Unified Representation of Geometric Primitives for Graph-SLAM Optimization Using Decomposed Quadrics

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Abstract—In Simultaneous Localization And Mapping (SLAM) problems, high-level landmarks have the potential to build compact and informative maps compared to traditional point-based landmarks. In this work, we focus on the parameterization of frequently used geometric primitives including points, lines, planes, ellipsoids, cylinders, and cones. We first present a unified representation based on quadrics, leading to a consistent and concise formulation. Then we further study a decomposed model of quadrics that discloses the symmetric and degenerated properties of a primitive. Based on the decomposition, we develop geometrically meaningful quadrics factors in the settings of a graph-SLAM problem. Then in simulation experiments, it is shown that the decomposed formulation has better efficiency and robustness to observation noises than baseline parameterizations. Finally, in real-world experiments, the proposed back-end framework is demonstrated to be capable of building compact and regularized maps.

I. INTRODUCTION

Geometric primitives such as points, lines, and planes have been widely used in SLAM to represent the 3D environment thanks to their simplicity. Many state-of-the-art graph-SLAM systems utilize one or a combination of those primitives to formulate the back-end optimization, estimating the states of the robot and landmarks simultaneously. Despite the simplicity, however, those primitives have limitations in representing more complex shapes in the environment, e.g. curved surfaces.

Recently, high-level landmarks embedded with semantic labels have been shown to significantly improve the performance of SLAM, localization, and place recognition [1] [2]. To include semantic information into the optimization framework of graph-SLAM, abstract shapes, such as cuboids [3] or ellipsoids [4], have been used to represent the geometry of objects. However, those shapes mainly capture the scene layout rather than the geometric details, resulting in less accurate metric representation. In fact, how to represent high-level geometric information in SLAM optimizations still remains an open problem [5].

In this work, we propose to use quadrics as a unified representation of geometric primitives. Quadrics, as a general algebraic representation of second-order surfaces, are able to represent 17 types of shapes [6] and have only been introduced to computer vision and SLAM very recently. We can roughly break down the ongoing research into two categories: Firstly, ellipsoid, as a special type of quadrics with a closed shape, is used to approximate the shape and pose of objects [4]. Secondly, the representation of low-level landmarks, namely points, lines and planes, can be unified using quadrics, leading to a compact formulation of graph-SLAM with heterogeneous landmarks [7].

Our work aligns with these two directions of research and extends the prior works in two aspects: Firstly, since quadrics have the power to represent various shapes, some of which are quite frequently seen in man-made environments (e.g. cylinders and cones), we can potentially include more types of primitives in SLAM and still keep a unified and concise formulation. Secondly, it is noticed that quadrics can be symmetric and degenerated, which could cause ambiguous estimation in SLAM. However, those properties are not readily available from quadrics representation. Therefore, we are particularly interested in finding out how the quadrics representation implicitly encodes the geometric properties, and hope the insights would lead us to a geometrically meaningful formulation of quadrics SLAM. Our main contribution can be summarized as:

• A unified representation of high-level geometric primitives using quadrics is proposed. A wider spectrum of shapes is included, while previous works only consider points, lines, planes, or ellipsoids for SLAM.
• A new decomposed representation of quadrics is proposed. The decomposed representation is geometrically

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Fig. 1: Top: A simple mock-up environment with cylinders and planes. Left: Map represented by compact high-level shapes. Right: Map represented by dense low-level points.
meaningful in that it explicitly models the degeneration and symmetry of quadrics.

- A novel decomposed quadrics factor is systematically formulated based on geometric error metrics.
- Experiments in simulation and the real world are conducted to show the proposed quadrics-based back-end framework is robust, efficient and lightweight.

The rest of this paper is structured as follows: Section II discusses the prior work on SLAM landmark representation. Section III covers the fundamentals of quadrics and Section IV details the formulation of quadrics factors. In Section V, experiments in simulation and real world are presented. Finally, conclusions are drawn in Section VI.

II. RELATED WORK

In this section, we review the low-level and high-level geometric landmark representations used in SLAM.

Low-level landmarks: Points are the most popular landmark representation in state-of-the-art SLAM systems [8] [9], providing a sparse feature-based or dense point cloud based representation of the environment. Differently, lines (edges) and planes are sometimes referred to as high-level landmarks and have been shown to improve the robustness and accuracy of SLAM [10]. Representation for lines include a point plus a direction [11], Plücker coordinates [12] and a pair of endpoints [13]. Planes are usually represented with a normal and a distance [14] as a non-minimal representation. Kaess [15] proposes to use unit quaternion as a minimal representation of planes and formulates the plane factors in a graph-SLAM problem. As another minimal representation, Geneva et al. [16] choose to use the closest point on a plane to the origin as the representation of planes. Although geometrically meaningful, each type of landmark requires a special implementation to be used in the factor graph framework.

To mitigate this issue, there are efforts to unify the representation of low-level landmarks. S"{P}map [17] is perhaps the earliest attempt to develop a generic framework for SLAM landmarks and showed how 2D line-segments representation can be unified. Closely related to our work, Nardi et al. [7] and Aloise et al. [18] introduce the concept of matchables as a unified representation of points, lines and planes in 3D. Differently, our representation extends to higher-order surfaces such as cylinders and cones and bridges the algebraic expression with the geometric meanings.

High-level landmarks: There is a vast literature on object-level or semantic SLAM, especially as deep learning is being used successfully for object detection. However, we realize a review of general semantic SLAM is beyond the scope of this work. Instead, we are more interested in the underlying geometry. Aligning with this line of research, Salas et al. [19] use pre-defined mesh models to represent detected objects which is difficult to generalize to unobserved objects. After that, more general shape representations are used. Yang et al. [3] fit cuboids as bounding boxes to describe objects. Papadakis et al. [20] extract predefined spheres while Nicholson et al. [4] propose to use ellipsoids to approximate size, position and orientation of objects. Tschopp et al. [5] demonstrate that supersquadrons have the advantage of physically meaningful parameterization. However, those methods assume bounded shapes, thus are not suitable to represent degenerated shapes such as a partially observed cylindrical structure. Different from those approaches, our work studies the degeneration behaviors of high-order shapes represented as quadrics.

III. QUADRICS BASICS

A. Quadrics Representation

Quadrics are defined implicitly by the zero contour of a two-degree algebraic function:

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fzx + 2Gxz + 2Hy + 2Iz + J = 0$$

(1)

There are 10 parameters but only 9 degrees of freedom due to the ambiguity of scale. The shape function (1) has a compact matrix form:

$$x^TQx = 0$$

(2)

where

$$Q = \begin{bmatrix} A & D & F & G \\ D & E & H & F \\ F & E & C & I \\ G & H & I & J \end{bmatrix}$$

Despite the 17 subtypes of quadrics, we consider four shapes, namely coincident planes, ellipsoids, elliptic cylinders and elliptic cones, that appear most frequently in man-made structured environments. Additionally, we also consider points and lines as degenerated ellipsoids and cylinders respectively.

B. Quadrics Composition

A given quadrics $Q$ contains three pieces of fundamental information: type (e.g. plane, cylinder etc.), size (e.g. radius of sphere and cylinders), and pose in 3D space, which can be encoded in three corresponding matrices.

1) Canonical Matrix $C$: The canonical form of a quadrics is obtained by aligning quadrics axes to the coordinate axes. In the canonical form, $Q$ is reduced to canonical matrix $C$. For quadrics discussed in this paper, $C$ is always a diagonal matrix, whose pattern uniquely determines the shape type. Table I summaries the canonical matrices of the considered quadrics in this paper.
2) Scale Matrix S: The canonical matrix C represents quadrics of unit length. For example, \( C = \text{diag}(1, 1, 1, -1) \) defines a unit sphere. To scale the unit quadrics, a diagonal scale matrix \( S \) is used. However, except ellipsoids, the other quadrics types in Table I are degenerated, meaning scaling in some directions won’t affect the geometric shape. For example, a plane can’t be scaled at all. Therefore, we use \( \mathbf{P} \in \{0, 1\}^3 \) to indicate the directions that can be scaled.

3) Transformation Matrix T: Let \( T \in SE(3) \) be the transform matrix between two frames. Then a given \( Q \) in one frame can be transformed to the other frame by:

\[
Q' = T^{-T}QT^{-1}
\]

4) Composition: Any quadrics \( Q \) can be constructed by the composition of the three matrices:

\[
Q = T^{-T}S^T CST^{-1}
\]

In preparation for the mathematical derivations later in this paper, we explicitly rewrite (4) as:

\[
Q = \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R}^T & -\mathbf{R}^T t \\ 0 & 1 \end{bmatrix} = \mathbf{R} \mathbf{D} \mathbf{R}^T + \mathbf{t} \mathbf{R} \mathbf{D} \mathbf{R}^T + d = \begin{bmatrix} \mathbf{E} & 1 \\ 0 & k \end{bmatrix}
\]

where \( \mathbf{D} \) and \( d \) are diagonal blocks of \( S^T \mathbf{C} S \). \( \mathbf{E}, \mathbf{l}, k \) are corresponding blocks of the resulting \( Q \).

C. Quadrics Decomposition

A given \( Q \) can be decomposed to disclose its geometric properties, which allows for an intuitive interpretation and eventually leads to a decomposed quadrics model.

1) Type Identification: In practice, we are more interested in identifying ellipsoids, cylinders and cones from a given \( Q \). Quadrics Shape Map (QSM) [21] can be used to determine the types of quadrics by analyzing the distribution of the eigenvalues of \( \mathbf{E} \). In simulation experiments, we assume the quadrics types are known, while in real-world experiments, quadrics types are determined using QSM.

2) Scale Identification: We first normalize the given \( Q \) to remove the scale ambiguity:

\[
Q = \begin{bmatrix} \prod \lambda^E_j \\ \prod \lambda^Q_i \end{bmatrix} \begin{bmatrix} \mathbf{Q} \end{bmatrix}
\]

where \( \lambda^E_j \) and \( \lambda^Q_i \) are nonzero eigenvalues of \( \mathbf{E} \) and \( \mathbf{Q} \) respectively. Specially, for cones, \( \mathbf{Q} \) is normalized by the negative eigenvalue of \( \mathbf{E} \). Then the scale parameters can be recovered by:

\[
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \sqrt{\begin{bmatrix} 1/\lambda^E_j \\ 1/\lambda^E_i \\ 1/\lambda^E_3 \end{bmatrix}}
\]

assuming \( a \leq b \leq c \). In degenerated cases, certain eigenvalues will be zeros. Then the scale along those directions becomes undefined as specified in \( \mathbf{P} \).

3) Pose Identification: Isolating pose information from the given \( Q \) is to find \( \mathbf{R} \) and \( t \) that represent the transform between the observation frame and the quadrics canonical frame, or local frame. According to (5), the rotation can be found from eigenvalue decomposition of \( \mathbf{E} = \mathbf{VDV}^T \), while recovering \( t \) involves solving \( \mathbf{Et} + 1 = 0 \). However, recovering \( \mathbf{R} \) and \( t \) needs to consider several special situations:

- \( \mathbf{V} \) is not necessarily a valid rotation matrix. The direction of eigenvector \( v \) can be identical or opposite to the column of \( \mathbf{R} \), due to the symmetry of quadrics.
- When \( \mathbf{E} \) has \text{nonzero eigenvalues} only, \( t \) can directly recovered as \( t = -\mathbf{E}^{-1} \)
- When \( \mathbf{E} \) has \text{zero eigenvalues}, \( t \) is only partially constrained.
- When \( \mathbf{E} \) has \text{two equal eigenvalues}, \( Q \) becomes revolution quadrics, where the rotation around the other axis becomes degenerated.

The above situations are caused by the degeneration and symmetry of quadrics. To systematically handle these issues, we model the pose of quadrics from the perspective of constraints, which will be further elaborated on in the next section. In Table II, we summarize all possible situations of degeneration and illustrate with examples.

IV. Quadrics in Factor Graphs

A. Pose-Quadrics Constraints

To constrain the rotation, we choose to align the columns of \( \mathbf{R} \) (noted as \( r_i \)) to corresponding non-degenerate eigenvectors \( v_i \). An rotation activation vector \( \mathbf{I}_R \in \{0, 1\}^3 \) is defined to mark the direction to be enforced (see Table II). Further more, to consistently handle the situations where \( v_i \) is opposite to \( r_i \), cross product is used to measure the unsigned direction alignment error:

\[
C(r_i) = v_i \times r_i = 0, \quad (\text{for } I^R = 1)
\]

As to translation, the constraint equation is:

\[
C(t) = \mathbf{Et} + 1 = \mathbf{VDV}^Tt + 1 = 0
\]
Similarly, translation degeneration indicator $I_i \in \{0,1\}^3$ can be defined and we can further decompose the equation and enforce the constraints explicitly:

$$C(t) = \lambda_i v_i^T t + v_i^T 1 = 0, \quad (\text{for } I_i = 1) \quad (10)$$

Equation (10) provides an geometric interpretation of translation constraints. One such equation defines a constraining plane with normal vector $v_i$ and distance $v_i^T 1/\lambda_i$. Therefore, $t$ is constrained to a point, line or plane due to the intersection of 3, 2 or 1 such constraining planes, respectively.

Finally, the scale constraints can be found by directly comparing to the eigenvalues:

$$C(s) = s_i^2 - \lambda_i = 0, \quad (\text{for } I_i^s = 1) \quad (11)$$

Equation (8) - (11) translate the observation of $Q$ into a set of constraints parameterized by the tuple $(I^R, I^t, I^s, V, D, I)$ where the geometric properties are explicitly represented.

### B. Error Function

Given the robot pose $(R_r, t_r)$ and the quadrics in the world frame $(R_q, t_q, s_q)$, the error function of observed quadrics in the robot body frame is defined as the residual vector of a constraint set:

$$e = \begin{pmatrix} e_R \\ e_t \\ e_s \end{pmatrix} = \begin{pmatrix} \text{diag}(I^R) (V \otimes \Delta R)^T \\ \text{diag}(I^t) (DV^T \Delta t + V^T 1) \\ \text{diag}(I^s) (s_i^2 - \Lambda) \end{pmatrix} \quad (12)$$

where $\otimes$ means column-wise cross product, $\Delta R = R_r^T R_q$ and $\Delta t = R_r^T (t_q - t_r)$ are the rotation and translation of quadrics pose transformed into the robot frame. $\Lambda = [\lambda_1, \lambda_2, \lambda_3]$ is the vector of eigenvalues stored in $D$. Here, $e_R$ is a $3 \times 3$ matrix and will be vectorized before being stacked into the error vector.

### C. Observation Uncertainty and Weighting

One direct benefit of using decomposed constraint representation is that it allows easy incorporation of uncertainties, or weights, to measure the strength of (8)-(11). We adopt a simple approach to compute the weight of a shape as $\tanh(N)$, where $N$ is the number of points. The adopted strategy reduces the weights of small shapes that tend to have higher uncertainty in fitted parameters.

### D. Solving the Factor Graph

Given a graph with quadrics, the cost function is constructed by accumulating the errors of each observation:

$$f = \sum e^T \Omega e \quad (13)$$

where $\Omega = \text{diag}(\Sigma_{t_q}^{-1}, \Sigma_{t_q}^{-1}, \Sigma_{s_q}^{-1})$ is the information matrix characterizing the weight of each component. In Algorithm 1, we report the basic steps of Levenberg–Marquardt (LM) method [22] for graph optimization. Sparsity is preserved by line 10 and 11, where only relative blocks of $b$ and $H$ are updated. Sparse Cholesky factorization is applied to solve line 13. We refer the readers to [8] [23] for more information about the sparse structure of factor graph and to Appendix I for the derivation of Jacobians for quadrics factors.

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**Algorithm 1** LM Algorithm for Quadrics Factor Graph

1. **Input:** Initial states $X_0 \in \mathbb{R}^{(9M+6N) \times 1}$ of $N$ poses, $M$ quadrics landmarks, and $K$ observations $\{Q_k\}$
2. **Output:** Updated states $X$
3. **Decomposition:** $Q_k \rightarrow (I^R_k, I^t_k, I^s_k, V_k, D_k, I_k)$
4. **Initialization:** $X \leftarrow X_0$
5. **while** not converged **do**
6. **for** each observation do
7. Pose Jacobian: $J_r = \left[ \frac{\partial e}{\partial e_R} \frac{\partial e}{\partial e_t} \frac{\partial e}{\partial e_s} \right]$  
8. Quadrics Jacobian: $J_q = \left[ \frac{\partial e}{\partial e_R} \frac{\partial e}{\partial e_t} \frac{\partial e}{\partial e_s} \right]$  
9. Evaluate observation error: $e = [e_R^T \quad e^T \quad e^T]$  
10. Update $b$: $b \leftarrow b + [\cdots J_q^T \Omega e \cdots J_q^T \Omega e \cdots ]$  
11. **Update** $H$: $H \leftarrow H + [\cdots J_q^T \Omega J_q \cdots J_q^T \Omega J_q \cdots ]$  
12. **end for**
13. Compute LM update: $\Delta = -(H + \lambda I)^{-1}b$  
14. **Apply** update: $X \leftarrow X + \Delta$  
15. **end while**
16. Return $X^* = X$

*$\lambda$ in line 13 is the LM damper updated in each iteration [24].

### E. Baseline Parameterizations

In this section, we discuss two baseline parameterizations as a comparison to the decomposed representation.

1) **Full Parameterization:** One could formulate the quadrics observation error using the full parameterization, namely the 10-D quadrics vector $q$:

$$q = [A \quad B \quad C \quad D \quad E \quad F \quad G \quad H \quad I \quad J]^T \quad (14)$$

Then the observation error is evaluated by first transforming the $q$ into robot body frame and then compute the difference with observation $\bar{q}$:

$$e = \bar{q} - (T_r^T (\bar{q})^\wedge T_r)^\vee \in \mathbb{R}^{10 \times 1} \quad (15)$$

where operator $(\cdot)^\wedge$ and $(\cdot)^\vee$ compute the quadrics vector and matrix respectively. About the full representation:

- The observation model has a simpler expression and easy to implement;
- The metric is algebra error instead of geometric error, which could introduce bias to estimation [25].
- It is difficult to interpret the uncertainties of $q$.

2) **Regularized Full Parameterization:** Inspired by [4], we implement another baseline where the structure of quadrics are explicitly modeled:

$$e = \bar{q} - (T_r^T QT_r)^\vee \in \mathbb{R}^{10 \times 1} \quad (16)$$

In here, $Q$ is constructed as in (4) from the quadrics states $(R_q, t_q, s_q)$. Equation (16) explicitly models rotation, translation and scale of quadrics, but still computes the algebra error. Compared to (15), the type of quadrics is
TABLE III: Perturbation Configurations

| Initialization Noise $\sigma_{x_0}$ | Observation Noise $\sigma_q$ |
|-------------------------------------|-------------------------------|
| Low (L) $(1^\circ, 0.1, 1^\circ, 0.1, 0.01)$ | $(1^\circ, 0.1, 0.01)$ |
| Medium (M) $(5^\circ, 0.5, 5^\circ, 0.5, 0.02)$ | $(2^\circ, 0.2, 0.02)$ |
| High (H) $(50^\circ, 5.0, 50^\circ, 5.0, 0.05)$ | $(5^\circ, 0.5, 0.05)$ |

Fig. 2: Convergence plot using decomposed and full quadrics factors at increasing initialization and observation error. Each configuration is repeated 10 times. Color codes: decomposed (blue), full (red), regularized-full parameterization (orange) now treated as prior knowledge and therefore the estimation is regularized. From now on, we use decomposed (D), full (F) and regularized-full (RF) parameterization to denote the proposed, baseline 1 and baseline 2 respectively.

V. EXPERIMENTS

A. Simulation

1) Synthetic Environment: The synthetic environment, as shown in Fig. 3, is a manhattan-like world that contains 15 quadrics landmarks of different types. The quadrics are randomly generated in a bounded space (6m × 6m × 1m). The simulated robot trajectory is shown as the red curve which contains 50 frames whose x-axis points to the origin. For each frame, the robot will sense the surrounding environment and the nearest $K = 10$ quadrics are observed.

2) Noise Simulation: There are 2 types of noise to be simulated. Firstly, the robot poses $\{R_r, t_r\}$ and quadrics parameters $\{R_q, t_q, s_q\}$ are perturbed according to Gaussian noise $\sigma_{x_0} = (\sigma_{\theta_r}, \sigma_{t_r}, \sigma_{\theta_q}, \sigma_{t_q}, \sigma_{s_q})$. This generates the initial guess for factor optimization. Secondly, each quadrics observation is perturbed in terms of rotation, translation and scale, according to Gaussian noise $\sigma_q = (\sigma_{\theta_q}, \sigma_{t_q}, \sigma_{s_q})$. This gives a set of noisy observations $\{Q\}$. Table III defines 3 levels of noise: low (L), medium (M) and high (H) which will be used to test the behaviors of different parameterizations.

3) Solving Factor Graph: To directly observe the behavior of the proposed quadrics factor, we choose to construct the factor graph only containing pose-quadrics factors and a prior factor of the first robot pose.

The convergence behavior under various noise levels using different parameterizations is reported in Fig. 2. The upper plot shows the error-iteration curves of increasing observation noise. In this test, the initialization noise is at a low level. In the lower plot, we report the convergence behavior under increasing initialization noise. In this test, the observation noise is kept at a low level. We observe that the decomposed representation has a faster convergence rate, especially at a high noise level. Besides, the curves of decomposed parameterization also tend to have fewer variations, which indicates the cost function using geometric error has better convexity.

We then qualitatively evaluate the converging basin for different parameterizations. In Fig. 3, we compare the optimized robot poses and quadrics to the ground truth under high initialization noise. It is observed that the optimized quadrics and robot poses stay closer to the ground truth when using decomposed parameterization, indicating a wider converging basin. Additionally, the optimized quadrics with full parameterization will change the type to compensate for noises, while shapes using the other two parameterizations are well regularized.

For the above 6 noise configurations, we also compare the final optimized states to the ground truth. For trajectories, we compute the absolute trajectory error (ATE). For quadrics, we directly compare the quadrics vector. In the case of decomposed representation, the quadrics vector is reconstructed using (4). Table IV shows the trajectory and quadrics errors in 6 noise configurations. Note that the errors are averaged across 10 tests sharing the same noise configurations. It is observed that decomposed parameterization consistently has smaller translation and quadrics errors. Although the regularized parameterization performs better in most cases in terms of rotation, the difference is small.

4) More Discussions: Through the experiments, we also found that the estimation accuracy of full and regularized-full parameterization is quite sensitive to the translation and scale perturbation of observation $Q$. Even a small perturbation would cause the final result to converge to a local minimum (see large errors of F and RF in Table IV). This can be explained by their correlation in $Q$. From (3), we can see that $t$ and $D$ are multiplied in $-D^T R (R - t) = 1$. In the case of small quadrics, small size noise will result in dramatic changes in the values of $D$ due to the inverse relationship. Then during optimization, $t$ tends to compensate for the amplified effects of scale noise thus leading to inaccurate estimation.
Fig. 3: Optimization results using decomposed (left), full (middle), and regularized-full (right) parameterizations. Ground truth trajectories and quadrics are visualized as solid curves and meshes respectively. Optimized poses and quadrics are drawn as frames and transparent surfaces respectively. Types of simulated shapes include points, lines, planes, ellipsoids, spheres, cylinders, and cones.

Fig. 4: Qualitative comparison of point cloud map before (left) and after (right) quadrics graph optimization.

B. Raw Data

To validate the proposed method using raw data, we use an Ouster OS1 LiDAR to map a room with cylinders and planes (see Fig. 1). As this work is focused on the backend, a simple front-end on top of a LiDAR odometry [26] is implemented. Firstly, shapes are extracted from selected laser scans (one scan per second) using the RANSAC method proposed in [27]. Then quadrics are associated incrementally by computing the Taubin distance [28] of shape points to existing quadrics in the map. If the averaged distance is smaller than a threshold, then two quadrics are matched. Otherwise, a new quadric is created and added to the map. Once all scans are processed, we obtain a list of quadrics each of which has a list of frame views. Quadrics are further pruned to only keep those with more than 6 views. Finally, a graph consist of robot poses and quadrics is obtained.

In the back-end stage, the graph is optimized using the LM algorithm presented in Algorithm 1. Mapping results are compared qualitatively with the LiDAR odometry and reported in Fig. 4. From the shown point clouds, we can see that the proposed quadrics-based back-end can generate better-aligned point clouds without any post-processing, meaning the robot trajectory is optimized. Additionally, the quadrics estimation is regularized and refined as well. For instance, in the zoomed-in views, the central axis (shown as the blue z-axis) of cylinders lies closer to the shape center in optimized maps, while the initial estimation is slightly off due to inaccurate shape fitting. Finally, although the visualization is using point clouds, the optimization only involves 9 quadrics (shown in Fig. 1 plus a hidden ceiling plane), making the framework lightweight.

It is worth mentioning that in this experiment, the number of scanned points on cylinders is much smaller than those on planes, limiting the contribution of cylinder observations to the pose optimization. However, those observations help to recover more accurate shapes, as shown in the zoomed-in views of Fig. 4.

VI. Conclusions and Future Work

In this paper, we unify the geometric primitive representation using quadrics, which generalizes to a wide spectrum of shapes. Additionally, we provide a decomposed representation of quadrics that explicitly discloses the geometric properties of shapes such as degeneration and symmetry. Then based on the decomposition, we show that the observation of quadrics can be translated into constraints to robot poses, and thus the formulation of quadrics factors in graph-SLAM is developed. In simulation experiments, we show that the decomposed quadrics factors utilize shape priors and optimize a geometric error, which makes it more stable and efficient than the baseline formulations. Finally, in a simple real-world environment, we demonstrate the map is more compact and regularized using the quadrics-based back-end framework.

Several unsolved questions could potentially be the directions of future work. Firstly, to make use of quadrics in a practical SLAM pipeline, the front-end still remains challenging. Instead of a simplistic front-end for the proof-of-concept, a practical one would need to solve quadrics extraction fastly and accurately. Secondly, it is not clear how to estimate the covariance matrix of quadrics fitting from partially observed data in a principled way that models the anisotropic nature of uncertainty. Finally, since high-level shapes have been shown to significantly reduce the number of landmarks in the map while still capture the overall layout, detecting loop-closure in lightweight maps would be another interesting direction to explore.

VII. Acknowledgement

The authors acknowledge the sponsorship of this work from the Shimizu Institute of Technology (Tokyo).
Following the notation convention of the paper, the quadric states in the world frame is represented by \( (R_q, t_q, s_q) \), and the robot pose is \( (R_r, t_r) \). Therefore, the state vector involved in a single observation is \( s = [R_r, t_r, R_q, t_q, s_q] \).

To simplify the presentation, we derive the Jacobian matrix based on a single observation, while the complete Jacobian can be constructed by filling in per observation Jacobians. As presented in the paper, the observation error is given by

\[
\mathbf{e} = \begin{pmatrix}
\mathbf{e}_R \\
\mathbf{e}_t \\
\mathbf{e}_s
\end{pmatrix} = 
\begin{pmatrix}
\text{diag}(\mathbf{I}^R) (\mathbf{V} \otimes \Delta R)^T \\
\text{diag}(\mathbf{I}^R) (\mathbf{D}^T \Delta t + \mathbf{V}^T \mathbf{1}) \\
\text{diag}(\mathbf{I}^s) ((\mathbf{s}^2 - \Lambda)^{-1})
\end{pmatrix} \in \mathbb{R}^{15}
\]

where \( \otimes \) means column-wise cross product. \( \Delta R = R^T_r R_q \) and \( \Delta t = R^T_r (t_q - t_r) \) are the rotation and translation of quadrics pose transformed into the robot frame. \( \Lambda = [\lambda_1, \lambda_2, \lambda_3] \) is the vector of eigenvalues which are stored as the diagonal elements of \( \mathbf{D} \). Here, \( \mathbf{e}_R \) is a \( 3 \times 3 \) matrix and will be vectorized and then stacked into the error vector.

Then we have the derivative \( \frac{\partial \mathbf{e}}{\partial \mathbf{x}} \in \mathbb{R}^{15 \times 15} \) as

\[
\frac{\partial \mathbf{e}}{\partial \mathbf{x}} =
\begin{pmatrix}
\frac{\partial \mathbf{e}_R}{\partial R_r} & \frac{\partial \mathbf{e}_R}{\partial t_r} & \frac{\partial \mathbf{e}_R}{\partial q} & \frac{\partial \mathbf{e}_R}{\partial t_q} & \frac{\partial \mathbf{e}_R}{\partial q} & \frac{\partial \mathbf{e}_R}{\partial t_q}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mathbf{e}_t}{\partial R_r} & \frac{\partial \mathbf{e}_t}{\partial t_r} & \frac{\partial \mathbf{e}_t}{\partial q} & \frac{\partial \mathbf{e}_t}{\partial t_q} & \frac{\partial \mathbf{e}_t}{\partial q} & \frac{\partial \mathbf{e}_t}{\partial t_q}
\end{pmatrix}
\begin{pmatrix}
\frac{\partial \mathbf{e}_s}{\partial R_r} & \frac{\partial \mathbf{e}_s}{\partial t_r} & \frac{\partial \mathbf{e}_s}{\partial q} & \frac{\partial \mathbf{e}_s}{\partial t_q} & \frac{\partial \mathbf{e}_s}{\partial q} & \frac{\partial \mathbf{e}_s}{\partial t_q}
\end{pmatrix}
\end{pmatrix}
\]

Note that the first dimension size 15 is the number of constraints or the error terms. The above Jacobian can be simplified by identifying zero blocks:

\[
\frac{\partial \mathbf{e}}{\partial \mathbf{x}} =
\begin{pmatrix}
0 & 0 & \frac{\partial \mathbf{e}_R}{\partial q} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{\partial \mathbf{e}_R}{\partial q} & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \frac{\partial \mathbf{e}_R}{\partial q} & 0 & 0 & 0
\end{pmatrix}
\]

Now we rewrite error terms explicitly to prepare for the

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to linearize the error terms w.r.t. rotation $R$

$\text{commutative rule of cross product and}$

$\begin{bmatrix}
\lambda_i u_i R_q^T(t_q - t) \\
+ u_i^T R_q^T R_q l
\end{bmatrix}$

$\text{approximation:}$

$\bar{e}_R = (v_i \times R_q^T R_q u_i) \in \mathbb{R}^{9 \times 1}$

$\text{and}$

$\begin{bmatrix}
\lambda_i u_i R_q^T(t_q - t) \\
+ u_i^T R_q^T R_q l
\end{bmatrix}$

$\bar{e}_q = (v_i \times R_q^T R_q u_i) \in \mathbb{R}^{3 \times 1}$

$\text{where } u_i \text{ are unit vectors:}$

$u_1 = (1, 0, 0)^T, u_2 = (0, 1, 0)^T, u_3 = (0, 0, 1)^T$

$\text{APPENDIX II}$

$\text{LINEARIZATION}$

Computing Jacobian involving $R_r$ and $R_q$ requires linearization which can be achieved by applying the small angle approximation:

$R_r = R_r \delta R_r, \quad \delta R_r \approx I + [w_r]_\times$

and

$R_q = R_q \delta R_q, \quad \delta R_q \approx I + [w_q]_\times$

where $[\cdot]_\times$ is the skew-symmetric operator:

$[w]_\times = 
\begin{bmatrix}
0 & -w_3 & w_2 \\
-w_3 & 0 & w_1 \\
w_2 & -w_1 & 0
\end{bmatrix}$

and $I$ is the identity matrix. Now we apply the anti-commutative rule of cross product

$a \times b = [a]_\times b = -b \times a = [-b]_\times a$  \hspace{1cm} (a, b \in \mathbb{R}^3)

$\text{to linearize the error terms w.r.t. rotation } R_r \text{ and } R_q:$

$\bar{e}_R|_{R_r} = (v_i \times (I + [w_r]_\times)^T R_q^T R_q u_i)$

$\text{and}$

$\bar{e}_q|_{R_q} = (v_i \times (I + [w_q]_\times)^T R_q^T R_q u_i)$

$\text{In the linearized cost function } \bar{e}_q|_{R_q}, \text{ constant terms not related to } x \text{ are omitted. The error terms are linearized in that now they are linear w.r.t. } w_r \text{ and } w_q. \text{ Finally, from the above linearized equations, we can have the Jacobian blocks:}$

$\frac{\partial e_R}{\partial R_r} = (v_i \times [R_q^T R_q u_i]_\times)$

$\frac{\partial e_q}{\partial R_r} = -u_i^T [R_q^T R_q l]_\times$

$\frac{\partial e_R}{\partial R_q} = (v_i \times [R_q^T R_q u_i]_\times)$

$\frac{\partial e_q}{\partial R_q} = -u_i^T [R_q^T R_q l]_\times$

As to Jacobian w.r.t. translation and scale, it is straight
forward:
\[
\frac{\partial e_t}{\partial t_r} = \begin{pmatrix}
\cdots \\
-\lambda_i u_i^T R_q^T \\
\cdots
\end{pmatrix}
\]
\[
\frac{\partial e_t}{\partial t_q} = \begin{pmatrix}
\cdots \\
\lambda_i u_i^T R_q^T \\
\cdots
\end{pmatrix}
\]
\[
\frac{\partial e_s}{\partial s_q} = \begin{pmatrix}
\cdots \\
2s_i \\
\cdots
\end{pmatrix}
\] (31)

The computed Jacobian blocks can then be filled into (19) and finally used to construct the complete Jacobian matrix used in the LM method (see Algorithm 1).