Hedging in a market with jumps – an FBSDE approach

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**Abstract**

We propose a model for hedging in a market with jumps for a large investor. The dynamics of the stock prices and the value process is governed by forward-backward SDEs driven by Teugels martingales. Unlike known FBSDE market models, ours accounts for jumps in stock prices. Moreover, it allows to find an optimal hedging strategy.

**Keywords:** Optimal hedging strategy, Forward-backward SDEs with jumps, incomplete market, orthonormalized Teugels martingales

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1. **Introduction**

In this article, we propose a forward-backward-SDE (FBSDE) model for hedging in a market with a large investor and where the stock prices are allowed to jump. The attempts to price contingent claims have their origin...
in the work of Black and Scholes (1973), where the authors give a formula to price a European call option assuming that the stock price evolves as a geometric Brownian motion. One of the assumptions of the classical Black and Scholes model is that no individual investor action is able to influence market prices. The importance of accounting for the existence of large investors, however, has been increasing, given the prevalence of electronic trading and, in particular, high frequency trading which makes possible to issue thousands of orders over short periods of time. This problem was addressed by Cvitanic and Ma (1996), where the authors present an FBSDE market model for a large investor, but in a Brownian market environment, where the stock prices, being modeled as geometric Brownian motions, are not allowed to have jumps. However, the documented evidence of jumps in the distribution of the returns (see, e.g., Eberlein and Keller (1995)) suggests that a geometric Brownian motion is not entirely suited to model the evolution of stock prices in real markets. In particular, in periods of heavy market turbulence, such as the “flash crash” in May 2010, when the main US indexes temporarily dropped by more than 9 per cent, hedging strategies resulting from Brownian models leave investors exposed to a significant downside risk.

It is known that markets, where the stock prices are modeled involving Lévy processes are, in general, incomplete, so contingent claims may not admit self-financing replicating strategies. The first attempt to define optimal strategies in the context of incomplete markets was made by Föllmer and Schweizer (1991), where the authors propose an optimal strat-
egy as the one that minimizes, in a certain sense, the injection of capital needed.

In our FBSDE market model, the evolution of the $d$-dimensional stock price $S_t = \{S^i_t\}_{i=1}^d$ is governed by an SDE driven by $m$ independent Brownian motions and $d - m$ martingales picked from the system of orthonormalized Teugels martingales $\{H^{(ik)}_t\}_{i=1, k \in \mathbb{N}}$ such that for each $i \in \{1, \ldots, l\}$, the family $\{H^{(ik)}_t\}_{k=1}^\infty$ is associated to a Lévy process $L^i_t$. All processes $L^i_t$ are assumed independent and purely discontinuous. We refer the reader to Nualart and Schoutens (2001) (p. 763) for details on the martingales $H^{(ik)}_t$.

Remark that different stock prices $S^i_t$ can jump at different times. Further, the value process $V_t$ and the portfolio process $\pi_t = \{\pi^i_t\}_{i=1}^d$ evolve according to a backward SDE with the final condition $h(S_T)$ which is the payoff at maturity $T$.

Our model involves the martingales $H^{(ik)}_t$ because they are independent, strongly orthonormal, purely discontinuous, but most importantly, the system $\{H^{(ik)}_t\}_{i=1, k \in \mathbb{N}}$, completed with the Brownian motions $\{B^i_t\}_{i=1}^m$, possesses the predictable representation property. The latter allows to decompose the discounted value process into a sum of the value of the hedging portfolio and a strongly orthogonal martingale. Therefore, our model allows to find a hedging strategy which is optimal in the sense of Schweizer (2008). It is worth to mention that due to the presence of $H^{(ik)}_t$, the SDEs representing the evolution of stocks become, in fact, driven by power-jump martingales built on the basis of the underlying Lévy processes (see Nualart and Schoutens...
(2001), p. 763). The presence of these “power-jump” terms may reflect “skewness”, “kurtosis”, and other volatile behavior or extremal movements of the market.

Thus, the main contribution of this work is introducing a model which accounts for jumps in stock prices and allows to find an optimal hedging strategy in the context of incomplete markets.

2. FBSDE model for hedging in a market with jumps

In what follows, we present our model. Let \((\Omega, \mathcal{F}, P)\) be a probability space, \(\{B_i^t\}_{i=1}^m\) be independent real-valued Brownian motions, and \(\{L^i_t\}_{i=1}^l\) be independent purely discontinuous real-valued Lévy processes with the Lévy measures \(\nu^i\) satisfying

\[
\int_{\mathbb{R}} (1 \wedge x^2) \nu^i(dx) < \infty \quad \text{and} \quad \int_{|x| \geq \varepsilon} e^{\lambda |x|} \nu^i(dx) < C
\]

for some positive \(\varepsilon, \lambda\), and \(C\). Define the filtration \(\mathcal{F}_t = \sigma\{B^i_s, 0 \leq s \leq t, 1 \leq i \leq m\} \vee \sigma\{L^i_s, 0 \leq s \leq t, 1 \leq i \leq l\} \vee \mathcal{N}\), where \(\mathcal{N}\) is the collection of all \(P\)-null sets. We agree that all Lévy measures \(\nu\) considered in this work, satisfy the condition \(\nu(\{0\}) = 0\).

Let, for each \(i \in \{1, \ldots, l\}\), \(\{H^{(i)}_t\}_{k=1}^\infty\) be the family of orthonormalized Teugels martingales associated to the Lévy process \(L^i_t\). Lemma \(\Box\) below provides a useful representation for the orthonormalized Teugels martingales \(H^{(i)}_t\) associated to an arbitrary one-dimensional Lévy process \(\ell_t\) with the
Lévy triple \((b,a,\lambda)\), \(a = (a_1, \ldots, a_M)\), and the Lévy measure \(\lambda\) satisfying (I) with \(\nu_i\) substituted by \(\lambda\). As in Nualart and Schoutens (2001), we introduce the polynomials \(q_{i-1}(x)\) obtained by the orthonormalization of the system \(\{1, x, x^2, \ldots\}\) with respect to the measure \(x^2\lambda(dx) + |a|^2\delta_0(dx)\), where \(\delta_0\) is the Dirac measure. Furthermore, we define \(p_i(x) = xq_{i-1}(x)\).

**Lemma 1.** Let \(\ell\) be a one-dimensional Lévy process with the Lévy-Itô decomposition \(\ell_t = bt + \sum_{i=1}^M a_i\beta_i(t) + \int_{|x|\leq 1} x\tilde{\mu}(t, dx) + \int_{|x|> 1} x\mu(t, dx)\), where \(\{\beta_i(t)\}_{i=1}^M\) are independent real-valued standard Brownian motions. Then, it holds that \(H^{(i)}_t = q_{i-1}(0) \sum_{j=1}^M a_j\beta_j(t) + \int_{\mathbb{R}} p_i(x)\tilde{\mu}(t, dx)\). In particular, if \(\ell_t\) is purely discontinuous, then \(H^{(i)}_t = \int_{\mathbb{R}} p_i(x)\tilde{\mu}(t, dx)\).

**Proof.** Define \(\tilde{p}_i(x) = p_i(x) - xq_{i-1}(0)\). We will use the following representation for \(H^{(i)}_t\) obtained in Nualart and Schoutens (2001) (p. 763):

\[
H^{(i)}_t = q_{i-1}(0)\ell_t + \sum_{0<s\leq t} \tilde{p}_i(\Delta\ell_s) - t\mathbb{E}\left[\sum_{0<s\leq t} \tilde{p}_i(\Delta\ell_s)\right] - tq_{i-1}(0)\mathbb{E}[\ell_1].
\]

Since \(\ell_t = \ell^c_t + \sum_{0<s\leq t} \Delta\ell_s\), where \(\ell^c_t\) is the continuous part of \(\ell_t\), we obtain

\[
H^{(i)}_t = q_{i-1}(0)\ell^c_t + \sum_{0<s\leq t} p_i(\Delta\ell_s) - \mathbb{E}\left[\sum_{0<s\leq t} \tilde{p}_i(\Delta\ell_s)\right] - q_{i-1}(0)\mathbb{E}[\ell_t] = q_{i-1}(0)\left[\ell^c_t - \mathbb{E}[\ell^c_t]\right] + \sum_{0<s\leq t} p_i(\Delta\ell_s) - \mathbb{E}\left[\sum_{0<s\leq t} \tilde{p}_i(\Delta\ell_s)\right] = q_{i-1}(0)\sum_{j=1}^M a_j\beta_j(t) + \int_{\mathbb{R}} p_i(x)\tilde{\mu}(t, dx). \tag*{\Box}
\]

Let us proceed with the description of the model. Fix a finite-time horizon \(T > 0\) and consider a market consisting of \(d\) risky assets (stocks) and risk-free money on a deposit. We assume that the price process of the risk-free
deposit evolves according to the equation

\[ \frac{dD_t}{D_t} = r(t, S_t, V_t, \pi_t) dt, \quad D_0 = 1, \quad (2) \]

where \( r : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) is the interest rate, \( S_t = \{S^i_t\}_{i=1}^d \) is the \( d \)-dimensional risky asset price process, \( V_t \) is the (real-valued) value process, and \( \pi_t = \{\pi^i_t\}_{i=1}^d \) is the portfolio process with \( \pi^i_t \) being the number assets of the \( i \)th stock. The evolution of \( S^i_t \) is assumed to be governed by the SDE

\[ dS^i_t = S^i_t \left\{ \tilde{f}_i(t, S_t, V_t, \pi_t) dt + \sum_{j=1}^d \sigma^{\alpha_j}_i(t, S_{t-}, V_{t-}) dM^{\alpha_j}_t \right\} \quad (3) \]

with the non-random initial condition \( S^i_0 > 0 \). In (3), \( \tilde{f}_i \) and \( \sigma^{\alpha_j}_i \) are real-valued functions defined on spaces of appropriate dimensions. Further, for \( j = 1, 2, \ldots, m, \alpha_j = j \) and \( M^{\alpha_j}_t = B^j_t \), while for \( j = m + 1, \ldots, d, \alpha_j \)'s are arbitrarily picked multiindexes from the set \( \{(ik), i = 1, \ldots, l, k = 2, 4, \ldots\} \) and \( M^{\alpha_j}_t = H^{\alpha_j}_t \). Remark, that the index \( k \) takes only even values.

The value process \( V_t \) represents the wealth of a “large” investor who holds \( d \) stocks and money on a deposit. The investor is assumed large, so the coefficients in our model would depend on \( V_t, S_t, \) and \( \pi_t \).

We define an admissible hedging strategy as a pair of predictable processes \((\pi_t, \pi^0_t)\) such that \( V_t = \sum_{i=1}^d \pi^i_t S^i_t + \pi^0_t D_t \) and \( V_T = h(S_T) \), where \( h(S_T) \) is the payoff at maturity \( T \). Note that the solution of (2) takes the form

\[ D_t = \exp\left\{ \int_0^t r_s ds \right\}, \quad \text{where} \quad r_s = r(s, S_s, V_s, \pi_s). \]

Let \( A_t = \exp\left\{ -\int_0^t r_s ds \right\} \).
Define $\hat{S}_t^i = A_t S_t^i$ and $\hat{V}_t = A_t V_t$ to be the discounted stock price and discounted value process, respectively. Furthermore, we define the cumulative cost process $C_t = \hat{V}_t - \sum_{i=1}^{d} \int_{0}^{t} \pi_s^i d\hat{S}_s^i$. We say that the strategy is optimal, if it is admissible and $C_t$ is a square-integrable martingale strongly orthogonal to the martingale part of each $\hat{S}_t^i$.

Lemma 2. The representations $\hat{V}_t = \sum_{i=1}^{d} \int_{0}^{t} \pi_s^i d\hat{S}_s^i + C_t$ and

$$V_t = V_0 + \sum_{i=1}^{d} \int_{0}^{t} \pi_s^i dS_t^i + \int_{0}^{t} \pi_s^0 dD_t + \int_{0}^{t} D_s dC_t$$

are equivalent.

Proof. Since $\langle V, A \rangle_t = \langle S^i, A \rangle_t = [V, A]_t = [S^i, A]_t = 0$, then by Itô’s product formula, $d\hat{S}_t^i = A_t dS_t^i - A_t r_t S_t^i dt$ and $d\hat{V}_t = A_t dV_t - r_t A_t V_t dt$. Substituting these expressions into the equation for $\hat{V}_t$ we obtain $d\hat{V}_t = \sum_{i=1}^{d} \pi_t^i dS_t^i + (V_t - \sum_{i=1}^{d} \pi_t^i S_t^i)r_t dt + D_t dC_t = \sum_{i=1}^{d} \pi_t^i dS_t^i + \pi_t^0 dD_t + D_t dC_t$. \( \square \)

Now we derive a backward SDE (BSDE) for the process $V_t$ with representation (4). First, we substitute $dD_t$ and $dS_t^i$ with the right-hand sides of equations (2) and (3), respectively. Since $V_T = h(S_T)$, from (4) we obtain

$$V_t - \mathbb{E}[h(S_T)] - \int_{t}^{T} g(s, S_s, V_s, Z_s^{(a)}) ds - \sum_{j=1}^{d} \int_{0}^{T} Z_s^{\alpha_j} dM_s^{\alpha_j} - \int_{0}^{T} D_s dC_s |\mathcal{F}_t]$$

$$= \sum_{j=1}^{d} \int_{0}^{t} Z_s^{\alpha_j} dM_s^{\alpha_j} + \int_{0}^{t} D_s dC_s = \sum_{k=1}^{l} \sum_{j=1}^{\infty} \int_{0}^{t} \hat{Z}_{s}^{(kj)} dH_s^{(kj)} ,$$

(5)
where \( g(t, s, v, z) = \tilde{g}(t, s, v, s^{-1}\sigma(t, s, v)^{-1}z) \) with \( \sigma(t, s, v) \) being the \( d \times d \) matrix with the element \( \sigma_{ij}^\alpha \) in the \( j \)th line and the \( i \)th column, \( \tilde{g}(t, s, v, \pi) = \sum_{i=1}^{d} s_i \pi_i f_i(t, s, v, \pi) + (v - \sum_{i=1}^{d} s_i \pi_i) r(t, s, v, \pi) \), \( \pi = (\pi_i)_{i=1}^{d} \), \( s = (s_i)_{i=1}^{d} \), and \( s = \text{diag}\{s_1, \ldots, s_n\} \). Further, \( (\alpha) \) denotes the set of multiindexes \( (\alpha) = \{\alpha_1, \ldots, \alpha_d\} \) and \( Z_t^{(\alpha)} = (Z_t^\alpha_1, \ldots, Z_t^\alpha_d) \). The last identity in (5) follows from the predictable representation property of the system \( \{H_t^{(kj)}\}_{j=1}^{\infty} \) for a fixed \( k \) (with \( \tilde{Z}_t^{(kj)} \) being predictable processes) and from the independence of the Lévy processes \( L_t^k, k = 1, \ldots, l \). The BSDE for \( V_t \) follows from (5):

\[
V_t = h(S_T) - \int_t^T g(s, S_s, V_s, Z_s^{(\alpha)}) ds - \sum_{j=1}^{d} \int_t^T Z_s^{\alpha_j} dM_s^{\alpha_j} - \sum_{(kj) \notin (\alpha)} \int_t^T \tilde{Z}_s^{(kj)} dH_s^{(kj)}. \tag{6}
\]

Making the change of variable \( \pi = s^{-1}\sigma(t, s, v)^{-1}z \) and introducing the functions \( f_i(t, s, v, z) = \tilde{f}_i(t, s, v, s^{-1}\sigma(t, s, v)^{-1}z) \), we transform SDE (3) to

\[
dS_t^i = S_t^i \left\{ f_i(t, S_t, V_t, Z_t^{(\alpha)}) dt + \sum_{j=1}^{d} \sigma_i^{\alpha_j}(t, S_t, V_t) dM_t^{\alpha_j} \right\}. \tag{7}
\]
Lemma 3. FBSDEs (6)–(7) are equivalent to

\[
\begin{cases}
S_i^t = S_i^0 + \int_0^t S^i_s f_i(s, S_s, V_s, Z_s, \hat{Z}_s(\cdot)) ds + \int_0^t S^i_s \sigma_i(s, S_s, V_s) dB_s \\
+ \int_0^t \int_{\mathbb{R}^l} S^i_s \psi_i(s, S_{s-}, V_{s-}, u) \tilde{N}(ds, du), & i = 1, \ldots, d, \\
V_t = h(S_T) - \int_t^T g(s, S_s, V_s, Z_s, \hat{Z}_s(\cdot)) ds - \int_t^T Z_s dB_s \\
- \int_t^T \int_{\mathbb{R}^l} \hat{Z}_s(u) \tilde{N}(ds, du),
\end{cases}
\]

(8)

where for \( u = (u_1, \ldots, u_l) \), \( \psi_i(t, s, v, u) = \sum_{q=m+1}^d \alpha^a_i(t, s, v)p_{\alpha_q}(u_{\alpha_q}) \) with \( u_{\alpha_q} = u_k \) if \( \alpha_q = (kj) \). Further, \( Z_t = (Z^1_t, \ldots, Z^m_t) \) and for each \( k \in \{1, \ldots, l\} \), \( \hat{Z}_t^{(kj)} \) are the components of the decomposition of \( \hat{Z}_t(0, \ldots, u_k, \ldots, 0) \) with respect to the basis of polynomials \( p_{(kj)}(u_k) \) in the space \( L_2(\nu^k(du_k)) \), while \( (Z^a_{t=m+1}, \ldots, Z^a_{t=d}) = \{\hat{Z}_t^{(kj)}\}_{(kj) \in (a)} \). Finally, \( \tilde{N} \) is the compensated Poisson random measure for the Lévy process \( (L^1_t, \ldots, L^l_t) \).

Remark 1. With a slight abuse of notation, in the coefficients \( f \) and \( g \), we write \( Z_t \) instead of \( (Z^1_t, \ldots, Z^m_t) \) and \( \hat{Z}_t(\cdot) \) instead of \( (Z^a_{t=m+1}, \ldots, Z^a_{t=d}) \). The dependence on \( \hat{Z}_t(\cdot) \) is understood as the dependence on its \( d - m \) components \( (Z^a_{t=m+1}, \ldots, Z^a_{t=d}) \).

Proof of Lemma 5. Note that for each \( k \), the system \( \{B_t^{(kj)}\}_{j=1}^\infty \) has the pre-
dictable representation property. Therefore,

\[
\int_t^T \int_{\mathbb{R}^l} \dot{Z}_s(u) \tilde{N}(ds, du) = \sum_{k=1}^l \int_t^T \int_{R_k} \dot{Z}_s(0, \ldots, u_k, \ldots, 0) \tilde{N}^k(ds, du_k)
\]

\[
= \sum_{k=1}^l \sum_{j=1}^\infty \int_t^T \dot{Z}^{(kj)}_s d\tilde{H}^{(kj)}_s,
\]

where \( R_k = \{te_k, t \in \mathbb{R}\} \) with \( \{e_k\}_{k=1}^l \) being an orthonormal basis in \( \mathbb{R}^l \), and \( \tilde{N}^k(t, \cdot) \) is the compensated Poisson random measure for \( L^k_t \), which, by the independence of \( L^k_t \)'s, is the restriction of \( \tilde{N}(t, \cdot) \) to \( R_k \). Since, by Lemma 1, \( H^{(kj)}_t = \int_{R_k} p_{(kj)}(u_k) \tilde{N}^k(t, du_k) \), we obtain that in \( L^2(\nu^k) \),

\[
\hat{Z}_t(0, \ldots, u_k, \ldots, 0) = \sum_{j=1}^\infty \hat{Z}^{(kj)}_t p_{(kj)}(u_k) \text{ a.s.}
\]

Moreover, for each \( k \), the system of polynomials \( \{p_{(kj)}\}_{j=1}^\infty \) is orthonormal in \( L^2(\nu^k) \) by the orthonormality of \( H^{(kj)}_t \)'s. Finally, since for each \( (kj) \), \( p_{(kj)}(0) = 0 \), we obtain

\[
\int_{\mathbb{R}^l} \sum_{q=m+1}^d \sigma_i^{\alpha_q}(t, x, y)p_{\alpha_q}(u_{\alpha_q}) \tilde{N}(dt, du)
\]

\[
= \sum_{q=m+1}^d \sigma_i^{\alpha_q}(t, x, y) \int_{R_{\alpha_q}} p_{\alpha_q}(u_{\alpha_q}) \tilde{N}^{\alpha_q}(dt, du_{\alpha_q}) = \sum_{k=m+1}^d \sigma_i^{\alpha_q}(t, x, y) d\tilde{H}^{\alpha_q}_t
\]

where \( R_{\alpha_q} = R_k \), \( \tilde{N}^{\alpha_q} = \tilde{N}^k \), and \( u_{\alpha_q} = u_k \) for \( \alpha_q = (kj) \).

**Remark 2.** Since \( \{\alpha_j\}_{j=m+1}^d \) are multiindexes picked from the set \( \{(ik), i = 1, \ldots, l, k = 2, 4, \ldots\} \), then each polynomial \( p_{\alpha_j} \) is of even degree, and, therefore, achieves a finite global minimum, which we denote by \( A_j \).

Assumption (A1) below guarantees the existence and uniqueness of solu-
tion to FBSDEs (6)–(7).

(A1) The coefficients of FBSDEs (8) satisfy the hypotheses of Theorem 3.1 in Wu (1999) (p. 436).

Lemma 4. Assume (A1). Then, there exists a unique \( F_t \)-adapted solution \((S_t, V_t, Z_t^{(a)})\) to FBSDEs (6)–(7) such that \((S_t, V_t)\) has càdlàg paths and \(Z_t^{(a)}\) is predictable.

Proof. Under (A1), Theorem 3.1 in Wu (1999) guarantees the existence of a unique \( F_t \)-adapted solution \((S_t, V_t, Z_t, \hat{Z}_t(\cdot))\) to FBSDEs (8) such that \((S_t, V_t)\) is càdlàg and \((Z_t, \hat{Z}_t(\cdot))\) is predictable. By Lemma 3, this is equivalent to the existence of a unique \( F_t \)-adapted solution \((S_t, V_t, Z_t^{(a)})\) to (6)-(7).

Assumptions (A2)–(A4) below guarantee the positivity of the prices \( S_t^i \), the non-negativity of \( V_t \), and the existence of the optimal strategy.

(A2) \( \det \{\sigma(t, s, v)\} \neq 0 \) for all \((t, s, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}\).

(A3) For all \((t, s, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}, i \in \{1, \ldots, d\}, \text{ and } j \in \{m + 1, \ldots, d\}, \sigma_i^{\alpha_j}(t, s, v) > 0. \) Moreover, if \( A_j < 0 \), then \( \sigma_i^{\alpha_j}(t, x, v)|A_j| < (d - m)^{-1} \).

(A4) If \((S_t, V_t, Z_t, \hat{Z}_t(\cdot))\) is the \( F_t \)-adapted solution to FBSDEs (8), then the random function \((\omega, t, y, z, \hat{z}) \mapsto g(t, S_t, y, z, \hat{z})\) satisfies condition \((A_\gamma)\) in Royer (2006) (p. 1362). Moreover, \( h(S_T) \geq 0 \) a.s.

The main result of our paper is the following.
Theorem 5. Let (A1)–(A4) hold, and let \((S_t, V_t, Z_{t}^{(\alpha)})\) be the solution to FB-SDEs (6)–(7). Then, \(S_t^i > 0, \ i = 1, \ldots, d,\) and \(V_t \geq 0\) a.s. Moreover, the pair of stochastic processes \((\pi_t, \pi_t^0)\), where \(\pi_t = \text{diag}\{S_t^1, \ldots, S_t^d\}^{-1} \sigma^{-1}(t, S_t, V_t)Z_{t}^{(\alpha)}\) and \(\pi_t^0 = \hat{V}_t - \sum_{i=1}^d \pi_s^i S_t^i\), is the optimal hedging strategy.

Proof. Note that the above representation for \(\pi_t\) holds by construction. Next, by the representation for the function \(\psi_i\), obtained in Lemma 3, and by (A3), \(\inf_{t>0} \psi_i(t, S_{t-}, V_{t-}, \Delta L_t) > -1\). Therefore, \(S_t^i\) can be represented by the Doléans-Dade exponential which is finite a.s.:

\[
S_t^i = S_0^i e^{\int_0^t \left(\tilde{f}(s, S_s, V_s, \pi_s) - \frac{\|\sigma_i(s, S_s, V_s)\|^2}{2}\right) ds + \int_0^t \sigma_i(s, S_s, V_s) dB_s + \int_0^t \int_{\mathbb{R}} \psi_i(s, S_{s-}, V_{s-}, u) \tilde{N}(ds, du) \prod_{0 \leq s \leq t} (1 + \psi_i(s, S_{s-}, V_{s-}, \Delta L_s)) e^{-\psi_i(s, S_{s-}, V_{s-}, \Delta L_s)},
\]

where \(\sigma_i = (\sigma_i^{\alpha_j})_{j=1}^d\). Therefore, for all \(i, S_t^i > 0\) a.s. Let us prove the non-negativity of \(V_t\). To this end, we apply the comparison theorem from Royer (2006) (Theorem 2.5, p. 1362) to the BSDE in (8), considered with respect to \((V_t, Z_t, \hat{Z}_t(\cdot))\), whereas the process \(S_t\) is fixed and assumed known from Lemma 4. Note that, by the definition, \(g(t, S_t, 0, 0, 0) = 0\). Therefore, we compare the solution \((V_t, Z_t, \hat{Z}_t(\cdot))\) with the identically zero solution to the BSDE whose generator is the same as in (8) but the final condition is zero. Remark that (A4) implies the assumptions of the comparison theorem in Royer (2006). Thus, by Theorem 2.5 in Royer (2006), \(V_t \geq 0\) for all \(t\) a.s.

Note that, by (5), \(C_t = V_0 + \sum_{(k,j) \in (\alpha)} \int_0^t \hat{A}_s \hat{Z}_s^{(k,j)} dH_s^{(k,j)}\), and, therefore, it is a square integrable martingale. Moreover, \(C_t\) is (weakly) orthogonal to the
stable subspace $S$ generated by $\{M_{t}^{\alpha_j}\}_{j=1}^{d}$, which follows from Theorem 35 of Protter (1992) (p.149) and from the strong orthogonality of the martingales $M_{t}^{\alpha_j}$. By Theorem 36 of Protter (1992) (p.150), $C_{t}$ is strongly orthogonal to $S$. It remains to note that the martingale parts of $\{\hat{S}_{t}^{i}\}_{i=1}^{d}$ belong to $S$. □

**Corollary 6.** The Föllmer-Schweizer decomposition of the discounted contingent claim $A_T h(S_T)$ takes the form

$$A_T h(S_T) = V_0 + \sum_{i=1}^{d} \int_{0}^{T} \pi_{t}^{i} d\hat{S}_{t}^{i} + \sum_{(kj) \notin (\alpha)} \int_{0}^{T} A_{t}^{(kj)} dH_{t}^{(kj)}.$$ 

**References**

E. Eberlein, U. Keller, 1995. Hyperbolic distributions in finance. *Bernoulli*, 1(3), 281–299.

F. Black, M. Scholes, 1973. The pricing of options and corporate liabilities. *J. Political Economy*, 81, 637–659.

H. Föllmer, M. Schweizer, 1991. Hedging of contingent claims under incomplete information. *Applied Stochastic Analysis*, 389–414.

J. Cvitanić, J. Ma, 1996. Hedging options for a large investor and Forward-Backward SDE’s. *The Annals of Applied Probability*, 6(2), 370–398.

D. Nualart, W. Schoutens, 2001. Backward stochastic differential equations and Feynman-Kac formula for Lévy processes with applications in finance. *Bernoulli*, 7(5), 761–776.
M. Royer, 2006. Backward stochastic differential equations with jumps and related non-linear expectations. *Stochastic Processes and their Applications*, 116, 1358–1376.

Z. Wu, 1999. Forward-Backward Stochastic Differential Equations with Brownian motion and Poisson process. *Acta Mathematicae Applicatae Sinica*, 15(4), 433–443.

M. Schweizer, 2008. Local risk-minimization for multidimensional assets and payment streams. *Advances in Mathematics of Finance*, Banach Center Publications, Institute of Mathematics, Polish Academy of Sciences, 83.

P. Protter, 1992. *Stochastic integration and differential equations, a new approach*, Springer.