On the rainbow antimagic coloring of vertex amalgamation of graphs

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Abstract. The purpose of this study is to develop rainbow antimagic coloring. This study is a combination of two notions, namely antimagic and rainbow concept. If every vertex of graph $G$ is labeled with the antimagic labels and then edge weight of antimagic labels are used to assign a rainbow coloring. The minimum number of colors for a rainbow path to exist with the condition satisfying the edge weights $w(x) \neq w(y)$ for any two vertices $x$ and $y$ is the definition of the rainbow antimagic connection number $rac(G)$. In this study, we use connected graphs and simple graphs in obtaining the rainbow antimagic connection number. This paper will explain the rainbow antimagic coloring on some graphs and get their formula of the rainbow antimagic connection number. We have obtained $rac(G)$ where $G$ is vertex amalgamation of graphs, namely path, star, broom, paw, fan, and triangular book graph.

1. Introduction

Over time, the topics in graph theory are developing and generating new ideas, for example rainbow antimagic coloring. The combination of two notions, namely rainbow connection and antimagic labeling is called rainbow antimagic coloring (RAC).

Hartsfield and Ringel [13] define that graph $G$ is said to be antimagic labeling if every edges are labeled with the integers $\{1, 2, \ldots, k\}$ so that vertex weight of two vertices is different, that is, all vertex on graph $G$ do not have the same vertex weight. The dense graph is proven to be antimagic according to Alon et al [2]. The antimagic labeling of trees has been discovered by Liang et al [20]. More study of antimagic labeling some graphs in [5, 15, 21, 23].

Chartrand, et al. [6] were the first to introduce the concept of rainbow connection. In the rainbow connection concept, there are adjacent edges that may have the same color where the color uses the integers $\{1, 2, \ldots, k\}$. A rainbow path in graph $G$ can be formed if there are no adjacent edges that have the same color. A graph $G$ is said to be rainbow connected if any two of its vertices are connected by a rainbow path. Rainbow connection can be defined as the edge coloring that makes the graph $G$ rainbow connected. Li et al [19] have discovered the exact value of 2-connected graph and chordal graph. The $rc$ and $src$ of fan and sun graph was invented by Sy et al [25]. Yandera et al [27] obtain the rainbow connection of $Amal_1(P_{m,2})$. More study of rainbow connection some graphs in [7, 10, 11, 12, 18].
A graph is said to be rainbow antimagic coloring if every vertex of graph $G$ is labeled with the antimagic labels and then edge weight of antimagic labels are used to assign a rainbow connection such that there is a rainbow path with all different edge weight [24]. The minimum number of colors for a rainbow path to exist with the conditional edge weights $w(x) \neq w(y)$ for any two vertices $x$ and $y$ is the definition of the rainbow antimagic connection number and denoted by $\text{rac}(G)$. Dafik et al. [8] obtained $\text{rac}(G)$ where $G$ is simple graph namely path, star, and cycle graph. Intan K et al. [16] found the rainbow antimagic connection number of related wheel graph. Sulistiono et al [24] made a theorem to prove the lower bound of $\text{rac}(G)$. More study on rainbow antimagic coloring can be seen at [1, 4, 9, 22].

We investigated about $\text{rac}(G)$ where $G$ is vertex amalgamation of path, star, broom, paw, fan, and triangular book graph. Lee, et al. [17] define that the graph $G$ is operated by vertex amalgamation if there is $v_0 \in V(G)$ as a fixed center point, then graph $G$ is multiplied by $n$ with $v_0$ as fixed center point and denoted by $\text{Amal}(G, v_0, n)$ with $n \geq 2$. The study of amalgamation product in [3, 12, 14, 15, 26]. For illustration of vertex amalgamation is provided in Figure 2.

![Figure 1. Vertex amalgamation of fan, Amal($F_n, v, m$)](image)

2. Previous Results

Some previous results related to the concept of rainbow antimagic coloring will be presented in this chapter.

**Preposition 1** [6] If $T$ is a tree with $|E(T)| = n$, then $rc(T) = n$.

**Theorem 1** [24] To proof lower bound of $\text{rac}(H)$ can use $\text{rac}(H) \geq \max\{\Delta(H), rc(H)\}$

**Theorem 2** [12] If $A$ is a amalgamation of fan graph, then $rc(A) = 3$.

**Theorem 3** [4] If $B$ is a lollipop graph with $m = 3$, then $\text{rac}(B) = n + 2$.

**Theorem 4** [1] If $C$ is a triangular book graph, then $\text{rac}(C) = n + 2$.

3. Results and Discussion

In this chapter, several theorems of rainbow antimagic coloring on vertex amalgamation products that have been obtained include:
Theorem 5 If $H_1$ is Amal($P_n, v, m$) graph with $m, n \geq 2$, then $rac(H_1) = mn - m$.

Proof. Let $H_1$ be a vertex amalgamation product of path with vertex set $V(H_1) = \{y\} \cup \{x_{a,b} : 1 \leq a \leq m \& \ 1 \leq b \leq n - 1\}$ and edge set $E(H_1) = \{yx_{a,1} : 1 \leq a \leq m\} \cup \{x_{a,b}x_{a,b+1} : 1 \leq a \leq m \& \ 1 \leq b \leq n - 2\}$. So, we get $|V(H_1)| = mn - m + 1$ and $|E(H_1)| = mn - m$.

We need to prove the correctness of the formula $rac(H_1)$ is $mn - m$ with $m, n \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $rac(H_1) \geq mn - m$ and $rac(H_1) \leq mn - m$.

First, we prove the correctness of the lower bound of $rac(H_1)$ is $rac(H_1) \geq mn - m$. We know that $H_1$ graph is classified as a tree graph. Based on Proposition 1 [6] and Theorem 1 [24], so that:

$$rac(H_1) \geq \max\{\Delta(H_1), rc(H_1)\}$$

$$mn - m \geq \max\{m, mn - m\}$$

$$mn - m \geq mn - m$$

So we got that the lower bound of $rac(H_1)$ is $rac(H_1) \geq mn - m$ with $m, n \geq 2$.

Next, we prove the correctness of the upper bound of $rac(H_1)$ is $rac(H_1) \leq mn - m$. We define the vertex function of $H_1$ graph with $f : V(I) \rightarrow \{1, 2, \ldots, |V(I)|\}$ are

$$f(y) = 1$$

$$f(x_{a,b}) = a + bm - m + 1, 1 \leq a \leq m \& \ 1 \leq b \leq n - 1$$

Obviously, we get the edge weights from a predefined vertex function where the edge weights will be used as edge coloring. The edge weights of $H_1$ graph are

$$w(yx_{a,1}) = a + 2, 1 \leq a \leq m$$

$$w(x_{a,b}x_{a,b+1}) = 2a + 2bm - m + 2, 1 \leq a \leq m \& \ 1 \leq b \leq n - 1$$

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $mn - m$. The rainbow path of $H_1$ graph that is formed can be seen in the Table 1. So we got that the upper bound of $rac(H_1)$ is $rac(H_1) \leq mn - m$.

Table 1. The rainbow path from $x$ to $y$ on $H_1$ graph.

| case | $x$    | $y$    | rainbow path |
|------|--------|--------|--------------|
| 1    | $x_{k,b}$ | $x_{l,c}$ | $x_{k,b}, \ldots, x_{k,1}, y, x_{l,1}, \ldots, x_{l,c}$ |

As a result of the explanation above, we have established that the lower and upper bounds of $rac(H_1)$ are $mn - m \leq rac(H_1) \leq mn - m$. So that the correctness of the formula $rac(H_1)$ is $mn - m$ with $m, n \geq 2$ has been proven. □

For illustration of $rac(H_1)$ is provided in Figure 2 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.
Figure 2. The RAC of $H_1$ graph with $n = 5$ and $m = 6$.

**Theorem 6** If $H_2$ is $\text{Amal}(S_n, v, m)$ graph with $m, n \geq 2$, then $\text{rac}(H_2) = mn$.

**Proof.** Let $H_2$ be a vertex amalgamation product of star with vertex set $V(H_2) = \{z\} \cup \{x_a : 1 \leq a \leq m\} \cup \{y_{a,b} : 1 \leq a \leq m & 1 \leq b \leq n - 1\}$ and edge set $E(H_2) = \{zx_a : 1 \leq a \leq m\} \cup \{x_ay_{a,b} : 1 \leq a \leq m & 1 \leq b \leq n - 1\}$. So, we get $|V(H_2)| = mn + 1$ and $|E(H_2)| = mn$.

We need to prove the correctness of the formula $\text{rac}(H_2)$ is $mn$ with $m, n \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $\text{rac}(H_2) \geq mn$ and $\text{rac}(H_2) \leq mn$.

First, we prove the correctness of the lower bound of $\text{rac}(H_2)$ is $\text{rac}(H_2) \geq mn$ with $m, n \geq 2$.

Next, we prove the correctness of the upper bound of $\text{rac}(H_2)$ is $\text{rac}(H_2) \leq mn$. We define the vertex function of $H_2$ graph with $f : V(H_2) \rightarrow \{1, 2, ..., |V(H_2)|\}$ are

$$
\begin{align*}
f(z) & = 1 \\
f(x_a) & = a + 1 , 1 \leq a \leq m \\
f(y_{a,b}) & = m + an - n - a + b + 2 , 1 \leq a \leq m & 1 \leq b \leq n - 1
\end{align*}
$$

Obviously, we get the edge weights from a predefined vertex function where the edge weights will be used as edge coloring. The edge weights of $H_2$ graph are

$$
\begin{align*}
w(zx_a) & = a + 2 , 1 \leq a \leq m \\
w(x_ay_{a,b}) & = m + an - n + b + 3 , 1 \leq a \leq m & 1 \leq b \leq n - 1
\end{align*}
$$

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $mn$. The rainbow path of $H_2$ graph that is formed can be seen in the Table 2. So we got that the upper bound of $\text{rac}(H_2)$ is $\text{rac}(H_2) \leq mn$. 
Table 2. The rainbow path from $x$ to $y$ on $H_2$ graph.

| case | $x$ | $y$ | rainbow path       |
|------|-----|-----|--------------------|
| 1    | $y_k,b$ | $y_l,c$ | $y_k,b, x_k,z, x_l, y_l,c$ |

As a result of the explanation above, we have established that the lower and upper bounds of $rac(H_2)$ are $mn \leq rac(H_2) \leq mn$. So that the correctness of the formula $rac(H_2)$ is $mn$ with $m,n \geq 2$ has been proven. □

For illustration of $rac(H_2)$ is provided in Figure 3 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.

![Figure 3](image_url)

Figure 3. The RAC of $H_2$ graph with $n = 4$ and $m = 6$.

**Theorem 7** If $H_3$ is $Amal(Br_d^n,v,m)$ graph with $n = 2d - 1$ and every integer $d,m \geq 2$, then $rac(H_3) = 2dm - 2m$.

**Proof.** Let $H_3$ be a vertex amalgamation product of broom with vertex set $V(H_3) = \{z\} \cup \{x_{a,b} : 1 \leq a \leq m, 1 \leq b \leq d - 1\} \cup \{y_{a,c} : 1 \leq a \leq m \& 1 \leq c \leq d - 1\}$ and edge set $E(H_3) = \{zx_{a,1} : 1 \leq a \leq m\} \cup \{x_{a,b}x_{a,b+1} : 1 \leq a \leq m, 1 \leq b \leq d - 2\} \cup \{x_{a,d-1}y_{a,c} : 1 \leq a \leq m \& 1 \leq c \leq d - 1\}$. So, we get $|V(H_3)| = 2dm - 2m + 1$ and $|E(H_3)| = 2dm - 2m$.

We need to prove the correctness of the formula $rac(H_3)$ is $2dm - 2m$ with $d,m \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $rac(H_3) \geq 2dm - 2m$ and $rac(H_3) \leq 2dm - 2m$.

First, we prove the correctness of the lower bound of $rac(H_3)$ is $rac(H_3) \geq 2dm - 2m$. We know that $H_3$ graph is classified as a tree graph. Based on Proposition 1 [6] and Theorem 1 [24], so that:

$$rac(H_3) \geq \max \{\Delta(H_3), rc(H_3)\}$$

$$2dm - 2m \geq \max \{m, 2dm - 2m\}$$

$$2dm - 2m \geq 2dm - 2m$$

So we got that the lower bound of $rac(H_3)$ is $rac(H_3) \geq 2dm - 2m$ with $d,m \geq 2$. 

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Next, we prove the correctness of the upper bound of $\text{rac}(H_3)$ is $\text{rac}(H_3) \leq 2dm - 2m$. We define the vertex function of $H_3$ graph with $f : V(H_3) \rightarrow \{1, 2, \ldots, |V(H_3)|\}$ are

$$f(z) = 1$$
$$f(x_{a,b}) = bm - m + a + 1, 1 \leq a \leq m, 1 \leq b \leq d - 1$$
$$f(y_{a,c}) = dm - m + (a - 1)(d - 1) + c + 1, 1 \leq a \leq m \& 1 \leq c \leq d - 1$$

Obviously, we get the edge weights from a predefined vertex function where the edge weights will be used as edge coloring. The edge weights of $H_3$ graph are

$$w(zx_{a,1}) = a + 2, 1 \leq a \leq m$$
$$w(x_{a,b}x_{a,b+1}) = 2bm - m + 2a + 2, 1 \leq a \leq m, 1 \leq b \leq d - 1$$
$$w(x_{a,d-1}y_{a,c}) = 2dm - 3m + (a - 1)(d - 1) + a + c + 2, 1 \leq a \leq m \& 1 \leq c \leq d - 1$$

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $2dm - 2m$. The rainbow path of $H_3$ graph that is formed can be seen in the Table 3. So we got that the upper bound of $\text{rac}(H_3)$ is $\text{rac}(H_3) \leq 2dm - 2m$.

**Table 3.** The rainbow path from $x$ to $y$ on $H_3$ graph.

| case | $x$ | $y$ | rainbow path |
|------|-----|-----|--------------|
| 1    | $y_{p,c}$ | $y_{q,d}$ | $y_{p,c}, y_{p,d}, x_{p,b}, \ldots, x_{p,1}, z, x_{q,1}, \ldots, x_{q,b}, y_{q,d}$ |

As a result of the explanation above, we have established that the lower and upper bounds of $\text{rac}(H_3)$ are $2dm - 2m \leq \text{rac}(H_3) \leq 2dm - 2m$. So that the correctness of the formula $\text{rac}(H_3)$ is $2dm - 2m$ with $d, m \geq 2$ has been proven. □

For illustration of $\text{rac}(H_3)$ is provided in Figure 4 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.

**Figure 4.** The RAC of $H_3$ graph with $d = 4$ and $m = 6$. 
Theorem 8 If $H_4$ is Amal$(P_{(3,n)}, v, m)$ graph with $m, n \geq 2$, then $rac(H_4) = mn + 2$.

Proof. Let $H_4$ be a vertex amalgamation product of paw with vertex set $V(H_4) = \{z\} \cup \{x_{a,b} : 1 \leq a \leq m \& 1 \leq b \leq n - 1\} \cup \{y_{a,c} : 1 \leq a \leq m, 1 \leq c \leq 2\}$ and edge set $E(H_4) = \{zx_{a,1} : 1 \leq a \leq m\} \cup \{x_{a,b}x_{a,b+1} : 1 \leq a \leq m \& 1 \leq b \leq n - 2\} \cup \{x_{a,n-1}y_{a,c} : 1 \leq a \leq m, 1 \leq c \leq 2\} \cup \{y_{a,1}y_{a,2} : 1 \leq a \leq m\}$. So, we get $|V(H_4)| = mn + m + 1$ and $|E(H_4)| = mn + 2m$.

We need to prove the correctness of the formula $rac(H_4)$ is $mn + 2$ with $m, n \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $rac(H_4) \geq mn + 2$ and $rac(H_4) \leq mn + 2$.

First, we prove the correctness of the lower bound of $rac(H_4)$ is $rac(H_4) \geq mn + 2$. We know that $H_4$ graph is $P_{(3,n)}$ graph which is copied as many as $m$. Based on Theorem 3 [4], so that $H_4$ graph requires as many colors as $mn + 2$. So we got that the lower bound of $rac(H_4)$ is $rac(H_4) \geq mn + 2$ with $m, n \geq 2$.

Next, we prove the correctness of the upper bound of $rac(H_4)$ is $rac(H_4) \leq mn + 2$. We define the vertex function of $H_4$ graph with $f : V(H_4) \rightarrow \{1, 2,..., |V(H_4)|\}$ are

$$
\begin{align*}
\text{f}(z) & = 1 \\
\text{f}(x_{a,b}) & = a + 1 + bm - m , 1 \leq a \leq m \& 1 \leq b \leq n - 1 \\
\text{f}(y_{a,1}) & = mn - m + a + 1 , 1 \leq a \leq m \\
\text{f}(y_{a,2}) & = mn + m - a + 2 , 1 \leq a \leq m
\end{align*}
$$

Obviously, we get the edge weights from a predefined vertex function where the edge weights will be used as edge coloring. The edge weights of $H_4$ graph are

$$
\begin{align*}
w(zx_{a,1}) & = a + 2 , 1 \leq a \leq m \\
w(x_{a,b}x_{a,b+1}) & = 2bm - m + 2a + 2 , 1 \leq a \leq m \& 1 \leq b \leq n - 2 \\
w(x_{a,n-1}y_{a,1}) & = 2mn - 3m + 2a + 2 , 1 \leq a \leq m \& 1 \leq b \leq n - 2 \\
w(x_{a,n-1}y_{a,2}) & = 2mn - m + 3 , 1 \leq a \leq m \& 1 \leq b \leq n - 2 \\
w(y_{a,1}y_{a,2}) & = 2mn + 3 , 1 \leq a \leq m \& 1 \leq b \leq n - 2
\end{align*}
$$

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $mn + 2$. The rainbow path of $H_4$ graph that is formed can be seen in the Table 4. So we got that the upper bound of $rac(H_4)$ is $rac(H_4) \leq mn + 2$.

Table 4. The rainbow path from $x$ to $y$ on $H_4$ graph.

| case | $x$  | $y$  | rainbow path | condition     |
|------|------|------|--------------|---------------|
| 1    | $y_{k,1}$ | $y_{l,c}$ | $y_{k,1}, x_{k,a}, \ldots, x_{k,1}, z, x_{l,1}, \ldots, x_{l,b}, y_{l,c}$ | $1 \leq c \leq 2$ |
| 2    | $y_{k,2}$ | $y_{l,c}$ | $y_{k,2}, x_{k,a}, \ldots, x_{k,1}, z, x_{l,1}, \ldots, x_{l,b}, y_{l,c}$ | $c = 1$ |
| 3    | $y_{k,2}$ | $y_{l,c}$ | $y_{k,1}, x_{k,a}, \ldots, x_{k,1}, z, x_{l,1}, \ldots, x_{l,b}, y_{l,c-1}, y_{l,c}$ | $c = 2$ |

As a result of the explanation above, we have established that the lower and upper bounds of $rac(H_4)$ are $mn + 2 \leq rac(H_4) \leq mn + 2$. So that the correctness of the formula $rac(H_4)$ is $mn + 2$ with $m, n \geq 2$ has been proven. \(\square\)

For illustration of $rac(H_4)$ is provided in Figure 5 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.
Figure 5. The RAC of $H_4$ graph with $n = 4$ and $m = 6$.

**Theorem 9** If $H_5$ is Amal($F_n, v, m$) graph with $m, n \geq 2$, then $\text{rac}(H_5) = mn + 1$.

**Proof.** Let $H_5$ be a vertex amalgamation product of star with vertex set $V(H_5) = \{y\} \cup \{x_{a,b} : 1 \leq a \leq m & 1 \leq b \leq n\}$ and edge set $E(H_5) = \{yx_{a,b} : 1 \leq a \leq m & 1 \leq b \leq n\} \cup \{x_{a,b}x_{a,b+1} : 1 \leq a \leq m & 1 \leq b \leq n - 1\}$. So, we get $|V(H_5)| = mn + 1$ and $|E(H_5)| = 2mn - m$.

We need to prove the correctness of the formula $\text{rac}(H_5)$ is $mn + 1$ with $m, n \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $\text{rac}(H_5) \geq mn + 1$ and $\text{rac}(H_5) \leq mn + 1$.

First, we prove the correctness of the lower bound of $\text{rac}(H_5)$ is $\text{rac}(H_5) \geq mn + 1$. We know that $H_5$ graph is $F_n$ graph which is copied as many as $m$. Based on Theorem 1 [24] and Theorem 2 [12], so that:

$$\text{rac}(H_5) \geq \max\{\Delta(H_5), \text{rc}(H_5)\}$$

$$mn + 1 \geq \max\{mn, 3\}$$

$$mn + 1 \geq mn$$

So we got that the lower bound of $\text{rac}(H_5)$ is $\text{rac}(H_5) \geq mn + 1$ with $m, n \geq 2$.

Next, we prove the correctness of the upper bound of $\text{rac}(H_5)$ is $\text{rac}(H_5) \leq mn + 1$. We define the vertex function of $H_5$ graph with $f : V(H_5) \rightarrow \{1, 2, ..., |V(H_5)|\}$ are

$$f(y) = \begin{cases} 
\frac{mn+m+2}{2}, & \text{if } n = \text{odd} \\
\frac{mn+n-2}{2}, & \text{if } n = \text{even}
\end{cases}$$

$$f(x_{a,b}) = \begin{cases} 
\frac{6a+b-5}{2}, & \text{if } n = \text{odd and } b = \text{odd} \\
\frac{2mn-4a-b+8}{2}, & \text{if } n = \text{odd and } b = \text{even} \\
\frac{8a+2b-6}{4}, & \text{if } n = \text{even and } b = \text{odd} \\
\frac{6mn-8a-2b+16}{4}, & \text{if } n = \text{even and } b = \text{even}
\end{cases}$$

Obviously, we get the edge weights from a predefined vertex function where the edge weights
will be used as edge coloring. The edge weights of $H_5$ graph are

$$w(yx_{a,b}) = \begin{cases} 
\frac{mn+m+6a+b-3}{2}, & \text{if } n = \text{odd and } b = \text{odd} \\
\frac{3mn+m-4a-b+10}{2}, & \text{if } n = \text{odd and } b = \text{even} \\
\frac{2mn+2n+8a+2b-10}{4}, & \text{if } n = \text{even and } b = \text{odd} \\
\frac{8mn-2n-8a-2b-4}{4}, & \text{if } n = \text{even and } b = \text{even} 
\end{cases}$$

$$w(x_{a,b}x_{a,b+1}) = \begin{cases} 
\frac{2mn+2a+2}{2}, & \text{if } n = \text{odd and } b = \text{odd} \\
\frac{2mn+2a+3}{2}, & \text{if } n = \text{odd and } b = \text{even} \\
\frac{mn+2}{2}, & \text{if } n = \text{even and } b = \text{odd} \\
\frac{mn+3}{2}, & \text{if } n = \text{even and } b = \text{even} 
\end{cases}$$

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $mn + 1$. The rainbow path of $H_5$ graph that is formed can be seen in the Table 5. So we got that the upper bound of $\text{rac}(H_5)$ is $\text{rac}(H_5) \leq mn + 1$.

**Table 5.** The rainbow path from $x$ to $y$ on $H_5$ graph.

| case | $x$ | $y$ | rainbow path |
|------|-----|-----|--------------|
| 1    | $x_{k,b}$ | $x_{l,c}$ | $x_{k,b}, y, x_{l,c}$ |

As a result of the explanation above, we have established that the lower and upper bounds of $\text{rac}(H_5)$ are $mn + 1 \leq \text{rac}(H_5) \leq mn + 1$. So that the correctness of the formula $\text{rac}(H_5)$ is $mn + 1$ with $m, n \geq 2$ has been proven. □

For illustration of $\text{rac}(H_5)$ is provided in Figure 6 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.

![Figure 6. The RAC of $H_5$ graph with $n = 5$ and $m = 4$.](image-url)
Theorem 10 If $H_6$ is Amal$(Tb_n, v, m)$ graph with $m, n \geq 2$, then $rac(H_6) = mn + 2m$.

Proof. Let $H_6$ be a vertex amalgamation product of star with vertex set $V(H_6) = \{z\} \cup \{x_a : 1 \leq a \leq m\} \cup \{y_{a,b} : 1 \leq a \leq m \& 1 \leq b \leq n\}$ and edge set $E(H_6) = \{zx_a : 1 \leq a \leq m\} \cup \{zy_{a,b} : 1 \leq a \leq m \& 1 \leq b \leq n\}$. So, we get $|V(H_6)| = mn + m + 1$ and $|E(H_6)| = 2mn + m$.

We need to prove the correctness of the formula $rac(H_6)$ is $mn + 2m$ with $m, n \geq 2$, it is necessary to prove it by using the lower and upper bounds, namely $rac(H_6) \geq mn + 2m$ and $rac(H_6) \leq mn + 2m$.

First, we prove the correctness of the lower bound of $rac(H_6)$ is $mn + 2m$. We know that $H_6$ graph is $Tb_n$ graph which is copied as many as $m$. Based on Theorem 4 [1], so that $H_6$ graph requires as many colors as $mn + 2m$. So we got that the lower bound of $rac(H_6)$ is $rac(H_6) \geq mn + 2m$ with $m, n \geq 2$.

Next, we prove the correctness of the upper bound of $rac(H_6)$ is $rac(H_6) \leq mn + 2m$. We define the vertex function of $H_6$ graph with $f : V(H_6) \rightarrow \{1, 2, \ldots, |V(H_6)|\}$ are

\[
\begin{align*}
  f(z) & = 1 \\
  f(x_a) & = a + 1, 1 \leq a \leq m \\
  f(y_{a,b}) & = m + (n - 1)(a - 1) + b + 1, 1 \leq a \leq m \& 1 \leq b \leq n
\end{align*}
\]

Obviously, we get the edge weights from a predefined vertex function where the edge weights will be used as edge coloring. The edge weights of $H_6$ graph are

\[
\begin{align*}
  w(zx_a) & = a + 2, 1 \leq a \leq m \\
  w(zy_{a,b}) & = m + (n - 1)(a - 1) + b + 2, 1 \leq a \leq m \& 1 \leq b \leq n \\
  w(x_{i}y_{a,b}) & = m + (n - 1)(a - 1) + a + b + 2, 1 \leq a \leq m \& 1 \leq b \leq n
\end{align*}
\]

Based on the edge weight function that has been obtained, a rainbow path can be formed if there are as many colors as $mn + 2m$. The rainbow path of $H_6$ graph that is formed can be seen in the Table 6. So we got that the upper bound of $rac(H_6)$ is $rac(H_6) \leq mn + 2m$.

**Table 6.** The rainbow path from $x$ to $y$ on $H_6$ graph.

| case | $x$   | $y$   | rainbow path   |
|------|-------|-------|----------------|
| 1    | $x_k$ | $x_l$ | $x_k, z, x_l$  |
| 2    | $y_{k,b}$ | $x_{l,c}$ | $y_{k,b}, z, x_{l,c}$ |
| 3    | $x_k$ | $x_{l,c}$ | $x_k, z, x_{l,c}$ |

As a result of the explanation above, we have established that the lower and upper bounds of $rac(H_6)$ are $mn + 2m \leq rac(H_6) \leq mn + 2m$. So that the correctness of the formula $rac(H_6)$ is $mn + 2m$ with $m, n \geq 2$ has been proven. $\square$

For illustration of $rac(H_6)$ is provided in Figure 7 and the vertex labels on the graph are black numbers, while the edge weights on the graph are blue numbers.
Figure 7. The RAC of $H_6$ graph with $n = 4$ and $m = 4$.

4. Conclusion

In this paper, we learned about the rainbow antimagic coloring of vertex amalgamation of graphs. We have concluded the exact value of $\text{rac}(G)$ where $G$ is vertex amalgamation of graphs namely path, star, broom, paw, fan, and triangular book graph. The open problems in this paper include.

Open Problem

(i) Determine $\text{rac}(G)$ where $G$ is graph with another operations in graph such as comb, corona, edge corona, etc.

(ii) Determine $\text{rac}(G)$ where $G$ is another special graph or another graph family.

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