Determination of the variable density of the rod from natural frequencies of longitudinal vibrations

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Abstract. Rods of various configurations are elements of many structures and machines. Therefore, the acoustic and vibration diagnostics of such parts has been widely developed. The paper considers the problem of determining the variable density of the rod from the natural frequencies of longitudinal vibrations. It is assumed that the density changes along the axis and is described by a polynomial function. This approach allows one to determine the law of density variation from a finite set of eigenvalues. The results of the study can find applications for finding hidden defects in steel and composite rods, which arise during the production process or due to corrosion.

1. Introduction
Rods of various configurations are elements of many structures and machines. Therefore, the acoustic and vibration diagnostics of such parts has been widely developed. These types of diagnostics are based on solving inverse problems. A feature of the problem of determining the variable density of a material is that the density cannot be visually determined or measured with a certain device. In special cases, it is possible to determine the heterogeneous structure inside the material using fluoroscopy. This problem is of the greatest relevance for bodies made of composite materials, where the density is unevenly distributed. Determination of damage and defects of rods by natural frequencies of longitudinal vibrations is especially important for long rods, since the longitudinal vibrations can propagate over longer distances than transverse ones. For example, such objects are drill rods, pipelines, power line wires, etc. A large number of works are devoted to the determination of the natural frequencies of longitudinal vibrations of elastic rods with variable density [1, 2, 3, 4]. As a rule, bar defects, for example, corrosion, are simulated as stepped bars [2, 4], that is, a bar consisting of two or more parts with different cross-sectional areas. In this paper, it is proposed to describe the variable density in the form of a polynomial function of the longitudinal coordinate. A problem close to the solution method was considered in [5], where the inverse problem for a rod with a variable cross-sectional area was considered. In the works [8, 9], the inverse coefficient problems on the recovery of the elastic modulus, density, and section function of a bar with a rigidly fixed left end are considered. The problem considered in this study work is also the inverse coefficient problem, since it is required to find not the natural frequencies, but the law of density variation.
2. Statement of the problem
The problem of determining the function that describes the law of variation of the density of the rod is considered. The natural frequencies of longitudinal vibrations will be used as input data. It is assumed that the density varies along the axis and is described by a polynomial of n degree. The polynomial was chosen due to the fact that any continuous function can be expanded in a power series. The stepwise change of the required function is not considered. It is proposed to solve the inverse problem by establishing the dependence of the natural vibration frequencies on the density distribution along the axis of the rod. This dependence can be obtained from the frequency equation if we assume that the coefficients of the polynomial density function are unknown. One-dimensional longitudinal vibrations of a rod with variable density are described by the equation [7, 9].

$$ES \frac{\partial^2 u}{\partial x^2} - \rho(x)S \frac{\partial^2 u}{\partial t^2} = 0, \quad (1)$$

where $u(x, t)$ is the displacement of the section with the $x$ coordinate, $\rho(x)$ is the density at the point $x$, the modulus of the elasticity $E$ and the sectional area $S$ of the rod are considered to be constant. The attachment to the ends of the rod can be different. As an example, we take a rigid boundary condition $u(0) = u(l) = 0$.

In this paper, it is proposed to take the variable density of the bar at the point $x$ in the form of a polynomial of $n$ degree

$$\rho(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n. \quad (2)$$

3. The solution method
The solution to equation (1) will be sought for following in the form, then the eigenvalue problem will take the form

$$y'' + \rho(x)\lambda^2 y = 0, \quad (3)$$
$$y(0) = y(1) = 0. \quad (4)$$

Equations (3) - (4) are the Sturm-Liouville problem, where $\lambda^2 = \frac{w^2}{E}$. The method for solving this problem is similar to [5, 6], that is, the general solution of equation (3) will be sought for in the form

$$y(x, \lambda) = C_1 y_1 + C_2 y_2, \quad (5)$$

here $y_1, y_2$ form the fundamental system of solutions (3). The polynomial functions $y_1, y_2$ will be constructed in the form of a Maclaurin series in the variables $x$ and $\lambda$, for which the conditions are fulfilled

$$y_1(0, \lambda) = 1, \quad y'_1(0, \lambda) = 0,$$
$$y_2(0, \lambda) = 0, \quad y'_2(0, \lambda) = 1. \quad (6)$$

This approach is similar to the sequential differentiation method. The difference is that in (5) each of the functions $y_1, y_2$ is constructed separately. The Cauchy data (6) ensure linear independence of the functions $y_1, y_2$ due to the properties of the Wronski determinant. Then $y_1, y_2$ of problem (3) - (4) under condition (6) are written down as

$$y_1(x, \lambda) = 1 - \lambda^2 \rho(0) \frac{x^2}{2!} - \rho'(0) \cdot \lambda^2 \frac{x^3}{3!} + \left(\lambda^4 \rho^2(0) - \lambda^2 \cdot \rho''(0)\right) \frac{x^4}{4!} + \ldots$$
$$y_2(x, \lambda) = x - \lambda^2 \rho(0) \frac{x^3}{3!} - 2 \lambda^2 \rho'(0) \frac{x^4}{4!} + \ldots$$
Further, substituting the general solution (5) into the boundary conditions (4), we obtain a nonlinear equality in which only the density and eigenvalues are present. This expression is called the frequency equation [5]

\[\Delta(\lambda) = y_2(1, \lambda) = 1 - \lambda^2 \rho(0) \frac{1}{3!} - 2\lambda^2 \rho'(0) \frac{1}{4!} + \ldots = 0. \tag{7}\]

Table 1 shows the form of the frequency equation in the case of various boundary conditions. Here, \(h, H\) denote the stiffness of springs at the left and the right ends, respectively.

| The left end boundary condition | The right end boundary condition | Frequency equation |
|-------------------------------|---------------------------------|--------------------|
| \(y(0) = 0\)                 | \(y'(0) = 0\)                  | \(\Delta(\lambda) = y_2'(1, \lambda) = 0\) |
| \(y'(0) = 0\)                | \(y''(0) = 0\)                 | \(\Delta(\lambda) = y_2'(1, \lambda) = 0\) |
| \(y(0) = 0\)                 | \(y'(0) + Hy(1) = 0\)          | \(\Delta(\lambda) = y_2'(1, \lambda) + H y_2(1, \lambda) = 0\) |
| \(y'(0) - Hy(0) = 0\)        | \(y(1) = 0\)                   | \(\Delta(\lambda) = -h y_2'(1, \lambda) - y_2(1, \lambda) = 0\) |
| \(y'(0) - Hy(0) = 0\)        | \(y'(1) = 0\)                  | \(\Delta(\lambda) = -h y_2'(1, \lambda) - y_2'(1, \lambda) = 0\) |

If the density function (2) is known (a direct problem), then solving (7) for \(\lambda\), we obtain the eigenvalues of problem (3) - (4). In the case when it is required to determine the density distribution law (an inverse problem), substituting the known eigenvalues in (7), we obtain a system of equations for the unknown coefficients \(a_0 \ldots a_n\). To reconstruct a polynomial of \(n\) degree, \(n + 1\) eigenvalues are required. Consequently, the more eigenvalues are known, the more accurate the solution of the inverse problem. It should be noted that in problem (3) - (4), the boundary conditions are symmetric and the inverse problem has two solutions. The second solution is obtained by swapping the ends of the rod in places. The uniqueness of the solution is achieved if additional information is known, for example, the density at the fixing point \(\rho(0)\).

Let us consider an example. It is required to determine the distribution law of the density of a rod of unit length. The left end of which is rigidly fixed \(y(0) = 0\), and the right end is elastically \(y'(1) + 2y(1) = 0\). Let there be known three eigenvalues \(\lambda_1 = 3.84808\), \(\lambda_2 = 8.17923\), \(\lambda_3 = 12.77029\), then we can restore the density function as a polynomial of second degree, that is, \(\rho(x) = a_0 + a_1 x + a_2 x^2\). Under these boundary conditions, the frequency equation takes the form \(\Delta(\lambda) = y_2'(1, \lambda) + 2y_2(1, \lambda) = 0\). Here

\[y_2 = x - \frac{1}{6} a_0 \lambda^2 x^3 - \frac{1}{12} a_1 \lambda^2 x^4 + \left( \frac{1}{120} a_0^2 \lambda^4 - \frac{1}{20} a_2 \lambda^2 \right) x^5 + \ldots,\]

\[y_2' = 1 - \frac{1}{2} a_0 \lambda^2 x^2 - \frac{1}{3} a_1 \lambda^2 x^3 + 5 \left( \frac{1}{120} a_0^2 \lambda^4 - \frac{1}{20} a_2 \lambda^2 \right) x^4 + \ldots.\]

For numerical calculations of the polynomials \(y_1\) and \(y_2\) take 40 orders.
Substituting the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ into the frequency equation, we obtain a nonlinear system

\[
\begin{align*}
\Delta(\lambda_1) &= y''_2(1, \lambda_1) + 2y_2(1, \lambda_1) = 0, \\
\Delta(\lambda_2) &= y''_2(1, \lambda_2) + 2y_2(1, \lambda_2) = 0, \\
\Delta(\lambda_3) &= y''_2(1, \lambda_3) + 2y_2(1, \lambda_3) = 0.
\end{align*}
\]

Using the Maple mathematical package, a numerical solution was obtained as $a_0 = 0.60402$, $a_1 = -0.70997$, $a_2 = 0.44534$. This solution was not exactly obtained (the exact solution is $a_0 = 0.8$, $a_1 = -1.0$, $a_2 = 0.4$), since in the search range $a_0 = -2.2$, $a_1 = -2.2$, $a_2 = -2.2$ there are several solutions. The program displays the first found solution.

Figure 1 shows the graphs of the distribution of a given density to be found numerically. Consider the eigenvalues of the found density function $\rho(x) = 0.44534x^2 - 0.70997x + 0.60402$. Using the above algorithm, it is easy to get $\lambda_1 = 3.84808$, $\lambda_2 = 8.17923$, $\lambda_3 = 12.95048$. If we compare the obtained eigenvalues from the initially given one (calculated under the assumption that $\rho(x) = 0.4x^2 - 1.0x + 0.8$), then we can see that the first two coincide exactly, the difference is only in the third eigenvalue. This suggests that the solution to the inverse problem is sensitive to the accuracy of the input data.

If we assume that the density on the left boundary ($x = 0$) is known, that is, $a_0 = 0.8$, then in the given search range we get a more accurate solution $a_1 = -1.00007$, $a_2 = 0.40011$.

4. Conclusion

The representation of the variable density in the form of a polynomial (2) allows one to determine the law of density variation along the axis using a finite set of eigenvalues. It can be seen from the given example that this inverse problem is sensitive to the input data error. When determining the coefficients of the density polynomial, a narrow search range must be specified, since the problem can have several solutions, including complex roots. The results of the study can find applications for finding the hidden defects in rods that arise during production, operation, or due to corrosion.

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