Differential rotation and convection in the Sun

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ABSTRACT

We show that the differential rotation profile of the solar convection zone, apart from inner and outer boundary layers, can be reproduced with great accuracy if the isorotation contours correspond to characteristics of the thermal wind equation. This requires that there be a formal quantitative relationship involving the entropy and the angular velocity. Earlier work has suggested that this could arise from magnetohydrodynamic stability constraints; here, we argue that purely hydrodynamical processes could also lead to such a result. Of special importance to the hydrodynamical solution is the fact that the thermal wind equation is insensitive to radial entropy gradients. This allows a much more general class of solutions to fit the solar isorotation contours, beyond just those in which the entropy itself must be a function of the angular velocity. In particular, for this expanded class, the thermal wind solution of the solar rotation profile remains valid even when large radial entropy gradients are present. A clear and explicit example of this class of solution appears to be present in published numerical simulations of the solar convective zone. Though hydrodynamical in character, the theory is not sensitive to the presence of weak magnetic fields. Thus, the identification of solar isorotation contours with the characteristics of the thermal wind equation appears to be robust, accommodating, but by no means requiring, magnetic field dynamics.

Key words: convection – instabilities – Sun: helioseismology – Sun: rotation.

1 INTRODUCTION

In a recent paper, Balbus (2009, hereafter B09) argued that the shape of the isorotation contours in the solar convection zone (SCZ) may be understood as a consequence of a dominant thermal wind balance in the vorticity equation, together with the near coincidence of surfaces of constant specific entropy $S$ (isentropes) and surfaces of constant angular velocity $\Omega$ (isotachs). With $S$ a function of $\Omega$, the thermal wind equation (TWE) may be solved analytically and exactly; the isorotation contours then explicitly correspond to the characteristics of the TWE. B09 found that the agreement between the characteristics and the observed $\Omega$ profile is very good, even under the simplest of assumptions. In this paper, we take a more systematic approach and find that the agreement is truly remarkable (cf. Fig. 1). The quality of the fit is strong evidence not only that thermal wind balance holds in the bulk of the SCZ (as noted previously in many numerical simulations), but also that there is some functional relationship between entropy and angular rotation that needs to be understood.

The hope, of course, is that this is telling us something important about the physics of the SCZ. B09 puts forth the case that the SCZ is in a state of marginal dynamic instability not only against convective instabilities but also against magnetobaroclinic instabilities. To understand why the distinction is important, recall that a weak magnetic field in a differentially rotating gas is a catalyst for destabilization. This effect is well known to the accretion disc community in the form of the magnetorotational instability, which renders fully turbulent what would otherwise be hydrodynamically stable Keplerian discs. B09 argues that in its magnetobaroclinic guise, this weak field instability can destabilize what would be hydrodynamically stable stratified configurations of the SCZ, driving the system to a condition of marginal instability. Such a state is reached when constant entropy surfaces and constant angular velocity surfaces nearly coincide, leading to an $S = f(\Omega^2)$ relationship.

In this paper, we examine another possible explanation for the striking fit evidenced by Fig. 1 in Section 2. We are motivated, in part, by the fact that some purely hydrodynamic calculations are able to reproduce rather well certain non-cylindrical features of the solar rotation profile (Miesch, Brun & Toomre 2006, hereafter MBT06; Miesch et al. 2008; Miesch & Toomre 2009). Those runs show that, provided there are sufficiently large latitudinal entropy gradients at the base of the SCZ, a very plausible fit to the solar

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isorotational contours can be found throughout the bulk of this region. In this current paper, we will argue that there may well be a generic hydrodynamical mechanism that explains both the observed helioseismology results and the numerical simulations.

An outline of our paper is as follows. In Section 2, we present an improved fit between an explicit solution of the TWE and the helioseismology data. We also introduce, following MBT06, the concept of ‘residual entropy’: the average entropy profile remaining after an underlying radial profile has been removed. Arguments are presented which suggest that convecting fluid elements will tend to move in surfaces of constant residual entropy. In Section 3, we further argue that the same fluid elements will also tend to move in surfaces of constant angular velocity and that these surfaces must therefore coincide with those of residual entropy. The coincidence of surfaces of residual entropy and angular velocity is necessary in our approach to obtain a solution of the TWE. In Section 4, we conclude with an evaluation and discussion of the relative merits of the hydrodynamical and magnetohydrodynamical approaches.

2 A SIMPLE MODEL FOR THE SOLAR ISOROTATION CONTOURS

2.1 Preliminaries

2.1.1 Coordinates and notation

Let \((R, \phi, z)\) be a standard cylindrical coordinate system and \((r, \theta, \phi)\) a standard spherical coordinate system. We consider an equilibrium flow in a state of azimuthal rotation in which the angular velocity \(\Omega\) is assumed to be independent of \(\phi\), but may depend upon \(R\) and \(z\). The background entropy profile \(S\) is also a function of \(R\) and \(z\). Our notation for the other fluid variables is standard: \(v\) is the velocity, \(P\) is the gas pressure and \(\rho\) is the mass density. Unless otherwise stated, all thermodynamic variables are understood to be \(\phi\)-independent azimuthal averages. The velocity \(v\) will in general contain (convective) fluctuating components; any azimuthal averaging will be treated explicitly. The SCZ gravitational field \(g\) is to
a good approximation $GM_\odot/r^2$, where $GM_\odot$ is the product of the Newtonian gravitational constant and a solar mass. Finally, we define a dimensionless entropy function $\sigma$:

$$\sigma \equiv \ln P \rho^{-\gamma},$$  \hfill (1)

where $\gamma$ is the adiabatic index. The thermodynamic entropy is then given by $S \equiv C_F \sigma$, where $C_F$ is the specific heat at constant pressure.

2.1.2 Review of the TWE solution

Our starting point is the partial differential equation arising from the assumption of a dominant thermal wind balance in the time-averaged (and thus azimuthally averaged) vorticity equation (e.g. Kitchatinov & Rüdiger 1995; Thompson et al. 2003; Miesch 2005, B09):

$$R \frac{\partial \Omega^2}{\partial \sigma} = \frac{g}{\gamma r} \frac{\partial r}{\partial \theta}$$  \hfill (2)

If, for some reason, there is a functional relationship of the form $\sigma = f(\Omega^2)$ (the sign of $\Omega$ presumably does not matter) then the TWE may be written in spherical coordinates as

$$\frac{\partial \Omega^2}{\partial r} - \frac{g f'}{\gamma r^2 \sin \theta \cos \theta} + \frac{\tan \theta}{r} \frac{\partial \Omega^2}{\partial \theta} = 0,$$  \hfill (3)

where $f' = df/d\Omega^2$. Note, however, that exactly the same equation obtains if the functional relationship between $\sigma$ and $\Omega$ takes the more general form

$$\sigma' \equiv \sigma - \sigma_r = f(\Omega^2),$$  \hfill (4)

where $\sigma'$ will be referred to as residual entropy and $\sigma_r$ is any function of $r$ alone. We will make important use of this ‘gauge freedom’ later in this paper.

The solution of equation (3) is that $\Omega^2$ is constant along the characteristic contours (B09):

$$R^2 = r^2 \sin^2 \theta = A - \frac{B}{r},$$  \hfill (5)

where $A$ is a constant of integration and

$$B = -\frac{2GM_\odot f'(\Omega^2)}{\gamma}$$  \hfill (6)

If we denote by the subscript '0' the fiducial starting point of the characteristic (at which $\Omega$ is specified) then the contour takes the form

$$R^2 = r^2 \sin^2 \theta = r_0^2 \sin^2 \theta_0 - B \left(\frac{1}{r} - \frac{1}{r_0}\right).$$  \hfill (7)

While $B$ is a constant along a given characteristic, its value can change from one characteristic to another depending upon the form of the function $f'$. B09 showed, however, that the simplest parameterizations of $f'$ (e.g. $f' = \gamma$ uniform constant or $f' = \gamma$ linear function of $\sin^2 \theta_0$) already give excellent qualitative fits to the helioseismology data. In fact, it is not difficult to do even better.

Fig. 1 shows a comparison between isorotational contours (in black) from recent GONG data1 and the characteristics of the TWE (overlayed in white). Here, we have carried out a more systematic approach than that of B09: we have integrated each characteristic from an interior starting point of 0.9 solar radii (as opposed to the less accurate surface fitting in B09) and chosen the constant $B$ to match the initial slope of the data at the fitting radius. The new result is remarkably accurate, not just in qualitative form but in quantitative detail as well. Expected departures of the black and white contours in the surface layers and tachocline are plainly visible, yet in the bulk of the SCZ the match is excellent.

We thank R. Howe for providing us with these beautiful results.

2.2 A functional relation between residual entropy and angular velocity

2.2.1 Theoretical considerations

One possible explanation for the striking resemblance of the analytic isorotation contours to the solar data was advanced in B09 by invoking a weak magnetic field. In the presence of such a field, disturbances associated with magnetobaroclinic modes lead to a condition of marginal instability if $\sigma = f(\Omega^2)$. This is a dynamical connection between $\sigma$ and $\Omega$, indeed a magnetohydrodynamic (MHD) connection.

We will review the merits and shortcomings of this scenario in more detail in Section 4. Here, we explore a more general connection between $\sigma$ and $\Omega$ relying neither upon magnetic fields nor even upon dynamical constraints. The idea relies instead on the gauge freedom expressed in equation (4).

Consider a fiducial, non-rotating, spherical, convecting star that maintains a well-defined, long-term average radial entropy profile $\sigma_r$, Convection continuously mixes high- and low-entropy fluid elements, yet $\sigma_r$ remains fixed. The reason for this is not mysterious. The system is heated from below, and this input tries to drive the gas into a state with a considerably steeper entropy profile. Convection counters this, allowing a balance to be struck between the external thermal driving and the heat transport by turbulent fluid elements. In particular, convective displacements do not need to be within constant entropy surfaces to preserve the long-term $\sigma_r$ profile, This profile is already being maintained by the external heating.

If we now introduce a small amount of rotation into the problem, we expect Coriolis forces to produce a more complex, but still well-defined, long-term entropy profile, $\sigma(\varpi, \theta)$. As before, convection will try its very best to change this profile. However, in contrast to the spherical case of the previous paragraph, it is very far from clear that radial heating from below can maintain a steady, non-spherical entropy profile. On the other hand, if the convective velocities $\varpi$ on average satisfy the constraint

$$\varpi \cdot \nabla \sigma' = 0,$$  \hfill (8)

then the entropy profile certainly can be maintained. In this case, the long-term maintenance of the spherically averaged component of the entropy $\sigma_r$ is assured by heating from below, while the residual

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1 We thank R. Howe for providing us with these beautiful results.

2 The inversion process by which the rotation contours are determined from helioseismology data is, it should be noted, less reliable near the poles.
entropy $\sigma'$ is preserved because convective displacements will tend on average to occur in surfaces in which this quantity is constant. Thus, the long-term average entropy profile $\sigma(r, \theta)$ remains intact. Convecting elements move not in constant entropy surfaces (which would preclude heat transport!), but in surfaces of constant residual entropy $\sigma'$.

Some care is needed for the proper interpretation of equation (8). The poloidal components of $\mathbf{v}$ (hereafter $v_p$) have a long-term average value much smaller than a typical fluctuation amplitude $|v_p|$. (This systematic velocity manifests itself principally as the meridional flow that has been measured at, and near, the surface.) A time average of equation (8) would produce contributions of second order in the fluctuations, including heretofore neglected $\sigma'$ fluctuations. On the other hand, the fact that the ‘up-down’ convective velocity fluctuations nearly cancel means that for a given convective cell there is very little to distinguish these two directions. (Because of rotation, ‘upward’ and ‘downward’ need not be precisely radial, of course.) These first-order velocity fluctuations in essence establish a well-defined axis relative to which $\nabla \sigma'$ is orthogonal; the positive and negative sense of the axis is unimportant. A more precise formulation of equation (8) is therefore

$$
\langle (v \cdot \nabla \sigma')^2 \rangle \ll \langle |v_p|^2 \rangle \langle |\nabla \sigma'|^2 \rangle, \tag{9}
$$

where the angle brackets $\langle \rangle$ denote time averaging. This is also a more robust formulation, since the quantity on the left is now dominated by the first-order velocity contribution $v_p$, and fluctuations in $\sigma'$ do not enter.

More physically, simulations and laboratory experiments suggest that convective transport is dominated by long-lived coherent structures. The updrafts and downward plumes characteristic of these structures are clearly associated with dominant ‘first-order’ velocity terms, and it is these structures that would exist within and maintain constant $\sigma'$ surfaces. If it could now be further argued that convecting fluid elements (or coherent structures) also lie within constant $\Omega$ surfaces, we would arrive at an understanding of why there is a functional relation between $\Omega$ and $\sigma'$: if elements are simultaneously moving in $r\theta$ surfaces of constant $\Omega$ and $\sigma'$, these must be the same surfaces. In other words, the two quantities are functionally related. In the next section, we will see that there is indeed a simple argument leading to the conclusion that displaced fluid elements will inevitably move in constant $\Omega$ surfaces.

Finally, we reiterate that although the original B09 derivation of the isorotation contours was based on the assumption that $\sigma = f(\Omega^2)$, exactly the same contours emerge if the functional relation is $\sigma' = f(\Omega^2)$. This very simple gauge invariance of the TWE allows for a considerably wider, more general class of thermal wind solutions, eliminating the constraint that constant (total) entropy and angular velocity surfaces coincide.

### 2.2.2 Comparison with simulations

Support for the line of argument advanced in the previous section can be found in the (purely hydrodynamical) MBT06 simulations. In these runs, the residual entropy $\sigma'$ is calculated relative to a background radial profile, corresponding to our $\sigma_r$. In Fig. 2, we show time-averaged contours of the angular velocity profile $\Omega$ from the MBT06 run designated AB3, of the residual entropy and of the total entropy $\sigma$. The agreement between the $\Omega$ and $\sigma'$ contours is apparent, as is the gross disagreement between $\Omega$ and $\sigma$. Evidently, there is a functional relation between $\sigma'$ and $\Omega$, whereas there is no such relation between $\Omega$ and $\sigma$. This is in good agreement with the above discussion. Note as well the small gradient of $\sigma'$ near the equator. This is as expected if $\sigma'$ represents an angular average of $\sigma$, which would be dominated by $\theta$ values near $\pi/2$.

### 3 Non-axisymmetric disturbances in a differentially rotating medium

#### 3.1 Comoving coordinates and wavenumbers

Consider the behaviour of convective disturbances in a differentially rotating gas $\Omega(R, z)$. The disturbances are embedded in this shearing medium, and since they are not propagated as waves, we may think of them as local in character. But ‘local’ refers to the shearing gas, not to an absolute space–time grid, so it is particularly useful to express flow quantities in Lagrangian coordinates tied to the shear flow. We designate these coordinates as the ‘primed’ system. This
transformation is given by
\[ R' = R, \quad \phi' = \phi - \Omega(R, z)t, \quad z' = z, \quad t' = t. \] (10)

For our present purposes, we need consider only the equation of mass conservation:
\[ \nabla \cdot (\rho v) = 0, \] (11)
where \( v \) is understood to be the velocity relative to the differential rotation \( R\Omega \). The gradients are of course the standard Eulerian derivatives taken at fixed time. We have not written the explicit \( \rho \) time derivative, since we are ultimately interested in long-term time averaging, and the term will disappear. Apart from this, the equation is exact; the mass flux is taken to have a full space time dependence.

Transforming to our comoving system,
\[ \frac{\partial}{\partial R} = \frac{\partial}{\partial R'} - t \frac{\partial}{\partial \Omega R} \frac{\partial}{\partial \phi'}, \] (12)
\[ \frac{\partial}{\partial z} = \frac{\partial}{\partial z'} - t \frac{\partial}{\partial \Omega} \frac{\partial}{\partial \phi'}, \] (13)
and the \( \phi \) derivative is unchanged. More compactly,
\[ \nabla = \nabla' - (t \nabla \Omega) \frac{\partial}{\partial \phi'}, \] (14)

a relation that holds for all three components.

The embedded mass flux field \( \rho v \) may be expressed as a superposition of Fourier components of the form
\[ \exp \left[ i \left( k'_{R} R' + m \psi' + k'_{z} z' - \omega t' \right) \right] \] (15)
in which all components of the \( k' \) wave vector are constants. The above coordinate transformation implies that in the mass conservation equation, the standard Eulerian spatial derivatives with respect to \( R \) and \( z \) are replaced, respectively, by \( ik'_{R}(t) \) and \( ik'_{z}(t) \), where
\[ k_{R}(t) = k'_{R} - mt \frac{\partial \Omega}{\partial R} \] (16)
\[ k_{z}(t) = k'_{z} - mt \frac{\partial \Omega}{\partial z}. \] (17)

Henceforth, the time dependence of \( k_{R} \) and \( k_{z} \) will be understood. A somewhat similar formalism has been developed for embedded disturbances in shear turbulence, where it is known as rapid distortion theory (Townsend 1980).

3.2 Mass flux morphology

The Fourier series for the mass flux is
\[ \rho v = \sum \mu(k', \omega) \exp \left[ i \left( k'_{R} R' + m \psi' + k'_{z} z' - \omega t' \right) \right], \] (18)
where \( \mu(k', \omega) \) is the amplitude of the indicated wavenumber and frequency. Mass conservation leads immediately to
\[ 0 = \nabla \cdot (\rho v) = \sum \mathbf{k} \cdot \mu(k', \omega) \exp \left[ i \left( k'_{R} R' + m \psi' + k'_{z} z' - \omega t' \right) \right]. \] (19)

Each Fourier term is linearly independent of the others, hence for all \( k', \omega \),
\[ (\mathbf{k} - mt \nabla \Omega) \cdot \mu(k', \omega) = 0 \] (20)
For non-vanishing \( m \), at sufficiently large \( t \),
\[ \mu(k', \omega) \cdot \nabla \Omega = 0. \] (21)

The mass flux is a superposition of the \( \mu \) vectors. Therefore, if the flux is not dominated by an \( m = 0 \) component, it will tend towards surfaces of constant \( \Omega \). Note that this statement is restricted neither to linearized displacements nor to the WKB limit. Finally, it has been noted that this tendency for turbulent convective structures to align with constant \( \Omega \) surfaces may also be supported on a dynamical basis via the effects of the Coriolis force on plumes (Brunnell, Hurlburt & Toomre 1996).

Alignment of embedded structures with constant \( \Omega \) surfaces is of course seen regularly in accretion disc simulations, where the shear is large and depends only upon \( R \). Disc features often appear nearly axisymmetric because of the effect of the shear. In the Sun, the \( R \) and \( z \) dependence of \( \Omega \) leads to a more structured response. This assumes, of course, that the rotational shear is strong enough to interact with the convective cells over the course of their lifetime. This is a reasonable assumption for the Sun, especially if the convection is dominated by long-lived coherent structures.

This, in principle, supplies the missing link of the argument of Section 2.2.1: the course of a displaced mass element will tend on average to follow a constant \( \Omega \) surface, essentially because differential rotation wraps the flow into sheets of constant \( \Omega \). Since fluid elements move at once in surfaces of constant \( \Omega \) and constant \( \sigma' \), these surfaces must in general coincide.

Note how the complete set of three hydrodynamical equations – mass conservation, entropy conservation and the equation of motion – has been incorporated in our solution. From mass conservation, we infer that fluid elements move in constant \( \Omega \) surfaces. From entropy conservation, we infer that fluid elements move in surfaces of constant \( \sigma' \). From these two inferences, it follows that constant \( \Omega \) and constant \( \sigma' \) surfaces coincide, and this, expressed as a functional dependence, is used in the equation of motion (thermal wind balance) to construct the isorotational contours of Fig. 1.

3.3 Heat transport

The convective heat transport is
\[ Q = \rho v \delta w, \] (22)
where \( \delta w \) represents the difference between the specific enalty of a convecting element and its surroundings. The reasoning we used in the last section shows that
\[ Q \cdot \nabla \Omega \to 0, \] (23)
as time progresses. In other words, the heat flux, like the mass flux, is predominantly in surfaces of constant \( \Omega \).

This offers a very plausible physical basis for what has long been regarded as a puzzling issue: why are the solar rotation contours quasi-radial? The answer we suggest is that the rotation contours are also the natural conduits for convective heat transport, and their near radial character simply reflects this property. The contours represent a compromise between strong radial thermal driving and the dynamical exigencies embodied in the TWE. Thus, they are nearly, but not perfectly, radial throughout the bulk of the SCZ. Near the equator and poles, the convective heat flow certainly does not follow isorotation contours, but as has already been noted in Section 2.1.1, these are precisely the regions in which the functional relationship between \( \sigma' \) and \( \Omega \) is maintained by mathematical symmetry.

3 The degenerate case in which the surfaces do not coincide would correspond to trivial circular azimuthal orbits of the fluid element.
It is interesting to speculate on how the isorotation contours would change in rapidly rotating stars. For example, when a solar-type star first arrives on the main sequence it spins much more rapidly than the Sun does now. The convection pattern is likely to be dominated by Coriolis effects, expressed through the Taylor–Proudman constraint (Busse & Simitev 2007). Numerical models confirm that convection takes the form of elongated ‘banana cells’ at low latitudes, outside the tangent cylinder that encloses the radiative zone (Ballot, Brun & Turck-Chièze 2007; Brown et al. 2008). In these rapidly rotating stars, the isochats tend to be roughly cylindrical and there are pole-equator differences in entropy and temperature that are significantly larger than those inferred for the Sun. In our model, the $1/Ω^2$ scaling of $f(Ω^2)$ suggests a much smaller value for the $B$ parameter in rapid rotators, which does indeed lead to more cylindrical isochats.

4 CONCLUDING DISCUSSION

What is novel about our approach is not the use of the TWE per se, which has provided important guidance to workers in this field for many years now. It is the suggestion that there is a functional relationship between the entropy and angular velocity, for it is this that allows an analytic deduction of the shape of the Sun’s isorotation contours to be made directly from the TWE.

As a formal matter, there is some freedom in choosing the precise form of the entropy–angular velocity relation, a type of gauge invariance. This is because the TWE involves only the $\partial_\sigma$ derivative of the entropy and the $\partial_\Omega$ derivative of the angular velocity. Different physical mechanisms for coupling $\sigma$ and $\Omega$ in principle entail a different choice of gauge.

In B09, an MHD process was invoked to couple $\sigma$ and $\Omega$. That paper noted that baroclinic modes in a weakly magnetized gas become marginally unstable if constant $\sigma$ and constant $\Omega$ surfaces coincide, i.e. $\sigma = f(Ω^2)$. In that approach, rotation and stratification are set on an equal footing in determining the dynamical stability of the Sun’s outer layers. Moreover, in the MHD model, the radial entropy gradient is restricted to be significantly less than the $\partial_\sigma$ gradient. This can be readily deduced from Fig. 1, since in the $\sigma = f(Ω^2)$ interpretation, the GONG data curves are isentropes as well as isorotation contours. Whether the actual radial entropy gradient is constrained in this way depends upon the scale of the effective mixing length. If this length is only a small fraction of the radial extent of the convection zone, the radial entropy gradient could significantly exceed the $\partial_\sigma$ gradient.

Finally, in the MHD model, the goodness of the fit between the theoretical and observed isorotation contours depends upon a very tight functional relationship between $\sigma$ and $\Omega$. But, were the $\sigma$ and $\Omega$ surfaces to coincide exactly, convection would be completely cutoff. Conversely, as we have noted above, if very vigorous convection is required, it becomes untenable to maintain a $\sigma = f(Ω^2)$ relation. Is the departure between $\nabla\sigma$ and $\nabla\Omega$ large enough to allow for vigorous convection, yet small enough to ensure that $\sigma = f(Ω^2)$ remains an excellent approximation in the TWE?

The purely hydrodynamical explanation for the solar isorotation contours avoids many of these potential difficulties. The restriction on the size of the radial entropy gradient is lifted by seeking a functional relationship of the form of equation (4). As the residual entropy $\sigma’$ can be significantly smaller than $\sigma$, the effects of a potentially large radial entropy gradient can be eliminated from the rotational dynamics of the problem. The physical basis for this functional form is related to the fact that convection should occur on average in surfaces of constant $\sigma’$, so as to maintain a well-defined time-averaged $\sigma$ (total entropy) profile. The kinematics of differential rotation and simple mass conservation ultimately confine mass motions to surfaces of constant $\Omega$. Together with equation (8), this ensures a relationship of the form $\sigma’ = f(Ω^2)$ and even suggests why the contours are quasi-radial (cf. equation 23). Without the need to restrict $\sigma’/\partial_\sigma$, convection can be as vigorous as needed (provided that it does not dominate the differential rotation), yet it does not directly affect the TWE solution for the isorotation contours.

Finally, the ‘hydrodynamical’ argument can easily accommodate MHD processes. Neither mass conservation nor entropy conservation, the two physical processes central to our argument, change in the presence of a magnetic field. The dynamical details, in particular the rotational couplings, may well be altered, but the functional relation between $\sigma’$ and $\Omega$ should remain robust. We therefore suggest that the simple physical arguments leading to equation (4) may prove to be a suitable basis for understanding much of the dynamics that gives rise to the solar isorotation contours.

The theory outlined in this paper is meant to be a viable alternative to an MHD model, and it is as yet far from complete. Our presentation falls well short of the standards of mathematical proof. We have relied on ‘tendencies’ for the fluid to move in surfaces of constant $\Omega$ and $\sigma’$ to be strong enough to firmly establish the relationship expressed compactly by equation (4). But, Fig. 1 speaks for itself. To understand at a deeper level why this approach seems to work so well, a more detailed elucidation of the dynamics is clearly desirable.

To conclude, in this paper we have advanced theoretical arguments to explain the striking correspondence between isentropes and isochats in azimuthally averaged models of the SCZ. These arguments need to be backed up by directed and detailed computations. To this end, we are currently developing an anelastic code that can be used to demonstrate explicitly that the relationship of equation (4) is valid for a variety of relevant configurations. Our principal aim will be to explore the connection between $\Omega$ and $\sigma$ (or $\sigma’$) as the overall rotation rate is varied. The effects on this relationship of varying the thermal boundary conditions, in order to represent the influence of the solar tachocline (Miesch & Toomre 2009), are also of great interest.

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