Physics of non-Abelian vortices in Bose-Einstein condensates

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Abstract. A wide order-parameter manifolds of Bose-Einstein condensates (BECs) with spin degrees of freedom enables various kinds of topological defects which never appear in conventional scalar BECs. In this paper, we focus on the characteristic example of them; non-Abelian vortices and their dynamics in the cyclic phase of a spin-2 BEC. Non-Abelian vortices shows the unique effect in their collision dynamics, i.e. unlike Abelian vortices, they do neither reconnect themselves nor pass through each other but create a rung vortex between them.

1. Introduction
Quantized vortices have been one of the main topics in superfluid systems, and had a long research history since their theoretical prediction [1] and experimental observations [2]. Quantized vortices have quantized circulations of the superfluid velocity, and a vortex in a scalar Bose-Einstein condensate, for example, has the circulation being integer multiple of $h/M$. Here $h$ is the Planck’s constant and $M$ is the mass of atoms. The quantized circulation has been observed in superfluid $^4$He [2], superfluid $^3$He [3], and atomic BECs [4, 5].

Quantized vortices are the topological defects of order parameters of superfluid, and can be classified as topological invariants defined by the topological structures of the order parameters. Topological invariants of quantized vortices (line defects) are defined by how the order parameters change along the closed path encircling the vortices. In the case of scalar BECs, for example, the order-parameter manifold is $U(1)$, and the $U(1)$ gauge changes by integer multiple of $2\pi$ along the closed path. As a result, topological invariants of vortices in scalar BECs are classified as the additive group of integers, which corresponds to the circulations of vortices being integer multiple of $h/M$.

For the case of multi-component BECs, a discrete symmetry enables vortices to have fractional circulations. Characteristic examples are half-quantized vortices in superfluid $^3$He-A [6], vortex cores in superfluid $^3$He-B, and the polar phase of a spin-1 spinor BEC. As other kind of vortices having interesting topological invariants, we discuss non-Abelian vortices here. For BECs with spin degrees of freedom, besides the $U(1)$ gauge, the direction of the $SO(3)$ spin rotates along the closed path encircling the vortices. Because $SO(3)$ is the non-Abelian group, topological invariants can be classified as discrete non-Abelian subgroup of $SO(3)$, defining non-Abelian vortices.

The non-Abelian property of non-Abelian vortices becomes remarkable in their collision dynamics. When two $U(1)$ vortices collide, they reconnect themselves. This reconnection of
vortices has been studied theoretically [10, 11, 12], and observed in superfluid \(^4\)He [13]. As other abnormal examples, collision of two \(U(1)\) vortices in the attractive BEC [14, 15] or two \(U(1) \times U(1)\) vortices in the attractive two-component BEC [16] results in the formation of a new vortex bridging two colliding vortices, like a rung of a ladder. We define this vortex as a rung vortex.

When the vortices are non-Abelian, the situation changes. It was predicted that the collision of two non-Abelian vortices produces a rung vortex [17, 18, 19]. In this paper, furthermore, we show that for two vortices with non-commutative topological invariants, both reconnection and passing through are topologically forbidden and only the formation of a rung vortex is allowed. As the other dynamics demonstrating the genuine non-Abelian character, we show that the two twisted vortices with commutative topological invariants can unravel, whereas those with non-commutative ones cannot.

Non-Abelian vortices and their collision dynamics can be realized in the cyclic phase of the spin-2 BEC which has the non-Abelian tetrahedral symmetry.

In this paper, we briefly overview our study of non-Abelian vortices. In Sec. 2, we define non-Abelian vortices based on homotopy theory and discuss their collision dynamics algebraically. In Sec. 3, we review the spin-2 BECs and non-Abelian vortices, and show our recent results for several collision dynamics of non-Abelian vortices.

2. Non-Abelian vortices and their collision dynamics

In this section, we define topological invariants of quantized vortices based on homotopy theory and non-Abelian vortices. After that, we discuss collision dynamics of vortices.

2.1. Order-parameter manifold

Quantized vortices appear in the symmetry broken system such as superfluid phase of liquid helium and ultracold atomic BECs. Let the Lie group \(G\) be a set of transformations which do not change the free energy of the system, and the subgroup \(H\) be a set of transformations which do not change the order parameter \(\psi\) of the symmetry broken system:

\[
H = \{ h \in G | h\psi = \psi \}. \tag{1}
\]

In cases of superfluid \(^4\)He, \(s\)-wave superconductors, and scalar BECs, the order parameter can be described by \(\psi = |\psi|e^{i\phi}\), where \(\phi\) is the phase of the order parameter. Because the free energy remains invariant under the \(U(1)\) gauge transformation: \(\phi \rightarrow \phi + \delta\phi\), then \(G \simeq U(1)\) and only the identity transformations remains \(\psi\) unchanged: \(H \simeq \{1\}\). The degrees of freedom of the order parameter can be described as the element of \(G\), whereas several elements in \(G\) are equivalent due to Eq. (1). Then, the exact degrees of freedom of the order parameter become the coset space: \(G/H\), which is called order-parameter manifold.

2.2. Topological invariants of quantized vortices

Let us consider the order parameter of the system \(\psi(\mathbf{x})\) as a continuous function of the coordinate \(\mathbf{x}\) in real space. A closed loop with a fixed base point in real space can be mapped into a loop in the order-parameter manifold (Fig. 1). If two loops in the order-parameter manifold can be transformed to each other through a continuous function, those loops are homotopic. If a loop in real space does not encircle a vortex, the corresponding loop in the order-parameter manifold is homotopic to the trivial loop: the base point. In contrast, a loop in real space encircles a vortex, the mapped loop in the order-parameter manifold is not homotopic to the base point. The topological invariant of a vortex can be defined by the sum of homotopic loops in the order parameter manifold.
When the order parameter can be described as a complex scalar function \( \psi(x) = |\psi(x)| e^{i \phi(x)} \), for example, the order-parameter manifold \( G/H \cong U(1) \) is isomorphic to a \( S^1 \) circle. Because loops encircling the \( S^1 \) circle \( n \) times for the nonzero integer \( n \) are not homotopic to a point, there exist vortices for such loops. Therefore, topological invariants of vortices can be classified as the integer \( n \). When a loop encircle two vortices in real space, the topological invariants of which are classified as \( n_1 \) and \( n_2 \) respectively, the mapped loop in the order-parameter manifold encircle the \( S^1 \) circle \( n_1 + n_2 \) times, which means that two vortices with topological invariants \( n_1 \) and \( n_2 \) can combine to one vortex with the topological invariant \( n_1 + n_2 \).

The topological invariant of a vortex can be defined by the map from the loop in real space to that in the order-parameter manifold, namely the fundamental group \( \pi_1[G/H] \). For \( G/H \cong U(1) \) case, the above discussion is summarized as

\[
\pi_1[G/H \cong U(1)] \cong \mathbb{Z}. \tag{2}
\]

When \( G \) is connected (\( \pi_0[G] \cong 1 \)), and simply connected (\( \pi_1[G] \cong 1 \)), then

\[
\pi_1[G/H] \cong \pi_0[H]. \tag{3}
\]

Here, the 0th homotopy set \( \pi_0[H] \) is the set of homotopic points. When \( H \) is discrete group, it satisfies that \( \pi_0[H] \cong H \).

2.3. Definition of non-Abelian vortices

For the term of non-Abelian vortices, there are three definitions:

(i) Vortices for non-Abelian \( G \),
(ii) Vortices for non-Abelian \( H \),
(iii) Vortices for non-Abelian \( \pi_1[G/H] \).

Definition (i) is the largest and includes definition (ii) which includes definition (iii). Although definition (ii) is widely used for non-Abelian vortices, we here adopt definition (iii).

When \( \pi_1[G/H] \) is non-Abelian, the topological invariant of a vortex depends on the base point of the loop in real space among the conjugacy class. To see this, we consider a vortex and paths as shown in Fig. 2. A loop \( P_1 : u \rightarrow v \rightarrow u' \rightarrow u \) with the base point \( u \) in real space encircles the vortex and is mapped to the loop in the order-parameter manifold which defines the topological invariant \( a \in \pi_1[G/H] \). Suppose the different point \( s \) from \( u \) in real space which is mapped to the same point as that mapped from \( u \) in the order-parameter manifold, and an unclosed path \( s \rightarrow u \) in real space which is mapped to the loop in the order-parameter manifold which defines
Figure 2. Paths encircling a non-Abelian vortex in real space (left) and mapped paths in the order-parameter manifold (right). A loop \( u \rightarrow v \rightarrow u' \rightarrow u \) is mapped to the loop giving the topological invariant \( a \), and the path \( s \rightarrow u \) (or \( u' \rightarrow s \)) is mapped to the loop giving the topological invariant \( b \).

The topological invariant \( b \in \pi_1[G/H] \). Then the unclosed path \( u' \rightarrow s \) gives the topological invariant \( b^{-1} \) and the loop \( P_2 : s \rightarrow u \rightarrow v \rightarrow u' \rightarrow s \) gives the topological invariant \( b^{-1}ab \). When \( \pi_1[G/H] \) is Abelian, paths \( P_1 \) and \( P_2 \) give the same topological invariant, i.e. \( a = b^{-1}ab \). However, non-commutative \( a \) and \( b \) for non-Abelian \( \pi_1[G/H] \) give different topological invariant for the two paths within the same conjugacy class.

2.4. Collision of non-Abelian vortices

Here, we consider algebraic and geometric structures of collisions of vortices. Let us consider two colliding vortices and four paths \( a, b, c, \) and \( d \), as shown in Fig. 3 (a), and assume that paths \( a \) and \( b \) define the topological invariants of vortices as \( A \) and \( B \), respectively. With the fixed base point, paths \( a \) and \( c \) are topologically equivalent. On the other hand, path \( d \) is topologically

Figure 3. (a) Four closed paths for two colliding vortices, where \( A \) and \( B \) denote the corresponding topological invariants. (b) Path homotopic to path \( d \) in (a), and topological invariants of vortices. (c) Y-shaped junction. (d)-(f) Collision patterns from two vortices starting from the configuration in (a). (d) Passing through. (e) Rung vortex \( AB \) between two vortices. (f) Rung vortex \( BA^{-1} \) between two vortices. (g) Twisted vortices. (h) Collision of twisted vortices and rung vortex \( BAB^{-1}A^{-1} \).
different from path b, but it can be continuously deformed to the path shown in Fig. 3 (b). As a result, path d defines the topological invariant as $ABA^{-1}$. When the invariants $A$ and $B$ are non-commutative for non-Abelian vortices, paths b and d give different topological invariants for the same vortex. Similarly, we can define the topological invariants of vortices for a Y-shaped junction as shown in Fig. 3 (c). Upon the collision of two vortices [Fig. 3(a)], three possible consequences [Figs. 3 (d) - 3 (f)] follow, where Fig. 3 (d) shows “passing through”, and Fig. 3 (e) and 3 (f) show formation of “rung vortices” that bridge the two vortices.

2.4.1. Same topological invariants: $A = B$ In this case, $A$ and $B$ are commutative, and all three cases are topologically allowed. In particular, Fig. 3 (f) reduces to reconnection because the rung vortex vanishes identically ($BA^{-1} = 1$). As other dynamics, Fig. 3 (e) shows the formation of the doubly quantized rung vortex $AB = A^2$ which is energetically unfavorable. Passing through in Fig. 3 (d) is also energetically unfavorable because the doubly quantized vortex is formed at just the moment of passing through. Therefore, reconnection is the most favorable dynamics in this case, except for specific situations such as the collision of attractive vortices as in type-I superconductors, where whether a rung vortex is formed or not is strongly dependent on the kinematic parameters of the collision [14, 15].

2.4.2. Different and commutative topological invariants: $A \neq B$ and $AB = BA$ In this case, both Figs. 3 (e) and 3 (f) reduce to the formation of a rung vortex which increases the total vortex line length and is energetically unfavorable. Therefore, passing through in Fig. 3 (d) is the most favorable dynamics. One exception is the formation of a rung vortex $AB$ for the attractive $U(1) \times U(1)$ BEC where a vortex $A$ and the other vortex $B$ are attractive each other and the rung vortex $AB$ (Fig. 3 (e)) or $BA^{-1}$ (Fig. 3 (f)) is formed [16].

2.4.3. Non-commutative topological invariants: $AB \neq BA$ When $A$ and $B$ are non-commutative for non-Abelian vortices, the transition from Fig. 3 (a) to 3 (e) is topologically forbidden because they are topologically distinct. Therefore, a rung vortex with the topological invariant of $AB$ or $BA^{-1}$ must be formed after the collision, regardless of the kinematic parameters such as the collision angles and the initial relative speed.

2.4.4. Collision of two twisted vortices Collision of two twisted vortices as shown in Fig. 3 (g) shows the genuine non-Abelian character. Figure 3 (h) shows the formation of the rung vortex $BAB^{-1}A^{-1}$ which always vanishes for Abelian vortices. Therefore, twisted vortices with commutative topological invariants can unravel, whereas twisted non-Abelian vortices with non-commutative ones cannot.

3. Non-Abelian vortices in spin-2 Bose-Einstein condensates
Spinor BECs with spin degrees of freedom admit various kind of topological excitations such as not only quantized vortices but also monopoles and skyrmions, reflecting their rich variety of order-parameter manifolds [20, 21, 22].

In this section, we show that non-Abelian vortices can exist in the cyclic phase of the spin-2 BEC. To do this, we first discuss the theory of the spin-2 BEC [23, 24, 25, 26, 27, 28, 29, 30, 31, 34, 35] and the cyclic phase which is one of possible ground states.
3.1. Spin-2 BEC

3.1.1. Hamiltonian

The system of spin-2 bosons can be described as 5-components Bose field \( \Psi = (\Psi_2, \Psi_1, \Psi_0, \Psi_{-1}, \Psi_{-2})^T \). The Hamiltonian becomes

\[
\hat{H} = \int dx \hat{\Psi}^\dagger_m(x) \left[ -\frac{\hbar^2}{2M} \nabla^2 + V^{\text{ext}}(x) \right] \hat{\Psi}_m(x) + \frac{1}{2} \int dx_1 dx_2 \hat{\Psi}^\dagger_{m_1}(x_1) \hat{\Psi}^\dagger_{m_2}(x_2) V^{\text{int}}_{m_1,m_2,m'_1,m'_2}(x_1-x_2) \hat{\Psi}_{m'_2}(x_2) \hat{\Psi}_{m'_1}(x_1),
\]

where \( M \) is the particle’s mass, \( V^{\text{ext}} \) is the external potential, and \( V^{\text{int}} \) is the particle interaction, and repeated indices are assumed to be summed over 2, 1, \( \cdots \), -2.

As an actual system, we consider spinor BECs of dilute cold atomic gases. In this system, the main inter-particle interaction is the two body \( s \)-wave scattering. When the dipole-dipole interaction is neglected, we obtain

\[
V^{\text{int}}_{m_1,m_2,m'_1,m'_2}(x_1-x_2) = \frac{4\pi\hbar^2}{M} \delta(x_1-x_2) \sum_{S=0,2,4} \sum_{M_s=-S}^{S} a_S C_{m_1,m_2}^{S,M_s} (C_{m'_1,m'_2}^{S,M_s})^*,
\]

Here, \( a_S \) is the two body \( s \)-wave scattering length for the total spin \( S \) channel and \( C_{m_1,m_2}^{S,M_s} \) is the Clebsch-Gordan coefficient. From Eqs. (4) and (5), we obtain

\[
\hat{H} = \int dx \hat{\Psi}^\dagger_m(x) \left[ -\frac{\hbar^2}{2M} \nabla^2 + V^{\text{ext}}(x) \right] \hat{\Psi}_m(x) + \frac{1}{2} \int dx \hat{N} \left[ c_0 \hat{n}^2(x) + c_1 \hat{F}^2(x) + c_2 \hat{A}_{20}(x) \hat{A}_{20}(x) \right],
\]

where,

\[
\hat{n}(x) = \sum_{m=-2}^{2} \hat{\Psi}^\dagger_m(x) \hat{\Psi}_m(x),
\]

\[
\hat{F}(x) = \sum_{m,n=-2}^{2} f_{m,n} \hat{\Psi}^\dagger_m(x) \hat{\Psi}_n(x),
\]

\[
\hat{A}_{20}(x) = \left[ 2 \hat{\Psi}_2(x) \hat{\Psi}_{-2}(x) + \hat{\Psi}_1(x) \hat{\Psi}_{-1}(x) + \hat{\Psi}_0^2(x) \right],
\]

are the density, spin density, and the pair-singlet operators, and \( \hat{N}[\cdots] \) is the operator for the normal ordering. \( f_{m,n} = (f^x_{m,n}, f^y_{m,n}, f^z_{m,n}) \) is the \( 5 \times 5 \) spin matrix:

\[
f^x = \frac{f^+ + (f^+)^T}{2}, \quad f^y = \frac{f^+ - (f^+)^T}{2i},
\]

Coefficients \( c_0, c_1, \) and \( c_2 \) satisfy

\[
c_0 = \frac{4\pi\hbar^2 4a_2 + 3a_4}{M}, \quad c_1 = \frac{4\pi\hbar^2 a_4 - a_2}{M}, \quad c_2 = \frac{4\pi\hbar^2 7a_0 - 10a_2 + 3a_4}{35}. \]
Here, we apply the mean-field approximation. Assuming that all particles occupy the single-particle ground state, we obtain [27]

\[
H = \int dx \left[ \frac{\hbar^2}{2M} |\nabla \Psi_m(x)|^2 + V^{\text{ext}}(x)n(x) \right] + \frac{1}{2} \int dx \left[ c_0 n^2(x) + c_1 F^2(x) + c_2 |A_{20}(x)|^2 \right],
\]

(10)

where,

\[
n(x) = \sum_{m=-2}^{2} |\Psi_m(x)|^2,
\]

\[
F(x) = \sum_{m,n=-2}^{2} f_{mn} \Psi_m^*(x) \Psi_n(x),
\]

(11)

\[
A_{20}(x) = \sum_{m=-2}^{2} (-1)^m \Psi_m(x) \Psi_{-m}(x),
\]

are the density, spin density, and singlet-pair amplitude. Although, for actual cold atomic BECs, we have to consider the optical trapping potential and the linear and quadratic Zeeman effects due to the external magnetic field as \(V^{\text{ext}}(x)\), we neglect these effects for simplicity and obtain the Hamiltonian density:

\[
\frac{H}{V} = \frac{1}{2} \left( c_0 n^2 + c_1 F^2 + c_2 |A_{20}|^2 \right),
\]

(12)

where \(V\) is the volume of the system.

3.1.2. Order-parameter manifold Here, we consider the transformation \(G\) which keeps the Hamiltonian in Eq. (12) invariant. For dynamical stability of BECs, \(c_0 > 0\) is required. Because the interactions proportional to \(c_0\), \(c_1\), and \(c_2\) have \(SU(5)\), \(SO(3)\), and \(SO(5)\) symmetries respectively [35], in addition to the \(U(1)\) gauge symmetry, \(G\) becomes

\[
G \simeq U(5)_{g+s} \simeq \frac{U(1)_g \times SU(5)_s}{Z_{g+s}} \quad \text{for } c_1 = c_2 = 0,
\]

\[
G \simeq U(1)_g \times SO(5)_s \quad \text{for } c_1 = 0 \text{ and } c_2 \neq 0,
\]

\[
G \simeq U(1)_g \times SO(3)_s \quad \text{for } c_1 \neq 0.
\]

(13)

Here, \(g\) and \(s\) denote the \(U(1)\) gauge and spin parts respectively.

To see the symmetry of the order parameter in detail, we define the density renormalized spinor order parameter \(\zeta = (\zeta_2, \zeta_1, \zeta_0, \zeta_{-1}, \zeta_{-2})^T\):

\[
\Psi_m = \sqrt{n} \zeta_m, \quad \sum_{m=-2}^{2} |\zeta_m|^2 = 1,
\]

(14)

and the following nematic tensor [23, 29]. Expanding \(\zeta\) by the rank-2 spherical harmonics \(Y_{2,m}(e)\) (\(e\) is the unit vector in spin space), we obtain

\[
\zeta^\Sigma(e) = \sum_{m=-2}^{2} \zeta_m Y_{2,m}(e) = \frac{1}{2} \sqrt{\frac{15}{8\pi}} e^T Q(\zeta) e,
\]

(15)
where
\[
Q(\zeta) = \begin{pmatrix}
\zeta_2 + \zeta_2 - \sqrt{\frac{2}{3}} \zeta_0 & i(\zeta_2 - \zeta_2) & -\z_1 + \zeta - 1 \\
i(\zeta_2 - \zeta_2) & -\zeta_2 - \zeta_2 - \sqrt{\frac{2}{3}} \zeta_0 & -i(\zeta_1 + \zeta - 1) \\
-\z_1 + \zeta - 1 & -i(\zeta_1 + \zeta - 1) & 2\sqrt{\frac{2}{3}} \zeta_0
\end{pmatrix}
\]  

(16)
is the traceless symmetric tensor. The spin rotation of the spinor \(\zeta\) and the \(SO(3)\) rotation of \(Q(\zeta)\) have 1 to 1 correspondence:
\[
Q(e^{-i\theta \cdot e} \zeta) = R(e, \theta) Q(\zeta) R^T(e, \theta).
\]  

(17)
Here, \(R(e, \theta)\) is the 3-dimensional rotation matrix with the rotation axis \(e\) and the rotation angle \(\theta\). Including the \(U(1)\) gauge transformation, we can show that the Hamiltonian is invariant when the arbitrary \(Q(\zeta)\) is transformed to
\[
H[\phi, R(e, \theta)] Q(\zeta) = e^{i\phi} R(e, \theta) Q(\zeta) R^T(e, \theta).
\]  

(18)
3.1.3. Ground state

The ground state can be obtained by finding the \(\zeta\) which minimize the Hamiltonian [23, 24, 27, 34].

(i) ferromagnetic-1 phase: \(c_1 < 0\) and \(c_2 > 4c_1\).

\[
\zeta_{F1} = (1, 0, 0, 0, 0)^T, \quad Q(\zeta_{F1}) = \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]  

(19)
There is a continuous transformation \(T[2\phi, R(\hat{z}, \phi)]\) keeping \(Q(\zeta_{F1})\) invariant, which reveals that the spin rotation (along the \(z\) axis) and the gauge transformation is equivalent, namely \(H_{F1} \simeq U(1)_{2g+s}\). The order-parameter manifold becomes
\[
\left(\begin{array}{c} G \\ H \end{array}\right)_{F1} \simeq \frac{U(1)_g \times SO(3)_s}{U(1)_{2g+s}} \simeq \frac{SO(3)_{g+s}}{(Z_2)_{g+s}}.
\]  

(20)
(ii) uniaxial nematic phase: \(c_1 > 0\) and \(c_2 < 0\).

\[
\zeta_{UN} = (0, 0, 1, 0, 0)^T, \quad Q(\zeta_{UN}) = \sqrt{\frac{2}{3}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix},
\]  

(21)
which remains invariant under a continuous transformation \(T[0, R(\hat{z}, \phi)]\) and a discrete transformation \(T[0, R(a\hat{x} + b\hat{y}, \pi)]\). These two transformations form the 2nd orthogonal group \(H_{UN} \simeq O(2)_s\) and the order-parameter manifold becomes
\[
\left(\begin{array}{c} G \\ H \end{array}\right)_{UN} \simeq \frac{U(1)_g \times SO(3)_s}{O(2)_s} \simeq U(1)_g \times \mathbb{RP}^2_s,
\]  

(22)
where \(\mathbb{RP}^2\) is the real projective plane.
(iii) biaxial nematic phase: $c_1 < 0$ and $c_2 < 4c_1$.

\[ \zeta_{BN} = \sqrt{\frac{1}{2}}(1, 0, 0, 0, 1)^T, \quad Q(\zeta_{BN}) = \sqrt{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \] (23)

There are seven discrete transformations $\hat{T}[0, R(\{\hat{x}, \hat{y}, \hat{z}\}, \pi)]$, $\hat{T}[\pi, R(\{\hat{y}, \{\pi/4, 3\pi/4\})]$, and $\hat{T}[\pi, R(\{\hat{x} \pm \hat{y}\}, \pi)]$ keeping $Q(\zeta_{BN})$ invariant. These transformations form the 4th dihedral group $H_{BN} \simeq (D_4)_{g+s}$ with the identity transformation and the order-parameter manifold becomes

\[ \left( \begin{array}{c} G \\ H \end{array} \right)_{BN} \simeq \frac{U(1)_g \times SO(3)_s}{(D_4)_{g+s}}. \] (24)

(iv) cyclic phase: $c_1 > 0$ and $c_2 > 0$.

\[ \zeta_C = \frac{1}{2}(i, 0, \sqrt{2}, 0, i)^T, \quad Q(\zeta_C) = \frac{\sqrt{3}}{2} \begin{pmatrix} e^{2\pi i/3} & 0 & 0 \\ 0 & e^{-2\pi i/3} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \] (25)

There are eleven discrete transformations $\hat{T}[0, R(\{\hat{x}, \hat{y}, \hat{z}\}, \pi)]$, $\hat{T}[2\pi/3, R(\{e_{1,2,3,4}\}, 2\pi/3)]$, and $\hat{T}[-2\pi/3, R(\{e_{1,2,3,4}\}, -2\pi/3)]$ keeping $Q(\zeta_C)$ invariant. Here $e_1 = (\hat{x} + \hat{y} + \hat{z})/\sqrt{3}$, $e_2 = (\hat{x} - \hat{y} - \hat{z})/\sqrt{3}$, $e_3 = (-\hat{x} + \hat{y} - \hat{z})/\sqrt{3}$, $e_4 = (-\hat{x} - \hat{y} + \hat{z})/\sqrt{3}$. These transformations form the tetrahedral group $H_C \simeq T_{g+s}$ with the identity transformation [19, 26, 30] and the order-parameter manifold becomes

\[ \left( \begin{array}{c} G \\ H \end{array} \right) \simeq \frac{U(1)_g \times SO(3)_s}{T_{g+s}}. \] (26)

(v) ferromagnetic-2 phase (This phase cannot exist as the ground state).

\[ \zeta_{F2} = (0, 1, 0, 0, 0)^T, \quad Q(\zeta_{F2}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -i \\ -1 & -i & 0 \end{pmatrix}. \] (27)

There is a continuous transformation $\hat{T}[\varphi, R(\hat{z}, \varphi)]$ keeping $Q(\zeta_{F2})$ invariant, namely $H_{F2} \simeq U(1)_{g+s}$. The order-parameter manifold becomes

\[ \left( \begin{array}{c} G \\ H \end{array} \right)_{F2} \simeq \frac{U(1)_g \times SO(3)_s}{U(1)_{g+s}} \simeq SO(3)_{g+s}. \] (28)

This state appear in the vortex core for example [19].

Each phase can also be characterized by the spin density $\mathbf{F}^2$, the singlet-pair amplitude $|A_{20}|^2$, and the singlet-trio amplitude $|A_{30}|^2$ defined as

\[ A_{30} = \frac{3\sqrt{6}}{2} \left( \Psi_0 \Psi_{-2} + 3 \Psi_{-1} \Psi_2 + \Psi_0 \Psi_{-1} - 6 \Psi_2 \Psi_{-2} \right). \] (29)

These values for each phase is summarized in TABLE 1 [32, 34].

The symmetry of each phase can be visualized by $\zeta^\Sigma$. Figure 4 (a) shows the $c_1 - c_2$ phase diagram and the shape of $\zeta^\Sigma(e)$. In the ferromagnetic-1 phase, $\zeta^\Sigma(e)$ shows the disk shape and
the phase changes by $4\pi$ around the disk. The rotation along the axis perpendicular to the disk and the $U(1)$ gauge transformation are equivalent which corresponds to $H_{01} \simeq U(1)_{2g+s}$. From the dumbbell and cloverleaf shapes of $\zeta^\Sigma(e)$ for the uniaxial nematic and biaxial nematic phases, it is easy to see that these phases have cylindrical symmetry $H_{un} \simeq O(2)_s$ and square symmetry $H_{bn} \simeq (D_1)_{g+s}$ respectively. In the cyclic phase, $\zeta^\Sigma(e)$ takes the triad shape which has the tetrahedral symmetry as shown in Fig. 4 (b).

Among four ground states, the biaxial nematic and cyclic phases gives the non-Abelian $\pi_1[G/H]$. However, the biaxial nematic and uniaxial nematic phases are energetically degenerate in the mean-field framework. The zero-point fluctuation lifts this degeneracy [32, 33, 34, 35]. In the mean-field theory, therefore, the biaxial nematic phase is difficult to study non-Abelian vortices and the cyclic phase is the best theoretical model to study those.

### 3.2. Non-Abelian vortices in the cyclic phase

We calculate the topological invariants of vortices in the cyclic phase from the order-parameter manifold of Eq. (26) [19, 26, 30]. Because $SO(3)$ is not simply connected, lets consider the mapping from $SO(3)$ to simply connected $SU(2)$:

$$\frac{U(1)_{g} \times SO(3)_s}{T_{g+s}} \simeq \frac{U(1)_{g} \times SU(2)_s}{T^*_g}$$

Here, $T^*$ is the lifting up from the tetrahedral group $T$ as the subgroup of $SO(3)$ to the double cover as the subgroup of $SU(2)$, and consists of 24 elements. To consider the action of the $SU(2)$ rotation to $\zeta$, we define the new tensor $Q(\zeta) \equiv Q(\zeta)\sigma \otimes \sigma$ and a new operation

$$T[\phi, U(e, \theta)] \equiv e^{i\phi}U(e, \theta) \otimes U(e, \theta)Q(\zeta)\left[U(e, \theta) \otimes U(e, \theta)\right]^\dagger,$$
where $\sigma$ is the Pauli matrix and $U(\mathbf{e}, \theta)$ is the $SU(2)$ rotation matrix.

Topological invariant defined as $\pi_1[[U(1)_{\mathcal{P}} \times SU(2)]_{\mathcal{T}}]$ can be described by $\mathcal{T}$. In the following, we concentrate on the energetically stable vortices and classify those by the conjugacy class. For one straight vortex along $z$-axis, we consider the order parameter $\Psi$ in the cylindrical coordinate $(r, \varphi, z)$ $(f_{1,2,3}(r) = 0$ and $f_{1,2,3}(r = \infty) = 1)$.

(i) $0 - 1/2$ vortex

Topological invariants can be written by $\mathcal{T}[0, \{\mathcal{I}_{x,y,z}\}]$, where $\mathcal{I}_{x,y,z} = U(\pi, \{\mathbf{x}, \mathbf{y}, \mathbf{z}\})$ are

$$\mathcal{I}_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \mathcal{I}_y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{I}_z = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \quad (32)$$

$\Psi$ is given by

$$\Psi(r, \varphi, z) = \hat{S} \frac{\sqrt{n}}{2} \begin{pmatrix} if_1(r)e^{-i\varphi}, 0, \sqrt{2} & 0, if_1(r)e^{i\varphi} \end{pmatrix}^T, \quad (33)$$

where $\hat{S}$ is the arbitrary $U(1)$ gauge transformation and $SO(3)$ spin rotation. 0 and 1/2 of the $0 - 1/2$ vortex denotes the mass and spin currents $\kappa_m$ and $\kappa_s$ defined as

$$\kappa_m = \kappa \int d\mathbf{x} \sum_n \text{Im}(\mathbf{\nabla}\Psi_m)\Psi_m^* \quad (34)$$

$$\kappa_s = \kappa \int d\mathbf{x} \sum_{m,n} \text{Im}(\mathbf{\nabla}\Psi_m)\mathbf{f}_{m,n}\Psi_n^*,$$

where $\kappa = h/M$. For $0 - 1/2$ vortex, $\kappa_m = 0$ and $\kappa_s \equiv \sqrt{(\kappa_m^2)^2 + (\kappa_s^2)^2 + (\kappa_s^2)^2} = \kappa/2$.

(ii) $0 - (1/2)$ vortex

Topological invariants can be written by $\mathcal{T}[0, -\{\mathcal{I}_{x,y,z}\}]$. $\Psi$ is given by

$$\Psi(r, \varphi, z) = \hat{S} \frac{\sqrt{n}}{2} \begin{pmatrix} if_1(r)e^{i\varphi}, 0, \sqrt{2} & 0, if_1(r)e^{-i\varphi} \end{pmatrix}^T. \quad (35)$$

Circulations become $\kappa_m = 0$ and $\kappa_s = \kappa/2$.

(iii) $1/3 - 1/3$ vortex

Topological invariants can be written by $\mathcal{T}[2\pi/3, \{\mathcal{E}, -\mathcal{I}_{x,y,z}\}]$, where $\mathcal{E}$ and $\mathcal{C}$ are

$$\mathcal{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{C} = U(\mathbf{e}_1, 2\pi/3) = \frac{1}{2} \left( \begin{array}{cc} 1 - i & -1 - i \\ 1 - i & 1 + i \end{array} \right) \quad (36)$$

By using the another form of the cyclic order parameter:

$$e^{i\cos^{-1}(1/\sqrt{3})}\mathbf{f}e^{i\varphi}\mathbf{f}^*/4\mathbf{\zeta}_C = \frac{1}{\sqrt{3}} \begin{pmatrix} 0, \sqrt{2}, 0, 0, 1 \end{pmatrix}$$

we obtain

$$\Psi(r, \varphi, z) = \hat{S} \sqrt{n/3} \begin{pmatrix} 0, \sqrt{2}, 0, 0, f_2(r)e^{i\varphi} \end{pmatrix}^T. \quad (38)$$

Circulations become $\kappa_m = \kappa/3$ and $\kappa_s = \kappa/3$. 
(iv) $(-1/3) - (-1/3)$ vortex
Topological invariants can be written by $T[-2\pi/3, \{-E, T_{x,y,z}\}C^2]$. $\Psi$ is given by

$$\Psi(r, \varphi, z) = \hat{S} \sqrt{\frac{n}{3}} \begin{pmatrix} 0, \sqrt{2}, 0, 0, f_2(r)e^{-i\varphi} \end{pmatrix}^T. \quad (39)$$

Circulations become $\kappa_m = -\kappa/3$ and $\kappa_s = \kappa/3$.

(v) $(-2/3) - 1/3$ vortex
Topological invariants can be written by $T[-4\pi/3, \{E, T_{x,y,z}\}C]$. $\Psi$ is given by

$$\Psi(r, \varphi, z) = \hat{S} \sqrt{\frac{n}{3}} \begin{pmatrix} 0, \sqrt{2}f_3(r)e^{-i\varphi}, 0, 0, 1 \end{pmatrix}^T. \quad (40)$$

Circulations become $\kappa_m = -2\kappa/3$ and $\kappa_s = \kappa/3$.

(vi) $2/3 - (-1/3)$ vortex
Topological invariants can be written by $T[4\pi/3, \{-E, T_{x,y,z}\}C^2]$. $\Psi$ is given by

$$\Psi(r, \varphi, z) = \hat{S} \sqrt{\frac{n}{3}} \begin{pmatrix} 0, \sqrt{2}f_3(r)e^{i\varphi}, 0, 0, 1 \end{pmatrix}^T. \quad (41)$$

Circulations become $\kappa_m = 2\kappa/3$ and $\kappa_s = \kappa/3$.

All other vortices classified by $\pi_1[U(1)_g \times SU(2)_s]/T_{g+s}^*$ are energetically unstable and split into vortices discussed above.

Order parameters of vortex cores are given by taking the limit of $r \rightarrow 0$:

(i) $0 - 1/2$ and $0 - (-1/2)$ vortices:

$$\Psi(r \rightarrow 0, \varphi, z) = \hat{S} \sqrt{\frac{n}{2}} \xi_{UN} \quad (42)$$

(ii) $1/3 - 1/3$ and $(-1/3) - (-1/3)$ vortices:

$$\Psi(r \rightarrow 0, \varphi, z) = \hat{S} \sqrt{\frac{2n}{3}} \xi_{F2} \quad (43)$$

(iii) $(-2/3) - 1/3$ and $2/3 - (-1/3)$ vortices:

$$\Psi(r \rightarrow 0, \varphi, z) = \hat{S} \sqrt{\frac{n}{3}} \xi_{F1} \quad (44)$$

Characteristics of all vortices are summarized in TABLE 2. Anti-vortex can be obtained by $T^{-1}$.

As shown in Fig. 4 (b), the tetrahedral symmetry of the cyclic phase can be visualized by the triad. Around the vortex core, the triad rotates. There are two kinds of the rotation which keeps the triad invariant: $\pi$ rotation along the lobe of the triad and $\pm 2\pi/3$ rotation along the axis crossing the center and the apex of the tetrahedron which correspond to $0 - (-1/2)$ vortices, and the other vortices respectively. For example, we plot $\zeta^\Sigma(e)$ for $0 - 1/2$ and $1/3 - 1/3$ vortices in Figs. 5 (a) and 5 (b) respectively.
In experiments, spinor BECs are realized as optically trapped cold dilute atomic gases. In this section, we discuss the cyclic phase in experiments.

### 3.3. Spin-2 BECs and the cyclic phase in experiments

In experiments, spinor BECs are realized as optically trapped cold dilute atomic gases. In this system, internal degrees of freedom are given by the hyperfine spin $F = I + S$, where $I$ and $S$ are the nuclear and electron spins. For spin-2 case, $F = 2^{87}$Rb BECs ($I = 3/2$, $S = 1/2$) are usually used because of their long lifetimes [36, 37, 38, 39, 40]. For interaction parameters of $F = 2^{87}$Rb, $c_0/(4\pi\hbar^2 a_B/M) = 106 \pm 4$, $c_1/(4\pi\hbar^2 a_B/M) = 0.99 \pm 0.06$, and $c_2/(4\pi\hbar^2 a_B/M) = -0.106 \pm 0.116$, and the uniaxial nematic phase is suggested for the ground state [39]. However, the large error bar for $c_2$ cannot exclude the possibility of the cyclic ground state because of complications arising from quadratic Zeeman effects and hyperfine-spin-exchanging relaxations [27, 34, 40].

### 3.4. Collision dynamics of non-Abelian vortices

Here, we perform the numerical simulation of vortices in the cyclic phase and show that the results satisfy the geometrical consideration discussed in the Sec. 2. To study the dynamics of vortices, we derive the Gross-Pitaevskii equation which describes the time evolution of the order parameter [27, 28]. The Gross-Pitaevskii equation can be obtained by the Hamilton equation $i\hbar \partial \Psi_m/\partial t = \delta H/\delta \Psi_m$ for the mean-field Hamiltonian:

\[
\begin{align*}
    i\hbar \frac{\partial \Psi \pm 2}{\partial t} &= \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 n \pm 2c_1 F^z \right] \Psi \pm 2 + c_1 F^z \Psi \pm 1 + c_2 A_{20} \Psi^*_{\pm 2} \\
    i\hbar \frac{\partial \Psi \pm 1}{\partial t} &= \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 n \pm c_1 F^z \right] \Psi \pm 1 + c_1 \left[ \frac{\sqrt{6}}{2} F^+ \Psi_0 + F^\pm \Psi_{\pm 2} \right] - c_2 A_{20} \Psi^*_{\pm 1} \\
    i\hbar \frac{\partial \Psi_0}{\partial t} &= \left[ -\frac{\hbar^2}{2M} \nabla^2 + c_0 n \right] \Psi_0 + \frac{\sqrt{6}}{2} c_1 \left[ F^- \Psi_0 + F^\pm \Psi_{\pm 1} + F^+ \Psi_1 \right] + c_2 A_{20} \Psi^*_0
\end{align*}
\]

### Table 2. Stable vortices in the cyclic phase. Each vortex is classified by the conjugacy class.

| Vortex | Topological Invariant | $\kappa_m/\kappa$ | $\kappa_m/\kappa$ | Vortex Core | Anti-vortex |
|--------|-----------------------|-------------------|-------------------|-------------|-------------|
| $0 - 1/2$ | $T[0, I_{x,y,z}]$ | 0 | 1/2 | $\propto \zeta_{UN}$ | $0 - (-1/2)$ |
| $0 - (-1/2)$ | $T[0, -I_{x,y,z}]$ | 0 | 1/2 | $\propto \zeta_{UN}$ | $0 - 1/2$ |
| $1/3 - 1/3$ | $T[2\pi/3, \{\mathcal{E}, -I_{x,y,z}\}]$ | $1/3$ | $1/3$ | $\propto \zeta_{F2}$ | $(-1/3) - (-1/3)$ |
| $(-1/3) - (-1/3)$ | $T[-2\pi/3, \{-\mathcal{E}, -I_{x,y,z}\}]$ | $-1/3$ | $1/3$ | $\propto \zeta_{F2}$ | $1/3 - 1/3$ |
| $(-2/3) - 1/3$ | $T[-4\pi/3, \{-\mathcal{E}, -I_{x,y,z}\}]$ | $-2/3$ | $1/3$ | $\propto \zeta_{F1}$ | $2/3 - (-1/3)$ |
| $2/3 - (-1/3)$ | $T[4\pi/3, -\{\mathcal{E}, I_{x,y,z}\}]$ | $2/3$ | $1/3$ | $\propto \zeta_{F1}$ | $(-2/3) - 1/3$ |

### Figure 5. Plot of $\Psi(x)$ around the vortex. (a) $0 - 1/2$ vortex. (b) $1/3 - 1/3$ vortex. Dashed lines in figures show the rotation axes of the triad.
Simulations are performed by using the initial state with vortices and numerically solving the Gross-Pitaevskii equation in the cubic box with the Neumann boundary [41]. Numerical parameters are:

(i) Interaction coefficients: \( c_0 > 0, \ c_1/c_0 = c_2/c_0 = 0.5 \).

(ii) The size of the box: \((64\xi_0)^3\) for the healing length \( \xi_0 \equiv \sqrt{\hbar^2/Mc_0n} \).

(iii) Number of numerical grids: 256\(^3\).

Interaction coefficients which we use make the cyclic state stable. Although these values are not realistic for the actual \(^{87}\text{Rb}\) BECs, we concentrate on the topological features for collisions of vortices, and do not pay attention to the consistency with actual experiments.

As initial states, we use two vortex configurations: (I) two straight vortices with an oblique angle and (II) two twisted vortices. As a pair of topological invariants, we use the following four patterns:

(i) \( T[2\pi/3, C] (1/3 - 1/3 \text{ vortex}) \) and \( T[2\pi/3, C] (1/3 - 1/3 \text{ vortex}) \) : same topological invariant

(ii) \( T[2\pi/3, C] (1/3 - 1/3 \text{ vortex}) \) and \( T[4\pi/3, -C^2] (2/3 - (-1/3) \text{ vortex}) \) : different and commutative topological invariants

(iii) \( T[2\pi/3, C] (1/3 - 1/3 \text{ vortex}) \) and \( T[2\pi/3, -L_y C] (1/3 - 1/3 \text{ vortex}) \) : non-commutative topological invariants

(iv) \( T[2\pi/3, C] (1/3 - 1/3 \text{ vortex}) \) and \( T[4\pi/3, -L_y C^2] (2/3 - (-1/3) \text{ vortex}) \) : non-commutative topological invariants

In the following, we denote the type of collision as (I)-(i) (two straight vortices with the same topological invariants).

Figure 6 shows the collisions of vortices with commutative topological invariants: (I)-(i) (Fig. 6 (a)-(c)) and (I)-(ii) (Fig. 6 (d)-(f)). Figures 6 (a)-(c) and 6 (d)-(f) show the reconnection and passing through, which corresponding to Fig. 3 (f) (the rung vortex vanishes: \( BA^{-1} = T[2\pi/3, C] (T[2\pi/3, C])^{-1} = T[0, \xi] \) ) and Fig. 3 (d).

Figure 7 shows the collisions of vortices with non-commutative topological invariants: (I)-(iii) (Fig. 6 (a)-(c)) and (I)-(iv) (Fig. 6 (d)-(f)). Both Figs. 7 (a)-(c) and 7 (d)-(f) show the formations of rung vortices, the topological invariants of which are \( BA^{-1} = \)}
Figure 7. Collisions of vortices with non-commutative topological invariants. (a)-(c) (I)-(iii) (rung vortex \( T[0, I_y] \)). (d)-(f) (I)-(iv) (rung vortex \( T[-2\pi/3, -I_z C^2] \)). Topological invariant of each vortex is labeled: 
\[ -I_{x,y,z} C \equiv T[2\pi/3, -I_{x,y,z} C], -I_z C^2 \equiv T[4\pi/3, -I_z C^2], I_y \equiv T[0, I_y], \text{ and } -I_z C^2 \equiv T[-2\pi/3, -I_z C^2] \]

\[ T[2\pi/3, C] \left( T[2\pi/3, -I_y C] \right)^{-1} = T[0, I_y] \] and \( BA^{-1} = T[2\pi/3, C] \left( T[4\pi/3, -I_z C^2] \right)^{-1} = T[-2\pi/3, -I_z C^2] \) being consistent with Fig. 3 (f).

We have performed numerical simulations with various combinations of topological invariants, relative velocities, and collision angles, and confirmed that passing through and reconnection occur only when the topological invariants of the two vortices are commutative, and that the formation of a rung vortex always occurs when the topological invariants of two vortices are non-commutative.

Figures 8 (a)-(c) and 8 (d)-(f) show the collision of two twisted vortices: (II)-(ii) and (II)-(iii). For commutative topological invariants: (II)-(ii), vortices unravel by the passing through as shown in Fig. 8 (a)-(c). On the other hand, twisted vortices with non-commutative topological
invariants cannot unravel because of the formation of the rung vortex as shown in Fig. 8 (d)-(f) which corresponds to Fig. 3 (h), i.e. the topological invariant of the rung vortex is $BA B^{-1} A^{-1} = T[2\pi/3, -J_x C] T[2\pi/3, C] (T[2\pi/3, -J_x C])^{-1} (T[2\pi/3, C])^{-1} = T[0, -J_x]$.

We finally describe a possible experimental manifestation of rung vortices. The phase-contrast imaging experiment [42] enables the measurement of local magnetization, and vortices with ferromagnetic-1 and ferromagnetic-2 cores like Fig. 7 (c)-(f) manifest themselves as bridged structures of localized magnetization.

4. Conclusion
In Sec. 2, we explain quantized vortices and their topological invariants based on homotopy theory, and discuss the definition of non-Abelian vortices and their collision dynamics. Non-Abelian vortices are defined as vortices with non-Abelian topological invariants given by the fundamental group of the order-parameter manifold. When two non-Abelian vortices with non-commutative topological invariant collide, they can neither reconnect nor path through each other, and form a rung vortex bridging two colliding vortices. In the case of two twisted vortices with non-commutative topological invariants, they cannot unravel because of the formation of the rung vortex.

In Sec. 3, we discuss the cyclic phase of spin-2 BECs as the system in which non-Abelian vortices can be realized. The fundamental group of the order-parameter manifold in the cyclic phase can be described by the non-Abelian tetrahedral group. We perform numerical simulations of the Gross-Pitaevskii equation for spin-2 BECs and show that collision dynamics of non-Abelian vortices discussed in Sec. 2 can be realized in this system.

In this paper, we concentrate on the cyclic phase of spin-2 BECs as the system for the study of non-Abelian vortices. Non-Abelian properties given by our results are, however, universal for any non-Abelian vortex in other system. We hope that our results lead further understanding of non-Abelian vortices in various kinds of system such as cosmic strings [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56], liquid crystals [17, 18, 57, 58, 59, 60, 61, 62], and so on.

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