Spin squeezing and entanglement in spinor-1 condensates

Özgür E. Müstecaplıoğlu, M. Zhang, and L. You

School of Physics, Georgia Institute of Technology, Atlanta GA 30332, USA

(Dated: November 3, 2018)

We analyze quantum correlation properties of a spinor-1 ($f = 1$) Bose-Einstein condensate using the Gell-Mann realization of SU(3) symmetry. We show that previously discussed phenomena of condensate fragmentation and spin-mixing can be explained in terms of the hypercharge symmetry. The ground state of a spinor-1 condensate is found to be fragmented for ferromagnetic interactions. The ground state of a spinor-1 condensate is found to be fragmented for antiferromagnetic interactions. Recent investigations reveal that such a system also possesses complex ground state structures and can exhibit novel dynamical effects.

The ground state of a spinor-1 condensate is found to be fragmented for ferromagnetic interactions. Recent investigations reveal that such a system also possesses complex ground state structures and can exhibit novel dynamical effects.

In this paper, we further explore quantum correlation properties such as spin squeezing and entanglement of a spinor-1 condensate \cite{13, 14}. For a spin half ($f = 1/2$) atomic system, a rotationally invariant Hamiltonian is known to not induce spin squeezing as the total spin is conserved \cite{13}. For a spinor-1 condensate, it was found that its Hamiltonian becomes rotationally invariant if the single (spatial) mode approximation is made to its order parameters, i.e. assuming $\psi_{m_f}(\vec{r}) = \phi(\vec{r})a_{m_f}$ with the same mode function $\phi(\vec{r})$ \cite{3, 4, 5, 6}. $a_{m_f}$ is the annihilation operator for atoms in the Zeeman state $m_f = \pm 0$. However, as we show in this work, various nonlinear processes do occur within different subspaces of the full SU(3) structure of a spinor-1 condensate, e.g. we find the existence of spin squeezing in the isospin sub-group \cite{3, 4}.

The generation and characterization of squeezing and entanglement of Bose-condensed atoms has recently emerged as an active research area. Historically, atomic squeezed states were first considered for a system of two level atoms. Even in this simple SU(2) case, it was found that some operational definitions of spin squeezing can become system dependent. Depending on the context in which the concept of squeezing is applied, different definitions arise \cite{14}. Squeezing in atomic variables was first introduced through reduced fluctuations in the atomic (Pauli) operators of the system \cite{15} such that atomic resonance fluorescence in the far-field zone is squeezed. In this case, it is useful and convenient to define the atomic squeezing parameter according to,

$$
\xi_h = \Delta J_i / \sqrt{|\langle J_j \rangle|/2}, \quad i \neq j \in \{x, y, z\}.
$$

This definition can be essentially read off from the Heisenberg uncertainty relation $\Delta J_i \Delta J_j \geq |\langle J_k \rangle|/2$ for the collective angular momentum components of the two level atomic system. When in such a squeezed state $\xi_h < 1$, the quantum fluctuation of one collective angular momentum component becomes lower than the Heisenberg limited value at the cost of increased fluctuation in the other component. The general family of two level atomic states satisfying this criterion was found to be Bloch states, or SU(2) coherent states. These “squeezed states” are obtained by simply rotating in space the collective (Dicke) state $|J, \pm J\rangle$ \cite{16}. Soon it became clear that neither $\xi_h$ nor the Bloch states are that useful in other applications of atomic squeezing. In particular, for Ramsey oscillatory field spectroscopy, a new squeezing parameter \cite{17},

$$
\xi_R = \sqrt{2 \Delta J_{\perp}} |\langle \vec{J} \rangle|
$$

is called for with $J_{\perp}$ the angular momentum component normal to the $\langle \vec{J} \rangle$, i.e. in the direction of the unit vector $\vec{n}$ along which $\Delta(\vec{n} \cdot \vec{J})$ is minimized. The squeezing condition $\xi_R < 1$ is not straightforwardly determined by the Heisenberg uncertainty relation. Instead, it is defined by requiring the improvement of signal to noise ratio in a typical Ramsey spectroscopy. It was later shown that the same criterion is also applicable for improving the phase sensitivity of a Mach-Zehnder interferometer \cite{18}. $\xi_R \equiv 1$ for Bloch states or SU(2) coherent states. An independent refinement of $\xi_h$ was suggested by Kitagawa and Ueda \cite{13} to make it independent of angular momentum coordinate system or specific measurement schemes. They emphasized that collective spin squeezing should reflect quantum correlations between individual atomic spins and defined a squeezing parameter

$$
\xi_R = \Delta J_{\perp} / \sqrt{J/2}
$$
to measure such correlations. The factor $J/2$ in the
denominator represents the variance of a Bloch state,
which comes from simply adding up the variance of each
individual spin (1/2). When quantum correlations ex-
ist among different atomic spins, the variance of certain
component of the collective spin can become lower than
$J/2$. This leads naturally to the criterion $\xi < 1$ for spin
squeezing. We note that this definition is directly re-
lated to the spectroscopic definition as $\xi_R = (J/\langle|J_z|\rangle)\xi_R$.
More recently, a particular type of quantum correlation,
namely the multi-particle entanglement, becomes impor-
tant for quantum information physics. A more stringent
criterion for atomic squeezing which combines the quan-
tum correlation definition with the inseparability require-
ment of system density matrix is given by Sørensen et al. [10].

$$\xi_e = \frac{(2J)\Delta(\vec{n}_1 \cdot \vec{J})^2}{(\vec{n}_2 \cdot \vec{J})^2 + (\vec{n}_3 \cdot \vec{J})^2} < 1,$$

with the $\vec{n}_i$ being mutually orthogonal unit vectors. $\xi_e$ is
in fact identical to $\xi_R$ along the direction $\langle\vec{n}_1 \cdot \vec{J}\rangle = 0$. It
was proven rigorously that when $\xi_e < 1$, the total state
of the N two level atoms becomes inseparable, i.e. en-
tangled in a general sense. All three definitions above
applies to a two component (two level) atomic system.

Many complications arise when attempt is made to ex-
tend spin squeezing to a spinor-1 SU(3) system. Under
certain restrictive conditions, $\xi_e$ has been used recently
to discuss a two mode entanglement in a spinor-1 conden-
sate [11].

A related problem to spin squeezing is its efficient gen-
eration and detection. In accordance with their respec-
tive definitions for SU(2) systems, several physical me-
chanisms have been proposed along this direction. Kite-
gawa and Ueda considered a model Hamiltonian $H_{\text{KU}} =
h\chi J_z^2$ that can be realized via the Coulomb interaction
between electrons in the two arms of an interferometer
[13]. Barnett and Dupertuis suggested that spin squeezing
can be achieved in a two-atom system described by
$H_{\text{BD}} = ihg(\vec{J}_{1z} + \vec{J}_{2z} - h.c.)$ [13]. Use of a pseudo-spin
two component atomic condensate system has also been
suggested [19]. Recently, two different groups considered
atomic (spin) squeezing and entanglement in a spinor-1 conden-
sate, under the assumption that one of the com-
ponent is highly populated such that quantum properties
are important only among the remaining two sparsely
populated components [12] [14]. Our aim in this study is
to remove such a restrictive condition, and consider the
full quantum correlations within a spinor-1 condensate.

Let us consider a general three component system la-
beled by $i, j \in \{+,-,0\}$. For spectroscopic and inter-
ferometric applications the observables of interest are
the relative number of particles $N_i - N_j$ (particle par-
titioning) and the corresponding phase differences $\phi_i -
\phi_j$ with their measurements limited by noises $\delta N_{ij} =
(\langle\Delta(N_i - N_j)\rangle)^{1/2}$ and $\delta \phi_{ij} = (\langle\Delta(\phi_i - \phi_j)\rangle)^{1/2}$. For
a two-component system, the particle partitioning be-
comes the collective angular momentum projection as

$N_+ - N_- = 2J_z$ in the standard Schwinger represen-
tation; the relative phase becomes the corresponding
azimuthal phase $\phi_x \equiv \phi_+ - \phi_-$, which is conjugate to
$J_x$. Quantum mechanically they satisfy $[J_x, \phi_x] = i$. Thus
from $\delta J_x \delta \phi_x \geq 1$ and $\delta N_{+} = 2(\Delta J_x^2)^{1/2}$, we find $\delta \phi_+ \approx \langle(\Delta \phi_x)^{1/2}\rangle/\langle|J_x|\rangle$ [13]. Therefore, for spin
squeezed states, one achieves higher angular resolution
and reduced particle partitioning noise. In a three com-
ponent system, we can similarly associate the three num-
ber difference $N_i - N_j$ with three subspace pseudo-spins
(each of spin-1/2) $\vec{U}$, $\vec{T}$, and $\vec{V}$ such that $N_+ - N_- =
2T_3, N_+ - N_0 = 2V_3, N_0 - N_3 = 2T_3$. The phase differ-
ces can then be similarly expressed in terms of com-
ponents $U_{x,y}$, $V_{x,y}$, and $T_{x,y}$. When demanding noise
reduction in such a SU(3) system, we need to consider
squeezing in the three spin-1/2 subsystems. One may
naively expect that results from the above discussed
SU(2) squeezing can be applied to each of the three sub-
systems, and collectively, one can simply demand that
$\xi_e < 1$ to be satisfied simultaneously. In reality this
does not work as the three-spin-1/2 subsystems do not
commute with each other. This is also the fundamen-
tal reason that makes it difficult to generate and de-
tect quantum correlations in a full SU(3) system. Fur-
thermore, due to the above non-commuting nature, the
three SU(2) sub-spins cannot be squeezed independently
of each other. Previous discussions of a spinor-1 conden-
sate entanglement are always limited to just one SU(2)
subspace, usually in the limit $N_0 \sim N$, i.e. one mode
is highly populated. Approximately, this limit destroys
the underlying non-commutative algebra among ($U, V, T$)
and simplifies the problem to that of a usual two-mode
SU(2) system.

One of the major results of this paper is that the ef-
fective Hamiltonian of a general spinor-1 condensate can be
decomposed as

$$H = h\chi_{\text{KU}} T_3^2 + h\chi_{\text{BD}} (U_V + h.c.);$$

which involves both the Kitagawa-Ueda (KU) and
Barnett-Dupertuis (BD) type of spin squeezing simultane-
ously. In other words, all three fictitious spins can
indeed be found squeezed in a spinor-1 condensate, as
the above two distinct nonlinearities commute with each
other, and therefore squeeze all three SU(2) subspaces
simultaneously. We find that the BD type interaction
dominate the SU(2) spin squeezing when $N_0$ is large; while in the opposite limit the
KU type squeezing governs. For intermediate values
of $N_0$ it is necessary to consider a generalized spin squiz-
ing for the three mode spinor-1 system. To achieve this,
we provide a new criterion for the $U-V$ spin squiz-
ing based on reduced quantum fluctuations imposed by
the BD type nonlinearity. When such a condition is satis-
fied, the state of a spinor-1 condensate as a macroscopic-
coherent quantum object becomes useful for three-mode
spectroscopic and interferometric applications. We fur-
ther show that this condition also corresponds to a two-
mode entanglement in terms of the Holstein-Primakoff
bosonic modes, and it reduces to previous results in the
large $N_0$ limit [1, 2]. Squeezing in $T$ spin is particularly useful for quantum information applications based on collective (Dicke) states $|J, J_z⟩$. These states are in fact stationary in a spinor-1 condensate and can be manipulated via external control fields [4]. Since $J_z = N_+ - N_- = 2T_3$, such $T$-squeezed states ensure well-defined Dicke states.

In our study of spin squeezing in a spinor-1 condensate as outlined in this paper, we present a systematic approach by recognizing the $(U, V, T)$ pseudo-spin subspaces as the Gell-Mann (quark) realization of the SU(3) algebra [20]. Similar recognitions are found useful in the recent discussions of quantum and semi-classical dynamics of three coupled atomic condensates [21], where the BD-type two-spin squeezing nonlinearity was absent. Earlier investigations of three level atomic systems also made efficient use of the density matrix and expressed atomic Hamiltonians in terms SU(3) generators [22, 23]. The main difference between our approach (on spinor-1 BEC) and those earlier studies are the enveloping Weyl-Heisenberg algebra of the bosonic operators, which leads to subsequently much larger Hilbert space of the system. In addition to spin squeezing, we also investigate other quantum correlation effects, e.g. condensate fragmentation with the new theoretical framework. We show that previous theories based upon the SO(2) rotational symmetry group cannot give a decomposition of angular momentum operator with nonlinearities that could easily be considered for spin squeezing. For instance, neither $H \propto L^2 - 2N$ [2] nor $H \propto N(N-1) - A^1 A$ [1] can lead to any simple recognition of the nonlinear coupling among various spin-components.

II. SU(3) FORMULATION FOR SPINOR-1 BEC

Under the single-mode approximation [7], a spinor-1 condensate is described by the Hamiltonian

$$H = \mu N - \lambda_\alpha N(N+1) + \lambda_\alpha (L^2 - 2N),$$

where $N = n_+ + n_- + n_0$ is the total number of atoms and the collective angular momentum ($L$) with the familiar raising (lowering) operator $L_+ = \sqrt{2}(a_+^\dagger a_0 + a_0^\dagger a_-)$ ($L_- = L_0^\dagger$), $L_z = n_+ - n_-$, and $n_1 = a_1^\dagger a_1$. $\mu$ is the chemical potential. $\lambda_{\alpha,\beta}$ are renormalized interaction coefficients, they are related to various s-wave scattering lengths and $\phi(\vec{r})$ [3, 4]. The validity of the single-mode approximation is now reasonably well understood [2, 3, 22], and for ferromagnetic interactions, it is in fact exact as shown recently [2]. $a_{\pm,0}$ can form a similar Schwinger representation of SU(3) in the following manner,

$$T_+ = a_+^\dagger a_-, \quad T_3 = \frac{1}{2}(n_+ - n_-),$$

$$\left( \begin{array}{c} V_+ \\ U_+ \end{array} \right) = a_+^\dagger a_0, \quad \left( \begin{array}{c} V_3 \\ U_3 \end{array} \right) = \frac{1}{2}(n_+ - n_-),$$

$$N = n_+ + n_- + n_0.$$  

FIG. 1: The action of $G_Y$ in $T_3 - Y$ space. Any point is coupled only to its next nearest neighbors along $T_3$-axis through a two step process $V_3U_+$ on the Y-line with $T_3$ unchanged in the end. Note that $V_3$ and $U_+$ commutes with each other and the conjugate process is also shown.

$$Y = \frac{1}{3}(n_+ + n_- - 2n_0).$$

The linear combinations $X_\pm = X_{\pm}(X = T, U, V)$ together with $T_3$ and $Y$ resemble the set of eight generators of SU(3) in spherical representation. $T_{\pm,3}, U_{\pm,3}$ and $V_{\pm,3}$ fulfill commutation relations $[X_+, X_-] = 2X_3$ and $[X_3, X_{\pm}] = \mp X_{\pm}$ of the SU(2) algebra. We call T-operators the isospin and Y operators the hypercharge only because of their formal resemblance [21]. $U$ and $V$ subalgebras will be called $U$- and $V$-spin, respectively. We then have $L_+ = \sqrt{2}(V_+ + U_-), L_3 = 2T_3,$ and

$$L^2 = 4T_3^2 + \frac{1}{2}(N - \epsilon_+)(N - \epsilon_-)$$

$$-2(Y - Y_0)^2 + G_Y,$$

$$G_Y = 2(V_3U_+ + h.c.),$$

with $\epsilon_\pm = -3/2 \pm \sqrt{2},$ and $Y_0 = -N/6 - 1/4$. For $N \gg 1$ this gives $Y_0 \approx -N/6$, which corresponds to $n_0 = N/2$. Using $[T_3, V_3] = \mp V_3/2$ and $[T_3, U_3] = \pm U_3/2$ we find $T_3, G_Y = 0$ consistent with $[H, L_\pm] = 0$. Hence the Hamiltonian (8) separates into two commuting parts $H = H_N[N] + H_T[T_3] + H_Y[Y]$. To our knowledge, this decomposition has not been discussed before. In Ref. [21], a model of $H = \chi(T_3^2 + 3Y^2)$ has been considered for both the quantum and semi-classical dynamics of $Y$ as well as for SU(3) coherent states. We note that the decomposition (11) differs from the Casimir relation for the two mode case [23]. In fact, with the spin singlet pair operator $A = (a_0^\dagger - 2a_+a_-)/\sqrt{3}$ as defined by Koashi and Ueda [8], we find $L^2 = N(N + 1) - A^\dagger A$.

Denote the simultaneous eigen-states of commuting operators $(N,Y,T_3)$ as $|N,T_3,Y⟩$, we find

$$\left( \begin{array}{c} V_3 \\ U_3 \end{array} \right) |N,T_3,Y⟩ = \sqrt{\left( \frac{N}{3} - Y \right) \left( \frac{N}{3} \pm T_3 + \frac{Y}{2} + 1 \right)} |N,T_3 \pm \frac{1}{2}Y, 1⟩,$$

where $|N,T_3,0⟩$ is a two step process $V_3U_+$ on the Y-line with $T_3$ unchanged in the end. Note that $V_3$ and $U_+$ commutes with each other and the conjugate process is also shown.
i.e., \(G_Y\) only couples next nearest neighbors along the \(Y\)-axis through off-axial hopping as depicted in Fig. 1. Perhaps it is not surprising that operators \(T_{\pm}, U_{\pm}\), and \(V_{\pm}\) are simply the off-diagonal elements of the single particle density operator \(\rho_{\mu\nu} = a_{\mu}^\dagger a_{\nu}\), while \(N, T_{3}\), and \(Y\) are related to the diagonal elements. \(T_{\pm}, V_{\pm}, U_{\pm}\) all raise and lower the \(T_{3}\) value by \((1\ or\ 1/2)\).

As an example we consider the simple case of the \(T_{3} = 0\) block along the line in Fig. 1 of the Hamiltonian (1) for \(\lambda'_m < 0\), the ferromagnetic case (as in \(^{87}\text{Rb}\)). The polar case of \(\lambda'_m > 0\) (as in \(^{23}\text{Na}\) BCP (1)) has been discussed in Ref. 3. Dropping the constant \(H_N[N]\) and write in units of \(|\lambda'_m|\), the Hamiltonian becomes

\[H = 2(Y - Y_0)^2 - 2t_Y (|Y| + 2|Y| + h.c.\),

where \(t_Y = (N/3 - Y)(N/3 + Y + 2)\). Following the tight-binding procedure for the restricted two-mode case (+ and -) discussed in Ref. 4, its eigenstates can be found by determining the \(\psi(Y)\) of \(|\psi\rangle = \sum_y Y \psi(Y)\rangle\) through a difference equation

\[E\psi(Y) = 2(Y - Y_0)^2\psi(Y) - 2[t_Y - \psi(Y - 2) + t_Y \psi(Y + 2)].\]

(14)

In the continuum limit and up to the first order in \(O(|Y|/N)\), the equivalent differential form becomes

\[
\left(\frac{E}{8Y_0^2} + 1\right) \psi = -2\frac{\partial^2 \psi}{\partial Y^2} - \frac{1}{Y_0} \frac{\partial \psi}{\partial Y} + \frac{(Y - Y_0)^2}{4Y_0^2} \psi\].

(15)

Its ground state is therefore

\[|\psi\rangle = \sum_y \exp \left(\frac{-\sqrt{2}(Y - Y_0)^2}{8Y_0^2} + \frac{Y - Y_0}{4Y_0}\right) |Y\rangle\],

(16)

which gives a diagonal \(\langle \rho_{\mu\nu} \rangle\) with \(\langle n_0 \rangle = N/2\) and \(\langle n_{\pm}\rangle = N/4\), i.e. a fragmented state 5 10. To check the validity of this approximate analytical result, we also solved the same problem for \(N = 10^3\) within the \(T_{3} = 0\) block by an exact diagonalization procedure. The results are compared in Fig. 4. We see that without any fitting parameter the analytical result agrees well with the exact numerical result. Since the Hamiltonian is block diagonal in even \(n_0\) and odd \(n_0\) spaces, we find two degenerate ground states with even and odd \(n_0\) components respectively. These even-odd ground states display opposite phases and can form a Schrödinger cat state 22. The approximate result here applies for a value of even \(N\), which leads to an even \(n_0\) within the \(T_{3} = 0\) block. Hence, only the even \(n_0\) block of the Hamiltonian is considered. We will also show below that under the single-mode approximation, the exact ground state for \(N \gg 1\) is generally a fragmented state with \(\langle n_0 \rangle = N/2\). More detailed studies and implications of the cat-like ground state in the even-odd number blocks and with their respective phases will be explored further and results presented elsewhere. We note that such cat states separated in the angular momentum \(L_z\) have been found in studies of Josephson type coupled condensates 23 24. In a spinor-1 condensate as considered here, we find that the dynamical behavior can be characterized by \(Y = N/3 - n_0\), which can be expressed as \(Y = 2(U_3 + V_3)/3\). Since the azimuthal phases are conjugate to angular momentum \(z\)-projection operators, the cat-like ground states predicted here resemble the angular momentum cat states of a two-component condensate in its conjugate phase spaces. Finally we note that the symmetry point \(Y_0\) can be adjusted by external control fields which contribute terms proportional to \(n_{\pm,0}\) to the Hamiltonian, and can be absorbed into the \((Y - Y_0)^2\) term through a new \(Y_0\) as \(n_\pm = N/3 + Y/2 + T_3\) and \(n_0 = N/3 - Y\).

We have now seen that the Hamiltonian of the system describes an effectively one dimensional dynamics along the \(Y\)-axis, similar to that of a diffusive random walk process but now with an attractor (for \(\lambda'_m < 0\)) \(Y_0\). Hence, we expect \(Y_0\) influence population dynamics in a similar manner it effects the fragmentation. For \(T_3 = 0\), it is known that populations oscillate around time-averaged values \(n_0 = N/2\) and \(n_{\pm} = N/4\) which are the same as the results we found for fragmented ground states. We conclude that steady-state values of population oscillations as well as fragmentation is determined by the hypercharge symmetry point \(Y_0\), which can be shifted by external fields.

III. TWO-SPIN AND ISOSPIN SQUEEZING

The form of \(G_Y\) suggests the existence of two mode squeezing as was also noted recently by Duan et al. 1, who studied a spinor-1 condensate initially prepared in the Fock state with only \(m_f = 0\) state populated. During the time when the total number of excitations into states \(m_f = \pm 1\) are negligible, the spin mixing term \((G_Y)\) in

\[\text{FIG. 2: The ground state expansion coefficients } \psi(Y) \equiv \psi(n_0) \text{ as a function of } n_0 \text{ for } N = 1000 \text{ atoms in the } T_{3} = 0 \text{ block. The solid curve is the approximate analytical result Eq. (15) without any fitting parameters while the other curves are obtained by an exact diagonalization procedure.}\]
the Hamiltonian simply reduces to a two-mode squeezing nonlinearity via $\langle n_0 \rangle(a_0^\dagger a_0^\dagger + h.c.)$. This creates a continuous variable type entanglement, or mode-entanglement in the second quantization form. In order to relate it to measurable spectroscopic spin squeezing and particle entanglement, Ref. [1] first showed that in the low excitation limit, the two-mode entanglement criterion can also be expressed in terms of spin squeezing parameters for $L_{x,y}$. In order to use the two-level SU(2) definition for spin squeezing of $L_{x,y}$, new pseudo-spins $J_\pm$ was introduced within the the two level subsystems $|+1\rangle \pm |-1\rangle$ and $|0\rangle$. They found that when $L_z \approx 0$, the system Hamiltonian becomes effectively $H = \lambda_0(L_x^2 - 2N) \approx \lambda_0(L_z^2 + L_x^2) \sim (J_{x,z}^2 + J_y^2)$, which causes each spin 1/2 subsystem to be squeezed via the single axis twisting scheme. For the independent single-axis twisting scheme to work efficiently in achieving substantial spin squeezing, the commutator $[J_{+x/y}, J_{-x/y}] = (T_x - T_y)/4 \propto T_y$, needs to be small. Hence, squeezing in the isospin is essential to achieve this two-mode squeezing goal. Without it, large quantum fluctuations in $T_y$ would destroy the two mode squeezing. Unfortunately, both the relationship between the two-mode squeezing and spin squeezing as well as the interpretation in terms of a dual single axis twisting fails to be adequate for higher excitations under more realistic situations. Indeed, for the extreme opposite case of $n_0 \ll n_\pm \approx N/2$, the Hamiltonian describes a single-mode amplitude squeezing as it reduces to $G_Y \sim (a_0^2 + h.c.)$. Anywhere in between of these two extreme limits, we propose a new type of squeezing, the two-spin squeezing as a generalization of single spin squeezing by taking into account quantum correlations for mode-entanglement applications. We first note that the two extreme types of squeezing in $G_Y$ can be handled at arbitrary levels of excitation by introducing a new two-spin squeezing operator via

$$K_+ = V_+ U_+ \sim \begin{cases} a_0^\dagger a_0^\dagger, & n_0 \gg n_\pm, \\ a_0^\dagger a_0^\dagger, & n_0 \ll n_\pm, \end{cases}$$

(17)

$$K_- = K_+^\dagger,$$

(18)

with $[K_-, K_+] = 2K_3$. The squeezing mechanism in Hamiltonian (14) is now understood to be a generalized Barnett-Dupuis (BD) type squeezing via the $V_+ U_+ + h.c.$ nonlinearity in $G_Y$. This is significantly more complicated than the two bosonic mode squeezing as the two spins $U$ and $V$ have a non-commuting algebra. The mode-entanglement of approximate bosonic modes $\alpha_\pm = a_0^\dagger a_0^\dagger/\sqrt{\langle n_0 \rangle}$ of Ref. [11] can in fact be generalized to mode-entanglement between exactly bosonic Holstein-Primakoff modes $29$ $a_{x=y,u,v}$ defined through $X_\pm = a_\pm \sqrt{\langle n_\pm \rangle - N_\pm}$ and $X_3 = N_a - S_3/2$, in the spin $S_a/2$ realization of corresponding SU(2) algebras of $U$ and $V$ spins with $N_a = a_0^\dagger a_0$. The squeezing treatment with the exact bosonic modes $a_U$ and $a_V$ remains to be more complicated than the usual two bosonic mode squeezing as it also suffers from the underline non-commutating algebra. This representation reduces to the usual SU(1,1) two-mode squeezing or amplitude squeezing in the appropriate limits. At low excitations when $n_0 \approx N$, we have $X_3 \approx -N/2$, $S_2 \approx \langle n_0 \rangle$, and $N_\pm \approx 0$. In this case, $X_\pm \approx \sqrt{\langle n_0 \rangle}a_\pm$ and $G_Y = 2\langle n_0 \rangle(a_0^\dagger a_0^\dagger + h.c.)$ demonstrates the two-mode [SU(1,1)] squeezing as in Ref. [1]. In the large $n_0$ scheme of Ref. [11], such modes are sparsely populated since, $a_0^\dagger a_0^\dagger = n_\pm (1 + n_0)/n_0$. In the opposite case of large $n_\pm$, we are in the strong excitation regime with $N_\pm \approx 1$, $S_{x/u} \approx (n_\pm)$ which gives effective modes to be $a_0^\dagger a_0^\dagger/\sqrt{\langle n_0 \rangle}$ with large occupations.

In order to define two-spin squeezing introduced via the $K$-operators in a similar way to the two-mode bosonic squeezing, we introduce Hermitian quadrature operators

$$X_\alpha^u = (e^{i\alpha} U_+ + h.c.)/\sqrt{2},$$

(19)

$$X_\alpha^v = (e^{i\alpha} V_+ + h.c.)/\sqrt{2},$$

(20)

$$Q_\alpha^{+\pi/2} = (X_\alpha^u + X_\alpha^v)/2,$$

(21)

$$Q_\alpha^{-+\pi/2} = (X_\alpha^{+\pi/2} - X_\alpha^{-+\pi/2})/2.$$  

(22)

From $J_{\pm} = \vec{V} - \vec{U}$ and $J_{x,y} = (3Y/2 \pm T_x)/2$ we find $Q_{\pm} = \langle \vec{n}(\alpha) \cdot J_{\pm} \rangle$ with $\vec{n}(\alpha) = (\cos \alpha, \sin \alpha, 0)$. If $U$ and $V$ were uncorrelated, their respective quantum noises would contribute to that of $J_\pm$ additively. Existence of quantum correlations between the $U$- and $V$-spins would reduce the quantum fluctuation in $J_\pm$. Thus, the $(J_\pm)$ spin squeezing is achieved by two-spin $(U-V)$ squeezing. From $[U_- , V_- ] = 0$, we find that

$$Q_{\alpha}^{+\pi/2} = (Q_{\alpha}^{+\pi/2} + Q_{\alpha}^{-+\pi/2})^2$$

$$= \sum_{u,v} [\langle \Delta X_{\alpha \dagger}^u \rangle^2 + \langle \Delta X_{\alpha \dagger}^v \rangle^2 + \langle \Delta X_{\alpha \dagger}^{+\pi/2} \rangle^2 + \langle \Delta X_{\alpha \dagger}^{-+\pi/2} \rangle^2] + C_{uv},$$

(23)

with the $U-V$ correlation function

$$C_{uv} = \langle \Delta X_{\alpha \dagger}^u \Delta X_{\alpha \dagger}^v \rangle / 4 + c.c.$$  

(24)

denote the correlations among $U$-$V$ spins to the quadrature noise, which reduces the uncertainty bound when two spin squeezing occurs. We find a lower bound for the quadrature noise $\sum_{u,v} [\langle \Delta X_{\alpha \dagger}^u \rangle^2 + \langle \Delta X_{\alpha \dagger}^v \rangle^2]$ by noting that $\langle X_{\alpha \dagger}^u X_{\alpha \dagger}^{+\pi/2} \rangle = 2V_3^2$, $\langle X_{\alpha \dagger}^u X_{\alpha \dagger}^{-+\pi/2} \rangle = -2iU_3$, and $| \langle U_3 \rangle | + | \langle V_3 \rangle | \geq | \langle U_3 + V_3 \rangle | = 3\langle Y \rangle / 2$. We finally find

$$Q_{\alpha}^{+\pi/2} + \langle \Delta X_{\alpha \dagger}^{+\pi/2} \rangle^2 \geq 3\langle Y \rangle / 4 + C_{uv},$$

(25)

Therefore, taking into consideration the important spin-spin correlation between different particles similar to the 1/2 case [13], we can introduce the $U$-$V$ squeezing condition as

$$C_{uv} = \frac{\langle \Delta X_{\alpha \dagger}^u \rangle^2 + \langle \Delta X_{\alpha \dagger}^{+\pi/2} \rangle^2}{\langle Y \rangle} < 3/4,$$

(26)

similar to the continuous variable system [31]. This is the major result of our paper on the two [SU(2)] spin squeezing within the SU(3) of a spinor-1 condensate. The significance of spin-spin correlation function to spin squeezing...
and entanglement for a two-mode system was previously discussed in Ref. [31], where they showed that a negative, finite correlation parameter causes spin squeezing and entanglement of the atomic states. With the Holstein-Primakoff relations, it is straightforward to show this condition contracts into $(\xi_\pm^2)^2 + (\xi_{\pm}^{\pi/2})^2 < 2$ when $n_0 \to N$. Thus Eq. (24) generalizes the two-mode entanglement criterion $(\xi_\pm^2)^2 + (\xi_{\pm}^{\pi/2})^2 < 2$ at low excitations [11] to arbitrary levels of excitation for two spin squeezing. For completeness, we note the squeezing parameter for $J_\pm$-spins are [11]

$$
(\xi_\pm^2)^2 = \frac{N((\Delta Q_\pm^2)^2)}{\langle Q_\pm^{\alpha+\pi/2} \rangle^2 + \langle J_{\pm3} \rangle^2},
$$
while the Heisenberg uncertainty relation gives $\Delta Q_\pm \Delta Q_\pm^{\pi/2} \geq |\langle J_{\pm3} \rangle|/2$. For many particle entanglement of three level atoms, the criterion is given by either $\xi_\pm < 1$.

The $U-V$ squeezing discussed above displays existence of nonlinear interactions within/among $T$, $U$, and $V$ subspace of (6). One may also contemplate for a one-axis isospin twisting (through $T_3^2$) of the particular form of $L^2$ [11]. However, the dynamics of spinor-1 BEC becomes considerably more complicated because of off-axis hopping processes along the hypercharge axis (as in Fig. 1). Due to the non-commutativity of sub-spin systems $(U, V, T)$, squeezing and entanglement appears even without essentially any axis-twisting. In fact, even when $T_3 = 0$, squeezing within the isospin subgroup can still happen as the $U-V$ two-spin squeezing interaction would redistribute the noise also for the isospin subspace, in addition to the $U-V$ spin space. To appreciate this fact, let us consider the rotation operator involving only $U-V$ spins and employ the SU(2) disentangling theorem to obtain

$$
R[\zeta] = e^{\zeta L_+ - \zeta^* L_-} = e^{\eta L_+} (1 + \eta \eta^*) L_+ e^{-\eta^* L_-},
$$
with $\eta = \zeta \tan \zeta/|\zeta|$. Using $[V_+, U_-] = T_+$ and $[V_+, T_+] = [U_-, T_+] = 0$, we find

$$
e^{\eta L_+} = e^{\eta^2 V_+} e^{\eta^* U_-} e^{-\eta^* T_-}/\sqrt{2}.
$$
Hence, we arrive at

$$
R[\zeta] = e^{\sqrt{2} \eta V_+} e^{\sqrt{2} \eta U_-} R_T[\eta] e^{-\sqrt{2} \eta^* V_+} e^{-\sqrt{2} \eta^* U_-},
$$
with a rotation operator within isospin space via $R_T = \exp(-\eta T_+ / \sqrt{2})(1 + \eta^2 T_+^2) \exp(-\eta^* T_- / \sqrt{2})$. This result reflects the nature of Euler-angle rotations in three dimensions for a spin-1 system. We thus conclude that squeezing in $J_3$ through redistributing the noise via rotations is always accompanied by a redistribution of the noise in the isospin subspace. Squeezing and many particle entanglement via the isospin can be checked using the usual spin squeezing criterion, which for both $T$-squeezing and the above derived $U-V$ squeezing are independent of their respective initial conditions. Hence, we have now greater freedom to consider a suitably prepared spinor-1 condensate to achieve many-particle and/or mode entanglement for quantum information applications as well as various type spin squeezing for atom interferometry and spectroscopy applications in the long time limit with more macroscopic populations in all $f = 1$ three component states can occur. In the limiting case discussed before either $n_0 \sim N$ or $n_\pm \sim N$ is required to be large, the quantum states (modes) of interest are always sparsely populated. More generally, one can use Raman coupled laser pulses on a spinor-1 condensate to generate states with arbitrary populations in each mode and with arbitrary initial phases. This allows then for the consideration of stationary states in the fully quantum mechanical framework for their use in squeezing-entanglement applications.

**IV. RESULTS AND DISCUSSIONS**

We now present some results on the numerical investigation of isospin squeezing. If the condensate atom number $N$ is fixed, a generic state $|\psi(0)\rangle = (a_0 a_+^\dagger + \alpha_+ a_-^\dagger + \alpha_- a_+^\dagger)^N|0, 0, 0\rangle/\sqrt{N!}$, can be prepared with Raman pulses [4] where $|0, 0, 0\rangle$ is the vacuum in the Fock basis $|n_0, n_-, n_+\rangle$ and $\alpha_j = |\alpha_j| e^{i\delta_j}$ complex. Using $m = n_+ - n_-$, we can write

$$
|\psi(0)\rangle = \sum_{mk} \psi_{Nmk}(\tilde{\alpha}) \left| 2k,\frac{N-m}{2} - k,\frac{N+m}{2} - k \right>,
$$
where $\tilde{\alpha} = (a_0, \alpha_-, \alpha_+, k) = 0, 1, \cdots$, $(N - |m|)/2$ for even $N + m, 2k = 1, 3, \cdots, (N - |m|)$ for odd $N + m$, and

$$
\psi_{Nmk} = \sqrt{C_{N}^{2k} C_{N-k}^{N-m}} a_0^{2k} \alpha_-^{N-m-k} \alpha_+^{N-m-k},
$$
where $C_n^m = \binom{n}{m}$ denotes the binomial coefficient. The basis transformation coefficients between angular momentum and Fock states are available from Ref. [33], written in more compact forms as

$$
|lm\rangle = \sum_k G_{lmk} \left| 2k,\frac{N-m}{2} - k,\frac{N+m}{2} - k \right> (31)
$$
with

$$
G_{lmk} = 2^k \frac{s_l}{4^r} \sum_r (-1)^r \binom{Nl}{kr} \left[ Nl \right] \left[ kr \right]
$$
and the symbolic notation

$$
\left[ \binom{Nl}{kr} \right] = \sqrt{C_{2k}^{2r} C_{2k+2r}^{2l} C_{N-l-2r}^{N-2k} C_{N-k-l}^{(N-1)/2-2r}} \times C_{l-2k-2r}^{(l-m)/2-k+r} / \sqrt{C_{N-2k}^{N-2k}}. (32)
$$
We note $l = N, N - 2, \cdots, N - 2|N/2|$ with $|n| = n, n - 1/2$ for $n$ even, odd, and $r = \max[0, k -
The population in the $m = 1$ state of Hamiltonian (6) as $\sum_{m=0}^{N} \rho_m(0) = \Lambda^2$, which takes an asymptotic form

$$G_{N0k} = \sqrt{C_{2N}^{2k}C_{2N-2k}^{N-m}} \sum_{k=0}^{N/2} \frac{1}{2^{k}} C_{2N}^{2k} \langle 0 \rangle = \Lambda^2 \rho_m(0) = \Lambda^2,$$

when $N \gg 1$. This is analogous to $\langle 0 \rangle = 1/2$ when $N = 2$. The special case of $\delta = 0$ and $\alpha_+ = \alpha_-$, we obtain $\alpha_0^2 = 1/2$, in complete agreement with earlier results for $\sum_{m=0}^{N} \rho_m(0) = \Lambda^2$. By defining $P_0 = |\alpha_0^2|$ as spin component populations, we find that stationary states require $P_0 = 1/2$ whenever $P_0 = P_+$. This is, however, not sufficient without establishing the phase constraint found above, which becomes particularly useful as it provides for more freedom in state preparation using Raman coupled laser fields. As an example, we now consider isospin squeezing with the same form of initial states as in Ref. [11] for $\alpha_0 = \sqrt{T_0} e^{i\theta/2}$ and $\alpha_\pm = \sqrt{T_0 - T_0^*}$. The $\sum_{m=0}^{N} \rho_m(0) = 1/2$ as the only non-vanishing spin component at $t = 0$. The population in the $m = 0$ component then acts as a knob between the two extreme squeezing type discussed earlier as well as between the $G_\alpha$ and $T_3$ terms. In the special case of within the $T_3 = 0$ block, we find that the dynamics of the system is determined only along the hypercharge $Y$ axis. Previous study in Ref. [11] with initial state $|0, N_0, 0\rangle$ results in spin-mixing dynamics, due to which $N_0$ was found to quickly reduce to some value without further oscillations or recovery. In our scheme, we find $n_{0,0}$, $n_{0,\pm}$ all exhibits collapse and revival patterns, so does $Y$ as $Y = N/3 - n_0$. Even for the $T_3 = 0$ block, we have seen redistribution of noise among the $U-V$ components affects fluctuations in the isospin as well. The squeezing parameter

$$\xi_\phi^2 = \frac{N(\Delta(T'_y)^2)}{\langle T'_y^2 \rangle + T_2^2},$$

is analogous to $\xi_\phi^2$ but for isospin $T' = R[\phi|T$ after rotated around $x$-axis by an angle $\phi$. Isospin squeezing is then characterized by $\xi_\phi < 1$. At $\phi = 2\pi/3$, this occurs after a very short time (see Fig. 3). It is especially interesting to note that $\xi_\phi$ exhibits collapse and revival patterns. The optimal angle $\phi_{\text{min}}$ for maximal squeezing (minimal $\xi_\phi$) is shown in Fig. 4. It oscillates around its time-averaged value $\approx 2\pi/3$. In general, we find $\xi_\phi$ achieves its minimum sooner and the minimum is smaller with decreasing values of $\theta$ or increasing values of $P_0$.

This effect is clearly unique to three-mode systems. In usual population spectroscopy (e.g. Ramsey type)
or in interferometry (e.g. of Mach-Zehnder type) for a
two-mode system, particle partitioning noise and phase
sensitivity can only be controlled by the modes involved
directly. Here, the \( m_f = 0 \) mode actually does not
belong to the isospin group, yet it still influences the
isospin noise properties. In contrast to the two-mode re-
result \( N_\pm = J \pm J_z = N/2 \pm 2T_3 \), a three-mode system has
\( N_{\pm} = N/3 + (U_3 + V_3 + 2T_3)/2 \). A direct measurement
of \( N_+ \) or \( N_- \) will uncover all noise terms due to quan-
tum correlations among the various spin components. A
measurement of \( N_+ - N_- \), on the other hand, is similar
to the two-mode case as the result is only affected by
the noise in the isospin. When \( T_3 = 0 \), the influence of the
\( m_f = 0 \) mode population is reflected in the two-spin
squeezing interaction between the \( U \) and \( V \)-spins, which
in turn also redistributes the noise in isospin.

In Fig. 6, results of two-spin squeezing are shown for
various initial Fock states \( |N_-, N_0, N_+ \rangle \) of a spinor-1 con-
densate. The lack of oscillations in Fig. 1(a) is due
to non-oscillatory behavior of \( n_0 \) for the particular ini-
tial conditions used here. The solid curves are for the
two-mode entanglement criterion of Ref. \[1\], valid only
when \( N_0 \gg N_\pm \). We see that when the initial states are
such that \( N_\pm \) modes are not near empty, the achievable
two-spin or two mode squeezing essentially diminishes.
However, there is a also turning point, when squeezing
is again recovered if \( N_\pm \) becomes significantly populated.
Hence, we have found a new squeezing regime when the
initial conditions are such that \( N_\pm \gg N_0 \). The results
are almost equivalent to the case \( N_0 \gg N_\pm \) considered
in Ref. \[1\]. This new initial condition generates the
two-mode entanglement via two spin squeezing between
the \( U-V \) spin modes, i.e. between the holstein-Primakoff
bosons. It should be noted that the two-mode entan-
glement criterion in terms of spin squeezing parameters
\( (\xi^2)_{UV} \) has been derived for \( N_0 \gg N_\pm \) in Ref. \[1\]. We
show here that this criterion is also satisfied in the op-
posite case of \( N_0 \ll N_\pm \). This observation emphasizes
that the \( U-V \) squeezing criterion and the corresponding
mode-entanglement can be sought for other initial condi-
tions when the criterion of Ref. \[1\] is no longer applicable.
For that aim, we consider an initial state \( |25, 0, 75 \rangle \)
as shown in Fig. 6 where the \( U-V \) squeezing is indeed

\[ \text{V. CONCLUSION} \]

We have provided a comprehensive treatment of quan-
tum correlations in a spinor-1 condensate. Although
no nonlinear interaction is apparent in the spinor con-
densate Hamiltonian when single mode approximation
is made, interesting quantum correlations do develop
within subgroups of the SU(3) system. We have an-
alyzed a spinor-1 condensate in terms of its \( T \)-, \( U \)-,
and \( V \)-spin components. We have found and character-
ized squeezing within one particular subgroup, similar
to that of the isospin structure and we have numerically
investigated its dynamics in terms of collapses and re-
vivals. We have developed the \( U-V \) spin squeezing as a
generalization of the often adopted spin (1/2) squeeze-
ing \[1\] to two-spin squeezing. Its relation to mode-
entanglement \[1\] in the Holstein-Primakoff representation
is also pointed out. We have presented new results
for condensate fragmentation and spin-mixing phenom-
ena in terms of the hypercharge symmetry and provided
general phase-amplitude conditions for stationary states
in the full quantum regime.

In a typical experiment, a small magnetic field gradi-
ent may be available \[3\], which results in an effective
Hamiltonian \[3\] \( H_B = (T_+ + T_-) + \beta T_3^2 - \gamma_B T_3 \), instead
of \[3\], with \( \alpha \), \( \beta \), and \( \gamma_B \) various renormalized parameters.
In this case isospin squeezing still occurs through the
one-axis twisting nonlinearity \[13\].

Spin squeezing parameters can be measured directly by

\[ \text{FIG. 5: Time-dependent } U-V \text{ squeezing parameter (dashed curve) and two-mode entanglement criterion (solid curve) for } N = 100 \text{ atoms initially prepared in a Fock state of } \psi(0) = |N_-, N_0, N_+ \rangle: |0, 100, 0 \rangle \text{ in (a), } |1, 98, 1 \rangle \text{ in (b), } |25, 50, 25 \rangle \text{ in (c), and } |50, 0, 50 \rangle \text{ in (d).} \]

\[ \text{FIG. 6: Same as Fig. 5 but now for the initial state } \psi(0) = |25, 0, 75 \rangle. \]
the interferometry or Ramsey spectroscopy [13]. Alternatively, the isospin (T) squeezing in spinor-1 condensate can also be observed experimentally with light scattering. Using Raman coupled laser fields, an interaction of the type \( H_R = g(T_+ J_+ + h.c.) \) can be engineered [34, 35], where \( J_- = \sqrt{2}(a_L a_S^\dagger + a_S^\dagger a_L) \) is an angular momentum operator, with \( a_S, a_L, a_A \) the annihilation operators for anti-Stokes, Stokes, and pump photons. The interaction \( H_R \) allows for the mapping of spin correlations into photon correlations as the total angular momentum \( T_3 + J_z \) is conserved. The solutions for \( J_-(t) \) depend on the initial conditions \( J_-(0) \) and \( T_-(0) \) [34, 35]. Therefore, the quadrature operators of scattered photons are directly related to initial condensate spin quadratures and a homo-
dyne measurement for Stokes parameters of the Raman field can reveal isospin squeezing [36].

VI. ACKNOWLEDGEMENTS

We thank Dr. Y. Su for helpful discussions. This work is supported by the NSF grant No. PHY-9722410 and by a grant from the National Security Agency (NSA), Advanced Research and Development Activity (ARDA), and the Defense Advanced Research Projects Agency (DARPA) under Army Research Office (ARO) Contract No. DAAD19-01-1-0667.