I. INTRODUCTION

Regardless of the dark energy nature, one of the most popular ad hoc theories to explain the current accelerated expansion of our universe is dark energy [1]. Necessity of dark energy comes from the fact that we need to have a negative pressure (repulsive action) to interpret cosmic expansion. Since the determination of dark energy nature is an important challenge for the physics communities, it is inevitable to look for an alternative theories scenario for dark energy to address the observational evidences. Modified gravity theory, instead of general relativity, is an alternative plan to describe the late time acceleration [2].

In recent years, variety of Modified theories of classical gravity have been proposed to solve some puzzles of standard general relativity. Amongst them the well-known $F(R)$ theory, whose Lagrangian density is an arbitrary function of the Ricci scalar, is quite special and received a growing attention (see for example [3] and references therein). $F(R)$ gravity provides a technically powerful tool to deal with the early time inflation [4], late time acceleration [5], the hierarchy and singularity problems [6, 7] and (the nature of) dark energy [8]. Holographic superconductor with linear and nonlinear Maxwell field in the frame of modified gravity has been studied [5, 10] and the condensation effects of nonlinearity in Maxwell field and curvature terms have been investigated in [10]. Although, the field equations of $F(R)$ theories are of fourth-order and solving them, directly, is so complicated, their valuable consequences motivate us to consider them and investigate their interesting properties. Using a suitable conformal transformation, it has been shown that $F(R)$ gravity models are equivalent to classical Einstein’s gravity with an extra scalar field. Also, we can apply some limitations on the model parameters to guarantee that the model follows the stability condition (the scalaron is not a tachyon) and has no ghosts [11, 12]. Some viable models of $F(R)$ theories have been widely investigated in the literature over the past few years [12–18].

In addition to the $F(R)$ theories, one of the interesting subjects for recent study is the investigation of three dimensional black holes [19]. Considering three dimensional spacetimes helps us to find some conceptual issues in the black hole properties, quantum view of gravity and string theory [20, 21]. Therefore, theoretical physicists have an interest in the $(2 + 1)$-dimensional manifolds and their properties [22]. Moreover, three dimensional solutions perform an essential role to improve our comprehension of gravitational interaction in low dimensional manifolds [23]. In addition, it is interesting to study the asymptotic behavior as well as near horizon solutions of BTZ black holes [24] and generalize its properties to higher dimensions [25]. In this work, we investigate some interesting solutions of $F(R)$ gravity in $(2 + 1)$-dimensions.

The organization of the paper is as follows: at first, we give a brief review of the field equations of $F(R)$ gravity. In the next section, we obtain the near horizon solution for $F(R)$ gravity in three dimensional static spacetime. After that, we look for the existence of exact solutions of some interesting models. We finish our paper with some conclusions.

II. BASIC FIELD EQUATIONS AND METRIC ANSÄTZ

Action of $F(R)$ gravity in the presence of matter field in arbitrary dimensions and its related field equations have been obtained before [26]. In addition, static and spherically symmetric solutions of F(R) gravity with constant Ricci scalar have been investigated [3, 26]. Following Refs. [26, 27], one finds that the action of 3-dimensional

Motivated by the well-known charged BTZ black holes, we look for $(2 + 1)$-dimensional solutions of $F(R)$ gravity. At first we investigate some near horizon solutions and after that we obtain asymptotically Lifshitz black hole solutions. Finally, we discuss about rotating black holes with exponential form of $F(R)$ theory.
$F(R) = R + f(R)$ gravity, in the presence of matter field has the form of

$$I = -\frac{1}{16\pi} \int d^3x \sqrt{-g} [R + f(R)] + I_{\text{matt}},$$  \hspace{1cm} (1)

where $I_{\text{matt}}$ is the action of matter fields. One can vary Eq. (1) with respect to the metric $g_{\mu\nu}$ to obtain the following field equations

$$R_{\mu\nu}(1 + f_R) - \frac{1}{2} g_{\mu\nu} [R + f(R)] + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla^2) f_R = T^\text{matt}_{\mu\nu},$$  \hspace{1cm} (2)

where $f_R \equiv df(R)/dR$ and $T^\text{matt}_{\mu\nu}$ is the standard matter stress-energy tensor which is derived from the matter action $I_{\text{matt}}$ in Eq. (1). Here, we want to obtain the 3-dimensional static and spherically symmetric solutions of Eq. (2). We assume the metric has the following ansatz

$$ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\phi^2.$$  \hspace{1cm} (3)

Considering the sourceless ($T^\text{matt}_{\mu\nu} = 0$) field equation (2) with the metric (3), one can obtain the following independent differential equations

$$2rg(r)f''_R + [rg'(r) + 2g(r)]f'_R + [rg''(r) + g'(r)]f_R - g'(r) = -rf(R),$$  \hspace{1cm} (4)

$$[rg'(r) + 2g(r)]f'_R + [rg''(r) + g'(r)]f_R - g'(r) = -rf(R),$$  \hspace{1cm} (5)

$$2rg(r)f''_R + 2rg'(r)f'_R + 2g'(r)f_R - rg''(r) = -rf(R),$$  \hspace{1cm} (6)

corresponding to $tt$, $rr$ and $\phi\phi$ components of Eq. (2), respectively. It is notable that the prime and double prime are, respectively, the first and second derivatives with respect to $r$. Here, we study black hole solutions with constant Ricci scalar (so $f''_R = f'_R = 0$), and therefore it is easy to show that the field equations (4-6) reduce to

$$[rg''(r) + g'(r)] f_R - g'(r) = -rf(R),$$  \hspace{1cm} (7)

$$2g'(r) f_R - rg''(r) = -rf(R),$$  \hspace{1cm} (8)

Equating the left hand sides of Eqs. (7) and (8), we obtain

$$[rg''(r) - g'(r)] (f_R + 1) = 0,$$  \hspace{1cm} (9)

with the trivial uncharged static BTZ solution $g(r) = \frac{r_+^2}{6} - M$ with arbitrary $f(R)$, and also $f(R) = -R$ for arbitrary $g(r)$. It is important to note that we are looking for a solution which satisfies both Eqs. (7) and (8), simultaneously, and the mentioned trivial BTZ solution is the solution of them for arbitrary $f(R)$. In addition, considering $f(R) = -R$ ($F(R) = 0$), both Eqs. (7) and (8) are satisfied for $g(r) = \frac{4\Lambda}{3} r^2 - M + \frac{C}{r}$, not arbitrary $g(r)$. As we will see, we can interpret $C$ as an electric charge parameter and so one can obtain charged solution of $F(R)$ gravity for the special case $f(R) = -R$. There are some interesting notes arising from this solution. We find that the field equations of $F(R)$ action admit a static solution as $g(r) = \frac{4\Lambda}{6} r^2 - M + \frac{C}{r}$ for $f(R) = -R$. On the other hand, one can obtain the same metric function $g(r) = -\Lambda r^2 - M + \frac{2}{r}$ for charged black hole Einstein-power Maxwell invariant (Einstein-PMI) gravity \cite{27} when the nonlinearity parameter is chosen $s = 3/4$ with following action

$$I = -\frac{1}{16\pi} \int d^3x \sqrt{-g} [R - 2\Lambda - (F_{\mu\nu} F^{\mu\nu})^s].$$

This result implies two interesting results. First, one can see the the effects of the cosmological constant and charge term of the PMI metric can be reproduced by $f(R) = -R$ in the $F(R)$ gravity. So one may like to interpret it as a kind of link between gravitational theory and a classical field theory like PMI. Indeed, $F(R)$ gravity provides a framework for putting the gravity and nonlinear electrodynamics in a unified context by using geometry. In other words, one may ask: in geometric point of view, can we consider PMI Lagrangian ($-F_{\mu\nu} F^{\mu\nu}$) \cite{28} equivalent to $(2\Lambda - R)$?

Second, from $F(R)$ gravity point of view of this solution, action become zero and there is not any well-defined thermodynamic potential for this theory \cite{28} and one can only talk about the temperature of the black hole. On the other hand, the metric of the Einstein-PMI theory is described by well-defined thermodynamic properties like entropy and so on. By this observation, one may ask: what is the role of the geometry of the spacetime in the black hole thermodynamics? Is it possible to define an entropy corresponding to the horizon of the black hole in the context of $F(R)$ gravity or not?
III. BLACK HOLE SOLUTIONS OF THE $F(R) = R + f(R)$ GRAVITY

A. Near horizon solution

The idea of studying the near horizon black hole has great appeal and a long history [29]. Considering some of viable complicated theories of gravity, one cannot find an (easy) exact solution. Therefore, we try to obtain approximate or numerical solutions with suitable boundary conditions [30]. Now, we consider near horizon solutions for some models of $F(R)$ gravity in three dimensional static spacetime.

1. Case I: $f(R) = -2\Lambda$

To demonstrate this method, here, we consider a trivial well-known case $f(R) = -2\Lambda$, with constant curvature i.e. $R = R_0$. Thus the equation of motion (2) reduces to

$$R_{\mu\nu} = g_{\mu\nu} \left( \frac{1}{2} R_0 - \Lambda \right), \quad (10)$$

Applying the metric (3) to Eq. (10), one may obtain

$$g'(r) = -2\Lambda r, \quad (11)$$
$$g''(r) = -2\Lambda. \quad (12)$$

Here, we would like to obtain the near horizon solution and compare it with the exact one. It is easy to show that the exact solution of Eqs. (11) and (12) is

$$g(r) = -\Lambda r^2 - M, \quad (13)$$

but the procedure is different for the near horizon black hole solution. According to the Hawking-Bekenstein temperature formula, if the metric (3) posses a black hole solution with an event horizon located at $r = r_+$, we can deduce

$$g(r_+) = 0$$
$$T = \frac{g'(r_+)}{4\pi}. \quad (14)$$

Expanding the metric function $g(r)$ near the horizon, one can obtain

$$g(r) = \frac{(r - r_+)}{1!}g'(r_+) + \frac{(r - r_+)^2}{2!}g''(r_+) + \frac{(r - r_+)^3}{3!}g'''(r_+) + \frac{(r - r_+)^4}{4!}g''''(r_+) + ..., \quad (15)$$

where in this $f(R)$ model, the nonvanishing terms of Eq. (15) are the first two terms. Thus, the near horizon solution of $g(r)$ is obtained as

$$g(r) = 4\pi T (r - r_+) - (r - r_+)^2\Lambda$$
$$= -r^2\Lambda + 2 (2\pi T + \Lambda r_+) r - r_+^2\Lambda - 4\pi T r_+. \quad (16)$$

Considering both Eqs. (11) and (13), one can show that $2\pi T = -\Lambda r_+$ and therefore Eq. (16) reduces to

$$g(r) = -\Lambda r^2 + \Lambda r_+^2. \quad (17)$$

In order to obtain the exact solution (13) from the near horizon solution, it is sufficient to set $M = -\Lambda r_+^2$ in Eq. (17). This adjustment may also come from Eq. (13), in which $g(r_+) = 0$. Hence, for the mentioned specific trivial model, the near horizon solution is matched to exact solution.

2. Case II: arbitrary $f(R)$ with constant $R$:

In this subsection, we apply the recent procedure to an arbitrary model of $f(R)$ with constant Ricci scalar. We can rewrite Eqs. (7) and (8) in the following forms

$$g''(r) = \frac{1}{r f_R} \left[ (1 - f_R) g'(r) - rf \right], \quad (18)$$
$$g''(r) = \frac{f + 2f_R}{r} g'(r). \quad (19)$$
Equating the right hand side of both Eqs. (18) and (19), one may obtain two solutions for $f_R$

$$f_R = -1, \frac{1}{2} - \frac{r f}{2g'(r)},$$

where we are not interested in the first solution ($f_R = -1$), here. One can use the second solution ($f_R = \frac{1}{2} - \frac{r f}{2g'(r)}$) and the definition of the black hole temperature $g'(r_+) = 4\pi T$ in Eq. (15) to obtain the near horizon solution

$$g(r) = 2\pi T \left(\frac{r^2 - r_+^2}{r_+}\right) = \frac{2\pi T}{r_+} r^2 - 2\pi T r_+.$$  \hspace{1cm} (21)

It is easy to set $\Lambda = -2\pi T/r_+$ and $M = 2\pi T r_+$ to obtain three dimensional solution of Einstein-Λ gravity. Therefore, the near horizon solution of arbitrary $F(R)$ gravity models with constant $R$ is the same as uncharged BTZ solution.

3. Case III: arbitrary $f(R)$ with nonconstant $R$:

Here, we take into account an arbitrary model of $f(R)$ with nonconstant Ricci scalar. One can consider Eqs. (14-16) and solve them near the horizon to obtain

$$g(r_+) = 0,$$  \hspace{1cm} (22)

$$g'(r_+) = \frac{f_+ [1 + f_{R+}] r_+}{1 - f_{R+} - f_{R+}' r_+ - 2f_{R+} f_{R+}' r_+},$$  \hspace{1cm} (23)

$$g''(r_+) = \frac{f_+ [1 + f_{R+} + f_{R+}' r_+]}{1 - f_{R+} - f_{R+}' r_+ - 2f_{R+} f_{R+}' r_+},$$  \hspace{1cm} (24)

where $f_+ = f(R)|_{r=r_+}$, $f_{R+} = f_R|_{r=r_+}$ and $f_{R+}' = \frac{df_{R+}}{dr}|_{r=r_+}$. Using the fact that $g'(r_+) = 4\pi T$ with Eq. (23), we achieve

$$f_+ = 4\pi T \left[1 - f_{R+} - f_{R+}' r_+ - 2f_{R+} f_{R+}' r_+\right] \frac{r_+}{1 + f_{R+}},$$  \hspace{1cm} (25)

$$g''(r_+) = 4\pi T \left[1 + f_{R+} + f_{R+}' r_+\right] \frac{r_+}{1 + f_{R+}}.$$  \hspace{1cm} (26)

and therefore the near horizon solution (Eq. (15) up to second order) reduces to

$$g(r) = 4\pi T (r - r_+) + \frac{2\pi T (r - r_+)^2 [1 + f_{R+} + f_{R+}' r_+]}{(1 + f_{R+}) r_+}$$

$$= \frac{2\pi T [1 + f_{R+} + f_{R+}' r_+] r_+^2}{r_+ (1 + f_{R+})} - \frac{4\pi T f_{R+}' r_+}{1 + f_{R+}} - \frac{2\pi T [1 + f_{R+} - f_{R+}' r_+] r_+}{1 + f_{R+}}.$$  \hspace{1cm} (27)

Eq. (27) is a second order function such as BTZ solution with additional linear term. In other word, we can compare Eq. (27) and BTZ solution with the following equalities

$$\frac{2\pi T [1 + f_{R+} + f_{R+}' r_+]}{r_+ (1 + f_{R+})} = -\Lambda,$$  \hspace{1cm} (28)

$$\frac{2\pi T [1 + f_{R+} - f_{R+}' r_+] r_+}{1 + f_{R+}} = M.$$  \hspace{1cm} (29)

After straightforward calculations, one can use Eqs. (23) and (29) to achieve

$$f_{R+}' = \frac{(\Lambda r_+^2 + M) (1 + f_{R+})}{(\Lambda r_+^2 - M) r_+},$$  \hspace{1cm} (30)

$$T = -\frac{(\Lambda r_+^2 - M)}{4\pi r_+^3}.$$  \hspace{1cm} (31)
Inserting Eqs. (30) and (31) in Eq. (27), we obtain
\[
g(r) = (r - r_+) \frac{(rr_+ + Ml^2)}{l^2 r_+} \\
= -\Lambda r^2 + \frac{(\Lambda r^2 + M)}{r_+} r - M, \tag{32}
\]
which is the BTZ solution with additional linear term. This linear term comes from the fact that we chose a nonconstant Ricci scalar solution. As we have seen in case II, this linear term vanishes for the solutions with constant Ricci scalar. As one can confirm, the linear term of Eq. (32) does not change the horizon class and asymptotic behavior of the spacetime and therefore, not only is it not necessary to remove it but also it will be interesting to think about its physical interpretation.

B. Hyperscaling violation and Lifshitz exact solutions:

In order to obtain the exact solutions, we should choose a specific model for \( F(R) \). We should consider \( F(R) \) models in which local gravity constraints are satisfied as well as cosmological and stability conditions. We know that some of the viable and interesting forms of \( F(R) \) gravity have been considered by Hu-Sawicki [14], Starobinsky [15] and its generalization [16], Appleby-Battye [17], Nojiri-Odintsov [12] and Tsujikawa [18]. In what follows, we consider two kinds of these models to obtain asymptotically Lifshitz solutions. For other models, the method is straightforward.

1. Hu-Sawicki model: \( F(R) = R - m^2 c_1 \left( \frac{R}{R_0} \right)^n \)

In this section, we are looking for the asymptotically Lifshitz solution with a hyperscaling overall factor for this kind of model. In order to achieve this goal, we consider the following ansatz
\[
ds^2 = r^\alpha \left[ - \left( \frac{r^2}{l^2} \right)^z g(r) dt^2 + \frac{l^2 dz^2}{r^2 g(r)} + r^2 d\phi^2 \right], \tag{33}
\]
where the constants \( z \) and \( \alpha \) are called dynamical and hyperscaling violation exponents, respectively. For simplicity, we can set \( \alpha = -2 \), now considering asymptotically Lifshitz metric (33) with the mentioned \( F(R) \) model. One can obtain the metric function \( g(r) \) with the following relation
\[
g(r) = (a + \frac{b}{r^{z-2}}) r^{-z} - \frac{l^2 R_0}{2 r^2 (z-2)^2}, \tag{34}
\]
Inserting the mentioned metric in the field equations, we obtain a set of algebraic equations for the model parameters as
\[
(n - 1) m^{2n} - c_2 R_0^n = 0, \tag{35}
\]
\[
\frac{m^2 c_1 R_0^{-n-1}}{m^{2n} + c_2 R_0^n} - 1 = 0, \tag{36}
\]
with the following solutions
\[
c_1 = \frac{nm^{2n-2} R_0^{n-1}}{R_0^n}, \tag{37}
\]
\[
c_2 = (n - 1) \frac{m^{2n}}{R_0^n}, \tag{38}
\]
where the constant \( R_0 \) is Ricci scalar. We calculate the nonvanishing components of Riemann tensor for obtained solution and find that
\[
R^{rtr} = -\frac{R_0 l^{2z-2}}{2 r^{2z-6}}, \tag{39}
\]
which confirm that for $z \neq 3$, there is an essential singularity at the origin ($r = 0$).

It is notable that existence of $r^{-2}$ factor in the last term of Eq. (34) excludes any Lifshitz solution with hyperscaling factor. In the case of $\alpha = 0$ with the mentioned model parameters, one can obtain an asymptotically Lifshitz solution

$$ds^2 = -\left(\frac{r^2}{l^2}\right)^z g(r) dt^2 + \frac{l^2 dr^2}{r^2 g(r)} + r^2 d\phi^2,$$

$$g(r) = \left(\frac{a r^{1/2 + \frac{1}{2z}} + b}{r^{1/2 + \frac{1}{2z}}}\right) r^{-\frac{1}{2z} - 1} - \frac{l^2 R_0}{2(z^2 + z + 1)}. \tag{41}$$

Moreover, in this situation, we can adjust $a = b = 0$ and $R_0 = -2(z^2 + z + 1)/l^2$ to find Lifshitz solution as a vacuum solution of the theory. It is easy to show that to the Kretschmann scalar diverges at $r \rightarrow 0$ and for large values of $r$ one obtains

$$\lim_{r \rightarrow \infty} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} = \frac{z^2 - z + 1}{z^2 + z + 1} R_0^2. \tag{42}$$

Here, we want to add angular momentum to static Lifshitz metric to obtain possible rotating solution. We take into account the following ansatz

$$ds^2 = -\left(\frac{r^2}{l^2}\right)^z g(r) dt^2 + \frac{l^2 dr^2}{r^2 g(r)} + r^2 [d\phi + h(r) dt]^2. \tag{43}$$

Inserting this rotating metric in the field equations with Eqs. (37) and (38), one can find the following solutions for the metric functions

$$h(r) = B r^{-\eta}, \tag{44}$$

$$g(r) = \left(\frac{a r^{1/2 + \frac{1}{2z}} + b}{r^{1/2 + \frac{1}{2z}}}\right) r^{-\frac{1}{2z} - 1} + \frac{B^2 \eta^2 [2z^2r^2 - 2\eta - 2\eta + 4 - 1]}{4 [2\eta^2 - 6\eta + \eta z - 2z + 5]} - \frac{l^2 R_0}{2(z^2 + z + 1)}. \tag{45}$$

We should note that for $B = 0$, this solution reduces to the static Lifshitz solution, as it should be. It has been shown that considering the dynamical field of Ricci scalar, the effective mass is related to $\frac{F’(R)}{R}$ \cite{31,32}. Therefore, in order to obtain a stable dynamical field, its effective mass must be positive. This requirement is known as the Dolgov-Kawasaki stability criterion \cite{32}. It is notable that the second derivative of the $F(R)$ function for this specific model is

$$F_{RR} = \frac{n-1}{R_0}, \tag{46}$$

which is positive for positive $R_0$ and $n > 1$.

2. Modified Starobinsky model [14]: $F(R) = R + \lambda \beta \left(\left[1 + \left(\frac{\beta}{n}\right)^2\right]^{-n} - 1\right) + \kappa \frac{R^2}{R}$

It is easy to show that taking into account the mentioned modified Starobinsky model with metrics (33), (40) and (43), one can obtain the same relation for $g(r)$ as presented in Eqs. (34), (41) and (45). We should note that in order to satisfy all field equations, we should set the model parameters

$$\kappa = -\frac{\beta}{2 R_0} - \frac{n \beta R_0}{2 (n + 1) R_0^2 + 2 \beta^2 \left(1 - \left[1 + \left(\frac{\beta}{n}\right)^2\right]^{(n+1)}\right)}, \tag{47}$$

$$\lambda = -\frac{\beta R_0}{2 (n + 1) R_0^2 + 2 \beta^2 \left[1 + \left(\frac{\beta}{n}\right)^2\right]^{n+1} - 2 \beta^2}. \tag{48}$$
In order to confirm the Dolgov-Kawasaki stability, we obtain

\[ F_{RR} = 1 - \frac{(2n+1)(n+1)(\Xi-1)^2 + (n+2)(\Xi-1)+1}{R_0 \left( \frac{1+(n+1)(\Xi-1) - 1}{2n+2} \right)}, \quad (49) \]

\[ \Xi = 1 + \left( \frac{R_0}{\beta} \right)^2. \]

It is easy to show that for special values of \( n \) and \( \beta \), we get positive \( F_{RR} \). It should also be noted that one can obtain the same exact solutions for most of viable models (or their generalizations) by setting the model parameters, suitably.

3. Rotating solution: case I:

Here, we consider a rotating spacetime as

\[ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 \left( d\phi + \frac{b}{r^2}dt \right)^2. \quad (50) \]

where \( b \) is a constant. Regarding Eq. (50) with the field equations (7) and (8), one may obtain

\[ g(r) = Ar^2 - M + \frac{b^2}{r^2}, \quad (51) \]

where \( A \) and \( M \) are integration constants and

\[ f(R) = \frac{g'(r)}{r} + \frac{2b^2}{r^4} + C \exp \left( \frac{-r^4R}{4b^2 + 2r^3g'(r)} \right). \quad (52) \]

Applying Eq. (51) in Eq. (52) and set \( A = -\Lambda \), one can obtain the exponential correction of Einstein gravity for the \( f(R) \) model. In other words, Eq. (52) reduces to

\[ f(R) = -2\Lambda + C \exp \left( \frac{R}{4\Lambda} \right). \quad (53) \]

In order to analyze the geometric properties of the solution, we calculate the nonzero components of Riemann tensor

\[ R^{trtr} = -\Lambda, \]

\[ R^{trt\phi} = \frac{-\Lambda b}{r^2}, \]

\[ R^{t\phi t\phi} = \frac{\Lambda}{Ar^4 + Mr^2 - b^2}, \]

\[ R^{t\phi r\phi} = \frac{-\Lambda (Ar^2 + M)}{r^2}, \]

and so there is a singularity located at \( r = 0 \) whose horizon is

\[ r_+ = \left( \frac{M + \sqrt{M^2 + 4\Lambda b^2}}{-2\Lambda} \right)^{1/2}. \quad (54) \]

Applying Dolgov-Kawasaki stability method, one obtains

\[ F_{RR} = \frac{C}{(4\Lambda)^2} e^{\frac{4\Lambda}{R}}, \quad (55) \]

which confirms that this model can be stable.
4. Rotating solution: case II:

Here, we generalize the recent rotating spacetime to the case of two unknown metric functions

\[ ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2 \left( d\phi + \frac{b(r)}{r^2} dt \right)^2. \]  

(56)

Inserting this metric into the field equations (7) and (8), one can obtain

\[ g(r) = -\Lambda r^2 - M - \frac{M^2}{4\Lambda r^2}, \]  

(57)

\[ b(r) = \pm \left( \sqrt{-\Lambda r^2} - \frac{M}{2\sqrt{-\Lambda}} \right) \]  

(58)

where \( \Lambda \) and \( M \) are integration constants which are related to the negative cosmological constant and mass parameter, respectively. In addition, there is a constraint on the \( f(R) \) model as

\[ f_R + \frac{f(R)}{4A} - \frac{1}{2} = 0, \]  

(59)

with the following solution

\[ f(R) = -2\Lambda + Ce^{\frac{r}{\Lambda}}. \]  

(60)

Indeed, the mentioned rotating solution is valid only for the exponential form of \( f(R) \) model. Calculating the nonzero components of the Riemann tensor shows that there is a singularity at \( r = 0 \) with the following horizon

\[ r_+ = \sqrt{-\frac{M}{2\Lambda}}. \]  

(61)

The stability discussion of the mentioned model is the same as the former rotating solution.

IV. CONCLUSIONS

In this paper, in order to better understanding of \((2 + 1)\)-dimensional gravity, we have considered \( F(R) \) theories of gravity and searched for either exact or near horizon solutions of general and specific models of \( F(R) \).

At first, we showed that the general \( F(R) \) gravity with constant Ricci scalar admits the uncharged static BTZ solution as an exact solution. Besides it, in the case \( F(R) = 0 \), where the action of the gravity vanishes on-shell, interestingly, there is a charged solution just the same as that in the Einstein-PMI gravity when the nonlinearity parameter is chosen \( s = 3/4 \).

We also focused attention on the near-horizon region by truncating the black hole metric to its leading terms close to the horizon. We started with a trivial \( F(R) \) model as an example and generalized our method to general models of \( F(R) \) gravity with (non)constant Ricci scalar and found that the near horizon metric functions are the same as exact uncharged BTZ solutions with an additional linear term for nonconstant Ricci scalar case.

Furthermore, we considered specific \( F(R) \) models to obtain exact solutions. We showed that one can obtain asymptotically Lifshitz solution with a hyperscaling overall factor and Lifshitz solution as a vacuum solution. We should note that in general we cannot obtain the mentioned exact solutions. For example, considering the Starobinsky model, we could not obtain asymptotically Lifshitz solution. In this case, we added an \( R^2 \) term [16] to obtain generalized Starobinsky model with (asymptotically) Lifshitz solution. Finally, we achieved two kinds of rotating solutions with rotating black hole interpretation. It is notable that in order to have rotating solutions, \( F(R) \) should be in exponential form.

In this paper, we obtained some near horizon and exact solutions of \( F(R) \) gravity and stress on the geometric properties of the solutions. Therefore it is worthwhile to think about thermodynamic properties of the obtained solutions.
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