Moore: Interval Arithmetic in Modern C++

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I. INTRODUCTION

INTERVAL arithmetic is a powerful tool, which can be used to solve practical and theoretical problems. However, implementing an interval arithmetic library for general use is difficult, and requires much attention to detail. These details involve the specification of interfaces, the implementation of efficient and accurate algorithms and dealing with bugs, which can be our own fault or the compiler’s.

Regarding interfaces, there are many design decisions to be made and each author has his own opinions, as one can see by comparing the libraries in [1]–[11]. Recently two documents [12], [13] were elaborated in order to provide a common interface for interval arithmetic libraries. Formally, the first document [12] is an active IEEE standard, while the second [13] is still a project. For simplicity, throughout the article we will refer to these documents as “The IEEE standards.” These standards are a significant step forward, but there is controversy in the interval arithmetic community regarding them. For instance, prof. U. Kulisch has made clear his disagreement with them.

This article presents the Moore library, which implements part of the IEEE standards in the most recent version of the C++ language, using new features of this language. The library was written mostly to be used on our own research, and we focus on improving the code more likely to be used often. For this reason, we do not fully implement the IEEE standards. However, we do hope that the library will be useful to others. People for which compliance with the standards is a priority would be better served by the libraries in [5] and [9]; people looking for better performance or more precise types of endpoints for their intervals may consider using our library. For instance, in Section VII we present experiments showing that our library is competitive in terms of speed with well known libraries, and it is significantly faster than the current reference implementation of the IEEE Standards in C++ [9] (the other reference implementation, [5], is implemented in Octave and is not comparable to our library.)

This article was written for people which are already familiar with interval arithmetic, who will understand us when we say that, when used properly, our library satisfies all the usual containment requirements of interval arithmetic. Our purpose is to describe the library, show that it is competitive with well known libraries, and expose its limitations (see the last section.) Even with these limitations, we would like to invite readers to experiment with the library. In fact, by reading this article one will have only a glimpse of the library, and the only way too fully understand it is to try it in practice. The Moore library is open source software, distributed under the Mozilla 2.0 license, and its source code can be obtained by sending an e-mail message to the author.

In the rest of this article we present the library, starting from the basic arithmetic operations and moving to more advanced issues. We also describe in which points our library deviates from the IEEE standards and the parts of the standards that we do implement.

II. HELLO INTERVAL WORLD

The Moore library can be used by people with varying degrees of expertise. Non experts can simply follow what is outlined in the code below:

```c++
#include "moore/minimal.h"
Moore::RaiiRounding r;
Moore::Interval<> x(2.0, 3.0);
Moore::Interval<> y("[-1,2]");
for(int i = 0; i < 10; ++i)
{
    y = (sin(x) - (y/x + 5.0) * y) * 0.05;
    cout << y << endl;
}
```

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In English, with the Moore library one can construct intervals by providing their endpoints as numbers or strings, and then use them in arithmetical expressions as if they were numbers. The library also provides the trigonometric and hyperbolic functions, their inverses, exponentials and logarithms, and convenient ways to read and write intervals to streams.

The file minimal.h in the code above contains the required declarations for using the library with double endpoints. The line

```
Moore::RaiiRounding r;
```

is required in order to use the library. It sets the rounding mode to upwards, and the rounding mode is restored when r is destroyed, following the Resource Acquisition is Initialization (Raii) pattern in C++. Our approach is similar to one option provided by the boost library [1]. However, the boost library is more flexible than ours: we only provide one rounding policy. In fact, providing fewer options instead of more is the usual choice made by the Moore library. We only care about concrete use cases motivated by our own research, instead of all possible uses of interval arithmetic. This is the main difference between the spirit of our library and the purpose of the IEEE standards for interval arithmetic. We prefer to provide a better library for a few users rather than trying to please a larger audience which we will never reach.

The intervals in the Moore library are parameterized by a single type. It does not contain class hierarchies, virtual methods or policy classes. On the one hand, users can only choose the type of the endpoints defining the intervals of the form \([a, b]\) with \(-\infty \leq a \leq b \leq +\infty\), or the empty interval. On the other hand, we do believe that our library goes beyond what is offered by other libraries in its support of generic endpoints and operations. As we explain in Section IV, the library can work with several types of endpoints “out of the box,” that is, it provides tested code in which several types of endpoints can be combined, as in this example:

```c++
RaiiRounding r;
Interval<> x("[-1,2]");
Interval<float> y("[-1/3,2/3]);
Interval<float128> z("[-inf,4]");
Interval<Real<256>> w("5?`);

auto h = hull(x, y, 0.3);
auto i = intersection(x, y, z, w);
auto j = sin(z * x/cos(y * z)) - exp(w);
```

The code above handles four kinds of endpoints:

- The interval x has endpoints of type double.
- y has endpoints of type float.
- The endpoints of z have quadruple precision.
- w has endpoints of type Real<256>, which represents floating point numbers with \(N = 256\) bits of mantissa, and the user can choose other values for \(N\).
- The compiler deduces that i is an interval with endpoints of type double, which is the appropriate type for storing the convex hull of x, y and 0.3.
- It also deduces that Real<256> is the appropriate type of endpoints for the intervals representing the intersection of x, y, z and w and the result of the evaluation of the expression assigned to j.

We ask the reader not to underestimate the code in the previous paragraph. It is difficult to develop the infrastructure required for users to handle intervals with endpoints of different types in expressions as natural as the ones in that code. In fact, there are numerous issues involved in dealing with intervals with generic endpoints, and simply writing generic code with this purpose is not enough. The code must be tested, and our experience shows that it may compile for some types of endpoints and may not compile for others.

There are two main points in which the Moore library does not follow the IEEE standards: decorations and exceptions. As we explain in Section V, we do not provide decorated intervals, because in our opinion decorations are a bad idea. We agree that it is quite useful to have standards, and we are glad to acknowledge the positive influence of the IEEE standards in our library. However, we also believe that standards should cover only the minimal number of features required to achieve the goals of the majority of the users, and decorations do not satisfy this basic criterion. We should also mention that we did implement a library which supported decorations, and that when we analyzed the result it was evident that this implementation should be thrown away.

We also treat exceptions differently. In the Moore library, the term exception applies to an error from which no recovery is possible (or desirable), and such that its occurrence leads to the termination of the program. For instance, as the excellent GMP library, we find it appropriate to terminate the program when we are not able to allocate memory. In
fact, any library relying on GMP would be subject to program termination in case of errors in memory allocation, unless precautions are taken. The Moore library uses GMP and did not take any precautions.

In our library, warnings are indicated by return values instead of raising flags. For instance, instead of raising an inexact flag when constructing an interval from the string "[1/3,2/3]" we provide a constructor with a second argument of type Accuracy which indicates the accuracy with which the string is converted to an interval, as in

```
Accuracy a;
Interval<> x("[1,2]", a);
Interval<> y("[1/3,2/3]", a);
Interval<> z("[1,2]");
Interval<> w("[1/3,2/3]");
Interval<> bad("disaster");
```

In the construction of \(x\), the variable \(a\) will be set to \(\text{Accuracy::Exact}\), and the construction of \(y\) yields \(a = \text{Accuracy::Tight}\). In the other constructor calls the user is not interested in knowing the accuracy and there is no reason to set a flag to inform him about something that he does not care. In debug mode, the construction of \(bad\) will cause the failure of an assertion, and the program will be stopped by the debugger. In release mode, the construction of \(bad\) will terminate the program, when calling an error handler which writes a message to the standard output stream for errors. This the default behavior, and the user can change it by changing the error handler, but we do not recommend it. We discuss Exception handling in more detail in Section VI.

Exceptions in the Moore library are reserved for truly exceptional cases, and when in doubt whether a string represents an interval the user should do something like the code below instead of trying to construct the interval directly from it:

```
Accuracy a;
Interval<> x;
if( a == \text{Accuracy::Invalid} ) {
    cout << "No, You are not an interval";
}
```

Finally, other than the cases mentioned above, one case mentioned in the next section, and the implementation of the functions cancel_minus and cancel_plus, the Moore library follows the IEEE standards for interval arithmetic closely, because they contain many good points. In particular, we implement all the functions mentioned in the simplified standard \[13\], including the ones which would qualify our library as a flavor of the full standard \[12\] in the absence of the points mentioned in the previous paragraph. We also implement the reverse and overlapping functions mandated by the full standard \[12\]. In the end, people not interested in decorations and exceptions will be able to use our library as they would use any other library conforming with the standards, with the additional functionality described in the next sections.

III. INPUT AND OUTPUT

Except for minor details which are still under discussion in working group for the simplified standard \[13\], we agree with all points mandated by the IEEE standards regarding input and output of plain intervals, and tried to implement them as faithfully as we could. We also provide several options for the formatted output of intervals, as an aid to the visual inspection of the results of interval calculations.

For instance, the code in the first page of this article outputs ten lines, and the first one is

\([-0.592944,0.345465]\]

Users may want to display this interval with more digits, or to use scientific notation. With the Moore library they could write

```
cout << std::setprecision(10);
cout << std::scientific;
```

before the for statement in that code, and obtain

\([-5.929439996e-1,3.4546487135e-1]\)
\([-8.2293866866e-2,1.9882190431e-1]\) ...

They could also introduce space between the numbers and the brackets and pad the numbers with zeros to the right, by including the lines

```
cout << Moore::Io::pad();
cout << Moore::Io::border_slack(2);
cout << Moore::Io::center_slack(1);
```

which lead to the better looking output

\([-5.929439996e-1, 3.4546487135e-1 ]
[-8.2293866866e-2, 1.9882190431e-1 ]
...

There are several options for controlling the way in which intervals are written to output streams, and they can also be read from input streams as in

```
Moore::Interval<> x;
cin >> x;
Accuracy a = try_scan(x, is);
```
In order to simplify its use, most options in the Moore library have reasonable default values, which are not always the most efficient. For instance, the convenience of io streams has a cost. Usually this overhead does not matter, but when reading and writing large files we can improve things a bit as in the next example, which reads intervals from a file and writes them to another file.

```cpp
std::vector<Interval<>> in, out;
...
std::ofstream os("file.txt");
write(out, os, Io::Format("A"));
os.close();

std::ifstream is("file.txt");
read(in, is);
is.close();

std::ifstream is2("file.txt");
Accuracy a = try_read(in, is);
assert( a == Accuracy::Exact );
```

In the code above we use the Format object in order to write the interval in hexadecimal format (as with "%A" in printf), and ignore the flags of the ostream os. This ensures that intervals will be read exactly as they were written. The example in Section II is more inefficient, because for each call of operator<< we would read the flags of std::cout, create a corresponding Format object and then write the interval according to this format, using dynamic allocated memory. In the example above there is only one Format object and only a few dynamic memory allocations.

IV. ENDPOINTS

There is only one interval class in the Moore library, and it has a single parameter: the type of the endpoints. The library provides a few types of endpoints “out of the box”, that is, endpoints which are ready to be used and have been tested. The library was designed, implemented and tested in order to provide the speed of arithmetics supported by the hardware, with the types float and double, efficient quadruple precision, with the type __float128, and high precision based on the MPFR library [14]. It also allows for operations mixing different types of endpoints. For instance, we could use a less precise and more efficient type for the $x$ coordinate and a more precise type of endpoint for the $y$ coordinate or in intermediate computations.

In summary, we currently allow the combination of following types of endpoints:

- The standard floating point types: float, double and long double.
- The quadruple precision type __float128 provided by the gcc’s quadmath library.
- Floating point numbers with mantissa of $N$ bits, where $N$ is a constant fixed at compile time. We provide a type Real<N>, which is a stack based wrapper of the __mpfr_struct from the MPFR library [14].

V. DECORATIONS

A significant part of the IEEE standards for interval arithmetic is devoted to decorations. Basically, a decoration is a tag attached to an interval in order to provide information regarding how it was obtained. The combination of the interval and the tag is called by decorated interval, and the standards have several requirements regarding decorated intervals.

In principle, decorated intervals would be a convenient way to propagate information about the evaluation of functions and exceptions in a thread safe way. However, we do not plan to use decorations, or to support them in the Moore library, because we believe that the cost involved in implementing, testing and maintaining the resulting code outweighs the benefits that decorations may bring, specially when we consider the use of various types of endpoints.

The problems with decorations start already in the simple case of double precision endpoints. With g++, a struct like

```cpp
struct Interval {
    double inf;
    double sup;
};
```

uses 16 bytes of memory, and adding a tag of type char leads to a decorated interval of type

```cpp
struct DecInterval {
    double inf;
    double sup;
    char decoration;
};
```

which occupies 24 bytes when compiled with g++, and this is 50% more than the memory used by plain intervals. Moreover, tags affect the alignment of the resulting struct: Today’s cache lines usually have 64 bytes, and can hold four Intervals, but only
two DecIntervals. Therefore, cache misses will happen more often when using decorated intervals, and access to memory will be more expensive, specially for vectors and matrices.

Another important issue is the combinatorial explosion of cases to be considered while planning, coding and testing libraries combining plain and decorated intervals. Each function with two arguments would require a version for each one the four possible kinds of pairs of inputs. Moreover, we would need to define the resulting decoration for all possible combinations of the decorations of the inputs, and there is no universal rule for defining the decoration of the output. For example, the functions convexHull and intersection in the IEEE standards have the decorations of the output defined in an ad-hoc way, and the users of the library will need to think about the decoration of the output of every function they write in order to keep the overall consistency of the decoration system. Finally, functions like fma and some reverse functions, which take three arguments, would require eight versions, and each one of them would need to be planned, coded and tested.

As a result of this combinatorial explosion on the code size caused by decorations, one of the two reference implementations of the IEEE standards [9] does not implement functions combining plain and decorated intervals with endpoints of different types. The other reference implementation [5] handles only endpoints of type double and is not as affected by the large number of possibilities entailed by the combination of plain interval, decorated interval and different types of endpoints. Therefore, we are not alone in our need to make strategic decisions regarding what should and should not be implemented. We decided to prioritize generic endpoints, the author of [5] chose to implement only endpoints of type double, and the author of [9] did as much as he found reasonable in order to support both decorations and generic endpoints. He also chose not to implement operations involving numbers and intervals, or the accumulation operators +=,-=,*= and /= for intervals. In the end, we believe that there is a place for each one of us among the users of interval arithmetic libraries.

VI. EXCEPTIONS

The Moore library handles exceptions depending on the mode in which the code is compiled. There are three modes: Debug, Fast and Safe.

Debug mode is in effect when NDEBUG is not defined. In this case asserts check the input to functions and whether the rounding mode is upwards (with other rounding modes the Moore library violates the usual rules of interval arithmetic.) In this mode the debugger stops the program when an inconsistency is found and the user will know in which point of the code the problem is. The downside of these safeguards is their cost. For instance, if statements and calls to std::fegetround in inner loops can have a noticeable negative effect. Debug mode is the safest one for the library and we suggest it for less experienced users.

The Fast mode is used when NDEBUG is defined and MOORE_IN_SAFE_MODE is not defined. It is appropriate for users secure about the correctness of their code, because it performs little checking, in cases in which we expect the overhead to be minimal. In this mode, if an error is detected then an error handler is called. The default error handler is a function on_error, declared in the file exception.h. The file on_error.cc contains a definition of this function, which writes a message to the standard output stream for errors and terminates the program. The user can replace this error handler by linking his own function on_error.

The Safe mode is used when both NDEBUG and MOORE_IN_SAFE_MODE are defined. In this mode we check for the same issues considered in Debug mode, and pay the same overhead, but instead of using assertions we call the error handler mentioned in the previous paragraph in case an inconsistency is detected. This mode is useful when one is trying to find bugs introduced by the optimizer. Dealing with this kind of bug is difficult because they disappear in Debug builds, but they are a part of life and we provide the Safe mode to help users and ourselves to deal with them.

VII. EXPERIMENTS

In this section we present the results of experiments comparing the Moore library with three other interval arithmetic libraries: Boost Interval [15], Filib [3] and libieeep1788 [9]. In summary, we show that our library is slightly faster than the Boost library, it is significantly faster than the libieeep1788 library, and it is faster than the Filib library in applications which rely only on arithmetic operations, but
TABLE I
NORMALIZED TIMES FOR THE LEBESGUE FUNCTION

|      | Moore | Filib | Boost | P1788 |
|------|-------|-------|-------|-------|
|      | 1     | 3.8   | 1.1   | 268.5 |

the elementary functions (sin, cos, etc.) in Filib are significantly faster than our library, the boost library and the libieeep1788 library. This difference in the evaluation of the elementary functions happens because the Boost, Moore and libieeep1788 libraries use the MPFR library, whereas Filib has its own implementation of the elementary functions, which was optimized for IEEE 754 double precision.

Besides the difference in speed, there is a difference in the accuracy of the elementary functions. When using IEEE754 double precision, due to the way in which argument reduction is performed, the Boost and Filib libraries can lead to errors of order of the square root of the machine precision \(10^{-8}\) in situations in which the Moore library and the libieeep1788 library lead to errors of the order of the machine precision \(10^{-16}\). Therefore, there is a trade off between accuracy and speed between the Moore and the Filib libraries. The accuracy of the elementary functions provided by the Filib library suffices for many applications and it would be a better choice than the Moore library for application in which the use of such functions with accuracy of order \(O(\sqrt{\epsilon})\) would suffice, and such functions would be called often.

The Moore library was implemented to be used in our research, and the experiments presented in this section reflect this. We present timings for situations related to our current research about the stability of barycentric interpolation [16]–[18]. In this research we look for parameters \(w_0, \ldots, w_n\) which minimize the maximum of the Lebesgue function

\[
\mathcal{L}(w; x, t) := \sum_{k=0}^{n} \frac{|w_k|}{|t-x_k|} \left/ \left| \sum_{k=0}^{n} w_k \right| \right.
\]

among all \(t \in [-1, 1]\), for a given vector \(x\) of nodes, and we use interval arithmetic to find such minimizers and validate them.

In our first experiment we measured the time to evaluate the Lebesgue function in (1) for 257 Chebyshev nodes of the second kind [16], with interval eights, at a million points \(t\). We obtained the normalized times in Table I (the time for the Moore library was taken as the unit.) This table indicates that for the arithmetic operations involved in the evaluation of the Lebesgue function (1), the Moore library is more efficient than the Boost, Filib and libieeep1788 libraries. The difference is slight between Moore and Boost (10%), more relevant between Moore and Filib (about 300%) and very significant between Moore and IeeeP1788 (about 25000%).

In our second experiment we considered computation of the roots of functions which use only arithmetic operations, like the Lebesgue function in Equation (1) and its derivatives with respect to its parameters. The data for this experiment was generated with an interval implementation of Newton’s method which can use any one of the four libraries mentioned above. We compared the times for the solution of random polynomial equations, with the polynomials and their derivatives evaluated by Horner’s method. We obtained the times in Figure 1, which corroborate the data in Table I.

Our first two experiments show that the Moore library is competitive for arithmetic operations, but they tell only part of the history about the relative efficiency of the four libraries considered. In order to have a more balanced comparison, in our third and last experiment we compared the times that the four libraries mentioned above take to evaluate of the elementary functions (sin, cos, exp, etc.) using the IEEE 754 double precision arithmetic. The results of this experiment are summarized in Table II below, which shows that in this case the Filib library is far more efficient that the other three alternatives.

Finally, we would like to say that we tried to be
TABLE II

| Function | Moore | Filib | Boost | P1788 |
|----------|-------|-------|-------|-------|
| sin      | 9.36  | 0.07  | 3.75  | 23.00 |
| cos      | 8.69  | 0.07  | 3.56  | 21.74 |
| tan      | 10.86 | 0.06  | 3.94  | 10.44 |
| asin     | 0.06  | 0.02  | 0.06  | 0.10  |
| acos     | 0.06  | 0.02  | 0.06  | 0.10  |
| atan     | 13.24 | 0.06  | 12.34 | 13.29 |
| exp      | 5.55  | 0.07  | 5.34  | 6.23  |
| log      | 5.96  | 0.06  | 5.69  | 6.49  |

fair with all libraries in the comparisons presented in this section. To the best of our knowledge, we used the faster options for each library. For instance, we used the boost library on its unprotected mode, which does not change rounding modes in order to evaluate arithmetical expressions. The code was compiled with gcc 6.2.0 with flag -O3 and NDEBUG defined (the flag -frounding-math should also be used when compiling the Moore library.)

VIII. LIMITATIONS

The Moore library was designed and implemented using a novel feature of the C++ language called concepts [19], and it pays the price for using the bleeding edge of this technology. The main limitations in the library are due to the current state of concepts in C++. For instance, only the latest versions of the gcc compiler support concepts, and today our library cannot be used with other compilers. Concepts are not formally part of C++ yet, and it will take a few years for them to reach their final form and become part of the C++ standard.

Additionally, several decisions regarding the library were made in order to get around bugs in gcc’s implementations of concepts and in the supporting libraries we use, and in order to reduce the compilation time. Our code would certainly be cleaner if we did not care about these practical issues, but without the compromises we took using the library would be more painful.

Another limitation is the need to guard the code by constructing an object of type RaiiRounding. In other words, the code must look like this

RaiiRounding r;
code using the Moore library

A similar requirement is made by the most efficient rounding policy for the boost library, but that library allows users to choose other policies for rounding, although the resulting code is less efficient. Things are different with the Moore library: as the buyers of Henry Ford’s cars in the 1920s, our users can choose any rounding mode as they want, so long as it is upwards. Users wanting to mix code from the Moore library with code requiring rounding to nearest will need to resort to kludges like this one:

```cpp
{    Moore::RaiiRounding r;
do some interval operations
}
back to rounding to nearest
{    Moore::RaiiRounding r;
do more interval operations
}
```

The experiments with elementary functions in Section VII show clearly that their implementation in the Moore library for IEEE double precision must be improved. In this specific case, we plan to replace the MPFR library by our own implementation of the elementary functions, but this replacement will take time, because there are several delicate issues to be considered. Hopefully, we will provide optimized elementary functions for IEEE double precision in the next version of the Moore library, which we expect to release in 2018.

Finally, the current version of the Moore library is the first one and it is quite likely that it contains bugs, although we did a significant effort to test it. Its design can certainly be improved, but we hope that it will evolve with time, better versions of C++ concepts and constructive criticism from users.

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REFERENCES

[1] H. Brönnimann, G. Melquiond, and S. Pion, “The design of the boost interval arithmetic library,” Theoretical Computer Science, vol. 351, no. 1, pp. 111–118, 2006.
[2] W. Hofschuster and W. Krämer, “C-XSC 2.0: A C++ library for extended scientific computing,” Lecture Notes in Computer Science, vol. 2991, pp. 15–35, 2004.
[3] M. Lerch, G. Tischler, and J. W. von Gudenberg, “filib++, a fast interval library supporting containment computations,” ACM Trans. Math. Software, vol. 32, no. 2, pp. 299–324, 2006.
[4] F. Goualard, “Gaol: Not just another interval library,” last accessed September 21, 2016. [Online]. Available: http://www.sourceforge.net/projects/gaol/
[5] O. Heimlich, “GNU octave interval package,” last accessed September 21, 2016. [Online]. Available: http://octave.sourceforge.net/interval/
[6] D. Nadezhin and S. Zhilin, “Interval library: principles, development, and perspectives,” Reliable Computing, vol. 19, pp. 229–247, 2014.
[7] B. Lambov, “Interval arithmetic using SSE-2,” in Reliable Implementation of Real Number Algorithms: Theory and Practice, ser. Lecture Notes in Computer Science, W. L. P. Hertling, C. M. Hoffmann and N. Revol, Eds., vol. 5045. Berlin, Germany: Springer, 2008, pp. 102–113.
[8] F. Rouillier and N. Revol, “Motivations for an arbitrary precision interval arithmetic and the mpfi library,” Reliable Computing, vol. 11, pp. 275–290, 2005.
[9] M. Nehmeier, “libieeep1788: A C++ implementation of the IEEE interval standard P1788,” in Norbert Wiener in the 21st Century (21CW).
[10] O. Knueppel, “PROFIL/BIAS - a fast interval library,” Computing, vol. 53, no. 3–4, pp. 277–287, 1994, last accessed September 21, 2016. [Online]. Available: http://www.ti3.tu-harburg.de/knueppel/profil/
[11] C++ Interval Arithmetic Programming Reference, MC68175/D, Sun Microsystems, 1996, last accessed September 21, 2016. [Online]. Available: http://docs.sun.com/app/docs/doc/819-3696-10
[12] 1788-2015 IEEE Standard for Interval Arithmetic, IEEE Std., 2015, last accessed September 21, 2016. [Online]. Available: https://standards.ieee.org/findstds/standard/1788-2015.html
[13] P1788.1-Standard for Interval Arithmetic (simplified), IEEE Std., 2015, last accessed September 21, 2016. [Online]. Available: https://standards.ieee.org/develop/project/1788.1.html
[14] L. Fousse, G. Hanrot, V. Lefèvre, P. Pélicier, and P. Zimmermann, “MPFR: a multiple-precision binary floating-point library with correct rounding,” ACM Trans. Math. Softw., vol. 33, no. 2, 2007, article 13.
[15] H. Brönnimann, G. Melquiond, and S. Pion, “The boost interval arithmetic library,” last accessed September 21, 2016. [Online]. Available: https://hal.inria.fr/inria-00348711/file/rnc.pdf
[16] W. F. Mascarenhas, “The stability of barycentric interpolation at the chebyshev points of the second kind,” Numer. Math., vol. 128, pp. 265–300, 2014.
[17] W. F. Mascarenhas and A. P. de Camargo, “The effects of rounding errors in the nodes on barycentric interpolation,” Numer. Math., 2016, doi:10.1007/s00211-016-0798-x.
[18] A. P. de Camargo and W. F. Mascarenhas, “The stability of extended floater–hormann interpolants,” Numer. Math., 2016, doi:10.1007/s00211-016-0840-z.
[19] W. Introduction to C++ concepts, last accessed September 21, 2016. [Online]. Available: https://en.wikipedia.org/wiki/Concepts_(C%2B%2B)