Spin Quantum Entanglement in a General Curved Static Space-Time

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Abstract

A general formalism of the spin quantum entanglement in a curved space-time represented. As examples Kerr and non commutative Reissner- Nordström models are considered. The behaviors of the concurrence and entanglement entropy as a function of the various parameters are also discussed.

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1 Introduction

During the last decade, a great interest has been devoted to quantum entanglement and information theory[12][4][18]. The spin quantum entanglement of a bipartite system plays an important role in most of the physical system such as condensed matter. Recently, the effect of the relativistic motion on the quantum spin states entanglement correlation has been the focus of many people[10][14][5]. More, interesting was the study of the quantum entanglement of a spin system in non inertial frames and a curved space-time[15][11][1][2].

In this paper, we discuss the effect of a certain static gravitational field (near to black hole) on the quantum spin entanglement (QSE) of a bipartite system. In section 2, we present the general mathematical formalism. In section 3 we consider the Kerr space-time. In section 4, the non commutative Reissner-Nordstrom space-time is considered and in section 5 we draw our conclusion.

2 Mathematical formalism

In order to study the spin of particle in a curved space-time one has to use an inertial local frame at each point. This can be done at the tangent at a point of curved space-time using the vierbien (or tetrad) $e^\mu_a$ ($\mu$ (resp. $a$) is a curved (resp. flat) index) defined by:

$$g_{\mu\nu}e^\mu_\alpha e^\nu_\beta = \eta_{\alpha\beta} \tag{1}$$

where $g_{\mu\nu}$ and $\eta_{\alpha\beta}$ are the metric of the curved and Minkowski space-time respectively. Let us introduce a one fermionic particle state $|P,\sigma\rangle$ with a 4-momentum $P^\mu$ and spin $\sigma(=\uparrow,\downarrow)$ at some point of the space-time. If we move from one point to another, this state becomes (in a local frame)[17][9]

$$\sum_{\sigma'} D_{\sigma'}(\Lambda,p) \left|\Lambda p,\sigma'\right\rangle \tag{2}$$

where $\Lambda$ is the Lorentz transformation matrix and $D_{\sigma'\sigma}$ the wigner rotation matrix elements [16].

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Let us consider a system of two non-interacting spin $\frac{1}{2}$ particles where its center of mass system can be described by an initial wave packet $|\Psi^i\rangle$ given in a local frame by:

$$|\Psi^i\rangle = \sum_{\sigma_1,\sigma_2} \int dp_1 dp_2 g_{\sigma_1,\sigma_2}(p_1,p_2) |p_1,\sigma_1;p_2,\sigma_2\rangle$$

with the normalisation condition:

$$\sum_{\sigma_1,\sigma_2} \int dp_1 dp_2 |g_{\sigma_1,\sigma_2}(p_1,p_2)|^2 = 1$$

Here $p_1$ and $p_2$ are the four-momentum of the particles 1 and 2 respectively. Now, it is easy to show that when the system reaches another point of the inertial local frame, the wave packet becomes $|\Psi^f\rangle$ such that :

$$|\Psi^f\rangle = \sum_{\sigma_1,\sigma_2,\sigma'_1,\sigma'_2} \int dp_1 dp_2 \sqrt{\frac{(\Lambda p_1)\cdot(\Lambda p_2)}{p_1^0 p_2^0}} g_{\sigma_1,\sigma_2}(p_1,p_2) \times D_{\sigma'_1,\sigma_1}(\Lambda, p_1)D_{\sigma'_2,\sigma_2}(\Lambda, p_2) |\Lambda p_1,\sigma'_1;\Lambda p_2,\sigma'_2\rangle$$

while the change in the vierbien $\delta e^\mu_\nu(x)$ is given by:

$$\delta e^\mu_\nu(x) = u^\nu(x) d\tau \nabla_\nu e^\mu_\nu(x) = -u^\nu(x) \omega^\alpha_\nu e^\mu_\alpha(x) d\tau = \chi^\mu_\nu(x) e^\mu_\nu(x) d\tau$$

where

$$\omega^\alpha_\nu = -e^\alpha_\nu(x) \nabla_\nu e^\alpha_\nu(x)$$

with

$$\chi^\mu_\nu(x) = -u^\nu(x) \omega^\alpha_\nu$$

Here $\omega^\alpha_\nu$ and $\chi^\mu_\nu(x)$ are the spin connection and the change of the spin connection along the direction of the 4-vector velocity $u^\nu(x)$ and $\nabla_\nu$ stands for the covariant derivative. It is worth to mention also that during this displacement, the change in the momentum $\delta p^\mu(x)$ is:

$$\delta p^\mu(x) = u^\nu(x) d\tau \nabla_\nu p^\mu(x) = ma^\mu(x) d\tau$$

where $\tau$ is the proper time and $a^\mu(x)$ the 4-vector acceleration given by:

$$a^\mu(x) = u^\nu(x) \parallel_\nu u^\mu(x)$$

Straightforward simplifications lead to:

$$\delta p^\alpha(x) = \chi^\alpha_\nu(x) p^\nu(x) d\tau$$

where the infinitesimal Lorentz transformation matrix elements $\chi^\alpha_\nu(x)$ have the form:

$$\chi^\alpha_\nu(x) = -\frac{1}{mc^2} [a^\alpha(x) p_\nu(x) - p^\alpha(x) a_\nu(x)] + \chi^\alpha_\nu(x)$$

and

$$a^\mu(x) = u^\nu [\parallel_\nu u^\mu + \Gamma^\nu_{\alpha\lambda} u^\lambda]$$

where $\Gamma^\nu_{\alpha\lambda}$ is the affline connection.

Now, let us consider a general static universe where the metric $ds^2$ has the form:

$$ds^2 = F(r)dt^2 + G(r)dr^2 + H(r,\theta)d\theta^2 + I(r,\theta)d\varphi^2$$

Using the spherical coordinate $(r, \theta, \varphi)$, and choosing the tetrad components:

$$e^0_0 = \frac{1}{\sqrt{F(r)}} \quad e^1_0 = \frac{1}{\sqrt{G(r)}} \quad e^\vartheta_0 = \frac{1}{\sqrt{H(r,\theta)}} \quad e^\varphi_0 = \frac{1}{\sqrt{I(r,\theta)}}$$

Thus, the non vanishing spin connection elements are:

$$\omega^0_{11} = \frac{1}{2} \frac{F'}{\sqrt{GF}} \quad \omega^0_{23} = \frac{1}{2\sqrt{IF}} \quad \omega^0_{32} = -\frac{1}{2\sqrt{GH}} H'$$

$$\omega^1_{03} = -\frac{1}{2} \frac{I'}{\sqrt{GI}} \quad \omega^2_3 = -\frac{1}{2\sqrt{HI}} \partial_\theta I$$

2
where \( \dot{t} = \frac{\partial t}{\partial \tau} \) and \( \dot{r} = \frac{\partial r}{\partial \tau} \). Furthermore the non vanishing components \( u^r, \chi^0 \) and \( \chi^a \), for a circular motion and constant angular velocity \( \frac{d\phi}{d\tau} \) on the equatorial plane where \( \theta = \frac{\pi}{2} \) are given by:

\[
\begin{align*}
\rho &= \left( \begin{array}{cc}
\cos \phi & \sin \phi \\
\sin \phi & -\cos \phi
\end{array} \right)
\end{align*}
\]

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\rho &= \left( \begin{array}{cc}
\cos \phi & \sin \phi \\
\sin \phi & -\cos \phi
\end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\lambda^0_1 &= -u^r \omega^0_1 \\
\lambda^3_1 &= -u^r \omega^3_1
\end{align*}
\]

and:

\[
\begin{align*}
\lambda^0_2 &= \frac{1}{mc^2} [p^0 a_1] + \lambda^0_1 \\
\lambda^2_2 &= \lambda^1_2 = -\frac{1}{mc^2} [a^1 p_3] + \lambda^3_3
\end{align*}
\]

It is important to mention that the two non vanishing components of the 4-vector velocity \( u^r \) and \( u^\phi \) can be rewritten as:

\[
\begin{align*}
\rho &= \left( \begin{array}{cc}
\cos \phi & \sin \phi \\
\sin \phi & -\cos \phi
\end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\lambda^0_1 &= -u^r \omega^0_1 \\
\lambda^3_1 &= -u^r \omega^3_1
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\[
\begin{align*}
\lambda^0_2 &= \frac{1}{mc^2} [p^0 a_1] + \lambda^0_1 \\
\lambda^2_2 &= \lambda^1_2 = -\frac{1}{mc^2} [a^1 p_3] + \lambda^3_3
\end{align*}
\]

where \( \xi \) is the rapidity in the local inertial frame such that \( \xi = \tanh \xi (\frac{\xi}{c} = \sqrt{\frac{2I}{mc^2}}, \gamma \) is the Lorentz factor) To quantify the spin entanglement of the two particles system, we use the Wootters concurrence\[19, 20\], for the mixed state \( |p_1, \uparrow, p_2, \downarrow \rangle \) defined by:

\[
C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \}
\]

where \( \rho \) is the state density matrix and \( \sqrt{\lambda_i} \) are the eigenvalues of \( \tilde{\rho} \) where \( \tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y) \) with \( \sigma_y \) is the Pauli matrix. If \( \lambda_i \) are positive real numbers, the entanglement can be quantified by the entanglement entropy \( E(\varrho) \) defined as\[5\]:

\[
E(\varrho) = \frac{\hbar}{2} \left( 1 + \sqrt{1 - C^2(\rho)} \right)
\]

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\]

In the case of a curved static space-time and spin singlet state, eq\[21\] can be shown to have the following expression\[11\]:

\[
C(\rho') = (\cos \Theta)^2 + (\sin \Theta)^2
\]

where:

\[
\langle \cos x \rangle = \int |f(p)|^2 \cos x \, dp
\]

\[
\langle \cos x \rangle = \int |f(p)|^2 \cos x \, dp
\]

\[
\langle \cos x \rangle = \int |f(p)|^2 \cos x \, dp
\]

and \( \Theta \) is a shorthand notation for \( \tau \Theta^1_3 (\Theta^1_3 \) is the only non vanishing components of \( \Theta^1_3 \)) with:

\[
\Theta^1_3 = \lambda^0_1 + \frac{\lambda^3_2 p_3 - \lambda^0_2 p^3}{p^0 + mc^2}
\]

\[
\Theta^1_3 = \lambda^0_1 + \frac{\lambda^3_2 p_3 - \lambda^0_2 p^3}{p^0 + mc^2}
\]

\[
\Theta^1_3 = \lambda^0_1 + \frac{\lambda^3_2 p_3 - \lambda^0_2 p^3}{p^0 + mc^2}
\]

For the general static metric of eq\[14\], \( \Theta \) can be rewritten as:

\[
\Theta = \Theta^1_3 \tau = \frac{-\alpha}{2G} \left( [AD^2 + B(D^2 - 1) - I' \sqrt{\frac{G}{HT}}] + \frac{D}{c^2} p [AD^2 + B(D^2 - 1) + AV(\sqrt{G})] \right)
\]

where \( A = \frac{L'}{\tau}J, B = \frac{L'}{\tau}J, D = \sqrt{1 + q^2}, C = 1 + \sqrt{1 + p^2} \) and \( \alpha = \frac{\pi}{2} \).
3 Kerr space-time

As a first application we consider the Kerr metric. In Boyer-Lindquits coordinates has the following expression:\[27\]:

\[ ds^2 = \frac{\Delta}{\rho^2} dt^2 - \frac{\rho^2}{\Delta} g(r) dr^2 - \rho^2 d\theta^2 - \frac{\sin^2 \theta}{\rho^2} d\phi^2 \]

(28)

where

\[ \rho = r^2 + a^2 \cos^2 \theta \quad \Delta = r^2 + r_s r + a^2 + r_s^2 \]

(29)

\[ a = \frac{J}{Mc} \quad r_s = \frac{2GM}{c^2} \quad r_s^2 = \frac{Q^2G}{4\pi\alpha c^4} \]

and

\[ d\bar{t}^2 = (dt - a \sin^2 \theta d\phi)^2 \]

\[ d\bar{\phi}^2 = ((r^2 + a^2) d\phi - adt)^2 \]

Here Mc, J and Q are the mass, angular momentum and charge Q of the blak hole, G, c, ζ0 are the Newton gravitational constant, velocity of light, vacuum permittivity respectively. In what follows we deal with a non charged black hole where \( Q = 0 \). In this case, we can show that the non vanishing components are:

\[ e_t^0 = \frac{\sqrt{\Delta}}{r^2} \quad e_r^0 = \frac{\sqrt{r^2}}{\Delta} \quad e_\theta^0 = \sqrt{r^2} \quad e_\phi^0 = \sqrt{\frac{1}{r^2}} \]

(30)

\[ u^t = r \frac{\gamma c}{\sqrt{\Delta}} \quad u^\phi = r \gamma v \]

(31)

\[ \Gamma_r^r = - \frac{r^2 r_s - 2ra^2}{2\Delta r^2} \quad \Gamma_t^r = \frac{\Delta(rr_s - 2a^2)}{2r^5} \quad \Gamma_r^\phi = \frac{\Delta}{r^5} \]

(32)

\[ \Gamma_\theta^r = - \frac{\Delta}{r} \quad \Gamma_r^\theta = \frac{r^2 r_s - 2ra^2}{2\Delta r^2} \quad \Gamma_r^\theta = \frac{1}{r} \]

\[ \omega_{11}^0 = \frac{rr_s - 2a^2}{2r^3} \quad \omega_{12}^1 = \sqrt{\Delta} \quad \omega_{13}^1 = \frac{\sqrt{\Delta}}{r^3} \]

(33)

\[ \lambda_1^0 = - \frac{\gamma c}{\sqrt{\Delta}} \frac{rr_s - 2a^2}{2r^2} \quad \lambda_3^1 = - \gamma v \frac{\sqrt{\Delta}}{r^2} \]

(34)

\[ a^t = \frac{r^2 c^2 (rr_s - 2a^2)}{2r^4} + \gamma^2 v^2 \frac{\Delta}{r^4} \]

(35)

\[ \lambda_1^0 = \frac{1}{Mc^2} \left( \frac{\gamma^2 c^2 (rr_s - 2a^2)}{2r^2} \right) \sqrt{\frac{r^2}{\Delta}} \]

\[ \lambda_3^1 = - \frac{1}{Mc} \left( \frac{\gamma^2 c^2 (-rr_s + 2r)}{2r^2} \right) \sqrt{\frac{r^2}{\Delta}} \]

(36)

Now, if we set \( q = \frac{\rho^2}{Me}, p = \frac{\rho^2}{Me}, z = \frac{t}{r}, \Sigma = \frac{r}{z}, \alpha = \frac{z}{r}, \) one has:

\[ \Theta = \alpha \sqrt{\frac{z^2}{z^2 - z + \Sigma^2} \left( 1 - 2z \right) q \sqrt{(q^2 + 1)(r^2 + 1)} - \frac{qp}{\sqrt{p^2 + 1} + 1}} \]

(37)

Figure 1 displays the variation of the concurrence as a function of the dimensionless parameter \( \Sigma \in [0,1] \) and fixed \( \alpha \approx 1, z = 1.5 \) and \( q \approx 0.1 \), the concurrence is an increasing function. This is due to the fact that the gravitational potential \( g_{00} \) decreases as \( J \) (or \( p \)) increases and thus information (or concurrence) increases until a saturated bound of the maximal entanglement \( (C^2 \approx 1) \). Figure 2 shows the
Figure 1: Variation of the concurrence as a function of $\Sigma$ with fixed $\alpha = 1$, $z = 1.5$ and $q = 0.1$.

Concurrence variation as a function of $q$ for fixed values of $\alpha = 2$, $z$ and $\Sigma$, at smaller value of $q$ ($q \rightarrow 0$), the entangled is max and if $q \rightarrow \infty$ the center of the wave packet travels more on the circular trajectory and therefore one has more decoherence (less entanglement) and consequently the concurrence decreases for example if $q = 1$, $C(\rho_f) = 0.7$ and if $q = 1.6$, $C(\rho_f) = 0.4$, the oscillator periodic behavior can be explained (as it was pointed out in ref[2]) by the fact that when $q$ increases, the exponential in the integral that present in the expression of the concurrence approaches unity, so the cosine and sine terms behavior dominates. It is worth to mention that this behavior (minima and maxima) changes if the other parameters such as $\Sigma$ and $z$ changes. Figure 2 shows that if $\Sigma$ decreases to 0.3 the number and shape of picks change and they become more pronounced, similar behavior is shown in Figure 3 if $z$ changes.

Figure 2: The concurrence as a function of $q$ for fixed $\alpha = 2$, $z = 1.5$.

Figure 3: The concurrence as a function of $q$ for fixed $\alpha = 1$, $\Sigma = 1$. 

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Figure 4 and 5 display the variation of the concurrence as a function of $z$ (or circular motion radius) for fixed values of $q, \Sigma, \alpha = 1$, notice that for smaller values of $r$ ($r \to 0$ near black hole singularity) where the gravitational field is infinite, the entanglement is minimal ($C(\rho') \sim 0$). If we go far from the singularity ($z$ increases) the gravitational field decreases and therefore the information increases and thus the concurrence ($C(\rho') \sim 1$). The shape and number of picks and minima depend strongly on the values of the parameters $q$ and $\Sigma$. Figures 4, 5 and 6 show the behavior of the concurrence with variation of $\Sigma$ and $q$ respectively.

Figure 4: the concurrence as a function of $z$ for fixed $q = 0.1, \alpha = 1$.

Figure 5: the concurrence as a function of $z$ for fixed $\Sigma = 1, \alpha = 1$.

Figure 6: the concurrence as a function of $z$ for fixed $q = 0.1, \alpha = 1$. 
4 Reissner-Nordström noncommutative space-time

As a second example we consider the Reissner-Nordström metric for a charged non rotating black hole in a commutative space-time. It is given by [6]:

\[ ds^2 = -(1 - \frac{2M}{r} + \frac{Q^2}{r^2}) c^2 dt^2 + \frac{dr^2}{(1 - \frac{2M}{r} + \frac{Q^2}{r^2})} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]  

(38)

with \( M \) and \( Q \) are mass and charge. Following ref[3], the Seiberg Witten vierbein \( \tilde{\omega}_\mu \) in a noncommutative gauge gravity is given by:

\[ \tilde{\omega}_\mu = \omega_\mu(x) - i\tilde{\eta}^{\mu \rho} \tilde{e}_\rho(x) + \tilde{\eta}^{\mu \rho} \tilde{\eta}^{\lambda \tau} \tilde{e}_\rho(x) \tau \mu \lambda \tau + O(\tilde{\eta}^3) \]  

(39)

where

\[ \tilde{e}_\rho = e_\rho(x) + i\tilde{\eta}^{\mu \rho} e_\mu(x) + \tilde{\eta}^{\mu \rho} \tilde{\eta}^{\lambda \tau} e_\mu(x) \tau \mu \lambda \tau + O(\tilde{\eta}^3) \]  

(40)

and

\[ e_\mu(x) = 1 \frac{[\omega^{ac}_\mu \partial_\mu e^d + (\partial_\mu \omega^{ac}_\mu + R^{ac}_{\mu \nu}) \epsilon^d_{\nu \rho}]}{\eta_{\rho \sigma} \eta_{\eta \tau}} \]  

(41)

where \( \tilde{\eta}^{\mu \nu} \) is noncommutativity anti-symmetric matrix elements defined as:

\[ [\tilde{x}^\mu, \tilde{x}^\nu] = i\tilde{\eta}^{\mu \nu} \]  

(42)

and \( \tilde{x}^\mu \) are the noncommutative space-time coordinates operators. Here \( \omega^{ab}_\mu \) (resp. \( D_\mu \)) is the commutative spin connection (resp. covariant derivative) and \( R^{ab}_\mu \) is the Riemann tensor. The noncommutative space-time vierbein and Minkowski metric are denoted by \( e_\mu \) and \( \eta_{\mu \nu} \) respectively. The noncommutative metric:

\[ \tilde{g}_{\mu \nu} = \frac{1}{2} (\tilde{\omega}_\mu \star \tilde{e}_{\nu} + \tilde{e}_\mu \star \tilde{\omega}_{\nu}) \]  

(43)

where "\( \star \)" is the Moyal star product [8]. Straightforward calculations using the Maple 13 and setting \( z = \frac{r}{c} \) and \( y = \frac{Q^2}{c^2} \), \( \tilde{\eta}^2 = \lambda \), (in the case \( \theta = \frac{\pi}{2} \)) one has:

\[ F = -(1 - \frac{1}{2} \frac{y}{z^2}) - \frac{(2z^3 - 9y^2z - \frac{11}{2} z^2 + 15yz - 14y^2)\lambda}{4z^6} \]  

(44)

\[ G = \frac{1}{(1 - \frac{1}{2} \frac{y}{z^2})} + \frac{(-z^3 + \frac{3}{2} z^2 + 3yz^2 - 2yz^2 + 2y^2)\lambda}{4z^2(z^2 - z + y)^2} \]  

\[ H = \frac{(z^2 + \frac{17}{2} z^3 + \frac{17}{2} z^2 + 27yz^2 - \frac{75}{2} z y + 30y^2)\lambda}{16z^2(z^2 - z + y)} \]  

\[ I = \frac{(2z^3 + 8y^2z^2 + z^2 - 8yz + y^2)\lambda}{16(z^2 - z + y)} \]  

Figure 7 displays the variation of the entanglement entropy \( E(\rho) \) as a function of the noncommutativity parameter \( \tilde{\eta}^2 \) for a non charged \( (Q = 0) \) black hole for fixed \( z = 1.5, y = 0, \alpha = 1, q = 0.01 \). Notice that if \( \tilde{\eta}^2 \) increases \( E(\rho) \) decreases. Thus, \( \tilde{\eta}^2 \) plays the role of a gravitational field (GF). In fact, as it was pointed out in ref[13], the non commutativity parameter \( \tilde{\eta} \) can be considered as like a magnetic field contributing to the matter density \( \rho \) and therefore affecting the curvature of the space-time through its contribution to GF. Consequently if \( \tilde{\eta}^2 \) increases the GF increases and the information decreases.
Including the contribution of NC of the space-time will generate an additional terms proportional to $\tilde{\eta}^2$. In fact the gravitational potential $\tilde{g}_{00}$ will be of the form:

$$\tilde{g}_{00} = \hat{A} + \hat{B}Q^2 + \tilde{\eta}^2(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F})$$

where

$$\hat{A} = -1 + \frac{1}{z}, \quad \hat{B} = -\frac{1}{z} r_s^2, \quad \hat{D} = \frac{7}{2z}, \quad \hat{C} = (9z - 15) \frac{1}{4z^5 r_s^4}, \quad \hat{F} = (-2z + \frac{11}{4}) \frac{1}{4z^4 r_s^4}$$

The behavior of the entanglement entropy $E(\rho)$ depends strongly on the sign of $(\hat{D}Q^4 + \hat{C}Q^2 + \hat{F})$ having $\hat{A}$ and $\hat{B}$ negative:

1) If $Q^2 \gg 1$ (in arbitrary unit) the term $\hat{D}Q^4$ dominates, since $\hat{D} \succ 0$, then if $\tilde{\eta}^2$ increases the GF decreases leading to an increases in $E(\rho)$ (as it is the case of Figure 7).

2) If $Q^2 \ll 1$, then the term $\hat{F}$ dominates and its sign will determine the behavior of $E(\rho)$ as a function of $\tilde{\eta}^2$. If $\hat{F} \succ 0$, then GF increases and $E(\rho)$ decreases, we return to case in the Figure 7. Figure 9 represents the variation of $E(\rho)$ as function of $z$ for fixed $Q = 0, \tilde{\eta} = 0, \alpha = 1, q = 0.01$, (case of Schwarzschild space-time in commutative space-time). Notice that we will reproduce the same behavior as in ref[11]. Figure 10 shows the variation of $E(\rho)$ as a function of $z$ for fixed $Q \neq 0, \tilde{\eta} = 0$ (case of Reissner Nordstrom in commutative space-time). Notice that the same behavior as in ref[5] was obtained. Figure 11 shows the variation of $E(\rho)$ as a function of $z$ and fixed $\lambda = 0.01, y = 0, \alpha = 1, q = 0.01$, this case is Schwarchild black hole in noncommutative space-time. Figure 12 represents the variation of $E(\rho)$ as a function of $z$ for fixed $\lambda = 0.1, y = 2, \alpha = 1, q = 0.01$, case of Reissner Nordstrom Black Hole in

Figure 7: $E(\rho)$ as a function of the $\lambda$ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$.

Figure 8: $E(\rho)$ as a function of $\lambda$ for fixed $z = 4, y = 0.6, \alpha = 1, q = 0.01$. 

Figure 9: $E(\rho)$ as a function of $\lambda$ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$. 

Figure 10: $E(\rho)$ as a function of $\lambda$ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$. 

Figure 11: $E(\rho)$ as a function of $\lambda$ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$. 

Figure 12: $E(\rho)$ as a function of $\lambda$ for fixed $z = 1.5, y = 0, \alpha = 1, q = 0.01$. 

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noncommutative space-time Figure 12 shows the variation of the quantum entanglement entropy (EE) $E(\varrho)$ as a function of $z$ (or $r$). Notice that far from the oscillatory behavior region, when $z$ (or $r$) increases, the GF $\hat{g}_{00}$ decreases until reaching a saturation value ($\approx 1$) where $E(\varrho)$ is maximal, the oscillatory behavior disappears when we enter the stability region where $E(\varrho) \sim 0.67$. The explanation of the oscillatory behavior is the same as for Figure 2. The number of picks and minima depend strongly on the values of the various parameters $\lambda, y, \alpha$ and $q$. Concerning the noncommutativity effect on the EE, it is clear that from eq(45) that for smaller values of $z$, as $\hat{\eta}$ increases the gravitational field $\hat{g}_{00}$ becomes more important (increases) and therefore $E(\varrho)$ decreases. For larger values of $z$, the effect is almost negligible since the terms $\sim \frac{1}{z^2}, \frac{1}{z^3}, \frac{1}{z^4}$ decreases faster than the commutative terms $\sim \frac{1}{z}$. Notice also that $y$ increases the GF increases (the term $\hat{\eta}^2 \hat{D}Q^4$ dominates at larger value of $Q$). Thus, the NC
effect on the EE becomes more imporatant for charged black hole than the neutral ones ( if the charge $Q$ increases EE decreases ). Table 1 Sumarizes the effect og the black hole charge on the EE. It worth to mention that in order to keep the perturbative expanion with respect to $\eta^2$ reliable, one has to have

$$|\eta^2 A_1| \prec |A_0|$$

where $A_0 = \hat{A} + \hat{B}Q^2$ and $A_1 = (\hat{D}Q^4 + \hat{C}Q^2 + \hat{F})$, this implies new constraints on the space parameters $\lambda, z, y, \alpha$

| $z$ | 2   | 4       | 5 | 6     |
|-----|-----|---------|---|------|
| $E(\rho)$, ($y = 0$) | 0.64694 | 0.664   | 0.6644 | 0.6646 |
| $E(\rho)$, ($y = 10$) | 0.6072 | 0.6567  | 0.6626 | 0.6641 |

Table 1: Illustrative values of EE as a function of $z$ for $y=0$ and $y=10$

5 Conclusion

Throughout this paper, we have studied in detail the singlet state of spin entanglement of two particles systems quantified by wootters concurrence and EE in a general static space-time. And applications, we have considered the Kerr and non commutative Reissner-Nordstrom space-time. In fact, in the first case we have studied the variation of the Wootters concurrence (WC) as a function of the various parameters such as q center of mass momentum of the wave packet, $\Sigma$ ( black hole rotation parameter), $z$ (distance from the black hole). It turns out that the behavior of the WC depends strongly on those parameters(see figures 1,2,3,4,5 and 6). Regarding the second case (see figures 7,8,9,10,11,12) the variation of the quantum EE as a function of $z$, the NC parameter $\lambda$, the black hole charge $y$ is discussed. We have noticed that the NC effect on the EE becomes more important in a charged black hole( more studies of triplet states and spin-momentum entanglement are under investigation)

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