Dressed coordinates: the path-integrals approach

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Abstract

The recent introduced dressed coordinates are studied in the path-integral approach. These coordinates are defined in the context of a harmonic oscillator linearly coupled to massless scalar field and, it is shown that in this model the dressed coordinates appear as a coordinate transformation preserving the path-integral functional measure. The analysis also generalizes the sum rules established in a previous work.

1 Introduction

In recent works, the concept of dressed coordinates and dressed states have been introduced in the context of a harmonic oscillator linearly coupled to a massless scalar field [1, 2, 3, 4]. As emphasized in these references, the introduction of the dressed (or renormalized [4, 5]) coordinate and state concepts is necessary in order to give physical consistence to the oscillator-field system as a toy model to describe an atom in interaction with the electromagnetic field, where the atom is roughly modeled by the harmonic oscillator. Also, as early stressed the introduction of dressed coordinates and states is twofold advantageous. From the physical view point, the dressed states behave as the one expected for the physically measurable states: excited atomic states are unstable whereas the atom in their ground state and no field quanta is stable. On the other hand, it allows exact computations for the probability amplitudes of the different radiation processes of the atom. Indeed, when the calculation is performed for weak coupling constant we obtain, for the spontaneous decay of the first excited state of the atom, the long know result: $e^{-\Gamma t}$ [6]. Furthermore, when applied to a confined atom, approximated by the harmonic oscillator, in a spherical cavity of sufficiently small diameter the method accounts for the experimentally observed inhibition of the decaying processes [7, 8]. Besides that, in Refs. [9, 5] the extension of the dressed coordinate and state concepts for non linear system have been addressed.

Nevertheless, in all previous works the approach used has been via the operatorial formalism of Quantum Mechanics. The aim of this paper is to develop a path-integral approach to the problem. We hope that the path integral approach will be more adequate in dealing with the problem of computing the reduced density matrix or to obtain the master equation for the atom as is the case in Caldeira-Legget type models [10, 11, 12]. As in previous works, because its exact integrability, the dressed coordinates were introduced in the context of a harmonic oscillator coupled linearly to a massless scalar field.

The paper is organized as follows: In section 2 we introduce the model and compute the propagator by exact diagonalization. Section 3 is devoted to establish the dressed coordinates. In section 4 we compute the probabilities associated to some physical processes and extend the sum rules found in [4]. Finally, in section 5 we give our concluding remarks. Through this paper we use natural units $c = \hbar = 1$.

2 The model and its exact diagonalization

We consider as a toy model of an atom-electromagnetic field system the system composed by a harmonic oscillator (the atom) coupled to a massless scalar field. By considering the dipole approximation and expanding in the field modes we get the following Hamiltonian [1]

$$H = \frac{1}{2} (p_0^2 + \omega_0^2 q_0^2) + \frac{1}{2} \sum_{k=1}^{N} (p_k^2 + \omega_k^2 q_k^2) - g_0 \sum_{k=1}^{N} c_k q_k + \frac{1}{2} \sum_{k=1}^{N} \frac{\eta^2}{\omega_k^2} q_k^2,$$

(1)

where $q_0$ is the oscillator coordinate and $q_k$ are the field modes with $k = 1, 2, \ldots ; \omega_k = 2\pi / L, c_k = \eta \omega_k, \eta = \sqrt{2g\Delta\omega}$, $\Delta \omega = \omega_{k+1} - \omega_k = 2\pi / L$. With $g$ being a frequency dimensional coupling constant and $L$ the diameter of the

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sphere in which we confine the oscillator-field system. In Eq. (1) the limit \( N \to \infty \) is to be understood. The last term in Eq. (1) can be seen as a frequency renormalization [1] and, it guarantees a positive-defined Hamiltonian. Due to the Hamiltonian (1) is quadratic in the momenta and there are not constraints we can write the propagator for the system as being

\[
K(\vec{q}_f, t; \vec{q}_i, 0) = \int Dq_0 \prod_{k=1}^{N} Dq_k \exp \left( i \int_0^t dt \ L \right),
\]

where \( \vec{q} = (q_0, q_1, \ldots, q_{N-1}, q_N)^T \) and \( L \), the Lagrangian, is given by

\[
L = \frac{1}{2} \left( \frac{\dot{q}_0^2}{\omega_0^2} - \frac{\dot{q}_k^2}{\omega_k^2} + 2q_0c_kq_k \right) = \frac{1}{2} \vec{q}^T \vec{q} - \frac{1}{2} \vec{q}^T A \vec{q},
\]

where

\[
\omega_k^2 = \omega_0^2 + \sum_{k=1}^{N} \frac{\dot{q}_k^2}{\omega_k^2},
\]

and \( A \) is a symmetric matrix whose components are given by

\[
A = \begin{pmatrix}
\omega_0^2 & -c_1 & -c_2 & \cdots & -c_{N-1} & -c_N \\
-c_1 & \omega_1^2 & & & & \\
-c_2 & \omega_2^2 & & & & \\
& \ddots & \ddots & \ddots & \\
-c_{N-1} & & & \omega_{N-1}^2 & \\
-c_N & & & & \omega_N^2
\end{pmatrix}.
\]

To compute the propagator, Eq. (2), we introduce the coordinate transformation

\[
\vec{q} = T\vec{Q} \quad , \quad q_\mu = \sum_{r=0}^{N} t_r^\mu Q_r \quad , \quad Q_r = \sum_{\mu=0}^{N} t_r^\mu q_\mu
\]

where \( T \) is an orthogonal matrix that diagonalize \( A \),

\[
D = T^T A T = \text{diag} \left( \Omega_0^2, \Omega_1^2, \Omega_2^2, \ldots, \Omega_{N-1}^2, \Omega_N^2 \right).
\]

It is easy to show [14] that the eigenvalues of \( A \), \( \Omega_r \) are obtained by solving the equation

\[
\omega_0^2 - \Omega^2 = \eta^2 \sum_{k=1}^{N} \Omega_k^2 - \Omega^2.
\]

It has shown that such equation has definite positive frequencies \( \Omega^2 \) as solutions [1]. The solutions of the characteristic equation [3] were used to describe radiation process in small cavities [2, 3] in good agreement with the experiment. In [3] this system is also used to describe a Brownian particle coupled to an ohmic environment.

Replacing Eq. (6) in Eq. (2) we get

\[
K(\vec{q}_f, t; \vec{q}_i, 0) = \prod_{r=0}^{N} \int DQ_r \exp \left[ i \int_0^t dt \left( \frac{1}{2} \dot{Q}_r^2 - \frac{1}{2} \Omega_r^2 Q_r^2 \right) \right].
\]

Note that in Eq. (9) the functional measure is maintained, this because \( \det T = 1 \). By using the known result for the propagator of a harmonic oscillator we get for Eq. (9),

\[
K(\vec{q}_f, t; \vec{q}_i, 0) = \prod_{\mu=0}^{N} \left( \frac{\Omega^2_{f,\mu}}{\Omega^2_{f,\mu} - \Omega^2_{i,\mu} \cos (\Omega_{f,\mu} t) - 2Q_{f,\mu} Q_{i,\mu}} \right)^{\frac{1}{2}} \exp \left[ \frac{i \Omega_{f,\mu}}{2 \sin (\Omega_{f,\mu} t)} \left( [Q_{f,\mu}^2 + Q_{i,\mu}^2] \cos (\Omega_{f,\mu} t) - 2Q_{f,\mu} Q_{i,\mu} \right) \right].
\]

\[\text{Note that in Eq. (10) the functional measure is maintained, this because } \det T = 1. \] By using the known result for the propagator of a harmonic oscillator we get for Eq. (10),

\[
K(\vec{q}_f, t; \vec{q}_i, 0) = \prod_{\mu=0}^{N} \left( \frac{\Omega^2_{f,\mu}}{\Omega^2_{f,\mu} - \Omega^2_{i,\mu} \cos (\Omega_{f,\mu} t) - 2Q_{f,\mu} Q_{i,\mu}} \right)^{\frac{1}{2}} \exp \left[ \frac{i \Omega_{f,\mu}}{2 \sin (\Omega_{f,\mu} t)} \left( [Q_{f,\mu}^2 + Q_{i,\mu}^2] \cos (\Omega_{f,\mu} t) - 2Q_{f,\mu} Q_{i,\mu} \right) \right].
\]
The spectral function is defined as being
\[ Y(t) = \int dq_0 dq_1 \cdots dq_N K(\vec{q}, t; \vec{q}, 0) \] (11)
which is easily computed expressing the integral in normal coordinates \( \{Q_r\} \), thus we obtain
\[ Y(t) = \prod_{r=0}^{N} \left( i2 \sin \left( \frac{\Omega_r t}{2} \right) \right)^{-1} = \sum_{n_0, \ldots, n_N=0}^{\infty} e^{-itE_{n_0, \ldots, n_N}} \] (12)
with the energy spectrum being given by
\[ E_{n_0, \ldots, n_N} = \sum_{r=0}^{N} \Omega_r \left( n_r + \frac{1}{2} \right) . \] (13)

In the \( \{Q_r\} \) coordinates, the ground state wave function is computed from the propagator given by Eq. (10) by taking the limit \( t \to -i\infty \)
\[ K(\vec{q}_f, t \to -i\infty; \vec{q}_i, 0) = \prod_{r=0}^{N} \left( \frac{\Omega_r}{\pi} \right)^{\frac{1}{4}} \exp \left( -\frac{1}{2} \Omega_r Q_r^2 \right) , \] (14)
thus the wave function of the ground state is
\[ \psi_{00 \ldots 0}(\vec{Q}) = \prod_{r=0}^{N} \left( \frac{\Omega_r}{\pi} \right)^{\frac{1}{4}} \exp \left( -\frac{1}{2} \Omega_r Q_r^2 \right) . \] (15)

We can write the ground state eigenfunction in the original coordinates \( q_\mu \), by using the third equation of (6)
\[ \psi_{00 \ldots 0}(\vec{q}) = \left( \frac{\Omega_0}{\pi} \right)^{\frac{1}{4}} \left( \frac{\Omega_1}{\pi} \right)^{\frac{1}{4}} \cdots \left( \frac{\Omega_N}{\pi} \right)^{\frac{1}{4}} \exp \left( -\frac{1}{2} \sum_{\mu, \nu=0}^{N} \sum_{r=0}^{N} \Omega_r t_\mu^r t_\nu^r q_\mu q_\nu \right) . \] (16)

3 The dressed coordinates

We have observed the vacuum wave function in the \( \{q_\mu\} \) and normal \( \{Q_r\} \) coordinates and we make a question: Is it possible to find some new set of coordinates \( \{\bar{q}_\mu\} \) relate to them what allow us to describe the oscillators with their non interacting characteristics? The answer is yes. In such one new set of coordinates the vacuum wave function must be given as
\[ \psi_{00 \ldots 0}(\vec{q}_0, \ldots, \vec{q}_N) = \text{cte.} \exp \left( -\frac{1}{2} \sum_{\mu=0}^{N} \omega_\mu \left( \bar{q}_\mu \right)^2 \right) \] (17)
and we named the set of coordinates \( \{\bar{q}_\mu\} \) as dressed coordinates\(^2\) which describe the oscillators as being non interacting. In a physical situation we can imagine the atom interacting with a bath of field modes, i.e., an electromagnetic field. The experience tell us that the atom does not change his energy spectrum (the energy spectrum when it is isolated) and only transitions between the energy levels are observed (absorption and emission process).

Thus we would have for the vacuum state wave functions the following relation
\[ \exp \left( -\frac{1}{2} \sum_{r=0}^{N} \Omega_r \left( Q_r \right)^2 \right) \propto \exp \left( -\frac{1}{2} \sum_{\mu=0}^{N} \omega_\mu \left( \bar{q}_\mu \right)^2 \right) \] (18)

First we will look for the matrix transformation between the normal \( \{Q_\mu\} \) and dressed coordinates \( \{\bar{q}_\mu\} \), thus we set
\[ \vec{q} = M \vec{\bar{q}} , \quad \bar{q}_\mu = \sum_{r=0}^{N} M_\mu^r Q_r \] (19)
the quadratic form (18) must be invariant under the linear transformation (19), then, to preserve the quadric form we set
\[ \sum_{\mu=0}^{N} \omega_\mu M_\mu^r M_\mu^s = \Omega_r \delta_{rs} \] (20)
\(^2\)The dressed coordinates are a new type of collective coordinates, which are defined from the normal coordinates, that allows a correct description for the absorption and radiation phenomena for a given physical system.
and to achieve the matrix $M$ we use the orthonormal matrix $T$ such that

$$M^r_\mu = \sqrt{\frac{\Omega_r}{\omega_\mu}} t^r_\mu$$

we can show that it satisfies the equation (20) by using the orthonormality condition of the matrix $T$. The determinant of the matrix $M$ is shown to be 1,

$$\det(M) = \sqrt{\frac{\Omega_0 \Omega_1 \cdots \Omega_N}{\omega_0 \omega_1 \cdots \omega_N}} \det(T) = 1$$

thus the transformation (19) preserve the path-integral measure defining the propagator (2) of the system. And the inverse transformation $\{Q_s \rightarrow \bar{q}_\mu\}$ is easily shown to be

$$Q_s = \sum_{\mu=0}^N \sqrt{\frac{\omega_\mu \Omega_s t_s^\mu \Omega_s t_s^\mu}{\Omega_s}} \bar{q}_\mu$$

Then, we come back to the functional integral defining the propagator (9) in terms of the normal coordinates and using the transformation (22) we can write

$$K(\bar{q}_f, t; \bar{q}_i, 0) = \int \left( \prod_{\mu=0}^N D\bar{q}_\mu \right) \exp \left\{ i \int_0^t dt \sum_{\mu, \nu=0}^N \left[ Z_{\mu\nu} \dot{\bar{q}}_\mu \dot{\bar{q}}_\nu - C_{\mu\nu} \omega_\mu \omega_\nu \bar{q}_\mu \bar{q}_\nu \right] \right\}$$

from which we can see the Lagrangian in terms of dressed coordinates is given by

$$L_d = \frac{1}{2} \sum_{\mu, \nu=0}^N \left[ Z_{\mu\nu} \dot{\bar{q}}_\mu \dot{\bar{q}}_\nu - C_{\mu\nu} \omega_\mu \omega_\nu \bar{q}_\mu \bar{q}_\nu \right]$$

where we have defined the matrices $Z_{\mu\nu}$ and $C_{\mu\nu}$

$$Z_{\mu\nu} = \sqrt{\omega_\mu \omega_\nu} \sum_{s=0}^N \frac{t_s^\mu t_s^\nu}{\Omega_s}, \quad C_{\mu\nu} = \frac{1}{\sqrt{\omega_\mu \omega_\nu}} \sum_{s=0}^N \Omega_s t_s^\mu t_s^\nu$$

such that

$$Z_{\mu\beta} C_{\beta\nu} = \delta_{\mu\nu}$$

They play the role of renormalization constants and the coordinates $\{\bar{q}_\mu\}$ are the renormalized coordinates such as it happened in a renormalized field theory.

To construct the dressed Hamiltonian we first define the dressed momentum $\tilde{p}_\mu$ canonically conjugate to the dressed coordinate $\bar{q}_\mu$, thus

$$\tilde{p}_\mu = \frac{\partial L_d}{\partial \dot{\bar{q}}_\mu} = \sum_{\nu=0}^N Z_{\mu\nu} \dot{\bar{q}}_\nu$$

from which we get

$$\dot{\bar{q}}_\mu = \sum_{\nu=0}^N C_{\mu\nu} \tilde{p}_\nu$$

Thus, the dressed Hamiltonian $H_d$ is computed to be

$$H_d = \sum_{\mu, \nu=0}^N \frac{1}{2} C_{\mu\nu} \left( \tilde{p}_\mu \tilde{p}_\nu + \omega_\mu \omega_\nu \bar{q}_\mu \bar{q}_\nu \right)$$

From (10) we write the propagator in dressed coordinates

$$K(\bar{q}_f, T; \bar{q}_i, 0) = \prod_{r=0}^N \left( \frac{\omega_r}{2\pi \sin(\Omega_r T)} \right)^{\frac{1}{2}} \times \exp \left\{ \frac{i}{2} \sum_{s=0}^N \sum_{\mu, \nu=0}^N \sqrt{\omega_\mu \omega_\nu} \frac{t_s^\mu t_s^\nu}{\sin(\Omega_s T)} \left[ (\bar{q}_f \mu \bar{q}_f \nu + \bar{q}_i \mu \bar{q}_i \nu) \cos(\Omega_s T) - 2\bar{q}_f \mu \bar{q}_i \nu \right] \right\}$$
Performing the Gaussian integrals in Eq. (38) we obtain
\[
\psi_{00.0}(q) = \left(\frac{\omega_0}{\pi}\right)^{1/4} \cdots \left(\frac{\omega_N}{\pi}\right)^{1/4} \exp \left(-\frac{1}{2} \sum_{\alpha=0}^{N} \omega_{\alpha} (\tilde{q}_\alpha)^2\right)
\]  
(31)

The Hamiltonian operator in dressed coordinates is expressed as
\[
H(\tilde{q}) = \sum_{\mu,\nu=0}^{N} C_{\mu\nu} \left(-\frac{1}{2} \frac{\partial}{\partial \tilde{q}_\mu} \frac{\partial}{\partial \tilde{q}_\nu} + \frac{1}{2} \omega_{\mu,\nu} \tilde{q}_\mu \tilde{q}_\nu\right).
\]  
(32)

with the coefficients \(C_{\mu\nu}\) are given in [25].

4 Computing the transition probabilities

In this section we show what to compute the transition amplitudes of the system using the exact dressed propagator [30], thus, we are interested in the following quantities,

\[
A_{m_0,m_1,\ldots,m_N}^{n_0,n_1,\ldots,n_N}(t) = \langle d(n_0,n_1,\ldots,n_N)|e^{-iHt}|m_0,m_1,\ldots,m_N\rangle_d,
\]  
(33)

that represents the probability amplitude of the system, initially prepared in the state \(|m_0,m_1,\ldots,m_N\rangle\), to be found at time \(t\) in the state \(|n_0,n_1,\ldots,n_N\rangle\).

Eq. (33) can be written in terms of the propagator as

\[
A_{m_0,m_1,\ldots,m_N}^{n_0,n_1,\ldots,n_N}(t) = \int d\chi d\xi \langle d(n_0,n_1,\ldots,n_N)|\chi\rangle K(\chi, t; \xi, 0) \langle \xi|m_0,\ldots,m_N\rangle_d.
\]  
(34)

where

\[
\langle \xi|m_0,\ldots,m_N\rangle_d = \psi_{m_0m_1\ldots m_N}(\xi') .
\]  
(35)

Using Eq. (35) we can write Eq. (34) as

\[
A_{m_0,m_1,\ldots,m_N}^{n_0,n_1,\ldots,n_N}(t) = \int d\chi d\xi \psi_{n_0n_1,\ldots,n_N}(\chi') K(\chi, t; \xi, 0) \psi_{m_0m_1\ldots m_N}(\xi')
\]  
(36)

First we compute \(A_{000,00,00,\ldots,0}^{000,00,00,\ldots,0}(t)\). Substituting \(\psi_{000,00,00,\ldots,0}(\chi'(\chi)), K(\chi, t; \xi, 0)\) and \(\psi_{000,00,00,\ldots,0}(\xi'(\xi))\) we have

\[
A_{000,00,00,\ldots,0}^{000,00,00,\ldots,0}(t) = \left[\prod_{r=0}^{N} \frac{1}{\pi \sqrt{2 t \sin(\Omega_r t)}}\right] \int d\chi \sum_{s=0}^{N} t_s^{\mu} \chi_r \propto \left[\frac{i}{2} \sum_{r=0}^{N} e^{i \Omega_r t} \chi_r^2 \right] F_{\mu}(\chi, t)
\]  
(37)

where

\[
F_{\mu}(\chi, t) = \int d\xi \frac{H_{\mu}}{\sqrt{2^{m_{\mu}} m_{\mu}}} \exp \left[\frac{i}{2} \sum_{s=0}^{N} \left(\frac{e^{i \Omega_s t} \xi_s^2 - 2 \chi_s}{\sin(\Omega_s t) \chi_s}\right)\right] \int d\xi \left[\frac{i}{2} \sum_{s=0}^{N} \left(\frac{e^{i \Omega_s t} \xi_s^2 - 2 \chi_s}{\sin(\Omega_s t) \chi_s}\right)\right].
\]  
(38)

Performing the Gaussian integrals in Eq. (38) we obtain

\[
F_{\mu}(\chi, t) = \left[\prod_{s=0}^{N} \frac{1}{\sqrt{2 \pi i \sin(\Omega_s t)} e^{\frac{1}{2} \Omega_s t}}\right] H_{\mu} \sum_{s=0}^{N} t_s^{\mu} \sin(\Omega_s t) \frac{\partial}{\partial \chi_s} \left[\frac{i}{2} \sum_{r=0}^{N} \left(\frac{e^{i \Omega_r t} \chi_r^2}{\sin(\Omega_r t) \chi_r}\right)\right].
\]  
(39)

Using the identity

\[
H_n \sum_{r=0}^{N} t_r^{\mu} \chi_r = n! \sum_{l_n \ldots l_1} \left(\frac{(l_0)}{l_0!} (l_1')_{l_1} \ldots (l_N')_{l_N} \right) H_{l_0} H_{l_1} \cdots H_{l_N}
\]  
(40)
we get

\[ A_{0,\ldots,0}^{1,\ldots,0} \]  

where

\[ I_{l,s_r} = \int d\chi_r \exp \left( i \frac{e^{i\Omega_r t}}{\sin(\Omega_r t)} \chi_r^2 \right) H_{l_r}(\chi_r) H_{s_r} \left( i \sin(\Omega_r t) \frac{\partial}{\partial \chi_r} \right) \exp \left( -i \frac{e^{-i\Omega_r t}}{2 \sin(\Omega_r t)} \chi_r^2 \right) \]

\[ = \int d\chi_r e^{-\chi_r^2} H_{l_r}(\chi_r) \left[ \exp \left( i \frac{e^{i\Omega_r t}}{\sin(\Omega_r t)} \chi_r^2 \right) H_{s_r} \left( i \sin(\Omega_r t) \frac{\partial}{\partial \chi_r} \right) \exp \left( -i \frac{e^{-i\Omega_r t}}{2 \sin(\Omega_r t)} \chi_r^2 \right) \right] . \]

If instead of integrating over coordinates \( \xi \) in Eq. \( \text{(37)} \) we first integrate over coordinates \( \chi \) we would get an expression similar to the one given in Eq. \( \text{(41)} \) but with \( I_{l,s_r} \) replaced with \( I'_{l,s_r} \):

\[ I'_{l,s_r} = \int d\xi_r e^{-\chi_r^2} H_{s_r}(\xi_r) \left[ \exp \left( i \frac{e^{i\Omega_r t}}{\sin(\Omega_r t)} \chi_r^2 \right) H_{l_r} \left( i \sin(\Omega_r t) \frac{\partial}{\partial \chi_r} \right) \exp \left( -i \frac{e^{-i\Omega_r t}}{2 \sin(\Omega_r t)} \chi_r^2 \right) \right] . \]

Then, since the final result must not depend on the order in which we perform the integrations we must have \( I_{l,s_r} = I'_{l,s_r} \), and from Eqs. \( \text{(42)} \) and \( \text{(43)} \) we conclude that \( I_{l,s_r} = I_{s,l_r} \).

To perform the integral given in Eq. \( \text{(42)} \) we have to use the following theorem

\[ \text{if } k < n \implies \int dx e^{-x^2} H_n(x) x^k = 0 . \]

Note that the expression in brackets in Eq. \( \text{(42)} \) is a polynomial of degree \( s_r \) in \( \xi_r \). Now, if \( l_r > s_r \), then by using theorem \( \text{(44)} \), we get \( I_{l,s_r} = 0 \). Because \( I_{l,s_r} = I_{s,l_r} \), we also get a vanishing result for \( l_r < s_r \). Then, the only non-vanishing result is obtained for \( l_r = s_r \). Using again theorem \( \text{(44)} \) we note that the only non-vanishing term of the polynomial in brackets is the one of highest power. Since the highest power of \( H_n(x) \) is given by \( 2^n x^n \) we have for Eq. \( \text{(42)} \)

\[ I_{l,s_r} = (2i \sin(\Omega_r t))^{s_r} \int d\chi_r e^{-\chi_r^2} H_{l_r}(\chi_r) \left[ \exp \left( i \frac{e^{i\Omega_r t}}{\sin(\Omega_r t)} \chi_r^2 \right) H_{s_r} \left( i \sin(\Omega_r t) \frac{\partial}{\partial \chi_r} \right) \exp \left( -i \frac{e^{-i\Omega_r t}}{2 \sin(\Omega_r t)} \chi_r^2 \right) \right] = e^{i s_r \Omega_r t} \int d\chi_r e^{-\chi_r^2} H_{l_r}(\chi_r) \left( 2 \right)^{s_r} \chi_r^{s_r} = e^{-i s_r \Omega_r t} \int d\chi_r e^{-\chi_r^2} H_{l_r}(\chi_r) H_{s_r}(\chi_r) = \sqrt{\pi} e^{-\frac{i s_r \Omega_r t}{2} \chi_r^2} |s_r! \delta_{l,s_r} . \]

Using Eq. \( \text{(55)} \) in Eq. \( \text{(41)} \) we get

\[ A_{0,\ldots,0}^{0,\ldots,0} \]  

where in passing to the last line we have used the identity

\[ \left( \sum_{r=0}^{N} X_r \right)^n = n! \sum_{l=n}^{\frac{N}{l}} \frac{X^n}{l_0! \cdots l_N!} . \]

In terms of

\[ f_{\mu \nu}(t) = \sum_{r=0}^{N} t^r_{\mu} t^r_{\nu} e^{-\alpha r t} , \]

Eq. \( \text{(40)} \) can be written as

\[ A_{0,\ldots,0}^{0,\ldots,0} \]  

(49)
It is straightforward to establish the following identity:

\[ \sum_{\mu=0}^{N} |f_{\mu}(t)|^2 = 1 \]  

(50)

The proof of the above identity follows trivially by using the orthonormality property of the matrix \( \{t_{\mu}^r \} \). Writing Eq. (50) for indexes 0 and \( k \) we have

\[ |f_{00}(t)|^2 + \sum_{k=1}^{N} |f_{0k}(t)|^2 = 1 \]

(51)

\[ |f_{k1,0}(t)|^2 + \sum_{k_2=1}^{N} |f_{k1,k_2}(t)|^2 = 1 \]  

(52)

The physical interpretation for the equations above is given as it follows. Let the initial state of the system given by \( |n,1_{k_1},1_{k_2}\rangle \), the atom in the \( n \)-th excited level and field quanta of frequencies \( \omega_{k_1}, \omega_{k_2} \), etc. The probability of this initial states to be found in a measurement performed at time \( t \) in the state \( |m,1_{k_1}',1_{k_2}'\rangle \) is denoted by \( P_{n,1_{k_1},1_{k_2}}(t) \). We know that \( P_{1,0,0}(t) = |f_{00}(t)|^2 \) is the probability of the oscillator to remain in the first excited level and \( P_{0,1,0}(t) = |f_{0k}(t)|^2 \) is the probability of the oscillator to decay from the first excited level to the ground state by emission of a field quanta of frequency \( \omega_{k} \). Obviously in this case we have

\[ P_{1,0,0}(t) + \sum_k P_{0,0,1}^{k}(t) = 1 \]  

(53)

that is nothing but Eq. (51). Also Eq. (52) can be written as

\[ P_{0,0,k_1}(t) + \sum_{k_2} P_{0,k_2}^{0}(t) = 1 \]  

(54)

where

\[ P_{0,0,k_1}^{1}(t) = |f_{k_1,0}(t)|^2 \]  

(55)

and

\[ P_{0,k_2}^{0}(t) = |f_{k_1,k_2}(t)|^2 \]  

(56)

With these identifications the physical meaning of Eq. (51) is clear: if initially we have a photon of frequency \( \omega_{k_1} \) and the oscillator is in its ground state, then at time \( t \), either the oscillator can go to its first excited level by absorption of the initial photon or can remain in its ground state scattering the initial photon to other photon of arbitrary frequency.

Note that in establishing the identities (50) and (51) it is used only the orthogonality property of the matrix \( \{t_{\mu}^r \} \). Then, it is natural to ask whether it is possible to compute other probabilities without doing a direct computation as performed in last section. The answer is yes. For example, if initially the oscillator is in its second excited level and there are no photons, at time \( t \) it can happen that the oscillator continues in their second excited level, it can go to their first excited level by emission of photon of arbitrary frequency \( \omega_{k_1} \) or it can decay to their ground state by emission of two photons of arbitrary frequencies \( \omega_{k_1} \) and \( \omega_{k_2} \). The respective probabilities are denoted by \( P_{2,0,0}(t), P_{2,0,1,1}(t) \) and \( P_{2,0,1,0,1}(t) \). Obviously we must have

\[ P_{2,0,0}(t) + \sum_{k_1} P_{2,0,1,1}(t) + \sum_{k_1,k_2} P_{2,0,1,0,1}(t) = 1 \]  

(57)

Taking the square of Eq. (57) we find

\[ \left( P_{1,0,0}^{1}(t) \right)^2 + 2P_{1,0,0}^{1}(t) \sum_{k_1} P_{1,0,1,1}(t) + \sum_{k_2} P_{1,0,1,0}(t)P_{1,0,1,2}(t) = 1 \]  

(58)

Identifying Eqs. (57) and (58) we obtain

\[ P_{2,0,0}(t) = \left( P_{1,0,0}^{1}(t) \right)^2 \]

\[ = |f_{00}(t)|^4 \]  

(59)

\[ P_{2,0,1,1}(t) = 2P_{1,0,0}^{1}(t)P_{1,0,1,1}(t) \]

\[ = 2|f_{00}(t)f_{0k_1}(t)|^2 \]  

(60)
and
\[ P^{0;1}_{2;0}(t) = P^{0;1}_{1;0}(t) P^{0;1}_{0;2}(t) = |f_{0_{k_1}}(t) f_{0_{k_2}}(t)|^2. \] (61)

As a second example we consider the oscillator is in its first excited state and there is one photon of frequency \( \omega_{k_1} \). At time \( t \) it can happen that: the oscillator go to its second excited level by absorbing the initial photon; or the oscillator remains in its first excited state and the initial photon is scattered to other photon of arbitrary frequency \( \omega_{k_2} \); or maybe the oscillator can be decay to its ground state by emission of a photon of arbitrary frequency \( \omega_{k_2} \) and the initial photon is scattered to other photon of frequency \( \omega_{k_3} \). The respective probabilities are denoted by \( P^{2;0}_{1;1_{k_1}}(t) \), \( P^{1;1_{k_2}}_{1;1_{k_1}}(t) \) and \( P^{0;1_{k_2}1_{k_3}}_{1;1_{k_1}}(t) \). Then, we must have

\[ P^{2;0}_{1;1_{k_1}}(t) = \sum_{k_2} P^{1;1_{k_2}}_{1;1_{k_1}}(t) + \sum_{k_2 k_3} P^{0;1_{k_2}1_{k_3}}_{1;1_{k_1}}(t). \] (62)

Taking Eq. 63 times Eq. 64 we have

\[ P^{1;0}_{1;0}(t) P^{1;0}_{0;1_{k_1}}(t) + \sum_{k_2} \left( P^{1;0}_{1;0}(t) P^{0;1_{k_2}}_{0;1_{k_1}}(t) + P^{1;0}_{1;0}(t) P^{0;1_{k_2}}_{1;0}(t) \right) + \sum_{k_2 k_3} P^{0;1_{k_2}}_{1;0}(t) P^{0;1_{k_3}}_{0;1_{k_1}}(t) = 1. \] (63)

From Eqs. 62 and 63 we have

\[ P^{1;0}_{1;1_{k_1}}(t) = P^{1;0}_{1;0}(t) P^{1;0}_{0;1_{k_1}}(t) = |f_{00}(t) f_{0 k_1}(t)|^2, \] (64)

\[ P^{1;1_{k_2}}_{1;1_{k_1}}(t) = P^{1;0}_{1;0}(t) P^{0;1_{k_2}}_{0;1_{k_1}}(t) + P^{1;0}_{1;0}(t) P^{0;1_{k_2}}_{1;0}(t) = |f_{00}(t) f_{k_2 k_1}(t)|^2 + |f_{0 k_1}(t) f_{0 k_2}(t)|^2, \] (65)

and

\[ P^{0;1_{k_2}1_{k_3}}_{1;1_{k_1}}(t) = \frac{1}{2} \left( P^{0;1_{k_2}}_{1;0}(t) P^{0;1_{k_3}}_{0;1_{k_1}}(t) + P^{0;1_{k_3}}_{1;0}(t) P^{0;1_{k_2}}_{0;1_{k_1}}(t) \right) \]
\[ = \frac{1}{2} \left( |f_{0 k_2}(t) f_{k_1 k_3}|^2 + |f_{0 k_3}(t) f_{k_1 k_2}|^2 \right). \] (66)

And so we can give all the probabilities associated to any decay or absorption process placing in the system.

5 Conclusions

In the present work is shown using the path-integral formalism that the dressed coordinates appear as a coordinate transformation preserving the quadric form that defines the ground state wave function of the system guaranteeing the vacuum stability and, they also leave invariant the functional measure of the path-integral. Within the Hamiltonian formalism it can be shown that such linear transformation, which defines the dressed coordinates, also leaves invariant the canonical form of the action. Thus, the dressed coordinates can be also defined via a canonical transformation.

The calculus of the transition amplitudes has been performed using the dressed propagator, being obtained the basic formula which defines the sum rules presented in [4], then, the rules have been extended to describe other physical processes. In spite of the computation seems very difficult, the dressed coordinates allow to use the properties of the Hermite polynomials simplifying greatly the calculus.

On the other hand, it has been made an extensive use of the model given by Eq. [1] to study different physical situations, such as the quantum Brownian motion, decoherence and other related problems in quantum optics. In such context, it is interesting the computation of the reduced matrix density for the model [1] in the framework of dressed coordinates; the advances in such direction will be reported elsewhere.

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A  The orthonormal matrix \( T = \left[ t^s_{\mu} \right] \)

Because the orthogonal character of the \( T \)-matrix its components satisfy

\[
\sum_{\mu=0}^{N} t^r_{\mu} t^s_{\mu} = \delta_{rs}, \quad \sum_{r=0}^{N} t^r_{\mu} t^r_{\nu} = \delta_{\mu \nu}
\]  

(67)

and from the Lagrangian \(^3\) expressed in terms of the normal \( \{ Q_r \} \) we get other important relation

\[
\sum_{\mu=0}^{N} \bar{\omega}_0^2 t^r_{\mu} t^s_{\mu} - \sum_{k=1}^{N} \eta \omega_k t^r_0 t^s_k = \Omega^2 \delta_{rs}
\]  

(68)

where \( \bar{\omega}_0^2 \) have been defined in \(^4\) and \( \bar{\omega}_k^2 = \omega_k^2 \). Using the equations above we can show the following sum

\[
\bar{\omega}_0^2 = \sum_{s=0}^{N} \Omega^2_s \left( t^s_0 \right)^2, \quad \eta \omega_k = - \sum_{s=0}^{N} \Omega^2_s t^s_k t^s_0, \quad \omega_k^2 = \sum_{s=0}^{N} \Omega^2_s \left( t^s_k \right)^2
\]  

(69)

and also compute the elements of the \( T \)-matrix

\[
t^s_{k} = \frac{\eta \omega_k}{\omega_k^2 - \Omega^2_s} t^s_0, \quad t^s_0 = \left[ 1 + \eta^2 \sum_{k=1}^{N} \frac{\omega_k^2}{\left( \omega_k^2 - \Omega^2_s \right)^2} \right]^{-\frac{1}{2}}
\]  

(70)

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