NEW INTERPRETATIONS FOR NONCROSSING PARTITIONS OF
CLASSICAL TYPES

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Abstract. We interpret noncrossing partitions of type $B$ and type $D$ in terms of noncrossing
partitions of type $A$. As an application, we get type-preserving bijections between noncrossing
and nonnesting partitions of type $B$, type $C$ and type $D$ which are different from those in the
recent work of Fink and Giraldo. We also define Catalan tableaux of type $B$ and type $D$, and
find bijections between them and noncrossing partitions of type $B$ and type $D$ respectively.

1. Introduction

A partition of a set $U$ is a collection of mutually disjoint nonempty subsets of $U$, called blocks,
whose union is equal to $U$. Let $\Pi(n)$ denote the set of partitions of $[n] = \{1, 2, \ldots, n\}$. For
$\pi \in \Pi(n)$, an edge of $\pi$ is a pair $(i, j)$ of integers $i$ and $j$ with $i < j$ such that $i$ and $j$ are in
the same block of $\pi$ and this does not contain any integer between them.

A partition $\pi \in \Pi(n)$ is called noncrossing (resp. nonnesting) if $\pi$ does not have two edges $(a, b)$
and $(c, d)$ satisfying $a < c < b < d$ (resp. $a < c < d < b$). We denote by $NC(n)$ (resp. $NN(n)$) the
set of noncrossing (resp. nonnesting) partitions of $[n]$.

Recently, noncrossing and nonnesting partitions have received great attention and have been
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For a partition $\sigma \in \Pi(n)$, a block $B$ of $\sigma$ is called nonnested (resp. nonaligned) if there is no edge $(i, j)$ of $\sigma$ with $i < \min(B) \leq \max(B) < j$ (resp. $\max(B) < i$). We denote by

\[ \#\Pi_B(n) = \sum_{k=1}^{n} S(n, k)t_{k+1}. \]
NNBK(σ) (resp. NABK(σ)) the set of nonnested (resp. nonaligned) blocks of σ. We define
\[ NC^{NN}(n) = \{ (σ, X) : σ \in NC(n), X \subseteq NNBK(σ) \}, \]
\[ NC^{NA}(n) = \{ (σ, X) : σ \in NC(n), X \subseteq NABK(σ) \}, \]
\[ NN^{NA}(n) = \{ (σ, X) : σ \in NN(n), X \subseteq NABK(σ) \}. \]

We denote by \( NC^{NN}_{\{0, \pm 1\}}(n) \) (resp. \( NC^{NA}_{\{0, \pm 1\}}(n) \) and \( NN^{NA}_{\{0, \pm 1\}}(n) \)) the set of triples \((σ, X, ϵ)\), where \((σ, X)\) is in \( NC^{NN}(n) \) (resp. \( NC^{NA}(n) \) and \( NN^{NA}(n) \)) and \( ϵ \in \{-1,0,1\} \) with the additional condition that if \( X = \emptyset \) then \( ϵ = 0 \).

By using our interpretation for \( Π_B(n) \), we obtain a bijection between \( NC_B(n) \) (resp. \( NN_B(n) \), \( NN_C(n) \)) and \( NC^{NN}(n) \) (resp. \( NN^{NA}(n) \), \( NN^{NA}(n) \)). Similarly we get a bijection between \( NC_D(n) \) (resp. \( NN_D(n) \)) and \( NC^{NN}_{\{0, \pm 1\}}(n-1) \) (resp. \( NN^{NA}_{\{0, \pm 1\}}(n-1) \)). Since \( NC^{NN}(n) \) and \( NC^{NN}_{\{0, \pm 1\}}(n-1) \) concern only type A noncrossing partitions, our interpretations have the advantage of understanding \( NC_B(n) \) and \( NC_D(n) \) as easily as \( NC(n) \).

To make a connection between noncrossing and nonnesting partitions in our interpretations we find an involution on \( NC(n) \) which interchanges the nonnesting blocks and the nonaligned blocks. Thus, as a byproduct, we get that the nonnesting blocks and the nonaligned blocks have a joint symmetric distribution on \( NC(n) \), in other words,
\[ \sum_{π \in NC(n)} x^{nn(π)} y^{na(π)} = \sum_{π \in NC(n)} x^{na(π)} y^{nn(π)}, \]
where \( nn(π) \) (resp. \( na(π) \)) denotes the number of nonnesting (resp. nonaligned) blocks of \( π \).

Combining our bijections together with the bijection between \( NC(n) \) and \( NN(n) \) due to Athanasiadis [2], we obtain type-preserving bijections, i.e. bijections preserving block sizes, between noncrossing and nonnesting partitions of classical types. Our type-preserving bijections are different from those of Fink and Giraldo [11].

We provide another interpretation for \( NC_B(n) \) and \( NC_D(n) \): a bijection between \( NC_B(n) \) and the set \( Μ(n) \) of pairs \((σ, x)\) where \( σ \in NC(n) \) and \( x \) is either \( \emptyset \), an edge of \( σ \) or a block of \( σ \), and a bijection between \( NC_D(n) \) and the set \( Ω(n) \) of pairs \((σ, x)\) where \( σ \in NC(n-1) \) and \( x \) is either \( \emptyset \), an edge of \( σ \) or a block of \( σ \) or an integer in \( \{-n+1, 0, n\} \). In fact, \( Μ(n) \) and \( Ω(n) \) are essentially the same as \( NC(n) \times [n+1] \) and \( NC(n-1) \times [3n-2] \) respectively. Using these interpretations, we give another proof of the formula for the number of noncrossing partitions of type \( B_n \) and type \( D_n \) with given block sizes.

It is well known that \( NC(n) \) is in bijection with the set of Dyck paths, i.e. lattice paths from \((0,0)\) to \((n,n)\) which do not go below the line \( y=x \). Using \( NC^{NA}(n) \) and \( NC^{NA}_{\{0, \pm 1\}}(n-1) \) we find a bijection between \( NC_B(n) \) and the set \( LP(n) \) of lattice paths from \((0,0)\) to \((n,n)\) and a bijection between \( NC_D(n) \) and the set \( LP(n) \) of lattice paths in \( LP(n) \) which do not touch \((n-1,n-1)\) and \((n-1,n-1)\) simultaneously.

Permutation tableaux were first introduced by Postnikov [16] in the study of the totally nonnegative Grassmannian. Catalan tableaux are special permutation tableaux. Permutation tableaux and Catalan tableaux are respectively in bijection with permutations and noncrossing partitions; see [7, 9, 15, 20]. Lam and Williams [14] defined permutation tableaux of type \( B_n \). In this paper we define Catalan tableaux of type \( B_n \) and \( D_n \) which are special permutation tableaux of type \( B_n \). Then we find bijections between them and \( NC_B(n) \) and \( NC_D(n) \).

The rest of this paper is organized as follows. In Section 2 we recall the definitions of noncrossing and nonnesting partitions of finite reflection groups and the combinatorial models for them for classical reflection groups. In Section 3 we define a map from \( Π_B(n) \) to the set of certain triples. In Section 4 we give new interpretations for \( NC_B(n) \), \( NC_D(n) \), \( NN_B(n) \), \( NN_C(n) \) and \( NN_D(n) \). In Section 5 we find type-preserving bijections between noncrossing and nonnesting partitions of classical types. In Section 6 we find a bijection between \( NC_B(n) \) (resp. \( NC_D(n) \)) and \( Μ(n) \) (resp. \( Ω(n) \)). In Section 7 we find a bijection between \( NC_B(n) \) (resp. \( NC_D(n) \)) and \( LP(n) \) (resp. \( LP(n) \)). In Section 8 we define the sets \( CT_B(n) \) and \( CT_D(n) \) of Catalan tableaux of type \( B_n \) and type \( D_n \), and find bijections between them and \( NC_B(n) \) and \( NC_D(n) \) respectively.
General definitions for noncrossing and nonnesting partitions. For a finite Coxeter system $(W, S)$ with the set $T = \{wsw^{-1} : s \in S, w \in W\}$ of reflections, the absolute length $\ell_T(w)$ of an element $w \in W$ is defined to be the smallest integer $i$ such that $w$ can be written as a product of $i$ reflections. The absolute order on $W$ is defined as follows: $u \preceq_T w$ if and only if $\ell_T(w) = \ell_T(u) + \ell_T(w^{-1}w)$. Then the noncrossing partition poset $NC(W)$ is defined to be the interval $\{w \in W : 1 \leq_T w \leq_T c\}$, where $c$ is a Coxeter element. It turns out that $NC(W)$ does not depend on the particular choice of $c$ up to isomorphism.

Nonnesting partitions are defined for crystallographic reflection groups. Suppose $W$ is a crystallographic reflection group and $\Phi^+$ is a positive root system of $W$. The root poset $(\Phi^+, \preceq)$ has the partial order $\alpha \leq \beta$ if and only if $\beta - \alpha$ can be written as a linear combination of the positive roots with nonnegative integer coefficients. A nonnesting partition of $W$ is an antichain in the root poset $(\Phi^+, \preceq)$. We denote by $NN(W)$ the set of nonnesting partitions of $W$.

For classical types, we will use the following root posets:

- $\Phi^+(A_{n-1}) = \{e_i - e_j : 1 \leq i < j \leq n\}$,
- $\Phi^+(B_n) = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{e_i : 1 \leq i \leq n\}$,
- $\Phi^+(C_n) = \{e_i \pm e_j : 1 \leq i < j \leq n\} \cup \{2e_i : 1 \leq i \leq n\}$,
- $\Phi^+(D_n) = \{e_i \pm e_j : 1 \leq i < j \leq n\}$.

2. Combinatorial models. We use the definitions in [11]. For type $D_n$, our definitions are stated in a slightly different way from those in [11], but one can easily check that they are equivalent.

For a partition $\pi$ of a finite set $U$ and a total order $a_1 \prec a_2 \prec \cdots \prec a_n$ of $U$, the standard representation of $\pi$ with respect to the order $a_1 \prec a_2 \prec \cdots \prec a_n$ is the drawing obtained as follows. Arrange $a_1, a_2, \ldots, a_n$ in a horizontal line. Draw an arc between $a_i$ and $a_j$ for each pair $(i, j)$ with $i < j$ such that $a_i, a_j \in B$ for a block $B$ of $\pi$ which does not contain $a_t$ with $i < t < j$. See Figure 4.

We say that $\pi$ is noncrossing (resp. nonnesting) with respect to the order $a_1 \prec a_2 \prec \cdots \prec a_n$ if $\pi$ satisfies the following condition: if $a_i, a_j \in B$ and $a_j, a_k \in B'$ (resp. $a_i, a_k \in B$ and $a_j, a_k \in B'$) for some blocks $B$ and $B'$ of $\pi$ and for some integers $i < j < k < \ell$, then we have $B = B'$. In other words, $\pi$ is noncrossing (resp. nonnesting) with respect to the order $a_1 \prec a_2 \prec \cdots \prec a_n$ if and only if the standard representation of $\pi$ with respect to this order does not have two arcs which cross each other (resp. two arcs one of which nests the other). For example, the partition in Figure 4 is noncrossing but not nonnesting with respect to the order written there.

A noncrossing partition (resp. nonnesting partition) is a partition of $[n]$ which is noncrossing (resp. nonnesting) with respect to the order $1 \prec 2 \prec \cdots \prec n$. See Figures 2 and 3 for an example.

Figure 1. The standard representation of $\{\{1, 3, 8\}, \{2\}, \{4, 5, 6\}, \{7\}, \{9, 10\}\}$ with respect to the order $4 \prec 3 \prec 8 \prec 1 \prec 5 \prec 2 \prec 6 \prec 7 \prec 10 \prec 9$.

Figure 2. A noncrossing partition of type $A_9$. 

2. Preliminaries

In this section we recall the definitions noncrossing and nonnesting partitions of finite reflection groups and the combinatorial models $NC_B(n), NC_D(n), NN_B(n), NNC(n)$ and $NN_D(n)$. 

- Figure 1.
- Figure 2.
- Figure 3.
We denote by NC($n$) (resp. NN($n$)) the set of noncrossing (resp. nonnesting) partitions of type $A_{n-1}$.

There is a natural bijection between NC($A_{n-1}$) and NC($n$). If we take $c = (1, 2, \ldots, n)$ for the Coxeter element, each element in NC($A_{n-1}$) can be written as a product of disjoint cycles of form $(a_1, a_2, \ldots, a_k)$ where $a_1 < a_2 < \cdots < a_k$. Then the bijection is simply changing each cycle $(a_1, a_2, \ldots, a_k)$ to the block $\{a_1, a_2, \ldots, a_k\}$. One can check that we always get a noncrossing partition. For example, $(1, 4, 10)(2, 3)(5, 6, 7, 9)(8) \in$ NC($A_9$) corresponds to the noncrossing partition in Figure 2. In fact, this bijection is a poset isomorphism if we order NC($n$) by refinement. Thus we have NC($A_{n-1}$) $\cong$ NC($n$).

Similarly, there is a natural bijection between NN($A_{n-1}$) and NN($n$). For an antichain $\pi$ of $\Phi^+(A_{n-1})$, we construct the corresponding nonnesting partition by making the edge $(i, j)$ for each element $e_i < e_j \in \pi$. For example, the nonnesting partition in Figure 3 corresponds to

$$\{e_1 - e_3, e_2 - e_4, e_4 - e_6, e_6 - e_9, e_5 - e_7, e_7 - e_{10}\} \in$\text{NN}(A_9).$$

Thus we have NN($n$) $\cong$ NN($A_{n-1}$).

In order to define combinatorial models for noncrossing and nonnesting partitions of other classical types, we need type $B$ partitions introduced by Reiner [17]. There is a natural way to identify $\pi \in \Pi(n)$ with an intersection of a collection of the following reflecting hyperplanes of type $A_{n-1}$:

$$\{x_i - x_j = 0 : 1 \leq i < j \leq n\}.$$

For example, $\{(1, 3, 4), (2, 6), (5)\}$ corresponds to

$$\{(x_1, \ldots, x_6) \in \mathbb{R}^6 : x_1 = x_3 = x_4, x_2 = x_6\}.$$  

With this observation Reiner [17] defined a partition of type $B_n$ to be an intersection of a collection of the following reflecting hyperplanes of type $B_n$:

$$\{x_i = 0 : 1 \leq i \leq n\} \cup \{x_i \pm x_j = 0 : 1 \leq i < j \leq n\}.$$

Note that we can also consider such an intersection as a partition of $[\pm n] = \{1, 2, \ldots, n, -1, -2, \ldots, -n\}$.

For example, the intersection

$$\{(x_1, \ldots, x_8) \in \mathbb{R}^8 : x_1 = -x_3 = x_6, x_5 = x_8, x_2 = x_4 = 0\}$$

corresponds to

$$\{\pm\{1, -3, 6\}, \{2, 4, -2, -4\}, \pm\{5, 8\}, \pm\{7\}\},$$

which means

$$\{\{1, -3, 6\}, \{-1, 3, -6\}, \{2, 4, -2, -4\}, \{5, 8\}, \{-5, -8\}, \{7\}, \{-7\}\}.$$  

Equivalently, we define a partition of type $B_n$ as follows.

A partition of type $B_n$ is a partition $\pi$ of $[\pm n]$ such that if $B$ is a block of $\pi$ then $-B = \{-x : x \in B\}$ is also a block of $\pi$, and there is at most one block, called a zero block, which satisfies $B = -B$. We denote by $\Pi_B(n)$ the set of partitions of type $B_n$.

Now we are ready to define combinatorial models for noncrossing and nonnesting partitions of other classical types.

A noncrossing partition of type $B_n$ is a partition $\pi \in \Pi_B(n)$ which is noncrossing with respect to the order $1 \prec 2 \prec \cdots \prec n \prec -1 \prec -2 \prec \cdots \prec -n$. See Figure 4 for an example. A noncrossing partition of type $D_n$ is a partition $\pi \in \Pi_B(n)$ such that

1. if $\pi$ has a zero block $B$, then $\{n, -n\} \subseteq B,$

2. otherwise, $\pi$ is a collection of the following reflecting hyperplanes

3. $\{x_i = 0 : 1 \leq i \leq n\} \cup \{x_i \pm x_j = 0 : 1 \leq i < j \leq n\}.$

4. a nonnesting partition of type $D_n$ is a partition $\pi \in \Pi_B(n)$ such that

5. if $\pi$ has a zero block $B$, then $\{n, -n\} \subseteq B,$
(2) $\pi' \in \text{NC}_B(n - 1)$, where $\pi'$ is the partition obtained from $\pi$ by taking the union of the blocks containing $n$ or $-n$ and removing $n$ and $-n$.

See Figure 5 for an example. We denote by $\text{NC}_B(n)$ (resp. $\text{NC}_D(n)$) the set of noncrossing partitions of type $B_n$ (resp. type $D_n$). Like type $A$, we have $\text{NC}_B(n) \cong \text{NC}(B_n)$ and $\text{NC}_D(n) \cong \text{NC}(D_n)$. We note that $\text{NC}_B(n)$ and $\text{NC}_D(n)$ can also be defined using circular representation, see [3][12][17]. However, the standard representation is more suitable for our purpose.

A nonnesting partition of type $B_n$ is a partition $\pi \in \Pi_B(n)$ such that $\pi_0$ is nonnesting with respect to the order $1 \prec \cdots \prec n \prec 0 \prec -n \prec \cdots \prec -1$, where $\pi_0$ is the partition of $[\pm n] \cup \{0\}$ obtained from $\pi$ by adding $0$ to the zero block if $\pi$ has a zero block, and by adding the singleton $\{0\}$ otherwise. See Figure 3 for an example. A nonnesting partition of type $C_n$ is a partition $\pi \in \Pi_B(n)$ which is nonnesting with respect to the order $1 \prec \cdots \prec n \prec -n \prec \cdots \prec -1$. See Figure 7 for an example. A nonnesting partition of type $D_n$ is a partition $\pi \in \Pi_B(n)$ such that

1. if $\pi$ has a zero block $B$, then $\{n, -n\} \subseteq B$,
2. $\pi' \in \text{NN}_B(n - 1)$, where $\pi'$ is the partition obtained from $\pi$ by taking the union of the blocks containing $n$ or $-n$ and removing $n$ and $-n$.

See Figure 8 for an example. We denote by $\text{NN}_B(n)$ (resp. $\text{NN}_C(n)$ and $\text{NN}_D(n)$) the set of nonnesting partitions of type $B_n$ (resp. type $C_n$ and type $D_n$). Then we have $\text{NN}_B(n) \cong \text{NN}(B_n)$, $\text{NN}_C(n) \cong \text{NN}(C_n)$ and $\text{NN}_D(n) \cong \text{NN}(D_n)$.

3. PARTITIONS OF CLASSICAL TYPES

For $\pi \in \Pi_B(n)$ and a block $B$ of $\pi$, let $B^+$ (resp. $B^-$) denote the set of positive (resp. negative) integers in $B$. Note that $(-B)^+ = -(B^-)$. We define $\alpha(\pi), \beta(\pi)$ and $\gamma(\pi)$ as follows:

- $\alpha(\pi)$ is the partition in $\Pi(n)$ such that $A \in \alpha(\pi)$ if and only if $A = B^+$ for some $B \in \pi$,
- $\beta(\pi)$ is the set of blocks $A \in \alpha(\pi)$ such that $\pi$ has a block containing $A$ and at least one negative integer,
- $\gamma(\pi)$ is the matching on $\beta(\pi)$ such that $\{A_1, A_2\} \in \gamma(\pi)$ if and only if $A_1 \neq A_2$ and $A_1 \cup (-A_2)$ is a block of $\pi$.

Example 3.1. If $\pi = \{\pm\{1, -3, 6\}, \{2, 4, -2, -4\}, \pm\{5, 8\}, \pm\{7\}\}$, we have $\alpha(\pi) = \{\{1, 6\}, \{2, 4\}, \{3\}, \{5, 8\}, \{7\}\}, \beta(\pi) = \{\{1, 6\}, \{2, 4\}, \{3\}\}$ and $\gamma(\pi)$ is the matching on $\beta(\pi)$ with the only one matching pair $\{\{1, 6\}, \{3\}\}$.

Assume that a block $A \in \beta(\pi)$ is not matched in $\gamma(\pi)$. If $B$ is the block of $\pi$ with $A = B^+$, we have $B^+ \cap (-B^-) \neq \emptyset$ because otherwise $A$ would be matched with another block $A' = -(B^-) = -(B^+)$. Thus we have an integer $i$ both in $B^+$ and $-(B^-)$, which implies $i, -i \in B$. Therefore $B$ is a zero block of $\pi$, which is unique. This argument shows that $\gamma(\pi)$ is a maximal matching on $\beta(\pi)$. In other words, if $|\beta(\pi)|$ is even, then $\gamma(\pi)$ is a complete matching on $\beta(\pi)$; and if $|\beta(\pi)|$ is odd, then there is a unique unmatched block $A \in \beta(\pi)$ in $\gamma(\pi)$, and in this case, $\pi$ has the zero block $A \cup (-A)$.

It is easy to see that $\pi$ can be reconstructed from $(\alpha(\pi), \beta(\pi), \gamma(\pi))$. Thus we get the following proposition.

Proposition 3.1. The map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ is a bijection between $\Pi_B(n)$ and the set of triples $(\sigma, X, Y)$, where $\sigma \in \Pi(n)$, $X$ is a set of blocks of $\sigma$ and $Y$ is a maximal matching on $X$.

Now we define $\alpha_0(\pi) = \alpha(\pi) \cup \{\{0\}\}$, which is a partition of $[n] \cup \{0\}$, and $\gamma_0(\pi)$ to be the matching on the blocks of $\alpha_0(\pi)$ defined as follows. If $\gamma(\pi)$ is a complete matching, then the matching pairs of $\gamma(\pi)$ and $\gamma_0(\pi)$ are the same. If there is an unmatched block $A$ in $\gamma(\pi)$, which is necessarily unique, then the matching pairs of $\gamma_0(\pi)$ are those in $\gamma(\pi)$ and $\{\{0\}, A\}$. Note that $\gamma_0(\pi)$ is not necessarily a maximal matching.

Example 3.2. If $\pi$ is the partition in Example 3.1 we have

$\alpha_0(\pi) = \{\{0\}, \{1, 6\}, \{2, 4\}, \{3\}, \{5, 8\}, \{7\}\}$,

and $\gamma_0(\pi)$ is the matching on $\alpha_0(\pi)$ with the two matching pairs $\{\{1, 6\}, \{3\}\}$ and $\{\{0\}, \{2, 4\}\}$. 
Since $\gamma_0(\pi)$ determines $\beta(\pi)$ and $\gamma(\pi)$, we get the following.

**Proposition 3.2.** The map $\pi \mapsto (\alpha(\pi), \gamma_0(\pi))$ is a bijection between $\Pi_B(n)$ and the set of pairs $(\sigma, X)$ where $\sigma \in \Pi(n)$ and $X$ is a matching on the blocks of the partition $\sigma \cup \{\{0\}\}$.

If $\alpha(\pi)$ has $k$ blocks, then $\alpha_0(\pi)$ has $k + 1$ blocks. Let $A_1, A_2, \ldots, A_{k+1}$ be the blocks of $\alpha_0(\pi)$ with $\max(A_1) < \max(A_2) < \cdots < \max(A_{k+1})$. By identifying the block $A_i$ with the integer $i$, we can consider $\gamma_0(\pi)$ as a matching on $[k + 1]$ or an involution on $[k + 1]$. Thus we get the following formula for the cardinality of $\Pi_B(n)$.

**Corollary 3.3.** The cardinality of $\Pi_B(n)$ is equal to

$$\sum_{k=1}^{n} S(n, k)t_{k+1},$$

where $S(n, k)$ is the Stirling number of the second kind and $t_n$ is the number of involutions on $[n]$.

Note that the formula in Corollary 3.3 is a type $B$ analog of $\#\Pi(n) = \sum_{k=1}^{n} S(n, k)$.

4. Interpretations for noncrossing and nonnesting partitions

The following terminologies will be used for the rest of this paper.

An integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a weakly decreasing sequence of positive integers. Each $\lambda_i$ is called part of $\lambda$ and $\ell$ is called length of $\lambda$. We define $|\lambda|$ to be the sum $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell$ of all parts of $\lambda$. We will also consider $\lambda$ as the multiset $\{1^{m_1}, 2^{m_2}, \ldots\}$, where $m_i$ is the number of parts equal to $i$ in $\lambda$.

For two multisets $A$ and $B$, let $A \uplus B$ denote the multiset union of $A$ and $B$.

For a subset $S$ of $[n]$ and a partition $\pi$ of $S$, the type $\text{type}(\pi)$ of $\pi$ is the integer partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ such that $m_i$ is equal to the number of blocks of size $i$ in $\pi$. The type $\text{type}(\pi)$ of a partition $\pi \in \Pi_B(n)$ is the integer partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ such that $m_i$ is equal to the number of unordered pairs $(B, -B)$ of nonzero blocks of size $i$ in $\pi$.

Recall the sets $\text{NC}^{\text{NN}}(n)$, $\text{NC}^{\text{NA}}(n)$, $\text{NN}^{\text{NA}}(n)$, $\text{NC}^{\text{NN}}_{\{0, \pm 1\}}(n)$, $\text{NC}^{\text{NA}}_{\{0, \pm 1\}}(n)$ and $\text{NN}^{\text{NA}}_{\{0, \pm 1\}}(n)$ in Definition 1.1.

**Notation.** From now on, if we write $\{A_1, A_2, \ldots, A_k\} <$, it is automatically assumed that $A_i$’s are sorted in increasing order by their largest elements, that is, $\max(A_1) < \max(A_2) < \cdots < \max(A_k)$.

For a set $X = \{A_1, A_2, \ldots, A_{2k}\} <$ of even number of blocks, we define $\text{pairing}(X)$ to be the following multiset:

$$\text{pairing}(X) = \{|A_1 \cup A_{2k}|, |A_2 \cup A_{2k-1}|, \ldots, |A_{k} \cup A_{k+1}|\}.$$

4.1. Noncrossing partitions. Let $\pi \in \text{NC}_B(n)$ and consider the map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ in the previous section. Since $\pi$ is noncrossing with respect to the order $1 < 2 < \cdots < n < -1 < -2 < \cdots < -n$, one can easily see that $\alpha(\pi) \in \text{NC}(n)$, all the blocks in $\beta(\pi)$ are nonnested, and the matching $\gamma(\pi)$ is uniquely determined by $\beta(\pi)$. For instance, if $\beta(\pi) = \{A_1, A_2, \ldots, A_k\} <$, then $\gamma(\pi)$ is the matching consisting of $\{A_i, A_{k+1-i}\}$ for all $1 \leq i \leq \lfloor k/2 \rfloor$.

For $\pi \in \text{NC}_B(n)$, we define $\phi_{B}^{\text{NC}}(\pi) = (\alpha(\pi), \beta(\pi))$. In other words, $\phi_{B}^{\text{NC}}(\pi)$ is the pair $(\sigma, X)$ where $\sigma$ is the partition obtained from $\pi$ by removing all the negative integers and $X$ is the set of blocks of $\sigma$ which are properly contained in some blocks of $\pi$. Note that we have $\phi_{B}^{\text{NC}}(\pi) \in \text{NC}^{\text{NN}}(n)$. 

![Figure 4. The standard representation of an element in NC_B(10) with respect to the order 1 < 2 < ... < 10 < -1 < -2 < ... < -10.](image)
Proposition 4.1. The map \( \phi^{\text{NC}} \) where 

\[ \sigma = \{\{1, 4, 5\}, \{2, 3\}, \{6\}, \{7, 9\}, \{8, 10\} \} \]

and \( X = \{\{1, 4, 5\}, \{7, 9\}, \{10\}\} \).

From the construction, one can easily prove the following proposition.

Proposition 4.2. The map \( \phi^{\text{NC}}_B : \text{NC}_B(n) \to \text{NC}^{\text{NN}}(n) \) is a bijection. Moreover, if \( \phi^{\text{NC}}_B(\pi) = (\sigma, X) \) and \( X = \{A_1, A_2, \ldots, A_k\} \_< \), then

\[ \text{type}(\pi) = \text{type}(\sigma \setminus X) \cup T, \]

where

\[ T = \begin{cases} \text{pairing}(X), & \text{if } k \text{ is even}, \\ \text{pairing}(X \setminus \{A_{(k+1)/2}\}), & \text{if } k \text{ is odd}. \end{cases} \]

Now we consider \( \pi \in \text{NC}_D(n) \). Let \( \pi' \) be the partition obtained from \( \pi \) by taking the union of the blocks containing \( n \) or \( -n \) and removing \( n \) and \( -n \). Note that \( \pi \) is uniquely determined by \( \pi' \) and the block of \( \pi \) containing \( n \). We define \( \phi^{\text{NC}}_D(\pi) = (\sigma, X, \epsilon) \), where \( \sigma, X \) and \( \epsilon \) are obtained as follows.

1. If \( \pi \) has the blocks \( \pm\{n\} \) or \( \pi \) has a zero block, then \( \phi^{\text{NC}}_B(\pi') \) and \( \epsilon = 0 \).
2. Otherwise, the block of \( \pi \) containing \( n \) can be written as

\[ \{a_1, a_2, \ldots, a_r, -b_1, -b_2, \ldots, -b_s, n\} \]

for some integers \( r, a_1, \ldots, a_r, b_1, \ldots, b_s \) with \( r, s \geq 0, r + s \geq 1, 1 \leq a_1 < \cdots < a_r < n \) and \( 1 \leq b_1 < \cdots < b_s < n \). Let \( \epsilon = 1 \) if \( s = 0 \), or \( r, s > 0 \) and \( a_r < b_s \); and \( \epsilon = -1 \) otherwise. Let \( \sigma \) be the partition of \( \{n-1\} \) such that \( A \in \sigma \) if and only if \( A = B^+ \setminus \{n\} \) for some \( B \in \pi \) with \( B^+ \neq \emptyset \). Let \( X \) be the set of blocks of \( \sigma \) which are properly contained in some blocks of \( \pi \).

Note that \( \phi^{\text{NC}}_D(\pi) \in \text{NC}^{\text{NN}}_{\{0, \pm 1\}}(n-1) \).

Example 4.2. Let \( \pi = \{\pm\{1, 2, -8\}, \pm\{-3, -5, 6, 7, 10\}, \pm\{4\}, \pm\{9\}\} \) as shown in Figure 5. Then \( \phi^{\text{NC}}_D(\pi) = (\sigma, X, \epsilon) \) where \( \sigma = \{\{1, 2\}, \{3, 5\}, \{4\}, \{6, 7\}, \{8\}\} \), \( X = \{\{1, 2\}, \{3, 5\}, \{6, 7\}, \{8\}\} \) and \( \epsilon = -1 \).

Proposition 4.2. The map \( \phi^{\text{NC}}_D : \text{NC}_D(n) \to \text{NC}^{\text{NN}}_{\{0, \pm 1\}}(n-1) \) is a bijection. Moreover, if \( \phi^{\text{NC}}_D(\pi) = (\sigma, X, \epsilon) \) and \( X = \{A_1, A_2, \ldots, A_k\}_< \), then \( \text{type}(\pi) = \text{type}(\sigma \setminus X) \cup T \), where

\[ T = \begin{cases} \text{pairing}(X) \cup \{\epsilon \}, & \text{if } \epsilon = 0 \text{ and } k = 2t, \\ \text{pairing}(X \setminus \{A_{t+1}\}), & \text{if } \epsilon = 0 \text{ and } k = 2t + 1, \\ \text{pairing}(X \setminus \{A_t, A_{t+1}\}) \cup \{|A_t| + |A_{t+1}| + 1\}, & \text{if } \epsilon 
eq 0 \text{ and } k = 2t, \\ \text{pairing}(X \setminus \{A_{t+1}\}) \cup \{|A_{t+1}| + 1\}, & \text{if } \epsilon 
eq 0 \text{ and } k = 2t + 1. \end{cases} \]

Proof. We will find the inverse map of \( \phi^{\text{NC}}_D \). Let \( (\sigma, X, \epsilon) \in \text{NC}^{\text{NA}}_{\{0, \pm 1\}}(n-1) \) and \( \pi' = (\phi^{\text{NC}}_B)^{-1}(\sigma, X) \in \text{NC}_B(n-1) \).

If \( \epsilon = 0 \), then \( \pi \in \text{NC}_D(n) \) is the partition obtained from \( \pi' \) by adding \( n \) and \( -n \) to the zero block if \( \pi' \) has a zero block; and by adding the two singletons \( \pm\{n\} \) otherwise.

Now assume \( \epsilon \neq 0 \). If \( k = 2t \), then \( \pi' \) has the blocks \( \pm\{A_t \cup (-A_{t+1})\} \). Then \( \pi \) is the partition obtained from \( \pi' \) by replacing \( \pm\{A_t \cup (-A_{t+1})\} \) with \( \pm\{\epsilon(A_t \cup (-A_{t+1})) \cup \{n\}\} \). Here for a set \( B \),
the notation $\epsilon B$ means the set $\{\epsilon \cdot x : x \in B\}$. If $k = 2t + 1$, then $\pi'$ has the blocks $\pm A_{t+1}$. Then 
$\pi$ is the partition obtained from $\pi'$ by replacing $\pm A_{t+1}$ with $\pm (\epsilon(A_{t+1}) \cup \{n\})$.

One can easily check that this is the inverse map of $\phi^\text{NC}_B$. The ‘moreover’ statement is obvious
from the construction of the inverse map. 

\section{Nonnesting partitions}
As we did for noncrossing partitions, we can find interpretations for nonnesting partitions of classical types.

Consider the map $\pi \mapsto (\alpha(\pi), \beta(\pi), \gamma(\pi))$ for $\pi \in \text{NN}_B(n)$. It is easy to see that $\alpha(\pi) \in \text{NN}(n)$, all the blocks in $\beta(\pi)$ are nonaligned and $\gamma(\pi)$ is determined from $\beta(\pi)$ as follows. Let $\beta(\pi) = \{A_1, A_2, \ldots, A_{2k}\}_<$ if $\beta(\pi)$ has even number of blocks; and $\beta(\pi) = \{A_0, A_1, A_2, \ldots, A_{2k}\}_<$ otherwise. Then $\gamma(\pi)$ is the matching consisting of \{A_i, A_{2k+1-i}\} for $i \in [k]$.

For $\pi \in \text{NN}_B(n)$, we define $\phi^\text{NN}_B(\pi) = (\alpha(\pi), \beta(\pi))$. In other words, $\phi^\text{NN}_B(\pi)$ is the pair $(\sigma, X)$ where $\sigma$ is the partition obtained from $\pi$ by removing all the negative integers and $X$ is the set of blocks of $\sigma$ which are properly contained in some blocks of $\pi$. Note that we have $\phi^\text{NN}_B(\pi) \in \text{NN}_A(n)$.

\begin{example}
Let $\pi = \{\{1, 3, 7, -7, -3, -1\}, \{2, 4\}, \{5, 9, -10, -6\}, \{8\}\} \in \text{NN}_B(10)$ as shown in Figure 6. Then $\phi^\text{NN}_B(\pi) = (\sigma, X)$ where $\sigma = \{\{1, 3, 7\}, \{2, 4\}, \{5, 9\}, \{6, 10\}, \{8\}\}$ and $X = \{\{1, 3, 7\}, \{5, 9\}, \{6, 10\}\}$.

From the construction, one can easily prove the following proposition.

\begin{proposition}
The map $\phi^\text{NN}_B : \text{NN}_B(n) \to \text{NN}_A(n)$ is a bijection. Moreover, if $\phi^\text{NN}_B(\pi) = (\sigma, X)$ and $X = \{A_1, A_2, \ldots, A_k\}_<$, then
\[
\text{type}(\pi) = \text{type}(\sigma \setminus X) \cup T,
\]

where
\[
T = \begin{cases}
\text{pairing}(X), & \text{if } k \text{ is even}, \\
\text{pairing}(X \setminus \{A_1\}), & \text{if } k \text{ is odd}.
\end{cases}
\]

Similarly, we define $\phi^\text{NN}_C(\pi) = (\alpha(\pi), \beta(\pi))$ for $\pi \in \text{NN}_C(n)$. Then we have $\phi^\text{NN}_C(\pi) \in \text{NN}_A(n)$. Note that if $\pi \in \text{NN}_C(n)$ and $\beta(\pi) = \{A_0, A_1, A_2, \ldots, A_k\}_<$, then $\gamma(\pi)$ is the matching consisting of \{A_i, A_{k+1-i}\} for all $i = 1, 2, \ldots, [k/2]$.

\begin{example}
Let $\pi = \{\{1, 3, 7, -10, -6\}, \{2, 4\}, \{5, 9, -10, -5\}, \{8\}\} \in \text{NN}_C(10)$ as shown in Figure 7. Then $\phi^\text{NN}_C(\pi) = (\sigma, X)$ where $\sigma = \{\{1, 3, 7\}, \{2, 4\}, \{5, 9\}, \{6, 10\}, \{8\}\}$ and $X = \{\{1, 3, 7\}, \{5, 9\}, \{6, 10\}\}$.

Then we get the following proposition in the same way.

\begin{proposition}
The map $\phi^\text{NN}_C : \text{NN}_C(n) \to \text{NN}_A(n)$ is a bijection. Moreover, if $\phi^\text{NN}_C(\pi) = (\sigma, X)$ and $X = \{A_1, A_2, \ldots, A_k\}_<$, then
\[
\text{type}(\pi) = \text{type}(\sigma \setminus X) \cup T,
\]

where
\[
T = \begin{cases}
\text{pairing}(X), & \text{if } k \text{ is even}, \\
\text{pairing}(X \setminus \{A_1\}), & \text{if } k \text{ is odd}.
\end{cases}
\]

\end{example}
Theorem 5.1. Figure 2 are represented below. completely determined by the largest elements and the sizes of the blocks. For example, the largest $B$ section we find type-preserving bijections between noncrossing and nonnesting partitions of types $C$ and $N$. Then there is a unique element $\sigma'$ in $\sigma$ such that $\phi_D(\sigma') = \phi_D(\sigma)$ is the same way as $\phi_D(\sigma)$ is even, $\phi_D(\sigma)$ is odd.

Now we consider nonnesting partitions of type $D_n$. Let $\pi \in NN_D(n)$ and let $\pi'$ be the partition obtained from $\pi$ by unioning the blocks containing $n$ or $-n$ and removing $n$ and $-n$. Then $\phi_D(\pi)$ is defined in the same way as $\phi_D(\pi)$. That is, we define $\phi_D(\pi) = (\sigma, X, \epsilon)$, where $\sigma$ and $X$ are constructed as follows.

1. If $\pi$ has the blocks $\pm\{n\}$ or $\pi$ has a zero block, then $(\sigma, X, \epsilon) = \phi_D(\pi')$ and $\epsilon = 0$.
2. Otherwise, the block of $\pi$ containing $n$ can be written as

   $$\{a_1, a_2, \ldots, a_r, -b_1, -b_2, \ldots, -b_s, n\}$$

   for some integers $r, s, a_1, \ldots, a_r, b_1, \ldots, b_s$ with $r, s \geq 0, r + s \geq 1, 1 \leq a_1 \leq \cdots < a_r < n$ and $1 \leq b_1 \cdots b_s < n$. Let $\epsilon = 1$ if $s = 0$ or $r, s > 0$ and $a_r < b_s$; and $\epsilon = -1$ otherwise. Let $\sigma$ be the partition of $|n - 1|$ such that $A \in \sigma$ if and only if $A = B^+ \setminus \{n\}$ for some $B \in \pi$ with $B^+ \neq \emptyset$. Let $X$ be the set of blocks of $\sigma$ which are properly contained in some blocks of $\pi$.

Note that $\phi_D(\pi) \in NN_{\sigma}(n - 1)$.

Example 4.5. Let $\pi = \{\pm\{1, 4, 7, -3, -6, 10\}, \pm\{2\}, \pm\{5, 9, -8\}\} \in NN_D(10)$ as shown in Figure 8. Then $\phi_D(\pi) = (\sigma, X, \epsilon)$ where $\sigma = \{\{1, 4, 7\}, \{2\}, \{3, 6\}, \{5, 9\}, \{8\}\}$, $X = \{\{3, 6\}, \{1, 4, 7\}, \{8\}, \{5, 9\}\}$ and $\epsilon = -1$.

Proposition 4.5. The map $\phi_D : NN_D(n) \rightarrow NN_{\sigma}(n - 1)$ is a bijection. Moreover, if $\phi_D(\pi) = (\sigma, X, \epsilon)$ and $X = \{A_1, A_2, \ldots, A_k\}_<$, then $type(\pi) = type(\sigma \setminus X) \cup T$, where

$$T = \begin{cases} pairing(X) \cup \{1\}, & \text{if } \epsilon = 0 \text{ and } k \text{ is even}, \\ pairing(X \setminus \{A_1\}), & \text{if } \epsilon = 0 \text{ and } k \text{ is odd}, \\ pairing(X \setminus \{A_1, A_2\}) \cup \{|A_1| + |A_2| + 1\}, & \text{if } \epsilon \neq 0 \text{ and } k \text{ is even}, \\ pairing(X \setminus \{A_1\}) \cup \{|A_1| + 1\}, & \text{if } \epsilon \neq 0 \text{ and } k \text{ is odd}. \end{cases}$$

Proof. The proof is similar to that of Proposition 4.2, hence we omit it.

5. Type-Preserving Bijections

In the previous section we have interpreted noncrossing and nonnesting partitions of types $B_n$, $C_n$, and $D_n$ in terms of noncrossing and nonnesting partitions of type $A_{n-1}$ or $A_{n-2}$. In this section we find type-preserving bijections between noncrossing and nonnesting partitions of types $B_n$, $C_n$, and $D_n$ using the following theorem as one of the building blocks.

Theorem 5.1. Theorem 3.1 Suppose $\{A_1, A_2, \ldots, A_k\}_<$ is the set of blocks of $\sigma \in NC(n)$. Then there is a unique element $\sigma' \in NN(n)$ such that $\{A_1', A_2', \ldots, A_k'\}_<$ is the set of blocks of $\sigma'$ with $max(A_i) = max(A_i')$ and $|A_i| = |A_i'|$ for all $i \in [k]$.

The above theorem follows from the observation that any partition in $NC(n)$ or $NN(n)$ is completely determined by the largest elements and the sizes of the blocks. For example, the largest elements (circled vertices) and the sizes (integers above vertices) of the blocks of the partition in Figure 2 are represented below.
Lemma 5.2. Suppose \{A_1, \ldots, A_k\}_{<} and \{A'_1, A'_2, \ldots, A'_k\}_{<} are the sets of blocks of \(\sigma \in \text{NC}(n)\) and \(\rho(\sigma) \in \text{NN}(n)\) respectively. Then \(A_i\) is a nonaligned block of \(\sigma\) if and only if \(A'_i\) is a nonaligned block of \(\rho(\sigma)\).

Proof. By definition, \(A_i\) is aligned if and only if there is an integer \(t\) such that \(\max(A_i) < t\) and \(t \neq \max(A_j)\) for all \(j \in [k]\). Thus \(A_{k-i}\) is nonaligned if and only if \(\max(A_{k-i}) = n - i\). Since \(\max(A_i) = \max(A'_i)\) for all \(i \in [k]\), we are done.

Now we define a map \(\overline{\pi} : \text{NC}^{\text{NA}}(n) \to \text{NN}^{\text{NA}}(n)\). For \((\sigma, X) \in \text{NC}^{\text{NA}}(n)\), suppose that \{\(A_1, A_2, \ldots, A_k\)\}_{<} is the set of blocks of \(\sigma\) and \(X = \{A_1, A_2, \ldots, A_k\}_{<}\). Suppose also that \{\(A'_1, A'_2, \ldots, A'_k\)\}_{<} is the set of blocks of \(\rho(\sigma)\) and \(X' = \{A'_1, A'_2, \ldots, A'_k\}_{<}\). Then we define \(\overline{\pi}(\sigma, X) = (\sigma', X')\). In other words, if we identify a block \(A\) with its largest element \(a = \max(A)\), then \(\overline{\pi}(\sigma, a, a_1, \ldots, a_k) = (\rho(\sigma), a, a_1, \ldots, a_k)\). For example, if \(\sigma\) is the partition in Figure 2 and \(X = \{\{8\}, \{1, 4, 10\}\}\) then \(\overline{\pi}(\sigma, X) = (\sigma', X')\), where \(\sigma'\) is the partition in Figure 3 and \(X' = \{\{8\}, \{5, 7, 10\}\}\). Note that the largest elements of the blocks in \(X\) are exactly those in \(X'\).

By Lemma 5.2, we have \(\overline{\pi}(\sigma, X) \in \text{NN}^{\text{NA}}(n)\). Thus we get the following proposition.

Proposition 5.3. The map \(\overline{\pi} : \text{NC}^{\text{NA}}(n) \to \text{NN}^{\text{NA}}(n)\) is a bijection such that if \(\overline{\pi}(\sigma, X) = (\sigma', X')\) and \(X = \{A_1, A_2, \ldots, A_k\}_{<}\), then \(\text{type}(\sigma) = \text{type}(\sigma')\) and \(X' = \{A'_1, A'_2, \ldots, A'_k\}_{<}\) with \(\max(A_i) = \max(A'_i)\) and \(|A_i| = |A'_i|\) for all \(i \in [k]\).

5.1. Interchanging nonnested blocks and nonaligned blocks. In this subsection we will construct an involution on \(\text{NC}(n)\) which interchanges nonnested blocks and nonaligned blocks. In order to do this we need several definitions.

For \(\pi \in \text{NC}(n)\) and \(S = \{a_1, a_2, \ldots, a_k\}\) with \(1 \leq a_1 < a_2 < \cdots < a_k \leq n\), we define \(\pi \cap S\) to be the partition of \([k]\) obtained from \(\pi\) by removing all the integers not in \(S\) and replacing \(a_i\) with \(i\) for each \(i \in [k]\).

For two partitions \(\sigma \in \text{NC}(n)\) and \(\tau \in \text{NC}(m)\), we define \(\sigma \uplus \tau\) to be the partition in \(\text{NC}(n+m)\) obtained from \(\sigma\) by adding all the blocks of \(\tau\) whose elements are increased by \(n\). Ignoring the labels, the standard representation of \(\sigma \uplus \tau\) looks as follows:

\[
\sigma \uplus \tau = \begin{array}{c}
\sigma \\
\tau
\end{array}
\]

If \(\pi \in \text{NC}(n)\) cannot be expressed as \(\pi = \sigma \uplus \tau\) for some \(\sigma \in \text{NC}(r)\) and \(\tau \in \text{NC}(s)\) with \(r, s \geq 1\), then we say that \(\pi\) is connected. Since \(\pi \in \text{NC}(n)\) is a noncrossing partition, \(\pi\) is connected if and only if \(1\) and \(n\) are in the same block.

For a connected partition \(\sigma \in \text{NC}(n)\) and any partition \(\tau \in \text{NC}(m)\), we define \(\sigma \ast \tau\) to be the partition in \(\text{NC}(n+m+1)\) obtained from \(\sigma \uplus \tau\) by adding \(n+m+1\) to the block containing \(n\). Thus the standard representation of \(\sigma \uplus \tau\) looks as follows (here a half-circle means a connected partition and a round-rectangle means any partition):
For example,

\[
\begin{array}{ccc}
  * & \cdots & \cdots
\end{array}
\]

We also consider \(\sigma \ast \tau\) when one (or both) of \(\sigma\) and \(\tau\) is the empty partition \(\emptyset\): \(\emptyset \ast \emptyset\) is the unique partition \(\{\{1\}\}\) in \(\Pi(1)\), \(\emptyset \ast \tau\) is \(\tau \cup \{m+1\}\) and \(\sigma \ast \emptyset\) is the partition obtained from \(\sigma\) by adding \(n+1\) to the block containing \(n\).

For \(\pi \in \text{NC}(n)\), we define two maps \(\text{decomp}_1(\pi)\) and \(\text{decomp}_2(\pi)\) as follows. If \(\{n\}\) is not a block of \(\pi\), then we can uniquely decompose \(\pi\) as \(\pi = \sigma \uplus (\tau \ast \upsilon)\), see the diagram below.

\[
\pi = \begin{array}{ccc}
  \sigma & \tau & \upsilon
\end{array}
\]

In this case, we define \(\text{decomp}_1(\pi) = \text{decomp}_2(\pi) = (\sigma, \tau, \upsilon)\). If \(\{n\}\) is a block of \(\pi\), then we define \(\text{decomp}_1(\pi) = (\pi \cap [n-1], \emptyset, \emptyset)\) and \(\text{decomp}_2(\pi) = (\emptyset, \emptyset, \pi \cap [n-1])\). Note that if \(\text{decomp}_1(\pi) = (\sigma, \tau, \upsilon)\) or \(\text{decomp}_2(\pi) = (\sigma, \tau, \upsilon)\), we always have \(\pi = \sigma \uplus (\tau \ast \upsilon)\). Moreover, if \(\text{decomp}_1(\pi) = (\sigma, \tau, \upsilon)\) and \(\tau = \emptyset\), then \(\upsilon = \emptyset\), whereas, if \(\text{decomp}_2(\pi) = (\sigma, \tau, \upsilon)\) and \(\tau = \emptyset\), then \(\sigma = \emptyset\).

Now we are ready to define a map \(\xi : \text{NC}(n) \to \text{NC}(n)\). First, we assume that \(\{n\}\) is not a block of \(\pi \in \text{NC}(n)\). Suppose also that \(\pi\) has \(r\) nonnested blocks and \(s\) nonaligned blocks.

For \(i \in [r]\), let \(\text{decomp}_1(\pi_i) = (\pi_{i+1}, \sigma_i, \sigma'_i)\), where \(\pi_1 = \pi\). Since \(\pi\) has \(r\) nonnested blocks, we have \(\pi_i \neq \emptyset\) for \(i \in [r]\) and \(\pi_{r+1} = \emptyset\). Thus

\[
\pi \ast = \pi_1 \uplus (\sigma_1 \ast \sigma'_1) = \pi_2 \uplus (\sigma_2 \ast \sigma'_2) \uplus (\sigma_1 \ast \sigma'_1) = \ldots
\]

Pictorially, the above decomposition of \(\pi\) can be represented as follows.

\[
\pi = \begin{array}{ccc}
  \sigma_1 & \cdots & \sigma_r
\end{array}
\]

Note that \(\sigma_1 \neq \emptyset\), and for \(2 \leq i \leq r\), if \(\sigma_i = \emptyset\), the \(\sigma'_i = \emptyset\). If \(\{N_1, N_2, \ldots, N_r\}_<\) is the set of all nonnested blocks of \(\pi\), then \(|N_i| - 1\) is equal to the size of the block of \(\sigma_{r+1-i}\) containing the largest integer if \(\sigma_{r+1-i} = \emptyset\); and \(0\) if \(\sigma_{r+1-i} = \emptyset\).

Similarly, for \(i \in [s]\), let \(\text{decomp}_2(v_i) = (\tau'_i, \tau_i, v_{i+1})\), where \(v_1 = \pi\). Since \(\pi\) has \(s\) nonaligned blocks, we have \(v_i \neq \emptyset\) for \(i \in [s]\) and \(v_{s+1} = \emptyset\). Thus

\[
\pi = v_1 = \tau'_1 \uplus (\tau_1 \ast v_2)
\]

Pictorially, the above decomposition of \(\pi\) can be represented as follows.

\[
\pi = \begin{array}{ccc}
  \tau'_1 & \cdots & \tau'_s
\end{array}
\]

Note that \(\tau_1 \neq \emptyset\), and for \(2 \leq i \leq s\), if \(\tau_i = \emptyset\), the \(\tau'_i = \emptyset\). If \(\{A_1, A_2, \ldots, A_s\}_<\) is the set of all nonaligned blocks of \(\pi\), then \(|A_i| - 1\) is equal to the size of the block of \(\tau_{s+1-i}\) containing the largest integer if \(\tau_{s+1-i} = \emptyset\); and \(0\) if \(\tau_{s+1-i} = \emptyset\).
Since \( \{n\} \) is not a block of \( \pi \), we have \( \text{decomp}_1(\pi) = \text{decomp}_2(\pi) \), thus \( \pi_2 = \pi'_1, \; \pi_1 = \tau_1 \) and \( \pi'_1 = \nu_2 \). Thus we get the following:

\[
\pi = (\sigma_r \star \pi'_1) \cup \cdots \cup (\sigma_2 \star \pi'_2) \cup (\tau_1 \star (\tau_2 \cup (\tau_3 \cup \cdots \cup (\pi_2 \cup (\pi_3 \cup \emptyset) \cup \cdots)) \cdots) \]

Then we define

\[
\xi(\pi) = (\tau_s \star \tau'_s) \cup \cdots \cup (\tau_2 \star \tau'_2) \cup (\sigma_1 \star (\sigma_2 \cup (\sigma_3 \cup \cdots (\sigma_r \cup (\pi_r \cup \emptyset) \cup \cdots)) \cdots)
\]

See Figure 9.

Now let \( \pi \) be any element in \( \text{NC}(n) \). If \( k \) is the largest integer such that \( k \leq n \) and \( \{k\} \) is not a block of \( \pi \), we define \( \xi(\pi) \) to be the partition obtained from \( \xi(\pi \cap [k]) \) by adding the blocks \( \{k + 1\}, \{k + 2\}, \ldots, \{n\} \). See Figure 10.

For \( \pi \in \text{NC}(n) \), let \( \text{nn}(\pi) \) (resp. \( \text{na}(\pi) \)) denote the number of nonnested (resp. nonaligned) blocks of \( \pi \). From the construction of \( \xi \), it is easy to see that the following theorem holds.

**Theorem 5.4.** The map \( \xi \) is a type-preserving involution on \( \text{NC}(n) \) satisfying \( \text{nn}(\xi(\pi)) = \text{na}(\pi) \) and \( \text{na}(\xi(\pi)) = \text{nn}(\pi) \). Moreover, if \( \{N_1, N_2, \ldots, N_r\}_< \), \( \{N'_1, N'_2, \ldots, N'_s\}_< \), \( \{A_1, A_2, \ldots, A_s\}_< \) and \( \{A'_1, A'_2, \ldots, A'_r\}_< \) are the set of nonnested blocks of \( \pi \) and \( \xi(\pi) \) and the set of nonaligned blocks of \( \pi \) and \( \xi(\pi) \) respectively, then \( |N_i| = |A'_i| \) and \( |A_j| = |N'_j| \) for all \( i \in [r] \) and \( j \in [s] \).

The following corollary is an immediate consequence of Theorem 5.4.

**Corollary 5.5.** We have

\[
\sum_{\pi \in \text{NC}(n)} x^{\text{nn}(\pi)} y^{\text{na}(\pi)} = \sum_{\pi \in \text{NC}(n)} x^{\text{na}(\pi)} y^{\text{nn}(\pi)}.
\]

In fact, we can find a formula for the following generating function:

\[
F(x, y, z) = \sum_{n \geq 0} \left( \sum_{\pi \in \text{NC}(n)} x^{\text{nn}(\pi)} y^{\text{na}(\pi)} \right) z^n.
\]
Let $NC'(n)$ denote the set of connected partitions in $NC(n)$. We define
\[ C(z) = \sum_{n \geq 0} \# NC(n) z^n = \frac{1 - \sqrt{1 - 4z}}{2z}, \quad B(z) = \sum_{n \geq 1} \# NC'(n) z^n, \]
\[ A(x, z) = \sum_{n \geq 0} \left( \sum_{\pi \in NC(n)} x^{\text{n}(\pi)} \right) z^n = \sum_{n \geq 0} \left( \sum_{\pi \in NC(n)} x^{\text{n}(\pi)} \right) z^n. \]

It is not difficult to see that
\[ C(z) = \frac{1}{1 - B(z)}, \quad A(x, z) = \frac{1}{1 - xB(z)}. \]

Using the decomposition $\pi = \sigma \cup (\tau \cup \mu)$, where $\mu$ is a partition consisting of singletons and $\tau$ is a connected partition, one can also show that
\[ F(x, y, z) = \frac{1}{1 - xyz} (1 + xyz A(x, z) A(y, z) B(z)). \]

Solving the above equations, we get the following generating function.

**Proposition 5.6.** We have
\[ F(x, y, z) = \frac{1}{1 - xyz} \left( 1 + \frac{2xyz(3 + \sqrt{1 - 4z})}{(1 - 3x - x\sqrt{1 - 4z})(1 - 3y - y\sqrt{1 - 4z})} \right). \]

We can naturally extend $\xi$ to the map $\overline{\xi} : NC_{NN}(n) \to NC_{NA}(n)$ defined as follows. Let $(\sigma, X) \in NC_{NN}(n)$ and $\sigma' = \xi(\sigma)$. Suppose $\{A_1, A_2, \ldots, A_k\}_c$ is the set of all nonnesting blocks of $\sigma$ and $\{A'_1, A'_2, \ldots, A'_k\}_c$ is the set of all nonaligned blocks of $\sigma'$. Then we can write $X = \{A_1, A_2, \ldots, A_k\}_c$ and $X' = \{A'_1, A'_2, \ldots, A'_k\}_c$. We define $\overline{\xi}(\sigma, X) = (\sigma', X')$, where $X' = \{A'_1, A'_2, \ldots, A'_k\}_c$. By Theorem 5.4 we get the following corollary.

**Corollary 5.7.** The map $\overline{\xi} : NC_{NN}(n) \to NC_{NA}(n)$ is a bijection. Moreover, if $\overline{\xi}(\sigma, X) = (\sigma', X')$, $X = \{A_1, A_2, \ldots, A_r\}_c$ and $X' = \{A'_1, A'_2, \ldots, A'_r\}_c$, then type($\sigma$) = type($\sigma'$), $r = s$ and $|A_i| = |A'_i|$ for all $i \in [r]$.

**5.2. Rearranging nonnesting blocks.** Let $(\sigma, X) \in NC_{NN}(n)$. Suppose $\{A_1, A_2, \ldots, A_k\}_c$ is the set of all nonnesting blocks of $\sigma$, $X = \{A_1, A_2, \ldots, A_k\}_c$, and $\sigma_j = \sigma \cap [\min(A_j), \max(A_j)]$. Then we have $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_k$. For a permutation $p = p_1 p_2 \cdots p_k$ of $[k]$, the rearrangement of $(\sigma, X)$ according to $p$ is defined to be the pair $(\sigma', X')$ of $\sigma' = \sigma_{p_1} \cup \sigma_{p_2} \cup \cdots \cup \sigma_{p_k}$ and $X = \{A'_{p_1}, A'_{p_2}, \ldots, A'_{p_k}\}$, where $a_j = j$ if $j \notin \{i_1, i_2, \ldots, i_k\}$; and $a_j = i_p$, if $j = i$, and $\{A'_1, A'_2, \ldots, A'_k\}_c$ is the set of all nonnesting blocks of $\sigma'$.

For $(\sigma, X) \in NC_{NN}(n)$ with $|X| = k$, we define $t_B(\sigma, X)$ to be the rearrangement of $(\sigma, X)$ according to
\[ p = \begin{cases} 12 \cdots k, & \text{if } k = 2t, \\ (t+1)12 \cdots t(t+2)(t+3)\cdots(2t+1), & \text{if } k = 2t+1. \end{cases} \]

For $(\sigma, X, \epsilon) \in NC_{NN_{(0, \pm 1)}}(n)$ with $|X| = k$, we define $t_D(\sigma, X, \epsilon)$ to be $(\sigma', X', \epsilon)$, where $(\sigma', X')$ is the rearrangement of $(\sigma, X)$ according to
\[ p = \begin{cases} 12 \cdots k, & \text{if } k = 2t \text{ and } \epsilon = 0, \\ (t+1)12 \cdots (t-1)(t+2)(t+3)\cdots(2t), & \text{if } k = 2t \text{ and } \epsilon \neq 0, \\ (t+1)12 \cdots t(t+2)(t+3)\cdots(2t+1), & \text{if } k = 2t+1. \end{cases} \]

Clearly, $t_B : NC_{NN}(n) \to NC_{NN}(n)$ and $t_D : NC_{NN_{(0, \pm 1)}}(n) \to NC_{NN_{(0, \pm 1)}}(n)$ are type-preserving bijections.

By the properties of the bijections we have defined so far, we get the following theorem.

**Theorem 5.8.** The composed maps $(\phi_{B}^{NC})^{-1} \circ \overline{\xi} \circ t_B \circ \phi_{B}^{NC}$, $(\phi_{C}^{NC})^{-1} \circ \overline{\xi} \circ t_D \circ \phi_{B}^{NC}$ and $(\phi_{D}^{NC})^{-1} \circ \overline{\xi} \circ t_D \circ \phi_{B}^{NC}$ are type-preserving bijections between noncrossing partitions and nonnesting partitions of type $B_n$, $C_n$ and $D_n$ respectively; see Figures 74 and 75.
Remark 5.1. Our type-preserving bijections are different from those of Fink and Giraldo [11] because our bijections do not preserve certain statistics preserved by their bijections. In fact, they showed that their bijections are the unique ones preserving those statistics. There are other bijections between noncrossing and nonnesting partitions of classical types due to Rubey and Stump [15] for type B and Conflitti and Mamede [8] for type D. However, their bijections preserve not the types but ‘openers’ and ‘closers’.

6. Another interpretation for noncrossing partitions of type B and type D

We denote by $\mathcal{B}(n)$ the set of pairs $(\sigma, x)$, where $\sigma \in \text{NC}(n)$ and $x$ is either $\emptyset$, an edge or a block of $\sigma$. Note that if a partition $\sigma$ of $[n]$ has $i$ edges, then there are $n - i$ blocks in $\sigma$. For each $\sigma \in \text{NC}(n)$, we have $n + 1$ choices for $x$ with $(\sigma, x) \in \mathcal{B}(n)$. Hence, $\mathcal{B}(n)$ is essentially the same as $\text{NC}(n) \times [n + 1]$.

We define a map $\varphi_B : \text{NC}(n) \rightarrow \mathcal{B}(n)$ as follows. For $(\sigma, X) \in \text{NC}(n)$ with $X = \{A_1, A_2, \ldots, A_k\}$, $\varphi_B(\sigma, X)$ is defined to be $(\sigma', x)$, where $\sigma'$ is the partition obtained from $\sigma$ by unioning $A_i$ and $A_{k+1-i}$ for $i = 1, 2, \ldots, [k/2]$, and

\[
x = \begin{cases} 
\emptyset, & \text{if } k = 0; \\
(\max(A_i), \min(A_{i+1})), & \text{if } k \neq 0 \text{ and } k = 2t; \\
A_{t+1}, & \text{if } k = 2t + 1.
\end{cases}
\]

Example 6.1. If $\sigma = \{\{1, 2\}, \{3\}, \{4, 7\}, \{5, 6\}, \{8, 9, 10\}, \{11\}\}$ and $X = \{\{1, 2\}, \{3\}, \{4, 7\}, \{8, 9, 10\}, \{11\}\}$, then $\varphi_B(\sigma, X) = (\sigma', x)$, where $\sigma' = \{\{1, 2, 11\}, \{3, 8, 9, 10\}, \{4, 7\}, \{5, 6\}\}$ and $x$ is the block $\{4, 7\}$.

Theorem 6.1. The map $\psi_B = \varphi_B \circ \phi^{\text{NC}}_B$ is a bijection between $\text{NC}(n)$ and $\mathcal{B}(n)$. Moreover, if $\psi_B(\pi) = (\sigma, x)$, then type($\pi$) = type($\sigma$) if $x$ is not a block; and type($\pi$) = type($\sigma \setminus \{x\}$) if $x$ is a block.

Proof. Since $\phi^{\text{NC}}_B : \text{NC}(n) \rightarrow \text{NC}(n)$ is a bijection, it is sufficient to show that $\varphi_B : \text{NC}(n) \rightarrow \mathcal{B}(n)$ is a bijection. Let us find the inverse map of $\varphi_B$.

Let $(\sigma, x) \in \mathcal{B}(n)$. Then we construct $\sigma'$ and $X$ as follows.

If $x = \emptyset$, then $\sigma' = \sigma$ and $X = \emptyset$.

If $x$ is an edge $(a, b)$, then let $E$ be the set of edges $(i, j)$ of $\sigma$ with $i \leq a < b \leq j$. Then $\sigma'$ is the partition obtained from $\sigma$ by removing the edges in $E$, and $X$ is the set of blocks of $\sigma'$ which contain an endpoint of an edge in $E$. Here the endpoints of an edge $(i, j)$ are the integers $i$ and $j$.

If $x$ is a block $B$, then let $E$ be the set of edges $(i, j)$ of $\sigma$ with $i < \min(B) \leq \max(B) < j$. Then $\sigma'$ is the partition obtained from $\sigma$ by removing the edges in $E$, and $X$ is the set of blocks of $\sigma'$ which are equal to $B$ or contain an endpoint of an edge in $E$.

It is easy to see that the map $(\sigma, x) \mapsto (\sigma', X)$ is the inverse of $\varphi_B$. The ‘moreover’ statement is clear from the construction of $\phi^{\text{NC}}_B$ and $\varphi_B$. \hfill $\square$

Since $\mathcal{B}(n)$ is the same as $\text{NC}(n) \times [n + 1]$, Theorem 6.1 gives a bijective proof of $\# \text{NC}(n) = \binom{2n}{n}$.

Remark 6.1. For $\pi \in \text{NC}(n)$, let $\text{Abs}(\pi)$ be the partition in $\text{NC}(n)$ such that $B$ is a block of $\text{Abs}(\pi)$ if and only if $B = \{i : i \in B'\}$ for some $B' \in \pi$. Biane et al. [5] Theorem in Subsection 14] proved that the map $\pi \mapsto \text{Abs}(\pi)$ is an $(n + 1)$-to-1 map from $\text{NC}(n)$ to $\text{NC}(n)$, thus proved $\# \text{NC}(n) = \binom{2n}{n}$ bijectively. In fact, they proved that $\text{NC}(n)$ is in bijection with the set of pairs $(\sigma, x)$ where $\sigma \in \text{NC}(n)$ and $x$ is a block of either $\sigma$ or the Kreweras complement $\text{Kr}(\sigma)$. The Kreweras complement has the property that the sum of the number of blocks of $\sigma$ and the number of blocks of $\text{Kr}(\sigma)$ is equal to $n + 1$. It is easy to check that if $\varphi_B \circ \phi^{\text{NC}}_B(\pi) = (\sigma, x)$, then $\sigma = \text{Abs}(\pi)$.

We denote by $\mathfrak{D}(n)$ the set of pairs $(\sigma, x)$ such that $\sigma \in \text{NC}(n - 1)$ and $x$ is either $\emptyset$, an edge of $\sigma$, a block of $\sigma$ or an integer in $[\pm(n - 1)]$. We can also easily see that $\mathfrak{D}(n)$ is essentially the same as $\text{NC}(n - 1) \times [3n - 2]$. 
We define a map $\varphi_D : \text{NC}_{\{0, \pm 1\}}^N(n-1) \rightarrow \mathcal{D}(n)$ as follows. Let $(\sigma, X, \epsilon) \in \text{NC}_{\{0, \pm 1\}}^N(n-1)$ and $X = \{A_1, A_2, \ldots, A_k\}$. Then $\varphi_D(\sigma, X, \epsilon)$ is defined to be $(\sigma', x)$, where $\sigma'$ is the partition obtained from $\sigma$ by unioning $A_i$ and $A_{k+1-i}$ for $i = 1, 2, \ldots, \lfloor k/2 \rfloor$, and

$$x = \begin{cases} \emptyset, & \text{if } \epsilon = 0 \text{ and } k = 0; \\ (\max(A_t), \min(A_{t+1})), & \text{if } \epsilon = 0, k = 2t \neq 0; \\ A_{t+1}, & \text{if } \epsilon = 0 \text{ and } k = 2t + 1, \\ \epsilon \cdot \max(A_{(k+1)/2}), & \text{if } \epsilon \neq 0. \end{cases}$$

**Theorem 6.2.** The map $\psi_D = \varphi_D \circ \phi_D^{\text{NC}}$ is a bijection between $\text{NC}_D(n)$ and $\mathcal{D}(n)$. Moreover, if $\psi_D(\pi) = (\sigma, x)$, then

$$\text{type}(\pi) = \begin{cases} \text{type}(\sigma) \cup \{1\}, & \text{if } x = \emptyset \text{ or } x \text{ is an edge}, \\ \text{type}(\sigma \setminus \{x\}), & \text{if } x \text{ is a block}, \\ \text{type}(\sigma \setminus B) \cup \{|B| + 1\}, & \text{if } x \in [\pm(n-1)] \text{ and } B \text{ is the block of } \sigma \text{ containing } |x|. \end{cases}$$

**Proof.** The proof is similar to that of Theorem 6.1; hence we omit it. \hfill \Box

Since $\mathcal{D}(n)$ is the same as $\text{NC}(n-1) \times [3n-2]$, Theorem 6.2 gives a bijective proof of $\# \text{NC}_D(n) = \frac{3n-2}{n-1}$. For an integer partition $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$, let $m_\lambda = m_1!m_2!\cdots$. Kreweras proved the following formula for the number of $\pi \in \text{NC}(n)$ with given block sizes.

**Theorem 6.3** ([13]). Let $\lambda$ be an integer partition with $|\lambda| = n$ and length $\ell$. Then the number of $\pi \in \text{NC}(n)$ with type$(\pi) = \lambda$ is equal to

$$\frac{n!}{m_\lambda(n-\ell+1)!}.$$

As an application of Theorems 6.1 and 6.2, we can give another proof of the following type $B$ and type $D$ analogs of Theorem 6.3.

**Theorem 6.4** ([2]). Let $\lambda$ be an integer partition with $|\lambda| \leq n$ and length $\ell$. Then the number of $\pi \in \text{NC}_B(n)$ with type$(\pi) = \lambda$ is equal to

$$\frac{n!}{m_\lambda(n-\ell)!}.$$

**Proof.** Let $|\lambda| = n-k$ and $\psi_B(\pi) = (\sigma, x) \in \mathcal{B}(n)$.

If $k = 0$, then $\pi$ does not have a zero block and $x$ is not a block. Since $\sigma$ has $\ell$ blocks and $n-\ell$ edges, there are $(n-\ell+1) \cdot \frac{n!}{m_\lambda(n-\ell+1)!} = \frac{n!}{m_\lambda(n-\ell)!}$ choices of $(\sigma, x) \in \mathcal{B}(n)$.

If $k \neq 0$, then $\pi$ has a zero block of size $2k$. Thus $x$ is a block of size $k$ in $\sigma$. Let $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ and $\lambda' = \text{type}(\sigma)$. Note that $\lambda' = \lambda \cup \{k\}$ and $m_{\lambda'} = m_{\lambda} \cdot \frac{(m_{\lambda} + 1)!}{(m_{\lambda} - k)!} = m_\lambda(m_k + 1)$. Thus, there are $\frac{n!}{m_{\lambda'}(n-k)!}$ choices for $\sigma \in \text{NC}(n)$ and for each $\sigma$ there are $(m_k + 1)$ choices for $x$. Thus we get the desired formula. \hfill \Box

**Theorem 6.5** ([3]). Let $\lambda = \{1^{m_1}, 2^{m_2}, \ldots\}$ be an integer partition with $|\lambda| \leq n$ and length $\ell$. Then the number of $\pi \in \text{NC}_D(n)$ with type$(\pi) = \lambda$ is equal to

$$\begin{cases} \frac{(n-1)!}{m_\lambda(n-\ell-1)!}, & \text{if } |\lambda| \leq n-2, \\ \frac{(m_1 + 2(n-\ell))! (n-1)!}{m_\lambda(n-\ell)!}, & \text{if } |\lambda| = n. \end{cases}$$

Note that if type$(\pi) = \lambda$ for $\pi \in \text{NC}_D(n)$, then $|\lambda|$ can not be $n-1$.

**Proof.** Let $|\lambda| = n-k$ and $\psi_D(\pi) = (\sigma, x)$.

If $k \geq 2$, then $x$ is a block of size $k$ and we can use the same argument in the proof of Theorem 6.4.

Assume $k = 0$. Then $x$ is either $\emptyset$, an edge of $\sigma$ or an integer in $[\pm(n-1)]$.

If $x = \emptyset$, then type$(\sigma) = \lambda \setminus \{1\} = \{1^{m_1-1}, 2^{m_2}, \ldots\}$. **
If $x$ is an edge, then the type of $\sigma$ is $\lambda \setminus \{1\}$. Since $\sigma$ has $\ell - 1$ blocks, there are $n - \ell$ choices of $x$.

Let $\lambda' = \lambda \setminus \{1\}$. Then there are \[
\frac{(n-1)!}{m_{\lambda'}((n-1)\ell - (\ell-1)\ell+1)!}
\] choices of $\sigma$ and $n - \ell + 1$ choices of $x$. Thus there are \[
\frac{(n-1)!}{m_\lambda(n-\ell)!} = m_1 \cdot \frac{(n-1)!}{m_\lambda(n-\ell)!}
\] possibilities when $x$ is either 0 or an edge.

Now assume that $x$ is an integer in $[\pm(n-1)]$. If $|x|$ is contained in a block of size $i$, then the corresponding block in $\sigma$ is of size $i + 1$. Thus \[
type(\sigma) = \lambda(i) = \{1^{m_1}, \ldots, (i - 1)^{m_{i-1}}, i^{m_i + 1}, (i + 1)^{m_{i+1} - 1}, (i + 2)^{m_{i+2}}, \ldots\}.
\] Note that $m_{\lambda(i)} = m_\lambda \cdot \frac{1 + m_i}{m_{i+1}}$. Thus there are \[
\frac{(n-1)!}{m_{\lambda(i)}((n-1)\ell - (\ell-1)\ell+1)!}
\] choices of $\sigma$. For each $\sigma$, there are $1 + m_i$ choices for the block containing $x$, and $2i$ choices for $x$. Thus in this case the number of possible $(\sigma, x)$'s is equal to \[
\sum_{i \geq 1} 2i(1 + m_i) \cdot \frac{(n-1)!}{m_{\lambda(i)}(n-\ell)!} = \frac{2(n-1)!}{m_\lambda(n-\ell)!} \sum_{i \geq 1} (1 + m_i) \cdot \frac{i \cdot m_{i+1}}{1 + m_i}.
\]

Since \[
\sum_{i \geq 1} i \cdot m_{i+1} = \sum_{i \geq 0} i \cdot m_{i+1} = \sum_{i \geq 0} (i + 1)m_{i+1} - \sum_{i \geq 0} m_{i+1} = \sum_{i \geq 1} i \cdot m_i - \sum_{i \geq 1} m_i = n - \ell,
\] \[\text{(2)}\] is equal to \[
(n - \ell) \cdot \frac{2(n-1)!}{m_\lambda(n-\ell)!}.
\] The sum of \[\text{(1)}\] and \[\text{(2)}\] gives the desired formula. \hfill \box

7. Lattice paths

Let $LP(n)$ denote the set of lattice paths from $(0, 0)$ to $(n, n)$ consisting of up step $(0, 1)$ and east step $(1, 0)$. A Dyck path of length $2n$ is a lattice path in $LP(n)$ which never goes below the line $y = x$.

It is well known that $NC(n)$ is in bijection with the set of Dyck path of length $2n$: the Dyck path corresponding to $\sigma \in NC(n)$ is determined as follows. The $(2i - 1)$th step and the $(2i)$th step are, respectively, $(0, 1)$ and $(0, 1)$ if $i$ is the minimum of a non-singleton block of $\sigma$: $(1, 0)$ and $(1, 0)$ if $i$ is the maximum of a non-singleton block of $\sigma$: $(0, 1)$ and $(1, 0)$ if $\{i\}$ is a block of $\sigma$: $(1, 0)$ and $(0, 1)$ otherwise.

Now let us find a bijection between $NC_B(n)$ and $LP(n)$. Since $NC_B(n)$ is in bijection with $NC_{NN}(n)$, we will use $NC_{NN}(n)$ instead of $NC_B(n)$.

Let $(\sigma, X) \in NC_{NN}(n)$. Suppose $P$ is the Dyck path corresponding to $\sigma$. Consider a block $B \in X$ with $\min(B) = i$ and $\max(B) = j$. Since $B$ is nonnested, the $(2i - 1)$th step starts at $(i - 1, i - 1)$ and the $(2j)$th step ends at $(j, j)$. Then we reflect the subpath of $P$ consisting of the $r$th steps for all $r \in [2i - 1, 2j]$ across the line $y = x$. Let $g(\sigma, X)$ be the lattice path obtained by this reflection for each $B \in X$.

**Example 7.1.** Let $\sigma = \{(1, 4, 5), (2, 3), (6), (7, 9), (8), (10)\}$ and $X = \{(1, 4, 5), (6), (10)\}$. Then $(\sigma, X) \in NC_{NN}(10)$. The lattice path $g(\sigma, X)$ is obtained from the Dyck path corresponding to $\sigma$ by reflecting the subpaths corresponding to the nonnested blocks in $X$. See Figure [11]

It is easy to see that the map $g$ is a bijection.

**Proposition 7.1.** The map $g : NC_{NN}(n) \to LP(n)$ is a bijection.

Thus we get $\# NC_B(n) = \# NC_{NN}(n) = \binom{2n}{n}$. Note that we did not use the number of Dyck paths. Since $\# NC_B(n) = 2B(n) = (n+1) \cdot \# NC(n)$, we get another combinatorial proof of the fact that the number of Dyck paths of length $2n$ is equal to the Catalan number $\frac{1}{n+1} \binom{2n}{n}$. 
Remark 7.1. Reiner [17, Proposition 17] also found a bijection between $\text{NC}_B(n)$ and $\text{LP}(n)$ which is different from ours. Ferrari [10, Proposition 2.5] considered the set $\overline{\text{NC}}(n)$ of ‘component-bicoloured’ noncrossing partitions of $[n]$ and found a bijection between this set and $\text{LP}(n)$. In fact, $\overline{\text{NC}}(n)$ is essentially the same as $\text{NC}^{NN}(n)$ and our bijection $g$ is identical with Ferrari’s bijection.

We can also find a bijection between $\text{NC}_D(n)$ and a subset of $\text{LP}(n)$. To do this, we need another interpretation for $\text{NC}_D(n)$.

We denote by $\text{NC}^{NN}(n)$ the set of elements $(\sigma, X) \in \text{NC}^{NN}(n)$ such that if $X$ has a block $A$ containing $n$, then $|A| \geq 2$.

For $(\sigma, X, \epsilon) \in \text{NC}^{NN}_{\{0, \pm 1\}}(n-1)$ with $X = \{A_1, A_2, \ldots, A_k\}_<$, we define $\kappa(\sigma, X, \epsilon)$ to be the pair $(\sigma', X')$, where $\sigma'$ and $X'$ are defined as follows:

- If $\epsilon = 0$, then let $\sigma'$ be the partition obtained from $\sigma$ by adding the singleton $\{n\}$ and let $X' = X$.
- If $\epsilon = 1$, then let $\sigma'$ be the partition obtained from $\sigma$ by adding $n$ to the block $A_k$ and let $X' = X$.
- If $\epsilon = -1$, then let $\sigma'$ be the partition obtained from $\sigma$ by adding $n$ to the block $A_k$ and let $X' = X \setminus \{A_k\}$.

One can easily check that this is a bijection.

**Proposition 7.2.** The map $\kappa : \text{NC}^{NN}_{\{0, \pm 1\}}(n-1) \rightarrow \overline{\text{NC}}^{NN}(n)$ is a bijection.

Let $\text{LP}(n)$ denote the set of lattice paths in $\text{LP}(n)$ which do not touch $(n-1, n-1)$ and $(n, n-1)$ simultaneously. Note that the cardinality of $\text{LP}(n)$ is equal to $\binom{2n}{n} - \binom{2n-2}{n-1}$. It is easy to see that $g(\sigma, X) \in \text{LP}(n)$ for each $(\sigma, X) \in \overline{\text{NC}}^{NN}(n)$, and the map $g : \overline{\text{NC}}^{NN}(n) \rightarrow \text{LP}(n)$ is a bijection.

**Proposition 7.3.** The map $g : \overline{\text{NC}}^{NN}(n) \rightarrow \text{LP}(n)$ is a bijection.

Thus we get a combinatorial proof of $\# \text{NC}_D(n) = \# \overline{\text{NC}}^{NN}(n) = \binom{2n}{n} - \binom{2n-2}{n-1}$.

8. **Catalan tableaux of classical types**

A **Ferrers diagram** is a left-justified arrangement of square cells with possibly empty rows and columns. The **length** of a Ferrers diagram is the sum of the number of rows and the number of columns. If a Ferrers diagram is of length $n$, then we label the steps in the border of the Ferrers diagram with $1, 2, \ldots, n$ from north-west to south-east. We label a row (resp. column) with $i$ if the row (resp. column) contains the south (resp. east) step labeled with $i$. The $(i,j)$-**entry** is the cell in the row labeled with $i$ and in the column labeled with $j$. See Figure 12.

For a Ferrers diagram $F$, a **permutation tableau** of shape $F$ is a $0,1$-filling of the cells in $F$ satisfying the following conditions:
Figure 12. A Ferrers diagram with labeled rows and columns.

(1) each column has at least one 1,
(2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row.

The length of a permutation tableau is defined to be the length of its shape. A Catalan tableau is a permutation tableau which has exactly one 1 in each column. Let $CT(n)$ denote the set of Catalan tableaux of length $n$. There is a simple bijection between $CT(n)$ and $NC(n)$ due to Burstein [7, Theorem 3.1]. His bijection can be described in the following way which is similar to that in the proof of Proposition 6 in [9].

Let $\sigma \in NC(n)$. We first make the Ferrers diagram $F$ as follows. The $i$th step of the border of $F$ is south if $i$ is the smallest integer in the block containing $i$; and west otherwise. We fill the $(i,j)$-entry with 1 if and only if $i$ and $j$ are in the same block whose smallest integer is $i$. One can easily check that this is a bijection. For more information of Catalan tableaux and permutation tableaux, see [20, 21].

Lam and Williams [14] defined permutation tableaux of type $B_n$. See [15] for the ‘alternative tableaux’ version. The definition of permutation tableaux of type $B_n$ in [14] can be written as follows.

Let $F$ be a Ferrers diagram with $k$ columns including empty columns. The shifted Ferrers diagram $\overline{F}$ of $F$ is the diagram obtained from $F$ by adding $k$ rows of size 1, 2, ..., $k$ above it in increasing order. The rightmost cell of an added row is called diagonal. We label the added rows as follows. If the diagonal of an added row is in the column labeled with $i$, then the row is labeled with $-i$. For example, see Figure 13 at this moment, ignore the 0’s and 1’s.

A permutation tableau of type $B_n$ is a 0, 1-filling of the cells in the shifted Ferrers diagram $\overline{F}$ for a Ferrers diagram $F$ of length $n$ satisfying the following conditions:

(1) each column has at least one 1,
(2) there is no 0 which has a 1 above it in the same column and a 1 to the left of it in the same row,
(3) if a 0 is in a diagonal, then it does not have a 1 to the left of it in the same row.

A Catalan tableau of type $B_n$ is a permutation tableau of type $B_n$ such that each column has exactly one 1. A Catalan tableau of type $D_n$ is a Catalan tableau of type $B_n$ with the following additional condition: if the last row is not empty, then the left most column does not have 1 in the topmost cell. Let $CT_B(n)$ and $CT_D(n)$ denote the set of Catalan tableaux of type $B_n$ and type $D_n$ respectively.

Now we will find a bijection between $NC^{NN}(n)$ and $CT_B(n)$.

Let $(\sigma, X) \in NC^{NN}(n)$. Suppose $F$ is the Ferrers diagram of length $n$ such that the $i$th step of the border of $F$ is south if $i$ is the smallest integer in a block of $\sigma$ which is not in $X$; and west otherwise. Let $T$ be the 0,1-filling of the shifted Ferrers diagram $\overline{F}$ obtained as follows. For each $i$ which is the smallest integer in a block in $X$, fill the $(-i,i)$-entry with 1. For each pair $(i, j)$ of distinct integers such that $i$ and $j$ are in the same block $B$ and $i = \min(B)$, fill the $(-i,j)$-entry with 1 if $B$ is in $X$; and fill the $(i,j)$-entry with 1 otherwise. Fill the remaining entries with 0’s.

We define $f(\sigma, X)$ to be $T$. For example, see Figure 13.

**Theorem 8.1.** The map $f$ is a bijection between $NC^{NN}(n)$ and $CT_B(n)$.

**Proof.** First, we will show that $T = f(\sigma, X) \in CT_B(n)$. By the construction, each column of $T$ contains exactly one 1, and the row of $T$ labeled with $-i$ has a 1 if and only if the diagonal entry
Question 9.1. From Burstein’s sequel paper [12], the interpretation of $\mathcal{B}(n)$ to $\mathcal{NC}(n)$ have two interpretations $\pi$. Theorem 8.2. The map for $\mathcal{NC}(n)$ to $\mathcal{CT}(n)$ has only one 1, this condition is equivalent to the following: there is no quadruple $(i,j,i',j')$ with $i < i'$, $j < j'$ and $|i'| < |i|$ such that both the $(i,j)$-entry and the $(i',j')$-entry are filled with 1, where $i$ and $i'$ can be negative. Note that we also have $|i| \leq |j|$ and $|i'| \leq |j'|$ because there are the $(i,j)$-entry and the $(i',j')$-entry.

Suppose that we have such a quadruple $(i,j,i',j')$. Then we have either $|i| < |i'| < j < j'$ or $|i'| < |i| \leq j < j'$. Let $B$ and $B'$ be the blocks of $\sigma$ with $|i|, j \in B$ and $|i'|, j' \in B'$. If $|i| < |i'| < j < j'$, then we must have $B = B'$ since $\sigma \in \mathcal{NC}(n)$. Then $|i| = \min(B) = \min(B') = |i'|$, which is a contradiction. If $|i'| < |i| \leq j < j'$, then $i < 0$. Thus $B$ is in $X$, which implies that $B$ is nonnested. However this is a contradiction because $|i'| < |i| \leq j < j'$ and $\sigma \in \mathcal{NC}(n)$, $B$ cannot be nonnested.

Now we define the inverse map of $f$. Let $T \in \mathcal{CT}(n)$. Define $\sigma$ to be the partition of $|n|$ such that $i$ and $j$ are in the same block $B$ with $\min(B) = i$ if and only if $i < j$ and either the $(i,j)$-entry or the $(-i,j)$-entry of $T$ is filled with 1. Define $X$ to be the set of blocks $B$ of $\sigma$ such that the row of $T$ labeled with $-\min(B)$ contains a 1. It is easy to see that the map $T \mapsto (\sigma,X)$ is the inverse of $f$.

Remark 8.1. Burstein’s bijection between $\mathcal{CT}(n)$ and $\mathcal{NC}(n)$ in [7] is a restriction of the ‘zigzag’ map for permutation tableaux in [20]. We will not go into the details but our map $f$ can also be expressed as a restriction of a type $B$ analog of the ‘zigzag’ map.

If we restrict $f$ to $\mathcal{NN}(n)$, we get the following theorem.

Theorem 8.2. The map $f : \mathcal{NN}(n) \rightarrow \mathcal{CT}(n)$ is a bijection.

9. Concluding remarks

Figures 14 and 15 illustrate the objects and the bijections between them in this paper. We have two interpretations $\mathcal{NC}(n)$ and $\mathcal{B}(n)$ for $\mathcal{NC}(n)$. Since both of them are closely related to $\mathcal{NC}(n)$, they may be useful to prove type $B$ analogs of interesting properties of $\mathcal{NC}(n)$. In the author’s sequel paper [12], the interpretation $\mathcal{B}(n)$ is used to study the poset structure of $\mathcal{NC}(n)$ and $\mathcal{NC}(n)$.

Since we have a bijection between $\mathcal{NC}(n)$ and $\mathcal{B}(n) = \mathcal{NC}(n) \times [n+1]$, one can ask the following question.

Question 9.1. Is there a natural bijection between $\mathcal{NN}(n)$ and $\mathcal{NN}(n) \times [n+1]$?
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