On the Duality Gap Convergence of ADMM Methods

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Abstract

This paper provides a duality gap convergence analysis for the standard ADMM as well as a linearized version of ADMM. It is shown that under appropriate conditions, both methods achieve linear convergence. However, the standard ADMM achieves a faster accelerated convergence rate than that of the linearized ADMM. A simple numerical example is used to illustrate the difference in convergence behavior.

1 Introduction

This paper considers the following optimization problem:

\[
\min_{w,v} \left( \phi(w) + g(v) \right)
\]

subject to \( Aw - Bv = c, \) \hspace{1cm} (1)

where \((w, v) \in \mathbb{R}^n \times \mathbb{R}^m\) are unknown vectors, \( A \in \mathbb{R}^{p \times n}, B \in \mathbb{R}^{p \times m} \) and \( c \in \mathbb{R}^p \) are known matrices and vector. In this paper, we assume that \( \phi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\} \) are convex functions.

A popular method for solving (1) is the Alternating Direction Method of Multipliers (ADMM) algorithm. It solves the problem by alternatively optimizing the variables in the Augmented Lagrangian function:

\[
L(w, v, \alpha, \rho) = \phi(w) + g(v) + \alpha^T(Aw - Bv - c) + \frac{\rho}{2} \|Aw - Bv - c\|^2; \hspace{1cm} (2)
\]

and the resulting procedure is summarized in Algorithm 1. In the algorithm, both \(G\) and \(H\) are symmetric positive semi-definite matrices. In the standard ADMM, we can set \(G = 0\) and \(H = 0\). The method of introducing the additional term \(\|v - v^{t-1}\|_G^2 = (v - v^{t-1})^T G (v - v^{t-1})\) is often referred to as preconditioning. If we let \(G = \beta I - B^T B\) for a sufficiently large \(\beta > 0\) such that \(G\) is positive semi-definite, then the minimization problem to obtain \(v^t\) in line 3 of Algorithm 1 becomes:

\[
v^t = \arg \min_v \left[ g(v) - (\alpha^{t-1} + \rho B^T A w^{t-1} + \rho G v^{t-1})^T v + \frac{\rho \beta}{2} v^T v \right], \hspace{1cm}
\]

which may be simpler to solve than the corresponding problem with \(G = 0\), since the original quadratic term \(v^T B^T Bv\) is now replaced by \(v^T v\). The additional term \(\|w - w^{t-1}\|_H^2\) can play a similar role of preconditioning.
Section 5. Theoretical analysis for both standard and linearized ADMM. Section 4 provides a simple numerical

\[ \phi \] strongly convex, and \( / \) linearized ADMM algorithm for solving (1). Under the assumption that

\[ g / \] linear convergence rate of \( 1 / \sqrt{\lambda \gamma} \) (with optimally chosen \( \rho \)) while the linearized ADMM achieves a slower worst case linear convergence rate of \( 1 / (1 + \Theta(\lambda \gamma)) \).

The paper is organized as follows. Section 2 reviews related work. Section 3 provides a theoretical analysis for both standard and linearized ADMM. Section 4 provides a simple numerical example to illustrate the difference in convergence behavior. Concluding remarks are given in Section 5.

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**Algorithm 1** Preconditioned Standard ADMM Algorithm

1: Choose \( w^0, v^0, \) and \( \alpha^0 \)
2: for \( t = 1, 2, \ldots \) do
3: \( v^t = \arg \min_v [g(v) - \alpha^{t-1}^T Bv + \frac{\rho}{2} \| Av^{t-1} - Bv - c \|^2_2 + \frac{\rho}{2} \| v - v^{t-1} \|^2_G]; \)
4: \( w^t = \arg \min_w [\phi(w) + \alpha^{t-1}^T Aw + \frac{\rho}{2} \| Aw - Bv^t - c \|^2_2 + \frac{\rho}{2} \| w - w^{t-1} \|^2_H]; \)
5: \( \alpha^t = \alpha^{t-1} + \rho(Aw^t - Bv^t - c); \)
6: end for
7: Output: \( w^t, v^t, \alpha^t. \)

---

**Algorithm 2** Preconditioned Linearized ADMM Algorithm

1: Choose \( w^0, v^0, \) and \( \alpha^0 \)
2: for \( t = 1, 2, \ldots \) do
3: \( v^t = \arg \min_v [g(v) - \alpha^{t-1}^T Bv + \frac{\rho}{2} \| Av^{t-1} - Bv - c \|^2_2 + \frac{\rho}{2} \| v - v^{t-1} \|^2_G]; \)
4: \( w^t = \arg \min_w [\phi_H(w^{t-1}, w) + \alpha^{t-1}^T Aw + \frac{\rho}{2} \| Aw - Bv^t - c \|^2_2]; \)
5: \( \alpha^t = \alpha^{t-1} + \rho(Aw^t - Bv^t - c); \)
6: end for
7: Output: \( w^t, v^t, \alpha^t. \)

For simplicity, this paper focuses on the scenario that \( g(\cdot) \) is strongly convex, and \( \phi(\cdot) \) is smooth. The results allow \( g(\cdot) \) to include a constraint \( v \in \Omega \) for a convex set \( \Omega \) by setting \( g(v) = +\infty \) when \( v \notin \Omega \). The same proof technique can also handle other three cases with one objective function being smooth and one being strongly convex.

The standard ADMM algorithm assumes that the optimization problem to obtain \( w^t \) is simple. If this optimization is difficult to perform, then we may also consider the linearized ADMM formulation which replaces \( \phi(w) \) by a quadratic approximation \( \phi_H(w) \) defined as

\[ \phi_H(w^{t-1}; w) = \phi(w^{t-1}) + \nabla \phi(w^{t-1})^T( w - w^{t-1}) + \frac{1}{2} (w - w^{t-1})^T H(w - w^{t-1}). \]

The resulting algorithm is described in Algorithm 2. Both \( H \) and \( G \) are symmetric positive semi-definite matrices. By setting \( H = \beta I - \rho A^T A \), we can replace the term \( w^T A^T A w \) by \( w^T w \) in the optimization of line 4 of Algorithm 2.


2 Related Work on ADMM and Linearized ADMM

In this section, we review some previous work on the convergence analysis of ADMM and Linearized ADMM, focusing mainly on linear convergence results.

2.1 Results for ADMM

Many authors have studied the linear convergence of ADMM in recent years. For example, the authors in [6] presented a novel proof for the linear convergence of the ADMM algorithm. Moreover, the analysis applies for the more general case in which the objective function can be the summation of more than two separable functions (\(\phi\) and \(g\) in our case). However, the assumption on each separable function is very complex, and no explicit rate is obtained. Therefore their results are not directly comparable to ours.

Another work is [4], which presented analysis for the linear convergence of generalized ADMM under certain conditions. More comprehensive results for the general form of constraint \(Aw - Bv = c\) were obtained later in [3] using similar ideas. In that paper, they presented an extension of ADMM algorithm called Relaxed ADMM, which leads to linear convergence in the following four cases (it also requires either \(A\) or \(B\) are invertible): \(\phi\) is strongly convex and smooth; \(g\) is strongly convex, and smooth; \(\phi\) is smooth, and \(g\) is strongly convex; \(g\) is smooth, and \(\phi\) is strongly convex. However, their analysis employs a technique for analyzing the dual objective of ADMM that may be regarded as a Relaxed Peacheman-Rachford splitting method. It can be used to prove the dual convergence. In contrast, our analysis uses a very different argument that can directly bound the convergence of primal objective function and the duality gap. Moreover, even when the required regularity conditions for linear convergence are not satisfied, our analysis immediately implies a sublinear \(1/t\) convergence of duality gap (assuming a finite solution exists for the underlying problem). Therefore the analysis of this paper contains a unified treatment that can simultaneously handle both linear and sublinear convergence depending on the regularity condition. In contrast, although sublinear results can be obtained using techniques similar to those of [3] (see results in [2]), they require specialized treatment and the obtained results are in different forms that are not compatible with the duality gap convergence of this paper. In this setting, the operator splitting proof techniques of [3, 2] and the objective function proof technique of this paper are complementary to each other.

Another advantage of our proof technique is that it can be directly applied to linearized ADMM with minimal modifications.

Our analysis employs a technique similar to that of [10] (note that neither linear convergence nor duality gap convergence was studied in [10]). At the conceptual level, the technique is also closely related to the analysis of [1], but the actual execution differs quite significantly. One may view the analysis of this paper as a refined version of those in [10], in that we simultaneously handle linear and sublinear cases depending on regularity conditions. Moreover, our analysis unifies the techniques used in [10] (which deals with primal objective convergence) and the techniques used in [1] (which deals with a special primal-dual objective convergence); our proof shows that the seemingly different results in these two papers can be proved using the same underlying argument. Although results similar to ours were presented in [1] for a procedure related to a specific form of preconditioned ADMM (see [1] for discussions), they did not analyze the standard ADMM (or its linearized version) under the general condition \(Aw - Bv = c\). Therefore results obtained in this paper for ADMM are different from those of [1].

Another result on the linear convergence of the standard ADMM can be found in a recent
paper [9], which uses a different technique than what’s presented in this paper and that of [4, 3]. Their results are not directly comparable to ours. Moreover, some other work on the convergence of ADMM like procedures include [7, 5, 12], which focused on different applications that are not related to our work.

### 2.2 Results for Linearized ADMM

One advantage of our proof technique is that it also handles linearized ADMM, with new results not available in the previous literature. Most of previous work on linearized ADMM does not consider linear convergence; a few that do consider impose strong assumptions on the matrices $A, B$, or the functions $f, g$.

There are several papers that considered linear convergence of Linearized ADMM. For example [6] considered linearized ADMM, but as mentioned earlier, their rate is not explicit and they impose complex conditions that are incompatible with our results. Similarly, a linear convergence result for linearized ADMM was also obtained in [5], but only under the assumption of $g = 0$ and some strong constraints on the matrices $A$ and $B$. Again their results are incompatible with ours.

Some other work considered Linearized ADMM in the general cases but without linear convergence. For example, in [11], the authors consider the convergence of Linearized ADMM on several different cases, and obtained sublinear convergence of $1/t$. Similar sublinear results can be found in [10] for stochastic ADMM. As we have pointed out, our proof technique is closely related to that of [10], which can handle both linearized and standard ADMM under the same theoretical framework.

### 3 Main Results

This section provides our main results for the standard ADMM and the linearized ADMM. We will derive upper bounds on their convergence rates, as well as the worst case matching lower bounds for some specific problems.

#### 3.1 Notations

Given any convex function $h$, we may define its convex conjugate

$$h^*(\beta) = \sup_u [\beta^T u - h(u)],$$

and define the Bregman divergence of a convex function $h(u)$ as:

$$D_h(u', u) = h(u) - h(u') - \nabla h(u')^T (u - u').$$

We will assume that $\phi$ is $1/\gamma$ smooth:

$$\forall w, w', \quad D_\phi(w', w) \leq \frac{1}{2\gamma} \|w' - w\|^2,$$

which also implies that

$$D_\phi(w', w) \geq \frac{\gamma}{2} \|\nabla \phi(w') - \nabla \phi(w)\|^2, \quad D_{\phi^*}(u', u) \geq \frac{\gamma}{2} \|u' - u\|^2.$$


We also assume that \( g \) is \( \lambda \) strongly convex:

\[
\forall v, v', \quad D_g(v', v) \geq \frac{\lambda}{2} \|v' - v\|^2.
\]

Assume also that \((w_*, v_*, \alpha_*)\) is an optimal solution of \((\text{II})\), which satisfies the equality:

\[
Aw_* - Bv_* - c = 0, \quad A^\top \alpha_* = -\nabla \phi(w_*), \quad w_* = \nabla \phi^*(-A^\top \alpha_*), \quad B^\top \alpha_* = \nabla g(v_*).
\]

Given any \( \alpha \), taking inf over \((w, v)\) with respect to the Lagrangian

\[
\phi(w) + g(v) + \alpha^\top (Aw - Bv - c),
\]

we obtain the dual

\[
D(\alpha) = -\phi^*(-A^\top \alpha) - g^*(B^\top \alpha) - \alpha^\top c.
\]

It is clear by definition that for any pair \((w, v)\) that are feasible (that is \(Aw - Bv - c = 0\)), and any \( \alpha \), we have \( \phi(w) + g(v) \geq D(\alpha) \). The value \( \phi(w) + g(v) - D(\alpha) \) is referred to as the duality gap. Duality gap is always larger than primal suboptimality \([\phi(w) + g(v)] - [\phi(w_*) + g(v_*)]\). Therefore if the duality gap is zero, then \((w, v)\) solves \((\text{II})\).

We may also introduce the concept of restricted duality gap as in \((\text{II})\). Consider regions \( B_1 \subset \mathbb{R}^p \), and \( B_2 \subset \mathbb{R}^m \). Given any \( \hat{\alpha}, \hat{v} \), we can define the restricted duality gap

\[
\mathcal{G}_{B_1 \times B_2}(\hat{\alpha}, \hat{v}) = \sup_{\alpha \in B_1; v \in B_2} \left[ \phi^*(-A^\top \alpha) + g(v) - \phi^*(-A^\top \alpha) - g(v) + \hat{\alpha}^\top (Bv + c) - \alpha^\top (B\hat{v} + c) \right].
\]

If we pick \((\alpha, v) = (\alpha_*, v_*)\), then

\[
D_{\phi^*}(-A^\top \alpha_*, -A^\top \hat{\alpha}) + D_g(v_*, \hat{\alpha}) = \phi^*(-A^\top \hat{\alpha}) + g(\hat{v}) - \phi^*(-A^\top \alpha_*) - g(v_*) + \hat{\alpha}^\top (Bv_* + c) - \alpha_*^\top (B\hat{v} + c).
\]

Therefore as long as \((\alpha_*, v_*) \in B_1 \times B_2\), we have

\[
D_{\phi^*}(-A^\top \alpha_*, -A^\top \hat{\alpha}) + D_g(v_*, \hat{\alpha}) \leq \mathcal{G}_{B_1 \times B_2}(\hat{\alpha}, \hat{v}).
\]

Assume \(AA^\top\) is invertible, and let

\[
A^+ = A^\top (AA^\top)^{-1}
\]

be the pseudo-inverse of \(A\), then we may let \( \hat{w} = A^+(B\hat{v} + c) \). It follows that \( Aw - B\hat{v} - c = 0 \). If we set \( B_1 \times B_2 = \mathbb{R}^p \times \mathbb{R}^m \), then we recover the unrestricted duality gap:

\[
\mathcal{G}_{\mathbb{R}^p \times \mathbb{R}^m}(\hat{\alpha}, \hat{v}) = [\phi(\hat{w}) + g(\hat{v})] - D(\hat{\alpha}),
\]

where the maximum over \((\alpha, v)\) is taken at \(-A^\top \alpha = \nabla \phi(\hat{w})\) and \(v = \nabla g^*(B^\top \hat{\alpha})\).

### 3.2 Standard ADMM

In general, we have the following result.
Theorem 3.1 Assume that $\phi$ is $1/\gamma$ smooth and $g$ is $\lambda$ strongly convex. Assume that we can write $H = A^T \bar{H} A$. Let $\sigma_{\max}(H)$ and $\sigma_{\max}(\bar{H})$ be the largest eigenvalues of $H$ and $\bar{H}$ respectively, $\sigma_{\min}(A)$ be the smallest eigenvalue value of $(A A^T)^{1/2}$, $\sigma_{\max}(B)$ be the largest singular value of $B$, $\sigma_{\max}(G)$ be the largest singular value of $G$. Consider $s \in (0, 1)$ and $\theta > 0$ such that

$$
\theta \leq \min \left( \frac{\gamma \rho \sigma_{\min}(A)^2}{\gamma \sigma_{\max}(H) + 1}, \frac{s \rho}{\sigma_{\max}(\bar{H})}, \frac{(1-s)\lambda}{(\rho + \sigma_{\max}(H))\sigma_{\max}(B)^2 + (1-s)\rho \sigma_{\max}(G)} \right).
$$

Let $\tilde{\alpha}^t = \alpha^t + \bar{H}A(w^t - w^{t-1})$. Then for all $(\alpha, v)$ and $w = \nabla \phi^*(-A^T\alpha)$, Algorithm 1 produces approximate solutions that satisfy

$$
\sum_{t=1}^T (1 + \theta)^{t-T} r_t \leq (1 + \theta)^{-T} \delta_0 - \delta_T,
$$

and

$$
\sum_{t=1}^T (1 + \theta)^{t-T} r_t^* \leq (1 + \theta)^{-T} \delta_0 - \delta_T,
$$

where

$$
\begin{align*}
 r_t &= \phi(w^t) + g(v^t) - \phi(w) - g(v) - \tilde{\alpha}^T (Aw - Bv - c) + \alpha^T (Aw^t - Bv^t - c), \\
 r_t^* &= \phi^*(-A^T \tilde{\alpha}^t) + g(v^t) - \phi^*(-A^T \alpha) - g(v) + \tilde{\alpha}^T (Bv + c) - \alpha^T (Bv^t + c), \\
 \delta_t &= \frac{\rho}{2} \|Aw^t - Bv - c\|^2 + \frac{1}{2} \|Aw^t - Bv - c\|_H^2 + \frac{1}{2} \|v^t - v\|^2 + \frac{1}{2} \|\alpha - \alpha^t\|^2.
\end{align*}
$$

For arbitrary $(\alpha, v)$, the left hand side of (5) and (6) can be difficult to understand. We may choose specific values of $(\alpha, v)$ so that the results are easier to interpret. By setting $(\alpha, w, v) = (\alpha_s, w_s, v_s)$ in Theorem 3.1 and using (3), we obtain the following corollary.

Corollary 3.1 Under the conditions of Theorem 3.1 we have

$$
\begin{align*}
\sum_{t=1}^T (1 + \theta)^{t-T} &\left[ \max(D_\phi(w_s, w^t), D_{\phi^*}(-A^T \alpha_s, -A^T \tilde{\alpha}^t)) + D_g(v_s, v^t) \right] \\
&\leq \frac{\rho}{2} \|A(w^t - w_s)\|^2_2 + \frac{1}{2} \|A(w^t - w_s)\|_H^2 + \frac{1 + \theta}{2\rho} \|\alpha^T - \alpha_s\|_2^2 + \frac{\rho(1 + \theta)}{2} \|v^t - v_s\|^2_G \\
&\leq \frac{(1 + \theta)^{-T}}{2} \left[ \rho \|A(w_0 - w_s)\|_2^2 + \|A(w_0 - w_s)\|_H^2 + \frac{1 + \theta}{\rho} \|\alpha^0 - \alpha_s\|_2^2 + \rho(1 + \theta) \|v^0 - v_s\|^2_G \right].
\end{align*}
$$

Using the definition of restricted duality gap, it is easy to see that (6) directly implies an upper bound of restricted duality gap, which is the same style as results of [1]. Our result is more general than those of [1] because the results can also be expressed in the form of Corollary 3.1 as well as in terms of unrestricted duality gap, as stated below.
Corollary 3.2  Under the conditions of Theorem 3.1, and let $A^+$ be the pseudo-inverse of $A$. Define

$$\delta^0_* = \left[ (\rho + \sigma_{\text{max}}(\tilde{H})) ||A(w^0 - w_*)||^2 + \frac{1 + \theta}{\rho} ||\alpha^0 - \alpha_*||^2 + \rho (1 + \theta) ||v^0 - v_*||^2 \right]$$

$$b_1(\delta) = \sup_u \left\{ ||\alpha^0 + A^{-T} \nabla \phi(A^+(B_u + c))||^2 : ||u - v_*||^2 \leq \frac{\delta}{\rho(1 + \theta)} \right\}$$

$$b_2(\delta) = \sup_{\beta} \left\{ ||v^0 - \nabla g^*(B^T \beta)||^2 : ||\beta - \beta_*||^2 \leq \sqrt{\frac{\rho \delta}{1 + \theta} + \sigma_{\text{max}}(\tilde{H}) \sqrt{2(2 + \theta) \delta}} \right\}$$

$$b(\delta) = \frac{1 + \theta}{2\rho} b_1(\delta) + \frac{1 + \theta}{\sigma_{\text{max}}(G)} b_2(\delta) + (\rho + \sigma_{\text{max}}(\tilde{H}))(\sigma_{\text{max}}(B)^2 b_2(\delta) + ||Aw^0 - Bv^0 - c||^2).$$

Then we have the following bound in duality gap

$$[\phi(A^+(Bv^T + c)) + g(v^T)] - D(\bar{\alpha}^T) \leq (1 + \theta)^{-T} b((1 + \theta)^{-T} \delta^0_*).$$

Moreover, define

$$\bar{v}^T = \frac{\sum_{t=1}^T (1 + \theta)^t v^t}{\sum_{t=1}^T (1 + \theta)^t}, \quad \bar{\alpha}^T = \frac{\sum_{t=1}^T (1 + \theta)^t \bar{\alpha}^t}{\sum_{t=1}^T (1 + \theta)^t}.$$ 

Then

$$[\phi(A^+(B\bar{v}^T + c)) + g(\bar{v}^T)] - D(\bar{\alpha}^T) \leq \frac{b(\delta^0_*)}{\sum_{t=1}^T (1 + \theta)^t}.$$ 

In the above results, we consider the simple case of $H = 0$. Then the optimal value of $\theta$ is achieved when we take

$$\rho = \frac{\sqrt{\sigma_{\text{max}}(B)^2 + \sigma_{\text{max}}(G)}}{\sigma_{\text{min}}(A)} \sqrt{\frac{\lambda}{\gamma}}, \quad \theta = \sigma_{\text{min}}(A) \sqrt{\frac{(\sigma_{\text{max}}(B)^2 + \sigma_{\text{max}}(G)) \gamma \lambda}. $$

When $\theta > 0$, this implies the following convergence from Corollary 3.1

$$\max |D_\phi(w_*, w^T), D_\phi^*(-A^T \alpha_*, -A^T \bar{\alpha}^T)| + D_g(v_*, v^T)
+ \frac{\rho}{2} ||A(w^T - w_*)||^2 + \frac{1}{2\rho} ||\alpha^T - \alpha_*||^2 + \frac{\rho}{2} ||v^T - v_*||^2
\leq \left( 1 + \sigma_{\text{min}}(A) \sqrt{\frac{\sigma_{\text{max}}(B)^2 + \sigma_{\text{max}}(G)) \gamma \lambda}{1 - T} \right) \left[ \frac{\rho}{2} ||A(w^0 - w_*)||^2 + \frac{1}{2\rho} ||\alpha^0 - \alpha_*||^2 + \frac{\rho}{2} ||v^0 - v_*||^2 \right].$$

This implies $||w_* - w^T||_2 = O((1 + \theta)^{-T}), ||v_* - v^T||_2 = O((1 + \theta)^{-T}),$ and $||\alpha_* - \alpha^T||_2 = O((1 + \theta)^{-T})$.

The linear convergence result holds when $\theta > 0$. However, even when $\theta = 0$ (and $H \neq 0$), we can still obtain the following sublinear convergence from Corollary 3.1

$$\max \left[ \frac{1}{T} \sum_{t=1}^T D_\phi(w_*, w^T), \frac{1}{T} \sum_{t=1}^T D_\phi^*(-A^T \alpha_*, -A^T \bar{\alpha}^T) \right] + D_g(v_*, v^T)
\leq \frac{1}{2T} \left[ (\rho + \sigma_{\text{max}}(\tilde{H}))(||A(w^0 - w_*)||^2 + \frac{1}{\rho} ||\alpha^0 - \alpha_*||^2 + \rho ||v^0 - v_*||^2 \right] ,$$
where \( \bar{w}^T = T^{-1} \sum_{t=1}^{T} w^t \), \( \bar{v}^T = T^{-1} \sum_{t=1}^{T} v^t \). This result does not require any assumption on \( \phi \), \( g \), \( A \), \( B \).

Similar results hold for unrestricted duality gap under the conditions of Corollary 3.2. For example, when \( \theta = 0 \), but \( AA^\top \) is invertible, we obtain the sublinear convergence of duality-gap below.

\[
[\phi(A^+(B\bar{v}^T + c)) + g(\bar{v}^T)] - D(\tilde{\alpha}^T) \leq \frac{b(s^0)}{T}.
\]

This bound can be compared to the main result of [1] stated in terms of the restricted duality gap (in which the authors studied a method that is related to, but not identical to ADMM). Their result did not imply a bound on the unrestricted duality gap because they did not obtain a counterpart of Corollary 3.1.

In the case of \( \phi \) being smooth but \( g \) is not a strongly convex function, given any \( \epsilon > 0 \), we can set \( \lambda = \epsilon \), and apply ADMM with \( g(v) \) replaced by the strongly convex function \( g(v) + \lambda v^\top v \). With \( \rho \) chosen optimally, this leads to

\[
\phi(A^+(Bv^T + c)) + g(v^T) - D(\alpha^T) = O(\epsilon)
\]

when we take \( T = \ln(1/(\gamma\epsilon))/\sqrt{\gamma\epsilon} \).

### 3.3 Linearized ADMM

For Linearized ADMM, we have the following counterpart of Theorem 3.1. Here we need to assume that \( A \) is invertible and \( H \) is sufficiently large so that \( \sigma_{\min}(H) \geq \gamma^{-1} \).

**Theorem 3.2** Assume that \( \phi \) is \( 1/\gamma \) smooth and \( g \) is \( \lambda \) strongly convex, and \( A \) is a square invertible matrix. Assume that we can write \( H = A^\topHA \). Let \( \sigma_{\min}(H) \) and \( \sigma_{\max}(H) \) be the smallest eigenvalue of \( H \) and the largest eigenvalue of \( H \) respectively, and we assume that \( \sigma_{\min}(H) \geq \gamma^{-1} \). Let \( \sigma_{\min}(A) \) be the smallest eigenvalue value of \( (AA^\top)^{1/2} \), \( \sigma_{\max}(B) \) be the largest singular value of \( B \), \( \sigma_{\max}(G) \) be the largest singular value of \( G \). Consider \( s \in [0,1] \) and \( \theta > 0 \) such that

\[
\theta \leq \min \left( \frac{\rho \sigma_{\min}(A)^2}{\sigma_{\min}(H)}, \frac{s \rho}{\sigma_{\max}(H)}, \frac{(1-s)\lambda}{(\rho + \sigma_{\max}(H))\sigma_{\max}(B)^2 + (1-s)\rho \sigma_{\max}(G)} \right).
\]

Let \( \tilde{\alpha}^t = \alpha^t + \tilde{H}A(w^t - w^{t-1}) \). Then for any \( (\alpha, v) \), and \( w = \nabla \phi^*(-A^\top \alpha) \), Algorithm 2 produces approximate solutions that satisfy

\[
\sum_{t=1}^{T}(1 + \theta)^{t-T}r_t \leq (1 + \theta)^{-T}\delta_0 - \delta_T \tag{7}
\]

\[
\sum_{t=1}^{T}(1 + \theta)^{t-T}r^*_t \leq (1 + \theta)^{-T}\delta_0 - \delta_T \tag{8}
\]

where

\[
r_t = \phi(w^{t-1}) + g(v^t) - \phi(w) - g(v) - \tilde{\alpha}^T(Aw - Bv - c) + \alpha^T(Aw^{t-1} - Bv^t - c),
\]

\[
r^*_t = \phi^*(-A^\top \tilde{\alpha}^t) + g(v^t) - \phi^*(-A^\top \alpha) - g(v) + \tilde{\alpha}^T(Bv + c) - \alpha^T(Bv^t + c),
\]

\[
\delta_t = \frac{1}{2} \left[ \|Aw^t - Bv - c\|^2_H + \rho \|Aw^t - Bv - c\|^2 + \rho(1 + \theta)\|v^t - v\|^2_G + \frac{1 + \theta}{\rho} \|\alpha^t - \alpha\|^2 \right].
\]
Similar to the corollaries of Theorem 3.1, we have the following three corollaries of Theorem 3.2.

**Corollary 3.3** Under the conditions of Theorem 3.2, we have

\[
\sum_{t=1}^{T} (1 + \theta)^{-T} \left[ \max(D_\phi(w_s, w^{t-1}), D_\phi^*(A^T \alpha_s, -A^T \bar{\alpha}^t)) + D_g(v_s, v^t) \right]
\]

\[+ \frac{1}{2} \|w^T - w_s\|^2_H + \frac{\rho}{2} \|A(w^T - w_s)\|^2_2 + \frac{1 + \theta}{2\rho} \|\alpha^T - \alpha_s\|^2_2 + \frac{\rho(1 + \theta)}{2} \|v^T - v_s\|^2_G \]

\[\leq \frac{(1 + \theta)^{-T}}{2} \left[ (\rho + \sigma_{\text{max}}(\widetilde{H})) \|A(w^0 - w_s)\|^2_2 + \frac{1 + \theta}{\rho} \|\alpha^0 - \alpha_s\|^2_2 + \rho(1 + \theta) \|v^0 - v_s\|^2_G \right].\]

**Corollary 3.4** Under the conditions of Theorem 3.2 and let \( A^+ \) be the pseudo-inverse of \( A \). If we define \( \delta_s^0, b(\delta), \tilde{v}^T \) and \( \bar{\alpha}^T \) as in Corollary 3.2, then

\[
[\phi(A^+(Bv^T + c)) + g(v^T)] - D(\alpha^T) \leq (1 + \theta)^{-T} b(1 + \theta)^{-T} \delta_s^0,
\]

\[
[\phi(A^+(B\tilde{v}^T + c)) + g(\tilde{v}^T)] - D(\bar{\alpha}^T) \leq \frac{b(\delta_s^0)}{\sum_{t=1}^{T} (1 + \theta)^t}.
\]

The requirement of \( \sigma_{\text{min}}(H) \geq \gamma^{-1} \) is the key difference between Theorem 3.1 and Theorem 3.2. The fast convergence of ADMM requires that \( H \) to be of order \( O(\rho) \), which may be smaller than \( \Theta(\gamma^{-1}) \). Consider the case that \( H = \Theta(\gamma^{-1})I \) for linearized ADMM, then the optimal \( \rho \) can be chosen as \( \rho = \Theta(\gamma^{-1}) \). This leads to a linear convergence with \( \theta = \Theta(\lambda\gamma) \). The rate is slower than that of the standard ADMM, which can achieve \( \theta = \Theta(\sqrt{\lambda\gamma}) \) at the optimal choice of \( \rho \).

Similar to the case of standard ADMM, we could take \( \theta = 0 \): as long \( \phi \) is \( 1/\gamma \) smooth, and \( H \) satisfies \( \sigma_{\text{min}}(H) \geq 2/\gamma \), we can achieve the following sublinear convergence without additional assumptions:

\[
\sum_{t=1}^{T} \left[ \max(D_\phi(w_s, w^{t-1}), D_\phi^*(A^T \alpha_s, -A^T \bar{\alpha}^t)) + D_g(v_s, v^t) \right]
\]

\[\leq \frac{1}{2} \left[ (\rho + \sigma_{\text{max}}(\widetilde{H})) \|A(w^0 - w_s)\|^2_2 + \rho \|v^0 - v_s\|^2_G + \rho^{-1} \|\alpha^0 - \alpha_s\|^2_2 \right].\]

A similar result holds for duality gap convergence when \( A \) is a square invertible matrix.

### 3.4 Lower Bounds

We consider the quadratic case that \( A = B = I \), \( c = 0 \), and

\[
\phi(w) = \frac{1}{2} w^T Qw, \quad g(v) = \frac{1}{2} v^T \Lambda v.
\]

The optimal solution is

\[
w_s = v_s = \alpha_s = 0.
\]

We show that with appropriately chosen \( Q \) and \( \Lambda \) so that \( Q \) is \( 1/\gamma \) smooth, and both \( \Lambda \) and \( Q \) are \( \lambda \) strongly convex, the convergence rate of ADMM can be \( 1 - \Theta(\sqrt{\gamma\lambda}) \) and the convergence rate of linearized ADMM can be \( 1 - \Theta(\gamma\lambda) \).
ADMM

We assume that $Q$ and $\Lambda$ are diagonal matrices.

The ADMM iterate satisfies the following equations (with $G = 0$):

$$
\begin{align*}
    v^t &= (\Lambda + \rho I)^{-1}(\alpha^{t-1} + \rho w^{t-1}) \\
    w^t &= (Q + \rho I)^{-1}(\rho v^t - \alpha^{t-1}) \\
    \alpha^t &= \alpha^{t-1} + \rho(w^t - v^t),
\end{align*}
$$

which implies

$$
\begin{align*}
    v^t &= (\Lambda + \rho I)^{-1}(\alpha^{t-1} + \rho w^{t-1}) \\
    w^t &= (\Lambda + \rho I)^{-1}(Q + \rho I)^{-1}(\rho^2 w^{t-1} - \Lambda \alpha^{t-1}) \\
    \alpha^t &= (\Lambda + \rho I)^{-1}(Q + \rho I)^{-1}Q(\Lambda \alpha^{t-1} - \rho^2 w^{t-1}).
\end{align*}
$$

We may write $[w^t; \alpha^t] = M[w^{t-1}; \alpha^{t-1}]$. Now we take $Q = \Lambda = \text{diag}(\lambda, 1/\gamma)$, where we assume that $\lambda \leq 1/\gamma$. Then the largest eigenvalue of $M$, which determines the rate of convergence of ADMM, is

$$
\max \left[ \frac{\rho^2 + \lambda^2}{(\rho + \lambda)^2}, \frac{\rho^2 \gamma^2 + 1}{(\rho + \lambda)(\rho + 1)^2} \right].
$$

The optimal $\rho$ to minimize the above is $\rho = \sqrt{\lambda/\gamma}$, and the maximum value is $(1 + \gamma \lambda)/(1 + \sqrt{\gamma \lambda})^2$. This special case matches the convergence rate behavior of $1 - \Theta(\sqrt{\gamma \lambda})$ we proved for the ADMM method.

Linearized ADMM

We assume that $H$, $Q$, and $\Lambda$ are diagonal matrices. The linearized ADMM iterate satisfies the following equations (with $G = 0$):

$$
\begin{align*}
    v^t &= (\Lambda + \rho I)^{-1}(\alpha^{t-1} + \rho w^{t-1}) \\
    w^t &= (H + \rho I)^{-1}((H - Q)w^{t-1} + \rho v^t - \alpha^{t-1}) \\
    \alpha^t &= \alpha^{t-1} + \rho(w^t - v^t),
\end{align*}
$$

which implies that

$$
\begin{align*}
    v^t &= (\Lambda + \rho I)^{-1}(\alpha^{t-1} + \rho w^{t-1}) \\
    w^t &= (\Lambda + \rho I)^{-1}(H + \rho I)^{-1}((\rho I + \Lambda)(H - Q) + \rho^2 w^{t-1} - \Lambda \alpha^{t-1}) \\
    \alpha^t &= (\Lambda + \rho I)^{-1}(H + \rho I)^{-1}H(\alpha^{t-1} + \rho(\Lambda - (\rho I + \Lambda)H^{-1}Q)w^{t-1}).
\end{align*}
$$

Now let $\lambda \leq 1/\gamma$, and we take $Q = \Lambda = \text{diag}(\lambda, 1/\gamma)$, and $H = \text{diag}(2/\gamma, 2/\gamma)$. It follows that the convergence rate of linearized ADMM is no faster than the largest eigenvalue of

$$
M = \begin{bmatrix}
\frac{1}{(\rho + h)(\rho + \lambda)} & \frac{\rho^2 + (\rho + \lambda)(h - q) - \lambda}{\rho(h - (\rho + \lambda)q)} & \frac{-\lambda}{\rho(h - (\rho + \lambda)q)} \\
\frac{\rho \lambda h - (\rho + \lambda)q}{\rho(h - (\rho + \lambda)q)} & \frac{\lambda h}{\rho(h - (\rho + \lambda)q)} & \frac{\lambda h}{\rho(h - (\rho + \lambda)q)}
\end{bmatrix}
$$

10
with $q = \lambda$ and $h = 2/\gamma$. When $\rho \leq h - \lambda$, the largest eigenvalue of $M$ is no less than

$$\frac{\rho^2 + (h - q)\rho + (h - q)\lambda}{\rho^2 + (h + \lambda)\rho + h\lambda} \geq \frac{h - \lambda}{h + \lambda} = 1 - O(\lambda\gamma).$$

Similarly, it is also not difficult to check that the eigenvalue is no less than $1 - O(\lambda\gamma)$ when $\rho \geq h - \lambda$. It follows that this special case matches the convergence rate behavior of $1 - \Theta(\gamma\lambda)$ we proved for the linearized ADMM method.

### 4 Numerical Illustration

Although we have obtained both the worst case upper bounds and matching lower bounds for ADMM and Linearized ADMM. The analysis shows that in the worst case ADMM converges at a faster rate of $1 - \Theta(\sqrt{\lambda\rho})$ while in the worst case Linearized ADMM converges at a slower rate of $1 - \Theta(\lambda\rho)$.

However, for any specific problem, both methods can converge faster than the corresponding worst case upper bounds obtained in this paper. In this section, we use a simple example to illustrate the real convergence behavior of ADMM versus linearized ADMM methods at different choices of $\rho$'s, to illustrate the phenomenon that the former can converge significantly faster than the latter.

Consider the following 1-dimensional problem:

$$\phi(w) = \frac{w}{\sqrt{\gamma}} \arctan\left(\frac{w}{\sqrt{\gamma}}\right) - \frac{1}{2} \ln(1 + \frac{w^2}{\gamma}) + \frac{\mu}{2} w^2, \quad g(v) = \frac{1}{12} v^4 + \frac{\lambda}{2} v^2,$$

with $A = B = I$ and $c = 0$. It can be checked that $\phi(w)$ is $1/\gamma + \mu$ smooth and $\mu$ strongly convex; $g(v)$ is $\lambda$-strongly convex.

We compare the convergence of ADMM versus linearized ADMM with different values of $\rho$. In linearized ADMM, and we set $h = 2(\mu + 1/\gamma)$. Note that for this problem, $w_* = v_* = 0$, and we can define the error of a solution $(w, v)$ as $\sqrt{w^2 + v^2}$.

Figure 1 shows the convergence behavior when $\gamma = 0.1$, and $\lambda = \mu = 0.2$. This is the situation that $\lambda\gamma = 0.02$ is relatively small. In this case, we compare three different values of $\rho$'s: $\rho = 0.2\sqrt{\lambda/\gamma}$, $\rho = \sqrt{\lambda/\gamma}$, and $\rho = 5\sqrt{\lambda/\gamma}$. The corresponding convergence rates for ADMM are 0.51, 0.21, and 0.41; the corresponding convergence rates for linearized ADMM are 0.51, 0.53, and 0.64. This shows that ADMM is superior to Linearized ADMM for $\rho$'s. Moreover, it achieves relatively fast convergence rate at the optimal choice of $\rho = \sqrt{\lambda/\gamma}$, while Linearized ADMM is relatively insensitive to $\rho$.

Figure 2 shows the convergence behavior when $\gamma = \lambda = \mu = 1$. This is the situation that $\lambda\gamma = 1$ is relatively large. We compare three different values of $\rho$'s: $\rho = 0.2\sqrt{\lambda/\gamma}$, $\rho = \sqrt{\lambda/\gamma}$, and $\rho = 5\sqrt{\lambda/\gamma}$. The corresponding convergence rates for ADMM are 0.78, 0.49, and 0.64; the corresponding convergence rates for linearized ADMM are 0.82, 0.69, and 0.82. The relatively convergence behaviors of ADMM and linearized ADMM are consistent with those of Figure 1.

### 5 Conclusion

This paper presents a new duality gap convergence analysis of standard ADMM versus linearized ADMM under conditions commonly studied in the literature. It is shown that in the worst case,
the standard ADMM converges with an accelerated rate that is faster than that of the linearized ADMM. Matching lower bounds are obtained for specific problems. A simple numerical example illustrates this behavior. One consequence of our analysis is that the standard ADMM does not require Nesterov’s acceleration scheme in theory because it already enjoys the squared root convergence rate for smooth-strongly convex problems. On the other hand, linearized ADMM may still benefit from extra acceleration steps. Finally the results obtained in this paper only show the worst case behaviors for both algorithms (under appropriate assumptions commonly used in the literature). In practice, both methods might converge faster, and it remains open to study such faster convergence rates under additional suitable assumptions.

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A Proof of Theorem 3.1

The fact that $w^t$ minimizes the objective function in line 4 of Algorithm 1 together with the relationship of $\alpha^t$ and $\alpha^{t-1}$ in line 5, implies that

$$\nabla \phi(w^t) + A^T \alpha^t = H(w^{t-1} - w^t).$$ (9)
We thus obtain
\[
\phi(w^t) - \phi(w) + (\alpha^t)\top A(w^t - w) + (Aw - Bv - c)^\top \tilde{H}A(w^{t-1} - w^t) \\
\leq -\frac{\gamma}{2}\|\nabla \phi(w^t) - \nabla \phi(w)\|_2^2 \\
+ \nabla \phi(w^t)^\top (w^t - w) + (\alpha^t)\top A(w^t - w) + (Aw - Bv - c)^\top \tilde{H}A(w^{t-1} - w^t) \\
= -\frac{\gamma}{2}\|A^\top(\alpha^t - \alpha) + H(w^t - w^{t-1})\|_2^2 \\
+ (w^{t-1} - w^t)^\top \tilde{H}A(w^t - w) + (Aw - Bv - c)^\top \tilde{H}A(w^{t-1} - w^t) \\
= -\frac{\gamma}{2}\|A^\top(\alpha^t - \alpha) + H(w^t - w^{t-1})\|_2^2 \\
+ \frac{1}{2}\left[\|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^t - Bv - c\|_H^2 - \|Aw^{t-1} - Aw^t - w^t\|_H^2\right].
\] (10)

In the above derivation, the inequality is a direct consequence of the smoothness of \(\phi\), which implies that for any \(w'\) and \(w\), \(\phi(w') + \nabla \phi(w')^\top (w - w') + 0.5\|\nabla \phi(w) - \nabla \phi(w')\|_2^2\). The first equality is due to (13), and \(\nabla \phi(w) + A^\top \alpha = 0\) (which follows from the assumption \(w = \nabla \phi^*(-A^\top \alpha)\) of the theorem). The second equality is algebra.

We also have from the optimality of \(v^t\) for minimizing the objective function in line 3 of Algorithm\(\underline{1}\) and the relationship of \(\alpha^t\) and \(\alpha^{t-1}\) in line 5:
\[
\nabla g(v^t) - B^\top \alpha^t = -\rho G(v^t - v^{t-1}) + \rho B^\top A(w^{t-1} - w^t).
\] (11)

Therefore
\[
g(v^t) - g(v) + \frac{\lambda}{2}\|v^t - v\|_2^2 - \alpha^t B(v^t - v) \\
\leq \nabla g(v^t)^\top (v^t - v) - \alpha^t B(v^t - v) \\
= \rho(v^t - v^{t-1})^\top G(v - v^t) + \rho(w^t - w^{t-1})^\top A^\top B(v - v^t) \\
= \frac{\rho}{2}\left[\|v - v^{t-1}\|_G^2 - \|v - v^t\|_G - \|v^t - v^{t-1}\|_G^2\right] \\
+ \frac{\rho}{2}\left[\|Aw^{t-1} - Bv - c\|_2^2 + \|Aw^t - Bv^t - c\|_2^2 - \|Aw^t - Bv - c\|_2^2 - \|Aw^{t-1} - Bv^t - c\|_2^2\right] \\
= \frac{\rho}{2}\left[\|v - v^{t-1}\|_G^2 - \|v - v^t\|_G - \|v^t - v^{t-1}\|_G^2\right] + \frac{1}{2\rho}\|\alpha^t - \alpha^{t-1}\|_2^2 \\
+ \frac{\rho}{2}\left[\|Aw^{t-1} - Bv - c\|_2^2 - \|Aw^t - Bv - c\|_2^2 - \|Aw^{t-1} - Bv^t - c\|_2^2\right].
\] (12)

In the above derivation, the first inequality is due to the strong convexity of \(g(\cdot)\). The first equality employs (11). The second equality is algebra, and the third equality is due to the relationship of \(\alpha^t\) and \(\alpha^{t-1}\) in line 5 of Algorithm\(\underline{1}\).

Finally we have
\[
- (\alpha^t - \alpha)^\top (Aw^t - Bv^t - c) \\
= - \frac{1}{\rho}(\alpha^t - \alpha)^\top (\alpha^t - \alpha^{t-1}) \\
= \frac{1}{2\rho}\left[\|\alpha - \alpha^{t-1}\|_2^2 - \|\alpha - \alpha^t\|_2^2 - \|\alpha^t - \alpha^{t-1}\|_2^2\right].
\] (13)
where the first equality uses the relationship of \( \alpha^t \) and \( \alpha^{t-1} \) in line 5 of Algorithm 1, and the second equality is algebra.

By adding (10), (12), (13), we obtain

\[
\phi(w^t) + g(v^t) - \phi(w) - g(v) = \alpha^t(\rho w^t - \rho w - c) + \alpha^t(\rho w^t - \rho w - c)
\]

\[
\leq -\frac{\gamma}{2} A^T(\alpha^t - \alpha) + H(w^t - w^{t-1})\|v^t - v\|_2^2 - \frac{\lambda}{2} \|v^t - v\|_2^2
\]

\[
+ \frac{\rho}{2} [\|v - v^{t-1}\|_G^2 - \|v - v\|_G^2 - \|v^t - v^{t-1}\|_G^2]
\]

\[
+ \frac{\rho}{2} [\|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^{t-1} - Bv - c\|_H^2]
\]

which can be rewritten as the following bound:

\[
r_t \leq -\frac{\gamma}{2} A^T(\alpha^t - \alpha) + H(w^t - w^{t-1})\|v^t - v\|_2^2 - \frac{1}{2} \|w^t - w^{t-1}\|_H^2 + \frac{\rho}{2} \|\alpha^t - \alpha\|_2^2
\]

\[
- \frac{\lambda}{2} \|v^t - v\|_2^2 + \frac{\rho\theta}{2} \|v - v\|_G^2 - \frac{\rho}{2} \|v^t - v^{t-1}\|_G^2
\]

\[
+ \frac{\rho\theta}{2(1 + \theta)} [\|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^{t-1} - Bv - c\|_H^2]
\]

\[
+ \frac{1}{2} [\frac{1}{1 + \theta} [\|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^{t-1} - Bv - c\|_H^2]
\]

\[
+ \frac{\rho}{2} [\|v - v^{t-1}\|_G^2 - (1 + \theta)\|v - v\|_G]
\]

\[
+ \frac{\rho}{2} [\frac{1}{1 + \theta} [\|Aw^{t-1} - Bv - c\|_H^2 - \|Aw^{t-1} - Bv - c\|_H^2]
\]

\[
+ \frac{1}{2\rho} [\|\alpha - \alpha^{t-1}\|_2^2 - (1 + \theta)\|\alpha - \alpha^{t-1}\|_2^2]
\]

\[
= X_t + Y_t - \frac{\rho}{2} \|v^t - v^{t-1}\|_G^2 + Z_t + (1 + \theta)^{-1} \delta_t - \delta_t.
\]

We can bound \( X_t \) as follows:

\[
X_t = -\frac{\gamma}{2} A^T(\alpha^t - \alpha) + H(w^t - w^{t-1})\|v^t - v\|_2^2 - \frac{1}{2} \|w^t - w^{t-1}\|_H^2 + \frac{\rho}{2} \|\alpha^t - \alpha\|_2^2
\]

\[
\leq -\frac{\gamma}{2} A^T(\alpha^t - \alpha) + H(w^t - w^{t-1})\|v^t - v\|_2^2 - \frac{1}{2\sigma\max(H)} [H(w^t - w^{t-1})\|v^t - v\|_2^2 + \frac{\rho}{2} \|\alpha^t - \alpha\|_2^2
\]

\[
\leq \frac{1}{2} \max u \left[ -\gamma A^T(\alpha^t - \alpha) + u\|\alpha^t - \alpha\|_2^2 - \sigma\max(H)^{-1}\|u\|_2^2 \right] + \frac{\rho}{2} \|\alpha^t - \alpha\|_2^2
\]

\[
= -\frac{\gamma}{2\sigma\max(H)} + 1 A^T(\alpha^t - \alpha)\|\alpha^t - \alpha\|_2^2 + \frac{\rho}{2\rho} \|\alpha^t - \alpha\|_2^2 \leq 0.
\]
Therefore (10) can be replaced by the following inequality:

\[
\theta \leq \frac{\theta(1 + \sigma_{\text{max}}(\tilde{H})/\rho)}{1 + \theta} \| Aw^{t-1} - Bv - c \|_2^2 + \frac{\rho \theta}{2(1 + \theta)} \| Aw^{t-1} - Bv - c \|_2^2 - \frac{\rho}{2} \| Aw^{t-1} - Bv^t - c \|_2^2
\]

Less than

\[
\leq \frac{\rho}{2} \left[ \frac{\theta(1 + \sigma_{\text{max}}(\tilde{H})/\rho)}{1 + \theta} \| Aw^{t-1} - (Bv + c) \|_2^2 - \| Aw^{t-1} - Bv^t - c \|_2^2 \right]
\]

Less than

\[
\leq \frac{\rho \theta (\rho + \sigma_{\text{max}}(\tilde{H}))}{2(\rho - \theta \sigma_{\text{max}}(\tilde{H}))} \| B(v^t - v) \|_2^2.
\]

where the second inequality uses the fact that

\[
\frac{\theta(1 + a)}{1 + \theta} \| u \|_2^2 - \| u' \|_2^2 \leq \frac{1 + a}{1 - \theta} \theta \| u - u' \|_2^2,
\]

when \( \theta a < 1 \) with \( a = \sigma_{\text{max}}(\tilde{H})/\rho \). Therefore

\[
Y_t + Z_t \leq -\frac{\lambda}{2} \| v^t - v \|_2^2 + \frac{\rho \theta}{2} \| v - v' \|_G + \frac{\theta \rho (\rho + \sigma_{\text{max}}(\tilde{H}))}{2(\rho - \theta \sigma_{\text{max}}(\tilde{H}))} \| B(v^t - v) \|_2^2
\]

\[
\leq \left[ -\frac{\lambda}{2} + \frac{\rho \theta}{2} \| G \|_G + \frac{\theta \rho (\rho + \sigma_{\text{max}}(\tilde{H}))}{2(1 - s)} \| \sigma_{\text{max}}(B) \|_2^2 \right] \| v^t - v \|_2^2 \leq 0.
\]

Therefore we obtain

\[
r_t \leq X_t + Y_t + Z_t + (1 + \theta)^{-1} \delta_t - \delta_t \leq (1 + \theta)^{-1} \delta_t_0 - \delta_t.
\]

Now by multiplying the above displayed inequality by \( (1 + \theta)^{t-T} \), and sum over \( t = 1, \ldots, T \), we obtain (6).

In order to obtain (9), we simply note that (9) implies that

\[
\nabla \phi^*(-A^\top \tilde{\alpha}^t) - w^t = 0.
\]

Therefore (11) can be replaced by the following inequality:

\[
\phi^*(-A^\top \tilde{\alpha}^t) - \phi^*(-A^\top \alpha) + (\alpha^t - \alpha)^\top Aw^t + (Bv - c)^\top \tilde{H} A(w^{t-1} - w^t)
\]

\[
\leq \nabla \phi^*(-A^\top \tilde{\alpha}^t)^\top (A^\top \alpha^t + (\alpha^t - \alpha)^\top Aw^t
\]

\[
\leq -\frac{\gamma}{2} \| A^\top \alpha^t + (\alpha^t - \alpha)^\top Aw^t \|_2^2 + (-Bv - c)^\top \tilde{H} A(w^{t-1} - w^t)
\]

\[
= -\frac{\gamma}{2} \| A^\top (\alpha^t - \alpha) + H(w^t - w^{t-1}) - A^\top \alpha \|_2^2 + (Bv - c)^\top \tilde{H} A(w^{t-1} - w^t)
\]

\[
= -\frac{\gamma}{2} \| A^\top (\alpha^t - \alpha) + H(w^t - w^{t-1}) \|_2^2
\]

\[
+ \frac{1}{2} \left[ \| Aw^{t-1} - Bv - c \|_H^2 - \| Aw^t - Bv - c \|_H^2 - \| Aw^t - Aw^{t-1} \|_H^2 \right],
\]

where the first inequality uses the fact \( \phi^* \) is \( \gamma \) strongly convex, which is a direct consequence of the fact that \( \phi \) is \( 1/\gamma \) smooth. The first equality is due to (13) and the definition of \( \tilde{\alpha}^t \). The second equality is algebra.

Now, we note that the right hand side of (15) is the same as that of (10). Therefore the remaining of the proof follows the same argument as that of (5), where we simply use the addition of (15), (12), and (13) to replace the addition of (10), (12), and (13). This leads to (6).
B Proof of Corollary 3.2

We have from (3)

\[
2(1 + \theta)^T \left[ \phi^*(-A^T \alpha) + g(v^T) - \phi^*(-A^T \alpha) - g(v) + (\tilde{a}^T)^T (Bv + c) - \alpha^T (Bv^T + c) \right]
\]

\[
\leq (\rho + \sigma_{\text{max}}(\tilde{H})) \|Aw^0 - Bv - c\|_2^2 + \rho (1 + \theta) \|v^0 - v\|_G^2 + \frac{1 + \theta}{\rho} \|\alpha - \alpha^0\|_2^2
\]

\[
\leq 2(\rho + \sigma_{\text{max}}(\tilde{H})) \|Aw^0 - Bv^0 - c\|_2^2 + (2(\rho + \sigma_{\text{max}}(\tilde{H})) \sigma_{\text{max}}(B)^2 + \rho (1 + \theta) \sigma_{\text{max}}(G)) \|v^0 - v\|_2^2 + \frac{1 + \theta}{\rho} \|\alpha - \alpha^0\|_2^2.
\]

Now we set \( \alpha = -(A^+)^T \nabla \phi(A^+(Bv^T + c)) \) and \( v = \nabla g^*(B^T \tilde{a}^T) \). This choice achieves the maximum value of the left hand side over \((\alpha, v)\). With this choice, and the definition of convex conjugate, we obtain

\[
2(1 + \theta)^T [\phi(A^+(Bv^T + c)) + g(v^T) - D(\tilde{a}^T)]
\]

\[
\leq (2(\rho + \sigma_{\text{max}}(\tilde{H})) \sigma_{\text{max}}(B)^2 + \rho (1 + \theta) \sigma_{\text{max}}(G)) \|v^0 - v\|_2^2 + \frac{1 + \theta}{\rho} \|\alpha - \alpha^0\|_2^2
\]

\[+ 2(\rho + \sigma_{\text{max}}(\tilde{H})) \|Aw^0 - Bv^0 - c\|_2^2. \tag{16} \]

From Corollary 3.1, we obtain

\[
\frac{\rho}{2} \|A(w^T - w_*)\|_2^2 + \frac{1 + \theta}{2\rho} \|\alpha^T - \alpha_*\|_2^2 + \frac{\rho(1 + \theta)}{2} \|v^T - v_*\|_G^2 \leq \frac{(1 + \theta)^T}{2} \delta_*^0. \tag{17} \]

Therefore

\[
\|A(w^T - w^{T-1})\|_2^2 \leq 2 \|A(w^T - w)\|_2^2 + 2 \|A(w^{T-1} - w)\|_2^2 \leq 2(2 + \theta)(1 + \theta)^{-T} \delta_*^0 / \rho.
\]

Moreover, (17) also implies \( \|\alpha^T - \alpha_*\|_2^2 \leq \rho (1 + \theta)^{-T} (1 + \theta)^{-T} \delta_*^0 \). Therefore

\[
\|\tilde{a}^T - \alpha_*\|_2 \leq \|\alpha^T - \alpha_*\|_2 + \sigma_{\text{max}}(\tilde{H}) \|A(w^T - w^{T-1})\|_2
\]

\[
\leq \sigma_{\text{max}}(\tilde{H}) \sqrt{2(2 + \theta)(1 + \theta)^{-T} \delta_*^0 / \rho + \rho (1 + \theta)^{-T} (1 + \theta)^{-T} \delta_*^0}.
\]

It follows from the definition of \( b_2(\cdot) \) that

\[
\|v - v^0\|_2^2 \leq b_2((1 + \theta)^{-T} \delta_*^0).
\]

Similarly, we obtain from (17) that \( \|v^T - v_*\|_G^2 \leq (1 + \theta)^{-T} \delta_*^0 / (\rho + \rho \theta) \). It implies that

\[
\|\alpha - \alpha^0\|_2^2 \leq b_1((1 + \theta)^{-T} \delta_*^0).
\]

Now the first desired bound of the theorem can be obtained by plugging in the estimates of \( \|v - v^0\|_2^2 \) and \( \|\alpha - \alpha^0\|_2^2 \) into (16).

For the second desired bound, we note from the Jensen’s inequality and (3) that

\[
\leq \frac{1}{\sum_{t=1}^T (1 + \theta)^{-1}} \left[ \frac{1}{2}(\rho + \sigma_{\text{max}}(\tilde{H})) \|Aw^0 - Bv - c\|_2^2 + \frac{\rho}{2} \|v^0 - v\|_G^2 + \frac{1}{2\rho} \|\alpha - \alpha^0\|_2^2 \right].
\]

Again we simply take the choice of \((\alpha, v)\) that achieves the maximum on the left hand side: \( \alpha = -(A^+)^T \nabla \phi(A^+(Bv^T + c)) \) and \( v = \nabla g^*(B^T \tilde{a}^T) \).

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C Proof of Theorem 3.2

The basic proof structure is the same as that of Theorem 3.1. The fact that $w^t$ minimizes the objective function in line 4 of Algorithm 2 together with the relationship of $\alpha^t$ and $\alpha^{t-1}$ in line 5, implies that

$$\nabla \phi(w^{t-1}) + A^T \alpha^t = H(w^{t-1} - w^t).$$ (18)

We thus obtain

$$\phi(w^{t-1}) - \phi(w) + (\alpha^t)^T A(w^t - w) + \alpha^T A(w^{t-1} - w^t)$$

$$+ (Aw - Bv - c)^T \tilde{H} A(w^{t-1} - w^t)$$

$$\leq \nabla \phi(w^{t-1})^T (w^{t-1} - w) + (\alpha^t)^T A(w^t - w) + \alpha^T A(w^{t-1} - w^t)$$

$$+ (Aw - Bv - c)^T \tilde{H} A(w^{t-1} - w^t) - \frac{\gamma}{2} \| \nabla \phi(w^{t-1}) - \nabla \phi(w) \|^2,$$

$$(H(w^{t-1} - w^t) - A^T \alpha^t)^T (w^{t-1} - w) + (\alpha^t)^T A(w^t - w) + \alpha^T A(w^{t-1} - w^t)$$

$$+ (Aw - Bv - c)^T \tilde{H} A(w^{t-1} - w^t) - \frac{\gamma}{2} \| A^T (\alpha^t - \alpha) + H(w^t - w^{t-1}) \|^2$$

$$= (w^t - w^{t-1})^T (A^T (\alpha^t - \alpha)) - \frac{\gamma}{2} \| A^T (\alpha^t - \alpha) + H(w^t - w^{t-1}) \|^2$$

$$+ \frac{1}{2} \left[ \| Aw^{t-1} - Bv - c \|^2_H - \| Aw^t - Bv - c \|^2_H + \| w^t - w^{t-1} \|^2_H \right],$$ (19)

where the derivation uses similar arguments as those of (10). The first inequality uses the smoothness of $\phi$, and the first equality uses (18). The second equality is algebra.

We also have from the optimality of $v^t$ for minimizing the objective function in line 3 of Algorithm 2 and the relationship of $\alpha^t$ and $\alpha^{t-1}$ in line 5, to obtain (12). Finally, we can also obtain (13).

By adding (19), (12), (13), and use the simplified notation $\Delta w = (w^t - w^{t-1})$, and $\Delta \alpha = \alpha^t - \alpha$, we obtain

$$r_t \leq \Delta w^T (A^T \Delta \alpha) - \frac{\gamma}{2} \| A^T \Delta \alpha + H \Delta w \|^2_H + \frac{1}{2} \| \Delta w \|^2_H + \frac{\theta}{2\rho} \| \Delta \alpha \|^2$$

$$- \frac{\lambda}{2} \| v^t - v \|^2_G + \frac{\rho \theta}{2} \| v - v^t \|^2_G - \frac{\rho}{2} \| v^t - v^{t-1} \|^2_G$$

$$+ \frac{\theta}{2(1 + \theta)} \| w^{t-1} - A^{-1}(Bv + c) \|^2_H + \frac{\rho \theta}{2(1 + \theta)} \| Aw^{t-1} - Bv - c \|^2_H - \frac{\rho}{2} \| Aw^{t-1} - Bv^t - c \|^2_H$$

$$+ \frac{1}{2} \left[ \frac{1}{1 + \theta} \| Aw^{t-1} - (Bv + c) \|^2_H - \| Aw^t - (Bv + c) \|^2_H \right]$$

$$+ \frac{\rho}{2} \| v - v^{t-1} \|^2_G - (1 + \theta) \| v - v^t \|^2_G$$

$$+ \frac{\rho}{2(1 + \theta)} \| Aw^t - Bv^t - c \|^2_H - \| Aw^t - Bv - c \|^2_H$$

$$+ \frac{1}{2\rho} \| \alpha - \alpha^{t-1} \|^2_G - (1 + \theta) \| \alpha - \alpha^t \|^2_H].$$
We can multiply $X_t$ as follows:

$$
X_t = -(H \Delta w)^\top (\gamma I - H^{-1})(A^\top \Delta \alpha) - \frac{\gamma}{2} (\|A^\top \Delta \alpha\|^2_H + \|H \Delta w\|^2_2) + \frac{1}{2} \|\Delta w\|^2_H + \frac{\theta}{2\rho} \|\Delta \alpha\|^2_2
$$

$$
\leq (\gamma - 1/\sigma_{\min}(H)) \|H \Delta w\|_2 \|A^\top \Delta \alpha\|_2 - \frac{\gamma - 1/\sigma_{\min}(H)}{2} \|H \Delta w\|^2_2 - \frac{\gamma}{2} \|A^\top \Delta \alpha\|^2_2 + \frac{\theta}{2\rho} \|\Delta \alpha\|^2_2
$$

$$
\leq -\frac{1}{2\sigma_{\min}(H)} \|A^\top \Delta \alpha\|^2_2 + \frac{\theta}{2\rho} \|\Delta \alpha\|^2_2 \leq 0.
$$

The first inequality uses the assumption that $\gamma - \sigma_{\min}(H)^{-1} \geq 0$ in the theorem, and norm inequalities. The second inequality is obtained by taking the maximum over $\|H \Delta w\|_2$. The last inequality uses the assumptions on $\theta$. We also can use the same derivation as that of Theorem 3.1 to show that $Y_t + Z_t \leq 0$. Therefore

$$
r_t \leq X_t + Y_t - \frac{\rho}{2} \|v^t - v^{t-1}\|^2_2 + Z_t + (1 + \theta)^{-1} \delta_{t-1} \leq (1 + \theta)^{-1} \delta_{t-1} - \delta_t.
$$

We can multiply the above by $(1 + \theta)^{t-T}$ and then sum over $t = 1, \ldots$ to obtain (7).

Similarly we can prove a dual version of (19) below. The equation in (18) and the definition of $\hat{\alpha}^t$ in the theorem imply that

$$
w^{t-1} = \nabla \phi^*(-A^\top \hat{\alpha}^t).
$$

We thus have

$$
\phi^*(-A^\top \hat{\alpha}^t) - \phi^*(-A^\top \alpha) + (\alpha^t - \alpha)^\top Aw^t + (-Bv - c)^\top \hat{H}A(w^{t-1} - w^t)
$$

$$
\leq \nabla \phi^*(-A^\top \hat{\alpha}^t)^\top (-A^\top \hat{\alpha}^t + A^\top \alpha) - \frac{\gamma}{2} \|A^\top (\hat{\alpha}^t - \alpha)\|^2_2
$$

$$
+ (\alpha^t - \alpha)^\top Aw^t + (-Bv - c)^\top \hat{H}A(w^{t-1} - w^t)
$$

$$
= (w^{t-1})^\top (-A^\top \alpha^t + H(w^t - w^{t-1}) + A^\top \alpha) + (\alpha^t - \alpha)^\top Aw^t
$$

$$
- \frac{\gamma}{2} \|A^\top (\alpha^t - \alpha) + H(w^t - w^{t-1})\|^2_2 + (-Bv - c)^\top \hat{H}A(w^{t-1} - w^t)
$$

$$
= (w^t - w^{t-1})^\top (A^\top (\alpha^t - \alpha)) - \frac{\gamma}{2} \|A^\top (\alpha^t - \alpha) + H(w^t - w^{t-1})\|^2_2
$$

$$
+ \frac{1}{2} \left[\|Aw^t - Bv - c\|^2_H - \|Aw^t - Bv - c\|^2_H + \|w^t - w^{t-1}\|^2_H\right].
$$

In the above derivation, the first inequality uses the strong convexity of $\phi^*$, which follows from the smoothness of $\phi$. The first equality uses the relationship of $\nabla \phi^*(-A^\top \hat{\alpha}^t)$ and $w^{t-1}$ and the relationship of $\hat{\alpha}^t$ and $\alpha^t$. The last equality uses algebra. Note that the right hand side of (19) and that of (20) are the same. Therefore by adding (20), (12), (13), we obtain

$$
r_t^* \leq X_t + Y_t - \frac{\rho}{2} \|v^t - v^{t-1}\|^2_2 + Z_t + (1 + \theta)^{-1} \delta_{t-1} - \delta_t \leq (1 + \theta)^{-1} \delta_{t-1} - \delta_t.
$$

We can multiply $(1 + \theta)^{t-T}$ to both sides, and then sum over $t = 1, \ldots$ to obtain (8).