Specifying Prior Distributions in Reliability Applications

Qinglong Tian†  Colin Lewis-Beck‡  Jarad B. Niemi∗  William Q. Meeker∗

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Abstract

Especially when facing reliability data with limited information (e.g., a small number of failures), there are strong motivations for using Bayesian inference methods. These include the option to use information from physics-of-failure or previous experience with a failure mode in a particular material to specify an informative prior distribution. Another advantage is the ability to make statistical inferences without having to rely on specious (when the number of failures is small) asymptotic theory needed to justify non-Bayesian methods. Users of non-Bayesian methods are faced with multiple methods of constructing uncertainty intervals (Wald, likelihood, and various bootstrap methods) that can give substantially different answers when there is little information in the data. For Bayesian inference, there is only one method—but it is necessary to provide a prior distribution to fully specify the model. Much work has been done to find default or objective prior distributions that will provide inference methods with good (and in some cases exact) frequentist coverage properties. This paper reviews some of this work and provides, evaluates, and illustrates principled extensions and adaptations of these methods to the practical realities of reliability data (e.g., non-trivial censoring).

Keywords: Bayesian inference, Default prior, Few failures, Fisher information matrix, Jeffreys prior, Noninformative prior, Reference prior

1 Background and Motivating Examples

1.1 Bayesian Methods in Reliability Applications

The use of probability plotting and maximum likelihood (ML) methods for the analysis of censored reliability data has matured over the past 30 years. These methods appear in numerous textbooks and are readily available in several widely-used commercial statistical software packages (e.g., JMP, MINITAB, and SAS). More recently, commercial statistical software that provides capabilities for doing Bayesian estimation has become available (e.g., JMP and SAS). Bayesian estimation requires the specification of a joint prior distribution for the model parameters. The purpose of this paper is to provide guidance on how prior specification should be done in reliability applications. We focus on

†Department of Statistics, Iowa State University
‡Department of Biostatistics and Medical Informatics, University of Wisconsin-Madison
§Eli Lilly and Company
applications requiring a single distribution (e.g., Weibull or lognormal). The basic ideas, however, can be applied to more complicated models, as described in our concluding remarks section.

1.2 Motivating Examples

Bearing cage field data

Figure 1(a) is an event plot of bearing-cage fracture times for six failed units as well as running times for 1,697 units that had accumulated various amounts of service time without failing. The data and an analysis appear in Abernethy et al. (1983). These data represent a population of units that had been introduced into service over time and the data are multiply censored (censoring at multiple points in time). There were concerns about the adequacy of the bearing-cage design. Analysts wanted to use these initial data to decide if a redesign would be needed to meet the design-life specification. Figure 1(b) is a Weibull probability plot. This requirement was that $t_{0.1}$ (sometimes referred to as B10) be at least 8,000 hours. Because of the small number of failures, the confidence interval for the fraction failing at 8,000 hours is wide and deciding whether the reliability goal is being met is difficult. A likelihood ratio confidence interval for the fraction failing at 8,000 hours is $[0.026, 0.9999]$, which is not useful.

Rocket motor field data

This example was first presented in Olwell and Sorell (2001). The US Navy had an inventory of approximately 20,000 missiles. Each included a rocket motor—one of five critical components. These missiles were subject, over time, to thermal cycling. Only 1,940 of the missiles had actually been put into use over a period of time up to 18 years subsequent to their manufacture. At their time of use, 1,937 of these motors performed satisfactorily; but there were three catastrophic launch failures. Responsible scientists and engineers believed that these failures were caused by the thermal cycling. In particular, it was believed that the thermal cycling resulted in failed bonds between the solid propellant and the missile casing.
The failures raised concern about the previously unanticipated possibility of a sharply increasing failure rate over time (i.e., rapid wearout) as the motors aged and were subjected to thermal cycling while in storage. If this were indeed the case, a possible—but costly—remedial strategy might be to replace aged rocket motors with new ones. Thus, to assess the magnitude of the problem, it was desired to quantify the rocket-motor failure probability as a function of the amount of thermal cycling to which a motor was exposed and to obtain appropriate confidence bounds around such estimates based on the results for the 1,940 rocket motors—assuming these to be a random sample from the larger population (at least concerning their failure-time distribution).

Because no information was directly available on the thermal cycling history of the individual motors, the age of the motor (i.e., time since manufacture) at launch was used as a surrogate. This was not an ideal replacement because the thermal cycling rate, or rate of accumulation of other damage mechanisms, varied across the population of motors, depending on an individual missile’s age and environmental storage history. Compared to a scale based on the number of thermal cycles, the effect of using time since manufacture is to increase the variability in the observed failure-time response, as described in Meeker et al. (2009). The failure probability 20 years after manufacture was of particular interest.

The specific age at failure of each of the three failed motors was not known—all that was known was that failure, in each case, had occurred sometime before the time of launch—thus making the time since manufacture at launch left-censored observations of the actual failure times. Similarly, the information of (eventual) failure age for the 1,937 successful motors is right-censored—all that is known is that the time to the yet-to-occur failure exceeds the calendar age at the time of launch. Thus, the available rocket-motor field-performance data, contained only left- and right-censored observations—but no known exact failure times (such data are known as “current-status data”). Figure 2(a) is an event plot that further illustrates the structure of the data.

Because failure times are only bounded (no exact failure times) and because of the very small number of known failures, the amount of information in the data is severely limited. Nevertheless, it is possible to estimate the Weibull parameters from these data. Figure 2(b) is a Weibull probability plot of the data. The ML estimates of the Weibull parameters are $\hat{\eta} = 21.23$ years and $\hat{\beta} = 8.126$. A likelihood ratio confidence interval for the fraction failing at 20 years is $[0.023, 0.9999]$, which, again, is not useful.

For most failure mechanisms operating in the field, the Weibull shape parameter $\beta$ will be less than 4. Using the surrogate years since manufacture in place of the unknown number of thermal cycles will further increase the relative variability in the data which would make $\beta$ even smaller. Thus the estimate $\hat{\beta} = 8.126$ was surprisingly large.

1.3 Literature Review

Books that focus on Bayesian methods for reliability data analysis include Martz and Waller (1982), Hamada et al. (2008), and Liu and Abeyratne (2019). Sander and Badoux (2012) contains six con-
tributions that describe early work on the application of Bayesian methods in different reliability applications. Numerous papers describing particular applications of Bayesian methods have appeared in the engineering and statistical literature over the past 30 years.

Gutiérrez-Pulido et al. (2005) suggest specifying fully informative prior distributions for a two-parameter distribution by specifying intervals for the mean and standard deviation or two quantiles for the failure-time distribution. Kaminskiy and Krivtsov (2005) suggest, for a Weibull distribution, specifying fully informative joint prior distributions for the parameters by specifying priors for two points on the cdf of the Weibull distribution. Krivtsov and Frankstein (2017) extend the method to other failure-time distributions. Section 5.2 reviews some of the literature on noninformative prior distributions.

1.4 Overview

The remainder of this paper is organized as follows. Section 2 provides a brief review and defines notation for reliability models, censoring, and likelihood. Section 3 briefly introduces the basic concepts of Bayesian inference and explains motivation for and mechanics of needed reparameterization. The commonly used noninformative prior distributions are derived from the Fisher information matrix (FIM). Section 4 describes how to obtain the elements of the FIM for different kinds of censoring. Section 5 reviews the commonly used noninformative prior distributions and describes extensions to Type 1 and Type 2 censoring. Section 6 explains how the results in Section 5 can be applied in situations where there is random censoring. Section 7 explains the importance of weakly informative prior distributions for some applications and illustrates the use of noninformative and weakly informative priors in the examples. In most reliability applications, engineers will have strong prior information on only one parameter (e.g., the Weibull shape parameter). Section 8 shows how to combine prior information for one parameter with a noninformative or weakly informative prior for the other parameter and applies the ideas to the examples. Section 9 describes a simulation that was conducted to study the coverage probabilities of credible intervals computed under different noninformative priors. Section 10 suggests
and illustrates methods of doing prior distribution sensitivity analysis. Section 11 gives concluding remarks and suggests extensions and areas for future research.

2 Review of Single Distribution Reliability Models, Censoring, and Likelihood

This section briefly reviews the commonly used models, censoring, and methods for fitting a single distribution to reliability data.

2.1 Log-Location-Scale Distributions

The most frequently used distributions for failure-time data are in the log-location-scale family of distributions. A random variable $T > 0$ belongs to the log-location-scale family if $Y = \log(T)$ is a member of the location-scale family. The cdf for a log-location-scale distribution is

$$F(t; \mu, \sigma) = \Phi \left[ \frac{\log(t) - \mu}{\sigma} \right], \ t > 0$$  \hspace{1cm} (1)

and the corresponding pdf is

$$f(t; \mu, \sigma) = \frac{1}{\sigma t} \phi \left[ \frac{\log(t) - \mu}{\sigma} \right],$$ \hspace{1cm} (2)

where $\Phi(z)$ and $\phi(z) = d\Phi(z)/dz$ are, respectively, the cdf and pdf for the particular standard location-scale distribution. The most common log-location-scale distributions are the lognormal ($\Phi(z) = \Phi_{\text{norm}}(z)$ is the standard normal cdf), Weibull ($\Phi(z) = \Phi_{\text{sev}}(z) = 1 - \exp[-\exp(z)]$), Fréchet ($\Phi(z) = \Phi_{\text{lev}}(z) = \exp[-\exp(-z)]$), and loglogistic distributions ($\Phi(z) = \Phi_{\text{logis}}(z) = 1/[1 + \exp(-z)]$).

Example 1: The lognormal distribution

If $T$ has a lognormal distribution, then $Y = \log(T) \sim \text{NORM}(\mu, \sigma)$, where $\mu$ is the mean and $\sigma$ is the standard deviation of the underlying normal distribution. For the lognormal distribution, $\sigma$ is the shape parameter and $\exp(\mu)$ is the median (and a scale parameter).

Example 2: The Weibull distribution

The Weibull cdf is often given as

$$\Pr(T \leq t; \eta, \beta) = F(t; \eta, \beta) = 1 - \exp \left[ - \left( \frac{t}{\eta} \right)^\beta \right], \ t > 0,$$ \hspace{1cm} (3)

where $\beta$ is a shape parameter and $\eta$ is a scale parameter—sometimes called “characteristic life” and is approximately the 0.63 quantile of the distribution.

The Weibull cdf can also be expressed by (1) with the parameters $\mu = \log(\eta)$ and $\sigma = 1/\beta$. Although results (e.g., from software) are typically presented in the more familiar ($\eta, \beta$) parameter-
ization, it is common practice to use the \((\mu, \sigma)\) parameterization for the development of theory and software for the entire (log-)location-scale families and this is especially true for regression models like those used for accelerated testing (e.g., Meeker et al., 2022, Chapters 18 and 19).

### 2.2 Quantities of Interest in Reliability Data Analysis

In reliability applications, the usual distribution parameters are not of primary interest. Instead, there is generally a need to estimate failure probabilities (computed using (1) at a specified time \(t_e\)) or a failure-time distribution \(p_e\) quantile. More generally, the \(p\) quantile for a distribution in the log-location-scale family can be expressed as

\[
t_p = \exp[\mu + \Phi^{-1}(p)\sigma].
\]

These quantiles also play an important role in prior elicitation, as described in Section 3.2.

### 2.3 Censoring

Censoring is ubiquitous in the analysis of reliability and other time-to-event data. There are different kinds of censoring that arise in applications.

- **Right censoring** arises when one or more units have not failed when the data are analyzed and occur for different reasons.
  - Time (Type 1) censoring arises in life tests where the test ends at a specified time. The number of failures is random (and could be zero).
  - Failure (Type 2) censoring arises in life tests where the test ends after a specified number of units have failed. The length of the test is random. Such tests are not common in practice because of the need to adhere to schedules.
  - Multiple right censoring (many different censoring times) is common in field data. Differing censoring times arise from some combination of staggered entry, differing use rates (when time is measured in real-time since entering service), and competing risks (e.g., failure modes unrelated to the one of primary interest).

- **Interval censoring** arises when failures are found at inspections times. All that is known is that a failure occurred between the most recent previous and the current inspection.

- **Left censored observations** arise when a failure has already occurred at the first time a unit is observed and is equivalent to an interval-censored observation that has zero as its beginning time.

An assumption of noninformative censoring (e.g., Lawless, 2003, pages 59–60) is generally required to use the common methods for analyzing censored data.
2.4 Log-likelihood

For data consisting of \( n \) exact failure times and right-censored observations, the likelihood is

\[
L(\text{DATA}|\mu, \sigma) = \prod_{i=1}^{n} L_i(\text{data}_i|\mu, \sigma) = \prod_{i=1}^{n} \left\{ \frac{1}{\sigma t_i} \phi \left[ \frac{\log(t_i) - \mu}{\sigma} \right] \right\}^{\delta_i} \times \left\{ 1 - \Phi \left[ \frac{\log(t_i) - \mu}{\sigma} \right] \right\}^{1-\delta_i},
\]  

(5)

where, for observation \( i \), \( \text{data}_i = (t_i, \delta_i) \), \( t_i \) is either a failure time or a right-censored time, \( \delta_i = 1 \) for an exact failure time, and \( \delta_i = 0 \) for a right-censored observation. For interval-censored observations with lower and upper interval endpoints \( t_{L,i} \) and \( t_{U,i} \), the factor on the left in (5) is replaced by

\[
\left\{ \Phi \left[ \frac{\log(t_{U,i}) - \mu}{\sigma} \right] - \Phi \left[ \frac{\log(t_{L,i}) - \mu}{\sigma} \right] \right\}.
\]  

(6)

For a left-censored observation (an interval that starts at zero), the term on the right in (6) is zero. For more information about likelihoods for censored data, see Chapters 2, 7, and 8 in Meeker et al. (2022). For many purposes (e.g., computational and for the development of theory), it is convenient to use the log-likelihood \( \mathcal{L}(\text{DATA}|\mu, \sigma) = \log[L(\text{DATA}|\mu, \sigma)] \).

3 Using Bayesian Methods and Prior Information in Reliability Applications

As we saw in the motivating examples in Section 1.2, the data in reliability applications often have few failures and thus contain little information. Engineers, however, often have additional useful, but imprecise information that can be combined with the limited data. For example, if failures are caused by a wearout mechanism, the hazard function would be increasing and thus the Weibull shape parameter would be greater than one. Previous experience with a particular failure mechanism in the same material may allow bounding a Weibull or lognormal shape parameter more precisely. On the other hand, there may not be information that can be used to set an informative prior distribution on a scale parameter. In such cases, the informative prior for the shape parameter needs to be used in conjunction with a noninformative or weakly information prior for the scale parameter.

3.1 Bayes’ Theorem

Bayes’ theorem for continuous parameters in \( \theta \) can be written as

\[
\pi(\theta|\text{DATA}) = \frac{L(\text{DATA}|\theta)\pi(\theta)}{\int L(\text{DATA}|\theta)\pi(\theta)d\theta}
\]  

(7)

where the joint prior distribution \( \pi(\theta) \) quantifies the available prior information about the unknown parameters in \( \theta \). The output of (7) is \( \pi(\theta|\text{DATA}) \), the joint posterior distribution for \( \theta \), reflecting
knowledge of \( \theta \) after the information in the data and the prior distribution have been combined. For (log-)location-scale distributions used here, \( \theta = (\mu, \sigma) \).

### 3.2 Reparameterization

In many situations, it is important to replace the traditional parameters (e.g., \( \mu \) and \( \sigma \) for log-location-scale distributions) with alternative parameters. For a log-location-scale distribution, it is useful to replace the usual scale parameter \( \exp(\mu) \) with a particular quantile \( t_{p_r} \) as an alternative scale parameter for a value of \( p_r \) that is chosen in a purposeful manner. Doing this has important advantages for the following reasons.

1. Elicitation of a prior distribution is facilitated because the parameters have practical interpretations and are familiar to practitioners.
2. When prior knowledge is accumulated based on experience involving heavy censoring the traditional parameters \( \mu \) and \( \sigma \) would have strong correlation. Using \( t_{p_r} \) and \( \sigma \) for a suitably specified value of \( p_r \) would allow specifying the joint prior density as a product of marginal densities.
3. The numerical performance of MCMC algorithms is generally improved.

A useful reparameterization for the Weibull distribution replaces \( \eta \) with a particular distribution quantile that could be suggested by available data. For heavily right-censored data from a high-reliability product, this would be a lower-tail quantile of the failure-time distribution. For example, if in certain applications one typically sees 10% of a population of units failing, then something like the 0.05 quantile would be a more appropriate scale parameter. Also, it is easier to elicit prior information for such a small quantile compared to the time at which a proportion 0.63 would fail.

The \( p_r \) quantile of the Weibull distribution is \( t_{p_r} = \eta \left[- \log(1 - p_r)\right]^{1/\beta} \). Replacing \( \eta \) with the equivalent expression \( \eta = t_{p_r} / \left[- \log(1 - p_r)\right]^{1/\beta} \) in (3) provides a reparameterized version of the Weibull distribution:

\[
\Pr(T \leq t; t_{p_r}, \beta) = F(t; t_{p_r}, \beta) = 1 - \exp \left[- \left( \frac{t}{t_{p_r} / \left[- \log(1 - p_r)\right]^{1/\beta}} \right)^\beta \right]
\]

\[
= 1 - \exp \left[ \log(1 - p_r) \left( \frac{t}{t_{p_r}} \right)^\beta \right], \ t > 0.
\] (8)

The latter expression shows that \( t_{p_r} \) is an alternative scale parameter.

Especially when there is heavy censoring (i.e., only a small fraction failing), estimation of \( (t_{p_r}, \beta) \) will be more stable than estimating \( (\eta, \beta) \), for some appropriately chosen value of \( p_r \). Thus one could choose \( p_r \), based on the data, by taking the largest value of the nonparametric estimate of \( F(t) \) and dividing it by two, assuring that the parameter \( t_{p_r} \) is within the data. Another alternative is to choose \( p_r \) based on engineering knowledge that would allow elicitation of an informative or weakly informative prior distribution for \( t_{p_r} \). Note that it is possible to have two separate reparameterizations (one for estimation and one for elicitation), as a prior with one parameterization can be easily translated into
a prior for another parameterization. Usually, however, the elicitation-motivated reparameterization will be sufficient for both purposes.

Figure 3 illustrates such reparameterizations for the bearing-cage and rocket-motor data. For these examples, \( p_r \) was chosen to provide a well-behaved likelihood surface. The lognormal and other log-location-scale distributions can be similarly reparameterized.

![Figure 3: Likelihood contour plots for the bearing-cage (top) and rocket motor data (bottom) in the traditional \((\eta, \beta)\) parameterization (left) and in the \((t_{0.005}, \beta)\) reparameterization (right).](image)

### 4 The Fisher Information Matrix

As described in Section 5 and Section B of the appendix, the Fisher information matrix (FIM) is used to define certain noninformative prior distributions. This section shows how to compute the FIM for (log-)location-scale distributions and different kinds of censoring.
4.1 Scaled Fisher Information Matrix Elements

The scaled FIM elements are:

\[
\begin{align*}
    f_{11}(z_c) &= \left[ \frac{\sigma^2}{n} \right] E \left[ -\frac{\partial^2 \log L(\mu, \sigma)}{\partial \mu^2} \right] \\
    f_{12}(z_c) &= \left[ \frac{\sigma^2}{n} \right] E \left[ -\frac{\partial^2 \log L(\mu, \sigma)}{\partial \mu \partial \sigma} \right] \\
    f_{22}(z_c) &= \left[ \frac{\sigma^2}{n} \right] E \left[ -\frac{\partial^2 \log L(\mu, \sigma)}{\partial \sigma^2} \right].
\end{align*}
\]

The \( \sigma^2/n \) term cancels with a term \( n/\sigma^2 \) arising from the expectations in (9) and thus these scaled FIM elements depend only on \( z_c \) and the assumed distribution (but not on \( n \) or \( \sigma \)).

For Type 1 (time) censoring, \( z_c = \left[ \log(t_c) - \mu \right]/\sigma \) is a standardized censoring times where \( t_c \) is the censoring time and \( p_c = \Phi(z_c) \) is the expected fraction failing. For Type 2 (failure) censoring \( z_c = \Phi^{-1}(r/n) \) where \( r/n \) is the given fraction failing. An algorithm to compute these elements is given by Escobar and Meeker (1994) and implemented in the function \texttt{lsinf} in the R package \texttt{lsinf} (Meeker, 2022).

4.2 Fisher Information for Type 1 and Type 2 Censored Samples

For a sample of \( n \) iid observations, singly censored (i.e., all censoring is at one time point) at time \( t_c \), the FIM for \((\mu, \sigma)\) is

\[
I_{(\mu, \sigma)} = \frac{n}{\sigma^2} \begin{bmatrix} f_{11}(z_c) & f_{12}(z_c) \\ \text{symmetric} & f_{22}(z_c) \end{bmatrix}.
\]

With Type 2 censoring, the scaled FIM \( (\sigma^2/n) I_{(\mu, \sigma)} \) depends on \( r \), the known number of failures, and has no unknown parameters. With Type 1 censoring, the scaled FIM depends on \( p_c = F(t_c) \), the unknown expected fraction failing at time \( t_c \). To simplify the presentation in the remainder of this paper, when it is possible, we will suppress the dependency of the \( f_{ij} \) elements on \( z_c \). That is, for example, we write \( f_{11} \) instead of \( f_{11}(z_c) \).

4.3 Fisher Information Matrix for Randomly Censored Samples

Random censoring arises for different reasons such as staggered entry, differing use rates in the population, and competing risks. The competing risk model provides a convenient model to describe or characterize random censoring. Suppose that \( T \) is a random failure time having a log-location-scale distribution with parameters \((\mu, \sigma)\) and \( C \) is a random censoring time for a unit. Then the unit fails if \( T \leq C \) and is censored if \( T > C \). If \( \log(C) \) has a pdf \( h(x) \), then, using conditional expectation, as
outlined in Escobar and Meeker (1998), the FIM for a sample of size \( n \) is

\[
I_{(\mu, \sigma)} = \frac{n}{\sigma^2} \begin{bmatrix}
\int_{-\infty}^{\infty} f_{11}(w)h(x)dx & \int_{-\infty}^{\infty} f_{12}(w)h(x)dx \\
\text{symmetric} & \int_{-\infty}^{\infty} f_{22}(w)h(x)dx
\end{bmatrix},
\]

where \( w = (x - \mu)/\sigma \).

5 Noninformative Prior Distributions for Log-Location-Scale Distributions

5.1 Motivation

In many (if not most) applications of Bayesian methods, there is a desire to analyze data without using any information that might be available to specify a prior distribution (known as objective-Bayesian analysis). This would be the case, for example, when there is not agreement among interested parties (e.g., engineers and managers or manufacturers and consumers) or when there is a need to avoid having to defend an assumed prior distribution (e.g., in legal or regulatory proceedings). Practitioners may be unable to specify their belief using a probability distribution due to lack of statistical expertise thus requiring the use of a default prior. In these situations, an alternative is to use what is generically called a noninformative prior distribution. Several different methods for specifying a noninformative prior distribution have been suggested.

5.2 Previous Work on Noninformative Prior Distributions

There is extensive literature concerning noninformative prior distributions. Here we mention work most closely related to ours. Bernardo (1979) reviews the early literature and describes criteria and methods for constructing reference prior distributions. Berger and Bernardo (1992a) extend previous work on reference prior distributions, focusing on multiparameter models and recommend a specification of order based on parameter importance. Berger and Bernardo (1992b) provide an updated literature review of this area and describe a general algorithm for finding a reference prior for continuous multiparameter models with a given parameter ordering. Sun (1997) reviews earlier work showing various conditions for second-order and third-order probability matching priors for two-parameter distributions and applies these results to the Weibull distribution demonstrating that certain reference priors meet the conditions. Sun and Berger (1998) consider informative priors when there is external information (e.g., for only one parameter), with the idea of using a reference prior distribution for other parameters, similar to what we suggest in Section 8. Abbas and Tang (2015, 2016) describe reference prior distributions for the Fréchet and loglogistic distributions. Ghosh et al. (2006, Chapter 5) provide a summary of methods and operational details for obtaining noninformative prior distributions.
5.3 Parameterization for Prior Distributions

In our log-location-scale distribution examples, when specifying a prior distribution we will use the parameterization \( (t_p, \sigma) \) (or \( (t_p, \beta = 1/\sigma) \) for the Weibull distribution) because these are the parameters that have a practical interpretation, allowing elicitation of informative or specifying weakly informative prior distributions, when needed. When using algorithms to compute MCMC draws, however, we use the \( (y_p = \log(t_p), \log(\sigma)) \) parameterization because the expressions for the priors tend to be simpler, MCMC algorithms work better in the unconstrained parameter space, and plots of the posterior draws tend to be easier to interpret on the log scales. Prior distributions for \( (t_p, \sigma) \) are easily translated into priors for \( (\log(t_p), \log(\sigma)) \). Examples are given in Section B of the appendix.

5.4 A Fundamental Principle for Specifying Noninformative Prior Distributions

There is an important fundamental principle for specifying noninformative prior distributions in situations where there is only a small amount of information in the data corresponding to the desired inference(s). The prior should put negligible density in parts of the parameter space that are impossible or clearly implausible and that would otherwise lead to substantial posterior probability in such parts of the parameter space. This is the often-stated justification for the use of weakly informative priors (Section 7). For example, using a prior \( \pi[\log(t_p), \log(\sigma)] \propto 1 \) (also known as “flat”) in situations with a small number of failures can result (because of the diffuseness of the likelihood) in non-negligible posterior probabilities in nonsensical regions of the parameter space. We have developed our recommendations (summarized in Section 8.2) to be consistent with this principle.

5.5 Jeffreys Prior Distributions

The Jeffreys prior can be derived as being proportional to the square root of the determinant of the FIM (defined in Section 4.2). For models with one parameter (e.g., the exponential distribution or the normal distribution with known standard deviation), the Jeffreys prior distribution has been shown to provide results (e.g., credible, tolerance, and prediction intervals) that are the same as or close to classical non-Bayesian methods for certain models.

The Jeffreys prior, even if there is more than one parameter, is invariant to reparameterization. This means if you use the square root of the determinant of the FIM definition for a given parameterization you will get a Jeffreys prior. For any other parameterization, the Jeffreys prior can be obtained by finding the square root of the determinant of the FIM in the new parameterization or by using the transformation of a random variable method. That is, either method will lead to the same Jeffreys prior distribution.

As shown in Section B of the appendix, the Jeffreys prior is \( \pi[\log(t_p), \log(\sigma)] \propto 1/\sigma \) for Type 2 censoring (or complete data). For Type 1 censoring, the Jeffreys prior is

\[
\pi[\log(t_p), \log(\sigma)] \propto \frac{1}{\sigma} \sqrt{f_{11} f_{22} - f_{12}^2}.
\]
Here the $f_{ij}$ values are scaled elements of the FIM defined in Section 4.1, which depend on the standardized censoring time $z_c = [\log(t_c) - \mu]/\sigma$. The priors for Type 1 censoring are more complicated because the $f_{ij}$ elements depend on the unknown parameters through $z_c$ (more specifically, they depend on $p_{\text{fail}} = \Phi(z_c)$, the unknown expected fraction failing at the censoring time $t_c$).

For (log-)location-scale distributions (and other distributions with more than one parameter), the Jeffreys priors have well-known deficiencies (Jeffreys, 1961, page 182). Also, in models with more than one parameter, Jeffreys priors may not have the desirable classical properties of agreeing with reference priors that are probability matching (e.g., Sun, 1997).

### 5.6 Independence Jeffreys Prior Distributions

The independence Jeffreys (IJ) prior (also known as the modified Jeffreys prior) is obtained by finding the Jeffreys prior for each parameter, assuming that it is the only unknown parameter, and then using the product of these conditional priors as the joint prior, as if the parameters were independent random variables (but notably, they are not independent). In contrast to the Jeffreys prior, for (log-)location-scale distributions, the IJ prior distribution has an appealing property. In particular, it provides, for complete and Type 2 censored data, the same exact inferences (i.e., the credible/confidence intervals procedures with coverage probabilities that are the same as the nominal credible/confidence level) as non-Bayesian pivotal-based methods (and approximately the same for other kinds of censoring). This result is given for complete data in DiCiccio et al. (2017) but is also true for Type 2 censored data.

#### Type 2 censoring IJ prior distributions

As shown in Section B.3 of the appendix, for Type 2 censoring, the conditional Jeffreys (CJ) prior for $\log(t_{pr})$ given $\log(\sigma)$ is $\pi[\log(t_{pr}) | \log(\sigma)] \propto 1$. The CJ prior for $\log(\sigma)$ given $\log(t_{pr})$ is $\pi[\log(\sigma) | \log(t_{pr})] \propto 1$. Thus the IJ prior is $\pi[\log(t_{pr}), \log(\sigma)] \propto 1$, which we (following common usage) call “flat.”

#### Type 1 censoring IJ prior distributions

As shown in Section B.3 of the appendix, for Type 1 censoring, the CJ prior for $\log(t_{pr})$ given $\log(\sigma)$ is $\pi[\log(t_{pr}) | \log(\sigma)] \propto \sqrt{f_{11}}$. The CJ prior for $\log(\sigma)$ given $\log(t_{pr})$ is $\pi[\log(\sigma) | \log(t_{pr})] \propto \sqrt{f_{11}[\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22}}$.

So, the IJ prior is

$$
\pi[\log(t_{pr}), \log(\sigma)] \propto \pi[\log(t_{pr}) | \log(\sigma)]\pi[\log(\sigma) | \log(t_{pr})] \\
\propto \sqrt{f_{11}\{f_{11}[\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22}\}}. \quad (11)
$$

where, again, the $f_{ij}$ values are scaled elements of the FIM defined in Section 4.1, which depend on the standardized censoring time $z_c = [\log(t_c) - \mu]/\sigma$ or $p_c = \Phi(z_c)$. 

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Relative to flat priors, these IJ priors tend to be more consistent with the fundamental principle described in Section 5.4, and provide advantages over traditional noninformative priors. This is explained in Section 5.10 and demonstrated in Section 9.5.

5.7 Reference Prior Distributions

Another well-studied approach to finding a noninformative prior distribution is to use a reference prior. Generally, a reference prior is the prior distribution that maximizes the Kullback-Leibler divergence between the prior and the expected posterior distribution. Different reference prior distributions can arise, depending on the specified order of importance of the parameters (or functions of the parameters). Interestingly, certain definitions of reference priors lead exactly to the IJ priors (and others lead to the Jeffreys prior).

As an example, with Type 2 censoring or complete data, the IJ prior for the log-location-scale distribution parameters \((t_p, \sigma)\) is (proportional to) \(1/(t_p\sigma)\) and for \((\log(t_p), \log(\sigma))\) the IJ prior is flat (i.e., uniform over the entire \((\log(t_p), \log(\sigma))\) plane for any \(p\)). As shown in Section B.6 of the appendix, these IJ priors are also reference priors when either \(t_p\) or \(\sigma\) is the parameter of first importance.

As described in Section B.4 of the appendix, expressions for computing reference priors are not readily available for Type 1 censoring. However, given the equivalence of IJ and ordered reference priors described above for Type 2 censoring, we expect that the IJ priors for Type 1 censoring (Section 5.6) would provide a good approximation to the corresponding reference priors.

5.8 Improper Priors and Posteriors

Noninformative priors are generally improper (i.e., their integrals over the parameter space are not finite). Thus, when using such prior distributions (e.g., flat, IJ, or CJ combined with a proper marginal for the other parameter), it is important to assure that there is a sufficient amount of information in the data that the posterior will be proper. Ramos et al. (2020) give conditions under which a Weibull posterior distribution will be proper for some improper priors, suggesting that having two failures is sufficient to result in a proper posterior. We did extensive numerical experiments with flat and IJ prior distributions and simulated Type 1-censored samples with different numbers of failures. These experiments indicated that the posterior is proper and reasonably well behaved when there are at least three failures. With only two failures and a flat prior, however, even when an appropriate parameterization was used, we were not able to find a stable sampler. For example, the sampler would often return large numbers or infinite values of the parameters, perhaps due to limitations in computer floating-point representation of real numbers (we also encountered infinite values in our experiments with three failures, but their occurrence was relatively rare). In summary, even if a proper posterior is assured theoretically with two failures, there are practical difficulties.
5.9 A Summary of Noninformative Priors for (Log-)Location-Scale Distributions

Table 1 summarizes Jeffreys, IJ, and reference noninformative prior distributions for (log-)location-scale distributions using different parameterizations. Section B of the appendix gives derivations of these priors. The relationships and correspondences (some of which have been noted previously in the literature) are interesting. For example,

- For all parameterizations, the usual Jeffreys prior is the same as the reference prior when no parameter-importance is specified (as noted by Kass and Wasserman, 1996, page 1,350).
- When the distribution parameters are defined as the shape parameter \( \sigma \) and a scale parameter (e.g., the traditional \( \exp(\mu) \) or the \( p \) quantile \( t_p \) for any \( p \)), the ordered reference prior is the same as the the IJ prior, irrespective of the ordering. This equivalence property also holds for one-to-one functions of the individual parameters (e.g., when \( \sigma \) is replaced by \( \log(\sigma) \)).
- Related to the previous point, although IJ priors for log-location-scale distributions are not in general invariant to reparameterization (in the sense described in Section 5.5), they are invariant to one-to-one monotone reparameterizations of either or both of the \( (t_p, \sigma) \) parameters. The proof of this result is given in Section B.10 of the appendix.
- This IJ/ordered-reference equivalence property does not hold when the distribution parameters are defined as the shape parameter \( \sigma \) and \( \zeta_e = (\log(t_e) - \mu)/\sigma \) and \( \sigma \) is in second order.

5.10 Implementing and Interpreting the IJ Prior Distributions

Implementing the IJ priors (e.g., \( \pi[\log(t_p), \log(\sigma)] \propto 1 \) or \( \pi[\log(t_p), \sigma] \propto 1/\sigma \)) that arise with complete data or Type 2 censoring is straightforward. There is, however, more than one way to implement the IJ priors that arise from Type 1 censoring. Here we describe some alternatives using the IJ prior for \( (\log(t_p), \log(\sigma)) \) in (11) as an example. Under all of the alternatives, the goal is to define the prior over the entire parameter space (e.g., the real plane for \( (\log(t_p), \log(\sigma)) \)). First, for the given censoring time \( t_c \), \( z_c = [\log(t_c) - \mu]/\sigma \) (the argument for the \( f_{ij} \) values) is computed as a function of \( (\log(t_p), \log(\sigma)) \) (i.e., by using \( \mu = \log(t_p) - \Phi^{-1}(p_r)\sigma \)). There are several choices for evaluating the \( \Phi^{-1}(p_{\text{lim}}) \) in (11).

- One option would be to take \( p_{\text{lim}} \) to be the reparameterization \( p_r \) in \( \log(t_p) \), as used in the computation of \( z_c \) needed to evaluate the basic FIM elements. If the reparameterization \( p_r \) is chosen based on the nonparametric estimate of \( F(t) \) (as suggested in Section 3.2), it could be viewed as a slight breach of the likelihood principle (i.e., letting the data influence the choice of the prior).
- Maximum likelihood estimators (of any functions of the parameters that might be of interest) are invariant to the choice of \( p_r \) in a reparameterization. Bayesian estimators are at least approximately invariant, as long as the reparameterization also applies to the prior distribution. This suggests that one could choose \( p_{\text{lim}} \), based on prior engineering knowledge, to be something
Table 1: Improper noninformative joint prior distributions $\pi(\theta_1, \theta_2)$ for different parameterizations for log-location-scale distributions based on Type 2 and Type 1 censored data

| Prior type and parameterization | Type 2 | Type 1 |
|---------------------------------|--------|--------|
| Jeffreys ($\mu, \sigma$)        | $1/\sigma^2$ | $1/\sigma^2 \sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Jeffreys ($\log(t_p), \sigma$)  | $1/\sigma^2$ | $1/\sigma^2 \sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Jeffreys ($t_p, \sigma$)        | $1/(t_p \sigma^2)$ | $1/((t_p \sigma^2) \sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Jeffreys ($\log(t_p), \log(\sigma)$) | $1/\sigma$ | $1/\sigma \sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Jeffreys ($\zeta_e, \sigma$)    | $1/\sigma$ | $1/\sigma \sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Jeffreys ($\zeta_e, \log(\sigma)$) | $1$ | $\sqrt{f_{11} f_{22} - f_{12}^2}$ |
| Lij ($\mu, \sigma$)             | $1/\sigma$ | $1/\sigma \sqrt{f_{11} f_{22}}$ |
| Lij ($\log(t_p), \sigma$)       | $1/\sigma$ | $1/(t_p \sigma) \sqrt{f_{11} \{\Phi^{-1}(p)\}^2 - 2 f_{12} \Phi^{-1}(p) + f_{22}}$ |
| Lij ($t_p, \sigma$)              | $1/(t_p \sigma)$ | $1/(f_{11} \{\Phi^{-1}(p)\}^2 - 2 f_{12} \Phi^{-1}(p) + f_{22})$ |
| Lij ($\log(t_p), \log(\sigma)$) | $1$ | $\sqrt{f_{11} \{\Phi^{-1}(p)\}^2 - 2 f_{12} \Phi^{-1}(p) + f_{22}}$ |
| Lij ($\zeta_e, \sigma$)         | $1/\sigma$ | $1/\sigma \sqrt{f_{11} \zeta_e^2 - 2 f_{12} \zeta_e + f_{22}}$ |
| Lij ($\zeta_e, \log(\sigma)$)  | $1$ | $\sqrt{f_{11} \zeta_e^2 - 2 f_{12} \zeta_e + f_{22}}$ |

The scaled FIM elements $f_{11}$, $f_{12}$, and $f_{22}$ depend on the standardized censoring time $z_c = [\log(t_c) - \mu]/\sigma$. Parameters shown within {...} indicate parameter-importance order, if there is one. The parameter $\zeta_e = [\log(t_e) - \mu]/\sigma$ where $t_e$ is the time at which a failure probability is to be estimated. The three reference priors for $(\mu, \sigma)$ are exactly the same as for $(\log(t_p), \sigma)$ and are thus not presented here.
like the proportion that would be expected to fail in the life test or a particular quantile of interest.

- Numerical examples in Sections 7.4 and 8.3 and simulation results presented in Section 9.5 show that estimation results (point estimates and credible intervals) are usually not highly sensitive to the choice of $p_{\text{lim}}$. Thus a simple alternative, requiring no inputs, would be to choose $p_{\text{lim}}$ such that $\Phi^{-1}(p_{\text{lim}}) = 0$, also resulting in a simpler form for the IJ prior. Then the IJ prior for $(\log(t_{p_r}), \log(\sigma))$ simplifies to

$$\sqrt{f_{11}f_{22}},$$

which, as a special case of (11), is the IJ prior for $(\mu, \log(\sigma))$.

Section C in the appendix provides a detailed description of the Type 1 censoring IJ prior distribution features and the reasons those features arise. Here we provide a brief summary of that material. Figure 4 illustrates the general shapes of the IJ priors for different values of $p_r$ with $p_{\text{lim}} = 0.01$. These IJ priors have been scaled to have a maximum of 1.0 and thus we refer to them as relative densities. Generally, the IJ prior for small $\sigma$ and $t_{p_r} < t_c$ is flat at a level of 1.0. For any small $\sigma$, as the value of $t_{p_r}$ crosses $t_c$, there is a steep cliff, with the level of the density dropping to near 0. Note

Figure 4: Independence Jeffreys prior densities for $p_r = 0.01$ (top left), $p_r = 0.10$ (top right), $p_r = 0.50$ (bottom left), and $p_r = 0.99$ (bottom right), all with $p_{\text{lim}} = 0.01$. 

...
that this is because, in bottom-right part of the parameter space, the probability of getting even one failure is negligible. Detailed examples are given in Section C.2 of the appendix. The importance of this result is that, relative to other noninformative priors, the IJ priors (and CJ priors when used in conjunction with an informative or weakly information marginal prior for the other parameter) follow the fundamental principle described in Section 5.4.

For larger values of $\sigma$, the IJ prior is approximately flat at a level less than 1.0 for all values of $t_{pr}$. The level is controlled by both $p_r$ and $p_{lim}$. Figures 13–15 in Section C.2 of the appendix give numerous examples.

6 Random Censoring IJ Prior Distributions

As described in Section 2.3, field reliability data almost always result in multiply time-censored data. The random censoring IJ prior, based on a competing risk model, can be obtained from the random censoring FIM described in Section 4.3. Suppose that $T$ is the failure time for a unit but that it will not be observed if the random censoring time $C < T$. There are two situations to consider, described in the following subsections.

6.1 IJ Prior Distributions for Limited-Time Random Censoring

High-reliability products or any product that has been in the field for a small amount of time (e.g., a smart phone) will have a largest value of a nonparametric estimate (e.g., Kaplan–Meier) of the marginal cdf of $T$ that is considerably less than 1. This situation is common and arises when data are analyzed at a particular data-freeze date when only a small fraction of units in the field has failed. This is similar to Type 1 (time) censoring, except that there will be some additional right-censored observations (e.g., due to staggered entry into service) before the censoring time $t_c$. The bearing cage field data (Section 1.2) provides an example of such data. The rocket motor field data (also Section 1.2) is different because all three failures were left censored. The nonparametric ML estimate of $F(t)$ jumps to 1.0 at the left-censored observation at 16.5 years because this is larger than 16 years, the largest right-censored observation (there is additional discussion of this point at the end of Section 8.3). However, the similarity of the shapes of the bearing cage and rocket motor likelihoods (see Figure 3), suggests a much smaller effective $t_c$ for the rocket motor data—perhaps 11 years.

It would be possible to define and compute an IJ prior for a competing risk model describing multiple censoring by using the scaled FIM elements inside the square brackets in (10) to replace the $f_{ij}$ values in (11). The pdf of $\log(C)$, $h(x)$ would describe the pattern up to the censoring time $t_c$, where all of the remaining mass would be concentrated. The shape of the resulting IJ prior in this situation will, however, be similar to the IJ prior for Type 1 censoring, described in Sections 5.6 and 5.10 and thus those can be used instead. We use this approach for our examples in Sections 7.4 and 8.3.
6.2 IJ Prior Distributions for Unlimited-Time Random Censoring

For products that have a substantial amount of field experience and are run until failure, after which they are replaced (e.g., single-use batteries), the nonparametric estimate of the marginal cdf of \( T \) will typically be close to 1. An example of this kind of data is given in the mechanical switch example in Nair (1984), where he focused on estimating the marginal distribution of failure mode A and the occurrence of failure mode B resulted in the random censoring.

Again, it would be possible to define and compute an IJ prior for this situation by using the scaled FIM elements inside the square brackets in (10) to replace the \( f_{ij} \) values in (11). The pdf of \( \log(C) \), \( h(x) \) could be obtained by looking at the distribution of censoring times. The shape of the resulting IJ prior in this situation will be approximately flat because there is not a single censoring time that ends the failure-observation process.

7 Motivation for Weakly Informative Prior Distributions

7.1 Potential difficulties with noninformative prior distributions

The noninformative prior distributions described in Section 5 have appealing theoretical properties under certain specified conditions (e.g., (log-)location-scale distributions with complete data or Type 2 censoring). In most applications, such conditions will not be met exactly. Noninformative prior distributions can put large amounts of relative density at unreasonable (e.g., impossible) parts of the parameter space. Then, with limited information in the data (e.g., a small number of failures), a noninformative prior distribution can strongly influence inferences and possibly result in misleading conclusions. In such situations, especially, it is better to use a prior distribution that rules out combinations of parameters that are impossible or nonsensical.

7.2 Previous Work on Weakly Informative Prior Distributions

While there is a vast literature on noninformative or reference prior distributions, less work has been done on weakly informative prior distributions. Weakly informative priors are constructed to be diffuse relative to the likelihood and known scale of the data. However, as opposed to noninformative priors, weakly informative priors put density on reasonable values of the parameters while down weighting nonsensical values. When there is a small amount of information in the data (e.g., few failures) or if fitting a complex model with many parameters, weakly informative priors can help stabilize estimation whereas a noninformative prior (e.g., a flat prior) can result in dispersed posterior distributions with probability mass on extreme parameter values. Gelman et al. (2008) recommend weakly informative priors for the parameters of logistic and other regression models. More recently, Gelman et al. (2017) provide a historical overview of the different classes of Bayesian priors such as, uniform, Jeffreys, reference, and weakly informative priors. The authors illustrate the dangers of using flat or default priors and recommend weakly informative prior distributions that are selected based on the data and subject-specific domain. Lemoine (2019) uses simulation to demonstrate how noninformative priors
can produce spurious parameter estimates, and advocates for weakly informative priors as the new default choice for Bayesian estimation.

7.3 Weakly Informative Prior Distributions for Log-Location-Scale Distributions

When there is little or no prior information about certain parameters or when there is need to present an objective analysis where results do not depend on prior information, a commonly-used alternative is to specify weakly informative marginal priors for those parameters. Commonly-used weakly informative priors include a normal distribution with a large variance for parameters that are unrestricted in sign or a lognormal distribution with a large value of the shape parameter (log-standard-deviation) for parameters that must be positive.

These choices can be motivated by the fact that a normal distribution prior density with any mean will approach a flat prior as the standard deviation of the normal distribution increases. Correspondingly, a lognormal distribution prior $\pi(t)$ with any log-mean will be proportional to $1/t$ as the log-standard-deviation increases. These results are illustrated in Figure 5 and proofs are in Section G of the appendix.

![Figure 5: Illustration that a normal distribution density approaches a uniform (flat prior) distribution as $\sigma$ increases (a) and that a lognormal density in $t$ is proportional to $1/t$ for large values of $\sigma$ (b).](image)

The parameters for these weakly informative marginal prior distributions can be chosen using knowledge about the scale of the response (i.e., depending on the units of the response), what values of the parameters are physically possible, and knowledge based on previous experience and engineering knowledge. The center of these distributions might be chosen in a conservative manner.

Instead of specifying the prior distribution parameters (especially for the lognormal distribution), it is generally better (and much easier for prior elicitation) to specify a range that contains a large proportion of the probability distribution. Here we use a 0.99 probability range which is defined as the 0.005 and the 0.995 quantiles of the prior distribution. For example, to specify that the Weibull shape parameter has a lognormal prior distribution with probability 0.99 between 1.5 and 5, we write $\beta \sim \text{LNORM}(1.5, 5)$.

The use of quantiles to elicit/specify prior distributions has been discussed previously (e.g., in...
Dey and Liu, 2007). Meyer and Booker (2001) point out that individuals tend to underestimate uncertainty. Thus, unless the prior interval is based on quantitative information (e.g., interval estimates on the same parameter from a previous study or studies) one might want to ask for a 99% interval and treat it as if it were a 95% interval. Mikkola et al. (2021) describe the current state of the art of prior elicitation and provide an extensive literature review, including other papers that use quantiles in elicitation.

7.4 Comparisons of ML and Bayesian Estimation using Noninformative or Weakly Informative Prior Distributions

Bearing cage field data

This is a continuation of the bearing cage examples in Sections 1.2 and 3.2 where we compare ML and Bayes estimation using alternative noninformative priors. Figure 6 compares ML and Bayes estimation with a flat (on log($t_{0.10}$) and log($\sigma$)) and IJ priors with $p_{\text{lim}} = 0.01$ and 0.05 (a) and $p_{\text{lim}} = 0.10$ and 0.50 (b). Recall that interest centers on estimating $F(t)$ at 8,000 hours. There is little difference among the IJ priors except when $p_{\text{lim}} = 0.01$, which provides more optimistic results (i.e., smaller failure probability estimates) in the upper tail of the distribution, where the flat prior is more pessimistic. The other IJ priors agree well with the ML results except for the upper confidence bounds on $F(t)$ in the lower tail of the distribution. The flat prior results are more optimistic in the upper tail of the distribution (the region of interest).

Rocket motor field data

This is a continuation of the rocket motor examples in Sections 1.2 and 3.2. Although ML estimates and associated confidence intervals exist for the rocket motor data, because the failures are left-censored observations, using Bayes estimation with flat or IJ priors apparently results in an improper posterior distribution. We avoid this problem by using a weakly informative prior $\beta \sim <\text{LNORM}>(0.2, 25)$ for the comparisons in this section (note that this is an extremely wide range relative to values of
Figure 7: Rocket motor comparison of estimation results for ML and Bayes weakly informative priors on $\beta$ with flat prior for $t_{pr}$ (a) and a CJ prior for $t_{pr}$ (b).

$\beta$ seen in typical applications). Figure 7(a) compares estimation results for ML, Bayes with a flat (uniform) prior for $\log(t_{pr})$, and Bayes with a with a weakly informative $t_{pr} \sim \text{LNORM}(0.2, 25)$. Figure 7(b) is similar but replaces the weakly informative prior for $\log(t_{pr})$ with a CJ prior. Note that the CJ prior for $\log(t_{pr})$ does not depend on $p_{lim}$. The estimates using the weakly informative $t_{pr} \sim \text{LNORM}(5, 400)$ in Figure 7(a) are considerably more optimistic than the CJ prior results in Figure 7(b). An important advantage of the CJ prior is that it does not require the user to specify any parameters.

8 Combining Informative with Noninformative or Weakly Informative Prior Distributions

8.1 Motivation for Partially Informative Prior Distributions

An important reason for using Bayesian methods is that they provide a formal mechanism for including prior information (i.e., knowledge beyond that provided by the data) into the analysis. When combining a CJ prior with an informative or weakly informative prior our approach is similar to that suggested in Sun and Berger (1998). For example, as described in Section 5.6, the CJ prior $\pi[\log(t_{pr})|\log(\sigma)]$ can be combined with an informative or weakly informative marginal prior $\pi(\log(\sigma))$ to give the joint prior $\pi[\log(t_{pr}), \log(\sigma)]$. In our software, for convenience, the prior is specified in terms of the familiar Weibull shape parameter $\beta = 1/\sigma$ (or the lognormal shape parameter $\sigma$) and then transformed into the marginal prior for $\log(\sigma)$.

Section 3.2 discussed the need for reparameterization. If there is prior information for one or more of the model parameters and if the definition of the parameters (i.e., the particular parameterization) has been chosen such that the information about parameters is approximately mutually independent, then one can specify a joint prior density as the product of marginal densities for each parameter.

Informative marginal prior distributions can be used for those parameters for which there is
appreciable prior information. As mentioned by Meyer and Booker (2001), individuals providing an informative marginal prior distribution will typically feel more comfortable expressing their knowledge about a parameter by using a symmetric distribution. The normal distribution (truncated below zero for a positive parameter) is a reasonable (approximately symmetric) choice. Then noninformative (e.g., flat or CJ) or weakly informative (e.g., normal with a large 99% range) marginal prior distributions can be specified for the other unrestricted parameters (e.g., \(\log(t_{pr})\) or \(\log(\sigma)\)).

A useful generalization of the normal distribution is the location-scale-\(t\) (LST) distribution. That is, for a specified degrees-of-freedom parameter \(r_d > 0\), there is a symmetric LST distribution having tails that are heavier (or much heavier) than the normal distribution. For large values of \(r_d\) (e.g., greater than 60), the LST distribution is approximately the same as a normal distribution. For \(r_d = 1\), the LST distribution is a Cauchy distribution.

### 8.2 A Summary of Recommended Log-Location-Scale Prior Distributions

Table 2 provides a summary of the recommended prior distributions for use with log-location-scale distributions. Some comments on these prior distributions and Table 2 are:

| Prior distributions for \(t_{pr}\) | Type of prior | Prior distribution inputs |
|-----------------------------------|---------------|---------------------------|
| Lognormal for \(t_{pr}\)          | Informative   | \((\mu_{\log(t_{pr})}, \sigma_{\log(t_{pr})})\) |
| Truncated (+) normal for \(t_{pr}\) | Informative   | \((\mu_{t_{pr}}, \sigma_{t_{pr}})\) |
| Log-location-scale-\(t\) for \(t_{pr}\) | Informative   | \((\mu_{\log(t_{pr})}, \sigma_{\log(t_{pr})}, r_d)\) |
| Truncated (+) location-scale-\(t\) for \(t_{pr}\) | Informative   | \((\mu_{t_{pr}}, \sigma_{t_{pr}}, r_d)\) |
| Flat for \(\log(t_{pr})\) | Noninformative | None |
| Conditional Jeffreys for \(\log(t_{pr})|\log(1/\beta)\) | Noninformative | \(t_c\) |

| Prior distributions for \(\beta = 1/\sigma\) (or \(\sigma\)) | Type of prior | Prior distribution inputs |
|-------------------------------------------------------------|---------------|---------------------------|
| Lognormal for \(\beta\)                                    | Informative   | \((\mu_{\log(\beta)}, \sigma_{\log(\beta)})\) |
| Truncated (+) normal for \(\beta\)                        | Informative   | \((\mu_{\beta}, \sigma_{\beta})\) |
| Log-location-scale-\(t\) for \(\beta\)                    | Informative   | \((\mu_{\log(\beta)}, \sigma_{\log(\beta)}, r_d)\) |
| Truncated (+) location-scale-\(t\) for \(\beta\)          | Informative   | \((\mu_{\beta}, \sigma_{\beta}, r_d)\) |
| Flat for \(\log(1/\beta)\)                                | Noninformative | None |
| Conditional Jeffreys for \(\log(1/\beta)|\log(t_{pr})\)   | Noninformative | \(t_c\) and \(p_{lim}\) |
The CJ priors referenced in Table 2 are for Type 1 censoring (Section 5.6) and limited-time random censoring (Section 6.1).

Table 2 gives priors for the Weibull failure-time distribution shape parameter $\beta = 1/\sigma$. The recommendations are similar for the lognormal failure-time distribution with shape parameter $\sigma$.

As described in Section 7.3, lognormal, truncated normal, log-location-scale-t, and truncated location-scale-t prior distributions for the interpretable Weibull parameters $t_p$ and $\beta = 1/\sigma$ are initially specified by a 0.99 probability range. This range is then translated into parameters for the marginal prior distributions for both $t_p$ and $\beta = 1/\sigma$.

Subsequently, the marginal priors for $t_p$ and $\beta = 1/\sigma$ are used to obtain (by standard methods for obtaining the distribution of a transformation of random variables), the marginal priors for $\log(t_p)$ and $\log(1/\beta) = \log(\sigma)$ that are used in the MCMC computations. Details are given in Section H of the appendix.

For the CJ priors, the censoring time $t_c$ is not a parameter, but it is a necessary input.

The noninformative flat and CJ priors do not require specification of any parameters and are thus given directly in terms of the $\log(t_p)$ or $\log(1/\beta)$ parameterization.

When the CJ priors for $\log(t_p) | \log(1/\beta)$ and for $\log(1/\beta) | \log(t_p)$ are used together, the result is a joint IJ prior for $(\log(t_p), \log(1/\beta))$. Note that when these two densities are used together, the result is not a joint distribution of independent random variables. This is the reason that we use the name Independence Jeffreys (IJ).

### 8.3 Comparisons of Bayesian Estimation using Noninformative and Partially Informative Prior Distributions

This section returns to the two motivating examples, comparing noninformative (or weakly informative) priors with partially informative priors.

**Bearing cage field data**

For the bearing cage field data, Figure 8 compares Bayesian estimation results for a noninformative prior (on the left) and a partially informative prior (on the right). For the noninformative prior, we used an IJ prior with $p_{\text{lim}} = 0.10$ (noting from Section 7.4 that the estimation results are insensitive to the choice of $p_{\text{lim}}$ between 0.05 and 0.50). For the partially informative prior, we combine the CJ prior for $y_p$ given $\log(\sigma)$ (which, from Section B.7 is $\pi(y_p | \tau) \propto \sqrt{f_{11}}$) with an informative truncated (to be positive) normal distribution prior $\beta \sim \langle \text{TNORM+} > (1.5, 3)$. Note that the CJ prior for $y_p$ given $\log(\sigma)$ does not depend on $p_{\text{lim}}$.

The probability plot on the top-left in Figure 8 gives Bayesian estimation results with the noninformative IJ prior and they are similar to the ML results given on the right in Figure 1. The credible interval for $F(8000)$ is [0.03, 0.99992], which is not useful for assessing whether the goal of
fraction failing less than 0.10 has been met.

The probability plot on the top-right gives Bayesian estimation results with the partially informative prior, showing the improved precision. A 95% credible interval for \( F(8000) \) is \([0.15, 0.92]\) indicating clearly that the goal of fraction failing of less than 0.10 has not been met.

The middle and bottom rows of plots show likelihood contours along with prior and posterior draws, respectively. As described more fully in Section D.2 of the appendix, prior draws were obtained by sampling from versions of the noninformative priors that were bounded by a large rectangle (much
larger than the boundaries of the contour plots where the draws are plotted) so that the resulting prior distributions are proper. These plots provide a visualization of how constraining the values of the Weibull shape parameter \( \beta \) to those that are consistent with engineering knowledge importantly improves precision for estimation.

**Rocket motor field data**

For the rocket motor field data, Figure 9 compares Bayesian estimation results for a noninformative/weakly informative prior (on the left) and a partially informative prior (on the right). As described in Section 7.4, because of the limited amount of information in the three left-censored observations, a weakly informative \( \beta \sim <\text{LNORM}>(0.2, 25) \) marginal prior was used for the noninformative part of the comparison and, as in Figure 7(b), this was paired with a noninformative CJ prior. Olwell and Sorell (2001) stated, “It is unusual in Weibull analysis to get shape parameters greater than 5.” Also, because thermal cycling is a wearout mode, \( \beta > 1 \). Thus for the informative part of the comparison, the weakly informative prior was replaced by a somewhat informative \( \beta \sim <\text{LNORM}>(1, 5) \).

The probability plot on the top-left in Figure 9 gives Bayesian estimation results with the noninformative/weakly informative prior, and they are similar to the ML results given on the right in Figure 2. The credible interval for \( F(20) \) is \([0.006, 0.98]\), which does not help assess whether there is a serious problem or not. The probability plot on the top-right gives Bayesian estimation results with the partially informative prior, showing considerably better precision for estimating \( F(20) \). A 95% credible interval for \( F(20) \) is \([0.005, 0.16]\) which would help assess the need for corrective action. Similar to Figure 8 for the bearing cage example, the plots in the middle and bottom rows of Figure 9 show likelihood contours along with prior and posterior draws, respectively. Again, these plots provide a visualization of how the partially informative prior for the Weibull shape parameter \( \beta \) improves estimation precision.

One might ask why the Bayesian estimate of \( F(t) \) in the top-right plot in Figure 9 differs so much from the nonparametric estimate represented by the three plotted points. The nonparametric ML estimate (NPMLE) of \( F(t) \) for current-status data can be obtained by using the “Pool Adjacent Violators” algorithm for the observed fraction failing as a function of years. For the rocket motor data, the NPMLE is a step function that jumps from 0 to \( 1/384 = 0.026 \) at 8.5 years, to \( 1/9 = 0.111 \) at 14.2 years, and to 1.0 at 16.5 years. Then the points are plotted at half of the jump height, as suggested in Lawless (2003, Section 3.3.1.2). The slope of the ML estimate of \( F(t) \) is the same as the ML estimate of \( \beta \), as shown in Meeker et al. (2022, Section 6.2.4) and this is approximately true for the Bayesian estimate. As mentioned in Section 1.2, the ML estimate \( \hat{\beta} = 8.126 \) which, as suggested by Olwell and Sorell (2001), is physically unreasonable for Weibull field data. Olwell and Sorell (2001) also mention that it is well known that the Weibull ML estimator for \( \beta \) has serious upward bias when there are few failures. The Bayesian estimate with the noninformative/weakly informative prior \( \hat{\beta} = 5.81 \), is an improvement but still physically unreasonable. Using the partially informative prior, described above, gives a much more reasonable \( \hat{\beta} = 2.83 \), but the change in slope results in the deviation between the
Figure 9: Rocket motor Bayesian estimation results comparing a weakly informative/noninformative prior (on the left) with a partially informative prior (on the right). For each prior there is a Weibull probability plot showing the point estimate and credible intervals for $F(t)$ (top), draws from the bounded joint prior and likelihood contours (middle), and draws from the joint posterior and likelihood contours (bottom).

Bayesian and the nonparametric estimates of $F(t)$ seen in the top-right plot in Figure 9.
9 Simulation Study to Evaluate Alternative Noninformative Prior Distributions

9.1 Goals of the Simulation

As mentioned in Section 5.6, for complete and Type 2 censored data from a (log-)location-scale distribution, when using an IJ prior distribution, Bayesian credible intervals have coverage probabilities that are the same as the nominal credible/confidence level (i.e., giving “exact” interval procedures). For Type 1 and random censoring, this result is approximate. This section describes a simulation study to compare coverage probabilities of credible intervals for the different noninformative prior distributions (flat and IJ with different \( p_{\text{lim}} \)) for the most commonly used failure-time distributions (Weibull and lognormal) under Type 1 censoring. Here we focus on the Weibull distribution simulation. Results for the lognormal distribution simulation were similar and are given in Section E of the appendix. The rest of this section describes the design of the study and the results.

9.2 Simulation Factors

We simulated a life test where all units are put on test simultaneously and are observed until a fixed censoring time, \( t_c \). Simulated failure times larger than \( t_c \) are right-censored at \( t_c \) (i.e., Type 1 censoring). Without loss of generality, we use fix the Weibull parameters \( \mu = 0 \) and \( \beta = 1/\sigma = 1 \.

The experimental factors for the simulation were:

- \( E(r) \): the expected number of failures before \( t_c \),
- \( p_{\text{fail}} \): the expected proportion of failures before time \( t_c \), and
- \( \pi(\log(t_{p_r}), \sigma) \): the joint prior distribution for the unknown parameters, where \( p_r \) is the reparameterization quantile and is always chosen to be \( r/(2n) \), resulting in a well-behaved likelihood (and posterior).

We use the expected number of failures instead of the sample size as an experimental factor because it is a better measure of the amount of information in a data set and avoids a strong interaction that arises when sample size and expected proportion censored are used as factors. This simulation is similar to that used in Jeng and Meeker (2000) except that instead of comparing ten different non-Bayesian confidence interval methods we compare the Bayesian method under different noninformative priors.

9.3 Simulation Factor Levels

In order to cover a range of situations and joint prior specifications, we use the following levels of the factors:

- \( E(r) = 10, 25, 35, 50, 100 \),
- \( p_{\text{fail}} = 0.01, 0.05, 0.10, 0.50 \), and
\[ \pi(\log(t_{pr}), \sigma) = \text{flat}, \quad \text{IJ}(p_{\text{lim}} = 0.01), \quad \text{IJ}(p_{\text{lim}} = 0.05), \quad \text{IJ}(p_{\text{lim}} = 0.10), \quad \text{IJ}(p_{\text{lim}} = 0.50). \]

For each factor-level combination, the sample size is computed as \( n = \text{E}(r)/p_{\text{fail}} \). The censoring time is computed as \( t_c = \exp(\mu + \Phi^{-1}(p_{\text{fail}})\sigma) \). We simulate \( t_i, i = 1 \ldots n \). Observations with \( t_i > t_c \) are coded as being censored at \( t_c \). As discussed in Section 5.8, with fewer than three failures, the posterior is either improper or poorly behaved. Thus we condition on at least 3 failures.

For each factor-level combination, we simulated 5,000 data sets. As outlined in Section 5.10, the IJ prior is specified in terms of the reparameterization \((\log(t_{pr}), \log(\sigma))\) and the IJ prior \(\pi[\log(t_{pr}), \log(\sigma)]\) also depends on the specification of \(p_{\text{lim}}\). We obtain posterior draws for each simulated data set, for five different noninformative prior distributions—flat and IJ with four different levels of \(p_{\text{lim}}\).

### 9.4 Estimation

The Weibull distribution models were fit using codes based on the R package `rstan` (Stan Development Team, 2020). R package `lsinf` was used to compute the scaled FIM needed for the IJ priors. Implementation details are described in Section D of the appendix. Four chains were run for each estimation run, resulting in 10,000 draws after warmup and thinning. Before doing the production simulation runs, extensive experiments were conducted to investigate the performance of the `rstan` NUTS sampler for our different factor-level combinations. The Gelman-Rubin potential scale reduction factor and close examination of select trace plots were used to check for adequate mixing of the four chains (Gelman and Rubin, 1992). Performance of the prior distributions was evaluated in terms of coverage probability, computed as the proportion of times the computed 95\% credible intervals cover the true value of the quantiles being estimated. Evaluations were done for credible intervals for the Weibull \(p_e = 0.01, 0.05, 0.10, \) and 0.50 quantiles.

### 9.5 Weibull Distribution Simulation Results and Conclusions

Figures 10 \((p_{\text{fail}} = 0.01 \text{ and } p_{\text{fail}} = 0.05)\) and 11 \((p_{\text{fail}} = 0.10 \text{ and } p_{\text{fail}} = 0.50)\) summarize the Weibull distribution simulation results. Section E of the appendix gives similar results for the lognormal distribution. When interpreting the results of the simulation, it is important to keep in mind that, as mentioned in Section 9.3, within each of the four plots in Figures 10 and 11, the results are based on the same set of 5,000 simulated data sets. Thus, for example, all of the points in a plot for a particular value of \(\text{E}(r)\) and estimated quantile tend to move together. Given the nominal credible level of 0.95 for the intervals, the standard error of the estimated coverage probabilities is approximately \(\sqrt{0.95(1 - 0.95)/5000} = 0.003\). Also, recall that the simulation results are conditional on observing at least three failures. The probabilities of fewer than three failures when \(\text{E}(r) = 10\) are 0.0027, 0.0023, 0.0019, and 0.0002, respectively, for \(p_{\text{fail}} = 0.01, 0.05 \text{ 0.10, and 0.50}\).

Some observations from Figures 10 and 11 are

- With a few exceptions (some of which can be explained by Monte Carlo (MC) error), the IJ priors, when compared to flat, tend to result in coverage probabilities closer to the nominal values.
Figure 10: Weibull distribution simulation results for $p_{\text{fail}} = 0.01$ on the top and $p_{\text{fail}} = 0.05$ on the bottom.
Figure 11: Weibull distribution simulation results for $p_{\text{fail}} = 0.10$ on the top and $p_{\text{fail}} = 0.50$ on the bottom.
Figure 12: Sensitivity analysis comparing different weakly informative priors for the Weibull shape parameter $\beta$.

- Again, with a few exceptions, the coverages probabilities for the IJ priors are not highly sensitive to the choice of $p_{\text{lim}}$.
- Taking into account MC error, the coverage probabilities for the IJ and flat priors are close to the 0.95 nominal credible level for $E(r) \geq 35$.
- There is a pattern in some of the exceptional cases. Particularly, exceptional cases arise when $E(r) = 10$, $p_{\text{fail}} \leq 0.10$, and $p_{\text{fail}} = p_{\text{lim}} = p_{e}$. Jeng and Meeker (2000) observed similar behavior in their simulation (particularly in the case of estimating a quantile close to the expected fraction failing when the expected number failing is small) and showed that the behavior was caused by the discreteness in the sample space resulting from the random number of failures with Type 1 censoring.

From these simulation results, and consistent with the comments at the end of Section 5.10, we conclude that the IJ prior can provide useful improvement in performance in Type 1 (and similar) applications where there will be few failures.

10 Prior Distribution Sensitivity Analysis for the Motivating Examples

As mentioned previously, with limited information in the data (e.g., a small number of failures due to censoring in reliability applications), the choice of a prior distribution could have a strong influence on inferences. When attempting to use weakly informative prior distributions, it is useful to experiment with different specifications to assess sensitivity.

The plots in Figure 12 show two different $2 \times 2$ factorial comparisons. In both cases, the baseline prior is the log-location-scale-$t$ distribution with 60 degrees of freedom, denoted by $\beta \sim <\text{LLST60}>(0.20, 25)$, which is essentially equivalent to the $\beta \sim <\text{LNORM}>(0.20, 25)$ used in the rocket motor example in Section 7.4.
The plot on the left varies the degrees of freedom from 5 to 60 (heavy versus lighter tails) and the quantile range from (0.20, 25) to (0.10, 50) (smaller to larger lognormal shape parameter). The plot on the right varies the lower endpoint of the quantile range from 0.10 to 0.20 and the upper endpoint from 25 to 50 (changing both the log-location-scale-t median and shape parameter). Note that the two plots have two common factor-level combinations.

There is not much difference among the different priors in the mid-years, where there is more information. Not surprisingly, when extrapolating beyond 16 years, the point estimates and the upper uncertainty bounds have what might be considered to be important differences. In practice, one would consider the plausible ranges for a parameter (even the narrower $\beta$ quantile range (0.20, 25) is extremely wide relative to $\beta$ values typically encountered in practice) and then perhaps choose a weakly informative prior that is a compromise or somewhat conservative.

## 11 Concluding Remarks and Areas for Future Research

This paper provides guidance for setting prior distributions for log-location-scale distributions used in reliability applications. We applied our recommendations to field data applications with complicated censoring and derived new noninformative priors that have good coverage-probability properties that will be especially useful in small-information (e.g., few failures) applications where the use of methods based on asymptotic theory can give misleading results.

The general principles and prior distributions we suggest are applicable in many other reliability models and in domains outside of reliability. For example, in accelerated testing, engineers often have useful prior information about the activation energy of a temperature-accelerated failure mode or regression coefficients in other kinds of acceleration models (but not the other parameters in the model). Xu et al. (2015) describe the use of noninformative prior distributions for accelerated test models, and their methods could be extended to the more commonly used Type 1 censoring. Bayesian methods are also being used for other types of reliability data, including degradation data and recurrent events data. Noninformative priors are needed for most, if not all, of the model parameters in such applications.

The further development and implementation (e.g., in widely available software) of default noninformative and partially informative prior distributions is important for making the advantages of Bayesian methods (described in our abstract) more accessible to practitioners. For example, after selecting a failure-time distribution, an analyst could be presented with options to override default noninformative priors and specify a weakly informative or an informative prior for each of the failure-time distribution parameters. Such functionality would allow users to easily perform a sensitivity analysis to see the effect that various noninformative/weakly informative priors have on the estimation results. Once this type of software is available, other than additional computational effort, there would be little justification for recommending non-Bayesian model fitting in reliability applications.
12 Acknowledgments

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Appendix

A Overview of the Appendix

The appendix provides derivations, proofs, additional detailed descriptions, and additional simulation results. Section B shows how to derive Jeffreys, Independence Jeffreys (IJ), and reference priors for different parameterizations. Section C describes features of the Type 1 censoring IJ priors and the reasons that those features arise. Section D describes some details of implementing a Conditional Jeffreys (CJ) or IJ prior using Stan. Section 9 in the main paper gives results for a simulation using the Weibull distribution. Simulations were also done for the lognormal distribution and they are described in Section E. Section F looks carefully at estimation results under different noninformative priors for two data sets with only three failures, based on two of the simulation factor-level combinations. G provides proofs for the limiting results for normal/lognormal weakly informative prior distributions that are described in Section 7.3. Section H gives expressions for the log-truncated and log-reciprocal-truncated pdfs that are used in our Stan implementation of truncated normal and truncated location-scale-t informative prior distributions.

B Derivations of the Noninformative Prior Distributions

B.1 Motivation for and Overview of the Derivations

Section 5.9 and Table 1 in the paper provides a summary of noninformative prior distributions (Jeffreys, IJ, and Reference) for different parameterizations of (log-)location-scale distributions. The different parameterizations are needed because

- In reliability applications, inferences are generally needed on alternative “parameters” such as quantiles or failure probabilities.
- Priors may be elicited for parameters that differ from the traditional parameters.
- MCMC computations may be conducted using a parameterization that differs from the parameterization where the prior is elicited/specified.

This section provides technical details and derivations of these prior distributions. Section B.2 shows how to compute the elements of the Fisher information matrix (FIM) for the ($\mu, \sigma$) parameterization. Section B.3 defines and gives general methods for obtaining the Jeffreys, CJ, and IJ priors for the ($\mu, \sigma$) parameterization for Type 2 and Type 1 censoring. Section B.4 defines and gives general methods for obtaining reference priors for the ($\mu, \sigma$) parameterization for Type 2 censoring. Sections B.5–B.9 derive the noninformative priors for other parameterizations.
B.2 The Fisher Information Matrix for \((\mu, \sigma)\)

The FIM plays an important role in computing noninformative priors such as Jeffreys, IJ, and reference priors. Suppose the random variable \(Y\) has a location-scale distribution with cdf

\[ G(y; \mu, \sigma) = \Phi(z), \]

and pdf \(g(y; \mu, \sigma) = \phi(z)/\sigma\), where \(z = (y - \mu)/\sigma\). \(\Phi(\cdot)\) and \(\phi(\cdot)\) are the standard cdf and pdf for the particular distribution. We use the results in Escobar and Meeker (1994). For Type 2 (failure) censoring after \(r\) out of \(n\) failures, the scaled elements of the FIM for the parameters \(\mu\) and \(\sigma\) are

\[
\begin{align*}
    f_{11}(z_c) &= \frac{\sigma^2}{n} E \left[ -\frac{\partial^2 \log(L)}{\partial \mu^2} \right] = \Psi_0(z_c), \\
    f_{12}(z_c) &= \frac{\sigma^2}{n} E \left[ -\frac{\partial^2 \log L}{\partial \mu \partial \sigma} \right] = \Psi_1(z_c), \\
    f_{22}(z_c) &= \frac{\sigma^2}{n} E \left[ -\frac{\partial^2 \log L}{\partial \sigma^2} \right] = \Psi_2(z_c),
\end{align*}
\]

(12)

where the largest \(n-r\) sample values are censored, \(\Phi^{-1}(p)\) is the \(p\)-quantile of \(\Phi(\cdot)\), and \(z_c = \Phi^{-1}(r/n)\).

Here \(L \equiv L(\mu, \sigma)\) is the likelihood function and

\[
\Psi_i(a) = \int_{-\infty}^{a} [1 + xH(x)]^i H(x)^{2-i} \phi(x) dx, \quad (i = 0, 1, 2)
\]

where

\[ H(x) = \frac{\phi'(x)}{\phi(x)} + \frac{\phi(x)}{1 - \Phi(x)}, \]

\(\phi'(x)\) is the derivative of \(\phi(x)\) and \(E\) is the expectation with respect to \(Y\). For Type 1 (time) censored data with fixed right censoring time \(y_c\), the \(f_{jk}\) (\(jk = 11, 12, 22\)) are still given by (12) but \(z_c = (y_c - \mu)/\sigma\) and the number of failures \(0 \leq r \leq n\) is random having a binomial distribution with probability \(p = \Phi(z_c)\). Then for either Type 1 or Type 2 censoring, the FIM for \((\mu, \sigma)\) is

\[
I_n(\mu, \sigma) = \frac{n}{\sigma^2} \begin{bmatrix} \Psi_0(z_c) & \Psi_1(z_c) \\ \Psi_1(z_c) & \Psi_2(z_c) \end{bmatrix} = \frac{n}{\sigma^2} \begin{bmatrix} f_{11}(z_c) & f_{12}(z_c) \\ \text{symmetric} & f_{22}(z_c) \end{bmatrix},
\]

(13)

To simplify the presentation in the remainder of this section we suppress the dependency of the \(f_{ij}\) elements on \(z_c\). That is, for example, we write \(f_{11}\) instead of \(f_{11}(z_c)\).

B.3 Jeffreys Priors

**Jeffreys Prior**

From (13), the Jeffreys prior is defined as

\[
\pi(\mu, \sigma) \propto \sqrt{|I_n(\mu, \sigma)|} = \frac{n}{\sigma^2} \sqrt{f_{11}f_{22} - f_{12}^2}.
\]

(14)
For Type 2 censoring or complete data, \( z_c \) is a known constant, so that the Jeffreys prior is \( \pi(\mu, \sigma) \propto 1/\sigma^2 \). For Type 1 censoring, \( z_c = (y_c - \mu)/\sigma \) is a function of parameters \((\mu, \sigma)\) and the Jeffreys prior is

\[
\pi(\mu, \sigma) \propto \frac{1}{\sigma} \sqrt{f_{11}f_{22} - f_{12}^2}.
\]

**Independence Jeffreys prior**

The IJ prior is the product of the CJ priors, which is obtained by computing the Jeffreys prior for each parameter, when all of the other parameters are known. For Type 2 censoring or complete data, because \( f_{11} \) and \( f_{22} \) are known constants, the CJ prior for \( \mu \) is \( \pi(\mu | \sigma) \propto \frac{1}{\sigma} \sqrt{f_{11}} \). The CJ prior for \( \sigma \) is \( \pi(\sigma | \mu) \propto \frac{1}{\sigma} \sqrt{f_{22}} \). So, the IJ prior for \((\mu, \sigma)\) under Type 2 censoring is

\[
\pi(\mu, \sigma) \propto \frac{1}{\sigma} \sqrt{f_{11}f_{22}}.
\]

**B.4 Reference Priors**

**General description**

The Jeffreys prior has good properties for one-parameter models, but has deficiencies for multi-parameter models. For example, for the Norm\((\mu, \sigma)\) distribution, the Jeffreys prior is \( \pi(\mu, \sigma) \propto 1/\sigma^2 \), which does not have the desirable properties of the right invariant prior \( \pi(\mu, \sigma) \propto 1/\sigma \), as described in Ghosh et al. (2006, Chapter 5). Also see Jeffreys (1961, pages 182–184). Bernardo (1979) proposed an alternative to the Jeffreys prior, called a reference prior. The basic idea is to find the prior that maximizes the Kullback-Leibler (KL) divergence between the prior and the expected posterior. These ideas have been further studied and extended by Berger and Bernardo (1989, 1992a,b). For a scalar parameter, the reference prior is the same as the Jeffreys prior.

For multiple parameters, in the case where all parameters are of same importance, the reference prior again leads to the Jeffreys prior (cf. Kass and Wasserman 1996, page 1350). But one can also allow parameters to have different importance. For example, for a two parameter vector \((\theta_1, \theta_2)\), \( \theta_1 \) might be of primary importance.

In this section, we employ the general methods in Ghosh et al. (2006) to derive reference priors for Type 2 censored data. We first obtain the conditional prior for \( \theta_2 \) given \( \theta_1 \), which is denoted by \( \pi(\theta_2 | \theta_1) \propto \sqrt{I_{22}(\theta)} \), where \( I_{22}(\theta) = \text{E} \left[ -\partial^2 \log \mathcal{L}(\theta)/\partial \theta_2^2 \right] \). The marginal prior for \( \theta_1 \), denoted by \( \pi(\theta_1) \) is defined as the maximizer of the KL-divergence between the prior \( \pi(\theta) = \pi(\theta_1)\pi(\theta_2 | \theta_1) \) and the expected posterior distribution. It should be noted that the computing of a reference prior, including the KL-divergence, is done over a sequence of compact sets \( K_i \) that increase in size, where the union \( \bigcup_{i=1}^{\infty} K_i \) is the parameter space. A reference prior is computed on each compact set \( K_i \), followed by a limiting operation. This approach is used to avoid improper prior distributions that would otherwise arise after the limiting operation. With two parameters, we use increasing rectangles \( K_i \).
Reference priors for Type 2 censoring

For Type 2 censoring (or complete data), we first consider the reference priors for parameter ordering $\theta_{(1)} = \mu$ and $\theta_{(2)} = \sigma$, where $\theta_{(1)}$ is more important than $\theta_{(2)}$ (later this ordering is denoted by $\{\mu, \sigma\}$). The conditional prior is $\pi_i(\sigma|\mu) \propto \sqrt{f_{22}/\sigma}$ on the rectangle $K_i = K_{1i} \times K_{2i}$. Because $\sqrt{f_{22}}$ is a constant, the conditional prior becomes $\pi_i(\sigma|\mu) = c_i/\sigma$, where $c_i$ is generic notation for a normalizing constant. The marginal prior for $\mu$ on $K_i$ that maximizes the KL-divergence is

$$
\pi_i(\mu) \propto \exp \left\{ \int_{K_{2i}} \frac{1}{2} \log [h_1(\theta)] \pi_i(\sigma|\mu) d\sigma \right\},
$$

(15)

where

$$
h_1(\theta) = \frac{|I_{11}(\mu, \sigma)|}{I_{22}(\mu, \sigma)} = \frac{1}{\sigma^2} \frac{n(f_{11}f_{22} - f_{12}^2)}{f_{22}}.
$$

(16)

So, the marginal prior for $\mu$ on rectangle $K_i$

$$
\pi_i(\mu) \propto \exp \left\{ \int_{K_{2i}} \frac{1}{2} \log \left( \text{const} \frac{c_i}{\sigma} \right) d\sigma \right\},
$$

(17)

is a constant. Thus, $\pi_i(\mu)$ is flat on $K_i$ and the reference prior for parameter ordering $\theta_{(1)} = \mu$, $\theta_{(2)} = \sigma$ is $\pi_i(\mu, \sigma) = \pi_i(\sigma|\mu)\pi_i(\mu) \propto 1/\sigma$. Therefore, the reference prior for parameter ordering $\{\mu, \sigma\}$ is $\pi(\mu, \sigma) \propto 1/\sigma$. We denote such a prior by $\pi(\{\mu, \sigma\})$ to indicate the order of parameters.

For parameter ordering $\theta_{(1)} = \sigma$ and $\theta_{(2)} = \mu$, the conditional prior is $\pi_i(\mu|\sigma) \propto \sqrt{n f_{11}}/\sigma$. Given the value of $\sigma$, the conditional prior for $\mu$ on $K_i$, which is denoted by $\pi_i(\mu|\sigma) = c_i$, is flat as $f_{11}$ is a known constant. The marginal prior for $\sigma$ on $K_i$ is

$$
\pi_i(\sigma) \propto \exp \left\{ \int_{K_{1i}} \frac{1}{2} \log [h_2(\theta)] \pi_i(\mu|\sigma) d\mu \right\},
$$

where

$$
h_2(\theta) = \frac{|I_{11}(\mu, \sigma)|}{I_{11}(\mu, \sigma)} = \frac{1}{\sigma^2} \frac{n(f_{11}f_{22} - f_{12}^2)}{f_{11}}.
$$

So, the marginal prior for $\sigma$ on rectangle $K_i$

$$
\pi_i(\sigma) \propto \exp \left\{ \int_{K_{1i}} \frac{1}{2} \log \left( \text{const} \frac{c_i}{\sigma} \right) d\mu \right\} = \frac{c_i}{\sigma}.
$$

Thus, the joint prior on the rectangle $K_i$ is $\pi_i(\mu, \sigma) = c_i/\sigma$. Then, as a limit of $\pi_i(\mu, \sigma)$, the reference prior for $\{\sigma, \mu\}$ is $\pi(\{\sigma, \mu\}) \propto 1/\sigma$.

Reference priors for Type 1 censoring

Although we could use the same procedure to compute reference priors for Type 1 censoring data, the computations are usually intractable and do not have closed forms. For Type 1 censoring, $h_1(\theta)$ is no longer proportional to $1/\sigma^2$ because $f_{11}$, $f_{12}$, and $f_{22}$ are no longer constant. Thus, the integral in (15) will not reduce to the simple integral in (17). Obtaining the analytical form of the integral in
(15) under Type 1 censoring is difficult and in the rest of this work, we only provide reference priors for Type 2 censoring (or complete) data. We will, however, provide results for Jeffreys, CJ and IJ priors for Type 1 censoring.

B.5 Using the Parameterization \((y_p, \sigma)\)

**FIM for \((y_p, \sigma)\)**

The \(p\) quantile of a location-scale random variable \(Y\) is \(y_p = \mu + \sigma \Phi^{-1}(p)\) so that \(z_c = \Phi^{-1}(r/n)\) for Type 2 censoring and \(z_c = (y_c - \mu)/\sigma = (y_c - y_p - \sigma \Phi^{-1}(p))/\sigma\) for Type 1 censoring. The large-sample approximate covariance matrices of ML estimators \((\hat{\mu}, \hat{\sigma})\) and ML estimators \((\hat{y}_p, \hat{\sigma})\) are, respectively, the inverse of the FIMs \(I_n^{-1}(\mu, \sigma)\) and \(I_n^{-1}(y_p, \sigma)\). The large-sample approximate covariance matrix for \((\hat{y}_p, \hat{\sigma})\) is \(I_n^{-1}(y_p, \sigma)\) and can be computed from the large-sample approximate covariance matrix for \((\hat{\mu}, \hat{\sigma})\) using the delta method:

\[
I_n^{-1}(y_p(\mu, \sigma), \sigma(\mu, \sigma)) = \nabla I_n^{-1}(\mu, \sigma) \nabla'
\]

\[
= \frac{\sigma^2}{n(f_{12}^2 - f_{11}f_{22})} \times \begin{bmatrix}
\Phi^{-1}(p) \left[ f_{12} - \Phi^{-1}(p)f_{11} \right] - f_{22} + \Phi^{-1}(p)f_{12} & f_{12} - \Phi^{-1}(p)f_{11} \\
\text{symmetric} & -f_{11}
\end{bmatrix},
\]

where the Jacobian \(\nabla\) is

\[
\nabla = \begin{bmatrix}
\frac{\partial y_p}{\partial \mu} & \frac{\partial y_p}{\partial \sigma} \\
\frac{\partial \sigma}{\partial \mu} & \frac{\partial \sigma}{\partial \sigma}
\end{bmatrix} = \begin{bmatrix}
1 & \Phi^{-1}(p) \\
0 & 1
\end{bmatrix}.
\]

Then the FIM for \((y_p, \sigma)\) is

\[
I_n(y_p, \sigma) = \frac{n}{\sigma^2} \begin{bmatrix}
f_{11} & f_{12} - \Phi^{-1}(p)f_{11} \\
\text{symmetric} & f_{11}[\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22}
\end{bmatrix}.
\]

(18)

**Jeffreys prior**

The determinant of the FIM (18) is

\[
|I_n(y_p, \sigma)| = \frac{n^2}{\sigma^4} (f_{11}f_{22} - f_{12}^2).
\]

Because the determinant is the same as in (14), the Jeffreys prior for \((y_p, \sigma)\) is the same as \((\mu, \sigma)\) for both Type 2 and Type 1 censoring.

**IJ prior**

For Type 2 censored or complete data, the elements of the FIM are known constants and thus as

\[
\pi(y_p|\sigma) \propto 1 \quad \text{and} \quad \pi(\sigma|y_p) \propto 1/\sigma.
\]

Then the IJ prior is \(\pi(y_p, \sigma) \propto 1/\sigma\). For Type 1 censoring, the CJ prior for \(y_p\) is

\[
\pi(y_p|\sigma) \propto \sqrt{f_{11}},
\]
and the CJ prior for $\sigma$ is
\[
\pi(\sigma|y_p) \propto \frac{1}{\sigma} \sqrt{f_{11}[\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22}},
\]
which is the same as $\pi(\sigma|\mu)$ in Section B.3 when $p$ is chosen such that $\Phi^{-1}(p) = 0$. Then the IJ prior for $(y_p, \sigma)$ is
\[
\pi(y_p, \sigma) \propto \pi(y_p|\sigma)\pi(\sigma|y_p)
\]
\[
\propto \frac{1}{\sigma} \sqrt{f_{11} \{f_{11}[\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22}\}.
\]

**Reference prior**

We consider Type 2 censored or complete data. For parameter ordering $\{y_p, \sigma\}$, the conditional Jeffreys prior for $\sigma$ is
\[
\pi(\sigma|y_p) \propto \frac{1}{\sigma}.
\]
The marginal prior for $y_p$ is
\[
\pi(y_p) \propto \exp \left\{ \int \frac{1}{2} \log \left[ \frac{|I_n(y_p, \sigma)|}{\text{const}/\sigma^2} \right] \frac{1}{\sigma} d\sigma \right\} \propto 1.
\]
Thus the reference prior for $\{y_p, \sigma\}$ is
\[
\pi(\{y_p, \sigma\}) \propto \frac{1}{\sigma}.
\]
For parameter ordering $\{\sigma, y_p\}$, the conditional Jeffreys prior is $\pi(y_p|\sigma) \propto 1$. The marginal prior for $\sigma$ is given by
\[
\pi(\sigma) \propto \exp \left\{ \int \frac{1}{2} \log \left[ \frac{I_n(y_p, \sigma)}{\text{const}/\sigma^2} \right] dy_p \right\} \propto \frac{1}{\sigma}.
\]
Thus the reference prior for $\{\sigma, y_p\}$ is
\[
\pi(\{\sigma, y_p\}) \propto \frac{1}{\sigma}.
\]

**B.6 Using the Parameterization $(t_p, \sigma)$**

The $p$ quantile of a log-location-scale random variable $T$ is $t_p = \exp(y_p) = \exp[\mu + \sigma \Phi^{-1}(p)]$.

**FIM for $(t_p, \sigma)$**

Using the delta method, the inverse of the FIM for $(t_p, \sigma)$ is
\[
\Gamma_n^{-1}(t_p, \sigma) = \nabla \Gamma_n^{-1}(\mu, \sigma) \nabla' = \frac{\sigma^2}{n(f_{12}^2 - f_{11}f_{22})} \times \begin{bmatrix}
\partial_{t_p} \Phi^{-1}(p) [f_{12} - \Phi^{-1}(p)f_{11}] - f_{22} + \Phi^{-1}(p)f_{12} & t_p [f_{12} - \Phi^{-1}(p)f_{11}]
\end{bmatrix}_{\text{symmetric}},
\]
where the Jacobian $\nabla$ is
\[
\nabla = \begin{bmatrix}
\partial_{t_p} & \partial_{t_p} \\
\partial_{\mu} & \partial_{\sigma} \\
\partial_{\mu} & \partial_{\sigma}
\end{bmatrix} = \begin{bmatrix}
t_p & t_p\Phi^{-1}(p) \\
0 & 1
\end{bmatrix}.
\]
Then the FIM for \((t_p, \sigma)\) is

\[
I_n(t_p, \sigma) = \frac{n}{\sigma^2} \begin{bmatrix}
  f_{11}/t_p^2 & [f_{12} - f_{11}\Phi^{-1}(p)]/t_p \\
  \text{symmetric} & f_{11}\Phi^{-1}(p) - 2f_{12}\Phi^{-1}(p) + f_{22}
\end{bmatrix}.
\]

**Jeffreys prior**

The determinant of the FIM is

\[
\det[I_n(t_p, \sigma)] = n^2 \frac{(f_{11}f_{12} - f_{12}^2)}{t_p^2\sigma^4}.
\]

For Type 2 censoring, the Jeffreys prior is

\[
\pi(t_p, \sigma) \propto \frac{1}{t_p\sigma^2}.
\]

For Type 1 censoring, the Jeffreys prior is

\[
\pi(t_p, \sigma) \propto \frac{1}{t_p\sigma^2} \sqrt{f_{11}/t_p}.
\]

**IJ prior**

The CJ prior for \(t_p\) given \(\sigma\) is \(\pi(t_p|\sigma) \propto \sqrt{f_{11}/(t_p\sigma)}\) and that for \(\sigma\) given \(t_p\) is

\[
\pi(\sigma|t_p) \propto \frac{1}{\sigma} \sqrt{(f_{11}\Phi^{-1}(p) - 2f_{12}\Phi^{-1}(p) + f_{22})}.
\]

For Type 2 censoring, we have \(\pi(t_p|\sigma) \propto 1/t_p\) and \(\pi(\sigma|t_p) \propto 1/\sigma\). Then the IJ prior is

\[
\pi(t_p, \sigma) \propto \frac{1}{t_p\sigma}.
\]

For Type 1 censoring, we have \(\pi(t_p|\sigma) \propto \sqrt{f_{11}/t_p}\) and

\[
\pi(\sigma|t_p) \propto \frac{1}{\sigma} \sqrt{(f_{11}\Phi^{-1}(p) - 2f_{12}\Phi^{-1}(p) + f_{22})}.
\]

Then the IJ prior is

\[
\pi(t_p, \sigma) \propto \frac{1}{t_p\sigma} \sqrt{f_{11}[f_{11}\Phi^{-1}(p) - 2f_{12}\Phi^{-1}(p) + f_{22}]}.
\]

**Reference prior**

We only consider complete or Type 2 censored data. For \(\{t_p, \sigma\}\), the conditional Jeffreys prior for \(\sigma\) is \(\pi(\sigma|t_p) \propto 1/\sigma\). Then the marginal prior for \(t_p\) is

\[
\pi(t_p) = \exp \left\{ \frac{1}{2} \log \left[ \frac{1/(t_p^2\sigma^4)}{1/\sigma^2} \right] \frac{1}{\sigma} d\sigma \right\} \propto \frac{1}{t_p}.
\]

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The reference prior for \( \{t_p, \sigma\} \) is
\[
\pi(\{t_p, \sigma\}) \propto \frac{1}{t_p \sigma}.
\]

For \( \{\sigma, t_p\} \), the conditional Jeffreys prior for \( t_p \) is \( \pi(t_p|\sigma) \propto 1/t_p \). The marginal prior for \( \pi(\sigma) \) is
\[
\pi(\sigma) \propto \exp \left\{ \frac{1}{2} \int \log \left[ \frac{1/(t_p^2 \sigma^4)}{1/(t_p^2 \sigma^2)} \right] dt_p \right\} \propto \frac{1}{\sigma}.
\]

The reference prior for \( \pi(\{\sigma, t_p\}) \) is \( \pi(\{\sigma, t_p\}) \propto \frac{1}{t_p \sigma} \).

### B.7 Using the Parameterization \((y_p, \log(\sigma))\)

As described in Section 5.3 of the paper, \((y_p, \log(\sigma))\) is the parameterization that we used for our MCMC computations.

#### FIM for \((y_p, \log(\sigma))\)

To simplify the notation in the derivation, let \( \tau \equiv \log(\sigma) \). Then the inverse of the FIM for \((y_p, \tau)\) is
\[
I_n^{-1}(y_p, \tau) = \nabla I_n^{-1}(\mu, \sigma) \nabla'
= \frac{1}{n [f_{12}^2 - f_{11} f_{22}]} \times \left[ \begin{array}{cc} \sigma^2 \{ \Phi^{-1}(p) [f_{12} - \Phi^{-1}(p) f_{11}] - f_{22} + \Phi^{-1}(p) f_{12} \} & \sigma [f_{12} - \Phi^{-1}(p) f_{11}] \\ \text{symmetric} & -f_{11} \end{array} \right],
\]
where the Jacobian is
\[
\nabla = \left[ \begin{array}{cc} \frac{\partial y_p}{\partial \mu} & \frac{\partial y_p}{\partial \sigma} \\ \frac{\partial y_p}{\partial \tau} & \frac{\partial y_p}{\partial \sigma} \end{array} \right] = \left[ \begin{array}{cc} 1 & \Phi^{-1}(p) \\ 0 & 1/\sigma \end{array} \right].
\]

Thus, the FIM for \((y_p, \tau)\) is
\[
I_n(y_p, \tau) = n \left[ \begin{array}{cc} f_{11}/\sigma^2 & [f_{12} - \Phi^{-1}(p) f_{11}] / \sigma \\ \text{symmetric} & f_{11} [\Phi^{-1}(p)]^2 - 2 f_{12} \Phi^{-1}(p) + f_{22} \end{array} \right] = n \left[ \begin{array}{cc} f_{11}/\exp(2\tau) & [f_{12} - \Phi^{-1}(p) f_{11}] / \exp(\tau) \\ \text{symmetric} & f_{11} [\Phi^{-1}(p)]^2 - 2 f_{12} \Phi^{-1}(p) + f_{22} \end{array} \right].
\]

#### Jeffreys prior

The determinant of the FIM is
\[
|I_n(y_p, \tau)| = \frac{n^2}{\exp(2\tau)} (f_{11} f_{22} - f_{12}^2) = \frac{n^2 \sigma^2}{\sigma^2} (f_{11} f_{22} - f_{12}^2).
\]
Then, for Type 2 censoring, the Jeffreys prior is $\pi(y_p, \tau) \propto 1/\sigma$ because the $f_{ij}$ elements are fixed constants. This result also can be obtained directly from Section B.5 using the fact that the Jeffreys prior is invariant to parameter transformation. For example, we already know that $\pi(y_p, \sigma) \propto 1/\sigma^2$. Then the Jeffreys prior for parameters $(y_p, \tau)$ can be computed as

$$
\pi(y_p, \tau) \propto \frac{1}{\exp(2\tau)} \left| \begin{bmatrix} \frac{\partial y_p}{\partial y_p} & \frac{\partial y_p}{\partial \tau} \\ \frac{\partial y_p}{\partial \sigma} & \frac{\partial \sigma}{\partial \tau} \end{bmatrix} \right| = \frac{1}{\exp(2\tau)} \left| \begin{bmatrix} 1 & 0 \\ 0 & \exp(\tau) \end{bmatrix} \right| = \frac{1}{\exp(\tau)} = \frac{1}{\sigma}.
$$

For Type 1 censoring, the Jeffreys prior is $\pi(y_p, \tau) \propto \exp(\tau) \sqrt{f_{11}f_{22} - f_{12}^2} = \frac{1}{\sigma} \sqrt{f_{11}f_{22} - f_{12}^2}.

**IJ prior**

For Type 2 censoring, the CJ prior for $y_p$ given $\tau$ is $\pi(y_p|\tau) \propto 1$; the CJ prior for $\tau$ given $y_p$ is $\pi(\tau|y_p) \propto 1$. Thus the IJ prior is $\pi(y_p, \tau) \propto 1$. For Type 1 censoring, the CJ prior for $y_p$ given $\tau$ is $\pi(y_p|\tau) \propto \sqrt{f_{11}}$; the CJ prior for $\tau$ given $y_p$ is

$$
\pi(\tau|y_p) \propto \sqrt{f_{11}} \left\{ f_{11} [\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22} \right\}.
$$

So, the IJ prior is

$$
\pi(y_p, \tau) \propto \pi(y_p|\tau)\pi(\tau|y_p)
\propto \sqrt{f_{11}} \left\{ f_{11} [\Phi^{-1}(p)]^2 - 2f_{12}\Phi^{-1}(p) + f_{22} \right\}.
$$

**Reference prior**

For Type 2 censoring or complete data, the reference prior for parameter ordering $\{y_p, \tau\}$ is computed by first deriving the conditional Jeffreys prior $\tau$

$$
\pi(\tau|y_p) \propto 1;
$$

the marginal prior for $y_p$ is

$$
\pi(y_p) \propto \exp \left\{ \frac{1}{2} \int \log \left[ \frac{1/\exp(2\tau)}{1} \right] d\tau \right\} \propto 1.
$$

Thus the reference prior for $\{y_p, \tau\}$ is given by

$$
\pi(\{y_p, \tau\}) \propto 1.
$$
For parameter ordering \{\tau, y_p\}, the conditional Jeffreys prior for \(y_p\) is \(\pi(y_p|\tau) \propto 1\) and the marginal prior for \(\tau\) is

\[
\pi(\tau) \propto \exp \left\{ \frac{1}{2} \int \log \left[ \frac{1}{\exp(2\tau)} \right] \pi(y_p|\tau) dy_p \right\} \propto 1.
\]

The reference prior for \(\{\tau, y_p\}\) is \(\pi(\{\tau, y_p\}) \propto 1\).

### B.8 Using the Parameterization \((\zeta_e, \sigma)\)

#### FIM for \((\zeta_e, \sigma)\)

Let \(\zeta_e = (y_e - \mu)/\sigma\), where \(y_e > 0\) is a given value. The parameter \(\zeta_e\) is important because inferences are often needed for \(\Pr(Y \leq y_e) = \Phi(\zeta_e)\). For Type 2 censoring, \(z_c = \Phi^{-1}(r/n)\); for Type 1 censoring, \(z_c = (y - y_e + \sigma \zeta_e)/\sigma\). The inverse of the FIM for \((\zeta_e, \sigma)\) is

\[
I_n^{-1}[\zeta_e(\mu, \sigma), \sigma(\mu, \sigma)] = \nabla I_n^{-1}(\mu, \sigma) \nabla'
\]

where the Jacobian \(\nabla\) is

\[
\nabla = \begin{bmatrix}
\frac{\partial \zeta_e}{\partial \mu} & \frac{\partial \zeta_e}{\partial \sigma}
\end{bmatrix} = \begin{bmatrix}
-\frac{1}{\sigma} & -\zeta_e
\end{bmatrix}.
\]

By replacing \((\mu, \sigma)\) with \((\zeta_e, \sigma)\) in \(I_n[\zeta_e(\mu, \sigma), \sigma(\mu, \sigma)]\), the FIM for \((\zeta_e, \sigma)\) is

\[
I_n(\zeta_e, \sigma) = \frac{n}{\sigma^2} \begin{bmatrix}
\sigma^2 f_{11} & \sigma(f_{11} \zeta_e - f_{12}) \\
\text{symmetric} & f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}
\end{bmatrix}.
\]  

#### Jeffreys prior

The determinant of the FIM (19) is

\[
|I_n(\zeta_e, \sigma)| = \frac{n^2}{\sigma^2} \left( f_{22}f_{11} - f_{12}^2 \right).
\]

For Type 2 censoring or complete data, the Jeffreys prior is \(\pi(\zeta_e, \sigma) \propto 1/\sigma\). For Type 1 censoring, the Jeffreys prior is

\[
\pi(\zeta_e, \sigma) \propto \frac{1}{\sigma} \sqrt{f_{22}f_{11} - f_{12}^2}.
\]

### IJ prior

For Type 2 censoring using the appropriate elements of (19), the CJ prior for \(\zeta_e\) given \(\sigma\) is \(\pi(\zeta_e|\sigma) \propto 1\); the prior for \(\sigma\) given \(\zeta_e\) is \(\pi(\sigma|\zeta_e) \propto 1/\sigma\). So, the IJ prior for \((\zeta_e, \sigma)\) is \(\pi(\zeta_e, \sigma) \propto 1/\sigma\). For Type 1 censoring, the CJ prior for \(\zeta_e\) given \(\sigma\) is \(\pi(\zeta_e|\sigma) \propto \sqrt{f_{12}}\) and the CJ prior for \(\sigma\) given \(\zeta_e\) is \(\pi(\sigma|\zeta_e) \propto \sqrt{f_{12}}\).
\[
(1/\sigma)\sqrt{f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22}}. \quad \text{So, the IJ prior for } (\zeta_e, \sigma) \text{ is}
\]
\[
\pi(\zeta_e, \sigma) \propto \frac{1}{\sigma}\sqrt{f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22}}.
\]

**Reference prior**

We consider the reference prior for Type 2 censoring and complete data. For parameter ordering \(\theta_1 = \zeta_e\) and \(\theta_2 = \sigma\), the conditional prior on \(K_i = K_{1i} \times K_{2i}\) is \(\pi(\sigma|\zeta_e) \propto 1/\sigma\) and it denoted by \(\pi_i(\sigma|\zeta_e) = c_i/\sigma\). Then the marginal prior for \(\zeta_e\) is
\[
\pi_i(\zeta_e) = c_i \left(\int_{K_{1i}} \frac{1}{2} \log \left| \frac{L_n(\zeta_e, \sigma)}{f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22}}\right| \frac{c_i}{\sigma} d\sigma \right)
\]
\[
= \frac{c_i}{\sqrt{f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22}}}
\]

Here \(c_i\) is the generic notation for a normalizing constant. Thus, the reference prior for \(\{\zeta_e, \sigma\}\) is
\[
\pi(\{\zeta_e, \sigma\}) \propto \frac{1}{\sigma\sqrt{f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22}}}
\]

For parameter ordering \(\{\sigma, \zeta_e\}\), the conditional prior on \(K_i = K_{1i} \times K_{2i}\) is \(\pi(\zeta_e|\sigma) \propto c_i\). The marginal prior for \(\zeta_e\) is
\[
\pi_i(\sigma) \propto \exp \left\{\int_{K_{1i}} \frac{1}{2} \log \left| \frac{L_n(\zeta_e, \sigma)}{f_{11}}\right| c_i d\zeta_e \right\} \propto \frac{1}{\sigma}
\]

So, the corresponding reference prior is \(\pi(\{\sigma, \zeta_e\}) \propto 1/\sigma\).

**B.9 Using the Parameterization** \((\zeta_e, \log(\sigma))\)

**FIM for** \((\zeta_e, \log(\sigma))\)

We define \(\tau = \log(\sigma)\); then the inverse of the FIM for \((\zeta_e, \tau)\) is
\[
I_n^{-1}[\zeta_e, \tau(\sigma)] = \nabla I_n^{-1}(\zeta_e, \sigma) \nabla'
\]
\[
= \frac{1}{n[f_{12} - f_{11}f_{22}]} \times \begin{bmatrix}
\zeta_e[f_{12} - \zeta_e f_{11}] - f_{22} + \zeta_e f_{11} & \zeta_e f_{11} - f_{12} \\
\text{symmetric} & -f_{11}
\end{bmatrix}
\]

where
\[
\nabla = \begin{bmatrix} 1 & 0 \\ 0 & 1/\sigma \end{bmatrix}
\]

Then the FIM for \((\zeta_e, \tau)\) is
\[
I_n(\zeta_e, \tau) = n \begin{bmatrix} f_{11} & \zeta_e f_{11} - f_{12} \\
\text{symmetric} & f_{11}\zeta_e^2 - 2f_{12}\zeta_e + f_{22} \end{bmatrix}
\]
Jeffreys prior

The determinant of the FIM in (20) is

\[ |I_n(\zeta_e, \tau)| = n^2 \left[ f_{11} f_{22} - f_{12}^2 \right]. \]

For Type 2 censoring or complete data, the Jeffreys prior is \( \pi(\zeta_e, \tau) \propto 1 \). For Type 1 censoring data, the Jeffreys prior is

\[ \pi(\zeta_e, \tau) \propto \sqrt{f_{11} f_{22} - f_{12}^2}. \]

IJ prior

For Type 2 censoring or complete data using the appropriate elements of (20), the CJ prior for \( \zeta_e \) is \( \pi(\zeta_e \mid \tau) \propto 1 \) and the CJ prior for \( \tau \) is \( \pi(\tau \mid \zeta_e) \propto 1 \); thus the IJ prior is \( \pi(\zeta_e, \tau) \propto 1 \). For Type 1 censoring, the prior for \( \zeta_e \) is \( \pi(\zeta_e \mid \tau) \propto \sqrt{f_{11}} \) and the prior for \( \tau \) is \( \pi(\tau \mid \zeta_e) \propto \sqrt{f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}} \). Then the IJ prior is

\[ \pi(\zeta_e, \tau) \propto \sqrt{f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}}. \]

Reference prior

We consider the reference prior for Type 2 or complete data. For parameter ordering \( \{\zeta_e, \tau\} \), the CJ prior for \( \tau \) is \( \pi(\tau \mid \zeta_e) \propto 1 \) and the marginal distribution for \( \zeta_e \) is

\[ \pi(\zeta_e) \propto \exp \left\{ \frac{1}{2} \int \log \left[ \frac{1}{f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}} \right] d\tau \right\} \propto \frac{1}{\sqrt{f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}}}. \]

Thus the reference prior for parameter ordering \( \{\zeta_e, \tau\} \) is

\[ \pi(\{\zeta_e, \tau\}) \propto \frac{1}{\sqrt{f_{11} \zeta_e^2 - 2f_{12} \zeta_e + f_{22}}}. \]

For \( \{\tau, \zeta_e\} \), the CJ prior for \( \zeta_e \) is \( \pi(\zeta_e \mid \tau) \propto 1 \) and the marginal prior for \( \tau \) is given by

\[ \pi(\tau) \propto \exp \left\{ \frac{1}{2} \int \log \left[ \frac{\|I\|}{n f_{11}} \right] d\zeta_e \right\} \propto 1. \]

Then the reference prior for \( \{\tau, \zeta_e\} \) is \( \pi(\{\tau, \zeta_e\}) \propto 1 \).

B.10 Proof of invariance of IJ priors to one-to-one reparameterizations

Here we show that IJ priors for log-location-scale distributions are invariant (in the sense described in Section 5.5 of the paper) to one-to-one monotone reparameterizations of either or both of the \( (t_p, \sigma) \) parameters.

Suppose that the original parameters are \( (t_p, \sigma) \). The new parameters are \( \theta_1 = A(t_p) \) and
\( \theta_2 = B(\sigma) \) where \( A \) and \( B \) are one-to-one monotone transformations. Denote the derivatives by

\[
a = \frac{d\theta_1}{dt_p} \quad \text{and} \quad b = \frac{d\theta_2}{d\sigma}.
\]

Here \( a \) is written as a function of \( \theta_1 \) as \( a = a(\theta_1) \) and \( b \) is written as a function of \( \theta_2 \) as \( b = b(\theta_2) \).

The inverse of the FIM (large-sample approximate covariance matrix) under parameterization \((t_p, \sigma)\) is (from Section B.6)

\[
I^{-1}(t_p, \sigma) = \frac{\sigma^2}{n(f_{12}^2 - f_{11}f_{22})} \times \begin{bmatrix}
    t_p^2 \{\Phi^{-1}(p) [f_{12} - \Phi^{-1}(p)f_{11}] - f_{22} + \Phi^{-1}(p)f_{12}\} & t_p [f_{12} - \Phi^{-1}(p)f_{11}] \\
    \text{symmetric} & -f_{11}
\end{bmatrix}.
\]

Using the delta method, the inverse FIM (large-sample approximate covariance matrix) for the \((\theta_1, \theta_2)\) parameterization

\[
I^{-1}(\theta_1, \theta_2) = \frac{[\sigma(\theta_2)]^2}{n(f_{12}^2 - f_{11}f_{22})} \times \nabla \begin{bmatrix}
    [t_p(\theta_1)]^2 \{\Phi^{-1}(p) [f_{12} - \Phi^{-1}(p)f_{11}] - f_{22} + \Phi^{-1}(p)f_{12}\} & t_p(\theta_1) [f_{12} - \Phi^{-1}(p)f_{11}] \\
    \text{symmetric} & -f_{11}
\end{bmatrix} \nabla',
\]

where

\[
\nabla = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}.
\]

Then the FIM for the \((\theta_1, \theta_2)\) parameterization is

\[
I(\theta_1, \theta_2) = \frac{n}{[t_p(\theta_1)]^2 [\sigma(\theta_2)]^2 a^2 b^2} \times \begin{bmatrix}
    b^2 f_{11} & ab \sigma(\theta_2) [f_{12} - \Phi^{-1}(p)f_{11}] \\
    \text{symmetric} & -a^2 [t_p(\theta_1)]^2 \{\Phi^{-1}(p) [f_{12} - \Phi^{-1}(p)f_{11}] - f_{22} + \Phi^{-1}(p)f_{12}\}
\end{bmatrix}
\]

For the CJ priors, we have

\[
\pi(\theta_1|\theta_2) \propto \frac{1}{at_p(\theta_1)} \quad \text{and} \quad \pi(\theta_2|\theta_1) \propto \frac{1}{b\sigma(\theta_2)}.
\]

So, the IJ prior is

\[
\pi_1(\theta_1, \theta_2) \propto \frac{1}{t_p(\theta_1)\sigma(\theta_2)} \frac{1}{ab}.
\]

To prove that the IJ prior is invariant to the transformations, we can first perform the variable transformation on the IJ prior using \((t_p, \sigma)\) and show that the prior after transformation is the same as \(\pi_1(\theta_1, \theta_2)\). From Section B.6, the IJ prior using \((t_p, \sigma)\) is \(1/(t_p \sigma)\). First replace \(t_p\) and \(\sigma\) with \(t_p(\theta_1)\) and \(\sigma(\theta_2)\), then finish the transformation with the Jacobian

\[
\pi_2(\theta_1, \theta_2) \propto \frac{1}{t_p(\theta_1)\sigma(\theta_2)} |J|,
\]

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where

\[ |J| = \begin{vmatrix} \frac{\partial t_p(\theta_1)}{\partial \theta_1} & \frac{\partial t_p(\theta_1)}{\partial \theta_2} \\ \frac{\partial \sigma(\theta_1)}{\partial \theta_1} & \frac{\partial \sigma(\theta_2)}{\partial \theta_2} \end{vmatrix} = \frac{1}{|J^*|}, \]

and

\[ J^* = \begin{bmatrix} \frac{\partial \theta_1}{\partial t_p} & \frac{\partial \theta_1}{\partial \sigma} \\ \frac{\partial \theta_2}{\partial t_p} & \frac{\partial \theta_2}{\partial \sigma} \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad \text{and} \quad |J^*| = ab. \]

Thus

\[ \pi_2(\theta_1, \theta_2) \propto \frac{1}{t_p(\theta_1)\sigma(\theta_2)} \frac{1}{ab}, \]

giving the desired result.

### C Understanding the Type 1 Censoring Independence Jeffreys Prior Distribution

#### C.1 Features of the Type 1 Censoring Independence Jeffreys Prior Distribution

Figures 13, 14, and 15 show contour plots of the Type 1 censoring IJ prior densities in (11) of the main paper for various values of \( p_r \) and \( p_{\text{lim}} \). That is,

\[ \pi(\log(t_{pr}), \log(\sigma)) \propto \sqrt{f_{11} \{ f_{11}[\Phi^{-1}(p_{\text{lim}})]^2 - 2f_{12}\Phi^{-1}(p_{\text{lim}}) + f_{22} \}} \]

where the \( f_{ij} \) values are scaled elements of the FIM defined in Section 4.1 of the main paper. These scaled elements depend on the standardized censoring time \( z_c = [\log(t_c) - \mu]/\sigma \) or the expected fraction failing \( p_c = \Phi(z_c) \). To make it easier to compare across different inputs, the densities in Figures 13, 14, and 15 were scaled to have a maximum of 1.0 and thus we refer to them as relative densities.

We use the lognormal distribution for the examples (because of the well-known interpretation of the usual \((\mu, \sigma)\) parameters and the median \( t_{0.50} = \exp(\mu) \). The results are largely similar for the Weibull distribution when presented in the \((t_{pr}, \sigma)\) parameterization (where the Weibull shape parameter is \( \beta = 1/\sigma \)).

Without any important loss of generality, all of these examples given here used \( t_c = 135 \). Note that the density is computed as a function of \( \log(t_{pr}) \) and \( \log(\sigma) \) but presented in the plots (for the sake of easier interpretation) using log axes for \( t_{pr} \) and \( \sigma \).

The prior joint relative densities in Figures 13, 14, and 15 have some common features. In particular,

- For any given value of \( \sigma \), \( \pi(\log(t_{pr}), \log(\sigma)) \) is a decreasing function of \( t_{pr} \).
- For small \( \sigma \) and \( t_{pr} < t_c \), the relative density is approximately flat at a level of 1.0.
• Following from the previous point, the joint densities are improper in the sense that the area under the density is infinite.

• For small $\sigma$ and $t_{pr} > t_c$, the level of the density is, relative to the flat region, negligible, for all values of $p_r$ and $p_{lim}$.

• Following from the features in the previous two points, there is a steep cliff as one crosses $t_c$ for small values of $\sigma$.

• For large values of $\sigma$ the level of the density is approximately flat, but at a level that is less than 1. The level in this region increases with increasing values of $p_r$ and is usually (but not always) increasing in $p_{lim}$.

• With $p_r = 0.99$ (or larger) the density is approximately flat except in the “Negligible density” region of the parameter space.

The reasons for these features are given in Section C.2.

C.2 Reasons for the Type 1 Censoring Independence Jeffreys Prior Distribution Features

Here we explore the reasons for the different IJ prior density shapes and features in Figures 13, 14, and and 15. Figure 16 shows the probability of having one or more failures in a Type 1-censored life test as a function the parameters $t_{pr}$ and $\sigma$ and sample size $n$. This binomial probability can be computed as

$$Pr(r > 0) = 1 - Pr(r = 0) = 1 - [1 - Pr(T < t_c)]^n$$

where

$$Pr(T < t_c) = \Phi_{\text{norm}} \left[ \frac{\log(t_c) - \mu}{\sigma} \right] = \Phi_{\text{norm}} \left[ \frac{\log(t_c) - \log(t_{pr})}{\sigma} \right] + \Phi_{\text{norm}}^{\text{-1}}(p_r).$$  \hspace{1cm} (21)$$

Figure 16 shows that the negligible-density region of the IJ prior densities in Figures 13, 14, and 15 corresponds to that part of the parameter space where the probability of having a failure is negligible.

Every point in the $(t_{pr}, \sigma)$ parameter space corresponds to a particular lognormal cdf. Figure 17 provides a visualization of the relationship for the particular case of $p_r = 0.10$ and $p_{lim} = 0.10$. The horizontal lines in the probability plots on the right are at $p_r = 0.10$. The vertical lines are at the censoring time $t_c = 135$. The horizontal position of the colored squares in the density plots on the left indicate the value of $t_{pr} = 80, 90, 100, 120, 135, 160,$ and 180, controlling the horizontal location of the cdf in the plots on the right. The vertical position of the squares indicate the value of $\sigma = 0.02, 0.4,$ and 2.0, which controls the slopes of the cdfs on the lognormal scales on the right of Figure 17 (the slope
Figure 13: IJ prior densities $p_r = 0.01$ (on the left) and $p_r = 0.05$ (on the right) for $p_{\text{fim}} = 0.01$ (top), 0.1 (middle), and 0.50 (bottom).
Figure 14: IJ prior densities $p_r = 0.10$ (on the left) and $p_r = 0.25$ (on the right) for $p_{\text{lim}} = 0.01$ (top), 0.1 (middle), and 0.50 (bottom).
Figure 15: IJ prior densities \( p_r = 0.50 \) (on the left) and \( p_r = 0.99 \) (on the right) for \( p_{\text{lim}} = 0.01 \) (top), 0.1 (middle), and 0.50 (bottom).
of the lognormal cdf is $1/\sigma$ on the linear axes underlying lognormal probability scales, as described in Meeker et al. (2022, Section 6.2.3)).

In the top-right probability plot in Figure 17, consider, for example, the cdf on the far right with $t_{pr} = 180$ and $\sigma = 0.02$ (corresponding to the red square on the far right in the top-left contour plot). Substituting these parameter values, $p_r = 0.10$, and $t_c = 135$ into (21) gives

$$\Pr(T < t_c) = \Phi_{\text{norm}}\left[\frac{\log(135) - \log(180)}{0.02} + \Phi^{-1}_{\text{norm}}(0.10)\right] = 1.298 \times 10^{-55}.$$ 

If $t_{pr} = 135$ (corresponding to the cyan square in the top-left contour plot), by definition of the distribution quantile, $\Pr(T < t_c) = 0.10$ (for any value of $\sigma$) and if $t_{pr} = 120$ and $\sigma = 0.02$ (corresponding to the green square in the top-left contour plot), $\Pr(T < t_c) = 0.999997$, showing the steepness of the cliff.

D Implementation of and Experiences Using Stan with a CJ or an IJ Prior Distribution

Section 5.6 of the main paper gives expressions for the CJ priors for both $y_p = \log(t_p)$ and $\log(\sigma)$ as well as the the IJ priors. These are easy to compute given the algorithms to compute the FIM elements described in Section 4 and could be used in conjunction with a standard MCMC method like the Metropolis–Hastings algorithm. In our work, we used two alternative methods—one based on rstan (Stan Development Team, 2020) and another that uses a simple rejection method described in Smith and Gelfand (1992). The rest of this section describes some implementation details and our experiences.

D.1 Implementation Using Rstan

We did not see a way to compute FIM elements within a Stan model. As an alternative, we sent down vectors (length 200 was the default) of values and then used a Stan-model function to compute the needed FIM element values with linear interpolation. Expressions for the IJ prior density were programmed directly in the model block of the Stan model (as if it were part of the likelihood function).

These codes were exercised extensively in the running of our simulation study. For some factor-level combinations we noticed excessively long run times (often with many treedepth exceedences) for some data sets. The root cause was found to be extremely small stepsize values obtained from the adaptation stage of the Stan NUTS sampler for an IJ prior and a small number of failures (e.g., fewer than ten). Because we never saw this behavior with flat priors (even with only three failures), we suspect that the problems arose because of our non-differentiable piecewise-linear approximation to the FIM elements. We avoided the problems by disabling the NUTS adaptation and specifying a stepsize. After some experimentation we found that, for a given data set, the flat prior stepsize divided by 100 worked well (i.e., fast sampling with no warnings).
Figure 16: Probability of one or more failures before the the censoring time $t_c = 135$ for sample sizes $n = 100$ (on the left) and $n = 1000$ (on the right) for reparameterization quantile $p_r = 0.01$ (top), 0.10 (middle), and 0.50 (bottom).
Figure 17: IJ prior densities with points in the parameter space indicated by the squares (on the left) and Lognormal cdfs on lognormal probability scales for the different points (on the right) for \( \sigma = 0.02 \) (top), 0.4 (middle), and 2.0 (bottom).
Section F.2 looks closely at estimation results for a data set with three failures from a sample of size 1000. The likelihood for these data has an interesting funnel shape with a sharp point at the top (Weibull) or bottom (lognormal). When running the Stan NUTS sampler, some divergent transitions were observed, although they were not concentrated in the tip. Changing the default adapt_delta to 0.995 eliminates the divergent transitions.

D.2 Implementation of Sampling from IJ/CJ Priors for Bayes Without Tears Plots

Smith and Gelfand (1992) describe a simple accept/reject posterior sampling method that accepts points from the prior with a probability equal to the value of the relative likelihood at the point. As illustrated in Figures 8 and 9 (also see the numerous plots in Section F), comparing plots of prior points and likelihood contours with a similar plot of posterior points and likelihood contours provides insight into how the prior and likelihood combine to produce a posterior distribution.

It is easy to compute values of the IJ prior using the expressions in Section 5.6 used, for example, to compute the contours in Figures 13–15 of the appendix. However, we saw no simple way to sample from these improper priors. Instead we used Stan to sample from the prior density, in a manner similar to what we describe in Section D.1, but with no data contributing to the likelihood. Because the Stan NUTS sampler cannot be used to sample from an improper distribution, sampling was constrained to a large rectangle (much larger than the boundaries of the contour plots where the draws are plotted) so that the resulting prior distributions are proper.

E Lognormal Distribution Simulation Results and Conclusions

Section 9.5 of the main paper presents the simulation results, evaluating the credible interval coverage probabilities using different noninformative priors for the Weibull distribution. Figures 18 and 19 summarize the results of the simulation using the lognormal distribution.

Similar to the results in Section 9.5, observations from Figures 18 and 19 are

• With a few exceptions the IJ priors, when compared to flat, tend to result in coverage probabilities closer to the nominal values.

• Again, with a few exceptions, the coverages probabilities for the IJ priors are not highly sensitive to the choice of $p_{\text{lim}}$.

• Taking into account MC error, the coverage probabilities for the IJ and flat priors are close to the 0.95 nominal credible level for $E(r) \geq 35$.

• The pattern for exceptional cases when $E(r) = 10$ and $p_{\text{fail}} = p_{\text{lim}} = p_e$ is still present, but not as strong.
Figure 18: Lognormal distribution simulation results for $p_{\text{fail}} = 0.01$ on the top and $p_{\text{fail}} = 0.05$ on the bottom.
Figure 19: Lognormal distribution simulation results for $p_{fail} = 0.10$ on the top and $p_{fail} = 0.50$ on the bottom.
F A Careful Look at Examples with Three Failures

This section looks carefully at Weibull and lognormal Bayesian estimation results for two data sets with only three failures under three different noninformative priors. The two data sets were chosen based on the two extreme simulation factor-level combinations described in Section 9.3 (sample sizes 20 and 1000).

In some examples, the Bayesian estimate of $F(t)$ does not appear to agree well with the non-parametric estimates. With Type 1 censoring, to a very high degree of approximation, the Weibull ML estimate at the censoring time is $\hat{F}(t_c) \approx r/n$ (Escobar, 2010). Thus there is an invisible pseudo data point at $(t_c, r/n)$ and this point is indicated in the probability plots in this section with the symbol □. This pseudo data point has more influence than the other visible data points (smaller order statistics have more variability). When the Bayesian estimate of $F(t)$ does not agree well with the points, this pseudo data point will help explain why. We also did experiments to assure that our conclusions in the section are not sensitive to the location of the three failure times before the censoring time.

The different point colors in the contour plots correspond to the four different MCMC chains that were used in generating the points. Recall that, as described in Section D.2 draws from an IJ/CJ prior are generated by using Stan. Clusters of one color would indicate problems with the NUTS sampler, but we do not see any such problems.

F.1 Example of Three Failures from a Sample of Size of Twenty

The results in this section correspond to the factor-level combination E($r$) = 10 and $p_{\text{fail}} = 0.50$ resulting in a sample of size $n = 20$ and a censoring time of 1. To illustrate an extreme, a simulated sample resulting in three failures was chosen. The failures were at times 0.414, 0.586, and 0.684. The value of $p_r$ for reparameterization was chosen to be $(3/20)/2 = 0.075$, resulting in likelihoods that are reasonably well behaved (except for the funnel shape with a sharp point). Figures 20–22 give results for flat, IJ with $p_{\text{lim}} = 0.01$, and IJ with $p_{\text{lim}} = 0.50$ priors, respectively.

F.2 Example of Three Failures from a Sample of Size of One Thousand

The results in this section correspond to the factor-level combination E($r$) = 10 and $p_{\text{fail}} = 0.01$ resulting in a sample of size $n = 1000$ and a censoring time of 0.0977. To illustrate an extreme, a simulated sample resulting in three failures was chosen. The failures were at times 0.0526, 0.0825, and 0.0836. The value of $p_r$ for reparameterization was chosen to be $(3/1000)/2 = 0.015$, resulting in likelihoods that are reasonably well behaved (except for the funnel shape with a sharp point). Figures 23–25 give results for flat, IJ with $p_{\text{lim}} = 0.01$, and IJ with $p_{\text{lim}} = 0.50$ priors, respectively.
Figure 20: Bayesian estimation results for data with $r = 3$ and $n = 20$ using the Weibull distribution (on the left) and the lognormal distribution (on the right) showing a the estimate of $F(t)$ on a probability plot (top), draws from the bounded joint prior and likelihood contours (middle), and posterior draws and likelihood contours (bottom) for a flat prior.

### G Proofs of Limiting Results for Weakly Informative Prior Distributions

As mentioned in Section 7.3 the normal (lognormal) distribution with a large standard deviation (log standard deviation) is often use to specify weakly informative prior distributions.
Figure 21: Bayesian estimation results for data with $r = 3$ and $n = 20$ using the Weibull distribution (on the left) and the lognormal distribution (on the right) showing a the estimate of $F(t)$ on a probability plot (top), draws from the bounded joint prior and likelihood contours (middle), and posterior draws and likelihood contours (bottom) for an IJ prior with $p_{\text{lim}} = 0.01$.

G.1 Limit of a Normal Distribution as its Standard Deviation Increases

As mentioned in Section 7.3 of the paper, a normal distribution prior density with any mean will approach a flat prior as the standard deviation of the normal distribution increases. First we consider
Figure 22: Bayesian estimation results for data with \( r = 3 \) and \( n = 20 \) using the Weibull distribution (on the left) and the lognormal distribution (on the right) showing the estimate of \( F(t) \) on a probability plot (top), draws from the bounded joint prior and likelihood contours (middle), and posterior draws and likelihood contours (bottom) for an IJ prior with \( p_{\text{lim}} = 0.50 \).

A NORM(\( \mu, \sigma \)) distribution truncated outside of \( \mu \pm A \) for a value \( A > 0 \)

\[
\lim_{\sigma \to \infty} \frac{1}{\sigma} \phi_{\text{norm}} \left( \frac{x - \mu}{\sigma} \right) \Phi_{\text{norm}} \left( \frac{A}{\sigma} \right) - \Phi_{\text{norm}} \left( -\frac{A}{\sigma} \right) = \frac{1}{2A}. \tag{22}
\]
for any $\mu$ and $\mu - A \leq x \leq \mu + A$. We use this truncated distribution so that the density remains proper in the limit.

Although it is possible to use L'Hospital's Rule to compute the limit in (22) directly, the needed derivatives are complicated, making the proof lengthy. Here we take an alternative simpler path. The
Figure 24: Bayesian estimation results for data with \( r = 3 \) and \( n = 1000 \) using the Weibull distribution (on the left) and the lognormal distribution (on the right) showing the estimate of \( F(t) \) on a probability plot (top), draws from the bounded joint prior draws and likelihood contours (middle), and posterior draws and likelihood contours (bottom) for an IJ prior with \( p_{\text{lim}} = 0.01 \).

denominator in (22) can be written as

\[
\Phi_{\text{norm}} \left( \frac{A}{\sigma} \right) - \Phi_{\text{norm}} \left( -\frac{A}{\sigma} \right) = \int_{-A/\sigma}^{A/\sigma} \phi_{\text{norm}}(w) \, dw
\]
Figure 25: Bayesian estimation results for data with \( r = 3 \) and \( n = 1000 \) using the Weibull distribution (on the left) and the lognormal distribution (on the right) showing a the estimate of \( F(t) \) on a probability plot (top), draws from the bounded joint prior and likelihood contours (middle), and posterior draws and likelihood contours (bottom) for an IJ prior with \( p_{\text{lim}} = 0.50 \).

for \( A > 0 \). The mean value theorem for integrals says that

\[
\int_{-A/\sigma}^{A/\sigma} \phi_{\text{norm}}(w) \, dw = \left[-\frac{A}{\sigma} \left( \frac{A}{\sigma} \right) \right] \phi_{\text{norm}}(\zeta) = \frac{2A}{\sigma} \phi_{\text{norm}}(\zeta),
\]
where $-A/\sigma \leq \zeta \leq A/\sigma$. Then

$$\frac{1}{\sigma} \phi_{\text{norm}} \left( \frac{x - \mu}{\sigma} \right) = \frac{1}{\sigma} \phi_{\text{norm}} \left( \frac{x - \mu}{\sigma} \right) = \frac{1}{2A} \phi_{\text{norm}} \left( \frac{x - \mu}{\sigma} \right).$$

For large $\sigma$, both $-A/\sigma$ and $A/\sigma$ are approximately zero, implying that $\zeta$ is approximately zero. Of course, $(x - \mu)/\sigma$ will also be approximately zero. Thus

$$\frac{1}{2A} \phi_{\text{norm}} \left( \frac{x - \mu}{\sigma} \right) \approx \frac{1}{2A} \phi_{\text{norm}}(0) = \frac{1}{2A},$$

giving the needed result.

G.2 Limit of a Lognormal Distribution as its Log Standard Deviation Increases

As mentioned in Section 7.3 of the paper, a lognormal distribution prior $f(t)$ with any log-mean will be proportional to $1/t$ as the log standard deviation increases. We start by noting that, for any values of $\mu$ and $t > 0$, the standard normal density has the limit

$$\lim_{\sigma \to \infty} \phi_{\text{norm}} \left( \frac{\log(t) - \mu}{\sigma} \right) = \lim_{\sigma \to \infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{\log(t) - \mu}{\sigma} \right)^2 \right] = \frac{1}{\sqrt{2\pi}} > 0.$$  

This implies that, for any fixed large value of $\sigma$, the lognormal density

$$\frac{1}{\sigma t} \phi_{\text{norm}} \left( \frac{\log(t) - \mu}{\sigma} \right) \approx \frac{1}{\sigma t \sqrt{2\pi}},$$

giving the needed result.

H Log-Truncated and Log-Reciprocal-Truncated Distributions

H.1 Motivation for the Distributions

As mentioned in Section 8.1 of the main paper, when specifying an informative prior distribution for a positive parameter (like a log-location-scale distribution shape parameter $\sigma$ or $\beta = 1/\sigma$ or a log-location-scale distribution quantile $t_p$), a normal distribution truncated below zero (denoted by TNORM) is often used. Although it is a slight abuse, to keep the notation simple and standard, we employ $\mu$ and $\sigma$ to denote the parameters of the TNORM distribution. The pdf for the TNORM($\mu, \sigma$) random variable $T > 0$ is

$$d_{\text{tnorm}}(t; \mu, \sigma) = \frac{1}{\sigma} \phi_{\text{norm}} \left( \frac{t - \mu}{\sigma} \right) \left( 1 - \Phi_{\text{norm}} \left( \frac{\mu}{\sigma} \right) \right), \quad t > 0.$$  

(23)
Informative (and perhaps weakly informative) priors are specified in terms of the distributions for parameters like \( T = \sigma \), \( T = \beta = 1/\sigma \), or \( T = t_p \). Then expressions for the prior pdfs of the unconstrained parameters \( S = \log(\sigma) \), \( S = \log(\beta) = \log(1/\sigma) \), or \( S = \log(t_p) \) are needed, as these transformed parameters are used in the MCMC sampling.

### H.2 Log-Truncated-Normal Distribution pdf

If \( T \sim \text{TNORM}(\mu, \sigma) \), the distribution of \( S = \log(T) \) can be obtained by using the standard random variable transformation methods described, for example, in Chapter 2 of Casella and Berger (2002). Specifically, because \( S \) is a monotone increasing function of \( T \) and the inverse of the transformation is \( T = \exp(S) \), the pdf for \( S \) is

\[
\text{dlnorm}(s; \mu, \sigma) = \text{dtnorm}(\exp(s); \mu, \sigma) \times \exp(s)
\]

\[
= \left( \frac{1}{\sigma} \right) \frac{\phi_{\text{norm}} \left( \frac{\exp(s) - \mu}{\sigma} \right)}{1 - \Phi_{\text{norm}} \left( -\frac{\mu}{\sigma} \right)} \times \exp(s),
\]

\[
= \left( \frac{1}{\sigma} \right) \frac{\phi_{\text{norm}} \left( \frac{\mu - \exp(s)}{\sigma} \right)}{\Phi_{\text{norm}} \left( \frac{\mu}{\sigma} \right)} \times \exp(s),
\]

\[-\infty < s < \infty.\]

We call this the log-truncated-normal (LTNORM) distribution. The simpler second expression is obtained by using the symmetry relationships \( \Phi_{\text{norm}}(z) = 1 - \Phi_{\text{norm}}(-z) \) and \( \phi_{\text{norm}}(z) = \phi_{\text{norm}}(-z) \). Figure 26(a) shows pdfs for the LTNORM distribution.
H.3 Log-Reciprocal-Truncated-Normal Distribution pdf

Similar to Section H.2, if \( T \sim \text{TNORM}(\mu, \sigma) \), then the distribution of \( W = \log(1/T) = -\log(T) \) can, again, be obtained by using the standard random variable transformation methods. Specifically, because \( W \) is a monotone decreasing function of \( T \) and the inverse of the transformation is \( T = \exp(-W) \), the pdf for \( W \) is

\[
dlrtnorm(w; \mu, \sigma) = dtnorm(\exp(-w); \mu, \sigma) \times \exp(-w)
\]

\[
= \left( \frac{1}{\sigma} \right) \phi_{\text{norm}} \left( \frac{\exp(-w) - \mu}{\sigma} \right) \times \exp(-w), \quad (25)
\]

\[
= \left( \frac{1}{\sigma} \right) \phi_{\text{norm}} \left( \frac{\mu - \exp(-w)}{\sigma} \right) \times \exp(-w) \quad -\infty < w < \infty.
\]

We call this the log-reciprocal-truncated-normal (LRTNORM) distribution. Figure 26(b) shows pdfs for the LRTNORM distribution.

H.4 Log-Truncated-Location-Scale-\( t \) and Log-Reciprocal-Truncated-Location-Scale-\( t \) Distribution pdfs

As mentioned in Section 8.1, a useful generalization of the normal distribution is the location-scale-\( t \) (LST) distribution. Similar to what is described in Section H.1, if priors are specified for the positive log-location-scale distribution parameters (e.g., \( \sigma, \beta = 1/\sigma \), or \( t_p \)) using a truncated LST distribution, pdfs of the unconstrained parameters (e.g., \( S = \log(\sigma), S = \log(\beta) = \log(1/\sigma) \), or \( S = \log(t_p) \)) are needed. This is because the pdfs of these transformed parameters are used to specify priors in the MCMC sampling. We refer to these distributions as LTLST and LRLST. Expressions for the LTLST and LRLST pdfs are obtained in the same manner as in Sections H.2 and H.3 except that the standard normal pdf and cdf are replaced by their LST counterparts and these depend on the specified degrees of freedom parameter. In particular, the standard LST pdf is the Student’s \( t \) pdf:

\[
\phi_{\text{lst}}(z; r_d) = \frac{1}{\sqrt{\pi r_d}} \frac{\Gamma([r_d + 1]/2)}{\Gamma(r_d/2)} \frac{1}{(1 + z^2/r_d)^{[r_d+1]/2}}, \quad -\infty < z < \infty.
\]

The corresponding LST cdf is \( \Phi_{\text{lst}}(z; r_d) \). Then, following the same path used in (24) and (25) (without giving all of the steps), the LTLST and LRLST pdfs are

\[
dltlst(s; \mu, \sigma, r_d) = \left( \frac{1}{\sigma} \right) \phi_{\text{lst}} \left( \frac{\mu - \exp(s)}{\sigma}; r_d \right) \times \exp(s), \quad -\infty < s < \infty,
\]
and

\[ \text{drltst}(w; \mu, \sigma, r_d) = \left( \frac{1}{\sigma} \right) \phi_{\text{lst}} \left( \frac{\mu - \exp(-w)}{\sigma}; r_d \right) \times \exp(-w) - \infty < w < \infty. \]

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