Dynamic bifurcation for a three-species cooperating model

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Abstract. A dynamic bifurcation analysis on a three-species cooperating model was presented and it was proved that the problem bifurcated an attractor as the parameter \(\lambda\) crossed the critical value \(\lambda_0\). The analysis was based on the attractor bifurcation theory together with the central manifold reduction.

1. Introduction

In this paper, we consider a three-species cooperating model in one-dimensional

\[
\begin{align*}
\frac{\partial u}{\partial t} &= p_1 \Delta u + u(a - bu + cv + dw), \\
\frac{\partial v}{\partial t} &= p_2 \Delta v + v(e + fu - gv + hw), \\
\frac{\partial w}{\partial t} &= p_3 \Delta w + w(r + lu + mv - nw),
\end{align*}
\]

with the initial and Neumann boundary conditions:

\[
\begin{align*}
  u(x, 0) &= u_0, \\
  v(x, 0) &= v_0, \\
  w(x, 0) &= w_0, \\
  \left. \frac{\partial u}{\partial n} \right|_{\partial \Omega} &= \left. \frac{\partial v}{\partial n} \right|_{\partial \Omega} = \left. \frac{\partial w}{\partial n} \right|_{\partial \Omega} = 0,
\end{align*}
\]

which indicates that the system is closed and the species have no communication with the outside world.

The unknown functions include the population density functions \(u, v, w\). The above problem (1) arises in a simple food chain describing three interacting species in a spatial habitat \(\Omega\). The spatial density of three species at time \(t\) is represented by \(u, v, w\). Where \(p_1, p_2, p_3\) and \(a, e, r\) represent the diffusivity and growth rate of the three species respectively. The parameters \(b, c, d, f, g, h, m, n\) are positive constants, where \(b, g, n\) are respective intraspecific competition and \(c, d, f, h, l, m\) are interspecific cooperation. For instance, \(cv\) represents the beneficial effect of the specie \(v\) on the specie \(u\), \(-gv\) represents the internal competition of \(v\).
There have been many studies on the reaction-diffusion system of three species. Dancer and Du [1,2] have given various sufficient conditions (and also a necessary condition) for the classical three-species system to have at least one positive solution. Their work in [1,2] more or less reveal the similarity between the three-species system and the two-species system. Kim [3] considered properties of solutions for a cooperating three-species food chain model, the work has shown that global solutions exist if the intraspecific competitions are strong whereas blow-up solutions exist under certain conditions if the intra-specific competitions are weak. For more research about the three-species system refer to [4–8].

Inspired by the above research and the dynamic theory, we will investigate the dynamic bifurcation for the three-species system. It worth noting that Ma and Wang [9,10,11] developed the new dynamic bifurcation theory based a notion of bifurcation called attractor bifurcation. The main theorem associated with attractor bifurcation states that if there are $m$ eigenvalues crossing the imaginary axis, then the system bifurcates from a trivial steady state solution to an attractor with dimension between $m - 1$ and $m$, as the control parameter crosses a certain critical value, provided the critical state is asymptotically stable. Using this new attractor bifurcation theory, Ma and Wang studied bifurcation and stability of the solution for many dynamical systems. For example, The Rayleigh-Bénard convection [12,13,14], The Taylor problem [15,16], they also discussed the dynamic phase transition theory for the PVT systems [17] and the Magneto-hydrodynamic Convection [11].

The main objective of this paper is to conduct bifurcation and stability analysis for the three-species cooperating model by using this new attractor bifurcation theory.

2. Eigenvalue problem

In this paper, we mainly focus on the dynamic bifurcation for the model (1). For simplicity, we assume the space $\Omega = [0, L] \subset \mathbb{R}^1$, and the let the population rates $a = \lambda$ be control parameters.

To get the abstract form of (1) and (2), we define the following spaces

$$ X = \left\{ U = (u, v, w) \mid u, v, w \in L^2(\Omega) \right\}, $$

$$ X_1 = \left\{ U = (u, v, w) \mid u, v, w \in H^2(\Omega) \text{ and } \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0 \right\}, $$

where

$$ L^2(\Omega) = \left\{ U : \Omega \to \mathbb{R}^3 \mid \left( \int_{\Omega} |U|^2 \right)^{\frac{1}{2}} < \infty \right\}, $$

$$ H^2(\Omega) = \left\{ U \in L^2(\Omega) \mid D^2 U \in L^2(\Omega) \right\}, \forall \alpha < 2. $$

Now we define the operators $L_2 = -A + B_1 : X_1 \to X$, $G : X_1 \to X$ by

$$ -AU = (p_1 \Delta u, p_2 \Delta v, p_3 \Delta w), $$

$$ B_1 U = (-\lambda u, ev, rw), $$

$$ GU = (-bu^2 + cuv + dvw, fuv - gw^2 + hvw, luv + mw^2 - nw^2), $$

where $U = (u, v, w) \in X_1$.

Therefore, the problems (1) and (2) are equivalent to the following abstract equation:

$$ \begin{cases} \frac{dU}{dt} = L_2 U + G(U), \\ U(x, 0) = U_0, \end{cases} \tag{3} $$

where the initial value $U_0 = (u_0, v_0, w_0)$, and $\lambda = a$ is the control parameters.

We consider the following eigenvalue equations of (3).
\[
\begin{cases}
p_1 \Delta u + \lambda u = \beta u, \\
p_2 \Delta v + ev = \beta v, \\
p_3 \Delta w + lw = \beta w, 
\end{cases}
\]  
(4)

with the boundary condition (2).

Let \( \psi_k \) and \( \rho_k \) be the eigenvalue and eigenvector of Laplacian with the Neumann condition

\[
\begin{cases}
-\Delta \psi_k = \rho_k \psi_k, \\
\frac{\partial \psi_k}{\partial n} |_{\partial \Omega} = 0, \\
\int_{\Omega} \psi_k^2 dx = 1.
\end{cases}
\]  
(5)

And \( \rho_k \) satisfy

\[
0 = \rho_0 < \rho_1 < \rho_2 < \cdots < \rho_k < \cdots,
\]

\[
\rho_k = \frac{k^2 \pi^2}{L^2}, \psi_0 = \frac{1}{\sqrt{L}} \psi_k = \frac{\sqrt{2}}{L} \cos \frac{k\pi x}{L}.
\]

Denote by \( M_k \) the matrix given by

\[
M_k = \begin{pmatrix}
-p_1 \rho_k + \lambda & 0 & 0 \\
0 & -p_2 \rho_k + e & 0 \\
0 & 0 & -p_3 \rho_k + l
\end{pmatrix}.
\]

It is clear that all eigenvalues \( \beta_i \) (i = 1, 2, 3) and eigenvectors \( e_i \) (i = 1, 2, 3) satisfy the following equations

\[
Le_i = M_k e_i = \beta_i e_i,
\]

\[
e_i = \xi_i \psi_k, i = 1, 2, 3,
\]

where \( \xi_i \in \mathbb{R}^3 \) are the eigenvectors of \( M_k \), \( \beta_i \) are the eigenvectors of \( L \), which are expressed as

\[
\beta_i = \lambda - p_i \rho_k, \beta_k = e - p_2 p_k, \beta_k = l - p_3 p_k.
\]

Obviously, the following theorem hold true:

**Theorem 2.1** Let \( \lambda_0 = p_k \rho_k, k \geq 1 \), then we have the following assertions: \( \beta_i (\lambda) \) is the first real eigenvalues of \( (3) \) near \( \lambda = \lambda_0 \) satisfying that

\[
\beta_i (\lambda) = \lambda - p_i \rho_k \begin{cases}
> 0, \lambda > \lambda_0, \\
= 0, \lambda = \lambda_0, \\
< 0, \lambda < \lambda_0,
\end{cases}
\]  
(6)

\[
\beta_i (\lambda_0) \neq 0, \forall i = 2, 3;
\]

\[
\beta_i (\lambda_0) \neq 0, \forall l \neq k, i = 1, 2, 3.
\]

It is easy to determine the dual eigenvector are given by

\[
e^*_i = e_i;\]
and satisfy
\[ L e_k^* = \beta_k e_k^*, i = 1, 2, 3, \]

3. Bifurcation analysis

Then, under the condition (6), for the system (1) with the initial and boundary condition (2), we have the following transition theorem.

**Theorem 3.1** Let \( H \) be the number given by
\[ H(\lambda) = \frac{b^2}{L(\lambda - \rho_{2k})}, \]

Then the transition of (1) and (2) is continuous if \( H(\lambda_0) < 0 \), and is jump if \( H(\lambda_0) > 0 \). Moreover, the following assertions hold true:

1. If \( H(\lambda_0) < 0 \), (1) has no bifurcation on \( \lambda < \lambda_0 \), and has extract two bifurcated solutions \( U_k^\pm(\lambda) \), which are attractors. In addition, there is a neighborhood \( A \subset X \) near \( U = 0 \), such that the stable manifold \( \Gamma \) of \( U = 0 \) divided \( A \) into two disjoint open sets \( A^+ \) and \( A^- \). Moreover, \( U^+ \subset A^+ \), \( U^- \subset A^- \), and \( U_k^\pm \) attract \( A^\pm \).

2. If \( H(\lambda_0) > 0 \), (1) has no bifurcation on \( \lambda > \lambda_0 \), has exact two bifurcated solutions \( U_k^\pm(\lambda) \), which are saddles.

3. The solutions \( U_k^\pm(\lambda) \) can be expressed as
\[ U_k^\pm(\lambda) = \pm \frac{\beta_k(\lambda)}{H(\lambda_0)} e_k + o \left( \frac{\beta_k(\lambda)}{H(\lambda_0)} e_k \right). \]

**Proof.** We will apply theorem 2.1 and the center manifold reduction to prove the theorem. We know that all eigenvector \( e_k \) of \( L_k \) constitute an orthogonal base of \( X \). Hence \( X_1 \) and \( X \) can be deposed into the direct sum:
\[ X_1 = E_1 + E_2, X = E_1 \oplus E_2, \]

where
\[ E_1 = span\{e_k\}, \]
\[ E_2 = span\{e_j, e_{j_k}, e_{j_1}\}, j \neq k, i = 1, 2, 3. \]

Then \( U = (u, v, w) \in X \) can be written as
\[ U = (u, v, w) = x_k e_k + \sum_{i \neq k} y_i e_i + \sum_{j=0}^{\infty} y_{j_k} e_{j_k} + \sum_{j=0}^{\infty} y_{j_1} e_{j_1}. \]

The function (3) can be written as
\[ \frac{d}{dt}(x_k e_k + \sum_{i \neq k} y_i e_i + \sum_{j=0}^{\infty} y_{j_k} e_{j_k} + \sum_{j=0}^{\infty} y_{j_1} e_{j_1}) = L(x_k e_k + \sum_{i \neq k} y_i e_i + \sum_{j=0}^{\infty} y_{j_k} e_{j_k} + \sum_{j=0}^{\infty} y_{j_1} e_{j_1}) + G(x_k e_k + \sum_{i \neq k} y_i e_i + \sum_{j=0}^{\infty} y_{j_k} e_{j_k} + \sum_{j=0}^{\infty} y_{j_1} e_{j_1}). \]

Equation of (3) on the center manifold is given by
\[
\begin{align*}
\frac{dx_k}{dt} &= \beta_k x_k + \langle G(x_k e_k + \Phi, e_k^*), \\
\frac{dy_k}{dt} &= \beta_k y_k + \langle G(x_k e_k + \Phi, e_k^*), \\
\frac{dy_{j_2}}{dt} &= \beta_{j_2} y_{j_2} + \langle G(x_k e_k + \Phi, e_{j_2}^*), \\
\frac{dy_{j_3}}{dt} &= \beta_{j_3} y_{j_3} + \langle G(x_k e_k + \Phi, e_{j_3}^*),
\end{align*}
\]

(7)

where \( \Phi \) is the center manifold function at \( \lambda = \lambda_0 \), which is

\[
\Phi(x) = \sum_{i=k} y_i e_i + \sum_{j=0}^{\infty} y_{j_2} e_{j_2} + \sum_{j=0}^{\infty} y_{j_3} e_{j_3}.
\]

(8)

By direct calculation from (7), we have

\[
y_{i_1} = \begin{cases} 
\frac{b}{\sqrt{2Lb_{ik}}}, & |x_k|^2, i_1 = 2k, \\
0(|x_k|^2), & i_1 \neq 2k, 
\end{cases}
\]

\[
y_{j_2} = o(|x_k|^2), \quad y_{j_3} = o(|x_k|^2).
\]

(9)

Hence, we obtain from (7), (8), (9) that

\[
\Phi(x, \lambda) = \frac{bx_k^2}{\sqrt{2Lb_{2k} \lambda}} e_{(2k)0} + o(|x_k|^2).
\]

(10)

Inserting (10) into (7), we have

\[
\frac{dx_k}{dt} = \beta_k x_k - H(\lambda)x_k^3 + o(|x_k|^3),
\]

(11)

where

\[
H(\lambda) = \frac{b^2}{L(\lambda - \rho_{2k}k)}.
\]

Hence, the theorem follow from (11). The proof is completed.

Remark: The local topological structure of the transitions of (1) and (2) is schematically shown in the centre manifold in Figure 1 and Figure 2.

![Topological structure of continuous transition of (1) and (2), when \( H(\lambda_0) < 0 \).](image-url)
Figure 2. Topological structure of continuous transition of (1) and (2), when \( H(\lambda_0) > 0 \).

4. Conclusion
In this article, we first prove the existence of bifurcation when the system parameter crosses critical number \( \lambda_0 \), which is the first eigenvalue of the eigenvalue problem of the linearized equation of (4). Second, we show that the type of bifurcation varies with the system parameter \( H(\lambda) \). If \( H(\lambda_0) < 0 \), the transition is continuous. However, if \( H(\lambda_0) > 0 \), the transition is jump. At last, the expression of bifurcated solution is also obtained.

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