Statistical Model with Measurement Degree of Freedom and Quantum Physics

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Abstract

This is an English translation of the manuscript[1] which was appeared in Surikaiseki Kenkyusho Kokyuroku No. 1055 (1998). The asymptotic efficiency of statistical estimate of unknown quantum states is discussed, both in adaptive and collective settings. Adaptive bounds are written in single letterized form, and collective bounds are written in limiting expression. Our arguments clarify mathematical regularity conditions.

1 Introduction

This is an English translation of the manuscript[1] which appeared in Surikaiseki Kenkyusho Kokyuroku No. 1055 (1998).

In the estimation of the unknown density operator by use of the experimental data, the error can be reduced by the improvement of the design of the experiment. Therefore, it is natural to ask what is the limit of the improvement. To answer the question, Helstrom[3] founded the quantum estimation theory, in analogy with classical estimation theory(in the manuscript, we refer to statistical estimation theory of probability distribution as ‘classical estimation theory’). Often, for simplicity, it is assumed that a state belongs to a family $\mathcal{M} = \{\rho_\theta | \theta \in \Theta \subset \mathbb{R}^m\}$ of states, which is called model and that the finite dimensional parameter $\theta$ is to be estimated statistically.

He considered the quantum analogue of Cramér-Rao inequality, which gives the lower bound of mean square error of locally unbiased estimate. This bound, however, is not achievable at all, when the number of the data is finite.

Let us assume that the number $n$ of the data tends to infinite. Then, if some regularity conditions are assumed, it is concluded that if the estimate is consistent, i.e., the estimate converges to the true value of parameter, the first order asymptotic term of mean square error satisfies the Cramér-Rao inequality, and that the bound is achieved for all $\theta \in \Theta$. This kind of discussion is called first order asymptotic theory.

The quantum version of first order asymptotic theory is started by H. Nagaoka[5][6]. He defined, in our terminology, the quasi-quantum Cramér-Rao type bound, and pointed out that the bound is achieved asymptotically and globally. The proof of achievability, however, is only roughly sketched in his paper. In this manuscript, the proof of the achievability of the bound is fully written out, and the regularity conditions for the achievability is revealed. In addition, we defined another bound, the quantum Cramér-Rao type bound, and showed that the new bound is also achievable, if the use of quantum correlation between samples are allowed.

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2 Preliminaries

An estimate $\hat{\theta}$ is obtained as a function $\hat{\theta}(\omega)$ of data $\omega \in \Omega$ to $\mathbb{R}^m$. The purpose of the theory is to obtain the best estimate and its accuracy. The optimization is done by the appropriate choice of the measuring apparatus and the function $\hat{\theta}(\omega)$ from data to the estimate.

Let $\sigma(\mathbb{R}^m)$ be a $\sigma$-field in the space $\mathbb{R}^m$. Whatever apparatus is used, the data $\omega \in \Omega$ lie in a measurable subset $B \in \sigma(\mathbb{R}^m)$ of $\Omega$ writes

$$\Pr\{\omega \in B|\theta\} = \text{tr} \rho(\theta)M(B), \quad (1)$$

when the true value of the parameter is $\theta$. Here, $M$, which is called positive operator valued measure (POM, in short), is a mapping from subsets $B \subset \Omega$ to non-negative Hermitian operators in $\mathcal{H}$, such that

$$M(\phi) = O, M(\Omega) = I, \quad M(\bigcup_{i=1}^{\infty} B_i) = \sum_{i=1}^{\infty} M(B_i), \quad (B_i \cap B_j = \phi, i \neq j), \quad (2)$$

(see Ref.[3],p.53 and Ref.[4],p.50.). Conversely, some apparatus corresponds to any POM $M$. Therefore, we refer to the measurement which is controled by the POM $M$ as 'measurement $M$'. A pair $(\hat{\theta}, M)$ is called an estimator.

The classical Fisher information matrix $J^M_\theta$ by the POM $M$ is defined, as in the classical estimation theory,

$$J^M_\theta := \left[ \int_{\omega \in \Omega} \partial_i \log \frac{dP^M_\theta}{d\nu} \partial_j \log \frac{dP^M_\theta}{d\nu} dP \right],$$

where $\partial_i = \partial / \partial \theta^i$, $P^M_\theta(B) := \text{tr} \rho_\theta M(B)$, and $\nu$ is some underlying measure (in the manuscript, we assume that for any POM $M$, there is a measure $\nu$ in $\Omega$ such that $P^M_\theta < \nu$ for all $\theta \in \Theta$). Denote the mean square error matrix of $(\hat{\theta}/M)$ by $V^M_\theta[M]$, and, as the measure of accuracy, let us take $\text{Tr} G V^M_\theta[M]$, where $G$ is nonnegative symmetric real matrix. If $G = \text{diag}(g_1, \cdots, g_m)$, $\text{Tr} G V^M_\theta[M]$ is weighed sum of mean square error of the estimate $\theta^j$ of each component $\theta^i$ of the parameter.

Let us define locally unbiased estimator $(\hat{\theta}, M)$ at $\theta$ by,

$$E_{\theta}[\hat{\theta}_n, M] := \int \hat{\theta}_n(\omega) \text{tr} \rho_\theta M(d\omega) = \theta^j, \quad (j = 1, \cdots, m). \quad (3)$$

$$\int \hat{\theta}_n(\omega) \text{tr} \partial_k \rho_\theta M(d\omega) = \delta^j_k, \quad (j, k = 1, \cdots, m). \quad (4)$$

Then, $J^M_\theta$ is caracterized by,

$$J^M_{\theta} = \inf\{V_{\theta}[\hat{\theta}, M] | \hat{\theta} : (\hat{\theta}, M) \text{ is locally unbiased}\},$$

and the quasi-quantum Cramér-Rao type bound $C_\theta(G)$ is defined by,

$$C_\theta(G) := \inf\{\text{Tr} G V_{\theta}[\hat{\theta}, M] | M \text{ is locally unbiased}\} = \inf\{\text{Tr} G J^{M-1}_{\theta} | M \text{ is a POM in } \mathcal{H}\}.$$
Nagaoka pointed out that the quasi-quantum Cramér-Rao type bound is achievable asymptotically for every $\theta \in \Theta$. $C_\theta(G)$ is calculated explicitly for several special cases [7][2].

Suppose $n$-i.i.d. pairs $\rho^n_\theta$ of the unknown state $\rho_\theta$ are given. The sequence $\{(\theta, M_n)\}$, where $M_n$ is a POM in $\mathcal{H}^\otimes n$, is said to be MSE consistent if the estimate $\hat{\theta}_n$ by converges to the true value of the parameter in the mean, i.e., $\lim_{n \to \infty} V_\theta[(\hat{\theta}_n, M_n)] = 0$.

### 3 The quasi-classical Cramér-Rao type bound

#### 3.1 The lower bound

Let $M(1), ..., M(n)$ be a sequence of the POMs in $\mathcal{H}$, and apply the measurement $M(1)$ to the first sample, and the measurement $M(2)$ to the second sample, and so on. The choice of $M(k)$ is dependent on the outcome $\bar{\omega}_{k-1} = (\omega(1), ..., \omega(k-1))$ of $M(1), ..., M(k-1)$. To reveal the dependency of $M(k)$ on $\bar{\omega}_{k-1}$, we write $M(k)[\bar{\omega}_{k-1}]$.

Let us define the POM $M_n$ in $\mathcal{H}^n$ which takes value in $\Omega^n$ by,

$$M_n(B) = \int_{\bar{\omega} \in B} \otimes_{k=1}^n M(k)[\bar{\omega}_{k-1}](d\omega(k)).$$

Then the data $\bar{\omega}_n$ is controled by the probability distribution $P_\theta^{M_n}(B) = \text{tr} \rho^{\otimes n}_\theta M_n(B)$.

The estimator is said to be asymptotically unbiased if

$$\left( B_n \right)_i = \left( B_\theta \left( \hat{\theta}_n, M \right) \right)_i := \int_{\Omega^n} \left( \hat{\theta}_n(\omega) - \theta \right)^i P_\theta^M(d\omega) \to 0 \text{ as } n \to \infty, \quad (5)$$

$$\left( A_n \right)_j = \left( A_\theta \left( \hat{\theta}_n, M \right) \right)_j := \frac{\partial}{\partial \theta_j} E_\theta^B[\hat{\theta}_n, M_n] \to \delta_j \text{ as } n \to \infty. \quad (6)$$

The MSE consistent estimator satisfies (5) always. Therefore, if appropriate regularity conditions are assumed so that the differential, the integral and the trace commute with each other, then (6) is also satisfied, and the estimator will be asymptotically unbiased.

**Theorem 1** If $\{(\hat{\theta}_n, M_n)\}$ is MSE consistent, and $\lim_{n \to \infty} nV_\theta[(\hat{\theta}_n, M_n)]$ exists,

$$\lim_{n \to \infty} n \text{Tr } G V_\theta[(\hat{\theta}_n, M_n)] \geq C_\theta(G), \quad (7)$$

**Proof** In the almost same manner as classical estimation theory, (6) leads to,

$$nV_\theta[\hat{\theta}_n, M_n] \geq nA_n \left( J_\theta^{M_n} \right)^{-1} t A_n \quad (8)$$

Elementary calculation leads to,

$$\frac{1}{n} J_\theta^{M_n} = J_\theta^{M_\theta}, \quad (9)$$

where $M_\theta \in \mathcal{M}$ is a POM in $\mathcal{H}$ which is defined by

$$M_\theta^{\otimes n} \left( \prod_{k=1}^n B_k \right) = \int_{\bar{\omega}_{n-1}} \sum_{k=1}^n M(k)[\bar{\omega}_{k-1}](B_k) P_\theta^M(d\omega_n).$$

(8) and (9) yield

$$\text{Tr } G nV_\theta[\hat{\theta}_n, M_n] \geq \text{Tr } G A_n \left( J_\theta^{M_\theta} \right)^{-1} t A_n \geq C_\theta(t A_n G A_n). \quad (10)$$

Passing both sides of (10) to the limit $n \to \infty$, we have the theorem.
3.2 Estimator which achieves the bound

The estimator defined in the following achieves the equality in the inequality (7) if the regularity conditions (B.1-4) are satisfied. The proof will be presented later in the subsection 3.4.

First, apply the measurement \( M_0 \) to \( \sqrt{n} \) samples of unknown state \( \rho_\theta \), and calculate \( \hat{\theta}_n \) which satisfies (12). Second, apply the measurement \( M_{\hat{\theta}_n} \) to the remaining \( n - \sqrt{n} \) samples, where \( M_\theta \) is defined by

\[
\text{Tr} \ G \left( J_\theta^{M_\theta} \right)^{-1} \leq C_\theta(G) + \epsilon', \tag{11}
\]

(11) is satisfied. Then, \( \hat{\theta}_n \) is defined to be \( \bar{\theta}_n(\hat{\theta}_n) \), where \( \bar{\theta}_n(\theta') \) is defined by,

\[
\bar{\theta}_n(\theta') = \arg\max_{\theta \in \Theta} \sum_{k=\sqrt{n}+1}^{n} \log \frac{dP_{\theta'}^{M_{\theta'}}}{d\nu}(\omega_k).
\]

3.3 Regularity conditions

(B.1) There is a POM \( M_0 \) and \( \hat{\theta}_n \) which satisfies

\[
\lim_{n \to \infty} P_{\theta}^{M_n}\{\|\theta - \hat{\theta}_n\| > \delta\} = 0, \quad \forall \delta > 0. \quad \tag{12}
\]

(B.2) \( K := \sup_{\theta \in \Theta} \|\theta\| \) is finite.

(B.3) \( \bar{\theta}_n(\theta') \) achieves the equality in classical asymptotic Cramér-Rao inequality of the family \( \{P_{\theta'}^{M_{\theta'}}\} \) of probability distributions.

(B.4) The higher order term of mean square error of \( \bar{\theta}_n(\theta') \) is uniformly bounded when \( \|\theta' - \theta\| < \delta_1 \) for some \( \delta > 0 \). In other words, for any \( \epsilon > 0, \theta \in \Theta \), there exists a positive real number \( \delta_1 > 0 \) and a natural number \( N \) such that,

\[
(n - \sqrt{n}) \text{Tr} G V_{\theta,n} - \text{Tr} G \left( J_\theta^{M_\theta} \right)^{-1} < \epsilon + \epsilon', \quad \forall n \geq N, \quad \forall \hat{\theta} \ s.t. \quad \|\theta - \hat{\theta}\| \leq \delta_1. \tag{13}
\]

where \( V_{\theta,n} \) is the conditional mean square error matrix of \( \hat{\theta}_n \) when \( \hat{\theta}_n \) is given.

(B.5) For any \( \epsilon > 0, \theta \in \Theta \), there exists \( \delta_2 > 0 \), such that,

\[
\left| \text{Tr} G \left( J_\theta^{M_\theta} \right)^{-1} - C_\theta(G) \right| < \epsilon + \epsilon', \quad \forall \theta, \forall \hat{\theta} \ s.t. \quad \|\theta - \hat{\theta}\| < \delta_2. \tag{14}
\]

(B.1) is satisfied almost always, and (B.2) is not restrictive. For \( \bar{\theta}_n(\theta') \) is the maximum likelihood estimator of the family \( \{P_{\theta'}^{M_{\theta'}}\} \) of probability distributions, (B.3) is satisfied in usual cases. The validity of (B.4), however, is hard to verify. Therefore, in the future, this condition needs to be replaced by other conditions. Obviously, (B.5) reduces to the following (B.5.1-2), both of which are natural.

(B.5.1) The map \( \theta \mapsto C_\theta(G) \) is continuous.

(B.5.2) For any \( \theta' \), the map \( \theta \mapsto [J_\theta^{M_{\theta'}}]^{-1} \) is continuous.
3.4 Proof of achievable

Theorem 2 If the model $M$ satisfy conditions (B.1-5) in the following, then we have,
\[
\lim_{n \to \infty} n \text{Tr} G V_{\hat{\theta}_n}(M_n) = C_{\theta}(G), \forall \theta \in \Theta,
\]
(15)

Proof Let us choose $\delta_1, \delta_2$ and $N$ so that (13 – 14) are satisfied, and define $\delta' := \min(\delta_1, \delta_2)$. Then, if $n \geq N$, we have,
\[
n \text{Tr} G V_{\hat{\theta}_n}(M_n) = n \int \text{Tr} G V_{\hat{\theta}_n}(\hat{\theta}_n, M_n) P_{\theta}^M(d\omega)
\]
\[
\leq n \int_{||\theta - \hat{\theta}_n|| \leq \delta'} \text{Tr} G V_{\hat{\theta}_n}(\hat{\theta}_n, M_n) P_{\theta}^M(d\omega) + K^2 \text{Tr} G \int_{||\theta - \hat{\theta}_n|| > \delta'} P_{\theta}^M(d\omega)
\]
\[
\leq \frac{n}{n - \sqrt{n}} \int_{||\theta - \hat{\theta}_n|| \leq \delta'} \left( C_{\theta}(G) + 2\epsilon + \epsilon' \right) P_{\theta}^M(d\omega) + nK^2 \text{Tr} G P_{\theta}^M \{ ||\theta - \hat{\theta}_n|| > \delta \}
\]
\[
\leq \frac{n}{n - \sqrt{n}} \int_{||\theta - \hat{\theta}_n|| \leq \delta'} \left( C_{\theta}(G) + 2\epsilon + \epsilon' \right) P_{\theta}^M(d\omega) + nK^2 \text{Tr} G P_{\theta}^M \{ ||\theta - \hat{\theta}_n|| > \delta \}.
\]
(B.1) implies that the third term of last end of the equation tends to $n$ as $n \to \infty$. Therefore, we have, for every $\epsilon' > 0$ and for every $\epsilon > 0$,
\[
\lim_{n \to \infty} n \text{Tr} G V_{\hat{\theta}_n}(M_n) \leq C_{\theta}(G) + 2\epsilon + \epsilon'.
\]
which leads to the theorem. \hfill \Box

4 Use of quantum correlation

In this section, we consider the minimization of asymptotic mean square error where $M_n$ runs every POM which satisfies MSE consistensy. Physically, this means we allow the use of interactions between samples.

So far, we considered POM which takes value in $\Omega$, or the totality of all the possible data. Instead, in this section, we consider POM with the values in $\mathbb{R}^d$, for if $M$ is a POM with values in $\Omega$, $M \circ \hat{\theta}^{-1}$ is POM with the values in $\mathbb{R}^d$. MSE consistensy is defined in the same way as the precedent sections.

Let $C_{\theta}^Q(G)$ denote the quasi-quantum Cramér-Rao type bound of the family $\{\rho_{\theta}^n | \theta \in \Theta\}$ of density operators in $\mathcal{H}^\otimes n$. Then, the quantum Cramér-Rao type bound $C_{\theta}^Q(G)$ is defined by,
\[
C_{\theta}^Q(G) := \lim_{n \to \infty} n C_{\theta}^n(G).
\]

For $C_{\theta}(G) \geq n C_{\theta}^n(G)$ holds true, we have,
\[
C_{\theta}(G) \geq C_{\theta}^A(G).
\]
Theorem 3 If the sequence \( \{M^n\}_{n=1}^\infty \) is MSE consistent, we have,
\[
\lim_{n \to \infty} n \text{ tr } G \theta (M^n) \geq C_{\theta}^Q(G).
\] (16)

Proof In the almost same manner as the proof of theorem 1, we have,
\[
V_\theta[M_n] \geq A_n \left(J_\theta^M\right)^{-1} t A_n,
\]
\[
n \text{Tr } G \theta[M_n] \geq n \text{ Tr } G A_n \left(J_\theta^M\right)^{-1} t A_n \geq nC_{\theta}^o \left(t A_n G A_n\right),
\]
which approaches (16) as \( n \to \infty \). \( \square \)

If the family \( \{\rho_\theta^\otimes n|\theta \in \Theta\} \) of density operators satisfies (B.1-5), we have the following theorem.

Theorem 4 There is a MSE consistent sequence \( \{M_n\} \) of POM such that \( \lim_{n \to \infty} n \text{ tr } G \theta [M_n] \leq C_{\theta}^Q(G) + \epsilon \) is satisfied for every \( \epsilon > 0 \) and for every \( \theta \in \Theta \).

Proof Let us divide \( n \) samples into \( n_2 \) groups each of which is consist of \( n_1 \) samples, and let \( M_{(1)}^{n_1}, \ldots, M_{(n_2)}^{n_1} \) be a sequence of POMs in \( \mathcal{H}_{\theta}^\otimes n_1 \). Apply the measurement \( M_{(j)}^{n_1} \) to the first group \( \rho_\theta^\otimes n_1 \) of samples, and apply \( M_{(1)}^{n_2} \) to the second samples, and so on. The choice of \( M_{(k)}^{n_1} \) is dependent on the outcome of the measurements \( M_{(1)}, \ldots, M_{(k-1)}^{n_1} \). With \( n_1 \) fixed, let us approach \( n_2 \) to \( \infty \). Then, theorem 2 implies the existence of a MSE consistent sequence \( \{M_n\} \) of POM which satisfies
\[
\lim_{n \to \infty} n \text{ tr } G \theta [M_n] = \lim_{n_2 \to \infty} n_1 n_2 \text{ tr } G \theta [M_n] = n_1 C_{\theta}^n(G). \] (17)

For any epsilon, if \( n_1 \) is sufficiently large, \( \lim_{n \to \infty} n \text{ tr } G \theta (M^n) \leq C_{\theta}^Q(G) + \epsilon \) is satisfied, and we have the theorem. \( \square \)

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