THE ERGODIC CLOSING LEMMA FOR NONSINGULAR ENDOMORPHISMS

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ABSTRACT. We generalize Mañé's Ergodic Closing Lemma to the context of $C^1$-Endomorphisms without singularities.

1. INTRODUCTION

Let $M$ be a finite dimensional compact Riemannian manifold and let $N\text{End}^1(M)$ denote the set of $C^1$ nonsingular endomorphisms defined in $M$. By $g : M \to M$ to be a nonsingular endomorphism we mean that the derivative $Dg(p)$ of $g$ in each point $p \in M$ is a linear isomorphism. We endow $N\text{End}^1(M)$ with the $C^1$ topology, and denote its corresponding metrics by $d_1$; therefore $N\text{End}^1(M)$ is an open subset of the complete space $C^1(M)$ whose elements are $C^1$—endomorphisms defined in $M$.

The main purpose of this paper is to prove

Theorem A. (Ergodic Closing Lemma for nonsingular endomorphisms.) Let $M$ be a compact manifold. Then, there exists a residual subset $\mathcal{R} \subset N\text{End}^1(M)$ such that for any $f \in \mathcal{R}$, the set of $f$—invariant probabilities $\mathcal{M}_1(f)$ is the closed convex hull of ergodic measures supported on periodic orbits of $f$.

For the proof of theorem above, we also prove another version of Ergodic Closing Lemma for endomorphisms (see Th. 2) in the next section.

Remark 1. The Ergodic Closing Lemma is a result about shadowing by periodic orbits. Although the classical notion of shadowing is not generic even among $C^1$-diffeomorphisms (see [5]), Ergodic Closing Lemma asserts that $C^1$—generically most orbits in a measure-theoretical point of view can be shadowed by periodic orbits. Using ideas in [2], [3], and the Ergodic Closing Lemma we also obtained a new criteria of generic Hyperbolicity/Expansion based in periodic sets [4].

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2. Proof of the Ergodic Closing Lemma for nonsingular endomorphisms

Let us start by fixing some notation. Given \( x \in M \), we define \( B_\epsilon(f, x) \) as an \( \epsilon \)-neighborhood of the orbit of \( x \). Define \( \Sigma(f) \) as the set of points \( x \in M \) such that for every neighborhood \( U \) of \( f \) and every \( \epsilon > 0 \), there exist \( g \in U \) and \( y \in M \) such that \( y \in \text{Per}(g) \), \( g = f \) on \( M \setminus B_\epsilon(f, x) \) and \( d(f^j(x), g^j(y)) \leq \epsilon \), \( \forall 0 \leq j \leq m \), where \( m \) is the \( g \)-period of \( y \).

The residual version of the Ergodic Closing Lemma (our Th. A) is a consequence of the following result:

**Theorem 2.** For any nonsingular endomorphism \( f \), \( \Sigma(f) \) is a total probability set, that is, \( \Sigma(f) \) is a full probability set for any \( f \)-invariant probability.

**Remark 3.** Theorem 2 above, with \( f \in \text{Diff}^1(M) \) instead of \( f \in \text{NEnd}^1(M) \) in its statement, was the former Ergodic Closing Lemma proved by Mañé. In fact, Mañé, in \[8\], did not explicitly give the proof of his corresponding classical residual version, although he stated such version in \[9\]. Recently, while we were writing this article, Abdenur et al in \[1\] filled out this gap.

**Definition 4.** \((\epsilon-\text{shadowing by a periodic point})\) Let \( f \) and \( g \) maps on a compact metric space \( \Lambda \). Given \( \epsilon > 0 \) and \( x \in \Lambda \), we say that a \( g \)-periodic point \( p \) with period \( n \) \( \epsilon \)-shadows \( x \) iff \( d(g^j(p), f^j(x)) < \epsilon \), \( \forall 0 \leq j \leq n \).

**Definition 5.** Let \( \epsilon > 0 \) and a neighborhood \( U \ni f, U \subset \text{NEnd}^1(M) \). We define \( \Sigma(f, U, \epsilon) \) as the set of points \( x \in M \) such that there exist \( g \in U \) and \( y \in M \) such that \( y \in \text{Per}(g) \), \( g = f \) on \( M \setminus B_\epsilon(f, x) \) and \( d(f^j(x), g^j(y)) \leq \epsilon \), \( \forall 0 \leq j \leq m \), where \( m \) is the \( g \)-period of \( y \). That is, \( \Sigma(f, U, \epsilon) \) is the set of points \( x \in M \) which are \( \epsilon \)-shadowed by a periodic point \( y \in \text{Per}(g) \), for some \( g \in U \). Everyday there is no chance of misunderstanding, we will just write \( \Sigma(U, \epsilon) \) instead of \( \Sigma(f, U, \epsilon) \).

If we take a nested neighborhood basis \( U_n \) of \( f \) in \( \text{NEnd}^1(M) \) then

\[
\Sigma(f) = \cap_{n \in \mathbb{N}} \Sigma(f, U_n, 1/n).
\]

Therefore, Th. 2 is an immediate consequence of

**Proposition 6.** For any nonsingular endomorphism \( f \), any neighborhood \( U \) of \( f \) and \( \epsilon > 0 \), \( \Sigma(U, \epsilon) = \Sigma(U, \epsilon) \) is a total probability set for \( f \).

The proof of Proposition 6 is quite long, and it is a consequence of lemmas and Theorems we prove in the sequel. Such proof has two main parts.

Part 1 consists in an improvement of Closing Lemma (see \[10\], \[11\], \[12\], \[13\]), stating that given a nonsingular endomorphism \( f \), for any point \( x \in M \) that returns sufficiently close to itself, there is an iterate \( y = f^{m(x)}(x) \) such that we can perturb \( f \) into a \( g \) for which there exists a periodic orbit that shadows \( y \). The precise statement of this part corresponds to lemma \[12\] whose proof we write down further in this paper. Note that this part is entirely topological, and it does not use measure/ergodic theoretical arguments.

Part 2 uses Birkhoff Theorem (ergodicity) and a Vitali’s Covering argument to prove that the set of points that are shadowed by a periodic point of some nearby endomorphism
For the statement of the Perturbation lemmas that we will need for the proof of Lemma [12] let us introduce some notation. As the manifold $M$ is compact, there is $\delta$ such that \{exp$_p$, p $\in$ M\} is an equilipschitz family of diffeomorphisms, such that each exponential map exp$_p$ embeds $B(0, \delta)$ in a neighborhood $B_p$ of $p$. Given $p \in M$, we define a metric $d' = d'_p : B_p \times B_p \rightarrow [0, +\infty)$ given by 

$$d'(x, y) := |\exp^{-1}_p(x) - \exp^{-1}_p(y)|.$$ 

Obviously, $d'$ is Lipschitz-equivalent to the manifold usual metric restricted to $B_p$. Setting $d'$ as the metric in $B_p$, then exp$_p$ isometrically maps $B(0, \delta)$ on $B_p = B'(p, \delta)$, where the quote ' signs the ball in the metric $d'$.

**Lemma 7.** [12] For any $\eta > 0$, there is an $\alpha > 0$ such that for any $f \in \text{NEnd}^1(M)$, any $q \in M$, any two points $v_1, v_2 \in T_q M$ with $\|v_1 - v_2\|/\alpha) \subset B(0, \delta) \subset T_q M$, there is a diffeomorphism $h = h_{q, \alpha, v_1, v_2} : M \rightarrow M$, called an $\alpha$-lift, such that:

1. $h(\exp_q(v_2)) = \exp_q(v_1)$;
2. The closure of set of points where $h$ differs to the identity is contained in $\exp_q(B(v_2, \|v_1 - v_2\|/\alpha));$
3. $d_1(hf, f) < \eta.$

**Definition 8.** (Dynamical neighborhood.) We say that a neighborhood $V$ of a point $p \in M$ is $N$--dynamical for $f$ if each connected component $\bigcup_{j=0}^N f^{-j}(V)$ contains exactly one point of $\bigcup_{j=0}^N f^{-j}(\{p\})$.

**Lemma 9.** [7, 13] Let $f \in \text{NEnd}^1(M)$, $p \in M$, $N \geq 1$ given such that all terms in $\bigcup_{j=0}^{N+1} f^{-j}(p)$ are distinct. Then, for any $\eta > 0$, there is a $\beta > 0$, and a map $f_1 \in \text{NEnd}^1(M)$, called a local linearization of $f$ with the following properties (1)-(5).

1. $B'(p, \beta)$ is $(N+1)$-dynamical for both $f$ and $f_1$, and $f^{-j}(B'(p, \beta)) = f_1^{-j}(B'(p, \beta))$ for $j = 1, \ldots, N + 1$.
2. For $q \in \bigcup_{j=1}^{N+1} f^{-j}(p)$, let $V(q)$ be the open connected component of $\bigcup_{j=1}^{N+1} f^{-j}(B'(p, \beta/4))$ containing $q$. Then, $f_1|_{V(q)} = \exp_{f(q)} \circ (T_q f) \circ \exp^{-1}_q$.
3. $f_1^{N+1}(x) = f(x), \forall x \in f^{-N-1}(B'(p, \beta)).$
4. $f_1 = f$ on $M \setminus \bigcup_{j=1}^{N+1} f^{-j}(B'(p, \beta)).$
5. $d_1(f_1, f) < \eta.$

**Remark 10.** For the sequel, we need to emphasize two trivial, but important consequences of the last lemma. The first one is that if for some $k \in \mathbb{N}$ we have that $x, f^k(x)$ are both out of $\bigcup_{j=0}^N f^{-j}(B'(p, \beta))$, then $f^k(x) = f_1^k(x)$. The second one is that, as $\eta$ goes to 0, one can take $\beta$ arbitrarily small in the last lemma.

**Theorem 11.** (Th. A in [13].) Let $(T, T_q)$ a complete tree of isomorphisms associated to the pre-orbit of a point $q_0 \in M$, that is, a collection of $n$--dimensional inner product spaces $E_q$ and isomorphisms $T_q : E_q \rightarrow E_{q_0}$ associated to each $q$ in the pre-orbit of $q_0$, with $T_{q_0}$
equal to identity. Given \( \alpha > 0 \), there are \( \rho > 0 \) and \( N \geq 1 \) such that: for any ordered set \( X = \{ x_0 < \cdots < x_i \} \subset E_{q_0} \), there is a point \( y \in X \cap B(x_t, \rho|x_0 - x_t|) \) such that for any branch \( \Gamma = \{ q_0, q_1, \ldots, \} \) of \( T \), there is a point \( w = w(\Gamma) \in X \cap B(x_t, \rho|x_0 - x_t|) \) which is before \( y \) in the order of \( X \), together with \( N + 1 \) points \( c_0(\Gamma) = c_0, \ldots, c_N(\Gamma) = c_N \in B(x_t, \rho|x_0 - x_t|) \) satisfying the following two conditions:

\[
\begin{align*}
\bullet & \quad c_0 = w, c_N = y; \text{ and } \\
\big| T_{q_0}^{-1}(c_j) - T_{q_0}^{-1}(c_{j+1}) \big| & \leq \alpha d(T_{q_0}^{-1}(c_{j+1}) - T_{q_0}^{-1}(A)), \text{ where } A := \{ x \in X, w < x < y \} \cup \partial B(x_t, \rho|x_0 - x_t|).
\end{align*}
\]

The next lemma is the main target in the first part of our Ergodic Closing Lemma. It is a topological result, and it has nothing to do with Measure/Ergodic theory. It implies in particular that, given \( \epsilon > 0 \) and any \( f \)-recurrent point \( x \), then \( x \) has an iterate which is \( \epsilon \)-shadowed by a periodic point of some \( g \) close to \( f \).

**Lemma 12.** Given \( f \in N\text{End}_1^1(M) \), \( p \in M \), \( \epsilon > 0 \) and a neighborhood \( U \) of \( f \), there exist \( r > 0 \), \( \rho' > 1 \) such that if for some natural \( t > 0 \), we have \( x, f^t(x) \in B_p'(p) \), with \( 0 < \tau \leq r \), then there exist \( 0 \leq t_1 < t_2 \leq t \) and \( g \in U \) such that:

\[
\begin{align*}
\bullet & \quad w = f^{t_1}(x), y = f^{t_2}(x) \in B_{\rho'}(p); \\
\bullet & \quad g^{t_2-t_1}(w) = w; \\
\bullet & \quad g(z) = f(z) \text{ for } z \notin B_{\epsilon}(f, x) \text{ and } d(g^j(w), f^j(w)) \leq \epsilon, \forall 0 \leq j \leq t_2 - t_1.
\end{align*}
\]

**Proof:** Take an \( \eta > 0 \) such that the \( \eta \)-ball with center \( f \) is contained in \( U \). Take \( 1 > \alpha > 0 \) such that \( d_1(h \circ f, f) < \eta/2 \), for any \( \alpha \)-lift \( h \).

Without loss of generality, we assume that \( \epsilon < \alpha^2 \). We also assume that \( \epsilon < \delta \).

We assume that \( p \) is not periodic for \( f \), otherwise, there is nothing to prove.

This implies that all points in the pre-orbit of \( p \) are distinct. Let \( \rho > 2 \) and \( N \geq 1 \) be the numbers provided by th. \( 11 \) for \( \alpha > 0 \) taken as above, and for \( q_0 = p \), each \( q_j \) to be some \( j \)-pre-image of \( p \), \( E_{q_j} = T_{q_j}M \) and \( T_{q_j} = Df^j(q_j) \).

So, take \( r > 0 \) such that \( r < \epsilon/(6\rho) \) and \( \text{diam}(f^{-j}(B^\prime(p, 3\rho r))) < \epsilon, \forall j = 0, \ldots, N + 1 \). We assume that each connected component of \( \cup_{j=0}^{N+1} f^{-j}(B^\prime(p, 3\rho r)) \) contains exactly one point \( q_j \in f^{-j}(p), j = 1, \ldots, N + 1 \). In particular, if \( z, f^j(z) \in B^\prime(p, 3\rho r), \) then \( \hat{t} > N + 1 \).

Now, assuming that \( x, f^t(x) \in B^\prime(p, \tau) \), for some \( 0 < \tau \leq r \) we can apply th. \( 11 \) to the set \( X = \{ x, f(x), \ldots, f^t(x) \} \cap B^\prime(p, 3\rho \tau) \) endowed with the order given by the iterate number: if \( f^k(x), f^k(x) \in X \), then \( f^k(x) < f^k(x) \Leftrightarrow k < \hat{k} \). Therefore, set \( \rho' = 3\rho \). We then obtain \( f^t(x) = y \in \{ x, \ldots, f^t(x) \} \cap B^\prime(f^t(x), \rho \cdot d'(f^t(x), x)) \subset B^\prime(p, \rho \tau) \) such that for any branch \( \Gamma = \{ p = p_0, p_1, \ldots, p_n, \ldots \} \) of the pre-orbit of \( p \), there is \( w = w(\Gamma) = f^{t_1}(x) \in \{ x, \ldots, f^t(x) \} \cap B^\prime(f^t(x), \rho \cdot d'(f^t(x), x)), \) with \( t_1 = t_1(\Gamma) < t_2 \) together with points \( c_0(c_0(\Gamma), \ldots, c_N(c_N(\Gamma) \in B^\prime(f^t(x), \rho \cdot d'(f^t(x), x)) \) such that:

\( a \) \( c_0 = w, c_N = y; \) and

\( b \) \( |T_{p_j}^{-1}(c_j) - T_{p_j}^{-1}(c_{j+1})| \leq \alpha d(T_{p_j}^{-1}(c_{j+1}) - T_{p_j}^{-1}(A)), \) where \( A := \{ f^j(x) \in X; t_1 < j < t_2 \} \cup \partial B^\prime(f^t(x), \rho \cdot d'(f^t(x), x)). \)

As \( w = w(\Gamma) \) and \( y \) are both in \( X \), there is a natural number \( k(\Gamma) \geq 1 \) such that \( f^k(\Gamma)(w(\Gamma)) = y \). Note that \( k(\Gamma) > N + 1 \), as \( \cup_{j=0}^{N+1} f^{-j}(\{ y \}) \cap B^\prime(p, 3\rho r) = y \), from our
choice of $r$. Setting $z := f^{k_{(\Gamma)}}(w(\Gamma))$, we see that $z$ does not depend on the branch $\Gamma$ of $p$, since $w(\Gamma)$ and $y$ are in $X$, $f^{k_{(\Gamma)}}(w(\Gamma)) = y$ and $N$ do not depend on $\Gamma$. By our choice of $r$, since $y \in B'(p, 3\rho r)$ there is a unique connected component $V_{n+1} \subset f^{-(n+1)}(B'(p, 3\rho r))$ such that $z \in V_{n+1}$. Also, there is a unique $p_{n+1} \in f^{-(n+1)}(p) \cap V_{n+1}$. From now on, we fix $\Gamma$ as some branch of $p$ containing $p_{n+1}$ (That is, $\Gamma = (p = p_0, \ldots , p_{n+1}, \ldots ))$, and we consider all constants $w, c_0, \ldots , c_N, k$ obtained by applying th. [11] with respect to such branch. For each $p_j, j = 0, \ldots , N - 1$, let $h_{p_j}$ be the $\alpha$-kernel lift obtained by treating in lemma [7] $q = p_j, v_1 = [Df^j(p_j)]^{-1}(c_j), v_2 = [Df^j(p_j)]^{-1}(c_{j+1})$. Defining a map $g : M \to M$ by

$$g := \begin{cases} h_{p_j} \circ f_1 & \text{on } V(p_{j+1}); \\ f_1 & \text{on the rest of } M, \end{cases}$$

we have that $g \in NEnd^l(M)$ and $d_1(g, f) < \eta$. Thus $g \in U$.

Due to condition (b) above, the $g$-orbit from $w$ to $z$ never touches the region in which $q \neq f_1$. Therefore, $g^{k-(n+1)}(w) = f_1^{k-(n+1)}(w)$. By remark [10] we also have that $f_1^{k-(n+1)}(w) = f_1^{k-(n+1)}(w)$, and thus

$$g^{k-(n+1)}(w) = f_1^{k-(n+1)}(w) = z.$$ 

Now, it is easy to see that $g^{N+1}(z) = w$ and then $g^k(w) = w$. In fact, $f_1^{N+1}(z) = y$, and the lifts $h_{p_{N-1}}, \ldots , h_0$ gradually and slightly modifies $f_1$-orbit segment joining $z$ and $y$, in such way that $g^{N+1}(z) = w$ and $d(g^j(z), f^j(z)) \leq d(g^j(z), f_1^j(z)) + d(f_1^j(z), f^j(z)) < \epsilon, \forall j = 1, \ldots , N + 1$.

□

Now we begin the second part of the proof of Proposition [6]. Although the main idea of this part is borrowed from [8], the proofs we have written are presented in an abstract setting for future use and bookkeeping purposes. This will also clarify the sort of arguments which are used. Such arguments are basically measure theoretical, and ergodic tools.

We start by introducing some notation. We say that a subset $C$ of the torus $T^s$ is a cube if it can be written as $A = I_1 \times \cdots \times I_s$, where the sets $I_i$ are intervals of same length in $S^1$ (containing both, none, or one of its boundary points). If $p_i$ is the middle point of $I_i$, we say that the point $(p_1, \ldots , p_s)$ is the center of $A$. The length of the intervals $I_i$ is called the side of the cube. For each $k \in \mathbb{N}^+$, let $(P_j^{(k)})_{j \in \mathbb{N}^+}$ be a sequence of partitions of $T^s$ by cubes whose side is $2\pi/k^j$. For every atom $P$ of a partition $P_j^{(k)}$, we can associate cubes $\hat{P}$ and $\tilde{P}$ having the same center of $P$, but with sides $2\pi/k^j-1$ and $6\pi/k^j-1$, respectively. If $x \in T^s$, denote by $P_j^{(k)}(x)$ the atom of $P_j^{(k)}$ containing $x$. Suppose that $M$ is isometrically embedded in $T^s$.

We recall the following useful fact on such kind of partitions:

**Lemma 13.** For every probability measure $\mu$ on the Borel sets of $T^s$, every $\delta > 0$ and for all odd natural $k$, the following inequalities holds for any $j \geq 1$:

$$\mu(\{x; \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x))\}) \geq 1 - \delta k^s$$
and
\[ \mu\{x; \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x))\} \geq 1 - 3^s k^s. \]

**Proof:** Done in [8]. \(\square\)

Let \( f \in NEnd^1(M) \), \( \epsilon > 0 \), a neighborhood \( U \) of \( f \) and an ergodic \( \mu \in \mathcal{M}_1(f) \) be given. Extend \( \mu \) to a measure on \( T^s \) by \( \mu(A) := \mu(A \cap M) \), for all Borel set \( A \subset T^s \).

Let \( \mathcal{U} \subset M \) be some Borelian set and suppose that \( \mathcal{U}(r, \rho) \), where \( r > 0, \rho > 1 \), is some Borelian set whose elements are points \( x \in M \) with the following property: if \( y \in B'_r(x) \) for some \( 0 < r' \leq r \) and \( f^t(y) \in B'_\rho(x) \), for some \( t > 0 \) then there exist \( 0 \leq t_1 < t \), such that \( f^{t_1}(y) \in \overline{B'_{\rho r}(x) \cap \mathcal{U}} \). Take \( r_i > 0, \rho_i > 1 \) two monotone sequences converging respectively to 0 and \( +\infty \).

Our first target in this second part is to obtain an abstract result (Th. 17) which will be essential in both proofs of Proposition 6 and Th. A. Such result says that, if \( \cup_i \mathcal{U}(r_i, \rho_i) = M \), then \( \mathcal{U} \) has total probability for \( f \).

**Remark 14.** All results from this point of the paper up to Th. 17 do not use much regularity of \( f \). In fact, specifically for the statements from Lemma 15 up to Th. 17 we only request \( f : M \to M \) to be a Borel map such that \( \mathcal{M}_1(f) \neq \emptyset \). This occurs, for instance, if \( f \) is a continuous map.

For each pair \((i, l)\), we can find an odd natural \( k = k(i, l) \) and \( j(i, l) \) such that \( \forall j \geq j(i, l) \) and \( x \in T^s \) there exists \( 0 \leq r \leq r_i \) satisfying
\[ P_j^{(k)}(x) \subset B_r(x) \]
and
\[ \hat{P}_j^{(k)}(x) \supset \overline{B_{\rho r}(x)}, \]
the balls here are taken in the torus.

The next lemma is where the \( \mu \)-ergodicity is necessary for the proof of Proposition 6.

**Lemma 15.** If \( x \in \mathcal{U}(r_i, \rho_i) \), \( j \geq j(i, l) \), \( k = k(i, l) \) and \( \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x)) \), we have:
\[ \mu(\hat{P}_j^{(k)}(x) \cap \mathcal{U}) \geq \delta \mu(\hat{P}_j^{(k)}(x)). \]

**Proof:** As \( \mu \) is ergodic, for \( \mu \)-typical \( y \in M \), we have that
\[ \mu(\hat{P}_j^{(k)}(x) \cap \mathcal{U}) = \lim_{n \to +\infty} \frac{1}{n} \#\{1 \leq t \leq n; f^t(y) \in \hat{P}_j^{(k)}(x) \cap \mathcal{U}\}, \]
and
\[ \mu(P_j^{(k)}(x)) = \lim_{n \to +\infty} \frac{1}{n} \#\{1 \leq t \leq n; f^t(y) \in P_j^{(k)}(x)\}. \]
By the definition of \( \mathcal{U}(r_i, \rho_i) \), between any pair of natural numbers \( n_1 \) and \( n_2 \) such that \( f^{n_1}(y), f^{n_2}(y) \in P_j^{(k)}(x) \subset B'_r(x) \), there exists \( n_1 \leq t_1 < n_2 \), such that \( f^{t_1}(y) \in \overline{B'_{\rho r}(x) \cap \mathcal{U}} \subset (\hat{P}_j^{(k)}(x) \cap \mathcal{U}) \). This implies that
\[ \#\{1 \leq t \leq n; f^t(y) \in \hat{P}_j^{(k)}(x) \cap \mathcal{U}\} \geq \]
\[ \#\{1 \leq t \leq n; f^t(y) \in P_j^{(k)}(x)\} - 1. \]

Hence

\[ \mu(\hat{P}_j^{(k)}(x) \cap \mathcal{O}) \geq \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x)). \]

Now define \( \Lambda_0^\delta(i, l) \), for \( \delta > 0 \), as the set of points \( x \in T^s \) such that for \( k = k(i, l) \), we have

\[ \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x)) \]

and

\[ \mu(P_j^{(k)}(x)) \geq \delta \mu(\hat{P}_j^{(k)}(x)), \]

for an infinite sequence \( \varsigma(x) \) of values of \( j \).

Define \( \Lambda_\delta(i, l) := \Lambda_0^\delta(i, l) \cap \mathcal{O}(r_i, p_l) \).

The next lemma, a kind of Vitali’s covering lemma, will be useful to estimate the measure of \( \mathcal{O}^c \):

**Lemma 16.** Given a neighborhood \( V \) of \( \mathcal{O}^c \cap \Lambda_\delta(i, l) \), there exist sequences \( x_q \in \mathcal{O}^c \cap \Lambda_\delta(i, l) \), \( (j_q) \), \( j_q \in \varsigma(x_q) \subset \mathbb{N} \), \( q = 1, 2, \ldots \), such that

1. The sets \( \hat{P}_j^{(k)}(x_q) \), \( q \in \mathbb{N} \) are disjoint and contained in \( V \);
2. \( \mu((\mathcal{O}^c \cap \Lambda_\delta(i, l)) \setminus \cup_{q \in \mathbb{N}} \hat{P}_j^{(k)}(x_q)) = 0. \)

**Proof:**

By standard measure theoretical arguments, a translation \( \tau : T^s \to T^s \) can be found in such way that

\[ \mu(\tau(\cup\{\partial \hat{A}; A \in P_j^{(k)}, k \geq 1, j \geq 1\})) = 0; \]

where \( \partial \hat{A} \) is the boundary of \( \hat{A} \in P_j^{(k)} \).

Denoting by \( \mathcal{F} \) the family of sets \( P_j^{(k)}(x) \) with \( x \in \Lambda_\delta(i, l) \cap \mathcal{O}^c \) and \( j \in \varsigma(x) \). Take a sequence \( A_u \in \mathcal{F} \) satisfying:

1. \( \hat{A}_u \subset V, \forall u \in \mathbb{N}, \) and \( \mu(\hat{A}_u \cap \hat{A}_e) = 0, \forall 1 \leq e < u, \)
2. \( \text{diam}(A_u) = \max\{\text{diam}(A); \hat{A} \subset V \) and \( \mu(\hat{A} \cap \hat{A}_e), \forall 1 \leq e < u\}. \)

Such properties imply that \( \lim_{u \to +\infty} \text{diam}(A_u) = 0 \) and

\[ \sum_u \mu(A_u) = \mu(\cup_u A_u) \leq 1. \]  \hspace{1cm} (1)

We claim that for \( N \geq 1 \)

\[ (\Lambda_\delta(i, l) \cap \mathcal{O}^c) \setminus \cup_{u=1}^N \hat{A}_u \subset \cup_{u=N}^\infty \hat{A}_u. \]  \hspace{1cm} (2)

In fact, if \( x \in (\Lambda_\delta(i, l) \cap \mathcal{O}^c) \setminus \cup_{u=1}^N \hat{A}_u \), there exist \( A \in \mathcal{F} \) with \( x \in A \) and

\[ \hat{A} \cap (\cup_{u=1}^N \hat{A}_u) = \emptyset. \]
Take $N_1 > N$ such that $\hat{A} \cap \hat{A}_u = \emptyset$, $\forall 1 \leq u < N_1$ and $\hat{A} \cap \hat{A}_{N_1} \neq \emptyset$. By item (2) above, it follows that $\text{diam}(\hat{A}) \leq \text{diam}(\hat{A}_{N_1})$. This implies that $\hat{A} \subset \hat{A}_{N_1}$ and then

$$x \in A \subset \hat{A}_{N_1} \subset \cup_{u > N} \hat{A}_u,$$

which concludes the proof of equation (2). By such equation and our assumption that partition elements borders have zero measure, we obtain that

$$\mu\left((\Lambda_{\delta}(i, l) \cap \mathcal{O}) \setminus \bigcup_{u=1}^{N} \hat{A}_u\right)\leq \mu\left(\left(\Lambda_{\delta}(i, l) \cap \mathcal{O}\right) \setminus \bigcup_{u=1}^{N} \hat{A}_u\right)\leq$$

$$\mu\left(\bigcup_{u > N} \hat{A}_u\right) \leq \sum_{u > N} \mu(\hat{A}_u) \leq \delta^{-1} \sum_{u > N} \mu(A_u).$$

Due to eq. (i) the tail sum above goes to zero as $N \to +\infty$, which implies the lemma. □

Lemmas [15] and [16] are the key ingredients in the

**Theorem 17.** Let $M$ be a compact Riemannian manifold and let $f : M \to M$ be a measurable Borelian map such that $\mathcal{M}_1(f) \neq \emptyset$. Let $\mathcal{O} \subset M$ and $\mathcal{O}(r, \rho)$ be Borelian subsets of $M$, where $r > 0$, $\rho > 1$.

Suppose that the points $x \in \mathcal{O}(r, \rho)$ have the following property: if $y \in B_{r'}(x)$ for some $0 < r' \leq r$ and $f^{t}(y) \in B_{r'}(x)$, for some $t > 0$ then there exist $0 \leq t_1 \leq t$, such that $f^{t_1}(y) \in B_{r'}(x) \cap B_{r'}(x)$. Suppose also that $r_i > 0$, $\rho_i > 1$ are two monotone sequences converging respectively to 0 and $+\infty$, such that $\cup l \mathcal{O}(r_i, \rho_i) = M$. Then, $\mathcal{O}$ has total probability with respect to the map $f$.

**Proof:** Consider $\Lambda_{\delta}^0(i, l)$ and $\Lambda_{\delta}^0(i, l) = \Lambda_{\delta}^0(i, l) \cap \mathcal{O}(r_i, \rho_i)$ the same sets defined above in our text.

By Lemma [13], this implies that

$$\mu(\Lambda_{\delta}^0(i, l)) \geq 1 - \delta(k^s + 3sk^s).$$

The last inequality implies that

$$\cup_{l=1}^{\infty} \Lambda_{1/n}(i, l) = \mathcal{O}(r_i, \rho_i) \mod (0).$$

Therefore, all we need is to prove that

$$\mu(\mathcal{O}^c \cap \Lambda_{\delta}(i, l)) = 0, \forall 0 < \delta < 1,$$

which implies that

$$\mathcal{O} \supset \mathcal{O}(r_i, \rho_i) \mod (0) \Rightarrow \mathcal{O} \supset \left(\cup_{l=1}^{\infty} \mathcal{O}(r_i, \rho_i) \right) \mod (0) = M \mod (0),$$

this last equality by hypothesis. We will then have $\mu(\mathcal{O}) = \mu(M) = 1$, and the proof of Th. 17 will be completed.

Fix $(i, l)$ and $\delta > 0$. Let $V$ a neighborhood of $\Lambda_{\delta}(i, l)$. By lemmas [13] and [16] it follows that

$$\mu(V) \geq \sum_{q} \mu\left(\hat{P}_{jq}(x_q)\right) \geq \frac{1}{1 - \delta} \sum_{q} \mu\left(\hat{P}_{jq}(x_q) \cap \mathcal{O}^c\right) =$$
\[
\frac{1}{1 - \delta} \mu \left( \bigcup_q \hat{P}_q^{(k)}(x_q) \right) \cap U^c \geq \frac{1}{1 - \delta} \mu(\Lambda_{\delta}(i, l) \cap U^c).
\]

But if \(\mu(\Lambda_{\delta}(i, l) \cap U^c) > 0\), one can take \(V\) satisfying
\[
\mu(V) < \frac{1}{1 - \delta} \mu(\Lambda_{\delta}(i, l) \cap U^c),
\]
contradicting the last inequality. Hence \(\mu(\Lambda_{\delta}(n, m) \cap U^c) = 0\).

Now, let us finish the proof of Proposition 6 (which implies Th. 2).

Proof: (Proposition 6)

Define \(\Sigma(U, \epsilon, r, \rho)\), where \(r > 0, \rho > 1\), as the set of points \(x \in M\) such that if \(y \in B_r(x)\) for some \(0 < r' \leq r\) and \(f^t(y) \in B_{r'}(t)\), for some \(t > 0\) then there exist \(0 \leq t_1 < t_2 \leq t\), \(g \in U\) and \(z \in M\) such that \(g = f\) on \(M \setminus B_r(f, x)\),
\[
g^{t_2 - t_1}(z) = z, d(g^j(z), f^j(f^{t_1}(y))) \leq \epsilon, \forall 0 \leq j \leq t_2 - t_1,
\]
and
\[
f^{t_1}(y) \in \overline{B_{r'}(x)}.
\]
(In particular, \(f^{t_1}(y) \in \Sigma(U, \epsilon)\).)

It is easy to see that \(\Sigma(U, \epsilon, r, \rho)\) is a Borelian set. Again, let \(r_i > 0\) and \(\rho_l > 1\) to be two monotone sequences converging respectively to 0 and \(+\infty\). We note that Lemma 12 implies that
\[
M = \bigcup_{i \geq 1} \bigcup_{l \geq 1} \Sigma(U, \epsilon, r_i, \rho_l),
\]
for every neighborhood \(U\) of \(f\) and every \(\epsilon > 0\).

So, taking \(\mathcal{U} = \Sigma(U, \epsilon)\) and \(\mathcal{U}(r_i, \rho_l) = \Sigma(U, \epsilon, r_i, \rho_l)\) in Th. 17, we conclude that \(\mu(\mathcal{U}) = \mu(\Sigma(U, \epsilon)) = 1\) for all \(f\)-ergodic probability. By Ergodic Decomposition Theorem, this implies that \(\Sigma(U, \epsilon)\) has total probability.

So far, we have proven the raw version of Ergodic Closing Lemma for Endomorphisms (Th. 2). The next lemma will be used in the proof of the residual version of Ergodic Closing Lemma. We denote by \(\mathcal{M}(M)\) the set of probabilities on \(M\) endowed with the weak-* topology.

Lemma 18. Let \(f : M \to M\) be an endomorphism. Suppose that, for \(x\) in a total probability set \(S \subset M\), given \(\epsilon > 0\) and \(U\) a neighborhood of \(f\), there exists \(g_{x, \epsilon} \in U\) and a \(g_{x, \epsilon}\)-periodic point \(p = p(x, \epsilon)\) which \(\epsilon\)-shadows \(x\). Then, given any ergodic measure \(\mu \in \mathcal{M}_1(f)\), there are \(g_k \to f\) and \(g_k\)-periodic points \(p_k\) such that \(\mu\) is the limit of the sequence \((\mu_k)\) of \(g_k\)-ergodic measures respectively supported in the orbit of \(p_k\). Moreover, each \(p_k\) can be taken to be a hyperbolic periodic point for \(g_k\).

Proof:

Let us consider an \(f\)-ergodic probability \(\mu\). We suppose, without loss of generality, that \(\mu\) is not supported in a periodic orbit, otherwise there is nothing to prove. For a \(\mu\)-typical
point \( x \in M \), we can assume that \( x \) is recurrent (by Poincaré’s Recurrence Theorem), has the shadowing property as in lemma’s statement, and that

\[
\frac{1}{n} \sum_{j=0}^{n-1} \delta f^j(x) \to_{\mathrm{weak-*}} \mu,
\]

as \( n \to +\infty \) (this, by the Ergodic Decomposition Theorem).

Set \( \epsilon_1 = 1 \) and \( n_k > 0 \) as the first return time of the orbit of \( x \) to \( B(x, \epsilon_k) \), where \( \epsilon_{k+1} := d(f^{n_k}(x), x)/2, \forall k \geq 1 \).

Therefore, \( n_k \to +\infty \) as \( k \to +\infty \). By hypothesis, one can take a sequence of \( g_k := g_{x, \epsilon_k} \), with \( g_k \to f \), exhibiting \( g_k \)-periodic points \( (p_k) \) such that each \( p_k \epsilon_k/3 \)-shadows the orbit of \( x \). In particular, the period \( t_{k+1} \) of \( p_{k+1} \) is, at least, \( n_k \) (otherwise, the orbit of \( x \) would return to \( B(x, \epsilon_k) \) before \( n_k \)). So, \( t_{k+1} \geq n_k \) implies that \( t_k \to +\infty \) as \( k \to +\infty \), and (up to take a subsequence) we can suppose that \( t_k \) are distinct. Note that slightly perturbing \( g_k \) in the neighborhood of \( p_k \), we can suppose that \( p_k \) is hyperbolic. Set \( \mu_k \) as the \( g_k \)-ergodic probability supported in the orbit of \( p_k \). We will show that \( \mu_k \to_{\mathrm{weak-*}} \mu \) as \( k \to +\infty \). From equation \( \ref{eq:3} \) we have that

\[
v_k = \frac{1}{t_k} \sum_{j=0}^{t_k-1} \delta f^j(x) \to_{\mathrm{weak-*}} \mu,
\]

as \( k \to +\infty \). Let \( \alpha > 0 \) and \( \{\varphi_1, \ldots, \varphi_s\} \subset C^0(M) \) be given. All we need to see is that there exists \( k_0 \in \mathbb{N} \) such that \( \mu_k \) belongs to the neighborhood

\[
V_{\varphi_1, \ldots, \varphi_s; \alpha} := \{\nu \in \mathcal{M}(M); |\int \varphi_i d\nu - \int \varphi_i \mu| < \alpha, \forall i = 1, \ldots, s\},
\]

for all \( k \geq k_0 \).

In fact, as \( \varphi_i, i = 1, \ldots, s \) are uniformly continuous, take \( \epsilon > 0 \) such that \( |\varphi_i(y) - \varphi_i(z)| < \alpha/2, \forall i = 1, \ldots, s, \forall y, z \in M \) such that \( d(y, z) < \epsilon \). Then, take \( k_0 \) such that \( \epsilon_k < \epsilon/2, \forall k \geq k_0, \forall i = 1, \ldots, s \). We conclude that

\[
|\int \varphi_i d\nu_k - \int \varphi_i d\mu| \leq |\int \varphi_i d\nu_k - \int \varphi_i \nu_k| + |\int \varphi_i \nu_k - \int \varphi_i d\mu| < \frac{1}{t_k} \sum_{j=0}^{t_k-1} |\varphi_i(f^j(x)) - \varphi_i(g^j_k(p_k))| + \alpha/2 \leq \alpha, \forall i = 1, \ldots, s;
\]

which implies the lemma.

\( \square \)

Now, we proceed with the proof of Th. 18 by deriving it from Th. 2 and lemma 18 above. The arguments here are basically the same as in Th. 4.2 in [1].

\textbf{Proof:} (Th. 18 - Ergodic Closing Lemma for Nonsingular Endomorphisms - Residual version.)

For \( m \in \mathbb{N} \) fixed, by standard transversality arguments, the collection \( K_m \) of endomorphisms \( f \) such that all periodic points of \( f \), with period up to \( m \) are hyperbolic is an open and dense subset \( NEnd_1^m(M) \).
So \( \hat{\mathcal{R}} := \cap_{m=1}^{+\infty} \mathcal{K}_m \) is a residual set. Let the set of probabilities \( \mathcal{M}(M) \) on \( M \) to be endowed with the weak-* topology and let \( \kappa \) be the collection of compact subsets of \( \mathcal{M}(M) \) endowed with Hausdorff distance. Given \( f \in \mathcal{R} \), denote by \( \mathcal{M}_{\text{per}}(f) \) the set of \( f \)-ergodic measures supported in \( f \)-periodic orbits.

Set \( \Upsilon : \hat{\mathcal{R}} \to \kappa \) given by

\[
\Upsilon(f) = \overline{\mathcal{M}_{\text{per}}(f)}
\]

Due to the robustness of hyperbolic periodic points, such \( \Upsilon \) is lower semicontinuous. This implies that there is a residual subset \( \mathcal{R} \subset \hat{\mathcal{R}} \) whose elements are continuity points for \( \Upsilon \).

From now on, let \( f \in \mathcal{R} \). Let us prove that \( \mathcal{M}_1(f) \) is the closed convex hull of \( f \)-ergodic measures supported in \( f \)-periodic orbits. By Ergodic Decomposition Theorem, all we need to prove is that any \( f \)-ergodic measure \( \mu \) is in \( \overline{\mathcal{M}_{\text{per}}(f)} \).

By Lemma 18, such measure \( \mu \) is accumulated by \( \mu_k \in \mathcal{M}_{\text{per}}(g_k) \), where \( g_k \to f \) as \( k \to +\infty \). As \( \mathcal{R} \) is residual, by means of a slight perturbation, we can suppose that \( g_k \in \hat{\mathcal{R}} \) (as we construct \( p_k \) to be hyperbolic in the proof of that Lemma 18, such \( p_k \) persist under any sufficiently small perturbation).

Since \( f \) is a continuity point for \( \Upsilon \), we have that \( \overline{\mathcal{M}_{\text{per}}(g_k)} \to \overline{\mathcal{M}_{\text{per}}(f)} \) as \( k \to +\infty \), and this implies that \( \mu \in \overline{\mathcal{M}_{\text{per}}(f)} \).

Therefore, \( \overline{\mathcal{M}_{\text{per}}(f)} \) contains all \( f \)-ergodic measures, and by Ergodic Decomposition Theorem, we conclude that \( \mathcal{M}_1(f) \) is the closed convex hull of \( f \)-ergodic measures supported in periodic orbits.

\[ \square \]

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