EXTENSION OF TRIPLE LAPLACE TRANSFORM FOR SOLVING FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. In this article, we extend the concept of triple Laplace transform to the solution of fractional order partial differential equations by using Caputo fractional derivative. The concerned transform is applicable to solve many classes of partial differential equations with fractional order derivatives and integrals. As a consequence, fractional order telegraph equation in two dimensions is investigated in detail and the solution is obtained by using the aforementioned triple Laplace transform, which is the generalization of double Laplace transform. The same problem is also solved by taking into account the Atangana-Baleanu fractional derivative. Numerical plots are provided for the comparison of Caputo and Atangana-Baleanu fractional derivatives.

1. Introduction. The applications of differential equation is one of the interesting and most essential area in the field of engineering, physics and other branches of applied disciplines. Although for the solution of such problems involving differential equations, there are no common techniques. The integral transform technique is one of the greatest known scheme used by numbers of researchers for the solution of ordinary and partial differential equations [7, 8]. The double Laplace transform and Sumudu transform were used in [15, 18] for the solution of wave equation and Poisson equation.

The area of fractional calculus, which has been attracted much attention in last few decades due to its enormous numbers of applications in almost all disciplines of applied sciences and engineering. The fractional calculus became an aspirant to find out the solution of complex systems exist in numerous fields in sciences, (see for detail [14, 26, 6]). In the theoretical and applied point of view large numbers of sweeping problems are existed in this region [10, 11, 27], which needs solution, for illustration, the physical significance of the fractional order derivative, etc. In the field of mathematical modeling having, partial derivatives of fractional order naturally seem in dealing with the generality of the current traditional models. Consequently there is a requirement to improve the existing literature from the traditional to the fractional calculus. At the primary view this procedure looks very simple and straight. In fact this is a complex procedure, which plentiful courtesy is

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requisite. Because the fractional calculus needs extra surroundings in order to be
defined appropriately.

Atangana and Baleanu (AB) [2] in their recent work introduced a new version
of fractional derivatives which uses the generalized Mittag-Leffler function as the
non-singular and non-local kernel and accepts all properties of fractional derivatives
(for more detail see [3, 4, 5]).

Kilbas et al. [14], recycled the Laplace transform technique to find out the
solution of fractional order ordinary differential equations. The Laplace transform
method was used by Oldham and Spanier to obtain the solution of the fractional
order homogeneous ordinary differential equations. This method was further used
by Dorta, Seikazieva, Miller and Ross for the solutions of such equations. Numbers
of authors used this approach to solve fractional orders partial differential equation
(for more detail see [12, 16, 17, 9, 28]). The triple Laplace transform hasn’t used
extensively and no data is available for solution of ODE’s or PDE’s of fractional
order using triple Laplace transform.

The significant solicitation of the telegraph equation can be realized as the ca-
pacity of the thermal diffusivity in polymers [19]. Similarly it seems in illustrating
pressure diffusion in porous medium. Usually the analytic solutions of fractional
telegraph equation are very difficult and yet haven’t be solved analytically. While
for the solution of this equation the numerical methods are frequently used. The
fractional order telegraph equation and its generalization are related to the nonlocal
spectacles and itself dowries it’s possess attention. Compelling into interpretation
the above stated results, we establish the triple Laplace transform formulas for the
partial derivatives having fractional order. We also solve the telegraph equation
of fractional order by applying the triple Laplace formulas with certain initial and
boundary conditions in two dimensions. In the recent past much emphasis have also
been given in the literature for numerical solution of two dimensional hyperbolic
telegraph equation [21]-[25].

The organization of the manuscript is stated as follows: The second section is
devoted to the fundamental definitions of the fractional order integrals and frac-
tional order derivatives as well as the double Laplace transform. In section 3, the
triple Laplace transforms formulas are introduced for the partial fractional integral-
s and Caputo derivatives. In section 4, we solve the proposed telegraph equation
in two dimensions by triple Laplace transform using Caputo’s fractional deriva-
tive. In section 5, solution of fractional telegraph equation has been obtained using
Atangana-Baleanu fractional derivative. In the same section we compar both the
obtained results. Finally, conclusion is presented in section 6.

2. Preliminaries. We recall some fundamental definitions and well known results.
Particularly, the definition of ordinary, partial fractional integral, derivatives, the
Laplace, double Laplace and triple Laplace transforms are included.

**Definition 2.1.** The Riemann-Liouville fractional integral of a function \( f \) on \((0, \infty)\)
having order \( \alpha > 0 \) is defined by

\[
I_{a}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-\tau)^{\alpha-1} f(\tau) d\tau,
\]

such that the integral on the right converges.
Definition 2.2. The left Riemann-Liouville fractional derivative of a function \( f(t) \) having order \( \alpha > 0 \) over \((0, \infty)\) is defined by

\[
D_0^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_0^t (t - \tau)^{n - \alpha - 1} f(\tau) d\tau, \quad \alpha \in (n - 1, n],
\]

where the integral on the right is point wise defined on \((0, \infty)\).

Definition 2.3. The left Caputo fractional derivative of order \( \alpha > 0 \) of a function \( f(t) \) over \((0, \infty)\) is defined by

\[
i^cD_0^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n - \alpha - 1} \left( \frac{d}{d\tau} \right)^n f(\tau) d\tau, \quad \alpha \in (n - 1, n],
\]

where the integral on the right is point wise defined on \((0, \infty)\).

Definition 2.4. The Laplace transform of a function \( f(t) \) is defined as

\[
F(s) = \int_0^\infty f(t)e^{-st} dt, \quad s > 0, \; t \geq 0.
\]

Definition 2.5. The double Laplace transform of \( f(x, t) \) is given by

\[
L_t L_x \{ f(x, t) \} = F(s_1, s_2) = \int_0^\infty e^{-s_1 t} \int_0^\infty e^{-s_2 x} f(x, t) dx dt, \quad s_1, s_2 > 0, x, \; t \geq 0.
\]

In view of Definition 2.4 and 2.5, triple Laplace transform is defined as follow:

Definition 2.6. The triple Laplace transform of \( f(x, y, t) \) is given by

\[
L_t L_y L_x \{ f(x, y, t) \} = F(s_1, s_2, s_3)
\]

\[
= \int_0^\infty e^{-s_3 t} \int_0^\infty e^{-s_2 y} \int_0^\infty e^{-s_1 x} f(x, y, t) dx dy dt, \quad s_1, s_2, s_3 > 0, x, y, \; t \geq 0.
\]

Definition 2.7. Let \( f \in H^1(a, b), \; a < b, \; \alpha \in [0, 1] \) then, the definition of Atangana-Baleanu fractional derivative is given by:

\[
\begin{align*}
\text{ABC}D_t^\alpha_a f(t) &= \frac{B(\alpha)}{1 - \alpha} \int_a^t f'(x) E_{\alpha}[-\alpha \frac{(t - x)^\alpha}{1 - \alpha}] dx,
\end{align*}
\]

where \( B(\alpha) \) is a normalization function such that \( B(0) = B(1) = 1 \). The Laplace transform of Atangana-Baleanu fractional derivative is given by:

\[
L_t \{ \text{ABC}D_t^\alpha_a f(t) \} (s) = \frac{B(\alpha)}{1 - \alpha} \frac{s^\alpha \{ f(t) \} (s) - s^{(\alpha - 1)} f(0)}{s^\alpha + \frac{\alpha}{1 - \alpha}}.
\]

Definition 2.8. The Mittag-Leffler function needed in this paper is defined as

\[
E_{\alpha, \beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha + \beta)}, \quad t, \beta \in \mathbb{C}, \; \Re(\alpha) > 0.
\]

The Laplace transform of the \( t^{\beta - 1} E_{\alpha, \beta}(\lambda t^\alpha) \) is provided as

\[
L_t \left[ \frac{t^{\beta - 1} E_{\alpha, \beta}(\lambda t^\alpha)}{s^\alpha - \lambda} \right] = \frac{s^{\alpha - \beta}}{|s|^\alpha}, \quad \text{for} \; |\lambda| < |s|.
\]
3. Preliminaries related to double and triple Laplace transform. We illustrate the Laplace transforms of partial integral and derivatives having fractional order, the Laplace transform formulas for the partial derivatives having integer order is given in the following theorem.

**Theorem 3.1.** [1] Let \( f \in C^l(R^+ \times R^+) \), \( l = \max\{m,n\} \) and \( \left| \frac{\partial^{m+j}f(x,t)}{\partial x^m \partial t^j} \right| \leq ke^{\pi_1+\pi_2} \) where \( k, \pi_1, \pi_2 \geq 0 \) and \( i = 0,1,...,m, j = 0,1,...n \), then the following formulas holds:

\[
L_t L_x \left\{ \frac{\partial^n f(x,t)}{\partial x^n} \right\} = s_t^n L_x L_t \left\{ f(x,t) \right\} - \sum_{i=0}^{n-1} s_t^{n-1-i} L_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\}, \quad (9)
\]

\[
L_t L_x \left\{ \frac{\partial^m f(x,t)}{\partial t^m} \right\} = s_t^m L_x L_t \left\{ f(x,t) \right\} - \sum_{j=0}^{m-1} s_t^{m-1-j} L_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\}, \quad (10)
\]

and

\[
L_t L_x \left\{ \frac{\partial^{m+n} f(x,t)}{\partial t^m \partial x^n} \right\} = s_t^m s_x^n \left[ L_t L_x \left\{ f(x,t) \right\} - \sum_{j=0}^{m-1} s_t^{m-1-j} L_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\} \right]
- \sum_{i=0}^{n-1} s_t^{n-i-1} L_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\} + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_t^{n-i-1} s_x^{m-j-1} \left\{ \frac{\partial^{i+j} f(0,0)}{\partial x^i \partial t^j} \right\}, \quad (11)
\]

where \( \frac{\partial^{i+j}}{\partial x^i \partial t^j} f(0,0) \) denoted the value of the mixed partial derivative at the point \((0,0)\).

**Theorem 3.2.** [1] Let \( \Re(\alpha), \Re(\beta) \geq 0 \) and \( f \in L_1([0,a) \times (0,b]) \) for any \( a,b \geq 0 \). Also let \( |f(x,t)| \leq ke^{\pi_1+\pi_2}, x > a > 0, t > b > 0 \) hold for constant \( k, \pi_1, \pi_2 > 0 \) . Then the double Laplace transform of fractional integrals is provided as

\[
L_t L_x \left\{ I_0^\alpha f(x,t) \right\}(s_1,s_2) = \frac{1}{s_t^\alpha} L_t L_x \left\{ f(x,t) \right\}(s_1,s_2), \quad (12)
\]

\[
L_t L_x \left\{ I_0^\beta f(x,t) \right\}(s_1,s_2) = \frac{1}{s_x^\beta} L_t L_x \left\{ f(x,t) \right\}(s_1,s_2), \quad (13)
\]

and

\[
L_t L_x \left\{ I_0^{\beta+\alpha} f(x,t) \right\}(s_1,s_2) = \frac{1}{s_t^\alpha s_x^\beta} L_t L_x \left\{ f(x,t) \right\}(s_1,s_2). \quad (14)
\]

**Theorem 3.3.** [1] Let \( \alpha, \beta > 0, n - 1 < \alpha \leq n, m - 1 < \beta \leq m, n \in N, \) be such that \( f \in C^l(R^+ \times R^+) \), \( l = \max\{m,n\} \), \( f^{(l)} \in L_1([0,a) \times (0,b]) \) for any \( a,b \geq 0 \). Also let \( |f(x,t)| \leq ke^{\pi_1+\pi_2}, x > a > 0, t > b > 0 \) hold, for constant \( k, \pi_1, \pi_2 > 0 \) , then

\[
L_t L_x \left\{ x^\alpha D_0^\alpha f(x,t) \right\} = s_t^\alpha \left[ L_t L_x \left\{ f(x,t) \right\} - \sum_{i=0}^{n-1} s_t^{n-1-i} L_t \left\{ \frac{\partial^i f(0,t)}{\partial x^i} \right\} \right], \quad (15)
\]

\[
L_t L_x \left\{ t^\beta D_0^\beta f(x,t) \right\} = s_x^\beta \left[ L_t L_x \left\{ f(x,t) \right\} - \sum_{j=0}^{m-1} s_x^{m-1-j} L_x \left\{ \frac{\partial^j f(x,0)}{\partial t^j} \right\} \right], \quad (16)
\]
Theorem 3.4. [13] Let $f \in C^l(R^+ \times R^+ \times R^+)$, $l = \max \{m, n, p\}$ and $i = 0, 1, \ldots, m, j = 0, 1, \ldots n, h = 0, 1, \ldots p$, then the following formulas hold:

$$L_t L_y L_x \left\{ \frac{\partial^m}{\partial x^m} f(x, y, t) \right\} = s_1^m L_t L_y L_x \left\{ f(x, y, t) \right\} - \sum_{k=0}^{m-1} s_1^{m-1-k} L_t L_y \left\{ \frac{\partial^k}{\partial x^k} f(0, y, t) \right\}. \quad (18)$$

and

$$L_t L_y L_x \left\{ \frac{\partial^n}{\partial y^n} f(x, y, t) \right\} = s_2^n L_t L_y L_x \left\{ f(x, y, t) \right\} - \sum_{k=0}^{n-1} s_2^{n-1-k} L_t L_y \left\{ \frac{\partial^k}{\partial y^k} f(x, 0, t) \right\}. \quad (19)$$

Theorem 3.5. [13] Let $\Re(\alpha), \Re(\beta), \Re(\gamma) \geq 0$ and $f \in L_1((0, a) \times (0, b) \times (0, c))$ for any $a, b, c \geq 0$. Also let $|f(x, y, t)| \leq k e^{\gamma x_1+y_2+t_3}$, where $x > a > 0, y > b > 0, t > c > 0$ holds for constants $k, \tau_1, \tau_2, \tau_3 > 0$. Then

$$L_t L_y L_x \left\{ I^\alpha_1 f(x, y, t) \right\} (s_1, s_2, s_3) = \frac{1}{s_1^\alpha} L_t L_y L_x \left\{ f(x, y, t) \right\} (s_1, s_2, s_3), \quad (20)$$

$$L_t L_y L_x \left\{ I^\beta_1 f(x, y, t) \right\} (s_1, s_2, s_3) = \frac{1}{s_2^\beta} L_t L_y L_x \left\{ f(x, y, t) \right\} (s_1, s_2, s_3), \quad (21)$$

$$L_t L_y L_x \left\{ I^\gamma_1 f(x, y, t) \right\} (s_1, s_2, s_3) = \frac{1}{s_3^\gamma} L_t L_y L_x \left\{ f(x, y, t) \right\} (s_1, s_2, s_3). \quad (22)$$

Theorem 3.6. [13] Let $\alpha, \beta, \gamma > 0, n - 1 < \alpha \leq m - 1 < \beta \leq m, \gamma \leq p, m, n, p \in N$, be such that $f \in C^l(R^+ \times R^+ \times R^+)$, $l = \max \{m, n, p\}, f^{(i)} \in L_1((0, a) \times (0, b) \times (0, c))$ for any $a, b, c \geq 0$. Also let $|f(x, y, t)| \leq k e^{\gamma x_1+y_2+t_3}$, $x > a > 0, y > b > 0, t > c > 0$ holds for constant $k, \tau_1, \tau_2, \tau_3 > 0$, then

$$L_{t y z} \left\{ D_1^\alpha f(x, y, t) \right\} (s_1, s_2, s_3) = s_1^\alpha \left[ L_{t y z} \left\{ f(x, y, t) \right\} - \sum_{i=0}^{n-1} s_i^{-1-i} L_t L_y \left\{ \frac{\partial^i f(0, y, t)}{\partial x^i} \right\} \right], \quad (23)$$

$$L_{t y z} \left\{ D_1^\beta f(x, y, t) \right\} (s_1, s_2, s_3) = s_2^\beta \left[ L_{t y z} \left\{ f(x, y, t) \right\} - \sum_{j=0}^{n-1} s_j^{-1-j} L_t L_x \left\{ \frac{\partial^j f(x, 0, t)}{\partial y^j} \right\} \right], \quad (24)$$

$$L_{t y z} \left\{ D_1^\gamma f(x, y, t) \right\} (s_1, s_2, s_3) = s_3^\gamma \left[ L_{t y z} \left\{ f(x, y, t) \right\} - \sum_{k=0}^{n-1} s_k^{-1-k} L_y L_x \left\{ \frac{\partial^k f(x, y, 0)}{\partial t^k} \right\} \right]. \quad (25)$$
where \( L_{1yz} = L_1L_yL_x \).

4. Solution of fractional telegraph equation using Caputo fractional derivatives. The aims of this section is devoted to establish the solution of the second-order linear two-space dimensional hyperbolic telegraph equation of fractional order [20], using triple Laplace transform. The telegraph equation of fractional order is given by

\[
D_t^{2\alpha}u(x,y,t) + 2aD_t^\alpha u(x,y,t) + b^2u(x,y,t) = (D_x^2 + D_y^2)u(x,y,t),
\]

along with initial and boundary conditions

\[
\begin{aligned}
&u(0,y,t) = u(x,0,t) = u_t(x,y,0) = 0, \\
&D_xu(0,y,t) = \pi E_\alpha(-y\alpha), \quad y > 0, t > 0, \\
&D_yu(x,0,t) = \pi E_\alpha(x\alpha), \quad u(x,0) = e^{-y}, \quad x > 0, y > 0, t > 0,
\end{aligned}
\]

where \( 0 < \alpha \leq 1 \) and \( D_t, D_x, D_y \) denote the corresponding partial derivative with respect to \( t, x, y \) respectively. The equation mostly models signal analysis for propagation and transmission of electrical signals. Using the definition of double Laplace transform on initial and boundary condition Eq.(27), we have

\[
\begin{aligned}
L_1L_y \{u(0,y,t)\} &= L_1L_x \{u(x,0,t)\} = L_yL_x \{u_t(x,y,0)\} = 0, \\
L_1L_y \{D_xu(0,y,t)\} &= \frac{\pi S_2^{\alpha-1}}{S_3(1+S_2^2)}, \quad L_1L_x \{D_yu(x,0,t)\} = \frac{\pi S_1^{\alpha-1}}{S_3(S_1^2 - 1)}, \\
L_yL_x \{u(x,y,0)\} &= \frac{1}{S_1(S_2 + 1)}.
\end{aligned}
\]

The application of triple Laplace transform on two dimension fractional telegraph equation and by the use of linearity property at Eq.(39) yields

\[
\begin{aligned}
L_1L_yL_x \{D_t^{2\alpha}u(x,y,t)\} + L_1L_yL_x \{2aD_t^\alpha u(x,y,t)\} + b^2L_1L_yL_x \{u(x,y,t)\}
&= L_1L_yL_x \{D_x^2u(x,y,t)\} + L_1L_yL_x \{D_y^2u(x,y,t)\}.
\end{aligned}
\]

Using the definition of triple Laplace transform as in Theorem 3.5, Eq.(29) reduce to the following equation

\[
\begin{aligned}
(1+2a)S_3^{\alpha-1}L_1L_yL_x \{u(x,y,t)\} + b^2L_1L_yL_x \{u(x,y,t)\} &= (1+2a)S_3^{\alpha-1}L_yL_x \{u(x,y,0)\} \\
+ S_1^2L_1L_yL_x \{u(x,y,t)\} - L_1L_y \{u(0,y,t)\} - S_1L_1L_y \{D_xu(0,y,t)\} \\
+ S_2^2L_1L_yL_x \{u(x,y,t)\} - L_1L_x \{u(x,0,t)\} - S_2L_1L_x \{D_yu(x,0,t)\}.
\end{aligned}
\]

Direct substitution of Eq.(28) in Eq.(30) with some algebraic manipulation and little re-arrangement yields

\[
\begin{aligned}
L_1L_yL_x \{u(x,y,t)\} &= \frac{(1+2a)S_3^{\alpha-1}}{S_1(S_2 + 1)((1+2a)S_3^2 + b^2 - S_1^2 - S_2^2)} \\
&\quad - \frac{\pi S_1S_2^{\alpha-1}}{S_3(1+S_2^2)((1+2a)S_3^2 + b^2 - S_1^2 - S_2^2)} \\
&\quad - \frac{\pi S_1^{\alpha-1}S_2}{S_3(S_1^2 - 1)((1+2a)S_3^2 + b^2 - S_1^2 - S_2^2)}.
\end{aligned}
\]
Before to apply the inverse triple Laplace transform on Eq. (31), we give the following results

\[(1 + x)^\alpha = \sum_{p=0}^{\infty} \frac{1}{p!} \frac{\Gamma(p - \alpha)}{\Gamma(\alpha)} (-x)^p, \quad (32)\]

The application of the above equation, i.e. Eq. (32) in the Eq. (31) leads to the following equation

\[L_{txy}\{u\} \]

\[
= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

where \(L_{txy}\{u\} = L_t L_y L_x\) and \(u = u(x, y, t)\). The application of the inverse triple Laplace on the above equation leads to the following equation

\[u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(m + n) \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
\times \frac{\pi^{2p+2q+2r+2}}{\Gamma(2v - 2p - 2q)} \Gamma(-2v - 2r) \Gamma(\alpha p + 1)
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(m + n) \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(m + n) \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

In Fox H-function

\[u(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(m + n) \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
\times H_{1,8}^{2,1} \left[ \left( \frac{x}{y} \right)^2 \right]_{(1 + p, 1), (1 - m - n, 0), (0, 1), (2, 0), (2, 0), (1 - p, 0), (1 + 2p + 2q, 2), (1 + 2r, 2), (\alpha p, 0)}
\]

\[
- \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^{q+r+v} \Gamma(m + n) \Gamma(v - p)\pi^{-(2p+2q+2r+2)} a^m n b^{2m} \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}{v! \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(-1) \Gamma(p) S_1^{2v-2p-2q} S_2^{2v-2r} S_3^{\alpha p + 1}}
\]

\[
\times H_{1,7}^{2,1} \left[ \left( \frac{x}{y} \right)^2 \right]_{(1 + p, 1), (1 - m - n, 0), (0, 1), (2, 0), (2, 0), (1 - p, 0), (2 + 2p, 2), (\alpha + \alpha s, 2), (-\alpha - \alpha p, 0)}
\]
Applying Laplace transform and using eq. (8) and (38), we get
\begin{equation}
\frac{-\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{t^{2\alpha} y^{2n} \pi^{1+2p}}{\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{t^{2\alpha} y^{2n} \pi^{1+2p}}} \times H_{1,6}^{2,1} \left[ \begin{array}{c}
-\left( \frac{x}{y} \right)^{2} \left( 1 + p, 1, (1 - m - n, 0), (0, 1), (2, 0), (1 - p, 0), (2p + \alpha, 2), (2, -2), (1 - \alpha - \alpha p, 0). \right) \end{array} \right] \right. \end{equation}

We used the following property of Fox H-function for obtaining Eq. (33)
\begin{equation}
H_{1,6}^{1,1} \left[ \begin{array}{c}
-\sigma \left( 1 - a_{1}, A_{1}, ..., (1 - a_{s}, A_{s}) \right) \left( 1, 0), (1 - b_{1}, B_{1}), ..., (1 - b_{t}, B_{t}) \right) \end{array} \right] = \sum_{r=0}^{\infty} \frac{\Gamma(a_{1} + A_{1} r) ... \Gamma(a_{s} + A_{s} r)}{r! \Gamma(b_{1} + B_{1} r) ... \Gamma(b_{t} + B_{t} r)} \sigma^{r}. \end{equation}

5. Solution of fractional telegraph equation using Atangana-Baleanu fractional derivative. In this section, the fractional order telegraph equation given by
\begin{equation}
D_{t}^{2\alpha} u(x, y, t) + 2a D_{t}^{\alpha} u(x, y, t) + b^{2} u(x, y, t) = (D_{x}^{2} + D_{y}^{2}) u(x, y, t),
\end{equation}
along with initial and boundary conditions
\begin{equation}
\begin{cases}
\begin{aligned}
u(0, y, t) &= \nu(x, 0, t) = \nu_{t}(x, y, 0) = 0, \\
u(x, y, 0) &= e^{-y}, \quad x > 0, y > 0, t > 0,
\end{aligned}
\end{cases}
\end{equation}
is solved by Fourier sine transform and Laplace transform using Atangana-Baleanu fractional derivative. To do this, we multiply both sides of Eq. (9) by \( \sin(\zeta y) \), and integrating the result with respect to \( \zeta \) from 0 to \( \infty \), (we do not need a specific boundary condition for \( u(\infty, t) \) but only require that, for a physically meaningful system, \( u \) and all of its derivatives in \( y \) vanish as \( y \to \infty \)) we attain the differential equation
\begin{equation}
D_{t}^{2\alpha} u_{n}(x, \zeta, t) + 2a D_{t}^{\alpha} u_{n}(x, \zeta, t) + b^{2} u_{n}(x, \zeta, t) = D_{x}^{2} u_{n}(x, \zeta, t) - \zeta^{2} u_{n}(x, \zeta, t), \quad \zeta > 0.
\end{equation}
where the Fourier sine transform \( u_{n}(x, \zeta, t) \) of \( u(x, y, t) \) has to satisfy the conditions
\begin{equation}
u_{n}(x, \zeta, 0) = \frac{2\zeta}{\pi(1 + \zeta^{2})} 	ext{ and } \frac{\partial u_{n}(x, \zeta, 0)}{\partial x} = 0.
\end{equation}
Applying Laplace transform and using eq. (8) and (38), we get
\begin{equation}
D_{\zeta}^{2} \tilde{u}_{n}(x, \zeta, s) - (\zeta^{2} + b^{2} + \frac{\Gamma(\alpha)(1 + 2a)}{(1 - \alpha)(s^{\alpha} + \frac{\alpha}{1 - \alpha})}) \tilde{u}_{n}(x, \zeta, s)
\end{equation}
\begin{equation}
= -s^{\alpha - 1}(1 + 2a)2\zeta^{\alpha - 1} \left( 1 - \alpha \right) \left( s^{\alpha} + \frac{\alpha}{1 - \alpha} \right) \left( 1 + \zeta^{2} \right),
\end{equation}
where \( \tilde{u}_{n}(x, \zeta, s) \) is the Laplace transform of \( u_{n}(x, \zeta, t) \). The solution of the above equation is
\begin{equation}
\tilde{u}_{n}(x, \zeta, s) = \frac{K}{H} e^{-\sqrt{\pi x} - 1}.
\end{equation}
where
\begin{equation}
H = \zeta^{2} + b^{2} + \frac{\Gamma(\alpha)(1 + 2a)}{(1 - \alpha)(s^{\alpha} + \frac{\alpha}{1 - \alpha})}.
\end{equation}
and
\[ K = \frac{s^{\alpha-1}(1+2a)2\zeta}{(1-\alpha)(s^{\alpha} + \frac{\alpha}{1-\alpha})\pi(1+\zeta^2)}. \] (42)

To get the analytical solution for \( u(x, y, t) \) and to avoid difficult calculations of contour integrals and residues, we will apply the discrete inverse Laplace transform method, but first we have to expressed Eq.(40) in series form as

\[
\bar{u}_n(x, \zeta, s) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{j+m+n}x^j \zeta^{-(2j+1)}a^{-j}b^n}{m!n!p!x^{-j-m}} \Gamma(-1)\Gamma(-j)\Gamma(-j-1)s^{j+p-\alpha n} \]
\[ - \frac{\Gamma(k-i)\Gamma(l+i)}{1+o-\alpha k+1} \]

Applying the discrete inverse Laplace transform to Eq.(43), we get

\[
\bar{u}_n(x, \zeta, t) = \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{j+m+n+p}x^j \zeta^{-(2j+1)}a^{-j}b^n}{m!n!p!x^{-j-m}} \Gamma(-1)\Gamma(-j)\Gamma(-j-1)(1+p-\alpha n)t^{j+p+\alpha n-1} \]

Finally, the inverse finite Fourier sine transform gives the analytical solution

\[
u(x, y, t) = \frac{2}{\pi} \sum_{\zeta=1}^{\infty} \sin(\zeta y) \sum_{i=0}^{\infty} \sum_{o=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+o+k+1}x^i \zeta^{-(2i+1)}a^{-i}b^n}{m!n!p!x^{-i-m}} \Gamma(-1)\Gamma(-i)\Gamma(-i+1)\Gamma(-1)\Gamma(-i+1)\Gamma(-i+1-\alpha n)t^{i+1-\alpha n+1} \]

To get Eq.(45) in a more compact form we use Fox H-function,

\[
u(x, y, t) = \frac{2}{\pi} \sum_{\zeta=1}^{\infty} \sin(\zeta y) \sum_{i=0}^{\infty} \sum_{o=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+o+k+1}x^i \zeta^{-(2i+1)}a^{-i}b^n}{m!n!p!x^{-i-m}} \Gamma(-1)\Gamma(-i)\Gamma(-i+1)\Gamma(-1)\Gamma(-i+1)\Gamma(-i+1-\alpha n)t^{i+1-\alpha n+1} \]

\[
\times H_{1,3}^3 \left[ \frac{1}{t} \left| \begin{array}{c} 1, (1-o+i, 0), (1-k+i, 0), (1-i, 1), \\ (2,0), (1-i,0), (1-i,0), (1+i,0), (0,1), (1+i-o+ak,-1). \end{array} \right| \right] 
- \frac{2}{\pi} \sum_{\zeta=1}^{\infty} \sin(\zeta y) \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{j+m+n+p}x^j \zeta^{-(2j+1)}a^{-j}b^n t^{-\alpha n-m}}{m!n!} \]

\[
\times H_{1,3}^3 \left[ t \left| \begin{array}{c} 1, (1-m+j, 0), (1-n-j, 0), (-j, 1), \\ (2,0), (1+j,0), (1-j,0), (2+j,0), (0,1), (\alpha n+m,1). \end{array} \right| \right]. \] (46)

Figure 1. The plot shows comparison between AB (lower surface) and Caputo (upper surface) for $u(x, y, t)$ at fixed $y = 0.5$.

Figure 2. The plot shows comparison between AB (dotted) and Caputo (solid) by considering solution profile of $u(x, y, t)$ at fixed $x = 0.5$ and $y = 0.5$. 
Figure 3. The plot shows comparison between AB (lower surface) and Caputo (upper surface) for $u(x,y,t)$ at fixed $t = 0.5$.

Figure 4. The plot shows comparison between AB (red/bottom curve) and Caputo (blue/top curve) by considering solution profile of $u(x,y,t)$ at fixed $y = 1$ and $t = 0.5$. 
Figure 5. The plot shows comparison between AB (upper surface) and Caputo (lower surface) for $u(x, y, t)$ at fixed $x=-1.5$.

Figure 6. The plot shows comparison between AB (dotted curve) and Caputo (solid curve) by considering solution profile of $u(x, y, t)$ at fixed $x = -1.5$ and $t = 1$.

6. Conclusion. The solution of fractional differential equations is an interesting tool, especially the partial fractional differential equations need more attention. In
this article, we have developed the triple Laplace transform for the solution of telegraph equation having fractional order in two dimension. We first establish the formulas of triple Laplace transform for partial derivatives, integral with fractional orders and applied it successfully to partial integrals and partial derivatives of fractional order. Finally, we have applied the triple Laplace transform and obtained the solution of generalized fractional telegraph equation. We have sureness in that the proposed technique is also appropriate to further types of fractional order partial differential equations.

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