On the Expectation of the Norm of Random Matrices with Non-Identically Distributed Entries

Stiene Riemer * Carsten Schütt *

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Abstract

Let $X_{i,j}$, $i,j = 1, \ldots, n$, be independent, not necessarily identically distributed random variables with finite first moments. We give estimates for the expectation of the norm of the random matrix $(X_{i,j})_{i,j=1}^n$. We improve a result by R. Latala.

Keywords: Random Matrix, Largest Singular Value, Orlicz Norm

1 Introduction and Notation

We study the order of magnitude of the expectation of the largest singular value, i.e. the norm of random matrices with independent entries

$$E \left( \left\| (a_{i,j} g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \right),$$

where $a_{i,j} \in \mathbb{R}$, $i,j = 1, \ldots, n$, $g_{i,j}$, $i,j = 1, \ldots, n$, are standard Gaussian random variables and $\left\| \cdot \right\|_{2 \to 2}$ is the operator norm on $\ell_2^n$. There are two cases with a complete answer. Chevet [2] showed for matrices satisfying $a_{i,j} = a_i b_j$ that the expectation is proportional to

$$\left\| a \right\|_2 \left\| b \right\|_{\infty} + \left\| a \right\|_{\infty} \left\| b \right\|_2,$$

where $\left\| a \right\|_2$ denotes the Euclidean of $a = (a_1, \ldots, a_n)$ and $\left\| a \right\|_{\infty} = \max_{1 \leq i \leq n} |a_i|$.

*Christian-Albrechts-Universität, Mathematisches Seminar, Kiel, Germany, email: lastname@math.uni-kiel.de.
For diagonal matrices with diagonal elements $d_1, \ldots, d_n$ we have that the expectation of the norm is of the order the Orlicz norm $\|\langle d_1, \ldots, d_n \rangle\|_{M}$ where the Orlicz function is given by $M(s) = \sqrt{2/\pi} \int_{0}^{s} e^{-t^2} dt$ [3]. This Orlicz norm is up to a logarithm of $n$ equal to to the norm $\max_{1 \leq i \leq n} |d_i|$. These two cases are of very different structure and seem to present essentially what might occur concerning the structure of matrices. This leads us to conjecture that the expectation for arbitrary matrices is up to a logarithmic factor equal to

$$\max_{i=1, \ldots, n} \| (a_{i,j})_{j=1}^{n} \|_{2} + \max_{j=1, \ldots, n} \| (a_{i,j})_{i=1}^{n} \|_{2}. \quad (1)$$

Latała [4] showed for arbitrary matrices

$$\mathbb{E} \left( \| (a_{i,j}g_{i,j})_{i,j=1}^{n} \|_{2 \to 2} \right) \lesssim \max_{i=1, \ldots, n} \| (a_{i,j})_{j=1}^{n} \|_{2} + \max_{j=1, \ldots, n} \| (a_{i,j})_{i=1}^{n} \|_{2} + \max_{i=1, \ldots, n} \| (a_{i,j})_{i,j=1}^{n} \|_{2} .$$

Seginer [11] showed for any $n \times m$ random matrix $(X_{ij})_{i,j=1}^{n,m}$ of independent identically distributed random variables

$$\mathbb{E} \left( \| (X_{ij})_{i,j=1}^{n,m} \|_{2 \to 2} \right) \leq c \left( \mathbb{E} \max_{1 \leq i \leq n} \| (X_{ij})_{j=1}^{m} \|_{2} + \mathbb{E} \max_{1 \leq i \leq n} \| (X_{ij})_{j=1}^{m} \|_{2} \right).$$

The largest singular value was first investigated by [12, 13]. The behavior of the smallest singular value has been determined in [1] [6] [7].

**Theorem 1.1.** There is a constant $c > 0$ such that for all $a_{i,j} \in \mathbb{R}$, $i, j = 1, \ldots, n$, and all independent standard Gaussian random variables $g_{i,j}$, $i, j = 1, \ldots, n$,

$$\mathbb{E} \left( \| (a_{i,j}g_{i,j})_{i,j=1}^{n} \|_{2 \to 2} \right) \leq c \left( \ln \left( e \left( \frac{\| (a_{i,j})_{i,j=1}^{n} \|_{1}}{\| (a_{i,j})_{i,j=1}^{n} \|_{\infty}} \right) \right) \mathbb{E} \left( \max_{1 \leq i \leq n} \| (a_{i,j}g_{i,j})_{j=1}^{n} \|_{2} \right) + \mathbb{E} \left( \max_{j=1, \ldots, n} \| (a_{i,j}g_{i,j})_{i=1}^{n} \|_{2} \right) \right).$$

In the same way we prove Theorem 1.1 we can show the similar formula

$$\mathbb{E} \left( \| (a_{i,j}g_{i,j})_{i,j=1}^{n} \|_{2 \to 2} \right) \leq c \left( \ln \left( e \left( \frac{\| (a_{i,j})_{i,j=1}^{n} \|_{1}}{\| (a_{i,j})_{i,j=1}^{n} \|_{\infty}} \right) \right) \left( \max_{1 \leq i \leq n} \| (a_{i,j})_{j=1}^{n} \|_{2} + \max_{j=1, \ldots, n} \| (a_{i,j})_{i=1}^{n} \|_{2} \right) \right).$$

This inequality is generalized to arbitrary random variables as in [3].
Theorem 1.2. Let $X_{i,j}$, $i, j = 1, ..., n$, be independent, mean zero random variables. Then

$$E \left( \left\| \left( X_{i,j} \right)_{i,j=1}^{n} \right\|_{2 \rightarrow 2} \right) \leq c \left( \ln(n) \right)^{\frac{3}{2}} \left( E \max_{i=1,...,n} \left\| (X_{i,j})_{j=1}^{n} \right\|_{2} + E \max_{j=1,...,n} \left\| (X_{i,j})_{i=1}^{n} \right\|_{2} \right).$$

Since $E \left( \left\| (a_{i,j} g_{i,j})_{i,j=1}^{n} \right\|_{2 \rightarrow 2} \right)$ is up to a logarithmic factor equal to (1) we investigate better estimate from below. On the other hand,

$$E \left( \max_{i=1,...,n} \left\| (a_{i,j} g_{i,j})_{j=1}^{n} \right\|_{2} \right) + E \left( \max_{j=1,...,n} \left\| (a_{i,j} g_{i,j})_{i=1}^{n} \right\|_{2} \right) \tag{2}$$

is obviously smaller than $E \left( \left\| (a_{i,j} g_{i,j})_{i,j=1}^{n} \right\|_{2 \rightarrow 2} \right)$. We show that the expression (2) is equivalent to the Musielak-Orlicz norm of the vector $(1, \ldots, 1)$, where the Orlicz functions are given through the coefficients $a_{i,j}$, $i, j = 1, \ldots, n$. Our formula (Theorem 3.1) enables us to estimate from below the expectation of the operator norm in many cases efficiently.

Moreover, we do not know of any matrix where the expectation of the norm is not of the same order as (2).

A convex function $M : [0, \infty) \rightarrow [0, \infty)$ with $M(0) = 0$ is called an Orlicz function [8]. Let $M$ be an Orlicz function and $x \in \mathbb{R}^{n}$ then the Orlicz norm of $x$, $\|x\|_{M}$, is defined by

$$\|x\|_{M} = \inf \left\{ t > 0 \left| \sum_{i=1}^{n} M \left( \frac{|x_i|}{t} \right) \leq 1 \right\}. \tag{3}$$

We say that two Orlicz functions $M$ and $N$ are equivalent ($M \sim N$) if there are strictly positive constants $c_1$ and $c_2$ such that for all $s \geq 0$

$$M(c_1 s) \leq N(s) \leq M(c_2 s).$$

If two Orlicz functions are equivalent, so are their norms: For all $x \in \mathbb{R}^{n}$

$$c_1 \|x\|_{M} \leq \|x\|_{N} \leq c_2 \|x\|_{M}.$$

In addition, let $M_i, i = 1, \ldots, n$, be Orlicz functions and let $x \in \mathbb{R}^{n}$ then the Musielak-Orlicz norm of $x$, $\|x\|_{(M_i)}$, is defined by

$$\|x\|_{(M_i)} = \inf \left\{ t > 0 \left| \sum_{i=1}^{n} M_i \left( \frac{|x_i|}{t} \right) \leq 1 \right\}. \tag{4}$$
2 The upper estimate

In this section we are going to prove the upper estimate. We require the following known lemma. In a more general form see e.g. ([10], Lemma 10).

**Lemma 2.1.** Let \( x^{(l)} = \frac{1}{\sqrt{l}} (1, \ldots, 1, 0, \ldots, 0) \), \( l = 1, \ldots, n \), and let \( B_T \) be the convex hull of \( (\varepsilon_1 x^{(l)}_{\pi(1)}, \ldots, \varepsilon_n x^{(l)}_{\pi(n)}) \), where \( \varepsilon_i = \pm 1, i = 1, \ldots, n \), and \( \pi \) denote permutations of \( \{1, \ldots, n\} \). Let \( \| \cdot \|_T \) be the norm on \( \mathbb{R}^n \) whose unit ball is \( B_T \). Then, for all \( x \in \mathbb{R}^n \)

\[
\| x \|_2 \leq \| x \|_T \leq \sqrt{\ln(en)} \| x \|_2.
\]

**Proof.** Let \( x \in \mathbb{R}^n \). Then \( x_1^*, \ldots, x_n^* \) denotes the decreasing rearrangement of the numbers \( |x_1|, \ldots, |x_n| \). Let \( a_k = \sqrt{k} - \sqrt{k-1} \) for \( k = 1, \ldots, n \). Then, for all \( x \in \mathbb{R}^n \)

\[
\| x \|_T = \sum_{k=1}^{n} x_k^* (\sqrt{k} - \sqrt{k-1}).
\]

Since \( \sqrt{k} - \sqrt{k-1} \leq \frac{1}{\sqrt{k}} \)

\[
\| x \|_T \leq \left( \sum_{k=1}^{n} (\sqrt{k} - \sqrt{k-1})^2 \right)^{\frac{1}{2}} \| x \|_2 \leq \left( \sum_{k=1}^{n} \frac{1}{k} \right)^{\frac{1}{2}} \| x \|_2 \leq \sqrt{\ln(en)} \| x \|_2.
\]

We denote

\[
S_{T}^{n-1} = \left\{ x = (x_1, \ldots, x_n) \in S^{n-1} \mid \exists i = 1, \ldots, n \left| \left\{ j = 1, \ldots, n \mid x_j = \pm \frac{1}{\sqrt{i}} \right\} = \{i\} \right. \right\}.
\]

Then by our previous lemma we have

\[
\| A \|_{2 \to 2} = \sup_{x \in S_{T}^{n-1}} \| Ax \|_2 \leq \sqrt{\ln(en)} \sup_{x \in S_{T}^{n-1}} \| Ax \|_2.
\] (3)

We use now the concentration of sums of independent gaussian random variables \( X = \sum_{i=1}^{n} g_i z_i \) in a Banach space ([5], Theorem 4.7): For all \( t > 0 \)

\[
P\{\| X \| - \mathbb{E}\| X \| \geq t\} \leq 2 \exp(-Kt^2/\sigma(X)^2),
\] (4)

where \( K = \frac{2}{\pi^2} \) and

\[
\sigma(X) = \sup_{\| \xi \| = 1} \left( \sum_{i=1}^{n} |\xi(z_i)|^2 \right)^{\frac{1}{2}}.
\] (5)

The following lemma is an immediate consequence.

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Lemma 2.2. For all $i, j = 1, \ldots, n$ let $a_{i,j} \in \mathbb{R}$, let $g_{i,j}$ be independent standard Gaussians, $G = (a_{i,j}g_{i,j})_{i,j=1}^n$ and let $x \in B_2^n$. For all $\beta \geq 1$ and all $x$ with $\max_{i=1,\ldots,n} \left( \sum_{j=1}^n a_{i,j}^2 x_j^2 \right) > 0$ we have

$$\mathbb{P} \left( \|Gx\|_2 > \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) + \mathbb{E} \left( \max_{j=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) \right) \right)$$

$$\leq 2 \exp \left( -K \left( \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) + \mathbb{E} \left( \max_{j=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) \right) - \left( \sum_{j=1}^n a_{i,j}^2 x_j^2 \right) \right)^2 \right).$$

where $K$ is the constant from [3].

Please note that

$$\|Gx\|_2 \leq \left( \sum_{i,j=1}^n a_{i,j}^2 x_j^2 \right)^{\frac{1}{2}} \quad \sigma(Gx) = \max_{i=1,\ldots,n} \left( \sum_{j=1}^n a_{i,j}^2 x_j^2 \right)^{\frac{1}{2}}.$$

Proposition 2.3. For all $i, j = 1, \ldots, n$ let $a_{i,j} \in \mathbb{R}$, let $g_{i,j}$ be independent standard Gaussian random variables and let $G = (a_{i,j}g_{i,j})_{i,j=1}^n$. For all $\beta$ with $\beta \geq \sqrt{n} / 2$

$$\mathbb{P} \left( \|G\|_{2 \to 2} > \beta \ln(n) \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) + \max_{j=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) \right)$$

$$\leq 2 \sum_{i=1}^n \exp \left( \ln(2n) - K \frac{2\beta^2}{\pi} \right),$$

where $C$ is an absolute constant. Furthermore, we get for $\beta$ such that $K \frac{2\beta^2}{\pi} = 3 \ln(2n)$ and $K = \frac{2}{\pi}$

$$\mathbb{P} \left( \|G\|_{2 \to 2} > \sqrt{\frac{3\pi^3}{4}} \ln(en) \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) + \max_{j=1,\ldots,n} \|a_{i,j}g_{i,j}\|_2 \right) \right)$$

$$\leq \frac{1}{n^2}.$$

Proof. We shall apply Lemma 2.2. We may assume that $\max_{i=1,\ldots,n} \left( \sum_{j=1}^n a_{i,j}^2 x_j^2 \right) > 0$. By (3)

$$\|G\|_{2 \to 2} \leq \sqrt{\ln(en)} \sup_{x \in S_+^{n-1}} \|Gx\|_2.$$
Therefore, for \( \beta \in \mathbb{R}_{>0} \), we have
\[
\mathbb{P} \left( \|G\|_{2 \to 2} > \beta \sqrt{\ln(en)} \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right) \\
\leq \mathbb{P} \left( \sup_{x \in S^{n-1}} \|Gx\|_2 > \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right).
\]

For all \( l = 1, \ldots, n \) let \( M_l \) be the set of \( x^{(l)} \in S^{n-1} \), such that \( x_j^{(l)} \in \{0, \pm \frac{1}{\sqrt{n}}\} \) for all \( j = 1, \ldots, n \). Now we apply Lemma 2.2 and get
\[
\mathbb{P} \left( \|G\|_{2 \to 2} > \beta \sqrt{\ln(en)} \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right) \\
\leq \mathbb{P} \left( \sup_{x \in S^{n-1}} \|Gx\|_2 > \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right) \\
\leq \sum_{l=1}^{n} \sum_{x^{(l)} \in M_l} \mathbb{P} \left( \|Gx^{(l)}\|_2 > \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right) \\
\leq 2 \sum_{l=1}^{n} \sum_{x^{(l)} \in M_l} \exp \left( -K \cdot l \right) \left( \beta \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{ij}g_{i,j})_{i=1}^{n} \right\|_2 \right) \right) \right) \\
\cdot \left( \max_{i=1,\ldots,n} \left( \sum_{j \in \{k : x_k^{(l)} \neq 0\}} a_{ij}^2 \right) \right)^{-\frac{1}{2}} \\
\cdot \left( \sqrt{\sum_{i=1}^{n} \sum_{j \in \{k : x_k^{(l)} \neq 0\}} a_{ij}^2} \right)^{-1} \\
\cdot \left( \max_{i=1,\ldots,n} \left( \sum_{j \in \{k : x_k^{(l)} \neq 0\}} a_{ij}^2 \right)^{-1} \right)^{-\frac{1}{2}} \left( \sum_{i=1}^{n} \sum_{j \in \{k : x_k^{(l)} \neq 0\}} a_{ij}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^{n} \sum_{j \in \{k : x_k^{(l)} \neq 0\}} a_{ij}^2 \right)^{-\frac{1}{2}}\]
we have for all $\beta$ with $\beta \geq \frac{\pi}{2}$

$$\beta \mathbb{E} \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})\|_2 - \frac{1}{\sqrt{l}} \left( \sum_{i=1}^{n} \sum_{j \in \{k|x_k^l| \neq 0\}} a_{ij}^2 \right)^{\frac{1}{2}} \geq 0.$$  

Thus

$$\mathbb{P}\left( \|G\|_{2 \rightarrow 2} > \beta \sqrt{\ln(en)} \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|(a_{ij}g_{ij})_{j=1}^{n}\|_2 + \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})_{i=1}^{n}\|_2 \right) \right) \right)$$

$$\leq 2 \sum_{l=1}^{n} \sum_{x(l) \in M_l} \exp \left( -K \cdot l \right) \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|(a_{ij}g_{ij})_{j=1}^{n}\|_2 + \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})_{i=1}^{n}\|_2 \right) \right)^{\frac{1}{2}}.$$  

Again, by (2) we have for all $\beta$ with $\beta \geq \sqrt{\frac{\pi}{2}}$

$$\mathbb{P}\left( \|G\|_{2 \rightarrow 2} > \beta \sqrt{\ln(en)} \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|(a_{ij}g_{ij})_{j=1}^{n}\|_2 + \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})_{i=1}^{n}\|_2 \right) \right) \right)$$

$$\leq 2 \sum_{l=1}^{n} \sum_{x(l) \in M_l} \exp \left( -K \cdot l \right) \leq 2 \sum_{l=1}^{n} 2 \cdot 2 \cdot \exp \left( -K \cdot l \right)$$

$$= 2 \sum_{l=1}^{n} \exp \left( l \ln(2n) - K \cdot l \right).$$

We choose $\beta$ such that $3 \ln(2n) = K \cdot \frac{n^2}{4}$. Then

$$\mathbb{P}\left( \|G\|_{2 \rightarrow 2} > \sqrt{3 \ln(2n)} \ln(n) \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|(a_{ij}g_{ij})_{j=1}^{n}\|_2 + \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})_{i=1}^{n}\|_2 \right) \right) \right)$$

$$\leq 2 \sum_{l=1}^{n} \exp \left( l \ln(2n) - 3 \ln(2n) \right) = 2 \sum_{l=1}^{n} \exp \left( -2l \ln(2n) \right) = 2 \sum_{l=1}^{n} \left( \frac{1}{4n^2} \right)^{l}$$

$$= 2 \left( 1 - \frac{\left( \frac{1}{4n^2} \right)^{n+1}}{1 - \frac{1}{4n^2}} \right) - 1 = \frac{1 - \left( \frac{1}{4n^2} \right)^{n}}{4n^2} \leq \frac{1}{n^2}. $$

Proposition 2.4. Let $a_{ij} \in \mathbb{R}$, $i,j = 1,\ldots,n$, and $g_{i,j}$, $i,j = 1,\ldots,n$, be independent standard Gaussian random variables, then

$$\mathbb{E} \left( \|(a_{ij}g_{ij})_{i=1}^{n}\|_{2 \rightarrow 2} \right)$$

$$\leq \left( 1 + \sqrt{\frac{3\pi^3}{4} \ln(en)} \right) \left( \mathbb{E} \left( \max_{i=1,\ldots,n} \|(a_{ij}g_{ij})_{j=1}^{n}\|_2 + \max_{j=1,\ldots,n} \|(a_{ij}g_{ij})_{i=1}^{n}\|_2 \right) \right).$
Proof. We divide the estimate of $\mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \right)$ into two parts. Let $M$ be set of all points with

$$
\left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} 
\leq \sqrt{\frac{3\pi^3}{4}} \ln(en) \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{i=1}^n \right\|_2 \right).
$$

Clearly,

$$
\mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \chi_M \right) 
\leq \sqrt{\frac{3\pi^3}{4}} \ln(en) \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{i=1}^n \right\|_2 \right).
$$

Furthermore, by Cauchy-Schwarz inequality and Proposition 2.3 we get

$$
\mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \chi_{M^c} \right) 
\leq \sqrt{\mathbb{P}(M^c)} \left( \mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2}^2 \right) \right)^{\frac{1}{2}}
\leq \frac{1}{n} \left( \mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \right) \right)^{\frac{1}{2}} 
\leq \frac{1}{n} \left( \int \sum_{i=1}^n \left( \sum_{j=1}^n |a_{i,j}g_{i,j}| \right)^2 \, d\mathbb{P} \right)^{\frac{1}{2}}
\leq \frac{1}{n} \left( \sum_{i,j=1}^n |a_{i,j}|^2 \right)^{\frac{1}{2}} 
\leq \max_{1 \leq i \leq n} \left( \sum_{j=1}^n |a_{i,j}|^2 \right)^{\frac{1}{2}}.
$$

Besides, we obviously have

$$
\max_{i=1,\ldots,n} \left\| (a_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j})_{i=1}^n \right\|_2 
\leq \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{i=1}^n \right\|_2 \right).
$$

Altogether, this yields

$$
\mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \chi_{M^c} \right) 
\leq \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{i=1}^n \right\|_2 \right).
$$

Summing up, we get

$$
\mathbb{E} \left( \left\| (a_{i,j}g_{i,j})_{i,j=1}^n \right\|_{2 \to 2} \right) 
\leq \left( 1 + \sqrt{\frac{3\pi^3}{4}} \ln(en) \right) \mathbb{E} \left( \max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{i=1}^n \right\|_2 \right).
$$
\[\square\]
Proof. (Theorem 1.1) W.l.o.g. we assume \( a_{i,j} \leq 1, i,j = 1,...,n \), and that there is a coordinate that equals 1. For all \( i,j \) we define
\[
a_{i,j}^k = \begin{cases} \frac{1}{2^k} & \text{if } \frac{1}{2^k} < a_{i,j} \leq \frac{1}{2^{k-1}} \\ 0 & \text{else.} \end{cases}
\]

Let \( G = (a_{i,j}g_{i,j})_{i,j=1}^n \) and \( G^k = (a_{i,j}^kg_{i,j})_{i,j=1}^n \). We denote by \( \phi(k) \) the number of nonzero entries of the matrix \((a_{i,j}^k)_{i,j=1}^n\) and we choose \( \gamma \) such that \( \|(a_{i,j})_{i,j=1}^n\|_1 = 2^\gamma \|(a_{i,j})_{i,j=1}^n\|_\infty \). Thus, we get \( \phi(k) \frac{1}{2^k} = \sum_{i,j=1}^n a_{i,j}^k \leq 2^\gamma \) and therefore \( \phi(k) \leq 2^{k+\gamma} \). Therefore, the non-zero entries of \( G^k \) are contained in a submatrix of size \( 2^{k+\gamma} \times 2^{k+\gamma} \). Taking this into account and applying Proposition 2.4 to \( G^k \)
\[
\mathbb{E} \left\| G^k \right\|_{2 \rightarrow 2} \leq \left( 1 + \sqrt{\frac{3\pi^3}{4} \ln(e^{2^{k+\gamma}})} \right) \mathbb{E} \left( \max_{i=1,...,n} \left\| (a_{i,j}^k g_{i,j})_{j=1}^n \right\|_2 + \max_{j=1,...,n} \left\| (a_{i,j}^k g_{i,j})_{i=1}^n \right\|_2 \right)
\]
\[
\leq 140(k + \gamma) \left( \mathbb{E} \sum_{i,j=1}^n |a_{i,j}^k g_{i,j}|^2 \right)^{\frac{1}{2}} \leq 140(k + \gamma) 2^{\frac{\gamma}{2} - \frac{k}{2}}
\]
Therefore,
\[
\sum_{k \geq 2\gamma} \mathbb{E} \left\| G^k \right\|_{2 \rightarrow 2} \leq 140 \sum_{k \geq 2\gamma} \frac{k + \gamma}{2^k} \leq 280 \sum_{k=1}^\infty \frac{k}{2^k}.
\]
Since one of the coordinates of the matrix is 1
\[
\mathbb{E} \left\| G^1 \right\|_{2 \rightarrow 2} \geq \int_{-\infty}^\infty |g| \, dt = \sqrt{\frac{2}{\pi}}.
\]
Therefore, there is a constant \( c \) such that
\[
\mathbb{E} \left\| G \right\|_{2 \rightarrow 2} \leq 2 \mathbb{E} \left\| \sum_{k \leq 2\gamma} G^k \right\|_{2 \rightarrow 2} + 2 \sum_{k > 2\gamma} \left\| G^k \right\|_{2 \rightarrow 2} \leq c \mathbb{E} \left\| \sum_{k \leq 2\gamma} G^k \right\|_{2 \rightarrow 2}.
\]
The matrix \( \sum_{k \leq 2\gamma} G^k \) has at most
\[
\sum_{k \leq 2\gamma} \phi(k) \leq \sum_{k \leq 2\gamma} 2^{\gamma+k} \leq 2^{3\gamma+1} \leq \left( \frac{\|(a_{i,j})_{i,j=1}^n\|_1}{\|(a_{i,j})_{i,j=1}^n\|_\infty} \right)^4
\]
entries that are different from 0. Therefore, all nonzero entries of \( \sum_{k \leq 2\gamma} G^k \) are contained in a square submatrix having less than \( 7 \) rows and columns.
We may apply Proposition 2.4 and get with a proper constant $c$

\[
E (\|G\|_{2 \to 2}) \leq c \left( 1 + \sqrt{\frac{3\pi^3}{4}} \ln \left( e \left( \frac{1}{\| (a_{i,j})_{i,j=1}^n \|_\infty} \right)^4 \right) \right) \times
\]

\[
E \left( \max_{i=1, \ldots, n} \left( \sum_{k \leq \gamma} (a_{i,j}^r g_{i,j})_{j=1}^n \right)_{j=1}^\gamma + \max_{j=1, \ldots, n} \left( \sum_{k \leq \gamma} (a_{i,j}^r g_{i,j})_{i=1}^n \right)_{i=1}^\gamma \right).
\]

\[\square\]

3 The lower estimate

Theorem 3.1. For all $i, j = 1, \ldots, n$ let $a_{i,j} \in \mathbb{R}$ and $g_{i,j}$ be independent standard Gaussians. For all $s \in \mathbb{R}_{\geq 0}$ and for all $i = 1, \ldots, n$ let

\[
N_i(s) = \begin{cases}
    s \max_{j=1, \ldots, n} |a_{i,j}| e^{-\frac{1}{\max_{j=1, \ldots, n} a_{i,j}}} & , s < \frac{1}{\| (a_{i,j})_{i=1}^n \|_2} \\
    \max_{i=1, \ldots, n} \frac{|a_{i,j}|}{\| (a_{i,j})_{i=1}^n \|_2} e^{-\frac{1}{\max_{i=1, \ldots, n} a_{i,j}}} + \frac{2}{e} \| (a_{i,j})_{j=1}^n \|_2 \left( s - \frac{1}{\| (a_{i,j})_{j=1}^n \|_2} \right) & , s \geq \frac{1}{\| (a_{i,j})_{i=1}^n \|_2}
\end{cases}
\]

respectively let for all $s \in \mathbb{R}_{\geq 0}$ and for all $j = 1, \ldots, n$

\[
\bar{N}_j(s) = \begin{cases}
    s \max_{i=1, \ldots, n} |a_{i,j}| e^{-\frac{1}{\max_{i=1, \ldots, n} a_{i,j}}} & , s < \frac{1}{\| (a_{i,j})_{j=1}^n \|_2} \\
    \max_{i=1, \ldots, n} \frac{|a_{i,j}|}{\| (a_{i,j})_{i=1}^n \|_2} e^{-\frac{1}{\max_{i=1, \ldots, n} a_{i,j}}} + \frac{2}{e} \| (a_{i,j})_{i=1}^n \|_2 \left( s - \frac{1}{\| (a_{i,j})_{i=1}^n \|_2} \right) & , s \geq \frac{1}{\| (a_{i,j})_{j=1}^n \|_2}
\end{cases}
\]

Then

\[
c_1 \left( \| (1)_{j=1}^n \|_{(N_i)_j} + \| (1)_{i=1}^n \|_{(\bar{N}_j)_i} \right)
\leq E \left( \max_{i=1, \ldots, n} \| (a_{i,j} g_{i,j})_{j=1}^n \|_2 + \max_{j=1, \ldots, n} \| (a_{i,j} g_{i,j})_{i=1}^n \|_2 \right)
\leq c_2 \left( \| (1)_{j=1}^n \|_{(N_i)_j} + \| (1)_{i=1}^n \|_{(\bar{N}_j)_i} \right),
\]

where $c_1$ and $c_2$ are absolute constants.

The following example is an immediate consequence of Theorem 3.1. It covers Toeplitz matrices.
Example 3.2. Let $A$ be an $n \times n$-matrix such that for all $i,=1,\ldots,n$ and $k=1,\ldots,n$

\[
\left(\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}} = \left(\sum_{j=1}^{n}|a_{j,k}|^2\right)^{\frac{1}{2}}
\]

and

\[\max_{1 \leq j \leq n}|a_{i,j}| = \max_{1 \leq j \leq n}|a_{j,k}|\]

Then

\[
E \left(\max_{i=1,\ldots,n} \left\| (a_{i,j}g_{i,j})_{j=1}^{n} \right\|_2 + \max_{j=1,\ldots,n} \left\| (a_{i,j}g_{j,i})_{i=1}^{n} \right\|_2\right)
\]

\[\sim \max \left\{ \left(\sum_{j=1}^{n}|a_{i,j}|^2\right)^{\frac{1}{2}}, \sqrt{\ln n} \max_{1 \leq j \leq n}|a_{i,j}| \right\}
\]

We associate to a random variable $X$ an Orlicz function $M$ by

\[
M(s) = \int_{0}^{s} \int_{\frac{1}{t} \leq |X|} |X| dP dt.
\] (8)

We have

\[
M(s) = \int_{0}^{s} \int_{\frac{1}{t} \leq |X|} |X| dP dt
\]

\[= \int_{0}^{s} \left( \int_{t}^{\frac{1}{t}} dP \left( |X| \geq \frac{1}{t} \right) + \int_{\frac{1}{t}}^{\infty} dP \left( |X| \geq u \right) du \right) dt.
\] (9)

Lemma 3.3. There are strictly positive constants $c_1$ and $c_2$ such that for all $n \in \mathbb{N}$, all independent random variables $X_1, \ldots, X_n$ with finite first moments and for all $x \in \mathbb{R}^n$

\[c_1 \|x\|_{(M_i)_n} \leq E \max_{1 \leq i \leq n} |x_i X_i| \leq c_2 \|x\|_{(M_i)_n},
\]

where $M_1, \ldots, M_n$ are the Orlicz functions that are associated to the random variables $X_1, \ldots, X_n$ (8).

Lemma 3.3 is a generalization of the same result for identically distributed random variables [3]. It can be generalized from the $\ell_\infty$-norm to Orlicz norms.

We use the fact [9] that for all $s > 0$

\[
\frac{\sqrt{2\pi}}{(\pi - 1)x + \sqrt{x^2 + 2\pi}} e^{-\frac{1}{2}x^2} \leq \sqrt{\frac{2}{\pi}} \int_{x}^{\infty} e^{-\frac{1}{2}t^2} dt \leq \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}x^2}.
\] (10)
Proof. (Theorem 3.1) We apply Lemma 3.3 to the random variables

\[ X_i = \left( \sum_{j=1}^{n} |a_{i,j}g_{i,j}|^2 \right)^{\frac{1}{2}} \quad i = 1, \ldots, n. \]

Now, it is enough to show that \( M_i \sim N_i \) for all \( i = 1, \ldots, n \). We have two cases.

We consider first \( s < \frac{1}{t} \left( \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \). There are constants \( c_1, c_2 > 0 \) such that for all \( u > 2\mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \)

\[ \exp \left( -c_1 \frac{u^2}{\max_{j=1,\ldots,n} a_{i,j}^2} \right) \leq \mathbb{P} \left( \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \geq u \right) \leq \exp \left( -c_2 \frac{u^2}{\max_{j=1,\ldots,n} a_{i,j}^2} \right). \]

The right-hand side inequality follows from (4). The left-hand side inequality follows from

\[ \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \geq a_{i,1}^2 g_{i,1}^2. \]

Since \( \frac{1}{t} > 2\mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \), we can apply (11). Therefore,

\[
M_i(s) = \int_{0}^{s} \left\{ \frac{1}{t} \mathbb{P} \left( \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \geq t \right) + \int_{t}^{\infty} \mathbb{P} \left( \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \geq u \right) \, du \right\} \, dt
\]

\[
\leq \int_{0}^{s} \left\{ \frac{1}{t} \exp \left( -\frac{c_2}{t^2 \max_{j=1,\ldots,n} a_{i,j}^2} \right) + \int_{t}^{\infty} \exp \left( -c_2 \frac{u^2}{\max_{j=1,\ldots,n} a_{i,j}^2} \right) \, du \right\} \, dt.
\]

By (10)

\[
M_i(s) \leq \int_{0}^{s} \left\{ \frac{1}{t} \exp \left( -\frac{c_2}{t^2 \max_{j=1,\ldots,n} a_{i,j}^2} \right) + t \max_{j=1,\ldots,n} a_{i,j} \exp \left( -\frac{c_2}{t^2 \max_{j=1,\ldots,n} a_{i,j}^2} \right) \right\} \, dt.
\]

Since \( \frac{1}{t} > 2\mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \geq \sqrt{\frac{2}{\pi}} \| (a_{i,j})_{j=1}^{n} \|_{2} \), we get

\[
\frac{1}{t} + t \max_{j=1,\ldots,n} a_{i,j}^2 \leq \frac{1}{t} + \sqrt{\frac{\pi}{2}} \frac{1}{\| (a_{i,j})_{j=1}^{n} \|_{2}} \| (a_{i,j})_{j=1}^{n} \|_{2}^2 \leq \frac{3}{t}.
\]
Thus,

\[
\frac{1}{t} \leq \frac{1}{t} + t \max_{j=1,\ldots,n} a_{i,j}^2 \leq \frac{3}{t}
\]

Altogether, we get

\[
M_i(s) \leq \int_0^s \frac{3}{t} \exp \left( -\frac{c_2^2}{t^2 \max_{j=1,\ldots,n} a_{i,j}^2} \right) \, dt = \int_{\frac{s}{3}}^\infty \frac{3}{u} \exp \left( -\frac{c_2^2 u^2}{\max_{j=1,\ldots,n} a_{i,j}^2} \right) \, du.
\]

Passing to a new constant \( c_2 \) and using (10) we get for all \( s \) with \( 0 \leq s < \frac{1}{2} \left( \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \)

\[
M_i(s) \leq 3 \int_{\frac{s}{2}}^\infty \exp \left( -\frac{c_2^2 u^2}{\max_{j=1,\ldots,n} a_{i,j}^2} \right) \, du \leq \frac{s}{c_2} \left( \max_{j=1,\ldots,n} |a_{i,j}| \right) \exp \left( -\frac{c_2^2}{s^2 \max_{j=1,\ldots,n} a_{i,j}^2} \right) \cdot
\]

From this and the definition of \( N_i \) we get that there is a constant \( c \) such that for all \( s \) with \( 0 \leq s < \frac{1}{2} \left( \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \)

\[
M_i(s) \leq N_i(cs).
\]

Indeed, the inequality follows immediately from (12) provided that \( \frac{1}{c_2} \leq \frac{1}{2} \left( \sum_{j=1}^n |a_{i,j}|^2 \right)^{-\frac{1}{2}} \). If \( \frac{c_2^2}{2} \left( \sum_{j=1}^n |a_{i,j}|^2 \right)^{-\frac{1}{2}} \leq s \leq \frac{1}{2} \left( \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \) then, by (12) and \( \sqrt{\frac{2\pi}{\max_{1 \leq j \leq n} a_{i,j}}} \leq \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \),

\[
M_i(s) \leq \frac{2}{c_2} \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \exp \left( -\frac{c_2^2 \left( \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^2}{4 \max_{j=1,\ldots,n} a_{i,j}^2} \right) \leq \frac{\sqrt{2\pi}}{c_2}.
\]

Moreover,

\[
\frac{\sqrt{2\pi}}{c_2} \leq N_i \left( \left( \frac{\sqrt{2\pi}}{c_2} + 1 \right) \left\| (a_{i,j})_{j=1}^n \right\|_2^{-1} \right).
\]

Therefore, with a universal constant \( c \) the inequality \( M_i(s) \leq N_i(cs) \) also holds for those values of \( s \). The inverse inequality is treated in the same way.

Now we consider \( s \) with \( s \geq \frac{1}{2} \left( \mathbb{E} \left( \sum_{j=1}^n a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \) and denote \( \alpha = \)
Now, we give a lower estimate. By (8), for all
\[
\int 1_{\alpha(s) \leq s(1)} dt = \int 1_{\alpha(s) \leq s(1)} dt + \int_{1/4}^{s} \mathbb{P} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \geq u \right) du dt
\]
\[
= \int_{1/4}^{s} \mathbb{P} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \geq u \right) du dt + \int_{1/4}^{s} \mathbb{P} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \geq u \right) du dt.
\]
By (12) the first summand is of the order
\[
\frac{\max_{j=1, \ldots, n} \left| a_{i,j} \right|}{\mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2}} \exp \left( -\left( \frac{\mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2}}{\max_{j=1, \ldots, n} a_{i,j}^2} \right)^2 \right). \]
We estimate the second summand. The second summand is less than or equal to
\[
\int_{1/4}^{s} \left\{ \frac{1}{t} + \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} \right\} dt \leq \int_{1/4}^{s} 3 \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} dt \leq 3 \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} s.
\]
Therefore, with a universal constant c we have for all s with \( s \geq \frac{1}{2} \left( \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} \right)^{-1}
\[
M_i(s) \leq (c - 1)s \left( \sum_{j=1}^{n} \left| a_{i,j} \right|^2 \right)^{1/2} \leq cs \left( \sum_{j=1}^{n} \left| a_{i,j} \right|^2 \right)^{1/2} - 1 \leq N_i(cs).
\]
Now, we give a lower estimate. By (8), for all s with \( s \geq \frac{2}{n}
\[
M_i(s) \geq \int_{1/4}^{s} \int_{1/4}^{s} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} d\mathbb{P} dt \geq \int_{1/4}^{s} \int_{1/4}^{s} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{1/2} d\mathbb{P} dt.
\]
By the definition of $\alpha$

$$M_i(s) \geq \frac{1}{2} \int_{\alpha}^{s} \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \, dt$$

$$= \frac{1}{2} \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \left( s - 2 \left( \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \right)^{-1} \right)$$

$$= \frac{1}{2} \mathbb{E} \left( \sum_{j=1}^{n} a_{i,j}^2 g_{i,j}^2 \right)^{\frac{1}{2}} \left( s - 1 \right).$$

The rest is done as in the case of the upper estimate.  

\[ \square \]

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