Spacetime metric from linear electrodynamics

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The Maxwell equations are formulated on an arbitrary \((1+3)\)-dimensional manifold. Then, imposing a (constrained) linear constitutive relation between electromagnetic field \((E, B)\) and excitation \((\mathcal{D}, \mathcal{H})\), we derive the metric of spacetime therefrom. file metric3.tex, 1999-05-30

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In Einstein’s gravitational theory, the metric \(g\) of spacetime is the fundamental field variable. The metric governs temporal and spatial distances and angles. All other geometrical notions can be derived from the metric. In particular, the affine properties of spacetime, i.e. those related to parallel transfer and represented by the linear connection \(\Gamma\), become subordinate to the metric. In Riemannian spacetime, the connection \(\Gamma\) can be exclusively expressed in terms of the metric, and the same is true for the curvature \(\text{Ric}\). In any case, the dichotomy of metric and affine properties of spacetime and attempts to understand it runs through much of present-day theorizing on the fundamental structure of spacetime. Already Eddington, e.g., tried through much of present-day theorizing on the fundamental structure of spacetime.

Thinking more from the point of view of a quantum substructure supposedly underlying classical spacetime, arguments were advanced \cite{7} that the metric is some kind of effective field which “froze out” during a cosmic phase transition in the early universe, cf. also \cite{4}. In other words, also here the metric would be a secondary structure comparable to the strain field in a solid.

The aim of this letter is, however, more modest: We formulate classical electrodynamics in the so-called metric-free version, see \cite{4, 5}, by taking recourse to the conservation laws of electric charge and magnetic flux and to the existence of a Lorentz force density. To complete the apparatus of the field equations, we have eventually to specify the constitutive relation between field strength \(F = (E, B)\) and excitation \(H = (\mathcal{D}, \mathcal{H})\).

We choose a linear law \(H_{ij} = \kappa_{ij}^{kl} F_{kl}/2\) with 21 independent constitutive functions \(\kappa_{ij}^{kl}(x)\). The linear law can be interpreted as a new kind of duality operation \# mapping 2-forms into 2-forms: \(H \sim \# F\). Let us stress that no metric has been used so far.

We impose a constraint on the duality operator, namely \(\# \# = -1\) (for Euclidean signature +1). This, together with two formulas of Urbanke \cite{6} which had been used in a Yang-Mills context, allows us to derive from \(\kappa_{ij}^{kl}\) a metric with pseudo-Euclidean signature. In this way we recognize how closely the concept of a metric is connected with the electromagnetic properties of spacetime itself or of material media embedded therein – a fact which, perhaps, doesn’t come as a surprise in view of the principle of the constancy of the velocity of light.

(1) Three axioms of electrodynamics: Spacetime is assumed to be a 4-dimensional differentiable manifold which allows a foliation into 3-dimensional submanifolds which can be numbered with a monotonically increasing parameter \(\sigma\). The existence of an electric current (3-form) \(J = -j \wedge d\sigma + \rho\) is postulated which, by axiom 1, is conserved:

\[
\oint_{C_3} J = 0 \quad \partial C_3 = 0 .
\] (1)

Here \(C_3\) is an arbitrary closed 3-dimensional submanifold of the 4-manifold. By de Rham’s theorem, the inhomogeneous Maxwell equation is a consequence therefrom,

\[
J = dH ,
\] (2)

with \(H = H_{ij} dx^i \wedge dx^j/2 = -\mathcal{H} \wedge d\sigma + \mathcal{D}\). The current \(J\), together with a force density \(f_\alpha\), originating from mechanics, allow to formulate the Lorentz force density as axiom 2:

\[
f_\alpha = (e_\alpha \cdot F) \wedge J .
\] (3)

Greek indices \(\alpha, \beta, \ldots = 0, 1, 2, 3\) are anholonomic or frame indices and \(e_\alpha\) is the local frame, the interior product is denoted by \(\cdot\). This axiom introduces the electromagnetic field strength (2-form) \(F = F_{ij} dx^i \wedge dx^j/2 =\)
\[ E \wedge d\sigma + B \text{ as a new concept. In axiom 3, the corresponding magnetic flux is assumed to be conserved,} \]
\[
\oint F = 0, \quad \partial C_2 = 0, \quad (4)
\]
for any closed submanifold \( C_2 \). As a consequence, we find the homogeneous Maxwell equation
\[
dF = 0. \quad (5)
\]

These equations are all diffeomorphism invariant and don’t depend on metric or connection. This is also true for the exterior and the interior product. Therefore these equations are valid in special and general relativity likewise and in non-Riemannian spacetimes of gauge theories of gravity, see \([8]\). Electric charges and, under favorable conditions, also magnetic flux quanta can be counted. This is why the metric is dispensible under those circumstances.

(2) Constitutive law as axiom 4: The simplest constitutive law is, of course, a linear law. If it is additionally isotropic, it yields in particular vacuum electrodynamics. However, isotropy can only be formulated if a metric is available which is not the case under the present state of our discussion. Therefore we assume only linearity:
\[
H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} (x) F_{kl}. \quad (6)
\]
Since \( H \) is an odd and \( F \) an even form, the constitutive matrix \( \kappa \) is odd. Therefore we split off the Levi-Civita symbol
\[
\tilde{\chi}_{ijkl} = \frac{1}{2} \kappa_{ij}^{mn} \epsilon_{mnkl} f \chi_{ijkl}, \quad (7)
\]
where \( f \) is a dimensionful scalar function such that \( \chi_{ijkl} \) is dimensionless. The tensor density \( \chi_{ijkl} \) carries the weight \(-1\). The Lagrangian of the theory is quadratic in \( F \). Thus we find \( \chi_{ijkl} = \tilde{\chi}_{ijkl} \), i.e., 21 independent functions. Since \( H_{ij} \) and \( F_{kl} \) can be measured independently, the constitutive functions \( \chi_{ijkl} \) can be experimentally determined.

It is convenient to write \( H \) and \( F \) as row vectors \( H_I = (H_{01}, H_{02}, H_{03}, H_{23}, H_{31}, H_{12}) \), where \( I \) runs from 1 to 6, etc. Then the constitutive law reads
\[
H_I = \kappa_{IK} F_K = \chi_{1M} \epsilon^{MK} f F_K \text{ with } \chi_{1M} = \chi_{MI}. \quad (8)
\]
Furthermore, in local coordinates, the basis of the 2-forms is represented by the six 2-forms \( dx^i \wedge dx^j \). They can be put into the column vector Cyrillic \( B \), namely \( B_I \). Then \( H = H_I B_I \) and \( F = F_I B_I \).

(3) Duality operator \( \# \) and its closure: We can define, by means of the linear constitutive law \([8]\), a new duality operator mapping 2-forms into 2-forms. Accordingly, we require for the 2-form basis
\[
\# B_I = (\chi_{KM} \epsilon^{MI}) B^K, \quad (9)
\]
i.e., the duality operator incorporates the constitutive properties specified in \([8]\). In particular, we have for the electromagnetic field two-forms \( H = f \# F \).

A duality operator, applied twice, should, up to a sign, lead back to the identity. By such a postulate we can constrain the number of independent components of \( \chi \) without using, say, a metric:
\[
\# \# = -1. \quad (10)
\]
We concentrate here on the minus sign; the rule \#\# = +1 would lead to Euclidean signature.

It is convenient to write the \( 6 \times 6 \) matrices, which define the duality operator, in terms of \( 3 \times 3 \) matrices:
\[
\chi_{IK} = \chi_{KI} = \left( \begin{array}{cc} A & C \\ C^T & B \end{array} \right), \quad \epsilon^{IK} = \epsilon_{KI} = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \quad (11)
\]
Here \( A = A^T, B = B^T \), the superscript \( T \) denoting transposition. The general non-trivial solution of \([10]\) is given by
\[
\chi_{IK} = \left( \begin{array}{cc} pB^{-1} + qN & B^{-1}K \\ -KB^{-1} & B \end{array} \right), \quad (12)
\]
Here \( B \) is a nondegenerate arbitrary symmetric matrix (6 independent components) and \( K \) an arbitrary antisymmetric matrix (3 components). Furthermore, we construct the symmetric matrix \( N \) as a solution of the homogeneous system \( KN = NK = 0 \). With \( K_{ab} = \epsilon_{abc} k^c \), we have explicitly:
\[
N = \left( \begin{array}{ccc} (k^1)^2 & k^1 k^2 & k^1 k^3 \\ k^1 k^2 & (k^2)^2 & k^2 k^3 \\ k^1 k^3 & k^2 k^3 & (k^3)^2 \end{array} \right). \quad (13)
\]
Finally \( q = -1/\det B, p = [\text{tr}(NB)/\det B] - 1 \).

(4) Selfduality and a triplet of 2-forms: With our new duality operator we can define the selfdual \((s)\) and the anti-selfdual \((a)\) of a 2-form. For the 2-form basis \( B \) we have
\[
\begin{aligned}
\{s\} B &:= \frac{1}{2} (B - i \# B), \quad \{a\} B := \frac{1}{2} (B + i \# B),
\end{aligned} \quad (14)
\]
with \( \# B = i B, \# a B = -i a B \). Thus the 6-dimensional space of 2-forms decomposes into two 3-dimensional invariant subspaces corresponding to the eigenvalues \( \pm i \) of the duality operator. With the decomposition into two 3-dimensional row vectors,
\[
B' = \left( \begin{array}{c} \alpha^a \\ \beta^a \end{array} \right), \quad a, b, \ldots = 1, 2, 3, \quad (15)
\]
we take care of this fact also in the 2-form basis.

One of the 3-dimensional invariant subspaces can be spanned by, say, \( \{s\} \), whereas \( \beta \) is obtained from it by means of the linear transformation \( \beta = (i + \)
$B^{-1}K)B^{-1}$. Therefore $(\gamma)^{a}$ subsumes the properties of this invariant subspace and so does the triplet of 2-forms

$$S^{(a)} := -(B^{-1})^{ab} \gamma^{b}.$$ 

(16)

Hereafter, $(B^{-1})^{ab}$ and $B_{cd}$ denote the matrix elements of $B^{-1}$ and $B$, respectively. Incidentally, the anti-self dual $(\gamma)^{a}$ spans the other invariant subspace. The whole information of the linear constitutive law (6) is now encoded into the triplet of 2-forms $S^{(a)}$. A direct calculation demonstrates that they satisfy the so-called completeness condition:

$$S^{(a)} \wedge S^{(b)} = \frac{1}{3} (B^{-1})^{ab} (B)_{cd} S^{(c)} \wedge S^{(d)}.$$ 

(17)

**Extracting the metric:** Urbantke [7] (see also the discussions in [9,10]) was able to derive, within $SU(2)$ Yang-Mills theory, a 4-dimensional metric $g_{ij}$ (with $i,j = 0,1,2,3$) from a triplet of 2-forms which are related to 2-plane elements of spacetime with certain distinguished properties. Since the completeness condition (17) is fulfilled, the Urbantke’s formulas

$$\sqrt{\det g} \; g_{ij} = \frac{2}{3} \sqrt{\det B} \epsilon^{klmn} S^{(a)}_{ik} S^{(b)}_{lm} \epsilon^{(c)ij},$$

(18)

$$\sqrt{\det g} = -\frac{1}{6} \epsilon^{klmn} B_{cd} S^{(c)}_{ik} S^{(d)}_{mn},$$

(19)

are also applicable in our case. Here the $S^{(a)}_{ij}$ are the components of the 2-form triplet according to $S^{(a)} = S^{(a)}_{ij} \, dx^i \wedge dx^j / 2$.

If we express $(\gamma)^{a}$ in terms of $\beta$ and $\gamma$ and then substitute (16) into (13), (19), we can, after a very involved computation, display the metric explicitly:

$$g_{ij} = \frac{1}{\sqrt{\det B}} \left( \frac{\det B}{-k_b} - k_a \right),$$

(20)

$$\text{Here we used the abbreviation } k_a := B_{ab} k^b = B_{ab} \epsilon^{bde} K_{cd}/2. \text{ The } 3 \times 3 \text{ matrix } B_{ab} \text{ can have any signature. Nevertheless, Eq. (20) always yields a metric with Minkowskian signature.}

This representation (20) of the metric is our basic result. Since the triplet $S^{(a)}$ is defined up to an arbitrary scalar factor, our procedure in general defines a conformal class of metrics rather than a metric itself.

As the most simple example, we will construct the Minkowski metric. Recall that (20) depends on the symmetric $3 \times 3$ matrix $B$ (not to be confused with the magnetic field $B$) and the antisymmetric $3 \times 3$ matrix $K$. If we choose $f^2 = \varepsilon_0 / \mu_0$, $B = (\varepsilon_0 \mu_0)^{-1/2} \mathbf{1}$ and $K = 0$, then, according to (2), this translates into the conventional vacuum relations

$$\mathbf{D} = \varepsilon_0 \mathbf{E} \quad \text{and} \quad \mathbf{H} = \mathbf{B} / \mu_0.$$ 

(21)

On the other hand, if we substitute it into (20), we immediately find (denoting $c := 1/\sqrt{\varepsilon_0 \mu_0}$)

$$g_{ij} = \frac{1}{c^2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

(22)

i.e. the Minkowski metric of special relativity (including its signature) has been derived from the conventional vacuum relation $H = (D, H)$ and field strength $F = (E, B)$.

(6) Outlook: In this letter we have demonstrated that the metric-free formulation of classical electrodynamics (in the spirit of the old Kottler-Cartan-van Dantzig approach, cf. [3]) naturally leads to the reconstruction of the spacetime metric from the constitutive law. It seems worthwhile to remind ourselves that the constitutive (or material) relation is a postulate which arises not from a pure mathematical considerations but rather from the analysis of experimental data [4]. A development of an alternative axiomatics of the Maxwell theory on the basis of the postulate of the well-posedness of Cauchy problem [11] (see also a related discussion in [3]) gives good reasons to assume that the crucial closure condition (1) is tantamount to a postulate of a well-posed Cauchy problem for Maxwell’s equations.

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**Note added in proof:** After this work was completed, we have learned that Schöenberg [12], in a not widely available journal, had already developed the approach to the spacetime metric on the basis of the constitutive relation (6) and the “closure” relation (11). We are grateful to Helmuth Urbantke and Ted Jacobson for corresponding remarks and to José W. Maluf for sending us a copy of Schöenberg’s paper and his curriculum vitae. However, our derivation of the general solution (12) of the “closure” relation, and the complete explicit construction of the metric (20) are new. Moreover, even earlier, Peres [13] investigated related structures, and thus he can be considered as a forerunner of Schöenberg. We thank Asher Peres for pointing out to us the relevance of his paper. In the meantime Guillermo Rubilar has checked our formula (13) by means of computer algebra. For further discussions of our preprint we would like to thank Ted Frankel, Yakov Itin, Bahram Mashhoon, and Eckehard Mielke.

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