Choptuik scaling and the scale invariance of Einstein’s equation

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Abstract

The relationship of Choptuik scaling to the scale invariance of Einstein’s equation is explored. Ordinary dynamical systems often have limit cycles: periodic orbits that are the asymptotic limit of generic solutions. We show how to separate Einstein’s equation into the dynamics of the overall scale and the dynamics of the “scale invariant” part of the metric. Periodicity of the scale invariant part implies periodic self-similarity of the spacetime. We also analyze a toy model that exhibits many of the features of Choptuik scaling. PACS 04.20.-q, 04.20.Fy, 04.40.-b

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I. INTRODUCTION

Recently Choptuik has found scaling phenomena in gravitational collapse. He numerically evolves a one parameter family of initial data for a spherically symmetric scalar field coupled to gravity. Some of the data collapse to form black holes while others do not. There is a critical value of the parameter separating those data that form black holes from those that do not. The critical solution (the one corresponding to the critical parameter) has the property of periodic self similarity: after a certain amount of logarithmic time the profile of the scalar field repeats itself with its spatial scale shrunk. For parameters slightly above the critical parameter the mass of the black hole formed scales like \((p - p^*)^\gamma\) where \(p\) is the parameter, \(p^*\) is its critical value and \(\gamma\) is a universal scaling exponent that does not depend on which family is being evolved. Numerical simulations of the critical gravitational collapse of other types of spherically symmetric matter were subsequently performed. These include complex scalar fields, perfect fluids, axions and dilatons and Yang-Mills fields. In addition scaling has been found in the collapse of axisymmetric gravity waves. Thus scaling seems to be a generic feature of critical gravitational collapse. In some of these systems the critical solution has periodic self-similarity while in other systems it has exact self-similarity.

These phenomena were discovered numerically, so one would like to have an analytic explanation for why systems that just barely undergo gravitational collapse behave in this way. Gundlach and Koike, Hara and Adachi have explained the scaling of black hole mass analytically subject to the following assumptions: (i) the critical solution is periodically self-similar and (ii) the critical solution has exactly one unstable mode. This still leaves unexplained the mystery of why the critical solution is periodically self-similar.

While periodic self-similarity is an unusual property for a dynamical system, periodicity is not. In fact, many dynamical systems have limit cycles, \(i.e.\) periodic trajectories that are approached asymptotically for a large class of initial conditions. What is it about Einstein’s equation that gives rise to periodically self-similar solutions rather than periodic ones? One
feature of Einstein’s equation is scale invariance: let $g_{ab}$ be a solution of the vacuum Einstein equation and let $k$ be a positive constant. Then $kg_{ab}$ is a solution of the vacuum Einstein equation. The same property holds for the Einstein-scalar equation (the system studied by Choptuik). Let $(g_{ab}, \phi)$ be a solution of the Einstein-scalar equation and let $k$ be a positive constant. Then $(kg_{ab}, \phi)$ is a solution of the Einstein-scalar equation. This feature of scale invariance suggests that in some sense the metric can be decomposed into an “overall scale” and a “scale invariant part” and that these two pieces have very different dynamics. In particular a periodically self-similar metric could be realized as a periodic scale invariant part of the metric. Also an exactly self-similar metric could be realized as a static scale invariant part of the metric. Thus if the dynamical system of the scale invariant part of the metric has a limit point i.e. a point in phase space that is approached asymptotically for a large class of initial conditions (a usual feature for dynamical systems) then the metric has a critical solution that is exactly self-similar.

This program of finding scale invariant dynamics for the metric and matter fields has been carried out in the spherically symmetric case for various kinds of matter. In spherical symmetry the metric takes the form

$$ds^2 = -\alpha^2(r, t) dt^2 + a^2(r, t) dr^2 + r^2 d\Omega^2 .$$ (1)

With $t$ or $r$ as the overall scale, the quantities $\alpha$ and $a$ are scale invariant. The dynamics then becomes a set of differential equations for $\alpha$, $a$ and the scale invariant matter variables. These equations have been found for radiation [8], a massless scalar field [7], scalar electrodynamics [9] and the Einstein-Yang-Mills system. [10]

Homogeneous cosmologies have also been treated using this sort of approach. Wainwright and Hsu [11] have found a set of scale invariant variables that describe the dynamics of homogeneous, anisotropic spacetimes. These variables have been useful both in the work of reference [11] and in recent work of Rendall. [12] The connection between scale invariance and discrete self similarity has been noted in a study by Traschen [13] of fluctuations about an extremal black hole.
One would like to carry out this program of finding scale invariant metric variables in cases where there are no symmetries. The renormalization group framework of reference [8] is sufficiently general to accommodate the case of scale invariance with no symmetries. However, the coordinate invariance of general relativity allows more than one possible set of scale invariant variables. One would like to find a set appropriate for critical gravitational collapse. In the Euclidean approach to quantum gravity [14] the metric is decomposed into a conformal factor and a determinant=1 piece. However, in the Lorentzian case Einstein’s equations are a dynamical system described by the ADM formalism. It therefore makes sense to have the overall scale determined by some dynamical condition.

In section 2 we treat the dynamics of a toy model that has many of the features of Choptuik scaling. In section 3 we modify the ADM formalism to write Einstein’s equation as a dynamical system of scale invariant quantities. Section 4 is a discussion of the implications of these results.

II. TOY MODEL

We now consider a toy model that has many of the features of Choptuik scaling. This model is constructed by adding an overall scale degree of freedom to a model known to have a limit cycle. The model is a dynamical system consisting of three functions of time: \( a, b \) and \( c \). Define the quantity \( s \) by

\[
s \equiv \frac{2 \dot{a}}{a} - \left( \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) \tag{2}
\]

where an overdot denotes derivative with respect to \( t \). Choose the equations of motion of this system to be

\[
3 \frac{d}{dt} \left( \frac{\dot{a}}{a} \right) = - \ln \left( \frac{a^2}{bc} \right) + s + \epsilon s \left( 1 - \frac{1}{3} \ln \left( \frac{a^2}{bc} \right) \right)^2 , \tag{3}
\]

\[
3 \frac{d}{dt} \left( \frac{\dot{b}}{b} \right) = \ln \left( \frac{b^2}{ac} \right) + s , \tag{4}
\]
\[
3 \frac{d}{dt} \left( \frac{\dot{c}}{c} \right) = 3 \ln \left( \frac{a}{b} \right) + s - \epsilon s \left( 1 - \left[ \frac{1}{3} \ln \left( \frac{a^2}{bc} \right) \right]^2 \right). \quad (5)
\]

Here \( \epsilon \) is a small positive constant. Note that this system has the property of scale invariance: if \((a, b, c)\) is any solutions of the equations of motion and \(k\) is any positive constant then \((ka, kb, kc)\) is also a solution of the equations of motion. This suggests that one might better understand this system by dividing its variables into an “overall scale” and a “scale invariant part.” To make this idea precise define the quantities \(N, \alpha\) and \(\beta\) by

\[
N \equiv (abc)^{1/3}, \quad (6)
\]

\[
\alpha \equiv \ln(a/N), \quad (7)
\]

\[
\beta \equiv \ln(b/N). \quad (8)
\]

Note that \(\alpha\) and \(\beta\) are scale invariant. Thus the quantity \(N\) is the overall scale of the system and \((\alpha, \beta)\) is the scale invariant part of the system. Expressing equations (3-5) in terms of these new variables we find

\[
\ddot{\alpha} + \frac{d}{dt} \left( \frac{\dot{N}}{N} \right) = -\alpha + \dot{\alpha} + \epsilon \dot{\alpha} \left( 1 - \alpha^2 \right), \quad (9)
\]

\[
\ddot{\beta} + \frac{d}{dt} \left( \frac{\dot{N}}{N} \right) = \beta + \dot{\alpha}, \quad (10)
\]

\[
-\ddot{\alpha} - \ddot{\beta} + \frac{d}{dt} \left( \frac{\dot{N}}{N} \right) = \alpha - \beta + \dot{\alpha} - \epsilon \dot{\alpha} \left( 1 - \alpha^2 \right) \quad , \quad (11)
\]

Adding equations (9-11) we find

\[
\frac{d}{dt} \left( \frac{\dot{N}}{N} \right) = \dot{\alpha}. \quad (12)
\]

Using this result in equations (9-10) we find

\[
\ddot{\alpha} = -\alpha + \epsilon \dot{\alpha} \left( 1 - \alpha^2 \right), \quad (13)
\]

\[
\ddot{\beta} = \beta \quad . \quad (14)
\]

Note that the system \((\alpha, \beta)\) is a dynamical system independent of the overall scale \(N\) and with equations of motion given by equations (13-14). The general solution of equation (14)
\[
\beta = \frac{1}{2} (\beta_0 + v_{\beta_0}) e^t + \frac{1}{2} (\beta_0 - v_{\beta_0}) e^{-t} 
\]

where \(\beta_0 \equiv \beta(0)\) and \(v_{\beta_0} \equiv \dot{\beta}(0)\). Note that as \(t \to \infty\) we have \(\beta \to \pm\infty\) or 0 depending on whether the quantity \(\beta_0 + v_{\beta_0}\) is respectively positive, negative or zero. Now consider a generic one parameter family of initial data for the system \((a, b, c)\) depending on parameter \(p\). In general there will be a range of \(p\) for which the long time evolution of the system gives \(\beta \to \infty\), a range of \(p\) for which \(\beta \to -\infty\) and a critical value \(p^*\) of \(p\) for which \(\beta \to 0\). This is analogous to the case of Choptuik scaling where there is a range of \(p\) for which the late time evolution gives a black hole, a range of \(p\) for which the late time evolution gives flat space and a critical parameter \(p^*\) separating these two ranges for which the late time evolution gives the Choptuik critical solution.

Now consider the evolution equation (13) for \(\alpha\). This is the van der Pol equation. [15] It is well known that this equation has a stable limit cycle. A solution of equation (13) to zeroth order in \(\epsilon\) is

\[
\alpha = 2 \cos(t + \phi_0) 
\]

where \(\phi_0\) is a constant. Furthermore any initial data sufficiently close to the trajectory given in equation (16) will approach that trajectory asymptotically as \(t \to \infty\). It then follows that for generic one parameter families of initial data for the \((\alpha, \beta)\) system (sufficiently close to the limit cycle) there is a critical value of the parameter for which the trajectory is asymptotically periodic.

Now consider the behavior, at late times, of the overall scale \(N\) in the critical solution. With \(\alpha\) given by equation (16) we find using equation (12)

\[
N = n \exp[\kappa t + 2 \sin(t + \phi_0)] 
\]

where \(n\) and \(\kappa\) are constants. That is, the overall scale is a periodic function multiplied by an exponential. The asymptotic behavior of the critical solution expressed in terms of the
variables \((a, b, c)\) is then

\[
a = n \exp \left[ \kappa t + 2\sqrt{2} \sin (t + \phi_0 + \pi/4) \right], \tag{18}
\]

\[
b = n \exp \left[ \kappa t + 2 \sin (t + \phi_0) \right], \tag{19}
\]

\[
c = n \exp \left[ \kappa t + 2\sqrt{2} \sin (t + \phi_0 - \pi/4) \right]. \tag{20}
\]

Thus the critical solution for \((a, b, c)\) is periodically self similar. After a certain amount of time the solution repeats its behavior with only its overall scale changed.

**III. SEPARATION OF EINSTEIN’S EQUATION**

We now apply the ideas developed in the previous section to Einstein’s equation. Start with the standard ADM formalism with, for simplicity, zero shift. The spacetime metric has the form

\[
ds^2 = -N^2 dt^2 + h_{ik} dx^i dx^k. \tag{21}
\]

Here \(h_{ik}\) is the intrinsic metric of the \(t = \text{const.}\) slices. The lapse \(N\) can be chosen arbitrarily. The extrinsic curvature of the \(t = \text{const.}\) slices is

\[
K_{ik} = \frac{1}{2N} \partial_t h_{ik}. \tag{22}
\]

(Here \(\partial_t\) denotes \(\partial/\partial t\)). We would like to keep the same time and space coordinates \((t, x^i)\) when the spacetime metric is multiplied by an overall constant. Thus under the transformation \(h_{ik} \rightarrow k h_{ik}\) for a constant \(k\) we want \(N \rightarrow \sqrt{k} N\). An evolution equation consistent with this condition is

\[
\partial_t N = \frac{1}{3} N^2 K. \tag{23}
\]

Actually any constant could be chosen to replace the \(1/3\); but as we will see later the choice of \(1/3\) has other nice features. We now wish to replace \(h_{ik}\) with a scale invariant quantity. Define

\[
\tilde{h}_{ik} \equiv N^{-2} h_{ik}. \tag{24}
\]
This $\tilde{h}_{ik}$ is the “scale invariant part” of the spatial metric. Using equations (22) and (23) we find

$$\partial_t \tilde{h}_{ik} = 2 \tilde{K}_{ik}$$  \hspace{1cm} (25)

where $\tilde{K}_{ik}$, the “scale invariant part” of the extrinsic curvature, is defined by

$$\tilde{K}_{ik} = N^{-1} \left( K_{ik} - \frac{1}{3} K h_{ik} \right)$$ \hspace{1cm} (26)

Note that $\tilde{K}_{ik}$ is trace free. This is due to the presence of the factor $1/3$ in equation (23). Thus the extrinsic curvature essentially has two parts: the trace and the scale invariant part. We now need to find an evolution equation for $\tilde{K}_{ik}$ and we would like to express this equation in terms of scale invariant quantities. One such quantity is

$$\omega_i \equiv N^{-1} \partial_i N$$ \hspace{1cm} (27)

The evolution equation for $K_{ik}$ is

$$\partial_t K_{ik} = D_i D_k N + N \left( 2 K_{ip} K_k^p - K K_{ik} + R_{ik} - (3) R_{ik} \right)$$ \hspace{1cm} (28)

Here $D_i$ and $(3) R_{ik}$ are respectively covariant derivative and Ricci tensor of the metric $h_{ik}$ and $R_{ik}$ is the spacetime Ricci tensor. From equations (22),(23) and (28) straightforward but tedious algebra gives an evolution equation for $\tilde{K}_{ik}$. Expressing that equation in terms of scale invariant quantities yields

$$\partial_t \tilde{K}_{ik} = -\frac{2}{3} (NK) \tilde{K}_{ik} + 2 \tilde{K}_{ip} \tilde{K}_k^p + R_{ik} - (3) \tilde{R}_{ik} + 2 \tilde{D}_i \omega_k - 2 \omega_i \omega_k$$
$$+ \frac{1}{3} \tilde{h}_{ik} \left[ (3) \tilde{R} + 2 \omega_p \omega^p - 2 \tilde{D}_p \omega^p - \tilde{h}^{pq} R_{pq} \right]$$ \hspace{1cm} (29)

Here $\tilde{D}_i$ and $(3) \tilde{R}_{ik}$ are respectively covariant derivative and Ricci tensor of the metric $\tilde{h}_{ik}$ and all indices are lowered and raised with $\tilde{h}_{ik}$ and its inverse.

We now extract from Einstein’s equation the equations of a dynamical system whose variables are scale invariant quantities. For the vacuum case these variables are $(\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)$. In the case where matter fields are present, and where the matter field equations are scale invariant, these metric variables must be supplemented with scale invariant matter fields. Is
the set \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\), along with the matter fields, complete? That is, can the time derivative of each variable be expressed in terms of the other variables? Clearly this holds for the time derivative of \(\tilde{h}_{ik}\). In equation (29) the only questionable term is the one proportional to \(NK\). However, we now show that the Hamiltonian constraint gives \(NK\) in terms of \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\) and the matter fields. The Hamiltonian constraint is

\begin{equation}
(3) R + K^2 - K_{ik}K^{ik} = 2G_{\mu\nu}n^\mu n^\nu .
\end{equation}

Expressing this constraint in terms of scale invariant quantities we have

\begin{equation}
NK = \pm \sqrt{\frac{3}{2}} \left[ \tilde{K}_{pq}\tilde{K}^{pq} + 2\omega_p\omega^p + 4\tilde{D}_p\omega^p - (3)\tilde{R} + 2\tilde{h}^{pq}R_{pq} - N^2\tilde{R} \right]^{1/2} .
\end{equation}

It is also helpful to have an evolution equation for \(NK\). From equations (22),(23) and (28) it follows that

\begin{equation}
\partial_t(NK) = -\frac{2}{3} (NK)^2 + 5\tilde{D}_p\omega^p + 4\omega_p\omega^p + \tilde{h}^{pq}R_{pq} - (3)\tilde{R} .
\end{equation}

In equations (31) and (32) all indices are lowered and raised with \(\tilde{h}_{ik}\) and its inverse. We also need an evolution equation for \(\omega_k\). From equations (23) and (27) it follows that

\begin{equation}
\partial_t\omega_k = \frac{1}{3} \tilde{D}_k (NK) .
\end{equation}

Equations (25), (29) and (33) are the evolution equations for the dynamical system of scale invariant quantities \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\). Here the auxiliary quantity \(NK\) is given by the constraint equation (31) and its evolution is given by equation (32).

Suppose that we have a solution for the dynamical system \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\). What else is needed to determine the metric? Clearly the quantity \(N\) along with \(\tilde{h}_{ik}\) is enough to determine the spacetime metric. However, the quantity \(\omega_k\) already contains most of the information about \(N\). The only missing piece of information is the value of \(N\) at a single point of space as a function of time. Pick a spatial point \(x^i_0\) and define \(N_0(t) \equiv N(t, x^i_0)\). Then the quantity \(N_0\) along with a solution of the scale invariant dynamical system determines the spacetime metric. How does the quantity \(N_0\) evolve? Applying equation (23) at the
point \( x_0^i \) we find
\[
\partial_t \ln N_0 = \frac{1}{3}(NK)(x_0^i) .
\] (34)

However, the quantity \( NK \) is determined by the dynamical system \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\). Therefore, up to an overall constant scale (the value of \( N_0 \) at some initial time \( t_0 \)) the spacetime metric is determined by the scale invariant dynamical system.

Now suppose that the variables \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\) are periodic functions of time. Then it follows that \( N_0 \) is an exponential function multiplied by a periodic function. It then follows that the spacetime is periodically self-similar. If instead \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k)\) are independent of time then \( N_0 \) is an exponential function of time and the spacetime is exactly self-similar.

We now consider how to specify the scale invariant equations for one particular type of matter: the massless, minimally coupled scalar field. Here the Ricci tensor is
\[
R_{\mu\nu} = 8\pi \nabla_\mu \phi \nabla_\nu \phi
\] (35)

and the scalar field satisfies the wave equation
\[
\nabla_\mu \nabla^\mu \phi = 0 .
\] (36)

The field \( \phi \) is scale invariant, so the full set of scale invariant quantities is \((\tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k, \phi)\). It follows that
\[
R_{ik} = 8\pi \tilde{D}_i \phi \tilde{D}_k \phi ,
\] (37)
\[
N^2 R = 8\pi \left[ \tilde{D}_i \phi \tilde{D}^i \phi - (\partial_\phi)^2 \right] .
\] (38)

Expressed in terms of scale invariant quantities equation (36) becomes
\[
-\partial_t \partial_t \phi - \frac{2}{3} NK \partial_i \phi + 2\omega^j \tilde{D}_j \phi + \tilde{D}_i \tilde{D}^i \phi = 0 .
\] (39)

IV. DISCUSSION

The analysis of the previous section shows that the Einstein-scalar equation can be separated into equations for scale invariant quantities and an equation for the overall scale.
Periodic self similarity of the Choptuik critical solution is then equivalent to the existence of a limit cycle for the scale invariant system. Thus the somewhat odd property of periodic self similarity is reduced to the more familiar property of limit cycles of dynamical systems. (Correspondingly the property of exact self-similarity is reduced to the property of limit points of dynamical systems).

What remains to be seen is whether the particular set of scale invariant variables used in section 3 is appropriate for the Einstein-scalar system. That is, we need to find out whether the variables \( \tilde{h}_{ik}, \tilde{K}_{ik}, \omega_k, \phi \) are periodic functions of time in the Choptuik critical solution. This question is presently under study. If the answer is no, then one would need to search for a different set of scale invariant variables to describe critical gravitational collapse.

One would also like to know what physical processes are responsible for the existence of limit cycles in the process of gravitational collapse. In the van der Pol equation, limit cycles come about due to energy dissipation (both positive and negative). The van der Pol system is essentially the harmonic oscillator with a small perturbation. For the harmonic oscillator energy \( E = (\dot{\alpha}^2 + \alpha^2)/2 \) one can show using equation (13) that on the average the energy increases if the amplitude of oscillations is below that of the limit cycle and decreases if the amplitude is above that of the limit cycle. Thus the process of energy dissipation drives arbitrary trajectories to the limit cycle. What is the analog of energy dissipation (both positive and negative) in the spherically symmetric Einstein-scalar system? When the scalar field is weakly gravitating it disperses at late times; so the energy in a fixed region tends to decrease. For a strongly self gravitating scalar field the self gravity tends to concentrate the energy of the field into ever smaller regions. It is these two competing effects that, depending on their relative strengths, combine to form field dispersion, black hole formation or the critical solution. What is needed is to find an “energy-like” quantity for the scale invariant system whose evolution corresponds to effects of concentration or dispersion of the field.

What about the more general case of systems with axisymmetry or no symmetry at all? Here the equations become much more complicated and proving the existence of limit
cycles becomes much more difficult. Here too a promising approach is to find an energy like quantity that is conserved on the limit cycle. An examination of the behavior of such a quantity should yield new intuitions on how scaling arises. It is therefore likely that an examination of the scale invariant dynamical system will be a powerful tool in understanding Choptuik scaling.

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