Delta Modulator Based Quantised State-Feedback Control of Networked Linear Systems

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ABSTRACT In this study, the design of quantised state-feedback controller using a Delta-Modulator (Δ-M) for linear networked systems is proposed. This modulator can be considered as one type of sliding mode quantiser (SMQ) and has several advantages such as lower design complexity, lower cost, and less noisy. In this study, the stability conditions of both the continuous-time and the discrete-time Δ-M based quantised control systems are derived. The switching function bounds, which ensure that the steady state behaviour of the system is periodic are derived. The effectiveness of the Δ-M based quantised control system is investigated using a ZigBee protocol based experimental communication network with inherent imperfections associated with real-time network. It is shown that the designed quantised state-feedback controller gives the desired performance.

INDEX TERMS Delta Modulator, State-Feedback Control, Switched Control

I. INTRODUCTION

During the past few years, sliding mode techniques have been popular in control applications such as controller design [1], [2], observer design [3] and sliding mode quantiser (SMQs) [4], [5] due to their robustness and simplicity in converting input signals to switching signals which are described by a sequence of binary values. Different types of SMQs can be used to generate these binary values such as delta modulator (Δ-M), sigma-delta modulator (ΣΔ-M), delta-sigma modulator (ΔΣ-M) [6]–[8], pulse-width-modulator (PWM) based sliding mode [9], where these are used in many control applications [10]–[18].

However, the switching components in SMQs add more complexities to the system. If these quantisers are to be used effectively in control applications, they should satisfy both the stability and equivalence conditions [9]. In [5], [9] detailed investigations on the stability of SMQs in continuous-time (CT) systems have been carried out where it has been found out that the SMQs can be used in high-speed switching. However, the gain of the switching function affects the stability of the closed loop system. Under the assumption of ideal sliding mode and high switching frequency, the SMQs are equivalent to the inputs. This ensures the equivalence conditions [17], [19].

Most of the modern controllers are often implemented in the discrete-time domain where discretisation would introduce delays into switching components. In real-time, the effects of discretisation and the factors which affect the implementation under high switching frequency (for SMQs) have been reported in [20]–[22]. However, the results presented in [20]–[22] do not explicitly address the issues related to equivalence conditions. Therefore this has been the primary motivation of this study where the performance of SMQs designed using equivalence conditions is investigated. Amongst various quantisers, this study focuses on using Delta-modulators (Δ-M) as a quantiser. This modulator is extensively being used in power converters [6], [17] and in applications where bandwidth utilisation of the communication channel is necessary. Some further advantages of , the Δ-M include its simplicity in implementation, requirements of fewer hardware resources, lower complexity, cost-effective operations, lower noise, and many others [23]–[26].
In this study, a \( \Delta \)-M based quantised state-feedback controller is designed for multi-input-multi-output (MIMO) systems; both in CT and discrete-time (DT) domains. The main contributions of this paper are highlighted as follows: (i.)

1) Established the stability conditions for both CT and DT \( \Delta \)-M based quantised state-feedback systems.

2) Derived sufficient conditions for the existence of periodic orbits (or zig-zag behaviour) of the switching function and states.

3) Determined the bound of the switching function which results in periodic orbits.

4) Validated the theoretical findings using a real ZigBee protocol based communication system.

The rest of the paper is organised as follows. Section-II and section-III discuss, CT-\( \Delta \)-M and DT-\( \Delta \)-M respectively and the stability conditions are derived. The effectiveness of the proposed control strategy and the theoretical findings are validated using two simulation examples in section-IV followed by conclusions on section-V.

For the rest of the paper, \([|x|]_n\) and \(\varrho(A) \triangleq \max_{1 \leq i \leq n} |\lambda_i|\) respectively denote the standard Euclidean norm of the vector \(x \in \mathbb{R}^n\) and the spectral radius of the matrix \(A \in \mathbb{R}^{n \times n}\).

II. CONTINUOUS-TIME SWITCHED CONTROL SYSTEM

A. REVIEW OF CONTINUOUS-TIME DELTA MODULATOR

The block diagram of a MIMO CT \( \Delta \)-M is shown in Figure 1.

![Fig 1: Continuous-time Delta-Modulator (\( \Delta \)-M).](image)

It consists of two integrators, one on the feedback path and the other in the decoder side. Input of the quantiser is the difference between the output of the integrator on the feedback path and the input signal. This is a static quantiser and the output \(\delta(t)\) is a single-bit signal which is transmitted through a communication channel before received by the decoder. Under ideal conditions, the output of the decoder \(\hat{x}(t)\) is equivalent to the input signal \(\bar{x}(t)\) [10], [19], [27].

The MIMO CT \( \Delta \)-M is described as:

\[
\begin{align*}
s(t) &= x(t) - \hat{x}(t) \quad (1a) \\
\hat{x}(t) &= \Theta \text{sgn}(s(t)) \quad (1b)
\end{align*}
\]

where \(x \in \mathbb{R}^n\), \(\hat{x} \in \mathbb{R}^n\), \(s \in \mathbb{R}^n\) and \(\Theta \in \mathbb{R}^{n \times n}\) respectively denote the input signal, the quantised signal, the switching signal and the gain of the 2-level quantiser. Further, 
\[
\text{sgn}(s(t)) \in \left\{ -1, 1 \right\} \quad (2)
\]

The communication channel between the encoder and the decoder carries a single-bit \(n\)-dimensional signal which is expressed as:

\[
\delta(t) = \frac{1}{2} \left[ 1_n + \text{sgn}(s(t)) \right] \quad (3)
\]

where \(\delta(t) \in \left\{ 0, 1 \right\}, \cdots, \left\{ 0, 1 \right\} \quad (4)
\]

\(T \in \mathbb{R}^n\).

The quantiser gain \(\Theta\) is often selected as a positive definite matrix for design simplicity. This is expressed as:

\[
\Theta = \text{diag}\{\theta_i\} \forall i = (1, 2, \cdots, n). \quad (4)
\]

For proper operation of the MIMO CT \( \Delta \)-M, the quantisation gain \(\Theta\) must be same in both the encoder and the decoder [10], [19]. In [28], it has been shown that the existence of the sliding mode (i.e. \(s(t)^T \delta(t) \leq 0\)) is ensured provided 
\[
\|\dot{x}(t)\|_\infty \leq \theta; \quad \theta = \min(\theta_i). \quad (5)
\]

B. CONTINUOUS-TIME QUANTISED STATE-FEEDBACK CONTROL SYSTEM

The present study focuses on stabilisation problem of linear time-invariant (LTI) systems via state feedback which is described by:

\[
\begin{align*}
\dot{x}(t) &= A \bar{x}(t) + B u(t) \quad (5a) \\
u(t) &= -K \tilde{x}(t) \quad (5b)
\end{align*}
\]

where \(\bar{x} \in \mathbb{R}^n\) and \(u \in \mathbb{R}^m\) respectively denote the system states and the control signal. The pair \((A, B)\) is assumed to be controllable and the closed loop system is expressed as:

\[
\dot{x}(t) = A_{cl} \bar{x}(t) \quad (6)
\]

where \(A_{cl} = A - BK\).

The feedback control gain \(K\) is designed such that the poles of the closed-loop system \(A_{cl}\) are at desired locations.

The solution of (6) gives:

\[
\bar{x}(t) = e^{A_{cl}t} \bar{x}(0) \quad (7)
\]

where \(\bar{x}(0)\) is the initial conditions of \(\bar{x}(t)\). It is assumed that

\[
\|e^{A_{cl}t}\| \leq ce^{-\lambda t} \quad (8)
\]

The solution of (7) is therefore bounded and satisfies the condition:

\[
\|\bar{x}(t)\| \leq ce^{-\lambda t} \|\bar{x}(0)\| \quad (9)
\]

Using (1) and (5), the quantised state-feedback control system can be described as:

\[
\begin{align*}
\dot{x}(t) &= A x(t) + B \hat{u}(t) \quad (10a) \\
\dot{\hat{x}}(t) &= \Theta \text{sgn}(s(t)) \quad (10b) \\
\hat{u}(t) &= -K \hat{x}(t) \quad (10c) \\
s(t) &= x(t) - \hat{x}(t) \quad (10d)
\end{align*}
\]

where \(s(t)\) denotes the quantisation error. For the sake of convenience let us re-write (10) as:

\[
\begin{align*}
\dot{x}(t) &= A_{cl} x(t) + BK s(t) \quad (11a) \\
\dot{s}(t) &= A_{cl} x(t) + BK s(t) - \Theta \text{sgn}(s(t)) \quad (11b)
\end{align*}
\]
The objective is to limit \( t \to \infty \ s(t) = 0 \) in the equivalent sliding mode.

**Proposition II.1.** Assume \( x(0) \) is known and (8) holds. If there exists a quantiser gain \( \Theta = \text{diag}\{\theta_{ii}\} \) such that
\[
c g(\Theta) e^{-\lambda t} \|x(0)\| < \tilde{\theta},
\]
then the quantised state feedback control systems (11), is stable and converges to the system (5).

Proof: From (11),
\[
s(t)^T \dot{s}(t) = s(t)^T (A_c x(t) + BK s(t) - \Theta \text{sgn}(s(t)))
\]
\[
\leq s(t)^T A_c x(t) + s(t)^T BK s(t) - \tilde{\theta} \|s(t)\|_1
\]
\[
\leq g(A_c) \|s(t)\| \|x(t)\| + \|BK\| \|s(t)\|_1^2
\]
\[
- \tilde{\theta} \|s(t)\|_1
\]
\[
\leq \|s(t)\| \{g(A_c) \|x(t)\| + \|BK\| \|s(t)\| - \tilde{\theta}\}
\]
(13)

Note that since, \( g(A) \triangleq \max \{\|\lambda\| : \lambda \in \text{spec}(A)\} \) and \( \|s(t)\|_2 \leq \|s(t)\|_1 \), the solution for (11) is given by:
\[
x(t) = e^{A_c t} \bar{x}(0) + \int_0^t BK e^{A_c(t-\tau)} s(\tau) d\tau
\]
(14)

where
\[
\int_0^t s(\tau) d\tau = \left[ \int_0^t s_0(\tau) d\tau \int_0^t s_1(\tau) d\tau \ldots \int_0^t s_n(\tau) d\tau \right]^T
\]

Considering (8), the bound of the solution can be found and is given by:
\[
\|x(t)\| \leq c e^{-\lambda t} \|x(0)\| + c \int_0^t \|BK\| e^{-\lambda(t-\tau)} s(\tau) d\tau
\]
\[
\leq c e^{-\lambda t} \|x(0)\| + \frac{c \|BK\|}{\lambda} \sup_{0 < \tau < t} \|s(t)\|
\]
(15)

Since \( \sup_{0 < t < \tau} \|s(t)\| \leq \|s(t)\|_1 \), manipulating (13) and (15) gives,
\[
c g(A_c) e^{-\lambda t} \|x(0)\| + \left( 1 + \frac{c}{\lambda} \right) \|BK\| \|s(t)\| \leq \tilde{\theta}
\]
(16)

Thus, \( s(t)^T \dot{s}(t) \leq 0 \) and the sliding mode is ensured.

Consider the case of ideal sliding mode and assume that \( \tilde{x}(0) = x(0) \) (i.e. \( s(0) = 0 \)). Then the switching function (11b) will initiate an equivalent sliding mode and remain there indefinitely (i.e. \( s(t) = 0 \)) [27]. Therefore, if the condition (12) is true, then using (11b) and (13), it can be proved that \( s(t)^T \dot{s}(t) \leq 0 \).

**Remark 1.** If \( \|s(0)\| \geq 0 \), then the switching function (11b) requires finite-time (also called reaching-time), to change its mode from the reaching-mode to equivalent-mode. However, this finite-time may destabilise the entire closed loop system. Hence, in this paper, initial values of the system is assumed to be known and \( \tilde{x}(0) \) is selected such that \( \|s(0)\| = 0 \).

**Remark 2.** Higher values of feedback control gain \( K \) will make the degree of stability of the system higher. However, it forces \( \theta_{ii} \) for all \( i = 1, 2, \ldots, n \) to be chosen high, which results in the chattering phenomenon.

**Remark 3.** In Proposition II.1, an ideal sliding mode is assumed where infinite sampling switch is considered. As a result of the imperfection of switch elements, the switching manifold can be described as a boundary layer. The width of the boundary layer is defined as:
\[
\|s(t)\| < \epsilon(h)
\]
(17)
where \( h \) denotes the sampling time. It is worth to note that, \( \lim_{h\to0} \epsilon(h) = 0 \). By choosing the optimal quantiser gain \( \Theta \) and reducing sampling frequency of the switching; the boundary layer (17) can be minimised, undesired oscillations (chattering phenomena) can be attenuated and the system can be stabilised.

### III. DISCRETE-TIME SWITCHED CONTROL SYSTEMS

#### A. Euler Discretised System Description

The discrete-time equivalent of the CT system (5), using Euler discretisation is expressed as:
\[
\tilde{x}(k + 1) = \Phi \tilde{x}(k) + \Gamma u(k)
\]
(18a)
\[
u(k) = -K \tilde{x}(k)
\]
(18b)

where \( \Phi = I_n + hA = \{\phi_{ij}\} \) and \( \Gamma = hB \). Note that \( \tilde{x}(k), u(k) \) denote respectively \( \tilde{x}(kh) \) and \( u(kh) \) \( \forall k \in [k(h), (k + 1)h] \), where \( h \) is the sampling period.

Simplifying (18) gives,
\[
\tilde{x}(k + 1) = \Phi_d \tilde{x}(k)
\]
(19)

where
\[
\Phi_d = I_n + hA_{cl}
\]
(20)

The sampling time \( h \in \mathbb{R}^+ \) is selected in the range [29]:
\[
0 < h < h_{max} = \frac{2}{g(A_c)}
\]
(21)

This will ensure that the eigenvalues of \( \Phi_d \) are at desired locations. Substituting (20) in (19) gives:
\[
\Delta \tilde{x}(k) = hA_{cl} \tilde{x}(k)
\]
(22)

where \( \Delta \tilde{x}(k) = \tilde{x}(k + 1) - \tilde{x}(k) \).

Re-write (22) as:
\[
\Delta \tilde{x}(k) = hA_{cl} (\Phi_d)^{k} \tilde{x}(0)
\]
(23)

where \( \tilde{x}(k) = (\Phi_d)^{k} \tilde{x}(0) \).

If the feedback gain matrix \( K \) is selected such that, \( \|\Phi_d\| < 1 \), the it can be shown that
\[
\|\Delta \tilde{x}(k)\| < \ldots < \|\Delta \tilde{x}(1)\| < \|\Delta \tilde{x}(0)\|.
\]
(24)

This proves,
\[
\|\Delta \tilde{x}(k)\| < h g(A_{cl}) \|\tilde{x}(0)\|
\]
(25)
B. DISCRETE-TIME DELTA MODULATOR

The block diagram of a MIMO discrete-time Δ-M is shown in Figure 2. From Figure 2, the dynamics of the discrete-time Δ-M can be expressed as:

\[\begin{align*}
\dot{x}(k+1) &= \dot{x}(k) + \Theta sgn(s(k)) \\
\dot{s}(k) &= x(k) - \dot{x}(k)
\end{align*}\]  

(26a)

(26b)

where \(\Theta\) is a positive diagonal matrix i.e., \(\Theta = diag\{\theta_{ii}\} \forall i = (1, 2, \ldots, n)\).

Further \(sgn(s(k)) \in \{-1, 1\}^T \in \mathbb{R}^n\), where,

\[sgn(s_i(k)) = \begin{cases} +1, & \text{if } s_i(k) \geq 0; \\ -1, & \text{if } s_i(k) < 0; \end{cases} \quad i = (1, 2, \ldots, n)
\]

(27)

The communication channel between the encoder and the decoder carries a single-bit \(n\)-dimensional signal which is expressed as:

\[\delta(k) = \frac{1}{2} [1_n + sgn(s(k))].\]  

(28)

This implies that \(\delta(k) \in \{-1, 1\}^T \in \mathbb{R}^n\).

For the proper operation of the MIMO DT Δ-M, the quantisation gain \(\Theta\) must be same in both the encoder and the decoder [10], [19]. In [28], it has been shown that the existence of the sliding mode (i.e., \(|s(k+1)| \leq |s(k)|\)) is ensured provided \(\|\Delta x(k)\|_\infty \leq \tilde{\theta}\) ∀ \(\Delta x(k) = x(k+1) - x(k)\).

C. DISCRETE-TIME QUANTISED STATE-FEEDBACK CONTROL SYSTEM

Using (18) and (26), the quantised state-feedback control system is described as:

\[\begin{align*}
x(k+1) &= \Phi cl x(k) + \Gamma \hat{u}(k) \\
\dot{x}(k+1) &= \dot{x}(k) + \Theta sgn(s(k)) \\
\hat{u}(k) &= -K \ddot{x}(k) \\
s(k) &= x(k) - \dot{x}(k)
\end{align*}\]  

(29a)

(29b)

(29c)

(29d)

Substituting (29c) and (29d) into (29a) gives:

\[x(k+1) = \Phi cl x(k) + \Gamma K s(k)\]  

(30)

Re-write (30) as:

\[\Delta x(k) = hA cl x(k) + \Gamma K s(k)\]  

(31)

Using (20), (29) and (31), the dynamics of the switching function \(s(k)\) can be described as:

\[s(k+1) = x(k+1) - \dot{x}(k+1) - \dot{x}(k)\]

\[= \Phi cl x(k) + \Gamma K s(k) - \dot{x}(k) - \Theta sgn(s(k))\]

\[= [\Phi cl - I_n] x(k) + [\Gamma K + I_n] s(k) - \Theta sgn(s(k))\]

\[= \Delta x(k) + s(k) - \Theta sgn(s(k))\]  

(32)

Definition III.1. If there exists \(\delta > 0\) and \(M > 0\) such that \(|f(h)| < M |g(h)|\) for \(h < \delta\), then the order of \(f(h)\) is equal \(g(h)\) provided \(h \to 0\). This is denoted by \(f(h) = O(g(h))\).

Lemma III.2. Let \(s_i(k)\) be defined as in (32). If \(|s(k+1)| < |s(k)|\) and

\[\sup_{n \geq 0} |\Delta x_i(k)| < \theta_{ii}(h) < \infty\]  

(33)

then,

\[|s_i(\infty)| \leq 2\theta_{ii}(h) < \varepsilon\]  

(34)

for \(i = 1, \ldots, n\) where \(\varepsilon\) is a positive constant.

Proof: Consider two cases: when \(s_i(k) > 0\) and \(s_i(k) \leq 0\). From (32), when \(s_i(k) > 0\),

\[s_i(k+1) = \Delta x_i(k) + s_i(k) + \theta_{ii}\]

\[< s_i(k) < 2\theta_{ii}\]  

(35)

Correspondingly, when \(s_i(k) \leq 0\),

\[s_i(k+1) = \Delta x_i(k) + s_i(k) + \theta_{ii}\]

\[> s_i(k) > 2\theta_{ii}\]  

(36)

From the above two cases \(|s_i(k)| < 2\theta_{ii}\). Further, as \(k \to \infty\), \(s_i(k)\) will be bounded as:

\[|s_i(\infty)| \leq 2\theta_{ii} < \varepsilon\]  

(37)

Using Definition III.1, (31) can be expressed as:

\[\Delta x_i(k) = \alpha_i x(k) + \beta_i n\]  

(38)

where \(\alpha_i\) and \(\beta_i\) respectively denote the \(i^{th}\) row of \(hA cl\) and \(\Gamma K\). If both \(\alpha_i\) and \(\beta_i\) can be expressed as \(\alpha_i = O(h)\) and \(\beta_i = O(h)\), (33) can be satisfied with arbitrary small \(h\). This also means that \(\theta_{ii}\) is a function of \(h\) (i.e., \(\theta_{ii}(h)\)) and \(\theta_{ii} \to 0\) as \(h \to 0\).

Remark 4. For an arbitrarily small value of \(h\), \(\Delta x(k)\) tends to its counterpart of the nominal system \(\Delta \hat{x}(k)\) in (22). Considering (25), this also implies that \(|\Delta x(k)| \leq h \hat{g}(A cl) \|\hat{x}(0)\| \leq \theta_{ii}(h)\).

Proposition III.3. If the condition in Lemma III.2 is true, then (30) is bounded as,

\[\frac{\sqrt{n} \theta_{ii}}{1 - \|\Phi cl\|} \|\Gamma K\|.\]  

(39)

Proof: From (30):

\[\|x(k+1)\| = \|\Phi cl x(k) + \Gamma K s(k)\|\]

\[\leq \|\Phi cl\| \|x(k)\| + \|\Gamma K\| \|s(k)\|\]  

(40)
If \( I_n - \Phi_{cl} \) is non-singular, iterating (40) \( \kappa \) times gives,
\[
\| x(\kappa) \| \leq \| \Phi_{cl} \|^\kappa \| x(0) \| + \| \Gamma K \| \sum_{i=0}^{\kappa-1} \| \Phi_{cl} \|^i \| s(\kappa - i - 1) \|
\]
\[
\leq \| \Phi_{cl} \|^\kappa \| x(0) \| + \sqrt{n} \theta_{ii} \| \Gamma K \| \sum_{i=0}^{\kappa-1} \| \Phi_{cl} \|^i
\]
\[
\leq \| \Phi_{cl} \|^\kappa \| x(0) \| + \frac{\sqrt{n} \theta_{ii} \| \Gamma K \| (1 - \| \Phi_{cl} \|^\kappa)}{1 - \| \Phi_{cl} \|}
\]
(41)

When \( \kappa \to \infty \), it can be shown that \( \| x(\infty) \| \leq \frac{\sqrt{n} \theta_{ii} \| \Gamma K \|}{1 - \| \Phi_{cl} \|} \)
which completes the proof.

**D. BOUNDARY LAYER OF SLIDING MODE AND PERIODICITY**

In the following, we will estimate the boundary width of sliding mode which is associated to the quantization error.

**Lemma III.4.** [30] Let \( x(k) \), \( f_s \) respectively denote the discretised samples of \( x(t) \) and the over-sampling frequency of \( \Delta M \) which satisfies
\[
\frac{f_s}{2B_X} > 2^\alpha.
\]
(42)

where \( X, B_X \) respectively denote the upper bound of \( x \) and the bandwidth of the closed-loop system and \( \alpha \) is a positive number. Then,
\[
\| \Delta x(k) \| \leq \frac{\pi}{2^{\alpha}} X
\]
(43)

The bound of \( s(k) \) in the equivalent-based sliding mode, is presented in the following [19], [31].

**Lemma III.5.** If the diagonal elements of \( \Theta \) is selected such that
\[
\hat{\theta} = \min(\theta_{ii}) \geq \frac{\pi}{2^\alpha} X \geq \| \Delta x(k) \|
\]
and the initial conditions satisfy \( s(0) = 0 \), then the system (26) exhibits quasi-sliding motion and the switching function \( s(k) \) remain inside the boundary layer \( \Omega \)
\[
\| s(k) \| \leq \Omega = \frac{\| \Delta x(k) \|^\alpha + \sqrt{n} \theta_{ii}}{2(\theta_{ii} - n\| \Delta x(k) \|)}
\]
indeefinitely.

Proof: Consider the dynamics of \( s(k) \) in (32). Let us choose a positive-definite, DT candidate Lyapunov function,
\[
\mathcal{V}(k) = s^T(k)s(k)
\]
(45)

Using (43), the first difference \( \Delta \mathcal{V}(k) \) of the Lyapunov function \( \mathcal{V}(k) \) can be written as:
\[
\Delta \mathcal{V}(k) = s^T(k+1)s(k+1) - s^T(k)s(k)
\]
\[
= [\Delta x(k) + s(k) - \Theta \text{sgn}(s(k))]^T \times [\Delta x(k) + s(k) - \Theta \text{sgn}(s(k))] - s^T(k)s(k)
\]
\[
= \Delta x^T(k)\Delta x(k) + \Delta s^T(k)\Delta s(k) - \Theta \text{sgn}(s(k))^T \Delta x(k)
\]
\[
+ \Theta \text{sgn}(s^T(k))\text{sgn}(s(k)) - s^T(k)\Theta \text{sgn}(s(k))
\]
\[
\]
where $\Psi = \Gamma K$. When $q(\Phi_{cl}) < 1$ and $s(k)$ exhibits periodic behaviour, then $x(k)$ and $\hat{x}(k)$ also exhibit similar behaviour and their periods will be similar [13].

**IV. SIMULATION RESULTS**

The effectiveness of the proposed $\Delta$-M based quantised state-feedback controller design is demonstrated in a practical networked environment considering examples of both discrete and continuous time systems. In this study, the wireless communication channel is implemented using two Arduino boards and two Zigbee modules as hardware in loop (HIL). The modulated signal is transmitted to the Zigbee module 2 in Arduino board 2 from the Zigbee module 1 in Arduino board 1. This board is connected to the computer and with the plant, the controller, and the quantiser. The Zigbee module 2 in Arduino board 2 acts as a hop device in another computer which transmits the signal back to the Zigbee module 1 in Arduino board 1 which then transmits the signal into the demodulator (see figure 3.)

![Networked control using ZigBee protocol based communication network.](image)

**A. EXAMPLE 1**

Consider a continuous-time linear system with three number of inputs described by:

$$\dot{x}(t) = A x(t) + B u(t)$$  \hspace{1cm} (49)

where $A$ and $B$ are given by,

$$A = \begin{bmatrix} -1 & 0 & -1 \\ -1 & -1 & -1 \\ -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The controller and the quantisation gain of the $\Delta$-M based quantiser are designed following the method described in section II for the continuous-time system. The initial conditions $x_0$ of the the plant is taken as $x_0 = [0.05; -0.5; 0]$, and the poles and the feedback gain $K$ is calculated by placing the closed loop poles at $\lambda_{pole, cl} = -1, -1, -1$. The performance of the controller in stabilising the states are shown in Figure 4 and the behaviour of the switching function is shown in Figure 5.

After designing the controller for continuous time systems, in the next phase, the $\Delta$-M based quantised controller is designed for the discrete-time system. The discrete equivalent of the continuous-time system described in (49), at a sampling rate of $h = 0.01$ seconds, is given by:

$$x(k + 1) = \Phi x(k) + \Gamma u(k)$$  \hspace{1cm} (50)

where

$$\Phi = \begin{bmatrix} 0.9900 & 0 & -0.0100 \\ -0.0100 & 0.9900 & -0.0100 \\ -0.0100 & 0.0100 & 0.9900 \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} 0.0100 & 0 & 0 \\ 0 & 0.0100 & 0 \\ 0 & 0 & 0.0100 \end{bmatrix}$$

The controller and the gain of the $\Delta$-M based quantiser are designed following the approaches described in section II and section-III. The initial conditions and $x_0$ and feedback gain $K$ are same as the continues-time system. The quantisation gain $\Theta$ is given by:

$$\Theta = \begin{bmatrix} 0.05 & 0 & 0 \\ 0 & 0.05 & 0 \\ 0 & 0 & 0.05 \end{bmatrix}$$

From Figure 6, it is observed that all the states converge to zero within finite time. Further, the switching function starts inside the region $\Omega$, as can be seen in Figure 7, and stays there indefinitely with a period of 2 (see Figure 8). It is worth to emphasis that although the period of the switching function depends on the choice of the quantisation gain and the sampling time, this remains always within the region $\Omega$. 
The rate of the continuous-time system described in (51) at a sampling time.

In this study, a delta modulator based single-bit quantised controller could maintain the stability of the overall system.

V. CONCLUSIONS

In this study, a delta modulator based single-bit quantised state-feedback controller is designed for linear networked control systems. The stability conditions Δ-M is derived for both in CT and DT domains. It is confirmed that the stability of delta Modulator is heavily dependent on the quantiser gain and on some properties of the input signals. The bound of the switching function is derived such that periodic behaviour in the steady-state is ensured. The effectiveness of the designed delta modulator based single-bit quantised control approach is illustrated using a practical ZigBee protocol based networked control system which inherent many network imperfections like packet losses, transmission delays, and bit-rate constraints. The results of the experimental simulations were carried out using two examples and the results of the simulations confirm the theoretical findings.

REFERENCES

[1] V. I. Utkin, Sliding Modes in Control and Optimization. Springer Berlin Heidelberg, 1992.
[2] D. Liberzon, "Hybrid feedback stabilization of systems with quantized signals," Automatica, vol. 29, pp. 1543–1554, 2003.
[3] C. Edward and S. Spurgeon, Sliding Mode Control: Theory And Applications. London, U.K.: Taylor & Francis, 1998.
[4] S. H. Yu, "Analysis and design of single-bit sigma-delta Modulators using the theory of sliding modes," IEEE Transactions on Control Systems Technology, vol. 14, no. 2, pp. 336–345, 2006.
FIGURE 7: Behaviour of the switching function for the example 1 for the discrete-time system.

FIGURE 8: Zig-zag (periodic) behaviour of the switching function for the example 1 for the discrete-time system.

FIGURE 9: Simulated example 2: Mechanical damper system.

FIGURE 10: Behaviour of the states for the Example 2 for the continuous-time system.

FIGURE 11: Behaviour of the switching function for the Example 2 for the continuous-time system.

[5] M. Salimi, J. Soltani, A. Zakipour, and N. R. Abjadi, “Hyper-plane sliding mode control of the DC-DC buck/boost converter in continuous and discontinuous conduction modes of operation,” IET Power Electronics, vol. 8, no. 8, pp. 1473–1482, 2015.

[6] H. Sira-Ramirez, “Sliding mode-$\Delta$; modulation control of a “buck” converter,” In Proceedings of 42nd IEEE Conference on Decision and Control, Dec. 2003.

[7] D. Almakhles, A. K. Swain, and A. Nasiri, “The dynamic behaviour of data-driven $\Delta$-M and $\Delta$Σ-M in sliding mode control,” International Journal of Control, vol. 90, no. 11, pp. 2406–2414, 2016.

[8] Y. S. Chou, C. C. Lin, and Y. L. Chang, “Robust H-infinity filtering in finite frequency domain for polytopic systems with application to in-band quantisation noise reduction in uncertain cascaded sigma-delta modulators,” IET Control Theory and Applications, vol. 6, no. 9, pp. 1155–
FIGURE 12: Behaviour of the states for the Example 2 for the discrete-time system.

1171, 2012.
[9] S. C. Tan, Y. M. Lai, and C. K. Tse, “General Design Issues of Sliding-Mode Controllers in DC-DC Converters,” IEEE Transactions on Industrial Electronics, vol. 55, no. 3, pp. 1160–1174, 2008.
[10] C. C. de Wit, F. Gomez-Estern, and F. Rubio, “Delta-modulation coding redesign for feedback-controlled systems,” IEEE Transactions on Industrial Electronics, vol. 56, no. 1, pp. 2684–2696, 2009.
[11] ——, “Adaptive delta modulation in networked controlled systems with bounded disturbances,” IEEE Transactions on Automatic Control, vol. 56, no. 1, pp. 129–134, 2011.
[12] R. Gai, X. Xia, and G. Chen, “Complex dynamics of systems under delta-modulated feedback,” IEEE Transactions on Automatic Control, vol. 51, no. 1, pp. 1888–1902, 2006.
[13] X. Xia and A. Zinober, “Periodic orbits from $\Delta$-modulation of stable linear systems,” IEEE Transactions on Automatic Control, vol. 49, no. 1, pp. 1376–1380, 2004.
[14] X. Xia, “Periodic orbits arising from two-level quantized feedback control,” Chaos, Solitons and Fractals, vol. 33, no. 4, pp. 1339–1347, 2007.
[15] X. Xia and G. Chen, “On delta-modulated control: A simple system with complex dynamics,” Chaos, Solitons and Fractals, vol. 33, no. 4, pp. 1314–1328, 2007.
[16] X. Xia, G. Chen, R. Gai, and A. Zinober, “Periodicity in delta-modulated feedback control,” Journal of Control Theory and Applications, vol. 6, no. 1, pp. 37–44, 2008.
[17] D. J. Almakhles, A. K. Swain, and N. D. Patel, “Stability and performance analysis of bit-stream-based feedback control systems,” IEEE Transactions on Industrial Electronics, vol. 62, no. 1, pp. 4319–4327, 2015.
[18] Y. S. Chou and Y. L. Chang, “Decentralised output-feedback controller syntheses with restricted frequency-domain specifications via generalised Kalman-Yakubovich-Popov lemma: A unified approach,” IET Control Theory and Applications, vol. 9, no. 10, pp. 1615–1628, 2015.
[19] C. Wanigasekara, “Identification of nonlinear systems and quantised feedback control of networked systems,” Ph.D. dissertation, The University of Auckland, 2020.
[20] Y. Xinghao, B. Wang, Z. Gaias, and G. Chen, “Discretization Effect on Equivalent Control-Based Multi-Input Sliding-Mode Control Systems,” IEEE Transactions on Automatic Control, vol. 53, no. 6, pp. 1563–1569, 2008.
[21] ——, “ZOH discretization effect on single-input sliding mode control systems with matched uncertainties,” Automatica, vol. 45, no. 1, pp. 118–1259, 2009.
[22] S. Qu, X. Xia, and J. Zhang, “Dynamical Behaviors of an Euler Discretized Sliding Mode Control Systems,” IEEE Transactions on Automatic Control, vol. 59, no. 9, pp. 2525–2529, 2014.
[23] T. Li and Y. Fujimoto, “Control system with high-speed and real-time communication links,” IEEE Transactions on Industrial Electronics, vol. 55, no. 1, pp. 1548–1557, 2008.
[24] U. Premaratne, S. Halmagne, and I. Mareels, “Event triggered adaptive differential modulation: A new method for traffic reduction in networked control systems,” IEEE Transactions on Automatic Control, vol. 58, no. 1, pp. 1696–1706, 2013.
[25] Z. Xi, “Dynamic sliding mode controller design for networked control systems with random packet loss and event driven quantisation,” IET Control Theory and Applications, vol. 12, no. 17, pp. 2433–2440, 2018.
[26] X. Wang, C. Wen, J. Yan, and Y. Xia, “Quantised stabilisation of continuous-time switched systems with time-delay,” IET Control Theory and Applications, vol. 12, no. 7, pp. 900–913, 2018.
[27] D. Almakhles, A. K. Swain, and A. Nasiri, “The dynamic behaviour of data-driven $\Delta$-M and $\Delta\Sigma$-M in sliding mode control,” International Journal of Control, vol. 90, no. 11, pp. 2406–2414, 2017.
[28] D. Almakhles, “Two-level quantised control systems: sliding mode approach,” International Journal of Control, pp. 1–9, 2018.
[29] D. Almakhles, A. K. Swain, A. Nasiri, and N. Patel, “An adaptive two-level quantizer for networked control systems,” IEEE Transactions on Control Systems Technology, vol. 25, no. 1, pp. 1084–1091, 2017.
[30] C. Wanigasekara, D. Almakhles, A. Swain, and S. K. Nguang, “Delta-Modulator-Based Quantised Output Feedback Controller for Linear Networked Control Systems,” IEEE Access, vol. 8, pp. 175 169–175 179, 2020.
[31] M. Pilloni, A. and Franceschelli, A. Pisano, and E. Usai, “Delta modulation ($\Delta$-M) via second-order sliding-mode control technique,” Control Engineering Practice, vol. 92, no. 1, pp. 104129, 2019.
[32] R. S. Esfandiar and B. Lu, CRC Press, Taylor & Francis Group. American Mathematical Society, 2018.
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**FIGURE 13:** Behaviour of the switching function for the Example 2 for the discrete-time system.

**FIGURE 14:** Zig-zag (periodic) behaviour of the switching function for the example 2 for the discrete-time system.

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