SHORT NOTE

Growth Estimates for $\exp(A^{-1}t)$ on a Hilbert Space

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Abstract

Let $A$ be the infinitesimal generator of an exponentially stable, strongly continuous semigroup on the Hilbert space $H$. Since $A^{-1}$ is a bounded operator, it is the infinitesimal generator of a strongly continuous semigroup. In this paper we show that the growth of this semigroup is bounded by a constant $t \log(t)$.

1. Introduction

Over the last five years there is a growing interest in the behavior of the semigroup generated by $A^{-1}$, where $A$ is the infinitesimal generator of a bounded semigroup. Note that throughout the paper we assume that $A^{-1}$ exists as a closed, densely defined operator. This interest was raised by three questions. The first one is coming from systems theory, see Curtain [2].

Within infinite-dimensional systems theory one studies the following set of equations:

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t),
\end{align*} \]

where $A : D(A) \subset H \to H$ is the infinitesimal generator of a $C_0$-semigroup on the Hilbert space $H$, $B$ is a bounded linear operator from the Hilbert space $U$ to the dual of the domain of $A^*$, i.e., $B \in \mathcal{L}(U, D(A^*)')$, $C \in \mathcal{L}(D(A), Y)$, where $Y$ is a third Hilbert space, and $D \in \mathcal{L}(U, Y)$. In [2] the following related system was introduced:

\[ \begin{align*}
\dot{x}_1(t) &= A^{-1}x_1(t) + A^{-1}Bu_1(t) \\
y_1(t) &= -CA^{-1}x_1(t) + (D - CA^{-1}B)u_1(t).
\end{align*} \]

This system has the nice property that $A^{-1}B \in \mathcal{L}(U, H), CA^{-1} \in \mathcal{L}(H, Y)$. Hence this system is a bounded linear system as studied in the text book by Curtain and Zwart [3]. Furthermore, the systems (1) and (2) share many properties. For instance, (1) is input-state stable if and only if (2) is. Here input-state stability means that for all inputs $u \in L^2((0, \infty); U)$ the solution of (1) exists and is (uniformly) bounded on $[0, \infty)$. 
The only stability property which is not known is the state-state stability, i.e., if the semigroup generated by $A$ is (strongly) stable if and only if the semigroup generated by $A^{-1}$ (strongly) stable.

The second motivation for our problem comes from numerical analysis. Consider the (abstract) differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0.$$  

A standard method for solving this differential equation is by using the Crank-Nicolson method. In this method the differential equation (3) is replaced by the difference equation

$$x_{d}(n+1) = (I + \Delta A/2)(I - \Delta A/2)^{-1}x_{d}(n), \quad x_{d}(0) = x(0),$$

where $\Delta$ is the time step.

If $H$ is finite-dimensional, and thus $A$ is a matrix, then it is easy to show that the solutions of (3) are bounded if and only if the solutions of (4) are bounded. In Azizov, Barsukov, and Dijksma [1], Gomilko [4], and in Guo and Zwart [6], it is shown that the solutions of (4) are bounded if both $A$ and $A^{-1}$ generate a uniformly bounded semigroup. The question whether the uniform boundedness of the semigroup generated by $A$ is sufficient is still open. The answer to this question will be positive, if the uniform boundedness of the semigroup generated by $A$ implies the uniform boundedness of the semigroup generated by $A^{-1}$.

The problem whether the inverse of the generator of a bounded semigroup is again a generator of a bounded semigroup was posed as an open problem by deLaubenfels in [7]. This serves as our third motivation.

In the above, we assumed that $A$ generates a uniformly bounded semigroup on a Hilbert space. In Zwart [8] it was shown that if $A$ generates a uniformly bounded semigroup on a Banach space, then the growth of $\exp(A^{-1}t)$ is bounded by a constant times $\sqrt{t}$. It is even shown that this estimate is sharp, i.e., there exists a Banach space and a generator of a nilpotent semigroup, such that $\|\exp(A^{-1}t)\| = m\sqrt{t}$ for some $m$. In this paper we show that for Hilbert spaces the situation is less dramatic. In Section 2 we prove that the growth is always bounded by a constant time $\log(t)$. However, before we can prove this, we need the following result on Lyapunov equations. For the proof we refer to Curtain and Zwart [3, section 4.1 and 5.1].

**Lemma 1.1.** Let $A$, with domain $D(A)$, generate a strongly continuous semigroup $(\exp(At))_{t \geq 0}$ on a Hilbert space $H$, and let $C$ be a bounded operator from $H$ to a Hilbert space $Y$. Then the following are equivalent:

1. There exists an $m \geq 0$ such that for all $x \in D(A)$ there holds

$$\int_0^\infty \|C \exp(At)x\|^2 dt \leq m\|x\|^2.$$
There exists a self-adjoint, non-negative $Q \in \mathcal{L}(H)$ such that
\[ \langle Ax, Qz \rangle_H + \langle Qx, Ay \rangle_H = -\langle Cx, Cz \rangle_Y, \quad x, z \in D(A). \] (5)

There exists a self-adjoint, non-negative $Q \in \mathcal{L}(H)$ such that $QD(A) \subset D(A^*)$
\[ A^*Q + QA = -C^*C \quad \text{on } D(A). \] (6)

Furthermore, the following additional results hold:
1. If item 2. or 3. holds, then
\[ \langle x, Qy \rangle \geq \int_0^\infty \langle C \exp(At)x, C \exp(At)y \rangle dt \] for all $x, y \in H$, where equality holds if $\exp(At)$ is strongly stable, i.e. when $\lim_{t \to 0} \exp(At)x = 0$ for all $x \in H$.

2. If item 1. holds, then one can choose $Q$ as
\[ \langle x, Qy \rangle = \int_0^\infty \langle C \exp(At)x, C \exp(At)y \rangle dt, \quad x, y \in H. \]

2. New results on bounded semigroups

We begin with a lemma which is based on Lemma 1.1.

**Lemma 2.1.** Let $A$ generate a bounded $C_0$-semigroup $\exp(At)$ on a Hilbert space $H$, and let $M$ equal $\sup_{t \geq 0} \| \exp(At) \|$. Then for all $\varepsilon > 0$, $\gamma > 0$, and all $x \in H$, we have that
\[ \int_0^\infty \| (\gamma A - \varepsilon I)^{-1} \exp((\gamma A - \varepsilon I)^{-1} t)x \|^2 dt \leq \frac{M^2}{\varepsilon} \| x \|^2. \]
The same estimate holds for the adjoint.

**Proof.** Since $\| \exp(At) \|$ is bounded by $M$, we have that for $\gamma, \varepsilon > 0$
\[ \int_0^\infty \| \exp((\gamma A - \varepsilon I)t)x_0 \|^2 dt \leq \frac{M^2}{\varepsilon} \| x_0 \|^2. \]
Hence by Lemma 1.1, there exists a non-negative bounded operator $Q_{\gamma, \varepsilon}$ satisfying $Q_{\gamma, \varepsilon} D(A) \subset D(A^*)$,
\[ (\gamma A - \varepsilon I)^* Q_{\gamma, \varepsilon} + Q_{\gamma, \varepsilon} (\gamma A - \varepsilon I) = -I \quad \text{on } D(A), \] (7)
and $\| Q_{\gamma, \varepsilon} \| \leq \frac{M^2}{\varepsilon}$.

Multiplying this equation from the right by $(\gamma A - \varepsilon I)^{-1}$ and from the left by $(\gamma A - \varepsilon I)^{-1}^* = (\gamma A^* - \varepsilon I)^{-1}$, we obtain
\[ (\gamma A^* - \varepsilon I)^{-1} Q_{\gamma, \varepsilon} + Q_{\gamma, \varepsilon} (\gamma A - \varepsilon I)^{-1} = -(\gamma A^* - \varepsilon I)^{-1} (\gamma A - \varepsilon I)^{-1}. \] (8)
By Lemma 1.1 this implies that
\[ \int_0^\infty \| (\gamma A - \varepsilon I)^{-1} \exp((\gamma A - \varepsilon I)^{-1} t)x \|^2 dt \leq \langle x, Q_{\gamma, \varepsilon} x \rangle \leq \frac{M^2}{2\varepsilon} \| x \|^2 \]
which proves the result.

Using this lemma, we obtain a growth estimate for the semigroup generated by \((\gamma A - I)^{-1}\).

**Theorem 2.2.** Let \( A \) be the infinitesimal generator of the bounded semigroup \((\exp(At))_{t \geq 0}\) on the Hilbert space \( H \), and let \( M = \sup_{t \geq 0} \| \exp(At) \| \). Then for all \( \gamma > 0 \),
\[ \| \exp((\gamma A - I)^{-1} t) \| \leq \begin{cases} 1 + \frac{M}{\sqrt{2}} \sqrt{1 + \frac{M^2(e-1)}{2\varepsilon} \log(t)} & t \geq e \\ 1 + \frac{M}{\sqrt{2}} \sqrt{t} & t \in [0, e]. \end{cases} \] (9)

**Proof.** The proof consists out of several steps. The estimate on \([0, e]\) is proved in Step 1. In the second step we compare the semigroups \( \exp((\gamma A - \varepsilon_1 I)^{-1} t_1) \) and \( \exp((\gamma A - \varepsilon_2 I)^{-1} t_1) \). This is used in the third step to compare \( \exp((\gamma A - I)^{-1} t_1) \) and \( \exp((\gamma A - e^{-N} I)^{-1} t_1) \), for \( N \in \mathbb{N} \). In the last step, we combine step 3 with step 1, and derive the estimates in (9) on \([e, \infty)\).

**Step 1.** For \( t \in \mathbb{R} \) we have that
\[ \exp((\gamma A - I)^{-1} t)x = x + \int_0^t (\gamma A - I)^{-1} \exp((\gamma A - I)^{-1} s)x ds. \]
Using Cauchy-Schwarz, and Lemma 2.1 we find
\[ \| \exp((\gamma A - I)^{-1} t)x \| \leq \| x \| + \sqrt{t} \frac{M}{\sqrt{2}} \| x \|. \]
Thus we have proved the estimate on \([0, e]\).

**Step 2.** Let \( t_1 > 0 \) be fixed, then by the variation of constant formula, we find
\[ \exp((\gamma A - \varepsilon_1 I)^{-1} t_1)x - \exp((\gamma A - \varepsilon_2 I)^{-1} t_1)x \]
\[ = \int_0^{t_1} \exp((\gamma A - \varepsilon_1 I)^{-1}(t_1 - s)) \left( (\gamma A - \varepsilon_1 I)^{-1} - (\gamma A - \varepsilon_2 I)^{-1} \right) \]
\[ \times \exp((\gamma A - \varepsilon_2 I)^{-1} s)x ds. \] (10)
Since
\[ (\gamma A - \varepsilon_1 I)^{-1} - (\gamma A - \varepsilon_2 I)^{-1} = (\gamma A - \varepsilon_1 I)^{-1} [\varepsilon_1 - \varepsilon_2] (\gamma A - \varepsilon_2 I)^{-1}, \]
we can use Lemma 2.1, to find that
\[
\langle y, \exp((\gamma A - \varepsilon_1 I)^{-1}t_1)x - \exp((\gamma A - \varepsilon_2 I)^{-1}t_1)x \rangle \\
= \int_0^{t_1} \langle y, \exp((\gamma A - \varepsilon_1 I)^{-1}(t_1 - s))(\gamma A - \varepsilon_1 I)^{-1}x \rangle \\
+ [\varepsilon_1 - \varepsilon_2] \exp((\gamma A - \varepsilon_2 I)^{-1}s)(\gamma A - \varepsilon_2 I)^{-1}x \rangle ds \\
= [\varepsilon_1 - \varepsilon_2] \int_0^{t_1} \langle \exp((\gamma A - \varepsilon_1 I)^{-1}(t_1 - s))(\gamma A - \varepsilon_1 I)^{-1}y, \\
\exp((\gamma A - \varepsilon_2 I)^{-1}s)(\gamma A - \varepsilon_2 I)^{-1}x \rangle ds \\
\leq |\varepsilon_2 - \varepsilon_1| \frac{M^2}{2\sqrt{\varepsilon_2 \cdot \varepsilon_1}} \| x \| \| y \| ,
\]
where we have used the Cauchy-Schwarz inequality. Since the norm of an operator \( S \) equals \( \sup_{x,y \neq 0} \frac{|\langle y, Sx \rangle|}{\| x \| \| y \|} \), we find that
\[
\| \exp((\gamma A - \varepsilon_1 I)^{-1}t_1) - \exp((\gamma A - \varepsilon_2 I)^{-1}t_1) \| \leq |\varepsilon_2 - \varepsilon_1| \frac{M^2}{2\sqrt{\varepsilon_2 \cdot \varepsilon_1}}. \tag{11}
\]

**Step 3.** Let \( N \in \mathbb{N} \) be given and choose \( \varepsilon_n = e^{-n}, n \in \{0, 1, 2, \ldots, N\} \). Then
\[
\| \exp((\gamma A - e^{-N}I)^{-1}t_1) - \exp((\gamma A - I)^{-1}t_1) \| \\
= \| \sum_{n=1}^{N} \exp((\gamma A - e^{-n}I)^{-1}t_1) - \exp((\gamma A - e^{-n+1}I)^{-1}t_1) \| \\
\leq \sum_{n=1}^{N} \frac{e - 1}{2e} M^2, \tag{12}
\]
where we have used (11). Hence we have that
\[
\| \exp((\gamma A - e^{-N}I)^{-1}t_1) - \exp((\gamma A - I)^{-1}t_1) \| \leq \frac{M^2(e - 1)}{2\sqrt{e}} \sqrt{N}. \tag{13}
\]

**Step 4.** Let \( t \in [e, \infty) \) and choose \( N \in \mathbb{N} \) such that \( e^N \leq t < e^{N+1} \). Furthermore, define \( t_1 \) as \( t * e^{-N} \). By the definition of \( N \), we see that \( 1 \leq t_1 < e \).

Since \( \exp((\gamma A - I)^{-1}t) = \exp((e^{-N}\gamma A - e^{-N}I)^{-1}t_1) \) we have that, see (13),
\[
\| \exp((\gamma A - I)^{-1}t) - \exp((e^{-N}\gamma A - I)^{-1}t_1) \| \leq \frac{M^2(e - 1)}{2\sqrt{e}} \sqrt{N} \\
\leq \frac{M^2(e - 1)}{2\sqrt{e}} \log(t). \tag{14}
\]
Now since \( t_1 \in [1,e) \) we may use step 1. to majorize \( \exp((e^{-N\gamma A} - I)^{-1}t_1) \).
Doing so, we find

\[
\|\exp((e^{-N\gamma A} - I)^{-1}t_1)\| \leq 1 + \sqrt{t_1} \frac{M}{\sqrt{2}} \leq \left[ 1 + M \frac{\sqrt{e}}{\sqrt{2}} \right].
\]

Combining this with (14), proves (9).

From this theorem we derive two corollaries. Since the inverse generator of an exponentially stable semigroup is a bounded operator, it is clear that this inverse is the infinitesimal generator of a strongly continuous semigroup. In the first corollary we show that this semigroup can grow at most like the logarithm of \( t \).

If \( A \) generates a bounded semigroup, then it is not a priori clear whether \( A^{-1} \) generates a strongly continuous semigroup, even when \( A^{-1} \) exists as a closed and densely defined operator. A natural approach to this problem, would be to consider \( A - \varepsilon I \), and letting \( \varepsilon \) approach zero. In the second corollary, we show that we only have the estimate \( \|\exp((A - \varepsilon I)^{-1})\| = O(|\log(\varepsilon)|) \) for \( \varepsilon \downarrow 0 \). Hence we cannot conclude that \( A^{-1} \) is the generator of a strongly continuous semigroup.

**Corollary 2.3.** Let \( A \) generate a strongly continuous semigroup \( (\exp(At))_{t \geq 0} \) on a Hilbert space \( H \). Assume further that this semigroup is exponentially stable and satisfies \( \|\exp(At)\| \leq Me^{-\omega t} \) with \( M \geq 1 \) and \( \omega > 0 \). Then

\[
\|\exp(A^{-1}t)\| \leq \begin{cases} 1 + \frac{M}{\sqrt{2}} \sqrt{e} + \frac{M^2(e-1)}{2\sqrt{e}} \log(t/\omega) & t \geq e\omega \\ 1 + \frac{M}{\sqrt{2}} \sqrt{\frac{e}{2}} & t \in [0,e\omega]. \end{cases}
\]

**Proof.** We define \( A_0 \) as \( A_0 = A + \omega I \). By the assumptions it is clear that \( A_0 \) generates a semigroup on \( H \) which is uniformly bounded by \( M \). Simple manipulation gives

\[
\exp(A^{-1}t) = \exp((A_0 - \omega I)^{-1}t) = \exp((\omega^{-1}A_0 - I)^{-1}t\omega^{-1}).
\]

Using Theorem 2.2 gives the desired result.

Hence if \( A \) generates an exponentially stable semigroup, then \( \exp(A^{-1}t) \) can grow at most like \( \log(t) \). We remark that similar result holds for the powers of \( A_d = (A + I)(A - I)^{-1} \), see Gomilko [4].

**Corollary 2.4.** Let \( A \) be the infinitesimal generator of a strongly continuous semigroup on the Hilbert space \( H \). Assume that the semigroup is uniformly
bounded by $M$. For $\varepsilon \in (0, e^{-1})$ we have that

\[ \| \exp((A - \varepsilon I)^{-1}) \| \leq 1 + \frac{M}{\sqrt{2}} \sqrt{\varepsilon} + \frac{M^2(e - 1)}{2\sqrt{e}} |\log(\varepsilon)| \] \quad (17)

**Proof.** We have that

\[ \exp((A - \varepsilon I)^{-1}) = \exp((\varepsilon^{-1} A - I)^{-1} \varepsilon^{-1}) \]

Applying Theorem 2.2 proves the assertion.

From the above result, we conclude that the problem whether $A^{-1}$ generates a strongly continuous semigroup is still open. In [5] Gomilko shows that by putting some additional conditions on the resolvent of $A$ the operator $A^{-1}$ generates a uniformly bounded, strongly continuous semigroup.

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