FANO VARIETIES WITH LARGE SESHADRI CONSTANTS IN POSITIVE CHARACTERISTIC

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Abstract. We prove that an \( n \)-dimensional Fano variety (with arbitrary singularities) in positive characteristic is isomorphic to \( \mathbb{P}^n \) if the Seshadri constant of the anti-canonical divisor at some smooth point is greater than \( n \). We also classify Fano varieties whose anti-canonical divisors have Seshadri constants \( n \).

1. Introduction

Let \( X \) be a normal projective variety and \( L \) an ample \( \mathbb{Q} \)-Cartier divisor on \( X \). The Seshadri constants of \( L \), originally introduced by Demailly \cite{Dem92}, serve as a measure of the local positivity of the divisor \( L \).

Definition 1. Let \( L \) be an ample \( \mathbb{Q} \)-Cartier divisor on a projective variety \( X \) and \( x \in X \) a smooth point. The Seshadri constant of \( L \) at \( x \) is defined as

\[
\epsilon(L, x) := \sup \{ t \in \mathbb{R}_{>0} \mid \sigma^* L - tE \text{ is ample} \},
\]

where \( \sigma : \text{Bl}_x X \to X \) is the blow-up of \( X \) at \( x \), and \( E \) is the exceptional divisor of \( \sigma \).

When \( X \) is Fano, i.e. \( -K_X \) is \( \mathbb{Q} \)-Cartier and ample, it is natural to look at the Seshadri constant of the anti-canonical divisor. It turns out that the choice of \( X \) is quite restricted if \( \epsilon(-K_X, x) \) is large. For example, Bauer and Szemberg \cite{BS09} showed that if \( X \) is a complex Fano manifold of dimension \( n \) with \( \epsilon(-K_X, x) > n \) for some \( x \in X \) then \( X \cong \mathbb{P}^n \). This is generalized by Y. Liu and the author \cite{LZ16, Zhu17} to complex Fano varieties with arbitrary singularities. However, these results ultimately relied on the Kawamata-Viehweg vanishing theorem, thus were restricted to characteristic zero. On the other hand, using Frobenius technique, Murayama \cite{Mur17} recently generalized the result of Bauer and Szemberg to positive characteristic, albeit under a stronger assumption:

Theorem 2. \cite[Theorem B]{Mur17} Let \( X \) be a smooth Fano variety of dimension \( n \) defined over an algebraically closed field of positive characteristic. Assume that \( \epsilon(-K_X, x) \geq n+1 \) for some \( x \in X \), then \( X \cong \mathbb{P}^n \).

For the rest of the paper, all varieties are defined over an algebraically closed field \( k \) of characteristic \( p > 0 \). The aim of this note is to provide an argument that generalizes all the aforementioned results to positive characteristic.

Theorem 3. Let \( X \) be a normal projective variety of dimension \( n \) and \( \Delta \) an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( L = -(K_X + \Delta) \) is \( \mathbb{Q} \)-Cartier and ample. Assume that \( \epsilon(L, x) > n \) for some smooth point \( x \in X \), then \( X \cong \mathbb{P}^n \).

By standard reduction mod \( p \) technique, combining with Mori’s characterization of projective space \cite{Mor79}, the theorem yields a different proof its characteristic zero analog:
Corollary 4. Let $X$ be a normal projective variety of dimension $n$ over $\mathbb{C}$ and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ such that $L = -(K_X + \Delta)$ is $\mathbb{Q}$-Cartier and ample. Assume that $\epsilon(L, x) > n$ for some smooth point $x \in X$, then $X \cong \mathbb{P}^n$.

The argument we introduce here has the additional bonus that it generalizes [LZ16, Theorem 3] (which classifies complex Fano varieties $X$) with $\epsilon(-K_X, x) = n$ to positive characteristic as well.

Theorem 5. Let $X$ be a normal projective variety of dimension $n$ and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$ such that $L = -(K_X + \Delta)$ is $\mathbb{Q}$-Cartier and ample. Assume that $\epsilon(L, x) = n$ for some smooth point $x \in X$ and that either $(L^n) > n^n$ or $\Delta \neq 0$. Then either $X \cong \mathbb{P}^n$ or $X$ is one of the following:

1. a degree $d + 1$ weighted hypersurface $(x_0x_{n+1} = f(x_1, \ldots, x_n)) \subset \mathbb{P}(1^{n+1}, d);
2. the blow-up of $\mathbb{P}^n$ along a hypersurface contained in a hyperplane;
3. a Gorenstein log Del Pezzo surface of degree $\geq 5$.

Note that the condition on Seshadri constant $\epsilon(L, x) = n$ already implies $(L^n) \geq n^n$. When equality holds, we have (by the above theorem, we may assume $\Delta = 0$):

Theorem 6. Let $X$ be a normal projective variety of dimension $n$ such that $-K_X$ is $\mathbb{Q}$-Cartier and ample. Assume that $\epsilon(-K_X, x) = n$ for some smooth point $x \in X$, $((-K_X)^n) = n^n$ and $p \neq 2$, then $X$ is one of the following:

1. a quartic weighted hypersurface $X_4 = (x^2_{n+1} + x_nh(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1}))$ ($h \neq 0$) or $(x_nx_{n+1} = f(x_0, \ldots, x_{n-1})) \subset \mathbb{P}(1^n, 2^2);
2. the quotient of the quadric $Q_k = \left( \sum_{i=0}^{k} x_i^2 = 0 \right) \subset \mathbb{P}^{n+1}$ ($2 \leq k \leq n + 1$) by an involution $\tau(x_i) = \delta_ix_i$ ($\delta_i = \pm 1$) that is fixed point free in codimension 1 and such that not all the $\delta_i$ ($i = 0, \ldots, k$) are the same;
3. a Gorenstein log Del Pezzo surface of degree 4.

In particular, every Fano variety $X$ with $\epsilon(-K_X, x) = \dim X$ lifts to characteristic zero at least when $p = \text{char}(k)$ is a prime different from 2.

We now outline the proof of these theorems. Let $\sigma : Y \to X$ be the blowup of $X$ at $x$ with exceptional divisor $E$ and consider $D = \sigma^*L - \epsilon(L, x)E$. In characteristic zero, the proof in [LZ16] goes by analyzing the morphism defined by $|mD|$ ($m \gg 0$). To adapt it to positive characteristic, we need to prove that $\epsilon(L, x) \in \mathbb{Q}$ and that $D$ is semiample (which are somewhat obvious over $\mathbb{C}$). We first show in [2] that our assumption on Seshadri constant implies the global $F$-regularity of the pair $(Y, \Delta)$, which suffices to conclude that $\epsilon(L, x) \in \mathbb{Q}$, as it is essentially a consequence of Kodaira vanishing on $Y$ by the argument in [BS09, Proposition 1.1]. The semiample-ness of $D$ is a bit more complicated and a key step is given by Lemma [17] (based on the ideas of [CTX15]) on the base locus of adjoint divisors. Once this is done, Theorem 3 follows from the same argument in [LZ16] while Theorem 5 (resp. Theorem 6) reduces to the classification in positive characteristic of varieties containing a projective space in the smooth locus (resp. Gorenstein conic bundles in the sense of Definition 24 containing the projective space as a double section) under certain conditions. These two topics are treated in [3] and [4] respectively. Finally we finish the proof of the main theorems in [5].
Acknowledgement. The author would like to thank his advisor János Kollár for constant support, encouragement and numerous inspiring conversations. He also wishes to thank Takumi Murayama for several useful comments on the earlier draft of this paper and Yuchen Liu for helpful discussion.

2. Global F-regularity

Definition 7. Let $X$ be a normal quasi-projective variety and $\Delta$ an effective $\mathbb{Q}$-divisor on $X$. The pair $(X, \Delta)$ is called globally $F$-regular if for all effective Weil divisor $D$, there exists an $e$ such that the composition

$$O_X \to F^e_*O_X \to F^e_*O_X([((p^e - 1)\Delta] + D)$$

splits as a map of $O_X$-modules. It is called strongly $F$-regular if the pair is globally $F$-regular after restricting to every affine charts.

Since $X$ is quasi-projective, any effective divisor is contained in the support of some ample divisor, hence in the above definition of global $F$-regularity, it suffices to check splitting of (1) when $D$ is Cartier and ample. It is also clear from the definition that if $(X, \Delta)$ is globally $F$-regular and $0 \leq \Delta' \leq \Delta$ then $(X, \Delta')$ is also globally $F$-regular. Moreover, if $H$ is another effective divisor then $(X, \Delta + \epsilon H)$ is also globally $F$-regular for $0 < \epsilon \ll 1$ and thus we can perturb the divisor $\Delta$ (preserving global $F$-regularity) so that no coefficient of $\Delta$ has a denominator divisible by $p$. For more background on global $F$-regularity, see [SS10].

It is well known (see e.g. [MR85, SZ15]) that for any divisor $D$

$$\text{Hom}_{O_X}(F^e_*O_X(D), O_X) \cong H^0(X, (1 - p^e)K_X - D)$$

and (1) splits if and only if the composition

$$F^e_*O_Y([((1 - q)(K_Y + \Delta)] - D) \to F^e_*O_Y([(1 - q)K_Y]) \to O_Y$$

splits (where the second arrow is given by the trace map). And the latter condition is equivalent to saying that (2) induces a surjective map on global sections.

The following criterion also turns out to be quite useful when verifying a given pair is globally $F$-regular.

Lemma 8. Let $(X, D = E + \Delta)$ be a pair such that $L = -(K_X + D)$ is nef and big, $E$ is a prime divisor contained in the smooth locus of $X$ and $E \not\subseteq \text{Supp}(\Delta)$. Assume that $(E, \Delta|_E)$ is globally $F$-regular and $L|_E$ is ample, then $(X, \Delta)$ is also globally $F$-regular.

Proof. We first make a few reductions. Since $L$ is nef and big, there exists an effective divisor $M$ such that $L - \epsilon M$ is ample for all $0 < \epsilon \ll 1$. As $L|_E$ is ample, $(L + \epsilon E)|_E$ is also ample for sufficiently small $\epsilon$, hence $L + \epsilon E$ is nef and big for $0 \leq \epsilon \ll 1$ (if $C$ is a curve such that $(L + \epsilon E \cdot C) < 0$ then since $L$ is nef we have $C \subseteq E$, but this contradicts the ampleness of $(L + \epsilon E)|_E$). Let $a \geq 0$ be the coefficient of $E$ in $M$, let $\lambda = \frac{1}{a+1}$ and $D' = D + \epsilon(\lambda M - (1 - \lambda)E)$, then $E$ still has coefficient one in $D'$ (i.e. $D' = E + \Delta'$ where $E \not\subseteq \text{Supp}(\Delta')$) and for sufficiently small $\epsilon$, $(E, \Delta'|_E)$ is still globally $F$-regular. We also have $-(K_X + D') = (1 - \lambda)(L + \epsilon E) + \lambda(L - \epsilon M)$, hence for $0 < \epsilon \ll 1$, $-(K_X + D')$ is ample. Since $\Delta' \geq \Delta$, we may replace $D$ by $D'$ and assume that $L = -(K_X + D)$ is ample in what follows. By perturbing the coefficients of components of $\Delta$, we may also assume that $(p^e - 1)\Delta$ has integral coefficients for some $e > 0$. 


Let $H$ be an ample Cartier divisor on $X$ such that $\Delta \cup \text{Sing}(X) \subseteq \text{Supp}(H) \not\subseteq E$. Consider the following commutative diagram

$$
\begin{array}{ccc}
F_\ast^e \mathcal{O}_X((1 - p^e)(K_X + E + \Delta) - H) & \rightarrow & F_\ast^e \mathcal{O}_E((1 - p^e)(K_E + \Delta_E) - H) \\
\n\n\mathcal{O}_X & \rightarrow & \mathcal{O}_E
\end{array}
$$

where the two vertical arrows are given by the trace map. By assumption, $\text{Tr}_E$ induces a surjection on global sections. As $L$ is ample, for sufficiently large and divisible $e$ we have $H^1(X, F_\ast^e \mathcal{O}_X((1 - p^e)(K_X + E + \Delta) - E - H)) = H^1(X, (p^e - 1)L - E - H) = 0$, thus by the long exact sequence of cohomology, $i$ also induces a surjection on global sections. It follows that $H^0(\text{Tr}_E^\ast)$ is surjective as well. By \cite[Theorem 3.9]{SS10}, this implies that $(X, \Delta)$ is globally F-regular.

**Corollary 9.** Let $(X, \Delta)$ be a pair such that $L = -(K_X + \Delta)$ is ample. Assume that $(L^n) > n^n$ and $\epsilon(L, x) \geq n$ for some smooth point $x \in X \setminus \Delta$. Let $Y$ be the blow up of $X$ at $x$ and $\Delta$ be also its strict transform on $Y$. Then $(Y, \Delta)$ is globally F-regular.

**Proof.** Let $E$ be the exceptional divisor of the blowup $\sigma : Y \rightarrow X$, then the pair $(Y, E + \Delta)$ satisfies all the assumptions of the Lemma \cite{SS10}.

**Remark 10.** Note that $\epsilon(L, x) \geq n$ already implies $(L^n) \geq n^n$. However, the assumption $(L^n) > n^n$ in the above corollary can not be removed in general (even in the boundary free case, i.e. when $\Delta = 0$). For example, consider the pair $(X = \mathbb{P}^n, H)$ where $H$ is a hyperplane, then clearly $\epsilon(-(K_X + H), x) = n$ for any smooth point $x \in X$, but $H$ is an $F$-pure center of the pair. As another example, consider the Fermat cubic surface $Y = (x^3 + y^3 + z^3 + w^3 = 0) \subseteq \mathbb{P}^3$, then $Y$ is not even globally F-split in characteristic 2, but $Y$ is also the blow up of a smooth del Pezzo surface of degree 4 whose anticanonical divisor has Seshadri constant 2 at the point we blow up.

One of the advantages of global F-regularity is that most vanishing results that hold in characteristic zero remain valid. In particular we have (see \cite[Theorem 6.8]{SS10} for the dual statement):

**Lemma 11.** Let $(Y, \Delta)$ be a globally F-regular pair and $D$ an effective Weil divisor on $Y$ such that $D - (K_Y + \Delta)$ is nef and big. Then $H^i(Y, \mathcal{O}_Y(D)) = 0$ for all $i > 0$.

**Proof.** We note that the assumption implicitly requires that $D - (K_Y + \Delta)$ is $\mathbb{Q}$-Cartier. We may perturb the pair as before and assume that $(p^e - 1)\Delta$ has integral coefficients for sufficiently divisible $e$ and that $D - (K_Y + \Delta)$ is ample. Let $q = p^e$. Since $(Y, \Delta)$ is globally F-regular, the trace map

$$
\text{Tr}^e : F_\ast^e \mathcal{O}_Y((1 - q)(K_Y + \Delta)) \rightarrow F_\ast^e \mathcal{O}_Y((1 - q)K_Y) \rightarrow \mathcal{O}_Y
$$

splits for every sufficiently divisible $e$. Taking the reflexive tensor with $\mathcal{O}_Y(D)$ we see that $H^i(Y, \mathcal{O}_Y(D))$ is a direct summand of $H^i(Y, \mathcal{O}_Y(D) \otimes F_\ast^e \mathcal{O}_Y((1 - q)(K_Y + \Delta))) = H^i(Y, \mathcal{O}_Y((1 - q)(K_Y + \Delta) + qD))$, but the latter group is zero when $i > 0$ and $q \gg 0$ by Serre vanishing, thus $H^i(Y, \mathcal{O}_Y(D)) = 0.$
Corollary 12. Let \((Y, \Delta)\) be a globally \(F\)-regular pair, \(f : Y \to X\) a proper morphism and \(D\) an Weil divisor on \(Y\) such that \(D - (K_Y + \Delta)\) is \(f\)-nef and \(f\)-big. Then \(R^i f_* O(D) = 0\) for all \(i > 0\).

Proof. It suffices to show that \(H^i(Y, O_Y(D + f^* H)) = 0\) for sufficiently ample divisor \(H\) on \(X\). But for such \(H\), \(D + f^* H - (K_Y + \Delta)\) is nef and big by assumption, so the statement follows directly from Lemma [1]. □

3. VARIETIES CONTAINING PROJECTIVE SPACE AS A DIVISOR

In [LZ16], an important step in the classification of varieties \(X\) with \(\epsilon(-K_X, x) \geq n\) is the classification of varieties (over \(\mathbb{C}\)) that contain a divisor \(D \cong \mathbb{P}^{n-1}\) in the smooth locus. In this section we carry out the parallel study of such varieties in positive characteristic. We start with the Picard number one case.

Lemma 13. Let \(X\) be a normal projective variety of dimension \(n\) and \(D \cong \mathbb{P}^{n-1}\) a divisor contained in its smooth locus. Assume that \(N_{D/X}\) is nef and \(n \geq 3\) if \(N_{D/X}\) is ample. Then the natural restriction \(\text{Cl}(X) \to \text{Cl}(D)\) is surjective.

Proof. Let \(d = \deg N_{D/X}\). If \(d > 0\), let \(Z \subseteq D\) be a smooth hypersurface of degree \(d\) and let \(\tilde{X}\) be the blow up of \(X\) along \(Z\). Note that since \(n \geq 3\), \(Z\) is connected. Let \(E\) be the exceptional divisor and \(\tilde{D}\) the strict transform of \(D\). Then we have \(\text{Cl}(\tilde{X}) \cong \text{Cl}(X) \oplus \mathbb{Z}[E]\) and the image of \(\text{Cl}(\tilde{X}) \to \text{Cl}(\tilde{D})\) is the same as the image of \(\text{Cl}(X) \to \text{Cl}(D) \cong \text{Cl}(\tilde{D})\). Since \(N_{D/X} \cong O_D\), we may replace \((X, D)\) by \((\tilde{X}, \tilde{D})\) and reduce to the case that \(d = 0\).

As \(D \cong \mathbb{P}^{n-1}\) and \(d = 0\), we have \(h^0(D, N_{D/X}) = 1\) and \(h^1(D, N_{D/X}) = 0\), hence the Hilbert scheme of \(X\) is smooth and of dimension 1 at the point \([D]\). It follows that there exists a curve \(C\) (not necessarily proper) and a family of divisors of \(X\)

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & X \\
\downarrow f & & \downarrow \\
C & & \\
\end{array}
\]

such that \(f\) is smooth, \(g\) identifies a fiber \(F\) of \(f\) with \(D\) and if \(F_s, F_t\) are fibers of \(f\) over \(s \neq t \in C\), then \(g(F_s) \neq g(F_t)\). As \(\mathbb{P}^{n-1}\) is rigid, after shrinking \(C\) we may assume that all fibers of \(f\) are isomorphic to \(\mathbb{P}^{n-1}\); moreover since \(C\) is a curve, \(f\) is indeed a \(\mathbb{P}^{n-1}\)-bundle by Tsen’s theorem. On the other hand, as \(N_{D/X} \cong O_D\), we have \(D' \cap D = \emptyset\) if \(D' \neq D\) is algebraically equivalent to \(D\), thus \(g\) is an isomorphism in a neighbourhood of \(D\). Therefore, after further shrinking of \(C\) we may assume that \(g\) is an open immersion. Then \(g^* : \text{Cl}(X) \to \text{Cl}(Y)\) is surjective. Since \(f : Y \to C\) is a \(\mathbb{P}^{n-1}\)-bundle, \(\text{Cl}(Y) \to \text{Cl}(F)\) is also surjective, so the lemma follows. □

Remark 14. The \(n \geq 3\) assumption in the above lemma is necessary if \(N_{D/X}\) is ample, since \(\text{Cl}(X) \to \text{Cl}(D)\) is not surjective when \(D\) is a conic in \(X = \mathbb{P}^2\). It is also not hard to see that the statement does not hold if \(N_{D/X}\) has negative degree. For example, consider a general surface \(S\) of degree \(d \geq 4\) that contains a conic curve \(C\), then \(\text{Pic}(S)\) is generated by \(C\) by [Lop91, Theorem II.3.1] and the hyperplane class \(H\). Since \((C \cdot H) = 2\) and \((C^2) = 2(3 - d)\), \(\text{Pic}(S) \to \text{Pic}(C)\) is not surjective and the image is an index 2 subgroup.
Lemma 15. Let $X$ be a normal projective variety of dimension $n \geq 2$ containing a divisor $D \cong \mathbb{P}^{n-1}$ in its smooth locus. Assume that $\rho(X) = 1$, then one of the following holds:

1. $X \cong \mathbb{P}(1^n, d)$ for some $d \in \mathbb{Z}_{>0}$ and $D$ is the hyperplane defined by the vanishing of the last coordinate; or
2. $n = 2$, $X \cong \mathbb{P}^2$ and $D$ is a smooth conic.

Proof. First consider the case $n \geq 3$. Let $X_0$ be the smooth locus of $X$. Since $\rho(X) = 1$, $D$ is ample, so $X$ has only isolated singularities and the natural map $\text{Cl}(X) \cong \text{Pic}(X_0) \to \text{Pic}(\hat{X})$ is an isomorphism by [Gro05, Exposé XI, Proposition 2.1], where $\hat{X}$ is the formal completion of $X$ along $D$. As $\hat{X} \cong \mathbb{P}^{n-1}$ and $n \geq 3$, we have $H^1(D, \mathcal{O}_D(-mD)) = 0$ for all $m$, hence by the exact sequence (c.f. [Gro05, Exposé XI, §1])

\[ H^1(D, \mathcal{O}_D(-mD)) \to \text{Pic}(D_{m+1}) \to \text{Pic}(D_m) \to H^2(D, \mathcal{O}_D(-mD)) \]

the restriction map $\text{Pic}(\hat{X}) \to \text{Pic}(D)$ is injective; on the other hand it is also surjective by Lemma [13], thus we have an isomorphism $\text{Cl}(\hat{X}) \cong \text{Pic}(D)$. In particular, $X$ is $\mathbb{Q}$-factorial and since $-(K_X + D)|_D = -K_D$ is ample, $-(K_X + D)$ is ample on $X$ itself. By Lemma [8] $X$ is globally F-regular.

Let $H$ be the ample generator of $\text{Cl}(X)$, then $\mathcal{O}_D(H) \cong \mathcal{O}_D(1)$ and there exists a positive integer $d$ such that $D \sim dH$. Consider the exact sequence

\[ H^0(X, \mathcal{O}_X(H - D)) \to H^0(X, \mathcal{O}_X(H)) \to H^0(D, \mathcal{O}_D(H)) \to H^1(X, \mathcal{O}_X(H - D)) \]

Since $X$ is globally F-regular and $H - (K_X + D)$ is ample, we have $H^1(X, \mathcal{O}_X(H - D)) = 0$ by Lemma [11]. If $d = 1$, then $H \sim D$ is Cartier and it follows from [1] that $H$ is globally generated and $h^0(X, H) = n + 1$, thus $|H|$ induces a morphism $X \to \mathbb{P}^n$ of degree $(H^n) = (H^{n-1} \cdot D) = 1$, which is an isomorphism $X \cong \mathbb{P}^n$. If $d > 1$, then $H^0(X, \mathcal{O}_X(H - D)) = 0$ and by [1] we have $h^0(X, H) = n$ and is globally generated in a neighbourhood of $D$. The global sections of $\mathcal{O}_X(H)$ and the canonical section of $\mathcal{O}_X(D) \cong \mathcal{O}_X(dH)$ then defines a morphism $X \to \mathbb{P}(1^n, d)$ of degree $(H^n \cdot D) = 1$, which is again an isomorphism $X \cong \mathbb{P}(1^n, d)$. By construction, $D$ is identified with the hyperplane defined by the vanishing of the last coordinate.

Next assume $n = 2$. By assumption $(D^2) > 0$, thus by Lemma [10] $X$ is $\mathbb{Q}$-factorial. As in the $n \geq 3$ case, we still have $-(K_X + D)$ is ample and $X$ is globally F-regular. If $\text{Cl}(X) \to \text{Pic}(D)$ is surjective then as before we have $X \cong \mathbb{P}(1, 1, d)$ for some $d > 0$. If $\text{Cl}(X) \to \text{Pic}(D)$ is not surjective then since $(K_X + D \cdot D) = -2$ we see that the image is generated by the restriction of $H = -(K_X + D)$. We also have $D \sim \mathbb{Q} \cdot dH$ for some $d > 0$. We now divide into three cases according to the value of $d$.

If $d = 1$, then by the same argument as in the $n \geq 3$ case, $|D|$ is base point free and identifies $X$ with a quadric in $\mathbb{P}^3$ (note that $(D^2) = 2d = 2$ and $h^0(X(D), 4)$ and $D$ a hyperplane section. Since $\rho(X) = 1$, $X$ is singular, but then $\text{Cl}(X) \to \text{Pic}(D)$ is surjective, contrary to our assumption.

If $d = 2$, then $(H^2) = \frac{1}{2}(H \cdot D) = 1$. As before, by [1] and the global F-regularity of $X$ we have $H^0(X, \mathcal{O}_X(H)) \cong H^0(D, \mathcal{O}_D(2))$ and $h^0(X, H) = 3$, hence for any $x \in D$, we may choose two different $H_1, H_2 \sim H$ passing through $x$. Clearly both $H_i$ are integral (otherwise $\text{Cl}(X) \to \text{Pic}(D)$ is surjective). Since $(H_1 \cdot H_2) = (H^2) = 1$, we see that $H_1$ only intersects $H_2$ at $x$. It follows that $H$ is Cartier, base point free and defines a
morphism $X \rightarrow \mathbb{P}^2$ of degree 1, which is an isomorphism that identifies $D$ with a smooth conic.

Finally if $d \geq 3$, we still have $H^0(X, \mathcal{O}_X(H)) \cong H^0(D, \mathcal{O}_D(2))$. Let $s_0$ be the canonical section of $H^0(X, \mathcal{O}_X(D))$. Choose $s_1, s_2 \in H^0(X, \mathcal{O}_X(H))$ whose restrictions on $D$ induce a separable morphism $D \rightarrow \mathbb{P}^1$ of degree 2. Then we can define a separable double cover $f : X \rightarrow Y = \mathbb{P}(1,1,d)$ sending $x \in X$ to $[s_1(x) : s_2(x) : s_0(x)]$. We have $K_X = f^*K_Y + R$ for some divisor $R$ supported in the branched locus of $f$. A direct calculation yields $R \sim Q H$, thus $f_*R \sim f_*H \sim 2L$ where $L$ is the ample generator of $\text{Cl}(Y) \cong \mathbb{Z}$. But since $d \geq 3$, $f_*R$ and thus $R$ cannot be integral. It follows that we have a decomposition $H \sim Q R_1 + R_2$ for some effective nonzero $\mathbb{Z}$-divisor $R_1, R_2$, but then $(R_1 \cdot D) + (R_2 \cdot D) = (H \cdot D) = 2$ and $\text{Cl}(X) \rightarrow \text{Pic}(D)$ is surjective. So this case cannot happen and the proof is now complete. □

The following lemma is used in the above proof.

**Lemma 16.** Let $X$ be a normal projective surface. Suppose there exists a smooth rational curve $C$ contained in the smooth locus of $X$ such that $(C^2) \geq 0$. Then $X$ has rational singularities. In particular, $X$ is $\mathbb{Q}$-factorial.

**Proof.** After possibly blowing up points on $C$ we reduce to the case that $(C^2) = 0$. Let $\tilde{X} \rightarrow X$ be the minimal resolution of $X$ and let $\tilde{C}$ also denote its strict transform on $\tilde{X}$. Since $C$ is a smooth rational curve we have $(K_{\tilde{X}} \cdot \tilde{C}) = -2$ by adjunction. By Riemann-Roch we have

$$\chi(\mathcal{O}_{\tilde{X}}(m\tilde{C})) = \frac{1}{2}(m\tilde{C} \cdot m\tilde{C} - K_{\tilde{X}}) + \chi(\mathcal{O}_{\tilde{X}}) = m + \chi(\mathcal{O}_{\tilde{X}})$$

On the other hand by Serre duality we have $h^2(\tilde{X}, m\tilde{C}) = h^0(\tilde{X}, K_{\tilde{X}} - m\tilde{C}) = 0$ when $m \gg 0$. It follows that $h^0(\tilde{X}, m\tilde{C}) \geq 2$ for sufficiently large $m$. Hence there exists an effective divisor $\Gamma \sim m\tilde{C}$ for some $m > 0$ such that $\tilde{C} \not\subseteq \text{Supp}(\Gamma)$. As $(\tilde{C} \cdot \Gamma) = m(C^2) = 0$, we see that $\Gamma$ is disjoint from $\tilde{C}$, thus $m\tilde{C}$ is base point free. Since $\tilde{C}$ is the pullback of $C$, $C$ is semiample and induces a morphism $p : X \rightarrow Y$ with connected fibers to a curve $Y$ such that the general fiber is isomorphic to $C$ (if $\Gamma \equiv mC$ is an irreducible fiber in the smooth locus of $X$, then $2p_a(\Gamma) - 2 = (K_X + \Gamma \cdot \Gamma) = -2m$, thus $p_a(\Gamma) = 0$ and $m = 1$). By [Che97, Theorem 2 and Remark 3], $X$ has rational singularities and hence is $\mathbb{Q}$-factorial by [Lip69, Proposition 17.1]. □

We next turn to the case when the Picard number is at least two. In [LZ16, Lemma 12], this is done by running MMP, which is not yet available in positive characteristic in general. Nevertheless, the following lemma serves as a substitute at least for the purpose of this note.

**Lemma 17.** Let $(Y, \Delta)$ be a strongly $F$-regular pair and $D$ a nef divisor such that $D - (K_Y + \Delta)$ is nef and big. Suppose that $y \in Y$ is contained in the stable base locus of $D$, then there exists a positive dimensional subvariety $V \subseteq Y$ containing $y$ such that $D|_V$ is numerically trivial.

**Proof.** This is indeed a consequence of the arguments in [CTX15, §3-4]. Namely, if $y \in Y$ is not contained in any positive dimensional subvariety $V \subseteq Y$ such that $D|_V$ is numerically trivial, then the same argument as in [CTX15, Theorem 3.7] creates a $\mathbb{Q}$-divisor $D^{(c)} =$
\[ \sum_{i=1}^{n} t_i(e)D_i \] and an isolated non-F-pure center \( W \) supported at \( y \) for which the proof of [CTX15, Theorem 1.1] can be used to show that \( y \) is not a base point of \( |mD| \) for \( m \gg 0 \).

We now briefly explain the idea for classifying varieties \( X \) containing a divisor \( D \cong \mathbb{P}^{n-1} \) such that \( \rho(X) \geq 2 \) and \( -(K_X + D) \) is ample. Instead of running the MMP, we consider divisors of the form \( L_\lambda = -(K_X + \lambda D) \) and hope that for some \( \lambda \), the corresponding divisor \( L_\lambda \) defines the contraction of the extremal ray we want. A natural idea is to take the largest \( \lambda \) such that \( L_\lambda \) is nef. To make the argument work, we need to show that \( \lambda \in \mathbb{Q} \) and that \( L_\lambda \) is semiample. Once this is done, it is quite straightforward to finish the classification.

For the next couple lemmas, we introduce the following notations. Let \( D \) be an effective divisor on \( X \), we define

\[ \rho(X, D) := \text{rank} \text{Im}(\text{Pic}(X) \to \text{Pic}(D)) \]

If in addition \( D \) is Cartier and \( L \) is an ample divisor on \( X \), we define

\[ \epsilon(L, D) = \sup \{ t | L - tD \text{ is ample} \} \]

and let \( s(L, D) \) be the largest integer \( s \) such that \( (L - sD)|_D \) is base point free and \( H^0(X, L - sD) \to H^0(D, L - sD|_D) \) is surjective.

**Lemma 18.** Let \( L \) be an ample divisor on \( X \) and \( D \) an effective Cartier divisor. Then for all \( m \geq 1 \) we have

\[ \frac{s(mL, D)}{m} \leq \epsilon(L, D) = \lim_{m \to \infty} \frac{s(mL, D)}{m} \]

**Proof.** The proof is similar to that of the analogous statement for Seshadri constants (where \( D \) is the exceptional divisor of a blow up). We first prove the inequality \( \frac{s(mL, D)}{m} \leq \epsilon(L, D) \). Let \( s = s(mL, D) \), it suffices to show that \( mL - sD \) is nef. Suppose it is not, then there exists a curve \( C \subseteq X \) such that \( (mL - sD \cdot C) < 0 \). Since \( L \) is ample, we have \( (D \cdot C) > 0 \) and therefore, \( C \) intersects \( D \). Choose \( x \in C \cap D \). By the definition of \( s(L, D) \), there exists a section \( u \in H^0(X, mL - sD) \) that does not vanish at \( x \). But this implies \( mL - sD \cdot C \geq 0 \), a contradiction.

Now let \( \lambda \) be any rational number such that \( \lambda < \epsilon(L, D) \). We will show \( s(mL, D) \geq \lfloor \lambda m \rfloor \) for \( m \gg 0 \), thus proving the equality part of the lemma. To this end fix \( m \gg 0 \) and let \( s = \lfloor \lambda m \rfloor \). By Lemma 19 \( mL - sD \) is very ample and \( H^1(X, mL - (s + 1)D) = 0 \). Therefore, \( (mL - sD)|_D \) is base point free and by the long exact sequence of cohomology, \( H^0(X, mL - sD) \to H^0(D, (mL - sD)|_D) \) is surjective. Thus \( s(mL, D) \geq \lfloor \lambda m \rfloor \) and we are done. \( \Box \)

Recall the following Fujita-type result that is used in the above proof (it will also be used later).

**Lemma 19.** Let \( L \) be an ample divisor on \( X \) and \( D \) a Cartier divisor. Let \( \lambda > 0 \) be such that \( L - \lambda D \) is still ample and let \( m, s \geq 0 \) be integers such that \( s \leq \lambda m \). Let \( F \) be a coherent sheaf on \( X \). Then for \( m \gg 0 \), \( mL - sD \) is very ample and \( H^1(X, F(mL - sD)) = 0 \).
Proof. We may assume $\lambda \in \mathbb{Q}$ (otherwise enlarge $\lambda$ slightly). Choose sufficiently large and divisible $N$ such that $H_1 = NL$ and $H_2 = N(L - \lambda D)$ are both very ample. Then since $\lambda$ is rational, there exists finitely many line bundles $L_i$ such that $mL_i - sD = L_i + a_1 H_1 + a_2 H_2$ for some $i$ and some integers $a_1, a_2 \geq 0$. As $m \gg 0$ we have $\max\{a_1, a_2\} \gg 0$, thus the lemma follows from [Hu83, Theorem 2 and Corollary 3].

Lemma 20. Let $(X, \Delta)$ be a globally F-regular pair and $D$ a prime Cartier divisor on $X$ such that $L = -(K_X + \Delta + D)$ is ample. Let $\lambda = \epsilon(L, D)$. Assume either $\rho(X, D) = 1$ or $(L - \lambda D)|_D$ is ample. Then $\lambda \in \mathbb{Q}$ and $L - \lambda D$ is semiample.

Proof. We first prove $\lambda \in \mathbb{Q}$. Suppose this is not the case. If $\rho(X, D) = 1$, then as $\lambda \not\in \mathbb{Q}$, $(L - \lambda D)|_D$ is nef but not numerically trivial, so we reduce to the case when $(L - \lambda D)|_D$ is ample. Choose $\mu > \lambda$ such that $L - \mu D|_D$ is still ample. We claim that $s(mL, D) \geq \lfloor \lambda (m + 1) \rfloor$ for sufficiently large and divisible $m$. To see this, let $m$ be fixed and let $s = \lfloor \lambda (m + 1) \rfloor$, then as $m \gg 0$ we have $s < \mu m$ and thus by Lemma [19] $(mL - sD)|_D$ is very ample. Moreover, since $\lambda \not\in \mathbb{Q}$, we have $s < \lambda (m + 1)$, hence $mL - (s + 1)D - (K_X + \Delta) \sim (m + 1)L - sD$ is ample and as $X$ is globally F-regular, $H^1(X, mL - (s + 1)D) = 0$ by Lemma [11] thus $H^0(X, mL - sD) \rightarrow H^0(D, (mL - sD)|_D)$ is onto. So $s(mL, D) \geq s = \lfloor \lambda (m + 1) \rfloor$, proving the claim. On the other hand, by Lemma [18] we have $s(mL, D) = \lfloor \lambda (m + 1) \rfloor \leq \lambda m$ (for sufficiently divisible $m$). As $\lambda > 0$ and $\lambda \not\in \mathbb{Q}$, this is a contradiction.

Thus we have $\lambda \in \mathbb{Q}$. Let $M = L - \lambda D$. Under either assumption of the lemma, $mM|_D$ is base point free for sufficiently divisible $m$. We also have $H^1(X, mM - D) = 0$ since $X$ is globally F-regular and $mM - D - (K_X + \Delta) = mM + L$ is ample, hence $H^0(X, mM) \rightarrow H^0(D, mM|_D)$ is onto and the stable base locus $B = Bs(M)$ of $M$ is disjoint from $D$. On the other hand, by Lemma [17], for any $x \in B$, there exists a positive dimensional subvariety $C \subseteq X$ containing $x$ such that $M|_C$ is numerically trivial. By taking hyperplane sections we may assume that $C$ is a curve. Clearly $C$ intersects $D$, for otherwise $M|_C = L|_C$ is ample. Since $x \in B$ and $(M \cdot C) = 0$, we have $C \subseteq B$, but then $B \cap D$ contains $C \cap D$ and in particular is nonempty, a contradiction. Thus $B = \emptyset$ and $M$ is semiample. \[ \square \]

The next two lemmas are natural generalizations of [LZ16, Lemma 4, 7] to pairs. We omit the proofs since the argument in [LZ16] works verbatim here.

Lemma 21. Let $\pi : S \rightarrow T$ be a proper birational morphism between normal surfaces and $\Delta$ an effective divisor on $S$. Let $C \subseteq S$ be a $K_S$-negative $\pi$-exceptional curve such that $C \not\subseteq \text{Supp} (\Delta)$. Then $-(K_S + \Delta) \cdot C \leq 1$, with equality if and only if $C$ is disjoint from $\Delta$ and $S$ has only Du Val singularities along $C$.

Lemma 22. Let $g : Y \rightarrow Z$ be a proper birational morphism between normal varieties and $\Delta$ an effective divisor on $Y$. Let $D$ be a smooth $g$-ample Cartier divisor on $Y$ such that $-(K_Y + \Delta + gD)$ is $g$-nef for some $\lambda \geq 1$. Assume that $Y$ is Cohen-Macaulay, $D \cap \Delta = \emptyset$ and $g|_D : D \rightarrow G = g(D)$ is an isomorphism, then $\lambda = 1$, $\text{Ex}(g)$ is disjoint from $\Delta$ and $Z$ is smooth along $G$.

We are ready to finish the second part of the classification of varieties containing the projective space as a smooth divisor.
Lemma 23. Let \((X, \Delta)\) be a pair and \(D \cong \mathbb{P}^{n-1}\) a prime divisor contained in the smooth locus of \(X\) such that \(L = -(K_X + \Delta + D)\) is ample. Assume that \(\rho(X) \geq 2\) and \(\Delta \cap D = \emptyset\). Then \(X\) is isomorphic to a \(\mathbb{P}^1\)-bundle \(\mathbb{P}(O \oplus O(-d))\) over \(\mathbb{P}^{n-1}\) for some \(d \in \mathbb{Z}_{\geq 0}\) and \(D\) is a section.

Proof. By Lemma \(\S\) and our assumption, \((X, \Delta)\) is globally \(F\)-regular. Since \(\rho(X) \geq 2\), we may an ample divisor \(H\) and \(0 < t \ll 1\) such that \((X, \Delta + tH)\) is still globally \(F\)-regular, \(L_1 = -(K_X + \Delta + D)\) is ample and that \(L_1\) and \(D\) are linearly independent in \(\text{Pic}(X)_\mathbb{Q}\). Let \(\lambda = \epsilon(L_1, D)\). Clearly \(\lambda > 0\). As \(D \cong \mathbb{P}^{n-1}\), we have \(\rho(X, D) = \rho(D) = 1\). Thus by Lemma \([20]\) \(\lambda \in \mathbb{Q}\) and \(M = L_1 - \lambda D\) is semiample. Since \(M \neq 0\) in \(\text{Pic}(X)_\mathbb{Q}\), it induces a morphism (with connected fibers) \(g : X \to Y\) such that \(\dim Y \geq 1\) and \(M = g^*H\) for some ample divisor \(H\) on \(Y\). We claim that \(M|_D\) is ample. Indeed, if \((L_1 - \lambda D)|_D = M|_D \sim_\mathbb{Q} 0\), then as \(L_1\) is ample, \(D|_D\) is ample as well. Let \(S\) be a surface in \(X\) given by a complete intersection of general hyperplanes, then we have \((D|_S^2) > 0\) and \((D|_S \cdot M|_S) = 0\), but then by Hodge index theorem, \((M|_S^2) < 0\) (\(M|_S\) is not numerically trivial since \(M\) is not), contradicting the fact that \(M\) is nef. Hence \(M|_D\) is ample and by the same argument as in Lemma \([20]\) we know that \(H^0(X, mM) \to H^0(D, mM|_D)\) is onto, therefore \(g|_D\) is a closed embedding. Since \((X, \Delta)\) is globally \(F\)-regular, \(X\) is Cohen-Macaulay by \([SZ15\) Theorem 1.18]. We also have \(-(K_X + \Delta + (\lambda + 1)D) \sim_\mathbb{Q} M + tH\) which is \(g\)-ample. By Lemma \([22]\) \(g\) cannot be birational, hence induces an isomorphism \(\mathbb{P}^{n-1} = D \cong Y\). If \(C\) is a scheme theoretic fiber of \(g\), then \(C\) has dimension one since \(\dim(C \cap D) = 0\). Since \(g|_D\) is an isomorphism and every component of \(C\) intersects \(D\), \(C\) is indeed an integral curve. Let \(\mathcal{I}_C\) be the ideal sheaf of \(C\). Consider the exact sequence

\[\cdots \to R^1g_*O_X \to H^1(C, O_C) \to R^2g_*\mathcal{I}_C \to \cdots\]

Since \(g\) has fiber dimension at most one we have \(R^2g_*\mathcal{I}_C = 0\) and as \(X\) is globally \(F\)-regular we also have \(R^1g_*O_X = 0\), thus \(H^1(C, O_C) = 0\) and \(C \cong \mathbb{P}^1\). It follows that \(g : X \to Y\) is a \(\mathbb{P}^1\)-fibration with a section \(D\). Thus \(X \cong \mathbb{P}_Y(O_Y \oplus O_Y(-d))\) for some \(d \geq 0\).

4. Conic bundles

In this section we study conic bundles in positive characteristic. Later we will apply these results to classify varieties \(X\) with \(\epsilon(-K_X) = n\) and \((-K_X)^n = n^n\).

Definition 24. Let \(f : X \to Y\) be a proper morphism between normal quasi-projective varieties. If the general fiber of \(f\) is a plane conic (so is either a \(\mathbb{P}^1\) or a double line in characteristic \(2\)), we call \(f\) a rational conic bundle. If \(X\) is Cohen-Macaulay, every fiber of \(f\) has pure dimension 1, \(f_*O_X = O_Y\) and there exists a Cartier divisor \(D\) on \(X\) such that \(-K_X \equiv_f D\) is \(f\)-ample, then we call \(f\) a Gorenstein conic bundle.

Lemma 25. Let \(C\) be a locally complete intersection (l.c.i.) curve over \(k\). Assume that \(\omega_C^{-1}\) is ample. Then the following are equivalent:

1. \(h^0(C, O_C) = 1\);
2. \(\deg \omega_C = -2\) and every irreducible component of \(C\) is isomorphic to \(\mathbb{P}^1\);
3. \(C\) is a plane conic.

Proof. We will show \((1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)\). By Riemann-Roch and Serre duality (see \([Liu02\) for Riemann-Roch formula on singular curves) we have \(\chi(O_C) = -\chi(\omega_C) = \)
− \deg \omega_C - \chi(\mathcal{O}_C)$, hence $− \deg \omega_C = 2 \chi(\mathcal{O}_C)$. On the other hand, since $\omega_C^{-1}$ is ample, $− \deg \omega_C > 0$, thus if (1) holds we have $0 < \chi(\mathcal{O}_C) = 1 − h^1(C, \mathcal{O}_C) \leq 1$, hence $\chi(\mathcal{O}_C) = 1$, $h^1(C, \mathcal{O}_C) = 0$ and $\deg \omega_C = −2$. Moreover, as $\dim C = 1$ the map $H^1(C, \mathcal{O}_C) \to H^1(C, \mathcal{O}_C)$ is surjective for every component $C_i$ of $C_{\text{red}}$, which implies (2).

Write $[C] = \sum a_i [C_i]$ as a 1-cycle where the $C_i$'s are irreducible components of $C$, then $\deg \omega_C = \sum a_i \deg(\omega_C|_{C_i})$. Since $\omega_C^{-1}$ is ample, $\deg(\omega_C|_{C_i}) < 0$. Hence if (2) holds we have either $C$ is reduced with at most two components or $[C] = 2[C_1]$. If $C$ is reduced, the same Riemann-Roch calculation as above yields $h^1(C, \mathcal{O}_C) = 0$, hence either $C \cong \mathbb{P}^1$ or $C$ is the union $C_1 \cup C_2$ of two $\mathbb{P}^1$. In the latter case, by the exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$ we have $h^0(\mathcal{O}_{C_1 \cap C_2}) = 1$ hence $C_1 \cap C_2$ (scheme-theoretic intersection) consists of only one point and $C$ is a reducible conic. If $[C] = 2[C_1]$, let $I$ be the ideal sheaf of $C_1$, then $I^2 = 0$ and we have an exact sequence $0 \to I \otimes \omega_C \to \omega_C \to \omega_C|_{C_1} \to 0$. As $\deg \omega_C = −2$ and $C_1 \cong \mathbb{P}^1$, we get $\deg(\omega_C|_{C_1}) = −1$ and $\chi(\omega_C|_{C_1}) = 0$. On the other hand, by Riemann-Roch we have $\chi(\omega_C) = \frac{1}{2} \deg \omega_C = −1$, hence $\chi(I \otimes \omega_C) = −1$ and $\deg I = −1$. It follows that $C$ is an infinitesimal extension (see [Har77, II, Ex 8.7]) of $C_1$ by $\mathcal{O}_{C_1}(-1)$, which is classified by $H^1(C_1, T_{C_1}(-1)) = 0$ by [Har77, III, Ex 4.10]. Since one such extension is given by the planar double line, it is isomorphic to $C$ and in particular $C$ is a plane conic. This proves (3). Finally it is clear that (3) implies (1). □

Lemma 26. Let $f : X \to Y$ be a proper morphism. Assume that $Y$ is smooth and $f$ is a Gorenstein conic bundle, then $f$ is a conic bundle.

Proof. By dimension reason the singular locus of $X$ cannot dominate $Y$, hence the general fiber $C$ of $f$ is l.c.i and $\omega_C^{-1}$ is ample by adjunction. Since $f_* \mathcal{O}_X = \mathcal{O}_Y$, we have $h^0(C, \mathcal{O}_C) = 1$, thus $C$ is a plane conic by Lemma 25. In particular, $\dim_x m_{C,x}/m_{C,x}^2 \leq 2$ for any $x \in C$ and the image of $m_{Y,f(x)}/m_{Y,f(x)}^2 \to m_{X,x}/m_{X,x}^2$ has dimension at least $n − 2$ (where $n = \dim X$). It follows that if $B \subseteq Y$ is a general complete intersection curve passing through a fixed point $y \in Y$, then the surface $S = f^{-1}(B)$ is generically reduced and $S_0$ (since $X$ is Cohen-Macaulay), thus is reduced. Moreover, $f$ is flat by [Har77, III, Ex 10.9] and since $(D \cdot C) = (−K_X \cdot C) = 2$ (where $D$ is a Cartier divisor on $X$ such that $−K_X \equiv D$), $f^{-1}(y)$ has at most 2 irreducible components (counting multiplicities), hence $S$ is smooth at every generic point of $f^{-1}(y)$, for otherwise $f^{-1}(y)$ contains a component of multiplicity $\geq 2^2 = 4$. We need to show that $f^{-1}(y)$ is a plane conic.

If $C$ is reduced, then it is a smooth rational curve, hence $S$ is smooth in codimension one and thus normal. By adjunction $S \to B$ is also a Gorenstein conic bundle, so by [LZ16, Lemma 15] (whose proof works in any characteristic), $f^{-1}(y) = f|_{S_0}^{-1}(y)$ is a plane conic. If $C$ is a double line (which only happens in characteristic 2), then we have $f^{-1}(y) = 2C_1$ as a 1-cycle. Let $\tilde{S} \to S$ be the normalization of $S$, $\Delta \subseteq \tilde{S}$ the conductor and $g : \tilde{S} \to B$ the Stein factorization of $\tilde{S} \to B$. Let $\tilde{C}_1$ be the strict transform of $C_1$. Then the general fiber of $g$ is a smooth rational curve, therefore $\tilde{B} \to B$ is purely inseparable of degree 2 and indeed every fiber of $g$ is irreducible and reduced. By [Kol96, II.2.8], $g$ is a $\mathbb{P}^1$-bundle. It follows that $2 = (−K_{\tilde{S}} \cdot \tilde{C}_1) = (−K_S \cdot C_1) + (\Delta \cdot C_1)$, but since $(−K_S \cdot C_1) = (D \cdot C_1) = 1$, we get $(\Delta \cdot C_1)_1 = 1$. Hence the conductor intersects $\tilde{C}_1$ transversally at a single point and $\tilde{C}_1 \to C_1$ is an isomorphism. In particular, $C_1 \equiv \mathbb{P}^1$. Note that $\chi(\mathcal{O}_{f^{-1}(y)}(-D)) = \chi(\mathcal{O}_{C}(-D)) = −1$, by the exact sequence $0 \to I_{C_1}(-D) \to \mathcal{O}_{f^{-1}(y)}(-D) \to \mathcal{O}_{C_1}(-1) \to 0$.
and the similar proof of (2) ⇒ (3) in Lemma 25 we see that $f^{-1}(y)$ is a planar double line.

We therefore conclude that in all cases $f^{-1}(y)$ is a plane conic. As $f^{-1}(y)$ is cut out by hypersurfaces, $X$ has only hypersurface singularities and in particular is Gorenstein. The lemma now follows from standard argument (i.e. $\mathcal{E} = f_*\omega_X^{-1}$ is a vector bundle of rank 3 on $Y$ and $X$ embeds into $\mathbb{P}(\mathcal{E})$, see e.g. [Sar82]).

The following corollary is well-known in characteristic zero by the work of [And85].

**Corollary 27.** Let $f : X \to Y$ be a proper morphism. Assume that every fiber of $f$ has dimension 1, $-K_X$ is $f$-ample and $f_*\mathcal{O}_X = \mathcal{O}_Y$. Then $f$ is a rational conic bundle. If in addition $X$ and $Y$ are both smooth, then $f$ is a conic bundle.

**Proof.** This is an immediate consequence of the above lemma.

The next lemma is essentially [LZ16, Lemma 17], with F-regularity in place of klt singularity.

**Lemma 28.** Let $f : X \to Y$ be a Gorenstein conic bundle and $\phi : \tilde{Y} \to Y$ a finite separable morphism. Let $\tilde{X}$ be the normalization of $X \times_Y \tilde{Y}$. Assume that $X$ is strongly F-regular and the branch divisor of $\phi$ is disjoint from the singular locus of $\tilde{Y}$ and $Y$. Then $\tilde{f} : \tilde{X} \to \tilde{Y}$ is also a Gorenstein conic bundle.

**Proof.** By shrinking $Y$ we may assume either $\phi$ is étale in codimension one or both $Y$ and $\tilde{Y}$ are smooth. In the first case $\tilde{X}$ is also strongly F-regular by [Wat91, Theorem 2.7] hence is Cohen-Macaulay by [SZ15, Theorem 1.18], and the other properties of Gorenstein conic bundles are preserved by a finite base change that is étale in codimension one. In the second case $f$ is a conic bundle by Lemma 26 hence the same holds for $\tilde{f}$. □

## 5. PROOF OF MAIN RESULTS

Before proving the main theorems, we make a few reductions and fix the following notations. After a base change, we first assume that the base field $k$ is uncountable. Since the Seshadri constant of a line bundle $L$ attains its maximum at a very general point of $X$, we may also assume that $x \not\in \text{Supp} (\Delta)$. Let $\sigma : Y \to X$ be the blow up of $X$ at $x$ and let $E$ be the exceptional divisor. Let $\Delta$ also denote its strict transform on $Y$.

**Proof of Theorem 3.** As $\epsilon(L, x) > n$, $-(K_Y + \Delta + E) = \sigma^*L - nE$ is ample. Clearly $\rho(Y) \geq 2$ and $\Delta \cap E = \emptyset$, thus by Lemma \[23\] $Y$ is isomorphic to a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-d))$ over $\mathbb{P}^{n-1}$ for some $d \in \mathbb{Z}_{>0}$ and $E$ is a section. But since $N_E/Y \cong \mathcal{O}_E(-1)$, we have $d = 1$ and $E$ is the unique negative section. It follows that $Y$ is the blowup of $\mathbb{P}^n$ at a point and therefore $X \cong \mathbb{P}^n$. □

**Proof of Theorem 5 when $(L^n) > n^n$.** By assumption, $D = -(K_Y + \Delta + E) = \sigma^*L - nE$ is nef and big and $(Y, \Delta)$ is globally F-regular by Corollary 9. We claim that $D$ is semiample. Note that $mD - E - (K_Y + \Delta) = (m+1)D$ is nef and big, so by Lemma \[11\] $H^1(Y, \mathcal{O}_Y(mD - E)) = 0$ for all $m \geq 0$ and

$$H^0(Y, \mathcal{O}_Y(mD)) \to H^0(E, \mathcal{O}_E(mD))$$

is surjective, hence $E$ is disjoint from the stable base locus $\text{Bs}(D)$ of $D$. Now if $y \in \text{Bs}(D)$, then by Lemma 17 there exists a curve $C$ containing $y$ such that $(D \cdot C) = 0$, but then
varieties (the characteristic zero assumption in [Kaw85] is only used to make the varieties 
D numerical dimension of m one of the following holds:

Z image in R (6)

\[ Z \cong \mathbb{P}(1^n, d) \] for some \( d \in \mathbb{Z}_{>0} \) and \( G \) is the hyperplane defined by the vanishing of the last coordinate;

(2) \( Z \) is isomorphic to a \( \mathbb{P}^1 \)-bundle \( \mathbb{P}((\mathcal{O} \oplus \mathcal{O}(-d))) \) over \( \mathbb{P}^{n-1} \) for some \( d \in \mathbb{Z}_{\geq 0} \) and \( G \) is a section; or

(3) \( n = 2, Z \cong \mathbb{P}^2 \) and \( G \) is a smooth conic.

We now show that \( Y \) is the blowup of \( Z \) along a hypersurface in \( G \cong \mathbb{P}^{n-1} \). This essentially follows from the argument of [LZ16] Lemma 11, once we have the vanishing (6)

\[ R^1g_*\mathcal{O}_Y(-mW) = 0 \]

for all \( m \geq 0 \), where \( W = g^*G - E \). But since \( -mW - (K_Y + \Delta) \sim_{g, \mathbb{Q}} (m + 1)E \) is \( g \)-ample, (6) follows from Corollary [12] and the global F-regularity of \( (Y, \Delta) \). We also notice that \( G \) is nef by [LZ16] Lemma 10. We can therefore apply the same argument of [LZ16] Lemma 13 to conclude the proof. □

**Proof of Theorem 3 and 4.** The proof of both theorems is intertwined and a bit lengthy, so we divide it into several steps. Let \( D = \sigma^*L - nE = -(K_Y + \Delta + E) \). Since the case \( (D^n) = (L^n) - n^n > 0 \) of Theorem 3 is already treated, we may assume \( (D^n) = 0 \).

**Step 1 (D is semiample).** By assumption \( -(K_Y + \Delta) \) is ample, \( D \) is nef and \( (D^n) = 0 \), hence by Riemann-Roch we have

\[
h^0(Y, mD) \geq h^0(Y, mL) - h^0(E, \mathcal{O}_{mnE}) = \frac{(D^n)n^m}{n!} + \frac{(-K_Y \cdot D^{n-1})}{2(n-1)!}m^{n-1} + O(m^{n-2}) \]

It follows that \( \nu(Y, D) = \kappa(Y, D) = n - 1 \) where \( \nu(Y, D) = \max\{d|D^d \neq 0\} \) is the numerical dimension of \( D \). By [Kaw85] Proposition 2.1, there exists a diagram of normal varieties (the characteristic zero assumption in [Kaw85] is only used to make the varieties in the diagram smooth)

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{\mu} & Y \\
\downarrow f & & \downarrow \\
Z_0 & & \\
\end{array}
\]

and a nef and big divisor \( D_0 \) on \( Z_0 \) such that \( \mu \) is birational, \( f \) is equi-dimensional and \( \mu^*D = f^*D_0 \). It follows that for every closed point \( y \in Y \) there exists a curve \( C_y \subseteq Y \)
(coming from a fiber of \( f \) that intersects \( \mu^{-1}(y) \)) such that \((D \cdot C_y) = 0 \) and \( C_y \) is unique if \( y \) is general. Since \( \kappa(Y, D) = n - 1 \), for sufficiently divisible \( m \) the linear system \(|mD|\) gives a rational map \( g : Y \dasharrow Z \) with \( \dim Z = n - 1 \). As \((D \cdot C_y) = 0 \), \( g \) is defined along \( C_y \) if \( y \notin \text{Bs}(D) \), hence \( C_y \) is the (at least set-theoretic) general fiber of \( g \) and we get a proper morphism \( g_1 : Y_1 \to Z_1 \) with \( g_1_*\mathcal{O}_{Y_1} = \mathcal{O}_{Z_1} \) by shrinking \( Z \) and taking Stein factorization. Let \( Z_1 \dasharrow \text{Chow}(Y) \) be the rational map induced by \( g_1 \) (see [Kol96, I.3-4] for the definition and basic properties of Chow varieties) and let \( Z' \) be the normalization of the closure of the image of \( Z_1 \) in \( \text{Chow}(Y) \). By Corollary 27 the general fiber \( L_z \) of \( g_1 \) is a plane conic.

Assume for the moment that \( L_z \) is a smooth conic (this is automatically satisfied when \( p = \text{char}(k) \neq 2 \) or \((\Delta \cdot C_y) > 0 \): in the latter case, \((D \cdot C_y) < -(K_Y - E \cdot C_y) \leq 0 \) if \( L_z \) is nonreduced). In particular, \( g_1 \) has reduced general fiber and we get a universal family \( q : U \to Z' \). Let \( u : U \to Y \) be the cycle map. We claim that \( u \) is an isomorphism.

To this end let \( C \subseteq U \) be a curve that is contracted by \( u \). Since \( u \) is injective on every fiber of \( q \), \( q(C) \) is not a point. Let \( S \) be the irreducible component of \( q^{-1}(q(C)) \) that contains \( C \). By construction \((u^*D \cdot C) = (u^*D \cdot F) = 0 \) where \( F \) is any component in a fiber of \( q \), thus by [BCE02, Proposition 2.5], \( u^*D|_S \) is numerically trivial. Let \( T = u(S) \), then \( T \) is a surface in \( Y \) such that \( D|_T = 0 \). As \( D = -(K_Y - \Delta) \) and \( -(K_Y + \Delta) \) is ample, \( T \) must intersect \( E \) and \( \dim(T \cap E) \geq 1 \), but then since \( D|_E \sim -nE|_E \) is ample, \( D|_{T \cap E} \) cannot be numerically trivial. Hence \( u \) is quasi-finite and is indeed an isomorphism since it is both birational and \( Y \) is normal.

Thus we get a rational conic bundle \( Y \cong U \to Z' \) with general fiber \( C_y \). As \((D \cdot C_y) = 0 \), any \( G \in |mD| \) can not dominate \( Z' \), thus as every fiber of \( q \) has pure dimension 1, \( G \) is in fact the pullback of an effective divisor on \( Z' \). On the other hand by [KM98, Lemma 5.16] applied to the finite morphism \( E \cong \mathbb{P}^{n-1} \to Z' \), we see that \( Z' \) is \( \mathbb{Q} \)-factorial of Picard number one. Hence \( D \) is semiample if \( L_z \) is reduced.

**Step 2** (Proof of Theorem 5 when \( \Delta \neq 0 \)). We claim that \( \Delta \) is \( \mathbb{Q} \)-Cartier. Using the notation and construction in Step 1, there are three cases to consider.

Suppose first that \((\Delta \cdot C_y) > 0 \). Then \(|mD| (m \gg 0)\) defines a morphism \( g : Y \to Z \) by Step 1. Moreover \((E \cdot C_y) = -(K_Y + \Delta) \cdot C_y < 2 \), thus \((E \cdot C_y) = 1 \) and \( E \to Z \) is an isomorphism. Since \( E \) intersects every component in the fiber of \( g \) (for otherwise this component would have zero intersection number with the ample divisor \(-(K_Y + \Delta)\)), we see that every fiber of \( g \) is generically irreducible and reduced. By [LZ16, Lemma 6], there exists a codimension \( \geq 2 \) subset \( W \subseteq Z \) such that \( Y \setminus g^{-1}(W) \) is isomorphic to a \( \mathbb{P}^1 \)-bundle over \( Z \setminus W \). It follows that the class group of \( Y \) is generated by \( E \) and \( g^*\text{Pic}(Z) \) and in particular \( Y \) is \( \mathbb{Q} \)-factorial. Thus \( \Delta \) is \( \mathbb{Q} \)-Cartier in this case.

Assume next that \((\Delta \cdot C_y) = 0 \) and \( L_z \) is a smooth conic. Again we have a rational conic bundle \( g : Y \to Z \) defined by \(|mD| (m \gg 0)\). Since \((\Delta \cdot C_y) = 0 \), \( \Delta \) is a divisor in \( Z \). As \((E \cdot C_y) = -(K_Y + \Delta) \cdot C_y = 2 \), every fiber of \( g \) has at most 2 components (counting multiplicity), thus by the same proof of [LZ16, Lemma 16], \( g^{-1}(u) \) is a plane conic where \( u \) is a generic point of \( \Delta \). We claim that \( \Delta \) is proportional to \( g^*\Delta \) over \( u \). Suppose not, then \( g^{-1}(u) \) is not irreducible and there exists a component \( F \) of \( g^{-1}(u) \) such that \((\Delta \cdot F) > 0 \). But we also have \(-(K_Y \cdot F) = 1 \leq (E \cdot F) \), hence \((D \cdot F) = -(K_Y + \Delta + E) \cdot F < 0 \), a contradiction. Therefore, we can find a \( \mathbb{Q} \)-divisor
\(\Delta_1\) supported on \(\Delta_Z\) such that \(\Delta = g^*\Delta_1\). Recall that \(Z\) is \(\mathbb{Q}\)-factorial, thus \(\Delta\) is also \(\mathbb{Q}\)-Cartier in this case.

Finally suppose that \((\Delta \cdot C_y) = 0\) and \(L\) is a nonreduced conic. In particular \(p = 2\). Taking the base change of \(g_1 : Y_1 \to Z_1\) by \(E_1 \to Z_1\) (where \(E_1\) is the preimage of \(Z_1\) in \(E\)), we get a family \(h_1 : U_1 \to E_1\) of reduced curves in \(Y\) with general member \(C_y\). As in Step 1 we may extend \(h_1\) to a universal family \(h : U \to V\) (where \(V\) is the closure of the image of \(E_1\) in Chow(\(Y\))) and the same argument there implies that the cycle map \(u : U \to Y\) is quasi-finite, thus is an inseparable double cover. It follows that the Frobenius map of \(Y\) factors through \(u\), hence \(u^{-1}(E)\) is \(\mathbb{Q}\)-factorial and \(\Delta\) is \(\mathbb{Q}\)-Cartier if and only if \(u^*\Delta\) is \(\mathbb{Q}\)-Cartier. But as \((E \cdot C_y) = (K_Y \cdot C_y) = 1\), every fiber of \(h\) is generically integral, thus \(u^*\Delta\) is the pullback of a divisor from \(V\). Since \(V\) is dominated by \(u^{-1}(E)\), it is \(\mathbb{Q}\)-factorial by [KM98] Lemma 5.16. Hence \(\Delta\) is \(\mathbb{Q}\)-Cartier in this last case.

Now that \(\Delta \neq 0\) is \(\mathbb{Q}\)-Cartier, we may replace \((X, \Delta)\) by \((X, (1-c)\Delta)\) for \(0 < c \ll 1\) and reduce to the case \((L^n) > n^3\) using [FKL16] Theorem B. This finishes the proof of Theorem 5.

In the remaining part of the proof, we assume that \(\Delta = 0\) and \(p > 2\). By Step 1, this implies that \(D\) is semiample and induces a morphism \(g : Y \to Z\). We have \(-K_Y \sim g_*\mathcal{O}_E\), thus \(g\) is a Gorenstein conic bundle if \(Y\) is Cohen-Macaulay.

**Step 3 (Surface case).** If \(Y\) is a surface, then by [LZ16] Lemma 15], \(Y\) has only Du Val singularity. It follows that \(X\) is a Gorenstein log del Pezzo surface of degree \((K_X^2) = 4\). Hence from now on, we assume that \(n = \dim X \geq 3\).

**Step 4 (\(Y\) is globally F-regular).** It is clear that \(E\) is a double section of \(g\). We may assume that \(E \to Z\) is ramified (the quasi-étale case is similar and even simpler), then \(Z \cong \mathbb{P}(1^n-1,2), \ g|_{E}\) is ramified along the hyperplane \(M \subseteq Z\) defined by the vanishing of the last coordinate and \(K_E = g^*(K_Z + \frac{1}{2}M)\). Since the general fiber of \(g\) is a smooth rational curve, we can choose an ample Cartier divisor \(H\) on \(Z\) such that \(Y\setminus g^{-1}H\) is smooth and globally F-regular. We then have a similar diagram as in the proof of Lemma 8

\[
F^e_*\mathcal{O}_X((1-p^e)(K_X + E) - g^*H) \xrightarrow{\text{Tr}^e} F^e_*\mathcal{O}_E((1-p^e)K_E - g|_E^*H)
\]

Although \(H^0(j)\) is not surjective, its image contains \(g^*H^0(Z, (1-p^e)(K_Z + \frac{1}{2}M) - H)\) for \(e \gg 0\), hence by the same argument in Lemma 8 it suffices to show that \(H^0(\text{Tr}^e_Z)\) is surjective for \(e \gg 0\) where \(\text{Tr}^e_Z : F^e_*\mathcal{O}_Z((1-p^e)(K_Z + \frac{1}{2}M) - H) \to \mathcal{O}_Z\) is the trace map. But it is clear that the toric pair \((Z, \frac{1}{2}M)\) is globally F-regular, so we are done.

**Step 5 (Analysis of the Gorenstein conic bundles).** Since \(Y\) is globally F-regular, it is Cohen-Macaulay by [SZ15] Theorem 1.18], thus \(g\) is a Gorenstein conic bundle. Let \(W\) be the normalization of \(Y \times_Z E\), then since \(E \cong \mathbb{P}^{n-1} \to Z\) is quasi-étale unless \(Z \cong \mathbb{P}(1^n-1,2)\) in which case the branch divisor is disjoint from \(\text{Sing}(Z)\), \(W \to E \cong \mathbb{P}^{n-1}\) is a also Gorenstein conic bundle by Lemma 28 and is indeed a conic bundle by Lemma 26 as \(E\) is smooth. The theorem now follows from the same calculation as in the proof of [LZ16] Lemma 19].
References

[And85] Tetsuya Ando. On extremal rays of the higher-dimensional varieties. Invent. Math., 81(2):347–357, 1985.

[BCE02] Thomas Bauer, Frédéric Campana, Thomas Eckl, Stefan Kebekus, Thomas Peternell, Sławomir Rams, Tomasz Szemberg, and Lorenz Wotzlaw. A reduction map for nef line bundles. In Complex geometry (Göttingen, 2000), pages 27–36. Springer, Berlin, 2002.

[BS09] Thomas Bauer and Tomasz Szemberg. Seshadri constants and the generation of jets. J. Pure Appl. Algebra, 213(11):2134–2140, 2009.

[Che97] I. A. Cheltsov. Del Pezzo surfaces with nonrational singularities. Mat. Zametki, 62(3):451–467, 1997.

[CTX15] Paolo Cascini, Hiromu Tanaka, and Chenyang Xu. On base point freeness in positive characteristic. Ann. Sci. Éc. Norm. Supér. (4), 48(5):1239–1272, 2015.

[Dem92] Jean-Pierre Demailly. Singular Hermitian metrics on positive line bundles. In Complex algebraic varieties (Bayreuth, 1990), volume 1507 of Lecture Notes in Math., pages 87–104. Springer, Berlin, 1992.

[FKL16] Mihai Fulger, János Kollár, and Brian Lehmann. Volume and Hilbert functions of $\mathbb{R}$-divisors. Michigan Math. J., 65(2):371–387, 2016.

[Fuj83] Takao Fujita. Vanishing theorems for semipositive line bundles. In Algebraic geometry (Tokyo/Kyoto, 1982), volume 1016 of Lecture Notes in Math., pages 519–528. Springer, Berlin, 1983.

[Gro05] Alexander Grothendieck. Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2), volume 4 of Documents Mathématiques (Paris) [Mathematical Documents (Paris)]. Société Mathématique de France, Paris, 2005. Séminaire de Géométrie Algébrique du Bois Marie, 1962, Augmenté d’un exposé de Michèle Raynaud. [With an exposé by Michèle Raynaud], With a preface and edited by Yves Laszlo, Revised reprint of the 1968 French original.

[Har77] Robin Hartshorne. Algebraic geometry. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Kaw85] Y. Kawamata. Pluricanonical systems on minimal algebraic varieties. Invent. Math., 79(3):567–588, 1985.

[KM98] János Kollár and Shigefumi Mori. Birational geometry of algebraic varieties, volume 134 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

[Kol96] János Kollár. Rational curves on algebraic varieties, volume 32 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 1996.

[Lip69] Joseph Lipman. Rational singularities, with applications to algebraic surfaces and unique factorization. Inst. Hautes Études Sci. Publ. Math., (36):195–279, 1969.

[Liu02] Qing Liu. Algebraic geometry and arithmetic curves, volume 6 of Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[Lop91] Angelo Felice Lopez. Noether-Lefschetz theory and the Picard group of projective surfaces. Mem. Amer. Math. Soc., 89(438):x+100, 1991.

[LZ16] Yuchen Liu and Ziquan Zhuang. Characterization of projective spaces by Seshadri constants. 2016. Preprint available at arXiv:1607.05743.

[Mor79] Shigefumi Mori. Projective manifolds with ample tangent bundles. Ann. of Math. (2), 110(3):593–606, 1979.

[MR85] V. B. Mehta and A. Ramanathan. Frobenius splitting and cohomology vanishing for Schubert varieties. Ann. of Math. (2), 122(1):27–40, 1985.

[Mur17] Takumi Murayama. Frobenius-Seshadri constants and characterizations of projective space. 2017. Preprint available at arXiv:1701.00511.
[Sar82] V. G. Sarkisov. On conic bundle structures. Izv. Akad. Nauk SSSR Ser. Mat., 46(2):371–408, 432, 1982.

[SS10] Karl Schwede and Karen E. Smith. Globally $F$-regular and log Fano varieties. Adv. Math., 224(3):863–894, 2010.

[SZ15] Karen E. Smith and Wenliang Zhang. Frobenius splitting in commutative algebra. In Commutative algebra and noncommutative algebraic geometry. Vol. I, volume 67 of Math. Sci. Res. Inst. Publ., pages 291–345. Cambridge Univ. Press, New York, 2015.

[Wat91] Keiichi Watanabe. $F$-regular and $F$-pure normal graded rings. J. Pure Appl. Algebra, 71(2-3):341–350, 1991.

[Zhu17] Ziquan Zhuang. Fano varieties with large Seshadri constants. 2017. Preprint available at arXiv:1707.02017

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