This paper presents a observer-based fault estimation method for a class of singularly perturbed systems subjected to parameter uncertainties and time-delay in state and disturbance signal with finite energy. To solve the estimation problem involving actuator fault and sensor fault for the uncertain disturbed singularly perturbed systems with time-delay, the problem we studied is firstly transformed into a standard $H_{\infty}$ control problem, in which the performance index $\gamma$ represents the attenuation of finite energy disturbance. By adopting Lyapunov function with the $\varepsilon$-dependence, a sufficient condition can be derived which enables the designed observer to estimate different kinds of fault signals stably and accurately, and the result obtained by dealing with small perturbation parameter in this way is less conservative. A novel multi-objective optimization scheme is then proposed to optimal disturbance attenuation index $\gamma$ and system stable upper bound $\varepsilon^*$, in this case, the designed observer can estimate the fault signals better in the presence of interference when the systems guarantee maximum stability bound. In the end, the validity and correctness of proposed scheme is verified by comparing the error between the estimated faults and the actual faults.

1. Introduction. The research on singularly perturbed systems attracts much attention owning to their wide application in practical engineering[32]. The research methods mainly include $H_{\infty}$ control[14, 25], adaptive control[6], fuzzy control[13, 15], sliding mode control[28, 4] and their combination[21, 29]. In practical engineering applications, due to insufficient understanding of system characteristics or links, unfamiliar with environment and modeling errors, etc., systems are affected by parameter inevitably, and the time-delay caused by signal transmission delays is ubiquitous as well. Hence, the analysis and research on uncertain singularly perturbed systems with time-delay become a hot issue. Currently, there are some research in the stability[12]-[3] and robust control[8, 22] or guaranteed cost control[16] under
certain performance constraints. In addition, the stability upper bound represented by $\varepsilon^*$ of singularly perturbed systems has attracted wide attention\[10]-[5].

In the process of actual engineering system control, the occurrence of actuator faults and sensor faults and control components faults in dynamic systems could greatly reduce the control performance and make the system inoperable, such that the fault diagnosis problem of singular perturbation systems is widely concerned. In singularly perturbed systems, faults may also cross time scales due to the existence of small perturbed parameter, which increases the difficulty of fault diagnosis. At the beginning of the study, it can only detect whether the fault occurs or not without obtaining the information about the shape and magnitude\[17, 18], and then the conservative fault diagnosis can be realized along with the further research\[19, 20]. Recently, with the gradual improvement of the theoretical system, the research on fault estimation has yielded fruitful results\[5],\[9]-[27]. In observer-based fault diagnosis methods, a fault diagnosis method based on Proportional-Integral(PI) observer is proposed for systems with sensor and actuator faults in [9], and then the fault-tolerant controller is designed based on optimal control theory. An observer-based fault estimation method is proposed for the Lipschitz nonlinear singular perturbation systems with sensor fault in [5], in which a optimal scheme is presented to estimate the maximum stability bound and the best uncertainty decay ability. A feasible scheme for estimating sensor fault vector by observer based on descriptor approach is adopted in paper [24]. Different from the above, the transient behavior of the system is improved to a satisfactory level by designing fault detection filters and pole assignment based on the finite frequency method in [1]. Generally speaking, the literature research systems discussed above are relatively single. However, for the fault estimation problem of uncertain time-delay singularly perturbed systems with faults and external disturbances, although it is significant, has not been researched yet.

In view of above, the fault estimation problem for a class of singularly perturbed systems with time-delay and uncertainties in the presence of disturbance is addressed in the paper. Inspired by the literature [5], a residual observer to exactly estimate actuator and sensor faults by taking into account the diversity of actual industrial system faults is designed. A multi-objective optimization scheme is then proposed and the stability upper bound and the $H_{\infty}$ performance index of the system are optimized simultaneously, in this way, the designed observer can estimate the fault signals better in the presence of interference when the systems guarantee maximum stability bound. The final numerical example will demonstrated the feasibility and correctness of this method.

2. Preliminary knowledge and problem description. Consider a class of disturbed singularly perturbed systems with uncertainty and time-delay as follows

$$\begin{cases}
E \dot{x}(t) = (A + \Delta A)x(t) + (Ah + \Delta Ah)x(t - h) + Bu(t) + Ww(t) + D_1f(t) \\
y(t) = Cx(t) + D_2f(t)
\end{cases}$$

\begin{equation}
E_x = \begin{bmatrix} I_{n1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \ x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{equation}

where $x(t) \in \mathbb{R}^n$ is the state vector of the system and satisfies $n_1 + n_2 = n$. $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the control input vector and measure output vector, respectively. $w(t) \in \mathbb{R}^l$ represents finite energy disturbance signal. $h \in (0, \bar{h}]$ is the system
constant time-delay, and $0 < \varepsilon \ll 1$ is the perturbation parameter of singular perturbation system. $\Delta A$ and $\Delta A_h$ are time-invariant matrices that describe the bounded uncertainty of the parameter

$$[\Delta A \Delta A_h] = MF(\sigma) [N N_h]$$

(2)

where $M, N, N_h$ are known constant real matrices with appropriate dimensions. The uncertain matrix $F(\sigma)$ is satisfied

$$F(\sigma)F^T(\sigma) \leq I$$

(3)

$\sigma \in \Theta$, $\Theta$ is a dense set of $R$. Suppose given arbitrary matrix $F: FF^T \leq I$, It exists $\sigma \in \Theta$ such that $F = F(\sigma)$.

If 2 and 3 are both established, $\Delta A$ and $\Delta A_h$ are said to be tolerable.

The fault $f(t)$ studied in this paper includes actuator fault $f_a(t)$ and sensor fault $f_s(t)$

$$f(t) = \begin{bmatrix} f_a(t) \\ f_s(t) \end{bmatrix}$$

(4)

Assuming the fault signals can be described by a dynamic system as

$$\dot{\varphi}(t) = G\varphi(t)$$

$$f(t) = F\varphi(t)$$

the expression 4 is an outer system that describes the fault signals, $\varphi(t) \in R^{(n_a+n_s)}$ is the system fault state vector, and

$$\varphi(t) = \begin{bmatrix} \varphi_a(t) \\ \varphi_s(t) \end{bmatrix}, G = \begin{bmatrix} G_a & 0 \\ 0 & G_s \end{bmatrix}, F = \begin{bmatrix} F_a & 0 \\ 0 & F_s \end{bmatrix}$$

**Remark 1.** The outer system 4 is a common expression for continuous faults, such as step faults, sinusoidal faults, attenuation faults, exponential faults and so on.

Set $z(t) = \begin{bmatrix} x(t) \\ \varphi(t) \end{bmatrix}$, an augmented system can be obtained

$$\begin{cases}
\dot{E}_z z(t) = (\bar{A} + \Delta \bar{A}) z(t) + (\bar{A}_h + \Delta \bar{A}_h) z(t-h) + \bar{B} u(t) + \bar{W} w(t) \\
y(t) = C z(t)
\end{cases}$$

(5)

where

$$E_z = \begin{bmatrix} E_x & 0 \\ 0 & I \end{bmatrix}, \bar{A} = \begin{bmatrix} A & D_1 F \\ 0 & G \end{bmatrix}, \Delta \bar{A} = \begin{bmatrix} \Delta A & 0 \\ 0 & 0 \end{bmatrix}, \bar{A}_h = \begin{bmatrix} A_h & 0 \\ 0 & 0 \end{bmatrix},$$

$$\Delta \bar{A}_h = \begin{bmatrix} \Delta A_h & 0 \\ 0 & 0 \end{bmatrix}, \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \bar{W} = \begin{bmatrix} W \\ 0 \end{bmatrix}, C = \begin{bmatrix} C & D_2 F \end{bmatrix}.$$
Lemma 2.4. [30] If the symmetric matrices $P_1, P_2, P_3, P_4$ and the appropriate dimension matrix $P_5$ are satisfied

$$P_1 > 0$$

$$\begin{bmatrix}
P_1 + \varepsilon^* P_3 & \varepsilon^* P_5^T \\
\varepsilon^* P_5 & P_2
\end{bmatrix} > 0$$

(6)

then, the following formula holds.

$$E \varepsilon^T P(\varepsilon) = P^T(\varepsilon)E \varepsilon > 0, \quad \forall \varepsilon \in (0, \varepsilon^*)$$

(7)

where

$$P(\varepsilon) = \begin{bmatrix}
P_1 + \varepsilon P_3 & \varepsilon P_5^T \\
P_5 & P_2 + \varepsilon P_4
\end{bmatrix}$$

Construct a residual observer as follows

$$\begin{cases}
\dot{\hat{z}}(t) = (\hat{\dot{A}} + \Delta \hat{A}) \hat{z}(t) + (\hat{A}_h + \Delta \hat{A}_h) \hat{z}(t - h) + \hat{B} u(t) + \hat{L}(y - \hat{y}) \\
y(t) = \hat{C} \hat{z}(t)
\end{cases}$$

(8)

where $\hat{z}(t), \hat{x}(t), \hat{\varphi}(t)$ represent the estimated value of the augmented system state vector $z(t)$, the original system state vector $x(t)$ and the fault state vector $\varphi(t)$, respectively. $\hat{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \in \mathbb{R}^{(n+n_n+n_f) \times P}$ is the observer gain matrix to be solved.

Define the estimation error $e_z(t) = z(t) - \hat{z}(t)$. $\Delta \hat{A}, \Delta \hat{A}_h$ and $w(t)$ are consistent with the uncertainty of the original system (1), and $w(t)$ is also the anti-interference ability of the observer. The expression about estimation error can be got

$$\begin{cases}
\dot{\hat{e}}_z(t) = (\hat{\dot{A}} + \Delta \hat{A} - \hat{L}\hat{C})e_z(t) + (\hat{A}_h + \Delta \hat{A}_h)e_z(t - h) + \hat{W} w(t) \\
e_y(t) = \hat{C}e_z(t)
\end{cases}$$

(9)

In order to realize the design goals of fault estimation and optimization, an observer is first designed and the input $u(t)$ and output $y(t)$ of the original systems are taken as the input vectors of the observer. The observer can accurately output the state vectors of the original system and outer system representing the fault as well as possesses the ability of anti-disturbance. Furthermore, the stability upper bound and the attenuation ability of finite energy disturbance are optimized jointly based on multi-objective optimization algorithm. Detailed description is as follows.

1) Fault estimation

1) When the disturbance does not exist ($w(t) = 0$), the estimation error $e_z(t)$ gradually tends to 0, it is shown that the observer can estimate the state vectors of the original system and the external system representing the fault.

2) When the disturbance exist ($w(t) \neq 0$), the observer can accurately output the state vectors of the original system and the external system representing the fault as well as possesses the ability of anti-disturbance, the estimation error $e_z(t)$ satisfies $H_{\infty}$ performance under zero initial conditions

$$\|e_z\|_2 \leq \gamma \|w(t)\|_2$$

(10)

which can be converted to the following form

$$J = \int_0^\infty (e_z(t)^T e_z(t) - \gamma^2 w(t)^T w(t)) dt < 0, \quad \forall \varepsilon \in (0, \varepsilon^*)$$

(11)

where $\gamma$ is the attenuation performance index of finite energy disturbance.
(2) Optimization
A multi-objective optimization algorithm is proposed to optimize $H_\infty$ performance index $\gamma$ and system perturbation upper bound $\varepsilon^*$, in which a compromise scheme can be found to ensure both larger stable upper bound and stronger anti-disturbance capability.

Remark 2. In engineering practice, finding an accurate non-conservatively stable upper bound can ensure that the controller proposed in the singular perturbation systems (including the residual estimator) exists, and the increase in the stability upper bound indicates that there is greater opportunity to obtain an effective controller[5].

3. Residual observer design and optimization.

3.1. Design of Residual Observer.

Theorem 3.1. Assuming that prerequisites of Lemmas 2.1-2.2 are satisfied in system 1, given parameters $\gamma > 0$, $0 < \varepsilon^* < 1$, $h \in (0, \bar{h}]$ and $Q = \text{diag}\{Q_1, Q_2\} > 0$. The residual observer can be constructed by equation 8 if it exists symmetric matrices $P, P_1, P_2, P_3, P_4$ and the appropriate dimension matrices $P_5, Y$ satisfying the linear matrix inequalities 6 and 12-13. And the estimation error 9 is robust stable and satisfies the $H_\infty$ performance index, the gain matrix can be obtained by $\bar{L} = \bar{P}^{-T}(\varepsilon)Y$.

\begin{equation}
\begin{bmatrix}
\Gamma(0) + I & \bar{P}^T(0)(\bar{A}_h + \Delta \bar{A}_h) & \bar{P}^T(0)\bar{W}^* \\
* & -Q & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0 \tag{12}
\end{equation}

\begin{equation}
\begin{bmatrix}
\Gamma(\varepsilon^*) + I & \bar{P}^T(\varepsilon^*)(\bar{A}_h + \Delta \bar{A}_h) & \bar{P}^T(\varepsilon^*)\bar{W}^* \\
* & -Q & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0 \tag{13}
\end{equation}

where $\Gamma(\varepsilon) = (\bar{A} + \Delta \bar{A})^T \bar{P}(\varepsilon) + \bar{P}^T(\varepsilon)(\bar{A} + \Delta \bar{A}) - \bar{C}^TY^T - Y\bar{C} + Q$, $\bar{P}(\varepsilon) = \begin{bmatrix} P(\varepsilon) & 0 \\ 0 & P \end{bmatrix}$, $\varepsilon \in (0, \varepsilon^*]$. 

Proof of Theorem 3.1. The system is strongly observable when the prerequisites in Lemmas 2.1-2.2 are satisfied, so the residual observer constructed by equation 8 exists. Then we demonstrate that the residual observer satisfies the design requirements of fault estimation.

Consider the Lyapunov function as follows

\begin{equation}
V(e_\varepsilon) = V_1(e_\varepsilon) + V_2(e_\varepsilon) \\
= e_\varepsilon^T(t)\bar{E}_\varepsilon^T \bar{P}(\varepsilon) e_\varepsilon(t) + \int_{t-h}^t e_\varepsilon^T(\alpha)Qe_\varepsilon(\alpha)d\alpha \tag{14}
\end{equation}

The following formula can be deduced from Lemma 2.4

\begin{equation}
\bar{E}_\varepsilon^T \bar{P}(\varepsilon) = \bar{P}(\varepsilon)\bar{E}_\varepsilon^T = \begin{bmatrix} E_\varepsilon^T P(\varepsilon) & 0 \\ 0 & P \end{bmatrix} > 0 \tag{15}
\end{equation}

obviously, $V$ is positive definite when $Q > 0$ holds.
Derivative of the Lyapunov function, we can get
\[
\dot{V}(e_z) = \dot{V}_1(e_z) + \dot{V}_2(e_z) \\
= \begin{bmatrix} \dot{e}_z \end{bmatrix}^T \bar{P}(e_z) e_z + e_z^T \bar{P}^T(e_z) \begin{bmatrix} \dot{e}_z \end{bmatrix} + \frac{1}{\gamma} \int_{t-h}^{t} [e_z^T(t) Q e_z(t-h)] d\alpha \\
\begin{bmatrix} \dot{e}_z(t) - e_z^T(t-h) Q e_z(t-h) \end{bmatrix} \\
= e_z^T(t) \Pi e_z(t) + 2e_z^T(t) \bar{P}^T(e_z) (\bar{A}_h + \Delta \bar{A}_h) e_z(t-h) \\
+ 2e_z^T(t) \dot{P}^T \dot{W} w(t) - e_z^T(t-h) Q e_z(t-h)
\] (16)

where \( \Pi = (\bar{A} + \Delta \bar{A} - \bar{L} \bar{C})^T \bar{P}^T(e) + \bar{P}^T(e) (\bar{A} + \Delta \bar{A} - \bar{L} \bar{C}) + Q \).

From Theorem 3.1 we know that the linear matrix inequality 17 holds
\[
\begin{bmatrix}
\Gamma(e) + I & \bar{P}^T(e) (\bar{A}_h + \Delta \bar{A}_h) & \bar{P}^T(e) \bar{W} \\
* & -Q & 0 \\
* & * & -\gamma^2 I
\end{bmatrix} < 0
\] (17)

By substituting \( \bar{L} = \bar{P}^{-T}(e) Y \) into equation 17 and using Schur complement, linear matrix inequality 18 can be derived as
\[
\begin{bmatrix}
\Gamma(e) + I & \bar{P}^T(e) (\bar{A}_h + \Delta \bar{A}_h) \\
* & -Q
\end{bmatrix} < 0
\] (18)

which can be derived to
\[
e_z^T(t) \Pi e_z(t) + 2e_z^T(t) \bar{P}^T(e_z) (\bar{A}_h + \Delta \bar{A}_h) e_z(t-h) - e_z^T(t-h) Q e_z(t-h) < 0
\]
so, the inequality \( \dot{V} < 0 \) holds when \( w(t) = 0 \), and the state estimation error \( e_z \) gradually tends to 0.

When \( w(t) \neq 0 \), the performance index function 9 is converted to the following form
\[
J = \int_{0}^{\infty} \left[ e_z^T(t) e_z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right] dt + V(t)_{t=0} - V(t)_{t=\infty}
\]
it exists that \( V(t)_{t=0} = 0, V(t)_{t=\infty} \geq 0 \) under zero initial condition, so
\[
J \leq \int_{0}^{\infty} \left[ e_z^T(t) e_z(t) - \gamma^2 w^T(t) w(t) + \dot{V}(t) \right] dt
\]

The above equation can be converted into
\[
J \leq \int_{0}^{\infty} \mu^T \Psi \mu dt
\]
where \( \mu = \begin{bmatrix} e_z^T(t) & e_z^T(t-h) & w^T(t) \end{bmatrix}^T \)
\[
\Psi = \begin{bmatrix}
\Pi(e) + I & \bar{P}^T(e) (\bar{A}_h + \Delta \bar{A}_h) & \bar{P}^T(e) \bar{W} \\
* & -Q & 0 \\
* & * & -\gamma^2 I
\end{bmatrix}
\] (19)

The equations 19 and 17 are equivalent by substituting \( \bar{L} = \bar{P}^{-T}(e) Y \), which means that \( \Psi < 0 \) holds for every \( e \) in the range of 0 to \( e^* \), so \( J < 0 \).

To sum up, system 9 is robust stable and satisfies the \( H_{\infty} \) performance index.

Proof is completed. \( \square \)
3.2. Multi-objective optimization. A multi-objective optimization algorithm is proposed to optimize both the stability upper bound $\varepsilon^*$ and the $H_\infty$ performance index $\gamma$ of the system simultaneously. The system maximum stability upper bound can be expressed by the maximum perturbation upper bound $\varepsilon^*$, while $\gamma$ represents the strength of the damping capacity of the finite energy disturbance. Consequently, the optimization problem can be transformed into the following mathematical form

$$
\min \gamma, \max \varepsilon^* \\
\text{s.t. } 6, 12, 13
$$

Optimization algorithm is as follows:

Step 1: Select the initial value of the stability bound $\varepsilon^*$ as $\varepsilon_0(0 < \varepsilon_0 \ll 1)$ and step length as $d(0 < d \ll 1)$.

Step 2: Set $i = 1, \varepsilon_1^* = \varepsilon_0$, calculate the result at this time and write it as $\gamma_1$:

$$
\min \gamma \\
\text{s.t. } 6, 12, 13
$$

Step 3: Set $i = i + 1, \varepsilon_i^* = \varepsilon_{i-1}^* + d$, substitute Step 2 loop and record the result as $\gamma_i$. Then, define the function to evaluate the optimization performance

$$
f = \delta_1 \frac{\varepsilon_{\max} - \varepsilon_i^*}{\varepsilon_{\max} - \varepsilon_{\min}} + \delta_2 \frac{\gamma_i - \gamma_{\min}}{\gamma_{\max} - \gamma_{\min}}
$$

where $\delta_1, \delta_2$ are the weight of the evaluation function and satisfy the equation $\delta_1 + \delta_2 = 1$. And the values of weight could be got from practical requirements.

Step 4: Go through all the combination solutions from 0 to 1, and find the value of $i$ that corresponds to the smallest evaluation function $f$. ie $i_{op} = \arg \min f_i$

Step 5: The optimized combination is

$$
\begin{align*}
\varepsilon_{\text{opt}}^* &= \varepsilon_i^* \\
\gamma_{\text{opt}} &= \gamma_{i_{op}}
\end{align*}
$$

and the optimal observer gain matrix $\bar{L}$ can be obtained after optimization value is substituted into the system.

4. Numerical example. The parameters of an uncertain disturbed singularly perturbed system with time-delay are as follows:

$$
E = \begin{bmatrix} 1 & 0 \\ 0 & \varepsilon \end{bmatrix}, A = \begin{bmatrix} -0.04 & 0.3 \\ -0.8 & -5 \end{bmatrix}, A_h = \begin{bmatrix} 0.25 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}^T,
$$

$$
D_1 = \begin{bmatrix} 0.8 & 0 \\ 0.6 & 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}^T, W = \begin{bmatrix} 0.2 \\ 0.1 \end{bmatrix}, M = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, F = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix},
$$

$$
N = \begin{bmatrix} 0.4 & 0.2 \\ 0.5 & 0.4 \end{bmatrix}, N_h = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, G = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}, \bar{h} = 0.8
$$

We select the initial value of the perturbation parameter and the step length as $\varepsilon_0 = 0.01, d = 0.01$. The weight coefficients data from paper [5] can be chosen as $\delta_1 = 0.2, \delta_2 = 0.8$ according to practical requirements.

The optimized perturbation parameter $\varepsilon_{\text{opt}}^* = 0.10$. Then the perturbation parameter are selected as $\varepsilon = \varepsilon_{\text{ver}1} = 0.11(\varepsilon_{\text{ver}1} \notin (0, \varepsilon_*)$ and $\varepsilon = \varepsilon_{\text{ver}2} = 0.09(\varepsilon_{\text{ver}2} \in (0, \varepsilon_*)$) in order to verify the superiority of multi-objective optimization algorithms. The gain matrix of residual observer

$$
\bar{L}_{\text{opt}} = \begin{bmatrix} -41.8274 & 13.6671 & -0.4818 & -0.6725 \end{bmatrix}^T,
$$

$$
\bar{L}_{\text{ver}1} = \begin{bmatrix} -14.6130 & 4.9119 & -0.6328 & -0.6403 \end{bmatrix}^T,
$$
\[ \bar{L}_{\text{ver}} = \begin{bmatrix} -50.1267 & 16.5708 & -0.4887 & -0.6731 \end{bmatrix}^T \]

Given a disturbance variable as \( w(t) = 2e^{-0.5t} \sin(3t) \) and initial conditions as 
\[
\begin{align*}
    e_x(0) &= \begin{bmatrix} 0.5 \\ 0.3 \end{bmatrix}, \\
    e_{\varphi}(0) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
    \varphi(0) &= \begin{bmatrix} 0.056 \\ 0.083 \end{bmatrix}
\end{align*}
\]

The simulation trajectories comparison between the estimated fault signals and the actual fault signals under three perturbation parameters are shown below.

**Figure 1.** Curves of estimated actuator fault

**Figure 2.** Curves of estimated actuator fault

Figs. 1-8 describe the comparison of actuator and sensor fault estimation signals and errors when the perturbation parameters take different values. Taking the actuator fault of the system as an example. The comparison between the estimated faults and the actual fault is shown in Fig. 1 when the perturbation parameters are 0.11 and 0.10, respectively. The simulation curve shows that when the perturbation parameters \( \varepsilon = \varepsilon^* = 0.10 \), the estimated fault basically coincides with the actual fault trajectory. However, when the value of the observer is not in the upper bound of maximum stability, i.e., \( \varepsilon = \varepsilon_{\text{ver}} = 0.11(\varepsilon_{\text{ver}} \notin (0, \varepsilon^*)) \), the deviation between the
Figure 3. Curves of the actuator fault estimation error

Figure 4. Curves of the actuator fault estimation error

Figure 5. Curves of estimated sensor fault
Figure 6. Curves of estimated sensor fault

Figure 7. Curves of the sensor fault estimation error

Figure 8. Curves of the sensor fault estimation error
estimated fault signal and the actual fault signal value is large. By comparing the estimated error of Fig. 3, it is concluded that the output of the observer is difficult to describe the actual fault signal and does not meet the design requirements. Fig. 2 depicts the comparison between the estimated fault and the actual fault signals when the perturbation parameters are 0.09 and 0.10. It can be seen that the three curves basically coincide at this time. It shows that the observer designed by selecting the parameters within the maximum stability upper bound meets the requirements of fault estimation. The estimated error under the two values described in Fig. 4 shows that the order of magnitude of error is kept at $10^{-3}$ and the difference between them is very small. It shows that the system has strong anti-disturbance ability at this time and the anti-disturbance ability of the two systems is almost same. Therefore, when the perturbation parameter is 0.10, the system has the maximum stability upper bound and the best anti-disturbance characteristic, which proves the correctness of this method. In addition, the state vector estimation error waveforms of the outer fault system and original system are respectively described in Figs. 9-10 when $\varepsilon = \varepsilon^* = 0.10$. From the estimated error trajectories, it can be seen that...
the state vector curves tend to zero quickly and the fluctuation range is very small and the magnitude of the curve is even in the order of $10^{-4}$. It implies that the observer can accurately estimate the state vectors of the outer fault system and original system with better anti-disturbance characteristics.

5. Conclusion. In this paper, the problem of optimal fault estimation for perturbed singular perturbation systems with actuator and sensor faults is considered. Firstly, $H_\infty$ performance index is used to characterize the attenuation ability of finite energy disturbance by reasonably constructing augmented system. And then the sufficient condition is given based on Lyapunov stability theory, such that the observer can accurately estimate the state and fault signals of the system. Furthermore, the maximum stability upper bound and the best anti-disturbance capability are guaranteed based on the multi-objective optimization algorithm. Last but not the least, a numerical example is given to compare the fault output trajectories in different perturbation parameter values. When the perturbation parameter takes the maximum stable upper bound, the stability and anti-disturbance performance of the system are both optimal, which proves the correctness of the scheme.

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