A Comparison of Different Divergence-free Solutions for 3D Anisotropic CSEM Modeling Using Staggered Finite Difference Method

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Abstract. This paper presents a staggered finite difference (FD) method for numerical modeling of 3D controlled-source electromagnetic (CSEM) data in an anisotropic conductive medium. The traditional Krylov subspace methods may not be convergence for EM modeling in strong electrical anisotropy due to the variation of the conductivity in the sea-bottom sediments over a range of frequencies (0.1-10 HZ). The algorithm is based on a modified version of the curl-curl equations with scaled grad-div operator (CCGD) in frequency domain. In this approach, we integrate the divergence correction term in the original system with its importance controlled by scaling factors (the same values as model resistivities are used in this paper) and avoid the application of the iterative divergence correction. The corresponding responses calculated by the CCGD and the traditional CC-DC approach for the VTI ocean canonical reservoir model are nearly identical. Then based on the case, we examine the numerical performance of the CCGD approach, and compare it with the CC-DC in terms of computing time and iteration number. The results indicate that the proposed CCGD approach is efficient and stable over different frequencies.

1. Introduction

The marine controlled-source electromagnetic method (MCSEM) was commonly used for the off-shore hydrocarbon exploration [2]. However, the subsurface conductivity structure could be very complex due to the existence of anisotropic mediums, which may result from the variation of the conductivity in the multiplex sea-bottom sediments, such as vertical transverse isotropic (VTI), horizontal transverse isotropic (HTI) and tilted transverse isotropic (TTI) medium. In such cases, if we use an isotropic model in EM data interpretation, significant errors may occur. The 3D electromagnetic modeling requires to solve the diffusive Maxwell's equations in a discretized grid for an anisotropic media. In this paper, we discuss the finite difference method, as it combines simple implementation and flexibility on anisotropic conductivity structures.

Due to the scale and property of the typical large linear system of equations generated from the FD discretization of the curl-curl equations, the traditional Krylov subspace methods may not convergence for EM modeling in strong electrical anisotropy medium. To overcome this problem, two common approaches are used to solve the problem. One is preconditioner [3,5] which improves the convergence of the solution and the other is divergence correction which can speed up the convergence of the solution...
based on curl-curl equations (CC-DC). In a more concise way, the authors [1,3] modify the original curl-curl equations with the divergence correction equation (CCGD). Dong & Egbert (2019) applied this method to solve magnetotelluric forward modeling problems, indicating that this approach is efficient and stable especially for low frequencies. Whether this approach is effective for CSEM forward modeling problems in an anisotropic medium, is investigated in this paper.

1.1. Curl-curl equations modified with grad-div operator (CCGD)

At the frequencies used in CSEM explorations, displacement currents in the earth can commonly be ignored, compared to conduction currents. An $e^{i\omega t}$ time dependence is assumed. EM fields $\mathbf{E}$ and $\mathbf{H}$ are diffused according to Maxwell’s equations given by

$$\nabla \times \mathbf{H} = -\sigma \mathbf{E} + \mathbf{J}_s,$$

$$\nabla \times \mathbf{E} = i \omega \mu \mathbf{H},$$

where $\omega$ is the angular frequency, $\mu$ is the free space magnetic permeability, $\mathbf{J}_s$ is the source current, and $\sigma$ is the conductivity. By eliminating the magnetic fields, we can obtain the second-order partial differential equations in terms of the electric field

$$\nabla \times \nabla \times \mathbf{E} + i \omega \mu \nabla \times \nabla \times \mathbf{E} = \mathbf{J}_s.$$  

3D finite difference numerical modeling of EM response based on the total field formulation typically requires a fine space discretization close to the current source to accurately capture the rapid change of the primary fields. This can be avoided by solving the modeling problem based on a secondary field equation, with primary fields obtained analytically. In this method, the total field is decomposed into the primary and secondary fields ($\mathbf{E}_b$ and $\mathbf{E}_a$)

$$\mathbf{E} = \mathbf{E}_b + \mathbf{E}_a,$$

Background fields $\mathbf{E}_b$ can be obtained from isotropic background medium (e.g., uniform half-space or layered medium) $\sigma_b$ and the anomalous conductivity $\Delta \sigma$ as

$$\sigma = \sigma_b + \Delta \sigma.$$  

Based on this decomposition, we can obtain the second order differential equation for the secondary electrical fields as

$$\nabla \times \nabla \times \mathbf{E}_a + i \omega \mu \nabla \times \nabla \times \mathbf{E}_a = -i \omega \mu \Delta \sigma \mathbf{E}_a.$$  

Equation (6) with appropriate boundary conditions can be solved numerically by either the integral or differential equation method, for example, the finite difference method used in this paper. Applying finite difference discretization, we can obtain

$$\mathbf{A} \mathbf{x} = \mathbf{b},$$  

Equation (7) can be solved by either direct or iterative solvers [6]. However, when the size of equation (7) is significantly large, direct solvers can be prohibitive in terms of both computing time and memory cost. Because of this reason, the iterative solvers (such as BiCGstab used in this paper) are commonly applied. However, the convergence of solving the equation (7) iteratively can be slow due to the abundant null space of the double-curl operator, which can be alleviated by the application of preconditioners. As $\omega$ becomes very small, the numerical solution cannot adequately model the charges accumulated at the discontinuities of conductivity and this can pose a challenge for iterative solvers to converge. This is commonly improved by enforcing the static divergence correction iteratively

$$\nabla \cdot \sigma \mathbf{E} = \nabla \cdot \sigma \mathbf{E}_b + \nabla \cdot \Delta \sigma \mathbf{E}_a = 0.$$  

Instead of enforcing the divergence correction iteratively, authors [1,3] developed an alternative approach, in which they add the div operator in equation (8) to the curl-curl equation in equation (6). However, the dimension between the current divergence operator and the curl-curl operator in equa-tion (6) is not consistent. For simplicity, we assume that the air conductivity $\sigma_a > 0$ (e.g., $10^{-10}$ S/m). Then, as in the early work [3,7], we add a scaled version of equation (8) $\lambda_1 \nabla (\nabla \cdot \sigma \mathbf{E}_a) + \lambda_2 \nabla (\nabla \cdot \Delta \sigma \mathbf{E}_a) = 0$ to Eq. (6) to obtain


\[ \nabla \times \nabla \times \mathbf{E}_a - \lambda_1 \nabla (\nabla \cdot \mathbf{a} \mathbf{E}_a) + i \omega \mu \sigma \mathbf{E}_a = -i \omega \mu \sigma \mathbf{E}_b + \lambda_2 \nabla (\nabla \cdot \mathbf{\Delta} \mathbf{E}_b) , \quad (9) \]

where \( \lambda_1 \) and \( \lambda_2 \) are scaling factors (possibly spatially variable). With an appropriate choice of \( \lambda_1 \) and \( \lambda_2 \), this modified system can adequately model the charges accumulation at the discontinuity of conductivity even when the third term vanishes as \( \omega \to 0 \). It is worth to note that \( \lambda_1 \) and \( \lambda_2 \) can also be moved before the divergence operator, since the actual constraints applied are \( \nabla \cdot \mathbf{\sigma} \mathbf{E}_a + \nabla \cdot \mathbf{\Delta} \mathbf{E}_b = 0 \).

In the air \( \mathbf{\Delta} \mathbf{E}_b = 0 \), and we set \( \lambda_2 = 1/\sigma_{\text{air}} \). In this region the curl-curl operator is reduced to the Laplacian, and (10) becomes \( \nabla^2 \mathbf{E}_a + i \omega \mu \sigma \mathbf{E}_a = 0 \), whose null space is reduced significantly. In the earth, we use \( \lambda_1 = 1/\sigma_{\text{earth}} \) and \( \lambda_2 = 1/\mathbf{\Delta} \sigma \). A detailed discussion on this issue can be found in the Dong & Egbert’s (2019) research.

1.2. VTI ocean canonical reservoir model

The deep water complex VTI ocean canonical reservoir model that has been considered is shown in Fig. 1. The sea water is isotropic, the water depth is 1000 m, the source height is 130 m, the thickness of the sediment is 1.5 km and the reservoir is 100 m thick. The model is discretized with a 60 × 60 × 35 cell mesh (58 by 58 by 27 km3, not including the air) which is sufficiently large for the EM fields to satisfy Dirichlet conditions at the boundaries when the source is located in the central part of the model. The model is discretized with an FD grid having uniform cell sizes of 100 m in the x- and y-directions and 80 m in the z-direction. Anisotropy in the sediment and reservoir is considered.

![Figure 1. Vertical cross-section of a VTI ocean canonical reservoir model.](image-url)
Figure 2. Comparison of (a) run time and (b) iteration number for our anisotropic 3-D FD method based on CC-DC and CCGD, required to converge to a tolerance of $10^{-8}$ for the VTI ocean canonical reservoir model.

Figure 3. Amplitude (top panel) and phase (bottom panel) responses of Ex at 1 Hz for the VTI ocean canonical reservoir model of Fig. 5. They are calculated using our anisotropic 3-D FD method based on CC-DC (solid lines) and CCGD (star lines).
Fig. 2 provides a more detailed view of the relative performance of CC-DC and CCGD, in terms of calculation time and iteration number for a range of frequencies. Curl-curl equations without divergence correction cannot achieve convergence in this frequency range, and the minimum relative residual is $2 \times 10^{-8}$ on the preset maximum iteration number. In addition, CCGD is slightly faster than CC-DC for this range of frequencies. As the frequency decreases, the CCGD gains a significant advantage, and the algorithm is three times faster than that of the standard approach (CC-DC) at 0.1Hz (37s vs. 101s). Note that the reduction of iteration count for the CCGD is less significant than the reduction in calculation time, as each iteration is faster due to the reduction in the number of non-zero elements in the coefficient matrix.

The solid lines and star lines respectively, show results in Fig.3, obtained with our anisotropic 3-D FD method based on CC-DC and CCGD. The corresponding responses computed with two approaches for the model are nearly identical, with the maximum errors of less than $10^{-8}$ for the amplitude and phase of the x-component electric field.

2. Conclusions
In this article, we develop an efficient and stable finite difference CSEM modeling algorithm in a general anisotropic media, which is based on curl-curl (CC) equations modified with the grad-div operator (CCGD) to eliminate the abundant null space. We integrate the divergence correction term in the curl-curl equations and avoid the iterative use of the divergence correction (traditional CC-DC) explicitly as commonly done.

The modified system results in a much sparser coefficient matrix with a significant smaller condition number than the original system, which is favored by iterative solvers. The modeling results for a VTI ocean canonical reservoir model indicate that our algorithm is accurate with maximum errors for the amplitude and the phase of the x-component electric field are less than $10^{-8}$, compared with CC-DC. The test also indicates that, in contrast to the traditional CC-DC approach, our approach converges slightly faster as the frequency decreases. And our algorithm outperforms CC-DC to a large degree both in terms of computing time and iteration number.

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