WEITZENBÖCK DERIVATIONS
OF FREE METABELIAN LIE ALGEBRAS

RUMEN DANGOVSKI, VESSELIN DRENSKY, AND ŞEHMUS FINDIK

Abstract. A nonzero locally nilpotent linear derivation \( \delta \) of the polynomial algebra \( K[X_d] = K[x_1, \ldots, x_d] \) in several variables over a field \( K \) of characteristic 0 is called a Weitzenböck derivation. The classical theorem of Weitzenböck states that the algebra of constants \( K[X_d]^\delta \) (which coincides with the algebra of invariants of a single unipotent transformation) is finitely generated. Similarly one may consider the algebra of constants of a locally nilpotent linear derivation \( \delta \) of a finitely generated (not necessarily commutative or associative) algebra which is relatively free in a variety of algebras over \( K \). Now the algebra of constants is usually not finitely generated. Except for some trivial cases this holds for the algebra of constants \( (L_d/L_d')^\delta \) of the free metabelian Lie algebra \( L_d/L_d' \) with \( d \) generators. We show that the vector space of the constants \( (L_d'/L_d'')^\delta \) in the commutator ideal \( L_d'/L_d'' \) is a finitely generated \( K[X_d]^\delta \)-module. For small \( d \), we calculate the Hilbert series of \( (L_d'/L_d'')^\delta \) and find the generators of the \( K[X_d]^\delta \)-module \( (L_d'/L_d'')^\delta \). This gives also an (infinite) set of generators of the algebra \( (L_d'/L_d'')^\delta \).

1. Introduction

A linear operator \( \delta \) of an algebra \( R \) over a field \( K \) is a derivation if \( \delta(uv) = \delta(u)v + u\delta(v) \) for every \( u, v \in R \). In this paper the base field \( K \) will be of characteristic 0. We fix also an integer \( d \geq 2 \) and a set of variables \( X_d = \{x_1, \ldots, x_d\} \). Let \( K[X_d] = K[x_1, \ldots, x_d] \) be the polynomial algebra in \( d \) variables. Every mapping \( \delta : X_d \to K[X_d] \) can be extended in a unique way to a derivation of \( K[X_d] \) which we shall denote by the same symbol \( \delta \). In our considerations \( \delta \) will act as a nonzero nilpotent linear operator of the vector space \( KX_d \) with basis \( X_d \). Such derivations are called Weitzenböck. The Jordan normal form \( J(\delta) \) of the matrix of \( \delta \)

\[
J(\delta) = \begin{pmatrix}
J_1 & 0 & \cdots & 0 \\
0 & J_2 & \cdots & 0 \\
0 & 0 & \ddots & \vdots \\
0 & 0 & \cdots & J_s
\end{pmatrix}
\]

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consists of Jordan cells with zero diagonals

\[ J_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i = 1, \ldots, s. \]

Hence for each dimension \( d \) there is only a finite number of essentially different Weitzenböck derivations. Up to a linear change of the coordinates, the Weitzenböck derivations \( \delta \) are in a one-to-one correspondence with the partition \((p_1+1, \ldots, p_s+1)\) of \( d \), where \( p_1 \geq \cdots \geq p_s \geq 0 \), \((p_1+1)+\cdots+(p_s+1) = d\), and the correspondence is given in terms of the size \((p_1+1) \times (p_1+1)\) of the Jordan cells \( J_i \) of \( \delta \). We shall denote the derivation corresponding to this partition by \( \delta(p_1, \ldots, p_s) \).

Clearly, any Weitzenböck derivation \( \delta \) is locally nilpotent, i.e., for any \( u \in K[X_d] \) there exists an \( n \) such that \( \delta^n(u) = 0 \). The linear operator

\[ \exp \delta = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots \]

acting on \( KX_d \) is unipotent. It is well known that the algebra of constants of \( \delta \)

\[ K[X_d]^\delta = \ker \delta = \{ u \in K[X_d] | \delta(u) = 0 \} \]

coinsides with the algebra of invariants of \( \exp \delta \)

\[ K[X_d]^{\exp \delta} = \{ u \in K[X_d] | \exp \delta(u) = u \}, \]

and the latter coincides also with the algebra of invariants \( K[X_d]^{UT_2(K)} \) of the unitriangular group \( UT_2(K) = \{ \exp(\alpha \delta) | \alpha \in K \} \). This allows to study the algebra \( K[X_d]^\delta \) with methods of classical invariant theory.

The classical theorem of Weitzenböck \([26]\) states that for any Weitzenböck derivation \( \delta \) the algebra of constants \( K[X_d]^\delta \) is finitely generated. See the book by Nowicki \([22]\) for more information on Weitzenböck derivations, including references and examples of explicit sets of generators of the algebra \( K[X_d]^\delta \) for concrete \( \delta \). For computational aspects see also the books by Derksen and Kemper \([2]\) and Sturmfels \([25]\).

The polynomial algebra \( K[X_d] \) is free in the class of all commutative algebras. Similarly, we may consider the relatively free algebra \( F_d(\mathfrak{U}) \) in a variety \( \mathfrak{U} \) of (not necessarily associative) algebras. For a background on varieties of associative and Lie algebras see, respectively, the books by Drensky \([11]\) and Bahturin \([2]\). As in the polynomial case, if \( F_d(\mathfrak{U}) \) is freely generated by the set \( X_d \), then every map \( X_d \to F_d(\mathfrak{U}) \) can be extended to a derivation of \( F_d(\mathfrak{U}) \). Again, we shall call the derivations \( \delta \) which act as nilpotent linear operators of the vector space \( KX_d \) Weitzenböck derivations and shall denote them in the same way \( \delta(p_1, \ldots, p_s) \) as in the polynomial case.

Drensky and Gupta \([14]\) studied Weitzenböck derivations \( \delta \) acting on relatively free associative and Lie algebras. In particular, if the polynomial identities of the variety \( \mathfrak{U} \) of associative algebras follow from the identity \([x_1, x_2] [x_3, x_4] = 0\) which is equivalent to the condition that \( \mathfrak{U} \) contains the algebra \( U_2(K) \) of \( 2 \times 2 \) upper triangular matrices, then the algebra of constants \( F_d(\mathfrak{U})^\delta \) is not finitely generated. If \( U_2(K) \) does not belong to \( \mathfrak{U} \) (which implies that \( \mathfrak{U} \) satisfies some Engel identity
\[ [x_2, x_1, \ldots, x_1] = x_2 \text{ad}^n x_1 = 0, \] a result of Drensky \cite{12} gives that the algebra \( F_d(\Omega) \) is finitely generated.

Although not finitely generated in the general case, the (associative) algebra \( F_d(\Omega) \) has some features typical for finitely generated (commutative) algebras. In particular, the Hilbert (or Poincaré) series of \( F_d(\Omega) \) is a rational function. This follows from results of Belov \cite{4} and Berele \cite{6,7} combined with ideas of classical invariant theory, see Drensky and Genov \cite{13} and Benenati, Boumova, Drensky, Genov, and Koev \cite{5}. Hence, it is interesting to know how far from finitely generated \( F_d(\Omega) \) is.

We consider this problem for free metabelian Lie algebras. Let \( L_d \) be the free Lie algebra with \( X_d \) as a set of free generators. The variety \( \mathfrak{A}_2 \) of metabelian (solvable of length 2) Lie algebras is defined by the polynomial identity \( [x_1, x_2], [x_3, x_4] = 0 \). The free metabelian algebra \( F_d(\mathfrak{A}_2) \) is the relatively free algebra in \( \mathfrak{A}_2 \) and is isomorphic to the factor algebra \( L_d/L'_d = L_d/[[L_d, L_d], [L_d, L_d]] \). We denote its free generators with the same symbols \( x_j \) as the generators of \( L_d \) and \( K[X_d] \). By Drensky and Gupta \cite{14}, if \( \delta \) is a Weitzenb"ock derivation of \( L_d/L''_d \), then the algebra of constants \( (L_d/L''_d)^\delta \) is finitely generated if and only if the Jordan normal form of \( \delta \) is equal to 1.

The commutator ideal \( L'_d/L''_d \) of the algebra \( L_d/L''_d \) has a natural structure of a \( K[X_d] \)-module. Our first result is that its vector subspace \( (L'_d/L''_d)^\delta \) is a Weitzenb"ock derivation of \( K[X_d] \)-module. Freely speaking, this means that the algebra of constants \( (L_d/L''_d)^\delta \) is very close to finitely generated.

Then, using the methods of \cite{5} we give an algorithm how to calculate the Hilbert series of \( (L_d/L''_d)^\delta \) and calculate it for small \( d \).

Let the Jordan form of \( \delta \) contain a \( 1 \times 1 \) cell. Then we may assume that \( \delta \) acts as a nilpotent linear operator on \( KX_{d-1} \) and \( \delta(x_d) = 0 \). It is well known that in the commutative case \( K[X_d] = (K[X_{d-1}]^\delta)[x_d] \) and this reduces the study of the algebra \( K[X_d]^\delta \) to the algebra of constants \( K[X_{d-1}]^\delta \) in the polynomial algebra in \( d - 1 \) variables. Using the methods of \cite{5} again, we establish a similar result for the algebra of constants \( (L_d/L''_d)^\delta \). The result is more complicated than in the polynomial case but we give an algorithm which expresses the generators of the \( K[X_d] \)-module \( (L'_d/L''_d)^\delta \) in terms of the generators of the \( K[X_{d-1}] \)-module \( (L'_{d-1}/L''_{d-1})^\delta \) and the generators of the algebra \( K[X_{d-1}]^\delta \).

Finally, we find the generators of the \( K[X_d]^\delta \)-module \( (L'_d/L''_d)^\delta \) for \( d \leq 4 \) and for \( d = 6, \delta = \delta(1, 1, 1) \). This gives also an explicit (infinite) set of generators of the algebra \( (L_d/L''_d)^\delta \).

2. Finite generation

We assume that all Lie commutators are left normed, e.g.,

\[ [x_1, x_2, x_3] = [[x_1, x_2], x_3] = [x_1, x_2] \text{ad} x_3. \]

It is well known, see \cite{2}, that the metabelian identity implies the identity

\[ [x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_n}] = [x_{j_1}, x_{j_2}, x_{j_3}, \ldots, x_{j_n}], \]
Hence the polynomial algebra $K[X] \otimes RUMEN DANGOVSKI, VESSELIN DRENSKY, ŞEHMUS FINDIK$

where $\sigma$ is an arbitrary permutation of $3, \ldots, k$ and that $L'_d/L''_d$ has a basis consisting of all

$$[x_{i_1}, x_{i_2}, x_{i_3}, \ldots, x_{i_k}], \quad 1 \leq j \leq d, \quad j_1 > j_2 \leq j_3 \leq \cdots \leq j_k.$$  

Hence the polynomial algebra $K[X]$ acts on $L'_d/L''_d$ by the rule

$$uf(x_1, \ldots, x_d) = uf(adx_1, \ldots, adx_d), \quad u \in L'_d/L''_d, f(X_d) \in K[X].$$

Recall the construction of abelian wreath products due to Shmel’kin [24]. Let $A_d$ and $B_d$ be the abelian Lie algebras with bases $\{a_1, \ldots, a_d\}$ and $\{b_1, \ldots, b_d\}$, respectively. Let $C_d$ be the free right $K[X_d]$-module with free generators $a_1, \ldots, a_d$. We give it the structure of a Lie algebra with trivial multiplication. The abelian ideal $\mathfrak{L}$ defined by $\varepsilon$ is a metabelian Lie algebra and every mapping $\delta : L_d/L''_d \to A_d\wr B_d$ can be extended to a homomorphism $\varepsilon : L_d/L''_d \to A_d\wr B_d$. As a special case of the embedding theorem of Shmel’kin [24], the homomorphism $\varepsilon : L_d/L''_d \to A_d\wr B_d$ defined by $\varepsilon(x_j) = a_j + b_j, \ j = 1, \ldots, d$, is a monomorphism. If

$$u = \sum [x_{i_1}, x_{i_2}]f_{i_3}(X_d), \quad f_{i_3}(X_d) \in K[X],$$

then

$$\varepsilon(u) = \sum (a_i x_j - a_j x_i) f_{i_3}(X_d).$$

An element $\sum_{i=1}^d a_i f_i(X_d) \in A_d\wr B_d$ is an image of an element from the commutator ideal $L'_d/L''_d$ if and only if $\sum_{i=1}^d x_i f_i(X_d) = 0$.

If $\delta$ is a Weitzenböck derivation of $L_d/L''_d$, we define an action of $\delta$ on $A_d\wr B_d$ assuming that

$$\delta(a_j) = \sum_{i=1}^d \alpha_{ij} a_i, \quad \delta(b_j) = \sum_{i=1}^d \alpha_{ij} b_i, \quad j = 1, \ldots, d,$$

where $\alpha_{ij} \in K, i, j = 1, \ldots, d,$ and

$$\delta(x_j) = \sum_{i=1}^d \alpha_{ij} x_i, \quad j = 1, \ldots, d.$$  

Obviously, the vector space $C_d^\delta$ of the constants of $\delta$ in the free $K[X_d]$-module $C_d$ is a $K[X_d]^{\delta}$-module. The following lemma is a partial case of [12, Proposition 3].

**Lemma 2.1.** The vector space $C_d^\delta$ is a finitely generated $K[X_d]^{\delta}$-module.
Clearly, if \( u \in L_d/L_d' \), then \( \varepsilon(\delta(u)) = \delta(\varepsilon(u)) \). To simplify the notation we shall omit \( \varepsilon \) and shall think that \( L_d/L_d' \) is a subalgebra of \( A_d \). Since the action of \( K[X_d]/ L_d' / L_d'' \) agrees with its action on \( C_d \), we shall also think that \( L_d'/L_d'' \) is a \( K[X_d] \)-submodule of \( C_d \).

**Theorem 2.2.** Let \( \delta \) be a Weitzenb"ock derivation of the free metabelian Lie algebra \( L_d/L_d' \). Then the vector space \( (L_d'/L_d'')^\delta \) of the constants of \( \delta \) in the commutator ideal \( L_d'/L_d'' \) of \( L_d/L_d'' \) is a finitely generated \( K[X_d]^\delta \)-module.

**Proof.** By Lemma 2.1 the \( K[X_d]^\delta \)-module \( C_d^\delta \) is finitely generated. Since the algebra \( K[X_d]^\delta \) is also finitely generated, all \( K[X_d]^\delta \)-submodules of \( C_d^\delta \), including \( (L_d'/L_d'')^\delta \), are also finitely generated. \( \square \)

### 3. Hilbert Series

Since the base field \( K \) is of characteristic 0, the relatively free algebra \( F_d = F_d(\mathfrak{U}) \) of (not necessarily associative or Lie) algebras is a graded vector space. If \( F_d^{(n)} \) is the homogeneous component of degree \( n \) of \( F_d \), then the Hilbert series of \( F_d \) is the formal power series

\[
H(F_d, z) = \sum_{n \geq 0} \dim F_d^{(n)} z^n.
\]

The algebra \( F_d \) is also multigraded, with a \( \mathbb{Z}^d \)-grading which counts the degree of each variable \( x_j \) in the monomials in \( F_d \). If \( F_d^{(n_1, \ldots, n_d)} \) is the multihomogeneous component of degree \( (n_1, \ldots, n_d) \), then the corresponding Hilbert series of \( F_d \) is

\[
H(F_d, z_1, \ldots, z_d) = \sum_{n_j \geq 0} \dim F_d^{(n_1, \ldots, n_d)} z_1^{n_1} \cdots z_d^{n_d}.
\]

Similarly, if \( \delta \) is a Weitzenb"ock derivation of \( F_d \), the algebra of constants \( F_d^\delta \) is graded and its Hilbert series is

\[
H(F_d^\delta, z) = \sum_{n \geq 0} \dim(F_d^\delta)^{(n)} z^n.
\]

As in the commutative case the algebra of constants \( F_d^\delta \) coincides with the algebra \( F_d^\delta UT_2(K) \) of \( UT_2(K) \)-invariants, where the action of \( UT_2(K) \) on \( F_d \) is defined by its realization as \( UT_2(K) = \{ \exp(\alpha \delta) \mid \alpha \in K \} \). There is an analogue of the integral Molien-Weyl formula due to Almkvist, Dicks and Formanek \( \text{[1]} \) which allows to calculate the Hilbert series of \( F_d^\delta UT_2(K) \) evaluating a multiple integral, if we know the Hilbert series \( H(F_d, z_1, \ldots, z_d) \) of \( F_d \). For varieties of associative algebras and for the variety of metabelian Lie algebras the Hilbert series of \( F_d \) is a rational function in \( d \) variables. Then the integral can be evaluated using the Residue Theorem, see the book \( \text{[9]} \) for details. Instead, in \( \text{[13]} \) and \( \text{[15]} \) another approach was suggested. It combines ideas of De Concini, Eisenbud, and Procesi \( \text{[8]} \), Berele \( \text{[6, 7]} \), and classical results of Elliott \( \text{[16]} \) and MacMahon \( \text{[21]} \). We give a short summary of the method. For details we refer to \( \text{[5]} \).

We assume that \( X_d \) is a Jordan basis of the vector space \( K X_d \) for the Weitzenb"ock derivation \( \delta = \delta(p_1, \ldots, p_n) \) of \( F_d \). First we define an action of the general linear group \( GL_2(K) \) on \( F_d \). Let \( Y_i = \{ x_j, x_{j+1}, \ldots, x_{j+p_i} \} \) be the part of the basis \( X_d \) corresponding to the \( i \)-th Jordan cell of \( \delta \). We identify the vector space \( KY_i \) with the vector space of the binary forms (homogeneous polynomials in two commuting...
variables $y_{11}$ and $y_{22}$ of degree $p_i$. We assume that $GL_2(K)$ acts on the two-dimensional vector space with basis $\{y_{11}, y_{22}\}$ and extend its action diagonally on the polynomial algebra $K[y_{11}, y_{22}]$. We want to synchronize the actions on $K[y_{11}, y_{22}]$ and of $UT_2$ as a subgroup of $GL_2(K)$. For this purpose we define an action of the derivation $\delta$ on $K[y_{11}, y_{22}]$ by $\delta(y_{11}) = 0$, $\delta(y_{22}) = y_{11}$. Then we identify $x_{j+p}$ with $y_{11}^{p_j}$ and $x_{j+p-k} = \delta^k(x_{j+p})$ with $\delta^k(y_{11}^{p_j}) = p_j(p_i - 1) \cdots (p_i - k + 1)y_{11}^{p_j-k}$, $k = 1, \ldots, p_i$. In this way the vector space $KX_d$ has a structure of a $GL_2(K)$-module and we extend diagonally the action of $GL_2(K)$ on the whole $F_d$. The basis $X_d$ consists of eigenvectors of the diagonal subgroup of $GL_2(K)$. If $g = \xi_1 e_{11} + \xi_2 e_{22}$, $\xi_1, \xi_2 \in K^*$, is a diagonal matrix, then $g(x_{j+p}) = \xi_1^{p_j-k} \xi_2^k$, $k = 0, 1, \ldots, p_i$. This defines a grading on $F_d$ assuming that the bidegree of $x_{j+p}$ is $(p_i - k, k)$. Now $F_d$ is a direct sum of irreducible polynomial $GL_2(K)$-submodules. The irreducible polynomial $GL_2(K)$-modules are indexed by partitions $\lambda = (\lambda_1, \lambda_2)$. If $W = W(\lambda)$ is an irreducible component of $F_d$, it contains a unique (up to a multiplicative constant) nonzero element $w$ of bidegree $(\lambda_1, \lambda_2)$. It is invariant under the action of $UT_2(K)$ and by [8] the algebra of $GL_2(K)$-invariants $F_d^{UT_2(K)} = F_d^\lambda$ is spanned by these vectors $w$. We express the Hilbert series of $F_d$ as a bigraded vector space. For this purpose we replace in the Hilber series $H(F_d, t_1, t_2)$ the variables $x_j, x_{j+1}, \ldots, x_{j+p_i-1}, x_{j+p_i}$ corresponding to each set $Y_i = \{x_j, x_{j+1}, \ldots, x_{j+p_i-1}, x_{j+p_i}\}$ by $t_1^{p_j}z, t_1^{p_j-1}t_2z, \ldots, t_1^{p_j-p_i}t_2^{p_i}z, t_1^{p_j-p_i}t_2^{p_i-1}z, t_1^{p_j-p_i}t_2^{p_i-2}z$, respectively, and obtain the Hilbert series

$$H_{GL_2}(F_d, t_1, t_2, z) = H(F_d, t_1^{p_j}z, t_1^{p_j-1}t_2z, \ldots, t_1^{p_j-p_i}t_2^{p_i}z, t_1^{p_j-p_i}t_2^{p_i-1}t_2z, \ldots, t_1^{p_j-p_i}t_2^{p_i-2}z).$$

The variable $z$ gives the total degree and $t_1, t_2$ count the bidegree induced by the action of the diagonal subgroup of $GL_2(K)$: The coefficient of $t_1^{p_j}t_2^{p_i}z^n$ in $H_{GL_2}(F_d, t_1, t_2, z)$ is equal to the dimension of the elements of $F_d$ which are linear combinations of products of length $n$ in the variables $X_d$ and are of bidegree $(n_1, n_2)$. The Hilbert series is an infinite linear combination with nonnegative integer coefficients of Schur functions

$$H_{GL_2}(F_d, t_1, t_2, z) = \sum_{n \geq 0} \sum_{(\lambda_1, \lambda_2)} m(\lambda_1, \lambda_2, n) S_{(\lambda_1, \lambda_2)}(t_1, t_2) z^n$$

and, by the representation theory of $GL_2(K)$, the multiplicity $m(\lambda_1, \lambda_2, n)$ is equal to the multiplicity of the irreducible $GL_2(K)$-module $W(\lambda_1, \lambda_2)$ in the homogeneous component $F_d^{(n)}$ of total degree $n$ of $F_d$. Hence the bigraded Hilbert series of the algebra $F_d^{UT_2(K)}$ of $UT_2(K)$-invariants is

$$H_{GL_2}(F_d^{UT_2(K)}, t_1, t_2, z) = \sum_{n \geq 0} \sum_{(\lambda_1, \lambda_2)} m(\lambda_1, \lambda_2, n) t_1^{\lambda_1}t_2^{\lambda_2} z^n$$

which is the so called multiplicity series of $H_{GL_2}(F_d, t_1, t_2, z)$ considered as a symmetric function in the variables $t_1, t_2$. In order to obtain the Hilbert series of $F_d^{UT_2(K)}$ as a $\mathbb{Z}$-graded vector space, it is sufficient to replace $t_1$ and $t_2$ with 1:

$$H(F_d^{UT_2(K)}, z) = \sum_{n \geq 0} (F_d^{UT_2(K)})^{(n)} z^n = H_{GL_2}(F_d^{UT_2(K)}, 1, 1, z).$$

To determine the multiplicity series of $H_{GL_2}(F_d, t_1, t_2, z)$ we follow the receipt of [6] [13, 5]. We consider the function

$$f(t_1, t_2, z) = (t_1 - t_2)H_{GL_2}(F_d, t_1, t_2, z)$$
which is skewsymmetric in \( t_1 \) and \( t_2 \) and consider the Laurent series

\[
f(t_1 \xi, t_2 / \xi, z) = \sum_{k=-\infty}^{+\infty} f_k(t_1, t_2, z) \xi^k.
\]

Then

\[
H_{GL_d}(F_{d''}(K), t_1, t_2, z) = \frac{1}{t_1} \sum_{k \geq 0} f_k(t_1, t_2, z).
\]

By the theorem of Belov \([1]\) the Hilbert series of the relatively free associative algebra \( F_d \) is a rational function with denominator which is a product of factors of the form \( 1 - z_1^q \cdots z_d^q \). Berele \([6, 7]\) calls such rational functions nice and proves that the multiplicity series of a nice rational symmetric function is nice again. The Hilbert series of the relatively free associative algebra \( F_d \) is also nice, see below. By \([5]\), when \( H_{GL_d}(F_d, t_1, t_2, z) \) is a nice rational function, its multiplicity series (which is equal to \( H_{GL_d}(F_{d''}(K), t_1, t_2, z) \)) can be evaluated by the method of Elliott \([16]\) and its further development by McMahon \([21]\), the so called partition analysis or \( \Omega \)-calculus. In \([5]\) an improvement of the \( \Omega \)-calculus is used, in the spirit of the algorithm of Xin \([27]\) which involves partial fractions and allows to perform computations with standard functions of Maple on a usual personal computer.

The next fact is well known, see, e.g., \([10]\).

**Lemma 3.1.** The Hilbert series of the free metabelian Lie algebra \( L_d / L_d'' \) is

\[
H(L_d / L_d'', z_1, \ldots, z_d) = 1 + (z_1 + \cdots + z_d) + (z_1 + \cdots + z_d - 1) \prod_{j=1}^{d} \frac{1}{1 - z_j}.
\]

Now we shall give the Hilbert series of the subalgebras of constants of Weitzenböck derivations of free metabelian Lie algebras with small number of generators. In some of the cases we give both Hilbert series, as graded and bigraded vector spaces, because we shall use the results in the last section of our paper. We do not give results for derivations with a one-dimensional Jordan cell because we shall handle them in the next section.

**Example 3.2.** Let \( \delta = \delta(p_1, \ldots, p_s) \) be the Weitzenböck derivation of the free metabelian Lie algebra \( L_d / L_d'' \) which has Jordan cells of size \( p_1 + 1, \ldots, p_s + 1 \). Then the Hilbert series of the algebra of constants \( (L_d / L_d'') \delta \) are:

\( d = 2, \delta = \delta(1) \):

\[
H_{GL_2}(L_2 / L_2'', t_1, t_2, z) = t_1 z + \frac{t_1 t_2 z^2}{1 - t_1 z}, \quad H((L_2 / L_2'') \delta, z) = z + \frac{z^2}{1 - z};
\]

\( d = 3, \delta = \delta(2) \):

\[
H_{GL_3}(L_3 / L_3'', t_1, t_2, z) = t_1^2 z + \frac{t_1^4 t_2 z^2}{(1 - t_1 z)(1 - t_1 t_2 z)}, \quad H((L_3 / L_3'') \delta, z) = z + \frac{z^2}{(1 - z)^2};
\]

\( d = 4, \delta = \delta(3) \):

\[
H_{GL_4}(L_4 / L_4'', t_1, t_2, z) = t_1^3 z + \frac{t_1^4 t_2 z^2(t_1^2 + t_2^2 + t_1^4 t_2 z^2 + t_1^6 t_2^2 z^3 - t_1^8 t_2^4 z^4)}{(1 - t_1 z)(1 - t_1 t_2 z)(1 - t_1^4 t_2 z^4)}.
\]
\[ H((L_4/L_4')^\delta, z) = z + \frac{z^2(2 + z^2 + z^3 - z^4)}{(1 - z)^2(1 - z^4)}; \]

\( d = 4, \delta = \delta(1, 1): \)

\[ H_{GL_2}((L_4/L_4')^\delta, t_1, t_2, z) = 2t_1 z + \frac{t_1 z^2(t_1 + 3t_2 - t_1^2 t_2 z^2)}{(1 - t_1 z)^2(1 - t_1 t_2 z^2)}; \]

\[ H((L_4/L_4')^\delta, z) = 2z + \frac{z^2(4 - z^2)}{(1 - z)^3(1 - z^3)}; \]

\( d = 5, \delta = \delta(4): \)

\[ H((L_5/L_5')^\delta, z) = z + \frac{z^2(2 + 2z + z^2 - 2z^4 + z^5)}{(1 - z)^2(1 - z^2)(1 - z^5)}; \]

\( d = 5, \delta = \delta(2, 1): \)

\[ H((L_5/L_5')^\delta, z) = 2z + \frac{z^2(4 + 2z^2 - 3z^3 + z^4)}{(1 - z)^3(1 - z^3)}; \]

\( d = 6, \delta = \delta(5): \)

\[ H((L_6/L_6')^\delta, z) = z + \frac{p(z)}{(1 - z)^2(1 - z^4)(1 - z^6)(1 - z^8)}, \]

\[ p(z) = z^2(3 + 3z + 7z^2 + 10z^3 + 11z^4 + 14z^5 + 13z^6 + 16z^7 + 12z^8 + 8z^9 + 10z^{10} + 3z^{11} + 5z^{12} - z^{13} + z^{14} - z^{16} + 2z^{17} - z^{18}); \]

\( d = 6, \delta = \delta(3, 1): \)

\[ H((L_6/L_6')^\delta, z) = 2z + \frac{z^2(5 + 6z + 8z^2 + 11z^3 + 5z^4 - 2z^5 + 3z^6 - 2z^7 + 2z^9 - z^{10})}{(1 - z)^2(1 - z^2)(1 - z^4)^2}; \]

\( d = 6, \delta = \delta(2, 2): \)

\[ H((L_6/L_6')^\delta, z) = 2z + \frac{z^2(5 + 8z - 6z^3 + 2z^4 + 2z^5 - z^6)}{(1 - z)^2(1 - z^2)^3}; \]

\( d = 6, \delta = \delta(1, 1, 1): \)

\[ H_{GL_2}((L_6/L_6')^\delta, t_1, t_2, z) = 3t_1 z + \frac{t_1 z^2 p(t_1, t_2, z)}{(1 - t_1 z)^3(1 - t_1 t_2 z^2)^3}, \]

\[ p = 3(t_1 + 2t_2) + t_1(-t_1 + t_2)z - 9t_1^2 t_2 z^2 + 3t_1^2 t_2(-3t_2 + t_1)z^3 + t_1^2 t_2(9t_1 - t_2)z^4 + 3t_1^2 t_2^2(t_2 - t_1)z^5 - 3t_1^3 t_2^3 z^6 + t_1^3 t_2^3 z^7); \]

\[ H((L_6/L_6')^\delta, z) = 3z + \frac{z^2(9 + 9z - 6z^3 + 2z^4 + 2z^5 - z^6)}{(1 - z)^2(1 - z^2)^3}. \]
4. Derivations with one-dimensional Jordan cell

In this section we assume that the Jordan form of $\delta$ contains a $1 \times 1$ cell, $\delta$ acts as a nilpotent linear operator on $KX_{d-1}$, and $\delta(x_d) = 0$. We fix a finite system $\{f_1(X_{d-1}), \ldots, f_l(X_{d-1})\}$ of generators of the algebra of constants $K[X_{d-1}]^\delta$. Without loss of generality we may assume that the polynomials $f_r(X_{d-1})$ are homogeneous, $r = 1, \ldots, l$. Also, we fix a system $\{c_1, \ldots, c_k\}$ of generators of the $K[X_{d-1}]^\delta$-module $(L_{d-1}^\delta/L_{d-1}^\delta)^\delta$. Our purpose is to find a generating set of the $K[X_d]^\delta$-module $(L_d/L_d^\delta)^\delta$.

Lemma 4.1. The Hilbert series of $(L'_d/L'_d)^\delta$, $(L'_{d-1}/L'_{d-1})^\delta$, and $K[X_{d-1}]^\delta$ are related by

$$H_{GL_2}((L'_d/L'_d)^\delta, t_1, t_2, z) = \frac{1}{1-z} H_{GL_2}((L'_{d-1}/L'_{d-1})^\delta, t_1, t_2, z)$$

$$+ \frac{z}{1-z} (H_{GL_2}(K[X_{d-1}]^\delta, t_1, t_2, z) - 1),$$

where $H_{GL_2}(K[X_{d-1}]^\delta, t_1, t_2, z)$ is the Hilbert series of $K[X_{d-1}]^\delta$. The Hilbert series $H_{GL_2}(L'_d/L'_d, z_1, \ldots, z_d) = 1 + (z_1 + \cdots + z_d - 1) \prod_{j=1}^d \frac{1}{1 - z_j}$

In the Hilbert series $H(L'_d/L'_d, z_1, \ldots, z_d)$ in the lemma we remove the summand $z_1 + \cdots + z_d$ which gives the contribution of the elements of first degree.) Following the procedure described in Section 3, we replace its variables $z_j$ with $t_1^{q_j} t_2^{r_j} z$, where the nonnegative integers $q_j, r_j$ depend on the size of the corresponding Jordan cell and the position of the variable $x_j$ in the Jordan basis of $KX_d$. In particular, we have to replace the variable $z_n$ with $z$. Hence

$$H_{GL_2}(L'_d/L'_d, t_1, t_2, z) = 1 + \left( \sum_{j=1}^{d-1} t_1^{q_j} t_2^{r_j} z - 1 \right) \prod_{j=1}^{d-1} \frac{1}{1 - t_1^{q_j} t_2^{r_j} z}$$

$$= \frac{1}{1-z} \left( \left( 1 + \sum_{j=1}^{d-1} t_1^{q_j} t_2^{r_j} z - 1 \right) \prod_{j=1}^{d-1} \frac{1}{1 - t_1^{q_j} t_2^{r_j} z} + z \left( \prod_{j=1}^{d-1} \frac{1}{1 - t_1^{q_j} t_2^{r_j} z} - 1 \right) \right)$$

$$= \frac{1}{1-z} H_{GL_2}(L'_{d-1}/L'_{d-1}, t_1, t_2, z) + \frac{z}{1-z} (H_{GL_2}(K[X_{d-1}], t_1, t_2, z) - 1).$$

The Hilbert series $H_{GL_2}((L'_d/L'_d)^\delta, t_1, t_2, z)$ is equal to the multiplicity series of $H_{GL_2}(L'_d/L'_d, t_1, t_2, z)$. Similar statements hold for the other two Hilbert series $H_{GL_2}((L'_{d-1}/L'_{d-1})^\delta, t_1, t_2, z)$ and $H_{GL_2}(K[X_{d-1}]^\delta, t_1, t_2, z)$. Hence

$$H_{GL_2}((L'_d/L'_d)^\delta, t_1, t_2, z) = \frac{1}{1-z} H_{GL_2}((L'_{d-1}/L'_{d-1})^\delta, t_1, t_2, z)$$

$$+ \frac{z}{1-z} (H_{GL_2}(K[X_{d-1}]^\delta, t_1, t_2, z) - 1)$$
which implies that
\[ H((L_d'/L_d'_{\delta}), z) = \frac{1}{1 - z}H((L_{d-1}'/L_{d-1}'_{\delta}), z) + \frac{z}{1 - z}(H(K[X_{d-1}]_{\delta}), z) - 1). \]

Let \( \omega = \omega(K[X_{d-1}]) \) be the augmentation ideal of \( K[X_{d-1}] \), i.e., the ideal of all polynomials without constant term. We define a \( K \)-linear map
\[ \pi : \omega(K[X_{d-1}]) \to L_d'/L_d' \]
by
\[ \pi(x_j, \cdots x_j) = \sum_{k=1}^{n} [x_d, x_{j_k}]x_{j_1} \cdots x_{j_{k-1}}x_{j_{k+1}} \cdots x_{j_n}, \quad x_{j_1} \cdots x_{j_n} \in K[X_{d-1}], n \geq 1. \]

**Lemma 4.2.** (i) The map \( \pi \) satisfies the equality
\[ \pi(uv) = \pi(u)v + \pi(v)u, \quad u, v \in \omega. \]
(ii) The derivation \( \delta \) and the map \( \pi \) commute.

**Proof.** (i) It is sufficient to show the equality for \( u, v \) being monomials only. Let \( u = x_{i_1} \cdots x_{i_m} \) and \( v = x_{j_1} \cdots x_{j_n} \). We use the standard notation \( x_{j_1} \cdots \hat{x}_{j_k} \cdots x_{j_n} \) to indicate that \( x_{j_k} \) does not participate in the product. Then
\[ \pi(uv) = \pi(x_{i_1} \cdots x_{i_m}x_{j_1} \cdots x_{j_n}) = \left( \sum_{i=1}^{m} [x_d, x_{i_i}]x_{i_1} \cdots \hat{x}_{i_i} \cdots x_{i_m} \right) (x_{j_1} \cdots x_{j_n}) \]
\[ + \left( \sum_{k=1}^{n} [x_d, x_{j_k}]x_{j_1} \cdots \hat{x}_{j_k} \cdots x_{j_n} \right) (x_{i_1} \cdots x_{i_m}) = \pi(u)v + \pi(v)u. \]

(ii) Again, it is sufficient to show the equality for monomials only. We proceed by induction on the length of the monomials. If \( u = x_j \) and
\[ \delta(x_j) = \sum_{i=1}^{d-1} \alpha_{ij}x_i, \quad \alpha_{ij} \in K, j = 1, \ldots, d - 1, \]
then
\[ \pi(\delta(x_j)) = \sum_{i=1}^{d-1} \alpha_{ij}\pi(x_i) = \sum_{i=1}^{d-1} \alpha_{ij}[x_d, x_i] = [x_d, \sum_{i=1}^{d-1} \alpha_{ij}(x_i)] = \delta(\pi(x_j)) \]
because \( \delta(x_d) = 0 \). Let the monomials \( u \) and \( v \) belong to \( \omega \). Using that \( \delta \) is a derivation of \( L_d/L_d' \) which by the inductive arguments commute with \( \pi \) on \( u \) and \( v \), and applying (i), we obtain
\[ \delta(\pi(uv)) = \delta(\pi(u)v + \pi(v)u) = \delta(\pi(u)v) + \delta(\pi(v)u) = \pi(\delta(u))v + \pi(\delta(v))u + \pi(\delta(v))u + \pi(v)\delta(u) = \pi(\delta(u)). \]

\[ \square \]

The next theorem and its corollary are the main results of the section.
Theorem 4.3. Let $X_d$ be a Jordan basis of the derivation $\delta$ acting on $KX_d$ and let $\delta$ have a $1 \times 1$ Jordan cell corresponding to $x_d$. Let \{\(v_i\mid i \in I\) and \{\(u_j\mid j \in J\) be, respectively, homogeneous bases of \((L'_{d-1}/L''_{d-1})^\delta\) and $\omega(K[X_{d-1}])^\delta$ with respect to both $\mathbb{Z}$- and $\mathbb{Z}^2$-gradings. Then \((L'_d/L''_d)^\delta\) has a basis

$$\{v_i x^n_d, \pi(u_j)x^n_d \mid i \in I, j \in J, n \geq 0\}.$$

Proof. The Hilbert series of \((L'_{d-1}/L''_{d-1})^\delta\) and $\omega(K[X_{d-1}])^\delta$ are equal, respectively, to the generating functions of their bases. Hence

$$H_{GL_d}((L'_{d-1}/L''_{d-1})^\delta, t_1, t_2, z) = \sum_{n \geq 0} t_1^n t_2^n z^n,$$

where $v_i$ is of bidegree $(p_i, q_i)$ and of total degree $m_i$. Since $x_d$ is of bidegree $(0, 0)$ and of total degree 1, the generating function of the set $V = \{v_i x^n_d \mid i \in I, n \geq 0\}$ is

$$G(V, t_1, t_2, z) = \sum_{n \geq 0} t_1^n t_2^n z^n = \frac{1}{1-z} H_{GL_d}((L'_{d-1}/L''_{d-1})^\delta, t_1, t_2, z).$$

The map $\pi$ sends the monomials of $\omega(K[X_{d-1}])$ to linear combinations of commutators with an extra variable $x_d$ in the beginning of each commutator. Hence, if the Hilbert series of $\omega(K[X_{d-1}])^\delta$ is

$$H_{GL_d}(\omega(K[X_{d-1}])^\delta, t_1, t_2, z) = H_{GL_d}(K[X_{d-1}]^\delta, t_1, t_2, z) - 1 = \sum_{n \geq 0} \sum_{j \in J} t_1^n t_2^n z^n,$$

where the bidegree of $u_j$ is $(k_j, l_j)$ and its total degree is $n_j$, then the generating function of the set $U = \{\pi(u_j)x^n_d \mid j \in J, n \geq 0\}$ is

$$G(U, t_1, t_2, z) = \sum_{n \geq 0} \sum_{j \in J} t_1^n t_2^n z^n = \frac{z}{1-z} (H_{GL_d}(K[X_{d-1}]^\delta, t_1, t_2, z) - 1).$$

Hence, by Lemma 4.1

$$H_{GL_d}((L'_d/L''_d)^\delta, t_1, t_2, z) = G(V, t_1, t_2, z) + G(U, t_1, t_2, z).$$

Since both sets $V$ and $U$ are contained in $(L'_d/L''_d)^\delta$, we shall conclude that $V \cup U$ is a basis of $(L'_d/L''_d)^\delta$ if we show that the elements of $V \cup U$ are linearly independent. For this purpose it is more convenient to work in the abelian wreath product $A_d \wr B_d$. The elements $v_i$ belong to $L'_{d-1}/L''_{d-1} \subset A_d \wr B_d$ and hence are of the form

$$v_i = \sum_{k=1}^{d-1} a_k g_{ki}(X_{d-1}), \quad g_{ki}(X_{d-1}) \in K[X_{d-1}].$$

Hence

$$v_i x_d^n = \sum_{k=1}^{d-1} a_k g_{ki}(X_{d-1}) x_d^n.$$ 

On the other hand, the elements $\pi(u_j)$ are of the form

$$\pi(u_j) = \sum_{k=1}^{d-1} [x_d, x_k] h_{kj}(X_{d-1}) = n_j a_d \sum_{k=1}^{d-1} x_k h_{kj}(X_{d-1}) - \sum_{k=1}^{d-1} a_k h_{kj}(X_{d-1}) x_d$$

$$= n_j a_d u_j - \sum_{k=1}^{d-1} a_k h_{kj}(X_{d-1}) x_d.$$
Hence
\[
\pi(u_j)x^n = (n_j a_d u_j - \sum_{k=1}^{d-1} a_k h_{kj}(X_{d-1})x_d)x_d^n.
\]

Let \( v = \sum \xi_{in} v_i x_d^n + \sum \eta_{jn} \pi(u_j)x^n = 0 \) for some \( \xi_{in}, \eta_{jn} \in K \). Since the elements \( u_j \) are linearly independent in \( \omega(K[X_{d-1}])^\delta \), comparing the coefficient of \( a_d \) in \( v \) we conclude that \( \eta_{jn} = 0 \). Then, using that the elements \( v_i \) are linearly independent in \( (L_{d-1}'/L_{d-1}'')^\delta \), we derive that \( \xi_{in} = 0 \). Hence the set \( V \cup U \) is a basis of \( (L_d'/L_d'')^\delta \). \( \square \)

**Corollary 4.4.** Let \( X_d \) be a Jordan basis of the derivation \( \delta \) acting on \( KX_d \) and let \( \delta \) have a \( 1 \times 1 \) Jordan cell corresponding to \( x_d \). Let \( \{c_1, \ldots, c_k\} \) and \( \{f_1, \ldots, f_l\} \) be, respectively, homogeneous generating sets of the \( K[X_{d-1}]^\delta \)-module \( (L_{d-1}'/L_{d-1}'')^\delta \) and of the algebra of constants \( K[X_{d-1}]^\delta \). Then the \( K[X_d]^\delta \)-module \( (L_d'/L_d'')^\delta \) is generated by the set \( \{c_1, \ldots, c_k\} \cup \{\pi(f_1), \ldots, \pi(f_l)\} \).

**Proof.** Clearly, the \( K[X_{d-1}]^\delta \)-module \( (L_{d-1}'/L_{d-1}'')^\delta \) is spanned by the elements \( c_j f_1^{j_1} \cdots f_l^{j_l} \). In particular, in this way we obtain all elements \( v_j \) from the basis of the vector space \( (L_{d-1}'/L_{d-1}'')^\delta \). Since \( x_d \in K[X_d]^\delta \), we obtain also all elements \( v_j x_d^n \).

By Lemma 4.2 (i) we obtain that the \( K[X_{d-1}]^\delta \)-module \( \pi(\omega(K[X_{d-1}])^\delta) \) is generated by \( \pi(f_1), \ldots, \pi(f_l) \). Hence all elements \( \pi(u_j) \), where \( \{u_j \mid j \in J\} \) is the basis of \( \omega(K[X_{d-1}])^\delta \), belong to the \( K[X_{d-1}]^\delta \)-module generated by \( \pi(f_1), \ldots, \pi(f_l) \).

In this way we obtain also the elements \( \pi(u_j)x_d^n \) and derive that \( \{c_1, \ldots, c_k\} \cup \{\pi(f_1), \ldots, \pi(f_l)\} \) generate the \( K[X_d]^\delta \)-module \( (L_d'/L_d'')^\delta \). \( \square \)

**Example 4.5.** Let \( d = 4 \) and let the Jordan normal form of \( \delta \) have two cells, of size \( 3 \times 3 \) and \( 1 \times 1 \), respectively. Hence \( \delta = \delta(2,0) \) in our notation. By Example 5.1 for \( d = 3 \) and \( \delta = \delta(2) \), the algebra \( K[X_3]^\delta \) is generated by \( f_1 = x_1 \) and \( f_2 = x_2^3 - 2x_3x_2 \). The \( K[X_3]^\delta \)-module \( (L_3'/L_3'')^\delta \) is generated by \( c_1 = [x_2, x_1] \) and \( c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2] \). Hence, by Corollary 4.4 the \( K[X_4]^\delta \)-module \( (L_4'/L_4'')^\delta \) is generated by \( c_1, c_2 \) and

\[
\pi(f_1) = [x_4, x_1], \quad \pi(f_2) = 2([x_4, x_2, x_2] - [x_4, x_1, x_3] - [x_4, x_3, x_1]).
\]

5. GENERATING SETS FOR SMALL NUMBER OF GENERATORS

In this section we shall find the generators of the \( K[X_d]^\delta \)-module \( (L_d'/L_d'')^\delta \) for \( d \leq 4 \) and for \( d = 6, \delta = \delta(1,1,1) \). By Corollary 4.4 we shall assume that \( \delta \) has no \( 1 \times 1 \) Jordan cells.

**Example 5.1.** Let \( d = 3, \delta = \delta(2) \), and let \( \delta(x_1) = 0, \delta(x_2) = x_1, \delta(x_3) = x_2 \). It is well known, see e.g., [22], that \( K[X_3]^\delta \) is generated by the algebraically independent polynomials \( f_1 = x_1, f_2 = x_2^3 - 2x_1x_3 \). Hence

\[
H_{GL_2}(K[X_3]^\delta, t_1, t_2, z) = \frac{1}{(1 - t_1^2z)(1 - t_1t_2^2z^2)}.
\]

By Example 5.2

\[
H_{GL_2}((L_3'/L_3'')^\delta, t_1, t_2, z) = \frac{t_1^3t_2z^2}{(1 - t_1^2z)(1 - t_1t_2^2z^2)} = \frac{t_3^2t_2z^2(1 + t_1t_2z)}{(1 - t_1^2z)(1 - t_1t_2^2z^2)}.
\]

It is easy to see that the Lie elements

\[
c_1 = [x_2, x_1], \quad c_2 = [x_3, x_1, x_1] - [x_2, x_1, x_2]
\]
belong to \((L'_4/L'_{40})^\delta\) and are of bidegree \((3,1)\) and \((4,2)\), respectively. If \(c_1\) and \(c_2\) generate a free \(K[X_3]^\delta\)-submodule of \((L'_4/L'_{40})^\delta\), its Hilbert series is

\[
(t_1^4t_2^2z^2 + t_1^4t_2^2z^3)H_{GL_2}(K[X_3]^\delta, t_1, t_2, z) = \frac{t_1^4t_2^2z^2(1 + t_1t_2z)}{(1 - t_1^2z)(1 - t_1t_2^2z^2)}
\]

\[
= H_{GL_2}(L'_3/L'_{30})^\delta, t_1, t_2, z).
\]

Then we can derive that \(c_1\) and \(c_2\) generate the whole \(K[X_3]^\delta\)-module \((L'_3/L'_{30})^\delta\). Hence it is sufficient to show that \(c_1\) and \(c_2\) generate a free \(K[X_3]^\delta\)-module. Let \(c_1u_1(f_1, f_2) + c_2u_2(f_1, f_2) = 0\) for some \(u_1(f_1, f_2), u_2(f_1, f_2) \in K[f_1, f_2]\). Working in the wreath product \(A_6 \wr B_3\) we obtain

\[
0 = (a_2x_1 - a_1x_2)u_1(f_1, f_2) + ((a_3x_1 - a_1x_3)x_1 - (a_2x_1 - a_1x_2)x_2)u_2(f_1, f_2)
\]

\[
= a_1(-x_2u_1(f_1, f_2) + (-x_1x_3 + x_2^2)u_2(f_1, f_2))
\]

\[
+ a_2(x_1u_1(f_1, f_2) - x_1x_2u_2(f_1, f_2)) + a_3x_1^2u_2(f_1, f_2).
\]

Since the coefficient \(x_1^2u_2(f_1, f_2)\) of \(a_3\) is equal to \(0\), we obtain that \(u_2(f_1, f_2) = 0\). Similarly, the coefficient of \(a_1\) gives that \(u_1(f_1, f_2) = 0\) and this shows that the \(K[X_3]^\delta\)-module \((L'_3/L'_{30})^\delta\) is generated by \(c_1\), \(c_2\). As a vector space \((L'_3/L'_{30})^\delta\) is spanned by the elements \(x_1, x_1^2 f_2^2,\) and \(c_1f_1^4 f_2^2, f_1^4, f_2^2, f_3 f_2^4\), \(q_t, r_j \geq 0\). This easily implies that the algebra \((L'_3/L'_{30})^\delta\) is generated by the infinite set

\[
\{x_1, c_1f_1^4, c_2f_2^2 \mid r_j \geq 0\}.
\]

**Example 5.2.** Let \(d = 4, \delta = \delta(3)\), and let \(\delta(x_1) = 0, \delta(x_2) = x_1, \delta(x_3) = x_2, \delta(x_4) = x_3\). Then, see [22], \(K[X_4]^\delta\) is generated by

\[
f_1 = x_1, \quad f_2 = x_2^2 - 2x_1x_3, \quad f_3 = x_3^2 - 3x_1x_2x_3 + 3x_1^2 x_4,
\]

\[
f_4 = x_2^2x_3^2 - 2x_2^2 x_4 + 6x_1x_2x_3x_4 - \frac{8}{3} x_1^2 x_3^2 - 3x_1^2 x_4^2.
\]

The generators of \(K[X_4]^\delta\) satisfy the defining relation

\[
f_3^2 = f_2 - 3f_1 f_4
\]

and the algebra \(K[X_4]^\delta\) has the presentation

\[
K[X_4]^\delta \cong K[f_1, f_2, f_3, f_4 \mid f_3^2 = f_2 - 3f_1 f_4].
\]

In particular, as a vector space \(K[X_4]^\delta\) has a basis

\[
\{f_1 f_2^2 f_3^4, f_1 f_2^2 f_3^4, f_3 f_2^4 \mid q_1, q_2, q_4 \geq 0\}
\]

and its Hilbert series is

\[
H_{GL_2}(K[X_4]^\delta, t_1, t_2, z) = \frac{1 + t_1^4t_2^2z^3}{(1 - t_1^2z)(1 - t_1t_2^2z^2)(1 - t_1t_2^4z^4)}.
\]

By Example 5.2 the Hilbert series of \((L'_4/L'_{40})^\delta\) is

\[
H_{GL_2}(L'_4/L'_{40})^\delta, t_1, t_2, z) = \frac{t_1^4t_2^2z^2(t_1^2 + t_2^2 + t_1^2t_2^2z^2 + t_1^4t_2^2z^3 - t_1^8t_2^4z^4)}{(1 - t_1^2z)(1 - t_1t_2^2z^2)(1 - t_1t_2^4z^4)}
\]

\[
= \frac{t_1^4t_2^2z^2(t_1^2 + t_2^2 + t_1^2t_2^2z^2 + t_1^4t_2^2z^3 - t_1^8t_2^4z^4)}{(1 - t_1^2z)(1 - t_1t_2^2z^2)(1 - t_1t_2^4z^4)}
\]

\[
= (t_1^5t_2 + t_1^3t_2^3)z^2(1 + t_1z + (t_1^2t_2 + t_1^5t_2)z^3 + \ldots)
\]

This suggests that the \(K[X_4]^\delta\)-module \((L'_4/L'_{40})^\delta\) has two generators \(c_1\) and \(c_2\) of bidegree \((5,1)\) and \((3,3)\), respectively. They together with \(c_1f_1\) and \(c_2f_1\) give the contribution \((t_1^5t_2 + t_1^3t_2^3)z^2(1 + t_1z)\). We also expect two generators \(c_3\) and \(c_4\) of
bidegree $(7, 2)$ and $(5, 4)$, respectively. By easy calculations we have found the explicit form of $c_1, c_2, c_3, c_4$:

$$c_1 = [x_2, x_1], \quad c_2 = [x_4, x_1] - [x_3, x_2],$$
$$c_3 = [x_3, x_1, x_1] - [x_2, x_1, x_2], \quad c_4 = 3[x_2, x_1, x_4] - 2[x_3, x_1, x_3] + [x_3, x_2, x_2].$$

For example, $c_4$ is a linear combination of all commutators of degree $3$ and bidegree $(5, 4)$: $[x_2, x_1, x_4], [x_4, x_1, x_2], [x_3, x_1, x_3]$ and $[x_3, x_2, x_2]$:

$$c_4 = \gamma_1[x_2, x_1, x_4] + \gamma_2[x_4, x_1, x_2] + \gamma_3[x_3, x_1, x_3] + \gamma_4[x_3, x_2, x_2], \quad \gamma_1, \gamma_2, \gamma_3, \gamma_4 \in K,$$
and the condition $\delta(c_4) = 0$ gives

$$0 = \gamma_1[x_2, x_1, x_3] + \gamma_2([x_3, x_1, x_2] + [x_4, x_1, x_1]) + \gamma_3([x_2, x_1, x_3] + [x_3, x_1, x_2]) + \gamma_4([x_3, x_1, x_2] + [x_3, x_2, x_1])$$
$$= (\gamma_1 + \gamma_3 - \gamma_4)[x_2, x_1, x_3] + (\gamma_2 + \gamma_3 + 2\gamma_4)[x_3, x_1, x_2] + \gamma_2[x_4, x_1, x_1],$$

Hence

$$\gamma_1 + \gamma_3 - \gamma_4 = \gamma_2 + \gamma_3 + 2\gamma_4 = \gamma_2 = 0$$
and, up to a multiplicative constant, the only solution is

$$\gamma_1 = 3, \quad \gamma_2 = 0, \quad \gamma_3 = -2, \quad \gamma_4 = 0.$$

Similarly, we obtain one more generator of bidegree $(7, 5)$:

$$c_5 = 3(-[x_3, x_1, x_1, x_4] + [x_2, x_1, x_2, x_4] + [x_3, x_1, x_2, x_3])$$
$$-4[x_2, x_1, x_3, x_3] - [x_3, x_2, x_2, x_2].$$

The Hilbert series of the free $K[X_4]^S$-module generated by five elements of bidegree $(5, 1), (3, 3), (7, 2), (5, 4)$, and $(7, 5)$ is

$$H_{GL_4}(t_1, t_2, z) = \frac{t_1^3 t_2 z^2 ((1 + t_1^2 t_2 z)(t_1^2 + t_2^2) + t_1^3 t_2 t_2 z^2)(1 + t_1^4 t_2 z^2)}{(1 - t_1 z)(1 - t_1 t_2 z)(1 - t_1^2 t_2 z^2)(1 - t_1^3 t_2 z^4)}.$$

Hence

$$H_{GL_4}(t_1, t_2, z) - H_{GL_4}((L_4^L/L_4^L)^S, t_1, t_2, z) = (t_1^3 - t_1^2)^S t_2^4 z^5 + \cdots$$

which suggests that there is a relation of bidegree $(11, 4)$ and a generator of bidegree $(8, 7)$. Continuing in the same way, we have found the generators

$$c_6 = -9[x_2, x_1, x_1, x_4, x_4] + 18[x_3, x_1, x_1, x_3, x_4] - 12[x_4, x_1, x_1, x_3, x_3]$$
$$-9[x_3, x_1, x_2, x_2, x_4] + 12[x_4, x_1, x_2, x_2, x_3] + 4[x_2, x_1, x_3, x_3]$$
$$-6[x_3, x_1, x_2, x_3, x_3] - 3[x_4, x_2, x_2, x_2, x_2] + 3[x_3, x_2, x_2, x_2, x_3],$$

$$c_7 = -18[x_3, x_1, x_1, x_4, x_4] + 18[x_4, x_1, x_1, x_3, x_4] + 18[x_2, x_1, x_1, x_2, x_4, x_4]$$
$$-9[x_4, x_1, x_1, x_2, x_2, x_4] - 18[x_2, x_1, x_1, x_3, x_3, x_4] + 18[x_4, x_1, x_1, x_3, x_3, x_3]$$
$$-18[x_4, x_1, x_1, x_2, x_3, x_3] + 8[x_3, x_1, x_1, x_3, x_3, x_3] - 9[x_2, x_1, x_2, x_2, x_3, x_4]$$
$$-3[x_3, x_1, x_2, x_2, x_3, x_4] + 15[x_4, x_1, x_2, x_2, x_2, x_3] + 10[x_2, x_1, x_2, x_3, x_3, x_3]$$
$$-12[x_3, x_1, x_2, x_2, x_3, x_3] - 3[x_4, x_2, x_2, x_2, x_2, x_2] + 3[x_3, x_2, x_2, x_2, x_2, x_3]$$

of bidegree $(8, 7)$ and $(10, 8)$, respectively. We have also found the relations

$$R_1(11, 4) : c_1 f_3 = -c_3 f_2 + c_4 f_1^3,$$
$$R_2(13, 5) : c_3 f_3 = -(c_1 f_2^2 + c_3 f_1^2),$$
$$R_3(11, 7) : c_4 f_3 = -(3c_1 f_4 + c_5 f_2),$$
$$R_4(11, 7) : c_6 f_1 = 3(c_1 f_4 - c_2 f_2^2 + c_5 f_2),$$

where $f_1, f_2, f_3, f_4, f_5$ are the generators of the free $K[X_4]^S$-module.
with unknowns $\bar{u}_1, \ldots, \bar{u}_7$. First, we substitute $x_2 = 0$. Then $f_1, f_2, f_4$ become

$$f_1 = x_1, \quad f_2 = -2x_1x_3, \quad f_4 = -\frac{8}{3}x_1^3 - 3x_1^2x_2^2.$$ 

Similarly, $c_j$ and $c_2f_3$ become

$$c_1 = a_2x_1, \quad c_2 = -a_1x_4 + a_2x_3 + a_4x_1, \quad c_3 = (-a_1x_3 + a_3x_1)x_1,$$

$$c_4 = 2a_1x_3^2 + 3a_2x_1x_4 - 2a_3x_1x_3, \quad c_5 = (3a_1x_3x_4 - 4a_2x_3^2 - 3a_3x_1x_4)x_1,$$

$$c_6 = (-6a_1x_3^2x_4 + a_2(-9x_1x_4^2 + 4x_3^3) + 18a_3x_1x_3x_4 - 12a_4x_1x_3^3)x_1,$$

$$c_7 = (-8a_1x_3^4 - 18a_3x_1x_3^2x_4 + 2a_3(-9x_1x_4^2 + 4x_3^3)x_1 + 18a_4x_1^2x_3x_4)x_1,$$

$$c_2f_3 = (-3a_1x_4 + 3a_2x_3 + 3a_4x_1)x_1^2x_4.$$ 

Direct calculations give that the coordinates $\bar{v}_i$ of $v = \sum_{j=1}^{7} c_j\bar{u}_j + c_2\bar{f}_3\bar{u}_8$ are

$$-x_1\bar{u}_2 - x_1x_3\bar{u}_3 + 2x_3^2\bar{u}_4 + 3x_1x_3x_4\bar{u}_5 - 6x_1x_3^2x_4\bar{u}_6 - 8x_1x_3^3\bar{u}_7 - 3x_1^2x_4^2\bar{u}_8 = 0,$$

$$x_1\bar{u}_1 + x_3\bar{u}_2 + 3x_1x_4\bar{u}_4 - 4x_1x_3^2\bar{u}_5 + (-9x_1x_4^2 + 4x_3^3)x_1\bar{u}_6 - 18x_1^2x_3^2x_4\bar{u}_7 + 3x_1^2x_3x_4\bar{u}_8 = 0,$$

$$(x_1\bar{u}_3 - 2x_3\bar{u}_4 - 3x_1x_3^2\bar{u}_5 + 18x_1x_3x_4\bar{u}_6 + 2(-9x_1x_4^2 + 4x_3^3)\bar{u}_7)x_1 = 0,$$

$$(\bar{u}_2 - 12x_1x_3^2\bar{u}_6 + 18x_1^2x_3x_4\bar{u}_7 + 3x_1^2x_4^2\bar{u}_8)x_1 = 0,$$

$$\bar{v}_1 = 0, \quad i = 1, \ldots, 4,$$

with unknowns $u_1, \ldots, u_8$. Similarly, $c_j$ can be replaced with a linear combination of other generators if $j \neq 2$. Similarly for $c_6f_1$ and $c_7f_1$. Hence the $K[X_d]^{\delta}$-module generated by $c_1, \ldots, c_7$ is spanned by

$$C = \{c_jf_1^{4\varepsilon_1}f_2^{4\varepsilon_2}f_3^{4\varepsilon_3} | q_j, r_j, s_j \geq 0, j = 1, 3, 4, 5\}$$

$$\cup\{c_jf_2^{4\varepsilon_1}f_3^{4\varepsilon_2} | r_j, s_j \geq 0, j = 6, 7\} \cup \{c_2f_1^{4\varepsilon_1}f_2^{4\varepsilon_2}f_3^{4\varepsilon_3}f_4^{4\varepsilon_4} | q_2, r_2, s_2 \geq 0, \varepsilon_1 = 0, 1\}.$$
where $\bar{u}_j = u_j(f_1, f_2, f_4)$ Since $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_4 v_4 = 0$, because $v$ belongs to the commutator ideal $L'_4/L''_4$, we have that $x_1 \bar{v}_1 + x_3 \bar{v}_3 + x_4 \bar{v}_4 = 0$. Hence we can remove the first equation and obtain

$$(x_1 \bar{u}_1 + x_3 \bar{u}_2 - 4x_1^2 \bar{u}_3 + (-9x_1 x_2^2 + 4x_3^2) \bar{u}_5) + 3(u_1 - 6x_1 x_2^2 \bar{u}_7 + x_3 \bar{u}_8) x_1 x_4 = 0,$$

$$(x_1 \bar{u}_3 - 2x_3 \bar{u}_4 + 2(9x_1 x_2^2 + 4x_3^2) \bar{u}_7) + 3(-\bar{u}_5 + 6x_3 \bar{u}_6)x_1 x_4 = 0,$$

$$(\bar{u}_2 - 12x_1^2 \bar{u}_6) + 3(6x_3 \bar{u}_7 + \bar{u}_8) x_1^2 x_4 = 0.$$ 

The variable $x_4$ participates in the polynomials $f_1, f_2, f_4$ in even degrees only. Hence $\bar{u}_1, \ldots, \bar{u}_8$ do not contain odd degrees of $x_4$. The only odd degrees of $x_4$ in the above equations come from $3(\bar{u}_4 - 6x_1 x_2^2 \bar{u}_7 + x_3 x_4) x_1 x_4$, $3(-\bar{u}_5 + 6x_3 \bar{u}_6)x_1 x_4$, and $3(6x_3 \bar{u}_7 + \bar{u}_8)x_1^2 x_4$. Hence

$$(-\bar{u}_5 + 6x_3 \bar{u}_6)x_1 = -f_1 \bar{u}_5(f_1, f_2, f_4) - 3f_2 \bar{u}_6(f_2, f_4) = 0,$$

$$(6x_3 \bar{u}_7 + \bar{u}_8)x_1 = -3f_2 \bar{u}_7(f_2, f_4) + f_1 \bar{u}_8(f_1, f_2, f_4) = 0.$$ 

Since $f_1, f_2, f_4$ are algebraically independent in $K[x_1, x_3, x_4]$, the equations

$$-f_1 \bar{u}_5(f_1, f_2, f_4) - 3f_2 \bar{u}_6(f_2, f_4) = -3f_2 \bar{u}_7(f_2, f_4) + f_1 \bar{u}_8(f_1, f_2, f_4) = 0$$

give that $\bar{u}_6 = \bar{u}_7 = 0$ and, as a consequence, $\bar{u}_j = 0$ for $j = 1, \ldots, 8$. Using again the algebraic independence of $f_1, f_2, f_4$ we obtain that $u_j = 0$ for $j = 1, \ldots, 8$. This completes the proof that the $K[X_4]^\delta$-module $(L'_4/L''_4)^\delta$ is generated by $c_1, \ldots, c_7$.

As in the previous example we obtain that the algebra $(L_4/L''_4)^\delta$ is generated by $\{x_1, c_j f_2^s f_4^{r_j}, c_2 f_2^s f_3 f_4^{r_j} | r_j, s_j \geq 0, j = 1, \ldots, 7\}$.

Nowicki [22] conjectured that if all Jordan cells of the Weitzeböck derivation $\delta$ are of size $2 \times 2$, i.e., $\delta = \delta(1, \ldots, 1)$, then $K[X_{24}]^\delta$ is generated by

$$\{x_{2j-1}, x_{2k-1} x_{2l} - x_{2k} x_{2l-1} | j = 1, \ldots, d, 1 \leq k < l \leq d\}.$$ 

There are several proofs of the conjecture based on different ideas. The unpublished proof by Derksen and the proof by Bedratyuk [3] show that the result follows from well known results of classical invariant theory. Khoury [17, 18] uses Gröbner bases techniques. The proof by Drensky and Makar-Limanov [15] is based on elementary ideas and the approach by Kuroda [20] exploits earlier ideas of Kurano [19] related with the Roberts counterexample to the Hilbert 14th problem [23]. In particular, [15] gives the Gröbner basis of the ideal of relations between the generators of the algebra $K[X_{24}]^\delta$ and a basis for $K[X_{24}]^\delta$ as a vector space. The next examples handle the cases $(L_4/L''_4)^\delta, \delta = \delta(1, 1)$ and $(L_6/L''_6)^\delta, \delta = \delta(1, 1, 1)$.

**Example 5.3.** Let $d = 4$, $\delta = \delta(1, 1)$, and let $\delta(x_1) = 0$, $\delta(x_2) = x_1$, $\delta(x_3) = 0$, $\delta(x_4) = x_3$. Then, see [22] and the comments above, $K[X_4]^\delta$ is generated by the algebraically independent polynomials $f_1 = x_1, f_2 = x_3, f_3 = x_1 x_4 - x_2 x_3$. Hence

$$H_{GL_2}(K[X_4]^\delta, t_1, t_2, z) = \frac{1}{(1 - t_1 z)^2(1 - t_1 t_2 z^2)}.$$ 

By Example 5.2

$$H_{GL_2}((L'_4/L''_4)^\delta, t_1, t_2, z) = \frac{t_1 z^2(t_1 + 3t_2 - t_1^2 t_2 z^2)}{(1 - t_1 z)^2(1 - t_1 t_2 z^2)}.$$ 

The Lie elements

$$c_1 = [x_3, x_1], \ c_2 = [x_2, x_1], \ c_3 = [x_4, x_3], \ c_4 = [x_4, x_1] - [x_3, x_2]$$
Hence, if we show that the elements of the set $\{c_1 f_1^{r_1} f_2^{r_2} f_3^{s_3} \mid q_1, r_1, q_2, r_2, s_j \geq 0, j = 2, 3, 4\}$ satisfy the relation

$C = \{c_1 f_1^{r_1} f_2^{r_2} f_3^{s_3} \mid q_1, r_1, q_2, r_2, s_j \geq 0, j = 2, 3, 4\}$

is

$G(C, t_1, t_2, z) = \frac{t_1^2 z^2}{(1-t_1 z)^2} + \frac{3t_1 t_2 z^2}{(1-t_1 z)^2(1-t_2 z^2)} = H_{GL_2}(\{(L_1/L_0)^{\delta}, t_1, t_2, z\})$.

Hence, if we show that the elements of the set $C$ are linearly independent we shall conclude that the $K[X_4]^{\delta}$-module $(L_1/L_0)^{\delta}$ is generated by $c_1, c_2, c_3, c_4$. Let

$c_1 u_1(f_1, f_2) + c_2 u_2(f_1, f_2, f_3) + c_3 u_3(f_1, f_2, f_3) + c_4 u_4(f_1, f_2, f_3) = 0$

for some $u_1(f_1, f_2), u_j(f_1, f_2, f_3) \in K[f_1, f_2, f_3], j = 2, 3, 4$. Working in the wreath product $A_4 \wr B_4$ we obtain

$0 = (a_3 x_1 - a_1 x_3) u_1(x_1, x_3) + (a_2 x_1 - a_1 x_2) u_2(x_1, x_3, x_1 x_4 - x_2 x_3)$

$= a_1 (-x_3 w_1(x_1, x_3) - x_2 w_2(x_1, x_3, x_1 x_4 - x_2 x_3) - x_4 w_4(x_1, x_3, x_1 x_4 - x_2 x_3))$

$+ a_2(x_1 u_2(x_1, x_3, x_1 x_4 - x_2 x_3) + x_3 u_4(x_1, x_3, x_1 x_4 - x_2 x_3))$

$+ a_3(x_1 u_1(x_1, x_3) - x_4 u_3(x_1, x_3, x_1 x_4 - x_2 x_3) - x_2 u_4(x_1, x_3, x_1 x_4 - x_2 x_3))$

$+ a_4(x_3 u_3(x_1, x_3, x_1 x_4 - x_2 x_3) + x_1 u_4(x_1, x_3, x_1 x_4 - x_2 x_3))$.}

In the coefficient of $a_1$ (which has to be equal to 0), the only expression which does not depend on $x_2$ and $x_4$ is $-x_3 u_1(x_1, x_3)$ and hence $u_1 = 0$. This implies that $-x_3 u_2 - x_4 u_4 = 0$ and $u_2 = x_3 u, u_4 = -x_2 u$ for some $u \in K[X_4]$. Similarly, from the coefficient of $a_2$ we derive $u_2 = x_3 v, u_4 = -x_1 v$ for some $v \in K[X_4]$. It follows from the equalities

$u_2 = x_3 v = x_3 u, u_4 = -x_2 u = -x_1 v$

that $u_2 = u_4 = 0$ which also implies that $u_3 = 0$. In this way the $K[X_4]^{\delta}$-module $(L_1/L_0)^{\delta}$ is generated by $c_1, c_2, c_3, c_4$. This also gives that the algebra $(L_1/L_0)^{\delta}$ is generated by

$\{x_1, c_1 f_1^{r_1}, c_2 f_2^{r_2}, c_3 f_3^{s_3} \mid r_1, r_2, s_j \geq 0, j = 2, 3, 4\}$.

**Example 5.4.** Let $d = 6$, $\delta = \delta(1,1,1)$, and let $\delta(x_1) = \delta(x_3) = \delta(x_5) = 0, \delta(x_2) = x_1, \delta(x_4) = x_3, \delta(x_6) = x_5$. Then, see [22] and [15], $K[X_6]^{\delta}$ is generated by the polynomials

$f_1 = x_1, \quad f_2 = x_3, \quad f_3 = x_5,$

$f_4 = x_1 x_4 - x_2 x_3, \quad f_5 = x_1 x_6 - x_2 x_5, \quad f_6 = x_3 x_6 - x_4 x_5$, with the only defining relation

$$\begin{vmatrix} x_1 & x_3 & x_5 \\ x_1 & x_3 & x_5 \\ x_2 & x_4 & x_6 \end{vmatrix} = f_1 f_6 - f_2 f_5 + f_3 f_4 = 0.$$
Hence we can replace $f_2 f_5$ with $f_1 f_6 + f_3 f_4$ and $K[X_6]^d$ has a basis
\[ \{ f_1^{q_1} f_2^{q_2} f_3^{q_3} f_4^{q_4} f_5^{q_5} f_6^{q_6}, f_1^{q_1} f_4^{q_4} f_6^{q_6} f_5^{q_5+1} f_6^{q_6} | q_j \geq 0 \}. \]

The Hilbert series of $K[X_6]^d$ is
\[ H_{GL_2}(K[X_6]^d, t_1, t_2, z) = \frac{1}{(1-t_1z)^3(1-t_1t_2z^2)^2} + \frac{t_1t_2z^2}{(1-t_1z)^3(1-t_1t_2z^2)^3}. \]

By Example 5.2,
\[ H_{GL_2}((L'_6/L''_6)^d, t_1, t_2, z) = \frac{t_1z^2p(t_1, t_2, z)}{(1-t_1z)^3(1-t_1t_2z^2)^3}, \]
\[ p = 3(t_1 + 2t_2) + t_1(-t_1 + t_2)z - 9t_1^2t_2z^2 + 3t_1^2t_2(3t_2 + t_1)z^3 + t_1^2t_2^2(9t_1 - t_2)z^4 + 3t_1^2t_2^2(t_2 - t_1)z^5 - 3t_1^4t_2z^6 + t_1^5t_2z^7. \]

Following the approach in Example 5.2 we have found a set of ten generators of the $K[X_6]^d$-module $(L'_6/L''_6)^d$:
\[
\begin{align*}
&c_1 = [x_3, x_1], \quad c_2 = [x_5, x_1], \quad c_3 = [x_5, x_3], \\
&c_4 = [x_2, x_1], \quad c_5 = [x_4, x_3], \quad c_6 = [x_6, x_5], \\
&c_7 = [x_4, x_1] - [x_3, x_2], \quad c_8 = [x_6, x_1] - [x_5, x_2], \quad c_9 = [x_6, x_3] - [x_5, x_4], \\
&c_{10} = [x_3, x_2, x_5] - [x_5, x_2, x_3] - [x_4, x_1, x_5] + [x_5, x_1, x_4],
\end{align*}
\]

and 21 relations between them:
\[
\begin{align*}
R_1(3, 0) :& \quad c_3 f_1 = -c_1 f_3 + c_2 f_2, \\
R_2(3, 1) :& \quad c_1 f_4 = -c_4 f_2 - c_5 f_1 + c_7 f_1 f_2, \\
R_3(3, 1) :& \quad c_1 f_5 = -c_2 f_4 - 2c_4 f_2 f_3 + c_7 f_1 f_3 + c_8 f_1 f_2 - c_9 f_1^2, \\
R_4(3, 1) :& \quad c_3 f_4 = -c_1 f_5 + 2c_5 f_1 f_3 - c_7 f_2 f_3 + c_8 f_2^2 - c_9 f_1 f_2, \\
R_5(3, 1) :& \quad c_3 f_5 = -c_2 f_4 - 2c_6 f_1 f_2 - c_7 f_2^2 + c_8 f_2 f_3 + c_9 f_1 f_3, \\
R_6(3, 1) :& \quad c_3 f_6 = -c_5 f_3^2 - c_6 f_2^2 + c_9 f_2 f_3, \\
R_7(3, 1) :& \quad c_5 f_5 = -c_4 f_2^2 - c_7 f_2^2 + c_8 f_1 f_3, \\
R_8(3, 1) :& \quad c_{10} f_1 = -c_1 f_5 - c_4 f_2 f_3 + c_8 f_1 f_2 - c_9 f_2^2, \\
R_9(3, 1) :& \quad c_{10} f_3 = -c_1 f_5 + c_4 f_2 f_3 + c_9 f_1 f_2, \\
R_{10}(3, 1) :& \quad c_{10} f_3 = -c_2 f_4 - c_6 f_1 f_2 - c_7 f_2^2 + c_8 f_2 f_3, \\
R_{11}(4, 1) :& \quad c_1 f_1 f_6 = -c_2 f_4 f_3 - c_4 f_2^2 f_3 + c_5 f_1^2 f_3 + c_8 f_1 f_2^2 - c_9 f_2 f_3, \\
R_{12}(4, 1) :& \quad c_1 f_3 f_6 = c_2 f_4 f_3 + c_5 f_1^2 f_3 + c_8 f_1 f_2^2 - c_9 f_1 f_3, \\
R_{13}(4, 1) :& \quad c_2 f_1 f_6 = -c_2 f_4 f_3 - c_4 f_2^2 f_3 - c_5 f_1^2 f_3 + c_8 f_1 f_2 f_3, \\
R_{14}(3, 2) :& \quad c_{10} f_4 = c_4 f_2 f_6 + c_5 f_1 f_5 - c_7 f_1 f_6 + f_1 f_4) + c_8 f_2 f_4 - c_9 f_1 f_4, \\
R_{15}(3, 2) :& \quad c_{10} f_5 = c_4 f_2 f_6 - c_7 f_1 f_5 - c_7 f_3 f_5 + c_8 f_3 f_4, \\
R_{16}(3, 2) :& \quad c_{10} f_6 = -c_5 f_3^2 - c_6 f_4 f_3 + c_9 f_3 f_4.
\end{align*}
\]

Hence the $K[X_6]^d$-module generated by $\{c_1, \ldots, c_{10}\}$ is spanned by
\[
\begin{align*}
&c_1 f_1^{q_1} f_2^{q_2} f_3^{q_3} f_4^{q_4} f_5^{q_5} f_6^{q_6}, \quad c_2 f_1^{q_1} f_2^{q_2} f_3^{q_3} f_4^{q_4} f_6^{q_6}, \quad c_2 f_1^{q_1} f_2^{q_2} f_3^{q_3} f_4^{q_4} f_6^{q_6};
\end{align*}
\]
We solve the system by the Gauss method keeping the entries of the matrix in the previous examples, we shall work in the abelian wreath product $A \wr B$. Clearly, we may assume that the linear dependence $H$ set is equal to the Hilbert series $H_{GL_2}((L_6/L_6'')^k, t_1, t_2, z)$. Hence, as in the other examples in this section, it is sufficient to show that the set consists of linearly independent elements. Let

$$\sum_{j=1}^{10} c_j u_j = 0,$$

where $u_j$ are polynomials in $f_1, \ldots, f_6$ of the form

$$u_1 = u'_1(f_1, f_2, f_3) + u''_1(f_2, f_6),$$
$$u_2 = u'_2(f_1, f_2, f_3) + u''_2(f_1, f_2, f_3)f_4 + u''_3(f_2, f_3)f_6,$$
$$u_3 = u_3(f_2, f_3),$$
$$u_4 = u'_4(f_1, f_2, f_3, f_4) + u''_4(f_1, f_2, f_3, f_4)f_5 + u'''_4(f_1, f_3, f_4, f_5)f_6,$$
$$u_j = u'_j(f_1, f_2, f_3, f_4, f_6) + u''_j(f_1, f_3, f_4, f_5, f_6)f_5, \quad j = 5, 6, 7, 8, 9,$$
$$u_{10} = \text{const.}$$

Clearly, we may assume that the linear dependence $\sum_{j=1}^{10} c_j u_j = 0$ is homogeneous.

Since there is no linear dependence of degree 3, we conclude that $u_{10} = 0$. As in the previous examples, we shall work in the abelian wreath product $A \wr B$. As in Example 6.2, we shall denote by $v_i$ the coordinate of $a_i$ of $v \in A \wr B$. The six coordinates $v_i$ of

$$v = \sum_{j=1}^{9} c_j u_j = \sum_{i=1}^{6} a_i v_i = 0$$

define a linear homogeneous system

$$v_i = 0, \quad i = 1, \ldots, 6,$$

with unknowns $u_1, \ldots, u_9$ and with a matrix

$$\begin{pmatrix}
  x_3 & -x_5 & 0 & -x_2 & 0 & 0 & -x_4 & -x_6 & 0 \\
  0 & 0 & 0 & x_1 & 0 & 0 & x_3 & x_5 & 0 \\
  x_1 & 0 & -x_5 & 0 & -x_4 & 0 & -x_2 & 0 & -x_6 \\
  0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5 \\
  0 & x_1 & x_3 & 0 & 0 & -x_6 & 0 & -x_2 & -x_4 \\
  0 & 0 & 0 & 0 & 0 & x_5 & x_1 & 0 & x_3
\end{pmatrix}.$$

We solve the system by the Gauss method keeping the entries of the matrix in $K[X_6]$. Since $v \in L_6/L_6''$ and $\sum_{i=1}^{6} x_i v_i = 0$, we can remove the first row of the matrix. Then we bring the matrix in a triangular form

$$\begin{pmatrix}
  x_1 & 0 & -x_5 & 0 & -x_4 & 0 & -x_2 & 0 & -x_6 \\
  0 & x_1 & x_3 & 0 & 0 & -x_6 & 0 & -x_2 & -x_4 \\
  0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5 & 0 \\
  0 & 0 & 0 & 0 & x_5 & 0 & x_1 & 0 & x_3
\end{pmatrix}.$$
We multiply the first row by $x_3$ and add to it the fourth row multiplied by $x_4$.
Similarly we multiply the second row by $x_5$ and add the fifth row multiplied by $x_6$:
\[
\begin{pmatrix}
  x_1 x_3 & 0 & -x_3 x_5 & 0 & 0 & 0 & x_1 x_4 - x_2 x_3 & 0 & -(x_3 x_6 - x_4 x_5) \\
  0 & x_1 x_5 & x_3 x_5 & 0 & 0 & 0 & x_1 x_6 - x_2 x_5 & x_3 x_6 - x_4 x_5 & 0 \\
  0 & 0 & 0 & x_1 & 0 & 0 & x_3 & x_5 & 0 \\
  0 & 0 & 0 & 0 & x_3 & 0 & x_1 & 0 & x_5 \\
  0 & 0 & 0 & 0 & 0 & x_5 & 0 & x_1 & x_3 \\
\end{pmatrix}.
\]

The second row of the matrix gives the equation
\[
x_1 x_5 u_2 + x_3 x_5 u_3 + (x_1 x_6 - x_2 x_5) u_8 + (x_3 x_6 - x_4 x_5) u_9 = 0.
\]

Since $u_3$ depends on $x_3, x_5$ only and the monomials of all other summands depend also on the other variables, we conclude that $u_3 = 0$. Let $w_j$ be the component of $u_j$ which does not depend on $x_2, x_4, x_6, j = 7, 8, 9$. Since $u_1, u_2$ depend linearly on $x_2, x_4, x_6$, the first two rows of the matrix give the system
\[
x_1 x_3 (u_1'' + (x_3 x_6 - x_4 x_5) u_1'') + (x_1 x_4 - x_2 x_3) w_7 - (x_3 x_6 - x_4 x_5) w_9 = 0 \\
x_1 x_5 (u_2'' + (x_1 x_4 - x_2 x_5) u_2' + (x_3 x_6 - x_4 x_5) u_2'') + (x_1 x_6 - x_2 x_5) w_8 + (x_3 x_6 - x_4 x_5) w_9 = 0.
\]
Since $u_1', u_1'', u_2', u_2'', w_7, w_8, w_9$ do not depend on $x_2, x_4, x_6$, we derive that $u_1' = u_2' = 0$. We rewrite the system in the form
\[
-x_3 w_7 x_2 + x_5 (-x_1 x_4 u_1'' + w_9) x_4 + x_3 (x_1 x_3 u_1'' - w_9) x_6 = 0 \\
x_3 (x_1 x_3 u_1'' + w_9) x_2 + x_5 (x_1 x_4 u_2'' - x_1 x_3 u_2'' - w_9) x_4 + (x_1 x_3 x_5 u_2'' + x_1 w_8 + x_3 w_9) x_6 = 0
\]
which implies
\[
w_7 = 0, \quad w_8 = -x_1 x_3 u_2'', \quad w_9 = x_1 x_3 u_2'' \quad x_3 u_1'' - x_1 u_2'' + x_5 u_2'' = 0.
\]

The latter equation gives that every monomial of $u_2''$ depends on $x_1$ or $x_5$ which is impossible because $u_1'' = u_2''(x_3)$. Hence $u_1' = 0, -x_1 u_2' + x_5 u_2'' = 0$, and $u_2''$ depends on $x_1$ which is also impossible. Again, $u_2' = u_2'' = 0$. Now the system of the system with unknowns $u_4, \ldots, u_9$ becomes
\[
\begin{pmatrix}
  x_1 & 0 & 0 & x_3 & x_5 & 0 \\
  0 & x_3 & x_5 & 0 & 0 & x_1 \\
  0 & 0 & x_5 & 0 & x_1 & x_3 \\
  0 & 0 & 0 & x_1 x_4 - x_2 x_3 & 0 & -x_3 x_6 - x_4 x_5 \\
  0 & 0 & 0 & 0 & x_1 x_6 - x_2 x_5 & x_3 x_6 - x_4 x_5 \\
\end{pmatrix}
\]
and the solution of the system is
\[
u_4 = f_2^3 v, \quad u_5 = f_2^3 v, \quad u_6 = f_4^2 v, \quad u_7 = -f_5 f_6 v, \quad u_8 = f_4 f_6 v, \quad u_9 = -f_4 f_5, \quad v \in K[x_6]^5.
\]

Hence $f_2^3$ divides
\[
u_4 = u_4'(f_1, f_2, f_3, f_4) + u_4''(f_1, f_2, f_3, f_4) f_6 + u_4''(f_1, f_3, f_4, f_5) f_5 + u_4''(f_1, f_3, f_4, f_5) f_6
\]
and therefore $f_6$ divides $u_4' + u_4'' f_5$. If we order the variables by $x_6 > x_4 > x_5 > x_1 > x_3$, then the leading monomial of $u_4' + u_4'' f_5$ with respect to the lexicographical order is the leading monomial of $u_4''(x_1, x_5, x_1 x_4, x_1 x_6) x_1 x_6$ which cannot be divisible by $f_6$ with leading monomial $x_3 x_6$. Hence $u_4' + u_4'' f_5 = 0$.

Again, $f_6$ does not divide $u_4' + u_4'' f_5$ and, as a result, $f_2^3$ cannot divide $u_4$. Hence $v = 0$ and this completes the proof.
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