Dissipative extension theory for linear relations

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Abstract

This work is devoted to dissipative extension theory for dissipative linear relations. We give a self-consistent theory of extensions by generalizing the theory on symmetric extensions of symmetric operators. Several results on the properties of dissipative relations are proven. Finally, we deal with the spectral properties of dissipative extensions of dissipative relations and provide results concerning particular realizations of this general setting.

Mathematics Subject Classification(2010): 47A06 47A45 47B44
Keywords: Closed linear relations; Dissipative extensions; Nondensely defined operators.

*Research partially supported by SEP-CONACYT 254062
1. Introduction

This paper deals with the theory of dissipative extensions of dissipative relations and it can be seen as a generalization of the classical von Neumann theory of symmetric extensions of symmetric operators [43]. The theory is presented thoroughly and the exposition goes along the lines of the classical texts on the von Neumann theory (see for instance [3, Chap. 7], [10, Chap. 4], [45, Chap. 8]), but in a more general setting.

In this work we obtain new results in the theory of dissipative relations. There are several results deemed to be folklore knowledge for which, to the best of our understanding, there were no proofs in the literature prior to this work. It is also worth remarking that the proofs of some classical results on dissipative operators, as well as their particular cases: symmetric and selfadjoint operators, are simplified and streamlined when considered in the more general framework of dissipative relations.

Our motivation for studying relations and its extensions comes from their use in the boundary triplet theory (see [16–18, 21]) and quasi boundary triplet theory (see [7, 8]). The theory of relations is also used in studying extensions of nondensely defined symmetric operators (see for instance [12] and cf. [28]). We remark that the examples given in Section 5 are related to this kind of applications. Relations are also relevant in other contexts; for instance in the theory of canonical systems (see [23, 24]).

It is not a coincidence that von Neumann was not only the pioneer in extension theory of operators, but also in the theory of linear relations. Indeed, the modern notion of linear relation goes back to [14]. The theory was later developed in [11, 19]. More recent accounts on the matter can be found in [13, 25]. Symmetric extension theory of symmetric relations were first developed in [19] (cf. [1, 20]). Various aspects of the theory of symmetric relations were studied in [13, 30]. The perturbation theory of linear relations is dealt with in [5, 14, 22, 46].

The theory of dissipative operator has its roots in the theory of contractions for which a seminal work is Sz. Nagy’s [11]. Contractive and dissipative operators are related via the Cayley transform (see [42, Chap. 4, Sec. 4]). One of the first works on dissipative operators is due to Philips [36]. The development of Sz. Nagy and Foiaş’s theory for dissipative operators was done in [34, 35] and later generalized in [31–33]. Dissipative extension theory was formulated in [36].

The theory presented here generalizes previous results in two directions. We consider relations which are dissipative extensions of dissipative relations. This general setting not only covers all earlier results, but also shed light on the peculiarities of dissipative relations that may be important in further developments and applications (as for instance in the context of boundary triplets for partial differential equations where the deficiency indices are infinite). Dissipative relations appear in applications in [16, 18] and are studied in [6, 19].

The paper is organized as follows. In Section 2 we give a general account on the theory of closed linear relations. Here we lay out the notation and introduce preparatory facts. Section 3 is concerned with the theory of dissipative relations. In this section, we extend some results which characterize dissipative operators to the case of relations (Theorems 3.1 and 3.6 and Proposition 3.5). Theorems 3.7 and 3.10 give criteria for maximal dissipativeness for sums of dissipative relations. Propositions 3.11 and 3.13 allow us to study the spectrum and the deficiency index of a dissipative relation in terms of the spectrum and the deficiency index of the operator part of it. In Section 4 we deal with dissipative extensions of dissipative relations. Here, instead of using the Cayley transform for relations (see [19, Sec. 2]), we recur to its modern countepart, the Z-transform, introduced in [20]. Theorem 4.7 provides the generalization of the von Neumann formula for which
Corollary 4.8 and Propositions 4.9 and 4.10 are related results. The spectral properties of dissipative relations are dealt with in Proposition 4.11 and Corollary 4.13. Finally, Section 5 presents examples of dissipative extensions for a Jacobi operator and the operator of multiplication in a de Branges space in a general setting including the case when they are not densely defined.

2. Spectral theory of closed linear relations

Let \( H \) be a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) being antilinear in the first argument. Consider the orthogonal sum of \( H \) with itself, i.e. \( H \oplus H \) (see [10, Chap. 2 Sec. 3.3]), and denote an arbitrary element of it as a pair \((f, g)\) with \( f, g \in H \). Thus,

\[
\left\langle \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} \right\rangle = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle .
\]

(2.1)

We shall use the norm

\[
\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\| = \|f\| + \|g\| ,
\]

which is equivalent to the norm

\[
\left\| \begin{pmatrix} f \\ g \end{pmatrix} \right\|^2 = \|f\|^2 + \|g\|^2 .
\]

(2.2)

generated by the inner product (2.1).

Define the operators \( U, W \) acting on \( H \oplus H \) by the rule

\[
U \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} g \\ f \end{pmatrix}, \quad W \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} -g \\ f \end{pmatrix} .
\]

(2.3)

One verifies that

\[
U^2 = I = -W^2, \quad UW = -WU ,
\]

where \( I \) is the identity operator in \( H \oplus H \). Moreover, for any linear subset \( G \) of \( H \oplus H \) the following holds

\[
(WG) = W(G), \quad \overline{WG} = \overline{WG}, \quad (UG) = U(G), \quad \overline{UG} = \overline{UG} .
\]

(2.4)

Throughout this paper, any linear set \( T \) in \( H \oplus H \) is called a linear relation or simply a relation. The graph of a linear operator is a relation, and thus any operator can be seen as a particular instance of a relation. Not all relations are graphs of operators since for a linear relation \( G \) to be the graph of an operator, it is necessary and sufficient that

\[
\left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in G : f = 0 \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} .
\]

(2.5)

A closed relation is a subspace in \( H \oplus H \). If a closed relation is an operator, then the operator is closed [10, Chap. 3, Sec. 2].
For a given relation \( T \), define the sets
\[
\text{dom} T := \left\{ f \in \mathcal{H} : \left( \begin{array}{c} f \\ g \end{array} \right) \in T \right\}, \quad \text{ran} T := \left\{ g \in \mathcal{H} : \left( \begin{array}{c} f \\ g \end{array} \right) \in T \right\},
\]
\[
\text{ker} T := \left\{ f \in \mathcal{H} : \left( \begin{array}{c} f \\ 0 \end{array} \right) \in T \right\}, \quad \text{mul} T := \left\{ g \in \mathcal{H} : \left( \begin{array}{c} 0 \\ g \end{array} \right) \in T \right\},
\]
which are linear sets in \( \mathcal{H} \). Moreover, if \( T \) is closed, then \( \text{ker} T \) and \( \text{mul} T \) are subspaces of \( \mathcal{H} \). According to (2.5) a relation is an operator if and only if \( \text{mul} T = \{ 0 \} \).

Let \( T \) and \( S \) be relations, and \( \zeta \in \mathbb{C} \). Consider the relations:
\[
T + S := \left\{ \left( \begin{array}{c} f \\ g + h \end{array} \right) : \left( \begin{array}{c} f \\ g \end{array} \right) \in T, \left( \begin{array}{c} h \\ g \end{array} \right) \in S \right\}, \quad \zeta T := \left\{ \left( \begin{array}{c} f \\ \zeta g \end{array} \right) : \left( \begin{array}{c} f \\ g \end{array} \right) \in T \right\},
\]
\[
ST := \left\{ \left( \begin{array}{c} f \\ k \end{array} \right) : \left( \begin{array}{c} f \\ g \end{array} \right) \in T, \left( \begin{array}{c} g \\ k \end{array} \right) \in S \right\}, \quad T^{-1} := \mathcal{U} T.
\]
Note that \( T^{-1} \) is the inverse of the relation \( T \). Clearly,
\[
\text{dom} T^{-1} = \text{ran} T, \quad \text{ran} T^{-1} = \text{dom} T, \quad \text{ker} T^{-1} = \text{mul} T, \quad \text{mul} T^{-1} = \text{ker} T,
\]
\[
(TS)^{-1} = S^{-1} T^{-1}.
\]
We also deal with the relations:
\[
T \oplus S := T + S, \quad \text{with} \ T \subset S^\perp, \quad T \ominus S := T \cap S^\perp.
\]
Clearly, \( T^\perp \) is a closed relation. Note that, in the last two definitions, we consider the orthogonal sum and difference in \( \mathcal{H} \oplus \mathcal{H} \). It will cause no confusion to use the same symbol \( \oplus \) when referring to subspaces of a Hilbert space and when forming the orthogonal sum of Hilbert spaces.

Define \( T^* := \left\{ \left( \begin{array}{c} h \\ k \end{array} \right) \in \mathcal{H} \oplus \mathcal{H} : \langle k, f \rangle = \langle h, g \rangle, \forall \left( \begin{array}{c} f \\ g \end{array} \right) \in T \right\} \). \( T^* \) is the adjoint of \( T \) and has the following properties:
\[
T^* = (\mathcal{W} T)^\perp, \quad S \subset T \Rightarrow T^* \subset S^*, \quad S \subset T \Rightarrow T^* \subset S^*, \quad (\alpha T)^* = \overline{\alpha} T^*, \quad \text{with} \ \alpha \neq 0
\]
\[
(T^*)^{-1} = (T^{-1})^*, \quad \ker T^* = (\text{ran} T)^\perp.
\]
The last item above implies
\[
\mathcal{H} = \overline{\text{ran} T} \oplus \ker T^*.
\]
Proposition 2.1. Let $T$ be a closed linear relation, then $\text{mul} T = (\text{dom} T^*)^\perp$.

Proof. It follows from (2.8) and (2.10) that
\[
\text{mul} T = \ker T^{-1} = \ker [(T^*)^{-1}]^* = [\text{ran}(T^*)^{-1}]^\perp = (\text{dom} T^*)^\perp.
\]

For the linear relations $S$ and $T$, one directly verifies
\[
S^* + T^* \subset (S + T)^*.
\] (2.12)

The conditions for the equality in the above inclusion are given by the next assertion which follows from the proof of [4, Thm. 3.41].

Proposition 2.2. If the domain of $T$ is in the domain of $S$ and the domain of $(T + S)^*$ is in the domain of $S^*$, then $(S + T)^* = S^* + T^*$.

We shall say that a relation $T$ is bounded if there exists $C > 0$ such that for all $(f, g) \in T$ it holds $\|g\| \leq C \|f\|$. It follows from this definition that a bounded relation is a bounded operator.

Remark 2.3. Repeating the proof of [10, Thm. 3.2.3], one verifies that if $T$ and $S$ are two closed relations such that $S$ is bounded, then $T + S$ is a closed relation.

Define the quasi-regular set of $T$ by
\[
\hat{\rho}(T) := \{ \zeta \in \mathbb{C} : (T - \zeta I)^{-1} \text{ is bounded} \}
\]

As in the case of operators, the quasi-regular set is open. It is a well-known fact, and a useful one, that a bounded operator $T$ is closed if and only if its domain is closed.

Proposition 2.4. For every $\zeta \in \hat{\rho}(T)$ it holds that $\text{ran}(T - \zeta I)$ is closed if and only if $T$ is closed.

Proof. We suppose that $\text{ran}(T - \zeta I) = \text{dom}(T - \zeta I)^{-1}$ is closed, then $(T - \zeta I)^{-1}$ is closed, whence $T - \zeta I$ and $T$ are simultaneously closed (see Remark 2.3). Conversely, closedness of $T$ implies both closedness of $(T - \zeta I)$ and $(T - \zeta I)^{-1}$. Therefore $\text{dom}(T - \zeta I)^{-1}$ is closed and by (2.8) the assertion follows.

Similar to what happens to operators, the deficiency index
\[
\eta_\zeta(T) := \dim [\mathcal{H} \ominus \text{ran}(T - \zeta I)]
\] (2.13)
is constant on each connected component of $\hat{\rho}(T)$ (cf. [10, Chp. 3 Sec. 7 Lem. 3]).

For a linear relation $T$ and $\zeta \in \mathbb{C}$, we introduce the deficiency space
\[
\mathbf{N}_\zeta(T) := \left\{ \left( \begin{array}{c} f \\ \zeta f \end{array} \right) \in T \right\},
\] (2.14)
which is a closed bounded relation and $\text{dom} \mathbf{N}_\zeta(T) = \ker(T - \zeta I)$. Hence, it follows from (2.10) and Proposition 2.2 that
\[
\eta_\zeta(T) = \dim \mathbf{N}_\zeta(T^*).
\]
Both the deficiency index and the deficiency space play a crucial role in the theory of dissipative extensions of symmetric relations developed in Section 3.

The connected components of \( \hat{\rho}(T) \) in which \( \eta_\zeta(T) = 0 \) constitute the regular set of \( T \) which is denoted by \( \rho(T) \). Thus

\[
\rho(T) = \{ \zeta \in \mathbb{C} : (T - \zeta I)^{-1} \in \mathcal{B}(H) \},
\]

where \( \mathcal{B}(H) \) is the class of bounded operators defined on the whole space \( H \). What was said before implies that if the relation is not closed, then its regular set is empty.

**Proposition 2.5.** Let \( T \) be a closed linear relation. If \( \zeta \in \rho(T) \), then \( \zeta \in \rho(T^*) \).

**Proof.** The fact that \( \zeta \in \rho(T) \) means that \( (T - \zeta I)^{-1} \in \mathcal{B}(H) \). This implies that \( [(T - \zeta I)^{-1}]^* \in \mathcal{B}(H) \) (see [10, Chap. 2 Sec. 4]) which yields \( (T^* - \zeta I)^{-1} \in \mathcal{B}(H) \) by (2.10).

In analogy to the operator case, the spectrum of the linear relation \( T \), denoted \( \sigma(T) \), and its spectral core, \( \hat{\sigma}(T) \), are the complements in \( \mathbb{C} \) of \( \rho(T) \) and \( \hat{\rho}(T) \), respectively. As in the case of operators, one has

\[
\hat{\sigma}(T) = \sigma_p(T) \cup \sigma_c(T),
\]

where the point spectrum, \( \sigma_p(T) \), and the continuous spectrum, \( \sigma_c(T) \) are given by

\[
\sigma_p(T) := \{ \zeta \in \mathbb{C} : \ker(T - \zeta I) \neq \{0\} \} \quad \text{and} \quad \sigma_c(T) := \{ \zeta \in \mathbb{C} : \text{ran}(T - \zeta I) \neq \text{ran}(T - \zeta I) \}.
\]

For a closed relation \( T \), define

\[
T_{\infty} := \left\{ \begin{pmatrix} 0 \\ g \end{pmatrix} \in T \right\}, \quad T_{\odot} := T \ominus T_{\infty}.
\]

Thus,

\[
T = T_{\odot} \oplus T_{\infty}. \tag{2.15}
\]

Note that \( \text{ran} T_{\infty} = \text{mul} T \) and \( T_{\odot} \) is a closed operator. Moreover, \( \text{dom} T_{\odot} \) coincides with \( \text{dom} T \) and \( T_{\odot} \subset T \). We say that \( T_{\odot} \) is the operator part of \( T \) and \( T_{\infty} \) is the multivalued part of \( T \).

Apart from (2.15), there are alternative decompositions of linear relations, not necessarily closed, into its regular and singular parts [25].

The decomposition (2.15) allows to study some spectral properties of the relation \( T \) by means of its operator part. The next results deal with this matter.

**Proposition 2.6.** If \( T \) is a closed relation, then \( \hat{\rho}(T) \subset \hat{\rho}(T_{\odot}) \).

**Proof.** Observe that \( (T_{\odot} - \zeta I)^{-1} \subset (T - \zeta I)^{-1} \), for any \( \zeta \in \mathbb{C} \). If \( \zeta \in \hat{\rho}(T) \), then \( (T - \zeta I)^{-1} \) is bounded. Thus \( (T_{\odot} - \zeta I)^{-1} \) is bounded and \( \zeta \in \hat{\rho}(T_{\odot}) \).

The condition for the equality in the above result is given by the next assertion.

**Proposition 2.7.** If \( T \) is a closed relation such that \( \text{dom} T \subset (\text{mul} T)^\perp \), then \( \hat{\rho}(T) = \hat{\rho}(T_{\odot}) \).

**Proof.** It suffices to show that \( (T - \zeta I)^{-1} \) is bounded when \( \zeta \in \hat{\rho}(T_{\odot}) \).
If \((\begin{pmatrix} h \\ k \end{pmatrix}) \in (T - \zeta I)^{-1}\), that is \((\begin{pmatrix} k \\ h + \zeta k \end{pmatrix}) \in T\), then there exist \((\begin{pmatrix} k \\ f \end{pmatrix}) \in T_\otimes\) and \((\begin{pmatrix} 0 \\ g \end{pmatrix}) \in T_\infty\) such that \((\begin{pmatrix} k \\ h + \zeta k \end{pmatrix}) = (\begin{pmatrix} k \\ f + g \end{pmatrix})\). Thus

\[ h = f - \zeta k + g. \]  
(2.16)

Note that \((\begin{pmatrix} f - \zeta k \\ k \end{pmatrix}) \in (T_\otimes - \zeta I)^{-1}\) and there exist \(C > 0\) such that

\[ \|k\| \leq C \|f - \zeta k\|. \]  
(2.17)

Since \(f\) and \(k\) are orthogonal to \(g\), one has

\[ \|f - \zeta k + g\|^2 = \|f - \zeta k\|^2 + \|g\|^2. \]  
(2.18)

Combining (2.16), (2.17), and (2.18), one obtains

\[ \|k\|^2 \leq C^2(\|f - \zeta k\|^2 + \|g\|^2) = C^2\|f - \zeta k + g\|^2 = C^2\|h\|^2. \]

Therefore \(\|k\| \leq C \|h\|\) which means that \((T - \zeta I)^{-1}\) is bounded.  

For the relations \(T\) and \(S\), define the relation \(T_S\) in the Hilbert space \((\text{mul} S)^\perp \oplus (\text{mul} S)^\perp\) (with inner product inherent from \(\mathcal{H} \oplus \mathcal{H}\)) by

\[ T_S := T \cap (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp. \]  
(2.19)

The relation \(T_S\) is a linear relation. In some cases it is useful to consider \(T_S\) as a linear relation in \(\mathcal{H} \oplus \mathcal{H}\) and then \(T_S \subset T\). Note that if \(T\) is closed, then \(T_S\) is also closed and if \(T\) is an operator then \(T_S\) is also an operator in \((\text{mul} S)^\perp\). Furthermore, one can verifies that \((T_S)^{-1} = (T^{-1})_S\).

**Proposition 2.8.** Let \(T\) and \(S\) be linear relations such that \(T\) is closed. If \(\text{mul} T \subset \text{mul} S\), then \(T_S = (T_\otimes)_S\) and, therefore, \(T_S\) is a closed operator in \((\text{mul} S)^\perp \oplus (\text{mul} S)^\perp\).

**Proof.** Since

\[ (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp \subset (\text{mul} T)^\perp \oplus (\text{mul} T)^\perp, \]
one has \(T_\infty \cap (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp = \{0\} \oplus \{0\}\). Hence

\[ T_S = T \cap (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp = (T_\otimes + T_\infty) \cap (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp = T_\otimes \cap (\text{mul} S)^\perp \oplus (\text{mul} S)^\perp = (T_\otimes)_S. \]
Remark 2.9. For a closed relation $T$ with $\text{dom}T \subset (\text{mul}T)\perp$ it follows that $\text{dom}T_\odot$ and $\text{ran}T_\odot$ are in $(\text{mul}T)\perp$. Thus by Proposition 2.8
\[ T_T = (T_\odot)_T = T_\odot \cap (\text{mul}T)\perp \oplus (\text{mul}T)\perp = T_\odot. \] (2.20)
This means that $T_\odot$ and $T_T$ have the same elements and, when $T_T$ is regarded as a relation in $\mathcal{H} \oplus \mathcal{H}$, one can write
\[ T = T_T \oplus T_\infty. \]

Besides, for any $\zeta \in \mathbb{C}$,
\[
T - \zeta I = (T_\odot - \zeta I) \oplus T_\infty \\
= (T_T - \zeta I) \oplus T_\infty. \tag{2.21}
\]

Theorem 2.10. If $T$ is a closed linear relation such that $\text{dom}T \subset (\text{mul}T)\perp$, then

(a) $\hat{\sigma}(T) = \hat{\sigma}(T_T)$.

(b) $\sigma(T) = \sigma(T_T)$.

(c) $\sigma_c(T) = \sigma_c(T_T)$.

(d) $\sigma_p(T) = \sigma_p(T_T)$

Proof. (a) The subspaces $(T_T - \zeta I)^{-1}$ and $(T_\odot - \zeta I)^{-1}$ coincide due to (2.20). Therefore $(T_T - \zeta I)^{-1}$ is bounded if and only if $(T_\odot - \zeta I)^{-1}$ is bounded. Thus $\hat{\rho}(T_\odot) = \hat{\rho}(T_T)$ and hence the assertion holds by Proposition 2.7.

(b) For $\zeta \in \rho(T_T)$, the operator $(T_T - \zeta I)^{-1}$ is bounded in the space $(\text{mul}T)\perp$. By the previous item $(T - \zeta I)^{-1}$ is also bounded. Thus, taking into account (2.21), one has
\[
\text{dom}(T - \zeta I)^{-1} = \text{ran}(T - \zeta I) \\
= \text{ran}(T_T - \zeta I) \oplus \text{mul}T \\
= \text{dom}(T_T - \zeta I)^{-1} \oplus \text{mul}T \\
= (\text{mul}T)\perp \oplus \text{mul}T = \mathcal{H}.
\]
Therefore $\zeta \in \rho(T)$. The other inclusion follows from a similar reasoning.

(c) It follows from (2.21) that
\[
\text{ran}(T - \zeta I) = \text{ran}(T_T - \zeta I) \oplus \text{mul}T.
\]
Thus, since mul $T$ is closed, ran$(T - \zeta I)$ is closed if and only if ran$(T_T - \zeta I)$ is closed. Therefore $\sigma_c(T) = \sigma_c(T_T)$.

(d) Since dom $T \subset (\text{mul}T)\perp$, one has
\[
\ker(T - \zeta I) = \ker(T_T - \zeta I).
\]
From this equation (d) follows at once. \qed
3. Dissipative relations

This section presents the theory of dissipative relations in a fashion similar to the theory of dissipative operators given in [42, Chap. 4, Sec. 4]. The theory of these operators was introduced by [36] (see further developments in [29]). This section is related to [19, Sec. 3] and extends some of its results (cf. [6]).

A linear relation \( L \) is said to be dissipative if
\[
\text{Im} \langle f, g \rangle \geq 0 \quad (3.1)
\]
holds for all \( (f, g) \) in \( L \). If the equality in (3.1) takes place for all \( (f, g) \) in \( L \), then \( L \) is said to be symmetric. Note that \( L \) is symmetric if and only if \( L \subset L^* \).

**Theorem 3.1.** The linear relation \( L \) is dissipative if and only if the lower half plane \( \mathbb{C}_- \) is contained in \( \hat{\rho}(L) \) and for all \( \zeta \in \mathbb{C}_- \),
\[
\| (L - \zeta I)^{-1} \| \leq -1 / \text{Im} \zeta .
\]

**Proof.** Suppose that \( L \) is dissipative and let \( \zeta \in \mathbb{C}_- \). If \( \left( \begin{array}{c} h \\ k \end{array} \right) \in (L - \zeta I)^{-1} \), i.e. \( \left( \begin{array}{c} k \\ h + \zeta k \end{array} \right) \in L \), then \( \text{Im}(k, h + \zeta k) \geq 0 \). Therefore
\[
0 \leq \text{Im}(k, h) + \text{Im} \zeta \| k \|^2 \\
\leq |\langle k, h \rangle| + \text{Im} \zeta \| k \|^2 \\
\leq \| h \| \| k \| + \text{Im} \zeta \| k \|^2.
\]
For \( k \neq 0 \) the last inequality yields
\[
\| k \| \leq - \frac{1}{\text{Im} \zeta} \| h \|. \quad (3.2)
\]
If \( k = 0 \), then (3.2) holds. Hence \( \| (L - \zeta I)^{-1} \| \leq -1 / \text{Im} \zeta \) and \( \zeta \in \hat{\rho}(L) \).

Conversely, if \( \left( \begin{array}{c} f \\ g \end{array} \right) \in L \) and \( \tau > 0 \), then \( \left( \begin{array}{c} g - (-i\tau)f \\ f \end{array} \right) \in [L - (-i\tau)I]^{-1} \) and, by hypothesis,
\[
\| f \|^2 \leq \frac{1}{\tau^2} \| g + \tau i f \|^2 \\
\leq \frac{1}{\tau^2} (\| g \|^2 + \tau^2 \| f \|^2 + 2\tau \text{Im}(f, g)).
\]
Thus,
\[
-\frac{1}{2\tau} \| g \|^2 \leq \text{Im}(f, g).
\]
Letting \( \tau \) tends to infinity, one arrives at \( \text{Im}(f, g) \geq 0 \) and hence \( L \) is dissipative. \( \square \)

**Remark 3.2.** Note that if \( A \) is symmetric so is \(-A\). Therefore, by Theorem 3.1 \( \mathbb{C}_- \subset \hat{\rho}(-A) \), which implies that the upper half plane \( \mathbb{C}_+ \) is contained in \( \hat{\rho}(A) \). Hence \( A \) is symmetric if and only
if \( \mathbb{C} \setminus \mathbb{R} \subset \hat{\rho}(A) \) and, for all \( \zeta \in \mathbb{C} \setminus \mathbb{R} \), the inequality

\[
\| (A - \zeta I)^{-1} \| \leq 1/|\text{Im}\, \zeta |
\]

holds.

Due to Theorem 3.1, for a closed dissipative relation \( L \), the set \( \mathbb{C} \) is a connected component of \( \hat{\rho}(L) \). Hence, the deficiency index (2.13) is constant on \( \mathbb{C} \). Let set \( \eta_{\pm}(L) := \eta_{\pm}(L) \) for any \( \zeta \in \mathbb{C} \). Then, in view of (2.14), one has

\[
\eta_{\pm}(L) = \dim \mathfrak{N}_{\pm}(L^*).
\]

Furthermore, if \( A \) is a closed symmetric relation, then \( \mathbb{C} \) is also a connected component of \( \hat{\rho}(A) \), and one can also set \( \eta_{\pm}(A) := \eta_{\pm}(A) \) for any \( \zeta \in \mathbb{C} \). Hence \( A \) has indices

\[
(\eta_{\pm}(A), \eta_{\pm}(A)) = (\dim \mathfrak{N}_{\pm}(A^*), \dim \mathfrak{N}_{\pm}(A^*)), \quad \zeta \in \mathbb{C}.
\]  

(3.3)

A dissipative relation \( L \) is said to be maximal if it is closed and \( \eta_{-}(L) = 0 \) (or equivalently \( \mathbb{C} \subset \rho(L) \)).

\( L \) is maximal in the following sense. If \( A \) is another dissipative relation such that \( L \subset A \), one verifies that \( \eta_{-}(A) \leq \eta_{-}(L) \). Then \( \overline{A} \) is also maximal. Thus, for \( \zeta \in \mathbb{C} \), one has \( (L - \zeta I)^{-1} \subset (\overline{A} - \zeta I)^{-1} \) and both are in \( \mathcal{B}(\mathcal{H}) \). This implies that \( (L - \zeta I)^{-1} = (\overline{A} - \zeta I)^{-1} \) and then \( L = \overline{A} \). Therefore \( L = A \).

**Proposition 3.3.** If \( L \) is a closed dissipative relation whose domain is the whole space, then \( L \) is a bounded maximal dissipative operator.

**Proof.** If \( \begin{pmatrix} f \\ i f \end{pmatrix} \in L^* \), then there exist \( \begin{pmatrix} f \\ g \end{pmatrix} \in L \) such that \( \langle f, g \rangle = \langle i f, f \rangle \). Thus \( -i\|f\|^2 = \langle f, g \rangle \) and

\[
-i\|f\|^2 = \text{Im}(-i\|f\|^2) = \text{Im}\langle f, g \rangle \geq 0.
\]

This implies that \( f = 0 \) and then \( \eta_{-}(L) = 0 \). Furthermore, by Proposition 3.11, \( \text{mul} L \subset (\text{dom} L)^\perp = \{0\} \).

Thereupon \( L \) is a closed operator defined on the whole space and therefore it is bounded.

**Proposition 3.4.** If \( L \) is a maximal dissipative relation, then

\[
\rho(L) \cap (\mathbb{C} \cup \mathbb{R}) = \hat{\rho}(L) \cap (\mathbb{C} \cup \mathbb{R}).
\]

**Proof.** Suppose that \( \zeta \in \hat{\rho}(L) \cap \mathbb{C} \). Since \( \eta_{-}(L) = 0 \), the set \( \text{ran}(L - \zeta I) \) coincides with the whole space. This means that \( \zeta \in \rho(L) \).

Now suppose that \( \zeta \in \hat{\rho}(L) \cap \mathbb{R} \). Since \( \hat{\rho}(L) \) is open, there exists an open neighborhood \( \mathcal{V}(\zeta) \) of \( \zeta \) in \( \hat{\rho}(L) \). Since \( \eta_{\zeta}(L) \) is constant on each connected component of \( \hat{\rho}(L) \), one has, for any \( \nu \in \mathcal{V}(\zeta) \cap \mathbb{C} \),

\[
\eta_{\zeta}(L) = \eta_{\nu}(L) = \eta_{-}(L) = 0.
\]

Thus \( \text{ran}(L - \zeta I) = \mathcal{H} \), which yields \( \zeta \in \rho(L) \).
From Proposition 3.4, one concludes that
\[ \sigma(L) \cap \mathbb{R} = \hat{\sigma}(L) \cap \mathbb{R}. \]  

(3.4)

Proposition 3.5. A linear relation \( L \) is dissipative if and only if \( -L^{-1} \) is dissipative.

Proof. Suppose that \( L \) is dissipative and let \( \begin{pmatrix} f \\ g \end{pmatrix} \in -L^{-1} \) then \( \begin{pmatrix} -g \\ f \end{pmatrix} \in L \) and
\[ 0 \leq \text{Im}(\langle -g, f \rangle) = \text{Im} -\langle g, f \rangle = \text{Im} \langle f, g \rangle, \]
thence \( -L^{-1} \) is dissipative. The converse can be established by noting that
\[ -(-L^{-1})^{-1} = L. \]  

(3.5)

Thus, the transform \( L \to -L^{-1} \) preserves the class of dissipative relations. Furthermore this transform also preserves the subclass of maximal, dissipative relations.

Theorem 3.6. If \( L \) is a maximal, dissipative relation, then \( -L^{-1}, -L^{*} \) and \( -L^\perp \) are maximal dissipative relations. Conversely, if \( -L^{-1}, -L^{*} \) or \( -L^\perp \) is a maximal dissipative relation, then \( L \) is a maximal dissipative relation.

Proof. It follows from Proposition 3.5 that \( -L^{-1} \) is dissipative and since \( L \) is closed, so is \( -L^{-1} \). One should show that \( \mathbf{N}_i((-L^{-1})^*) \) is the trivial relation. Let
\[ \begin{pmatrix} g \\ ig \end{pmatrix} \in (-L^{-1})^*. \]  

(3.6)

The maximality of \( L \) means that \( \text{ran}(L + iI) = \mathcal{H}, \) so there exists \( \begin{pmatrix} h \\ ig \end{pmatrix} \in (L + iI) \), which implies that \( \begin{pmatrix} ig - ih \\ -h \end{pmatrix} \in -L^{-1}. \) Taking into account (3.6), one has
\[ \langle g, -h \rangle = \langle ig, ig - ih \rangle = \|g\|^2 + \langle g, -h \rangle. \]

Thus \( g = 0 \) and \( -L^{-1} \) is maximal dissipative.

Now consider \( \zeta \in \mathbb{C}_- \) and let \( \begin{pmatrix} h \\ k \end{pmatrix} \in (-L^*-\zeta I)^{-1}, \) that is \( \begin{pmatrix} k \\ -h \end{pmatrix} \in (L + \zeta I)^* \). Since \( \eta_-(L) = 0, \) \( \text{ran}(L + \zeta I) = \mathcal{H}, \) so there exists \( \begin{pmatrix} f \\ k \end{pmatrix} \in (L + \zeta I). \) Therefore it should hold that
\[ \|k\|^2 = \langle f, -h \rangle. \]  

(3.7)

Observe that \( \begin{pmatrix} k \\ f \end{pmatrix} \in (L - (-\zeta))^{-1} \) and from Proposition 3.1
\[ \|f\| \leq \frac{1}{\text{Im} \zeta} \|k\|. \]  

(3.8)
Then, by (3.7) and (3.8),
\[ \|k\|^2 = \langle f, -h \rangle \leq \|f\| \|h\| \leq -\frac{1}{\text{Im} \zeta} \|k\| \|h\|. \]

For \( k \neq 0 \) the last inequality yields
\[ \|k\| \leq -\frac{1}{\text{Im} \zeta} \|h\|. \] (3.9)

If \( k = 0 \) then (3.9) is trivial. Hence, by Proposition 3.1, \(-L^*\) is dissipative. Maximality, i.e. the fact that \( \mathbb{C}^- \subset \rho(-L^*) \), follows from Proposition 2.5.

Observe that
\[ -L^\perp = -(WWL)^\perp = -(WL)^* = -(-L^{-1})^*. \]

Thus by what has been proven \(-L^\perp\) is maximal dissipative. The converse assertions follow from (3.5), \(-(-L^*)^* = L\), and \(-(-L^\perp)^\perp = L\).

Let us turn to the question of when the sum of maximal dissipative relations is a maximal dissipative relation.

**Theorem 3.7.** Let \( A \) and \( V \) be maximal dissipative relations. If \( \text{dom} \ V = \mathcal{H} \), then \( L = A + V \) is a maximal dissipative relation.

**Proof.** The fact that \( L \) is dissipative follows directly from (3.1). Closedness is a consequence of Remark 2.3. It remains to be proven that \( L \) is maximal which in turn is reduced to showing that \( \mathcal{N}_i(L^*) \) is trivial. Observe that Proposition 3.3 ensures \( V \in \mathcal{B}(\mathcal{H}) \) and therefore \( V^* \in \mathcal{B}(\mathcal{H}) \). By Proposition 2.2 if \( \begin{pmatrix} f \\ i_f \\ i_s \end{pmatrix} \in L^* \), then there is \( \begin{pmatrix} f \\ s \\ t \end{pmatrix} \in A^* \) and \( \begin{pmatrix} f \\ t \end{pmatrix} \in V^* \) such that \( i_f = t + s \). Thus
\[ \begin{pmatrix} f \\ s \\ t \end{pmatrix} \in V^* \] and \( \begin{pmatrix} f \\ t \end{pmatrix} \in A^* \). (3.10)

On the other hand, since \(-i \in \rho(A)\), there exists \( \begin{pmatrix} t \\ k \\ k \end{pmatrix} \in (A+iI)^{-1} \), which implies that \( \begin{pmatrix} k \\ t \end{pmatrix} \in A \). This inclusion and the second one in (3.10) yield \( \langle i_f - t, k \rangle = \langle f, t - i_k \rangle \) and therefore \( \text{Im} \langle k, t \rangle = \text{Im} \langle f, t \rangle \). Thus, one obtains from the dissipativity condition that
\[ 0 \leq \text{Im} \langle k, t - i_k \rangle \leq \text{Im} \langle k, t \rangle = \text{Im} \langle f, t \rangle. \] (3.11)

By Theorem 3.6 \(-V^*\) is dissipative. Using this fact and the first inclusion in (3.10) one arrives at
\[ \text{Im} \langle f, t \rangle = -\text{Im} \langle f, -t \rangle \leq 0, \]
which, together with \((3.11)\), yields \(\text{Im}(f, t) = 0\). To conclude the proof, use the dissipativity of \(-A^*\) (Theorem 3.6) and the second inclusion in \((3.10)\) to obtain

\[
0 \leq \text{Im}(f, -if + t) = -\|f\|^2,
\]

which implies \(f = 0\). 

Let us introduce the concept of relatively boundedness for relations in a way analogous to the same concept for operators [27, Chap. 4, Sec. 1].

A relation \(S\) is said to be \(T\)-bounded if \(\text{dom} T \subset \text{dom} S\) and there exists \(c > 0\) such that for all \(\left( \begin{array}{c} f \\ h \end{array} \right) \in T\) and \(\left( \begin{array}{c} f \\ g \end{array} \right) \in S\) the following holds

\[
\|g\| \leq c \left\| \left( \begin{array}{c} f \\ h \end{array} \right) \right\|.
\]

(3.12)

Observe that if \(S\) is \(T\)-bounded, then \(S\) is an operator. Furthermore, \(S\) is said to be strongly \(T\)-bounded when \(c < 1\) in \((3.12)\). Note that our definition of strongly relatively boundedness is formally stronger than the definition given in [27, Chap. 4, Sec. 1], however it can be proven to be equivalent by following the argumentation of the proof of [10, Thm. 3 Sec. 4 Chap 3].

**Lemma 3.8.** Let \(S\) be strongly \(T\)-bounded. The relation \(T\) is closed if and only if \(T + S\) is closed.

**Proof.** Since \(S\) is strongly \(T\)-bounded, it follows from the triangle inequality that there exists \(0 < c < 1\) such that for all \(\left( \begin{array}{c} f \\ h \end{array} \right) \in T\) and \(\left( \begin{array}{c} f \\ g \end{array} \right) \in S\),

\[
(1 - c) \left\| \left( \begin{array}{c} f \\ h \end{array} \right) \right\| \leq \left\| \left( \begin{array}{c} f \\ h + g \end{array} \right) \right\| \leq (1 + c) \left\| \left( \begin{array}{c} f \\ h \end{array} \right) \right\|.
\]

(3.13)

Suppose that \(T\) is closed and let \(\left( \begin{array}{c} f \\ s \end{array} \right) \in T + S\), then there are sequences \(\left\{ \left( \begin{array}{c} f_n \\ h_n \end{array} \right) \right\}_{n \in \mathbb{N}}\) in \(T\) and \(\left\{ \left( \begin{array}{c} f_n \\ g_n \end{array} \right) \right\}_{n \in \mathbb{N}}\) in \(S\) such that

\[
\left( \begin{array}{c} f_n \\ h_n + g_n \end{array} \right) \rightarrow \left( \begin{array}{c} f \\ s \end{array} \right).
\]

It follows from \((3.13)\) and the fact that \(\left\{ \left( \begin{array}{c} f_n \\ h_n + g_n \end{array} \right) \right\}_{n \in \mathbb{N}}\) is a Cauchy sequence, that \(\left\{ \left( \begin{array}{c} f_n \\ h_n \end{array} \right) \right\}_{n \in \mathbb{N}}\) converges to some \(\left( \begin{array}{c} f \\ h \end{array} \right) \in T\). Thereupon, there exists \(\left( \begin{array}{c} f \\ g \end{array} \right) \in S\) such that \(\left( \begin{array}{c} f \\ h + g \end{array} \right) \in T + S\). Thus,
again by (3.13), one obtains
\[
\left\| \begin{pmatrix} f \\ h + g \end{pmatrix} - \begin{pmatrix} f \\ s \end{pmatrix} \right\| = \lim_{n \to \infty} \left\| \begin{pmatrix} f \\ h + g \end{pmatrix} - \begin{pmatrix} f_n \\ h_n + g_n \end{pmatrix} \right\|
\]
\[
= \lim_{n \to \infty} \left\| \begin{pmatrix} f - f_n \\ h - h_n + (g - g_n) \end{pmatrix} \right\|
\]
\[
\leq \lim_{n \to \infty} (1 + c) \left\| \begin{pmatrix} f - f_n \\ h - h_n \end{pmatrix} \right\| = 0.
\]

Hence \( \begin{pmatrix} f \\ s \end{pmatrix} \in T + S \), which establishes that \( T + S \) is closed. The proof of the converse assertion is carried out analogously. □

The requirement of \( S \) being strongly \( T \)-bounded in the last result cannot be relaxed (see a counterexample in [10, Sec. 4 Chap 3]).

**Lemma 3.9.** Let \( T \) be a closed linear relation. If \( S \) and \( S^* \) are strongly \( T \)-bounded and strongly \( T^* \)-bounded, respectively, then

\[
(T + S)^* = T^* + S^*. 
\]  

(3.14)

**Proof.** Due to (2.12), \( T^* + S^* \subset W(T+S)^\perp \). It follows from Lemma 3.8 that \( W(T+S) \) and \( (T^* + S^*) \) are closed. Thus, for proving (3.14), it suffices to show that

\[
W(T + S) \oplus (T^* + S^*) = \mathcal{H} \oplus \mathcal{H}. 
\]  

(3.15)

By hypothesis, there exist 0 < \( b < 1 \) such that, for any \( \begin{pmatrix} f \\ h \end{pmatrix} \in T \), \( \begin{pmatrix} f \\ g \end{pmatrix} \in S \), \( \begin{pmatrix} l \\ t \end{pmatrix} \in T^* \) and \( \begin{pmatrix} l \\ s \end{pmatrix} \in S^* \), one has

\[
\|g\|^2 \leq b \left\| \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2, 
\]

\[
\|s\|^2 \leq b \left\| \begin{pmatrix} l \\ t \end{pmatrix} \right\|^2. 
\]  

(3.16)

One obtains from (2.10), using the fact that \( T \) is closed, that

\[
WT \oplus T^* = \mathcal{H} \oplus \mathcal{H}. 
\]  

(3.17)

Thus, for every \( \begin{pmatrix} r \\ k \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} \), there exist \( \begin{pmatrix} f \\ h \end{pmatrix} \in T \) and \( \begin{pmatrix} f \\ g \end{pmatrix} \in S \) such that \( \begin{pmatrix} r \\ k \end{pmatrix} = \begin{pmatrix} -h + l \\ f + t \end{pmatrix} \).

Since \( \text{dom} \ T \subset \text{dom} \ S \) and \( \text{dom} \ T^* \subset \text{dom} \ S^* \), one can find \( g, s \in \mathcal{H} \) such that \( \begin{pmatrix} f \\ g \end{pmatrix} \in S \) and...
\((l/s) \in S^*\). Define the linear relation \(Q\) in \((H \oplus H) \oplus (H \oplus H)\) as follows

\[
Q := \left\{ \begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} : \tilde{r} = \begin{pmatrix} r \\ k \end{pmatrix} \text{ and } \tilde{s} = \begin{pmatrix} -g \\ s \end{pmatrix} \right\}.
\]

Due to the fact that the norm in \(H \oplus H\) is equivalent to (2.2), it follows from (3.16) and (3.17) that, for any \(\begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} \in Q\),

\[
\|\tilde{s}\|^2 = \|g\|^2 + \|s\|^2
\]

\[
\leq b \left( \left\| \begin{pmatrix} f \\ h \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} l \\ t \end{pmatrix} \right\|^2 \right)
\]

\[
\leq b \left( \left\| \begin{pmatrix} -h \\ f \end{pmatrix} \right\|^2 + \left\| \begin{pmatrix} l \\ t \end{pmatrix} \right\|^2 \right)
\]

\[
= b \left( \left\| \begin{pmatrix} -h + l \\ f + t \end{pmatrix} \right\|^2 \right) = b\|\tilde{r}\|^2.
\]

Then \(Q \in B(H \oplus H)\) with \(\|Q\| < 1\), which implies that

\[
\text{ran}(Q + I) = H \oplus H.
\]

Therefore, for any \(\begin{pmatrix} v \\ w \end{pmatrix} \in H \oplus H\), there exists \(\begin{pmatrix} \tilde{r} \\ \tilde{s} \end{pmatrix} \in Q\) such that

\[
\begin{pmatrix} v \\ w \end{pmatrix} = \tilde{s} + \tilde{r} = \begin{pmatrix} -g - h + l \\ s + f + t \end{pmatrix}
\]

\[
= W\begin{pmatrix} f \\ h + g \end{pmatrix} + \begin{pmatrix} l \\ t + s \end{pmatrix} \in W(T + S) \oplus (T^* + S^*),
\]

whence (3.15) follows.

To state the following assertion, one requires a certain class of symmetric relations. A relation \(A\) is said to be positive (denoted by \(A \geq 0\)) whenever

\[
\langle f, g \rangle \geq 0, \ \text{for all } \begin{pmatrix} f \\ g \end{pmatrix} \in A.
\]

**Theorem 3.10.** Let \(A\) and \(B\) be two selfadjoint relations such that \(B\) is positive and strongly \(A\)-bounded. Then \(A + iB\) is a maximal dissipative relation.

**Proof.** By a direct verification of (3.1), one establishes that \(A + iB\) is dissipative. The closedness follows from Lemma 3.8 after noting that \(iB\) is also strongly \(A\)-bounded.

It remains to prove that \(\mathcal{N}_i((A + iB)^*)\) is a trivial relation. By Lemma 3.9, one has \((A + iB)^* = \)
A − iB. For an arbitrary 
\[
\begin{pmatrix} f \\ if \end{pmatrix} \in A − iB,
\]
there exist \( \begin{pmatrix} f \\ h \end{pmatrix} \in A \) and \( \begin{pmatrix} f \\ g \end{pmatrix} \in B \) such that \( if = h − ig \). Thus, \( \begin{pmatrix} f \\ i(f + g) \end{pmatrix} \in A \) and, due to the selfadjointness of A, one concludes
\[
\|f\|^2 + \langle f, g \rangle = \text{Im}\langle f, i(f + g) \rangle = 0.
\]
Since B is positive the last equality yields that \( f = 0 \). Therefore \( A + iB \) is maximal dissipative.

The following assertion sheds light on the interrelationship of \( \text{dom} \ L \) and \( \text{mul} \ L \) for a dissipative relation \( L \).

**Proposition 3.11.** If \( L \) is a closed dissipative relation, then \( \text{dom} \ L \subset (\text{mul} \ L)^\perp \).

**Proof.** For \( \begin{pmatrix} f \\ g \end{pmatrix} \in L \), there exist elements \( f_1, g_1 \in (\text{mul} \ L)^\perp \) and \( f_2, g_2 \in \text{mul} \ L \) such that \( \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ g_1 + g_2 \end{pmatrix} \). To prove the statement, it suffices to show that \( f_2 = 0 \).

Suppose \( f_2 \neq 0 \). This implies that there exits \( \tau > 0 \) such that
\[
\text{Im}\langle f_1, g_1 \rangle − \tau\|f_2\|^2 < 0.
\]
(3.18)
Thus \( \begin{pmatrix} f_1 + f_2 \\ g_1 + g_2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ −(i\tau f_2 + g_2) \end{pmatrix} \in L \). Since \( L \) is dissipative, one has
\[
0 \leq \text{Im}\langle f_1 + f_2, g_1 − i\tau f_2 \rangle
= \text{Im}\langle f_1, g_1 \rangle − i\tau\|f_2\|^2
= \text{Im}\langle f_1, g_1 \rangle − \tau\|f_2\|^2,
\]
which contradicts (3.18). Therefore \( f_2 = 0 \).

**Remark 3.12.** Due to last proposition, the spectrum of any closed dissipative relation satisfies the conditions of Theorem 2.10. Moreover, one can verify that the operator part of a closed dissipative relation is a closed dissipative operator. Conversely, for a closed relation \( L \) such that \( \text{dom} \ L \subset (\text{mul} \ L)^\perp \), if \( L_\circ \) is dissipative, then \( L \) is dissipative.

**Proposition 3.13.** Let \( L \) be a closed linear relation. If \( L \) is (maximal) dissipative, then \( L_\circ \) is (maximal) dissipative operator in \( (\text{mul} \ L)^\perp \oplus (\text{mul} \ L)^\perp \) and
\[
\eta_−(L_\circ) = \eta_−(L).
\]
(3.19)
Conversely, if \( \text{mul} \ L \subset (\text{dom} \ L)^\perp \) and \( L_\circ \) is (maximal) dissipative, then \( L \) is (maximal) dissipative and, therefore, (3.19) holds.
Proof. Suppose that $L$ is closed dissipative. It follows from Proposition 3.11 and (2.20) that $L_L$ is a closed, dissipative operator in $(\text{mul } L)^\perp \oplus (\text{mul } L)^\perp$. Moreover, (2.21) implies that $H \ominus \text{ran}(L - \zeta I) = H \ominus \text{ran}(L_L - \zeta I_L)$.

Whence (3.19) follows.

For the converse assertion, one again uses (2.20) to conclude that $L_\ominus$ is dissipative. Thus, taking into account Remark 3.12, one has that $L$ is dissipative.

Note that $L$ is maximal if and only if $L_L$ is maximal due to (3.19). \qed

4. Dissipative extensions of dissipative relations

This section is devoted to the development of the theory of extensions of dissipative relations. We consider only extensions without exit to a larger space (cf. [3, Appendix 1]). Our approach is similar to the one used in the von Neumann theory. There are other ways for dealing with extensions of operators (see for instance [38, Sec. 14]).

A relation $V$ is a contraction if it is bounded (and then it is actually an operator) with $\|V\| \leq 1$. It is known that if a relation $V$ satisfies $V^{-1} \subset V^*$, then $V$ is a particular kind of contraction called isometric operator for which $\|V\| = 1$ holds. Moreover if $V^{-1} = V^*$ the operator $V$ is said to be unitary.

Denote $\mathbb{T}_e := \{\zeta \in \mathbb{C} : |\zeta| > 1\}$. For any contraction $V$, one verifies $\mathbb{T}_e \subset \hat{\rho}(V)$. Whence the deficiency index $\eta_{\zeta}(V)$ (see (2.13)) is constant for $\zeta \in \mathbb{T}_e$. Define

$$
\eta_e(V) := \eta_{\zeta}(V), \quad \zeta \in \mathbb{T}_e.
$$

Following the argumentation of [10, Thm. 4.2.2] (which deals with isometric operators), it can be proven for any contraction $V$ that

$$
\eta_e(V) = \dim (\mathcal{H} \ominus \text{dom } V). \quad (4.1)
$$

If $V$ and $\hat{V}$ are closed contractions such that $V \subset \hat{V}$, then

$$
\eta_e(V) = \eta_e(\hat{V}) + \eta_0, \quad (4.2)
$$

where $\eta_0 = \dim(\text{dom } \hat{V} \ominus \text{dom } V)$.

A contraction $V$ it is said to be maximal if it is closed and $\eta_e(V) = 0$.

Compare the following statement with [10, Sec. 4.4] and [19, Thm. 5.1].

**Theorem 4.1.** Let $V$ be a closed contraction. The operator $\hat{V}$ is a closed contraction extension of $V$ if and only if there exists a unique closed contraction $W$ such that

$$
\hat{V} = V \oplus W, \quad (4.3)
$$

and

$$
2|\text{Re}(\langle f, h \rangle - \langle g, k \rangle)| \leq (\|f\|^2 - \|g\|^2) + (\|h\|^2 - \|k\|^2), \quad (4.4)
$$

16
for all \((f g) \in V\) and \((h k) \in W\). Note that the right-hand side of (4.3) is an orthogonal sum of relations (see (2.9)). Moreover, if \(V\) is isometric, then the condition (4.4) turns into the condition that either
\[
dom V \perp \dom W \quad \text{or} \quad \ran V \perp \ran W
\]
holds. In view of (4.3), the conditions in (4.5) hold simultaneously.

Proof. Suppose that \(\hat{V}\) is a closed contraction extension of \(V\) and consider \(W = \hat{V} \ominus V\) then \(W\) is a closed contraction and one verifies that \(\hat{V} = V \oplus W\).

For every \((f g) \in V\) and \((h k) \in W\), one has that \(\left(\alpha f + h \alpha g + k\right) \in \hat{V}\) and \(||\alpha g + k|| \leq ||\alpha f + h||\) with \(\alpha \in \mathbb{C}\). Then
\[
|\alpha|^2 ||g||^2 + ||k||^2 + 2 \Re \overline{\alpha} \langle g, k \rangle = ||\alpha g + k||^2 \\
\leq ||\alpha f + h||^2 \\
= |\alpha|^2 ||f||^2 + ||h||^2 + 2 \Re \overline{\alpha} \langle f, h \rangle,
\]
whence
\[
-2 \Re \overline{\alpha} \langle f, h \rangle - \langle g, k \rangle \leq |\alpha|^2 (||f||^2 - ||g||^2) + (||h||^2 - ||k||^2).
\]
Thus, setting \(\alpha := \pm 1\), the inequality (4.4) holds.

If \(V\) is isometric then \(||f|| = ||g||\). It turns out that in this case
\[
\langle f, h \rangle = \langle g, k \rangle
\]
(4.7)
since otherwise there would exist \(\tau > 0\) such that
\[
\tau ||f, h \rangle - \langle g, k \rangle|| > ||h||^2 - ||k||^2.
\]
This inequality contradicts (4.6) when \(\alpha = -\tau ||\langle f, h \rangle - \langle g, k \rangle||/(\langle h, f \rangle - \langle k, g \rangle)\). Therefore since \(V\) and \(W\) are orthogonal, it follows from (4.7) that
\[
0 = \left\langle \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \right\rangle \\
= \langle f, h \rangle + \langle g, k \rangle = 2 \langle f, h \rangle = 2 \langle g, k \rangle.
\]
The uniqueness of the decomposition is trivial. The converse assertion is straightforward.

Note that under the assumption that \(V\) is isometric in (4.3), the number \(\eta_0\) in (4.2) is given by \(\eta_0 = \dim \dom W\). Moreover, in this case, \(\hat{V}\) is isometric if and only if \(W\) is isometric.

We now turn to the question of extending closed dissipative relations and, in particular, closed symmetric relations. To this end, we introduce a fractional linear transformation of a relation as follows.

Definition 4.2. Following \([19]\), for a relation \(T\) and \(\zeta \in \mathbb{C}\), define the Cayley transform of \(T\) by
\[
C_{\zeta}(T) := \left\{ \begin{pmatrix} g - \zeta f \\ g - \zeta f \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in T \right\} = I + (\overline{\zeta} - \zeta)(T - \overline{\zeta}I)^{-1}
\]
Also, let us define the $Z$ transform of $T$ (cf. [20])

$$Z_\zeta(T) := \zeta C_\zeta(T)$$

This is a linear relation which satisfies

$$\text{dom} Z_\zeta(T) = \text{ran}(T - \zeta I), \quad \text{ran} Z_\zeta(T) = \text{ran}(T - \zeta I),$$
$$\text{mul} Z_\zeta(T) = \ker(T - \zeta I), \quad \ker Z_\zeta(T) = \ker(T - \zeta I).$$

(4.8) (4.9)

The $Z$ transform has the following properties (see [19, Lems. 2.6, 2.7] and [20, Props. 3.6, 3.7]).

For any $\zeta \in \mathbb{C}$:

(i) $Z_\zeta(Z_\zeta(T)) = T$.

(ii) $Z_\zeta(T) \subset Z_\zeta(S) \iff T \subset S$.

(iii) $Z_{-\zeta}(T) = -Z_\zeta(-T)$.

(iv) If $|z| = 1$, then $Z_\zeta(T^{-1}) = Z_{\bar{\zeta}}(T) = (Z_\zeta(T))^{-1}$.

For any $\zeta \in \mathbb{C}\setminus\mathbb{R}$:

(vii) $Z_\zeta(T + S) = Z_\zeta(T) + Z_\zeta(S)$.

(viii) If $\zeta = \pm i$, then $Z_\zeta(T \oplus S) = Z_\zeta(T) \oplus Z_\zeta(S)$.

(ix) $Z_\zeta(T^*) = (Z_{\bar{\zeta}}(T))^*$.

(x) $Z_\zeta(T)$ is closed $\iff T$ is closed.

**Proposition 4.3.** Under the assumption that $\zeta \in \mathbb{C}_+$ and $|\zeta| = 1$, the linear relation $L$ is (closed, maximal) dissipative if and only if $V = Z_\zeta(L)$ is a (closed, maximal) contraction.

**Proof.** Suppose that $L$ is a dissipative relation and let $\left(\frac{g - \zeta f}{\bar{\zeta} g - f}\right) \in Z_\zeta(L) = V$, with $\left(\frac{f}{g}\right) \in L$. Then

$$\|g - \bar{\zeta} f\|^2 - \|\zeta g - f\|^2 = 2 \text{Re}(\zeta \langle f, g \rangle) + 2(\text{Re} \langle \zeta f, g \rangle),$$
$$= 4 \text{Im} \zeta \text{Im} \langle f, g \rangle \geq 0.$$ (4.10)

Thus $V$ is a contraction.

Conversely, let $\left(\frac{g - \zeta f}{\bar{\zeta} g - f}\right) \in Z_\zeta(V) = L$, with $\left(\frac{f}{g}\right) \in V$. Then

$$\text{Im} \langle g - \bar{\zeta} f, \bar{\zeta} g - f \rangle = \text{Im} \langle \zeta \|g\|^2 + \|f\|^2 - 2 \text{Re} \langle g, f \rangle \rangle$$
$$= \text{Im} \zeta (\|f\|^2 - \|g\|^2) \geq 0,$$ (4.11)

therefore $L$ is dissipative. Note that $L$ and $V$ are simultaneously closed due to (x). As regards the maximality,

$$\eta_e(V) = \dim(\mathcal{H} \ominus \text{dom } V)$$
$$= \dim(\mathcal{H} \ominus \text{dom } Z_\zeta(L)) \quad \text{(due to (4.8))}$$
$$= \dim(\mathcal{H} \ominus \text{ran}(L - \zeta I)) = \eta_e(L).$$
Remark 4.4. It follows from (4.10) and (4.11) that, for all $|\zeta| = 1$, a relation is symmetric if and only if its $Z$ transform is isometric. Moreover, Proposition 4.3 shows that the $Z$ transform gives a one-to-one correspondence between contractions and dissipative relations.

The next assertion can be found in [19, Thm. 6.1] and corresponds to the first von Neumann formula. It characterizes the adjoint of a closed symmetric relation by means of its deficiency space (2.14). We omit the proof since it can be obtained by the same argumentation used in the proof of the first von Neumann formula (cf. [10, Thm. 4.4.1]).

Theorem 4.5. For a closed symmetric relation $A$, one has

$$A^* = A + N_\zeta(A^*) + N_\zeta(A^*)^*, \quad \zeta \in \mathbb{C}\setminus \mathbb{R}. \quad (4.12)$$

For $\zeta \in \{i, -i\}$, the direct sum in (4.12) is orthogonal.

The proof of the next proposition repeats the argumentation given in the first part of the proof of Theorem 4.1.

Proposition 4.6. Let $L$ be a closed dissipative relation. Then $\hat{L}$ is a closed dissipative extension of $L$ if and only if there exist a unique closed dissipative relation $S$ such that

$$\hat{L} = L \oplus S,$$

and, for all $\begin{pmatrix} f \\ g \end{pmatrix} \in L$ and $\begin{pmatrix} h \\ k \end{pmatrix} \in S$,

$$\text{Im}(\langle f, g \rangle + \langle h, k \rangle) \geq |\text{Im}(\langle f, g \rangle - \langle h, k \rangle)|.$$

The following extends the so-called second von Neumann formula (cf. [19, Thm. 6.2]).

Theorem 4.7. Let $A$ be a closed symmetric relation. $\hat{A}$ is a closed dissipative (symmetric) extension of $A$ if and only if, for a fixed $\zeta \in \mathbb{C}_+ \ (\zeta \in \mathbb{C}\setminus \mathbb{R})$,

$$\hat{A} = A + (\mathbf{V} - \mathbf{I})D, \quad (4.13)$$

where $D \subset N_\zeta(A^*)$ is a closed bounded relation and $\mathbf{V} : D \to N_{-\zeta}(A^*)$ is a closed contraction (isometry) in $(\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})$. For $\zeta = i$, the direct sum in (4.13) is orthogonal.

Proof. It follows from Proposition 4.3 that $Z_{\zeta/|\zeta|}(|\zeta|^{-1}A)$ and $Z_{\zeta/|\zeta|}(|\zeta|^{-1}\hat{A})$ are, respectively, a closed isometric and a closed contraction (isometry) whenever $\zeta \in \mathbb{C}_+$ ($\zeta \in \mathbb{C}\setminus \mathbb{R}$). Moreover, since $A \subset \hat{A}$, one has $Z_{\zeta/|\zeta|}(|\zeta|^{-1}A) \subset Z_{\zeta/|\zeta|}(|\zeta|^{-1}\hat{A})$ in view of property (11).

Theorem 4.1 implies the existence of a closed contraction (isometry) $W$ such that

$$Z_{\zeta/|\zeta|}(|\zeta|^{-1}\hat{A}) = Z_{\zeta/|\zeta|}(|\zeta|^{-1}A) \oplus W, \quad (4.14)$$

with

$$\text{dom} W \subset \mathcal{H} \oplus \text{dom} Z_{\zeta/|\zeta|}(|\zeta|^{-1}A) = \mathcal{H} \oplus \text{ran}(A - \zeta I) = \ker(A^* - \zeta I),$$

$$\text{ran} W \subset \mathcal{H} \oplus \text{ran} Z_{\zeta/|\zeta|}(|\zeta|^{-1}A) = \mathcal{H} \oplus \text{ran}(A - \zeta I) = \ker(A^* - \overline{\zeta} I). \quad (4.15)$$
By applying the Z transform to (4.14), using (i), one obtains

\[ \hat{A} = A + |\zeta| Z_{\zeta/|\zeta|}(W). \]  

(4.16)

Observe that \( \text{dom } W \) is closed. Consider the linear relation

\[ D = \left\{ \left( \frac{v}{\zeta v} \right) : v \in \text{dom } W \right\}, \]  

(4.17)

whence, in view of (4.15), \( D \subseteq \mathcal{N}_{\zeta}(A^*) \). Thus \( D \) is bounded and then closed.

For every \( \left( \frac{v}{w} \right) \in W \), define the relation \( V \) in \((\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})\) with \( \text{dom } V = D \) such that

\[ V \left( \frac{v}{\zeta v} \right) = \left( \frac{w}{\zeta w} \right). \]

It follows from (4.15) that \( VD \subseteq \mathcal{N}_{\zeta}(A^*) \) and, since \( W \) is a contraction (isometry):

\[ \left\| \frac{\zeta}{|\zeta|} \left( \frac{w}{\zeta w} \right) \right\| = \|w\| + \|\zeta w\| \]
\[ \leq \|v\| + \|\zeta v\| = \left\| \left( \frac{v}{\zeta v} \right) \right\|. \]

(4.18)

Thus \( V \) is a closed contraction (isometry because the equality holds in (4.18) when \( W \) is an isometry). Hence

\[ |\zeta| Z_{\zeta/|\zeta|}(W) = \left\{ \left( \frac{\zeta}{|\zeta|} \frac{w - v}{w - \zeta v} \right) : \left( \frac{\zeta}{|\zeta|} \frac{v}{w} \right) \in D \right\} \]
\[ = \left\{ V \left( \frac{v}{\zeta v} \right) - \left( \frac{v}{\zeta v} \right) : \left( \frac{v}{\zeta v} \right) \in D \right\} = (V - I)D. \]  

(4.19)

Therefore (4.16) is transformed into \( \hat{A} = A + (V - I)D \). For \( \zeta = i \), the orthogonality of the direct sum in (4.16) follows from property (viii).

We now prove the converse assertion. Define

\[ W = \left\{ \left( \frac{\zeta}{|\zeta|} \frac{w}{w} \right) : \left( \frac{v}{\zeta v} \right) \in D \text{ and } \left( \frac{w}{\zeta w} \right) \in VD \right\}. \]

Since \( V \) is a contraction (isometry), one has

\[ \left\| \frac{\zeta}{|\zeta|} \frac{v}{\zeta v} \right\| - \left\| \frac{\zeta}{|\zeta|} \frac{w}{\zeta w} \right\| = \frac{1}{1 + |\zeta|} \left[ (1 + |\zeta|) \|v\| - (1 + |\zeta|) \|w\| \right] \]
\[ = \frac{1}{1 + |\zeta|} \left( \left\| \frac{v}{\zeta v} \right\| - \left\| \frac{w}{\zeta w} \right\| \right) \geq 0. \]

(4.20)

From this, taking into account that \( \text{dom } W = \text{dom } D \), one concludes that \( W \) is a closed contraction (isometry because the equality holds in (4.20) when \( V \) is an isometry).

Also, reading (4.19) backwards, one arrives at (4.16). Now, multiply (4.16) by \( |\zeta|^{-1} \) and apply
\( \mathcal{Z}_{\zeta/|\zeta|}(\cdot) \) to both sides of the resulting equality. This yields (4.14), where the orthogonality is a consequence of

\[
\begin{align*}
\text{dom } W & \subset \ker(A^* - \zeta I) = \mathcal{H} \ominus \text{ran}(A - \overline{\zeta} I) = \mathcal{H} \ominus \text{dom}\mathcal{Z}_{\zeta/|\zeta|}(|\zeta|^{-1} A), \\
\text{ran } W & \subset \ker(A^* - \overline{\zeta} I) = \mathcal{H} \ominus \text{ran}(A - \zeta I) = \mathcal{H} \ominus \text{ran}\mathcal{Z}_{\zeta/|\zeta|}(|\zeta|^{-1} A).
\end{align*}
\]

The assertion then follows from (4.14) in view of Theorem 4.1 and Proposition 4.3.

As a consequence of (4.13), any dissipative extension \( S \) of a symmetric relation \( A \) satisfies

\[
A \subset S \subset A^*.
\]

Corollary 4.8. If \( A \) is a closed symmetric relation and \( \hat{A} \) is a closed dissipative extension of \( A \), then

\[
\eta_-(A) = \eta_-(\hat{A}) + \dim[\hat{A}/A]. \tag{4.21}
\]

Proof. In the proof of Theorem 4.7 one verifies that \( (V-I) \) gives a one-to-one correspondence. Thus, by (4.13) and by (4.17),

\[
\dim[\hat{A}/A] = \dim[(V-I)D] = \dim(\text{dom } W).
\]

Hence, taking into account (4.8), it follows from (4.14) and (4.2) that

\[
\eta_-(A) = \eta_-(\hat{A}) + \dim(\text{dom } W)
\]

\[
= \eta_-(\hat{A}) + \dim[\hat{A}/A].
\]

Since \( \mathcal{N}_\zeta(A^*) = \mathcal{N}_{-\zeta}(-A^*) \), for a closed symmetric relation \( A \), the equality \( \eta_+(A) = \eta_-(A) \) holds. This, together with Corollary 4.8 yields that if \( \hat{A} \) is a closed symmetric extension of \( A \), then

\[
\eta_\pm(A) = \eta_\pm(\hat{A}) + \dim[\hat{A}/A]. \tag{4.22}
\]

A closed symmetric relation \( A \) is selfadjoint if and only if it has indices \((0,0)\) (see (3.3)). For this reason, the selfadjoint relations are maximal dissipative.

There is another way to construct maximal dissipative extensions of symmetric relations on the basis of formula (4.12).

Proposition 4.9. Let \( A \) be a closed symmetric relation with finite deficiency index \( \eta_-(A) = n \). Then, for every \( \zeta \in \mathbb{C}_+ \) fixed, the relation

\[
\hat{A} := A \oplus \mathcal{N}_\zeta(A^*) \tag{4.23}
\]

is the unique maximal dissipative extension of \( A \) such that \( \dim \mathcal{N}_\zeta(\hat{A}) = n \).

Proof. Fix \( \zeta \in \mathbb{C}_+ \). Note that (4.12) implies

\[
A \cap \mathcal{N}_\zeta(A^*) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.
\]
Also, (4.12) and (4.23) yield $\mathbb{N}_\zeta(A^*) = \mathbb{N}_\zeta(\hat{A})$.

Appealing to (3.1), one verifies that $\hat{A}$ is dissipative. Since $A$ is closed and $\dim \mathbb{N}_\zeta(A^*) = \eta_-(A)$ is finite, $\hat{A}$ is closed. Besides, from Corollary 4.8 one has

$$\eta_-(\hat{A}) = \eta_-(A) - \dim[\hat{A}/A] = \eta_-(A) - \dim \mathbb{N}_\zeta(A^*) = 0.$$  

Thus, $\hat{A}$ is a maximal dissipative extension of $A$.

To prove uniqueness, let $L$ be a maximal dissipative extension of $A$ such that $\mathbb{N}_\zeta(L) = n$. Since $L \subset A^*$, $\mathbb{N}_\zeta(L) \subset \mathbb{N}_\zeta(A^*)$ holds, so taking into account the dimension of the spaces, one concludes that $\mathbb{N}_\zeta(L) = \mathbb{N}_\zeta(A^*)$. Therefore $\hat{A} = A + \mathbb{N}_\zeta(A^*) \subset L$. To complete the proof, it only remains to recall that $\hat{A}$ is maximal. □

Note that $\hat{A}$ is a maximal dissipative, nonselfadjoint relation. The next assertion complements the previous one.

**Proposition 4.10.** Let $A$ be a closed symmetric relation with finite deficiency indices $(n,n)$. If $\alpha \in \hat{\rho}(A) \cap \mathbb{R}$, then

$$L := A + \mathbb{N}_\alpha(A^*)$$

(4.24)

is the unique maximal dissipative extension of $A$ such that $\dim \mathbb{N}_\alpha(L) = n$. Moreover $L$ is selfadjoint.

Proof. Since $A$ is symmetric, it follows that $(\mathbb{C}\setminus\mathbb{R}) \cup \{\alpha\}$ is in a connected component of $\hat{\rho}(A)$. Thus $\dim \mathbb{N}_\alpha(A^*) = n$.

If one assumes that $\begin{pmatrix} f \\ \alpha f \end{pmatrix} \in A$, then $\begin{pmatrix} 0 \\ f \end{pmatrix} \in (A - \alpha I)^{-1}$. It follows from the fact that $(A - \alpha I)^{-1}$ is an operator that $f = 0$. Hence $A$ and $\mathbb{N}_\alpha(A^*)$ are linearly independent.

Taking into account that $\alpha \in \mathbb{R}$, one verifies that $L$ is symmetric and closed directly from (4.24). Hence, $L \subset A^*$. Using again (4.24), one concludes that $\mathbb{N}_\alpha(L) = \mathbb{N}_\alpha(A^*)$.

As in the proof of Proposition 4.9 one obtains, on the basis of (4.22), that $\eta_\pm(L) = 0$. Uniqueness can also be proved along the lines of the proof of Proposition 4.9. □

Similar to the operator case, one can characterize the spectrum of a selfadjoint extension of the symmetric relation $A$ in the intervals intersecting $\hat{\rho}(A)$.

**Proposition 4.11.** Let $A$ be a closed symmetric relation with finite deficiency indices $(n,n)$ (see (3.3)) and $L$ be a selfadjoint extension of $A$. If a real interval $\Delta$ is in $\hat{\rho}(A)$, then the spectrum of $L$ in $\Delta$ only has isolated eigenvalues of multiplicity at most $n$.

Proof. Fix $\zeta \in \sigma(L) \cap \Delta$. Since $\zeta \in \hat{\rho}(A)$, $\dim \mathbb{N}_\zeta(A^*) = n$ and, in view of Proposition 2.4, $\text{ran}(A - \zeta I)$ is closed. Now, due to the fact that $L \subset A^*$, one can define

$$K := \text{ran}(L - \zeta I) \ominus \text{ran}(A - \zeta I).$$

(4.25)

Thus $K \subset \ker(A^* - \zeta I)$ and

$$\dim K \leq \dim[\ker(A^* - \zeta I)] = \dim \mathbb{N}_\zeta(A^*) = n.$$  

(4.26)
Then by (4.25) and (4.26) ran($L - \zeta I$) is closed. This implies that $\zeta \notin \sigma_c(L)$. Furthermore, by Proposition 3.13 $L_L$ is a selfadjoint operator and then, recurring to Theorem 2.10 one obtains that $\zeta$ is an isolated eigenvalue.

Let us compute the multiplicity of the eigenvalue $\zeta$. To this end, observe that $NNN\zeta(L) \subset NNN\zeta(A^*)$. Also,

$$\dim[\ker(L - \zeta I)] = \dim NNN\zeta(L) \leq \dim NNN\zeta(A^*) = n.$$ 

\[\square\]

Definition 4.12. A relation $T$ is said to be regular if its quasi-regular set is the whole complex plane, that is $\hat{\rho}(T) = \mathbb{C}$.

Corollary 4.13. Let $A$ be a closed, regular, symmetric relation with $\eta_-(A) = n$. Assume that $L$ is a maximal dissipative extension of $A$.

(i) If $L$ is selfadjoint, then its spectrum consists of isolated eigenvalues of multiplicity at most $n$.

(ii) If $L$ is not selfadjoint, then its spectral core consists of eigenvalues of multiplicity at most $n$.

(iii) If $n = 1$, then every number in $\mathbb{C}_+ \cup \mathbb{R}$ is an eigenvalue of one, and only one, realization of $L$.

Proof. First note that, due to the regularity of $A$, its deficiency indices are equal. Now, (i) is a consequence of Proposition 4.11 since $\mathbb{R} \subset \hat{\rho}(A)$. For proving (ii), consider $\zeta \notin \hat{\rho}(L)$ and repeat the argumentation of the proof of Proposition 4.11 to show that ran($L - \zeta I$) is closed so $\zeta \notin \sigma_c(L)$. The multiplicity of any eigenvalue is computed as in the proof of Proposition 4.11. (iii) follows from Propositions 4.9 and 4.10. \[\square\]

Remark 4.14. Since by , any closed regular symmetric operator is unitarily equivalent to the multiplication operator is a dB space (see Section 5.2), the spectrum of $L$ in the above corollary is discrete even in the nonselfadjoint case (see Theorem 5.3).

5. Applications to nondensely defined operators

5.1. Jacobi operators

Consider the Hilbert space $l_2(\mathbb{N})$, i.e. the space of square-summable sequences. Fix two real sequences $\{b_k\}_{k=1}^\infty$ and $\{q_k\}_{k=1}^\infty$ such that $b_k > 0$ for $k \in \mathbb{N}$ and let $J$ be the operator whose matrix representation with respect to the canonical basis $\{\delta_k\}_{k=1}^\infty$ of $l_2(\mathbb{N})$ is

$$\begin{pmatrix}
 q_1 & b_1 & 0 & 0 & \cdots \\
 b_1 & q_2 & b_2 & 0 & \cdots \\
 0 & b_2 & q_3 & b_3 & \cdots \\
 0 & 0 & b_3 & q_4 & \cdots \\
 \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}; \quad (5.1)
$$

see [3] Sec. 47] for the definition of a matrix representation for an unbounded closed symmetric operator.
Consider the difference equation

\[ b_{k-1} \phi_{k-1} + q_k \phi_k + b_k \phi_{k+1} = \zeta \phi_k, \quad \zeta \in \mathbb{C}, \quad (5.2) \]

for \( k \in \mathbb{N} \) with \( b_0 = 0 \). Setting \( \phi_1 = 1 \), one solves recurrently \((5.2)\) and \( \phi_k \) is a polynomial of degree \( k - 1 \) in \( \zeta \), denoted here by \( \pi_k(\zeta) \), and known as the \( k - 1 \)-th polynomial of the first kind associated to \((5.1)\). Similarly, \( \phi_k \) is a polynomial of degree \( k - 2 \) if one sets \( \phi_1 = 0 \) and \( \phi_2 = 1/b_1 \), in \((5.2)\). In this case \( \phi_k \) is the \( k - 1 \)-th polynomial of the second kind associated to \((5.1)\) and denoted by \( \theta_k(\zeta) \).

Remark 5.1. The symmetric operator \( J \) has deficiency indices \((0, 0)\) or \((1, 1)\). The first case is characterized by the divergence of the series \( \sum_k |\pi_k(\zeta)|^2 \) for all \( \zeta \in \mathbb{C} \setminus \mathbb{R} \), while the second case by the convergence of it (see [2, Chap. IV] and [9, Chap. VII]).

Suppose that \( J \) is selfadjoint and consider the linear operator

\[ B = J |_{\text{dom } J \cap \text{span} \{ \delta_1 \}}. \]

The operator \( B \) is closed, non-densely defined, and symmetric. By \((4.22)\), \( B \) has indices \((1, 1)\).

Proposition 5.2. The maximal dissipative extensions are in one-to-one correspondence with \( \tau \in \mathbb{C} \cup \mathbb{R} \cup \{ \infty \} \) and they are perturbations of \( J \) given by

\[ J(\tau) = \left\{ \begin{pmatrix} f \\ g + \tau \langle \delta_1, f \rangle \delta_1 \end{pmatrix} : \begin{pmatrix} f \\ g \end{pmatrix} \in J \right\}, \quad \tau \neq \infty, \quad (5.4) \]

and

\[ J(\infty) = B + \text{span} \left\{ \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\}, \quad (5.5) \]

where, for \( \tau \in \mathbb{R} \cup \{ \infty \} \), \( J(\tau) \) is selfadjoint. Furthermore,

\[ B^* = J + \text{span} \left\{ \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} \right\}. \quad (5.6) \]

Proof. Fix \( \zeta \in \mathbb{C} \), then \( \pi(\zeta) \) and \( \theta(\zeta) \) do not belong to \( l_2(\mathbb{N}) \) in view of Remark 5.1 and \((5.3)\). According to [2, Chap.1 Sec. 3] (see also [9, Chap. VII]), there exists a unique function \( m(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C} \) satisfying \( m(\zeta) = m(\overline{\zeta}) \) and \( (\text{Im } \zeta)(\text{Im } m(\zeta)) > 0 \) such that

\[ \psi(\zeta) = \theta(\zeta) + m(\zeta) \pi(\zeta) \in \text{dom } J. \]

By a straightforward computation, one obtains

\[ J\psi(\zeta) = \delta_1 + \zeta \psi(\zeta) \quad (5.7) \]
so that, for every \( f \in \text{dom} \, B \), one has
\[
\langle f, \zeta \psi(\zeta) \rangle = \langle f, \delta_1 + \zeta \psi(\zeta) \rangle = \langle f, J(\psi(\zeta)) \rangle
\]
\[
= \langle Jf, \psi(\zeta) \rangle = \langle Bf, \psi(\zeta) \rangle,
\]
which means that \( \begin{pmatrix} \psi(\zeta) \\ \zeta \psi(\zeta) \end{pmatrix} \in B^* \). Therefore
\[
N_\zeta(B^*) = \text{span} \left\{ \begin{pmatrix} \psi(\zeta) \\ \zeta \psi(\zeta) \end{pmatrix} \right\} \quad \text{and} \quad N_{\overline{\zeta}}(B^*) = \text{span} \left\{ \begin{pmatrix} \psi(\overline{\zeta}) \\ \overline{\zeta} \psi(\overline{\zeta}) \end{pmatrix} \right\},
\]
(5.8)
since \( \eta(\pm (B)) = 1 \). By Theorem 4.5, it holds that
\[
B^* = B + \text{span} \left\{ \begin{pmatrix} \psi(\zeta) \\ \zeta \psi(\zeta) \end{pmatrix} \right\} + \text{span} \left\{ \begin{pmatrix} \psi(\zeta) \\ \zeta \psi(\zeta) \end{pmatrix} \right\},
\]
(5.9)
Now, since \( \text{dom} \, B \) as well as \( \psi(\zeta) \) and \( \psi(\zeta) \) belong to \( \text{dom} \, J \), it follows that \( \text{dom} \, B^* \subset \text{dom} \, J \).
Hence \( J \) and \( B^* \) have the same domain. Observe that \( J \) and \( \{0\} \oplus \text{span} \{\delta_1\} \) are linearly independent so that \( J + Z \subset B^* \). On the other hand, (5.9) implies that there exist \( f \in \text{dom} \, B \) and \( a, b \in \mathbb{C} \) such that
\[
\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f + a \psi(\zeta) + b \psi(\overline{\zeta}) \\ Bf + a \zeta \psi(\zeta) + b \zeta \overline{\psi(\zeta)} \end{pmatrix}
\]
for every \( \begin{pmatrix} h \\ k \end{pmatrix} \in B^* \). Then by (5.7)
\[
\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f + a \psi(\zeta) + b \psi(\overline{\zeta}) \\ Bf + a \zeta \psi(\zeta) + b \zeta \overline{\psi(\zeta)} \end{pmatrix} + \begin{pmatrix} 0 \\ -(a + b) \delta_1 \end{pmatrix} = \begin{pmatrix} f + a \psi(\zeta) + b \psi(\overline{\zeta}) \\ Bf + a \zeta \psi(\zeta) + b \zeta \overline{\psi(\zeta)} \end{pmatrix} + \begin{pmatrix} 0 \\ -(a + b) \delta_1 \end{pmatrix} \in J + Z.
\]
We have proven 5.6. Now we turn to the proof of (5.4) and (5.5). Note that these equations yield closed dissipative extensions of \( B \) which are therefore maximal. Theorem 4.7 asserts that every maximal dissipative extension \( J(\beta) \) of \( B \) is given by
\[
J(\beta) = B + (V_\beta - I)N_\zeta(B^*),
\]
(5.10)
with \( V_\beta : N_\zeta(B^*) \to N_{\overline{\zeta}}(B^*) \) being a closed contraction. On the basis of (5.8), one concludes that all the contraction mappings are in one-to-one correspondence with \( \beta \in \mathbb{T} \cup \mathbb{T}i \) given by
\[
V_\beta \left( \begin{pmatrix} \psi(\zeta) \\ \zeta \psi(\zeta) \end{pmatrix} \right) = \beta \left( \begin{pmatrix} \psi(\overline{\zeta}) \\ \overline{\zeta} \psi(\overline{\zeta}) \end{pmatrix} \right),
\]
whence, by means of (5.10), one arrives at
\[
J(\beta) = B + \text{span} \left\{ \begin{pmatrix} \beta \psi(\overline{\zeta}) - \psi(\zeta) \\ \overline{\zeta} \beta \psi(\overline{\zeta}) - \zeta \psi(\zeta) \end{pmatrix} \right\},
\]
(5.11)
25
The last equality implies that \( \text{dom } J(\beta) \subset \text{dom } J \). Take the Möbius transformation

\[
\beta_r = \frac{1 + \tau m(\zeta)}{1 + \tau m(\zeta)}.
\]

Since \( m(\zeta) \in \mathbb{C}_+ \), one has that \( \beta_r \) maps \( \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\} \) onto \( \mathbb{T} \cup \mathbb{T}_1 \), with \( \beta_\infty = m(\zeta)/m(\zeta) \). Then for \( \tau \neq \infty \) it follows from (5.11) that, for any \( \begin{pmatrix} h \\ k \end{pmatrix} \in J(\beta_r) \), there exists \( f \in \text{dom } B \) such that

\[
\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} f + \alpha(\beta_r \psi(\zeta) - \psi(\zeta)) \\ Bf + \alpha(\beta_r \zeta \psi(\zeta) - \zeta \psi(\zeta)) \end{pmatrix} = \begin{pmatrix} Jf + \alpha[\beta_r(\delta_1 + \zeta \psi(\zeta)) - (\delta_1 + \zeta \psi(\zeta))] + \alpha(1 - \beta_r) \delta_1 \\ J[f + \alpha(\beta_r \psi(\zeta) - \psi(\zeta))] + \alpha(1 - \beta_r) \delta_1 \end{pmatrix}.
\]

(5.12)

Note that \( \psi_1(\zeta) = m(\zeta) \), so

\[
\tau(\delta_1, h) = \tau(\delta_1, f + \alpha(\beta_r \psi(\zeta) - \psi(\zeta))) = \alpha \tau(\beta_r m(\zeta) - m(\zeta)) = \alpha \tau \left( \frac{1 + \tau m(\zeta)}{1 + \tau m(\zeta)} m(\zeta) - m(\zeta) \right) = \alpha \left( 1 - \frac{1 + \tau m(\zeta)}{1 + \tau m(\zeta)} \right) = \alpha(1 - \beta_r).
\]

Thus (5.12) yields \( \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h \\ Jh + \tau(\delta_1, h) \delta_1 \end{pmatrix} \subset J(\tau) \). Due to maximality it follows that \( J(\beta_r) = J(\tau) \).

For \( \beta_\infty = m(\zeta)/m(\zeta) \), the expression \( \beta_\infty \psi(\zeta) - \psi(\zeta) \) vanishes. Thus, by (5.11), it follows that \( \text{dom } J(\beta_\infty) \subset \text{dom } B \). Thence, according to (5.12), for every \( \begin{pmatrix} h \\ k \end{pmatrix} \in J(\beta_\infty) \),

\[
\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h \\ Bh \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha(1 - \beta_r) \delta_1 \end{pmatrix} \in B + Z = J(\infty).
\]

Therefore \( J(\beta_\infty) = J(\infty) \). \( \square \)

From what has been said, all the maximal dissipative extensions (5.4) of \( B \) have the representation

\[
J(\tau) = \begin{pmatrix}
q_1 + \tau & b_1 & 0 & 0 & \cdots \\
b_1 & q_2 & b_2 & 0 & \cdots \\
0 & b_2 & q_3 & b_3 & \cdots \\
0 & 0 & b_3 & q_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
5.2. Operator of multiplication in dB spaces

There are two essentially different ways of defining a de Branges space (dB space) [15, Chap. 2]. The following one has an axiomatic structure:

A nontrivial Hilbert space of entire functions \( B \) is said to be a dB space when for every function \( f(z) \) in the space, the following holds:

(A1) For every \( w \in \mathbb{C} \setminus \mathbb{R} \), the linear functional \( f(\cdot) \mapsto f(w) \) is continuous;

(A2) for every non-real zero \( w \) of \( f(z) \), the function \( f(z)(z - \overline{w})(z - w)^{-1} \) belongs to \( B \) and has the same norm as \( f(z) \);

(A3) the function \( f^\#(z) = \overline{f(z)} \) also belongs to \( B \) and has the same norm as \( f(z) \).

Due to the polarization identity, (A3) implies

\[ \langle f(\zeta), g(\zeta) \rangle = \langle g^\#(\zeta), f^\#(\zeta) \rangle \]  

for every \( f(z), g(z) \in \mathcal{B} \).

By the Riesz lemma, (A1) is equivalent to the existence of a unique reproducing kernel \( k(z,w) \) that belongs to \( B \) for every \( w \in \mathbb{C} \setminus \mathbb{R} \) and satisfies

\[ \langle k(\zeta,w), f(\zeta) \rangle = f(w), \]  

for every \( f(z) \in \mathcal{B} \). Besides, \( k(w,w) = \langle k(\zeta,w), k(\zeta,w) \rangle > 0 \) as a consequence of (A2) (see the proof of [15, Thm. 23]). Note also that \( k(z,w) = k(w,z) \). Finally, in view of (5.13), for every \( f(z) \in \mathcal{B} \),

\[ \langle k^\#(\zeta,w), f(\zeta) \rangle = \overline{\langle k(\zeta,w), f^\#(\zeta) \rangle} = f^\#(w) = \langle k(\zeta,\overline{w}), f(\zeta) \rangle, \]

whence \( k(z,w) = k(z,\overline{w}) \).

The operator of multiplication by the independent variable in \( \mathcal{B} \) is defined by the relation

\[ S = \left\{ \left( \begin{array}{l} f(z) \\ z f(z) \end{array} \right) \mid f(z), z f(z) \in \mathcal{B} \right\}. \]

Clearly it is an operator and [26, Prop. 4.2,Cors. 4.3 and 4.7] show that \( S \) is closed, regular, symmetric, with deficiency indices (1,1), and not necessarily densely defined.

Fix \( w \in \mathbb{C}_+ \). It follows from (5.14) that for every \( \left( \begin{array}{l} f(z) \\ (z - w)f(z) \end{array} \right) \in S - wI \)

\[ \langle k(\zeta,w), (\zeta - w)f(\zeta) \rangle = (w - w)f(w) = 0, \]

which implies that \( k(z,w) \in \ker(S^* - \overline{w}I) = \text{dom} N_{\overline{w}}(S^*) \). Since \( \eta_\pm(S) = 1 \), one has

\[ N_{\overline{w}}(S^*) = \text{span} \left\{ \left( \begin{array}{l} k(z,w) \\ \overline{w} k(z,w) \end{array} \right) \right\}; \quad N_w(S^*) = \text{span} \left\{ \left( \begin{array}{l} k(z,\overline{w}) \\ w k(z,\overline{w}) \end{array} \right) \right\}. \]

Equation (4.12) now reads

\[ S^* = S + \text{span} \left\{ \left( \begin{array}{l} k(z,w) \\ \overline{w} k(z,w) \end{array} \right) \right\} + \text{span} \left\{ \left( \begin{array}{l} k(z,\overline{w}) \\ w k(z,\overline{w}) \end{array} \right) \right\}. \]
Furthermore, by Theorem 4.7, every maximal dissipative extension $S_\tau$ of $S$ is given by

$$S_\tau = S + (V_\tau - I)N_w(S^*)$$

(5.18)

with $V_\tau : N_w(S^*) \to N_w(S^*)$ being a closed contraction given by

$$V_\tau \left( \begin{pmatrix} k(z, \overline{w}) \\ wk(z, \overline{w}) \end{pmatrix} \right) = \tau \left( \begin{pmatrix} k(z, w) \\ \overline{w}k(z, w) \end{pmatrix} \right),$$

where $|\tau| \leq 1$. Note that the form of $V_\tau$ has been deduced from (5.16). Whence, by (5.18), one has

$$S_\tau = S + \text{span} \left\{ \left( \tau k(z, w) - k(z, \overline{w}) \right) \right\}.$$  

(5.19)

Notice that for any $\tau \in \mathbb{C}$ such that $|\tau| \leq 1$, $S_\tau$ has the spectral properties given in Corollary 4.13. Moreover, for $|\tau| = 1$, $V_\tau$ is isometric and, as a consequence of Theorem 4.7, $S_\tau$ is a selfadjoint extension of $S$.

The other definition of dB space requires the Hardy space

$$\mathcal{H}_2(\mathbb{C}_+) := \left\{ f(z) \text{ holomorphic in } \mathbb{C}_+ : \sup_{y > 0} \int_{\mathbb{R}} |f(x + iy)|^2 < \infty \right\}$$

as well as an Hermite-Biehler function, which is an entire function $e(z)$ satisfying

$$|e(z)| > |e^\#(z)|, \quad z \in \mathbb{C}_+,$$

whence it follows that $e(z)$ is a function without zeros in the half-plane $\mathbb{C}_+$.

The dB space associated with an Hermite-Biehler function $e(z)$ is the linear manifold

$$\mathcal{B}(e) := \left\{ f(z) \text{ entire : } \frac{f(z)}{e(z)}, \frac{f^\#(z)}{e(z)} \in \mathcal{H}_2(\mathbb{C}_+) \right\},$$

equipped with the inner product

$$\langle f(t), g(t) \rangle_e := \int_{\mathbb{R}} \frac{f(t)g(t)}{|e(t)|^2} dt.$$  

Without loss of generality, let us assume that $e(z)$ not only has no zeros in $\mathbb{C}_+$, but also in $\mathbb{R}$. In [26, Sec. 5] (see also [15, Sec. 19]), it is shown that, for any $w \in \mathbb{C}$, the expression

$$k(z, w) = \frac{e^\#(z)e(\overline{w}) - e(z)e^\#(\overline{w})}{2\pi i(z - \overline{w})}$$

(5.20)

is the reproducing kernel of $\mathcal{B}(e)$. Moreover, since $e(z)$ does not have zeros on $\mathbb{C}_+ \cup \mathbb{R}$, $k(z, w)$ has no zeros in $\mathbb{C}_+ \cup \mathbb{R}$ for every $w \in \mathbb{C}_+$.

For a given dB space $\mathcal{B}$ with reproducing kernel $k(z, w)$, if one defines

$$e_{w_0}(z) := \frac{\pi(z - \overline{w}_0)}{(\text{Im} w_0)k(w_0, w_0)} k(z, w_0),$$

(5.21)

28
then \( e_{w_0}(z) \) is an Hermite-Biehler function \([3] \text{ Sec. 2}\) and \( B = B(e_{w_0}) \) isometrically \([4] \text{ Thm. 7}\). The reproducing kernel of \( B(e_{w_0}) \) can be computed using (5.20):

\[
k_{w_0}(z, w_0) = \frac{\# e_{w_0}(z)e_{w_0}(\overline{w_0}) - e_{w_0}(z)e_{w_0}(\overline{w_0})}{2\pi i(z - \overline{w_0})} = \frac{-e_{w_0}(z)}{2\pi i(z - \overline{w_0})} \left( \frac{\pi(\overline{w_0} - w_0)}{(\overline{\text{Im} w_0})k(w_0, w_0)}K(w_0, w_0) \right) = e_{w_0}(z) \frac{w_0}{z - \overline{w_0}}.
\]

Therefore

\[
e_{w_0}(z) = (z - \overline{w_0})k_{w_0}(z, w_0).
\]

(5.22)

The set of associated functions \( \text{Assoc} \) of a dB space \( B \) is given by

\[\text{Assoc} B = B + zB.\]

Therefore, for a \( \tau \in \mathbb{C} \), define

\[\varphi_\tau(z) := \tau e(z) - e^\#(z).\]

(5.23)

These entire functions belongs to \( \text{Assoc} B \) and determine the maximal dissipative extensions of the multiplication operator.

**Theorem 5.3.** Fix \( w \in \mathbb{C}_+ \) and consider the dB space \( B(e_w) \) with \( e_w(z) \) given in (5.21). All the maximal dissipative extension of the operator of multiplication \( S \) (see (5.15)) are in one-to-one correspondence with the set of entire functions \( \varphi_\tau(z) \), \( |\tau| \leq 1 \). These maximal dissipative extensions are given by

\[
S_\tau = \left\{ \left( h_\alpha(z) \right) : h_\alpha(z) = f(z) + \alpha(\tau k_w(z, w) - k_w(z, \overline{w})), \quad f(z) \in \text{dom} \ S \right\}.
\]

(5.24)

Moreover, \( \sigma(S_\tau) = \{ \lambda \in \mathbb{C}_+ \cup \mathbb{R} : \varphi_\tau(\lambda) = 0 \} \). The eigenfunction corresponding to \( \lambda \in \sigma(S_\tau) \) is \( h^{(\tau)}(z) = \frac{\varphi_\tau(z)}{z - \lambda} \).

**Proof.** All dissipative extensions \( S_\tau \) are given by (5.19). If \( \left( \begin{array}{c} h(z) \\ g(z) \end{array} \right) \in S_\tau \), then there exist \( \left( \begin{array}{c} f(z) \\ z f(z) \end{array} \right) \in S \) and \( \alpha \in \mathbb{C} \) such that

\[
\left( \begin{array}{c} h(z) \\ g(z) \end{array} \right) = \left( \begin{array}{c} f(z) + \alpha(\tau k_w(z, w) - k_w(z, \overline{w})) \\ z f(z) + \alpha(\tau k_w(z, w) - w k_w(z, \overline{w})) \end{array} \right).
\]

It follows from (5.22) and (5.23) that

\[
g(z) = zh(z) - \alpha[\tau(z - \overline{w})k_w(z, w) - (z - w)k_w(z, \overline{w})]
= zh(z) - \alpha[\tau e_w(z) - e_w^\#(z)]
= zh(z) - \alpha \varphi_\tau(z),
\]

whence (5.24) follows.

By Corollary 4.13 for every \( \lambda \in \sigma(S_\tau) \) (which is a subset of \( \mathbb{C}_+ \cup \mathbb{R} \) by Theorem 3.1), \( N_\lambda(S_\tau) \) is
one-dimensional. Thus, in view of (5.24), \( \left( \frac{h_\alpha(z)}{\lambda h_\alpha(z)} \right) \in \mathbf{N}_\lambda(S_\tau) \) with \( \alpha \neq 0 \) and
\[
\lambda h_\alpha(z) = zh_\alpha(z) - \alpha \varphi_\tau(z),
\]
so \( \varphi_\tau(\lambda) = 0 \).

On the other hand, if \( \lambda \notin \sigma(S_\tau) \), then \( (S_\tau - \lambda I)^{-1} \in \mathcal{B}(\mathcal{H}) \). Thus
\[
\begin{pmatrix}
g(z) \\
k_w(z,w) + \lambda g(z)
\end{pmatrix}
\in S_\tau.
\]
which implies
\[
\begin{pmatrix}
g(z) \\
k_w(z,w) + \lambda g(z)
\end{pmatrix}
\in S_\tau.
\]
Using again (5.24), one arrives at
\[
k_w(z,w) + \lambda g(z) = zg(z) - \alpha \varphi_\tau(z).
\]
Therefore, for \( z = \lambda \), \( k_w(\lambda, w) = -\alpha \varphi_\tau(\lambda) \). Since \( k_w(z, w) \) has no zeros in \( \mathbb{C}_+ \cup \mathbb{R} \), one concludes that \( \lambda \notin \{ \lambda \in \mathbb{C}_+ \cup \mathbb{R} : \varphi_\tau(\lambda) = 0 \} \).

We have proven that
\[
\hat{\sigma}(S_\tau) \subset \{ \lambda \in \mathbb{C}_+ \cup \mathbb{R} : \varphi_\tau(\lambda) = 0 \} \subset \sigma(S_\tau).
\]
Since the zeros of a nonzero entire function is a discrete set, the spectral core is a discrete set. This implies that \( \rho(S_\tau) = \hat{\rho}(S_\tau) \) because of the maximality of \( S_\tau \).

The fact that \( h(\tau)(z) \) is an eigenfunction of \( S_\tau \) corresponding to the eigenvalue \( \lambda \) is a consequence of (5.25). \( \square \)

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