LIE HIGHER DERIVATIONS ON GENERALIZED MATRIX ALGEBRAS

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Abstract. In this paper, at first the construction of Lie higher derivations and higher derivations on a generalized matrix algebra were characterized; then the conditions under which a Lie higher derivation on generalized matrix algebras is proper are provided. Finally, the applications of the findings are discussed.

1. INTRODUCTION

Let us recall some basic facts related to (Lie) higher derivations on a general algebra. Let $A$ be a unital algebra, over a unital commutative ring $\mathbb{R}$, $\mathbb{N}$ be the set of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

(a) A sequence $\mathcal{D} = \{D_k\}_{k \in \mathbb{N}_0}$ (with $D_0 = id_A$) of linear maps on $A$ is called a higher derivation if

$$D_k(xy) = \sum_{i+j=k} D_i(x)D_j(y),$$

for all $x, y \in A$ and $k \in \mathbb{N}_0$.

(b) A sequence $\mathcal{L} = \{L_k\}_{k \in \mathbb{N}_0}$ (with $L_0 = id_A$) of linear maps on $A$ is called a Lie higher derivation if

$$L_k([x, y]) = \sum_{i+j=k} [L_i(x), L_j(y)],$$

for all $x, y \in A$ and $k \in \mathbb{N}_0$, where $[\cdot, \cdot]$ stands for a commutator defined by $[x, y] = xy - yx$.

Note that $\mathcal{D}_1$ (resp. $\mathcal{L}_1$) is a derivation (resp. Lie derivation) when $\{D_k\}_{k \in \mathbb{N}_0}$ (resp. $\{L_k\}_{k \in \mathbb{N}_0}$) is a higher derivation (resp. Lie higher derivation). Let $D$ (resp. $L$) be a derivation (resp. Lie derivation) on $A$, then $\mathcal{D} = \{\frac{D_k}{k!}\}_{k \in \mathbb{N}_0}$ (resp. $\mathcal{L} = \{\frac{L_k}{k!}\}_{k \in \mathbb{N}_0}$) is a higher derivation (resp. Lie higher derivation) on $A$, where $D^0 = id_A$ (resp. $L^0 = id_A$), the identity mapping of $A$. These kind of higher derivations (resp. Lie higher derivations) are called ordinary higher derivations (resp. Lie higher derivations). Trivially, every higher derivation is a Lie higher derivation, but the converse is not true, in general. If $\mathcal{D} = \{D_k\}_{k \in \mathbb{N}_0}$ is a higher derivation on $A$ and $\tau = \{\tau_k\}_{k \in \mathbb{N}}$ is a sequence of linear maps on $A$ which is center valued (i.e. $\tau_k(A) \subseteq Z(A)$= the center of $A$), then $\mathcal{D} + \tau$ is a Lie higher derivation if and only if $\tau$ vanishes at commutators, i.e. $\tau_k([x, y]) = 0$, for all $x, y \in A$ and $k \in \mathbb{N}$. Lie higher derivations of this form are called proper Lie higher derivations. We say that an algebra $A$ has Lie higher derivation (LHD for short) property if every Lie higher derivation on it is proper. A main problem in the realm of Lie higher derivations is that, under what conditions a Lie higher derivation on an algebra is proper. Many authors have studied the problem for various algebras; see [5, 6, 7, 8, 15, 16, 17, 21, 22, 23] and references therein.

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Han [7] studied Lie-type higher derivations on operator algebras. He showed that every Lie (triple) higher derivation on some classical operator algebras is proper. Wei and Xiao [21] have examined innerness of higher derivations on triangular algebras. They also discussed Jordan higher derivations and nonlinear Lie higher derivations on a triangular algebra in [22] and [23], respectively. Qi and Hou [17] showed that every Lie higher derivation on a nest algebra is proper. Li and Shen [8] and also Qi [16] have extended the main result of [17] for a triangular algebra by providing some sufficient conditions under which a Lie higher derivation on a triangular algebra is proper.

In this paper we investigate the LHD property for a generalized matrix algebra. Generalized matrix algebras were first introduced by Sands [18]. Here, we offer definition of a generalized matrix algebra. A Morita context \((A, B, M, N, \Phi_{MN}, \Psi_{NM})\) consists of two unital algebras \(A, B\), an \((A, B)\)-module \(M\), a \((B, A)\)-module \(N\), and two module homomorphisms \(\Phi_{MN} : M \otimes_B N \rightarrow A\) and \(\Psi_{NM} : N \otimes_A M \rightarrow B\) satisfying the following commutative diagrams:

\[
\begin{array}{ccc}
M \otimes_B N \otimes_A M & \xrightarrow{\Phi_{MN} \otimes I_M} & A \otimes_A M \\
\downarrow_{I_M \otimes \Psi_{NM}} & & \downarrow_{\cong} \\
M \otimes_B B & \xrightarrow{\cong} & M
\end{array}
\]

and

\[
\begin{array}{ccc}
N \otimes_A M \otimes_B N & \xrightarrow{\Psi_{NM} \otimes I_N} & B \otimes_B N \\
\downarrow_{I_N \otimes \Phi_{MN}} & & \downarrow_{\cong} \\
N \otimes_A A & \xrightarrow{\cong} & N.
\end{array}
\]

For a Morita context \((A, B, M, N, \Phi_{MN}, \Psi_{NM})\), the set

\[
\mathcal{G} = \left( \begin{array}{ccc} A & M \\ N & B \end{array} \right) = \left\{ \begin{pmatrix} a & m \\ n & b \end{pmatrix} \bigg| a \in A, \ m \in M, \ n \in N, \ b \in B \right\}
\]

forms an algebra under the usual matrix operations, where at least one of two modules \(M\) and \(N\) is nonzero. The algebra \(\mathcal{G}\) is called a generalized matrix algebra. In above definition if \(N = 0\), then \(\mathcal{G}\) becomes the triangular algebra \(\text{Tri}(A, M, B)\), whose (Lie) derivations and its properties are extensively examined by Cheung [3].

Let \(\mathcal{G} = \left( \begin{array}{ccc} A & M \\ N & B \end{array} \right)\) be a generalized matrix algebra. We are dealing with various types of faithfulness.

1. The \((A, B)\)-module \(M\) is called left (resp. right) faithful if \(aM = \{0\}\) (resp. \(Mb = \{0\}\)) necessities \(a = 0\) (resp. \(b = 0\)), for all \(a \in A\) (resp. \(b \in B\)). If \(M\) is both left and right faithful it is called faithful. The left and right faithfulness of \(N\) can be defined in a similar way.

2. The \((A, B)\)-module \(M\) is called strongly faithful if

\[
\text{either } M \text{ is faithful as a right } B\text{-module and } am = 0 \text{ implies } a = 0 \text{ or } m = 0 \text{ for all } a \in A, \ m \in M; \text{ or}
\]

\[
M \text{ is faithful as a left } A\text{-module and } mb = 0 \text{ implies } m = 0 \text{ or } b = 0 \text{ for all } m \in M, \ b \in B.
\]

The strong faithfulness for \(N\) can be defined similarly.

3. The generalized matrix algebra \(\mathcal{G}\) is called weakly faithful if

\[
aM = \{0\} = Na \text{ implies } a = 0,
\]

\[
Mb = \{0\} = bN \text{ implies } b = 0.
\]

(1.1)
It is evident that if $M$ is strongly faithful then $M$ is faithful and either $A$ or $B$ has no zero devisors. It is also trivial that if either $M$ or $N$ is faithful, then $G$ is weakly faithful. 

It is worth mentioning that in the case $G$ is a triangular algebra the weak faithfulness of $G$ is nothing more than faithfulness of $M$.

By a standard argument one can check that the center $Z(G)$ of $G$ is

$$Z(G) = \{a \oplus b | a \in Z(A), b \in Z(B), \ am = mb, \ na = bn \ \text{for all} \ m \in M, n \in N\},$$

where $a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$. Consider two natural projections $\pi_A : G \longrightarrow A$ and $\pi_B : G \longrightarrow B$ by

$$\pi_A : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{pmatrix} a & m \\ n & b \end{pmatrix} \mapsto b.$$

Clearly $\pi_A(Z(G)) \subseteq Z(A)$ and $\pi_B(Z(G)) \subseteq Z(B)$. Moreover, if $G$ is weakly faithful then $\pi_A(Z(G))$ is isomorphic to $\pi_B(Z(G))$.

Clearly, there exists a unique algebra isomorphism $\varphi : \pi_A(Z(G)) \longrightarrow \pi_B(Z(G))$ such that $am = m\varphi(a)$ and $\varphi(a)n = na$ for all $m \in M, n \in N$; or equivalently, $a \oplus \varphi(a) \in Z(G)$ for all $a \in A$, (see [1, Proposition 2.1] and [3, Proposition 3]).

This paper is organized as follows; in section 2, we characterize the structure of Lie higher derivations and higher derivations on the generalized matrix algebra $G$. The LHD property for the generalized matrix algebra $G$ is investigated in section 3. In section 4, we offer some alternative sufficient conditions ensuring the LHD property for $G$ (Theorems 4.1, 4.2, 4.3). We then come to our main result, Theorem 4.4, collecting some sufficient conditions ensuring the LHD property for a generalized matrix algebra. Section 5 includes some applications of our conclusions to some main examples of a generalized matrix algebra such as: trivial generalized matrix algebras, triangular algebras, unital algebras with a nontrivial idempotent, the algebra $B(X)$ of operators on a Banach space $X$ and the full matrix algebra $M_n(A)$ on a unital algebra $A$. Since the proof of Theorems 2.2 and 2.3 are too long we devote section 6 to them.

2. THE STRUCTURE OF (LIE) HIGHER DERIVATIONS ON $G$

We start this section with the following result of [11] which describes the structure of derivations and Lie derivations on a generalized matrix algebra.

**Proposition 2.1** ([11, Propositions 4.1, 4.2]). Let $G$ be a generalized matrix algebra.

- If $A$ and $B$ are 2–torsion free then a linear map $L_1 : G \longrightarrow G$ is a Lie derivation if and only if it has the presentation

  $$L_1 \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} p_{11}(a) + p_{12}(b) - mn_1 - m_1n & am_1 - m_1b + f_{13}(m) \\ n_1a - bm_1 + g_{14}(n) & q_{11}(a) + q_{12}(b) + n_1m + nm_1 \end{pmatrix},$$

  where $m_1 \in M, n_1 \in N$ and $p_{11} : A \longrightarrow A, \ p_{12} : B \longrightarrow A, \ q_{11} : A \longrightarrow B, \ q_{12} : B \longrightarrow B, \ f_{13} : M \longrightarrow M, \ g_{14} : N \longrightarrow N$ are linear maps satisfying the following properties:

  1. $p_{11}$ and $q_{12}$ are Lie derivations.
  2. $p_{12}([b, b']) = 0, q_{11}([a, a']) = 0$.
  3. $p_{12}(B) \subseteq Z(A), q_{11}(A) \subseteq Z(B)$.
  4. $f_{13}(am) = p_{11}(a)m - mq_{11}(a) + af_{13}(m), \ f_{13}(mb) = mq_{12}(b) - p_{12}(b)m + f_{13}(m)b$.
  5. $g_{14}(na) = np_{11}(a) - q_{11}(a)n + g_{14}(n)a, \ g_{14}(bn) = q_{12}(b)n - np_{12}(b) + bg_{14}(n)$.
(6) $p_{11}(mn) - p_{12}(nm) = mg_{14}(n) + f_{13}(m)n, \quad q_{12}(nm) - q_{11}(mn) = g_{14}(n)m + nf_{13}(m)$.

- A linear map $D_1 : G \to G$ is a derivation if and only if it has the presentation

$$D_1 \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{ccc} p_{11}(a) - mn_1 - m_1n & am_1 - m_1b + f_{13}(m) \\ n_1a - bn_1 + g_{14}(n) & q_{12}(b) + n_1m + nm_1 \end{array} \right),$$

where $m_1 \in M, n_1 \in N$ and $p_{11} : A \to A, \quad p_{12} : B \to B, \quad q_{11} : A \to B, \quad q_{12} : B \to M, \quad g_{14} : N \to N$ are linear maps satisfying the following properties:

(a) $p_{11}$ and $q_{12}$ are derivations.
(b) $f_{13}(am) = p_{11}(a)m + af_{13}(m), \quad f_{13}(mb) = mq_{12}(b) + f_{13}(m)b$.
(c) $g_{14}(na) = np_{11}(a) + g_{14}(n)a, \quad g_{14}(bn) = q_{12}(b)n + bg_{14}(n)$.
(d) $p_{11}(mn) = mg_{14}(n) + f_{13}(m)n, \quad q_{12}(nm) = g_{14}(n)m + nf_{13}(m)$.

We are preparing to describe the structure of Lie higher and higher derivations on a generalized matrix algebra. We start with fixing some notations which will be needed in the sequel.

2.1. Some notations. Before we proceed for the result, for more convenience, we fix some notations. Throughout $\mathbb{N}$ stands for the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For each $k \in \mathbb{N}$, we define $\eta_k$ and $\nu_k$ by

$$\eta_k := \begin{cases} (k-1)/2 & \text{if } k \text{ is odd} \\ k/2 & \text{if } k \text{ is even} \end{cases} \quad \text{and} \quad \nu_k := \begin{cases} (k-1)/2 & \text{if } k \text{ is odd} \\ k/2 - 1 & \text{if } k \text{ is even} \end{cases}.$$

Let $k \in \mathbb{N}, \alpha_i, \beta_i \in \{1, 2, \cdots, k\}, (1 \leq i \leq r)$ and $n_{\alpha_i} \in N, m_{\beta_i} \in M$ we define

$$\begin{align*}
(\alpha + \beta)_r := \sum_{i=1}^r (\alpha_i + \beta_i), \\
(n_{\alpha_i}m_{\beta_i})^r := n_{\alpha_i}m_{\beta_i} \cdots n_{\alpha_i}m_{\beta_i}, \text{ and } (n_{\alpha_i}m_{\beta_i})_r := n_{\alpha_i}m_{\beta_i} \cdots n_{\alpha_i}m_{\beta_i}, \\
(m_{\beta_i}n_{\alpha_i})^r := m_{\beta_i}n_{\alpha_i} \cdots m_{\beta_i}n_{\alpha_i}, \text{ and } (m_{\beta_i}n_{\alpha_i})_r := m_{\beta_i}n_{\alpha_i} \cdots m_{\beta_i}n_{\alpha_i}. 
\end{align*}$$

We also define $N_k$ and $M_k$ so that $N_1 := n_1, \quad M_1 := m_1, \quad N_2 := n_2, \quad M_2 := m_2$ and for each $k \geq 3$, by

$$\begin{align*}
N_k &= \sum_{r=1}^{\eta_k} \sum_{(\alpha+\beta)_r+\gamma=k} \left( \prod_{\rho=1}^r n_{\alpha_{\rho}}m_{\beta_{\rho}}n_{\gamma_\rho} \right) + n_k, \\
M_k &= \sum_{r=1}^{\nu_k} \sum_{(\alpha+\beta)_r+\gamma=k} \left( \prod_{\rho=1}^r m_{\beta_{\rho}}n_{\alpha_{\rho}}n_{\gamma_\rho} \right) + m_k.
\end{align*}$$

For example, for $k = 3$ and $k = 4$ we have:

$$\begin{align*}
N_3 &= n_1m_1n_1 + n_3, \quad M_3 = m_1m_1m_1 + m_3, \\
N_4 &= n_1m_1n_2 + n_2m_1n_1 + n_2m_1n_1 + n_4, \quad M_4 = m_1m_1m_2 + m_1n_2m_1 + m_2n_1m_1 + m_4.
\end{align*}$$

2.2. The structure of Lie higher derivations on $G$. It is easy to check that every sequence \{${\mathcal{L}_k}$\}_{k \in \mathbb{N}_0} of linear mappings on a generalized matrix algebra $G = \left( \begin{array}{cc} A & M \\ N & B \end{array} \right)$ enjoys the presentation

$$\mathcal{L}_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{ccc} p_{11}(a) + p_{22}(b) + p_{33}(n) + p_{44}(n) & f_{12}(a) + f_{22}(b) + f_{33}(m) + f_{44}(n) \\ g_{11}(a) + g_{22}(b) + g_{33}(m) + g_{44}(n) & q_{12}(b) + q_{22}(b) + q_{33}(m) + q_{44}(n) \end{array} \right)$$

for each $k \in \mathbb{N}_0$, where the entries mappings $p_{1k} : A \to A, \quad p_{2k} : B \to B, \quad p_{3k} : M \to A, \quad p_{4k} : N \to A, \quad q_{1k} : A \to B, \quad q_{2k} : B \to B, \quad q_{3k} : M \to B, \quad q_{4k} : N \to B, \quad f_{1k} : A \to M, \quad f_{2k} : B \to M, \quad f_{3k} : M \to M, \quad f_{4k} : N \to M, \quad g_{1k} : A \to N, \quad g_{2k} : B \to N, \quad g_{3k} : M \to N \quad g_{4k} : N \to N$ are linear. For each $k \in \mathbb{N}$ we set $m_k := f_{1k}(1)$ and $n_k := g_{1k}(1)$.
Before proceeding for the structure of Lie higher derivations on $G$ in Theorem 2.2, we need to fix some more notations and also define some auxiliary mappings by setting $P_0 = P'_0 = id_A, P''_0 := 0, Q_0 = Q'_0 := 0$ and $p_1 = p'_{1} = q''_{1} := 0, q_1 = q'_{1} = q''_{1} := 0$. Further for each $k \in \mathbb{N}$, we set $P_k := p_{1k} + p_k, P'_k := p_{1k} + p'_k, P''_k := p_{2k} - p''_k, Q_k := q_{2k} + q_k, Q'_k := q_{2k} + q'_k$ and $Q''_k := q_{1k} - q''_k$ where for each $k \geq 2$

\[
p_k(a) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (P_1(a)m_{\beta_1} - m_{\beta_1}Q'_i(a))n_{\alpha_1}(m_{\beta}n_{\alpha})^r,
\]

\[
p'_k(a) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (m_{\beta}n_{\alpha})_r m_{\beta_1}(n_{\alpha_1}P'_i(a) - Q'_i(a)n_{\alpha_1}),
\]

\[
p''_k(b) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (m_{\beta}Q_1(b) - P'_i(b)m_{\beta_1})n_{\alpha_1}(m_{\beta}n_{\alpha})^r,
\]

\[
= \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (m_{\beta}n_{\alpha})_r m_{\beta_1}(Q'_i(b)n_{\alpha_1} - n_{\alpha_1}P'_i(b)),
\]

\[
q_k(b) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (n_{\alpha}m_{\beta})_r n_{\alpha_1}(m_{\beta}Q_1(b) - P''_i(b)m_{\beta_1}),
\]

\[
q'_k(b) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (Q'_i(b)n_{\alpha_1} - n_{\alpha_1}P''_i(b))m_{\beta_1}(n_{\alpha}m_{\beta})^r,
\]

\[
q''_k(a) := \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (n_{\alpha}m_{\beta})_r n_{\alpha_1}(P_1(a)m_{\beta_1} - m_{\beta}Q''_i(a))
\]

\[
= \sum_{r=1}^{n_k} \sum_{i+(\alpha + \beta)_r = k, i \leq k-2} (n_{\alpha}m_{\beta})_r n_{\alpha_1}(P'_i(a) - Q''_i(a)n_{\alpha_1})m_{\beta_1}(n_{\alpha}m_{\beta})^r.
\]

For example; for $k = 2$ we get:

$p_2(a) = am_{1n_1}, p'_2(a) = m_{1n_1}, p''_2(b) = m_{1b}, q_2(b) = n_1m_{1b}, q'_2(b) = bm_{1}m_{1}$ and $q''_2(a) = n_1am_{1}$.

Similarly, for $k = 3$ one can check that

\[
p_3(a) = am_{1n_2} + am_{2n_1} + P_1(a)m_{1n_1} - m_{1}Q''_1(a)n_1
\]

\[
p'_3(a) = m_{1n_2}a + m_2n_{1a} + m_{1n_1}P_1(a) - m_{1}Q''_1(a)n_1,
\]

\[
p''_3(b) = m_{1bn_2} + m_{2 bn_1} + m_{1}Q_1(b)n_1 - P''_1(b)m_{1n_1}
\]

\[
= m_{1bn_2} + m_{2 bn_1} + m_{1}Q_1(b)n_1 - m_{1n_1}P''_1(b),
\]

\[
q_3(b) = n_1m_{2b} + n_2m_{1b} + n_1m_{1}Q_1(b) - n_1P''_1(b)m_{1}
\]

\[
q'_3(b) = bm_{1n_2} + bn_2m_{1n_1} + Q_1(b)n_1m_{1} - n_1P''_1(b)m_{1}
\]

\[
q''_3(a) = n_1am_{2} + n_2am_{1} + n_1P_1(a)m_{1} - m_{1n_1}Q''_1(a)
\]

\[
= n_1am_{2} + n_2am_{1} + n_1P'_1(a)m_{1} - Q''_1(a)n_1m_{1}.
\]

Now, we are ready to present the following result describing the structure of Lie higher derivations on the generalized matrix algebra $G$. 

Theorem 2.2. Let $\mathcal{G} = \left( \begin{array}{cc} A & M \\ N & B \end{array} \right)$ be a generalized matrix algebra such that $A, B$ are 2-torsion free. Then a sequence $\mathcal{L} = \{L_k\}_{k \in \mathbb{N}_0} : \mathcal{G} \rightarrow \mathcal{G}$ of linear mappings (as presented in $(\star)$) is a Lie higher derivation if and only if

1. $\{P_k\}_{k \in \mathbb{N}_0}, \{Q_k\}_{k \in \mathbb{N}_0}$ are Lie higher derivations on $A$, $\{Q'_k\}_{k \in \mathbb{N}_0}$ are Lie higher derivations on $B$, $Q''_k(A) \subseteq Z(B), P''_k(B) \subseteq Z(A)$, and $Q''_k([a, a']) = 0, P''_k([b, b']) = 0$, for all $k \in \mathbb{N}$.

2. $g_{1k}(a) = \sum_{i+j=k, j \neq k} (n_i P'_i(a) - Q''_i(a) n_i)$ and $f_{1k}(a) = \sum_{i+j=k, j \neq k} (P'_i(a) m_i - m_i Q''_i(a))$,

3. $g_{2k}(b) = - \sum_{i+j=k, j \neq k} (Q'_j(b) n_i - n_i P''_j(b))$ and $f_{2k}(b) = - \sum_{i+j=k, j \neq k} (m_i Q'_j(b) - P''_j(b) m_i)$,

4. $f_{3k}(am) = \sum_{i+j=k} (P'_i(a) f_{3j}(m) - f_{3j}(m) Q''_i(a))$,

5. $f_{3k}(mb) = \sum_{i+j=k} (f_{3j}(m) Q'_i(b) - P''_i(b) f_{3j}(m))$,

6. $g_{4k}(na) = \sum_{i+j=k} (g_{4j}(n) P'_i(a) - Q''_i(a) g_{4j}(n))$,

7. $g_{4k}(bn) = \sum_{i+j=k} (Q'_i(b) g_{4j}(n) - g_{4j}(n) P''_i(b))$,

8. $g_{6k}(m) = - \sum_{i+j+r=k} N_i f_{3r}(m) N_j$ and $f_{4k}(m) = - \sum_{i+j+r=k} M_i g_{4r}(n) M_j$,

9. $p_{3k}(m) = \sum_{i+j=k} - f_{3i}(m) N_j$ and $q_{3k}(m) = \sum_{i+j=k} N_j f_{3i}(m)$,

10. $p_{4k}(n) = \sum_{i+j=k} - M_j g_{4i}(n)$ and $q_{4k}(n) = \sum_{i+j=k} g_{4i}(n) M_j$,

11. $p_{1k}(mn) - p_{2k}(mn) = \sum_{i+j=k} (p_{3i}(m) p_{4j}(n) + f_{3i}(m) g_{4j}(n) - p_{4j}(n) p_{3i}(m) - f_{4j}(n) q_{3i}(m))$,

12. $q_{1k}(mn) - q_{2k}(mn) = \sum_{i+j=k} (g_{3i}(m) f_{4j}(n) + q_{3i}(m) q_{4j}(n) - g_{4j}(n) f_{3i}(m) - q_{4j}(n) q_{3i}(m))$.

2.3. The structure of higher derivations on $\mathcal{G}$. Similar to $(\star)$ let $\mathcal{D} = \{D_k\}_{k \in \mathbb{N}_0} : \mathcal{G} \rightarrow \mathcal{G}$ be a sequence of linear mappings with the following presentation

$$D_k \left( \begin{array}{ccc} a & m & n \\ n & b \end{array} \right) = \left( \begin{array}{ccc} p_{1k}(a) + p_{2k}(b) + p_{3k}(m) + p_{4k}(n) & f_{1k}(a) + f_{2k}(b) + f_{3k}(m) + f_{4k}(n) \\ g_{1k}(a) + g_{2k}(b) + g_{3k}(m) + g_{4k}(n) & q_{1k}(a) + q_{2k}(b) + q_{3k}(m) + q_{4k}(n) \end{array} \right),$$

whose entries maps are linear. As before for each $k \in \mathbb{N}_0$ we set $m_k := f_{1k}(1)$ and $n_k := g_{1k}(1)$.

Before proceeding for the structure of higher derivations on $\mathcal{G}$ in Theorem 2.3, we need to fix some more notations and also define some auxiliary mappings by setting $P_0 = P'_0 := id_A$, $Q_0 = Q'_0 := id_B$, and $p_1 := p'_1 := 0$, $q_1 := q'_1 := 0$. Further for each $k \in \mathbb{N}$, we set $P_k := p_{1k} + p_k$, $P'_k := p_{1k} + p'_k$, $Q_k := q_{2k} + q_k$, $Q'_k := q_{2k} + q'_k$, where for each $k \geq 2$

$$p_k(a) := \sum_{r=1}^k \sum_{i+(\alpha + \beta)_r=k, i \leq k-2} P'_i(a) m_{\beta} n_{\alpha}(m_{\beta} n_{\alpha})^r,$$

$$p'_k(a) := \sum_{r=1}^k \sum_{i+(\alpha + \beta)_r=k, i \leq k-2} m_{\beta} n_{\alpha}(m_{\beta} n_{\alpha})^r P'_i(a),$$
\[ q_k(b) := \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} (n_\alpha m_\beta)_r n_{\alpha_1} m_{\beta_1} Q_i(b), \]
\[ q'_k(b) := \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} Q'_i(b) (n_\alpha m_\beta)_r n_{\alpha_1} m_{\beta_1}. \]

In particular for the case \( k = 2 \) we get
\[ p_2(a) = a m_1 n_1, \quad p'_2(a) = m_1 n_1 a, \quad q_2(b) = n_1 m_1 b \text{ and } q'_2(b) = b n_1 m_1. \]

Similarly for the case \( k = 3 \) it is easy to check that
\[ p_3(a) = a m_1 n_2 + a m_2 n_1 + P_1(a) m_1 n_1, \]
\[ p'_3(a) = m_1 n_2 a + m_2 n_1 a + m_1 n_1 P'_1(a), \]
\[ q_3(b) = n_1 m_2 b + n_2 m_1 b + n_1 m_1 Q_1(b), \]
\[ q'_3(b) = b n_1 m_2 + b n_2 m_1 + Q'_1(b) n_1 m_1. \]

Parallel to Theorem 2.2, in the following result we characterize the structure of higher derivations on the generalized matrix algebra \( G \).

**Theorem 2.3.** Let \( G = \left( \begin{array}{cc} A & M \\ N & B \end{array} \right) \) be a generalized matrix algebra. Then a sequence \( D = \{D_k\}_{k \in \mathbb{N}_0} : G \rightarrow G \) of linear mappings (as presented in \( \bullet \)) is a higher derivation if and only if

(a) \( \{P_k\}_{k \in \mathbb{N}_0}, \{P'_k\}_{k \in \mathbb{N}_0} \) are higher derivations on \( A \) and \( \{Q_k\}_{k \in \mathbb{N}_0}, \{Q'_k\}_{k \in \mathbb{N}_0} \) are higher derivations on \( B \).

(b) \( g_{1k}(a) = \sum_{i+j=k, j \neq k} n_i P'_i(a) \) and \( f_{1k}(a) = \sum_{i+j=k, j \neq k} P_j(a) m_i, \)

(c) \( g_{2k}(b) = -\sum_{i+j=k, j \neq k} Q'_j(b) n_i \) and \( f_{2k}(b) = -\sum_{i+j=k, j \neq k} m_i Q_j(b), \)

(d) \( f_{3k}(am) = \sum_{i+j=k} P_i(a) f_{3j}(m) \) and \( f_{3k}(mb) = \sum_{i+j=k} f_{3j}(m) Q_i(b), \)

(e) \( g_{4k}(na) = \sum_{i+j=k} g_{4j}(n) P'_i(a) \) and \( g_{4k}(bn) = \sum_{i+j=k} Q'_j(b) g_{4i}(n), \)

(f) \( g_{3k}(m) = -\sum_{i+j+r=k} N_i f_{3r}(m) N_j \) and \( f_{4k}(n) = -\sum_{i+j+r=k} M_i g_{4r}(n) M_j, \)

(g) \( p_{3k}(m) = -\sum_{i+j=k} f_{3k}(m) N_j \) and \( q_{3k}(m) = \sum_{i+j=k} N_j f_{3k}(m), \)

(h) \( p_{4k}(n) = -\sum_{i+j=k} M_j g_{4i}(n) \) and \( q_{4k}(n) = \sum_{i+j=k} g_{4i}(n) M_j, \)

(i) \( q_{1k}(a) = \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} (n_\alpha m_\beta)_r n_{\alpha_1} P_i(a) m_{\beta_1} \)
  \( = \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} n_{\alpha_1} P'_i(a) m_{\beta_1} (n_\alpha m_\beta)_r, \)

(j) \( p_{2k}(b) = \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} m_{\beta_1} Q_i(b) n_{\alpha_1} (m_\beta n_\alpha)_r \)
  \( = \sum_{r=1}^{\eta} \sum_{i+(\alpha + \beta) = k, i \leq k-2} (m_\beta n_\alpha)_r m_{\beta_1} Q'_i(b) n_{\alpha_1}, \)
2.4. The center valued mappings on $G$. In the next result we characterize the structure of center valued mappings on $G$ vanishing at commutators of $G$.

**Proposition 2.4.** A sequence $\tau = \{\tau_k\}_{k \in \mathbb{N}}$ of linear maps on $G$ is center valued and vanishes at commutators if and only if for each $k \in \mathbb{N}$ the map $\tau_k$ has the presentation

$$
\tau_k \left( \begin{array}{ccc}
  a & m \\
  n & b
\end{array} \right) = \left( \begin{array}{ccc}
  \ell_k(a) + P_k^n(b) & Q_k^n(a) \\
  Q_k^n(b) & \ell_k(b)
\end{array} \right),
$$

where $\ell_k : A \rightarrow Z(A)$, $P_k^n : B \rightarrow Z(A)$, $Q_k^n : A \rightarrow Z(B)$ and $\ell_k' : B \rightarrow Z(B)$ are linear maps vanishing at commutators and satisfying the following properties:

(i) $\ell_k(a) \oplus Q_k^n(a) \in Z(G)$ and $P_k^n(b) \oplus \ell_k'(b) \in Z(G)$, for all $a \in A, b \in B$ and $k \in \mathbb{N}$.
(ii) $\ell_k(mn) = P_k^n(mn)$ and $\ell_k'(nm) = Q_k^n(mn)$, for all $m \in M, n \in N$ and $k \in \mathbb{N}$.

3. Proper Lie higher derivations

Hereinafter, suppose that the modules $M$ and $N$ appeared in definition of the generalized matrix algebras are 2–torion free; ($M$ is said to be 2–torsion free if $2m = 0$ implies $m = 0$ for all $m \in M$).

According to the Cheung’s method [3, Theorem 6], in the following theorem we give a necessary and sufficient condition under which Lie higher derivations on the generalized matrix algebra $G$ are proper.

**Theorem 3.1.** Let $G$ be a generalized matrix algebra. A Lie higher derivation $L$ on $G$ of the form $(\star)$ is proper if and only if there exist two sequences of linear mappings $\{\ell_k\}_{k \in \mathbb{N}} : A \rightarrow Z(A)$ and $\{\ell'_k\}_{k \in \mathbb{N}} : B \rightarrow Z(B)$ satisfying the following three properties:

(A) $\{P_k - \ell_k\}_{k \in \mathbb{N}}$ and $\{Q_k - \ell'_k\}_{k \in \mathbb{N}}$ are higher derivations on $A$ and $B$, respectively.
(B) $\ell_k(a) \oplus Q_k^n(a) \in Z(G)$ and $P_k^n(b) \oplus \ell_k'(b) \in Z(G)$, for all $a \in A, b \in B$ and $k \in \mathbb{N}$.
(C) $\ell_k(mn) = P_k^n(mn)$ and $\ell_k'(nm) = Q_k^n(mn)$, for all $m \in M, n \in N$ and $k \in \mathbb{N}$.

**Proof.** For sufficiency by induction on $k$, we know that for $k = 1$ the result is true by [3]. Now let the result holds for any integer less than $k$, we prove this for $k$. By using induction hypothesis and Theorems 2.2 and 2.3, like step 1 appeared in [4], without loss of generality we can consider structure of $L_k$ as

$$
L_k \left( \begin{array}{ccc}
  a & m \\
  n & b
\end{array} \right) = \left( \begin{array}{ccc}
  p_{1k}(a) + p_{2k}(b) + q_{1k}(a) + q_{2k}(b) & \ell_k(a) + \ell_k'(b) + f_{1k}(m) \\
  g_{1k}(a) + g_{2k}(b) + g_{4k}(n) & q_{1k}(a) + q_{2k}(b)
\end{array} \right),
$$

where $p_{1k}, p_{2k}, q_{1k}, q_{2k}, f_{1k}, f_{2k}$ and $g_{4k}$ have the properties (1), (2), (6), (7), (8) and (9) in Lemma 2.2. Replace $p_{1k}, p_{2k}, q_{1k}, q_{2k}$ with $P_k - \ell_k, Q_k - \ell'_k, P_k^n + P_k', Q_k^n + Q_k'$, respectively, then we have

$$
L_k \left( \begin{array}{ccc}
  a & m \\
  n & b
\end{array} \right) = \left( \begin{array}{ccc}
  P_k(a) - \ell_k(a) + P_k^n(b) + p_{1k}(a) + p_{2k}(b) + q_{1k}(a) + q_{2k}(b) \\
  g_{1k}(a) + g_{2k}(b) + g_{4k}(n) & q_{1k}(a) + q_{2k}(b)
\end{array} \right),
$$

(3.1)
By the induction hypothesis as $L_i$ is proper for all $i < k$ we may write $p_k, p'_k, p''_k, q_k, q'_k, q''_k, f_{1k}, f_{2k}, g_{1k}$ and $g_{2k}$ as follows

$$
p_k(a) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} (P_i(a) - \ell_i(a))m_{\beta_{i1}}n_{\alpha_1} \ldots m_{\beta_{i\alpha_r}}n_{\alpha_r},$$

$$
p'_k(a) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} m_{\beta_{i1}}n_{\alpha_1} \ldots m_{\beta_{i\alpha_r}}(P'_i(a) - \ell'_i(a)),$$

$$
p''_k(b) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} m_{\beta_{i1}}(Q_i(b) - \ell'_i(b))n_{\alpha_1} \ldots m_{\beta_{i\alpha_r}}n_{\alpha_r},$$

$$
q_k(b) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} n_{\alpha_1}m_{\beta_{i1}} \ldots n_{\alpha_r}m_{\beta_{i\alpha_r}}(Q_i(b) - \ell'_i(b)),
$$

$$
q'_k(b) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} (Q'_i(b) - \ell'_i(b))n_{\alpha_1}m_{\beta_{i1}} \ldots n_{\alpha_r}m_{\beta_{i\alpha_r}},
$$

$$
q''_k(a) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} n_{\alpha_1}m_{\beta_{i1}} \ldots n_{\alpha_r}(P'_i(a) - \ell_i(a))m_{\beta_{i\alpha_r}}n_{\alpha_r},$$

$$
q''_k(b) = \sum_{r=1}^{n_k} \sum_{i+(\alpha+\beta)r = k, i \leq k-2} n_{\alpha_1}(P'_i(a) - \ell_i(a))m_{\beta_{i1}} \ldots n_{\alpha_r}m_{\beta_{i\alpha_r}},$$

$$
f_{1k}(a) = \sum_{i+j=k, i \neq k} (P_i(a)m_j - m_jQ''_i(a)) = \sum_{i+j=k, i \neq k} (P_i(a) - \ell_i(a))m_j,$$

$$
f_{2k}(b) = -\sum_{i+j=k, i \neq k} (m_jQ_i(b) - P''_i(b)m_j) = -\sum_{i+j=k, i \neq k} m_j(Q_i(b) - \ell'_i(b)),$$

$$
g_{1k}(a) = \sum_{i+j=k, i \neq k} (n_jP'_i(a) - Q'_i(a)n_j) = \sum_{i+j=k, i \neq k} n_j(P'_i(a) - \ell_i(a)),$$

$$
g_{2k}(b) = -\sum_{i+j=k, i \neq k} (Q'_i(b)n_j - n_jP''_i(b)) = -\sum_{i+j=k, i \neq k} (Q'_i(b) - \ell'_i(b))n_j,$$

i.e. $p_k(a), p'_k(a), p''_k(b), q_k(b), q'_k(b), q''_k(a), f_{1k}(a), f_{2k}(b), g_{1k}(a)$ and $g_{2k}(b)$ are the same as those appeared in the structure of higher derivations in Theorem 2.3. So we can present (3.1) to the simpler form

$$
L_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{cc} P_k(a) & P''_k(b) \\ g_{1k}(n) & Q''_k(a) + Q_k(b) \end{array} \right) \cdot f_{1k}(m),
$$

Set

$$
D_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{cc} P_k(a) - \ell_k(a) \\ g_{1k}(n) \end{array} \right) \left( \begin{array}{cc} f_{3k}(m) \\ Q_k(b) - \ell'_k(b) \end{array} \right),
$$

and

$$
\tau_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{cc} \ell_k(a) + P''_k(b) \\ Q''_k(a) + \ell'_k(b) \end{array} \right).$$

For more convenience set

(i) $P_k(a) = P_k(a) - \ell_k(a)$, $Q_k(b) = Q_k(b) - \ell'_k(b)$

(ii) $\gamma_k(a, b) = \ell_k(a) + P''_k(b)$ and

(iii) $\gamma'_k(a, b) = Q''_k(a) + \ell'_k(b)$. 
Considering the above relations we have
\[ \mathcal{L}_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{cc} P_k(a) + \gamma_k(a,b) & f_{3k}(m) \\ g_{4k}(n) & Q_k(b) + \gamma_k'(a,b) \end{array} \right). \]
Apply \( \mathcal{L}_k \) on commutator \( \left( \begin{array}{cc} 0 & 0 \\ n & -nm \end{array} \right) \), we get:
\[ P_k(mn) = \sum_{i+j=k} f_{3i}(m)g_{4j}(n) - \gamma_k(mn,-nm), \tag{3.2} \]
\[ Q_k(nm) = \sum_{i+j=k} g_{4j}(n)f_{3i}(m) - \gamma_k'(mn,-nm). \tag{3.3} \]
From the assumption \((\mathcal{C})\), since \( \ell_k(mn) = P'_k(mn) \) and \( \ell'_k(nm) = Q'_k(nm) \), then \( \gamma_k(mn,-nm) = 0 \) and \( \gamma'_k(mn,-nm) = 0 \), it follows that
\[ P_k(mn) = \sum_{i+j=k} f_{3i}(m)g_{4j}(n) \quad \text{and} \quad Q_k(nm) = \sum_{i+j=k} g_{4j}(n)f_{3i}(m) \]
for all \( m \in M, n \in N \) and \( k \in \mathbb{N} \). A direct verification reveals that \( \mathcal{D} = \{ D_k \}_{k \in \mathbb{N}} \) is a higher derivation and \( \tau = \{ \tau_k \}_{k \in \mathbb{N}} \) is a sequence of center valued maps.

For necessity, let \( \mathcal{L} \) be proper i.e. \( \mathcal{L} = \mathcal{D} + \tau \) for some higher derivation \( \mathcal{D} \) and a sequence of center valued maps \( \tau \). Applying the presentations \((\bigstar), (\heartsuit)\) for \( \mathcal{L} \) and \( \mathcal{D} \), respectively, we have \( \tau = \mathcal{L} - \mathcal{D} \) as
\[ \tau_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( (P_k - P_k)(a) + P'_k(b) \right) Q'_k(a) + (Q_k - Q_k)(b) \]
We set \( \ell_k = P_k - P_k, \ell'_k = Q_k - Q_k \), one can directly check that \( \{ \ell_k \}_{k \in \mathbb{N}}, \{ \ell'_k \}_{k \in \mathbb{N}} \) are two sequences of maps satisfying the required properties.

**Remark 3.2.** It is worthwhile mentioning that in the case where \( M \) is a faithful \((A,B)\)-module then;

(i) In Theorem 2.2, the conditions \( Q'_k([a,a']) = 0 \) and \( P'_k([b,b']) = 0 \) for all \( k \in \mathbb{N} \), are superfluous as those can be acquired from (1), (4) and (5). Indeed, by induction on \( k \) we know that for \( k = 1 \) this is true by [2]. Suppose that the result holds for any integer less than \( k \). For \( a, a' \in A, m \in M \), from (4) we get
\[ f_{3k}([a,a']m) = \sum_{i+j=k, j \neq 0} (P_i([a,a'])f_{3j}(m) - f_{3j}(m)Q'_i([a,a'])). \tag{3.4} \]
On the other hand, employing (c) and then (a), we have
\[ f_{3k}([a,a']m) = f_{3k}(aa'm - a'am) \]
\[ = \sum_{i+j=k, j \neq 0} (P_i(a)f_{3j}(a'm) - f_{3j}(a'm)Q'_i(a)) \]
\[ - \left( \sum_{i+j=k, j \neq 0} (P_i(a')f_{3j}(am) - f_{3j}(am)Q'_i(a')) \right) \]
\[ = \sum_{r+s+t=k} (P_r(a)P_s(a')f_{3t}(m) - P_r(a)f_{3t}(m)Q'_t(a')) \]
\[ - \sum_{k+s+t=k} (P_s(a)f_{3t}(m)Q'_k(a) - f_{3t}(m)Q'_s(a')Q'_t(a)) \]
\[ - \sum_{r+s+t=k} (P_r(a')P_s(a)f_{3t}(m) - P_r(a')f_{3t}(m)Q'_t(a)) \]
\[ + \sum_{r+s+t=k} (P_s(a) f_{3t}(m) Q^m_r(a') - f_{3t}(m) Q^m_r(a) Q''_r(a')) \]
\[ = \sum_{r+s+t=k} [P_s(a), P_s(a)] f_{3t}(m). \quad (3.5) \]

Comparing the equations (3.4) and (3.5) along with assumption of induction indicates that 
\[ mQ''_k([a, a']) = 0, \text{ for any } m \in M, \text{ thus the equality } Q''_k([a, a']) = 0 \]
follows from the faithfulness of M (as a right B–module). Similarly one can check that 
\[ P''_k([b, b']) = 0 \quad \text{for all } k \in \mathbb{N}. \]

(ii) In Theorem 2.3, the assertion (a) can also be removed as it can be acquired from (d) and (e) 
by a similar reasoning as in (i), (see [3, Page 303]).

(iii) In Theorem 3.1, the same reason as in (ii) indicates that the assertion (A) in Theorem 3.1, 
staking that \( P_k - \ell_k \) and \( Q_k - \ell_k \) are higher derivations, is extra.

In the next corollary we offer the criterion characterizing LHD property for the generalized matrix 
algebra \( \mathcal{G} \) as a conclusion of Theorem 3.1.

**Corollary 3.3.** Let \( \mathcal{G} \) be a generalized matrix algebra and \( \mathcal{L} \) be a Lie higher derivation on \( \mathcal{G} \) of the 
form stated in Theorem 2.2. If \( \mathcal{L} \) is proper then

(A') \( Q''_k(A) \subseteq \pi_B(Z(\mathcal{G})), \ P''_k(B) \subseteq \pi_A(Z(\mathcal{G})) \) and,

(B') \( P''_k((nm)) \oplus Q''_k((mn)) \in Z(\mathcal{G}), \)

for all \( m \in M, n \in N \) and \( k \in \mathbb{N}. \) The converse is valid when \( \mathcal{G} \) is weakly faithful.

**Proof.** Let \( \mathcal{L} \) be proper, then Theorem 3.1 ensures that (A') and (B') hold. Conversely, suppose that 
\( \mathcal{G} \) is weakly faithful. Let \( \varphi : \pi_A(Z(\mathcal{G})) \to \pi_B(Z(\mathcal{G})) \) be the isomorphism satisfying \( a \oplus \varphi(a) \in Z(\mathcal{G}) \)
for all \( a \in A, \) whose existence guaranteed by the weak faithfulness of \( \mathcal{G} \) and [1]. By using assumption 
(A'), we define \( \ell_k : A \to Z(A) \) and \( \ell'_k : B \to Z(B) \) by \( \ell_k = \varphi^{-1} \circ Q''_k \) and \( \ell'_k = \varphi \circ P''_k. \) Obviously 
\( \ell_k(a) \oplus Q''_k(a) \in Z(\mathcal{G}) \) and \( P''_k(b) \oplus \ell'_k(b) \in Z(\mathcal{G}), \) for all \( a \in A, b \in B. \) Further, (B') follows that 
\[ \ell_k(mn) = \varphi^{-1}(Q''_k(mn)) = P''_k(mn) \quad \text{and} \quad \ell'_k(mn) = \varphi(P''_k(mn)) = Q''_k(mn). \]

Now properness of \( \mathcal{L} \) follows from Theorem 3.1 and part (iii) of Remark 3.2. \( \square \)

4. **Some Sufficient Conditions and the Main Result**

By using Corollary 3.3 in the next theorem we give the “higher” version of a modification of Du 
and Wang’s result [4, Theorem 1], (see also [20, Corollary 1] and [19, Theorem 2.1] in the case \( n = 2 \)).

**Theorem 4.1.** Let \( \mathcal{G} \) be a weakly faithful generalized matrix algebra. If

(i) \( \pi_A(Z(\mathcal{G})) = Z(A), \pi_B(Z(\mathcal{G})) = Z(B), \) and

(ii) either \( A \) or \( B \) does not contain nonzero central ideals,

then \( \mathcal{G} \) has LHD property.

**Proof.** By Corollary 3.3, it is enough to show that \( P''_k((nm)) + Q''_k((mn)) \in Z(\mathcal{G}) \) for all \( m \in M, n \in N. \)

Without loss of generality suppose that \( A \) has no nonzero central ideal. Put
\[ \gamma_k(a, b) = \ell_k(a) + P''_k(b) \quad (a \in A, \ b \in B, \ k \in \mathbb{N}), \]
where, as in the proof of the above corollary, \( \ell_k = \varphi^{-1} \circ Q''_k \) and that \( P_k = P_k - \ell_k \) is a higher derivation.

Now equation (3.2) implies that
\[ P_k(amn) = \sum_{i+j=k} f_{3i}(am) g_{4j}(n) - \gamma_k(ann, -nam). \]
The latter equation with the fact that \( P_k \) is a higher derivation follows that

\[
\sum_{i+j=k} P_i(a)P_j(mn) = \sum_{i+j=k} P_i(a)f_{3r}(m)g_{4j}(n) - \gamma_k(amn, -nam).
\]

By using assumption of induction we get

\[
aP_k(mn) = \sum_{r+j=k} af_{3r}(m)g_{4j}(n) - \gamma_k(amn, -nam).
\]

Multiply equation (3.2) from the left by \( a \), then we have

\[
aP_k(mn) = a \sum_{i+j=k} f_{3r}(m)g_{4j}(n) - a\gamma_k(mn, -nm),
\]

for all \( a \in A \), \( m \in M \) and \( n \in N \). The two last equations imply that the set \( A\gamma_k(mn, -nm) \) is a central ideal of \( A \) for each pair of elements \( m \in M \), \( n \in N \). Hence \( \ell_k(mn) - P_k''(mn) = \gamma_k(mn, -nm) = 0 \) and so \( P_k''(mn) \oplus Q_k''(mn) = \ell_k(mn) \oplus Q_k''(mn) \in Z(G) \).

As some examples of an algebra that has no nonzero central ideal we can mention to a noncommutative unital prime algebra with a nontrivial idempotent, in particular \( B(X) \), the algebra of operators on a Banach space \( X \) with \( \dim(X) > 1 \), and the full matrix matrix algebra \( M_n(A) \) with \( n \geq 2 \) (see [4, Lemma 1]). Also in [4, Theorem 2] it is shown that in the generalized matrix algebra \( G \) with loyal \( M \), \( A \) does not contain central ideal if \( A \) is noncommutative.

Parallel to the results of [14] we have the three following theorems which all of them can be proved by induction and using techniques of [14] for step 1.

Recall that an algebra \( A \) is called domain if it has no zero devisors or equivalently if \( aa' = 0 \) implies \( a = 0 \) or \( a' = 0 \) for every two elements \( a, a' \in A \).

**Theorem 4.2.** Let \( G \) be a weakly faithful generalized matrix algebra. Then \( G \) has LHD property if

(i) \( \pi_A(Z(G)) = Z(A), \pi_B(Z(G)) = Z(B) \) and

(ii) \( A \) and \( B \) are domain.

**Theorem 4.3.** The generalized matrix algebra \( G \) has LHD property if

(i) \( \pi_A(Z(G)) = Z(A), \pi_B(Z(G)) = Z(B) \) and

(ii) either \( M \) or \( N \) is strongly faithful.

It’s remarkable that, we do not know when one can withdraw the assertion strong faithfulness in Theorem 4.3.

Now, by gathering the above observations and combination of the assertions in Theorems 4.1, 4.2 and 4.3, we are able to give the main result of this paper providing several sufficient conditions that guarantee the LHD property for a generalized matrix algebra, which one part of its is a generalization of [13, Theorem 3.3].

**Theorem 4.4.** Let \( G \) be a weakly faithful generalized matrix algebra. If the following two conditions hold:

(I) \( \pi_A(Z(G)) = Z(A), \pi_B(Z(G)) = Z(B) \)

(II) one of the following conditions holds:

(i) either \( A \) or \( B \) does not contain nonzero central ideals

(ii) \( A \) and \( B \) are domain

(iii) either \( M \) or \( N \) is strongly faithful,
then \( \mathcal{G} \) has LHD property.

5. Applications

In this section we investigate LHD property for some main examples of a generalized matrix algebra which includes: trivial generalized matrix algebras, triangular algebras, unital algebras with a non-trivial idempotent, the algebra \( B(X) \) of operators on a Banach space \( X \) and the full matrix algebra \( M_n(A) \) on a unital algebra \( A \).

LHD property of trivial generalized matrix algebras and \( \text{Tri}(A, M, B) \). The generalized matrix algebra \( \mathcal{G} \) is called trivial when \( MN = 0 \) and \( NM = 0 \) in its definition. It can be pointed out to the triangular algebra \( \text{Tri}(A, M, B) \) as a main example of a trivial generalized matrix algebra that whose LHD property has been studied in \([12, 13, 16, 21, 23]\). As a urgent consequence of Corollary 3.3 and Theorem 4.4 we achieve the next result which characterizing the LHD property for trivial generalized matrix algebras.

Corollary 5.1. Let \( \mathcal{G} \) be a trivial generalized matrix algebra and \( \mathcal{L} \) be a Lie higher derivation on \( \mathcal{G} \) of the form stated in (\( \star \)). If \( \mathcal{L} \) is proper then \( Q_k'(A) \subseteq \pi_B(Z(\mathcal{G})) \), \( P_k''(B) \subseteq \pi_A(Z(\mathcal{G})) \). The converse is valid when \( \mathcal{G} \) is weakly faithful.

Specifically, a trivial generalized matrix algebra \( \mathcal{G} \) has LHD property if the following two conditions hold:

(I) \( \mathcal{G} \) is weakly faithful,

(II) \( \pi_A(Z(\mathcal{G})) = Z(A) \) and \( \pi_B(Z(\mathcal{G})) = Z(B) \).

In the next example which has been raised by Benković [1, Example 3.8] and modified in [14] we give a trivial generalized matrix algebra, which is not triangular, without the LHD property.

Example 5.2. Let \( M \) be a commutative unital algebra of dimension 3, on the commutative unital ring \( R \), with base \( \{1, m, m'\} \) such that \( m^2 = m'^2 = mm' = m'm = 0 \). Put \( N = M \) and let \( A = \{r + r'm \mid r, r' \in R\} \) and \( B = \{u + u'm' \mid u, u' \in R\} \) be the subalgebras of \( M \). Consider the generalized matrix algebra \( \mathcal{G} = \begin{pmatrix} A & M \\ N & B \end{pmatrix} \) under the usual addition, usual scalar multiplication and the multiplication defined by

\[
\begin{pmatrix} a & m \\ n & b \end{pmatrix} \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = \begin{pmatrix} aa' + mn' & am' + mb' \\ na' + bn' & bb' \end{pmatrix}.
\]

The generalized matrix algebra \( \mathcal{G} \) is trivial since \( MN = 0 = NM \). The linear map \( \mathcal{L} : \mathcal{G} \to \mathcal{G} \) defined by

\[
\mathcal{L} \begin{pmatrix} r + r'm \\ t + t'm + t''m' \end{pmatrix} \begin{pmatrix} s + s'm + s''m' \\ u + u'm' \end{pmatrix} = \begin{pmatrix} u'm - s''m - s'm' \\ -t''m - t'm' r'm' \end{pmatrix},
\]

where all coefficients are in the ring \( R \), is an improper Lie derivation. Now the ordinary Lie higher derivation induced by \( \mathcal{L} \) is improper.

Applying Theorems 2.2 and 2.3 for the special case \( N = 0 \) we arrive to the following characterizations of (Lie) higher derivations for the triangular algebra \( \text{Tri}(A, M, B) \) which have already presented in [13].

Corollary 5.3. Let \( \mathcal{L} = \{\mathcal{L}_k\}_{k \in \mathbb{N}} \) be a sequence of linear maps on \( \text{Tri}(A, M, B) \), then \( \mathcal{L} \) is a Lie higher derivation if and only if \( \mathcal{L}_k \) can be presented in the form

\[
\mathcal{L}_k \begin{pmatrix} a & m \\ b & \end{pmatrix} = \left( p_{1k}(a) + p_{2k}(b) \sum_{i+j=k, i \neq k} (p_{1i}(a) + p_{2i}(b))m_j - m_j(q_{2i}(b) + q_{1i}(a)) + f_{3k}(m) \right) q_{1k}(a) + q_{2k}(b).
\]
where \( \{m_j\}_{j \in \mathbb{N}} \subseteq M \), and for each \( k \in \mathbb{N} \), \( q_{1k} : A \to Z(B) \), \( p_{2k} : B \to Z(A) \), \( f_{3k} : M \to M \) are linear maps satisfying:

1. \( \{p_{1k}\}_{k \in \mathbb{N}}, \{q_{2k}\}_{k \in \mathbb{N}} \) are Lie higher derivations on \( A, B \), respectively,
2. \( q_{1k}[a, a'] = 0 \) and \( p_{2k}[b, b'] = 0 \) for all \( a, a' \in A, b, b' \in B \), and
3. \( f_{3k}(am) = \sum_{i+j=k} (p_{1i}(a)f_{3j}(m) - f_{3j}(m)q_{1i}(a)) \), \( f_{3k}(mb) = \sum_{i+j=k} (f_{3j}(m)q_{2i}(b) - p_{2i}(b))f_{3j}(m) \) for all \( a \in A, b \in B, m \in M \).

• Let \( D = \{D_k\}_{k \in \mathbb{N}} \) be a sequence of linear maps on \( \text{Tri}(A, M, B) \), then \( D \) is a higher derivation if and only if, \( D_k \) can be presented in the form

\[
D_k \left( \begin{array}{cc} a & m \\ b & 0 \end{array} \right) = \left( \begin{array}{c} p_{1k}(a) + p_{2k}(b) \sum_{i+j=k, i \neq k} ((p_{1i}(a) + p_{2i}(b))m_j - m_j(q_{2i}(b) + q_{1i}(a))) + f_{3k}(m) \\ q_{1k}(a) + q_{2k}(b) \end{array} \right)
\]

where \( \{m_j\}_{j \in \mathbb{N}} \subseteq M \), and for each \( k \in \mathbb{N} \), \( q_{1k} : A \to Z(B) \), \( p_{2k} : B \to Z(A) \), \( f_{3k} : M \to M \) are linear maps satisfying:

1. \( \{p_{1k}\}_{k \in \mathbb{N}}, \{q_{2k}\}_{k \in \mathbb{N}} \) are Lie higher derivations on \( A, B \), respectively,
2. \( q_{1k}[a, a'] = 0 \) and \( p_{2k}[b, b'] = 0 \) for all \( a, a' \in A, b, b' \in B \), and
3. \( f_{3k}(am) = \sum_{i+j=k} (p_{1i}(a)f_{3j}(m) - f_{3j}(m)q_{1i}(a)) \), \( f_{3k}(mb) = \sum_{i+j=k} (f_{3j}(m)q_{2i}(b) - p_{2i}(b))f_{3j}(m) \) for all \( a \in A, b \in B, m \in M \).

LHD property of unital algebras with a nontrivial idempotent. Let \( A \) be a unital algebra with a nontrivial idempotent \( e \) and \( f = 1 - e \). From the Peirce decomposition we can presented \( A \) as \( A = \begin{pmatrix} eAe & eAf \\ fAe & fAf \end{pmatrix} \). By using Theorem 4.4 for the generalized matrix algebra \( A \) we get the next result which is the "higher" version of [14, Corollary 4.3].

Corollary 5.4. Let \( A \) be a 2-torsion free unital algebra with a nontrivial idempotent \( e \) satisfying

\[
eae \cdot eAf = 0 \text{ implies } eae = 0, \quad \text{and } eAf \cdot faf = 0 \text{ implies } faf = 0,
\]

for any \( a \in A \), where \( f = 1 - e \). If the following conditions hold:

(I) \( Z(fAf) = Z(A)f, \ Z(eAe) = Z(A)e \)

(II) one of the following three conditions holds:

(i) either \( eAe \) or \( fAf \) does not contain nonzero central ideals
(ii) \( eAe \) and \( fAf \) are domain
(iii) either \( eAf \) or \( fAe \) is strongly faithful,

then \( A \) has LHD property.

As urgent consequences of Corollary 5.4 in the next results we obtain LHD property of the full matrix algebra \( M_n(A) \) and \( B(X) \), the algebra of all operator on Banach space \( X \) with \( \dim(X) \geq 2 \). The LHD property of \( B(X) \) with \( \dim(X) > 1 \) was proved by Han [7, Corollary 3.3] by a completely different method. Also the Lie derivation property of \( B(X) \) was proved by Lu and Jing [9] for Lie derivable maps at zero and idempotents. In addition, for properness of nonlinear Lie derivations on \( B(X) \) see [10].

Corollary 5.5. The algebra \( B(X) \) of bounded operators on a Banach space \( X \) with \( \dim(X) \geq 2 \) has LHD property.

Proof. It follows from Corollary 5.4 and the proof appeared in [14, Corollary 4.4]. \( \square \)

Corollary 5.6. Let \( A \) be a 2-torsion free unital algebra. The full matrix algebra \( \mathfrak{A} = M_n(A) \) with \( n \geq 3 \) enjoys the LHD property.
Proof. Consider nontrivial idempotents $e = e_{11}$ and $f = e_{22} + \cdots + e_{nn}$. It is obvious that $eAe = A, fAf = M_{n-1}(A)$. From $Z(A) = Z(A)1_A$ we conclude that $Z(eAe) = Z(A)e$ and $Z(fAf) = Z(A)f$, so assumption (I) of Corollary 5.4 holds. Moreover, [4, Lemma 1] guarantees that the algebra $fAf = M_{n-1}(A)$ does not contain nonzero central ideals, so part (i) of condition (II) in Corollary 5.4 is fulfilled. Hence by the mentioned corollary $M_n(A)$ has the LHD property.

It’s remarkable that Corollary 5.6 is the “higher” version of [4, Corollary 1].

6. PROOFS OF THEOREMS 2.2 AND 2.3

Proof of Theorem 2.2.

Proof. We proceed the proof by induction on $k$. The case $k = 1$ follows from Proposition 2.1. Suppose that the conclusion holds for any integer less than $k$. By (\dagger), $\mathcal{L}_k$ has the presentation

$$\mathcal{L}_k \left( \begin{array}{cc} a & m \\ n & b \end{array} \right) = \left( \begin{array}{cc} p_{1k}(a) + p_{2k}(b) + p_{3k}(m) + p_{4k}(n) & f_{1k}(a) + f_{2k}(b) + f_{3k}(m) + f_{4k}(n) \\ q_{1k}(a) + q_{2k}(b) + q_{3k}(m) + q_{4k}(n) & \end{array} \right),$$

for each $\left( \begin{array}{cc} a & m \\ n & b \end{array} \right) \in \mathcal{G}$. Applying $\mathcal{L}_k$ for the commutator $\left[ \left( \begin{array}{cc} 0 & m \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & b \end{array} \right) \right]$, we have

$$\left( \begin{array}{cc} p_{3k}(mb) & f_{3k}(mb) \\ q_{3k}(mb) & \end{array} \right)$$

$$= \sum_{i+j=k, i, j \neq 0} \left[ \left( \begin{array}{cc} p_{1i}(a) + p_{3i}(m) & f_{1i}(a) + f_{3i}(m) \\ q_{1i}(a) + q_{3i}(m) & \end{array} \right), \left( \begin{array}{cc} p_{2j}(b) & f_{2j}(b) \\ q_{2j}(b) & \end{array} \right) \right]$$

$$+ \left[ \left( \begin{array}{cc} p_{1k}(a) + p_{3k}(m) & f_{1k}(a) + f_{3k}(m) \\ q_{1k}(a) + q_{3k}(m) & \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & b \end{array} \right) \right]$$

$$+ \left[ \left( \begin{array}{cc} a & m \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} p_{2k}(b) & f_{2k}(b) \\ q_{2k}(b) & \end{array} \right) \right].$$

(6.1)

Use the equalities $p_{1i} = P_i - p_i, q_{2j} = Q_j - q_j, q_{1i} = Q''_i + q''_i, p_{2j} = P''_j + p''_j$, it follows that,

$$f_{3k}(mb) = f_{1k}(a)b + f_{3k}(m)b + af_{2k}(b) + mq_{2k}(b) - p_{2k}(b)m$$

$$+ \sum_{i+j=k, i, j \neq 0} \left( (p_{1i}(a) + p_{3i}(m))f_{2j}(b) + (f_{1i}(a) + f_{3i}(m))q_{2j}(b) \right)$$

$$- \sum_{i+j=k, i, j \neq 0} \left( p_{2j}(b)(f_{1i}(a) + f_{3i}(m)) + f_{2j}(b)(q_{1i}(a) + q_{3i}(m)) \right)$$

(6.2)

$$= f_{1k}(a)b + f_{3k}(m)b + af_{2k}(b) + mq_{2k}(b) - p_{2k}(b)m$$

$$+ \sum_{i+j=k, i, j \neq 0} \left( P_i(a)f_{2j}(b) - f_{2j}(b)Q''_i(a) \right) - \sum_{i+j=k, i, j \neq 0} p_i(a)f_{2j}(b)$$

$$+ \sum_{i+j=k, i, j \neq 0} \left( f_{1i}(a)Q_j(b) - P''_j(b)f_{1i}(a) \right) + \sum_{i+j=k, i, j \neq 0} p_{3i}(m)f_{2j}(b)$$

$$+ \sum_{i+j=k, i, j \neq 0} \left( f_{3i}(m)Q_j(b) - P''_j(b)f_{3i}(m) \right) - \sum_{i+j=k, i, j \neq 0} f_{1i}(a)q_{j}(b)$$

$$- \sum_{i+j=k, i, j \neq 0} f_{3i}(m)q_{j}(b) - \sum_{i+j=k, i, j \neq 0} p''_j(b)f_{3i}(m) - \sum_{i+j=k, i, j \neq 0} p''_j(b)f_{1i}(a).$$
\[- \sum_{i+j=k, i,j \neq 0} f_{2j}(b)q''_i(a) - \sum_{i+j=k, i,j \neq 0} f_{2j}(b)q_3(m) \]

(6.3)

In (6.3), if we put \( m = 0, a = 1, b = 1 \) and use the definition of \( p_i, p''_i, q_i \) and \( q''_i \) appeared in page 5 for \( a = 1, b = 1 \) then we arrive at,

\[
0 = f_{1k}(1) + f_{2k}(1) - \sum_{i+j=k, i,j \neq 0} p_i(1)f_{2j}(1) - \sum_{i+j=k, i,j \neq 0} f_{2j}(1)q''_i(1) \\
- \sum_{i+j=k, i,j \neq 0} f_{1i}(1)q_j(1) - \sum_{i+j=k, i,j \neq 0} p''_i(1)f_{1i}(1) \\
= f_{1k}(1) + f_{2k}(1) + \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = i} m_jn_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r}m_j \\
+ \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = i} m_jn_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r}m_j \\
- \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = j} m_jn_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r}m_j \\
- \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = j} m_jn_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r}m_j;
\]

from which we get \( f_{1k}(1) = -f_{2k}(1) \). Note that in the recent calculations, by induction hypothesis we have \( f_{1j}(1) = -f_{2j}(1) = m_j \) for all \( j < k \) and

\[
\sum_{i+j=k, i,j \neq 0} (P_i(1)m - mQ''_i(1)) = 0, \quad \sum_{i+j=k, i,j \neq 0} (mQ_j(1) - P''_j(1)m) = 0
\]

for all \( m \in M \).

Next, if we apply (6.2) for \( m = 0, b = 1 \) and use the equations \( q_{2j} = Q_j - q_j, \ p_{2j} = P''_j + p''_j \) then we have

\[
f_{1k}(a) = a f_{1k}(1) + \sum_{i+j=k, i,j \neq 0} (p_{1i}(a)f_{1j}(1) - f_{1i}(1)q_{2j}(1) + p_{2j}(1)f_{1i}(a) - f_{1j}(1)q_{1i}(a)) \\
= \sum_{i+j=k, j \neq 0} (p_{1i}(a)m_j - m_jq_{1i}(a)) + \sum_{i+j=k, i,j \neq 0} (p''_i(1)f_{1i}(a) + f_{1i}(1)q_j(1)) \\
= \sum_{i+j=k, j \neq 0} (p_{1i}(a)m_j - m_jq_{1i}(a)) \\
+ \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = i} m_{\beta_1}n_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r} \sum_{s+t=i} (P_i(a)m_s - m_sQ''_i(a)) \\
+ \sum_{i+j=k, i,j \neq 0} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = j} n_{\alpha_1}m_{\beta_1}\ldots n_{\alpha_r}m_{\beta_r} \sum_{s+t=i} m_jn_{\alpha_1}\ldots m_{\beta_r}n_{\alpha_r}m_j \\
= \sum_{i+j=k, j \neq 0} (P_i(a)m_j - m_jQ''_i(a)).
\]
It’s remarkable that $\sum_{i+j=k, i, j \neq 0} (P''_{ij}(1)f_{1i}(a) - f_{1i}(a)Q_{ji}(1)) = 0$ by induction hypothesis. Similarly we can show that $g_{1k}(1) = -g_{2k}(1)$ and similar equations hold for $g_{1k}(a), g_{2k}(b), f_{2k}(b)$. On the other hand when we set $a = 0$ in (6.2) we get

$$f_{3k}(mb) = f_{3k}(m)b + m q_{2k}(b) - p_{2k}(b)m + \sum_{i+j=k, i, j \neq 0} (p_{3i}(m)f_{2j}(b) + f_{3k}(m)q_{2j}(b))$$

$$- \sum_{i+j=k, i, j \neq 0} (p_{2j}(b)f_{3i}(m) + f_{2j}(b)q_{3i}(m))$$

$$= \sum_{i+j=k} (f_{3i}(m)q_{2j}(b) - p_{2j}(b)f_{3i}(m)) + \sum_{i+j=k, i, j \neq 0} (p_{3i}(m)f_{2j}(b) - f_{2j}(b)q_{3i}(m)).$$  \(6.4\)

We calculated the phrase appeared in the last Sigma of (6..4).

$$p_{3i}(m)f_{2j}(b) - f_{2j}(b)q_{3i}(m)$$

$$= \sum_{s+t=i} \sum_{\lambda+\mu=j} f_{3i}(m)N_{s}(m_{\lambda}Q_{\mu}(b) - P''_{\mu}(b)m_{\lambda}) + \sum_{s+t=i} \sum_{\lambda+\mu=j} (m_{\lambda}Q_{\mu}(b) - P''_{\mu}(b)m_{\lambda})N_{s}f_{3i}(m)$$

$$= f_{3i}(m) \sum_{s+t=i} \sum_{\lambda+\mu=j} \sum_{(\alpha+\beta)+\gamma+s} (n_{\alpha_{1}}m_{\beta_{1}} \ldots n_{\alpha_{s}}m_{\beta_{s}}n_{\gamma} + n_{s})(m_{\lambda}Q_{\mu}(b) - P''_{\mu}(b)m_{\lambda})$$

$$+ \sum_{s+t=i} \sum_{\lambda+\mu=j} \sum_{(\alpha+\beta)+\gamma+s} (m_{\lambda}Q_{\mu}(b) - P''_{\mu}(b)m_{\lambda})(n_{\alpha_{1}}m_{\beta_{1}} \ldots n_{\alpha_{s}}m_{\beta_{s}}n_{\gamma} + n_{s})f_{3i}(m)$$

$$= f_{3i}(m) \sum_{r' = 1}^{\eta_{j}} \sum_{(\alpha+\beta)+\mu+\gamma+s} n_{\alpha_{1}}m_{\beta_{1}} \ldots n_{\alpha_{s}}m_{\beta_{s}}Q_{\mu}(b) - P''_{\mu}(b)m_{\beta_{s}}$$

$$+ \sum_{r' = 1}^{\eta_{j}} \sum_{(\alpha+\beta)+\mu+\gamma+s} (m_{\beta_{s}}Q_{\mu}(b) - P''_{\mu}(b)m_{\beta_{s}})(n_{\alpha_{1}} \ldots n_{\alpha_{s}}m_{\beta_{s}}).$$  \(6.4\)

The indices $s, t$ appeared in the first equation of above relations acquire all values between 0 to $i$ and $i$ takes all values between 1 to $k - 1$, because of all these changes is symmetric the second equation of above relations hold.

Note that one of the maximum length for $(n_{\alpha_{1}}m_{\beta_{1}}) \ldots (n_{\alpha_{s}}m_{\beta_{s}})$ is at $\mu = 0, \lambda = 1$, in this case $j = s + 1$ and the length of $(n_{\alpha_{1}}m_{\beta_{1}}) \ldots (n_{\alpha_{s}}m_{\beta_{s}})(n_{\gamma_{1}}m_{1})$ is $\nu_{s} + 1 = \nu_{j+1} = \begin{cases} j/2 : & j - 1 \in \mathbb{D} \\ (j - 1)/2 : & j - 1 \in \mathbb{E} \end{cases}$  \(6.4\)

It is remarkable that the length of

$$(n_{\alpha_{1}}m_{\beta_{1}}) \ldots (n_{\alpha_{s}}m_{\beta_{s}})(n_{\gamma_{1}}m_{1}), \sum_{i=1}^{\nu_{s}}(\alpha_{i} + \beta_{i}) + \gamma + \lambda = j,$$

in the case where $\lambda \neq 1$ is the same as length of $(n_{\alpha_{1}}m_{\beta_{1}}) \ldots (n_{\alpha_{s}}m_{\beta_{s}})(n_{\gamma_{1}}m_{1})$, or equivalently it is $\eta_{j}$. Now by replacing this relation in (6.4) we get

$$f_{3k}(mb) = \sum_{i+j=k} (f_{3i}(m)Q_{j}(b) - P''_{j}(b)f_{3i}(m)).$$

Similarly one can check similar equations for $f_{3k}(am), g_{1k}(na), g_{4k}(bn)$. Again from (6.1) we get,

$$p_{3k}(mb) = ap_{2k}(b) + mp_{2k}(b) - p_{2k}(b)a + \sum_{i+j=k, i, j \neq 0} ((p_{3i}(a) + p_{3i}(m))p_{2j}(b) + (f_{1i}(a) + f_{3i}(m))g_{2j}(b)).$$
If we put \( m = 0 \), \( b = 1 \) in (6.5), then we have

\[
p_{3k}(m) = -mn_k - \sum_{i+j=k, i,j \neq 0} \left( f_{3i}(m)n_j + m_j \sum_{s+l+t=1} N_j f_{3l}(m)N_s \right)
\]

\[
+ \sum_{i+j=k, i,j \neq 0} p_{3i}(m) \sum_{r=1} \sum_{(\alpha + \beta)_r = j} m_{\beta_1}n_{\alpha_1} \ldots m_{\beta_r}n_{\alpha_r}
\]

\[
- \sum_{i+j=k, i,j \neq 0} \sum_{r=1} n_j f_{3i}(m)N_r m_{\beta_1}n_{\alpha_1} \ldots m_{\beta_r}n_{\alpha_r}
\]

\[
- \sum_{i+j=k, i,j \neq 0} \sum_{r=1} m_{\beta_1}n_{\alpha_1} \ldots m_{\beta_r}n_{\alpha_r} f_{3i}(m)N_r
\]

\[
- \sum_{i+j=k, i,j \neq 0} m_{\alpha_1} \ldots m_{\alpha_r} f_{3i}(m)N_r
\]

Also it is not difficult to check similar relations for \( q_{3k}(m), p_{4k}(n), q_{4k}(n) \). From (2,1)-entry equation (6.1) we have

\[
g_{3k}(mb) = -b(g_{1k}(a) + g_{3k}(m)) - g_{2k}(b) a
\]

\[
+ \sum_{i+j=k} \left( (g_{1i}(a) + g_{3i}(m))p_{2j}(b) + (q_{1i}(a) + q_{3i}(m))g_{2j}(b) \right)
\]

\[
- \sum_{i+j=k} \left( g_{2j}(b)(p_{1i}(a) + p_{3i}(m)) + q_{2j}(b)(g_{1i}(a) + g_{3i}(m)) \right).
\]

Set \( m = 0 \), \( b = 1 \) then we have

\[
2g_{3k}(m) = \sum_{i+j=k, i,j \neq 0} \left( g_{3i}(m)p_{2j}(1) - q_{2j}(1)g_{3i}(m) - q_{3i}(m)n_j + n_j p_{3i}(m) \right)
\]

\[
= \sum_{i+j=k, i,j \neq 0} \left( g_{3i}(m)p_{2j}''(1) + q_j(1)g_{3i}(m) \right)
\]

\[
- \sum_{s+t+j=k} \left( N_j f_{3i}(m)n_j + n_j f_{3i}(m)N_s \right)
\]
\[
= - \sum_{s+t+j=k} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r=j} N_s f_{3t}(m) N_t m_{\beta_1} n_{\alpha} \ldots m_{\beta_r} n_{\alpha_r} \\
- \sum_{s+t+l+j=k} \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r=j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} m_{\beta_1} N_s f_{3t}(m) N_l \\
- \sum_{s+t+j=k} (N_s f_{3t}(m) n_j + n_j f_{3t}(m) N_s) \\
= - \sum_{s+t+j=k} (N_s f_{3t}(m)(n_j + \sum_{r=1}^{\nu_j} \sum_{(\alpha+\beta)_r+l=j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} m_{\beta_1} n_l)) \\
- \sum_{s+t+j=k} ((\sum_{r=1}^{\nu_j} \sum_{(\alpha+\beta)_r+l=j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} m_{\beta_1} n_l + n_j) f_{3t}(m) N_s) \\
= -2 \sum_{s+t+j=k} N_s f_{3t}(m) N_j.
\]

As \( A \) is 2-torsion free we get \( q_{3k}(m) = - \sum_{s+t+j=k} N_s f_{3t}(m) N_j \). By similar argument and 2-torsion freeness of \( B \), it follows that \( f_{4k}(n) = - \sum_{s+t+j=k} M_s q_{4t}(n) M_j \).

From (2,2)-entry of equation (6.1) we have

\[
q_{3k}(mb) = (q_{1k}(a) + q_{3k}(m)b - b(q_{1k}(a) + q_{3k}(m))) - g_{2k}(m)b \\
+ \sum_{i+j=k, i,j \neq 0} ((g_{1i}(a) + g_{3i}(m)) f_{2j}(b) + (q_{1i}(a) + q_{3i}(m)) q_{2j}(b)) \\
- \sum_{i+j=k, i,j \neq 0} (g_{2j}(b)(f_{1i}(a) + f_{3i}(m)) + q_{2j}(b)(q_{1i}(a) + q_{3i}(m)))
\]

Set \( m = 0 \) in the last equation then we get

\[
0 = q_{1k}(a)b - b q_{1k}(a) + \sum_{i+j=k, i,j \neq 0} (g_{1i}(a) f_{2j}(b) + q_{1i}(a) q_{2j}(b) - g_{2j}(b) f_{1i}(a) - q_{2j}(b) q_{1i}(a))
\]

From replacement \( q_{1i}, q_{2j} \) in the last equation with \( Q_i'' + q_i'', Q_j - q_j \) respectively and assumption of induction we have

\[
0 = \sum_{s+t+\mu+\lambda=k} (n_s P'_t(a) - Q'_t(a)n_s)(P''_\mu(b)m_\lambda - m_\lambda Q'_\mu(b)) \\
+ \sum_{i+j=k, i,j \neq 0} (Q'_t(a) + q'_t(a))(Q_j(b) - q_j(b)) \\
- b q_{1k}(a) - \sum_{s+t+\mu+\lambda=k} (n_s P''_\mu(b) - Q'_\mu(b)n_s)(P_t(a)m_\lambda - m_\lambda Q'_\mu(a)) \\
- \sum_{i+j=k, i,j \neq 0} (Q_j(b) - q_j(b))(Q''_t(a) + q''_t(a))
\]

hence

\[
0 = q_{1k}(a)b - \sum_{s+t+\lambda=k} (n_s P'_t(a) - Q'_t(a)n_s)m_\lambda b \\
- \sum_{i+j=k, i,j \neq 0} q''_t(a) \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r=j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} m_{\beta_1} b \\
- b q_{1k}(a) + \sum_{s+t+\lambda=k} b m_s (P_t(a)m_\lambda - m_\lambda Q'_\mu(a))
\]
+ \sum_{i+j=k, i, j \neq 0} \sum_{r=1}^{m} \sum_{(\alpha+\beta)=j} b_{n_{\alpha}, m_{\beta}, \ldots, n_{\alpha}, m_{\beta}, q''}(a)
= \sum_{s+i+k=k} \sum_{i, j=0}^{m} (n_{s}, P_{s}'(a) - Q''_{s}(a) n_{s}) m_{\lambda} b
- \sum_{i+j=k} \sum_{r=1}^{m} \sum_{(\alpha+\beta)=j} \sum_{(\alpha+\beta)=j} (n_{\alpha}, P_{s}'(a) - Q''_{s}(a)n_{\alpha}, m_{\beta}, n_{\alpha}, m_{\beta}, \ldots, n_{\alpha}, m_{\beta}, b
- b q_{1k}(a) + \sum_{s+i+k=k} b_{n_{s}}(P_{1}(a)m_{\lambda} - m_{\lambda} Q''_{1}(a))
+ b q_{1k}(a) - \sum_{r=1}^{m} \sum_{(\alpha+\beta)=k, i \leq k} (n_{\alpha}, P_{s}'(a) - Q''_{s}(a)n_{\alpha}, m_{\beta}, n_{\alpha}, m_{\beta}, \ldots, n_{\alpha}, m_{\beta}, Q''_{s}(a))
= [Q''_{k}(a), b]

i.e. \( Q''_{k}(a) := q_{1k}(a) - q''_{k}(a) \in Z(A) \) for all \( a \in A \). By similar argument one can check that
\( P''_{k}(b) := p_{2k}(b) - p''_{k}(b) \in Z(A) \) for all \( b \in B \). Now apply \( L_{k} \) on commutator \( \left( \begin{array}{cc} 0 & 0 \\ 0 & b' \end{array} \right) \) we have

\[
\left( \begin{array}{cc}
p_{2k}(b, b') & 0 \\ \ast & q_{2k}(b, b') \end{array} \right) = \sum_{i+j=k} \left( \begin{array}{cc}
p_{2k}(b, b') & f_{2i}(b) \\ \ast & q_{2k}(b, b') \end{array} \right) \left( \begin{array}{cc}
p_{2k}(b, b') & f_{2j}(b) \\ \ast & q_{2k}(b, b') \end{array} \right)
\]

From (1,1)-entry of above equation and assumption of induction we have

\[
p_{2k}(b, b') = \sum_{i+j=k} \left[ p_{2i}(b), p_{2j}(b') \right] + \sum_{i+j=k} \left( f_{2i}(b) q_{2i}(b') - f_{2j}(b) q_{2j}(b) \right)
= \sum_{i+j=k} \left[ p_{2i}(b), p_{2j}(b') \right] + \sum_{\alpha+\beta=\xi+\zeta=k} (P''_{\xi}(b)m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, P''_{\zeta}(b) - Q_{\xi}(b)) n_{\alpha}
- \sum_{\alpha+\beta=\xi+\zeta=k} (P''_{\xi}(b)m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, P''_{\zeta}(b) - Q_{\xi}(b)) n_{\alpha}
= \sum_{i+j=k} \sum_{r=1}^{m} \sum_{(\alpha+\beta)=j} \sum_{(\alpha+\beta)=i} \sum_{(\alpha+\beta)=j} \sum_{(\alpha+\beta)=j} (m_{\beta}, Q_{\xi}(b)) n_{\alpha}, \ldots, m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, m_{\beta}, Q_{\xi}(b) - P''_{\xi}(b)m_{\beta}) n_{\alpha} \ldots m_{\beta}, n_{\alpha}
- P''_{\xi}(b)m_{\beta}, n_{\alpha}, \ldots, m_{\beta}, n_{\alpha} (m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, m_{\beta}, Q_{\xi}(b) - P''_{\xi}(b)m_{\beta}) n_{\alpha} \ldots m_{\beta}, n_{\alpha}
- \sum_{\alpha+\beta=\xi+\zeta=k} (m_{\beta}, Q_{\xi}(b)) n_{\alpha}, \ldots, m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, m_{\beta}, Q_{\xi}(b) - P''_{\xi}(b)m_{\beta}) n_{\alpha} \ldots m_{\beta}, n_{\alpha}
+ \sum_{\alpha+\beta=\xi+\zeta=k} (m_{\beta}, Q_{\xi}(b)) n_{\alpha}, \ldots, m_{\beta}, Q_{\xi}(b)) (n_{\alpha}, m_{\beta}, Q_{\xi}(b) - P''_{\xi}(b)m_{\beta}) n_{\alpha} \ldots m_{\beta}, n_{\alpha}.
By replacing $Q'$ with $q_{2\ast} + q'_{\ast}$ in following sentences of relation (6.7) we have

$$
\sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} (m_{\beta_1} Q_{(b)} q_i (b') n_{\alpha_1} \ldots m_{\beta_r} n_{\alpha_r} - P''_{\xi} (b) m_{\beta_1} q_i (b') n_{\alpha_1} \ldots m_{\beta_r} n_{\alpha_r})
$$

$$
- \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} m_{\beta_1} Q_{(b)} q_j (b) n_{\alpha_1} \ldots m_{\beta_r} n_{\alpha_r},
$$

$$
+ \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} P''_{\xi} (b) m_{\beta_1} q_j (b) n_{\alpha_1} \ldots m_{\beta_r} n_{\alpha_r},
$$

$$
+ \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} m_{\beta_1} Q_{(b)} q_{2i} (b') n_{\alpha_1} - \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} m_{\beta_1} Q_{(b)} q_{2j} (b) n_{\alpha_1},
$$

$$
+ \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} m_{\beta_1} Q_{(b)} q_{2i} (b') n_{\alpha_1} - \sum_{i+j=k} \sum_{\zeta + (\alpha + \beta) = j}^{\eta_j} m_{\beta_1} Q_{(b)} q''_{2j} (b) n_{\alpha_1},
$$

$$
+ \sum_{\alpha_1 + \beta_1 + \zeta + \xi = k} m_{\beta_1} Q_{(b)} n_{\alpha_1} P''_{\xi} (b) - \sum_{\alpha_1 + \beta_1 + \zeta + \xi = k} P''_{\xi} (b) m_{\beta_1} Q'_{(b)} n_{\alpha_1},
$$

$$
+ \sum_{\alpha_1 + \beta_1 + \zeta + \xi = k} P''_{\xi} (b) m_{\beta_1} Q'_{(b)} n_{\alpha_1} - \sum_{\alpha_1 + \beta_1 + \zeta + \xi = k} m_{\beta_1} Q_{(b)} n_{\alpha_1} P''_{\xi} (b). \quad (6.7)
$$
By a similar way on following sentences of relation (6.7) we have

\[- \sum_{i+j=k} \sum_{\alpha_1+\beta_1+\xi=i} m_{\beta_1} Q_{\xi}(b') q_{2j}(b) n_{\alpha_1} \]

\[- \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} m_{\beta_1} Q_{\xi}(b') q_{j}(b) n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[= - \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} m_{\beta_1} Q_{\xi}(b') Q_{\zeta}(b) n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[+ \sum_{r'=2}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} m_{\beta_1} Q_{\xi}(b') n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r}. \]

Now consider following sentences of relation (6.7)

\[\sum_{i+j=k} \sum_{\alpha_1+\beta_1+\zeta=k} m_{\beta_1} Q_i(b') n_{\alpha_1} P_{\zeta}''(b) \]

\[- \sum_{i+j=k} \sum_{\alpha_1+\beta_1+\zeta=k} P_{\zeta}''(b) m_{\beta_1} q_i(b') n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[- \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} P_{\zeta}''(b) m_{\beta_1} Q_i(b') n_{\alpha_1} \]

\[= - \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} P_{\zeta}''(b) m_{\beta_1} q'_i(b') n_{\alpha_1} \]

\[\vdots \]

\[- \sum_{r'=2}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} P_{\zeta}''(b) m_{\beta_1} Q_i(b') n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r}. \]

Similarly consider following sentences of relation (6.7) we have

\[- \sum_{\alpha_1+\beta_1+\zeta=k} m_{\beta_1} Q_{\zeta}(b) n_{\alpha_1} P_{\xi}''(b') \]

\[+ \sum_{i+j=k} \sum_{\alpha_1+\beta_1+\zeta=k} P_{\xi}'(b') m_{\beta_1} q_j(b) n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[+ \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} P_{\xi}'(b') m_{\beta_1} Q_{\zeta}(b) n_{\alpha_1} \]

\[= \sum_{r'=2}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k} P_{\xi}'(b') m_{\beta_1} Q_{\zeta}(b) n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r}. \]

Gather above relations and replace in relation (6.7), from this and assumption of induction we have

\[p_{2k}[b', b'] = \sum_{r'=1}^{\eta_k} \sum_{\xi+\zeta+(\alpha+\beta),r'=k, \zeta+\xi=k-2} m_{\beta_1} [Q_{\zeta}(b), Q_{\xi}(b')] n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[= \sum_{r'=1}^{\eta_k} \sum_{i+(\alpha+\beta),r'=k} m_{\beta_1} Q_i[b, b'] n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r} \]

\[= \sum_{r'=1}^{\eta_k} \sum_{i+(\alpha+\beta),r'=k, i\leq k-2} (m_{\beta_1} Q_i[b, b'] - P_{\xi}'(b') m_{\beta_1}) n_{\alpha_1} \ldots m_{\beta_r}, n_{\alpha_r}. \]
i.e. $P^{r\prime}_{\beta\gamma}(b, b') := p_2(b, b') - p_{r\prime}(b, b') = 0$ for all $b, b' \in B$. Similarly, $Q^{r\prime\prime}_{\kappa\lambda}(a, a') := q_{1k}(a, a') - q_{r\prime\prime}(a, a') = 0$ for all $a, a' \in A$. From (2, 2)-entry of equation (6.6) we have

$$q_{2k}(b, b') = \sum_{i+j=k} [q_{2i}(b), q_{2j}(b')] + \sum_{i+j=k} (g_{2i}(b)f_{2j}(b') - g_{2j}(b')f_{2i}(b)).$$

Replace $q_{2k}, q_{2i}, q_{2j}$ with $Q_k - q_k, Q_i - q_i, Q_j - q_j$ respectively, then we have

$$Q_k[b, b'] - q_k[b, b'] = \sum_{i+j=k} [Q_i(b) - q_i(b), Q_j(b') - q_j(b')] + \sum_{\alpha_1 + \beta_1 + \xi + \zeta = k} (n_{\alpha_1} P^{\alpha_1}_{\xi}(b) - Q_\xi(b)n_{\alpha_1})(P^{\beta_1}_{\xi}(b')m_{\beta_1} - m_{\beta_1}Q_\xi(b'))$$

$$- \sum_{\alpha_1 + \beta_1 + \xi + \zeta = k} (n_{\alpha_1} P^{\alpha_1}_{\xi}(b') - Q_\xi(b')n_{\alpha_1})(P^{\beta_1}_{\xi}(b)m_{\beta_1} - m_{\beta_1}Q_\xi(b)).$$

To show that $Q_k$ is a Lie higher derivation by assumption of induction, it is enough to check following equation

$$q_k[b, b'] = \sum_{i+j=k} ([Q_i(b), q_j(b')] + [q_i(b), Q_j(b')] - [q_i(b), q_j(b')])$$

$$+ \sum_{\alpha_1 + \beta_1 + \xi + \zeta = k} (n_{\alpha_1} P^{\alpha_1}_{\xi}(b)m_{\beta_1}Q_\xi(b') + Q_\xi(b)n_{\alpha_1} P^{\beta_1}_{\xi}(b)m_{\beta_1} - Q_\xi(b)n_{\alpha_1}m_{\beta_1}Q_\xi(b'))$$

$$- \sum_{\alpha_1 + \beta_1 + \xi + \zeta = k} (n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}Q_\xi(b) + Q_\xi(b')n_{\alpha_1} P^{\beta_1}_{\xi}(b)m_{\beta_1} - Q_\xi(b')n_{\alpha_1}m_{\beta_1}Q_\xi(b)).$$

From definition of $q_k$ and equations $Q_k = q_{2k} + q_k$, $Q'_k = q_{2k} + q'_k$ we have

$$\sum_{r=1}^{\eta_k} \sum_{i+j=(\alpha_1 + \beta_1), \xi = k} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1} Q_i[b, b'] = \sum_{i+j=k} Q_i(b) \sum_{r=1}^{\eta_1} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1} Q_\xi(b')$$

$$- \sum_{i+j=k} Q_i(b) \sum_{r=1}^{\eta_2} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}$$

$$- \sum_{i+j=k} \sum_{r=1}^{\eta_3} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b)Q_i(b)$$

$$+ \sum_{i+j=k} \sum_{r=1}^{\eta_4} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}Q_i(b)$$

$$+ \sum_{i+j=k} \sum_{r=1}^{\eta_5} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b)Q_j(b')$$

$$- \sum_{i+j=k} \sum_{r=1}^{\eta_6} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}Q_j(b')$$

$$- \sum_{i+j=k} Q_j(b') \sum_{r=1}^{\eta_7} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1} Q_\xi(b)$$

$$+ \sum_{i+j=k} Q_j(b') \sum_{r=1}^{\eta_8} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}$$

$$- \sum_{i+j=k} Q_j(b) \sum_{r=1}^{\eta_9} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b')$$

$$+ \sum_{i+j=k} Q_j(b) \sum_{r=1}^{\eta_{10}} \sum_{\xi = (\alpha_1 + \beta_1), \xi = j} n_{\alpha_1}m_{\beta_1} \ldots n_{\alpha_1} P^{\alpha_1}_{\xi}(b')m_{\beta_1}.$$
By omitting similar sentences we have

\[
0 = \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} m_{\beta_1} Q_\xi(b') \\
- \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') m_{\beta_1} \\
+ \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') Q_\xi(b) \\
- \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') m_{\beta_1} \\
+ \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') Q_\xi(b) \\
+ \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') m_{\beta_1} \\
+ \sum_{i+j=k} \sum_{r=1}^{n_j} \sum_{\xi+(\alpha+\beta)_r = j} n_{\alpha_r} m_{\beta_r} \ldots n_{\alpha_1} P_\xi''(b') Q_\xi(b)
\]
Now consider commutator

\[ \sum_{i+j=k, \xi+\alpha_1=i, \xi+\beta_1=j} q'_i(b) n_{\alpha_1} m_{\beta_1} Q_{\xi}(b') \]

\[ - \sum_{i+j=k, \xi+\beta_1=i, \xi+\alpha_1=j} q'_i(b) P'_{\xi}(b) m_{\beta_1} \]

\[ + \sum_{i+j=k, \xi+\beta_1=i, \xi+\alpha_1=j} q'_i(b) n_{\alpha_1} m_{\beta_1} Q_{\xi}(b) \]

Hence \( Q_k \) is a Lie higher derivation on \( B \). By similar techniques one can show that \( P_k \) and \( P'_k \) are Lie higher derivations on \( A \) and also \( Q'_k \) is a Lie higher derivation on \( B \).

Now consider commutator \( \left[ \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \right] \), apply \( L_k \) on it then we have

\[
\left( p_{1k}(mn) - p_{2k}(nm) \right) = \sum_{i+j=k} \left[ L_i \left( \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \right), L_j \left( \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix} \right) \right]
\]

Hence

\[
p_{1k}(mn) - p_{2k}(nm) = \sum_{i+j=k} \left( p_{3i}(m)p_{4j}(n) + f_{3i}(m)g_{4j}(n) - p_{4j}(n)p_{3i}(m) - f_{4j}(n)g_{3i}(m) \right),
\]

and

\[
q_{1k}(mn) - q_{2k}(nm) = \sum_{i+j=k} \left( g_{3i}(m)f_{4j}(n) + q_{3i}(m)q_{4j}(n) - g_{4j}(n)q_{3i}(m) - q_{4j}(n)q_{3i}(m) \right).
\]

\[ \square \]

**Proof of Theorem 2.3.**

**Proof.** From the fact that “every higher derivation is a Lie higher derivation” and Proposition 2.2 follow that items (3), (8), (9) and (10) hold automatically. For other items, we proceed the proof by induction on \( k \). The case \( k = 1 \) follows from [4]. Suppose that the conclusion is true for any integer less than \( k \), and \( L_k \) has the presentation

\[ D_k \left( \begin{pmatrix} a & m \\ n & b \end{pmatrix} \right) = \begin{pmatrix} p_{1k}(a) + p_{2k}(b) + p_{3k}(m) + p_{4k}(n) & f_{1k}(a) + f_{2k}(b) + f_{3k}(m) + f_{4k}(n) \\ g_{1k}(a) + g_{2k}(b) + g_{3k}(m) + g_{4k}(n) & q_{1k}(a) + q_{2k}(b) + q_{3k}(m) + q_{4k}(n) \end{pmatrix}, \]
in which the maps appeared in the entries are linear.

Apply $\mathcal{D}_k$ on equation $\begin{pmatrix} a & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & mb \\ 0 & 0 \end{pmatrix}$ then we have

\[
\begin{pmatrix} p_{3k}(mb) \\ g_{3k}(mb) \\ q_{3k}(mb) \end{pmatrix} = \begin{pmatrix} p_{1k}(a) + p_{3k}(m) \\ g_{1k}(a) + g_{3k}(m) \\ q_{1k}(a) + q_{3k}(m) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} a & m \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p_{2k}(b) \\ g_{2k}(b) \\ q_{2k}(b) \end{pmatrix} + \sum_{i+j=k, i,j \neq 0} \begin{pmatrix} p_{1i}(a) + p_{3i}(m) \\ g_{1i}(a) + g_{3i}(m) \\ q_{1i}(a) + q_{3i}(m) \end{pmatrix} \begin{pmatrix} p_{2j}(b) \\ g_{2j}(b) \\ q_{2j}(b) \end{pmatrix}.
\]

From (1,2)-entry of above equation we have

\[
f_{3k}(mb) = f_{1k}(a)b + f_{3k}(m)b + af_{2k}(b) + mq_{2k}(b) + \sum_{i+j=k, i,j \neq 0} \left( (p_{1i}(a) + p_{3i}(m))f_{2j}(b) + (f_{1i}(a) + f_{3i}(m))q_{2j}(b) \right).
\]

(6.8)

Set $m = 0$, $b = 1$ in equation (6.8) then we have

\[
f_{1k}(a) = am_k + \sum_{i+j=k, i,j \neq 0} (p_{1i}(a)m_j - f_{1i}(a)q_{2j}(1))
\]

\[
= \sum_{i+j=k, j \neq 0} p_{1i}(a)m_j + \sum_{i+j=k, i,j \neq 0} f_{1i}(a)q_{j}(1)
\]

\[
= \sum_{i+j=k, j \neq 0} p_{1i}(a)m_j + \sum_{i+j=k, i,j \neq 0} f_{1i}(a)\sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r = j} n_{\alpha,m_{\beta_1}, \ldots, n_{\alpha_1,m_{\beta_1}}}
\]

\[
= \sum_{i+j=k, j \neq 0} p_{1i}(a)m_j + \sum_{i+j=k, i,j \neq 0} P_i(a)m_j n_{\alpha_1,m_{\beta_1}, \ldots,n_{\alpha_m,m_{\beta_1}}}
\]

\[
= \sum_{i+j=k} P_i(a)m_j.
\]

Similarly one can check that similar equations for $g_{1k}(a), g_{2k}(b), f_{2k}(b)$. Now, set $m = 0$ in (6.8) hence

\[
f_{3k}(mb) = f_{3k}(m)b + mq_{2k}(b) + \sum_{i+j=k, i,j \neq 0} (p_{3i}(m)f_{2j}(b) + f_{3i}(m)q_{2j}(b))
\]

\[
= \sum_{i+j=k} f_{3i}(m)q_{2j}(b) + \sum_{i+j=k, i,j \neq 0} p_{3i}(m)f_{2j}(b).
\]

(6.9)

As

\[
p_{3i}(m)f_{2j}(b) = \sum_{s+t=1, \lambda+\mu=j} f_{3i}(m)N_{s}\lambda M_{\mu}(b)
\]
\[
= \sum_{s+t=1,\mu+j=\lambda} \sum_{r=1}^{\mu_s} \sum_{n_s} \mathbf{f}_{3l}(m)(n_{\alpha_1}m_{\beta_1} \cdots n_{\alpha_r}m_{\beta_r}n_{r+}n_5) m_\lambda Q_\mu(b)
\]
\[
= \mathbf{f}_{3l}(m) \sum_{r=1}^{\eta_j} \sum_{(\alpha+\beta)_r+\mu=j,\mu\leq k-2} n_{\alpha_1}m_{\beta_1} \cdots n_{\alpha_r}m_{\beta_r}Q_\mu(b)
\]
\[
= \mathbf{f}_{3l}(m)q_j(b),
\]
we can replace this relation in (6.9) and get
\[
\mathbf{f}_{3k}(mb) = \sum_{i+j=k} \mathbf{f}_{3l}(m)Q_j(b).
\]
Similarly one can check similar equations for \( \mathbf{f}_{3k}(am), \mathbf{g}_{4k}(na), \mathbf{g}_{4k}(bn) \). Now apply \( \mathcal{D}_k \) on equation
\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & b' \\
0 & b & \text{bb'}
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \text{bb'}
\end{pmatrix},
\]
then
\[
\begin{pmatrix}
p_{2k}(bb') & * \\
* & q_{2k}(bb')
\end{pmatrix} = \sum_{i+j=k} \begin{pmatrix}
p_{2i}(b) & \mathbf{f}_{2i}(b) \\
\mathbf{g}_{2i}(b) & q_{2i}(b)
\end{pmatrix} \begin{pmatrix}
p_{2j}(b') & \mathbf{f}_{2j}(b') \\
\mathbf{g}_{2j}(b') & q_{2j}(b')
\end{pmatrix}. \quad (6.10)
\]
From (1,1)-entry of above equation and assumption of induction we have
\[
p_{2k}(bb') = \sum_{i+j=k} (p_{2i}(b)p_{2j}(b') + f_{2i}(b)g_{2j}(b'))
\]
\[
= \sum_{i+j=k} \sum_{r=1}^{\eta_i} \sum_{r'=1}^{\eta_j} \sum_{(\alpha+\beta)_r+\mu_1=\lambda_1} m_{\beta_1}Q_\lambda(b)n_{\alpha_1} \cdots m_{\beta_1}n_{\alpha_r}m_{\beta_r}n_{\alpha_r'}m_{\beta_r'}n_{\alpha_r'} \cdots m_{\beta_1}Q_\lambda'(b')n_{\alpha_1}
\]
\[
+ \sum_{i+j=k} \sum_{\zeta + \beta_1 = i} \sum_{\zeta + \alpha_1 = j} m_{\beta_1}Q_\lambda(b)Q_\lambda'(b')n_{\alpha_1}.
\]
Set \( b' = 1 \) it follows that
\[
p_{2k}(b) = \sum_{i+j=k} \sum_{r=1}^{\eta_i} \sum_{r'=1}^{\eta_j} \sum_{(\alpha+\beta)_r+\mu_1=\lambda_1} m_{\beta_1}Q_\lambda(b)n_{\alpha_1} \cdots m_{\beta_1}n_{\alpha_r}m_{\beta_r}n_{\alpha_r'}m_{\beta_r'}n_{\alpha_r'} \cdots m_{\beta_1}Q_\lambda(b)Q_\lambda'(b)Q_\lambda'(b')n_{\alpha_1}
\]
\[
+ \sum_{i+j=k} \sum_{\zeta + \beta_1 = i} \sum_{\zeta + \alpha_1 = j} m_{\beta_1}Q_\lambda(b)n_{\alpha_1} \cdots m_{\beta_1}n_{\alpha_r}m_{\beta_r}n_{\alpha_r'}m_{\beta_r'}n_{\alpha_r'} \cdots m_{\beta_1}Q_\lambda(b)Q_\lambda'(b')(b' \in B).
\]
Note that if we set \( b = 1 \) then
\[
p_{2k}(b') = \sum_{r=1}^{\eta_k} \sum_{\zeta + (\alpha+\beta)_r = k} m_{\beta_1}n_{\alpha_r} \cdots m_{\beta_1}Q_\lambda'(b')n_{\alpha_1}, \quad (b' \in B).
\]
A similar argument reveals that
\[
q_{1k}(a) = \sum_{r=1}^{\eta_k} \sum_{i+j=(\alpha+\beta)_r, i \leq k-2} n_{\alpha_r}m_{\beta_r} \cdots n_{\alpha_1}P_i(a)m_{\beta_1}
\]
By assumption of induction, cancelling similar sentences and replacing 

\[ q_{2k}(bb') = \sum_{i+j=k} (g_{2i}(b)q_{2j}(b') + q_{2i}(b)q_{2j}(b')). \]

From (2, 2)-entry of equation (6.10) we have

\[ Q_k(bb') - q_k(bb') = \sum_{i+j=k} (Q_i(b) - q_i(b))(Q_j(b') - q_j(b')) \]

+ \[ \sum_{i+j=k} \sum_{i \neq 0} \sum_{\xi + \alpha_1 = i \xi + \beta_1 = j} Q_\xi(b)n_{\alpha_1}m_{\beta_1}Q_\xi(b'). \]

In sequel we show that

\[ q_k(bb') = \sum_{i+j=k} (Q_i(b)q_j(b') + q_i(b)Q_j(b') - q_i(b)q_j(b')) \]

- \[ \sum_{i+j=k} \sum_{i \neq 0} \sum_{\xi + \alpha_1 = i \xi + \beta_1 = j} Q_\xi(b)n_{\alpha_1}m_{\beta_1}Q_\xi(b'). \]

As

\[ \sum_{r=1}^{\eta_k} \sum_{i+j=k} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_i(bb') = \sum_{i+j=k} Q_i(b) \sum_{r=1}^{\eta_j} \sum_{\zeta + (\alpha + \beta)_r = j} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b)Q_j(b') \]

- \[ \sum_{i+j=k} \sum_{r=1}^{\eta_j} \sum_{\zeta + (\alpha + \beta)_r = j} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

+ \[ \sum_{i+j=k} q_{2i}(b) \sum_{r=2}^{\eta_j} \sum_{\zeta + (\alpha + \beta)_r = j} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

- \[ \sum_{i+j=k} \sum_{\xi + \alpha_1 = i \xi + \beta_1 = j} Q_\xi(b)n_{\alpha_1}m_{\beta_1}Q_\xi(b'). \]

By assumption of induction, cancelling similar sentences and replacing \( Q_\xi \) with \( q_{2i} + q_i' \) we have

\[ 0 = + \sum_{i+j=k} q_{2i}(b) \sum_{r=1}^{\eta_j} \sum_{\zeta + (\alpha + \beta)_r = j} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

- \[ \sum_{i+j=k} Q_i'(b) \sum_{\xi + \alpha_1 + \beta_1 = j} n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

= \[ + \sum_{i+j=k} q_{2i}(b) \sum_{r=2}^{\eta_j} \sum_{\zeta + (\alpha + \beta)_r = j} n_{\alpha_r}m_{\beta_r} \ldots n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

- \[ \sum_{i+j=k} q_i'(b) \sum_{\zeta + \alpha_1 + \beta_1 = j} n_{\alpha_1}m_{\beta_1}Q_\xi(b') \]

= \[ \vdots \]
\[= \sum_{(\alpha+\beta)\eta_k=k} bm_{\alpha \eta_k} m_{\beta \eta_k} \ldots n_{\alpha_1 m_{\beta_1}} b' - \sum_{(\alpha+\beta)\eta_k=k} bm_{\alpha \eta_k} m_{\beta \eta_k} \ldots n_{\alpha_1 m_{\beta_1}} b'.\]

Hence \(\{Q_k\}_{k \in \mathbb{N}_0}\) is a higher derivation on \(B\). By similar techniques one can be shown that \(\{P_k\}_{k \in \mathbb{N}_0}, \{P'_k\}_{k \in \mathbb{N}_0}\) are higher derivations on \(A\) and also \(\{Q'_k\}_{k \in \mathbb{N}_0}\) is a higher derivation on \(B\).

Now consider equation \((0 \ 0 \ m \ 0 \ 0) (0 \ 0 \ n \ 0 \ 0) = (mn \ 0 \ 0 \ 0 \ 0)\), apply \(D_k\) on it then we have,

\[
\begin{pmatrix}
p_{1k}(mn) & * & q_{1k}(mn) \\
* & & \\
\end{pmatrix} = \sum_{i+j=k} \begin{pmatrix}
p_{3i}(m) & f_{3i}(m) \\
g_{3i}(m) & q_{3i}(m) \\
\end{pmatrix} \begin{pmatrix}
p_{4j}(n) & f_{4j}(n) \\
g_{4j}(n) & q_{4j}(n) \\
\end{pmatrix}.
\]

Hence

\[p_{1k}(mn) = \sum_{i+j=k} (p_{3i}(m)p_{4j}(n) + f_{3i}(m)g_{4j}(n)),\]

and

\[q_{1k}(mn) = \sum_{i+j=k} (g_{3i}(m)f_{4j}(n) + q_{3i}(m)q_{4j}(n)).\]

Similarly one can check similar equations for \(p_{2k}(nm)\) and \(q_{2k}(nm)\). \(\square\)

**Proof of Proposition 2.4.**

**Proof.** Let \(\tau\) maps into the center of \(G\) and vanishes at commutators. Suppose that linear map \(\tau_k : G \rightarrow Z(G)\) has general form as;

\[
\tau_k \begin{pmatrix} a & m \\ n & b \end{pmatrix} = \begin{pmatrix} p_{1k}(a) + p_{2k}(b) + p_{3k}(m) + p_{4k}(n) \\ q_{1k}(b) + q_{2k}(b) + q_{3k}(m) + q_{4k}(n) \end{pmatrix},
\]

(6.11)

for all \(k \in \mathbb{N}\). As \(\tau\) vanishing at commutators we have

\[
0 = \tau_k \begin{pmatrix} a & m \\ n & b \end{pmatrix}, \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = \tau_k \begin{pmatrix} [a, a'] + mn' - m'n \\ na' + bn' - n'a - b'n \end{pmatrix} \begin{pmatrix} am' + mb' - a'm - m'b \\ b', \ b' + mm' - n'm' \end{pmatrix},
\]

(6.12)

for all \(k \in \mathbb{N}\). Now by relations (6.11) and (6.12) we have

\[p_{3k}(am' + mb' - a'm - m'b) = 0, \quad q_{3k}(am' + mb' - a'm - m'b) = 0,\]

if we set \(a = a' = 0, b = 0, b' = 1\) in the above commutator we have \(p_{3k}(m) = 0\) and \(q_{3k}(m) = 0\) for all \(m \in M\) and \(k \in \mathbb{N}\). Similarly, \(p_{4k}(n) = 0, q_{4k}(n) = 0\) for all \(n \in N, k \in \mathbb{N}\). Moreover if \(m' = 0, n = 0\) we have \(p_{1k}(mn') = p_{2k}(n'm), q_{1k}(mn') = q_{2k}(n'm)\) for all \(m \in M, n' \in N, k \in \mathbb{N}\). One more time set \(b = 0, m = 0, n = 0\) in relation (6.12) then we have \(p_{1k}[a, a'] = 0\) and \(q_{1k}[a, a'] = 0\) for all \(a, a' \in A, k \in \mathbb{N}\). By a similar way one can check that \(p_{2k}[: b, b'] = 0, q_{2k}[b, b'] = 0\) for all \(b, b' \in B, k \in \mathbb{N}\). As \(\tau\) maps into \(Z(G)\) we have;

\[
\begin{pmatrix} p_{1k}(a) + p_{2k}(b) & 0 \\ q_{1k}(a) + q_{2k}(b) & \end{pmatrix} = \begin{pmatrix} a' & m' \\ n' & b' \end{pmatrix} = 0,
\]

for all \(a, a' \in A, b, b' \in B, m' \in M, n' \in N\) and \(k \in \mathbb{N}\). From this, a direct verification reveals that the remainders hold. The reverse argument is trivial. \(\square\)
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