Virial theorem and hypervirial theorem in a spherical geometry

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Abstract

The virial theorem in the one- and two-dimensional spherical geometry are presented in both classical and quantum mechanics. Choosing a special class of hypervirial operators, the quantum hypervirial relations in the spherical spaces are obtained. With the aid of the Hellmann–Feynman theorem, these relations can be used to formulate a perturbation theorem without wavefunctions, corresponding to the hypervirial-Hellmann–Feynman theorem perturbation theorem of Euclidean geometry. The one-dimensional harmonic oscillator and two-dimensional Coulomb system in the spherical spaces are given as two sample examples to illustrate the perturbation method.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The virial theorem (VT) has been known for a long time in both classical mechanics and quantum mechanics. In the classical case, it provides a general equation relating the average over time of the kinetic energy \(\langle T \rangle\) with that of the function of potential energy \(\langle \vec{r} \cdot \nabla V \rangle\). The VT was given its technical definition by Clausius in 1870 [1]. Mathematically, the theorem states that

\[
2\langle T \rangle = \langle \vec{r} \cdot \nabla V \rangle. \tag{1}
\]

If the potential takes the power function \(V(r) = \alpha r^n\) with \(r = |\vec{r}|\), the VT adopts a simple form as

\[
2\langle T \rangle = n\langle V \rangle. \tag{2}
\]

Thus, twice the average kinetic energy equals \(n\) times the average potential energy. The VT in quantum mechanics has the same form as the classical one, except for the average over time...
in equations (1) and (2) being replaced by the average over an energy eigenstate of the system. It dates back to the old papers of Born, Heisenberg and Jordan [2] and is derived from the fact that the expectation value of the time-independent operator \( \vec{F} \cdot \vec{p} \) under an eigenstate is a constant [3]:

\[
\frac{d}{dt} \langle \psi | (\vec{F} \cdot \vec{p}) | \psi \rangle = \langle \psi | [\vec{F} \cdot \vec{p}, H] | \psi \rangle = 0,
\]

where \( H = \frac{\hat{p}^2}{2m} + V \) is the Hamiltonian and \( | \psi \rangle \) is an eigenket of \( H \).

In 1960, Hirschfelder [4] generalized the relationship by pointing out that \( \vec{F} \cdot \vec{p} \) could be replaced by any other operators which were not dependent on time explicitly. In this way, he established the hypervirial theorem (HVT). For example, in a one-dimensional system, one can replace \( \vec{F} \cdot \vec{p} = xp \) by the hypervirial operator \( x^p \) and obtain the recurrence relation of \( \langle x^k \rangle \):

\[
2kE\langle x^{k-1} \rangle = 2k\langle x^{k-1}V \rangle + \left( x^k \frac{dV}{dx} \right) - \frac{1}{4}k(k-1)(k-2)\langle x^{k-3} \rangle,
\]

where \( k \) is an integer and \( E \) is the eigenenergy.

The Hellmann–Feynman (HF) theorem is another important theorem in quantum mechanics, which has been applied to the force concept in molecules by using the internuclear distance as a parameter [5, 6]. Let the Hamiltonian \( H(\xi) \) of a system be a time-independent operator that depends explicitly upon a continuous parameter \( \xi \) and \( \langle \psi(\xi) \rangle \) be a normalized eigenfunction of \( H(\xi) \) with the eigenvalue \( E_m(\xi) \), i.e., \( H(\xi)|\psi(\xi)\rangle = E_m(\xi)|\psi(\xi)\rangle \). The HF theorem states that

\[
\frac{\partial E_m(\xi)}{\partial \xi} = \langle \psi(\xi) | \frac{\partial H(\xi)}{\partial \xi} | \psi(\xi) \rangle.
\]

If the potential takes the power function \( V(r) = a r^n \), the HF gives an equation representing the relation between the eigenenergy \( E_m \) and the mean value of \( r^n \),

\[
\frac{\partial E_m}{\partial \alpha} = (r^n).
\]

Based on the relations in equations (4) and (6), the hypervirial-Hellmann–Feynman theorem (HVHF) perturbation theorem is established [7, 8]. It provides a very efficient algorithm for the generation of perturbation expansions to large order, replacing the formal manipulation of Fourier series expansions with recursion relations. This perturbation method just needs the energy instead of the wavefunctions of the system, and it is easy to achieve on the computer.

These results are well known, but have not, to our knowledge, been exploited in a curved space. In this work, we focus on the one- and two-dimensional spherical geometry. The coordinate systems adopted in this paper are shown in figure 1(a). (i) An intuitive way to describe a two-dimensional sphere is to embed it in a three-dimensional Euclidean space. Each pair of independent variables \((q_1, q_2)\) of the three-dimensional Cartesian coordinates \((q_1, q_2, q_3)\), with the origin \(O_q\) in the figure, under the constraint

\[
q_1^2 + q_2^2 + q_3^2 = \frac{1}{\lambda},
\]

where \( \lambda \) is the curvature of the sphere. The points on the sphere can also be described by the spherical polar coordinate \((R, \chi, \theta)\) defined by \((q_1, q_2, q_3) = (R \sin \chi \cos \theta, R \sin \chi \sin \theta, R \cos \chi)\) with \( R = 1/\sqrt{\lambda} \) being a constant. (ii) The Cartesian coordinates \((x_1, x_2)\) of the two-dimensional gnomonic projection, which is the projection onto the tangent plane from the center of the sphere in the embedding space, are given by

\[
q_1 = \frac{x_1}{\sqrt{1+\lambda r^2}}, \quad q_2 = \frac{x_2}{\sqrt{1+\lambda r^2}},
\]
where $r^2 = x_1^2 + x_2^2$ and the point of tangency $O_x$ in the figure being the origin. And the polar coordinate $(r, \theta)$ of the projection is defined by $r = R \tan \chi$ and $(x_1, x_2) = (r \cos \theta, r \sin \theta)$. In this work, we mainly adopt the two coordinate systems, $(x_1, x_2)$ and $(r, \theta)$, considering the results of Higgs [9] introduced in the following.

In 1979, Higgs [9] introduced a generalization of the hydrogen atom and harmonic oscillator in a spherical space. He demonstrated that, in the gnomonic projection as shown in figure 1(a), the orbits of the motion on a sphere can be described by

$$\frac{1}{2}L^2 \left[ r^{-4} \left( \frac{dr}{d\theta} \right)^2 + r^{-2} \right] + V(r) = E - \frac{1}{2} \lambda L^2, \quad (9)$$

where the angular momentum $L = x_1 p_2 - x_2 p_1$ is an invariant quantity with the potential $V(r)$ being radial symmetric. The Hamiltonian can be written as

$$H = \frac{\pi^2}{2} + \frac{1}{2} \lambda L^2 + V(r), \quad (10)$$

where the conservation vector $\vec{\pi} = \vec{p} + \frac{1}{2} \lambda \vec{x} (\vec{x} \cdot \vec{p}) + (\vec{p} \cdot \vec{x}) \vec{x}$ is the conserved vector in free particle motion on the sphere. Since the curvature appears only in the right combination $E - \frac{1}{2} \lambda L^2$ of equation (9), the projected orbits are the same, for a given $V(r)$, as in Euclidean geometry. Consequently, according with the Bertrand theorem [10, 11], the orbits are closed only if the potential takes the Coulomb or isotropic oscillator form, i.e. $V(r) = -\frac{\kappa}{r}$ or $V(r) = \frac{1}{2} \omega^2 r^2$, with $\kappa$ and $\omega$ being constants. Therefore, the systems described by equation (10) with the two mentioned potentials are defined as the Kepler problem and isotropic oscillator in a spherical geometry in [9]. The algebraic relations of their conserved quantities reveal that the dynamical symmetries of the two systems are described by the $SO(3)$ and $SU(2)$ Lie groups, respectively. These results are the beginning of the so-called Higgs algebra, which has been studied in a variety of directions [12–16].

The concept of symmetry is one of the cornerstones in the modern physics, and dynamical symmetry plays an important role in many important physical models. Since the dynamical symmetries of the Kepler problem and isotropic oscillator on a 2-sphere described by equation (10) adhere to the behaviors in two-dimensional Euclidean geometry, our question is: do more qualities of being homogeneous exist? This paper is aimed at constructing the VT and the HVT for the spherical geometry and studying their applications. In this work, we focus on the two- and one-dimensional cases for simplicity. On the other hand, the motion of a charged particle on a 2-sphere is not trivial, which is related to the famous fractionally quantized Hall states [17, 18]. We provide a general equation relating the average of the kinetic energy with that of the potential energy in the spherical geometry. We also give a generalized HVHF, which could propose to solve a class of problems the sense of perturbation.
The paper is organized as follows. In section 2, the VT in both classical mechanics and quantum mechanics is constructed. In section 3, we generalize the VT to HVT and give the quantum hypervirial relation. Two examples are taken to demonstrate the perturbation method which is combined HVT with HF theorem in section 4. We end this paper with some relevant discussions in the last section.

2. Virial theorem

2.1. Classical mechanics

To obtain the classical VT in the two-dimensional spherical geometry, two special orbits are listed in the following for examples. (i) The first one is the uniform circular motion with \( \ddot{r} = 0 \) as shown by the curve \( s_1 \) in figure 1(a). The kinetic energy of this case is given by

\[
T = \frac{1}{2} (R \sin \chi_0)^2 \dot{\theta}^2,
\]

where \( R \sin \chi_0 \) is the radius of the path. The corresponding centripetal force is

\[
F = \frac{2T}{R \sin \chi_0} = \frac{(1 + \lambda r^2) \vec{r} \cdot \nabla V}{R \sin \chi_0}.
\] (11)

Hence, one can obtain

\[
2 \langle T \rangle = \langle (1 + \lambda r^2) \vec{r} \cdot \nabla V \rangle,
\] (12)

which can be considered as the VT under the case of uniform circular motion. (ii) The orbit \( s_2 \) in figure 1(b) depicts the case where the angular momentum \( L \) is zero. In the same way, the relationship between kinetic energy and potential energy can be obtain as

\[
2 \langle (1 + \lambda r^2) T \rangle = \langle (1 + \lambda r^2) \vec{r} \cdot \nabla V \rangle.
\] (13)

These serve a good inspiration for us to presume that the VT in a spherical geometry is

\[
2 \langle (1 + \lambda r^2) T_r \rangle + 2 \langle T_\theta \rangle = \langle (1 + \lambda r^2) \vec{r} \cdot \nabla V \rangle,
\] (14)

where \( T_r \) and \( T_\theta \) are the radial and rotational kinetic energy, respectively.

In the appendix, we give a proof that equation (14) is satisfied for an arbitrary orbit in the spherical space, and it is equivalent to

\[
\langle (1 + \lambda r^2) \vec{r} \cdot \nabla V \rangle = \langle (1 + \lambda r^2) \pi^2 \rangle = \langle (1 + \lambda r^2) (2T - \lambda L^2) \rangle,
\] (15)

where \( L = x_1 p_2 - x_2 p_1 \) is the angular momentum. It is easy to find that, when the curvature \( \lambda \to 0 \), the above result reduces to equation (1).

2.2. Quantum mechanics

In the literature [9], to construct the the conserved quantities on the sphere, Higgs replaced the momentum \( \vec{p} \) in the generators on the plan by the vector \( \vec{\pi} \). This enlightens us on the subject that we can replace \( \vec{r} \cdot \vec{p} \) in equation (3) by \( \vec{r} \cdot \vec{\pi} + \vec{\pi} \cdot \vec{r} \) to obtain the VT on the sphere. The expected value of the commutator is

\[
\langle [\vec{r} \cdot \vec{\pi} + \vec{\pi} \cdot \vec{r}, H] \rangle = 0.
\] (16)

For the system in the one-dimensional curve, whose Hamiltonian is given by \( H = \pi^2/2 + V \) with \( \pi = p + \lambda (x^2 p + px^2) \)/2, the above relation leads to

\[
\left( 1 + \lambda x^2 \right) \frac{\pi^2}{2} + \frac{\pi^2}{2} (1 + \lambda x^2) + \frac{1}{2} \lambda (1 + \lambda x^2) (1 + 3 \lambda x^2) = \left( 1 + \lambda x^2 \right) \frac{dV}{dx}.
\] (17)
And in the two-dimensional case, from equations (10) and (16), we obtain
\[
\left(1 + \lambda r^2\right)\frac{\pi^2}{2} + \frac{\pi^2}{2}(1 + \lambda r^2)} + \frac{1}{2} \langle \lambda (1 + \lambda r^2)(2 + 3\lambda r^2) \rangle = \langle (1 + \lambda r^2)\vec{r} \cdot \nabla V \rangle.
\] (18)

In the polar coordinate, the Hamiltonian (10) can be written as
\[
H_0 = T_r + T_\theta + V,
\]
\[
T_r = -\frac{1}{2} \left[ 3\lambda + \frac{15}{4}\lambda^2 r^2 + \frac{1}{r} \left(1 + \lambda r^2\right) \left(1 + 5\lambda r^2\right) \frac{\partial}{\partial r} + (1 + \lambda r^2)^2 \frac{\partial^2}{\partial r^2} \right],
\]
\[
T_\theta = -\frac{1}{2} \left[ \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \lambda \frac{\partial^2}{\partial \theta^2} \right],
\]
where \(T_r\) and \(T_\theta\) denote the radial and rotational kinetic energy. The relation of equation (18) is equivalent to
\[
\langle (1 + \lambda r^2)T_r + T_\theta (1 + \lambda r^2) \rangle + 2(T_\theta) + \frac{1}{2} \langle \lambda (1 + \lambda r^2)(2 + 3\lambda r^2) \rangle = \langle (1 + \lambda r^2)\vec{r} \cdot \nabla V \rangle.
\] (20)

These results have the same form as the classical mechanical counterparts in equations (14) and (15), but with a term \(\frac{1}{2} \langle \lambda (1 + \lambda r^2)(2 + 3\lambda r^2) \rangle\) in addition. And the term is different from the corresponding one in the one-dimensional case (17), which comes from the commutation relation of \(\vec{r}\) and \(\vec{p}\).

3. Hypervirial theorems

In the above, we have got the VT in both classical mechanics and quantum mechanics. We will discuss the quantum HVT in the present part. A natural candidate of the hypervirial operator is \(r^2 \pi + \pi r^3\) according to \(r^2 p\) in the plane we mentioned in section 1, with \(k\) being integers.

3.1. One dimensional

In the one-dimensional case, one can calculate directly the commutation relation in the expected value
\[
\langle [x^k \pi + \pi x^k, H] \rangle = 0
\] (21)
and obtain
\[
[x^k \pi + \pi x^k, H] = k^2 \left[ (1 + \lambda x^2)x^{k-1}\pi^2 + 2\pi^2(1 + \lambda x^2)x^{k-1}
\right.
\]
\[
+ (1 + \lambda x^2)(k - 1)(k - 2)x^{k-3} + 2\lambda k^2 x^{k-1} + \lambda^2 (k + 1)(k + 2)x^{k+1}] - 2\lambda k^2 (1 + \lambda x^2)\frac{dV}{dx},
\] (22)

Because of \(\pi^2/2 = H - V\) and \(\langle H \rangle = E_n\) being the eigenvalues of the eigenstate, equation (21) turns to
\[
2kE_n \langle x^{k-1}\rangle_\lambda - 2k \langle x^{k-1}V \rangle_\lambda - \left( x^k \frac{dV}{dx} \right)_\lambda + \frac{k}{4} (k + 1)(k + 2)\lambda^2 \langle x^{k+1} \rangle_\lambda + 2k^2 \lambda \langle x^{k-1} \rangle_\lambda
\]
\[
+ (k - 1)(k - 2)\langle x^{k-3} \rangle_\lambda = 0,
\] (23)
in which we denote \(\langle f \rangle_\lambda = \langle (1 + \lambda x^2) f \rangle\). Hence, we obtain the recurrence formula of \(\langle x^k \rangle_\lambda\), which is the quantum hypervirial relations in the one-dimensional sphere.
3.2. Two dimensional

We now consider the HVT in the two-dimensional spherical geometry. For a radial potential \( V = V(r) \) in the Hamiltonian (19), the eigenfunction of energy can be written as
\[
\Psi(r, \theta) = e^{i m \theta} \psi(r),
\]
with \( m = 0, \pm 1, \pm 2 \ldots \) is the eigenvalue of the conserved angular momentum \( L \). The Schrödinger equation
\[
H_0 \Psi(r, \theta) = E \Psi(r, \theta)
\]
reduces to the radial equation as
\[
H_1 \psi(r) = E \psi(r),
\]
where the Hamiltonian \( H_1 \) is given by
\[
H_1 = -\frac{1}{2} \left[ (1 + \lambda r^2)^2 \frac{d^2}{dr^2} + \frac{(1 + \lambda r^2)(1 + 5\lambda r^2)}{r} \frac{d}{dr} - \frac{1 + \lambda r^2}{r^2} m^2 + 3\lambda + \frac{15}{4} \lambda^2 r^2 \right] + V.
\]

It can be written as
\[
H_1 = \frac{\pi^2}{2} + V_1,
\]
where the radial component of \( \pi \) is \( \pi_r = -i \left[ (1 + \lambda r^2)^2 \frac{d}{dr} + \frac{1}{2} \pi_x + \frac{3}{2} \lambda r \right] \) and \( V_1 = V - \frac{1}{2} \left[ (1 + m^2)\lambda - \frac{m^2 - 1/4}{r^2} \right] \). Choosing the hypervirial operator as \( r^k \pi_r + \pi_r r^k \), one can obtain the recurrence relation
\[
2kE_n (r^{k-1})_\lambda - 2k (r^{k-1} V_1)_\lambda - \left( r^k \frac{dV_1}{dx} \right)_\lambda + \frac{k}{4} ((k + 1)(k + 2)\lambda^2 (r^{k+1})_\lambda + 2k^2 \lambda (r^{k-1})_\lambda
+ (k - 1)(k - 2)(r^{k-3})_\lambda = 0,
\]
from
\[
\langle [r^k \pi_r + \pi_r r^k, H_1] \rangle = 0.
\]

Here, the notation \( \langle f \rangle_\lambda = \langle (1 + \lambda r^2) f \rangle \). It is the two-dimensional quantum hypervirial relation that we will discuss in this work. And when \( \lambda \to 0 \), it reduces to the result in the 2-plane case [19].

4. Application of the HVTs

In this section, we will generalize the HVHF theorem to the spherical space based on the hypervirial relations in the above. When the perturbation of potential \( V(r) \) takes the form as \( r^l (1 + \lambda r^2) \) with \( l \) being integers, we can determine the eigenenergies in the various orders of approximation without calculating the wavefunction, as the HVHF theorem in the Euclidean geometry. In the following, we will give two sample examples to illustrate this method.

4.1. One-dimensional harmonic oscillator

The Hamiltonian of the one-dimensional harmonic oscillator in the spherical geometry with a perturbation potential is
\[
H = \frac{\pi^2}{2} + \frac{1}{2} \alpha x^2 + \beta x^4 (1 + \lambda x^2).
\]
where $\alpha$ and $\beta$ are real numbers, $l$ is an integer and $\lambda$ is the curvature of the sphere. The perturbation $\beta x^2(1 + \lambda x^2)$ has to be very small, and $\beta$ is the smallness parameter.

Then, the HVHF recurrence relation in equation (23) becomes

$$
(k + 1)\alpha - \frac{k}{4} (k + 1)(k + 2)\lambda^2 \langle x^k \rangle_{\lambda} = 2kE_n \langle x^{k-1} \rangle_{\lambda} + \frac{k^3}{2} \lambda \langle x^{k-1} \rangle_{\lambda} + \frac{k}{4} (k - 1)(k - 2) \langle x^{k-3} \rangle_{\lambda} - \beta (2k + l) \langle x^{k+l-1} \rangle_{\lambda} - \beta \lambda (2k + l + 2) \langle x^{k+l+1} \rangle_{\lambda}.
$$

The above equation establishes precisely regarding the $n$th energy level. In order to obtain the approximate solution of the energy eigenvalues $E_n$, we expand both $E_n$ and desired expectation values $\langle x^k \rangle_{\lambda}$ in powers of the perturbation parameter $\beta$ as

$$
E_n = E_n^{(0)} + \beta E_n^{(1)} + \beta^2 E_n^{(2)} + \cdots = \sum_{j=0}^{\infty} \beta^j E_n^{(j)},
$$

$$
\langle x^k \rangle_{\lambda} = \langle x^k \rangle_{\lambda,0} + \beta \langle x^k \rangle_{\lambda,1} + \beta^2 \langle x^k \rangle_{\lambda,2} + \cdots = \sum_{j=0}^{\infty} \beta^j Q^k_j,
$$

where we introduce the notation $Q^k_j = \langle x^k \rangle_{\lambda,j}$ for convenience. We now insert the series in (33) into (32) and order in power of $\beta$. It is straightforward to obtain the relation

$$
(k + 1)\alpha - \frac{k}{4} (k + 1)(k + 2)\lambda^2 Q^k_{\lambda} = 2k \sum_{j=0}^{\nu} E_n^{(j)} Q^{k-j}_{\lambda,\nu-j} + \frac{k^3}{2} \lambda Q^{k-1}_{\lambda,\nu} + \frac{k}{4} (k - 1)(k - 2) Q^{k-3}_{\lambda,\nu} - \beta (2k + l) Q^{k+l-1}_{\lambda,\nu-1} - \beta \lambda (2k + l + 2) Q^{k+l+1}_{\lambda,\nu-1}.
$$

In addition, by the HF theorem, we know that

$$
\frac{\partial E_n}{\partial \beta} = \frac{\partial H}{\partial \beta} = \langle x^l \rangle_{\lambda},
$$

which gives another relationship of the coefficient of $\beta$:

$$
E_n^{(j)} = \frac{1}{j} Q^l_{j-1}.
$$

In other words, the $j$th approximate of the energy eigenvalue $E_n^{(j)}$ is determined by the $(j-1)$th approximate of the desired values $Q^l_{j-1}$.

In the following, we give an explicit example. We let $l = 1$ in the equations (34) and (36) and obtain, respectively,

$$
(k + 1)\alpha - \frac{k}{4} (k + 1)(k + 2)\lambda^2 Q^k_{\lambda} = 2k \sum_{j=0}^{\nu} E_n^{(j)} Q^{k-j}_{\lambda,\nu-j} + \frac{k^3}{2} \lambda Q^{k-1}_{\lambda,\nu} + \frac{k}{4} (k - 1)(k - 2) Q^{k-3}_{\lambda,\nu} - \beta (2k + 1) Q^{k}_{\lambda,\nu-1} - \beta \lambda (2k + 3) Q^{k+2}_{\lambda,\nu-1},
$$

$$
E_n^{(j)} = \frac{1}{j} Q^l_{j-1}.
$$

One can start from

$$
\langle x^k \rangle_{\lambda} = (1 + \lambda x^2) = 1 + \lambda \langle x^2 \rangle
$$

(39)
to obtain $Q_0^0$. By the HF theorem

$$\frac{\partial E_n}{\partial \alpha} = \left( \frac{\partial H}{\partial \alpha} \right) = \frac{1}{2} \langle x^2 \rangle, \quad (40)$$

one can find that

$$\frac{1}{2} \langle x^2 \rangle = \sum_{j=0}^{\infty} \beta_j \frac{\partial E_n^0}{\partial \alpha}, \quad (41)$$

Substituting it into equation (39), the expectation value $\langle x^0 \rangle$ expansion will be denoted as

$$\langle x^0 \rangle = 1 + \frac{1}{2} \beta_0 \frac{\partial E_n^0}{\partial \alpha} + \beta_1 \frac{\partial E_n^1}{\partial \alpha} + \beta_2 \frac{\partial E_n^2}{\partial \alpha} + \cdots \quad (42)$$

Ordering in power of $\beta$, it is easy to find the first term of the recursion:

$$Q_0^0 = 1 + 2 \lambda \frac{\partial E_n^0}{\partial \alpha} = 1 + \frac{(2n + 1) \lambda}{\sqrt{\lambda^2 + 4 \alpha}}, \quad (43)$$

$$Q_1^0 = 2 \lambda \frac{\partial E_n^1}{\partial \alpha},$$

$$Q_2^0 = 2 \lambda \frac{\partial E_n^2}{\partial \alpha},$$

$$\vdots$$

The eigenenergy of the one-dimensional harmonic oscillator in a spherical geometry is $E_n^0 = (n + \frac{1}{2}) \sqrt{\lambda^2 + 4 \alpha} + \frac{1}{2} \lambda$ [20, 21].

When $\gamma = 0$, one can substitute $Q_0^0$ into equation (37) and obtain the values of $Q_0^1$, $k = 0$ $Q_0^1 = 0, \quad (44)$

$k = 1$ $Q_0^1 = \frac{(2n + 1) \lambda + \sqrt{\lambda^2 + 4 \alpha}}{(4 \alpha - 3 \lambda^2) \sqrt{\lambda^2 + 4 \alpha}} \frac{(2n + 1) \sqrt{\lambda^2 + 4 \alpha} + (2n^2 + 2n + 3) \lambda}{(2n^2 + 2n + 3) \lambda}, \quad (45)$

$$\vdots$$

Using equations (44) and (38), we can obtain the first-order perturbation of $E_n$.

$$E_n^{(1)} = Q_0^1 = 0, \quad (46)$$

And from equations (43) and (46), we have

$$Q_1^0 = 2 \lambda \frac{\partial E_n^1}{\partial \alpha} = 0. \quad (47)$$

In the case of $\gamma = 1$, using $Q_0^1$ and equation (37), we can derive the values of $Q_1^1$ and consequently the second approximation of energy level

$$E_n^{(2)} = - \frac{\sqrt{\lambda^2 + 4 \alpha} + (2n + 1) \lambda}{2 \lambda \sqrt{\lambda^2 + 4 \alpha}} \frac{3 \lambda [(2n + 1) \lambda + \sqrt{\lambda^2 + 4 \alpha}] - 2 \alpha (4 \lambda - 3 \lambda^2) \sqrt{\lambda^2 + 4 \alpha}}{2 \alpha \sqrt{\lambda^2 + 4 \alpha}} \times [(2n + 1) \sqrt{\lambda^2 + 4 \alpha} + (2n^2 + 2n + 3) \lambda]. \quad (48)$$

In this way, we can obtain the expectation value expansions $Q_j^\gamma$ and the energy values $E_n^{(j)}$ in the various orders of approximation as

$$E_n^{(3)} = 0, \quad (49)$$

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\[ E_n^{(4)} = -\frac{1}{4\alpha} \left( \frac{6\lambda^2(2E_0^{(0)} + \lambda)}{4\alpha - 3\lambda} + \kappa \right) \frac{\partial Q_1}{\partial \alpha} + 2E_n^{(2)} Q_0^{(0)} - 3Q_1^{(0)} \]

\[ - \frac{5\lambda}{3\alpha - 6\lambda^2} \left[ (4E_0^{(0)} + 4\lambda) Q_1^{(0)} - \left( 5 + \frac{21\lambda E_0^{(0)}}{2\alpha - 15\lambda^2} \right) Q_0^{(0)} - \frac{21\lambda}{8\alpha - 60\lambda} Q_0^{(0)} \right] \].

(50)

In the limit \( \lambda \to 0 \), \( E_n^{(2)} \) tends to \(-1/(2\alpha)\) and the other \( E_n^{(i)} \) tends to zero which correspond to the exact result in the Euclidean space.

It is worth mentioning that, alien from Euclidean space, (i) the HF theorem has been used twice in this HVHF perturbative method. (ii) Only when the exponent \( l \) in the perturbation potential is a positive integer, we can obtain \( Q_0^{(i)} \) from equations (43), (37) and (38).

4.2. Two-dimensional coulomb system

Here, we wish to show that the HVHF perturbation method can be easily applied to treat the Coulomb system with a perturbation in the two-dimensional sphere which is described by the Hamiltonian

\[ H = \frac{\pi^2}{2} + \frac{1}{2} \lambda L^2 - \frac{k}{r} + \beta r'(1 + \lambda r^2), \]  

(51)

where \( k \) is a real number, and \( \beta \) is the perturbation parameter. Hence, the potential in the radial Hamiltonian (28) is

\[ V_l = -\frac{k}{r} + \beta r'(1 + \lambda r^2) - \frac{1}{2} \left[ \left( \frac{1}{2} - m^2 \right) \lambda - \frac{m^2 - 1/4}{r^2} \right]. \]  

(52)

The hypervirial relation equation (29) turns to

\[ \frac{1}{4} \left[ \kappa (k - 2) - (k - 1) (4m^2 - 1) \right] \langle \kappa^{k-1}_{\lambda} \rangle + \frac{\lambda k}{2} \left( k^2 + 2 - 4m^2 \right) \langle \kappa^{k-1}_{\lambda} \rangle \]

\[ + 2kE_0 \langle \kappa^{k-1}_{\lambda} \rangle + 2(k - 1) \kappa \langle \kappa^{k-2}_{\lambda} \rangle + \frac{k}{2} (k + 1) (k + 2) \langle \kappa^{k+1}_{\lambda} \rangle \]

\[ - \beta (2k + 1) \langle \kappa^{k+l-1}_{\lambda} \rangle - \beta \lambda (2k + l + 2) \langle \kappa^{k+l+1}_{\lambda} \rangle = 0. \]  

(53)

Considering the angular quantum number \( m^2 \) as a parameter of the potential \( V_l \), one can obtain the expansion coefficients for \( \langle r^{-2} \rangle_{\lambda} \), by using the HF theorem,

\[ \langle r^{-2} \rangle_{\lambda} = \langle r^{-2} \rangle + \lambda \langle 1 \rangle = 2 \frac{\partial E_n}{\partial m^2}. \]  

(54)

From this starting point, as we show in the one-dimensional case, we can obtain any order perturbation on the energy level, with the precondition that \( l \) is a negative integer.

Taking \( l = -3 \) for example, in the first approximation, the eigenvalue \( E_n \) is

\[ E_n = -\frac{k^2}{2(n + \sqrt{m^2} + \frac{1}{2})^2} + \frac{\lambda}{2} \left( n + \sqrt{m^2} \right) \left( n + \sqrt{m^2} + \frac{1}{2} \right) \]

\[ + \beta \frac{8k^3}{\sqrt{m^2}(4m^2 - 1)(n + \sqrt{m^2} + \frac{1}{2})} + \beta \frac{2k\lambda}{\sqrt{m^2}(4m^2 - 1)(4n + 4\sqrt{m^2} + 1)}. \]  

(55)

When \( \lambda \to 0 \), this result is coincided with the literature [22].
5. Conclusion and discussion

The VT in a spherical geometry has been proved in both classical and quantum conditions. We have also considered the HVT and obtained the hypervirial relations. The HVT and HF theorems have been shown to provide a powerful method of generating perturbation expansions. We have taken the Coulomb problem and harmonic oscillator for instances to illustrate this method. When the curvature $\lambda$ is zero, the results reduce to the counterpart of Euclidean space.

In this paper, we only give attention to one- and two-dimensional systems. Since Higgs’ results have extended to the $N$-dimensional spherical geometry directly [23], we can foretell that our treatment can be generalized to the $N$-sphere and suggest that the VT is given by

$$\langle (1 + \lambda r^2) \pi^2 + \frac{\pi^2}{2} (1 + \lambda r^2) \rangle + \frac{1}{2} \langle \lambda (1 + \lambda r^2) (N + 3 \lambda r^2) \rangle = n \langle (1 + \lambda r^2) \vec{r} \cdot \nabla V \rangle.$$  

Some researchers have discussed the superintegrable potentials in the hyperbolic plane [24], it is interesting and possible to study the VT, HVT and HVHF in the situation of the curvature $\lambda < 0$. On the other hand, the systems in the curved space that we investigate in this work can also be considered as the problems with position-dependent effective mass, which are widely applied in various areas of material science and condensed matter [25, 21, 26, 27]. We hope to find the applications of our results in these directions in further research.

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Appendix. Proof of the VT in classical mechanics

In this part, we will give the strict proof for the classical VT in equations (14) and (15). We adopt the subscripts $p$ and $s$ to distinguish the systems on a plane and on a sphere, respectively. From equation (9), we know that, for a given $V(r)$, when

$$E_s = \frac{1}{2} \lambda L_s^2 = E_p, \quad L_s = L_p, \quad \text{ (A.1)}$$

the projected orbit of a spherical system is the same as the orbit of a system in Euclidean geometry. It is easy to find that, for the corresponding points $(r_s, \theta_s) = (r_p, \theta_p) = (r, \theta)$, the velocities satisfy

$$\vec{v}_s = (1 + \lambda r^2) \vec{v}_p, \quad \text{ (A.2)}$$

where $\vec{v}_s = (\dot{r}_s, r_s \dot{\theta}_s)$ and $\vec{v}_p = (\dot{r}_p, r_p \dot{\theta}_p)$. For the system in a flat space whose Hamiltonian is given by $H = \frac{p^2}{2} + V$, the two terms in equation (1) are

$$\langle \vec{r}_p \cdot \nabla V \rangle = \frac{1}{\tau_p} \int_0^{\tau_p} \vec{r}_p \cdot \nabla V \, d\tau_p = \frac{1}{\tau_p} \int_c \vec{r}_p \cdot \nabla V \frac{1}{v_p^2} \vec{v}_p \cdot d\vec{s}_p, \quad \text{ (A.3)}$$

$$\langle T_p \rangle = \frac{1}{\tau_p} \int_0^{\tau_p} \frac{1}{2} v_p^2 \, d\tau_p = \frac{1}{\tau_p} \int_c \frac{1}{2} v_p^2 \frac{1}{v_p^2} v_p \cdot d\vec{s}_p$$

where $d\vec{s}_p = (d\tau_p, r_p d\theta_p)$, $c$ denotes the orbit of motion, and $\tau_p$ is the period (for the aperiodic case $\tau_p \to +\infty$). Suppose the period of the system with the same orbit $c$ in the sphere described by equation (10) is $\tau_s$. Then, considering the relations in equations (A.1) and (A.2), one can
find
\[ \langle \mathbf{r}_p \cdot \nabla V \rangle = \frac{\tau_s}{\tau_p} \big[ (1 + \lambda \mathbf{r}_s^2) \mathbf{r}_p \cdot \nabla V \big], \]
\[ \langle T_p \rangle = \frac{\tau_s}{\tau_p} \big[ (1 + \lambda \mathbf{r}_s^2) \mathbf{T}_{ir} + \langle T_{ir} \rangle \big] = \frac{\tau_s}{\tau_p} \left( 1 + \lambda \mathbf{r}_s^2 \right) \frac{\lambda \mathbf{r}_s^2}{2}, \]
where the radial kinetic energy \( T_{ir} = R^2 \mathbf{r}_s^2 / 2 = \lambda \mathbf{r}_s^2 / \left[ 2 \left( 1 + \lambda \mathbf{r}_s^2 \right) \right] \) and the rotational kinetic energy \( T_{ir} = R^2 \sin^2 \chi \mathbf{r}_s^2 / 2 = \tau_r^2 \mathbf{r}_s^2 / \left[ 2 \left( 1 + \lambda \mathbf{r}_s^2 \right) \right] \). Therefore, relation (14) is the VT in a spherical geometry, and it is equivalent to equation (15). Here, the proof comes to an end.

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