HOW TO DISENTANGLE TWO BRAIDED HOPF ALGEBRAS

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Abstract. We show how to define the tensor product of two braided Hopf algebras.

Introduction

In this short note we show how to endow with a canonical Hopf algebra structure the tensor product \( H \otimes L \) of two braided Hopf algebras living in a monoidal category, provided that there is an isomorphism \( c : L \otimes H \to H \otimes L \) satisfying suitable conditions (see Proposition 2.1). In particular, the square tensor \( H \otimes H \) is always a braided Hopf algebra with this structure. Also, the tensor product of two Hopf algebras living in a braided category is again a braided Hopf algebra (although in a new braided category). When seen in this context, the key point is to replace the braid \( c_{HL} \) with \( c_{LH}^{-1} \).

1. Preliminaries

We work in a monoidal category \( C \), for instance, the category of vector spaces over a field \( k \). We write \( \otimes \) and \( I \) for the tensor product and the unit of \( C \), respectively. The associativity and unit constraints are assumed without referring to them. We assume the reader is familiar with the notions of algebras and coalgebras in monoidal categories. All the algebras are associative unitary and the coalgebras are coassociative counitary. Given an algebra \( A \) and a coalgebra \( C \), we let \( \mu : A \otimes A \to A \), \( \eta : I \to A \), \( \Delta : C \to C \otimes C \) and \( \varepsilon : C \to I \) denote the multiplication, the unit, the comultiplication and the counit, respectively, specified with a subscript if necessary. We are going to use the nowadays well known graphic calculus for monoidal and braided categories. As usual, morphisms will be composed downwards and tensor products will be represented by horizontal concatenation in the corresponding order. The identity map of an object will be represented by a vertical line. Given an algebra \( A \), the diagrams \( \begin{array}{c} \mu' \\ \downarrow \end{array} \) and \( \begin{array}{c} \eta' \\ \downarrow \end{array} \) stand for the multiplication map and the unit of \( A \) respectively. Given a coalgebra \( C \), the comultiplication and the counit of \( C \) will be represented by the diagrams \( \begin{array}{c} \Delta' \\ \downarrow \end{array} \) and \( \begin{array}{c} \varepsilon' \\ \downarrow \end{array} \) respectively.

Let \( V, W \) be objects in \( C \) and let \( c : V \otimes W \to W \otimes V \) be an arrow.

- If \( V \) is an algebra, then we say that \( c \) is compatible with the algebra structure of \( V \) if \( c(\eta \otimes W) = W \otimes \eta \) and \( c(\mu \otimes W) = (W \otimes \mu)(c \otimes V)(V \otimes c) \).
- If \( V \) is a coalgebra, then we say that \( c \) is compatible with the coalgebra structure of \( V \) if \( (W \otimes \varepsilon)c = \varepsilon \otimes W \) and \( (W \otimes \Delta)c = (c \otimes V)(V \otimes c)(\Delta \otimes W) \).

Of course, there are similar compatibilities when \( W \) is an algebra or a coalgebra.
2. Tensor products of braided Hopf algebras

Recall that a braided bialgebra in \( C \) is an object \( H \) of \( C \) endowed with an algebra structure, a coalgebra structure and an isomorphism \( c_H \in \text{End}_C(H^2) \) (called the braid of \( H \)) satisfying the Braid Equation

\[
(\mu_H \otimes H)\circ (H \otimes \mu_H) = (\mu_H \otimes \mu_H) \circ (\mu_H \otimes c_H \otimes H) \circ (H \otimes \mu_H) = (\mu_H \otimes \mu_H) \circ (H \otimes \mu_H) = (\varepsilon_H \otimes \varepsilon_L) \circ (\Delta_H \otimes \Delta_L),
\]

such that: \( c_H \) is compatible with the algebra and coalgebra structures of \( H \), \( \eta \) is a coalgebra morphism, \( \varepsilon \) is an algebra morphism and \( \Delta = (\mu \otimes \mu) \circ (\mu \otimes c_H \otimes H) \circ (\Delta \otimes \Delta) \). If moreover there exists a map \( S : H \to H \) which is the inverse of the identity in the monoid \( \text{Hom}(H, H) \) with the convolution product, then we say that \( H \) is a braided Hopf algebra and we call \( S \) the antipode of \( H \).

Let \( H \) be a braided bialgebra in \( C \). It is well known that if the braid of \( H \) is involutive (i.e., \( c_H^2 = 1 \)), then \( H \otimes H \) is a braided bialgebra in a natural way. The following Proposition is the main result in this note. It shows in particular that the involutivity hypothesis can be removed.

**Proposition 2.1.** Let \( H \) and \( L \) braided bialgebras in \( C \) and let \( c_{LH} : L \otimes H \to H \otimes L \) be an invertible arrow. If

\[
(\mu_L \otimes H)(H \otimes \mu_H) = (\mu_H \otimes \mu_H) \circ (\mu_L \otimes H) \circ (H \otimes \mu_H) \circ (\mu_L \otimes H) = (\mu_L \otimes \mu_H) \circ (\mu_L \otimes \mu_H) \circ (\mu_L \otimes \mu_H) = (\mu_H \otimes \mu_H) \circ (\mu_H \otimes \mu_H) = (\varepsilon_H \otimes \varepsilon_L) \circ (\Delta_H \otimes \Delta_L),
\]

and \( c_{LH} \) is compatible with the bialgebras structures of \( H \) and \( L \), then \( H \otimes L \) is a braided bialgebra, via

- \( \mu_H \otimes L = (\mu_H \otimes \mu_L) \circ (H \otimes c_{LH} \otimes L) \),
- \( \eta_H \otimes L = \eta_H \otimes \eta_L \),
- \( \Delta_H \otimes L = (\mu \otimes c_H \otimes H) \circ (\Delta_H \otimes \Delta_L) \).

Moreover, if \( H \) and \( L \) are braided Hopf algebras, then so is \( H \otimes L \), with antipode \( S_{H \otimes L} = S_H \otimes S_L \).

**Proof.** The fact that \( H \otimes L \) is an algebra and a coalgebra is well-known and standard. We check now the compatibility between multiplication and comultiplication:

![Diagram](attachment:image.png)

We leave to the reader to prove that \( c_{H \otimes L} \) satisfies the Braid Equation, and that it is compatible with \( \mu_{H \otimes L}, \Delta_{H \otimes L}, \eta_{H \otimes L} \) and \( \varepsilon_{H \otimes L} \). The proof of the last assertion in the statement is also straightforward. \( \square \)

3. Braided families and compatible maps

The aim of this section is to give a more categorical proof of Proposition 2.1. The methods presented here could be useful in generalizing this result to braided versions of bicrossproducts, matched pairs, etc. (see \[112\]). Let \( V = (V_i)_{i \in \mathbb{I}} \) be a family of objects in \( C \). A family \( C \) of isomorphisms

\[
c_{ij} : V_i \otimes V_j \to V_j \otimes V_i \quad (i, j \in \mathbb{I})
\]
is said to be braided if \( \forall i, j, k \in \mathbb{S} \) the Braid Equations
\[
(V_k \otimes c_{ij}) (c_{ik} \otimes V_j) (V_i \otimes c_{jk}) = (c_{ik} \otimes V_j) (V_i \otimes c_{jk}) (c_{ij} \otimes V_k).
\]
are satisfied. Given such a family and a string \( i = (i_1, \ldots, i_n) \) of elements in \( \mathbb{S} \), we call \( n \) the length of \( i \), we let \( i_{<n} \) denote the string \( (i_1, \ldots, i_{n-1}) \) and we put \( V_i = V_{i_1} \cdots \otimes V_{i_n} \). For each pair \( i, j \) of such strings we define the map \( c_{ij}: V_i \otimes V_j \to V_j \otimes V_i \), recursively by
- If \( i = (i) \) and \( j = (j) \), then \( c_{ij} = c_{ij} \).
- If \( i = (i) \) and \( \text{length}(j) = m > 1 \), then \( c_{ij} = (V_{i_{<m}} \otimes c_{ij_m}) \circ (c_{ij_{<m}} \otimes V_{j_m}) \).
- If \( \text{length}(i) = n > 1 \), then \( c_{ij} = (c_{i_{<n}, j} \otimes V_n) \circ (V_{<n} \otimes c_{i_{<n}j}) \).

We say that a map \( f: V_i \to V_j \) is compatible with \( C \) if
\[
(f \otimes V_i) c_{ij} = c_{ij} (V_i \otimes f) \quad \text{and} \quad (V_i \otimes f) c_{ij} = c_{ij} (f \otimes V_i)
\]
for each \( l \in \mathbb{S} \).

Let \( \mathcal{D} = (V, C, M) \), where \( M \) is a family of maps compatible with \( C \). We want to embed this datum into a braided category in a natural way. To this end, let \( B_{\mathcal{D}} \) be the category whose objects are pairs \( W = (W, \lambda^W) \), where \( W \) is an object in \( C \) and \( \lambda^W = (\lambda^W_{ij})_{i,j} \) is a family of isomorphisms \( \lambda^W_{ij}: V_i \otimes W \to W \otimes V_i \), subject to
- \((W \otimes c_{ij}) \circ \lambda^W_{ij} = \lambda^W_{ij} \circ (c_{ij} \otimes W)\),
- \((W \otimes f) \circ \lambda^W_{ij} = \lambda^W_{ij} \circ (f \otimes W)\)

for each map \( f: V_i \to V_j \) in \( M \), where \( \lambda^W_{ij}: V_i \otimes W \to W \otimes V_i \) is recursively defined as follows:
- If \( i = (i) \), then \( \lambda^W_{ii} = \lambda_i^W \).
- If \( n = \text{length}(i) > 1 \), then \( \lambda^W_{ii} = (\lambda^W_{i_{<n} \otimes V_n}) \circ (V_{<n} \otimes \lambda^W_{i_{<n} i}) \).

A morphism \( \phi: W \to Z \) in \( B_{\mathcal{D}} \) is a map \( \phi: W \to Z \) in \( C \) such that
\[
(\phi \otimes V_i) \circ \lambda^W_{ij} = \lambda^Z_{ij} \circ (V_i \otimes \phi) \quad \text{for all } i \in \mathbb{S}.
\]

The category \( B_{\mathcal{D}} \) is monoidal, with
- tensor product given by \( W \otimes Z = (W \otimes Z, (W \otimes \lambda^Z_{ij}) \circ (\lambda^W_{ij} \otimes Z)) \),
- unit \( I = (I, \lambda^I_{ij}) \), where \( \lambda^I_{ij}: V_i \otimes I \to I \otimes V_i \) is the canonical map in \( C \).
- associativity and unit constraints induced from those of \( C \).

Remark 3.1. The family \( V \) can be embedded in \( B_{\mathcal{D}} \) by means of \( V_j = (V_j, \lambda^V_{ij}) \), where \( \lambda^V_{ij} = c_{ij} \).

We consider \( Z(B_{\mathcal{D}}) \), the center of \( B_{\mathcal{D}} \), which, we recall from [Ma] Cor. 9.1.6, has as objects the pairs \( (W, \gamma_{W, -}) \), where \( W \) is an object in \( B_{\mathcal{D}} \) and \( \gamma_{W, -} \) is a natural isomorphism \( (W \otimes -) \to (- \otimes W) \) such that
\[
\gamma_{W, I} \quad \text{is the canonical isomorphism in } B_{\mathcal{D}},
\gamma_{W, X \otimes Y} = (X \otimes \gamma_{W, Y}, (\gamma_{W, X} \otimes Y)) \quad \forall X, Y \in B_{\mathcal{D}}.
\]
An arrow \( \theta: (W, \gamma_{W, -}) \to (Z, \gamma_{Z, -}) \) is an arrow \( \theta: W \to Z \) in \( B_{\mathcal{D}} \) such that
\[
(X \otimes \theta) \circ \gamma_{W, X} = \gamma_{Z, X} \circ (\theta \otimes X) \quad \forall X \in B_{\mathcal{D}}.
\]

**Proposition 3.2.** The initial datum \( \mathcal{D} \) is included in the center via the identification \( i: \mathcal{D} \to Z(B_{\mathcal{D}}) \) given by
- \( iV_j = (V_j, \gamma_{V_j, -}) \), where \( \gamma_{V_j, W} = \lambda^V_{ij} \) for all \( V_j \in V \).
- \( ic_{ij} = \gamma_{V_j, V_i} = \lambda^V_{ij} = c_{ij} \) for all \( c_{ij} \in C \).
- \( i.f = f \) for all \( f \in M \).

**Proof.** Straightforward. \( \square \)
3.1. Alternative proof of Proposition 2.1

Proof. We take \( \mathbf{V} = (V_1 = H, V_2 = L, V_0 = I) \). \( \mathbf{C} \) consists of the arrows \( c_{11} = c_H, c_{12} = c_{HL}, c_{21} = (c_{HL})^{-1}, c_{22} = c_L \) and the canonical maps \( V_0 \otimes V_i \to V_i \otimes V_0, V_i \otimes V_0 \to V_0 \otimes V_i \). We take also \( \mathcal{M} = \{ \varepsilon_H, \varepsilon_L, \eta_H, \eta_L, \Delta_H, \Delta_L, \mu_H, \mu_L \} \). By Proposition 3.2 these data are included, via a map \( \iota \), in a braided category. Moreover, the braiding between \( \iota V_1 \) and \( \iota V_2 \) is involutive, i.e., \( c_{\iota V_1, \iota V_2} c_{\iota V_2, \iota V_1} = \text{id} \). Hence, the first item of [BD, Corollary 2.17] applies to give the desired result. \( \square \)

References

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