Quantum Processors and Controllers

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Abstract

In this paper is presented an abstract theory of quantum processors and controllers, special kind of quantum computational network defined on a composite quantum system with two parts: the controlling and controlled subsystems. Such approach formally differs from consideration of quantum control as some external influence on a system using some set of Hamiltonians or quantum gates. The model of programmed quantum controllers discussed in present paper is based on theory of universal deterministic quantum processors (programmable gate arrays). Such quantum devices may simulate arbitrary evolution of quantum system and so demonstrate an example of universal quantum control.

Keywords: quantum, computer, control, processor, universal

1 Introduction

Let us consider simple example of control using a Hilbert space of composite quantum system with two parts:

\[ \mathcal{H} = \mathcal{H}_c \otimes \mathcal{H}_d. \]  

Here \( \mathcal{H}_d \) is a Hilbert space of quantum system considered as data, (controlled variables, subject of control) and \( \mathcal{H}_c \) is Hilbert space of control (“manager”, program of changes). The approach is close analogue of conditional quantum dynamics [1].

In quantum networks bits are changed to qubits (quantum bits) and yet another quantum gate with two qubits is controlled-\( U \) gate \[2\], when quantum gate \( U \) is applied to second qubit if first one is \( |1\rangle \) ((\( a \)) is notion for state of qubit in Dirac notation), but if first qubit is \( |0\rangle \) then second one is unchanged.

It is possible to write controlled-\( U \) as 4 \( \times \) 4 matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & U_{11} & U_{12} \\
0 & 0 & U_{21} & U_{22}
\end{pmatrix},
\]

where \( U_{ij} \) are components of quantum gate for second qubit, i.e., 2 \( \times \) 2 matrix \( U \).

To describe development of the idea, programmable quantum gate arrays are used in Sec. 2. Such quantum devices also are called quantum processors, but may be used as quantum controllers as well, it is discussed in Sec. 3. More formal mathematical description of programmable quantum controllers provided in Sec. 4. Universality of quantum computations and control are briefly recollected in Sec. 5. Some discussion on universal control with continuous quantum variables is presented in Sec. 6.

2 Programmable quantum gate arrays

It is possible to use decomposition Eq. 1 with more general quantum networks and describe process of control as

\[ \mathcal{C}_{\text{int}}: (|C\rangle \otimes |\Psi\rangle) \mapsto |C'\rangle \otimes (u_C|\Psi\rangle), \]

i.e., \( \mathcal{C}_{\text{int}} \) is some fixed network and different control strategies ensured by different states \( |C'\rangle \) of control registers: for each such state different operator \( u_C \) acts on second system. The expression Eq. 5 coincides with definition of special kind of quantum network, programmable quantum gate array [3].

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[1] or any other introduction in quantum information science.

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Let us use notation $C_{\text{avl}}^m$ for $\dim H_c = m$ and $\dim H_d = n$.

It should be mentioned, that $C_{\text{trl}}$ as any other quantum computational network with pure states must be described as linear unitary operator acting on the Hilbert space Eq. (1) (similarly with example Eq. (2) above for simplest case with two qubits). Here the quantum laws have serious implications denoted already in [3]: any two states of “program” (first, control register) must be orthogonal, i.e., maximal number of different operators $u_{\text{trl}}$ available for application to second, controlled system is equal to dimension of Hilbert space $H_c$, i.e., for universal control $\dim H_c = \infty$, because number of different quantum gates is infinite.

To explain this result, let us consider two different “control strategies” $|A\rangle$ and $|B\rangle$

$$C_{\text{trl}}(|A\rangle \otimes |\Psi\rangle) = |A'\rangle \otimes (u_A|\Psi\rangle),$$
$$C_{\text{trl}}(|B\rangle \otimes |\Psi\rangle) = |B'\rangle \otimes (u_B|\Psi\rangle),$$

but because $C_{\text{trl}}$ is unitary operator, it may not change scalar product of two vectors, i.e.,

$$\langle A|B\rangle = \langle A'|B'\rangle \langle \Psi|u_A^{\dagger}u_B|\Psi\rangle$$  \hspace{1cm} (4)

In Eq. (1) $\langle A|B\rangle$ and $\langle A'|B'\rangle$ are fixed numbers, but for $u_A \neq u_B$ term $\langle \Psi|u_A^{\dagger}u_B|\Psi\rangle$ depends on $|\Psi\rangle$. But Eq. (1) must be satisfied for any $|\Psi\rangle$ and so

$$\langle A|B\rangle = \langle A'|B'\rangle = 0,$$  \hspace{1cm} (5)

i.e., states corresponding to different programs are orthogonal.

For example even for one controlled qubit, set of all possible gates may be described by continuous three-dimensional family, i.e., even for this simple case with $\dim H_d = 2$, for universal control it is necessary to have $\dim H_c = \infty$ with control register described by three continuous quantum variables ($C^{\infty}_{\text{avl}}^3$).

It is interesting, that such enormous difference between size of control and controlled system is rather subtle property of quantum dynamics, for example, it may be found linear, but non-unitary operator like Eq. (3) for universal control and with size of control register only in two times bigger than for controlled quantum system $C_{\text{avl}}^2$, it is simply operator of multiplication of a matrix on a vector written as formal linear map $\{(H \otimes H^* \otimes H \rightarrow (H \otimes H^*) \otimes H; A \otimes v \mapsto 1 \otimes Av)\}$, but it would contradict to laws of quantum mechanics.

3 Quantum processors as controllers

It is clear, that such approach has some difference with other methods of consideration of quantum control [9, 10, 11, 12, 13, 14], there controlled system is also described as some state, but control is introduced as set of “external” controlling operators; gates or Hamiltonians, i.e., control and data are described from different points of view (semiclassical coherent control).

Note Another attempt of joint quantum description of control and controlled system, using same term “quantum controller,” was included in [9], as some perspective for above-mentioned semiclassical coherent control. It was not suggested a general model of such joint quantum description, but few illustrative examples were presented. But here is discussed an alternative approach, it is enough to recall some distinctive principles of consistent framework for quantum control with pure states considered in present paper:

1. Control and controlled system must not be entangled. It follows directly from definition Eq. (3).

2. The consequence of such definition is impossibility to use superposition of states in control register.

3. So, there is noticeable asymmetry between control and controlled system in such approach.

4. Final development of the principles is original three-level design of programmable quantum controller discussed below in Sec. 4, Fig. 1.

The construction of programmable quantum gate arrays, or quantum processors [3, 4, 10, 11, 12, 13, 14] provides more unified description of control and controlled system. It is in agreement with principles just mentioned above. Term “quantum controller” may be also justified for such a system, because for universal quantum processor on controlled system $H_d$ formally may be simulated practically any physical process, if to use tradition of consideration of such systems suggested by R. Feynman and D. Deutsch [15, 16]. For classical processors difference in sizes of program and data is not such a radical and this new property of quantum processor (controller) is related with infinite amount of different quantum programs (algorithms of control).

Despite of discussed above result about infinite-dimensional controlling register, universal quantum controllers with finite dimension of control space $H_c$ are also quite useful. It should be mentioned first so-called stochastic quantum processors [3, 4, 12, 13]. Such quantum processor does not produce correct
answer each time, but provides special “check bit” displaying if answer is correct or not. If answer is not correct, it is suggested to perform calculations again and again. Probability of correct answer is reduced with size of data register and increases with number of tries.

Seems idea of stochastic quantum processor quite interesting, but has lot of problems, for example it is not even clear if it is possible to use composition of such networks for few-steps process due to unspecified time of each step and it is certainly some problem for application of such system as quantum controllers. It is also not quite clear, if it is always possible to “discard” incorrect result of action for general controller and start all again.

In addition, the “ideal limit” of such design resembles non-possible linear (but non-unitary) operator discussed earlier and it is similar with some other known models of quantum systems (“relaxation” gate, “instantaneous” reduction, etc.), then balance between “archous” and “impossible” is too fine and linked with deep problems of quantum mechanics.

Anyway, the idea of stochastic gates seems useful, for example in [13] was shown, that continuous limit of some special stochastic network discussed in [4] coincides with continuous limit of some “deterministic” quantum gate, despite of very different behavior in finite, discrete case. It should be mentioned also, that main efforts of many authors last time were applied rather to the stochastic design, but deterministic one seems more appropriate for present consideration of quantum controllers.

Another construction with finite control register uses “universality in approximate sense”. It is quite reasonable approach and based on idea, that in realistic tasks always possible instead of continuous infinite space of parameters to use only finite set of points for approximation. The more dense set, the more accurate such a method. Some basic papers about universality in quantum computation uses such approach [10, 11, 12, 13].

4 More rigor mathematical description

Let us consider quantum processors and controllers with more details [10, 11]. It was already mentioned, that all different states of controlling register must be orthogonal. Let us use for simplicity finite controlling register and choose such orthogonal states as new basis. It is possible to denote it simply as $|0\rangle$, $|1\rangle$, $|2\rangle$, . . . , i.e., “no operation,” “operation #1,” “operation #2,” . . .

If dim $\mathcal{H}_c = m$ and dim $\mathcal{H}_d = n$ in Eq. (1), then dim $\mathcal{H} = mn$ and in suggested new basis $\mathcal{C}_{trl}$ may be written as block-diagonal $mn \times mn$ matrix

$$\mathcal{C}_{trl} = \begin{pmatrix} u_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & u_{m-1} \end{pmatrix},$$

where $u_k$ are $n \times n$ matrices and it is convenient to choose $u_0 = 1$ (“no operation”). It is example of conditional quantum dynamics described in [10] and using Dirac notation it may be rewritten as

$$\mathcal{C}_{trl} = |0\rangle\langle 0| \otimes u_0 + |1\rangle\langle 1| \otimes u_1 + \cdots$$

Such approach may be appropriate for simple quantum controller, but for more difficult operations it is reasonable to consider an advanced design [10, 11] of quantum processor that can be used as a programmable quantum controller.

Here $\mathcal{H}_p, \mathcal{H}_c, \mathcal{H}_d$ are Hilbert spaces of program, controller and data, or pseudo-classical, intermediate and quantum buses respectively (see Fig 1).

The idea is to use composition of two operators. First one was described earlier, it is quantum controller $\mathcal{C}_{trl}$ acting on intermediate and quantum buses $\mathcal{H}_c \otimes \mathcal{H}_d$ and second one acts on $\mathcal{H}_p \otimes \mathcal{H}_c$ and on each step provides intermediate bus $\mathcal{H}_c$ with new state $|k\rangle$ used as index $k$ of operator $u_k$ by quantum controller.

Let us consider simplest example with “cyclic memory (ROM)”. Let dim $\mathcal{H}_c = m$ and it is necessary to perform program with $p$ steps. Then dim $\mathcal{H}_p = m^{p-1}$ and element $|K\rangle$ of $m^p$-dimensional Hilbert space $\mathcal{H}_p \otimes \mathcal{H}_c$ may be described as

$$|K\rangle \equiv |k_p, \ldots, k_2; k_1\rangle$$

and “program” is simply operator of cyclic shift

$$\mathcal{S}_{\text{cyc}}: |K\rangle \mapsto |k_1, k_p, \ldots, k_3; k_2\rangle.$$

Finally, for $p$ steps of the programmable quantum controller with cyclic ROM ($\mathcal{C}_{\text{cyc}}^{m^p}$), it is possible to write

$$\left(\mathcal{S}_{\text{cyc}} \mathcal{C}_{trl}\right)^p (|K\rangle |\Psi\rangle) \mapsto |K\rangle (u_{k_p} \cdots u_{k_1} |\Psi\rangle)$$

and because set of operators $u_k$ contains identity (unit), it is possible to implement any sequence with up to $p$ operators using different programs $|K\rangle$. 
One problem here is huge size of program register. A method to reduce the size is to use instead of shifted array more complicated algorithm for generation of indexes. For example, instead of each sequence of $n$ equivalent indexes $k$ it could use pair $(n, k)$. It should be mentioned yet, that only reversible algorithms are appropriate here due to common principles of quantum computations — otherwise dynamics would not be unitary. Really there are some methods of automatic conversion of any algorithm to reversible one, but in such a case each step generates a “garbage” and size of program register may be even bigger, than for ROM. So the area is related with classical theory of optimal reversible computations.

On the other hand, it was already mentioned earlier, that all states of program register are orthogonal. It is not necessary to use superposition of different states. It was a reason to call the register “pseudo-classical.” Such systems may be more simple for implementation and so problem with size may be not such essential, as for data register.

Yet another advantage of such pseudo-classical program register is possibility to use “halt bit” and algorithms with variable length. It is mentioned here, because such an opportunity is not very common for general quantum algorithms due to quantum parallelism and interference of different branches.

5 Universality of quantum control

When method of generation of arbitrary sequence of operators like Eq. (11) is given, ideas of implementation of universal control follows to standard procedures [2, 5, 6, 10, 11, 16, 17, 18, 20, 21].

Let us consider case with finite size of control register. For good approximation it is possible to choose $u_k$ near identity operator, i.e.

$$u_k(\varepsilon) = \exp iH_k\varepsilon \approx 1 + iH_k\varepsilon \quad (\varepsilon \to 0).$$  (12)

Here $H_k$ are Hermitian operator and corresponds to Hamiltonians in some other approach to quantum control [3]. Then small parameter $\varepsilon$ is analogue of minimal time of action of some external influence by the control Hamiltonian.

Due to general theory it is enough to have possibility to generate full Lie algebra of Hermitian operators as linear span of $H_k$ and all possible consequent commutators, but this part coincides with general theory of universal quantum computations and control and does not discussed here in details [2, 5, 6, 10, 11, 16, 17, 18, 20, 21, 22, 23].

Algorithms of generation of indexes for application of different gates $u_k$ often may be described using few nested cycles with repeating series of states-indexes $|k\rangle$ [10, 22, 23] due to general algebraic approach with Lie algebras and commutators mentioned above.
Despite of such analogue in mathematical expressions, discussed approach has some advantages due to closed description of controlling and controlled systems. Really the quantum controller uses only one fixed Hamiltonian $H_C$, $C_{\text{ctrl}} = e^{iH_C dt}$ and $H_k$ are rather formal operators.

On the other hand, such quantum description has some difficulties, because despite of pseudo-classical character of program register, it is anyway some special kind of quantum system. It is not look reasonable, that for control of such program register may be used some standard silicon chip. It was already mentioned, that for presented model only reversible programs are compatible with laws of quantum mechanics. Really it could be simply shown, that irreversible operator for some state of quantum controller has absurd property: it “shrinks” to zero wave vector of the system (and all environment, due to linearity). One method of more correct modeling is to use mixed states and density matrices. But it is not only problem of given approach, most other models of interface between classical devices and quantum system always provides some challenge and may be much more nontrivial [24].

6 Continuous quantum variables

Quantum computations with continuous variables is also active area of research [25]. The ideas presented here also possible to use in case of control described by continuous quantum variables. For such a case direct sum in Eq. \(7\) should be changed to direct integral [14]. For such a system quantum control variables are continuous, but controlled system is described by finite-dimensional Hilbert space. It is particular case of hybrid quantum computing [26].

Here pseudo-classical character of program bus provides some simplification. It may be described using classical terms and it is in agreement with relative success of usual semiclassical description of quantum control. But intermediate and quantum buses may not be considered using only classical ideas. Here intermediate bus could provide some challenge as an “interface” between classical and quantum world. In presented approach it is not so critical, because pseudo-classical bus is also described as a quantum system and was called so due to “recommendation” to use here only orthogonal set of states. It is principally possible to apply any superposition of states to such “pseudo-classical” bus, but in such a case states of control and controlled system became entangled after application of $C_{\text{ctrl}}$ Eq. \(3\) and it is not considered as prescribed functioning of considered device.

One simple method of description of quantum computation with continuous variables is to consider some functional spaces and space of linear differential operators. Well known example are operators of coordinate and momentum $p$, $q$. Draft of universal quantum controller, based on design of hybrid quantum processor [14] depicted on Fig. 2.

Here controlled system is anyway finite-dimensional and only some subset of controlling variables described by infinite-dimensional Hilbert space. Interesting question is problem of universal control of continuous variables. Such models were described yet only in semiclassical approach to quantum computation and control. It was shown, that Hamiltonian with (bi)linear combinations of coordinate and momentum are not enough [25] for universal quantum computation (control) and so some nonlinear (third-order) [26] or exponential [27] expressions may be used instead.

It is clear from previous consideration, that it is simpler to use some analogue of universality in approximate sense for control of quantum continuous variables — it was discussed earlier, dimension of Hilbert space for universal control must coincide with cardinality (“number of points”) in Hilbert space of controlled system and so for controlled system with continuous variables ($C^\infty_{\infty}$) such idea would produce...
very different problems related with mathematical
time of infinite cardinal numbers.

So, (precisely) universal control of $N$-dimensional system is possible using continuous quantum variables ($C^{\infty}_N$), but quite likely, that control of continuous quantum system ($C^\infty$) may be universal only in approximate sense. On the other hand, distinction between approximate and rigor universality in last case has rather theoretical significance, because it is not clear, how to find a difference between such $C^\infty$-controllers during any finite amount of time. Anyway, both tasks discussed below are difficult and out of scope of presented paper.

In addition, more accurate consideration of models of quantum computations and control with continuous variables is not complete without necessary attention to principles of quantum field theory. This difficult area is still in state of development, especially because correct description of quantum fields is possible only using relativistic theory.

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