ON THE THEORY OF THE KOLMOGOROV OPERATOR
IN THE SPACES $L^p$ AND $C_\infty$. I

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ABSTRACT. We obtain the basic results concerning the problem of constructing operator realizations of the formal differential expression $\nabla \cdot a \cdot \nabla - b \cdot \nabla$ with measurable matrix $a$ and vector field $b$ having critical-order singularities as the generators of Markov semigroups in $L^p$ and $C_\infty$.

INTRODUCTION

Let $\Omega$ be an open set in $\mathbb{R}^d$, $d \geq 3$. Our object of study is the formal differential operator

$$(-\nabla \cdot a \cdot \nabla + b \cdot \nabla)u(x) = -\sum_{i,k=1}^d \frac{\partial}{\partial x_i} \left(a_{ik}(x) \frac{\partial u(x)}{\partial x_k}\right) + \sum_{i=1}^d b_i(x) \frac{\partial u(x)}{\partial x_i}, \quad x \in \Omega.$$ 

Under rather general assumptions on the matrix $a$,

$$a(x) = a(x)^*: \Omega \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad a \in \left[L^1_{\text{loc}}(\Omega, \mathcal{L}^d)\right]^{d \times d},$$

$$\sigma I \leq a(x)$$

for some constant $\sigma > 0$ and $\mathcal{L}^d$ a.e. $x \in \Omega$, \hfill (H$_1$)

we will construct by means of the theory of the quadratic forms in Hilbert spaces, three operator realizations $A_D, A_{iD}, A_N$ of $-\nabla \cdot a \cdot \nabla$ in $L^2 = L^2(\Omega, \mathcal{L}^d)$. Each of these realizations is the (minus) generator of a symmetric Markov semigroup and inherits some basic properties of the classical Dirichlet and Neumann extensions of $-\Delta$. Let $A$ denote one of these operators and let $\{e^{-tA_r}, t > 0\}_{1 \leq r < \infty}$ be the collection of consistent (and bounded) $L^p \cap L^q = e^{-tA_1} \cap L^q$, $1 \leq p, q < \infty$, $A_2 \equiv A$) $C_0$ semigroups on the scale $[1, \infty]$ of $L^r$ spaces.

For any $\mathcal{L}^d$ measurable $b: \Omega \to \mathbb{R}^d$ we define in $L^r$ the maximal operator $B_r \supset b \cdot \nabla$ of domain

$$\{f \in L^r \cap W^{1,1}_{\text{loc}}(\Omega) \mid b \cdot \nabla f \in L^r\}.$$ 

1. Assuming that

$$b_a := \sqrt{b \cdot a^{-1} \cdot b} \in L^d + L^\infty,$$ \hfill (C$_d$)

we will prove, essentially using specific properties of the symmetric Markov semigroup $e^{-tA}$ and the structure of $B_r$, that $B_r$ is $A_r$ bounded (with relative bound zero) in $L^r$ for all $r \in ]1, 2 \wedge \frac{d}{2}[$. Simple examples show that this result cannot be improved in terms of the $L^p$ spaces. Moreover, the interval $]1, 2 \wedge \frac{d}{2}[$ cannot be enlarged to $]1, 2 \wedge \frac{d}{2}[$ under the assumption (C$_d$).

By means of the standard tools of the perturbation theory for linear operators one concludes that the algebraic sum $A_p + B_p$, $p \in ]1, 2 \wedge \frac{d}{2}[$ is the (minus) generator of a quasi bounded $C_0$ semigroup in $L^p$, say $A_p \equiv A_p(a, b)$. Moreover, essentially using specific properties of the symmetric Markov semigroup $e^{-tA}$ we will prove that $T^p_t := e^{-tA_p}$ is a Markov semigroup (i.e. positivity preserving.

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Let \(-\Lambda_r\) denote the generator of \(T_r^t\). Then

1) \(\Lambda_r \supset A_r + B_r, 1 < r < \infty\) and \(\Lambda_r = A_r + B_r\) only for \(r \in ]1, 2 \wedge \frac{1}{2}[,\)

2) In place of \(A\) one can substitute \(A \upharpoonright V\) provided that \(V f(x) = V(x)f(x), V\ is \mathcal{L}^d\ measurable, \ 0 \leq V\ and \ D(A^\frac{1}{2}) \cap D(V^\frac{1}{2})\ is dense in \(L^2\). The simplest sufficient condition is \(V \in L^1_{loc}, see, however, the example in [21].\)

\[\text{2.} \text{In order to treat more singular } b's\ we introduce the class } \mathbf{F}_\delta(A) \text{ of form-bounded vector fields. We say that } b : \Omega \to \mathbb{R}^d \text{ belongs to } \mathbf{F}_\delta(A), \text{ and write } b \in \mathbf{F}_\delta(A) \text{ if and only if } b \text{ is } \mathcal{L}^d \text{ measurable, } b^2_a \in L^1_{loc} \text{ and there are constants } \delta > 0, 0 \leq \lambda < \infty \text{ such that }\]

\[\|b_a(\lambda + A)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}.\]

Under the following assumptions on \((a, b)\):

\[a \in (H_1), b \in \mathbf{F}_\delta(A), \delta < 4, \text{ and } b^2_a \in L^1 + L^\infty\]  

(*)

we will construct an operator realization \(\Lambda_r(a, b)\) of \(-\nabla \cdot a \nabla + b \cdot \nabla\ in L^r\ for every \(r \in I_c := ]\frac{2}{2 - \sqrt{\delta}}, \infty[,\) such that \(-\Lambda_r(a, b)\ is the generator of a Markov semigroup. This semigroup is holomorphic for \(r \in I_c^0 := ]\frac{2}{2 - \sqrt{\delta}}, \infty[,\) for every \(r \in I_c^0;\)

\[\text{The interval } I_c \text{ is called the interval of contraction solvability.}\]

Note that any \(b\ satisfying the assumption } (C_d)\ belongs also to } \mathbf{F}_0(A) := \bigcap_{\delta > 0} \mathbf{F}_\delta(A). \text{ In particular, for each } n = 1, 2, \ldots, b_n \text{ satisfies } (C_d) \text{ and belongs to } \mathbf{F}_\delta(A). \text{ Again in place of } A \text{ one can substitute } A \upharpoonright V \text{ provided that } V \text{ is } \mathcal{L}^d \text{ measurable, } 0 \leq V\ and \ D(A^\frac{1}{2}) \cap D(V^\frac{1}{2})\ is dense in \(L^2\).\]

Next, we will prove that, for more regular matrices the constraint \(b^2_a \in L^1 + L^\infty\ in (*)\) is superfluous. This is true in particular for any \(a \in (H_1), the class of uniformly elliptic matrices (i.e. } a \in (H_1) \text{ and } a(x) \leq \xi I \text{ for some constant } \xi \text{ and } \mathcal{L}^d \text{ a.e. } x \in \Omega).\]

Moreover, it will be shown that if

\[(a, b) = (a \in (H_u), b \in \mathbf{F}_\delta(A), \delta > 4),\]

then, \(e^{-t\Lambda_r(a,b)} \upharpoonright L^p \cap I^p, \ p \in I_c^0,\) extends to a bounded holomorphic semigroup in \(L^r\ for each \(r \in I_m - I_c, where \ I_m := ]\frac{2}{2 - \sqrt{\delta}}, \infty[,\]  

\[\text{We will present examples of } (a, b) \text{ which show that } I_c, I_m \text{ are maximal, as well as examples of } (a, b) = (a \in (H_u), b \in \mathbf{F}_\delta(A), \delta > 4)\ which show that the constraint } \delta < 4 \text{ has direct bearing on the subject matter.}\]

The class \(\mathbf{F}_\delta(A)\ contains vector fields } b \text{ having critical-order singularities: the basic properties of } \Lambda_r(a, b) \text{ (including the smoothness of } D(\Lambda_r(a, b))\) exhibit a strong quantitative dependence on

\[\text{1The maximal interval of quasi bounded solvability for } A - V, \text{ with } 0 \leq V \leq \delta A + c(\delta), 0 < \delta < 1, \text{ is } I_m := ]r(\delta), r'(\delta)[, r'(\delta) := \left[\frac{2}{2 - \sqrt{\delta}}, \frac{1}{\sqrt{\delta}}\right]. \text{ This was proved in 1995 by Yu.A. Semenov, based on ideas set forth in [19]. The fact that the semigroup associated with the Schrödinger operator } -\Delta - V, \ V \in L^{\frac{4}{2}} \text{, can be extended to a } C_0 \text{ semigroup on } L^r(\mathbb{R}^d) \text{ for every } r \in I_m \text{ was first observed in [12].}\]
the value of \( \delta \) (note that \( b \in \mathbf{F}_\delta(A) \) if and only if \( cb \in \mathbf{F}_{c\delta}(A) \), \( c > 0 \), so \( \delta \) effectively plays the role of a “coupling constant” for \( b \cdot \nabla \)). See examples in section 4.

Now consider the following assumption on \( (a,b) \):

\[
a \in (H_1) \quad \text{and} \quad b \in \mathbf{F}_\delta(A), \quad \delta < 1.
\]

(removing in (\*) the constraint \( b^2 \in L^1 + L^\infty \) but restricting the range of \( \delta \).

Using old ideas of J.Lions and E.Hille we will construct in \( L^2 \) a Markov semigroup \( e^{-t\Lambda(a,b)} \), which possesses some important properties:

\[
e^{-t\Lambda(a,b)} = sLr^r \lim_{n \to \infty} e^{-t\Lambda(a,b_n)}, \quad (2 \leq r < \infty)
\]

whenever \( \{b_n\} \subset \mathbf{F}_\delta(A) \) and \( b_n \to b \) \( L^d \) a.e.

\[
\|e^{-t\Lambda(a,b)}\|_{r \to q} \leq ccLr \left( \frac{1}{r} - \frac{1}{q} \right)^{\frac{d}{2}} \omega_r = \frac{\lambda d}{2(r - 1)}, \quad 2 \leq r < q \leq \infty.
\]

\( \Lambda \supset A + B, \quad D(\Lambda) \subset D(A^+ \phi) \), the resolvent set of \( -\Lambda \) contains \( O := \{ \zeta \mid \text{Re} \zeta > \lambda \} \).

\[
(\zeta + \Lambda)^{-1} = (\zeta + A)^{-\frac{1}{2}} (1 + T_\zeta)^{-1} (\zeta + A)^{-\frac{1}{2}}, \quad \|T_\zeta\|_{2 \to 2} \leq \sqrt{\delta} \quad (\zeta \in O).
\]

Under more restrictive assumption on \( (a,b) \):

\[
a \in (H_u) \quad \text{and} \quad b \in \mathbf{F}_\delta(A), \quad \delta < 1.
\]

the results above obtained by different techniques are unified. In particular, the interval \([2,\infty[\)

extends to \( I_u \) and \( I_m \).

It is useful (in some cases necessary) to have the convergence

\[
e^{-t\Lambda(a,b)} = sLr^r \lim_{n \to \infty} e^{-t\Lambda(a_n,b_n)},
\]

where \( a, a_n \in (H_u) \), \( b, b_n \in \mathbf{F}_\delta(A_n) \), \( \delta < 4 \), \( a_n, b_n \) bounded and smooth, and \( a_n \to a, \ b_n \to b \) \( L^d \) a.e.

We succeeded in proving this convergence, though under the additional constraints \( a \in (H_u) \)

and \( \delta < 1 \). However, to make this result \textit{unconditional} we face the following problem: Given

\( (a,b) = (a \in (H_u), b \in \mathbf{F}_\delta(A)) \) to construct \( (a_n, b_n) \) with the claimed properties. In the simplest case \( \Omega = \mathbb{R}^d \), \( a = a_n = I \) we solve the problem simply putting \( b_n = E_n(b1_{\Omega_n}) \), where \( \Omega_n = B(0,n) \)

and \( E_n f := e^{\epsilon_n} f \) or \( E_n f := \gamma_{\epsilon_n} * f, \ \gamma_{\epsilon_n} \) is the K. Friedrichs mollifier, with evident modifications

for \( \Omega \subset \mathbb{R}^d \). \( (\Omega = \bigcup_{n} \Omega_n, \ \Omega_n \subset \Omega_{n+1}, \ \Omega_n \) are open and bounded).

We also mention the following result: Set \( (\nabla a)_k = \sum_{i=1}^d (\partial_i a_{ik}) \) and \( b := (\nabla a) \), so formally

\[\nabla \cdot a \cdot \nabla + b \cdot \nabla = -a \cdot \nabla^2 (\equiv -\sum_{i,k} a_{ik} \partial_i \partial_k). \]

Set \( a_n = E_n a, \ b_n := (\nabla a_n). \) Fix \( \delta < \infty \). Then \( b \in \mathbf{F}_\delta(A) \Rightarrow b_n \in \mathbf{F}_\delta(A_n) \) provided that the matrix \( a \in (H_u) \) is diagonal and \( |b| \in L^2 + L^\infty \).

3. Let \( C_\infty = C_\infty(\mathbb{R}^d) \) denote the space of all continuous functions vanishing at infinity endowed with the sup-norm. Our next concern is: to find a subclass of \( (H_u) \) and constraints on \( \delta \) which allow to construct in \( C_\infty \) a \( C_0 \) semigroup associated with \( -\nabla \cdot a \cdot \nabla + b \cdot \nabla \) whenever \( b \in \mathbf{F}_\delta(A) \).

As the first step we consider the case

\[
(a,b) = (I, b \in \mathbf{F}_\delta) \quad \text{with} \quad \sqrt{\delta} < 1 \wedge \frac{2}{d-2}, \ d \geq 3.
\]
The constraint on $\delta$ allows to establish the following fundamental fact \cite{13}:

\[(\zeta + \Lambda q)^{-1}L^q \subset W^{1,q}j, \quad \zeta \in \rho(-\Lambda q), \quad j = \frac{d}{d-2}, \quad (*)\]

whenever $q \in ]2 \vee d - 2, \frac{2}{\sqrt{d}}]$ (\(< P_0\)). In particular, $e^{-t\Lambda r}L^r \subset C^{0,1-\frac{d}{d-2}}$ for any $r \in ]\frac{2}{2\sqrt{d}}, q\] [$.

Note that $(*)$ is a “trivial” fact only for $(d,q) = (3,2)$. Indeed, $(b \in F_\delta, \delta < 1)$ implies that $\Lambda_2 = -\Delta + b \cdot \nabla, D(\Lambda_2) = W^{2,2} \subset W^{1,2}$. Thus, if $d = 3$, then $(\zeta + \Lambda_2)^{-1}L^2 \subset W^{1,6} \subset C^{0,4}$. However, already for $d = 4$, $(\zeta + \Lambda_2)^{-1}L^2 \subset W^{1,p}$, $p = d$, not $p > d$.

We emphasize that the assumption $b \in F_\delta$ does not guarantee the validity of $W^{2,p}$ estimates on $(\zeta + \Lambda r)^{-1}L^r$ for $r(d - 2)$ large enough to conclude that $e^{-t\Lambda r}L^r \subset C^{0,\alpha}$.

We will also discuss the analogue of $(*)$ for $(a, b \in F_\delta(A)), a(x) = I + c|x|^{-2}x^\prime \cdot x, c > -1$.

Armed with $(*)$ we will prove that

\[s-C_\infty \lim_{n \to \infty} e^{-t\Lambda C_\infty} (b_n), \quad b_n = E_n(b1_{B(0,n)})\]

exists uniformly in $t \in [0,1]$, and hence determines a $C_0$ semigroup $e^{-t\Lambda C_\infty}$ (a Feller semigroup).

(\text{This result has been established in \cite{13} under the additional assumption $|b| \in L^2 + L^\infty$.})

4. Next, consider the case $(\Omega = \mathbb{R}^d)$

\[(a, b) = (I, b \in F_\delta^\frac{1}{2}) \text{ with } \sqrt{\delta} < 1, \quad d \geq 3.\]

By definition, $b \in F_\delta^{1/2}$ if and only if $|b| \in L_{loc}^1$ and there exists $0 \leq \lambda = \lambda_\delta$ and

\[||b|^\frac{1}{2}(\lambda - \Delta)^{-\frac{1}{2}}||_{2 \to 2} \leq \sqrt{\delta}.\]

Recalling the definition of the Kato class $K^{d+1}_\delta$: $\|b|(\lambda - \Delta)^{-\frac{1}{2}}\|_{1 \to 1} \leq \delta$, it is a simple matter to conclude that $K^{d+1}_\delta \subset F_\delta^{1/2}$ while $\bigcup_{\delta > 0} K^{d+1}_\delta = \bigcup_{\delta < \infty} F_\delta = \emptyset$. Taking into account that by interpolation, $F_{\delta^2} \subset F_\delta^{1/2}$, it is clear that $F_{\delta^2} \subset F_\delta^{1/2}$.

To deal with such general class of vector fields we will again use ideas of E. Hille and J. Lions (alternatively, ideas of E. Hille and H.F. Trotter) to construct the generator $-\Lambda \equiv -\Lambda(b)$ (an operator realization of $\Delta - b \cdot \nabla$) of a quasi bounded holomorphic semigroup in $L^2$. This operator has some remarkable properties. Namely, $\rho(-\Lambda) \supset \mathcal{O} := \{\zeta \mid \text{Re} \zeta > \lambda\}$, and, for every $\zeta \in \mathcal{O}$,

\[(\zeta + \Lambda)^{-1} = J_\zeta^3(1 + H_\zeta^2 S_\zeta)^{-1}J_\zeta = J_\zeta^3 - J_\zeta^3 H_\zeta^2 (1 + S_\zeta H_\zeta^2)^{-1} S_\zeta J_\zeta;\]

\[\|H_\zeta S_\zeta\| \leq \delta, \quad \|(\zeta + \Lambda)^{-1}\|_{2 \to 2} \leq |\zeta|^{-1} (1 - \delta)^{-1};\]

\[e^{-t\Lambda r} \|_{r \to q} \leq e^{t\Lambda r - \frac{r(q - \frac{1}{2})}{q - \frac{1}{2}}}, \quad 2 \leq r < q \leq \infty;\]

where $J_\zeta := (\zeta - \Delta)^{-\frac{1}{2}}, H_\zeta := |b|^\frac{1}{2} J_\zeta; S := |b|^\frac{1}{2} \cdot \nabla J_\zeta^3, b^\prime := |b|^{-\frac{1}{2}} b$.

As in the case of $b \in F_\delta$, it is reasonable to expect that there exists a strong dependence between the value of $\delta$ and smoothness of the solutions of the equation $(\zeta + \Lambda r)u = f, \zeta \in \rho(-\Lambda r), f \in L^r$.

Such a dependence does exist. Set

\[m_d := \pi^\frac{1}{2}(2e)^{-\frac{1}{2}}(d - 1)^{-\frac{d}{d+2}}, \quad \kappa_d := \frac{d}{d-1}, \quad r_\pm := \frac{2}{1 \pm \sqrt{1 - m_d}}.\]

It will be established that if $b \in F_\delta^{1/2}$ and $m_d \delta < 1$, then $(e^{-t\Lambda r}(b), r \in ]2, \infty[)$ extends by continuity to a quasi bounded $C_0$ semigroup in $L^r$ for all $r \in ]r_\pm, \infty[$. For every $r \in I_\delta := ]r_-, r_+[,$ the semigroup
ON THE THEORY OF THE KOLMOGOROV OPERATOR IN THE SPACES \( L^p \) AND \( C_\infty \).

is holomorphic, the resolvent set \( \rho(-\Lambda_r(b)) \) contains the half-plane \( \mathcal{O} := \{ \zeta \in \mathbb{C} \mid \Re \zeta > \kappa_d \lambda_d \} \), and the resolvent admits the representation
\[
(\zeta + \Lambda_r(b))^{-1} = (\zeta - \Delta)^{-1} - Q_r(1 + T_r)^{-1} G_r, \quad \zeta \in \mathcal{O},
\]
where \( Q_r, G_r, T_r \) are bounded linear operators on \( L^r \); \( D(\Lambda_r(b)) \subset W^{1+\frac{d}{q},r} (q > r) \).

In particular, for \( m_d \delta < 4 \frac{d-2}{(d-1)^2} \), there exists \( r \in I_s, \ r > d - 1 \), such that \((\zeta + \Lambda_r(b))^{-1} L^r \subset C^{0,\gamma}, \gamma < 1 - \frac{d-1}{r}\).

The results above yield the following: Let \( b \in F_{\delta}^{1/2} \) for some \( \delta \) such that \( m_d \delta < 4 \frac{d-2}{(d-1)^2} \). Let \( \{b_n\} \) be any sequence of bounded smooth vector fields, such that \( b_n \to b \) strongly in \( L^1_{\text{loc}} \), and, for a given \( \varepsilon > 0 \) and some \( \delta_1 \in ]\delta, \delta + \varepsilon] \), \( \{b_n\} \subset F_{\delta_1}^{1/2} \). Then
\[
\text{s-C}_\infty \lim_{n \uparrow \infty} e^{-t \Lambda b_n} \quad (**) \]
exists uniformly in \( t \in [0,1] \), and hence determines a \( C_0 \) semigroup \( e^{-t \Lambda b_n} \).

The results \((**), (***)) can be obtained via direct investigation in \( L^r \) of the operator-valued function \( \Theta_r(\zeta, b) \) defined by the right hand side of \((**)) without appealing to \( L^2 \) theory (but again appealing to the ideas of E. Hille and H. F. Trotter) [9].

In the next parts of this work we extend our results concerning the \( C_\infty \)-theory to the operator \( A + b \cdot \nabla \), and with time-dependent coefficients \( a, b \) [10].

Contents

Introduction
1. Markov generators associated with \( -\nabla \cdot a \cdot \nabla \)
2. \( b \cdot \nabla \) is \( A_r \) bounded
3. \( b \cdot \nabla \) is \( A \) form-bounded
3.1. Uniformly elliptic case I
3.2. The maximal interval of bounded \( L^p \) solvability for \( -\nabla \cdot a \cdot \nabla + b \cdot \nabla \)
3.3. Uniformly elliptic case II. The Hille-Lions approach
3.4. Non-divergent form operators
3.5. \( W^{1,s} \)-estimates on solutions to \((\mu - \Delta + b \cdot \nabla) u = f, b \in F_{\delta} \)
3.6. \( L^r \)-strong Feller semigroup on \( C_\infty \) corresponding to \( -\Delta + b \cdot \nabla \), \( b \in F_{\delta} \)
4. \( b \cdot \nabla \) is \( -\Delta \) weakly form-bounded
4.1. A variant of the Hille-Lions approach. \( L^2 \)-theory
4.2. The Hille-Trotter approach. \( L^2 \)-theory
4.3. \((L^p \to L^q) \) estimates
4.4. \( L^r \)-theory and \( W^{1+\frac{d}{2},r} \)-estimates on solutions to \((\lambda - \Delta + b \cdot \nabla) u = f, b \in F_{\delta}^{1/2} \)
4.5. \( L^r \)-strong Feller semigroup on \( C_\infty \) corresponding to \( -\Delta + b \cdot \nabla \), \( b \in F_{\delta}^{1/2} \)
Appendix A. Lattices
Appendix B. Composition of Lipschitz functions and functions in Sobolev spaces
Appendix C. Monotone Convergence Theorem for sesquilinear forms
Appendix D. The criteria of Phillips and Stampacchia
Appendix E. Trotter’s Approximation Theorem
1. Markov generators associated with $-\nabla \cdot a \cdot \nabla$

Throughout the paper we denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \to Y$, endowed with the operator norm $\| \cdot \|_{X \to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\| \cdot \|_{p} := \| \cdot \|_{L^{p}(\mu)}$.

1. Let $X$ be a set and $\mu$ a measure on $X$. Fix $p \in [1,\infty]$. A $C_{0}$ semigroup $T^{t}$, $t \geq 0$, of quasi contractions on $L^{p} = L^{p}(X,\mu)$ (i.e. $\| T^{t}f \|_{p} \leq e^{ctqT} \| f \|_{p}$, $f \in L^{p}$) is called Markov if, for each $t > 0$,

$$T^{t}L^{p}_{+} \subset L^{p}_{+}, \quad (f \in L^{p}, |f| \leq 1) \Rightarrow |T^{t}f| \leq 1.$$  

With each Markov semigroup $T^{t}$ we associate a collection $\{ T^{t}_{r} \}_{p \leq r < \infty}$ of consistent quasi contraction $C_{0}$ semigroups on the scale $[p,\infty]$ of $L^{r}$ spaces as follows.

Since $\| T^{t} \|_{p \to p} \leq 1$ and $\| T^{t}f \|_{p} \leq \| f \|_{p}$ ($f \in L^{p} \cap L^{\infty}$), by the Riesz Interpolation Theorem, $\| T^{t}f \|_{r} \leq \| f \|_{r}$ ($f \in L^{p} \cap L^{\infty}$) for all $r \in [p,\infty]$. Since $L^{p} \cap L^{\infty}$ is a dense subspace of $L^{r}$ for each $r \in [p,\infty]$, $T^{t} \mid L^{p} \cap L^{\infty} : L^{r} \to L^{r}$ extends by continuity to a semigroup in $L^{r}$:

$$T^{t}_{r} := (T^{t} \mid L^{p} \cap L^{\infty})_{clos \rightarrow L^{r}}.$$  

Next, by Hölder’s inequality:

$$\| (T^{t} - 1)f \|_{r} \leq 2 \| f \|_{\infty}^{1-\frac{1}{r}} \| (T^{t} - 1)f \|_{p}^{\frac{1}{r}} \quad (f \in L^{p} \cap L^{\infty}),$$

and hence we see that $T^{t}_{r}$ is strongly continuous.

Set $T^{t}_{p} := T^{t}$. Then $T^{t}_{r}$ and $T^{t}_{q}$, $r, q \in [2,\infty]$ are consistent:

$$T^{t}_{r} \mid L^{r} \cap L^{q} = T^{t}_{q} \mid L^{r} \cap L^{q}.$$  

Let $-A_{r}$ denote the generator of $T^{t}_{r}$. Then

$$A_{r} \upharpoonright \mathcal{D}(A_{r}) \cap \mathcal{D}(A_{q}) = A_{q} \upharpoonright \mathcal{D}(A_{r}) \cap \mathcal{D}(A_{q}) \quad (2 \leq r, q < \infty),$$

because $E := (1 + A_{r})^{-1}[L^{1} \cap L^{\infty}]$ is a core of $A_{r}$ for all $r \in [p,\infty]$.

**Definition.** Let $A \geq 0$ be a self-adjoint operator in $L^{2}(X,\mu)$. The semigroup $T^{t} = e^{-tA}$, $t \geq 0$, is called symmetric Markov if, for all $t > 0$,

$$T^{t}L^{2}_{+} \subset L^{2}_{+}, \quad (f \in L^{2}, |f| \leq 1) \Rightarrow |T^{t}f| \leq 1.$$  

With each symmetric Markov semigroup $T^{t}$ we associate the collection $\{ T^{t}_{r} \}_{1 \leq r \leq \infty}$ of contraction semigroups defined by

$$T^{t}_{r} := (T^{t} \mid L^{1} \cap L^{\infty})_{clos \rightarrow L^{r}} \quad (r \in [1,\infty]),$$

$$T^{t}_{\infty} := (T^{t})^{\ast}.$$
It is easily seen that the semigroup $T_t^1$ is strongly continuous for each $r \in ]1, \infty[$. (It is also known that $T_t^1$ is strongly continuous as well.) We have $A_r \cap D(A_r) \cap D(A_q) = A_q \cap D(A_r) \cap D(A_q)$ $(1 \leq r, q < \infty)$.

2. Let $\Omega$ be an open set in $\mathbb{R}^d$, $d \geq 3$,

$$a(x) = a(x)^* : \Omega \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d, \quad a(x) \in [L^1_{\text{loc}}(\Omega, \mathcal{L}^d)]^{d \times d},$$

$$\sigma I \leq a(x)$$

for some constant $\sigma > 0$ and $\mathcal{L}^d$ a.e. $x \in \Omega$. \hspace{1cm} (H1)

One can define at least three realizations of the differential expression $-\nabla \cdot a \cdot \nabla$ in $L^2 = L^2(\Omega, \mathcal{L}^d)$ as (minus) symmetric Markov generators: $A_D, A_{iD}, A_N$, the Dirichlet, intermediate Dirichlet, and generalized Neumann \cite{20};

Let $\mathcal{T}$ denote a collection of all closed, symmetric non-negative quadratic forms in $L^2$. Define the pre-Dirichlet form $\varepsilon$ by setting

$$\varepsilon[u, v] := \langle \nabla u \cdot a \cdot \nabla v \rangle, \quad D(\varepsilon) = C^1_c(\Omega) \times C^1_c(\Omega),$$

Here

$$\nabla u \cdot a \cdot \nabla v(x) := \sum_{i,k=1}^d a_{ik}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_k}, \quad \langle f \rangle := \int_\Omega f d\mathcal{L}^d.$$ 

Let

$$\mathcal{T}_M := \{ \tau \in \mathcal{T} \mid \tau \leftrightarrow T, \ e^{-st}, s > 0, \ \text{is a symmetric Markov semigroup} \}$$

($\tau \leftrightarrow T$ denotes the one to one correspondence (Appendix \[C\]).)

Let

$$\mathcal{T}_M(\varepsilon) := \{ \tau \in \mathcal{T}_M \mid \tau \supset \varepsilon \}.$$ 

It is said that $\tau \in \mathcal{T}_M$ is local if $\tau[f, g] = 0$ whenever $(f, g) \in D(\tau)$, $f, g \geq 0$, and $f \land g = 0$.

Consider the following extensions of $\varepsilon$ :

$$\tau_D := \varepsilon^\text{clo}$$(the closure of $\varepsilon$),

$$\tau_{iD} \supset \tau_D, \quad D(\tau_{iD}) := \{ u \in W^{1,2}_0(\Omega) \mid \langle \nabla u \cdot a \cdot \nabla u \rangle < \infty \},$$

$$\tau_N \supset \tau_{iD}, \quad D(\tau_N) := \{ u \in W^{1,2}(\Omega) \mid \langle \nabla u \cdot a \cdot \nabla u \rangle < \infty \}.$$ 

Lemma 1.1. $\tau_D, \tau_{iD}, \tau_N \in \mathcal{T}_M(\varepsilon)$, and are local.

Proof. Define

$$a^n(\cdot) := \sigma I + (a(\cdot) - \sigma I) \left( I + \frac{1}{n} a(\cdot) \right)^{-1}, \quad n = 1, 2, \ldots$$

Clearly, $\sigma I \leq a^n(\cdot) \leq (n + \sigma)I$ and $a^n(\cdot) \leq a^{n+1}(\cdot) \leq a(\cdot) \mathcal{L}^d$ a.e. .

Let $E = W^{1,2}_0(\Omega)$ or $W^{1,2}(\Omega)$. Define

$$\tau^n[u, v] := \langle \nabla u \cdot a^n \cdot \nabla v \rangle, \quad D(\tau^n) = E,$$

$$\varepsilon^n[u, v] := \langle \nabla u \cdot a^n \cdot \nabla v \rangle, \quad D(\varepsilon^n) = D(\varepsilon).$$
It is well known that \( \tau^n \in \mathcal{T}_M(\varepsilon^n) \), and are local (by verifying the Beurling-Dini conditions, using properties of Sobolev’s spaces, such as “\( C^\infty(\Omega) \cup W^{1,2}(\Omega) \) is dense in \( W^{1,2}(\Omega') \)”, and Fukushima’s test function \( f_\varepsilon = \varepsilon \phi(1) \)), \( \phi \) is a \( C^\infty \) function from \( \mathbb{R} \) to \([0, \infty[, \phi(0) = 0, |\phi'(x)| \leq 1 \) and \( \phi(x) = |x| - 1 \) if \( |x| \geq 2 \); otherwise, see [5 Sect. 1.4] or [4 Theorems 1.3.5, 1.3.9]).

Define \( \tau \) by

\[
\tau[u, v] := \lim_n \tau^n[u, v],
\]

\[
D(\tau) := \{ u \in E \mid \sup_n \langle \nabla u \cdot a^n \cdot \nabla u \rangle < \infty \}
\]

\[
= \{ u \in E \mid \langle \nabla u \cdot a \cdot \nabla u \rangle < \infty \}.
\]

By the Monotone Convergence Theorem for a non-decreasing sequence of closed symmetric non-negative quadratic forms (Appendix [C]), \( \tau \in \mathcal{T} \) and, obviously, \( \tau \in \mathcal{T}_M(\varepsilon) \) and is local. Since \( \tau \supset \varepsilon \), \( \varepsilon \) is closable. Thus \( \tau_D \) is well defined. Now it is clear that \( \tau_D \) belongs to \( \mathcal{T}_M(\varepsilon) \) and is local.

**Remarks.** 1. The operators \( A_D, A_iD, A_N \) associated with \( \tau_D, \tau_i, \tau_N \), respectfully, possess some nice properties. \( C^\infty(\Omega) \) is a form core for \( A_D \). If \( A_2(a) \) stands for \( A_iD \) or \( A_N \), then \( e^{-sA_p(a)} = \lim_n e^{-sA_p(a^n)} \), \( p \in ]1, \infty[ \). At least \( e^{-sA_D} \) and \( e^{-sA_{iD}} \) have nice embedding properties \( (L^p \to L^q) \).

2. \( \tau_D \) is the maximal element of \( \mathcal{T}_M(\varepsilon) \) endowed with the semi-order \( \tau_1 < \tau_2 \iff D(\tau_1) \supset D(\tau_2), \tau_1[u] \leq \tau_2[u], u \in D(\tau_2) \).

### 2. \( b \cdot \nabla \) is \( A_r \) bounded

Let \( \Omega \) be an open set in \( \mathbb{R}^d \), \( d \geq 3 \), and \( L^p = L^p(\Omega, \mathcal{L}^d) \). Let \( a : \Omega \to \mathbb{R}^d \) be a symmetric \( \mathcal{L}^d \) measurable strictly positive \((a \in (H_1)) \) matrix. Let \( b : \Omega \to \mathbb{C}^d \) be \( \mathcal{L}^d \) measurable and define \( B_r = B_r(b) \) to be the maximal operator of \( b \cdot \nabla \) in \( L^r \), \( 1 \leq r < \infty \), of domain

\[
D(B_r) := \{ h \in L^r \mid |\nabla h| \in L^1_{\text{loc}} \text{ and } b \cdot \nabla h \in L^r \}.
\]

Throughout this section \( A \) denotes \( A_D, A_iD, A_N \). In case of \( A_N \) we also assume that \( \partial \Omega \in C^{0,1} \).

**Definition.** We say that \( b \cdot \nabla \) is \( A_r \) bounded if \( \|B_r(\lambda + A_r)^{-1}\|_{r \to r} < \infty, \lambda > 0 \).

1. Set \( j := \frac{d}{d+2} \) and \( j' = \frac{d}{2} \).

**Proposition 2.1.** Assume that \( b_a^2 := b \cdot a^{-1} \cdot \overline{b} \leq W + C \mathcal{L}^d \) a.e. for a function \( W \in L^j' \) and a non-negative constant \( C \). Then, for every \( r \in ]1, j' \land 2[, \) the operator \( B_r \) is \( A_r \)-bounded and the inequality

\[
\|B_r(\lambda + A_r)^{-1}\|_{r \to r} \leq 4 \left( \frac{C}{(r-1)\lambda} \right)^{1/2} + 4c(r, j)||W||^{1/2}_{j'} \quad (\lambda > 0)
\]

holds with the constant \( c(r, j) \to \infty \) as \( r \to 1 \). If \( b_a^2 \in L^\infty \), then the assertion above is valid for all \( 1 < r \leq 2 \). Moreover, for a given \( \eta > 0 \), we can choose \( W_1, C_1 \) and then \( \lambda > 0 \) such that \( b_a^2 \leq W_1 + C_1 \) and

\[
4 \left( \frac{C_1}{(r-1)\lambda} \right)^{1/2} + 4c(r, j)||W_1||^{1/2}_{j'} < \eta.
\]
Proof. Set \( u = (\lambda + A_r)^{-1}h \), \( h \in L^1 \cap L^\infty \). Let \( 1 < p \leq 2 \). Since \( u \in D(A_p) \) \( \Rightarrow u_p := u^{\frac{p}{2}} \in D(A^{1/2}) \) (Appendix G Theorem G.1.1), we conclude that \( \nabla u_{p} \in L^2 \). Since \( 2 \geq p \) and \( u \in L^\infty \),
\[
\nabla u = \nabla u_{p}^{2/p} = (2/p)u_{p}^{2/p - 1} \left( \frac{\lambda}{p} \right) \nabla u_{p} = (2/p)u_{p}^{2/p - 1} \nabla u_{p}
\]
(Appendix [B]. Now, \( |b \cdot \nabla u|^2 \leq b_a^2 \nabla u \cdot a \cdot \nabla \bar{u} = (2/p)^2 b_a^2 u^{2/p} \nabla u_p \cdot a \cdot \nabla \bar{u}_p \), and so by Hölder’s inequality,
\[
\|b \cdot \nabla u\|_r^2 \leq (2/p)^2 \|b_a^2 u^{2/p} \|_{r/p} \langle A^{1/2} u_p, A^{1/2} u_p \rangle \leq (p - 1)^{-1} \|b_a^2 u^{2/p} \|_{r/p} \langle A_p u, u^{p-1} \rangle.
\]
If \( b_a^2 \leq C \), then \( \|b \cdot \nabla u\|_r^2 \leq \frac{C}{r-1} \|u\|_r^{2-r} \langle A_p u, u^{r-1} \rangle \), and since \( \|u\|_r \leq \lambda^{-1} \|h\|_r \) and \( \langle A_p u, u^{r-1} \rangle \leq \|h\|_r \|u\|_r^{r-1}, \)
\[
\|b \cdot \nabla u\|_r \leq \sqrt{\frac{C}{(r-1)\lambda}} \|h\|_r. \tag{\textbullet}\]
Similarly, if \( b_a^2 \leq W \), then the same argument yields
\[
\|b \cdot \nabla u\|_r^2 \leq (p - 1)^{-1} \|h\|_r \|u\|_{r/p-1}^{p-1} \|W^{2/p} \|_{r/p}, \quad \text{where} \quad (p - 1)^{-1} = r'(r^{-1} - j^{-1}).
\]
Therefore, by Hölder’s inequality and the embedding \( (\lambda + A_q)^{-1} : L^q \to L^{s_r} \), \( q^{-1} - s^{-1} = j^{-1} \), we have
\[
\|b \cdot \nabla u\|_r \leq c(r, j) \|W\|_{j'}^{1/2} \|h\|_r, \quad h \in L^1 \cap L^\infty.
\]
It remains to pass in this inequality and in (\textbullet) to an arbitrary \( h \in L^r \). Using the facts that the weak gradient is closed in \( L^r \), and \( L^1 \cap L^\infty \) is dense in \( L^r \), we have \( \|1_{\{|h| \leq a\}} b \cdot \nabla u\|_r \leq c(r, j) \|W\|_{j'}^{1/2} \|h\|_r \) for all \( h \in L^r_a \). The use of Fatou’s Lemma now completes the proof.

Remarks. 1. The implication \( h \in D(A_r) \Rightarrow \nabla h \in \left[ L^r \right]^d \) for \( r \in ]1, 2] \) follows now e.g. from a priori estimate \( \|\nabla (1 + A)^{-1}h\|_r \leq C(r) \|h\|_r, \quad h \in L^1 \cap L^\infty \), which in turn is a simple corollary of (\textbullet). We also mention that, in general, even the condition \( b_a^2 \in L^\infty \) allows \( b \) to be unbounded.

2. In terms of the \( L^p \) spaces, the assumption \( W \in L^j \) cannot be weakened even in the case \( a = I. \) Also, it does not guarantee that \( b \cdot \nabla \) is \( A_1 \) bounded.

3. Let us recall a simple corollary of the Hille Perturbation Theorem (see e.g. [8] Ch. IX, sect. 2.2)):

**Theorem 2.1.** Let \( e^{-tA} \) be a symmetric Markov semigroup, \( K \) a linear operator in \( L^r \) for some \( r \in ]1, \infty[ \). If for some \( \lambda > 0 \) \( \|K(\lambda + A_r)^{-1}\|_{r \to r} < \frac{1}{2} \), then \( -\Lambda_r := -A_r - K \) of domain \( D(A_r) \) is the generator of a quasi bounded holomorphic semigroup on \( L^r \).

Thus, Proposition 2.1 implies that the (minus) algebraic sum \( A_r + B_r \) is the generator of a holomorphic semigroup in \( L^r \) for every \( r \in ]1, j' \wedge 2[ \).

Instead, one can employ the Miyadera Perturbation Theorem [18] (see also [22]):

**Theorem 2.2.** Let \( e^{-tA} \) be a symmetric Markov semigroup, \( K \) a linear operator in \( L^r \) for some \( r \in ]1, \infty[ \). If for some \( \lambda > 0 \) \( \|K(\lambda + A_r)^{-1}\|_{r \to r} < 1 \), then \( -\Lambda_r := -A_r - K \) of domain \( D(A_r) \) is the generator of a quasi bounded \( C_0 \) semigroup on \( L^r \).
Thus, again Proposition [21] implies that the (minus) algebraic sum $A_r + B_r$ is the generator of a quasi bounded $C_0$ semigroup in $L^r$ for every $r \in [1, 2]$.  

2. Now we specify the results of previous subsection for $b : \Omega \rightarrow \mathbb{R}^d$. Denote $T^t_r = e^{-t(\lambda + \Lambda_r(b))}$, where $\Lambda_r(b) = A_r + B_r$ of domain $D(\Lambda_r(b)) = D(A_r)$. Since $b$ is real valued, $T^t_r \text{Re} L^r \subset \text{Re} L^r$. We claim that $T^t_r$ is a positivity preserving $L^\infty$ contraction semigroup. We prove this here only for $b_a \in L^\infty$ by verifying the criteria of R. Phillips and G. Stampacchia for $T^t_2$ (Appendix D). 

Let $b_a^2 \leq C$, $2C < \lambda$. Set $\Gamma = \lambda + \Lambda_2(b)$. It is seen that, for $f \in D(A)$, $\text{Re} \langle \Gamma f, f \rangle \geq 0$, and so $T^t_2$ is a contraction. For any $f \in \text{Re} L^2$ define $f_+ = f \vee 0$. If $f = \text{Re} f \in D(A)$, then 

$$\langle \Gamma f, f_+ \rangle = \lambda \langle f_+^2 \rangle + \langle A f, f_+ \rangle + \langle b \cdot \nabla f, f_+ \rangle.$$ 

Taking into account that $A$ is local and $f_+ \in D(A^{1/2}) \subset W^{1,2}_0$, we obtain 

$$\langle A f, f_+ \rangle = \langle \nabla f \cdot a \cdot \nabla f_+ \rangle = \langle \nabla f_+ \cdot a \cdot \nabla f_+ \rangle,$$

$$\langle b \cdot \nabla f, f_+ \rangle \leq \langle b \cdot a^{-1} \cdot b f_a^2 \rangle^{1/2} \langle \nabla f_+ \cdot a \cdot \nabla f_+ \rangle^{1/2} \leq C \langle f_+^2 \rangle + \langle \nabla f_+ \cdot a \cdot \nabla f_+ \rangle.$$ 

Thus $\langle \Gamma f, f_+ \rangle \geq 0$ for all $f = \text{Re} f \in D(A) = D(\Gamma)$. 

Next, let $f \in D(A)$. Clearly, $f_\wedge := (1 \wedge |f|) \text{sgn} f$ and $f - f_\wedge$ are from $D(A^{1/2})$,

$$\nabla (f - f_\wedge) = 1_{|f| > 1} [\nabla f f_\wedge + f f_\wedge] \text{ and } \text{Re} \nabla f = |f| \nabla |f|.$$ 

Setting $\psi = 1_{|f| > 1} (|f| - 1)$ we have 

$$\text{Re} \langle \Gamma f, f - f_\wedge \rangle = \lambda \langle |f| \psi \rangle + \langle \nabla \psi \cdot a \cdot \nabla \psi \rangle + \langle b \cdot \nabla \psi, \psi \rangle \geq 0.$$ 

Thus $\text{Re} \langle \Gamma f, f - f_\wedge \rangle \geq 0$ for all $f \in D(A) = D(\Gamma)$. 

Corollary 2.1. $T^t_2$ is a Markov semigroup. 

Now let $r \in [1, 2]$. By the construction of $\Lambda_r$ and $\Lambda_2$, $T^t_r f = T^t_2 f$ whenever $f \in L^r \cap L^2$. Therefore, for each $r \in [1, 2]$, the semigroup $T^t_r$ preserves positivity and is $L^\infty$ contraction. 

3. $b \cdot \nabla$ is $A$ form-bounded 

Throughout this section we are assuming that $\Omega$ is an open set of $\mathbb{R}^d$, $d \geq 3$, $a : \Omega \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$, a symmetric $L^d$ measurable strictly positive ($a \in (H_1)$) matrix. 

By $A$ we denote one of the $A_D, A_{iD}, A_N$. In the results concerning $(L^P, L^Q)$ estimates for $A_N$ we also assume that $\partial \Omega \in C^{0,1}$. 

Definition. We say that a $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ belongs to $F_\delta(A)$, the class of $A$ form-bounded vector fields, and write $b \in F_\delta(A)$, if $b_a := \sqrt{b \cdot a^{-1} \cdot b} \in L^2_{\text{loc}}$ and there exists a constant $0 \leq \lambda = \lambda_\delta < \infty$ such that

$$\|b_a(\lambda + A)^{-1/2}\|_{2 \rightarrow 2} \leq \sqrt{\delta}.$$ 

(Equivalently, $b_a^2 \leq \delta A + c(\delta)$ in the sense of quadratic forms for some constant $c(\delta) (= \lambda \delta)$.)
Remark. If $K$ is a positivity preserving linear operator on $L^p$ or $C_\infty$ (e.g. a Markov semigroup or the resolvent of its generator), then
\[ |Kf| \leq K|f|, \quad (f \in L^p \text{ or } C_\infty, \text{ respectively}) \]
(Appendix A). This fact will be extensively used below.

If $b \in F_\delta(A)$, $0 < \delta < 1$, then the Hille-Lions theory (section 3.3) yields the following: In $L^2$, there exists an operator realization $\Lambda \equiv \Lambda(b)$ of the formal differential operator $A + b \cdot \nabla$ such that $D(\Lambda) \subset D(A^{\frac{1}{2}})$, $\rho(-\Lambda) \supset O := \{ \zeta \in \mathbb{C} \mid \text{Re} \zeta > \lambda_0 = \frac{\lambda}{2(1+\sqrt{1-\delta})} \}$. If $\zeta \in O$, then
\[ R_\zeta = R_{a,\zeta}^{\frac{1}{2}}(1+T)^{-1}R_{a,\zeta}^{\frac{1}{2}}, \quad (i_0) \]
where $R_\zeta \equiv R_\zeta(b) := (\zeta + \Lambda)^{-1}$, $R_{a,\zeta} := (\zeta + A)^{-1}$, $T := R_{a,\zeta} b \cdot \nabla R_{a,\zeta}$. \[ e^{-t\Lambda(b_a)} \xrightarrow{\mathcal{S}} e^{-t\Lambda} \text{ as } n \uparrow \infty, \quad (ii_0) \]

\[ (1 + A)^{\frac{1}{2}}(R_\zeta(b_n) - R_\zeta(b)) \xrightarrow{\mathcal{S}} 0 \quad \text{as } n \uparrow \infty, \quad (iii_0) \]
where $b_n := 1_n b$, $1_n$ is the indicator of $\{ x \in \Omega : |b_a(x)| \leq n \}$.

Remark.
\[ e^{-t(\lambda_0 + \Lambda)} \text{ is a Markov semigroup.} \quad (i) \]
For all $2 \leq r < q \leq \infty$ and $t > 0$,
\[ \|e^{-t\Lambda_r}\|_{r \to q} \leq c e^{2\lambda_0 t} t^{\frac{d}{2} - 1 - \frac{d}{q}}. \quad (ii) \]
Indeed, $(i)$ follows from $(ii_0)$ and the results of section 2.2. Also, slightly modifying the arguments from section 2.2, it is not difficult to verify directly that $\lambda_0 + \Lambda(b)$ obeys the conditions of R. Phillips and G. Stampacchia. In turn, $(i_0) + (i)$ entails $(ii)$ by modifying the arguments from the proof of Theorem 4.1 below.

Now, we develop the semigroup theory of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ in the $L^r \equiv L^r(\Omega, C_\delta)$ spaces in the case $0 < \delta < 4$.

Theorem 3.1. Assume that $b \in F_\delta(A)$ for some $0 < \delta < 4$. If $1 \leq \delta < 4$ also assume that $b_a \in L^2 + L^\infty$. Set $r_\delta := \frac{2}{2-\sqrt{\delta}}$. Then $\nabla \cdot a \cdot \nabla - b \cdot \nabla$ has an operator realization $-\Lambda_r(b)$ in $L^r$ for any $r \in I_\delta := [r_\delta, \infty]$ as the generator of a positivity preserving, $L^\infty$ contraction, quasi contraction $C_0$ semigroup on $L^r$. In full:

Let $1_n$ denote the indicator of $\{ x \in \Omega \mid |b(x)| \leq n \}$ and set $b_n := 1_n b$. Then
\[ e^{-t\Lambda_r(b)} := s\cdot L^r \cdot \lim_{n \to \infty} e^{-t\Lambda_r(b_n)} \quad (r \in I_\delta^c := [r_\delta, \infty]); \quad (*) \]

\[ e^{-t\Lambda_{r_\delta}(b)} := \left[ e^{-t\Lambda_r(b)} \mid L^1 \cap L^r \right]_{L^r \to L^{r_\delta}}^{\text{clos}} \quad (r \in I_\delta^c); \quad (**) \]

\[ \|e^{-t\Lambda_r(b)}\|_{r \to r} \leq e^{t\omega_r}, \quad \omega_r = \frac{\lambda_\delta}{2(r - 1)}, \quad (r \in I_\delta); \]
There is a constant $c = c(\delta, d)$ such that the $(L^r, L^q)$ estimate

$$\|e^{-t\Lambda_r(b)}\|_{r \to q} \leq c e^{\lambda_{t,r}} t^{-\frac{d}{2} \left(\frac{1}{r} - \frac{1}{q}\right)}$$

is valid for all $r_\delta < r < q \leq \infty$, $t > 0$.

For each $r \in I_c^0$, $e^{-t\Lambda_r(b)}$ is a holomorphic semigroup of quasi contractions on the sector

$$|\arg t| \leq \frac{\pi}{2} - \theta_r, \quad 0 < \theta_r < \frac{\pi}{2}, \quad \tan \theta \leq \mathcal{K}(2 - r' \sqrt{\delta})^{-1},$$

where $K = \frac{r^2 - 2}{\sqrt{r - 1}} + r' \sqrt{\delta}$ if $r \leq 2r_\delta$ and $K = \frac{r^2 - 2}{\sqrt{r - 1}}$ if $r > 2r_\delta$.

Remarks. 1. The additional assumption $b_0 \in L^2 + L^\infty$ in case $1 \leq \delta < 4$, for a general $a \in (H_1)$, seems to be close to optimal. For more regular $a$'s no such assumption is needed. See Theorem 3.2 below and the remark after.

2. The interval $I_c = [\frac{\delta}{2 - \sqrt{\delta}}, \infty[$ is called the interval of contraction solvability.

The interval $I_m := [\frac{\delta}{2 - \sqrt{\delta}}, \infty[$ is called the interval of bounded solvability. We will show below (Therem 3.3) that if the matrix $a$ is uniformly elliptic, then, for any $r \in I_m - I_c$, one can still define $e^{-t\Lambda_r}$ as a quasi bounded holomorphic semigroup. We also note that $I_m \ni d$, whilst $I_c \ni d$ whenever $\sqrt{\delta} \leq 2d - 1$.

3. The example of $\Lambda_r(b) \supset L = -\Delta + b \cdot \nabla$, $b(x) = c|x|^2 x$, $c = \frac{d^2 - 2}{d} \sqrt{\delta} (\delta < 4)$ in $\mathbb{R}^d$, $d \geq 3$, shows that both intervals $I_c, I_m$ are maximal. Any $c < d - 2$ is admissible according to Theorem 3.1. In turn, $c = d - 2$ makes $I_c = \emptyset$, while $c = d$ makes $I_m = \emptyset$ even formally ($c = d \Rightarrow \delta = 4\left(\frac{d}{d - 2}\right)^2 > 4$).

In $\Omega = B(0, 1) := \{x \in \mathbb{R}^d \mid |x| < 1\}$ consider the Dirichlet problem

$$Lu = 0, \quad u = 0 \text{ on } \partial \Omega.$$  \hspace{1cm} (D)

Obviously, $u_1 = 0$, $u_2 = |x|^{\alpha} - 1$, $\alpha = c - (d - 2)$, are solutions of (D); $u_2 \in L^r$ for all $0 < r < \frac{d}{\delta} = \frac{2}{2 - \sqrt{\delta}} \frac{d}{d - 2} \equiv r_\delta j$. In particular, $u_2 \in L^r$ for all $r_\delta < r < r_\delta j$, and satisfies the maximum principle. If $u_2$ would be a solution to $\Lambda_r v = 0$ (equivalently $(\mu + \Lambda_r)v = \mu v (\mu > 0)$) for some $r \in I_c^0$, then $v$ should belong to $L^j \cap L^\infty$. Thus $u_2 \notin D(\Lambda_r), r \in I_c$.

If $\delta > 4$, then $\alpha > 0$, and so the problem (D) has two bounded weak solutions. By the way, now $u_2$ does not satisfy the maximum principle. See also an example in 3.4.

Thus, the assumption $\delta > 4$ (even for $a = I$) destroys the uniqueness of (accordingly defined) weak solutions.

Proof. 1. First of all, let us prove that for all $r \in I_c, n \geq 1$, $t \geq 0$,

$$\|e^{-t\Lambda_r(b_n)}\|_{r \to r} \leq e^{\lambda_{t,r}}.$$  \hspace{1cm} (*)

Set $u \equiv u_n := e^{-t\Lambda_r(b_n)}[h] = e^{-t\Lambda_{2r}(b_n)}[h]$, $h \in L^1 \cap L^\infty$. By the results of section 2.2, $u \in D(A) \cap L^\infty_+$. If $r \geq 2$, then by Theorem G.1 (Appendix G), $u, u^u, u^r \in D(A^u)$, and since $D(A^u) \subset E (E = W^1_2)$ if $A = A_D$ or $A_{c,D}$, and $E = W^{1,2}$ if $A = A_N$, $\nabla u^u = \frac{1}{r} u^r$, $\nabla u^r = \frac{1}{r} u^r - 1 \nabla u^u$ because $r \geq 2$ and $u \in L^\infty$; if $1 < r \leq 2$, then $u^{r - 1} \nabla u = \frac{1}{r} u^r = \frac{1}{r} u^r - 1 \nabla u$ because $u \in D(A_r)$ and $u^{r - 1}, u^r \in L^\infty$. In this connection see Appendix B.

Noticing that $\Lambda_r(b_n)u = A_r u + b_n \cdot \nabla u$ if $1 < r \leq 2$, and $\Lambda_r(b_n)u = \Lambda_2(b_n)u = Au + b_n \cdot \nabla u$, $Au \in L^2, u^{r - 1} \in L^2$, $\langle Au, u^{r - 1} \rangle \geq \frac{d}{r^{\frac{3}{2}}} \|A^u u^r\|_2^2$ if $r \geq 2$, we have

$$-\frac{d}{dt} \langle u^{r - 1}, u^{r - 1} \rangle = \langle \Lambda_r(b_n)u, u^{r - 1} \rangle = \langle A_r u, u^{r - 1} \rangle + \langle b_n \cdot \nabla u, u^{r - 1} \rangle, \quad 1 < r \leq 2,$$
\[-\frac{d}{dt} u, u^{r-1}\rangle = \langle \Lambda_r(b_n)u, u^{r-1}\rangle = \langle Au, u^{r-1}\rangle + \langle b_n \cdot \nabla u, u^{r-1}\rangle, \ r \geq 2.\]

Therefore,
\[-\frac{d}{dt} \|u\|_r^r \geq \frac{4}{r'} \langle A_{\frac{r}{2}}^\frac{1}{2} u^\frac{r}{2}, A_{\frac{r}{2}}^\frac{1}{2} u^\frac{r}{2}\rangle + 2 \langle u^\frac{r}{2} b_n \cdot \nabla u^\frac{r}{2}\rangle.\]

Using the conditions \(r \in I_c, \ b \in F_\delta(A)\) and completing the quadratic estimate
\[2\langle u^\frac{r}{2} b_n \cdot \nabla u^\frac{r}{2}\rangle \leq \varepsilon \|b_n u^\frac{r}{2}\|_2^2 + \varepsilon^{-1} \|A_{\frac{r}{2}}^\frac{1}{2} u^\frac{r}{2}\|_2^2 \leq (\varepsilon \delta + \varepsilon^{-1}) \|A_{\frac{r}{2}}^\frac{1}{2} u^\frac{r}{2}\|_2^2 + \varepsilon \|b_n\|_r \|u\|_r^r,\]

we obtain (choosing \(\varepsilon = \frac{r'}{2}\)) and taking into account that \(\delta \leq \frac{\delta}{r'}\) for \(r \in I_c\)
\[2\langle u^\frac{r}{2} b_n \cdot \nabla u^\frac{r}{2}\rangle \leq \frac{4}{r'} \|A_{\frac{r}{2}}^\frac{1}{2} u^\frac{r}{2}\|_2^2 + \frac{c(\delta) r'}{2} \|u\|_r^r.\]

Thus \(\frac{d}{dt} \|u\|_r^r \leq \frac{c(\delta) r'}{2} \|u\|_r^r\). The desired bound (\(\ast\)) follows.

Since the mapping \(I_c \ni p \to \|f\|_p, \ f \in L^1 \cap L^\infty\) is continuous, (\(^\ast\)) yields the bound
\[\|e^{-t \Lambda_r(b_n)}h\|_\infty \leq \|h\|_\infty \quad (h \in L^1 \cap L^\infty).\]

Next, if \(\delta < 1\), then \(u_n \overset{\delta}{\to} u\), and hence \(u_n \to u\) strongly in \(L^r, r \in I_c^0\).

2. To justify (\(\ast\)) when \(1 \leq \delta < 4\), we use the direct method: setting \(g = u_m - u_n\), we will prove that, for each \(r \in I_c^0\), \(\|g\|_r \to 0\) as \(n, m \uparrow \infty\) uniformly on \(t \in [0, 1]\). Obviously, the latter combined with (\(\ast\)) will yield (\(\ast\)).

Since \(-\frac{d}{dt} g = A g + b_m \cdot \nabla g + (b_m - b_n) \cdot \nabla u, \ u \equiv u_n\), we have, multiplying both sides of the equation by \(|g|^{r-2} g\), integrating over \(\Omega\), using the assumptions \(b \in F_\delta(A), \ b_a^2 = V_1 + V_\infty, \ V_1 \in L^1, \) and \(r > r_\delta,\)
\[\frac{d}{dt} \|g\|_r^r \leq \frac{c(\delta)}{\sqrt{\delta}} \|g\|_r^r + r \langle 1 - 1_m, b \sqrt{\nabla u \cdot a \cdot \nabla u} |g|^{r-1}\rangle \leq \frac{c(\delta)}{\sqrt{\delta}} \|g\|_r^r + r \langle 1 - 1_m, V_1 \rangle \|g\|_\infty^{r-1} \langle Au, u \rangle.\] (\(\ast\))

To get a suitable bound on \(\|g\|_r\) we need the following estimates
\[\int_0^T e^{-ct} \langle Au, u \rangle dt \leq \|h\|_2^2 + \|V_1 \|_1 \|h\|_\infty^2 \quad (c := \|V_\infty\|_\infty).\]

Evidently, \(-\frac{d}{dt} \|u\|_2^2 = 2 \langle Au, u \rangle + 2 \langle ub_n \cdot \nabla u \rangle \leq \langle Au, u \rangle - \langle b_a^2 u^2 \rangle.\) Setting \(\psi(t) = e^{-ct} \|u\|_2,\) and using the bound \(\|u\|_\infty \leq \|h\|_\infty,\) we have \(e^{-ct} \langle Au, u \rangle \leq -\frac{d}{dt} \psi(t) + e^{-ct} \|V_1 \|_1 \|h\|_\infty^2,\) and so \(\int_0^T e^{-ct} \langle Au, u \rangle dt \leq \psi(0) + \frac{1}{c} \|V_1 \|_1 \|h\|_\infty^2 \leq \|h\|_2^2 + \|V_1 \|_1 \|h\|_\infty^2.\) The required estimate is proved.

Now using this estimate, the bound \(\|g\|_\infty \leq 2 \|h\|_\infty\) and the equality \(g|_{t=0} = 0\) one can easily integrate (\(\ast\)), obtaining sup\(t \in [0, T]\) \(\|g(t)\|_2^2 \leq C(\|1 - 1_m\|V_1 \|_1,\) where the constant \(C\) depends on \(T, r, \delta, \|h\|_2, \|h\|_\infty, \|V_1 \|_1, \|V_\infty\|_\infty\) only. The latter makes \(\|g\|_r \to 0\) as \(n, m \uparrow \infty\) uniformly in \(t \in [0, T].\) Thus the sup-\(\langle Au, u \rangle \) does define a \(C_0\) semigroup. We have proved (\(\ast\)). Now it should be clear that
\[\|e^{-t \Lambda_r(b_n)}f\|_{r_\delta} \leq e^{k \omega r_\delta} \|f\|_{r_\delta} \quad (r_\delta \in I_c, \ f \in L^1 \cap L^\infty)\] (\(\ast\))
and \( e^{-t\Lambda_{r}(b)} \) defined by \((**)\) is indeed a semigroup. The fact that it is strongly continuous follows from \((*)\) and \( \|f - e^{-t\Lambda_{r}(b)}f\|_{r} \to 0 \) as \( t \downarrow 0 \) for any \( r \in I_{c}^{0} \) and any \( f \in L^{1} \cap L^{\infty} \) by employing Fatou’s Lemma.

3. A proof of \((***)\) presented below is based on the embedding property of \( A_{\delta}^{2} \) and \((*)\), the \( L^{r} \) quasi contraction property of \( e^{-t\Lambda_{r}(b_{n})} \). In view of \((*)\), it suffices to treat the case \( \Lambda(b_{n}) \). We have for \( u := e^{-t\Lambda(b_{n})}f, f \in L^{1}_{+} \cap L^{\infty}, r = 2\delta, \)

\[
-\frac{d}{dt} \|u\|_{r}^{2} \geq \left( \frac{4}{t^{r}} - 2\sqrt{\delta} \right) \|A_{\delta}^{2}u\|_{2}^{2} - \frac{c(\delta)}{\sqrt{\delta}} \|u\|_{r}^{2}.
\]

Note that \( \frac{2}{r} > \sqrt{\delta} \). Using the Nash inequality \( \|A_{\delta}^{2}h\|_{2}^{2} \geq C_{N}\|h\|_{2}^{2} - \frac{4}{\sqrt{\delta}} \|h\|_{1} - \frac{1}{\sqrt{\delta}} \) and \((*)\), we have setting \( w := \|u\|_{r}^{2}, \)

\[
-\frac{d}{dt}w \geq -c_{2}w^{\frac{2}{r}} + c_{1}e^{-c_{2}t} \|f\|_{r}^{2},
\]

where \( c_{1} = C_{N}\frac{2}{\delta^{2}} \left( \frac{4}{r^{2}} - 2\sqrt{\delta} \right), c_{2} = \frac{2}{r}c(\delta), c_{3} = \frac{2}{r}\omega_{r}. \)

Integrating this inequality yields

\[
\|e^{-t\Lambda_{r}(b_{n})}\|_{r} \to 2\delta \leq c_{1}e^{-c_{2}t}t^{-\frac{2}{r}} \left( \frac{r}{2} - \frac{1}{r} \right), \quad t > 0.
\]

By duality, \( \|e^{-t\Lambda_{r}(b_{n})}\|_{2 \to 2\delta} \leq \) the RHS of \((***)\), and

\[
\|e^{-t\Lambda_{r}(b_{n})}\|_{1 \to 2\delta} = \|e^{-t\Lambda_{r}(b_{n})}\|_{r} \to \infty \leq C_{1}(d, \delta)e^{c_{2}t}t^{-\frac{2}{r}} \frac{1}{\delta r},
\]

by the Extrapolation Theorem (Appendix II). Interpolating the latter, we arrive at the desired estimate.

4. Proof of \((***)\). Set \( z := 2\lambda \sqrt{\delta} \). We need to prove that

\[
|\text{Im} \langle \Lambda_{r}(b)u, u|u|^{-2} \rangle| \leq \mathcal{K}(2 - \sqrt{\delta}r^{-1})\text{Re} \left( (z + \Lambda_{r}(b))u, u|u|^{-2} \right), \quad u \in D(\Lambda_{r}(b)), r \in I_{c}^{0}. \quad (\bullet)
\]

a) First we prove \((\bullet)\) for \( b = b_{n} \) and \( u = u_{n} \in D(\Lambda_{r}(b_{n})), \) where \( b_{n} := b_{1n}, 1_{n} \) is the indicator of \( \{x \in \Omega \mid b_{n}(x) \leq n\}. \) Note that \( D(\Lambda_{r}) \cap L^{\infty} \) is a core of \( \Lambda_{r}(b_{n}) \) for \( r \in \{r \geq 2 \} \cap I_{c}^{0} \), and \( D(\Lambda_{r}) \cap L^{\infty} \) is a core of \( \Lambda(b_{n}) \) for \( r \in \{r \geq 2 \} \cap I_{c}^{0}. \) Thus, we will take \( u_{n} \) from these cores. Then \( \Lambda_{r}(b_{n})u_{n} = \tilde{A}u_{n} + b_{n} \cdot \nabla u_{n}, \) where \( \tilde{A}v = \Lambda_{r}v \) if \( v \in \Lambda_{r} \cap L^{\infty} \) and \( \tilde{A}v = Av \) if \( v \in D(\Lambda_{r}) \cap L^{\infty}. \)

Using the assumption \( b_{n} \in F_{\delta}(A), \) inequality \( \|A_{\delta}^{2}u\|_{2} \leq \|A_{\delta}^{2}u\|_{1}^{\frac{1}{2}} \), equality \( X \equiv (\nabla u \cdot a \cdot \nabla \tilde{u}, |u|^{-2}) = \text{Re} \langle \tilde{A}u, u|u|^{-2} \rangle - \frac{d(r-2)}{r^{2}}\|A_{\delta}^{2}u\|_{2}^{2}, \) and completing quadratic estimates we obtain

\[
|\text{Re} \langle b_{n} \cdot \nabla u, u|u|^{-2} \rangle| = |\langle b_{n} \cdot \nabla u, u|u|^{-1} \rangle| \leq \frac{2}{r^{2}}\|b_{n}\|_{2}^{2}\|A_{\delta}^{2}u\|_{2}\|u\|_{r},
\]

\[
\leq \frac{2\sqrt{\delta}}{r}\|A_{\delta}^{2}u\|_{2}^{2} + \frac{\lambda\sqrt{\delta}}{r}\|u\|_{r};
\]

\[
|\text{Im} \langle b_{n} \cdot \nabla u, u|u|^{-2} \rangle| \leq |\langle b_{n} \cdot \nabla u, u|u|^{-2} \rangle| \leq \|b_{n}\|_{2}\|u\|_{2}\|X_{\frac{1}{2}}\|,
\]

\[
\leq \frac{1}{4\epsilon}\text{Re} \langle \tilde{A}u, u|u|^{-2} \rangle + \left( \delta\epsilon - \frac{r-2}{r^{2}\epsilon} \right)\|A_{\delta}^{2}u\|_{2}^{2} + \lambda\delta\epsilon\|u\|_{r}^{2}, \quad (\epsilon > 0);\]
\[
\text{RHS} \equiv \text{Re} \langle (z + \Lambda_r(b_n))u, u|u|^{r-2}\rangle \\
\geq \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle - \frac{2\sqrt{\delta}}{r} \|A\frac{1}{2}|u|^2 + \left( z - \frac{\lambda\sqrt{\delta}}{r} \right) \|u\|^r; \\
\text{RHS} \geq (1 - \frac{r'\sqrt{\delta}}{2}) \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle + \left( z - \frac{\lambda\sqrt{\delta}}{r} \right) \|u\|^r; \\
\text{LHS} \equiv |\text{Im} \langle \Lambda_r(b_n)u, u|u|^{r-2}\rangle| \\
\leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle + (\delta\varepsilon - \frac{r - 2}{r^2}\varepsilon) \left(\frac{A\frac{1}{2}|u|^2}{2} + \lambda\delta\varepsilon \|u\|^r; \\
(\star \star) \right) \\
\text{LHS} \leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \text{RHS} + \left\{ \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \frac{2\sqrt{\delta}}{r} + \delta\varepsilon - \frac{r - 2}{r^2}\varepsilon \right\} \|A\frac{1}{2}|u|^2\|_2^2 \\
- \left\{ \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] (z - \frac{\lambda\sqrt{\delta}}{r}) - \lambda\delta\varepsilon \right\} \|u\|^r; \\
(\star \star \star) \right)
\]

Note that \( r \leq 2r_\delta \) makes \( \frac{1}{4\varepsilon} \frac{2\sqrt{\delta}}{r} + \delta\varepsilon - \frac{r - 2}{r^2}\varepsilon > 0 \) for any \( \varepsilon > 0 \). Thus, setting \( \varepsilon = \frac{1}{r\sqrt{\delta}} \) and \( z = 2\lambda\sqrt{\delta} \), we obtain from (\( \star \star \star \)) and (\( \star \))

\[
\text{LHS} \leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \text{RHS} + \left\{ \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \frac{2\sqrt{\delta}}{r} + \delta\varepsilon - \frac{r - 2}{r^2}\varepsilon \right\} \frac{rr'}{4} \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle \\
- \left\{ \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] (z - \frac{\lambda\sqrt{\delta}}{r}) - \lambda\delta\varepsilon \right\} \|u\|^r; \\
\leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \text{RHS} + \left\{ \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \frac{\sqrt{\delta}}{2} + (\delta\varepsilon - \frac{r - 2}{r^2}\varepsilon) \frac{rr'}{4} \right\} (1 - \frac{r'\sqrt{\delta}}{2})^{-1} \text{RHS} \\
= \left( \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right) \left( \frac{\sqrt{\delta}}{2} + \delta\varepsilon \right) 2(2 - r'\sqrt{\delta})^{-1} \text{RHS} \\
= \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{\sqrt{\delta}}{r\sqrt{\delta}} \right] (2 - r'\sqrt{\delta})^{-1} \text{RHS}. 
\]

If \( r > 2r_\delta \), then setting \( \varepsilon = \frac{\sqrt{r - 1}}{r\sqrt{\delta}} \) and \( z = 2\lambda\sqrt{\delta} \) we obtain from (\( \star \star \)) and (\( \star \))

\[
\text{LHS} \leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} \right] \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle + \delta\varepsilon \|A\frac{1}{2}|u|^2\|^2 + \lambda\delta\varepsilon \|u\|^r \\
\leq \left[ \frac{|r - 2|}{2\sqrt{r - 1}} + \frac{1}{4\varepsilon} + \delta\varepsilon \frac{rr'}{4} \right] \text{Re} \langle \tilde{A}u, u|u|^{r-2}\rangle + \lambda\delta\varepsilon \|u\|^r \\
\leq \frac{|r - 2| + r\sqrt{\delta}}{2\sqrt{r - 1}} (2 - r'\sqrt{\delta})^{-1} \text{RHS}. 
\]

b) Finally, employing the Privalov-Vitali Convergence Theorem in the same way as in [17 p. 21], we end the proof of (\( \star \)).

3.1. Uniformly elliptic case I. We say that a matrix \( a \) is uniformly elliptic and write \( a \in (H_u) \)

if \( a \in (H_1) \) and there is a constant \( \xi < \infty \) such that \( a(x) \leq \xi I \) for \( L^d \) a.e \( x \in \Omega \).

In the uniformly elliptic case and \( 1 \leq \delta < 4 \) the assumption \( b_a \in L^2 + L^\infty \) is superfluous:
Theorem 3.2. Let $a \in (H_\alpha), A = A_D$ or $A_N$, and $b : \Omega \to \mathbb{R}^d, b \in F_\delta(A)$ for some $1 \leq \delta < 4$. Then the limit
\[
e^{-tA_\alpha(b)} := s-L^* \lim_n e^{-tA_\alpha(b_n)}, \quad r > r_\delta := \frac{2}{2 - \sqrt{\delta}}
\]
exists uniformly in $t \in [0, T]$ for each $T < \infty$, and determines a positivity preserving, $L^\infty$ contraction, quasi contraction $C_0$ semigroup on $L^r$.

Proof. 1. Let $k > 2$. Define
\[
\eta(t) := \begin{cases} 
0, & \text{if } t < k, \\
(k^2 - 1)^k, & \text{if } k \leq t \leq 2k, \\
1, & \text{if } 2k < t,
\end{cases}
\]
and $\zeta(x) = \eta(\frac{|x-\bar{o}|}{R})$, $R > 0$.

Note that $|\nabla \zeta| \leq R^{-1} 1_{\sqrt{\delta}} \zeta^{1-\frac{1}{r}}$.

By $u_n$ we denote the solution to $(\frac{d}{dt} + \Lambda_r(b_n))u_n = 0, u_n(0) = f \in L^\infty \cap L^2_\mu$. Set $v := \zeta u_n$. Clearly,
\[
\left\langle \zeta (\frac{d}{dt} + A + b_n \cdot \nabla) u_n, v^{r-1}\right\rangle = 0.
\]

Set $[F, G]_\mu := FG - GF$ and $\langle Av, v^{r-1} \rangle := \langle \nabla v \cdot a \cdot \nabla v^{r-1} \rangle$. Since the matrix is uniformly elliptic, $\langle A^\mu \phi, A^\mu \psi \rangle = \langle \nabla \phi \cdot a \cdot \nabla \psi \rangle$ for all $\phi, \psi \in D(A^\mu) = E,$ where $E = W^{1, 2}_2$ and $E = W^{1, 2}$ for $A = A_D$ and $A = A_N$. Also $u_n \in D(A) \cap L^\infty, D(A) \subset D(A^\mu) = E.$ Thus $v \in W^{1, 2}_2$ as well as $v^\mu$ and $\zeta v^{r-1}$ since $r \geq 2$. Therefore $\langle Av_n, \zeta v^{r-1} \rangle = \langle \nabla u_n \cdot a \cdot \nabla (\zeta v^{r-1}) \rangle$. Now it is easy to justify the following equation and equality.
\[
\left\langle \frac{d}{dt} + A + b_n \cdot \nabla, v, v^{r-1}\right\rangle = \left\langle [A, \zeta] u_n + u_n b_n \cdot \nabla, v, v^{r-1}\right\rangle = \left\langle [A, \zeta] u_n, v, v^{r-1}\right\rangle \tag{*}
\]

By the quadratic estimates
\[
\langle u_n b_n \cdot \nabla, v^{r-1}\rangle = \langle b_n \cdot \nabla, v^{r}\rangle 
\leq \frac{\mu \sqrt{\delta}}{r} || A^\mu v^{\frac{r}{2}} ||_2^2 + \frac{r \sqrt{\delta}}{4 \mu} \langle \nabla \cdot a \cdot \nabla \zeta, v^{r}\rangle + \frac{\mu c(\delta)}{r \sqrt{\delta}} || v ||_r^r, \quad (\mu > 0),
\]
\[
\frac{2(r - 2)}{r} \langle \nabla v^\mu \cdot a \cdot \nabla, v^\mu \rangle \leq \frac{\mu \sqrt{\delta}}{r} || A^\mu v^\mu ||_2^2 + \frac{(r - 2)^2}{r \mu \sqrt{\delta}} \langle \nabla \cdot a \cdot \nabla \zeta, v^{r}\rangle,
\]
we get from (*)
\[
\frac{d}{dt} || v ||_r^r + 2 \left( \frac{2}{r^2} - (1 + \mu) \sqrt{\delta} \right) || A^\mu v^\mu ||_2^2 \leq \left( \frac{(r - 2)^2}{\mu \sqrt{\delta}} + \frac{r^2 \sqrt{\delta}}{4 \mu} + r \right) \langle \nabla \cdot a \cdot \nabla \zeta, v^{r}\rangle + \frac{r + \mu}{\sqrt{\delta}} c(\delta) || v ||_r^r.
\]
Recalling that $\frac{2}{r^2} \geq \sqrt{\delta}$, we can find $\mu > 0$ such that $\frac{2}{r^2} - (1 + \mu) \sqrt{\delta} \geq 0$. Thus
\[
\frac{d}{dt} || v ||_r^r \leq \left( \frac{4(r - 2)^2 + r^2 \delta}{4 \mu \sqrt{\delta}} + r \right) \langle \nabla \cdot a \cdot \nabla \zeta, v^{r}\rangle + \frac{r + \mu}{\sqrt{\delta}} c(\delta) || v ||_r^r \tag{**}
\]
Next, \( \langle \nabla \zeta \cdot \frac{a}{\zeta^2} \cdot \nabla \zeta, v^r \rangle \leq \xi R^{-2} \| 1_{\nabla \zeta} \zeta^{-29} v^r \|_1 \), where \( \theta = k^{-1} \) and \( 1_{\nabla \zeta} \) denotes the indicator of the support of \( |\nabla \zeta| \). Since \( \| u_n \|_\infty \leq \| f \|_\infty \), \( \| 1_{\nabla \zeta} \|_\infty \leq c(d, \theta) R \frac{2d}{r} \), and
\[
\| 1_{\nabla \zeta} \zeta^{-29} v^r \|_1 \leq \| 1_{\nabla \zeta} u_n^{29} \|_\infty \| v \|_{r-29} \|_r \leq \| 1_{\nabla \zeta} \zeta^{-29} \|_\infty \| v \|_{r-29},
\]
we obtain, using the Young inequality, the crucial estimate (on which the whole proof rests)
\[
\langle \nabla \zeta \cdot \frac{a}{\zeta^2} \cdot \nabla \zeta, v^r \rangle \leq \frac{2\theta}{r} [\xi \zeta(d)] \frac{r}{2} R^{d-\frac{4}{r}} \| f \|_{\infty}^r + \frac{r - 2\theta}{r} \| v \|_{\infty}^r.
\]
Fix \( \theta \) by \( 0 < \theta < \frac{r}{d+2r} \). Now from (**) we obtain the inequality
\[
\frac{d}{dt} \| v \|_r^r \leq N(r, d, \delta) \| v \|_r^r + M(r, d, \delta) R^{-\gamma} \| f \|_\infty^r, \quad \gamma = \frac{r}{\theta} - d > 0,
\]
from which we conclude that, for given \( T, f \in L^2 \cap L^\infty, \) \( \varepsilon > 0 \), there exists \( R \) such that
\[
\sup_{t \in [0, T], n} \| \zeta u_n(t) \|_r \leq \varepsilon.
\]

2. Let \( k \) be as above. Define
\[
\eta(t) := \begin{cases} 
1, & \text{if } t < 2k, \\
(1 - \frac{1}{k} (t - 2k))^k, & \text{if } 2k \leq t \leq 3k; \quad \text{and } \zeta(x) = \eta\left(\frac{|x - b|}{R}\right), \quad R > 0.
\end{cases}
\]

Note that \( |\nabla \zeta| \leq R^{-1} 1_{\nabla \zeta} \zeta^{-1} \frac{1}{k} \).

Set \( g := u_n - u_m \) and \( v := \zeta g \). Let \( (Av, v|v|^{r-2}) := \langle \nabla v \cdot a \cdot \nabla (v|v|^{r-2}) \rangle \). Clearly
\[
\langle \zeta \left( \frac{d}{dt} + A + b_n \cdot \nabla \right) g + \zeta(b_n - b_m) \cdot \nabla u_m, v|v|^{r-2} \rangle = 0,
\]
\[
\langle \left( \frac{d}{dt} + A + b_n \cdot \nabla \right) v, v|v|^{r-2} \rangle = \langle [A, \zeta] - b_n \cdot \nabla u_n, v|v|^{r-2} \rangle + \langle \zeta(b_n - b_m) \cdot \nabla u_m, v|v|^{r-2} \rangle,
\]
\[
\langle [A, \zeta] - b_n \cdot \nabla u_n, v|v|^{r-2} \rangle = 2 \left( \frac{r - 2}{r} \| \nabla |v|^{\frac{r}{2}} \cdot \alpha \cdot \nabla \zeta, |v|^{\frac{r}{2}} + \| \nabla \zeta \cdot \frac{a}{\zeta^2} \cdot \nabla \zeta, |v|^{r-2} \rangle,
\]
\[
\frac{d}{dt} \| v \|_r^r \leq \left( \frac{4(r - 2)^2 + r^2\delta}{4\mu\sqrt{\delta}} + r \right) \| \nabla \zeta \cdot \frac{a}{\zeta^2} \cdot \nabla \zeta, v^r \rangle + \frac{r + \mu}{\sqrt{\delta}} \| v \|_r^r
\]
\[
+ r \langle \zeta(b_n - b_m) \cdot \nabla u_m, v|v|^{r-2} \rangle \text{ with the same } \mu \text{ as in (**),}
\]
\[
\langle \zeta(b_n - b_m) \cdot \nabla u_m, v|v|^{r-2} \rangle \leq \langle \zeta(b_n - b_m) \cdot a^{-1} \cdot (b_m - b_n) \| \nabla u_m \cdot a \zeta \cdot \nabla u_m \| (2 \| f \|_\infty)^{r-1}.
\]

In order to estimate \( \int_0^T \langle \nabla u_m(t) \cdot a \zeta \cdot \nabla u_m(t) \rangle dt \) note that \( \langle \frac{d}{dt} u_m + Au_m + b_n \cdot \nabla u_m, \zeta u_m \rangle = 0 \), or
\[
\frac{1}{2} \frac{d}{dt} \| \zeta u_m^2 \| + \langle \nabla u_m \cdot a \zeta \cdot \nabla u_m \rangle + \langle \nabla u_m \cdot a u_m \cdot \nabla \zeta \rangle + \langle b_m \cdot \nabla u_m, \zeta u_m \rangle = 0,
\]
and so
\[
\frac{d}{dt} \| \zeta u_m^2 \| + \langle \nabla u_m \cdot a \zeta \cdot \nabla u_m \rangle \leq 2 \left( \langle \nabla \zeta \cdot \frac{a}{\zeta} \cdot \nabla \zeta \rangle + \langle \zeta b \cdot a^{-1} \cdot b \rangle \right) \| f \|_\infty^2.
\]
\[\int_0^T \langle \nabla u_m(t) \cdot a \cdot \nabla u_m(t) \rangle dt \leq \|f\|_2^2 + 2T \left( \langle \nabla \zeta \cdot \frac{a}{\zeta} \cdot \nabla \zeta \rangle + \langle \zeta b \cdot a^{-1} \cdot b \rangle \right) \|f\|_\infty^2 \]

\[= \|f\|_2^2 + TL(R) \|f\|_\infty^2.\]

Now it should be clear that the above is sufficient for concluding that the following inequality analogous to (**) holds for all \(n, m\) and \(t \in [0, T]\),

\[e^{-NT} \|g\|_r^r \leq tMR^{-\gamma} \|f\|_\infty^r + t(2 \|f\|_\infty)^{r-1} \left( \|f\|_2^2 + TL(R) \|f\|_\infty^2 \right)^{\frac{r}{2}} \langle \zeta(b_n - b_m) \cdot a^{-1} \cdot (b_n - b_m) \rangle^{\frac{r}{2}}.\]

3. It follows from Step 1 and Step 2 that, for each \(0 < T < \infty, \ r > r_\delta, f \in L^2_+ \cap L^\infty\) and \(\epsilon > 0\), we can find \(R < \infty\) and \(M < \infty\) such that

\[\sup_{t \in [0, T], n, m \geq 1} \|(1 - \mathbf{1}_{B(o, 2kR)})(u_n(t) - u_m(t))\|_r < \epsilon; \quad \sup_{t \in [0, T], n, m \geq M} \|\mathbf{1}_{B(o, 2kR)}(u_n(t) - u_m(t))\|_r < \epsilon.\]

The proof of Theorem 3.2 is completed.

Remark. Let \(a \in (H_1)\). Set \(A := A_\delta D\) or \(A_N\). Fix any compact \(K\) in \(\Omega\). The proof of Theorem 3.2 shows that the matrix \(a(x)\) admits the \((|x|^\gamma, \gamma > 2)\)-growth at infinity while on \(K\) being just from \([L^1]^{d \times d}\). Indeed, for instance, let \(A = A_\delta D\). By the definition of \(A_\delta D, \psi \in D(A^{\frac{1}{2}})\) if and only if \(\psi \in W^{1,2}_0\) and \(\langle \nabla \psi \cdot a \cdot \nabla \psi \rangle < \infty\). It is a simple matter to check that

\[\langle \nabla (\zeta|v|^r-2) \cdot a \cdot \nabla (\zeta|v|^r-2) \rangle < \infty,\]

and hence to conclude that

\[\langle Ag, \zeta|v|^r-2 \rangle = \langle \nabla g \cdot a \cdot \nabla (\zeta|v|^r-2) \rangle.\]

If \(A = A_\delta D\) one has to modify \(\zeta\) to \(C_0^1\) function and use the fact that \(C_0^1\) is a core of \(A_\delta D\).

3.2. The maximal interval of bounded \(L^p\) solvability for \(-\nabla \cdot a \cdot \nabla + b \cdot \nabla\). The example of \(\Lambda_r := -\Delta + b \cdot \nabla\) with \(b(x) := c|x|^{-2}x \in F_\delta(-\Delta), \ c = \frac{(d-2)\sqrt{\delta}}{2-\sqrt{\delta}}, \ \text{in} \ \Omega = B(0, R) \text{ or } \mathbb{R}^d, \) can be used to show that both intervals \(I_c := [\frac{2}{2-\sqrt{\delta}}, \infty[ \) and \(I_m := [\frac{2}{2-\sqrt{\delta}} \frac{2}{2-\sqrt{\delta}}, \infty[ \) are maximal, i.e. for \(q \notin I_c \ (q \notin I_m)\) the constructed \(C_0\) semigroup \(e^{-t\Lambda_r}\) can not be extended as a quasi contraction (respectively, quasi bounded) \(C_0\) semigroup on \(L^q\).

**Theorem 3.3.** Let \(a \in (H_u)\). Assume that \(b \in F_\delta(A)\) for some \(0 < \delta < 4\). Then \(e^{-t\Lambda_r(b)}, \ r \in I_c,\) extends to a positivity preserving, \(L^\infty\) contraction, quasi bounded holomorphic semigroup on \(L^r\) for every \(r \in I_m - I_c\).

**Proof.** Let first \(\Omega = \mathbb{R}^d\). Set \(b_n := \gamma_{\epsilon_n} \ast (b1_n), \) where \(\gamma_{\epsilon_n}\) is the K.Friedrichs mollifier, \(1_n\) is the indicator of \(\{x \in \Omega : |x| \leq n, |b(x)| \leq n\}\). Since our assumptions on \(\delta\) involve strict inequalities only, by selecting sufficiently small \(\epsilon_n \downarrow 0\), we may assume that \(b_n \in F_\delta(A)\) for all \(n\) with \(\lambda = \lambda_\delta\) independent of \(n\).

It suffices to prove that

\[\|(z + \Lambda_q^*(b_n))^{-1}\|_{qj \rightarrow qj} \leq c|z - z_0|^{-1}, \quad \text{Re} \ z > z_0, \quad n = 1, 2, \ldots, \quad j = \frac{d}{d-2}, \quad (\bullet)\]

where \(\Lambda_q^*(b_n)\) is the adjoint of \(\Lambda_r(b_n), \ q = \frac{r}{r-1} \in ]1, \frac{2}{\sqrt{\delta}}[, \) for some \(c, z_0 > 0\) independent of \(n\).
For $1 < p \leq 2$, let $B_{p,n} := -\nabla \cdot b_n = -b_n \cdot \nabla - \text{div } b_n$, $D(B_{p,n}) := D(A_p)$. Then the operator $\Lambda_p^* = A_p + B_{p,n}$ of domain $D(A_p)$ is the (minus) generator of a quasi bounded holomorphic semigroup on $L^p$.

Set $u \equiv u_n := (z + \Lambda_p^*)^{-1} h$, $h \in L^1 \cap L^\infty$, $\text{Re } z > \frac{\delta}{2(1-\frac{2}{p})}$. By consistency, $u = (z + \Lambda^*)^{-1} h$, so $u \in D(\Delta^\frac{1}{2})$, and, clearly, $\|u\|_\infty < \infty$ for every $n$.

a) $0 < \delta < 1$. Fix any $2 \leq q < \frac{2}{\sqrt{\delta}}$ close to $\frac{2}{\sqrt{\delta}}$. Then

$$\langle (z + \Lambda^*) u, u|u|^{q-2} \rangle = \langle (z + A - \nabla \cdot b_n) u, u|u|^{q-2} \rangle,$$

$$-\langle \nabla \cdot (b_n u), u|u|^{q-2} \rangle = \langle b_n \cdot u \nabla \bar{u}, |u|^{q-2} \rangle + (q-2) \langle b_n \cdot \nabla |u|, |u|^{q-2} \rangle,$$

and so

$$z\|u\|^q_q + \langle Au, u|u|^{q-2} \rangle + \langle b_n \cdot u \nabla \bar{u}, |u|^{q-2} \rangle + (q-2) \langle b_n \cdot \nabla |u|, |u|^{q-1} \rangle = \langle h, u|u|^{q-2} \rangle.$$

Taking the real and imaginary parts of this identity, we have

$$\text{Re } z\|u\|^q_q + \text{Re } \langle Au, u|u|^{q-2} \rangle + (q-1) \langle b_n \cdot \nabla |u|, |u|^{q-1} \rangle \leq \langle h, u|u|^{q-2} \rangle,$$

$$\alpha (\text{Im } z\|u\|^q_q - \text{Im } \langle Au, u|u|^{q-2} \rangle - \langle b_n \cdot u \nabla \bar{u}, |u|^{q-2} \rangle) \leq \alpha \langle h, u|u|^{q-2} \rangle \quad (0 < \alpha < 1).$$

Adding these inequalities and using the inequality $|\text{Im } \langle Au, u|u|^{q-2} \rangle| \leq \frac{q-2}{2\sqrt{q-1}} \text{Re } \langle Au, u|u|^{q-2} \rangle$, we obtain

$$\alpha |z - z_0|\|u\|^q_q + (1 - \frac{q-2}{2\sqrt{q-1}}) \text{Re } \langle Au, u|u|^{q-2} \rangle \leq \alpha \langle b_n \cdot u \nabla \bar{u}, |u|^{q-2} \rangle + (q-1) \langle b_n \cdot \nabla |u|, |u|^{q-1} \rangle + (1 + \alpha) \|h\|_q \|u\|^{q-1}.$$

Recalling that $\langle b_n \cdot u \nabla \bar{u}, |u|^{q-2} \rangle \leq \langle b^2, |u|^{q} \rangle X$, where

$$X \equiv \langle \nabla u \cdot a \cdot \nabla \bar{u}, |u|^{q-2} \rangle = \text{Re } \langle Au, u|u|^{q-2} \rangle - \frac{4(q-2)}{q^2} \|A^\frac{1}{2} |u|^{\frac{q}{2}} \|_2^2,$$

(see the proof of Theorem 3.1 steps 1, 4) and performing quadratic estimates, we obtain

$$|z - z_0|\|u\|^q_q + \|A^\frac{1}{2} |u|^{\frac{q}{2}} \|_2^2 \leq C_{\alpha} \|h\|_q \|u\|^{q-1},$$

where the constant $C_{\alpha} < \infty$ for appropriate chosen $\alpha$ and $z_0$.

The latter and the Sobolev Embedding Theorem yield

$$\|u\|_{qj} \leq c(q,d) |z - z_0|^{-\frac{1}{qj}} \|h\|_q, \quad \text{Re } z > z_0.$$  \hfill (\star)

(b) Let $1 \leq \delta < 4$, so the interval $[1, \frac{2}{\sqrt{\delta}}]$ does not contain 2. Fix any $1 < q < \frac{2}{\sqrt{\delta}}$ close to $\frac{2}{\sqrt{\delta}}$. Following the arguments above, but using Theorem C.1(i),(ii) (Appendix C) in place of Theorem C.1(iv),(v), we obtain (\star) in the case.

We are in position to complete the proof of Theorem 3.3 for $\Omega = \mathbb{R}^d$. Fix $1 < q < \frac{2}{\sqrt{\delta}}$ close to $\frac{2}{\sqrt{\delta}}$. Set $R_n^*(z) := (z + \Lambda^*_q(b_n))^{-1}$. Our goal is to prove that $\|R_n^*(z)\|_{qj} \leq c |z - z_0|^{-1}, \text{Re } z > z_0.$ Given $l = (l^1, \ldots, l^d) \in \mathbb{Z}^d$, define a cube

$$Q_{l} := \{x \in \mathbb{R}^d \mid |l^m| |z - z_0|^{-\frac{1}{2}} - x^m| \leq (4 |z - z_0|)^{-\frac{1}{2}}, \quad m = 1, \ldots, d\}.$$  

Given $k \in \mathbb{Z}^d$, subdivide $\mathbb{R}^d$ into $Q_k + \sum_{i \in \mathbb{Z}^d - \{k\}} Q_i$. Fix $k.$
1) Let $i \in \mathbb{Z}^d$ be such that $|k - i| \leq \alpha := \frac{\sqrt{d}}{2}(1 + \frac{1}{\gamma}), \frac{\sqrt{d}}{2} < \gamma < \frac{1}{d}$. Then

$$\|1_k R_n(z) 1_i\|_{qj \to qj} \leq c_1 |z - z_0|^{-1}, \quad c_1 = c_1(q, d),$$

where $1_i$ denotes the indicator function of $Q_i$. [Indeed, by Hölder's inequality, $\|1_k R_n(z) 1_i h\|_{qj} \leq \|R_n(z)\|_{qj \to qj}\|1_i\|_1 \|h\|_{qj}$, $j' = \frac{d}{2}$, so (\#) and $\|1_i\|_1 = c_d |z - z_0|^{-\frac{d}{2}}$ yield the required].

2) Let $i \in \mathbb{Z}^d$ be such that $|k - i| \leq \alpha$. Then $(\xi \text{ the ellipticity constant})$

$$\|1_k R_n(z) 1_i\|_{qj \to qj} \leq c_2 |k - i|^{-\frac{1}{2}} |z - z_0|^{-1}, \quad c_2 = c_2(q, d, \gamma, \xi).$$

(The proof of the inequality, which we call the separating property, is given below.)

1) and 2) combined yield

$$\|\langle R_n^*(z) h, g \rangle\| \leq \sum_{i, k \in \mathbb{Z}^d} |\langle 1_k R_n^*(z) 1_i h, 1_k g \rangle| \leq \sup_{i \in \mathbb{Z}^d} \left( \sum_{i \in \mathbb{Z}^d} |\langle 1_i R_n^*(z) 1_i h, 1_k g \rangle| \right);$$

$$\|R_n^*(z)\|_{qj \to qj} \leq c_1 |z - z_0|^{-1} + c_2 |z - z_0|^{-1} \sup_{k \in \mathbb{Z}^d} \sum_{i \in \mathbb{Z}^d} |k - i|^{-\frac{1}{2}}$$

$$\leq c_1 |z - z_0|^{-1} + c_2 |z - z_0|^{-1} \sup_{k \in \mathbb{Z}^d} \int_{\alpha}^{\infty} t^{-d - \frac{1}{2}} dt$$

$$\leq c_3 |z - z_0|^{-1},$$

which ends the proof of the theorem in the case $\Omega = \mathbb{R}^d$.

Now we come to the proof of the inequality from 2). Let $i \in \mathbb{Z}^d - \{k\}$ be such that $|k - i| \geq \alpha$. Define functions $\zeta_i(x)$ by

$$\zeta_i(x) := \eta \left( \frac{|z - z_0|^{-\frac{1}{2}}k - x|}{|k - i|} \right) \left( 1 + \frac{1}{\gamma} \right),$$

where

$$\eta(s) = \begin{cases} 
1, & s \leq 1 \\
(1 - \frac{s - m}{m})^m, & 1 < s < m + 1 \\
0, & m + 1 \leq s \quad (m = \frac{1}{\gamma}).
\end{cases}$$

We list the following properties of $\zeta_i : \zeta_i \mid Q_k = 1_k, \zeta_i \mid Q_i = 0$. Define

$$\Gamma_i(x) := \nabla \zeta_i(x) \cdot \frac{a(x)z}{\zeta_i(x)^2} \cdot \nabla \zeta_i(x), \quad \epsilon(x) = \frac{k|z - z_0|^{-\frac{1}{2}} - x}{|k|z - z_0|^{-\frac{1}{2}} - x|}.$$}

Using the inequalities $-\partial_s \eta(s) \leq \eta(s)^{-1 - \gamma}$ and $\epsilon \cdot a \cdot \epsilon \leq \xi$, we have

$$\Gamma_i \leq \left( 1 + \frac{\gamma}{\gamma} \right)^2 \xi |k - i|^{-2} |z - z_0| \gamma c^{-2 \gamma}. \quad (\xi)$$

**Lemma 3.4.** Define $v_i := R_n^*(z) 1_i f, u := \zeta_i v_i, f \in L^1 \cap L^\infty$. Then

$$|z - z_0| ||u||_q^q + ||u||_{qj}^q \leq c(q) (\Gamma_i ||u||^q). \quad (i)$$

$$|z - z_0|^{-2\gamma/q} ||u||_{q-2\gamma}^q ||u||_{qj}^{2\gamma} \leq c(q, \gamma) |k - i|^{-2} (||v_i||^{2\gamma} ||u||^{q-2\gamma}). \quad (ii)$$
Due to $\zeta_i \uparrow Q_k = 1_k$, (iii) implies 2).

Proof of Lemma. (ii) follows from (i), Young’s inequality and (v). In turn, (iii) follows from (ii) by applying Hölder’s inequality to $|v_i|^{2\gamma}|u|^{q-2\gamma}$, so that $\|u\|_{q_j} \leq c(q, \gamma)\zeta \frac{1}{\delta} |k-i|^{-\frac{1}{\delta}} |z-z_0|^{-1} \|f\|_{q_j}$.

We are left to prove (i). We have $(z + A_q - \nabla \cdot b_n)v_i = \mathbf{1}_f$ if $q \geq 2$, we write $A \equiv A_2$ in place of $A_q$ and, since $\zeta \uparrow Q_i = 0$, $(z + A_q + \nabla \cdot b_n)v_i, \zeta_i u |u|^{q-2} = 0$. One can easily check that $g \in D(A^\frac{1}{2}) \Rightarrow \zeta_i g \in D(A^\frac{1}{2})$. Since $v_i \in D(A)$, both $u$ and $u |u|^{q-2}$ belong to $D(A^\frac{1}{2})$ (Appendix C). Thus

$$z\|u\|_{r^i} + \langle A_q u, u |u|^{q-2} \rangle - \langle \nabla \cdot b_n v_i, \zeta_i u |u|^{q-2} \rangle = \langle [A_q, \zeta_i] v_i, u |u|^{q-2} \rangle,$$

where

$$\langle [A_q, \zeta_i] v_i, u |u|^{q-2} \rangle := \langle \nabla \zeta_i \cdot a \nabla u |u|^{q-2} \rangle - \langle \zeta_i \cdot a \nabla v_i, u |u|^{q-2} \rangle
$$

$$= \langle \nabla \zeta_i \cdot a u |u|^{q-2} \rangle - \langle \zeta_i \cdot a \nabla u |u|^{q-2} \rangle + \langle \zeta_i \cdot a \nabla \zeta_i, u |u|^{q-2} \rangle.$$

The rest of the proof resembles what we have already done. Taking the real and imaginary parts of the last equation and performing quadratic estimates we arrive at (i). □

The same proof works for an arbitrary open $\Omega \subset \mathbb{R}^d$ with $\zeta_i \uparrow \Omega$ in place of $\zeta_i$. □

3.3. Uniformly elliptic case II. The Hille-Lions approach. Let $a \in (H_0), b \in F_{\delta}(A), 0 < \delta < 4,$ and let $\Lambda_\omega(a, b) \equiv \Lambda_\omega(b)$ be the operator defined in Theorem 3.2. It is useful (in some cases necessary) to have the convergence

$$e^{-t \Lambda_\omega(a, b)} = \text{s-L}^\bullet \lim_n e^{-t \Lambda_\omega(a_n, b_n)},$$

where $a_n$ and $b_n$ have smooth and bounded entries.

Theorem 3.5. Fix $\delta < 1$. Let $a, a_n \in (H_0), b, b_n : \Omega \rightarrow \mathbb{R}^d, n = 1, 2, \ldots$. Assume that

$$b \in F_{\delta}(A(a)), b_n \in F_{\delta}(A(a_n)) \text{ with fixed } \lambda = \lambda_{\delta} \text{ for all } n = 1, 2, \ldots.$$  \hspace{2cm} (i)

$$a_n \rightarrow a \text{ strongly in } [L^2_{\text{loc}}]^{d \times d}, \quad b_n \rightarrow b \text{ strongly in } [L^2_{\text{loc}}]^d. \hspace{2cm} (ii)$$

Then $s-L^\bullet \lim_{n \uparrow r_{\delta}} e^{-t \Lambda_\omega(a_n, b_n)} = e^{-t \Lambda_\omega(a, b)}$ whenever $r \in I^0_{\delta} = ]r_{\delta}, \infty[$ (recall $r_{\delta} = \frac{2}{2 - \sqrt{\delta}}$).

Proof. 1. Set $\mathcal{H} = L^2(\Omega, \mathcal{L}^d), \mathcal{H}_+ = \langle \mathcal{H}_+, \mathcal{H}_- \rangle$ denote the pairing between $(\mathcal{H}_+, \mathcal{H}_-)$ which coincides with $\langle f, g \rangle_{\mathcal{H}}$ for $f \in \mathcal{H}$. Then $\mathcal{H}_+ \subset \mathcal{H} \subset \mathcal{H}_-$ is the standard triple of Hilbert spaces w.r.t. $\langle, \rangle_{\mathcal{H}}$. By $\hat{\omega}$ denote the extension by continuity of $A \equiv A(a)$ to the operator from $\mathcal{H}_+ \rightarrow \mathcal{H}_-$. Then $\hat{\omega} \in B(\mathcal{H}_+, \mathcal{H}_-)$ and $\|f, (\zeta + \hat{\omega} f)\| \geq ||f||_{\mathcal{H}}, f \in \mathcal{H}_+, \Re \zeta > \lambda$. Thus $\zeta + \hat{\omega}$ is a bijection. Clearly $\langle \zeta + \hat{\omega} \rangle^{-1} \mathcal{H} = (\zeta + \hat{\omega})^{-1}$. Consider $\bar{B} \equiv b \cdot \nabla : \mathcal{H}_+ \rightarrow \mathcal{H}_-$. By $b \in F_{\delta}(A), \bar{B} \in B(\mathcal{H}_+, \mathcal{H}_-)$ and

$$|\langle f, (\zeta + \hat{\omega} + \bar{B} f) \rangle| \geq (1 - \sqrt{\delta})|\langle f, (\mu + \hat{\omega} f) \rangle|, \quad \mu = \frac{2\zeta - \lambda \sqrt{\delta}}{2(1 - \sqrt{\delta})} > 0$$
whenever \( \text{Re } \zeta > \frac{\lambda \sqrt{\delta}}{2} \).

Thus, for \( \hat{\Lambda} \equiv \hat{\Lambda}(a, b) := \hat{A} + \hat{B} \) and every \( \zeta \in \mathcal{O}_a := \{ z \mid \text{Re } z > \frac{\lambda \sqrt{\delta}}{2} \} \), \( \zeta + \hat{\Lambda} \) is a bijection.

In \( \mathcal{H} \) define the operators
\[
H = b_a(\zeta + A)^{-\frac{1}{2}}, \quad S = \frac{b}{b_a} \cdot \nabla (\zeta + A)^{-\frac{1}{2}} \quad \text{and } H^*S.
\]

Clearly, due to \( b \in \mathbf{F}_0(A) \), for each \( \zeta \) with \( \text{Re } \zeta \geq \lambda \),
\[
\|H^*\|_{2\rightarrow 2} = \|H\|_{2\rightarrow 2} \leq \sqrt{\delta}.
\]

Since \( (\mathcal{L}^d \text{ a.e.}) |Sf| \leq \frac{1}{\sqrt{\delta}} \cdot a^{-1} \cdot \frac{1}{\sqrt{\delta}} \sqrt{\nabla (\zeta + A)^{-\frac{1}{2}} f \cdot a \cdot \nabla (\zeta + A)^{-\frac{1}{2}} f} \), and so \( \|Sf\|_2 \leq \|f\|_2, \ f \in L^2 \), we conclude that \( \|H^*S\|_{2\rightarrow 2} \leq \sqrt{\delta} \).

Set \( \hat{R}_\zeta \equiv \hat{R}_\zeta(a, b) := (\zeta + \hat{A})^{-1}, \ \hat{F}_\zeta \equiv \hat{F}_\zeta(a) := (\zeta + \hat{A})^{-1}, \ P_\zeta \equiv P_\zeta(a, b) := H^*S. \)

Clearly, for all \( \zeta, \eta \in \mathcal{O} := \{ z \mid \text{Re } z > \lambda \}, \)
\[
\hat{R}_\zeta - \hat{R}_\eta = (\eta - \zeta) \hat{R}_\zeta \hat{R}_\eta; \\
\hat{R}_\zeta - \hat{F}_\zeta = -\hat{F}_\zeta \hat{B} \hat{R}_\zeta = -\hat{F}_\zeta \hat{B} \hat{F}_\zeta \hat{F}_\zeta \hat{B} \hat{F}_\zeta - \ldots; \\
\hat{F}_\zeta \hat{B} \hat{F}_\zeta \mid \mathcal{H} = F_\zeta^2 P_\zeta F_\zeta^2, \ \|P_\zeta\|_{2\rightarrow 2} \leq \sqrt{\delta}; \\
R_\zeta := \hat{R}_\zeta \uparrow \mathcal{H} = F_\zeta^2 (1 + P_\zeta)^{-1} F_\zeta^2; \\
\|R_\zeta\|_{2\rightarrow 2} \leq |\zeta|^{-1} (1 - \sqrt{\delta})^{-1}. \tag{p1}
\]

Now, we employ Hille’s theory of pseudo-resolvents. By (p1), \( R_\zeta \) is a pseudo-resolvent on \( \mathcal{O} \).

By (p2), the common null set of \( \{ R_\zeta \mid \zeta \in \mathcal{O} \} \) is \( \{0\} \). Also, from (p2) it follows that \( \nu R_\nu \xrightarrow{\ast} 1 \) as \( \nu \uparrow \infty \). (Indeed, in view of (p3) and (p2), we have only to prove that \( \nu M_\nu f \equiv \nu F_\nu^2 H_\nu^*(1 + S_\nu H_\nu^*)^{-1} S_\nu F_\nu^2 f \xrightarrow{\ast} 0 \) for every \( f \in D(A_\nu^2) \). Since \( \|S_\nu F_\nu^2 f\|_2 \leq \|F_\nu A_\nu^2 f\|_2 \leq \nu^{-1} \|A_\nu^2 f\|_2 \), it is seen that \( \|M_\nu f\|_2 \leq \nu^{-\frac{1}{2}} \|f\|_2 \). Therefore, the range of \( R_\zeta \) is dense in \( \mathcal{H} \), and \( R_\zeta \) is the resolvent of a densely defined closed operator \( \Lambda \equiv \Lambda(a, b) \) (Appendix \[F\] Theorem \[E1\]).

Finally, by (p3), \( -\Lambda \) is the generator of a quasi bounded holomorphic semigroup.

**Remarks.** 1. The above construction of \( \Lambda(a, b) \) works for \( a \in (H_1) \), \( A = A_D, A_N \) or \( A_{iD} \). The usage of (p2) leads to the convergence \( (1 + A)^{\frac{1}{2}} (R_\zeta(b_n) - R_\zeta(b)) \xrightarrow{\ast} 0 \) claimed in section \[E\] almost immediately.

2. If \( a \in (H_u) \), \( A = A_D \), then \( \mathcal{H}_+ = W_0^{1,2}(\Omega), \ \mathcal{H}_- = W^{-1,2}(\Omega) \). If \( a \in (H_u) \), \( A = A_N \), then \( \mathcal{H}_\pm = W^{\pm 1,2}(\Omega) \). \[\square\]

2. In view of (p3) it suffices to prove the convergence \( R_\zeta(a_n, b_n) \xrightarrow{\ast} R_\zeta(a, b) \) for \( \zeta = \lambda_\delta \). For brevity, set \( F_n = F_\lambda(a_n), \ F = F_\lambda(a) \).

Note, that \( F_n \xrightarrow{\ast} F \) if and only if \( \langle (F_n - F)f, F_nf \rangle \to 0 \) and \( \langle (F - F_n)f, Ff \rangle \to 0 \ (f \in \mathcal{H}) \). In turn, \[
\|\langle (F_n - F)f, F_nf \rangle\| = \|\nabla F f \cdot (a_n - a) \cdot \nabla F_n^2 f\| \leq \|\nabla F_n^2 f\|_2 \|(a_n - a) \cdot \nabla F f\|_2,
\]
\[ \| \nabla F^2_n f \|_2 \leq \sigma^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} \| f \|_2 \]

and \( a_n \to a \) strongly in \( L^2_{\text{loc}} \). Thus \( F_n \overset{s}{\to} F \).

Next, employing the formula
\[ F^{\frac{1}{2}}(a) = \pi^{-1} \int_0^\infty t^{-\frac{1}{2}} F_{t+\lambda}(a) dt, \]

it is seen that
\[ \int_T^\infty t^{-\frac{1}{2}} \| (F_{t+\lambda}(a) - F_{t+\lambda}(a_n)) f \|_2 dt \leq 2T^{-\frac{1}{2}} \| f \|_2 \]

and
\[ \lim_n \int_0^T t^{-\frac{1}{2}} \| (F_{t+\lambda}(a) - F_{t+\lambda}(a_n)) f \|_2 dt = 0. \]

Thus \( F^{\frac{1}{2}} \overset{s}{\to} F^{\frac{1}{2}} \) and, since \( \nabla \) is a closed operator in \( H, \nabla F^{\frac{1}{2}}_n \overset{s}{\to} \nabla F^{\frac{1}{2}} \). In turn, the latter and the fact that \( b_n \to b \) strongly in \( L^2_{\text{loc}} \) yield \( H^* S_n \overset{s}{\to} H^* S \). (Indeed, \( s \cdot L^2 - H^* S_n = s \cdot L^2 - F^{\frac{3}{2}}_n b_n \cdot \nabla F^{\frac{1}{2}} \).

Therefore, it suffices to establish the convergence \( \| F_n^{\frac{1}{2}} b_n \cdot \nabla F^{\frac{1}{2}} f - H^* b_n \cdot \nabla F^{\frac{1}{2}} f \|_2 \to 0 \), or \( \| F_n^{\frac{1}{2}} (b_n \cdot \varphi) - H^* (\frac{b_n}{t_n} \cdot \varphi) \|_2 \to 0 \), only for all \( \varphi \in \mathbb{C}^d \).

3.4. **Non-divergent form operators.** The following theorem is a by-product of Theorem 3.1 and Theorem 3.3.

**Theorem 3.6.** Set \( b := (\nabla a), (\nabla a)_k = \sum_{i=1}^d (\partial_i a_k), \) and \( b^2_a = b \cdot a^{-1} \cdot b. \)

(i) If \( a \in (H_1), b \in F_\delta(A) \) for some \( \delta < 4 \) and also \( b^2_a \in L^1 + L^\infty \) if \( 1 \leq \delta < 4 \), then \( a \cdot \nabla^2 \) has an operator realization \( \Lambda(a,b) \) in \( L^r \) for every \( r \in I_0^\circ \) as the generator of the positivity preserving, \( L^r \) quasi contraction, \( L^\infty \) contraction \( C_0 \) semigroup \( e^{-t \Lambda(a,b)} = s \cdot L^r \cdot \lim_n e^{-t \Lambda(a,b)_n}. \)

(ii) If \( a, a_n \in (H_n), n = 1, 2, \ldots, (\nabla a) \in F_\delta(A), (\nabla a_n) \in F_\delta(A_n) \) for some \( \delta < 1 \) and all \( n \), and if \( a_n \to a, (\nabla a_n) \to (\nabla a) \) strongly in \( L^2_{\text{loc}}, \) then, for every \( r \in I_0^\circ, \)
\[ e^{-t \Lambda(a,b)} = s \cdot L^r \cdot \lim_n e^{-t \Lambda(a_n,b_n)}, \quad b = (\nabla a), b_n = (\nabla a_n). \]

In particular, for every \( r \geq d \), there exists a constant \( 0 < \alpha < 1 \) such that
\[ (\lambda + \Lambda_r)^{-1} L^r \cap L^\infty \subset C^{0,\alpha}, \quad \lambda > \omega_r. \]

(iii) If \( \Omega = \mathbb{R}^d, a \in (H_n), b := (\nabla a) \in F_\delta \equiv F_\delta (-\Delta) \) with \( \delta_1 = \sigma^2 \delta, \delta < 1, \) and \( |b| \in L^2 + L^\infty, \)
then, for every \( r \in I_0^\circ, \)
\[ e^{-t \Lambda(a,b)} = s \cdot L^r \cdot \lim_n e^{-t \Lambda(a_n,b_n)}, \quad a_n := e^{\Delta} a, b_n := (\nabla a_n). \]

(iv) If \( \Omega = \mathbb{R}^d, a \) is a uniformly elliptic diagonal matrix, \( b := (\nabla a) \in F_\delta(A) \) with \( \delta < 1, \)
\[ |b| \in L^2 + L^\infty, \]
then, for each \( r \in I_0^\circ, \)
\[ e^{-t \Lambda(a,b)} = s \cdot L^r \cdot \lim_n e^{-t \Lambda(a_n,b_n)}, \quad a_n := e^{\Delta} a, b_n := (\nabla a_n). \]
Proof. The claimed convergence in (i) (respectively, (ii)) is a direct consequence of Theorem 3.1
(Theorem 3.5).

1. The important thing in (iii) is the fact that \(b_n \in F_\delta(A_n)\), \(0 < \delta < 1\), uniformly in \(n\), and so \(\Lambda_r(a_n, b_n)\) are well defined for \(r \in I_c^0\). Set \(E_n f := \frac{\delta}{n} f\). Alternatively, we may set \(E_n f := \gamma_n * f\), \(\gamma_n\) denotes the K. Friedrichs mollifier. Note the following elementary pointwise inequalities

\[
(\nabla E_n a)^2 \leq E_n(\nabla a)^2, \quad |E_n(fg)|^2 \leq (E_n|f|^2)|E_n|g|^2, \quad f, g \in L^2 + L^\infty.
\]

Clearly, \(b = (\nabla a) \in F_\delta(A)\) with \(c(\delta) = \frac{c(\delta_1)}{\delta}\). Thus we only need to show that \(b_n \in F_\delta(A_n)\) in order to conclude that \(b_n \in F_\delta(A_n)\). Set \(|f|_\varepsilon := |f| + \varepsilon e^{-x^2}\). We have for \(f \in W^{1, 2}\),

\[
\|(\nabla E_n a)^2\|_2 = \|(E_n(\nabla a)^2|f|^2\|_1 = \lim_{\varepsilon \downarrow 0} \|(E_n(\nabla a)^2|f|_\varepsilon|^2\|_1,
\]

\[
\|(E_n(\nabla a)^2)|f|^2\|_1 = \|(\nabla a)\sqrt{E_n|f|^2}\|_2^2 \leq 
\delta_1 \|(\sqrt{E_n|f|^2})\|_2^2 + c(\delta_1)\|E_n|f|^2\|_1
\]

\[
= \delta_1 \left\| \frac{E_n(|f|_\varepsilon \nabla |f|_\varepsilon)}{\sqrt{E_n|f|^2}} \right\|_2^2 + c(\delta_1)\|E_n|f|^2\|_1
\]

\[
\leq \delta_1 \|(\nabla|f|_\varepsilon)^2\|_2 + c(\delta_1)\||f|_\varepsilon|^2\|_1
\]

\[
\leq \delta_1 \|(\nabla|f|_\varepsilon)^2\|_2 + c(\delta_1)\||f|_\varepsilon|^2\|_2^2.
\]

Since

\[
\lim_{\varepsilon \downarrow 0} \left( \delta_1 \|(\nabla|f|_\varepsilon)^2\|_2 + c(\delta_1)\||f|_\varepsilon|^2\|_2^2 \right) = \delta_1 \|(\nabla|f|)^2\|_2 + c(\delta_1)\||f|^2\|_2 \leq \delta_1 \|(\nabla|f|^2\|_2 + c(\delta_1)\||f|^2\|_2^2,
\]

we have proved that \(b_n \in F_\delta(A)\).

2. Now we claim that under assumptions on \(a\) in (iv)

\[
b \in F_\delta(A) \Rightarrow E_n b \in F_\delta(A_n).
\]

Only for simplicity we treat a particular case: \(a = \kappa^2 I\). Since \(b = (\nabla a) = 2\kappa \nabla \kappa\), \(b_a = 4(\nabla \kappa)^2\). The assumption \(b \in F_\delta(A)\) means that

\[
4\langle(\nabla \kappa)^2|f|^2\rangle \leq \delta \|\kappa \nabla f\|_2^2 + c(\delta)\|f\|_2^2 \quad f \in D(A^{\frac{1}{2}}).
\]

Set \(a_n := E_n a\). Then \((\nabla a_n) = 2E_n(\kappa \nabla \kappa), \ (\nabla a_n) \cdot a_n^{-1} \cdot (\nabla a_n) = \frac{4|E_n(a \nabla \kappa)|^2}{E_n(\kappa^2 \kappa^2)}\). Note that

\[
[E_n(\kappa \nabla \kappa)]^2 \leq (E_n(\kappa \kappa))^2 E_n(\nabla \kappa)^2,
\]

and so

\[
\langle(\nabla a_n) \cdot a_n^{-1} \cdot (\nabla a_n), |f|^2\rangle \leq 4\langle E_n(\nabla \kappa)^2, |f|^2\rangle = 4\langle(\nabla \kappa)^2, E_n|f|^2\rangle.
\]

But

\[
4\langle(\nabla \kappa)^2, E_n|f|_\varepsilon|^2\rangle \leq \delta \|\kappa \nabla E_n|f|_\varepsilon|^2\|_2^2 + c(\delta)\|E_n|f|_\varepsilon|^2\|_2^2 = \delta(\frac{\kappa^2|E_n|f|^2|_\varepsilon|^2}{4E_n|f|_\varepsilon|^2}) + c(\delta)\|E_n|f|_\varepsilon|^2\|_2^2
\]

\[
\frac{(\nabla E_n|f|_\varepsilon|^2)^2}{4E_n|f|_\varepsilon|^2} \leq E_n|\nabla|f|_\varepsilon|^2\|.\]

Thus,

\[
4\langle(\nabla \kappa)^2, E_n|f|^2\rangle \leq \delta(\kappa^2 E_n|\nabla|f|^2\|_2^2 + c(\delta)\|f|^2\|_2^2) = \delta\langle (E_n(\kappa \kappa)^2)|\nabla|f|^2\|_2^2 + c(\delta)\|f|^2\|_2^2\rangle.
\]
and
\[
\langle (\nabla a_n) \cdot a_n^{-1} \cdot (\nabla a_n), |f|^2 \rangle \leq \delta \|A_n^\frac{1}{2} f\|^2_2 + c(\delta) \|f\|^2_2 \leq \delta \|A_n^\frac{1}{2} f\|^2_2 + c(\delta) \|f\|^2_2.
\]
In other words, \(E_n b \in F_\delta(A_n)\) as required.

3. By Theorem 3.5, steps 1 and 2 entail the claimed convergences in (iii), (iv).

4. Finally, in (ii) the assertion concerning \(C^{\alpha,\alpha}_\infty\) smoothness follows now from the Krylov-Safonov a priori Hölder continuity of \((\lambda + \Lambda(a_n, b_n))^{-1} f, f \in C_\infty\) [15, Sect. 4.2]. □

**Remarks.** 1. Let \(\{\varepsilon_n\}\) be a sequence such that \(\varepsilon_n \downarrow 0\) as \(n \uparrow \infty\). The proof of (iii) yields the following.

For any \(b : \mathbb{R}^d \rightarrow \mathbb{R}^d\), \(|b| \in L^2_{\text{loc}}\) define \(b_n \equiv E_n b \equiv \gamma_{\varepsilon_n} \ast (b\mathbf{1}_{B(0, n)})\) and chose \(\{\varepsilon_n\}\) such that \(b_n \rightarrow b\) strongly in \(L^2_{\text{loc}}\) as \(n \uparrow \infty\). Fix \(\delta < \infty\). Then
\[
b \in F_\delta \Rightarrow (b_n \in F_\delta, b_n \rightarrow b \text{ strongly in } L^2_{\text{loc}}, \text{ and hence } e^{-t\Lambda(b_n)} \xrightarrow{\ast} e^{-t\Lambda(b)}).
\]

2. The same arguments used in step 2 yield the following.

If \(a \in (H_1), a_{ik} = \kappa^2_{ik} d, |\kappa_i| + |\nabla_i \kappa_i| \in L^2 + L^\infty, i, l = 1, 2, \ldots, d\), then
\[
b \in F_\delta(A) \Rightarrow b_n \in F_\delta(A_n) \quad (b = (\nabla a), b_n = e^{\Delta} b).
\]

**Example.** In \(\mathbb{R}^d, d \geq 3\) consider the matrix
\[
a(x) = I + c|x|^{-2} x^t \cdot x, \quad a_{ik}(x) = \delta_{ik} + c|x|^{-2} x_i x_k \quad \text{with } c = \frac{d - 1}{1 - \alpha} - 1, \quad \alpha < 1.
\]
Thus \(a(x)\) is strictly positive, \((\nabla a) = (d - 1)c|x|^{-2} x\), \(a^{-1} = I - \frac{c}{d - 1} |x|^{-2} x^t \cdot x\) and \(b^2 = \frac{|(d - 1)c|^2}{c + 1} |x|^{-2}\). The following Hardy type inequality (with the best possible constant) will be proved below:
\[
(c + 1) \frac{(d - 2)^2}{4} \| |x|\cdot 1 h \|^2 \leq \langle \nabla h \cdot a \cdot \nabla h \rangle \quad (h \in W^{1,2}((\mathbb{R}^d))).
\]

(\ast) implies that \(b \in F_\delta(A)\) with \(\delta = 4(1 + \frac{c}{d - 1})^2\) and \(c(\delta) = 0\). In particular, \(\delta < 4\) if and only if \(\alpha \in ]2(d - 2), 0[\) or equivalently \(c \in ]\frac{d - 1}{d - 2}, 0[\cup]0, d - 2[\). Armed with (\ast) and Theorem 3.1, one can reconsider the conclusions in [16 Ch. I, sect. 3, Example 4].

Consider the following problem in \(L^p((\mathbb{R}^d, \mathbb{L}^d)), p > 0, d \geq 3:\n\[
a \cdot \nabla^2 u = 0, \quad u(x) \uparrow \{ |x| \geq 1 \} = 0.
\]
If \(\alpha \neq 0\), then the problem has two solutions \(u_1 = 0\) and \(u_2 = |x|^{\alpha} - 1\). If \(\delta < 4\), then \(\alpha < 0\) and the unbounded solution to \(\Lambda_p u = 0\), \(p > \frac{2}{2 - \sqrt{\delta}},\) is inadmissible according to Theorem 3.1 \((\Lambda_p \supset -\nabla \cdot a \cdot \nabla + (\nabla a) \cdot \nabla\) only formally equals to \(-a \cdot \nabla^2\)). If \(\alpha > 0\) (so that \(\delta > 4\)), then the problem has two (bounded) solutions.

**Conclusion.** The condition \(\delta < 4\) of Theorem 3.1 can not be substantially strengthened.

**Proof of (\ast).** Let \(c > 0\). Since \(\langle \phi, x \cdot \nabla \phi \rangle = -\frac{d}{2} \langle \phi, \phi \rangle, \phi \in C_c^\infty\), we have
\[
\langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle = c(\|x \cdot \nabla(|x|^{-1} \phi)\|_2^2 - (d - 1)\| |x|^{-1} \phi\|_2^2).
\]
Next, the following inequality (with the sharp constant) is valid:
\[
\|x \cdot \nabla f\|_2 \geq \frac{d}{2} \|f\|_2, \quad (f \in D(\mathbb{G})), \quad (\nabla)
\]
where \( D \upharpoonright C_c^\infty = \frac{\sqrt{-1}}{2} (x \cdot \nabla + \nabla \cdot x) \).

Indeed, since the operator \( D = (D \upharpoonright C_c^\infty)_{L^2 \to L^2} \) is self-adjoint, \( \|(D - \lambda)^{-1}\| = \frac{1}{|\mathrm{Im} \lambda|} \) for \( \mathrm{Re} \lambda = 0 \).

Therefore
\[
\|x \cdot \nabla f\|_2 = \| \frac{1}{2} (x \cdot \nabla + \nabla \cdot x - d) f\|_2 = \|(D - \sqrt{-1} \frac{d}{2}) f\|_2 \geq \frac{d}{2} \|f\|_2, \quad (f \in C_c^\infty)
\]
and \((\nabla)\) is proved. But then
\[
\langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle \geq c \frac{(d - 2)^2}{4} \|x|^{-1} \phi\|_2^2 \quad (\phi \in C_c^\infty).
\]

(*) follows now from the equality \( \langle \phi, -\nabla \cdot a \cdot \nabla \phi \rangle = \langle \phi, -\nabla \cdot (a - 1) \cdot \nabla \phi \rangle + \langle \phi, -\Delta \phi \rangle \) and Hardy’s inequality \( \langle \phi, -\Delta \phi \rangle \geq \frac{(d-2)^2}{4} \|x|^{-1} \phi\|_2^2 \).

Finally, the obvious inequality \((1 + c) \langle \phi, -\Delta \phi \rangle \geq \langle \phi, -\nabla \cdot a \cdot \nabla \phi \rangle\) clearly shows that the constant in (*) is sharp.

If \(-1 < c \leq 0\), (*) is a trivial consequence of Hardy’s inequality. \(\square\)

**Remark.** The Krylov-Safonov a priori estimates yield the uniqueness of a “good” solution to \(-a \cdot \nabla^2 = f\) in \(L^d\) provided that \(a \in (H^s)\) is continuous outside of a “sufficiently small” set \([2]\).

The assumption \((\nabla a) \in F_s(A)\) does not guarantee the validity of \(W^{2,r}\) estimates on \((\zeta + \Lambda_\nu (a, \nabla a))^{-1} L^r\) for some \(r > (d-2) \vee 2\). The same is true even for \((\nabla a) \in F_0(A)\). (See also remark 5 in the next section.)

3.5. \(W^{1,s}\)-estimates on solutions to \((\mu - \Delta + b \cdot \nabla) u = f, b \in F_\delta\). Let \(d \geq 3\), \(L^p = L^p(\mathbb{R}^d, \mathcal{L}^d), b : \mathbb{R}^d \to \mathbb{R}^d, 0 < \delta < 4\). Define \(b_n := E_{\varepsilon_n} b := \gamma_{\varepsilon_n} * (b 1_{B(0,n)})\), \(n = 1, 2, \ldots\).

Set \(\Lambda_{p}(b_n) := -\Delta + b_n \cdot \nabla, D_{\Lambda_{p}(b_n)} = W^{2,p}(\mathbb{R}^d), 1 < p < \infty\). Clearly, \(-\Lambda_{p}(b_n)\) is the generator of a holomorphic semigroup in \(L^p\). According to Theorem 3.4 and the fact \(b_n \in F_\delta\) (see Remark 1 after the proof of Theorem 3.6), for each \(p \in I_c = [\frac{2}{2 - \sqrt{\delta}}, \infty[,\)
\[
\|e^{-t\Lambda_{p}(b_n)}\|_{p \to p} \leq e^{\omega_p t}, \quad \omega_p = \frac{c(\delta)}{2(p - 1)},
\]
and by Theorem 3.2, for each \(p \in I_c^0\), \(s\)-\(L^p\)-\(\lim_{n} e^{-t\Lambda_{p}(b_n)}\) exists and determines the \(C_0\) semigroup \(e^{-t\Lambda_{p}}\); \(\Lambda_{p} = \Lambda_{p}(b)\) is an operator realization of \(-\Delta + b \cdot \nabla\) in \(L^p\).\n
**Theorem 3.7.** Let \(d \geq 3\). Fix any \(\delta \in [0, 1 \wedge (\frac{2}{7 - 2})^2\]. Assume that \(b \in F_\delta\). Let \(q \in [\frac{2}{2 - \sqrt{\delta}}, \infty[\]
\[
Set u = (\mu + \Lambda_{q}(b))^{-1} f, where f \in L^q, \mu > \omega_q. Then there exist constants \(\lambda_0 = \lambda_0(\delta, q)\) and \(K_l = K_l(\delta, q), l = 1, 2\), such that, for all \(\mu > \lambda_0 \vee \omega_q,\)
\[
\|u\|_{W^{1,1}} \leq K_1 (\mu - \lambda_0)^{-\frac{1}{2}} \|f\|_q;
\]
\[
\|u\|_{W^{1,2}} \leq K_2 (\mu - \lambda_0)^{-\frac{2}{d} + \frac{1}{2}} \|f\|_q, \quad j = \frac{d}{d - 2}.
\]
In particular, \((\mu + \Lambda_{q}(b))^{-1} : L^q \to C^{0,1 - \frac{d}{2}}\) whenever \(q \in [2 \vee (d - 2), \frac{2}{\sqrt{\delta}}[\) and \(\mu > \omega_q\).

**Proof.** 1. Let \(q \in [\frac{2}{2 - \sqrt{\delta}}, \infty[\]. Clearly, \(b_n \cdot \nabla \upharpoonright W^{2,q}\) is the Miyadera perturbation of \(-\Delta : \|b_n \cdot \nabla (\lambda_0 - \Delta)^{-1}\|_{q \to q} < 1, \sqrt{\lambda_0} \geq m_{n,d}\) for some constant \(m_{n,d}\). Therefore, by the resolvent identity, \((\mu + \Lambda_{q}(b_n))^{-1} L^q \subset W^{2,q}, \mu > \omega_q\).
2. Let $q \in [2, \frac{2}{\sqrt{\omega}}]$. Set $u_n = (\mu + \Lambda_q(b_n))^{-1} f$, $0 \leq f \in C^\infty_c$. We will use the following notations.

$$w := \nabla u_n, \ w_i := \nabla_i u_n, w_{ik} := \nabla_i w_k, \ \Delta u_n = \nabla \cdot w = \sum_{i=1}^d w_{ii};$$

$$\phi := -\nabla \cdot (w|w|^{q-2}) \equiv -\sum_{i=1}^d \nabla_i (w_i|w|^{q-2}).$$

Note that $u_n \in \mathcal{S}$, the L. Schwartz space of test functions. Then in view of

$$\langle -\Delta u_n, \phi \rangle = \langle -\Delta w, w|w|^{q-2} \rangle = \sum_{i,k=1}^d \langle w_{ik}, w_{ik}|w|^{q-2} + (q-2)|w|^{q-3} w_{ik} \nabla_i |w| \rangle$$

$$= -|w|^{q-2} \Delta u_n - (q-2)|w|^{q-3} w \cdot \nabla |w| =: \phi_1 + \phi_2, \ \phi_1 = -|w|^{q-2}(\mu u_n + b \cdot w - f)$$

we obtain

$$\mu(|w|^q) + I_q + (q-2)J_q = \langle b_n \cdot w - f, |w|^{q-2}(\mu u_n + b_n \cdot w - f) + (q-2)|w|^{q-3} w \cdot \nabla |w| \rangle, \quad (*)$$

where

$$I_q = \sum_{i=1}^d \langle |\nabla w_i|^2, |w|^{q-2} \rangle, \quad J_q = \langle |\nabla |w||^2, |w|^{q-2} \rangle.$$

Now we bound the terms from the RHS of $(*)$ by $J_q$, $B_q := \langle |b_n \cdot w|^2 |w|^{q-2} \rangle$, $\|w\|^{q-2}, \|f\|_q^2$ and $(\frac{\mu}{\mu - \omega_q})^2 \|f\|_q^2$ as follows.

1. $\langle b_n \cdot w, |w|^{q-2} \mu u_n \rangle \leq \frac{\mu}{\mu - \omega_q} B_{q} \|w\|_{q}^{q-2} \|f\|_q \cdot \left(\frac{2}{2-\sqrt{\omega}} < q \Rightarrow \|u_n\|_q \leq (\mu - \omega_q)^{-1} \|f\|_q\right).$
2. $\langle b_n \cdot w, |w|^{q-2} b_n \cdot w \rangle = B_{q}^2.$
3. $\|\langle b_n \cdot w, |w|^{q-2} (f) \rangle\| \leq B_{q} \|w\|_{q}^{q-2} \|f\|_q.$
4. $(q-2)\langle b_n \cdot w, |w|^{q-3} w \cdot \nabla |w| \rangle \leq (q-2)B_{q} \frac{1}{2} J_{q}.$
5. $\langle -f, |w|^{q-2} \mu u_n \rangle \leq 0.$
6. $\langle -f, |w|^{q-2} b_n \cdot w \rangle \leq B_{q} \frac{1}{2} \|w\|_{q}^{q-2} \|f\|_q.$
7. $\langle f, |w|^{q-2} f \rangle \leq \|w\|_{q}^{q-2} \|f\|_q^2.$
8. $(q-2)\langle -f, |w|^{q-3} w \cdot \nabla |w| \rangle \leq (q-2)B_{q} \frac{1}{2} \|w\|_{q}^{q-2} \|f\|_q.$

(Below we will get rid of the terms arising in the RHS of 1), 3), 5), 8) using Young’s inequality, at expense of increasing the coefficients $K_1, K_2$ of $\|f\|_q.$)

By means of 1)-8) we have $(\varepsilon, \varepsilon_0 > 0)$

$$\text{RHS of } (*) \leq (q-2)\varepsilon J_q + (q-2)(\varepsilon_0 J_{q} + \frac{1}{4\varepsilon_0} B_q) + (1 + 3\varepsilon)B_q + \left(1 + \frac{\mu}{\mu - \omega_q} + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon} \frac{\mu^2}{(\mu - \omega_q)^2}\right) \|w\|^{q-2} \|f\|_q^2.$$
Setting \( \varepsilon_0 = \frac{q \sqrt{2}}{4} \) we have

\[
\text{RHS of (⋆)} \leq \left[ (q - 2) \frac{q \sqrt{2}}{2} + \frac{\delta q^2}{4} + (q - 2)\varepsilon + 3\varepsilon \frac{\delta q^2}{4} \right] J_q \\
+ \left( 1 + 3\varepsilon + \frac{q - 2}{q \sqrt{2}} \right) c(\delta) \|u\|_q^q + \left( 1 + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon} \frac{\mu^2}{(\mu - \omega_q)^2} \right) \|w\|_{q-2}^2 \|f\|_q^2.
\]

Thus

\[
\mu \|w\|_q^q + I_q \geq \left[ (q - 2)(1 - \frac{q \sqrt{2}}{2}) - \frac{\delta q^2}{4} \right] J_q - \left[ (q - 2)\varepsilon + 3\varepsilon \frac{\delta q^2}{4} \right] J_q \\
\leq \left( 1 + 3\varepsilon + \frac{q - 2}{q \sqrt{2}} \right) c(\delta) \|w\|_q^q + \left( 1 + \frac{q}{4\varepsilon} + \frac{1}{4\varepsilon} \frac{\mu^2}{(\mu - \omega_q)^2} \right) \|w\|_{q-2}^2 \|f\|_q^2.
\]

Since \( I_q \geq J_q \) and \( q - 1 - (q - 2) \frac{q \sqrt{2}}{2} - \frac{\delta q^2}{4} > 0 \) due to \( q \in [2, \frac{4}{\sqrt{2}}] \), we conclude that, for suitable \( \varepsilon \), there are \( C_1 = C_1(\varepsilon, \delta, q) > 0 \) and \( C_2 = (\varepsilon, \delta, q, \omega_q) < \infty \) such that

\[
(\mu - \tilde{\omega}_\varepsilon) \|w\|_q^q + C_1 J_q \leq C_2 \|w\|_{q-2}^2 \|f\|_q^2,
\]

where \( \tilde{\omega}_\varepsilon = (1 + 3\varepsilon + \frac{q - 2}{q \sqrt{2}})c(\delta) \).

The latter, the Sobolev and the Young inequalities combined imply that

\[
\|\nabla u_n\|_{qj} \leq K_2 (\mu - \lambda_0)^{\frac{1}{2} - \frac{1}{2}} \|f\|_q\text{ whenever }\mu > \lambda_0 := \tilde{\omega}_\varepsilon > \omega_q.
\]

It is also seen that \( \|\nabla u_n\|_q \leq C_1^\frac{1}{2} (\mu - \lambda_0)^{-\frac{1}{2}} \|f\|_q \). Since \( u_n \rightarrow u \) strongly in \( L^q_+ \) and hence in \( L^q_+ \), and since \( (C^\infty_\geq)_+ \) is dense in \( L^q_+ \), we have \( u \in W^{1,q}_+ \), and the inequality \( \|\nabla u\|_{qj} \leq K_2 (\mu - \lambda_0)^{\frac{1}{2} - \frac{1}{2}} \|f\|_q \) holds for all \( f \in L^q_+ \).

We have established the assertions of the theorem for all \( f \in L^q_+ \), and hence for all \( f \in L^q \) (with \( K_l \rightarrow 4K_l, l = 1, 2 \)). \( \square \)

**Remarks.**

1. There is an obvious analogue of Theorem 3.7 for \( \Omega \subset \mathbb{R}^d \) with the additional assumption \( \partial \Omega \in C^{0,1} \) in case of \( \Delta = \Delta_N \).

2. Only for \( d = 3 \) and \( q = 2 \) the result of Theorem 3.7 is “trivial”. Indeed, by Miyadera’s Perturbation Theorem, the assumption \( b \in \mathbf{F}_\delta, \delta < 1 \), implies that \( -\Lambda_2(b) = \Delta - b \cdot \nabla \) of domain \( W^{2,2} \) is the generator of a \( C_0 \) semigroup in \( L^2 \), and hence, for \( \mu > \omega_2, (\mu + \Lambda_2(b))^{-1} : L^2 \rightarrow W^{1,2j} \), \( 2j = 6 = 2d \). However, already for \( d = 4 \), \( W^{2,2} \subset W^{1,2j} \), \( 2j = d \) but not crucial \( 2j > d \).

3. Let \( |b| \in L^d + L^\infty, d \geq 3 \). Then, by the Sobolev Embedding Theorem, for any \( q \in [\frac{d}{2}, d] \), there exists a constant \( \lambda_{d,b} > 0 \), such that

\[
(\mu + \Lambda_q(b))^{-1} : L^q \rightarrow W^{2,q} \subset C^{0,2-d} \text{ for all } \mu > \lambda_{d,b}.
\]

In particular,

\[
(\mu + \Lambda_q(b))^{-1} : L^q \cap L^d \rightarrow C^{0,\alpha} \text{ for any } \alpha < 1.
\]

By Theorem 3.7 the last embedding also holds for \( b \in \mathbf{F}_0 = \bigcap_{\delta > 0} \mathbf{F}_\delta \).

4. In fact, for \( b \in [L^{d,\infty}]^d \) (\( \subseteq \mathbf{F}_\delta \) with \( \delta = c_d \|b\|_{d,\infty} \), see examples in section 1 for details), one has the following characterization of \( W^{2,q} \) smoothness of \( u = (\mu + \Lambda_q(b))^{-1}f, f \in L^q \).
We will need:

\[ |\nabla (-\Delta)^{-1}(x,t)| \leq m^*_d(-\Delta)^{-\frac{1}{2}}(x,y), \quad x,y \in \mathbb{R}^d \]

with \( m^*_d := \frac{d-2}{2} \sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right) \).

Let \( \Omega_d = \pi^\frac{d}{2} \Gamma\left(\frac{d}{2} + 1\right) \) denote the volume of the unit ball in \( \mathbb{R}^d \). Now, for every \( 1 < q < d \)

\[ \|b \cdot \nabla (-\Delta)^{-1}\|_{q \to q} \leq m^*_d \|b(-\Delta)^{-\frac{1}{2}}\|_{q \to q} \quad (\text{we apply} \ [12 \ \text{Prop.} \ 2.5]) \]

\[ \leq m^*_d \|b\|_{d,\infty} \Omega_d^{-\frac{1}{d}} \|x^{-1}(-\Delta)^{-\frac{1}{2}}\|_{q \to q} \quad (\text{we apply} \ [12 \ \text{Lemma} \ 2.7]) \]

\[ = m^*_d \|b\|_{d,\infty} \Omega_d^{-\frac{1}{d}} R_{d,q}, \quad R_{d,q} := \frac{1}{2} \frac{\Gamma\left(\frac{d}{2q}\right) \Gamma\left(\frac{d}{2q} - \frac{1}{2}\right)}{\Gamma\left(\frac{d+1}{2} - \frac{d}{2q}\right)} \]

Thus, if \( \|b\|_{d,\infty} < (m^*_d \Omega_d^{-\frac{1}{d}} R_{d,q})^{-1} \) then by Miyadera’s Perturbation Theorem \( D(\Lambda_q(b)) \subset W^{2,q}, \ 1 < q < d \) (Note that \( q = d \) is unavailable for \( b \in [L^{d,\infty}]^d \).

5. Let \( d \geq 3 \) and \( b(x) = c|x|^{-2}x, \ |c| < 1 \). Clearly, \( b \in F \) with \( \delta = \frac{4c^2}{(d-2)^2} \) satisfies the assumptions of Theorem 3.7 if \( d \geq 4 \). For \( d = 3 \) however \( |c| \) has to be strictly less than \( \frac{1}{2} \).

In fact, for this vector field the constraints on \( \delta \) in Theorem 3.7 can be substantially relaxed (for \( d \geq 6 \) if \( c > 0 \) and for \( d \geq 3 \) if \( c < 0 \), see Theorem 3.8 and Theorem 3.9 below).

**Theorem 3.8.** Let \( \Omega = \mathbb{R}^d, \ d \geq 3 \). Let \( b(x) = -c|x|^{-2}x, \ c = \frac{d-2}{2} \sqrt{\delta}, \ \delta < \infty \).

Then the interval of contraction solvability for \( \Lambda_q(b) \supset -\Delta + b \cdot \nabla \) is \([1, \infty[\).\n
1) \( d \geq 4 \). Let \( u = (\mu + \Lambda_q(b))^{-1}f, \ \mu > 0, \ f \in L^q \). Then, for any \( \delta < \infty \), there exist \( q > d - 2 \) and constants \( K_l = K_l(\delta, q), \ l = 1, 2, \) such that

\[ \|u\|_{W^{1,q}} \leq K_1 \mu^{-\frac{1}{2}}\|f\|_q; \]

\[ \|u\|_{W^{1,q}} \leq K_2 \mu^{-\frac{1}{2}}\|f\|_q. \]

2) \( d = 3 \). Then 1) holds but for any \( \sqrt{\delta} < \frac{2}{q}, \ q \geq 2 \).

In particular, \( (\mu + \Lambda_q(b))^{-1} : L^q \to C^{0,1-\frac{2}{d}} \).

**Proof.** Set \( |x|^2 := |x|^2 + \epsilon, \ \epsilon > 0 \), \( b\epsilon(x) := -c|x|^2x, \ u \equiv u\epsilon = (\mu + \Lambda_q(b\epsilon))^{-1}f, \ w := \nabla u \). We follow the proof of Theorem 3.7. We have

\[ \mu(|w|^q) + I_q + (q-2)J_q = c(|x|^2x \cdot w, \phi) + \langle f, \phi \rangle, \]

where \( \phi := -\nabla \cdot (w|w|^{q-2}) \). Integrating by parts, we have

\[ \langle |x|^2x \cdot w, \phi \rangle = Z + \langle |x|^2x \cdot \nabla |w|, \ |w|^{q-1} \rangle, \]

where

\[ Z := \langle |x|^{-2} |w|^q - 2 |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle, \]

and so

\[ \langle |x|^2x \cdot w, \phi \rangle \leq \langle |x|^{-2} |w|^q \rangle + \langle |x|^2x \cdot \nabla |w|, \ |w|^{q-1} \rangle, \]

we obtain the inequality

\[ \mu(|w|^q) + I_q + (q-2)J_q \leq c(|x|^2 |w|^q) + c(|x|^2x \cdot \nabla |w|, |w|^{q-1}) + \langle f, \phi \rangle. \]
Noticing that
\[ \langle |x|^{-2} x \cdot \nabla |w|, |w|^{q-1} \rangle = -\frac{d-2}{q} \langle |x|^{-2} |w|^q \rangle - \frac{2\varepsilon}{q} \langle |x|^{-4} |w|^q \rangle, \]

\[ I_q \geq J_q \quad \text{and} \quad \langle |x|^{-2} |w|^q \rangle = \langle |x|^{-2} |w|^q \rangle - \varepsilon \langle |x|^{-2} |x|^{-2} |w|^q \rangle, \]
we have
\[ \mu \langle |w|^q \rangle + (q-1)J_q \leq c \left( 1 - \frac{d-2}{q} \right) \langle |x|^{-2} |w|^q \rangle - c \left( 1 - \frac{d-2}{q} \right) \varepsilon \langle |x|^{-2} |x|^{-2} |w|^q \rangle + \langle f, \phi \rangle \]
\[
\text{(here we are using Hardy’s inequality)}
\]
\[ \leq c \left( 1 - \frac{d-2}{q} \right) \frac{q^2}{(d-2)^2} J_q - c \left( 1 - \frac{d-2}{q} \right) \varepsilon \langle |x|^{-2} |x|^{-2} |w|^q \rangle + \langle f, \phi \rangle. \]

Therefore, for \( q > d - 2 \),
\[ \mu \langle |w|^q \rangle + \left[ q - 1 - \frac{q(q-d+2)}{(d-2)^2} \right] J_q \leq \langle f, \phi \rangle. \]

Now, arguing as in the proof of Theorem \( 3.7 \) (see estimates 5)-8) there), we bound \( \langle f, \phi \rangle \) as follows:
\[ \langle f, \phi \rangle \leq \varepsilon J_q + C(\varepsilon) \|w\|_q^{q-2} \|f\|_q^2, \quad 0 < C(\varepsilon) < \infty, \]
where \( \varepsilon > 0 \) is to be chosen sufficiently small.

Finally, applying the Sobolev and the Young inequalities as in the last step of the proof of Theorem \( 3.7 \) we obtain the required estimates on \( \|u\|_{W^{1,q}}, \|u\|_{W^{1,q}} \).

### Theorem 3.9

Let \( \Omega = \mathbb{R}^d, \ d \geq 5 \). Let \( b(x) := c|x|^{-2}x, \ c = \frac{d-2}{2} \sqrt{\delta}, \ \sqrt{\delta} < \frac{d-2}{2}. \) There exist \( q > 2 \vee d-2 \) and constants \( K_l = K_l(\delta, q), \ l = 1, 2, \) such that \( u = (\mu + \Lambda_q(b))^{-1} f, \mu > 0, \ f \in L^q, \) satisfies
\[ \|u\|_{W^{1,q}} \leq K_1 \mu^{-\frac{1}{2}} \|f\|_q; \]
\[ \|u\|_{W^{1,q}} \leq K_2 \mu^{\frac{1}{4} - \frac{1}{2}} \|f\|_q. \]

In particular, \( (\mu + \Lambda_q(b))^{-1} : L^q \to C^{0,1-\frac{d}{\delta}} \).

**Proof.** Modifying the proof of Theorem \( 3.7 \) we have
\[ \mu \langle |w|^q \rangle + I_q + (q-2)J_q = -c \langle |x|^{-2} x \cdot w, \phi \rangle + \langle f, \phi \rangle, \]

Integrating by parts, we have
\[ \langle |x|^{-2} x \cdot \nabla |w|, |w|^{q-1} \rangle = Z + \langle |x|^{-2} x \cdot \nabla |w|, |w|^{q-1} \rangle, \]
where \( Z = \langle |x|^{-2} |w|^q \rangle - 2 \langle |x|^{-4} (x \cdot w)^2 |w|^{q-2} \rangle, \)
\[ \langle |x|^{-2} x \cdot \nabla |w|, |w|^{q-1} \rangle = -\frac{d-2}{q} \langle |x|^{-2} |w|^q \rangle - \frac{2\varepsilon}{q} \langle |x|^{-4} |w|^q \rangle. \]
Clearly,

\[
\langle |x|^{-4}(x \cdot w)^2|w|^{q-2}\rangle \leq \langle |x|^{-4}|x|^{-2}|w|^q\rangle \\
= \langle |x|^{-2}|w|^q\rangle - 2\varepsilon \langle |x|^{-2}|x|^{-2}|w|^q\rangle + \varepsilon^2 \langle |x|^{-4}|x|^{-2}|w|^q\rangle,
\]

\[
\varepsilon \langle |x|^{-4}|x|^{-2}|w|^q\rangle \leq \langle |x|^{-2}|x|^{-2}|w|^q\rangle.
\]

Therefore,

\[
\mu \langle |w|^q\rangle + I_q + (q - 2)J_q \\
\leq c \frac{q + d - 2}{q} \langle |x|^{-2}|w|^q\rangle - c\varepsilon \frac{q + d - 2}{q} \langle |x|^{-2}|x|^{-2}|w|^q\rangle + \langle f, \phi \rangle \\
\leq c \frac{q(q + d - 2)}{(d - 2)^2} J_q + \langle f, \phi \rangle,
\]

and so

\[
\mu \langle |w|^q\rangle + \left[ q - 1 - c\frac{q(q + d - 2)}{(d - 2)^2} \right] J_q \leq \langle f, \phi \rangle.
\]

It is seen that if \( \sqrt{d} < \frac{d - 2}{d - 2} \), then \( q - 1 - c\frac{q(q + d - 2)}{(d - 2)^2} > 0 \) for some \( q > d - 2 \). The rest of the proof repeats the end of the proof of Theorem 3.8. \( \square \)

**Remark.** Consider the formal differential operator \( L = -\Delta + b \cdot \nabla \), \( b(x) := \frac{d - 2}{2} \sqrt{d} |x|^{-2} x \). If \( \delta > 4 \), then the Dirichlet problem for \( L \) in \( \{|x| < 1\} \) has two distinct bounded weak solutions, moreover, one of these solutions does not satisfy the maximum principle; see remark 3 after Theorem 3.1. In view of Theorem 3.9, this observation cannot be used to justify discarding \( b(x) = \frac{d - 2}{2} \sqrt{d} |x|^{-2} x \) regardless of the value of \( \delta \) (and thus the whole class \( F_\delta - [L^d + L^\infty]^d \)) as an argument for the “optimality” of the assumption \( b \in [L^d + L^\infty]^d \) (a recurrent theme in the literature).

We generalize and detailize Theorem 3.9 as follows.

**Theorem 3.10.** Let \( \Omega = \mathbb{R}^d \), \( d \geq 3 \). Assume that \( b \in F_\delta \), \( \delta < 4 \), \( b^2 \in L^1 + L^\infty \), and \( G \equiv G(b), G_{ik} := \nabla_k b_i \), satisfies the inequality

\[
|\langle (Gh, h) \rangle| \leq \delta_1 (|\nabla |h||^2) + c_1(\delta_1) (|h|^2), \quad h \in C_0^\infty (\mathbb{R}^d, \mathbb{R}^d)
\]

for some \( 0 < \delta_1 \leq 1 - \frac{\sqrt{d}}{2} \), \( 0 \leq c_1(\delta_1) < \infty \). Let \( u = (\mu + \Lambda_q(b))^{-1} f \), \( f \in L^q \), \( \mu > \omega_q \), \( q \in ]2 \vee q^-, q^+[ \), where

\[
q^+ := \frac{2 - \sqrt{d} + \sqrt{(2 - \sqrt{d})^2 - 4\delta_1}}{\delta_1}.
\]

There exist constants \( \lambda_0 = \lambda_0(\delta, q) \) and \( K_l = K_l(\delta, q) \), \( l = 1, 2 \), such that, for all \( \mu > \lambda_0 \vee \omega_q \),

\[
\|u\|_{W^{1,q}_l} \leq K_1 (\mu - \lambda_0)^{-\frac{1}{2}} \|f\|_q;
\]

\[
\|u\|_{W^{1,q}_l} \leq K_2 (\mu - \lambda_0)^{\frac{1}{2} - \frac{d}{2}} \|f\|_q.
\]

In particular, if \( 2 - \sqrt{d} + \sqrt{(2 - \sqrt{d})^2 - 4\delta_1} > (2 \vee d - 2)\delta_1 \) then there exists \( q \in ]2 \vee d - 2, q^+[ \) such that \( (\mu + \Lambda_q(b))^{-1} : L^q \to C^{0,1 - \frac{d}{2q^+}} \).
Proof. Set \( b_n := E_n b \), where \( E_n := e^{\frac{b}{n}} \). Then \( b_n \in \Phi_\delta \) (see the arguments in section 3.4) and \( E_n G(b) \) satisfies \( (\star) \) with the same \( \delta_1 \) for all \( n \). Indeed,

\[
\langle |E_n G(b)| h, h \rangle \leq \langle G(b) \tilde{h}, \tilde{h} \rangle, \quad \tilde{h}_1 := (E_n h_1^2)^{\frac{1}{2}},
\]

and by \( (\star) \),

\[
\langle G(b) \tilde{h}, \tilde{h} \rangle \leq \delta_1 \| \nabla (E_n |h|^2)^{\frac{1}{2}} \|_1 + c_1(\delta_1) \| E_n |h|^2 \|_1.
\]

Noticing that

\[
\langle \nabla (E_n |h|^2)^{\frac{1}{2}} \rangle = \frac{E_n(|h|\nabla |h|)}{\sqrt{E_n|h|^2}} \leq \| E_n (\nabla |h|)^2 \|_1 = \| \nabla |h| \|_2^2, \quad \text{and} \quad \| E_n |h|^2 \|_1 = \| h \|_2^2,
\]

we obtain the required.

Now, we have

\[
\mu \langle |w|^q \rangle + I_q + (q - 2) J_q + \langle b_n \cdot w, \phi \rangle = \langle f, \phi \rangle.
\]

Integrating by parts, we obtain

\[
\langle b_n \cdot w, \phi \rangle = \langle |E_n G(b)| w, w |w|^{q-2} \rangle + \langle b_n \cdot \nabla |w|, |w|^{q-1} \rangle.
\]

Thus,

\[
\langle b_n \cdot w, \phi \rangle \leq \delta_1 \| \nabla |w| \|_2 \| \nabla |w| \|_2 + \alpha \langle |w|^q \rangle + \frac{1}{4\alpha} J_q \quad (\alpha = \frac{1}{q\sqrt{\delta}})
\]

\[
\leq \left[ \frac{q^2}{4} \delta_1 + \frac{q^2}{4} \delta_\alpha + \frac{1}{4\alpha} \right] J_q + \alpha (c_1(\delta_1) + c(\delta)) \| w \|_q^q
\]

\[
= \left[ \frac{q^2}{4} \delta_1 + \frac{q}{2\sqrt{\delta}} \right] J_q + \frac{c_1(\delta_1) + c(\delta)}{q\sqrt{\delta}} \| w \|_q^q.
\]

Applying \( I_q \geq J_q \), we obtain

\[
\left( \mu - \frac{c_1(\delta_1) + c(\delta)}{q\sqrt{\delta}} \right) \langle |w|^q \rangle + \left( q - 1 - \frac{q^2}{4} \delta_1 - \frac{q}{2\sqrt{\delta}} \right) J_q \leq \langle f, \phi \rangle.
\]

(It is seen that if \( q \in [2 \vee q^-, q^+] \), then \( q - 1 - \frac{q^2}{4} \delta_1 - \frac{q}{2\sqrt{\delta}} > 0 \).) The rest of the proof repeats the end of the proof of Theorem 3.8 \( \square \)

Remarks. 1. The requirement \( b^2 \in L^1 + L^\infty \) is not essential: one can get rid of it by defining \( b_n \) as \( b_n := e^{\varepsilon_n A} \zeta_n b \), where \( \zeta_n(x) = \eta_n(|x|) \),

\[
\eta_n(t) := \begin{cases} 1, & \text{if } t < n, \\ 2 - \frac{t}{n}, & \text{if } n \leq t \leq 2n, \\ 0, & \text{if } 2n < t, \end{cases}
\]

and \( \varepsilon \downarrow 0 \) are chosen sufficiently small. (Albeit this works for \( \delta_1 < 1 - \frac{\sqrt{\delta}}{2} \). The latter does not affect the result since the interval \( [q^-, q^+] \) is open.)

2. It is easy to modify the proof above to work on an arbitrary open set \( \Omega \subset \mathbb{R}^d \).

3. For the vector field \( b(x) = |x|^{-2} x \), \( c > 0 \), one has \( h^i G h = c(|x|^{-2} h^2 - 2x \cdot h^2 |x|^{-4}) \), and so \( \langle G h, h \rangle \rangle \leq c(|x|^{-2} |h|^2) \). Thus, for this vector field the conditions of Theorem 3.10 are satisfied with \( \delta_1 = \frac{4}{(d - 2)^2} c, \delta = \frac{4}{(d - 2)^2} c^2 \). In particular, one recovers the assertion of Theorem 3.9.
In the forthcoming paper [11] we extend Theorem 3.7 to the operators $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ and $-a \cdot \nabla^2 + b \cdot \nabla$. Here we only mention the following

**Theorem (11).** Let $\Omega = \mathbb{R}^d, d \geq 3, c > -1, a(x) = I + c|x|^{-2}x^t \cdot x, x, \in \mathbb{R}^d$. Let $b \in F_\delta(-\Delta)$, or $b \in F_\delta(A)$, where $A = A_D.$

(i) [Divergence form] If $\sqrt{\delta} \in [0, 1 \wedge \frac{2}{d+2}]$ and $|c|$ is sufficiently small, or $c \in \left[-\frac{1}{2 + \frac{2}{d+2}}, 2(d-3)(d-1)\right]$, $c \in [-1/9, 4/13]$, $c = d = 3$ and $\delta$ is sufficiently small, then, for $q > d - 2$ sufficiently close to $d - 2$ the operator realization $\Lambda_q(a, b)$ of $-\nabla \cdot a \cdot \nabla + b \cdot \nabla$ is well defined, and there exist constants $\lambda_0 = \lambda_0(c, \delta, q)$ and $K_i = K_i(c, \delta, q), i = 1, 2, \text{ such that } u = (\mu + \Lambda_q(a, b))^{-1}f \ (\mu \in \rho(-\Lambda_q), f \in L^q)$ satisfies the inequalities

$$
\|u\|_{W^{1,q}} \leq K_1(\mu - \lambda_0)^{-\frac{1}{2}}\|f\|_q,
$$

$$
\|u\|_{W^{1,q_j}} \leq K_2(\mu - \lambda_0)^{\frac{1}{2} - \frac{1}{j}}\|f\|_q, \quad j = \frac{d}{d+2}.
$$

In particular, $(\mu + \Lambda_q(a, b))^{-1} : L^q \rightarrow C^{0,1-\frac{d}{d+2}},$ for all $\mu > \lambda_0 \vee \omega_q$.

(ii) [Non-divergence form] If $\sqrt{\delta} \in [0, 1 \wedge \frac{2}{d+2}]$ and $|c|$ is sufficiently small, or $c \in \left[-\frac{1}{1 + \frac{(d-4)^2}{4(d-3)(2d-5)}}, \frac{d-3}{2}\right]$, $c \in [-1, 1/3]$, $c = d = 3$ and $\delta$ is sufficiently small, then, for all $q > d - 2$ sufficiently close to $d - 2$ the operator realization $\Lambda_q(a, (\nabla a) + b)$ of $a \cdot \nabla^2 - b \cdot \nabla$ in $L^q$ is well defined, and (*) holds for $u = (\mu + \Lambda_q(a, (\nabla a) + b))^{-1}$ $(\mu \in \rho(-\Lambda_q), f \in L^q)$.

3.6. $L^r$-strong Feller semigroup on $C_\infty$ corresponding to $-\Delta + b \cdot \nabla$, $b \in F_\delta$. Armed with Theorem 3.2 and Theorem 3.7, we establish

**Theorem 3.11.** Let $d \geq 3, b : \mathbb{R}^d \rightarrow \mathbb{R}^d, b \in F_\delta, \delta \in \left[0, 1 \wedge \frac{4}{(d-2)^2}\right]$. Then:

(i) The limit

$$
e^{-t\Lambda_C(b)} := \text{s} \cdot C_\infty \lim_{n \rightarrow 0} e^{-t\Lambda_{C_\infty}(b_n)} \quad (t \geq 0)
$$

exists and determines a positivity preserving contraction $C_0$ semigroup on $C_\infty$. Here $\{b_n\}$ is defined in section 3.2. $\Lambda_C(b_n) := -\Delta + b_n \cdot \nabla, D(\Lambda_C(b_n)) = (1 - \Delta)^{-1}C_\infty$.

(ii) [The $L^r$-strong Feller property] $((\mu + \Lambda_C(b))^{-1} \uparrow L^r \cap C_\infty)^{\text{clos}}_{L^r \rightarrow C_\infty} \in \mathcal{B}(L^r, C^{0,1-\frac{d}{r}})$ whenever $r \in \left[\frac{2}{2-\sqrt{\delta}}, \frac{2}{\sqrt{\delta}}\right]$, $\frac{2}{\sqrt{\delta}} \left[\text{ and } \mu > \omega_q\right]$.

(iii) The integral kernel of $e^{-t\Lambda_C(b)}$ determines the transition probability function of a Hunt process.

**Proof.** We will need the following auxiliary results. Set

$$
g := u_m - u_n, \quad u_n = (\mu + \Lambda_r(b_n))^{-1}f, \quad f \in L^1 \cap L^\infty, \quad \mu > \frac{c(\delta)}{\delta}.
$$

**Lemma 3.12.** There are positive constants $C = C(d), k = k(\delta)$ such that

$$
\|g\|_{rj} \leq \left(C\delta \|\nabla u_n\|_{rj}^2\right)^{\frac{1}{2}} \left(r^2k\right)^{\frac{1}{2}} \|g\|_{r'(r-2)}^{1-\frac{2}{r'}}.
$$
where \( q \in \left[ \frac{2}{2 - \sqrt{\delta}}, \frac{2}{\sqrt{\delta}} \right] \cap (d - 2) \), \( 2x = qj \), \( j = \frac{d}{x - 2} \), \( x' := \frac{x}{x - 1} \) and \( x'(r - 2) > \frac{2}{2 - \sqrt{\delta}} \).

**Proof of Lemma 1.** Note that \( g \) satisfies the equation

\[
(\mu + \Lambda_q(b_n))g = F, \quad F = (b_m - b_n) \cdot \nabla u_n.
\]

Let \( \psi = g|g|^{r - 2}, v = g|g|^{\frac{r - 2}{2}} \). Taking the scalar product of the equation by \( \psi \), we have

\[
\mu |v|^2 + \frac{4}{rr'}|\nabla v|^2 = \frac{2}{r} \langle v, b_m \cdot \nabla v \rangle + \langle (b_m - b_n) \cdot \nabla u_n, v|v|^{1 - \frac{2}{r}} \rangle.
\]

By the quadratic estimates

\[
\langle v, b_m \cdot \nabla v \rangle \leq \varepsilon |b_m v|^2 + (4\varepsilon)^{-1} |\nabla v|^2
\]

\[
\leq (\varepsilon \delta + (4\varepsilon)^{-1}) |\nabla v|^2 + \varepsilon c(\delta)|v|^2
\]

\[
= \sqrt{\delta} |\nabla v|^2 + (2\sqrt{\delta})^{-1} c(\delta)|v|^2 \quad (\varepsilon = (2\sqrt{\delta})^{-1}),
\]

\[
\langle (b_m - b_n) \cdot \nabla u_n, v|v|^{1 - \frac{2}{r}} \rangle \leq \langle (|b_n| + |b_m|)|v|, |v|^{1 - \frac{2}{r}} |\nabla u_n| \rangle
\]

\[
\leq \eta \delta |\nabla v|^2 + \eta c(\delta)|v|^2 + \eta^{-1} |v|^{1 - \frac{2}{r}} |\nabla u_n|^2 \quad (\eta > 0),
\]

we obtain the inequality

\[
\left[ \mu - \left( \frac{1}{r} \sqrt{\delta} + \eta \right) c(\delta) \right] |v|^2 + \left( \frac{4}{rr'} - \frac{r}{r'} \sqrt{\delta} - \eta \delta \right) |\nabla v|^2 \leq \eta^{-1} |v|^{1 - \frac{2}{r}} |\nabla u_n|^2.
\]

Since \( r > \frac{2}{2 - \sqrt{\delta}} \Leftrightarrow \frac{2}{r} - \sqrt{\delta} > 0 \), we choose \( k > 1 \) so large that \( \frac{4}{rr'} - \frac{2}{r} \sqrt{\delta} = \frac{2}{r} (\frac{2}{r} - \sqrt{\delta}) > 2r^{-k} \). Fix \( \eta \) by \( \eta \delta = \frac{4}{rr'} - \frac{2}{r} \sqrt{\delta} - r^{-k} \). Thus

\[
\left[ \mu - \frac{c(\delta)}{\delta} \left( \frac{4}{rr'} - \frac{1}{r} \sqrt{\delta} - r^{-k} \right) \right] |v|^2 + r^{-k} |\nabla v|^2 \leq \delta r^k |v|^{1 - \frac{2}{r}} |\nabla u_n|^2.
\]

Our choice of \( \mu \) ( \( \mu > \frac{c(\delta)}{\delta} \) ) ensures that the expression contained in square brackets is strictly positive. Thus

\[
|\nabla v|^2 \leq \delta r^{2k} |\nabla u_n|^2 |v|^{2(1 - \frac{2}{r})}.
\]

Finally, applying the uniform Sobolev inequality \( c_d |v|^2_{2j} \leq |\nabla v|_{2}^2 \), we end the proof of Lemma 3.12. \( \square \)

**Lemma 3.13.** In the notation of Lemma 3.12, for any \( r_0 > \frac{2}{2 - \sqrt{\delta}} \)

\[
|g|_{\infty} \leq B |g|^{\gamma}_{r_0}, \quad \mu \geq 1 + \lambda_0 \sqrt{\frac{c(\delta)}{\delta}},
\]

where \( \gamma = \left( 1 - \frac{d}{2} \right) \left( 1 - \frac{d}{2} + 2r^{-2} \right)^{-1} > 0 \), and \( B = B(d, \delta, K_2) < \infty \) is a constant (\( K_2 \) is the constant in Theorem 3.7).

**Proof of Lemma 3.13.** Let \( D := C \delta \sup_n |\nabla u_n|^2_{qj} \). By Theorem 3.7, \( D < \infty \). We iterate the inequality

\[
|g|_{rj} \leq D^j |g|^{2k}_{rj} |g|^{1 - \frac{2}{r'}}_{x'(r - 2)},
\]

(*)
as follows. Successively setting \( x'(r_1 - 2) = r_0, \) \( x'(r_2 - 2) = jr_1, \) \( x'(r_3 - 2) = jr_2, \ldots \) so that \( r_n = (t - 1)^{-1} \left( t^n \frac{\gamma}{\gamma' + 2} - t^{n-1} \frac{\gamma}{\gamma' + 2} - 2 \right), \) where \( t = \frac{\gamma}{\gamma'} > 1, \) we get from (*)

\[
\|g\|_{r_n} \leq D^{\alpha_n} \Gamma_n \|g\|_{r_0}^{\gamma_n},
\]

where

\[
\alpha_n = \frac{1}{r_1} \left(1 - \frac{2}{r_2}\right) \left(1 - \frac{2}{r_3}\right) \ldots \left(1 - \frac{2}{r_n}\right) + \frac{1}{r_2} \left(1 - \frac{2}{r_3}\right) \ldots \left(1 - \frac{2}{r_n}\right) + \ldots + \frac{1}{r_{n-1}} \left(1 - \frac{2}{r_n}\right) + \frac{1}{r_n}; \quad \gamma_n = \left(1 - \frac{2}{r_1}\right) \left(1 - \frac{2}{r_2}\right) \ldots \left(1 - \frac{2}{r_n}\right); \quad \Gamma_n = \left[r_{n-1}^{-1} r_{n-2}^{-1} \ldots r_1^{-1} \right]^{2k}.
\]

Since \( \alpha_n = t^n - r_n^{-1}(t - 1)^{-1} \) and \( \gamma_n = r_0 t^n - 1 (x' r_n)^{-1}, \)

\[
\alpha_n \leq \alpha = \left(\frac{r_0}{x'} + 2 - \frac{r_0}{j}\right)^{-1},
\]

and

\[
\inf_{n} \gamma_n > \gamma = \left(1 - \frac{x'}{j}\right) \left(1 - \frac{x'}{j} + \frac{2 x'}{r_0}\right)^{-1} > 0, \quad \sup_{n} \gamma_n < 1.
\]

Note that \( \|g\|_\rho \to 0 \) as \( n, m \uparrow \infty \) since \( r_0 \in P_{\rho}, \) and so \( \|g\|_{\rho}^{\gamma_n} \leq \|g\|_{\rho}^{\gamma} \) for all large enough \( n, m. \)

Finally, since

\[
\frac{1}{r_n} \Gamma_n = r_{n-1}^{-1} r_{n-2}^{-1} \ldots r_1^{-1} \text{ and } b^{t^n} \leq r_n \leq a t^n,
\]

where \( a = r_1(t - 1)^{-1}, b = r_1 t^{-1}, \) we have

\[
\frac{1}{r_n} \Gamma_n \leq (a t^n)^{(b t^n)^{-1}} (a t^{n-1})^{(b t^{n-1})^{-1}} \ldots (a t)^{(b t)^{-1}}
\]

\[
= \left[a^{t^n - t^{n-1}} \sum_{i=1}^{n} t^{-i} \right]^{\frac{1}{b}} \leq \left[a^{(t-1)^{-1}} b^{t(t-1)^2} \right]^{\frac{1}{b}}.
\]

The proof of Lemma 3.13 is completed. \( \square \)

**Remark.** The fact that \( \gamma > 0 \) is the main concern of the iterative procedure.

**Lemma 3.14.** Let \( U_n := (\mu + \Lambda_{r_0}(b_n))^{-1} F, \mu > \frac{c(\delta)}{\gamma'}, \) \( F := b_n \cdot \nabla (\mu - \Delta)^{-1} f, f \in C^1. \) There are constants \( 0 < \tilde{\gamma} \leq 1, \tilde{B} \) and \( \hat{B} \) independent of \( n \) such that

\[
\|U_n\|_{\infty} \leq \tilde{B} \|U_n\|_{r_0}^{\tilde{\gamma}},
\]

\[
\|\mu U_n\|_{\infty} \leq \hat{B} \|\mu U_n\|_{r_0}^{\hat{\gamma}}
\]

whenever \( r_0 > \frac{2}{2 - \sqrt{\delta}}. \)

**Proof.** Proceeding exactly as in the proof of Lemma 3.12, we obtain the inequalities

\[
\|U_n\|_{r_j} \leq \left(C \delta \|\nabla (\mu - \Delta)^{-1} f\|_{q_j}^2 \right) \left(r^{2k}\right)^{\frac{1}{2}} \|U_n\|_{x'(r_j)}^{1 - \frac{2}{k}},
\]

\[
\|U_n\|_{r_j} \leq \left(C \delta \|\nabla (\mu - \Delta)^{-1} f\|_{q_j}^2 \right) \left(r^{2k}\right)^{\frac{1}{2}} \|U_n\|_{x'(r_j)}^{1 - \frac{2}{k}},
\]
∥µU_n∥_r ≤ (Cδ∥∇f∥_2^2)^\frac{1}{r} (r^{2k})^\frac{1}{r} ∥µU_n∥_r^{1-\frac{2}{r}} ;

their iteration provides the required result. □

Lemma 3.15. In the notation of Lemma 3.14 we have

∥µU_n∥_r ≤ \left( \frac{8}{r} \left( \frac{2}{rr^r} - \sqrt{\delta} \right) \right)^{-\frac{1}{2}} \left( \mu - \frac{c(\delta)}{\delta} \right)^{-\frac{1}{2}} ∥∇f∥_r

whenever r > \frac{2}{2-\sqrt{\delta}}.

Proof. Proceeding again as in Lemma 3.12 we obtain the inequality \( (\eta > 0) \)

\[
\left[ \mu - \left( \frac{1}{r} \sqrt{\delta} + \eta \right) c(\delta) \right] ||v||_2^2 + \left( \frac{4}{rr^r} - \frac{2}{r} \sqrt{\delta} - \eta \delta \right) ||∇v||_2^2 \leq (4\eta)^{-1} ||v||^{1-\frac{2}{r}} ||f_\mu||^2_2,
\]

where v := U_n |U_n|^{\frac{2}{rr^r}} \text{ and } f_\mu := \nabla (\mu - \Delta)^{-1} f. Setting here \( \eta \delta = \frac{4}{rr^r} - \frac{2}{r} \sqrt{\delta} \) and noticing that

\[
\delta \left[ \mu - \left( \frac{1}{r} \sqrt{\delta} + \eta \right) c(\delta) \right] = \left[ \mu - \frac{c(\delta)}{\delta} \left( \frac{4}{rr^r} - \sqrt{\delta} \right) \right] \geq \left[ \mu - \frac{c(\delta)}{\delta} \right]
\]

we have

\[
\frac{8}{r} \left( \frac{2}{rr^r} - \sqrt{\delta} \right) \left[ \mu - \frac{c(\delta)}{\delta} \right] ||v||_2^2 \leq \delta ||v||_2^{2\left(1-\frac{2}{r}\right)} ||f_\mu||^2_2.
\]

It remains to note that ||f_\mu||_r ≤ \mu^{-1} ||∇f||_r. □

Lemma 3.16. \( s-C_\infty - \lim_{\mu \to \infty} \mu (\mu + \Lambda_{C_\infty} (b_n))^{-1} = 1 \) uniformly in n.

Proof. We only need to show that

\[
\lim_{\mu \to \infty} \sup_n ||\mu \left[ (\mu + \Lambda_r (b_n))^{-1} - (\mu - \Delta)^{-1} \right] f||_\infty = 0 \quad \text{for all } f \in C^1_c.
\]

Indeed, since \( -[(\mu + \Lambda_r (b_n))^{-1} - (\mu - \Delta)^{-1}] f = (\mu + \Lambda_r (b_n))^{-1} b_n \cdot \nabla (\mu - \Delta)^{-1} f = U_n \), we obtain by Lemma 3.14 and Lemma 3.15 that

\[
||µU_n||_\infty \leq \hat{B}||µU_n||_r^{\frac{2}{5}} \leq \hat{B} \left( \mu - \frac{c(\delta)}{\delta} \right)^{-\frac{2}{5}} ||∇f||_r^{\frac{2}{5}},
\]

which yields the required. □

We are in position to complete the proof of Theorem 3.11. (i) follows from Lemmas 3.13, 3.16 and Theorem 3.2 by applying the Trotter Approximation Theorem (Appendix D). (ii) is Theorem 3.7. The proof of (iii) is standard.

The proof of Theorem 3.11 is completed. □

Remark. Theorem 3.11 is valid for any \( \{b_n\} \subset C^\infty \cap \mathbf{F}_\delta, b_n \overset{a.e.}{\to} b \mathbf{L} \overset{d}{\to} \text{ a.e.} \)
4. *b · ∇ is −△ weakly form-bounded*

Let \( L^p = L^p(\mathbb{R}^d, \mathcal{L}^d) \), \( p \in [1, \infty] \), be the standard (complex) Lebesgue spaces.

Consider the following classes of vector fields:

**Definition.** (1) A \( b : \mathbb{R}^d \to \mathbb{R}^d \) belongs to the Kato class \( K_{\delta}^{d+1} \) (write \( b \in K_{\delta}^{d+1} \)) if \( |b| \in L^1_{\text{loc}} \) and there exists \( \lambda = \lambda_{\delta} > 0 \) such that

\[
\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{1 \to 1} \leq \delta.
\]

(2) A \( b : \mathbb{R}^d \to \mathbb{R}^d \) belongs to \( F_{\delta}^{1/2} \equiv F_{\delta}^{1/2}(-\Delta) \), the class of weakly form bounded vector fields (write \( b \in F_{\delta}^{1/2} \)) if \( |b| \in L^1_{\text{loc}} \) and there exists \( \lambda = \lambda_{\delta} > 0 \) such that

\[
\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}.
\]

**Example 1.** 1. The inclusions \( |b| \in L^p + L^\infty \) (\( p > d \)) \( \Rightarrow b \in K_{0}^{d+1} := \bigcap_{\delta > 0} K_{\delta}^{d+1} \), and \( |b| \in L^d + L^\infty \) \( \Rightarrow b \in F_0 := \bigcap_{\delta > 0} F_{\delta} \) follow from Hölder’s inequality and the Sobolev Embedding Theorem, respectively.

2. For \( |b| \in L^{d,\infty} \), one can verify, using [12, Prop. 2.5, 2.6, Cor. 2.9]:

\[
b \in F_{\delta_1}, \quad \text{with } \sqrt{\delta_1} = \|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2}
\]

\[
\leq \|b\|_{d,\infty} \Omega_d^{-\frac{1}{2}} \|x|^{-1}\|\lambda - \Delta\|_{2 \to 2}^{-\frac{1}{2}}
\]

\[
\leq \|b\|_{d,\infty} \Omega_d^{-\frac{1}{2}} \|x|^{-2}\|\lambda - \Delta\|^2\Gamma(\frac{d}{2})\Gamma(\frac{d}{2} + 1)
\]

\[
\leq \|b\|_{d,\infty} \Omega_d^{-\frac{1}{2}} \frac{2}{d-2}.
\]

where \( \Omega_d = \pi^{d/2} \Gamma(\frac{d}{2} + 1) \) is the volume of the unit ball in \( \mathbb{R}^d \).

3. By Hardy’s inequality, \( b(x) := \sqrt{\delta_1 \frac{d-2}{d} x}^{-2} x \in F_{\delta_1} - K_{\delta}^{d+1} \) for any \( \delta, \delta_1 > 0 \). (And, of course, \( b \notin F_{\delta_2} \) if \( \delta_2 < \delta_1 \).)

4. In turn,

\[
b(x) := e 1_{|x| < 1} |x|^{s-1}, \quad \frac{1}{2} < s < 1,
\]

where \( e = (1, \ldots, 1) \in \mathbb{R}^d \), \( x = (x_1, \ldots, x_d) \), is in \( K_{0}^{d+1} - F_{\delta} \) for any \( \delta > 0 \).

Combining this with the previous example, we obtain that \( K_{0}^{d+1} - F_{\delta} \neq \emptyset \), and \( F_{\delta_1} - K_{\delta}^{d+1} \neq \emptyset \) for any \( \delta, \delta_1 > 0 \).

5. An example of a \( b \in K_{\delta}^{d+1} - K_{0}^{d+1}, \delta > 0 \), can be obtained as follows.

Fix \( e \in \mathbb{R}^d, |e| = 1 \). Let \( z_n := (2^{-n}, 0, \ldots, 0) \in \mathbb{R}^d, n \geq 1 \). Set

\[
b(x) := eF(x), \quad F(x) := \sum_{n=1}^{\infty} 8^n 1_{B(z_n, 8^{-n})}(x), \quad x \in \mathbb{R}^d,
\]

where \( B(z_n, 8^{-n}) \) is the open ball of radius \( 8^{-n} \) centered at \( z_n \) and \( 1_{B(z_n, 8^{-n})} \) is its indicator.

Then \( b \in K_{\delta}^{d+1} - K_{0}^{d+1} \) for appropriate \( \delta > 0 \). [9].

6. For every \( \varepsilon > 0 \) one can find \( b \in F_{\delta} \) such that \( |b| \notin L^2_{\text{loc}}^{2+\varepsilon} \), e.g. consider

\[
b^2(x) = C \frac{1_{B(0,1) + a} - 1_{B(0,1) - a}}{|x|^{-1}(-\ln |x| - 1)^b}, \quad b > 1, \quad 0 < a < 1.
\]
7. Let $W_s$ denote the class of vector fields $b$ such that $|b|^2$ is in the class of Chang-Wilson-Wolff $(s > 1)$:

$$v \in L^s_{\text{loc}} : \|v\|_{W_s} := \sup_Q \frac{1}{|Q|} \int_Q |v(x)| l(Q)^2 \varphi(|v(x)| l(Q)^2) dx < \infty$$

where $|Q|$ and $l(Q)$ are the volume and the side length of a cube $Q$, respectively, $\varphi : [0, \infty] \to [1, \infty]$ is an increasing function such that $\int_1^\infty \frac{dx}{x \varphi(x)} < \infty$. By [1], if $b \in W_s$, then $b \in F_\delta$, $\delta = \delta(\|b^2\|_{W_s}) < \infty$.

The class $W_s$ contains, in particular, the vector fields $b$ with $|b|^2$ in the Campanato-Morrey class $(s > 1)$

$$\left\{ v \in L^s_{\text{loc}} : \frac{1}{|Q|} \left( \int_Q |v(x)|^s dx \right)^{\frac{1}{s}} \leq c_s l(Q)^{-2} \text{ for all cubes } Q \right\}$$

(write $b \in C_s$).

**Remark.** The classes $F_\delta$, $K_\delta^{d+1}$ cover singularities of $b$ of critical order (i.e. sensitive to multiplication by a constant) at isolated points or along hypersurfaces, respectively, as follows from Example [2] (3-5). The classes $K_0^{d+1}$, $F_0$ and, thus, $[L^d + L^\infty]^d$, do not contain vector fields having critical-order singularities.

The class $F_\delta^{1/2}$ is the largest:

$$K_\delta^{d+1} \subsetneq F_\delta^{1/2}, \quad F_\delta_1 \subsetneq F_\delta^{1/2} \quad \text{for } \delta = \sqrt{\delta_1},$$

$$\left( b \in F_\delta, \text{ and } f \in K_\delta^{d+1} \right) \implies \left( b + f \in F_\delta^{1/2}, \sqrt{\delta} = \frac{4}{\sqrt{\delta_1} + \sqrt{\delta_2}} \right) \quad (1)$$

**Proof.** Indeed, for $b \in K_\delta^{d+1}$, $|||b|(\lambda - \Delta)^{-\frac{1}{2}}||_{1 \to 1} \leq \delta$. By duality, $||(\lambda - \Delta)^{-\frac{1}{2}}b||_{\infty} \leq \delta$, and so by interpolation, $|||b|(\lambda - \Delta)^{-\frac{1}{2}}||_{2 \to 2} \leq \delta$. Therefore, $b \in F_\delta^{1/2}$. The second inclusion $F_\delta_1 \subsetneq F_\delta^{1/2}$, $\delta = \sqrt{\delta_1}$ is the consequence of the Heinz inequality [3]. The last assertion now follows from

$$b \in F_\delta^{1/2}, f \in F_\delta^{1/2} \implies b + f \in F_\delta^{1/2},$$

where we have used $(||b|| + ||f||)^{\frac{1}{s}} \leq ||b||^{\frac{1}{s}} + ||f||^{\frac{1}{s}}$.

4.1. **A variant of the Hille-Lions approach. $L^2$-theory.** Let $W^{\alpha,p} = W^{\alpha,p}(\mathbb{R}^d, \mathcal{L}^d)$, $\alpha > 0$, be the Bessel space endowed with norm $\|u\|_{p,\alpha} := \|g\|_{p}$, $u = (1 - \Delta)^{-\frac{1}{2}}g$, $g \in L^p$, and $W^{-\alpha,p'}$, $p' = \frac{p}{p-1}$, the anti-dual of $W^{\alpha,p}$.

**Theorem 4.1.** Let $d \geq 3$, $b : \mathbb{R}^d \to \mathbb{R}^d$. Assume that $b \in F_\delta^{1/2}$, $\delta < 1$. Then there is an operator realization $-\Lambda$ of $\Delta - b \cdot \nabla$ in $L^2$ generating a quasi bounded holomorphic semigroup $e^{-t\Lambda}$. We have:

(i) The resolvent set $\rho(-\Lambda)$ contains the half-plane $\mathcal{O} := \{ \zeta \in \mathbb{C} \mid \text{Re } \zeta \geq \lambda = \lambda_\delta \}$. The resolvent admits the representation

$$(\zeta + \Lambda)^{-1} = (\zeta - \Lambda)^{-\frac{1}{2}}(1 + H^sS)^{-1}(\zeta - \Lambda)^{-\frac{1}{2}},$$

For instance, the uniqueness of weak solution to the Cauchy problem for $\partial_t - \Delta + b \cdot \nabla$ can fail if $b \in F_\delta$ is replaced with $\epsilon b \in F_{\epsilon, \delta}$ for a sufficiently large constant $\epsilon$. [13 Example 5].
Proof. $H := |b|^\frac{2}{d}(\zeta - \Delta)^{-\frac{1}{4}}, \quad b_+^\frac{1}{2} := |b|^\frac{1}{2}b, \quad S := b_+^\frac{1}{2} \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}}, \quad \text{and } \|H^* S\|_{2 \to 2} \leq \delta;$

(ii) 

$$(\zeta + \Lambda)^{-1} = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{3}{2}} H^*(1 + S H^*)^{-1} S(\zeta - \Delta)^{-\frac{1}{4}};$$

(iii) 

$$\| (\zeta + \Lambda)^{-1} \|_{2 \to 2} \leq (1 - \delta)^{-1} |\zeta|^{-1} \quad (\zeta \in \mathcal{O});$$

(iv) $\Lambda$ is related to $-\Delta + b \cdot \nabla$ as follows. If $f \in D(\Lambda)$, then 

$$\langle \Lambda f, g \rangle = \langle \nabla f, \nabla g \rangle + \langle b_+^\frac{1}{2} \cdot \nabla f, |b|^\frac{1}{2} g \rangle, \quad (g \in W^{1,2});$$

(v) If $f \in D(\Lambda)$, then $b \cdot \nabla f \in L_{\text{loc}}^1, \quad D(\Lambda) \subset W_{\text{loc}}^{2,1}, \quad \text{and}$ 

$$\langle \Lambda f, \phi \rangle = \langle f, -\Delta \phi \rangle + \langle b \cdot \nabla f, \phi \rangle \quad (\phi \in C_\infty^1).$$

Proof. 1. Set $\mathcal{H}_0 := L^2$. Define $A := \lambda - \Delta$ of domain $D(A) = W^{2,2}, \quad \mathcal{H}_0 := \{D(A^\alpha), \langle f, g \rangle_{\mathcal{H}_0} = \langle A^\alpha f, A^\alpha g \rangle \} (\alpha \geq 0)$ and $J_\lambda := (\lambda - \Delta)^{-\frac{1}{2}}, \quad \text{the } \frac{1}{2} \text{ power of Bessel’s potential. Clearly, } \mathcal{H}_0 = W^{2,0,2}$ and $\mathcal{H}_{-\frac{1}{2}} = W^{-\frac{1}{2},2}$. Consider the chain of Hilbert spaces 

$$\mathcal{H}_1 \subset \mathcal{H}_{\frac{1}{2}} \subset \mathcal{H}_{\frac{3}{4}} \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{-\frac{1}{2}}.$$ 

Then $J_\lambda : \mathcal{H}_l \to \mathcal{H}_{l + \frac{1}{2}}, \quad l = -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{4}, \frac{3}{4}$ are bijections, 

$$\mathcal{H}_{\frac{1}{2}} \subset \mathcal{H}_{\frac{3}{4}} \subset \mathcal{H}_{-\frac{1}{2}}$$

is the standard triple of Hilbert spaces (so that $\mathcal{H}^*_{-\frac{1}{2}} = \mathcal{H}_{\frac{1}{2}}$ with respect to $\langle f, g \rangle_{\mathcal{H}_{\frac{1}{2}}}$.}

Figure 1.
By \( \langle f, g \rangle_{\frac{1}{2}} \), \( f \in \mathcal{H}_{-\frac{1}{2}}, \ g \in \mathcal{H}_{\frac{1}{2}} \) we denote the pairing between \( \mathcal{H}_{-\frac{1}{2}} \) and \( \mathcal{H}_{\frac{1}{2}} \). Then

\[
\langle f, g \rangle_{\frac{1}{2}} = \langle f, g \rangle_{\mathcal{H}_{\frac{1}{2}}} \quad \text{whenever} \quad f \in \mathcal{H}_{\frac{1}{2}}.
\]

By \( \hat{A} \) we denote the extension of \( A - \lambda \) to a bounded map from \( \mathcal{H}_{\frac{1}{2}} \) into \( \mathcal{H}_{-\frac{1}{2}} \). Then

\[
\| \langle \zeta + \hat{A} \rangle f, f \rangle_{\frac{1}{2}} \| \geq \| f \|_{\mathcal{H}_{\frac{1}{2}}}^2 \quad (f \in \mathcal{H}_{\frac{1}{2}}, \ \text{Re} \ \zeta \geq \lambda),
\]

and so \( \zeta + \hat{A} \) is a bijection; \( \| \zeta + \hat{A} \|_{\mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{-\frac{1}{2}}} \geq 1 \). Clearly

\[
D(A) = \{ f \in \mathcal{H}_{\frac{1}{2}} \mid \hat{A}f \in \mathcal{H}_0 \} \quad \text{and} \quad A^{-1} = (\lambda + \hat{A})^{-1} \mid \mathcal{H}_0 = J_{\lambda}^1.
\]

The operator \( \hat{B} := b \cdot \nabla : \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{-\frac{1}{2}} \) is bounded. Indeed, according to \( b \in \mathbf{F}^{1/2} \),

\[
b^{\frac{1}{2}} \cdot \nabla : \mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_0, \ |b|^{\frac{1}{2}} : \mathcal{H}_0 \to \mathcal{H}_{-\frac{1}{2}}, \ \hat{B} \in \mathcal{B}(\mathcal{H}_{\frac{1}{2}}, \mathcal{H}_{-\frac{1}{2}}) \quad \text{with} \quad \| \hat{B} \|_{\mathcal{H}_{\frac{1}{2}} \to \mathcal{H}_{-\frac{1}{2}}} \leq \delta.
\]

Thus \( \| \langle \zeta + \hat{A} + \hat{B} \rangle f, f \rangle_{\frac{1}{2}} \| \geq (1 - \delta)\| f \|_{\mathcal{H}_{\frac{1}{2}}}^2 \), and so \( \zeta + \hat{A} + \hat{B} \) is a bijection.

Set \( \hat{R}_\zeta := (\zeta + \hat{A})^{-1} \) and \( \hat{F}_\zeta := (\zeta + \hat{A})^{-1}(= J_{\zeta}^1) \). Clearly,

\[
\hat{R}_\zeta = \hat{R}_\eta + (\eta - \zeta)\hat{R}_\zeta \hat{R}_\eta, \quad \zeta, \eta \in \mathcal{O}; \quad (p_1)
\]

\[
\hat{R}_\zeta = \hat{F}_\zeta - \hat{F}_\eta \hat{R}_\zeta = \hat{F}_\zeta - \hat{F}_\zeta \hat{B} \hat{F}_\zeta + \hat{F}_\zeta \hat{B} \hat{F}_\zeta \hat{B} \hat{F}_\zeta - \ldots;
\]

\[
\hat{F}_\zeta \hat{B} \hat{F}_\zeta = J_{\zeta}^3 J_{\zeta} \hat{B} J_{\zeta}^3 J_{\zeta};
\]

\[
\hat{F}_\zeta \hat{B} \hat{F}_\zeta \mid \mathcal{H}_0 = J_{\zeta}^3 J_{\zeta} |b|^{\frac{1}{2}} b^{\frac{1}{2}} \cdot \nabla J_{\zeta}^3 J_{\zeta} \mid \mathcal{H}_0 = J_{\zeta}^3 H^* S J_{\zeta};
\]

\[
R_\zeta := \hat{R}_\zeta \mid \mathcal{H}_0 = J_{\zeta}^3 (1 + H^* S)^{-1} J_{\zeta}. \quad (p_2)
\]

Since \( |J_{\zeta} g| \leq J_{\lambda} |g| \), and hence

\[
\| H^* S f \|_{\mathcal{H}_0} \leq \| H \|_{\mathcal{H}_0 \to \mathcal{H}_0} \| b |^{\frac{1}{2}} J_{\lambda} | \nabla J_{\zeta}^2 f \|_{\mathcal{H}_0} \leq \delta \| \nabla J_{\zeta}^2 f \|_{\mathcal{H}_0} \leq \delta \| f \|_{\mathcal{H}_0}, \ f \in \mathcal{H}_0,
\]

it follows from \( (p_2) \) that

\[
\| R_\zeta \|_{\mathcal{H}_0 \to \mathcal{H}_0} \leq (1 - \delta)^{-1} |\zeta|^{-1}. \quad (p_3)
\]

We conclude from \( (p_1) \) that \( R_\zeta \) is a pseudo-resolvent, and from \( (p_2) \) that its null-set is \( \{0\} \). Therefore, \( R_\zeta \) is the resolvent of some closed operator \( \Lambda \) in \( \mathcal{H}_0 \), and \( \Lambda = R_\zeta^{-1} - \zeta \) (Appendix [F] Theorem [F.1]).

It is also seen that \( D(\Lambda) := \hat{R}_\zeta \mathcal{H}_0, \ \Lambda f := \hat{A} f, \ f \in D(\Lambda) \).

Next, let us show that \( \Lambda \) is a densely defined operator. Indeed, by the construction, \( \mathcal{H}_0 \) is a dense subspace of \( \mathcal{H}_{-\frac{1}{2}} \), and hence \( \hat{R}_\zeta \mathcal{H}_0 \) is a dense subspace of \( \mathcal{H}_{\frac{1}{2}} \) and of \( \mathcal{H}_0 \) for \( \mathcal{H}_{\frac{1}{2}} \) is a dense subspace of \( \mathcal{H}_0 \). Thus \( D(\Lambda) \) is dense in \( \mathcal{H}_0 \).

Taking into account \( (p_3) \), we conclude that \( -\Lambda \) is the generator of a quasi bounded holomorphic semigroup. Thus \( (i) \) and \( (iii) \) are established. \( (ii) \) is an easy consequence of \( (i) \).
2. In order to justify (iv) and (v), define $P_\zeta(b) := -J_\zeta \tilde{B} J_\zeta^3 \upharpoonright L^2 = -H^* S \ (\text{Re}\ \zeta \geq \lambda)$. Let \{b_n\} be given by

$$b_n := 1_n b, \quad n = 1, 2, \ldots,$$

(2)

where $1_n$ is the indicator of \{x \in \mathbb{R}^d \mid |x| \leq n, |b(x)| \leq n\}. Then $b_n \in F^{1/2}_\delta$, and so $\Lambda(b_n)$ is well defined. Since for every $f \in L^2$

$$\|P_\zeta(b) f - P_\zeta(b_n) f\|_2 \leq \|H^*\|_{2 \to 2} \|1_n - b_n\|_2 \|\nabla J_\zeta^3 f\|_2$$

$$\leq \sqrt{\delta} \|1_n f\|_2, \quad F = |b|^2 |\nabla J_\zeta^3 f| \in L^2,$$

it is seen that

$$s-L^2 \lim_{n \to \infty} (\zeta + \Lambda(b_n))^{-1} = s-L^2 \lim_{n \to \infty} J_\zeta^3 (1 + P_\zeta(b_n))^{-1} J_\zeta = J_\zeta^3 (1 + P_\zeta(b))^{-1} J_\zeta = (\zeta + \Lambda(b))^{-1}. \quad (3)$$

Let $f \in L^2$, $g \in W^{1,2}$, $\phi \in C_c^\infty$. Clearly, $b_n^2 \cdot \nabla (\zeta + \Lambda(b))^{-1} f \in L^2$, $|b|^2 \in L^2$, and

$$\langle \Lambda(b_n)(\zeta + \Lambda(b_n))^{-1} f, g \rangle \to \langle \Lambda(b)(\zeta + \Lambda(b))^{-1} f, g \rangle;$$

$$\langle \Lambda(b_n)(\zeta + \Lambda(b_n))^{-1} f, g \rangle = \langle \nabla (\zeta + \Lambda(b_n))^{-1} f, \nabla g \rangle + \langle b_n^2 \cdot \nabla (\zeta + \Lambda(b_n))^{-1} f, |b|^2 g \rangle$$

$$\to \langle \nabla (\zeta + \Lambda(b))^{-1} f, \nabla g \rangle + \langle b^2 \cdot \nabla (\zeta + \Lambda(b))^{-1} f, |b|^2 g \rangle;$$

$$\langle \Lambda(b_n)(\zeta + \Lambda(b_n))^{-1} f, \phi \rangle = \langle (\zeta + \Lambda(b_n))^{-1} f, -\Delta \phi \rangle + \langle b_n^2 \cdot \nabla (\zeta + \Lambda(b_n))^{-1} f, |b|^2 \phi \rangle$$

$$\to \langle (\zeta + \Lambda(b))^{-1} f, -\Delta \phi \rangle + \langle b \cdot \nabla (\zeta + \Lambda(b))^{-1} f, |b|^2 \phi \rangle$$

$$= \langle (\zeta + \Lambda(b))^{-1} f, -\Delta \phi \rangle + \langle b \cdot \nabla (\zeta + \Lambda(b))^{-1} f, \phi \rangle.$$

Thus $\langle u, \Delta \phi \rangle = \langle -\Lambda u + b \cdot \nabla u, \phi \rangle$ whenever $u \in D(\Lambda)$. This means that $D(\Lambda) \subset W^{2,1}_{loc}$. \hfill \Box

4.2. The Hille-Trotter approach. $L^2$-theory. We give an alternative proof of Theorem 4.1.

Let \{b_n\} be given by (2). Since $b_n \in F^{1/2}_\delta$, we have \|b_n \cdot \nabla (\eta - \Delta)^{-1}\|_{2 \to 2} < 1$ whenever $\eta > n^2$. Therefore, by the Miyadera Perturbation Theorem, $-\tilde{\Lambda}(b_n) := \Delta - b_n \cdot \nabla$ of domain $W^{2,2}$ generates a $C_0$ semigroup in $L^2$.

We can construct $\Lambda(b)$ by showing first that the resolvent set of $-\tilde{\Lambda}(b_n)$ contains $\mathcal{O} = \{\zeta \mid \text{Re}\ \zeta \geq \lambda\}$, and that there is a constant $c$ such that, for all $n = 1, 2, \ldots$,

$$\|(\zeta + \tilde{\Lambda}(b_n))^{-1}\|_{2 \to 2} \leq c|\zeta|^{-1}, \quad \zeta \in \mathcal{O}.$$

We accomplish this as follows. Define

$$\Theta(\zeta, b_n) := J_\zeta^3 (1 + P_\zeta(b_n))^{-1} J_\zeta, \quad \zeta \in \mathcal{O},$$

where $P_\zeta(b_n) := J_\zeta |b_n|^2 \hat{b} \cdot \nabla J_\zeta^3 \in \mathcal{B}(L^2)$. We prove consecutively that the operator-valued function $\Theta(\zeta, b_n)$ possesses the following properties:

$$\Theta(\eta, b_n) f = (\eta + \tilde{\Lambda}(b_n))^{-1} f, \quad f \in L^2, \text{ whenever } \eta > n^2 \lor \lambda; \quad (p_1)$$

$$\|\Theta(\zeta, b_n)\|_{2 \to 2} \leq c|\zeta|^{-1} \text{ for some constant } c \text{ and all } n = 1, 2, \ldots; \quad (p_2)$$
\[ \Theta(\zeta, b_n) - \Theta(\eta, b_n) = (\eta - \zeta) \Theta(\zeta, b_n) \Theta(\eta, b_n), \quad \eta \in \mathcal{O}. \]  

Note that (p1) follows from the definitions of \( \Theta(\zeta, b_n) \), \( \tilde{\Lambda}(b_n) \), and from the obvious equality

\[ \Theta(\eta, b_n)f = J_\eta^1 f - J_\eta^1 b_n \cdot \nabla J_\eta^1 f + J_\eta^1 b_n \cdot \nabla J_\eta^1 f + \cdots = (\eta + \tilde{\Lambda}(b_n))^{-1} f, \quad f \in L^2, \]

while (p2) follows from the definition of \( \Theta(\zeta, b_n) \). (p3) says that \( \Theta(\zeta, b_n) \) is a pseudo-resolvent. But then the range of \( \Theta(\zeta, b_n) \) equals to the range of \( \Theta(\eta, b_n) \) for all \( \eta \in \mathcal{O} \), and hence is dense in \( L^2 \) by (p1). Thus the properties (p1), (p3) mean that

\[ \Theta(\zeta, b_n) = (\zeta + \tilde{\Lambda}(b_n))^{-1}, \]

and hence \( \mathcal{O} \subset \rho(-\tilde{\Lambda}(b_n)) \) and the semigroup \( e^{-t\tilde{\Lambda}(b_n)} \) is holomorphic (due to (p2)).

Finally, on the basis of the Trotter Approximation Theorem, by proving that \( \mu \Theta(\mu, b_n) \xrightarrow{\delta} 1 \) as \( \mu \uparrow \infty \) uniformly in \( n \), we conclude, using \( \Theta(\zeta, b_n) \xrightarrow{\delta} \Theta(\zeta, b) \) (see [3]), that \( \Theta(\zeta, b) \) is indeed the resolvent of an operator \(-\Lambda(b)\), which generates a holomorphic (\( \|\Theta(\zeta, b)\|_{2 \to 2} \leq c|\zeta|^{-1} \)) semigroup.

The proof that \( \mu \Theta(\mu, b_n) \xrightarrow{\delta} 1 \) as \( \mu \uparrow \infty \) uniformly in \( n \) is carried out as follows. Set \( H_n = |b|^2 J_\mu \) and \( S_n = b_n \cdot \nabla J_\mu^3 \). Since \( S_\mu \)-\text{Lip}_{\mu} \leq 1 \) and \( \sup_n \|\Theta(\mu, b_n)\|_{2 \to 2} \leq c \mu^{-1} \),

\[ \lim_{\mu \uparrow \infty} \mu \sup_n \|J_\mu^3 H_n^*(1 + S_n H_n^*)^{-1} S_n J_\mu f\|_2 = 0 \quad (f \in C_c) \]

needs only to be proved (see assertion (ii) of the theorem). Noticing that

\[ \|S_n J_\mu f\|_2 = \|b_n \cdot \nabla J_\mu^4 f\|_2 \leq \sqrt{\delta \mu^{-\frac{3}{2}}} \|f\|_2 \]  and \( \|H_n^*(1 + S_n H_n^*)^{-1}\|_{2 \to 2} \leq \sqrt{\delta}(1 - \delta)^{-1} \),

we obtain

\[ \|J_\mu^3 H_n^*(1 + S_n H_n^*)^{-1} S_n J_\mu f\|_2 \leq \delta(1 - \delta)^{-1} \mu^{-\frac{3}{2}} \|f\|_2. \]

The desired convergence follows.

It remains to prove (p3).

**Proof of (p3).** Set \( F_\zeta := (\zeta - \Delta)^{-1} \) and define

\[ N_\zeta^k := (-1)^k F_\zeta b_n \cdot \nabla F_\zeta \cdots b_n \cdot \nabla F_\zeta, \quad 1 \leq k := \# b_n \text{'s}, \quad N_\zeta^0 := F_\zeta. \]

Obviously,

\[ \Theta(\zeta, b_n) = \sum_{k=0}^{\infty} N_\zeta^k \]  (absolutely convergent series in \( L^2 \)),

\[ \Theta(\zeta, b_n) \Theta(\eta, b_n) = \sum_{k=0}^{\infty} \sum_{i=0}^{l} N_\zeta^i N_\eta^{l-i}, \quad \zeta, \eta \in \mathcal{O}. \]  (\(*\)

Define

\[ I_{l,m}^k(\zeta, \eta) := F_\zeta b_n \cdot \nabla F_\zeta \cdots b_n \cdot \nabla F_\zeta F_\eta b_n \cdot \nabla F_\eta \cdots b_n \cdot \nabla F_\eta, \]

\[ l := \# \zeta \text{'s}, \quad m := \# \eta \text{'s}, \quad k := \# b_n \text{'s}. \]

Substituting the identity \( F_\zeta F_\eta = (\eta - \zeta)^{-1} (F_\zeta - F_\eta) \) inside the product

\[ N_\zeta^k N_\eta^m = (-1)^{k+m} F_\zeta b_n \cdot \nabla F_\zeta \cdots b_n \cdot \nabla F_\zeta F_\eta b_n \cdot \nabla F_\eta \cdots b_n \cdot \nabla F_\eta, \]
we obtain \( N_k^l N_m^r = (\eta - \zeta)^{-1} (1)^{k+m} \left[ I_{k+1,m}^{l+m} - I_{k,m+1}^{l+m} \right] \). Therefore,

\[
(\eta - \zeta) \sum_{i=0}^{l} N_k^l N_m^r = (1)^{l} \left[ I_{1,l}^{l} - I_{l+1,l+1}^{l} + I_{2,l-1}^{l+1} - I_{l+1,l}^{l+1} + \cdots + I_{l+1,0}^{l} - I_{l,1}^{l} \right]
\]

\[
= (1)^{l} (I_{l+1,0}^{l} - I_{0,l+1}^{l}).
\]

Substituting the last identity in the RHS of (\(*\)) we obtain

\[
(\eta - \zeta) \Theta(\zeta, b_n) \Theta(\eta, b_n) = \sum_{l=0}^{\infty} (1)^{l} (I_{l+1,0}^{l} - I_{0,l+1}^{l}) = \Theta(\zeta, b_n) - \Theta(\eta, b_n).
\]

The proof of (p3) is completed. \( \square \)

It follows from (\[\text{3}\]) that

\[
-\Lambda(b_n) = -\tilde{\Lambda}(b_n) \quad (= \Delta - b_n \cdot \nabla, D(-\Lambda(b_n)) = W^{2,2}).
\]

The latter and (\[\text{3}\]) yield

**Corollary 4.1.** In the assumptions of Theorem 4.4, for \( b_n \)'s given by (\[\text{2}\]), we have

\[
(\zeta + \Lambda(b_n))^{-1} \stackrel{\text{s}}{\rightarrow} (\zeta + \Lambda(b))^{-1} \quad (\zeta \in \mathcal{O}).
\]

4.3. \((L^p \to L^q)\) estimates.

**Theorem 4.2.** Let \( d \geq 3, b : \mathbb{R}^d \to \mathbb{R}^d, b \in \mathcal{F}_{\delta}^{1/2}, \delta < 1 \). Let \( \Lambda(b) \) be the operator realization of \( -\Delta + b \cdot \nabla \) in \( L^2 \) constructed in Theorem 4.7. Let \( \{b_n\} \) be given by (\[\text{2}\]). There is a family \( \{e^{-t\Lambda(b)} t \geq 0\}_{2 \leq r < \infty} \) of consistent, positivity preserving, \( L^\infty \) contraction \( C_0 \) semigroups such that

\[
e^{-t\Lambda(b)} \upharpoonright L^r \cap L^2 = e^{-t\Lambda(b)} \upharpoonright L^r \cap L^2;
\]

\[
\textbf{-}\text{-} L^r \lim_{n} e^{-t\Lambda(b)}(b_n) = e^{-t\Lambda(b)} \quad \text{(uniformly on every compact interval of} t); \quad (\text{i})
\]

\[
\|e^{-t\Lambda} f\|_r \leq c_r e^{\omega r} t^{-\frac{d}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \|f\|_r, \quad c_r = c_r(\delta, c_2), \quad \omega_r = \frac{2\lambda \delta}{r} \quad (f \in L^r, 2 \leq r < \infty). \quad (\text{ii})
\]

**Proof.** The bounds \[
\|e^{-t\Lambda(b)}\|_{2 \to 2} \leq c_2 e^{\omega r} \quad \text{and} \quad \|e^{-t\Lambda(b)} f\|_\infty \leq \|f\|_\infty, f \in L^2 \cap L^\infty,
\]

yield via the Riesz Interpolation Theorem

\[
\|e^{-t\Lambda(b)} f\|_r \leq c_2^2 e^{\omega r} t^{-\frac{d}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \|f\|_r, \quad f \in L^2 \cap L^\infty.
\]

and so we obtain a family \( \{e^{-t\Lambda(b)}\}_{1 \leq p < \infty} \) of consistent \( C_0 \) semigroups by setting

\[
e^{-t\Lambda(b)} := \left[ e^{-t\Lambda(b)} \upharpoonright L^2 \cap L^\infty \right] \text{clos} \upharpoonright L^r \to L^r.
\]

(i) By Corollary 4.1 (i) holds for \( r = 2 \), and hence for each \( r > 2 \). Since the \( C_0 \) semigroups \( e^{-t\Lambda(b)} \) are positivity preserving and \( L^\infty \) contractions, so is \( e^{-t\Lambda(b)} \).

(ii) Our strategy is as follows: 1) We prove that, for some \( r \in ]2, \infty[, \)

\[
\|e^{-t\Lambda} \|_{2 \to r} \leq C_T t^{-\frac{d}{2} \left( \frac{1}{r} - \frac{1}{q} \right)} \quad (0 < t < T < \infty). \quad (\star)
\]
2) Using the Extrapolation Theorem (Appendix III) between \( \|e^{-t\Lambda}f\|_\infty \leq \|f\|_\infty, \ f \in L^2 \cap L^\infty \) and (\(*\)), we conclude that
\[
\|e^{-t\Lambda}\|_{2 \to \infty} \leq C_T t^{-\frac{d}{2}}.
\]
Then (ii) follows from the Riesz Interpolation Theorem.

Since (2) is straightforward, we need only to prove (\(*\)). Without loss of generality we may suppose that \( \lambda \geq 1 \). Set \( \Gamma_0 = \lambda - \Delta \) and \( \Gamma = \lambda + \Delta \). Due to \( |(\eta + \Gamma_0)^{-\beta}| \leq \Gamma_0^{-\beta} |f| \) \( \) and \( \|\eta + \Gamma_0\|_{-2 \to 2} \leq (1 + |\eta|)^{-\beta} \) (0 < \( \beta < 1 \), Re \( \eta \geq 0 \), we have
\[
\|\eta + \Gamma\|_{-1 \to 2} \leq (1 - \delta)^{-1} \|\eta + \Gamma_0\|^{-\frac{2}{3}} \|\eta + \Gamma_0\|^{-\frac{1}{2}} \|\eta + \Gamma_0\|^{-\frac{1}{2}} \leq (1 - \delta)^{-1} (1 + |\eta|)^{-1}.
\]
Also, for any \( r \in [2, \infty] \) and \( \mu \geq 0 \), \( \|\eta + \Gamma_r\|_{-1 \to r} \leq c_r (1 + \mu)^{-1} \).

Let \( \Gamma_{\mu}^{-\beta} \) and \( \Gamma_r^{-\beta} = (\Gamma^{-\beta})_{\mu} \) be the fractional powers of \( \Gamma_r \). Below we will need the following well known results:
\[
\Gamma_r^{-\beta} = \frac{\sin \pi \beta}{\pi} \frac{1}{1 - \beta} \int_0^\infty \mu^{1 - \beta} (\mu + \Gamma_r)^{-2} d\mu;
\]
\[
\|\Gamma_{\mu}^{-\beta}\|_{2 \to 2} \leq c (1 + \mu)^{-1 + \beta};
\]
\[
\|\Gamma_{\mu}^{-\beta}e^{-t\Gamma}\|_{2 \to 2} \leq c(\beta)t^{-\beta}
\]
(see e.g. [14] Ch. 4).

Fix \( \beta = \frac{1}{4}, r = \frac{2d}{d - 1} \). Let \( F_t = \Gamma_{\mu}^{2 \beta} e^{-t\Gamma} f, \ f \in L^2 \cap L^r \). We have
\[
\|e^{-t\Gamma} f\|_r = \|\Gamma_{\mu}^{-2\beta} F_t\|_r \leq \frac{2}{\pi} \int_0^\infty \mu^{1 - 2\beta} (\mu + \Gamma_r)^{-2} F_t d\mu.
\]
By the embedding \( (\mu + \Gamma_0)^{-\beta} L^2 \subset L^r \),
\[
\|\mu + \Gamma_r\|^{-2} F_t \|_r \leq c(d) \|\mu + \Gamma_0\|^{-\frac{1}{2}} (1 + P_\mu(b))^{-1} (\mu + \Gamma_0)^{-\frac{1}{2}} (\mu + \Gamma)^{-1} F_t \|_2,
\]
and hence \( \|\mu + \Gamma_r\|^{-2} F_t \|_r \leq c(d) (1 - \delta)^{-1} \mu^{-3\beta} (\mu + \Gamma)^{-1} F_t \|_2 \).

Thus
\[
\|e^{-t\Gamma} f\|_r \leq C \int_0^\infty \mu^{-\beta} \|\mu + \Gamma\|^{-1} F_t \|_2 \ d\mu
\]
\[
\leq C_1 \left( \int_0^{\frac{1}{t}} \mu^{-\beta} (1 + \mu)^{2\beta - 1} d\mu \|f\|_2 + \int_0^\infty \mu^{-\beta - 1} d\mu \|\Gamma_{\mu}^{2\beta} e^{-t\Gamma} f\|_2 \right)
\]
\[
\leq C_2 \left( \int_0^{\frac{1}{t}} \mu^{-\frac{2}{d}} \ d\mu + \int_0^\infty \mu^{-\beta - \frac{2}{d}} d\mu \ t^{-2\beta} \right) \|f\|_2 = 4C_2 t^{-\beta} \|f\|_2,
\]
which ends the proof of (\(*\)).

4.4. \( L^r \)-theory and \( W^{1+\frac{1}{q}r} \) -estimates on solutions to \( (\lambda - \Delta + b \cdot \nabla) u = f, \ b \in F^{1/2}_\delta \). As in the case of \( b \in F_\delta \), it is reasonable to expect that there is a strong dependence between the value of \( \delta \) and the smoothness of \( D(\Lambda_r(b)) \).
Set
\[ m_d := \pi^2 (2e)^{-\frac{d}{2}} e^d (d - 1)^{-\frac{d-1}{2}}, \quad c_r := \frac{r r'}{4}, \quad \kappa_d := \frac{d}{d-1}, \]
\[ r_{\mp} := \frac{2}{1 \pm \sqrt{1 - m_d \delta}}. \]

**Theorem 4.3.** Let \( d \geq 3 \), \( b : \mathbb{R}^d \to \mathbb{R}^d \). Assume that \( b \in F^{\frac{1}{2}}_\delta \), \( m_d \delta < 1 \). Then \( (e^{t\Lambda_r(b)}, r \in [2, \infty[) \) extends by continuity to a quasi bounded \( C_0 \) semigroup in \( L^r \) for all \( r \in ]r_-, \infty[. \)

For every \( r \in I_s := ]r_-, r_+[ \) we have:

(i) The resolvent set \( \rho(-\Delta_r(b)) \) contains the half-plane \( \mathcal{O} := \{ \zeta \in \mathbb{C} \mid \Re \zeta \geq \kappa_d \lambda_\delta \} \), and the resolvent admits the representation
\[ (\zeta + \Lambda_r(b))^{-1} = \Theta_r(\zeta, b), \quad \zeta \in \mathcal{O}, \]
where
\[ \Theta_r(\zeta, b) := (\zeta - \Delta)^{-1} - Q_r(1 + T_r)^{-1} G_r, \]
the operators \( Q_r, G_r, T_r \in \mathcal{B}(L^r) \),
\[ \|Q_r\|_{r \to r} \leq C_1 |\zeta|^{-\frac{d}{2}} \cdot \frac{1}{\sqrt{\pi}}, \quad \|G_r\|_{r \to r} \leq C_2 |\zeta|^{-\frac{d}{2}} \cdot \frac{1}{\sqrt{\pi}}, \quad \|T_r\|_{r \to r} \leq m_d c_r \delta, \quad (\ast) \]
\[ G_r \equiv G_r(\zeta, b) := b^\frac{1}{r} \cdot \nabla (\zeta - \Delta)^{-1}, \quad b^\frac{1}{r} := |b|^{\frac{1}{r}} b, \]
\( Q_r, T_r \) are the extensions by continuity of densely defined (on \( \mathcal{E} := \bigcup_r e^{-t|b|L^r} \)) operators
\[ Q_r \upharpoonright \mathcal{E} \equiv Q_r(\zeta, b) \upharpoonright \mathcal{E} := (\zeta - \Delta)^{-1} |b|^\frac{1}{r}, \quad T_r \upharpoonright \mathcal{E} \equiv T_r(\zeta, b) \upharpoonright \mathcal{E} := b^\frac{1}{r} \cdot \nabla (\zeta - \Delta)^{-1} |b|^\frac{1}{r}. \]

(ii) It follows from (i) that \( e^{t\Lambda_r(b)} \) is holomorphic: there is a constant \( C_r \) such that
\[ \|(\zeta + \Lambda_r(b))^{-1}\|_{r \to r} \leq C_r |\zeta|^{-1}, \quad \zeta \in \mathcal{O}. \]

(iii) For every \( 1 \leq p < r < q \) and \( \zeta \in \mathcal{O} \), define
\[ G_r(p) \equiv G_r(p, \zeta, b) := b^\frac{1}{r} \cdot \nabla (\zeta - \Delta)^{-\frac{1}{r}} \cdot \frac{1}{\sqrt{\pi}}, \]
\[ Q_r(q) \equiv Q_r(q, \zeta, b) := (\zeta - \Delta)^{-\frac{1}{2q}} |b|^\frac{1}{2q} \quad \text{on} \ \mathcal{E}. \]

Then \( G_r(p) \in \mathcal{B}(L^r) \). \( Q_r(q) \) extends by continuity to a bounded operator in \( L^r \). Its extension we denote again by \( Q_r(q) \).

(iv) For every \( \zeta \in \mathcal{O} \),
\[ \Theta_r(\zeta, b) = (\zeta - \Delta)^{-1} - (\zeta - \Delta)^{-\frac{1}{r}} \cdot \frac{1}{\sqrt{\pi}} Q_r(q)(1 + T_r)^{-1} G_r(p)(\zeta - \Delta)^{-\frac{1}{2q}}; \]
\( \Theta_r(\zeta, b) \) extends by continuity to an operator in \( \mathcal{B}(W^{-\frac{1}{p}} \bigwedge W^{1+\frac{1}{2}r}) \).

(v) By (i) and (iv), \( D(\Lambda_r(b)) \subset W^{1+\frac{1}{2}r} (q > r) \). In particular, if \( m_d \delta < 4 d^{-2} (d - 1)^2 \), then there exists \( r \in I_s, \ r > d - 1 \), such that \( D(\Lambda_r(b)) \subset C^{0, \gamma}, \gamma < 1 - \frac{d-1}{r} \).

(vi) \( D(\Lambda_r(b)) \subset W^{1}_{loc} \) and
\[ \langle \Lambda_r(b) u, v \rangle = \langle u, -\Delta v \rangle + \langle b \cdot \nabla u, v \rangle, \quad u \in D(\Lambda_r(b)), \quad v \in C_c^\infty(\mathbb{R}^d). \]
(vii) Let \{b_n\} be given by (2), then
\[
e^{-t\lambda_r(b)} = sL^r \lim_n e^{-t\lambda_r(b_n)} \quad \text{(uniformly on every compact interval of } t \geq 0),
\]

(viii) \((e^{-t\lambda_r(b)}, t > 0)\) preserves positivity and
\[
\|e^{-t\lambda_r(b)}f\|_p \leq C_0 t^{\frac{q}{2}(1 - \frac{1}{p})}\|f\|_r, \quad 0 < t \leq 1, \quad r_- < r < p \leq \infty, \quad f \in L^r.
\]

**Proof.** 1. Suppose that (\(2\)) is established.

Then \(\Theta_r(\zeta, b) \upharpoonright L^2 \cap L^r = \Theta_2(\zeta, b) \upharpoonright L^2 \cap L^r, \quad \zeta \in \mathcal{O}, \) and (i), (vi) follow immediately from Theorem 4.1.

(vii) is a simple consequence of Corollary 4.1 and the bound \(\|e^{-t\lambda_r}\|_{r \to r_0} \leq C_{r_0} = C_{r_0}(\delta, r_- - r_0) (t \leq 1, \quad r_- < r_0 < 2). \) The latter is also needed for the proof of (viii) (cf. Theorem 4.2).

(iv) is an obvious consequence of (i) + (iii).

It remains to prove (\(2\)) and (iii).

2. **Proof of (2).** Let \(r \in [1, \infty[. \) We will need

(a) \(\mu \geq \lambda \Rightarrow \|b\|^{\frac{1}{2}}(\mu - \Delta)^{-\frac{1}{2}}\|f\|_{r \to r} \leq (c_r\delta)^{\frac{1}{2}}\mu^{-\frac{1}{2r}} \) (recall \(c_r = \frac{\mu^2}{4}\)).

Indeed, in \(L^2\) define \(A = (\mu - \Delta)^{-\frac{1}{2}}, \quad D(A) = W^{1,2}. \) Since \(-(A - \mu^2)\) is a symmetric Markov generator, for any \(r \in [1, \infty[,
\[
0 \leq u \in D(A_r) \Rightarrow v = u^{\frac{1}{2}} \in D(A^{\frac{1}{2}}) \) and \(c_r^{-1}\|A^{\frac{1}{2}}v\|_r^2 \leq \langle A_r u, u^{-1}\rangle.
\]

Now, let \(u = A_r^{-1}|f|, \quad f \in L^r. \) Clearly, \(\|u\|_r \leq \mu^{-\frac{1}{2}}\|f\|_r. \) Since \(b \in F_{\delta, 2}^1\) we have
\[
(c_r\delta)^{-1}\|b\|^{\frac{1}{2}}v\|_r^2 \leq \langle A_r u, u^{-1}\rangle = \langle f, u^{-1}\rangle,
\]
and so \(\|b\|^{\frac{1}{2}}v\|_r^2 \leq c_r\delta\|f\|_r\|u\|_r^{-1}, \) \(\|b\|^{\frac{1}{2}}A_r^{-1}f\|_r \leq c_r\delta\|f\|_r, \quad f \in \mathcal{E}.
\]

(b) \(\mu \geq \lambda \Rightarrow \|b\|^{\frac{1}{2}}\mu^{-\frac{1}{2}}\|f\|_r \leq c_r\delta\|f\|_r, \quad f \in \mathcal{E}.
\]

Indeed, let \(u = A_r^{-1}|b|^\frac{1}{2}|f|, \quad f \in \mathcal{E}. \) Then arguing as in (a) we have
\[
\|b\|^{\frac{1}{2}}u\|_r^2 \leq c_r\delta\|f\|_r\|b\|^{\frac{1}{2}}u\|_r^{-1}, \text{ or } \|b\|^{\frac{1}{2}}u\|_r \leq c_r\delta\|f\|_r.
\]

Thus (b) is proved.

(c) \(\mu \geq \lambda \Rightarrow \|(|\mu - \Delta|^{-\frac{1}{2}}|b|^{\frac{1}{2}}f\|_r \leq (c_r\delta)^{\frac{1}{2}}\mu^{-\frac{1}{2r}}\|f\|_r, \quad f \in \mathcal{E}.
\]

Indeed, (c) follows from (a) by duality.

We are in position to complete the proof of (\(2\)). Using (b) and the pointwise bound
\[
|\nabla(\zeta - \Delta)^{-\frac{1}{2}}(x, y)| \leq m_d(\lambda - \Delta)^{-\frac{1}{2}}(x, y), \quad \text{Re } \zeta \geq \kappa_d\lambda \quad (A_1)
\]
(see a proof of (2A1) below), we have, for every \(r \in I_s\) and \(\mu = \lambda, \)
\[
\|T_r f\|_r \leq m_d\|b\|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{2}}|b|^{\frac{1}{2}}f\|_r \leq m_d c_r\delta\|f\|_r, \quad f \in \mathcal{E}.
\]

Note that \(m_d c_r\delta < 1\) due to \(r \in I_s.\)

Next, using (a) and the pointwise bound
\[
|\nabla(2\kappa_d\zeta - \Delta)^{-\frac{1}{2}}(x, y)| \leq 2^{\frac{d}{2}}m_d(|\zeta| - \Delta)^{-\frac{1}{2}}(x, y), \quad \text{Re } \zeta > 0 \quad (A_2)
\]
(see a proof of (A.3) below), we have, for every $r \in I_s$, $\zeta \in \mathcal{O}$ and $\mu = |\zeta|$, 
\[ \|G_r(2\kappa_d \zeta, b)\|_{r \to r} \leq 2^{\frac{1}{2}}(c_r \delta)^{\frac{1}{2}} m_d |\zeta|^{-\frac{1}{2r}}. \]
Now, using the identity $(\zeta - \Delta)^{-1} = (2\kappa_d \zeta - \Delta)^{-1}(1 + (2\kappa_d - 1)\zeta(\zeta - \Delta)^{-1})$, we obtain 
\[ \|G_r(\zeta, b)\|_{r \to r} \leq 2^{\frac{1}{2}}(c_r \delta)^{\frac{1}{2}} m_d |\zeta|^{-\frac{1}{2r}} \equiv C_2 |\zeta|^{-\frac{1}{2r}}. \]
Similarly, using (c) and the pointwise bound 
\[ |(2\zeta - \Delta)^{-\frac{1}{2}}(x, y)| \leq 2^{\frac{1}{2}}(\frac{1}{2})^\frac{1}{2}(x, y), \ \text{Re} \zeta > 0 \quad (A.4) \]
(see a proof of (A.4) below), we have, for every $r \in I_s$ and $\text{Re} \zeta \geq \lambda$, 
\[ \|Q_r(2\zeta, b)f\|_r \leq \|(2\zeta - \Delta)^{-\frac{1}{2}}\|_{r \to r} \|(2\zeta - \Delta)^{-\frac{1}{2}}|b|^{\frac{1}{2}}f\|_r \leq \|2\zeta - \frac{1}{2} 2^{\frac{1}{2}}(c_r \delta)^{\frac{1}{2}} |\zeta|^{-\frac{1}{2r}} \|f\|_r, \ f \in \mathcal{E}. \]
Finally, using the identity $(\zeta - \Delta)^{-1} = (1 + \zeta(\zeta - \Delta)^{-1})(2\zeta - \Delta)^{-1}$, we obtain 
\[ \|Q_r(\zeta, b)f\|_r \leq \|1 + \zeta(\zeta - \Delta)^{-1}\|_{r \to r} \|Q_r(2\zeta, b)f\|_r \leq 2^{\frac{1}{2}}(c_r \delta)^{\frac{1}{2}} |\zeta|^{-\frac{1}{2r}} \|f\|_r, \quad \|\|f\|_r \| \leq C_1 |\zeta|^{-\frac{1}{2r}} \|f\|_r, \quad f \in \mathcal{E}, \]
completing the proof of (A).

3. **Proof of (iii).** We have to deduce the bounds on $\|Q_r(g)\|_{r \to r}, \|G_r(p)\|_{r \to r}$. Let $\text{Re} \zeta \geq \lambda$ and $r < q$. Using (c) and the formula 
\[ (z - \Delta)^{-\alpha} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty t^{-\alpha}(t + z - \Delta)^{-1} dt \quad 0 < \alpha < 1, \ \text{Re} z > 0, \]
we obtain 
\[ \|Q_r(q)f\|_r \leq \|(\text{Re} \zeta - \Delta)^{-\frac{1}{2r}}|b|^{\frac{1}{2}}|f\|_r \leq \|(\lambda - \Delta)^{-\frac{1}{2r}}|b|^{\frac{1}{2}}|f\|_r \leq s_q \int_0^\infty t^{-\frac{1}{2r}} \| \|t + \lambda - \Delta\|^{-\frac{1}{2}} |b|^{\frac{1}{2}}|f\|_r dt \left( s_q = \frac{\sin \frac{\pi}{2q}}{\pi} \right) \]
\[ \leq s_q \int_0^\infty t^{-\frac{1}{2r}} \| \|t + \lambda - \Delta\|^{-\frac{1}{2}} |b|^{\frac{1}{2}}|f\|_r dt \leq K_{1,q} \|f\|_r, \quad f \in \mathcal{E}, \]
where $K_{1,q} < \infty$ due to $q > r$.

Let $\zeta \in \mathcal{O}$ and $1 \leq p < r$. Similarly, using (a) and the pointwise bound 
\[ |\nabla(\zeta - \Delta)^{-1+\frac{1}{2r}}(x, y)| \leq m_{d, r}(\kappa_d^{-1}\text{Re} \zeta - \Delta)^{-1+\frac{1}{2r}}(x, y) \quad (A.2) \]
(see proof of (A.2) below), we obtain 
\[ \|G_r(p)f\|_r \leq m_{d,p} |b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{2}} |f| \|_r \leq m_{d,p} s_p(c_r \delta)^{\frac{1}{2}} \int_0^\infty t^{-\frac{1}{2r}}(t + \lambda)^{-\frac{1}{2}} \|f\|_r dt = K_{2,p} \|f\|_r, \quad f \in L^r, \]
where \( K_{2,p} < \infty \) due to \( p < r \). The proof of (iii) is completed.

4. We now give a proof of (A1)-(A4).

**Proof of (A1).** Let \( \alpha \in (0,1] \). Set \( c(\alpha) = \sup_{\zeta > 0} \zeta e^{-(1-\alpha)\zeta^2} = (2(e(1-\alpha))^{-\frac{1}{2}}) \). Then, for all \( \zeta > 0 \),

\[
\zeta e^{-\zeta^2} \leq c(\alpha) e^{-\alpha \zeta^2}
\]

(\*)

Using the formula \( (\zeta - \Delta)^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\zeta t} t^{-\beta-1} (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4t}} dt \), \( 0 < \beta \leq 1 \), first with \( \beta = 1 \), and then with \( \beta = \frac{1}{2} \), we have

\[
|\nabla(\zeta - \Delta)^{-1}(x,y)| \leq \int_0^\infty e^{-\text{Re}\, \zeta\, t}\, \left(\frac{d}{2}\right)^\frac{1}{2} |x-y|^2 \, \frac{2t}{e^{-\frac{|x-y|^2}{4t}} dt}
\]

\[
\leq c(\alpha) \int_0^\infty e^{-\text{Re}\, \zeta\, t} t^{-\frac{1}{2}} (4\pi t)^{-\frac{d}{2}} e^{-\alpha \frac{|x-y|^2}{4t}} dt
\]

(by (\*) with \( \zeta = \frac{|x-y|}{2\sqrt{t}} \))

The minimum of \( c(\alpha) \alpha^{-\frac{d-1}{4}} \Gamma(\frac{1}{2})(\alpha \text{Re} \, \zeta - \Delta)^{-\frac{1}{2}}(x,y) \) in \( \alpha \in (0,1] \) is attained at \( \alpha = \frac{d-1}{4} (= k_1^{-1}) \), and equals \( m_d \).

The proof of (A2) is similar.

**Proof of (A3).** First suppose that \( \text{Im} \, \zeta \leq 0 \). By Cauchy’s theorem,

\[
\int_0^\infty e^{-\text{Re}\, \zeta\, t}\, \left(\frac{d}{2}\right)^\frac{1}{2} |x-y|^2 \, \frac{2t}{e^{-\frac{|x-y|^2}{4t}} dt}
\]

(i.e. we have changed the contour of integration from \( \{ t \mid t \geq 0 \} \) to \( \{ re^{i\varphi} \mid r \geq 0 \} \)). Thus

\[
|\nabla(\zeta - \Delta)^{-1}(x,y)| \leq \int_0^\infty \left| e^{-\text{Re}\, \zeta\, r}\, \left(\frac{d}{2}\right)^\frac{1}{2} |x-y|^2 \, \frac{2r}{e^{-\frac{|x-y|^2}{4r}} dr}\right|
\]

\[
\leq e^{-r\frac{1}{\sqrt{2}}} (\text{Re} \, \zeta - \text{Im} \, \zeta), \quad \left| e^{-\frac{|x-y|^2}{4r}} \right| \leq e^{-\frac{|x-y|^2}{4r}}, \quad \text{Re} \, \zeta - \text{Im} \, \zeta \geq |\zeta|,
\]

\[
|\nabla(\zeta - \Delta)^{-1}(x,y)| \leq \int_0^\infty e^{-\text{Re}\, \zeta\, r}\, \left(\frac{d}{2}\right)^\frac{1}{2} |x-y|^2 \, \frac{2r}{e^{-\frac{|x-y|^2}{4r}} dr},
\]

\[
\leq \frac{2\pi m_d}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-\text{Re}\, \zeta\, r}\, \left(\frac{d}{2}\right)^\frac{1}{2} |r| (4\pi r)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4r}} dr,
\]

\[
= \frac{2\pi}{\Gamma(\frac{1}{2})} m_d (k_1^{-1}) (|\zeta| - \Delta)^{-\frac{1}{2}}(x,y),
\]

and so (A3) for \( \text{Im} \, \zeta \leq 0 \) is proved. The case \( \text{Im} \, \zeta > 0 \) is treated analogously.

**Proof of (A4).** First suppose that \( \text{Im} \, \zeta \leq 0 \). By Cauchy’s theorem,

\[
\int_0^\infty e^{-\text{Re}\, \zeta\, t} \left(\frac{d}{2}\right)^\frac{1}{2} |x-y|^2 \, \frac{2t}{e^{-\frac{|x-y|^2}{4t}} dt}
\]

so we estimate as above:

\[
|\nabla(\zeta - \Delta)^{-\frac{1}{2}}(x,y)| \leq \int_0^\infty e^{-r\frac{1}{\sqrt{2}}} (4\pi r)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{4r}} dr,
\]

\[
= \frac{2\pi m_d}{\Gamma(\frac{1}{2})} (2^{-1} |\zeta| - \Delta)^{-\frac{1}{2}}(x,y),
\]

The case \( \text{Im} \, \zeta > 0 \) is treated analogously.
Remarks. 1. In the proof of $G_r(p)$, $Q_r(q) \in B(L^p)$ we appeal to the $L^p$ inequalities between the operator $(\lambda - \Delta)^{\frac{1}{2}}$ and the “potential” $|b|$ (Appendix C). This is the reason for the symmetry of the interval $I_s$ in spite of $-\Delta + b \cdot \nabla$ being a non-symmetric operator.

2. In the proof of Theorem 4.3 carried out for the Kato class $K^{d+1}_\delta$ the interval $I_s$ transforms into $[1, \infty]$, and the dependence of the properties of $D(\Lambda_r(b))$ on $\delta$ gets lost. The latter indicates the smallness of $K^{d+1}_\delta$ as a subclass of $F^{1/2}_\delta$.

3. Consider the subclass of $F^{1/2}_\delta$ consisting of vector fields such that

$$\|\mathbf{b}\|_2^{1/2} (\Delta)^{-\frac{1}{2}} \mathbf{f}\|_{2,2} \leq \sqrt{\delta}. $$

For such vector fields the constant $m_d$ in Theorem 4.3 can be replaced with

$$m^*_d := \frac{d - 2}{2} \frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)}. $$

Note that $m^*_d < m_d$ for all $d \geq 3$, while $m_d/m^*_d \to 1$ as $d \to \infty$.

This subclass contains e.g. $b \in [L^{d,\infty}]^d$ with $\sqrt{\delta} = \left(\|\mathbf{b}\|_{d,\infty} \Omega_d^{-\frac{1}{2}}\right)^{\frac{1}{2}} 2^{-\frac{1}{2}} \Gamma\left(\frac{d-1}{2}\right) / \Gamma\left(\frac{d+1}{2}\right)$, where $\Omega_d = \pi^\frac{d}{2} \Gamma\left(\frac{d}{2} + 1\right)$ denotes the volume of the unit ball in $\mathbb{R}^d$. [12] Prop. 2.5, 2.6, Cor. 2.9.

4. For $b \in [L^{d,\infty}]^d$, the assertion of Theorem 4.3(c) (i.e. $D(\Lambda_r(b)) \subset W^{1+\frac{1}{r},q}$, $q > r$) can be strengthened: $D(\Lambda_r(b)) \subset W^{1+\frac{1}{r},s}$, $s < dr$. Indeed, appealing to [12] Prop. 2.5, 2.6, Cor. 2.9, we have, for every $f \in \mathcal{E}$ (taking into account the previous remark),

$$\|\mathbf{b}\|_2^{\frac{1}{2}} \cdot \nabla (-\Delta)^{-\frac{1}{2}} \mathbf{f}\|_s \leq c_1 \|f\|_s, \quad c_1 := m^*_d \left(\Omega_d^{-\frac{1}{2}} \|b\|_{d,\infty}\right)^{\frac{1}{2}} c(r,d),$$

$$\|(-\Delta)^{-\frac{1}{2}} |b|^{\frac{1}{2}} \mathbf{f}\|_s \leq c_2 \|f\|_s, \quad c_2 := \left(\Omega_d^{-\frac{1}{2}} \|b\|_{d,\infty}\right)^{\frac{1}{2}} c(r',d),$$

where $c(r,d) := 2^{-\frac{1}{2}} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2} + 1\right)} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d-1}{2} + 1\right)}$, so

$$\|\mathbf{b}\|_2^{\frac{1}{2}} \cdot \nabla (-\Delta)^{-1} |b|^{\frac{1}{2}} \mathbf{f}\|_s \leq c_3 \|f\|_s, \quad c_3 := m^*_d \Omega_d^{-\frac{1}{2}} \|b\|_{d,\infty} c(r,d)c(r',d).$$

Now, we can estimate $\|Q_r(r)\|_{s \to s}$, $\|G_r(r)\|_{s \to s}$, $\|T_r\|_{s \to s}$, concluding that $\|\Theta_r(\zeta, b)\|_{s \to s} < \infty$, $s < dr$. In view of Theorem 4.3(i), the last estimate implies the required.

5. We obtain, using [12] Cor. 2.9] (see examples in the beginning of the section):

$$|x|^{-2} x \in F^{\frac{1}{d}}_\delta, \quad \sqrt{\delta} = 2^{-\frac{1}{2}} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)},$$

$$|x|^{-2} x \in F^{\frac{1}{d_1}}_\delta, \quad \sqrt{\delta_1} = \frac{2}{d - 2},$$

and so $\delta < \sqrt{\delta_1}$.

6. Theorem 4.3 compared to Theorem 3.7 covers a larger class of vector fields, and at the same time establishes stronger smoothness properties of $D(\Lambda_r(b))$: $D(\Lambda_r(b)) \subset W^{1+\frac{1}{r},q}$, $q > r$, $r \in \mathcal{I}_s$, while in Theorem 3.7 $D(\Lambda_r(b)) \subset W^{1+\frac{1}{d'},q}$, $r \in \left[\frac{1}{2} - \frac{2}{\sqrt{\delta}}, \frac{2}{\sqrt{\delta}}\right]$.

Nevertheless, despite the inclusion $F^{\frac{1}{d}}_\delta \subset F^{1/2}_\delta$, $\delta = \sqrt{\delta_1}$, cf. (11), the difference in the admissible values of $\delta, \delta_1$ shows that these classes are essentially incomparable.
7. In order to define \( \Lambda(b, V) \supset -\Delta + b \cdot \nabla - V, \) \( 0 \leq V, \) with \( b \in \mathbf{F}_{\delta} \), it is enough to assume that 
\[
\| V \frac{1}{4} (\lambda - \Delta)^{-\frac{3}{2}} \|_{2 \to 2} \leq \sqrt{\delta V} \text{ with } \delta_b + \delta_V < 1.
\]
However, if \( b \in \mathbf{F}_{\delta}^{1/2} \), then \( \Lambda(b, V) \) can be defined only for \( V \) such that 
\[
\| V \frac{1}{4} (\lambda - \Delta)^{-\frac{3}{2}} \|_{2 \to 2} \leq (\delta V)^{\frac{3}{2}} \text{ with } \delta_b + \delta_V < 1.
\]

4.5. \( L' \)-strong Feller semigroup on \( C_{\infty} \) corresponding to \( -\Delta + b \cdot \nabla, \) \( b \in \mathbf{F}_{\delta}^{1/2} \). In Theorem 4.3 one can use the following approximation of \( L \) by smooth vector fields:
\[
b_n := e^{\varepsilon_n \Delta} 1_n b, \quad \varepsilon_n \downarrow 0,
\]
where \( 1_n \) is the indicator of \( \{ x \in \mathbb{R}^d : |x| \leq n, |b(x)| \leq n \} \) (alternatively, one can use the K. Friedrichs’ mollifier).

For any \( \tilde{\delta} > \delta \) we can select a sequence \( \varepsilon_n \downarrow 0 \) such that \( b_n \in \mathbf{F}_{\delta}^{1/2} \) with the same \( \lambda = \lambda_{\tilde{\delta}} \). Indeed, we have (\( f \in L^2 \))
\[
\| b_n \frac{1}{2} (\lambda - \Delta)^{-\frac{1}{2}} f \|^2 \leq \| 1_n b \frac{1}{2} (\lambda - \Delta)^{-\frac{1}{2}} f \|^2 + \| b_n - 1_n b \frac{1}{2} (\lambda - \Delta)^{-\frac{1}{2}} f \|^2
\]

In turn, by Hölder’s inequality,
\[
\| b_n - 1_n b \frac{1}{2} (\lambda - \Delta)^{-\frac{1}{2}} f \|^2 \leq \| b_n - 1_n b \|_d \| (\lambda - \Delta)^{-\frac{1}{2}} f \|^2_{2d, \frac{1}{d}} \leq C_d \| b_n - 1_n b \|_d \| f \|^2,
\]
where \( C_d > 0 \) is the constant in the uniform Sobolev inequality. Since \( 1_n b \in L^\infty \) and has compact support (and hence \( e^{\varepsilon \Delta} 1_n b \to 1_n b \) in \( L^d \) as \( \varepsilon \downarrow 0 \)), for every \( \tilde{\delta} > \delta \), we can select \( \varepsilon_n, n = 1, 2, \ldots \) sufficiently small so that 
\[
\| b_n - 1_n b \|_d < \frac{\tilde{\delta} - \delta}{C_d}, \quad \text{and hence } \| b_n - 1_n b \|_d \| (\lambda - \Delta)^{-\frac{1}{2}} f \|^2_{2d, \frac{1}{d}} < (\tilde{\delta} - \delta) \| f \|^2.
\]
Therefore, 
\[
\| b_n \frac{1}{2} (\lambda - \Delta)^{-\frac{1}{2}} f \|^2 \leq \tilde{\delta} \| f \|^2,
\]
as needed.

Since the assumptions on \( \delta \) involve strict inequalities only, we may assume without loss of generality that \( b_n \) defined by (4) are in \( \mathbf{F}_{\delta}^{1/2} \) with the same \( \lambda_{\tilde{\delta}} \).

**Theorem 4.4.** Let \( d \geq 3, b : \mathbb{R}^d \to \mathbb{R}^d, b \in \mathbf{F}_{\delta}^{1/2}, m_d \delta < 4 \frac{d-2}{(d-1)^2} \). Fix \( r \in ](d - 1, \frac{2}{1 - \sqrt{1-m_d}} \]. By \( S \) denote the L. Schwartz space of test functions. Then:

(i) \( e^{-t \Lambda_{C_{\infty}}(b)} := (e^{-t \Lambda_r(b)} \upharpoonright S)_{C_{\infty} \to C_{\infty}} \) (after a change on a set of measure zero), \( t \geq 0 \), determines a positivity preserving contraction \( C_0 \) semigroup on \( C_{\infty} \) (Feller semigroup), where the semigroup \( e^{-t \Lambda_r(b)} \) has been constructed in Theorem 4.3.

(ii) [The \( L' \)-strong Feller property] \( (\mu + \Lambda_{C_{\infty}}(b))^{-1} \upharpoonright L^r \in \mathcal{B}(L^r, C^{0, \alpha}), \mu > 0, \alpha < 1 - \frac{d-1}{r} \).

(iii) The integral kernel \( e^{-t \Lambda_{C_{\infty}}(b)}(x, y) \) of \( e^{-t \Lambda_{C_{\infty}}(b)} \) determines the transition probability function of a Hunt process.

(iv) Let \( \{ b_n \} \) be given by (4), then
\[
e^{-t \Lambda_{C_{\infty}}(b)} = s_{C_{\infty}} \lim_n e^{-t \Lambda_{C_{\infty}}(b_n)} \quad \text{(uniformly on every compact interval of } t \geq 0),
\]
where \( \Lambda_{C_{\infty}}(b_n) := -\Delta + b_n \cdot \nabla, D(\Lambda_{C_{\infty}}(b_n)) = (1 - \Delta)^{-1} C_{\infty} \).

**Proof.** (i), (ii). Let \( \Theta_r(\mu, b) \) be the operator-valued function introduced in Theorem 4.3

1) For every \( \mu \geq \kappa_d \lambda, \Theta_r(\mu, b) \subseteq C_{\infty} \), and \( \| \Theta_r(\mu, b) f \|_{\infty} \leq \mu^{-1} \| f \|_{\infty}, f \in S \).
Indeed, by Theorem 4.3, since \( r > d - 1 \), \( \Theta_r(\mu, b)L^r \subset C_\infty \), which yields the first assertion. Since \( e^{-t\Lambda(b)} \) is an \( L^\infty \)-contraction, the second assertion follows.

In view of 1), we can define \((\mu \geq \kappa_d \lambda)\)

\[
\Theta_{C_\infty}(\mu, b) := (\Theta_r(\mu, b) \upharpoonright S)_{C_\infty} \in \mathcal{B}(C_{\infty}) \quad (\text{after a change on a set of measure zero}).
\]

2) \( \mu \Theta_{C_\infty}(\mu, b) \to 1 \) as \( \mu \uparrow \infty \) in \( C_{\infty} \).

Indeed, since \( \| \mu \Theta_{C_\infty}(\mu, b) \|_{r \to \infty} \leq 1 \), and \( S \) is dense in \( C_{\infty} \), it suffices to prove that

\[
\mu \Theta_r(\mu, b)f \to f \quad \text{as} \quad \mu \uparrow \infty \quad \text{in} \quad C_{\infty}, \quad \text{for every} \quad f \in S.
\]

Put \( \Theta_r := \Theta_r(\mu, b) \), \( T_r := T_r(\mu, b) \). Since \( \mu(\mu - \Delta)^{-1}f \to f \) in \( C_{\infty} \), it suffices to show that

\[
\| \mu \Theta_r f - \mu(\mu - \Delta)^{-1}f \|_r \to 0.
\]

For each \( f \in S \) there is \( h \in S \) such that \( f = (\lambda - \Delta)^{-\frac{1}{2}}h \), where \( \lambda = \lambda_\delta > 0 \). Let \( q > r \). Write

\[
\Theta_r f - (\mu - \Delta)^{-1}f = -(\mu - \Delta)^{-\frac{1}{2}}Q_r(q)(1 + T_r)^{-1}b^2(\lambda - \Delta)^{-\frac{1}{2}} \cdot (\mu - \Delta)^{-1}\nabla h.
\]

Using estimates \( \|(1 + T_r)^{-1}\|_{r \to r} \leq (1 - m_d \delta)^{-1} \), \( \|Q_r(q)\|_{r \to r} \leq K_{1,q} < \infty \) (cf. proof of Theorem 4.3 (iii)) and

\[
\|(\mu - \Delta)^{-\frac{1}{2}}\|_{r \to \infty} \leq c \mu^{-\frac{1}{2} + \frac{d}{2r} - \frac{1}{2q}} \quad c < \infty,
\]

we obtain

\[
\|\Theta_r f - (\mu - \Delta)^{-1}f\|_r \leq C \mu^{-\frac{1}{2} + \frac{d}{2r} - \frac{1}{2q}} \mu^{-1} \|
abla h\|_r.
\]

Since \( r > d - 1 \), choosing \( q \) sufficiently close to \( r \), we obtain

\[
-\frac{1}{2} + \frac{d}{2r} - \frac{1}{2q} - 1 < -1,
\]

so \( \mu \Theta_r - \mu(\mu - \Delta)^{-1} \to 0 \) in \( C_{\infty} \). The proof of 2) is completed.

\( \Theta_r(\mu, b) \) satisfies the resolvent identity for \( \mu \geq \kappa_d \lambda \) (Theorem 4.3 (i)), and so does \( \Theta_{C_\infty}(\mu, b) \upharpoonright S \).

Therefore, \( \Theta_{C_\infty}(\mu, b) \) is a pseudo-resolvent for \( \mu \geq \kappa_d \lambda \). The latter and 2) yield (by Theorems F.1 and F.2 Appendix F): \( \Theta_{C_\infty}(\mu, b) \) is the resolvent of a densely defined closed operator \( \Lambda_{C_\infty}(b) \).

By 1), \( \|\mu + \Lambda_{C_\infty}(b)\| \cdot \mu \leq -1 \), so \( -\Lambda_{C_\infty}(b) \) is the generator of a contraction \( C_0 \) semigroup \( e^{-t\Lambda_{C_\infty}(b)} \).

Clearly, \( e^{-t\Lambda_{C_\infty}(b)} \) is positivity preserving, so we have proved (i). Now (ii) follows from Theorem 4.3 (iv). The proof of (iii) is standard.

(iv) Note that

\[
(\mu + \Lambda_{C_\infty}(b))^{-1} \upharpoonright S = \Theta_r(\mu, b) \upharpoonright S, \quad n = 1, 2, \ldots, \quad \mu \geq \kappa_d \lambda,
\]

The latter, combined with

\[
\Theta_r(\mu, b)f \to \Theta_r(\mu, b)f \quad \text{in} \quad C_{\infty}, \quad \mu \geq \kappa_d \lambda, \quad f \in S,
\]

yields \( (\mu + \Lambda_{C_\infty}(b))^{-1} \to (\mu + \Lambda_{C_\infty}(b))^{-1} \) in \( C_{\infty} \), \( \mu \geq \kappa_d \lambda \Rightarrow (iv) \).

Proof of (iii). It suffices to prove that

\[
(\mu - \Delta)^{-\frac{1}{2} - \frac{1}{2q}}Q_r(q,b_n)(1 + T_r(b_n))^{-1}G_r(b_n) \to (\mu - \Delta)^{-\frac{1}{2} - \frac{1}{2q}}Q_r(q,b)(1 + T_r(b))^{-1}G_r(b) \quad \text{on} \quad S \quad \text{in} \quad C_{\infty}.
\]
We choose $q$ close to $d - 1$ so that $\left(\mu - \Delta\right)^{-\frac{1}{2}} \frac{1}{\sqrt{r}} L^r \subset C_\infty$. Thus it suffices to prove that
\[
G_r(b_n) \xrightarrow{\mathcal{S}} G_r(b), \quad (1 + T_r(b_n))^{-1} \xrightarrow{\mathcal{S}} (1 + T_r(b))^{-1}, \quad Q_r(q, b_n) \xrightarrow{\mathcal{S}} Q_r(q, b) \text{ in } L^r.
\]
In turn, since \((1 + T_r(b_n))^{-1} - (1 + T_r(b))^{-1} = (1 + T_r(b_n))^{-1}(T_r(b) - T_r(b_n))(1 + T_r(b))^{-1}\), it suffices to prove that $T_r(b_n) \xrightarrow{\mathcal{S}} T_r(b)$. Finally,
\[
T_r(b_n) - T_r(b) = T_r(b_n) - b_n^1 \cdot \nabla(\zeta - \Delta)^{-1}\|\| + b_n^1 \cdot \nabla(\zeta - \Delta)^{-1}\|\| - T_r(b),
\]
and hence we have to prove that
\[
\frac{1}{n} \cdot \nabla(\zeta - \Delta)^{-1}\|\| - T_r(b) := J_n^{(1)} \xrightarrow{\mathcal{S}} 0 \text{ and } T_r(b_n) - b_n^1 \cdot \nabla(\zeta - \Delta)^{-1}\|\| := J_n^{(2)} \xrightarrow{\mathcal{S}} 0.
\]
Now, by the Dominated Convergence Theorem, $G_r(b_n) \xrightarrow{\mathcal{S}} G_r(b)$, $J_n^{(1)}|_{\mathcal{E}} \xrightarrow{\mathcal{S}} 0$. Also
\[
\|J_n^{(2)}\|_r = \|G_r(b_n)(\|b_n^1\| - \|b\|)f\|_r \leq \|G_r(b_n)\|_{r \to r}\|\|b_n^1\| - \|b\|f\|_r \leq m_\delta(1 + \delta)\|\|\|b_n^1\| - \|b\|f\|_r, \quad (f \in \mathcal{E}).
\]
Thus, $J_n^{(2)}|_{\mathcal{E}} \xrightarrow{\mathcal{S}} 0$. Since $\|J_n^{(2)}\|_{r \to r}, \|J_n^{(1)}\|_{r \to r} \leq m_\delta \delta$, we conclude that $T_r(b_n) \xrightarrow{\mathcal{S}} T_r(b)$. It is clear now that $Q_r(q, b_n) \xrightarrow{\mathcal{S}} Q_r(q, b)$.

The proof of (2) is completed. \hfill $\Box$

The proof of Theorem 4.4 is completed. \hfill $\Box$

Remarks. 1. The assertion (iv) of Theorem 4.4 holds for any \(\{b_n\} \subset C^1(\mathbb{R}^d, \mathbb{R}^d) \cap F_\delta^{1/2}, b_n \to b \mathcal{L}^d\) a.e., in particular, for the $b_n$’s given by (1), but not for $b_n$’s as in Theorem 3.11.

2. Theorem 4.3 allows us to move the proof of convergence in $C_\infty$ to $L^r$, $r > d - 1$, a space having much weaker topology (locally). The same idea has been realized in the proof of Theorem 3.11.

3. In comparison with the construction of a Feller semigroup in section 3.6 here the relative easiness of the construction stems from the fact that one already has the limiting object, i.e. $\Theta_p(\mu, b)$, $p > d - 1$, while in section 3.6 one has to work with Cauchy’s sequences.

4. In view of the remark 4 after Theorem 4.3 for $b \in [L^{d, \infty}]^d$ instead of “$m_\delta \delta < \frac{4}{(d-1)^2}$” it suffices to require
\[
m_\delta^\ast \Omega_d^{-\frac{1}{2}} \|b\|_{d, \infty} 2^{-1} \frac{\sqrt{\pi} \Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d-1}{2}\right)} < 1
\]
(then $\|T_r(b)\|_r < 1$ for all $r > d - 1$ sufficiently close to $d - 1$).

Appendix A. Lattices

Let $X$ be a set, let $\{f : X \to \mathbb{R}\}$ be the collection of all real-valued functions on $X$. 

Theorem B.1. Let $\sup K f$, $\inf K f$ be the supremum and infimum of the set of functions $f \in L$ and all constants $c \in [0, \infty[$.

a) $f + g, \inf \{f, g\} \in L$.

b) $cf, \inf \{c, f\} \in L$.

c) $f \geq g :\Rightarrow \ f - g \in L$.

Here $f \geq g :\Leftrightarrow f(x) \geq g(x)$ on $X$; $\inf \{f, g\}(x) = \inf \{f(x), g(x)\}$.

Claim A.1. $(f, g \in L$ and $g \geq 0) \Rightarrow \sup \{f, g\} \in L$. In particular, $f^+ \in L$, where $f^+ := \sup \{f, 0\}$, $f^- := f^+ - f$.

Proof. $f + g = \inf \{f, g\} + \sup \{f, g\}$, and so $\sup \{f, g\} = f + g - \inf \{f, g\}$. By c), $\sup \{f, g\} \in L$. By $g \geq 0$, $\sup \{f, g\} \in L^+ := L \cap \{f \geq 0\}$. Thus, $f^+ \in L^+$ and $L = L^+-L^+ := \{f-g|f,g \in L^+\}$.

Let $L_C \equiv L+iL := \{f = u+iv | u,v \in L, i = \sqrt{-1}, L \text{ is a lattice}\}$. An operator $K : L_C \to L_C$ is called positivity preserving if $K[L^+] \subset L^+$.

Claim A.2. If a linear operator $K : L_C \to L_C$ preserves positivity, then

$$|Kf| \leq K|f| \quad (f \in L_C).$$

Proof. Let $f = u + iv, u,v \in L$. Then $u \leq |u| \leq |f|$ and

$$Ku = K(u^+ - u^-) = Ku^+ - Ku^- \leq Ku^+ \leq K|u| \leq K|f|.$$ 

Set $Kf = \rho e^{i\phi}, \rho > 0, 0 \leq \phi < 2\pi$, and $e^{-i\phi}f = u_1 + iv_1, u_1,v_1 \in L$. Clearly, $|Kf| = \rho = e^{-i\phi}Kf = Ke^{-i\phi}f = Ku_1 + iKv_1 = Ku_1$ and $Ku_1 \leq K|u_1| \leq K|f|$.

Appendix B. Composition of Lipschitz functions and functions in Sobolev spaces

Theorem B.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function and $u \in W^{1,p}, p \geq 1$. If $f(u) \in L^p$, then $f(u) \in W^{1,p}$ and $L^d$ a.e. $\nabla f(u) = f'(u)\nabla u$.

See e.g. [21] Theorem 2.1.11.

Appendix C. Monotone Convergence Theorem for sesquilinear forms

Let $H$ be a (complex) Hilbert space with the inner product $\langle f, g \rangle$ and norm $||f|| = \langle f, f \rangle^{1/2}$. Let $\mathcal{T}$ denote a family of all closed, symmetric, non-negative, densely defined sesquilinear forms in $H$. If $t \in \mathcal{T}$, then there exist a unique self-adjoint operator $T \geq 0$ such that

$$t[u, v] = \langle Tu, v \rangle, \quad u \in D(T) \subset D(t), \quad v \in D(t).$$

Then

$$t[u, v] = \langle T^{1/2} u, T^{1/2} v \rangle, \quad D(T^{1/2}) = D(t).$$

In this case we say that $T$ is associated with $t$ and write $T \leftrightarrow t$ or/and $t = T_T$.

Let $a, b \in \mathcal{T}$ be such that $D(a) \cap D(b)$ is dense in $H$. Then $a + b \in \mathcal{T}$. Thus, for $a, b, a + b \in \mathcal{T}$ and $A \leftrightarrow a, B \leftrightarrow b$, we write $A \oplus B \leftrightarrow a + b$. $A \oplus B$ is called the form sum of $A$ and $B$. $A \oplus B$, a self-adjoint extension of the algebraic sum $A + B$, possesses some exclusive properties described below.
We endow $\mathcal{T}$ with a semi-order $\prec$:

$$a \prec b \iff D(a) \supset D(b) \text{ and } a[u] \leq b[u] \ (u \in D(b)).$$

Here and below $a[u] \equiv a[u, u]$. If $a, b \in \mathcal{T}$ and $A \leftrightarrow a, B \leftrightarrow b$, we write $A \leq B$ if and only if $a \prec b$. In this case $(\lambda + B)^{-1} \leq (\lambda + A)^{-1}$ $(\lambda > 0)$ in the sense that

$$\| (\lambda + B)^{-\frac{1}{2}} f \| \leq \| (\lambda + A)^{-\frac{1}{2}} f \| \ (f \in \mathcal{H}).$$

**Theorem C.1** (Convergence from below, see e.g. [3] Ch. VIII, sect. 3) or [3]). Let $\{a_n\}_{n=1}^\infty \subset \mathcal{T}$ be such that

$$a_1 \prec a_2 \prec \ldots .$$

Define $a$ by

$$a[u] := \lim_{n \to \infty} a_n[u], \quad D(a) := \left\{ u \in \bigcap_{n=1}^\infty D(a_n) \mid \text{the finite } \lim_{n \to \infty} a_n[u] \text{ exists} \right\}. $$

Suppose that $D(a)$ is dense in $\mathcal{H}$. Then $a \in \mathcal{T}$ and, for all $u, v \in D(a)$,

$$a_n[u, v] \to a[u, v].$$

Let $A \leftrightarrow a$ and $A_n \leftrightarrow a_n$. Then

$$(\lambda + A_n)^{-1} \overset{\lambda}{\to} (\lambda + A)^{-1} \quad (\text{Re } \lambda > 0),$$

$$(\lambda + A_n)^{\frac{1}{2}} u \overset{\lambda}{\to} (\lambda + A)^{\frac{1}{2}} u \quad (u \in D(a), \ \lambda > 0).$$

**Remark.** This theorem is extremely useful in constructing operator realizations of formal differential expressions. See e.g. section 1. Also, it is a proper tool for perturbation theory of self-adjoint operators.

**Appendix D. The criteria of Phillips and Stampacchia**

There is a characterization of Markov semigroups in terms of their generators.

Let $X$ be a set and $\mu$ a measure on $X$. Recall that a $C_0$ semigroup $T^t, t \geq 0,$ of contractions on $L^p = L^p(X, \mu), p \in [1, \infty[$, is called Markov if, for each $t > 0$,

$$T^t L^p_+ \subset L^p_+,$$

$$f \in L^p, |f| \leq 1 \Rightarrow |T^t f| \leq 1.$$  

Let $A$ denote the generator of $T^t$. Let us introduce the following conditions:

$$[Af, f^+] \geq 0 \quad (f \in D(A) \cap \text{Re } L^p) \quad (\text{R. Phillips}),$$

$$\text{Re } [Af, f - f_\lambda] \geq 0 \quad (f \in D(A)) \quad (\text{G. Stampacchia}).$$

Here $\text{sgn } z := z/|z|$ if $z \neq 0$, and $\text{sgn } 0 = 0$; $f^+ := f \vee 0$; $f_\lambda := (|f| \wedge 1)\text{sgn } f$; $[f, g] := \langle f, |g|^{p-1} \text{sgn } g \rangle \|g\|_p^{2-p}$, $f, g \in L^p$, the semi-inner product in $L^p$. Here and elsewhere $f \wedge g \equiv \inf\{f, g\}$, $f \vee g \equiv \sup\{f, g\}$. 

It follows from Proposition 1 and Proposition 2 below (see [17] Sect. 1) that
\[(i) + (ii) \iff (i') + (ii')\]

This equivalence is useful first of all for actual verification of (i), (ii) in the case when \(A\) is an operator realization of a formal partial differential expression.

**Proposition D.1.** Let \(e^{-tA}, t \geq 0\), be a contraction \(C_0\) semigroup in \(L^p\) such that \(e^{-tA}R_{L^p} \subset \text{Re } L^p\). Then
\[(i) \iff (i').\]

**Proposition D.2.** Let \(e^{-tA}, t \geq 0\), be a contraction \(C_0\) semigroup in \(L^p\). Then
\[\|e^{-tA}v\|_\infty \leq \|v\|_\infty \quad (v \in L^p \cap L^\infty; t \geq 0)\]
if and only if
\[\text{Re } [Af, f - f_\wedge] \geq 0 \quad (f \in \mathcal{D}(A)).\]

**Appendix E. Trotter’s Approximation Theorem**

Consider a sequence of \(C_0\) semigroups \(e^{-tA_k}\) on a (complex) Banach space \(Y\).

**Theorem E.1** (H.F. Trotter [8] Ch. IX, sect. 2). Let \(\sup_k \| (\mu + A_k)^{-1} \|_{Y \to Y} \leq \mu^{-1}, \mu > \omega, or\sup_k \|(z + A_k)^{-1} \|_{Y \to Y} \leq C|z|^{-1}, \text{Re } z > \omega, and let s-lim_{\mu \to \infty} \mu(\mu + A_k)^{-1} = 1 \text{ uniformly in } k. \text{Let} s-lim_k (z + A_k)^{-1}\) exist for some \(\zeta\) with \(\text{Re } \zeta > \omega. \text{Then there is a } C_0 \text{ semigroup } e^{-tA}\) such that
\[(z + A_k)^{-1} \xrightarrow{s} (z + A)^{-1} \quad \text{for every } \text{Re } z > \omega,\]
and
\[e^{-tA_k} \xrightarrow{s} e^{-tA}\]
uniformly in any finite interval of \(t \geq 0\).

**Appendix F. Hille’s Theorems on Pseudo-resolvents**

Let \(Y\) be a (complex) Banach space. A pseudo-resolvent \(R_\zeta\) is a function defined on a subset \(\mathcal{O}\) of the complex \(\zeta\)-plane with values in \(\mathcal{B}(Y)\) such that
\[R_\zeta - R_\eta = (\eta - \zeta)R_\zeta R_\eta, \quad \zeta, \eta \in \mathcal{O}.\]
Clearly, \(R_\zeta\) have common null-set.

**Theorem F.1.** If the null-set of \(R_\zeta\) is \(\{0\}\), then \(R_\zeta\) is the resolvent of a closed linear operator \(A\), the range of \(R_\zeta\) coincides with \(D(A)\), and \(A = R_\zeta^{-1} - \zeta.\)

**Proof.** Put \(A := R_\zeta^{-1} - \zeta.\) Since \(R_\zeta\) is closed, so is \(R_\zeta^{-1}\) and \(A\). A straightforward calculation shows that \((\zeta + A)R_\zeta f = f, f \in Y, \text{ and } R_\zeta(\zeta + A)g = g, g \in D(A),\) as needed. \(\square\)
**Theorem F.2.** If there exists a sequence of numbers \( \{\mu_k\} \subset \mathcal{O} \) such that \( \lim_{k} |\mu_k| = \infty \) and 
\[ \sup_k \|\mu_k R_{\mu_k}\|_{Y \to Y} < \infty, \]
then the set \( \{y \in Y : \lim_k \mu_k R_{\mu_k} y = y\} \) is contained in the closure of the range of \( R_{\mathcal{C}} \).

**Proof.** Indeed, let \( \lim_k \mu_k R_{\mu_k} y = y \). That is, for every \( \varepsilon > 0 \), there exists \( k \) such that \( \|y - \mu_k R_{\mu_k} y\| < \varepsilon \), so \( y \) belongs to the closure of the range of \( R_{\mathcal{C}} \). \( \square \)

See [7 Sect. 5.2], [23 Ch. VIII, sect. 4].

**Appendix G.** \( L^r \)-inequalities for symmetric Markov generators

Let \( X \) be a set and \( \mu \) a \( \sigma \)-finite measure on \( X \). Let \( T^t = e^{-tA} \), \( t \geq 0 \), be a symmetric Markov semigroup in \( L^2(X, \mu) \).

**Theorem G.1** ([17 Theorem 2.1]). If \( f \in D(A_r) \) for some \( r \in ]1, \infty[ \), then \( f(r) := |f|^\frac{1}{r} \text{sgn } f, |f|^\frac{1}{r} \in D(A_{\frac{1}{r}}) \) and

\[
\frac{4}{rr'} \|A_{\frac{1}{r}} f(r)\|^2 \leq \text{Re} \langle A_r f, |f|^{r-1} \text{sgn } f \rangle \leq \varkappa(r) \|A_{\frac{1}{r}} f(r)\|^2, \tag{i}
\]

where \( \varkappa(r) := \sup_{s \in [0,1]} \{(1 + s^\frac{1}{r})(1 + s^\frac{1}{r'}) \} \), \( r' = r(r-1) \);

\[
|\text{Im} \langle A_r f, |f|^{r-1} \text{sgn } f \rangle| \leq \frac{|r-2|}{2\sqrt{r-1}} \text{Re} \langle A_r f, |f|^{r-1} \text{sgn } f \rangle; \tag{ii}
\]

If \( 0 \leq f \in D(A_r) \), then

\[
\frac{4}{rr'} \|A_{\frac{1}{r}} f\|^2 \leq \langle A_r f, f^{r-1} \rangle \leq \|A_{\frac{1}{r}} f\|^2, \tag{iii}
\]

If \( r \in [2, \infty[ \) and \( f \in D(A) \cap L^\infty \), then \( |f|^\frac{1}{r}, f(r) \in D(A_{\frac{1}{r}}) \) and

\[
\frac{4}{rr'} \|A_{\frac{1}{r}} f(r)\|^2 \leq \text{Re} \langle A f, |f|^{r-1} \text{sgn } f \rangle \leq \varkappa(r) \|A_{\frac{1}{r}} f(r)\|^2, \tag{iv}
\]

\[
|\text{Im} \langle A f, |f|^{r-1} \text{sgn } f \rangle| \leq \frac{|r-2|}{2\sqrt{r-1}} \text{Re} \langle A f, |f|^{r-1} \text{sgn } f \rangle; \tag{v}
\]

If \( r \in [2, \infty[ \) and \( 0 \leq f \in D(A) \cap L^\infty \), then \( f^\frac{1}{r} \in D(A_{\frac{1}{r}}) \) and

\[
\frac{4}{rr'} \|A_{\frac{1}{r}} f^\frac{1}{r}\|^2 \leq \langle A f, f^{r-1} \rangle \leq \|A_{\frac{1}{r}} f^\frac{1}{r}\|^2. \tag{vi}
\]

(The proof of (iii) works in the assumptions of (vi); the proof of (i), (ii) works in the assumptions of (iv), (v).)

**Appendix H.** Extrapolation Theorem

**Theorem H.1** (T. Coulhon-Y. Raynaud). Let \( U^{t,\tau} : L^1 \cap L^\infty \to L^1 + L^\infty \) be a two-parameter evolution family of operators:

\[
U^{t,\tau} = U^{t,\tau} U^{\tau, s}, \quad 0 \leq s < \tau < t \leq \infty.
\]
Suppose that, for some $1 \leq p < q < r \leq \infty$, $\nu > 0$, $M_1$ and $M_2$, the inequalities

$$
\|U^{t,s}f\|_p \leq M_1 \|f\|_p \quad \text{and} \quad \|U^{t,s}f\|_r \leq M_2(t-s)^{-\nu}\|f\|_q
$$

are valid for all $(t,s)$ and $f \in L^1 \cap L^\infty$. Then

$$
\|U^{t,s}f\|_r \leq M(t-s)^{-\nu/(1-\delta)}\|f\|_p,
$$

where $\delta = \frac{r-1}{r-1 - \frac{q-1}{q}}$ and $M = 2^{\nu/(1-\delta)^2} M_1 M_2^{1/(1-\delta)}$.

**Proof.** Set $2t_s = t + s$. The hypotheses and Hölder’s inequality imply

$$
\|U^{t,s}f\|_r \leq M_2(t - t_s)^{-\nu}\|U^{t_s,s}f\|_q
$$

$$
\leq M_2(t - t_s)^{-\nu}\|U^{t_s,s}f\|_r \|U^{t_s,s}f\|_p^{1-\delta}
$$

$$
\leq M_2 M_1^{1-\delta}(t - t_s)^{-\nu}\|U^{t_s,s}f\|_r \|f\|_p^{1-\delta},
$$

and hence

$$(t - s)^{\nu/(1-\delta)}\|U^{t,s}f\|_r / \|f\|_p \leq M_2 M_1^{1-\delta} 2^{\nu/(1-\delta)} [(t_s - s)^{-\nu/(1-\delta)}\|U^{t_s,s}f\|_r / \|f\|_p]^{\delta}.$$ 

Setting $R_{2T} := \sup_{t,s \in [0,T]} [(t - s)^{\nu/(1-\delta)}\|U^{t,s}f\|_r / \|f\|_p]$, we obtain from the last inequality that $R_{2T} \leq M^{1-\delta}(R_T)^{\delta}$. But $R_T \leq R_{2T}$, and so $R_T \leq M$. \qed

**Corollary H.1.** Let $U^{t,s} : L^1 \cap L^\infty \to L^1 + L^\infty$ be an evolution family of operators. Suppose that, for some $1 \leq p < q < r \leq \infty$, $\nu > 0$, $M_1$ and $M_2$, the inequalities

$$
\|U^{t,s}f\|_r \leq M_1 \|f\|_r \quad \text{and} \quad \|U^{t,s}f\|_q \leq M_2(t-s)^{-\nu}\|f\|_p
$$

are valid for all $(t,s)$ and $f \in L^1 \cap L^\infty$. Then

$$
\|U^{t,s}f\|_r \leq M(t-s)^{-\nu/(1-\delta)}\|f\|_p,
$$

where $\delta = \frac{r-1}{r-1 - \frac{q-1}{q}}$ and $M = 2^{\nu/(1-\delta)^2} M_1 M_2^{1/(1-\delta)}$.

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