Performance of some numerical Laplace inversion methods on American put option formula

I Octaviano¹, A R Yuniar¹, L Anisa¹, S D Surjanto¹ and E R M Putri¹

¹Department of Mathematics, Institut Teknologi Sepuluh Nopember, Surabaya, Indonesia

Abstract. Numerical inversion approaches of Laplace transform is used to obtain a semianalytic solution. Some of the mathematical inversion methods such as Durbin-Crump, Widder, and Papoulis can be used to calculate American put options through the optimal exercise price in the Laplace space. The comparison of methods on some simple functions is aimed to know the accuracy and parameters which used in the calculation of American put options. The result obtained is the performance of each method regarding accuracy and computational speed. The Durbin-Crump method has an average error relative of $2.006e-004$ with computational speed of $0.04871$ seconds, the Widder method has an average error relative of $0.0048$ with computational speed of $3.100181$ seconds, and the Papoulis method has an average error relative of $9.8558e-004$ with computational speed of $0.020793$ seconds.

1. Introduction

Laplace transform is a technique for solving problems by transforming from time-space to Laplace space. The ability to search Laplace transform from a statement and then inverse it makes Laplace transform very useful for solving differential equations [1]. Inverse Laplace transforms for complex functions are difficult to solve analytically, so a numerical approach is required to obtain the solution [2].

Various numerical approaches have been developed over the last few years to obtain a semianalytical solution of inverse Laplace transform, including the Gaver-Stehfest method, Durbin-Crump method, Widder method, and Papoulis method [3]. The numerical solution of inverse Laplace transform is advantageous in various aspects, one of them is financial planning [4]. In the capital market, there are several derivative products that are widely used, one of them is an option.

The option is an agreement or contract between the seller and the buyer of the option, whereby the option seller guarantees the right (but not the obligation) of the option buyer to buy or sell specific assets at a predetermined time and price. There are two types of an option based on the execution time, namely European style and American style. The European style option is a type which option can be exercised when the lifetime of the options ends, whereas American style is a type which option can be applied at any time during the lifetime of options [5].

One of the factors that may affect the price of an option is dividends. Dividends are the distribution of profits to shareholders based on the number of shares owned. The existence of dividend causes decreased the value of the stock price. The reduced amount of the stock price causes the reduced value of the call option. Otherwise, the put option will be increased [6].

The value of the option can be obtained through the optimal exercise price formula in the Laplace space so that the inverse Laplace transform is needed to achieve the solution of the problem [7]. In this research, there will be discussed the performance of the Durbin-Crump method, Widder method, and
Papoulis method on the calculation of the American put option value. The performance is assessed from two aspects, namely the accuracy and computational speed.

2. Methods
2.1. Durbin-Crump method
The Durbin-Crump method approaches the Laplace inverse transform by constructing an infinite series into even periodic functions and odd periodic functions. The Durbin-Crump is given as follows:

\[
\begin{align*}
    f_N(t) &\approx \frac{e^{\nu t}}{T} \left[ -\frac{1}{2} \text{Re}\{F(\nu)\} + \sum_{k=0}^{N} \left( \text{Re}\left\{ F\left( \nu + \frac{ik\pi}{T} \right) \right\} \cos \frac{k\pi}{T} t - \text{Im}\left\{ F\left( \nu + \frac{ik\pi}{T} \right) \right\} \sin \frac{k\pi}{T} t \right] \\
    &\approx \frac{e^{\nu t}}{T} \left[ -\frac{1}{2} \text{Re}\{F(\nu)\} + \sum_{k=0}^{N} \left( \text{Re}\left\{ F\left( \nu + \frac{ik\pi}{T} \right) \right\} \cos \frac{k\pi}{T} t - \text{Im}\left\{ F\left( \nu + \frac{ik\pi}{T} \right) \right\} \sin \frac{k\pi}{T} t \right]
\end{align*}
\]

The Durbin-Crump method approaches the time functions \( f_N(t) \) with a periodic function with \( T \) period. The Laplace variables in this method are \( \nu \) and \( \nu + \frac{ik\pi}{T} \) with \( \nu = \alpha - \frac{\ln E}{2\pi} \) and \( E \) is an error tolerance which value is \( 1 \times 10^{-6} \). While \( \alpha \) is the pole of the \( F(s) \) functions. If the functions don't have any pole, then the value of \( \alpha \) is zero [8].

2.2. Widder method
The characteristic of positive operators series is defined by the Laplace transformation formula, which learned in obtaining the methods and use to solve the inverse Laplace transformation given as follows [9]:

\[
    f(t) = \lim_{k \to \infty} (-1)^k \frac{1}{k!} \left( \frac{k+1}{T} \right)^{k+1} P^{(k)} \left( \frac{k}{T} \right)
\]

In the formula of Widder method was defined \( P^{(k)} \) as the \( k \)th derivative function of \( F \). The Laplace variable \( s \) on Widder method is substituted into \( \frac{k}{T} \) with \( t > 0 \). The \( f(t) \) functions are continuous on \( t \) with intervals at \( 0, \infty \).

2.3. Papoulis method
Papoulis method is based on extension of \( f(t) \) into the following exponential series [10]:

\[
    f(t) = \sum_{n=0}^{N-1} \alpha_n P_{2n}(e^{-\rho t})
\]

Where \( P_{2n}(e^{-\rho t}) \) is a \( 2n \)th degree Legendre polynomial in \( e^{-\rho t} \). Then, \( \alpha_n \) is the coefficient of Legendre polynomial which can be calculated from the following recursion formula [8]:

\[
    \rho f((2k+1)\rho) = \sum_{m=0}^{k} \frac{(k-m+1)m_{m+1}}{2 \left( k+\frac{1}{2} \right)_{m+1}} \alpha_m
\]

with \( k = n \) and \( (j)_m \) are Pochammer symbols [8]:

- \( (j)_m = 1 \) untuk \( m = 0 \)
- \( (j)_m = f(j+1) \ldots (j+m-1) \) untuk \( m > 0 \).

The parameters in this method are \( N \) and \( \rho \), with \( N \) is a positive integer and \( \rho \) is a real number.

3. Result and Discussion
3.1. Simple function tested
Ideally, a robust inversion method should depend very little on the parameters involved in the numerical inversion, i.e., the results of numerical inversion should not be sensitive to the variation of these parameters. However, due to the inherently unstable nature of the numerical Laplace inversion, parametric tests for the sensitivity of a method to different parameter values always need to be performed.

We examine the sensitivity of Durbin-Crump, Widder, and Papoulis method by running five simple functions as given below:

\[
\begin{align*}
\mathcal{L}^{-1} \left( \frac{2}{t} \right) &= 5 & \text{(3.1)} \\
\mathcal{L}^{-1} \left( \frac{1}{s-0.1} \right) &= e^{0.1t} & \text{(3.2)} \\
\mathcal{L}^{-1} \left( \frac{1}{s^2} \right) &= t & \text{(3.3)} \\
\mathcal{L}^{-1} \left( \frac{1}{s^2+1} \right) &= \sin t & \text{(3.4)} \\
\mathcal{L}^{-1} \left( \frac{s}{s^2+1} \right) &= \cos t & \text{(3.5)}
\end{align*}
\]

The inverse value of Laplace transformation of some of these simple functions is calculated at some time point to know the accuracy of the method. To see the suitability of the method with the options problem, then the time point taken at a little time which is \( t = 0.2, 0.4, 0.6, 0.8, 1 \). Several different parameters in each method will be selected to be tested on a simple function that has been selected. The range of parameters to be used in option problem is selected based on the lowest value of average error relative to simple function test.

From the test results of the five simple functions, it can be concluded that if the value of parameter \( N \) in the Durbin-Crump method increase, then the average error relative become smaller. However, average error relative to the \( f(t) = t \) and \( f(t) = \sin(t) \) decrease until the value of \( N \) reach 200000, then increase again. Thus, the value of parameter \( N = 200000 \) is selected for the case of calculating the optimal exercise price of American put option.

For \( f(t) = \sin t \), \( f(t) = \cos t \), \( f(t) = e^{0.1t} \) and \( f(t) = t \) functions, the results of Widder method shows that if the value of parameter \( k \) increase, then the average error relative becomes smaller. In the \( f(t) = 5 \) function, it has the lowest value of average error relative compared to the other simple functions tested.

The parameter of Papoulis method uses \( N = 15, 16, 17, ..., 23 \) and \( \rho = 0.5, 0.6, 0.7, ..., 2 \). There are 28 pair values of the parameter \((N, \rho)\) which have average error relative below 5% for all of five simple functions based on results of the trials from five simple function above and that shown in Table 1.

| \( (N, \rho) \) | \( (N, \rho) \) | \( (N, \rho) \) | \( (N, \rho) \) |
|---------------|---------------|---------------|---------------|
| (15,1.9)      | (19,1.5)      | (21,1.5)      | (22,1.3)      |
| (16,1.2)      | (19,1.8)      | (21,1.8)      | (22,1.5)      |
| (16,1.6)      | (19,1.9)      | (21,1.9)      | (22,1.6)      |
| (17,1.6)      | (20,1.3)      | (21,2)        | (22,1.8)      |
| (18,1.3)      | (20,1.8)      | (22,0.5)      | (22,2)        |
| (18,1.7)      | (21,1.2)      | (22,0.8)      | (23,0.8)      |
| (19,0.6)      | (21,1.4)      | (22,1)        | (23,1.6)      |

This 28 pair values of the parameter \((N, \rho)\) will be selected and used in the application of calculating the optimal exercise price of American put option.
3.2. Application of Optimal Exercise Formula of American Put Option without Dividends

To test the accuracy of Durbin-Crump, Widder, and Papoulis method on the calculation of the optimal exercise price of American put option, first, we will compare the numerical result of that method with the analytical inversion has been successfully performed by Zhu [7]. This analytical inversion is shown in the case of calculating the optimal exercise price of American put option without dividend payments. So, we first adopt the same example used by Zhu.

The normalized option parameters in this example are $\gamma = 2.222$ and $D = 0$ with $\sigma = 30\%$, which corresponds to an option with 1-year lifetime. The data of analytical inversion that Zhu has been performed shown in Table 2. The accuracy level can be obtained by the average error relative of the analytic inversion results in Table 2.

| $\tau$  | Analytical Inversion Result |
|--------|-----------------------------|
| 0.005  | 0.84830692                  |
| 0.010  | 0.81923386                  |
| 0.015  | 0.80187124                  |
| 0.020  | 0.78790613                  |
| 0.025  | 0.77928012                  |
| 0.030  | 0.77144913                  |
| 0.035  | 0.76493095                  |
| 0.040  | 0.75938297                  |
| 0.045  | 0.75458026                  |

The value of parameter $N = 200000$ on the Durbin-Crump method will be used in the calculation of the optimal exercise price of American Put Option without dividends based on the results of a simple function test. The Durbin-Crump method with parameter $N = 200000$ has an average error relative of $2.006e-004$ when compared to Zhu's analytic results (Table 3).

| $(N, \rho)$ | AvgER  | $(N, \rho)$ | AvgER  |
|------------|--------|------------|--------|
| (15,1.9)   | 0.0015 | (21,1.5)   | 0.0023 |
| (16,1.2)   | 0.0019 | (21,1.8)   | 0.0098 |
| (16,1.6)   | 0.0016 | (21,1.9)   | 0.0536 |
| (17,1.6)   | 0.0013 | (21,2)     | 0.0259 |
| (18,1.3)   | 0.0015 | (22,0.5)   | 0.1147 |
| (18,1.7)   | 0.0012 | (22,0.8)   | 0.2110 |
| (19,0.6)   | 0.0030 | (22,1)     | 0.0452 |
| (19,1.5)   | 0.0017 | (22,1.3)   | 0.0242 |
| (19,1.8)   | 0.0010 | (22,1.5)   | 0.0810 |
| (19,1.9)   | 9.8558e-004 | (22,1.6) | 0.6059 |
| (20,1.3)   | 0.0011 | (22,1.8)   | 0.1302 |
| (20,1.8)   | 9.9331e-004 | (22,2)  | 0.2669 |
| (21,1.2)   | 0.0034 | (23,0.8)   | 1.3807 |
| (21,1.4)   | 0.0024 | (23,1.6)   | 3.9174 |

On the Widder method, it is necessary to determine the value of parameter $k$. On the Widder method, if $k$ goes to infinity based on Widder formula in the initial conditions, then the calculation of the optimal exercise price will take a long time. To solve this problem, we use the small value of $k$ as an input value to get the optimal exercise price of American Put Option without dividends. The result obtained that Widder method with the value of parameter $k = 5$ provides a
reasonably fast computing speed, and when compared to Zhu's analytic results, it has an average error relative of 0.0048.

On the other hand, the average error relative of the 28 pair values of Papoulis' parameters ($N, \rho$) on the calculation of optimal exercise price of American put option without dividends shown in Table 3. Table 3 shows that when $N = 19$ and $\rho = 1.9$, Papoulis method indicates the smallest value of average error relative.

The small value of average error relative indicates that the Durbin-Crump method with $N = 200000$, Widder method with $k = 5$, and Papoulis method with $N = 19$ and $\rho = 1.9$ works very well in the application of calculating the optimal exercise price of American put option.

From the results discussed above, performance comparison on the Intel Core i5, 2.5 GHz machines of the three methods shown in Table 4. We use MATLAB R2017a software to obtain the performance of the three methods on the calculation of American put option without dividends.

**Table 4. Performance Comparison between Durbin-Crump, Widder, and Papoulis Methods**

| Methods      | AER                     | Computational Speed |
|--------------|-------------------------|---------------------|
| Durbin-Crump | 2.006e – 004            | 0.04871             |
| Widder       | 0.0048                  | 3.100181            |
| Papoulis     | 9.8558e – 004           | 0.020793            |

From Table 4, it can be seen that the Durbin-Crump method has the best accuracy and the Papoulis method has the best computational speed on the calculation of optimal exercise price of American put option without dividends.

### 3.3. Application of Optimal Exercise Formula of American Put Option with Dividends

In this section, Widder method is not applied to the calculation of optimal exercise price of American put option with a dividend. This is because the derivative function of the American put option's exercise price with a dividend in Laplace space is difficult to be solved. The Durbin-Crump method also not applied to the calculation of exercise price of American put option with a dividend. This is because there are two Laplace variables of the Durbin-Crump method, so the optimal exercise equation also yields two root values. The first root value can be obtained easily by setting the $\nu$ as a Laplace variable. However, it is difficult to calculate the root value for the Laplace variable $\nu + \frac{ik\pi}{T}$ because the equation contains an imaginary number. So in this section, only Papoulis method can be applied to calculate optimal exercise price of American put option with a dividend (Table 5).

**Table 5. Performance Comparison between Papoulis and Gaver-Stehfest Methods**

| Gaver-Stehfest | Papoulis |
|---------------|----------|
| $S_T$ ($)     | Computational Speed (s) | $S_T$ ($) | Computational Speed (s) |
| 70.31         | 0.00994  | 69.89    | 0.014067              |

First, we will compare the numerical result of Papoulis method with the numerical result of the Gaver-Stehfest method that Jin Zhang [7] has been done. The reason that we choose Gaver-Stehfest method to compare because it has been successfully applied on the calculation of optimal exercise price of American put option with a dividend. So, we first adopt the same example used by Jin Zhang. Table 5 shows the performance comparison of optimal exercise price ($S_T$) calculation on Intel Core i5, 2.5 GHz machines, using the Gaver-Stehfest method and Papoulis method. The normalized option parameters in this example are: strike price $X = \$100$, interest rate $r = 5\%$, dividend $D_0 = 5\%$, and volatility of underlying asset $\sigma = 30\%$ which corresponds to an option with 1-year lifetime. The optimal exercise price shown in Table 5 is calculated when $\tau = 0.006116$ ($t = 0.864$). We use
MATLAB R2017a software to get the performance of both methods on calculating the value of American put option with a dividend.

From Table 5, it can be seen that the computational speed of Papoulis method is slightly slower than the computational speed of Gaver-Stehfest method. However, optimal exercise price generated by the calculation of Papoulis method is lower than the optimal exercise price generated by the Gaver-Stehfest method.

4 Conclusion

Based on analysis and discussion that has been presented in previous, it can be concluded that the calculation of optimal exercise price of American put option without dividends, the Durbin-Crump method has the best accuracy, and the Papoulis method has the best computational speed. But, in the case of computational speed, Papoulis method is slightly slower than the Gaver-Stehfest method. However, optimal exercise price generated by the calculation of Papoulis method is lower than the Gaver-Stehfest method.

References

[1] Ruggeri R 2004 *The Laplace Transform in Option Pricing* (Venezia: Universita CaFoscari)
[2] Zhan H and Wang Q 2015 *Advances in Water Resources* **75** 80
[3] Zhu S 2006 *A New Analytical Approximation Formula for The Optimal Exercise Boundary of American Put Options* (Australia: International Journal of Theoretical and Applied Finance)
[4] Hon Y C, J Zhang, and S Zhu 2006 *Numerical Valuation of American Puts on a Dividend Paying Asset*. Financial System Engineering IV-Vol 9 (Hongkong: Global-Link Publisher)
[5] Muzzioli S and Reynaerts H 2008 *American Option Pricing with Imprecise Risk-Neutral Probabilities* (Belgium: Department of Applied Mathematics and Computer Science, Ghent University)
[6] Hull J C 2002 *Option Futures and other Derivatives Seventh Edition* (New Jersey: Prentice Hall).
[7] Zhang J 2007 *Some Innovative Numerical Approaches for Pricing American Options* (Australia: School of Mathematics and Applied Statistics, University of Wollongong)
[8] Cheng A and P Sidoruk 1994 *The Matematica Journal* (Miller Freeman Publication)
[9] Baumer B and Neubrander F 1995 *Laplace Transform Methods for Evolution Equations* (USA: Louisiana State University)
[10] Cohen A M 2007 *Numerical Methods for Laplace Transform Inversion*. (NewYork: Springer Science+Business Media, LLC)