RAMSEY SUBSETS OF THE SPACE OF INFINITE BLOCK SEQUENCES OF VECTORS.

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Abstract. We study families of infinite block sequences of elements of the space FIN_k. In particular we study Ramsey properties of such families and Ramsey properties localized to a selective or semiselective coideal. We show how the stable ordered-union ultrafilters defined by Blass, and Matet-adequate families defined by Eisworth in the case k = 1 fit in the theory of the Ramsey space of infinite block sequences of finite sets of natural numbers.

1. Introduction

In this article we study the Ramsey property of subsets of the space FIN_k^∞ of infinite block sequences of elements of FIN_k. The case k = 1 deals with block sequences of finite sets of natural numbers. We consider coideals contained in these spaces and the Ramsey property localized on such a coideals.

Let k be a positive integer, FIN_k = \{p : \mathbb{N} \to \{0, 1, \ldots, k\} : \{n : p(n) \neq 0\} \text{ is finite and } k \in \text{range}(p)\}. For p \in FIN_k, supp(p) = \{n : p(n) \neq 0\}. FIN_k is a partial semigroup under the partial semigroup operation of addition of elements with disjoint support. A block sequence of elements of FIN_k is a (finite or infinite) sequence (p_n) with supp(p_n) < supp(p_{n+1}) for every n \in \mathbb{N} (i.e. the maximal element of supp(p_n) is strictly below the minimal element of supp(p_{n+1}) for every n).

The relation between FIN_k and the positive part of the unitary sphere of the Banach space c_0 is well known, see for example [20], page 37. It permits to identify elements of FIN_k with vectors.

The operation \(T : \text{FIN}_k \to \text{FIN}_{k-1}\) is defined by \(T(p)(n) = \max\{p(n) - 1, 0\}\).

Given an infinite block sequence \(A = (p_n)\) of elements of FIN_k, the subsemigroup \([A]\) of FIN_k generated by A is the collection of elements of FIN_k of the form

\[T^{(i_0)}(p_{n_0}) + \cdots + T^{(i_l)}(p_{n_l})\]

for some sequence \(n_0 < \cdots < n_l\) and some choice \(i_0, \ldots, i_l \in \{0, 1, \ldots, k\}\). Notice that for the sum to remain in FIN_k at least one of the numbers \(i_0, \ldots, i_l\) must be 0.

We denote by FIN_k^∞ the space of infinite block sequences of elements of FIN_k. If k = 1 we simply write FIN^∞; in this case, \([A]\) is the subsemigroup of FIN^∞ formed by all the block sequences whose elements are finite unions of elements of A.

To define the Ramsey property of subsets of FIN_k^∞, we first recall the definition of the Ramsey property for subsets of the space \(\mathbb{N}^{[\infty]}\) of all infinite sets of natural numbers. With the product topology (the topology inherited from the product topology on \(2^{\mathbb{N}}\)), this space is homeomorphic to \(\mathbb{R} \setminus \mathbb{Q}\), the irrational numbers. The exponential topology of this space,

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also called the Ellentuck topology, is finer than the product topology and it is generated by
the basic sets of the form
\[ [a, A] = \{ X \in \mathbb{N}^{[\omega]} : a \sqsubseteq X \subseteq A \}, \]
where \( a \) is a finite set of natural numbers, \( A \) is an infinite subset of \( \mathbb{N} \), and \( a \sqsubseteq X \) means that \( a \) is an initial segment of \( X \) in its increasing order.

A subset \( A \subseteq \mathbb{N}^{[\omega]} \) is Ramsey, or has the Ramsey property, if for every \([a, A]\) there is an
infinite subset \( B \) of \( A \) such that \([a, B] \subseteq A\) or \([a, B] \cap A = \emptyset\). Silver proved that all analytic
subsets of \( \mathbb{N}^{[\omega]} \) have the Ramsey property. His proof has a metamathematical character, as
opposed to the combinatorial proof of Galvin and Prikry for the Borel sets. Ellentuck \[7\]
gave a topological proof of Silver’s result by showing that a subset of \( \mathbb{N}^{[\omega]} \) is Ramsey if and
only if it has the property of Baire with respect to the exponential topology.

We present below similar results for the space \( \text{FIN}_k^{\infty} \) of infinite block sequences of elements
of \( \text{FIN}_k \). We also consider certain subfamilies of \( \text{FIN}_k^{\infty} \) to define coideals, selective coideals
and semiselective coideals; and study some forcing notions related to these subfamilies.
Previous work in this subject was done in \[1, 5, 9, 14\]. More recently, Zhang \[21\] studies
the preservation of selective ultrafilters on \( \text{FIN} \) under Sacks forcing, proves that selective
ultrafilters on \( \text{FIN} \) localize their parametrized Milliken theorem, and also proves that that
those selective ultrafilters are Ramsey.

García Ávila, in \[9\], considers several forcing notions related to the space \( \text{FIN}^{\infty} \), and in
particular a forcing notion analogous to Mathias forcing adapted to this space. She proves
that this notion has a pure decision property (a Prikry property) and asks if it has a property
analogous to the fact that an infinite subset of a Mathias generic real is also a Mathias generic
real (hereditary genericity, or the Mathias property). This question was answered positively
in \[2\], and here we extend this answer to the forcing localized on a semiselective coideal.

In this article we study these forcing notions and their relation to some classes of ultrafilters
introduced by Blass and Hindman (see \[1\]). Stable ordered-union ultrafilters on the space \( \text{FIN} \) of finite sets of natural numbers were defined by Blass (\[1\]); these ultrafilters are related
to Hindman’s theorem on partitions of \( \text{FIN} \) in the same way selective ultrafilters on \( \omega \) are
related to Ramsey’s theorem. We show that stable ordered-union ultrafilters are closely
related to selective ultrafilters on the Ramsey space \( \text{FIN}^{\infty} \). In his study of forcing and stable
ordered-union ultrafilters (\[5\]) Eisworth isolates the concept of Matet-adequate families of
elements of \( \text{FIN}^{\infty} \), and proves that forcing with such a family adds a stable ordered-union ultrafilter. We show that Matet-adequate families correspond to selective coideals of the
topological Ramsey space \( \text{FIN}^{\infty} \).

We also address the problem of the consistency of the statement all subsets of \( \text{FIN}_k^{\infty} \) have
the Ramsey property localized with respect to a semiselective coideal. This presentation is
formulated in the context of topological Ramsey spaces as presented in \[20\].

2. Block sequences of elements of \( \text{FIN}_k \).

As defined above, for a positive integer \( k \), \( \text{FIN}_k^{\infty} \) denotes the collection of all infinite block
sequences of elements of \( \text{FIN}_k \), that is to say, all sequences \( p_0, p_1 \ldots \) where, for all \( i \in \omega \),
\( p_i \in \text{FIN}_k \) and \( \max(\text{supp}(p_i)) < \min(\text{supp}(p_{i+1})) \).

For a positive integer \( d \), \( \text{FIN}_k^{[d]} \) denotes the collection of all finite block sequences of length
\( d \).
We define the approximation space of $\text{FIN}_k^\infty$ as the set

$$\mathcal{AFIN}_k^\infty := \bigcup \{ \text{FIN}_k^{[d]} : d \in \omega \} = \text{FIN}_k^{<\infty}$$

the collection of finite block sequences of elements of $\text{FIN}_k$.

For each $m \in \omega$, we define the approximation function $r_m : \text{FIN}_k^\infty \to \text{FIN}_k^{[m]}$ that sends an infinite block sequence to its first $m$ blocks. We define $r : \text{FIN}_k^\infty \times \omega \to \text{FIN}_k^{<\infty}$ by $r(X, n) = r_n(X)$.

For $A \in \text{FIN}_k^\infty$ we use the symbols

$$\text{FIN}_k^\infty \upharpoonright A := \{ B \in \text{FIN}_k^\infty : B \subseteq [A] \} \text{ and } \text{FIN}_k^{<\infty} \upharpoonright A := \{ s \in \text{FIN}_k^{<\infty} : s \subseteq [A] \}$$

Consider the binary relation defined on $\text{FIN}_k^\infty$ by $X \leq Y$ if $X$ is a condensation of $Y$, that is, every element of $X$ belongs to the subsemigroup $[Y]$ generated by $Y$.

The triple $(\text{FIN}_k^\infty, \leq, r)$ satisfies the following properties A1-A4.

(A.1) [Metrization]

(A.1.1) For any $A \in \text{FIN}_k^\infty$, $r_0(A) = \emptyset$.

(A.1.2) For any $A, B \in \text{FIN}_k^\infty$, if $A \neq B$ then $(\exists n) (r_n(A) \neq r_n(B))$.

(A.1.3) If $r_n(A) = r_m(B)$ then $n = m$ and $(\forall i < n) (r_i(A) = r_i(B))$.

Take the discrete topology on $\text{FIN}_k^{<\infty}$ and endow $(\text{FIN}_k^{<\infty})^N$ with the product topology; this is the metric space of all the sequences of elements of $\text{FIN}_k^{<\infty}$. Notice that $\text{FIN}_k^\infty$ is a closed subspace of $(\text{FIN}_k^{<\infty})^N$.

With this notation, the basic open sets generating the metric topology on $\text{FIN}_k^\infty$ are of the form

$$[s] = \{ B \in \text{FIN}_k^\infty : (\exists n) (s = r_n(B)) \}$$

where $s \in \text{FIN}_k^{<\infty}$. Let us define the length of $s$, as the unique integer $|s| = n$ such that $s = r_n(A)$ for some $A \in \text{FIN}_k^\infty$.

We will consider another topology on $\text{FIN}_k^\infty$ which we will call the Ellentuck (or exponential) topology. The Ellentuck type neighborhoods are of the form:

$$[a, A] = \{ B \in [a] : B \leq A \} = \{ B \in \text{FIN}_k^\infty : (\exists n) a = r_n(B) \& B \leq A \}$$

where $a \in \text{FIN}_k^{<\infty}$ and $A \in \text{FIN}_k^\infty$.

We will use $[n, A]$ to abbreviate $[r_n(A), A]$.

Notice that

$$\text{FIN}_k^{<\infty} \upharpoonright A = \{ a \in \text{FIN}_k^{<\infty} : [a, A] \neq \emptyset \}.$$ 

Given a neighborhood $[a, A]$ and $n \geq |a|$, let $r_n[a, A]$ be the image of $[a, A]$ by the function $r_n$, i.e.,

$$r_n[a, A] = \{ r_n(B) : B \in [a, A] \}.$$ 

Given $a, b \in \text{FIN}_k^{<\infty}$, write

$$a \sqsubseteq b \text{ iff } (\exists A \in \text{FIN}_k^\infty) (\exists m, n \in \mathbb{N}) m \leq n, a = r_m(A) \text{ and } b = r_n(A).$$

By A.1, $\sqsubseteq$ can be proven to be a partial order on $\text{FIN}_k^{<\infty}$.

The relation $\leq_{fin}$ on $\text{FIN}_k^{<\infty}$ is defined in a similar way as the relation $\leq$ on $\text{FIN}_k^\infty$, and we have the following.
(A.2) [Finitization] The quasi order $\leq_{\text{fin}}$ on $\text{FIN}_k^\infty$ satisfies:

(A.2.1) $A \leq B$ iff $(\forall n) \left( \exists m \right) \left( r_n(A) \leq_{\text{fin}} r_m(B) \right)$.

(A.2.2) $\{ b \in \text{FIN}_k^\infty : b \leq_{\text{fin}} a \}$ is finite, for every $a \in \text{FIN}_k^\infty$.

(A.2.3) If $a \leq_{\text{fin}} b$ and $c \subseteq a$ then there is $d \subseteq b$ such that $c \leq_{\text{fin}} d$.

Given $A \in \text{FIN}_k^\infty$ and $a \in \text{FIN}_k^\infty \upharpoonright A$, we define the depth of $a$ in $A$ as

$$\text{depth}_A(a) := \min\{ n : a \leq_{\text{fin}} r_n(A) \}$$

(A.3) [Amalgamation] Given $a$ and $A$ with $\text{depth}_A(a) = n$, the following holds:

(A.3.1) $(\forall B \in [n, A]) \left( [a, B] \neq \emptyset \right)$.

(A.3.2) $(\forall B \in [a, A]) \left( \exists A' \in [n, A] \right) \left( [a, A'] \subseteq [a, B] \right)$.

(A.4) [Pigeonhole Principle (essentially Gowers’ Theorem [11], Hindman’s Theorem [12] for $k = 1$)] Given $a$ and $A$ with $\text{depth}_A(a) = n$, for every $\mathcal{O} \subseteq \text{FIN}_k^{|a|+1}$ there is $B \in [n, A]$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \subseteq \mathcal{O}^c$.

Definition 1. A set $\mathcal{X} \subseteq \text{FIN}_k^\infty$ is Ramsey if for every neighborhood $[a, A] \neq \emptyset$ there exists $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq \text{FIN}_k^\infty$ is Ramsey null if for every neighborhood $[a, A]$ there exists $B \in [a, A]$ such that $[a, B] \not\subseteq \mathcal{X}$.

A set $\mathcal{X} \subseteq \text{FIN}_k^\infty$ has the abstract Baire property if for every neighborhood $[a, A] \neq \emptyset$ there exists $\emptyset \neq [b, B] \subseteq [a, A]$ such that $[b, B] \subseteq \mathcal{X}$ or $[b, B] \cap \mathcal{X} = \emptyset$. A set $\mathcal{X} \subseteq \mathcal{R}$ is nowhere dense if for every neighborhood $[a, A]$ there exists $[b, B] \subseteq [a, A]$ such that $[b, B] \cap \mathcal{X} = \emptyset$.

Theorem 1. [20] $(\text{FIN}_k^\infty, \leq, r)$ is a topological Ramsey space. In other words, a subset $\mathcal{X} \subseteq \text{FIN}_k^\infty$ is Ramsey if and only if it has the Baire property with respect to the Ellentuck topology, and Ramsey null sets coincide with nowhere dense sets.

Proof. The result follows from the fact that $(\text{FIN}_k^\infty, \leq, r)$ satisfies A1, A2, A3 and A4 (see [20]). \qed

Definition 2. Given $\mathcal{H} \subseteq \text{FIN}_k^\infty$, we say that $\mathcal{H}$ is a coideal if it satisfies the following:

(a) $\mathcal{H}$ is closed under finite changes, i.e. if $A \in \mathcal{H}$ and $A \triangle B$ is finite, then $B \in \mathcal{H}$.

(b) For all $A, B \in \text{FIN}_k^\infty$, if $A \in \mathcal{H}$ and $A \leq B$ then $B \in \mathcal{H}$.

(c) (A3 mod $\mathcal{H}$) For all $A \in \mathcal{H}$ and $a \in \text{FIN}_k^\infty \upharpoonright A$, the following holds:

- $[a, B] \neq \emptyset$ for all $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$.
- If $B \in \mathcal{H} \upharpoonright A$ and $[a, B] \neq \emptyset$ then there exists $A' \in [\text{depth}_A(a), A] \cap \mathcal{H}$ such that $\emptyset \neq [a, A'] \subseteq [a, B]$.

(d) (A4 mod $\mathcal{H}$) Let $A \in \mathcal{H}$ and $a \in \text{FIN}_k^\infty \upharpoonright A$ be given. For all $\mathcal{O} \subseteq \text{FIN}_k^{|a|+1}$ there exists $B \in [\text{depth}_A(a), A] \cap \mathcal{H}$ such that $r_{|a|+1}[a, B] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, B] \cap \mathcal{O} = \emptyset$.

For a coideal $\mathcal{H}$, and $A \in \text{FIN}_k^\infty$,

$$\mathcal{H} \upharpoonright A = \{ B \in \mathcal{H} : B \leq A \}.$$

The Ramsey property and the Baire property are localized to a coideal in the following fashion.

Definition 3. $\mathcal{X} \subseteq \text{FIN}_k^\infty$ is $\mathcal{H}$-Ramsey if for every $[a, A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \subseteq \mathcal{X}^c$. If for every $[a, A] \neq \emptyset$, there exists $B \in [a, A] \cap \mathcal{H}$ such that $[a, B] \subseteq \mathcal{X}^c$; we say that $\mathcal{X}$ is $\mathcal{H}$-Ramsey null.
Definition 4. $\mathcal{X} \subseteq \text{FIN}^\infty_k$ is $\mathcal{H}$-Baire if for every $[a,A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b,B] \subseteq [a,A]$, with $B \in \mathcal{H}$, such that $[b,B] \subseteq \mathcal{X}$ or $[b,B] \subseteq \mathcal{X}^c$. If for every $[a,A] \neq \emptyset$, with $A \in \mathcal{H}$, there exists $\emptyset \neq [b,B] \subseteq [a,A]$, with $B \in \mathcal{H}$, such that $[b,B] \subseteq \mathcal{X}^c$; we say that $\mathcal{X}$ is $\mathcal{H}$-meager.

It is clear that if $\mathcal{X} \subseteq \text{FIN}^\infty_k$ is $\mathcal{H}$-Ramsey then $\mathcal{X}$ is $\mathcal{H}$-Baire. Theorem 1 can be extended to the properties $\mathcal{H}$-Ramsey and $\mathcal{H}$-Baire when the family $\mathcal{H}$ is a semiselective coideal. We define this notion now.

Definition 5. Given $A \in \text{FIN}^\infty_k$ and a sequence $A = (A_n)_{n \in \mathbb{N}} \subseteq \text{FIN}^\infty_k$, we say that $B \in \text{FIN}^\infty_k$ is a diagonalization of $A$ within $A$ if for every $b \in \text{FIN}^\infty_k \upharpoonright B$ with depth$_A(b) = n$ we have $[b,B] \subseteq [a,A]$.

Definition 6. A coideal $\mathcal{H} \subseteq \text{FIN}^\infty_k$ is selective if given $[a,A] \neq \emptyset$ with $A \in \mathcal{H}$, for every sequence $A = (A_n)_{n \in \mathbb{N}} \subseteq \mathcal{H} \upharpoonright A$ such that $A_n \supseteq A_{n+1}$ and $[a,A_n] \neq \emptyset$, there exists $B \in \mathcal{H} \cap [a,A]$ which diagonalizes $A$ within $A$.

Notice that $\text{FIN}^\infty_k$ is a selective coideal.

We will see that, for the case $k = 1$, stable ordered-union ultrafilters (see definitions 13 and 14) give rise to other examples of selective coideals on $\text{FIN}$.

Definition 7. Let $\mathcal{H} \subseteq \text{FIN}^\infty_k$ be a coideal. Given sets $\mathcal{D}, \mathcal{S} \subseteq \mathcal{H}$, we say that $\mathcal{D}$ is dense open in $\mathcal{S}$ if the following hold:

1. $(\forall A \in \mathcal{S}) (\exists B \in \mathcal{D}) B \leq A$.
2. $(\forall A \in \mathcal{S}) (\forall B \in \mathcal{D}) [A \leq B \rightarrow A \in \mathcal{D}]$.

Definition 8. Given $A \in \mathcal{H}$ and a collection $\mathcal{D} = \{\mathcal{D}_a\}_{a \in \text{FIN}^\infty_k \upharpoonright A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a),A]$, we say that $B \leq A$ is a diagonalization of $\mathcal{D}$ if there exists a family $A = \{A_a\}_{a \in \text{FIN}^\infty_k \upharpoonright A}$, with $A_a \in \mathcal{D}_a$, such that for every $a \in \text{FIN}^\infty_k \upharpoonright B$ we have $[a,B] \subseteq [a,A_a]$.

Definition 9. We say that a coideal $\mathcal{H} \subseteq \text{FIN}^\infty_k$ is semiselective if for every $A \in \mathcal{H}$, every collection $\mathcal{D} = \{\mathcal{D}_a\}_{a \in \text{FIN}^\infty_k \upharpoonright A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a),A]$ and every $C \in \mathcal{H} \upharpoonright A$, there exists $B \in \mathcal{H} \upharpoonright C$ such that $B$ is a diagonalization of $\mathcal{D}$.

Lemma 1. Given a coideal $\mathcal{H}$ of $\text{FIN}^\infty_k$ and $A \in \mathcal{H}$, for every $(\mathcal{D}_a)_{a \in \text{FIN}^\infty_k \upharpoonright A}$ such that each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a),A]$ there exists $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{H} \upharpoonright A$ such that $A_n \in \mathcal{D}_a$ for all $a \in \text{FIN}^\infty_k \upharpoonright A$ with depth$_A(a) = n$.

Proof. For every $n \in \mathbb{N}$, list

$$\{a_1, a_2, \ldots, a_{k_n}\} \subseteq \{a \in \text{FIN}^\infty_k \upharpoonright A : \text{depth}_A(a) = n\}$$

Since each $\mathcal{D}_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a),A]$, for each $i = 1, \ldots, k_n$, $D_{a_i}$ is dense in $[n,A]$. So, we can choose $A^1 \in D_{a_1} \upharpoonright A$, $A^2 \in D_{a_2} \upharpoonright A^1$, $\ldots$, $A^{k_n} \in D_{a_{k_n}} \upharpoonright A^{k_n-1}$. Clearly, $A^{k_n} = \bigcap_{i=1}^{k_n} D_{a_i}$. Put $A_n = A^{k_n}$; then $(A_n)_{n \in \mathbb{N}}$ is the desired sequence. We have that for every $n$, $A_n \in \mathcal{H} \upharpoonright A$, and $A_n \in \mathcal{D}_a$ for every $a \in \text{FIN}^\infty_k \upharpoonright A$ with depth$_A(a) = n$. \hfill \square

Proposition 1. A coideal $\mathcal{H} \subseteq \text{FIN}^\infty_k$ is semiselective if for every $A \in \mathcal{H}$ and every set $\mathcal{D} = \{D_n : n \in \omega\}$, each $D_n$ dense open subset of $\mathcal{H} \cap [n,A]$, there exists $B \in \mathcal{H} \upharpoonright A$ such that for each $s \in \text{FIN}^\infty_k \upharpoonright B$, $B/s \in \mathcal{D}_{\text{depth}_A(s)}$. 

Proof. Given $A \in \mathcal{H}$, and $(D_n)_{n \in \mathbb{N}}$, we define $D_a = D_n$ for every $a \in \text{FIN}_k^\infty \upharpoonright A$ with $\text{depth}_A(a) = n$. A diagonalization of $(D_a)$ diagonalizes $(D_n)$ within $A$.

Conversely, given $A \in \mathcal{H}$, and $\mathcal{D} = \{D_a\}_{a \in \text{FIN}_k^\infty \upharpoonright A}$ such that each $D_a$ is dense open in $\mathcal{H} \cap \text{[depth}_A(a), A]$, for every $n \in \mathbb{N}$, $D_n = \bigcap_{\text{depth}_A(a) = n} D_a$. For every $C \in \mathcal{H} \upharpoonright A$, there is $B \in \mathcal{H} \upharpoonright C$ such that $B/b \in D_n$ for every $b \in \text{FIN}_k^\infty \upharpoonright B$ with $\text{depth}_A(b) = n$. So, defining $A_a$ so that $r_{\text{[a,B]}}(A_a) = a$ and $A_a/a = B/a$ if $a \in \text{FIN}_k^\infty \upharpoonright B$, and choosing any $A_a \in \mathcal{D}_a$ if $a \notin \text{FIN}_k^\infty \upharpoonright B$, the family $\mathcal{A} = \{A_a\}_{a \in \text{FIN}_k^\infty \upharpoonright A}$ verifies that $B$ is a diagonalization of $(D_a)$. □

If $A, B \in \text{FIN}_k^\infty$, we write $A \leq^* B$ to express that $A$ is almost a condensation of $B$, that is, except for a finite number of its elements, every element of $A$ belongs to $[B]$.

**Lemma 2.** A coideal $\mathcal{H}$ is semiselective iff it is $\sigma$-distributive with respect to $\leq^*$.

Proof. Let $\mathcal{H}$ be a semiselective coideal. Let $\mathcal{D} = \{D_n : n \in \omega\}$ be a family of $\leq^*$-dense open subsets of $\mathcal{H}$ and let $W_A := \{B \in \mathcal{H} : B \leq^* A\}$. Notice that each $D_n$ is dense open in $\mathcal{H}$. Clearly, a diagonalization $B \in \mathcal{H}$ of the $D_n$'s within $A$ is an element of both $\bigcap \mathcal{D}$ and $W_A$.

Conversely, given $A \in \mathcal{H}$ and $\mathcal{D}$ as in Proposition, let $B \in \mathcal{H}$ such that $B \in \bigcap \mathcal{D}$ and $B \leq^* A$. We can assume $B \in \mathcal{H} \upharpoonright A$ without a loss of generality. Then, for any $s \in \text{FIN}_k^\infty \upharpoonright B$ with $\text{depth}_A(s) = n$, it holds that $B/s \in D_n$ since $B \in D_n$ and $B/s \leq^* B$. □

The next theorem is also a consequence of Lemma.

**Theorem 2.** If $\mathcal{H} \subseteq \text{FIN}_k^\infty$ is a selective coideal then $\mathcal{H}$ is semiselective.

Proof. Consider $A \in \mathcal{H}$ and let $\mathcal{D} = (D_a)_{a \in \text{FIN}_k^\infty \upharpoonright A}$ be such that each $D_a$ is dense open in $\mathcal{H} \cap [\text{depth}_A(a), A]$. Fix $B \in \mathcal{H} \cap [a, A]$. Then $D_a$ is dense open in $\mathcal{H} \cap [a, B]$, for all $a \in \text{FIN}_k^\infty \upharpoonright B$. Using lemma we can build $\mathcal{A} = (A_n)_{n \in \mathbb{N}} \subseteq \mathcal{H} \upharpoonright B$ such that $A_n \in \mathcal{D}_a$ for every $a \in \text{FIN}_k^\infty \upharpoonright B$ with $\text{depth}_A(a) = n$. By selectivity, there exists $C \in \mathcal{H} \cap [a, B]$ which diagonalizes $\mathcal{A}$. Hence, $\mathcal{H}$ is semiselective. □

**Definition 10.** A family $\mathcal{U} \subseteq \text{FIN}_k^\infty$ is an ultrafilter if it satisfies the following:

(a) $\mathcal{U}$ is a filter on $(\text{FIN}_k^\infty, \leq)$ invariant under finite changes. That is:
    (1) If $A \in \mathcal{U}$ and $A \Delta B$ is finite, then $B \in \mathcal{U}$.
    (2) For all $A, B \in \text{FIN}_k^\infty$, if $A \in \mathcal{U}$ and $A \leq B$ then $B \in \mathcal{U}$.
    (3) For all $A, B \in \mathcal{U}$, there exists $C \in \mathcal{U}$ such that $C \leq A$ and $C \leq B$.

(b) If $\mathcal{U}' \subseteq \text{FIN}_k^\infty$ is a filter on $(\text{FIN}_k^\infty, \leq)$ and $\mathcal{U} \subseteq \mathcal{U}'$ then $\mathcal{U}' = \mathcal{U}$. That is, $\mathcal{U}$ is a maximal filter on $(\text{FIN}_k^\infty, \leq)$.

(c) (A3 mod $\mathcal{U}$) For all $A \in \mathcal{U}$ and $a \in \text{FIN}_k^\infty \upharpoonright A$ with $\text{depth}_A(a) = n$, the following holds:
    (c.1) $\forall B \in [n, A] \cap \mathcal{U}$, $([a, B] \cap \mathcal{U} \neq \emptyset)$.
    (c.2) $\forall B \in [a, A] \cap \mathcal{U}$, $\exists A' \in [n, A] \cap \mathcal{U}$ such that $[a, A'] \subseteq [a, B]$.

(d) (A4 mod $\mathcal{U}$) Let $A \in \mathcal{U}$ and $a \in \text{FIN}_k^\infty \upharpoonright A$ be given. For all $\mathcal{O} \subseteq \text{FIN}_k^{[a]+1}$ then there exists $B \in \text{[depth}_A(a), A] \cap \mathcal{U}$ such that $r_{[a]+1}[a, B] \subseteq \mathcal{O}$ or $r_{[a]+1}[a, B] \cap \mathcal{O} = \emptyset$.

An ultrafilter is in particular a coideal, so using definitions and we can consider semiselective and selective ultrafilters on $\text{FIN}_k^\infty$. □
Will show below, in section 5, that for ultrafilters on \( \text{FIN}_k^\infty \) semiselectivity is equivalent to selectivity. This is also the case for ultrafilters on the space \( \mathbb{N}^{[\infty]} \) as was shown by Farah in [8].

3. RAMSEY SUBSETS OF \( \text{FIN}_k^\infty \)

Let \( \mathcal{H} \) be a semiselective coideal in \( \text{FIN}_k^\infty \).

**Lemma 3.** Let \( \mathcal{O} \subseteq \text{FIN}_k^\infty \) be open in the metric topology. Then, \( \mathcal{O} \) is \( \mathcal{H} \)-Ramsey.

We adapt the ideas of Nash-Williams, Galvin and Prikry and Farah to this context. Before proceeding with the proof, we give some definitions.

**Definition 11.** Fix a family \( \mathcal{F} \subseteq \text{FIN}_k^\infty \).

1. \( A \in \mathcal{H} \) accepts \( s \in \text{FIN}_k^{<\infty} \) if for every \( B \in [s, A] \cap \mathcal{H} \) there is \( n \) such that \( r_n(B) \in \mathcal{F} \).
2. \( A \) rejects \( s \) if no \( B \in [s, A] \cap \mathcal{H} \) accepts \( s \).
3. \( A \) decides \( s \) if it accepts \( s \) or rejects \( s \).

The following facts follow from the definition.

**Lemma 4.** Fix a family \( \mathcal{F} \subseteq \text{FIN}_k^{<\infty} \).

1. \( A \in \mathcal{H} \) accepts and rejects \( s \in \text{FIN}_k^{<\infty} \) implies that \( [s, A] \) is empty.
2. \( A \) accepts \( s \) and \( B \leq A, B \in \mathcal{H} \), then \( B \) accepts \( s \).
3. \( A \) rejects \( s \) and \( B \leq A, B \in \mathcal{H} \), then \( B \) rejects \( s \).
4. For every \( A \in \mathcal{H} \) and for every \( s \in \text{FIN}_k^{<\infty} \) with \( s \sqsupset a \), there is \( B \in [s, A] \cap \mathcal{H} \) that decides \( s \).
5. If \( A \) accepts \( s \), then it accepts every \( t \in r_{|s|+1}([s, A]) \).
6. If \( A \) rejects \( s \), then there exists \( B \in [s, A] \cap \mathcal{H} \) such that \( A \) does not accept any \( t \in r_{|s|+1}([s, B]) \).

We only prove the last fact, using that if \( c: \text{FIN}_k^n \to 2 \) there is \( X \in \mathcal{H} \) such that \( c \) is constant on \( r_n[X] \). Suppose \( A \) rejects \( s \), and let \( \mathcal{O} = \{ t \in \text{FIN}_k^{<\infty} : A \) accepts \( t \} \). By A.4 mod \( \mathcal{H} \), there exists \( B \in \mathcal{H} \cap [s, A] \) such that \( r_{|s|+1}[s, B] \subseteq \mathcal{O} \) or \( r_{|s|+1}[s, B] \subseteq \mathcal{O}^c \). If the first alternative holds then take \( C \in \mathcal{H} \cap [s, B] \). Let \( b = r_{|s|+1}(C) \). Then \( b \in \mathcal{O} \) and therefore \( A \) accepts \( b \). Since \( C \in [b, A] \) then there exists \( n \) such that \( r_n(C) \in \mathcal{F} \). Therefore \( B \) accepts \( b \), because \( C \) is arbitrary. But this contradicts that \( A \) rejects \( s \). Hence, \( r_{|s|+1}[s, B] \subseteq \mathcal{O}^c \) and we are done.

**Claim 1.** Given \( A \in \mathcal{H} \), there exists \( D \in \mathcal{H} \setminus A \) which decides every \( b \in \text{FIN}_k^{<\infty} \setminus D \).

**Proof.** For every \( a \in \text{FIN}_k^{<\infty} \setminus A \) define

\[
\mathcal{D}_a = \{ C \in \mathcal{H} \cap [\text{depth}_A(a), A] : C \text{ decides } a \}
\]

By parts \([2, 3]\) and \([4]\) of Lemma \([4]\) each \( \mathcal{D}_a \) is dense open in \( \mathcal{H} \cap [\text{depth}_A(a), A] \). By semiselectivity, there exists \( D \in \mathcal{H} \setminus A \) which diagonalizes the collection \( (\mathcal{D}_a)_{a \in \text{FIN}_k^{<\infty} \setminus A} \). By parts \([2, 3]\) of Lemma \([4]\) \( D \) decides every \( a \in \text{FIN}_k^{<\infty} \setminus D \). \(\square\)

The following is an abstract version of the semiselective Galvin lemma (see [9, 8]).

**Lemma 5** (Semiselective Galvin’s lemma for \( \text{FIN}_k^\infty \)). Given \( \mathcal{F} \subseteq \text{FIN}_k^{<\infty} \), a semiselective coideal \( \mathcal{H} \subseteq \text{FIN}_k^\infty \), and \( A \in \mathcal{H} \), there exists \( B \in \mathcal{H} \setminus A \) such that one of the following holds:
Proof. Consider $D$ as in the Claim. If $D$ accepts $\emptyset$ part (2) holds and we are done. So assume that $D$ rejects $\emptyset$ and for $a \in {\text{FIN}}_{k}^{\infty} \mid D$ define

\[ D_a = \{ C \in H \cap [\text{depth}_A(a), D] : C \text{ rejects every } b \in r_{|a|+1}([a, C]) \} \]

if $D$ rejects $a$, and $D_a = H \cap [\text{depth}_A(a), D]$, otherwise. By parts 3 and 4 of Lemma 4 each $D_a$ is dense open in $H \cap [\text{depth}_A(a), D]$. By semiselectivity, choose $B \in H \mid D$ such that for all $a \in {\text{FIN}}_{k}^{\infty} \mid B$ there exists $C_a \in D_a$ with $[a, B] \subseteq [a, C_a]$. For every $a \in {\text{FIN}}_{k}^{\infty} \mid B$, $C_a$ rejects all $b \in r_{|a|+1}([a, C_a])$. So $B$ rejects all $b \in r_{|a|+1}([a, B])$: given one such $b$, choose any $\hat{B} \in H \cap [b, B]$. Then $\hat{B} \in H \cap [b, C_a]$. Therefore, since $C_a$ rejects $b$, $\hat{B}$ does not accept $b$.

Hence, $B$ satisfies that ${\text{FIN}}_{k}^{\infty} \mid B \cap F = \emptyset$. This completes the proof of the Lemma. □

Notation. $FIN_k^\infty \mid [a, B] = \{ b \in FIN_k^\infty : a \sqsubseteq b \& (\exists n \geq |a|)(\exists C \in [a, B]) b = r_n(C) \}$. In a similar way we can prove the following generalization of lemma 5.

Lemma 6. Given a semiselective coideal $H \subseteq FIN_k^\infty$, $F \subseteq FIN_k^\infty$, $A \in H$ and $a \in FIN_k^\infty \mid A$, there exists $B \in H \cap [a, A]$ such that one of the following holds:

1. $FIN_k^\infty \mid [a, B] \cap F = \emptyset$, or
2. $\forall C \in [a, B] (\exists n \in \mathbb{N}) (r_n(C) \in F)$.

Now, we proceed to prove Lemma 5. Recall that the basic metric open subsets of $FIN_k^\infty$ are of the form $[b] = \{ A \in FIN_k^\infty : b \sqsubseteq A \}$, where $b \sqsubseteq A$ means $(\exists n \in \mathbb{N}) (r_n(A) = b)$.

Proof. (of lemma 5) Let $X$ be a metric open subset of $\mathcal{R}$ and fix a nonempty $[a, A]$ with $A \in H$. Without a loss of generality, we can assume $a = \emptyset$. Since $X$ is open, there exists $F \subseteq FIN_k^\infty$ such that $X = \bigcup_{b \in F} [b]$. Let $B \in H \mid A$ be as in Lemma 5. If (1) holds then $[0, B] \subseteq X$ and if (2) holds then $[0, B] \subseteq X$. □

4. Ultrafilters and Forcing

4.1. Ultrafilters on $\mathbb{N}$ and on $FIN$. We recall some definitions related to ultrafilters on $\omega$.

A non-principal ultrafilter $U$ on $\mathbb{N}$ is a **P-point** if for every partition $\mathbb{N} = \bigcup_{i \in \omega} A_i$ into sets not belonging to $U$ there is $B \in U$ such that $|B \cap A_i| < \omega$ for every $i \in \omega$. $U$ is said to be a **Q-point** if for every partition $\mathbb{N} = \bigcup_{i \in \omega} A_i$ into finite sets there is $B \in U$ such that $|B \cap A_i| \leq 1$ for every $i \in \omega$. An ultrafilter $U$ is **selective** if it is a P-point and a Q-point.

Definition 12. An ultrafilter $U$ on $\mathbb{N}$ is **strongly summable** if for every $A \in U$ there is a strictly increasing sequence of positive integers $\{n_k : k \in \omega\}$ such that $FS(\{n_k\})$ is an element of $U$ and $FS(\{n_k\}) \subseteq A$.

Here, $FS(\{n_k\}) = \{ \sum_{k \in F} n_k : F \in [\omega]^{<\infty} \}$. That is, $FS(\{n_k\})$ is the set of finite sums of elements of $\{n_k\}$ with no repetitions.

Strongly summable ultrafilters are related to Hindman’s theorem as selective ultrafilters are related to Ramsey’s theorem.

For ultrafilters on $FIN$, the following definitions due to Blass ([1]) give the corresponding ultrafilters related to the finite unions version of Hindman’s theorem.
Definition 13. ([1]) An ultrafilter $U$ on $FIN$ is a union ultrafilter if it has a basis of sets of the form $FU(\{a_n : n \in \omega\})$ where $\{a_n : n \in \omega\}$ is a sequence of pairwise disjoint elements of $FIN$.

$U$ is an ordered-union ultrafilter if it has a basis of sets of the form $FU(\{a_n : n \in \omega\})$ where $\{a_n : n \in \omega\}$ is a block sequence of pairwise disjoint elements of $FIN$ (i.e. for every $n \max(a_n) < \min(a_{n+1})$).

Here, $FU(\{a_n : n \in \omega\}) = \{\cup_{k \in F}a_k : F \in [\omega]^{<\infty}\}$.

Recall the following fact about forcing with coideals on $\mathbb{N}^{[\omega]}$. If $H$ is a selective coideal on $\mathbb{N}^{[\omega]}$, forcing with the partial order $(H, \subseteq^*)$ adds a selective ultrafilter on $\mathbb{N}$ contained in $H$ ([15]). Farah, in [8], proved the same for $\omega$-selective ultrafilter on $a$ finite number of its elements, every element of $A \in H$ contains a coideal contained in $H$.

Definition 14. ([1]) An ordered-union ultrafilter $U$ on $FIN$ is stable if for every sequence $\{D_n : n \in \omega\} \subseteq FIN^\infty$ such that $FU(D_n) \in U$ for every $n$, there is $E \in FIN^\infty$ such that $FU(E) \in U$ and for every $n E \leq^* D_n$.

As we will see, if $H$ is a semiselective coideal, then the partial order $(H, \leq^*)$ adds a stable ordered-union ultrafilter. This result is due to Eisworth, who proved it in [6] using the concept of Matet-adequate families $H \subseteq FIN^\infty$.

Definition 15. [6] A family $H \subseteq FIN^\infty$ is Matet-adequate if

1. $H$ is closed under finite changes,
2. For all $A, B \in FIN^\infty$, if $A \in H$ and $A \leq B$ then $B \in H$.
3. $(H, \leq^*)$ is $\sigma$-closed,
4. If $A \in H$ and $FU(A)$ is partitioned into 2 pieces then there is $B \leq A$ in $H$ so that $FU(B)$ is included in a single piece of the partition (this is called the Hindman property).

Semiselective coideals and Matet-adequate subfamilies of $FIN^\infty$ are clearly related. Both classes of families share the two properties of being closed under finite changes and being closed upwards; they share also the Hindman property, which is equivalent to $A4 \pmod{H}$. A Matet adequate family is $\sigma$-closed as a partial order under $\leq^*$, and thus $\sigma$-distributive; while a semiselective coideal $H$, has the diagonalization property of definition [6] which also implies that $(H, \leq^*)$ is $\sigma$-distributive. We will prove that in fact Matet-adequate families are just semiselective coideals in $FIN^\infty$.

4.2. Ultrafilters on $FIN^\infty_k$. We will work now in the context of the space $FIN^\infty_k$, with generalizations of the mentioned concepts to this context. So, we fix $k \geq 1$ from now on.

The poset $(FIN^\infty_k, \leq)$ (and therefore $(FIN^\infty_k, \leq^*)$) has a maximal element $1$ consisting of the set of functions $k \cdot \chi_{\{n\}}$ varying $n \in \omega$.
Definition 16.  (1) An ultrafilter $\mathcal{U}$ on $\text{FIN}_k$ is ordered-$T$ if it has a basis of sets of the form $[A]$ with $A \in \text{FIN}_k^\infty$.

(2) An ordered-$T$ ultrafilter is stable if for every sequence $\{[A_n] : n \in \omega \} \subseteq \mathcal{U}$, with each $A_n \in \text{FIN}_k^\infty$, there exists $E \in \text{FIN}_k^\infty$ such that $E \leq^* A_n$ for each $n \in \omega$ and $[E] \in \mathcal{U}$.

(3) An ultrafilter $\mathcal{U}$ on $\text{FIN}_k$ has the Ramsey property for pairs if for every partition $c : \text{FIN}_k^{[2]} \to 2$

there exists $X \in \mathcal{U}$ such that $|X|_2^c$ is monochromatic.

These notions coincide with those of ordered-union, stable, and Ramsey property for pairs as in $[1]$ when $k = 1$.

Theorem 3. ($[6]$) Let $\mathcal{H}$ be a $\sigma$-distributive coideal on $\text{FIN}_k^\infty$. If $G$ is $(\mathcal{H}, \leq^*)$-generic over $V$, then

$$\mathcal{U}_G = \{A \subseteq \text{FIN}_k : \exists B \in G ([B] \subseteq A)\}$$

is a stable ordered-$T$ ultrafilter.

Proof. Clearly, $\mathcal{U}_G$ is an ordered-$T$ filter. The stability follows from the $\sigma$-distributivity as follows. Let $\{A_n : n \in \omega\}$ be a sequence of elements of $\mathcal{U}_G$. Take a sequence $\{B_n : n \in \omega\}$ of elements of $G$ such that for every $n \in \omega$, $[B_n] \subseteq A_n$. For every $n$, let

$$D_n = \{B \in \text{FIN}_k : B \perp B_n \text{ or } B \leq^* B_n\}.$$ 

Each $D_n$ is dense in $(\mathcal{H}, \leq^*)$, and so, by $\sigma$-distributivity of $\mathcal{H}$ the intersection $\bigcap_n D_n$ is dense and is in the ground model. Thus there exists $E \in \mathcal{H} \cap (\bigcap_n D_n)$. Then $[E] \in \mathcal{U}_G$ and $E$ is an almost condensation of $B_n$ for every $n$. For every partition of $\text{FIN}_k$ in the ground model, Gowers property $A4$ of $\mathcal{H}$ gives $A \in G$ such that $[A]$ is included in one part of the partition. But since $\mathcal{H}$ is $\sigma$-distributive, there are no new partitions of $\text{FIN}_k$ in the extension and thus $\mathcal{U}_G$ is an ultrafilter. \qed

Definition 17. If $\mathcal{U}$ is an ordered-$T$ ultrafilter on $\text{FIN}_k$, then

$$\mathcal{U}^\infty = \{A \in \text{FIN}_k^\infty : [A] \in \mathcal{U}\}.$$ 

The next definition was first proposed by Krautzberger in $[13]$ and will be used in the proof of Theorem $[4]$.

Definition 18. Let $A \in \text{FIN}_k^\infty$ and $f \in [A]$. We say that $n \in \omega$ is an $A$-splitting point of $f$ if both $f \upharpoonright (n+1)$ and $f \upharpoonright \omega \setminus (n+1)$ are elements of $[A]$ and there exists $g \in [A]$ such that

$$f \upharpoonright (n+1) < g < f \upharpoonright \omega \setminus (n+1).$$ 

We define $\pi_A : [A] \to \omega$ as $\pi_A(f) := |\{n \in \omega : n \text{ is an } A\text{-splitting point of } f\}|$.

Theorem 4. ($[1]$) For any ordered-$T$ ultrafilter ultrafilter $\mathcal{U}$ on $\text{FIN}_k$, the following are equivalent.

(1) $\mathcal{U}$ is stable.

(2) $\mathcal{U}^\infty$ is selective

(3) $\mathcal{U}$ has the Ramsey property for pairs: Whenever $[\text{FIN}_k]_2^c$ is partitioned into two pieces, there is $X \in \mathcal{U}$ with $[X]_2^c$ included in one piece of the partition. (Here $[X]_2^c$ is the set of pairs $(s,t)$ of elements of $X$ with $s < t$).
Eisworth and Krautzberger (for the case \( k \))

We prove the equivalence between the first three properties following ideas of Blass.

**Proof.** We prove the equivalence between the first three properties following ideas of Blass, Eisworth and Krautzberger (for the case \( k = 1 \)) in [11, 12] and [13] respectively.

1 \( \rightarrow \) 2 Suppose \( U \) is a stable ordered-T ultrafilter on \( \text{FIN}_k \). To see that \( U^\infty \) is invariant under finite changes it is enough to notice that for \( A \in U^\infty \), and \( s \) is an element of \( A \), the set \( \{ t \in [A] : \text{supp}(s) \subseteq \text{supp}(t) \} \) is not in \( U \). This is so because this set does not contain any set of the form \( [X] \) with \( X \in \text{FIN}^\infty \). Once we have this, is is clear that the same applies to \( \{ t \in [A] : \exists s \in F(\text{supp}(s) \subseteq \text{supp}(t) \} \) for any finite subsequence \( F \) of \( A \). Finally, if \( B \) differs from \( A \) in a finite set, then \( [B] \) contains \( [C] \) for some \( C \) obtained deleting finitely many elements from \( A \).

It is easy to verify that \( U^\infty \) is closed upwards, since if \( A \leq B, [A] \subseteq [B] \).

If \( A, B \in U^\infty \), then \( [A] \) and \( [B] \) are in \( U \) and so their intersection is also in \( U \). Then, there is \( C \in U^\infty \) with \( [C] \) contained in this intersection, and therefore \( C \leq A, B \). We have then that \( U^\infty \) is a filter. That \( U^\infty \) is a maximal filter follows from the maximality of \( U \) as an ultrafilter on \( \text{FIN}_k \). A3 (mod \( U^\infty \)) is obvious. And A4 (mod \( U^\infty \)) is just a consequence of the fact that stable ordered-T ultrafilters contain homogeneous sets for partitions as in Gowers’ theorem.

We show now that \( U^\infty \) is stable. Let \( A \in U^\infty \) and \( \{ A_n : n \in \omega \} \subseteq U^\infty \upharpoonright A \) as in definition [8]. As \( U \) is stable let \( E \in U^\infty \upharpoonright A \) such that \( E \leq* A_n \) and let \( j : \omega \to \omega \) such that \( E/j(n) \leq A_n \) and \( j(n) \) is minimal with such a property (this guarantees that the function \( j \) is increasing).

First of all let us see that \( \{ f \in [E] : j(\min(f)) < \max(f) \} \in U \). Note that given \( B \in U^\infty \upharpoonright E \), there exists \( f < g \) both in \( [B] \) such that \( j(\min(f)) < \max(g) \) and therefore

\[
j(\min(f + g)) = j(\min(f)) < \max(g) = \max(f + g)
\]

that is, the set above intersect any element of \( U \). Let \( C \in U^\infty \upharpoonright E \) such that \( [C] \) is contained in that set.

Let us see now that \( \{ f \in [C] : \pi_C(f) \text{ is odd} \} \in U \). For this note that any condensation of \( C \) contains \( f < g < h \) and \( \pi_C(f + h) = \pi_C(f) + \pi_C(h) + 1 \) so at least one between \( f, h \) and \( f + h \) has an odd number of \( C \)-splitting points. Take now \( B \in U^\infty \upharpoonright C \) such that \( [B] \) is contained in the set above and lets see that \( B \) is a diagonalization of the \( A_n \)’s within \( A \). Given \( f \in [B] \) and \( h \in B/f \) we have that there exists \( g \in [C] \) with \( f < g < h \), otherwise \( \pi_C(f + h) = \pi_C(f) + \pi_C(h) \), that is an even number. We have then that

\[
\min(h) > \max(g) > j(\min(g)) \geq j(\max(f))
\]

and therefore \( h \in A_{\max(f)} \leq A_{\text{depth}_A(f)} \) since \( \text{depth}_A(f) \leq \max(f) \).

2 \( \rightarrow \) 3 Let \( c : [\text{FIN}_k]_2 \to 2 \) a partition and for each \( f \in \text{FIN}_k \) consider the induced partition \( c_f : \text{FIN}_k/f \to 2 \) given by \( c_f(g) = c(f, g) \). As \( U \) has the Gowers property (for being ordered-T) let \( A_f \in U^\infty \) such that \( [A_f] \) is monochromatic for \( c_f \). By a counting argument there exists an infinite \( F \subseteq \text{FIN}_k \) and \( i \in 2 \) such that for all \( f \in F \), \( c_f^*[A_f] = i \). Let \( G_n := \{ f \in F : \max(f) = n \} \). Take \( A_0 = 1 \) and, since \( G_n \) is finite, let \( A_{n+1} \in U^\infty \upharpoonright A_n \) such that \( A_{n+1} \leq A_f \) for each \( f \in G_{n+1} \). By selectivity let
$B \in \mathcal{U}^\infty$ be a diagonalization of the $A_n$'s within $\mathcal{I}$. We have then that, for $f \in [B]$,

$$B/f \leq A_{\text{depth}_1(f)} = A_{\text{max}(f)} \leq A_f$$

and therefore, if $f < g$ are both in $[B]$, $c(f, g) = c_f(g) = i$.

3 → 1 Let $\{A_n : n \in \omega\} \subseteq \mathcal{U}^\infty$ and consider the partition of $[\text{FIN}_k]^2_<$ in two pieces given by

$$c(f, g) = \begin{cases} 1 & (\forall n \leq \text{max}(f))(g \in A_n) \\ 0 & \text{otherwise} \end{cases}$$

By the Ramsey property for pairs let $B \in \mathcal{U}^\infty$ such that $[[B]]^2_<$ is monochromatic and note that such a set must meet the color 1. To see this let $f \in [B]$ and choose some $f < g$ with

$$g \in [B] \cap \bigcap \{[A_n] : n \leq \text{max}(f)\} \in \mathcal{U}.$$ 

Now fix $n \in \omega$ and take $f \in B$ with $n \leq \text{max}(f)$. We have then that $B/f \leq A_n$ thus $B \leq^* A_n$.

□

We go back now to the case $k = 1$ to show that Matet-adequate families are the same as selective coideals.

**Theorem 5.** Every Matet-adequate family is a selective coideal in $\text{FIN}^\infty$. Therefore, Matet-adequate families and selective coideals coincide.

**Proof.** (inspired on [6]) Let $\mathcal{H}$ be a Matet-adequate family, and let $[a, A] \neq \emptyset$ with $A \in \mathcal{H}$, and $\{A_n : n \in \omega\}$ a $\leq$-decreasing sequence in $\mathcal{H}$, with each $A_n$ a condensation of $A$ such that $[a, A_n] \neq \emptyset$. Let $C \in \mathcal{H}$ be such that $C \leq^* A_n$ for every $n \in \omega$. Force with $(\mathcal{H} | C, \leq^*)$, the collection of all condensations of $C$ which are in $\mathcal{H}$. This adds a stable ordered-union ultrafilter on $\text{FIN}$ such that $\text{FU}(C) \in \mathcal{U}$. In fact, if $G$ is a generic subset of $\mathcal{H} \upharpoonright C$, then $\mathcal{U}_G = \{A \subseteq \text{FIN} : \exists B \in G (\text{FU}(B) \subseteq A)\}$ is a stable union-ordered ultrafilter and $\text{FU}(C)$ belongs to it.

Notice that for every $n$, $\text{FU}(A_n) \in \mathcal{U}$. Applying Theorem 4 we get $B \leq C$ with $\text{FU}(B) \in \mathcal{U}$, which diagonalizes $\{A_n : n \in \omega\}$ within $A$.

$B$ is in the ground model since no reals are added by this forcing. And $B \in \mathcal{H}$ by definition of $\mathcal{U}_G$ and the fact that $\mathcal{H}$ is upwards closed. Using the fact that every selective coideal is $\sigma$-closed the proof is finished. □

5. **SELECTIVE ULTRAFILTERS AND STABLE ORDERED-T ULTRAFILTERS**

In this section we complete our remarks concerning the relation between stable ordered-union ultrafilters on $\text{FIN}$ and selective ultrafilters on the topological Ramsey space $\text{FIN}^\infty$.

Recall that for an ultrafilter $\mathcal{U}$ on $\text{FIN}_k$,

$$\mathcal{U}^\infty = \{A \in \text{FIN}_k^\infty : [A] \in \mathcal{U}\}.$$

By Theorem 4 if $\mathcal{U}$ is a stable ordered-T ultrafilter on $\text{FIN}_k$, then $\mathcal{U}^\infty$ is a selective ultrafilter on $\text{FIN}_k^\infty$. The following is a sort of reverse implication.

**Theorem 6.** If $\mathcal{V}$ is a selective ultrafilter on $\text{FIN}_k^\infty$, then the filter $\mathcal{U}$ on $\text{FIN}_k$ generated by $\{[A] : A \in \mathcal{V}\}$ is a stable ordered-T ultrafilter.
Proof. Let $\mathcal{V}$ be a selective ultrafilter on $\text{FIN}_k^\infty$, and let $\mathcal{U}$ be the filter on $\text{FIN}_k$ generated by $\{[A] : A \in \mathcal{V}\}$. $\mathcal{U}$ is in fact a filter because if $A, B \in \mathcal{V}$ then there exists $C \in \mathcal{V}$ such that $C \leq A$ and $C \leq B$, and it is obvious that it is an ordered-$T$ filter. To see that it is an ultrafilter on $\text{FIN}_k$, suppose that there is an ordered-$T$ ultrafilter $\mathcal{U}'$ properly extending $\mathcal{U}$. Take $X \in \mathcal{U}' \setminus \mathcal{U}$. Then there is $A \in \text{FIN}_k^\infty$ such that $[A] \subseteq X$, and $A \notin \mathcal{V}$. But then $(\mathcal{U}')^\infty$ is a filter properly containing $\mathcal{V}$.

Let us now verify that $\mathcal{U}$ is stable. Given a sequence $\{A_n : n \in \omega\} \subseteq \text{FIN}_k^\infty$ such that $[A_n] \in \mathcal{U}$ for every $n$, we use the selectivity of $\mathcal{V}$ to obtain $E$ which is an almost condensation of each $A_n$. For every $n$, since $[A_n] \in \mathcal{U}$, there is $B_n \in \mathcal{V}$ such that $[B_n] \subseteq [A_n]$, and therefore $B_n \leq A_n$. Now we work with the $B_n$'s, and using the fact that they belong to the ultrafilter $\mathcal{V}$, we construct a descending sequence $\{C_n : n \in \omega\}$ as follows, $C_0 = B_0$ and if $C_n$ has been defined, we let $C_{n+1}$ be an element $C$ of $\mathcal{V}$ such that $C \leq C_n$ and $C \leq B_n$. The selectivity of $\mathcal{V}$ gives us a diagonalization $E$ of the sequence $\{C_n : n \in \omega\}$ in the ultrafilter. Then $E$ is an almost condensation of each $A_n$.

We will now prove that any semiselective ultrafilter on $\text{FIN}_k^\infty$ is in fact selective.

**Theorem 7.** If $\mathcal{V}$ is a semiselective ultrafilter on $\text{FIN}_k^\infty$, then it is selective.

**Proof.** Notice that the ultrafilter $\mathcal{U}$ on $\text{FIN}_k$ generated by $\{[A] : A \in \mathcal{V}\}$ is an ordered-$T$ ultrafilter. Let us show that it has the Ramsey property for pairs. Let $A$ be an almost condensation of each $\text{FIN}_k^\infty$ defined, we let $C$ of $V$ such that $C$ are the product topology regarding $2 = \bigotimes_{n \in \omega} \text{FIN}_k$. Some notation is needed. For $x = (x_n)_{n \in 2^\omega}$, $x|_k$ denotes the finite sequence $(x_0, x_1, \ldots, x_{k-1})$. For $u \in 2^{< \omega}$, let $[u] = \{x \in 2^\omega : (\exists k)(u = x|_k)\}$ and let $|u|$ be the length of $u$. Given a perfect set $Q \subseteq 2^\omega$, let $T_Q$ be its associated perfect tree. For $n \in \mathbb{N}$, let $T_P | n = \{u \in T_P : |u| = n\}$. Also, for $u, v = (v_0, v_1, \ldots, v_{|v|-1}) \in 2^{< \omega}$ we write $u \sqsubseteq v$ to mean $(\exists k \leq |v|)(u = (v_0, v_1, \ldots, v_{k-1}))$. For each $u \in 2^{< \omega}$, let $Q(u) = Q \cap [u(Q)]$, where $u(Q) \in T_Q$ is define inductively, as follows: $\emptyset(Q) = \emptyset$. Suppose $u(Q)$ is defined. Find $\sigma \in T_Q$ such that $\sigma$ is the $\sqsubseteq$-extension of $u(Q)$ where the first ramification extending $u(Q)$ occurs. Then, set $(u * i)(Q) = \sigma \ast i$, $i = 0, 1$. Here $"\ast"$ denotes concatenation. Note that for each $n$, $Q = \bigcup\{Q(u) : u \in 2^n\}$.

**Definition 19.** Let $\mathcal{H} \subseteq \text{FIN}_k^\infty$ be a semiselective coideal. We say that a set $X \subseteq 2^\omega \times \text{FIN}_k^\infty$ is **perfectly $\mathcal{H}$-Ramsey** if for every perfect set $Q \subseteq 2^\omega$ and every neighborhood $[a, A] \neq \emptyset$ in
\( FIN_k^\infty \) with \( A \in \mathcal{H} \) there exist a perfect set \( S \subseteq Q \) and \( B \subseteq [a, A] \cap \mathcal{H} \) such that \( S \times [a, B] \subseteq \mathcal{X} \) or \( S \times [a, B] \cap \mathcal{X} = \emptyset \). A set \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \) is perfectly \( \mathcal{H} \)-Ramsey null if for every perfect set \( Q \subseteq 2^\infty \) and every neighborhood \([a, A] \neq \emptyset \) in \( FIN_k^\infty \) with \( A \in \mathcal{H} \) there exist a perfect set \( S \subseteq Q \) and \( B \subseteq [a, A] \cap \mathcal{H} \) such that \( S \times [a, B] \cap \mathcal{X} = \emptyset \).

Also, we will need to adapt the notion of abstract Baire property (see \([18]\)) to this context:

**Definition 20.** Let \( \mathbb{P} \) be the family of perfect subsets of \( 2^\infty \) and let \( \mathcal{H} \subseteq FIN_k^\infty \) be a semiselective coideal. We will say that a set \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \) has the \( \mathbb{P} \times Exp(\mathcal{H}) \)-Baire property if for every perfect set \( Q \subseteq 2^\infty \) and every neighborhood \([a, A] \neq \emptyset \) in \( FIN_k^\infty \) with \( A \in \mathcal{H} \) there exist a perfect set \( S \subseteq Q \) and a nonempty neighborhood \([b, B] \subseteq [a, A] \) with \( B \in \mathcal{H} \) such that \( S \times [b, B] \subseteq \mathcal{X} \) or \( S \times [b, B] \cap \mathcal{X} = \emptyset \). A set \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \) is \( \mathbb{P} \times Exp(\mathcal{H}) \)-meager if for every perfect set \( Q \subseteq 2^\infty \) and every neighborhood \([a, A] \neq \emptyset \) in \( FIN_k^\infty \) with \( A \in \mathcal{H} \) there exist a perfect set \( S \subseteq Q \) and a nonempty neighborhood \([b, B] \subseteq [a, A] \) with \( B \in \mathcal{H} \) such that \( S \times [b, B] \cap \mathcal{X} = \emptyset \).

Next, we state the main result of this section:

**Theorem 8.** Let \( \mathbb{P} \) be the family of perfect subsets of \( 2^\infty \) and let \( \mathcal{H} \subseteq FIN_k^\infty \) be a semiselective coideal. The following are true:

(a) \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \) is perfectly \( \mathcal{H} \)-Ramsey iff \( \mathcal{X} \) has the \( \mathbb{P} \times Exp(\mathcal{H}) \)-Baire Property.

(b) \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \) is perfectly \( \mathcal{H} \)-Ramsey null iff \( \mathcal{X} \) is \( \mathbb{P} \times Exp(\mathcal{H}) \)-meager.

In order to prove Theorem 8 we will need to introduce two combinatorial forcings and prove a series of lemmas related to them. Before doing that, we will state the following definition and a related lemma which will be useful in the sequel.

**Definition 21.** Let \( S \) be a nonempty subset of \( \mathbb{P} \times FIN_k^\infty \). A set \( \mathcal{D} \subseteq S \) is dense open in \( S \) if:

(i) For every \((Q, B) \in S \) there exists \((P, A) \in \mathcal{D} \) such that \( P \times A \subseteq Q \times B \); and

(ii) If \((P, A) \in \mathcal{D} \) and \( Q \times B \subseteq P \times A \) then, \((Q, B) \in \mathcal{D} \).

**Lemma 8.** Let \( \mathcal{H} \subseteq FIN_k^\infty \) be a semiselective coideal and let \( S \) be a nonempty subset of \( \mathbb{P} \times \mathcal{H} \). If \( \mathcal{D}_n, n \in \mathbb{N} \), are dense open subsets of \( S \) then, the set

\[ \mathcal{D}_\infty = \{(P, A) : (\forall u \in T_P \uparrow n) (\forall a \in [A] \text{ with } \text{depth}_A(a) \leq n) (P \cap [u], A/a) \in \mathcal{D}_n \} \]

is dense open in \( S \).

**Proof.** The proof of Lemma 8 follows the same argument used in the proof of Lemma 2.5 of \([8]\) so we will leave it to the reader. \( \square \)

Now, fix a semiselective coideal \( \mathcal{H} \subseteq FIN_k^\infty \).

**Combinatorial Forcing 1.** Fix \( \mathcal{F} \subseteq 2^{<\infty} \times FIN_k^{<\infty} \). For a perfect \( Q \subseteq 2^\infty \), \( A \in \mathcal{H} \) and a pair \((u, a) \in 2^{<\infty} \times FIN_k^{<\infty} \), we say that \((Q, A)\) accepts \((u, a)\) if for every \( x \in Q(u) \) and for every \( B \subseteq [a, A] \cap \mathcal{H} \) there exist integers \( k \) and \( m \) such that \((x\upharpoonright_k, r_m(B)) \in \mathcal{F} \). We say that \((Q, A)\) rejects \((u, a)\) if for every perfect \( S \subseteq Q(u) \) and every \( B \subseteq [a, A] \cap \mathcal{H} \), \((S, B)\) does not accept \((u, a)\). Also, we say that \((Q, A)\) decides \((u, a)\) if it accepts or rejects it.

**Combinatorial Forcing 2.** Fix \( \mathcal{X} \subseteq 2^\infty \times FIN_k^\infty \). For a perfect \( Q \subseteq 2^\infty \), \( A \in \mathcal{H} \) and a pair \((u, a) \in 2^{<\infty} \times FIN_k^{<\infty} \), we say that \((Q, A)\) accepts \((u, a)\) if \( Q(u) \times [a, A] \subseteq \mathcal{X} \).
that \((Q, A)\) rejects \((u, a)\) if for every perfect \(S \subseteq Q(u)\) and every \(B \in [a, A] \cap \mathcal{H}\), \((S, B)\) does not accepts \((u, a)\). And as before, we say that \((Q, A)\) decides \((u, a)\) if it accepts or rejects it.

**Note:** Lemmas 9, 10, and 11 below hold for both combinatorial forcings defined above.

**Lemma 9.** Let a perfect \(Q \subseteq 2^\omega\), \(A \in \mathcal{H}\) and a pair \((u, a) \in 2^{<\omega} \times \text{FIN}^{\infty}_k\) be given. The following are true:

- (a) If \((Q, A)\) accepts (rejects) \((u, a)\) then \((S, B)\) also accepts (rejects) \((u, a)\), for every perfect \(S \subseteq Q(u)\) and every \(B \in [a, A] \cap \mathcal{H}\).
- (b) If \((Q, A)\) accepts (rejects) \((u, a)\) then \((Q, B)\) also accepts (rejects) \((u, a)\), for every \(B \in [a, A] \cap \mathcal{H}\).
- (c) There exist a perfect set \(S \subseteq Q\) and \(B \in [a, A] \cap \mathcal{H}\) such that \((S, B)\) decides \((u, a)\).
- (d) If \((Q, A)\) accepts \((u, a)\) then \((Q, A)\) accepts \((u, b)\) for every \(b \in r_{|a|+1}[a, A]\).
- (e) If \((Q, A)\) rejects \((u, a)\) then there exist \(B \in [\text{depth}_A(a), A] \cap \mathcal{H}\) such that \((Q, A)\) does not accept \((u, b)\) for every \(b \in r_{|a|+1}[a, B]\).
- (f) \((Q, A)\) accepts (rejects) \((u, a)\) iff \((Q, A)\) accepts (rejects) \((v, a)\), for every \(v \in 2^{<\omega}\) with \(u \subseteq v\).

**Proof.** (a), (b), (c), (d) and (f) follow from the definitions. Now to proof (e), take \((u, a)\) with \(|a| = m\) and suppose \((Q, A)\) rejects it. Define \(\phi : \text{FIN}^{m+1}_k \rightarrow 2\) such that \(\phi(b) = 1\) iff \((Q, A)\) accepts \((u, b)\). Let \(n = \text{depth}_A(a)\). By A4 mod \(\mathcal{H}\), there exists \(B \in [n, A] \cap \mathcal{H}\) such that \(\phi\) is constant on \(r_{m+1}[a, B]\).

If \(\phi\) takes value 1 on \(r_{m+1}[a, B]\) then \((Q, B)\) accepts \((u, a)\). So, in virtue of part (b), \(\phi\) must take value 0 on \(r_{m+1}[a, B]\) since \((Q, A)\) rejects \((u, a)\). Then \(B\) is as required. \(\square\)

**Lemma 10.** For every perfect \(P \subseteq 2^\omega\) and \(A \in \mathcal{H}\) there exist a perfect \(Q \subseteq P\) and \(B \leq A\) in \(\mathcal{H}\) such that \((Q, B)\) decides \((u, a)\), for every \((u, a) \in 2^{<\omega} \times \text{FIN}^{\infty}_k \upharpoonright B\) with \(\text{depth}_B(a) \leq |u|\).

**Proof.** Let

\[
\mathcal{D}_n = \{(Q, B) \in \mathbb{P} \times \mathcal{H} : Q \subseteq P, B \leq A \text{ and } (\forall (u, a) \in T_Q \times \text{FIN}^{\infty}_k \upharpoonright B) \text{ with } \text{depth}_B(b) \leq |u| = n) \ (Q, B) \text{ decides } (u, a)\}
\]

Let \(\mathcal{S} = \{(Q, B) \in \mathbb{P} \times \mathcal{H} : Q \subseteq P, B \leq A\}\). Then \(\mathcal{D}_n\) is dense open in \(\mathcal{S}\). Let \(\mathcal{D}_\infty\) be as in Lemma 8 and choose \((Q, B) \in \mathcal{D}_\infty\). Then \(Q\) and \(B\) are as required. \(\square\)

**Lemma 11.** Let \(Q\) and \(B\) be as in Lemma 10. Suppose \((Q, B)\) rejects \((<> , \emptyset)\). Then there exists \(D \leq B\) in \(\mathcal{H}\) such that \((Q, D)\) rejects \((u, b)\), for every \((u, b) \in 2^{<\omega} \times \text{FIN}^{\infty}_k \upharpoonright D\) with \(\text{depth}_D(b) \leq |u|\).

**Proof.** For \(n \in \mathbb{N}\), let

\[
\mathcal{D}_n = \{C \in \mathcal{H} \upharpoonright B : (Q, C) \text{ rejects every } (u, b) \in 2^n \times \text{FIN}^{\infty}_k \upharpoonright C \text{ with } \text{depth}_C(b) = n\}.
\]

**Claim.** Every \(\mathcal{D}_n\) is dense open in \(\mathcal{H} \upharpoonright B\).

**Proof of Claim.** By induction on \(n \geq 1\). Case \(n = 1\): Let \(D \in \mathcal{H} \upharpoonright B\) be given. Note that \((Q, D)\) rejects \((<> , \emptyset)\), by Lemma 9(b). Therefore, by parts (b), (e) and (f) of Lemma 9 by the choice of \((Q, B)\), and by the fact that \(|b| \leq \text{depth}_D(b)\), for every \(b \in \text{FIN}^{\infty}_k \upharpoonright D\), we
can find $C_1 \in \mathcal{D}_1$ with $C_1 \leq D$. On the other hand, obviously, if $D \in \mathcal{D}_1$ and $C \leq D$ then, $C \in \mathcal{D}_1$. That is, $\mathcal{D}_1$ is dense open in $\mathcal{H} \upharpoonright B$.

Now suppose $\mathcal{D}_n$ is dense open in $\mathcal{H} \upharpoonright B$ and, again, let $D \in \mathcal{H} \upharpoonright B$ be given. Choose $C_n \in \mathcal{D}_n$ with $C_n \leq D$. Let $u_0, u_1, \ldots, u_{2^n+1-1}$ be a list of the elements of $2^{n+1}$, and let $b_0, b_1, \ldots, b_m$ be a list of the $b \in \text{FIN}^{\leq \infty}_k \upharpoonright C_n$ such that $\text{depth}_{C_n}(b) = n$. By Lemma \[8\](f), $(Q, C_n)$ rejects $(u_i, b_j)$ for every $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, m\}$. Now, by Lemma \[8\](e) there exists $C_{n,0} \in [n, C_n] \cap \mathcal{H}$ such that $(Q, C_{n,0})$ rejects $(u_0, b)$ for every $b \in r_{|b_0|+1}[b_0, C_{n,0}]$.

In the same way, for every $(i, j) \in \{0, 1, \ldots, 2^{n+1} - 1\} \times \{0, 1, \ldots, m\}$, we can find $C_{n,j}^{i,j}$ satisfying the following:

1. $C_{n,j}^{i,j} + 1 \in [n, C_n] \cap \mathcal{H}$,
2. $C_n + 1,0 \in [n, C_n] \cap \mathcal{H}$, and
3. $(Q, C_n)_{j+1}$ rejects $(u_i, b)$ for every $b \in r_{|b_j|+1}[b_j, C_n^{i,j}]$.

Let $C_{n+1} = C_n^{2^n+1,m}$. Notice that $C_{n+1} \in \mathcal{D}_{n+1}$ and $C_{n+1} \leq D$. And, obviously, as in the case $n = 1$, if $D \in \mathcal{D}_{n+1}$ and $C \leq D$ then, $C \in \mathcal{D}_{n+1}$. Thus, $\mathcal{D}_{n+1}$ is dense open in $\mathcal{H} \upharpoonright B$. This completes the induction argument and the proof of the Claim. \[\square\]

Then, by semiselectivity, there exists a diagonalization $D \in \mathcal{H} \upharpoonright B$ of the sequence $(\mathcal{D}_n)_n$. It is easy to see that $D$ is as required. \[\square\]

The next theorem is inspired by Theorem 2.3 of \[8\] and Theorem 3 of \[17\].

**Theorem 9.** Let $\mathcal{H} \subseteq \text{FIN}^{\leq \infty}_k$ be a semiselective coideal. For every $\mathcal{F} \subseteq 2^{\leq \infty} \times \text{FIN}^{\leq \infty}_k$, perfect $P \subseteq 2^{\leq \infty}$ and $A \in \mathcal{H}$ there exist a perfect $S \subseteq P$ and $D \leq A$ in $\mathcal{H}$ such that one of the following holds:

- (a) for every $x \in S$ and every $C \subseteq D$ in $\mathcal{H}$ there exist integers $l$ and $m > 0$ such that $(x|_l, r_m(C)) \in \mathcal{F}$.
- (b) $(T_S \times \text{FIN}^{\leq \infty}_k \upharpoonright D) \cap \mathcal{F} = \emptyset$.

**Proof.** Given $\mathcal{F} \subseteq 2^{\leq \infty} \times \text{FIN}^{\leq \infty}_k$, perfect $P \subseteq 2^{\leq \infty}$ and $A \in \mathcal{H}$, consider the combinatorial forcing 1. Let $Q \subseteq P$ and $B \leq A$ be as in Lemma \[10\]. If $(Q, B)$ accepts $(\prec, \emptyset)$ then part (a) of Theorem \[9\] holds by the definition of “accepts”. So suppose $(Q, B)$ does not accept (and hence, rejects) $(\prec, \emptyset)$. By Lemma \[10\] find $D \subseteq B$ in $\mathcal{H}$ such that $(Q, D)$ rejects $(u_0, b)$, for every $(u, b) \in 2^{\leq \infty} \times \text{FIN}^{\leq \infty}_k \upharpoonright D$ with $\text{depth}_D(b) \leq |u|$. Suppose towards a contradiction that there exist $(t, b)$ in $(T_Q \times \text{FIN}^{\leq \infty}_k \upharpoonright D) \cap \mathcal{F}$. Find $u \in 2^{\leq \infty}$ such that $Q(u) \subseteq Q \cap [t]$. Then $(Q, D)$ accepts $(u_0, b)$: for $x \in Q(u)$ and $C \subseteq [b, D]$, let $l = |t|$ and $m$ be such that $r_m(C) = b$. Then $(x|_l, r_m(C)) = (t, b) \in \mathcal{F}$. But then, by Lemma \[10\](f), $(Q, D)$ accepts $(u, b)$, for every $v \in 2^{\leq \infty}$ such that $u \subseteq v$ and $|v| \geq \text{depth}_D(b)$. This is a contradiction with the choice of $D$. Therefore, for $S = Q$ and $D$ part (b) of Theorem \[9\] holds. \[\square\]

Now we are ready to prove the main result of this section:

**Proof of Theorem \[8\]** (a) The implication from left to right is obvious. So suppose $\mathcal{X} \subseteq 2^{\leq \infty} \times \text{FIN}^{\leq \infty}_k$ has the $\mathbb{P} \times \text{Exp}(\mathcal{H})$-Baire Property, and let $P \times [a, A]$ be given, with $A \in \mathcal{H}$. In order to make the proof notationally simpler, we will assume $a = \emptyset$ without a loss of generality.
Claim 2. Given $\hat{X} \subseteq 2^\omega \times \text{FIN}_k^\omega$, perfect $\hat{P} \subseteq 2^\omega$ and $\hat{A} \in \mathcal{H}$, there exist $Q \subseteq \hat{P}$ and $B \leq \hat{A}$ in $\mathcal{H}$ such that for each $(u, b) \in 2^{<\omega} \times \text{FIN}_k^{<\omega} \upharpoonright B$ with $|u| \geq \text{depth}_{B_1}(b)$ one of the following holds:

i.) $Q(u) \times [b, B] \subseteq \hat{X}$

ii.) $R \times [b, C] \not\subseteq \hat{X}$, for every $R \subseteq Q(u)$ and every $C \leq B$ compatible with $b$.

Proof of Claim 2. Consider the Combinatorial Forcing 2 and apply Lemma 10.

Apply the Claim 2 to $\mathcal{X}$, $P$ and $A$ to find $Q_1 \subseteq P$ and $B_1 \leq A$ in $\mathcal{H}$ such that for each $(u, b) \in 2^{<\omega} \times \text{FIN}_k^{<\omega} \upharpoonright B_1$ with $|u| \geq \text{depth}_{B_1}(b)$ one of the following holds:

1.) $Q_1(u) \times [b, B_1] \subseteq \mathcal{X}$ or
2.) $R \times [b, C] \not\subseteq \mathcal{X}$, for every $R \subseteq Q_1(u)$ and every $C \leq B_1$ compatible with $b$.

For each $t \in T_{Q_1}$, choose $u_1^t \in 2^{<\omega}$ such that $u_1^t(Q_1) \subseteq t$.

Let

$$\mathcal{F}_1 = \{ (t, b) \in T_{Q_1} \times \text{FIN}_k^{<\omega} \upharpoonright B_1 : Q_1(u_1^t) \times [b, B_1] \subseteq \mathcal{X} \}$$

Now, pick $S_1 \subseteq Q_1$ and $D_1 \leq B_1$ in $\mathcal{H}$ satisfying Theorem 9. If (a) of Theorem 9 holds then $S_1 \times [0, D_1] \subseteq \mathcal{X}$ and we are done. So suppose (b) holds. Apply the Claim 2 to $\mathcal{X}^c$, $S_1$ and $D_1$ to find $Q_2 \subseteq S_1$ and $B_2 \leq D_1$ in $\mathcal{H}$ such that for each $(u, b) \in 2^{<\omega} \times \text{FIN}_k^{<\omega} \upharpoonright B_2$ with $|u| \geq \text{depth}_{D_2}(b)$ one of the following holds:

3.) $Q_2(u) \times [b, B_2] \subseteq \mathcal{X}^c$ or
4.) $R \times [b, C] \not\subseteq \mathcal{X}^c$, for every $R \subseteq Q_2(u)$ and every $C \leq B_2$ compatible with $b$.

As before, for each $t \in T_{Q_2}$, choose $u_2^t \in 2^{<\omega}$ such that $u_2^t(Q_2) \subseteq t$.

Let

$$\mathcal{F}_2 = \{ (t, b) \in T_{Q_2} \times \text{FIN}_k^{<\omega} \upharpoonright B_2 : Q_2(u_2^t) \times [b, B_2] \subseteq \mathcal{X}^c \}$$

Again, pick $S_2 \subseteq Q_2$ and $D_2 \leq B_2$ in $\mathcal{H}$ satisfying Theorem 9. If (a) of Theorem 9 holds then $S_2 \times [0, D_2] \cap \mathcal{X} = \emptyset$ and we are done. So suppose (b) holds again. Let us see that this contradicts the fact that $\mathcal{X}$ has the $\mathbb{P} \times \text{Exp}(\mathcal{H})$-Baire Property.

Note that for every $(t, b) \in T_{S_2} \times \text{FIN}_k^{<\omega} \upharpoonright D_2$ the following holds:

(i) $Q_1(u_1^t) \times [b, B_1] \not\subseteq \mathcal{X}$, and

(ii) $Q_2(u_2^t) \times [b, B_2] \not\subseteq \mathcal{X}^c$.

So, suppose there is a nonempty $R \times [b, C] \subseteq S_2 \times [0, D_2] \cap \mathcal{X}$, with $C \in \mathcal{H}$, and pick $t \in T_R$ with $|u_1^t| \geq \text{depth}_{B_3}(b)$. Note that $R \cap [t] \subseteq Q_1(u_1^t)$. On the one hand we have that $R \cap [t] \times [b, C] \subseteq R \times [b, C] \subseteq \mathcal{X}$. But in virtue of (i), $Q_1(u_1^t) \times [b, B_1] \not\subseteq \mathcal{X}$ and hence by (2) above we have that $R \cap [t] \times [b, C] \not\subseteq \mathcal{X}$. If we suppose that there is a nonempty $R \times [b, C] \subseteq S_2 \times [0, D_2] \cap \mathcal{X}^c$, with $C \in \mathcal{H}$, we reach to a similar contradiction in virtue of (ii) and (4) above. So there is neither $R \times [b, C] \subseteq S_2 \times [0, D_2] \cap \mathcal{X}$ nor $R \times [b, C] \subseteq S_2 \times [0, D_2] \cap \mathcal{X}^c$.

But this is impossible because $\mathcal{X}$ has the $\mathbb{P} \times \text{Exp}(\mathcal{H})$-Baire Property.
(b) Again, the implication from left to right is obvious. Conversely, the result follows easily from part (a) and the fact that $X$ is $\mathbb{P} \times \text{Exp}(\mathcal{H})$-meager. This completes the proof of Theorem 

6.1. Closedness Under the Souslin Operation.

**Lemma 12.** Let $\mathcal{H}$ be a semiselective coideal in $\text{FIN}_k^\infty$. The perfectly $\mathcal{H}$-Ramsey null subsets of $2^\infty \times \text{FIN}_k^\infty$ form a $\sigma$-ideal.

**Proof.** Let $(\mathcal{X}_n)_n$ be a sequence of perfectly $\mathcal{H}$-Ramsey null subsets of $2^\infty \times \text{FIN}_k^\infty$ and fix $P \times [a, A]$. We can assume $a = \emptyset$. Also, it is easy to see that the finite union of perfectly $\mathcal{H}$-Ramsey null sets yields a perfectly $\mathcal{H}$-Ramsey null set; so we will assume $(\forall n) \mathcal{X}_n \subseteq \mathcal{X}_{n+1}$. The rest of the proof is similar to the proof of Lemma [10]. For $n \in \mathbb{N}$, let

$$D_n = \{(Q, B) : Q \subseteq P, B \leq A, (\forall b \in \text{FIN}_k^\infty \setminus B \text{ with depth}_B(b) = n) Q \times [b, B] \cap \mathcal{X}_n = \emptyset\}.$$ 

Let $\mathcal{S} = \{(Q, B) \in \mathbb{P} \times \mathcal{H} : Q \subseteq P, B \leq A\}$. Every $D_n$ is dense open in $\mathcal{S}$, so let $D_\infty$ be as in Lemma [8] and choose $(Q, B) \in D_\infty$. Then, $Q \times [0, B] \cap \bigcup_n \mathcal{X}_n = \emptyset$: take $(x, C) \in Q \times [0, B]$ and fix arbitrary $n$. To show that $(x, C) \notin \mathcal{X}_n$ let $l$ be large enough so that depth$_B(r_l(C)) = m \geq n$. Then by construction $Q \times [r_l(C), B] \cap \mathcal{X}_m = \emptyset$ and hence, since $\mathcal{X}_n \subseteq \mathcal{X}_m$, we have $(x, C) \notin \mathcal{X}_n$. This completes the proof. □

Now, we borrow some terminology from [19]: Let $\mathcal{A}$ be a family of subsets of a set $\mathcal{Z}$. We say that $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{Z}$ are compatible (with respect to $\mathcal{A}$) if there exists $\mathcal{W} \in \mathcal{A}$ such that $\mathcal{W} \subseteq \mathcal{X} \cap \mathcal{Y}$. Also, we say that $\mathcal{A}$ is $M$-like if for any $\mathcal{B} \subseteq \mathcal{A}$ such that $|\mathcal{B}| < |\mathcal{A}|$, every member of $\mathcal{A}$ which is not compatible with any member of $\mathcal{B}$ is compatible with $\mathcal{Z} \setminus \bigcup \mathcal{B}$.

The families $\mathbb{P}$ of perfect subsets of $2^\infty$ and $\text{Exp}(\mathcal{H})$ are $M$-like. Therefore, according to Lemma 2.7 in [19], the family $\mathbb{P} \times \text{Exp}(\mathcal{H}) = \{P \times [n, A] : P \in \mathbb{P} \text{ and } A \in \mathcal{H}\}$ is also $M$-like. This lead us to the following:

**Corollary 1.** The family of perfectly $\mathcal{H}$-Ramsey subsets of $2^\infty \times \text{FIN}_k^\infty$ is closed under the Souslin operation.

**Proof.** Theorem [8] states that the family of perfectly $\mathcal{H}$-Ramsey subsets of $2^\infty \times \text{FIN}_k^\infty$ coincides with the family of subsets of $2^\infty \times \text{FIN}_k^\infty$ which have the $\mathbb{P} \times \text{Exp}(\mathcal{H})$-Baire property. As pointed out in the previous paragraph, $\mathbb{P} \times \text{Exp}(\mathcal{H})$ is $M$-like. So the proof follows from Lemma [12] above and Lemmas 2.5 and 2.6 of [19] (which refer to a well-known result of Marczewski [16]). □

7. Mathias forcing on $\text{FIN}_k^\infty$

We recall the definition of Mathias forcing with respect to a coideal $\mathcal{H}$ ([15]). Given a coideal $\mathcal{H} \subseteq \mathbb{N}^{<\infty}$, 

$$\mathbb{M}_\mathcal{H} = \{[a, A] : a \in [\mathbb{N}]^{<\infty}, A \in \mathcal{H}, \max(a) < \min(A)\},$$

with the order relation $[a, A] \leq [b, B]$ if $b$ is an initial segment of $a$, $A \subseteq B$ and $a \setminus b \subseteq B$. 

Lemma 6 applied to $D \in \mathcal{V}$ is a general version for topological Ramsey spaces appears in [3].

Given $s \in \text{FIN}_k^\infty$ and $A \in \text{FIN}_k^\infty$, we define the set

$$[s, A] := \{B \in \text{FIN}_k^\infty : s \subseteq B \text{ and } B \leq A\}$$

where $s \subseteq B$ means that $s$ is an initial segment of $B$ in its increasing order.

**Definition 22.** Let $\mathcal{H}$ be a coideal in $\text{FIN}_k^\infty$. The **Mathias forcing localized at $\mathcal{H}$** is the partially ordered set

$$\mathcal{M}_\mathcal{H} := \{(s, A) \in \text{FIN}_k^{<\infty} \times \mathcal{H} : s < A\}$$

ordered by $(s, A) \leq (t, B)$ if $[s, A] \subseteq [t, B]$, i.e., $t \subseteq s$, $A \leq B$ and $s \setminus t \subseteq [B]$.

We will show some facts about this forcing notion before continuing in our study of the consistency (relative to $ZF$) of the $\mathcal{H}$-Ramseyness of every subset of $\text{FIN}_k^\infty$ when $\mathcal{H}$ is in a suitable class of coideals. For an ultrafilter $\mathcal{U}$ we use the special notation $\mathcal{M}_\mathcal{U} := \mathcal{M}_\mathcal{U}^\infty$.

We say that $\mathcal{X} \in \text{FIN}_k^\infty$ is $\mathcal{M}_\mathcal{H}$-**generic** over a model $V$ if for every dense open subset $\mathcal{D} \in V$ of $\mathcal{M}_\mathcal{H}$, there exists a condition $(a, A) \in \mathcal{D}$ such that $cX \in [a, A]$.

If $G$ is a $\mathcal{M}_\mathcal{H}$-generic filter over $V$, then

$$\mathcal{X} = \bigcup \{s : (\exists A \in \mathcal{H})(s, A) \in G\} \in (\text{FIN}_k^\infty)^{V[G]}$$

is a $\mathcal{M}_\mathcal{H}$-generic block sequence over $V$.

**Definition 23.** It is said that $\mathcal{H}$ has the **pure decision property** (the Prikry property) if for every sentence of the forcing language $\phi$ and every condition $(a, A) \in \mathcal{M}_\mathcal{H}$ there exists $B \in [a, A] \cap \mathcal{H}$ such that $(a, B)$ decides $\phi$.

It is said that $\mathcal{H}$ has the **hereditary genericity property** (the Mathias property) if it satisfies that if $\mathcal{X}$ is $\mathcal{M}_\mathcal{H}$-generic over a model $V$, then every $\mathcal{Y} \leq \mathcal{X}$ is $\mathcal{M}_\mathcal{H}$-generic over $V$.

The following result was proved in [9, 10] for the particular case of $\mathcal{H} = \text{FIN}_k^\infty$. A more general version for topological Ramsey spaces appears in [3].

**Theorem 10.** If $\mathcal{H} \subseteq \text{FIN}_k^\infty$ is a semiselective coideal then it has the pure decision property.

**Proof.** Suppose $\mathcal{H} \subseteq \text{FIN}_k^\infty$ is a semiselective coideal, and fix a sentence $\varphi$ if the forcing language and a condition $(a, A) \in \mathcal{M}_\mathcal{H}$. For every $b \in \text{FIN}_k^{<\infty}$ with $a \subseteq b$, let

$$\mathcal{D}_b = \{B \in \mathcal{H} \cap \text{depth}_A(b), A) : (b, B) \text{ decides } \varphi \text{ or } (\forall C \in \mathcal{H} \cap [b, B]) (b, C) \text{ does not decide } \varphi\}.$$ 

and set $\mathcal{D}_b = \mathcal{H} \cap \text{depth}_A(b), A)$, for all $b \in \text{FIN}_k^{<\infty} \upharpoonright A$ with $b \not\subseteq a$.

Each $\mathcal{D}_b$ is dense open in $\mathcal{H} \cap \text{depth}_A(b), A]$.

Fix a diagonalization $B \in \mathcal{H} \upharpoonright A$. For every $b \in \text{FIN}_k^{<\infty} \upharpoonright A$. Let

$$\mathcal{F}_0 = \{b \in \text{FIN}_k^{<\infty} \upharpoonright B : a \subseteq b \land (b, B) \text{ forces } \varphi\},$$

$$\mathcal{F}_1 = \{b \in \text{FIN}_k^{<\infty} \upharpoonright B : a \subseteq b \land (b, B) \text{ forces } \neg \varphi\}.$$ 

Let $\dot{C} \in \mathcal{H} \upharpoonright B$ as in Lemma 6 applied to $a$, $B$ and $\mathcal{F}_0$. And let $C \in \mathcal{H} \upharpoonright \dot{C}$ be as in Lemma 6 applied to $a$, $\dot{C}$ and $\mathcal{F}_1$. Let us prove that $(a, C)$ decides $\varphi$. So let $(b_0, C_0)$ and $(b_1, C_1)$ be two different arbitrary extensions of $(a, C)$. Suppose that $(b_0, C_0)$ forces $\varphi$ and $(b_1, C_1)$ forces $\neg \varphi$. Then $b_0 \in \mathcal{F}_0$ and $b_1 \in \mathcal{F}_1$. But $b_0, b_1 \in \text{FIN}_k^{<\infty} \upharpoonright C$, so by the choice
of \( C \) this means that every element of \( \mathcal{H} \cap [a, C] \) has an initial segment in \( \mathcal{F}_0 \) and an initial segment in \( \mathcal{F}_1 \). So there exist two compatible extensions of \((a, C)\) such that one forces \( \varphi \) and the other forces \( \neg \varphi \). A contradiction. So either both \((b_0, C_0)\) and \((b_1, C_1)\) force \( \varphi \) or both \((b_0, C_0)\) and \((b_1, C_1)\) force \( \neg \varphi \). Therefore \((a, C)\) decides \( \varphi \).

\[ \square \]

Now we will prove that if \( \mathcal{H} \subseteq \text{FIN}_k^\infty \) is semiselective then it has the hereditary genericity property (see Theorem \([12]\) below).

Given a selective ultrafilter \( \mathcal{U} \subseteq \text{FIN}_k^\infty \), let \( M_\mathcal{U} \) be set of all pairs \((a, A)\) such that \( A \in \mathcal{U} \) and \([a, A] \neq \emptyset \). Order \( M_\mathcal{U} \) with the same ordering used before.

Extending to the context of \( \text{FIN}_k \) the notion of capturing devised by Mathias in \([15]\) is essential in what follows.

**Definition 24.** Let \( \mathcal{U} \subseteq \text{FIN}_k^\infty \) be a selective ultrafilter, \( \mathcal{D} \) a dense open subset of \( M_\mathcal{U} \), and \( a \in \text{FIN}_k^\infty \). We say that \( A \) captures \((a, D)\) if \( A \in \mathcal{U} \), \([a, A] \neq \emptyset \), and for all \( B \in [a, A] \) there exists \( m > |a| \) such that \((r_m(B), A) \in \mathcal{D}\).

**Lemma 13.** Let \( \mathcal{U} \subseteq \text{FIN}_k^\infty \) be a selective ultrafilter and \( \mathcal{D} \) a dense open subset of \( M_\mathcal{U} \). Then, for every \( a \in \text{FIN}_k^\infty \) there exists \( A \in \mathcal{U} \) which captures \((a, D)\).

**Proof.** Given \( a \in \text{FIN}_k^\infty \), we can choose \( B \in \mathcal{U} \) such that \([a, B] \neq \emptyset \), for example \( 1 \). We define a collection \((C_b)_{b \in \text{FIN}_k^\infty \cup B} \) with \([b, C_b] \neq \emptyset \), such that:

1. For all \( b_1, b_2 \in \text{FIN}_k^\infty \cup B \), if \( \text{depth}_B(b_1) = \text{depth}_B(b_2) \) then \( C_{b_1} = C_{b_2} \).
2. For all \( b_1, b_2 \in \text{FIN}_k^\infty \cup B \), if \( b_1 \sqsubseteq b_2 \) then \( C_{b_1} \geq C_{b_2} \).
3. For all \( b \in \text{FIN}_k^\infty \cup B \) with \( a \sqsubseteq b \) either \((b, C_b) \in \mathcal{D}\) or if such a \( C_b \in \mathcal{D}\) does not exist then \( C_b = B \).

For every \( b \in \text{FIN}_k^\infty \cup B \), let \( C_n = C_b \) if \( \text{depth}_B(b) = n \). Notice that \( C_n \geq C_{n+1} \), for all every \( n \in \mathbb{N} \). By selectivity, let \( C \in \mathcal{U} \cap [a, B] \) be a diagonalization of \((C_n)_{n \in \mathbb{N}} \). Then, for all \( b \in \text{FIN}_k^\infty \cup C \) with \( a \sqsubseteq b \), if there exists a \( \hat{C} \in \mathcal{U} \) such that \((b, \hat{C}) \in \mathcal{D}\), we must have \((b, C) \in \mathcal{D}\).

Let \( \mathcal{X} = \{D \in \text{FIN}_k^\infty : D \leq C \to (\exists b \in \text{FIN}_k^\infty \cup D) a \sqsubseteq b \& (b, C) \in \mathcal{D}\} \). \( \mathcal{X} \) is a metric open subset of \( \text{FIN}_k^\infty \) and therefore, by Lemma \([13]\) it is \( \mathcal{U} \)-Ramsey. Take \( \hat{C} \in \mathcal{U} \cap [\text{depth}_C(a), C] \) such that \([a, \hat{C}] \subseteq \mathcal{X} \) or \([a, \hat{C}] \cap \mathcal{X} = \emptyset \). We will show that the first alternative holds: Pick \( A \in \mathcal{U} \cap [a, \hat{C}] \) and \((a', A') \in \mathcal{D} \) such that \((a', A') \leq (a, A) \). Notice that \( a \sqsubseteq a' \) and therefore, by \((3)\), we have \((a', C) \in \mathcal{D} \). By the definition of \( \mathcal{X} \), we also have \( A' \in \mathcal{X} \). Now choose \( A'' \in \mathcal{U} \cap [a', A'] \). Then \((a', A'') \) is also in \( \mathcal{D} \) and therefore \( A'' \in \mathcal{X} \). But \( A'' \in [a', A'] \subseteq [a, A] \subseteq [a, C] \). This implies \([a, C] \subseteq \mathcal{X} \). Finally, that \( A \) captures \((a, D)\) follows from the definition of \( \mathcal{X} \) and the fact that \([a, A] \subseteq [a, C] \subseteq [a, C] \). This completes the proof. \( \square \)

**Theorem 11.** Let \( \mathcal{U} \subseteq \text{FIN}_k^\infty \) be a selective ultrafilter in a given transitive model \( V \) of \( ZF + DC_{\mathbb{R}} \). Forcing over \( V \) with \( M_\mathcal{U} \) adds a generic \( g \in \text{FIN}_k^\infty \) with the property that \( g \leq^* A \) for all \( A \in \mathcal{U} \). In fact, \( B \in \text{FIN}_k^\infty \) is \( M_\mathcal{U} \)-generic over \( V \) if and only if \( B \leq^* A \) for all \( A \in \mathcal{U} \).

**Proof.** Suppose that \( B \in \text{FIN}_k^\infty \) is \( M_\mathcal{U} \)-generic over \( V \). Fix an arbitrary \( A \in \mathcal{U} \). Let us show that the set \( \{(c, C) \in M_\mathcal{U} : C \leq^* A\} \) is dense open. Note that this set is in
V. Fix \((a, A') \in \mathbb{M}_U\). Since \(U\) admits finite changes, choose \(A'' \leq^* A\) in \(U\) such that \([a, A''] \neq \emptyset\). Clearly, \([a, A'] \cap [a, A''] \neq \emptyset\), and then there is \(n \in \mathbb{N}\) and \(C_1 \in U\) such that 
\[\exists [n, C_1] \subseteq [a, A'] \cap [a, A''].\]
Let \(c = r_n(C_1)\). By \(A_3\) mod \(U\), there exists \(C_2 \in U \cap \text{depth}_{A'}(c, A')\) such that \(\emptyset \neq [c, C_2] \subseteq [c, C_1]\). It is clear that \([c, C_2] \subseteq [a, A'']\) and therefore \(A'' \leq^* A'\). Also, since \(\text{depth}_{A'}(c) \geq \text{depth}_{A'}(a)\), we have \([a, C_2] \neq \emptyset\). Thus, \((a, C_2) \leq (a, A')\). That is, \(D\) is dense. It is obviously open. So, by genericity, there exists \((c, C) \in D\) such that \(B \in [c, C]\). Hence \(B \leq^* A\).

Now, suppose that \(B \in \text{FIN}_k^\infty\) is such that \(B \leq^* A\) for all \(A \in U\), and let \(D\) be a dense open subset of \(\mathbb{M}_U\). We need to find \((a, A) \in D\) such that \(B \in [a, A]\). In \(V\), by using Lemma \(13\) iteratively, we can define a sequence \((A_n)_n\) such that \(A_n \in U\), \(A_{n+1} \leq A_n\), and \(A_n\) captures \((r_n(B), D)\). Since \(U\) is in \(V\) and selective, we can choose \(A \in U\), \(V\), such that \(A \leq^* A_n\) for all \(n\). By our assumption on \(B\), we have \(B \leq^* A\). So there exists an \(a \in \text{FIN}_k^\infty\) such that \([a, B] \subseteq [a, A]\). Let \(m = \text{depth}_B(a)\). By \(A_3\) mod \(U\), we can assume that \(a = r_m(B) = r_m(A)\), and also that \(A \in [r_m(B), A_m]\). Therefore, \(B \in [m, A]\) and \(A\) captures \((r_m(B), D)\). Hence, the following is true in \(V\):

\[(\forall C \in [m, A])((\exists n > m)((r_n(C), A) \in D)).\]

Let \(F = \{b : (\exists n > m)(b \in r_n[m, A]\) \& \((b, A) \notin D\}\) and give \(F\) the strict end-extension ordering \(\sqsubset\). Then the relation \((F, \sqsubset)\) is in \(V\), and by equation \(1\) \((F, \sqsubset)\) is well-founded. Therefore, by a well-known argument due to Mostowski, equation \(1\) holds in the universe. Hence, since \(B \in [m, A]\), there exists \(n > m\) such that \((r_n(B), A) \in D\). But \(B \in [r_n(B), A]\), so \(B\) is \(\mathbb{M}_U\)-generic over \(V\).

Corollary 2. If \(B\) is \(\mathbb{M}_U\)-generic over some model \(V\) and \(A \leq B\) then \(A\) is also \(\mathbb{M}_U\)-generic over \(V\). In other words, \(U\) has the hereditary genericity property.

Lemma 14. Let \(H \subseteq \text{FIN}_k^\infty\) be a semiselective coideal. Consider the forcing notion \(P = (H, \leq^*)\) and let \(\hat{U}\) be a \(P\)-name for a \(P\)-generic ultrafilter. Then the iteration \(P \ast \mathbb{M}_U\) is equivalent to the forcing \(\mathbb{M}_H\).

Proof. Recall that \(P \ast \mathbb{M}_U = \{(B, (\hat{a}, \hat{A})) : B \in H \& B \vdash (\hat{a}, \hat{A}) \in \mathbb{M}_U\}\), with the ordering \((B, (\hat{a}, \hat{A})) \leq (B_0, (\hat{a}_0, \hat{A}_0)) \iff B \leq^* B_0 \& (\hat{a}, \hat{A}) \leq (B_0, (\hat{a}_0, \hat{A}_0)).\) The mapping \((a, A) \mapsto (\hat{a}, \hat{A})\) is a dense embedding from \(\mathbb{M}_H\) to \(P \ast \mathbb{M}_U\) (here \(\hat{a}\) and \(\hat{A}\) are the canonical \(P\)-names for \(a\) and \(A\), respectively): It is easy to show that this mapping preserves the order. So, given \((B, (\hat{a}, \hat{A})) \in P \ast \mathbb{M}_U\), we need to find \((d, D) \in \mathbb{M}_H\) such that \((D, (\hat{d}, \hat{D})) \leq (B, (\hat{a}, \hat{A}))\). Since \(P\) is \(\sigma\)-distributive, there exists \(a \in \text{FIN}_k^\infty\), \(A \in H\) and \(C \leq^* B\) in \(H\) such that 
\[C \vdash (\hat{a} = \hat{a} \& \hat{A} = \hat{A})\]
(so we can assume \(a \in \text{FIN}_k^\infty \cap C\)). Notice that \((C, (\hat{a}, \hat{A})) \in P \ast \mathbb{M}_U\) and \((C, (\hat{a}, \hat{A})), (\hat{C}, (\hat{a}, \hat{A})) \leq (B, (\hat{a}, \hat{A}))\). So, \(C \vdash \hat{C} \in \hat{U}\) and \(C \vdash \hat{A} \in \hat{U}\). Then, \(C \vdash (\exists x \in \hat{U}) \langle x \in [\hat{a}, \hat{A}] \& x \in [\hat{a}, \hat{C}]\rangle\). So there exists \(D \in H\) such that \(D \in [a, A] \cap [a, C]\). Hence, \((D, (\hat{a}, \hat{D})) \leq (B, (\hat{a}, \hat{A}))\). This completes the proof.

The next theorem follows immediately from Corollary 2 and Lemma 14.

Theorem 12. If \(H \subseteq \text{FIN}_k^\infty\) is a semiselective coideal then it has the hereditary genericity property.
For the next lemma it will be useful to have the following notion. Given $P \in \text{FIN}_k^\infty$, $P = (p_0, p_1, \ldots)$, every element of $\text{FIN}_k^\infty \upharpoonright X$ is obtained from a finite subsequence of $X$ by

$$T^{(i_0)}(p_{n_0}) + \cdots + T^{(i_l)}(p_{n_l})$$

for some increasing sequence $n_0 < \cdots < n_l$ and some choice $i_0, \ldots, i_l \in \{0, 1, \ldots, k\}$ with at least one of the numbers $i_0, \ldots, i_l$ equal to 0. We can thus define a well ordering of $\text{FIN}_k^\infty \upharpoonright X$ in the following way.

Let $s = T^{(i_0)}(p_{n_0}) + \cdots + T^{(i_l)}(p_{n_l})$ and $t = T^{(j_0)}(p_{m_0}) + \cdots + T^{(j_h)}(p_{m_h})$, then

$s <_{\text{lex}} t$ if

$$s <_{\text{lex}} t \iff \begin{cases} (n_0, \ldots, n_l) <_{\text{lex}} (m_0, \ldots, m_h), & \text{or} \\ (n_0, \ldots, n_l) = (m_0, \ldots, m_h) \text{ and } (i_0, \ldots, i_l) <_{\text{lex}} (j_0, \ldots, j_h). \end{cases}$$

Lemma 15. ([S] for coideals in the space $\mathbb{N}[\infty]$) For a coideal $H$ on $\text{FIN}_k^\infty$, the following are equivalent:

1. $H$ is semiselective,
2. $M_H$ has the pure decision property,
3. $M_H$ has the hereditary genericity property.

Proof. We prove first (2) implies (1). Suppose $H$ is not semiselective. Then there is $A \in H$ and a sequence $(D_a : a \in \text{FIN}_k^{<\infty} \upharpoonright A)$ of dense open subsets of $(H, \leq^*)$ such that no element of $H \upharpoonright A$ is a diagonalization of the sequence.

We now work with $\{B : B \leq A\}$, and consider the forcing notion $M_H/A$.

Pick a sequence $(A_a : a \in \text{FIN}_k^{<\infty} \upharpoonright A)$ of maximal antichains of $(H \upharpoonright A, \leq^*)$ with $A_a \subseteq D_a$, such that no element of $H \upharpoonright A$ is a diagonalization of the sequence of dense open sets determined by the antichains.

Each antichain $A_a$ determines a maximal antichain in $M_H/A$, namely, $\{[\emptyset, A] : A \in A_a\}$. Let $\dot{x}$ be a canonical $M_H/A$-name for a generic sequence, and let $\tau_a$ be a name of the unique element of the antichain $A_a$ such that $\dot{x} \leq^* \tau_a$ is forced.

Let us show that $\dot{x}$ is forced not to be a diagonalization of $(A_a)$. Suppose the contrary, and let $[s, B]$ be a condition that forces that for every $a \in \text{FIN}_k^{<\infty} \upharpoonright \dot{x}$, $[a, \dot{x}] \subseteq [a, \tau_a]$. Now, $B/s$ is not a diagonalization of the sequence $(A_a)$ and so there is $t \in B/s$ be such that $[t, B/t]$ is not contained in $[t, A]$ of $A_a$. There is an element $A_t$ of $A_t$ such that there is $C \in H$ with $C \leq B$ and $C \leq A_t$. Since $[t, B/t]$ is not contained in $[t, A_t]$, there is some $r \in \text{FIN}_k^{<\infty} \upharpoonright B/t$ not in $\text{FIN}_k^{<\infty} \upharpoonright A_t$. The condition $[s \sim t \sim r, B]$ forces that $\tau_t = A_t$, and forces that $[t, \dot{x}/t]$ is not contained in $[t, \tau_t]$. But this contradicts that being an extension of $[s, B]$, it must force $[t, \dot{x}/t] \subseteq [t, \tau_t]$.

Consider the formula

$$\phi : s$$

is the $<_{\text{lex}}$-first element of $\text{FIN}_k^{<\infty} \upharpoonright \dot{x}$ such that $\dot{x}/s \not\leq s$, and the number of blocks of $\dot{x}$ below $s$ is even.

Since $\dot{x}$ is forced not to diagonalize the sequence $(A_a)$, $s$ is well defined.

If $M_H$ has the pure decision property, there is $\dot{A} \in H$ such that $[\emptyset, \dot{A}]$ decides $\phi$.

Now, we find $s_1 < t_1 < s_2 < t_2$ in $\text{FIN}_k^{<\infty}$ as follows:

$s_1$ is the least $s \in \text{FIN}_k^{<\infty} \upharpoonright \dot{A}$ such that $[s, \dot{A}]$ is not contained in $[s, X]$ for any member $X$ of $A_{s_1}$. Such $s_1$ exists because $\dot{A}$ is not a diagonalization of $A_{s_1}$.

Let $A_1 \in A_{s_1}$ be such that there is $C \in H$ with $C \leq \dot{A}$ and $C \leq A_1$, and pick $t_1 > s_1$ such that $t_1 \in (\text{FIN}_k^{<\infty} \upharpoonright A) \setminus (\text{FIN}_k^{<\infty} \upharpoonright A_1)$.
Set now \( A' \in \mathcal{H} \) such that \( A' \subseteq A \) and \( A' \leq A_1/t_1 \). Since \( A' \) does not diagonalize the sequence \( (A_\alpha)_{\alpha < \kappa} \), let \( s_2 \) be the least \( s \in \text{FIN}_k^{<\infty} \cup A' \) such that \( [s_2, A'] \) is not contained in \([s_2, X]\) for any element \( X \) of \( A_{s_2} \). Take \( A_2 \in A_{s_2} \) such that there is \( C \in \mathcal{H} \) with \( C \leq A' \) and \( C \leq A_2 \), and pick \( t_2 > s_2 \) such that \( t_2 \in (\text{FIN}_k^{<\infty} \cup A') \setminus (\text{FIN}_k^{<\infty} \cup A_2) \).

Take \( A'' \in \mathcal{H} \) below \( A' \) and below \( A_2/t_2 \).

Then,

\[
[s_1, t_1], A'] \models s = s_1 \text{ and } \hat{x} \text{ does not have elements below } s.
\]

So, this condition forces \( \phi \), while

\[
[s_1, s_2, t_2], A'' \models s = s_2 \text{ and } \hat{x} \text{ has one element below } s.
\]

thus, this second condition forces \( \neg \phi \), a contradiction since both conditions extend \([\emptyset, A]\).

Now, we prove (3) implies (1).

As in the previous case, suppose \( \mathcal{H} \) is not semiselective and thus there is \( A \in \mathcal{H} \) and a sequence \( (D_a : a \in \text{FIN}_k \cup A) \) of dense open subsets of \( (\mathcal{H}, \leq^*) \) such that no element of \( \mathcal{H} \upharpoonright A \) is a diagonalization of the sequence. Pick a sequence \( (A_a : a \in \text{FIN}_k \cup A) \) of maximal antichains of \( (\mathcal{H} \upharpoonright A, \subseteq^*) \) with \( A_a \subseteq D_a \), such that no element of \( \mathcal{H} \upharpoonright A \) is a diagonalization of the sequence.

We know that \( \hat{x} \) is forced not to be a diagonalization of the sequence \( (A_a) \). But since for every \( a \) \( \hat{x} \subseteq^* \tau_a \) is forced. there is \( y \leq \hat{x} \) which diagonalizes \( (A_a) \) and thus this \( y \) cannot be in \( \mathcal{H} \).

**Theorem 13.** Suppose \( \lambda \) is a Mahlo cardinal and let \( V[G] \) be a generic extension by \( \text{Col}(\omega, \lambda) \). If \( \mathcal{H} \) is a semiselective coideal on \( \text{FIN}_k^\infty \) in \( V[G] \), then every subset of \( \text{FIN}_k^\infty \) in \( L(\mathbb{R}) \) of \( V[G] \) is \( \mathcal{H} \)-Ramsey.

**Proof.** Let \( \mathcal{H} \subseteq \text{FIN}_k^\infty \) be a semiselective coideal in \( V[G] \); and let \( M_{\mathcal{H}} \) be the corresponding forcing with respect to \( \mathcal{H} \). Let \( A \subseteq \text{FIN}_k^\infty \) in \( L(\mathbb{R})^{V[G]} \); in particular, \( A \) is defined in \( V[G] \) by a formula \( \varphi \) from a sequence of ordinals.

Let \( \hat{\mathcal{H}} \) be a \( \text{Col}(\omega, < \lambda) \)-name for \( \mathcal{H} \). Notice that \( \hat{\mathcal{H}} \subseteq \mathcal{V}_\lambda \); also, elements of \( \text{FIN}_k^\infty \) in \( V[G] \) have \( \text{Col}(\omega, < \lambda) \)-names in \( V_\lambda \). Therefore \( \hat{\mathcal{H}} \) is interpreted in \( V_\lambda[G] \) as \( \mathcal{H} \).

Since \( \lambda \) is a Mahlo cardinal, the set of inaccessible cardinals below \( \lambda \) is stationary, and we can find an inaccessible \( \kappa < \lambda \) such that

\[
\langle V_\kappa, \in, \hat{\mathcal{H}} \cap V_\kappa, \text{Col}(\omega, < \lambda) \cap V_\kappa \rangle < \langle V_\lambda, \in, \hat{\mathcal{H}}, \text{Col}(\omega, < \lambda) \rangle.
\]

It follows that

\[
\langle V_\kappa[G_\kappa], \in, \hat{\mathcal{H}} \cap V_\kappa[G_\kappa], \text{Col}(\omega, < \lambda) \cap V_\kappa[G_\kappa] \rangle < \langle V_\lambda[G], \in, \hat{\mathcal{H}}, \text{Col}(\omega, < \lambda) \rangle,
\]

where \( G_\kappa = G \cap \text{Col}(\omega, < \kappa) \). This can be verified as follows. Let \( \phi(\tau_1, \ldots, \tau_n) \) be a formula of the forcing language of \( \text{Col}(\omega, < \kappa) \). If this formula is valid in \( V_\kappa[G_\kappa] \), interpreting \( \tau_1, \ldots, \tau_n \) by \( G_\kappa \), then there is a condition \( p \in G_\kappa \) that forces \( \phi \). Since \( p \) is also in \( G \) and \( \tau_1, \ldots, \tau_n \) are also \( \text{Col}(\omega, < \lambda) \)-names, \( \phi \) is also valid in \( V_\lambda[G] \). Conversely, if \( \phi(\tau_1, \ldots, \tau_n) \) is valid in \( V_\lambda[G] \) there is \( q \in G \) that forces it (in \( V_\lambda \)). Since the names \( \tau_1, \ldots, \tau_n \) are \( \text{Col}(\omega, < \kappa) \)-names, \( p \upharpoonright (\omega \times \kappa) \) forces in \( V_\lambda \) the formula \( \phi \). By elementarity this also holds in \( V_\kappa \), and therefore \( \phi \) interpreted by \( G_\kappa \) is valid in \( V_\kappa[G_\kappa] \).

We want to show that the coideal \( \mathcal{H} \cap V_\kappa[G_\kappa] \) is semiselective in \( V_\kappa[G_\kappa] \). We will show that in the model \( V_\kappa[G_\kappa] \) the partial order \( M_{\mathcal{H} \cap V_\kappa[G_\kappa]} \) has the pure decision property.
We now consider the forcing notion $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$ in $V_κ(G_κ)$. Let $ψ$ be a formula in the forcing language of $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$, and let $[a, A]$ be a condition in $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$. Since this is also a condition in $\mathbb{M}_H$, in $V_κ(G_κ)$ there is $B ∈ \mathcal{H}$, $B ⊆ A$, such that $[a, B]$ decides $ψ$. But by elementarily this statement also holds in $V_κ[G_κ]$ and so in $V_κ[G_κ]$ there is $B ∈ \mathcal{H} \cap V_κ[G_κ]$, $B ⊆ A$ such that $[a, B]$ decides $ψ$.

Since this happens for every formula of the forcing language of $\mathcal{H} \cap V_κ[G_κ]$ has the pure decision property and thus it is semiselective and it also has the hereditary genericity property.

Now the proof can be finished as in [15]. Let $x$ be the canonical name of a $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$ generic sequence, and consider the formula $φ(x)$ in the forcing language of $V[G_κ]$. By the pure decision property of $\mathcal{H} \cap V[G_κ]$, there is $A ′ ≤ A$, $A′ ∈ \mathcal{H} \cap V[G_κ]$, such that $[a, A′]$ decides $φ(x)$. Since $2^{2^{β_1}}$ computed in $V[G_κ]$ is countable in $V[G]$, there is $x ∈ [a, A′]$ a $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$-generic sequence $x$ over $V[G_κ]$ such that $x ∈ [a, A′]$. To see that there is such a generic sequence in $\mathcal{H}$ we argue as in 5.5 of [15] using the semiselectivity of $\mathcal{H}$ and the fact that $\mathcal{H} \cap V[G_κ]$ is countable in $V[G]$ to obtain an element of $\mathcal{H}$ which is generic. By the hereditary genericity property of $\mathcal{H} \cap V[G_κ]$, every $y ∈ [a, x \setminus a]$ is also $\mathbb{M}_{\mathcal{H}\cap V_{κ}(G_κ)}$-generic over $V[G_κ]$, and also $y ∈ [a, A′]$. Thus $φ(x)$ if and only if $[a, A′] ⊩ φ(\dot{x})$, if and only if $φ(y)$. Therefore, $[a, x \setminus a]$ is contained $\mathcal{A}$ or is disjoint from $\mathcal{A}$. □

As in [15], we obtain the following.

**Corollary 3.** If ZFC is consistent with the existence of a Mahlo cardinal, then so is the statement that for every semiselective co-ideal $\mathcal{H}$ on $FIN_{κ}^∞$, every subset of $FIN_{κ}^∞$ in $L(\mathbb{R})$ is $\mathcal{H}$-Ramsey.

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