Coherence optimised subarray partition for hybrid MIMO phased array radar

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Abstract
Hybrid MIMO phased array radar (HMPAR) is widely recognized in the research of radar systems owing to its capacity to provide a balance or trade-off between transmit coherent gain and waveform diversity gain. The balance or trade-off between these two gains is closely related to spatial coherence between different arrays, represented by the degree of cross-correlation between steering vectors over different angles. Proper subarray partitions can provide an extra degree of freedom for optimising the coherence of the array manifold. A coherence optimised subarray partition for HMPAR is proposed with an analysis of connections between coherence and subarray partitions. First, the definition of coherence of array manifold is presented, and its connection between beampattern and the signal to interference plus noise ratio (SINR) is analysed. Next, the problem of optimising array manifold coherence is transformed into the discrete coherence optimization of the steering vector codebook. By properly selecting different codewords with the proper degree of correlation between each other, coherence between the target and interference steering vectors is optimised. The original combinatorial problem is approximated by a series of subproblems, which is solved in iteratively. Numerical results revealed that the proposed subarray partition provides lower coherence, lower beampattern peak sidelobes and higher SINR compared with other subarray configurations.

1 | INTRODUCTION

As a prominent representative of new radar systems, multiple input multiple output (MIMO) radar has been the focus of research and is receiving enduring attention owing to its beampattern design flexibility, increased parameter identifiability and virtual array extensibility [1], all of which come from its inherent diversity in transmitting independent waveforms and receiving echoes through matched filters. Such diversity in MIMO radar favours target detection and interference suppression, but it is counterproductive to target tracking, resolution and parameter estimation because of the lack of robustness against noise as a result of its omnidirectional transmission of waveforms [2]. To adapt MIMO radar to those unfavourable tasks, more degrees of freedom have been exploited from its beampattern by designing partially correlated waveforms between orthogonal waveforms in MIMO radar and fully correlated ones in phased array radar [3]. However, to achieve beampattern or spatial energy distributions while meeting other requirements in MIMO radar, the design of waveform covariance matrix and corresponding partially correlated waveforms often involves solving optimization problems with high computational complexity. Worse, implementing an MIMO radar system with partially correlated waveforms requires each element in the MIMO radar array to be equipped with particular phase shifters, amplifiers, transceivers, matched filters and other electronic devices. For a modern radar system with hundreds or even thousands of array elements uniformly placed on a regular grid, the cost and complexity of assigning one waveform for each element are unaffordable and unnecessary [4]. Therefore, methods to reduce the cost and complexity of MIMO radar system while preserving the diversity of MIMO radar are receiving attention.

To reduce the complexity of the system while also exploiting more degree of freedom (DOF) from the array configuration of an MIMO radar, a radar system that combines the merits of both an MIMO radar and phased array radar is proposed, called hybrid MIMO phased array radar (HMPAR)
MIMO radar with beamforming on the receive side still has limitations. In fact, Tang et al. [21] shows that partially correlated beamforming matrix design has much more DOFs than directly designing array structures. For example, Fuhrmann et al. [8] uses disjoint subarrays to transmit waveforms with special temporal and spatial properties to achieve an arbitrary shape of a beampattern at the transmitting side while retaining a phased-array like resolution at the receiving side. To reduce complexity and aperture loss from disjoint subapertures, other authors [9–12] propose a uniform overlapped subarray partition to expand subarray apertures while producing beampatterns with lower sidelobes as well as higher signal to interference plus noise ratios (SINRs). To exploit the DOFs from subarray partition further, non-uniform subarray structures were developed [13–15], which further promote beampattern and SINR performance. These enhanced performances come from the overlapped subarray partition structure of the system. However, research on transmit array partitioning concentrates on validating the advantage of certain partitions in terms of beampattern design [16], interference cancelation [17] and direction finding [18]. In fact, new DOFs from array configurations have been under-used, because partitioning methods for transmit arrays can neither determine from the value of two gains whether a particular array partition is suitable for a certain environment nor modify the configuration of a transmit array by adjusting the value of two gains. Connections between performance, the transmit array partition and the method for designing a partition based on these connections, however, have not been reported in the open literature.

Transmit coherent gain and waveform diversity gain are ways to describe cross-correlation properties of the waveforms of HMPAR, and the balance between these gains is usually achieved by adjusting the degree of correlation between spatial waveforms. Because the waveforms transmitted from different subarrays are orthogonal from each other, the correlation between different waveforms in space is closely related to the coherence of an array manifold matrix, which is determined by the specific partition of the array [19].

One way to partition an array is to design a beamforming matrix, which also serves the purpose of changing the degree of correlation between spatial waveforms and realize a beampattern with a certain energy distribution. Thus, the design of a beamforming matrix usually aims to match beampatterns, whereas even with identical beampatterns with HMPAR, MIMO radar with beamforming on the receive side still has serious SINR degradation owing to the effect of correlated noise [20]. This contrast indicates that although beampattern matching is important to the design of a beamforming matrix, beamforming matrix design has much more DOFs that can be used for applications other than beampattern matching, such as interference cancelation, direction finding and array partitioning. In fact, Tang et al. [21] shows that partially correlated waveforms in the space are equivalent to those formed by transmitting orthogonal ones from multiple subarrays through the proper design of a beamforming matrix, which suggests that an array partition based on a beamforming matrix provides more DOFs than directly designing array structures. In this way, cross-correlation between different waveforms, the coherence of an array manifold matrix, the partition of subarrays and beamforming matrix, and selection of a suitable target of optimization are crucial.

As a key performance index that has an important role in designing a compressive sensing matrix [22] and reducing beampattern sidelobes [23], the coherence of an array manifold matrix measures the degree of correlation among different columns of a matrix. An array partitioning scheme with the target of optimising beamforming matrix is proposed that achieves more flexible trade-off between transmit coherent gain and waveform diversity gain by adjusting the degree of correlation among different columns of array manifold matrix. To obtain optimised coherence, the proposed method transforms the original optimization of beamforming matrix into a problem that finds multiple elements in the Grassmannian manifold with a certain distance. To tackle the non-convex and non-differentiable nature of the problem, the transformed problem is approximated by a series of differentiable subproblems that can be solved iteratively by update factors deduced from theoretical derivation. Numerical results have shown that by optimising the coherence of the array manifold matrix directly, the proposed method can exploit DOF in the array configuration and has better beampattern sidelobe and SINR performance compared with other array partitions.

The contributions of this work are thus:

(1) Current research on array partitions or subarray configurations is concentrated on the performance analysis of the proposed design, whose advantages are attributed to the trade-off between two gains without an in-depth discussion on the essential advantage brought by proposed design. Although theoretical derivations are provided in Hassanien and Vorobyov [6] and Li and Wang [15], those propositions focus on validating the improved performance of some particular metrics without depicting the connection between the array configuration and performance. Here, the connections among the trade-off, cross-correlations among different waveforms, cross-correlations among steering vectors and the coherence of array manifold matrix are exhibited so that the effect of array partitioning can be better characterised and the DOF of array configuration can be exploited more flexibly to obtain improved system performance.

(2) After characterising the connection between trade-off and the feature of array configuration, a method for optimising the coherence of an array manifold matrix is proposed, which is equivalent to finding multiple lines with an optimised minimum distance between them, which is equivalent to the Grassmannian line packing problem. Unfortunately, it is difficult to obtain a solution to the solution to the proposed problem owing to its non-convex
and non-differentiable nature. To tackle these issues, the original problem is decomposed into multiple subproblems by the successive approximation of ρ-norm to ∞-norm. Moreover, a coherence-based metric is proposed as the target in each subproblem instead of the distance-based metric commonly used in other studies to obtain an array manifold with optimised coherence. Using the Lagrange function technique, the constrained problem is turned into its unconstrained duality forms, and the equilibrium conditions are solved to obtain the update factors on the array manifold matrix, which is optimised in iteratively.

(3) Although the proposed scheme has optimised coherence of the array manifold matrix, the degree of cross-correlation among steering vectors over different angles is not necessarily optimal owing to the discrete nature of the proposed scheme. To deal with this problem, the dimensions of the beamforming matrix and the array manifold matrix are considered separately, which achieves balance between the problem dimensions and flexibility in adjusting coherence. In addition, update factors are considered for the codebook composed of target steering vectors and the one containing interference steering vectors so that two codebooks are updated separately, guaranteeing the optimised degree of cross-correlation between target and interference steering vectors.

The proposed coherence based scheme is neither optimal nor the best existing possible solution for array structure optimization. Rather, the proposed scheme opens the possibility for coherence optimization through adjusting the transmit array structure. Making both the transmit and receive array sparse provides even more degrees of freedom and reduces computational and hardware costs; improved coherence optimization on a sparse array is an interesting topic for future research.

The rest of this work is organised as follows: the signal model of HMPAR, is given in Section 2. Section 3 provides the analysis of connection between the trade-off of gains, correlation between waveforms, coherence and subarray partitions. Section 4 mainly focuses on coherence-based optimization and its iterative approximation to obtain solutions. Numerical results are provided in Section 5, and Section 6 concludes the work with future prospects.

## SIGNAL MODEL

Assume there are $M_r$ elements in the transmit array and $M_c$ elements in the receive array of HMPAR. The whole transmit array aperture is partitioned into $N$ subarrays corresponding to $N$ waveforms. According to the definition in Hassanien and Vorobiov [6], each subarray may contain different numbers of elements ranging from 1 to $M_r$, and each orthogonal waveform is coherently transmitted from one subarray, whereas different subarrays overlap. In this way, different waveforms are transmitted from superimposed subarrays rather than array elements so that the number of beamforming controls, which is related to the cost and complexity of the system, is reduced.

Without a loss of generality, the distance between two elements are assumed to be half the wavelength, to simplify the subsequent analysis. The partition of the whole array is usually expressed in an $M_r \times N$ beamforming matrix $W$, each column of which corresponds to the specific partition of one subarray, reflected in its corresponding weight vector $w_n$, with zeros denoting deactivated array elements and non-zero values indicating activated ones.

The complex envelope of the signal transmitted by the $n$-th subarray is modelled as:

$$s_n(t) = \rho w_n^T \phi_n(t)$$ (1)

where $\rho$ is coefficient denoting energy allocated to each subarray; $\phi_n(t)$ is the orthogonal waveform transmitted from the $n$-th subarray, with the weight vector expressed as $w_n$. For orthogonal waveforms, there are $\int \phi_n(t) \phi_m^H(t) \, dt = 0 (n \neq m)$.

After being transmitted from $n$-th subarray, the waveform due to the target at the direction $\theta$ is:

$$p_n(\theta, t) = s_n^T(\theta, t) a(\theta) = \rho w_n^H \phi_n(\theta)$$ (2)

where $a(\theta) = [1, e^{-j \pi \sin \theta}, \ldots, e^{-j \pi (M-1) \sin \theta}]^T$ is the steering vector of the whole transmit array. The echo reflected from the direction $\theta$ consists of all waveforms from different subarrays:

$$r(\theta, t) = \rho \beta(\theta) \sum_{n=1}^{N} w_n^H a(\theta) \phi_n(t)$$ (3)

where $\beta(\theta)$ is the reflection coefficient from hypothetical target in direction $\theta$. Using matrix notation, Equation (3) can be rewritten as:

$$r(\theta, t) = \rho \beta(\theta) (W^H a(\theta))^T \phi(t)$$ (4)

where $W = [w_1, w_2, \ldots, w_N]$ is an $M_r \times N$ transmit beamforming matrix whose $n$-th column corresponds to the weight vector for the $n$-th subarray. Here, $\phi(t)$ is a vector and $\phi(t) = [\phi_1(t), \phi_2(t), \ldots, \phi_N(t)]^T$.

Suppose a spatial target is located at direction $\theta$, with reflection coefficient as $\beta(\theta)$ and multiple interference sources at directions $\theta_i (i = 1, 2, \ldots, D)$ with reflection coefficients $\beta(\theta_i)$, respectively. The received signal waveform at the receiver array can be expressed as:

$$x(t) = r(\theta, t) b(\theta) + \sum_{i=1}^{D} r(\theta_i, t) b(\theta_i) + n(t)$$ (5)

where $b(\theta_i)$ and $b(\theta)$ are both $M_r \times 1$ receive steering vector and $n(t)$ is a noise vector with the same dimension, which has zero mean and covariance matrix $R_n \in \mathbb{C}^{M_r \times M_r}$. 
The output data after being processed through matched filters against different waveforms at the receivers can be formulated as an $NM_r \times 1$ virtual data vector:

$$y(t) = \rho \beta(\theta) u(\theta) + \rho \sum_{i=1}^{D} \beta(\theta_i) u(\theta_i) + n(t)$$ \hspace{1cm} (6)

where $u(\theta) = (W^H a(\theta)) \otimes b(\theta)$ is an $NM_r \times 1$ virtual steering vector.

In the case of disjoint subarray partition, the matrix $W_{dis}$ can be expressed as:

$$W_{dis} \triangleq \begin{bmatrix}
    w_{1,1} & 0 & 0 \\
    \vdots & \ddots & \vdots \\
    w_{1,K_1} & \cdots & w_{3,1} \\
    0_{K_2} & \cdots & 0_{K_2} \\
    \vdots & \ddots & \vdots \\
    0_{K_N} & 0_{K_N} & 0_{K_N} \\
    \vdots & \cdots & w_{K,1} \\
    \vdots & \cdots & \vdots \\
    \vdots & \cdots & \vdots \\
    \vdots & \cdots & \vdots \\
    \vdots & \cdots & \vdots \\
    0_{K_{M-K+1}} & w_{K,M-K+1} & \cdots \\
    0 & 0 & w_{K,M-K+1}
\end{bmatrix}$$ \hspace{1cm} (7)

where $0$ denotes the vector of all zeros and $w_{ij}$ represents the weight corresponding to the $j$-th element in the $i$-th subarray; $K_i$ is the number of elements in the $i$-th subarray, and there are $\sum_i K_i = N$.

In the case of an equally overlapped subarray partition, the matrix $W_{opp}$ can be denoted as:

$$W_{opp} \triangleq \begin{bmatrix}
    w_{1,1} & 0 & 0 \\
    \vdots & \ddots & \vdots \\
    w_{1,K_1} & \cdots & w_{K-2} \\
    0_{K-2} & \cdots & w_{K,1} \\
    \vdots & \ddots & \vdots \\
    0_{M-K+1} & w_{K,M-K+1} & \cdots \\
    0 & 0 & w_{K,M-K+1}
\end{bmatrix}$$ \hspace{1cm} (8)

where $0$ denotes the vector of all zeros and $w_{ij}$ represents the weight corresponding to the $j$-th element in the $i$-th subarray.

Similarly, in the case of an unequally overlapped subarray partition, matrix $W_{uopp}$ has the form:

$$W_{uopp} \triangleq \begin{bmatrix}
    w_{1,1} & w_{2,1} & w_{K,1} \\
    \vdots & \ddots & \vdots \\
    w_{1,M-K+1} & \cdots & w_{K,3} \\
    0_{K-2} & \cdots & w_{K,M-K+1} \\
    0 & 0 & w_{K,M}
\end{bmatrix}$$ \hspace{1cm} (9)

where $0$ denotes the vector of all zeros and $w_{ij}$ denotes the weight corresponding to the $j$-th element in the $i$-th subarray.

These examples show that a phased MIMO radar with different subarray partitions has different beamforming matrices, which have an effect on system performance because they determine how the output from each element is combined in the subarrays. In fact, even with the same number of subarrays, different array partitions contribute to different degrees of cross-correlation between spatial waveforms, which in turn lead to different system performance. This suggests that direct array partitioning can be expressed in beamforming matrices, yet optimization over beamforming matrices is more flexible than direct array partitioning because the former strategy determines both the distribution of ones and zeros and the excitation value of non-zero elements, but the latter does not.

Therefore, more examples of designing beamforming matrix $W$ are found in Hassanien and Vorobyov [16] and Tang et al. [21], in which the matrices are not determined by assigning the index of zeros and non-zeros, but are solved from semidefinite optimization problems with the target of energy focussing and direction finding. However, although the resultant matrices are more flexible in focussing energy in certain spatial sectors while suppressing potential interferences in others, these problems have computational complexity and are often prone to phase distortion, which makes the system not robust to noise and other factors. To exploit DOF further from the partition of array, connections among specific system performance, spatial waveform correlations, and array partitions are explored. Therefore, to provide insights into designing array partitions, these connections are discussed in the next section.

## 3 | CONNECTION BETWEEN TRADE-OFF AND SUBARRAY PARTITION

In this section, the trade-off between transmit coherent gain and waveform diversity gain, cross-correlation between waveforms, the coherence of array manifold matrix and the way to partition subarrays is discussed to depict their connections. Balance between the two gains are represented by the normalized beampattern and SINR performance; different array partitions are expressed by their corresponding beamforming matrices.

### 3.1 | Trade-off between two gains

In this subsection, we consider the beampattern of an HMPAR radar with conventional receive beamforming for the sake of simplicity. A beampattern with adaptive beamforming has similar advantages without a loss of generality [24].

The definition of a beampattern varies in the literature. Here, we adopt the normalized transmit-receive beampattern, given by following:

$$G(\theta) = \frac{\left| w_{ij}^H u(\theta) \right|^2}{\left| w_{ij}^H u(\theta) \right|} = \frac{\left| u(\theta) \right|^2}{\left| u(\theta) \right|} \quad (10)$$
where \( w_r \) is an \( NM \times 1 \) receive beamforming vector, which is the same as the transmit-receive virtual steering vector, according to the definition of conventional beamforming:

\[
\mathbf{w}_r = \mathbf{u}(\theta) = (\mathbf{c}(\theta) \odot \mathbf{d}(\theta)) \otimes \mathbf{b}(\theta)
\]

where

\[
\mathbf{u}(\theta) \triangleq (\mathbf{c}(\theta) \odot \mathbf{d}(\theta)) \otimes \mathbf{b}(\theta)
\]

The transmit coherent processing vector can be expressed as:

\[
\mathbf{c}(\theta) \triangleq \left[ \mathbf{w}_1 \mathbf{a}_1(\theta), ..., \mathbf{w}_N \mathbf{a}_N(\theta) \right]^T
\]

where \( \mathbf{a}_n(\theta) \) is the \( n \)-th steering vector including only active elements in the transmit array and \( \mathbf{w}_n \) is the corresponding weight vector. Without the loss of generality, the conventional beamforming is also adopted at the transmit side, which means the transmit weight vector is the same as the steering vector. In other words, \( \mathbf{w}_n = \mathbf{a}_n(\theta) \).

Similarly, the waveform diversity vector can be expressed as:

\[
\mathbf{d}(\theta) \triangleq \left[ e^{-j\tau_1(\theta)}, ..., e^{-j\tau_N(\theta)} \right]^T
\]

where \( e^{-j\tau_n(\theta)} \) is the phase difference incurred by time travelling from the first element of the first subarray to the first element of the \( n \)-th subarray.

Taking in Equations (11) and (10) can be rewritten as:

\[
D(\theta) = \frac{|\mathbf{d}^H(\theta)\mathbf{d}(\theta)|^2}{||\mathbf{d}(\theta)||^4} \tag{17}
\]

And the third is the receive beampattern:

\[
R(\theta) = \frac{|\mathbf{b}^H(\theta)\mathbf{b}(\theta)|^2}{||\mathbf{b}(\theta)||^4} \tag{18}
\]

Because the beampattern represents the gain of the array antenna in different directions, Equations (16) and (17) are also called transmit coherent processing gain and waveform diversity gain, respectively. In Deligiannis et al. [10], the target of the array partition is to achieve balance or a trade-off between these two gains through proper array partition. However, the metric of the trade-off neither gives a proper criterion to determine a good array partition nor provides ways to partition the array according to the value of two gains. Because the value of the two gains reflects the degree of correlation between spatial waveforms, the cross-correlation of waveforms will be discussed in the next subsection.

### 3.2 Cross-correlation between waveforms

The beamforming matrix satisfies that \( \mathbf{W} = [\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_N] \). Thus, the virtual steering vector can also be expressed as:

\[
\mathbf{w}_r = \mathbf{u}(\theta) = (\mathbf{W}^H \mathbf{a}(\theta)) \otimes \mathbf{b}(\theta) \tag{19}
\]

In this way, Equation (16) can also be reformulated as:

\[
G(\theta) = \frac{|(\mathbf{W}^H \mathbf{a}(\theta) \otimes \mathbf{b}(\theta))^H(\mathbf{W}^H \mathbf{a}(\theta) \otimes \mathbf{b}(\theta))|^2}{||\mathbf{W}^H \mathbf{a}(\theta) \otimes \mathbf{b}(\theta)||^4} \tag{20}
\]

\[
= \sum_{k=1}^{N} \frac{||\mathbf{W}^H \mathbf{a}(\theta)||^2}{||\mathbf{W}^H \mathbf{a}(\theta)||^4} \cdot \frac{||\mathbf{d}(\theta)||^2}{||\mathbf{d}(\theta)||^4} \cdot \frac{||\mathbf{b}(\theta)||^2}{||\mathbf{b}(\theta)||^4}
\]

\[
= \sum_{k=1}^{N} \frac{|\mathbf{a}_k^H(\theta)\tilde{\mathbf{a}}_k(\theta)|^2}{||\mathbf{a}_k(\theta)||^4} \cdot \frac{||\mathbf{d}(\theta)||^2}{||\mathbf{d}(\theta)||^4} \cdot \frac{||\mathbf{b}(\theta)||^2}{||\mathbf{b}(\theta)||^4}
\]

\[
= \frac{1}{N} \sum_{k=1}^{N} \frac{|\mathbf{a}_k^H(\theta)\tilde{\mathbf{a}}_k(\theta)|^2}{||\mathbf{a}_k(\theta)||^4} \cdot \frac{||\mathbf{d}(\theta)||^2}{||\mathbf{d}(\theta)||^4} \cdot \frac{||\mathbf{b}(\theta)||^2}{||\mathbf{b}(\theta)||^4}
\]

\[
(15)
\]

It is evident from Equation (15) that the overall beampattern of HMPAR can be seen as a product of three terms, the first of which is the transmit coherent beampattern:

\[
C(\theta) = \sum_{k=1}^{N} \frac{|\mathbf{a}_k^H(\theta)\tilde{\mathbf{a}}_k(\theta)|^2}{||\mathbf{a}_k(\theta)||^4} \tag{16}
\]

The second is the waveform diversity beampattern:

\[
R(\theta) = \frac{|\mathbf{b}_k^H(\theta)\mathbf{b}(\theta)|^2}{||\mathbf{b}(\theta)||^4} \tag{22}
\]
Here, we focus on comparing the transmit coherent beampattern because the DOF of HMPAR mainly comes from transmit side. The comparison between overall beampatterns is thus mainly a comparison of the first term with different partitions.

Equation (4) can also be expressed as:

\[
\begin{align*}
\rho(\theta, t) &= \rho(\theta)(W^H a(\theta))^{T} \phi(t) \\
&= \rho(\theta)(a(\theta))^T W^* \phi(t)
\end{align*}
\]

Equation (23) reveals that orthogonal waveforms transmitted through subarray partitions expressed in beamforming matrix \( W \) are equivalent to partially correlated signal waveforms determined by the matrix, namely:

\[
s(t) = W^* \phi(t)
\]

The waveform covariance matrix or cross-correlation matrix between different waveforms can be expressed as:

\[
R = E\{ss^H\} = WW^H
\]

Equation (21) can be therefore reformulated as:

\[
C(\theta) = \frac{|a^H(\theta) Ra(\theta)|^2}{||W^H a(\theta)||^4}
\]

Equation (26) indicates that rather than measuring the property of beampattern by two gains, the effect of these gains is unified in matrix \( R \), which reflects the degree of correlation between different waveforms; it also determines the features of the beampattern. In fact, by designing matrix \( W \) properly, a beampattern with certain characteristics can be achieved. A specific application of optimising \( W \) is designing a beampattern with low sidelobes, which contributes to SINR performance.

The sidelobe is defined to be the beampattern gain value outside the main lobe region, or the sectors of interest. In accord with this definition, the beampattern within the sidelobe region is:

\[
S(\theta) = C(\theta) R(\theta), \theta \in \Theta
\]

where \( \Theta \) denotes the complement set of angles of interest.

The transmit coherent beampattern within the sidelobe region is:

\[
C(\theta) = \frac{|a^H(\theta) Ra(\theta)|^2}{||W^H a(\theta)||^4}
\]

Therefore, low beampattern sidelobes can be achieved by designing matrix \( R \) or \( W \) with certain cross-correlation properties.

The cross-correlation between different waveforms also affects the output SINR performance as follows.

The output SINR of HMPAR is defined as the ratio of energy of the signal over the energy of interference and noise:

\[
\text{SINR} = \frac{\frac{M_i}{N} \sigma_i^2 |w^H u(\theta_i)|^2}{w^H R_{i+n} w_r}
\]

\( R_{i+n} \) can be expressed as:

\[
R_{i+n} = \sum_{i=1}^{D} \frac{M_i}{N} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2 + \sigma_n^2 I
\]

Therefore, the output of SINR can be reformulated as:

\[
\text{SINR} = \frac{\frac{M_i}{N} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2}{u^H(\theta_i) \left( \sum_{i=1}^{D} \frac{M_i}{N} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2 + \sigma_n^2 I \right) u(\theta_i)}
\]

\[
= \frac{\frac{M_i}{N} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2}{u^H(\theta_i) \left( \sum_{i=1}^{D} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2 + \sigma_n^2 u^H(\theta_i)u(\theta_i) \right)}
\]

where \( \sigma_i^2, \sigma_n^2 \) denote the power of signal, interference and noise, respectively.

Assume that all interference comes from angles in the sidelobe region with dominant power compared with noise, the output SINR can be rewritten as:

\[
\text{SINR} = \frac{\sigma_i^2 \| u(\theta_i) \|^4}{\sum_{i=1}^{D} \sigma_i^2 |u^H(\theta_i)u(\theta_i)|^2}
\]

In the same way, the output SINR is related to the cross-correlation of waveforms or the waveform covariance matrix. The cross-correlation property is brought by the partitioning of subarrays, which is determined by the beamforming matrix. Cross-correlation among different waveforms is closely connected to the subarray partitions, which is discussed in the next subsection.
3.3 Cross-correlation among steering vectors

As shown in the last subsection, cross-correlation among different waveforms is brought by matrix \( W \), which reflects both subarray partitions and combination coefficients of each subarray. It is therefore necessary to consider the effect of subarray properties on the performance of beampatterns and output SINRs.

Because partially correlated waveforms are equivalent to orthogonal waveforms emitted from an equivalent array, letting \( c(\theta) = W^H a(\theta) \), which can be seen as an equivalent steering vector, Equation (17) can be reformulated as:

\[
C(\theta) = \frac{|c^H(\theta)c(\theta)|^2}{\|c(\theta)\|^4}, \quad \theta \in \Theta
\]  

(33)

In this way, cross-correlation among different waveforms is integrated in the degree of correlation among equivalent steering vectors over different sets of angles. In designing a low sidelobe of beampattern, the most important target to be optimised is the highest sidelobe. Equation (18) shows that the highest sidelobe of the transmit beampattern is determined by the largest value of the term \( c^H(\theta)c(\theta) \), which denotes the degree of cross-correlation among equivalent steering vectors over different sets of angles. To reduce the peak value of \( c^H(\theta)c(\theta) \), cross-correlation among equivalent steering vectors over different sets of angles is to be investigated.

Similar observations can be obtained for output SINR:

\[
u^H(\theta)u(\theta)
\]

\[
= (W^H a(\theta) \otimes b(\theta))^H (W^H a(\theta) \otimes b(\theta))
\]

\[
= \left[ (W^H a(\theta))^H \otimes b^H(\theta) \right] \left( W^H a(\theta) \otimes b(\theta) \right)
\]

\[
= \left[ (W^H a(\theta))^H W^H a(\theta) \right] \otimes \left( b^H(\theta)b(\theta) \right)
\]

\[
= a^H(\theta)WW^H a(\theta)b^H(\theta)b(\theta)
\]

\[
= c^H(\theta)c(\theta)b^H(\theta)b(\theta)
\]

Therefore:

\[
\text{SINR} = \frac{\sigma^2}{\sum_{l=1}^{D} |c^H(\theta)c(\theta)|^2 \left| b^H(\theta)b(\theta) \right|^2}
\]

(35)

Equation (35) shows that the output SINR is inversely relevant to the beampattern when power of sources and interference are determined in advance. According to Equation (24), the value of the denominator is largely determined by the term \( c^H(\theta)c(\theta) \). In the same way, cross-correlation among equivalent steering vectors over different sets of angles in the potential location of targets and interference is considered for maximising the SINR.

3.4 Coherence of array manifold matrix

In the earlier description, the term \( c^H(\theta)c(\theta) \) is important in the performance of the beampattern and SINR of an HMPAR. If the maximum inner product between two equivalent transmit steering vectors, \( c(\theta) \) and \( c(\theta) \), is smaller, the beampattern value in the sidelobe region is smaller and the denominator in the expression of SINR is larger. Because the maximum inner product is obtained from steering vectors over different directions, which can be grouped into a matrix describing the array manifold, the coherence of the array manifold matrix is considered. To facilitate the discussion afterward, the following definition of coherence is considered:

**Definition 1**: Coherence of a matrix \( C \) is defined as the maximum inner product between the normalized columns of \( C \). For a matrix \( C = [c_1, c_2, \ldots, c_D] \), where \( c_n = [c_{1,n}, c_{2,n}, \ldots, c_{N,n}]^T \), coherence is defined to be:

\[
\mu(C) = \max_{n \neq l} \frac{|\langle c_n, c_l \rangle|}{\|c_n\| \|c_l\|}
\]

(36)

where \( |\langle c_n, c_l \rangle| = c_n^H c_l, n, l \in \{1, 2, \ldots, D\} \).

Using the notation \( c(\theta) = W^H a(\theta) \), Equation (6) can be rewritten as:

\[
y(t) = \rho \beta(\theta)(c(\theta) \otimes b(\theta)) + n(t)
\]

\[
+ \rho \sum_{i=1}^{D} \beta_i(\theta)(c(\theta) \otimes b(\theta)) + n(t)
\]

\[
\mu(C) = \max_{n \neq l} \frac{|\langle c_n, c_l \rangle|}{\|c_n\| \|c_l\|}
\]

(37)

Let \( v(\theta) = c(\theta) \otimes b(\theta), V = [v(\theta_1), v(\theta_2), \ldots, v(\theta_D)] \) and \( X = \rho[\beta(\theta_1), \beta(\theta_1), \ldots, \beta(\theta_D)]^T \). Equation (25) can be rewritten as:

\[
y(t) = VX + n(t)
\]

(38)

Based on the simplified model in Equation (27), gram matrix \( Q = V^H V \), which has an important role in compressive sensing, also provides some insight into the coherence and beampattern. The element on the \( i \)-th row and \( l \)-th column of matrix \( Q \) can be represented as:

\[
Q(i, l) = \frac{v^H(\theta_i)v(\theta_l)}{\|v(\theta_i)\| \|v(\theta_l)\|}
\]

\[
= \frac{1}{M_iM_l} \sum_{m=1}^{M_i} \sum_{n=1}^{M_l} \exp(j\pi(m + n)(\sin(\theta_i) - \sin(\theta_l)))
\]

(39)

The peak value matrix \( Q \) on the diagonal, which denotes the largest value of the inner products between steering vectors on the same direction, is called the main lobe. The peak value of matrix \( Q \) off the diagonal, denoting the dot product of steering vectors on different directions, is called the sidelobe. It is therefore apparent that the term \( \frac{v^H(\theta_i)v(\theta_l)}{\|v(\theta_i)\| \|v(\theta_l)\|} \) has the role of the sidelobe.
Because \( v(\theta) = c(\theta) \otimes b(\theta) \), it is also seen from the property of the Kronecker product that:

\[
\begin{align*}
\mathbf{v}^H(\theta_1)\mathbf{v}(\theta_2) &= (c(\theta_1) \otimes b(\theta_1))^H(c(\theta_2) \otimes b(\theta_2)) \\
&= c^H(\theta_1)c(\theta_2)b^H(\theta_1)b(\theta_2) \\
&= c^H(\theta_1)c(\theta_2)\mathbf{H}(\theta_1)\mathbf{b}(\theta_1)
\end{align*}
\] (40)

Equation (40) suggests that the coherence of different virtual steering vectors is associated with the coherence of different transmit steering vectors as well as that of different receiving steering vectors. Moreover, the coherence of \( \mathbf{V} \), or the largest sidelobe on Gram matrix \( \mathbf{Q} \), can be reduced by decreasing the coherence of either the transmit manifold matrix or the receiving manifold matrix. The goal is to partition the transmit subarray to achieve lower coherence; the design on receive array or the joint design of both transmit and receive array is a topic to be considered in future.

In this way, the relation between the coherence and beampattern is clear. To minimise the highest sidelobe of the beampattern of \( \mathbf{Q} \) and maximise the SINR of the system, coherence reflected in the inner product term is minimised.

To perform optimization over beampattern side lobes and the SINR, the steering vector over different regions of interest is considered. Because the peak sidelobe of the beampattern corresponds to the maximum of \( c^H(\theta_i)c(\theta) \), whereas the largest denominator of SINR expression is also related to the maximum of \( c^H(\theta_i)c(\theta) \), it is necessary to design \( c(\theta) \) so that the maximum inner products among steering vectors over different angles is minimised. In other words,

\[
\begin{align*}
\min_{\theta_i \neq \theta_j} \frac{\|c(\theta_i)\|}{\|c(\theta_j)\|} \\
\text{Or}
\end{align*}
\]

\[
\begin{align*}
\min_{\theta_i \neq \theta_j} \mu(c),
\end{align*}
\] (42)

The \( N \times D \) codebook matrix \( \mathbf{C} \) can be constructed by putting steering vectors over different angles into its columns as codewords, where \( N \) is the number of subarrays and \( D \) is the number of angles of interest. This shows that to minimise the coherence of matrix \( \mathbf{C} \), the inner product of each two columns of the manifold matrix needs to be minimised. Therefore, by putting steering vector \( c(\theta) \) over different angles in the different columns of a matrix, the term \( c^H(\theta_i)c(\theta) \) can be measured and analysed using the metric of coherence.

The \( N \times D \) matrix \( \mathbf{C} \) can be regarded as a rank-\( D \) codebook with each column as an \( N \times 1 \) codeword in the complex Grassmannian manifold of \( \mathbb{C}^N \). In this way, each codeword can be seen as a line in the Grassmannian manifold, where coherence among different vectors in the subspace of manifold is well-defined. In the Grassmannian manifold, minimisation of coherence of a codebook is equivalent to maximisation of the minimum distance between lines, all of which are represented by codewords in the codebook. Therefore, the problem of finding \( N \) lines with minimum coherence or maximum distance between each other is called Grassmannian line packing problems.

However, it is difficult to obtain the solution to a Grassmannian line packing problem for three reasons: first, neither the objective function of the problem \( \min_{\theta_i \neq \theta_j} \mu(c) \) nor the manifold on which it lies is convex. Second, the \( \min \) operator is not differentiable, so the gradient cannot be calculated for the possible use of unconstrained optimization techniques. Worse, although alternative metrics are available for measuring the distance of different lines, these metrics for the manifold are both non-convex and non-differentiable. To tackle this problem, the method for discrete coherence optimization is proposed in Section 4.

4 | COHERENCE OPTIMIZED SUBARRAY PARTITION

4.1 | Discrete coherence optimization of steering vector codebook

4.1.1 | Decomposition of subproblems

Because of the non-convexity of the target function, the problem is inherently non-convex. Worse, the minimisation-maximisation structure of the target function makes it unsuitable for heuristic methods because its derivative is not smooth. To tackle this problem, we take the sequential optimization technique, decomposing a problem into a series of subproblems with smooth derivatives to be solved using heuristic methods such as gradient descent, conjugate gradient and Newton’s method.

To mitigate the effect of norm of vector on the measurement of coherence, the constraint of unit length is added. Therefore, the problem of minimising vector inner product or coherence is:

\[
\begin{align*}
\min_{\theta_i \neq \theta_j} \max_{\mathbf{C}} \|\mathbf{f}(\mathbf{c}_i)\| \quad s.t. \quad \|\mathbf{c}_i\|^2 = 1 \\
\end{align*}
\] (43)

One difficulty in solving Equation (28) is fact that operator \( \max(*) \) is neither convex nor differentiable over the manifold. To achieve at least a smooth derivative for the target function, the smooth approximation of \( p \)-norm to the \( \infty \)-norm is considered. In other words, the \( \infty \)-norm is:

\[
\begin{align*}
\max_{\mathbf{C}} \|\mathbf{f}(\mathbf{c}_i)\| \\
\end{align*}
\] (44)

where \( f(*) \) is the target function on the manifold and \( \{\mathbf{c}_i\} \) is a set from Grassmannian manifold \( G_{N,1} \), and can be approximated by \( p \)-norm.
where $p \geq 1$ and $p \to \infty$ asymptotically. In this way, approximation of $\infty$-norm of the target function on vectors is achieved by its $p$-norm. Therefore, the original problem in Equation (28) can be reformulated as a series of subproblems:

$$
\min \left( \sum_{\{c_i\} \in \mathbb{G}_{N,1}} |f(c_i)|^p \right)^{\frac{1}{p}} \text{ s.t. } \|c_i\|^2 = 1 \tag{46}
$$

where $p \geq 1$. In this way, the original problem can be approximately solved by solving multiple subproblems described in Equation (31). Specifically, the solution to the original problem in Equation (28) can be approximately obtained by taking an initial value of $p$ and a set of vectors $C$, both of which are exploited in the adaptation of a conventional optimization process such as gradient descent, conjugate gradient and Newton’s method to solve one subproblem on $C$ with parameter $p$. A locally optimised solution is produced by such a process, after which the solution to one subproblem is taken as the initial value for the next subproblem, with increasing $p$. After the approximation of $\infty$-norm by $p$-norm, the outer wrapping function of the objective is differentiable, whereas the inner dot product between vectors still needs to be considered.

### 4.1.2 Selection of metric

Coherence among different vectors is equivalent to the distance between vectors [25, 26], with larger intervector distance implying lower coherence and smaller distance indicating higher coherence. Multiple metrics and corresponding methods are proposed based on the intervector distance [27–30]. However, these distance-based metrics require a large amount of calculations rendered by the approximation of integral to summation. Worse, the distance metric often causes instability in determining the convergence of solution to subproblems. To provide a more stable and fast solution to each subproblem, we adopt a coherence-based metric and accompanying methods in the following.

The coherence-based metric is proposed as:

$$
|\langle c_n, c_i \rangle|^2 - \mu_c \tag{47}
$$

where $\mu_c$ is the target value, which serves as a benchmark to evaluate the difference between the inner product of any two vectors from the manifold, and is the lowest achievable bound of such product. Such a difference is used to adjust each component of the vector in the update process. The target value is usually chosen as $\mu_c = \beta \mu_e(N, D)$, where $\beta$ is the scaling coefficient and $\mu_e$ is the benchmark value of coherence, denoting the achievable lower coherence bound. The value of $\mu_c$ is usually given by [30]:

$$
\mu_c(N, D) = \begin{cases} 
\sqrt{\frac{D - N}{N(D - 1)}} & \text{if } D \leq N^2 \\
\max \left( \sqrt{\frac{1}{N}}, \frac{2D - N^2 - N}{(N + 1)(D - N)} \right) & \\
\text{if } N^2 \leq D \leq 2(N^2 - 1) & \\
\max \left( \sqrt{\frac{2D - N^2 - N}{(N + 1)(D - N)}}, 1 - 2N^{-\frac{1}{2}} \right) & \\
\text{if } 2(N^2 - 1) < D, 
\end{cases}
$$

Or, alternatively, the mean value of all inner products from the previous subproblem.

As $p$ increases, the differences among different codeword vectors are more intricate and harder to fathom, reducing the efficacy of the iterative method. To avoid potential instability brought by the constant bound, it is necessary to reduce the value by choosing $0 < \beta < 1$.

Therefore, the problem in Equation (46) can be reformulated as:

$$
\min \left( \sum_{n \neq l} \left( |\langle c_n, c_l \rangle|^2 - \mu_c^2 \right)^p \right)^{\frac{1}{p}} \text{ s.t. } \|c_n\|^2 = 1 \tag{49}
$$

### 4.1.3 Solution to subproblems

To satisfy the constraint of unit vector length, the Lagrange function technique is used to turn a constrained problem into an unconstrained one. Considering the unit-length constraints, the Lagrange function is defined as:

$$
f(C, \lambda) = \left( \sum_{n \neq l} \left( |\langle c_n, c_l \rangle|^2 - \mu_c^2 \right)^p \right)^{\frac{1}{p}} + \sum_{n=1}^{D} \lambda_n (\|c_n\|^2 - 1) \tag{50}
$$

where $\lambda_n \{n = 1, 2, \ldots, D\}$ are Lagrange multipliers for enforcing the unit-length constraints.

The solution to the constrained problem in Equation (33) can be found by deducing the local minimum of the
while Optimum beamforming matrix Output: Initial parameters: Input: algorithm Algorithm 1 Discrete coherence optimization

1: Randomly generate $W$
2: while $p < p_{\text{max}}$ do
3: while $q \leq q_{\text{max}}$, and any $\|c_n(q) - c_n^{(q-1)}\| \geq \varepsilon$ do
4: produce $G^{(q)}$ by Equation (52);
5: update $c_n(q)$ by Equation (53);
6: $p = 2p$,
7: $C^{(q)}$ produces the minimum coherence
8: Obtain $W$ from Equation (54).

The optimised set of vectors, or codebook, can be found by taking those equilibrium conditions as the constraint to construct gradients to perform gradient descent method on matrix $A$. Specifically, the codebook can be iteratively updated by adding a matrix of update factors multiplied by a gain coefficient, usually taking decaying values as the process is ongoing:

$$C^{(q)} = C^{(q-1)} + \alpha G^{(q)} \frac{\|G^{(q-1)} + \alpha G^{(q)}\|}{\|C^{(q-1)} + \alpha G^{(q)}\|}, q = 1, \ldots, q_{\text{max}}$$

Matrix $C$ converges to a fixed point as $q \to \infty$ when $\alpha$ is small enough.

Algorithm 1 Discrete coherence optimization algorithm

| Input: Initial parameters: $M$, $D$, $\alpha^{(1)}$, $\varepsilon$, $\mu$, $p_{\text{max}}$, $q_{\text{max}}$ |
| Output: Optimum beamforming matrix $W$ |
| 1: Randomly generate $C^{(0)}_{N \times D}$ |
| 2: while $p < p_{\text{max}}$ do |
| 3: while $q \leq q_{\text{max}}$, and any $\|c_n(q) - c_n^{(q-1)}\| \geq \varepsilon$ do |
| 4: produce $G^{(q)}$ by Equation (52);
5: update $c_n(q)$ by Equation (53);
6: $p = 2p$,
7: $C^{(q)}$ produces the minimum coherence
8: Obtain $W$ from Equation (54). |

After codebook $C$ with optimised coherence is obtained from the iterative optimization, a corresponding array partition matrix can be developed by:

$$W = (CA^\dagger)^H$$

where $A$ is the manifold matrix for the whole array at $D$ different angles.

The discrete coherence optimization algorithm is summarised in Algorithm 1.

4.1.4 Computational complexity analysis

The computational complexity of Algorithm 1 mainly consists of the multiplications in the process, or specifically, of the multiplications in the pseudoinverse operation of Equation (54) and multiple iterations or while loops; in each iteration, the complexity mainly lies in the update process of matrix $C$. For each $c_n$, the update process stated in Equation (52) is mainly composed of the dot product of two vectors. Because both of these two vectors are $N$-dimensional, the dot product takes $O(N)$ multiplications and their power operation takes $O(p_{\text{max}})$ multiplications. For the coherence of the matrix $C$ to be minimised, $D$ vectors in different columns of matrix $C$ go through the process stated in Equation (52), which takes $D(D - 1)$ combinations. Therefore, the complexity of multiplications in each loop is $O(D^2 \cdot \max\{N, p_{\text{max}}\})$.

The number of inner loops is $q_{\text{max}}$ because $q$ is increased by 1; the number of outer loops is $\log_2(p_{\text{max}})$ because $p$ is increased by the power of 2.

Therefore, the complexity of the update process is:

$$O(q_{\text{max}}D^2 \log_2(p_{\text{max}}) \cdot \max\{N, p_{\text{max}}\})$$

The complexity of the pseudoinverse of the matrix $C$ is $O(D^2)$, where $D$ denotes the number of angles in the process.

The main computational complexity of Algorithm 1 is $O(q_{\text{max}}D^2 \log_2(p_{\text{max}}) \cdot \max\{N, p_{\text{max}}\} + D)$. It is implied in the expression that the complexity is related to $q_{\text{max}}$, $p_{\text{max}}$, $N$, and $D$. Among them, the order of $D$ is important to the complexity because the value of $D$ is of the power of 3.

4.1.5 Problems with discrete coherence optimization

The partitioned array produce by discrete coherence optimization is used for direction finding with good performance, although the method still has beampattern distortion at some angle sector between only $D$ arbitrary different angles are selected in the manifold matrix. Worse, simply increasing the number of sampled angles does not suffice because the problem of increased sampling angles becomes ill-conditioned, leading to no viable solution. Distortions in the beampattern, especially those between preselected angles to minimise the steering vector on these points, suggest that it is far from
adequate to consider only these angles. However, it does not mean that the more angles, the lower the coherence among steering vectors on these angles and the better the beampattern performance. Although it is important to achieve minimised coherence among different steering vectors over the different angle sectors, it is even more important to refine the quality of the beampattern by selecting the proper number of angles to form the manifold matrix and decide which angles to choose. In other words, after choosing the metric and iterative update method, two more factors need to be determined:

1. How to select these angles to minimise coherence correctly, and
2. What the proper number of angles is that balances the quality of beampattern sampling and computational complexity.

To tackle these problems and reduce beampattern distortion, the sampled angles and corresponding steering vectors in the manifold matrix should not be chosen arbitrarily; rather, they ought to be selected considering relations among coherence, beampattern sidelobe level and SINR performance. Selection of the proper number of angles in the manifold matrix will be discussed in the next subsection.

4.2 | Oversampled coherence optimization

4.2.1 | Reformulation of coherence optimization

A problem of the discrete optimization problem is fact that the manifold matrix considers only discrete $D$ angles. Although codebook $\mathbf{C}$ is able to achieve the lowest possible coherence among codewords, coherence in angles between those preset discrete ones is not necessarily satisfactory. Consider manifold matrix $\mathbf{C}$, with $D$ differently sampled angles over the whole scanning region. The discrete optimization in the last subsection achieves the lowest possible coherence among different columns of manifold matrix $\mathbf{C}$. However, although coherence among different steering vectors is optimised, the sidelobe level in the beampattern is associated with the coherence of two steering vectors, one with target direction $\theta$, and another with the direction outside the sector of interest, that is $\theta \notin \Theta$. Such a connection between coherence and the sidelobe level is not considered in discrete coherence optimization. To guarantee that optimised coherence fully serves for the optimised sidelobe level, it is necessary to transform the original problem into a minimisation of coherence among steering vectors within the sectors of interest and those outside the sectors of interest. Therefore, the optimization problem in Equation (43) can be reformulated as:

$$\min_{\mathbf{c}} \max_{\theta \in \Theta} \left| \mathbf{c}^H(\theta_s)\mathbf{c}(\theta_t) \right|, s.t. \| \mathbf{c}(\theta_t) \| = 1$$

(55)

where $\theta_t$ is the steering angle, $\Theta$ is directions in the sectors of interest, and $\Theta$ is regions outside the sectors of interest. In this way, only coherence among target steering vectors and sidelobe steering vectors are considered, which is productive for reducing effective sidelobes and enhancing SINRs.

4.2.2 | Selection of angle numbers and sections

The problem in Equation (55) suggests that an infinite number of angles resides both in the sector of interest and outside the sector of interest. Although it is inadequate to choose only a few angles, the complexity of optimisation over an infinite number of angles is computationally unaffordable. It is therefore necessary to reach a trade-off between the performance of the sidelobe and complexity. The equivalent steering vector $\mathbf{c}(\theta)$ and the physical steering vector $\mathbf{a}(\theta)$ are connected by array partition matrix $\mathbf{W}^H$, which is of $N \times M_t$ dimensions. Matrix $\mathbf{W}$ is complex, which means the DOF in optimising $\mathbf{W}$ is $2NM_t$, so it is reasonable to select $D = 2NM_t$ as the number of sampled angles to exert the DOF fully for optimization.

After choosing the proper number of angles, the distribution of these angular grids in different sectors also matters. To obtain optimised output SINR performance, it is reasonable to divide the whole beampattern into main lobe region $\Theta$, where the spatial target probably resides, and sidelobe region $\Theta$, where interference may exist. After the division of beampattern regions, $2NM_t$ grid points are uniformly distributed in the whole angular region, whereas the number of grid points within the main lobe and that in the sidelobe are determined by the ratio between the width of main lobe and the width of the sidelobe. In other words, suppose the number of grid points in the main lobe region with width $L_t$ is denoted as $N_t$ and $N_t$ angles are within the sidelobe region with width $L_s$. Then, the connection holds:

$$\frac{N_t}{N_s} = \frac{L_s}{L_t}$$

(56)

whereas

$$N_t + N_s = 2NM_t$$

(57)

4.2.3 | Solution to subproblems

The target function does not have smooth derivatives; it can be turned into a series of subproblems through the approximation of $\infty$-norm by $p$-norm. Let $\mathbf{c}_t$ and $\mathbf{c}_s$ represent $\mathbf{c}(\theta_t)$ and $\mathbf{c}(\theta_s)$, respectively. The optimization in Equation (55) can be reformulated as:

$$\min_{\mathbf{c}} \max_{\theta \in \Theta} \left| \mathbf{c}^H(\theta_s)\mathbf{c}(\theta_t) \right|, s.t. \| \mathbf{c}(\theta_t) \| = 1$$

(55)
where $\mu_0$ is the benchmark value of coherence among different steering vectors.

As $p$ increases, $p$-norm approaches a more accurate approximation of $\infty$-norm, whereas the numerical stability is preserved by taking only even $p$ values. The subproblem in Equation (58) can be solved using the Lagrange function technique similar with that adopted in discrete coherence optimization. In particular, because the target in this subsection Equation (58) can be solved using the Lagrange function technique similar with that adopted in discrete coherence optimization. In particular, because the target in this subsection.

where

$$G(q) = \left[ \begin{array}{c} a(q) \\ a(q) \\ \vdots \\ a(q) \end{array} \right], \quad n = 1, \ldots, N_t$$

denotes the matrix of factors for updating the interference codebook.

For the interference codebook, there is:

$$c_i^{(q)} = \frac{c_i^{(q-1)} + \alpha_i G_i^{(q)}}{\|c_i^{(q-1)} + \alpha_i G_i^{(q)}\|}, \quad q = 1, \ldots, q_{\text{max}}$$

(62)

where $G_i^{(q)} = \left[ \begin{array}{c} a(q) \\ a(q) \\ \vdots \\ a(q) \end{array} \right], \quad m = 1, \ldots, N_t$ denotes the matrix of factors for updating the target codebook.

Similarly, update factors for the interference steering vector codebook can be denoted as:

$$g_m^{(q)} = \frac{\left[ \begin{array}{c} \alpha_i \\ \alpha_i \\ \vdots \\ \alpha_i \end{array} \right]}{\left[ \begin{array}{c} \alpha_i \\ \alpha_i \\ \vdots \\ \alpha_i \end{array} \right]} \sum_{\theta_i, \theta_j, \theta_k} \left( \left| \left( c_i^{(q)}, c_j^{(q)} \right)^2 - \mu_i^2 \right|^p \right)^{\frac{1}{p-1}}$$

$$\times 2 \sum_{\theta_i, \theta_j, \theta_k} \left[ \left( \left| \left( c_i^{(q)}, c_j^{(q)} \right)^2 - \mu_i^2 \right|^p \left( c_i^{(q)}, c_j^{(q)} \right) \cdot c_i^{(q)} \right)^{\frac{1}{p-1}} \right]$$

(59)

In the same way with the discrete optimization problem, both the optimised target and interference codebooks can be iteratively updated by adding a matrix of update factors multiplied by a gain coefficient, usually taking decaying values as the ongoing process. For the target codebook, there is:

$$c_i^{(q)} = \frac{c_i^{(q-1)} + \alpha_i G_i^{(q)}}{\|c_i^{(q-1)} + \alpha_i G_i^{(q)}\|}, \quad q = 1, \ldots, q_{\text{max}}$$

(61)

where $G_i^{(q)} = \left[ \begin{array}{c} a(q) \\ a(q) \\ \vdots \\ a(q) \end{array} \right], \quad m = 1, \ldots, N_t$ denotes the matrix of factors for updating the target codebook.

4.2.4 Computational complexity analysis

The computational complexity of Algorithm 2 mainly consists of multiplications in the process, or specifically, of multiplications in the pseudoinverse operation of Equation (54) and multiple iterations or while loops; in each iteration, the complexity mainly lies in the update process of matrix $C$. For each pair of $c_i$ and $c_j$, the update process stated in Equation (52) is mainly composed of the dot product of two
vectors. Because both of these two vectors are $N$-dimensional, the dot product takes $O(N)$ multiplications and their power operation takes $O(p_{\text{max}})$ multiplications. For the coherence of the matrix $C$ to be minimised, $N_i$ vectors $c_i$ in the sector of interest and $N_i$ vectors $c_i$ in the sector of interference are going through the process stated in Equation (52), which takes $N_iN_i$ combinations. Therefore, the complexity of multiplications in each loop is $O(N_iN_{\text{max}}\{N, p_{\text{max}}\})$.

The number of inner loops is $q_{\text{max}}$ because $q$ is increased by 1; the number of outer loops is $\log_2(p_{\text{max}})$ because $p$ is increased by the power of 2. In this way, the complexity of the update process is $O(q_{\text{max}}N_iN_i\log_2(p_{\text{max}})\max\{N, p_{\text{max}}\})$. The complexity of the pseudoinverse of the matrix is $O(D^3)$, where $D = 2M_rN$ denotes the number of angles in the coherence optimization.

The main computational complexity of Algorithm 2 is $O(q_{\text{max}}N_iN_i\log_2(p_{\text{max}})\max\{N, p_{\text{max}}\} + M_t^2N^3)$. Different from the complexity of Algorithm 1, the complexity of Algorithm 2 is related not only to $q_{\text{max}}$, $p_{\text{max}}$ and $N$, but also to $N_i$, $N_i$ and $M_t$. The finer the grid of angles is, the greater of complexity because $N_i$ and $N_i$ are worse. The number of transmit elements and subarrays is vital to the increase in computations, both of which are an order of power of 3. This means that the complexity of the proposed schemes grows much faster than the does number of transmit elements and subarrays, indicating high complexity. Therefore, the proposed algorithm is far from optimal, which achieves coherence optimization at the cost of complexity.

5 | NUMERICAL RESULTS

In this section, numerical results are presented comparing the proposed partition and other subarray partitions appeared in existing literature, including the equally overlapped partition [6], unequally overlapped partition [13], and centre spanned partition [15]. For the sake of a fair comparison, in all subarray partitions, $M_t = 10$ and $M_r = 10$ are assumed for the whole transmit and receive array, respectively. To achieve a fair comparison, in all subarray partitions, the number of subarrays is assumed to be $N = 5$. Additive white Gaussian noise with normal distribution $\mathcal{N}(0, \sigma^2)$ with identical independent distribution across array elements is also assumed.

To depict the advantage of methods proposed in Section 4, multiple numerical comparisons are presented between the equally overlapped partition, unequally overlapped partition, centre spanned partition and proposed partition. First, the array partition obtained from the discrete coherence optimization problem is compared with other partitions in terms of measured coherence. Then, to demonstrate further the effect of coherence optimization on beampattern sidelobes and output SINR performance, the array partition obtained from oversampled coherence optimization is compared with different subarray partition schemes in both conventional and adaptive beamforming.

### Table 1: Discrete coherence comparison of different subarray partitions

| Partition schemes | Discrete coherence value |
|-------------------|--------------------------|
| Equal             | 0.7098                   |
| Unequal           | 0.5995                   |
| Centre spanned    | 0.5572                   |
| Proposed          | 0.5434                   |

#### 5.1 | Coherence comparison

In this example, coherence values of the manifold matrix in different subarray partitions are compared to verify the advantage of the proposed scheme over classical designs, such as equally fully overlapped subarrays, unequally fully overlapped ones and centre spanned subarrays. To compare coherence, $D = 10$ spatial angles are chosen between $-55^\circ$ and $55^\circ$, which are used in the discrete coherence optimization. The coherence-optimised subarray partition is obtained solving the discrete problem in Equation (49). The coherence values of different subarray partitions are listed in Table 1.

Table 1 shows that the coherence of the proposed subarray partition scheme is lower than all other subarray configurations, followed by that of the centre-spanned subarray partition. Coherence of the unequally fully overlapped subarray is higher than for the previous two, which is lower than coherence of the equally fully overlapped one. As discussed in Sections 3 and 4, the coherence of the manifold matrix reflects similarity among the different columns of the matrix, which are steering vectors in different angles. Within this perspective, this comparison reveals that the obtained codewords in the codebook on the Grassmannian manifold is more dissimilar than the steering vectors in the manifold matrices that reflect subarray partitions such as equally fully overlapped subarrays, unequally fully overlapped subarrays and centre spanned subarrays. Such dissimilarity among steering vectors occurs because the proposed subarray partition can exploit more degrees of freedom from the structure of subarrays than other partitions, even though the number of subarrays is the same.

Although the proposed subarray partition can achieve lower coherence than other configurations, the corresponding dissimilarity among different steering vectors does not necessarily translate to dissimilarity among steering vectors on angles within sectors of interest (target region) and outside sectors of interest (interference region). To examine the effect of coherence on specific performance, such as the beampattern and SINR, more comparisons need to be conducted in subsequent subsections.

#### 5.2 | Conventional beampattern comparison

In this example, conventional beampatterns of different subarray partitions show the advantage of the proposed partition, which deals with the degree of cross-correlation among
steering vectors over different sectors of interest. For comparison with a conventional beampattern of different partitions, a spatial target is assumed to be located at 10° and two interference sources are at −30° and −10° to construct both target and interference steering vectors, respectively. Because the peak of sidelobes is associated with cross-correlation between steering vectors on target angles and angles outside the sectors of interest, the proposed partition is developed by the oversampled optimization problem in Equation (58) rather than in Equation (49). Specifically, \(2NM_i = 100\) grid points are uniformly distributed within the whole angular region. Moreover, in accordance with the assumption of target and interference, the target region is assumed to be \([0°, 20°]\) and the interference region is \([-90°, 0°]\) and \([20°, 90°]\) so that \(N_t = 12\) and \(N_i = 88\) are determined for Equation (58). The proposed coherence-optimised partition can therefore be obtained from the solution to Equation (58) and the comparison of the beampattern of the proposed scheme with that of different subarray partitions is shown in Figure 1. Correspondingly, the peak sidelobe levels and main lobe width of different subarray partitions are shown in Table 2.

Figure 1 shows that although the number of subarrays in all partitions is the same, the beampatterns of different partitions exhibit different peak sidelobe levels. More specifically, Table 2 shows that the proposed partition has the lowest peak sidelobe level, which is 6.54 dB lower than that of the centre-spanned partition. The peak sidelobe of the centre spanned partition is 2.42 dB lower than that of the unequal partition, which is 5.92 dB higher than that of the equal partition. Table 2 also demonstrates that although the proposed partition has the lowest sidelobe among all schemes, its main lobe width is the narrowest, which is 4.95° narrower than the widest width of main lobes in other three partitions. These observations show that the beampattern of proposed partitions has the advantage of a lower sidelobe compared with previous designs, which is attributed to more degrees of freedom in directly optimising steering vectors of the manifold rather than a prefixed structure of array.

### 5.3 Adaptive beampattern comparison

To provide the beampattern performance of the proposed scheme of subarray partition more comprehensively, adaptive beampatterns of different subarray partitions are compared to demonstrate the advantage of proposed partition schemes in this example. Conditions and assumptions for numerical results are identical to those in the last example, and the number of subarrays is identical for all partitioning schemes for a fair comparison. The coherence optimised subarray partition is obtained from solving Equation (58) with the same division of angular regions in the last example. Adaptive beamforming in the receive side usually adopts minimum variance distortionless response (MVDR) beamforming, expressed as:

\[
w = \frac{R_{i+n}^{-1}u(\theta_i)}{u^H(\theta_i)R_{i+n}u(\theta_i)}
\]  

(63)

where \(R_{i+n}\) is the covariance matrix of interference plus noise, which is often approximated by \(\hat{R} \triangleq \sum_{t=1}^{T}Y_Ty_H^T\), where \(y_t\) represents data snapshots from different radar pulses during a coherent processing interval. \(u(\theta_i)\) is the virtual steering vector on the target direction.

The adaptive beampatterns of different subarrays are depicted in Figure 2, which shows that all beampatterns have irregular shapes compared with more regular shapes in the last example. Similar to conventional beamforming, Figure 2 demonstrates that the peak sidelobe level on the beampattern of the proposed partition is lower than that of other subarray partitions, and the proposed partition has a narrower main lobe width. Specifically, numerical values in Table 3 indicate that the peak sidelobe level of the proposed partition is 3.24 dB lower than that of centre-spanned partition, which is 0.17 dB higher than that of unequal partition and 6.01 dB lower than that of the equal partition. In addition, the main lobe width of the proposed partition is the narrowest, which

| Partition schemes | Peak sidelobe level (dB) | Main lobe width (degree) |
|--------------------|-------------------------|--------------------------|
| Equal [6]          | −24.89                  | 19.15                    |
| Unequal [13]       | −30.81                  | 18.65                    |
| Centre spanned [15]| −33.28                  | 18.68                    |
| Proposed           | −39.82                  | 14.2                     |

**TABLE 2** Peak sidelobe level and main lobe width of different subarray partitions (conventional beamforming)
is 3.58° narrower than the widest width of the equal partition.

Table 3 shows that both the peak sidelobe levels and main lobe width of all beampatterns are increased compared with those in conventional beamforming in the last example. The reason for this phenomenon is fact that compared with the case of conventional beamforming, the shape of the adaptive beampattern as well as the degree of cross-correlation in space is further influenced by adaptive training data, which counteracts the effect of subarray partitioning. However, despite the reduced sidelobe suppression performance from adaptive beamforming, the peak sidelobe level of the proposed scheme is still lower than that of other subarray partitioning schemes and the main lobe width of the proposed scheme is narrower than that of other partitions, which demonstrates the advantage of optimising the coherence of the array manifold over other methods for partitioning subarrays.

5.4 Conventional signal to interference plus noise ratio comparison

In this subsection, the SINRs of different partitions are investigated against different a signal to noise (SNR) with interference to noise ratio (INR) set at 30 dB to validate the effect of optimising subarray partitions on the output SINR. As in the case of the beampatterns, the SINR is associated with coherence between steering vectors on target angles and angles outside the sectors of interest. Therefore, the proposed partition is developed by Equation (58) rather than Equation (49), with other conditions and assumptions remaining the same as in Subsection 5.2. The output SINRs of different partitions are depicted in Figure 3. More specifically, the output SINRs of different partitions at 0 dB SNR are shown in Table 4.

Figure 3 and Table 4 show that the output SINR of proposed partitions is higher than that of the centre-spanned partition, followed by that of unequally overlapped subarrays. The output SINR of the equally overlapped subarrays is the lowest among all subarray partitions. Specifically, Table 4 indicates that the output SINR of proposed partition is 2.95 dB higher than that of the centre spanned partition, which is 8.19 dB higher than that of the unequal partition and 10.84 dB higher than that of the equal partition. These results agree with the discussion in Section 2 and the observation in Subsection 5.1, where a lower sidelobe level means a higher output SINR.

5.5 Adaptive signal to interference plus noise ratio comparison

Similar to the case of the beampattern, the output SINR of different partitions with adaptive beamforming is considered in this example, where the INR is set as 30 dB for all partitions.
To compare the output SINR performance of different schemes, the coherence-optimised subarray partition is obtained by solving Equation (58), and the adaptive beamforming on the receive side adopts MVDR beamforming, as with the condition in Subsection 5.3. Other conditions and assumptions remain the same; the output SINRs with adaptive beamforming of different subarray partitions are revealed in Figure 4. More specifically, the output SINRs of different partitions at 0 dB SNR are shown in Table 5.

Figure 4 shows that the output SINR of the proposed partition is the higher than the SINR of other partitions, followed by that of the centre-spanned partition, unequal and equal partitions. Specifically, the output SINR of the proposed partition is 0.82 dB higher than that of the centre spanned partition, 3.77 dB higher than the unequal partition, and 6.21 dB higher than the equal partition. Both the output SINR and the difference between the SINRs of different partitions are reduced, which corresponds to the fact that adaptive beamforming counteracts the effect of the proposed partition on the beampattern performance in Subsection 5.3. However, the output SINR of the proposed partition is still higher than that of other partitions, which demonstrates the advantage of coherence-based optimization, even in the case of adaptive beamforming.

### 6 | CONCLUSION

A subarray partition scheme is proposed to improve the beampattern and SINR by optimising subarray partitions for HMPAR. The system is improved by finding a codebook with the lowest coherence on the Grassmannian manifold, featuring the metric of coherence rather than other metrics in classical methods. To provide a solution for the optimization problem, the original problem is decomposed into multiple subproblems with smooth derivatives, which facilitates the production of update factors to adjust different codewords for an optimised codebook iteratively. The obtained codebook and codewords are then used to derive the proper formulation of subarrays to achieve a system with an improved beampattern sidelobe and SINR performance. Compared with previous methods, which focused on classical regular partitions, the proposed partition is implemented through manifold coherence, which captures the essence of the problem and therefore has the advantage of even larger degrees of freedom. These advantages are verified in the numerical results, which reveal that the array partition realized using the proposed scheme has a better beampattern sidelobe and output SINR performance. The proposed method is limited in its way of obtaining an array partition matrix and is far from optimal because the complexity in determining the optimised codebook is not computationally efficient. Nevertheless, this work depicts the possibility of lowering the coherence of the manifold matrix of the whole array only by exploiting the available DOF at the transmit side. In fact, applying the proposed scheme on optimising both the transmit and receive array or a sparse array structure is a more promising prospect because more degrees of freedom are available while computational and hardware costs are reduced. Therefore, future research may include coherence measurements of different array configurations and sparse array designs based on a coherence metric.

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