§1. Introduction

1.1. This is the first of two papers addressing a Cauchy type inequality for the geometric intersection number between two 1-dimensional submanifolds in a surface. As a consequence, we reestablish some of the basic results in Thurston’s theory of measured laminations. In this paper, we consider surfaces with non-empty boundary using ideal triangulations. In the sequel, we establish the inequality for closed surfaces using Dehn-Thurston coordinates.

1.2. Let us begin with a brief review of Thurston’s theory (see [Bo], [FLP], [Mo], [PH], [Th1], [Th2] and others). Given a compact orientable surface Σ with possibly non-empty boundary, a curve system on Σ is a proper 1-dimensional submanifold so that each component of it is not null homotopic and not relatively homotopic into the boundary ∂Σ. The space of all isotopy classes of curve systems on Σ is denoted by \( \text{CS}(Σ) \). This space was introduced by Max Dehn in 1938 [De] who called it the arithmetic field of the topological surface. Given two classes α and β in \( \text{CS}(Σ) \), their geometric intersection number \( I(α, β) \), is defined to be \( \min\{ |a ∩ b| : a ∈ α, b ∈ β \} \). Thurston observed that the pairing \( I(,): \text{CS}(Σ) × \text{CS}(Σ) → \mathbb{Z} \) behaves like a non-degenerate “bilinear” form in the sense that (1) given any α in \( \text{CS}(Σ) \) there is β in \( \text{CS}(Σ) \) so that their intersection number \( I(α, β) \) is non-zero, and (2) \( I(k_1α_1, k_2α_2) = k_1k_2I(α_1, α_2) \) for \( k_i ∈ \mathbb{Z}_{≥0}, α_i ∈ \text{CS}(Σ) \) where \( k_iα_i \) is the collection of \( k_i \) copies of \( α_i \). In linear algebra, given a non-degenerate quadratic form \( ω \) on a lattice \( L \) of rank \( r \), one can form a completion of \( (L, ω) \) by canonically embedding \( L \) into \( \mathbb{R}^r \) so that the form \( ω \) extends continuously on \( \mathbb{R}^r \). Thurston’s construction is the exact analogy. Thurston’s space of measured laminations on the surface Σ, denoted by \( \text{ML}(Σ) \) is defined to be the completion of the pair \( (\text{CS}(Σ), I) \) in the following sense. Given α in \( \text{CS}(Σ) \), let \( π(α) \) be the map sending β to \( I(α, β) \). This gives an embedding of \( π : \text{CS}(Σ) → \mathbb{R}^{\text{CS}(Σ)} \) where the target has the the product topology. The space \( \text{ML}(Σ) \) is define to be the closure of \( Q_{>0} × π(\text{CS}(Σ)) = \{ rπ(x) : r ∈ Q_{>0}, x ∈ \text{CS}(Σ) \} \). Using the notion of train-tracks, Thurston showed that \( \text{ML}(Σ) \) is homeomorphic to a Euclidean space and that the intersection pairing \( I(,): \text{ML}(Σ) × \text{ML}(Σ) → \mathbb{R} \) extends to a continuous homogeneous map from \( \text{ML}(Σ) × \text{ML}(Σ) \) to \( \mathbb{R} \). See [Bo], [FLP], [Mo], [PH], [Th2] and others for a proof of the first statement and [Bo],[Re] and others for a proof of the continuity of the the extension. In fact Bonahon generalized the intersection number to the geodesic currents and proved the continuity of the pairing in the generalized situation.

One of our goals is to derive the continuity of the intersection pairing on \( \text{ML}(Σ) \) from a simple Cauchy inequality. This inequality also implies that the space \( \text{ML}(Σ) \) is a Euclidean space (see §4). Our main observation is that Thurston’s theory of measured laminations is based on a simple geometric fact that the intersection of two distinct line segments in a disk consists of at most one point and the intersection number being one is detected by the relative locations of the end points of the line segments.
1.3. Given a compact surface $\Sigma$ with non-empty boundary, recall that an ideal triangulation of a surface $\Sigma$ is a maximal collection of pairwise disjoint, pairwise non-isotopic essential arcs in $\Sigma$. Fix an ideal triangulation $t = t_1 \cup t_2 \cup \ldots \cup t_N$ of the surface, L. Mosher [Mo] proved that the space of closed curve systems $CS_0(\Sigma) = \{[s] \in CS(\Sigma) : \partial s = \emptyset\}$ can be parametrized using the ideal triangulation where $[s]$ denotes the isotopy class of a submanifold $s$. Namely there is an injective map $T : CS_0(\Sigma) \to \mathbb{Z}_{\geq 0}^N$ sending $\alpha$ to $(I(\alpha, [t_1]), \ldots, I(\alpha, [t_N]))$. Our main result is the following.

**Theorem.** Suppose $t = t_1 \cup \ldots \cup t_N$ is an ideal triangulation of a compact surface. Then for any three classes $\alpha, \beta, \gamma \in CS_0(\Sigma)$, the following inequality holds:

$$|I(\alpha, \beta) - I(\alpha, \gamma)| \leq |\alpha||\beta - \gamma|$$

where $|\alpha| = \sum_{i=1}^{N} I(\alpha, [t_i])$ and $|\beta - \gamma| = \sum_{i=1}^{N} |I(\beta, [t_i]) - I(\gamma, [t_i])|$. Furthermore, the constant 1 in the inequality is optimal.

The basic idea of the proof is as follows. Given two classes $\beta, \gamma$ so that $|\beta - \gamma| = n$, we produce a sequence of 1-dimensional submanifolds $\beta_i$ for $i = 0, 1, \ldots, n$ starting from $\beta$ and ending at $\gamma$ so that $|\beta_i - \beta_{i+1}| = 1$. Thus, we may assume that $|\beta - \gamma| = 1$. We prove the theorem in this special case by analyzing the surgery procedure relating $\beta$ and $\gamma$.

We remark that a similar result that $|I(\alpha, \beta) - I(\alpha, \gamma)| \leq K|\alpha||\beta - \gamma|$ was obtained earlier by M. Rees [Re] using train-tracks. The constant $K$ in her theorem is big and depends on the train-tracks.

1.4. Let us derive the continuity of $I(\cdot, \cdot)$ on the space of compactly supported measured laminations $ML_0(\Sigma)$ which is the closure of $Q_{\geq 0} \times \pi(CS_0(\Sigma))$ using the above inequality. One first extends the pairing $I(\cdot, \cdot)$ to $(Q_{\geq 0} \times \pi(CS_0(\Sigma))^2$ by linearity $I(k_1, \alpha_1, k_2, \alpha_2) = k_1 k_2 I(\alpha_1, \alpha_2)$. Thus the inequality still holds for $\alpha, \beta$ and $\gamma$ in $Q_{\geq 0} \times CS_0(\Sigma)$. Since the product space $R^{CS(\Sigma)}$ is metrizable, the continuity of the pairing $I(\cdot, \cdot)$ on $ML_0(\Sigma) \times ML_0(\Sigma)$ follows by showing that if $(\alpha_n, \beta_n) \in (Q_{\geq 0} \times CS_0(\Sigma))^2$ converges, then $I(\alpha_n, \beta_n)$ converges. Now since $\alpha_n$ and $\beta_n$ converge, both limit$I(\alpha_n, [t_i])$ and limit$I(\beta_n, [t_i])$ exist for all $t_i$. Thus, $\lim_{n, m}|\alpha_n - \alpha_m| = 0$ and $\lim_{n, m}|\beta_n - \beta_m| = 0$. By the inequality, $|I(\alpha_n, \beta_n) - I(\alpha_m, \beta_m)| \leq |I(\alpha_n, \beta_n) - I(\alpha_n, \beta_m)| + |I(\alpha_n, \beta_m) - I(\alpha_m, \beta_m)| \leq |\alpha_n||\beta_n - \beta_m| + |\beta_m||\alpha_n - \alpha_m|$ which converges to 0.

As a consequence of the continuity, we see that the Cauchy inequality still holds for $\alpha, \beta$ and $\gamma$ in $ML_0(\Sigma)$. Thus we deduce a result of Mosher that each element $\alpha$ in $ML_0(\Sigma)$ is determined by the $N$-tuple of intersection numbers $T(\alpha) = (I(\alpha, [t_1]), \ldots, I(\alpha, [t_N]))$. Furthermore the Cauchy inequality implies that the space $ML_0(\Sigma)$ is locally compact and the map $T : ML_0(\Sigma) \to R^N$ is proper. Indeed, if a sequence $\alpha_n$ in $ML_0(\Sigma)$ is bounded under $T$, then for any $\beta \in CS(\Sigma)$, the Cauchy inequality implies that $I(\alpha_n, \beta) \leq |T(\alpha_n)||T(\beta)|$ is bounded in $n$ for each fixed $\beta$. Since there are at most countably many $\beta$’s, by the standard diagonalization argument, there is a subsequence $\alpha_{n_i}$ so that $I(\alpha_{n_i}, \beta)$ converges for all $\beta$. This simply says that
\{\alpha_n\} contains a convergent subsequence. To see that \(T\) is proper, we note that if \(T(\alpha_n)\) converges to a point in \(\mathbb{R}^N\), then \(T(\alpha_n)\) is bounded. Thus \(\alpha_n\) contains a convergent subsequence. This shows that \(T\) is proper and \(T : ML_0(\Sigma) \to \mathbb{R}^N\) is an embedding whose image is a closed subset. In fact, Mosher showed that the image of \(ML_0(\Sigma)\) is homeomorphic to a Euclidean space. This result will also be derived from theorem 1.3 in §4.

1.5. In a forthcoming paper [LS], we shall establish an analogous Cauchy inequality for the geodesic length of the geodesic length function of the surface. Namely, given a complete hyperbolic metric, let \(l_d(\alpha)\) be the sum of the lengths of the system of \(d\)-geodesics representing \(\alpha\). We prove that \(|l_d(\alpha) - l_d(\beta)| \leq d||\alpha - \beta\|\) and \(|l_d(\alpha) - l_{d'}(\alpha)| \leq |\alpha||d - d'|\) with respect to a fixed ideal triangulation of the surface. As a consequence, we reestablish a result of Thurston that the geodesic length function extends continuously to a function on \(Teich(\Sigma) \times ML_0(\Sigma) \to \mathbb{R}\). See [Bo] for a written proof of this result.

1.6. Part of the work is supported by the NSF.

§2. Preliminaries

2.1. We begin by introducing some notations. Let \(\Sigma = \Sigma_{g,r}\) be a compact orientable surface of genus \(g\) with \(r\) \((\geq 0)\) many boundary components. We shall assume that the Euler characteristic of the surface \(\Sigma\) is negative. Isotopies of the surface leave the boundary invariant. Given a 1-submanifold \(s\), we denote the isotopy class of \(s\) by \([s]\) and a small regular neighborhood of \(s\) by \(N(s)\). The interior of a manifold \(X\) will be denoted by \(int(X)\). The geometric intersection number \(I([a],[b])\) will also be denoted by \(I(a,b)\) and \(I([a],b)\).

A simple loop (or a proper arc) in a surface \(\Sigma\) is called trivial if it is null homotopic (or relatively homotopic into \(\partial \Sigma\)). A 1-submanifold in a surface \(\Sigma\) is essential if each component of it is non-trivial. We shall enlarge the space of curve systems \(CS(\Sigma)\) to the space of all isotopy classes of essential 1-submanifolds, denoted by \(ES(\Sigma)\). The intersection pairing \(I(\cdot, \cdot)\) is defined similarly on \(ES(\Sigma)\). Note that \(CS(\Sigma) \subset ES(\Sigma)\) and \([\partial \Sigma] \in ES(\Sigma) - CS(\Sigma)\).

Our first result is to give a parametrization of the space \(ES(\Sigma)\) for surfaces with non-empty boundary using ideal triangulations of the surface. To achieve this, we begin by parametrizing arc systems on polygons.

2.2. Let \(P_n\) be an \(n\)-sided polygon. An arc in \(P_n\) is a proper embedding of a closed interval into \(P_n - \{\text{vertices of } P_n\}\). An arc in \(P_n\) is called trivial if its endpoints either lie in one side of \(P_n\) or in two adjacent sides of \(P_n\). An arc system in \(P_n\) is a finite disjoint union of non-trivial arcs in \(P_n\). Let \(ES(P_n)\) be the set of all isotopy classes of arc systems in \(P_n\) where isotopies leave each side invariant. Given two classes \(\alpha\) and \(\beta\) in \(ES(P_n)\), we define their intersection number to be \(I(\alpha, \beta) = min\{||a \cap b| : a \in \alpha, b \in \beta\}\). We say a non-trivial arc \(s\) in \(P_n\) is parallel to a side if one of the components of \(P_n - s\) is a quadrilateral.

We first give a parametrization of \(ES(P_6)\). Let the six sides of the hexagon \(P_6\) be \(A_1, B_3, A_2, B_1, A_3, B_3\) labeled cyclically. A parametrization of \(ES(P_6)\) using the
A-sides is as follows. Take $\alpha = [a]$ in $ES(P_6)$. Let $x_i = I(\alpha, A_i) = |a \cap A_i|$ and $x'_i$ be the number of components of $a$ which are parallel to $A_i$. Evidently $x_ix'_i = 0$. We call $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$ the $t$-coordinate of $\alpha$ with respect to the A-sides of the hexagon.

2.3. Lemma. Let $\Delta = \{(a_1, a_2, a_3) \in \mathbb{Z}_0^3 : a_i + a_j \geq a_k, \text{ for all } i \neq j \neq k \neq i\}$. The map $T : ES(P_6) \to \{(x_1, x_2, x_3, x'_1, x'_2, x'_3) \in \mathbb{Z}_0^6 : x_ix'_i = 0, \text{ if } (x_1, x_2, x_3) \in \Delta \text{ then } x_1 + x_2 + x_3 \text{ is even} \}$ sending an element to its $t$-coordinate is a bijection.

Proof. Clearly $T$ is well defined. To see that $T$ is onto, we construct the arc system $a$ with a given vector $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$ as the coordinate according to the following five cases: (1) $(x'_1, x'_2, x'_3) = (0, 0, 0)$, and $(x_1, x_2, x_3) \in \Delta$; (2) $(x'_1, x'_2, x'_3) = (0, 0, 0)$ and $(x_1, x_2, x_3) \notin \Delta$; (3) $x'_1 = x'_2 = 0, x'_3 > 0$; (4) $x'_1 = 0$ and $x'_2x'_3 > 0$ and $x'_1x'_2x'_3 > 0$. The corresponding arc systems are listed in the figure 2.1 below.

The arc system $a$ can be described as follows. Let $a_i$ (resp. $b_i$) be an arc parallel to $A_i$ (resp. $B_i$) and $c_i$ be an arc joining $A_i$ to $B_i$. We use $kx$ to denote $k$ parallel copies of an arc $x$. The in the case (1), $a = \bigcup_{k=1}^{3}(x_i+\frac{x_j-x_k}{2})b_k$; in the case (2) say $x_k \geq x_i + x_j$, then $a = x_i b_i \cup x_j b_j \cup (x_k - x_i - x_j)c_k$; in the case (3), say $x_i \geq x_j$, then $a = x_k a_k \cup x_j b_j \cup (x_i - x_j)c_i$; in the case (4) $a = x'_j a_j \cup x'_k a_k \cup x_i c_i$; and in the last case (5) $a = \bigcup_{i=1}^{3} x'_i a_i$. Since two non-trivial arcs are isotopic if and only if their end points lands on the same set of sides on the polygon, the map $T$ is injective. □.

![Figure 2.1](image-url)

2.4. Remark. The parametrization of the arc systems in $P_6$ whose ends are in A-sides is well known.
2.5. To parametrize the arc systems on any polygon $P_{2n}$ of even sides, we use disjoint non-trivial arcs to decompose $P_{2n}$ into hexagons. Let the $A$-sides of the hexagons correspond to the decomposing arcs. Then a parametrization of $P_{2n}$ is given by taking the $t$-coordinates of the hexagons with respect to the $A$-sides.

2.6. One of the key ingredients in the proof of theorem 1.3 is to understand the surgery procedure relating two elements in $ES(P_6)$ whose $t$-coordinates differ by a basis vector. For simplicity, a class $\alpha$ in $ES(P_6)$ is called even if all components of its $t$-coordinates are even numbers. We shall describe the surgery procedure relating two even arc systems $\alpha$ and $\beta$ so that their $t$-coordinates $(x_1, x_2, x_3, x'_1, x'_2, x'_3)$ and $(y_1, y_2, y_2, y'_1, y'_2, y'_3)$ are related by $(x_1, x_2, x_3, x'_1, x'_2, x'_3) = (y_1, y_2, y_2, y'_1, y'_2, y'_3) + (2, 0, ..., 0)$.

Note that $x'_1 = y'_1$. Since $x_1 > 0$, it follows that $x'_1 = y'_1 = 0$.

Take a standard representative $a$ for $\alpha$. To obtain a standard representative $b$ for $\beta$, we perform the following surgery operation on $a$. If $a$ contains arcs $b_2$ and $b_3$, we replace $a$ by $(a - b_2 \cup b_3) \cup b_1$ to obtain $b$; if $a$ contains an arc parallel to $c_1$, then since $\alpha$ is even, $a$ contains two copies of $c_1$. We replace $a$ by $a - 2c_1$ to obtain $b$. In the remaining case, $a$ is disjoint from either $c_2$ or $c_3$, say $a \cap c_2 = \emptyset$. Since $x_1 \geq 2$, $a$ contains at least 2 copies of $b_3$. In this case, replace $a$ by $(a - 2b_3) \cup 2c_2$ to obtain $b$. Note that the arcs created lie in a small regular neighborhood of the boundary and the arcs deleted. See figure below for the illustration.
To obtain a standard representative of \( a \) from \( b \), we perform the following surgery operation on \( b \). If \( b \) contains some copies of \( b_1 \) but no \( c_2 \) or \( c_3 \), replace \( b \) by \((b - b_1) \cup b_2 \cup b_3\) to obtain \( a \). If \( b \) contains no \( b_1 \), \( c_2 \) and \( c_3 \), replace \( b \) by \( b \cup 2c_1 \) to obtain \( a \). If \( b \) contains some \( c_2 \) or \( c_3 \), say \( c_2 \subset b \), then \( b \) contains even number of copies of \( c_2 \). Replace \( b \) by \((b - 2c_2) \cup 2b_3\).

§3. Geometric Intersection Numbers on Surfaces with Boundary

We prove the main theorem 1.3 in this section. The basic idea of the proof is as follows. We enlarge the classes of curve systems to essential submanifolds and parametrize the space of all isotopy classes of essential submanifolds by using an ideal triangulation. The reason for doing so is due to the fact that given two isotopy classes of essential submanifolds whose \( t \)-coordinates are distance-\( k \) apart, there is a sequence of \( k + 1 \) essential 1-submanifolds starting and ending at these two given classes so that the \( t \)-coordinates of adjacent elements in the sequence are distance-1 apart. This property does not hold for the space of curve systems. Thus, it reduces the proof of the Cauchy inequality to the case of \( t \)-coordinates being distance-1 apart. By the surgery operation relating distance-1 1-submanifolds, we prove the theorem.

3.1. We give a parametrization of the space \( ES(\Sigma) \) as follows. Fix a maximal collection \( t = t_1 \cup ... \cup t_N \) of pairwise disjoint, non-isotopic essential arcs (an ideal triangulation) of the surface \( \Sigma \). Thus the components of \( \Sigma - \cup_{i=1}^{N} \operatorname{int}(N(t_i)) \) are hexagons. Let the \( A \)-sides of the hexagons correspond to \( t_i \)’s. Given \( \alpha \) in \( ES(\Sigma) \), let \( t(\alpha) \) be the \( t \)-coordinate of \( \alpha \) which is the collection of \( t \)-coordinates of \( \alpha \) in each hexagon. Namely, \( t(\alpha) = (x_1, ..., x_N, x'_1, ..., x'_N) \) where \( x_i = I(\alpha, t_i) \) and \( x'_i \) is the number of components of \( \alpha \) equal to \( [t_i] \). Clearly \( x_ix'_i = 0 \).

3.2. Lemma. (see also [Mo]) Fix an ideal triangulation \( t \) of \( \Sigma \). Then the map \( T : ES(\Sigma) \to X = \{(x_1, ..., x_N, x'_1, ..., x'_N) \in \mathbb{Z}_{\geq 0}^N : x_ix'_i = 0, \text{ if } t_i, t_j \text{ and } t_k \text{ bound a } \triangle \text{ and } (x_i, x_j, x_k) \in \Delta, \text{ then } x_i + x_j + x_k \text{ is even} \} \) sending an element to its
t-coordinate is a bijection. In particular, the image of $T$ contains the set of even vectors $L = \{(x_1, ..., x_N, x_1', ..., x_N') \in (2\mathbb{Z}_{\geq 0})^N : x_i x_i' = 0\}$.

Proof. To see that the map $T$ is onto, take an element $(x_1, ..., x_N, x_1', ..., x_N')$ in the set $X$. Let $H$ be a hexagonal component of $\Sigma - \bigcup_{i=1}^{N} \text{int}(N(t_i))$ with three A-sides parallel to $t_i, t_j$ and $t_k$ (it may occur that $t_i = t_j$). By lemma 2.2, we construct an arc system in $H$ with the $t$-coordinate $(x_i, x_j, x_k, x_i', x_j', x_k')$. Now glue these arc systems across $N(t_i) = t_i \times [-1, 1]$ by adding parallel arcs $\{(p_1, ..., p_n) \times [-1, 1]\}$. We obtain a 1-submanifold $s$ properly embedded in $\Sigma$. By the construction, there are no Whitney discs in $s \cup t$ and $s \cup \partial \Sigma$. Thus the submanifold $s$ is essential and its $t$-coordinate is the given vector $(x_1, ..., x_N, x_1', ..., x_N')$. We call $s$ a standard representative. To see that the map $T$ is injective, given $\alpha$ in $ES(\Sigma)$, choose a representative $a \in \alpha$ so that $I(\alpha, t) = |a \cap t|$. Thus $a \cap H$ is an arc system in each hexagonal component of $\Sigma - \bigcup \text{int}(N(t_i))$. Since each non-trivial arc in the quadrilateral $N(t_i)$ is parallel to a side, it follows that $a$ is isotopic to a standard representative. It follows that the map $T$ is injective. □

Having introduced the coordinate, let us now restate theorem 1.3 in its most general form which we will prove.

3.3. Theorem. Suppose $t = t_1 \cup ... \cup t_N$ is an ideal triangulation of a compact surface. Then for any three classes $\alpha, \beta, \gamma \in ES(\Sigma)$ with $t$-coordinates $(x_1, ..., x_N, x_1', ..., x_N'), (y_1, ..., y_N, y_1', ..., y_N')$ and $(z_1, ..., z_N, z_1', ..., z_N')$, the following inequality holds:

$$|I(\alpha, \beta) - I(\alpha, \gamma)| \leq 2|\alpha||\beta - \gamma|$$

where $|\alpha| = \sum_{i=1}^{N} x_i + x_i'$ and $|\beta - \gamma| = \sum_{i=1}^{N} (|y_i - z_i| + |y_i' - z_i'|)$. Furthermore, if $\alpha$ is in $CS_0(\Sigma)$, then

$$|I(\alpha, \beta) - I(\alpha, \gamma)| \leq |\alpha||\beta - \gamma|.$$

These inequalities are optimal.

3.4. We now begin the proof of theorem 3.3 for classes in $ES(\Sigma)$. Since the intersection pairing $I(\cdot, \cdot)$ is homogeneous, it suffices to prove the inequality for $2\alpha$, $2\beta$ and $2\gamma$ in $ES(\Sigma)$. The $t$-coordinate of $2\alpha$ is an even vector in $L$. For simplicity, we call a class $\alpha \in ES(\Sigma)$ even if $T(\alpha) \in L$ (see lemma 3.2). Thus it suffices to prove theorem 3.3 for even classes.

Given two even vectors $u = (u_1, ..., u_{2N})$ and $v = (v_1, ..., v_{2N})$ in $L$ so that their distance $|u - v| = \sum_{i=1}^{2N} |u_i - v_i|$ is $2n$, there is a sequence of $n + 1$ even vectors $w_j, j = 0, ..., n$ so that $w_0 = u$, $w_n = v$ and $|w_{i+1} - w_i| = 2$. Thus given two even classes $\beta, \gamma$ in $ES(\Sigma)$ so that $|\beta - \gamma| = 2n$, by lemma 3.2, there exists a sequence of $n + 1$ even classes starting from $\beta$ and ending at $\gamma$ so that the adjacent elements are of distance-2 apart. Thus it suffices to prove theorem 3.3 for even classes $\beta$ and $\gamma$ so that $|\beta - \gamma| = 2$. Without loss of generality, we may assume that $T(\gamma) = T(\beta) + (0, ..., 0, 2, 0, ..., 0)$, i.e., $(z_1, ..., z_N, z_1', ..., z_N') = ...$
\((y_1, \ldots, y_N, y'_1, \ldots, y'_N) + (0, \ldots, 0, 2, 0, \ldots, 0)\). We need to consider two cases: (1) \(z'_i = y'_i + 2\) and (2) \(z_i = y_i + 2\) for some \(i\).

In the first case that \(z'_i = y'_i + 2\), the class \(\gamma\) is obtained from \(\beta\) by adding two copies of \([t_i]\). Thus \(I(\alpha, \beta) = I(\alpha, \gamma) - 2x_i\). The inequality follows.

In the second case, we shall prove the Cauchy inequality by showing the following three inequalities:

1. \(I(\alpha, \beta) \leq I(\alpha, \gamma) + 4|\alpha|\)
   and
2. \(I(\alpha, \gamma) \leq I(\alpha, \beta) + 2|\alpha|\).

Furthermore, if \(\alpha\) is a closed curve system in \(CS_0(\Sigma)\), we shall prove
3. \(I(\alpha, \beta) \leq I(\alpha, \gamma) + 2|\alpha|\).

Let us assume for simplicity that \(i = 1\). Let \(H_1\) and \(H_2\) be the closures of the hexagonal components of \(\Sigma - (\bigcup_{i=2}^N\text{int}(N(t_i)) \cup t_1)\) lying on two sides of \(t_1\) (it may occur that \(H_1 = H_2\)). See figure 3.1. If \(H_1 \neq H_2\), then \(H_1 \cap H_2 = t_1\) and \(H_1 \cup H_2\) is an octagon. In this case we assume that \(H_1 \cup H_2\) is a convex octagon.

![Figure 3.1](image)

Take a standard representative \(a\) of \(\alpha\) so that \(a \cap (H_1 \cup H_2)\) consists of straight line segments when \(H_1 \cup H_2\) is an octagon.

3.5. To prove \(I(\alpha, \beta) \leq I(\alpha, \gamma) + 4|\alpha| = I(\alpha, \gamma) + 2|\alpha||\beta - \gamma|\), we find a standard representative \(c\) of \(\gamma\) so that \(|a \cap c| = I(\alpha, \gamma)\). By lemma 3.2, a standard representative \(b\) of \(\beta\) can be constructed as follows. The submanifold \(b\) coincides with \(c\) outside \(H_1 \cup H_2\). In each of the hexagon \(H_i - \text{int}(N(t_i))\), \(b\) is obtained from \(c\) by one of the three surgery operations \(S_1^-, S_2^-\) or \(S_3^-\) described in §2.6, figure 2.2. Inside the regular neighborhood \(N(t_1)\) of \(t_1\), the submanifold \(b\) is obtained from \(c\) by a switch operation as shown in figure 3.2.
Note that the dotted arcs in figure 3.2 indicate the arcs to be deleted in the surgery or arcs whose end points are isotoped into $\partial \Sigma$. Also note that the submanifold $b$ is obtained from $c$ by a surgery construction in $N(t_1)$ using at most four copies of $t_1$. By definition, $I(\alpha, \beta) \leq |a \cap b|$. We estimate the intersection number $|a \cap b|$ by considering the locations of intersection points. Since the surgery operations in figure 2.2 show that the new arcs created are in a small neighborhood of the deleted arcs and $t_1$, $|a \cap b \cap (H_i - N(t_1))| \leq |a \cap c \cap (H_i - N(t_1))|$. By the construction, $|a \cap b \cap N(t_1)| \leq 4|a \cap t_1|$. Thus $I(\alpha, \beta) \leq |a \cap b| \leq |a \cap c| + 4|a| = I(\alpha, \gamma) + 4|\alpha|$. If $\alpha \in CS_0(\Sigma)$, then the end points of $a \cap H_i$ are all in the $A$-sides. It follows that $2|a \cap t_1| \leq |\alpha|$. Thus inequality (3) holds.

3.6. To prove $I(\alpha, \gamma) \leq I(\alpha, \beta) + 2|\alpha| = I(\alpha, \beta) + |\alpha||\beta - \gamma|$, we find a standard
representative $b$ of $\beta$ so that $|a \cap b| = I(\alpha, \beta)$ and $b \cap (H_1 \cup H_2)$ consists of straight line segments when $H_1 \cup H_2$ is an octagon. By lemma 3.2, a standard representative $c$ for $\gamma$ can be constructed as follows. Let $c$ be the 1-submanifold coincide with $b$ outside $H_1 \cup H_2$. In each of the hexagon $H_i - \text{int}(N(t_1))$, $c$ is obtained from $b$ by one of the three surgeries $S_1^+, S_2^+, S_3^+$ described in §2.6 figure 2.3. Inside the regular neighborhood $N(t_1)$ of $t_1$, the 1-submanifold $c$ is obtained from $b$ by a switching operation as shown in figure 3.4.

![Figure 3.4](image)

Surgery from $b$ to $c$ in a neighborhood of $t_1$.

There are six types of surgeries relating $b$ to $c$. See figure 3.5.

![Figure 3.5](image)

We need to consider the two cases that $H_1 \neq H_2$ or $H_1 = H_2$ separately. Let us focus on the primary case that $H_1 \neq H_2$, i.e., $H_1 \cup H_2$ is an octagon. In this case, let
$c'$ be the representative of $\gamma$ so that 1) $c'$ is equal to $c$ outside the octagon $H_1 \cup H_2$ and 2) $c' \cap (H_1 \cup H_2)$ consists of line segments. By definition $I(\alpha, \gamma) \leq |a \cap c'|$. Thus to establish inequality (2), it suffices to show that inside the octagon $H_1 \cup H_2$,

\[(4) \quad |a \cap c' \cap (H_1 \cup H_2)| \leq |a \cap b \cap (H_1 \cup H_2)| + 2|a|.
\]

To show (4) and to save notation, consider only the local problem where $a$, $b$, and $c'$ are line segments in a single octagon $H_1 \cup H_2$. Thus we can write merely $a$ instead of $a \cap (H_1 \cup H_2)$, etc. By the surgery construction relating $b$ and $c'$ (see figure 3.5), we can express $b = b_1 \cup b_2$ and $c' = b_1' \cup b_2'$ as disjoint unions of line segments so that 1) $b_2$ consists of parallel arcs crossing $t_1$, 2) $\partial b_2 \subset \partial b_2'$, $|\partial b_2' - \partial b_2| \leq 4$ and, 3) for each component $x$ of $b_2$ there is a component $x'$ of $b_2'$ so that $x$ and $x'$ share one end point and the other end points of $x, x'$ bound an open interval containing at most one point of $\partial b_2$. See figure 3.6.

![Figure 3.6](image)

Now for each line segment $t$ in $H_1 \cup H_2$ whose end points are in $\partial(H_1 \cup H_2)$, we have $|b_2' \cap t| - |b_2 \cap t| \leq 2$. Indeed, the intersection of two line segments $s$ and $t$ in an octagon $H_1 \cup H_2$ is completely determined by the relative positions of their end points $\partial s$ and $\partial t$. It follows that inside $H_1 \cup H_2$, $|a \cap c' \cap (H_1 \cup H_2)| \leq |a \cap b \cap (H_1 \cup H_2)| + 2n$ where $n$ is the number of components of $a \cap (H_1 \cup H_2)$. Since the number of components of an arc system having $t$-coordinate $(x_1, x_2, x_3, y_1, y_2, y_3)$ inside a hexagon is $x_1 + x_2 + x_3 + y_1 + y_2 + y_3$, it follows that $n \leq |a|$. (To see this there are several cases need to be verified. Namely one should discuss the cases where pair of $A$-sides of $H_1 \cup H_2$ corresponds to a single $t_i$.) Thus inequality (4) follows.

The second case that $H_1 = H_2$ is an annulus is simple. We simply note that there are three surgeries relating $c$ to $b$ as shown in figure 3.7. The three surgeries
depend on the number $n$ of components of arcs jointing $t_2$ to $\partial \Sigma$ in $H_1$. In the first case, $n \geq 4$, in the second case $n = 0$ and in the last case $n = 2$. In the first case $n \geq 4$, we remove four such arcs and replace them by four arcs going around the boundary component of $\partial \Sigma$. In the second case, we add two parallel copies of the boundary components. In the last case of $n = 2$, we remove these two arcs and replace them by a parallel copy of the boundary component and an arc going around the boundary. Evidently inequality (2) holds.

Figure 3.7

§4. The Topology of the Space of Measured Laminations

In this section, we derive the known fact ([Mo], [Th]) that the spaces of all closed measured laminations $ML_0(\Sigma)$ is homeomorphic to a Euclidean space.

To see that $ML_0(\Sigma)$ is a Euclidean space, we fix an ideal triangulation $t = t_1 \cup ... \cup t_N$ of the surface. By theorem 1.3, we see that the map $T : ML_0(\Sigma) \to \mathbb{R}_{\geq 0}^N$ sending an element $\alpha$ to its $t$-coordinate is an embedding into a closed subset. It remains to find the image of the map $T$. To this end, let us find the images under $T$ of the space of all closed curve systems $CS_0(\Sigma)$. Given a $t$-coordinate $x = (x_1, ..., x_N)$ subject to the condition that when $t_i, t_j,$ and $t_k$ form the $A$-sides of a hexagonal component of $\Sigma - \cup_r t_r$, then $(x_i, x_j, x_k) \in \Delta$, one constructs an essential submanifold $s$ with $x$ as its $t$-coordinate by lemma 3.2. This essential submanifold $s$ is a closed curve system if and only if the submanifold $s$ contains no loop parallel to $\partial \Sigma$. This is the same as saying that at least one of the hexagons incident on $\partial_i \Sigma$ does not contain an arc parallel to the $B$-side corresponding to $\partial_i \Sigma$, i.e., for each boundary component $\partial_i \Sigma$,

$$(8) \quad \min_H \{x_j + x_k - x_i\} = 0$$

where the minimum runs over all hexagons $H$ incident on $\partial_i \Sigma$ and $H$ is formed by the arcs $t_j, t_k,$ and $t_i$ with $t_i$ opposite to a $B$-side in $\partial_i \Sigma$. Suppose $r$ is the number
of boundary components of the surface $\Sigma$. There are $r$ many equations (8). Thus we see that $CS_0(\Sigma)$ can be described as a finite union of regions, each of which is described by integer coefficient linear equations (coming from (8)) in the $x_i$ and triangle inequalities saying that certain linear combinations of the $x_i$ with integer coefficients are nonnegative. Thus the set of rational solutions to these equations is dense in the set of real solutions. This shows that the image $T(ML_0(\Sigma))$ is equal to the subspace $S$ of $\mathbb{R}_\geq 0^N$ subject to $r$ equations (8) and the triangular inequalities:

$$x_j + x_k \geq x_l$$

where $t_j, t_k, t_l$ form the $A$-sides of a hexagonal component of $\Sigma - \bigcup_{r=1}^{N} t_r$.

One may see the topological type of the space defined by equations (8) and inequalities (9) as follows. Let us make a change of variable $s$ by letting $y_i = (x_j + x_k - x_l)/2$ in (9). Geometrically, $y_i$ is the number of copies of arcs parallel to the $B$-side of the hexagon corresponding to a boundary component of $\partial \Sigma$. Then (9) becomes $(y_1, \ldots, y_M) \in \mathbb{R}_\geq 0^M$. The equations (8) become

$$\min\{y_i | i \in B_j\} = 0,$$

where the index set $B_j$ consists of indices $i$ so that $y_i$ is around the $j$-th boundary component of the surface. Finally we have a new set of equation defined on each edge of the form

$$y_i + y_j = y_k + y_l$$

for each edge $t_n$ of the ideal triangulation so that $y_i, y_j, y_k, y_l$ are adjacent to $t_n$. These are exactly the switching equations in the train-track dual to the ideal triangulation $t$ ([Mo], [Th]). We claim that the equations (10) and (11) define a space $S$ in $\mathbb{R}^M$ homeomorphic to a Euclidean space of dimension $6g - 6 + 2r$. To this end, consider the linear subspace $V$ of $\mathbb{R}^M$ spanned by the vectors $\sum_{i \in B_j} e_i$ where $e_i$ is the vector with $y_i = 1$ and all other $y_j = 0$. Let $W$ be the linear subspace defined by equations (11). Let $P : \mathbb{R}^M \to \mathbb{R}^M/V$ be the quotient map. We claim that the restriction map $P|_S : S \to P(W)$ is a homeomorphism. Since $S$ is closed and locally compact, it suffices to show that the restriction map $P|_S$ is one-to-one and onto. To see the map is onto, given a vector $y = (y_1, \ldots, y_M)$ in $W$, by adding the vector $-\sum_{j=1}^{r} \sum_{i \in B_j} \min\{y_i | i \in B_j\} e_i$ to $y$, we see that the new vector is in the space $S$. On the other hand, if $y$ and $y'$ are two vectors in $S$ so that $y - y' \in V$, then by looking at the components around each boundary $\partial_j \Sigma$, we conclude that $y = y'$. This shows that the space $S$ and hence $ML_0(\Sigma)$ is homeomorphic to a Euclidean space. To find the dimension of the Euclidean space, we note that the two linear subspaces $W$ and $V$ intersect transversely at 0 in $\mathbb{R}^M$. Assuming this, since the dimension of $W$ is $6g + 3r - 6$ and the dimension of $V$ is $r$, one finds the dimension of the quotient space to be $6g + 2r - 6$. 

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It remains to show that the subspaces $W$ and $V$ intersect transversely at 0. This follows from a little bit of combinatorics, a linear combination of equations of type (10) may be regarded as a linear combination of the duals to the $t_i$ (suitably directed). Suppose a sum of these is a sum of equations of type (12). Then duals to consecutive $t_i$ around a boundary component must get weights which differ by a constant. But since the boundary edges cycle, this says that the duals to the $t_i$ incident to a particular boundary component all get the same weight. Since every boundary component is joined in a connected graph by the $t_i$, we conclude that all duals get the same weight (up to sign for orientation). However looking at a single hexagon shows that the orientations cannot be compatible unless all the weights are zero and therefore all the weights are zero. Thus the only linear combination which vanishes is the trivial one. This establishes the assertion.

Finally, we remark that the same argument shows that the closures of $Q_{\geq 0} \times T(ES(\Sigma))$ and $Q_{\geq 0} \times T(CS(\Sigma))$ are Euclidean spaces. Their images in $\mathbb{R}^{2N}$ are given by $\{(x_1,\ldots,x_N,y_1,\ldots,y_N) \in \mathbb{R}^{2N} \mid x_i \geq 0, y_i \geq 0 \text{ and } x_i y_i = 0 \text{ for all } i\}$ and $\{(x_1,\ldots,x_N,y_1,\ldots,y_N) \in \mathbb{R}^{2N} \mid x_i \geq 0, y_i \geq 0, x_i y_i = 0 \text{ for all } i, \text{ and equation (8)}\}$.

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