THE WHITTAKER PERIOD FORMULA ON $\widetilde{SL}_2$

Abstract. We decompose the global Whittaker period functional on a cuspidal representation of $\widetilde{SL}_2(\mathbb{A}_F)$ as an Euler product of local period integrals of matrix coefficients. The decomposition was previously proved for a totally real number field $F$ using relative trace formula. We find a direct comparison of the global and local Whittaker periods for representations of $\widetilde{SL}_2$ and $PGL_2$ in the context of theta lifting, and use it to get a new argument working for all number fields $F$. Our main tools are inner product formula and the isometry property of quadratic Fourier transform derived from certain estimates for Bruhat-Schwartz functions.

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In [16], Waldspurger established an identity relating the Fourier coefficient of a half integral weight modular form to the special value of the $L$-function of a classical modular form. This Fourier coefficient identity and its many later versions (for example, [7, 15, 6, 8]) are stated with certain subtle conditions in term of the classical language of modular forms and has important number-theoretic applications (for example, [5, 11, 13, 4]).

From the view of representation theory, the Fourier coefficient identity is about decom-posing the global Whittaker period functional on a genuine irreducible cuspidal representation of the metaplectic group $\widetilde{SL}_2(\mathbb{A}_F)$, where $F$ is a number field and $\mathbb{A}_F$ is the ring of $F$-adels. The ideal identity would have the global Whittaker period functional on one side and the product of local integrals of matrix coefficients on the other side, with the $L$-value as the proportion constant. There were two approaches to such a representation-theoretic identity.

The first approach is via Jacquet’s relative trace formula. Baruch and Mao [3] proved a “Basic Waldspurger Formula” based on their very technical comparison of local Bessel distributions on $GL_2$ and $\widetilde{SL}_2$ over $p$-adic fields and the field of real numbers [11, 2]. They did not complete the local comparison of distributions over the complex field and hence their decomposition formula was stated only for totally real number fields.

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The second approach is by descent. Lapid and Mao \cite{9, 10} proposed a project to relate the Whittaker functional on $\widetilde{Sp}_n$-representation to the Whittaker functional on $GL_m$-representations. Since the decomposition of the Whittaker functional of cuspidal $GL_m$-representations are known by the Rankin-Selberg method, they reduce the problem of decomposing the global Whittaker functional on $\widetilde{Sp}_n$-representations to a set of local conjectural identities. While this descent approach is very inspiring, the local conjectural identities involve integrals that are divergent and difficult to define or deal with at the present moment, even for the beginning case of $\widetilde{Sp}_1 = \widetilde{SL}_2$. They currently only verify the local conjectures for supersingular representations.

In this paper, we prove the decomposition formula for an arbitrary number field via the theta correspondence between $\widetilde{SL}_2$ and $PGL_2$, where $PGL_2$ is identified with $SO(2, 1)$. We actually use the inner product formula to directly transfer the global and local Whittaker periods of $\widetilde{SL}_2$-representations to $PGL_2$. For the local period transfer, we also encounter the issue of divergent local integrals as in the approach of descent, but we are able to provide a clean treatment using the notion of distribution and an isometry property of quadratic Fourier transform; the isometry property is based on an estimate of the quadratic Fourier transform of Bruhat-Schwartz functions (c.f. section 2). In general, we believe that the treatment of divergent local integrals should be based on specific estimates for the integrands involved.

**Theorem.** Let $\sigma$ be a genuine irreducible cuspidal representation of $\widetilde{SL}_2(\mathbb{A}_F)$ that is orthogonal to elementary theta series. Let $\psi$ be a character of $\mathbb{A}_F/F$. Suppose that the $\psi$-Whittaker functional $\ell_{\sigma,\psi}(\varphi) = \int_{\mathbb{A}_F/F} \varphi(1, x) \psi(-x) dx$ is non-vanishing on $\sigma$ and write $\pi = \Theta_{\widetilde{SL}_2 \times PGL_2}(\sigma, \psi)$ for the global theta lift of $\sigma$ to $PGL_2(\mathbb{A})$ with respect to $\psi$. Then for decomposable vectors $\varphi_i = \otimes \varphi_{i,v} \in \sigma$, there is
\[
\ell_{\sigma,\psi}(\varphi_1) \ell_{\sigma,\psi}(\varphi_2) = \frac{L(\frac{1}{2}, \pi) \xi(2)}{L(1, \pi, \text{ad})} \prod_v \frac{L(1, \pi_v, \text{ad})}{L(\frac{1}{2}, \pi_v, \xi_{F_v}(2))} \int_{F_v} (\sigma_v(1, x_v) \varphi_{1,v} \varphi_{2,v})_{\sigma_v} \psi_v(-x_v) dx_v.
\]

Here $(,)_{\sigma_v}$ denotes the local Hermitian pairing on $\sigma_v$ and their product is the global pairing $(\varphi_1, \varphi_2)_\sigma := \int_{\widetilde{SL}_2(F) \backslash \widetilde{SL}_2(\mathbb{A}_F)} \varphi_1(g) \varphi_2(g) dg$. The local integral on $F_v$ in the above formula may not be absolutely convergent; so we interpret its meaning in term of distribution: the function $(\sigma_v(1, x_v) \varphi_{1,v} \varphi_{2,v})_{\sigma_v}$ defines a distribution on the space of Bruhat-Schwartz functions $\mathcal{S}(F_v)$ and the Fourier transform of this distribution can be shown to be represent by a smooth function $W_{\varphi_{1,v} \varphi_{2,v} \psi_v}(a_v)$ on $F_v^\times$, then we define $\int_{F_v} (\sigma_v(1, x_v) \varphi_{1,v} \varphi_{2,v})_{\sigma_v} \psi_v(-x_v) dx_v$ as $W_{\varphi_{1,v} \varphi_{2,v} \psi_v}(-1)$. When $v$ is a finite place, this definition agrees with the one given in \cite{9, 10} in term of stable integrals (c.f. remark 4).

We sketch our argument for the theorem. By multiplicity one results of local Whittaker functionals, the global Whittaker functionals on $\sigma$ and $\pi$ can be decomposed as products of normalized local integrals of matrix coefficients up to constants $c_\sigma, c_\pi$ (c.f. sections 3.2 and 4.2.3). Let $f_i = \Theta(\phi_i, \varphi_i) \in \pi$ be the lifts of the forms $\varphi_i \in \sigma$ via decomposable Bruhat-Schwartz functions $\phi_i$. Then the global Whittaker functional $\ell_{\pi, \psi}$ on $f_i$ can be easily expressed using the global Whittaker functional $\ell_{\sigma, \psi}$ on $\varphi_i$ (c.f. Lemma 5). Similarly,
The transfer for local period integrals of matrix coefficients is the key part of this paper. Our tool for doing it is the isometry property of local quadratic Fourier transform \( f_v(x_v) \to \mathcal{F}_2 f_v(t_v) := \int_{F_v} f_v(x_v) \psi(t_v x_v^2) d x_v, \ f_v \in \mathcal{L}^1(F_v) \). We prove the isometry property (c.f. Proposition 2) for p-adic fields and archimedean local fields in the same way, and hence get the theorem for arbitrary number fields \( F \). It is this local period transfer that replaces the local comparison of Bessel distributions in the approach of relative trace formula.

In general, we expect the same type of local and global period transfer concerning the dual reductive pair \((\widetilde{Sp}_n, SO_{2m+1})\). When \( n = m = 1 \), besides the relation discussed here, there is also a transfer relation between the Fourier period of \( \widetilde{SL}_2 \)-representations and the toric period of \( SO_3 \)-representations; see [12] for the case of anisotropic \( SO_3 \). When \( n = 1 \) and \( m = 2 \), we also found an explicit transfer relation between the Whittaker period of \( \widetilde{SL}_2 \)-representations and the Bessel period of Saito-Kurokawa representations; we explicate this relation in another paper and use it to do the refined Gross-Prasad conjecture for Bessel periods of Saito-Kurokawa representations.

1. Notations

Let \( F \) be a number field and \( \mathbb{A} := \mathbb{A}_F \) the ring of \( F \)-adels. When \( v \) is a non-archimedean place of \( F \), let \( \mathcal{O}_{F_v} \) be the ring of \( v \)-adic integers in \( F_v \) and \( \varpi_v \) a uniformizer of \( \mathcal{O}_{F_v} \); for a non-trivial character \( \psi_v \) of \( F_v \), its conductor is defined as the largest \( \varpi_v^n \mathcal{O}_{F_v} \) on which \( \psi_v \) is trivial; for a quasi-character \( \chi_v \) of \( F_v^\times \), define its conductor \( \text{Cond}(\chi_v) \) as \( \mathcal{O}_{F_v} \) if \( \chi_v \) is unramified and as \( \varpi_n \mathcal{O}_{F_v} \) is \( n \) the smallest positive integer such that \( \chi_v \) is trivial on \( 1 + \varpi_n \mathcal{O}_{F_v} \).

Let \( \psi : \mathbb{A}/F \to S^1 \) be a non-trivial character. At each place \( v \) of \( F \), let \( d a_v \) be the self-dual Haar measure of \( F_v \) with respect to \( \psi_v \), then \( d a = \prod_v d a_v \) is the Tamagawa measure on \( \mathbb{A} \). Write \( d^* a_v = \frac{d a_v}{\varpi_v} \) and \( d^* a = \prod_v d^* a_v \). The Tamagawa measure on \( \mathbb{A}^\times \) is \( d^* a = \frac{1}{\text{Res}_{s=1} \zeta_F(s)} \prod_v \zeta_{F_v}(1) d^* a_v \); we usually decompose \( d^* a \) locally as \( \prod_v d^* a_v \), where \( d^* a_v \) is such that \( \int_{\mathcal{O}_{F_v}} 1 d^* a_v = 1 \) for almost all \( v \).

For \( \delta \in F^\times \), write \( \psi_\delta = \psi(\delta \cdot) \) for the \( \delta \)-twist of \( \psi \) and \( \chi_\delta \) for the quadratic character \( < \delta, > \) on \( \mathbb{A}^\times \), where \( <, > \) is the Hilbert symbol.

For a quadratic space \( (V, q) \) over \( F \), set \( q(X, Y) = q(X + Y) - q(X) - q(Y) \). \( \mathcal{S}(V_{F_v}) \) denotes the space of Bruhat-Schwartz functions on \( V(F_v) \) and \( \mathcal{S}(V_{\mathbb{A}}) \) the space of Bruhat Schwartz functions on \( V(\mathbb{A}) \).

2. Quadratic Fourier Transform

Let \( k \) be a local field of characteristic zero, \( \psi \) a non-trivial character of \( k \). The Fourier transform with respect to \( \psi \) is defined on \( \mathcal{L}^1(k) \) by \( \phi(x) \xrightarrow{\mathcal{F}} \hat{\phi}(x) := \int_k \phi(y) \psi(xy) dy \), where
Proof. We estimate $F(x) := \chi(x)|x| \int_k \phi(x-y) \chi^{-1}(y)|y|^{-1}dy = \int_k \phi(t) \chi^{-1}(1 - \frac{t}{x})|1 - \frac{t}{x}|^{-1}dt$. 

(i) $k$ is $p$-adic. One easily sees that under the condition $\text{Supp}(\hat{\phi}) \subseteq x \cdot (\text{Cond}(\chi) \cap \varpi \mathcal{O}_k)$, 
the integrand $\hat{\phi}(t) \chi^{-1}(1 - \frac{t}{x})|1 - \frac{t}{x}|^{-1} = \hat{\phi}(t)$ and hence $F(x) = \int_k \hat{\phi}(t)dt = \phi(0)$. 

(ii) $k$ is archimedean. We break $F(x)$ into three parts to do the estimate, 

$$F(x) = \int_{|t| \leq \frac{|a|}{2}} + \int_{|t| < \frac{3|a|}{2}} + \int_{|t| \geq \frac{3|a|}{2}} \hat{\phi}(t) \chi^{-1}(1 - \frac{t}{x})|1 - \frac{t}{x}|^{-1}dt$$ 

$$= F_1(x) + F_2(x) + F_3(x).$$

Concerning $F_1(x)$, we have 

$$|F_1(x) - \phi(0)| = \left| \int_{|t| \leq \frac{|a|}{2}} \hat{\phi}(t) \chi^{-1}(1 - \frac{t}{x})|1 - \frac{t}{x}|^{-1}dt - \int_k \hat{\phi}(t)dt \right|$$ 

$$\leq \int_{|t| > \frac{|a|}{2}} |\hat{\phi}(t)|dt + \int_{|t| \leq \frac{|a|}{2}} |\hat{\phi}(t)||\chi^{-1}(1 - \frac{t}{x})|1 - \frac{t}{x}|^{-1} - 1|dt.$$
When $|t| \leq |x|/2$, $(1 - \frac{t}{x})$ falls in a compact neighborhood of 1 that does not contain 0. By mean value theorem, when $e(\chi) \in [e_1, e_2]$, there is a constant $c_1$ such that

$$|\chi^{-1}(1 - \frac{t}{x})| 1 - \frac{t}{x}|^{-1} - 1| \leq c_1 \frac{|t|}{|x|}.$$

Hence $|F_1(x) - \phi(0)| \leq \int_{|t| > \frac{3|x|}{2}} |\hat{\phi}(t)||\chi^{-1}(1 - \frac{t}{x})| 1 - \frac{t}{x}|^{-1} dt + \int_{|t| < \frac{3|x|}{2}} |\hat{\phi}(t)||1 - \frac{t}{x}|^{-1} dt$.

Concerning $F_2(x)$, we make a change of variable $u = t/x$ and rewrite

$$F_2(x) = \int_{1/2 < |u| < 3/2} x \hat{\phi}(xu) \chi^{-1}(1 - u)|1 - u|^{-1} du.$$

Then $|F_2(x)| \leq aA_x$ with

$$A_x = \sup_{1/2 < |u| < 3/2} |x \hat{\phi}(xu)|, \quad a = \sup_{e_1 \leq e(\chi) \leq e_2} \int_{1/2 < |u| < 3/2} |1 - u|^{-e(\chi) - 1} du.$$

Note that $\lim_{x \to \infty} A_x = 0$ because $\phi$ is rapidly decreasing.

Concerning $F_3(x)$, we observe that when $|t| > \frac{3|x|}{2}$, one has $|x - t| < c_2|t|$ and $|1 - \frac{t}{x}|^{-1} < c_2$ for certain constant $c_2 > 1$. Hence

$$|F_3(x)| \leq \int_{|t| > \frac{3|x|}{2}} |\hat{\phi}(t)||\chi^{-1}(1 - \frac{t}{x})| 1 - \frac{t}{x}|^{-1} dt$$

$$\leq c_2^{-e(\chi) + 1} |x|^{e(\chi)} \int_{|t| > \frac{3|x|}{2}} |\hat{\phi}(t)||1 - \frac{t}{x}|^{-1} dt$$

$$\leq c_1^{-e(\chi) + 1} |x|^{e(\chi)} \int_{|t| > \frac{3|x|}{2}} |\hat{\phi}(t)||1 - \frac{t}{x}|^{-1} dt.$$

where $b = c_1^{-e_2 + 1}$ and $B_x = |x|^{e_2} \int_{|t| > \frac{3|x|}{2}} |\hat{\phi}(t)||1 - \frac{t}{x}|^{-1} dt$. Note that $\lim_{x \to \infty} B_x = 0$ because $\phi$ is rapidly decreasing.

Given $\epsilon$, we can choose $\delta$ depending on $\epsilon$, $\hat{\phi}$, $e_1$, $e_2$ such that when $|x| > \delta$, $|F_1(x) - \phi(0)|$, $|F_2(x)|$, and $|F_3(x)|$ are all smaller than $\frac{\epsilon}{3}$. Then $|F(x) - \phi(0)| < \epsilon$ follows. \hfill $\Box$

**Proposition 1.** Suppose that $\phi \in \mathcal{S}(k)$.

(i) $k$ is $p$-adic. For $\chi$ with $e(\chi) > -1$, $\hat{\phi}(x) = \phi(0)\gamma(0, \chi^{-1}, \psi)\chi^{-1}(x)|x|^{-1}$ when $\text{Supp}(\hat{\phi}) \subseteq x \cdot (\text{Cond}(\chi) \cap \mathfrak{o}_k)$.

(ii) $k$ is archimedean. For $\chi$ with $e(\chi)$ varying over a compact subset of $(-1, 0)$, the expression $\gamma(0, \chi^{-1}, \psi)\chi^{-1}(x)|x|\hat{\phi}(x)$ uniformly converges to $\phi(0)$ when $x \to \infty$.

**Proof.** (i) $k$ is $p$-adic. By the local functional equation, one has $Z(\phi \psi x, \chi, s) = \gamma(1 - s, \chi^{-1}, \psi) \int_k \hat{\phi}(x - y)\chi^{-1}(y)|y|^{1-s}d^xy$ when $\text{Re}(s) < 1 - e(\chi)$. Now suppose $\text{Supp}(\hat{\phi}) \subseteq x \cdot (\text{Cond}(\chi) \cap \mathfrak{o}_k)$. Lemma 1(ii) implies that $Z(\phi \psi x, \chi, s) = \phi(0)\gamma(1 - s, \chi^{-1}, \psi)\chi^{-1}(x)|x|^{-s}$
when \( \text{Re}(s) < 1 - e(\chi) \); by meromorphic continuation, this equation holds for all \( s \). Setting \( s = 1 \), we obtain that \( \hat{\phi}(x) = Z(\phi \psi, \chi, s)|_{s=1} = \phi(0) \gamma(0, \chi^{-1}, \psi) \chi^{-1}(x)|x|^{-1} \).

(ii) \( k \) is archimedean. The assertion directly follows from lemma [1] (ii).

\[ \hat{\phi}(x) = Z(\phi \psi, \chi, s)|_{s=1} = \phi(0) \gamma(0, \chi^{-1}, \psi) \chi^{-1}(x)|x|^{-1} \]

Remark 1. When \( k \) is archimedean and \( e(\chi) \geq 0 \), there are more cases for the asymptotic formulas of \( \hat{\phi}(x) \) because of the many poles of \( L(s, \chi) \). We do not discuss them here because we only need the formula when \( e(\chi) \in (-1, 0) \) for the proof of Proposition [2].

2.2. Quadratic Fourier transform. For \( f(x) \in L^1(k) \), we define its quadratic Fourier transform with respect to \( \psi \) as \( \mathcal{F}_2 f(t) = \int_k f(x) \psi(tx^2) dx \). Obviously, \( \mathcal{F}_2 f = 0 \) when \( f \) is odd. When \( f \) is even, we associate a function \( \tilde{f} \) supported on \( \mathbb{K} \) by setting

\[
\tilde{f}(x) = \begin{cases} 
  f(\sqrt{x}), & x \in \mathbb{K}, \\
  0, & x \notin \mathbb{K}, 
\end{cases}
\]

then \( \mathcal{F}_2 f = 2|2|^{-1} \mathcal{F}(|\tilde{f}|) \).

Because the Fourier transform \( \mathcal{F} \) respects the \( L^2 \)-pairing, we expect the following equation

\[
\int_k \mathcal{F}_2 f_1(t) \mathcal{F}_2 f_2(-t) dt = 2|2|^{-1} \int_k f_1(x) f_2(x) |x|^{-1} dx.
\]

when certain good conditions are met. The next proposition provides a sufficient condition.

Proposition 2. Suppose the \( W(x) \in L^1(k) \) is an even function and \( \phi \in \mathcal{S}(k) \). If \( W^2(x)|x|^{-1} \), \( W(x)|x|^{-1} \) and \( \mathcal{F}_2 W(t)|t|^{-1/2} \) are integrable, then

1. \( \int_k \mathcal{F}_2 W(t) \mathcal{F}_2 \phi(-t) dt = 2|2|^{-1} \int_k W(x) \phi(x) |x|^{-1} dx \);  
2. \( \int_k \mathcal{F}_2 W(\delta t) \mathcal{F}_2 \phi(-t) dt = 0 \) for \( \delta \in \mathbb{K} \setminus \mathbb{K} \).

Proof. The condition \( W^2(x)|x|^{-1} \in L^1(k) \) means that \( \mathcal{F}_2 W \in L^2(k) \). If \( \mathcal{F}_2 \phi \in L^2(k) \), then the stated equalities follow from the \( L^2 \)-isometry of \( \mathcal{F} \). However, \( \mathcal{F}_2 \phi \in L^2(k) \) if and only \( \phi(0) = 0 \); the condition \( \mathcal{F}_2 W(t)|t|^{-1/2} \in L^1(k) \) is imposed for the situation \( \phi(0) \neq 0 \). The detailed argument is presented below.

We may assume that \( \phi \) is even. Write \( \phi = \phi_1 + \phi_2 \), where \( \phi_1 = \phi - \phi_2 \) and \( \phi_2 \) is chosen as i) \( \phi(0) 1_{\mathbb{K}} \) when \( k \) is \( p \)-adic, ii) \( \phi(0) \lambda(|x|) \) when \( k \) is archimedean, where \( \lambda(r) \) is an even, smooth, compactly supported function on \( \mathbb{R} \) and takes constant value 1 in an open neighborhood of 0. We will prove equalities (1) and (2) for \( \phi_1 \) respectively.

i) Concerning \( \phi_1 \), the following observation shows that \( \tilde{\phi}_1 \cdot |\cdot|^{-1/2} \in L^2(k) \).

(a) When \( k \) is \( p \)-adic, \( \phi_1 \) is locally constant and compactly supported on \( \mathbb{K} \) and thus so is \( \tilde{\phi}_1 \cdot |\cdot|^{-1/2} \).

(b) When \( k \) is real, \( \tilde{\phi}_1 \cdot |\cdot|^{-1/2} \) vanishes on \( \mathbb{R} \) and for \( x > 0 \), \( \tilde{\phi}_1(x)|x|^{-1/2} = \frac{\phi_1(\sqrt{x}) - \phi(0)}{\sqrt{x}} \) is bounded near 0 because of the mean value theorem.

(c) When \( k \) is complex, \( \tilde{\phi}_1(z)|z|^{-1/2} = (\phi(\sqrt{z}) - \phi(0))|z|^{-1/2} \) and, by the mean value theorem, is bounded near 0 by \( c|z|^{-1/4} \) for certain constant \( c \).
Applying the $L^2$-isometry property of Fourier transform, we get
\[
\int_k \mathcal{F}_2 W(t) \mathcal{F}_2 \phi_1(-t) dt = 2^2 |2|^{-2} \int_k \mathcal{F}(\tilde{W} | \cdot |^{-1/2}) (t) \mathcal{F}(\tilde{\phi}_1 | \cdot |^{-1/2}) (-t) dt
\]
\[
= 2^2 |2|^{-2} \int_k \tilde{W}(x) |x|^{-1/2} \cdot \tilde{\phi}_1(x) |x|^{-1/2} dx
\]
\[
= 2^2 |2|^{-2} \int_{k^2} W(\sqrt{x}) \phi_1(\sqrt{x}) |x|^{-1} dx
\]
\[
= 2 |2|^{-1} \int_k W(x) \phi_1(x) |x|^{-1} dx.
\]
Similarly, \( \int_k \mathcal{F}_2 W(\delta t) \mathcal{F}_2 \phi_1(-t) dt = 2^2 |2|^{-2} \int_k \tilde{W}(\delta^{-1} x) |\delta^{-1} x|^{-1/2} \cdot \tilde{\phi}_1(x) |x|^{-1/2} dx \) when \( \delta \in k^\times \setminus k^{\times 2} \), the right hand side vanishes because \( \tilde{W}(\delta^{-1} x) \tilde{\phi}_1(x) \) vanishes on \( k^{\times} \).

ii) Concerning \( \phi_2 \), we observe that there is a function \( f(x) \in S(k) \) that is constant in a neighborhood of 0 and satisfies \( \tilde{\phi}_2 = \sum_i c_i \chi_i f(x) \) on \( k^{\times} \), where \( \chi_i \) are quadratic characters of \( k^{\times} \) and \( c_i \) are constants.

(a) When \( k \) is \( p \)-adic, \( \tilde{\phi}_2 |_{k^{\times}} \) is the characteristic function of \( \mathcal{O}_k \cap k^{\times 2} \). Let \( \{ \chi_i \} \) be the finite set of quadratic characters of \( k^{\times} \), then there are constants \( c_i \) such that \( \tilde{\phi}_2 = \sum_i c_i \chi_i 1_{\mathcal{O}_k} \) on \( k^{\times} \). Take \( f(x) = 1_{\mathcal{O}_k} (x) \).

(b) When \( k \) is real, \( \tilde{\phi}_2(x) = (1 + sgn(x)) \lambda (|x|^{1/2}) \). Take \( f(x) = \lambda (|x|^{1/2}) \).

(c) When \( k \) is complex, \( \tilde{\phi}_2(x) = \lambda (|x|^{1/2}) \). Take \( f(x) = \lambda (|x|^{1/2}) \).

We claim the following:
\[
\int_k \mathcal{F}(\tilde{W} | \cdot |^{-1/2}) (t) \mathcal{F}(f_{\chi_i} | \cdot |^{-1/2}) (-t) dt = \int_k \tilde{W}(x) f(x) \chi_i(x) |x|^{-1} dx. \tag{*}
\]

To prove it, we consider a family of functions \( \mathcal{F}(f_{\chi_i} | \cdot |^{-1/2+s}) (t), s \in [0, \frac{1}{4}] \). They are uniformly bounded and, by lemma \( \Pi \) there exist constants \( \delta \) and \( C \) such that when \( |x| > \delta \),
\[
\mathcal{F}(f_{\chi_i} | \cdot |^{-1/2+s}) (t) \leq C |\gamma(\frac{1}{2} - s, \chi_i, \psi)||t|^{-\frac{1}{4} - s}, \quad s \in [0, \frac{1}{4}].
\]

Thus, \( \mathcal{F}(f_{\chi_i} | \cdot |^{-1/2+s}) (t) \in L^2(k) \) and, because of the condition \( \mathcal{F}_2 W | \cdot |^{-\frac{1}{2}} \in L^1(k) \), \( \mathcal{F}(\tilde{W} | \cdot |^{-1/2}) \mathcal{F}(f_{\chi_i} | \cdot |^{-1/2+s}) \) are dominated by an integrable function. By the \( L^2 \)-isometry property of \( \mathcal{F} \),
\[
\int_k \mathcal{F}(\tilde{W} | \cdot |^{-1/2})(t) \mathcal{F}(f_{\chi_i} | \cdot |^{-1/2+s}) (-t) dt = \int_k \tilde{W}(x) f(x) \chi_i(x) |x|^{-1+s} dx.
\]

Letting \( s \to 0^+ \) and applying the dominated convergence theorem, we get equation \( (*) \).

Because \( \tilde{\phi}_2 = \sum_i c_i \chi_i f \) on \( k^{\times} \), we then have
\[
\int_k \mathcal{F}(\tilde{W} | \cdot |^{-1/2})(t) \mathcal{F}(\tilde{\phi}_2 | \cdot |^{-1/2}) (-t) dt = \int_k \tilde{W}(x) \tilde{\phi}_2(x) |x|^{-1} dx,
\]
whence \( \int_k \mathcal{F}_2 W(t) \mathcal{F}_2 \phi(-t) dt = 2 |2|^{-1} \int_k W(x) \phi(x) |x|^{-1} dx. \)
By the same argument,
\[ \int_k \mathcal{F}_2 W(\delta t) \mathcal{F}_2 \phi_2(-t) dt = 2|2|_k^{-2} \int_k \tilde{W}(\delta^{-1} x) |\delta^{-1} x|^{-1/2} \cdot \tilde{\phi}_2(x) |x|^{-1/2} dx; \]
when \( \delta \in k^\times \backslash k^{x^2} \), it is zero because \( \tilde{W}(\delta^{-1} x) \tilde{\phi}_2(x) \) vanishes on \( k^\times \). \( \square \)

**Remark 2.** Suppose that \( W_1, W_2 \in L^1(k) \), \( \phi \in \mathcal{S}(k) \) and \( \delta \in k^\times / k^{x^2} \). In order to have
\[ \int_k (\mathcal{F}_2 W_1(t) + \mathcal{F}_2 W_2(\delta t)) \mathcal{F}_2 \phi(-t) dt = 2|2|_k^{-1} \int_k W_1(x) \phi(x) |x|^{-1} dx, \]
it suffices to check that \( W_i^2(x) |x|^{-1}, W_i(x) |x|^{-1} \) (\( i = 1, 2 \)) and \( (\mathcal{F}_2 W_1(t) + \mathcal{F}_2 W_2(\delta t)) |t|^{-\frac{1}{2}} \)
are integrable. There is no need to check the integrability of \( \mathcal{F}_2 W_i(t) |t|^{-\frac{1}{2}} \) separately. The argument for this fact is the same as for Proposition 2.

2.3. **Fourier transform of distributions.** The space of Bruhat-Schwartz function \( \mathcal{S}(k^m) \) consists of rapidly decreasing functions when \( k \) is archimedean and of locally constant compactly supported functions when \( k \) is nonarchimedean. It carries a standard complete topology when \( k \) is archimedean.

Let \( \mathcal{S}'(k^m) \) denote the space of continuous linear maps \( T : \mathcal{S}(k^m) \rightarrow \mathbb{C} \) when \( k \) is archimedean and the space of linear maps when \( k \) is nonarchimedean. It is called the space of tempered distributions on \( k^m \). (When \( k \) is archimedean, this is the classical notion of tempered distribution.) The Fourier transform on \( L^1(k^m) \) is extended to \( \mathcal{S}'(k^m) \) by setting \( \mathcal{F}T := T \circ \mathcal{F} \) for \( T \in \mathcal{S}'(k^m) \).

A functional \( T \in \mathcal{S}'(k^m) \) is said to be represented over an open subset \( U \subset k^m \) by a function \( f \in L^1_{\text{loc}}(U) \) if \( T(\phi) = \int_U f(X) \phi(X) dX \) for all \( \phi \in C^\infty_c(U) \). When \( f \) is required to be continuous, it is unique if it exists.

3. **The Whittaker functional on representations of \( GL_2 \)**

In this section, we review the relation between the Whittaker functional and inner product pairing of a local or global representation of \( GL_2 \). Let \( P \) denote the subgroup of \( GL_2 \) consisting of upper triangular matrices, \( U \) the unipotent radical of \( P \), and \( M \) the subgroup of \( GL_2 \) consisting of diagonal matrices.

3.1. **Local theory.** Let \( k \) be a local field of characteristic zero and \( \psi \) a non-trivial character of \( k \). For an infinite dimensional admissible representation \( \pi \) of \( GL_2(k) \), a Whittaker functional on \( \pi \) with respect to \( \psi \) is a linear functional \( \ell_\psi : \pi \rightarrow \mathbb{C} \) satisfying \( \ell_\psi(\pi(u)f) = \psi(u)\ell_\psi(f) \) for \( f \in \pi \). Obviously, if \( \ell_\psi \) is a Whittaker functional with respect to \( \psi \), then \( f \rightarrow \ell_\psi(\pi(a_1) f) \) is a Whittaker functional with respect to \( \psi_a \). When \( \pi \) is irreducible, the space of Whittaker functionals on \( \pi \) with respect to \( \psi \) is 1-dimensional.

Now suppose that \( \pi \) is irreducible unitary and \( (, ) \) is the inner product pairing on \( \pi \otimes \pi \). Choose a non-zero Whittaker functional \( \ell_\psi \) on \( \pi \) and put \( W_{f, \psi}(h) = \ell_\psi(h \circ v) \) for \( f \in \pi \) and \( h \in GL_2(k) \). It is well-known that there exists a non-zero constant \( c \neq 0 \) such that
\[ (f_1, f_2) = c \int_{k^\times} W_{f_1, \psi} \overline{W_{f_2, \psi}}((a_1)) d^\times a, \quad f_1, f_2 \in \pi. \] (1)
Note that the function \( W_{f_1, \psi} \overline{W_{f_2, \psi}}(\pi) \) belongs to \( L^1(k) \) because \( \pi \) is unitary, and that \( (\pi(u)f_1, f_2) \) is its Fourier transform.

On the other hand, let \( T_{(\pi(u)f_1, f_2)} \) be the distribution on \( k \) represented by \( (\pi(u)f_1, f_2) \). Since \( \pi \) is unitary, \( (\pi(u)f_1, f_2) \) is bounded and smooth, whence \( T_{(\pi(u)f_1, f_2)} \) belongs to \( S'(k) \). It is easy to see that \( FT_{(\pi(u)f_1, f_2)} \) is represented over \( k^* \) by a smooth function, which we denote by \( W_{f_1, f_2, \psi}(a) \). Though \( (\pi(u)f_1, f_2) \) is not integrable on \( k \), we define \( \int_k (\pi(u)f_1, f_2) \psi(-u)du \) as \( W_{f_1, f_2, \psi}(-1) \). Furthermore, Equation (1) implies that \( W_{f_1, f_2, \psi}(a) = c \cdot W_{f_1, \psi} W_{f_2, \psi}((-a_1)) \) for \( |a|^{-1} \), whence

\[
\int_k (\pi(u)f_1, f_2) \psi(-u) = c \ell_{\psi}(f_1) \ell_{\psi}(f_2).
\]

(2)

### 3.2. Global theory

Fix a non-trivial character \( \psi \) of \( \mathbb{A}/F \). Let \( \pi \) be an irreducible unitary cuspidal representation of \( GL_2(\mathbb{A}) \). There is a standard inner product pairing and a standard Whittaker functional:

\[
(f_1, f_2) = \int_{PGL_2(F) \backslash PGL_2(\mathbb{A})} f_1(h) \overline{f_2(h)} dh, \quad \ell_{\psi}(f) = \int_{U(F) \backslash U(\mathbb{A})} f(u) \psi^{-1}(u) du.
\]

We shall explain the relation between \(( , \) and \( \ell_{\psi} \).

Choose a maximal compact subgroup \( K = \prod_v K_v \) of \( GL_2(\mathbb{A}) \). Write \( \pi = \otimes_v \pi_v \) as the restricted tensor product of irreducible unitary representations \( \pi_v \) of \( GL_2(F_v) \), where almost all \( \pi_v \) are spherical with respect to \( K_v \). Choose a spherical vector \( f_v,0 \) in each spherical \( \pi_v \), then \( \pi \) is spanned by decomposable vectors of the form \( f = \otimes f_v \), where \( f_v = f_v,0 \) at almost all places. At each place \( v \), we fix a choice of non-zero Whittaker functional \( \ell_{\psi_v} \) on \( \pi_v \), requiring that \( \ell_{\psi_v}(f_v,0) = 1 \) for almost all spherical \( \pi_v \) and that \( \ell_{\psi} = \prod_v \ell_{\psi_v} \); we also fix a choice of local inner product pairing \(( , \) on \( \pi_v \otimes \pi_v \), requiring that \( (f_v,0, f_v,0) = 1 \) for almost all spherical \( \pi_v \) and that \( ( , ) = \prod_v ( , )_v \). The measures on \( \mathbb{A}, \mathbb{A}_v^* \) are chosen as in section 11.

At each place \( v \) of \( F \), according to the local theory, there is a local constant \( c_v \) such that

\[
(f_{1,v}, f_{2,v})_v = c_v \cdot \int_{F_v^x} (\pi_v(u) f_{1,v} f_{2,v}) \psi(^{-u_v}) du_v,
\]

\[
\ell_{\psi_v}(f_{1,v}) \overline{\ell_{\psi_v}(f_{2,v})} = \frac{1}{c_v} \int_k (\pi_v(u) f_{1,v} f_{2,v}) \psi(-u_v) du_v, \quad f_{1,v}, f_{2,v} \in \pi_v.
\]

Using the formula for spherical Whittaker functions, one easily sees that \( c_v = \frac{\zeta_{F_v}(2)}{L(1, \pi_v, \psi)} \) at almost all places.

We introduce two normalized local pairings,

\[
(f_{1,v}, f_{2,v})^* = \frac{\zeta_{F_v}(2)}{L(1, \pi_v, \psi)} \int_{F_v^x} (\pi_v(u) f_{1,v} f_{2,v}) \psi^{-1}(u_v) du_v
\]

\[
\mathcal{L}^*_{\psi}(f_{1,v}, f_{2,v}) = \frac{L(1, \pi_v, \psi)}{\zeta_{F_v}(2)} \int_{U_v} (\pi_v(u) f_{1,v} f_{2,v}) \psi^{-1}(u_v) du_v.
\]
Then one can immediately write down two global formulas,

\[
(f_1, f_2) = \frac{1}{c_\pi} \cdot \frac{L(1, \pi, \text{ad})}{\zeta_F(2)} \prod_v \left( f_{1, v}, f_{2, v} \right)^{\gamma},
\]

\[
\ell_\psi(f_1) \ell_\psi(f_2) = c_\pi \cdot \frac{\zeta_F(2)}{L(1, \pi, \text{ad})} \prod_v \mathcal{L}_\psi^2(f_{1, v}, f_{2, v}),
\]

where the constant \( c_\pi = \frac{L(1, \pi, \text{ad})}{\zeta_F(2)} \prod_v \zeta_F(2, v). \)

The following proposition is more or less known and we include its proof for completeness. For the assertion on general \( GL_m \), see [10]; note that the local integrals at archimedean places are defined in a different ad hoc way in [10].

**Proposition 3.** \( c_\pi = 1. \)

**Proof.** Let \( Z \) be the center of \( GL_2 \) and put \( \overline{P} = Z \setminus P, K = Z(\mathbb{A}) \setminus KZ(\mathbb{A}). \) Set \( \text{Vol}(K) = 1, \) then the Tamagawa measure on \( PGL_2(\mathbb{A}) \) can be decomposed as \( dh = c|a|^{-1}dx|\pi|k \) for \( h = (1, \frac{1}{2}) \), where \( dx, \pi, k \) are Tamagawa measures and \( c \) is a constant.

Let \( \text{Ind}_{PGL_2^1(\mathbb{A})}^* \) be the representation of \( PGL_2(\mathbb{A}) \) unitarily induced from the quasi-character \( (a_1, a_2) \rightarrow |\frac{a_1}{a_2}|^s \) and \( F_s \) the spherical section satisfying \( F_s(\mathbb{A}) = 1, k \in K. \) The associated Eisenstein series

\[
E(F_s)(h) = \sum_{\gamma \in F(\mathbb{A}) \setminus GL_2(\mathbb{A})} F_s(\gamma h)
\]

is absolutely convergent when \( \text{Re}(s) > \frac{1}{2} \) and has meromorphic continuation to \( \mathbb{C}; \) \( E(F_s) \) has a simple pole at \( s = \frac{1}{2} \) with constant residue \( \kappa. \) It is known that \( \kappa = c. \)

One has

\[
(f_1, f_2) = \frac{1}{\kappa} \int_{[PGL_2]} f_1(h) \overline{f_2(h)} \left[ \text{Res}_{s = \frac{1}{2}} E(F_s) \right](h) dh
\]

\[
= \frac{1}{\kappa} \text{Res}_{s = \frac{1}{2}} \int_{[PGL_2]} f_1(h) \overline{f_2(h)} E(F_s)(h) dh.
\]

Using the Whittaker expansion of \( f_1, f_2 \) and the unfolding-folding technique, one can write

\[
\int_{[PGL_2]} f_1(h) \overline{f_2(h)} E(F_s)(h) dh = \int_{\mathbb{A}^\times} W_{f_1, \psi} W_{f_2, \psi} (\omega_1) \overline{F_s(h)} |a|^{s - \frac{1}{2}} d^* \pi dh
\]

\[
= c \int_{K} \int_{\mathbb{A}^\times} W_{f_1, \psi} W_{f_2, \psi} (\omega_1) \overline{F_s(h)} |a|^{s - \frac{1}{2}} d^* \pi dh
\]

\[
= c \prod_v \int_{K_v} \int_{\mathbb{A}^\times_v} W_{f_1, \psi_v} W_{f_2, \psi_v} (\omega_v) \overline{F_s(h_v)} |a_v|^{s - \frac{1}{2}} d^* \pi_v dh_v.
\]

Note that the local integral \( \int_{K_v} \int_{\mathbb{A}^\times_v} \) equals \( \frac{L(s, \pi_v, \text{ad}) \zeta_F(s + \frac{1}{2})}{\zeta_F(2s + 1)} \) at almost all places. Thus,

\[
(f_1, f_2) = \frac{c}{\kappa} \text{Res}_{s = \frac{1}{2}} \left[ \frac{L(s + \frac{1}{2}, \pi, \text{ad}) \zeta_F(s + \frac{1}{2})}{\zeta_F(2s + 1)} \right] \prod_v (f_{1, v}, f_{2, v}),
\]
with

\[(f_1, f_2) := \left[ \zeta_{F_v}(2s + 1) \int_{K_v} \int_{F_v^x} W_{f_1, \psi_v} \overline{W_{f_2, \psi_v}} \left( (a_v \ 1) k_v \right) |a_v|^s - \frac{1}{2} d^* a_v d k_v \right]_{s = \frac{1}{2}} \]

\[= \frac{\zeta_{F_v}(2)}{L(1, \pi_v, \text{ad}) \zeta_{F_v}(1)} \int_{K_v} \int_{F_v^x} W_{f_1, \psi_v} \overline{W_{f_2, \psi_v}} \left( (a_v \ 1) k_v \right) d^* a_v d k_v \]

\[= \int_{K_v} (k_v \circ f_1, k_v \circ f_2)^z \]

\[= (f_1, f_2)^z. \]

So \((f_1, f_2) = \frac{c}{\kappa} \cdot \frac{L(1, \pi_v, \text{ad}) \text{Res}_{s = 1} \zeta_{F_v}(s)}{\zeta_{F_v}(2)} \prod_v (f_1, f_2)^z. \) Therefore, \(c_\pi = \kappa/c = 1. \)

4. The Whittaker Functional on Representations of \(\widehat{SL}_2\)

In this section, we study the relation between the Whittaker functional and inner product pairing of a local or global representation of \(\widehat{SL}_2\). The local theory is parallel to that of \(GL_2\); the global theory is similar in the statement of the relation but the determination of the global constant is reduced to the global constant for \(PGL_2\) via the transfer of Fourier coefficient in the setting of theta lifting. Let \(B\) denote the subgroup of \(SL_2\) consisting of upper triangular matrices, \(N\) the unipotent radical of \(B\), and \(A\) the subgroup of \(SL_2\) consisting of diagonal matrices.

4.1. Local Theory. Let \(k\) be a local field of characteristic zero and fix a non-trivial character \(\psi\) of \(k\).

4.1.1. The group. \(\widehat{SL}_2(k)\) is the non-trivial two-fold cover of \(SL_2(k)\) when \(k \neq \mathbb{C}\) and the trivial two-fold cover of \(SL_2(k)\) when \(k = \mathbb{C}\). Identify \(\widehat{SL}_2(k)\) with \(SL_2(k) \times \{\pm 1\}\), then the group law can be given as \([g_1, \epsilon_1][g_2, \epsilon_2] = [g_1 \cdot g_2, \epsilon(g_1, g_2) \epsilon_1 \epsilon_2]\), where \(\epsilon(g_1, g_2) = \langle j(g_1) j(g_1 g_2) j(g_2) j(g_1 g_2) >\) and the \(j\) function is defined on \(SL_2(k)\) by

\[j \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} c, & \text{if } c \neq 0 \\ a, & \text{if } c = 0. \end{cases} \]

The topology on \(\widehat{SL}_2(k)\) is the product topology when \(k\) is \(p\)-adic or \(\mathbb{C}\); when \(k = \mathbb{R}\), \(\widehat{SL}_2(k)\) is homeomorphically identified with \(B(\mathbb{R}) \times (SO_2(\mathbb{R}) \times \{\pm 1\})\), where the topology on the second factor is defined with regard to the following isomorphism,

\[\tilde{\gamma} : \mathbb{R}/4\pi \mathbb{Z} \rightarrow SO_2(\mathbb{R}) \times \mathbb{Z}/2\mathbb{Z}, \]

\[\theta \mapsto [(\begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array}), 1_{\{\theta, 3\pi \} + 4\pi \mathbb{Z}}(\theta) - 1_{\{\pi, 3\pi \} + 4\pi \mathbb{Z}}(\theta)].\]

Let \(K\) be a maximal compact subgroup of \(SL_2(k)\). \(K\) has a canonical lift to \(\widehat{SL}_2(k)\) when \(k\) is not dyadic and we still denote it by \(K\). For a subset \(X\) of \(SL_2(k)\), let \(\tilde{X}\) be its preimage in \(\widehat{SL}_2(k)\). Note that \(\tilde{N}(k)\) is split over \(N(k)\) but \(\tilde{A}(k)\) is not split over \(A(k)\) unless \(k = \mathbb{C}\). For an element \(g\) of \(SL_2(k)\), we take it as \([g, 1]\) when regarding it as an
element of \(\widetilde{SL}_2(k)\). For \(a \in k^\times\), write \(a\) for \([(a^{-1}_a), 1]\); for \(n \in k\), use the same notation \(n\) for the element \([(1/\bar{1}), 1]\) when there is no confusion.

4.1.2. Induced representations. For a quasi-characters \(\mu\) of \(k^\times\), let \(\overline{\rho}(\mu) = \widetilde{\rho}_\psi(\mu)\) be the right translation action of \(\widetilde{SL}_2(k)\) on the space of \(\tilde{K}\)-finite functions \(\varphi : \widetilde{SL}_2(F) \to \mathbb{C}\) satisfying

\[
\varphi([(a^{-1}_a), \epsilon]g) = \epsilon\chi(\mu(a))|a|f(g).
\]

Here \(\chi\varphi\) is a function on \(k^\times\) valued in \(S^1\) and occurs in the Weil representation (c.f. section 4.2.2 or [17]). The dual of \(\overline{\rho}(\mu)\) is \(\overline{\rho}(\chi_{-1}^{-1})\) and \(<\varphi_1, \varphi_2> = \int_k \varphi_1\varphi_2(wn)dn\) is a natural \(\widetilde{SL}_2(k)\)-invariant pairing on \(\overline{\rho}(\mu) \otimes \overline{\rho}(\chi_{-1}^{-1})\).

When \(k\) is \(p\)-adic, \(\psi\) is of conductor \(O_k\), and \(\overline{\rho}(\mu)\) is spherical, the spherical matrix coefficient of \(\overline{\rho}(\mu)\) takes the following value,

\[
\langle a \circ \varphi_0, \varphi_\varphi \rangle = \left|a\right|\chi(a)\left|\mu(a)\right| \cdot \left[\frac{1 - \mu^{-2}(a)}{1 - \mu^{-2}(a)} + \mu^{-1}(a) \cdot \frac{1 - \mu^{-2}(a)}{1 - \mu^{-2}(a)}\right]; \quad |a| \leq 1.
\]

4.1.3. The Whittaker functional. For an infinite dimensional admissible representation \(\sigma\) of \(SL_2(k)\), a Whittaker functional on \(\sigma\) with respect to \(\psi\) is a linear functional \(\ell_\psi : \sigma \to \mathbb{C}\) satisfying \(\ell_\psi(\sigma(n)\varphi) = \psi(n)\ell_\psi(\varphi)\) for \(\varphi \in \sigma\). Obviously, if \(\ell_\psi\) is a Whittaker functional with respect to \(\psi\), then \(\varphi \mapsto \ell_\psi(\sigma(a)\varphi)\) is a Whittaker functional with respect to \(\psi_a\psi\). When \(\sigma\) is irreducible, the space of Whittaker functionals on \(\sigma\) with respect to \(\psi\) is 1-dimensional. Specifically, when \(k\) is \(p\)-adic, \(\psi\) is of conductor \(O_k\), and the induced representation \(\overline{\rho}(\mu)\) is spherical, let \(\varphi_0\) be a spherical vector with \(\ell_\psi(\varphi_0) = 1\), then

\[
\ell_\psi(a \circ \varphi_0) = \chi(\mu(a))a \cdot \left(\frac{1 - \mu(a)}{1 - \mu(a)^2}\right)^T1_{O_k}(a)\mu^{-1}(a) + \frac{1 - \mu(a)^{-1}}{1 - \mu(a)^{-2}}\cdot \mu(a)1_{O_k}(a).
\]

Now suppose that \(\sigma\) is irreducible unitary and (, ) is the inner product pairing on \(\sigma \otimes \sigma\). Let \(\{\delta_i\}\) be a set of representatives of \(k^\times/k^\times\); for each \(\delta_i\), choose a non-zero Whittaker functional \(\ell_{\psi_{\delta_i}}\) on \(\sigma\) with respect to \(\psi_{\delta_i}\) if such functionals exist, or set \(\ell_{\psi_{\delta_i}} = 0\). Put \(W_{\varphi_1, \psi_{\delta_i}}(g) = \ell_{\psi_{\delta_i}}(g \circ \varphi)\) for \(\varphi \in \sigma\) and \(g \in \widetilde{SL}_2(k)\). Note that the function \(W_{\varphi_1, \psi_{\delta_i}}\) belongs to \(L^1(k)\) because \(\sigma\) is irreducible unitary.

Lemma 2 ([1][2]). \((\varphi_1, \varphi_2) = \frac{|2\pi|}{2} \sum_i c_{\sigma, \psi_{\delta_i}} \cdot \int_{k^\times} W_{\varphi_1, \psi_{\delta_i}} W_{\varphi_2, \psi_{\delta_i}}(a)d^\times a\), where each \(c_{\sigma, \psi_{\delta_i}}\) is a constant determined by \(\ell_{\psi_{\delta_i}}\) and (, ). \(c_{\sigma, \psi_{\delta_i}}\) is zero when \(\ell_{\psi_{\delta_i}} = 0\) and non-zero when \(\ell_{\psi_{\delta_i}} \neq 0\).

Proof. It is shown in [1] when \(k\) is \(p\)-adic and in [2] when \(k\) is \(R\). When \(k = \mathbb{C}\), \(\widetilde{SL}_2(\mathbb{C})\) is the trivial 2-fold cover of \(SL_2(\mathbb{C})\) and the equality follows from equation [1] for irreducible representations of \(GL_2(\mathbb{C})\).

We would like to define a local Whittaker functional by integrating matrix coefficients. Generally, \(\int_k \sigma(n)\varphi_1, \varphi_2, n)dn\) is integrable only when \(\sigma\) is a square-integrable representation. Let \(T_{(\sigma(n)\varphi_1, \varphi_2)}\) be the distribution on \(k\) represented by \(\sigma(n)\varphi_1, \varphi_2\). Since \(\sigma\) is unitary, \((\sigma(n)\varphi_1, \varphi_2)\) is bounded and smooth, whence \(T_{(\sigma(n)\varphi_1, \varphi_2)}\) belongs to \(\mathcal{S}(k)\). It is
Lemma 3. $W_{\varphi_1,\varphi_2,\psi}(-a) = |\delta_1|^{-1} c_{\sigma,\psi_\delta_1} W_{\varphi_1,\psi_\delta_1} \frac{W_{\varphi_2,\psi_\delta_1}(b)}{|b|^{-1}}$ for $a = \delta_1 b^2 \in \delta_1 k^{\times^2}$.

Proof. By lemma 2, $(\sigma(n)\varphi_1, \varphi_2) = \frac{|2|_k}{2} \sum_k c_{\sigma,\psi_\delta_1} \cdot \int_{k^{\times^2}} W_{\varphi_1,\psi_\delta_1} \frac{W_{\varphi_2,\psi_\delta_1}(a)}{a} \psi(\delta_1^2 n) d^2 a$. Recall the notations in section 2.2 and let $\widehat{W}_i$ be the function associated to $W_i(a) := W_{\varphi_1,\psi_\delta_1} \frac{W_{\varphi_2,\psi_\delta_1}(a)}{|a|^{-1}}$. Then $(\sigma(n)\varphi_1, \varphi_2) = \sum_i \frac{|2|_k c_{\sigma,\psi_\delta_1}}{2} F_2 W_i(\delta_1 n) = \sum_i c_{\sigma,\psi_\delta_1} F(\widehat{W}_i | \cdot |^{-\frac{1}{2}})(\delta_1 n)$. Hence, for $\phi \in S(k)$,

$$F T_{(\sigma(n)\varphi_1, \varphi_2)}(\phi) = \int_k (\sigma(n)\varphi_1, \varphi_2) \widehat{\phi}(n) dn = \sum_i \int_k c_{\sigma,\psi_\delta_1} F(\widehat{W}_i | \cdot |^{-\frac{1}{2}})(\delta_1 n) \widehat{\phi}(n) dn = \sum_i \int_k |\delta_1|^{-\frac{1}{2}} c_{\sigma,\psi_\delta_1} \overline{W}_i(-\delta_1^{-1} a) |a|^{-\frac{1}{2}} \phi(da)$$

Therefore, $T_{(\sigma(n)\varphi_1, \varphi_2)}$ is represented by $W_{\varphi_1,\varphi_2,\psi}(a) = \sum_i |\delta_1|^{-\frac{1}{2}} c_{\sigma,\psi_\delta_1} \overline{W}_i(-\delta_1^{-1} a) |a|^{-\frac{1}{2}}$. Since $\overline{W}_i$ are all supported on $k^{\times^2} \cup \{0\}$, we have $W_{\varphi_1,\varphi_2,\psi}(-\delta_1 b^2) = |\delta_1|^{-1} c_{\sigma,\psi_\delta_1} \overline{W}_i(b^2) |b|^{-1} = |\delta_1|^{-1} c_{\sigma,\psi_\delta_1} W_{\varphi_1,\psi_\delta_1} \frac{W_{\varphi_2,\psi_\delta_1}(b)}{|b|^{-1}}$.

Remark 3. As a consequence, there is

$$\ell_{\psi_\delta_1}(\varphi_1) \ell_{\psi_\delta_1}(\varphi_2) = \frac{|\delta_1|}{c_{\sigma,\psi_\delta_1}} \int_k (\sigma(n)\varphi_1, \varphi_2) \psi(-\delta_1 n) dn.$$
Proof. As in the proof of lemma 3, we have $(\sigma(n)\varphi_1, \varphi_2) = \sum_{\delta j} 2^{-1}\int_{k} |a|^2 c_{\sigma, \psi_j} F_j(\delta j n),$

where $W_j(a) = W_{\varphi_1, \psi_j} W_{\varphi_2, \psi_j}(a)|a|^{-1}$. Hence

$$\int_{k} (\sigma(n)\varphi_1, \varphi_2) F_2(\Phi(n)) dn = \sum_{\delta j} 2^{-1}\int_{k} |a|^2 c_{\sigma, \psi_j} F_2(\delta j n) F_2(\Phi(n)) dn.$$ 

Since $\sigma$ is unitary, we know that $(\sigma(n)\varphi_1, \varphi_2)|n|^{-\frac{1}{2}}$ is integrable by looking at the asymptotic behavior of matrix coefficients, and that $W_{\sigma, \psi_1}(a)|a|^{-1}, W_{\sigma, \psi_2}(a)|a|^{-1}$ are integrable by looking at the asymptotic behavior of Whittaker functions. Applying Proposition 2 and remark 2, we get the stated formula. \qed

Remark 5. We briefly describe an estimate of the Whittaker function $W_{\varphi, \psi}(a)$ when $\varphi$ belongs to an irreducible unitary representation $\sigma$ of $\tilde{SL}_2(k)$. We did not find a simple reference for it, but ingredients of the proof can be gathered from the literature. First, $W_{\varphi, \psi}(a)$ vanishes near $\infty$ when $k$ is $p$-adic and is rapidly decreasing near $\infty$ when $k$ is archimedean. Second, for the behavior when $a$ is near 0, one needs to distinguish different classes of $\sigma$: (i) When $\sigma = \tilde{\rho}_\psi(\mu)$, the unitaricity of $\sigma$ requires that either $\mu$ is unitary or $\mu = |\cdot|^{s_0}$ with $s_0 \in (-\frac{1}{2}, \frac{1}{2})$, there is the bound $W_{\varphi, \psi}(a) = O(|a|^{1+|s_0(\mu)|})$ for both cases; (ii) When $\sigma$ is not an induced representation, there are two situations: (ii.a) $k$ is $p$-adic and $\sigma$ is supercuspidal or a special representation $\tilde{Sp}_\psi(\mu) \subset \tilde{\rho}_\psi(\mu)$ with $\mu^2 = |\cdot|$, then the function $W_{\varphi, \psi}(a)$ belongs to $S(k^\times)$ in the former case and is of type $O(|a|^{\frac{3}{2}})$ near 0 in the latter case; (ii.b) $k$ is real and $\sigma$ is a subquotient of $\tilde{\rho}_\psi(\cdot|^{\frac{3}{2}})$ with $n \in 2\mathbb{Z}_{\geq 0}$, in this case $\sigma$ is square-integrable and there is the bound $W_{\varphi, \psi}(a) = O(|a|^{-\frac{n+3}{2}})$ near 0.

4.2. Global Theory. $\tilde{SL}_2(\mathbb{A})$ denotes the two-fold metaplectic cover of $SL_2(\mathbb{A})$. At each place $v$, choose a maximal compact subgroup $K_v$ of $SL_2(F_v)$ and lift it to $\tilde{SL}_2(F_v)$ when $v$ is not dyadic. Let $\prod_v \tilde{SL}_2(F_v)$ be the restricted product of $\tilde{SL}_2(F_v)$ with respect to $K_v$ and $I$ the subgroup consisting of $[1, (\epsilon_v)]$, where $\epsilon_v \in \{\pm 1\}$ equals 1 at almost all places and satisfies $\prod_v \epsilon_v = 1$. Then $\tilde{SL}_2(\mathbb{A})$ is the quotient group $\left(\prod_v \tilde{SL}_2(F_v)\right)/I$.

4.2.1. Cuspidal representations of $\tilde{SL}_2(\mathbb{A})$. We fix a non-trivial character $\psi$ of $\mathbb{A}/F$ and describe cuspidal representations of $\tilde{SL}_2(\mathbb{A})$.

(i) For each $\delta \in F^\times$, there is a Weil representation $\omega_{\psi_\delta}$ of $\tilde{SL}_2(\mathbb{A})$ on $S(\mathbb{A})$ associated to $\psi_\delta$; two representations $\omega_{\psi_\delta_1}, \omega_{\psi_\delta_2}$ are isomorphic if and only if $\delta_1 \delta_2^{-1} \in F^\times$. Theta functions associated to these representations $\omega_{\psi_\delta}$ are called elementary theta series on $\tilde{SL}_2(\mathbb{A})$; they are of the form $\Theta_{\phi, \psi_\delta}(g) = \sum_{\xi \in F} \omega_{\psi_\delta}(g) \phi(\xi), \phi \in S(\mathbb{A})$.

(ii) Let $A_0(\tilde{SL}_2)$ be the space of genuine cuspidal automorphic forms on $\tilde{SL}_2(\mathbb{A})$ and $A_{00}(\tilde{SL}_2)$ the subspace of $A_0(\tilde{SL}_2)$ consisting of forms that are orthogonal to all elementary theta series. Then $A_{00}(\tilde{SL}_2)$ is a disjoint union of Waldspurger packets $Wd_\psi(\pi),$

$$A_{00}(\tilde{SL}_2) = \sqcup \pi Wd_\psi(\pi),$$
where $\pi$ runs over irreducible cuspidal $PGL_2(\mathbb{A})$-representations and $Wd_\psi(\pi)$ consists of irreducible cuspidal $\widetilde{SL}_2(\mathbb{A})$-representations $\sigma$ such that $\sigma = \Theta(\pi \otimes \chi_\delta, \psi_\delta)$ for certain $\delta \in F^\times/F^{\times 2}$. The dependence of the Waldspurger packet on the chosen additive character $\psi$ is given by $Wd_\psi(\pi) = Wd_{\psi_\delta}(\pi \otimes \chi_\delta)$.

4.2.2. Weil representation of the pair $\widetilde{SL}_2 \times PGL_2$. In order to have an direct perception of the Waldspurger packet, we briefly describe the Weil representation and the associated theta lifting. Identify $PGL_2$ with $SO(V, q)$ with $V = \{X \in M_{2 \times 2}(F) : \text{Tr}(X) = 0\}$, $q(X) = -\det X$, and $PGL_2$ acting on $V$ by $h \circ X = hXh^{-1}$.

For a non-trivial additive character $\psi$ of $\mathbb{A}/F$, there is a Weil representation $\omega_\psi = \otimes \omega_{\psi_v}$ of $SL_2(\mathbb{A}) \times O(V)_\mathbb{A}$ on $S(V_\mathbb{A})$; at each place $v$, the action $\omega_{\psi_v}$ on $S(V_{F_v})$ is given by

$$
\omega_v(h_v)\varphi_v(X_v) = \varphi_v(h_v^{-1}X_v), \quad h_v \in O(V)_{F_v},
$$

$$
\omega_v(n_v, \epsilon)\varphi_v(X_v) = \epsilon\varphi_v(n_v q(X_v))\varphi_v(X_v), \quad n_v \in N(F_v),
$$

$$
\omega_v(a_v, \epsilon)\varphi_v(X_v) = \epsilon\chi_{\psi_v}(a_v)\varphi_v(a_v^{-1}X_v), \quad a_v \in F_v^\times,
$$

$$
\omega_v(w, \epsilon)\varphi_v(X_v) = \epsilon \gamma(\psi_v, V_{F_v}) \int_{V_{F_v}} \phi_v(Y_v)\varphi_v(q(X_v, Y_v))dY_v.
$$

Here $w = (\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix})$, $q(X, Y) = q(X + Y) - q(X) - q(Y)$, $\gamma(\psi_v, V_{F_v})$ is a constant of norm 1, and $\chi_{\psi_v}$ is a function on $F_v^\times$ with values in $S^1$ satisfying $\chi_{\psi_v}(a_1a_2) = \chi_{\psi_v}(a_1)\chi_{\psi_v}(a_2) < a_1, a_2 >$.

Let $\sigma$ be an irreducible genuine cuspidal representation of $\widetilde{SL}_2(\mathbb{A})$. For a form $\varphi \in \sigma$, its lift to $PGL_2$ with respect to $\omega_\psi$ and via a function $\phi \in S(V_\mathbb{A})$ is defined as

$$
\Theta_\psi(\phi, \varphi)(h) = \int_{[SL_2]} \overline{\varphi(g)} \Theta_{\phi, \psi}(g, h)dg, \quad h \in PGL_2(\mathbb{A}).
$$

Here $\Theta_{\phi, \psi}(g, h) = \sum_{\xi \in V_\mathbb{A}} \omega_{\psi_v}(g, h)\phi(\xi)$ is the theta kernel attached to $\phi$. The global theta lift of $\sigma$ to $PGL_2$ with respect to $\psi$ is then $\Theta(\sigma, \psi) = \{\Theta(\phi, \varphi) | \phi \in S(V_\mathbb{A}), \varphi \in \sigma\}$.

Similarly, for an irreducible cuspidal representation $\pi$ of $PGL_2(\mathbb{A})$, we define the lift of a form $f \in \pi$ by $\Theta_{\pi}(\phi, f)(g) = \int_{[PGL_2]} \overline{f(h)} \Theta_{\phi, \psi}(g, h)dh$, and the lift of the representation by $\Theta(\pi, \psi) = \{\Theta(\phi, f) | \phi \in S(V_\mathbb{A}), f \in \pi\}$.

When the additive character $\psi$ is fixed, we may choose skip the subscripts $\psi$.

4.2.3. The global Whittaker functional. Let $\sigma$ be an irreducible cuspidal $\widetilde{SL}_2(\mathbb{A})$-representation contained in $A_{00}(\widetilde{SL}_2)$. There is a standard inner product pairing and a standard Whittaker functional with respect to $\psi_\delta$ for $\delta \in F^\times$:

$$
(\varphi_1, \varphi_2) = \int_{[SL_2]} \varphi_1(g)\overline{\varphi_2(g)}dg, \quad \ell_{\psi_\delta}(\varphi) = \int_{\mathbb{A}/F} \varphi(n)\psi(-\delta n)dn.
$$

Supposing that $\ell_{\psi}$ is non-zero, we will study the relation between $(.,.)$ and $\ell_{\psi}$. Note that when $\sigma \in Wd_\psi(\pi)$, $\ell_{\psi_\delta}$ is non-zero if and only if $\sigma = \Theta(\pi \otimes \chi_\delta, \psi_\delta)$.

Here is the initial setup. Write $\sigma = \otimes \sigma_v$ as a restricted tensor product of irreducible $\widetilde{SL}_2(F_v)$-representations $\sigma_v$; for each spherical $\sigma_v$, let $\varphi_{v, 0}$ be the spherical vector chosen for
Proposition 4. Write \( \overline{\ell}_{\psi_{\delta_v}}(\varphi) = 1 \) for almost all spherical \( \sigma_v \) and that \( (\cdot)_v = \prod_v (\cdot)_v \). Pick up a set of representatives \( \{ \delta_{v,i} \} \) of \( F_v^x / F_v^{x^2} \) with \( \delta_{v,1} = 1 \); for each \( \delta_{v,i} \), choose a non-zero local Whittaker functional \( \ell_{\psi_{\delta_{v,i}}} \) on \( \sigma_v \) with respect to \( \psi_{\delta_{v,i}} \) if such functionals exist, or set \( \ell_{\psi_{\delta_{v,i}}} = 0 \); additionally, we require that \( \ell_{\psi}(\varphi_v,0) = 1 \) for almost all spherical \( \sigma_v \) and that \( \ell_{\psi} = \prod_v \ell_{\psi_v} \); set \( W_{\varphi_v,\psi_{\delta_{v,i}}}(g_v) = \ell_{\psi_{\delta_{v,i}}} (\sigma_v(g_v) \varphi_v) \). The measures on \( \mathbb{A}, \mathbb{A}^\times \) are chosen as in section [I].

At each place \( v \) of \( F \), according to the local theory, there is a local constant \( c_{\sigma_v,\psi_v} \) such that

\[
(\varphi_{1,v}, \varphi_{2,v})_v = \frac{|2|_{F_v} c_{\sigma_v,\psi_v}}{2} \int_{F_v^x} W_{\varphi_{1,v},\psi_v}(a_v) d^x a_v + \sum_{i \neq 1} \frac{|2|_{F_v} c_{\sigma_v,\psi_v,\delta_{v,i}}}{2} \int_{F_v^x} W_{\varphi_{1,v},\psi_{\delta_{v,i}}}(a_v) d^x a_v,
\]

\[
\ell_{\psi_v}(\varphi_{1,v}) \ell_{\psi_v}(\varphi_{2,v}) = \frac{1}{c_{\sigma_v,\psi_v}} \int_k (\sigma_v(n_v) \varphi_{1,v}, \varphi_{2,v}) \psi(-n_v) d\nu_v, \quad \varphi_{i,v} \in \pi_v.
\]

Using the formula for spherical matrix coefficients, one can easily see that \( c_{\sigma_v,\psi_v} = \frac{L(\frac{1}{2}, \pi_v) \zeta_{F_v}(2)}{L(1, \pi_v, ad)} \) at almost all places. So we introduce the normalized local pairing

\[
\mathcal{L}_{\psi_v}^* (\varphi_{1,v}, \varphi_{2,v}) = \frac{L(1, \pi_v, ad)}{L(\frac{1}{2}, \pi_v)} \frac{L(\frac{1}{2}, \pi_v) \zeta_{F_v}(2)}{\zeta_{F_v}(2)} \int_k (\sigma_v(n_v) \varphi_{1,v}, \varphi_{2,v}) \psi(-n_v) d\nu_v.
\]

**Proposition 4.** Write \( c_\sigma = \frac{L(1, \pi, ad)}{L(\frac{1}{2}, \pi)} \prod_v c_{\sigma_v,\psi_v} L(1, \pi_v, ad) \), then

\[
\ell_{\psi_1}(\varphi_1) \ell_{\psi_2}(\varphi_2) = c_\sigma \cdot \frac{L(\frac{1}{2}, \pi) \zeta_{F}(2)}{L(1, \pi, ad)} \prod_v \mathcal{L}_{\psi_v}^* (\varphi_{1,v}, \varphi_{2,v}).
\]

**Proof.** It directly follows from the local equations. \( \square \)

## 5. Transfer of Whittaker Functionals

Suppose that \( \sigma \in Wd_\psi(\pi) \) and \( \ell_{\psi} \) is non-zero on \( \sigma \), then \( \sigma = \Theta(\pi, \psi) \) and \( \pi = \Theta(\sigma, \psi) \). We will use theta lifting to express the Whittaker functional on \( \pi \) in term of the Whittaker functional on \( \sigma \), and hence obtain the relation between the global constants \( c_\pi, c_\sigma \). When necessary, we add subscripts to distinguish the functionals on \( \pi \) and \( \sigma \).

Recall the Weil representation of \( SL_2(\mathbb{A}) \times PGL_2(\mathbb{A}) \) on \( S(V_\mathbb{A}) \). Choose a basis \( \{ e_+, e_0, e_- \} \) of \( V(F) \), where \( e_+ = (\frac{1}{0} 1), e_0 = (1 -1), e_- = (0 -1) \). Define the partial Fourier transform from \( S(V_\mathbb{A}) \) to \( S(\mathbb{A}e_0 + \mathbb{A}^2) \) as

\[
\hat{\phi}(x_0; x_-, y_-) = \int_{\mathbb{A}} \phi(x_+ e_+ + x_0 + x_- e_-) \psi(x_+ y_-) dx_+.
\]

**Lemma 5.** If \( f = \Theta(\phi, \varphi) \in \pi \), then \( \ell_{\pi,\psi_{\omega}}(f) = \int_{N(\mathbb{A}) \setminus SL_2(\mathbb{A})} \overline{W_{\phi,\psi}(g)} \omega(g) \hat{\phi}(e_0; 0, 1) dg \).
Proof. We first observe that
\[
\Theta_\phi(g) = \sum_{\xi_0 \in F_{\mathrm{e}}} \sum_{\eta \in F^2} \omega(g) \hat{\phi}(\xi_0; \eta)
\]
\[
= \sum_{\xi_0 \in F_0} \hat{\phi}(\xi_0; 0) + \sum_{\gamma \in N(F) \setminus SL_2(F)} \sum_{\xi_0 \in F_0} \omega(\gamma g) \hat{\phi}(\xi_0; 0, 1).
\]
Because \(\sigma \in A_{00}(\tilde{SL}_2)\) is orthogonal to elementary theta series, there is
\[
\Theta(\phi, \varphi) = \int_{[SL_2]} \overline{\varphi(g)} \sum_{\gamma \in N(F) \setminus SL_2(F)} \sum_{\xi_0 \in F_0} \omega(\gamma g, h) \hat{\phi}(\xi_0; 0, 1) dg
\]
\[
= \int_{N(\mathbb{A}) \setminus SL_2(\mathbb{A})} W_{\varphi, \psi(\xi_0)}(g) \sum_{\xi_0 \in F_0} \omega(g, h) \hat{\phi}(\xi_0; 0, 1).
\]
Hence, for \(\alpha \in F^\times\) and \(f = \Theta(\phi, \varphi)\), one has
\[
\ell_{\pi, \psi_0}(f) = \int_{A/\mathbb{F}} f(n) \psi(-\alpha n) dn
\]
\[
= \int_{A/\mathbb{F}} \int_{N(\mathbb{A}) \setminus SL_2(\mathbb{A})} W_{\varphi, \psi(\xi_0)}(g) \sum_{\xi_0 \in F_0} \omega(g, n) \hat{\phi}(\xi_0; 0, 1) \psi(-\alpha n) dg dn
\]
\[
= \int_{N(\mathbb{A}) \setminus SL_2(\mathbb{A})} W_{\varphi, \psi(\xi_0)}(g) \sum_{\xi_0 \in F_0} \omega(g) \hat{\phi}(\xi_0; 0, 1) \int_{A/\mathbb{F}} \psi(-q(n e_0, \xi_0) - \alpha n) dn] dg
\]
\[
= \int_{N(\mathbb{A}) \setminus SL_2(\mathbb{A})} W_{\varphi, \psi(\xi_0)}(g) \omega(g) \hat{\phi}(\xi_0; 0, 1) \frac{\alpha}{2} d e_0.
\]
\[
\end{proof}

Now we work on the relation between local Whittaker functionals on \(\pi_v\) and \(\sigma_v\). For this purpose, one needs the global inner product formula concerning the lifting \(\sigma \to \pi\) and the normalized local theta correspondences derived from it.

**Proposition 5.** \cite{14} Suppose that \(\sigma \subset A_{00}(\tilde{SL}_2)\) and \(\pi = \Theta(\sigma, \psi)\) is non-zero. For decomposable vectors \(\varphi_i \in \sigma, \phi_i \in \mathcal{S}(V_{\mathbb{A}}) \ (i = 1, 2)\), one has
\[
(\Theta(\phi_1, \varphi_1), \Theta(\phi_2, \varphi_2))_\pi = \frac{L(1, \pi)}{\zeta_F(2)} \prod_v \frac{\zeta_{F_v}(2)}{L(1, \pi_v)} \int_{SL_2(F_v)} (\sigma_v(g_v)\varphi_{1,v}, \varphi_{2,v})(\omega_v(g_v)\phi_{1,v}, \phi_{2,v}) dg_v.
\]

By local Howe duality, at each place \(v\) of \(F\), the \(\tilde{SL}_2(F_v) \times PGL_2(F_v)\)-invariant homomorphism \(\theta_v : \mathcal{S}(V_{F_v}) \to \sigma_v \otimes \pi_v\) form a vector space of dimension 1. We choose a \(\theta_v\) so that with respect to the associated local theta lifting \(\theta_v(\phi_v, \varphi_v) = (\theta_v(\phi_v), \varphi_v)_{\sigma_v}\) there is
\[
(\theta_v(\phi_{1,v}, \varphi_{1,v}), \theta_v(\phi_{1,v}, \varphi_{1,v}))_{\pi_v} = \frac{L(1, \pi_v)}{\zeta_{F_v}(2)} \int_{SL_2(F_v)} (\omega_v(g_v)\phi_{1,v}, \phi_{2,v})(\sigma_v(g_v)\varphi_{1,v}, \varphi_{2,v})_{\sigma_v} dg_v.
\]
Such a normalized \(\theta_v\) is unique up to multiplication by a constant of norm 1. We further require that at almost all places, \(\theta_v(1 \Sigma(\sigma_{F_v}), \varphi_{v,0}) = f_{v,0}\). Then there is a global constant
Proof. \(\omega\) satisfying \(\Theta(\phi, \varphi) = c_{\sigma, \pi} \prod_v \theta_v(\phi_v, \varphi_v)\). The global inner product formula amounts to say that \(|c_{\sigma, \pi}|^2 = \frac{L(\frac{1}{2}, \pi)}{\zeta_F(2)}\).

Furthermore, at each place \(v\), there is a homeomorphism \(N(F_v) \setminus SL_2(F_v) \rightarrow F_v^2 \setminus \{(0, 0)\}\) given by \(N(F_v)g_v \rightarrow (0, 1)g_v\), the additive Haar measure on \(F_v^2\) is of the form \(c'_v d\hat{g}_v\), where \(c'_v\) is a local constant depending on the quotient measure \(d\hat{g}_v\). Because the maximal compact subgroup \(K_v\) of \(SL_2(F_v)\) has measure 1 for almost all \(v\), one has \(c'_v = \frac{1}{\zeta_F(2)}\) for almost all \(v\); because \(SL_2(\mathbb{A})\) is given the Tamagawa measure, one has \(\prod_v c'_v = 1\).

Recall the local constant \(c_{\sigma, \psi_v}\) defined in lemma 2 and section 4.2.3.

Lemma 6. If \(f_{i, v} = \theta_v(\phi_{i, v}, \varphi_{i, v}) \in \pi_v, i = 1, 2\), then

\[
\mathcal{L}_{\pi_v, \psi, 2, v}(f_{1, v}, f_{2, v}) = \frac{c'_v c_{\sigma, \psi_v} L(1, \pi_v, \text{ad})}{|2|^v L(\frac{1}{2}, \pi_v)} J_v(\varphi_{1, v}, \phi_{1, v}) J_v(\varphi_{2, v}, \phi_{2, v}).
\]

where \(J_v(\varphi, \phi) = \int_{N(F_v) \setminus SL_2(F_v)} W_{\varphi, \psi, \psi}(g_v) \omega_v(g_v) \hat{\phi}_v(\sigma_v^0; 0, 1) d\hat{g}_v\) equals 1 at almost all places.

Proof. We first observe that

\[
(\phi_{1, v}, \phi_{2, v}) = \int_{F_v \times F_v^2} \hat{\phi}_{1, v}(a_v e_0; y_v) \bar{\phi}_{2, v}(a_v e_0; y_v) d\hat{g}_v
\]

\[
= c'_v \int_{F_v \times [N(F_v) \setminus SL_2(F_v)]} \omega(g'_v) \hat{\phi}_{1, v}(a_v e_0; 0, 1) \omega(g'_v) \hat{\phi}_{2, v}(a_v e_0; 0, 1) d\hat{g}_v.
\]

Because \(\omega_v(n_v) \hat{\phi}_v(a_v e_0; 0, 1) = \hat{\phi}_v(a_v e_0; 0, 1) \psi(-2n_v a_v)\), the following holds for \(\lambda \in \mathcal{S}(F_v)\):

\[
\int_{F_v} (\omega_v(n_v) \hat{\phi}_{1, v}, \phi_{2, v}) \hat{\lambda}(-n_v) d\hat{n}_v = c'_v \int_{F_v \times [N(F_v) \setminus SL_2(F_v)]} \omega(g'_v) \hat{\phi}_{1, v}(a_v e_0; 0, 1) \omega(g'_v) \hat{\phi}_{2, v}(a_v e_0; 0, 1) \lambda(-2a_v) d\hat{a}_v d\hat{g}_v.
\]

Now we compute \(W_{f_{1, v}, f_{2, v}, \psi}(a_v)\). For a function \(\lambda \in C^\infty_c(F_v^\times)\), one has

\[
\int_{F_v^\times} \mathcal{W}_{f_{1, v}, f_{2, v}, \psi}(a_v) \lambda(a_v) d\hat{a}_v
\]

\[
= \int_{F_v} \mathcal{W}_{f_{1, v}, f_{2, v}, \psi}(a_v) \hat{\lambda}(-n_v) d\hat{n}_v
\]

\[
= \frac{\zeta_{F_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{F_v} \int_{SL_2(F_v)} \omega_v(g_v, n_v) \phi_{1, v}, \phi_{2, v}) \bar{(\sigma(g_v) \varphi_{1, v}, \varphi_{2, v})} \hat{\lambda}(-n_v) d\hat{g}_v d\hat{n}_v
\]

\[
= \frac{\zeta_{F_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{SL_2(F_v)} \left(\int_{F_v} \omega_v(g_v, n_v) \phi_{1, v}, \phi_{2, v}) \hat{\lambda}(-n_v) d\hat{n}_v\right) d\hat{g}_v
\]

\[
= \frac{c'_v \zeta_{F_v}(2)}{L(\frac{1}{2}, \pi_v)} \int_{SL_2(F_v)} \int_{F_v \times [N(F_v) \setminus SL_2(F_v)]} \omega_v(g'_v) \hat{\phi}_{1, v}(a_v; 0, 1) \lambda(-2a_v) d\hat{a}_v d\hat{g}_v.
\]
The triple integral on the right hand side of the above equation is absolutely convergent for all \( \lambda \in C_c^\infty(F_v^\times) \). Hence for \( a_v \in F_v^\times \),

\[
W_{f_1,v,f_2,v}(a_v) = \frac{c'_v\zeta_{F_v}(2)}{|2_v L(\frac{1}{2}, \pi_v)|} \int_{SL_2(F_v)} \int_{N(F_v) \setminus SL_2(F_v)} \frac{(\sigma_v(g_v')\varphi_{1,v}, \varphi_{2,v}) \omega_v(g_v')}{\phi_{1,v}(-\frac{a_v}{2};0,1) \omega_v(g_v') \phi_{2,v}(-\frac{a_v}{2};0,1)} dg_v dg'_v.
\]

Setting \( a_v = -2 \) and doing the decomposition \( g_v = n_{a_v} k_1 \) and \( g'_v = b_v k_2 \), we can rewrite

\[
W_{f_1,v,f_2,v}(a_v) = \frac{c'_v\zeta_{F_v}(2)}{|2_v L(\frac{1}{2}, \pi_v)|} \int_{K_v \times K_v} \int_{F_v^\times} I_v(a_v \circ \varphi_{1,v}, \varphi_{2,v}, a_v \circ \phi_{1,v}, \phi_{2,v}) |a_v|^{-2} d^* a_v,
\]

where

\[
I_v(\varphi_{1,v}, \varphi_{2,v}, \phi_{1,v}, \phi_{2,v}) = \int_{F_v^\times} \int_{F_v^\times} (n_v \circ \varphi_{1,v}, \varphi_{2,v}) \omega(n_v) \phi_{1,v}(b_v; 0, b_v^{-1}) \phi_{2,v}(b_v; 0, b_v^{-1}) |b_v|^{-1} d^* b_v d n_v.
\]

By lemma 4

\[
I_v(\varphi_{1,v}, \varphi_{2,v}, \phi_{1,v}, \phi_{2,v}) = c_{\sigma_v, \psi_v} \int_{F_v^\times} W_{\varphi_{1,v}, \psi_v} W_{\varphi_{2,v}, \psi_v} (b_v) \phi_{1,v}(b_v; 0, b_v^{-1}) \phi_{2,v}(b_v; 0, b_v^{-1}) |b_v|^{-3} d^* b_v.
\]

It follows that

\[
\int_{F_v^\times} I_v(a_v \circ \varphi_{1,v}, \varphi_{2,v}, a_v \circ \phi_{1,v}, \phi_{2,v})
= c_{\sigma_v, \psi_v} \int_{F_v^\times} W_{\varphi_{1,v}, \psi_v} (a_v) \phi_{1,v}(a_v; 0, 1) |a_v|^{-\frac{3}{2}} d^* a_v \cdot \int_{F_v^\times} W_{\varphi_{2,v}, \psi_v} (b_v) \phi_{2,v}(b_v; 0, 1) |b_v|^{-\frac{3}{2}} d^* b_v.
\]

Therefore \( W_{f_1,v,f_2,v}(a_v) = \frac{c'_v c_{\sigma_v, \psi_v} \zeta_{F_v}(2)}{|2_v L(\frac{1}{2}, \pi_v)|} J_v(\varphi_{1,v}, \phi_{1,v}) J_v(\varphi_{2,v}, \phi_{2,v}) \) and

\[
L_{\pi_v, \psi_{-2,v}}(f_1,v, f_2,v) = L(1, \pi_v, \ad) \zeta_{F_v}(2) W_{f_1,v,f_2,v}(a_v)
= c'_v c_{\sigma_v, \psi_v} L(1, \pi_v, \ad) J_v(\varphi_{1,v}, \phi_{1,v}) J_v(\varphi_{2,v}, \phi_{2,v}).
\]

Proposition 6. \( c_\sigma = c_\pi, \text{ hence } c_\sigma = 1. \)

Proof. Consider decomposable vectors \( \varphi_i = \otimes \varphi_{i,v} \in \sigma \) and \( \phi_i = \otimes \phi_{i,v} \in S(V)_R \), \( i = 1, 2 \). Write \( f_i = \Theta(\varphi_i, \phi_i) \) and \( f'_i,v = \theta_v(\varphi_{i,v}, \phi_{i,v}) \), then \( f_i = c_{\sigma_v}(\otimes f'_i,v) \). By lemma 5

\[
\ell_{\pi_v, \psi_{-2}}(f_1) \ell_{\pi_v, \psi_{-2}}(f_2) = \prod_v J_v(\varphi_{1,v}, \phi_{1,v}) J_v(\varphi_{2,v}, \phi_{2,v}).
\]
On the other hand, we apply lemma 6 and get

\[ \prod_v L_{\pi_v, \psi_{-2, v}}(f_1, v, f_2, v) = |c_{\pi, \psi}|^2 \prod_v L_{\pi_v, \psi_{-2, v}}(f_1', v, f_2', v) \]

\[ = L(\frac{1}{2}, \pi) \prod_v c_{\pi_v, \psi_v} L(1, \pi_v, \text{ad}) \frac{c_{\pi_v, \psi_v} L(1, \pi_v, \text{ad})}{\zeta_F(2)} J_v(\varphi_{1, v}, \phi_{1, v}) J_v(\varphi_{2, v}, \phi_{2, v}) \]

\[ = L(\frac{1}{2}, \pi) \prod_v c_{\pi_v, \psi_v} L(1, \pi_v, \text{ad}) \frac{c_{\pi_v, \psi_v} L(1, \pi_v, \text{ad})}{\zeta_F(2)} J_v(\varphi_{1, v}, \phi_{1, v}) J_v(\varphi_{2, v}, \phi_{2, v}) \]

\[ = \frac{1}{c_{\pi}} L(1, \pi, \text{ad}) \prod_v J_v(\varphi_{1, v}, \phi_{1, v}) J_v(\varphi_{2, v}, \phi_{2, v}). \]

So \( \ell_{\pi, \psi_{-2}}(f_1) \ell_{\pi, \psi_{-2}}(f_2) = c_{\pi} \cdot \frac{\zeta_F(2)}{L(1, \pi, \text{ad})} \prod_v L_{\pi_v, \psi_{-2, v}}(f_1, v, f_2, v). \) Therefore \( c_{\pi} = c_{\sigma}. \)

**Theorem 1.** Suppose that \( \sigma \in A_{00}(\tilde{SL}_2) \) is in the Waldspurger packet \( Wd_\psi(\pi) \), where \( \pi \) is an irreducible cuspidal representation of \( \text{PGL}_2(A) \). If \( \ell_\psi \) is non-vanishing on \( \sigma \), then

\[ \ell_\psi \otimes \ell_{\psi} = \frac{L(\frac{1}{2}, \pi)\zeta_F(2)}{L(1, \pi, \text{ad})} \prod_v L_{\psi_v}^\sharp. \]

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