BETTI SPLITTING OF COMPONENTWISE LINEAR IDEALS

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Abstract. We say that a monomial ideal $I$ admits a Betti splitting $I = J + K$ if the Betti numbers of $I$ can be determined in terms of the Betti numbers of the ideals $J, K$ and $J \cap K$. Given a componentwise linear ideal $I$, we prove that $I = J + K$ is a Betti splitting of $I$, provided $J$ and $K$ are componentwise linear too. Applications are given showing that this result is suitable for recursive procedures. We get information on the Alexander dual of vertex-decomposable and shellable simplicial complexes and we determine the graded Betti numbers of the defining ideal of three general fat points in the projective space.

1. Introduction

Our aim is to pursue the spirit of Mayer and Vietoris, Eliahou and Kervaire, Francisco Hà and Van Tuyl in order to find suitable decomposition of the Betti table of a monomial ideal, possibly available for recursive procedures.

Let $K$ be a field and let $I \subseteq R = K[x_1, \ldots, x_n]$ be a monomial ideal. Consider $J, K \subseteq I$ monomial ideals such that the set of minimal generators of $I$ is the disjoint union of the minimal generators of $J$ and $K$. We say that $I = J + K$ is a Betti splitting of $I$ if

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K),$$

for all $i, j \geq 0$,

where $\beta_{i,j}(\cdot)$ denotes the graded Betti numbers of a minimal $R$-free graded resolution.

This approach was used by Eliahou and Kervaire [5], giving an explicit formula for the total Betti numbers of stable ideals. Fatatibi [7] specified the formula for graded Betti numbers. Many authors wrote papers applying the Eliahou-Kervaire technique to the resolution of special classes of monomial ideals (see [6], [7], [8], [10], [16], [18], [19], [20], [21]). In [9] Francisco, Hà and Van Tuyl proved that if $J$ and $K$ have both a linear resolution, then $I = J + K$ is a Betti splitting of $I$ [9 Corollary 2.4]. If the ideal $I$ has a linear resolution, then the converse holds (Proposition 2.4). In this paper we prove that, if $I$ is a componentwise linear ideal, then $I = J + K$ is a Betti splitting of $I$, provided $J$ and $K$ are componentwise linear (Theorem 3.1).

Componentwise linear ideals have been studied by several authors (see [13], [8], [10], [12]). Stable ideals, ideals with linear quotients and ideals with linear resolution are componentwise linear ideals.

Notice that, up to polarization, componentwise linear monomial ideals can be considered squarefree (Proposition 3.2) and the action preserves the numerical invariants of the minimal free resolution. By the Stanley-Reisner correspondence, monomial squarefree ideals correspond to simplicial complexes $\Delta$. This is an important bridge between Commutative Algebra and Combinatorics. In particular, by Hochster’s formula (see [12],[14]), the graded Betti numbers of the squarefree monomial ideal $I_\Delta$ reflect geometric and topological information on $\Delta$. This gives to our paper one more motivation coming from shape recognition.
Topological features of the objects are captured by the study of simplicial homology. One of the main challenges in homology computation is to be able to deal with currently available data sets, thus leading to high-dimensional complexes with a large number of vertices. In order to reduce computational costs, the strategy is to decompose these shapes in smaller ones, make computations on pieces and then to recover the information about the original shape.

As a consequence of the main result we recover a result of Moradi and Kosh-Ahang [16], proving that the Alexander dual of a vertex-decomposable simplicial complex admits \( x_i \)-splitting (Corollary 4.2). We can prove a Betti splitting for shellable simplicial complexes (Corollary 4.4), showing that in general they do not admit \( x_i \)-splitting (Example 4.3). A further application is an extension of a result proved by Valla in [21]. By using a recursive approach we can compute explicitly the graded Betti numbers of the defining ideal of three general fat points in the projective space (Corollary 5.2, Corollary 5.3).

Acknowledgements. The results presented in this paper are part of my PhD thesis and I would like to thank my advisors, Maria Evelina Rossi, Leila De Floriani and Emanuela De Negri for their encouragements and their constant care about my work. The author is grateful to Tai Huy Hà for his valuable comments and remarks about a preliminary version of this paper and to Aldo Conca, Ulèrico Fugacci and Matteo Varbaro for helpful discussions and suggestions.

2. Preliminaries

Let \( \mathbb{K} \) be a field, \( R = \mathbb{K}[x_1, ..., x_n] \), \( \mathfrak{M} \) the maximal homogeneous ideal of \( R \), \( I \subseteq R \) an homogeneous ideal. Denote by \( \beta_{i,j}(I) = \dim_\mathbb{K} \text{Tor}_i(I, \mathbb{K})_j \) the graded Betti numbers of \( I \) and by \( \beta_i(I) = \sum_{j \in \mathbb{N}} \beta_{i,j}(I) \) the total i-th Betti numbers of \( I \). Denote by \( \deg(m) \) the degree of a monomial \( m \), by \( G(I) \) the minimal system of monomial generators of a monomial ideal \( I \) and by \( \text{indeg}(I) \) the lowest degree of a generator in \( G(I) \).

Definition 2.1. Let \( I, J \) and \( K \) be monomial ideals such that \( I = J + K \) and \( G(I) \) is the disjoint union of \( G(J) \) and \( G(K) \). Then \( J + K \) is a Betti splitting of \( I \) if

\[
\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K), \text{ for all } i, j \in \mathbb{N}.
\]

By [9 Proposition 2.1] the previous definition can be given in terms of the vanishing of some maps between \( \text{Tor} \)-modules. In some cases we focus our attention in a special splitting of a monomial ideal \( I \). Let \( J \) be the ideal generated by all monomials of \( G(I) \) divided by a variable \( x_i \) and let \( K \) be the ideal generated by the remaining monomials of \( G(I) \). If \( I = J + K \) is a Betti splitting, we call \( I = J + K \) a \( x_i \)-splitting of \( I \).

If \( I \) is generated in degree \( d \) we say that \( I \) has a \( d \)-linear resolution if \( \beta_{i,i+j}(I) = 0 \) for each \( i \in \mathbb{N} \) and \( j \neq d \). When the context is clear, we simply write that \( I \) has a linear resolution. The Castelnuovo-Mumford regularity of \( I \) is defined by \( \text{reg}(I) = \max\{j - i | \beta_{i,j}(I) \neq 0\} \). An ideal \( I \) generated in degree \( d \) has a \( d \)-linear resolution if and only if \( \text{reg}(I) = d \).

Componentwise linear ideals were introduced by Herzog and Hibi in [13]. Denote by \( I_{<j>} \) the ideal generated by all the homogeneous polynomials of degree \( j \) belonging to \( I \). In the monomial case, \( I_{<j>} \) is simply the ideal generated by monomials of degree \( j \) belonging to \( I \).
Definition 2.2. \( I \) is componentwise linear if \( I_{<j>} \) has a linear resolution, for each \( j \in \mathbb{N} \).

If \( I \) is a monomial squarefree ideal, denote by \( I_{[j]} \) the ideal generated by squarefree monomials of degree \( j \) belonging to \( I \) (for \( j > n \) we have \( I_{[j]} = 0 \)). Note that \( I \) is a componentwise linear ideal if and only if \( I_{[j]} \) has a linear resolution, for each \( j \in \mathbb{N} \) [13, Proposition 1.5]. The resolution of a monomial squarefree componentwise linear ideal \( I \) can be computed from the resolutions of its graded pieces, as stated in the following result.

**Proposition 2.3.** ([13, Corollary 1.6]) Let \( I \) be a monomial squarefree componentwise linear ideal. Then \((\mathfrak{M}I)_{[j]}\) has a linear resolution for each \( i, j \in \mathbb{N} \) we have \( \beta_{i,i+j}(I) = \beta_i(I_{[j]}) - \beta_i((\mathfrak{M}I)_{[j]}) \).

Let \( I, J \) and \( K \) be monomial ideals, with \( I = J + K \), \( G(I) = G(J) \cup G(K) \) and \( G(J) \cap G(K) = \emptyset \). Francisco, Hà and Van Tuy réal proved in [9, Corollary 2.4] that if \( J \) and \( K \) have a linear resolution, then \( I = J + K \) is a Betti splitting. This condition is actually a characterization of the Betti splitting, provided \( I \) has a linear resolution.

**Proposition 2.4.** Let \( d \) be a positive integer, \( I \) be a monomial ideal with a \( d \)-linear resolution, \( J, K \neq 0 \) monomial ideals such that \( I = J + K \), \( G(I) = G(J) \cup G(K) \) and \( G(J) \cap G(K) = \emptyset \). Then the following facts are equivalent:

1. \( I = J + K \) is a Betti splitting;
2. \( J \) and \( K \) have \( d \)-linear resolutions.

When this is the case, then \( J \cap K \) has a \((d + 1)\)-linear resolution.

**Proof.** Assume 1. holds. Then \( \beta_{i,i+j}(I) = \beta_{i,i+j}(J) + \beta_{i,i+j}(K) + \beta_{i-1,i+j}(J \cap K) \) for all \( i, j \geq 0 \). Let \( i \geq 0 \). For \( j \neq d \) we have \( \beta_{i,i+j}(I) = \beta_{i,i+j}(J) = \beta_{i,i+j}(K) = 0 \), thus \( J \) and \( K \) have a \( d \)-linear resolution. By [9, Corollary 2.4] one has that 2. implies 1.

By [9, Corollary 2.2] we have \( \text{reg}(J \cap K) \leq \text{reg}(I) + 1 = d + 1 \). Since \( \text{indeg}(J \cap K) \geq d + 1 \), then \( \text{reg}(J \cap K) \geq \text{indeg}(J \cap K) \geq d + 1 \), thus \( J \cap K \) has a \((d + 1)\)-linear resolution.

3. **Main Theorem**

In this section we prove our main theorem.

**Theorem 3.1.** Let \( I \) be a monomial componentwise linear ideal, \( J, K \) monomial ideals such that \( I = J + K \), \( G(I) = G(J) \cup G(K) \) and \( G(J) \cap G(K) = \emptyset \). If \( J \) and \( K \) are componentwise linear, then \( I = J + K \) is a Betti splitting of \( I \).

To prove the theorem, we need some preliminary results. First of all we show that we may reduce the problem to squarefree monomial ideals. Let \( I \) be a monomial ideal. Denote \( \mathcal{P}(I) \) its polarization (see for instance [17]). Since polarization is a particular case of a distraction operator (see [1]), using [10, Lemma 2.10] and [11, Corollary 2.10], it can be easily proved the following well-known fact.

**Proposition 3.2.** Let \( I \subseteq R = \mathbb{K}[x_1, \ldots, x_n] \) be a componentwise linear monomial ideal. Then \( \mathcal{P}(I) \) is squarefree componentwise linear.

Notice that graded Betti numbers are preserved under polarization [12, Corollary 1.6.3]. Throughout this section we will assume:
• I is a monomial squarefree componentwise linear ideal.
• J, K ≠ 0 are monomial squarefree componentwise linear ideals such that I = J + K, G(I) = G(J) ∪ G(K) and G(J) ∩ G(K) = ∅.

Let d be a positive integer. Define the ideal
\[ \overline{K}_{[d]} := \{ g \in G(K_{[d]}) | g \notin G(J_{[d]}) \}. \]

We remark that \( \overline{K}_{[d]} \) can be the zero ideal in the case \( K_{[d]} \subseteq J_{[d]} \). If for a certain d we have \( J_{[d]} = \overline{K}_{[d]} = 0 \), then \( J_{[d]} = K_{[d]} \). By eventually exchanging J and K we may assume that only two cases can happen: \( J_{[d]} = \overline{K}_{[d]} = 0 \) or \( \overline{K}_{[d]} \neq 0 \).

Accordingly to the above definition, consider the ideal
\[ (\overline{MK})_{[d]} := \{ g \in G((\overline{MK})_{[d]}) | g \notin G((\overline{MJ})_{[d]}) \}. \]

Note that \( \overline{K}_{[d]} \subseteq K_{[d]} \) and \((\overline{MK})_{[d]} \subseteq (\overline{MK})_{[d]} \).

With the previous notation, we prove the following technical results.

**Lemma 3.3.** Let d be a positive integer. Assume \( \overline{K}_{[d]} \) has a linear resolution. Then \((\overline{MK})_{[d+1]} \) has a linear resolution.

**Proof.** If \( \overline{K}_{[d]} = 0 \) one has \((\overline{MK})_{[d+1]} = (\overline{MK})_{[d]})_{[d+1]} \subseteq (\overline{MJ})_{[d+1]} = (\overline{MJ})_{[d+1]} \), then \((\overline{MK})_{[d+1]} = 0 \). We may assume \((\overline{MK})_{[d+1]} \neq 0 \), \( \overline{K}_{[d]} \neq 0 \). We prove that \((\overline{MK})_{[d+1]} \subseteq K_{[d]} \).

Let \( m \in G((\overline{MK})_{[d+1]}). \) Suppose \( m \notin \overline{K}_{[d]} \). Then for each variable \( x|m \), \( m \in J_{[d]} \). Hence \( m \in G((\overline{MJ})_{[d+1]} \), a contradiction.

Since \((\overline{MK})_{[d+1]} \) is a monomial squarefree ideal, then there are variables \( x_{i_1}, \ldots, x_{i_r} \) such that \((\overline{MK})_{[d+1]} = (x_{i_1}, \ldots, x_{i_r})\overline{K}_{[d]} \) and \( x_{i_j} \) does not divide any generator of \( K_{[d]} \), for each j. Then \( x_{i_1}, \ldots, x_{i_r} \) is a \( K_{[d]} \)-regular sequence. Since \( \overline{K}_{[d]} \) has a linear resolution, by [3 Theorem 2.2], \( \text{reg}((\overline{MK})_{[d+1]}) = \text{reg}((x_{i_1}, \ldots, x_{i_r})\overline{K}_{[d]}) \leq \text{reg}(\overline{K}_{[d]}) + 1 = d + 1 \). Since this ideal is generated in degree \( d + 1 \), it follows \( \text{reg}((\overline{MK})_{[d+1]}) = d + 1 \). \( \square \)

**Lemma 3.4.** For every positive integer d, we have

1. \( \overline{K}_{[d]} + (\overline{MK})_{[d]} = K_{[d]} \). In particular if \( \overline{K}_{[d]} = 0 \), then \( \beta_{i,i+d}(K) = 0 \) for each \( i \geq 0 \).
2. \( (\overline{MK})_{[d]} = \overline{K}_{[d]} \cap (\overline{MK})_{[d]} \).

**Proof.** We prove 1. Clearly \( \overline{K}_{[d]} + (\overline{MK})_{[d]} \subseteq K_{[d]} \). Let \( m \in G(K_{[d]}) \). We prove that if \( m \notin \overline{K}_{[d]} \), then \( m \notin (\overline{MK})_{[d]} \). Since \( (g \in G(K)|\text{deg}(g) = d) \subseteq \overline{K}_{[d]} \), we have \( m \notin G(K) \). Then there are \( h \in G(K) \) and a squarefree monomial \( r \), such that \( m = rh \) with \( \text{lcm}(r, h) = 1 \).

Hence \( m \in (\overline{MK})_{[d]} \). If \( \overline{K}_{[d]} = 0 \) then \( (\overline{MK})_{[d]} = K_{[d]} \) and by Proposition 2.3, \( \beta_{i,i+d}(K) = 0 \) for each \( i \geq 0 \).

We prove 2. First we show the inclusion \( (\overline{MK})_{[d]} \subseteq \overline{K}_{[d]} \). If \( (\overline{MK})_{[d]} = 0 \) we have nothing to prove, then we may assume \( (\overline{MK})_{[d]} \neq 0 \). Let \( m \in G((\overline{MK})_{[d]}) \). By definition \( m \in G((\overline{MK})_{[d]}) \) and \( m \notin G((\overline{MJ})_{[d]}) \). Since \( (\overline{MK})_{[d]} \subseteq K_{[d]} \) thus \( m \in G(K_{[d]}) \). Suppose \( m \in G(J_{[d]}) \). By the first part of the proof, we would have \( m \in G(J_{[d]}) \), a contradiction. Then \( m \in G(K_{[d]}) \). Since \( (\overline{MK})_{[d]} \subseteq (\overline{MK})_{[d]} \), the first inclusion is proved.
If \( \overline{K}_{[d]} = 0 \) or \((\mathfrak{M} K)_{[d]} = 0 \) then \( \overline{(\mathfrak{M} K)}_{[d]} = 0 \). We may assume \( \overline{K}_{[d]} \neq 0 \) and \((\mathfrak{M} K)_{[d]} \neq 0 \). We claim that the ideal \( \overline{K}_{[d]} \cap (\mathfrak{M} K)_{[d]} \) is generated in degree \( d \). Let \( m \in G(\overline{K}_{[d]} \cap (\mathfrak{M} K)_{[d]}) \) and suppose \( \deg(m) > d \). Then there would be a variable \( x \) dividing \( m \) such that \( \frac{m}{x} \in K_{[d]} \). By the first part of the proof \( \frac{m}{x} \in \overline{K}_{[d]} \) or \( \frac{m}{x} \in (\mathfrak{M} K)_{[d]} \). In the first case \( m \in \mathfrak{M} K_{[d]} \cap \overline{K}_{[d]} \). In both cases \( m \notin G(\overline{K}_{[d]} \cap (\mathfrak{M} K)_{[d]}) \), a contradiction. Then \( \deg(m) = d \) and \( m \in G(\overline{K}_{[d]} \cap G((\mathfrak{M} K)_{[d]}) \). Since \( m \notin G(J_{[d]}) \), then \( m \notin G((\mathfrak{M} J)_{[d]} \) and hence \( m \in G((\overline{\mathfrak{M} K})_{[d]} \).

\[ \square \]

**Proposition 3.5.** For every positive integer \( d \), the ideals \( K_{[d]} \) and \((\mathfrak{M} K)_{[d]} \) have a linear resolution. Moreover, \( \beta_{i,i+d}(K) = \beta_{i}(\overline{K}_{[d]}) - \beta_{i}((\overline{\mathfrak{M} K})_{[d]} \) for each \( i \geq 0 \).

**Proof.** If \( K_{[d]} = 0 \) by Lemma 3.4 one has \( \beta_{i,i+d}(K) = 0 \) and the result follows. Assume \( K_{[d]} \neq 0 \). First we prove that \( K_{[d]} \) has a \( d \)-linear resolution. We proceed by induction on \( d \geq \text{indeg}(K) \). Assume \( d = \text{indeg}(K) \). Then \((\mathfrak{M} K)_{[d]} = 0 \), hence by Lemma 3.3 one has \( K_{[d]} = K_{[d]} \); since \( K \) is componentwise linear this ideal has a \( d \)-linear resolution. Assume \( K_{[d]} \) has a \( d \)-linear resolution. We prove that \( K_{[d+1]} \) has a \((d+1)\)-linear resolution. Consider the sequence

\[
0 \to \overline{(\mathfrak{M} K)}_{[d+1]} \overset{\phi}{\to} K_{[d+1]} \oplus (\mathfrak{M} K)_{[d+1]} \overset{\psi}{\to} K_{[d+1]} \to 0
\]

defining \( \phi(a) = (a, -a) \) and \( \psi(a, b) = a + b \). By Lemma 3.4 it is exact. By inductive hypothesis and by Lemma 3.3 one has that \((\overline{\mathfrak{M} K})_{[d+1]} \) has a linear resolution. Then by [4, Corollary 20.19] and componentwise linearity of \( K \) we have

\[
\text{reg}(K_{[d+1]} \oplus (\mathfrak{M} K)_{[d+1]}) \leq \max\{\text{reg}(\overline{(\mathfrak{M} K)}_{[d+1]}), \text{reg}(K_{[d+1]})\} = d + 1.
\]

Then \( \text{reg}(K_{[d+1]} = d+1 \). By Lemma 3.3 \((\overline{\mathfrak{M} K})_{[d]} \) has a \( d \)-linear resolution for \( d \geq \text{indeg}(K) + 1 \).

Now we prove the second part. By Proposition 2.3 the ideals \( \mathfrak{M} I \), \( \mathfrak{M} J \) and \( \mathfrak{M} K \) are componentwise linear. Moreover \( \mathfrak{M} I = \mathfrak{M} J + \mathfrak{M} K \). By the first part of the proof \((\overline{\mathfrak{M} K})_{[d]} \) and \( K_{[d]} \cap (\overline{\mathfrak{M} K})_{[d]} \) have a linear resolution. Since all the modules involved have a linear resolution, by the Tor long exact sequence associated to the short exact sequence above we have \( \beta_{i}(K_{[d]} \cap (\overline{\mathfrak{M} K})_{[d]} = \beta_{i}((\overline{\mathfrak{M} K})_{[d]} \) for each \( i \geq 0 \) and \( d \geq 1 \). By Proposition 2.3 the statement follows. \[ \square \]

**Lemma 3.6.** For every positive integer \( d \) we have

1. \((J_{[d]} \cap K_{[d]} + (\mathfrak{M}(J \cap K)_{[d+1]} = (J \cap K)_{[d+1]} \).
2. \((\mathfrak{M} J)_{[d]} \cap (\overline{\mathfrak{M} K})_{[d]} = (J_{[d]} \cap K_{[d]} \cap (\mathfrak{M}(J \cap K)_{[d+1]} \).

**Proof.** For \( d \geq n \) or \( d < \max\{\text{indeg}(J), \text{indeg}(K)\} \), one has

\[
J_{[d]} \cap K_{[d]} = (\mathfrak{M}(J \cap K)_{[d+1]} = (J \cap K)_{[d+1]} = (\mathfrak{M} J)_{[d]} \cap (\overline{\mathfrak{M} K})_{[d]} = 0.
\]

We may assume \( \max\{\text{indeg}(J), \text{indeg}(K)\} \leq d \leq n - 1 \).

We prove 1. First we show \((J_{[d]} \cap K_{[d]} \subseteq (J \cap K)_{[d+1]} \) if \( K_{[d]} = 0 \) we have nothing to prove. Let \( K_{[d]} \neq 0 \) and \( m \in G(J_{[d]} \cap K_{[d]} \). Then \( m \in J \cap K \). Consider the splitting \( I_{[d]} = J_{[d]} + K_{[d]} \). Clearly \( G(I_{[d]} = G(J_{[d]} \cap G(K_{[d]} = \emptyset \). By our assumptions and by

\[
\square \]
Proposition 3.5. $J_{[d]}$ and $\overline{K}_{[d]}$ have a $d$-linear resolution. Then by Proposition 2.4, $J_{[d]} \cap \overline{K}_{[d]}$ is generated in degree $d + 1$. Then $\deg(m) = d + 1$. Since $(\mathfrak{m}(J \cap K))_{[d] + 1} \subseteq (J \cap K)_{[d] + 1}$ the first statement is proved.

Let $m \in G((J \cap K)_{[d + 1]})$. Clearly $m \in J \cap K$. Suppose first $m \notin G(J \cap K)$. Then $m \in (\mathfrak{m}(J \cap K))_{[d + 1]}$. Assume now $m \in G(J \cap K)$. By our assumptions $m \notin G(J)$ and $m \notin G(K)$. Suppose in fact $m \in G(J)$. Then $m \notin G(K)$ and there is $h \in G(K)$ such that $m \in (h)$. Since both $m$ and $h$ are minimal generators of $I$, this is a contradiction. The proof is the same for $K$. Then $m \in J_{[d]}$. Moreover there are $a \in G(K)$ and a squarefree monomial $h$, $\deg(h) \geq 1$ such that $m = ah$. Suppose $m \notin \overline{K}_{[d]}$. Then for each variable $x|m$ we would have $\frac{m}{x} \in J_{[d]}$. In particular, for each $x|a$, $\frac{m}{x} \in J_{[d]} \cap K \subset J \cap K$, a contradiction.

We prove 2. For $d = \max\{\indeg(J), \indeg(K)\}$ one has $(\mathfrak{m}(J \cap K))_{[d]} = (\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} = 0$, then we may assume $d > \max\{\indeg(J), \indeg(K)\}$. If $(\mathfrak{m}K)_{[d]} = 0$ we have nothing to prove. Assume $(\mathfrak{m}J)_{[d]} \neq 0$. Clearly $(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} \subseteq J_{[d]} \cap \overline{K}_{[d]}$. Note that, by the proof of Lemma 3.3 $(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} \subseteq J_{[d]} \cap \overline{K}_{[d]}$. By the first part of the proof one has $J_{[d - 1]} \cap \overline{K}_{[d - 1]} \subseteq (J \cap K)_{[d - 1]}$. By Proposition 3.3, the ideal $(\mathfrak{m}K)_{[d]}$ has a $d$-linear resolution and by Proposition 2.4, the ideal $(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]}$ is generated in degree $d + 1$, hence

$$(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} \subseteq (\mathfrak{m}(J \cap K))_{[d + 1]} = (\mathfrak{m}(J \cap K))_{[d + 1]}.$$ 

For the other inclusion, note that $\mathfrak{m}(J \cap K) \subseteq \mathfrak{m}^2 J$. Then

$$(J_{[d]} \cap \overline{K}_{[d]}) \cap (\mathfrak{m}(J \cap K))_{[d + 1]} \subseteq (\mathfrak{m}(J \cap K))_{[d + 1]} \subseteq (\mathfrak{m}^2 J)_{[d + 1]} = (\mathfrak{m}(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} \subseteq (\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]}).$$

Analogously one proves that $(\mathfrak{m}(J \cap K))_{[d + 1]} \subseteq (\mathfrak{m}K)_{[d]}$. By Lemma 3.4

$$(J_{[d]} \cap \overline{K}_{[d]}) \cap (\mathfrak{m}(J \cap K))_{[d + 1]} \subseteq J_{[d]} \cap (\mathfrak{m}(J \cap K))_{[d + 1]} \subseteq K_{[d]} \cap (\mathfrak{m}K)_{[d]} = (\mathfrak{m}K)_{[d]}.$$

\textbf{Proposition 3.7.} Let $I$ be a monomial squarefree componentwise linear ideal, $I = J + K$, with $J, K$ monomial squarefree componentwise linear ideals, $G(I) = G(J) \cup G(K)$ and $G(J) \cap G(K) = \emptyset$. Then $J \cap K$ is monomial squarefree componentwise linear. Moreover

$$\beta_{i-1,i+d}(J \cap K) = \beta_{i-1}(J_{[d]} \cap \overline{K}_{[d]}) - \beta_{i-1}(\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]}$$

for each $i, d \in \mathbb{N}$.

\textbf{Proof.} We prove first that $J \cap K$ is componentwise linear. If $J = 0$ or $K = 0$ then $J \cap K = 0$ and we have nothing to prove. Assume $J, K \neq 0$. Let $r = \max\{\indeg(J), \indeg(K)\}$. Then $J \cap K$ is generated in degree at least $r + 1$. We proceed by induction on $d \geq r$. Let $d = r$. Since $(\mathfrak{m}(J \cap K))_{[r + 1]} = 0$, by Lemma 3 one has $J_{[r]} \cap \overline{K}_{[r]} = (J \cap K)_{[r + 1]}$. If $\overline{K}_{[r]} = 0$ then we have nothing to prove. Suppose $\overline{K}_{[r]} \neq 0$. By the same argument of Lemma and by Proposition 2.4, $J_{[r]} \cap \overline{K}_{[r]}$ has a $(r + 1)$-linear resolution, then $(J \cap K)_{[r + 1]}$ has a $(r + 1)$-linear resolution.

Assume $(J \cap K)_{[d]}$ has a $d$-linear resolution. Consider the sequence

$$0 \rightarrow (\mathfrak{m}J)_{[d]} \cap (\mathfrak{m}K)_{[d]} \overset{\phi}{\rightarrow} (J_{[d]} \cap \overline{K}_{[d]}) \oplus (\mathfrak{m}(J \cap K))_{[d + 1]} \overset{\psi}{\rightarrow} (J \cap K)_{[d + 1]} \rightarrow 0$$

with $\phi(a) = (a, -a)$ and $\psi(a, b) = a + b$. By Lemma 3 it is exact. Since $(\mathfrak{m}(J \cap K))_{[d + 1]} = (\mathfrak{m}(J \cap K))_{[d + 1]}$, hence by inductive hypothesis and by Proposition 2.4 $(\mathfrak{m}(J \cap K))_{[d + 1]}$
has a \((d+1)\)-linear resolution. Note that by Proposition 2.3 the ideals \((\mathfrak{M} J)[d] \cap (\mathfrak{M} K)[d] \cap K[d]\) are zero or have a \((d+1)\)-linear resolution. By Corollary 20.19
\[
\text{reg}((J \cap K)[d+1]) \leq \max\{\text{reg}((\mathfrak{M} J)[d] \cap (\mathfrak{M} K)[d]) - 1, \text{reg}((J[d] \cap K[d]) + (\mathfrak{M} (J \cap K))[d+1])\} = d+1.
\]
Since \((J \cap K)[d+1]\) is generated in degree \(d+1\) then \(\text{reg}((J \cap K)[d+1]) = d+1\).

Since all the modules involved have a linear resolution, by the long exact sequence associated to the short exact sequence above we have
\[
\beta_i((J \cap K)[d+1]) + \beta_i((\mathfrak{M} J)[d] \cap (\mathfrak{M} K)[d]) = \beta_i(J[d] \cap K[d]) + \beta_i((\mathfrak{M} (J \cap K))[d+1]), \text{ for } i \geq 0, d \geq 1.
\]
By the first part of the proof \(J \cap K\) is componentwise linear, then by Proposition 2.3 the statement follows.

Now we are able to prove Theorem 3.1.

**Proof of Theorem 3.1** As before, we denote by \(P(I)\) the polarization of \(I\). Note that \(P(I) = P(J + K) = P(J) + P(K)\) and \(G(P(I)) = G(P(J)) \cup G(P(K))\), \(G(P(J)) \cap G(P(K)) = 0\). By Corollary 1.6.3 the graded Betti numbers are preserved under polarization, then by Proposition 3.2 it is enough to prove the theorem for monomial squarefree componentwise linear ideals.

Let \(d\) be a positive integer. First assume \(J[d] = K[d] = 0\), then \(J[d] = K[d]\) and thus \(I[d] = J[d]\). By Lemma 3.4 \(\beta_{i,d}(J) = \beta_{i,d}(K) = 0\) for each \(i \in \mathbb{N}\) and by Lemma 3 one has \((\mathfrak{M} (J \cap K))[d+1] = (J \cap K)[d+1]\). By Proposition 3.1 \(J \cap K\) is componentwise linear and then, by Proposition 2.3 \(\beta_{i-1,d}(J \cap K) = 0\). We have to prove \(\beta_{i,d}(I) = 0\). Since \((\mathfrak{M} J)[d] = (\mathfrak{M} K)[d] = 0\), then \((\mathfrak{M} J)[d] = (\mathfrak{M} K)[d]\) and \((\mathfrak{M} J)[d] = (\mathfrak{M} (J + K))[d] = (\mathfrak{M} J + \mathfrak{M} K)[d] = (\mathfrak{M} J)[d] + (\mathfrak{M} K)[d] = (\mathfrak{M} J)[d]\). Since \(I[d] = J[d]\), hence by Proposition 2.3 we conclude.

Assume \(K[d] \neq 0\). By Proposition 3.5 one has \(K[d]\) has a \(d\)-linear resolution. Note that \(G(I[d])\) is the disjoint union of \(G(J[d])\) and \(G(K[d])\). By componentwise linearity of \(I\) and \(J\), \(I[d]\) and \(J[d]\) have a \(d\)-linear resolution. Then by Proposition 2.4 \(I[d] = J[d] + K[d]\) is a Betti splitting of \(I[d]\) and
\[
\beta_i(I[d]) = \beta_i(J[d]) + \beta_i(K[d]) + \beta_{i-1}(J[d] \cap K[d]), \text{ for each } i \geq 0.
\]
By Proposition 2.3 \(\mathfrak{M} I, \mathfrak{M} J\) and \(\mathfrak{M} K\) are componentwise linear ideals. Note that \(G((\mathfrak{M} I)[d])\) is the disjoint union of \(G((\mathfrak{M} J)[d])\) and \(G((\mathfrak{M} K)[d])\). Clearly \((\mathfrak{M} J)[d]\) has a \(d\)-linear resolution and by Proposition 3.5 the ideal \((\mathfrak{M} K)[d]\) has a \(d\)-linear resolution. Then, by Proposition 2.4 the splitting \((\mathfrak{M} I)[d] = (\mathfrak{M} J)[d] + (\mathfrak{M} K)[d]\) is a Betti splitting and
\[
\beta_i((\mathfrak{M} I)[d]) = \beta_i((\mathfrak{M} J)[d]) + \beta_i((\mathfrak{M} K)[d]) + \beta_{i-1}((\mathfrak{M} J)[d] \cap (\mathfrak{M} K)[d]), \text{ for each } i \geq 0.
\]

By Proposition 2.3 for \(I\) and \(J\), by Proposition 3.5 for \(K\) and Proposition 3.1 for \(J \cap K\), we get
\[
\beta_{i,d}(I) = \beta_{i,d}(J) + \beta_{i,d}(K) + \beta_{i-1,d}(J \cap K), \text{ for all } i, d \in \mathbb{N}.
\]

**Remark 3.8.** We don’t know if the converse of Theorem 3.1 holds in general. It can be proved by standard homological arguments that the converse holds provided \(J \cap K\) is componentwise linear.
4. BETTI SPLITTING FOR SIMPLICIAL COMPLEXES

In this section we present some applications of Theorem 3.1 to simplicial complexes with the hope that it could be useful to deal with large amount of data as we mentioned in the introduction. For definitions and terminology about simplicial complexes and their properties (sequentially Cohen-Macaulayness, shellability, vertex decomposability, etc...) we refer to [15] Chapter 3. For the Stanley-Reisner correspondence and Alexander dual of a monomial squarefree ideal see [12], Chapter 1. Denote by $I_\Delta^*$ the Alexander dual ideal of a simplicial complex $\Delta$ and by $\mathcal{F}(\Delta)$ its set of facets.

In [9], Theorem 2.3 Francisco, Hà and Van Tuyl give conditions on $J, K$ and $J \cap K$ forcing $I = J + K$ to be a Betti splitting of $I$. We show an ideal which admits a Betti splitting following Theorem 3.1 but it is not a consequence of [9] Theorem 2.3.

**Example 4.1.** Let $\mathbb{K}$ be a field of characteristic zero and let $I \subseteq \mathbb{K}[x_1, \ldots, x_{12}]$ be the Alexander dual ideal of the sequentially Cohen-Macaulay simplicial complex $\Delta$ defined by the following facets:

$$\{[1, 2, 3, 4], [1, 2, 3, 12], [3, 4, 6], [3, 4, 5], [4, 5, 6], [3, 5, 6],$$
$$[5, 6, 7], [5, 7, 8], [4, 9], [9, 10], [10, 11], [6, 9], [8, 12]\}.$$

By [12], Proposition 8.2.20, $I$ is componentwise linear. Let $\Delta = \Delta_1 \cup \Delta_2$, with $\mathcal{F}(\Delta_1) = \{[1, 2, 3, 4], [1, 2, 3, 12], [3, 4, 5], [3, 4, 6], [4, 9]\}$ and $\mathcal{F}(\Delta_2) = \{[3, 5, 6], [4, 5, 6], [5, 6, 7], [5, 7, 8], [6, 9], [8, 12], [10, 11], [9, 10]\}$.

Let $J_i$ be the Alexander dual ideal of $\Delta_i$, for $i = 1, 2$. Both $\Delta_1$ and $\Delta_2$ are shellable. Then $J_1$ and $J_2$ are componentwise linear and by Theorem 3.1 $I = J_1 + J_2$ is a Betti splitting of $I$. The assumptions of [9] Theorem 2.3 are not satisfied since $\beta_{1,11}(J_1 \cap J_2) > 0$ and both $\beta_{1,11}(J_1)$ and $\beta_{1,11}(J_2)$ are not zero.

We recover in a simpler way the following known result.

**Corollary 4.2.** ([16] Theorem 2.8, Corollary 2.11) If $\Delta$ is a vertex decomposable simplicial complex then there is $x_i \in V(\Delta)$ such that $I_\Delta^*$ admits $x_i$-splitting.

**Proof.** By vertex-decomposability, there is a shedding vertex $v = x_i \in V(\Delta)$ such that $	ext{del}_\Delta(v)$ and $	ext{link}_\Delta(v)$ are both vertex-decomposable. By [16], Lemma 2.2 we have $I_\Delta^* = x_i I^*_{\text{del}_\Delta(v)} + I^*_{\text{link}_\Delta(v)}$. A vertex-decomposable simplicial complex is shellable and hence sequentially Cohen-Macaulay by [12], Corollary 8.2.19. Then $I_\Delta^* = x_i I^*_{\text{del}_\Delta(v)}$ and $I^*_{\text{link}_\Delta(v)}$ are componentwise linear by [12], Theorem 8.2.20. By Theorem 3.1 we conclude. \qed

In Corollary 4.2 vertex-decomposability cannot be replaced by shellability, as it is shown in the next example.

**Example 4.3.** Consider the simplicial complex $\Delta$ defined by the following facets.

$$\{[2, 3, 4], [2, 4, 7], [1, 2, 7], [1, 6, 7], [2, 3, 5], [1, 2, 5], [1, 2, 6], [2, 3, 6], [3, 5, 6], [5, 6, 7], [4, 5, 7], [1, 4, 5],$$
$$[1, 3, 4], [1, 3, 7], [3, 5, 7], [4, 6, 7], [4, 6, 12], [6, 11, 12], [6, 8, 12], [8, 9, 12], [6, 8, 9], [6, 9, 10], [6, 10, 11],$$
$$[9, 10, 11], [8, 9, 11], [4, 11, 12], [4, 8, 11], [4, 9, 12], [4, 9, 10], [4, 8, 10], [8, 10, 12], [10, 11, 12]\}.$$

The given order of the facets of $\Delta$ is a shelling, thus $\Delta$ is shellable. Let $I$ be the Alexander dual ideal of $\Delta$. By [12], Proposition 8.2.5 the Alexander dual ideal of a shellable complex
has linear quotients, thus by [12, Proposition 8.2.15] $I$ componentwise linear. Since $I$ is generated in degree 9, $I$ has a 9-linear resolution. Consider the splittings $I = x_iJ + K$, for each $x_i$, $1 \leq i \leq 12$. The resolutions of $x_iJ$ is not linear, for each $1 \leq i \leq 12$. By Proposition 2.4 $I$ does not admits $x_i$-splitting.

Nevertheless it is a consequence of the main result that the Alexander dual of a shellable simplicial complex always admits Betti splitting.

**Corollary 4.4.** Let $\Delta$ be a shellable simplicial complex. Then $I_\Delta^*$ admits Betti splitting.

**Proof.** By the assumption, $I_\Delta^* = (f_1, \ldots, f_k)$ has linear quotients in the given order and then it is componentwise linear. The ideal $J = (f_1, \ldots, f_{k-1})$ has linear quotients and the ideal $K = (f_k)$ has a linear resolution. Then $J$ and $K$ are componentwise linear. By Theorem 3.1 the result follows.

Let $\Delta$ be the complex in Example 4.3. Since the resolutions of $I_{\Delta_1}^*$ and $I_{\Delta_2}^*$ are linear, $I_\Delta = I_{\Delta_1}^* + I_{\Delta_2}^*$ is a Betti splitting of $I_\Delta^*$.

The analogous result can be proved for constructible simplicial complexes, not necessarily pure (this result in the pure case is [18, Corollary 3.4]).

Corollary 4.4 does not hold for sequentially Cohen-Macaulay simplicial complexes, as it is shown in [2, Example 4.1].

5. THE RESOLUTION OF THE IDEAL OF THREE GENERAL FAT POINTS IN $\mathbb{P}^{n-1}$

Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme consisting of three general fat points in $\mathbb{P}^{n-1}$, with $n \geq 4$ and $1 \leq a \leq b \leq c$. After a change of coordinates, we may assume that $P_1 = [1 : 0 : \ldots : 0]$, $P_2 = [0 : 1 : 0 \ldots : 0]$ and $P_3 = [0 : 0 : 1 : 0 \ldots : 0]$. Then the defining ideal of $X$ is

$$I_{n,a,b,c} = (x_2, \ldots, x_n)^a \cap (x_1, x_3, \ldots, x_n)^b \cap (x_1, x_2, x_4, \ldots, x_n)^c \subseteq \mathbb{K}[x_1, \ldots, x_n].$$

In the case of two fat points we denote $I_{n,0,b,c}$ by $I_{n,b,c}$, with $b \leq c$. By convention $I_{n,b,c} = 0$ if $n \leq 2$. If $a = b = c$ we denote $I_{n,a,a,a}$ by $I_{n,a}$. Francisco proved in [8] that the defining ideal of the zero-dimensional schemes of $r \leq n + 1$ general fat points in $\mathbb{P}^n$ is componentwise linear. In general the ideals $I_{n,a,b,c}$ are not stable (even if $a = b = c = 1$). Valla computed the graded Betti numbers of the defining ideal of two general fat points in $\mathbb{P}^n$, $n \geq 2$ [21, Corollary 3.5] by using a Betti splitting argument. We prove a splitting result for $I_{n,a,b,c}$ and, as a consequence, we give a recursive procedure to compute the graded Betti numbers of $I_{n,a,b,c}$ in the case $a \neq c$.

**Theorem 5.1.** Let $n \in \mathbb{N}$, $n \geq 4$. Let $X = \{(P_1, a), (P_2, b), (P_3, c)\}$ be the 0-dimensional scheme defined by three general fat points in $\mathbb{P}^{n-1}$, with $1 \leq a \leq b \leq c$. Suppose $c \neq a$. Then $I = I_{n,a,b,c}$ admits $x_1$-splitting.

**Proof.** Let $J = x_1I_{n,a,b-1,c-1}$ and $K = (x_3, \ldots, x_n)^b \cap (x_2, x_4, \ldots, x_n)^c$. We show $I = J + K$. Let $f \in G(I)$. Suppose $f \in (x_1)$. Then $f \in (x_1) \cap I$. Since $(x_1) \cap (x_1, x_3, \ldots, x_n)^b = x_1(x_1, x_3, \ldots, x_n)^{b-1}$ and $(x_1) \cap (x_1, x_2, x_4, \ldots, x_n)^c = x_1(x_1, x_2, x_4, \ldots, x_n)^{c-1}$ hence

$$(x_1) \cap I = x_1(x_2, \ldots, x_n)^a \cap x_1(x_1, x_3, \ldots, x_n)^{b-1} \cap x_1(x_1, x_2, x_4, \ldots, x_n)^{c-1} = x_1I_{n,a,b-1,c-1} = J.$$
Let \( f \notin (x_1), \) then \( f \in (x_2, \ldots, x_n)^a \cap (x_3, \ldots, x_n)^b \cap (x_2, x_4, \ldots, x_n)^c. \) Since \((x_3, \ldots, x_n) \subseteq (x_2, x_3, \ldots, x_n) \) and \( a \leq b, \) hence \( f \in (x_3, \ldots, x_n)^b \cap (x_2, x_4, \ldots, x_n)^c = K. \) For the other inclusion \( J = (x_1) \cap I \subseteq I \) and \( K = (x_3, \ldots, x_n)^b \cap (x_2, x_4, \ldots, x_n)^c = (x_2, x_n)^a \cap (x_3, \ldots, x_n)^b \cap (x_2, x_4, \ldots, x_n)^c \subseteq I. \)

We prove now that \( G(I) = G(J) \cup G(K) \) and \( G(J) \cap G(K) = \emptyset. \) Let \( g \in G(I). \) Then \( g \in J \) or \( g \in K. \) Suppose \( g \in J. \) If \( g \notin G(J), \) there would be \( m \in G(J) \) such that \( g \in (m). \) Since \( J \subseteq I \) and \( g \in G(I) \) this is a contradiction. The proof works also for \( K, \) then the first inclusion is clear.

For the other inclusion, we prove first that \( K \subseteq \mathcal{M}I_{n,a,b-1,c-1}. \) Let \( h \in K. \) If there is a variable \( x_i, \) with \( x_i|h \) and \( 4 \leq i \leq n, \) then \( h \in x_i[(x_3, \ldots, x_n)^b-1 \cap (x_2, x_4, \ldots, x_n)^c-1] \) and \( K \subseteq \mathcal{M}[(x_3, \ldots, x_n)^b-1 \cap (x_2, x_4, \ldots, x_n)^c-1]. \) Since \( a \leq c-1 \) and \( (x_2, x_4, \ldots, x_n) \subseteq (x_2, \ldots, x_n) \) hence
\[
K \subseteq \mathcal{M}[(x_2, \ldots, x_n)^a \cap (x_3, \ldots, x_n)^b-1 \cap (x_2, x_4, \ldots, x_n)^c-1] \subseteq \mathcal{M}I_{n,a,b-1,c-1}.
\]

Otherwise \( h = x_2^{c-2}x_3^c = x_2x_3(x_2^{b-1}x_3^{c-1}) \in \mathcal{M}I_{n,a,b-1,c-1}. \)

Let \( g \in G(J). \) By definition there is \( s \in G(I_{n,a,b-1,c-1}) \) such that \( g = x_1s. \) Suppose \( g \notin G(I). \) Then there would be \( h \in G(I) \) and a monomial \( r \) such that \( g = x_1s = hr. \) Since \( G(I) \subseteq G(J) \cup G(K) \) and \( g \in G(J), \) hence \( h \in G(K). \) Then \( x_1|r \) and \( s = r/h, \) where \( r = \frac{x_1}{x_1}. \)

Since \( h \in \mathcal{M}I_{n,a,b-1,c-1} \) this is a contradiction.

Let \( g \in G(K). \) Suppose \( g \notin G(I). \) Then there is \( m \in G(I) \) such that \( g \in (m). \) Since \( I \subseteq G(K), \) hence \( m \in J = (x_1) \cap I. \) Then \( x_1|g, \) a contradiction.

Clearly one has \( G(J) \cap G(K) = \emptyset. \) By [5, Theorem 4.6] \( J \) and \( K \) are componentwise linear. By Theorem 2.3 we have \( I = J + K \) is a Betti splitting of \( I. \)

Notice that in general our splitting does not satisfy the assumption of [5, Theorem 2.3] (see for instance the case \( n = 4, a = b = 1, c = 2). \)

In the next corollary we compute explicitly the graded Betti numbers of \( I_{n,a,b,c} \) in the case \( a + b \leq c \) by a recursive procedure.

**Corollary 5.2.** Let \( n \in \mathbb{N}, n \geq 4. \) Let \( X = \{(P_1, a), (P_2, b), (P_3, c)\} \) be the 0-dimensional scheme consisting of three general fat points in \( \mathbb{P}^{n-1}, \) with \( 1 \leq a \leq b \leq c \) and \( I = I_{n,a,b,c}. \) Assume \( a + b \leq c. \) Then
\[
\beta_{i,i+c}(I) = \beta_{i,i+c-b}(I_{n,a,c-b}) + \sum_{r=0}^{b-1} \beta_{i,i+c-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,i+c-r-1}(I_{n-1,b-r,c-r})\]
\[
\beta_{i,j}(I) = \begin{cases} 
\binom{n-2}{i} \binom{n-3+c-a+j+i}{n-3} + \binom{n-3+c+b-j+i}{n-3} & \text{if } c+1+i \leq j \leq a+c+i \\
\binom{n-2}{i} \binom{n-3+c-a-j+i}{n-3} + \binom{n-3+c+b-j+i}{n-3} & \text{if } a+c+1+i \leq j \leq b+c+i \\
0 & \text{if } j \geq b+c+1+i.
\end{cases}
\]

**Proof.** Since \( a + b \leq c, \) the assumptions of Theorem 5.1 are satisfied, then \( I_{n,a,b,c} \) admits \( x_1 \)-splitting. Let \( J \) and \( K \) be as in the proof of Theorem 5.1. Clearly \( J \cap K = x_1K. \) Note that \( \beta_{i,j}(J) = \beta_{i,j-1}(I_{n,a,b-1,c-1}) \) and \( \beta_{i-1,j}(J \cap K) = \beta_{i-1,j-1}(K), \) for each \( i, j \geq 0. \) We remark that, after a relabeling of the variables, \( K \) is the ideal of two general fat points in \( \mathbb{P}^{n-2}, \) i.e. \( K = I_{n-1,b,c}. \) Then we get the formula
\[
\beta_{i,j}(I) = \beta_{i,j-1}(I_{n,a,b-1,c-1}) + \beta_{i,j}(I_{n-1,b,c}) + \beta_{i-1,j-1}(I_{n-1,b,c}).
\]
Since \( c - r \neq a \), for \( 0 \leq r \leq c - a - 1 \) and \( b - 1 \leq c - a - 1 \), we can apply recursion to \( I_{n,a,b-r,c-r} \) for \( 0 \leq r \leq b - 1 \) to get

\[
(1) \quad \beta_{i,j}(I) = \beta_{i,j-b}(I_{n,a,c-b}) + \sum_{r=0}^{b-1} \beta_{i,j-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,j-1-r}(I_{n-1,b-r,c-r}).
\]

By [21, Corollary 3.5] and by the relation \( \sum_{r=0}^{h} \binom{s}{c} = \binom{s+1}{c+1} - \binom{h}{c} \) we get the formula. \( \square \)

Theorem 5.1 allows us to apply a recursive procedure for computing the Betti numbers of \( I = I_{n,a,b,c} \) with \( a + b > c \) and \( c \neq a \). This formula has the limit that the Betti numbers of \( I_{n,a} \) could be involved. These ideals are studied in [7]. An explicit formula for the graded Betti numbers of \( I_{n,a} \) is given only for \( a = 2 \) [8, Proposition 3.2] and \( a = 3 \) [8, Proposition 3.3]. In [8], Francisco suggest to use a result of Gasharov, Hibi and Peeva [11] to compute the resolution of \( I_{n,a} \).

**Corollary 5.3.** Let \( n \in \mathbb{N} \), \( n \geq 4 \). Let \( X = \{(P_1, a), (P_2, b), (P_3, c)\} \) be the 0-dimensional scheme consisting of three general fat points in \( \mathbb{P}^{n-1} \), with \( 1 \leq a \leq b \leq c \) and \( I = I_{n,a,b,c} \). Assume \( a + b > c \) and \( c \neq a \). Let \( k = a + b - c \) and \( B_{n,a,b,c}^{i,j} := \binom{n-3+c+a-j+i}{n-3} + \binom{n-3+a+b-j+i}{n-3} - 2\binom{n-3+a+b-j+i}{n-3} \), for \( i, j \in \mathbb{N} \). Moreover for \( s, t \in \mathbb{N} \) let \( \gamma_{n,i}^{s,t} := \beta_{i,i+t}(I_{n-1,s,t}) + \beta_{i-1,i+t-1}(I_{n-1,s,t}) \). Then

\[
\beta_{i,i+k}(I_{n,k}) = \sum_{r=0}^{c-a-1} \gamma_{n,r,c-r}^{n,i} + \sum_{r=0}^{c-b-1} \gamma_{n,r,a-r}^{n,i}.
\]

\[
\beta_{i,j}(I) = \begin{cases} 
\beta_{i,j+k}(I_{n,k}) + \binom{n-2}{i} B_{n,a,b,c}^{i,j} & \text{if } c + 1 + i \leq j \leq a + b + i. \\
\binom{n-2}{i} \left[ \binom{n-3+c+a-j+i}{n-3} + \binom{n-3+a+b-j+i}{n-3} \right] & \text{if } a + b + 1 + i \leq j \leq a + c + i. \\
\binom{n-2}{i} \left( \binom{n-3+c+a-j+i}{n-3} - \binom{n-3+a+b-j+i}{n-3} \right) & \text{if } a + c + 1 + i \leq j \leq b + c + i. \\
0 & \text{if } j \geq c + b + 1 + i. 
\end{cases}
\]

**Proof.** Note that \( c < a + b \leq 2c \) and \( c \geq 2 \). By Theorem 5.1 \( I_{n,a,b,c} \) admits \( x_1 \)-splitting. The main difference with Corollary 5.2 is that \( c - r \neq a \) for \( 0 \leq r \leq c - a \) but \( c - a < b \). Then, following the proof of Corollary 5.2 and using (1) we get

\[
\beta_{i,j}(I) = \beta_{i,j+a-c}(I_{n,a,a+b-c,a}) + \sum_{r=0}^{c-a-1} [\beta_{i,j-r}(I_{n-1,b-r,c-r}) + \beta_{i-1,j-1-r}(I_{n-1,b-r,c-r})].
\]

We focus our attention only on the first term \( \beta_{i,j+a-c}(I_{n,a,a+b-c,a}) \) of the equation. Now one has \( a + b - c \leq a \leq a \). If \( b = c \) we are done and the formula is proved. If \( b < c \), then \( a + b - c \neq a \) and the assumptions of Theorem 5.1 are satisfied. Since we consider the new order of the ideals of the intersection given by multiplicities then \( I_{n,a,a+b-c,a} \) admits \( x_2 \)-splitting. One has \( a - r \neq a + b - c \) for \( 0 \leq r \leq c - a - b - 1 \). Using (1) and \( k = a + b - c \) we get

\[
\beta_{i,j+a-c}(I_{n,k,a,a}) = \beta_{i,j+k-c}(I_{n,k}) + \sum_{r=0}^{c-b-1} [\beta_{i,j+a-c-r}(I_{n-1,a-r,a-r}) + \beta_{i-1,j+a-c-r-1}(I_{n-1,a-r,a-r})].
\]

By [8, Proposition 3.1] the ideal \( I_{n,k} \) is generated in degree at most \( 2k \). By [21, Corollary 3.5] we conclude. \( \square \)
We prove that in Theorem 5.1 the assumption $c \neq a$ is essential.

**Example 5.4.** Consider three double points in $\mathbb{P}^3$. The defining ideal $I = I_{4,2}$ admits the decomposition $I = J + K$ of Theorem 5.1 where $J = x_1I_{2,1,1}$ and $K = (x_3, x_4)^2 \cap (x_2, x_4)^2$. Unfortunately $G(I) \neq G(J) \cup G(K)$, since we have $x_1x_4^2 \in G(J)$ that is not a minimal generator of $J$. The same problem arise if we choose $x_2$ or $x_3$. It can be proved that, in general, $I_{n,a}$ admits $x_n$-splitting. To be more precise we have $I_{n,a} = x_nI_{n,a-1} + I_{n-1,a}$, with $G(I_{n,a}) = G(x_nI_{n,a-1}) \cup G(I_{n-1,a})$ and $G(x_nI_{n,a-1}) \cap G(I_{n-1,a}) = \emptyset$, but we are not able to take advantage from this decomposition, since the resolutions of both $I_{n,a-1}$ and $I_{n-1,a}$ are in general unknown.

**Remark 5.5.** If $n = 3$, the projective dimension of the ideal $I = I_{3,a,b,c}$ of three general fat points in $\mathbb{P}^2$ is 1 then the resolution is easy to compute. In the case $a + b \leq c$ is the same that in Corollary 5.2, in the case $a + b > c$ define $k = \left\lfloor \frac{a+b+c}{2} \right\rfloor$ and it is a computation to prove that, for $0 \leq i \leq 1$, graded Betti numbers are

$$
\beta_{i,j}(I) = \begin{cases} 
2 - i + (-1)^{a+b-c+1} & \text{if } j = c + k + i \\
3 & \text{if } c + k + 1 + i \leq j \leq a + b + i. \\
2 & \text{if } a + b + 1 + i \leq j \leq a + c + i. \\
1 & \text{if } a + c + 1 + i \leq j \leq b + c + i. \\
0 & \text{if } j \geq b + c + 1 + i.
\end{cases}
$$

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