GROUP ACTIONS WITH COMMENSURATED SUBSETS,
WALLINGS AND CUBINGS

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ABSTRACT. We study commensurating actions of groups and the associated properties FW and PW, in connection with wallings, median graphs, CAT(0) cubings and multi-ended Schreier graphs.

1. INTRODUCTION

1.A. Commensurated subsets and associated properties.

1.A.1. Context. The source of CAT(0) cubings can be found in several originally partly unrelated areas, including median graphs, group actions on trees, ends of Schreier graphs, Coxeter groups, cubulations of 3-dimensional manifolds. The link with Kazhdan’s Property T was gradually acknowledged, first in the case of trees, then for finite-dimensional CAT(0) cubings and then for general CAT(0) cubings; the same arguments were also found at the same time in different languages, notably in terms of wall spaces. The present paper is an attempt to give a synthesis of those different point of views, and especially to advertise the most elementary approach, namely that of group actions with commensurated subsets.

Let us emphasize that actions on CAT(0) cubings have now reached a considerable importance in geometric group theory. However, maybe partly because of the scattering of points of view and the elaborateness of the notion of CAT(0) cubing, it is sometimes considered as a intermediate tool, for instance to prove that some group does not have Property T or has the Haagerup Property. This is certainly unfair, and CAT(0) cubings and consorts are worth much better than being subcontractors of those analytic properties, and therefore we introduce the following terminology.

1.A.2. Actions with commensurated subsets. Consider an action of a group $G$ on a discrete set $X$ (we assume throughout this introduction that groups are discrete, but will address the topological setting as well). We say that a subset $M \subset X$ is commensurated by the $G$-action if

$$\ell_M(g) = \#(M \triangle gM) < \infty, \quad \forall g \in G,$$

where $\ell_M(g)$ denotes the minimal number of elements of $M$ in the symmetric difference $M \triangle gM$. This notion is closely related to the walling property FW, and we refer to the introduction for details and further references.
where $\triangle$ denotes the symmetric difference$^1$. We say that $M$ is *transfixed* if there exists a $G$-invariant subset $N$ with $M \triangle N$ finite. Transfixed subsets are the trivial instances of commensurated subsets.

1.A.3. *Property FW.*

**Definition 1.1.** We say that $G$ has Property FW if for every $G$-set, every commensurated subset is transfixed.

For every $G$-set $X$ with a commensurated subset $M$, define the function $\ell_M(g) = \#(gM \triangle M)$; every such function on $G$ is called a *cardinal definite function*$^2$ on $G$.

Property FW for the group $G$ turns out to have several equivalent characterizations, both geometric and combinatorial, see Section \ref{sec:characterizations} for the relevant definitions and Proposition \ref{prop:equivalent_chara} for the proofs.

(i) $G$ has Property FW;
(ii) every cardinal definite function on $G$ is bounded;
(iii) every cellular action on any CAT(0) cube complex has bounded orbits for the $\ell^1$-metric (we allow infinite-dimensional cube complexes);
(iv) every cellular action on any CAT(0) cube complex has a fixed point;
(v) every action on a connected median graph has bounded orbits;
(vi) every action on a nonempty connected median graph has a finite orbit;
(vii) (if $G$ is finitely generated and endowed with a finite generating subset) every Schreier graph of $G$ has at most 1 end;
(viii) for every set $Y$ endowed with a walling and compatible action on $Y$ and on the index set of the walling, the action on $Y$ has bounded orbits for the wall distance;
(ix) every isometric action on an “integral Hilbert space” $\ell^2(X, \mathbb{Z})$ ($X$ any discrete set), or equivalently on $\ell^2(X, \mathbb{R})$ preserving integral points, has bounded orbits;
(x) for every $G$-set $X$ we have $H^1(G, \mathbb{Z}X) = 0$.

The implication (iii) $\Rightarrow$ (iv) looks like a plain application of the center lemma; however this is not the case in general since the CAT(0) cube complex is not complete and we would only deduce, for instance, a fixed point in the $\ell^2$-completion. The argument (which goes through median graphs) is due to Gerasimov and is described in \S3.C.

Note that FW means “fixed point property on walls”, in view of (viii).

It follows from (ix) that Property FH implies Property FW. Recall that Property FH means that every isometric action on a real Hilbert space has bounded

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$^1$The notion of commensurated subset is the set analogue of the notion of commensurated subgroup which is not considered in this paper.

$^2$When $X$ is replaced by an arbitrary measure space and cardinality is replaced by the measure, we obtain the notion of measure definite function addressed by Robertson and Steger in \cite{RS}.
orbits, and is equivalent (for countable groups) to Kazhdan’s Property T by Delorme-Guichardet’s Theorem [BHV]. Also, Property FH was characterized by Robertson-Steger [RS] in a way very similar to the definition above of Property FH, namely replacing the discrete set $X$ and cardinality of subsets by a measured space and cardinality of its measurable subsets.

Also, since trees are the simplest examples of CAT(0) cube complexes, it follows from [iii] that Property FW implies Serre’s Property FA: every action on a tree has bounded orbits. This can also be directly viewed with commensurating actions: if a group $G$ acts on a tree, then it acts on the set of oriented edges, and for each fixed vertex $x_0$, the set of oriented edges pointing towards $x_0$ is a commensurated subset, which is transfixed only if the original action has bounded orbits.

Thus FW is a far-reaching generalization of Property FA, while weaker than Property FH. Note that Property FH is of much more analytical nature, but is has no combinatorial characterization at this time. A considerable work has been done to prove weak converse to the implication $FW \Rightarrow FA$, the first of which being Stallings’ characterization of finitely generated groups with several ends. However, there are a lot of groups satisfying FA but not FW, see Example 5.12.

1.A.4. Property PW. In view of [iii], it is natural to introduce the opposite property PW (which was explicitly introduced in [CSVa]):

**Definition 1.2.** The group $G$ has Property PW if it admits a proper commensurating action, in the sense that the cardinal definite function $\ell_M$ is proper on $G$.

(Recall that $f : G \to \mathbb{R}$ proper means $\{x : |f(x)| \leq r\}$ is finite for all $r < \infty$.) Obviously, a group has both Properties PW and FW if and only if it is finite. We say that an isometric action of a discrete group on a metric space $X$ is proper if for some $x \in X$, the function $g \mapsto f(x, gx)$ is proper; then this holds for all $x \in X$. Property PW has, in a similar fashion, equivalent restatements:

(iii') there exists a proper cellular action on a (possibly infinite-dimensional) complete CAT(0) cube complex with the $\ell^1$-metric;

(v') there is a proper isometric action on a connected median graph;

(viii') there exists a set $Y$ endowed with a walling and compatible actions on this set and on the index set of the walling, such the action on $Y$ endowed with the wall distance is metrically proper;

(ix') there exists a proper isometric action on an “integral Hilbert space” $\ell^2(X, \mathbb{Z})$ (for some discrete set $X$), actually extending to $\ell^2(X, \mathbb{R})$.

Here we enumerate in accordance with the corresponding characterizations of Property FW; for instance we omit the analogue of [iii] because it would be tautological, while [vii] has no trivial restatement. Actually I do not know any purely combinatorial definition of Property PW (not explicitly involving any kind of
properness); nevertheless it is a very important feature in geometric group theory, and its refinements (such as proper cocompact actions on finite-dimensional proper cube complexes) play an essential role in the understanding of 3-manifold groups, among others.

Similarly as above, we see that Property PW implies the Haagerup Property, which for a countable group asserts the existence of a proper isometric action on a Hilbert space. One of the main strengths of the Haagerup Property for a group $G$ is that it implies that $G$ satisfies the Baum-Connes conjecture in its strongest form, namely with coefficients $[HK]$.

1.A.5. How to read this paper? The long section 3 contains a synthesis of previous work, using, insofar as possible, a unified language. After a possible touchdown at the examples in Section 2, the beginner is rather advised to start with Section 4 (where all groups can be assumed to be discrete in a first reading), which studies commensurating actions in general, and Section 5, which uses the background of Section 4 to the study of Properties FW and PW. The material of Section 3 is only used in §5.D to justify the equivalences stated in the introduction. Section 6 applies basic results on commensurating actions from Section 4 to the study of cardinal definite functions on abelian groups, with applications to properties FW and PW.

Warning. A significant part of this paper consists of non-original results (see §7); however the way they are stated here can differ from the classical (and divergent) points of view. A large part of the paper (Section 3) is an attempt to a synthesis of these points of view.

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2. Classical examples

2.6. Examples with Property PW. Let us give examples of groups satisfying Property PW. It can be hard to give accurate references, insofar as the link different characterizations of Property PW were not originally well-understood, and also because Property PW was often obtained accidentally. Therefore, in the following enumeration, I use a chronological order taking into account the availability of the methods rather than any findable explicit assertion. From this prospective, we can agree that Property PW for the trivial group was realized soon after the Big Bang and that probably a few years later Lucy considered as folklore that it is also satisfied by \( \mathbb{Z} \).

The first next examples are groups acting properly on trees; however a finitely generated group with this property is necessarily \textit{virtually free}, so this gives a small class of groups. However, unlike the property of acting properly on a tree, Property PW is stable under taking \textit{direct products} and \textit{overgroups of finite index}. It follows, for instance, that every finitely generated, \textit{virtually abelian} group has Property PW; interestingly this provides the simplest counterexamples to the implication \( \text{FA} \Rightarrow \text{FW} \) (e.g., a nontrivial semidirect product \( \mathbb{Z}^2 \rtimes (\mathbb{Z}/3\mathbb{Z}) \) does the job). All the previous examples are groups acting properly on a finite product of trees. Other instances of such groups are \textit{lattices} (or discrete subgroups) \textbf{in products of rank one simple groups} over non-Archimedean local fields, and also Burger-Mozes’ groups, which are infinite, finitely presented and simple. A typical example that is not cocompact is the \textit{lamplighter} group \( F \wr \mathbb{Z} \).
(where $F$ is any finite group), which acts properly on the product of two trees, or a non-cocompact lattice such as $\text{SL}_2(F_p[t, t^{-1}])$.

In chronological order, the next examples seem to be Coxeter groups. This was proved by Bozajko, Januszkiewicz and Spatzier [BJS]. Actually, they use a certain action on a space with walls which was explicitly provided (modulo the language of walls), along with all necessary estimates, by Tits [Tit 2.22]. They actually do more, by providing an explicit cubulation of the Cayley graph; it seems to be the first construction of this type. Thus they constructed a proper cocompact action of any finitely generated Coxeter group on a $\text{CAT}(0)$ cube complex, but claimed as main result the much weaker assertion that their infinite subgroups of Coxeter groups do not have Kazhdan’s Property T.

Let us point out that at that time, the Haagerup Property, explicitly studied and characterized in Akemann-Walter [AW], was not yet popular as it became after being promoted by Gromov [Gro 7.A,7.E] and then in the book by Valette and the other authors of [CCJJV].

Let us give some more recent examples.

- Wise proved in [Wi04] that every group with a finite small cancelation presentation of type $C'(1/6)$ or $C'(1/4)-T(4)$ acts properly cocompactly on a finite-dimensional, locally finite cube complex.
- Bergeron and Wise then obtained a cubulation which, combined with a result of Kahn and Markovic [KM], implies that every torsion-free cocompact lattice of $\text{SL}_2(\mathbb{C})$ admits a proper cocompact action on a finite-dimensional, locally finite $\text{CAT}(0)$ cubing.
- Hsu and Wise [HsW] proved that the fundamental group of a graph of free groups with cyclic edge groups, provided it has no non-Euclidean Baumslag-Solitar subgroups, has a proper cocompact action on a locally finite $\text{CAT}(0)$ cubing.
- Ollivier and Wise [OW] proved that Gromov random groups at density $d < 1/6$ act properly cocompactly on a finite dimensional, locally finite $\text{CAT}(0)$ cube complex.
- Gautero [Gau] proved that the Formanek-Procesi group, a non-linear semidirect product of two non-abelian free groups of finite rank, has a proper action on a finite-dimensional $\text{CAT}(0)$ cubing.

All the previous examples act properly on a finite-dimensional, locally finite cube complex. A very different example was discovered by Farley, namely Thompson’s groups [Far] and more generally diagram groups over finite semigroup presentations. This was extended by Hughes to suitable groups of local similarities of compact ultrametric spaces [Hui].

\footnote{Note that Hughes and Farley calls a proper commensurating action (i.e., whose associated cardinal definite function is proper) a zipper action. One reason for which we do not follow this terminology is that in order to understand a (possibly proper) commensurating action, it can be useful to decompose the set into orbits and analyse individually each orbit; when passing
The author, Stalder and Valette proved in [CSVa] that Property PW is stable under taking standard wreath products. In general (e.g., in the case of $\mathbb{Z} \wr \mathbb{Z}$), the methods outputs an infinite-dimensional cube complex even in both groups act on finite-dimensional ones. The statement given in [CSVa] was weaker, because the authors had primarily the Haagerup Property in mind, but the method in [CSVb] allows to get the general case. See [LG] and Proposition 5.16.

There are natural strengthenings of Property PW, such as, in order of strength: having a proper action on a finite-dimensional CAT(0) cube complex, having a proper action on a finite-dimensional locally finite CAT(0) cube complex, having a proper cocompact action on a finite-dimensional CAT(0) cube complex. We do not address them in this paper, although they are of fundamental importance. These classes are much more restricted. For instance, it was observed in [CSVa] that the wreath product $(\mathbb{Z}/2\mathbb{Z}) \wr F_2$ has Property FW but does not have any proper action on a finite-dimensional CAT(0) cube complex. N. Wright’s result that finite-dimensional CAT(0) cube complexes have finite asymptotic dimension [Wr] provides a wealth of other examples, including the wreath product $\mathbb{Z}/\mathbb{Z}$, which has Property PW by [CSVa] (completed in Proposition 5.16), or Thompson’s group $F$ of the interval, which has Property PW by Farley [Far]. I do not know if having a proper cellular action on a (finite-dimensional / proper / finite-dimensional and proper) CAT(0) cube complex are distinct properties. [For non-proper actions, there exists [ABJLMS, ChK] a finitely generated group $Q$ with no non-trivial cellular action on any locally finite finite-dimensional cube complex; so the free product $Q * Q$ also shares this property, but it also acts with unbounded orbits on a (not locally finite) tree.]

2.7. Examples with the negation of Property FW. If $G$ is a finitely generated group and $H$ is a subgroup, let us say that $H$ is **coforked** in $G$ if $G/H$ has at least 2 ends. This is called “codimension 1 subgroup” by Sageev [Sa95] and in some subsequent papers but this terminology can be confusing as there is no underlying notion of dimension, and because it is not well-reflected by the geometry of $H$ inside $G$, as shown by the following example:

**Example 2.1.** Consider the infinite dihedral group $G_1 = \langle a, b \mid a^2, b^2 \rangle$ with generating set $\{a, b, aba\}$ and the group $G_2 = \langle t, u \mid [t, u], u^2 \rangle \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ with generating set $\{tu, u\}$. Then there exists a isomorphism between the Cayley graphs of these groups mapping $\{1, a\}$ to $\{1, u\}$. On the other hand, $\{1, a\}$ is not coforked in $G_1$, while $\{1, u\}$ is coforked in $G_2$.

Note that the trivial group is coforked in the infinite dihedral, showing that an overgroup of finite index of a coforked subgroup may not be coforked. On the other hand, a finite index of a coforked subgroup is always coforked, and a coforked subgroup is coforked in any coforked subgroup containing it.

to an orbit we generally lose the properness. Commensurated subsets are called **immobile** in [Ner].
All the above examples of Property PW groups, provided they are infinite, fail to have Property FW, i.e., admit a coforked subgroup (assuming they are finitely generated). Note that any group having a quotient without Property FW also fails to have Property FW. In particular, any finitely generated group virtually admitting a homomorphism onto \( \mathbb{Z} \), or equivalently having an infinite virtually abelian quotient, fails to have Property FW. In this case, the kernel of a homomorphism of a finite index subgroup onto \( \mathbb{Z} \) is a coforked subgroup.

Also, having in mind that Property FW is inherited by finite index subgroups, all groups having a finite index subgroup splitting as a nontrivial amalgam fails to have Property FW, a coforked subgroup being the amalgamated subgroup. Also, countable infinitely generated groups fail to have Property FW, since they do not have Property FA [Ser].

Wise [Wi04] proved that every finitely generated group with a (possibly infinite) small cancelation presentation of type \( C'(1/6) \) or \( C'(1/4)-T(4) \) fails to have Property FW (and have Property PW in the finitely presented case). In the infinitly presented case, it has been recently announced by Arzhantseva and Osajda [AO] that these groups also have Property PW.

Ollivier and Wise [OW] proved that Gromov random groups at density \( d < 1/5 \) admit a coforked subgroup.

A wreath product \( A \wr_S B = A^{(S)} \rtimes B \), with \( A \) nontrivial, \( S \) an infinite \( B \)-set and \( B \) countable never has Property FW (Proposition 5.8). Provided \( S \) admits an infinite orbit \( Bs \), a careful look at the construction shows that \( A_S \setminus \{1\} \) is a coforked subgroup. Of course if \( A \) or \( B \) does not have Property PW, the wreath product does not have Property PW.

The first Grigorchuk group and the Basilica group are examples of self-similar groups with a natural action on a regular rooted tree. The first has subexponential growth and thus is amenable [Gr] while the second has exponential growth [GrZ] but yet is amenable [BV]. Both admit Schreier graphs with 2 ends; for the Grigorchuk group this is established in [GrK] and for the Basilica group this is obtained in [DDMN]. In both cases, the coforked subgroup is obtained as the stabilizer of a suitable boundary point of the rooted tree. Accordingly, these groups do not have Property FW; it is not known if they have Property PW.

The derived subgroup of the full topological group associated to an infinite minimal subshift is, by a result of Matui [Ma], an infinite, finitely generated simple group; it was subsequently shown to be amenable by Juschenko and Monod [JM]. This group admits a transitive action on \( \mathbb{Z} \) for which every element acts with bounded displacement. In particular, it commensurates \( \mathbb{N} \); this shows that this group does not have Property FW. It is not known whether it has Property PW.

2.8. Examples with the negation of Property PW. For a time the only known source of non-PW (countable) groups was the class of groups without the Haagerup Property, and the only known source of groups without the Haagerup Property
was groups with an infinite subgroup with relative Property T. Other examples appeared in Cor1, yet, as observed in Cor, they have an infinite subset with relative Property T.

The question whether any group with the Haagerup Property has Property PW has circulated for many years until solved in the negative by Haglund, who proved that a group with Property PW has all its infinite cyclic subgroups undistorted (a more direct approach of Haglund’s theorem is given as Corollary 6.2). In particular, plenty of amenable groups do not have Property FW, although all amenable groups have the Haagerup Property [AW].

Another approach can be used by using Houghton’s characterization of coforked subgroups of virtually polycyclic groups: it follows that a virtually polycyclic group has Property PW if and only if it is virtually abelian (see Section 6.C).

2.9. Examples with Property FW. Groups with Kazhdan’s Property T are the initial source of examples of groups with Property FW. The question of finding FW groups without Property T was asked in the second arxiv version of [CDH] and no example seems to be written. One example can be derived from Haglund’s theorem and a bounded generation argument due to Carter and Keller, namely $\text{SL}_2(\mathbb{Z}[\sqrt{k}])$, where $k$ is an arbitrary non-square positive integer. This group actually has the Haagerup Property. In general, I conjecture:

**Conjecture 2.2.** Let $S$ be a connected semisimple Lie group with no compact simple factor, whose Lie algebra has $\mathbb{R}$-rank at least 2. Then every irreducible lattice in $S$ has Property FW.

(Irreducible means that the projection modulo any simple factor is dense.) The conjecture holds for $S$ when it has Kazhdan’s Property T since this is inherited by the lattice. It also includes the above case of $\text{SL}_2(\mathbb{Z}[\sqrt{k}])$, which is an irreducible lattice in $\text{SL}_2(\mathbb{R})^2$. Further instances of the conjecture will be given in Cor2.

2.10. Between Property FW and PW. The main general question is to describe cardinal definite functions on a given group. Since this is a bit too much as it is sensitive on the choice of a commensurated subset in its commensurability class, it can be relaxed to: determine cardinal definite functions up to bounded functions (which has a complete answer for finitely generated abelian groups, see Proposition 6.9). It would be simplistic to reduce this to the study to Properties PW and FW, considering the groups in between only as mixtures from these two classes (see the Examples in §2.7). Also, even if a group has Property PW, there is still an interesting work to understand in which ways it can act properly. The natural study, given a group, say finitely generated, is to consider the class of commensurating actions: the first step being to characterize its coforked subgroups among its subgroups, and then to describe the commensurated subsets in
the associated coset spaces (or more generally the finite partitions by commensurated subsets), which essentially amounts to describing the space of ends of the corresponding Schreier graphs.

3. Commensurating actions and other properties

If $G$ is a group, we say that a function $G \rightarrow \mathbb{R}$ is cardinal definite\(^4\) if there exists a $G$-set $X$ and a $G$-commensurated subset $M \subset X$ such that $f(g) = \ell_M(g) = \#(M \triangle gM)$ for all $g \in G$. When $G$ is topological, we require, in addition, that $G$ is a continuous discrete $G$-set and $M$ has an open stabilizer. That is, the pointwise stabilizers of elements of $x$ are open subgroup, and the stabilizer of $M$ is open. We actually check (Lemma 4.5) that $f$ is cardinal definite on the topological group $G$, if and only if it is a continuous function and is cardinal definite as function on $G^\delta$, which denotes $G$ endowed with the discrete topology.

3.A. Wallings. There is a well-known close connection between commensurated actions and wall spaces of Haglund-Paulin [HP]. The suitable definition to make this connection a perfect dictionary is the following.

**Definition 3.1.** Given a set $V$, a *walling* on $V$ with index set $I$ is a map

$$I \xrightarrow{\mathcal{W}} 2^V$$

$$i \mapsto W_i$$

such that for all $u, v \in V$, the number $d_W(u, v)$ of $i$ such that $W_i$ separates $u$ and $v$ (in the sense that $\#(W_i \cap \{u, v\}) = 1$) is finite. The $W_i$ are called halfspaces.

Given a group $G$ and a $G$-set $V$, a $G$-walling (or ($G$-)equivariant walling) is the data of an index $G$-set $I$ and a walling $I \rightarrow 2^V$, which is $G$-equivariant, $G$ acting by left multiplication on the power set $2^V$.

If $f : V_1 \rightarrow V_2$ is a map and $\mathcal{W}$ a walling on $V_2$, define the *pull-back* of $\mathcal{W}$ to $V_1$ by $f$ as the composite map $f^*\mathcal{W} = (f^{-1}) \circ \mathcal{W}$ where $(f^{-1})$ is the inverse image map $2^{V_2} \rightarrow 2^{V_1}$.

Note that $d_W(u, v)$ is a pseudo-distance on $V$ and, in the equivariant case, is preserved by $G$. Also note that the pull-back $f^*\mathcal{W}$ is clearly a walling, and that the map $f$ such as above is an isometry $(V_1, d_{f^*\mathcal{W}}) \rightarrow (V_2, d_\mathcal{W})$; moreover if $f$ is a $G$-map between $G$-set and $\mathcal{W}$ is equivariant, then so is $f^*G$.

**Proposition 3.2.** Given a $G$-set $X$, there is a canonical bijection

$$\{\text{comm. subsets of } X\} \leftrightarrow \{\text{G-wallings on } G \text{ indexed by } X\}$$

$$A \mapsto \{h \in G \mid x \in hA\}_{x \in X};$$

$$\{x \in X \mid 1 \in W_x\} \leftrightarrow (W_x)_{x \in X}.$$

\(^4\)The terminology “cardinal definite” is in analogy with the notion of measure definite function introduced by Robertson and Steger [RS], where discrete set with the counting measure are replaced by general measure spaces.
If the commensurated subset $A \subset X$ and the walling $W$ correspond to each other under this bijection, then the pseudo-distances $d_A$ and $d_W$ are equal.

**Proof.** These maps are reciprocal, defining an even larger bijection between subsets of $X$ and $G$-equivariant maps $X \to 2^G$. Given corresponding $A \subset X$ and the $G$-equivariant mapping $W : X \to 2^G$, a straightforward verification shows that $A$ is commensurated if and only if $W$ is a walling, also showing the last statement. □

**Corollary 3.3.** Let $G$ be a group and $f : G \to \mathbb{R}$ a function. Equivalences

1. $f$ is cardinal definite;
2. there exists a $G$-set $V$, a $G$-walling $W$ on $V$ and $v \in V$ such that $f(g) = d_W(v, gv)$ for all $g$;
3. there is a $G$-walling $W$ on $G$ such that $f(g) = d_W(1, g)$ for all $g$.

**Proof.** The equivalence between (1) and (3) is immediate from Proposition 3.2 and (3) $\Rightarrow$ (2) is trivial, while (2) $\Rightarrow$ (3) is obtained by pulling back the walling to $G$ by the orbital map $g \mapsto gv$. □

**Remark 3.4.** The original definition of wall spaces involves nontrivial partitions by two subsets rather than subsets of $V$. There are several minor nuances with the definition used here:

1. we allow $\emptyset$ and $V$ among the halfspaces;
2. we allow multiple halfspaces in the sense that $i \mapsto W_i$ is not required to be injective (although clearly, each halfspace occurs with finite multiplicity, except possibly $\emptyset$ and $V$);
3. we use an explicit $G$-structure on the indexing family of the walling, rather than considering a subset of the power set (as in the original definition) or a discrete integral measure on the power set (as in [CSVb]).
4. we consider subsets (that is, half-spaces, as in [CSVb]) rather than partitions into two subsets (as in the original definition). Unlike in [NRo, HrWi], we do not assume stability under complementation.

Among those nuances, the most essential is probably the one in (4). The necessity of considering half-spaces rather than partitions was observed in many places, including [HrWi, CSVb]. The construction of [CSVa, CSVb], which maps a walling (or a measured walling) on a group $\Gamma$ to a walling on the wreath product $A \wr \Gamma$, is very far from symmetric with respect to complementation! (For instance, it maps the walling by singletons to a bounded walling, but not maps the walling by co-singletons to an unbounded walling as soon as $\Gamma$ is infinite.)

**Remark 3.5.** A variant of the above notion of walling is when the halfspaces are self-indexed. Here, this corresponds to the assumption that the family of halfspaces is $x \mapsto W_x$ is injective. This condition is pretty unnatural, although easier at first sight. Also, it does not seriously change the definitions. For instance, if $G$ is finitely generated, then for any $G$-invariant walling on $G$, the multiplicity
of any halfspace is actually bounded (see Corollary 4.9), so removing redundant halfspaces preserves, for instance, the properness of the wall distance.

Let us provide a topological version of Proposition 3.2. Let $G$ be a topological group. A continuous discrete $G$-set is a $G$-set in which all pointwise stabilizers $G_x$, $x \in X$, are open subgroups of $G$. We call topological commensurated subset of $X$ a commensurated subset $A$ whose global stabilizer is open in $G$, or equivalently if the cardinal definite function $\ell_A$ is continuous. We will see in Lemma 4.11 that if $G$ is locally compact, then any commensurated subset of $X$ is actually a topological commensurated subsets, but this is not true for an arbitrary topological group.

**Proposition 3.6.** Let $G$ be a topological group and $X$ a discrete $G$-set. The bijection of Proposition 3.2 actually is a bijection

$$\{\text{comm. subsets of } X\} \sim \rightarrow \{\text{clopen } G\text{-wallings on } G\text{ indexed by } X\}$$

If the commensurated subset $A \subset X$ and the walling $W$ correspond to each other under this bijection, then the pseudo-distances $d_A$ and $d_W$ are equal. In particular, the above bijection restricts to a bijection

$$\{\text{top. comm. subsets of } X\} \sim \rightarrow \{\text{clopen } G\text{-wallings of } X\text{ with continuous wall pseudodistance}\}$$

**Proof.** Since the stabilizer of $x$ is an open subgroup $G_x$, we have $G_x W_x = W_x$ and therefore $W_x$ is a union of right cosets of $G_x$ and thus is clopen. Thus the bijection of Proposition 3.2 actually has the above form. The second statement is contained in Proposition 3.2 and the third follows. □

**Remark 3.7.** The data of a commensurated subset $A$ in $G/H$ is obviously equivalent to that of a subset $B \subset G$ which is left-$H$-invariant and satisfies $\#(H \setminus (H \triangle H g)) < \infty$ for all $g \in G$ (these conditions are precisely those considered by Sageev in [Sa95, Theorem 2.3]). Namely, if $\pi$ is the projection $G \rightarrow G/H$ and $\iota$ is the inverse map $g \mapsto g^{-1}$ in $G$, then $B = \iota(\pi^{-1}(A))$ and $A = \pi(\iota(B))$. In this language, the halfspace $W_g$ associated to $g \in G/H$ in Proposition 3.2 is just $W_g = gB$.

3.B. Ends of Schreier graphs and coforked subgroups. We here extend the notion given above for finitely generated discrete groups.

**Definition 3.8.** We say that an open subgroup $H$ of a topological group $G$ is coforked if there is in $G/H$ a commensurated subset, infinite and coinfinite, with open stabilizer.

The openness of the stabilizer is automatic when $G$ is locally compact (see Corollary 4.13), so in that case, if $H$ is an open subgroup of $G$, it is coforked in the topological group $G$ if and only it is coforked in $G_\delta$, where $G_\delta$ denotes $G$ endowed with the discrete topology. There are natural extensions of this definition for the
subgroups that are not supposed open, but they are not relevant here (see Remark 3.14).

**Definition 3.9.** If $G$ is a group and $S$ a symmetric generating subset of $G$, and $X$ is a $G$-set, the Schreier graph $\text{Sch}(X, S)$ is the graph whose vertex set is $X$, and with an oriented edge $(x, sx)$ labeled by $s \in S$ for each $(s, x) \in S \times X$. We say that $(X, S)$ is of finite type if the 0-skeleton of the Schreier graph is locally finite (i.e., every vertex is linked to finitely many vertices). The $S$-boundary of a set $Y \subset X$ of vertices is the set of elements in $Y$ joined by an edge to an element outside $Y$.

Note that $(X, S)$ is of finite type when $S$ is finite, but also holds in the important case where $G$ admits a group topology for which $S$ is compact and $X$ is a continuous discrete $G$-set. (In practice, it means that when $G$ is a topological group with a compact generating set, we can apply the following to the underlying discrete group $G^\delta$.)

We have the following, which for $S$ finite is essentially the contents of [Sa95, Theorem 2.3].

**Proposition 3.10.** If $(X, S)$ is of finite type, a subset of $X$ with finite $S$-boundary is commensurated. Conversely, if $G$ is a topological group, $X$ is a continuous discrete $G$-set, $S$ is compact and $M$ is a commensurated $G$-set with open stabilizer, then $M$ has a finite $S$-boundary.

In particular, if $G$ is locally compact and $S$ a compact generating subset, and $X$ is a continuous discrete $G$-set, the set of subsets of $X$ with finite $S$-boundary is equal to the set of $G$-commensurated subsets in $X$ (which in particular does not depend on $S$).

**Proof.** If $(X, S)$ is of finite type and $s \in S$, then $M \triangle sM$ is contained in the union of $M \setminus sM$ and $s(M \setminus s^{-1}M)$, which are both finite, so $M$ is commensurated.

Conversely, under the additional assumptions and assuming $M$ commensurated, the function $g \mapsto gM$ is locally constant and thus has a finite image in restriction to the compact set $S$. So $S$-boundary of $M$, which is the union $\bigcup_{s \in S} M \setminus sM$ and thus is a finite union of finite sets, is finite.

The second statement follows modulo the fact that for a locally compact group and continuous discrete $G$-set, the stabilizer of a commensurated subset is automatically open, see Corollary 4.13. \qed

We here define the space of ends as follows. First, given a graph (identified with its set of vertices) and a subset $Y$, we define $\pi_0(Y)$ as the set of components of $Y$, where two vertices in $Y$ are in the same component if and only if they can be joined by a sequence of edges with all vertices in $Y$. Then the space of ends of a graph $L$ is the (filtering) projective limit of $\pi_0(L \setminus F)$, where $F$ ranges over finite subsets of $L$; endowing $\pi_0(L \setminus F)$ with the discrete topology, the projective limit is endowed with the projective limit topology; if $L$ has finitely
many components, this is a projective limit of finite discrete sets and thus is a compact totally disconnected space, and is metrizable if \( L \) is countable.

Moreover, the space of ends is nonempty if and only if \( L \) is infinite, and has at least two points if and only if there exists a finite set of vertices such that the complement has two infinite components.

Also note that the end space of \( X \) the disjoint union of ends of spaces of its connected components.

**Corollary 3.11.** If \( G \) is locally compact and compactly generated, if \( H \) is an open subgroup in \( G \), then it is coforked in \( G \) if and only if the Schreier graph of \( G/H \), with respect to some/any generating compact subset of \( G \), has at least 2 ends. \( \square \)

The space of ends can be conveniently described using Boolean algebras. Given a Boolean algebra \( A \), define an ultrafiltering as a unital ring homomorphism onto \( \mathbb{Z}/2\mathbb{Z} \). Thus, an ultrafiltering on a power set \( 2^X = (\mathbb{Z}/2\mathbb{Z})^X \) is the same as an ultrafilter on \( X \), and an ultrafiltering on \( 2^X /2^X \) is the same as a non-principal ultrafilter on \( X \). The set of ultrafilterings \( \chi(A) \) of \( A \) is a compact set, for the topology of pointwise convergence. A basic result is the following

**Proposition 3.12 (Stone’s representation theorem).** For any Boolean algebra \( A \), we have \( \bigcap_{\xi \in \chi(A)} \text{Ker}(\xi) = \{0\} \). In particular, we have \( \chi(A) = \emptyset \) if and only if \( A = \{0\} \).

*Proof.* Clearly, a Boolean algebra admits no nonzero nilpotent element. On the other hand, if \( A \) is an arbitrary commutative ring, an elementary application of Zorn’s lemma shows that the intersection of all prime ideals is the set of nilpotent elements [Mat, Theorem 1.2]. Thus if \( A \) is a Boolean algebra, this intersection is trivial. Moreover, a Boolean algebra which is a domain is necessarily equal to \( \{0,1\} \); in other words, the prime ideals in a Boolean algebra are exactly the kernels of ultrafilterings. So the proof is complete. \( \square \)

Let \( X \) be a discrete \( G \)-set, and let \( \text{Comm}_G(X) \) the set of topological \( G \)-commensurated subsets of \( X \), which is a Boolean subalgebra of \( 2^X \).

**Proposition 3.13.** If \( X \) has finitely many \( G \)-orbits, the natural compactification

\[
X \rightarrow \chi(\text{Comm}_G(X))
\]

\[
x \mapsto \delta_x,
\]

where \( \delta_x(A) \) is 1 or 0 according to whether \( x \in A \), coincides with the end compactification of the Schreier graph \( \text{Sch}(X,S) \), and restricts to a natural homeomorphism between the set of ends of the graph \( \text{Sch}(X,S) \) and the space of ultrafilterings \( \chi(\text{Comm}_G(X)/2^X) \).

\footnote{Ultrafilterings of Boolean algebras are more commonly (but abusively) called ultrafilters, because ultrafilters on a set \( X \) canonically correspond to ultrafilterings on the power set \( 2^X \).}
Proof. We can view $\chi(\text{Comm}_G(X)/2^{(X)})$ as a closed subset of $\chi(\text{Comm}_G(X))$, consisting of those ultrafilterings vanishing on $2^{(X)}$. An easy verification shows that its complement consists of the Dirac ultrafilterings $(\delta_x)_{x \in X}$. This is an open discrete, and dense subset in $\chi(\text{Comm}_G(X))$.

To check that this is exactly the end compactification of $\text{Sch}(X,S)$, we just have to show that given a locally finite graph with vertex set $L$ and finitely many components, if its set of ends is defined as above, then it is naturally identified with $\chi(Q(L)/2^{(L)})$, where $Q(L)$ is the set of subsets of $L$ with finite boundary.

So $Q(L)/2^{(L)}$ is the inductive limit, over finite subsets $F$ of $L$, of the $Q_F(L)/2^F$, where $Q_F(L)$ is the set of subsets of $L$ with boundary contained in $F$; thus $\chi(Q(L)/2^{(L)})$ is the projective limit of all $\chi(Q_F(L)/2^F)$. The Boolean algebra $Q_F(L)/2^F$ can be described as the set of unions of connected components of $L \setminus F$. Since the latter is finite, $\chi(Q_F(L)/2^F)$ can be thus identified with the set of connected components of $L \setminus F$. Thus $\chi(Q(L)/2^{(L)})$ is identified with the set of ends of $L$. \[ \square \]

We will prove in the sequel that for a compactly generated locally compact group $G$, Property FW can be tested on transitive $G$-sets (see Proposition 5.3[1]). In particular, such a group $G$ has Property FW if and only it has no coforked open subgroup.

**Remark 3.14.** In this paper we only consider coset spaces of open subgroups. But ends of pairs of groups $H \subset G$ with $G$ locally compact compactly generated and $H$ closed in $G$ can also be considered. We do not give the general definition here (due to Houghton \cite{Ho74}), but in case $G$ is a connected Lie group, it corresponds to the ends of the manifold $G/H$. For instance, the upper unipotent subgroup $U$ of $\text{SL}_2(\mathbb{R})$ has a 2-ended coset space $\text{SL}_2(\mathbb{R})/U$. More generally, for $n \geq 2$, in $\text{SL}_n(\mathbb{R})$, the stabilizer of a nonzero vector in $\mathbb{R}^n$ has a 2-ended coset space, namely $\mathbb{R}^n \setminus \{0\}$ (which is homeomorphic to $\mathbb{R} \times S^{n-1}$). Since $\text{SL}_n(\mathbb{R})$ has Property T for $n \geq 3$, this shows that the present study does not carry over this context.

3.C. From median graphs to commensurating actions. On a metric space $D$, define the full segment $[x, y]$, for $x, y \in D$ as the set of $t$ such that $d(x, t) + d(t, y) = d(x, y).$ The metric space $D$ is called median if for all $x, y, z$, the intersection $[x, y] \cap [y, z] \cap [z, x]$ is a single point, called the median of $(x, y, z)$ and denoted by $xyz$ or $m(x, y, z)$ if the previous notation is awkward. A subset $E$ of a median space $D$ is called fully convex if $[x, y] \subset E$ for all $x, y \in E$, and biconvex\footnote{A biconvex subset is sometimes called a halfspace. Here halfspace refers to a certain walling, which is a distinct meaning, although in the sequel we will indeed prove that in median graphs, there is a walling for which the halfspaces are the biconvex subsets.} if both $E$ and $E^c$ are fully convex. Define $\mathcal{H}(x, y)$ as the set of biconvex subsets of $D$ containing $x$ and not $y$. 
A graph (not oriented, without multi-edges and self-loops and identified with its set of vertices) is median if each of its connected components (as a graph) is a median metric space.

**Example 3.15.** Let $X$ be a set. Endow the power set $2^X$ with a graph structure by calling $N$, $N'$ adjacent if $\#(N \triangle N') = 1$; thus $N$, $N'$ are in the same component if and only if they are commensurable (in other words, the connected component of $N$ is $\text{Comm}_N(X)$) and their graph distance is then $\#(N \triangle N')$. Then this graph is median. Indeed, the full segment $[N, N']$ between any commensurable $N, N'$ is the set of subsets trapped between $N \cap N'$ and $N \cup N'$, from which it easily follows that the intersection $[N, N'] \cap [N', N''] \cap [N'', N]$ is the singleton $\{(N \cap N') \cup (N' \cap N'') \cup (N'' \cap N)\}$ (more symmetrically described as the set of elements belonging to at least two of the three subsets $N, N', N''$) and thus the graph is median.

**Example 3.16.** Let $X$ be a set. Endow $Z^X$ with a graph structure by calling $f$, $f'$ adjacent if $f - f'$ or $f' - f$ is the Dirac function at some $x \in X$. Similarly as in Example 3.15, this is a median graph; the connected component of $f$ is the set of $f'$ such that $f' - f$ has a finite support, and the median of $f, f', f''$ is the function mapping $x$ to the median point in $(f(x), f'(x), f''(x))$.

**Proposition 3.17.** Let $G$ be a topological group and $f$ a cardinal definite function on $G$. Then there exists a connected median graph with a continuous isometric action of $G$ and a vertex $v$ such that $f(g) = d(v, gv)$ for all $g \in G$.

**Proof.** Let $X$ be a discrete continuous $G$-set and $M$ a commensurated subset with open stabilizer such that $f = \ell_M$. Consider the action of $G$ on the power set $2^X$, endowed with the median graph structure given in Example 3.15. Then by assumption, this action preserves the connected component $D = \text{Comm}_M(X)$. Moreover, since points and $M$ have open stabilizers, so do all elements in $\text{Comm}_M(X)$. So the action of $G$ on $X$ is continuous. If $v = M$, then we have $f(g) = d(v, gv)$ for all $g \in G$. $\Box$

There is no uniqueness statement in Proposition 3.17. Here the median graph constructed is, in a certain sense, huge (e.g., it is not locally finite unless $X$ is finite). There are indeed improved versions of the proposition, see [3.11]. Before this, we will show (Corollary 3.27) that the converse of proposition holds, i.e. every action on a connected median graph and choice of vertex gives rise to a cardinal definite function.

### 3.D. Walls in median graphs.

Define in any metric space $D$ and $x, y \in D$

$$P_{x,y} = \{z \in X \mid d(z, \{x, y\}) = d(z, x)\} = \{z \in X \mid d(z, x) \leq d(z, y)\}.$$ 

**Lemma 3.18** (Chepoi). If $D$ is a connected median graph (identified with the set of vertices), then for all adjacent $x, y \in D$

1. $D$ is the disjoint union of $P_{x,y}$ and $P_{y,x}$.
(2) [Che, Lemma 6.3] $P_{x,y}$ is biconvex.

Proof. In (1), it is trivial that $D = P_{x,y} \cup P_{y,x}$ so we only have to check that they are disjoint. By definition, we have $[x,y] = \{x,y\}$. So for any $z \in D$, the unique median point $xyz$ is either $x$ or $y$. In the first case $z \notin P_{y,x}$ and in the second case $z \notin P_{x,y}$.

We need an intermediate result in order to prove (2). Let us say that $E \subset D$ is 2-convex if $[x,y] \subset E$ whenever $x,y \in E$ and $d(x,y) = 2$. We begin by the following:

Sublemma [Che, Lemma 4.2]. In a connected median graph, every 2-convex connected subgraph is fully convex.

Proof of the sublemma. The statement is that if $E$ is 2-convex, then for any $x,y \in E$ at distance $\ell$ in $D$ and distance $k$ in $E$, we have $[x,y] \subset E$. This is trivial if $\ell \leq 1$ and true by 2-convexity if $\ell = 2$. We argue by induction on $n = 2k + \ell$. So the statement holds for $n \leq 6$. Assume $\ell \geq 3$ and $n \geq 7$ and assume the statement known for all lesser $n$. Consider two points $x_0, x_\ell$ in $E$ at distance $k$ in $E$, along with a geodesic statement $(x_0, x_1, \ldots, x_\ell)$ in $D$, we have to show that all $x_i$ belong to $E$. Denote by $(e_0 = x_0, e_1, \ldots, e_\ell = x_\ell)$ a path in $E$. Because of (1), we have $d(e_1, x_\ell) \in \{\ell + 1, \ell - 1\}$. We discuss accordingly.

If $d(e_1, x_\ell) = \ell + 1$, then $(e_1, x_0, \ldots, x_\ell)$ is a geodesic segment of length $\ell' = \ell + 1$ in $E$ joining $e_1, x_\ell \in E$, which have distance $k' = k - 1$ in $E$; here $n' = 2k' + \ell' = n - 1$, so we can apply the induction hypothesis, implying that $e_1, x_0, \ldots, x_\ell$ are all in $E$, proving what is required. Assume now that $d(e_1, x_\ell) = \ell - 1$. Define $y = m(x_1, e_1, x_\ell)$. Since $d(x_\ell, e_1) = d(x_\ell, x_1)$ and $e_1 \neq x_1$, we have $y \notin \{e_1, x_1\}$. So $d(y, e_1) = d(y, x_1) = 1$ and $d(y, x_\ell) = \ell - 2$. Since the distance of $e_1$ and $x_\ell$ in $E$ is $\ell'' = \ell - 1$ and in $E$ is $k'' = k - 1$, we have $n'' < n$ and thus by induction $[e_1, x_\ell] \subset E$. In particular, $y \in E$. Since $x_0 \in E$, by 2-convexity, it follows that $x_1 \in E$; moreover $x_1$ is connected to $x_\ell$ in $E$ by a path $(x_\ell, y, \ldots)$ of length $\ell''' = \ell - 1 \leq k - 1$ and once again the induction hypothesis shows that $[x_1, x_\ell] \subset E$, concluding the proof of the sublemma.

Let us now prove (2). By the sublemma, consider $a, b \in P_{x,y}$ at distance 2 and $c$ with $ac = bc = 1$ (we write $ab$ for $d(a,b)$, etc.), and let us prove that $c \in P_{x,y}$; suppose by contradiction the contrary. Define $\delta = ax$, so $ay = \delta + 1$. Since $cx = cy + 1$, $|cx - ax| \leq 1$ and $|cy - ay| \leq 1$, then necessarily $cx = \delta + 1$ and $cy = \delta$ and in turn, $bx = \delta$ and $by = \delta + 1$. In particular, $\delta \geq 1$ (otherwise $a = b$ contradicting $ab = 2$).

Define $m = m(a, b, x)$. Then $ma = mb = 1$ and $mx = \delta - 1$, so $my = \delta$. In particular, $m = m(a, b, y)$. On the other hand, $cy = \delta$ and $ca = cb = 1$ and thus $c$ is a median for $(a, b, y)$, thus by uniqueness $c = m$ and hence $cx = \delta - 1$, a contradiction. So $c \in P_{x,y}$. \hfill $\square$

Proposition 3.19. If $D$ is a connected median graph and $x, y \in D$, then $\mathcal{H}(x, y)$ is finite of cardinality $d(x, y)$. More precisely, for any geodesic $x = x_0, x_1, \ldots, x_n = y$, we have $\mathcal{H}(x, y) = \{P_{x_i, x_{i+1}}, 0 \leq i < n\}$. 
Proof. Let us first check the inclusion \( \mathcal{H}(x, y) \subset \{D_{x_i, x_{i+1}}, 0 \leq i < n\} \). Clearly any element \( E \) in \( \mathcal{H}(x, y) \) belongs to \( \bigcup_{i=1}^{n-1} \mathcal{H}(x_i, x_{i+1}) \) for some \( i \). If \( z \in E \), by full convexity of \( E \) we have \([x_i, z] \subset E\) and in particular \( m(x_i, x_{i+1}, z) \in E\); since the latter belongs to \( \{x_i, x_{i+1}\}\), it is equal to \( x_i\), so \( z \in P_{x_i, x_{i+1}}\). Since \( E \) is fully convex, the same argument shows that if \( z \notin E \) then \( z \in P_{x_i, x_{i+1}}\), which is the complement of \( P_{x_i, x_{i+1}}\) by Lemma 3.18. So \( E = P_{x_i, x_{i+1}}\).

For the converse (harder) inclusion, Lemma 3.18 shows that \( P_{x_i, x_{i+1}}\) is biconvex for all \( 0 \leq i < n \); it contains \( x_0 \) and not \( x_n \).

Remark 3.20. Going back to \( \text{Comm}_M(X) \) as in Example 3.15 if \( N, N' \) are adjacent, say \( N' = N \cup \{x\} \), then \( P_{N', N} \) is the set of subsets commensurable to \( M \) containing \( x \).

If \( x \in D \), let \( \mathcal{H}_x \) be the set of biconvex subsets containing \( x \).

Corollary 3.21. If \( D \) is a nonempty median graph, then the map \( x \mapsto \mathcal{H}_x \) is a canonical isometric embedding of \( (D, 2d) \) into \( \text{Comm}_{\mathcal{H}_0}(\mathcal{H}_D) \), for some choice \( x_0 \in D \); this embedding does not depend on \( x_0 \).

Proof. The symmetric difference \( \mathcal{H}_x \triangle \mathcal{H}_y \) is equal to \( \mathcal{H}(x, y) \cup \mathcal{H}(y, x) \) and thus by Proposition 3.19 has exactly \( 2d(x, y) \) elements.

Consider an (unoriented) connected median graph. Thus every edge \( \{x, y\} \) defines two oriented edges, \( (x, y) \) and \( (y, x) \). We have seen that the biconvex sets are the \( P_{x, y} \), where \( (x, y) \) ranges over oriented edges. Also, if \( (x, y) \) is an edge in a median graph, we define \( P_{x, y} \) as the part \( P_{x, y}^C \) of the component \( C \) containing \( x \) and \( y \).

Definition 3.22. In a median graph, we say that oriented edges \( (x, y) \) and \( (z, t) \) are parallel if \( P_{x, y} = P_{z, t} \); this is an equivalence relation. We say that \( (x, y) \) and \( (z, t) \) are elementary parallel if they form a square so that \( (x, y) \) and \( (z, t) \) are parallel (in the usual sense) and have the same direction, or equivalently, if \( \{x, z\} \) and \( \{y, t\} \) are edges.

Proposition 3.23. The parallelism equivalence relation between oriented edges is generated by elementary parallelism.

Proof. We first check that if \( (x, y) \) and \( (z, t) \) are elementary parallel then they are parallel. Indeed, this implies that \( x \in P_{z, t} \) and \( y \notin P_{z, t} \), so \( P_{z, t} \) is a biconvex subset by Lemma 3.18 which contains \( x \) but not \( y \), by Proposition 3.19 the only biconvex subset with these properties is \( P_{x, y} \), so \( P_{x, y} = P_{z, t} \).

Let us prove the generation property. Namely, let us prove by induction on \( n \geq 1 \) that if \( (x, y) \) and \( (z, t) \) are oriented edges with \( P_{x, y} = P_{z, t} \) and \( d(x, z) = n \), then there exist sequences \( x = x_0, x_1, \ldots, x_n = z \) and \( x = x_0, x_1, \ldots, x_n = z \) such that \( (x_{i-1}, y_{i-1}) \) and \( (x_i, y_i) \) are elementary parallel for all \( i = 1, \ldots, n \). This holds by definition for \( n = 1 \); assume \( n > 1 \) and that it holds for \( n - 1 \). Pick a path \( y = y_0, y_1, \ldots, y_n = t \). Define \( y_0 = y \) and consider the median \( y_1 = m(x_1, y_1, t) \).
We see that \( y_1 \notin \{ x_1, y \} \): indeed since \( P_{y,x} \) is biconvex and contains both \( t \) and \( y \), it contains \( y_1 \), so \( y_1 \neq x_1 \); also \( s \neq y_1 \) because \( d(t, x_1) \leq d(t, z) + d(z, x_1) = n \) and \( y_1 = y \) would imply that \( y_1 \) is on the total segment joining \( x_1 \) to \( t \) and thus imply \( y = x_1 \) which is not possible (because \( x_1 \in P_{x,y} \)). So \((x, y)\) and \((x_1, y_1)\) form a square so are elementary parallel and hence parallel by the first part of the proof. So we conclude by induction. □

**Definition 3.24.** An orientation on a median graph is called median if any two elementary parallel edges have the same orientation. An oriented median graph is a median graph endowed with a median orientation.

**Example 3.25.** The median structure on \( 2^X \) (Example 3.15) has a canonical orientation, where \((N, N')\) is an oriented edge if \( N' = N \cup \{ x \} \) for some \( x \notin N \); this orientation is obviously median and preserved by the action of the permutation group of \( X \).

Similarly, the median structure on \( Z^X \) (Example 3.16) has a canonical median orientation, where \((f, f')\) is an oriented edge if \( f' - f \) is the Dirac function at some \( x \in X \). This orientation is preserved by both the action of the permutation group of \( X \) and the action of \( Z^X \) on itself by translations.

An immediate consequence of Proposition 3.23 is the following:

**Corollary 3.26.** Let a group \( G \) act by automorphisms on a median graph. Equivalences:

1. \( G \) preserves a median orientation;
2. the action of \( G \) has no wall inversion, i.e., for every biconvex subset \( B \) and \( g \) we have \( gB \neq B^c \).

□

**Corollary 3.27.** Let \( G \) be a topological group. A function \( f : G \to \mathbb{R} \) is cardinal definite if and only if there is a continuous action of \( G \) by automorphisms on a connected oriented median graph \( X \) and a vertex \( v \) such that \( f(g) = d(v, gv) \) for all \( g \).

Moreover, if there is a continuous action of \( G \) by automorphisms on a connected median graph \( X \) and a vertex such that \( f(g) = d(g, gv) \) for all \( g \) then \( 2f \) is cardinal definite.

**Proof.** The forward implication is Proposition 3.17, except that there is no wall inversion; which immediately follows from the description of biconvex subsets in Remark 3.20.

Conversely, if the condition is satisfied, let \( \mathcal{H} \) be the set of biconvex subsets of \( X \); as a \( G \)-set, it is a quotient of the set of oriented edges of \( X \) and thus is a continuous discrete \( G \)-set. So by Corollary 3.21, the self-indexed family \( \mathcal{H} \) is a \( G \)-walling on \( X \) inducing 2 times the graph distance. By Corollary 3.3, we deduce that \( 2f \) is cardinal definite. If there is no wall inversion, we can split the set of biconvex into two \( G \)-invariant subset \( \mathcal{H}' \) and \( \mathcal{H}'' \), such that for every biconvex subset \( B \) we have \( \#(\{ B, B^c \} \cap \mathcal{H}') = 1. \) Then the self-indexed family \( \mathcal{H}' \) is a
3.E. Gerasimov’s theorem. The following theorem was proved by Gerasimov [Ger] for finitely generated groups. The following generalization, which relaxes the finite generation assumption, uses in a more or less hidden way several of his arguments, although the final layout of the proof is substantially simplified.

**Theorem 3.28.** Let a group $G$ act isometrically on a connected median graph $D$ with a bounded orbit. Then it has a finite orbit.

**Proof.** Let $B$ be the set of biconvex subsets of $D$. For $x \in D$, define $\mathcal{H}(x)$ as the set of biconvex subsets of $D$ containing $x$. Thus for all $x, y \in D$ we have $\mathcal{H}(x) \smallsetminus \mathcal{H}(y) = \mathcal{H}(x, y)$, which is finite by the easier inclusion of Proposition 3.19. The map $x \mapsto \mathcal{H}(x)$ is equivariant; by the more difficult inclusion (based on Lemma 3.18(2) of Proposition 3.19) it is injective. Also it is a “median homomorphism” in the sense that $\mathcal{H}(m(x, y, z)) = m(\mathcal{H}(x), \mathcal{H}(y), \mathcal{H}(z))$, where $m(A, B, C) = (A \cap B) \cup (B \cap C) \cup (C \cap A)$ for any $A, B, C$ sets of subsets of $D$.

The group $G$ naturally acts on $B$. For $g \in G$ and $x \in X$, we have $g\mathcal{H}(x) \triangle \mathcal{H}(x) = \mathcal{H}(gx) \triangle \mathcal{H}(x) = \mathcal{H}(x, y) \cup H(y, x)$, and thus $\mathcal{H}(x)$ is commensurated by the $G$-action.

Since $\ell_{\mathcal{H}(x)}$ is moreover bounded, $\mathcal{H}(x)$ is transfixed by Theorem 4.16. This implies that the set $\mathcal{E}(x)$ of $G$-orbits $Z$ such that $Z \cap \mathcal{H}(x) \notin \emptyset$ is finite; let $n(x)$ be the number of infinite orbits among $\mathcal{E}(x)$.

If $n(x) = 0$, then the orbit of $\mathcal{H}(x)$ is finite, since it ranges over subsets $Y$ of $X$ such that $Y \triangle X \subset \bigcup_{Z \in \mathcal{E}(x)} Z$, and thus (by injectivity of $x \mapsto \mathcal{H}(x)$) the orbit of $x$ itself is finite, as required.

So to prove the theorem, it is enough to show that whenever $n(x) \geq 1$ is finite, there exists $x' \in D$ such that $n(x') < n(x)$. Indeed, let $Z \in \mathcal{E}(x)$ be an infinite orbit. By Theorem 4.16, there exists a finite subset $N$ of $Z$ such that $Z \cap \mathcal{H}(x)$ is either equal to $N$ or $Z \setminus N$. By the general Lemma 3.29, there exist $g, g' \in G$ such that $N, gN$ and $g'N$ are pairwise disjoint. Therefore $m(N, gN, g'N)$ is either equal to $\emptyset$ or $Z$. It follows, defining $x' = m(x, gx, g'x)$ that $m(\mathcal{H}(x), g\mathcal{H}(x), g'\mathcal{H}(x)) = \mathcal{H}(x')$, we see that $n'(x) < n(x)$. □

We used the following basic lemma, extracted from Gerasimov’s paper:

**Lemma 3.29.** Let a group $G$ act transitively on an infinite set $X$. Then for every finite subset $F$ of $G$ there exist $g, g' \in G$ with $F, gF$ and $g'F$ pairwise disjoint.

**Proof.** Let us first find $g \in G$ with $F \cap gF$ disjoint. The set $P$ of $g$ such that $F \cap gF \neq \emptyset$ is precisely $\bigcup_{x, y \in F} \{g : gx = y\}$. Fix $x_0 \in X$, let $H$ be its stabilizer and fix a finite set $K \subset G$ such that $F = Kx_0$. Then

$$P = \bigcup_{h, k \in K} \{g : ghx_0 = kx_0\} = \bigcup_{h, k \in K} \{g : k^{-1}gh \in H\} = \bigcup_{h, k \in K} (kg^{-1})gHg^{-1}.$$
This is a finite union of left cosets of subgroups of infinite index; by B.H. Neumann [Ne54], it follows that \( P \neq G \). So taking \( g \notin P \) we have \( F \cap gF = \emptyset \).

Now let us prove the lemma. By the previous case, find \( g \) such that \( F \cap gF = \emptyset \). Then apply the previous case again to \( F \cup gF \): there exists \( g' \) such that \( F \cup gF \) and \( g'F \cup g'gF \) are disjoint. In particular, \( F, gF \) and \( g'F \) are pairwise disjoint. \( \Box \)

See Corollary 3.38 for an improvement of Theorem 3.28.

3.F. Involutive commensurating actions and the Sageev graph.

3.F.1. Involutions.

Definition 3.30. If \((E, \leq)\) is any partially preordered set endowed with a order-reversing involution \( \sigma \) (involutive preposet), define an ultraselection\(^7\) on \( E \) as a subset \( S \subset E \) satisfying

- \( S \) is a selection, i.e., namely \( x < y \) and \( x \in S \) implies \( y \in S \) (\( x < y \) means \( x \leq y \) and \( x \neq y \)).
- \( \sigma(S) = S^c \) (i.e., we have a partition \( E = S \sqcup \sigma(S) \)).

An equivalent data is that of a function \( j : E \to \{0, 1\} \) such that \( j(x) + j(\sigma(x)) = 1 \) for all \( x \in E \), and \( j \) is non-decreasing, namely \( x \leq y \) and \( j(x) = 1 \) implies \( j(y) = 1 \). Given \( j \), we get \( S \) by \( S = j^{-1}(\{1\}) \subset E \), and given \( S \), we obtain \( j = 1_S \). Note that the set of ultraselections is obviously compact under the pointwise convergence topology.

Say that two ultraselections \( S, T \) on the involutive pre-poset \( E \) are incident if \( \#(S \triangle T) = 2 \). In this case, there exists \( z \in E \) such that \( S \triangle T = \{z, \sigma(z)\} \); moreover \( z \) is a minimal element of \( S \). Conversely, if \( S \) is an ultraselection and \( z \) is a minimal element of \( S \), then \( S \cup \{\sigma(z)\} \setminus \{z\} \) is an ultraselection, incident to \( S \) by definition. This incidence relation defines a graph structure (non-oriented, with no self-loop and with no multiple edges), denoted by \( \text{Sel}(E, \leq, \sigma) \). The next two lemmas are straightforward generalizations of Nica’s Lemma 4.3 and Proposition 4.5 in [Nic]; on the other hand, the idea of associating a cubing to an abstract poset was introduced byNiblo-Reeves in [NRe] (with a few restrictions).

Lemma 3.31. Two ultraselections \( S, T \) on the involutive pre-poset \( E \) are in the same connected component if and only if \( S \triangle T \) is finite. Moreover, the inclusion of connected components of \( \text{Sel}(E, \leq, \sigma) \) into \( \text{Sel}(E, =, \sigma) \) is isometric.

Proof. This is clearly necessary. Let us prove it is sufficient. By induction on \( 2n = \#(S \triangle T) \) (which is even because \( S \setminus T = \sigma(T \setminus S) \). The case \( n = 0 \) is clear. Otherwise, find an minimal element \( z \) in \( S \setminus T \). If by contradiction there exists \( x \in S \) with \( x < z \), then by minimality of \( z \), necessarily \( x \in T \), but since \( T \)

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\(^7\)Ultraselections are sometimes called ultrafilters, but this terminology is incoherent, partly because there is no natural way to characterize ultrafilters as ultraselections (ultrafilters require a condition on intersections, which is not reflected here). Actually in his original paper, Sageev mentioned accurately that these satisfy “certain ultrafilter-like properties”.

is an ultraselection this forces $z \in T$, contradiction. Thus $S' = S \cup \{\sigma(z)\} \setminus \{z\}$ is an ultraselection incident to $j$, and $\#(S' \Delta T) = 2n - 2$, so $S'$ is in the same component as $T$ by induction.

The above proof actually proves the isometric statement as well. \[ \square \]

We have the following lemma (see §3.C for the definition of median graph).

**Lemma 3.32.** The ultraselection graph of any involutive pre-poset is median.

**Proof.** Begin by the case of the discrete preposet $\text{Sel}(E, =, \sigma)$. Thus an ultraselection here is just a subset $S \subset E$ such that $E = S \sqcup \sigma(S)$. Then the function $2^A \to 2^X$ mapping $B$ to $B \sqcup \sigma(B)$ is a bijection from $2^A$ to the set of ultraselections of $(E, =, \sigma)$; if $2^A$ is endowed with its median graph structure from Example 3.15, this is a graph isomorphism, so $\text{Sel}(E, =, \sigma)$ is median.

Now let the preposet be arbitrary. By Lemma 3.31, the embedding of the graph $\text{Sel}(E, \leq, \sigma)$ into the graph $\text{Sel}(E, =, \sigma)$ is isometric (if we allow infinite distances). This shows that full segments $[x, y]$ in the first graph are contained in full segments $[x, y]'$ in the second one. In particular, it immediately follows that for all $S_1, S_2, S_3 \in \text{Sel}(E, \leq, \sigma)$, the intersection $[S_1, S_2] \cap [S_2, S_3] \cap [S_3, S_1]$ contains at most one point; we only have to check that the median point $T = (S_1 \cap S_2) \cup (S_2 \cap S_3) \cup (S_3 \cap S_1)$ belongs to $\text{Sel}(E, \leq, \sigma)$, i.e., is an ultraselection. Clearly the set of selections is stable under taking finite unions and intersections, and therefore $T$ is a selection. Moreover, keeping in mind that $\sigma(S_i) = S_i^c$, we see that $\sigma(T) = T^c$ and thus $T$ is an ultraselection and the proof is complete. \[ \square \]

Let now $X$ be a $G$-set with commensurated subset $A$ and $\sigma$ a $G$-equivariant involution of $X$ such that $X = A \sqcup \sigma(A)$. Define, as in Proposition 3.2, the corresponding walling $W_x = \{h \in G \mid x \in hA\}$. Endow $X$ with the partial order given by $x \leq y$ if $W_x \leq W_y$. The above graph structure on $X$ is clearly $G$-invariant. On the other hand, $A$ itself is an ultraselection, as well as any translate $gA$. This is clear by observing that $gA = \{x \in X \mid g \in W_x\}$; thus translates of $A$ are called principal ultraselections.

A first observation is that all principal ultraselections belong to the same connected component of the graph, by Lemma 3.31.

**Definition 3.33.** The Sageev graph associated to $(X, A, \sigma)$ is the component of $\text{Sel}(X, \leq, \sigma)$ containing principal ultraselections.

The Sageev graph connected by definition and is median by Lemma 3.32. It contains an isometric copy of the set of translates of $A$, with the symmetric difference metric.

**Remark 3.34.** Actually, Sageev [Sa95] directly cubulated the set of ultraselections and the link with median graphs was brought out later by Chepoi [Che]. The point of view given here is closer to that of Nica [Nic].
The Sageev graph can be viewed as an isometrically embedded subgraph of \( \text{Comm}_A(X) \), by Lemma 3.31. In general, finding a cubulation associated to a commensurating action amounts to finding “small” invariant connected median subgraphs of \( \text{Comm}_A(X) \). Here small can mean that the graph is locally finite, (more weakly) that the underlying cubing is finite-dimensional.

A construction, assuming in addition that the stabilizers of points in \( X \) are finitely generated, is done by Niblo, Sageev, Scott and Swarup [NSSS]. It consists in replacing the partial preorder on \( X \) by a larger one, requiring that \( x \preceq y \) if \( W_x \setminus W_y \) is “small” in a suitable sense. This can be done consistently under the assumption that the walling \((W_x)\) is in “good” position; they actually show that this can always be supposed at the cost of replacing \( A \) by a commensurable subset (and using crucially that the acting group and stabilizers are finitely generated).

3.G. CAT(0) cube complexes. Let us briefly present CAT(0) cube complexes. These spaces are in canonical correspondence with median graphs and therefore

We define a cube complex as the data of a set \( V \) and, for each \( d \geq 1 \), a set \( C_d \) of \( 2^d \)-element subsets (called \( d \)-cubes), each \( d \)-cube \( C \) coming with a bijection \( \phi_C \) with the vertex set of the standard \( d \)-cube, bijection defined modulo composition by an automorphism of the standard \( d \)-cube, with the assumptions that \( C_0 = V \) and for every \( d \), if \( C \subset V \) is a \( d \)-cube and \( \emptyset \neq C' \subset C \), then \( C' \) is a cube if and only if \( \phi_C(C') \) is a face in the standard \( d \)-cube.

A cube complex has a natural unoriented graph structure (with no self-edge and no multiple edge), where \( C_1 \) is the set of edges.

Denote by \( S^1(v) \) the set of vertices adjacent to \( v \) and \( C(v) \) the set of cubes containing \( v \). If \( v \) is a vertex, the link \( L_v \) at \( v \) is the simplicial complex (possibly with multiple simplices) whose underlying set is \( S^1(v) \), and for which the family of simplices is given by the \( C \cap S^1(v) \), when \( C \) ranges over \( C(v) \).

The cube complex is called (combinatorially) locally CAT(0) if for every \( v \), the link \( L_v \) is a flag complex in the sense that it has no double edge (and hence no double simplex, i.e. the family of simplices is injective) and whenever it contains the 1-skeleton of a simplex, it also contains the whole simplex.

Every CAT(0) cube complex has a “topological” realization; given as gluing of cubes; the word “topological” is ambiguous because there can be several nonequivalent choices of topologies; one of these is the topology given as inductive limits of the topologies on finite unions of cells; others are obtained by endowing cubes with the \( \ell^p \)-metric of the (half) unit cube and the the associated length distance, which is actually geodesic.

Whether the obtained topological space is simply connected does not depend on the choice among those topologies; this can be characterized combinatorially in terms of the 2-skeleton and we then simply say that the cube complex is simply connected. A cube complex is called (combinatorially) CAT(0) if it is simply connected and combinatorially locally CAT(0).
If the cubes are endowed with the Euclidean metric, it can be shown (see [Hag, Section 2]) that the topological realization of the cube complex is CAT(0) if and only if the complex is combinatorially CAT(0). Note that if the dimension is unbounded, the topological realization, with the Euclidean metric, can fail to be complete; its completion is still CAT(0) but it can be delicate to work within.

**Theorem 3.35** (Chepoi [Che]). For every CAT(0) cube complex, the 1-skeleton is a median graph. Conversely, every median graph is the 1-skeleton of a unique CAT(0) cube complex, in which $d$-cubes are precisely the subsets of vertices forming a subgraph (and actually full subgraph) isomorphic to the $d$-cube.

We refer to the latter as the canonical cubulation of a median graph. For Corollary 3.27 we derive:

**Corollary 3.36.** A continuous function $f : G \to \mathbb{R}$ is cardinal definite if and only if there is a cellular continuous action of $G$ on a CAT(0) cube complex and a vertex $v$ such that $f(g) = d_1(v, gv)$, where $d_1$ denotes the $\ell^1$-metric on the CAT(0) cube complex.

From Theorem 3.28 we also get

**Corollary 3.37.** Fix $p \in [1, \infty[$. If a group acts cellularly on a CAT(0) cube complex and has a bounded orbit in the $\ell^p$-metric, then it has a fixed point.

**Proof.** It it has a bounded orbit, then it has a bounded vertex orbit. Now a set of vertices is bounded in the $\ell^1$-metric if and only if it is bounded in the $\ell^p$-metric: the forward implication is trivial, and conversely if the $\ell^1$-diameter is $\geq n$ then the $\ell^p$-diameter is $\geq n^{1/p}$.

So vertex orbits are bounded in the $\ell^1$-metric. By Theorem 3.28 there is a finite vertex orbit. Now we use the fact that every finite subset of a connected median graph is contained in a finite fully convex subset: this is true in the median graph $\text{Comm}_M(X)$ for every set $X$ and $M \subset X$ and for every connected median graph, since any such graph embeds isometrically into a median graph of the form $\text{Comm}_M(X)$ by an obvious variant of Corollary 3.21 (where we rather take the map $x \mapsto \mathcal{H}(x) \cap \mathcal{H}(x_0)$).

Now we have an invariant fully convex subset $E \subset D$, we use the canonical cubulation of $E$, which is contained in the canonical cubulation of $D$, so by the Bruhat-Tits fixed point lemma (applied to a finite cube complex with the $\ell^2$-distance), there is a fixed point.

In turn, this provides the following improvement of Theorem 3.28.

**Corollary 3.38.** Let a group $G$ act isometrically on a connected median graph $D$ with a bounded orbit. Then there is an invariant cube (for the canonical cubulation). If moreover the action on $D$ preserving a median orientation, then there is a fixed vertex.
Proof. By Corollary 3.37 there is a fixed point, say in the $k$-skeleton with $k$ minimal. So the $k$-cube containing it is invariant. Now assume that there is no wall inversion. Identify this cube to $[-1,1]^k$ and transfer the $G$-action. Let $(e_i)$ be the canonical basis in $\mathbb{R}^k$. Every edge in this cube has, up to orientation, the form $(x, x + 2e_i)$, and we have $H(x, x + 2e_i) = H - e_i, e_i)$. It follows that the set of $2k$ biconvex subsets
\[ \{ H(-\varepsilon e_i, \varepsilon e_i) \mid \varepsilon \in \{-1, 1\}, 1 \leq i \leq k \} \]
is $G$-invariant. Since there is no wall inversion, there is a function $i \mapsto \varepsilon_i$ such that the set of $k$ biconvex subsets
\[ \{ H(-\varepsilon_i e_i, \varepsilon_i e_i) \mid 1 \leq i \leq k \} \]
is $G$-invariant. In particular, the face $\{ x_i = \varepsilon_i \}$ is invariant for all $i$, and it follows that the point $(\varepsilon_1, \ldots, \varepsilon_k)$ is fixed. □

3.H. On the language of “almost invariant” subsets. In the literature, the following language is sometimes used: let $G$ be a group and $H$ a subgroup. An $H$-almost invariant subset of $G$ means a left-$H$-invariant subset of $G$ whose projection on $H \setminus G$ is commensurated by the right action of $G$.

The usual data in this context is the following: a finite family $(H_i)$ of subgroups of $G$ and an $H_i$-almost invariant subset $X_i$ of $G$, and $E$ is the family of left $G$-translates of all $H_i$.

We should emphasize that these data are equivalent to that of a $G$-set $X$ with a finite family $(x_1, \ldots, x_n)$ including one element in each orbit, and a commensurated subset $M \subset X$, namely given the above date, consider the disjoint union $\bigsqcup_{i=1}^n H_i$ and $M = \bigcup X_i^{-1}/H_i$; conversely given $X$, $M$, and $x_i$ as above, we can define $H_i$ as the stabilizer of $x_i$ and $X_i = \{ g \in G \mid g^{-1}x_i \in M \}$.

Inasmuch as the indexing of orbits and choice of representative points is artificial, the data of the $G$-set $X$ and the commensurated subset $A$ seem enough. Actually, the family of the left translates of the $X_i^{-1}$ are nothing else than the walling $(W_x)$.

Remark 3.39. Given a $G$-invariant involution $\sigma : X \to X$, the condition that $W_{\sigma(x)} = W_x^c$ for all $x$ is equivalent to the requirement that $A$ is a fundamental domain for $\sigma$, i.e., $X = A \sqcup \sigma(A)$.

When $X$ is a $G$-set with commensurated subset $A$, it can canonically (with respect to the data of $(X, A)$) be embedded in a $G$-set with commensurated subset with such an involution, namely $X' = X \times \{0, 1\}$ with the involution $\sigma : (x, 0) \leftrightarrow (x, 1)$ and the commensurated subset $A' = (A \times \{0\}) \cup (A^c \times \{1\})$.

The Kropholler conjecture, usually termed in the previous language can be restated as follows (bi-infinite means infinite and co-infinite):

Conjecture 3.40 (Kropholler). Let $G$ be a finitely generated group and $G/H$ a transitive $G$-set. Assume that $G$ commensurates an bi-infinite subset $A \subset$
$G/H$. Assume in addition that $A$ is $H$-invariant. Then $G$ splits over a subgroup commensurable (in the group sense) to $H$.

The conjecture holds when $H$ is finite by Stallings’ theorem. The assumption that $A$ is $H$-invariant is essential as otherwise Properties FW and FA would be equivalent (which is not the case, see Example 5.12).

Dunwoody extends the conjecture to arbitrary discrete groups; the conclusion being replaced by the existence of an unbounded action of $G$ on a tree for which each edge orbit contains an edge whose stabilizer is commensurable (in the group sense) to $H$.

Let us also mention, in the language of commensurated subsets, a basic lemma of Scott [Sc98, Lemma 2.3]. Let $G$ be a group acting on a set $X$ with a commensurated subset $M$. Define (as in Proposition 3.2) $W_x = \{g \in G \mid x \in gM\}$. Denote by $p_x$ the anti-orbital map $g \mapsto g^{-1}x$; note that it maps $W_x$ into $M$.

**Proposition 3.41.** Let $G$ be a compactly generated locally compact group and $X$ a continuous discrete $G$-set with commensurated subset $M$. Let $x, y \in X$ be points such that $M \cap Gx$ and $M \cap Gy$ are infinite. Then $p_x(W_x \cap W_y)$ is finite if and only if $p_y(W_x \cap W_y)$ is finite.

**Proof.** The equivalence is symmetric, so we only have to prove the forward implication. Fix a compact symmetric generating subset $S$ of $G$. Assume that $p_x(W_x \cap W_y)$ is finite. Since $M \cap Gx$ is not transfixed, there exists $z \in M$ not in $p_x(W_x \cap W_y)$. Since the latter is finite, there exists $k$ such that for every $x' \in p_x(W_x \cap W_y)$ there exists $s \in S^k$ such that $z = sx$. In other words, for every $g \in W_x \cap W_y$, there exists $s \in S^k$ such that $z = sgx = p_x(g^{-1}s^{-1})$. Since $z \in M \cap Gx$ and $z \notin p_x(W_x \cap W_y)$, we have $g^{-1}s^{-1} \in W_x$ and it follows that $g^{-1}s^{-1} \notin W_y$, i.e. $sgy \notin M$. Thus we have proved that for every $v \in p_y(W_x \cap W_y)$ (which is contained in $M$), there exists $s \in S^k$ such that $sv \notin M$. So the $p_y(W_x \cap W_y)$ is at bounded distance to the boundary of $M$ in the Schreier graph of $X$ with respect to $S$. Since the latter is locally finite and $M$ has a finite boundary, we deduce that $p_y(W_x \cap W_y)$ is finite. \[\square\]

If $M \cap Gx$ is finite but not $M \cap Gy$, the conclusion of the previous proposition fails, as shown by elementary examples in [Sc98, Remark 2.4].

4. **Commensurating actions of topological groups**

4.A. **The commensurating symmetric group.** If $X$ is a set, let $\mathcal{S}(X)$ be the group of its permutations. It is endowed with its usual Polish topology, which is a group topology, for which a basis as neighborhoods of the identity is given by pointwise stabilizers of singletons.

Given a topological group $G$ and an abstract $G$-set $X$ (endowed with the discrete topology), it follows from the definitions that the following conditions are equivalent
• for every $x \in X$, the stabilizer $G_x$ is open in $G$;
• the structural map $G \times X \to X$ is continuous;
• the structural homomorphism $G \to S(X)$ is continuous.

We then say that $X$ is a continuous discrete $G$-set.

Consider now a set $X$ and a subset $M$. Let $S(X, M)$ be the group of permutations of $X$ commensurating $M$. It acts faithfully on the set $\text{Comm}_M(X)$ of subsets of $X$ commensurable to $M$. We endow $S(X, M)$ with the group topology induced by the inclusion in $S(\text{Comm}_M(X))$.

**Lemma 4.1.** A basis of neighborhoods of the identity in $S(X, M)$ is given by the subgroups $H_M(F)$ for $F$ finite subset of $G$, where $H_M(F)$ is the pointwise stabilizer of $F$ in the stabilizer of $M$. In particular, the inclusion $S(X, M) \to S(X)$ is continuous.

**Proof.** Let us first observe that the action of $S(X, M)$ on $X$ is continuous. Indeed, the stabilizer of $x \in X$ contains the intersection of the stabilizers of $M \cup \{x\}$ and $M \setminus \{x\}$, and therefore is open. This shows that all $H_M(F)$ are open in $S(X, M)$.

Let $P$ be a finite subset of $\text{Comm}_M(X)$ and $H$ its pointwise stabilizer. Define $F = \bigcup_{N \in P} M \triangle N$. Then $F$ is finite and $H$ contains $H_M(F)$. This shows that the $H_M(F)$ form a basis of neighborhoods of the identity. \qed

Given a continuous discrete $G$ set and commensurated subset, a natural requirement is that $M$ has an open stabilizer. This is (obviously) automatic if $G$ is discrete, but also, more generally, when $G$ is locally compact, or has an open Polish and separable subgroup, see §4.C. It follows from Lemma 4.1 that this holds if and only if the homomorphism $G \to S(X, M)$ is continuous.

We should not that the automatic continuity does not hold in general. The simplest counterexample is the tautological one: if $X$ is any infinite set and $M$ an infinite subset with infinite complement, then the stabilizer of $M$ in $S(M)$ is not open. Therefore, if we define $G$ as the group $S(X, M)$ endowed with the topology induced by the inclusion into $S(X)$, then $X$ is a continuous discrete $G$-set and $M$ is commensurated but does not have an open stabilizer.

**Remark 4.2.** There is a natural faithful action of $S(X, M)$ on the two-point compactification of $X$ given by the disjoint union of the one-point compactifications of $M$ and its complement. The compact-open topology then coincides with the topology of $S(X, M)$ described above.

**Definition 4.3.** Let $G$ be a topological group. We define a cardinal definite function on $G$ as a function of the form $\ell_M$ for some continuous discrete $G$-set $X$ and commensurated subset $M$ with open stabilizer.

**Lemma 4.4.** On a topological group, a sum of finitely many cardinal definite functions is cardinal definite. More generally, an arbitrary sum of cardinal definite functions, if finite and continuous, is cardinal definite.
Proof. Let \((\ell_i)_{i \in I}\) be cardinal definite functions on \(G\). Write \(\ell_i = \ell_M\) with \(M_i \subset X_i\) is a commensurated subset with open stabilizer, \(X_i\) being a continuous discrete \(G\)-set. Assume that \(\sum \ell_i\) is everywhere finite and continuous. Let \(X\) be the disjoint union of the \(X_i\) and \(M \subset X\) the union of the \(M_i\). Then \(X\) is a continuous discrete \(G\)-set, \(M\) is commensurated by \(G\) and \(\ell_M = \sum \ell_i\); since it is continuous, \(M\) has an open stabilizer (namely \(\{\ell_M = 0\}\)) and thus \(\ell_M\) is cardinal definite. \(\square\)

Let \(G\) be a topological group. Denote by \(G_\delta\) the group \(G\) endowed with the discrete topology.

**Lemma 4.5.** A function \(G \to \mathbb{R}\) is cardinal definite if and only if it is cardinal definite on \(G_\delta\) and is continuous on \(G\).

**Proof.** The only if condition is clear. Conversely, suppose that \(\ell\) is continuous and cardinal definite on \(G_\delta\). So \(\ell = \ell_M\) for some \(G\)-set \(X\) and commensurated subset \(M\). Note that \(X\) may not be a continuous \(G\)-set. Decomposing \(X\) into \(G\)-orbits and in view of Lemma 4.4, we can suppose that \(X\) is \(G\)-transitive. We can suppose that \(M\) is not invariant; thus \(\bigcup_{g,h \in G} gM \triangle hM\) is not empty; being \(G\)-invariant, it is therefore equal, by transitivity, to all of \(X\). The Boolean algebra generated by the \(gM \triangle hM\) when \(g, h\) range over \(G\) defines an invariant partition of \(X\) by finite subsets; by transitivity all these subsets have the same cardinal \(n\).

Note that \(M\) is a union of components of the partition. Define \(X'\) as the quotient of \(X\) by this partition and \(M'\) the image of \(M\) in \(X'\). The stabilizer of any element of \(X'\) is the stabilizer of some component of the partition of \(X\) and thus, as a finite intersection of subsets of the form \(gM \triangle hM\), is open. So \(X'\) is a continuous discrete \(G\)-set. Since \(\ell_M\) is open, the stabilizer of \(M\), and hence of \(M'\), is open. So \(\ell_{M'}\) is cardinal definite. Since \(\ell_M(g) = n\ell_{M'}(g)\), we deduce from Lemma 4.4 that \(\ell_M\) is cardinal definite. \(\square\)

**Remark 4.6.** If \(M\) is infinite, the commensurating symmetric group \(S(X,M)\) can be identified with the self-homeomorphism group, with the compact-open topology, of the disjoint union \(\hat{M} \sqcup (X \setminus M)\), where \(\hat{M}\) is the one-point compactification of \(M\).

4.B. **Cofinality \(\neq \omega\).** Here we give some results on commensurating action of topological groups with uncountable cofinality. They are useful even in the special case of a finitely generated acting group.

We say that a topological group has **uncountable cofinality** (or **cofinality \(\neq \omega\)**) if it cannot be written as the union of an infinite (strictly) increasing sequence of open subgroups, or equivalently if any continuous action of \(G\) on any ultrametric space has bounded orbits. For instance, a countable discrete group has cofinality \(\neq \omega\) if and only if it is finitely generated, and more generally a \(\sigma\)-compact locally compact group has cofinality \(\neq \omega\) if and only if it is compactly generated. There also exist uncountable discrete groups with cofinality \(\neq \omega\).
Proposition 4.7. Let $G$ be a topological group with uncountable cofinality. Let $X$ be a discrete continuous $G$-set. Let $M \subset X$ be a commensurated subset with an open stabilizer, and let $(X_i)_{i \in I}$ be the orbit decomposition of the $G$-set $X$. Then $X_i \cap M$ is $G$-invariant for all but finitely many $i$'s.

Proof. Let $J$ be the set of $i$ such that $X_i \cap M$ is not $G$-invariant. We need to show that $J$ is finite. Otherwise, there exists a decreasing sequence of non-empty subsets $J_n \subset J$ with $\bigcap J_n = \emptyset$. Define

$$G_n = \{ g \in G : \forall i \in J_n : g(M \cap X_i) = M \cap X_i \}.$$ 

Note that $G_n$ contains the stabilizer of $M$, which is open by assumption. So $(G_n)$ is a non-decreasing sequence of open subgroups, and $\bigcup G_n = G$ because for a fixed $g$, if $n$ is large enough, the finite subset $M \triangle gM$ does not intersect $\bigcup_{i \in J_n} X_i$. So by the cofinality assumption, $G = G_n$ for some $n$, i.e., $M \cap X_i$ is $G$-invariant for all $i \in J_n$. This contradicts the definition of $J$. \hfill \Box

Corollary 4.8. Let $G$ be a topological group with uncountable cofinality. Then every cardinal definite function on $G$ is a finite sum of cardinal definite functions associated to transitive actions of $G$.

Using the dictionary between commensurating actions and wallings (Proposition 3.6), we get

Corollary 4.9. Let $G$ be a topological group with uncountable cofinality. Then for every clopen $G$-walling on $G$, there are finitely many $G$-orbits of halfspaces $\neq \emptyset, G$. In particular, halfspaces $\neq \emptyset, G$ have a bounded multiplicity. \hfill \Box

Proposition 4.7 also has a geometric interpretation for actions on median graphs.

Corollary 4.10. Let $G$ be a topological group with uncountable cofinality and let $G$ act continuously by automorphisms on a connected median graph $V$; fix a vertex $x_0 \in V$. Let $\mathcal{H}$ be the set of biconvex subsets in $V$. Let $\mathcal{H}(Gx_0)$ be the set of biconvex subsets containing the whole orbit $Gx_0$ and $\mathcal{H}'(Gx_0)$ the set of biconvex subsets whose complement contains the entire orbit $Gx_0$ (these are clearly disjoint $G$-invariant subsets of $\mathcal{H}$). Then

$$\mathcal{H} \setminus (\mathcal{H}(Gx_0) \sqcup \mathcal{H}'(Gx_0))$$

consists of finitely many $G$-orbits for the action of $G$ on $\mathcal{H}$. \hfill \Box

4.C. Automatic continuity. Given a topological space $X$, a family of subsets $(Y_i)_{i \in I}$ has locally bounded multiplicity if the function $f = \sum_{i \in I} 1_{Y_i}$ is finite and for every $x \in X$ there exists a neighborhood $V$ of $x$ such that $f$ is bounded on $V$.

For a topological group $G$, consider the following two properties:

- (*) every family of nonempty clopen subsets of $G$ with locally bounded multiplicity is countable;
• (**) if \( o(G) \) is the intersection of all open subgroups of \( G \), then \( G/o(G) \) is zero-dimensional in the sense that it admits a basis of neighborhoods consisting of open subgroups.

**Proposition 4.11.** Let \( G \) be a Baire topological group with an open subgroup satisfying either (*) or (**) . Suppose that \( G \) acts continuously on a discrete set \( X \) with a commensurated subset \( M \). Then the corresponding homomorphism \( G \to S(X, M) \) is continuous (i.e., \( M \) has an open stabilizer, or, still equivalently, \( t_M \) is continuous).

**Proof.** Let us first prove the result when assuming that \( X \) is countable, only making use of the fact that \( G \) is Baire. Then the stabilizer of \( M \) is closed and has an at most countable index, hence is open by the Baire property.

Let us now prove the general case. Since we need to prove the continuity of a homomorphism, we can pass to an open subgroup and suppose that \( G \) satisfies either (*) or (**). Observe that 
\[
\ell_M(g) = \#(M \triangle gM) = \#(gX \setminus M) + \#(g^{-1}M \setminus M)
\]
\[
= \sum_{x \notin M} 1_{gM}(x) + 1_{g^{-1}M}(x).
\]
Since each \( x \) has an open stabilizer, each function \( x \mapsto u_x(g) = 1_{gM}(x) + 1_{g^{-1}M}(x) \) is continuous, as well as each finite sum of these. It follows that \( \ell_M \), as a filtering supremum of continuous functions, is lower semicontinuous. Hence for every \( r \), \( K_r = \{ x \in G : \ell(x) \leq r \} \) is closed. By the Baire property, there exists \( r \) such that \( K_r \) has non-empty interior. Note that \( K_r \) is symmetric and \( K_r K_r \subset K_{2r} \). It follows that \( K_{2r} \) is a neighborhood of \( 1 \) in \( G \).

Now we conclude in two different manners. If \( G \) satisfies (*), define \( U_x = \{ g : u_x(g) \neq 0 \} \); the above shows that the family \( (U_x)_{x \in X} \) By (*), it follows that the set \( X' \) of \( x \) such that \( U_x \) is non-empty is at most countable; it is clearly \( G \)-invariant. Thus the stabilizer of \( M \) equals the stabilizer of \( M \cap X' \) and therefore is open by the countable case.

Suppose that \( G \) satisfies (**) instead. The action of \( o(G) \) on \( X \) is trivial and \( \ell_M \) is \( o(G) \)-invariant; thus \( \ell_M \) is bounded on \( o(G)K_{2r} \). By (**), the latter contains an open subgroup \( L \) of \( G \). Since \( \ell_M \) is bounded on \( L \), the subset \( M \) is \( L \)-transfixed (by Theorem 4.16), and hence has the same stabilizer as some finite subset of \( X \). By continuity of the action on \( X \), we deduce that \( M \) has an open stabilizer. \( \square \)

It is obvious that if \( X, Y \) are topological spaces with a continuous injection of \( X \) into \( Y \) with dense image and \( X \) satisfies Condition (*) of Lemma 4.11, then so does \( Y \). In particular, separable topological spaces satisfy Condition (*). This is also true for \( \sigma \)-compact locally compact groups:

**Lemma 4.12.** Let \( G \) be a \( \sigma \)-compact locally compact group. Then every family of nonempty open subsets with locally bounded multiplicity is countable.
Note that, in contrast, the Stone-Cech boundary of $\mathbb{N}$ is a compact space containing continuum many pairwise disjoint clopen subsets and thus fails to satisfy (*) of Lemma 4.11.

**Proof of Lemma 4.12.** Fix a left Haar measure $\lambda$ and write $G = \bigcup_n \Omega_n$, where each $\Omega_n$ is an open subset with compact closure. Suppose that $(U_i)_{i \in I}$ is an uncountable family of nonempty open subsets in $G$ with locally bounded multiplicity.

Then for some $n$, the set $J_n$ of $i$ such that $V_i = U_i \cap \Omega_n$ is nonempty is uncountable. Since the multiplicity is locally bounded, it is bounded by a certain number $k$ on $\Omega_n$. Therefore, for every finite subset $J$ of $I$ we have

$$\sum_J \lambda(V_i) = \sum_J \int_{\Omega_n} 1_{V_i} = \int_{\Omega_n} \sum_J 1_{V_i} \leq \lambda(\Omega_n)k$$

thus the family $(\lambda(V_i))$ is summable and thus has a countable support, contradicting $\lambda(V_i) > 0$ for all $i \in J_n$. \qed

Note that locally compact group also satisfy (**).

**Corollary 4.13.** Let $G$ be a locally compact group. Then for every continuous discrete $G$-set, every commensurated subset has an open stabilizer. In other words, for every pair of sets $M \subset X$ and homomorphism $f : G \to S(X, M)$, the continuity of the composite map $G \to S(X)$ implies the continuity of $f$. \qed

**4.D. Affine $\ell^p$ action.** Let $X$ be a set and $M$ a subset. We denote by $R^X$ the space of all functions $X \to \mathbb{R}$. By $p$ we denote any real number in $[1, \infty]$. Define

$$\ell^p_M(X) = \{ f \in R^X : f - 1_M \in \ell^p(\mathbb{R}) \}.$$ 

It is endowed with a canonical structure of an affine space over $\ell^p(X)$ and the corresponding $\ell^p$-distance. It only depends on the commensurability class of $M$. Also define

$$\ell^0_M(X) = \{ f \in R^X : f - 1_M \in \ell^0(\mathbb{R}) \},$$

where $\ell^0(X) = R^{(X)}$ is the space of finitely supported functions. So $\ell^p_M(X) \subset \ell^p_M(X) \subset \ell^q_M(X)$ for all $p \leq q$. By the symbol $\ast$, we mean either $p$ or $\circ$. We also denote, for $I \subset \mathbb{R}$, the subset $\ell^\ast_M(X)_I$ as the set of elements in $\ell^\ast_M(X)$ with values in $I$.

There is a natural action of $S(X)$ on $R^X$, which preserve the subspaces $\ell^\ast(X)$. The stabilizer of each of the spaces $\ell^\ast_M(X)$ is precisely $S(X, M)$.

**Lemma 4.14.** For every $p$, the action of $S(X, M)$ on $\ell^p_M(X)$ is continuous.

**Proof.** Since this action is isometric, it is enough to check that the orbital map $i_x : g \mapsto gx$ is continuous for every $x$ ranging over a dense subset, namely $\ell^p_M(X)$. For such an $x$, the stabilizer is open and thus the continuity of the orbital map follows. \qed
Let us observe that the normed affine spaces \( \ell_{M}^p(X) \) as well as the actions of \( \mathcal{S}(X, M) \) depend only on the commensurability class of \( M \).

We endow \( \mathcal{S}(X, M) \) with the left-invariant pseudo-metric \( \ell^p_M(g, h) = \#(gM \triangle hM) \). Note that this is continuous, but does not define the topology of \( \mathcal{S}(X, M) \) since it is not Hausdorff (for \( \#(X) \geq 3 \)). We have \( \ell^p_M(g, h) = \ell^p_M(g^{-1}h) \), with the length \( \ell_M \) defined by \( \ell_M(g) = \#(M \triangle gM) \).

**Proposition 4.15.** The action of \( \mathcal{S}(X, M) \) on \( \ell^p_M(X) \) is faithful, continuous and metrically proper. Moreover, the injective homomorphism \( \alpha_p : \mathcal{S}(X, M) \to \text{Isom}(\ell^p_M(X)) \) has a closed image, namely the set \( \Xi^p_M(X) \) of affine isometries of \( \ell^p_M(X) \) that preserve the set of points in \( \{0, 1\}^X \cap \ell^p_M(X) \), and whose linear part preserves the closed cone \( \ell^p(X) \cap [0, \infty]^{X} \).

**Proof.** Note that the set \( \{0, 1\}^X \cap \ell^p_M(X) \) is equivariantly identified with the set of indicator functions of elements of \( \text{Comm}_M(X) \). Since \( \mathcal{S}(X, M) \) acts faithfully on \( \text{Comm}_M(X) \), it follows that the action on \( \ell^p_M(X) \) is faithful.

Another consequence is that if both \( M \) and its complement are infinite, \( \alpha_p(\mathcal{S}(X, M)) \) and \( \Xi^p_M(X) \) are both transitive on \( \{0, 1\}^X \cap \ell^p_M(X) \). If \( M \) or its complement is finite, it still holds that \( \alpha_p(\mathcal{S}(X, M)) \) and \( \Xi^p_M(X) \) have the same orbits on \( \{0, 1\}^X \cap \ell^p_M(X) \), by an argument left to the reader.

Let us check that \( \alpha_p(\mathcal{S}(X, M)) = \Xi^p_M(X) \). The inclusion \( \subseteq \) is clear; conversely, given \( \phi \in \Xi^p_M(X) \), after composition by an element of \( \alpha_p(\ell^p_M(X)) \) (using the previous observation about orbits), we obtain an element \( \phi_1 \) with \( \phi_1(1_M) = 1_M \).

The 1-sphere in \( \{0, 1\}^X \cap \ell^p_M(X) \) can be described as the disjoint union \( A \sqcup B \), wherein \( A \) consists of elements of the form \( 1_{M \cup \{x\}} = 1_M + \delta_x \) for \( x \notin M \) and \( B \) of elements of the form \( 1_{M \smallsetminus \{x\}} = 1_M - \delta_x \) for \( x \in M \). Note that \( \phi_1 \) preserves this 1-sphere. Since it moreover satisfies the condition on the linear part, it preserves both \( A \) and \( B \). Thus the actions of \( \phi_1 \) on \( A \) and \( B \) defines a permutation \( \sigma \) of \( X \) preserving \( M \) by \( \phi_1(1_M + \delta_x) = 1_M + \delta_{\sigma(x)} \) for \( x \notin M \) and \( \phi_1(1_M - \delta_x) = 1_M - \delta_{\sigma(x)} \) for \( x \in M \). Thus \( \alpha_p(\sigma) \) and \( \phi_1 \) coincide on \( 1_M \) and \( A \cup B \), which together generate affinely \( \ell^p_M(X) \). Thus \( \phi_1 = \alpha_p(\sigma) \) and we deduce that \( \phi \in \alpha_p(\mathcal{S}(X, M)) \).

Finally, we see the metric properness as a consequence of the equality \( \| g1_M - h1_M \|^p = \ell^p_M(g, h) \). \( \square \)

**4.E. Boundedness and commensurability.** Note that if \( N \) is commensurable to \( M \), then \( d_M - d_N \) is bounded. In particular, the bornology on \( \mathcal{S}(X, M) \) defined by \( d_M \) is canonical.

The affine action gives a short proof of the following combinatorial result of Brailovsky, Pasechnik and Praeger [BPP]. Recall from the introduction that in a \( G \)-set \( X \), a subset \( M \) is **transfixed** if there is a \( G \)-invariant subset \( N \) commensurable to \( M \), i.e. satisfying \( \#(M \triangle N) < \infty \).
Theorem 4.16. Let $G$ be a subgroup of $S(X, M)$. Then $\ell_M(G)$ is bounded if and only if $M$ is transfixed.

Proof. Obviously if $N$ is commensurable to $M$ and $G$-invariant then $d_M$ is bounded by $2\#(M \triangle N)$ on $G$. Conversely, assume that $d_M$ is bounded on $G$. Then the action of $G$ on $\ell^2_M(X)$ has bounded orbits. By the center lemma, it has a fixed point $f$. Then, since $f$ is $G$-invariant, the subset $\{ f \geq 1/2 \}$ is $G$-invariant; moreover since $f \in \ell^2_M(X)$, this subset is commensurable to $M$. \qed

This provides an analogue of Corollary 4.18 for bounded cardinal definite functions, relaxing the cofinality hypothesis.

Corollary 4.17. Let $G$ be a topological group. Then every bounded cardinal definite function on $G$ is a finite sum of (bounded) cardinal definite functions associated to transitive actions of $G$, and cannot be written as an infinite sum of nonzero cardinal definite functions.

Proof. If $\ell = \ell_M$ is cardinal definite and bounded, then $M$ being transfixed, $M \cap V$ is invariant for all but finitely many orbits $V$, whence the result. \qed

Remark 4.18. Brailovsky, Pasechnik and Praeger [BPP] proved that if $\sup_G \ell_M < \infty$ then $G$ preserves a subset $N$ commensurable to $M$ (with an explicit but non-optimal bound on $\#(N \triangle M)$. An almost optimal result was subsequently provided by P. Neumann [Neu]: if $\sup_G \#(gM \setminus M) = m < \infty$, then there exists $N$ $G$-invariant with $\#(N \triangle M) \leq \max(0, 2m - 1)$.

This can be restated with only symmetric differences. First note that because of the existence of $N$, $gM \setminus M$ and $M \setminus gM$ have the same cardinality for all $g$. Therefore Neumann’s result can be restated as: if $\sup_G \ell_M = m < \infty$, then $m$ is even and there exists a $G$-invariant subset $N$ of $X$, commensurable to $M$ with $\#(N \triangle M) \leq \max(0, m - 1)$.

Remark 4.19. Let $s(m)$ be the optimal bound in the above result, so that Neumann’s result can be stated as: $s(m) \leq m - 1$ for all $m \geq 1$ and $s(m) = s(m - 1)$ for odd $m$, so we can focus on $s(m)$ for even $m$.

The inequality $s(m) \leq m - 1$ is maybe an equality for all even $m$. This holds when $m = 2^d \geq 1$ is a power of 2, taking $X$ to be the projective space $P^d(F_2)$ and $M$ a hyperplane, so $\#(M) = 2^d - 1$, $\#(X \triangle M) = 2^d$; if $G$ is any subgroup of $\text{GL}_{d+1}(F_2)$ transitive on $X$, then $\sup_{g \in G} \ell_M(g) = 2^d$ and the only $G$-invariant subsets $N$ are $\emptyset$ and the whole projective space, so the one minimizing $\#(M \triangle N)$ is $N = \emptyset$, which satisfies $\#(M \triangle N) = \#(M) = 2^d - 1$.

In general, write $m = \sum_{j \in J} 2^j$ (since $m$ is even, $J$ is a finite subset of the positive integers), define $X = \bigcup_{j \in J} P^j(F_2)$ and $M = \bigcup H_j$, where $H_j$ is a hyperplane in $P^j(F_2)$. Then $\#(M) = m - \#(J)$, $\sup_{g \in G} \ell_M(g) = m$, and the $G$-invariant subset $N$ minimizing $M \triangle N$ is $N = \emptyset$. So if $j_m$ is the number of digits 1 in the binary writing of $m$ (so $j_m \leq \log_2(m) + 1$), then we have $m - j_m \leq s(m) \leq m - 1$ for all $m \geq 1$ (thus Neumann’s upper bound is “asymptotically optimal”).
The left bound \( m - j_m \) is not always attained. For \( m = 6 \) (where \( m - j_m = 4 \)), consider the transitive action of \( G = X = \mathbb{Z}/10\mathbb{Z} \) on itself. Consider the subset \( M = \{0, 1, 2, 5, 7\} \). Then a direct verification shows that \( \#(M \cap g+M) \geq 2 \) for all \( g \), so \( \#(M \triangle g+M) \leq 6 \) for all \( g \); thus \( s(6) = 5 \). In general, I do not know if for odd \( n \), \( \mathbb{Z}/2n\mathbb{Z} \) always contains an \( n \)-element subset \( M \) such that \( \#(M \cap q+M) \geq (n - 1)/2 \) for all \( q \in \mathbb{Z}/2n\mathbb{Z} \). (For \( n = 9 \) the subset \( \{0, 1, 2, 4, 5, 9, 11, 15, 17\} \) works, thus \( s(10) = 9 \).)

4.F. **Induction.** Let \( G \) be a group and \( H \) a subgroup. Let \( X \) be a \( G \)-set. Endow \( G \times X \) with left and right commuting actions of \( G \) and \( H \) by

\[
g(g_0, x_0)h = (gg_0h, h^{-1}x_0),
\]

and define the (additive) induced action

\[
\text{ind}^G_H(X) = (G \times X)/H,
\]

which naturally inherits from the structure of a left \( G \)-set. For instance, for every subgroup \( L \) of \( H \), we have a natural identification \( \text{ind}^G_H(H/L) = G/L \).

Denote by \( \pi \) the projection \( G \times X \to (G \times X)/H \). Note that if \( F \) is a right transversal (so that \( G \) is set-wise the product \( FH \)) then \( \pi \) restricts to a bijection from \( F \times X \) to \( (G \times X)/H = \text{ind}^G_H(X) \). Also, note that \( \pi \) is injective in restriction to \( \{1\} \times X \), giving rise an \( H \)-equivariant embedding of \( X \) into \( \text{ind}^G_H(X) \).

**Lemma 4.20.** Suppose that \( G \) is a topological group, \( H \) is open in \( G \) and that \( X \) is a continuous discrete \( H \)-set. Then \( \text{ind}^G_H(X) \) is a continuous discrete \( G \)-set.

**Proof.** We need to show that the stabilizer in \( G \) of \( \pi(g_0, x_0) \) is open. An element \( g \in G \) belongs to this stabilizer if and only there exists \( h \in H \) such that \( g(g_0, x_0) = (g, x_0)h \), that is, \( h^{-1}x_0 = x_0 \) and \( g = g_0h^{-1} \). Thus the stabilizer of \( \pi(g_0, x_0) \) is equal to \( g_0Hx_0g_0^{-1} \), which is open in \( H \) and hence in \( G \). \( \square \)

**Proposition 4.21.** Assume that \( H \) has finite index in \( G \). Suppose that \( M \) is an \( H \)-commensurated subset of \( X \) and \( F \) is a right transversal of \( G \) modulo \( H \), with \( 1 \in F \). Identify \( M \) to \( \pi(\{1\} \times M) \) and define \( M' = \bigcup_{f \in F} fM \). Then \( M' \) is commensurated by the \( G \)-action and \( M' \cap X = M \). In particular, the restriction of \( \ell_{M'} \) to \( H \) is \( \geq \ell_M \).

If \( G \) is a topological group, \( H \) is open and \( M \) has an open stabilizer in \( H \), then \( M' \) has an open stabilizer in \( G \).

**Proof.** Fix \( g \in G \). For every \( f \in F \), there exists a unique \( f' \in F \) such that \( gf \in f'H \). Write \( gf = f'h_f. \) Then \( gfM = f'h_fM \subset f'M \cup (M \triangle h_fM) \), where

\[\text{Ind}^G_H(X) = \{\xi : G \to X \mid \forall h \in H, g \in G, \xi(gh) = h^{-1} \cdot \xi(g)\},\]

where \( G \) acts by \( g \cdot \xi(x) = \xi(g^{-1}x) \), which is notably used in representation theory; we do not consider it here.
\( M \triangle h_f M \) is finite. Thus, using that \( F \) is finite, \( gM' \setminus M' \) is finite for all \( g \in G \). Since \( G \) is a group, it follows that \( M' \triangle gM' \) is finite for all \( g \).

Since \( 1 \in F \), we have \( M' = M \cup \bigcup_{f \in F \setminus \{1\}} fM \), while if \( f \notin H \) we have \( fM \cap X = \emptyset \). Hence \( M' \cap X = M \).

If \( L \) is the stabilizer of \( M \) in \( H \), then \( L \) is open by assumption. We then see that \( M' \) is stabilized by \( \bigcap_{f \in F} fLf^{-1} \), which is open. \( \square \)

**Proposition 4.22.** Let \( G \) be a topological group and \( N \) a normal subgroup. Let \( \ell \) be a cardinal definite function on \( G \) whose restriction to \( N \) is bounded. Then there exists a cardinal definite function \( \bar{\ell} \) on \( G/N \) such that, denoting by \( \pi : G \to G/N \) the natural projection, we have \( \ell - \bar{\ell} \circ \pi \) is bounded.

**Lemma 4.23.** Let \( G \) be a topological group and \( N \) a normal subgroup. Let \( \ell \) be a cardinal definite function on \( G \) vanishing on \( N \) (and hence \( N \)-invariant). Then the resulting cardinal definite function \( \bar{\ell} \) on \( G/N \) is cardinal definite.

**Proof.** Note that the function \( \bar{\ell} \) on \( G/N \) is continuous by definition of the quotient topology.

Let \( X \) be a continuous discrete \( G \)-set and \( M \subset X \) a \( G \)-commensurated subset with open stabilizer such that \( \ell_M = \ell \). We begin by the case when \( X \) is \( G \)-transitive. Let \( X' \) be the quotient of \( X \) by the \( N \)-action and \( M' \) the image of \( M \) in \( X' \). Then all fibers of \( X \to X' \) have the same cardinal \( \alpha \), and the inverse image of \( gM' \triangle M' \) is \( gM \triangle M \). In follows that either \( M \) is \( G \)-invariant (in which case \( \ell = 0 \) and there is nothing to prove), or that \( \alpha \) is finite. Then \( \ell_M = \alpha \ell_{M'} \).

Since the action on \( X' \) factors through a continuous action of \( G/N \), we see that \( \ell_{M'} \) is cardinal definite on \( G/N \), and hence \( \bar{\ell} = \alpha \ell_{M'} \) is cardinal definite on \( G/N \) as well (by Lemma 4.24).

In general, assume \( X \) arbitrary. Decompose \( X \) into \( G \)-orbits as \( X = \bigcup X_i \), yielding a decomposition \( \ell = \sum \ell_i \). Since \( M \cap X_i \) is \( N \)-invariant, \( \ell_i \) is \( N \)-invariant and hence factors, by the transitive case, \( \bar{\ell}_i \). Then since \( \bar{\ell} = \sum \bar{\ell}_i \) is finite and continuous, it is cardinal definite by Lemma 4.24. \( \square \)

**Proof of Proposition 4.22.** Let \( X \) be a continuous discrete \( G \)-set and \( M \subset X \) a \( G \)-commensurated subset with open stabilizer such that \( \ell_M = \ell \). By Theorem 4.16, \( M \) is commensurable to an \( N \)-invariant subset \( M' \). Then \( \ell_{M'} \) factors through a cardinal definite function on \( G/N \) by Lemma 4.23 proving the proposition. \( \square \)

4.G. Wreath products. If \( H \) is a discrete group, \( G \) is a topological group, \( Y \) is a continuous discrete \( G \)-set, the wreath product \( H \wr_Y G \) is by definition the semidirect product \( H^{(Y)} \rtimes G \), where \( G \) acts by shifting the direct sum (or restricted direct product) \( H^{(Y)} = \bigoplus_{y \in Y} H \). Since the action of \( G \) on the discrete group \( H^{(Y)} \) is continuous, this semidirect product is a topological group with the product topology.

There is a trivial way to define commensurating actions of \( H \wr_Y G \) out of commensurating actions of \( H \). Let \( H \) act on a set \( X \), commensurating a subset
Then $H \wr Y G$ acts on $X \times Y$, where the action of the $y$-th summand of $H$ in $H^Y$ is the given one on $X \times \{y\}$ and is the trivial action on $X \times (Y \setminus \{y\})$, and the action of $G$ permutes the components. Note that this action is continuous, the stabilizer of a point $(x, y)$ being the open subgroup $H_{Y \setminus \{y\}} H_y G_y$. This action commensurates $M \times Y$, which has an open stabilizer (as it contains the open subgroup $G$), and the length is given by

$$\ell_{M \times Y}((h_y)_{y \in Y} g) = \sum_{y \in Y} \ell_M(h_y).$$

Interestingly, this length is usually unbounded even if $\ell_M$ is bounded. For record:

**Proposition 4.24.** For every cardinal definite function $\ell$ on $H$, the function $(h_y)_{y \in Y} g \mapsto \sum_{y \in Y} \ell_M(h_y)$ is cardinal definite on $H \wr Y G$. In particular, the function $hg \mapsto 2\#(\text{Supp}(h))$ is cardinal definite on $H \wr Y G$.

**Proof.** The second statement is the particular case where $\ell = 21_{H \setminus \{1\}}$. It is indeed cardinal definite, associated to the left action of $H$ on itself and the commensurated subset $M = \{1\}$. \hfill \Box

We now proceed to describe another more elaborate construction due to the author, Stalder and Valette [CSVa]. The construction is described in [CSVa] in terms of wallings so we need to translate it into commensurating actions. We here deal with a standard wreath product $H \wr G$ (i.e., $G$ is discrete and $Y$ is $G$ with the left action by translation).

Start from a $G$-set $X$ with a commensurated subset. For $x \in X$, define $A_x = \{h \in G \mid x \in hM\}$. Let $Z_X$ be the set of pairs $(x, \mu)$, where $x \in X$ and $\mu$ is a finitely supported function from the complement $A_x^c$ to $H$. Let $H \wr G$ act on $Z_X$ as follows:

$$g \cdot (x, \mu) = (gx, g \cdot \mu); \quad \lambda \cdot (x, \mu) = (x, \lambda|_{A_x^c} \mu),$$

where $g \cdot \mu(\gamma) = \mu(g^{-1}\gamma)$. Define $N = M \times \{1\} \subset Z_X$.

**Proposition 4.25.** The subset $N$ of $Z_X$ is commensurated by the $G$-action and we have the following two lower bounds

$$\ell_N(wg) \geq \#(M \triangle gM); \quad \ell_N(wg) \geq \sup_{\gamma \in \text{Supp}(w)} \#(M \setminus \gamma M)$$

and the upper bound

$$\ell_N(wg) \leq \#(M \triangle gM) + \sum_{\gamma \in \text{Supp}(x)} \#(M \setminus \gamma M) + \sum_{\gamma \in g^{-1}\text{Supp}(x)} \#(M \setminus \gamma M).$$

**Proof.** This actually follows from Proposition 3.2 and the verifications in [CSVa], but it is instructive to provide a direct proof.

For $w \in H^G$ and $g \in G$, let us describe $N \setminus wgN$. Elements $(x, \mu)$ in this set satisfy $x \in M$, $\mu = 1$, and also $g^{-1}w^{-1}(x, 1) \notin N$. The latter condition means
Let us also observe that if $X$ is $G$-transitive, then $Z_X$ is $(H \wr G)$-transitive, and if $L \subset G$ is the stabilizer of $x_0 \in X$, then the stabilizer of $(x_0, 1) \in Z_X$ is $H^{(A_{x_0})} L$.

5. **Property FW etc.**

**Definition 5.1.** Let $G$ be a topological group. We say that $G$ has **Property FW** if for every continuous discrete $G$-set, any commensurated subset with open stabilizer is transfixed (i.e., is commensurable to an invariant subset).

By Theorem 4.16, this amounts to saying that every cardinal definite (Definition 4.3) function on $G$ is bounded. This allows the following generalization: if $L \subset G$, we say that $(G, L)$ has relative **Property FW** if every cardinal definition function on $G$ is bounded on $L$. In case $L$ is a subgroup, Theorem 4.16 shows that this means that for every continuous discrete $G$-set $X$ and commensurated subset $M$ with open stabilizer, $M$ is transfixed in restriction to $L$.

If in Definition 5.1 we restrict to transitive actions, we get the following a priori weaker notion.

**Definition 5.2.** Let $G$ be a topological group. We say that $G$ has **Property FW’** if for every continuous discrete transitive $G$-set, any commensurated subset is either finite or cofinite.

The negation of Property FW’ is also known as “semisplittable”.

We use the notion of topological groups with uncountable cofinality (or cofinality $\neq \omega$) from §4.B; important examples of such groups are compactly generated locally compact groups and in particular finitely generated discrete groups.

**Proposition 5.3.** Let $G$ be a topological group.

(1) If $G$ has uncountable cofinality, then $G$ has **Property FW’** if and only if $G$ has **Property FW’**;

(2) if $G$ has countable cofinality, then $G$ does not have **Property FW’**.
Proof. The first part immediately follows from Proposition 4.7. For the second, if $G = \bigcup G_n$ with $(G_n)$ an nondecreasing union of proper open subgroups, then if $T = \bigsqcup G/G_n$ is endowed with the natural $G$-action and $x_n$ is its base-point, then $\{g_n | n \geq 0\}$ is commensurated but not transfixed.

Remark 5.4. If $G$ has countable cofinality (e.g. is discrete, infinitely generated and countable), it does not have Property FW by Proposition 5.3(2), while $G$ may have either Property FW’ or not. For instance, no infinite countable locally finite group has Property FW’ [Coh] (using the action of $G$ on itself), while the group $\text{SL}_n(\mathbb{Q})$ has Property FW’ for all $n \geq 3$ [Cor2].

5.B. Features of Property FW.

Proposition 5.5. Let $G$ be a topological group and $H$ an open subgroup of finite index. Then $G$ has Property FW if and only if $H$ has Property FW.

Proof. We begin by the easier implication. If $H$ has Property FW and $\ell$ is a cardinal definite function on $G$, write $G = FH$ with $F$ finite; if $m$ is an upper bound for $\ell$ on $F \cup H$ then $2m$ is an upper bound for $\ell$.

Conversely suppose that $G$ has Property FW and let $\ell$ be a cardinal definite function on $H$. Then by Proposition 4.21 (which uses additive induction of actions), there exists a cardinal definite function $\ell'$ on $G$ such that $\ell'|_H \geq \ell$. By Property FW, $\ell'$ is bounded and hence $\ell$ is bounded.

Corollary 5.6. For every finitely generated group $\Gamma$ with Property FW, every finite index subgroup of $\Gamma$ has a finite abelianization.

Property FW is obviously stable under taking quotients. The following proposition shows it is also stable by taking extensions.

Proposition 5.7. Let $G$ be a topological group and $N$ a normal subgroup. Suppose that $(G, N)$ has relative Property FW and $G/N$ has Property FW. Then $G$ has Property FW.

Proof. Let $\ell$ be a cardinal definite function on $G$. Then by relative Property FW, $\ell$ is bounded on $N$. Hence by Proposition 4.22, there exists a function $\ell'$ on $G$ factoring through a cardinal definite function on $G/N$ such that $\ell - \ell'$ is bounded. By Property FW for $G/N$, $\ell'$ is bounded. So $\ell$ is bounded as well.

Proposition 5.8. Let $G$ be a topological group, $Y$ an infinite continuous discrete $G$-set and $H$ a nontrivial discrete group. Then the wreath product $H \wr_Y G$ does not have Property FW. In particular, if $H, G$ are discrete groups with $H$ nontrivial and $G$ infinite then $H \wr_Y G$ does not have Property FW.

Proof. By Proposition 4.24, the function $wg \mapsto 2\#(\text{Supp}(w))$ is cardinal definite on $H \wr_Y G$; it is unbounded as soon as $Y$ is infinite. (The proof of Proposition 4.24 also shows that if $Y$ has an infinite $G$-orbit then $H \wr_Y G$ does not have Property FW'.)
Proposition 5.8 was obtained for standard wreath products with a similar argument in [CMV, Theorem 3], although claiming a weaker statement.

**Remark 5.9.** A more careful look at the proof of Proposition 5.8 (see the stabilizer computations before Proposition 4.24) shows that if $H$ is a nontrivial finitely generated discrete group, $G$ is a compactly generated locally compact group and $y \in Y$ has an infinite $G$-orbit, then $H^{(Y \setminus \{y\})}G_y$ is coforked in $H \wr y G$. In particular, if $G$ is an infinite finitely generated discrete group, then $H^{(G \setminus \{1\})}$ is coforked in the standard wreath product $H \wr G$.

We now recall some geometric rigidity properties

- A topological group is strongly bounded (SB) if every continuous isometric action on a metric space has bounded orbits, or equivalently if every continuous subadditive nonnegative real-valued function is bounded (strongly bounded is sometimes called Bergman Property, strong Bergman Property, strong uncountable cofinality).
- A topological group has Property FH if for every continuous isometric action on a Hilbert space there is a fixed point, or equivalently (by the center lemma) orbits are bounded. For $\sigma$-compact locally compact groups, the Delorme-Guichardet Theorem [BHV, §2.12] states that Property FH is equivalent to Kazhdan’s Property T (defined in terms of unitary representations).
- A topological group has Property FA if for every continuous isometric action on the 1-skeleton of a tree there is a fixed point, or equivalently (by the center lemma) orbits are bounded. By Bass-Serre Theory [Ser], a topological group has Property FA if and only if it satisfies the following three conditions
  - it has no continuous homomorphism onto $\mathbb{Z}$;
  - it has no decomposition as a nontrivial amalgam over open subgroups;
  - it has uncountable cofinality (as a topological group).
- Cofinality $\neq \omega$ can also be characterized as: every continuous isometric action on an ultrametric space has bounded orbits.

**Proposition 5.10.** For a topological group $G$, we have the following implications

$$\text{SB} \Rightarrow \text{FH} \Rightarrow \text{FW} \Rightarrow \text{FA} \Rightarrow \text{cofinality} \neq \omega$$

**Proof.** The left implication is trivial. The last implication is due to Serre [Ser, §6.1]: let $(G_n)$ be a nondecreasing sequence of subgroups whose union is $G$ and define $T$ as the $G$-set given as the disjoint union $\bigsqcup G/G_n$; endow it with the graph structure joining any $g \in G/G_n$ to its image in $G/G_{n+1}$. Then $T$ is an unbounded tree on which $G$ acts transitively by automorphisms. So $G$ does not have Property FA.

The implication FH$\Rightarrow$FW is a consequence of Proposition 4.15 (for $p = 2$).

For the implication FW$\Rightarrow$FA, assume that $G$ has Property FW and let $G$ act continuously on a tree. Consider the action of $G$ on the set $X$ of oriented
edges. This action is continuous, as the stabilizer of a given oriented edge is the pointwise stabilizer of a pair of vertices. Fix a vertex \( x_0 \) and let \( M \) be the set of edges pointing towards \( x_0 \). Then the stabilizer of \( M \) is open, since it contains the stabilizer of \( x_0 \). Moreover, \( M \) is commensurated and \( \ell_M(g) = d(x_0, gx_0) \) for all \( g \in G \). By Property FW, \( \ell_M \) is bounded. Thus the orbit of \( x_0 \) is bounded and hence there is a fixed point in the 1-skeleton. \( \square \)

**Remark 5.11.** The implications of Proposition 5.10 are not equivalences, even for countable discrete groups. Let us begin by the easiest:

- (uncountable cofinality) \( \not\Rightarrow \) FA: \( \mathbb{Z} \) is a counterexample;
- FH \( \not\Rightarrow \) SB: consider any infinite discrete (finitely generated) group with Property T, e.g. \( \text{SL}_3(\mathbb{Z}) \);
- That FA \( \not\Rightarrow \) FW is now well-known; this was first mentioned in [ChN, Example 2]. See Example 5.12;
- FW \( \not\Rightarrow \) FH: a counterexample is \( \text{SL}_2(\mathbb{Z}[1/2]) \), see Example 5.4 (although the existence of such an example is by no ways surprising, it does not seem to appear anywhere in the literature).

**Example 5.12.** Let us provide two types of finitely generated groups with Property FA but not Property FW (the first examples appeared in [Ser, Example 2.5]).

1. Finitely generated groups with Property FA but without hereditary Property FA, i.e. with a finite index subgroup without Property FA. By Proposition 5.5, such groups do not have Property FW. There are a lot of such groups. For instance, any infinite finitely generated Coxeter group whose Coxeter graph has no \( \infty \)-label has this property. This includes the examples in [ChN, Example 2] (which are lattices in the group of isometries of the Euclidean plane). Such groups (as all finitely generated Coxeter groups) actually have Property PW [BJS], and have Property FA, as any group generated by a finite set \( S \) such that \( S \cup S^2 \) consists of torsion elements [Ser, Corollaire 2, p. 90]. Another elementary example is the following: let \( D \) be a finitely generated group with finite abelianization with a finite index subgroup \( D_1 \), with infinite abelianization (e.g. the infinite dihedral group) and \( F \) a nontrivial finite group. Then the standard wreath product \( D \wr F \) has Property FA by the general criterion of the author and A. Kar [CoK, Theorem 1.1]; besides it admits \( D_1^F \) as a subgroup of finite index with infinite abelianization and thus failing to have Property FA. Also, if \( D \) has finite abelianization but splits as a nontrivial amalgam, then \( D \wr F \) has Property FA again by [CoK, Theorem 1.1] but admits the finite index subgroup \( D^F \), which fails to have Property FA. In case every finite index subgroup of \( D \) has a finite abelianization (e.g. \( D \) is the free product of two infinite finitely generated simple groups), \( D \wr F \) is then an example of a finitely generated group with Property FA,
without hereditary Property FA but all of whose finite index subgroups have trivial abelianization.

(2) Finitely generated groups with hereditary Property FA and without Property FW. For instance, if $B$ is any nontrivial finitely generated group with finite abelianization (e.g. a nontrivial finite group) and if $\Gamma$ is any infinite finitely generated group with hereditary Property FA (e.g. with Property FW), then by the criterion of \[\text{CoK}, \text{Theorem 1.6}\], the standard wreath product $B \wr \Gamma$ has hereditary Property FA, while it does not have Property FW by Proposition 5.8. Other examples with hereditary Property FA and without Property FA are Grigorchuk’s groups $\text{Gr}$, as well as topological full groups of minimal Cantor systems (see \[2.7\]).

Remark 5.13. Proposition 5.3(1) can be extended to relative versions. Namely, if $L \subset G$ is any subset and $G$ has cofinality $\neq \omega$, then relative Property FW for $(G, L)$ can be tested on transitive actions of $G$.

Also, Proposition 5.10 extends to the relative case: just define relative Property FH, FA, cofinality $\omega$ by saying that every continuous isometric action of $G$ on a Hilbert space (resp. tree, resp. ultrametric space) is bounded in restriction to $L$.

Concerning Proposition 5.5 it is obvious that relative Property FW for $(G, L)$, when $L$ is a subgroup, does not change if $L$ varies in its group commensurability class. Also, its proof shows that, for any open subgroup $H$ of $G$ and subset $L$ of $H$, $(G, L)$ has relative Property FW if and only $(H, L)$ has Property FW.

Proposition 5.7 also extends the following relative version: if $(G, N)$ has relative Property FW, if $L$ is an $N$-invariant subset of $G$ and $(G/N, L/N)$ has relative Property FW then $(G, L)$ has relative Property FW.

5.C. Property PW.

Definition 5.14. A locally compact group $G$ has Property PW if it has a proper cardinal definite function.

Observe that if a locally compact group $G$ has both Properties PW and FW then it is compact. More generally, if it has Property PW and $(G, L)$ has relative Property FW for some subset $L$, then $G/L$ is compact. Also observe that from the bare existence of a proper continuous real-valued function, every locally compact group with Property PW is $\sigma$-compact.

Observe that Property PW for locally compact group is stable by taking closed subgroups.

Proposition 5.15. Let $G$ be a locally compact group and $H$ an open subgroup of finite index. Then $G$ has Property PW if and only if $H$ has Property PW.

Proof. Leaving aside the trivial implication, assume that $H$ has Property PW. Let $\ell$ be a proper cardinal definite function on $H$. By Proposition 4.21 there exists a cardinal definite function $\ell'$ on $G$ with $\ell'|_H \geq \ell$. In particular, $\ell'|_H$ is proper. Now it is true for an arbitrary length function that properness on an open
finite index subgroup implies properness; thus $\ell'$ is proper and $G$ has Property PW. □

Let us now mention the following slight generalization of the main result from [CSVa]. There, $H$ was assumed finite, but the easy trick to carry over arbitrary $H$ was used in [CSVb] in a similar context.

**Theorem 5.16.** Let $H, G$ be discrete groups with Property PW. Then the standard wreath product \( H \wr G \) has Property PW.

**Proof.** Start from an action of $G$ on a set $X$ with a commensurated subset $M$ such that $\ell_M$ is proper, and consider the action on $X' \times \{0, 1\}$ with commensurated subset $M' = M \times \{0\} \cup M^c \times \{1\}$. Then perform, out of the latter action, the construction $Z_{X'}$ of Proposition 4.25 to obtain a cardinal definite function $\ell$ on $H \wr G$ satisfying

$$\ell(wg) \geq \sup_{\gamma \in \{g\} \cup \text{Supp}(w)} \ell_M(\gamma).$$

By properness of $\ell_M$, for every $n$ the subset $F_n = \ell^{-1}_M([0, n])$ of $G$ is finite. By the above inequality, for every $wg$ such that $\ell(wg) \leq n$, we have $\{g\} \cup \text{Supp}(w) \subset F_n$. (Note that in case $H$ is finite, this shows that $\ell$ is proper.)

Now let us also use that $H$ has Property PW, let $\ell_0$ be a proper cardinal definite function on $H$; by Proposition 4.24, the function $wg \mapsto \ell'(wg)$, defined as $\ell'(wg) = \sum_{\gamma \in G} \ell_0(h_\gamma)$, is cardinal definite. Define $F'_n = \ell^{-1}_0([0, n])$; this is a finite subset of $H$.

Defining $\ell_1 = \ell + \ell'$, if $\ell_1(wg) \leq n$, then $\ell(wg) \leq n$ so $\{g\} \cup \text{Supp}(w) \subset F_n$ by the above, and $\ell'(wg) \leq n$, so $w_\gamma \in F'_n$ for all $\gamma \in G$. Thus $w \in (F'_n)^{F_n}$ and $g \in F_n$, which leaves finitely many possibilities for $wg$. Thus the cardinal definite function $\ell_1$ is proper. □

### 5.D. Link with other properties

Let us check the equivalences of the introduction for Property FW and PW. The following proposition summarizes several of the previous results.

**Proposition 5.17.** Let $G$ be a topological group and $f : G \to \mathbb{R}$ a function. Equivalences:

1. $f$ is cardinal definite
2. There exists continuous action of $G$ on a CAT(0) cube complex and a vertex $x_0$ such that $f(g) = d(x_0, gx_0)$ for all $g$;
3. There exists continuous action of $G$ on median graph and a vertex $x_0$ such that $f(g) = d(x_0, gx_0)$ for all $g$;
4. There exists a topological space $Y$ with a continuous $G$-action (i.e., the map $G \times Y \to Y$ is continuous), a continuous discrete $G$-set $X$, a $G$-walling $\mathcal{W} = (W_x)_{x \in X}$ of $Y$ by clopen subsets, and $y_0 \in Y$ such that $f(g) = d_\mathcal{W}(y_0, gy_0)$ for all $g$. 


(V) there exists an isometric action on an “integral Hilbert space” \( \ell^2(X, \mathbb{Z}) \) (\( X \) any discrete set), or equivalently on \( \ell^2(X, \mathbb{R}) \) preserving integral points and \( v_0 \in \ell^2(X, \mathbb{Z}) \) such that \( f(g) = \|v_0 - gv_0\|_1 \) for all \( g \in G \).

Proof. The equivalence \( \mathbf{(I)} \iff \mathbf{(III)} \) follows from Corollary 3.27.

The equivalence \( \mathbf{(II)} \iff \mathbf{(III)} \) follows from Chepoi’s result (Theorem 3.35) that median graphs are precisely the 1-skeleta of CAT(0) cube complexes.

The implication \( \mathbf{(I)} \implies \mathbf{(IV)} \) follows from Proposition 3.6 by taking \( Y = G \). Let us show the converse: assume \( \mathbf{(IV)} \) and consider a \( G \)-wallowing \( W \) on \( Y \) and \( y_0 \in Y \) as in \( \mathbf{(IV)} \). Fix \( y_0 \in Y \) and pull the walling back to a walling \( W' \) on \( G \) by the orbital map \( g \mapsto gy_0 \). Then for all \( g \in G \) we have \( d_{W'}(1, g) = d_W(y_0, gy_0) \). By Proposition 3.6 this is a cardinal definite function.

Let us now consider \( \mathbf{(V)} \). First note that since the closed affine subspace generated by \( \ell^2(X, \mathbb{Z}) \) is \( \ell^2(X, \mathbb{R}) \), by the GNS construction isometries of \( \ell^2(X, \mathbb{Z}) \) have a unique extension to \( \ell^2(X, \mathbb{R}) \), which is affine by the Mazur-Ulam theorem. In particular, both versions of \( \mathbf{(V)} \) are equivalent.

We have \( \mathbf{(V)} \implies \mathbf{(III)} \), by observing that the obvious graph structure on \( \ell^2(X, \mathbb{Z}) \), obtained by joining any two points at distance 1, is connected median (it also implies \( \mathbf{(II)} \) directly).

The implication \( \mathbf{(I)} \implies \mathbf{(V)} \) is provided by the Niblo-Roller construction, as formulated in Proposition 4.15. \( \square \)

Lemma 5.18 (Gerasimov). If a group action on a CAT(0) cube complex has bounded orbits, then it has a fixed point.

Proof. By Theorem 3.28 there is a finite orbit of vertices. The convex hull of this finite orbit is a CAT(0) cube complex with finitely many cells; we can then apply the Bruhat-Tits fixed point lemma. \( \square \)

Lemma 5.19. Let \( G \) be a topological group and \( X \) a continuous discrete \( G \)-set. Let \( A \) be an arbitrary discrete nonzero abelian group. Suppose that \( H^1(G, A^{(X)}) = 0 \). Then every \( G \)-commensurated subset of \( X \) with open stabilizer is transfixed.

Proof. Let \( M \) a commensurated subset with an open stabilizer; fix \( a \in A \setminus \{0\} \). For \( N \subset X \), and \( x \in X \), define \( 1^a_M(x) \) as equal to \( a \) if \( x \in N \) and 0 otherwise.

Then \( g \mapsto 1^a_M - 1^a_{gM} \) is a continuous 1-cocycle of \( G \) in \( A^{(X)} \) and hence by the vanishing of \( H^1(G, A^{(X)}) \) is a 1-coboundary, i.e. has the form \( f - gf \) for some \( f \in A^{(X)} \). Hence the function \( h = 1^a_M - f \) is \( G \)-invariant. So \( \{x \in X : h(x) = a\} \) is \( G \)-invariant and since \( f \) has finite support, it is commensurable to \( M \). \( \square \)

Proposition 5.20. Let \( G \) be a topological group. Equivalences:

(i) \( G \) has Property FW;
(ii) every cardinal definite function on \( G \) is bounded;
(iii) every continuous cellular action on any CAT(0) cube complex has bounded orbits (we allow infinite-dimensional cube complexes);
(iv) every continuous cellular action on any CAT(0) cube complex has a fixed point;
(v) every continuous action on a median graph has bounded orbits;
(vi) every action on a nonempty median graph has a finite orbit;
(vii) (if \( G \) is compactly generated and endowed with a compact generating subset) for every open subgroup \( H \subset G \), the Schreier graph \( G/H \) has at most 1 end;
(viii) for every topological space \( Y \) with a continuous \( G \)-action (i.e., the map \( G \times Y \to Y \) is continuous), every continuous discrete \( G \)-set \( X \) and every \( G \)-walling \( W = (W_x)_{x \in X} \) of \( Y \) by clopen subsets, the \( G \)-orbits in \( (Y,d_W) \) are bounded.
(ix) every isometric action on an “integral Hilbert space” \( \ell^2(X,Z) \) (\( X \) any discrete set), or equivalently on \( \ell^2(X,R) \) preserving integral points, has bounded orbits.
(x) for every continuous discrete \( G \)-set \( X \) we have \( H^1(G,ZX) = 0 \).

Proof. Proposition \ref{5.17} immediately entails the equivalences (iii) \( \iff \) (vi) \( \iff \) (viii) \( \iff \) (ix).

One direction in the equivalence (i) \( \iff \) (ii) is trivial and the other follows from Theorem \ref{4.16}.

The equivalence (ii) \( \iff \) (vii) follows from Proposition \ref{3.10}.

One direction in the equivalence (v) \( \iff \) (vi) is trivial and the other follows from Theorem \ref{3.28}

(iv) \( \iff \) (iii) is trivial and the converse follows from Lemma \ref{5.18}.

Let us finally prove (i) \( \iff \) (x). The implication \( \Rightarrow \) follows from Lemma \ref{5.19}.

Conversely, assume that \( G \) has Property FW and let \( X \) be a continuous discrete \( G \)-set. Any continuous cocycle of \( G \) in \( ZX \) defines a continuous affine action of \( G \) on the discrete abelian group \( ZX \), which lifts the projection \( ZX \rtimes G \to G \). Note that the action of \( ZX \rtimes G \) on \( ZX \) is generated by the linear action of \( G \) and the action by translations. In particular, it preserves the oriented median connected graph structure of \( ZX \) (Examples \ref{3.16} and \ref{3.25}). Hence by Property FW and Corollary \ref{3.38}, the affine action of \( G \) fixes an element of \( ZX \). This means that it is defined by a 1-coboundary. Hence \( H^1(G,ZX) \).

Remark 5.21. In Proposition \ref{5.20} (x), the ring \( Z \) cannot be replaced by an arbitrary unital nonzero ring. While the proof carries over to show that if \( R \) is an arbitrary unital nonzero ring, \( X \) is a discrete continuous \( G \)-set with a commensurated subset \( M \) with an open stabilizer, \( H^1(G,RX) = 0 \) implies that \( X \) is transfixed, the converse does not always hold. More precisely, consider \( n \geq 2 \), the group \( G = Z/nZ \) and the \( G \)-set \( X = \{ x \} \) reduced to a singleton. Since \( G \) is finite, it has property FW. On the other hand, it is straightforward that for \( R = Z/nZ \) we have \( H^1(G,RX) \) nonzero (since it consists of the group of group endomorphisms of \( Z/nZ \)).

The 1-cohomology of \( G \) in \( ZX \) has been studied in \cite{Ho78}.
Note that Proposition 5.17 immediately implies the equivalence between Property PW and its various reformulations given in the introduction.

**Remark 5.22.** For a topological group $G$, Property FW is also equivalent to the property of having bounded orbits on space with walls (in its most popularized sense, not allowing multiplicities of walls). Obviously it implies this property; to see the converse, assume that $G$ does not have Property FW. Then it has an unbounded cardinal definite function $f$. According to an orbit decomposition, write $f = \sum_{i \in I} n_i f_i$, where $(n_i)$ is a family of positive integers and $(f_i)$ is an injective family of nonzero cardinal definite functions. Define $f' = \sum_{i \in I} f_i$. Then $f' \leq f$ and $f'$ is cardinal definite. We claim that $f'$ is unbounded as well. If $I$ is finite, then $f \leq nf'$ where $n = \max_i n_i$, so this is clear. If $I$ is infinite, we evoke the contraposition of Corollary 4.17 to infer that $f'$ is unbounded.

Note that we assume above that no $f_i$ is a proper multiple, i.e. has the form $kf''$ with $k \geq 2$ integer and $f''$ cardinal definite (otherwise we decompose again).

Now write $f_i = \ell M_i$, where $M_i$ is a commensurated subset in a set $X_i$ with open stabilizer, if $(W_i(x))_{x \in X_i}$ is the associated walling on $G$ given by Proposition 3.2. Since $f_i$ is a not proper multiple, the walling $x \mapsto W_i(x)$ is injective. Also since the $f_i$ are distinct, we have $W_i(x) \neq W_j(y)$ for any $i \neq j$, $x \in X_i$ and $y \in X_j$. So the structure of space with walls on $G$ for which the walls are the $W_i(x)$ for $i \in I$ and $x \in X_i$ (and their complement if necessary) gives rise to an unbounded distance on $G$.

**Remark 5.23.** There is an analogue of Property FW and its cousins for actions by group automorphisms. Namely, a group $G$ has Property FG if whenever it acts by automorphisms on a discrete group $H$ commensurating (in the group sense) a subgroup $P$ also commensurated by $H$, there exists a subgroup $P'$ commensurable to $P$ and invariant by $G$ (there is an obvious relative version). The existence of $P'$ can also be characterized by the boundedness of the length $L_P(g) = \log([P : P \cap g(P)]/[g(P) : P \cap g(P)])$, by a result of Schlichting (rediscovered by Bergman and Lenstra [BLe]). By a projective limit construction [ShW], these properties can also be characterized in terms of actions by topological automorphisms on totally disconnected locally compact group, namely by the condition that every such action preserves a compact open subgroup. Property FG implies Property FW (since any action on a set $X$ commensurating a subset $M$ induces an action on the group $C^{(X)}$ commensurating the subgroup $C^{(M)}$, where $C$ is a 2-element group), but the converse does not hold, since $\text{SL}_3(\mathbb{Z}[1/2])$ does not have Property FG, as we see by using its action by conjugation on $\text{SL}_3(\mathbb{Q}_2)$. In particular, Property FG does not follow from Property T. On the other hand, by [ShW, Theorem 1.3], $\text{SL}_d(\mathbb{Z})$ for $d \geq 3$ as well as many other non-uniform lattices have Property FG; the argument also carrying over $\text{SL}_2(\mathbb{Z}[\sqrt{2}])$. The only source of Property FG I know uses distortion of abelian subgroups. I plan to write more on Property FG and the similarly defined Property PG in a subsequent paper.
6. CARDINAL DEFINITE FUNCTION ON ABELIAN GROUPS AND APPLICATIONS

6.A. Cyclic groups. Let $\mathbb{N}$ be the set of nonnegative integers.

**Proposition 6.1.** Let $\ell$ be an unbounded cardinal definite function on $\mathbb{Z}$ associated to a transitive $\mathbb{Z}$-set $X$. Then there exists a bounded function $b : \mathbb{Z} \to \mathbb{N}$ such that we have $\ell(n) = |n| + b(n)$ for all $n \in \mathbb{Z}$.

**Proof.** Since $\ell = \ell_M$ is unbounded, $X$ is infinite and hence we can identify $X$ to $\mathbb{Z}$. Then $M \subset \mathbb{Z}$ is commensurated by translations and hence has a finite boundary in the standard Cayley graph of $\mathbb{Z}$. It follows that $\ell(n) = |n| + b(n)$ with $b : \mathbb{Z} \to \mathbb{Z}$ bounded. We claim that $b(\mathbb{Z}) \subset \mathbb{N}$; indeed, for all $k \geq 1$ and $n$, we have, by subadditivity $|kn| + b(kn) = \ell(kn) \leq k\ell(n) = k|n| + kb(n)$, whence, dividing by $k$ we obtain $0 \leq b(n)$. \qed

Let $G$ be a compactly generated locally compact group and $Z$ an infinite discrete cyclic subgroup. Let $| \cdot |$ be the word length in $G$ with respect to a compact generating subset. If $g$ is a generator of $Z$, the limit $\lim_{n \to \infty} |g^n|/n$ exists; $Z$ is called distorted if this limit is zero and undistorted otherwise. This does not depend on the choice of the compact generating subset of $G$.

Let us also say that $G$ has uniformly undistorted discrete cyclic subgroups (the word “discrete” can be dropped if $G$ is discrete) if $\inf_{g} \lim_{n \to \infty} |g^n|/n$, where $g$ ranges over non-elliptic elements, i.e., generators of infinite discrete cyclic subgroups.

The following corollary was first obtained indirectly by F. Haglund by studying the dynamics of isometries of CAT(0) cube complexes.

**Corollary 6.2.** Let $G$ be a compactly generated locally compact group and $Z$ an infinite discrete cyclic subgroup. If $Z$ is distorted then $(G, Z)$ has relative Property FW. In particular, if $G$ has Property PW then $Z$ is undistorted; actually $G$ then has uniformly undistorted cyclic subgroups.

**Proof.** Let $f$ be a cardinal definite function on $G$. Then $f$ is a length function, and in particular $f$ is asymptotically bounded by the word length of $G$. If $Z$ is distorted, it follows that $f$ is sublinear on $Z$ with respect to the usual word length on $Z \simeq \mathbb{Z}$. By Lemma 6.1, it follows that $f$ is bounded on $Z$. For the uniform statement, assume that $f$ is proper; then for every non-elliptic element $g$, Lemma 6.1 implies $f(g^n) \geq |n|$ for all $n \geq 1$. For some constant $c > 0$, we have $f \leq c| \cdot |$, where $| \cdot |$ is the word length on $G$. Hence $\lim_{n \to \infty} |g^n|/|n| \geq 1/c$, which is independent of $g$. \qed

**Example 6.3.** The central generator $z$ of the discrete Heisenberg group $H$ is quadratically distorted, in the sense that $|z^n| \simeq \sqrt{n}$, it follows that $H$ does not have Property PW (this example was noticed in [Hag]).
If \( u \in \text{GL}_4(\mathbf{Z}) \) is a matrix whose characteristic polynomial is irreducible over \( \mathbb{Q} \) with exactly 2 complex eigenvalues on the unit circle, then the corresponding semidirect product \( \mathbf{Z}^4 \rtimes_u \mathbf{Z} \) has its cyclic subgroups undistorted, but not uniformly. Indeed, if \( p \) is the projection to the sum in \( \mathbf{R}^4 \) of eigenvalues of modulus \( \neq 1 \), then for \( g \in \mathbf{Z}^4 \setminus \{ 0 \} \) (written multiplicatively), we have \( \lim_{n \to \infty} |g^n|/n \simeq \|p(g)\| \), which is never zero but accumulates at zero. It follows that \( \mathbf{Z}^4 \rtimes_u \mathbf{Z} \) does not have Property PW (this also follows from Proposition 6.20).

**Example 6.4.** Let \( A \) be the ring of integers a number field which is not \( \mathbb{Q} \) or an imaginary quadratic extension, e.g., \( A = \mathbf{Z}[\sqrt{2}] \). Then \( \text{SL}_2(A) \) has Property FW. Note that it contrast, it has the Haagerup Property, as a discrete subgroup (actually an irreducible non-uniform lattice) in some product \( \text{SL}_2(\mathbb{R})^{n_1} \times \text{SL}_2(\mathbb{C})^{n_2} \) with \( n_1 + n_2 \geq 2 \). For instance, \( A = \mathbf{Z}[\sqrt{2}] \) is a lattice in \( \text{SL}_2(\mathbb{R})^2 \). To see Property FW, first observe that the condition on \( A \) implies that \( A^\times \) has an infinite order element; a first consequence is that, denoting by \( U_{12} \) and \( U_{21} \) the upper and lower unipotent subgroups in \( A \), that \( U_{12} \) and \( U_{21} \) are both exponentially distorted. In particular, since they are finitely generated abelian groups and are thus boundedly generated by cyclic subgroups, it follows from Corollary 6.2 that \((\text{SL}_2(A), U_{12} \cup U_{21})\) has relative Property FW. Now a more elaborate consequence of the condition on \( A \) is the theorem of Carter, Keller and Paige, see Witte Morris [Wil]; \( \text{SL}_2(A) \) is boundedly generated its two unipotent subgroups. It follows, using the trivial observation that \((G, L)\) has relative Property FW implies \((G, L^n)\) has relative Property FW for all \( n \geq 1 \), that \( \text{SL}_2(A) \) has Property FW.

**Proposition 6.5.** Let \( G \) be a compactly generated locally compact abelian group. Let \( Z \) be a normal infinite cyclic discrete subgroup. Then either \((G, Z)\) has relative Property FW, or \( Z \) is a topological direct factor in some open normal subgroup of finite index of \( G \) containing \( Z \).

**Proof.** Assume that \((G, Z)\) does not have relative Property FW. Let \( H_1 \) be the centralizer of \( Z \) in \( G \), which is open of index at most 2. Let \( X \) be a continuous discrete \( G \)-set and \( M \) a commensurated subset with open stabilizer such that \( \ell_M \) is unbounded on \( Z \). Decompose \( X \) into \( Z \)-orbits as \( \bigcup_{i \in I} X_i \). Note that \( G \) permutes the \( Z \)-orbits and thus naturally acts on \( I \); since the stabilizer of \( X_i \) contains the stabilizer of any \( x \in X_i \) this action is continuous. Let \( J \) be the set of \( i \) such that \( M_i = X_i \cap M \) is infinite and coincfinite in \( X_i \). Then \( X_J = \bigcup_{j \in J} X_i \) is \( G \)-invariant. Define \( K = I \setminus J \), \( M_J = M \cap X_J \), \( M_K = M \setminus X_J \). Then \( \ell_M = \ell_{M_J} + \ell_{M_K} \). Since \( Z \) is compactly generated, it follows from Proposition 4.7 that \( M_K \) is transfixed by \( A \), so \( \ell_{M_K} \) is bounded on \( A \). Thus \( \ell_{M_J} \) is proper on \( A \). Again by Proposition 4.7 \( J \) is finite. So some open subgroup of finite index \( H_2 \) of \( G \) fixes \( J \) pointwise. Define \( H = H_1 \cap H_2 \).

Since \( \ell_M \) is unbounded on \( Z \), the set \( J \) is nonempty; let us pick \( j \in J \). Then \( Z \) acts on \( X_j \) with a single infinite orbit; therefore if \( L \) is the pointwise stabilizer of \( X_j \) in \( H \), we have \( H = Z \times L \) as an abstract group; thus the continuous
homomorphism $Z \times L \to H$ is bijective; since $Z \times L$ is $\sigma$-compact, it is a topological
group isomorphism.

Example 6.6. Let $\Gamma$ be a cocompact lattice in $\text{PSL}_2(\mathbb{R})$ and $\tilde{\Gamma}$ its inverse image
in $\tilde{\text{PSL}}_2(\mathbb{R})$. Then $\tilde{\Gamma}$ does not have Property PW. Indeed, its center $Z \approx \mathbb{Z}$ is not
virtually a direct factor. To see the latter well-known fact, we use that for any
finite index subgroup $\Lambda$ of $\tilde{\Gamma}$ there exist $g \geq 2$ and $2g$ elements $x_1, y_1, \ldots, y_g \in \tilde{\Gamma}$
(actually generating a finite index subgroup of $\Lambda$, namely $\Lambda$ itself if $\Lambda$ is torsion-free)
such that $\prod_{i=1}^g [x_i, y_i]$ is a nonzero element of $Z$.

On the other hand, $\tilde{\Gamma}$ has its infinite cyclic subgroups undistorted. Note that
it also has the Haagerup Property, because $\tilde{\text{PSL}}_2(\mathbb{R})$ has the Haagerup Property,
by [CCJJV, Chap. 4].

6.B. Abelian groups.

Lemma 6.7. Let $A$ be a compactly generated locally compact abelian group and
$X$ a continuous discrete transitive $A$-set with a non-transfixed commensurated
subset. Then there is a continuous surjective homomorphism $\chi : A \to \mathbb{Z}$ (unique
up to multiplication by $-1$) such that the kernel $K$ of the action of $A$ on $X$ is
an open finite index subgroup of $\text{Ker}(\chi)$. In particular, the associated cardinal
definite function on $A$ has the form $g \mapsto |\text{Ker}(\chi) : K| |\chi(g)| + b(g)$ where $b : A \to \mathbb{N}$ is a bounded continuous function.

Proof. Denote $A' = A/K$. In particular, the action of $A'$ is simply transitive; since
point stabilizers are open it follows that $A'$ is discrete, and hence finitely generated
by assumption. Since the action is simply transitive, we also deduce that $A'$ is
multi-ended, and therefore is virtually infinite cyclic. Write $A' = \mathbb{Z} \times F$ with
$F$ finite abelian. Define $M' = \bigcup_{g \in F} gM$; it is $F$-invariant and commensurable
to $M$. By Lemma 6.1, $\ell_M(n, f) = |F||n| + b'(n)$ with $b'$ bounded and hence
$\ell_M(n, f) = |F||n| + b(n)$ with $b$ bounded; the same argument as in the proof of
Lemma 6.1 shows that $b \geq 0$. Letting $\chi$ denote the composite homomorphism
$A \to A' \to \mathbb{Z}$, observe that $|F| = |\text{Ker}(\chi) : K|$ and $\chi$ is determined up to the
sign, so the proof is complete.

Lemma 6.8. Let $A$ be an abelian group. Let $(\chi_i)_{i \in I}$ be a family of pairwise
non-proportional nonzero homomorphisms $A \to \mathbb{R}$. Then the family $(|\chi_i|)$ of
functions $A \to \mathbb{R}$ is linearly independent over $\mathbb{R}$.

Proof. We can suppose $I$ finite. Then, replacing $A$ by its quotient by the intersection of the $\text{Ker}(\chi_i)$, we can suppose that $A = \mathbb{Z}^k$. Write $V = \mathbb{R}^k$, so that
$A \subset V$.

Arguing by contradiction, we can suppose that $I = \{1, \ldots, k\}$ ($k$ a positive
integer) and that $g \mapsto f(g) = \sum_{i=1}^k \lambda_i |\chi_i(g)|$ is bounded, where each $\lambda_i$ is a nonzero real. Extend $\chi_i$ to a continuous homomorphism $\hat{\chi}_i : V \to \mathbb{R}$. Writing
each element in $V$ as the sum of an element of $A$ and a bounded element, we
see that the function $g \mapsto \hat{f}(g) = \sum_{i \in I} \lambda_i |\hat{\chi}_i(g)|$ is still bounded on $V$. Since $\hat{f}(g^n) = n\hat{f}(g)$ for all $g \in V$ and $n \geq 0$, by computing $\lim_{n \to \infty} f(g^n)/n$, we see that actually $\hat{f}(g) = 0$ for all $g \in V$. We thus have

$$|\hat{\chi}_1(g)| = -\sum_{i>1} \frac{\lambda_i}{\lambda_1} |\hat{\chi}_i(g)| \quad \forall g \in V;$$

let $V_1 \subset V$ be the hyperplane $\{ \hat{\chi}_1 = 0 \}$. The right-hand term is a smooth function outside $\bigcup_{i>1} V_i$, which does not contain $V_1$ because the hyperplanes $V_i$ are pairwise distinct. But the left-hand term is smooth at no point of $V_1$. This is a contradiction. \qed

**Proposition 6.9.** Let $A$ be a compactly generated locally compact abelian group and $f$ a continuous cardinal definite function on $A$. Then there exist finitely many continuous homomorphisms $\chi_i : A \to \mathbb{Z}$ and a bounded function $b : A \to \mathbb{N}$ such that, for all $g \in A$ we have

$$f(g) = \sum_i |\chi_i(g)| + b(g).$$

Moreover, if the $\chi_i : A \to \mathbb{Z}$ are required to be surjective, this decomposition is unique modulo the ordering of the $\chi_i$ and changing $\chi_i$ into $-\chi_i$. We call the term $f_0(g) = \sum_i |\chi_i(g)|$ the homogeneous part of $f$; it is given by $f_0(g) = \lim_{n \to \infty} f(g^n)/n$, and satisfies $f_0(ng) = |n|f_0(g)$ for all $n \in \mathbb{Z}$.

**Proof.** The uniqueness immediately follows from Lemma 6.8

To prove the existence, first by Corollary 4.8 we have $f = \sum_{i=1}^k f_i$ where $f_i$ is a cardinal definite function associated to a transitive continuous action. By Lemma 6.7, we have $f_i(g) = n_i|\chi_i(g)| + b_i(g)$, where $n_i$ is a nonnegative integer and $\chi_i$ is a homomorphism to $\mathbb{Z}$ (which we can suppose to be surjective if $f_i$ is unbounded and 0 otherwise).

The last statement is clear. \qed

The following corollary generalizes the second statement in Corollary 6.2.

**Corollary 6.10.** If $G$ is a compactly generated locally compact group with Property PW, then any compactly generated closed abelian subgroup $A$ is undistorted.

**Proof.** We begin by the following claim: let $f$ be a proper cardinal definite function on $\mathbb{Z}^k$. Then for some norm on $\mathbb{R}^k$ we have $f \geq \| \cdot \|$. Granted the claim, any compactly generated locally compact abelian group $A$ has a cocompact lattice isomorphic to $\mathbb{Z}^k$ for some $k$, so we can suppose $A \simeq \mathbb{Z}^k$. Let $f$ be a proper cardinal definite function on $G$. Let $\lambda$ be the word length in $G$ with respect to a compact generating subset. Then $f \leq c\lambda$ for some $c > 0$. So, taking a norm as in the claim, we have $\| \cdot \| \leq c\lambda$ on $A$. Since $\| \cdot \|$ is equivalent to the word length in $\mathbb{Z}^k$, we deduce that $A$ is undistorted.

Let us now prove the claim. In view of Proposition 6.9, we can suppose that $f = \sum_{i \in I} |\chi_i| + b$ with $\chi_i : \mathbb{Z}^k \to \mathbb{Z}$ a surjective homomorphism and $b \geq 0$ a bounded
function. Denote by $\chi_i$ the unique extension $\hat{\chi}_i$ as a continuous homomorphism $\mathbb{R}^k \to \mathbb{R}$. Define $\hat{f} = \sum \hat{\chi}_i$. Then $\hat{f}$ is a seminorm on $\mathbb{R}^k$; since its restriction to $\mathbb{Z}^k$ is proper, it is actually a norm. Since $f \geq \hat{f}$ on $\mathbb{Z}^k$, the claim is proved. 

Remark 6.11. Proposition 6.9 shows that if $G$ is an abelian compactly generated locally compact group and $f$ a cardinal definite function on $G$, there exists a (unique) least element $f_0$ in the set of cardinal definite functions $f'$ such that $f - f'$ is bounded. I do not know if such a statement holds when $G$ an arbitrary compactly generated locally compact group.

Let us mention in the abelian case that if $f = \ell_M$, then $f_0$ has, by construction (see the proof of Lemma 6.7), the form $\ell_{M'}$ with $M'$ commensurable to $M$.

Proposition 6.12. Let $A$ be a finitely generated abelian group and $f$ a proper cardinal definite function on $A$. Then there exists $m$ such that for every abelian group of finite index $B \supset A$ and measure definite function $f'$ on $B$ extending $f$, we have $[B/T_B : A/T_A] \leq m$, where $T_A$ and $T_B$ denote the torsion groups in $A$ and $B$. In particular, if $B$ is torsion-free then $[B : A] \leq m$.

Proof. Define $V = A \otimes \mathbb{Z} \mathbb{R}$ and let $\Gamma_A$ be the image of $A$ in $V$; it is a lattice. Let $f_0$ be the homogeneous part of $f$, which extends naturally to $V$; since $f$ is proper, $f_0^{-1}(\{0\}) = \{0\}$. Consider the open polyhedron $\Omega = \{x \in V : |f_0(x)| < 1\}$. There exists a positive lower bound for the covolume of a lattice $\Lambda$ in $V$ such that $\Lambda \cap \Omega = \{0\}$; in particular, there is an upper bound $m$ for the index of an overgroup $\Lambda$ of $\Gamma_A$ such that $\Lambda \cap \Omega = \{0\}$.

Let now $f'_0$ be the homogeneous part of $f'$. Since for $g \in B$ we have $f'_0(g) = \lim_{n \to \infty} f'(g^n)/n!$ and $g^n \in A$ for large $n$, we have $(f'_0)|_A = f_0$. Also define $\Gamma_B$ as the image $B$ in $V$. So $\Gamma_B$ is a lattice in $V$ containing the lattice $\Gamma_A$, and since $f_0$ takes integer values on $B$, we have $\Gamma_B \cap \Omega = \{0\}$. So $[\Gamma_B : \Gamma_A] \leq m$. But $\Gamma_B = B/T_B$ and $\Gamma_A = A/T_A$, so $[\Gamma_B : \Gamma_A] = [B/T_B : A/T_A] \leq m$. 

Corollary 6.13. Let $G$ be a locally compact group with Property PW. Then every discrete torsion-free abelian subgroup of finite $\mathbb{Q}$-rank is free abelian. 

Lemma 6.14. Let $G$ be a compactly generated locally compact group with a closed normal discrete subgroup $A$ isomorphic to $\mathbb{Z}^d$. Suppose that $A$ is undistorted in $G$. Then the homomorphism $G \to \text{GL}_d(\mathbb{Z})$ has a finite image.

Proof. Let $S$ be the image in $\text{GL}_d(\mathbb{Z})$ of a compact, symmetric generating subset of $G$. Since $\mathbb{Z}^d$ is undistorted, for every $x \in \mathbb{Z}^d$ there exists a constant $C$ such that, in $\mathbb{Z}^d$, for all $n \in \mathbb{N}$ and $g \in S^n$, we have $\|g a^n g^{-1}\| \leq Cn + C$ (where we write the law of $\mathbb{Z}^d$ multiplicatively). Rewriting this additively gives $\|ng \cdot x\| \leq Cn + C$; dividing by $n \geq 1$, this yields $\|g \cdot x\| \leq C + C/n \leq 2C$ for all $g \in S^n$. This bound does not depend on $n$, and this shows that for every $x \in \mathbb{Z}^d$, the subset $\{g \cdot x : g \in G\}$ is bounded. In particular, the union of orbits of all basis elements in $\mathbb{Z}^d$ is finite, so there is some finite index subgroup of $G$ fixing all basis elements. Thus $G \to \text{GL}_d(\mathbb{Z})$ has finite image.
The following proposition partly generalizes Proposition 6.5.

**Proposition 6.15.** Let $G$ be a compactly generated locally compact abelian group with Property PW. Let $A$ be a closed normal abelian subgroup; suppose that $A$ is discrete and free abelian of finite rank. Then $A$ is a topological direct factor in some open normal subgroup of finite index of $G$ containing $A$.

**Proof.** By Property PW, the identity component of $G$ is compact. By Corollary 6.10, $A$ is undistorted in $G$. By Lemma 6.14, the centralizer $H_1$ of $A$ in $G$ (which is closed and contains $A$) has finite index in $G$, hence is open in $G$. Let $X$ be a continuous discrete $G$-set and $M$ a commensurated subset with open stabilizer such that $\ell_M$ is proper (we will only use that $\ell_M$ is proper on $A$).

Arguing exactly as in the proof of Proposition 6.5, we can suppose that $I = J$ in the notation therein, i.e., we can suppose that $I = J$ is finite and for every $j \in J$ we have $M_j$ and $X_j \setminus M_j$ both infinite. Let $L_j$ be the kernel of the action of $H$ on $X_j$. Then $X_j$ can be identified to $A/A_{L_j}$, where $A$ acts by left translations. The centralizer of the group of left translations in any group is the group of right translations, which means left translations in the abelian case. Since $H$ centralizes $A$, we deduce that $H$ acts on $X_j$ by left translations. Let $L_j$ be the kernel of the action of $H$ on $X_j$. Let $\chi_j : A \to \mathbb{Z}$ be the surjective homomorphism associated to the action of $A$ on $X_j$; pick $j_1, \ldots, j_k$ so that $\chi_{j_1}, \ldots, \chi_{j_k}$ are independent, where $A \simeq \mathbb{Z}^k$ (they exist by properness of $\ell_{M_j}$ on $A$). Define $L = \bigcap_{i=1}^k L_{j_i}$. We claim that $H$ is the direct product $A \times L$ as a topological group. Indeed, we have $A \cap L = \{1\}$, because since $\bigcap_{i=1}^k \ker(\chi_{j_i}) = \{1\}$ on $A$; moreover $AL = H$ because $L$ is the kernel of the image of the action homomorphism $H \to \mathcal{S}(\bigcup_{i=1}^k X_{j_i})$ and the image of this homomorphism coincides with its restriction to $A$. Thus $A \times L \to H$ is a bijective continuous homomorphism; since $A \times L$ is $\sigma$-compact it follows that is a topological group isomorphism. $\blacksquare$

6.C. Polycyclic groups.

**Theorem 6.16** (Houghton [Ho82]). Let $\Gamma$ be a virtually polycyclic group of Hirsch length $k$. Then a subgroup $\Lambda$ is coforked if and only if it has Hirsch length $k - 1$ and has a normalizer of finite index.

**Definition 6.17.** We say that a discrete group is Euclidean if it finitely generated, virtually abelian without nontrivial normal subgroup. We say that a discrete group $\Gamma$ is residually Euclidean if the intersection $\vd(\Gamma)$ of normal subgroups $N$ with $\Gamma/N$ Euclidean, is trivial.

By Bieberbach’s theorem, a discrete group is Euclidean if and only it admits a faithful, cocompact proper action on a Euclidean space. This justifies the terminology, although we will not use this fact.
Lemma 6.18. Let $\Gamma$ be a virtually polycyclic group with Hirsch length $k$ and $\Lambda$ a subgroup. If $\Lambda$ has Hirsch length $k - 1$ and has its normalizer has finite index in $\Gamma$, then it contains a finite index subgroup of $VD(\Gamma)$.

Proof. Let $L_1$ be the normalizer of $\Lambda$; it has finite index. The group $L_1/H$ has Hirsch length 1 so has an infinite cyclic subgroup $L/H$ of finite index. The subgroup $g\Lambda g^{-1}$ only depends on the class of $g$ in $G/L$. Consider the finite intersection $N = \bigcap_{g \in G/L} g\Lambda g^{-1}$. Since $N = \bigcap_{g \in G} g\Lambda g^{-1}$, it is normal in $G$. Also, the diagonal map $L \to \prod_{g \in G/L} L/g\Lambda g^{-1} \simeq \mathbb{Z}^{\mathbb{Z}/L}$ has kernel equal to $N$ and it follows that $L/N$ is a finitely generated abelian group. Thus $\Gamma/N$ is virtually abelian. If $N'/N$ is its maximal finite normal subgroup, then $\Gamma/N'$ is Euclidean and thus $VD(\Gamma) \subset N'$. So $VD(\Gamma) \cap N$ has finite index in $VD(\Gamma)$ and is contained in $\Lambda$. □

Corollary 6.19. Under the hypotheses of Theorem 6.16, if $\Lambda$ is coforked then it contains a finite index subgroup of $VD(\Gamma)$. □

Proposition 6.20. Let $\Gamma$ be a virtually polycyclic group. Then $(\Gamma, VD(\Gamma))$ has relative Property FW.

Proof. Consider a transitive commensurating action of $\Gamma$ on a set $X$ with commensurated subset $A$, and $x \in X$, and let $H$ be the stabilizer of $x$. Then by Corollary 6.19, $H$ contains a finite index subgroup $Q_1$ of $VD(\Gamma)$. Since $VD(\Gamma)$ is finitely generated, it contains a characteristic subgroup of finite index $Q$ contained in $Q_1$. So $Q$ is normal in $\Gamma$ and fixes a point in the transitive $\Gamma$-set $X$. It follows that the $Q$-action on $X$ is identically trivial. Thus the action of $VD(\Gamma)$ on $X$ factors through a finite group, which has Property FW, so it leaves invariant a subset commensurable to $A$. This shows that $(\Gamma, VD(\Gamma))$ has relative Property FW. □

Note that Proposition 6.20, which has just been proved using Theorem 6.16, easily implies Theorem 6.16 (precisely, it boils down the proof of Theorem 6.16 to the case when $\Gamma$ is virtually abelian). Let us now provide a proof of Proposition 6.20 not relying on Theorem 6.16 but relying instead on the results of Section 6.

Alternative proof of Proposition 6.20. We argue by induction on the Hirsch length of $VD(\Gamma)$. If it is zero, there is nothing to prove. Otherwise, $VD(\Gamma)$ contains a an infinite $\Gamma$-invariant subgroup $\Lambda$ of minimal nonzero Hirsch length. Passing to a finite index characteristic subgroup, we can suppose that $\Gamma$ is abelian and torsion-free. Let us show that $(\Gamma, \Lambda)$ has relative Property FW. Otherwise, by contradiction there is a cardinal definite function $f$ on $\Gamma$ such that $f$ is unbounded on $\Lambda$. Let $P \subset \Lambda$ be the maximal subgroup of $\Lambda$ on which $f$ is bounded (which exists by an easy argument using that $\Lambda$ is finitely generated abelian). By minimality of $\Lambda$, the action of $\Gamma$ on $\Lambda \otimes Q$ is irreducible. It follows that the intersection of $\Gamma$-conjugates of $P$ is zero. Thus there exist $\gamma_1, \ldots, \gamma_k \in \Gamma$ such that $\bigcap_{i=1}^k \gamma_i P \gamma_i^{-1} = \{0\}$. Therefore, defining $f'(g) = \sum_i f(\gamma_i^{-1} x \gamma_i)$, the function
$f'$ is cardinal definite and is not bounded on any nontrivial subgroup of $\Lambda$. By Proposition 6.9 it follows that $f$ is proper on $\Lambda$. By Proposition 6.15 (which, as indicated in its proof, only uses the properness of the cardinal definite function on $\Lambda$), $\Lambda$ is a direct factor of some normal finite index subgroup $H$ of $\Gamma$; thus clearly $\Lambda \cap \mathcal{V}D(H) = \{1\}$. Thus the image of $\Lambda$ in $\Gamma/\mathcal{V}D(H)$ is infinite. Since $H$ has finite index in $\Gamma$, the group $\Gamma/\mathcal{V}D(H)$ is virtually abelian thus its quotient by some finite normal subgroup is a Euclidean quotient of $\Gamma$ in which $\Lambda$ has an infinite image. Thus the image of $\Lambda$ in $\Gamma/\mathcal{V}D(\Gamma)$ is infinite. This contradicts $\Lambda \subset \mathcal{V}D(\Gamma)$.

Thus $(\Gamma, \Lambda)$ has relative Property FW. Clearly $\mathcal{V}D(\Gamma/\Lambda) = \mathcal{V}D(\Gamma)/\Lambda$. By induction, $(\Gamma/\Lambda, \mathcal{V}D(\Gamma)/\Lambda)$ has relative Property FW. So by the relative version of Proposition 5.7 (see Remark 5.13), $(\Gamma, \mathcal{V}D(\Gamma))$ has relative Property FW. □

If $\Gamma$ is a group, recall that its first Betti number $b_1(\Gamma)$ is the $\mathbb{Q}$-rank of $\text{Hom}(\Gamma, \mathbb{Z})$, which is either a finite integer or $+\infty$ (this definition is questionable when $\Gamma$ is infinitely generated but this is the one we use; for instance $b_1(\mathbb{Q}) = 0$ with this definition). Also, define its first virtual Betti number $vb_1(\Gamma) = \sup_{\Lambda} b_1(\Lambda)$, where $\Lambda$ ranges over finite index subgroups of $\Gamma$.

Lemma 6.21. Let $\Gamma$ be a discrete group with $vb_1(\Gamma) < \infty$. Then $\Gamma/\mathcal{V}D(\Gamma)$ is Euclidean.

Proof. Equivalently, we have to show that if $\Gamma$ is residually Euclidean and $k = vb_1(\Gamma) < \infty$ then $\Gamma$ is Euclidean.

Let $P$ be a finite index subgroup surjecting onto $\mathbb{Z}^k$; replacing if necessary $P$ by the intersection of its conjugates, we can suppose that $P$ is normal. So $[P, P]$ is normal and $\Gamma/[P, P]$ is virtually $\mathbb{Z}^k$. We claim that $[P, P] = 1$. Indeed, by contradiction take $x \in [P, P] \setminus 1$. By assumption, there exists $N$ such that $\Gamma/N$ is Euclidean and $x \notin N$. Replacing $N$ by $N \cap [P, P]$, we can suppose $N \subset [P, P]$. Since $k = vb_1(\Gamma)$, it follows that $\Gamma/N$ is virtually $\mathbb{Z}^k$. So the kernel of the surjection $\Gamma/N \to \Gamma/P$ is finite. Since $\Gamma/N$ is Euclidean, this kernel is trivial, i.e., $N = P$, so $x \in N$, a contradiction. This shows that $\Gamma$ is virtually abelian. Clearly, if it has a nontrivial finite normal subgroup then it is not residually Euclidean, so $\Gamma$ is Euclidean. □

Thus we have the following corollary of Proposition 6.20.

Corollary 6.22. Let $\Gamma$ be a virtually polycyclic group. Then $\Gamma$ has Property PW if and only if it is virtually abelian.

Proof. Clearly $\mathbb{Z}^k$ has Property PW and hence, by Proposition 5.15 every finitely generated virtually abelian group has Property PW.

Conversely, if $\Gamma$ is virtually polycyclic with Property PW, it follows from Proposition 6.20 and Lemma 6.21 that $\Gamma$ is finite-by-(virtually abelian). Since any virtually polycyclic group is residually finite, it follows that $\Gamma$ is virtually abelian. □
7. About this paper

Most of the paper consists of a “digest” of known results but also contain some new ones; the purpose of this final paragraph is to clarify this. Maybe the main goal of this paper is to start at the beginning with the very elementary notion of commensurating action, rather than the more elaborate notion of CAT(0) cube complex. The introduction of this point of view in the measurable context in relation to Property T is due to Robertson and Steger [RS]. Let us now point out the new results of the paper.

• Maybe the most apparent contribution is the use of topological groups rather than discrete ones. However, this change is mainly secondary and was certainly not the main motivation for writing this paper; it nevertheless provides a more coherent setting, for instance for the study of Property FW for irreducible lattices [Cor2]. It involves some technical issues, which are addressed in §4.C. Also, part of this generalization is to remove finite generation assumptions, even in the case of discrete groups.

• The notion of $G$-walling is very similar to many definitions appearing in various places. It was cooked up so that the (almost tautological) Proposition 5.2 holds. Although the correspondence between actions on (various forms of) spaces with walls and commensurating actions (sometimes dressed up as almost invariant subsets) is known in both directions, such a simple statement as Proposition 5.2 was not previously extracted.

• The finiteness result of §4.B and its corollaries (including Proposition 5.3(1)) seem to be new, even when specified to finitely generated discrete groups.

• The notion of Property FW was briefly addressed in Sageev’s question “which classes of finitely generated groups admit a coforked subgroup?” [Sa97, Q1]; the underlying property would rather be FW’, which is less convenient to deal with for infinitely generated groups. Anyway none of these properties was subsequently studied for its own right and the propositions following the definition (from Proposition 5.3 to Proposition 5.7) can be termed as new. For instance the stability by extensions is trickier than we could expect.

• Most equivalences in Proposition 5.20 (i.e., those of the introduction, restated for topological groups) are classical or essentially classical, but the characterizations (ix) and (x) are new (note that Proposition 5.20(x) is not as obvious as it may seem at first sight, see Remark 5.21).

• A part of Section 6 is a direct proof of Haglund’s result on distortion of cyclic subgroup. The application of Example 6.4 is new. Proposition 6.5 is entirely new as well as its application Example 6.6. I am not comfortable to decide whether to call Proposition 6.20 a new result or an immediate application of Houghton’s Theorem 6.16 (for which we also provide an alternative proof); however, had this this observation been made earlier,
the question of finding a group with the Haagerup Property with no proper action on a CAT(0) cube complex (or space with walls) would not have been considered as open between the late nineties and Haglund’s paper [Hag]. Section 6 also emphasizes new properties (uniform non-distortion, $\mathcal{VD}(\Gamma)$) which hopefully could prove relevant in other contexts.

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