Statistical Phases and Momentum Spacings for One-Dimensional Anyons

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Anyons and fractional statistics\(^1\,\,\,^2\) are by now well established in two-dimensional systems. In one dimension, fractional statistics has been established so far only through Haldane’s fractional exclusion principle\(^3\), but not via a fractional phase the wave function acquires as particles are interchanged. At first sight, the topology of the configuration space appears to preclude such phases in one dimension. Here we argue that the crossings of one-dimensional anyons are always unidirectional, which makes it possible to assign phases consistently and hence to introduce a statistical parameter \(\theta\). The fractional statistics then manifests itself in fractional spacings of the single-particle momenta of the anyons when periodic boundary conditions are imposed. These spacings are given by \(\Delta p = 2\pi \hbar / L (|\theta| / \pi + \text{non-negative integer})\) for a system of length \(L\). This condition is the analogue of the quantisation of relative angular momenta according to \(l_z = \hbar (-\theta / \pi + 2 \cdot \text{integer})\) for two-dimensional anyons.

The concept of fractional statistics, as introduced by Leinaas and Myrheim\(^4\) and Wilczek\(^5\), has generically been associated with identical particles in \textit{two space dimensions}. It is intimately related to the topology of the configuration space, or the existence of fractional relative angular momentum. Angular momentum does not exist in one dimension (1D), and is quantised in units
of $\hbar/2$ in three dimensions, due to the commutation relations of the three generators of rotations. In two dimensions (2D), however, there is only one generator, $L_z$, which may have arbitrary eigenvalues $l_z$. Wilczek proposed that two-dimensional anyons with statistical parameter $\theta$ and relative angular momenta $l_z = \hbar(-\theta/\pi + 2 \cdot \text{integer})$ may be realized by particle flux-tube composites, attaching magnetic flux $\Phi = 2\theta\hbar c/e = \theta/\pi \cdot \Phi_0$ to bosons of charge $e$. The choices $\theta = 0$ and $\theta = \pi$ correspond to bosons and fermions, respectively.

More fundamentally, the possibility of fractional statistics arises in 2D because one can associate a winding number with paths interchanging particles. The sum over paths in the many-particle path integral consists of infinitely many topologically distinct sectors, which correspond to the different winding configurations of the particles around each other. By the rules of quantum mechanics, one is allowed to assign different weights to distinct sectors, provided these weights satisfy the composition principle. In particular, one may assign a phase factor $e^{\pm i\theta}$ for each (counter-) clockwiwise interchange of two particles. This choice corresponds to Abelian anyons with statistical parameter $\theta$ if the bare particles are bosons. The implicit assumption that the world lines never cross, i.e., the particles do not pass through each other, holds automatically for all values $\theta \neq 0 \mod 2\pi$ due to the non-vanishing relative angular momentum alluded to above. In three or higher dimensions, the only topologically inequivalent sectors correspond to interchanges of particles, and the only consistent choices for the statistics are bosons and fermions. In 1D, the situation is alike if particles are allowed to pass through each other, and trivial if they are not. In either case, the topology appears to preclude the possibility of one-dimensional anyons.
The association of anyons with 2D, however, was challenged by Haldane in 1991, who generalised the notion of fractional statistics to arbitrary dimensions by defining statistics through a fractional and hence generalised Pauli exclusion principle. According to his definition, the statistics of anyons is given by a rational “exclusion” parameter \( g = p/q \) (with \( p, q \) integer) which states that the creation of \( q \) anyons reduces the number of single particle states additional anyons could be placed in by \( p \). In particular, Haldane showed that the creation of \( m \) quasiholes in a \( \nu = 1/m \) Laughlin state reduces the number of available single-quasiparticle states by 1, which implies \( g = 1/m \). This result is fully consistent with the statistical parameter \( \theta = \pi/m \) obtained by Halperin and Arovas.

Most strikingly, however, Haldane showed that the spinons in the Haldane–Shastry model (HSM)\(^{9-12}\), a spin 1/2 chain with Heisenberg interactions which fall off as \( 1/r^2 \) with the distance, obey half-Fermi exclusion statistics. Haldane observed that for a chain with \( N \) sites, the number of single-particle states available to additional spinons is given by \( M + 1 \), where \( M \) is the number of up or down spins in the uniform singlet liquid, which in the presence of \( N_{sp} \) spinons is given by \( M = (N - N_{sp})/2 \). The creation of 2 spinons hence reduces the number of available states by 1, which implies \( g = 1/2 \). (Note that since there are always fewer single-spinon states as there are sites, localised spinon states cannot be orthogonal.) Haldane further demonstrated that the dimension of the Hilbert space spanned by the ground state and all possible many-spinon eigenstates of the model is \( 2^N \), as required for a spin 1/2 system with \( N \) sites. The concept of fractional statistics hence was established in a one-dimensional system, but it appeared that it could only be defined through an exclusion principle. Moreover, Haldane observed that the two
definitions of statistics do not always match, as hard-core bosons in 2D with magnetic flux-tubes attached would be classified as anyons according to winding phases, but as fermions according to his exclusion principle.

Let us briefly summarise: Fractional statistics is fundamentally associated with phases the many body wave functions acquire as particles are interchanged or wind around each other, and can hence, by the rules of quantum mechanics for identical particles as we know them, only exist in 2D. Nonetheless, according to an alternative definition in terms of a generalised exclusion principle, fractional statistics can be defined independently of the dimension. This alternative definition does not always match the original one. There would not be much reason to pay attention to it, or even use the fractional exclusion of states as a definition of fractional statistics, if there were not a concrete example of a one-dimensional system (the HSM) which supports excitations with, at least according to this definition, fractional statistics.

In this Letter, we resolve the apparent conflict between the two definitions. The argument consists of several parts. First, we show that in the one-dimensional system obeying a fractional exclusion principle, the HSM, an analog of a winding phase, i.e., a statistical phase acquired by the wave function as the anyons go through each other, exists. The conflict with the topological considerations explained above is circumvented in that the crossing of the spinons occurs in one direction only. The statistical phase of $\pi/2$ acquired by the wave function as the spinons cross manifests itself in a fractional shift for the spacings of the single-spinon momenta.

Second, we show that a fractional shift for the momentum spacings, and hence a statistical
**Fractional statistics in 2D:**

interchange through counterclockwise winding

\[ |\psi\rangle \rightarrow e^{i\theta}|\psi\rangle \]

relative angular momentum

\[ l_z \rightarrow l_z - \frac{\hbar}{\pi} \]

**Fractional statistics in 1D:**

relative motion of anyons is unidirectional (e.g. 2 moves clockwise relative to 1)

when anyons cross:

\[ |\psi\rangle \rightarrow e^{i\theta}|\psi\rangle \]

momentum spacing

\[ p_1 - p_2 = \Delta p \rightarrow \Delta p - \frac{2\pi \hbar}{L} \theta \]

Figure 1: **Fractional statistics in two and in one dimension.** In 2D, a fractional phase \( \theta \) acquired when anyons are interchanged through winding around each other manifests itself in a fractional shift in the relative angular momentum. In 1D, a fractional phase when anyons cross manifests itself in a fractional shift in momentum spacing. Consistency requires that the relative motion of 1D anyons is unidirectional, *i.e.*, that they always cross in the same direction.
phase of $\pi/2$ acquired by the wave function, also exists for the holons in the Kuramoto–Yokoyama model (KYM)\textsuperscript{13}, the supersymmetrically extended HSM allowing for itinerant holes. This suggests that the holons are half-fermions, a conclusion reached previously by Ha and Haldane\textsuperscript{14} using the asymptotic Bethe ansatz (ABA), by Kuramoto and Kato\textsuperscript{15} from thermodynamics, and by Arikawa, Saiga, and Kuramoto\textsuperscript{16} from the electron addition spectral function of the model. Since the $N$ localised single-holon states of the KYM are orthogonal, however, they appear to be fermions according to Haldane’s exclusion statistics. As a resolution of the conflict, we propose that the exclusion principle yields the correct statistics only when applied to energy eigenstates of a given model.

Finally, we argue that the picture we propose—crossings in only one direction, statistical phases acquired by the wave function as anyons go through each, fractionally spaced single anyon momenta—holds for 1D anyons in general.

The subtleties involved are best explained by looking closely at two-spinon and two-holon eigenstates of the KYM. The model is conveniently formulated by embedding the one-dimensional chain with PBCs into the complex plane by mapping it onto the unit circle with the sites located at complex positions $\eta_\alpha = \exp\left(i\frac{2\pi}{N} \alpha\right)$, where $N$ is the number of sites and $\alpha = 1, \ldots, N$. Each site can be occupied either by an up- or down-spin electron or a hole (\textit{i.e.}, the site is empty). The Hamiltonian is given by

$$H_{\text{KY}} = -\frac{2\pi^2}{N^2} \sum_{\alpha \neq \beta}^N \frac{P_{\alpha\beta}}{\eta_\alpha - \eta_\beta^2},$$  \hspace{1cm} (1)

where the graded permutation operator $P_{\alpha\beta}$ exchanges particles on sites $\eta_\alpha$ and $\eta_\beta$, multiplied by
a minus sign if both particles are fermions \(\text{i.e.},\) neither of them a hole). In the absence of holes, Eq. (1) reduces to the HSM, which possesses the exact ground state

\[ \Psi_0[z_i] = \prod_{i<j}^M (z_i - z_j)^2 \prod_{i=1}^M z_i \]  

(2)

for \(N\) even, \(M = N/2\), and \([z_i] \equiv (z_1, \ldots, z_M)\). The \(z_i\)’s denote the positions of the up spins. The greatly simplifying feature of the HSM (and the KYM) is that the spinons (and the holons) are free in the sense that they only “interact” through their half-Fermi statistics\(^{17–19}\).

Let us now turn to the two-spinon eigenstates. A momentum basis for spin-polarised two-spinon states is given by

\[ \Psi_{mn}[z_i] = \sum_{\alpha,\beta} (\bar{\eta}_\alpha)^m (\bar{\eta}_\beta)^n \prod_{i=1}^M (\eta_\alpha - z_i)(\eta_\beta - z_i) \Psi_0[z_i], \]  

(3)

where \(M = (N - 2)/2\) and \(M \geq m \geq n \geq 0\). For \(m\) or \(n\) outside this range, \(\Psi_{mn}\) vanishes identically, reflecting the overcompleteness of the position space basis. Acting with Eq. (1) on Eq. (3) yields\(^{20}\)

\[ H_{\text{KY}} \left| \Psi_{mn} \right\rangle = E_{mn} \left| \Psi_{mn} \right\rangle + \sum_{l=1}^{l_{\text{max}}} V_{mn}^l \left| \Psi_{m+l,n-l} \right\rangle \]  

(4)

with \(l_{\text{max}} = \min(M-m,n)\), \(V_{mn}^l = -\frac{2\pi^2}{N^2}(m-n+2l)\), and

\[ E_{mn} = E_0 + \epsilon(q_m) + \epsilon(q_n). \]  

(5)

\(E_0 = -\frac{\pi^2}{4N}\) is the ground state energy,

\[ \epsilon(q) = \frac{1}{2} q (\pi - q) + \frac{\pi^2}{8N^2}, \]  

(6)

and we have identified the single-spinon momenta for \(m \geq n\) according to

\[ q_m = \pi - \frac{2\pi}{N} \left( m + \frac{1}{2} + s \right), \quad q_n = \pi - \frac{2\pi}{N} \left( n + \frac{1}{2} - s \right), \]  

(7)
with a statistical shift \( s = 1/4 \). Since the “scattering” of the non-orthogonal basis states \( |\Psi_{mn}\rangle \) in Eq. (4) only occurs in one direction, increasing \( m - n \) while keeping \( m + n \) fixed, the eigenstates of \( H_{KY} \) have energy eigenvalues \( E_{mn} \).

The relevant feature for our present purposes is the shift \( s \) in the single-spinon momenta Eq. (7), which we will elaborate on now. The state Eq. (3) tells us unambiguously that the sum of both spinon momenta is given by \( q_m + q_n = 2\pi - 2\pi N (m + n + 1) \), and hence Eq. (7). The shift \( s \) is determined by demanding that the excitation energy Eq. (5) of the two-spinon state is a sum of single-spinon energies, which in turn is required for the explicit solution here to be consistent with the ABA results\(^{17-19}\).

The appearance of this shift, which decreases the momentum \( q_m \) of spinon 1 and increases momentum \( q_n \) of spinon 2, is somewhat surprising, given that the basis states Eq. (3) are constructed symmetrically with regard to interchanges of \( m \) and \( n \). To understand this asymmetry, note that \( M \geq m \geq n \geq 0 \) implies \( 0 < q_m < q_n < \pi \). The dispersion Eq. (6) implies that the group velocity of the spinons is given by

\[
v_g(q) = \partial_q \epsilon(q) = \frac{\pi}{2} - q,
\]

which in turn implies that \( v_g(q_m) > v_g(q_n) \). The physical significance of this result can hardly be overstated. It means that the relative motion of spinon 1 (with \( q_m \)) with respect to spinon 2 (with \( q_n \)) is always counterclockwise on the unit circle. Then, however, the shifts in the individual spinon momenta can be explained by simply assuming that the two-spinon state acquires a statistical phase \( \theta = 2\pi s \) whenever the spinons pass through each other. This phase implies that \( q_m \) is
shifted by $-\frac{2\pi}{N}s$ since we have to translate spinon 1 counterclockwise through spinon 2 and hence counterclockwise around the unit circle when obtaining the allowed values for $q_m$ from the PBCs. Similarly, $q_n$ is shifted by $+\frac{2\pi}{N}s$ since we have to translate spinon 2 clockwise through spinon 1 and hence clockwise around the unit circle when obtaining the quantisation of $q_n$. (The fact that the “bare” ($s = 0$) values for $q_m$ and $q_n$ are quantised as $\frac{2\pi}{N}(\frac{1}{2} + \text{integer})$ is related to the bosonic representation of the “bare” spinons. If we had chosen a fermionic representation, they would be quantised as $\frac{2\pi}{N} \cdot \text{integer}$.)

That the crossing of the spinons occurs only in one direction is not just a peculiarity, but a necessary requirement for fractional statistics to exist in 1D at all. If the spinons could cross in both directions, the fact that paths interchanging them twice (i.e., once in each direction) are topologically equivalent to paths not interchanging them at all would imply $2\theta = 0 \mod 2\pi$ for the statistical phase, i.e., only allow for the familiar choices of bosons or fermions. With the scattering occurring in only one direction, arbitrary values for $\theta$ are possible. The one-dimensional anyons neither break time-reversal symmetry (T) nor parity (P).

We now turn to the two-holon eigenstates of the KYM\textsuperscript{21}, which are highly instructive with regard to Haldane’s exclusion principle as a definition of fractional statistics. A momentum basis for two-holon states is given by

$$\Psi_{mn}^{\text{ho}}[z_i, h_j] = \phi_{mn}(h_1, h_2) \prod_{i=1}^{M} (h_1 - z_i)(h_2 - z_i)\Psi_0[z_i], \quad (9)$$

where $M = (N - 2)/2$ and $[z_i, h_j] \equiv (z_1, \ldots, z_M; h_1, h_2)$. The $z_i$’s denote the positions of the up spins again, and $h_1, h_2$ the positions of the holes. $\phi_{mn}(h_1, h_2)$ is an internal holon-holon wave
function. Using the educated guess \( \phi_{mn}(h_1, h_2) = (h_1 - h_2)(h_1^m h_2^n + h_1^n h_2^m) \), we obtain

\[
H_{\text{KY}}^{\text{ho}} \left| \Psi_{mn}^{\text{ho}} \right\rangle = E_{mn}^{\text{ho}} \left| \Psi_{mn}^{\text{ho}} \right\rangle + \sum_{l=1}^{l_{\text{max}}} V_l^{mn} \left| \Psi_{m-l,n+l}^{\text{ho}} \right\rangle
\]

(10)

for \( 0 \leq n \leq m \leq M + 1 \). If this condition is violated, the basis states \( \left| \Psi_{mn}^{\text{ho}} \right\rangle \) do not vanish identically, but it is not possible to construct eigenstates from them. In Eq. (10), \( l_{\text{max}} \) is the largest integer \( l \leq \frac{m-n}{2} \), \( V_l^{mn} = \frac{2\pi^2}{N^2} (m-n) \), and

\[
E_{mn}^{\text{ho}} = E_0 + \epsilon^{\text{ho}}(p_m) + \epsilon^{\text{ho}}(p_n).
\]

(11)

The single-holon energies are given by

\[
\epsilon^{\text{ho}}(p) = \frac{1}{2} p (\pi + p) - \frac{\pi^2}{8N^2},
\]

(12)

and we have identified the single-holon momenta for \( m \geq n \) according to

\[
p_m = -\pi + \frac{2\pi}{N} (m+s), \quad p_n = -\pi + \frac{2\pi}{N} (n-s),
\]

(13)

with \( s = 1/4 \). The “scattering” occurs again only in one direction, this time decreasing \( m - n \) while keeping \( m + n \) fixed, which implies both that the basis states \( \left| \Psi_{mn}^{\text{ho}} \right\rangle \) are not orthogonal and that the two-holon eigenstates of \( H_{\text{KY}} \) have energy eigenvalues \( E_{mn}^{\text{ho}} \). The statistical shift \( s \) is once again determined by demanding that the holons are free, which in turn is required by consistency with the ABA solutions\(^{18}\).

The momenta are again limited to about half of the Brillouin zone, \(-\pi - \frac{\pi}{2N} \leq p_n < p_m \leq \frac{\pi}{2N}\). With the holon group velocity

\[
\nu^{\text{ho}}_g(p) = \partial_p \epsilon^{\text{ho}}(p) = \frac{\pi}{2} + p,
\]

(14)
we obtain $v_{g}^{ho}(p_m) > v_{g}^{ho}(p_n)$. The crossing of the holons occurs again only in one direction, and the momentum shifts as well as the half-Fermi statistics emerges as in the case of the spinons, except that the state now acquires the phase $\theta = -2\pi s$, with the result that the momentum $p_m$ of the holon with the larger group velocity $v_{g}^{ho}(p_m)$ is shifted by $+\frac{2\pi}{N} s$, and $p_n$ shifted by $-\frac{2\pi}{N} s$. Physically, this reversal in the sign reflects that the holon is created by annihilation of an electron at a spinon site, \textit{i.e.}, by removing a fermion from a half-fermion. The spacing between $p_m$ and $p_n$, however, is quantised as for the spinons above. Note that the hard-core constraint of the holons is irrelevant here.

Let us now reconcile this result with the exclusion principle. As mentioned, the hard-core condition for holons effects that they are fermions according to Haldane’s exclusion principle applied to states localised in position space. When applied to exact eigenstates of the model, however, the result is different. Since the creation of 2 holons decreases the number of up or down spins in the uniform liquid $M$ by 1, the number of single-holon states (labelled by $m$ or $n$ above) available for additional holons decreases by 1. This implies half-Fermi statistics, and is consistent with the momentum spacings. \textit{The exclusion principle hence yields the correct statistics only if applied to eigenstates of the model.} The wave function for localised holons is really a superposition of a holon state (onto which we project in Eq. (9)) and a holon surrounded by an incoherent spinon cloud in a singlet configuration.

So far, our discussion has been limited to a particular model. The conclusions, however, hold in general. As noted above, the KYM is special in that the spinon and holon excitations are free.
The single spinon and holon momenta are hence good quantum numbers. The eigenstates of the model can be labelled in terms of these momenta, which we have shown to be fractionally spaced. Any other model of a one dimensional spin chain can be described as a KYM supplemented by additional terms, which give rise to an interaction between the spinons and holons. This interaction will scatter the basis states of free spinons and holons, the eigenstates of the KYM, into each other. The eigenstates of the interacting model will hence be superpositions of states with different single spinon and holon momenta, all of which, however, will be fractionally spaced. In other words, the fractional shifts in Eq. (7), Eq. (13) (and also Eq. (15), Eq. (16) below) will still be good quantum numbers, while the integers $n$ and $m$ will turn into “superpositions of integers”.

This argument shows that whenever we have spinons and holons in a one-dimensional spin chain, we have fractionally spaced single particle momenta as a consequence of their fractional statistics. Is it reasonable to assume that this picture holds for anyons in 1D in general? We believe there are very good reasons to do so. First, spinons and holons are the only known examples of anyons in 1D. This picture hence holds for all examples of 1D systems with fractional statistics. Second, the picture resolves a profound conflict, as topology precludes the existence of one dimensional anyons in a conventional framework of indistinguishable particles. The conflict is circumvented here in that the anyons become distinguishable through their dynamics, and cross in one direction only. If the picture we propose here were not of general validity, another resolution to this conflict would have to exist. This does not appear to be the case. In any event, the picture we propose is the only consistent picture available at present. It is hence only reasonable to assume general validity.
We conclude with a summary. We propose that the statistics of identical particles is always reflected in the quantisation condition of an observable quantity. For anyons with statistical parameter $\theta$ in 2D, the kinematical relative angular momentum between two anyons is quantised as

$$l_z = \hbar \left( -\frac{\theta}{\pi} + 2m \right),$$  \hspace{1cm} (15)$$

where $-\pi < \theta \leq \pi$ and $m$ is integer.

For anyons with statistical parameter $\theta$ in a one-dimensional system with length $L$ and periodic boundary conditions—and this is the central message of this Letter—the allowed values for the spacings between the kinematical (linear) momenta are quantised as

$$p_{i+1} - p_i = \Delta p = \frac{2\pi \hbar}{L} \left( \frac{|\theta|}{\pi} + n \right)$$

for $p_{i+1} - p_i \geq 0$, where $-\pi < \theta \leq \pi$ and $n$ is a non-negative integer. The spacing condition Eq. (16) holds for many-anyon states with single-particle momenta $p_1 \leq p_2 \leq \ldots \leq p_N$ in any interval $p_i \in I$, provided that the anyon group velocity $v_g(p) = \partial_p \epsilon(p)$ is a strictly increasing ($\theta < 0$) or decreasing ($\theta > 0$) function of $p$ in this interval. This condition is required for the anyons to cross in one direction only. In an interacting many particle system, the quantum numbers $m$ and $n$ in Eq. (15) and Eq. (16) are not expected to be good quantum numbers. The fractional shifts $-\theta/\pi$ and $|\theta|/\pi$, however, are topological invariants.

Note that Eq. (15) and Eq. (16) hold only between the *physical* or *kinematical* statistics of the anyons and the *kinematical* angular or linear momenta, as canonical momenta are gauge dependent. In particular, one may change the canonical momenta while simultaneously changing
the canonical statistics of the fields (i.e., the statistics imposed when canonically quantising the fields) used to describe the anyons via a “singular” gauge transformation. The canonical statistics may either be bosonic, as in the case of the spinons in the analysis above, or fermionic, as in the case of the holons above.

Our analysis further demonstrates that particular care must be exercised when defining statistics using Haldane’s exclusion principle. The fact that it gives the correct result for the statistics of holons in the KYM when applied to eigenstates of the model but an incorrect result when applied to holon states localised in position space leads us to conjecture that in general, the exclusion principle yields correct results only when applied to eigenstates of a given model.

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