Global existence of solutions to semilinear damped wave equation with slowly decaying initial data in exterior domain

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Abstract. In this paper, we discuss the global existence of weak solutions to the semilinear damped wave equation

\[
\begin{align*}
\partial_t^2 u - \Delta u + \partial_t u &= f(u) \quad \text{in } \Omega \times (0, T), \\
u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(0) = u_0, \partial_t u(0) &= u_1 \quad \text{in } \Omega,
\end{align*}
\]

in an exterior domain \( \Omega \) in \( \mathbb{R}^N \) \((N \geq 2)\), where \( f: \mathbb{R} \to \mathbb{R} \) is a smooth function behaves like \( f(u) \sim |u|^p \). From the viewpoint of weighted energy estimates given by Sobajima–Wakasugi \[25\], the existence of global-in-time solutions with small initial data in the sense of \( \langle x \rangle^\lambda u_0, \langle x \rangle^\lambda \nabla u_0, \langle x \rangle^\lambda u_1 \in L^2(\Omega) \) with \( \lambda \in (0, \frac{2}{N}) \) is shown under the condition \( p \geq 1 + \frac{4}{N+2} \).

The sharp lower bound for the lifespan of blowup solutions with small initial data \((\varepsilon u_0, \varepsilon u_1)\) is also given.

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1 Introduction

In this paper, we consider the initial-boundary value problem of the semilinear damped wave equation

\[
\begin{align*}
\partial_t^2 u(x, t) - \Delta u(x, t) + \partial_t u(x, t) &= f(u(x, t)), \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \\
\partial_t \nu(x, 0) &= u_1(x), \quad x \in \Omega,
\end{align*}
\]  

(1.1)

where \( \partial_t = \frac{\partial}{\partial t}, \Delta = \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} \) and \( \Omega \subset \mathbb{R}^N \) \((N \in \mathbb{N}, N \geq 2)\) is an exterior domain (that is, \( \mathbb{R}^N \setminus \Omega \) is bounded) having a smooth boundary \( \partial \Omega \). The function \( u: \Omega \times [0, T] \to \mathbb{R} \) is unknown, the pair \((u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)\) is given and \( f \in C^1(\mathbb{R}) \) satisfies that there exist constants \( p > 1 \) and \( C_f \geq 0 \) such that

\[ f(0) = 0, \quad |f(\xi) - f(\eta)| \leq C_f (|\xi| + |\eta|)^{p-1} |\xi - \eta|, \quad \xi, \eta \in \mathbb{R}. \]

(1.2)

The damped wave equation was introduced by Cattaneo \[1\] and Vernotte \[29\] to discuss an model of heat conduction with finite propagation property. The equation is derived by combining “balance law” \( u_t = \text{div} q \) and “time-delayed Fourier law” \( \tau q + q = \nabla u \), where \( q \) is the heat flux

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and \( \tau \) is small enough. Therefore one can expect that the behavior of solution to (1.1) can be approximated by the one of solution to heat equation.

The aim of this study is to give global existence of solutions to (1.1) under the smallness of initial data in the sense of weighted norm

\[
\int_{\Omega} \left( |\nabla u_0|^2 + |u_0|^2 + |u_1|^2 \right) \langle x \rangle^{2\lambda} \, dx
\]

for fixed \( \lambda \geq 0 \), by using the idea of weighted energy estimates including Kummer’s confluent hypergeometric functions, originated in [25].

For the case \( \Omega = \mathbb{R}^N \), there are many previous works dealing with the analysis of critical exponent in the following sense: \( p_c \) is the critical exponent if \( 1 < p < p_c \), then a blowup solution with sufficiently small initial data exists, on the other hand, if \( p > p_c \), then smallness of initial data provides the existence of global-in-time solutions. The critical exponent for compactly supported initial data was given by Todorova–Yordanov [27] with \( p_c = 1 + \frac{2}{N} \), by introducing the energy estimates with exponential type weight function. Similar philosophy can be found in Ikehata–Tanizawa [13] for non-compactly supported initial data.

It should be noticed that the critical exponent is exactly the same as Fujita exponent for semilinear heat equation found in the pioneering work in Fujita [4]. The critical case \( p = p_c \) was discussed in Zhang [34] and blowup result for small initial data is proved.

For the framework of weak solutions in \( H^1 \) with non-compactely supported initial data, Nakao–Ono [18] found that the existence of global-in-time solutions with sufficiently small initial data when \( 1 + \frac{4}{N} \leq p \leq \frac{N+2}{N-2} \). Later, Ikehata–Ohta [13] discussed the critical exponent of (1.1) with initial data \( (u_0, u_1) \in (H^1 \cap L^\infty) \times (L^2 \cap L^\infty) \). It is proved that the critical exponent for this problem is \( p_c = 1 + \frac{N}{4} \); if \( 1 < p < 1 + \frac{N}{4} \), then nonexistence of global-in-time solutions occurs, and if \( 1 + \frac{N}{4} < p < \frac{N+2}{N-2} \), then global-in-time solutions exist for sufficiently small initial data.

The initial data in the class \( (H^{a,0} \cap H^{0,\delta}) \times (H^{a-1,0} \cap H^{0,\delta}) \) are discussed by Hayashi–Kaiikina–Naumkin [5], where \( H^{a,m} = \{ \phi \in L^2 : \| \langle x \rangle^m \langle \partial_x \rangle^a \phi \|_{L^2} < \infty \} \) with the Fourier multiplier \( (i\partial_x)^m = F^{-1} \langle \xi \rangle^m F \). They proved the existence of global-in-time solutions (in \( L^1 \)) to (1.1) with \( p > 1 + \frac{N}{4} \) and a heat-like asymptotic profile of solutions. The analysis of [7] is generalized by Ikeda–Inui–Wakasugi [7] in the framework of \( (H^{a,0} \cap H^{0,\delta}) \times (H^{a-1,0} \cap H^{0,\delta}) \) which can be embedded into \( L^r \)-space \( (r \in (1, 2)) \). In their paper the critical exponent is determined as \( 1 + \frac{2}{N} \) which is the same as [13]. Recently, Inui–Ikeda–Okamoto–Wakasugi [8] discussed the critical case \( p = 1 + \frac{2}{N} \) under some restriction on \( r \), which is required by a derivative loss of \( L^p \)-estimates for high frequency parts of solutions to the linear damped wave equation. We note that for the analysis of the Cauchy problem of the equation \( \partial_t^2 u - \Delta u + \frac{\mu}{(1+\ell)^\tau} \partial_t u = f(u) \) (with time-dependent damping term), a similar study can be found in the literature (see e.g., Wirth [31, 32, 33], Nishihara [19], Lin–Nishihara–Zhai [17], Wakasugi [30], Lai–Takamura–Wakasa [16], and Ikeda–Sobajima [9] and their reference therein).

For the case of damped wave equation in an exterior domain, Ono [20] discussed the existence of global-in-time solutions to (1.1) under \( 2 \leq N \leq 6, 1 + 4/N + 2 < p < 1 + 2/(N-2) \) by using the result of Dan–Shibata [3]. On the other hand, Ikehata [10, 11] proved the existence of global-in-time solutions for \( N = 2, 2 < p < \infty \) by using weighted energy estimates. Takeda–Ogawa [26] proved non-existence of global-in-time solutions to (1.1) when \( N \geq 2, 1 < p < 1 + 2/N \) by employing the method of Kaplan [15] and Fujita [4]. Note that in the analysis of weighted energy estimates of the linear problem with a class of space-dependent damping term in exterior domain can be found in Ikehata [12], Todorova–Yordanov [28], Radu–Todorova–Yordanov [21, 22] and
Wakasugi–Sobajima [23, 24], however, their weight function forms $e^{c(1+|x|^2)^{2-\alpha}/(1+t)^{1+\beta}}$ and therefore the initial data must have an exponential decay property.

Recently, Wakasugi–Sobajima [25] found a framework of weighted energy estimates with a weight function of polynomial type. In [25], the weight function is taken as the inverse of the positive solution of heat equation $\partial_t \Phi = \Delta \Phi$ including the Kummer confluent hypergeometric function (see Section 2.1 below). This enables us to obtain the weighted energy estimate of polynomial type.

The purpose of the present paper is to discuss the nonlinear problem of damped wave equation in exterior domain in view of weighted energy estimate of polynomial type introduced by [25].

To state the main result, we first give the definition of the solutions to (1.1) in this paper.

**Definition 1.1** (Weak solution). The function $u : \Omega \times [0, T) \rightarrow \mathbb{R}$ is called a weak solution of (1.1) in $(0, T)$ if $u$ belongs to the class

$$S_T = \{ u \in C([0,T); H^1_0(\Omega)) \cap C^1([0,T); L^2(\Omega)) : f(u(\cdot)) \in C([0,T); L^2(\Omega)) \}$$

and $U = (u(t), \partial_t u(t))$ satisfies the following integral equation in $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega)$:

$$U(t) = e^{-tA}U_0 + \int_0^t e^{-(t-s)A}[\mathcal{N}(U(s))]ds, \quad t \in [0, T),$$

where $U_0 = (u_0, u_1)$, $A = \begin{pmatrix} 0 & -1 \\ -\Delta & 1 \end{pmatrix}$ with domain $D(A) = (H^2(\Omega) \times H^1_0(\Omega)) \times H^1_0(\Omega)$ and $\mathcal{N}(u,v) = (0, f(u))$.

The existence of local-in-time solutions to (1.1) is well-known (see e.g., Ikawa [1] and Cazenave–Haraux [2]).

**Proposition 1.1.** Assume that $f$ satisfies (1.2) with $1 < p \leq \frac{N}{N-2}$. Then for every $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$, there exists a positive constant $T > 0$ depending only on $N, p, C_0, \|u_0\|_{H^1}, \|u_1\|_{L^2}$ such that there exists a unique weak solution $u$ in $(0, T)$.

The notion of lifespan is the following.

**Definition 1.2** (Lifespan). For the solution $u$ of (1.1) with initial data $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$, we define lifespan $T_* = T_*(u_0, u_1) > 0$ (the maximal existence time of solution $u$) as follows:

$$T_* = \sup\{ T > 0 : (1.1) \text{ has a unique weak solution in } (0, T) \}.$$ 

Now we are in a position to state the main result of the present paper.

**Theorem 1.2.** Assume that $f$ satisfies (1.2) with $1 < p \leq \frac{N}{N-2}$. Then for every $\lambda \in [0, \frac{N}{2})$, there exists a positive constant $\delta^*_\lambda > 0$ such that the following assertion holds:

For every $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ satisfying

$$\int_\Omega \left( |\nabla u_0(x)|^2 + |u_0(x)|^2 + |u_1(x)|^2 \right)^\lambda dx \leq \delta^*_\lambda,$$
one has $T_*(u_0, u_1) = \infty$ when $p \geq 1 + \frac{4}{N+2\lambda}$. Namely, there exists a global weak solution of (1.1) with initial data $(u_0, u_1)$. Moreover, $u$ satisfies the following weighted estimates: there exists a positive constant $M_* > 0$ such that

$$
\int_{\Omega} (|\nabla u(x, t)|^2 + |\partial_t u(x, t)|^2) (1 + t + |x|^2)^\lambda \, dx \leq \frac{M_*}{1 + t},
$$

$$
\int_{\Omega} |u_1(x, t)|^2 (1 + t + |x|^2)^\lambda \, dx \leq M_*^*,
$$

$$
\int_{0}^{\infty} \int_{\Omega} |\nabla u(x, s)|^2 (1 + s + |x|^2)^\lambda \, dx \, ds \leq M_*^*,
$$

$$
\int_{0}^{\infty} (1 + s) \int_{\Omega} |\partial_t u(x, s)|^2 (1 + s + |x|^2)^\lambda \, dx \, ds \leq M_*^*.
$$

On the other hand, for the case $1 < p < 1 + \frac{4}{N+2\lambda}$, one has the following lower estimate of lifespan

$$
T_*(\varepsilon u_0, \varepsilon u_1) \geq C \varepsilon^{-(\frac{1}{p-1} - \frac{N+2\lambda}{4})^{-1}}
$$

for some $C > 0$ (independent of $\varepsilon$) and sufficiently small $\varepsilon > 0$.

Remark 1.1. In the case $\Omega = \mathbb{R}^N$, the global-in-time solution of (1.1) with slowly decaying initial data (like $(x)^{-\mu}$) was constructed in Hayashi–Kaihina–Naumkin [5] (for $p > 1 + \frac{2}{N}$) under a weaker assumption than ours. In the case of exterior domain, it is already discussed when $\lambda = 1$. However, the case $\lambda \in (0, \frac{N}{2})$ is not dealt with so far. The global existence for weighted-$L^2$-type initial data is now established.

Remark 1.2. For $L^r$-type initial data, Ikeda–Inui–Okamoto–Wakasugi [8] proved the case $p = 1 + \frac{2}{N}$ under some restriction on $r$, which is critical in this situation and related to our critical case $p = 1 + \frac{4}{N+2\lambda}$. Although their aspect is quite far from ours, it should be noticed that the framework in [8] is difficult to apply to the case of exterior domain because of the use of a deep Fourier analysis.

Remark 1.3. For the lifespan estimates, Ikeda–Inui–Wakasugi [7] provided upper bound of lifespan estimates with a specific situation

$$
\Omega = \mathbb{R}^N, \quad u_0 + u_1 \geq \max\{1, |x|\}^{-\frac{N}{2}-\lambda}, \quad \lambda < \frac{2}{p-1} - \frac{N}{2}, \quad f(u) = |u|^p.
$$

They proved $T_*(\varepsilon u_0, \varepsilon u_1) \geq C \varepsilon^{-(\frac{1}{p-1} - \frac{N+2\lambda}{4})^{-1}}$. Combining their result, we can assert that the lower estimate in Theorem 1.2 is almost sharp.

As a corollary of Theorem 1.2, we can deduce the existence of global-in-time solutions to (1.1) with $p > 1 + \frac{2}{N}$ for polynomially decaying initial data.

**Corollary 1.3.** Assume that $f$ satisfies (1.2). Then for every $1 + \frac{2}{N} < p \leq \frac{N}{N-2}$, there exists a positive constant $\delta_p^{**} > 0$ such that the following assertion holds: For every $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$
\int_{\Omega} \left( |\nabla u_0(x)|^2 + |u_0(x)|^2 + |u_1(x)|^2 \right) (x)^N \, dx \leq \delta_p^{**},
$$

one has $T_*(u_0, u_1) = \infty$. 

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Of course we can proceed the same argument in the one-dimensional case $\Omega = \mathbb{R}$. However, the lack of the validity of (weighted) Hardy’s inequality causes, and some difficulty appears. To avoid the use of Hardy’s inequality, we use a solution of heat equation with some modification. As a result, we lose the result of the critical situation $p = 1 + \frac{4}{1+2\lambda}$. The precise statement for the case $\Omega = \mathbb{R}$ is written in the end of the last section.

This paper is organized as follows. In Section 2, we state the properties of a family of self-similar solutions to the heat equation including Kummer’s Confluent hypergeometric functions and collect some functional inequalities we need in the derivation of weight energy estimates. Section 3 is devoted to prove Theorem 1.2. Finally, we give a remark about the weighted energy estimates and global existence for one-dimensional case in Section 4.

2 Preliminaries

2.1 The weight functions including confluent hypergeometric functions

For $t_0 \geq 1$ and $\beta \geq 0$, define

$$\Phi_\beta(x, t : t_0) = (t_0 + t)^{-\beta} e^{-\frac{|x|^2}{4(t_0 + t)}} M\left(\frac{N}{2} - \beta, \frac{N}{2}, \frac{|x|^2}{4(t_0 + t)}\right), \quad (x, t) \in \mathbb{R}^N \times [0, \infty),$$

where $M(a, c; z)$ is Kummer’s confluent hypergeometric function defined as

$$M(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} z^n,$$

with the Pochhammer symbol $(d)_0 = 1$ and $(d)_n = \prod_{k=0}^{n-1} (d + k - 1)$. These functions are given by Sobajima–Wakasugi [25] as a family of self-similar solution of linear heat equation $\partial_t \Phi - \Delta \Phi = 0$. Then we have the following lemma.

Lemma 2.1 ([25]).

(i) for every $\beta \geq 0$, $\partial_t \Phi_\beta(x, t) = \Delta \Phi_\beta(x, t)$ for $(x, t) \in \mathbb{R}^N \times [0, \infty),$

(ii) for every $\beta \geq 0$, $\partial_t \Phi_\beta(x, t) = -\beta \Phi_{\beta+1}(x, t)$ for $(x, t) \in \mathbb{R}^N \times [0, \infty),$

(iii) for every $\beta \geq 0$, there exists a positive constant $C_\beta$ such that

$$|\Phi_\beta(x, t)| \leq C_\beta \left( t_0 + t + \frac{|x|^2}{4} \right)^{-\beta},$$

(iv) for every $0 \leq \beta < \frac{N}{2}$, there exists a positive constant $c_\beta$ such that

$$\Phi_\beta(x, t) \geq c_\beta \left( t_0 + t + \frac{|x|^2}{4} \right)^{-\beta}.$$

2.2 Functional inequalities with weights

In view of Lemma 2.1 for the same constant $t_0 \geq 1$ as $\Phi_\beta$, we also introduce

$$\Psi(x, t : t_0) = t_0 + t + \frac{|x|^2}{4}, \quad (x, t) \in \mathbb{R}^N \times [0, \infty).$$

The following Hardy type inequality with $\Psi$ is also needed.
Lemma 2.2. Let $\lambda > -\frac{N-2}{2}$. For every $w \in C^1_0(\Omega)$,
\[
4K(\lambda)^2 \int_{\Omega} |w|^2 \Psi^{\lambda-1} \, dx \leq \int_{\Omega} |\nabla w|^2 \Psi^\lambda \, dx
\]  
(2.1)
with $K(\lambda) = \min\{\frac{N}{2} + \lambda - 1, \frac{N}{2}\}$. That is, if $N \geq 2$, then (2.1) holds for every $\lambda > 0$ and every $w \in H^1_0(\Omega)$ satisfying $w\Psi^\lambda, (\nabla w)\Psi^\lambda \in L^2(\Omega)$.

Proof. Noting that
\[
\text{div} \left( \frac{x}{2} \Psi^{\lambda-1} \right) = \frac{N}{2} \Psi^{\lambda-1} + (\lambda - 1) \frac{|x|^2}{4} \Psi^{\lambda-2} \geq K \Psi^{\lambda-1}
\]
with $K(\lambda) = \min\{\frac{N}{2} + \lambda - 1, \frac{N}{2}\} > 0$, we see from integration by parts and Hölder inequality that
\[
K(\lambda) \int_{\mathbb{R}^N} |w|^2 \Psi^{\lambda-1} \, dx \leq \int_{\mathbb{R}^N} |w|^2 \text{div} \left( \frac{x}{2} \Psi^{\lambda-1} \right) \, dx
\]
\[\leq -2 \int_{\mathbb{R}^N} w \left( \nabla w \cdot \frac{x}{2} \right) \Psi^{\lambda-1} \, dx
\]
\[\leq 2 \left( \int_{\mathbb{R}^N} |w|^2 \Psi^{\lambda-1} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\nabla w|^2 \frac{|x|^2}{4} \Psi^{\lambda-1} \, dx \right)^{\frac{1}{2}}
\]
\[\leq 2 \left( \int_{\mathbb{R}^N} |w|^2 \Psi^{\lambda-1} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\nabla w|^2 \Psi^{\lambda} \, dx \right)^{\frac{1}{2}}.
\]
The last assertion can be verified by the standard approximation argument with the mollifier and cut-off functions. $\square$

The following lemma is well-known Gagliardo–Nirenberg inequality.

Lemma 2.3 (Gagliardo–Nirenberg inequality). If $1 < p \leq \frac{N+2}{N-2}$, then there exists a constant $C_{GN,p} > 0$ such that for every $w \in H^1_0(\Omega)$,
\[
\|w\|_{L^{p+1}(\Omega)} \leq C_{GN,p} \|w\|^{\frac{N(p-1)}{2(p+1)}}_{L^2(\Omega)} \|\nabla w\|^{\frac{N(p-1)}{2(p+1)}}_{L^2(\Omega)}.
\]

Next we give a weight version of Gagliardo-Nirenberg inequality, which we will exactly need in the treatment of nonlinear term $f(u)$ in (1.1).

Lemma 2.4. If $N \geq 2$ and $1 < p \leq \frac{N+2}{N-2}$ and $\lambda > 0$, then there exists a constant $\tilde{C}_p > 0$ such that for every $w \in H^1_0(\Omega)$ satisfying $v\Psi^\lambda, (\nabla v)\Psi^\lambda \in L^2(\Omega)$,
\[
\|v \Psi^\frac{\lambda}{p+1}\|_{L^{p+1}(\Omega)} \leq \tilde{C}_p (t_0 + t) \frac{\lambda^{(p-1)}}{2(p+1)} \|v \Psi^\frac{\lambda}{2(p+1)}\|_{L^2(\Omega)} \|\nabla v\Psi^\frac{\lambda}{2(p+1)}\|_{L^2(\Omega)}.
\]

Proof. Note that by assumption, we have $v \Psi^\frac{\lambda}{p+1} \in H^1_0(\Omega)$. Therefore applying Lemma 2.3 to $w = v \Psi^\frac{\lambda}{p+1}$ and using $|\nabla \Psi| \leq \Psi^\frac{\lambda}{2}$ imply
\[
\|v \Psi^\frac{\lambda}{p+1}\|_{L^{p+1}(\Omega)} \leq C_{GN,p} \|v \Psi^\frac{\lambda}{p+1}\|^{\frac{N(p-1)}{2(p+1)}}_{L^2(\Omega)} \|\nabla (v \Psi^\frac{\lambda}{p+1})\|^{\frac{N(p-1)}{2(p+1)}}_{L^2(\Omega)}.
\]
\[\leq C_{GN,p} \|v \Psi^\frac{\lambda}{p+1}\|^{\frac{N(p-1)}{2(p+1)}}_{L^2(\Omega)} \left( \|\nabla v\Psi^\frac{\lambda}{2(p+1)}\|_{L^2(\Omega)} + \frac{\lambda}{p+1} \|v \Psi^\frac{\lambda}{2(p+1)}\|_{L^2(\Omega)} \right)^{\frac{N(p-1)}{2(p+1)}}.\]
Combining the above inequality with Lemma 2.2 with \(v\) and \(\frac{\lambda}{p+1}\), we have

\[
\|v\Psi^{\frac{\lambda}{p+1}}\|_{L^{p+1}(\Omega)} \leq C_{GN,p} \left( 1 + \frac{\lambda K(\frac{\lambda}{p+1})}{p+1} \right)^{\frac{N(p-1)}{2(p+1)}} \|v\Phi^{\frac{\lambda}{p+1}}\|_{L^2(\Omega)} \|\nabla v\Psi^{\frac{\lambda}{p+1}}\|_{L^2(\Omega)}. 
\]

Using the inequality \(\Psi^{-1} \leq (t_0 + t)^{-1}\), we deduce the desired inequality. \(\Box\)

Thirdly, we give an inequality related to integration by parts formula with non-uniform weight. Although its proof is essentially stated in [25], we would give a proof for reader’s convenience.

**Lemma 2.5.** Assume that \(\Phi \in C^2(\Omega)\) is positive and \(\delta \in (0, \frac{1}{2})\). Then for every \(u \in H_0^1(\Omega)\) having a compact support,

\[
\int_{\Omega} u \Delta u \Phi^{-1+2\delta} \, dx \leq -\frac{\delta}{1-\delta} \int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx + \frac{1 - 2\delta}{2} \int_{\mathbb{R}^N} u^2(\Delta \Phi)\Phi^{-2+2\delta} \, dx
\]

**Proof.** Set \(v = \Phi^{-1+\delta} u\). Then we have

\[
\int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx = \int_{\Omega} |\nabla v|^2 \Phi \, dx + 2(1-\delta) \int_{\Omega} v(\nabla v \cdot \nabla \Phi) \, dx + (1-\delta)^2 \int_{\Omega} v^2 \frac{|\nabla \Phi|^2}{\Phi} \, dx.
\]

We see from integration by parts that

\[
\int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx \geq -(1-\delta) \int_{\Omega} u^2(\Delta \Phi)\Phi^{-2+2\delta} \, dx + (1-\delta)^2 \int_{\Omega} \frac{|\nabla \Phi|^2}{\Phi} \, dx.
\]

Using the above inequality with integration by parts twice, we deduce

\[
\int_{\Omega} u \Delta u \Phi^{-1+2\delta} \, dx = - \int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx + (1-2\delta) \int_{\Omega} u(\nabla u \cdot \nabla \Phi)\Phi^{-2+2\delta} \, dx
\]

\[
= - \int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx - \frac{1 - 2\delta}{2} \int_{\Omega} u^2(\Delta \Phi)\Phi^{-2+2\delta} \, dx
\]

\[
+ (1-\delta)(1-2\delta) \int_{\Omega} u^2 \frac{|\nabla \Phi|^2}{\Phi} \, dx
\]

\[
\leq \left( \frac{1 - 2\delta}{1-\delta} - 1 \right) \int_{\Omega} |\nabla u|^2 \Phi^{-1+2\delta} \, dx + \frac{1 - 2\delta}{2} \int_{\Omega} u^2(\Delta \Phi)\Phi^{-2+2\delta} \, dx.
\]

The proof is complete. \(\Box\)

### 3 Proof of main theorem (Theorem 1.2)

Since the weak solution \(u\) of (1.1) can be approximated by the one with smooth compactly supported initial data, in this section we may assume that \(u_0 \in H^2(\Omega) \cap H_0^1(\Omega)\) and \(u_1 \in H_0^1(\Omega)\) are compactly supported without loss of generality. By finite propagation property, we also can assume that the solution \(u(t) \in H^2(\Omega) \cap H_0^1(\Omega)\) \((t \geq 0)\) is also compactly supported for every \(t \in [0, T_*)\).

The proof of Theorem 1.2 is based on the following proposition which is well-known, and so-called blowup alternative.
Proposition 3.1. Assume that \( f \) satisfies (1.2) with \( 1 < p \leq \frac{N}{N-2} \). Let \( u \) be the weak solution of (1.1) in \((0, T_*)\) with the corresponding lifespan \( T_* \). If \( T_* < \infty \), then one has

\[
\lim_{t \to T_*} (\|u\|_{H^1_0(\Omega)} + \|u\|_{L^2(\Omega)}) = \infty.
\]

For \( t_0 \geq 1 \) and \( \lambda > 0 \), define the following weighted energy functional for the weak solution \( u \) as follows:

\[
E_\lambda(t : t_0) := (t_0 + t) \int_\Omega \left( |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) \Psi(x,t : t_0) \lambda^2 \, dx, \quad t \geq 0. \tag{3.1}
\]

Then the following lemma holds.

Lemma 3.2. Let \( E_\lambda \) be given in (3.1). Then for every \( t_0 \geq 1 \) and \( \lambda \geq 0 \),

\[
\frac{d}{dt} E_\lambda(t : t_0) \leq (\lambda^2 + \lambda + 1) \int_\Omega |\nabla u|^2 \lambda^2 \, dx + (\lambda + 1 - t_0 - t) \int_\Omega |\partial_t u|^2 \lambda^2 \, dx
\]

\[
+ \frac{d}{dt} \left[ 2(t_0 + t) \int_\Omega F(u) \Psi \lambda^2 \, dx + 2(\lambda + 1) \int_\Omega F(u) \Psi \lambda^2 \, dx \right], \quad \text{a.a. } t \in (0, T_*),
\]

where \( F(\xi) = \int_0^\xi f(\eta) \, d\eta \) for \( \xi \in \mathbb{R} \).

Proof. Observe that \( \partial_t u(t) \in H^1_0(\Omega) \) and \( \Psi = t_0 + t + \frac{|x|^2}{4} \). By integration by parts we have

\[
\frac{d}{dt} E_\lambda(t : t_0) = \int_\Omega \left( |\nabla u|^2 + |\partial_t u|^2 \right) \left[ \Psi + (t_0 + t) \lambda \Psi^{-1} \right] \, dx
\]

\[
+ 2(t_0 + t) \int_\Omega \left( \nabla \partial_t u \cdot \nabla u + \partial_t u \partial_t^2 u \right) \lambda \Psi \, dx
\]

\[
\leq (\lambda + 1) \int_\Omega \left( |\nabla u|^2 + |\partial_t u|^2 \right) \lambda \Psi \, dx
\]

\[
+ 2(t_0 + t) \int \partial_t u(-\Delta u + \partial_t^2 u) \Psi - \lambda \partial_t u (\nabla u \cdot \nabla \Psi) \Psi^{-1} \, dx.
\]

Since \( u \) satisfies (1.1), the Schwarz inequality and the inequality \((t_0 + t)^\frac{1}{2} |\nabla \Psi| \leq \Psi \) yield

\[
\frac{d}{dt} E_\lambda(t : t_0) \leq (\lambda + 1) \int \left( |\nabla u|^2 + |\partial_t u|^2 \right) \lambda \Psi \, dx - 2(t_0 + t) \int |\partial_t u|^2 \lambda \Psi \, dx
\]

\[
+ 2(t_0 + t) \int \partial_t u(f(u)) \lambda \Psi \, dx
\]

\[
+ 2\lambda(t_0 + t)^\frac{1}{2} \left( \int |\partial_t u|^2 \lambda \Psi \, dx \right)^\frac{1}{2} \left( \int |\nabla u|^2 \Psi \, dx \right)^\frac{1}{2}.
\]

Noting that

\[
\int \partial_t u(f(u)) \lambda \Psi \, dx = \frac{d}{dt} \left[ (t_0 + t) \int F(u) \lambda \Psi \, dx \right] + \int F(u) \lambda \Psi \, dx \leq \int F(u) \lambda \Psi \, dx,
\]

we deduce the desired inequality. \( \square \)
Next we assume $\lambda \in \left[0, \frac{N}{4} \right)$. Set $\delta = \frac{N - 2\lambda}{4N} \in \left(0, \frac{1}{4}\right)$. Define

$$\beta := \frac{\lambda}{1 - 2\delta} = \frac{2\lambda N}{N + 2\lambda} \in \left(\lambda, \frac{N}{2}\right)$$

and

$$\tilde{E}_\lambda(t : t_0) := \int_{\mathbb{R}^N} \left(2u(x,t)\partial_t u(x,t) + |u(x,t)|^2\right)\Phi_\beta(x,t : t_0)^{-1+2\delta} \, dx, \quad t \in [0, T_*). \quad (3.2)$$

Then the following inequality holds.

**Lemma 3.3.** Let $\tilde{E}_\lambda$ be as in (3.2). Then for every $t_0 \geq 1$ and $\lambda \in \left[0, \frac{N}{4} \right)$ with $\delta = \frac{N - 2\lambda}{4N}$ and $\beta = \frac{2\lambda N}{N + 2\lambda}$, one has

$$\frac{d}{dt} \tilde{E}_\lambda(t : t_0) \leq \frac{1}{c^1_{\beta^2}} \left(2 + \frac{(1 - 2\delta)\beta C_{\beta+1}}{c_{\beta} K(\lambda)}\right) \int_{\mathbb{R}^N} |\partial_t u|^2 \Psi^\lambda \, dx$$

$$+ \left(\frac{(1 - 2\delta)\beta C_{\beta+1}}{c^2_{\beta^2} t_0} - \frac{2\delta}{(1 - \delta)C^1_{\beta^2}}\right) \int_{\mathbb{R}^N} |\nabla u|^2 \Psi^\lambda \, dx + \frac{2}{c_{\beta}^1} \int_{\mathbb{R}^N} |uf(u)|\Psi^\lambda \, dx.$$

**Proof.** Since $u$ is a solution of (1.1), we have

$$\frac{d}{dt} \tilde{E}_\lambda(t : t_0) = 2 \int_{\mathbb{R}^N} |\partial_t u|^2 \Phi_\beta^{-1+2\delta} \, dx + 2 \int_{\mathbb{R}^N} u(\partial_t^2 u + \partial_t u)\Phi_\beta^{-1+2\delta} \, dx$$

$$- (1 - 2\delta) \int_{\mathbb{R}^N} \left(2u\partial_t u + |u|^2\right)\Phi_\beta^{-2+2\delta} \partial_t \Phi_\beta \, dx$$

$$= 2 \int_{\mathbb{R}^N} |\partial_t u|^2 \Phi_\beta^{-1+2\delta} \, dx + 2 \int_{\mathbb{R}^N} u(\Delta u)\Phi_\beta^{-1+2\delta} \, dx + 2 \int_{\mathbb{R}^N} uf(u)\Phi_\beta^{-1+2\delta} \, dx$$

$$- 2(1 - 2\delta) \int_{\mathbb{R}^N} u\partial_t u\Phi_\beta^{-2+2\delta} \partial_t \Phi_\beta \, dx - (1 - 2\delta) \int_{\mathbb{R}^N} |u|^2 \Phi_\beta^{-2+2\delta} \partial_t \Phi_\beta \, dx.$$

Employing Lemma 2.5 with $\Phi = \Phi_\beta$, we have

$$\frac{d}{dt} \tilde{E}_\lambda(t : t_0) \leq 2 \int_{\mathbb{R}^N} |\partial_t u|^2 \Phi_\beta^{-1+2\delta} \, dx - \frac{2\delta}{1 - \delta} \int_{\mathbb{R}^N} |\nabla u|^2 \Phi_\beta^{-1+2\delta} \, dx + 2 \int_{\mathbb{R}^N} uf(u)\Phi_\beta^{-1+2\delta} \, dx$$

$$- 2(1 - 2\delta) \int_{\mathbb{R}^N} u\partial_t u\Phi_\beta^{-2+2\delta} \partial_t \Phi_\beta \, dx - (1 - 2\delta) \int_{\mathbb{R}^N} |u|^2 \Phi_\beta^{-2+2\delta} (\partial_t \Phi_\beta - \Delta \Phi_\beta) \, dx.$$

Here we use the profile of $\Phi_\beta$ stated in Lemma 2.1. Then

$$\frac{d}{dt} \tilde{E}_\lambda(t : t_0) \leq \frac{2}{c^1_{\beta^2}} \int_{\mathbb{R}^N} |\partial_t u|^2 \Psi^\lambda \, dx - \frac{2\delta}{(1 - \delta)C^1_{\beta^2}} \int_{\mathbb{R}^N} |\nabla u|^2 \Psi^\lambda \, dx + 2 \int_{\mathbb{R}^N} uf(u)\Psi^\lambda \, dx$$

$$+ \frac{2(1 - 2\delta)\beta C_{\beta+1}}{c^2_{\beta^2}} \int_{\Omega} |u| |\partial_t u| \Psi^{\lambda - 1} \, dx.$$

By Lemma 2.2, the last term on the right-hand side of the above inequality can be estimated as follows:

$$\int_{\Omega} |u| |\partial_t u| \Psi^{\lambda - 1} \, dx \leq \left(\int_{\Omega} |u|^2 \Psi^{\lambda - 1} \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |\partial_t u|^2 \Psi^{\lambda - 1} \, dx\right)^{\frac{1}{2}}$$

$$\leq \frac{1}{K(\lambda)(t_0 + t)^{\frac{3}{2}}} \left(\int_{\Omega} |\nabla u|^2 \Psi \, dx\right)^{\frac{1}{2}} \left(\int_{\Omega} |\partial_t u|^2 \Psi \, dx\right)^{\frac{1}{2}}.$$
Hence we obtain the desired inequality.

To the end of this section we will give an estimate for the following weighted total energy functional:

\begin{align*}
m(t: t_0) := (t_0 + t) \int_{\mathbb{R}^N} & \left( |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) \Psi(x, t: t_0)^\lambda \, dx \\
& + \int_{\mathbb{R}^N} |u(x,t)|^2 \Psi(x, t: t_0)^\lambda \, dx, \quad t \in [0, T_*).
\end{align*}

**Lemma 3.4.** for every \( \nu > 0 \) there exists positive constants \( \gamma_\nu > 0 \) and \( \Gamma_\nu > 0 \) such that if \( t_0 \geq c_\nu^{-1+2\delta} \), then

\[ \gamma_\nu m(t: t_0) \leq E(t: t_0) + \nu \tilde{E}(t: t_0) \leq \gamma_\nu m(t: t_0), \quad t \in (0, T_*). \]

**Proof.** By the Schwarz inequality and Lemma 2.1 (iv) we see that

\[ \left| \int_{\Omega} u \partial_t u \Phi^{-1+2\delta} \, dx \right| \leq \left( \int_{\Omega} |\partial_t u|^{2\Phi^{-1+2\delta}} \, dx \right)^{\frac{1}{2}} \left( \int_{\Omega} |u|^{2\Phi^{-1+2\delta}} \, dx \right)^{\frac{1}{2}} \]

\[ \leq \frac{1}{c_\beta^{-\delta}(t_0 + t)^\frac{1}{2}} E(t: t_0) \]

\[ \leq \frac{1}{c_\beta^{-2\delta} t_0} E(t: t_0) + \frac{1}{4} \int_{\Omega} |u|^{2\Phi^{-1+2\delta}} \, dx \]

and therefore

\[ E(t: t_0) + \nu \tilde{E}(t: t_0) \geq \left( 1 - \frac{\nu}{c_\beta^{-1+2\delta} t_0} \right) E(t: t_0) + \frac{\nu}{2} \int_{\Omega} |u|^{2\Phi^{-1+2\delta}} \, dx \]

\[ E(t: t_0) + \nu \tilde{E}(t: t_0) \leq \left( 1 + \frac{\nu}{c_\beta^{-1+2\delta} t_0} \right) E(t: t_0) + \frac{3\nu}{2} \int_{\Omega} |u|^{2\Phi^{-1+2\delta}} \, dx. \]

In view of Lemma 2.1 (iii), this means that the assumption \( t_0 \geq c_\nu^{-1+2\delta} \) implies the assertion of this lemma.

Furthermore we set

\[ Y(t: t_0) := \int_0^t \int_{\Omega} |\nabla u(x,s)|^2 \Psi(s: t_0)^\lambda \, dx \, ds, \quad t \in [0, T_*) \]

\[ Z(t: t_0) := \int_0^t (t_0 + s) \int_{\Omega} |\partial_t u(x,s)|^2 \Psi(s: t_0)^\lambda \, dx \, ds, \quad t \in [0, T_*) \]

**Proposition 3.5.** There exists positive constants \( t^*_0 \geq 1 \) and \( \eta^* > 0 \) such that

\[ \eta^*(m(t: t_0^*) + Y(t: t_0^*) + Z(t: t_0^*)) \leq m(0: t^*_0) + \int_{\mathbb{R}^N} |F(u_0)| \Psi^*(0)^\lambda \, dx \]

\[ + (t^*_0 + t) \int_{\mathbb{R}^N} |F(u(t))| \Psi^*(t)^\lambda \, dx + \int_0^t \int_{\mathbb{R}^N} (|F(u(s))| + |u(s)| f(u(s))) \Psi^*(s)^\lambda \, dx \, ds, \]

where \( \Psi^*(\cdot, t) = \Psi(\cdot, t: t^*_0) \).
Remark 3.1. If \( f \equiv 0 \), then (1.1) is linear problem of damped wave equation in exterior domain. In this case Proposition 3.5 provides the following energy decay estimates

\[
\int_{\mathbb{R}^N} |\nabla u(t)|^2 \, dx + \int_{\mathbb{R}^N} |\partial_t u(t)|^2 \, dx \leq C(1 + t)^{-\lambda - 1},
\]

\[
\int_{\mathbb{R}^N} |u(t)|^2 \, dx \leq C(1 + t)^{-\lambda}
\]

under the assumption \( (|\nabla u_0|^2 + |u_0|^2 + |u_1|^2)x^\lambda \in L^1(\Omega) \) with \( \lambda \in [0, \frac{N}{2}) \), which is slightly weaker than that of [25].

Proof. We see from Lemmas 3.2 and 3.3 that if there exists a constant \( t_1 > 0 \) such that if \( t_0 \geq t_1 \), then we have

\[
\frac{d}{dt} E_{\lambda}(t : t_0) = (\lambda^2 + \lambda + 1) \int_{\Omega} |\nabla u|^2 \Psi^\lambda \, dx - \frac{1}{2} (t_0 + t) \int_{\Omega} |\partial_t u|^2 \Psi^\lambda \, dx
\]

\[
+ \frac{d}{dt} \left[ 2(t_0 + t) \int_{\Omega} F(u) \Psi^\lambda \, dx \right] + 2(\lambda + 1) \int_{\Omega} F(u) \Psi^\lambda \, dx
\]

\[
c_\beta^{1-2\delta} \frac{d}{dt} E_{\lambda}(t : t_0) \leq (2 + \frac{(1 - 2\delta)\beta c_{\beta + 1}}{c_{\beta} K(\lambda)}) \int_{\mathbb{R}^N} |\partial_t u|^2 \Psi^\lambda \, dx - \frac{\delta c_\beta^{1-2\delta}}{(1 - \delta)c_\beta^{1-2\delta}} \int_{\mathbb{R}^N} |\nabla u|^2 \Psi^\lambda \, dx
\]

\[
+ 2 \int_{\mathbb{R}^N} |uf(u)| \Psi^\lambda \, dx.
\]

Therefore by choosing \( \nu = \delta^{-1}(1 - \delta)c_\beta^{1-2\delta}(\lambda^2 + \lambda + 2) \) and \( t_0^* \) sufficiently large, we have

\[
\frac{d}{dt} \left[ E_{\lambda}(t : t_0) + \nu E_{\lambda}(t : t_0) \right] \leq \left[ \nu \left( 2 + \frac{(1 - 2\delta)\beta c_{\beta + 1}}{c_{\beta} K(\lambda)} \right) - \frac{1}{2} (t_0 + t) \right] \int_{\Omega} |\partial_t u|^2 \Psi^\lambda \, dx
\]

\[- \int_{\Omega} |\nabla u|^2 \Psi^\lambda \, dx + \frac{d}{dt} \left[ 2(t_0 + t) \int_{\Omega} F(u) \Psi^\lambda \, dx \right]
\]

\ [+ 2(\lambda + 1) \int_{\Omega} F(u) \Psi^\lambda \, dx + 2\nu \int_{\mathbb{R}^N} |uf(u)| \Psi^\lambda \, dx \leq - \frac{1}{4} (t_0 + t) \int_{\Omega} |\partial_t u|^2 \Psi^\lambda \, dx - \int_{\Omega} |\nabla u|^2 \Psi^\lambda \, dx
\]

\[+ \frac{d}{dt} \left[ 2(t_0 + t) \int_{\Omega} F(u) \Psi^\lambda \, dx \right]
\]

\[+ 2(\lambda + 1) \int_{\Omega} F(u) \Psi^\lambda \, dx + 2\nu \int_{\mathbb{R}^N} |uf(u)| \Psi^\lambda \, dx \]

Integrating it over \([0, t]\) and applying Lemma 3.4, we have

\[
\gamma_{\nu} m_{\lambda}(t : t_0) + \int_0^t \int_{\Omega} |\nabla u(s)|^2 \Psi(s) \, dx \, ds + \frac{1}{4} \int_0^t (t_0^* + s) \int_{\Omega} |\partial_t u(s)|^2 \Psi(s) \, dx \, ds
\]

\[\leq \Gamma_{\nu} m_{\lambda}(0 : t_0) - 2t_0 \int_{\Omega} F(u_0) \Psi(0) \, dx
\]

\[+ 2(t_0 + t) \int_{\Omega} F(u(t)) \Psi(t) \, dx + \int_0^t \int_{\Omega} \left( 2(\lambda + 1) F(u(s)) + 2\nu |u(s) f(u(s))| \right) \Psi(s) \, dx \, ds,
\]

where \( \Psi(s) = \Psi(\cdot, t : t_0^*) \). This yields the desired inequality. \( \square \)
Proof of Theorem 1.3. Put \( m_\lambda(t) = m_\lambda(t : t_0^*) \) and \( Y_\lambda(t) = Y_\lambda(t : t_0^*) \). By Proposition 3.3 and 1.2, we deduce

\[
\eta^*(m_\lambda(t) + Y_\lambda(t)) \leq m_\lambda^* + \frac{C_f}{p + 1} (t_0^* + t) \int_{\Omega} |u(t)|^{p+1} \Psi_*(t)^\lambda \, dx \\
+ \frac{(p+2)C_f}{p + 1} \int_0^t \int_{\Omega} |u(s)|^{p+1} \Psi_*(s)^\lambda \, dx \, ds,
\]

where

\[
m_\lambda^* := m_\lambda(0) + \frac{C_f}{p + 1} \int_{\Omega} |u_0|^p \Psi_*(0)^\lambda \, dx < \infty.
\]

(The supercritical case \( 1 + \frac{4}{N+2\lambda} < p \leq \frac{N}{N-2} \)) Observe that Lemma 2.4 that

\[
\int_{\Omega} |u(t)|^{p+1} \Psi_*(t)^\lambda \, dx \\
\leq \tilde{C}_p^{p+1}(t_0^* + t)^{-\frac{2}{p-1}} \left( \int_{\Omega} |u(t)|^{2} \Psi_*(t)^\lambda \, dx \right)^{\frac{p+1}{2}} \left( \int_{\Omega} \nabla u(t)^2 \Psi_*(t)^\lambda \, dx \right)^{\frac{N(p-1)}{2}}
\]

\[
\leq \tilde{C}_p^{p+1}(t_0^* + t)^{-\frac{N+2\lambda(p-1)}{4}} m_\lambda(t)^{\frac{p+1}{2}}.
\]

Therefore from (3.4) we obtain the following integral inequality for \( m_\lambda(t) \):

\[
\eta^* m_\lambda(t) \leq m_\lambda^* + \frac{C_f \tilde{C}_p^{p+1}}{p + 1} (t_0^* + t)^{1 - \frac{(N+2\lambda)(p-1)}{4}} m_\lambda(t)^{\frac{p+1}{2}} \\
+ \frac{(p+2)C_f \tilde{C}_p^{p+1}}{p + 1} \int_0^t (t_0^* + t)^{\frac{(N+2\lambda)(p-1)}{4}} m_\lambda(s)^{\frac{p+1}{2}} \, ds, \quad t \in [0, T_*).
\]

Consequently, setting

\[
M_\lambda(t) := \sup_{0 \leq \tau \leq t} m_\lambda(\tau), \quad t \in [0, T_*)
\]

we see from the assumption \( p > 1 + \frac{4}{N+2\lambda} \) that \( (N+2\lambda)(p-1) > 1 \)

\[
\eta^* M_\lambda(t) \leq m_\lambda^* + \frac{C_f \tilde{C}_p^{p+1}}{p + 1} \left( 1 + \frac{p+2}{1 - \frac{(N+2\lambda)(p-1)}{4}} \right) M_\lambda(t)^{\frac{p+1}{2}}.
\]

It is worth noticing that \( M_\lambda \) is continuous. This implies that there exist constants \( \delta_\lambda > 0 \) and \( M_\lambda^* > 0 \) such that if \( m_\lambda^* \leq \delta_\lambda \), then \( M_\lambda(t) \leq M_\lambda^* \) for every \( t \in [0, T_*) \), that is, we obtain

\[
(t_0 + t) \int_{\Omega} \left( |\nabla u(x,t)|^2 + |\partial_t u(x,t)|^2 \right) \Psi(x,t)^\lambda \, dx + \int_{\Omega} |u(x,t)|^2 \Psi(x,t)^\lambda \, dx \leq M_\lambda^*, \quad t \in (0, T_*).
\]

(The critical case \( p = 1 + \frac{4}{N+2\lambda} \)) In this case, \( Y_\lambda \) plays an important role. Note that Hölder inequality yields

\[
\int_{\Omega} |u|^{p+1} \Psi^\lambda \, dx \leq \left( \int_{\Omega} |u|^2 \Psi^{\lambda-1} \, dx \right)^{1-\theta} \left( \int_{\Omega} |u|^{p+1} \Psi^{\frac{p+1}{2}\lambda} \, dx \right)^{\theta}
\]

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with $\theta = \frac{N}{N+2\lambda} \in (0,1)$ and $q = 1 + \frac{4}{N}$. By Lemma 2.3 with $p = q$ and Lemma 2.2 we deduce
\[
\int_{\Omega} |u|^q \Psi^\frac{q}{p+1} dx \leq C_* \int_{\Omega} |\nabla u|^2 \Psi^\lambda dx \left( \int_{\Omega} |u|^2 \Psi^\lambda dx \right)^{\frac{2}{N}}.
\]
Combining the above two estimates and using Lemma 2.2 again, we have
\[
\int_{\Omega} |u|^{p+1} \Psi^\lambda dx \leq C^{\theta} \left( \int_{\Omega} |\nabla u|^2 \Psi^\lambda dx \right)^{\frac{1-\theta}{\theta}} \left( \int_{\Omega} |u|^2 \Psi^\lambda dx \right)^{\frac{\theta}{2}}.
\]
Therefore we see from 3.4 that
\[
\eta^* \left( m_\lambda(t) + Y_\lambda(t) \right) \leq m^*_\lambda + \frac{C^\theta C_f}{p+1} m_\lambda(t)^{\frac{p+1}{2}} + \frac{(p+3)C^\theta C_f}{p+1} \int_0^t m_\lambda(s)^{\frac{p+1}{2}} \int_{\Omega} |\nabla u(s)|^2 \Psi^\lambda \right) dx ds, \tag{3.5}
\]
Choosing $\tilde{M}_\lambda(t) := M_\lambda(t) + Y_\lambda(t)$, $t \in [0, T_*)$
and noting that
\[
\int_0^t m_\lambda(s)^{\frac{p+1}{2}} \int_{\Omega} |\nabla u(s)|^2 \Psi^\lambda \right) dx ds \leq M_\lambda(t)^{\frac{p+1}{2}} Y_\lambda(t) \leq \tilde{M}_\lambda(t)^{\frac{p+1}{2}},
\]
we have
\[
\tilde{M}_\lambda(t) \leq m^*_\lambda + \frac{(p+3)C^\theta C_f}{p+1} \tilde{M}_\lambda(t)^{\frac{p+1}{2}}.
\]
The rest of the proof is exactly the same as the supercritical case. The proof is complete. \qed

Remark 3.2. Similar argument as the critical case also works when $1 + \frac{4}{N+2\lambda} < p < \frac{N}{N-2}, 1 + \frac{2}{\lambda}$. 

4 Remark on one-dimensional case

In this section we consider the one-dimensional case
\[
\left\{ \begin{array}{ll}
\partial_t^2 u(x, t) - \partial_x^2 u(x, t) + \partial_t u(x, t) = f(u(x, t)), & (x, t) \in \mathbb{R} \times (0, T), \\
u(x, 0) = u_0(x), & x \in \mathbb{R}, \\
\partial_t u(x, 0) = u_1(x), & x \in \mathbb{R}.
\end{array} \right. \tag{4.1}
\]
In this case, we can also discuss the weighted energy estimate of polynomial type for $\lambda \in [0, \frac{1}{2})$. However, the lack of validity of Hardy type inequality (Lemma 2.2), we take a small modification of the weight function in (3.2) as follows:
\[
\tilde{\Phi}_\beta(x, t : t_0) = \left( 2 - \frac{1}{t_0 + t} \right) \Phi_\beta(x, t : t_0)
\]
with \( \delta = \frac{1-2\lambda}{4} \) and \( \beta = \frac{2\lambda}{1+2\lambda} \). Then by virtue of the properties of \( \Phi_\beta \) in Lemma 2.1, we have

\[
\partial_t \tilde{\Phi}_\beta \geq \Delta \tilde{\Phi}_\beta + \frac{1}{(1+t)^2} \tilde{\Phi}_\beta
\]

and hence we can proceed an argument similar to the one in Section 3 with \( p > 1 + \frac{4}{N+2\lambda} = 1 + \frac{4}{1+2\lambda} \). It should be noticed that the case \( p = 1 + \frac{4}{N+2\lambda} \) cannot be treated because of the lack of the validity of weighted Hardy inequality. Therefore we have the following estimate

\[
\eta^* M_\lambda(t) \leq m^*_\lambda + \tilde{C}_\varepsilon M_\lambda(t) \frac{p+1}{2} \quad t \in (0, T^*_\varepsilon).
\]

Consequently, we can obtain

**Theorem 4.1.** Assume that \( N = 1 \) and \( f \) satisfies (1.2) with \( 1 < p < \infty \). Then for every \( \lambda \in [0, \frac{1}{2}] \), there exists a positive constant \( \delta^*_\lambda > 0 \) such that the following assertion holds: For every \( (u_0, u_1) \in H^2(\mathbb{R}) \times H^1(\mathbb{R}) \) satisfying

\[
\int_{\mathbb{R}} \left( |\nabla u_0|^2 + |u_0|^2 + |u_1|^2 \right) (x)^{2\lambda} dx \leq \delta^*_\lambda,
\]

one has \( T^*_\varepsilon(u_0, u_1) = \infty \) when \( p > 1 + \frac{4}{1+2\lambda} \). Namely, there exists a global weak solution \( u \) of (1.1) with initial data \((u_0, u_1)\). Moreover, \( u \) satisfies the following weighted energy estimate: there exists a positive constant \( M^*_\lambda > 0 \) such that

\[
\int_{\mathbb{R}} \left( |\nabla u(t)|^2 + |\partial_t u(t)|^2 \right) (1 + t + |x|^2)^\lambda dx \leq \frac{M^*_\lambda}{1+t},
\]

\[
\int_{\mathbb{R}} |u_1(t)|^2 (1 + t + |x|^2)^\lambda dx \leq M^*_\lambda.
\]

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