A Multilateral Bailey Lemma and Multiple Andrews–Gordon Identities

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Abstract

A multilateral Bailey Lemma is proved, and multiple analogues of the Rogers–Ramanujan identities and Euler’s Pentagonal Theorem are constructed as applications. The extreme cases of the Andrews–Gordon identities are also generalized using the multilateral Bailey Lemma where their final form is written in terms of determinants of theta functions.

Keywords: multilateral Bailey Lemma, multiple Andrews–Gordon identities, determinant evaluations, theta functions

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1. Introduction

Let \( B_{k,i}(n) \) denote the number of partitions of \( n \) of the form \((b_1 \ldots b_s)\) where \( b_j - b_{j+k-1} \geq 2 \), and at most \( i-1 \) of \( b_j \) equal 1. Let \( A_{k,i}(n) \) denote the number of partitions of \( n \) into parts that are not equivalent to \( 0, \pm i \pmod{2k+1} \). B. Gordon \cite{Gordon} proved that \( A_{k,i}(n) = B_{k,i}(n) \) for all \( n \), and thus gave a combinatorial generalization for the Rogers–Ramanujan identities.

G. E. Andrews \cite{Andrews} gave an analytic counterpart of Gordon’s result, the Andrews–Gordon identities, and extended Rogers–Ramanujan identities to all odd moduli. These identities may be written in the form

\[
\sum_{n_1 \geq \ldots \geq n_{k-1} \geq 0} \frac{q^{n_1^2 + \ldots + n_{k-1}^2 + n_1 \ldots + n_{k-1}}}{(q)_{n_1-n_2}(q)_{n_2-n_3} \ldots (q)_{n_{k-1}}} = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)} \quad \text{(1)}
\]

where \( i \in [k], \ k \geq 2 \) and \(|q| < 1\). Here, the \( q \)-Pochhammer symbol \((a;q)_\alpha\) is defined formally by

\[
(a;q)_\alpha := \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \quad \text{(2)}
\]

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where the parameters $a, q, \alpha \in \mathbb{C}$, and $(a; q)_\infty$ denotes the infinite product $(a; q)_\infty := \prod_{i=0}^{\infty} (1 - aq^i)$.

The so-called extreme cases ($i = 1$ and $i = k$) of the Andrews–Gordon identities correspond to the $k$–th iteration of the one–parameter Bailey Lemma. The full Andrews–Gordon identities for any $i \in [k]$ may be proved by using the two–parameter Bailey Lemma [1].

This paper is devoted to a multiple analogue of Andrews–Gordon identities associated to the root systems of rank $n$. We first construct a multilateral version of the $BC_n$ Bailey Lemma [10], and then write a multiple analogue of Andrews–Gordon identities. A brief review of the $BC_n$ Bailey Lemma is presented in the next section. First, however, we recall the main properties of the well–poised Macdonald functions $W_{\lambda/\mu}(z; q, p, t; a, b)$ and Jackson coefficients $\omega_{\lambda/\mu}(z; r, q, p, t; a, b)$ used in the construction of the multiple Bailey Lemma [11]. These functions are first introduced in the author’s thesis [9] supervised by R. A. Gustafson.

2. Background

Let $V$ denote the space of infinite lower–triangular matrices whose entries are rational functions over the field $\mathbb{F} = \mathbb{C}(q, p, r, a, b)$ as in [11]. The condition that a matrix $u \in V$ is lower triangular with respect to the partial inclusion ordering $\subseteq$ defined by

$$\mu \subseteq \lambda \Leftrightarrow \mu_i \leq \lambda_i, \quad \forall i \geq 1. \tag{3}$$

can be stated in the form

$$u_{\lambda \mu} = 0, \quad \text{when } \mu \nsubseteq \lambda. \tag{4}$$

The multiplication operation in $V$ is defined by the relation

$$(uv)_{\lambda \mu} := \sum_{\mu \subseteq \nu \subseteq \lambda} u_{\lambda \nu} v_{\nu \mu} \tag{5}$$

for $u, v \in V$. The matrices used in the definition of $BC_n$ Bailey Lemma involve the elliptic well–poised Macdonald functions $W_{\lambda/\mu}$ and elliptic well–poised Jackson coefficients $\omega_{\lambda/\mu}$ on $BC_n$.

2.1. Well–poised Macdonald functions

Recall that an elliptic analogue of the basic factorial is defined in [12] in the form

$$(a; q, p)_m := \prod_{k=0}^{m-1} \theta(aq^m) \tag{6}$$

where $a \in \mathbb{C}$, $m$ is a positive integer and the normalized elliptic function $\theta(x)$ is given by

$$\theta(x) = \theta(x; p) := (x; p)_\infty (p/x; p)_\infty \tag{7}$$
for $x, p \in \mathbb{C}$ with $|p| < 1$. The definition is extended to negative $m$ by setting $(a; q, p)_m = 1/(aq^m; q, p)_{-m}$. Note that when $p = 0$, $(a; q, p)_m$ reduces to the standard (trigonometric) $q$–Pochhammer symbol.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $t \in \mathbb{C}$, define

$$
(a)\lambda = (a; q, p, t)\lambda := \prod_{k=1}^{n} (at^{1-i}; q, p)_{\lambda_i}.
$$

(8)

Note that when $\lambda = (\lambda_1) = \lambda_1$ is a single part partition, then $(a; q, p, t)\lambda = (a; q, p)\lambda_1 = (a)\lambda_1$. The following notation will also be used.

$$(a_1, \ldots, a_k)\lambda = (a_1, \ldots, a_k; q, p, t)\lambda := (a_1)\lambda \cdots (a_k)\lambda.
$$

(9)

Now let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions of at most $n$ parts for a positive integer $n$ such that the skew partition $\lambda/\mu$ is a horizontal strip; i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \lambda_n \geq \mu_n \geq \lambda_{n+1} = \lambda_{n+1} = 0$. Following [11], define

$$
H_{\lambda/\mu}(q, p, t, b) := \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{(\mu_i+\lambda_j)1-i}; q^{(\mu_j+\lambda_i)1-i})_{\mu_i-1-\lambda_j}}{(q^{(\mu_j+\lambda_i)1-i}; q^{(\mu_j+\lambda_i)1-i})_{\mu_i-1-\lambda_j}} \right\} \prod_{1 \leq i < (j-1) \leq n} \frac{(q^{(\mu_i+\lambda_j)1-(j-1)i}; q^{(\mu_j+\lambda_i)1-(j-1)i})_{\mu_i-1-\lambda_j}}{(q^{(\mu_j+\lambda_i)1-(j-1)i}; q^{(\mu_j+\lambda_i)1-(j-1)i})_{\mu_i-1-\lambda_j}}
$$

(10)

and

$$
W_{\lambda/\mu}(x; q, p, t, a, b) := H_{\lambda/\mu}(q, p, t, b) \frac{(x^{-1}, ax)_\lambda(qbx/t, qbx/(ax))_\mu}{(x^{-1}, ax)_\mu(qbx, qbx/(ax))_\lambda} \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{1-2i}q^{2\mu_i})}{\theta(bt^{1-2i})} \frac{(bt^{1-2i})_{\mu_i+\lambda_i+1}}{(bt^{1-2i})_{\mu_i+\lambda_i+1}} \right\} \nu(x)_{\lambda/\mu}
$$

(11)

where $q, p, t, x, a, b \in \mathbb{C}$. The function $W_{\lambda/\mu}(y, z_1, \ldots, z_\ell; q, p, t, a, b)$ is extended to $\ell + 1$ variables $y, z_1, \ldots, z_\ell \in \mathbb{C}$ through the following recursion formula

$$
W_{\lambda/\mu}(y, z_1, z_2, \ldots, z_\ell; q, p, t, a, b) = \sum_{\nu < \lambda} W_{\lambda/\nu}(yt^{-\ell}; q, p, t, at^{2\ell}, bt^{\ell}) W_{\nu/\mu}(z_1, \ldots, z_\ell; q, p, t, a, b).
$$

(12)

We will also need the elliptic Jackson coefficients below. Let $\lambda$ and $\mu$ be again partitions of at most $n$–parts such that $\lambda/\mu$ is a skew partition. Then the
Jackson coefficients $\omega_{\lambda/\mu}$ are defined by

$$\omega_{\lambda/\mu}(x; r, q, p, t; a, b) := \frac{(x^{-1}, ax)_\lambda}{(qbr^{-2}, q^n-1)_\mu} \cdot \frac{(br^{-1}t^{2-2i})}{(br^{-1}t^{2-2i})} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{2i-2j})}{(qt^{2i-2j})} \right\}$$

where $x, r, q, p, t, a, b \in \mathbb{C}$. From here and on, $\delta(n)$ denotes the $n$-tuple $\delta(n) := (n-1, \ldots, 1, 0)$ and $q^{\lambda t\delta(n)}$ denotes $q^{\lambda t\delta(n)} := (q^{\lambda t^{n-1}}, q^{\lambda t^{n-2}}, \ldots, q^{\lambda t})$.

Using the recurrence relation (14) the definition of $\omega_{\lambda/\mu}$ yields a recursion formula for Jackson coefficients in the form

$$\omega_{\lambda/\mu}(y, z; r; a, b) := \sum_{\mu} \omega_{\lambda/\mu}(r^{-k}y, r; ar^{-2k}, br^{-k}) \omega_{\lambda/\mu}(z; r; a, b)$$

where $y = (x_1, \ldots, x_{n-k}) \in \mathbb{C}^{n-k}$ and $z = (x_{n-k+1}, \ldots, x_n) \in \mathbb{C}^k$.

A key result used in the development of the $BC_n$ Bailey Lemma, the cocycle identity for $\omega_{\lambda/\mu}$, is written in [11] in the form

$$\omega_{\lambda/\mu}((uv)^{-1}; uv, q, p, t; a(uv)^2, buv) = \sum_{\mu \leq \lambda \leq \nu} \omega_{\mu/\nu}(v^{-1}; v, q, p, t; a(vu)^2, buv) \omega_{\lambda/\mu}(u^{-1}; u, q, p, t; au^{-2}, bu)$$

where the summation index $\lambda$ runs over partitions.

Using the recurrence relation (14) the definition of $\omega_{\lambda/\mu}(x; r; a, b)$ can be extended from the single variable $x \in \mathbb{C}$ case to the multivariable function $\omega_{\lambda/\mu}(z; r; a, b)$ with arbitrary number of variables $z = (x_1, \ldots, x_n) \in \mathbb{C}^n$. That $\omega_{\lambda/\mu}(z; r; a, b)$ is symmetric is also proved in [11] using a remarkable elliptic $BC_n$ $10\varphi_9$ transformation identity.

### 2.2 Limiting Cases

The limiting cases of the basic (the $p = 0$ case of the elliptic) $W$ functions $W_{\lambda/\mu}(x; q, 0, t, a, b) = W_{\lambda/\mu}(x; q, 0, t, a, b)$ will be used in computations in what follows. To simplify the exposition, some more notation would be helpful. Set

$$W_{\lambda/\mu}^b(x; q, t, a) := \lim_{b \to 0} W_{\lambda/\mu}(x; q, t, a, b)$$

and

$$W_{\lambda/\mu}^{\infty}(x; q, t, b) := \lim_{a \to 0} a^{-|\lambda|+|\mu|} W_{\lambda/\mu}(x; q, t, a, b)$$
and, finally
\[ W_{\lambda/\mu}^{ab}(x; q, t, s) := \lim_{a \to 0} W_{\lambda/\mu}(x; q, t, a, as) \quad (18) \]

The existence of these limits can be seen from \((p = 0)\) case of the definition \((24)\), the recursion formula \((12)\) and the limit rule
\[ \lim_{a \to 0} a^{|\mu|} (x/a) = (-1)^{|\mu|} x^{|\mu|} t^{-n(\mu)} q^{n(\mu')} \quad (19) \]

where \(|\mu| = \sum_{i=1}^n \mu_i\) and \(n(\mu) = \sum_{i=1}^n (i-1)\mu_i\), and \(n(\mu') = \sum_{i=1}^n \left( \frac{\mu_i}{2} \right)\). These functions are closely related to the Macdonald polynomials \([17]\), interpolation Macdonald polynomials \([20]\) and \(BC_{n}\) abelian functions \([22]\).

We now make these definitions more precise. Let \(H_{\lambda/\mu}(q, t, b) = H_{\lambda/\mu}(q, 0, t, b)\), and for \(x \in \mathbb{C}\)

\[ W_{\lambda/\mu}^a(x; q, t, b) := \lim_{a \to 0} a^{-|\lambda|+|\mu|} W_{\lambda/\mu}(x; q, t, a, b) \]

\[ = (-qb/x)^{-|\lambda|+|\mu|} q^{-n(\lambda')+n(\mu')} H_{\lambda/\mu}(q, t, b) \]

\[ \quad \cdot \frac{(x^{-1})^2 (qbx/t)^\mu}{(x^{-1})^2 (qbx)^\lambda} \prod_{i=1}^n \left\{ \frac{(1 - bt^{1-2i} q^{2\mu_i}) (bt^{1-2i})_{\mu_i+\lambda_i+1}}{(1 - bt^{1-2i}) (bt^{1-2i})_{\mu_i+\lambda_i+1}} \right\} \quad (20) \]

Using \((12)\) we get the following recurrence formula for \(W_{\lambda/\mu}^a\) function
\[ W_{\lambda/\mu}^a(y; z, q, t, b) = \sum_{\nu < \lambda} t^{2(|\lambda| - |\nu|)} W_{\lambda/\nu}^a(y t^{-\ell}; q, t, bt^\ell) W_{\nu/\mu}^a(z; q, t, b) \quad (21) \]

for \(y \in \mathbb{C}\) and \(z \in \mathbb{C}^\ell\). Similarly, let
\[ H_{\lambda/\mu}(q, t) := \lim_{b \to 0} H_{\lambda/\mu}(q, t, b) \]

\[ = \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{i+j-1} t^{j-i})_{\mu_j-1 - \lambda_i}}{(q^{i+j-1} t^{j-i})_{\mu_j-1 - \lambda_i}} \frac{(q^{j-i} t^{i-j})_{\mu_j-1 - \lambda_i}}{(q^{j-i} t^{i-j})_{\mu_j-1 - \lambda_i}} \right\} \quad (22) \]

Sending \(b \to 0\) we define the \(W_{\lambda/\mu}^b\) functions in the form
\[ W_{\lambda/\mu}^b(x; q, t, a) := \lim_{b \to 0} W_{\lambda/\mu}(x; q, t, a, b) \]

\[ = t^{-n(\lambda)+|\mu|+n(\mu)} H_{\lambda/\mu}(q, t) \frac{(x^{-1}, ax)^\lambda}{(x^{-1}, ax)^\mu} \quad (23) \]

The recurrence formula for \(W_{\lambda/\mu}^b\) would then be
\[ W_{\lambda/\mu}^b(y; z; q, t, a) = \sum_{\nu < \lambda} W_{\lambda/\nu}^b(y t^{-\ell}; q, t, at^{2\ell}) W_{\nu/\mu}^b(z; q, t, a) \quad (24) \]

for \(y \in \mathbb{C}\) and \(z \in \mathbb{C}^\ell\).
Finally, by setting $b = as$ in the definition of $W_{\lambda/\mu}$ function, and sending $a \to 0$ we define another family of symmetric rational functions $W_{ab\lambda/\mu}$ in the form

$$W_{ab\lambda/\mu}(x; q, t, s) := \lim_{a \to 0} W_{\lambda/\mu}(x; q, t, a, as)$$

$$= t^{-n(\lambda)+|\mu|+n(\mu)} H_{\lambda/\mu}(q, t) \frac{(x^{-1})^\lambda}{(x^{-1})^\mu} \frac{(qs/(xt))^\mu}{(qs/x)^\lambda}$$

(25)

Using (12) again we get the following recurrence formula for $W_{ab\lambda/\mu}$ function

$$W_{ab\lambda/\mu}(y, z; q, t, s) = \sum_{\nu \prec \lambda} W_{ab\lambda/\nu}(yt^{-\ell}; q, t, st^{-\ell}) W_{ab\nu/\mu}(z; q, t)$$

(26)

where $y \in \mathbb{C}$ and $z \in \mathbb{C}^\ell$ as before. We will often use two more limiting cases $W_{s\uparrow\lambda/\mu}$ and $W_{s\downarrow\lambda/\mu}$ defined as follows.

$$W_{s\uparrow\lambda/\mu}(x; q, t) := \lim_{s \to \infty} s^{|\lambda|-|\mu|} W_{ab\lambda/\mu}(x; q, t, s)$$

$$= (-q/x)^{-|\lambda|+|\mu|} q^{-n(\lambda')+(n(\mu') H_{\lambda/\mu}(q, t) \frac{(x^{-1})^\lambda}{(x^{-1})^\mu}}$$

(27)

The recurrence formula for $W_{s\uparrow\lambda/\mu}$ function turns out to be

$$W_{s\uparrow\lambda/\mu}(y, z; q, t) = \sum_{\nu \prec \lambda} S^\ell(|\lambda|-|\nu|) W_{s\uparrow\lambda/\nu}(yt^{-\ell}; q, t) W_{s\uparrow\nu/\mu}(z; q, t)$$

(28)

for $y \in \mathbb{C}$ and $z \in \mathbb{C}^\ell$. Similarly,

$$W_{s\downarrow\lambda/\mu}(x; q, t) := \lim_{s \to 0} W_{ab\lambda/\mu}(x; q, t, s)$$

$$= t^{-n(\lambda)+|\mu|+n(\mu)} H_{\lambda/\mu}(q, t) \frac{(x^{-1})^\lambda}{(x^{-1})^\mu}$$

(29)

The recurrence formula for $W_{s\downarrow\lambda/\mu}$ function may be written as

$$W_{s\downarrow\lambda/\mu}(y, z; q, t) = \sum_{\nu \prec \lambda} W_{s\downarrow\lambda/\nu}(yt^{-\ell}; q, t) W_{s\downarrow\nu/\mu}(z; q, t)$$

(30)

for $y \in \mathbb{C}$ and $z \in \mathbb{C}^\ell$.

2.3. Bailey Lemma

Bailey Lemma was introduced first by W. N. Bailey [6] in 1944 as a special case of a general series transformation argument, as he successfully attempted to clarify the mechanism behind Rogers’ proof [23] of the famous Rogers–Ramanujan identities. G. E. Andrews [2] gave a stronger version of the
Lemma emphasizing its iterative nature in 1984. P. Paule [21] independently presented a bilateral version of the Lemma around the same time.

A. K. Agarwal, G. E. Andrews and D. M. Bressoud [1] introduced an abstract matrix formulation of the one–parameter Bailey Lemma and extended the notion of a Bailey chain to that of a Bailey lattice. Consequently, this formulation gave rise to a two parameter Bailey lattice in [8]. The most recent one dimensional extension of the Bailey Lemma was given by Andrews [2], where he defined the concept of a well–poised Bailey chain and a Bailey tree.

Bressoud, Ismail and Stanton [8] established variants of Bailey Lemma where they replaced the base \( q \) to \( q^k \), instead of changing the parameters during the iteration. These versions of the Bailey Lemma proved to be very effective as well. They also introduced a method of inserting linear factors into the Bailey chain which eliminated the need for the Bailey lattice in the Bailey Lemma proof of Andrews–Gordon identities.

The general matrix formulation of the Bailey Lemma may be given as follows. For \( q, a \in \mathbb{C} \), the infinite lower triangular matrices

\[
M(a) = \left( \frac{1}{(q)_{i-j}(aq)_{i+j}} \right)
\]

and

\[
M^{-1}(a) = \left( \frac{(-1)^{i-j}(aq)_{i+j}q^{(i-j)}}{(q)_{i-j}} \right)
\]

are inverses of each other. This result is equivalent to a terminating \( 4 \varphi_3 \) summation theorem. Now if \( \alpha \) and \( \beta \) form a Bailey pair, that is if \( \beta = M(a) \alpha \), then the pair \( \beta' \) and \( \alpha' \) defined by \( \alpha' = S(a) \alpha \) and \( \beta' = N(a) \beta \) form a Bailey pair, where \( S(a) \) is the infinite diagonal matrix

\[
S(a) := \left( \frac{(\rho)_{i-j}(\sigma)_{j}(aq/\rho\sigma)^i}{(aq/\rho)_{i}(aq/\sigma)_{j}} \delta_{ij} \right)
\]

and

\[
N(a) := \left( \frac{(aq/\rho\sigma)_{i-j}(\sigma)_{j}(aq/\rho\sigma)^i}{(q)_{i-j}(aq/\rho)_{i}(aq/\sigma)_{j}} \right)
\]

where \( \rho, \sigma \in \mathbb{C} \).

The limiting case as \( \sigma, \rho \to \infty \) of this result, which we call terminating weak Bailey Lemma, may be written [11] explicitely as

\[
\sum_{m=0}^{n} \frac{b^m a^{m^2}}{(q)_{n-m}} \cdot \sum_{k=0}^{m} \frac{1}{(q)_{m-k}(qb)_{m+k}} \cdot \alpha_k = \sum_{k=0}^{n} \frac{b^k q^{k^2}}{(q)_{n-k}(qb)_{n+k}} \cdot \alpha_k \tag{31}
\]

Sending, in addition, \( n \to \infty \) gives [2] the non–terminating weak Bailey Lemma as a series identity in the form

\[
\sum_{m=0}^{\infty} \frac{b^m a^{m^2}}{(q)_{m-k}(qb)_{m+k}} \cdot \alpha_k = \frac{1}{(qb)_{\infty}} \sum_{k=0}^{\infty} \frac{b^k q^{k^2}}{(qb)_{k}} \alpha_k \tag{32}
\]
for any Bailey pair \((\alpha, \beta)\). This is in one dimensional case the version given by Paule. Sending \(n \to \infty\) gives bilateral non–terminating Bailey Lemma as follows.

\[
\sum_{m=0}^{\infty} b^m q^{m^2} \sum_{k=0}^{m} \frac{1}{(q)_{m-k}(q)_{m+k}} \cdot \alpha_k = \frac{1}{(qb)_\infty} \sum_{k=0}^{\infty} b^k q^{k^2} \alpha_k \tag{33}
\]

Using these limiting cases, Paule \[21\] gave a bilateral version of the Bailey Lemma, where he defines a Bailey pair \((\alpha, \beta)\) by

\[
\beta_n = \sum_{k=\infty}^{-\infty} \frac{\alpha_k}{(q)_{n-k}(q)_{n+k+\delta}}
\]

for \(\delta \in \{0, 1\}\), and shows that if

\[
\beta'_n = \sum_{j=0}^{\infty} \frac{q^{j^2+\delta j}}{(q)_{n-j}} \beta_j
\]

and \(\alpha'_n = q^{j^2+\delta j} \alpha_n\), then \((\alpha', \beta')\) also form a Bailey pair. We give multiple analogues of these limiting versions of Bailey Lemma in this paper.

As pointed out above, the one–parameter Bailey Lemma depends on a single parameter \(b\) which remains unchanged throughout the chain. However, by replacing the matrix \(M(a)\) by the two–parameter matrix \[8\]

\[
M(a, b) = \left( \frac{(b/a)_{i-j}(b)_{i+j}(1 - aq^{2j})a^{i-j}}{(q)_{i-j}(aq)_{i+j}(1 - a)} \right)
\]

the Bailey Lemma is extended to a two parameter identity. Iteration of this version, extending the Bailey chain concept, generates the so–called Bailey lattice. The matrix \(M(a, b)\) has interesting properties such as

\[
M(b, c)M(a, b) = M(a, c) \quad \text{and} \quad M^{-1}(a, b) = M(b, a).
\]

These properties follow as special cases from a powerful result given by Bressoud \[8\]. Namely,

\[
S^{-1}(c) M(c, d) S(c) = S^{-1}(b) M(b, d) S(b) M(c, b),
\]

for \(qbc = d\rho^2\). With this extension the two–parameter Bailey Lemma states that, if the infinite sequences \(a\) and \(\beta\) form a Bailey pair with respect to \(M(a, b)\), that is if \(\beta = M(a, b) \alpha\), then the pair \(\beta'\) and \(\alpha'\) defined by

\[
\beta' = S(c)S^{-1}(b)M(b, d)S(b)\beta
\]

and

\[
\alpha' = S(a)M(a, c)\alpha
\]

also form a Bailey pair with respect to \(M(c, d)\) provided that \(qbc = d\rho^2\).
First multiple series generalizations of the Bailey Lemma were apparently given by Milne and Lilly \cite{16} for the root systems $A_l$ and $C_l$ of rank $l$. They generalized matrix formulation of the Bailey Lemma in these cases and gave numerous applications of their generalizations. A different $A_2$–type Bailey Lemma was presented more recently by Andrews, Schilling and Warnaar \cite{3} where they used supernomial coefficients to replace the one dimensional $q$–binomial coefficient in bilateral Bailey Lemma.

Gustafson also gave a multiple analogue of the Bailey Lemma in terms of his rational Schur functions $S_\lambda$ in an unpublished manuscript.

The Bailey Lemma proof of the extreme cases of Andrews–Gordon identities may be summarized as follows. Start with the elementary non–trivial sequence $\beta_n = \delta_{0n}$, and using the inverse matrix $M^{-1}(b)$ compute the corresponding $\alpha_n$. Iterate the Bailey Lemma with this pair $k$ times to write the generalized Watson transformation. Send the parameters $n, \sigma_i, \rho_i \to \infty$ in the Watson transformation to get the generalized Rogers–Selberg identity. Specialize the parameter $b$ by setting $b = 1$ and $b = q$, and use the Jacobi triple product identity

$$(q, q/z, z)_{\infty} := \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}} z^m, \quad |q| < 1$$

(35)

to write the final form \(\square\) of the identities. For the full Andrews–Gordon identities one also changes the parameter $b$ during the iteration.

2.4. One parameter $BC_n$ Bailey Lemma

We now give a review of our multiple \cite{10} Bailey Lemma starting with the definitions of the multiple analogues of $M(b)$ and $S(b)$ matrices.

**Definition 1.** Let $\lambda$ be a partition of at most $n$–parts and $b \in \mathbb{C}$. Define

$$M_{\nu\lambda}(b) := (-1)^{|\lambda|} q^{2|\lambda|+n(\lambda') n(\lambda)+2(1-n)|\lambda|} \frac{t}{q^{n-1}} \frac{(bt^1-n)_{\lambda}}{(bt^1-n)_{\lambda}}$$

$$\cdot \prod_{1 \leq i < j \leq n} \left( \frac{(qt^{i-j})_{\lambda_i-\lambda_j} (bt^{i-j})_{\lambda_i+\lambda_j+1}}{(qt^{i-j})_{\lambda_i-\lambda_j} (bt^{i-j})_{\lambda_i+\lambda_j+1}} \right) W_{\lambda}^{\sigma}(q^{\nu} t^{\delta(n)}; q, t, bt^{1-n})$$

(36)

and the infinite diagonal matrix $S(b)$ with diagonal entries

$$S_{\lambda}(b) := (qb/\sigma \rho)^{|\lambda|} \frac{(\sigma, \rho)_{\lambda}}{(qb/\sigma, qb/\rho)_{\lambda}}$$

(37)

where $|\lambda| = \sum_{i=1}^{n} \lambda_i$ and $n(\lambda) = \sum_{i=1}^{n} (i-1) \lambda_i$, and $n(\lambda') = \sum_{i=1}^{n} \binom{\lambda_i}{2}$.

The properties of these matrices are investigated in \cite{11}. It is shown, for example, that $M(b)$ is lower triangular and is independent of different representations of $\lambda$ in any dimension $n$. It was also shown that $M(b)$ is invertible.
where the inverse $M^{-1}(b)$ is an infinite dimensional lower triangular matrix with entries

$$M_{\lambda\mu}^{-1}(b) = \frac{q^{-|\lambda|+|\mu|}t^{2n(\mu)}}{(qb,qt^{n-1})_{\mu}} \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})} \right\} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \right\} W_{\mu}(q^{\lambda}t^{\delta(n)}; q, t, bt^{2-2n}) \quad (38)$$

Next, we recall the notion of a Bailey pair.

**Definition 2.** The infinite sequences $\alpha$ and $\beta$ of rational functions $\alpha_\lambda, \beta_\lambda$ over the field $\mathbb{C}(q, t, r, a, b)$ form a Bailey pair relative to $b$ if they satisfy

$$\beta_\lambda = \sum_{\mu} M_{\lambda\mu}(b) \alpha_\mu \quad (39)$$

where the sum is over partitions.

The one parameter $BC_n$ Bailey Lemma given in [10] may now be stated as follows.

**Theorem 1.** Suppose that the infinite sequences $\alpha$ and $\beta$ form a Bailey pair relative to $b$. Then $\alpha'$ and $\beta'$ also form a Bailey pair relative to $b$ where

$$\alpha'_\lambda = S_\lambda(b) \alpha_\lambda \quad (40)$$

and

$$\beta'_\lambda = \sum_{\mu} N_{\lambda\mu}(b) \beta_\mu \quad (41)$$

where the sum is over partitions, and the entries of the matrix are given by

$$N_{\mu\lambda}(b) = q^{|\mu|}t^{2n(\mu)} \frac{(qb,qb/\sigma\rho)_{\mu}}{(qb/\sigma,qb/\rho)_{\mu}} \frac{(\sigma,\rho)_{\mu}}{(qb,qt^{n-1})_{\mu}} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \right\} W_{\mu}(q^{\nu}t^{\delta(n)}; q, t, \sigma\rho t^{n-1}/qb) \quad (42)$$

The power of Bailey Lemma comes from its potential for indefinite iteration. The lemma can be applied to a given Bailey pair $(\alpha, \beta)$ repeatedly producing an infinite sequence of Bailey pairs $(\alpha, \beta) \rightarrow (\alpha', \beta') \rightarrow (\alpha'', \beta'') \rightarrow \cdots$, what is called a Bailey chain. In fact, a stronger result says that it is possible to walk along the Bailey chain in every direction as depicted in the following figure.

The lower triangular matrices $M(b)$, $N(b)$ and the diagonal matrix $S(b)$, having no zero entries on their diagonal, are all invertible. One move forward and backward in the first line in Figure 1 by $S(b)$ and $S^{-1}(b)$, in the second line by $N(b)$ and $N^{-1}(b)$ and move up and down by $M(b)$ and $M^{-1}(b)$. That $N(b)$ is invertible follows immediately from the construction $N(b) = M(b)S(b)M^{-1}(b)$. 

10
Therefore, the entire Bailey chain is uniquely determined by a single node $\alpha^{(i)}$ or $\beta^{(i)}$ for any $i \in \mathbb{Z}$ in the chain.

This powerful iteration mechanism allows one to prove numerous multiple basic hypergeometric series and multiple $q$-series identities. We will write a multilateral version of the multiple Bailey Lemma for the extreme cases of the Andrews–Gordon identities.

The Bailey pair $(\alpha, \beta)$ corresponding to the simplest non–trivial sequence $\beta$ defined by $\beta_{\lambda} = \delta_{\lambda \lambda}$ is called the unit Bailey pair. The corresponding $\alpha$ sequence is easily be computed to be

\[
\alpha_{\lambda} = \sum_{\mu} M_{\lambda \mu}^{-1}(b) \beta_{\mu} = q^{-|\lambda|} \prod_{i=1}^{n} \left( \frac{(1 - bt^{2-2i}q^{2\lambda_i})}{(1 - bt^{2-2i})} \right)
\]

using the the inverse matrix $M^{-1}(b)$.

Iterating the multiple Bailey Lemma $N$ times, starting with the unit Bailey pair yields the generalized Watson transformation in the form

\[
\frac{(qb, qb/\rho_0\sigma_\nu N\lambda N)}{(qb, qb/\rho_0)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{1-j})_{\nu_i-\mu_i} - \mu_i}{(qt^{1-j})_{\nu_i-\mu_i} - \mu_i} \right\} W_{\mu_0}^{ab}(q^{k-1} \delta(n); q, t, \rho_0\sigma_\nu N\lambda N) \prod_{i=1}^{n} \left( \frac{(1 - bt^{2-2i}q^{2\mu_i})}{(1 - bt^{2-2i})} \right)
\]

\[
= \sum_{\mu \leq \mu_0} (-1)^{|\mu|} q^{|\mu|+n(\mu')} \prod_{i=1}^{n} \left( \frac{(qt^{1-j})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}{(qt^{1-j})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}} \right) W_{\mu_0}^{a}(q^{k-1} \delta(n); q, t, bt^{1-n}) \prod_{k=1}^{N} \left( \frac{\sigma_{\nu_{N-k+1}, \rho N-k+1}^{\mu}}{(qb/\rho_0\sigma_\nu N\lambda N, qb/\rho N-k+1)^{\mu}} \frac{qb}{\sigma_{\nu_{N-k+1}, \rho N-k+1}^{\mu}} \right)_{|\nu|}
\]

where

\[
\sum_{\nu_n \leq \nu_{n-1} \leq \cdots \leq \nu_1 \leq \nu_0} := \sum_{\nu_1 \leq \nu_0} \sum_{\nu_2 \leq \nu_1} \cdots \sum_{\nu_n \leq \nu_{n-1}}
\]

It was shown [10] that the $N = 1$ case of the generalized Watson transformation [14] reduces to a multiple analogue of the terminating $6\phi_5$ summation formula, and the $N = 2$ case reduces to that of the Watson transformation.
Sending $\sigma_i, \rho_i \to \infty$ and $\mu^0 \to \infty$ in the identity (14) gives the generalized Rogers–Selberg identity in the next result.

**Lemma 2.** With notation as above, we have

\[
(qb)_{\infty}^n \sum_{\mu^1 \leq \cdots \leq \mu^n, \ell(\mu^1) \leq n} q^{\mu_1^1 + 2n(\mu^1')}(1-n)|\mu^1|b^{\mu^1^1}
\]

\[
= \prod_{1 \leq i < j \leq n} \left\{ (t^{j-i})_{\mu_i^1 - \mu_j^1} \frac{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\},
\]

\[
\prod_{k=2}^{N-1} \left\{ q^{\mu_k^1}p^{2n(\mu^k)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\} \right\} \frac{1}{(qt^{n-1})_{\mu_1^1}} \sum_{\mu^1, \ell(\mu^1) \leq n} q^{\mu_1^1 + 2n(\mu^1')} (1-n)|\mu^1|b^{\mu^1^1}
\]

\[
= \prod_{1 \leq i < j \leq n} \left\{ (t^{j-i})_{\mu_i^1 - \mu_j^1} \frac{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\},
\]

\[
= \prod_{k=2}^{N-1} \left\{ q^{\mu_k^1}p^{2n(\mu^k)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\} \right\} \frac{1}{(qt^{n-1})_{\mu_1^1}} \sum_{\mu^1, \ell(\mu^1) \leq n} q^{\mu_1^1 + 2n(\mu^1')} (1-n)|\mu^1|b^{\mu^1^1}
\]

**Proof.** First we set $\mu^0 = M^n$ in the generalized Watson transformation (14) and use the analogue of the Weyl degree formula (14) and the limit rule (10), and send $\sigma_i, \rho_i \to \infty$ to get

\[
(qb)_M^n \sum_{\mu^1 \leq \cdots \leq \mu^n, \ell(\mu^1) \leq M^n} q^{(M+1)|\mu^1|+2n(\mu^1')} (1-n)|\mu^1|b^{\mu^1^1}
\]

\[
= \prod_{1 \leq i < j \leq n} \left\{ (t^{j-i})_{\mu_i^1 - \mu_j^1} \frac{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\},
\]

\[
\prod_{k=2}^{N-1} \left\{ q^{\mu_k^1}p^{2n(\mu^k)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\} \right\} \frac{1}{(qt^{n-1})_{\mu_1^1}} \sum_{\mu^1, \ell(\mu^1) \leq M^n} q^{\mu_1^1 + 2n(\mu^1')} (1-n)|\mu^1|b^{\mu^1^1}
\]

\[
= \prod_{1 \leq i < j \leq n} \left\{ (t^{j-i})_{\mu_i^1 - \mu_j^1} \frac{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\},
\]

\[
\prod_{k=2}^{N-1} \left\{ q^{\mu_k^1}p^{2n(\mu^k)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu_i^1 - \mu_j^1}}{(qt^{j-i-1})_{\mu_i^1 - \mu_j^1}} \right\} \right\} \frac{1}{(qt^{n-1})_{\mu_1^1}} \sum_{\mu^1, \ell(\mu^1) \leq M^n} q^{\mu_1^1 + 2n(\mu^1')} (1-n)|\mu^1|b^{\mu^1^1}
\]

We next send $M \to \infty$ as we apply the dominated convergence theorem as stated in (14) to get the generalization of the $BC_n$ Rogers–Selberg identity (16) to be proved. The convergence theorem applies, because for $a, b \in \mathbb{C}$ such that
the denominator never vanishes there exists a constant $C$ independent of $\alpha$ such that
\[
\left| \frac{(aq^n)_{\infty}}{(bq^n)_{\infty}} \right| < C \tag{48}
\]
when $|q| < 1$. This observation also implies, in the view of the recurrence relation (12), that for any partitions $\lambda, \mu$ of at most $n$–parts and $q, t \neq 0, a \neq 0, b \in \mathbb{C}$ such that $|q| < 1$, we have
\[
\left| W_{\mu}(q^{\lambda^d(n)}; q, t; a, b) \right| < C_w \tag{49}
\]
where $C_w$ is a constant independent of $\lambda$ and $\mu$.

Before writing the extreme cases of our multiple Andrews–Gordon identities, we multilateralize the summand in the well–poised right hand side of (46).

**Lemma 3.** The summand in the well–poised right hand side of the generalized Rogers–Selberg identity (46) is symmetric under the standard hyperoctahedral group action of permutations and sign changes.

**Proof.** Set $q^{z_i} = b_{i}^{1/2}t_{i}^{-1}$ in (46) and use the definition (2) and standard properties of the $q$–Pochhammer symbol to write the summand in the form
\[
\prod_{i=1}^{n} q^{-(N-1)z^{2}} \prod_{i=1}^{n} \frac{(q^{1+2zi})_{\infty}(q^{1-2zi})_{\infty}}{(qt^{n-i})_{\infty}(q^{1-2zi})_{\infty}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+z_i-z_j})_{\infty}(q^{1-z_i+z_j})_{\infty}}{(t^{-1}qt^{-1})_{\infty}(t^{-1}qt^{-1})_{\infty}} \right\} \prod_{\mu, \ell(\mu) \leq n} \sum_{i=1}^{n} \frac{n^{(N-1)(\mu_i+z_i)^2} \prod_{i=1}^{n} \frac{(q^{1-z_i+(\mu_i+z_i)t^{n-i}})_{\infty}(q^{1-z_i+(\mu_i+z_i)t^{n-i}})_{\infty}}{(q^{1+2z_i+2\mu_i})_{\infty}(q^{1-2z_i-2\mu_i})_{\infty}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+z_i-z_j+\mu_i-\mu_j})_{\infty}(q^{1-z_i+z_j-\mu_i+\mu_j})_{\infty}}{(t^{-1}qt^{-1})_{\infty}(t^{-1}qt^{-1})_{\infty}} \right\} \right\} \tag{50}
\]
It is now obvious that the summand has the hyperoctahedral group symmetries
\[q^{z+\mu} \leftrightarrow q^{w(z+\mu)}.
\]

We now write a $D_n$ generalization of the Andrews–Gordon identities (11) in the two extreme cases ($i = 1$ and $i = k$) in the next theorem.
Theorem 4. Let $|q| < 1$ and $N$ be a positive integer. The well–poised side of the Andrews–Gordon identities can be written in the form

$$
\prod_{i=1}^{n-1} \left\{ \frac{1}{(1 + q^{n-i})} \right\} \prod_{1 \leq i < j \leq n} \frac{1}{\left( 1 - q^{j-i-2}(1 - q^{2n-i-j})^2 \right)} \frac{1}{2} (-1)^{\binom{n}{2}} \prod_{i=1}^{n} q^{(n-i)^2}
$$

$$
\det_{1 \leq i, j \leq n} \left( q^{(j-1)(n-i)} (q^{2N+1}, q^{(2n-2i+1)N+(j-1)}, q^{2N+2-(2n-2i+1)N-j}; q^{2N+1})_\infty \right.
$$

$$
+ q^{-(j-1)(n-i)} (q^{2N+1}, q^{(2n-2i+1)N-(j-1)}, q^{2N-(2n-2i+1)N+j}; q^{2N+1})_\infty \right)
$$

when $b = t^{2n-2}$. For the specialization $b = qt^{2n-2},$ we get

$$
\prod_{i=1}^{n} \left\{ \frac{q^{(n-i)(n-i+1/2)}}{(1 - q^{2n-2i+1})} \right\} \prod_{1 \leq i < j \leq n} \frac{1}{(1 - q^{j-i-2}(1 - q^{2n-i-j})^2)} \frac{1}{2} (-1)^{\binom{n}{2}} \det_{1 \leq i, j \leq n} \left( q^{(j-1)(n-i+1/2)} (q^{2N+1}, q^{(2n-2i+2)N+(j-1)}, q^{2N+2-(2n-2i+2)N-j}; q^{2N+1})_\infty \right.
$$

$$
+ q^{-(j-1)(n-i+1/2)} (q^{2N+1}, q^{(2n-2i+2)N-(j-1)}, q^{2N-(2n-2i+2)N+j}; q^{2N+1})_\infty \right)
$$

PROOF. By a routine application of the multilateralization lemma from [10], we can write the well–poised side of the (46) in the form

$$
\prod_{i=1}^{n-1} \left\{ \frac{1}{(1 + q^{n-i})} \right\} \prod_{1 \leq i < j \leq n} \frac{1}{\left( 1 - q^{j-i-2}(1 - q^{2n-i-j})^2 \right)}
$$

$$
\sum_{\mu \in \mathbb{Z}^n} \prod_{i=1}^{n} (-1)^{\mu_i} q^{(2Nn-N+1-n-(2N-1)(i-1))\mu_i} q^{(2N+1)}^{(\nu_{i-1})} \prod_{1 \leq i < j \leq n} \left\{ (1 - q^{j-i+\mu_i-\mu_j})(1 - q^{2n-i-j+\mu_i+\mu_j}) \right\}
$$

when $b = t^{2n-2}$ corresponding to $z_i = n - i$ specialization. Similarly, we write

$$
\prod_{i=1}^{n} \left\{ \frac{1}{(1 - q^{2n-2i+1})} \right\} \prod_{1 \leq i < j \leq n} \frac{1}{(1 - q^{j-i-2}(1 - q^{2n+1-i-j})^2)}
$$

$$
\sum_{\mu, \ell(\mu) \leq n} \prod_{i=1}^{n} (-1)^{\mu_i} q^{(2nN+(1-n)-(2N-1)(i-1))\mu_i} q^{(2N+1)}^{(\nu_{i-1})} \prod_{1 \leq i < j \leq n} \left\{ (1 - q^{j-i+\mu_i-\mu_j})(1 - q^{2n+1-i-j+\mu_i+\mu_j}) \right\}
$$

for $b = qt^{2n-2}$ corresponding to the $z_i = n - i + 1/2$ specialization. Next, we
employ the determinant evaluations
\[
\prod_{1 \leq i < j \leq n} (1 - x_i x_j^{-1})(1 - x_i x_j)
\]
\[
= \frac{1}{2} (-1)^{\binom{n}{2}} \prod_{i=1}^{n} x_i^{n-1-i} \det_{1 \leq i, j \leq n} \left( x_i^{-1} + x_i^{-j+1} \right)
\]  
(55)
for the root system \(D_n\) of rank \(n\), and apply the Jacobi triple product identity (35) to write the specializations (53) and (54) in the forms to be proved. Note that the balanced left hand side of the series can be put into a form where the sum runs over all \(n\)-tuples of non-negative integers. However, we will not pursue it here.

3. Multilateral Bailey Lemma

I would like to present our multilateralization argument in one dimensional case to make the multiple analogue easier to read. The series formulation of the classical Bailey Lemma may be written in explicit form as follows.

\[
\sum_{m=0}^{n} \frac{(\sigma, \rho)_m(qb/\sigma \rho)_{n-m}(qb/\sigma, qb/\rho)_n m}{(qb/\sigma, qb/\rho)_n m} \sum_{k=0}^{m} \frac{1}{(qb/\sigma \rho)_k (qb/\sigma)_k (qb/\rho)_k} \alpha_k
\]
\[
= \sum_{k=0}^{n} \frac{1}{(qb/\sigma)_k (qb/\rho)_k} \frac{(\sigma)_k (\rho)_k (qb/\sigma \rho)_k^k}{(qb/\sigma \rho)_k (qb/\sigma)_k (qb/\rho)_k} \cdot \alpha_k
\]  
(56)

Note that the identity may be written as

\[
\sum_{m=0}^{n} \frac{(\sigma, \rho)_m(q^{1+\delta}/\sigma \rho)_{n-m}(q^{1+\delta}/\sigma, q^{1+\delta}/\rho)_n m}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho)_n m} \sum_{k=0}^{m} \frac{1}{(q^{1+\delta}/\sigma \rho)_k (q^{1+\delta}/\sigma)_k (q^{1+\delta}/\rho)_k} \alpha_k
\]
\[
= \sum_{k=-\infty}^{n} \frac{1}{(q^{1+\delta}/\sigma \rho)_k (q^{1+\delta}/\sigma)_k (q^{1+\delta}/\rho)_k} \frac{(\sigma)_k (\rho)_k (q^{1+\delta}/\sigma \rho)_k^k}{(q^{1+\delta}/\sigma \rho)_k (q^{1+\delta}/\sigma)_k (q^{1+\delta}/\rho)_k} \cdot \alpha_k
\]  
(57)
where \(\delta \in \{0, 1\}\). Since the maps \(k \leftrightarrow -k - \delta\) generate the set of all integers, we write the sum over all integers.

This is what we call ‘strong bilateral’ Bailey Lemma. If we send the parameters \(\sigma, \rho \to \infty\), we get the weak version

\[
\sum_{m=0}^{n} \frac{q^{m(m+\delta)}}{(q)_{n-m} (q^{1+\delta})_{n-m}} \sum_{k=-\infty}^{n} \frac{1}{(q)_{n-k} (q^{1+\delta})_{n-k} \alpha_k}
\]
\[
= \sum_{k=-\infty}^{n} \frac{q^{k(k+\delta)}}{(q^{1+\delta})_{n-k} \alpha_k}
\]  
(58)

If we also send \(n \to \infty\), we get

\[
\sum_{m=0}^{\infty} \frac{q^{m(m+\delta)}}{(q)_{n-m} (q^{1+\delta})_{n-m}} \sum_{k=-\infty}^{n} \frac{1}{(q^{1+\delta})_{n-k} \alpha_k}
\]
\[
= \frac{1}{(q^{1+\delta})_{\infty}} \sum_{k=-\infty}^{\infty} q^{k(k+\delta)} \alpha_k
\]  
(59)
which, under standard converging conditions, gives the non–terminating bilateral Bailey Lemma listed above. It should be noted that this technique may be applied to write bilateral version for many well–poised hypergeometric series identities that satisfy the invariance property under the action of sign changes $z + \delta/2 \leftrightarrow w(z + \delta/2)$ where $w \in \mathbb{Z}_2$. We will illustrate this below for multiple analogues of the very–well poised $_6\varphi_5$ and Jackson sum identities.

We now give a multilateral version of multiple Bailey Lemma [10]. It was already shown [10] that the matrix entries $M_{\lambda\nu}(b)$ and $S_{\lambda}(b)$ are invariant under the hyperoctahedral group action of permutations an sign changes when $\lambda$ is a rectangular partition $\lambda = k^n$. More precisely, it was shown that under the specialization $t = q^k$ and $b = q^{2z_i+2k(i-1)}$ where $z_i \in \mathbb{C}$ and $k \geq 0$ is a non–negative integer, the matrix entries are invariant under the action $(\mu_i + z_i) \leftrightarrow w(\mu_i + z_i)$ for all elements $w \in W$, the hyperoctahedral group or rank $n$. It was further verified that this action generates the full weight lattice $\mathbb{Z}^n$ only if $z_i = m/2 + k(n-i)$ for some non–negative integers $m, k \geq 0$. Here, we extend these results for an arbitrary partition $\lambda$.

**Theorem 5.** The specialized matrix entries $M_{\lambda\nu}(b)$ are invariant under the hyperoctahedral group action of sign changes and permutations for the specializations $b = q^{-2z_i+2k(n-i)}$ and $t = q^k$ where $q \in \mathbb{C}$ and $m, k \geq 0$ when $\nu$ is an arbitrary partition.

**Proof.** It was shown [10] that $W_\lambda$ functions are well–defined for any $\lambda \in \mathbb{Z}^n$ extending the original definition given for partitions. Therefore, we only need to verify the invariance for the $W_\lambda$ function that enters the definition of $M_{\lambda\nu}(b)$.

The duality formula [10] for $W_\lambda$ functions states that

$$W_\lambda\left(k^{-1}q^{\mu t}\,; q, t, k^2\alpha, kb\right) \cdot \frac{(qb^{n-1})_{\lambda}(qb/a)_{\lambda}}{(k)_\lambda(k\alpha t^{n-1})_{\lambda}} \prod_{1 \leq i < j \leq n} \frac{\left((t^{-i})_{\lambda_i-\lambda_j} (qa^{t^{2n-i-j}+j+1})_{\lambda_i+\lambda_j}\right)}{\left((t^{-i})_{\lambda_i-\lambda_j} (qa^{t^{2n-i-j}})_{\lambda_i+\lambda_j}\right)}$$

$$= W_\nu\left(h^{-1}q^{\lambda t^\delta}\,; q, t, h^2\alpha', h\beta\right) \cdot \frac{\left((qt^{n-1})_{\nu}(qb/a')_{\nu}\right)}{(h)_{\nu}(ha^{t^{n-1}})_{\nu}} \prod_{1 \leq i < j \leq n} \frac{\left((t^{-i})_{\nu_i-\nu_j} (qa^{t^{2n-i-j}+j+1})_{\nu_i+\nu_j}\right)}{\left((t^{-i})_{\nu_i-\nu_j} (qa^{t^{2n-i-j}})_{\nu_i+\nu_j}\right)}$$  \hspace{1cm} (60)

where $k = a't^{n-1}/b$ and $h = a't^{n-1}/b$. Since $W_\nu$ on the right is a symmetric function, the left hand side is invariant under the permutations of $k^{-1}q_i^{t^{n-i}}$, or that of $q^{-m/2+k(n-i)+\nu}$ upon setting $k = -m/2$ and $t = q^k$. This is precisely what we need for the multilateralization of $BC_n$ Bailey Lemma. Moreover, the identity [10]

$$a^{1/\lambda}b^{-\lambda}q^{-\lambda}\,t^{-n/\lambda+(n-1)/\lambda}W_\lambda(x_1^{-1}, \ldots, x_n^{-1}, q^{-1}, p, t^{-1}, a^{-1}, b^{-1})$$

$$= a^{-1/\lambda}b^{1/\lambda}q^{1/\lambda}\,t^{n/\lambda-(n-1)/\lambda}W_\lambda(x_1, \ldots, x_n; q, p, t, a, b)$$  \hspace{1cm} (61)
shows that the $W_{\lambda}$ is invariant under sign changes too. This is true, in particular, if we set $x_i = q^{\lambda_i} t^{n-i}$ as needed in Bailey Lemma. The symmetries for the diagonal $S_{\nu}(b)$ is verified in [10] for arbitrary partitions $\nu$.

Now we give our multilateral definitions for the multiple $M(b)$ and $S(b)$ matrices.

**Definition 3.** Let $\lambda$ be a partition of at most $n$--parts and $b = q^{m+2k(n-1)}$ and $t = q^k$ for $q \in \mathbb{C}$ and $m, k \geq 0$. Define

$$M_{\nu\lambda}(m, k) := q^{\lambda} q^{2kn(\lambda)+k(1-n)|\lambda|+m|\lambda|} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+k(j-i)})_{\lambda_i-\lambda_j}}{(q^{1+k(j-i-1)})_{\lambda_i-\lambda_j}} \frac{(q^{m+k(2n-1-i-j)})_{\lambda_i+\lambda_j}}{(q^{m+k(2n-1-i-j)})_{\lambda_i+\lambda_j}} \right\}$$

$$W^a_{\lambda}(q^r \delta(n); q, q^k, q^{m+k(n-1)})$$

and

$$S_{\lambda}(m, k) := \frac{(q^{1+m+2k(n-1)}/\rho)^{|\lambda|}(\sigma, \rho)_{\lambda}}{(q^{1+m+2k(n-1)}/\sigma, q^{1+m+2k(n-1)}/\rho)_{\lambda}}$$

where $|\lambda| = \sum_{i=1}^{n} \lambda_i$ and $n(\lambda) = \sum_{i=1}^{n} (i-1) \lambda_i$, and $n(\lambda') = \sum_{i=1}^{n} \binom{\lambda_i}{2}$ as before. We also set

$$N_{\nu\mu}(m, k)$$

$$= (q^{1+m+2k(n-1)}, q^{1+m+2k(n-1)}/\rho)^{\mu} q^{\nu+2k(n-1)(\sigma, \rho)_{\mu}} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+k(j-i)-1})_{\mu_i-\mu_j}}{(q^{1+k(j-i-1)})_{\mu_i-\mu_j}} \right\} W^{ab}_{\mu}(q^r \delta(n); q, q^k, q^{1-m-k(n-1)})$$

With these matrices, the strong multilateral Bailey Lemma can be stated exactly as before. We will give the Bailey Lemma in the special case when $m = \delta \in \{0, 1\}$ as in the classical one dimensional case, and for $k = 1$ or $t = q$. For clarity, we will drop $k$ from the notation when $k = 1$ in the discussion below. $M_{\nu\lambda}(m, 1)$, for example, denotes $M_{\nu\lambda}(m, 1)$. In particular, for $k = 1$ and $m = \delta$ we get

$$M_{\nu\lambda}(\delta) := q^{2n(\lambda)+|\delta+2-n|\lambda)} W^a_{\lambda}(q^r \delta(n); q, q^\delta+n-1) \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{\delta^{1+j-i})_{\lambda_i-\lambda_j} (q^{\delta+2n+1-i,j})_{\lambda_i+\lambda_j}}{(q^{\delta+2n+1-i,j})_{\lambda_i+\lambda_j}} \right\}$$

where $\delta \in \{0, 1\}$.

**Lemma 6.** Let the infinite sequences $\alpha = \{\alpha_{\mu} : \mu \in \mathbb{Z}^n\}$ and $\beta = \{\beta_{\lambda} : \lambda$ is a partition of at most $n$--part $\}$ of rational functions over the field $\mathbb{C}(q, t, r, a, b)$ form a Bailey pair relative to $b$. That is, they satisfy

$$\beta_{\lambda} = \sum_{\mu \in \mathbb{Z}^n} M_{\lambda\mu}(\delta) \alpha_{\mu}$$
where the sum terminates above at \( \lambda \). Then \( BC_n \) Bailey Lemma implies that \( \alpha' \) and \( \beta' \) also form a Bailey pair where

\[
\alpha'_\mu = S_\mu(\delta) \alpha_\mu
\]

(67)

for \( \mu \in \mathbb{Z}^n \), and

\[
\beta'_\nu = \sum_{\lambda} N_{\nu\lambda}(\delta) \beta_\lambda
\]

(68)

where the sum is over partitions.

**Proof.** The proof is an immediate application of Lemma 1.

This is our strong multilateral Bailey Lemma. By sending \( \sigma, \rho \rightarrow \infty \) and/or \( \lambda \rightarrow \infty \) using the dominated convergence theorem both in \( BC_n \) Bailey Lemma of Theorem 1 and in the multilateral \( BC_n \) Bailey Lemma of Lemma 6, we can write the terminating and non–terminating weak Bailey Lemmas. We will only state multilateral terminating weak Bailey Lemma here.

**Lemma 7.** Let \( M_{\nu\lambda}(\delta) \) matrix be defined as above in (65). Set

\[
S_\lambda(\delta) = q^{(\delta+2(n-1))\lambda+n_2(\lambda)-2n(\lambda)}
\]

(69)

where \( n_2(\lambda) := |\lambda| + 2n(\lambda') = \sum_{i=1}^n \lambda_i^2 \), and

\[
N_{\nu\mu}(\delta) = \frac{q^{(\delta+n)\mu+n_2(\mu)}q^{2n-1}\nu}{(q^n+2n-1,q^n)_{\mu}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(1-q^{i+j+\mu_i-\mu_j})}{(1-q^{j-i})} \right\} W_{\mu}^{s\uparrow}(q^{\nu+2(n)\lambda}; q, q)
\]

(70)

The Bailey Lemma of Lemma 6 holds true with these definitions.

**Proof.** The proof follows immediately from Theorem 6 as \( \sigma, \rho \rightarrow \infty \).

Now, we write the inverse of \( M(b) \) matrix.

**Lemma 8.** For partitions \( \lambda \) and \( \mu \) of at most \( n \)-part, the inverse of \( M(b) \) may be written as

\[
M_{\lambda\mu}^{-1}(b) := (-1)^{\lambda(\mu)(n-1)-n(\lambda)\lambda} q^{n(\lambda')} \frac{(bt^{1-n})_\lambda}{(qt^{n-1})_\lambda} \prod_{i=1}^n \left\{ \frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})} \right\} \cdot \frac{q^{[\mu]t^{2n(\mu)}}}{(qt^{n-1})_{\mu}(qb)_{\mu}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(qt^{j-i})_{\mu-i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \right\} W_{\mu}^{b}(q^{\lambda}\delta(n); q, t, bt^{2n})
\]

(71)

**Proof.** This follows immediately form the cocycle identity given in [10].
Note that under the specialization $b = q^{m+2k(n-1)}$ and $t = q^k$ for $m, k \geq 0$, we can write the inverse matrix as

$$M_{\lambda\mu}^{-1}(m, k) = (-1)^{|\lambda|} q^{k(n-1)|\lambda|-kn(\lambda)+n(\lambda')} \prod_{i=1}^{n} \frac{(1 - q^{m+2k(n-i)+2\lambda_i})}{(1 - q^{m+2k(n-i)})} \frac{q^{\mu|+2kn(\mu)}}{(q^1+k(n-1))_{\mu}(q^1+m+2k(n-1))_{\mu}} \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+k(j-i)})_{\mu_{-i}-\mu_{j}}}{(q^{1+k(j-i-1)})_{\mu_{i}-\mu_{j}}} \right\} W_{\mu}(q^{\lambda'+k\delta(n)}; q, q, q^m)$$

(72)

In particular, when $m = \delta$ and $k = 1$, we get

$$M_{\lambda\mu}^{-1}(\delta) = (-1)^{|\lambda|} q^{(n-1)|\lambda|-n(\lambda)+n(\lambda')} \frac{(q^{\delta+n-1})_{\lambda}}{(q^\delta)_{\lambda}} \prod_{i=1}^{n} \frac{(1 - q^{\delta+2(n-i)+2\lambda_i})}{(1 - q^{\delta+2(n-i)})} \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+j-i})_{\mu_{-i}-\mu_{j}}}{(q^{1+j-i-1})_{\mu_{i}-\mu_{j}}} \right\} W_{\mu}(q^{\lambda'+\delta(n)}; q, q, q^\delta)$$

(73)

With these definitions, we start iterating the multilateral Bailey Lemma now. Note that if we iterate the strong multilateral Bailey Lemma, we generate multiple multilateral versions of basic hypergeometric series identities such as \(\phi_5\) or Jackson sum. However, we will present the weak versions here.

The simplest non-trivial Bailey pair corresponds to $\beta_\lambda = \delta_{\lambda 0}$. In this case the corresponding $\alpha_\lambda$ sequence becomes

$$\alpha_\lambda = \sum_{\mu} M_{\lambda\mu}^{-1}(h) \beta_\mu = (-1)^{|\lambda|} q^{(n-1)|\lambda|-n(\lambda)} q^n(\lambda') f(\delta)$$

(74)

where

$$f(\delta) := \frac{1}{n!} \prod_{i=1}^{n-1} \frac{1}{(1 + q^{n-i})}, \quad \text{if } \delta = 0$$

(75)

and

$$f(\delta) := \frac{1}{n!} \prod_{i=1}^{n} \frac{1}{(1 - q^{1+2n-2i})}, \quad \text{if } \delta = 1$$

(76)

Writing the first iteration explicitly gives

$$(q^{\delta+2n-1}) = \sum_{\lambda \in \mathbb{Z}^n} W_\lambda^n(q^{\delta+\delta(n)}; q, q, q^{\delta+n-1})$$

$$\times \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+j-i})_{\lambda_{i}-\lambda_{j}}}{(q^{1+j-i-1})_{\lambda_{i}-\lambda_{j}}} \frac{(q^{\delta+2n-1-i-j})_{\lambda_{i}+\lambda_{j}}}{(q^{\delta+2n-1-i-1-j})_{\lambda_{i}+\lambda_{j}}} \right\} q^{(2\delta+2n-1)|\lambda|} (-1)^{|\lambda|} q^{n(\lambda')+n(\lambda)-n(\lambda)} f(\delta)$$

(77)
This identity yields the Euler’s Pentagonal Number Theorem in the limit as \( \nu \to \infty \) when \( \delta = 0 \). Using the identity \([10]\)

\[
\lim_{k \to \infty} W_{\mu}^{\nu} (q^{k+\delta(n)}; q, q^{\delta+2n-2}q^{1-n}) = (q^{\delta+n})^{-|\mu|} \prod_{1 \leq i < j \leq n} \frac{(q^{j-i+1})_{\mu_i-\mu_j}(q^{\delta+1+2n-i-j})_{\mu_i+\mu_j}}{(q^{j-i})_{\mu_i-\mu_j}(q^{\delta+2n-i-j})_{\mu_i+\mu_j}}
\]

and taking the limit, we get

\[
(q^{\delta+2n-1})_{\infty n} = \sum_{\lambda \in \mathbb{Z}^*} q^{(\delta+n-1)|\lambda|} (-1)^{|\lambda|} q^{n(\lambda') + n_2(\lambda) - n(\lambda)} n! f(\delta) \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(1 - q^{j-i+i+\lambda_i+\lambda_j}) (1 - q^{\delta+2n-i-j+\lambda_i+\lambda_j})}{(1 - q^{j-i})^2 (1 - q^{\delta+2n-i-j})^2} \right\}
\]

which is the special \( t = q \) case of the Euler’s Pentagonal Number Theorem given in \([10]\). Iterate the Bailey Lemma for a second time to get

\[
\sum_{\mu \leq \nu} q^{(\delta+n)|\mu| + n_2(\mu)} (q^{\delta+2n-1})_{\mu} \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(1 - q^{j-i+i+\mu_i+\mu_j})}{(1 - q^{j-i})} \right\} W_{\mu}^{\nu} (q^{\nu+\delta(n)}; q, q) \cdot (q^{\delta+2n-1})_{\mu}
\]

\[
= \sum_{\lambda \in \mathbb{Z}^*} q^{2n(\lambda) + (\delta+2-n)|\lambda|} W_{\lambda}^{\delta} (q^{\nu+\delta(n)}; q, q, q^{\delta+n-1}) \cdot \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{1+j-i})_{\lambda_i-\lambda_j} (q^{\delta+2n+1-i-j})_{\lambda_i+\lambda_j}}{(q^{j-i})_{\lambda_i-\lambda_j} (q^{\delta+2n-i-j})_{\lambda_i+\lambda_j}} \right\} \cdot (q^{\delta+2(n-1)|\lambda| + n_2(\lambda) - 2n(\lambda)})^2 \cdot (-1)^{|\lambda|} q^{(n-1)|\lambda| - n(\lambda)} q^{(n(\lambda')) f(\delta)}
\]

This is a multiple analogue of specialized Rogers–Selberg identity \([11]\). Note also that although the series on the right hand side written over \( \mathbb{Z}^n \), it actually terminates from above by \( \nu \) and from below by \( -(\nu - 2n - 2i - \delta) \). In the limit \( \nu \to \infty \), this identity gives the multiple analogues of the celebrated first \((\delta = 0)\) and the second \((\delta = 1)\) Rogers–Ramanujan identities. Recall the identity \([10]\) that

\[
\lim_{k \to \infty} W_{\mu}^{\nu} (q^{k+\delta(n)}; q, q) = q^{-|\mu|} \prod_{1 \leq i < j \leq n} \left\{ \frac{(q^{j-i+1})_{\mu_i-\mu_j}}{(q^{j-i})_{\mu_i-\mu_j}} \right\}
\]
Therefore, in the limit we get

\[
\sum_{\mu \in \mathbb{Z}^n} q^{(\delta + n-1)|\mu| + n_2(\mu)} \prod_{1 \leq i < j \leq n} \left\{ \frac{(1 - q^{i-j+\mu_i - \mu_j})}{(1 - q^{i-j})^2} \right\} = \frac{1}{\left( q^{\delta + 2n-1} \right)_\infty} \sum_{\lambda \in \mathbb{Z}^n} \prod_{1 \leq i < j \leq n} \left\{ \frac{(1 - q^{i-j+\lambda_i - \lambda_j})}{(1 - q^{i-j})^2} \frac{(1 - q^{\delta + 2n-i-j+\lambda_i + \lambda_j})}{(1 - q^{\delta + 2n-i-j})^2} \right\} \cdot \left( (-1)^{||\lambda||} q^{(2\delta + 2(n-1)||\lambda|| + 2n_2(\lambda) - 3n(\lambda) + n(\lambda')} \right) n! f(\delta)
\]

This is precisely our multiple Rogers–Ramanujan identities \[11\]. Repeating the iteration \(N\) times in the same way generates the multiple Andrews–Gordon identities given above in Theorem \[3\] for the extreme cases.

4. Conclusion

The full version of the Andrews–Gordon identities can be written in a similar way by using the multilateral version of the two-parameter Bailey Lemma. We will write the full version of these identities in another publication.

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