Analytical solutions for cosmological perturbations in a one-component universe with shear stress

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Abstract

In this paper we construct explicit solutions for scalar, vector and tensor perturbations in a less known setting, a flat universe filled by an isotropic elastic solid with pressure and shear modulus proportional to energy density. The solutions generalize well known formulas for cosmological perturbations in a universe filled by an ideal fluid.

1 Introduction

In standard cosmological model the anisotropic stress is usually supposed to be zero. However, the presence of the anisotropic stress significantly modifies the evolution of cosmological perturbations. We study this effect in a model which contains an additional elastic solid continuum and we present analytical solutions of the equations of motion of perturbations in the case of a one-component universe. The effect naturally depends on the properties of the added continuum, therefore we need a theory describing elastic continuum in the framework of the theory of general relativity at first. Such theory is called relasticity (short name for relativistic elasticity).

2 Equations of motion

The dynamics of a spacetime containing relastic continuum is governed by Einstein equations $2G_{\mu\nu} = T_{\mu\nu}$ (we use units $c^4/16\pi\kappa = 1$ and $c = 1$), which relate Einstein tensor $G_{\mu\nu}$ to the energy-momentum tensor $T_{\mu\nu}$, and equations of state $p_i = p_i(\rho_i)$, which relate pressure $p_i$ of the $i$-th component of the universe to its energy density $\rho_i$. We consider a homogeneous and isotropic flat universe (Robertson-Walker metric with $k = 0$) with perturbations in the form of planar waves imposed on the metric and energy density. Einstein equations are then reduced to Friedmann equations (zero order equations determining evolution of the scale parameter $a$) and linearized equations describing evolution of cosmological perturbations. Friedmann equations are

$$\dot{a}^2 = a^{-1}\varepsilon/6, \quad \partial\varepsilon_i/\partial a = -3a^{-1}\sigma_i,$$

where we have defined energy per particle $\varepsilon \equiv \rho a^3$ and pressure energy per particle $\sigma \equiv p a^3$.

The quantities $\varepsilon_i$ and $\sigma_i$ entering the second equation are fractions of $\varepsilon$ and $\sigma$ going to the $i$-th component of the universe, $\varepsilon_i \equiv \rho_i a^3$ and $\sigma_i \equiv p_i a^3$.

The linearized equations governing cosmological perturbations are more complicated and can be found in the textbooks on cosmology, for example in [2] or [4]. As a result of linearization, they can be separated into three independent sets of equations, each set defining one type of perturbations: scalar (longitudinal waves), vector (transversal waves) and tensor (gravitational waves).

Usually the continuum is supposed to be an ideal fluid, therefore only the isotropic stress is considered. We extend this standard model by allowing one or more components of the universe to be an isotropic elastic solid. At this point two new parameters $\lambda$ and $\mu$, defined analogically as Lame coefficients in classical elasticity, enter the calculations (see [3]).
The standard cosmology is formulated in terms of the pressure energy per particle, which is not used in classical elasticity. Given the dependence of \( \sigma \) on \( a \), the \( \lambda \) coefficient is redundant as seen from the equation (valid for each component of the universe separately, if they do not interact with each other)

\[
\frac{\partial \sigma}{\partial a} = -a^{-1}(2\sigma + 3\lambda + 2\mu)
\]

(2)

Instead of the coefficient \( \mu \) it is more convenient to work with the shear modulus of the continuum \( \mu_s = \mu + \sigma \) (shear modulus of an ideal fluid is zero). Other interesting quantities are sound speeds. Let us define an auxiliary speed of sound \( c_s \), which is an analogue of the speed of sound in an ideal fluid at constant entropy, and two physical speeds of sound, \( c_s^\parallel \) (longitudinal) and \( c_s^\perp \) (transversal, see a hint of the derivation in [11]),

\[
c_s^2 \equiv \frac{dp}{d\rho} = \frac{1}{3} \frac{5\sigma + 3\lambda + 2\mu}{\varepsilon + \sigma}, \quad c_s^\parallel \equiv \frac{2\mu + \lambda + 3\sigma}{\varepsilon + \sigma}, \quad c_s^\perp \equiv \frac{\mu + \sigma}{\varepsilon + \sigma}.
\]

(3)

The relation \( 3c_s^2 = 3c_s^\parallel^2 - 4c_s^\perp^2 \) follows immediately from these definitions. The definitions of sound speeds are again valid for each component of the universe separately and characterize sound waves in the respective component. Combined sound speeds (with each variable in the definition summed over the components of the universe) enter the equations for perturbations to the metric.

In this paper we analytically solve equations of motion for perturbations in comoving and proper time gauge, which were derived in [3]. Later we will show connection of the variables characterizing perturbations in this gauge with the more common gauge invariant variables introduced in [2]. Equations for scalar perturbations from [3], rewritten in a more convenient way (prime denotes differentiation with respect to conformal time \( \eta \)), are

\[
\frac{a}{a}S_0 = 3c_s^2s_0 - 3c_s^2s_1, \quad \frac{a}{a}s_1' = \frac{1}{3}k^2s_0 + \frac{1}{4} \left( \varepsilon + \sigma \right) (s_0 - s_1),
\]

(4)

where \( s_0 \equiv 6y_{01}\dot{a}/a \) and \( s_1 \equiv y_{11} \). The vector perturbations satisfy

\[
\frac{a}{a}v_\alpha' = 3c_s^2v_\alpha + 4c_s^2v_\alpha, \quad \frac{a}{a}v_\alpha' = \frac{1}{4}k^2v_\alpha - \frac{1}{4} \left( \varepsilon + \sigma \right) (v_\alpha - v_\alpha),
\]

(5)

where \( v_\alpha \equiv -\frac{1}{4}k^2v_\alpha \) and \( v_\alpha \equiv h_{\alpha\alpha} \). Index \( \alpha \) refers to the transversal direction (for \( k = (k, 0, 0) \) it assumes values 2, 3). Later on we omit this index because the evolution of \( v_\alpha \), \( v_\alpha \) is independent of the choice of \( \alpha \) (the set of equations for these quantities consists of two independent identical sets, one for each value of \( \alpha \)). The tensor perturbations (transversal and traceless part of the metric perturbations denoted as \( h_{ij} \) in [3]) have two degrees of freedom (two polarizations) which both satisfy

\[
h'' + 2\frac{a'}{a}h' + \left( k^2 + c_s^2\frac{\varepsilon + \sigma}{a} \right) h = 0.
\]

(6)

## 3 One component universe

The equations governing cosmological perturbations are solvable only numerically in a general multicomponent universe, however analytical solutions do exist in a one-component universe in case the dimensionless parameters \( w \equiv \sigma/\varepsilon \) and \( \xi = \mu_s/\varepsilon \) are both constant. The parameter \( w \) is a standard parameter used in the description of ideal fluid components of the universe and the parameter \( \xi \) quantifies the magnitude of the shear stress of the continuum.

Friedmann equations are easy to solve in a one-component universe, with the solution \( \varepsilon \propto a^{-3w} \) and \( a \propto \xi^{2/3(1+w)} \propto \eta^{2/(1+3w)} \). The sound speeds are constants and simplify as well to

\[
c_s^2 = w, \quad c_s^\parallel = w + \frac{4\xi}{3(1+w)}, \quad c_s^\perp = \frac{\xi}{1+w}.
\]

(7)
After all these simplifications, one obtains from the sets of equations for scalar and vector perturbations and the equation for tensor perturbation three second order differential equations for spherical Bessel functions multiplied by an appropriate power of $\eta$:

$$\eta^2 s''_0 + \frac{4}{1 + 3w} \eta s'_0 + \left[ c_{s||}^2 k^2 \eta^2 + 4 c_{s\perp}^2 \frac{6(1 + w)}{(1 + 3w)^2} \right] s_0 = 0.$$  \hspace{1cm} (8)

$$\eta^2 v''_0 - 2 \eta v'_0 + \left[ c_{s\perp}^2 k^2 \eta^2 + 3 c_{s||}^2 \frac{6(1 + w)}{(1 + 3w)^2} \right] v_0 = 0.$$  \hspace{1cm} (9)

$$\eta^2 h'' + \frac{4}{1 + 3w} \eta h' + \left[ k^2 \eta^2 + 4 c_{s\perp}^2 \frac{6(1 + w)}{(1 + 3w)^2} \right] h = 0.$$  \hspace{1cm} (10)

To express the solutions in a compact way let us define a new parameter $n_\xi$ by the equation

$$n_\xi(n_\xi + 1) \equiv 2 - 3 c_{s||}^2 \frac{6(1 + w)}{(1 + 3w)^2}.$$  \hspace{1cm} (11)

It will be the order of spherical Bessel functions. Both solutions to equation (11) are admissible, we can choose for definiteness that with the plus sign in front of the square root of the discriminant. The parameter $n_\xi$ obtained in this way depends on $\xi$ because the sound speed does so, and in the special case $\xi = 0$ it has the value $n_0 = (1 - 3w)/(1 + 3w)$ (for $w > 1$ or $w < -1/3$ the value is $n_0 = -2/(1 + 3w)$). The solutions of equations (8), (9) and (10) are:

$$s_0 = \eta^{-n_0} \left[ A_s j_{n_\xi}(\phi_{||}) + B_s y_{n_\xi}(\phi_{||}) \right],$$  \hspace{1cm} (12)

$$v_0 = \eta^2 \left[ A_v j_{n_\xi}(\phi_{\perp}) + B_v y_{n_\xi}(\phi_{\perp}) \right],$$  \hspace{1cm} (13)

$$h = \eta^{-n_0} \left[ A_h j_{n_\xi}(\phi) + B_h y_{n_\xi}(\phi) \right].$$  \hspace{1cm} (14)

The coefficients $A$, $B$ are dependent on the initial conditions and the arguments of the spherical Bessel functions $j_{n_\xi}$ and $y_{n_\xi}$ are $\phi_{||} \equiv c_{s||} k \eta$, $\phi_{\perp} \equiv c_{s\perp} k \eta$ and $\phi \equiv k \eta$. The solution enters oscillatory mode after the perturbation crosses the respective sound horizon. This behavior of the perturbations, observed also in a universe filled with ideal fluid, follows from the property of spherical Bessel functions that they are maximal approximately when the argument equals one and then start to oscillate. To complete the picture it remains to write down the functions $s_1$ and $v_1$,

$$s_1 = -\eta^{-n_0} \frac{k^2 \eta^2}{3} \frac{1 + 3w}{2} \left[ A_s j_{n_\xi}(\phi_{||}) + B_s y_{n_\xi}(\phi_{||}) \right],$$  \hspace{1cm} (15)

$$v_1 = \frac{k^2 \eta^2}{4} \left( \frac{1 - 3w}{2} v_0 + \eta^2 \frac{k^2 \eta^2}{4} \frac{1 + 3w}{2} \left[ A_s j_{n_\xi}(\phi_{\perp}) + B_s y_{n_\xi}(\phi_{\perp}) \right] \right),$$  \hspace{1cm} (16)

where we have used auxiliary functions

$$\hat{j}_{n_\xi}(x) \equiv \frac{1}{x} \left( \partial_x j_{n_\xi} - \frac{1}{x} j_{n_\xi} \right), \hspace{1cm} \hat{y}_{n_\xi}(x) \equiv \frac{1}{x} \left( \partial_x y_{n_\xi} - \frac{1}{x} y_{n_\xi} \right).$$  \hspace{1cm} (17)

## 4 Special cases

A special care is required in the case $w = -1$. The parameter $w$ assumes this value throughout and deeply withing dark energy dominated era as well. For such $w$, the scale parameter grows exponentially, $a \propto e^{Ht}$ ($H$ is inflationary Hubble parameter, which is constant because energy density stays constant). Granted $\sigma = -\varepsilon$, the equations (11) and (15) simplify considerably to

$$s_1 = 0, \hspace{1cm} H s_1 = \frac{1}{3 a^2} s_0, \hspace{1cm} v_1 = 0, \hspace{1cm} H v_1 = -\frac{1}{4 a^2} v_0.$$  \hspace{1cm} (18)
with trivial solutions \( s_0 = 0 \), \( s_1 = 0 \) and \( v_0 = 0 \), \( v_1 = 0 \). The equation governing tensor perturbations stays unchanged and its solution is obtained by performing the limit \( w \to -1 \) in the general solution (13), which can be easily done \((n_\xi = -1/2 + \sqrt{9-24\xi}/2 \) and \( n_0 = 1 \)).

Another special case is \( w = -1/3 \). It is the case of cosmic strings (since the linear energy density of the string is constant), and also the curvature term satisfies this equation of state. The problem is the choice of conformal time throughout the calculations, because the scale parameter depends exponentially and not polynomially on it. The dependence is \( a \propto e^{H\eta} = Ht \), where \( H \equiv a'/a \) is a constant parameter. Considering this dependence we get simplified sets of equations

\[
\frac{1}{H} \dot{s}'_0 = (1 - 6\xi) s_1 - s_0, \quad Hs'_1 = \frac{1}{3} k^2 s_0 + H^2 (s_0 - s_1),
\]

\[
\frac{1}{H} \dot{v}'_0 = 6\xi v_1 - v_0, \quad Hv'_1 = \frac{1}{4} k^2 v_0 - H^2 (v_0 - v_1),
\]

\[
h'' + 2Hh' + (k^2 + 6\xi H^2) h = 0.
\]

The solutions of these equations are no longer combinations of Bessel functions. Instead, they are polynomials when expressed in cosmological time,

\[
s_0 = A_s t^{m|| - 1} + B_s t^{-m|| - 1}, \quad v_0 = A_s t^{m\perp} + B_s t^{-m\perp}, \quad h = A_h t^{m\times - 1} + B_h t^{-m\times - 1},
\]

where the three parameters \( m \equiv (m||, m\perp, m\times) \) are defined in terms of the respective sound speeds \( c \equiv (c||, c\perp, 1) \) as

\[
m_p = (1 - c_p^2 k^2 H^{-2} - 6\xi)^{1/2}.
\]

The remaining variables \( s_1 \) and \( v_1 \) are to be derived from equations (19) of (20) respectively (the calculations are more complicated when \( \xi = 0 \) or \( \xi = -1/6 \)). We do not show these solutions here.

The last special case occurs when \( w \) is arbitrary and \( \xi = -3w(1+w)/4 \) or \( \xi = 0 \). In this case the independent variable \( \phi|| \) or \( \phi\perp \) is identically zero \((c|| = 0 \) or \( c\perp = 0 \)). It is the limit case of general solutions, the respective spherical Bessel functions in the solution stay constant.

### 5 Gauge invariant variables

So far we have calculated perturbations in the specific gauge proposed in [3]. However, it is more convenient to work with gauge invariant variables. Tensor perturbations are invariant by themselves, but \( s_0, s_1, v_0 \) and \( v_1 \) are gauge dependent, therefore we must find relations between them and the gauge invariant variables \( \Phi, \Psi \) and \( V_\alpha \) introduced in [2].

According to the definitions of the variables \( s_0 \) and \( s_1 \) (following from definitions of \( y_{01}, y_{11} \) in [3]), the expressions for the potentials \( \Phi \) and \( \Psi \) are

\[
\Psi = \frac{1}{8} \frac{a^2}{k^2} \frac{\varepsilon + \sigma}{a^3} (s_0 - s_1), \quad \Phi = \Psi + \frac{1}{2} \frac{a^2}{k^2} \frac{\mu_s}{a^3} s_1
\]

The limit of zero shear stress, for which the relation \( \Phi = \Psi \) is satisfied (standard scenario of a universe filled by an ideal fluid [2]), is a good back-check of the calculations.

Vector perturbations are described by the gauge invariant variable \( V_\alpha \), for which we have (see the definition of \( h_{1\alpha} \) in [3])

\[
V_\alpha = -\frac{a}{k} \frac{\varepsilon + \sigma}{a^3} v_0 \alpha.
\]

After inserting here from equations (12), (13) and (15) we obtain for scalar perturbations:

\[
\Psi = -\frac{1}{2\eta^m (\phi||)^2} \frac{1 + w}{1 + 3w} \left\{ \phi|| [A_s j_{n_\xi + 1} (\phi||) + B_s y_{n_\xi + 1} (\phi||)] - \left[ (n_\xi - n_0) + (n_0 + 1) \frac{4\xi}{1 + w} \right] [A_s j_{n_\xi} (\phi||) + B_s y_{n_\xi} (\phi||)] \right\}.
\]
\[ \Phi = -\frac{1}{2 \eta n_0 (\phi_\parallel)^2} \left\{ \left( 1 - \frac{4 \xi}{1 + w} \right) \phi_\parallel \left[ A_s j_{n_\xi+1} (\phi_\parallel) + B_s y_{n_\xi+1} (\phi_\parallel) \right] - \right. \\
\left. - \left[ \left( 1 - \frac{4 \xi}{1 + w} \right) (n_\xi - n_0) + \frac{8 \xi}{1 + w} \right] \left[ A_s j_{n_\xi} (\phi_\parallel) + B_s y_{n_\xi} (\phi_\parallel) \right] \right\}, \quad (27) \]

and for vector perturbations:

\[ \frac{a}{k} V = -\frac{12}{k^2 (1 + 3w)^2} \left[ A_v j_{n_\xi} (\phi_\perp^\xi) + B_v y_{n_\xi} (\phi_\perp^\xi) \right]. \quad (28) \]

6 Conclusion

We have found explicit solutions for scalar, vector and tensor perturbations in a one-component universe with shear stress. The solutions are combined of spherical Bessel functions and their form in the specific case \( \mu_s = 0 \) can be easily verified by comparing them with the formulas to be found in the literature [2].

References

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