Superconductivity-Induced Anderson Localisation.

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We have studied the effect of a random superconducting order parameter on the localization of quasi-particles, by numerical finite size scaling of the Bogoliubov-de Gennes tight-binding Hamiltonian. Anderson localization is obtained in $d=2$ and a mobility edge where the states localize is observed in $d=3$. The critical behavior and localization exponent are universal within error bars both for real and complex random order parameter. Experimentally these results imply a suppression of the electronic contribution to thermal transport from states above the bulk energy gap.

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During the past few years phase-coherent transport in hybrid superconducting structures has emerged as a new field of study, bringing together the hitherto separate areas of superconductivity and mesoscopic physics. Recent experiments have revealed a variety of unexpected phenomena, including zero-bias anomalies, re-entrant and long-range behaviour and phase-periodic transport. These experiments can all be described by combining traditional quasi-classical Green’s function techniques with boundary conditions derived initially by Zaitsev and simplified by Kuprianov and Lukichev or alternatively by generalised current-voltage relations based on a multiple scattering approach to phase-coherent transport. The latter approach focusses attention to Andreev scattering, whereby an electron can coherently evolve into a hole and vice versa, without phase breaking.

The aim if this Letter is to address a new phenomenon, not describable by quasi-classical techniques, namely the onset of quasi-particle Anderson localisation due to spatial fluctuations in a superconducting order parameter. In contrast with all of the above experiments, where the superconducting order parameter $\Delta(r)$ is typically homogeneous, there are many situations in which $\Delta(r)$ varies randomly in space, even though the underlying normal potential is perfectly ordered. One example is provided by the melting of a flux lattice in an otherwise perfectly crystalline high $T_c$ superconductor. Another should occur in anisotropic superconductors, where by analogy with $^3$He-A, disordered textures can arise when an anisotropic phase is nucleated from a more symmetric phase such as $^3$He-B. In the first of these examples, the order parameter is not quenched. Nevertheless, close to the melting curve, the time scale for changes in $\Delta(r)$ can be made arbitrarily long and therefore in the spirit of the Born - Oppenheimer approximation, it is reasonable to freeze the disorder and when necessary, treat any temporal fluctuations as a contribution to the inelastic scattering lifetime.

In one dimension, it is straightforward to demonstrate that fluctuations in $\Delta(r)$ alone can localise the excitations, even at energies high above the bulk energy gap. However, localisation in strictly one-dimension is of little interest experimentally and therefore in this Letter, we address the question of whether or not superconductivity induced Anderson localisation occurs in higher dimensions. Early analytic work by suggested that in the presence of time reversal symmetry, states of energy $E=0$ are localised for dimensions $d \leq 2$, while in the absence of time reversal symmetry such states are localised in all dimensions. However calculations using a numerical finite size scaling approach were inconclusive and to date there has been no experimental confirmation of these predictions. In this Letter we provide the first firm numerical evidence for superconductivity induced Anderson localisation in $d=2$ and $d=3$ dimensions and for the first time compute the exponent $\nu$ controlling the divergence of the localisation length $\xi$ at the mobility edge in $d=3$.

To address the question of superconductivity induced Anderson localisation, we analyze the tight-binding Bogoliubov-de Gennes equations

$$E\psi_i(E) = \epsilon_i \psi_i(E) - \gamma \sum_j \psi_j(E) + \Delta \phi_i(E),$$

$$E\phi_i(E) = -\epsilon_i \phi_i(E) + \gamma^* \sum_j \phi_j(E) + \Delta^* \psi_i(E),$$

where $\psi_i(E)$ ($\phi_i(E)$) indicates the particle (hole) wavefunction of energy $E$ on site $i$ and $j$ sums over the neighbours of $i$. Since only scaling behaviour near a critical point is of interest, we examine the simplest possible model of a system with no normal disorder, but a spatially fluctuating order parameter, obtained by choosing $\epsilon_i$ equal to a constant $\epsilon_0$ for all sites $i$ and to set the energy scale, choose $\gamma = 1$. Two models of disorder will be examined. In model 1, (which preserves time reversal symmetry), we choose $\Delta_i = \Delta_0[1 + \delta \Delta_i]$ and in model 2, (which breaks time reversal symmetry), we choose $\Delta_i = \Delta_0[1 + \delta \Delta_i] + \nu[1 + \delta \Delta_i]$, where $\delta \Delta_i$ and $\delta \Delta_i'$ are random numbers uniformly distributed between $-\Delta$ and $+\Delta$. In what follows, we choose $\epsilon_0 = 0$.

For each model, we compute the transfer matrix $T$ for...
a long strip \((d = 2)\) and a long bar \((d = 3)\) of length \(L\) sites and cross-section \(M^{d-1}\) sites, respectively and identify the localisation length \(\xi_M\) with the inverse of the corresponding smallest Lyapunov exponent. The results are, of course, sensitive to the chosen energy \(E\) and since, in the absence of disorder (ie \(\delta \Delta = 0\)), there exists an energy-gap at \(E = 0\), the usual choice of \(E = 0\) adopted in the absence of superconductivity is inappropriate. As a guide to a reasonable choice of \(E\), we consider the related problem of a system with normal disorder but with a uniform order parameter. In this case \(c_i\) is chosen randomly from a uniform probability distribution but \(\Delta_i = \Delta_0\), for all \(i\). As noted in \([2]\) if \(\psi_i^0(E_0)\) is a solution of

\[E = \sqrt{|E_0^2 + |\Delta_0|^2|}\].

This means that if in the absence of superconductivity a state at energy \(E_0\) is localised by normal disorder, then in the presence of a uniform order parameter \(\Delta_0\), quasi-particle states at energy \(E\) are localised with the same localisation length. As a consequence all critical properties are unchanged, provided \(E_0\) is replaced by \(E\). In the normal state problem the least localised states occur at \(E_0 = 0\) and therefore in the presence of normal disorder and a uniform superconducting order parameter these states occur at \(E = |\Delta_0|\).

Of course, in what follows we are interested in the opposite limit of a spatially fluctuating order parameter with no normal disorder. Nevertheless, guided by the above observation we choose \(E = c\langle |\Delta_i| \rangle\), where \(c\langle |\Delta_i| \rangle\) is the ensemble averaged order parameter, which gives \(E = \Delta_0\) for model 1 and \(E = \sqrt{2}\Delta_0\) for model 2.

The raw data for \(\xi_M/M\) versus \(M\), for model 1 with \(E = \Delta_0\), \(c_0 = 0\) and \(\delta \Delta = 0.1\), are shown in figures 1(a) and (b) for two and three dimensions, respectively. The strength of disorder in the order parameter is \(W = 2\Delta_0\delta \Delta\), whose critical value is denoted \(W_c\), and is varied by changing \(\Delta_0\), with fixed \(\delta \Delta\). In two dimensions \(\xi_M/M\) decreases with increasing \(M\) indicating that all states are localised with \(W_c = 0\), whereas in three dimensions there is a cross-over from localised to extended behaviour at around \(\Delta_0 \approx 12\) which for the adopted value of \(\delta \Delta = 0.1\) corresponds to \(W_c \approx 2.4\).

To quantify the critical behaviour in 3 dimensions, we linearize the data about \(W_c\) by writing \(\log(\xi_M/M) = \alpha_M + \beta_M \log W\) and obtain the coefficients \(\alpha_M\) and \(\beta_M\) for various \(M\). In terms of the fixed point values \(\log(\xi_M/M)_c\) and \(\log W_c\), we note that \(\alpha_M = \log(\xi_M/M)_c - \beta_M \log W_c\). Thus, a graph of \(\alpha_M\) versus \(\beta_M\) yields \(\log(\xi_M/M)_c\), \(\log W_c\) and hence the critical disorder \(W_c\). The critical exponent \(\nu\) for the divergence of the localization length \(\xi\) of the infinite system is obtained by substituting \(\alpha_M\) into the first linear relation, which yields \(\log(\xi_M/M) = \log(\xi_M/M)_c + \beta_M \log(W/W_c)\). Moreover, near the critical point \(\log(W/W_c) \sim (W - W_c)/W_c\) and \(\xi \sim |W - W_c|^{-\nu}\), so that \(\log(\xi_M/M) = \log(\xi_M/M)_c + \xi^{-1/\nu}\beta_M\), where the + (−) sign refers to \(W > W_c\) (\(W < W_c\)). The finite size scaling requirement \(\xi_M/M = f(\xi/M)\) immediately implies \(\beta_M \sim M^{1/\nu}\), which permits the computation of the exponent \(\nu\).

Figure 2 shows a graph of \(\log(\xi_M/M)\) versus \(\log \Delta_0\), from which \(\alpha_M\) and \(\beta_M\) for the chosen widths \(M\) can be extracted. The top-right insert shows the resulting plot of \(\alpha_M\) versus \(\beta_M\) whose slope is \(\log W_c\) and the corresponding intercept is \(\log(\xi_M/M)_c\). This yields \(W_c = 2.36 \pm 0.04\) which corresponds to \(\Delta_0 = 11.73 \pm 0.12\) and \((\xi_M/M)_c = 0.58 \pm 0.02\). The lower-left insert shows \(\log \beta_M\) versus \(\log M\) whose slope yields the critical exponent \(\nu = 1.64 \pm 0.06\).
For model 2, where time reversal invariance is broken due to the presence of a complex order parameter, all states are localized in $d = 2$. In contrast, figure 3(a) shows the corresponding plots of $\xi_M/M$ versus $M$ in 3 dimensions which clearly show a cross-over from extended to localised behaviour. Results from a more accurate calculation are presented in figure 3(b), where the upper-right figure yields $W_c = 5.57 \pm 0.12$ and $(\xi_M/M)_c = 0.58 \pm 0.02$. The lower-left insert shows $\log \beta_M$ versus $W_c$, the slope of which leads to the value for the exponent $\nu = 1.69 \pm 0.06$. The errors in the calculation of $\xi_M/M$ are monitored as a function of length $L$ and chosen to be less than about 0.01 by taking long strips of lengths $L = 250000$ and bars of $L = 200000$ ($L = 500000$) for the real (complex) case. The errors for $W_c$ and $\nu$ are estimated from the corresponding least-square fits. We have also repeated our calculations by taking points closer to the critical value $W_c$, where the above analysis holds, with no significant change of our results.

The first important feature of the above calculation is the unambiguous prediction of superconductivity-induced quasi-particle localization in $d = 2$ and the presence of a mobility edge in $d = 3$. Localization arises from fluctuations in the superconducting order parameter alone, without the need for additional normal disorder. A second key result is the observation that for both models we find $(\xi_M/M)_c \sim 0.58$ and $\nu \sim 1.6$, which are remarkably close to the values reported for normal $d = 3$ real systems [21], and also consistent with reported data for ordinary disordered critical systems with and without time-reversal invariance [22,23]. Recently, slightly different scaling behavior is obtained with and without time-reversal by an alternative data analysis based on polynomial fits [24]. Our study in $d = 3$ cannot distin-

From an experimental point of view, it is worth noting that the absence of quasi-particle diffusion does not imply the vanishing of the electrical conductance, because Andreev scattering does not conserve quasi-particle charge. It does, however, imply a vanishing of the electronic contribution to thermal transport from certain states above the gap. In a clean superconductor at a finite temperature $T$, this varies as $\exp(-\Delta/k_bT)$, where $\Delta$ is the bulk energy gap. In contrast, in the presence of a fluctuation-induced quasi-particle mobility edge $E_c$, this will be replaced by $\exp(-E_c/k_bT)$. Thus, for example, the melting of a flux lattice in a high temperature superconductor

![3D real](image1)

**FIG. 2.** Log-log plot of $\xi_M/M$ versus $\Delta_0$ where the intersection defines $W_c$ and $(\xi_M/M)_c$. The upper-right insert shows the coefficients $\alpha_M$ versus $\beta_M$ whose slope is $-\log W_c$. The lower-left insert shows a log-log plot of $\beta_M$ versus $M$ which yields the value for the exponent $\nu = 1.64 \pm 0.06$.

![3D complex](image2)

**FIG. 3.** (a) Log-log plot of $\xi_M/M$ versus $M$ for various values of $\Delta_0$ when time reversal symmetry is broken (model 2) which shows a cross-over from extended to localised states. (b) Log-log plot of $\xi_M/M$ versus $\Delta_0$ for model 2 where the intersections define $W_c$ and $(\xi_M/M)_c$. The upper-right insert shows the coefficients $\alpha_M$ versus $\beta_M$ whose slope is $-\log W_c$. The lower-left insert shows a log-log plot of $\beta_M$ versus $M$ which yields the value for the exponent $\nu = 1.69 \pm 0.06$. 


should be accompanied by an exponential change in the
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