Ramanujan in Computing Technology*

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1 Introduction

This paper is a tribute to the genius of the legendary Indian mathematician Srinivasa Ramanujan (22 December 1887 - 26 April 1920) in the centenary year of his death. The life story of Ramanujan is so well known that it needs no elaboration not even a summarisation. In his short life period he made substantial contributions to mathematical analysis, number theory, infinite series, and continued fractions, including solutions to mathematical problems then considered unsolvable. Ramanujan independently compiled nearly 3,900 results in the form of identities and equations. Many were completely novel; his original and highly unconventional results, such as the Ramanujan prime, the Ramanujan theta function, partition formulae and mock theta functions, have opened entire new areas of work and inspired a vast amount of further research. Nearly all his claims have now been proven correct.

The focus of the paper is the increasing influence of the ideas propounded by Ramanujan in the development of computing technology. We shall discuss the application of certain infinite series discovered by Ramanujan in computing the value of the mathematical constant $\pi$. We shall also consider certain special graphs known as Ramanujan graphs and the reason for designating them as such. We shall examine how certain researchers are attempting to create an abstract machine which they call Ramanujan machine which is thought of as simulating the hypothesised thought process of Ramanujan. We shall also have a brief look at the applications of Ramanujan’s discoveries in signal processing.

2 Some milestones in the history of computation of $\pi$

The value of the ratio of the circumference of a circle to its diameter has been of interest to mankind since the beginning of civilisations. However, the modern notation for the value of the ratio of the circumference of a circle to its diameter, namely $\pi$, was first used in print in a work of an English mathematician William Jones (1675-1749) published in 1706. However, it became a widely accepted
notation only when Euler used it in his famous 1748 publication “Introductio in analysin infinitorum” (Introduction to the Analysis of the Infinite).

2.1 Ancient times

The Rhind Papyrus dated around 1650 BCE says: “Cut off 1/9 of a diameter and construct a square upon the remainder; this has the same area as the circle”. This gives a value of

\[4\left(\frac{8}{9}\right)^2 = 3.16049\]

to the ratio of the circumference of a circle to its diameter, which is fairly accurate. The Shulba Sutras composed during the period 800 - 200 BCE and considered to be appendices to the Vedas give several different approximations to the ratio the circumference of a circle to its diameter and among them the value closest to the actual value is 3.088 (see [5]). In the earliest existing Chinese mathematical texts, Zhoubi suanjing (The Mathematical Classic of the Zhou Gnomon), a book dated to the period of the Zhou dynasty (1046 - 256 BCE) and Jiuzhang suanshu (Nine Chapters on the Mathematical Art), the ratio of the circumference of the circle to its diameter was taken to be three (see [11]).

2.2 Geometric algorithms

![Archimedes' idea for computing the value of \(\pi\)](image)

Figure 1: Archimedes’ idea for computing the value of \(\pi\)

The first algorithm for calculating the value of the ratio of the circumference of a circle to its diameter was a geometrical approach using polygons, devised around 250 BCE by the Greek mathematician Archimedes. Archimedes computed upper and lower bounds of the value of this ratio by drawing a regular hexagon inside and outside a circle, and successively doubling the number of sides until he reached a 96-sided regular polygon. By calculating the perimeters of these polygons, he proved that the ratio lies between 223/71 and 22/7. Mathematicians using polygonal algorithms calculated 39 digits of the value of the ratio in 1630, a record only broken in 1699 when infinite series were used to reach 71 digits (see [3]).

2.3 Sangamagrama Madhava

Sangamagrama Madhava (c.1340 - c.1425), an Indian mathematician and astronomer considered the founder of the Kerala school of astronomy and mathematics, calculated the value of \(\pi\) correct to 11 decimal places as

\[
\pi = 3.14159265359.
\]
Madhava obtained this value probably by taking the first 21 terms in the following infinite series discovered by him (see [11]):

\[
\pi = \sqrt{12} \sum_{k=0}^{\infty} \frac{(-3)^{-k}}{2k + 1} = \sqrt{12} \left( 1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \cdots \right)
\]

2.4 William Shanks

The British amateur mathematician William Shanks spending over 20 years attempted to calculate \( \pi \) to 707 decimal places. He did calculate 707 digits, but only the first 527 were correct. This was accomplished in 1873 and this was the longest expansion of \( \pi \) until the advent of the electronic digital computer.
To compute the value of $\pi$ Shanks used the following result known as Machin formula (discovered by John Machin, an English astronomer, in 1706)

$$\frac{\pi}{4} = 4 \tan^{-1}\left(\frac{1}{5}\right) - \tan^{-1}\left(\frac{1}{239}\right),$$

and the Maclaurin’s series expansion

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

This formula has a significantly increased rate of convergence, which makes it a much more practical method of calculation and it remained as the primary tool of $\pi$ calculations for centuries (well into the computer era). Machin himself used this formula to compute $\pi$ to 100 decimal places.

### 2.5 Computer programmes

Shank’s computation is only child’s play in the computer era as illustrated by the fact that the following code written in the C language prints out as much as 800 digits of $\pi$ accurately (for a detailed analysis of why the program does as claimed, see [18]):

```c
int a=10000,b,c=2800,d,e,f[2801],g;
main()
{
  for( ;b-c;) f[b++]=a/5;
  for( ;d=0,g=c*2;c=14,printf("%.4d",e+d/a),e=d%a)
    for(b=c; d+=f[b]*a,f[b]=d%--g,d/=g--;--b;d*=b);
}
```

It is even simpler if we use a computer algebra system to get 800 digits of $\pi$. For example, in Maxima we need only issue the following commands to get 800 digits of $\pi$:

```maxima
fpprintprec:800$
set_display(ascii)$
bfloat(%pi);
```

The current world record for the number of calculated digits of $\pi$ is 50 trillion and it was created by Timothy Mullican on 29 January 2020 (see [39]).

### 2.6 Do we ever need all these digits?

The largest number of digits of $\pi$ that we will ever need is 42, at least for computing circumferences of circles (see [30]). To compute the circumference of the known universe with an error less than the diameter of a proton, we need only 42 digits of $\pi$, assuming that the diameter of the known universe is 93 billion light years and that the diameter of a proton is $1.6^{-15}$ metres. Thus, in the fifty trillion digits of $\pi$ computed for the current record, all digits beyond the 42nd have no practical value. Here are all the digits of $\pi$ we will ever need:

$$\pi = 3.141592653589793238462643383279502884197169.$$
3 Ramanujan’s series for computing $\pi$

In the long journey covering the milestones in the computation of digits of $\pi$, Srinivasa Ramanujan’s name appears sometime around the late 1970’s and early 1980’s and his name appears via a much celebrated infinite series known as the “Ramanujan series”.

3.1 Ramanujan’s series

In 1903, or perhaps earlier, while in school, Ramanujan began to record his mathematical discoveries in notebooks. He continued this practice till his departure to England in 1914. There are three such notebooks the originals of which are now preserved in Madras University, Chennai. A fourth notebook,
which had been thought to have been lost and hence referred to as “The Lost Notebook” was later unearthed in 1974. It turned out to be not a notebook but a collection of loose sheets of paper. In the last page of the third of these notebooks Ramanujan listed a large number of series expansions for $\frac{1}{\pi}$ (see Figure 4). After his arrival in England, Ramanujan published a paper in 1914 (see [26]) containing all these expansions for $\frac{1}{\pi}$ (see Figure 5). Interestingly all of them remained unproven for nearly half a century.

The following is the series given by Ramanujan and used by Gosper to compute the digits of $\pi$:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{98} \sum_{k=0}^{\infty} \frac{(4k)! 26390k + 1103}{k!^4 396^{4k}}$$

### 3.2 Using Ramanujan’s series for $\pi$ computation

In 1977, Bill Gosper took the last of Ramanujan’s series from the list referred to, and used it to compute a record number of digits of $\pi$. There soon followed other computations, all based directly on Ramanujan’s idea.

Ralph William Gosper Jr (born April 26, 1943), known as Bill Gosper, is an American mathematician and programmer. Gosper was one among the first few persons to realize the possibilities of symbolic computations on a computer as a mathematics research tool whereas computer methods were previously limited to purely numerical methods. He made major contributions to Macsyma, Project MAC’s computer algebra system.

In November, 1985, Gosper employed a lisp machine at Symbolics and Ramanujan’s series to calculate 17,526,100 digits of $\pi$, which at that time was a world record (see [4]). It was only in 1987, the Ramanujan series for $1/\pi$ was finally proved and the proof was accomplished by Jonathan and Peter Borwein (see [4]).

### 3.3 Comments on the computations

There were a few interesting things about Gosper’s computation. To put these computations in perspective, it is better to quote verbatim from a reference
First, when he decided to use that particular formula, there was no proof that it actually converged to \( \pi \)! Ramanujan never gave the math behind his work, and the Borweins had not yet been able to prove it, because there was some very heavy math that needed to be worked through. It appears that Ramanujan simply observed the equations were converging to the 1103 in the formula, and then assumed it must actually be 1103. (Ramanujan was not known for rigor in his math, or for providing any proofs or intermediate math in his formulas.) The math of the Borwein’s proof was such that after he had computed 10 million digits, and verified them against a known calculation, his computation became part of the proof. Basically it was like, if you have two integers differing by less than one, then they have to be the same integer.

The second interesting thing is that he chose to use continued fractions to do his calculations. Most calculations before and since were done by direct calculation to the desired precision. Before you did any calculations, you had to decide how many digits you wanted, and later if you wanted more, you had to start over from the beginning. By using continued fractions, and a novel coding style, he was able to make his resumable. He could add more digits any time he felt like it and had the spare time. This was a major breakthrough at the time, because all previous efforts required starting over from the beginning if you wanted more.

The third interesting thing about his calculations was the other reason he chose to use infinite simple continued fractions. It is still not known whether \( \pi \) has any ‘structure’ or patterns to it. It is known that it’s irrational and transcendental, but it still might have some pattern to it that would allow us more easily calculate its digits. We just don’t know. And patterns show up more readily as a continued fraction rather than in some arbitrary base that we humans call ‘base 10’. As an example, ‘e’ and the square root of two both have very simple, obvious patterns in their continued fractions, even though they appear to be non-repeating in base 10.

4 Chudnovsky algorithm

4.1 Chudnovski’s formula for computing \( \pi \)

Further developing the ideas on which the Ramanujan series was proved, Chudnovsky brothers (David Volfovich Chudnovsky (born 1947) and Gregory Volfovich Chudnovsky (born 1952)) developed another series for \( 1/\pi \):

\[
\frac{1}{\pi} = 12 \sum_{q=0}^{\infty} \frac{(-1)^q (6q)! (545140134q + 13591409)}{(3q)!(q!)^3 (640320)^{3q+3/2}}
\]

This formula was used in the world record calculations of 2.7 trillion digits of \( \pi \) in December 2009, 10 trillion digits in October 2011, 22.4 trillion digits in November 2016, 31.4 trillion digits in September 2018 - January 2019 (see [9]), and 50 trillion digits on 29 January 2020 (see [39]).
The computer algebra system *Mathematica* widely used in many scientific, engineering, mathematical, and computing fields uses this formula to compute $\pi$ (see [3], [38]).

4.2 Chudnosky brothers

![Figure 7: Chudnovski brothers](image)

David Volfovich Chudnovsky and Gregory Volfovich Chudnovsky are American mathematicians and engineers known for their world-record mathematical calculations and developing the Chudnovsky algorithm used to calculate the digits of $\pi$ with extreme precision. They were born in Kiev in Ukraine which at the time of their births was part of the erstwhile Soviet Union. In part to avoid religious persecution and in part to seek better medical care for Gregory, who had been diagnosed with myasthenia gravis, a neuromuscular disease, the Chudnovsky family emigrated to the United States in 1977-78 and settled in New York. Gregory works on problems of diophantine geometry and transcendence theory, high-performance computing and computing architecture, and mathematical physics and its applications. A 1992 article in The New Yorker quoted the opinion of several mathematicians that Gregory Chudnovsky was one of the world’s best living mathematicians (see [24]).

4.3 The computational procedure

For a high performance iterative implementation, Chudnovski’s series may have to be put in different forms. We present below one of these approaches (see [37]). In this approach, firstly, the series is recast in the following form.

\[
\frac{426880\sqrt{10005}}{\pi} = \sum_{q=0}^{\infty} \frac{(6q)!(545140134q + 13591409)}{(3q)!(q!)^3} (-262537412640768000)^q
\]

There are three big integer terms in the series, namely, a multinomial term $M_q$, a linear term $L_q$, and an exponential term $X_q$ as shown below:

\[
\pi = C \left( \sum_{q=0}^{\infty} \frac{M_q \cdot L_q}{X_q} \right)^{-1}
\]
where

\[ C = 426880\sqrt{10005} \]
\[ M_q = \frac{(6q)!}{(3q)!((q)!^3}} \]
\[ L_q = 545140134q + 13591409 \]
\[ X_q = (-26537412640768000)^q. \]

The terms \( M_q, L_q, \) and \( X_q \) can be computed by using the following recurrence relations

\[ L_{q+1} = L_q + 545140134 \]
\[ X_{q+1} = X_q \cdot (-26537412640768000) \]
\[ M_{q+1} = M_q \cdot \left( \frac{(12q+2)(12q+6)(12q+10)}{(q+1)^3} \right) \]

with

\[ L_0 = 13591409, \quad X_0 = 1, \quad M_0 = 1. \]

The computation of \( M_q \) can be further optimized by introducing an additional term \( K_q \) as follows:

\[ K_{q+1} = K_q + 12 \]
\[ M_{q+1} = M_q \cdot \left( \frac{K_q^3 - 16K_q}{(q+1)^3} \right) \]

with

\[ K_0 = 6, \quad M_0 = 1. \]

5 Ramanujan graphs

Ramanujan graphs are graphs having a very special property. The terminology “Ramanujan graph” was first coined by Lubotzky, Phillips and Sarnak in a paper published in 1988 (see [17]). Though Ramanujan had no special interest in graph theory and these graphs are not due to Ramanujan, they are named after Srinivasa Ramanujan because the Ramanujan-Petersson conjecture is used in proving that an important class of graphs have the special property. First let us see what are Ramanujan graphs.

Let \( G = (V, E) \) be an undirected graph, \( V \) being the set of vertices and \( E \) the set of edges. The degree of a vertex \( v \) in \( G \) is the number of edges connecting it to other vertices in the graph. We say that a graph \( G \) is \( k \)-regular if every vertex \( v \) of \( G \) has degree \( k \). A path in a graph is a sequence of edges which joins a sequence of vertices. A graph is connected if there is a path between every pair of vertices.

We can associate a matrix called an adjacency matrix with every graph \( G \). Let \( V = \{v_1, \ldots, v_n\} \). The adjacency matrix of \( G \) is the \( n \times n \) matrix \( A = [a_{ij}] \) where

\[ a_{ij} = \begin{cases} 1 & \text{if an edge joining } v_i \text{ and } v_j \text{ exists} \\ 0 & \text{otherwise.} \end{cases} \]

Since \( G \) is undirected, \( A \) is a symmetric matrix and all of its eigenvalues are real. Moreover, if \( G \) is \( k \)-regular, the largest eigenvalue of \( A \) is \( k \).
5.1 Definition
A $k$-regular connected graph $G$ is called a Ramanujan graph if the second largest (in absolute value) eigenvalue of the adjacency matrix $A$ of $G$, denoted by $\lambda(G)$, satisfies the property
\[
\lambda(G) \leq 2\sqrt{k-1}.
\]
The graphs shown in Figure 8 are examples of Ramanujan graphs.

5.2 Ramanujan conjecture
Ramanujan studied the function $\Delta(z)$ defined by
\[
\Delta(z) = q \prod_{n>0} (1 - q^n)^{24}
= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - \cdots
= \sum_{n>0} \tau(n)q^n
\]
where $q = e^{2\pi iz}$. The function $\tau(n)$ is called the Ramanujan $\tau$-function. Ramanujan computed the values of $\tau(p)$ when $p$ is assigned the first 30 prime numbers as values. These values are shown in Table 1.

Based on these values of $\tau(p)$ Ramanujan made the following observations regarding the properties of the function $\tau(n)$ (see [27]):

1. If $m$ and $n$ are relatively prime, then
\[
\tau(mn) = \tau(m)\tau(n).
\]

2. For a prime number $p$ and an integer $j$ we have:
\[
\tau(p^{j+1}) = \tau(p^j)\tau(p) - p^{11}\tau(p^{j-1}).
\]

3. For a prime number $p$ we have:
\[
|\tau(p)| \leq 2p^{11}.
\]

---

1The terminology Ramanujan Graph was suggested by Peter Sarnak (see [34]).
Table 1: Values of $\tau(p)$ calculated by Ramanujan

| $p$ | $\tau(p)$ | $p$ | $\tau(p)$ | $p$ | $\tau(p)$ |
|-----|-----------|-----|-----------|-----|-----------|
| 2   | -24       | 31  | -52,843,168 | 73  | 1,463,791,322 |
| 3   | 252       | 37  | -182,213,314 | 79  | 38,116,845,680 |
| 5   | 4830      | 41  | 308,120,442  | 83  | -29,335,099,668 |
| 7   | -16744    | 43  | -17,125,708   | 89  | -24,992,917,110 |
| 11  | 534,612   | 47  | 2,687,48,496  | 97  | 75,013,568,546  |
| 13  | -577,738  | 53  | -1,596,055,698 | 101 | 81,742,959,102 |
| 17  | -6,905,934 | 59  | -5,189,203,740 | 103 | -225,755,128,648 |
| 19  | 10,661,420 | 61  | 6,956,478,662  | 107 | 90,241,258,356  |
| 23  | 18,643,272 | 67  | -15,481,826,884 | 109 | 73,482,676,310 |
| 29  | 128,406,630 | 71  | 9,791,485,272  | 113 | -85,146,862,638 |

Ramanujan did not prove these results. In 1917, L. Mordell proved the first two observations using techniques from complex analysis. The third observation remained unproven for a long time and it came to be referred to as the *Ramanujan conjecture* (see [21]). It was finally proven in 1974 by Deligne[2].

Let $p, q$ be prime numbers and $x_1, x_2, x_3, x_4$ be integers. Consider the expression

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + (2qx_2)^2 + (2qx_3)^2 + (2qx_4)^2$$

Given a nonnegative integer $n$, let $r_Q(n)$ denote the number of representations of $n$ in the form $Q(x_1, x_2, x_3, x_4)$. There is no explicit formula for $r_Q(n)$. But using the Ramanujan conjecture we get a good approximation for $r_Q(n)$ in the form

$$r_Q(p^k) = C(p^k) + O(p^{k(1/2+\epsilon)})$$

where

$$C(p^k) = c p^{k+1} - 1 \over p - 1$$

for some constant $c$, and $O$ denote the well known Big-O notation.

6 Construction of Ramanujan graph

6.1 Cayley graph

Let $G$ be a group and $S$ be some symmetric set of elements of $G$ (that is, $S$ is a subset of $G$ such that if $x \in S$ then $x^{-1} \in S$). Then the Cayley graph generated by $(G, S)$ is the graph whose vertices are the elements of $G$, and given $g, h \in G$, the edge $(g, h)$ exists if and only if there exists some $s \in S$ such that $gs = h$.

As an example, consider the group whose set of elements is

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

[2] Pierre Deligne is a Fields medal winning Belgian mathematician best known for his work on the Weil conjectures. For a layman’s introduction to Deligne’s work, one may read [10].
Figure 9: Cayley graphs with the group \( \mathbb{Z}_6 \) and the binary operation is addition modulo 6. If we choose \( S = \{-1, 1\} \) we get the graph in Figure 9(a) and if we set \( S = \{-2, 2\} \) we get the graph in Figure 9(b).

6.2 The groups \( \text{PGL}(2, q) \) and \( \text{PSL}(2, q) \)

Let \( q \) be a prime number. We denote by \( \mathbb{Z}_q \) the field whose set of elements is the set \( \{0, 1, 2, \ldots, q - 1\} \), the addition operation is addition modulo \( q \) and multiplication operation is multiplication modulo \( q \). Let \( \mathbb{Z}_q^2 \) be the 2-dimensional vector space over \( \mathbb{Z}_q \) and \( \text{GL}(2, q) \) be the set of nonsingular linear transformations of \( \mathbb{Z}_q^2 \) (GL for General Linear). Each such linear transformation can be represented by a square matrix of order 2 whose elements are in \( \mathbb{Z}_q \) and whose determinant is nonzero (modulo \( q \)). The set \( \text{GL}(2, q) \) is a group under operation of multiplication of matrices.

For \( A, B \in \text{GL}(2, q) \), we write \( A \sim B \) if there is a scalar matrix \( C \in \text{GL}(2, q) \) such that \( A = BC \). (A scalar matrix is a diagonal matrix with equal-valued elements along the diagonal.) This is an equivalence relation in \( \text{GL}(2, q) \) and the set of equivalence classes is denoted by \( \text{PGL}(2, q) \) (PGL for Projective General Linear). It can be easily seen that \( \text{PGL}(2, q) \) is also a group under an obvious multiplication operation.

The subset of \( \text{PGL}(2, q) \) consisting of matrices with determinant equal to 1 is also a group and is denoted by \( \text{PSL}(2, q) \) (PSL for Projective Special Linear).

6.3 A set for generating a Cayley graph of \( \text{PGL}(2, q) \) and \( \text{PSL}(2, q) \)

Let us now assume that \( p, q \) satisfy the conditions

\[
p \equiv 1 \pmod{4}, \quad q \equiv 1 \pmod{4}.
\]

The second of these conditions imply that there is an integer \( i \) such that

\[
i^2 \equiv -1 \pmod{q}.
\]

We choose such an \( i \) and keep it fixed.

Next we consider the equation:

\[
a_0^2 + a_1^2 + a_2^2 + a_3^2 = p.
\]
The condition \( p \equiv 1 \pmod{4} \) implies that there are precisely \( p + 1 \) solutions to this equation with \( a_0 > 0 \) and \( a_1, a_2, a_3 \) even. We now form the set \( S \) of \( p + 1 \) matrices with the following form using these \( p + 1 \) solutions:

\[
\alpha = \begin{bmatrix}
a_0 + ia_1 & a_2 + ia_3 \\
-a_2 + ia_3 & a_0 - ia_1
\end{bmatrix}
\]

It can be seen that \( S \) is a symmetrical set in \( PGL(2, q) \).

### 6.4 The graph \( X^{p,q} \)

Let \( m, n \) be positive integers. We say that \( m \) is a quadratic residue \( \pmod{n} \) if there is an integer \( x \) such that

\[
x^2 \equiv m \pmod{n}.
\]

If there is no such integer \( x \) we say that \( m \) is a quadratic nonresidue \( \pmod{n} \). For example, \( 1, 3, 4, 5, 9 \) are quadratic residues \( \pmod{11} \) and \( 2, 6, 7, 8, 10 \) are quadratic nonresidues \( \pmod{11} \).

The graph \( X^{p,q} \) is defined as follows:

- If \( p \) is a quadratic residue \( \pmod{q} \), then the Cayley graph of \( PSL(2, q) \) with \( S \) as described above as the generating set is defined as \( X^{p,q} \). In this case, we have a \((p + 1)\)-regular graph on \( q(q^2 - 1)/2 \) vertices.

- If \( p \) is a quadratic non-residue \( \pmod{q} \), then the Cayley graph of \( PGL(2, q) \) with \( S \) as described above as the generating set is defined as \( X^{p,q} \). This gives a \((p + 1)\)-regular graph on \( q(q^2 - 1) \) vertices.

### 6.5 \( X^{p,q} \) is a Ramanujan graph

A. Lubotzky, R. Phillips and P. Sarnak, in 1988 (see [17]) proved that \( X^{p,q} \) is a Ramanujan graph, that is,

\[
\lambda(X^{p,q}) \leq 2\sqrt{p + 1}.
\]

The proof of this required an application of the Ramanujan conjecture. So, A. Lubotzky, R. Phillips and P. Sarnak named any graph having such a property as a Ramanujan graph.

### 6.6 Numerical example

First of all we have to choose two prime numbers satisfying the following conditions:

- \( p \equiv 1 \pmod{4} \)
- \( q \equiv 1 \pmod{4} \)
• $p$ is a quadratic residue mod $q$, that is, there exists an integer $x$ such that $x^2 \equiv p \pmod{q}$.

It can be shown that smallest such prime numbers are $p = 5$ and $q = 29$. Note that $11^2 \equiv 5 \pmod{29}$.

Next we consider the diophantine equation

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 = 5.$$ 

The $p + 1 = 6$ solutions to this equation with $a_0 > 0$ and $a_1, a_2, a_3$ even are

$$\{(1, \pm 2, 0, 0), (1, 0, \pm 2, 0), (1, 0, 0, \pm 2)\}.$$ 

We have to choose an integer $i$ such that $i^2 \equiv -1 \pmod{q}$. We may choose $i = 12$ because $12^2 \equiv -1 \pmod{29}$. (A second possible value of $i$ is 17.)

The elements of the generating set $S$ of $PGL(2, 29)$ are given by

$$\begin{bmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - ia_1 \end{bmatrix}.$$ 

Substituting for $a_0, a_1, a_2, a_3$ and $i$, and making computations modulo 29 we get the following generating set for $PGL(2, 29)$:

$$S = \left\{ \begin{bmatrix} 25 & 0 \\ 0 & 6 \end{bmatrix}, \begin{bmatrix} 6 & 0 \\ 0 & 25 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 27 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 27 \\ 24 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 24 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 1 & 24 \end{bmatrix} \right\}.$$ 

To get a generating set for $PSL(2, 29)$, we need a set of matrices whose determinants are equal to 1 modulo 29. To get such a set we have to multiply the elements of the set $S$ by $\frac{1}{\sqrt{p}}$ (mod $q$. There are two values for $\sqrt{5}$ (mod 29, namely, 11 and 18. We choosing 11 arbitrarily, we have

$$\frac{1}{\sqrt{5}} \equiv \frac{1}{\sqrt{11}} \equiv 8 \pmod{29}.$$ 

Multiplying the elements in $S$ by 8 modulo 29 we get the following generating set for $PSL(2, 29)$.

$$S' = \left\{ \begin{bmatrix} 26 & 0 \\ 0 & 19 \end{bmatrix}, \begin{bmatrix} 19 & 0 \\ 0 & 26 \end{bmatrix}, \begin{bmatrix} 8 & 16 \\ 13 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 13 \\ 16 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 18 \\ 18 & 8 \end{bmatrix}, \begin{bmatrix} 8 & 11 \\ 11 & 8 \end{bmatrix} \right\}.$$ 

The number of elements in the set $PSL(5, 29)$ is

$$\frac{1}{2}q(q^2 - 1) = \frac{1}{2} \times 29(29^2 - 1) = 12180.$$ 

Thus the Ramanujan graph $X^{5, 29}$ is a 6-regular graph with 12180 vertices!

7 Applications of Ramanujan graphs

Most applications of Ramanujan graphs are a consequence of the fact that Ramanujan graphs are good examples of a certain class of graphs known as expander graphs. So we give here a quick review of the concept of expander graphs.
7.1 Expander graphs

Let $X = (V,E)$ be an undirected connected graph where $V$ is the set of vertices and $E$ is the set of edges. Let $F$ be a set of vertices in $X$. The boundary $\partial F$ is the set of edges connecting vertices in $F$ to vertices in $V - F$. For example, in the graph shown in Figure 10 let $F$ be the vertices surrounded by squares. The set of edges shown by thick line segments is the boundary $\partial F$ of $F$.

The expanding constant, or isoperimetric constant of $X$, is

$$h(X) = \inf \left\{ \frac{|\partial F|}{\min\{|F|,|V - F|\}} : F \subseteq V, \ 0 < |F| < +\infty \right\}.$$ 

If we view $X$ as a network transmitting information (where information retained by some vertex propagates, say in 1 unit of time, to neighboring vertices), then $h(X)$ measures the “quality” of $X$ as a network: if $h(X)$ is large, information propagates well. For example, if $X = K_n$ is the highly connected complete graph on $n$ vertices then $h(X) \sim \frac{n}{2}$ which grows proportionally with the number of edges. On the other hand if $X = C_n$ is the minimally connected cyclic graph on $n$ vertices then $H(X) \leq \frac{2}{n}$ which tends to zero as the $n$ increases to infinity. Thus $h(X)$ is a measure of the connectivity of $X$ as a network.

An expander family of graphs is defined as follows.

Let $(X_m)_{m \geq 1}$ be a family of graphs $X_m = (V_m,E_m)$ indexed by $m \in \mathbb{N}$ (set of natural numbers). Furthermore, fix $k \geq 2$. Such a family $(X_m)_{m \geq 1}$ of finite, connected, $k$-regular graphs is a family of expanders if $|V_m| \to +\infty$ for $m \to +\infty$, and if there exists $\epsilon > 0$, such that $h(X_m) \geq \epsilon$ for every $m \geq 1$.

7.2 Network theory

Consider the construction of a new telephone network and we want the network to have a high degree of connectivity. However, laying lines will be expensive, so we want to achieve this high degree of connectivity using as few edges in the network as possible. This problem can be modeled as one related to expander graphs. An expander graph is a sparse graph (a graph having only a few edges) that has strong connectivity properties, quantified using vertex, edge or spectral
expansion. Ramanujan graphs have applications in the construction and study
of expander graphs.

7.3 Pseudo-random number generators

The following text summarised from a patent application publication (see [15])
clearly explains the application of Ramanujan graphs in the design of pseudo-
random number generators.

Pseudorandom numbers may be generated using expander graphs. Expander
graphs are a collection of vertices that are interconnected via edges. Generally, a
walk around an expander graph is determined responsive to an input seed, and
a pseudorandom number is produced based on vertex names. Generally, any
type of expander graph may be employed to generate pseudorandom numbers.
Expander graphs are usually characterized as having a property that enables
them to grow quickly from a given vertex to its neighbors and onward to other
vertices. An example of a family of graphs that are considered to have good
expansion properties are the so-called Ramanujan graphs. Expander graphs that
are so-called k-regular graphs are particularly amenable for use in generating
pseudorandom numbers. These k-regular graphs are graphs that have the same
number of edges emanating from each vertex. The k-regular Ramanujan graphs
are particularly amenable for use in generating pseudorandom numbers.

7.4 Construction of hash functions

Ramanujan graphs have also been used in constructing secure cryptographic
hash functions. If finding cycles in such graphs is hard, then these hash func-
tions are provably collision resistant. As an example, there is a specific family
of optimal expander graphs for provable collision resistant hash function con-
structions, namely, the family of Ramanujan graphs constructed by Pizer (see
[14]).

8 Ramanujan machines

The Ramanujan machine is not actually a machine; it is only a concept. It
exists as a network of computers running algorithms and the algorithms outputs
conjectures! In his published papers and in his famous Note Books, Ramanujan
has stated a large number of results and formulas in the form of conjectures. Most
of these conjectures have been proved true by later mathematicians. Because of
his propensity to produce conjectures as if from thin air, Ramanujan has been
sometimes called the “conjecture machine” (see [40]). So it is only appropriate
that a machine, whether actual or conceptual, that produces conjectures be
called the “Ramanujan machine”.

Presently what the algorithms do is to come up with probable infinite con-
tinued fraction expansions of the constants e and π. The creators of the machine
claim that the algorithm have already generated several such expansions which
are conjectured to be true. So, we begin our discussion by explaining what a
continued fraction is and then having a look at some of the continued frac-
tions conjectured by Ramanujan. Finally we shall state some of the continued
fractions conjectured by the Ramanujan machine.
9 Continued fractions

9.1 Definitions

An expression of the following form is called a continued fraction:

\[
a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}\]

It is often expressed in the following form:

\[
a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots}}}\]

Here \(a_0, a_1, a_2, a_3, \cdots\) and \(b_1, b_2, b_3, \cdots\) may in general be integers, real numbers, complex numbers, or functions.

If \(a_1, a_2, a_3, \cdots\) are positive integers and \(b_1 = b_2 = b_3 = \cdots = 1\) and then it is called a simple continued fraction. In a simple continued fraction \(a_0\) may be any integer negative, zero or positive.

A continued fraction may have a finite or an infinite number of terms. For example, we have

\[
\frac{100}{2.54} = 39 + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2} + \frac{1}{\frac{1}{2}}}}}}}}}}
\]

Also, it can be shown that the golden ratio \(\phi\) has the following infinite simple
continued fraction expansion:
\[ \phi = \frac{\sqrt{5} + 1}{2} = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \cdots}}} \]

9.2 Continued fraction expressions for \( \pi \) and \( e \)

Several continued fraction expressions for \( \pi \) are well known. For example, we have the following simple continued fraction expression for \( \pi \):
\[ \pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292 + \frac{1}{1 + \cdots}}}} \]

Unfortunately, the numbers 7, 15, 1, 292, 1, \ldots have no obvious pattern! But there are continued fraction expressions which are not simple in which the numbers have simple patterns. The oldest such expression is the following one due to William Brouncker (ca. 1660’s):
\[ \pi = \frac{4}{1 + \frac{1^2}{1 + \frac{2^2}{1 + \frac{3^2}{1 + \cdots}}}} \]

There is an infinite continued fraction expression for \( e \) discovered by Euler (see [8, 22]), namely
\[ e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + 6 + \cdots}}}}} \]

10 Ramanujan and continued fractions

Ramanujan in his life time published only one continued fraction, the famous Rogers-Ramanujan continued fraction. But his notebooks and the letters he sent to G. H. Hardy before his departure to England contain enough materiel indicating Ramanujan’s early interest in continued fractions. These provide ample testimony to the fact that he had studied and used continued fractions extensively (see [1]).

10.1 Some early results

In his first letter to Hardy dated 16 January 1913 Ramanujan has stated the following continued fractions without proof (see [6]). These also appear in his Note Book 2.

1. We have
\[ \left( \frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+2}{4}\right)} \right)^2 = \frac{4}{x} + \frac{1^2}{2x} + \frac{3^2}{2x} + \frac{5^2}{2x} + \frac{7^2}{2x} + \cdots \]

2. If \( u = x + m + n + 1 \) and
\[ P = \frac{\Gamma\left(\frac{1}{4}u\right)\Gamma\left(\frac{1}{4}(u - 2n)\right)\Gamma\left(\frac{1}{4}(u - 2(m - 1))\right)\Gamma\left(\frac{1}{4}(u - 2(m + n - 1))\right)}{\Gamma\left(\frac{1}{4}(u + 2)\right)\Gamma\left(\frac{1}{4}(u - 2m)\right)\Gamma\left(\frac{1}{4}(u - 2(n - 1))\right)\Gamma\left(\frac{1}{4}(u - 2(m + n))\right)} \]
then
\[
\frac{1 - P}{1 + P} = \frac{m}{x} + \frac{1^2 - n^2}{x} + \frac{2^2 - m^2}{x} + \frac{3^2 - n^2}{x} + \frac{4^2 - m^2}{x} + \cdots
\]

3. If
\[
z = 1 + \left(\frac{1}{2}\right)^2 x + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \cdots
\]
\[
y = \frac{\pi}{2} 1 + \left(\frac{1}{2}\right)^2 (1 - x) + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 (1 - x)^2 \cdots
\]
then
\[
\frac{1}{(1 + a^2) \cosh y} + \frac{1}{(1 + 9a^2) \cosh 3y} + \frac{1}{(1 + 25a^2) \cosh 5y} + \cdots
\]
\[
= \frac{1}{2} z \sqrt{x} + \frac{(az)^2}{1} + \frac{(2az)^2}{1} + \frac{(3az)^2}{1} + \frac{(4az)^2}{1} + \cdots
\]
where \(a\) is any quantity.

4. If
\[
u = \frac{x}{1 + \frac{x^5}{1 + \frac{x^{10}}{1 + \frac{x^{15}}{1 + \cdots}}}}
\]
\[
u = \frac{\sqrt{x}}{1 + \frac{x^2}{1 + \frac{x^3}{1 + \cdots}}}
\]
then
\[
v^5 = u \cdot \frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}.
\]

5. We have
\[
\frac{1}{1} + \frac{e^{-2\pi}}{1} + \frac{e^{-4\pi}}{1} + \frac{e^{-6\pi}}{1} + \cdots = \left(\sqrt{\frac{5 + \sqrt{5}}{2}} - \sqrt{\frac{5 - 1}{2}}\right) \sqrt{e^{2\pi}}.
\]

10.2 Rogers-Ramanujan continued fraction

The Rogers-Ramanujan continued fraction was first discovered by Rogers and published with proof in 1888 and then rediscovered independently and published without proof by Ramanujan in 1914. If we write
\[
G(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}
\]
\[
H(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(1 - q)(1 - q^2) \cdots (1 - q^n)}
\]
then the following continued fraction is known as the Rogers-Ramanujan continued fraction: can

\[ R(q) = q^{1/5} \frac{H(q)}{G(q)} \]

\[ = q^{1/5} \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ldots}}} \]

In his very first letter to Hardy, Ramanujan has stated several properties of the function \( R(q) \) (see [2]).

### 11 Conjectures by Ramanujan machine

The Ramanujan machine was initially designed to output conjectures regarding continued fraction expressions for the mathematical constants \( \pi \) and \( e \). The machine has produced several such conjectures and most of the early ones have been proved to be correct. The capability of the machine has now been expanded to generate such expressions for other constants like the Catalan constant, values of the Reimann zeta function, etc.

“Fundamental constants like \( e \) and \( \pi \) are ubiquitous in diverse fields of science, including physics, biology, chemistry, geometry, and abstract mathematics. Nevertheless, for centuries new mathematical formulas relating fundamental constants are scarce and are usually discovered sporadically by mathematical intuition or ingenuity.” (see [25])

1. Conjectures for \( \pi \)

   The following is a conjecture involving \( \pi \) discovered by the ramanujan machine and which has been proved to be true.

   \[
   \frac{4}{3\pi - 8} = 3 + \frac{1}{6} + \frac{-2 \cdot 3}{9} + \frac{-3 \cdot 5}{12} + \frac{-4 \cdot 7}{15} + \ldots
   \]

2. Conjectures for \( e \)

   Here is an elegant continued fraction representation for \( e \) discovered by the Ramanujan machine.

   \[
   e = 3 + \frac{-1}{4} + \frac{-2}{5} + \frac{-3}{6} + \frac{-4}{7} + \ldots
   \]

3. Conjectures for \( \log 2 \):

   The following conjecture involving \( \log 2 \) was discovered after 1 May 2020 and has not yet been proven.

   \[
   \frac{1}{1 - \log 2} = a_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots
   \]

   \[
   = 4 + \frac{-8}{14} + \frac{-72}{30} + \ldots
   \]

   where

   \[
   a_n = 3n^2 + 7n + 4, \quad b_n = -2n^2(n + 1)^2.
   \]
4. Conjectures on Catalan’s constant

The Catalan’s constant $G$ is defined by

$$G = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

It is not known whether $G$ is irrational, let alone transcendental. There
is Ramanujan connection to this constant. Ramanujan has extensively
studied the following inverse tangent integral function

$$Ti(x) = \int_0^x \frac{\tan^{-1} x}{x} dx.$$ 

It can be shown that $G = Ti(1)$. The Ramanujan machine has produced
several conjectures regarding $G$ most of which have not yet been proven.
The following is a typical conjecture on $G$ discovered by the machine:

$$\frac{1}{2G} = \frac{a_0}{b_1} + \frac{a_1}{b_1} \cdot \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots = 1 + \frac{-2}{7} \cdot \frac{-32}{19} + \cdots$$

where

$$a_n = -2n^4, \quad b_n = 3n^2 + 3n + 1.$$ 

5. Conjectures on Apery’s constant

The Reimann zeta function, $\zeta(s)$, is defined by

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

The number $\zeta(3)$ is defined as the Apery’s constant. It is named after
Roger Apéry (1916 - 1994), a Greek-French mathematician. This constant
appears in several physical problems including in the study of quantum
electrodynamics.

The Ramanujan machine has generated several conjectures regarding the
Apery’s constant. For example,

$$\frac{1}{\zeta(3)} = \frac{a_0}{b_0} + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n} + \cdots$$

where

$$a_n = -n^6, \quad b_n = n^3 + (n + 1)^3.$$ 

12 Ramanujan in digital signal processing

First some background regarding digital signal processing. A signal is an electrical
or electromagnetic current that is used for carrying data from one device to
another. It is the key component behind virtually all communication comput-
ing, networking and electronic devices. Any quality, such as a physical quantity
that exhibits variation in space or time can be used as a signal. A signal can be
audio, video, speech, image, sonar and radar-related and so on.
Signal processing is a subfield of electrical engineering that deals with analysing, modifying, and synthesizing signals such as sound, images, and scientific measurements. Digital signal processing (DSP) is the process of analyzing and modifying a signal to optimize or improve its efficiency or performance. It involves applying various mathematical and computational algorithms to analog and digital signals to produce a signal that is of higher quality than the original signal. Digital signal processing is primarily used to detect errors, and to filter and compress analog signals in transit.

13 Ramanujan sum

13.1 Definition

In a paper published in 1918 (see [29]), Ramanujan introduced the following sum where \( n, d, k \) are positive integers and where \( (k, d) \) denotes the greatest common divisor of \( k \) and \( d \):

\[
c_q(n) = \sum_{k=1}^{n} \cos \left( \frac{2kn\pi}{q} \right).
\]

The expression on the right can be rewritten in the following form:

\[
c_q(n) = \sum_{k=1}^{n} e^{\frac{2\pi i kn}{q}}.
\]

The complex number \( e^{-\frac{2\pi i}{q}} \) is a \( q \)-th root of unity and it is sometimes denoted by \( W_q \). Using this notation, we also have

\[
c_q(n) = \sum_{k=1}^{n} W_q^{-kn}.
\]

The sum \( c_q(n) \) is now generally referred to as the Ramanujan sum.

For example let \( q = 6 \). The integers \( k \) lying in the range 1, 2, 3, 4, 5, 6 and satisfying \( (k, q) = 1 \) are 1, 5 only. Hence we have

\[
c_6(n) = \cos \frac{4\pi n}{6} + \cos \frac{10\pi n}{6} = e^{\frac{4\pi i n}{6}} + e^{\frac{10\pi i n}{6}}.
\]

A few values of \( c_q(n) \) are given in Table 2. It may be noted that \( c_q(n+6) = c_q(n) \) for all \( n \).

| \( n \) | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | \ldots |
|--------|----|----|----|----|----|----|----|----|----|----|----|----|--------|
| \( c_q(n) \) | 2  | 1  | -1 | -2 | -1 | 1  | 2  | 1  | -1 | -2 | -1 | 1  | \ldots |

Table 2: Values of \( c_6(n) \)
13.2 Some properties

A few of the many interesting properties of the Ramanujan sum are listed below.

1. The Ramanujan sum \( c_q(n) \) is a period function of \( n \) with period \( q \), that is,
   \[ c_q(n + q) = c_q(n) \quad \text{for all } n. \]

2. \( c_q(n) \) is an integer valued function.

3. The function \( c_q(n) \) is a multiplicative function of \( q \), that is, if \( (q_1, q_2) = 1 \)
   (in other words, if \( q_1 \) and \( q_2 \) are relatively prime), then
   \[ c_{q_1q_2}(n) = c_{q_1}(n)c_{q_2}(n). \]

4. Using the Mobius function \( \mu(n) \) there is an explicit formula for \( c_q(n) \):
   \[ c_q(n) = \sum_{\text{d divisor of } (q,n)} \mu(q/d)d. \]
   Note that the Mobius function is defined as follows:
   \[ \mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct prime numbers} \\ 0 & \text{otherwise} \end{cases} \]

5. Let \( q_1 \neq q_2 \) and let \( l \) be any common multiple of \( q_1 \) and \( q_2 \). Then
   \[ \sum_{n=0}^{l-1} c_{q_1}(n)c_{q_2}(n) = 0. \]

6. The Euler’s totient function \( \phi(n) \) is an arithmetic function which takes
   the number of positive integers not exceeding \( n \) and relatively prime to \( n \)
   as its value. Then
   \[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} c_r(n)c_s(n) = \begin{cases} c_r(h) & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \]
   \[ \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} c_r(n)c_s(n + h) = \begin{cases} \phi(r) & \text{if } r = s \\ 0 & \text{otherwise} \end{cases} \]

13.3 Ramanujan-Fourier series

An arithmetic function is a function defined on the set of positive integers which
takes real or complex numbers as values. For example, the divisor function \( d(n) \)
whose value is the number of divisors of \( n \) is an arithmetic function. Ramanujan’s principal objective in studying the properties of the function \( c_q(n) \) was to obtain an expression for an arithmetic function \( x(n) \) in the form of a series as follows.

\[ x(n) = \sum_{q=1}^{\infty} a_q c_q(n). \]
A series of this form is called Ramanujan series (or Ramanujan expansion or sometimes Ramanujan-Fourier series or Ramanujan-Fourier Transform) (see [20]).

As an example, Ramanujan discovered the following expansion for the divisor function \( d(n) \):

\[
d(n) = -\sum_{q=1}^{\infty} \frac{\log q}{q} c_q(n).
\]

As another example, consider the arithmetic function \( \sigma(n) \) which takes the sum of the divisors of \( n \) as its value. Thus

\[
\sigma(6) = 1 + 2 + 3 + 6 = 12, \quad \sigma(7) = 1 + 7 = 8.
\]

Ramanujan obtained the following series expansion for the function \( \sigma(n) \):

\[
\sigma(n) = \frac{\pi^2 n}{6} \sum_{q=1}^{\infty} \frac{c_q(n)}{q^2}.
\]

### 13.4 An early application of Ramanujan sums to signal processing

The discrete and fast Fourier transforms are suited for the analysis of periodic and quasiperiodic sequences. But they are not suitable to discover the features of aperiodic sequences such as low-frequency noise. The Ramanujan-Fourier transform has been applied to examine whether there are any hidden patterns in such sequences. The general idea is to compare the observed patterns and properties in experimental data with the properties of the Ramanujan-Fourier transforms of well known arithmetical functions in number theory and to see whether there is any match between the two sets of properties.

This idea was tested using certain observed data from the study of black holes in astronomy and the general conclusion was that many of these processes may be described using prime number theory. It has also been suggested that Ramanujan-Fourier transform could also be used in the study of radio-frequency oscillators close to phase locking (see [23]).

### 14 Ramanujan spaces

For a fixed positive integer \( q \), consider the matrix:

\[
B_q = \begin{bmatrix}
c_q(0) & c_q(1) & c_q(2) & \cdots & c_q(q-1) \\
c_q(1) & c_q(0) & c_q(1) & \cdots & c_q(2) \\
c_q(2) & c_q(1) & c_q(0) & \cdots & c_q(3) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_q(q-1) & c_q(q-2) & c_q(q-3) & \cdots & c_q(0)
\end{bmatrix}
\]

Let \( \mathbb{C}^q \) be the \( q \)-dimensional complex vector space whose vectors are represented as column vectors. The subspace of \( \mathbb{C}^q \) generated by the columns of \( B_q \) is called the Ramanujan space (or sometimes, the Ramanujan subspace) and is denoted by \( S_q \). It is shown that the space \( S_q \) has dimension \( \phi(q) \) where \( \phi \) is the Euler’s
totient function. Also, any \( \phi(q) \) consecutive columns in \( B_q \), in particular the first \( \phi(q) \) columns of \( B_q \), form a basis for \( S_q \).

As a concrete example, let us consider \( B_6 \) using the values of \( c_6(n) \) given in Table 2.

\[
B_6 = \begin{bmatrix}
2 & 1 & -1 & -2 & -1 & 1 \\
1 & 2 & 1 & -1 & -2 & -1 \\
-1 & 1 & 2 & 1 & -1 & -2 \\
-2 & -1 & 1 & 2 & 1 & -1 \\
-1 & -2 & -1 & 1 & 2 & 1 \\
1 & -1 & -2 & -1 & 1 & 2 
\end{bmatrix}
\]

It may be noted that \( \phi(6) = 2 \) and the first two columns are linearly independent, and further any other column is a linear combination of the first two columns.

We defined \( S_q \) as a \( q \)-dimensional subspace of vectors in \( \mathbb{C}^q \), spanned by columns of \( B_q \). However, the elements of \( S_q \) are sometimes regarded as period-\( q \) sequences. So, when we say that \( x(n) \in S_q \), it is understood that the vector of size \( q \) is extended periodically to obtain the sequence \( x(n) \).

The Ramanujan spaces has an important periodicity property. A discrete-time signal \( x(n) \) is said to be periodic if there exists an integer \( R \) such that

\[ x(n) = x(n + R) \]

for all \( n \), and the integer \( R \) is called a repetition interval. The period \( P \) (an integer) is the smallest positive repetition interval. It can be shown that any repetition interval is an integer multiple of \( P \). If \( x_1(n) \) and \( x_2(n) \) have periods \( P_1 \) and \( P_2 \), their sum \( x_1(n) + x_2(n) \) has a repetition interval \( R = \text{lcm}(P_1, P_2) \), so its period \( P \) is either this lcm or a proper divisor of it.

It has been shown that all vectors in the Ramanujan space \( S_q \) has period exactly equal to \( q \), in particular, it cannot be smaller than \( q \). More generally, let us consider a set of \( K \) Ramanujan spaces \( S_{q_m} \), \( m = 1, \ldots, K \) and \( x_m(n) \in S_{q_m} \). Let \( N \) be the lcm of \( q_1, \ldots, q_K \). Then the vector

\[ x(n) = \sum_{m=1}^{K} x_m(n) \]

is periodic with period exactly equal to \( N \); in particular, it cannot be smaller (see [31]).

Let \( x(n) \) be finite sequence of length. Let us assume that the sequence is extended for all values of \( n \) by stipulating that

\[ x(n + N) = x(n) \], for all \( n \).

It can be shown that such a sequence, considered as a vector \( x \) in \( \mathbb{C}^N \), can be expressed as a sum of vectors in the Ramanujan spaces \( S_{q_1}, S_{q_2}, \ldots, S_{q_K} \) where \( q_1, q_2, \ldots, q_K \) are the divisors of \( N \):

\[ x = \sum_{q_i|N} x_{q_i}, \text{ where } x_{q_i} \in S_{q_i} \].

This has been called the Ramanujan FIR Representation\(^3\) of the signal \( x(n) \). These representations can be used to determine hidden periodicities in signals and also for denoising of signals. For more details the reader may refer to [32].

\(^3\)FIR is an acronym for Finite Impulse Response.
14.1 Applications

The following quote from the concluding section of Vaidyanathan’s paper on Srinivasa Ramanujan and signal-processing problems (see [33]) gives an indication of the various fields where the ideas of Ramanujan sums and Ramanujan subspaces have been successfully applied.

“...In this paper, we presented an overview of the impact of Ramanujan sums in signal processing, especially integer-period estimation in real or complex signals. These methods have recently been used in identification of integer periodicities in DNA molecules and in protein molecules. These new methods are quite competitive and often work better than existing state of the art methods. Ramanujan filter banks have also been shown to be applicable in the identification of epileptic seizures in patients, which are characterized by sudden appearance of periodic waveforms in the measured EEG records. Other interesting applications have recently been reported by a number of authors such as, for example in source-separation, RF communications, ECG signal processing and brain-computer interfacing. More recently, a well-known algorithm called the MUSIC algorithm, which is popularly used for identifying sinusoids in noise, has been extended to the case of integer period identification using Ramanujan-subspace ideas. This method, known as iMUSIC, has also been compared with other well-known methods for multipitch estimation.”

15 Conclusion

“It is satisfying indeed when one finds that a well-known mathematical concept has practical impact in engineering, even though the original mathematical ideas may not have been inspired by any such application. Engineers have seen this happening over and over again in disciplines such as information theory, coding, digital communications, system theory and machine learning. Indeed, the classical view that pure mathematics of the highest quality is bound to be ‘useless’ for real-life applications is evidently not valid as the last several decades of science and engineering have amply demonstrated. What is often regarded as pure mathematics sometimes impacts engineering in wonderful ways. A classic example is the theory of finite fields and rings, which has impacted the practice of error-correction coding in digital communications, data compression, and digital storage. Other examples include graph theory which has had many engineering, network and signal processing applications. Yet another is number theory which has impacted nearly all aspects of science and engineering such as acoustical hall designs, computer music, and so forth.” (see [33])

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