RELATIVE TENSOR PRODUCTS AND KOSZUL DUALITY IN MONOIDAL $\infty$-CATEGORIES

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Abstract. This semi-expository work covers central aspects of the theory of relative tensor products as developed in [13], as well as their application to Koszul duality for algebras in monoidal $\infty$-categories. Part of our goal is to expand on the rather condensed account of loc. cit. Along the way, we generalize various aspects of the theory. For instance, given a monoidal $\infty$-category $\mathcal{C}$, an $\infty$-category $\mathcal{M}$ which is left-tensored over $\mathcal{C}$, and an algebra $A$ in $\mathcal{C}$, we construct an action of $\mathcal{A}$-$\mathcal{A}$-bimodules $N$ in $\mathcal{C}$ on left $\mathcal{A}$-modules $M \in \mathcal{M}$ by an external relative tensor product $N \otimes_\mathcal{A} M$. (Up until now, even the special (“internal”) case $\mathcal{C} = \mathcal{M}$ appears to have escaped the literature.)

As an application, we generalize the Koszul duality of loc. cit. to include modules. Our straightforward approach requires that we at this point assume certain compatibilities between tensor products and limits; these assumptions have recently been shown to be unnecessary in [5 §3].

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1. Introduction

Let $\mathcal{C}^\otimes$ be a monoidal $\infty$-category and let $A$ be an algebra object in $\mathcal{C}^\otimes$. Koszul duality for algebras in monoidal $\infty$-categories aims to set up an adjunction

$$(A) \quad \text{Alg}^\text{aug}(\mathcal{C}) \overset{\mathcal{D}_\text{Koszul}}{\rightleftarrows} \text{coAlg}^\text{aug}(\mathcal{C})$$

between the category of augmented algebras in $\mathcal{C}$ and augmented coalgebras in $\mathcal{C}$ which generalizes the classical bar-cobar adjunction used, for instance, in delooping and recognition theorems in classical topology [14], and developed in the setting of dga’s, for instance, by Loday-Vallette in [11]. In particular, if $A$ is an augmented algebra object in $\mathcal{C}$, then the object of $\mathcal{C}$ underlying $\mathcal{D}_\text{Koszul}(A)$ should be given by the relative tensor product $1 \otimes_A 1$.

The adjunction above can be upgraded to include modules. This generalizes the classical constructions which, indeed, include path-spaces (and not only loop-spaces) in their scope. It also relates to the Koszul duality for modules of Positselski [15], but in the general abstract setting of monoidal $\infty$-categories. For example, for a reasonably nice monoidal $\infty$-category $\mathcal{C}^\otimes$, we obtain adjoint functors

$$(M) \quad \text{LMod}^\text{aug}(\mathcal{C}) \overset{\mathcal{D}_\text{Koszul}}{\rightleftarrows} \text{colMod}^\text{aug}(\mathcal{C})$$

Here $\text{LMod}^\text{aug}(\mathcal{C})$ is the $\infty$-category of pairs $(A, X)$ consisting of an augmented algebra $A \to 1$ and an $A$-module $X$, and $\text{colMod}^\text{aug}(\mathcal{C}) = \text{LMod}^\text{aug}(\mathcal{C}^{op})^{op}$, see construction 8.4.1 for the precise definition. Moreover, recall that an object $(A, X)$ of $\text{LMod}^\text{aug}(\mathcal{C})$ witnesses a left-action of an augmented algebra $A$ on an object $X \in \mathcal{C}$. We show that in our construction

$\mathcal{D}_\text{Koszul}(A, X)$

witnesses a left coaction of the augmented coalgebra $\mathcal{D}_\text{Koszul}(A) = 1 \otimes_A 1$ on the object $1 \otimes_A X \in \mathcal{C}$.

We learned how to construct the adjunction $(A)$ from section 5.2.2 of Lurie [13]. Lurie’s construction (and hence ours), revolves around an augmented algebra

$$M = (A \to 1)$$

---

1 More recent related works which should be compared to the present one include [6], [16], [15], [5].
2 Let $A \in \text{Alg}^\text{aug}(\mathcal{C})$. We warn the reader that some authors, who restrict attention to situation where a colagebra can be turned into an algebra simply by taking linear duals, use the term “Koszul” to refer to the linear dual of our $\mathcal{D}_\text{Koszul}(A)$.
3 In fact, we may work more generally with an $\mathcal{L} \mathcal{M}$-monoidal category; see corollary 8.4.3 for the precise statement.
in \( \mathcal{C} \) (in which \( \mathbb{1} \) is temporarily assumed to be terminal\(^4\)), regarded as an algebra object of the twisted arrow category \( \mathcal{M} = \text{TwArr}(\mathcal{C}) \). Heavy use is made of (at least certain aspects of) a monoidal structure on the category \( _M \mathcal{BMod}_M(\mathcal{M}) \) of \( M \)-\( M \)-bimodules in the twisted arrow category. In our approach, for the upgrade to modules, we also need a left action of this category on the category \( L\mathcal{Mod}_M(\mathcal{M}) \) of left \( M \)-modules in the twisted arrow category. Many of the pages below are devoted to the construction of this expected action.

As we learned from the work of Brantner–Campos–Nuiten [5, §3], it is in fact possible to dodge this foundational issue, and to derive the Koszul duality for modules (\( M \)) from the Koszul duality for algebras (\( A \)). Moreover, in their indirect approach, it’s possible to avoid certain technical assumptions concerning the compatibility of tensor products with limits, which we have been unable to avoid in our direct approach. Nevertheless, we hope that our foundational work, and our straightforward approach to the case of modules, helps to paint a pleasing and conceptually satisfying picture, even if one that becomes unavailable (or undesirable) beyond a certain level of generality.

Unfortunately, the main theorems here are somewhat too technical to include precise statements in the introduction. Indeed, the very definition of the functorial desiderata which are supposed to characterize the objects to be constructed (in Lurie’s treatment, and hence in ours) require some work to formulate precisely. Moreover, on the face of it, those say nothing about the bar construction or the relative tensor product. Relative tensor products first appear in a far corner of the devissage that ensues, and when the construction is complete, comparison with the bar construction presents a further nontrivial task. In short, the definitions are technical and the proofs are indirect.

To compensate for this, our paper includes two overviews. One, which begins presently, explains the structure of the paper in broad outline: what are the main results, and how they fit together. The second, in section 2, provides an informal overview of core aspects of Lurie’s construction and of ours; this section functions as a second introduction and is not logically needed for what follows.

In section 3 we refine Lurie’s treatment of the relative tensor product and its associativity structure. Recall that when \( \mathcal{C}^\otimes \) is a monoidal category in which \( \otimes \) respects realizations, Lurie constructs a certain double \( \infty \)-category

\[
\mathcal{BMod}(\mathcal{C})^\otimes \to \Delta^{op}
\]

whose fiber over \([n]\) is the \( \infty \)-category of sequences of \( n \) algebras and bimodules between them. We check that this construction can be refined in the following way: dropping the blanket assumption on compatibility with realizations, we show that

\(^4\)More precisely, we work with an \( \mathcal{LM} \)-monoidal category \( \mathcal{C}^\otimes \) in which \( \mathbb{1} \) is (temporarily) assumed to be terminal in \( \mathcal{C}_a \).
one may restrict attention to the class $\mathcal{A}$ of algebras in $\mathcal{C}$ which are bar compatible, in the sense that for any $A \in \mathcal{A}$ and $A$-bimodules $X, Y$, the geometric realization of the bar construction $\text{Bar}_A(X, Y)_\bullet$ exists and is compatible with $\otimes$. In this way we obtain a double $\infty$-category $\text{BMod}_A(\mathcal{C})^\otimes \to \Delta^{op}$ whose higher vertical morphisms are “$A$-multimodules” (sequences of algebras belonging to $\mathcal{A}$ and bimodules between them). The main result of this section, which amounts to a mild refinement of [13, prop. 4.4.3.9], is our corollary 3.4.5. Given $A \in \mathcal{A}$, we use $\text{BMod}_A(\mathcal{C})^\otimes$ in section 4 to endow $A\text{BMod}_A(\mathcal{C})$ with a monoidal structure given by relative tensor product over $A$. We also construct an $\mathcal{L}\mathcal{M}$-monoidal category $\text{BLMod}_A(\mathcal{C})^\otimes \to \mathcal{L}\mathcal{M}^\otimes$ which witnesses a left action of the monoidal $\infty$-category $A\text{BMod}_A(\mathcal{C})^\otimes$ of $A$-$A$-bimodules on the category $\text{LMod}_A(\mathcal{C})$ of left $A$-modules, the action being given on the level of individual objects by the relative tensor product.  

Section 5 concerns a variant of the results of sections 3 and 4 in which $\mathcal{C}^\otimes$, instead of being monoidal, is $\mathcal{L}\mathcal{M}$-monoidal, and $A \in \text{Alg}(\mathcal{C}_a)$. Under suitable assumptions on $A$, we construct a left-action of the monoidal $\infty$-category $A\text{BMod}_A(\mathcal{C}_a)$ of $A$-$A$-bimodules in $\mathcal{C}_a$ on the $\infty$-category $\text{LMod}_A(\mathcal{C})$ of left $A$-modules in $\mathcal{C}_m$. Strictly speaking, this construction generalizes the analogous construction from the previous two sections, and it is this generalized version that is needed in the sequel. Indeed, even if one is interested in applying Koszul duality for modules to a monoidal category $\mathcal{C}^\otimes \to \text{Ass}^\otimes$, if the unit object $1 \in \mathcal{C}$ is not terminal, the methods of section 7 and section 8 necessitate that we work with an associated $\mathcal{L}\mathcal{M}$-monoidal category which witnesses $\mathcal{C}$ as left-tensored over the monoidal slice category $\mathcal{C}^\otimes_{/1}$.  

In section 6 we review $\mathcal{L}\mathcal{M}$-monoidal algebras and $\mathcal{L}\mathcal{M}$-monoidal pairings in preparation for section 7. Theorem 7.5.1 which takes place in the setting of general $\mathcal{L}\mathcal{M}$-monoidal pairings and generalizes Proposition 5.2.2.27 of [13], may reasonably be regarded as the main theorem on Koszul duality for modules, but in an abstract,
axiomatic form. As hinted above, section 2 is devoted to an overview of its proof, preceded by an overview of the proof of the analogous Theorem 5.2.2.7 of [13].

In section 3 we specialize from the generality of LM-monoidal pairings to the case of the twisted arrow category. We show that if $C^\otimes$ is an LM-monoidal $\infty$-category and $M$ is an algebra in the twisted arrow category of $C_a$ which lies over algebras $A$ in $C_a$ and $B$ in $C_a^{op}$ with $B$ trivial, then $\text{TwArr}(C)$ admits realizations of $M$-bar constructions $\otimes$-compatibly (see definition 3.3.2).

We arrive at our LM-monoidal version of [13, thm. 5.2.2.17] in our theorem 8.3.5. This completes the construction of the LM-monoidal bar-cobar adjunction under the assumption that the unit object of $C$ is both initial and terminal. It remains to dispense with this assumption and, more significantly, to compare the result with the relative tensor product $1 \otimes_A X$ mentioned above. The final result is summarized in corollary 8.4.3.

Let us indicate how corollary 8.4.3 may be applied, for instance, to the study of commutative algebras in motives. This is meant only to point in the direction of future applications — more significant applications will require more work. Let $Z$ be a Noetherian scheme and let $D^\otimes = DA^{et}(Z, \mathbb{Q})^\otimes$ denote the monoidal derived $\infty$-category of etale motives over $Z$ with $\mathbb{Q}$ coefficients; see, for instance, Ayoub [2, §2.1.1] for a model-categorical construction. The $\infty$-category $D^\otimes$ is presentably symmetric monoidal; hence $C^\otimes = \text{CAlg} D$ with its coCartesian symmetric monoidal structure admits geometric realizations of bar constructions $\otimes$-compatibly, and corollary 8.4.3 applies to produce a functor

$$D_{\text{Koszul}} = D_{\text{LMod}(\Lambda)}^\otimes : \text{LMod}^{\text{aug}}(\mathcal{C}) \to \text{coLMod}^{\text{aug}}(\mathcal{C}).$$

Let $A \in \mathcal{C}$ be a commutative algebra in $D$ and let

$$x, y : A \to 1$$

be two augmentations. In view of the equivalence

$$\text{Alg} \mathcal{C} \simeq \mathcal{C},$$

$(A, x)$ may be lifted to an object of $\text{Alg}^{\text{aug}}(\mathcal{C})$. We may then view $1$ as a left module via $y$ — we write $y1$ for this structure. Together, we obtain an object

$$(A \xrightarrow{x} 1, y1)$$

of $\text{LMod}^{\text{aug}}(\mathcal{C})$. Applying $D_{\text{Koszul}}$, we obtain an augmented coalgebra structure on $1 \otimes_{x,A,x} 1$ together with a coaction on $1 \otimes_{x,A,x} 1$ which makes the latter into a torsoric comodule in $\mathcal{C} = \text{CAlg} D$.

Suppose $A$ is the algebra associated to a smooth $Z$-scheme $X$ and $x, y$ are the augmentations associated to $Z$-points of $X$. Suppose, moreover, that all motives in sight
belong to a triangulated subcategory possessing a *motivic* t-structure (alternatively, replace motives with any realization). Then

\[ \pi^\text{un}_1(X, x) = \text{Spec } H^0(1 \otimes_{x,A,x} 1) \]

is the unipotent fundamental group of \(X\) at \(x\) and

\[ \pi^\text{un}_1(X, x, y) = \text{Spec } H^0(1 \otimes_{x,A,y} 1) \]

is the torsor of unipotent homotopy classes of paths from \(x\) to \(y\); see, for instance [7], [10]. Thus

\[ \mathcal{D}_{\text{Koszul}}(A \rightarrow 1, y \downarrow) \]

lifts the torsor structure of \(\pi^\text{un}_1(X, x, y)\) to the level of highly structured algebras in motives.

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2. Overview

Our goal in this section is to give an informal overview of the proof of Theorem 5.2.2.27 of Lurie [13], as well as of our proof of our analogous theorem 7.5.1.

2.1. **Review: Koszul duality and bimodules.** Let us first explain the relationship between the coalgebra structure of the Koszul dual and the monoidal structures on bimodules. Let \(C^\otimes\) be a symmetric monoidal \(\infty\)-category, and assume that \(C\) admits geometric realizations of simplicial objects and that the tensor product functor

\[ \otimes: C \times C \rightarrow C \]

preserves geometric realizations of simplicial objects separately in each variable. Under these assumptions, bimodules in \(C\) admit relative tensor products (see [13, sec. 4.4.3]). It is then easy to describe the underlying object of \(\mathcal{D}_{\text{Koszul}}(A)\); it is given by the relative tensor product \(1 \otimes_A 1\). To see the coalgebra structure, consider

\[ 1 \otimes_A 1 \simeq 1 \otimes_A A \otimes_A 1 \]

as the image of \(A\) under the functor

\[ 1 \otimes_A - \otimes_A 1: \text{AAlgMod}_A(C)^{\text{op}} \rightarrow C^{\text{op}}. \]

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8In this setting, following Deligne [8], “Spec” just means that we view the given object as an object of the opposite Tannakian category.
Being the unit of a monoidal category, $A \in \mathcal{A} \text{BMod}_A(C)^{op}$ admits an essentially unique algebra structure. If the functor (1) is lax monoidal then the induced functor $\mathbb{1} \otimes_A - \otimes_A \mathbb{1} : \text{Alg}(\mathcal{A} \text{BMod}_A(C)^{op}) \to \text{Alg}(C^{op})$, equips the Koszul dual $\mathbb{1} \otimes_A A \otimes_A \mathbb{1} \simeq \mathbb{1} \otimes_A \mathbb{1}$ with a coalgebra structure. Let us point out two crucial facts.

1. Being lax monoidal is additional structure on the functor (1), which depends on the choice of augmentation $A \to \mathbb{1}$. Its construction amounts to the main part of [13, sec. 5.2.2].

2. It is not at all clear how to combine the above construction into a functor $\mathcal{D}_{\text{Koszul}} : \text{Alg}_{\text{aug}}(C)^{op} \to \text{Alg}_{\text{aug}}(C^{op})$, since different algebras require us to examine different bimodule categories. To that end, it is better to describe each of the above constructions as solutions to representation problems. This is achieved via the theory of pairings of $\infty$-categories and its monoidal variants, developed in [13, sec. 5.2.1].

2.2. Review: functorial characterization of Koszul duality for algebras. Let $C^\otimes$ be a monoidal $\infty$-category. Lurie begins by formulating a functorial characterization of the coalgebra $\mathcal{D}_{\text{Koszul}}(A)$ associated to an algebra $A$ in $C$, which may be summarized as follows. After possibly replacing $C$ by $C/\mathbb{1}$ we may assume the unit object is terminal. There’s a right fibration

$$\lambda : \text{TwArr}(C) \to C \times C^{op}$$

which classifies the Hom-functor $\lambda^{-1}(A, B) = \text{Hom}_C(A, B)$. Informally, we think of an object of $\text{TwArr}(C)$ over $(A, B) \in C \times C^{op}$ as classifying a map $A \to B$ in $C$. Formally, $\lambda$ is an example of a pairing, i.e. a right fibration of the form

$$\mathcal{M} \to C \times \mathcal{D}.$$ 

The monoidal structure on $C$ endows the twisted arrow category $\text{TwArr}(C)$ with a monoidal structure, and $\lambda$ may be upgraded to a pairing between categories of algebras

$$\text{Alg}(\lambda) : \text{Alg} \text{TwArr}(C) \to \text{Alg}(C) \times \text{Alg}(C^{op}).$$

The pairing $\text{Alg}(\lambda)$ induces a functor

$$\hat{\mathcal{D}}_{\text{Alg}(\lambda)} : \text{Alg}(C)^{op} \to \text{PSh Alg}(C^{op})$$

to the category of presheaves, and the Koszul dual $\mathcal{D}_{\text{Koszul}}(A)$ of $A$ may be defined as the object of $\text{Alg}(C^{op})$ representing $\hat{\mathcal{D}}_{\text{Alg}(\lambda)}(A)$ (unique up to essentially unique equivalence), provided that such an object exists.
If $1 \in \mathcal{C}$ is initial, we may reverse direction. The pairing $\text{Alg}(\lambda)$ also induces a functor

$$\text{PSh Alg}(\mathcal{C}) \leftarrow \text{Alg}(\mathcal{C}^{op})^{op} : \hat{\mathcal{D}}_{\text{Alg}(\lambda)},$$

and the Koszul dual $\mathcal{D}'_{\text{Koszul}}(B)$ of $B$ may be defined as the object of $\text{Alg}(\mathcal{C})$ representing $\hat{\mathcal{D}}_{\text{Alg}(\lambda)}(B)$, provided that such an object exists.

If $1 \in \mathcal{C}$ is both initial and terminal, and if Koszul duals in both directions are representable, then it follows formally that $\mathcal{D}'_{\text{Koszul}}$ is left adjoint to $\mathcal{D}_{\text{Koszul}}$.

### 2.3. Review: construction of Koszul duality for algebras.

The construction of the functors $\mathcal{D}_{\text{Koszul}}, \mathcal{D}'_{\text{Koszul}}$ departs from the confines of the twisted arrow category. A pairing of $\infty$-categories

$$\lambda : \mathcal{M} \rightarrow \mathcal{C} \times \mathcal{D}$$

is left representable if the associated functor

$$\hat{\mathcal{D}}_{\lambda} : \mathcal{C}^{op} \rightarrow \text{PSh } \mathcal{D}$$

factors through the Yoneda embedding of $\mathcal{D}$. For example, the functor $\lambda$ above is left representable. Consider a general monoidal pairing of monoidal $\infty$-categories

$$\lambda^{\otimes} : \mathcal{M}^{\otimes} \rightarrow \mathcal{C}^{\otimes} \times_{\text{Ass}^{\otimes}} \mathcal{D}^{\otimes}$$

(a right fibration of underlying categories subject to conditions which we suppress throughout the introduction), and assume that the induced pairing of underlying categories is left representable. Then $\lambda^{\otimes}$ gives rise to a pairing $\text{Alg}(\lambda)$ of categories of algebras and further, to a functor

$$\hat{\mathcal{D}}_{\text{Alg}(\lambda)} : (\text{Alg} \mathcal{C})^{op} \rightarrow \text{PSh}(\text{Alg} \mathcal{D}).$$

Theorem 5.2.2.17 of [13] states that under certain assumptions, $\mathcal{D}_{\text{Alg}(\lambda)}(A)$ is representable for all $A$, hence $\hat{\mathcal{D}}_{\text{Alg}(\lambda)}$ factors as precomposition of the Yoneda embedding of $\text{Alg} \mathcal{D}$ and a functor

$$\mathcal{D}_{\text{Alg}(\lambda)} : (\text{Alg} \mathcal{C})^{op} \rightarrow \text{Alg} \mathcal{D}.$$

A fortiori, the algebra $\mathcal{D}_{\text{Alg}(\lambda)}(A)$ in $\mathcal{D}$ will be determined by the Koszul dual $\mathcal{D}_{\text{Koszul}}(A)$ as follows: the pairing $\lambda^{\otimes}$ in fact gives rise to a lax-monoidal functor

$$\mathcal{D}_{\lambda^{\otimes}} : (\mathcal{C}^{op})^{\otimes} \rightarrow \mathcal{D}^{\otimes}$$

and $\mathcal{D}_{\text{Alg}(\lambda)}(A) = \mathcal{D}_{\lambda^{\otimes}}(\mathcal{D}_{\text{Koszul}}(A))$.

One condition on $\lambda^{\otimes}$ ensures the existence of an essentially unique algebra $M \in \text{Alg}(\mathcal{M})$ lifting $(A, 1_\mathcal{D})$. In the case of the twisted arrow category, $M$ classifies the given augmentation (unique after replacing $\mathcal{C}$ with $\mathcal{C}/_1$). Continuing with Lurie (loc. cit.), there is then an induced pairing

$$\lambda_M : \mathcal{M}_{\text{BMod}}(\mathcal{M})^{\otimes} \rightarrow A_{\text{BMod}}(\mathcal{C})^{\otimes} \times_{\text{Ass}^{\otimes}} \mathcal{D}^{\otimes}.$$
If \( F = A \otimes E \otimes A \) is the free \( A \)-\( A \)-bimodule generated by \( E \in \mathcal{C} \), then its \( \lambda_M \)-dual \( \mathcal{D}_{\lambda_M}(F) \in \text{Psh} \mathcal{D} \) with respect to the pairing \( \lambda_M \) is represented by
\[
\mathcal{D}_{\lambda_M}(F) = \mathcal{D}_{\lambda}(E).
\]
However, while \( A \) is the unit object of \( \text{BMod}_A \), it is not free as \( A \)-\( A \)-bimodule. Using free resolutions, one constructs an \( M \)-\( M \)-bimodule \( M' \) lying over a pair \( (A,B) \) which witnesses the fact that \( B \in \mathcal{D} \) represents the presheaf \( \mathcal{D}_{\lambda_M}(A) \in \text{PSh}(\mathcal{D}) \). Since \( A \) is the unit object and since \( \mathcal{D}_{\lambda_M} \) is lax-monoidal, \( B = \mathcal{D}_{\lambda_M}(A) \in \mathcal{D} \) inherits an algebra structure, as witnessed by an algebra structure on \( M' \). Dispensing with the spent bimodule structures, we obtain an algebra object \( M' \) of \( \mathcal{M}^\otimes \) witnessing the fact that \( B \) represents \( \mathcal{D}_{\text{Alg}(\lambda)}(A) \).

2.4. **Review: comparison with the relative tensor product.** We now return to the case of the twisted arrow category, and relate the functor \( \mathcal{D}_{\text{Alg}(\lambda)} \) to the description of section 2.1. Suppose \( \mathcal{D} = \mathcal{C}^{\text{op}}, \mathcal{M} = \text{TwArr}(\mathcal{C}) \) is the twisted arrow category, and \( M \in \text{Alg}(\mathcal{M}) \) witnesses an augmentation \( A \to \mathbb{1} \). Then there’s a natural augmentation \( M \to \mathbb{1} \). Pulling back an object of \( \mathcal{M} \) regarded as \( \mathbb{1}\mathbb{1} \)-bimodule along the augmentation gives rise to a morphism \( f \) of pairings
\[
\begin{array}{ccc}
\mathcal{M} & \to & \mathbb{1}\mathbb{1}\text{BMod}_M \\
\downarrow & & \downarrow \\
\mathcal{C} \times \mathcal{D} & \to & \mathbb{1}\mathbb{1}\text{BMod}_A \times \mathcal{D}.
\end{array}
\]
A certain compatibility with induced duality functors (\( f \) is a “right representable morphism of right representable pairings”) leads to a commuting square
\[
\begin{array}{ccc}
\mathcal{C} & \leftarrow & \mathcal{D}^{\text{op}} \\
\downarrow & & \downarrow \\
\mathbb{1}\mathbb{1}\text{BMod}_A & \leftarrow & \mathcal{D}_{\lambda_M}^{\text{op}}.
\end{array}
\]
This shows that \( \mathcal{D}_{\lambda_M}' \) is given by pulling back bimodules along the augmentation. Hence its left adjoint \( \mathcal{D}_{\lambda_M} \) is given by
\[
\mathcal{D}_{\lambda_M}(X) = \mathbb{1} \otimes_A X \otimes_A \mathbb{1}
\]
so \( B = \mathbb{1} \otimes_A \mathbb{1} \).

2.5. **\( \mathcal{L}\mathcal{M} \)-variant: functorial characterization of Koszul duals.** Recall that the \( \infty \)-operad \( \mathcal{L}\mathcal{M}^\otimes \) has two colors \( a \) and \( m \), that an \( \mathcal{L}\mathcal{M} \)-monoidal category
\[
\mathcal{C}^\otimes \to \mathcal{L}\mathcal{M}^\otimes
\]
witnesses the category $\mathcal{C}_m$ as left-tensored over the monoidal category $\mathcal{C}_a^\otimes$, and that an $\mathcal{LM}$-algebra $X$ is a morphism of $\infty$-operads

$$X : \mathcal{LM}^\otimes \to \mathcal{C}^\otimes$$

over $\mathcal{LM}^\otimes$ which models a pair $(X_a, X_m)$ consisting of an algebra object $X_a$ in $\mathcal{C}_a$ and an $X_a$-module $X_m$ in $\mathcal{C}_m$. Given $X \in \text{LMod}(\mathcal{C})$ we say that $X$ endows $X_m$ with the structure of a left $X_a$-module.

Suppose given an $\mathcal{LM}$-monoidal pairing

$$(*) \quad \lambda^\otimes : \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times \mathcal{LM}^\otimes \mathcal{D}^\otimes$$

where the unit object of $\mathcal{D}_a$ is initial. We assume the underlying pairings $\lambda_a, \lambda_m$ give rise to duality functors

$$\mathcal{D}_{\lambda_a} : \mathcal{C}_a^{op} \to \mathcal{D}_a, \quad \mathcal{D}_{\lambda_m} : \mathcal{C}_m^{op} \to \mathcal{D}_m.$$

The pairing (*) may be upgraded to a pairing of left-module categories

$$\text{LMod}(\lambda) : \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})$$

which subsequently gives rise to a functor

$$\hat{\mathcal{D}}_{\text{LMod}(\lambda)} : \text{LMod}(\mathcal{C})^{op} \to \text{PSh LMod} \mathcal{D}$$

to the category of presheaves on $\text{LMod} \mathcal{D}$. The functor $\hat{\mathcal{D}}_{\text{LMod}(\lambda)}$ has the expected compatibility with the functor

$$\hat{\mathcal{D}}_{\text{Alg}(\lambda_a)} : \text{Alg}(\mathcal{C}_a)^{op} \to \text{PSh Alg}(\mathcal{D}_a^{op})$$

associated to the pairing

$$\lambda_a : \mathcal{M}_a^\otimes \to \mathcal{C}_a^\otimes \times \text{Ass}^\otimes \mathcal{D}_a^\otimes$$

so that if $X = (X_a, X_m)$ is a left-module object of $\mathcal{C}^\otimes$ and $\hat{\mathcal{D}}_{\text{LMod}(\lambda)}(X) = (Y_a, Y_m)$ is its candidate Koszul dual regarded as a left-module object in the category of presheaves, then

$$\hat{\mathcal{D}}_{\text{Alg}(\lambda_a)}(X_a) = Y_a.$$

Mirroring [13, 5.2.2.27], our theorem 7.5.1 states that under certain assumptions, the functor $\hat{\mathcal{D}}_{\text{LMod}(\lambda)}$ factors through a functor

$$\mathcal{D}_{\text{LMod}(\lambda)} : \text{LMod}(\mathcal{C})^{op} \to \text{LMod} \mathcal{D}.$$
2.6. $\mathcal{LM}$-variant: construction of Koszul duality functors. We adapt Lurie’s techniques from section 2.3 to the $\mathcal{LM}$-monoidal settings. Let $\mathcal{M} \otimes \to \mathcal{LM} \otimes$ be an $\mathcal{LM}$-monoidal category, and let $M$ be an algebra object of the monoidal category $\mathcal{M}_a$. In construction 4.3.2 we construct (under certain assumptions) an $\mathcal{LM}$-monoidal category $\text{BLMod}_M(\mathcal{M})\otimes$ witnessing a left action of the category of bimodules $\text{BLMod}_M(\mathcal{M})\otimes \otimes \text{Mod}_M(\mathcal{M}_a)$ on the category of left modules $\text{BLMod}_M(\mathcal{M})_m = \text{LMod}_M(\mathcal{M}_m)$.

We show that an $\mathcal{LM}$-algebra $Z = (Z_a, Z_m) \in \text{LMod BLMod}_M(\mathcal{M})$ is nothing more than an $M$-algebra $Z_a$ and a $Z_a$-module $Z_m$. We refer to such an object as an $\mathcal{LM}$-algebra over $M$. The characters above each play a role analogous to a character in the previous play as in the following table.

| $\mathcal{LM}$ | $\mathcal{LM}$ |
|----------------|----------------|
| $\text{Mod}_M(\mathcal{M})\otimes$ | $\text{Mod}_M(\mathcal{M}_a), \text{LMod}_M(\mathcal{M}_m)$ |
| $\text{Mod}_M(\mathcal{M})\otimes$ | $\text{BLMod}_M(\mathcal{M})\otimes$ |
| $\text{Alg}_M(\text{Mod}_M(\mathcal{M}))$ | $\text{LMod}(\text{BLMod}_M(\mathcal{M}))$ |

Returning to the $\mathcal{LM}$-monoidal pairing above, fix a left-module object $X = (X_a, X_m)$ of the $\mathcal{LM}$-monoidal category $\mathcal{C}^\otimes$. To construct the left-module object $D_\text{LMod}(\lambda)^\otimes$ of $\text{LMod} \mathcal{D}$ (and hence prove the theorem), we first lift $X_a$ to an algebra object $M$ of $\mathcal{M}_a$ lying above the pair of algebras $(X_a, 1_{D_a})$ just as in the monoidal case of [13, sec. 5.2.2]. We then construct an $\mathcal{LM}$-monoidal pairing $\lambda_M^\otimes : \text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_X(\mathcal{C})^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes$.

The $\mathcal{LM}$-monoidal pairing $\lambda_M^\otimes$ plays the role of the monoidal pairing of bimodule categories in the monoidal case. In fact, it includes a monoidal pairing $\lambda_M^\otimes : \text{BLMod}_M(\mathcal{M})^\otimes \xrightarrow{\text{BLMod}_X(\mathcal{C})^\otimes \times \mathcal{D}^\otimes}$, as well as a pairing of $\infty$-categories of left modules $\lambda_M^\otimes : \text{LMod}_M(\mathcal{M}_m) \xrightarrow{\text{LMod}_X(\mathcal{C}_m) \times \mathcal{D}_m}$.

\[\lambda_M^\otimes : \text{BLMod}_M(\mathcal{M})^\otimes \xrightarrow{\text{BLMod}_X(\mathcal{C})^\otimes \times \mathcal{D}^\otimes} \text{LMod}_M(\mathcal{M}_m) \xrightarrow{\text{LMod}_X(\mathcal{C}_m) \times \mathcal{D}_m},\]

So, while the construction of the $\mathcal{LM}$-monoidal category $\text{BLMod}_M(\mathcal{M})^\otimes$ is complicated, its category of $\mathcal{LM}$-algebras is predetermined.
and enables us to construct a pairing of ∞-categories

\[ \lambda_M : \text{LMod} \text{BLMod}_M(M) \to \text{LMod} \text{BLMod}_{X_a}(C) \times \text{LMod} D \]

of left modules over

\[ M \mapsto (X_a, 1_D). \]

Using free resolutions as above, we construct an \( M-M \)-bimodule \( Z_a \) lying over a pair

\[ (X_a, Y_a) \in X_a \text{BMod}_{X_a}(C_a) \times D_a \]

witnessing the fact that \( Y_a \in D_a \) represents the presheaf \( \hat{\mathcal{D}}_{\lambda_{M,a}}(X_a) \in \text{PSh}(D_a) \). Similarly, if \( F = X_a \otimes E \) is the free left \( X_a \)-module generated by \( E \in C_m \), then its dual \( \hat{\mathcal{D}}_{\lambda_{M,m}}(F) \in \text{PSh} D_m \) with respect to the pairing \( \lambda_{M,m} \) is represented by the object

\[ \mathcal{D}_{\lambda_{M,m}}(F) = \mathcal{D}_{\lambda_m}(E). \]

Applying this observation to a free resolution, one obtains a left \( M \)-module \( Z_m \) lying over a pair

\[ (X_m, Y_m) \in \text{LMod}_{X_a}(C_m) \times D_m \]

witnessing the fact that \( Y_m \in D_m \) represents the presheaf \( \hat{\mathcal{D}}_{\lambda_{M,m}}(X_m) \in \text{PSh}(D_m) \). Since \( X_a \) is the unit object of \( X_a \text{BMod}_{X_a}(C_a) \) and since \( \hat{\mathcal{D}}_{\lambda_M} \) is lax-\( \mathcal{L}\mathcal{M} \)-monoidal, \( Y = (Y_a, Y_m) \) inherits a left module structure, as witnessed by a left module structure on \( Z = (Z_a, Z_m) \). Dispensing with the spent \( M \)-algebra structure of \( Z_a \), we obtain a left module \( Z \) in \( \mathcal{M}^\otimes \) witnessing the fact that \( Y \) represents \( \mathcal{D}_{\text{LMod}(\lambda)}(X) \).

This completes our overview of the proof of theorem \( 7.5.1 \). Comparison with the bar construction in the case of the twisted arrow category is achieved similarly to §2.4 above, and we refer the reader to the proof of theorem \( 8.3.5 \) for the details.

### 3. Restricted double ∞-categories of bimodules in a monoidal ∞-category

In [13, lem. 4.4.3.9] Lurie constructs an ∞-categorical version of the double Morita category of algebras and bimodules in a monoidal ∞-category \( C \), under the assumption that tensor products in \( C \) are compatible with geometric realizations. Our goal in this section is corollary \( 3.4.5 \) a modification of Lurie’s construction which weakens the compatibility assumption: we only assume compatibility with a restricted collection of geometric realizations, prescribed by a class of algebras \( \mathcal{A} \). This section closely follows [13, sec. 4.4.3] with needed modifications; no new ideas are required. The results of this section will be used in section 4 to package the relative tensor product of an \( \mathcal{A}-\mathcal{A} \)-bimodule and a left \( \mathcal{A} \)-module into an \( \mathcal{L}\mathcal{M} \)-monoidal ∞-category.
3.1. Multimodules. We introduce some notation. The combinatorics of the associativity of relative tensor products up to coherent higher homotopies are encoded by the $\Delta^\text{op}$-family of $\infty$-operads

\[(*) \quad \text{Tens}^\circ \to \Delta^\text{op} \times \text{Fin}_\ast\]

of [13, Definition 4.4.1.1] or definition 5.2.3. We refer to $\text{Tens}^\circ$ as the *Morita $\infty$-operad family.* If $\alpha : S \to \Delta^\text{op}$ is a map of simplicial sets, we denote the pullback of $(*)$ composed with the projection $\Delta^\text{op} \times \text{Fin}_\ast \to \Delta^\text{op}$ along $\alpha$ by $\text{Tens}^\circ_S$ or by $\text{Tens}^\circ_{\alpha}$ if we wish to emphasize the dependence on the map $\alpha$. In particular, if $\alpha : s \to s'$ is a morphism in $\Delta^\text{op}$, we write $\text{Tens}^\circ_{\alpha}$ for the pullback along the associated morphism $\alpha : \Delta^1 \to \Delta^\text{op}$.

Recall from [13, Notation 4.4.1.9] that for each $n \geq 1$, the $\infty$-operad $\text{Tens}^\circ_{[n]}$ has colors $a_0, a_1, a_2, \ldots, a_n$ and $m_{0,1}, m_{1,2}, \ldots, m_{n-1,n}$. If $M : \text{Tens}^\circ_{[n]} \to \text{C}^\circ$ is a $\text{Tens}^\circ_{[n]}$-algebra over $\text{Ass}^\circ$ in a monoidal $\infty$-category $\text{C}^\circ$, then associated to each color $a_i$ is an algebra $A_i$ in $\text{C}$, and associated to each color $m_{i,i+1}$ is an $A_i$-$A_{i+1}$-bimodule in $\text{C}$.

**Terminology 3.1.1 ($\mathcal{A}$-multimodules).** We refer to a $\text{Tens}^\circ_{[n]}$-algebra

\[M : \text{Tens}^\circ_{[n]} \to \text{C}^\circ\]

in a monoidal category $\text{C}^\circ$ over $\text{Ass}^\circ$ as a *multimodule* and denote it by

\[M = A_0 M_{0,1}^{0,1} A_1 M_{1,2}^{1,2} \cdots A_{n-1} M_{n-1,n}^{n-1,n}\]

with the algebras as subscripts and the bimodules between them placed on the line.

An $(A_0, \ldots, A_n)$-*multimodule* (or a multimodule over $(A_0, \ldots, A_n)$), is simply a multimodule whose algebras are $(A_0, \ldots, A_n)$. If we wish to specify the number $n$ we say $n$-*module* instead of multimodule. Thus, for instance, a 2-module looks like so:

\[A_0 M_{A_1} N_{A_2}\]

Let $\mathcal{A} \subset \text{Alg}(\text{C})$ be a class of algebras closed under equivalences. By an $\mathcal{A}$-multimodule, we mean a multimodule

\[M = A_0 M_{0,1}^{0,1} A_1 M_{1,2}^{1,2} \cdots A_{n-1} M_{n-1,n}^{n-1,n}\]

whose algebras $A_0, \ldots, A_n$ belong to $\mathcal{A}$.

**Terminology 3.1.2 (Relative tensor product of $\mathcal{A}$-multimodules).** Let

\[q : \text{C}^\circ \to \text{Ass}^\circ\]
be a monoidal ∞-category, and let \( \mathcal{A} \subset \text{Alg}(\mathcal{C}) \) be a class of algebras closed under equivalences. We say that \( \mathcal{C} \) admits operadic left Kan extensions from \( \mathcal{A} \)-multimodules if for every edge \( \alpha : s \to s' \) in \( \Delta^\text{op} \) and every \( \mathcal{A} \)-multimodule \( F_0 \in \text{Alg}_{\text{Tens}_s / \text{Ass}^\otimes}(\mathcal{C}) \) there exists an operadic \( q \)-left Kan extension

\[
F \in \text{Alg}_{\text{Tens}_\alpha / \text{Ass}^\otimes}(\mathcal{C})
\]

of \( F_0 \), as in the following diagram:

\[
\begin{array}{ccc}
\text{Tens}_s & \xrightarrow{F_0} & \mathcal{C}^\otimes \\
\downarrow & & \downarrow \\
\text{Tens}_\alpha & \xrightarrow{F} & \text{Ass}^\otimes.
\end{array}
\]

We also say that \( \mathcal{C} \) admits relative tensor products of \( \mathcal{A} \)-multimodules; this generalizes [13, def. 4.4.2.3].

3.2. The Morita pre-double ∞-category. We will construct the Morita double ∞-category \( \text{BMod}_\mathcal{A}(\mathcal{C}) \) as a simplicial subset of the Morita double ∞-category \( \text{BMod}(\mathcal{C})^\otimes \) [13, cor. 4.4.3.2 and def. 4.4.3.10]:

**Definition 3.2.1.** For variable \( [n] \in \Delta^\text{op} \), the ∞-categories of \( n \)-multimodules in \( \mathcal{C} \) arrange themselves into a map of simplicial sets

\[
B\text{Mod}(\mathcal{C})^\otimes \to \Delta^\text{op}
\]

defined by the following universal property:

For every map of simplicial sets \( K \to \Delta^\text{op} \), we have a bijection

\[
\text{Hom}_{\text{Set}/\Delta^\text{op}}(K, B\text{Mod}(\mathcal{C})^\otimes) \cong \text{Alg}_{\text{Tens}_K/\text{Ass}^\otimes}(\mathcal{C})_0
\]

between the set of simplicial maps \( K \to B\text{Mod}(\mathcal{C})^\otimes \) over \( \Delta^\text{op} \) and the \text{Set}\[\text{Id}\] of \( \text{Tens}_K^\otimes \)-algebras in \( \mathcal{C}^\otimes \) over \( \text{Ass}^\otimes \), functorial in \( K \).

Functoriality in \( K \) implies that a diagram of simplicial sets

\[
L \longrightarrow B\text{Mod}(\mathcal{C})^\otimes \\
\downarrow \\
K \longrightarrow \Delta^\text{op}
\]

\[\text{We emphasize that the right hand side is considered as a set by adding the subscript 0}\]
commutes if and only if the corresponding diagram

\[
\begin{array}{ccc}
\text{Tens}_L^\otimes & \longrightarrow & C^\otimes \\
\downarrow & & \downarrow \\
\text{Tens}_K^\otimes & \longrightarrow & \text{Assoc}^\otimes
\end{array}
\]

commutes. We refer to $\text{BMod}(C)^\circledast$ as the Morita pre-double $\infty$-category of $C$\textsuperscript{11}

In the following paragraphs we assemble a few first properties of $\text{BMod}(C)^\circledast$.

**Lemma 3.2.2.** Let $q: C^\otimes \to \text{Ass}^\otimes$ be a monoidal $\infty$-category, and $p: \text{BMod}(C)^\circledast \to \Delta^{\text{op}}$ as above. Then $p$ is an inner fibration.

**Proof.** We verify that $p$ has the right lifting property with respect to all inner anodyne maps. By definition of $\text{BMod}(C)^\circledast$ checking the lifting condition against an inner anodyne map $A \to B$ is equivalent to solving the following lifting problem

\[
\begin{array}{ccc}
\text{Tens}_A^\otimes & \longrightarrow & C^\otimes \\
\downarrow & & \downarrow \\
\text{Tens}_B^\otimes & \longrightarrow & \text{Ass}^\otimes
\end{array}
\]

By [13 thm. 4.4.3.1] (flatness of $\text{Tens}^\otimes \to \Delta^{\text{op}}$) and [13 prop. B.3.14] the monomorphism $\text{Tens}_A^\otimes \to \text{Tens}_B^\otimes$ is a trivial cofibration in the Joyal model structure ([12 thm. 2.2.5.1]). It follows that the solution to the above lifting problem exists, as the right vertical morphism $C^\otimes \to \text{Ass}^\otimes$ is a categorical fibration. \hfill \square

**Definition 3.2.3.** Let $C^\otimes$ be a monoidal $\infty$-category and let $\mathcal{A} \subset \text{Alg}(C)$ be a class of algebras closed under equivalences. We say that $\text{BMod}(C)^\circledast \to \Delta^{\text{op}}$ admits coCartesian lifts from $\mathcal{A}$-multimodules if whenever $\alpha: s \to s'$ is an edge in $\Delta^{\text{op}}$ and $F_0$ is an $\mathcal{A}$-multimodule, there exits a $p$-coCartesian lift of $\alpha$ extending $F_0$.

**Proposition 3.2.4.** Let $q: C^\otimes \to \text{Ass}^\otimes$ be a monoidal $\infty$-category. Let $\alpha: s \to s'$ be an edge in $\Delta^{\text{op}}$ and let $F_0$ be a vertex of $\text{Alg}_{\text{Tens}_s^\otimes / \text{Ass}^\otimes}(C)$. The vertex $F_0$ corresponds to an element of $\text{Hom}_{\text{Set}/\Delta^{\text{op}}}(\{s\}, \text{BMod}(C)^\circledast)$,

\textsuperscript{11}This is the same as [13 Definition 4.4.3.10], except that we do not assume that the tensor product in $C$ is compatible with geometric realizations (in the sense of [13 def. 3.1.1.18]).

\textsuperscript{12}Here we consider $s$ as a map of simplicial sets $s: \{s\} = \Delta^0 \to \Delta^1 \to \Delta^{\text{op}}$.\hspace{1cm} 15
and hence to a vertex of $\text{BMod}(C)^\otimes$ lying over $s$. Let $F$ be a vertex of $\text{Alg}_{\text{Tens}_\alpha/\text{Ass}}(C)$ extending $F_0$; equivalently, $F$ is an edge of $\text{BMod}(C)^\otimes$ lying over $\alpha$ with source $F_0$. If $F$, regarded as

$$\text{Tens}_\alpha^\otimes \to C^\otimes,$$

is an operadic $q$-left Kan extension of $F_0$, then $F$ regarded as an edge of $\text{BMod}(C)^\otimes$ is $p$-coCartesian. In particular, if $C$ admits operadic left Kan extensions from $A$-multimodules (terminology 3.1.2), then $\text{BMod}(C)^\otimes$ admits coCartesian lifts from $A$-multimodules (definition 3.2.3).

This is essentially Corollary 4.4.3.2 of [13]. We nevertheless repeat the proof here for the convenience of the reader, since we found the argument in loc. cit. a bit condensed.

**Proof.** It’s enough to solve the lifting problem of [12, rem. 2.4.1.4] (dualized):

$$\begin{array}{ccc}
\Delta^{[0,1]} & \xrightarrow{F} & \Delta_0^n \\
\text{For } n \geq 2: & \downarrow & \downarrow^p \\
\Lambda_0^n & \xrightarrow{\gamma} & \Delta_0^n \\
\downarrow & \uparrow & \uparrow \sigma \\
\Delta^n & \xrightarrow{} & \Delta^\text{op}. \\
\end{array}$$

Base-changing $\text{Tens}_\sigma^\otimes$ along $\sigma$, we have a $\Delta^n$-family of $\infty$-operads

$$r: \text{Tens}_\sigma^\otimes \to \text{Fin}_* \times \Delta^n.$$

Let $\gamma$ be the composite map

$$\text{Tens}_\sigma^\otimes = \text{Tens}_\sigma^\otimes \to \text{Tens}^\otimes \to \text{Ass}^\otimes.$$

Diagram (*) gives rise to a commuting diagram of $\infty$-operad family maps (solid-arrow diagram below).

$$\begin{array}{ccc}
\text{Tens}_{\Delta_0^{[0,1]}} & \xrightarrow{F_0} & \text{Tens}_{\Delta_0^n} \\
\downarrow & \xrightarrow{F} & \downarrow f \\
\text{Tens}_{\Lambda_0^n} & \xrightarrow{p} & \text{Tens}_{\Delta_0^n} \\
\downarrow & \xrightarrow{\gamma} & \downarrow \sigma \\
\text{Tens}_{\Delta^n} & \xrightarrow{} & \text{Ass}^\otimes. \\
\end{array}$$

(***)
By assumption, the functor $F$ appearing in diagram (***) is an operadic $q$-left Kan extension of $F_0$. By [13, thm. 3.1.2.3 (B)] there exists a map $f$ of generalized operads making diagram (***) commute. This solves the lifting problem (**). 

3.3. Review of the bar construction. We very briefly review [13, Construction 4.4.2.7], mostly for the sake of establishing notation. Following [13, not. 4.4.2.1], we denote the pullback of $\text{Tens}\otimes$ along the morphism

$$\Delta^1 \to \Delta^{op}$$

which corresponds to $[1] \simeq \{0, 2\} \hookrightarrow [2]$ in $\Delta$ by $\text{Tens}_\otimes$. In [13, not. 4.4.2.4] Lurie constructs a functor

$$((\Delta_+)^{op} \to \text{Tens}_\otimes$$

which encodes the combinatorics of the bar construction; we denote this functor by $\text{bar}^+$ and its restriction to $\Delta^{op}$ by

$$\text{bar}: \Delta^{op} \to ((\Delta_+)^{op} \xrightarrow{\text{bar}^+} \text{Tens}_\otimes.$$ 

Note that the image of $\text{bar}$ is contained in $\text{Tens}_{[2]} \subset \text{Tens}_\otimes$.

**Definition 3.3.1.** Let $q: \mathcal{C} \to \text{Ass}_\otimes$ be a monoidal $\infty$-category. Let $F_0 = A M_B N_C$ be an $(A, B, C)$-module in $\mathcal{C}$. The operadic bar construction on $(M, N)$ over $B$ is given by

$$\text{Bar}_B^\otimes(M, N)_\bullet \colon \Delta^{op} \xrightarrow{\text{bar}} \text{Tens}_{[2]}^{\otimes} \xrightarrow{F_0} \mathcal{C}^{\otimes}.$$ 

The commuting diagram of simplicial sets

$$\begin{array}{ccc}
\Delta^{op} & \xrightarrow{\text{bar}} & \text{Tens}_{[2]}^{\otimes} \\
\downarrow & & \downarrow \\
\Delta_+^{op} & \xrightarrow{\text{bar}^+} & \text{Tens}_{\otimes} \xrightarrow{q} \text{Ass}_\otimes.
\end{array}$$

gives rise to a commuting square of simplicial sets (solid arrow diagram)

$$\begin{array}{ccc}
\Delta^{\{0\}} & \xrightarrow{\text{Bar}_B^\otimes(M, N)_\bullet} & \text{Fun}(\Delta^{op}, \mathcal{C}^{\otimes}) \\
\downarrow \quad \text{Bar}_B^\otimes(M, N)_\bullet & \swarrow & \downarrow \\
\Delta^1 & \xrightarrow{\beta} & \text{Fun}(\Delta^{op}, \text{Ass}_\otimes). \\
\downarrow \quad \sigma & \swarrow & \downarrow \\
\Delta^1 & \xrightarrow{\text{Bar}_B^\otimes(M, N)_\bullet} & \text{Fun}(\Delta^{op}, \text{Ass}_\otimes). \\
\end{array}$$

By [12, prop. 3.1.3.1], $q^{\Delta^{op}}$ is again a coCartesian fibration. Hence the edge $\sigma$ in $\text{Fun}(\Delta^{op}, \text{Ass}_\otimes)$ admits a coCartesian lift $\beta$ as indicated, extending the vertex $\text{Bar}_B^\otimes(M, N)_\bullet$ of $\text{Fun}(\Delta^{op}, \mathcal{C}^{\otimes})$. We define the (underlying) bar construction on $(M, N)$ over $B$ by

$$\text{Bar}_B(M, N)_\bullet := \beta(1).$$
The construction provides a natural transformation
\[ \beta : \text{Bar}_B^\circ(M, N) \to \text{Bar}_B(M, N) \]
of functors \( \Delta^\circ \to C^\circ \).

**Definition 3.3.2.** Let \( A \in \text{Alg}(C) \) be an algebra. We say that \( C \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly if the following conditions hold:

1. \( C \) admits realizations of the bar complexes \( \text{Bar}_A(M, N) \).
2. For every \( X, Y \in C \) the canonical map
   \[ |\text{Bar}_A(X \otimes M, N \otimes Y)| \to X \otimes |\text{Bar}_A(M, N)| \otimes Y \]
is an equivalence.

Let \( A \) be a class of algebras in \( C \). We say that \( C \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly if \( C \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly for every \( A \in A \). When \( A \) is the class of all algebras, we simply say that \( C \) admits realizations of bar constructions \( \otimes \)-compatibly.

Dually, let \( A \in \text{coAlg}(C) = \text{Alg}(C^{op})^{op} \) be a coalgebra. We say that \( C \) admits totalizations of \( A \)-cobar constructions \( \otimes \)-compatibly if \( C^{op} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly, and similarly for “admits totalizations of \( A \)-cobar constructions \( \otimes \)-compatibly” and “admits totalizations of cobar constructions \( \otimes \)-compatibly”.

**Proposition 3.3.3** (Refinement of HA.4.4.2.8 (1)). Let

\[ q : C^\circ \to \text{Ass}^\circ \]
be a monoidal \( \infty \)-category (so in particular a coCartesian fibration). Let \( F_0 = A_{M_B N_C} \) be an \( (A, B, C) \)-module in \( C \). Assume \( \text{Bar}_B(M, N) \) admits a geometric realization compatibly with \( \otimes \). Suppose given a commutative diagram of generalized \( \infty \)-operads (solid arrow diagram below)

![Diagram](image-url)

Then there exists an operadic \( q \)-left Kan extension \( F \) of \( F_0 \) (dotted arrow) making the diagram commute.

In the situation of proposition 3.3.3 we say that \( F \) exhibits \( X = F|_{\text{Tens}^\circ_{[1]}} \) as a relative tensor product of \( M \) and \( N \) (see [13] def. 4.4.2.3}).
Proof. By assumption, the underlying bar construction $\text{Bar}_B(M, N)_\bullet$ extends to a colimit diagram

$$\text{Bar}_B(M, N)_\bullet^+ : \Delta_+^{op} \to C.$$ 

Moreover, since the colimit is assumed to be compatible with tensor product, the composite

$$\Delta_+^{op} \xrightarrow{\text{Bar}_B(M, N)_\bullet^+} C \subset C^\otimes$$

is in fact an operadic colimit diagram in $C^\otimes$, see [12, prop. 3.1.1.16]. Denote the colimit by $P \in C$. We now upgrade the augmented underlying bar construction $\text{Bar}_B(M, N)_\bullet$ to a diagram

$$\text{Bar}^\otimes_B(M, N)_\bullet^+ : \Delta_+^{op} \to C^\otimes$$

with cone point $P$, such that

$$\text{Bar}^\otimes_B(M, N)_\bullet^+|_{\Delta_+^{op}} = \text{Bar}_B(M, N)_\bullet^+,$$

along with a natural transformation

$$\beta^+ : \text{Bar}^\otimes_B(M, N)_\bullet^+ \to \text{Bar}_B(M, N)_\bullet^+$$

extending the given natural transformation

$$\beta : \text{Bar}^\otimes_B(M, N)_\bullet \to \text{Bar}_B(M, N)_\bullet.$$ 

The natural transformation $\beta^+$ will be used to verify that $\text{Bar}^\otimes_B(M, N)_\bullet^+$ too is an operadic colimit diagram.

Let $a = \langle 1 \rangle$ denote the unique color of $\text{Ass}^\otimes$. The map

$$\Delta_+^{op} \xrightarrow{\text{bar}^+} \text{Tens}_\otimes \to \text{Ass}^\otimes$$

corresponds to a map

$$\alpha^+_o : \Delta_+^{op} \to \text{Ass}^\otimes / a.$$

Since $a$ is terminal in the overcategory (and since $\text{Ass}^\otimes$ is the nerve of a 1-category), there’s a unique natural transformation

$$\alpha^+ : \Delta_+^{op} \times \Delta^1 \to \text{Ass}^\otimes / a$$

from $\alpha^+_o = \alpha^+|_{\Delta_+^{op} \times \{0\}}$ to the constant diagram $\alpha^+|_{\Delta_+^{op} \times \{1\}}$ with value $a$. Direct verification shows that the square solid-arrow diagram of simplicial sets

$$\Delta_+^{op} \times \{1\} \xrightarrow{\text{Bar}_B(M, N)_\bullet^+} C^\otimes_{/ P}$$

$$\Delta_+^{op} \times \Delta^1 \xrightarrow{(\beta, \alpha^+)} C^\otimes \times_{\text{Ass}^\otimes} \text{Ass}^\otimes / a$$

commutes. By [12] prop. 2.1.2.1, \( \phi \) is a right fibration. By [12] cor. 2.1.2.7 \( \kappa \) is right anodyne. Hence there exists a dotted arrow \( \beta^+ \) as indicated, making both triangles of simplicial sets commute. This completes the construction of the promised natural transformation \( \beta^+ : \Delta^\op \times \Delta^1 \to C^\otimes \).

We now verify that \( \text{Bar}^\otimes_B(M, N) \) is an operadic colimit diagram. By construction, for every vertex \( x \in \Delta^\op \), the restriction \( \beta^+|_{\{x\} \times \Delta^1} = \beta|_{\{x\} \times \Delta^1} \) is a \( q \)-coCartesian edge in \( C^\otimes \). Moreover, \( \beta^+|_{\{-1\} \times \Delta^1} \) is equal to the identity morphism of \( P \), hence is again \( q \)-coCartesian. Hence [13] prop. 3.1.1.15 applies to show that the augmented operadic bar construction \( \text{Bar}^\otimes_B(M, N) \) is an operadic \( q \)-colimit diagram. The theorem follows by applying [13] prop. 4.4.2.5. \( \square \)

**Proposition 3.3.4** (Refinement of Lemma HA.4.4.3.9 (1)). Let \( q : C^\otimes \to \text{Ass}^\otimes \) be a monoidal \( \infty \)-category and let \( A \) be a class of algebra objects of \( C \) closed under equivalences. Assume that \( C \) admits realizations of \( A \)-bar construction \( \otimes \)-compatibly. Let

\[
p : \text{BMod}(C)^\otimes \to \Delta^\op
\]

be the associated Morita pre-double \( \infty \)-category (definition 3.2.1). Then \( \text{BMod}(C)^\otimes \) admits coCartesian lifts from \( A \)-multimodules (definition 3.2.3).

**Proof.** By proposition 3.2.4 it will suffice to show that for every map

\[
\alpha : [n] \to [m]
\]
in \( \Delta \) and every \( A \)-multimodule \( F_0 \in \text{Alg}_{\text{Tens}^\otimes_{[n]} / \text{Ass}^\otimes}(C) \), there exists an operadic \( q \)-left Kan extension \( F \in \text{Alg}_{\text{Tens}_A / \text{Ass}^\otimes}(C) \) of \( F_0 \). For each \( r \geq 0 \), we define

\[
\alpha_i : [r] \to [r + 1]
\]
to be the order-preserving map with image \([r + 1] \setminus \{i\}\). We can factor \( \alpha \) as a composition

\[
[n] \xrightarrow{\beta} [k] \to [k + 1] \to \cdots [m]
\]
where \( \beta \) has convex image and the remaining maps are of the form

\[
\alpha_i : [p - 1] \to [p]
\]
for some \( 0 < i < p \). Using [13] thm. 4.4.3.1 (flatness of \( \text{Tens}^\otimes \to \Delta^\op \)) and [13] thm. 3.1.4.1 (transitivity of operadic left Kan extensions) it’s enough to show that if \( \alpha \) is of either of these forms, then there exists an operadic \( q \)-left Kan extension \( F \) as above such that the associated output multimodule

\[
F_1 : \text{Tens}^\otimes_{[n]} \to \text{Tens}_A^\otimes \xrightarrow{F} \text{C}^\otimes
\]
is again an $A$-multimodule. When $\alpha$ has convex image this follows from [13, lem. 4.4.3.5.(1)]. Indeed, let $f$ denote the map

$$\text{Tens}^\otimes_{[n]} \to \text{Ass}^\otimes.$$ 

In the notation of that lemma, for every object $X \in \text{Tens}_{[n]}$, there exists a map

$$F_0(v_\alpha X) \to C$$

in $C^\otimes$ lying over the map

$$(*) \quad f(v_\alpha X) \to f(X)$$

given by an operadic $q$-colimit diagram $\Delta^1 \to C^\otimes$; this holds trivially since the map $(*)$ is necessarily the identity map of the unique color of $\text{Ass}^\otimes$. Hence [13, lem. 4.4.3.5(1)] implies that there exists an operadic $q$-left Kan extension $F$ of $F_0$. Moreover, by construction, if $F_0$ corresponds to an $A$-multimodule

$$A_0, M^{0,1}_{A_1}, M^{1,2}_{A_2}, \ldots, M^{n-1,n}_{A_n}$$

then $F_1$ corresponds to a two-sided truncation of the above, and hence to an $A$-multimodule.

When $\alpha$ is of the form $\alpha_i$ we construct the desired operadic $q$-left Kan extension $F$ using [13, lem 4.4.3.8]. Conditions (a), (b) and (c) hold trivially (since in the notation of that lemma the maps $f(a_{\alpha(j)}) \to f(b_j), f(m_{j-1,j}) \to f(n_{j-1,j}), f(m_{j,j+1}) \to f(n_{j,j+1})$ are all the identity of the unique object of $\text{Ass}^\otimes_{<1>}$). Condition (d) holds by proposition 3.3.3. Condition (a’) shows that $F_1$ is again an $A$-multimodule.

**Corollary 3.3.5.** In the situation and the notation of proposition 3.3.4, let

$$\text{BMod}_A(C)^\otimes \subset \text{BMod}(C)^\otimes$$

be the full subcategory supported on $A$-multimodules. Assume that $C$ admits realizations of $A$-bar constructions $\otimes$-compatibly. Then the restriction of $p$ to

$$\text{BMod}_A(C)^\otimes \to \Delta^{op}$$

is a coCartesian fibration.

**Proof.** The restriction

$$p' : \text{BMod}_A(C)^\otimes \to \text{BMod}(C)^\otimes \to \Delta^{op}$$

is an inner fibration as a composition of inner fibrations. Let $X \in \text{BMod}_A(C)^\otimes$ and let $\alpha : x \to y$ be a map in $\Delta^{op}$ with $x = p(X)$. By proposition 3.3.4 there exists a $p$-coCartesian lift $f : X \to Y$ of $\alpha$ in $\text{BMod}_A(C)^\otimes$. By [12, prop. 2.4.1.3(3)] the edge $f$ is $p'$-coCartesian. □
3.4. The Morita double ∞-category of A-multimodules.

**Terminology 3.4.1.** We say that a morphism \( \alpha : [n] \to [m] \) in \( \Delta^\text{op} \) is convex if the associated map of totally ordered sets \([n] \leftarrow [m]\) has convex image. Noting that identity morphisms are convex, and that the composite of two convex morphisms is again convex, we define the **convex ordinal category** \( \Delta^\text{op}_{\text{Convex}} \subset \Delta^\text{op} \) to be the subcategory whose objects are the same as those of \( \Delta^\text{op} \) and whose morphisms are the convex morphisms in \( \Delta^\text{op} \).

**Proposition 3.4.2** (or \( \text{BMod}(C)^\otimes \)). Let
\[ q: C^\otimes \to \text{Ass}^\otimes \]
be a monoidal ∞-category and let
\[ p: B^\otimes = \text{BMod}(C)^\otimes \to \Delta^\text{op} \]
be the associated Morita pre-double ∞-category (section 3.2). Then the pullback \( p_{\text{Convex}} \) of \( p \) to \( \Delta^\text{op}_{\text{Convex}} \) as in the following diagram
\[
\begin{array}{ccc}
B^\otimes_{\text{Convex}} & \longrightarrow & B^\otimes \\
\downarrow^{p_{\text{Convex}}} & & \downarrow^{p} \\
\Delta^\text{op}_{\text{Convex}} & \longrightarrow & \Delta^\text{op}
\end{array}
\]
is a coCartesian fibration. Moreover, for every \( n > 1 \), the induced map
\[ B^\otimes_{[n]} \to B^\otimes_{[1]} \times B^\otimes_{[0]} \times \cdots \times B^\otimes_{[0]} B^\otimes_{[1]} \]
is an equivalence of ∞-categories.

**Proof.** Let \( \alpha : [m] \to [n] \) in \( \Delta^\text{op} \) be convex, and let \( p(X_0) = [m] \). Together, \( \alpha \) and \( X_0 \) give rise to a commuting square (solid arrow diagram)
\[
\begin{array}{ccc}
\text{Tens}_{[m]}^\otimes & \longrightarrow & C^\otimes \\
\downarrow^{X_0} & & \downarrow^{X} \\
\text{Tens}_{\alpha}^\otimes & \longrightarrow & \text{Ass}^\otimes
\end{array}
\]
The condition of [13 lem. 4.4.3.5(1)] applies trivially to show that there exists an operadic \( q \)-left Kan extension \( X \) of \( X_0 \) as indicated. By proposition [3.2.4], \( X \) corresponds to a coCartesian edge in \( B^\otimes \) with source \( X_0 \) lifting \( \alpha \). By [12, prop. 2.4.1.3(2)] it follows that \( p_{\text{Convex}} \) is a coCartesian fibration. Using [13 prop. 4.4.1.11] (Segal condition for \( \text{Tens}^\otimes \)), we see that for each \( n \geq 0 \), the inclusions \([1] \simeq \{i - 1, i\} \leftrightarrow [n]\) induce an equivalence of ∞-categories as stated. \( \square \)
**Terminology 3.4.3.** We say a class $\mathcal{A}$ of algebras in a monoidal $\infty$-category $\mathcal{C}$ is **capable of relative tensor products** if $\mathcal{A}$ is closed under equivalences and if $\mathcal{C}$ admits geometric realizations of $\mathcal{A}$-bar constructions $\otimes$-compatibly.

**Definition 3.4.4** ([9, def. 2.1.4]). A double $\infty$-category is a *coCartesian fibration* $\mathcal{M} \to \Delta^{op}$ such that for all $n > 1$, the induced functor of $\infty$-categories $\mathcal{M}_{[n]} \to \mathcal{M}_{[1]} \times \mathcal{M}_{[0]} \times \cdots \times \mathcal{M}_{[0]} \times \mathcal{M}_{[1]}$ is an equivalence.

**Corollary 3.4.5.** Let $q: \mathcal{C}^\otimes \to \text{Ass}^\otimes$ be a monoidal $\infty$-category, and let $\mathcal{A}$ be a class of algebras in $\mathcal{C}$ which is capable of relative tensor products. Let

$$\text{BMod}_\mathcal{A}(\mathcal{C})^\otimes \subset \text{BMod}(\mathcal{C})^\otimes$$

denote the full subcategory supported on the class of $\mathcal{A}$-multimodules. Then

$$p_A : \text{BMod}_\mathcal{A}(\mathcal{C})^\otimes \to \Delta^{op}$$

is a double $\infty$-category.

**Proof.** According to corollary 3.3.5, $p_A$ is a coCartesian fibration. According to proposition 3.4.2, $p_A$ satisfies the Segal condition. □

In the situation of the above corollary, we refer to $\text{BMod}_\mathcal{A}(\mathcal{C})^\otimes$ as the *Morita double $\infty$-category of $\mathcal{C}$ over $\mathcal{A}$*. We end this section with a remark which, we hope, will help to clarify its structure.

**Remark 3.4.6.** Let $q: \mathcal{C}^\otimes \to \text{Ass}^\otimes$ be a monoidal $\infty$-category, and assume for simplicity that $\mathcal{C}$ is capable of relative tensor products so that

$$p : \text{BMod}(\mathcal{C})^\otimes \to \Delta^{op}$$

is a double $\infty$-category and if

$$\alpha : [m] \to [n]$$

is a morphism in $\Delta^{op}$, we have an induced functor

$$\alpha^! : \text{BMod}(\mathcal{C})^\otimes_{[m]} \to \text{BMod}(\mathcal{C})^\otimes_{[n]}.$$
(in which \( f \) is the composite \( \text{Tens}_\alpha^\otimes \to \text{Tens}^\otimes \to \text{Ass}^\otimes \)) is an operadic \( q \)-left Kan extension, then
\[
\alpha_1 F = F_\rightarrow \circ \iota_1.
\]

Two cases are of particular interest.

1. Suppose the morphism \( \alpha^{op} : [m] \leftarrow [n] \) in \( \Delta \) has convex image. Then \( \iota_0 \) admits a natural section \( V_\alpha \) and
\[
F_\rightarrow = F \circ V_\alpha
\]
is an operadic \( q \)-left Kan extension. Moreover,
\[
V_\alpha \circ \iota_1 = v_\alpha
\]
is given by composition with \( \alpha \). It follows that the \( n \)-module \( \alpha_1 F \) is a projection of the \( m \)-module \( F \) onto those components singled out by \( \alpha \).

2. Suppose \( n = m - 1 \) and \( \alpha^{op} : [m] \leftarrow [n] \) is a face map. Then the operadic \( q \)-left Kan extension \( F_\rightarrow \) is constructed via an associated operadic \( q \)-left Kan extension along
\[
\text{Tens}^\otimes_{[2]} \to \text{Tens}^\otimes
\]
which defines the relative tensor product. In this case, the \( n \)-module \( \alpha_1 F \) is given by a relative tensor product of neighboring bimodules in the \( m \)-module \( F \) as indicated by \( \alpha \).

To assemble these two building blocks into a natural formula for general \( \alpha \), it’s useful to change notation. We denote the colors of \( \text{Tens}^\otimes_{[k]} \) by
\[
f_i = f_{i,i} = a_i \quad \text{and} \quad f_{ij} = m_{ij}.
\]

If \( F : \text{Tens}^\otimes_{[k]} \to C^\otimes \) is a \( k \)-module in the monoidal \( \infty \)-category \( C \), we denote the associated algebras and bimodules by
\[
F_{ij} = F(f_{ij}) \quad \text{and} \quad F_i = F(f_i).
\]

Thus, for each \( i = 1, \ldots, k \), \( F_{i-1,i} \) is an \( F_{i-1}-F_i \)-bimodule. Also, if \( \alpha : [m] \to [n] \) is a morphism in \( \Delta^{op} \) as above, and \( i \in [n] = \{0, \ldots, n\} \), we denote \( \alpha^{op}(i) \) by \( \alpha_i \). In these notations, we have for \( i = 0, \ldots, m \),
\[
(\alpha_1 F)_i = F_{\alpha_i},
\]
and for \( 0 \leq i < i + 1 = j \leq m \),
\[
(\alpha_1 F)_{ij} = F_{\alpha_i,\alpha_i+1} \otimes_{F_{\alpha_i+1}} F_{\alpha_i+1,\alpha_i+2} \otimes_{F_{\alpha_i+2}} \cdots \otimes_{F_{\alpha_j-1}} F_{\alpha_j-1,\alpha_j}.
\]
4. ACTION OF ENDOMORPHISM CATEGORIES ON MORPHISM CATEGORIES IN A DOUBLE $\infty$-CATEGORY

In this section we construct the $\mathcal{LM}$-monoidal $\infty$-category $\text{BLMod}_A(C)$ (see construction 4.3.2), which exhibits the $\infty$-category of left $A$-modules as left tensored over $A$-bimodules. We assume only that $C$ admits bar constructions over $A$ which are compatible with tensor product. In particular, we do not assume that $C$ admits geometric realizations in general. As explained in the introduction, this level of generality is needed in order to handle relative tensor products in the twisted arrow category (see section 8).

In fact, our construction takes place in the general setting of a double $\infty$-category. If $M$ is a double $\infty$-category and $A$, $B$ are objects, we define the $\infty$-category $\text{Map}_{M}^h(B, A)$ of horizontal morphisms from $B$ to $A$ in $M$ in definition 4.1.4. We then construct an $\mathcal{LM}$-monoidal category $L\text{Map}_{M}^h(B, A)^{\otimes}$ which exhibits the $\infty$-category $\text{Map}_{M}^h(A, B)$ as left-tensored over the monoidal $\infty$-category $L\text{Map}_{M}^h(A, A)^{\otimes}$ (construction 4.2.7). We then obtain the desired action of bimodules on left modules by applying this construction to the double $\infty$-category $\text{BMod}_A(C)^{\otimes} \to \Delta^{op}$ of lemma 3.2.2.

4.1. The $\infty$-category of horizontal morphisms.

Definition 4.1.1. For $\mathcal{Y}$ an $\infty$-category we let $(\text{Set}_{\Delta}^+)_{/\mathcal{Y}}$ denote the simplicial model category of marked simplicial sets over $\mathcal{Y}$ with coCartesian model structure ([12, prop. 3.1.3.7, rem. 3.1.3.9, cor. 3.1.4.4]). We let $\text{Cat}_{\text{coCart}}^{\mathcal{Y}}$ be the simplicial nerve $N((\text{Set}_{\Delta}^+)_{/\mathcal{Y}}^\lozenge)$ of the full subcategory $(\text{Set}_{\Delta}^+)_{/\mathcal{Y}}$ of fibrant-cofibrant objects.

Remark 4.1.2. There is a natural bijection between vertices of $\text{Cat}_{\text{coCart}}^{\mathcal{Y}}$ and coCartesian fibrations of simplicial sets $\mathcal{X} \to \mathcal{Y}$. Given two such, $\mathcal{X}$, $\mathcal{X}'$, the set of edges $\mathcal{X} \to \mathcal{X}'$ in $\text{Cat}_{\text{coCart}}^{\mathcal{Y}}$ is in canonical bijection with the set of morphisms of simplicial sets over $\mathcal{Y}$ which send coCartesian edges to coCartesian edges.

Remark 4.1.3. The forgetful functor

$$\text{Cat}_{\text{coCart}}^{\mathcal{Y}} \to \text{Cat}_{\infty}$$

is a right adjoint, hence preserves limits. To see this, use marked simplicial sets to model $\text{Cat}_{\infty}$ ([13, rem. 2.1.4.10]), and apply [14, cor. 6.2.1] with $\pi : \mathcal{Y} \to \ast$ and $\rho = \text{id}_{\ast}$ to construct a right Quillen functor

$$\text{Set}_{\Delta}^{+}_{/\mathcal{Y}} \to \text{Set}_{\Delta}^{+}_{/\ast}$$

\[\text{We are indebted to Dan Petersen for suggesting that we work in the setting of double } \infty \text{-categories.}\]
modeling the forgetful functor.

**Definition 4.1.4.** Let $\mathcal{M}$ be a double $\infty$-category (definition 3.4.4). Given objects $A, B \in \mathcal{M}_{[0]}$, we define the $\infty$-category of horizontal morphisms $B \to A$ in $\mathcal{M}$, $\text{Map}_{\mathcal{M}}^{h}(B, A)$, by the pullback diagram

$$
\begin{array}{ccc}
\text{Map}_{\mathcal{M}}^{h}(B, A) & \longrightarrow & \mathcal{M}_{[1]} \\
\downarrow & & \downarrow \delta_0 \times \delta_1 \\
\{\ast\} & \longrightarrow & \mathcal{M}_{[0]} \times \mathcal{M}_{[0]}
\end{array}
$$

in $\text{Cat}_{\infty}$, where the $\delta_i$ are induced by the two face maps $[0] \to [1]$.

**Example 4.1.5.** Let $\mathcal{C}^{\otimes} \to \text{Ass}^{\otimes}$ be a monoidal $\infty$-category and let $\mathcal{A}$ be a class of algebras in $\mathcal{C}$ capable of relative tensor products (terminology 3.4.3); we denote the associated full subcategory of $\text{Alg} \mathcal{C}$ by $\mathcal{A}$. Then

$$
\text{BMod}(\mathcal{C})^{\otimes}_{[0]} = \mathcal{A}
$$

and given $A, B \in \mathcal{A}$ we have equivalences of $\infty$-categories

$$
\text{Map}_{\text{BMod}(\mathcal{C})}^{h}(B, A) \simeq \text{A BMod}_{B}(\mathcal{C}).
$$

4.2. **Left actions of horizontal endomorphisms.** In construction 4.2.7 we will endow $\text{Map}_{\mathcal{M}}^{h}(B, A)$ with a left-action of $\text{Map}_{\mathcal{M}}^{h}(A, A)$. The monoidal structure of $\text{Map}_{\mathcal{M}}^{h}(A, A)$ is constructed in [9, def. 3.4.2 and proof of prop. 3.4.8]. We recall its construction after fixing some relevant notation.

**Notation 4.2.1.** Let $\mathcal{X}_{\bullet}$ be a simplicial object in an $\infty$-category $\mathcal{C}$ admitting limits. Recall that the 0-coskeleton $\text{cosk}_0 \mathcal{X}_{\bullet}$ is the simplicial object given by the right Kan extension

$$
\begin{array}{ccc}
\{\ast\} & \xrightarrow{\mathcal{X}_0} & \mathcal{C} \\
\cap & \xrightarrow{\text{cosk}_0 \mathcal{X}_{\bullet}} & \Delta^{op}
\end{array}
$$

(see, e.g.,[12, not. 6.5.3.1]). In particular this applies to a diagram of $\infty$-categories $\mathcal{X}_{\bullet} : \Delta^{op} \to \text{Cat}_{\infty}$.

We now use straightening/unstraightening ([12, thm. 3.2.0.1]) to define the 0-coskeleton of a coCartesian fibration over $\Delta^{op}$. Given a coCartesian fibration

$$
\mathcal{X} \to \Delta^{op}
$$

we let $\mathcal{X}_{\bullet} : \Delta^{op} \to \text{Cat}_{\infty}$ denote its straightening. Define the coCartesian fibration

$$
\text{cosk}_0 \mathcal{X} \to \Delta^{op}
$$
as the un-straightening of \( \cosk_0 \mathcal{X} : \Delta^{op} \to \text{Cat}_\infty \). Note that

\[
(\cosk_0 \mathcal{X})_{[n]} \simeq \mathcal{X}_{[0]} \times \cdots \times \mathcal{X}_{[0]}
\]

\((n + 1 \text{ copies})\).

**Construction 4.2.2** ([9, def. 3.4.2]). Let \( \mathcal{M} \to \Delta^{op} \) be a double \( \infty \)-category (definition 3.4.4). By construction, \( \cosk_0 \mathcal{M} \) comes equipped with a canonical morphism of coCartesian fibrations over \( \Delta^{op} \)

\[
\kappa : \mathcal{M} \to \cosk_0 \mathcal{M}
\]

in which the map of fibers over \([n] \in \Delta^{op}\) is the map

\[
\mathcal{M}_{[n]} \to \mathcal{M}_{[0]} \times \cdots \times \mathcal{M}_{[0]}
\]

induced by the \( n + 1 \) face maps \([n] \leftarrow [0]\) in \( \Delta \). Given \( A \in \mathcal{M}_{[0]} \), we construct a section \( A^* : \Delta^{op} \to \cosk_0 \mathcal{M} \) which sends

\[
\Delta^{op} \ni [n] \mapsto (A, \ldots, A) \in (\cosk_0 \mathcal{M})_{[n]}
\]

as follows. Let \( \text{Const}(\ast) \) denote the constant simplicial \( \infty \)-category

\[
\text{Const}(\ast) : \Delta^{op} \to \text{Cat}_\infty
\]

with value \( \ast \in \text{Cat}_\infty \), and let \( \iota_{[0]} \) denote the functor \( \ast \to \Delta^{op} \) classified by \([0] \in \Delta^{op}\). The inclusion \( A \in \mathcal{M}_{[0]} \) gives rise to a natural transformation

\[
\text{Const}(\ast) \cdot \iota_{[0]} \to \mathcal{M}_{[0]}
\]

of functors \( \ast \to \text{Cat}_\infty \), hence to a natural transformation

\[
\text{Const}(\ast) \to \cosk_0 \mathcal{M}_\ast
\]

of functors \( \Delta^{op} \to \text{Cat}_\infty \), since the right Kan extension \( \cosk_0 \) is right adjoint to

\[
\iota_{[0]}^* : \text{Fun}(\ast, \text{Cat}_\infty) \to \text{Fun}(\Delta^{op}, \text{Cat}_\infty).
\]

Unstraightening defines a section \( A^* \) as hoped.

We define \( \text{End}^h_{\mathcal{M}}(A)^{\otimes} \) by the pullback diagram

\[
\begin{array}{ccc}
\text{End}^h_{\mathcal{M}}(A)^{\otimes} & \rightarrow & \mathcal{M} \\
\downarrow^{\kappa_{\ast}} & & \downarrow^{\kappa} \\
\Delta^{op} & \xrightarrow{A^*} & \cosk_0 \mathcal{M}.
\end{array}
\]

in the category \( \text{Cat}_{\text{coCart}/\Delta^{op}} \) of definition 4.1.1. The associated simplicial \( \infty \)-category

\[
\text{End}^h_{\mathcal{M}}(A)^{\otimes} : \Delta^{op} \to \text{Cat}_\infty
\]
is a monoid object in the sense of [13] def. 4.1.2.5. Hence by [13] rem. 4.1.2.6, prop. 2.4.2.5 and rem. 2.4.2.6 we obtain a monoidal ∞-category
\[
\Map^h_M(A, A)^\otimes \to \Ass^\otimes
\]
with underlying ∞-category equivalent to \(\Map^h_M(A, A)\).

**Example 4.2.3.** Let \(C^\otimes \to \Ass^\otimes\) be a monoidal ∞-category and let \(A\) be a class of algebras in \(C\) capable of relative tensor products (terminology [3.4.3]). If \(\mathcal{M} = \BMod_A(C)^\otimes\) is the Morita double ∞-category of \(A\)-multimodules constructed in corollary [3.4.5] then the map \(\kappa\) of construction [4.2.2] diagram (*) corresponds via straightening to a morphism of coherently commutative simplicial ∞-categories given on the level of objects and 1-morphisms by a diagram like so, in which \(A\) is regarded as a full subcategory of \(\text{Alg} C\) (we omit degeneracy maps):

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\BMod_A(C) \times _A BMod_A(C) & \longrightarrow & A \times A \times A \\
\downarrow & & \downarrow \\
BMod_A(C) & \longrightarrow & A \times A \\
\downarrow & & \downarrow \\
\mathcal{A} & \longrightarrow & \mathcal{A}.
\end{array}
\]

Given \(A \in \mathcal{A}\) then, \(\text{End}^h_{\BMod_A(C)^\otimes}(A)\) corresponds to a coherently commutative simplicial ∞-category given on the level of objects and 1-morphisms by

\[
\begin{array}{ccc}
\vdots & \\
\overset{A}{\BMod_A(C)} \times _A \BMod_A(C) & \\
\downarrow & \\
\overset{A}{\BMod_A(C)} & \\
\downarrow & \\
\{\ast\}.
\end{array}
\]

We now turn to the construction of the action of the ∞-category \(\Map^h_M(A, A)\) of horizontal endomorphisms of \(A\) in \(\mathcal{M}\) on the ∞-category \(\Map^h_M(B, A)\) of horizontal
morphisms from $B$ to $A$. We first import the classical construction of the décalage of a simplicial set to the context of coCartesian fibrations over $\Delta^\text{op}$.

**Construction 4.2.4.** Let $+^\text{op}$ denote the functor $\Delta \rightarrow \Delta$ defined on objects by

$$S \mapsto S_+ := S \amalg \{\infty\}$$

and on morphisms in the obvious way. Let $+$ denote the induced functor $\Delta^{\text{op}} \rightarrow \Delta^{\text{op}}$. Suppose $\mathcal{Y} \rightarrow \Delta^{\text{op}}$ is a coCartesian fibration. Then the décalage $\text{Dec} \mathcal{Y} \rightarrow \Delta^{\text{op}}$ of $\mathcal{Y}$ is the coCartesian fibration obtained from $\mathcal{Y}$ via pullback along the functor $+$. We let $\check{\mathcal{Y}} := \Delta^{\text{op}} \times \Delta^1$. Consider the two functors of 1-categories

$$\Delta^{\text{op}} \xrightarrow{+} \Delta^{\text{op}} \xleftarrow{\text{Id}}$$

The natural inclusions $S \subset S_+$ give rise to a natural transformation $+ \Rightarrow \text{Id}$, hence to a functor

$$\delta: \check{\mathcal{Y}} \rightarrow \Delta^{\text{op}}.$$ 

Similarly, let $\epsilon^{\text{op}}$ denote the composite functor $\Delta \rightarrow \{\ast\} \rightarrow \Delta$ sending all objects to a one-element totally ordered set, and let $\epsilon$ denote the induced functor on opposite categories. The maps of sets $\{\ast\} \rightarrow S_+$ sending $\ast \mapsto \infty$ give rise to a natural transformation $+ \Rightarrow \epsilon$, hence to a functor

$$\pi: \check{\mathcal{Y}} = \Delta^{\text{op}} \times \Delta^1 \rightarrow \Delta^{\text{op}}.$$ 

Returning to the coCartesian fibration $\mathcal{Y} \rightarrow \Delta^{\text{op}}$, the pullbacks

$$\delta^* \mathcal{Y} \rightarrow \check{\mathcal{Y}}, \quad \pi^* \mathcal{Y} \rightarrow \check{\mathcal{Y}}$$

correspond to morphisms

$$\text{Dec} \mathcal{Y} \rightarrow \mathcal{Y}$$

$$\mathcal{Y}|_0 \times \Delta^{\text{op}}$$

of coCartesian fibrations over $\Delta^{\text{op}}$.

**Proposition 4.2.5.** Let $\mathcal{Y} \rightarrow \Delta^{\text{op}}$ be a coCartesian fibration of simplicial sets. Then the décalage of $\text{cosk}_0 \mathcal{Y}$ decomposes as a product

$$\text{Dec} \text{cosk}_0 \mathcal{Y} \simeq (\text{cosk}_0 \mathcal{Y}) \times_{\Delta^{\text{op}}} (\mathcal{Y}|_0 \times \Delta^{\text{op}})$$

of coCartesian fibrations over $\Delta^{\text{op}}$, where $\mathcal{Y}|_0 \times \Delta^{\text{op}}$ denotes the constant fibration with fiber $\mathcal{Y}|_0$. 29
Proof. Applying construction 4.2.4 to \( \text{cosk}_0 \mathcal{Y} \), and noting that

\[
(\text{cosk}_0 \mathcal{Y})_{[0]} = \mathcal{Y}_{[0]},
\]

we obtain a map \( \psi \) from left to right. Additionally, taking fibers over \([n] \in \Delta\), we find that both sides are equivalent to \((\mathcal{Y}_{[0]})^{\times (n+2)}\) compatibly with \(\psi\). It follows that \(\psi\) is an equivalence. \(\square\)

Example 4.2.6. We apply construction 4.2.4 and proposition 4.2.5 to the Morita double \(\infty\)-category \(\text{BMod}_A(C)^{\otimes} \to \Delta^{op}\) of corollary 3.4.5. The morphism of coCartesian fibrations

\[
\text{Dec} \text{BMod}_A(C)^{\otimes} \to \text{BMod}_A(C)^{\otimes}
\]
corresponds via straightening to a diagram of \(\infty\)-categories

\[
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\text{BMod}_A(C) \times_A \text{BMod}_A(C) & \longrightarrow & \text{BMod}_A(C) \times_A \text{BMod}_A(C)
\end{array}
\]

\[
(D)
\begin{array}{ccc}
\downarrow & & \downarrow \\
\text{BMod}_A(C) \times_A \text{BMod}_A(C) & \longrightarrow & \text{BMod}_A(C)
\end{array}
\]

\[
\text{BMod}_A(C) \longrightarrow A
\]

in which the horizontal maps forget the rightmost algebra as well as the rightmost bimodule. The morphism of coCartesian fibrations

\[
\text{Dec} \text{cosk}_0 \text{BMod}_A(C)^{\otimes} \to \text{cosk}_0 \text{BMod}_A(C)^{\otimes}
\]
corresponds via straightening to a diagram of \(\infty\)-categories

\[
\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathcal{A} \times \mathcal{A} \times \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \times \mathcal{A} \times \mathcal{A}
\end{array}
\]

\[
(DC)
\begin{array}{ccc}
\downarrow & & \downarrow \\
\mathcal{A} \times \mathcal{A} \times \mathcal{A} & \longrightarrow & \mathcal{A} \times \mathcal{A}
\end{array}
\]

\[
\text{B} \times \mathcal{A} \longrightarrow \mathcal{A}
\]
The proof of proposition 4.2.5 shows that, regarded as a morphism of simplicial objects, this is just projection from a product.

We are now ready to construct the left action of $\text{Map}_h^M(A, A)$ on $\text{Map}_h^M(B, A)$.

**Construction 4.2.7.** Fix $M \to \Delta^{op}$ a double $\infty$-category and two objects $A, B \in M_0$ as in definition 4.1.4. Let $A^* : \Delta^{op} \to \cosk_0 M$ denote the section of $\cosk_0 M \to \Delta^{op}$ of construction 4.2.2. Combining $A^*$ with the constant section of $M_0 \times \Delta^{op} \to \Delta^{op}$ determined by $B$, we obtain a section of $(\cosk_0 M) \times \Delta^{op} (M_0 \times \Delta^{op}) \to \Delta^{op}$.

Composing with the equivalence of proposition 4.2.5, we obtain a section $(A^*, B)$ of $\text{Dec cosk}_0 M \to \Delta^{op}$.

Noting the Cartesian squares (solid arrow diagram)

\[
\begin{array}{ccc}
\Delta^{op} \times \{0\} & \xrightarrow{\delta^*} & \Delta^{op} \times \Delta^1 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\Delta^{op} \times \{0\} & \to & \Delta^{op} \\
\end{array}
\]

we obtain a section (denoted $A^*B \setminus A^*$) of the $\Delta^1$-family of coCartesian fibrations $\delta^* \cosk_0 M$ over $\Delta^{op}$ with fibers $(A^*, B)$ over $\Delta^{op} \times \{0\}$ (left-hand dotted arrow) and $A^*$ over $\Delta^{op} \times \{1\}$ (right-hand dotted arrow = pullback of $A^*B \setminus A^*$ along the inclusion $\Delta^{op} \times \{1\} \subset \Delta^{op} \times \Delta^1 = \vec{\nabla}$). We define $\text{LMap}_h^M(B, A)^\otimes$ by the pullback diagram in $\text{Cat}_{\text{coCart}/\vec{\nabla}}$ (definition 4.1.1)

\[
\begin{array}{ccc}
\text{LMap}_h^M(B, A)^\otimes & \xrightarrow{\delta^*} & \delta^* M \\
\downarrow \quad \downarrow \quad \downarrow \\
\vec{\nabla} & \to & \delta^* \cosk_0 M.
\end{array}
\]

The associated natural transformation in $\text{Fun}(\Delta^{op}, \text{Cat}_\infty)$ is a left-action object of $\text{Cat}_\infty$ in the sense of [13, def. 4.2.2.2].

The construction of [13, prop. 4.2.2.9] provides us with an equivalence of $\infty$-categories

$$\text{Mon}_{\Lambda M}(\text{Cat}_\infty) \to \text{LMon}(\text{Cat}_\infty)$$
from \(\mathcal{LM}\)-monoid objects in \(\mathcal{C}_{\infty}\) to left action objects in \(\mathcal{C}_{\infty}\). According to [13, prop. 2.4.2.5 and rem. 2.4.2.6], \(\mathcal{LM}\)-monoid objects of \(\mathcal{C}_{\infty}\) can be identified with \(\mathcal{LM}\)-algebra objects of \(\mathcal{C}_{\infty}\), which, in turn, may be identified with \(\mathcal{LM}\)-monoidal categories. Applying these constructions consecutively to the left action object of \(\mathcal{C}_{\infty}\) constructed above, we obtain an \(\mathcal{LM}\)-monoidal category

\[
LMap^h_{\mathcal{M}}(B, A)^\otimes \to \mathcal{LM}^\otimes
\]

which exhibits \(\text{Map}^h_{\mathcal{M}}(B, A)\) as left-tensored over \(\text{Map}^h_{\mathcal{M}}(A, A)^\otimes\).

4.3. The \(\mathcal{LM}\)-monoidal category of bimodules and left modules.

**Example 4.3.1.** Applying construction 4.2.7 to the double \(\infty\)-category \(\text{BMod}_A(\mathcal{C})^\otimes\), we find that the straightening of \(\delta^* \text{BMod}_A(\mathcal{C})^\otimes\) corresponds to a diagram of \(\infty\)-categories as in example 4.2.6(D), that \(\delta^* \cosk_0 \text{BMod}_A(\mathcal{C})^\otimes\) corresponds to a diagram of \(\infty\)-categories as in example 4.2.6(DC), and so the map \(\delta^* \kappa\) corresponds to a morphism of diagrams

\[
\text{example 4.2.6(D)} \to \text{example 4.2.6(DC)}.
\]

Thus, if \(\mathcal{A}\) is a class of algebras in \(\mathcal{C}\) capable of relative tensor products and \(A, B \in \mathcal{A}\) then \(LMap^h_{\text{BMod}_A(\mathcal{C})^\otimes}(B, A)^\otimes\) corresponds to a diagram like so:

\[
\begin{array}{ccc}
A\text{BMod}_A(\mathcal{C}) \times A\text{BMod}_B(\mathcal{C}) & \to & A\text{BMod}_A(\mathcal{C}) \\
\downarrow \downarrow \downarrow \downarrow & & \downarrow \downarrow \downarrow \\
A\text{BMod}_B(\mathcal{C}) & \to & A\text{BMod}_A(\mathcal{C})
\end{array}
\]

in which the horizontal maps are projections.

**Construction 4.3.2.** Let \(\mathcal{C}^\otimes \to \text{Ass}^\otimes\) be a monoidal category and let \(A \in \text{Alg}(\mathcal{C})\) be an algebra object. Assume \(\mathcal{C}\) admits realizations of \(A\)-bar construction \(\otimes\)-compatibly. Let \(\mathcal{A}\) denote the class of those algebras which are equivalent to \(A\) or to \(1\). Then \(\mathcal{A}\) is capable of relative tensor products (terminology 3.4.3), so that by corollary 3.4.5, \(\text{BMod}_A(\mathcal{C})^\otimes\) is a double \(\infty\)-category. We define

\[
\text{BMod}_A(\mathcal{C})^\otimes \to \mathcal{LM}^\otimes
\]
to be the $\mathcal{LM}$-monoidal category $\text{LMap}_M^A(1, A)^\otimes$ of construction 4.2.7 applied to $\mathcal{M} = \text{BMod}_A(C)^\otimes$.

**Remark 4.3.3.** The $\mathcal{LM}$-monoidal category

$$\text{BMod}_A(C)^\otimes \to \mathcal{LM}^\otimes$$

of construction 4.3.2 exhibits $\text{LMod}_A(C)$ as left tensored over $\text{BMod}_A(C)^\otimes$. Moreover, unwinding definitions, we find that the induced tensor product functor is given by the relative tensor product

$$\otimes_A : \text{BMod}_A(C) \times \text{LMod}_A(C) \to \text{LMod}_A(C),$$

$$(M, X) \mapsto M \otimes_A X := |(M \otimes A \otimes \cdots \otimes A) \otimes X|.$$

5. **External relative tensor products in $\mathcal{BM}^\otimes$-monoidal categories**

Let $\mathcal{C}^\otimes \to \mathcal{LM}^\otimes$ be an $\mathcal{LM}$-monoidal $\infty$-category exhibiting $\mathcal{C}_m$ as left-tensored over $\mathcal{C}_a$ and $A \in \text{Alg}(\mathcal{C}_a)$ an algebra. Our goal in this section is to construct an $\mathcal{LM}$-monoidal $\infty$-category $\text{BMod}_A(C)^\otimes \to \mathcal{LM}^\otimes$ exhibiting $\text{LMod}_A(\mathcal{C}_m)$ as left-tensored over $\text{BMod}_A(\mathcal{C}_a)$, see construction 5.4.5. In order to leverage the results of [13, sec. 4.4.1] we consider the slightly more general context of $\mathcal{BM}$-monoidal $\infty$-categories. Example 5.1.6 will allow us to apply our $\mathcal{BM}$-monoidal constructions to $\mathcal{LM}$-monoidal categories. The reader who wishes to move on to the central constructions of Koszul duality and is willing to accept the action of bimodules on left modules on faith, will only need definition 5.3.15 (compatibility of tensor product with $A$-bar constructions) for the sequel.

5.1. **Promoting $\mathcal{LM}$-monoidal $\infty$-categories to $\mathcal{BM}$-monoidal $\infty$-categories.**

We first recall the relevant notation from [13, sec. 4.3.1].

**Notation 5.1.1.** Let $\mathcal{BM}^\otimes$ be the $\infty$-operad of [13, def. 4.3.1.6]. We can describe $\mathcal{BM}^\otimes$ as the nerve of the following category (see [13, ex. 4.4.1.7]):

1. An object of $\mathcal{BM}^\otimes$ is a tuple $\langle (n), c_-, c_+ \rangle$ where $\langle n \rangle \in \text{Fin}$, is a finite pointed set and $c_-, c_+: \langle n \rangle^\circ \to [1]$ are a pair of maps satisfying $c_-(i) \leq c_+(i)$ for $i \in \langle n \rangle^\circ$.

2. Let $\langle (n), c_-, c_+ \rangle$ and $\langle (n'), c'_-, c'_+ \rangle$ be objects of $\mathcal{BM}^\otimes$. A morphism from $\langle (n), c_-, c_+ \rangle$ to $\langle (n'), c'_-, c'_+ \rangle$ consists of a morphism $\alpha: \langle n \rangle \to \langle n' \rangle$ in $\text{Ass}^\otimes$ satisfying the following conditions: for each $j \in \langle n' \rangle^\circ$ with fiber $\alpha^{-1}(j) = \{i_0 \prec \cdots \prec i_m\}$:

$$c_-(i_0) = c'_-(j), \quad c_+(i_m) = c'_+(j),$$

$$c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = c_-(i_2) \leq \cdots \leq c_+(i_m-1) = c_-(i_m) \leq c_+(i_m).$$

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The $\infty$-operad $BM^\otimes$ has three colors, given by objects $(\langle 1 \rangle, c_-, c_+) \in BM_{<1>}^\otimes$ with
\[ c_0 = c_1 = 0, \quad c_- = 0, c_+ = 1, \quad c_- = c_+ = 1. \]
We refer to these colors as $a_-, m$ and $a_+$ respectively.

**Definition 5.1.2.** A $BM$-monoidal $\infty$-category is by definition a coCartesian fibration $C^\otimes \rightarrow BM^\otimes$. We let $C_- = C_0, C_m, C_+ = C_1$ denote the fibers over $a_-, m, a_+$ respectively, so that $C^\otimes$ exhibits $C_m$ as bitensored over $C_-$ and $C_+$ ([13] def. 4.3.1.17).

The following statement allows us to promote an $LM$-monoidal $\infty$-category to a $BM$-monoidal $\infty$-category.

**Notation 5.1.3.** Let $O^\otimes$ be an $\infty$-operad. Following [13, rem. 2.4.2.6], write $Cat^O_{\infty} \subset (Op_{\infty})/O^\otimes$ for the subcategory spanned by $O$-monoidal $\infty$-categories and $O$-monoidal functors.

**Remark 5.1.4.** The terminal $\infty$-operad $Fin_\ast \rightarrow Fin_\ast$ defines a symmetric monoidal structure on the terminal $\infty$-category $\ast \in Cat_{\infty}$. We can therefore consider it as an object of the $\infty$-category of monoidal $\infty$-categories $\ast \in Cat_{\infty}^{Ass}$.

**Proposition 5.1.5.** The forgetful functor induces an equivalence of $\infty$-categories
\[ Cat^{BM}_{\infty} \times Cat^{Ass}_{\infty} \{\ast\} \sim \rightarrow Cat^{LM}_{\infty}. \]

**Proof.** Endow $Cat_{\infty}$ with the Cartesian monoidal structure of [13] prop. 2.4.1.5]. The fiber product $Cat_{\infty} \times_{Fin} BM^\otimes \rightarrow BM^\otimes$ exhibits $Cat_{\infty}$ as bitensored over itself, so we can consider $Cat_{\infty}$ as a $BM$-monoidal $\infty$-category. Applying [13] cor.4.3.2.8] to the $BM$-monoidal category $M = Cat_{\infty}$ and the trivial algebra $B = \ast \in Alg(Cat_{\infty})$ shows that the forgetful functor
\[ BMod(Cat_{\infty}) \times_{Alg(Cat_{\infty})} \{\ast\} \rightarrow LMod(Cat_{\infty}) \]
is an equivalence. The result follows from the equivalences
\[ Cat_{\infty}^{Ass} \simeq Alg(Cat_{\infty}), \quad Cat_{\infty}^{LM} \simeq LMod(Cat_{\infty}), \quad Cat_{\infty}^{BM} \simeq BMod(Cat_{\infty}), \]
which are all cases of the equivalence $Alg_{/O}(Cat_{\infty}) \simeq Cat^O_{\infty}$ of [13] rem. 2.4.2.6].

**Example 5.1.6.** By choosing a homotopy inverse to the equivalence of proposition [5.1.5] we can define a functor
\[ Cat_{\infty}^{LM} \sim \rightarrow Cat_{\infty}^{BM} \times Cat_{\infty}^{Ass} \{\ast\} \rightarrow Cat_{\infty}^{BM} \]
where the first functor is the chosen homotopy inverse and the second is the projection. Informally this functor extends an $LM$-monoidal $\infty$-category $C$ to a $BM$-monoidal $\infty$-category by defining a right action of the terminal category $\ast$ on $C_m$. 

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5.2. The generalized ∞-operad $\text{Tens}_\nabla$. Recall that $\nabla = \Delta^{op} \times \Delta^1$. In this subsection we construct a $\nabla$-family of ∞-operads $\text{Tens}_\nabla \to \nabla \times \text{Fin}_*$ (notation 5.2.5) and a functor $\varphi : \text{Tens}_\nabla \to \mathcal{B}M^\otimes$ (construction 5.2.8), which will be used to construct a $\nabla$-family $\text{BLMod}_{\mathcal{A}}(\mathcal{C})$ associated to a $\mathcal{B}M$-monoidal ∞-category $\mathcal{C}$ in the next subsection.

We recall notation from construction 4.2.4 associated to the décalage construction.

Notation 5.2.1. Let $(-)_+ : \Delta \to \Delta$ be the functor sending a finite totally ordered set $I$ to $I_+ = I \cup \{\infty\}$, where $i < \infty$ for all $i \in I$. The natural inclusion $I \to I_+$ defines a natural transformation $id_\Delta \Rightarrow (\cdot)_+$ of functors $\Delta \to \Delta$, or equivalently a natural transformation $(\cdot)_+ \Rightarrow id_{\Delta^{op}}$ of functors $\Delta^{op} \to \Delta^{op}$. Let $\delta : \nabla \to \Delta^{op}$, $(I, 0) \mapsto [k]_+ \subset [k]$, $(I, 1) \mapsto I$ be the associated functor.

We often denote the objects of $\Delta^{op}$ by $[k] = \{0 < 1 < \cdots < k\}$, where $i < \infty$ for all $i \in I$. The isomorphism $[k]_+ \cong [k_+ + 1]$ identifies $\infty \in [k]_+ \subset [k] = \{0 < 1 < \cdots < k < \infty\}$ with $k + 1 \in [k + 1]$. Given $[k] \in \Delta$ we write

\[0_+ : [k]_+ \to [0]_+ \cong [1], \quad 0_+(i) = \begin{cases} 0 & i \neq \infty \\ 1 & i = \infty \end{cases}\]

for the map induced by applying $(\cdot)_+$ to the zero map $[k] \to [0]$.

Remark 5.2.2. A morphism in $\nabla$ is of one of the forms

\[(\lambda, id_0), (\lambda, id_1), (\lambda, 0 \to 1)\]

where $\lambda : [k'] \to [k] \in \Delta$ is a map of linearly ordered nonempty finite sets. The functor $\delta$ sends $([k], 0)$ to $[k]_+ \cong [k+1]$, sends $([k], 1)$ to $[k]$, and acts on morphisms as follows.

1. $\delta$ sends $(\lambda, id_0)$ to $\lambda : [k'] \to [k] \in \Delta$.
2. $\delta$ sends $(\lambda, id_1)$ to

\[\lambda_+ : [k']_+ \to [k]_+ \in \Delta, \quad \lambda_+(i) = \begin{cases} \lambda(i) & i \neq \infty \\ \infty & i = \infty \end{cases}\]

3. $\delta$ sends $(\lambda, 0 \to 1)$ to $\lambda : [k'] \to [k] \subset [k]_+ \in \Delta$.

The following constructions make specific use of the $\Delta^{op}$-family of ∞-operads $\text{Tens}^\otimes$ from [13] def. 4.4.1.1. We first review its definition.

Definition 5.2.3. Let $\text{Ass}^\otimes$ be the ∞-operad of [13] def. 4.1.1.3. The category $\text{Tens}^\otimes$ is defined as follows:
(1) An object of $\text{Tens}^\otimes$ is a tuple $(\langle n \rangle, [k], c_-, c_+)$ where $\langle n \rangle \in \text{Fin}_*, [k] \in \Delta^{op}$ and the maps $c_-, c_+: \langle n \rangle \to [k]$ satisfy the following condition:

$$\text{for all } i \in \langle n \rangle^\circ : \quad c_-(i) \leq c_+(i) \leq c_--1(i) + 1.$$ 

In other words, either $c_+(i) = c_-(i)$ or $c_+(i) = c_--1(i) + 1$.

(2) Let $(\langle n \rangle, [k], c_-, c_+)$ and $(\langle n' \rangle, [k'], c'_-, c'_+)$ be two objects of $\text{Tens}^\otimes$. A morphism from $(\langle n \rangle, [k], c_-, c_+)$ to $(\langle n' \rangle, [k'], c'_-, c'_+)$ is a tuple $(\alpha, \lambda)$ where

$$\alpha: \langle n \rangle \to \langle n' \rangle \in \text{Ass}^\otimes, \quad \lambda: [k'] \to [k] \in \Delta$$

satisfy the following condition. For every $j \in \langle n' \rangle^\circ$ with totally ordered fiber $\alpha^{-1}(j) = \{i_0 \prec i_1 \prec \cdots \prec i_m\}$ the following equalities hold:

$$c_-(i_0) = \lambda(c'_-(j)), \quad c_+(i_m) = \lambda(c'_+(j)),$$

$$c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = \cdots = c_-(i_m) \leq c_+(i_m).$$

Remark 5.2.4. The category $\vec{\nabla}$ may be depicted as follows

\begin{equation*}
\begin{array}{c}
\vdots \quad \vdots \\
\vdots \quad \vdots \\
(\langle 2 \rangle, 0) \longrightarrow \longrightarrow \longrightarrow (\langle 2 \rangle, 1) \\
\vdots \quad \vdots \\
(\langle 1 \rangle, 0) \longrightarrow \longrightarrow \longrightarrow (\langle 1 \rangle, 1) \\
\downarrow \quad \downarrow \\
(\langle 0 \rangle, 0) \longrightarrow \longrightarrow \longrightarrow (\langle 0 \rangle, 1). \\
\end{array}
\end{equation*}

The functor $\delta: \vec{\nabla} \to \Delta^{op}$ restricts to the identity on the right column, identifies the left column with the full subcategory of $\Delta^{op}$ corresponding to the totally ordered sets $[k], k \geq 1$, and morphisms in $\Delta$ which preserve maximal elements (dashed arrows

---

\[14\] corresponding to $a_{c-(i)}$ of section 3
\[15\] corresponding to $m_{c-(i), c_+(i)}$
and maps the horizontal arrows of $\nabla$ to those face maps which do not preserve maximal elements (squiggly arrows).

The arrow $\tau$ in $\nabla$, which corresponds to the morphisms $\{0\} \to \{0,1\}$ $0 \mapsto 0$ in $\Delta$, $1 \xmapsto{id} 1$ in $\Delta^1$, will induce the external relative tensor product (see definition 5.3.17) $\tau^*: \text{BMod}(C_-) \times_{\text{Alg}(C_-)} \text{BMod}(C_m) \to \text{BMod}(C_m)$ and hence deserves special attention.

**Notation 5.2.5.** Let $\text{Tens}_\nabla$ be the category of [R def. 4.4.1.1]. We denote by $\text{Tens}_\nabla \to \text{BMod}(C_m)$ the pullback of $\text{Tens}_\nabla \to \Delta^ op$ along $\delta$ (see [R not. 4.4.1.15]). By construction, we can identify $\text{Tens}_\nabla |_{\Delta^ op \times \{1\}}$ with $\text{Tens}_\nabla \to \Delta^ op$, and $\text{Tens}_\nabla |_{\Delta^ op \times \{0\}}$ with the pullback of $\text{Tens}_\nabla \to \Delta^ op$ along $(-)_+: \Delta^ op \to \Delta^ op$.

**Remark 5.2.6.** We can unwind the definition of the category $\text{Tens}_\nabla$ as follows.

1. An object of $\text{Tens}_\nabla$ is a tuple $(b, \langle n \rangle, [k], c_-, c_+)$ where $b \in \Delta^1$, $\langle n \rangle \in \text{Fin}_*$, $[k] \in \Delta^ op$ and

   
   $$c_-, c_+: \langle n \rangle^* \to \delta([k], b) = \begin{cases} 
   [k]_+ & b = 0 \\
   [k] & b = 1 
   \end{cases}$$

   are maps satisfying $c_-(i) \leq c_+(i) \leq c_-(i) + 1$ for every $i \in \langle n \rangle^*$.
(2) Let \((b, \langle n \rangle, [k], c_-, c_+)\) and \((b', \langle n' \rangle, [k'], c'_-, c'_+)\) be two objects of \(\text{Tens}_{\nabla}^\otimes\). A morphism from \((b, \langle n \rangle, [k], c_-, c_+)\) to \((b', \langle n' \rangle, [k'], c'_-, c'_+)\) is a tuple \((\beta, \alpha, \lambda)\) where

\[
\beta: b \to b' \in \Delta^1, \quad \alpha: \langle n \rangle \to \langle n' \rangle \in \text{Ass}^\otimes, \quad \lambda: [k'] \to [k] \in \Delta
\]
satisfy the following condition. For every \(j \in \langle n' \rangle^0\) with totally ordered fiber \(\alpha^{-1}(j) = \{i_0 \prec i_1 \prec \cdots \prec i_m\}\) the following equalities hold:

\[
c_-(i_0) = \delta(\lambda, \beta)(c'_-(j)), \quad c_+(i_m) = \delta(\lambda, \beta)(c'_+(j)),
\]

\[
c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = \cdots = c_-(i_m) \leq c_+(i_m),
\]

where in the first two equalities we consider \(\delta(\lambda, \beta)\) as defining a morphism of totally ordered sets \(\delta([k'], b') \to \delta([k], b)\).

Since \(\text{Tens}_{\nabla}^\otimes\) and \(\text{BM}^\otimes\) are nerves of 1-categories we will be able to define a functor \(\text{Tens}_{\nabla}^\otimes \to \text{BM}^\otimes\) by its action on objects and morphisms. In its construction we use a certain natural transformation \(\eta\) from \(\delta\) to the constant functor \(\nabla \to \Delta^\text{op}\) with value \([1]\).

**Construction 5.2.7.** Let \(\delta: \nabla \to \Delta^\text{op}\) be the map of notation \([5.2.1]\). The totally ordered set \(\delta([k], b)\) admits a canonical map to \([1] = \{0 < 1\}\), constructed as follows. Note that \(\delta\) carries the initial object \(([0], 0) \in \Delta^\text{op} \times \Delta^1\) to \([0]_+ \cong [1]\), and hence induces a map on the slice categories

\[
\Delta^\text{op} \times \Delta^1 \cong (\Delta^\text{op} \times \Delta^1)_{([0], 0)/} \xrightarrow{\delta} (\Delta^\text{op})_{[1]/} \cong (\Delta_{[1]})^\text{op}.
\]

Given \(([k], b) \in \nabla\) on the left, we define

\[
\eta: \delta([k], b) \to [1]
\]
to be the image of \(([k], b)\) in \((\Delta_{[1]})^\text{op}\) on the right. Explicitly the map \(\eta\) is given by

\[
\eta: \delta([k], 0) = [k]_+ \to [1], \quad \eta(i) = \begin{cases} 0 & i \neq \infty \\ 1 & i = \infty \end{cases}
\]

\[
\eta: \delta([k], 1) = [k] \to [1], \quad \eta(i) = 0.
\]

**Construction 5.2.8.** We define a functor

\[
\varphi: \text{Tens}_{\nabla}^\otimes \to \text{BM}^\otimes
\]

by its action on objects and morphisms (see remark \([5.2.6]\)).

1. Let \(\varphi\) send an object \((b, \langle n \rangle, [k], c_-, c_+)\) of \(\text{Tens}_{\nabla}^\otimes\) to \((\langle n \rangle, \eta \circ c_-, \eta \circ c_+)\), where the maps

\[
\eta \circ c_-, \eta \circ c_+: \langle n \rangle^0 \to \delta([k], b) \xrightarrow{\eta} [1]
\]

are defined by composition with the canonical map \(\eta\) of construction \([5.2.7]\).
Let 
\((\beta, \alpha, \lambda): (b, \langle n \rangle, [k], c_-, c_+) \rightarrow (b', \langle n' \rangle, [k'], c'_-, c'_+)\) be a morphism in \(\text{Tens}_\nabla^\otimes\). Let \(\varphi\) send \((\beta, \alpha, \lambda)\) to \(\alpha: \langle n \rangle \rightarrow \langle n' \rangle\). One has to verify that the conditions of \([13, \text{def. 4.3.1.5}]\) are satisfied to ensure that \(\alpha\) is a morphism of \(\mathcal{BM}^\otimes\). Indeed, by the description of morphisms in remark \(5.2.6\) we have
\[c_-(i_0) = \delta(\lambda, \beta)(c'_-(j)), \quad c_+(i_m) = \delta(\lambda, \beta)(c'_+(j)),\]
so by naturality of the canonical map \(\eta\) we have
\[\eta \circ c_-(i_0) = \eta \circ c'_-(j), \quad \eta \circ c_+(i_m) = \eta \circ c'_+(j)\]
Finally, by the description of the object \((b, \langle n \rangle, c_-, c_+)\) in \(\text{Tens}_\nabla^\otimes\) we also have
\[c_-(i_0) \leq c_+(i_0) = c_-(i_1) \leq c_+(i_1) = \cdots = c_-(i_m) \leq c_+(i_m)\]
Applying the order preserving map \(\eta: \delta([k], b) \rightarrow [1]\) verifies the remaining conditions.

### 5.3. Construction of the \(\nabla\)-family \(\text{BLMod}_{\mathcal{A}}(\mathcal{C})\)

Let \(\mathcal{C}^\otimes\) be a \(\mathcal{BM}\)-monoidal \(\infty\)-category which admits realizations of \(\mathcal{A}\)-bar constructions \(\otimes\)-compatibly for an appropriate class \(\mathcal{A}\) of algebras in \(\mathcal{C}_-\). Our goal in this subsection is to construct a certain coCartesian fibration
\[p_\mathcal{A}: \text{BLMod}_{\mathcal{A}}(\mathcal{C})^\otimes \rightarrow \nabla\]
which will play a role similar to the role played by the Morita double \(\infty\)-category of example \(4.2.3\). In section \(5.4\) we will fix a algebras \(A\) in \(\mathcal{C}_-\) and \(B \in \mathcal{C}_+\) and use \(\text{BLMod}_{\mathcal{A}}(\mathcal{C})^\otimes\) to extract a left action of \(A\)-\(A\)-bimodules in \(\mathcal{C}_-\) on \(A\)-\(B\)-bimodules in \(\mathcal{C}_m\).

Throughout this subsection, we fix a \(\mathcal{BM}\)-monoidal category
\[q: \mathcal{C}^\otimes \rightarrow \mathcal{BM}^\otimes\].

**Notation 5.3.1.** For any map of simplicial sets \(f: K \rightarrow \nabla\) we let
\[\text{Tens}_f^\otimes \rightarrow K\]
denote the pullback of \(\text{Tens}_\nabla^\otimes \rightarrow \nabla\) along \(f\). Equivalently, \(\text{Tens}_f^\otimes\) is the pullback of \(\text{Tens}^\otimes \rightarrow \Delta^{op}\) along the composite
\[K \rightarrow \nabla \xrightarrow{\delta} \Delta^{op}\].

We allow ourselves to use the notation
\[\text{Tens}_f^\otimes = \text{Tens}_K^\otimes\]
when we see no danger of confusion. We consider $\text{Tens}_K^\otimes$ as a simplicial set over $\mathcal{B}\mathcal{M}^\otimes$ by the map

$$\text{Tens}_K^\otimes = \text{Tens}_f^\otimes \to \text{Tens}_\vec{\nabla}^\otimes \to \mathcal{B}\mathcal{M}^\otimes$$

where $\varphi$ is the functor of construction 5.2.8.

**Construction 5.3.2.** In terms of our fixed $\mathcal{B}\mathcal{M}$-monoidal category $q: \mathcal{C}^\otimes \to \mathcal{B}\mathcal{M}^\otimes$ we define a map of simplicial sets

$$p: \text{BLMod}(\mathcal{C})^\otimes \to \vec{\nabla}$$

by the following universal property:

(*) For every map of simplicial sets $K \to \vec{\nabla}$ there is a canonical bijection$^{16}$

$$\text{Hom}_{\text{Set}_{\Delta/\vec{\nabla}}} (K, \text{BLMod}(\mathcal{C})^\otimes) \cong \text{Alg}_{\text{Tens}_K^\otimes / \mathcal{B}\mathcal{M}(\mathcal{C})}.$$ 

In particular, a diagram of simplicial sets

$$\begin{array}{ccc}
L & \longrightarrow & \text{BLMod}(\mathcal{C})^\otimes \\
\downarrow & & \downarrow^p \\
K & \longrightarrow & \vec{\nabla}
\end{array}$$

commutes if and only if the corresponding diagram

$$\begin{array}{ccc}
\text{Tens}_L^\otimes & \longrightarrow & \mathcal{C}^\otimes \\
\downarrow & & \downarrow^q \\
\text{Tens}_K^\otimes & \longrightarrow & \mathcal{B}\mathcal{M}^\otimes
\end{array}$$

commutes.

**Lemma 5.3.3.** The map of simplicial sets of construction 5.3.2

$$p: \text{BLMod}(\mathcal{C})^\otimes \to \vec{\nabla}$$

is an inner fibration.

**Proof.** We verify that $p$ has the right lifting property with respect to all inner anodyne maps. By definition of $\text{BLMod}(\mathcal{C})^\otimes$ checking the lifting condition against an inner anodyne map $A \to B$ is equivalent to solving the following lifting problem

$$\begin{array}{ccc}
\text{Tens}_A^\otimes & \longrightarrow & \mathcal{C}^\otimes \\
\downarrow \text{dotted} & & \downarrow^q \\
\text{Tens}_B^\otimes & \longrightarrow & \mathcal{B}\mathcal{M}^\otimes
\end{array}$$

$^{16}$The subscript ‘0’ on the right refers to the set of 0-simplices.
By [13, thm. 4.4.3.1] the functor $\text{Tens}^\otimes \to \Delta^{op}$ is a flat categorical fibration, and by [13, rem. B.3.12] so is its pullback $\text{Tens}^\otimes_{\nabla} \to \nabla$. By [13, prop. B.3.14] the monomorphism $\text{Tens}^\otimes_{A} \to \text{Tens}^\otimes_{B}$ is a trivial cofibration in the Joyal model structure ([12, thm. 2.2.5.1]). It follows that the solution to the above lifting problem exists, as the right vertical morphism $C^\otimes \to BM^\otimes$ is a categorical fibration. □

Applying [13, cor. 4.4.3.2] to $O^\otimes = BM^\otimes$ and $S = \nabla \nabla \to BM^\otimes$ we get the following proposition.

**Proposition 5.3.4.** Let $\alpha: s \to s'$ be an edge in $\nabla = \Delta^{op} \times \Delta^{1}$ and let $F_0$ be a vertex of $\text{Alg}_{\text{Tens}_s/BM}(C)$. The vertex $F_0$ corresponds to an element of $\text{Hom}_{\text{Set}_{\Delta/\nabla}}(\{s\}, \text{BLMod}(C)^\otimes)$, and hence to a vertex of $\text{BLMod}(C)^\otimes$ lying over $s$. Let $F$ be a vertex of $\text{Alg}_{\text{Tens}_s/BM}(C)$ extending $F_0$; equivalently, $F$ is an edge of $\text{BLMod}(C)^\otimes$ lying over $\alpha$ with source $F_0$. If $F$, regarded as a map of generalized $\infty$-operads $\text{Tens}_\alpha^\otimes \to C^\otimes$ over $BM^\otimes$, is an operadic $q$-left Kan extension of $F_0$, then $F$ regarded as an edge of $\text{BLMod}(C)^\otimes$ is $q$-coCartesian.

**Proof.** Mutatis mutandis the same as proposition 3.2.4. □

Our next goal is to introduce the notion of an $A$-multimodule in a $BM$-monoidal $\infty$-category (definition 5.3.8 and definition 5.3.8).

**Notation 5.3.5.** For $[k] \in \Delta^{op}$ and $i = 0, \ldots, k$ let

$$
\epsilon_i: \text{Ass}^\otimes \to \text{Tens}_{[k]}^\otimes
$$

be the map of [13, rem. 4.4.1.8]. The map $\epsilon_i$ identifies $\text{Ass}^\otimes$ with the full subcategory of $\text{Tens}_{[k]}^\otimes$ spanned by objects $([n], [k], c_-, c_+)$ where $c_-(j) = c_+(j) = i$ for every $j \in \langle n \rangle^c$.

Let $\text{Ass}_-^\otimes \subset \text{BM}^\otimes$ be the full subcategory of objects of the form $([n], c_-, c_+)$ where $c_-(i) = c_+(i) = 0$ for every $i \in \langle n \rangle^c$. The forgetful functor $\text{BM}^\otimes \to \text{Ass}^\otimes$ restricts to an isomorphism $\text{Ass}_-^\otimes \xrightarrow{\sim} \text{Ass}^\otimes$, as in [13, rem. 4.3.1.10].

**Definition 5.3.6.** Fix $s \in \nabla = \Delta^{op} \times \Delta^{1}$. An $s$-multimodule in $C$ is a map of $\infty$-operads $\text{Tens}_s^\otimes \to C^\otimes$ over $\text{BM}^\otimes$. A multimodule in $C$ is an $s$-multimodule in $C$ for some $s \in \nabla$.
Remark 5.3.7. The \([k,i]\)-multimodules of definition 5.3.6 come in two varieties, depending on whether \(i = 0\) or \(1\). Note that we have canonical equivalences

\[
\text{Tens}_{([k],1)} \simeq \text{Tens}_{[k]} \quad \text{and} \quad \text{Tens}_{([k],0)} \simeq \text{Tens}_{[k]}_{+} \simeq \text{Tens}_{[k+1]}.
\]

Starting with the case \(i = 1\), if \(s = ([k], 1) \in \vec{\nabla}\) then

\[
\text{Tens}_{s} \to \text{Tens}_{\vec{\nabla}} \xrightarrow{\varphi} \text{BM}^\otimes
\]

factors through \(\text{Ass}_{\ast} \subset \text{BM}^\otimes\). Therefore we can identify a \([k,1]\)-multimodule in \(\mathcal{C}\) with a \([k]\)-multimodule in \(\mathcal{C}_{-}\).

Turning to the case \(i = 0\), a \([k,0]\)-multimodule \(M : \text{Tens}_{([k],0)} \to \mathcal{C}^\otimes\) in \(\mathcal{C}\) corresponds to a \([k]\)-multimodule in \(\mathcal{C}_{-}\) and an \(A_k - A_{\infty}\)-bimodule \(M^k_{\ast k}\) in \(\mathcal{C}_{m}\), where \(A_{\infty}\) is an algebra in \(\mathcal{C}_{+}\). This correspondence is furnished as follows. The inclusion

\[
\alpha : [k] \subset [k]_{+} = \{0 < \cdots < k < \infty\}
\]

gives rise to a map of \(\infty\)-operads [13, not. 4.4.1.10]

\[
v_{\alpha} : \text{Tens}_{[k]} \to \text{Tens}_{[k]_{+}}
\]

given by

\[
v_{\alpha}(\langle n \rangle, [k], c_{-}, c_{+}) = \langle n \rangle, [k]_{+}, \alpha \circ c_{-}, \alpha \circ c_{+}.
\]

The composition

\[
\text{Tens}_{[k]} \xrightarrow{v_{\alpha}} \text{Tens}_{[k]_{+}} \simeq \text{Tens}_{([k],0)} \to \text{Tens}_{\vec{\nabla}} \xrightarrow{\varphi} \text{BM}^\otimes
\]

factors through \(\text{Ass}_{\ast} \subset \text{BM}^\otimes\). Therefore, we can identify the composition

\[
\text{Tens}_{[k]} \xrightarrow{v_{\alpha}} \text{Tens}_{([k],0)} \to \mathcal{C}^\otimes
\]

with a \([k]\)-multimodule in \(\mathcal{C}^\otimes\) as in (*). On the other hand, the inclusion \(\beta : [1] \simeq \{k < \infty\} \subset [k]_{+}\) gives rise to a map of \(\infty\)-operads

\[
v_{\beta} : \text{BM}^\otimes \to \text{Tens}_{[k]_{+}}
\]

whose composition with \(\phi\) is the identity map of \(\text{BM}^\otimes\). Thus, the composition

\[
\text{BM}^\otimes \xrightarrow{v_{\beta}} \text{Tens}_{([k],0)} \to \mathcal{C}^\otimes
\]

may be identified with a bimodule in \(\mathcal{C}_{m}\) over algebras \(A'_{-}\) in \(\mathcal{C}_{-}\) and \(A_{\infty}\) in \(\mathcal{C}_{+}\). A straightforward verification shows that \(A'_{-} = A_{k}\).
Definition 5.3.8. Let $\mathcal{A} \subseteq \text{Alg}(\mathcal{C}_-)$ be a full subcategory, closed under equivalences. We say that a $([k], 1)$-multimodule in $\mathcal{C}$ is an $\mathcal{A}$-multimodule if the corresponding $[k]$-multimodule in $\mathcal{C}_-$ is an $\mathcal{A}$-multimodule. We say that a $([k], 0)$-multimodule $F \in \text{Alg}_{\text{Tens}(k), 0}/\text{BM}(\mathcal{C})$ is an $\mathcal{A}$-multimodule if the corresponding $[k]$-multimodule in $\mathcal{C}_-$ is an $\mathcal{A}$-multimodule in $\mathcal{C}_-$.

Definition 5.3.9. Let $\mathcal{A} \subseteq \text{Alg}(\mathcal{C}_-)$ be a full subcategory, stable under equivalences. We say that $\mathcal{C}$ admits operadic left Kan extensions from $\mathcal{A}$-multimodules if for every edge $\alpha: s \to s'$ of $\vec{\nabla}$ and $\mathcal{A}$-multimodule $F_0 \in \text{Alg}_{\text{Tens}_s/\text{BM}}(\mathcal{C})$ there exists a $q$-operadic left Kan extension $F \in \text{Alg}_{\text{Tens}_\alpha/\text{BM}}(\mathcal{C})$ of $F_0$ as in the following diagram:

\[
\begin{array}{ccc}
\text{Tens}_s^\otimes & \xrightarrow{F_0} & \mathcal{C}^\otimes \\
\downarrow & & \downarrow q \\
\text{Tens}_\alpha^\otimes & \xrightarrow{F} & \mathcal{B}M^\otimes.
\end{array}
\]

We also say that $\mathcal{C}$ admits relative tensor products of $\mathcal{A}$-multimodules, generalizing [13, def. 4.4.2.3].

In the terminology of proposition 5.3.4, we immediately deduce the following statement from proposition 5.3.4.

Corollary 5.3.10. Let $p: \text{BLMod}(\mathcal{C}) \to \vec{\nabla}$ be the map of construction 5.3.2. If $\mathcal{C}$ admits operadic left Kan extensions from left $\mathcal{A}$-multimodules, then $\text{BLMod}(\mathcal{C}) \to \vec{\nabla}$ admits coCartesian lifts from left $\mathcal{A}$-multimodules: for any edge $\alpha: s \to s'$ of $\Delta^{op} \times \Delta^1$ and $\mathcal{A}$-multimodule $F_0 \in \text{Alg}_{\text{Tens}_s/\text{BM}}(\mathcal{C})$ over $\alpha$, there exists a lift $F$ of $\alpha$ extending $F_0$.

Definition 5.3.11. Let us temporarily use the alternative notation $\mathcal{C}_0 = \mathcal{C}_-$ and $\mathcal{C}_1 = \mathcal{C}_+$ for the two monoidal categories associated to our fixed $\text{BM}$-monoidal category $\mathcal{C}$. If $A_{i_0}, A_{i_1}$ are algebras in $\mathcal{C}_{i_0}, \mathcal{C}_{i_1}$ with $0 \leq i_0 \leq i_1 \leq 1$, then we may speak unambiguously of $A_{i_0}$-$A_{i_1}$-bimodules in $\mathcal{C}$; we denote the $\infty$-category of such by $A_{i_0} \text{BMod}_{A_{i_1}}(\mathcal{C})$ or simply by $A_{i_0} \text{BMod}_{A_{i_1}}$. More generally, given algebras $A_{i_0}, A_{i_1}, \ldots, A_{i_n}$, with $0 \leq i_0 \leq i_1 \leq \cdots \leq i_n \leq 1$, we refer to a tuple $(M_{01}, M_{12}, \ldots, M_{n-1,n})$ where $M_{k,k+1}$ is an $A_{i_k}$-$A_{i_{k+1}}$-bimodule as an $(A_{i_0}, A_{i_1}, \ldots, A_{i_n})$-module.
Remark 5.3.12. By remark 5.3.7, a \([2,1]\)-module consists of algebras \(A, B, C\) in \(\mathcal{C}_-\), an \(A-B\)-bimodule \(M\) and a \(B-C\)-bimodule \(N\), hence, in our terminology and notation from example 5.1.6, an \((A,B,C)\)-module \(A\mbox{-mod}_{B\mbox{-mod}}^N\) in \(\mathcal{C}_-\). On the other hand, a \(([1],1)\)-module consists of algebras \(A, B\) in \(\mathcal{C}_-\), an algebra \(C\) in \(\mathcal{C}_+\), an \(A-B\)-bimodule \(M\) in \(\mathcal{C}_-\), and a \(B-C\)-bimodule \(N\) in \(\mathcal{C}_m\). For fixed \(A, B, C\), we refer to such an object again as an \((A,B,C)\)-module and we again use the notation \(A\mbox{-mod}_{B\mbox{-mod}}^N\).

Notation 5.3.13. Let \(\tau : \Delta^1 \to \vec{\nabla}\) denote the functor associated to the morphism
\([[1],0) \to ([0],0)\)
denoted by the same symbol \(\tau\) above (remark 5.2.4). Note that the composition \(\Delta^1 \to \vec{\nabla} \to \Delta^\text{op}\) corresponds to the morphism \([1] \to [2]\) in \(\Delta\) used to define \(\text{Tens}_\succ\) in \([13\text{ not. }4.4.2.1]\). Let \(\tau\) also denote the induced morphism
\(\text{Tens}_\succ \to \text{Tens}_\succ^\text{op}\).

Let \(\phi^{001}_{[\sigma]}\) and \(\phi^{001}_{[2]}\) denote the composites
\(\phi^{001}_{[\sigma]} : \text{Tens}_\succ^\text{op} \to \text{Tens}_\succ^\text{op} \to \text{BM}^\text{op},\)
\(\phi^{001}_{[2]} : \text{Tens}_\succ^\text{op} \to \text{Tens}_\succ^\text{op} \to \text{Tens}_\succ^\text{op} \to \text{BM}^\text{op}\).

We define a \(\phi^{001}_{[2]}\)-algebra to be a morphism of \(\infty\)-operads
\(F : \text{Tens}_\succ^\text{op} \to \text{C}^\text{op}\)
over \(\text{BM}^\text{op}\), or equivalently a \(([1],1)\)-module, i.e. an \((A,B,C)\)-module where \(A, B\) are algebras in \(\mathcal{C}_-\) and \(C\) is an algebra in \(\mathcal{C}_+\).

Definition 5.3.14. Let \(F : \text{Tens}_\succ^\text{op} \to \text{C}^\text{op}\) be a \(\phi^{001}_{[2]}\)-algebra in \(\text{C}^\text{op}\) with associated algebras and bimodules
\(A^M B^N C\).

We define the operadic bar construction \(\text{Bar}^\text{op}_{[2]}(F) = \text{Bar}_{[2]}(M,N)\) to be the composite functor
\(\Delta^\text{op} \to \text{Tens}_\succ^\text{op} \to \text{C}^\text{op}\).

The associated underlying bar construction
\(\text{Bar}(F) = \text{Bar}_{[2]}(M,N) : \Delta^\text{op} \to \text{C}^\text{op}\)
and the natural transformation
\(\beta : \text{Bar}^\text{op}_{[2]}(F) \to \text{Bar}(F)\)
are then constructed in the same way as for a monoidal category, see definition 3.3.4.
**Definition 5.3.15.** Let \( A \in \text{Alg}(\mathcal{C}_-) \) be an algebra. We say that \( \mathcal{C}^\otimes \to B\mathcal{M}^\otimes \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly if (in the terminology of remark 5.3.12) for any \((B, A, C)\)-module \( F : \text{Tens}_{[2]}^\otimes \to C^\otimes \) with \( B \in \text{Alg}(\mathcal{C}_-) \) and \( C \in \text{Alg}(\mathcal{C}_i) \) \((i = - \text{ or } +)\) the following conditions hold:

1. The appropriate \( \infty \)-category \( (\mathcal{C}_- \text{ if } i = - \text{ and } \mathcal{C}_m \text{ if } i = +) \) admits realizations of the bar constructions \( \text{Bar}_A(M, N)_\bullet \).
2. For every \( X, Y \) (for which the tensor products below are defined) the canonical map

\[
|X \otimes \text{Bar}_A(M, N)_\bullet \otimes Y| \to X \otimes |\text{Bar}_A(M, N)_\bullet| \otimes Y
\]

is an equivalence.

Let \( \mathcal{A} \) be a class of algebras in \( \mathcal{C}_- \). We say that \( \mathcal{C} \) admits realizations of \( \mathcal{A} \)-bar constructions \( \otimes \)-compatibly if \( \mathcal{C} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly for every \( A \in \mathcal{A} \).

Dually, let \( A \in \text{coAlg}(\mathcal{C}_-) = \text{Alg}(\mathcal{C}_-^{\text{op}})^{\text{op}} \) be a coalgebra. We say that \( \mathcal{C} \) admits totalizations of \( A \)-cobar constructions \( \otimes \)-compatibly, and similarly for “admits totalizations of \( A \)-cobar constructions \( \otimes \)-compatibly”.

**Proposition 5.3.16.** (Generalization of proposition 3.3.3 and [13, prop. 4.4.2.8(1)].) Let \( F_0 = AM_BN_C \) be an \((A, B, C)\)-module in \( \mathcal{C} \) with \( A, B \in \text{Alg}(\mathcal{C}_-) \) and \( C \in \text{Alg}(\mathcal{C}_+) \).

Assume the underlying bar construction \( \text{Bar}_B(M, N)_\bullet \) admits a geometric realization compatibly with \( \otimes \). Suppose given a commutative diagram of generalized \( \infty \)-operads (solid arrow diagram below)

\[
\begin{array}{ccc}
\text{Tens}_{[2]}^\otimes & \xrightarrow{F_0} & \mathcal{C}^\otimes \\
\downarrow & & \downarrow^q \\
\text{Tens}_{\otimes}^\otimes & \xrightarrow{\phi_{001}} & B\mathcal{M}^\otimes \\
\end{array}
\]

Then there exists an operadic \( q \)-left Kan extension \( F \) of \( F_0 \) (dotted arrow) making the diagram commute.

**Proof.** In view of the generality in which [13, prop. 4.4.2.5] is stated, this is mutatis mutandis the same as proposition 3.3.3.

**□**

**Definition 5.3.17.** In the situation of proposition 5.3.16 we say that \( F \) exhibits \( F_1 := F|_{\text{Tens}_{[1]}^\otimes} \) as an external relative tensor product of \( M \) and \( N \) over \( A \).

We define the **convex subcategory** of \( \vec{\nabla} \) by

\[
\vec{\nabla}_{\text{convex}} := \Delta_{\text{convex}}^{\text{op}} \times \Delta^1.
\]
Proposition 5.3.18 (Segal conditions for BLMod(C)\(^\circ\)). Let 
\[
p: B^\circ = \text{BMod}(C)^\circ \to \vec{\nabla}
\]
be the map of simplicial sets of construction 5.3.2. Then the pullback \(p_{\text{Convex}}\) of \(p\) to \(\vec{\nabla}_{\text{Convex}}\) as in the following diagram

\[
\begin{array}{ccc}
B^\circ_{\text{Convex}} & \rightarrow & B^\circ \\
\downarrow^{p_{\text{Convex}}} & \downarrow^{p} \\
\vec{\nabla}_{\text{Convex}} & \rightarrow & \vec{\nabla}
\end{array}
\]
is a coCartesian fibration. Moreover, for every \(n > 1\), the induced maps
\[
B^\circ_{([n],1)} \rightarrow B^\circ_{([1],1)} \times B^\circ_{([0],1)} \cdots \times B^\circ_{([0],1)} \simeq \text{BMod}(C_-) \times_{\text{Alg}(C_-)} \cdots \times_{\text{Alg}(C_-)} \text{BMod}(C_-)
\]
and
\[
B^\circ_{([n],0)} \rightarrow B^\circ_{([n],0)} \times B^\circ_{([0],0)} \simeq \text{BMod}(C_-) \times_{\text{Alg}(C_-)} \cdots \times_{\text{Alg}(C_-)} \text{BMod}(C_-) \times_{\text{Alg}(C_-)} \text{BMod}(C_m)
\]
are equivalences of \(\infty\)-categories.

Proof. Let \(\alpha : \Delta^1 \to \vec{\nabla}\) denote the map of simplicial sets associated to the arrow \(([k],0 \to 1)\) in \(\vec{\nabla}\), and consider a square of generalized \(\infty\)-operads (solid arrow diagram)

\[
\begin{array}{ccc}
\text{Tens}^\circ_{([k],0)} & \xrightarrow{X_0} & C^\circ \\
\downarrow & & \downarrow \\
\text{Tens}^\circ_{\alpha} & \xrightarrow{X} & \text{BM}^\circ
\end{array}
\]
An argument similar to [13, 4.4.3.5] shows that there exists an operadic \(q\)-left Kan extension as in the diagram. The remainder of the proof is similar to 3.3.4. \(\Box\)

Proposition 5.3.19. Let \(A\) be a class of algebra objects of \(C_-\) closed under equivalences. Assume that \(C\) admits realizations of \(A\)-bar construction \(\otimes\)-compatibly (definition 5.3.15). Let 
\[
p: \text{BLMod}(C)^\circ \to \vec{\nabla}
\]
be the map of construction 5.3.2. Then BLMod(C)^\circ admits coCartesian lifts from \(A\)-multimodules (corollary 5.3.10).

Proof. In view of proposition 5.3.18, this is similar to proposition 3.3.4. \(\Box\)
Theorem 5.3.20. In the situation and the notation of proposition 5.3.19, let

$$\text{BLMod}_A(C) \subset \text{BLMod}(C)$$

be the full subcategory supported on $A$-multimodules. Assume that $C$ admits realizations of $A$-bar constructions $\otimes$-compatibly. Then the restriction of $p$ to

$$p_A : \text{BLMod}_A(C) \to \vec{\nabla}$$

is a coCartesian fibration. Moreover, $p_A$ obeys the Segal conditions of proposition 5.3.18.

Proof. Mutatis mutandis the same as corollary 3.3.5. \qed

In the situation of theorem 5.3.20 with $A = \text{Alg}(C_-)$ for simplicity, the coCartesian fibration

$$p : \text{BLMod}(C) \to \vec{\nabla}$$

corresponds to a $\vec{\nabla}$-shaped diagram of $\infty$-categories which includes the vertices and arrows:

\[
\begin{array}{ccc}
\vdots & \downarrow & \vdots \\
\text{BMod}(C_-) \times \text{Alg}(C_-) & \to & \text{BMod}(C_-) \\
\downarrow & \downarrow & \downarrow \\
\text{BMod}(C_-) \times \text{Alg}(C_-) & \to & \text{BMod}(C_-) \\
\downarrow & \downarrow & \downarrow \\
\text{BMod}(C_-) & \to & \text{Alg}(C_-).
\end{array}
\]

5.4. Extracting the $\mathcal{LM}$-monoidal $\infty$-category $\text{BLMod}_A(C)$. We construct the $\mathcal{LM}$-monoidal $\infty$-category $\text{BLMod}_A(C)$ (construction 5.4.5). We will make use of a certain sub-family of $\infty$-operads $\text{MAss} \subset \text{Tens}$ which forgets those colors of $\text{Tens}$ which index bimodules.

Definition 5.4.1. Let $\text{MAss}$ denote the full subcategory of $\text{Tens}$ whose objects

$$c_\pm : \langle n \rangle \Rightarrow [k]$$

satisfy $c_- = c_+$. Thus, an object is a triple $c = (\langle n \rangle, [k], c)$ with $\langle n \rangle \in \text{Ass}$, $[k] \in \Delta$, and $c : \langle n \rangle \to [k]$ a map of sets, and a morphism $c \to c'$ is simply a commuting
Our results concerning the generalized $\infty$-operad $\text{MAss}^\otimes$ and the associated double $\infty$-categories (propostion [5.4.4] below) amount to an essentially trivial portion of the results concerning $\text{Tens}^\otimes$ and the associated double $\infty$-categories $\text{BMod}(\mathcal{C})^\otimes$ obtained in [13]; we limit ourselves here to an outline interspersed with references to loc. cit.

**Proposition 5.4.2.** The inclusion

$$\epsilon : \text{MAss}^\otimes \subset \text{Tens}^\otimes$$

is a morphism of $\Delta^{op}$-families of $\infty$-operads. The composite

$$\text{MAss}^\otimes \to \text{Tens}^\otimes \to \Delta^{op}$$

is flat [13].

**Proof.** The first statement is clear. Fix morphisms $\lambda = \lambda' \circ \lambda''$

$$[k] \xleftarrow{\lambda'} [k'] \xleftarrow{\lambda''} [k'']$$

in $\Delta$ and a morphism

$$(\alpha, \lambda) : c \to c''$$

in $\text{MAss}^\otimes$ over $\lambda$. The category

$$(*) \quad \text{MAss}^\otimes \times_{(\Delta^{op})_{[k]/[k']}} \{ (\lambda', \lambda'') \}$$

may be described concretely as the category whose objects are tuples $(\langle n' \rangle, \alpha', \alpha'', c')$ which fit into a commuting diagram like so

$$
\begin{array}{ccc}
\langle n \rangle \twoheadrightarrow \langle n' \rangle & \xrightarrow{\alpha} & \langle n'' \rangle \\
\downarrow{c} & & \downarrow{c''} \\
k \xleftarrow{\lambda} k' & \xleftarrow{\lambda'} k' & \xleftarrow{\lambda''} k'',
\end{array}
$$

and whose morphisms

$$((\langle n'_1 \rangle, \alpha'_1, \alpha''_1, c'_1) \to (\langle n'_2 \rangle, \alpha'_2, \alpha''_2, c'_2))$$
are those morphisms \( \langle n'_1 \rangle \to \langle n'_2 \rangle \) in \( \text{Ass}^\otimes \) which make the evident triangles commute. The object \( (\langle n'' \rangle, id, \alpha, \lambda'' \circ c'') \) is terminal. It follows that this category is weakly contractible.\(^{17}\) The proposition follows by \cite[Remark B.3.9]{13}.

Given an object \( I \in \Delta^{op} \), we denote by \( \text{MAss}^\otimes_I \) the fiber of \( \text{MAss}^\otimes \to \Delta^{op} \) over \( I \). For any totally ordered finite set \( I \), we have an equivalence of \( \infty \)-operads

\[
\text{MAss}^\otimes_I \simeq (\text{Ass}^\otimes)_I.
\]

Indeed, this follows from the proof of \cite[Proposition 4.4.1.11]{13} (Segal condition for \( \text{Tens}^\otimes \)) and can also easily be checked directly.

We denote by

\[
\epsilon_\nabla : \text{MAss}^\otimes_\nabla \to \text{Tens}^\otimes_\nabla
\]

the pullback of the inclusion \( \epsilon : \text{MAss}^\otimes \subset \text{Tens}^\otimes \) along the map

\[
\delta : \nabla \to \Delta^{op}
\]

of notation \ref{notation-5.2.1}. Given a map of simplicial sets \( f : K \to \nabla \) we denote by

\[
\text{MAss}^\otimes_f = \text{MAss}^\otimes_K
\]

the pullback of \( \text{MAss}^\otimes_\nabla \) along \( f \).

**Definition 5.4.3.** Let \( q : C^\otimes \to \text{BM}^\otimes \) be a \( \text{BM} \)-monoidal \( \infty \)-category. We define a map of simplicial sets

\[
q_a : \text{LAlg}(C)^\otimes \to \nabla
\]

by the following universal property:

\((*)\) For every map of simplicial sets \( K \to \nabla \) there is a canonical bijection

\[
\text{Hom}_{\text{Set}_\Delta/\phi}(K, \text{LAlg}(C)^\otimes) \cong \text{Alg}_{\text{MAss}_{K/\text{BM}}(C)_0}.
\]

Here the subscript ‘0’ on the right refers to the collection of 0-simplices, and the subscript “\( \text{MAss}_{K/\text{BM}} \)” refers to the composite morphism of generalized \( \infty \)-operads

\[
\text{MAss}^\otimes_\nabla \xrightarrow{\epsilon_\nabla} \text{Tens}^\otimes_\nabla \xrightarrow{\phi} \text{BM}^\otimes
\]

where \( \phi \) is the map constructed in construction \ref{construction-5.2.8}.

**Proposition 5.4.4.** Let \( q : C^\otimes \to \text{BM}^\otimes \) be a \( \text{BM} \)-monoidal category. Then the map \( q_a : \text{LAlg}(C)^\otimes \to \nabla \) of definition \ref{definition-5.4.3} is a coCartesian fibration.

\(^{17}\)We recall that an ordinary (respectively \( \infty \)-category) is said to be weakly contractible if the geometric realization of its nerve (respectively, of its underlying simplicial set) is contractible.
Proof. The map $\text{MAss}^\otimes \rightarrow \vec{\nabla}$ is flat since it is a pullback of the flat map

$$\text{MAss}^\otimes \rightarrow \Delta^{op},$$

see proposition [5.4.2]. As in lemma [5.3.3] it follows that $q_a$ is an inner fibration. Fixing a square (solid arrow diagram below)

(*)

```
\begin{tikzpicture}
  \node (A) {$\Delta^0$};
  \node (B) at (2,0) {$\text{LAlg}(C)^\otimes$};
  \node (C) at (0,-1) {$\Delta^1$};
  \node (D) at (2,-1) {$\vec{\nabla}$};
  \draw[->] (A) to node {$F_0$} (B);
  \draw[->] (A) to node [left] {$\{0\}$} (C);
  \draw[->] (B) to node [right] {$q_a$} (D);
  \draw[->] (C) to node [left] {$\lambda$} (D);
\end{tikzpicture}
```

we must establish the existence of a coCartesian lift $F$ of $\lambda$ extending $F_0$ as indicated. If we regard $\lambda$ also as a morphism $\lambda : s \rightarrow s'$ in $\vec{\nabla}$, the square diagram (*) corresponds to a diagram

(**)

```
\begin{tikzpicture}
  \node (A) {$\text{MAss}_s^\otimes$};
  \node (B) at (2,0) {$C^\otimes$};
  \node (C) at (0,-1) {$\text{MAss}_\lambda^\otimes$};
  \node (D) at (2,-1) {$\vec{\nabla}$};
  \draw[->] (A) to node {$F_0$} (B);
  \draw[->] (A) to node [left] {$\iota_0$} (C);
  \draw[->] (B) to node [right] {$q$} (D);
  \draw[->] (C) to node [left] {$v_\mu$} (D);
\end{tikzpicture}
```

In view of the flatness of $\text{MAss}^\otimes \rightarrow \vec{\nabla}$, the proof of proposition [5.3.4] applies with little change to show that it suffices to establish the existence of an operadic $q$-left Kan extension $F$ as in diagram (**).

If $\mu : [k] \rightarrow [k']$ is any morphism in $\Delta$ then $\text{MAss}_\mu^\otimes$ regarded a correspondence of $\infty$-operads from $\text{MAss}_{[k']}^\otimes$ to $\text{MAss}_{[k]}^\otimes$ is associated with a map of $\infty$-operads

$$v_\mu : \text{MAss}_{[k]}^\otimes \rightarrow \text{MAss}_{[k']}^\otimes$$

given by composition with $\mu$ (compare [13] Notation 4.4.1.10], where the analogous correspondences are nontrivial, except when $\mu$ has convex image). Consequently, the map $\iota_0$ of diagram (**) admits a section $V_\lambda$ and (similarly to [13] Remark 4.4.3.6]),

$$F := F_0 \circ V_\lambda$$

is an operadic $q$-left Kan extension as required. \qed
We remark that the diagram of $\infty$-categories associated to the coCartesian fibration of proposition 5.4.4 includes the vertices and arrows

\[
\begin{array}{ccc}
\vdots & \vdots & \\
\downarrow & \downarrow & \\
\Alg(C_-)^3 \times \Alg(C_+) & \longrightarrow & \Alg(C_-)^3 \\
\downarrow & \downarrow & \\
\Alg(C_-)^2 \times \Alg(C_+) & \longrightarrow & \Alg(C_-)^2 \\
\downarrow & \downarrow & \\
\Alg(C_-) \times \Alg(C_+) & \longrightarrow & \Alg(C_-) \\
\end{array}
\]

The map $\epsilon_\nabla : \text{MAss}_\nabla \rightarrow \text{Tens}_\nabla$ gives rise to a map $\rho$ of simplicial sets over $\nabla$

\[
\begin{array}{ccc}
\text{BLMod}(C) \rightarrow & \longrightarrow & \text{LAlg}(C) \\
\rho & \longrightarrow & q_\alpha \\
\Downarrow & \Downarrow & \\
\nabla & \nabla & \\
\end{array}
\]

The map

\[
\varphi : \text{Tens}_\nabla \rightarrow \text{BM}^\otimes
\]
restricts to a map

\[
\varphi : \text{MAss}_\nabla \rightarrow (\text{Ass}^\otimes) \cap \subset \text{BM}^\otimes.
\]

If $A$ is an algebra in $C_-$ and $B$ is an algebra in $C_+$, we let $A*B$ be the section of $q_\alpha$ associated to the object

\[
\text{MAss}_\nabla \rightarrow (\text{Ass}^\otimes) \cap \subset \text{BM}^\otimes.
\]

of $\text{Alg}_{\text{MAss}_\nabla / \text{BM}}(C)$.

**Construction 5.4.5.** Let $C^\otimes \rightarrow \text{BM}^\otimes$ be a $\text{BM}$-monoidal $\infty$-category and let $A \in \Alg(C_-)$, $B \in \Alg(C_+)$ be algebra objects. Let $A \subset \Alg(C_-)$ be the full subcategory spanned by all algebras equivalent to $A$ and assume $C^\otimes$ admits realizations of $A$-bar constructions $\otimes$-compatibly, so that by theorem 5.3.20 we have an associated coCartesian fibration

\[
p_A : \text{BLMod}_A(C)^\otimes \rightarrow \nabla
\]
which obeys the Segal conditions of proposition 5.3.18. We define the coCartesian fibration

\[
A \text{BLMod}_B(C)^\otimes \rightarrow \nabla
\]
by the pullback
\[
\begin{array}{ccc}
\mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})^\otimes & \longrightarrow & \mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})^\otimes \\
\downarrow & & \downarrow \\
\mathcal{V} & \overset{A^*B}{\longrightarrow} & \mathcal{L}\mathcal{A}\mathcal{G}(\mathcal{C})^\otimes.
\end{array}
\]

in \( \mathcal{C}at^{coCart} \). Applying Propositions 4.2.2.9 and 2.4.2.5 of [13] as in construction 4.2.7, we obtain an \( \mathcal{L}\mathcal{M} \)-monoidal category

\( \mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})^\otimes \rightarrow \mathcal{L}\mathcal{M}^\otimes \)

witnessing \( \mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})_m \) as left-tensored over \( \mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})_+ \).

Now suppose given additional algebras \( A \in \mathcal{A}lg(\mathcal{C}_-) \), \( B \in \mathcal{A}lg(\mathcal{C}_+) \) and morphisms of algebras

\( f : A' \rightarrow A \quad \text{and} \quad g : B' \rightarrow B. \)

Then Remark 3.4.9 of Haugseng [9] applies mutatis mutandis to provide a lax-monoidal functor

\( A'_{\mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})^\otimes} \leftarrow A_{\mathcal{B}_{\mathcal{L} \mathcal{M}}(\mathcal{C})^\otimes} \)

given informally by restriction of scalars along \( f \) and \( g \).

6. \( \mathcal{L}\mathcal{M} \)-algebras and \( \mathcal{L}\mathcal{M} \)-monoidal pairings

In this section we collect a few preliminaries in preparation for section 7.

6.1. \( \mathcal{L}\mathcal{M} \)-monoidal \( \infty \)-categories and \( \mathcal{L}\mathcal{M} \)-algebras. We review \( \mathcal{L}\mathcal{M} \)-monoidal \( \infty \)-categories, mainly in order to fix the notation.

The \( \infty \)-operad \( \mathcal{L}\mathcal{M}^\otimes \rightarrow \mathcal{F}in_* \) is constructed in [13] def. 4.2.1.7. We denote the two objects of the underlying category \( \mathcal{L}\mathcal{M}^\otimes \) by \( a \), \( m \). An \( \mathcal{L}\mathcal{M} \)-monoidal category \( \mathcal{C}^\otimes \) is a coCartesian fibration \( \mathcal{C}^\otimes \rightarrow \mathcal{L}\mathcal{M}^\otimes \) of \( \infty \)-operads (see [13] def. 2.1.2.13]). Following [13] sec. 4.2.1] we denote:

\[
\mathcal{C}_a := \mathcal{C}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \mathcal{A}ss^\otimes, \quad \mathcal{C}_a := \mathcal{C}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \{a\}, \quad \mathcal{C}_m := \mathcal{C}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \{m\}.
\]

Note that \( \mathcal{C}_a^\otimes \rightarrow \mathcal{A}ss^\otimes \) defines a monoidal structure on the \( \infty \)-category \( \mathcal{C}_a \) (see [13] rem. 4.2.1.10]). Following [13] def. 4.2.1.19 we say that \( \mathcal{C}^\otimes \rightarrow \mathcal{L}\mathcal{M}^\otimes \) exhibits \( \mathcal{C}_m \) as left tensored over \( \mathcal{C}_a \).

We denote the \( \infty \)-category of \( \mathcal{L}\mathcal{M} \)-algebras in \( \mathcal{C} \) by \( \mathcal{L}\mathcal{M}(\mathcal{C}) = \mathcal{A}lg_{/\mathcal{L}\mathcal{M}}(\mathcal{C}) \). Explicitly, an object \( X \in \mathcal{L}\mathcal{M}(\mathcal{C}) \) is a map of \( \infty \)-operads

\( \mathcal{L}\mathcal{M}^\otimes \rightarrow \mathcal{C}^\otimes \)

over \( \mathcal{L}\mathcal{M}^\otimes \). Precomposition with the inclusion \( \mathcal{A}ss^\otimes \rightarrow \mathcal{L}\mathcal{M}^\otimes \) defines a functor

\( \mathcal{L}\mathcal{M}(\mathcal{C}) \rightarrow \mathcal{A}lg(\mathcal{C}_a) \).
Given an algebra \( A \in \text{Alg}(\mathcal{C}_a) \) we define the \( \infty \)-category \( \text{LMod}_A(\mathcal{C}_m) \) of \( A \)-modules in \( \mathcal{C}_m \) as the fiber of
\[
\text{LMod}(\mathcal{C}) \to \text{Alg}(\mathcal{C}_a)
\]
over \( A \). We call an object of \( \text{LMod}_A(\mathcal{C}_m) \) a left \( A \)-module in \( \mathcal{C}_m \). Given an \( \mathcal{LM} \)-algebra \( X \in \text{LMod}(\mathcal{C}) \), we denote by \( X_a \in \text{Alg}(\mathcal{C}_a) \) its image under
\[
\text{LMod}(\mathcal{C}) \to \text{Alg}(\mathcal{C}_a)
\]
(so \( X \) is a left \( X_a \)-module) and we denote by \( X_m \in \mathcal{C}_m \) the object corresponding to
\[
\{m\} \to \mathcal{LM}^\otimes \xrightarrow{X} \mathcal{C}.
\]

Let \( \mathcal{C}^\otimes \) be an \( \mathcal{LM} \)-monoidal \( \infty \)-category and \( A \in \text{Alg}(\mathcal{C}_a) \) an algebra. The \( \infty \)-category
\[
\text{Alg}(\text{BLMod}_A(\mathcal{C}_a)) = \text{Alg}(\_A\text{BMod}_A(\mathcal{C}_a))
\]
of algebras in \( A \)-\( A \)-bimodules is equivalent to the category \( \text{Alg}(\mathcal{C}_a)_{A/} \) of algebras under \( A \). In view of this fact (under the assumption that \( \mathcal{C} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly), the following proposition provides a concrete interpretation of an \( \mathcal{LM} \)-algebra in the \( \mathcal{LM} \)-monoidal category \( \text{BLMod}_A(\mathcal{C})^\otimes \) as a pair consisting of an \( A \)-algebra \( B \) in \( \mathcal{C}_a \) and a left \( B \)-module in \( \mathcal{C}_m \).

**Proposition 6.1.1.** Let \( \mathcal{C}^\otimes \) be an \( \mathcal{LM} \)-monoidal \( \infty \)-category and \( A \in \text{Alg}(\mathcal{C}_a) \) an algebra. Assume \( \mathcal{C} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly. Then we have a pullback square of \( \infty \)-categories
\[
\begin{array}{c}
\text{LMod}(\text{BLMod}_A(\mathcal{C})) \\
\downarrow \text{fgt} \\
\text{LMod}(\mathcal{C})
\end{array}
\xrightarrow[	ext{fgt}]{\gamma} \begin{array}{c}
\text{Alg}(\_A\text{BMod}_A(\mathcal{C}_a)) \\
\downarrow \text{fgt} \\
\text{Alg}(\mathcal{C}_a)
\end{array}
\]

**Proof.** By [13, def. 4.2.1.13] the forgetful functors
\[
\text{LMod}(\text{BLMod}_A(\mathcal{C})) \to \text{Alg}(\_A\text{BMod}_A(\mathcal{C}_a)), \quad \text{LMod}(\mathcal{C}) \to \text{Alg}(\mathcal{C}_a)
\]
are categorical fibrations. Therefore it is enough to prove that for every
\[
B \in \text{Alg}(\_A\text{BMod}_A(\mathcal{C}_a))
\]
the induced functor between the fibers
\[
(3) \quad \text{LMod}_B(\text{BLMod}_A(\mathcal{C}_m)) \to \text{LMod}_B(\mathcal{C}_m)
\]
is an equivalence (we abuse notation by writing \( B \) again for the projection of \( B \) to \( \text{Alg}(\mathcal{C}_a) \) along the forgetful functor). This will follow from a simple application of the Lurie-Barr-Beck theorem ([13, cor. 4.7.3.16]).
Consider the commuting square

\[
\begin{array}{c}
\text{LMod}_B(\text{BLMod}_A(C)_m) \longrightarrow \text{LMod}_B(C_m) \\
\downarrow \quad \downarrow \\
\text{BLMod}_A(C)_m = \text{LMod}_A(C_m) \longrightarrow C_m,
\end{array}
\]

where the top horizontal functor is the induced functor \(3\). The vertical functors and the bottom horizontal functor are monadic by \([13\text{ Lurie-Barr-Beck, thm. } 4.7.3.5]\), and in particular conservative right adjoints. Therefore the composition

\[
\text{LMod}_B(\text{BLMod}_A(C)) \to \text{LMod}_A(C_m) \to C_m
\]
is a conservative right adjoint functor, with a left adjoint

\[
C_m \to \text{LMod}_A(C_m) \to \text{LMod}_B(\text{BLMod}_A(C)), \quad X \mapsto A \otimes X \mapsto B \otimes_A (A \otimes X).
\]

Note that the canonical map

\[
B \otimes_A (A \otimes X) \to B \otimes X
\]
is an equivalence in \(C_m\). Therefore by \([13\text{ cor. } 4.7.3.16 \text{ and rem } 4.7.3.17]\) the functor \(3\) is an equivalence, as claimed. \(\square\)

6.2. LM-monoidal pairings. We review the notion of a left representable pairing of \(\infty\)-categories from \([13\text{ sec. } 5.2.1]\).

A pairing of \(\infty\)-categories is a right fibration

\[
\lambda: \mathcal{M} \to \mathcal{C} \times \mathcal{D}.
\]

Let \(\chi: \mathcal{C}^{op} \times \mathcal{D}^{op} \to \mathcal{S}\) be the associated functor. We can view \(\chi\) also as a contravariant functor from \(\mathcal{C}\) to presheaves on \(\mathcal{D}\),

\[
\hat{\mathcal{D}}_\lambda: \mathcal{C}^{op} \to \text{Psh}(\mathcal{D}).
\]

An object \(M \in \mathcal{M}\) over \((X, Y) \in \mathcal{C} \times \mathcal{D}\) is left universal if it is a terminal object of the fiber \(\mathcal{M} \times_\mathcal{C} \{X\}\) (see \([13\text{ def. } 5.2.1.8]\)). We say that \(M \in \mathcal{M}\) is a left universal lift of \(X \in \mathcal{C}\). We say that the pairing \(\lambda\) is left representable if every \(X \in \mathcal{C}\) has a left universal lift. If the pairing \(\lambda\) is left representable then \(\hat{\mathcal{D}}_\lambda\) factors through the Yoneda embedding, and hence defines a functor

\[
\mathcal{D}_\lambda: \mathcal{C}^{op} \to \mathcal{D}.
\]

We refer to \(\mathcal{D}_\lambda: \mathcal{C}^{op} \to \mathcal{D}\) as the left duality functor associated to \(\lambda\).

Similarly, the pairing \(\lambda\) defines a functor \(\mathcal{D}_\lambda^{op}: \mathcal{D}^{op} \to \text{Psh}(\mathcal{C})\). An object \(M \in \mathcal{M}\) over \((X, Y) \in \mathcal{C} \times \mathcal{D}\) is right universal if it is terminal in \(\mathcal{M} \times_\mathcal{D} \{Y\}\). A pairing

\[\text{loc. cit.}\]

\[\text{loc. cit.}\]
is right representable if every \( Y \in \mathcal{D} \) has a right universal lift. If the pairing \( \lambda \) is right representable then \( \mathcal{D}_\lambda' \) factors through the Yoneda embedding \( \mathcal{C} \to \text{Psh}(\mathcal{C}) \), and defines a right duality functor

\[
\mathcal{D}_\lambda': \mathcal{D}^{\text{op}} \to \mathcal{C}.
\]

Moreover, if the pairing \( \lambda \) is both left and right representable then the duality functors associated to \( \lambda \) define an adjunction

\[
\mathcal{D}^{\text{op}}_\lambda: \mathcal{C} \rightleftarrows \mathcal{D}^{\text{op}}: \mathcal{D}_\lambda',
\]

\[
\text{Map}_\mathcal{D}(Y, \mathcal{D}_\lambda(X)) \simeq \chi(X,Y) \simeq \mathcal{M} \times_{\mathcal{C} \times \mathcal{D}} \{ (X,Y) \} \simeq \text{Map}_\mathcal{C}(X, \mathcal{D}_\lambda'(Y)).
\]

Our definition of “\( \mathcal{LM}\)-monoidal pairing” is a special case of [13, def. 5.2.2.20]

**Definition 6.2.1.** A pairing of \( \mathcal{LM}\)-monoidal \( \infty \)-categories, or simply an \( \mathcal{LM}\)-monoidal pairing, is given by three \( \mathcal{LM}\)-monoidal categories \( \mathcal{C}^\otimes, \mathcal{D}^\otimes, \mathcal{M}^\otimes \) and an \( \mathcal{LM}\)-monoidal functor

\[
\lambda: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes
\]

which is a categorical fibration (best understood using [12, cor. 2.4.6.5]) and which induces right fibrations

\[
\lambda_a: \mathcal{M}_a \to \mathcal{C}_a \times \mathcal{D}_a, \quad \lambda_m: \mathcal{M}_m \to \mathcal{C}_m \times \mathcal{D}_m
\]

after taking the fibers over \( a, m \in \mathcal{LM}^\otimes_{(1)} \). We define the associated monoidal pairing

\[
\lambda^\otimes_a: \mathcal{M}^\otimes_a \to \mathcal{C}^\otimes_a \times_{\text{Ass}^\otimes} \mathcal{D}^\otimes_a
\]

by the pullback of \( \lambda \) along \( \text{Ass}^\otimes \to \mathcal{LM}^\otimes \). If the underlying pairings \( \lambda_a, \lambda_m \) are both left representable, we say the monoidal pairing \( \lambda \) is left representable.

**Remark 6.2.2.** If the pairing \( \lambda_a \) is left representable, then the duality functor

\[
\mathcal{D}_{\lambda_a}: \mathcal{C}_a^{\text{op}} \to \mathcal{D}_a
\]

is lax monoidal. We can therefore view \( \mathcal{D}_m \) as left tensored over \( \mathcal{C}_m^{\text{op}} \) via \( \mathcal{D}_{\lambda_m} \). If in addition the pairing \( \lambda_m \) is left representable (so that in the terminology of definition 6.2.1 \( \lambda \) itself is left representable), then the duality functor

\[
\mathcal{D}_{\lambda_m}: \mathcal{C}_m^{\text{op}} \to \mathcal{D}_m
\]

is “lax left tensored over \( \mathcal{C}_m^{\text{op}} \)”: for every \( x \in \mathcal{C}_a, y \in \mathcal{C}_m \) there exists a natural morphism in \( \mathcal{D}_m \)

\[
\mathcal{D}_{\lambda_a}(x) \otimes \mathcal{D}_{\lambda_m}(y) \to \mathcal{D}_{\lambda_m}(x \otimes y),
\]

and these morphisms are coherently compatible.
Let
\[ \lambda: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{CM}^\otimes} \mathcal{D}^\otimes \]
be an \( \mathcal{LM} \)-monoidal pairing. By \cite[rem. 5.2.2.26]{...} \( \lambda \) induces a pairing of \( \infty \)-categories
\[ \text{LMod}(\lambda): \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}). \]
The goal of section 7 is to show that \( \text{LMod}(\lambda) \) is left representable (under suitable assumptions), and hence defines a duality functor
\[ \mathcal{D}_{\text{LMod}(\lambda)}: \text{LMod}(\mathcal{C})^{\text{op}} \to \text{LMod}(\mathcal{D}). \]
We will produce left universal lifts by studying \( \mathcal{LM} \)-monoidal pairings of bimodules.

6.3. \( \mathcal{LM} \)-monoidal pairings of bimodules. Let \( \lambda^\otimes: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes \) be an \( \mathcal{LM} \)-monoidal pairing, and let \( M \in \text{Alg}(\mathcal{M}_m) \) be an object over \((A,B) \in \text{Alg}(\mathcal{C}_m) \times \text{Alg}(\mathcal{D}_m)\). Assume \( \mathcal{M} \) admits geometric realizations of \( M \)-bar constructions \( \otimes \)-compatibly, and similarly for \( A \) and \( B \). By its functoriality, construction \[5.4.5\] gives rise to a map of \( \mathcal{LM} \)-monoidal \( \infty \)-categories
\[ \lambda_M^\otimes: \text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_A(\mathcal{C})^\otimes \times_{\mathcal{LM}^\otimes} \text{BLMod}_B(\mathcal{D})^\otimes. \]
However, in that construction we lose track of individual simplices. Consequently, we are unable to tell if \( \lambda_M^\otimes \) is a categorical fibration. Thus, a direct analog of \cite[lem. 5.2.2.29]{...} is lacking. Instead, in proposition \[6.3.1\] we take a fibrant replacement of \( \lambda_M^\otimes \) in the model category of preoperads.

**Proposition 6.3.1.** In the situation and the notation described above, let \( M \in \text{Alg}(\mathcal{M}_m) \) be an object over \((A,B) \in \text{Alg}(\mathcal{C}_m) \times \text{Alg}(\mathcal{D}_m)\). Assume \( \mathcal{M} \) admits realizations of \( M \)-bar constructions \( \otimes \)-compatibly, and similarly for \( A \) and \( B \). Then there exists an \( \mathcal{LM} \)-monoidal pairing
\[ \tilde{\lambda}_M^\otimes: \text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_A(\mathcal{C})^\otimes \times_{\mathcal{LM}^\otimes} \text{BLMod}_B(\mathcal{D})^\otimes, \]
and an equivalence of \( \mathcal{LM} \)-monoidal categories
\[ \tau: \text{BLMod}_M(\mathcal{M})^\otimes \xrightarrow{\sim} \text{BLMod}_M^\otimes(\mathcal{M}), \]
such that \( \tilde{\lambda}_M^\otimes \circ \tau = \lambda_M^\otimes. \)

**Proof.** We let \( \text{POp}_\infty \) denote the category of \( \infty \)-preoperads of \cite[def. 2.1.4.2]{...} endowed with the model structure of \cite[prop. 2.1.4.6]{...}. Given an \( \infty \)-operad \( \mathcal{O}^\otimes \) we let \( \mathcal{O}^{\otimes;2} \) denote the associated \( \infty \)-preoperad of \cite[not. 2.1.4.5]{...}. We recall that the overcategory \( \text{POp}_\infty/\mathcal{LM}^{\otimes;2} \) inherits a model structure in which weak equivalences,
fibrations, and cofibrations are precisely the morphisms that become such after forgetting the morphism to $\mathcal{L}\mathcal{M}^{\otimes,2}$. Proposition 2.1.4.6 of [13] shows that the object

$$\text{BLMod}_A(C)^{\otimes,2} \times_{\mathcal{L}\mathcal{M}^{\otimes,2}} \text{BLMod}_B(D)^{\otimes,2}$$

of $\mathcal{P}\text{Op}_\infty/\mathcal{L}\mathcal{M}^{\otimes,2}$ is fibrant. We factor the induced map $\lambda^{\otimes,2}$ as

$$\text{BLMod}_M(M) \rightarrow X \rightarrow \text{BLMod}_A(C)^{\otimes,2} \times_{\mathcal{L}\mathcal{M}^{\otimes,2}} \text{BLMod}_B(D)^{\otimes,2}$$

where $\tau$ is a trivial cofibration and $\mu$ is a fibration. Referring again to [13, prop. 2.1.4.6], we find that there exists a fibration of $\infty$-operads $\tilde{\lambda}^{\otimes,2}$ as in the lemma such that $\mu$ is induced by a weak equivalence $X \simeq \widetilde{\text{BLMod}}_M(M)^{\otimes,2}$ in $\mathcal{P}\text{Op}_\infty/\mathcal{L}\mathcal{M}^{\otimes,2}$. In fact, $\widetilde{\text{BLMod}}_M(M)^{\otimes}$ is obtained from $X$ simply by forgetting the marked edges. We will not distinguish notationally between

$$\tau: \text{BLMod}_M(M)^{\otimes} \rightarrow \text{BLMod}_M(M)^{\otimes}$$

and the induced map on marked simplicial sets. Composing with $\tau$ we obtain a weak equivalence

$$\text{BLMod}_M(M)^{\otimes,2} \simeq \widetilde{\text{BLMod}}_M(M)^{\otimes,2}.$$

By [13, prop. 2.1.4.6], the map $\tilde{\lambda}_M^{\otimes,2}$ is a categorical fibration.

We claim that $\tau$ induces an equivalence of $\mathcal{L}\mathcal{M}^{\otimes}$-monoidal categories as in the lemma. By [13, prop. 2.1.4.6] a fibrant object in $(\mathcal{P}\text{Op}_\infty)_{/\mathcal{L}\mathcal{M}^{\otimes,2}}$ is given by an $\infty$-operad $O^{\otimes}$ and a categorical fibration

$$O^{\otimes,2} \rightarrow \mathcal{L}\mathcal{M}^{\otimes,2}.$$

If $O^{\otimes}$, $O'^{\otimes}$ are $\infty$-operads over $\mathcal{L}\mathcal{M}^{\otimes}$ then the Kan complex

$$\text{Map}_{(\mathcal{P}\text{Op}_\infty)_{/\mathcal{L}\mathcal{M}^{\otimes,2}}}(O^{\otimes,2}, O'^{\otimes,2})$$

is a model for the mapping space $\text{Map}_{\mathcal{P}\text{Op}_\infty/\mathcal{L}\mathcal{M}^{\otimes}}(O^{\otimes}, O'^{\otimes})$. Therefore, by the Yoneda lemma, a weak equivalence

$$f: O^{\otimes,2} \rightarrow O'^{\otimes,2}$$

in $(\mathcal{P}\text{Op}_\infty)_{/\mathcal{L}\mathcal{M}^{\otimes,2}}$ is induced by an equivalence of $\infty$-operads $f: O^{\otimes} \rightarrow O'^{\otimes}$ over $\mathcal{L}\mathcal{M}^{\otimes}$. If $O^{\otimes}$ is an $\mathcal{L}\mathcal{M}$-monoidal category (i.e $O^{\otimes} \rightarrow \mathcal{L}\mathcal{M}^{\otimes}$ is a coCartesian fibration), then by the abstract characterization of coCartesian edges [12, prop. 2.4.4.3] we see that $O'^{\otimes} \rightarrow \mathcal{L}\mathcal{M}^{\otimes}$ is also an $\mathcal{L}\mathcal{M}$-monoidal category. Our claim follows by taking $f = \tau$. 57
In particular, $\tau$ induces equivalences of underlying $\infty$-categories

$$
\bar{\operatorname{BMod}}_M(\mathcal{M}_a) \simeq \text{BLMod}_M(\mathcal{M})_a \simeq \text{BLMod}_M(\mathcal{M})^\otimes,
$$

$$
\bar{\operatorname{LMod}}_M(\mathcal{M}_m) \simeq \text{BLMod}_M(\mathcal{M})_m^\otimes \simeq \text{BLMod}_M(\mathcal{M})^\otimes.
$$

Taking fibers over the two colors of $\mathcal{L}\mathcal{M}^\otimes$, it remains to show that associated maps

$$
\tilde{\lambda}^\otimes_a : \bar{\operatorname{BLMod}}_M(\mathcal{M}^\otimes) \rightarrow \text{BLMod}_A(\mathcal{C}^\otimes) \times \text{BLMod}_B(\mathcal{D}^\otimes) = \text{BMod}_A(\mathcal{C}) \times \text{BMod}_B(\mathcal{D}),
$$

$$
\tilde{\lambda}^\otimes_m : \bar{\operatorname{BLMod}}_M(\mathcal{M}^\otimes) \rightarrow \text{BLMod}_A(\mathcal{C}^\otimes) \times \text{BLMod}_B(\mathcal{D}^\otimes) = \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D})
$$

are right fibrations.

It suffices to show that the compositions

$$
\bar{\operatorname{BMod}}_M(\mathcal{M}_a) \xrightarrow{\sim} \bar{\operatorname{BLMod}}_M(\mathcal{M})^\otimes \xrightarrow{\tilde{\lambda}^\otimes_a} \text{BMod}_A(\mathcal{C}) \times \text{BMod}_B(\mathcal{D}),
$$

$$
\bar{\operatorname{LMod}}_M(\mathcal{M}_m) \xrightarrow{\sim} \bar{\operatorname{BLMod}}_M(\mathcal{M})^\otimes \xrightarrow{\tilde{\lambda}^\otimes_m} \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D})
$$

obtained using the factorization of $\lambda^\otimes$ above, are right fibrations. The first is a right fibration by the proof of [13, lem. 5.2.2.29]. The second is a right fibration by the same argument *mutatis mutandis*; we nevertheless give the details.

The various categories of algebras and left modules at play form a diagram as follows,

\[
\begin{array}{ccc}
\text{LMod}_M(\mathcal{M}_m) & \longrightarrow & \text{LMod}(\mathcal{M}) \\
\downarrow & & \downarrow \\
\text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}) & \longrightarrow & \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}) \\
\end{array}
\]

\[
\begin{array}{ccc}
\{\ast\} & \longrightarrow & \text{Alg}(\mathcal{M}_a) \times \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a) \\
\downarrow & & \downarrow \\
\text{LMod}_M(\mathcal{M}_m) & \longrightarrow & \text{LMod}(\mathcal{M}) \\
\end{array}
\]

in which the lower and left rectangles are Cartesian. Consequently, the induced square

\[
\begin{array}{ccc}
\text{LMod}_M(\mathcal{M}_m) & \longrightarrow & \text{LMod}(\mathcal{M}) \\
\downarrow & & \downarrow \theta \\
\text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}) & \longrightarrow & \text{Alg}(\mathcal{M}_a) \times \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a) \left( \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}) \right)
\end{array}
\]

is Cartesian. We will show that $\theta$ is a right fibration. Let $\theta'$ be the projection

$$
\text{Alg}(\mathcal{M}_a) \times \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a) \left( \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}) \right) \rightarrow \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}).$$
Since $\theta$ is a categorical fibration, it will suffice to show that $\theta$ and 
$$\theta \circ \theta' = \text{LMod}(\lambda) : \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})$$
are right fibrations. The map $\theta'$ is a pullback of 
$$\text{Alg}(\lambda_\alpha) : \text{Alg}(\mathcal{M}_\alpha) \to \text{Alg}(\mathcal{C}_\alpha) \times \text{Alg}(\mathcal{D}_\alpha).$$
We are therefore reduced to showing that $\text{Alg}(\lambda_\alpha)$ and $\text{LMod}(\lambda)$ are right fibrations. To fix ideas, let us focus on the latter. Let us denote 
$$\mathcal{E}^\otimes := \mathcal{C}^\otimes \times \mathcal{D}^\otimes \quad \text{and} \quad q := \text{LMod}(\lambda).$$
Since $\lambda_\alpha$ is by assumption a categorical fibration, it follows that $\text{Alg}(\lambda_\alpha)$ is an inner fibration, and hence that $\theta'$ is an inner fibration. Since $\theta$ is an inner fibration as well, it follows that 
$$q : \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{E})$$
is an inner fibration. According to [12, prop. 2.4.2.4], to show that $q$ is a right fibration we must check two properties:

(A) for any morphism in $\text{LMod}(\mathcal{E})$, any lift of the target may be extended to a Cartesian lift of the morphism,

(B) every morphism in $\text{LMod}(\mathcal{M})$ is $q$-Cartesian.

Recall the characterization of a Cartesian edge as relative colimit from [12, ex. 4.3.1.4]. We apply [13, cor. 3.2.2.3] with $O^\otimes := \mathcal{L}\mathcal{M}^\otimes$,

$$p := \lambda : \mathcal{M}^\otimes \to \mathcal{E}^\otimes,$$

and $K := \Delta^0$ (noting that the objects $X \in O$ considered there have as two possible values the two colors $a, m$ of the $\infty$-operad $\mathcal{L}\mathcal{M}^\otimes$). The condition of the Corollary holds because of our assumption that $\lambda_\alpha$ and $\lambda_m$ are right-fibrations, and the conclusions (1) and (2) of the Corollary imply the two properties (A), (B) as required. □

7. Koszul duality for $\mathcal{L}\mathcal{M}$-monoidal pairings

Let 
$$\lambda : \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \mathcal{D}^\otimes$$
be a left-representable $\mathcal{L}\mathcal{M}$-monoidal pairing (definition 6.2.1). Our goal in this section is to prove that under certain assumptions, the induced pairing

$$\text{LMod}(\lambda) : \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})$$
is left representable (theorem 7.5.1 — our analog of [13, 5.2.2.27]). An outline of the proof was already given in §6. However, armed as we are now with the constructions of sections 5 and 6, we can be more precise.
The proof revolves around the commutative diagram:

\[
\begin{array}{c}
\text{LMod}(\text{BLMod}_M(\mathcal{M})) \quad \text{LMod}(\mathcal{M}) \\
\text{LMod}(\text{BLMod}_A(\mathcal{C})) \times \text{LMod}(\text{BLMod}_1(\mathcal{D})) \quad \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}),
\end{array}
\]

where \( A = X_a \in \text{Alg}(C_a) \) is the underlying algebra of \( X \) and \( M \in \text{Alg}(M_a) \) a lift of \((A, \mathbb{1})\), whose existence requires further assumptions on the pairing \( \lambda_a \).\footnote{In fact, we will modify this diagram using proposition 6.3.1 to ensure that the vertical maps are categorical fibrations.}

We start by lifting \( X \in \text{LMod}(\mathcal{C}) \) to \( X' \in \text{LMod}(\text{BLMod}_A(\mathcal{C})) \) using proposition 7.1.1. The advantage of \( X' \) over \( X \) is that its underlying algebra is the unit object of \( \text{BMod}_A(\mathcal{C}) \). This makes it easy to find a left universal lift \( Z' \in \text{LMod}(\text{BLMod}_M(\mathcal{M})) \) of \( X' \), provided that the underlying pairings of

\[
\text{BLMod}_M(\mathcal{M}) \otimes \rightarrow \text{BLMod}_A(\mathcal{C}) \otimes \times \text{BLMod}_1(\mathcal{D}) \otimes
\]

are left representable, see proposition 7.2.1. Showing the left representability of these underlying pairings is the main step of the proof.

The left representability of

\[
\text{LMod}_M(\mathcal{M}) \rightarrow \text{LMod}_A(\mathcal{C}) \times \text{LMod}_1(\mathcal{D})
\]

was proved in [13, lem. 5.2.40] using free resolutions of bimodules. In proposition 7.3.3 we use free resolutions of left modules to show the left representability of

\[
\text{LMod}_M(\mathcal{M}) \rightarrow \text{LMod}_A(\mathcal{C}) \times \text{LMod}_1(\mathcal{D}).
\]

To complete the proof we show that the forgetful functor

\[
\text{LMod}(\text{BLMod}_M(\mathcal{M})) \rightarrow \text{LMod}(\mathcal{M})
\]

preserves left representable objects, see proposition 7.4.1. Applying the forgetful functor to \( Z' \in \text{LMod}(\text{BLMod}_M(\mathcal{M})) \) produces the desired left universal lift \( Z \in \text{LMod}(\mathcal{M}) \) of \( X \).

7.1. **Lifting left modules to left modules over algebras in bimodules.** Let \( \mathcal{C}^\otimes \rightarrow \mathcal{LM}^\otimes \) be an \( \mathcal{LM} \)-monoidal category and \( A \in \text{Alg}(C_a) \) an algebra. Let \( \text{BLMod}_A(\mathcal{C}) \) be the \( \mathcal{LM} \)-monoidal category of construction 5.4.5. Consider the forgetful functor

\[
\text{LMod}(\text{BLMod}_A(\mathcal{C})) \rightarrow \text{LMod}(\mathcal{C}).
\]
**Proposition 7.1.1.** Assume \( \mathcal{C} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly. Then every object \( X \in \text{LMod}(\mathcal{C}) \) over \( A \in \text{Alg}(\mathcal{C}_a) \) admits a lift to \( \text{LMod}(\text{BLMod}_A(\mathcal{C})) \) over the trivial algebra \( A \in \text{Alg}(\mathcal{C}_a) \).

**Proof.** The statement immediately follows from proposition 6.1.1 using the universal property of the pullback

\[
\begin{array}{ccc}
\text{LMod}(\text{BLMod}_A(\mathcal{C})) & \longrightarrow & \text{Alg}(\mathcal{C}_a) \\
\downarrow \text{fgt} & & \downarrow \text{fgt} \\
\text{LMod}(\mathcal{C}) & \longrightarrow & \text{Alg}(\mathcal{C}_a).
\end{array}
\]

\( \square \)

### 7.2. Koszul duals of modules over trivial algebras.

The next proposition, which is analogous to [13, prop. 5.2.2.28], concerns the special case of our Koszul duality in which the algebra in question is trivial. In theorem 7.5.1 we apply this to an arbitrary module \((A, X)\) by regarding \( A \) as the trivial algebra in the category of \( A \)-bimodules. Thus, we will apply proposition 7.2.1 to an \( \mathcal{L}\mathcal{M} \)-monoidal pairing of the form

\[
\text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_A(\mathcal{C})^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \text{BLMod}_1(\mathcal{D})^\otimes \simeq \text{BLMod}_A(\mathcal{C})^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \mathcal{D}^\otimes.
\]

**Proposition 7.2.1.** Let

\[\lambda: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{L}\mathcal{M}^\otimes} \mathcal{D}^\otimes\]

be an \( \mathcal{L}\mathcal{M} \)-monoidal pairing. Assume the underlying pairings

\[\lambda_a: \mathcal{M}_a \to \mathcal{C}_a \times \mathcal{D}_a \quad \text{and} \quad \lambda_m: \mathcal{M}_m \to \mathcal{C}_m \times \mathcal{D}_m\]

are left representable. Consider the induced pairing of \( \infty \)-categories

\[\text{LMod}(\lambda): \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D}).\]

Let \( A \in \text{Alg}(\mathcal{C}_a) \) be a trivial algebra object of \( \mathcal{C}_a \). Then:

1. Every left \( A \)-module \( X \in \text{LMod}(\mathcal{C}) \) over \( A \in \text{Alg}(\mathcal{C}_a) \) has a left universal lift, i.e. a left universal object \( Y \in \text{LMod}(\mathcal{M}) \) over \( X \in \text{LMod}(\mathcal{C}) \).
2. Let \( X \in \text{LMod}(\mathcal{C}) \) be a left module over \( A \in \text{Alg}(\mathcal{C}_a) \) and let \( Y \in \text{LMod}(\mathcal{M}) \) be an object over \( X \). Then \( Y \in \text{LMod}(\mathcal{M}) \) is a left universal lift of \( X \) if its images \( Y_m \in \mathcal{M}_m, Y_a \in \mathcal{M}_a \) are left universal lifts of \( X_m \in \mathcal{C}_m, A \in \mathcal{C}_a \) respectively.

**Proof.** We have to show that \( \text{LMod}(\mathcal{M}) \times_{\text{LMod}(\mathcal{C})} \{X\} \) has a terminal object, and that an object \( Y \in \text{LMod}(\mathcal{M}) \times_{\text{LMod}(\mathcal{C})} \{X\} \) is terminal if and only if the objects \( Y_m \in \mathcal{M}_m \times_{\mathcal{C}_m} \{X_m\} \) and \( Y_a \in \mathcal{M}_a \times_{\mathcal{C}_a} \{A\} \) are terminal.
Consider \( \mathcal{M}^\otimes, \mathcal{C}^\otimes \) and \( \mathcal{LM}^\otimes \) as coCartisian fibrations over \( \mathcal{LM}^\otimes \). Since \( A \) is a trivial algebra, the functor \( X : \mathcal{LM}^\otimes \to \mathcal{C}^\otimes \) carries coCartesian edges to coCartesian edges. Define \( \mathcal{N}^\otimes \to \mathcal{LM}^\otimes \) as the pullback

\[
\begin{array}{ccc}
\mathcal{N}^\otimes & \longrightarrow & \mathcal{M}^\otimes \\
\downarrow & & \downarrow \\
\mathcal{LM}^\otimes & \xrightarrow{X} & \mathcal{C}^\otimes
\end{array}
\]

in \( (\mathsf{Cat}_\infty)^{\text{coCart}}_{/\mathcal{LM}^\otimes} \) (definition 4.1.1). Verification of the Segal conditions, which is straightforward, shows that \( \mathcal{N}^\otimes \to \mathcal{LM}^\otimes \) is an \( \mathcal{LM} \)-monoidal category. Note that the \( \infty \)-category \( \mathcal{N}^\otimes \) is equivalent to the fibered product \( \mathcal{M} \otimes \times \mathcal{C} \otimes \mathcal{LM} \otimes \), since the forgetful functor \( (\mathsf{Cat}_\infty)^{\text{coCart}}_{/\mathcal{LM}^\otimes} \to \mathsf{Cat}_\infty \) preserves limits (see remark 4.1.3). Unwinding definitions, we find that we have equivalences

\[
\text{LMod}(\mathcal{N}) \cong \text{LMod}(\mathcal{M}) \times_{\text{LMod}(\mathcal{C})} \{X\}, \quad \mathcal{N}_a \cong \mathcal{M}_a \times_{\mathcal{C}_a} \{A\}, \quad \mathcal{N}_m \cong \mathcal{M}_m \times_{\mathcal{C}_m} \{X_m\}.
\]

The categories \( \mathcal{N}_a \cong \mathcal{M}_a \times_{\mathcal{C}_a} \{A\} \) and \( \mathcal{N}_m \cong \mathcal{M}_m \times_{\mathcal{C}_m} \{X_m\} \) each possess a terminal object by assumption. Since the forgetful functor preserves and reflects limits [13, cor. 3.2.2.5 applied with \( K = \emptyset \)], LMod(\( \mathcal{N} \)) has a terminal object, and, moreover, \( Y \in \text{LMod}(\mathcal{N}) \) is terminal if and only if the underlying objects \( Y_m \in \mathcal{M}_m \times_{\mathcal{C}_m} \{X_m\} \) and \( Y_a \in \mathcal{M}_a \times_{\mathcal{C}_a} \{A\} \) are both terminal objects, as claimed. □

7.3. Left representability for left modules. Let

\[
\lambda : \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes
\]

be an \( \mathcal{LM} \)-monoidal pairing and \( M \in \text{Alg}(\mathcal{M}_a) \) be an algebra over

\[
(A, B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a).
\]

Consider the \( \mathcal{LM} \)-monoidal pairing

\[
\tilde{\lambda}_M^\otimes : \text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_A(\mathcal{C})^\otimes \times \text{BLMod}_B(\mathcal{D})^\otimes
\]

of proposition 6.3.1. Our next goal is to show that the underlying pairings of \( \tilde{\lambda}_M^\otimes \) are left representable (see proposition 7.3.3). We will need the following statement, which is an addendum to [13 lem. 5.2.2.32].

**Lemma 7.3.1.** In the situation and the notation above, assume that \( B \) is a trivial algebra, that \( \mathcal{M} \) admits realizations of \( M \)-bar constructions \( \otimes \)-compatibly, and that \( \mathcal{C} \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly. Let

\[
F : \mathcal{M}_m \to \text{LMod}_M(\mathcal{M}), \quad F : X \mapsto M \otimes X
\]

be an \( \mathcal{LM} \)-monoidal pairing.
be a left adjoint to the forgetful functor. Then $F$ carries left universal objects of $\mathcal{M}_m$ with respect to the pairing of $\infty$-categories

$$\lambda_m: \mathcal{M}_m \to \mathcal{C}_m \times \mathcal{D}_m$$

to left universal objects of $\text{LMod}_\mathcal{M}(\mathcal{M})$ with respect to the pairing of $\infty$-categories

$$(\lambda_M)_m: \text{LMod}_\mathcal{M}(\mathcal{M}) \to \text{LMod}_\mathcal{A}(\mathcal{C}) \times \text{LMod}_\mathcal{B}(\mathcal{D}).$$

The proof will require the following categorical lemma.

**Lemma 7.3.2.** Let

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow p & & \downarrow q \\
\overline{\mathcal{C}} & \xrightarrow{F} & \overline{\mathcal{D}}
\end{array}$$

be a commutative diagram of $\infty$-categories. Assume that $p, q$ are Cartesian fibrations, and that $F, \overline{F}$ have right adjoints $F \dashv U, \overline{F} \dashv \overline{U}$ such that $p \circ \overline{U} = U \circ q$. Let $X \in \mathcal{C}$ be an object, and denote the fiber of $p$ over $X$ by $\mathcal{C}_X$ and the fiber of $q$ over $FX$ by $\overline{\mathcal{D}}_{FX}$. Then the restriction of $F$ to the fibers over $X$ and $FX$,

$$F_X: \mathcal{C}_X \to \overline{\mathcal{D}}_{FX},$$

has a right adjoint given by the composition

$$\overline{\mathcal{D}}_{FX} \xrightarrow{\eta_X} \mathcal{C}_X,$$

where $\eta_X$ is the functor induced by the Cartesian fibration $p$ and the unit map $\eta_X: X \to UFX$.

**Proof.** With $X \in \mathcal{C}_X$ and $\overline{Y} \in \overline{\mathcal{D}}_{FX}$ fixed, an elementary calculation with adjoints and Cartesian fibrations yields a homotopy equivalence of mapping spaces

$$(\ast) \quad \text{Map}_{\mathcal{C}_X}(X, \eta_X UFX \overline{Y}) \simeq \text{Map}_{\overline{\mathcal{D}}_{FX}}(FX, \overline{Y}).$$

A straightforward but tedious construction, based, for instance, on the twisted arrow category, upgrades (\ast) to an equivalence of coCartesian fibrations over $\mathcal{C}_X^\text{op} \times \overline{\mathcal{D}}_{FX}$. Alternatively (by interpreting $F, \overline{F}$ as a morphism of diagrams of $\infty$-categories), this is a special case of \cite[prop. 2.1.7]{Pi}. \hfill \square

**Proof of lemma 7.3.1.** Let

$$\begin{array}{ccc}
F': & \mathcal{C}_m & \to \text{LMod}_\mathcal{A}(\mathcal{C}) \\
& & \\
& & \\
& & \\
G': & \text{LMod}_\mathcal{A}(\mathcal{C}) & \to \mathcal{C}_m
\end{array}$$

be left adjoints to the forgetful functors

$$\begin{array}{ccc}
G': & \text{LMod}_\mathcal{A}(\mathcal{C}) & \to \mathcal{C}_m \\
& & \\
& & \\
& & \\
G': & \text{LMod}_B(\mathcal{D}) & \to \mathcal{D}_m
\end{array}$$

be a left adjoint to the forgetful functor. Then $F$ carries left universal objects of $\mathcal{M}_m$ with respect to the pairing of $\infty$-categories

$$\lambda_m: \mathcal{M}_m \to \mathcal{C}_m \times \mathcal{D}_m$$

to left universal objects of $\text{LMod}_\mathcal{M}(\mathcal{M})$ with respect to the pairing of $\infty$-categories

$$(\lambda_M)_m: \text{LMod}_\mathcal{M}(\mathcal{M}) \to \text{LMod}_\mathcal{A}(\mathcal{C}) \times \text{LMod}_B(\mathcal{D}).$$

The proof will require the following categorical lemma.

**Lemma 7.3.2.** Let

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow p & & \downarrow q \\
\mathcal{C} & \xrightarrow{\overline{F}} & \mathcal{D}
\end{array}$$

be a commutative diagram of $\infty$-categories. Assume that $p, q$ are Cartesian fibrations, and that $F, \overline{F}$ have right adjoints $F \dashv U, \overline{F} \dashv \overline{U}$ such that $p \circ \overline{U} = U \circ q$. Let $X \in \mathcal{C}$ be an object, and denote the fiber of $p$ over $X$ by $\mathcal{C}_X$ and the fiber of $q$ over $FX$ by $\overline{\mathcal{D}}_{FX}$. Then the restriction of $F$ to the fibers over $X$ and $FX$,

$$F_X: \mathcal{C}_X \to \overline{\mathcal{D}}_{FX},$$

has a right adjoint given by the composition

$$\overline{\mathcal{D}}_{FX} \xrightarrow{\overline{F}_X} \mathcal{C}_X,$$

where $\overline{F}_X$ is the functor induced by the Cartesian fibration $p$ and the unit map $\eta_X: X \to UFX$.

**Proof.** With $X \in \mathcal{C}_X$ and $\overline{Y} \in \overline{\mathcal{D}}_{FX}$ fixed, an elementary calculation with adjoints and Cartesian fibrations yields a homotopy equivalence of mapping spaces

$$(\ast) \quad \text{Map}_{\mathcal{C}_X}(X, \eta_X UFX \overline{Y}) \simeq \text{Map}_{\overline{\mathcal{D}}_{FX}}(FX, \overline{Y}).$$

A straightforward but tedious construction, based, for instance, on the twisted arrow category, upgrades (\ast) to an equivalence of coCartesian fibrations over $\mathcal{C}_X^\text{op} \times \overline{\mathcal{D}}_{FX}$. Alternatively (by interpreting $F, \overline{F}$ as a morphism of diagrams of $\infty$-categories), this is a special case of \cite[prop. 2.1.7]{Pi}. \hfill \square

**Proof of lemma 7.3.1.** Let

$$\begin{array}{ccc}
F': & \mathcal{C}_m & \to \text{LMod}_\mathcal{A}(\mathcal{C}) \\
& & \\
& & \\
& & \\
G': & \text{LMod}_\mathcal{A}(\mathcal{C}) & \to \mathcal{C}_m
\end{array}$$

be left adjoints to the forgetful functors

$$\begin{array}{ccc}
G': & \text{LMod}_\mathcal{A}(\mathcal{C}) & \to \mathcal{C}_m \\
& & \\
& & \\
& & \\
G': & \text{LMod}_B(\mathcal{D}) & \to \mathcal{D}_m
\end{array}$$
Modifying $F$ by a homotopy if necessary, we may assume that the diagram

\[
\begin{array}{ccc}
\mathcal{M}_m & \xrightarrow{F} & \text{LMod}_M(\mathcal{M}) \\
\downarrow & & \downarrow \\
\mathcal{C}_m \times \mathcal{D}_m & \xrightarrow{F' \times F''} & \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D})
\end{array}
\]

is commutative. For each $X \in \mathcal{C}_m$, $F$ induces a functor

\[
f : \mathcal{M}_m \times \mathcal{C}_m \{X\} \to \text{LMod}_M(\mathcal{M}) \times_{\text{LMod}_A(\mathcal{C})} \{F'(X)\}, \quad f : Z \mapsto M \otimes Z.
\]

We have to show that $F$ preserves left universal objects over $X$, or equivalently that $f$ preserves terminal objects. We will show that $f$ is an equivalence of $\infty$-categories.

By lemma 7.3.2, the functor $f$ has a right adjoint $g$, given by the composition

\[
g : \text{LMod}_M(\mathcal{M}) \times_{\text{LMod}_A(\mathcal{C})} \{F'(X)\} \to \mathcal{M}_m \times \mathcal{C}_m \{G''F''X\} \xrightarrow{\eta_X^*} \mathcal{M}_m \times \mathcal{C}_m \{X\}
\]

of the forgetful functor and the functor $\eta_X^*$ induced by the unit map $\eta_X : X \to G''F''X$.

We claim that the unit and counit of $f \dashv g$ are both natural equivalences.

Let $u : \text{id} \to g \circ f$ be the unit map of $f \dashv g$. For every object $Z \in \mathcal{M}_m \times \mathcal{C}_m \{X\}$ with image $Y \in \mathcal{D}_m$, the unit map $u_Z : Z \to (g \circ f)(Z)$ has image in $\mathcal{D}_m$ equivalent to the unit map

\[
Y \to (G'' \circ F'')(Y) \simeq B \otimes Y.
\]

Since $B$ is a trivial algebra, the latter is an equivalence. Since $\mathcal{M}_m \times \mathcal{C}_m \{X\} \to \mathcal{D}_m$ is a right fibration it follows from [12, prop. 2.1.1.5] that $u_Z$ is an equivalence.

The same argument shows that the counit of $f \dashv g$ is an equivalence, proving our claim.

□

With lemma 7.3.1 at hand, we return to showing that the underlying pairings of $\tilde{\lambda}^\otimes_M$ are left representable. This statement is an addendum to [13, lem. 5.2.2.40].

**Proposition 7.3.3.** Let $M \in \text{Alg}(\mathcal{M}_a)$ be an object over $(A,B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a)$ and assume $\mathcal{M}$ admits realizations of $M$-bar constructions $\otimes$-compatibly, and similarly for $A$ and $B$. Assume in addition that:

1. The object $B \in \text{Alg}(\mathcal{D}_a)$ is a trivial algebra.
2. The underlying pairings $\lambda_a, \lambda_m$ are left representable.
3. The $\infty$-categories $\mathcal{D}_a, \mathcal{D}_m$ admit totalizations of cosimplicial objects.

Then the induced pairings of $\infty$-categories

\[
(\lambda_M)_a : _M\text{BMod}_M(\mathcal{M}) \to _A\text{BMod}_A(\mathcal{C}) \times _B\text{BMod}_B(\mathcal{D}),
\]

\[
(\lambda_M)_m : \text{LMod}_M(\mathcal{M}) \to \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D})
\]

are both left representable.
Proof. The right fibration \((\lambda_M)_a\) is left representable by \cite[prop. 5.2.2.40]{13}. We will prove that \((\lambda_M)_m\) is left representable. Our proof rests on the following facts:

1. \cite[ex. 4.7.2.5]{13}: Let \(U : \text{LMod}_A(C_m) \to C_m\) be the forgetful functor and let \(X_* \in \text{LMod}_A(C)\) be a \(U\)-split simplicial object. Then \(X_*\) admits a colimit in \(\text{LMod}_A(C)\) which is preserved by \(U\).

2. \cite[cor. 4.7.2.11]{13}: Let \(q : C \to C\) be a right fibration of \(\infty\)-categories and \(X_*\) a simplicial object of \(\overline{C}\). If \(q(X_*)\) is split in \(C\) then \(X_*\) is split in \(\overline{C}\).

Fix an object \(X \in \text{LMod}_A(C)\). Let \(U_A : \text{LMod}_A(C_m) \to C_m\) denote the forgetful functor and \(f\) its left adjoint. By Lurie-Bar-Beck \cite[thm. 4.7.3.5]{13}, the functor \(U_A\) exhibits \(\text{LMod}_A(C)\) as monadic over \(C_m\). Invoking \cite[prop. 4.7.3.14]{13} we deduce that there exists a \(U_A\)-split simplicial object (in the sense of \cite[def 4.7.2.2]{13}): 

\[
\Delta^{op} \xrightarrow{\sim} (\Delta^{op})^b \xrightarrow{X_*} \text{LMod}_A(C) \\
\downarrow \quad \quad \downarrow U_A \\
\Delta_{-\infty}^{op} \xrightarrow{\sim} \to C_m
\]

such that \(\overline{X}_{-\infty} = X\) and \(\overline{X}_*\) is a colimit diagram witnessing an equivalence \(X \simeq \underleftarrow{\text{colim}} X_*\) between \(X\) and the colimit of \(X_* := \overline{X}|_{\Delta^{op}}\), and such that each \(X_n \in \text{LMod}_A(C), n \neq -\infty\), belongs to the essential image of the functor \(f\) (i.e. \(X_*\) is a free resolution of \(X \in \text{LMod}_A(C)\)).

The pairing \((\lambda_M)_m\) is classified by functors

\[
\chi : \text{LMod}_B(D)^{op} \to \text{PSh}(\text{LMod}_A(C)), \quad \chi : Y \mapsto \chi_Y \\
\chi' : \text{LMod}_A(C)^{op} \to \text{PSh}(\text{LMod}_B(D)), \quad \chi' : X \mapsto \chi'_X.
\]

In particular, \(\chi_Y\) is classified by the right fibration

\[
(\lambda_M)_Y : \text{LMod}_M(M) \times_{\text{LMod}_B(D)} \{Y\} \to \text{LMod}_A(C) \times \{Y\}.
\]

We claim that the augmented cosimplicial diagram

\[
\Delta^+ \xrightarrow{X_*^{op}} \text{LMod}_A(C)^{op} \xrightarrow{\chi'} \text{PSh}(\text{LMod}_B(D))
\]

is a colimit diagram, witnessing an equivalence of presheaves

\[
\chi'_X \xrightarrow{\sim} \varprojlim \chi'_{X_n}.
\]

Before establishing this claim, we explain how the proposition follows from the claim. We wish to prove that \(\chi'_X\) is representable. Since \(\mathcal{D}\) admits totalizations of cosimplicial objects, it is enough to show that each \(\chi'_{X_n}\) is representable, which follows from condition (2) together with lemma \cite[7.3.1]{73.1}.
To establish the claim, it will suffice to check that for arbitrary $Y \in \text{LMod}_B(D)$ the induced map

$$\chi'_X(Y) \to \lim \chi'_{X_\bullet}(Y)$$

is a homotopy equivalence. Since $\chi_Y(X)$ and $\chi'_X(Y)$ are both equivalent to the fiber of $(\lambda_M)_m$ over $(X,Y)$, we may equally show that

$$(7) \quad \chi_Y(X) \to \lim \chi_Y(X_\bullet)$$

is a homotopy equivalence of spaces. Therefore we fix $Y \in \text{LMod}_B(D)$ for the rest of the proof. We will prove that eq. (7) is a homotopy equivalence using [13, prop 5.2.2.39], which requires some preparation.

Write $Y_m \in D_m$ for the image of $Y$ under the forgetful functor $\text{LMod}_B(D) \to D_m$. We first construct an $\mathcal{LM}$-monoidal $\infty$-category $\mathcal{M}_Y$ exhibiting $\mathcal{M}_m \times D_m \{Y_m\}$ as left tensored over $\mathcal{M}_a \times D_a \{B\}$. Let $\mathcal{M}^\otimes_Y$ be the pullback

$$\begin{array}{ccc}
\mathcal{M}^\otimes_Y & \longrightarrow & \mathcal{M}^\otimes \\
\downarrow & \searrow \gamma & \downarrow \\
\mathcal{LM}^\otimes & \longrightarrow & D^\otimes.
\end{array}$$

Since $B$ is a trivial algebra we see that $\mathcal{M}^\otimes_Y \to \mathcal{LM}^\otimes$ is a coCartesian fibration. Also, note that the horizontal top morphism $\mathcal{M}_Y \to \mathcal{M}$ is an $\mathcal{LM}$-monoidal functor.

Next we express the source of (4) as a category of left modules in $\mathcal{M}_Y$. Regard $M$ as an algebra in $(\mathcal{M}_Y)_a \simeq \mathcal{M}_a \times D_a \{B\}$, and consider the $\infty$-category

$$\text{LMod}_M(\mathcal{M}_Y) \simeq \text{LMod}_M(\mathcal{M}) \times_{\text{LMod}_B(D)} \{Y\}.$$  

We have a commuting square

$$\begin{array}{ccc}
\text{LMod}_M(\mathcal{M}_Y) & \xrightarrow{U_{\mathcal{M}_Y}} & \mathcal{M}_m \times \mathcal{C}_m \{Y_m\} \\
\downarrow q & & \downarrow q' \\
\text{LMod}_A(\mathcal{C}) & \xrightarrow{U_A} & \mathcal{C}_m
\end{array}$$

where

$$U_{\mathcal{M}_Y} : \text{LMod}_M(\mathcal{M}_Y) \to (\mathcal{M}_Y)_m = \mathcal{M}_m \times D_m \{Y_m\}$$

is the forgetful functor, and $q', q$ are the compositions

$$q' : \mathcal{M}_m \times \mathcal{C}_m \{Y_m\} \to \mathcal{M}_m \xrightarrow{\lambda_m} \mathcal{C}_m \times D_m \to \mathcal{C}_m,$$

$$q : \text{LMod}_M(\mathcal{M}_Y) \to \text{LMod}_M(\mathcal{M}) \xrightarrow{(\lambda_M)_m} \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{C}) \to \text{LMod}_A(\mathcal{C}).$$
Moreover, the right fibration \( q \) is equivalent to the right fibration \((\lambda_M)_Y\) of (II) and hence classifies the presheaf \( \chi_Y \). To see this, consider the following diagram:

\[
\begin{array}{ccc}
\text{LMod}_M(\mathcal{M}_Y) & \xrightarrow{\gamma} & \text{LMod}_M(\mathcal{M}) \\
\downarrow^{(\lambda_M)_Y} & & \downarrow^{(\lambda_M)_m} \\
\text{LMod}_A(\mathcal{C}) \times \{Y\} & \xrightarrow{\gamma} & \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D}) \rightarrow \text{LMod}_A(\mathcal{C})
\end{array}
\]

and note that the composition of the middle row is the identity, and that \( q \) is given by the composition of the top path from \( \text{LMod}_M(\mathcal{M}_Y) \) to \( \text{LMod}_A(\mathcal{C}) \).

We are therefore in the situation of [13, cor. 5.2.2.39], i.e. the following conditions are equivalent:

1. The map (II) is a homotopy equivalence.
2. For every simplicial object \( N_\bullet \) in \( \text{LMod}_M(\mathcal{M}_Y) \) making the following solid diagram commute

\[
\begin{array}{ccc}
\Delta^{op} & \xrightarrow{N_\bullet} & \text{LMod}_M(\mathcal{M}_Y) \\
\downarrow & \exists \downarrow & \downarrow^{q} \\
(\Delta^{op})^p & \xrightarrow{\bar{N}_\bullet} & \text{LMod}_A(\mathcal{C})
\end{array}
\]

there exists an extension to a \( q \)-colimit diagram \( \bar{N}_\bullet^+ \) as indicated. Equivalently (since \( \bar{N}_\bullet \) itself is a colimit diagram) there exists a geometric realization \(|N_\bullet|\) which is preserved by the forgetful functor \( q \).

Observe that \( U_{\mathcal{M}_Y}(N_\bullet) \) is a simplicial object in \( \mathcal{M}_m \times \mathcal{C}_m \{Y_m\} \) and the functor \( q' : \mathcal{M}_m \times \mathcal{C}_m \{Y_m\} \rightarrow \mathcal{C}_m \) is a right fibration. Since the image

\[
q'(U_{\mathcal{M}_Y}(N_\bullet)) = U_A(q(N_\bullet)) = U_A(X_\bullet|\Delta^{op})
\]

is a split simplicial object and \( q' \) is a right fibration, it follows from [13, cor. 4.7.2.11] that \( U_{\mathcal{M}_Y}(N_\bullet) \) is a split simplicial object in \( \mathcal{M} \times \mathcal{C} \{Y\} \).

We therefore have a \( U_{\mathcal{M}_Y} \)-split simplicial object \( N_\bullet \) of

\[
\text{LMod}_M(\mathcal{M}_Y) = \text{LMod}_M(\mathcal{M} \times \mathcal{D} \{Y\}),
\]

so [13, ex. 4.7.2.5] implies that \( N_\bullet \) admits a colimit in \( \text{LMod}_M(\mathcal{M}_Y) \) which is preserved by \( U_{\mathcal{M}_Y} \).

We claim that the colimit of \( \text{LMod}_M(\mathcal{M}_Y) \) is preserved by \( q \). To see this, note that \( U_{\mathcal{M}_Y}(N_\bullet) \) is split, so the colimit of \( N_\bullet \) is preserved by \( U_{\mathcal{M}_Y} \circ q' = q \circ U_A \). Applying [13, cor. 4.7.2.11] again, we conclude that the colimit of \( N_\bullet \) is preserved.
by the right fibration $q$. It follows that the augmented cosimplicial diagram (3) is a colimit diagram, as claimed. This completes the proof of the proposition. □

7.4. Forgetting left module structures and left universal objects. Let $\mathcal{C}^\otimes$ be an $\mathcal{LM}$-monoidal category and let $A$ be an algebra in $\mathcal{A}_a$. Assume $\mathcal{C}^\otimes$ admits realizations of $A$-bar constructions $\otimes$-compatibly. Then the unit map $1 \to A$ gives rise to a lax $\mathcal{LM}$-monoidal map of $\mathcal{LM}$-monoidal categories

$$\text{BLMod}_A(\mathcal{C})^\otimes \to \text{BLMod}_1(\mathcal{C})^\otimes \simeq \mathcal{C}^\otimes$$

(construction 5.4.5), hence to a functor

$$\text{LMod} (\text{BLMod}_A(\mathcal{C})) \to \text{LMod}(\mathcal{C})$$

which corresponds under the equivalence of proposition 6.1.1 to forgetting $A$.

We return to the situation of section 7.3 given by al $\mathcal{LM}$-monoidal pairing

$$\lambda^\otimes: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}} \mathcal{D}^\otimes,$$

and an algebra $M \in \text{Alg}(\mathcal{M}_a)$ over algebras $(A, B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a)$. Assume $\mathcal{M}$ admits realizations of $M$-bar constructions $\otimes$-compatibly, and similarly for $A$. Applying functorial fibrant replacement in the model category $\text{POP}_{\infty/\mathcal{LM}^\otimes}^\otimes$ of $\infty$-preoperads over $\mathcal{LM}^\otimes$ as in the proof of proposition 6.3.1, we obtain an equivalence of $\mathcal{LM}$-monoidal pairings

$$\mathcal{M}^\otimes \to \tilde{\mathcal{M}}^\otimes$$

(in the sense of [13 Construction 5.2.1.14]) and a functor

$$\text{BLMod}_M(\mathcal{M})^\otimes \to \tilde{\mathcal{M}}^\otimes,$$

giving rise to a morphism of pairings

$$\text{LMod} \left( \text{BLMod}_M(\mathcal{M}) \right) \to \text{LMod}(\tilde{\mathcal{M}}),$$

The following proposition, concerning this map of pairings, mirrors [13 prop. 5.2.2.30].
**Proposition 7.4.1.** Let \( M \in \text{Alg}(\mathcal{M}) \) be an object over \((A, B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a)\). Assume that \( B \in \text{Alg}(\mathcal{D}_a) \) is a trivial algebra. Assume that \( \mathcal{M} \) admits realizations of \( M \)-bar constructions \( \otimes \)-compatibly, and similarly for \( A \). Assume that for every \( C \in \mathcal{C}_a \), the Kan complex

\[
\lambda_a^{-1} \{(C, B)\} \subseteq \mathcal{M}
\]

is contractible. Then (with tilde denoting functorial fibrant replacement in the model category of \( \infty \)-preoperads as above) the forgetful functor

\[
\text{LMod} \left( \text{BLMod}_M(\mathcal{M}) \right) \rightarrow \text{LMod}(\widetilde{\mathcal{M}})
\]
carries left universal objects to left universal objects.

**Proof.** In order to lighten notation, we drop the tildes throughout the proof. Let \( X \in \text{LMod}(\text{BLMod}_A(\mathcal{C})) \) and let \( X_0 \in \text{LMod}(\mathcal{C}) \) be its image under the forgetful functor \( \text{LMod}(\text{BLMod}_A(\mathcal{C})) \rightarrow \text{LMod}(\mathcal{C}) \). Consider the commutative diagram

\[
\begin{array}{ccc}
\text{LMod}(\text{BLMod}_M(\mathcal{M})) \times_{\text{LMod}(\text{BLMod}_A(\mathcal{C}))} \{X\} & \longrightarrow & \text{LMod}(\mathcal{M}) \times_{\text{LMod}(\mathcal{C})} \{X_0\} \\
\downarrow & & \downarrow \\
\text{LMod}(\text{BLMod}_B(\mathcal{D})) \times \{X\} & \longrightarrow & \text{LMod}_B(\mathcal{D}) \times \{X_0\}.
\end{array}
\]

We have to show that the top horizontal functor preserves terminal objects. We will show it is an equivalence. Since \( B \in \text{Alg}(\mathcal{D}_a) \) is a trivial algebra, the bottom horizontal map is an equivalence. Since the vertical maps are right fibrations, it will suffice to show that for each \( Y \in \text{LMod}(\text{BLMod}_B(\mathcal{D})) \) with image \( Y_0 \in \text{LMod}_B(\mathcal{D}) \), the induced map

\[
\text{LMod}(\text{BLMod}_M(\mathcal{M})) \times_{\text{LMod}(\text{BLMod}_A(\mathcal{C})) \times \text{LMod}(\text{BLMod}_B(\mathcal{D}))} \{(X, Y)\}
\]

(8)

\[
\text{LMod}(\mathcal{M}) \times_{\text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})} \{(X_0, Y_0)\}
\]

is a homotopy equivalence. We leverage Lurie’s proof of [13, prop. 5.2.2.30] to verify this claim.

By proposition [6.1.1] we have an equivalence of \( \infty \)-categories

\[
\text{LMod}(\text{BLMod}_A(\mathcal{C})) \simeq \text{LMod}(\mathcal{C}) \times_{\text{Alg}(\mathcal{C}_a)} \text{Alg}(\mathcal{C}_a)_{A/},
\]

and similarly for \( \text{LMod}(\text{BLMod}_B(\mathcal{D})) \) and \( \text{LMod}(\text{BLMod}_M(\mathcal{M})) \). Let

\[
A' \in \text{Alg}(\mathcal{C}_a)_{A/}, \quad B' \in \text{Alg}(\mathcal{D}_a)_{B/}
\]

be the images of

\[
X \in \text{LMod}(\mathcal{C}) \times_{\text{Alg}(\mathcal{C}_a)} \text{Alg}(\mathcal{C}_a)_{A/}, \quad Y \in \text{LMod}(\mathcal{D}) \times_{\text{Alg}(\mathcal{D}_a)} \text{Alg}(\mathcal{D}_a)_{B/},
\]

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respectively, and let $A'_0 \in \text{Alg}(\mathcal{C}_a)$, $B'_0 \in \text{Alg}(\mathcal{D}_a)$ be the images of $A', B'$, respectively. Thus for instance,

$$A' = (A \rightarrow A'_0)$$

is an $A$-algebra, and $X_0$ records the action of $A'_0$ on a left module. With this notation, we can identify the domain of (8) with the pullback of the diagram

$$\text{LMod}(\mathcal{M}) \times_{\text{Alg}(\mathcal{M}_a)} \text{Alg}(\mathcal{M}_a)_{M/}$$

by first taking pullbacks along the rows. Reversing the order of the limits, we obtain a homotopy pullback square

$$\text{LMod}(\text{BLMod}_M(\mathcal{M})) \times_{\text{BLMod}_M(\mathcal{C}) \times \text{BLMod}_M(\mathcal{D})} \{(X, Y)\}$$

where the left vertical map is equivalent to (8). By the proof of [13, prop. 5.2.2.30], the right vertical map is a homotopy equivalence. It follows that (8) is a homotopy equivalence, as claimed.

□
7.5. Koszul duality for $\mathcal{LM}$-monoidal pairings. The following proposition corresponds to [13, prop. 5.2.2.27].

**Theorem 7.5.1.** Let

$$\lambda^\otimes: \mathcal{M}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes$$

be an $\mathcal{LM}$-monoidal pairing which satisfies the following properties:

1. If $1$ denotes the unit object of $\mathcal{D}_a$, then the right fibration $\mathcal{M}_a \times \mathcal{D}_a \{1\} \to \mathcal{C}_a$ is a categorical equivalence.
2. The underlying pairings $\lambda_a, \lambda_m$ are left representable.
3. The $\infty$-categories $\mathcal{D}_a, \mathcal{D}_m$ admit totalizations of cosimplicial objects.
4. If $B \in \text{Alg}(\mathcal{D}_a)$ is a trivial algebra and $M \in \text{Alg}(\mathcal{M}_a)$ lies over $(A,B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a)$, then $\mathcal{C}$ admits realizations of $A$-bar constructions $\otimes$-compatibly and $\mathcal{M}$ admits realizations of $M$-bar constructions $\otimes$-compatibly.

Then the induced pairing

$$\text{LMod}(\lambda): \text{LMod}(\mathcal{M}) \to \text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})$$

is left representable.

**Proof.** Fixing $X \in \text{LMod}(\mathcal{C})$ a left module over $A \in \text{Alg}(\mathcal{C}_a)$, our goal is to show that there exists a left universal object in $\text{LMod}(\mathcal{M})$ lying over $X$. By proposition 7.1.1, we can lift $X \in \text{LMod}(\mathcal{C})$ to a left module $X' \in \text{LMod}(\text{BLMod}_A(\mathcal{C}))$ over the trivial algebra $A \in \text{Alg}(\text{BLMod}_A(\mathcal{C}))$.

Let $B$ be a trivial algebra object of $\mathcal{D}_a$. Then the fiber $\mathcal{M}_a \times \mathcal{D}_a \{B\}$ has an induced monoidal structure and condition (1) implies that the right fibration

$$\mathcal{M}_a \times \mathcal{D}_a \{B\} \to \mathcal{C}_a,$$

which upgrades naturally to a monoidal functor, is an equivalence of monoidal $\infty$-categories. It follows that the pair $(A,B)$ can be lifted to an object $M \in \text{Alg}(\mathcal{M}_a)$ in an essentially unique way. By construction 7.4.5 and proposition 6.3.1 we then have an $\mathcal{LM}$-monoidal category $\text{BLMod}_M(\mathcal{M})^\otimes$ which witnesses the action of the monoidal $\infty$-category

$$\text{BLMod}_M(\mathcal{M})^\otimes \simeq_M \text{BMod}_M(\mathcal{M})^\otimes$$

on the $\infty$-category

$$\text{BLMod}_M(\mathcal{M})_m \simeq \text{LMod}_M(\mathcal{M})$$

by relative tensor products, as well as an $\mathcal{LM}$-monoidal pairing

$$\lambda_M: \text{BLMod}_M(\mathcal{M})^\otimes \to \text{BLMod}_A(\mathcal{C})^\otimes \times_{\mathcal{LM}^\otimes} \text{BLMod}_B(\mathcal{D})^\otimes.$$
By proposition 7.3.3, the underlying pairings

\[(\lambda_M)_a : \text{BMod}_A(\mathcal{M}) \to \text{BMod}_A(\mathcal{C}) \times \text{BMod}_B(\mathcal{D}),\]

\[(\lambda_M)_m : \text{LMod}_A(\mathcal{M}) \to \text{LMod}_A(\mathcal{C}) \times \text{LMod}_B(\mathcal{D})\]

are both left-representable. Since \(A\) is a trivial algebra of \(\text{BMod}_A(\mathcal{C})\), we may therefore apply proposition 7.2.1 by substituting the \(\mathcal{LM}\)-monoidal pairing denoted \(\lambda_M\) here for the \(\mathcal{LM}\)-monoidal pairing denoted \(\lambda\) there. In this way, we obtain a left universal object

\[Z' \in \text{LMod}(\widetilde{\text{BMod}}_M(\mathcal{M}))\]

lying over \(X'\).

Let

\[\mathcal{M}^\otimes \to \widetilde{\mathcal{M}}^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} \mathcal{D}^\otimes\]

denote the factorization induced by functorial fibrant replacement in the model category of \(\infty\)-operads as in the preamble to proposition 7.4.1. By proposition 7.4.1, the forgetful functor

\[\text{LMod}(\widetilde{\text{BMod}}_M(\mathcal{M})) \to \text{LMod}(\widetilde{\mathcal{M}})\]

carries \(Z'\) to a left universal object \(\tilde{Z} \in \text{LMod}(\widetilde{\mathcal{M}})\) over \(X \in \text{LMod}(\mathcal{C})\). Since the induced functor

\[\text{LMod}(\widetilde{\mathcal{M}}) \to \text{LMod}(\mathcal{M})\]

ever \(\text{LMod}(\mathcal{C}) \times \text{LMod}(\mathcal{D})\) is an equivalence of pairings, it follows that \(\tilde{Z}\) lifts to a left universal object \(Z \in \text{LMod}(\mathcal{M})\) over \(X\), as hoped. \(\square\)

8. The case of the twisted arrow category

Theorem 7.5.1 applies in a straightforward way to the twisted arrow category of a suitable symmetric monoidal category in which the unit object is terminal. Our first goal in this section (theorem 8.3.5) is to compare the underlying objects of the resulting Koszul duals with the bar construction. Our second goal is to dispense with the assumption regarding the unit object. The final result is stated in corollary 8.4.3.

8.1. The right duality functor. As mentioned in section 2, we use the right duality functor to identify the underlying objects produced by the left duality functor. Our results concerning the right duality functor are best developed in the general setting of \(\mathcal{LM}\)-monoidal pairings. We thus continue with \(\mathcal{LM}\)-monoidal pairings in this subsection before specializing to the twisted arrow category in the next subsection. We begin with an \(\mathcal{LM}\)-monoidal addendum to [13 lem. 5.2.2.33].
Lemma 8.1.1. Let $\lambda^\otimes: \mathcal{M}^\otimes \to C^\otimes \times_{\mathcal{L} \mathcal{M}^\otimes} \mathcal{D}^\otimes$ be an $\mathcal{L} \mathcal{M}$-monoidal pairing and let $M \in \text{Alg}(\mathcal{M}_a)$ be an algebra over $(A,B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{D}_a)$, where $B$ is a trivial algebra object of $\mathcal{D}_a$. Let $\mathbb{1}$ denote the unit object of $\mathcal{M}_a$, and suppose we are given an augmentation $\epsilon: M \to \mathbb{1}$. Then the induced map

$$M = \mathcal{M}_a \coprod M_m$$

carries right universal objects of $M_m$ to right universal objects of $\text{LMod}_M(M_m)$.  

Proof. Fix an object $D \in \mathcal{D}_m$. Let $M = \mathcal{M}_a \coprod M_m$ denote the fiber of $\mathcal{M}^\otimes \to \mathcal{L} \mathcal{M}^\otimes \to \text{Fin}$, over $\langle 1 \rangle \in \text{Fin}^*$, and consider the full subcategory

$$\mathcal{M}_D \subseteq \mathcal{M}, \quad \mathcal{M}_D := \mathcal{M}_a \times_{D_a} \{B\} \coprod M_m \times_{D_m} \{D\}.$$ 

By [13, Proposition 2.2.1.1] the $\infty$-category $\mathcal{M}_D$ is the underlying $\infty$-category of an $\mathcal{L} \mathcal{M}$-monoidal subcategory $\mathcal{M}_D^\otimes \subseteq \mathcal{M}^\otimes$, which exhibits $M_m \times_{D_m} \{D\}$ as left tensored over $\mathcal{M}_a \times_{D_a} \{B\}$. We are interested in the restriction of the induced map $\mathcal{M}_m \to \text{LMod}_M(M_m)$ to the fibers over $D$. Note that $M$ is an algebra object of $\mathcal{M}_a \times_{D_a} \{B\}$, and we have an equivalence

$$\text{LMod}_M(M_D) \simeq \text{LMod}_M(M_m) \times_{\text{LMod}_B(D_m)} \{D\}.$$ 

Moreover, the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{M}_m & \simeq & \text{LMod}_1(M_m) \\
\downarrow & & \downarrow \\
M_m \times_{D_m} \{D\} & \simeq & \text{LMod}_1(M_m) \times_{\text{LMod}_B(D_m)} \{D\} \\
\downarrow & & \downarrow \\
\text{LMod}_1(M_D) & \to & \text{LMod}_1(M_D).
\end{array}
$$

The bottom map is the map induced by the augmentation $\epsilon: M \to \mathbb{1}$, considered as a map of associative algebras in the fiber $\mathcal{M}_a \times_{D_a} \{B\}$. By [13, cor. 4.2.3.3], this map preserves all limits, and in particular terminal objects. $\Box$

Following [13] variant 5.2.1.16, we say that a morphism of right representable pairings

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\gamma} & \mathcal{M}' \\
\downarrow^{\lambda} & & \downarrow^{\lambda'} \\
\mathcal{C} \times \mathcal{D} & \to & \mathcal{C}' \times \mathcal{D}'
\end{array}$$

22Here we identify the trivial algebra $B \in \text{Alg}(\mathcal{D}_a)$ with its underlying object $B \simeq \mathbb{1}_{D_a} \in \mathcal{D}_a$.  

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is right representable, if $\gamma$ carries right universal objects to right universal objects.

We recall from [13, Proposition 5.2.1.17] that under this assumption, the induced diagram of $\infty$-categories

$$
\begin{array}{ccc}
\mathcal{C} & \xleftarrow{\mathcal{D}^{op}} & \mathcal{D} \\
\downarrow & & \downarrow \\
\mathcal{C}' & \xleftarrow{\mathcal{D}'^{op}} & \mathcal{D}'
\end{array}
$$

commutes up to canonical homotopy.

The following proposition mirrors [13, cor. 5.2.2.34].

**Corollary 8.1.2.** In the situation and the notation of lemma 8.1.1, if $\lambda_m : \mathcal{M}_m \to \mathcal{C}_m \times \mathcal{D}_m$ is right representable, then the induced pairing

$$
\text{LMod}(\lambda) : \text{LMod}_M(\mathcal{M}_m) \to \text{LMod}_A(\mathcal{C}_m) \times \text{LMod}_B(\mathcal{D}_m)
$$

is right representable, and the duality functor $\mathcal{D}'_{\text{LMod}(\lambda)}$ can be identified with the composition

$$
\text{LMod}_B(\mathcal{D}_m)^{op} \simeq \mathcal{D}_m^{op} \xrightarrow{\lambda'} \mathcal{C}_m \to \text{LMod}_A(\mathcal{C}_m),
$$

where the last map is induced by the augmentation $\epsilon$.

**Proof.** We begin with the first statement. Consider the morphism of pairings given by

$$
\begin{array}{ccc}
\mathcal{M}_m & \xrightarrow{\lambda_m} & \text{LMod}_M(\mathcal{M}_m) \\
\downarrow & & \downarrow \text{LMod}(\lambda) \\
\mathcal{C}_m \times \mathcal{D}_m & \to & \text{LMod}_A(\mathcal{C}_m) \times \text{LMod}_B(\mathcal{D}_m),
\end{array}
$$

where the horizontal maps are induced by the augmentation $\epsilon$. Given arbitrary $D \in \mathcal{D}_m$, there exists a right universal object $Z \in \mathcal{M}_m$ lying over $D$ by assumption. By lemma 8.1.1 the image of $Z$ is a right universal object in $\text{LMod}_M(\mathcal{M}_m)$. Since $B \in \text{Alg}(\mathcal{D}_a)$ is a trivial algebra, the map $\mathcal{D}_m \to \text{LMod}_B(\mathcal{D}_m)$ is an equivalence. It follows that $\text{LMod}(\lambda)$ is right representable, as stated.

Moreover, it follows that diagram (*) is a right representable morphism of right representable pairings, hence by [13 prop. 5.2.1.17] the diagram

$$
\begin{array}{ccc}
\mathcal{D}_m^{op} & \xrightarrow{\mathcal{D}'_\lambda} & \mathcal{C}_m \\
\downarrow & & \downarrow \\
\text{LMod}_B(\mathcal{D}_m)^{op} & \xrightarrow{\mathcal{D}'_{\text{LMod}(\lambda)}} & \text{LMod}_A(\mathcal{C}_m)
\end{array}
$$
commutes up to canonical homotopy. Since $B \in \text{Alg}(D_a)$ is a trivial algebra, the left vertical map is an equivalence, and the proof is complete. □

8.2. The $\mathcal{LM}$-monoidal twisted arrow category. Let $\mathcal{C}$ be an $\infty$-category and $\text{TwArr}(\mathcal{C})$ the twisted arrow category of [13, sec. 5.2.1]. In order to apply theorem 7.5.1 we construct an $\mathcal{LM}$-monoidal structure on the twisted arrow category, extending [13, ex. 5.2.2.23].

**Proposition 8.2.1.** Let $\mathcal{C}^\otimes \to \mathcal{LM}^\otimes$ be an $\mathcal{LM}$-monoidal $\infty$-category. Then there exists an $\mathcal{LM}$-monoidal pairing

$$\text{TwArr}(\mathcal{C})^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} (\mathcal{C}^{op})^\otimes,$$

where the $\mathcal{LM}$-monoidal category $\text{TwArr}(\mathcal{C})^\otimes$ witnesses $\text{TwArr}(\mathcal{C}_m)$ as left tensored over the monoidal category $\text{TwArr}(\mathcal{C}_a)$.

**Proof.** Identify the $\mathcal{LM}$-monoidal category $\mathcal{C}^\otimes$ with with a left action object in the $\infty$-category $\text{Cat}_{\infty}$, exhibiting the left action of $\mathcal{C}_a$ on $\mathcal{C}_m$. As in [13, ex. 5.2.2.23], we can apply the functor $\mathcal{C}at_{\infty} \to \text{CPair}_{\text{perf}}$, $\mathcal{E} \mapsto (\text{TwArr}(\mathcal{E}) \to \mathcal{E} \times \mathcal{E}^{op})$ of [13, rem. 5.2.1.20] to the underlying $\infty$-categories $\mathcal{C}_a$ and $\mathcal{C}_m$. Following [13, prop. 5.2.1.10], we see that the pairing $\text{TwArr}(\mathcal{C}_m) \to \mathcal{C}_m \times \mathcal{C}_m^{op}$, considered as an object of $\text{CPair}_{\text{perf}}$, admits a left action of $\text{TwArr}(\mathcal{C}_a) \to \mathcal{C}_a \times \mathcal{C}_a^{op}$, considered as a monoid object in $\text{CPair}_{\text{perf}}$. We can now identify this left action object with the desired $\mathcal{LM}$-monoidal pairing $\text{TwArr}(\mathcal{C})^\otimes \to \mathcal{C}^\otimes \times_{\mathcal{LM}^\otimes} (\mathcal{C}^{op})^\otimes$. □

**Remark 8.2.2.** The coCartesian fibration $(\mathcal{C}^{op})^\otimes \to \mathcal{LM}^\otimes$ appearing in the previous proposition can be identified with the “fiberwise opposite” of $\mathcal{C}^\otimes \to \mathcal{LM}^\otimes$. Explicitly, the fiberwise opposite is given by applying the dualizing construction of [3] to the coCartesian fibration $\mathcal{C}^\otimes \to \mathcal{LM}^\otimes$, then taking the opposite functor.

8.3. Bar constructions in the twisted arrow category. Denote by $s$ and $t$ the **source** and **target** maps

$$s: \text{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}, \quad t: \text{TwArr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}^{op}$$

of the twisted $\infty$-category.

**Proposition 8.3.1.** Let $\mathcal{C}$ be an $\infty$-category, and let $K$ be a weakly contractible simplicial set.

1. Let

$$p: K \to \text{TwArr}(\mathcal{C})$$

be a diagram in the twisted arrow category. If $sp$ has a colimit in $\mathcal{C}$ and $tp$ is equivalent to a constant diagram, then $p$ admits a colimit in $\text{TwArr}(\mathcal{C})$.
(2) Let
\[ \overline{p} : K^\circ \to \text{TwArr}(\mathcal{C}) \]
be a diagram in the twisted arrow category. If \( s\overline{p} \) is a colimit diagram and \( t\overline{p} \) is equivalent to a constant diagram, then \( \overline{p} \) is a colimit diagram.

**Proof.** We begin with part (2). Without loss of generality, assume that \( t\overline{p} \) is constant with value \( E \). We can therefore consider
\[ \overline{p} : K^\circ \to \text{TwArr}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \]
as a diagram in the fiber of \( t \) over \( E \). By [13, prop. 5.2.1.8] applied to \( id_E \in \text{TwArr}(\mathcal{C}) \) and [12, prop. 4.4.4.5], \( \text{TwArr}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \) is equivalent to the slice category \( \mathcal{C}/E \). Therefore, by [12, prop. 1.2.13.8], \( \overline{p} \) is a colimit diagram in the fiber \( \text{TwArr}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \).

By [13, prop. 5.2.1.3], \( t \) is a right fibration. So by [12, cor. 4.3.1.16] \( \overline{p} \) is a \( t \)-colimit diagram in \( \text{TwArr}(\mathcal{C}) \). Since \( K \) is weakly contractible, the constant diagram \( t \circ \overline{p} \) is a colimit diagram in \( \text{TwArr}(\mathcal{C}) \) (see [12, cor. 4.4.4.10]). Therefore \( \overline{p} \) is a colimit diagram in \( \text{TwArr}(\mathcal{C}) \) by combining [12, prop. 4.3.1.5 (2)] with [12, ex. 4.3.1.3].

We now show (1). The projection \( s \circ p \) has a colimit in \( \mathcal{C} \) by assumption, which by [12, prop. 1.2.13.8] lifts to
\[ \overline{p} : K^\circ \to \text{TwArr}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \],
a colimit of \( p \) in \( \mathcal{C}/E \simeq \text{TwArr}(\mathcal{C}) \times_{\mathcal{C}} \{E\} \). By (2), \( \overline{p} \) is also a colimit diagram in \( \text{TwArr}(\mathcal{C}) \).

**Corollary 8.3.2.** Let \( \mathcal{C}^\circ \to \mathcal{L}\mathcal{M} \) be an \( \mathcal{L}\mathcal{M} \)-monoidal \( \infty \)-category, and let
\[ M \in \text{Alg}(\text{TwArr}(\mathcal{C}_a)) \]
be an algebra lying over \((A, B) \in \text{Alg}(\mathcal{C}_a) \times \text{Alg}(\mathcal{C}_a^\circ)\). Assume \( B \) is a trivial algebra and \( \mathcal{C}^\circ \) admits realizations of \( A \)-bar constructions \( \otimes \)-compatibly (definition 5.3.15 applied to \( \mathcal{C}^\circ \) regarded as a \( \mathcal{B}\mathcal{M} \)-monoidal category as in example 5.1.6). Then \( \text{TwArr}(\mathcal{C})^\circ \) admits realizations of \( M \)-bar constructions \( \otimes \)-compatibly.

**Proof.** We regard all \( \mathcal{L}\mathcal{M} \)-monoidal categories as \( \mathcal{B}\mathcal{M} \)-monoidal (example 5.1.6). In order to simplify the ensuing division into cases, we introduce some notation. We define a partial composition law on the set \( C = \{0, m, 1\} \) of colors of \( \mathcal{B}\mathcal{M}^\circ \) by
\[ 0^2 = 0 \quad 0m = m \quad m1 = m \quad 1^2 = 1 \]
and a relation by \( i < j \) if the product \( ij \) is defined.

We first verify condition (1) of definition 5.3.15 (existence of realizations). Let \((X, Y) \) be an \((L, M, R)\)-module in \( \text{TwArr}(\mathcal{C})^\circ \) with \( L \in \text{Alg}(\text{TwArr}(\mathcal{C}_0)) \) and \( R \in \text{Alg}(\text{TwArr}(\mathcal{C}_r)) \) \((r = 0 \text{ or } 1)\). There’s a unique color \( i \in C \) with \( 0 < i < r \). Moreover,
be a colimit diagram extending the bar construction \( \text{Bar}_M \) diagram \( R \to Y \) is trivial. On the other hand, the diagram \( s \circ \text{Bar}_M(X,Y) \cdot \) admits a colimit in \( C \) by assumption, since it is equivalent to an \( A \)-bar construction in \( C \). Therefore \( \text{Bar}_M(X,Y) \cdot \) has a colimit in \( \text{TwArr}(C) \) by proposition 8.3.1.

Next we verify condition (2) (realizations of \( M \)-bar constructions are compatible with tensor products). We continue to work with an \((L,M,R)\)-module \((X,Y)\) with \( R \in C_r \) and \( Y \in C_i \) as above. Let \( j > i \) and fix \( X' \in \text{TwArr}(C_a), Y' \in \text{TwArr}(C_j) \) so that the tensor product \( Y \otimes Y' \) is defined and lies in \( C_{ij} \). Let

\[
\text{Bar}_M(X,Y)^+ : \Delta_+^{op} \to \text{TwArr}(C_i)
\]

be a colimit diagram extending the bar construction \( \text{Bar}_M(X,Y) \cdot \). Consider the diagram

\[
\bar{\rho} : \Delta_+^{op} \xrightarrow{\text{Bar}_M(X,Y)^+} \text{TwArr}(C_i) \xrightarrow{X' \otimes - \otimes Y'} \text{TwArr}(C_{ij})
\]

defined by applying the functor \( X' \otimes - \otimes Y' \) to \( \text{Bar}_M(X,Y)^+ \). We claim that \( \bar{\rho} \) is a colimit diagram. Since \( B \) is a trivial algebra, the diagram \( \text{Bar}_B(t(X), t(Y))^+ \) is equivalent to a constant diagram with value \( t(X) \otimes t(Y) \), hence \( t \circ \bar{\rho} \) is equivalent to the constant diagram with value \( t(X') \otimes t(X) \otimes t(Y) \otimes t(Y') \). Since realizations of \( A \)-bar constructions in \( C \) are by assumption \( \otimes \)-compatible, \( s \circ \bar{\rho} \) is a colimit diagram. By 8.3.1(2), \( \bar{\rho} \) is a colimit diagram, as desired.

In the proof of theorem 8.3.5 below, we compare the Koszul dual \( \mathcal{O}_{\text{Koszul}}(A,X) \) with the relative tensor product \( 1 \otimes_A X \) by showing that (after forgetting the comodule structure on the former) both fulfill the same universal mapping property in \( C_m \). For this purpose, following [13, Def. 5.2.2.1], we define the “bar object” \( \text{Bar}(A,X) \) purely in terms of the universal mapping property. As noted by Lurie, the bar object can exist even in the absence of relative tensor products.

**Definition 8.3.3.** Let \( C^\otimes \) be an \( LM \)-monoidal category with unit object \( 1 \in C_a \) and let

\[
\epsilon : A \to 1
\]

be a morphism in \( \text{Alg}(C_a) \). Then there are induced functors

\[
\rho_\epsilon^A : C_a \simeq \mathbf{BMod}_1(C_a) \to \mathbf{BMod}_A(C_a), \quad \rho_\epsilon^m : C_m \simeq \mathbf{LMod}_1(C_m) \to \mathbf{LMod}_A(C_m).
\]

Let \( C \in C_a \). A morphism \( f : A \to \rho_\epsilon^A(C) \) in \( \mathbf{BMod}_A(C_a) \) is said to exhibit \( C \) as the bar object on \( A \to 1 \) (or simply on \( A \)) if, for every object \( D \in C_a \), composition with
induces a homotopy equivalence

\[ \text{Map}_{C_a}(C, D) \cong \text{Map}_{A \text{BMod}_A(C_a)}(A, \rho'_A(D)). \]

When such an object \( C \) exists it is uniquely determined up to equivalence and denoted by \( \text{Bar}(A) \).

Similarly, if \( X \in \text{LMod}_A(C_m) \) and \( Y \in C_m \), we say that a morphism \( g : X \to \rho'_m(Y) \) in \( \text{LMod}_A(C_m) \) exhibits \( Y \) as the bar object on \( (A, X) \) if, for every object \( Z \in C_m \), composition with \( g \) induces a homotopy equivalence

\[ \text{Map}_{C_m}(Y, Z) \cong \text{Map}_{\text{LMod}_A(C_m)}(X, \rho'_m(Z)). \]

When such a \( Y \) exists it is uniquely determined up to equivalence and we denote it by \( \text{Bar}(A, X) \).

**Remark 8.3.4.** Let \( C \) be an \( \mathcal{LM} \)-monoidal \( \infty \)-category which admits realizations of bar constructions \( \otimes \)-compatibly. It follows from [13, prop. 4.6.2.17] that if \( \epsilon : A \to 1 \) is an augmented associative algebra object of \( C \), then the forgetful functor

\[ \rho'_m : C_m \simeq \text{LMod}_1(C) \to \text{LMod}_A(C) \]

admits a left adjoint, given by the relative tensor product

\[ X \mapsto 1 \otimes_A X. \]

It follows that the bar object \( \text{Bar}(A, X) \) exists and is given by the relative tensor product

\[ \text{Bar}(A, X) = 1 \otimes_A X. \]

The next theorem mirrors [13, thm. 5.2.2.17].

**Theorem 8.3.5.** Let \( C \) be a \( \mathcal{LM} \)-monoidal \( \infty \)-category. According to proposition [8.2.1] that there is an associated \( \mathcal{LM} \)-monoidal pairing

\[ \lambda : \text{TwArr}(C) \otimes \to C \otimes_{\mathcal{LM} \otimes} (C^\text{op}) \otimes. \]

Then:

1. The induced map

\[ \text{LMod}(\lambda) : \text{LMod}(\text{TwArr}(C)) \to \text{LMod}(C) \times \text{LMod}(C^\text{op}) \]

is a pairing of \( \infty \)-categories.

2. Assume that the unit object \( 1 \in C_a \) is final and that \( C_a, C_m \) admit geometric realizations of simplicial objects. Assume, moreover, that \( C \) admits realizations of bar constructions \( \otimes \)-compatibly (definition 5.3.15). Then the pairing \( \text{LMod}(\lambda) \) is left representable, and therefore determines a functor

\[ \mathcal{D}^\text{op}_{\text{Koszul}} := \text{D}_{\text{LMod}(\lambda)} : \text{LMod}(C)^\text{op} \to \text{LMod}(C^\text{op}). \]
Moreover, the composition with the forgetful functor

\[ \text{LMod}(C)^{\text{op}} \xrightarrow{\mathcal{D}_{\text{Koszul}}^{\text{op}}} \text{LMod}(C^{\text{op}}) \rightarrow C^{\text{op}}_a \times C^{\text{op}}_m \]

is given by the bar objects

\[(A, X) \mapsto (\text{Bar}(A), \text{Bar}(A, X))\]

as defined in definition 8.3.3.

(3) Dually, assume that the unit object \(1 \in C_a\) is initial and that \(C\) admits totalizations of cosimplicial objects. Assume, moreover, that \(C\) admits totalizations of cobar constructions \(\otimes\)-compatibly (definition 5.3.13). Then the pairing \(\lambda\) is right representable, and therefore determines a functor

\[ \mathcal{D}_{\text{Koszul}}^{\text{op}} := \mathcal{D}_{\text{LMod}(\lambda)}^{\text{op}} : \text{LMod}(C^{\text{op}})^{\text{op}} \rightarrow \text{LMod}(C) \]

Moreover, the composition with the forgetful functor

\[ \text{LMod}(C^{\text{op}})^{\text{op}} \xrightarrow{\mathcal{D}_{\text{Koszul}}^{\text{op}}} \text{LMod}(C) \rightarrow C^{\text{op}}_a \times C^{\text{op}}_m \]

is given by

\[(C, Y) \mapsto (\text{CoBar}(Y), \text{CoBar}(C, Y)),\]

where \(\text{CoBar}(Y)\) and \(\text{CoBar}(C, Y)\) denote bar objects in \(C^{\text{op}}\).

**Proof of Theorem 8.3.5.** (1) is a special case of [13, rem. 5.2.2.26]. We turn to (2). Let \(\mathcal{D} = C^{\text{op}}\). We claim that the \(\text{LM}\)-monoidal pairing \(\lambda^\otimes : \text{TwArr}(C)^{\otimes} \rightarrow C^{\otimes} \times_{\mathcal{LM}} D^{\otimes}\)

satisfies the hypotheses of theorem 7.5.1.

7.5.1(1): The map \(\mathcal{M}_a \times D_{a}\{1\} \rightarrow C_a\) is an equivalence since \(1 \in C_a\) is terminal by assumption.

7.5.1(2): The underlying pairings \(\lambda_a, \lambda_m\) are left representable by [13, prop. 5.2.1.10].

7.5.1(3): The \(\infty\)-categories \(D_a, D_m\) admit totalizations by assumption.

7.5.1(4): Let \(B \in \text{Alg}(D_a)\) be a trivial algebra and \(M \in \text{Alg}(\mathcal{M}_a)\) an algebra lying over \((A, B) \in \text{Alg}(C_a) \times \text{Alg}(D_a).\) By assumption, \(C\) admits realizations of \(A\)-bar constructions \(\otimes\)-compatibly. Since \(B \in \text{Alg}(C^{\text{op}}_a)\) is a trivial algebra, proposition 8.3.1 implies that \(\text{TwArr}(C)\) admits realizations of \(M\)-bar constructions \(\otimes\)-compatibly.

We thus have a duality functor \(\mathcal{D}_{\text{LMod}(\lambda)}\) as above.

Let \((A, X)\) be a left module in \(C\) and let

\[(B, Y) = \mathcal{D}_{\text{LMod}(\lambda)}(A, X) \in \text{LMod}(C^{\text{op}})\]
By [13, thm. 5.2.2.17 (2)], the image of $B$ in $C_a$ is equivalent to $\text{Bar}(A)$. We must show that the image of $Y$ in $C_m$ is equivalent to $\text{Bar}(A, X)$. Since the unit object $1 \in C_a$ is final, there exists an essentially unique augmentation

$$\epsilon : A \to 1.$$ 

We can then identify $\epsilon$ with an algebra object of the monoidal $\infty$-subcategory

$$(C_a)_/1 \simeq \text{TwArr}(C_a) \times_{C_a^{op}} \{1\} \subset \text{TwArr}(C_a).$$

Let $M$ denote the associated algebra object in $\text{TwArr}(C_a)$:

$$\text{TwArr}(C_a) \to C_a \times C_a^{op}, \quad M \mapsto (A, 1).$$

By proposition 7.3.3 and corollary 8.1.2, the induced pairing $L\text{Mod}_M(\lambda)$:

$$L\text{Mod}_M(\text{TwArr}(C_m)) \to L\text{Mod}_A(C_m) \times L\text{Mod}_{/BD}(C_m^{op}) \simeq L\text{Mod}_A(C_m) \times C_m^{op}$$

is both left and right representable, hence induces adjoint duality functors

$$D_{L\text{Mod}_M(\lambda)} : L\text{Mod}_A(C_m) \rightleftharpoons (C_m^{op})^{op} \simeq C_m : D'_{L\text{Mod}_M(\lambda)}$$

as in section 6.2. By corollary 8.1.2, $D'_{L\text{Mod}_M(\lambda)}$ is the functor induced by the augmentation of $A$. It follows that its left adjoint $D_{L\text{Mod}_M(\lambda)}$ is given by

$$X' \mapsto \text{Bar}(A, X').$$

By the construction of the $\mathcal{LM}$-Koszul duality functor $D_{L\text{Mod}(\lambda)}$ (outlined in the introduction and carried out in the proof of theorem 7.5.1), as an object of the $\infty$-category $C_a^{op} \times C_m^{op}$, we can identify $(B, Y)$ with

$$(D_{L\text{Mod}_M(\lambda)}(A), D_{L\text{Mod}_M(\lambda)}(X)), $$

hence with $(\text{Bar}(A), \text{Bar}(A, X))$ as claimed. Statement (3) follows by symmetry. □

8.4. Dispensing with the assumption that the unit object is terminal.

Construction 8.4.1. Let $C^\otimes$ be an $\mathcal{LM}$-monoidal category, and let $1 \in C_a$ denote the unit object. We now construct an $\mathcal{LM}$-monoidal category $C^\otimes_\alpha \to \mathcal{LM}^\otimes$ exhibiting $C_m$ as left tensored over $(C_a)_/1$: informally, the action of $(A \to 1) \in (C_a)_/1$ on $X \in C_m$ is given by the action of $A \in C_a$ on $X \in C_m$.

Consider the forgetful functor from $\mathcal{LM}$-monoidal categories to monoidal categories induced by pulling back coCartesian fibrations along $\text{Ass}^\otimes \to \mathcal{LM}^\otimes$. Let $(B^\otimes \to \text{Ass}^\otimes) \mapsto (B^\otimes_+ \to \mathcal{LM}^\otimes)$ denote the right adjoint to the forgetful map. The coCartesian fibration $(B^\otimes_+ \to \mathcal{LM}^\otimes)$ is an $\mathcal{LM}$-monoidal category exhibiting the
terminal ∞-category as left tensored over \( B \). Given a monoidal functor \( B \rightarrow C \), we can form the following pullback
\[
\begin{array}{ccc}
C^\otimes & \longrightarrow & C^\otimes \\
\downarrow & & \downarrow \\
B^\otimes_+ & \longrightarrow & (C_a)^\otimes_+
\end{array}
\]
where the right vertical map is the counit of the above adjunction. Then
\[
C^\otimes_+ \rightarrow C^\otimes \rightarrow \mathcal{LM}^\otimes
\]
exhibits \( C_m \) as left tensored over \( B \).

Applying this construction to the forgetful functor \( (C_a)^\otimes_1 \rightarrow C^\otimes_1 \) produces the desired \( \mathcal{LM} \)-monoidal category \( C^\otimes_1 \rightarrow \mathcal{LM}^\otimes \). The reader is cautioned that our notation is not compatible with that of [13, sec. 2.2.2]. We can informally describe an \( \mathcal{LM} \)-algebra in \( C^\otimes_{/1} \) as a pair \( (A \rightarrow 1, X) \), where \( A \rightarrow 1 \) is an augmented algebra in \( C_a \) and \( X \in C_m \) has the structure of a left module over \( A \). We denote the \( \infty \)-category of \( \mathcal{LM} \)-algebras in \( C^\otimes_{/1} \) by
\[
\text{LMod}^{\text{aug}}(C) := \text{LMod}(C_{/1}).
\]
Evidently, the unit object in \( (C_a)_{/1} \) is terminal.

Similarly, applying the construction above to (the opposite of) the symmetric monoidal functor
\[
B = (C_a)_{1/1} := \Big( (C_a)_{/1} \Big)_{1/} \rightarrow (C_a)_{/1}
\]
constructs an \( \mathcal{LM} \)-monoidal category \( C^\otimes_{1/1} \) which exhibits \( C_m \) as tensored over \( C_{1/1} \).

**Definition 8.4.2.** Define \( \text{coLMod}^{\text{aug}}(C) := \text{LMod}^{\text{aug}}(C^{op})^{op} \).

Informally, one can describe an object of \( \text{coLMod}^{\text{aug}}(C) \) as a pair \( (C, Y) \) of an augmented coalgebra \( C \) in \( C_a \) and a left comodule \( Y \) over \( C \).

Combining these constructions with theorem 8.3.5 and remark 8.3.4, we arrive at the following corollary.

**Corollary 8.4.3.** Let \( C^\otimes \rightarrow \mathcal{LM}^\otimes \) be an \( \mathcal{LM} \)-monoidal \( \infty \)-category. There is an associated \( \mathcal{LM} \)-monoidal pairing
\[
\lambda: \text{TwArr}(C) \rightarrow C^\otimes \times_{\mathcal{LM}^\otimes} (C^{op})^{\otimes}
\]
and an induced pairing of \( \infty \)-categories
\[
\text{LMod}(\lambda): \text{LMod}(\text{TwArr}(C)) \rightarrow \text{LMod}(C) \times \text{LMod}(C^{op}).
\]
If $\mathcal{C}_a, \mathcal{C}_m$ admit geometric realizations of simplicial objects and $\mathcal{C}^\otimes$ admits geometric realizations of bar constructions $\otimes$-compatibly, then the pairing $\text{LMod}(\lambda)$ is left representable, and therefore determines a functor

$$\mathcal{D}_\text{Koszul} = \mathcal{D}^\text{op}_{\text{LMod}(\lambda)} : \text{LMod}^{\text{aug}}(\mathcal{C}) \rightarrow \text{coLMod}^{\text{aug}}(\mathcal{C}),$$

whose composition with the forgetful functor

$$\text{coLMod}^{\text{aug}}(\mathcal{C}) \rightarrow \mathcal{C}_a \times \mathcal{C}_m$$

is given by

$$(A, X) \mapsto (1 \otimes_A 1, 1 \otimes_A X).$$

If $\mathcal{C}_a, \mathcal{C}_m$ admit both realizations of simplicial objects and totalization of cosimplicial objects, and if $\mathcal{C}^\otimes$ admits realizations of bar constructions and totalizations of cobar constructions $\otimes$-compatibly, then the induced duality functors form an adjunction

$$\mathcal{D}_\text{Koszul} : \text{LMod}^{\text{aug}}(\mathcal{C}) \cong \text{coLMod}^{\text{aug}}(\mathcal{C}) : \mathcal{D}_\text{Koszul}'.$$

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