A LARGE DEVIATIONS PRINCIPLE FOR WIGNER MATRICES
WITHOUT GAUSSIAN TAILS

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Abstract. We consider \( n \times n \) hermitian matrices with i.i.d. entries \( X_{ij} \) whose tail probabilities \( P(|X_{ij}| \geq t) \) behave like \( e^{-at^\alpha} \) for some \( a > 0 \) and \( \alpha \in (0, 2) \). We establish a large deviations principle for the empirical spectral measure of \( X/\sqrt{n} \) with speed \( n^{1+\alpha/2} \) with a good rate function \( J(\mu) \) that is finite only if \( \mu \) is of the form \( \mu = \mu_{sc} \boxplus \nu \) for some probability measure \( \nu \) on \( \mathbb{R} \), where \( \boxplus \) denotes the free convolution and \( \mu_{sc} \) is Wigner’s semicircle law. We obtain explicit expressions for \( J(\mu_{sc} \boxplus \nu) \) in terms of the \( \alpha \)-th moment of \( \nu \). The proof is based on the analysis of large deviations for the empirical distribution of very sparse random rooted networks.

1. Introduction

Let \( \mathcal{H}_n(\mathbb{C}) \) denote the set of \( n \times n \) hermitian matrices. The empirical spectral measure of a matrix \( A \in \mathcal{H}_n(\mathbb{C}) \) is the probability measure on \( \mathbb{R} \) defined by

\[
\mu_A = \frac{1}{n} \sum_{k=1}^{n} \delta_{\lambda_k(A)},
\]

where \( \lambda_1(A) \geq \ldots \geq \lambda_n(A) \) denote the eigenvalues of \( A \) counting multiplicity. Below, we consider the empirical spectral measure of a Wigner random matrix \( X \) described as follows. Let \( (X_{ij})_{1 \leq i < j} \) be i.i.d. complex random variables with variance \( \mathbb{E}|X_{12} - \mathbb{E}X_{12}|^2 = 1 \), and let \( (X_{ii})_{i \geq 1} \) be i.i.d. real random variables. Extend this array by setting \( X_{ij} = X_{ji} \) for \( 1 \leq j < i \), and consider the sequence of \( n \times n \) Hermitian random matrices

\[
X(n) = (X_{ij})_{1 \leq i,j \leq n}.
\] (1)

For ease of notation, we often drop the argument \( n \) and simply write \( X \) for \( X(n) \).

The space \( \mathcal{P}(\mathbb{R}) \) of probability measures on \( \mathbb{R} \) is endowed with the topology of weak convergence: a sequence of probability measures \( (\mu_n)_{n \geq 1} \) converges weakly to \( \mu \) if for any bounded continuous function \( f : \mathbb{R} \to \mathbb{R} \), \( \int f d\mu_n \to \int f d\mu \) as \( n \) goes to infinity. We denote this convergence by \( \mu_n \rightharpoonup \mu \). Wigner’s celebrated theorem asserts that almost surely,

\[
\mu_{X/\sqrt{n}} \rightharpoonup \mu_{sc},
\] (2)

where \( \mu_{sc} \) is the semicircle law, i.e. the probability measure with density \( \frac{1}{\pi \sqrt{4-x^2}} \) on \([-2, 2]\); see e.g. \([4, 8, 12]\).

We consider large deviations, i.e. events of the form \( \mu_{X/\sqrt{n}} \in B \) where \( B \) is a measurable set in \( \mathcal{P}(\mathbb{R}) \) whose closure does not contain the limiting law \( \mu_{sc} \). Clearly, (2) implies that \( \mathbb{P}(\mu_{X/\sqrt{n}} \in B) \to 0 \) as \( n \to \infty \). It follows from known concentration estimates that if the entries \( X_{ij} \) are bounded, or if they satisfy a logarithmic Sobolev inequality, then \( \mathbb{P}(\mu_{X/\sqrt{n}} \in B) \) decays to 0 as fast as \( e^{-cn^2} \) for some constant \( c > 0 \); see Guionnet and Zeitouni \([11]\), or \([3]\). Further, if
the $X_{ij}$ have a gaussian law such that $X$ belongs to the gaussian unitary ensemble GUE or the gaussian orthogonal ensemble GOE, then a full large deviations principle for $\mu_{X/\sqrt{n}}$ with speed $n^2$ has been established by Ben Arous and Guionnet in [5]. However, apart from the GUE and GOE cases, we are not aware of any case for which the large deviations principle for $\mu_{X/\sqrt{n}}$ has been obtained.

In this paper we prove a large deviations principle under the assumption that $X_{ij}$ has tail probabilities $\mathbb{P}(|X_{ij}| \geq t)$ of order $e^{-at^\alpha}$ for some $a > 0$ and $\alpha \in (0, 2)$. Before stating our assumptions and results in detail, let us make some preliminary remarks.

By considering events of the form $|X_{ij}| \sim \sqrt{n}$, $(i, j) \in I$, for suitable sets $I$ of pairs of indices, it is not hard to see that a nontrivial large deviation can be achieved with probability at least as large as $e^{-cn^{1+\alpha/2}}$, for some $c > 0$. For instance, the case when $I$ is the diagonal $(i, i)$, $i = 1, \ldots, n$, can be used to produce a global shift of the spectral measure $\mu_{sc}$ at a cost

$$-\log \mathbb{P}(|X_{ii}| \sim \sqrt{n}, i = 1, \ldots, n) = O(n^{1+\alpha/2}),$$

on the exponential scale. Similarly, one expects to be able to produce more general deformations of $\mu_{sc}$ at a cost of order $n^{1+\alpha/2}$. It turns out that this picture is correct, provided the deformations of $\mu_{sc}$ are of the form $\mu = \mu_{sc} \boxplus \nu$ for some $\nu \in \mathcal{P}(\mathbb{R})$, where $\boxplus$ denotes the free convolution. Roughly speaking, the idea is that the entries of $X$ that are visible on a scale $\sqrt{n}$ form a very sparse weighted random graph or random network $G_n$ that is asymptotically independent from the rest of the matrix, and a large deviations principle for $\mu_{X/\sqrt{n}}$ can be deduced from a large deviations principle for the law of the random network $G_n$. This approach also allows us to obtain explicit expressions for the rate function.

The strategy of proof developed in the present work for Wigner matrices could certainly be generalized to other models such as random covariance matrices or random band matrices with the same type of tail assumptions on the entries. We also believe that our strategy might extend to other tail assumptions such as power laws $\mathbb{P}(|X_{ij}| \geq t) \sim 1/t^\alpha$, with exponent $\alpha > 2$. The analysis of large deviations for the associated random network is however more delicate in this case.

**Main result.** We recall that a sequence of random variables $(Z_n)_{n \geq 1}$ with values in a topological space $\mathcal{X}$ with Borel $\sigma$-field $\mathcal{B}$, satisfies the large deviations principle (LDP) with rate function $J$ and speed $v$, if $J : \mathcal{X} \mapsto [0, \infty]$ is a lower semi-continuous function, $v : \mathbb{N} \mapsto [0, \infty)$ is a function which increases to infinity, and for every $B \in \mathcal{B}$:

$$-\inf_{x \in B^0} J(x) \leq \liminf_{n \to \infty} \frac{1}{v(n)} \log \mathbb{P}(Z_n \in B) \leq \limsup_{n \to \infty} \frac{1}{v(n)} \log \mathbb{P}(Z_n \in B) \leq -\inf_{x \in \overline{B}} J(x),$$

where $B^0$ denotes the interior of $B$ and $\overline{B}$ denotes the closure of $B$. We recall that the lower semi-continuity of $J$ means that its level sets $\{x \in \mathcal{X} : J(x) \leq t\}$ are closed for all $t \geq 0$. When the level sets are compact the rate function $J$ is said to be *good*.

We now introduce our statistical assumption. Let $a, \alpha \in (0, \infty)$. We say that a complex random variable $Y$ belongs to the class $\mathcal{S}_\alpha(a)$, and write $Y \in \mathcal{S}_\alpha(a)$, if

$$\lim_{t \to \infty} -t^{-\alpha} \log \mathbb{P}(|Y| \geq t) = a,$$

and if $Y/|Y|$ and $|Y|$ are independent for large values of $|Y|$, i.e. there exists $t_0 > 0$ and a probability $\vartheta \in \mathcal{P}(\mathbb{S}^1)$ on the unit circle $\mathbb{S}^1$ such that for all $t \geq t_0$, all measurable sets $U \subset \mathbb{S}^1$, one has

$$\mathbb{P}(Y/|Y| \in U \text{ and } |Y| \geq t) = \vartheta(U)\mathbb{P}(|Y| \geq t).$$
For instance, if $Y$ is Weibull, i.e., $Y$ is a nonnegative random variable with distribution function $F(t) = 1 - e^{-at^\alpha}$, with $\alpha > 0$, and $a > 0$, then $Y \in S_\alpha(a)$, with $\theta = \delta_1$, the unit mass at the point 1. Clearly, if $Y \in S_\alpha(a)$ is real valued, then the associated measure $\vartheta$ must have support in $\{-1, 1\}$. Moreover, for all $\alpha > 0$ we write $Y \in S_\alpha(\infty)$ whenever $\square$ holds with $a = \infty$. Thus, with the above notation one has that if $Y$ is subgaussian then $Y \in S_\beta(\infty)$ for all $\beta \in (0, 2)$ and if $Y \in S_\alpha(a)$ for some $\alpha, a > 0$, then $Y \in S_\beta(\infty)$ for all $\beta \in (0, \alpha)$.

Throughout the paper, we assume that the array $X_{ij}$ is given as above, i.e. $X_{ij}, i < j$, are i.i.d. copies of a complex random variable $X_{12}$ with unit variance, and $X_{ii}$ are i.i.d. copies of a real random variable $X_{11}$. Moreover, the following main assumption will always be understood without explicit mention.

**Assumption 1.** There exist $\alpha \in (0, 2)$ and $a, b \in (0, \infty]$ such that $X_{12} \in S_\alpha(a)$ and $X_{11} \in S_\alpha(b)$.

The main result can be formulated as follows.

**Theorem 1.1.** The measures $\mu_{X/\sqrt{n}}$ satisfies the LDP with speed $n^{1+\alpha/2}$ and good rate function

$$J(\mu) = \begin{cases} \Phi(\nu) & \text{if } \mu = \mu_{\text{sc}} \oplus \nu \text{ for some } \nu \in \mathcal{P}(\mathbb{R}) \\ \infty & \text{otherwise}, \end{cases}$$

where $\Phi : \mathcal{P}(\mathbb{R}) \mapsto [0, \infty]$ is a good rate function.

The proof of Theorem 1.1 consists of two main parts. The first part, the “random matrix theory part” of the work, is discussed in Section 2. Here, we show that at speed $n^{1+\alpha/2}$ the large deviations are governed by the sparse $n \times n$ random matrix $C = C(n)$ defined by

$$C_{ij} = \begin{cases} \frac{X_{ij}}{\sqrt{n}} & \text{if } \varepsilon(n) \leq \frac{X_{ij}}{\sqrt{n}} \leq \varepsilon(n)^{-1} \\ 0 & \text{otherwise} \end{cases}$$

where $\varepsilon(n)$ is a cutoff sequence that for convenience will be set equal to $1/\log n$. In particular, we show that as far as the LDP with speed $n^{1+\alpha/2}$ is concerned, $\mu_{X/\sqrt{n}}$ behaves as $\mu_{\text{sc}} \oplus \mu_C$, where $\mu_C$ is the spectral measure of the matrix $C$; see Proposition 2.1 below. As a consequence, the LDP for $\mu_{X/\sqrt{n}}$ can be obtained by contraction if one has the LDP for $\mu_C$ with speed $n^{1+\alpha/2}$ and rate function $\Phi$.

The second part, the “random graph theory part” of the work, is presented in Section 3. Here, we prove the above mentioned LDP for the spectral measures $\mu_C$. This requires the analysis of large deviations for sparse random networks, and some use of the theory of local convergence for random networks that was recently developed by Benjamini and Schramm [6], Aldous and Steele [2], and Aldous and Lyons [1]. Let us briefly sketch the main ideas. Let $G_n$ be the sparse random network naturally associated to the $n \times n$ matrix $C$, and let $\rho_n$ denote the law of the equivalence class (under rooted isomorphisms) of the connected component of $G_n$ at the root, when the root is chosen uniformly at random. The law $\rho_n$ is regarded as an element of the space $\mathcal{P}(\mathcal{G}_v)$ of probability measures on $\mathcal{G}_v$, where $\mathcal{G}_v$ is the space of equivalence classes of connected rooted networks. We introduce a suitable weak topology on $\mathcal{P}(\mathcal{G}_v)$, and prove that the measures $\rho_n$ satisfy a LDP with speed $n^{1+\alpha/2}$ and a good rate function $I(\rho)$. The latter is finite only if $\rho$ belongs to the so called sofic measures, i.e. if $\rho$ is a limit of finite networks, and if the support of $\rho$ satisfies some natural constraints. We call $\mathcal{P}_s(\mathcal{G}_v)$ the set of such probability measures. We find that for $\rho \in \mathcal{P}_s(\mathcal{G}_v)$, one has

$$I(\rho) = b \mathbb{E}_\rho |\omega_{G}(a)|^{\alpha} + \frac{a}{2} \mathbb{E}_\rho \sum_{v \in V_{G} \setminus o} |\omega_{G}(a, v)|^{\alpha},$$

(7)
where $E_{\rho}$ denotes expectation w.r.t. $\rho$, the law of the equivalence class of a connected rooted network $(G, o)$, $o$ denoting the root; $\omega_G(o)$ denotes the weight of the loop at the root, and $\omega_G(o, v)$ denotes the weight of the edge $(o, v)$ if $v$ is an element of the vertex set $V_G$ of the network. We refer to Biane [7] for the precise result.

It turns out that the choice of a “myopic” topology on $\mathcal{P}(\mathcal{G}_*)$ is crucial to have the desired result. On the other hand we want this topology to be fine enough to have that the map $\rho \mapsto \mu_{\rho}$ defining the “spectral measure” associated to $\rho$ is continuous. If all this is satisfied, then a LDP for the spectral measure $\mu_C = \mu_{\rho_n}$ can be obtained by contraction from the LDP for $\rho_n$; see Proposition 3.13. In particular, we find that the function $\Phi$ in Theorem 1.1 is given by

$$\Phi(\nu) = \inf\{I(\rho) : \rho \in \mathcal{P}_s(\mathcal{G}_*) : \mu_{\rho} = \nu\}. \quad (8)$$

We turn to more explicit characterizations of the rate function in Theorem 1.1. First, the rate function $\Phi$ depends on the laws of $X_{11}$ and $X_{12}$ only through $a, a, b$ and the supports of the associated measures on $S^1$. While the variational principle (8) is not always explicitly solvable, there is a large class of $\nu \in \mathcal{P}(\mathbb{R})$ for which $\Phi(\nu)$ can be computed. This allows us to give explicit expressions for the rate function $J(\nu)$ in Theorem 1.1. Recall that the free convolution with $\mu_{sc}$ is injective: for any $\mu \in \mathcal{P}(\mathbb{R})$ there is at most one $\nu \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_{sc} \boxplus \nu$. Let $\mathcal{P}_{sym}(\mathbb{R})$ denote the set of symmetric probability measures on $\mathbb{R}$. If $\mu = \mu_{sym} \boxplus \nu$, then $\mu \in \mathcal{P}_{sym}(\mathbb{R})$ is equivalent to $\nu \in \mathcal{P}_{sym}(\mathbb{R})$. For more details on free convolution with the semi-circular distribution, we refer to Biane [7]. For $\nu \in \mathcal{P}(\mathbb{R})$ we use the notation

$$m_\alpha(\nu) = \int |x|^\alpha d\nu(x) \quad (9)$$

for the $\alpha$-th moment of $\nu$. If $X_{11} \in S_\alpha(b)$ for some $b < \infty$, then we write $\theta_b$ for the associated measure on $\{-1, 1\}$. The following theorem summarizes the main facts we can establish about the rate function.

**Theorem 1.2.** a) For any $\nu \in \mathcal{P}(\mathbb{R})$,

$$\Phi(\nu) \geq \left(\frac{a}{2} \wedge b\right) m_\alpha(\nu).$$

b) If $\text{supp}(\theta_b) = \{-1, 1\}$, then for any $\nu \in \mathcal{P}(\mathbb{R})$:

$$\Phi(\nu) \leq b m_\alpha(\nu).$$

c) If $\text{supp}(\theta_b) = \{-1, 1\}$, and $\nu \in \mathcal{P}_{sym}(\mathbb{R})$, then

$$\Phi(\nu) = \left(\frac{a}{2} \wedge b\right) m_\alpha(\nu).$$

Some remarks about Theorem 1.2. Part a) shows clearly that $J$ is a good rate function and that $J(\mu) = 0$ is equivalent to $\mu = \mu_{sc}$. Concerning the remaining statements, the fact that the moments $m_\alpha(\nu)$ appear naturally in the rate function and the special role played by symmetric measures $\nu$ can be understood as follows. Let $D$ denote the diagonal matrix with entries $X_{11}, \ldots, X_{nn}$ and, for $n$ even, let $A$ denote the block diagonal matrix with $2 \times 2$ blocks defined by $A_{2i-1, 2i} = X_{i,i+1}, A_{2i, 2i-1} = X_{i,i+1}$, $i = 1, \ldots, n/2$, and with $A_{i,j} = 0$ for all other entries. Then it is straightforward to see that the empirical spectral measures of $D/\sqrt{n}$ and $A/\sqrt{n}$ are given by

$$\mu_{\mathcal{D}/\sqrt{n}} = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_{i,i}/\sqrt{n}}, \quad \mu_{\mathcal{A}/\sqrt{n}} = \frac{1}{n} \sum_{i=1}^{n/2} \left(\delta_{|X_{i+1,i+1}/\sqrt{n}|} + \delta_{-|X_{i+1,i+1}/\sqrt{n}|}\right).$$
Our results will show in particular that if the variables $X_{ij}$ are as in Assumption 1, and $\text{supp}(\vartheta^b) = \{-1, 1\}$, then:

1) $\mu_{D/\sqrt{n}}$ satisfies a LDP on $\mathcal{P}(\mathbb{R})$ with speed $n^{1+\alpha/2}$ and rate function $I(\nu) = bm_\alpha(\nu)$, for all $\nu \in \mathcal{P}(\mathbb{R})$;

2) $\mu_{A/\sqrt{n}}$ satisfies a LDP on $\mathcal{P}(\mathbb{R})$ with speed $n^{1+\alpha/2}$ and rate function equal to $I(\nu) = \frac{a}{2} m_\alpha(\nu)$, for all $\nu \in \mathcal{P}_{\text{sym}}(\mathbb{R})$, and $I(\nu) = +\infty$ if $\nu \not\in \mathcal{P}_{\text{sym}}(\mathbb{R})$.

The statements above can be seen as extremal instances of Sanov’s theorem for variables with exponential tails of the form (4). Thus, roughly speaking, part b) in Theorem 1.2 says that for $\mu_{X/\sqrt{n}}$ it is always possible to realize a deviation $\mu_{\text{sc}} \boxplus \nu$ by tilting diagonal entries only, i.e. using the deviation $\nu$ for $\mu_{D/\sqrt{n}}$. When $b \leq a/2$, this is sharp, and indeed part a) and part b) above yield the expression $\Phi(\nu) \leq bm_\alpha(\nu)$. The general bound in part a) then shows that this is actually the best strategy.

If the support of $\vartheta^b$ is only $\{+1\}$ (or $\{-1\}$) then the above scenario changes in that one can use the diagonal matrix $D$ only to reach deviations $\nu$ whose support is $\mathbb{R}^+$ (or $\mathbb{R}^-$). In this case we have the following estimates. Without loss of generality, we restrict to $\text{supp}(\vartheta^b) = \{+1\}$.

**Theorem 1.3.** Suppose $\text{supp}(\vartheta^b) = \{+1\}$.

a) If $\text{supp}(\nu) \subset \mathbb{R}^+$, then

$$\Phi(\nu) \leq bm_\alpha(\nu).$$

b) Suppose $\alpha \in (1, 2)$. If $\nu \in \mathcal{P}_{\text{sym}}(\mathbb{R})$, then

$$\Phi(\nu) = \frac{a}{2} m_\alpha(\nu).$$

c) Suppose $\alpha \in (1, 2)$. If $\int x d\nu(x) < 0$ then $\Phi(\nu) = +\infty$.

The above result can be interpreted as before by appealing to the large deviations of $\mu_{D/\sqrt{n}}$ and $\mu_{A/\sqrt{n}}$. In particular, part b) shows that since one cannot realize a symmetric deviation $\nu \in \mathcal{P}_{\text{sym}}(\mathbb{R})$ using the matrix $D$ only, it is less costly to realize it using the matrix $A$ only. Similarly, in part c) one has that neither $D$ nor $A$, nor any other matrix with vanishing trace, can be used to produce a measure $\nu$ with $\int x d\nu(x) < 0$, and therefore the rate function must be $+\infty$. We believe that results in parts b) and c) above should hold without the additional condition $\alpha \in (1, 2)$.

The proofs of Theorems 1.2 and 1.3 are given in Subsection 3.10.

2. Exponential equivalences

Throughout the rest of the paper, we fix the cutoff sequence $\varepsilon(n)$ as

$$\varepsilon(n) = \frac{1}{\log n}. \quad (10)$$

For ease of notation, we often write simply $\varepsilon$ in place of $\varepsilon(n)$. We decompose the matrix $X$ as

$$\frac{X}{\sqrt{n}} = A + B + C + D, \quad (11)$$
where the matrices $A, B, C, D$ are defined by

\[ A_{ij} = 1_{|X_{ij}|<(\log n)^{2/\alpha}} \frac{X_{ij}}{\sqrt{n}} \]

\[ B_{ij} = 1_{(\log n)^{2/\alpha} \leq |X_{ij}| \leq \epsilon n^{1/2}} \frac{X_{ij}}{\sqrt{n}} \]

\[ C_{ij} = 1_{\epsilon n^{1/2} \leq |X_{ij}| \leq \epsilon^{-1} n^{1/2}} \frac{X_{ij}}{\sqrt{n}} \]

\[ D_{ij} = 1_{\epsilon^{-1} n^{1/2} < |X_{ij}|} \frac{X_{ij}}{\sqrt{n}} \]

We define the distance on $\mathcal{P}(\mathbb{R})$ as

\[ d(\mu, \nu) = \sup \{|g_\mu(z) - g_\nu(z)| : \Im(z) \geq 2\}, \quad (12) \]

where $g_\mu$ is the Cauchy-Stieltjes transform of $\mu$, i.e. for $z \in \mathbb{C}_+ = \{z \in \mathbb{C} : \Im(z) > 0\}$,

\[ g_\mu(z) = \int \frac{\mu(dx)}{x - z}. \quad (13) \]

Recall that this distance is a metric for the weak convergence, see e.g. [3, Theorem 2.4.4]. Let also $d_{KS}$ denote the Kolmogorov-Smirnov distance and let $W_1$ denote the $L^1$-Wasserstein distance, see Section 2 below for the relevant definitions. From (72) and (74) one has

\[ d(\mu, \nu) \leq d_{KS}(\mu, \nu) \wedge W_1(\mu, \nu). \quad (14) \]

The following proposition is the first major step on the way to prove Theorem 1.1

**Proposition 2.1.** The random probability measures $\mu_{sc} \boxplus \mu_C$ and $\mu_{X/\sqrt{n}}$ are exponentially equivalent: for any $\delta > 0$,

\[ \lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P} \left( d(\mu_{X/\sqrt{n}}, \mu_{sc} \boxplus \mu_C) \geq \delta \right) = -\infty. \]

2.1. **Preliminary estimates.** The strategy of proof of Proposition 2.1 is in 3 steps: we start by showing that the contribution of $D$ in (11) can be neglected (Lemma 2.2), then we show that $B$ can also be neglected (Lemma 2.3). The main step will then consist in proving that $\mu_{A+C}$ and $\mu_{sc} \boxplus \mu_C$ are exponentially equivalent.

**Lemma 2.2** (Very large entries). The random probability measures $\mu_{A+B+C}$ and $\mu_{X/\sqrt{n}}$ are exponentially equivalent: for any $\delta > 0$,

\[ \lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P} \left( d(\mu_{X/\sqrt{n}}, \mu_{A+B+C}) \geq \delta \right) = -\infty. \]

**Proof.** From (14), it is sufficient to prove that for any $\delta > 0$,

\[ \lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P} \left( d_{KS}(\mu_{X/\sqrt{n}}, \mu_{A+B+C}) \geq \delta \right) = -\infty. \]

Then, using the rank inequality Lemma 3.1 it is sufficient to prove that

\[ \lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\text{rank}(D) \geq \delta n) = -\infty. \]

However, the rank is bounded by the number of non-zeros entries of a matrix:

\[ \mathbb{P}(\text{rank}(D) \geq 2\delta n) \leq \mathbb{P} \left( \sum_{1 \leq i \leq j \leq n} 1(|X_{ij}| \geq \epsilon^{-1} n^{1/2}) \geq \delta n \right). \]

The Bernoulli variables $1(|X_{ij}| \geq \epsilon^{-1} n^{1/2}), 1 \leq i \leq j \leq n$, are independent. Also, by assumption (11), their mean value $p_{ij} = \mathbb{P}(|X_{ij}| \geq \epsilon^{-1} n^{1/2})$ satisfies

\[ p_{ij} \leq p = e^{-c_\nu \alpha n^{1/2}}. \]
for some \( c > 0 \). For our choice of \( \varepsilon \) in (10), \( p = o(1/n^2) \). Hence it is sufficient to prove that

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}\left( \sum_{1 \leq i \leq j \leq n} \left( 1(|X_{ij}| \geq \varepsilon^{-1}n^{1/2}) - p_{ij} \right) \geq \delta n \right) = -\infty.
\]

Recall that from Bennett’s inequality, if \( W_i \), \( i = 1, \ldots, m \) are independent Bernoulli(\( p_i \)) variables, and \( h(x) = (x + 1) \log(x + 1) - x \), then one has

\[
\mathbb{P}\left( \sum_{i=1}^{m} (W_i - p_i) \geq t \right) \leq \exp \left( -\sigma^2 h\left( \frac{t}{\sigma^2} \right) \right)
\]

with \( \sigma^2 = \sum_{i=1}^{m} p_i (1 - p_i) \). In our case, for all \( n \) large enough,

\[
\sigma^2 = \sum_{1 \leq i \leq j \leq n} p_{ij} (1 - p_{ij}) \leq \frac{n(n + 1)p}{2}.
\]

Therefore, using \( h(x) \sim x \log x \) as \( x \to \infty \),

\[
\mathbb{P}\left( \sum_{1 \leq i \leq j \leq n} \left( 1(|X_{ij}| \geq \varepsilon^{-1}n^{1/2}) - p_{ij} \right) \geq \delta n \right) \leq \exp \left( -\sigma^2 h\left( \frac{n\delta}{\sigma^2} \right) \right)
\]

\[
\leq \exp \left( -c_0 n \log \left( \frac{1}{np} \right) \right),
\]

for some constant \( c_0 > 0 \) depending on \( \delta \). Now, since \( n = o(p^{-1}) \), we find that for some \( c_1 > 0 \), for all \( n \) large enough the last expression is upper bounded by

\[
\exp \left( \frac{1}{2} c_0 n \log p \right) \leq \exp \left( -c_1 n^{1+\alpha/2} \epsilon^{-\alpha} \right).
\]

This proves the claim. \( \square \)

We now show that the contribution of \( B \) in (11) is also negligible.

**Lemma 2.3 (Moderately large entries).** The random probability measures \( \mu_{A+C} \) and \( \mu_{X/\sqrt{n}} \) are exponentially equivalent: for any \( \delta > 0 \),

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(d(\mu_{X/\sqrt{n}}, \mu_{A+C}) \geq \delta) = -\infty.
\]

**Proof.** By Lemma 2.2 and the triangle inequality, it is sufficient to check that for any \( \delta > 0 \),

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(W_2(\mu_{A+B+C}, \mu_{A+C}) \geq \delta) = -\infty,
\]

where \( W_2 \geq W_1 \) is the \( L^2 \)-Wasserstein distance defined by (73). From Hoffman-Wielandt inequality Lemma 2.2, it is sufficient to prove that for any \( \delta > 0 \),

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}\left( \frac{1}{n} \text{tr}(B^2) \geq \delta \right) = -\infty.
\]

We write

\[
\frac{1}{n} \text{tr}(B^2) \leq \frac{2}{n^2} \sum_{1 \leq i \leq j \leq n} |X_{ij}|^2 \mathbb{1}((\log n)^{2/\alpha} \leq |X_{ij}| \leq \varepsilon n^{1/2}).
\]

Thus, from Chernoff’s bound, for any \( \lambda > 0 \),

\[
\mathbb{P}\left( \frac{1}{n} \text{tr}(B^2) \geq 2\delta \right) \leq e^{-\lambda \delta} \prod_{1 \leq i \leq j \leq n} \mathbb{E}\left[ e^{-\varepsilon^2 \lambda |X_{ij}|^2 \mathbb{1}((\log n)^{2/\alpha} \leq |X_{ij}| \leq \varepsilon n^{1/2})} \right].
\]
To estimate the last expectation, we use the integration by part formula, for \( \mu \in \mathcal{P}([0, \infty)) \) and \( g \in C^1([0, \infty)) \),
\[
\int_a^b g(x) d\mu(x) = g(a)\mu([a, \infty)) - g(b)\mu((b, \infty)) + \int_a^b g'(x)\mu([x, \infty)) dx.
\]
(16)
Define the function
\[
f(x) = n^{-2} \lambda x^2 - cx^\alpha.
\]
(17)
Let \( \mu \) denote the law of \( |X_{ij}| \), and \( g(x) = e^{-2\lambda x^2} \). By Assumption (11) there exists a constant \( c > 0 \) such that
\[
\mu(t, \infty) = \mathbb{P}(|X_{ij}| > t) \leq \exp(-ct^\alpha),
\]
(18)
for all \( t \) large enough. In particular, \( g(t)\mu(t, \infty) \leq e^{f(t)} \). From (16) it follows that
\[
\mathbb{E}\left[ e^{-2\lambda |X_{ij}|^2} \mathbb{1}_{(|\log n|^{2/\alpha} \leq |X_{ij}| \leq 2n^{1/2}} \right] \leq 1 + \int_{|\log n|^{2/\alpha}}^{2n^{1/2}} g(x) d\mu(x)
\leq 1 + e^{f(|\log n|^{2/\alpha})} + \int_{|\log n|^{2/\alpha}}^{2n^{1/2}} \frac{2\lambda x}{n^2} e^{f(x)} dx.
\]
(19)
We choose \( \lambda = \frac{1}{2} c \varepsilon^{-2} n^{1+\alpha/2} \), with the constant \( c > 0 \) given in (18). Simple computations show that \( f(x) \) reaches its maximum for \( x \in [|\log n|^{2/\alpha}, 2n^{1/2}] \) at \( x = (\log n)^{2/\alpha} \), where it is equal to
\[
\frac{1}{2} c \varepsilon^{-2} n^{1+\alpha/2} (\log n)^{4/\alpha} - c(\log n)^2.
\]
Using (10), for \( n \geq n_0 \) this is smaller than \(-\frac{c}{2}(\log n)^2\). Therefore, using \( 1 + x < e^x \), \( x \geq 0 \), one has that (19) is bounded by \( e^{\frac{1}{n} \text{tr}(B^2)} \) for \( n \) large enough. It follows that
\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}\left( \frac{1}{n} \text{tr}(B^2) \geq 2\delta \right) \leq -\frac{1}{2} c \delta \varepsilon^{-2} + \frac{n^{1-\alpha/2}}{2} e^{-\frac{1}{4}(\log n)^2}.
\]
The desired conclusion follows.

For \( s > 0 \), we define the compact set for the weak topology
\[
K_s = \{ \mu \in \mathcal{P}([0, \infty)) : \int x^2 d\mu \leq s \}.
\]
For a suitable choice of \( s \), we now check that \( \mu_C \) is in \( K_s \) with large probability.

**Lemma 2.4** (Exponential tightness estimates).
\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}\left( \mu_C \notin K_{(\log n)^2} \right) = -\infty.
\]
Moreover, if \( I = \{(i,j) : |X_{ij}| > (\log n)^{2/\alpha}\} \), for any \( \delta > 0 \),
\[
\lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(|I| \geq \delta n^{1+\alpha/2}) = -\infty.
\]

**Proof.** Notice that
\[
\int x^2 d\mu_C \leq \frac{1}{n} \text{tr}(C^2) \leq \frac{2}{n^2} \sum_{1 \leq i,j \leq n} |X_{ij}|^2 \mathbb{1}_{(\varepsilon n^{1/2} < |X_{ij}| \leq \varepsilon^{-1} n^{1/2})}.
\]
We may repeat the argument in the proof of Lemma 2.3. This time we take $\lambda = \frac{1}{4}c\varepsilon^2 - \alpha - n^{1+\alpha/2}$, where $c$ is as in (15), and then define $f$ as in (17). For any $s > 0$ one has

$$P(\mu_C \notin K_{2s}) \leq e^{-\lambda s}(1 + e^{s\varepsilon n}) + \frac{1}{2}c\varepsilon n^{1/2}\varepsilon - \alpha \max_{x \in [\varepsilon n^{1/2}, \varepsilon^{-1}n^{1/2}]} e^{f(x)}s^2.$$

Simple considerations show that $f(x)$, for $x \in [\varepsilon n^{1/2}, \varepsilon^{-1}n^{1/2}]$ is maximized at $x = \varepsilon n^{1/2}$, where it satisfies $f(\varepsilon n^{1/2}) \leq -\frac{1}{2}c\varepsilon\varepsilon n^{n/2}$. This gives, for $n$ large enough,

$$\frac{1}{n^{1+\alpha/2}} \log P(\mu_C \notin K_{2s}) \leq -\frac{1}{2}c\varepsilon^2 - \alpha s - n^{1-\alpha/2}c - \frac{1}{2}c\varepsilon\varepsilon n^{n/2}.$$

We choose finally $s = 1/(2\varepsilon^2)$. For our choice of $\varepsilon$ in (10), this implies the first claim.

For the second claim, we have

$$P(|I| \geq 2\delta n^{1+\alpha/2}) \leq P\left(\sum_{1 \leq i < j \leq n} 1(|X_{ij}| \geq (\log n)^{2/\alpha}) \geq \delta n^{1+\alpha/2}\right).$$

The Bernoulli variables $1(|X_{ij}| \geq (\log n)^{2/\alpha})$, $1 \leq i \leq j \leq n$, are independent. Also, by Assumption [1] their average $p_{ij} = P(|X_{ij}| \geq (\log n)^{2/\alpha})$ satisfies

$$p_{ij} \leq p = e^{-c(\log n)^2},$$

for some $c > 0$. We argue as in the proof of Lemma 2.2. From Bennett’s inequality (15),

$$P\left(\sum_{1 \leq i < j \leq n} 1(|X_{ij}| \geq (\log n)^{2/\alpha}) - p_{ij}) \geq \delta n^{1+\alpha/2}\right) \leq \exp\left(-c_0 n^{1+\alpha/2}\log\left(\frac{n^{\alpha/2-1}}{p}\right)\right),$$

for some constant $c_0 = c_0(\delta) > 0$. Since $p = o(n^{\alpha/2-1})$, this gives the claim. $\square$

2.2. Auxiliary estimates. To complete the proof of Proposition 2.1 we shall need two extra results. The first is due to Guionnet and Zeitouni [11, corollary 1.4].

**Theorem 2.5** (Concentration for matrices with bounded entries). Let $\kappa \geq 1$, let $Y \in \mathcal{H}_n(\mathbb{C})$ be a random matrix with independent entries $(Y_{ij})_{1 \leq i < j \leq n}$ bounded by $\kappa$, and let $M \in \mathcal{H}_n(\mathbb{C})$ be a deterministic matrix such that $\int x^2d\mu_M \leq \kappa^2$. There exists a universal constant $c > 0$ such that for all $(c\kappa^2/n)^{2/5} \leq t \leq 1$,

$$P(W_1\left(\mu_{Y/\sqrt{n}+M}, \mathbb{E}\mu_{Y/\sqrt{n}+M} \geq t\right) \leq \frac{c\kappa}{t^{1/2}} \exp\left(-\frac{n^2t^5}{c\kappa^4}\right).$$

In [11] corollary 1.4], the result is stated for matrices $Y$ in $\mathcal{H}_n(\mathbb{C})$ such that the entries have independent real and imaginary parts. The extension to our setting follows by using a version of Talagrand’s concentration inequality for independent bounded variables in $\mathbb{C}$. Also, the matrix $M$ is not present in [11]. It is however not hard to check that its presence does not change the argument in [11] page 132], since one can use the bound

$$\int x^2d\mu_{Y/\sqrt{n}+M} \leq 2\int x^2d\mu_{Y/\sqrt{n}} + 2\int x^2d\mu_M \leq 4\kappa^2.$$

The latter is an easy consequence of e.g. Lemma 2.2.

The second result we need is a uniform bound on the rate of the convergence of the empirical spectral measure of sums of random matrices.
Theorem 2.6 (Uniform asymptotic freeness). Let $Y = (Y_{ij})_{1 \leq i, j \leq n} \in \mathcal{H}_n(\mathbb{C})$ be a Wigner random matrix with $\text{Var}(Y_{12}) = 1$, $\mathbb{E}|Y_{12}|^3 < \infty$ and $\mathbb{E}|Y_{11}|^2 < \infty$. There exists a universal constant $c > 0$ such that for any integer $n \geq 1$ and any $M \in \mathcal{H}_n(\mathbb{C})$,

$$d\left(\mathbb{E}\mu_{Y/\sqrt{n}+M}, \mu_{sc} \boxplus \mu_M\right) \leq c \frac{\sqrt{\mathbb{E}|Y_{11}|^2} + \mathbb{E}|Y_{12}|^3}{\sqrt{n}}.$$ 

A striking point of the above theorem is that the constant $c$ does not depend on $M$. The proof of Theorem 2.6 is given in Section A below. We are now ready to finish the proof of Proposition 2.1.

2.3. Proof of Proposition 2.1. By Lemma 2.2 and 2.3 it is sufficient to prove that $\mu_{A+C}$ and $\mu_{sc} \boxplus \mu_C$ are exponentially equivalent: for any $\delta > 0$,

$$\lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(d(\mu_{sc} \boxplus \mu_C, \mu_{A+C}) \geq \delta) = -\infty. \tag{20}$$

Let $\mathcal{F}$ be the $\sigma$-algebra generated by the random variables

$$\{X_{ij} : (i, j)\}$$

such that $|X_{ij}| \geq (\log n)^{2/\alpha}$. Then $C$ is $\mathcal{F}$-measurable and, given $\mathcal{F}$, $A$ is a random matrix with independent entries $(A_{ij})_{1 \leq i, j \leq n}$ bounded by $(\log n)^{2/\alpha}$. Define the event

$$E = \left\{ \int x^2 d\mu_C \leq (\log n)^2 \right\}.$$ 

Then $E \in \mathcal{F}$. Lemma 2.4 implies that for some sequence $s_1(n) \to \infty$ and all $n$ large enough,

$$\mathbb{P}(E^c) \leq e^{-s_1(n)n^{1+\alpha/2}}. \tag{21}$$

Also, using (14) and Theorem 2.5 applied to $\kappa = (\log n)^2 \vee (\log n)^{2/\alpha}$, for some sequence $s_2(n) \to \infty$, for all $n$ large enough,

$$\mathbb{1}_E \mathbb{P}_\mathcal{F}\left(d(\mathbb{E}_\mathcal{F}\mu_{A+C}, \mu_{A+C}) \geq \delta/3\right) \leq e^{-s_2(n)n^{1+\alpha/2}}. \tag{22}$$

where $\mathbb{P}_\mathcal{F}$ and $\mathbb{E}_\mathcal{F}$ are the conditional probability and expectation given $\mathcal{F}$. From (21) and (22), using the triangle inequality one has that (20) follows once we prove that for any $\delta > 0$:

$$\lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(d(\mu_{sc} \boxplus \mu_C, \mathbb{E}_\mathcal{F}\mu_{A+C}) \geq \delta) = -\infty. \tag{23}$$

We now use a coupling argument to remove the dependency between $A$ and $C$. Let $P_n$ be the law of $X_{12}$ conditioned on $\{|X_{12}| < (\log n)^{2/\alpha}\}$, and $Q_n$ be the law of $X_{11}$ conditioned on $\{|X_{11}| < (\log n)^{2/\alpha}\}$. We also define $I = \{(i, j) : |X_{ij}| \geq (\log n)^{2/\alpha}\}$. Given $\mathcal{F}$, if $(i, j) \in I$, then $A_{ij} = 0$ while, if $(i, j) \notin I$ and $1 \leq i \leq j \leq n$, then $\sqrt{n}A_{ij}$ has conditional law $P_n$ or $Q_n$ depending on whether $i < j$ or $i = j$.

On our probability space, we now consider $Y$ an independent hermitian random matrix such that $(Y_{ij})_{1 \leq i, j \leq n}$ are independent, and for $1 \leq i \leq n$, $Y_{ii}$ has law $Q_n$, while for $1 \leq i < j \leq n$, $Y_{ij}$ has law $P_n$. We form the matrix

$$A'_{ij} = \mathbb{1}(\{(i, j) \notin I\})A_{ij} + \mathbb{1}(\{(i, j) \in I\})\frac{Y_{ij}}{\sqrt{n}}.$$
By construction, $\sqrt{n}A'$ and $Y$ have the same distribution and are independent of $\mathcal{F}$. Also, by Lemma B.2 and Jensen’s inequality,

$$\mathbb{E}_F d(\mu_{A+C}, \mu_{A'+C}) \leq \sqrt{\frac{\mathbb{E}_F \text{tr}(A - A')^2}{n}} \leq \sqrt{\frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mathbb{E}_F \mathbf{1}((i, j) \in I) |Y_{ij}|^2} \leq c_0 \sqrt{\frac{|I|}{n^2}},$$

where we have used the fact that, for some constant $c_0 > 0$,

$$\max(\mathbb{E}|Y_{11}|^2, \mathbb{E}|Y_{12}|^2) \leq c_0^2.$$

Define the event

$$F = \{|I| \leq \delta n^2/c_0^2\}.$$

Then $F \in \mathcal{F}$ and

$$\mathbf{1}_F \mathbb{E}_F d(\mu_{A+C}, \mu_{A'+C}) \leq \delta. \quad (24)$$

From Lemma 2.4, for some sequence $s_3(n) \to \infty$, for all $n$ large enough,

$$\mathbb{P}(F^c) \leq e^{-s_3(n)n^{1+\alpha/2}}. \quad (25)$$

Observe that by definition of the distance (12),

$$d(\mathbb{E}_F \mu_{A'+C}, \mathbb{E}_F \mu_{A+C}) \leq \mathbb{E}_F d(\mu_{A'+C}, \mu_{A+C}).$$

Since $A'$ and $Y/\sqrt{n}$ have the same distribution, we deduce from (24), (25) and the triangle inequality that the proof of (23) can be reduced to the proof of

$$\lim_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}\left(d(\mu_{sc} \oplus \mu_C, \mathbb{E}_F \mu_{Y/\sqrt{n}+C}) \geq \delta\right) = -\infty. \quad (26)$$

Clearly, $\mathbb{E}|Y_{12}|^3 \leq c_0 (\log n)^{6/\alpha}$ and $\mathbb{E}|Y_{12}|^2 \to 1$. Hence (26) follows immediately from the uniform estimate of Theorem 2.6, applied to $M = C$, which is $\mathcal{F}$-measurable. Indeed, Theorem 2.6 implies that for $\delta > 0$,

$$\mathbb{P}\left(d(\mu_{sc} \oplus \mu_C, \mathbb{E}_F \mu_{Y/\sqrt{n}+C}) \geq \delta\right) = 0,$$

for all $n \geq n_0(\delta)$ where $n_0(\delta)$ is a constant depending only on $\delta$. This concludes the proof of Proposition 2.1.

### 3. Large Deviations of Very Sparse Rooted Networks

In this section, we start by adapting to our setting the notion of local weak convergence of rooted networks, introduced in [6], [2], and [1]. Next, we introduce a suitable projective limit topology on the space of networks. Then we prove the LDP for the network $G_n$ induced by the very sparse matrix $C$. Finally, we introduce the spectral measure associated to a network and project the LDP for networks onto a LDP for spectral measures.
3.1. Locally finite hermitian networks. Let $V$ be a countable set, the vertex set. A pair $(u, v) \in V^2$ is an oriented edge. A network or weighted graph $G = (V, \omega)$ is a vertex set $V$ together with a map $\omega$ from $V^2$ to $\mathbb{C}$. We say that a network is hermitian, if for all $(u, v) \in V^2$,
\[
\omega(u, v) = \overline{\omega(v, u)}.
\]
For ease of notation, we sometimes set $\omega(v) = \omega(v, v)$ for the weight of the loop at $v$. The degree of $v$ in $G$ is defined by
\[
\deg(v) = \sum_{u \in V} \vert \omega(v, u) \vert^2.
\]
The network $G$ is locally finite if for any vertex $v$, $\deg(v) < \infty$.

A path $\pi$ from $u$ to $v$ in $V$ is a sequence $\pi = (u_0, \cdots, u_k)$ with $u_0 = u$, $u_k = v$ and, for $1 \leq i \leq k$, $\vert \omega(u_{i-1}, u_i) \vert > 0$. If such $\pi : u \to v$ exists, then one defines the $\ell_2$ distance
\[
D_\pi(u, v) = \left(\sum_{i=1}^k \vert \omega(u_{i-1}, u_i) \vert^2\right)^{1/2}.
\]
The distance between $u$ and $v$ is defined as
\[
D(u, v) = \inf_{\pi : u \to v} D_\pi(u, v).
\]
Notice that weights are thought of as inverse of distances. If there is no path $\pi : u \to v$, then the distance $D(u, v)$ is set to be infinite. A network is connected if $D(u, v) < \infty$ for any $u \neq v \in V$.

All networks we consider below will be hermitian and locally finite, but not necessarily connected. We call $G$ the set of all such networks. For a network $G \in \mathcal{G}$, to avoid possible confusion, we will often denote by $V_G$, $\omega_G$, $\deg_G$ the corresponding vertex set, weight and degree functions.

Clearly, any $n \times n$ hermitian matrix $H_n \in \mathcal{H}_n(\mathbb{C})$ defines a finite network $G = G(H_n)$ in a natural way, by taking
\[
V_G = \{1, \ldots, n\}, \quad \omega_G(i, j) = H_n(i, j).
\]
For simplicity, we often write simply $H_n$ instead of $G(H_n)$.

3.2. Rooted networks. Below, a rooted network $(G, o) = (V, \omega, o)$ is a hermitian, locally finite and connected network $(V, \omega)$ with a distinguished vertex $o \in V$, the root. For $t > 0$, we denote by $(G, o)_t$ the rooted network with vertex set $\{u \in V : D(o, u) \leq t\}$, and with the weights induced by $\omega$. Two rooted networks $(G_1, o_1) = (V_1, \omega_1, o_1), i \in \{1, 2\}$, are isomorphic if there exists a bijection $\sigma : V_1 \to V_2$ such that $\sigma(o_1) = o_2$ and $\sigma(G_1) = G_2$, where $\sigma$ acts on $G_1$ through $\sigma(u, v) = (\sigma(u), \sigma(v))$ and $\sigma(\omega) = \omega \circ \sigma$.

We define the semi-distance $d_{\text{loc}}$ between two rooted networks $(G_1, o_1)$ and $(G_2, o_2)$ to be
\[
d_{\text{loc}}((G_1, o_1), (G_2, o_2)) = \frac{1}{1 + T},
\]
where $T$ is the supremum of those $t > 0$ such that there is a bijection $\sigma : V_{(G_1, o_1)_t} \to V_{(G_2, o_2)_t}$ with $\sigma(o_1) = o_2$ and such that the function $\omega_{G_1} - \omega_{G_2} \circ \sigma$ is bounded by $1/t$ on $V_{(G_1, o_1)_t}^2$.

The rooted network isomorphism defines a space $\mathcal{G}_*$ of equivalence classes of rooted networks $(G, o)$. On the space $\mathcal{G}_*$, $d_{\text{loc}}$ is a proper distance. The associated topology will be referred to as the local topology. We write $g$ for an element of $\mathcal{G}_*$. We shall denote the convergence on $(\mathcal{G}_*, d_{\text{loc}})$ by $d_{\text{loc}}(g_n, g) \to 0$ or $g_n \xrightarrow{\text{loc}} g$. 

The space \((\mathcal{G}_s, d_{\text{loc}})\) is separable and complete \([1]\). Let \(\mathcal{P}(\mathcal{G}_s)\) denote the space of probability measures on \(\mathcal{G}_s\). For \(\mu, \mu_n \in \mathcal{P}(\mathcal{G}_s)\), we write \(\mu_n \overset{\text{loc}}{\rightharpoonup} \mu\) when \(\mu_n\) converges weakly, i.e. when \(\int f d\mu_n \to \int f d\mu\) for every bounded continuous function \(f\) on \((\mathcal{G}_s, d_{\text{loc}})\). This notion of weak convergence is often referred to as local weak convergence. See \([1]\) for more details and examples.

For a network \(G \in \mathcal{G}\), and \(v \in V_G\), one writes \(G(v)\) for the connected component of \(G\) at \(v\), i.e. the largest connected network \(G' \subset G\) with \(v \in V_{G'}\). If \(G \in \mathcal{G}\) is finite, i.e. \(V_G\) is finite, one defines the probability measure \(U(G) \in \mathcal{P}(\mathcal{G}_s)\) as the law of the equivalence class of the root network \((G(o), o)\) where the root \(o\) is sampled uniformly at random from \(V_G\):

\[
U(G) = \frac{1}{V_G} \sum_{v \in V_G} \delta_{g(v)},
\]

where \(g(v)\) stands for the equivalence class of \((G(v), v)\). If \(G_n, n \geq 1\), is a sequence of finite networks from \(\mathcal{G}\), we shall say that \(G_n\) has local weak limit \(\rho \in \mathcal{P}(\mathcal{G}_s)\) if \(U(G_n) \overset{\text{loc}}{\rightharpoonup} \rho\).

### 3.3. Sofic measures.

Following \([1]\), a measure \(\rho \in \mathcal{P}(\mathcal{G}_s)\) is called sofic if there exists a sequence of finite networks \(G_n, n \geq 1\), whose local weak limit is \(\rho\). All sofic measures are unimodular, the converse is open; see \([1]\). We shall need to identify a subset of these measures. Let \(\vartheta_a, \vartheta_b\) denote the laws of \(X_{12}/|X_{12}|\) and \(X_{11}/|X_{11}|\) respectively, for \(X_{12} \in \mathcal{S}_a(a)\) and \(X_{11} \in \mathcal{S}_a(b)\), see Assumption \([1]\) and let \(S_a, S_b \subset \mathbb{S}^1\) denote their supports. Let \(\mathcal{A}_n \subset \mathcal{H}_n(\mathbb{C})\) be the set of \(n \times n\) hermitian matrices \(H\) such that either \(H_{ij} = 0\) or \(|H_{ij}| \in S_a\) for all \(i < j\), and such that either \(H_{ii} = 0\) or \(|H_{ii}| \in S_b\) for all \(i\). We say that \(\rho \in \mathcal{P}(\mathcal{G}_s)\) is admissible sofic if there exists a sequence of matrices \(H_n \in \mathcal{A}_n\) such that \(U(H_n) \overset{\text{loc}}{\rightharpoonup} \rho\), where \(H_n\) is identified with the associated network \(G(H_n)\) as in \([27]\). We denote by \(\mathcal{P}_s(\mathcal{G}_s)\) the set of admissible sofic probability measures. Measures in \(\mathcal{P}_s(\mathcal{G}_s)\) will often be called simply sofic if no confusion can arise.

Let \(g_0\) stand for the trivial network consisting of a single isolated vertex (the root) with zero weights. We refer to \(g_0\) as the empty network. Clearly, the Dirac mass at the empty network \(\rho = \delta_{g_0}\) is sofic (it suffices to consider matrices with zero entries). Let us consider some more examples.

**Example 3.1.** Suppose that \(S_b = \{-1, +1\}\). Let \(Y_1, Y_2, \ldots\) be i.i.d. random variables with distribution \(\nu \in \mathcal{P}(\mathbb{R})\). Consider the random diagonal matrix \(H_n\) with \(H_n(i, i) = Y_i\). Then, by the law of large numbers, almost surely \(U(H_n) \overset{\text{loc}}{\rightharpoonup} \rho\), where \(\rho\) is given by

\[
\rho = \int_{\mathbb{R}} \delta_{g_x} d\nu(x),
\]

if \(g_x\) is the network consisting of a single vertex (the root) with loop weight equal to \(x\).

**Example 3.2.** Suppose that \(Z_1, Z_3, Z_5, \ldots\) are i.i.d. complex random variables with law \(\mu \in \mathcal{P}(\mathbb{C})\) such that \(\mu\)-a.s. one has either \(Z_1 = 0\), or \(Z_1/|Z_1| \in S_a\). Consider the \(n \times n\) matrix \(H\) such that \(H_n(j, j + 1) = Z_j, H_n(j + 1, j) = \bar{Z}_j\), for all odd \(1 \leq j \leq n - 1\), and all other entries of \(H_n\) are zero. By construction, \(H_n \in \mathcal{A}_n\) almost surely. From the law of large numbers, almost surely \(U(H_n) \overset{\text{loc}}{\rightharpoonup} \rho\), where \(\rho\) is given by

\[
\rho = \frac{1}{2} \int_{\mathbb{C}} (\delta_{g_1} + \delta_{g_3}) d\mu(z),
\]

if \(g_x\) denotes the the equivalence class of the two vertex network \((V, \omega, o)\), with \(V = \{o, 1\}\), \(\omega(1, o) = x\) and \(\omega(o, o) = \omega(1, 1) = 0\).
Lemma 3.5. (Continuity of projections) Let $g \in G$ be the equivalence class of $G$. If $g \in G$ to $G = \{g\}$, then $U(A_m) = \frac{n}{m} U(H_n) + \frac{1}{1+4m} \delta_{g_0}$. As $m \to \infty$, $r/k \to 0$, $kn/r \to \infty$, and therefore $U(A_m)$ converges to $U(H_n)$.

3.4. Truncated networks. It will be important to work with suitable truncations of the weights. To this end we consider, for $0 < \theta < 1$, networks $G \in G$ such that for any $(u, v) \in V_G^2$,\[ \deg_{G}(v) \leq \theta^{-2}, \quad \text{and} \quad |\omega_G(u, v)| \geq \theta \mathbf{1} (\omega_G(u, v) \neq 0). \] We call $G^\theta$ the set of all such networks. Clearly, any $G \in G^\theta$ is locally finite and has at most $\theta^{-4}$ outgoing nonzero edges from any vertex. As before, one defines the space $G^\theta_\rho$ by taking equivalence classes of connected rooted networks from $G^\theta$. We define $\mathcal{P}(G^\theta)$ as the sets of $\rho \in \mathcal{P}(G_\ast)$ with support in $G^\theta_\rho$, and set $\mathcal{P}_s(G^\theta) = \mathcal{P}(G^\theta) \cap \mathcal{P}_s(G_\ast)$. The following lemma follows from routine diagonal extraction arguments.

Lemma 3.4. (i) For any $\theta > 0$, $G^\theta_\ast$ is a compact set for the local topology.
(ii) $\mathcal{P}_s(G_\ast)$ is closed for the local weak topology.

Next, we describe a canonical way to obtain a network in $G^\theta$ by truncating a network from $G$. For $0 < \theta < 1$, define the two continuous functions\[ \chi_\theta(x) = \begin{cases} 0 & \text{if } x \in [0, \theta) \\ (x - \theta) / \theta & \text{if } x \in [\theta, 2\theta) \\ 1 & \text{if } x \in [2\theta, \infty) \end{cases} \] and\[ \tilde{\chi}_\theta(x) = \begin{cases} 1 & \text{if } x \in [0, \theta^{-2} - 1) \\ \theta^{-2} - x & \text{if } x \in [\theta^{-2} - 1, \theta^{-2}) \\ 0 & \text{if } x \in [\theta^{-2}, \infty) \end{cases} \] that will serve as approximations for the indicator functions $\mathbf{1}(x \geq \theta)$ and $\mathbf{1}(x \leq \theta^{-2})$.

If $G = (V, \omega)$, we define $\tilde{G}_\theta = (V, \tilde{\omega}_\theta)$ as the network with vertex set $V$ and, for all $u, v \in V$,
\[ \tilde{\omega}_\theta(u, v) = \omega(u, v) \tilde{\chi}_\theta(\deg_G(u) \lor \deg_G(v)). \] (29)

Next, we define $G_\theta = (V, \omega_\theta)$ as the network with vertex set $V$ and, for all $u, v \in V$,
\[ \omega_\theta(u, v) = \omega_\theta(u, v) \chi_\theta(\tilde{\omega}_\theta(u, v)). \] (30)

Clearly, $G_\theta$ satisfies $\mathbf{28}$, and for any $u, v \in V$,
\[ \deg_{G_\theta}(u) \leq \deg_{G}(u) \quad \text{and} \quad |\omega_{G_\theta}(u, v)| \leq |\omega_{G}(u, v)|. \] (31)

If $g \in G_\ast$ and the network $(G(o), o)$ is in the equivalence class $g$, then $g_\theta \in G^\theta_\ast$ is defined as the equivalence class of $(G_\theta(o), o)$, where $G_\theta$ is defined by $\mathbf{28}$. This defines a map $g \mapsto g_\theta$ from $G_\ast$ to $G^\theta_\ast$. If $\rho \in \mathcal{P}(G_\ast)$ and $g$ has law $\rho$, the law of $g_\theta$ defines a new measure $\rho_\theta \in \mathcal{P}(G^\theta_\ast)$.

The next lemma follows easily from the continuity of $\chi_\theta, \tilde{\chi}_\theta$ and the fact that as $\theta \to 0$, for any for $x > 0$, $\chi_\theta(x) \to 1$ and $\tilde{\chi}_\theta(x) \to 1$.

Lemma 3.5 (Continuity of projections).

(i) For $\theta > 0$, the map $g \mapsto g_\theta$ from $G_\ast \to G^\theta_\ast$ is continuous for the local topology.
(ii) For $\theta > 0$, the map $\rho \mapsto \rho_\theta$ from $\mathcal{P}(G_\ast)$ to $\mathcal{P}(G^\theta_\ast)$ is continuous for the local weak topology.
(iii) As $\theta \to 0$, one has $g_\theta \overset{\text{loc}}{\to} g$ and $\rho_\theta \overset{\text{loc}}{\to} \rho$, for any $g \in G_\ast$ and $\rho \in \mathcal{P}(G_\ast)$. 

3.5. Projective topology for locally finite rooted networks. In order to circumvent the lack of compacity of \( \mathcal{P}_s(\mathcal{G}_s) \) w.r.t. local weak topology, we now introduce a new topology, the projective topology. For integers \( j \geq 1 \), set 

\[
\theta_j = 2^{-j}.
\]

Let \( p_j : \mathcal{G}_s \to \mathcal{G}^\theta_j \) be defined by \( p_j(g) = g_{\theta_j} \). Similarly, for \( 1 \leq i \leq j \), \( p_{ij} : \mathcal{G}^\theta_j \to \mathcal{G}^\theta_i \) is the map \( p_{ij}(g) = g_{\theta_i}, \ g \in \mathcal{G}^\theta_j \). The collection \((p_{ij})_{1 \leq i \leq j}\) is a projective system in the sense that for any \( 1 \leq i \leq j \leq k \),

\[
p_{ik} = p_{ij} \circ p_{jk}.
\]

The latter follows from \( 2\theta_{j+1} \leq \theta_j \) and \( \theta_j^{-2} \leq \theta_{j+1}^{-2} - 1 \).

Define the projective space \( \tilde{\mathcal{G}}_s \subset \prod_{j \geq 1} \mathcal{G}^\theta_j \) as the set of \( y = (y_1, y_2, \ldots) \in \prod_{j \geq 1} \mathcal{G}^\theta_j \) such that for any \( i \leq j \), \( p_{ij}(y_j) = y_i \); see e.g. [10, Appendix B] for more details on projective spaces. One can identify \( \mathcal{G}_s \) and \( \tilde{\mathcal{G}}_s \): 

**Lemma 3.6.** The map \( \iota(g) = (p_j(g))_{j \geq 1} \) from \( \mathcal{G}_s \) to \( \tilde{\mathcal{G}}_s \) is bijective.

**Proof.** The fact that \( \iota \) is injective is a consequence of Lemma 3.5 part (iii). It remains to prove that the map \( \iota \) is surjective. Let \( y = (y_j) \in \tilde{\mathcal{G}}_s \). One can represent the \( y_j \)'s by rooted networks \((G_j, o) = (V_j, \omega_j, o)\) such that \( V_j \subset V_{j+1} \). Set \( V := \bigcup_{j \geq 1} V_j \). By adding isolated points, one can view \((G_j, o)\) as the connected component at the root of the network \( \tilde{G}_j = (V, \omega_j) \), where \( \omega_j(u, v) = 0 \) whenever either \( u \) or \( v \) (or both) belong to \( V \setminus V_j \). Moreover, one has that \( \tilde{G}_i = (\tilde{G}_j)^{\theta_i} \) for all \( i < j \). This sequence of networks is monotone in the sense of \[ 3.11 \].

For fixed \( u, v \in V \), and \( j \in \mathbb{N} \), if \( \omega_j(u, v) \neq 0 \) then the degree of \( u \) and the degree of \( v \) is bounded by \( 2^{2j} \) in any network \( \tilde{G}_k, k \geq j \) and therefore \( \omega_k(u, v) = \omega_{j+1}(u, v) \) for all \( k \geq j + 1 \). In particular, for all \( u, v \in V \) the limit

\[
\omega(u, v) = \lim_{j \to \infty} \omega_j(u, v)
\]

exists and is finite. The same argument shows that for any \( u \in V \), \( \lim_{j \to \infty} \deg_{\tilde{G}_j}(u) \) exists and equals

\[
\sum_{v \in V} |\omega(u, v)|^2 < \infty.
\]

To prove surjectivity of the map \( \iota \), it suffices to take the network \( G = (V, \omega) \), and observe that it satisfies \( G_{\theta_j} = \tilde{G}_j \) for all \( j \in \mathbb{N} \).

With a slight abuse of notation, we will from now on write \( \mathcal{G}_s \) in place of \( \tilde{\mathcal{G}}_s \). The projective topology on \( \mathcal{G}_s \) is the topology induced by the metric

\[
d_{\text{proj}}(g, g') = \sum_{j \geq 1} 2^{-j} d_{\text{loc}}(g_{\theta_j}, g'_{\theta_j}).
\]

The metric space \((\mathcal{G}_s, d_{\text{proj}})\) is complete and separable. Also, \( g_n \xrightarrow{\text{proj}} g \), i.e. \( d_{\text{proj}}(g_n, g) \to 0 \), if and only if for any \( \theta > 0 \), \( (g_n)_{\theta} \xrightarrow{\text{loc}} g_{\theta} \). The projective weak topology is the weak topology on \( \mathcal{P}(\mathcal{G}_s) \) associated to continuous functions on \((\mathcal{G}_s, d_{\text{proj}})\). We denote the associated convergence by \( \xrightarrow{\text{proj}} \). Notice that \( \rho_n \xrightarrow{\text{proj}} \rho \) if and only if for any \( \theta > 0 \), \( (\rho_n)_{\theta} \xrightarrow{\text{loc}} \rho_{\theta} \). The topology generated by \( d_{\text{proj}} \) is coarser than the topology generated by \( d_{\text{loc}} \), and the weak topology associated to \( \xrightarrow{\text{proj}} \) is coarser than the weak topology associated to \( \xrightarrow{\text{loc}} \).
Example 3.7. Consider the star shaped rooted network \( (G_n, 1) = (V_n, \omega_n, 1) \) where \( V_n = \{1, \cdots, n\} \), with \( \omega_n(u,v) = \omega_n(v,u) = 1 \) if \( u = 1 \) and \( v \neq 1 \), and \( \omega(u,v) = 0 \) otherwise. Let \( g_n \) denote the associated equivalence class in \( G_* \). Then \( g_n \) does not converge in \( (G_*, d_{\text{loc}}) \) because of the diverging degree at the root. However, in \( (G_*, d_{\text{proj}}) \), \( g_n \xrightarrow{\text{proj}} g_0 \) where \( g_0 \) is the empty network. Moreover, \( U(G_n) \) does not converge in \( \mathcal{P}(G_*) \) for \( \xrightarrow{\text{loc}} \) however \( U(G_n) \xrightarrow{\text{proj}} \delta_{\emptyset} \).

Lemma 3.8. (i) \( G_* \) is compact for the projective topology.

(ii) \( \mathcal{P}_s(G_*) \) is compact for the projective weak topology.

Proof. Statement (i) is a consequence of Tychonoff theorem and Lemma 3.4(i). It implies that \( \mathcal{P}(G_*) \) is compact for projective weak topology. Hence, to prove statement (i), it is sufficient to check that \( \mathcal{P}_s(G_*) \) is closed. Assume that \( \rho_n \in \mathcal{P}_s(G_*) \) and \( \rho_n \xrightarrow{\text{proj}} \rho \). Then for any \( \theta > 0 \), \( (\rho_n)_\theta \in \mathcal{P}_s(G_*) \) and \( (\rho_n)_\theta \xrightarrow{\text{loc}} \rho_\theta \). By Lemma 3.4(ii), we deduce that \( \rho_\theta \in \mathcal{P}_s(G_*) \). However, as \( \theta \to 0 \), using Lemma 3.3 we find \( \rho_\theta \xrightarrow{\text{loc}} \rho \). By appealing to Lemma 3.4(iii) again we get \( \rho \in \mathcal{P}_s(G_*) \).

3.6. Large deviations for the network \( G_n \). For a rooted network \( (G,o) \), \( G = (V_G, \omega_G) \), define the functions

\[
\psi(G,o) = |\omega_G(o)|^a \quad \text{and} \quad \phi(G,o) = \frac{1}{2} \sum_{v \in V_G \setminus \emptyset} |\omega_G(o,v)|^a.
\]

(33)

Since these functions are invariant under rooted isomorphisms one can take them as functions on \( G_* \). Then, if \( \rho \in \mathcal{P}(G_*) \) we write \( E_\rho \psi \) and \( E_\rho \phi \) to denote the corresponding expectations. We remark that for any \( \theta > 0 \), the restriction of \( \phi, \psi \) to \( (G^\theta_*, d_{\text{loc}}) \) gives two bounded continuous functions. Therefore, as functions on \( (G_*, d_{\text{proj}}) \), \( \phi \) and \( \psi \) are lower semi-continuous.

We now come back to the random matrix \( C = C(n) \) defined in (11). For integer \( n \geq 1 \), consider the associated network

\[
G_n = (V_n, \omega_n), \quad \text{with} \quad V_n = \{1, \cdots, n\} \quad \text{and} \quad \omega_n(i,j) = C_{ij}.
\]

(34)

From the first Borel Cantelli lemma, almost surely the matrix \( C \) has no nonzero entry for \( n \) large enough. Therefore, almost surely, \( U(G_n) \xrightarrow{\text{loc}} \delta_{\emptyset} \), the Dirac mass at the empty network. The next proposition gives the large deviation principle for \( U(G_n) \) for the projective weak topology.

Proposition 3.9. \( U(G_n) \) satisfies an LDP on \( \mathcal{P}(G_*) \) equipped with the projective weak topology, with speed \( n^{1+\alpha/2} \) and good rate function \( I : \mathcal{P}(G_*) \to [0, \infty) \) defined by

\[
I(\rho) = \begin{cases} 
 b E_\rho \psi + a E_\rho \phi & \text{if} \quad \rho \in \mathcal{P}_s(G_*) \\
+\infty & \text{if} \quad \rho \notin \mathcal{P}_s(G_*)
\end{cases}
\]

(35)

If \( a \) or \( b \) is equal to \( \infty \), the above formula holds with the convention \( \infty \times 0 = 0 \).

Proof. For ease of notation, we define the random probability measure

\[
\rho_n = U(G_n).
\]

By construction, \( \rho_n \in \mathcal{P}_s(G_*) \), see Example 3.3, and therefore it is sufficient to establish the LDP on the space \( \mathcal{P}_s(G_*) \) with good rate function \( I(\rho) = b E_\rho \psi + a E_\rho \phi, \rho \in \mathcal{P}_s(G_*) \).

Let \( B_{\text{proj}}(\rho, \delta) \) (resp. \( B_{\text{loc}}(\rho, \delta) \)) denote the closed ball with radius \( \delta > 0 \) and center \( \rho \in \mathcal{P}_s(G_*) \) for the Lévy metric associated to the projective weak topology (resp. local weak topology).
By definition, we have for some continuous function and the fact that \( \phi, \psi \)

From (37), it follows that (36) holds under the assumption that both \( \rho \in \mathcal{P}_s(G_s) \)

Assume first that \( \mathbb{E}_\rho \psi \) and \( \mathbb{E}_\rho \phi \) are finite. From standard properties of weak convergence, and the fact that \( \phi, \psi \) are lower semi-continuous on \( (G_s, d_{\text{proj}}) \), it follows that the maps \( \mu \mapsto \mathbb{E}_\mu \psi \)
and \( \mu \mapsto \mathbb{E}_\mu \phi \) are lower semi-continuous on \( \mathcal{P}_s(G_s) \) w.r.t. the projective weak topology. Hence, we have for some continuous function \( h(\delta) \) with \( h(0) = 0 \),

By definition,

\[
\mathbb{E}_{\rho_n} \psi = \frac{1}{n^{1+\alpha/2}} \sum_{i=1}^{n} |X_{ii}|^{\alpha} 1(\varepsilon \sqrt{n} \leq |X_{ii}| \leq \varepsilon^{-1} \sqrt{n}),
\]

and

\[
\mathbb{E}_{\rho_n} \phi = \frac{1}{n^{1+\alpha/2}} \sum_{1 \leq i < j \leq n} |X_{ij}|^{\alpha} 1(\varepsilon \sqrt{n} \leq |X_{ij}| \leq \varepsilon^{-1} \sqrt{n}),
\]

are independent random variables. Therefore,

\[
P(\rho_n \in B_{\text{proj}}(\rho, \delta)) \leq P(\mathbb{E}_{\rho_n} \psi \geq \mathbb{E}_\rho \psi - h(\delta) ; \mathbb{E}_{\rho_n} \phi \geq \mathbb{E}_\rho \phi - h(\delta)).
\]

To prove the part of the bound involving \( \phi \), one may assume \( \mathbb{E}_\rho \phi > 0 \). Take \( \delta \) small enough, so that \( s := \mathbb{E}_\rho \phi - h(\delta) > 0 \). From Chernoff’s bound, for any \( 0 < a_1 < a \),

\[
P(\mathbb{E}_{\rho_n} \phi \geq s) \leq e^{-a_1 n^{1+\alpha/2}} \left( \mathbb{E} \exp \left( a_1 |X_{12}|^{\alpha} \mathbb{1}_{\varepsilon \sqrt{n} \leq |X_{12}| \leq \varepsilon^{-1} \sqrt{n}} \right) \right)^{n(n-1)/2}.
\]

By assumption, there exists \( a_2 \in (a_1, a) \), such that for all \( t > 0 \) large enough,

\[
P(|X_{12}| \geq t) \leq \exp(-a_2 t^\alpha).
\]

Using [16], one deduces that

\[
\mathbb{E} \exp \left( a_1 |X_{12}|^{\alpha} \mathbb{1}_{\varepsilon \sqrt{n} \leq |X_{12}| \leq \varepsilon^{-1} \sqrt{n}} \right) \leq 1 + e^{-(a_2-a_1) e^{\alpha n^{\alpha/2}}} + \alpha a_1 \int_{\varepsilon \sqrt{n}}^{\varepsilon^{-1} \sqrt{n}} x^{\alpha-1} e^{-(a_2-a_1)x^\alpha} dx
\]

\[
\leq 1 + \frac{a_2}{a_2 - a_1} e^{-(a_2-a_1) e^{\alpha n^{\alpha/2}}}.
\]

Therefore,

\[
P(\mathbb{E}_{\rho_n} \phi \geq s) \leq \exp \left( - a_1 n^{1+\alpha/2}s + \frac{a_2}{2(a_2 - a_1)} n^2 e^{-(a_2-a_1) e^{\alpha n^{\alpha/2}}} \right).
\]

We have thus proved that

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log P(\mathbb{E}_{\rho_n} \phi \geq s) \leq - a_1 (\mathbb{E}_\rho \phi - h(\delta)).
\]

Since the above inequality is true for any \( a_1 < a \), it also holds for \( a_1 = a \). Similarly, one has

\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log P(\mathbb{E}_{\rho_n} \psi \geq s) \leq -b(\mathbb{E}_\rho \psi - h(\delta)).
\]

From (37), it follows that (36) holds under the assumption that both \( \mathbb{E}_\rho \psi, \mathbb{E}_\rho \phi \) are finite. However, if either \( \mathbb{E}_\rho \psi \) or \( \mathbb{E}_\rho \phi \) is infinite, a straightforward adaptation of the above argument shows that the left hand side of (36) is \(-\infty\).
Lower bound. It is sufficient to prove that for any \( \rho \in P_s(G^\theta) \) and any \( \delta > 0 \),
\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\rho_n \in B_{\text{proj}}(\rho, \delta)) \geq -b \mathbb{E}_\rho \psi - a \mathbb{E}_\rho \phi.
\]
(38)

In order to prove (38), we may assume without loss of generality that \( I(\rho) = b \mathbb{E}_\rho \psi + a \mathbb{E}_\rho \phi < \infty \).

By monotonicity one has that
\[
\lim_{j \to \infty} I(\rho_{\theta_j}) = I(\rho).
\]

Therefore, since the projective topology is generated from the product topology on \( \prod_{j \geq 1} G^\theta_j \),

it is sufficient to prove (38) for all \( \rho \in P_s(G^\theta) \), for all \( 0 < \theta < 1 \). Finally, since the local weak topology is finer than the projective weak topology, it is enough to prove that for any \( 0 < \theta < 1 \),

\( \rho \in P_s(G^\theta) \) and \( \delta > 0 \),
\[
\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\rho_n \in B_{\text{loc}}(\rho, \delta)) \geq -b \mathbb{E}_\rho \psi - a \mathbb{E}_\rho \phi.
\]
(39)

Let us start with some simple consequences of Assumption 1. From (4), there exists a positive sequence \( \eta_n \) converging to 0 such that, for any \( s \geq \varepsilon(n) = 1/\log n \),
\[
e^{-\varepsilon(n) \alpha n^{\alpha/2}} \leq \mathbb{P}(|X_{12}| \geq s \sqrt{n}) \leq e^{-(a+\eta_n) \alpha n^{\alpha/2}}.
\]
(40)

In particular, if \( s \geq \varepsilon(n) \), then for any \( \gamma > 0 \), for all \( n \) large enough,
\[
\mathbb{P}(|X_{12}| \in [s, s + \gamma) \sqrt{n}) \geq \frac{1}{2} e^{-(a+\eta_n) \alpha n^{\alpha/2}}.
\]

Therefore, using (8), one finds that there exists a sequence \( a_n \to a \) such that for every \( \gamma > 0 \),

for all \( n \) large enough, for every \( z \in C \), with \( |z| \geq \varepsilon(n) \), \( z/|z| \in S_a \),
\[
\mathbb{P}(X_{12}/\sqrt{n} \in B_C(z, \gamma)) \geq e^{-a_n |z| \alpha n^{\alpha/2}},
\]
(41)

where \( S_a \) denotes the compact support of the measure \( \vartheta_a \in P(S^3) \) associated to \( X_{12} \), and \( B_C(z, \gamma) \) is the euclidean ball in \( C \), with center \( z \) and radius \( \gamma > 0 \).

Similarly, there exists a sequence \( b_n \to b \) such that for every \( \gamma > 0 \), for all \( n \) large enough, for every \( x \in \mathbb{R} \), with \( |x| \geq \varepsilon(n) \), \( x/|x| \in S_b \),
\[
\mathbb{P}(X_{11}/\sqrt{n} \in B_R(x, \gamma)) \geq e^{-b_n |x| \alpha n^{\alpha/2}}.
\]
(42)

Since \( \rho \in P_s(G^\theta) \), there exists a sequence of matrices \( H_n \in \mathcal{A}_n \), such that the associated network as in (27) is in \( G^\theta \) and such that \( U(H_n) \to^\text{loc} \rho \). In particular, for \( n \) sufficiently large one has
\[
U(H_n) \in B_{\text{loc}}(\rho, \delta/2).
\]

From Lemma 3.10 there exists \( \gamma = \gamma(\delta, \theta) > 0 \) such that if \( |\omega_{G_n}(i) - H_n(i, i)| \leq \gamma \) and \( |\omega_{G_n}(i, j) - H_n(i, j)| \leq \gamma \) for all \( 1 \leq i \leq j \leq n \), then \( \rho_n = U(G_n) \in B_{\text{loc}}(U(H_n), \delta/2) \). Then, by the triangle inequality, for all \( n \) large enough,
\[
\mathbb{P}(\rho_n \in B_{\text{loc}}(\rho, \delta)) \geq \mathbb{P}(\rho_n \in B_{\text{loc}}(U(H_n), \delta/2)) \geq \mathbb{P}\left( \max_{1 \leq i \leq n} |\omega_{G_n}(i) - H_n(i, i)| \leq \gamma, \max_{1 \leq i < j \leq n} |\omega_{G_n}(i, j) - H_n(i, j)| \leq \gamma \right).
\]

Independence of the weights \( \omega_{G_n}(i, j) = C_{i,j} \), \( 1 \leq i \leq j \leq n \) then gives
\[
\mathbb{P}(\rho_n \in B_{\text{loc}}(\rho, \delta)) \geq \prod_{i=1}^{n} \mathbb{P}(|C_{ii} - H_n(i, i)| \leq \gamma) \prod_{1 \leq i < j \leq n} \mathbb{P}(|C_{ij} - H_n(i, j)| \leq \gamma).
\]
Notice that whenever $H_n(i,j) \neq 0$ one has $|H_n(i,j)| \geq \theta$, and thus using (40) and (42) one has for all $i = 1, \ldots, n$:

$$
\mathbb{P}(|C_{ij} - H_n(i,j)| \leq \gamma) \geq e^{-b_n n^{\alpha/2} |H_n(i,j)|^\alpha} \left(1 - e^{-c_n n^{\alpha/2}}\right) + \left(1 - e^{-c_n n^{\alpha/2}}\right)
$$

where the constant $c$ satisfies $c \geq b/2 > 0$. Similarly, using (41), for all $i \leq j$ and for some $c \geq a/2 > 0$:

$$
\mathbb{P}(|C_{ij} - H_n(i,j)| \leq \gamma) \geq e^{-a_n n^{\alpha/2} |H_n(i,j)|^\alpha} \left(1 - e^{-c_n n^{\alpha/2}}\right).
$$

Observe that

$$
\frac{1}{n} \sum_{1 \leq i \leq n} |H_n(i,i)|^\alpha = \mathbb{E}_{U(H_n)} \psi, \quad \frac{1}{n} \sum_{1 \leq i < j \leq n} |H_n(i,j)|^\alpha = \mathbb{E}_{U(H_n)} \phi,
$$

Summarizing, using $(1 - e^{-c_n n^{\alpha/2}})^2 \geq 1/2$ for $n$ large enough, one finds

$$
\mathbb{P}(\rho_n \in B_{loc}(\rho, \delta)) \geq \frac{1}{2} e^{-b_n n^{1+\alpha/2} \mathbb{E}_{U(H_n)} \psi} e^{-a_n n^{1+\alpha/2} \mathbb{E}_{U(H_n)} \phi}.
$$

Since $\psi$ and $\phi$ are continuous and bounded on $G_0$, one has $\mathbb{E}_{U(H_n)} \psi \to \mathbb{E}_\rho \psi$ and $\mathbb{E}_{U(H_n)} \phi \to \mathbb{E}_\rho \phi$, as $n \to \infty$. Moreover, $a_n \to a$ and $b_n \to b$. Therefore, (43) implies the desired bound (39). This concludes the proof of the lower bound. \qed

The next lemma was used in the proof of the lower bound of Proposition 3.9. While the estimate is somewhat rough, it is crucial that it is uniform in the cardinality $n$ of the vertex set.

**Lemma 3.10.** Let $0 < \theta < 1$ and $\delta > 0$. There exists $\gamma = \gamma(\delta, \theta) > 0$ such that for any integer $n \geq 1$, for any networks $G \in \mathcal{G}$, $H \in \mathcal{G}_\theta$ with common vertex set $V = \{1, \ldots, n\}$ such that

$$
\max_{(u,v) \in V^2} |\omega_G(u,v) - \omega_H(u,v)| \leq \gamma,
$$

then

$$
\max_{u \in V} d_{loc} (\omega_G(u,u), (H(u), u)) \leq \delta.
$$

In particular,

$$
U(G) \in B_{loc}(U(H), \delta).
$$

**Proof.** Each edge of $H$ has a weight bounded by $\theta^{-1}$. This implies that in $H$ each path whose total length is bounded by $t > 0$, contains at most $t^2/\theta^2$ edges. Moreover, $H$ has at most $\theta^{-4}$ outgoing edges from any vertex. Hence, $H$ has at most $m = \theta^{-4} t^2/\theta^2$ vertices at distance less than $t$ from any given vertex. Fix the root $u \in V$ and $t > 0$. From the pigeonhole principle, there exists $t_0 > 0$ such that $t/2 < t_0 < t$, and an interval $I = [t_0 - t/(8m), t_0 + t/(8m)]$, such that there is no vertex within distance $s \in I$ from $u$ in $H$.

If $e_1, \ldots, e_k$ are the edges on a path in $H$, then provided that $0 < \gamma < \theta/2$,

$$
\left(\sum_{i=1}^k |\omega_H(e_i)|^{-2}\right)^{1/2} - \left(\sum_{i=1}^k |\omega_G(e_i)|^{-2}\right)^{1/2} \leq \sum_{i=1}^k (|\omega_H(e_i)|^{-1} - |\omega_G(e_i)|^{-1})^2 \leq \frac{4 \gamma^2 k}{\theta^4},
$$

where the first inequality follows from the joint convexity of $[0, \infty)^2 \ni (x, y) \mapsto (\sqrt{x} - \sqrt{y})^2$ and the second inequality follows from $|\omega_H(e_i)| \geq \theta$ and the assumption (44). In the worst possible case one can take $k = t^2/\theta^2$ for the number of edges at distance $t_0$ from $u$. Together with the previous observation, this shows that if $2\gamma \sqrt{k}/\theta^2 \leq t/(8m)$, i.e. $\gamma \leq \theta^3/(16m)$, then
the neighborhood of $u$ consisting of vertices within distance $t_0$ in $G$ and in $H$ have the same vertex set. From the definition of $d_{\text{loc}}$, this choice of $\gamma$ in (14) implies that

$$d_{\text{loc}}((G(u), u), (H(u), u)) \leq \frac{1}{1 + \gamma^{-1} \wedge t_0} \leq \frac{2}{t}.$$  

Thus, taking $t = 2/\delta$, one has (15), as soon as e.g. $\gamma \leq \theta^3/(16m) = \theta^3/(\delta \theta^2)/16$. From the definition of the Lévy distance, it immediately follows that $U(G) \in B_{\text{loc}}(U(H), \delta)$.

3.7. Spectral measure. For a network $G = (V, \omega) \in \mathcal{G}^\theta$, we may define the bounded linear operator $T$ on the Hilbert space $\ell^2(V)$ by

$$Te_v = \sum_{u \in V} \omega(u, v)e_u,$$

for any $v \in V$, where $\{e_u, u \in V\}$ denotes the canonical orthonormal basis of $\ell^2(V)$. $T$ is bounded since

$$\|Te_v\|^2 = \sum_{u \in V} |\omega(v, u)|^2 = \text{deg}(v) \leq \theta^{-2}.$$  

(47)

Also, since $G$ is hermitian, $T$ is self adjoint. We may thus define the spectral measure at vector $e_v$, as the unique probability measure $\mu_T$ on $\mathbb{R}$ such that for any integer $k \geq 1$,

$$\int x^k d\mu_T = \langle e_v, T^k e_v \rangle.$$  

(48)

Notice that for rooted networks $(G, o)$ with $G \in \mathcal{G}^\theta$, then the associated spectral measure $\mu_T$ is constant on the equivalence class of $(G, o)$, so that $\mu_T$ can be defined as a measurable map from $\mathcal{G}^\theta_\ast$ to $\mathcal{P}(\mathbb{R})$. Thus, if $\rho \in \mathcal{P}(\mathcal{G}^\theta_\ast)$, one can define the spectral measure of $\rho$ as

$$\mu_\rho = \mathbb{E}_\rho \mu_T^0.$$  

(49)

In general, if $\rho \in \mathcal{P}(\mathcal{G}_\ast)$, then (49) allows one to define the spectral measures $\mu_{\rho_\theta}$, where the truncated network $\rho_\theta$ is defined as in Lemma 3.5. When $\rho \in \mathcal{P}(\mathcal{G}_\ast)$, it is possible to define a notion of spectral measure $\mu_\rho$ as the limit of $\mu_{\rho_\theta}$ as $\theta \to 0$. More precisely, for a rooted network $(G, o)$, $G \in \mathcal{G}$, and for $\beta > 0$, let

$$\xi_\beta(G, o) = \sum_{v \in V_G} |\omega_G(o, v)|^\beta.$$  

Since $\xi_\beta$ is constant on the equivalence class of $(G, o)$, it can be seen as a function on $\mathcal{G}_\ast$. For $\beta > 0$, define

$$\mathcal{P}_{\beta}\mathcal{G}_\ast = \{ \rho \in \mathcal{P}(\mathcal{G}_\ast) : \mathbb{E}_\rho \xi_\beta < \tau \}.$$  

Lemma 3.11 and Lemma 3.12 below are suitable extensions of analoguous statements in [8, 9].

The first result allows one to define the spectral measure $\mu_\rho$ of any $\rho \in \mathcal{P}_{\beta}\mathcal{G}_\ast$.

**Lemma 3.11.** Let $0 < \beta < 2$, $\tau > 1$ and $\rho \in \mathcal{P}_{\beta}\mathcal{G}_\ast$. Then the weak limit

$$\mu_\rho := \lim_{\theta \to 0} \mu_{\rho_\theta}$$

exists in $\mathcal{P}(\mathbb{R})$.

**Proof.** To prove the lemma we are going to show that the sequence $\mu_{\rho_\theta}$, $\theta \to 0$, is Cauchy w.r.t. the metric (12).
By assumption, there exists a sequence $G_n$ of networks on $\{1, \cdots, n\}$ such that $\rho_n \xrightarrow{\text{loc}} \rho$, where $\rho_n = U(G_n)$. Call $T_n$ the associated hermitian matrix. The empirical distribution of the eigenvalues of $T_n$ satisfies, by the spectral theorem,

$$
\mu_{T_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(T_n)} = \frac{1}{n} \sum_{u=1}^{n} \mu_n^u = \mu_{\rho_n},
$$

(50)

where $\mu_n^u$ stands for the spectral measure at $u$; see [13].

The truncations $(\rho_n)_\theta$ and $\rho_\theta$ satisfy $(\rho_n)_\theta \xrightarrow{\text{loc}} \rho_\theta$ by Lemma 3.5(ii). Moreover for all $\theta > 0$,

$$
\mu_{(\rho_n)_\theta} \xrightarrow{\text{loc}} \mu_{\rho_\theta}.
$$

(51)

To prove (51), let $T^\theta$ denote the random bounded self-adjoint operator associated to $\rho_\theta$ via (29), and let $T_n^\theta$ be the matrices associated to $(\rho_n)_\theta$. One can realize these operators on a common Hilbert space $\ell^2(V)$. Since $(\rho_n)_\theta \xrightarrow{\text{loc}} \rho_\theta$, from the Skorokhod representation theorem one can define a common probability space such that the associated networks converge locally almost surely, so that a.s. $T_n^\theta \rightarrow T^\theta$, in $\ell^2(V)$, for any $v \in V$. This implies the strong resolvent convergence, see e.g. [13] Theorem VIII.25(a), and in particular that for any $v \in V$, a.s.

$$
\mu_{T_n^\theta} \xrightarrow{v} \mu_{T^\theta}.
$$

Then (51) follows by applying this to $v = o$ and taking expectation.

Let $T_n^\theta, \tilde{T}_n^\theta$ be the matrices associated to $(G_n)_\theta$ and $(\tilde{G}_n)_\theta$ respectively, where $(\tilde{G}_n)_\theta$ is defined according to (29), and $(G_n)_\theta$ according to (30). From (14), using the triangle inequality, Lemma 3.1 and Lemma 3.2,

$$
d(\mu_{T_n^\theta}, \mu_{T_n}) \leq \frac{1}{n} \text{rank}(\tilde{T}_n^\theta - T_n) + \left( \frac{1}{n} \text{tr}(\tilde{T}_n^\theta - T_n)^2 \right)^{1/2}.
$$

From the definition (29) one has

$$
\frac{1}{n} \text{rank}(\tilde{T}_n^\theta - T_n) \leq \frac{2}{n} \sum_{i=1}^{n} 1(\text{deg}_{G_n}(i) \geq \theta^{-2} - 1) = 2 \mathbb{P}_{\rho_n}(\text{deg}_G(o) \geq \theta^{-2} - 1).
$$

From (30) one finds

$$
\frac{1}{n} \text{tr}(\tilde{T}_n^\theta - T_n)^2 \leq \frac{1}{n} \sum_{i,j=1}^{n} |\omega_{G_n}(i,j)|^2 1(|\omega_{G_n}(i,j)| \leq 2\theta) 1(\text{deg}_G(i) \leq \theta^{-2})
$$

$$
= \mathbb{E}_{\rho_n} 1(\text{deg}_G(o) \leq \theta^{-2}) \sum_v |\omega_G(o,v)|^2 1(|\omega_G(o,v)| \leq 2\theta).
$$

Letting $n$ go to infinity, using $\mu_{T^\theta} = \mu_{(\rho_n)_\theta}$, and (51), one has $d(\mu_{T^\theta_n}, \mu_{T_n}) \rightarrow d(\mu_{\rho_n}, \mu_{\rho_\theta})$. Therefore, by the triangle inequality and the dominated convergence theorem, for any $0 < \theta' < \theta < 1/\sqrt{2}$,

$$
d(\mu_{\rho_n}, \mu_{\rho_{\theta'}}) \leq 4 \mathbb{P}_\rho(\text{deg}_G(o) \geq \theta^{-2}/2)
$$

$$
+ 2 \left( \mathbb{E}_\rho 1(\text{deg}_G(o) \leq \theta^{-2}) \sum_v |\omega_G(o,v)|^2 1(|\omega_G(o,v)| \leq 2\theta) \right)^{1/2}.
$$

Notice that, for $\beta \in (0, 2)$

$$
\text{deg}_G(o)^{\beta/2} = \left( \sum_v |\omega_G(o,v)|^2 \right)^{\beta/2} \leq \sum_v |\omega_G(o,v)|^\beta = \xi_\beta(G, o),
$$

(52)
where we use that \( \sum_{i=1}^{k} a_i^r \leq \left( \sum_{i=1}^{k} a_i \right)^r \) for all \( a_i \geq 0, r \geq 1 \) and \( k \in \mathbb{N} \). Moreover, 
\[
\sum_v |\omega_G(o,v)|^21(|\omega_G(o,v)| \leq \theta) \leq \theta^{2-\beta} \varepsilon_{\beta}(G, o).
\]

Hence, from Markov’s inequality,
\[
d(\mu_{\rho_0}, \mu_{\rho'}) \leq 4\theta^3 \mathbb{E}_p \xi_\beta + 2\theta^{1-\frac{\beta}{2}} (\mathbb{E}_p \xi_\beta)^{1/2}.
\]

By assumption \( \mathbb{E}_p \xi_\beta \) is finite. Hence, the sequence \( \mu_{\rho_0} \) is Cauchy.

\[\square\]

**Lemma 3.12.** For any \( \beta \in (0, 2) \), \( \tau > 0 \), the map \( \rho \mapsto \mu_\rho \) from \( \mathcal{P}_{s,\beta,\tau}(G_*) \) to \( \mathcal{P}(\mathbb{R}) \) is continuous for the projective weak topology.

**Proof.** For any \( \theta > 0 \), from (50),
\[
d(\mu_{\rho_0}, \mu_\rho) \leq c(\theta^3 + \theta^{1-\frac{\beta}{2}}),
\]
with a constant \( c = c(\tau) > 0 \). Hence from the triangle inequality, if \( \rho, \rho' \in \mathcal{P}_{s,\beta,\tau}(G_*) \),
\[
d(\mu_\rho, \mu_{\rho'}) \leq 2c(\theta^3 + \theta^{1-\frac{\beta}{2}}) + d(\mu_{\rho_0}, \mu'_{\rho_0}).
\]

Consider a sequence \( \rho' \) such that \( \rho' \xrightarrow{\text{proj}} \rho \). If \( \rho' \xrightarrow{\text{proj}} \rho \) then \( \rho' \xrightarrow{\text{loc}} \rho_0 \) and therefore, with the same argument used in the proof of (51) above one finds
\[
\mu_{\rho_0} \xrightarrow{\text{proj}} \mu_{\rho}.\]

We deduce that
\[
\limsup_{\rho' \xrightarrow{\text{proj}} \rho} d(\mu_\rho, \mu_{\rho'}) \leq 2c(\theta^3 + \theta^{1-\frac{\beta}{2}}).
\]

Since \( \theta > 0 \) is arbitrarily small, the statement of the lemma follows. \[\square\]

3.8. **Large deviations for the empirical spectral measure** \( \mu_C \). We can apply the previous results to the empirical spectral measure \( \mu_C \), where \( C = C(n) \) is the random matrix defined in (11). So far we have defined \( \mu_\rho \) for every \( \rho \in \bigcup_{\beta < 2} \bigcup_{\tau > 1} \mathcal{P}_{s,\beta,\tau}(G_*) \). If \( \rho \in \mathcal{P}_s(G_*) \) but \( \rho \notin \bigcup_{\beta < 2} \bigcup_{\tau > 1} \mathcal{P}_{s,\beta,\tau}(G_*) \), then we set
\[
\mu_\rho = \delta_0.
\]

**Proposition 3.13.** The empirical spectral measures \( \mu_C \) satisfy an LDP on \( \mathcal{P}(\mathbb{R}) \) equipped with the weak topology, with speed \( n^{1+\alpha}/2 \) and good rate function \( \Phi \) given by
\[
\Phi(\nu) = \inf \{ I(\rho), \rho \in \mathcal{P}_s(G_*): \mu_\rho = \nu \},
\]
where \( I(\rho) \) is the good rate function in Proposition 3.9.

**Proof.** Recall that by (50) the network \( G_n \) in (31) satisfies \( \rho_n = U(G_n) \) and
\[
\mu_{\rho_n} = \mu_C.
\]

Notice that if \( c = (\frac{4}{2} \wedge b) \), then
\[
I(\rho) \geq c \mathbb{E}_p \xi_\alpha.
\]

Hence, by Lemma 3.12 the map \( \rho \mapsto \mu_\rho \) is continuous on the domain of \( I(\rho) \). It is thus possible to apply a contraction principle to get the LDP for \( \mu_{\rho_n} \) from the LDP for \( \rho_n \). To be more precise, if \( B \) is a Borel set in \( \mathcal{P}(\mathbb{R}) \), we write for any \( \tau > 1 \),
\[
\mathbb{P}(\mu_{\rho_n} \in B; \rho_n \in \mathcal{P}_{s,\alpha,\tau}(G_*)) \leq \mathbb{P}(\mu_{\rho_n} \in B) \leq \mathbb{P}(\mu_{\rho_n} \in B; \rho_n \in \mathcal{P}_{s,\alpha,\tau}(G_*)) + \mathbb{P}(\rho_n \notin \mathcal{P}_{s,\alpha,\tau}(G_*))
\]
We start with the lower bound. Assume that $B$ is an open set. For each $\tau > 0$, by Lemma 3.12 the function $f_\tau : \rho \mapsto \mu_\rho$ from $\mathcal{P}_{s,\alpha,\tau}(G_*) \to \mathcal{P}(\mathbb{R})$ is continuous. Hence $f_\tau^{-1}(B)$ is an open subset of $\mathcal{P}_{s,\alpha,\tau}(G_*)$. By Proposition 3.9 it follows that

$$- \inf_{\rho \in \mathcal{P}_{s,\alpha,\tau}(G_*) : \mu_\rho \in B} I(\rho) \leq \liminf_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\mu_{\rho_n} \in B).$$

Using (56) one has

$$- \inf_{\rho \in \mathcal{P}_{s,\alpha,\tau}(G_*)} I(\rho) \leq (-c \tau) \vee \liminf_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\mu_{\rho_n} \in B).$$

Letting $\tau$ tend to infinity, we obtain the desired lower bound:

$$- \inf_{\nu \in B} \Phi(\nu) \leq \limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\mu_{\rho_n} \in B).$$

To prove the upper bound, assume that $B$ is closed. By Lemma 3.12, $f_\tau^{-1}(B)$ is a closed subset of $\mathcal{P}_{s,\alpha,\tau}(G_*)$. Proposition 3.9 yields

$$\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\mu_{\rho_n} \in B : \rho_n \in \mathcal{P}_{s,\alpha,\tau}(G_*)) \leq - \inf_{\rho \in \mathcal{P}_{s,\alpha,\tau}(G_*)} I(\rho),$$

and

$$\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\rho_n \notin \mathcal{P}_{s,\alpha,\tau}(G_*)) \leq -c \tau.$$

We have checked that

$$\limsup_{n \to \infty} \frac{1}{n^{1+\alpha/2}} \log \mathbb{P}(\mu_{\rho_n} \in B) \leq - \left( (c \tau) \wedge \inf_{\mu \in B} \Phi(\mu) \right).$$

Letting $\tau$ tend to infinity, we obtain the upper desired bound. The function $\Phi$ is a good rate function (see e.g. [10, Theorem 4.2.1-(a)] or Lemma 3.14 below). □

3.9. Proof of Theorem 1.1. Thanks to Proposition 2.1, all we have to show is that the sequence of measures $\mu_{sc} \boxplus \mu_C$ satisfies a LDP in $\mathcal{P}(\mathbb{R})$ with speed $n^{1+\alpha/2}$, with the good rate function $\Phi$ defined in Proposition 3.13. Since the map $\nu \mapsto \mu_{sc} \boxplus \nu$ is continuous in $\mathcal{P}(\mathbb{R})$, the above is an immediate consequence of Proposition 3.13 and the standard contraction principle. This ends the proof of Theorem 1.1.

3.10. On the rate function $\Phi$. We turn to a proof of the properties of the rate function listed in Theorem 1.2 and Theorem 1.3.

Lemma 3.14. For any $\beta \in (0, 2)$, $\tau > 1$, for any $\rho \in \mathcal{P}_{s,\beta,\tau}(G_*)$, one has

$$\int |x|^\beta d\mu_\rho(x) \leq \mathbb{E}_\rho \xi_\beta. \quad (57)$$

Proof. We use the following Schatten bound: for all $0 < p \leq 2$,

$$\int |x|^p d\mu_A(x) \leq \frac{1}{n} \sum_{k=1}^n \left( \sum_{j=1}^n |A_{kj}|^2 \right)^{p/2} \quad (58)$$

for every hermitian matrix $A \in \mathcal{H}_n(\mathbb{C})$. For a proof, see Zhan [13, proof of Theorem 3.32]. For $\rho \in \mathcal{P}_{s,\beta,\tau}(G_*)$, there exists a sequence of matrices $H_n$ such that $\rho_n = U(H_n) \overset{loc}{\to} \rho$. Let $T_n^\theta$ be
the hermitian matrix associated to $(H_n)_o$, the truncated network. From (58) and (52), one has for all $\theta > 0$:
\[
\int |x|^\beta d\mu_{T_n}(x) \leq \mathbb{E}_{\rho_n} \left( \theta^{-2} \wedge \sum_v |\omega(o,v)|^2 \right)^{\frac{\beta}{2}} \leq \mathbb{E}_{\rho_n}(\theta^{-\beta} \wedge \xi_\beta(G,o)).
\]
For $\theta > 0$ the spectral measures $\mu_{T_n} = \mu_{(\rho_n)_o}$ have compact support uniformly in $n$. Thus, letting $n$ go to infinity, from (51) one has
\[
\int |x|^\beta d\mu_{\rho_0}(x) \leq \mathbb{E}_{\rho}\xi_\beta.
\]
(59)
On the other hand, by definition of $\mu_\rho$, see Lemma 3.11 one has $\mu_{\rho_0} \tilde{\rightsquigarrow} \mu_\rho$, $\theta \to 0$, and therefore
\[
\int |x|^\beta d\mu_{\rho_0}(x) \leq \liminf_{\theta \to 0} \int |x|^\beta d\mu_{\rho}(x).
\]
This proves the claim (57).

**Proof of Theorem 1.2 (a).** The proof is an immediate consequence of Lemma 3.11. Indeed, from (50) and the definition of $\Phi$, it suffices to show that for any $\tau > 1$, for any $\rho \in \mathcal{P}_{s,\alpha,\tau}(G_s)$, one has
\[
\int |x|^\alpha d\mu_{\rho}(x) \leq \mathbb{E}_{\rho}\xi_\alpha.
\]
(60)
This is the case $\alpha = \beta$ in (57).

**Proof of Theorem 1.2 (b).** For $x \in \mathbb{R}$, let $g_x \in G_s$ denote the network consisting of a single vertex $o$ with weight $\omega(o,o) = x$. If $\nu \in \mathcal{P}(\mathbb{R})$, let $\rho \in \mathcal{P}(G_s)$ denote the law $\rho = \int_{\mathbb{R}} \delta_{g_x} d\nu(x)$. Notice that
\[
\mathbb{E}_{\rho}\xi_\alpha = \int_{\mathbb{R}} |x|^\alpha d\nu(x) = m_\alpha(\nu).
\]
Thus, we can assume $\mathbb{E}_{\rho}\xi_\alpha < \infty$, otherwise there is nothing to prove. Since we assume $\text{supp}(\delta_b) = \{-1, +1\}$, one has that $\rho$ is admissible sofic, see Example 3.1, and $\rho \in \mathcal{P}_{s,\alpha,\tau}(G_s)$ for some $\tau > 1$. The spectral measure $\mu_{\rho}$ of $\rho$, defined as in Lemma 3.11 is easily seen to be $\mu_\rho = \nu$. Then $\Phi(\nu) \leq I(\rho) = b \mathbb{E}_{\rho}\xi_\alpha = b m_\alpha(\nu)$.

**Proof of Theorem 1.2 (c).** Thanks to part (a) and part (b), all we need to prove is that
\[
\Phi(\nu) \leq \frac{a}{2} m_\alpha(\nu),
\]
(61)
for all symmetric probabilities $\nu$ on $\mathbb{R}$.

For $z \in \mathbb{C}$, let $\hat{g}_z \in G_s$ denote the equivalence class of the two vertex network $(V,\omega,o)$, with $V = \{o,1\}$, $\omega(o,1) = z$, $\omega(1,o) = \bar{z}$ and $\omega(o,o) = \omega(1,1) = 0$. Fix some $e^{i\varphi} \in S_\alpha = \text{supp}(\delta_a)$, let $T$ be a nonnegative random variable with some distribution $\mu_+$ on $[0, \infty)$, and let $\mu \in \mathcal{P}(\mathbb{C})$ denote the law of $Te^{i\varphi}$. The law
\[
\rho = \frac{1}{2} \int_{\mathbb{C}} (\delta_{g_z} + \delta_{g_\bar{z}}) d\mu(z),
\]
is sofic, see Example 3.2. A simple computation shows that the spectral measure of $\rho$ satisfies $\mu_\rho = \mu_{\text{sym}}$, where $\mu_{\text{sym}}$ denotes the symmetric probability on $\mathbb{R}$ such that
\[
\int_{\mathbb{R}} f(x) d\mu_{\text{sym}}(x) = \frac{1}{2} \int_0^\infty (f(x) + f(-x)) d\mu_+(x)
\]
for all bounded measurable $f$. 

To prove (61), let $\nu \in \mathcal{P}_{\text{sym}}(\mathbb{R})$ and write $\mu_+$ for the law of $|X|$ when $X$ has law $\nu$. Then $\nu = \mu_{\text{sym}}$ and the associated $\rho$ satisfies $\mu_\rho = \nu$. Therefore
\[ \Phi(\nu) \leq I(\rho) = \frac{a}{2} \int_0^\infty x^\alpha d\mu_+(x) = \frac{a}{2} \mu^\alpha(\nu). \]

**Proof of Theorem 1.3 (a).** We proceed as in the proof of Theorem 1.3 (b). Here $S_b = \{+1\}$ and thus the law $\rho = \int \delta_{x} d\nu(x)$ that we used there is not necessarily admissible sofic. However, it is so if one assumes supp($\nu$) $\subset \mathbb{R}_+$. The rest of the argument applies with no modifications.

For the remaining statements, we use the following observation.

**Lemma 3.15.** If $\rho \in \mathcal{P}_{s,\beta,\tau}(\mathcal{G}_s)$ for some $\beta \in (1,2)$, $\tau > 1$, then
\[ \int_{\mathbb{R}} x \, d\mu_\rho(x) = \mathbb{E}_\rho \omega_G(o). \]  

**Proof.** By definition of the spectral measure $\mu_{\rho_\theta}$, see (48), for every $\theta > 0$ one has
\[ \int_{\mathbb{R}} x \, d\mu_{\rho_\theta}(x) = \mathbb{E}_{\rho_\theta} \omega_G(o) = \mathbb{E}_\rho \omega_G(o), \]
where $G_\theta$ is the truncation of $G$, see (30). The weights $\omega_{G_\theta}(o)$ satisfy $|\omega_{G_\theta}(o)| \leq |\omega_G(o)|$ and, since $\beta > 1$, $\mathbb{E}_\rho |\omega_G(o)| \leq (\mathbb{E}_\rho \xi_\beta)^{1/\beta} < \tau^{1/\beta}$. Thus, by the dominated convergence theorem,
\[ \lim_{\theta \to 0} \int_{\mathbb{R}} x \, d\mu_{\rho_\theta}(x) = \mathbb{E}_\rho \omega_G(o). \]
From (59), and the fact that $\beta > 1$, we know that the identity map $x \mapsto x$ is uniformly integrable for $(\mu_{\rho_\theta})_{\theta > 0}$. Therefore, by definition of $\mu_\rho$, see Lemma 3.11, the limit above also equals $\int_{\mathbb{R}} x \, d\mu_\rho(x)$.

**Proof of Theorem 1.3 (b).** In view of the bound (61), it suffices to show that if $\rho \in \mathcal{P}_{s}(\mathcal{G}_s)$ with $\mu_\rho = \nu$, then
\[ \frac{a}{2} \int |x|^\alpha d\mu_\rho(x) \leq I(\rho). \]  

Thanks to (56), one may assume that $\rho \in \mathcal{P}_{s,\alpha,\tau}(\mathcal{G}_s)$ for some $\tau > 1$. Moreover, by (56) and (60), we know that (63) holds if $b > a/2$. If $b < a/2$ we proceed as follows. Since $\alpha > 1$ here, we may apply Lemma 3.15 and obtain that
\[ 0 = \int_{\mathbb{R}} x \, d\nu(x) = \mathbb{E}_\rho \omega_G(o), \]
where we use the symmetry assumption on $\nu$. Since $S_b = \{+1\}$, one has that $\omega_G(o) \geq 0$ and therefore $\omega_G(o) = 0 \rho$-a.s. In conclusion $I(\rho) = a \mathbb{E}_\rho \phi = \frac{a}{2} \mathbb{E}_\rho \xi_\alpha$, and the claim (63) follows from (57).

**Proof of Theorem 1.3 (c).** Suppose that $I(\rho) < \infty$. Then by (56) one has $\rho \in \mathcal{P}_{s,\alpha,\tau}(\mathcal{G}_s)$ for some $\tau > 1$. Since $\alpha > 1$, Lemma 3.15 yields $\int_{\mathbb{R}} x \, d\nu(x) = \mathbb{E}_\rho \omega_G(o)$ which, together with the assumption $\int_{\mathbb{R}} x \, d\nu(x) < 0$, implies $\mathbb{E}_\rho \omega_G(o) < 0$. However, $S_b = \{+1\}$ implies that $\mathbb{E}_\rho \omega_G(o) \geq 0$, a contradiction. Thus, $I(\rho) = +\infty$, for all $\rho \in \mathcal{P}_s(\mathcal{G}_s)$ such that $\mu_\rho = \nu$. 

\[ \square \]
A.1. Proof of Theorem 2.6. Recall the definition (13) of the function \( g_\mu : \mathbb{C}_+ \to \mathbb{C}_+ \), for a given \( \mu \in \mathcal{P}(\mathbb{R}) \). Theorem 2.6 is a consequence of the following result.

**Theorem A.1** (Uniform bound in subordination formula). Let \( Y = (Y_{ij})_{1 \leq i, j \leq n} \in \mathcal{H}_n(\mathbb{C}) \) be a Wigner random matrix with \( \text{Var}(Y_{12}) = 1 \), \( \mathbb{E}|Y_{12}|^3 < \infty \) and \( \mathbb{E}|Y_{12}|^2 < \infty \). There exists a universal constant \( c > 0 \), such that for any integer \( n \geq 1 \), any \( M \in \mathcal{H}_n(\mathbb{C}) \), any \( z \in \mathbb{C}_+ \), \( \text{Im}(z) \geq 1 \),

\[
\left| \frac{g_\mu(z) - g_{\mu_M}(z + \overline{g}(z))}{\text{Im}(z)/\sqrt{n} + 1} \right| \leq c \frac{(\mathbb{E}|Y_{12}|^2)^{1/2} + \mathbb{E}|Y_{12}|^3}{n^{1/2}}.
\]

where \( \overline{g}(z) = \mathbb{E}g_{\nu/\sqrt{n} + M}(z) \).

Theorem A.1 is a small generalization of Pastur and Shcherbina [12, Theorem 18.3.1]. We postpone its proof to the next subsection. We first check that it implies Theorem 2.6. This is done by a standard contraction argument. For \( z \in \mathbb{C}_+ \), we define the \( \mathbb{C}_+ \to \mathbb{C}_+ \) map,

\[
\phi_z : h \mapsto g_{\mu_M}(z + h).
\]

It is Lipschitz with constant \( 1/\text{Im}(z)^2 \). In particular, if \( \text{Im}(z) \geq 2 \), \( \phi_z \) is a contraction with Lipschitz constant \( 1/4 \). Now, it is well known that if \( \mu = \mu_M \boxplus \mu_{\text{sc}} \), we have for all \( z \in \mathbb{C}_+ \) the subordination formula,

\[
g_\mu(z) = g_{\mu_M}(z + g_\mu(z)) = \phi_z(g_\mu(z)),
\]

see Biane [7]. In particular, if for some probability measure \( \nu \in \mathcal{P}(\mathbb{R}) \) and \( \varepsilon \geq 0 \),

\[
|g_\nu(z) - g_{\mu_M}(z + g_\nu(z))| \leq \varepsilon,
\]

then

\[
|g_\mu(z) - g_\nu(z)| \leq \varepsilon + |\phi_z(g_\mu(z)) - \phi_z(g_\nu(z))| \leq \varepsilon + \frac{1}{\text{Im}(z)^2}|g_\mu(z) - g_\nu(z)|.
\]

So that, if \( \text{Im}(z) \geq 2 \),

\[
|g_\mu(z) - g_\nu(z)| \leq \frac{4}{3}\varepsilon.
\]

Hence from the definition of the distance \( d(\mu, \nu) \) in (12), we see that Theorem 2.6 is a corollary of Theorem A.1.

A.2. Proof of Theorem A.1 the Gaussian case. In this subsection, we assume that

1. \( G = (\Re(Y_{12}), \Im(Y_{12})) \) is a centered Gaussian vector in \( \mathbb{R}^2 \) with covariance \( K \in \mathcal{H}_2(\mathbb{R}) \), \( \text{tr}(K) = 1 \).
2. \( Y_{11} \) is a centered Gaussian in \( \mathbb{R} \) with variance 1.

The proof is a variant of Pastur and Shcherbina [12, Lemma 2.2.3]. We first recall the Gaussian integration by part formula: for any continuously differentiable function \( F : \mathbb{R}^2 \to \mathbb{R} \), with \( \mathbb{E}||\nabla F(G)||_2 < \infty \),

\[
\mathbb{E}F(G)G = K \mathbb{E}\nabla F(G).
\]

We identify \( \mathcal{H}_n(\mathbb{C}) \) with \( \mathbb{R}^{n^2} \). Then, if \( \Phi : \mathcal{H}_n(\mathbb{C}) \to \mathbb{C} \) is a continuously differentiable function, we define \( D_{jk}\Phi(X) \) as the derivative with respect to \( \Re(X_{jk}) \), and for \( 1 \leq j \neq k \leq n \), \( D'_{jk}\Phi(X) \) as the derivative with respect to \( \Im(X_{jk}) \).

Define the resolvent \( R(X) = (X - z)^{-1}, z \in \mathbb{C}_+ \). From the resolvent formula

\[
R(X + A) - R(X) = -R(X + A)AR(X),
\]

(66)
valid for any matrix $A \in \mathcal{H}_n(\mathbb{C})$, a standard computation shows that if $1 \leq j, k \leq n$, and $1 \leq a \neq b \leq n$, then
\[
D_{ab}R_{jk} = -(R_{ja}R_{bk} + R_{jb}R_{ak}) \quad \text{and} \quad D'_{ab}R_{jk} = -i(R_{ja}R_{bk} - R_{jb}R_{ak}),
\]
while if $1 \leq a \leq n$, then
\[
D_{aa}R_{jk} = -R_{ja}R_{ak}.
\]
Set $X = Y/\sqrt{n} + M$, so that
\[
R = (Y/\sqrt{n} + M - z)^{-1}.
\]
Using (65) we get, for $0 \leq a \neq b \leq n$, and all $j, k$:
\[
\mathbb{E}R_{jk}Y_{ab} = \frac{1}{\sqrt{n}} \mathbb{E} [K_{11}D_{ab}R_{jk} + K_{12}D'_{ab}R_{jk} + iK_{21}D_{ab}R_{jk} + iK_{22}D'_{ab}R_{jk}]
\]
\[
= -\frac{1}{\sqrt{n}} \mathbb{E} [(K_{11} - K_{22} + iK_{12} + iK_{21})R_{ja}R_{bk} + (K_{11} + K_{22} - iK_{12} + iK_{21})R_{jb}R_{ak}]
\]
\[
= -\frac{1}{\sqrt{n}} \mathbb{E} (\gamma R_{ja}R_{bk} + R_{jb}R_{ak}), \tag{67}
\]
where at the last line, we have used the symmetry of $K$ and $\text{tr}(K) = 1$, together with the notation
\[
\gamma = K_{11} - K_{22} + 2iK_{12} = \mathbb{E}Y_{ab}^2.
\]
Notice that $|\gamma| \leq 1$. Similarly, for $a = b$ one has
\[
\mathbb{E}R_{jk}Y_{aa} = -\frac{1}{\sqrt{n}} \mathbb{E} R_{ja}R_{ak}. \tag{68}
\]
Next, set
\[
G(z) = (M - z)^{-1}.
\]
Notice that in this case the dependency of $G(z)$ on $z$ is explicit in our notation. From the resolvent formula (66)
\[
R = G(z) - \frac{1}{\sqrt{n}} RYG(z).
\]
Hence, for $1 \leq j, k \leq n$, using (67)-(68),
\[
\mathbb{E}R_{jk} = G(z)_{jk} - \frac{1}{\sqrt{n}} \sum_{1 \leq a, b \leq n} \mathbb{E}[R_{ja}Y_{ab}]G(z)_{bk}
\]
\[
= G(z)_{jk} + \frac{\gamma}{n} \sum_{1 \leq a \neq b \leq n} \mathbb{E}[R_{ja}R_{ba}]G(z)_{bk} + \frac{1}{n} \sum_{1 \leq a, b \leq n} \mathbb{E}[R_{jb}R_{aa}]G(z)_{bk}.
\]
We set
\[
g = g_{\nu\sqrt{\pi} + \nu\ell}(z) = \frac{1}{n} \sum_{a=1}^{n} R_{aa}, \quad \overline{g} = \mathbb{E}g, \quad g = g - \mathbb{E}g,
\]
and consider the diagonal matrix $D$ with $D_{jk} = 1_{j=k}R_{jk}$. We find
\[
\mathbb{E}R = G(z) + \mathbb{E}[gR][G(z) + \frac{\gamma}{n} \mathbb{E}[R(R^T - D)]G(z)].
\]
Multiplying on the right hand side by $G(z)^{-1} = M - z$ and subtracting $\overline{g}R$ one has
\[
\mathbb{E}R(M - z - \overline{g}) = I + \mathbb{E}gR + \frac{\gamma}{n} \mathbb{E}R(R^T - D).
\]
Multiplying on the right hand side by $G(z + \overline{g})$
\[
\mathbb{E}R = G(z + \overline{g}) + \mathbb{E}gRG(z + \overline{g}) + \frac{\gamma}{n} \mathbb{E}R(R^T - D)G(z + \overline{g}).
\]
Finally, multiplying by $\frac{1}{n}$ and taking the trace,
\[
\overline{y} = g_{\mu_M}(z + \overline{y}) + \frac{1}{n} \mathbb{E}g_{\overline{r}}[RG(z + \overline{y})] + \frac{\gamma}{n^2} \mathbb{E} \text{tr}[R(R^T - D)G(z + \overline{y})].
\]
As a function of the entries of $Y$, $g$ has Lipschitz constant $O(n^{-1} \text{Im}(z)^{-2})$. This fact can be seen e.g. as in [8, Lemma 2.3.1]. Since the entries of $Y$ satisfy a Poincaré inequality, a standard concentration bound implies
\[
\mathbb{E}|g| = O(n^{-1} \text{Im}(z)^{-2}).
\]
Also, since $|\text{tr}(AB)| \leq n\|A\|\|B\|$, we find
\[
\left|\frac{1}{n} \text{tr} R G(z + \overline{y})\right| \leq \text{Im}(z)^{-2}\quad\text{and}\quad\left|\text{tr} R (R^T - D)G(z + \overline{y})\right| \leq 2n \text{Im}(z)^{-3}.
\]
This concludes the proof of Theorem A.1 in the Gaussian case.

A.3. Proof of Theorem A.1: the general case. Let $Y'_{ij} = Y_{ij} - \mathbb{E}Y_{12}$. Then $Y' - Y$ has rank at most 1. Hence by Lemma B.1
\[
|g_{\mu_Y/\sqrt{\pi_M}}(z) - g_{\mu_{Y'/\sqrt{\pi_M}}}(z)| \leq O((n \text{Im}(z))^{-1}),
\]
where we have used (14) and the fact that $f(x) = (x - z)^{-1}$ has a bounded variation norm of order $\text{Im}(z)^{-1}$. Also, we recall that the map $\phi_z$ defined by (64) is Lipschitz with constant $1/\text{Im}(z)^2$. Hence in order to prove Theorem A.1 we assume without loss of generality that the off-diagonal entries of the matrix are centered: $\mathbb{E}Y_{12} = 0$.

We now check that the diagonal entries of $Y$ are negligible. Let $Y'$ be the matrix obtained from $Y$ by setting the diagonal equal to zero: $Y'_{ij} = 1_{i \neq j} Y_{ij}$.

Lemma A.2 (Diagonal entries are negligible). For $z \in \mathbb{C}_+$, $\text{Im}z \geq 1$,
\[
|\mathbb{E}g_{\mu_Y/\sqrt{\pi_M}}(z) - \mathbb{E}g_{\mu_{Y'/\sqrt{\pi_M}}}(z)| = O\left(\left(\mathbb{E}|Y_{11}|^2/n\right)^{1/2}\right).
\]

Proof. From (14), we find
\[
|\mathbb{E}g_{\mu_Y/\sqrt{\pi_M}}(z) - \mathbb{E}g_{\mu_{Y'/\sqrt{\pi_M}}}(z)| \leq \frac{\mathbb{E}W_1(\mu_Y/\sqrt{\pi_M}, \mu_{Y'/\sqrt{\pi_M}})}{(\text{Im}z)^2} \leq \frac{\mathbb{E}W_2(\mu_Y/\sqrt{\pi_M}, \mu_{Y'/\sqrt{\pi_M}})}{(\text{Im}z)^2}.
\]
Then by Lemma B.2 using Jensen inequality,
\[
\mathbb{E}W_2(\mu_Y/\sqrt{\pi_M}, \mu_{Y'/\sqrt{\pi_M}}) \leq \frac{1}{n} \left(\sum_{i=1}^{n} \mathbb{E}|Y_{ii}|^2\right)^{1/2} = \frac{1}{\sqrt{n}} \left(\mathbb{E}|Y_{11}|^2\right)^{1/2}.
\]

As a consequence of Lemma A.2, we can assume without loss of generality that the diagonal entries of $Y$ are independent centered Gaussian with variance 1. By Subsection A.2 the conclusion of Theorem A.1 holds for the matrix $\tilde{Y}$ whose off-diagonal entries are centered Gaussian random variables with covariance is $K$, where $K$ is the covariance of $Y$, and with diagonal entries centered Gaussian with variance 1. Therefore, since the map $\phi_z$ defined by (64) is Lipschitz, in order to prove Theorem A.1 it is sufficient to establish that
\[
|\mathbb{E}g_{\mu_Y/\sqrt{\pi_M}}(z) - \mathbb{E}g_{\mu_{Y'/\sqrt{\pi_M}}}(z)| \leq c \frac{\mathbb{E}|Y_{12}|^3}{n^{1/2}}. \tag{69}
\]
We may repeat verbatim the interpolation trick in Pastur and Shcherbina [12] Theorem 18.3.1].
Consider the random matrix $\hat{Y}$, independent of $Y$, and for $0 \leq t \leq 1$, define the matrix
\[
Y(t) = \sqrt{t}Y + \sqrt{1-t}\hat{Y}.
\]
Set $R(t) = (Y(t)/\sqrt{n} + M - zI)^{-1}$. Then, using the resolvent equation (65)
\[
g_{\mu_Y/\mu,\mu}(z) - g_{\mu_Y/\mu,\mu}(z) = \frac{1}{n} \int_0^1 \frac{d}{dt} R(t) dt
\]
\[
= -\frac{1}{n^{3/2}} \int_0^1 \left( \frac{Y}{\sqrt{t}} - \frac{\hat{Y}}{\sqrt{1-t}} \right) R(t) dt
\]
\[
= -\frac{1}{2n^{3/2}} \int_0^1 \left( \text{tr} R(t) \frac{Y}{\sqrt{t}} - \text{tr} R(t) \frac{\hat{Y}}{\sqrt{1-t}} \right) dt.
\]
(70)
Next, consider the extension of (65) to arbitrary centered random variable $G$ with covariance $K$. Namely, for any twice continuously differentiable function $F: \mathbb{R}^2 \to \mathbb{R}$, with $\mathbb{E}\|\nabla F(G)\|_2 < \infty$ and $\sup_{x \in \mathbb{R}^2}\|\text{Hess} F(x)\| < \infty$, a Taylor expansion gives
\[
\mathbb{E}F(G)G = K\mathbb{E}\nabla F(G) + O\left( \mathbb{E}\|\nabla F(G)\|_2^2 \sup_{x \in \mathbb{R}^2}\|\text{Hess} F(x)\| \right).
\]
Since $Y$ and $\hat{Y}$ have the same first two moments, we get for all $t \in [0,1]$
\[
\mathbb{E}\text{tr} R^2(t) \frac{Y}{\sqrt{t}} = \mathbb{E}\text{tr} R^2(t) \frac{\hat{Y}}{\sqrt{1-t}} = \sum_{1 \leq j, k \leq n} \mathbb{E} R^2(t)_{kj} \frac{Y_{jk}}{\sqrt{t}} - \mathbb{E} R^2(t)_{kj} \frac{\hat{Y}_{jk}}{\sqrt{1-t}}
\]
\[
\leq c \frac{\mathbb{E}|Y_{12}|^3}{n} \sum_{1 \leq j, k \leq n} \sup_{X \in \mathcal{H}_n(C), \epsilon, \epsilon'} |D_{\epsilon}^j D_{\epsilon'}^k (R(X)^2)_{kj}|,
\]
where $c > 0$ is a constant, and $D_{\epsilon}^j D_{\epsilon'}^k$ ranges over $D_{\epsilon}^j D_{\epsilon'}^k$ and $D_{\epsilon}^j D_{\epsilon'}^k$. However, it follows from (67)-(68) that
\[
|D_{\epsilon}^j D_{\epsilon'}^k (R(X)^2)_{kj}|
\]
is a finite linear combination of products of 4 resolvent entries of the form $\prod_{i=1}^4 R(X)_{u_i, v_i}$. Since for any $X \in \mathcal{H}_n(C)$, $|R(X)_{ij}| \leq (3\pi)^{-1}$, one has, for some new constant $c > 0$ and for all $t \in [0,1]$: \[
|\mathbb{E}\text{tr} R^2(t) \frac{Y}{\sqrt{t}} - \mathbb{E}\text{tr} R^2(t) \frac{\hat{Y}}{\sqrt{1-t}}| \leq cn \frac{\mathbb{E}|Y_{12}|^3}{(3\pi)^4}
\]
Plugging this last upper bound in (70) concludes the proof (69) and of Theorem A.1

APPENDIX B.

In this section we collect some standard facts that are repeatedly used in the main text. For probability measures $\mu, \mu' \in \mathcal{P}(\mathbb{R})$, the Kolmogorov-Smirnov (KS) distance is defined by
\[
d_{KS}(\mu, \mu') = \sup_{t \in \mathbb{R}} |\mu(-\infty, t] - \mu'(-\infty, t]|.
\]
(71)
The KS distance is closely related to functions with bounded variations. More precisely, for \( f : \mathbb{R} \mapsto \mathbb{R} \) the bounded variation norm is defined as
\[
\|f\|_{BV} = \sup \sum_{k \in \mathbb{Z}} |f(x_{k+1}) - f(x_k)|,
\]
where the supremum is over all sequence \((x_k)_{k \in \mathbb{Z}}\) with \(x_n \leq x_{n+1}\). If \( f = 1((-\infty, t)) \) then \( \|f\|_{BV} = 1 \) while if the derivative of \( f \) is in \( L^1(\mathbb{R}) \), we have \( \|f\|_{BV} = \int |f'(x)|dx \). The KS distance is also given by the variational formula
\[
d_{KS}(\mu, \mu') = \sup \left\{ \int f d\mu - \int f d\mu' : \|f\|_{BV} \leq 1 \right\}.
\]

For \( p \geq 1 \) and \( \mu, \mu' \in \mathcal{P}(\mathbb{R}) \) such that \( \int |x|^p d\mu(x) \) and \( \int |x|^p d\mu'(x) \) are finite, their \( L^p \)-Wasserstein distance is defined as
\[
W_p(\mu, \mu') = \left( \inf_{\pi} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p d\pi(x, y) \right)^{\frac{1}{p}}
\]
where the infimum is over all coupling \( \pi \) of \( \mu \) and \( \mu' \) (i.e. \( \pi \) is probability measure on \( \mathbb{R} \times \mathbb{R} \) whose first marginal is equal to \( \mu \) and second marginal is equal to \( \mu' \)). Hölder’s inequality implies that for \( 1 \leq p \leq p' \), \( W_p \leq W_{p'} \).

For any \( p \geq 1 \), if \( W_p(\mu_n, \mu) \) converges to 0 then \( \mu_n \rightharpoonup \mu \). This follows for example from the Kantorovich-Rubinstein duality
\[
W_1(\mu, \mu') = \sup \left\{ \int f d\mu - \int f d\mu' : \|f\|_{Lip} \leq 1 \right\},
\]
where \( \|f\|_{Lip} \) denotes the Lipschitz constant of \( f \).

The following inequality is a standard consequence of interlacing, see e.g. [4, Theorem A.43].

**Lemma B.1** (Rank inequality). If \( A, B \in \mathcal{H}_n(\mathbb{C}) \), then
\[
d_{KS}(\mu_A, \mu_B) \leq \frac{1}{n} \text{rank}(A - B).
\]

Next, we recall a very useful estimate which allows one to bound eigenvalue differences in terms of matrix entries. For a proof see e.g. [3, Lemma 2.1.19].

**Lemma B.2** (Hoffman-Wielandt inequality). If \( A, B \in \mathcal{H}_n(\mathbb{C}) \), then
\[
W_2(\mu_A, \mu_B) \leq \sqrt{\frac{1}{n} \text{tr}((A - B)^2)}.
\]

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