ON ENDO-PRIME AND ENDO-COPRIME MODULES

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Abstract. The aim of this paper is to investigate properties of endo-prime and endo-coprime modules which are generalizations of prime and simple rings, respectively. Various properties of endo-coprime modules are obtained. Duality-like connections are established for endo-prime and endo-coprime modules.

1. Introduction and preliminaries

Throughout all rings are associative with non-zero identity elements and modules are unital. We know that $R$ is a prime ring if and only if any non-zero ideal of $R$ has zero left annihilator, or stated otherwise, any non-zero fully invariant submodule of $_RR$ is faithful over the endomorphism ring $\text{End}_R(R) \simeq R$. Haghany and Vedadi [5] generalized this property to modules: An $R$-module $M$ with $S = \text{End}_R(M)$ is called endo-prime if for any non-zero fully invariant submodule $K$ of $M$, $\text{Ann}_S(K) = 0$. They show that being endo-prime is a Morita invariant property, and an endo-prime module has a prime endomorphism ring. An $R$-module $M$ is a direct sum of isomorphic simple modules if and only if each non-zero element of $\sigma[M]$ is an endo-prime module. The dual notion of endo-prime modules defined by Wijayanti [11]: An $R$-module $M$ with $S = \text{End}_R(M)$ is called endo-coprime if for any proper fully invariant submodule $K$ of $M$, $\text{Ann}_S(M/K) = 0$. In the special case, $_RR$ is endo-coprime if and only if $R$ is a simple ring. It is shown in [11] that endo-coprime modules have prime endomorphism rings.

In section 2, we give sufficient conditions for an endo-prime module to be fully prime and polyform. Some general properties of endo-coprime $R$-module $M$ are obtained. We see that for endo-coprime $f$-coretractable module $_RM$, $\text{Ann}_R(M)$ is a prime ideal of $R$. Now consider comodules over coalgebra $C$ over a commutative ring $R$, provided that the coalgebra $C$ satisfies the $\alpha$-condition. In this context one of the questions one may ask is when the dual algebra $C^* = \text{Hom}_R(C,R)$ of an $R$-coalgebra $C$ is a prime algebra. The first paper to consider this was by Xu, Lu, and Zhu [14] who observed that this is the case if $C$ is a coalgebra over a field $k$ and $(C^* * f) \rightarrow C = C$ for any non-zero element $f \in C^*$. Another approach in this direction can be found in Jara et. al. [6] and Nekooei-Torkzadeh [8] where coprime coalgebras (over fields) are defined by using the wedge product. We show that a coalgebra $C$ over a base field is fully coprime if and only if its dual algebra $C^*$ is prime. It turns out that each non-zero element of $\sigma[M]$ is an endo-prime module if and only if each non-zero element of $\sigma[M]$ is an endo-coprime module. Being endo-coprime is also a Morita invariant property.

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In section 3, we shall deal with \( R \)-duals. It is proved that for a finitely generated module \( M \) over a quasi-Frobenius ring \( R \), \( M \) is an endo-prime (resp. endo-coprime) \( R \)-module if and only if \( M^* = \text{Hom}_R(M, R) \) is an endo-coprime (resp. endo-prime) \( R \)-module.

As before, \( R^M \) is a non-zero left module over the ring \( R \), its endomorphism ring \( \text{End}_R(M) \) will act on the right side of \( R^M \), in other words, \( R^M \text{End}_R(M) \) will be studied mainly. For the convenience of the readers, some definitions of modules that will be used in the next sections are provided. Let \( M \) be a left \( R \)-module. We say that \( N \in R\text{-Mod} \) is subgenerated by \( M \) if \( N \) is a submodule of an \( M \)-generated module (see the [12]). The category of \( M \)-subgenerated modules is denoted by \( \sigma[M] \). For submodule \( N \) of \( M \), we write \( N \leq_M M \) when \( N \) is a superfluous (or small) submodule of \( M \). In the category of left \( R \)-modules there are various notions of prime objects which generalize the well-known notion of a prime associative (commutative) ring \( R \). For the notions of (co)primeness of modules we refer to [11] and [13].

**Definition 1.1.** Recall that \( R^M \) is

- **prime** if for any non-zero (fully invariant) submodule \( K \) of \( M \), \( \text{Ann}_R(K) = \text{Ann}_R(M) \).
- **coprime** if for any proper (fully invariant) submodule \( K \) of \( M \), \( \text{Ann}_R(M/K) = \text{Ann}_R(M) \).
- **fully prime** if for any non-zero fully invariant submodule \( K \) of \( M \), \( M \) is \( K \)-cogenerated.
- **fully coprime** if for any proper fully invariant submodule \( K \) of \( M \), \( M \) is \( M/K \)-generated.
- **strongly coprime** if for any proper fully invariant submodule \( K \) of \( M \), \( M \) is sub-generated by \( M/K \), i.e., \( M \in \sigma[M/K] \).

\( M \) is called **retractable** (resp. **fi-retractable**) if for any non-zero submodule (resp. fully invariant submodule) \( K \) of \( M \), \( \text{Hom}_R(M, K) \neq 0 \).

Dually, \( M \) is called **coretractable** (resp. **fi-coretractable**) if for any proper submodule (resp. fully invariant submodule) \( K \) of \( M \),

\[
\pi_K \circ \text{Hom}_R(M/K, M) = \{ f \in S \mid (K)f = 0 \} \neq 0.
\]

We use \( \circ \) for the composition of mappings written on the right side. The usual composition is denoted by \( \circ \). Thus, from now on, we use \( (u)f \circ g = g \circ f(u) \).

2. **Endo-Prime and Endo-Coprime Modules**

We begin with investigating the relation between endo-prime and fully prime modules. For any fully invariant submodules \( K, L \) of \( M \), consider the product

\[
K \ast_M L := K\text{Hom}_R(M, L).
\]

According to [11, 1.6.3], \( M \) is fully prime if and only if for any fully invariant submodules \( K, L \) of \( M \), the relation \( K \ast_M L = 0 \) implies \( K = 0 \) or \( L = 0 \).

**Lemma 2.1.** Let \( R^M \) be a fi-retractable module and \( S = \text{End}_R(M) \). If \( S \) is prime, then \( M \) is fully prime.

**Proof.** Let \( K, L \) be non-zero fully invariant submodules of \( M \) which satisfy \( K \ast_M L = K\text{Hom}_R(M, L) = 0 \). By assumption we have

\[
M\text{Hom}_R(M, K)\text{Hom}_R(M, L) = 0.
\]
Since $S$ is prime, $\text{Hom}_R(M, K) = 0$ or $\text{Hom}_R(M, L) = 0$, a contradiction, because $M$ is fi-retractable. Consequently $M$ is fully prime.

A submodule $U$ of $R$-module $N$ is called $M$-rational in $N$ if for any $U \subseteq V \subseteq N$, $\text{Hom}_R(V/U, M) = 0$. $M$ is called polyform if any essential submodule is rational in $M$. The dual notions are: A submodule $X$ of $N$ is called $M$-corational in $N$ if for any $Y \subseteq X \subseteq N$, $\text{Hom}_R(M, X/Y) = 0$. $M$ is called copolyform if any superfluous submodule is corational in $M$.

An $R$-module $E$ is called pseudo-injective in $\sigma[M]$ if any diagram in $\sigma[M]$ with exact row

\[
\begin{array}{c}
0 & \longrightarrow & L & \overset{f}{\longrightarrow} & N \\
& & \downarrow{g} & & \downarrow{h} \\
& & E & & \downarrow{E} \\
& & & & \downarrow{E} \\
\end{array}
\]

can be extended nontrivially by some $s \in \text{End}_R(E)$ and $h : N \rightarrow E$ to the commutative diagram

\[
\begin{array}{c}
0 & \longrightarrow & L & \overset{f}{\longrightarrow} & N \\
& & \downarrow{g} & & \downarrow{h} \\
& & E & \overset{s}{\longrightarrow} & E \\
\end{array}
\]

that is, $gs = fh \neq 0$ (see [3]).

The following result shows sufficient conditions for endo-prime modules to be fully prime and polyform.

**Proposition 2.2.** Let $M$ be an endo-prime $R$-module with $S = \text{End}_R(M)$. Then the following statements hold:

(1) If $M$ is fi-retractable, then $M$ is fully prime.

(2) If $M$ is semi-injective, then the center of $S$ is a field.

(3) If $M$ is pseudo-injective in $\sigma[M]$ with $\text{Soc}(M) \neq 0$, then $M$ is polyform.

**Proof.** (1) Since $R$ is endo-prime, then $S$ is prime. Hence the assertion follows from Lemma 2.1.

(2) Let $f$ be a non-zero central element of $S$. Then $\text{Ker } f$ is a fully invariant submodule of $M$. Because $M$ is semi-injective, $fS = \text{Ann}_S(\text{Ker } f)$. Now since $M$ is endo-prime, we must have $\text{Ker } f = 0$ which implies that $fS = S$. Consequently $f$ is an invertible element of $S$.

(3) Because $M$ is endo-prime, we see that for any non-zero fully invariant submodule $K$ of $M$, $\text{Hom}_R(M/K, M) = 0$. Since $\text{Soc}(M)$ is a non-zero fully invariant submodule of $M$, $\text{Hom}_R(M/\text{Soc}(M), M) = 0$. For any essential submodule $L \leq M$, $\text{Soc}(M) \leq L$. Thus $\text{Hom}_R(M/L, M) = 0$ and by pseudo-injectivity of $M$ in $\sigma[M]$, $\text{Hom}_R(L'/L, M) = 0$ for any $L \subseteq L'$, i.e., $L$ is $M$-rational.

**Corollary 2.3.** The following statements hold on a prime ring $R$:

(1) If $R$ is semi-injective, then the center of $R$ is a field.

(2) If $R$ is pseudo-injective with $\text{Soc}(R) \neq 0$, then $R$ is polyform.
In continuation, we study endo-coprime modules. The following examples show that the concepts endo-coprime and coprime are different in general.

**Example 2.4.** (1) Let $F$ be a field, $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M = Re$. Then $\text{End}_R(M) \cong eRe \cong F$ as rings, so $R$ is endo-coprime by [11, 1.5.2 part (1)]. Consequently $M$ is not coprime by [11, Lemma 1.3.3 part (1)].

Now consider two left ideals $I = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ of $R$. We can easily see that $I, J \subseteq \text{Ann}_R(M)$ and $IJ = 0$. Thus $\text{Ann}_R(M)$ is not a prime ideal of $R$. Then $R$ is not coprime, by [11, Lemma 1.3.3 part (1)].

(2) Notice that $\text{End}_R(\mathbb{Q}) \cong \mathbb{Q}$ and hence $\mathbb{Q}$ has no nontrivial fully invariant submodules. Then $\mathbb{Q}$ is a coprime $\mathbb{Z}$-module. Also any non-zero factor module $\mathbb{Z}_{p^{\infty}}/K$ of $\mathbb{Z}_{p^{\infty}}$ (for any prime number $p$) is isomorphic to $\mathbb{Z}_{p^{\infty}}$ itself. Thus $\mathbb{Z}_{p^{\infty}}$ is a coprime $\mathbb{Z}$-module. Therefore $\mathbb{Z}_{p^{\infty}} \oplus \mathbb{Q}$ is a coprime $\mathbb{Z}$-module (see [11, Lemma 1.3.9]). But $\mathbb{Z}_{p^{\infty}} \oplus \mathbb{Q}$ is not endo-coprime, because $\text{Hom}_\mathbb{Z}(\mathbb{Z}_{p^{\infty}}, \mathbb{Q}) = 0$ and thus $\text{End}_\mathbb{Z}(\mathbb{Z}_{p^{\infty}} \oplus \mathbb{Q})$ cannot be a prime ring.

For any fully invariant submodules $K, L \subseteq M$, consider the inner coproduct

$$K :_M L = \bigcap \{ (L)f^{-1} \mid f \in \text{End}_R(M), K \subseteq \text{Ker } f \}$$

$$= \text{Ker}(\pi_K \circ \text{Hom}_R(M/K, M) \circ \pi_L),$$

where $\pi_K : M \to M/K$ and $\pi_L : M \to M/L$ denote the canonical projections. By [11, Proposition 1.7.3], $M$ is fully coprime if and only if for any fully invariant submodules $K, L$ of $M$, the relation $K :_M L = M$ implies $K = M$ or $L = M$. Notice that such a coproduct is considered in Bican et.al. [1] for any pair of submodules $K, L \subseteq M$ (not necessary fully invariant) and then a definition of “coprime modules” is derived from this coproduct.

In [11], Wijayanti introduced the following condition for an $R$-module $M$:

(**) For any proper (fully invariant) submodule $K$ of $M$, $\text{Ann}_R(K) \not\subseteq \text{Ann}_R(M)$.

**Proposition 2.5.** Let $M$ be an $R$-module with $S = \text{End}_R(M)$.

1. If $R$ is coprime and satisfies (**), then it is endo-coprime.
2. If $R$ is commutative and $R$ is endo-coprime, then $R$ is coprime.
3. If $R$ is endo-coprime and fi-coretractable, then for every left ideal $I$ of $R$ either $IM = 0$ or $IM = M$.
4. If $R$ is fi-coretractable and $S$ is prime, then $R$ is fully coprime.

**Proof.** (1) Let $R$ be coprime and $K$ be a proper fully invariant submodule of $M$. If there exists $0 \neq f \in \text{Ann}_S(M/K)$, then we have

$$\text{Ann}_R(M) = \text{Ann}_R(M/\text{Ker } f) = \text{Ann}_R((M)f),$$

by coprimeness of $M$. Thus $\text{Ann}_R(K) \subset \text{Ann}_R(M)$, a contradiction since $M$ satisfies (**).

(2) Let $K$ be a proper fully invariant submodule of $M$ and $r \in \text{Ann}_R(M/K)$. Since $R$ is commutative, multiplication on $M$ by $r$ (say $f_r$) is indeed an $R$-endomorphism of $M$. Therefore $(M)f_r \subseteq K$ and endo-coprimeness of $M$ implies that $f_r = 0$. Consequently $r \in \text{Ann}_R(M)$ as desired.

(3) Suppose that $pM$ is endo-coprime and fi-coretractable. Let $0 \neq IM \neq M$ for some left ideal $I$ of $R$. Since $R$ is fi-coretractable, there is $0 \neq f \in S$ such that $(IM)f = 0$, consequently $I(M)fS = 0$ which implies that $(M)fS \subseteq M$. Since
$RM$ is $fi$-coretractable, there is $0 \neq g \in S$ such that $fSg = 0$, contradicting the primeness of $S$.
(4) For proper fully invariant submodules $K, L$ of $M$, by $fi$-coretractibility we have
$\pi_K \circ \text{Hom}_R(M/K, M) \neq 0$, $\pi_L \circ \text{Hom}_R(M/L, M) \neq 0$ and
$$\pi_K \circ \text{Hom}_R(M/K, M) \circ \pi_L \circ \text{Hom}_R(M/L, M) \neq 0,$$
because $S$ is prime. Thus $K : M L \neq M$. Consequently $M$ is fully coprime. □

As a consequence we obtain:

**Corollary 2.6.** For $RM$ suppose that at least one of the following conditions hold:
(1) $RM$ is $fi$-coretractable and satisfies the (***) condition.
(2) $RM$ satisfies the (***) condition and $R$ is commutative.
Then the following statements are equivalent:
(a) $RM$ is coprime.
(b) $RM$ is endo-coprime.
(c) $RM$ is fully coprime.
(d) For every left ideal $I$ of $R$ either $IM = 0$ or $IM = M$.
(e) $\text{Ann}_R(M)$ is a prime ideal of $R$.

**Proof.** Suppose $RM$ is $fi$-coretractable with (***) condition. Then (a) ⇒ (b) by part (1) of Proposition 2.5.
(b) ⇒ (c). Since $S$ is prime, we can apply part (4) of 2.5.
(c) ⇒ (a) is trivial.
(b) ⇒ (d) by part (3) of Proposition 2.5, and (d) ⇒ (e) is clear.
(e) ⇒ (a) by [11, Lemma 1.3.3 part (2)].
We now assume (2). It is easy to see that $RM$ is $fi$-coretractable and so the conditions are equivalent for $RM$ by the previous case. □

Now we offer a short outline of basic facts about coalgebras and comodules referring to Brzeziński and Wisbauer [2] for details, and then we will state Proposition 2.5 and Corollary 2.6 for comodules and coalgebras (as in Jara et al. [6], Nekooei and Torkzadeh [8], Wijayanti [11] and Xu et al. [14]).

Let $C$ be a coalgebra over a commutative ring $R$ with counit $\varepsilon: C \to R$. Its dual algebra is $C^* = \text{Hom}_R(C, R)$ with multiplication defined as
$$(f \ast g)(c) = \mu \circ (f \otimes g) \circ \Delta(c) = \sum f(e_1)g(e_2)$$
for any $c \in C$ and $f, g \in C^*$, where $\Delta(c) = \sum e_1 \otimes e_2$ (Sweedler’s notation).
A right $C$-comodule is an $R$-module $M$ with an $R$-linear map $g^M: M \to M \otimes_R C$
called a right $C$-coaction, with the properties
$$(I_M \otimes \Delta) \circ g^M = (g^M \otimes I_C) \circ g^M \text{ and } (I_M \otimes \varepsilon) \circ g^M = I_M.$$ Denote by $\text{Hom}_C^R(M, N)$ the set of $C$-comodule morphisms from $M$ to $N$. The class of right comodules over $C$ together with the comodule morphisms form an additive category, which is denoted by $M_C$.
Similarly to the classical Hom-tensor relations, there are Hom-tensor relations in $M_C$ (see [2]). For any $M \in M_C$ and $X \in M_R$, there is an $R$-linear isomorphism
$$\phi: \text{Hom}_C^R(M, X \otimes_R C) \to \text{Hom}_R(M, X), \ f \mapsto (I_X \otimes \varepsilon) \circ f,$$
For $X = R$ and $M = C$ the map $\phi$ yields an algebra (anti-)isomorphism $\text{End}^C(C) \simeq C^*$. 
Any $M \in M^C$ is a (unital) left $C^*$-module by
$$\rightarrow: C^* \otimes_R M \rightarrow M, \quad f \otimes m \mapsto (I_M \otimes f) \circ \phi^M(m),$$
and any morphism $h: M \rightarrow N$ in $M^C$ is a left $C^*$-module morphism, i.e.,
$$\text{Hom}^C(M, N) \subset C^* \text{Hom}(M, N).$$
$C$ is a subgenerator in $M^C$, that is all $C$-comodules are subgenerated by $C$ as $C$-comodules and $C^*$-modules. Thus we have a faithful functor from $M^C$ to $C^* M$, where the latter denotes the category of left $C^*$-modules.
$C$ satisfies the $\alpha$-condition if the following map is injective for every $N \in M_R$,
$$\alpha_N : N \otimes_R C \rightarrow \text{Hom}_R(C^*, N), \quad n \otimes c \mapsto [f \mapsto f(c)n].$$
$M^C$ is a full subcategory of $C^* M$ if and only if $C$ satisfies the $\alpha$-condition, and then $M^C$ is isomorphic to $\sigma[C^*, C]$. 
Throughout, $C$ will be an $R$-coalgebra which satisfies the $\alpha$-condition.

For the following definition and more information on the topic, see [11].

**Definition 2.7.** Recall that a right $C$-comodule $M$ with $S = \text{End}_C(M)$ is
- coprime, fully coprime and strongly coprime if $M$ is coprime, fully coprime and strongly coprime as a left $C^*$-module, respectively.
- endo-coprime if for any proper fully invariant subcomodule $K$ of $M$, $\text{Ann}_S(M/K) = 0$.

**Proposition 2.8.** Let $M$ be a right $C$-comodule such that at least one of the following conditions hold on $M$:
1. $M$ is fi-coretractable and satisfies the (**) condition as a left $C^*$-module.
2. $M$ satisfies the (**) condition and $C^*$ is a commutative algebra.
Then the following statements are equivalent:
(a) $M$ is a coprime comodule.
(b) $M$ is an endo-coprime comodule.
(c) $M$ is a fully coprime comodule.
(d) For every left ideal $I$ of $C^*$ either $I \hookrightarrow M = 0$ or $I \hookrightarrow M = M$.
(e) $\text{Ann}_{C^*}(M)$ is a prime ideal of $C^*$.

**Proof.** By Corollary 2.6. $\Box$

For $M = C$, the assertions in 2.8 yield the following theorem that is a part of the main result 2.11.7 of [11], that concluded from [11, 1.7.11]. But [11, 1.7.11 part (i)] has an incorrect proof, because in its proof for finitely generated right ideals $I, J \subseteq \text{End}_R(M)$, submodules $K = \text{Ker} I$ and $L = \text{Ker} J$ are not fully invariant. If we consider $I$ and $J$ as finitely generated left or two sided ideals, then we cannot apply [11, 1.1.9 part (2)].

**Theorem 2.9.** If $C$ is a coalgebra over a field $k$, then the following statements are equivalent:
(a) $C$ is coprime as a right $C$-comodule.
(b) $C$ is coprime as a left $C$-comodule.
(c) $C$ is endo-coprime as a right $C$-comodule.
(d) $C$ is fully coprime as a right $C$-comodule.
For every left ideal $I$ of $C^*$ either $I \twoheadrightarrow C = 0$ or $I \rightarrowtail C = C$.

(f) $C^*$ is a prime algebra.

Proof. Over a field, $C$ is self-cogenerator and so $fi$-coretractable. According to [11, Lemma 2.2.9], $C$ satisfies condition $(**)$ as a left $C^*$-module. Also by [11, 2.6.3 part (i)], $C$ is coprime as a left $C$-comodule if and only if $C$ is endo-coprime. On the other hand $C$ is a faithful $C^*$-module (see [2, 4.6 part (2)]).

The following Proposition shows that a self-injective coprime module (in the sense of Bican et.al. [1]) is endo-coprime.

**Proposition 2.10.** Let $RM$ be self-injective. If every non-zero factor module of $M$ generates $M$, then $\text{End}_R(M)$ is a prime ring and $RM$ is endo-coprime.

Proof. Let $f \in S := \text{End}_R(M)$ and $I := fS$. By assumption $M$ is generated by $M/\text{Ker} I$, i.e., there is a short exact sequence $(M/\text{Ker} I)^{(A)} \rightarrowtail M \rightarrow 0$. Applying $\text{Hom}_R(-, M)$ to this exact sequence yields the commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_R(M, M) \\
\downarrow & & \downarrow \cong \\
0 & \longrightarrow & S \\
\end{array}
$$

since $\text{Hom}_R((M/\text{Ker} I)^{(A)}, M) \cong \text{Hom}_R(M/\text{Ker} I, M)^A$ and, by injectivity of $M$, $\pi_{\text{Ker} I} \circ \text{Hom}_R(M/\text{Ker} I, M) = I$ (see [12, 28.1 part (4)]). Thus $I$ is a faithful right $S$-module. Consequently $\text{End}_R(M)$ is a prime ring.

By assumption, $\text{Tr}(M/K, M) = M$ for all proper submodule $K \subset M$. Thus $\text{Hom}_R(M/K, M) \neq 0$, i.e., $M$ is coretractable. Then by [11, 1.5.2 part (iii)], $M$ is endo-coprime.

Let $N$ be a fully invariant submodule of $M$ and $f \in \text{End}_R(M)$, then we define $\overline{f} \in \text{End}_R(M/N)$ with $(m + N)\overline{f} = (m)f + N$ (or $f\pi_N = \pi_N\overline{f}$, where $\pi_N$ is the canonical projection $M \rightarrow M/N$).

We shall need the following Proposition that is a slight modification of 1.5.3 of [11].

**Lemma 2.11.** Let $M$ be an $R$-module and $N$ be a proper fully invariant submodule of $M$. Consider the ring homomorphism $\psi : \text{End}_R(M) \rightarrow \text{End}_R(M/N)$ with $f \mapsto \psi(f) := \overline{f}$, where $\overline{f}$ defined in the above statement.

(1) If $M$ is endo-coprime or $M$ is copolyform and $N \ll M$, then $\psi$ is injective.

(2) If $M$ is self-projective, then $\psi$ is surjective.

Proof. We only need to show that if $M$ is copolyform and $N \ll M$, then $\psi$ is injective, the other statements are proved in [11, 1.5.3]. Suppose that $\psi(f) = \overline{f} = 0$. Then $(M)f \subseteq N$, so $(M)f \ll M$. Since $M$ is copolyform, $(M)f$ is $M$-corational. Consequently $f \in \text{Hom}_R(M, (M)f) = 0$.

Further properties of endo-coprime modules are collected in the following.

**Proposition 2.12.** Let $RM$ be an endo-coprime module. Then the following statements hold:

(e) For every left ideal $I$ of $C^*$ either $I \twoheadrightarrow C = 0$ or $I \rightarrowtail C = C$.

(f) $C^*$ is a prime algebra.
(1) $M$ cannot have a nontrivial fully invariant submodule which is a direct summand. Consequently $R/\text{Ann}_R(M)$ has no nontrivial central idempotents.

(2) If $M = K \oplus L$ for non-zero submodules $K$ and $L$ of $M$, then $\text{Hom}_R(K, L) \neq 0$.

(3) If $M$ is coretractable, then any proper fully invariant submodule $N \subset M$ is superfluous and $M/N$ is coretractable.

(4) For any proper fully invariant submodule $N \subset M$, $M/N$ is endo-coprime provided that $M$ is self-projective.

(5) $M$ is Dedekind finite if and only if there exists a proper fully invariant submodule $N \subset M$ for which $M/N$ is Dedekind finite.

(6) If $M$ is semi-projective, then the center of $S = \text{End}_R(M)$ is a field.

Proof. (1) Since $M$ is endo-coprime, then $S = \text{End}_R(M)$ is a prime ring. Similar to the proof of [5, Proposition 1.9 part (1)], $M$ cannot have a fully invariant submodule which is a direct summand. Let $e$ be a nontrivial central idempotent element of $R/\text{Ann}_R(M)$. Thus $M$ can be decomposed to $M = eM \oplus (1 - e)M$. Then $eM = 0$ or $(1 - e)M = 0$. Consequently $e = 0$ or $e = 1$, since $M$ is a faithful $R/\text{Ann}_R(M)$-module.

(2) Assume that $M = K \oplus L$ for non-zero submodules $K$ and $L$ of $M$ such that $\text{Hom}_R(K, L) = 0$. Then $\text{End}_R(M) = \begin{bmatrix} \text{End}_R(K) & \text{Hom}_R(L, K) \\ 0 & \text{End}_R(L) \end{bmatrix}$. It follows that $(0 \oplus L) | \text{End}_R(M) \subseteq (0 \oplus L)$. Thus $(0 \oplus L)$ is a fully invariant submodule of $M$. By part (1), $K = 0$ or $L = 0$, a contradiction.

(3) Suppose $N$ is not superfluous in $M$ and $N + K = M$ for some proper submodule $K$. Since $M$ is coretractable, there exists a non-zero $f \in S = \text{End}_R(M)$ such that $(K)f = 0$. Then $(M)f \subseteq N$, which contradicts the assumption that $M$ is endo-coprime. Now to show that $M/N$ is coretractable, let $K/N$ be a proper submodule of $M/N$. Then there exists $0 \neq f \in S = \text{End}_R(M)$ with $(K)f = 0$. Now define the map $g \in \text{End}_R(M/N)$ by $(m + N)g = (m)f + N$. If $g = 0$, then $(M)f \subseteq N$, which contradicts the assumption that $M$ is endo-coprime. On the other hand $(K/N)g = 0$. Consequently $M/N$ is coretractable.

(4) Let $K/N$ be a proper fully invariant submodule of $M/N$. Then by [11, Corollary 1.1.21], $K$ is a proper fully invariant submodule of $M$. Suppose there exists $\overline{f} \in \text{End}_R(M/N)$ with $(M/N)\overline{f} \subset K/N$. By Lemma 2.11 there is $f \in \text{End}_R(M)$ with $f\pi_N = \pi_N \overline{f}$. Thus $(M)f \subseteq K$, which implies $f = 0$, because $M$ is endo-coprime. Consequently $\overline{f} = 0$.

(5) The necessity is clear. Conversely, let $N \subset M$ be a proper fully invariant submodule which $M/N$ is Dedekind finite. Since $M$ is endo-coprime, $\text{End}_R(M)$ is isomorphic to a subring of $\text{End}_R(M/N)$, by Lemma 2.11. The remainder of proof is similar to [5, Proposition 1.9 part (4)].

(6) Suppose that $f$ is a non-zero central element of $S$. Then $(M)f$ is a fully invariant submodule of $M$. If $(M)f \neq M$, then $f \in \text{Ann}_S(M/(M)f) = 0$, which contradicts our assumption. Thus $(M)f = M$, and because $M$ is semi-projective, we have $S = Sf$ which means that $f$ is invertible.

$\square$

Corollary 2.13. Let $R M$ be self-projective copolyform with $N$ a superfluous fully invariant submodule. Then $R M$ is endo-coprime if and only if so is $M/N$.

Proof. $(\Rightarrow)$. By part (3) of Proposition 2.12.

$(\Leftarrow)$. Let $M/N$ be endo-coprime and $K$ be a proper fully invariant submodule of $M$, with $(M)f \subseteq K$ for some $f \in \text{End}_R(M)$. Then $K + N \neq M$, and $(K + N)/N$ is
fully invariant in $M/N$ by [11, Lemma 1.1.20 part (ii)]. We have $(M)f \subseteq K + N$, hence $(M/N)f \subseteq (K + N)/N$ for $f$ which defined in prior of Lemma 2.11. Since $M/N$ is an endo-coprime module we must have $f = 0$. But then again by Lemma 2.11 we deduce that $f = 0$. □

Remark 2.14. A direct sum of endo-coprime modules need not be endo-coprime. To see this we recall that the endomorphism rings of the quasi-cyclic group $\mathbb{Z}/\mathbb{Z}$ and the group of $p$-adic integers $\mathbb{Q}_p^*$ are isomorphic commutative domains. On the other hand $\mathbb{Z}/\mathbb{Z}$ is a self-cogenerator $\mathbb{Z}$-module, so by [11, 1.5.2 part (iii)], $\mathbb{Z}/\mathbb{Z}$ is endo-coprime. But $\text{End}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \simeq \prod_p \mathbb{Q}_p^*$ is not a prime ring, thus $\mathbb{Q}/\mathbb{Z}$ is not an endo-coprime $\mathbb{Z}$-module.

A non-zero module is called **endo-simple** if it has no nontrivial fully invariant submodules.

**Lemma 2.15.** Let $R_M$ be a projective module in $\sigma[M]$. Then the following statements are equivalent:

(a) $M$ is fully coprime.

(b) $M$ is strongly coprime.

(c) $M$ is endo-simple.

Proof. (a) $\iff$ (c). See [9, Corollary 4.6], but notice that two concepts coprime modules in [9] and fully coprime modules in [11] are coincide.

(b) $\iff$ (c). Assume that $M$ is projective in $\sigma[M]$. Then for any non-zero fully invariant submodule $U$ of $M$, $M \not\subseteq \sigma[M/U]$ (see [10, Lemma 2.8]). In this case if $M$ is also strongly coprime, then $M$ has no nontrivial fully invariant submodules. □

The following result generalizes [11, 1.7.4].

**Corollary 2.16.** The following statements are equivalent for the ring $R$:

(a) $R$ is coprime.

(b) $R$ is endo-coprime.

(c) $R$ is fully coprime.

(d) $R$ is strongly coprime.

(e) $R$ is a simple ring.

**Corollary 2.17.** Let $R_M$ be projective in $\sigma[M]$. If $M$ is strongly coprime or fully coprime, then $M$ is copolyform.

Proof. By Lemma 2.15 and [11, 1.9.15]. □

Endo-coprimeness is preserved under isomorphism. We further have:

**Proposition 2.18.** Being endo-coprime is a Morita invariant property.

Proof. Assume that $A$ and $B$ are Morita equivalent rings with inverse category equivalences $\alpha : A\text{-Mod} \to B\text{-Mod}$ and $\beta : B\text{-Mod} \to A\text{-Mod}$. Let $M$ be an endo-coprime $A$-module, we want to show that $(M)\alpha$ is an endo-coprime $B$-module. Suppose that $N$ is a proper fully invariant submodule of $(M)\alpha$ and $h \in \text{End}_B((M)\alpha)$ such that $((M)\alpha)h \subseteq N$. Let $i$ denotes the inclusion map from $N$ to $(M)\alpha$, $g := (i)\beta$ and $N' := (N)\beta$. We have $(M)\alpha\beta(h)\beta \subseteq (N')g$. Since $(N')g$ is a proper fully invariant submodule of $(M)\alpha\beta$, by the endo-coprimeness of the latter, we deduce that $(h)\beta = 0$, and consequently $h = 0$. This shows that $(M)\alpha$ is an endo-coprime $B$-module. □
The following proposition is a generalization of [5, 1.15].

**Proposition 2.19.** The following statements are equivalent on a module $RM$:
(a) $RM$ is homogeneous semisimple.
(b) Each non-zero element of $\sigma[M]$ is an endo-prime module.
(c) Each non-zero element of $\sigma[M]$ is an endo-coprime module.
(d) Each non-zero element of $\sigma[M]$ is an endo-simple module.

**Proof.** (a) $\Leftrightarrow$ (b). [5, Proposition 1.15].
(d) $\Rightarrow$ (b) and (d) $\Rightarrow$ (c) are obvious.
(b) $\Rightarrow$ (d). Let $N \in \sigma[M]$ with $S = \text{End}_R(N)$ and $K$ be a nontrivial fully invariant submodule of $N$. Since $(N/K) \bigoplus N$ is endo-prime, $\text{End}_R((N/K) \bigoplus N)$ is a prime ring and so $\text{Ann}_S(K) = \pi_K \circ \text{Hom}_R(N/K, N) \neq 0$, that contradicts endo-primeness of $N$.
(c) $\Rightarrow$ (d). Let $N \in \sigma[M]$ with $S = \text{End}_R(N)$ and $K$ be a nontrivial fully invariant submodule of $N$. Since $\text{End}_R(N \bigoplus K)$ is a prime ring, $\text{Ann}_S(N/K) = \text{Hom}_R(N, K) \neq 0$, which is a contradiction with our hypothesis. □

**Proposition 2.20.** The following statements are equivalent on a ring $R$:
(a) $R$ is a simple Artinian ring.
(b) Each non-zero left $R$-module is endo-prime.
(c) Each non-zero finitely generated left $R$-module is endo-prime.
(d) Each non-zero left $R$-module is endo-coprime.
(e) Each non-zero finitely generated left $R$-module is endo-coprime.
(f) Each non-zero left $R$-module is endo-simple.

**Proof.** By 2.19 the statements (a), (b), (d) and (f) are equivalent.
(b) $\Rightarrow$ (c) and (d) $\Rightarrow$ (e) are trivial.
(c) $\Rightarrow$ (a). See [5, Proposition 1.16].
(e) $\Rightarrow$ (a). Clearly $R$ must be prime ring. For any simple module $RN$ and proper left ideal $I$ of $R$, since $N \bigoplus R$ and $R/I \bigoplus R$ are endo-coprime, their endomorphism rings are prime and consequently $\text{Hom}_R(N, R)$ and $\text{Hom}_R(R/I, R)$ cannot be zero. The reminder of the proof is similar to [5, Proposition 1.16]. □

3. Duality for Endo-Prime and Endo-Coprime Modules

In this section, we shall investigate $R$-duals. If $M$ is an $R$-module, then the $R$-dual of $M$ is $\text{Hom}_R(M, R)$, and this will be denoted by $M^*$.

Notice that $M^*$ is a right $R$-module and so its endomorphism ring $\text{End}_R(M^*)$ will act on the left side of $M^*$. The double dual of $M$, that is, $(M^*)^*$, will be denoted by $M^{**}$. If the natural map $\iota: M \rightarrow M^{**}$ given by

$$[(m)\iota](f) = (m)f \ (m \in M, f \in M^*),$$

is injective, $M$ will be called torsionless. It is clear that $M$ is torsionless if and only if the reject $\text{Rej}(M, R)$ is zero. A torsionless module $RM$ is said to be reflexive if the injection map $\iota$ is in fact an isomorphism (of left $R$-modules). For any submodule $K \subseteq M$, put $K^\perp := \{f \in M^* \mid (K)f = 0\}$, which is a submodule of $M^*$. Similarly, for any submodule $I \subseteq M^*$, put $I^\perp := \bigcap\{\ker f \mid f \in I\}$ (a submodule of $M$). We recall that, for any $f \in \text{End}_R(M)$, the map $f^* : M^* \rightarrow M^*$ given by $f^*(g) = fg$ is a right $R$-module homomorphism. If $M$ is a reflexive $R$-module, then the mapping $()^* : \text{End}_R(M) \rightarrow \text{End}_R(M^*)$ is a ring anti-isomorphism (see [4, Proposition 2.2]).

We recall from [7, Theorem 15.11] that, over quasi-Frobenius (QF) ring $R$ any
finitely generated module $R M$ is reflexive. Furthermore, under this assumption, for any submodules $K \subseteq M$ and $I \subseteq M^*$ we have $K^{\perp \perp} = K$ and $I^{\perp \perp} = I$. The reader is referred to Lam [7] and Wisbauer [12].

**Theorem 3.1.** Let $M$ be a finitely generated left module over QF ring $R$. Then the following statements hold:

1. $R M$ is endo-prime if and only if $M^*$ is endo-coprime (as a right $R$-module).
2. $R M$ is endo-coprime if and only if $M^*$ is endo-prime (as a right $R$-module).

**Proof.** (1) ($\Rightarrow$). Let $I$ be a proper fully invariant submodule of $M^*$. If $I^{\perp} = 0$, then $I = I^{\perp \perp} = M^*$, which contradicts our assumption. So $I^{\perp}$ is a non-zero fully invariant submodule of $M$. Let $f^* \in \text{End}_R(M^*)$ with $f^*(M^*) \subseteq I$. Then for any $g \in M^*$, we have $f g = f^*(g) \in I$. Hence $(I^{\perp}) f \subseteq \text{Rej}(M, R) = 0$. Consequently from endo-prime properties of $M$ we deduce that $f = 0$ and so $f^* = 0$.

($\Leftarrow$). Let $K$ be a non-zero fully invariant submodule of $M$. Then for any $f^* \in \text{End}_R(M^*)$ and any $g \in K^{\perp}$ we have $(K) f^*(g) = (K) f g \subseteq (K) g = 0$. Thus $K^{\perp}$ is a proper fully invariant submodule of $M^*$. Now let $h \in \text{End}_R(M)$ with $(K) h = 0$. Then for any $g \in M^*$, $(K) h^*(g) = (K) h g = 0$, i.e., $h^*(M^*) \subseteq K^{\perp}$. Consequently endo-coprime properties of $M^*$ implies that $h^* = 0$. Thus $\text{Im} h \subseteq \text{Rej}(M, R) = 0$.

(2) ($\Rightarrow$). Let $I$ be a non-zero fully invariant submodule of $M^*$ and $f^* \in \text{End}_R(M^*)$ such that $f^*(I) = 0$. Then for any $g \in I$, $(M) f \subseteq \text{Ker} g$, i.e., $(M) f \subseteq I^{\perp}$. Thus $f = 0$, because $M$ is endo-coprime and $I^{\perp}$ is a proper fully invariant submodule of $M$.

($\Leftarrow$). Now assume that $M^*$ is endo-prime and $K$ is a proper fully invariant submodule of $M$. If $K^{\perp} = 0$, then $K = K^{\perp \perp} = M$, a contradiction. Hence $K^{\perp}$ is a non-zero fully invariant submodule of $M^*$. Let $f \in \text{End}_R(M)$ with $(M) f \subseteq K$, then for any $g \in K^{\perp}$, $(M) f^*(g) = (M) f g \subseteq (K) g = 0$. Therefore $f^*(K^{\perp}) = 0$ and so $f^* = 0$, because $M^*$ is endo-prime. Since $M$ is torsionless $f$ must be zero. □

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