Optimal H2 order-one reduction by solving eigenproblems for polynomial equations

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Abstract

A method is given for solving an optimal H2 approximation problem for SISO linear time-invariant stable systems. The method, based on constructive algebra, guarantees that the global optimum is found; it does not involve any gradient-based search, and hence avoids the usual problems of local minima. We examine mostly the case when the model order is reduced by one, and when the original system has distinct poles. This case exhibits special structure which allows us to provide a complete solution. The problem is converted into linear algebra by exhibiting a finite-dimensional basis for a certain space, and can then be solved by eigenvalue calculations, following the methods developed by Stetter and Möller [29], [34]. The use of Buchberger’s algorithm is avoided by writing the first-order optimality conditions in a special form, from which a Gröbner basis is immediately available. Compared with our previous work [18], the method presented here has much smaller time and memory requirements, and can therefore be applied to systems of significantly higher McMillan degree. In addition, some hypotheses which were required in the previous work have been removed. Some examples are included.

I. INTRODUCTION

In this paper we consider the problem of approximating a stable linear dynamic system by one of lower McMillan degree. We take the L2 norm as the measure of approximation, namely we solve the problem

\[ \min_{h \in \mathcal{M}(n)} \int_0^\infty |h(t) - \hat{h}(t)|^2 dt \]

where \( h \in \mathcal{M}(N) \) is the impulse response of the original system, \( \hat{h} \) is the impulse response of the approximating system, and \( \mathcal{M}(N) \) denotes the set of impulse responses of minimal stable systems of McMillan degree \( N \). This problem is equivalent to the problem of finding an approximation which minimizes the \( H_2 \) norm of the error in the frequency response:

\[ \min_{\hat{H} \in \mathcal{H}(n)} \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega) - \hat{H}(\omega)|^2 d\omega \]

where \( H \) and \( \hat{H} \) are the frequency responses of the original and the approximating systems, respectively, and \( \mathcal{H}(N) \) denotes the set of Fourier transforms of elements of \( \mathcal{M}(N) \). Throughout this paper we consider SISO systems only, and we solve the \( H_2 \) problem for \( n = N - 1 \). We assume mostly that the ‘true’ system has distinct poles. From section II onwards we will work with the set \( \Sigma S_N \) of rational transfer functions, whose impulse responses are elements of \( \mathcal{M}(N) \) and frequency responses are elements of \( \mathcal{H}(N) \), and we will look for approximants in the set \( \Sigma S_n \).

The \( H_2 \) problem has many applications and connections to other problems in systems and control theory, including model simplification, system identification, and approximate model matching. Many publications treat this problem, such as [2], [28] and the references cited therein. An early publication on this problem, possibly the oldest, is [1]. We investigate the \( H_2 \) approximation problem by means of constructive algebra, in particular by exploiting the theory of polynomial ideals. There is an increasing use of computer algebra in systems theory, see e.g. [14], [25], [31], [33], [37], [38]. This paper makes a further contribution to this trend.

We believe that the significance of this paper lies in its introduction of a promising new approach to model reduction problems. We emphasise that this approach does not involve gradient-based search methods, and hence avoids the usual problems associated with local minima. Our use of constructive algebra leads to an algorithm with the important attribute that the solution found is guaranteed to be the global optimum. In [18] two of the present authors already applied constructive algebra to the \( H_2 \) approximation problem, taking an approach based on state-space realizations of the linear systems involved. By contrast, the approach here is based on a form of the first-order necessary conditions for optimality which arises from transfer function descriptions of both the original and the approximating systems. The solution method which we develop here is quite different from that developed in [18]. Computationally it is much more efficient, as regards both memory and time requirements. This allows us to tackle problems with significantly larger values of \( N \), as can be seen from the examples. Furthermore, [18] required some technical hypotheses relating to the finiteness of the number of critical points, which are not needed in this paper.

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In addition to finding the global optimum, our approach gives important new theoretical insight into the structure of the reduction-by-one problem. In particular, we show that the number of critical points is finite, and in fact no greater than $2^N - 1$. The computational complexity is high, and the method involves some delicate numerical steps, so we do not claim that our approach is a rival, at this stage, for conventional numerical approaches in routine applications to engineering problems. But even now it has some practical uses, for example as a generator of ‘benchmark’ solutions against which other methods can be tested. Since, as will be seen, it relies on eigenvalue calculations for a set of matrices which can be constructed in a rather straightforward manner, our approach is in some ways comparable with Glover’s method for solving the Hankel-norm approximation problem [16]. Promising developments which combine the current approach with numerical methods for solving large eigenvalue problems in related applications are reported in [7].

In the next section we obtain a special representation of the first-order necessary conditions for optimality. This representation is in the form of a set of quadratic equations, which take a special form which we call diagonal quadratic. The following section investigates such diagonal quadratic equations. It is shown that the polynomials which define these equations form a Gröbner basis for the ideal generated by themselves. It is further shown that these equations have a finite set of solutions, and that in consequence a certain space is finite-dimensional. Furthermore a basis for this space is identified, which allows a solution method based on linear algebra.

We then present such a method of solving a system of polynomial equations. This method relies on obtaining a Gröbner basis, but in the application to the specific $H_2$ problem considered here, such a basis is immediately available. This method of solving polynomial equations is of general use and it is known in the computer algebra community, see [10], [29], [34] and the references therein. The development here is self-contained and starts with constructing a matrix solution of the system of polynomial equations, from which the desired solutions can be found by solving a collection of eigenvalue-eigenvector problems. These eigenproblems can be solved either by numerical methods or by symbolic methods. We believe from a system theoretic point of view it is very natural to start with the construction of a matrix solution; in fact the matrices obtained are generalised companion matrices.

A section then applies this method to the solution of the $H_2$ problem, for the case $n = N - 1$ and distinct poles of the original system. How to treat repeated poles is outlined in a short section. This is followed by two examples.

II. A SPECIAL REPRESENTATION OF THE FIRST ORDER CONDITIONS.

In this section the first order conditions for a class of $H_2$ model order reduction problems will be considered. Studying the outcomes of a computer algebra calculation in which a set of symbolic first order conditions for the $H_2$ model order reduction problem was brought into a recursive form, it was observed that the occurrence of multiple poles in the original system gave rise to a certain singularity in the first order equations. This was the motivation for investigating the class of problems. These eigenproblems can be solved either by numerical methods or by symbolic methods. We believe from a system theoretic point of view it is very natural to start with the construction of a matrix solution; in fact the matrices obtained are generalised companion matrices.

Consider a continuous-time stable SISO linear system. Without loss of generality we assume the system to be strictly proper, because if it is not then the direct feedthrough term of the optimal $H_2$ approximant will be equal to the direct feedthrough term of the original system, and the strictly proper part of the optimal approximant will not be influenced at all (nor will the strictly proper part of any of the critical points) by the value of the direct feedthrough term. Let the transfer function of the original system (i.e. the system that is to be reduced in order) be given by $e(s)/d(s)$, where $e$ is some polynomial with real coefficients of degree at most $N - 1$, and $d$ is a monic polynomial with real coefficients of degree $N$ with all its zeroes (i.e. poles of the transfer function) $\delta_1, \delta_2, \ldots, \delta_N$, within the open left half of the complex plane. Assume that $e$ and $d$ are coprime.

Consider the rational function $\frac{b(s)}{a(s)}$. It is an element of the Hardy space $H_2$ of square summable functions on the imaginary axis which are analytic on the open right halfplane and satisfy a certain continuity requirement on the imaginary axis(cf. [26]).

In this paper we work with the subspace of real rational functions in $H_2$. This subspace consists of all strictly proper real rational functions which have the property that all the poles lie in the open left half plane. The space $H_2$ is in fact a Hilbert space with corresponding norm $\|\cdot\|_2$ of a function $t \in H_2$ given by

$$\|t\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |t(i\omega)|^2 d\omega.$$ 

Consider the differentiable manifold $\Sigma S_n$ of all real rational functions $\frac{b(s)}{a(s)}$ in $H_2$ such that $b(s)$ and $a(s)$ are coprime, the coefficients of $a(s)$ and $b(s)$ are real and $a(s)$ is a Hurwitz polynomial of degree $n$. For more information about the structure of this differentiable manifold see for example [6] and [23] and the references given there. The $H_2$ model order reduction problem can now be formulated as the following optimization problem:

$$\min_{\frac{b(s)}{a(s)} \in \Sigma S_n} \left\| \frac{e(s)}{d(s)} - \frac{b(s)}{a(s)} \right\|_2^2.$$
Remark. It is well-known that the distance squared \( \left\| \frac{e(s)}{a(s)} - \frac{b(s)}{a(s)} \right\|_2^2 \) is in fact a rational function of the coefficients of the numerator and denominator polynomials (see the literature, e.g. \([21]\); in order to obtain explicit rational function formulas one could use the methods proposed in \([24]\)).

A well-known first order necessary condition for optimality of an \( n \)-th order transfer function \( b(s)/a(s) \) with real coefficients, as an approximant in \( H_2 \) is the following. First let us present a geometric formulation.

If \( \frac{b(s)}{a(s)} \) is an optimal approximant of the transfer function \( \frac{e(s)}{a(s)} \) with respect to the \( H_2 \)-norm, then the difference \( \frac{e(s)}{a(s)} - \frac{b(s)}{a(s)} \) is perpendicular to the tangent plane at the manifold of transfer functions of order \( n \) at the point \( \frac{b(s)}{a(s)} \).

It is well-known (and not hard to show) that the tangent space consists of all strictly proper rational functions of the form \( \frac{p(s)}{a(s)} \), where \( p \) is a polynomial of degree at most \( 2n-1 \). From the theory of Hardy spaces it follows that the orthogonal complement in \( H_2 \) of this vector space is given by \( a(-s)^2H_2 \), i.e. all \( H_2 \)-functions which can be written as the product of the function \( a(-s)^2 \) and an arbitrary \( H_2 \) function. Combining this with the first order conditions given above, it follows that the numerator of the difference \( \frac{e(s)}{a(s)} - \frac{b(s)}{a(s)} \) has to be divisible by \( a(-s)^2 \). (Cf. \([28]\), see also \([2]\), \([3]\)). Algebraically this can be written down as follows:

Let \( n < N \). If \( \frac{b(s)}{a(s)} \) is an optimal approximant within the class of transfer functions of order \( n \) in \( H_2 \), of the transfer function \( \frac{e(s)}{d(s)} \) in \( H_2 \), with respect to the \( H_2 \)-norm, then there exists a polynomial \( q(s) \) of degree at most \( N-(n+1) \) such that

\[
e(s)a(s) - b(s)d(s) = a(-s)^2q(s).
\]

(3)

Let us now specialise to the case in which \( n = N-1 \) and the original system has distinct poles, i.e. the multiplicity of each of the \( N+1 \) poles \( \delta_1, \ldots, \delta_N \) is one. The rest of this paper concentrates mostly on this case. Now the polynomial \( q(s) \) has degree zero, so it reduces to a constant \( q(s) = q_0 \). The unknowns in the polynomial equation are the polynomials \( b(s), a(s) \) and the number \( q_0 \). Although \( q_0 \) is only an auxiliary variable we will not eliminate it. Note that once the polynomial \( a \) and the number \( q_0 \) are known, the polynomial \( b \) follows from the formula

\[
b(s) = \frac{e(s)a(s) - q_0 a(-s)^2}{d(s)}.
\]

(4)

Substituting \( s = \delta_i, i = 1, \ldots, N \) in the polynomial equation \( (5) \) one obtains:

\[
e(\delta_i)a(\delta_i) = a(-\delta_i)^2q_0, i = 1, \ldots, N.
\]

(5)

Note that the polynomials appearing here do not depend on the polynomial \( b \), due to the fact that \( d(\delta_i) = 0 \) for each \( i = 1, \ldots, N \). Further note that the possibility \( q_0 = 0 \) can be excluded on the grounds that if \( q_0 = 0 \) then either \( e(\delta_i) = 0 \) for some value of \( i \in \{1, \ldots, N\} \), which implies that there is pole-zero cancellation in the original transfer function and the order of the transfer function will be smaller than \( N \), which can be ruled out without loss of generality, or otherwise it would follow that \( a(s) = 0 \) in \( N \) different points, namely at \( s = \delta_i, i = 1, \ldots, N \), which together with the fact that \( a \) has degree \( n = N-1 \) would imply that \( a = 0 \), which is in contradiction with the assumption that \( a \) is monic. It follows that \( q_0 \neq 0 \) for each value of \( q_0 \) that corresponds to a solution of the first order equations. Therefore multiplying both sides of the polynomial equation with \( q_0 \) the first order conditions can be rewritten as

\[
e(\delta_i)a(\delta_i)q_0 = (a(-\delta_i)q_0)^2, i = 1, \ldots, N, q_0 \neq 0.
\]

(6)

The polynomial \( a \) is monic, so \( q_0 \) is the leading coefficient of the non-zero polynomial \( \bar{a} := q_0a \). Using this notation the first order equations can be rewritten as

\[
e(\delta_i)\bar{a}(\delta_i)q_0 = (a(-\delta_i)q_0)^2, i = 1, \ldots, N, \bar{a} \neq 0.
\]

(7)

The idea is now to consider this as an equation in the unknowns \( \bar{a}(-\delta_i), i = 1, \ldots, N \). In order to do this explicitly we need to express the sequence of numbers \( \bar{a}(\delta_i), i = 1, \ldots, N \) in terms of the sequence of numbers \( \bar{a}(-\delta_i), i = 1, \ldots, N \). This can be done by relating both sequences to the coefficients \( \tilde{a}, j = 0, \ldots, N-1 \) of the polynomial \( \tilde{a}(s) = \tilde{a}_{N-1}s^{N-1} + \tilde{a}_{N-2}s^{N-2} + \ldots + \tilde{a}_0s^0 \). Let \( V(\delta_1, \ldots, \delta_N) \) denote the Vandermonde matrix

\[
V(\delta_1, \ldots, \delta_N) := \begin{pmatrix}
1 & \delta_1 & \delta_1^2 & \cdots & \delta_1^{N-1} \\
1 & \delta_2 & \delta_2^2 & \cdots & \delta_2^{N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \delta_N & \delta_N^2 & \cdots & \delta_N^{N-1}
\end{pmatrix}.
\]

(8)

Using matrix-vector notation the following linear relations are obtained:

\[
\begin{pmatrix}
\bar{a}(\delta_1) \\
\vdots \\
\bar{a}(\delta_N)
\end{pmatrix} = V(\delta_1, \ldots, \delta_N) \begin{pmatrix}
\tilde{a}_0 \\
\vdots \\
\tilde{a}_{N-1}
\end{pmatrix}.
\]

(9)
and

\[
\begin{bmatrix}
\tilde{a}(\delta_1) \\
\vdots \\
\tilde{a}(\delta_N)
\end{bmatrix}
= V(\delta_1, \ldots, -\delta_N)
\begin{bmatrix}
\tilde{a}_0 \\
\vdots \\
\tilde{a}_{N-1}
\end{bmatrix}.
\] (10)

It follows that

\[
\begin{bmatrix}
\tilde{a}(\delta_1) \\
\vdots \\
\tilde{a}(\delta_N)
\end{bmatrix}
= V(\delta_1, \ldots, \delta_N)V(\delta_1, \ldots, -\delta_N)^{-1}
\begin{bmatrix}
\tilde{a}(\delta_1) \\
\vdots \\
\tilde{a}(\delta_N)
\end{bmatrix}.
\] (11)

Note that \(V(\delta_1, \ldots, -\delta_N)\) is an invertible matrix because, by assumption, for all \(i = 1, \ldots, N, j = 1, \ldots, N,\) if \(i \neq j\) then \(\delta_i \neq \delta_j\) and therefore we have \(\det(V(\delta_1, \ldots, -\delta_N)) = \prod_{1 \leq i < j \leq N}(\delta_i - \delta_j) \neq 0\) (cf. e.g. [27], p.35).

The first order equations can now be rewritten as

\[
\begin{bmatrix}
\tilde{a}(-\delta_1)^2 \\
\vdots \\
\tilde{a}(-\delta_N)^2
\end{bmatrix}
= \text{diag}(e(\delta_1), \ldots, e(\delta_N))V(\delta_1, \ldots, \delta_N)V(\delta_1, \ldots, -\delta_N)^{-1}
\begin{bmatrix}
\tilde{a}(-\delta_1) \\
\vdots \\
\tilde{a}(-\delta_N)
\end{bmatrix}.
\]

\[\tilde{a}(-\delta_1, \ldots, \tilde{a}(-\delta_N)) \neq 0\] (12)

where \(\text{diag}(e(\delta_1), \ldots, e(\delta_N))\) denotes the diagonal matrix with \(e(\delta_i)\) in the \((i, i)\)–entry, \(i = 1, \ldots, N\). This means that these first order equations can be written as

\[
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
\vdots \\
x_N^2
\end{bmatrix}
= M
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{bmatrix}, \quad x \neq 0
\] (13)

where \(x_i = \tilde{a}(-\delta_i), i = 1, \ldots, N\), \(x = (x_1, \ldots, x_N)^T\) and

\[
M = \text{diag}(e(\delta_1), \ldots, e(\delta_N))V(\delta_1, \ldots, \delta_N)V(\delta_1, \ldots, -\delta_N)^{-1}.
\] (14)

In the next section the solution of equations of the form found here will be treated in general.

III. DIAGONAL-QUADRATIC SYSTEMS OF EQUATIONS

In this section we will present results about an arbitrary system of polynomial equations of the form

\[
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
\vdots \\
x_N^2
\end{bmatrix}
= M
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_N
\end{bmatrix} + \mu,
\] (15)

where \(\mu \in \mathbb{C}^N\) is a constant \(N\)–vector. This will be called a diagonal-quadratic system of equations.

Remark. A quadratic equation in \(x\) can be written as \(x^T A x + c x + d\) for some symmetric matrix \(A\), a row vector \(c\) and a scalar \(d\). If \(A = e_i e_i^T\), for some \(i \in \{1, \ldots, N\}\), then the equation is one of the form described above. If there are \(N\) quadratic equations and the corresponding \(A\)–matrices are all diagonal, and these diagonal matrices form a basis of the linear vector space of all diagonal \(N \times N\) matrices then such a system can (obviously) be rewritten in the form above. That is the motivation for the terminology ‘diagonal-quadratic’.

In this paper use will be made of Gröbner basis theory and constructive algebra. For an exposition of this theory one can refer to e.g. [11]. In Gröbner basis theory an important role is played by the so-called monomial orderings. Let \(\alpha = (\alpha_1, \ldots, \alpha_N)\) denote an arbitrary vector of nonnegative integers, which will be called a multi-index in the sequel, then \(x^\alpha\) will denote the monomial \(x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_N^{\alpha_N}\). The multi-index \(\alpha\) is called the multi-degree of the monomial \(x^\alpha\). The corresponding total degree is defined as \(|\alpha| := \alpha_1 + \alpha_2 + \ldots + \alpha_N\). For a general definition of monomial ordering we refer to [11], p.54, Definition 1.

A partial ordering of monomials is defined by \(x^\alpha \succ x^\beta\) if \(|\alpha| > |\beta|\). Such an ordering is called a total degree ordering. For our purposes any complete ordering which is a refinement of the total degree ordering would do. For definiteness we choose to work with the graded lexicographic ordering, which refines the total degree ordering as follows: if \(|\alpha| = |\beta|\) then \(x^\alpha \succ x^\beta\) if \(\alpha_i > \beta_i\) for the smallest integer \(i \in \{1, \ldots, N\}\) for which \(\alpha_i \neq \beta_i\).

The total degree of a polynomial is defined as follows. Each polynomial is a unique linear combination of monomials with nonzero coefficients. The maximal total degree of these monomials is called the total degree of the polynomial. If we denote
Let $g_i(x_1, \ldots, x_N) := x_i^2 - m_i x - \mu_i, i = 1, \ldots, N$, then we are looking for the zeros of the ideal $I$ spanned by $G := \{g_1, g_2, \ldots, g_N\}$.

Let $< g_1, \ldots, g_N >$ denote the ideal generated by the set of polynomials $g_1, \ldots, g_N$. For a polynomial $f$, let $LT(f)$ denote the leading term of $f$, and for an ideal $I$ of polynomials, let $LT(I)$ denote the set of all leading terms of the polynomials in $I$.

**Definition 3.1:** For a fixed monomial ordering, a finite subset $\Gamma = \{\gamma_1, \ldots, \gamma_\nu\}$ of an ideal $I$ is a Gröbner basis if

\[ LT(\gamma_1), \ldots, LT(\gamma_\nu) \supseteq LT(I) > . \]

**Theorem 3.1:** The set $G$ is a Gröbner basis with respect to total degree ordering.

**Proof.** With respect to any ordering which is a refinement of partial ordering by total degree, the leading terms of $G$ are monomials of the form $x_i^2$. These are clearly pairwise coprime. But it is known that this implies that $G$ is a Gröbner basis [12, p.333, Ex.15.20]. □

An alternative but longer proof is available in [19].

This result is very important because to apply the results of Gröbner basis theory one needs a Gröbner basis. Usually one needs to apply an algorithm like Buchberger’s algorithm to bring a set of polynomials that generates the ideal in which one is interested in Gröbner basis form. In fact in many cases this is the most difficult part of the calculations. In the case at hand however the set of polynomials of which we want to find the zeros itself forms a Gröbner basis.

But that is not all. We can say more. We know that $G = \{g_1, \ldots, g_N\}$ forms a Gröbner basis and that the leading monomial of $g_i$ is $x_i^2$ for each $i = 1, \ldots, N$. Let $C[x_1, \ldots, x_N]$ denote the ring of polynomials with complex coefficients. Let $R$ denote the set of all multi-indices $R := \{(0, 1)^N\}$. In other words, $R$ is the set of all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_N)$ with the property that for each $i = 1, \ldots, N$ one has either $\alpha_i = 0$ or $\alpha_i = 1$. Let $Q$ denote the set of all multi-indices outside $R$. For each polynomial $p = p(x)$ there exists a unique additive decomposition $p = p_R + p_Q$, where the polynomial $p_R$ is a linear combination of monomials with multi-degree in $R$ and $p_Q$ is a linear combination of monomials with multi-degree in $Q$.

**Lemma 3.1:** Let $I$ denote the ideal generated by $G$.

(i) The set $V = V(I)$ of zeros in $C^N$ of the system of polynomial equations $g_i(x) = 0, i = 1, \ldots, N$, is finite.
(ii) The $C$–vector space $S = \text{Span}(x^{\alpha} : x^{\alpha} \notin LT(I) >)$ is finite-dimensional.
(iii) The $C$–vector space $C[x_1, \ldots, x_N]/I$ is finite-dimensional.
(iv) The set of monomials $\{x^{\alpha} : \alpha \in R\}$ forms a basis for the vector space $S$.
(v) The dimension of the vector space $S$ is $2^N$.
(vi) The dimension of the vector space $C[x_1, \ldots, x_N]/I$ is $2^N$.

**Proof.** ad(i)–(iii), (i)–(iii) follow immediately from [11, Chapter 5, Theorem 6].

ad (iv). Because $G$ is a Gröbner basis the ideal $LT(I)$ is equal to the ideal generated by the leading terms of the elements of $G$, i.e. the ideal $< x_1^2, \ldots, x_N^2 >$. The monomials in this ideal are precisely those which have multi-degree in the set $Q$. Therefore the monomials in $S$ are the all the monomials with multi-degree in $R$.

ad (v). From (iv) it follows that the dimension of $S$ is equal to the cardinality of $R$, which is $\text{card}(R) = 2^N$.

ad (vi). According to Proposition 4 of Chapter 5 of [11] the vector space $C[x_1, \ldots, x_N]/I$ is isomorphic to $S$ and therefore has the same dimension as $S$. □

From [11], Chapter 5, Section 3, Proposition 1 it follows that every polynomial in $C[x_1, \ldots, x_N]$ can be written in a unique way as the sum of an element of $S$ and an element of $I$. In other words, each equivalence class $f + I$, where $f$ is an arbitrary polynomial in $C[x_1, \ldots, x_N]$, has a unique representative in $S$. Let this representative be denoted by $\pi(f) \in S$. Given $f$, the polynomial $\pi(f)$ can be obtained by a general method from Gröbner basis theory, namely the so-called division algorithm with respect to the Gröbner basis $G$ as described in e.g. [11]. However, for diagonal quadratic equations, the division algorithm simplifies considerably and one can describe in direct terms how one can obtain $\pi(f)$ from $f$. The ‘reduction procedure’ can be described as follows. Using the same notation as above, one can write $f = f^Q + f^R$, where $f^R \in S$ and the monomials of $f^Q$ all have multi-degree in $Q$. This additive decomposition is obviously unique. If $f^Q = 0$ then $f = f^R \in S$ in which case $\pi(f) = f$ and we are done. If $f^Q \neq 0$ then consider any monomial of $f^Q$ with total degree equal to the total degree of $f^Q$. By construction each such monomial is divisible by at least one of the monomials $x_1^2, x_2^2, \ldots, x_N^2$. If it is divisible by $x_i^2$ then replacing it by the polynomial that is obtained by multiplying the monomial by $\frac{\partial}{\partial x_i^2}$ the result is a polynomial $\tilde{f}$ that is in the equivalence class $f + I$ and which has the following property. Either the total degree of $\tilde{f}^Q$ is smaller than the total degree of $f^Q$, or otherwise the total degree of $\tilde{f}^Q$ is equal to the total degree of $f^Q$ but the number of monomials in $f^Q$ with total degree equal to the total degree of $f^Q$ is one less than the number of monomials in $f^Q$ with total degree equal to the total degree of $f^Q$. Such a replacement of $f$ by $\tilde{f}$ will be called a ‘reduction step’. It follows that after a finite number of such reduction steps one arrives at a polynomial in the equivalence class $f + I$ with the property that it lies in $S$. This is then the unique polynomial $\pi(f)$ that was sought for.

The importance of this reduction procedure in our application will become clear in the examples section.
In this section a method to obtain the solutions of a system of polynomial equations in several variables will be outlined. A method of this kind was originally developed by [29], [34]. A similar approach, but differing in some details, was developed by the authors of the present paper, is available in [19], and is the approach which will be summarized here. All proofs are omitted from this section since they are available in the works cited above.

We will consider the situation in which the system of polynomial equations will have a finite number of solutions over the field of complex numbers \( \mathbb{C} \). In the modern constructive algebra approach to the problem of finding the roots of a system of polynomial equations the theory of Gröbner bases plays an important role. For this theory we refer, as before, to [11]. A fundamental theorem of the theory of Gröbner bases is that for any polynomial ideal given by a finite number of polynomials which generate it, a Gröbner basis can be computed with respect to any admissible monomial ordering (like the lexicographical ordering or the total degree ordering) in a finite number of steps. It can for example be obtained by Buchberger’s algorithm. However the number of steps required by such an algorithm can be huge. In the literature it is suggested that in order to obtain the roots of a system of polynomial equations, one can construct a Gröbner basis with respect to a lexicographical ordering [11, p.233], [15, pp. 459-462]. Also in the paper [18] this approach was followed to show that under two hypotheses described in that paper, the \( H_2 \) model order reduction problem can be solved in a finite number of steps. However only examples of reduction of third order models were presented in that paper. The bottle-neck in the calculations was the construction of a Gröbner basis. In the previous section it was shown that for the problem of reduction of the model order by one with respect to the \( H_2 \) norm, in case of an original model with distinct poles, the first order equations found already are in the form of a total degree Gröbner basis, so no Gröbner basis construction at all is required in the application at hand.

The idea is first to construct a commutative matrix solution for a system of polynomial equations which is in Gröbner basis form.

**Definition 4.1:** Let \( N \) be a positive integer. Let \( f \in \mathbb{C}[x_1, \ldots, x_N] \) be a polynomial in the variables \( x_1, \ldots, x_N \). Let \( M \) be a positive integer and consider an \( N \)-tuple \( (A_1, A_2, \ldots, A_N) \) of \( M \times M \) matrices that commute with each other, i.e. \( A_iA_j = A_jA_i \) for each pair \( (i,j) \), \( i = 1, \ldots, N \), \( j = 1, \ldots, N \). Then \( (A_1, A_2, \ldots, A_N) \) will be called a commutative matrix solution of the polynomial equation \( f = 0 \) if \( f(A_1, A_2, \ldots, A_N) = 0_M \), where the symbol \( 0_M \) denotes the \( M \times M \) zero matrix. In the following, an \( M \times M \) zero matrix will often be denoted by the symbol \( 0 \), as is usual, instead of the symbol \( 0_M \). The size of the zero matrix should then be clear from the context.

A \( M \times M \) tuple which is in fact a scalar solution of the system of polynomial equations involved. The commutative matrix solution that will be constructed here for the case of ideals with zero dimensional variety, has the property that ALL (scalar) solutions can be obtained in this way.

From a commutative matrix solution a scalar solution can be obtained by considering any common eigenvector of the matrices. The corresponding eigenvalues form an \( N \)-tuple which is in fact a scalar solution of the system of polynomial equations involved. The commutative matrix solution that will be constructed here for the case of ideals with zero dimensional variety, has the property that ALL (scalar) solutions can be obtained in this way.

It will first be explained how such a commutative matrix solution can be constructed. Then it will be shown how the (scalar) solutions of the system of polynomial equations can be derived from the matrix solution by eigenvalue-eigenvector calculations.

If \( \mathcal{F} \) is a field containing all the coefficients of the polynomial equations then all the entries of the matrix solution that will be constructed will be contained in \( \mathcal{F} \); in other words, only additions, subtractions, multiplications and divisions are required to obtain a matrix solution.

We start with two results which hold for an arbitrary polynomial ideal. For these results to hold, the ideal does not have to have the property that the number of common zeros of the polynomials in the ideal is finite. The two results consist of a number of observations concerning the operation ‘multiplication by \( x_i \) modulo the ideal’, for \( i \in \{1, \ldots, N \} \).

**Theorem 4.1:** Let \( N \) be a positive integer. Let \( I \subseteq \mathbb{C}[x_1, \ldots, x_N] \) be an ideal and let \( \mathcal{V} := \mathbb{C}[x_1, \ldots, x_N]/I \) denote the corresponding residue class ring. Let \( i \in \{1, \ldots, N \} \) be fixed. Let \( f_1, f_2 \in \mathbb{C}[x_1, \ldots, x_N] \). If \( f_1 \) and \( f_2 \) are equal modulo \( I \), then \( x_i f_1 \) and \( x_i f_2 \) are equal modulo \( I \). The mapping \( X_i : \mathcal{V} \rightarrow \mathcal{V}, f + I \mapsto x_i f + I \), is a linear endomorphism.

**Theorem 4.2:** Let \( N, I, \mathcal{V} \) and \( X_i, i = 1, \ldots, N \) be as given in the previous theorem. For any polynomial \( f \in \mathbb{C}[x_1, \ldots, x_N] \) the linear mapping \( f(X_1, X_2, \ldots, X_N) : \mathcal{V} \rightarrow \mathcal{V} \) is well-defined.

The following two statements are equivalent,

(1) \( f \in I \),

(2) \( f(X_1, \ldots, X_N) \) is equal to the zero mapping \( \mathcal{V} \rightarrow \mathcal{V}, f + I \mapsto 0 + I \).
Now we will specialize to systems of polynomial equations with finitely many common solutions. We will make extensive use of the results from section 3 of Chapter 5 of [11], pp. 228-235, especially Propositions 1 and 4 and Theorem 6 of that section.

Let \( g_1(x_1, \ldots, x_N) = 0, \ldots, g_N(x_1, \ldots, x_N) = 0 \) denote a system of \( N' \) polynomial equations with complex coefficients in the \( N \) variables \( x_1, \ldots, x_N \). The complex vector \( (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N \) is a root of the system of polynomial equations if for each \( j = 1, \ldots, N' \),

\[
g_j(\xi_1, \ldots, \xi_N) = 0.
\]

Let \( I = \langle g_1, \ldots, g_{N'} \rangle \subset \mathbb{C}[x_1, \ldots, x_N] \) denote the ideal generated by the polynomials \( g_1(x_1, \ldots, x_N), \ldots, g_{N'}(x_1, \ldots, x_N) \).

Suppose that \( G = \{g_1, \ldots, g_{N'}\} \) is in fact a Gröbner basis for \( I \), with respect to some fixed monomial ordering. Similarly to what was noted in the previous section for the special case of diagonal-quadratic systems of polynomial equations, the following can be said for this more general case. Each polynomial \( f \in \mathbb{C}[x_1, \ldots, x_N] \) is congruent modulo \( I \) to a polynomial \( r \) with leading term that cannot be reduced by any of the leading terms of the polynomials in the Gröbner basis; for each \( f \) the associated polynomial \( r \) is unique [11, Chapter 5, Section 3, Proposition 1] and will be denoted by \( \bar{r} \). The set \( V \) of all polynomials \( r \) obtained in this way forms a finite dimensional vector space if and only if the number of roots of the system of polynomial equations is finite. If this set is indeed a finite dimensional vector space, then it has a basis consisting of monomials, namely all monomials that cannot be reduced by any of the leading terms of the polynomials in the Gröbner basis. This result is due to Macaulay [12, Theorem 15.3, p.325]. Given the monomial ordering it is a straightforward task to list these monomials ( [11]). Let this basis be denoted by \( B \). Given the monomial ordering it is a straightforward task to list these monomials ( [11]). Let this basis be denoted by \( B \). The mapping \( V \rightarrow V, r \mapsto r + I \), is a linear bijection of vector spaces. In case \( V \) is finite dimensional, let \( B \) denote the basis of \( V \) obtained as the image of \( B \) under this mapping. Let \( D \) denote the dimension of \( V \).

For each \( i \in \{1, \ldots, N\} \) let \( A_{X_i} \) denote the \( D \times D \)-matrix of the endomorphism \( X_i \) with respect to the basis \( B \).

Using this set-up the following fundamental result can be obtained.

**Theorem 4.3:** Let a monomial ordering be fixed and let \( G \) be a Gröbner basis of the ideal \( I \). Let the associated linear space \( V \) be finite dimensional with dimension \( D \). Let \( f \in \mathbb{C}[x_1, \ldots, x_N] \) be given. Let the mappings \( X_i, i = 1, \ldots, N \) and \( f(X_1, X_2, \ldots, X_N) \) be as given in the previous theorems.

The matrix of the linear mapping \( f(X_1, X_2, \ldots, X_N) : V \rightarrow V \) with respect to the basis of monomials \( B \) of \( V \) is equal to \( f(A_{X_1}, A_{X_2}, \ldots, A_{X_N}) \).

The following two statements are equivalent,

(i) \( f \in I \),

(ii) \( f(A_{X_1}, A_{X_2}, \ldots, A_{X_N}) = 0 \), i.e. this matrix is the \( D \times D \) zero matrix.

This theorem tells us that the \( N \)-tuple of matrices \( (A_{X_1}, \ldots, A_{X_N}) \) is in fact a commutative matrix solution of any system of polynomial equations that generates \( I \).

The entries of the \( k \)-th column of the matrix \( A_{X_i} \), are obtained as follows. Let the \( k \)-th element of the basis \( B \) of monomials be denoted by \( b_k \). The monomial \( x_k b_k \) is either itself in the basis \( B \) or otherwise \( x_k b_k \neq x_k b_k \). In both cases \( x_k b_k \) can be written as a unique linear combination of the elements of \( B \). The coefficients of the linear combination are the entries of the \( k \)-th column of the matrix \( A_{X_i} \). If \( x_k b_k \) is itself in the basis \( B \) then the \( k \)-th column of the matrix \( A_{X_i} \) is a standard basis vector.

In the case \( N = 1 \) there exists a unique monic polynomial \( g \) such that \( I \) is generated by \( g \). In that case the matrix \( A_{X_1} \) is a companion matrix of \( g \) (cf. e.g. [27, p. 68]).

Now suppose that the vector \( v \) is a common eigenvector of the matrices \( A_{X_1}, \ldots, A_{X_N} \) with corresponding eigenvalues \( \xi_1, \xi_2, \ldots, \xi_N \), respectively, i.e. for each \( i \in \{1, \ldots, N\} \) the equality \( A_{X_i} v = \xi_i v \) holds and \( v \neq 0 \). Then for each \( f \in I \) one has

\[
0 = f(A_{X_1}, \ldots, A_{X_N})v = f(\xi_1, \ldots, \xi_N)v
\]

and therefore \( f(\xi_1, \ldots, \xi_N) = 0 \). It follows that \( (x_1, \ldots, x_N) = (\xi_1, \ldots, \xi_N) \) is a root of any system of polynomial equations that generates the ideal \( I \).

The following fundamental result states that in fact each of the finite number of roots is obtained in this way.

**Theorem 4.4:** Let \( N \) be a positive integer and let \( I \) be an ideal in the ring \( \mathbb{C}[x_1, \ldots, x_N] \) such that the corresponding set \( Z \subset \mathbb{C}^N \) of common zeros of all the polynomials in \( I \) is finite. Let \( X_i, i = 1, \ldots, N \) be as defined above. Then for each vector \( \xi = (\xi_1, \ldots, \xi_N) \in Z \) there exists a polynomial \( w \in \mathbb{C}[x_1, \ldots, x_N], w \not\in I \), with the property that for each \( i = 1, \ldots, N \), the following equality holds:

\[
X_i(w + I) = \xi_i w + I,
\]

i.e. \( w \) is a common eigenvector of the mappings \( X_1, X_2, \ldots, X_N \), with corresponding eigenvalues \( \xi_1, \ldots, \xi_N \), respectively.

From this theorem we have the following important corollary.

**Corollary 4.1:** Let \( N, I \) and \( Z \) be as given in the previous theorem. Let \( X_i, i = 1, \ldots, N \) be as defined above. Let a monomial ordering be given and let \( G \) be a Gröbner basis of \( I \) with respect to this monomial ordering. Let \( B \) denote the basis of all monomials in \( \mathbb{C}[x_1, \ldots, x_N] \) that are not included in the ideal \( \langle LT(G) \rangle \) generated by the leading terms of the
elements of \(G\) and let \(\mathcal{B}\) denote the corresponding basis of the residue class ring \(\mathbb{C}[x_1, \ldots, x_N]/I\), as before. Let \(A_{x_1}, \ldots, A_{x_N}\) denote the matrices of the linear endomorphisms \(X_1, \ldots, X_N\), respectively, with respect to the basis \(\mathcal{B}\). Then the following two statements are equivalent.

(i) \(\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{Z}\).

(ii) There exists a common eigenvector \(v \in \mathbb{C}^N \setminus \{0\}\) of the (commutative) matrices \(A_{x_1}, \ldots, A_{x_N}\) with corresponding eigenvalues \(\xi_1, \ldots, \xi_N\) respectively, i.e. there exists a nonzero vector \(v\) with the property

\[A_{x_i}v = \xi_i v, \quad i = 1, \ldots, N.\]

Various alternatives arise as to how to exploit the theory presented here to solve a system of polynomial equations, starting with a Gröbner basis. The commutative matrix solution presented can be calculated in symbolic form if the original system of equations is in symbolic form and it can be calculated exactly in numerical form if the coefficients of the original system of polynomials are given numerically. From the commutative matrix solution the roots of the system of polynomial equations can be obtained either by exact algebraic methods or by numerical methods that involve round-off errors. The exact algebraic approach will not be worked out here.

A (nonexact) numerical approach can be based on numerical calculation of the eigenvalues and eigenvectors of the matrices involved. In the examples section this approach will be applied to the \(H_2\) model order reduction problem.

The possibility of using a mixture of exact and symbolic calculations with numerical calculations is very promising for obtaining practically useful results. The matrices involved will tend to become huge (in terms of numbers of entries) if the number of variables involved grows; however eigenvalue calculation can be done numerically for quite big matrices. In section VII matrices with several hundreds of rows and columns are used. One can expect that usage of more refined numerical techniques will make it possible to push the limits quite a bit further.

Let \(f \in \mathbb{C}[x_1, \ldots, x_N]\) and let \(F\) be the corresponding linear endomorphism of \(\mathbb{C}[x_1, \ldots, x_N]/I\) defined by \(g + I \mapsto f.g + I\). If the number of common zeros of the polynomials in \(I\) is finite, and we have a basis \(\mathcal{B}\) of \(\mathbb{C}[x_1, \ldots, x_N]/I\) as before, then we can represent \(F\) with respect to this basis by a matrix \(A_F\). It is now straightforward to see that \(A_F = f(A_{x_1}, A_{x_2}, \ldots, A_{x_N})\).

More generally if \(f = \sum f_d x_d\), \(f_d \in \mathbb{C}[x_1, \ldots, x_N]\) and \(f_d(\xi) \neq 0\) for each common zero \(\xi\) of the polynomials in \(I\), then \(F\) and \(A_F\) are again well-defined and \(A_F = f_n(A_{x_1}, \ldots, A_{x_N}):(f_d(A_{x_1}, \ldots, A_{x_N}))^{-1}\). The eigenvalues of this matrix \(A_F\) are \(\{f(\xi) | \xi \in \mathbb{Z}\}\). For example in optimization problems in which the criterion function \(f\), say, is a rational function this can be used to obtain the matrix \(A_F\) which has as its eigenvalues the critical values of \(f\). (The values that a function takes on its set of critical points are called the critical values.) The matrix \(A_F\) could be called a critical value matrix and its characteristic polynomial a critical value polynomial. This is related to Theorem 9 and the subsequent Remark 10 in [18] concerning the existence and usage of a univariate polynomial which has the critical values of the criterion function as its zeros.

V. Model order reduction by one in \(H_2\)

Recall the formulation of the \(H_2\) model reduction problem from Section II. In order to facilitate the statement of the following theorem let us define the set \(\Xi\) as follows. Let \(\frac{\mathbf{f}}{\mathbf{a}} \in \Sigma S_N\) have \(N\) distinct poles \(\delta_1, \ldots, \delta_N \in \mathbb{C}\). Let the matrix \(M\) be as given in equation (14) and let \(\Xi\) denote the set of solutions in \(\mathbb{C}^N \setminus \{0\}\) of equation (13). The diagonal quadratic system of equations (13) is shown to form a total degree Gröbner basis in Theorem 5.1. In Lemma 3.1 a basis of 2\(N\) monomials of the corresponding vector space \(S\) is presented. This basis consists of the monomials outside the ideal generated by the leading terms of all polynomials in the ideal corresponding to the diagonal quadratic equations. Let this basis be denoted by \(\mathcal{B}\). Then Corollary 4.1 can be applied to (13) using the basis of monomials \(\mathcal{B}\). The implication is that in this case the set \(\Xi\) just defined is equal to the set \(Z\) of that Corollary, except that the zero vector is removed:

\[\Xi = Z \setminus \{0\}\]

It follows that \(\Xi\) contains at most 2\(N\)-1 elements, each of which is a vector of \(N\) entries that can be found as the eigenvalues corresponding to any common eigenvector of the matrices \(A_{x_1}, \ldots, A_{x_N}\) from Corollary 4.1. We therefore have the following theorem

**Theorem 5.1:** Let \(\frac{\mathbf{f}}{\mathbf{a}} \in \Sigma S_N\) have \(N\) distinct poles \(\delta_1, \ldots, \delta_N \in \mathbb{C}\).

(i) The number of critical points of the criterion function \(f : \Sigma S_N \rightarrow [0, \infty), \frac{\mathbf{f}}{\mathbf{a}} \mapsto \| \frac{\mathbf{f}}{\mathbf{a}} - \frac{\mathbf{b}}{\mathbf{a}} \|_2^2\) is finite and not greater than 2\(N\)-1.

(ii) If the rational function \(\frac{\mathbf{f}}{\mathbf{b}} \in \Sigma S_N\) is a critical point of \(f\) then there exists a number \(q_0\) and a vector \(\xi \in \mathbb{C}^N \setminus \{0\}\) such that \(q_0a(-\delta_i) = \xi_i, \quad i = 1, \ldots, N\). For given \(q_0\) and \(\xi\) the polynomial \(a\) is uniquely determined by this linear system of equations and \(\mathbf{b}\) is uniquely determined by equation (14).

Of course the solutions that will be found for the first order equations will in general not all correspond to rational functions \(\frac{\mathbf{f}}{\mathbf{a}} \in \Sigma S_N+1\); it is certainly possible that some will not correspond to real systems; some may correspond to real but unstable systems.

An algorithm to obtain all the critical points of the criterion function of \(H_2\) model reduction by one could now be constructed as follows.
VII we present some results obtained by calculations of the latter type.

(1) Construct the matrix $M$.
(2) Construct the matrices $A_{X_1}, \ldots, A_{X_N}$.
(3) Calculate the eigenvalues of these matrices that correspond to a common eigenvector of all these matrices. The result is a vector $\xi \in \mathbb{C}^N$. All nonzero vectors $\xi$ obtained in this way form the (finite) set $\Xi$.
(4) For each element of $\Xi$ solve equation (13) for $a$ and $q_0$, and select those $a$ that are real and Hurwitz.
(5) For those $a$ selected in the previous step, solve equation (4) for $b$.

Note that steps (1) and (2) can be done purely symbolically. Apart from considerations of memory storage and perhaps calculation time, it is not necessary to specify the original system; one can present it symbolically by its poles and the (non-zero) values of the numerator polynomial in these poles.

If the original system is specified numerically then step (3) can be worked out by either constructive algebra algorithms (using e.g. methods of isolation of zeros of polynomials) or by numerical algorithms that admit round-off errors. In section VII we present some results obtained by calculations of the latter type.

Let us first prove part 1 of the theorem. Due to the first order condition (3), combined with the equality $q(s) = q_0$ and the assumption that $e, d, a, b$ are real polynomials, we have

$$\lim_{s \to \infty} s^2 \left( \frac{a(s)}{d(s)} \right) = 1$$

Note that for any rational function $f$ defined in Theorem 5.1. This follows from the fact that this criterion function is not Hurwitz then the $H_2$-norm squared computed above is strictly greater than the global minimum of the criterion function $f$ as defined in Theorem 5.1.

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large segment of the imaginary axis. This is a standard argument in complex analysis that we will not repeat here (see e.g. [30]). The residue theorem now tells us that the integral is equal to

\[ q_0^2 \sum_{i=1}^{N} \text{Res}_{s=\delta_i} \left( \frac{a(-s)a(s)}{d(s)d(-s)} \right) = \]

\[ q_0^2 \sum_{i=1}^{N} \lim_{s \to \delta_i} \left( \frac{(s-\delta_i)a(-s)a(s)}{d(s)d(-s)} \right) = \]

\[ q_0^2 \sum_{i=1}^{N} \left( \frac{a(-\delta_i)a(\delta_i)}{d'(\delta_i)d(-\delta_i)} \right) = \]

\[ q_0^2 \sum_{i=1}^{N} \left( \frac{\tilde{a}(-\delta_i)\tilde{a}(\delta_i)}{d'(\delta_i)d(-\delta_i)} \right) \]

The first order conditions (??) can be rewritten as

\[ \tilde{a}(\delta_i) = \frac{\tilde{a}(-\delta_i)^2}{e(\delta_i)}, \quad i = 1, \ldots, N, \quad \tilde{a} \neq 0. \]

Substituting this and using \( x_i = \tilde{a}(-\delta_i) \) it follows that

\[ \| \frac{e(s)}{d(s)} - \frac{b(s)}{a(s)} \|^2_2 = \sum_{i=1}^{N} \frac{x_i^2}{e(\delta_i)d'(\delta_i)d(-\delta_i)}. \]

This shows 1.

Part 2 of the Lemma follows immediately from the fact that the \( L_2 \) norm and the \( H_2 \) norm coincide for all elements in \( H_2 \). (See also the remark made above after the definition of the \( L_2 \)-norm).

Proof of part 3: Suppose that \( a \) is not Hurwitz. Then it can be factored uniquely as \( a = a_1a_2 \), where \( a_1 \) and \( a_2 \) are monic and \( a_1(s) \) and \( a_2(s) \) are Hurwitz polynomials in the variable \( s \), with \( \text{deg}(a_1) < n \). There are corresponding polynomials \( b_1, b_2 \) with \( \text{deg}(b_1) < \text{deg}(a_1) \) and \( \text{deg}(b_2) < \text{deg}(a_2) \) such that \( \frac{b(s)}{a_1(s)} = \frac{b_1(s)}{a_1(s)} + \frac{b_2(s)}{a_2(s)} \). As is well-known (and following from Cauchy’s theorem in complex analysis)

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{b_1(i\omega)b_2(-i\omega)}{a_1(i\omega)a_2(-i\omega)} d\omega = 0 \]

and similarly

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e(i\omega)b_2(-i\omega)}{d(i\omega)a_2(-i\omega)} d\omega = 0. \]

From this well-known orthogonality property in \( L_2 \) it follows that

\[ \| \frac{e(s)}{d(s)} - \frac{b(s)}{a(s)} \|^2_2 = \]

\[ \| \frac{e(s)}{d(s)} - \frac{b_1(s)}{a_1(s)} \|^2_2 + \| \frac{b_2(s)}{a_2(s)} \|^2_2 \geq \]

\[ \| \frac{e(s)}{d(s)} - \frac{b_1(s)}{a_1(s)} \|^2. \]

This number is larger than the global minimum of the function \( f \) of Theorem [5,1] because \( \frac{b_1(s)}{a_1(s)} \) is the transfer function of a system of order \( < n \). As noted before it is well-known that the \( H_2 \)-norm squared of the difference between the original system and an approximant of order \( < n \), is always larger than the global minimum of the \( H_2 \)-norm squared of the difference between the original system and an approximant of order \( n \). This finishes the proof of part 3 and of the Theorem.

For any complex polynomial \( p \in \mathbb{C}[s] \) let \( \overline{p} \) denote the polynomial that is obtained when the coefficients of \( p \) are replaced by their complex conjugates. I.e. \( \overline{p} \) is the polynomial with the property that \( \overline{p}(r) = \overline{p}(r) \) for all \( r \in \mathbb{R} \), where \( s \) denotes the complex conjugate of a complex number \( s \).

**Lemma 5.1:** Let \( \frac{7}{2} \in \Sigma S_N \) have \( N \) distinct poles \( \delta_1, \ldots, \delta_N \in \mathbb{C} \).

Let \( a(s), b(s), q_0 \) be a complex solution of the polynomial equations [4],[5]. Then \( \overline{a}(s), \overline{b}(s), \overline{q_0} \) is also a solution.

The corresponding numbers \( \sum_{i=1}^{N} \frac{q_0a(-\delta_i)^3}{e(\delta_i)d'(\delta_i)d(-\delta_i)} \) and \( \sum_{i=1}^{N} \frac{q_0\tilde{a}(-\delta_i)^3}{e(\delta_i)d'(\delta_i)d(-\delta_i)} \) form a complex conjugate pair. In particular this implies that if one of these numbers is real the numbers are equal.
Proof. Consider a complex solution \(a(s), b(s), q_0\) of the first order equations \(e(s)a(s) - b(s)d(s) = a(-s)^2 q_0\). Because polynomials are completely determined by their restriction to the real numbers, an equivalent formulation of the first order equations is \(e(r)a(r) - b(r)d(r) = a(-r)^2 q_0\) for all \(r \in \mathbb{R}\). Complex conjugation of these equations gives \(e(r)\bar{a}(r) - b(r)d(r) = \bar{a}(-r)^2 \bar{q}_0\), which shows that \(a(s), b(s), \bar{q}_0\) is also a solution.

Because \(h\) is a real polynomial with distinct zeros the set of zeros of \(h\) consists of an even number, \(2l\), say, of complex solutions and \(n - 2l\) real solutions. The \(2l\) complex solutions can be partitioned into \(l\) pairs of complex conjugate solutions. It is easy to see that for each real zero \(\delta\) of \(h\),

\[
\frac{q_0 a(\delta)^3}{e(\delta)d'(\delta)d(-\delta)}
\]

and

\[
\frac{q_0 \bar{a}(\delta)^3}{e(\delta)d'(\delta)d(-\delta)}
\]

is a complex conjugate pair. And if \(\delta, \bar{\delta}\) is a complex conjugate pair of zeros of \(h\), then the complex conjugate of

\[
\frac{q_0 a(\delta)^3}{e(\delta)d'(\delta)d(\delta)} + \frac{q_0 \bar{a}(\delta)^3}{e(\delta)d'(\delta)d(-\delta)}
\]

is equal to

\[
\frac{\bar{q}_0 a(-\delta)^3}{e(\delta)d'(\delta)d(-\delta)} + \frac{\bar{q}_0 \bar{a}(-\delta)^3}{e(\delta)d'(\delta)d(-\delta)}.
\]

Combining this it follows that

\[
\sum_{i=1}^{N} \frac{q_0 a(-\delta_i)^3}{e(\delta_i)d'(\delta_i)d(-\delta_i)}
\]

and

\[
\sum_{i=1}^{N} \frac{q_0 \bar{a}(-\delta_i)^3}{e(\delta_i)d'(\delta_i)d(-\delta_i)}
\]

form a complex conjugate pair.

For ease of reference, let \(\phi\) be defined by \(\phi: \Xi \to \mathbb{C}, x \mapsto \sum_{i=1}^{N} \frac{x^3}{e(\delta_i)d'(\delta_i)d(-\delta_i)}\).

Using the results above one can find the global minimum of the criterion function as follows. For each of the at most \(2^N - 1\) elements of \(\Xi\), evaluate the numbers \(\phi(x) \in \mathbb{C}\). At least one of these numbers will be real and positive. Let \(k\) denote the number of distinct real positive numbers obtained in this way and let us denote these numbers by \(m_1, \ldots, m_k\) where \(m_1 < \ldots < m_k\). Consider the set \(\phi^{-1}(m_1)\). If each \(\xi \in \phi^{-1}(m_1)\) corresponds to a complex non-real solution \(a(s), b(s), q_0\) of the polynomial equations \((4), (5)\), there must be an even number of such solutions, as a result of Lemma \(5.1\). If any of the solutions is real then according to Theorem \(5.2\) the global minimum is equal to \(m_1\) and all real solutions \(a(s), b(s), q_0\) that correspond to this number are global approximants. If none of the solutions that correspond to \(\xi \in \phi^{-1}(m_1)\) are real then consider the set \(\phi^{-1}(m_2)\). If any of the corresponding solutions \(a(s), b(s), q_0\) is real then \(m_2\) is the global minimum, otherwise consider the solutions that correspond to \(m_3\) etc. One of the numbers \(m_1, \ldots, m_k\) is the global minimum and therefore the global minimum will be found in this way. It follows from Theorem \(5.2\) that all real solutions \(a(s), b(s), q_0\) that correspond to the global minimum are in fact admissible, i.e. \(a(s)\) is Hurwitz and \(a(s)\) and \(b(s)\) are coprime.

Remark. Note that the function \(\phi\) is a polynomial and therefore continuous and smooth. Depending on the size of the coefficients \(\frac{1}{e(\delta_i)d'(\delta_i)d(-\delta_i)}\), a perturbation in \(x\) due to numerical round-off error may cause a limited perturbation in the corresponding value of \(\phi\). This implies that if the size of the coefficients just mentioned is not too big, and the perturbations in \(x\) are limited then the effects of round-off error on the calculated critical values are limited. This can be contrasted with the possible effect of perturbations on the calculation of the critical points. Especially if a critical point \(\frac{\partial \phi}{\partial x} \in \Sigma S_{N-1}\) has poles near the imaginary axis, a small perturbation may produce a denominator polynomial with one or more right half-plane zeros, and therefore an inadmissible system, outside the manifold \(\Sigma S_{N-1}\). Note that even if due to round-off error our algorithm would not produce a reliable global approximant, knowledge of the value of the global minimum of the criterion function could be used to evaluate the performance of other algorithms for the \(H_2\) model order reduction problem.

Remark. The formula for \(\phi\) in the Theorem can be used to build the critical value matrix \(A_F\) that was mentioned at the end of the previous section, by taking the polynomial \(f\) mentioned there equal to \(\phi\). Note that because \(\phi\) is a polynomial no matrix inversion is required in the calculation of \(A_F = \phi(A_{X_1}, \ldots, A_{X_N})\). The matrix \(A_F\) can also be built up by direct construction of the matrix of the endomorphism \(F\) with respect to the basis \(B\) of monomials defined earlier.

\(\square\)
VI. REPEATED POLES

In this section we briefly outline how the development is changed if any of the poles of the original system are repeated, and indicate the additional difficulty which arises in that case. For simplicity of exposition we assume that one pole has multiplicity two: \( \delta_1 = \delta_2 \), and the other poles are distinct. In this case \((10)\) gives only \( N - 1 \) independent equations. An additional equation is obtained by differentiating \((3)\), which leads to

\[
e(\delta_1)\hat{a}'(\delta_1) + e'(\delta_1)\hat{a}(\delta_1) = -2\hat{a}(-\delta_1)\hat{a}'(-\delta_1)
\]

(Note that we have used \( d(\delta_1) = d'(\delta_1) = 0 \) here.) Taking \( x_1 = \hat{a}(-\delta_1), x_2 = \hat{a}'(-\delta_1), x_i = \hat{a}(-\delta_i) \) for \( i = 3, \ldots, N \), one obtains again a system of \( N \) quadratic polynomial equations in \( x_1, \ldots, x_N \) representing the first-order conditions.

This system of equations will not yet be in Gröbner basis form, in contrast to the case of distinct poles. So at this point it is necessary to employ Buchberger’s algorithm to obtain a Gröbner basis for the corresponding ideal. Subsequently the Stetter-Möller matrix method can again be used to find the critical points and hence the global optimum, provided that the number of critical points is finite. As far as we are aware, there is as yet no guarantee that this is the case.

If \( \delta_1 \) has multiplicity greater than two then higher-order differentiation of \((3)\) is needed, but otherwise the generalization is rather straightforward. If there are several repeated poles a similar approach can be followed.

VII. EXAMPLES

A. General

This section presents two examples on solving the \( H_2 \) model reduction problem and discusses several computational issues. The following is an outline of the algorithm implemented:

1) For the given \( N \)-th order transfer function to be reduced, construct the \( N \)-by-\( N \) matrix \( M \) (see equation \((13)\)).

2) For \( i = 1, \ldots, N \), construct the \( 2^N \)-by-\( 2^N \) matrix \( A_{X_i} \) from \( M \) (see Theorem \((4,5)\) and the following paragraph, and note that the reduction procedure of section \((III)\) is crucial in enabling this to be done).

3) Compute the eigenvalues and eigenvectors of all the \( A_{X_i} \)'s. Assume, for simplicity, that each \( A_{X_i} \) has a simple Jordan structure. Arrange these eigenvalues and eigenvectors such that the \( j \)-th eigenvector of \( A_{X_{i_1}} \) corresponds to the \( j \)-th eigenvector of \( A_{X_{i_2}} \) for all \( j = 1, \ldots, 2^N \) and \( i_1, i_2 = 1, \ldots, N \). Letting \( \xi_{i,j} \) denote the \( j \)-th eigenvalue of \( A_{X_{i_1}} \), form the \( N \)-tuples \((\xi_{1,j}, \ldots, \xi_{N,j})\) for \( j = 1, \ldots, 2^N \). Now each of these \( N \)-tuples contains the eigenvalues that correspond to one of the common eigenvalues of the set \( \{ A_{X_i} \} \). Our current implementation of this step uses numerical methods, so there are potential problems which can arise if eigenvalues and/or eigenvectors are repeated, or nearly so. We have not attempted to cope with all such eventualities.

4) Solve for \( \bar{a}_i \), using equation \((10)\), by making the association

\[
[\bar{a}(-\delta_1), \ldots, \bar{a}(-\delta_N)] = [\xi_{1,j}, \ldots, \xi_{N,j}].
\]

Normalise the coefficients such that \( a_{N-1} = 1 \) to obtain \( a_i \). Eliminate those polynomials \( a(s) = s^{N-1} + a_{N-2} s^{N-2} + \ldots + a_0 \) which are not admissible pole polynomials of an approximating system, because they are not real Hurwitz.

5) For each admissible pole polynomial \( a(s) \), obtain the zero polynomial \( b(s) \) from equation \((4)\). In practice the equation does not hold exactly, so a least-squares solution is found.

All the above steps except that of computing eigenvalues and eigenvectors can in principle be performed symbolically. Two different implementations have been attempted and they differ only in whether step 4 is performed symbolically or numerically; note that steps 3 and 4 are done numerically here. For the symbolic implementation of step 2 the \( A_{X_i} \)'s are computed from a symbolic definition of \( M = [m_{j,k}] \) using computer algebra software\(^1\) and the resulting symbolic expressions for the \( A_{X_i} \)'s (see the Appendix) are stored in a file to be read in by numerical software\(^2\) later. This has the advantage that the symbolic computation only has to be performed once for a given model order. Unfortunately, the length of these symbolic expressions soon becomes very large as the model order increases; the size of the file storing these expressions comes to more than 5 Mbytes for model order equal to 7 and this thus presents a practical limit to this implementation. Alternatively, due to the simplicity of the reduction procedure (see section \((III)\), step 2) can be implemented in a straightforward manner in a numerical package\(^3\). In this case, the highest model order that we could reduce is 9, which involves storing 9 \( 512 \times 512 \) matrices, and we ran into memory problems for model orders higher than this. The computer we used was a Sun Ultra 10, 300 MHz processor with 640 MByte RAM.

There are a number of numerical issues pertaining to this algorithm. Some of these issues are well known, e.g. possible ill-conditioning of Vandermonde matrices and the computation of eigenvalues and eigenvectors. These numerical problems will also cause difficulty in later steps of the algorithm. For example, numerical error may cause us to regard a real polynomial as complex in step 4 and as a result, a true local minimum of the problem may be mistakenly considered as inadmissible. The current implementation of this algorithm does not strive to overcome nor detect these problems. It is also beyond the

\(^1\)In our case, Maple.  
\(^2\)In our case, Matlab.
scope of this paper to give full numerical analysis of the proposed algorithm of this paper. A rudimentary check that we have
employed is to examine the least-squares error in step 5; however, this error must be interpreted with care as a small residual
to does not necessarily indicate an accurate solution [17]. Moreover, this check will not be able to tell us whether a correct
solution has been rejected. We have applied our algorithm to the third order systems that were investigated in [18] where
a symbolic algorithm was used to reduce them to second order systems. In this case, symbolic computation ensures that all
stationary points of the problem are computed and we find that the algorithm of this paper is able to find the same sets of
critical points as those reported in [18]. This comparison may indicate that our algorithm is likely to return the entire set of
stationary points when the model order is small.

B. Example 1: An easily reduced system

The system to be reduced is a 9th order transfer function and it is the highest order model that we could reduce thus far.
This system has Hankel singular values 9, 8, . . . , 2, 1 and its transfer function is
\[ s^8 + 1.4240 s^7 + 1.0946 s^6 + 0.2371 s^5 + 0.0134 s^4 + 1.1781 s^3 + 0.4557 s^2 + 0.0627 s + 0.0028. \]

The system to be reduced is taken from p.162 of [39] and is given by
\[ G(s) = \sum_{j=1}^{N} \frac{A^2_j}{s + A^2_j} \quad \text{with} \quad \alpha > 0. \]  

It is shown in [39] that all the Hankel singular values of this system tend to \( \frac{1}{2} \) as \( \alpha \to \infty \). On the other hand, when \( \alpha \approx 1 \)
and \( N > 1 \), the system is close to non-minimality as \( \alpha = 1 \) gives rise to a first order system. Our algorithm has numerical
difficulty when \( \alpha \) is chosen either too large or too close to 1. In both cases, the Vandermonde matrix becomes ill-conditioned:
the rows contain entries of drastically different magnitude in the first case and the poles are too close to each other in the
second.

Since the poles of this system are all real, this gives rise to a real \( M \) matrix and in turn real \( A_X \)'s. Due to the form of
Gröbner basis defined by \( M \), zero is always an eigenvalue of \( A_X \), (independent of whether \( M \) is real or complex). Since
the dimension of \( A_X \) is \( 2^N \) — an even number — and \( A_X \), is real, \( A_X \) must have at least one other non-zero real eigenvalue.
For \( \alpha \) close to zero or unity, we find in our examples there is a real eigenvalue that is approximately zero and the eigenvectors
corresponding to this eigenvalue and the zero eigenvalue are almost parallel to each other. This gives rise to difficulty in
matching the eigenvectors.

For model order \( N = 5 \), our algorithm succeeded in finding an approximant for systems with \( \alpha \) in the interval \([0.38, 0.79]\)
but failed in the intervals \((0, 0.38)\) and \((0.79, 1)\). For \( \alpha \) in the intervals \((0, 0.38)\) and \((0.84, 1)\), our algorithm returns no solution as
it either has difficulty in matching the eigenvectors or has rejected the admissible solutions because they are not real Hurwitz.
Our algorithm does return a solution for \( \alpha \in (0.79, 0.84) \) but a closer examination of the obtained approximant shows that it is
not a relaxation system. Since the system in equation \((18)\) is a relaxation system and it is proved in [4] that \( H_2 \) approximants
of relaxation systems are also relaxation systems, it implies that the solution given by our algorithm for this range of \( \alpha \)
is unacceptable.

It is also shown in [4] that any stable relaxation system, whose poles all have modulus smaller than \( \frac{1}{\sqrt{2}} \approx 0.707 \), has only
one admissible solution of the first-order optimality conditions. For \( \alpha = 0.78 \), the largest pole is \( 0.6084 \) and there should
therefore be only one such solution. For this case our algorithm returns precisely one admissible solution, in accordance with
this theory. It has absolute error 0.0334, which can be compared to the norm 1.6980 of the original system to give a relative
error of 1.96%. The transfer function of this approximant is
\[ 1.4240 s^7 + 1.0946 s^6 + 0.2371 s^5 + 0.0134 s^4 + 1.1781 s^3 + 0.4557 s^2 + 0.0627 s + 0.0028, \]
which can be shown to be a relaxation system.

As an alternative to the algorithm described at the beginning of this section, we have also treated Example 2 using an
algorithm based on building up the critical value matrix using \((16)\). The same results were obtained with both algorithms,
except when \( \alpha \) was very close to 1. For example with \( N = 2 \) and \( \alpha = 0.999 \) the first algorithm continued to give the correct
result (which was checked using exact algebraic calculation) but the second did not, because of numerical imprecision.

C. Example 2: A relaxation system

The system to be reduced is taken from p.162 of [39] and is given by
\[ G(s) = \sum_{j=1}^{N} \frac{A^2_j}{s + A^2_j} \quad \text{with} \quad \alpha > 0. \]  

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one admissible solution of the first-order optimality conditions. For \( \alpha = 0.78 \), the largest pole is \( 0.6084 \) and there should
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As an alternative to the algorithm described at the beginning of this section, we have also treated Example 2 using an
algorithm based on building up the critical value matrix using \((16)\). The same results were obtained with both algorithms,
except when \( \alpha \) was very close to 1. For example with \( N = 2 \) and \( \alpha = 0.999 \) the first algorithm continued to give the correct
result (which was checked using exact algebraic calculation) but the second did not, because of numerical imprecision.
The application of constructive algebra methods to the $H_2$ approximation problem offers the possibility of guaranteed location of the globally optimal approximant, despite the fact that this is a non-convex problem. Furthermore, the location of this optimal approximant could, in principle, be computed to any desired precision, by employing ‘symbolic’ methods throughout. One can envision, however, that these methods could be used in conjunction with more conventional numerical methods in a number of ways, to obtain various precision/efficiency trade-offs. One possibility is the one used by us to solve the examples in this paper, namely to employ conventional numerical eigenvalue solvers from the point at which the matrices $A_X$, have been determined. Another possibility would be to use constructive algebra methods to obtain an upper bound for the number of admissible critical points, and/or the value of the criterion function at the optimal approximant (which can be done without computing the optimal approximant itself), and to use these results to check the candidate optima obtained by a conventional numerical optimization approach.

It should be kept in mind that constructive algebra also offers the possibility of dealing with purely symbolic problem specifications — that is, of producing ‘generic’ results (for all transfer functions of a given order, say) rather than results for one specific system. This can be done in principle, although in practice the complexity of the required computations is well beyond current possibilities.

The commutative matrix approach which we have used to solve the system of critical-point (polynomial) equations is currently the subject of intense research activities in the computer algebra community, and in the systems theory community [7], [8] with good prospects of much more efficient algorithms being developed. We therefore expect that it will soon be possible to approximate higher-order systems than the ones we have been able to tackle in this paper, using essentially the same methods. Also, we expect that such developments will make constructive algebra methods attractive and feasible tools for tackling a wider range of problems in systems and control theory.

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