MINIMAL GROUP DETERMINANTS AND THE LIND-LEHMER
PROBLEM FOR DIHEDRAL GROUPS

TON BOERKOEL AND CHRISTOPHER PINNER

Abstract. We find the minimal non-trivial integer variable group determinant for any dihedral group of order less than $3.79 \times 10^{47}$. We think of this as the Lind-Lehmer problem for the dihedral group. We give a complete description of the determinants for some dihedral groups including $D_{2p}$ and $D_{4p}$.

1. Introduction

For a finite group $G = \{g_1, \ldots, g_n\}$ we assign a variable $x_g$ for each $g$ in $G$ and define the group determinant to be the $n \times n$ determinant
\[ D_G(x_{g_1}, \ldots, x_{g_n}) = \det \left( x_{g_i g_j^{-1}} \right). \]
Plainly this is a homogeneous polynomial of degree $n$ in the $n$ variables $x_g$. We are interested here in the smallest non-trivial value this can take when the $x_g$ are all integers:
\[ \lambda(G) = \min \{ |D_G(x_{g_1}, \ldots, x_{g_n})| : (x_{g_1}, \ldots, x_{g_n}) \in \mathbb{Z}^n \}. \]
In the cyclic group case $D_G(x_{g_1}, \ldots, x_{g_n})$ reduces to a circulant determinant. An old problem of Olga Taussky-Todd is to determine what values these can take for integer variables, see for example [17, 18, 10, 9]. For simplicity we shall use $\mathbb{Z}_n$, rather than $\mathbb{Z}/n\mathbb{Z}$, to denote the integers mod $n$, the basic cyclic group of order $n$. Studying the related Lind-Lehmer problem for cyclic groups, the value of $\lambda(\mathbb{Z}_n)$ was obtained by Kaiblinger [8] for $n$ not a multiple of $420$, with [19] extending this to $n$ not a multiple of $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. Here we do the same for $D_{2n}$, the dihedral group of order $2n$, when $n$ is not a multiple of $2^2 \cdot 3^2 \cdot \prod_{5 \leq p \leq 113} p > 1.89 \times 10^{47}$.

Dedekind showed that for abelian $G$ the group determinant can be factored over $\mathbb{C}[x_{g_1}, \ldots, x_{g_n}]$ into linear factors using the group of characters $\hat{G}$ on $G$
\[ \mathcal{D}_G(x_{g_1}, \ldots, x_{g_n}) = \prod_{\chi \in \hat{G}} (\chi(g_1)x_{g_1} + \cdots + \chi(g_n)x_{g_n}), \]
see for example Lang [11 §3.6]. Dedekind observed that for non-abelian groups one could have non-linear factors, leading Frobenius to develop representation theory.

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to describe the factorisation. In particular if \( \hat{G} \) denotes a complete set of non-isomorphic irreducible representations of \( G \) we have, see for example [7],

\[
D_G(x_{g_1}, \ldots, x_{g_n}) = \prod_{\rho \in \hat{G}} \det \left( \sum_{g \in G} x_g \rho(g) \right)^{\deg \rho}.
\]

See Conrad [3] for an account of the historical development of (2) and (3). Notice that the group determinant preserves multiplication in \( \mathbb{Z}[G] \), in the sense that if

\[
\left( \sum_{g \in G} a_g \right) \left( \sum_{g \in G} b_g \right) = \left( \sum_{g \in G} c_g \right), \quad c_g = \sum_{uv = g} a_u b_v,
\]

then

\[
D_G(a_{g_1}, \ldots, a_{g_n})D_G(b_{g_1}, \ldots, b_{g_n}) = D_G(c_{g_1}, \ldots, c_{g_n}),
\]

since plainly \( c_{g_{i-1}} = \sum_{u \in G} a_u b_{g_{i-1}} \). Finally, we note the trivial bound

\[
\lambda(G) \leq \max \{2, |G| - 1\}
\]

since, taking \( g_1 \) to be the identity element,

\[
D_G(0,1,\ldots,1) = (-1)^{n-1}(n-1),
\]

an easy evaluation viewed as a Lind Mahler measure \( M_{\mathbb{Z}[G]}(-1 + (x^n - 1)/(x - 1)) \).

2. Lind Mahler Measure

For a polynomial \( f(x) = \sum_{j=0}^N a_j x^j \) in \( \mathbb{Z}[x] \) one defines the traditional logarithmic Mahler measure \( m(f) \) by

\[
m(f) = \int_0^1 \log |f(e^{2\pi i \theta})| d\theta,
\]

with the classical Lehmer problem [12] asking whether there is a constant \( c > 0 \) such that \( m(f) = 0 \) or \( m(f) \geq c \). Lind [14] viewed \( f(e^{2\pi i \theta}) = \sum_{j=0}^N a_j e^{2\pi i j \theta} \) as a linear sum of characters on the group \( [0,1) = \mathbb{R}/\mathbb{Z} \), and extended the concept of Mahler measure to an arbitrary compact abelian group \( G \) with Haar measure \( \mu \) and group of characters \( \hat{G} \), defining

\[
m_G(f) = \int_G \log |f| d\mu
\]

where \( f \) is an element in \( \mathbb{Z}[\hat{G}] \). For example, for a finite abelian group

\[
G = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}
\]

the characters take the form

\[
\hat{G} = \{ \chi_u(x_1, \ldots, x_r) = e^{2\pi i x_1 u_1/n_1} \cdots e^{2\pi i x_r u_r/n_r} : u = (u_1, \ldots, u_r) \in G \},
\]
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the integral becoming an average over the group \( G \)

\[
m_G \left( \sum_{u \in G} a_u \chi_u \right) = \frac{1}{|G|} \sum_{x = (x_1, \ldots, x_r) \in G} \log \left| \sum_{u \in G} a_u \chi_u(x_1, \ldots, x_r) \right|
\]

\[
= \frac{1}{|G|} \log \prod_{x \in G} \left| \sum_{u \in G} a_u e^{2\pi i u_1 x_1/n_1} \cdots e^{2\pi i u_r x_r/n_r} \right|
\]

\[
= \frac{1}{|G|} \log \prod_{x \in G} \left| a_u \chi_x(u) \right|
\]

\[
= \frac{1}{|G|} \mathcal{D}_G(a_{g_1}, \ldots, a_{g_n}).
\]

Thus, as observed by Vipismakul \[22\] in his thesis, the Lind-Mahler measure for a finite abelian group essentially corresponds to evaluating the group determinant. In particular the corresponding Lehmer problem for the group, that is determining the minimal Lind-Mahler measure \( m(f) > 0 \), is the same as finding \( \frac{1}{|G|} \log \lambda(G) \).

As mentioned above cyclic groups were considered in \[14, 8, 19\], with \( G = \mathbb{Z}_p \) and various other groups investigated in \[5, 20, 6, 2\]. In those papers the Lind-Mahler measure was mostly viewed as the measure of a polynomial, representing the average of the polynomial over appropriate roots of unity; that is for a group \( G \) of the form \([3]\) and polynomial \( F \) in \( \mathbb{Z}[x_1, \ldots, x_r] \), we define

\[
m_G(F) = \frac{1}{|G|} \log |M_G(F)|,
\]

where

\[
M_G(F) := \prod_{u_1 = 0}^{n_1 - 1} \cdots \prod_{u_r = 0}^{n_r - 1} F(e^{2\pi i u_1/n_1}, \ldots, e^{2\pi i u_r/n_r}) \in \mathbb{Z}.
\]

Of course this measure is really defined on \( \mathbb{Z}[x_1, \ldots, x_r]/\langle x_1^{n_1} - 1, \ldots, x_r^{n_r} - 1 \rangle \), and writing

\[
F(x_1, \ldots, x_r) = \sum_{u \in G} a_u x_1^{u_1} \cdots x_r^{u_r} \mod \langle x_1^{n_1} - 1, \ldots, x_r^{n_r} - 1 \rangle
\]

we have

\[
M_G(F) = \mathcal{D}_G(a_{g_1}, \ldots, a_{g_n}).
\]

Notice that for a polynomial \( f \) in \( \mathbb{Z}[x] \) not vanishing on the unit circle the traditional Mahler measure is a limit of Lind measures

\[
m(f) = \lim_{n \to \infty} m_{\mathbb{Z}_n}(f).
\]

It is not immediately clear how to extend Lind’s original definition to a non-abelian finite group. See Dasbach and Lalín \[4\] for one approach. Here we suggest that the group determinant formulation provides a natural way to do this.

For a finite group \( G \) with generators \( \alpha_1, \ldots, \alpha_r \) and relations \( \mathcal{U}(\alpha_1, \ldots, \alpha_r) \), and a polynomial \( F \) in \( \mathbb{Z}[x_1, \ldots, x_r]/\mathcal{U}(x_1, \ldots, x_r) \)

\[
F(x_1, \ldots, x_r) = \sum_{g = \alpha_1^{m_1} \cdots \alpha_r^{m_r} \in G} a_g x_1^{m_1} \cdots x_r^{m_r} \mod \mathcal{U}(x_1, \ldots, x_r),
\]
we define

$$m_G(F) = \frac{1}{|G|} \log |M_G(F)|, \quad M_G(F) := \mathcal{D}_G(a_{g_1}, \ldots, a_{g_n}).$$

If multiplication of monomials follows the group relations then, as in (4), we recover the usual multiplicative property of Mahler measures

$$M_G(fh) = M_G(f)M_G(h).$$

For example, for the dihedral group

$$D_{2n} = \langle R, F \mid R^n = 1, F^2 = 1, RF = FR^{n-1} \rangle = \{1, R, \ldots, R^{n-1}, F, FR, \ldots, FR^{n-1} \},$$

we define the measure of a polynomial in \( \mathbb{Z}[x, y]/\langle x^n - 1, y^2 - 1, xy - yx^{n-1} \rangle \), reduced to the form

$$F(x, y) = \sum_{i=0}^{n-1} a_i x^i + \sum_{i=0}^{n-1} b_i yx^i,$$

to be

$$M_{D_{2n}}(F) = \mathcal{D}_{D_{2n}}(a_1, \ldots, a_n, b_1, \ldots, b_n).$$

3. THE DIHEDRAL GROUP

For \( G = D_{2n} \) the representations are fairly straightforward, see for example Serre [21, §5.3]. We have the degree 1 representations, the basic characters \( \chi \) with \( \chi(F) = \pm 1 \) and \( \chi(R) = 1 \) and when \( n \) is even \( \chi(R) = -1 \), giving us two or four linear factors as \( n \) is odd or even. The other representations all have degree 2 and arise from

$$\rho(R^j) = \begin{pmatrix} w^j & 0 \\ 0 & w^{-j} \end{pmatrix}, \quad \rho(FR^j) = \begin{pmatrix} 0 & w^{-j} \\ w^j & 0 \end{pmatrix},$$

where the \( w \) run through the complex \( n \)th roots of unity.

Writing \( a_i \) for the \( xR^i \) and \( b_i \) for the \( xFR^i \), these lead to

$$\sum_g x_g \rho(g) = \begin{pmatrix} \sum_{i=0}^{n-1} a_i w^i & \sum_{i=0}^{n-1} b_i w^{-i} \\ \sum_{i=0}^{n-1} b_i w^i & \sum_{i=0}^{n-1} a_i w^{-i} \end{pmatrix},$$

and the real quadratic factors, with coefficients in \( \mathbb{Q}(\cos(2\pi/n)) \),

$$Q(w) := \begin{pmatrix} \sum_{i=0}^{n-1} a_i w^i \\ \sum_{i=0}^{n-1} a_i w^{-i} \end{pmatrix} - \begin{pmatrix} \sum_{i=0}^{n-1} b_i w^i \\ \sum_{i=0}^{n-1} b_i w^{-i} \end{pmatrix}.$$

Thus with \( w_n \) denoting the primitive \( n \)th root of unity

$$w_n := e^{2\pi i/n},$$

the group determinant factors in \( \mathbb{C}[a_1, \ldots, a_n, b_1, \ldots, b_n] \) as follows:

When \( n = 2k + 1 \) is odd
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\( \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \) factors as two linear and the square of \( k \) quadratics

\[
\left( \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} b_i \right) \left( \sum_{i=0}^{n-1} a_i - \sum_{i=0}^{n-1} b_i \right) \prod_{j=1}^k Q(w_n^j)^2.
\]

When \( n = 2k \) is even

\( \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) \) factors as the product of four linear factors

\[
\left( \sum_{i=0}^{n-1} a_i + \sum_{i=0}^{n-1} b_i \right) \left( \sum_{i=0}^{n-1} a_i - \sum_{i=0}^{n-1} b_i \right) \left( \sum_{i=0}^{n-1} (-1)^i a_i + \sum_{i=0}^{n-1} (-1)^i b_i \right) \left( \sum_{i=0}^{n-1} (-1)^i a_i - \sum_{i=0}^{n-1} (-1)^i b_i \right)
\]

and the square of \( k - 1 \) quadratics

\[
\prod_{j=1}^{k-1} Q(w_n^j)^2.
\]

Note that in both cases we can write

\[
\mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}) = \prod_{j=0}^{n-1} Q(w_n^j).
\]

Writing these expressions in terms of values at roots of unity thus motivates us to define the Lind-Mahler measure of a polynomial,

\[
F(x, y) = \sum a_{ij} y^i x^j \in \mathbb{Z}[x, y],
\]

or more generally in \( \mathbb{Z}[x, x^{-1}, y, y^{-1}] \), relative to the dihedral group \( G = D_{2n} \) to be

\[
m_G(F) = \frac{1}{2n} \log |M_G(F)|,
\]

with

\[
M_G(F) := \prod_{j=0}^{n-1} \left( F(w_n^j, 1)F(w_n^{-j}, -1) + F(w_n^{-j}, 1)F(w_n^j, -1) \right),
\]

\[
= M_{\mathbb{Z}_n} \left( \frac{1}{2} \left( F(x, 1)F(x^{-1}, -1) + F(x, -1)F(x^{-1}, 1) \right) \right),
\]

where if we want the usual multiplicative property \( M_G(fg) = M_G(f)M_G(g) \) we must use the dihedral relationship on monomials, \( x^i y = yx^{n-i} \), when multiplying two polynomials.

Notice for an \( F(x, y) = f(x) + yg(x) \) with \( f(x), g(x) \) in \( \mathbb{Z}[x, x^{-1}] \) we have

\[
M_{D_{2n}}(F) = M_{\mathbb{Z}_n} \left( f(x) f(x^{-1}) - g(x)g(x^{-1}) \right).
\]

If \( f(x) = x^k f(x^{-1}), g(x) = x^k g(x^{-1}) \) for some \( k \) we have

\[
M_{D_{2n}}(F) = M_{\mathbb{Z}_n \times \mathbb{Z}_2}(F),
\]

a reciprocal property similar to Dasbach and Lalín’s [4, Theorem 12].

We also have

\[
M_{D_{2n}}(f(x)) = M_{D_{2n}}(yf(x)) = M_{\mathbb{Z}_n}(f(x))^2,
\]

and so \( \lambda(D_{2n}) \leq \lambda(\mathbb{Z}_n)^2 \).
4. Minimal Values of Dihedral Determinants and Measures

In this section we obtain some restrictions on the values taken by a dihedral group determinant with integer variables. These will be enough to determine the minimal non-trivial determinant for any dihedral group $G = D_{2n}$ of order less than $3.79 \times 10^{47}$.

**Theorem 4.1.**

$$\lambda(D_{2n}) = \begin{cases} 
3, & \text{if } 3 \nmid n, \\
4, & \text{if } n = 3m, \quad 2 \nmid m, \\
5, & \text{if } n = 2 \cdot 3m, \quad 5 \nmid m, \\
7, & \text{if } n = 2 \cdot 3 \cdot 5m, \quad 7 \nmid m, \\
11, & \text{if } n = 2 \cdot 3 \cdot 5 \cdot 7m, \quad 11 \nmid m, \\
13, & \text{if } n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11m, \quad 13 \nmid m, \\
16, & \text{if } n = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13m, \quad 2 \nmid m, \\
17, & \text{if } n = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13m, \quad 17 \nmid m, \\
19, & \text{if } n = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17m, \quad 19 \nmid m, \\
23, & \text{if } n = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19m, \quad 23 \nmid m, \\
27, & \text{if } n = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23m, \quad 3 \nmid m, \\
29, & \text{if } n = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23m, \quad 29 \nmid m, 
\end{cases}$$

and for primes $p = 31$ to 113

$$\lambda(D_{2n}) = p, \quad \text{if } n = 2^2 \cdot 3^2 \cdot \left( \prod_{5 \leq q < p} q \right) \cdot m, \quad p \nmid m.$$

This just leaves the groups $D_{2n}$ when $n$ is a multiple of

$$N = 2^2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot \cdots \cdot 107 \cdot 109 \cdot 113,$$

where $\lambda(D_{2N}) = 125$ or 127. Some multiples of $N$ can be determined. For example when $n$ a multiple of $5N$ we can rule out 125 and for $p = 127$ to 241

$$\lambda(D_{2n}) = p, \quad \text{if } n = 2^2 \cdot 3^2 \cdot 5^2 \cdot \left( \prod_{7 \leq q < p} q \right) \cdot m, \quad p \nmid m,$$

with $\lambda(D_{2n}) = 243$ or 251 when $n = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot \cdots \cdot 241$. For $n$ a multiple of $15N$

$$\lambda(D_{2n}) = \begin{cases} 
251, & \text{if } n = 2 \cdot 3^3 \cdot 5^2 \cdot \left( \prod_{7 \leq q \leq 241} q \right) \cdot m, \quad 251 \nmid m, \\
256, & \text{if } n = 2 \cdot 3^3 \cdot 5^2 \cdot \left( \prod_{7 \leq q \leq 251} q \right) \cdot m, \quad 2 \nmid m, \\
p, & \text{if } n = 2^2 \cdot 3^3 \cdot 5^2 \cdot \left( \prod_{7 \leq q < p} q \right) \cdot m, \quad p \nmid m, 
\end{cases}$$

for primes $p$ from 257 to 337, with $\lambda(D_{2n}) = 343$ or 347 when $n = 2^3 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot \cdots \cdot 337$.

**Theorem 4.1** will follow immediately from two lemmas. Laquer [10] and Newman [17] proved that $\mathcal{D}_2(a_0, \ldots, a_{n-1})$ achieves all integers coprime to $n$. See Mahoney and Newman [15] for other abelian groups. We similarly show for $G = D_{2n}$ that any positive integer coprime to $2n$ will be the absolute value of an integer determinant.

In particular we can achieve any odd prime $p \nmid n$: 
Lemma 4.1. Let $G = D_{2n}$.
If $m = 2t + 1$ with $(m, n) = 1$ and $0 \leq t < n$ then
\[ |\mathcal{D}_G(1, \ldots, 1, 0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)| = m. \]
For $t \geq n$ we can still achieve $m$ by wrapping around the remaining 1’s;
\[ a_i = \{|0 \leq j \leq t : j \equiv i \mod n\}, \quad b_i = \{|0 \leq j \leq t - 1 : j \equiv i \mod n\}. \]
If $2 \nmid n$ then $\mathcal{D}_G(1, 1, 0, \ldots, 0) = 2^2$.
If $2\mid n$ then $\mathcal{D}_G(1, 0, 1, 0, \ldots, 0) = 2^4$.
If $2^2\mid n$ then $\mathcal{D}_G(1, 0, 0, 0, 1, 0, 0, \ldots, 0) = 2^8$.
If $3\mid n$ then $|\mathcal{D}_G(1, 0, 0, 1, 0, 0, 0, \ldots, 0)| = 3^3$.

More generally if $p$ is odd and $p^\alpha \mid n$ then we can achieve $p^{\alpha^2}$ as the absolute value of an integer group determinant, if $2^\alpha \mid n$ then we can achieve $2^{2\alpha+1}$.

Without absolute values we show:

Theorem 4.2. Let $G = D_{2n}$ and $m$ be an integer coprime to $2n$.
If $n$ is odd then $m$ is a $\mathcal{D}_G(a_1, \ldots, a_n, b_0, \ldots, b_n)$ for some integers $a_1, \ldots, a_n, b_0, \ldots, b_n$.
If $n$ is even then either $m$ or $-m$ is a determinant, whichever is 1 mod 4.
If $n$ is even the odd determinants are all 1 mod 4.

Next, we obtain a condition on divisibility by primes dividing $2n$:

Lemma 4.2. Let $G = D_{2n}$ and $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ be in $\mathbb{Z}$.
If $p^{\alpha} \mid n$ and $p \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ then $p^{2\alpha+1} \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$.
If $2 \nmid n$ and $2 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ then $4 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$.
If $2 \mid n$ and $2 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ then $2^4 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ or
$2^6 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$.
If $2^\alpha \mid n$, $\alpha \geq 2$, and $2 \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$ then $2^{2\alpha+4} \mid \mathcal{D}(a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})$.

We observe that
\[ M_{D_{2p}} \left( \frac{x^{(p+1)/2} - 1}{x - 1} + (x - 1) + y \left( \frac{x^{(p-1)/2} - 1}{x - 1} + x^{-1}(x - 1) \right) \right) \]
\[ = M_{\mathbb{Z}_p} \left( x^{-(p-1)/2} \Phi_p(x) + x^{-1}(x - 1)^2 \right) = p^3, \]
so extra conditions would be needed on $n$ to rule out $p^3$ when $p\mid n$.
A similar problem occurs in the cyclic case where
\[ M_{\mathbb{Z}_p} (\Phi_p(x) + (1 - x)) = p^2. \]

5. The Dihedral Groups of Order $2p$ and $4p$

So far we have concentrated on just finding the smallest non-trivial group determinant or measure. Laquer [10] and Newman [17] obtained a complete description of the group determinants for $G = \mathbb{Z}_p$, showing that one achieves anything of the form $p^a m$, $p \nmid m$ with $a = 0$ or $a \geq 2$. Here we similarly give a complete description of the measures for the dihedral group $D_{2p}$:

Theorem 5.1. Suppose that $G = D_{2p}$ with $p$ an odd prime. Then the values achieved as integer group determinants take the form $2^a p^b m$ for any integer $m$ with $(m, 2p) = 1$, $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 3$. 
For $G = \mathbb{Z}_{2p}$ with $p$ an odd prime, Laquer \cite{Laquer} showed that the values take the form $2^a p^b m$ for any $(m, 2p) = 1$ and $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 2$. Likewise we can give a complete description for $D_{4p}$.

**Theorem 5.2.** Suppose that $G = D_{4p}$ with $p$ an odd prime.

The odd values achieved as integer group determinants are the $m \equiv 1 \mod 4$ with $p \nmid m$ or $p^3 \mid m$.

The even values are the integers of the form $2^a p^b m$ for any integer $m$ with $(m, 2p) = 1$, $a = 4$ or $a \geq 6$ and $b = 0$ or $b \geq 3$.

For $G = \mathbb{Z}_{2k}$ the odd values are all determinants. The even determinant values are $\mathcal{E} = 4\mathbb{Z}$, $16\mathbb{Z}$ and $32\mathbb{Z}$ when $k = 1, 2$ or $3$, but only upper and lower set inclusions $2^{2k-1}\mathbb{Z} \subseteq \mathcal{E} \subseteq 2^{k+2}\mathbb{Z}$ are known for $k \geq 4$, see Kaiblinger \cite{Kaiblinger} Theorem 1.1. Similarly for the dihedral groups $D_{2k}$, we give a complete description of the determinants for $k \leq 4$, and upper and lower containments for $k \geq 5$.

**Theorem 5.3.** For $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ the group determinant with integer variables are

$$4m + 1 \quad \text{and} \quad 2^4(2m + 1) \quad \text{and} \quad 2^6m, \ m \in \mathbb{Z}.$$ 

For $G = D_8$ the determinant takes the values $4m + 1$ and $2^8m$, $m$ in $\mathbb{Z}$.

For $G = D_{16}$ the determinant takes the values $4m + 1$ and $2^{10}m$, $m$ in $\mathbb{Z}$.

For $G = D_{2k}$ with $k \geq 4$ the odd values achieved are the integers $1 \mod 4$. The set $\mathcal{E}$ of even values achieved satisfies

$$2^{3k}\mathbb{Z} \subseteq \mathcal{E} \subseteq 2^{2k+2}\mathbb{Z},$$

and contains values with $2^{2k+2}\|M_G(F)$.

For $G = \mathbb{Z}_p^k$ with $p$ odd and $k \geq 2$ the values coprime to $p$ are all achievable while the set of values that are multiples of $p$ were shown by Newman \cite{Newman} to satisfy

$$p^{2k}\mathbb{Z} \subseteq \mathcal{P} \subseteq p^{k+1}\mathbb{Z}$$

with $\mathcal{P} = 27\mathbb{Z}$ when $k = 2$ and $p = 3$ but $\mathcal{P} \neq p^{k+1}\mathbb{Z}$ for $p \geq 5$.

Similarly for $G = D_{2p^k}$ we can obtain a complete description for a few primes when $k = 2$, and in general upper and lower set inclusions.

**Theorem 5.4.** For $G = D_{2p^k}$ with $p = 3, 5$ or $7$ the measures take the form $2^a p^b m$, $(m, 2p) = 1$ with $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 5$.

In general, if $G = D_{2p^k}$ with $p$ an odd prime and $k \geq 2$ the measures must take the form $2^a p^b m$, $(m, 2p) = 1$, with $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 2k + 1$. We can achieve everything of this form with $b = 0$ or $b \geq 3k$. There are measures with $b = 2k + 1$.

6. Proofs

Recall that

$$x^n - 1 = \prod_{m|n} \Phi_m(x), \quad \Phi_m(x) := \prod_{\gcd(j,m) = 1} (x - e^{2\pi ij/m}),$$

where $\Phi_m(x)$ is the $m$th cyclotomic polynomial, an irreducible polynomial in $\mathbb{Z}[x]$ whose roots are the primitive $m$th roots of unity. We write $\text{Res}(F, G)$ for the
Proof of Lemma 4.1. We begin with powers of 2. If we set \( f \) even we also have
\[
\text{Res}(\Phi_n, \Phi_m) = \begin{cases} 
\rho^{\phi(n)}, & \text{if } m = np^\alpha, \\
1, & \text{else.}
\end{cases}
\]

Observe that if \( 2 \nmid n \) we also have
\[
M_{2^n}(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}) = M_{2^n}(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1})^2
= \text{Res}(a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}, x^n - 1)^2.
\]

If \( n \) is odd the two linear terms give us
\[
\text{Res}(x + 1, x^n - 1) = 2, \quad \text{and if } 2^\alpha | n \text{ we have}
\]
\[
\text{Res}(x^{2^\alpha} + 1, x^n - 1) = \prod_{j=0}^{\alpha} \prod_{j=0}^{\alpha} \text{Res}(\Phi_{2^{n+j}}, \Phi_{2^{j+1}}) = 2^{2^\alpha}.
\]

Taking these polynomials and reducing mod \( x^n \) as necessary we get the squares of 2 if \( 2 \nmid n \) and \( 2^{2^\alpha} \) if \( 2^\alpha | n \). For odd \( p \) we could similarly get \( p^{2^\alpha} \) when \( p^\alpha | n \).

Instead suppose that \( m = 2t + 1 \) with \((m, n) = 1\) and consider the polynomial
\[
F(x, y) = \sum_{i=0}^{t} x^i + \sum_{i=0}^{t-1} y x^i = f(x) + yg(x), \quad f(x) = \frac{x^{t+1} - 1}{x - 1}, \quad g(x) = \frac{x^t - 1}{x - 1}
\]

where we reduce \( F \) mod \( x^n - 1 \) to make it of the form \( F \) if \( t \geq n \). Then
\[
M_{G}(F(x, y)) = M_{2^n} (f(x)f(x^{-1}) - g(x)g(x^{-1})) = M_{2^n} \left( x^{-t} \left( \frac{x^{n-1}}{x - 1} \right) \right).
\]

Since \((n, m) = 1\) the squared quadratic terms assemble to give
\[
\prod_{k | n, k \neq 1, 2} \prod_{d | m, d \neq 1} |\text{Res}(\Phi_d, \Phi_k)| = 1.
\]

For \( n \) odd the two linear terms give us \( f(1)^2 - g(1)^2 = m \) and \( M_{G}(F) = m \). For \( n \) even we also have \( f(-1)^2 - g(-1)^2 = (-1)^t \) and \( M_{G}(F) = m \) if \( m \equiv 1 \text{ mod } 4 \) and \( M_{G}(F) = -m \) if \( m \equiv 3 \text{ mod } 4 \).

If \( p = 2t + 1 \) and \( p^\alpha | n \) we consider
\[
F = \sum_{i=0}^{t} x^{ip^\alpha} + \sum_{i=0}^{t-1} y x^{ip^\alpha} = f(x^{p^\alpha}) + yg(x^{p^\alpha}),
\]

reducing mod \( x^n - 1 \) if necessary. We similarly obtain
\[
M_{G}(F) = M_{2^n} \left( x^{-ip^\alpha} \Phi_{p^{n+1}} \right),
\]

where \( |\text{Res}(\Phi_d, \Phi_{p^{n+1}})| = 1 \) for the \( d | n \), except for the \( d = p^j \) which contribute
\[
|M_{G}(F)| = |\text{Res}(x - 1, \Phi_{p^{n+1}})| \prod_{j=1}^{\alpha} |\text{Res}(\Phi_p^j, \Phi_{p^{n+1}})| = p \prod_{j=1}^{\alpha} p^{\phi(p^j)} = p^{p^\alpha}.
\]

\[\square\]
Proof of Theorem 4.2. From the proof of Lemma 4.1 we can obtain any \((m, 2n) = 1\) when \(n\) is odd and those with \(m \equiv 1 \mod 4\) when \(n\) is even. Note switching the roles of \(f(x)\) and \(g(x)\) switches the sign of \(\mathcal{M}_G(f(x) + yg(x))\) when \(n\) is odd but not for \(n\) even. So it just remains to show the odd \(D_{2n}\) measure of an \(F(x, y) = f(x) + yg(x)\) must be 1 mod 4 when \(n\) is even. The square terms will of course be 1 mod 4, so that leaves the contribution from the four linear terms \(f(1)^2 - g(1)^2\) and \(f(-1)^2 - g(-1)^2\). Since \(f(-1) \equiv f(1) \mod 2\) we have \(f(-1)^2 \equiv f(1)^2 \mod 4\) and \((f(1)^2 - g(1)^2) - (f(-1)^2 - g(-1)^2) \equiv (f(1)^2 - g(1)^2)^2 \equiv 1 \mod 4\). \(\square\)

Proof of Lemma 4.2. We write \(\mathcal{D} := \mathcal{D}_G(a_0, \ldots, a_n, b_0, \ldots, b_{n-1}).\)

Suppose first that \(p\) is odd with \(p^\alpha \| n\). If \(\alpha = 0\) then there is nothing to show, so assume that \(\alpha \geq 1\). If \(n = mp^\alpha\), then, writing \(sm + tp^\alpha = 1\) and observing that \(w_n = w_{\alpha,n}w_{m}^t\), and that \(Q(w) = Q(w^{-1})\), we have

\[
\prod_{j=1}^{[(n-1)/2]} Q(w_n^j)^2 = \prod_{j=0}^{p^\alpha-1} \prod_{u=0}^{m-1} Q(w_{m,u}^{w_{\alpha,n}}) = A_1 \prod_{d|m, d \neq 1,2} A_d^2,
\]

where, splitting into the order of the \(p^\alpha\)th and \(m\)th roots of unity, and re-pairing the \(d\)th roots with their conjugates, we get,

\[
A_1 := \prod_{v=1}^{\alpha} A_1(v), \quad A_d := A_d(0) \prod_{v=1}^{\alpha} A_d(v),
\]

where

\[
A_1(v) := \prod_{j=1}^{(p^\alpha - 1)/2} Q(w_{p,v}^j) \text{ if } m \text{ is odd}, \quad A_1(v) := \prod_{j=1}^{(p^\alpha - 1)/2} Q(w_{p,v}^j)Q(-w_{p,v}^j) \text{ if } m \text{ is even},
\]

and

\[
A_d(0) := \prod_{t, d = 1}^{d/2} Q(w_d^t), \quad A_d(v) := \prod_{t=1}^{(p^\alpha - 1)/2} \prod_{t, d = 1}^{d/2} Q(w_d^t w_{p,v}^t).
\]

Note, since \(Q(w) = Q(w^{-1})\), our products are over complete sets of conjugates and the \(A_d(i)\) are all integers. Notice also that our linear factors equal

\[
A_0 := \begin{cases} 
Q(1), & \text{if } m \text{ is odd,} \\
Q(1)Q(-1), & \text{if } m \text{ is even,}
\end{cases}
\]

and

\[
\mathcal{D} = A_0A_1^2 \prod_{d|m, d \neq 1,2} A_d^2.
\]

Setting \(\pi_n = 1 - w_{p,v}^\alpha\), observe that the \(p\)-adic absolute value on \(\mathbb{Q}\) extended to \(\mathbb{Q}(w_n)\) has \(|\pi_n|_p = p^{-1/\phi(p^\alpha)} < 1\). Since all the \(w_{p,v}^j \equiv 1 \mod \pi_n\) in \(\mathbb{Z}[w_n]\), we gain the following congruences mod \(\pi_n\) in \(\mathbb{Z}[w_n]\) and, since all the terms are integers, mod \(p\) in \(\mathbb{Z}^\times\):

\[
A_d(v) \equiv A_d(0)^{\phi(p^\alpha)} \mod p, \quad A_1(v) \equiv A_0^{\phi(p^\alpha)/2} \mod p.
\]
Hence if \( p \mid \mathcal{D} \) then either we have \( p \mid A_d(v) \) for some \( d \) and \( 0 \leq v \leq \alpha \), and hence \( p \mid A_d(v) \) all \( 0 \leq v \leq \alpha \), giving \( p^{\alpha+1} \mid A_d \) and \( p^{2\alpha+2} \mid \mathcal{D} \), or \( p \mid A_0 \) or \( p \mid A_1(v) \) for some \( 1 \leq v \leq \alpha \) and \( p \) divides all of these, giving \( p^\alpha \mid A_1 \), \( p \mid A_0 \) and \( p^{2\alpha+1} \mid \mathcal{D} \).

Suppose first that \( 2 \nmid n \). Notice that the two linear factors are congruent mod 2 so that their product is either odd or a multiple of 4, while the integer from the quadratic factors is squared. Thus \( 2 \mid \mathcal{D} \) implies that \( 2^2 \mid \mathcal{D} \).

So suppose that \( 2^\alpha \mid n \) with \( \alpha \geq 1 \) and \( 2 \mid \mathcal{D} \). Similar to the \( p \) odd case we write \( n = 2^\alpha m \) and end up with

\[
\mathcal{D} = Q(1)Q(-1)A_1^2 \prod_{\substack{d|m \\ d \neq 1}} A_d^2,
\]

with \( A_d \) as before and

\[
A_1 = \prod_{v=2}^\alpha A_1(v), \quad A_1(v) := \prod_{j=1 \atop j \text{ odd}} Q(w_{2v}).
\]

Again from 2-adic considerations we have

\[
A_1(v) \equiv Q(1)2^{\alpha-2} \mod 2, \quad A_d(v) \equiv A_d(0) \mod 2.
\]

Hence if \( 2 \mid A_d(v) \) for some \( d \) and \( 0 \leq v \leq \alpha \) then 2 divides them all and \( 2^{\alpha+1} \mid A_d \) and \( 2^{2\alpha+2} \mid \mathcal{D} \). Likewise, if \( 2 \mid Q(1), Q(-1) \) or \( A_1(v) \) for some \( 2 \leq v \leq \alpha \) then 2 divides all of them, so \( 2^{\alpha-1} \mid A_1 \) and, since the four linear factors are all congruent mod 2, also \( 2^4 \mid Q(1)Q(-1) \), giving \( 2^{2\alpha+2} \mid \mathcal{D} \). Moreover, writing \( Q(1) = f(1)^2 - g(1)^2 \) and \( Q(-1) = f(-1)^2 - g(-1)^2 \), if \( Q(1) \) is even then \( f(1), g(1), f(-1), g(-1) \) all have the same parity. If all are odd then \( 2^3 \mid Q(1), Q(-1) \) and \( 2^6 \mid Q(1)Q(-1) \). If all are even then \( (f(1)/2)^2 - (g(1)/2)^2 \) and \( (f(-1)/2)^2 - (g(-1)/2)^2 \) are either odd or 0 mod 4 and \( 2^4 \mid Q(1)Q(-1) \) or \( 2^9 \mid Q(1)Q(-1) \).

For \( \alpha \geq 2 \) we can extract two further 2’s. If \( 2^2 \mid A_d(v) \) for some \( 0 \leq v \leq \alpha \) or \( 2^2 \mid A_1(v) \) for some \( 2 \leq v \leq \alpha \) then, since these terms are squared, we gain two extra 2’s. This leaves us to consider the cases \( 2 \mid A_d(2) \) or \( 2 \mid A_1(2) \).

Suppose first that \( 2 \mid A_d(2) \). Suppose that \( f(x) = \sum_{j=0}^{n-1} a_j x^j \), \( g(x) = \sum_{j=0}^{n-1} b_j x^j \) and write

\[
H(x) = \prod_{(j,d)=1}^d Q(xw_d^j), \quad Q(x) = f(x)f(x^{-1}) - g(x)g(x^{-1}).
\]

Since we run over a full set of conjugates \( H(x) \in \mathbb{Z}[x,x^{-1}] \). Moreover \( H(x^{-1}) = H(x) \) and, writing \( x^j + x^{-j} \) as a polynomial in \( x + x^{-1} \), we have \( H(x) \in \mathbb{Z}[x+x^{-1}] \).

\[
H(x) = \sum_{j=0}^J B_j (x + x^{-1})^j.
\]

Note that

\[
A_d(0)^2 = H(1) \equiv B_0 + 2B_1 + 4B_2 \mod 8,
\]

\[
A_d(1)^2 = H(-1) \equiv B_0 - 2B_1 + 4B_2 \mod 8,
\]

\[
A_d(2) = H(i) = B_0.
\]
Hence if $2 || A_d(2)$ we have
\[ A_d(0)^2 + A_d(1)^2 \equiv 2B_0 \equiv 4 \mod 8, \]
ruling out having $2 || A_d(0)$ and $A_d(1)$. Thus $2^2 | A_d(0)$ or $A_d(1)$ and $2^4 | A_d(0)A_d(1)A_d(2)$ and $2^{\alpha+4} | \varnothing$.

Similarly if $2 || A_1(2)$ we have
\[ Q(1)Q(-1)A_1(2)^2 = Q(1)Q(-1)Q(i)^2 \]
and by the same argument, with $Q(x)$ in place of $H(x)$, we have
\[ Q(1) + Q(-1) \equiv 2Q(i) \mod 8 \]
with $4 | Q(\pm 1)$, and if $2 || Q(i)$ we can’t have $2^2 || Q(\pm 1)$. That is, for either $\epsilon = 1$ or $-1$ we must have $8 | Q(\epsilon) = f(\epsilon)^2 - g(\epsilon)^2$ and $2^4 || Q(-\epsilon)$. Note $f(\epsilon)$ and $g(\epsilon)$ must be the same parity and can’t both be odd, else $f(-\epsilon)$ and $g(-\epsilon)$ would both be odd and $8 | Q(-\epsilon)$. Hence we can write $f(\epsilon) = 2\alpha_0$ and $g(\epsilon) = 2\beta_0$ and $Q(\epsilon) = 4(\alpha_0^2 - \beta_0^2)$.

Since $2 | \alpha_0^2 - \beta_0^2$ we must have $4 | \alpha_0^2 - \beta_0^2$ and $2^3 | Q(1)Q(-1)Q(i)^2$, and again $2^{2\alpha+4} | \varnothing$.

**Proof of Theorem 4.7.** For $n$ of each given form Lemma 4.1 says that we can achieve the value claimed as $\lambda(D_{2n})$, while Lemma 4.2 rules out anything smaller.

**Proof of Theorem 5.3.** Suppose that $G = D_{2p^k}$ with $p$ an odd prime and $k \geq 1$. From Theorem 4.2 we can obtain all values coprime to $2p$. Products of
\[ M_G(x + 1) = M_{Z_{2p^k}}(x + 1)^2 = 2^2, \]
\[ M_G(x^2 + x + 1) = M_{Z_{2p^k}}(x^-2(x + 1)^2(x^2 + 1)) = 2^3, \]
give any $2^a$ with $a \geq 2$, and for $\ell \geq k$
\[
M_G \left( x^{(\frac{x^\ell}{x-1})-1} \frac{x^\ell}{x-1} + (x-1) + y \left( x^{(\frac{x^\ell}{x-1})-1} \frac{x^\ell}{x-1} + x^{-1}(x-1) \right) \right)
\]
\[
= M_{Z_{p^k}} \left( x^{-\ell} \left( \frac{x^\ell}{x-1} \right) + x^{-1}(x-1)^2 \right) = p^{\ell+2k}
\]
gives any power $p^b$ with $b \geq 3k$. Products of these achieve anything of the form $2^ap^bm$ with $(m, 2b) = 1$, $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 3k$. Lemma 4.2 shows that measures must be of this form with $a = 0$ or $a \geq 2$, and $b = 0$ or $b \geq 2k + 1$. For $k = 1$ these coincide and we have a complete description of the measures.

**Proof of Theorem 5.4.** From Lemma 4.2 and Theorem 4.2 we know that the values must be of the stated form and that we can obtain all values coprime to $2p$ that are $1 \mod 4$. To deal with the powers of $p$ we show that for any $k \geq 1$ there is an $F(x, y) = f(x) + yg(x)$ with
\[ M_G(F) = \delta p^{k+2}, \quad \delta := \begin{cases} 1, & \text{if } p^k \equiv 1 \mod 4, \\ -1, & \text{if } p^k \equiv -1 \mod 4. \end{cases} \]
Taking
\[ A = \frac{1}{2}(p^k + \delta), \quad B = \frac{1}{4}(p^k - \delta), \quad f(x) = \frac{x^A - 1}{x - 1}, \quad g(x) = (x^p + 1) \left( \frac{x^B - 1}{x - 1} \right), \]
we have \( M_G(F) = M_{\mathbb{Z}_2}(H(x)) \) where \( H(x) = f(x)f(x^{-1}) - g(x)g(x^{-1}) \). Plainly
\[ H(1) = A^2 - (2B)^2 = \delta p^k. \]
If \( x^p = -1 \) we have \( H(x) = x^{-(A-1)} (x^A - 1) / (x - 1)^2 \) and, since \( A \) is odd and coprime to \( p \), we have
\[ H(-1) = 1, \quad |\text{Res}(H(x), \Phi_{2p})| = 1. \]
If \( x^p = 1, x \neq 1 \), we have \( H(x) = K(x)/(x - 1)(x^{-1} - 1) \) where
\[ K(x^4) = (x^{4A} - 1)(x^{-4A} - 1) - 4(x^{4B} - 1)(x^{-4B} - 1) \]
\[ = (x^{2\delta} - 1)(x^{-2\delta} - 1) - 4(x^{-\delta} - 1)(x^\delta - 1) \text{ mod } (x^p - 1) \]
\[ = -(x^{\delta} - 1)^2(x^{-\delta} - 1)^2. \]
Since \( \text{Res}(K(x), \Phi_p) = \text{Res}(K(x^4), \Phi_p) \) and \( \text{Res}(x-1, \Phi_p) = p \) we see that \( \text{Res}(H(x), \Phi_p) = p^2 \) and \( M_G(F) = \delta p^{k+2} \).

To obtain the necessary powers of 2 we have
\[ M_G(x^2 + 1) = 2^4, \]
\[ M_G((x^p + 1) + y(x - 1)) = \prod_{x^p = -1} -|x - 1|^2 \prod_{x^p = 1} (x + 1)(x^{-1} + 1) = -2^4, \]
\[ M_G(1 + x^2 + x(1 + x^p)) = \prod_{x^p = -1} |x^2 + 1|^2 \prod_{x^p = 1} |x + 1|^4 = 2^6, \]
\[ M_G(1 - x^{p+2} + y(x^p + 1)(x + 1)) = \prod_{x^p = -1} |x^2 + 1|^2 \prod_{x^p = 1} -(x + 1)^2(x^{-1} + 1)^2 = -2^6. \]

To add an additional \( \pm 2^l \) to the \( 2^6 \) we choose \( m = 1 \) or \( 3 \) so that \( mp \pm 2^l \equiv 1 \) mod \( 4 \), set \( mp \pm 2^l = 2t + 1 \), where \( t \) is even, and take \( F(x, y) = f(x) + yg(x) \) with
\[ f(x) = (1 + x^2 + x(1 + x^p)) \left( \frac{x^{t+1} - 1}{x - 1} \right) - m \left( \frac{x^{2p} - 1}{x - 1} \right), \]
\[ g(x) = (1 + x^2 + x(1 + x^p)) \left( \frac{x^t - 1}{x - 1} \right) - m \left( \frac{x^{2p} - 1}{x - 1} \right). \]
Then \( H(x) = f(x)f(x^{-1}) - g(x)g(x^{-1}) \) has
\[ H(1) = (4(t + 1) - 2mp)^2 - (4t - 2mp)^2 = 2^4(2t + 1 - mp) = \pm 2^{l+4}. \]
Since
\[ (x^{t+1} - 1)(x^{-(t+1)} - 1) - (x^t - 1)(x^{-t} - 1) = x^{-t}(x^{2t+1} - 1)(x^{-1} - 1), \]
for \( x^p = -1 \) we have
\[ H(x) = |x^2 + 1|^2x^{-t} \left( \frac{x^{2t+1} - 1}{x - 1} \right), \quad H(-1) = 2^2, \quad \text{Res}(H(x), \Phi_{2p}) = 1. \]
For \( x^p = 1 \) with \( x \neq 1 \),
\[ H(x) = |1 + x|^4x^{-t} \left( \frac{x^{2t+1} - 1}{x - 1} \right), \quad \text{Res}(H(x), \Phi_p) = 1. \]
Hence $M_G(F) = \pm 2^{k+6}$. Products then achieve all the stated forms.

\[\square\]

Proof of Theorem 5.3. Suppose that $G = D_{2k}$ with $k \geq 2$. From Theorem 4.2, we know that the odd values taken are exactly the integers 1 mod 4.

Corresponding to $k = 2$ we have $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. From Lemma 4.2, the even measures satisfy $2^4 | M_G(F)$ or $2^5 | M_G(F)$. Writing $F(x, y) = f(x) + yg(x)$ for we have $M_G(F) = (f(1)^2 - g(1)^2)(f(-1)^2 - g(-1)^2)$ and the measures are readily achieved with

$$M_G(1 + m(x + 1) + ym(x + 1)) = 1 + 4m,$$

$$M_G(2 + m(x + 1) + ym(x + 1)) = 2^4(2m + 1),$$

$$M_G(3 + (m - 1)(x + 1) + y(1 + (m - 1)(x + 1))) = 2^6m.$$

For $G = D_{2k}$ with $k \geq 3$ the upper bound on the even values $\mathcal{E} \subseteq 2^{2k+2}\mathbb{Z}$ follows from Lemma 4.2. For the lower bound $2^{3k}\mathbb{Z} \subseteq \mathcal{E}$ take

$$f(x) = 2 + m \left( \frac{x^{2k-1} - 1}{x - 1} \right), \quad g(x) = (x + 1) - m \left( \frac{x^{2k-1} - 1}{x - 1} \right).$$

Writing $H(x) = f(x)f(x^{-1}) - g(x)g(x^{-1})$, we have $H(1) = 2^{k+2}m$, with $H(x) = (1 - x)(1 - x^{-1})$ when $x^{2k-1} = 1$, $x \neq 1$, and

$$M_G(f(x) + yg(x)) = M_{2^{2k-1}}(H(x)) = 2^{3k}m.$$

It is not hard to see that $2^{2k+2} | |M_G(2 + (1 - x))$.

For $G = D_8$ this gives $2^9 \mathbb{Z} \subseteq \mathcal{E} \subseteq 2^8 \mathbb{Z}$. For the missing multiples of $2^8$:

$$M_G \left( 2 + k \frac{x^4 - 1}{x - 1} + yk \frac{x^4 - 1}{x - 1} \right) = 2^8(4k + 1),$$

$$M_G \left( (x^2 + 1)(x - 1) + k \frac{x^4 - 1}{x - 1} + y \left( (x + 1) + k \frac{x^4 - 1}{x - 1} \right) \right) = -2^8(4k + 1).$$

For $G = D_{16}$ we have $2^{12} \mathbb{Z} \subseteq \mathcal{E} \subseteq 2^{10} \mathbb{Z}$. The remaining multiples of $2^{10}$ are readily obtainable using:

$$M_G \left( (1 + x^2)(1 + x^4) - (1 - x) \right) = 2^{10},$$

$$M_G \left( (1 + x^2)(1 + x^4) + (x - 1)(x^2 + 1) + y(x - 1) \right) = -2^{10},$$

$$M_G \left( (1 + x^2) - \left( \frac{x^8 - 1}{x - 1} \right) + y(x + 1) \right) = 2^{11},$$

$$M_G \left( (1 + x^2) + y \left( (x + 1) - \left( \frac{x^8 - 1}{x - 1} \right) \right) \right) = -2^{11}.$$

\[\square\]

Proof of Theorem 5.4. From the proof of Theorem 5.1, we know that the measures are of the form $2^ap^{b+m}$, $(m, 2p) = 1$ with $a = 0$ or $a \geq 2$ and $b = 0$ or $b \geq 2k + 1$ where we can achieve anything of this type with $b = 0$ or $b \geq 3k$. For $p | AB$ it is readily seen that $p^{2k+1} | |M_G(pA + B(x - 1))$.

For $k = 2$ this just leaves the measures with $b = 5$. For $p = 3, 5$ or 7 we have

$$M_{D_{2p^3}} \left( \frac{x^{(p+1)/2} - 1}{x - 1} + y \left( \frac{x^{(p+1)/2} - 1}{x - 1} - x^{p-1} \right) \right) = p^5.$$
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Department of Mathematics, DigiPen Institute of Technology, Redmond, WA 98052, USA
E-mail address: aboerkoel@digipen.edu

Department of Mathematics, Kansas State University, Manhattan, KS 66506, USA
E-mail address: pinner@math.ksu.edu