JONES INDEX THEOREM REVISITED

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Abstract. We prove the Jones Index Theorem using the K-theory of a cluster \( C^* \)-algebra of the Riemann sphere with two boundary components.

1. Introduction

The Jones Index Theorem is an analog of the Galois theory for the von Neumann algebras [Jones 1991] [5]. Recall that the factor is a von Neumann algebra \( \mathcal{M} \) with the trivial center. A subfactor \( \mathcal{N} \) of the factor \( \mathcal{M} \) is a subalgebra, such that \( \mathcal{N} \) is a factor. The index \([\mathcal{M} : \mathcal{N}]\) of a subfactor \( \mathcal{N} \) of a type II factor \( \mathcal{M} \) is a positive real number \( \dim_{\lambda \psi}(L^2(\mathcal{M})) \), where \( L^2(\mathcal{M}) \) is a representation of \( \mathcal{N} \) obtained from the canonical trace on \( \mathcal{M} \) using the GNS construction. We refer the reader to [Jones 1991] [5, Section 2.5] for the missing definitions and details. The Jones Index Theorem says that such subfactors exist only if:

\[ [\mathcal{M} : \mathcal{N}] \in [4, \infty) \bigcup \{4 \cos^2 \left( \frac{\pi}{n} \right) \mid n \geq 3 \}. \]  

The cluster algebra \( \mathcal{A}(\mathbf{x}, B) \) of rank \( n \) is a subring of the field of rational functions in \( n \) variables depending on a cluster of variables \( \mathbf{x} = (x_1, \ldots, x_n) \) and a skew-symmetric matrix \( B = (b_{ij}) \in M_n(\mathbb{Z}) \) [Fomin & Zelevinsky 2002] [3]. The pair \( (\mathbf{x}, B) \) is called a seed. A new cluster \( \mathbf{x}' = (x_1, \ldots, x'_k, \ldots, x_n) \) and a new skew-symmetric matrix \( B' = (b'_{ij}) \) is obtained from \( (\mathbf{x}, B) \) by the exchange relations:

\[
x_k x'_k = \prod_{i=1}^n x_i^{\max(b_{ik}, 0)} + \prod_{i=1}^n x_i^{\max(-b_{ik}, 0)},
\]

\[
b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{b_{ik}b_{kj} + b_{ik}b_{kj}}{2} & \text{otherwise}. \end{cases}
\]
The seed \((x', B')\) is said to be a mutation of \((x, B)\) in direction \(k\), where \(1 \leq k \leq n\); the algebra \(\mathcal{A}(x, B)\) is generated by cluster variables \(\{x_i\}_{i=1}^{\infty}\) obtained from the initial seed \((x, B)\) by the iteration of mutations in all possible directions \(k\). The Laurent phenomenon says that \(\mathcal{A}(x, B) \subset \mathbb{Z}[x^\pm 1]\), where \(\mathbb{Z}[x^\pm 1]\) is the ring of the Laurent polynomials in variables \(x = (x_1, \ldots, x_n)\) depending on an initial seed \((x, B)\). The \(\mathcal{A}(x, B)\) is a commutative algebra with an additive abelian semigroup consisting of the Laurent polynomials with positive coefficients. In particular, it has an order satisfying the Riesz interpolation property, so that \(\mathcal{A}(x, B)\) becomes a dimension group [Effros 1981] [2, Theorem 3.1]. A cluster \(C^*\)-algebra \(\mathbb{A}(x, B)\) is an AF-algebra, such that \(K_0(\mathbb{A}(x, B)) \cong \mathcal{A}(x, B)\), where \(\cong\) is an isomorphism of the dimension groups [6, Section 4.4].

An annulus in the complex plane will be denoted by
\[
\mathcal{D} = \{z = x + iy \in \mathbb{C} \mid r \leq |z| \leq R\}.
\] (1.3)

Recall that the Riemann surfaces \(\mathcal{D}\) and \(\mathcal{D}'\) are conformally equivalent if and only if \(R/r = R'/r' := t\). By \(T_\mathcal{D} = \{t \in \mathbb{R} \mid t > 1\}\) we understand the Teichmüller space of the annulus \(\mathcal{D}\). The Penner coordinates on \(T_\mathcal{D}\) are encoded by the cluster algebra \(\mathcal{A}(x, B)\), where
\[
B = \begin{pmatrix}
0 & 2 \\
-2 & 0
\end{pmatrix},
\] (1.4)

see [Fomin, Shapiro & Thurston 2008] [4, Example 4.4] and [Williams 2014] [8, Section 3]. The corresponding cluster \(C^*\)-algebra \(\mathbb{A} (\mathcal{D})\) is given by the Bratteli diagram shown in Figure 1. The latter is known as a GICAR (Gauge Invariant Canonical Anticommutation Relations) algebra [Davidson 1996] [1, Example III.5.5] and [Effros 1980] [2, p.13(e)]. Moreover, there exists an embedding of \(\mathbb{A} (\mathcal{D})\) into the UHF-algebra given by the formula:
\[
\mathbb{A}(\mathcal{D}) \hookrightarrow M_{2^\infty} := \bigotimes_{i=1}^{\infty} M_2(\mathbb{C}).
\] (1.5)

The CAR (Canonical Anticommutation Relations) algebra \(M_{2^\infty}\) plays an outstanding rôle in the theory of subfactors [Jones 1991] [5, Section 5.6]. In this note we use (1.5) and geometry of \(\mathbb{A}(\mathcal{D})\) to give a new shorter proof of the Jones Index Theorem:

**Theorem 1.1.** There is a subfactor \(\mathcal{N}\) of the hyperfinite \(II_1\) factor \(\mathcal{M}\) only if \(\frac{[\mathcal{M} : \mathcal{N}]}{4} \in [4, \infty) \cup \{4 \cos^2\left(\frac{\pi}{n}\right) \mid n \geq 3\}\).
The article is organized as follows. Section 2 contains a brief review of preliminary results. Theorem 1.1 is proved in Section 3.

2. Preliminaries

2.1. Cluster algebras of rank 2. Let $x_1$ and $x_2$ be independent variables of a cluster algebra. For a pair of positive integers $b$ and $c$, we define elements $x_i$ by the exchange relations

$$x_{i-1}x_{i+1} = \begin{cases} 
1 + x_i^b & \text{if } i \text{ odd}, \\
1 + x_i^c & \text{if } i \text{ even}.
\end{cases} \quad (2.1)$$

By a cluster algebra rank 2 we denote the algebra $\mathcal{A}(b, c)$ generated by the cluster variables $x_i$ [Sherman & Zelevinsky 2004] [7, Section 2].

Let $B$ be a basis of the algebra $\mathcal{A}(b, c)$.

**Theorem 2.1.** ([7, Theorem 2.8]) Suppose that $b = c = 2$ or $b = 1$ and $c = 4$. Then $B = \{x_i^p x_{i+1}^q \mid p, q \geq 0\} \cup \{T_n(x_1 x_4 - x_2 x_3) \mid n \geq 1\}$, where $T_n(x)$ are the Chebyshev polynomials of the first kind.

Let $r < R$ and consider an annulus $\mathcal{D}$ of the form (1.3) having one marked point on each boundary component. The cluster algebra $\mathcal{A}(b, c)$ associated to an ideal triangulation of $\mathcal{D}$ is given by the matrix (1.4) [Fomin, Shapiro & Thurston 2008] [4, Example 4.4]. The exchange relations in this case can be written as $x_{i-1}x_{i+1} = 1 + x_i^2$ and $B' = -B$. Comparing with the relations (2.1), we conclude that the $\mathcal{A}(b, c)$ is a cluster algebra of rank 2 with $b = c = 2$. Therefore the basis $B$ of the cluster algebra $\mathcal{A}(b, c)$ is described by Theorem 2.1. On the other hand, the cluster algebra $\mathcal{A}(2, 2)$ is known to encode the Penner coordinates on the Teichmüller space $T_\mathcal{D} = \{t \in \mathbb{R} \mid t > 1\}$ of the annulus $\mathcal{D}$ [Williams 2014] [8, Section 3].

Let $A(2, 2)$ be an AF-algebra, such that $K_0(A(2, 2)) \cong \mathcal{A}(2, 2)$. The Bratteli diagram of the cluster $C^*$-algebra $A(2, 2)$ has the form of a Pascal triangle shown in Figure 1 [6, Section 4.4]. Thus $A(2, 2)$ is a
GICAR algebra [Effros 1980] [2, p. 13(e)]. Consider a group of the modular automorphisms

\[ \sigma_t : A(2, 2) \to A(2, 2) \]  

constructed in [6, Section 4]. Such a group is generated by the geodesic flow on the Teichmüller space \( \mathcal{T}_g \), \( ibid. \).

2.2. Powers state. Let \( M_{2\infty} = \bigotimes_{i=1}^{\infty} M_2(\mathbb{C}) \) be the GICAR algebra [Davidson 1996][1, Example III.5.5] and [Effros 1980] [2, p. 13(c1)]. For \( 0 < \lambda < 1 \) consider the Powers state \( \varphi_\lambda \) on the tensor product \( M_{2\infty} \) given by the formula:

\[ \varphi_\lambda(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \ldots) = \prod_{i=1}^{n} Tr \left( \frac{1}{1 + \lambda} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} x_i \right). \]  

(2.3)

Applying the GNS construction to the pair \( (M_{2\infty}, \varphi_\lambda) \) one gets a factor \( R_\lambda \). The product \( \{ \bigotimes_{i=1}^{\infty} \exp \left( \sqrt{-1} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \right) \mid 0 < \lambda < 1 \} \) gives rise to a group of the modular automorphisms of \( R_\lambda \), see e.g. [Jones 1991][5, Section 1.10].

The GICAR algebra \( A(2, 2) \) embeds into the factor \( R_\lambda \) [Davidson 1996][1, Example III.5.5]. Moreover, a restriction of the modular automorphisms of \( R_\lambda \) coincides with the \( \sigma_t : A(2, 2) \to A(2, 2) \) constructed in [6, Section 4].

2.3. Basic construction. Denote by \( e_{ij} \) the matrix units of the algebra \( M_2(\mathbb{C}) \). Then \( e_t = \frac{1}{1+t} (e_{11} \otimes e_{11} + te_{21} \otimes e_{21} + \sqrt{t} (e_{12} \otimes e_{21} + e_{21} \otimes e_{12})) \) is a projection of the algebra \( M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) for each \( t \in \mathbb{R} \). Proceeding by induction, one can define projections \( e_i(t) = \theta^i(e_t) \in M_{2^n} \), where \( \theta \) is the shift automorphism of the UHF-algebra \( M_{2\infty} \). The \( e_i := e_i(t) \) satisfy the following relations

\[ \begin{cases} 
  e_i e_j = e_j e_i, & \text{if } |i - j| \geq 2 \\
  e_i e_{i \pm 1} e_i = \frac{t}{(1+t)^2} e_i,
\end{cases} \]  

(2.4)

so that \( Tr (xe_{n+1}) = [\mathcal{M} : \mathcal{N}]^{-1} Tr (x) \) [Jones 1991][5, Section 5.6]. The \( e_i(t) \) generate a subfactor \( \mathcal{N} \) of the type II von Neumann algebra \( \mathcal{M} \), such that

\[ [\mathcal{M} : \mathcal{N}]^{-1} = \frac{t}{(1+t)^2}. \]  

(2.5)
3. Proof

We shall use a simple analysis of the cluster algebra $\mathcal{A}(\mathcal{D}) \cong K_0(\mathcal{A}(\mathcal{D}))$ using the Sherman-Zelevinsky Theorem. Namely, such an algebra has a canonical basis of the form

$$B = \{x_i^p x_{i+1}^q \mid p, q \geq 0\} \cup \{T_n(x_1 x_4 - x_2 x_3) \mid n \geq 1\}, \quad (3.1)$$

where $T_n(x)$ are the Chebyshev polynomials of the first kind, see Theorem 2.1. We split the proof in two lemmas corresponding (roughly) to the cases $|B| = \infty$ and $|B| < \infty$, respectively.

Lemma 3.1. There exists a subfactor $\mathcal{N}$ of the hyperfinite type $II_1$ factor $\mathcal{M}$ whenever $[\mathcal{M} : \mathcal{N}] \in (4, \infty)$.

Proof. (i) Let us return to the inclusion (1.5) and consider the Powers state $\varphi_\lambda$ on $M_{2\infty}$. The Powers modular automorphism of the factor $R_\lambda$ induces a modular automorphism $\sigma_t : \mathcal{A}(\mathcal{D}) \to \mathcal{A}(\mathcal{D})$. The Penner coordinate $t = R/r > 1$ on $T_{\mathcal{D}}$ and the Powers parameter $0 < \lambda < 1$ are related by the formula:

$$t = \frac{1}{2} \left( \lambda + \frac{1}{\lambda} \right). \quad (3.2)$$

In other words, the Penner coordinates give the Powers states, i.e. for each $t > 1$ the evaluation map produces a positive homomorphism of $K_0(\mathcal{A}(\mathcal{D}))$ to $\mathbb{R}$, which correlates with a trace on the GICAR algebra $\sigma_t(\mathcal{A}(\mathcal{D}))$.

(ii) If $|B| = \infty$, then the Bratteli diagram of $\mathcal{A}(\mathcal{D})$ (Figure 1) is an infinite tower. The hyperfinite type $II_1$ factor $\mathcal{M}$ is obtained from a factor $\mathcal{N}$ by adjoining the Jones projections $e_i(t)$ using the basic construction (Section 2.3). The Penner coordinate $t > 1$ on $T_{\mathcal{D}}$ corresponds to the values of index $[\mathcal{M} : \mathcal{N}] = \frac{(1+t)^2}{t} > 4$ in view of formula (2.5). In other words, $[\mathcal{M} : \mathcal{N}] \in (4, \infty)$. Lemma 3.1 is proved. □

Lemma 3.2. There exists a subfactor $\mathcal{N}$ of the hyperfinite type $II_1$ factor $\mathcal{M}$ whenever $[\mathcal{M} : \mathcal{N}] \in \{4 \cos^2 \left( \frac{\pi n}{n} \right) \mid n \geq 3\} \cup \{4\}$.

Proof. (i) Recall that the Chebyshev polynomials satisfy the following relations:

$$T_0 = 1 \quad \text{and} \quad T_n \left[ \frac{1}{2} (t + t^{-1}) \right] = \frac{1}{2} (t^n + t^{-n}). \quad (3.3)$$
In view of 2.1, we choose $\frac{1}{2}(t + t^{-1}) = x_1 x_4 - x_2 x_3$. (Such a parametrization is always possible since the Penner coordinates [Williams 2014] [8, Section 3.2] on $T_D$ are given by the cluster $(x_1, x_2)$, where each $x_i$ is a function of $t$.)

The exchange relations (1.2) for $A(D)$ can be written as $x_{i-1} x_{i+1} = x_i^2 + 1$. It is easy to calculate that $x_1 x_4 - x_2 x_3 = \frac{x_1^2 + x_2^2}{x_1 x_2}$. An explicit resolution of cluster variables $x_1$ and $x_2$ is given by the formulas:

$$
\begin{align*}
    x_1 &= \frac{\sqrt{3}}{2} \sqrt{t + t^{-1}} \\
    x_2 &= \frac{\sqrt{3}}{2} \sqrt{t - t^{-1}}
\end{align*}
$$

(3.4)

The reader can verify, that equations (3.4) imply $x_1 x_4 - x_2 x_3 = \frac{1}{2}(t + t^{-1})$. The parametrization of the ordered $K_0$-group of the GICAR algebra $A(\mathcal{D})$ in this case differs from (3.2) in the sense that $t$ is allowed to be a complex number. As we shall see, such an extension does not affect the property of the index to be a real number. The compatibility of traces under the embedding (1.5) is preserved.

(ii) If $|\mathcal{B}| < \infty$, then the Bratteli diagram of $A(\mathcal{D})$ (Figure 1) is a finite tower. In particular, the formulas (3.1) and (3.3) imply

$$
T_n(x_1 x_4 - x_2 x_3) = T_0 = 1
$$

(3.5)

for some integer $n \geq 1$. But $x_1 x_4 - x_2 x_3 = \frac{1}{2}(t + t^{-1})$ and using formula (3.3) for the Chebyshev polynomials, one gets an equation

$$
t^n + t^{-n} = 2
$$

(3.6)

for (possibly complex) values of $t$. Since (3.6) is equivalent to the equation $t^{2n} - 2t^n + 1 = (t^n - 1)^2 = 0$, one gets the $n$-th root of unity

$$
t \in \{e^{\frac{2\pi i}{n}} \mid n \geq 1\}.
$$

(3.7)

The value

$$
[\mathcal{M} : \mathcal{N}] = \frac{(1 + t)^2}{t} = 1 + 2 + t = 2 \left[ \cos \left( \frac{2\pi}{n} \right) + 1 \right] = 4 \cos^2 \left( \frac{\pi}{n} \right)
$$

(3.8)

is a real number. We must exclude the case $n = 2$ corresponding to the value $t = -1$, because otherwise one gets a division by zero in (2.4).

Lemma 3.2 is proved.

Theorem 1.1 follows from lemmas 3.1 and 3.2.
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