Dynamical symmetry breaking in the Nambu-Jona-Lasinio model with external gravitational and constant electric fields

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Abstract

An investigation of the Nambu-Jona-Lasinio model with external constant electric and weak gravitational fields is carried out in three- and four-dimensional spacetimes. The effective potential of the composite bifermionic fields is calculated keeping terms linear in the curvature, while the electric field effect is treated exactly by means of the proper-time formalism. A rich dynamical symmetry breaking pattern, accompanied by phase transitions which are ruled, independently, by both the curvature and the electric field strength is found. Numerical simulations of the transitions are presented.

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1 Introduction.

The Nambu-Jona-Lasinio (NJL) model \[1\] has long been considered one of the most suitable approximations providing a correct description of low-energy strong interactions physics. It has been seen that dynamical symmetry breaking (DSB)\[1\] and dynamical fermions mass generation take place in the models with four-fermion interactions, those of NJL and Gross-Neveu \[4\].

External conditions, such as a non-zero temperature, a finite chemical potential or classical gauge fields, have been shown to enrich the phase structure of the NJL model essentially (for reviews and complete lists of references see, for instance, \[3\]-\[20\]).

The influence of external magnetic and electric fields in these models has been studied for some time \[8\]-\[20\]. It has been found out that a non-zero magnetic field always breaks the chiral symmetry, while the presence of an electric field (EF) tries to restore it.

Investigations of the influence of a classical gravitational field on the DSB phenomenon in the NJL model have been carried out for some years. It has been shown that curvature-induced phase transitions exist and that they must be incorporated into any realistic scenario of the early Universe \[21\]-\[24\] (for a general introduction to quantum field theory in curved spacetime see \[25\] and for a recent review of the NJL model in curved spacetime, \[20\]). It turns out that, in spite of the relatively small value of the curvature-dependent corrections at the low energy scale to be investigated within the NJL model, these corrections appear to be inescapable, in the sense that they must be taken into account when one performs the necessary "fine tuning" of the different cosmological parameters \[21\].

On the other hand, an external electromagnetic field plays an important role in the early Universe, which both can contain primodial magnetic fields \[28, 29\] and have a very large electrical conductivity \[28, 30\]. Investigations of the influence of an external magnetic field within the DSB in curved spacetime have been carried out recently \[31\]-\[33\]. Therefore, it seems natural to study the effect of a constant EF upon the DSB phenomenon in a curved spacetime extending the results of \[12, 13, 18\], which have been done for the case of flat spacetime.

In this paper we investigate the behaviour of the NJL model in the presence of an external constant EF treated nonperturbatively in the proper-time formalism \[9\]. The linear-curvature corrections to the effective potential (EP) of composite bifermionic fields are calculated and the phase structure of the model is investigated for negative values of the coupling constant. Both the four- and threedimensional cases are discussed. It is well known that in an external EF particle creation takes place and that the EP acquires an imaginary part \[9\]-\[13\]. That is why we study the small EF limit when the particle creation velocity is negligible. The phase transitions accompanying the DSB process on

\[1\] This phenomenon and its application in high energy physics have been described in \[2, 3\].
the spacetime curvature, as well as the values of EF strength will be described numerically in detail.

2 The fermion Green function in an external constant electromagnetic field with linear-curvature accuracy.

Our starting point is the Nambu-Jona-Lasinio model in curved space-time of arbitrary dimension, \( d \), as given by the following action \([1]\):

\[
S = \int d^d x \sqrt{-g} \left\{ i \bar{\psi} \gamma^\mu(x) D_\mu \psi + \frac{\lambda}{2N} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 \right] \right\},
\]

where the covariant derivative \( D_\mu \) includes the electromagnetic potential \( A_\mu \):

\[
D_\mu = \partial_\mu - i e A_\mu + \frac{1}{2} \omega^a_{\mu \sigma} \sigma^{ab}.
\]

The local Dirac matrices \( \gamma_\mu(x) \) are expressed through the usual flat ones \( \gamma_a \) and the tetrads \( e^a_\mu \):

\[
\gamma^\mu(x) = \gamma^a e^a_\mu(x),
\]

\[
\sigma^{ab} = \frac{1}{4} [\gamma_a, \gamma_b].
\]

The spin connection has the form

\[
\omega^a_{\mu \sigma} = \frac{1}{2} e^a_{\sigma \nu} (\partial_\mu e^b_\nu - \partial_\nu e^b_\mu) + \frac{1}{4} e^a_{\nu \rho} e^{\nu \rho} (\partial_\mu e^c_\nu - \partial_\nu e^c_\mu)
\]

\[
- \frac{1}{2} e^{b \nu} (\partial_\mu e^a_\nu - \partial_\nu e^a_\mu) - \frac{1}{4} e^{b \nu} e^{\nu \rho} e^{\nu \rho} (\partial_\mu e^c_\nu - \partial_\nu e^c_\mu),
\]

where \( N \) is the number of bispinor fields \( \psi_a \). The spinor representation dimension is supposed to be four. Greek and Latin indices correspond to the curved and flat tangent spacetimes, respectively.

Introducing the auxiliary fields:

\[
\sigma = -\frac{\lambda}{N} \bar{\psi} \psi, \quad \pi = -\frac{\lambda}{N} \bar{\psi} i \gamma_5 \psi,
\]

we can rewrite the action (1) as:

\[
S = \int d^d x \sqrt{-g} \left\{ i \bar{\psi} \gamma^\mu D_\mu \psi - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - \bar{\psi} (\sigma + i \pi \gamma_5) \psi \right\}.
\]

Then, the effective action in the large-\( N \) expansion is given by:

\[
\frac{1}{N} \Gamma_{eff}(\sigma, \pi) = -\int d^d x \sqrt{-g} \frac{\sigma^2 + \pi^2}{2\lambda} - i \ln \det \{ i \gamma^\mu(x) D_\mu - (\sigma + i \gamma_5 \pi) \}. \tag{8}
\]
Here we can put $\pi = 0$, because the final expression will depend on the combination $\sigma^2 + \pi^2$ only. This means that we are actually considering the Gross-Neveu model [4].

Defining the EP as $V_{eff} = -\Gamma_{eff}/N \int d^4x \sqrt{-g}$, for constant configurations of $\sigma$ and $\pi$ we obtain:

$$V_{eff} = \frac{\sigma^2}{2\lambda} + i\text{Sp} \ln\langle x |[i\gamma^\mu(x) D_\mu - \sigma]|x\rangle$$

(9)

By means of the usual Green function (GF), which obeys the equation

$$(i\gamma^\mu D_\mu - \sigma) G(x, x', \sigma) = \delta(x - x')$$

(10)

we obtain the following formula

$$V'_{eff}(\sigma) = \frac{\sigma}{\lambda} - i\text{Sp} G(x, x, \sigma).$$

(11)

To calculate the linear curvature corrections the local momentum expansion formalism is the most convenient one [34]. In the special Riemannian normal coordinate framework we have:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\sigma\nu} y^\rho y^\sigma,$$

(12)

$$e^\mu_a(x) = \delta^\mu_a + \frac{1}{6} R^\mu_{\rho\sigma a} y^\rho y^\sigma,$$

(13)

$$\omega_{\mu\nu}^{ab} = \frac{1}{2} R_{\mu\nu}^{ab} y^\lambda y^{\sigma},$$

(14)

$$y = x - x'.$$

(15)

The vector potential of the external electromagnetic field is chosen to be of the form:

$$A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x^\nu,$$

(16)

where $F_{\mu\nu}$ is the constant matrix of the electromagnetic field strength tensor. Substituting Eqs. (12)-(16) into Eq. (10), we arrive at the following equation for the GF:

$$\left[i\gamma^a (\delta^\mu_a + \frac{1}{6} R^\mu_{\rho\sigma a} y^\rho y^\sigma ) (\partial_\mu + \frac{1}{4} R_{bc\mu\lambda} y^\lambda y^{bc} - i e A_\mu) - \sigma \right] G(x, x', \sigma) = \delta(x - x')$$

(17)

Fulfilling the expansion on the different spacetime curvature monomials

$$G = G_0 + G_1 + ..., $$

(18)

where $G_0$ is the GF in flat spacetime, $G_1 \sim R$, and so on, we obtain the iterative sequence of equations:

$$\left[i \partial + e A(x) - \sigma \right] G_0(x, x') = \delta(x - x')$$

(19)

$$\left[i \partial + e A(x) - \sigma \right] G_1(x, x', \sigma) + \left[ i \gamma^a R^\mu_{\rho\sigma a} y^\rho y^\sigma (\partial_\mu - i e A_\mu(x)) \right] G_0(x, x') = \delta(x - x')$$

(20)
Here and below we can already forget about the difference between the two kinds of indices, Greek and Latin, because it only shows up beyond the linear curvature approximation in Eq. (20). We introduce the factor:

$$\Phi(x, x') = \exp \left[ ie \int_{x'}^x A^\mu(x'')dx'' \right],$$

which is the solution of the equation

$$(\partial_\mu - ieA_\mu)\Phi(x, x') = 0$$

and assume that, just as for flat spacetime, the GF has the form [9]:

$$G(x, x', \sigma) = \Phi(x, x') \tilde{G}(x - x', \sigma).$$

Then, the evident dependence on $A_\mu(x)$ disappears and we obtain the equation determining the GF $\tilde{G}_1(x - x', \sigma)$:

$$(i \not{\partial} - \sigma) x \tilde{G}_1(x - x', \sigma) = -\frac{i}{6} \gamma^\rho R^\mu_{\rho\sigma a}y^\rho y^\sigma \partial_\mu \tilde{G}_0(x - x', \sigma)
- \frac{i}{4} \gamma^a \sigma^{bc} R_{bca\lambda} y^\lambda \tilde{G}_0(x - x', \sigma).$$

The GF for the case of an external electromagnetic field in flat spacetime is supposed to be known. Denoting by $G_{00}(x - x', \sigma)$ the GF that satisfies the equation:

$$(i \not{\partial} - \sigma) G_{00}(x - x', \sigma) = \delta(x - x'),$$

we get:

$$\int dx'' G_{00}(x - x'', \sigma) \tilde{G}_1(x'' - x', \sigma) =$$

$$-\frac{i}{6} \gamma^\rho R^\mu_{\rho\sigma a}(x - x')^\rho (x - x')^\sigma \sigma^\sigma \partial_\mu \tilde{G}_0(x - x', \sigma)
- \frac{i}{4} \gamma^a \sigma^{bc} R_{bca\lambda}(x - x')^\lambda \tilde{G}_0(x - x', \sigma).$$

Finally,

$$\tilde{G}_1(x - x', \sigma) = \int dx'' G_{00}(x - x'', \sigma)
\times \left[ -\frac{i}{6} \gamma^\rho R^\mu_{\rho\sigma a}(x'' - x')^\rho (x'' - x')^\sigma \sigma^\sigma \partial_\mu \tilde{G}_0(x'' - x', \sigma)
- \frac{i}{4} \gamma^a \sigma^{bc} R_{bca\lambda}(x'' - x')^\lambda \tilde{G}_0(x'' - x', \sigma) \right].$$
We actually need the coincidence limit \( x \to x' \) in order to calculate the EP (11). This provides us with the opportunity to simplify (27), especially for constant curvature spacetime, where
\[
R_{\mu\sigma\kappa\lambda} = \frac{R}{d(d-1)} \left( \eta_{\mu\kappa} \eta_{\sigma\lambda} - \eta_{\mu\lambda} \eta_{\kappa\sigma} \right).
\]

Thus, our basic expression for the GF \( \tilde{G}_1 \) in a space-time of arbitrary dimension \( d \), is the following:
\[
\tilde{G}_1(0, \sigma) = -iR \frac{1}{12d(d-1)} \int dz G_{00}(-z, \sigma) \left[ 2 \partial_\mu \tilde{G}_0(z, \sigma) \right.
- \left. 2z^2 \gamma_\mu \partial_\mu \tilde{G}_0(z, \sigma) + 3(d-1) \partial_\mu \tilde{G}_0(z, \sigma) \right].
\]

This expression can be substituted into Eq. (11) directly, because \( \Phi(x, x) = 1 \).

### 3 Effective potential and phase transitions in the NJL model in curved spacetime.

The GF for the case of flat spacetime is found to be in the proper-time representation:
\[
\tilde{G}_0(z, \sigma) = e^{-i\pi d/4} \int_0^\infty \frac{ds}{(4\pi s)^{d/2}} e^{-is\sigma^2} \exp \left( -i \frac{1}{4s} z_\mu C_{\mu\nu} z_\nu \right) \times \left( \sigma + \frac{1}{2s} \gamma_\mu C_{\mu\nu} z^\nu - \frac{e}{2} \gamma_\mu F_{\mu\nu} z^\nu \right) \left[ \tau \coth \tau - \frac{e s}{2} \gamma_\mu \gamma_\nu F_{\mu\nu} \right],
\]

\[
G_{00}(-z, \sigma) = e^{-i\pi d/4} \int_0^\infty \frac{dt}{(4\pi t)^{d/2}} e^{ipt} \left[ -i(\sigma^2 t + \frac{z^2}{4t}) \left( \sigma + \frac{1}{2} \gamma_\mu z_\mu \right) \right],
\]

where:
\[
C_{\mu\nu} = \eta_{\mu\nu} - F_{\mu}^\lambda F_{\lambda\nu} \frac{1 - (eEs) \coth(eEs)}{E^2}.
\]

Let us now consider the 3D case. After a Wick rotation \( is \to s, it \to t \), we have the following expression:
\[
\text{Sp} \tilde{G}_1(0, \sigma) = \frac{iR\sigma}{72\pi^{3/2}} \int dt ds e^{-(t+s)\sigma^2} \frac{1}{(t + s)^{3/2}(1 + \kappa \cot \tau)^2}
\times \left[ -2\kappa(\kappa + \tau) + (9\tau + 5\kappa) \cot \tau + \kappa(\tau - 3\kappa) \cot^2 \tau \right],
\]

where \( \tau = eEs, \kappa = eEt \). Performing the integration over \( \sigma \), we get the EP:
\[
V_{\text{eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \int_{1/\Lambda^2}^\infty \frac{ds}{s^{5/2}} e^{-s\sigma^2} \tau \cot \tau
\]
\[ -\frac{R}{144 \pi^{3/2}} \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} ds dt e^{-(t+s)\sigma^2} \frac{ds dt e^{-(t+s)\sigma^2}}{(t + s)^{5/2}(1 + \kappa \cot \tau)^2} \times \left[ -2\kappa (\kappa + \tau) + (9\tau + 5\kappa) \cot \tau + \kappa (\tau - 3\kappa) \cot^2 \tau \right]. \] (34)

This expression coincides exactly, in the limit \( E \to 0 \), with previous results [31]:

\[ V_{\text{eff}}^{(E=0)}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{\Lambda^3}{6\pi^{3/2}} \left\{ (1 - 2\frac{\sigma^2}{\Lambda^2}) \exp\left(-\frac{\sigma^2}{\Lambda^2}\right) + 2\sqrt{\pi}\frac{\sigma^3}{\Lambda^3} \text{erfc}\left(\frac{\sigma}{\Lambda}\right) \right\} - \frac{R}{4\Lambda^2} \left[ \exp\left(-\frac{\sigma^2}{\Lambda^2}\right) - \sqrt{\pi}\frac{\sigma}{\Lambda} \text{erfc}\left(\frac{\sigma}{\Lambda}\right) \right], \] (35)

where

\[ \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} dt. \] (36)

There are two ways of justifying the introduction of the \( \Lambda \) parameter in equations (34) and (35) for the EP. The first one is the standard renormalization procedure, by means of the UV cutoff method. Then, in the limit \( \Lambda \to \infty \), after the renormalization of the coupling constant

\[ \frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{\Lambda}{\pi^2}, \] (37)

we have the well known expression for the renormalized EP in flat spacetime [13, 14] with linear curvature corrections [31]:

\[ V_{\text{eff},R}^{(E=0)}(\sigma) = \frac{\sigma^2}{2\lambda_R} + \frac{\sigma^3}{3\pi} + \frac{R\sigma}{24\pi}. \] (38)

Three-dimensional four-fermions models have been shown to be renormalizable in the leading large-\( N \) order [5]. Furthermore, external electromagnetic and gravitational fields do not interfere with the renormalization procedure, because the only divergence that appears in the expression for the EP (34) has already been removed by the substitution (37). This is due to the obvious fact that the local feature of renormalizability cannot be spoiled by the weak curvature of global spacetime or by an external EF. Formally this can be proved by looking at the calculations of the leading terms of the integrand in the limit \( s \to 0, t \to 0 \), which are the only essential ones in order to determine the UV-divergences of the EP (34). Thus, the renormalization has been performed by the formula (37) completely.

However, in general, four-fermion models may be considered as low energy effective theories being derived from a more complete version of a quantum field theory (QCD, for example). In this case the parameter \( \Lambda \) can be treated as a natural characteristic scale limiting the range where our low-energy approximation is valid, and then our model
will describe some phenomenological effects of elementary particle physics. Bearing these reasons in mind, we can otherwise maintain $\Lambda$ finite and investigate the DSB phenomenon in our model with this fixed cut-off.

The integrand contains the function $\cot(e Es)$, that periodically goes to $\infty$. It develops an infinite set of poles on the integration path and obliges us to take into account the contribution of the corresponding residua, given by the imaginary part of the EP. The presence of imaginary terms in the EP means that particle creation takes place and that our vacuum is actually unstable. The solution to this problem lies outside the limits of our present investigation. The simplest possibility seems to be to consider a comparatively small $\text{EF}$ strength, with specific values that would provide an exponentially depressed particle creation rate.

To start with we shall consider the case of a 3D flat spacetime:

$$V_{\text{flat eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{5/2}} e^{-s\sigma^2} \tau \cot \tau,$$

where the integrand function has its poles at the points

$$s_n = \frac{\pi n}{eE}, \quad n = 1, 2, ...$$

The standard residue technique yields:

$$V_{\text{flat eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \text{P.v.} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{5/2}} e^{-s\sigma^2} \tau \cot \tau - i \left(\frac{eE}{4\pi^2}\right)^{3/2} \sum_{n=1}^{\infty} n^{-3/2} \exp\left(-\frac{\pi n}{eE}\sigma^2\right),$$

where P.v. means "principal value." Using the analytical continuation of the Hurwitz $\zeta$-function, we find the following expression for the renormalized EP:

$$V_{\text{eff,R}}(\sigma) = \frac{\sigma^2}{2\lambda_R} - \frac{(2ieE)^{3/2}}{4\pi} \left[2\zeta\left(-\frac{1}{2}, \frac{\sigma^2}{2ieE}\right) - \left(\frac{\sigma^2}{2ieE}\right)^{1/2}\right].$$

A numerical analysis of $\text{Re} V_{\text{eff,R}}(\sigma)$ for negative coupling constant gives the typical behaviour of a first-order phase transition, as shown in figure 1. The critical values are defined as usual: $E_{c1}$ corresponds to the strength of $\text{EF}$ for which a local nonzero minimum appears, $E_{c1}$, when the real part of EP is equal at zero and at the local minimum, and $E_{c2}$, when the zero extremum becomes a maximum. For all figures, an arbitrary dimensional parameter, $\mu$, defining a typical scale in the model, is introduced in order to perform the plots in terms of dimensionless variables. $\Lambda$ obviously does not appear anywhere because after renormalization (37) it must be set such that $\Lambda \to \infty$. 
The curvature-dependent part of the effective potential does not contain new UV-divergences but, unfortunately, $\Lambda$ must be kept finite here, because the integrals cannot be calculated in the general case. Therefore, an analytical continuation cannot be performed. However, as mentioned above, the limit of weak electric field is the most suitable one here in order to investigate DSB correctly. The most natural way is to keep the flat part of EP in the same nonpertubative form (42) meanwhile in the curvature-dependent terms which are small already by themselves only the weak EF limit is taken into account. Thus, neglecting the next order exponentially depressed terms in curvature corrections, the renormalized EP turns out to be the following:

$$
V^{(3D)}_{\text{eff},R}(\sigma) = \frac{\sigma^2}{2\lambda_R} - \frac{(2ieE)^{3/2}}{4\pi} \left[ 2\zeta(-\frac{1}{2}) - \frac{\sigma^2}{2ieE} \right] + \frac{R\sigma}{24\pi} + \frac{iR(eE)^{1/6}}{2\pi^23^{2/3}} \exp\left(-\frac{\sigma^2}{eE}\right) \Gamma\left(\frac{2}{3}\right) \sigma^{2/3}.
$$

(43)

The shape of $\text{Re} V^{(3D)}_{\text{eff},R}(\sigma)$ is shown in figure 2, for the case of a fixed curvature and coupling constant, and in figure 3, for the case of fixed EF strength and coupling constant. The shapes are absolutely typical of first-order phase transition pictures.

For a finite cut-off scale $\Lambda$, the effective potential is given by:

$$
\text{Re} V^{(3D)}_{\text{eff}}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} P.v. \int_{1/\Lambda^2}^{\infty} \frac{ds}{s^{5/2}} e^{-s\sigma^2} \tau \cot \tau
$$

$$
- \frac{R}{144\pi^{3/2}} P.v. \int_{1/\Lambda^2}^{\infty} \int_{1/\Lambda^2}^{\infty} \frac{dsdt}{(t+s)^{5/2}(1+\kappa \cot \tau)^2} e^{-(t+s)s^2} \tau \cot \tau + \kappa(\tau - 3\kappa) \cot^2 \tau,
$$

(44)

$$
\text{Im} V^{(3D)}_{\text{eff}} = \frac{(eE)^{3/2}}{4\pi^2} \sum_{n=1}^{\infty} n^{-3/2} \exp\left(-\frac{\pi n}{eE}\sigma^2\right)
$$

$$
+ \frac{R}{144} \frac{(eE)^{1/2}}{\pi} \sum_{n=1}^{\infty} \exp\left(-\frac{\pi n}{eE}\sigma^2\right) \int_{\pi n - \varphi}^{\infty} dy \frac{\exp\left(\frac{\sigma^2}{eE}(\varphi - y)\right)}{(1+y^2)^2(y+\pi n - \varphi)^{5/2}}
$$

$$
\times \left[ 2y(y^2+4)(\pi n - \varphi + y)\frac{\sigma^2}{eE} + 3(\pi n - \varphi + y)(3-y^2) + y(8-y^2) \right],
$$

(45)

where

$$
\varphi = \arccos \frac{1}{\sqrt{1+y^2}}.
$$

(46)

In contrast to the case of $D = 3$, the 4D four-fermion models are not renormalizable. Thus, we have to keep $\Lambda$ finite, and do just the same type of calculations as for the
previous 3D situation:

\[
\begin{aligned}
\text{Sp} \tilde{G}_1(0, \sigma) &= \frac{iR\sigma}{96\pi^2} \int dt ds \frac{e^{-(t+s)\sigma^2}}{(t+s)^2(1 + \kappa \cot \tau)^2} \\
&\times \left[ -\kappa(\kappa + \tau) + 2(\kappa + 3\tau) \cot \tau + 2\kappa(\tau - \kappa) \cot^2 \tau \right].
\end{aligned}
\]

We arrive at the EP with a finite cut-off scale:

\[
\text{Re} V_{\text{eff}}^{(4D)} = \frac{\sigma^2}{2\lambda} + \frac{1}{8\pi^2} P.v. \int_{1/\Lambda^2}^{\infty} ds s e^{-\sigma^2} \cot \tau
\]

\[
- \frac{R}{192\pi^2} P.v. \int_{1/\Lambda^2}^{\infty} dt \int_{1/\Lambda^2}^{\infty} ds e^{-(t+s)\sigma^2}
\]

\[
\times \left[ -\kappa(\kappa + \tau) + 2(\kappa + 3\tau) \cot \tau + 2\kappa(\tau - \kappa) \cot^2 \tau \right],
\]

\[
\text{Im} V_{\text{eff}}^{(4D)} = -\frac{(eE)^2}{16\pi^3} \sum_{n=1}^{\infty} n^{-2} \exp\left(-\frac{\pi n}{eE} \sigma^2\right)
\]

\[
+ \frac{R(eE)}{192\pi} \sum_{n=1}^{\infty} \exp\left(-\frac{\pi n}{eE} \sigma^2\right) \int_{eE/\Lambda^2}^{\infty} dy \frac{\exp\left(\frac{\sigma^2}{eE}(\varphi - y)\right)}{(1 + y^2)^2(\varphi + \pi n - \varphi)^3}
\]

\[
\times \left[ y(y^2 + 4)(\pi n - \varphi + y) \frac{\sigma^2}{eE} + 6(\pi n - \varphi) - 2y(y^2 - 5) \right].
\]

(47)

However this cut-off regularization with a finite \(\Lambda\) is not the only one that can be performed. Instead of it, it is also possible to send \(\Lambda \to \infty\) and simultaneously rewrite our formula for the EP in \(4 - 2\epsilon\) dimension, introducing therefore some kind of dynamical regularization which seems to be more useful for numerical simulations. Then, for weak \(\text{EF}\) the EP can be expressed in the following, more convenient form:

\[
V_{\text{eff}}^{(4D)}(\sigma) = \frac{\sigma^2}{2\lambda} - \frac{(eE)^2}{8\pi^2} \Gamma(-1 + \epsilon) \left[ 4\zeta(-1 + \epsilon, -i \frac{\sigma^2}{2eE}) + i \frac{\sigma^2}{eE} \right]
\]

\[
- \frac{R\sigma^2}{96\pi^2} \Gamma(-1 + \epsilon) + \frac{iR(eE)^{2/3}}{48\pi^3 3^{1/3}} \exp\left(-\frac{\pi}{eE} \sigma^2\right) \Gamma\left(\frac{2}{3}\right) \sigma^{2/3}.
\]

(50)

Figure 4 presents the real part of EP in the \(4 - 2\epsilon\) -dimensional case in flat spacetime. It has been found by the numerical analysis that the final results are almost independent of the special \(\epsilon\) values. The phase transition is of second-order, because there exists a unique critical \(\text{EF}\) strength, defined as the value for which the zero extremum becomes a maximum and, simultaneously, a nonzero absolute minimum appears. This critical value separates the situations where the zero extremum is a maximum and a non-zero absolute
minimum of the potential exists, from the ones where the origin is the only and absolute minimum.

Figures 5 and 6 show the shapes of the real part of EP in curved $(4 - 2\epsilon)$-dimensional spacetime for fixed EF strength and spacetime curvature on the abscissa axis. In both cases, a second-order phase transition occurs.

It should be emphasized that in the presence of an imaginary part in the EP, the criterion for restoration of the chiral symmetry must be modified \[11, 13, 18\]. In fact, in that case the order parameter has to be chosen as $\sigma \langle \bar{\psi} \psi \rangle$, but not simply as $\sigma$ because, although the later vanishes at the critical point, the former might still break the chiral symmetry through the corresponding term in the action (7). Formula (11) gives that

$$\langle \bar{\psi} \psi \rangle = iN\text{Im} V'_{\text{eff}}(\sigma) - \frac{N}{\lambda} \sigma$$

and, therefore, we should only check whether our EP obeys the criterion:

$$\lim_{\sigma \to 0} \sigma \text{Im} V'_{\text{eff}}(\sigma) = 0.$$  \hspace{1cm} (52)

Formulae (43) and (50) make us sure that this criterion is satisfied here for both the 3D and 4D cases.

4 Conclusions.

Phenomenological models with four-fermion interaction seem to be useful for the description of low- and intermediate-energy physics of strong interactions \[7\]. The fermions are usually treated as quarks and the composite particles generated by the dynamical symmetry breaking as mesons, despite of some simplifications making the models under consideration more calculable.

The nonzero vacuum expectation value of composite bifermionic operator $\langle \bar{\psi} \psi \rangle$ plays in DSB scheme just the same role as $\langle \varphi \rangle$ in the traditional Higgs one and defines the dynamically generated mass of fermions in our case.

Usually the symmetry to be broken under the DSB mechanism is the chiral one. A dynamical version of fermions mass generation and dynamical chiral symmetry breaking have been investigated very carefully and some fruitful applications for the real high-energy physics \[5\]-\[15\] have been found.

In particular, the NJL model in external electromagntetical field has been studied from different points of view to realize the crucial role of that field in the DSB for flat spacetime \[9\]-\[20\]. The effect of dynamical chiral symmetry breaking by external magnetic field at any attractive interaction of the fermions has been shown to be universal and model independent phenomenon \[13\].

\[2\]More detailed information and complete literature list can be found in \[7\], for example
In our paper we have extended these investigations to the case of the presence of an electric field and non-zero spacetime curvature. Both these values try to restore the chiral symmetry in contrast to the magnetic field and their competition with the DSB catalysis by the magnetic field has to be understood more precisely.

The Gross-Neveu model with the simplest four-fermionic interaction $(\bar{\psi} \psi)^2$ is invariant under the discrete chiral transformation

$$\psi \rightarrow \gamma_5 \psi$$

Because of the discrete type of this symmetry Goldstone modes do not appear.

However NJL model has a continuous chiral invariance under the transformations

$$\psi \rightarrow e^{i\theta \gamma_5} \psi$$

In contrast to the previous case Goldstone bosons have to appear if only the symmetry is being broken. In the phenomenological applications of NJL model π field is treated as Goldstone modes corresponding to the massless pions, which obtain the kinetic terms through the quantum corrections. Of course, if we take into account the quarks’ current masses these modes become pseudo-Goldstone ones \[5, 7\]. We only split them out of our consideration from the very beginning by the appropriate choice of the direction in the $\sigma – \pi$ space and use this choice’s advantage that DSB is ruled by $\sigma$ itself.

The investigation of the phenomenon realization in the presence of magnetic field in flat spacetime has been performed both for NJL model and for quantum electrodynamics and quantum excitations of $\sigma$ and $\pi$ fields dispersion laws have been found \[15\]. To generalize this calculations for the curved spacetime case is an interesting problem we are dealing with now.

However, here we have studied the phase structure of the NJL model in curved spacetime with external constant pure electric field only. In three dimensions, a first-order phase transition takes place, both on the electric field strength and on the spacetime curvature. In the four-dimensional case – or, more precisely, in $4 - 2\epsilon$ dimensions with negligible $\epsilon$ – the phase transition is a typical second-order one, on both of these external parameters.

Our approximation is quite consistent, because all of the critical values are very small in comparison with the characteristic scale of the model $\mu$. It should be noted that the consideration, in a nonperturbative way, of an external electric field provides very interesting effects even in flat spacetime, and that the linear curvature terms induce some additional processes of chiral symmetry restoration.

We clearly observe that a positive spacetime curvature tries to restore chiral symmetry, as an external electric field does. This is in contrast with what happens in the case of a magnetic filed, where chiral symmetry is broken for any value of the field strength although positive curvature might restore chiral symmetry again for some critical value.
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Figure 1: The behaviour in 3D of the real part of the effective potential $\text{Re}V_{\text{eff}, R/\mu^3}^{\text{flat}}$ as a function of $\sigma/\mu$ is depicted for $R = 0$ and fixed $\lambda\mu = -100$. The curves from the upper part to the bottom of the plot correspond to the following electric field strengths: $eE/\mu^2 = 0.0003; 0.0023; 0.00020; 0.00017; 0.00005$, respectively. The critical values, defined as usual, are given by: $eE_{c1}/\mu^2 = 0.00023; eE_{c2}/\mu^2 = 0.00017; eE_{c2}/\mu^2 = 0.00005$. 
Figure 2: Behaviour in 3D of Re $V_{\text{eff},R}/\mu^3$ as a function of $\sigma/\mu$ for fixed $eE/\mu^2 = 0.00005$ and $\lambda\mu = -100$. From above to below, the curves in the plot correspond to the following values of $R/\mu^2 = 0.006; 0.005; 0.004; 0.0032; 0$, respectively. The critical values, defined as usual, are given by: $R_{c1}/\mu^2 = 0.005; R_{c2}/\mu^2 = 0.0032; R_{c2}/\mu^2 = 0$. 
Figure 3: Behaviour in 3D of $\text{Re} V_{\text{eff},R}/\mu^3$ as a function of $\sigma/\mu$, for fixed $R/\mu^2 = 0.0032$ and $\lambda \mu = -100$. Starting from above, the curves correspond to the following values of $eE/\mu^2 = 0.00015; 0.00011; 0.00007; 0.00005; 0.00001$, respectively. The critical values, defined as usual, are given by: $eE_{c1}/\mu^2 = 0.00011; eE_{c2}/\mu^2 = 0.00005$, while $eE_{c2}/\mu^2$ does not exist for the given values of $\mu \lambda$ and $R/\mu^2$. 
Figure 4: Behaviour of $\text{Re} V_{\text{eff}}/\mu^{(4-2\epsilon)}$ in a $(4-2\epsilon)$-dimensional spacetime as a function of $\sigma/\mu$, for fixed $R = 0$, $\epsilon = 0.02$ and $\lambda\mu = -1000$. Starting from above, the curves in the plot correspond to the following values of $eE/\mu^2 = 0.02; 0.01; 0.005; 0.001$, respectively. The critical value is given by: $eE_c/\mu^2 = 0.02$. 
Figure 5: Behaviour of $\text{Re} V_{\text{eff}}/\mu^{(4-2\epsilon)}$ in a $(4-2\epsilon)$-dimensional spacetime as a function of $\sigma/\mu$, for $eE/\mu^2 = 0.001$, $\epsilon = 0.005$ and fixed $\lambda \mu = -1000$. Starting from above, the curves correspond to the following values of the curvature: $R/\mu^2 = 0.0045; 0.0035; 0.02; 0.01; 0; -0.001$, respectively. The critical value is obtained for $Rc/\mu^2 = 0.0035$. 
Figure 6: Behaviour of $\text{Re} V_{eff}/\mu^{(4-2\epsilon)}$ in a $(4-2\epsilon)$-dimensional spacetime, as a function of $\sigma/mu$, depicted for $R/\mu^2 = 0.002$, $\epsilon = 0.005$ and fixed $\lambda\mu = -1000$. From above, the curves in the plot correspond to the following values of the electric field strength: $eE/\mu^2 = 0.004; 0.0028; 0.02; 0.015; 0.001$, respectively. The critical value is reached at $eE_c/\mu^2 = 0.0028$. 