cdot |q|.
\]

It implies \( \left| \frac{f(x) - f(a) - \frac{\Delta f}{\Delta x}(a)(x - a)}{x - a} \right| < (x - a) \cdot |q| \) for all \( q \in FQ \), and thus,

\[
\frac{f(x) - f(a)}{x - a} - \frac{\Delta f}{\Delta x}(a) \equiv 0.
\]

\( \square \)

A proof of Proposition 5. Put \( \alpha_i(x) = \left[ \frac{a}{\varepsilon_i} \right] \varepsilon_i \) for \( x \in [0,1]_Q \equiv \{ z \in Q; 0 \leq z \leq 1 \} \) and \( i \in \{ 1, 2 \} \). Then, for any \( a \in [0,1]_Q \) the next property holds by Theorem 6

\[
\left( (\exists \eta \in \text{mon}(0)) \ (y - \alpha_i(a) > |\eta|) \wedge (y \equiv \alpha_i(a)) \Rightarrow \left( \frac{f_i(y) - f_i(\alpha_i(a))}{y - \alpha_i(a)} - \frac{\Delta f_i}{\Delta x}(\alpha_i(a)) = 0 \right) \right).
\]

By Proposition 4, the following is met:

\[
\left( (\exists \delta \in \text{mon}(0)) \ (x - a > |\delta|) \wedge (x \equiv a) \Rightarrow \left( \frac{f_i(\alpha_i(x)) - f_i(\alpha_i(a))}{x - a} - \frac{\Delta f_i}{\Delta x}(\alpha_i(a)) = 0 \right) \right).
\]

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Theorem 7. Let
\[
\left( \exists \delta \in \text{mon} \left( 0 \right) \right) \left( x - a > |\delta| \right) \land \left( x \equiv a \right)
\Rightarrow \left( \frac{\Delta f}{\Delta x}(\alpha_1(a)) - \frac{\Delta f}{\Delta x}(\alpha_2(a)) \equiv 0 \right).
\]
Then, the following indiscernibility is met
\[
\text{there exists a rational sequence } (a_x) \quad \text{that there always exists } j > i.
\]
Hence, \(\frac{\Delta f}{\Delta x}(\alpha_1(a)) \equiv \frac{\Delta f}{\Delta x}(\alpha_2(a))\) is fulfilled. \(\Box\)

Now, a real function \(F\) on a real interval \([0, 1]\) is said to be differentiable if it is represented by \((f, [0, 1])\) and its difference quotient function \(\frac{\Delta f}{\Delta x}\) is continuous. The real function represented by \(\left( \frac{\Delta f}{\Delta x}, [0, 1] \right)\) is a derivative of \(F\) denoted by \(F'\).

It may seem unclear whether it is compatible with the derivative derived by an ordinary manner. As a matter of fact, it is. Specifically, its representation is indiscernible with the difference quotient function \(\frac{\Delta f}{\Delta x}\) to \(F\).

Let \((\text{mon} (a_n))_{n \in \mathbb{N}}\) and \((a_n)_{n \in \mathbb{N}}\) denote sequences of real numbers and their positions, which consist of \(\tau \in N \setminus FN\) bounded rational numbers, respectively. A real sequence \(\text{mon} (a_n)_{n \in FN}\) is said to converge to \(\text{mon} (x)\), or equivalently that of rationals \((a_n)_{n \in FN}\) to \(x\), iff
\[
(\forall q \in FQ) (\exists i \in FN) (\forall j > i) (|a_j - x| < |q|).
\]
It is simply denoted as \(\lim_{i \in FN} \text{mon} (a_i) = \text{mon} (x)\), or equivalently \(\lim_{i \in FN} a_i \equiv x\), iff \((\text{mon} (a_n))_{n \in FN}\) converges to \(\text{mon} (x)\). Let us simply denote \(\lim_{i \to 0} t \equiv 0\) iff there exists a rational sequence \((a_n)_{n \in FN}\) which converges to 0, or \(\lim_{i \in FN} a_i \equiv 0\).

**Theorem 7.** Let \(F\) be a differentiable real function and \(f\) be its representation.
Then, the following indiscernibility is met
\[
\lim_{t \to 0} \frac{f(x + t) - f(x)}{t} \equiv \frac{\Delta f}{\Delta x}(x).
\]

Proof. Let \((a_n)_{n \in \tau}\), in which \(\tau \in N \setminus FN\), be a decreasing sequence on \([0, 1]\), such that \(a_n \neq 0\) for all \(n \in FN\) and \(a_n \equiv 0\) but \(a_n \neq 0\) otherwise. Choose \(x \in [0, 1]\) arbitrarily. Suppose there exists \(q \in FQ\) which satisfies for all \(i \in FN\) that there always exists \(j > i\) fulfilling the inequality
\[
\left| \frac{f(x + a_j) - f(x)}{a_j} - \frac{\Delta f}{\Delta x}(x) \right| > |q|.
\]
Then, by the axiom of prolongation, there exists \(a \in \tau \setminus FN\) which satisfies
\[
\left| \frac{f(x + a_j) - f(x)}{a_j} - \frac{\Delta f}{\Delta x}(x) \right| > |q|.
\]
But by Theorem \(\Box\) if \(a_\alpha > |q| > 0\) is satisfied for some \(\eta \in \text{mon} (0)\), the following indiscernibility equivalence must be satisfied
\[
\frac{f(x + a_\alpha) - f(x)}{a_\alpha} = \frac{f \mid [0, 1] \ | (x + a_\alpha) - f \mid [0, 1] \ | (x)}{a_\alpha} \equiv \frac{\Delta f}{\Delta x}(x).
\]
It is a contradiction. \(\Box\)

See Sakahara and Sato \(\cite{2}\) for the statement under continua in general.
The basic properties of derivatives such as chain rule or inverse function theorem are also preserved (see 8.4 and 8.5 of Tsujishita [3], respectively). Higher order derivative and its differentiability are drawn in a similar way (see 8.6 and 8.7 of Tsujishita [3]).

The rational-valued function \( \Sigma f \Delta x \) on \([0,1]\), which is continuous and finite, is given as

\[
(\Sigma f \Delta x)(u) \equiv \sum_{0 \leq x \leq u} f(x) \Delta x.
\]

An indefinite integral of \( F \), denoted as \( \int_0^t F(x) \, dx \), is the real function represented by \( \Sigma f \Delta x \). Its continuity and finiteness are verified by Proposition 8.8.1 and its independence from the choice of the representation by Proposition 8.8.3 of Tsujishita [3].

Then the fundamental theorem of calculus is established.

**Proposition 8** (Proposition 8.8.4 of Tsujishita [3]). Suppose \( F \) is a real function on \([0,1]\). Then the real function \( \int_0^t F(x) \, dx \) on \([0,1]\) is differentiable and its derivative is \( F \).

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