To the problem of decomposition of the initial boundary value problems in mechanics

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Abstract. We consider the questions on the decomposition of the initial boundary value problems of elasticity theory and thin bodies for some anisotropic media. In particular, the initial boundary problems of the micropolar (and classical) theory of elasticity are presented, i.e. to introduce tensor-block matrix operators (tensor-operators). In the case of an isotropic micropolar elastic medium (isotropic and transversally isotropic classical media) tensor-block matrix operators (tensor-operators) of cofactors corresponding to the tensor-block matrix operators (tensor-operators) of of the initial boundary value problems are obtained, which are splitting the initial boundary value problems. From three-dimensional decomposed initial boundary value problems the corresponding decomposed initial boundary value problems for the theory of thin bodies are obtained.

1. Equations of motion relative to the displacement and rotation vectors for an elastic material without a center of symmetry

The constitutive relations for the linear elastic nonhomogenous anisotropic material without a symmetry center for small displacements and rotations and also isothermal processes have the form

\[\mathbf{P} = \mathbf{A} \otimes \mathbf{\gamma} + \mathbf{B} \otimes \mathbf{\kappa}, \quad \mathbf{\mu} = \mathbf{C} \otimes \mathbf{\gamma} + \mathbf{D} \otimes \mathbf{\kappa} \quad (\mathbf{\gamma} = \nabla \mathbf{u} - \mathbf{C} \mathbf{\cdot} \mathbf{\varphi}, \quad \mathbf{\kappa} = \nabla \mathbf{\varphi}),\]

(1)

where \(\mathbf{P}\) and \(\mathbf{\mu}\) are tensors of stresses and couple stresses, respectively, \(\mathbf{\gamma}\) is the strain tensor, \(\mathbf{\kappa}\) is the bending-torsion tensor, \(\mathbf{u}\) is the displacement vector, \(\mathbf{\varphi}\) is the rotation vector, \(\mathbf{A}, \mathbf{C} = \mathbf{B}^T\) and \(\mathbf{D}\) are the material tensors of the 4th rank, \(\mathbf{C}\) is the discriminant 3rd rank tensor, the superscript \(T\) in the upper right corner of the quantities denotes transposition, \(\otimes\) is the inner 2-product [1–10].

Substituting (1) in the equations of motion for small displacements and rotations

\[\nabla \cdot \mathbf{P} + \rho \mathbf{F} = \rho \partial^2_t \mathbf{u}, \quad \nabla \cdot \mathbf{\mu} + \mathbf{C} \otimes \mathbf{P} + \rho \mathbf{m} = \mathbf{J} \partial^2_t \mathbf{\varphi}\]

and introducing the tensor-block matrix operator of the equations of motion and the vector-columns of displacement and rotation vectors and vectors of volume forces and volume moments

\[\mathbf{M} = \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}, \quad \mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{\varphi} \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} \rho \mathbf{F} \\ \rho \mathbf{m} \end{pmatrix},\]

(2)
we obtain the equations of motion in displacement and rotation vectors
\[ \mathbb{M} \cdot \mathbf{U} + \mathbf{X} = 0, \]  
(3)
where the differential tensor-operators \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) and \( \mathbf{D} \) have the following expressions:
\[ \mathbf{A} = \mathbf{A}' - \mathbf{E} \rho \partial^2, \quad \mathbf{A}' = r_{ij} r_{kl} (A^{ijkl} \nabla_i + \nabla_i A^{ijkl}) \nabla_k, \]
\[ \mathbf{B} = r_{ij} r_{kl} [(B^{ijkl} \nabla_i + \nabla_i B^{ijkl}) \nabla_k - C^{ijkl}_{mn} A^{mn} \nabla_i A^{mn}], \]
\[ \mathbf{C} = r_{ij} r_{kl} [(B^{ijkl} \nabla_i + \nabla_i B^{ijkl}) + C^{ijkl}_{mn} A^{mn} \nabla_k], \quad \mathbf{D} = \mathbf{D}' - \mathbf{A} \partial^2, \]
\[ \mathbf{D}' = r_{ij} r_{kl} [(B^{ijkl} \nabla_i + \nabla_i B^{ijkl}) + (g^i_{kl} + g^i_{jl} - g^i_{rl} - g^i_{kj}) C^{ijkl}_{mn} B^{mn} \nabla_k - C^{ijkl}_{pq} (A^{pkmn} C^{lm} + A^{kn} B^{pq})]. \]

Here we take \( \mathbf{E} \) as the second-rank identity tensor, \( t \) is the time, \( \partial_t \) is the operator of the partial derivative with respect to time.

2. On static boundary conditions in the linear three-dimensional micropolar theory of elasticity. Tensor-boundary conditions
Taking into account (1), the static boundary conditions can be represented as follows
\[ \mathbf{T}^{(1)} \cdot \mathbf{u} + \mathbf{T}^{(2)} \cdot \mathbf{\varphi} = \mathbf{P}, \quad \mathbf{T}^{(3)} \cdot \mathbf{u} + \mathbf{T}^{(4)} \cdot \mathbf{\varphi} = \mathbf{\mu}, \]  
(5)
where \( \mathbf{P} \) and \( \mathbf{\mu} \) are the stress and couple stress vectors, respectively, given on the surface of the body and we introduce the following differential tensor operators
\[ \mathbf{T}^{(1)} = r_{ij} r_{kl} A^{ijkl} \nabla_i, \quad \mathbf{T}^{(2)} = r_{ij} r_{kl} B^{ijkl} \nabla_i - \mathbf{n} \cdot \mathbf{A} \otimes \mathbf{C}, \]
\[ \mathbf{T}^{(3)} = r_{ij} r_{kl} C^{ijkl} \nabla_i, \quad \mathbf{T}^{(4)} = r_{ij} r_{kl} D^{ijkl} \nabla_i - \mathbf{n} \cdot \mathbf{C} \otimes \mathbf{C}. \]

Introducing the tensor-block matrix operator of stress and couple stress and also the vector-column of stress and couple stress vectors
\[ \mathbf{T} = \begin{pmatrix} \mathbf{T}^{(1)} \\ \mathbf{T}^{(2)} \\ \mathbf{T}^{(3)} \\ \mathbf{T}^{(4)} \end{pmatrix}, \quad \mathbf{Q} = \begin{pmatrix} \mathbf{P} \\ \mathbf{\mu} \end{pmatrix}, \]
the static boundary conditions (5) taking out of the second relation (2) we can write by the form
\[ \mathbf{T} \cdot \mathbf{U} |_{S} = \mathbf{Q}. \]  
(6)

We note that the kinematic boundary conditions can be written as
\[ \mathbf{U} |_{S} = \mathbb{H} \quad \left( \mathbb{H} = \begin{pmatrix} \mathbf{f} \\ \mathbf{\psi} \end{pmatrix} \right), \]  
(7)
mixed boundary conditions can be written as
\[ \mathbf{T} \cdot \mathbf{U} |_{S_1} = \mathbf{Q}; \quad \mathbf{U} |_{S_2} = \mathbb{H}, \]
and the initial conditions have the form
\[ \mathbf{U} |_{t=t_0} = \mathbf{U}_0 \quad \mathbf{V} |_{t=t_0} = \mathbf{V}_0, \]  
(8)
where
\[ \mathbf{U}_0 = \begin{pmatrix} \mathbf{u}_0 \\ \mathbf{\varphi}_0 \end{pmatrix}, \quad \mathbf{V}_0 = \frac{d \mathbf{U}}{d t} |_{t=t_0} = \begin{pmatrix} \mathbf{v}_0 \\ \mathbf{\omega}_0 \end{pmatrix}, \quad \mathbf{v}_0 = \frac{d \mathbf{u}}{d t} |_{t=t_0}, \quad \mathbf{\omega}_0 = \frac{d \mathbf{\varphi}}{d t} |_{t=t_0}. \]

Here \( \mathbf{f} \) and \( \mathbf{\psi} \) are the displacement and rotation vectors given on the body surface; \( \mathbf{u}_0 \) and \( \mathbf{\varphi}_0 \) are the displacement and rotation vectors given for \( t = t_0 \); \( \mathbf{v}_0 \) and \( \mathbf{\omega}_0 \) are the velocity and angular velocity vectors given for \( t = t_0 \); \( S \) is the body surface, \( S_1 \cup S_2 = S, S_1 \cap S_2 = S \).
3. Formulation of initial-boundary value problems

We introduce the definitions.

**Definition 1.** If the vectors of displacements and rotations (kinematic boundary conditions) are given on the boundary of the body \( S \), then such conditions are called the first boundary conditions, and the problem of micropolar mechanics of deformable solids (MDS) using these conditions, as well as the initial conditions are called the first initial-boundary value problem.

In this case, the first initial-boundary value problem includes: the equations of motion (3), the kinematic boundary conditions (7) and the initial conditions (8).

**Definition 2.** If static boundary conditions (stress and couple stress vectors) are given on the body’s boundary \( S \), then such boundary conditions are called the second boundary conditions, and the problem of micropolar MDS, using them and the initial conditions, is called the second initial-boundary value problem.

In this case, the second initial-boundary value problem includes: the equations of motion (3), the static boundary conditions (6) and the initial conditions (8).

**Definition 3.** If static boundary conditions are given on one part of the body boundary \( S_1 \), and static boundary conditions are given on the rest of its part \( S_2 \), then such boundary conditions are called mixed boundary conditions, and the problem of micropolar MDS using them and initial conditions are called mixed initial-boundary value problem.

In this case, the mixed (third) initial-boundary value problem includes: the equations of motion (3), the kinematic boundary conditions (7) on one part of the body boundary and the static boundary conditions (6) on the other part of the body boundary and the initial conditions (8).

We note that, excluding the characteristics of the micropolar theory from the above definitions, we obtain the corresponding definitions for classical MDS.

We note also that the kinematic boundary conditions and the initial conditions do not need to be decompose, since they are set in a decomposed form. Hence, to decompose the first initial-boundary value problem, it is sufficient to decompose only the equations of motion, since, as already mentioned in the previous sentence, the kinematic boundary conditions and the initial conditions are decomposed. In this connection, the decomposition of the static boundary conditions represent of great interest. If the equations of motion (3) and the static boundary conditions (6) can be decompose under some conditions, then under the same conditions all the initial-boundary value problems formulated above can be decompose. Hence, it is necessary to establish the conditions under which the equations of motion (3) and the static boundary conditions (6) are decomposed.

4. Decomposition the equation of motion in case of a homogeneous isotropic micropolar medium

In this case, when many authors (see, for example, [11, 12]) considered, \( B = 0 \) and differential operator tensors \( A, B, C \) and \( D \) (see (4)) have form

\[
A = EQ_2 + d \nabla \nabla, \quad B = C = -2 \alpha C \cdot \nabla, \quad D = EQ_4 + m \nabla \nabla,
\]

\[
Q_2 = b \Delta - \rho \partial_t^2, \quad Q_4 = g \Delta - l - J \partial_t^2, \quad Q_1 = Q_2 + d \Delta, \quad Q_3 = Q_4 + m \Delta, \quad J = JE,
\]

\[
d = \lambda + \mu - \alpha, \quad l = 4 \alpha, \quad b = \mu + \alpha, \quad g = \delta + \beta, \quad m = \gamma + \delta - \beta,
\]

where \( Q_1, Q_2, Q_3, Q_4 \) are the wave operators, and the elasticity tensors have expressions

\[
A = a_1 C_{(1)} + a_2 C_{(2)} + a_3 C_{(3)}, \quad D = d_1 C_{(1)} + d_2 C_{(2)} + d_3 C_{(3)}.
\]

Here \( C_{(1)}, C_{(2)} \) and \( C_{(3)} \) — are basic isotropic tensors of the 4th rank, and the notations for material constants are \( a_1 = \lambda, a_2 = \mu, a_3 = \alpha, b_1 = \gamma, b_2 = \delta, b_3 = \beta \).
By
\[
\mathcal{M}_* = \begin{pmatrix} \hat{A} & \hat{B}^{(1)} \\ \hat{B}^{(2)} & \hat{C} \end{pmatrix}
\]
we denote the tensor-block matrix operator of cofactors for the tensor-block matrix operator \( \mathcal{M} \) of the equation (3), and, with simple calculations, we obtain

\[
\hat{A} = Q_3(Q_2Q_4 + 4\alpha^2\Delta)|\mathbf{E}Q_1Q_4 - (dQ_4 - 4\alpha^2)\nabla \nabla| \quad (\hat{A}^T = \hat{A}),
\]

\[
\hat{B} = \hat{B}^{(1)} = \hat{B}^{(2)} = -2\alpha Q_1Q_3(Q_2Q_4 + 4\alpha^2\Delta)\mathbf{C} \cdot \nabla \quad (\hat{B}^T = -\hat{B}),
\]

\[
\hat{C} = Q_1(Q_2Q_4 + 4\alpha^2\Delta)|\mathbf{E}Q_2Q_3 - (mQ_2 - 4\alpha^2)\nabla \nabla| \quad (\hat{C}^T = \hat{C}).
\]

Let us introduce tensor-block matrix operators

\[
\mathcal{N}^{(1)} = \begin{pmatrix} \mathbf{R} & \mathbf{S}^{(2)} \\ \mathbf{S}^{(1)} & \mathbf{T} \end{pmatrix}, \quad \mathcal{N}^{(2)} = \begin{pmatrix} \mathbf{R} & \mathbf{S}^{(1)} \\ \mathbf{S}^{(2)} & \mathbf{T} \end{pmatrix},
\]

\[
\mathbf{R} = \mathbf{E}Q_1Q_4 - (dQ_4 - 4\alpha^2)\nabla \nabla, \quad \mathbf{S}^{(1)} = Q_3\mathbf{B}, \quad \mathbf{S}^{(2)} = Q_1\mathbf{B},
\]

\[
\mathbf{B} = -2\alpha \mathbf{C} \cdot \nabla, \quad \mathbf{T} = \mathbf{E}Q_2Q_3 - (mQ_2 - 4\alpha^2)\nabla \nabla.
\]

Then if we will look for a solution of the equation (3) in the form (similar Galerkin)

\[
\mathbf{U} = \mathcal{N}^{(1)}^T \cdot \mathbf{V} \quad \left( \mathbf{U} = \begin{pmatrix} \mathbf{u} \\ \mathbf{\varphi} \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} \mathbf{v} \\ \mathbf{\psi} \end{pmatrix} \right),
\]

then we obtain the following decomposed equations:

\[
Q_1(Q_2Q_4 + 4\alpha^2\Delta)\mathbf{v} + \rho \mathbf{F} = 0, \quad Q_3(Q_2Q_4 + 4\alpha^2\Delta)\mathbf{\psi} + \rho \mathbf{m} = 0.
\]

If we apply the operator \( \mathcal{N}^{(2)}^T \) to the equation (3) on the left, then we have decomposed equations

\[
Q_1[(Q_2Q_4 + 4\alpha^2\Delta)\mathbf{u} + 2\alpha (\mathbf{C} \cdot \nabla) \cdot (\rho \mathbf{m})] + |\mathbf{E}Q_1Q_4 - (dQ_4 - 4\alpha^2)\nabla \nabla| \cdot (\rho \mathbf{F}) = 0,
\]

\[
Q_3[(Q_2Q_4 + 4\alpha^2\Delta)\mathbf{\varphi} + 2\alpha (\mathbf{C} \cdot \nabla) \cdot (\rho \mathbf{F})] + |\mathbf{E}Q_2Q_3 - (mQ_2 - 4\alpha^2)\nabla \nabla| \cdot (\rho \mathbf{m}) = 0. \tag{10}
\]

For \( \alpha = 0 \) (the case of a reduced medium), the classical equation follows from the first equation (10), and the second equation has a similar form.

5. Decomposition of static boundary conditions

In case of an isotropic micropolar material without a center of symmetry, according to (9) and \( \mathbf{B} = b_1\mathbf{C}^{(1)} + b_2\mathbf{C}^{(2)} + b_3\mathbf{C}^{(3)} \) we have

\[
\mathbf{T}^{(1)} = a_2\mathbf{E} \cdot \nabla + a_1\mathbf{n} \nabla + a_3(\mathbf{n} \nabla)^T, \quad \mathbf{T}^{(2)} = b_2\mathbf{E} \cdot \nabla + b_1\mathbf{n} \nabla + b_3(\mathbf{n} \nabla)^T - (a_2 - a_3)\mathbf{n} \cdot \mathbf{C},
\]

\[
\mathbf{T}^{(3)} = b_2\mathbf{E} \cdot \nabla + b_1\mathbf{n} \nabla + b_3(\mathbf{n} \nabla)^T, \quad \mathbf{T}^{(4)} = d_2\mathbf{E} \cdot \nabla + d_1\mathbf{n} \nabla + d_3(\mathbf{n} \nabla)^T - (b_2 - b_3)\mathbf{n} \cdot \mathbf{C}.
\]

We note that some authors (see, for example, [11, 12]) consider \( \mathbf{B} \) is an asymmetric tensor, so in the case of an isotropic medium it is zero, as was done above. However, some authors (see, for example, [13]) prove that \( \mathbf{B} \) is a symmetric tensor, therefore in the case of an isotropic medium it is not equal to zero and this tensor, like any isotropic tensor of the fourth rank, is generally determined by three parameters, as is customary in this case.
Next, it is easy to see, that
\[
T^{(2)} = T^{(3)} - (a_2 - a_3) n \cdot C, \quad T^{(4)} = T'^{(4)} - (b_2 - b_3) n \cdot C, \quad T'^{(4)} = d_2 E n \cdot \nabla + d_1 n \nabla + d_3 (n \nabla)^T.
\]

Assuming that the body has a piecewise-plain boundary and denoting by \( T_s^{(1)} \) and \( |T^{(1)}| \), \( T_s^{(3)} \) and \( |T^{(3)}| \), \( T_s^{(4)} \) and \( |T^{(4)}| \) differential tensor-operators of cofactors and determinants for the tensors-operators \( T^{(1)} \), \( T^{(3)} \) and \( T^{(4)} \), after simple calculations we obtain
\[
T_s^{(1)} = \begin{bmatrix}
(a_1 + a_2)(a_2 + a_3)|E n \cdot \nabla - a_3(a_1 + a_2)n \nabla - \\
- a_1(a_2 + a_3)(n \nabla)^T n \cdot \nabla + a_1 a_3|\nabla + (n n - E) \Delta,
\end{bmatrix}
\]
\[
|T^{(1)}| = a_2[(a_1 + a_2)(a_2 + a_3) n n - 1 \cdot a_3A n \cdot \nabla] = \\
a_2[(a_1 + a_2)(a_2 + a_3) n n - 1 \cdot a_3A n \cdot \nabla],
\]
\[
T_s^{(3)} = \begin{bmatrix}
b_1 + b_2)(b_2 + b_3)|E n \cdot \nabla - b_3(b_1 + b_2)n \nabla - \\
- b_1(b_2 + b_3)(n \nabla)^T n \cdot \nabla + b_1 b_3|\nabla + (n n - E) \Delta,
\end{bmatrix}
\]
\[
|T^{(3)}| = b_2[(b_1 + b_2)(b_2 + b_3) n n - 1 \cdot b_3A n \cdot \nabla],
\]
\[
T_s^{(4)} = \begin{bmatrix}
d_1 + d_2)(d_2 + d_3)|E n \cdot \nabla - d_3(d_1 + d_2)n \nabla - \\
- d_1(d_2 + d_3)(n \nabla)^T n \cdot \nabla + d_1 d_3|\nabla + (n n - E) \Delta,
\end{bmatrix}
\]
\[
|T^{(4)}| = d_2[(d_1 + d_2)(d_2 + d_3) n n - 1 \cdot d_3A n \cdot \nabla].
\]

Note that we want to obtain boundary conditions separately for \( u \) and \( \varphi \). Let us consider the case when \( b_2 = b_3, a_2 = a_3 \). Then \( T^{(2)} = T^{(3)}, T^{(4)} = T^{(4)} \) and the boundary conditions (6) can be written in form
\[
T^{(1)} u + T^{(3)} \cdot \varphi = P, \quad T^{(3)} u + T^{(4)} \cdot \varphi = \mu.
\]

In this case it is easy to obtain the boundary conditions separately for \( u \) and \( \varphi \)
\[
\left(|T^{(4)}| T^{(1)} T^{(1)} - |T^{(3)}| T^{(3)} T^{(3)} \right) \cdot u = \left(|T^{(4)}| T^{(3)} T^{(3)} - |T^{(3)}| T^{(3)} T^{(3)} \right) \cdot \varphi = |T^{(3)}| T^{(1)} T^{(1)} P - |T^{(1)}| T^{(1)} T^{(1)} \mu.
\]

6. Characteristic equation for a tensor-block matrix operator of second rank
The characteristic equation for the tensor-block matrix \( \mathbb{M} \) has the form
\[
\eta^6 - I_1(\mathbb{M}) \eta^5 + I_2(\mathbb{M}) \eta^4 - I_3(\mathbb{M}) \eta^3 + I_4(\mathbb{M}) \eta^2 - I_5(\mathbb{M}) \eta + I_6(\mathbb{M}) = 0,
\]
\[
S_k = I_k(\mathbb{M}) = \frac{1}{k!} \begin{vmatrix}
s_1 & 1 & \cdots & 0 & 0 \\
s_2 & s_1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
s_{k-1} & s_{k-2} & \cdots & 0 & 0 \\
s_k & s_{k-1} & \cdots & s_2 & s_1 \end{vmatrix}, \quad k = \overline{1, 6},
\]
\[
S_k = I_k(\mathbb{M}), \quad s_k = I_1(\mathbb{M}^k), \quad k = \overline{1, 6}, \quad \mathbb{M}^k = \overline{\mathbb{M}, \mathbb{M} \cdot \mathbb{M} \cdot \cdots \cdot \mathbb{M}},
\]
where \( I_k(\mathbb{M}), k = \overline{1, 6} \), we denote invariants of the tensor-block matrix operator \( \mathbb{M} \).
It is easy to see, that the relations inverse to (11) are represented in the form

\[ s_k = I_1(M^k) = \begin{bmatrix} S_1 & 1 & 0 & \cdots & 0 \\ 2S_2 & S_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ kS_k & S_{k-1} & S_{k-2} & \cdots & S_1 \end{bmatrix}, \quad k = 1, 6. \]

Replacing here \( M \) by the tensor-block matrix operators of the equations of motion in displacements and rotations under various anisotropic media, we obtain the characteristic equations for them. If we find the roots (eigenvectors) of the obtained characteristic equations for the tensor-block matrix operators indicated above, then their determinants we can write as a product of eigenoperators (simple operators). Thus, the equations are split.

7. Decomposition of the equation of the classical theory of elasticity for a transversely-isotropic body

In this case the equations and boundary conditions are represented in the form

\[ \mathbf{L} \cdot \mathbf{u} + \rho \mathbf{F} = 0, \quad \mathbf{T} \cdot \mathbf{u} = \mathbf{P} \quad (\mathbf{L} = r_i r_j A^{ijkl} \nabla_i \nabla_j \mathbf{v}, \quad \mathbf{T} = r_i r_j r_l A^{ijkl} \nabla_l \mathbf{v}), \]

where the elastic tensor \( \mathbf{A} \), the tensor-operator \( \mathbf{L} \), the tensor-operator of cofactors \( \mathbf{L}_c \) and the determinant \( |\mathbf{L}| \) for \( \mathbf{L} \) have the form

\[ \mathbf{A} = A_1 (e_1 e_1 + e_2 e_2 + e_3 e_3) + A_2 (e_1 e_2 + e_2 e_1 - e_3 e_6) + \]
\[ + A_3 (e_1 e_3 + e_2 e_1 + e_2 e_3 + e_3 e_2) + A_4 e_3 e_3 + 2A_5 (e_4 e_4 + e_5 e_5), \]
\[ \mathbf{L} = (e_1 e_1 e_1 e_1 + e_2 e_2 e_2 e_2) + e_3 e_3 e_3 e_3 + e_4 e_4 e_4 e_4 + e_5 e_5 e_5 e_5, \]
\[ \mathbf{L}_c = L_{11} e_1 e_1 + L_{22} e_2 e_2 + L_{33} e_3 e_3 + L_{44} e_4 e_4 + \]
\[ + L_{23} (e_2 e_3 + e_3 e_2) + L_{34} (e_3 e_4 + e_4 e_3) + \partial_i = \partial / \partial x_i, \quad i = 1, 2, 3, \]
\[ 2L_{11} = 2A_1 A_5 \Delta^2 + (A_1 + A_2) A_5 \Delta \partial^2 + \{2[A_1 A_4 - A_3 (A_3 + 2A_5)] \Delta - \]
\[ - [(A_1 + A_2) A_4 - 2(A_3 + A_3)^2 \partial^2 + 2A_4 A_5 \partial^3, \]
\[ 2L_{12} = \{-(A_1 + A_2) A_5 \Delta + [2(A_3 + A_3)^2 - (A_1 + A_2) A_4 \partial^2] \partial_1 \partial_2, \]
\[ 2L_{13} = -(A_1 + A_2) A_4 - 2(A_3 + A_3)^2 \partial^2 + 2A_4 A_5 \partial^3, \]
\[ 2L_{22} = 2A_1 A_5 \Delta^2 - (A_1 + A_2) A_5 \Delta \partial^2 + \{2[A_1 A_4 - A_3 (A_3 + 2A_5)] \Delta - \]
\[ - [(A_1 + A_2) A_4 - 2(A_3 + A_5)^2 \partial^2 + 2A_4 A_5 \partial^3, \]
\[ 2L_{23} = -(A_1 + A_2) A_3 + A_3 \Delta + 2A_3 A_5 \partial^3 \partial_2 \partial_3, \]
\[ 2L_{33} = [(A_1 - A_2) A_1 \Delta^2 + (3A_1 - A_1) A_5 \Delta \partial^2 + 2A_4 A_5 \partial^3, \]
\[ 2(|\mathbf{L}|) = (A_1 - A_2) A_1 A_5 \Delta^3 + \{(A_1 - A_2) [A_1 A_4 + A_2 (A_3 + 2A_5)] + 2A_1 A_5 \Delta \partial^2 \partial_3 + \]
\[ + A_1 A_2) A_4 A_5 - 2A_3 A_5 (A_3 + 2A_5) \Delta \partial^2 + 2A_4 A_5 \partial^3, \]
\[ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3, \quad \mathbf{e}_m \otimes \mathbf{e}_n = \delta_{mn}, \quad m, n = 1, 2, \ldots, 6, \]
\[ \mathbf{e}_1 = e_1 e_1, \quad \mathbf{e}_2 = e_2 e_2, \quad \mathbf{e}_3 = e_3 e_3, \quad \mathbf{e}_4 = (1/\sqrt{2})(e_1 e_2 + e_2 e_1), \]
\[ \mathbf{e}_5 = (1/\sqrt{2})(e_2 e_3 + e_3 e_2), \quad \mathbf{e}_6 = (1/\sqrt{2})(e_3 e_1 + e_1 e_3). \]
Here $\delta_{ij}$ is the Kronecker’s delta-symbol, $e_i, i, j = 1, 2, 3$ is the orthonormal basis.

Let us also note that
\[
2I_1(\mathbf{L}) = (3A_1 - A_2 + 2A_3)\Delta + (A_4 + 2A_5)\partial^2_3,
\]
\[
2I_2(\mathbf{L}) = [(3A_1 - A_2)A_3 + A_1(A_1 - A_2)]\Delta^2 + [(3A_1 - A_2)A_4 - 2A_3(A_3 + 2A_5) + (2A_4 + 3A_1 - A_2)A_5]\Delta\partial^2_3 + 2(2A_4 + A_3)A_5\partial^3_3, \quad I_3(\mathbf{L}) = |\mathbf{L}|.
\]

It is easy to see that the determinant of the tensor-operator $\mathbf{L}$ can be written in the form
\[
|\mathbf{L}| = A\Delta^3 + B\Delta^2\partial^2_3 + C\Delta\partial^3_3 + D\partial^4_3 = k(\Delta + k_1\partial^2_3)(\Delta + k_2\partial^3_3)(\Delta + k_3\partial^3_3);
\]

$k = A, \quad k_1 + k_2 + k_3 = B/A, \quad k_1k_2 + k_1k_3 + k_2k_3 = C/A, \quad k_1k_2k_3 = D/A,
\]

$A = (1/2)(A_1 - A_2), A_1A_5, \quad B = (1/2)[(A_1 - A_2)[A_1A_4 + A_3(A_3 - 2A_5)] + 2A_1A_5],
\]

$C = (1/2)[(3A_1 - A_2)A_4A_5 - 2A_3A_5(A_3 + 2A_5)], \quad D = A_4A_5^5.
\]

The split equations will have the form
\[
|\mathbf{L}|u + |\mathbf{L}|u \cdot (\rho\mathbf{F}) = 0 \quad \text{or} \quad \text{if we will look for} \ u \ \text{in form} \ u = |\mathbf{L}|v, \ \text{then} \ |\mathbf{L}|v + \rho\mathbf{F} = 0. \quad \text{(12)}
\]

Applying, for example, to the split equations in brackets from (12), the operator of moments of the $k$-th order [3, 8], the equations for prismatic bodies in moments with respect to any system of orthogonal polynomials have the form
\[
A\Delta^3\mathbf{v} + B\Delta^2\partial^{(k)}_3 + C\Delta\partial^{(k)}_3 + D\partial^{(k)}_3 = (\Delta + k_1\partial^2_3)(\Delta + k_2\partial^3_3)(\Delta + k_3\partial^3_3)
\]

where in the application of the system of Legendre polynomials, the expressions for $\partial^{(k)}_3, \partial^{(k)}_3$ and $\partial^{(k)}_3$ are defined by the following relation:
\[
\partial^{(k)}_3(x) = (2n + 1)\sum_{k=1}^{\infty} C^{2m-1}_{k+2m-2} \sum_{s=1}^{2m-1} (2n + 2k + 2s - 1)^{(n+2k+2m-2)} \partial^{(k)}_3, \quad n \in \mathbb{N}_0, \quad m \in \mathbb{N}. \quad \text{(13)}
\]

8. Decomposition of static boundary conditions of the classical theory of elasticity in the case of a transversely isotropic body

In this case, the boundary conditions can be written in the form $T \cdot u = P \quad (T = r, r, n_i n_j A^{ijkl} \nabla_k)$, where the tensor operator of stress operator $T$, the tensor-operator of the cofactors $T_{*}$ for $T$ and the determinant $|T|$ have the form
\[
T = e_1e_1[A_1n_1\partial_1 + 1/2(A_1 - A_2)n_2\partial_2 + A_5n_3\partial_3] +
+ e_1e_2[1/2(A_1 - A_2)n_2\partial_1 + A_1n_2\partial_2 + e_1e_3(A_3n_1\partial_3 + A_5n_3\partial_3)] +
+ e_2e_1[1/2(A_1 - A_2)n_1\partial_2 + A_1n_2\partial_2] + e_2e_2[A_1n_2\partial_2 + 1/2(A_1 - A_2)n_1\partial_1] +
+ A_5n_3\partial_3 + e_2e_3(A_3n_3\partial_3 + A_5n_3\partial_3) + e_1e_1(A_3n_3\partial_3 + A_5n_3\partial_3) +
+ e_3e_2(A_3n_3\partial_2 + A_5n_3\partial_2) + e_3e_3[A_4n_3\partial_3 + A_5(n_1\partial_1 + n_2\partial_2)], \quad T^T \neq \mathbf{T};
\]

$|T| = AA_5\{[A_2 + (A_2 - A_3)n_3^2]\Delta + (A_4 + 2A_5)\partial^3_3, \quad + \{A_2(A_3^2 - A_4) + A_4A_5^2 - [A_2 + A_4]A_3^2 + A_3(A_3^2 - A_4)\}n_3\Delta\partial_3 +
+ [2A_2(A_2^2 - A_4A_5) + (A_1 + A_2)A_4A_5]n_3^3\partial^3_3 +
+ A_5[A_3^2 - A_2(A_3 + 2A_4)n_3^2\partial^3_3 - A_3^2[A_3^2 - A_4]n_3^2\partial^3_3,
\]

$T_{11} = A_5(A_1n_1\partial_1 + A_1n_2\partial_2)n_3^2\partial^3_3 + A_5[A_3^2 - A_2(A_3 + 2A_4)n_3^2\partial^3_3 +
+ A_5^2 - A_3A_4]A_5\partial_3^3,
\]

$T_{12} = -A_5[A_2(A_2n_1\partial_2 + + A_2n_2\partial_2) + (A_2 - A_4)A_2n_2\partial_2]n_3\partial^3_3 + A_3A_5(A_3^2 - A_2A_4)n_3^2\partial^3_3 + A_3A_5A_5^2\partial^3_3 + A_3A_5n_2\partial^3_3 + n_2\partial^3_3,
\]

$T_{13} = (A_2A_5n_3\partial_2 - A_3A_5n_3\partial_3)C^2\partial^3_3(A_5\partial^3_3 - A_5n_3\partial_3) + A_3A_5n_3\partial^3_3$,
To consider the case of an isotropic medium. In this case, the tensor-operators and the
operators $A_{ab}$ are expressed as

$$\tilde{A}_{ab} = \frac{1}{2} \left( \tilde{A}_{ab} + \tilde{A}_{ba} \right),$$

$$\tilde{A}_{ab} = \frac{1}{2} \left( \tilde{A}_{ab} - \tilde{A}_{ba} \right),$$

for any symmetrical tensor $\tilde{A}_{ab}$.

The split boundary conditions have the form: $|T|u = T^s \cdot P$.

We note that the dynamic equations (the equilibrium equations) in displacements for any homogeneous isotropic body have the form

$$\mathbf{M} \cdot \mathbf{u} + \rho \mathbf{F} = 0 \quad (\mathbf{L} \cdot \mathbf{u} + \rho \mathbf{F} = 0), \quad \mathbf{M} = \mathbf{L} - \mathbf{E} \rho \partial^2, \quad \mathbf{L} = A_{\rho} e_i e_j \partial_i \partial_j. \quad (14)$$

It is easy to see, that

$$I_3(\mathbf{M}) = |\mathbf{M}| = I_3(\mathbf{M}) | \mathbf{M} - I_2(\mathbf{M}) \rho \partial^2 + I_1(\mathbf{M}) \rho^2 \partial^4 - \rho^3 \partial^6;$$

$$M_s = M^2 - I_1(\mathbf{M}) M + I_2(\mathbf{M}) E = L_s + [\mathbf{L} - I_1(\mathbf{L}) \mathbf{E}] \rho \partial^2 + [\mathbf{E} \rho^2 \partial^4;$$

$$= L^2 - L[I_1(\mathbf{L}) - \rho \partial^2] + E[I_2(\mathbf{L}) - I_1(\mathbf{L}) \rho \partial^2 + \rho^2 \partial^4], \quad L_s = L^2 - I_1(\mathbf{L}) \mathbf{L} + I_2(\mathbf{L}) \mathbf{E}.$$
The characteristic equation and its solutions have the form
\[
\det(M - \lambda I) = 0 \iff (\lambda^3 - I_1(M)\lambda^2 + I_2(M)\lambda - I_3(M) = 0);
\]
\[
I_1(M) = Q_1 + 2Q_2, \quad I_2(M) = Q_2(2Q_1 + Q_2), \quad I_3(M) = Q_1Q_2^2, \quad \lambda_1 = Q_1, \quad \lambda_2 = \lambda_3 = Q_2.
\]
It is clear that the wave operators are the eigenoperators for the tensor-operator \( M \). In this case, the second wave operator is the double root of the characteristic equation. To determine the eigenvectors, we obtain equations
\[
(\mathbf{E} \Delta - \nabla \nabla) \cdot \mathbf{a}_1 = 0, \quad \nabla \nabla \cdot \mathbf{a}_2 = 0,
\]
whose solutions are the following eigenvectors
\[
\mathbf{a}_1 = \nabla \varphi, \quad \mathbf{a}_2 = \nabla \times \psi + 1/3 \mathbf{r}.
\]

9. Quasistatic problem of the micropolar theory of elasticity in displacements and rotations
Let us consider an isotropic material with a center of symmetry. Then in case of quasistatics from (10) we have equations
\[
[(\lambda + 2\mu)(\mu + \alpha)(\delta + \beta)\Delta^3 - 4\alpha\mu(\lambda + 2\mu)\Delta^2]\mathbf{u} + \mathbf{S}^* = 0,
\]
\[
\{(\gamma + 2\delta)(\mu + \alpha)(\delta + \beta)\Delta^3 - 4\alpha[\mu(\gamma + 2\delta) + (\mu + \alpha)(\delta + \beta)]\Delta^2 + 16\alpha^2\mu\Delta\}\varphi + \mathbf{H}^* = 0, \quad (15)
\]
\[
\mathbf{S}^* = 2\alpha Q_1^*(\mathbf{C} \cdot \nabla) \cdot (\rho \mathbf{m}) + [\mathbf{E} Q_1^* Q_2^* - (dQ_1^* - 4\alpha^2)\nabla \nabla] \cdot (\rho \mathbf{F}),
\]
\[
\mathbf{H}^* = 2\alpha Q_3^*(\mathbf{C} \cdot \nabla) \cdot (\rho \mathbf{F}) + [\mathbf{E} Q_2^* Q_3^* - (mQ_3^* - 4\alpha^2)\nabla \nabla] \cdot (\rho \mathbf{m}),
\]
\[
Q_1^* = (b + d)\Delta, \quad Q_2^* = b\Delta, \quad Q_3^* = (g + m)\Delta - l, \quad Q_4^* = g\Delta - l,
\]
\[
d = \lambda + \mu - \alpha, \quad l = 4\alpha, \quad b = \mu + \alpha, \quad m = \gamma + \delta - \beta, \quad g = \delta + \beta. \quad (16)
\]
For \( \alpha = 0 \), that is, in the case of a reduced medium from (15) and (16) we obtain
\[
\Delta^2 \mathbf{u} + \mathbf{G} = 0, \quad \mathbf{G} = \frac{1}{\mu(\lambda + 2\mu)}[\mathbf{E}(\lambda + 2\mu) - (\lambda + \mu)\nabla \nabla] \cdot (\rho \mathbf{F});
\]
\[
\Delta^2 \varphi + \mathbf{H} = 0, \quad \mathbf{H} = \frac{1}{(\delta + \beta)(\gamma + 2\delta)}[\mathbf{E}(\gamma + 2\delta)\Delta - (\gamma + \delta - \beta)\nabla \nabla] \cdot (\rho \mathbf{m}). \quad (17)
\]
Note that the first of the equations (17) is a classical equation, and the second equation has a similar form.

10. The quasistatic problem of the micropolar theory of prismatic bodies of constant thickness in displacements and rotations and in moments of displacement and rotation vectors
Let us consider a prismatic body of constant thickness \( 2h \), and let us take the middle plane as the base plane. Then in this case \( g_M^P = \delta_M^P \), \( g_P^3 = 0 \), \( g^{33} = h^{-2} \) and nabla-operator and Laplacian \( \Delta \) and \( \Delta^2 \) and \( \Delta^3 \) will be presented in the form
\[
\hat{\nabla} \mathbf{F} = (r^P \partial_P + r^3 \partial_3) \mathbf{F} = (r^P \partial_P + h^{-1} \mathbf{n} \partial_3) \mathbf{F}, \quad -1 \leq x^3 \leq 1,
\]
\[
\hat{\Delta} \mathbf{F} = \hat{\nabla}^2 \mathbf{F} = (g^{PQ} \nabla_P \nabla_Q + g^{33} \partial_3^2) \mathbf{F} = (\Delta + h^{-2} \partial_3^2) \mathbf{F}, \quad \hat{\Delta} = g^{PQ} \nabla_P \nabla_Q,
\]
\[
\Delta^2 = \Delta^2 + 2h^{-2} \Delta \partial_3^2 + h^{-4} \partial_3^4, \quad \Delta^3 = \Delta^3 + 3h^{-2} \Delta^2 \partial_3^2 + 3h^{-4} \Delta \partial_3^4 + h^{-6} \partial_3^6. \quad (18)
\]
By the corresponding formulas (18), the equations (15) for the theory of prismatic bodies of constant thickness in displacements and rotations have the form

\[
\begin{align*}
\Delta^3 + A\Delta^2 + h^{-2}(3\Delta + 2A)\Delta\phi^1 + h^{-4}(3\Delta + A)\phi^2 + h^{-6}\phi^3 \vec{u} + \vec{S}^{**} &= 0, \\
\Delta^3 + (B\Delta + A)\Delta + h^{-2}(3\Delta + 2B)\Delta + C\phi^2 + h^{-4}(3\Delta + B)\phi^3 + h^{-6}\phi^4 \vec{v} + \vec{H}^{**} &= 0;
\end{align*}
\]  
(19)

\[
\vec{S}^{**} = \frac{\vec{S}^*}{(\lambda + 2\mu)(\mu + \alpha)(\delta + \beta)}, \quad \vec{H}^{**} = \frac{\vec{H}^*}{(\gamma + 2\delta)(\mu + \alpha)(\delta + \beta)}, \quad A = -\frac{4\alpha\mu}{(\mu + \alpha)(\delta + \beta)}, \quad B = -\frac{4\alpha[\mu(\gamma + 2\delta) + (\mu + \alpha)(\delta + \beta)]}{(\gamma + 2\delta)(\mu + \alpha)(\delta + \beta)}.
\]

Applying the operator of moments of the k-th order of some system of orthogonal polynomials (Legendre, Chebyshev) to the equations (19), we obtain the following equations for the micropolar theory of prismatic bodies of constant thickness in the moments of the displacement and rotation vectors:

\[
\begin{align*}
[\Delta^3 + (B\Delta + A)\Delta][k] \phi + h^{-2}(3\Delta + 2B)\Delta + C][k] \phi + h^{-4}(3\Delta + B)\phi + h^{-6}\phi \vec{u} + \vec{S}^{**} &= 0, \\
[\Delta^3 + A\Delta^2][k] \vec{u} + h^{-2}(3\Delta + 2A)\Delta \vec{u} + h^{-4}(3\Delta + A)\vec{u} + h^{-6}\vec{u} \vec{v} + \vec{S}^{**} &= 0, \quad k \in \mathbb{N}_0.
\end{align*}
\]  
(20)

Having the equations (20), by the formula (13), in which the vector \( \vec{v} \) is replaced by \( \vec{u} \), it is easy to obtain systems of equations of any approximation at moments with respect to the system of Legendre polynomials. The equations of the 5th (in the classical case) and the 8th (in the micropolar case) approximations in the moments are obtained in the articles [8,9] for an isotropic material.

We note that using the canonical expressions for the tensor and the tensor-block matrix of any even rank \([3, 9, 10]\), the initial-boundary value problems can be represented in canonical form and then split. In addition, I want to note that, for the theory of thin bodies, the split equations at equilibrium are equations of elliptic type of high order [9]. Using the method of I.N. Vekua [15], for them it is possible to write out analytical solutions.

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