QUASI-ISOMETRY AND FINITE PRESENTATIONS OF LEFT CANCELLATIVE MONOIDS.

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ABSTRACT. We show that being finitely presentable, and being finitely presentable with solvable word problem are quasi-isometry invariants of finitely generated left cancellative monoids. Our main tool is an elementary, but useful, geometric characterisation of finite presentability for left cancellative monoids. We also give examples to show that this characterisation does not extend to monoids in general, and indeed that properties such as solvable word problem are not isometry invariants for general monoids.

1. INTRODUCTION

A focus of much recent research has been the extent to which geometric methods developed in group theory can be applied to wider classes of monoids and semigroups. A key concept in geometric group theory is that of quasi-isometry: an equivalence relation on the class of metric spaces which captures their “large-scale” geometry.

In recent papers [4, 5], we have introduced notions of quasi-isometry for semimetric spaces (spaces equipped with asymmetric, partially defined distance functions), and hence for monoids. We proved a semigroup-theoretic analogues of the Švarc-Milnor lemma, showing that a monoid acting in a suitably controlled way by isometric embeddings on a semimetric space must be quasi-isometric to that space (see [5, Theorem 4.1] for a precise statement).

One of the main reasons quasi-isometry is important in geometric group theory, is that the quasi-isometry type of a group is a geometric invariant which encapsulates many important algebraic and combinatorial properties of the group. The aim of this note is to show that this is also true for left cancellative monoids. For example, just as for groups, the existence of a finite presentation is a quasi-isometry invariant of left cancellative, finitely generated monoids:

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Theorem A. Let $M$ and $N$ be left cancellative, finitely generated monoids which are quasi-isometric. Then $M$ is finitely presentable if and only if $N$ is finitely presentable.

Similarly, it is known that solvability of the word problem is a quasi-isometry invariant of finitely presented groups [1] (although it remains open if it is a quasi-isometry invariant of more general finitely generated groups [2, Section 3.7]). It transpires that the same holds for left cancellative monoids.

Theorem B. Let $M$ and $N$ be left cancellative, finitely presentable monoids which are quasi-isometric. Then $M$ has solvable word problem if and only if $N$ has solvable word problem.

Our proofs, which are given in Section 3 below, are in spirit similar to those known in the group case. They are not, however, entirely straightforward generalisations, since much of the standard geometric machinery used in the group case must be replaced with “directed” analogues, the theory of which is less well developed.

Our proof methods do not readily generalise to non-left-cancellative monoids, which seem to be more fundamentally “non-geometric” objects, in the sense that relatively little of their structure can be discerned even from their exact Cayley graphs, let alone from their quasi-isometry types. To illustrate this, in Section 4 we exhibit an uncountable family of finitely generated monoids which are pairwise non-isomorphic and differ in important respects (such as for example solvability of the word problem), but which share exactly the same unlabelled Cayley graph. It remains an open question whether finite presentability is even an isometry invariant, let alone a quasi-isometry invariant, for finitely generated monoids in general.

2. Preliminaries

In this section, we briefly recall some basic definitions which are essential for considering monoids as geometric objects.

Let $\mathbb{R}^\infty$ denote the set $\mathbb{R}^{\geq 0} \cup \{\infty\}$ of non-negative real numbers with $\infty$ adjoined. We equip it with the obvious order, addition and multiplication, leaving $0\infty$ undefined. Now let $X$ be a set. A function $d : X \times X \to \mathbb{R}^\infty$ is called a semimetric on $X$ if:

(i) $d(x, y) = 0$ if and only if $x = y$; and

(ii) $d(x, z) \leq d(x, y) + d(y, z)$;

for all $x, y, z \in X$. A set equipped with a semimetric is called a semimetric space. A useful example of a semimetric space is a directed graph, with the distance between two vertices defined to be the length of the shortest directed path between them, or $\infty$ if there is no such path.

Now let $f : X \to Y$ be a map between semimetric spaces $X$ and $Y$. Write $d_X$ and $d_Y$ for the semimetrics on $X$ and $Y$ respectively. If $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$ then $f$ is called an isometric embedding; if in addition $f$ is surjective then $f$ is an isometry. More generally, let $1 \leq \lambda < \infty$, $0 \leq \mu < \infty$ and $0 < \epsilon < \infty$ be constants. The map $f$ is called a $(\lambda, \epsilon)$-quasi-isometric embedding, and $X$ embeds quasi-isometrically in $Y$, if

$$\frac{1}{\lambda} d_X(x, y) - \epsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \epsilon$$
for all $x, y \in X$.

A subset $Z \subseteq Y$ is called $\mu$-quasi-dense if for every $y \in Y$ there exists a $z \in Z$ with $d_Y(y, z) \leq \mu$ and $d_Y(z, y) \leq \mu$. If $f : X \to Y$ is a $(\lambda, \epsilon)$-quasi-isometric embedding and its image is $\mu$-quasi-dense, then $f$ is called a $(\lambda, \epsilon, \mu)$-quasi-isometry, and the spaces $X$ and $Y$ are said to be quasi-isometric. Quasi-isometry forms an equivalence relation on the class of semimetric spaces $[4$, Proposition 1$]$. A semimetric space is called quasi-metric if it is quasi-isometric to a metric space, or equivalently $[4$, Proposition 2$]$ if there are constants $\lambda, \mu < \infty$ such that $d(x, y) \leq \lambda d(y, x) + \mu$ for all points $x$ and $y$.

Now let $M$ be a monoid generated by a finite subset $S$. Then $M$ is naturally endowed with the structure of a directed graph, with vertices the elements of $M$, and an edge from $x$ to $y$ if and only if there is a generator $s \in S$ such that $xs = y$ in $M$. This graph is called the (right) Cayley graph of $M$ with respect to the generating set $S$. The Cayley graph in turn has the structure of a semimetric space, as described above, with $d_S(x, y)$ being the shortest length of a word $w$ over the generating set $S$ such that $xw = y$ in $M$, or $\infty$ if there is no such word.

Of course different choices of finite generating set for $M$ will lead to different graphs and different semimetric spaces, but two different finite generating sets for the same monoid will always give rise to quasi-isometric spaces $[4$, Proposition 4$]$. In other words, provided a monoid admits a finite generating set, its quasi-isometry class is an invariant, and so it makes sense to speak of two abstract finitely generated monoids being quasi-isometric.

Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \prec g$ if there exists a constant $a$ such that $f(j) \leq ag(aj) + aj$ for all $j$. The functions $f$ and $g$ are said to be of the same type, written $f \sim g$, if $f \prec g$ and $g \prec f$.

Now fix a monoid presentation $\langle A \mid R \rangle$. If $u$ and $v$ are equivalent words then the area $A(u, v)$ is the smallest number of applications of relations from $R$ necessary to transform $u$ into $v$. The Dehn function of a presentation $\langle A \mid R \rangle$ is the function $\delta : \mathbb{N} \to \mathbb{N}$ given by

$$\delta(n) = \max \{ A(u, v) \mid u, v \in A^*, u \equiv_R v, |u| + |v| \leq n \}.$$  

The Dehn function is a measure of the complexity of transformations between equivalent words. The Dehn function depends on the presentation, but if $\delta$ and $\gamma$ are Dehn functions of different finite presentations for the same monoid then $\delta \sim \gamma$.

3. Geometric Nature of Finite Presentability

In this section we describe an elementary, but very useful, geometric property which, when applied to Cayley graphs, characterises finite presentability for left cancellative monoids. We then show that this property is invariant under quasi-isometry, from which follows the result that finite presentability is a quasi-isometry invariant of finitely generated left cancellative monoids.

We shall need the notion of a directed 2-complex, which was introduced by Guba and Sapir $[9]$. For every directed graph $\Gamma$ let $P(\Gamma)$ be the set of all directed paths in $\Gamma$, including the empty paths. We write $\iota p$ and $\tau p$ for the
start and end vertex respectively of a path \( p \). A pair of paths \( p, q \in P(\Gamma) \) are said to be \textit{parallel}, written \( p \parallel q \), if \( \iota p = \iota q \) and \( \tau p = \tau q \).

A \textit{directed 2-complex} is a directed graph \( \Gamma \) equipped with a set \( F \) (called the \textit{set of 2-cells}), and three maps \( \lfloor \cdot \rceil : F \to P, \lceil \cdot \rceil : F \to P, \) and \( -1 : \to F \) called \textit{top}, \textit{bottom}, and \textit{inverse} such that

- for every \( f \in F \), the paths \( \lfloor f \rceil \) and \( \lceil f \rceil \) are parallel;
- \( -1 \) is an involution without fixed points, and \( \lceil f^{-1} \rceil = \lceil f \rceil, \lfloor f^{-1} \rceil = \lfloor f \rceil \) for every \( f \in F \).

If \( K \) is a directed 2-complex, then paths on \( K \) are called \textit{1-paths}. The initial and terminal vertex of a 1-path \( p \) are denoted by \( \iota(p) \) and \( \tau(p) \), respectively. For every 2-cell \( f \in F \), the vertices \( \iota([f]) = \iota(\lfloor f \rceil) \) and \( \tau([f]) = \tau(\lceil f \rceil) \) are denoted \( \iota(f) \) and \( \tau(f) \), respectively.

An \textit{atomic 2-path} is a triple \( (p, f, q) \), where \( p, q \) are 1-paths in \( K \), and \( f \in F \) such that \( \tau(p) = \iota(f) \), \( \tau(f) = \iota(q) \). If \( \delta \) is an atomic 2-path then we use \( [\delta] \) to denote \( p \lfloor f \rceil q \) and \( [\delta] \) is denoted by \( p \lfloor f \rceil q \), these are the top and bottom 1-paths of the atomic 2-path. A 2-path \( \delta \) in \( K \) of length \( n \) is then a sequence of atomic paths \( \delta_1, \ldots, \delta_n \), where \( [\delta_i] = [\delta_{i+1}] \) for every \( 1 \leq i < n \). The top and bottom 1-paths of \( \delta \), denoted \( \lceil \delta \rceil \) and \( \lfloor \delta \rceil \) are then defined as \( \lceil \delta_1 \rceil \) and \( \lfloor \delta_n \rceil \), respectively.

We use \( \delta \circ \delta' \) to denote the composition of two 2-paths. We say that 1-paths \( p, q \) in \( K \) are \textit{homotopic} if there exists a 2-path \( \delta \) such that \( [\delta] = p \) and \( [\delta] = q \). We say that a directed 2-complex \( K \) is \textit{directed simply connected} if for every pair of parallel paths \( p \parallel q, p, q \) are homotopic in \( K \).

Let \( K \) be a directed 2-complex with underlying directed graph \( \Gamma \) and set of 2-cells \( F \). Let \( p \) and \( q \) be parallel paths in \( \Gamma \), and let \( K' \) be the 2-complex obtained from \( K \) by adjoining two new elements \( f \) and \( f' \) to \( F \) satisfying \( \lfloor f \rceil = \lfloor f' \rceil = p, \lceil f \rceil = \lceil f' \rceil = q \). We call \( K' \) the directed 2-complex \textit{obtained from} \( K \) by adjoining cells for the paths \( p \) and \( q \).

Given a directed graph \( \Gamma \) and natural number \( n \) we define a directed 2-complex \( K_n(\Gamma) \) with face set

\[ F = \{ (p, q) \mid p \text{ and } q \text{ are parallel paths in } \Gamma \text{ with } |p| + |q| \leq n \} \]

and \( \lfloor (p, q) \rceil = p, \lceil (p, q) \rceil = q \) and \( (p, q)^{-1} = (q, p) \). For \( n \in \mathbb{N} \), we say that a directed graph \( \Gamma \) is \textit{n-quasi-simply-connected} if \( K_n(\Gamma) \) is directed simply connected. We say that \( \Gamma \) is \textit{quasi-simply-connected} if it is \( n \)-quasi-simply-connected for some \( n \in \mathbb{N} \).

The directed 2-complex \( K_n(\Gamma) \) is the natural directed analogue of the \textit{Rips complex}, and Theorem 3.1 below is the cancellative monoid analogue of the well-known result in geometric group theory which states that a group \( G \) with generating set \( A \) is finitely presented if and only if the Rips complex \( Rips_r(G, A) \) is simply connected for \( r \) large enough; see [3, Chapter 4].

Let \( K_n(\Gamma) \) be a directed simply connected 2-complex. For each pair of parallel paths \( p \parallel q \) in \( \Gamma \) define the area \( A_{K_n(\Gamma)}(p, q) \) to be the minimum length of a 2-path from \( p \) to \( q \) in \( K_n(\Gamma) \). The Dehn function \( \gamma : \mathbb{N} \to \mathbb{Z}^+ \cup \{ \infty \} \) of \( K_n(\Gamma) \) is defined by

\[ \gamma(i) = \sup\{ A_{K_n(\Gamma)}(p, q) \text{ in } K_n(\Gamma) : p, q \in \Gamma, p \parallel q, |p| + |q| \leq i \}, \]

where the supremum of an unbounded set is taken to be \( \infty \).
Theorem 3.1. Let \( S \) be a left cancellative monoid generated by a finite set \( A \). Then \( S \) is finitely presented if and only if the right Cayley graph \( \Gamma_r(S, A) \) is quasi-simply-connected. Moreover, if \( \Gamma_r(S, A) \) is \( n \)-quasi-simply-connected then \( K_n(\Gamma_r(S, A)) \) has Dehn function equivalent to the Dehn function of \( S \).

Proof. First suppose that \( S \) is presented by a finite presentation \( (A \mid R) \), with Dehn function \( \delta : \mathbb{N} \rightarrow \mathbb{N} \). Then it is not hard to see that the right Cayley graph \( \Gamma = \Gamma_r(S, A) \) is quasi-simply-connected with

\[
    n = \max \{|u| + |v| : (u = v) \in R\}.
\]

Indeed, let \( p, q \in P(\Gamma) \) with \( p \parallel q \) and let \( w_p \) and \( w_q \) be the words labelling the paths \( p \) and \( q \) respectively. Since \( S \) is left cancellative, \( w_p = w_q \) in \( S \), and so there is a finite sequence of applications of relations from \( R \) that transforms \( w_p \) into \( w_q \). Moreover, this sequence can be chosen to have length at most \( \delta(|w_p| + |w_q|) = \delta(|p| + |q|) \). This sequence gives rise in a natural way to a 2-path, of the same length, in \( K_n(\Gamma) \) from \( p \) to \( q \). Thus, \( K_n(\Gamma) \) is directed simply connected with Dehn function bounded above by \( \delta \).

Conversely, suppose we are given that \( \Gamma = \Gamma_r(S, A) \) is \( n \)-quasi-simply-connected. Suppose \( K_n(\Gamma) \) has Dehn function bounded above by \( \omega : \mathbb{N} \rightarrow \mathbb{N} \). Let \( R \) be the set of all relations \( u = v \) over \( A \) holding in \( S \) with \( |u| + |v| \leq n \). Since \( A \) is finite, \( R \) is finite. We claim that \( (A \mid R) \) defines the monoid \( S \). By definition all of these relations hold in \( S \), so we need only show this set of relations is sufficient to define \( S \). Given \( \alpha, \beta \in A^* \) such that \( \alpha = \beta \) in \( S \), let \( p_\alpha, p_\beta \) be the paths in \( \Gamma \) labelled by \( \alpha, \beta \) respectively and with \( \omega p_\alpha = \omega p_\beta = 1 \) and \( \tau p_\alpha = \tau p_\beta = \alpha = \beta \). By assumption \( K_n(\Gamma) \) is directed simply connected, so there is a 2-path from \( p_\alpha \) to \( p_\beta \) in \( K_n(\Gamma) \), of length at most \( \omega(|p_\alpha| + |p_\beta|) = \omega(|\alpha| + |\beta|) \).

Now for any face \( f \) in this 2-path, \( |f| \) and \( [f] \) are parallel paths in \( \Gamma \), which since \( S \) is left cancellative means that their labels represent the same element of \( S \). Moreover, since \( f \) is a face in \( K_n(\Gamma) \), their labels have total length less than \( n \), and hence form the two sides of a relation in \( R \). It follows that the 2-path corresponds to a sequence of applications of relations from \( R \) which transforms the word \( \alpha \) into the word \( \beta \).

Moreover, this sequence has length at most \( \omega(|\alpha| + |\beta|) \). Thus, \( S \) is finitely presented with Dehn function bounded above by \( \omega \). Finally, since \( S \) is finitely presented, we may now apply the first part of the proof again to deduce that \( \omega \) is bounded above by the Dehn function for the presentation, which means that the two Dehn functions are equal.

It is natural to ask to what extent the left cancellativity assumption in the above theorem really is necessary. As it turns out, for finitely generated monoids in general being quasi-simply connected is neither a necessary nor a sufficient condition for the existence of a finite presentation. In Section 4 below, we shall see an example of a finitely generated monoid with Cayley graph which is a directed tree, but which is not finitely presented. This shows that being quasi-simply-connected is not sufficient to imply a finite presentation in general. Also in Section 4 we shall construct an example of a finitely presented monoid whose Cayley graph is not quasi-simply connected.
We shall now show that quasi-simply-connectedness is a quasi-isometry invariant of directed graphs, from which it will follow that finite presentability is a quasi-isometry invariant of finitely generated left cancellative monoids.

The following general lemma will prove useful for us.

**Lemma 3.2 (Quasi-inverses).** Let \( X \) and \( Y \) be quasi-isometric semimetric spaces. Then there exist constants \( \lambda, \epsilon \) and \( \mu \) and a pair of \((\lambda, \epsilon, \mu)\)-quasi-isometries \( f : X \rightarrow Y \) and \( g : Y \rightarrow X \) satisfying the following properties:

(i) \( d(y, fg(y)) \leq \mu \) and \( d(fg(y), y) \leq \mu \) for all \( y \in Y \);
(ii) \( d(x, gf(x)) \leq \mu \) and \( d(gf(x), x) \leq \mu \) for all \( x \in X \);
(iii) \( gfg(y) = g(y) \) for all \( y \in Y \);
(iv) \( fgf(x) = f(x) \) for all \( x \in X \).

**Proof.** Let \( g : Y \rightarrow X \) be a \((\lambda', \epsilon', \mu')\)-quasi-isometry. For every point \( x \in X \) choose and fix \( \hat{x} \in \text{im}(g) \) satisfying \( d(x, \hat{x}) \leq \mu' \) and \( d(\hat{x}, x) \leq \mu' \). Now define a map \( f : X \rightarrow Y \) by choosing for each \( z \in \text{im}(g) \) a point \( f(z) \) such that \( g(f(z)) = z \), and then extend to the whole of \( X \) by setting \( f(x) = f(\hat{x}) \) for all \( x \in X \).

Then straightforward calculations show that for all \( a, b \in X \) we have

\[
d(f(a), f(b)) \geq \frac{1}{\lambda'} d(a, b) - \frac{(\epsilon' + 2\mu')}{\lambda'},
\]

and

\[
d(f(a), f(b)) \leq \lambda' d(a, b) + \lambda'(\epsilon' + 2\mu')
\]

so that \( f \) is a quasi-isometric embedding. Also, for all \( y \in Y \),

\[
d(y, f(g(y))) \leq \epsilon' \quad \text{and} \quad d(f(g(y)), y) \leq \epsilon'
\]

therefore \( \text{im} f \) is quasi-dense and \( f \) is a \((\lambda', \sigma, \epsilon' + 1)\)-quasi-isometry where

\[
\sigma = \max \left( \frac{(\epsilon' + 2\mu')}{\lambda'}, \lambda'(\epsilon' + 2\mu') \right).
\]

Moreover, for all \( x \in X \) and \( y \in Y \) we have

\[
d(x, gf(x)) \leq \mu', \quad d(gf(x), x) \leq \mu',
\]

\[
gfg(y) = g(y) \quad \text{and} \quad fgf(x) = f(x).
\]

It follows that \( f \) and \( g \) are \((\lambda, \epsilon, \mu)\)-quasi-isometries satisfying the conditions given in the statement of the lemma where

\[
\lambda = \lambda', \quad \epsilon = \max(\epsilon', \sigma), \quad \text{and} \quad \mu = \max(\mu', \epsilon' + 1),
\]

as required. \( \square \)

**Lemma 3.3.** Let \( \Gamma \) and \( \Delta \) be simple directed graphs and let \( f : \Gamma \rightarrow \Delta \) and \( g : \Delta \rightarrow \Gamma \) be \((\lambda, \epsilon, \mu)\)-quasi-isometries satisfying (i)-(iv) from Lemma 3.2. Suppose \( K_m(\Gamma) \) is directed simply connected. Then \( K_m(\Delta) \) is directly simply connected where \( m = \max(\lambda^2 + (\lambda + 1)\epsilon + 2\mu + 1, (\lambda + \epsilon)n) \).

If, moreover, \( K_m(\Gamma) \) and \( K_m(\Delta) \) have Dehn functions \( \gamma \) and \( \delta \) respectively, then \( \delta \sim \gamma \).
Proof. For each arc \( a \) from \( a \) to \( b \) in \( \Gamma \), where \( a \) and \( b \) are vertices, choose and fix a geodesic path \( \pi(\gamma) \) in \( \Delta \) from \( f(a) \) to \( f(b) \); note that \( d(f(a), f(b)) \leq \lambda + \varepsilon \), so such a geodesic exists. The map \( \pi \) extends naturally to a map from \( P(\Gamma) \) to \( P(\Delta) \) which we also denote by \( \pi \). Let \( F \) and \( E \) be the sets of 2-cells of \( K_n(\Gamma) \) and \( K_m(\Delta) \) respectively. By definition of \( m \) (considering the right hand term) for all \( e \in F \) there exists \( e \in E \) such that \( [e] = \pi([f]) \), \( [e] = \pi([f]) \). For each \( f \in F \) choose and fix such an \( e \in E \) and define \( \pi(f) = e \). For each atomic 2-path \((p, f, q)\) of \( K_n(\Gamma) \), define \( \pi(p, f, q) = (\pi(p), \pi(f), \pi(q)) \). Clearly this is an atomic 2-path of \( K_m(\Delta) \). This now extends in an obvious way to a mapping \( \pi \) from 2-paths of \( K_n(\Gamma) \) to 2-paths of \( K_m(\Delta) \). In other words, \( \pi : K_n(\Gamma) \to K_m(\Delta) \) is a morphism of directed 2-complexes (in the sense of [\( \square \] Section 5)).

Now let \( p, q \in P(\Gamma) \) with \( p \parallel q \). By assumption there is a 2-path in \( K_n(\Gamma) \) from \( p \) to \( q \). The image of this 2-path under \( \pi \) is then a 2-path in \( K_m(\Delta) \) from \( \pi(p) \) to \( \pi(q) \), of the same length.

Next suppose \( r \) and \( s \) are parallel paths in \( \Delta \). Suppose the vertices of these paths, in order, are \( r_0, r_1, \ldots, r_c \) and \( s_0, s_1, \ldots, s_d \) respectively. Then \( r_0 = s_0 \), \( r_c = s_d \). For each \( i \) let \( \sigma_i \) be a path in \( \Delta \) from \( r_i \) to \( f g(r_i) \) with \( |\sigma_i| \leq \mu \) and let \( \sigma_i^{-1} \) be a path in \( \Delta \) from \( f g(r_i) \) to \( r_i \) with \(|\sigma_i^{-1}| \leq \mu \). For \( 0 \leq i < c \), choose a geodesic \( \tau_i \) in \( \Gamma \) from \( g(r_i) \) to \( g(r_{i+1}) \). Similarly, for each \( 0 \leq j < d \), choose a geodesic \( \zeta_j \) in \( \Gamma \) from \( g(s_j) \) to \( g(s_{j+1}) \).

Let \( \tau = \tau_0 \ldots \tau_{c-1} \) and \( \zeta = \zeta_0 \ldots \zeta_{d-1} \). Since \( g \) is a \((\lambda, \varepsilon, \mu)\)-quasi-isometry and we have \( |\tau_i| = d(g(r_i), g(s_{i+1})) \leq \lambda + \varepsilon \), and hence \( |\tau| \leq c(\lambda + \varepsilon) \). Similarly, \( |\zeta| \leq d(\lambda + \varepsilon) \). Now \( \tau \) and \( \zeta \) are parallel paths of length \( \tau \) in \( \Gamma \) and hence in \( K_n(\Gamma) \). Since \( K_n(\Gamma) \) is directed simply connected, this means there is a 2-path \( \phi \) from \( \tau \) to \( \zeta \). And since \( \delta \) is the Dehn function of \( K_n(\Gamma) \), we may choose \( \phi \) of length at most \( \delta(|\tau| + |\zeta|) \leq \delta((c + d)(\lambda + \varepsilon)) \).

Now by the above observations, \( \pi(\phi) \) is a 2-path from \( \pi(\tau) \) to \( \pi(\zeta) \), of length at most \( \delta((c + d)(\lambda + \varepsilon)) \).

Moreover, by definition of \( m \) (considering the left hand term) there exists \( e_0 \in E \) with
\[
[e_0] = s_0 \pi(\tau_0) \sigma_1^{-1}, \quad [e_0] = (r_0, r_1),
\]
and for \( 1 \leq i \leq c - 1 \), there exist \( e_i \in E \) such that
\[
[e_i] = \pi(\tau_i) \sigma_i^{-1}, \quad [e_i] = \sigma_i^{-1} \circ (r_i, r_{i+1}).
\]
These combine, as illustrated in Figure\( [\Pi] \) to give an atomic 2-path of length \( c \) from \( r \) to \( s \). An entirely similar argument shows that there is a 2-path of length \( d \) from \( s_0 \pi(\zeta) \sigma_c^{-1} \) to \( s \), and we have already seen that there is a 2-path of length at most \( \delta((c + d)(\lambda + \varepsilon)) \) from \( \pi(\tau) \) to \( \pi(\zeta) \), and hence there is a 2-path of the same length from \( s_0 \pi(\tau) \sigma_c^{-1} \) to \( s_0 \pi(\zeta) \sigma_c^{-1} \). Thus, there is a 2-path of length at most
\[
c + d + \delta((c + d)(\lambda + \varepsilon))
\]
where \( c + d = |r| + |s| \). This shows that \( K_m(\Delta) \) is directed simply connected and \( \delta \prec \gamma \) as required. Moreover, now we know that \( K_m(\Delta) \) is directed simply connected, we may apply what we have proved with \( \Gamma \) and \( \Delta \), to yield \( \gamma \prec \delta \) and hence \( \gamma \sim \delta \).

We are now ready to prove our main theorems.
\textbf{Theorem A.} Let $M$ and $N$ be left cancellative, finitely generated monoids which are quasi-isometric. Then $M$ is finitely presentable if and only if $N$ is finitely presentable.

\textit{Proof.} It follows from Lemmas 3.2 and 3.3 that the property of being quasi-simply-connected is a quasi-isometry invariant of directed graphs. The result then follows by applying Theorem 3.1. \hfill $\Box$

\textbf{Theorem B.} Let $M$ and $N$ be left cancellative, finitely presentable monoids which are quasi-isometric. Then $M$ has solvable word problem if and only if $N$ has solvable word problem.

\textit{Proof.} Let $\langle A \mid R \rangle$ and $\langle B \mid S \rangle$ be finite presentations for $M$ and $N$ respectively, and let $\delta$ and $\gamma$ be the Dehn functions of these presentations respectively. Then by Theorem 3.1 there is an $n$ such that $K_n(\Gamma_r(A, R))$ and $K_n(\Gamma_r(B, S))$ are directed simply connected and moreover, if we let $\delta'$ and $\gamma'$ be the Dehn functions of these graphs, then $\delta \sim \delta'$ and $\gamma \sim \gamma'$. Now by Lemmas 3.2 and 3.3 we have $\gamma \sim \gamma' \prec \delta' \sim \delta$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{An illustration of the proof of Lemma 3.3.}
\end{figure}

There is a 2-path from $r$ to $\sigma_0 \pi(\tau) \sigma_c^{-1}$ given by first replacing $(r_0, r_1)$ by $\sigma_0 \pi(\tau_0) \sigma_1^{-1}$, then $\sigma_1^{-1}(r_1, r_2)$ by $\pi(\tau_1) \sigma_2^{-1}$, and so on. In a similar way one constructs a 2-path from $s$ to $\sigma_0 \pi(\zeta) \sigma_c^{-1}$. 

Theorem A. Let $M$ and $N$ be left cancellative, finitely generated monoids which are quasi-isometric. Then $M$ is finitely presentable if and only if $N$ is finitely presentable.

Proof. It follows from Lemmas 3.2 and 3.3 that the property of being quasi-simply-connected is a quasi-isometry invariant of directed graphs. The result then follows by applying Theorem 3.1. \hfill $\Box$

Theorem B. Let $M$ and $N$ be left cancellative, finitely presentable monoids which are quasi-isometric. Then $M$ has solvable word problem if and only if $N$ has solvable word problem.

Proof. Let $\langle A \mid R \rangle$ and $\langle B \mid S \rangle$ be finite presentations for $M$ and $N$ respectively, and let $\delta$ and $\gamma$ be the Dehn functions of these presentations respectively. Then by Theorem 3.1 there is an $n$ such that $K_n(\Gamma_r(A, R))$ and $K_n(\Gamma_r(B, S))$ are directed simply connected and moreover, if we let $\delta'$ and $\gamma'$ be the Dehn functions of these graphs, then $\delta \sim \delta'$ and $\gamma \sim \gamma'$. Now by Lemmas 3.2 and 3.3 we have $\gamma \sim \gamma' \prec \delta' \sim \delta$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{An illustration of the proof of Lemma 3.3.}
\end{figure}
Since $M$ has solvable word problem, $\delta$ is bounded above by a recursive function. Hence so is $\gamma$, and so $N$ has solvable word problem. \qed

4. The Non-Cancelative Case

In this section we present some examples showing that the theory developed above is very far from being extendable to arbitrary finitely generated monoids. We begin by giving examples which show that, for finitely generated monoids in general, being quasi-simply-connected is neither a necessary nor sufficient condition for the existence of a finite presentation.

First, let us see how to construct an example of a finitely presented monoid whose Cayley graph is not quasi-simply connected. This serves as an instructive example of how intuitions from group theory can fail in a more general setting. It would seem intuitively clear that if a monoid is finitely presented then one should be able to use the relations to build 2-paths between arbitrary pairs of parallel paths in the Cayley graph. Indeed this is true for 2-paths whose origin is the identity element of the monoid, but in general there are many more 2-paths than that in the Cayley graph, and without left cancellativity the idea of filling in parallel paths with relations loses sense since one has to “trace back” to the identity of the monoid in order to find two words that are equal before one can start applying relations from the presentation.

Before presenting the example we shall need to introduce a some basic notions from the structure theory of semigroups. Recall that Green’s relations $\mathcal{L}$ and $\mathcal{R}$ are defined on any semigroup $S$ by $a \mathcal{L} b$ [respectively, $a \mathcal{R} b$] if either $a = b$ or there exist elements $c, d \in S$ with $a = cb$ and $b = da$ [respectively, $a = bc$ and $b = ad$]. Green’s relation $\mathcal{H}$ is defined by $a \mathcal{H} b$ if and only if $a \mathcal{L} b$ and $a \mathcal{R} b$. All three relations are equivalence relations. Notice it is immediate from the definitions that the $\mathcal{R}$-classes of a finitely generated monoid (that is, equivalence classes of the relation $\mathcal{R}$) are exactly the strongly connected components of the Cayley graph. It is well known that the $\mathcal{H}$-class of any idempotent is a maximal subgroup. The notion of Schützenberger group gives a useful way to associate a group to an $\mathcal{H}$-class not containing an idempotent. Let $H$ be an $\mathcal{H}$-class of $S$, and let $\text{Stab}(H) = \{s \in S : sH = H\}$ denote the (left) stabilizer of $H$ under the action of $S$. We define an equivalence $\sigma = \sigma(H)$ on the stabilizer by $(x, y) \in \sigma$ if and only if $xh = xy$ for all $h \in H$. It is straightforward to verify that $\sigma$ is a congruence, and that $\mathcal{G}(H) = \text{Stab}(H)/\sigma$ is a group, called the left Schützenberger group of $H$. One can also define the right Schützenberger group of $H$ in the natural way, and it turns out that the left and right Schützenberger groups are isomorphic to one another. For information about the basic properties of Schützenberger groups we refer the reader to [7, Section 2.3].

Let $R$ be an $\mathcal{R}$-class of $H$. The (right) Schützenberger graph $\Gamma(R, A)$ of $R$, with respect to $A$, is the strongly connected component of $h \in H$ in $\Gamma(M, A)$. It is easily seen to consist of those vertices which are elements of $R$, together with edges connecting them, and so can be obtained by beginning with a directed graph with vertex set $R$ and a directed labelled edge from $x$ to $y$ labelled by $a \in A$ if and only if $xa = y$. From its construction it is clear
that for any generating set $A$ of $M$, $\Gamma(R, A)$ may be viewed as a connected geodesic semimetric space.

The following observation results from the fact that the property of being quasi-simply-connected is inherited by the strongly connected components of a directed graph (by which we mean the subdigraphs induced on the equivalence classes of vertices where two vertices $u$ and $v$ are in the same class if there is a directed path from $u$ to $v$ and a directed path back from $v$ to $u$).

**Proposition 4.1.** Let $S$ be a monoid generated by a finite set $A$ and let $R$ be an $R$-class of $S$ with Schützenberger graph $\Delta(R)$. If $\Gamma_r(S, A)$ is $n$-quasi-simply-connected then $\Delta(R)$ is $n$-quasi-simply-connected.

**Proof.** By definition $\Delta(R)$ is a strongly connected component of the digraph $\Gamma_r(S, A)$. Suppose $p$ and $q$ are parallel paths in $\Delta(R) \subseteq \Gamma_r(S, A)$. Since $\Gamma_r(S, A)$ is $n$-quasi-simply connected there is a 2-path $\delta$ in $K_n(\Gamma_r(S, A))$ from $p$ to $q$. Since $\Delta(R)$ is a strongly connected component of $\Gamma_r(S, A)$, all 1-paths featuring in $\delta$ lie inside $\Delta(R)$ and hence in $K_n(\Delta(R))$. But now by the definition of $K_n(\Delta(R))$, all faces featuring in $\delta$ lie in $K_n(\Delta)$, and so $\delta$ is also a 2-path in $K_n(\Delta(R))$ from $p$ to $q$. \qed

**Corollary 4.2.** Let $S$ be a monoid generated by a finite set $A$ and let $H$ be an $H$-class of $S$. If $S$ is quasi-simply connected and the $R$-class $R$ of $H$ contains only finitely many $H$-classes, then the Schützenberger group $G(H)$ is finitely presented.

**Proof.** The Schützenberger group $G(H)$ acts naturally on the Schützenberger graph $\Delta(R)$ in such a way that applying the Švarc-Milnor lemma for groups acting on semimetric spaces [11 Theorem 1] it follows that $G(H)$ is a finitely generated group which is quasi-isometric to $\Delta(R)$; see [4 Section 5]. By Proposition 4.1 $\Delta(R)$ is quasi-simply connected. Since they are quasi-isometric, the group $G(H)$ is quasi-simply connected by Lemma 3.3 which, by Theorem 5.1 implies that the group $G(H)$ is finitely presented. \qed

**Proposition 4.3.** There exists a finitely presented monoid $S$ whose Cayley graph is not quasi-simply connected.

**Proof.** Let $A$ be the alphabet

$$\{a_1, a_2, a_3, a_4, a'_1, a'_2, a'_3, a'_4, b, c, d\}$$

and consider the presentation

$$\langle A \mid a_ja'_j = a'_ja_j = \epsilon, a_1a_2 = a_3a_4, \quad a_jb = ba_j^2, \quad cb^2 = cb, \quad a_jd = da_j, \quad cbsd = a_jcbd (j = 1, 2, 3, 4) \rangle.$$ 

Let $S$ be the monoid defined by this presentation, and let $H$ be the $H$-class of $h \equiv cbd$. In [10] using Reidemeister-Schreier rewriting methods it is shown that the Schützenberger group $G(H)$ is defined by the following group presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1^{a_i}a_2^{a'_i} = a_3^{a'_i}a_4^{a_i} (i = 0, 1, 2, \ldots) \rangle$$

which is not finitely presented since it is an amalgamated product of two free groups of rank two with a free group of infinite rank amalgamated. It is also
shown in [10] that the $H$-class $H$ is the unique $H$-class in its $R$-class. Therefore, by the contrapositive to Corollary 4.2 the finitely presented monoid $S$ is not quasi-simply connected.

Next, we exhibit an uncountable family of (non-left-cancelative) finitely generated monoids which are pairwise non-isomorphic, but nevertheless all share the same unlabelled Cayley graph. This Cayley graph will turn out to be isomorphic to a directed rooted tree with all vertices having out-degree 4 or 5. We show that this family contains examples of monoids with solvable word problem, and monoids with word problem neither recursively enumerable nor co-recursively enumerable. This shows that the solvability of the word problem for general monoids is a fundamentally “non-geometric” property which cannot be seen in a Cayley graph.

We take the set $\mathbb{N}$ of natural numbers including 0. For each non-empty proper subset $X$ of $\mathbb{N}$, we define a finitely generated monoid $M(X)$ by the following infinite presentation:

$$\langle a, b, c, d, e \mid ab^i c = ab^i d \ (i \in X), \ ab^j c = ab^j e \ (j \notin X) \rangle.$$ 

We remark that since the monoids $M(X)$ are given by homogeneous presentations with finitely many generators, they are residually finite. Indeed, any finite set of elements can be separated by a Rees quotient factoring out the ideal consisting of all elements whose representatives have length $n$ or more, for some sufficiently large $n$.

It is easy to show from the definition that the word problems for monoids in this class can belong to a broad range of computability and complexity classes:

**Proposition 4.4.** The word problem for $M(X)$ is linear-time Turing equivalent to the membership problem for $X$ in unary coding. In particular, the word problem for $M(X)$ is decidable if and only if $X$ is a recursive subset of $\mathbb{N}$.

**Proof.** Given a solution to the word problem for $M(X)$, one may decide whether $n \in \mathbb{N}$ by simply checking whether $ab^n c = ab^n d$ in $M(X)$.

Conversely, it is easy to see that the defining presentation for $M(X)$ yields an (infinite) terminating, convergent writing system:

$$ab^i c \rightarrow ab^i d \text{ for } i \in X, \quad ab^j c \rightarrow ab^j e \text{ for } j \notin X.$$ 

Clearly, it is an easy matter to check if a given word is left-hand-side of a rule. Given an algorithm for membership of $X$, which can check of which rule a given word is the left-hand-side. Thus, we can compute a normal form for a word $u$ by iteratively checking its (finitely many) factors to see if any is the left-hand-side of a rule, and if so applying the rule. In fact, the complete lack of overlap between left-hand-sides and right-hand-sides of rules means that this procedure can be performed by a single pass from left to right across $u$, and the sum of all the values for which membership of $X$ must be checked will not exceed the length of $u$.

**Proposition 4.5.** The word problem for $M(X)$ is recursively enumerable if and only if it is co-recursively enumerable.
Proof. If the word problem for $M(X)$ is recursively enumerable, then by the same argument as in the first part of the proof of Proposition 4.4, so is $X$. Now notice that $M(X)$ is isomorphic to $M(\mathbb{N} \setminus X)$, via the map exchanging $d$ and $e$. Since a recursively enumerable word problem is an isomorphism invariant, the word problem for $M(\mathbb{N} \setminus X)$ is recursively enumerable, and by the same argument as above, so is $\mathbb{N} \setminus X$. Thus, $X$ is recursive, and so by Proposition 4.4 $M(X)$ has solvable word problem. A dual argument shows that $M(X)$ is co-recursively enumerable if and only if it is recursive. □

**Proposition 4.6.** For any subsets $X$ and $Y$ of $\mathbb{N}$, the semigroups $M(X)$ and $M(Y)$ are isometric to each other, and to a directed rooted tree in which every vertex has outdegree 4 or 5.

Proof. It follows from the confluence and termination of the rewriting system in the proof of Proposition 4.4 that $N = A^* \setminus (A^*ab^icA^*)$ ($i \in \mathbb{N}$), where $A = \{a, b, c, d, e\}$ is a set of unique normal forms for the elements of $M(X)$. Note that this set is independent of the choice of the $X$. Moreover, given a normal form $u$, the normal forms for elements of the form $ux$ with $x$ a generator are:

- $ua, ub, ud, ue$ if $u = u'ab^i$ for some $u' \in A^*$ and $i \in \mathbb{N}$; or
- $ua, ub, uc, ud$ and $ue$, otherwise.

From this it is immediate that the Cayley graph of $M(X)$ is a rooted directed tree in which every vertex has outdegree 4 or 5. Notice, moreover, that the normal forms to which the normal form $u$ is connected in the Cayley graph are independent of $X$. It follows that the identity map on normal forms induces an isometry between $M(X)$ and $M(Y)$ for any subsets of $X$ and $Y$ of $\mathbb{N}$. □

We can also use this example to show that finite presentability is not a quasi-isometry invariant of finitely generated monoids considered with the (symmetric) metric induced by its Cayley graph regarded as an undirected graph. An immediate corollary of Proposition 4.6 is that the undirected Cayley graphs of the monoids of the form $M(X)$ are all isometric, and are all trees in which every vertex has degree 5 or 6.

It is a well-known, if at first a little surprising, fact in geometric group theory that any free group of finite rank exceeding 2 is quasi-isometric to the free group of rank 2. This is actually a special case of a more general phenomenon involving quasi-isometries between locally finite trees.

The simplest non-elementary Gromov hyperbolic metric spaces are homogeneous simplicial trees $T$ of constant valency $\geq 3$. One interesting feature of such geometries is that all trees with constant valency $\geq 3$ are quasi-isometric to each other. Indeed, as observed in [9, Section 2.1] and [8], each such tree is quasi-isometric to any tree $T$ satisfying the following properties:

- $T$ has bounded valency, meaning that vertices have uniformly finite valency; and
- $T$ is bushy, meaning that each point of $T$ is a uniformly bounded distance from a vertex having at least 3 unbounded complementary components.
Lemma 4.7. Let $T_1$ and $T_2$ be locally finite graph theoretic trees. If $T_1$ and $T_2$ have bounded degree and every vertex in $T_1$ and in $T_2$ has degree at least 3 then $T_1$ and $T_2$ are quasi-isometric.

Proof. This follows from the fact that $T_1$ and $T_2$ are both bushy trees. □

Theorem 4.8. For every subset $X$ of the natural numbers, the undirected Cayley graph of the finitely generated monoid $M(X)$ (defined above) is quasi-isometric to the Cayley graph of the free group $F_2$.

In particular finite presentability is not an undirected quasi-isometry invariant of finitely generated monoids.

Proof. We observed above that the right directed Cayley graph $\Gamma(M(X))$ is a directed rooted tree in which every vertex has in degree 1, and out degree either 4 or 5. It follows that the corresponding undirected Cayley graph is a tree in which every vertex has degree 5 or 6. Now the result follows by Lemma 4.7 since the undirected Cayley graphs of $M(X)$ and the free group $F_2$ are both bushy trees. □

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