Research Article

Zero-divisor graphs of Catalan monoid

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Abstract

Let \( C_n \) be the Catalan monoid on \( X_n = \{1, \ldots, n\} \) under its natural order. In this paper, we describe the sets of left zero-divisors, right zero-divisors and two sided zero-divisors of \( C_n \); and their numbers. For \( n \geq 4 \), we define an undirected graph \( \Gamma(C_n) \) associated with \( C_n \) whose vertices are the two sided zero-divisors of \( C_n \) excluding the zero element \( \theta \) of \( C_n \) with distinct two vertices \( \alpha \) and \( \beta \) joined by an edge in case \( \alpha \beta = \theta = \beta \alpha \). Then we first prove that \( \Gamma(C_n) \) is a connected graph, and then we find the diameter, radius, girth, domination number, clique number and chromatic numbers and the degrees of all vertices of \( \Gamma(C_n) \). Moreover, we prove that \( \Gamma(C_n) \) is a chordal graph, and so a perfect graph.

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1. Introduction

The zero-divisor graph was introduced by Beck on commutative rings in [3]. In Beck’s definition zero element is a vertex in the graph too, later the standard definition of zero-divisor graphs on commutative rings was given by Anderson and Livingston in [1]. Let \( R \) be commutative ring, let \( Z(R) \) be the set of the zero-divisors of \( R \). The zero-divisor graph of \( R \) is an undirected graph \( \Gamma(R) \) with vertices \( Z(R) \setminus \{0\} \), where distinct vertices \( x \) and \( y \) of \( \Gamma(R) \) are adjacent if and only if \( xy = 0 \). Demeyer et. al. have considered this definition for semigroups and they defined and found some basic properties of the zero-divisor graph of a commutative semigroup with zero in [5, 6]. In particular, they proved that the zero-divisor graph of a commutative semigroup with zero is connected. Since then, the zero-divisor graphs of some special classes of commutative semigroups with zero have been researched (see [4, 17]). For non-commutative rings, a directed zero-divisor graph and some undirected zero-divisor graphs were defined by Redmond in [14]. For a ring \( R \) let \( Z_R(T) \) be the set of all two sided zero-divisor elements of \( R \). Then Redmond defines an undirected zero-divisor graph \( \Gamma(R) \) with vertices \( Z_R(T) \setminus \{0\} \), where distinct vertices \( x \) and \( y \) are adjacent with a single edge if and only if \( xy = 0 = yx \) (see [14, Definition 3.4.]). If \( R \) is a non-commutative ring, then \( \Gamma(R) \) need not to be connected (for an example see [14, Figure 9.]) and if \( R \) is a commutative ring then \( \Gamma(R) \) coincide with standard zero-divisor graph of \( R \) in [14]. As Demeyer et. al. we can consider this definition for non-commutative semigroups which have zero.

Let \( X_n = \{1, \ldots, n\} \) finite set with its natural order. Let \( T_n \) be the full transformation semigroup on \( X_n \). We call a transformation \( \alpha : X_n \rightarrow X_n \) order-preserving if \( x \leq y \)
We define a function $f : \mathbb{Z}^d \to \mathbb{Z}^d$ such that each consecutive difference $v_i - v_{i-1}$ lies in $S$ for every $i = 1, \ldots, k$. In the two-dimensional space $\mathbb{Z}^2$, let $(x_1, y_1)$ and $(x_2, y_2)$ be two points with $x_1 \leq x_2$ and $y_1 \leq y_2$. Then a North-East (NE) lattice path from $(x_1, y_1)$ to $(x_2, y_2)$ is a lattice path in $\mathbb{Z}^2$ with steps in $S = \{(0, 1), (1, 0)\}$. (0, 1) steps are called North steps, and (1, 0) steps are called East steps. For $n \in \mathbb{Z}^+$ a Dyck path is a NE lattice path from $(0, 0)$ to $(n, n)$ that lies below but may touch the diagonal $y = x$ (see, for example [13,15]). It is a well-known fact that the number of all the Dyck paths from $(0, 0)$ to $(n, n)$ is

$$C_n = \frac{1}{n+1} \binom{2n}{n},$$

the $n$-th Catalan number. (Andre’s) reflection principle is a tool to prove this fact (see [8]). Moreover, Higgins proved that the cardinality of $\mathbb{C}_n$ is equal to $C_n$ in [10]. Thus there are bijections from $\mathbb{C}_n$ to $DP_n$ which is the set of Dyck paths from $(0, 0)$ to $(n, n)$. Since bijections from $\mathbb{C}_n$ to $DP_n$ are useful throughout this paper, we state and prove this well-known fact with defining a bijection from $\mathbb{C}_n$ to $DP_n$.

**Proposition 2.1.** Let $DP_n$ be the set of Dyck paths from $(0, 0)$ to $(n, n)$. Then there is a bijection from $\mathbb{C}_n$ to $DP_n$.

**Proof.** We define a function $f : \mathbb{C}_n \to DP_n$ as follows: For the zero element $\theta$, let $\theta f = P_\theta$ where

$$P_\theta : (0, 0) - (1, 0) - \cdots - (n, 0) - (n, 1) - \cdots - (n, n),$$

implies $x\alpha \leq y\alpha$ for all $x, y \in X_n$, and order-decreasing if $x\alpha \leq x$ for all $x \in X_n$. Some properties of the semigroup $\mathbb{C}_n$, which consists of all order-preserving and order-decreasing transformations have been investigated over the last forty years (see, for example [2,10,12]). Moreover, $\mathbb{C}_n$ is called Catalan monoid too. Since $1\alpha = 1$ for every $\alpha \in \mathbb{C}_n$, if we take and fix $\theta \in \mathbb{C}_n$ as the unique constant map, then $\theta\alpha = \alpha\theta = \alpha$ for every $\alpha \in \mathbb{C}_n$, and so $\theta$ is the zero element of $\mathbb{C}_n$. Moreover, it is clear that $\mathbb{C}_n$ is a non-commutative semigroup for $n \geq 3$.

For $n \geq 2$ let $\mathbb{C}_n^* = \mathbb{C}_n \setminus \{\theta\}$. Then we define the following sets

$$L = L(\mathbb{C}_n) = \{\alpha \in \mathbb{C}_n^* | \alpha\beta = \theta \text{ for some } \beta \in \mathbb{C}_n\},$$
$$R = R(\mathbb{C}_n) = \{\alpha \in \mathbb{C}_n^* | \gamma\alpha = \theta \text{ for some } \gamma \in \mathbb{C}_n\}$$
$$T = T(\mathbb{C}_n) = \{\alpha \in \mathbb{C}_n^* | \alpha\beta = \gamma\alpha \text{ for some } \beta, \gamma \in \mathbb{C}_n\} = L \cap R$$

which are called the set of left zero-divisors, right zero-divisors and (two sided) zero-divisors of $\mathbb{C}_n$, respectively. It is known that the cardinality of $\mathbb{C}_n$ is $\frac{1}{n+1} \binom{2n}{n}$ which is called $n$-th Catalan number (see, for example [10]). In this paper we find the left zero-divisors, right zero-divisors and zero-divisors of $\mathbb{C}_n$, and then we find their numbers.

For a semigroup $S$ with zero 0 if $T(S) \setminus \{0\} \neq \emptyset$ where $T(S) = \{z \in S | zx = 0 = yz \text{ for } x, y \in S \setminus \{0\}\}$, then we similarly define the (undirected) zero-divisor graph $\Gamma(S)$ associated with $S$ whose the set of vertices is $T(S) \setminus \{0\}$ with distinct two vertices joined by an edge in case $xy = 0 = yz$ for some $x, y \in T(S) \setminus \{0\}$. Notice that $\theta \in T(\mathbb{C}_n)$ for all $n \geq 2$, but $T(\mathbb{C}_n) \setminus \{\theta\} \neq \emptyset$ if $n \geq 3$. Moreover, $\Gamma(\mathbb{C}_3)$ is a graph with exactly one vertex and no edge. In this paper, we prove that $\Gamma(\mathbb{C}_n)$ is a connected graph and then we find the diameter, radius, girth, domination number, clique and chromatic numbers and the degrees of all vertices of $\Gamma(\mathbb{C}_n)$ for $n \geq 4$. Moreover, we prove that $\Gamma(\mathbb{C}_n)$ is a chordal graph, and so a perfect graph for $n \geq 4$.

For semigroup terminology see [9,11] and for graph theoretical terminology see [16].
which is clearly a Dyck path. For any $\alpha \in \mathcal{C}_n \setminus \{\theta\}$ and for any $i \in X_n$ let $h_i(\alpha)$ be the horizontal line from $(i-1, i\alpha - 1)$ to $(i, i\alpha - 1)$. For every $i \in X_n$, since $i\alpha \leq i$, $h_i(\alpha)$ does not cross the diagonal. Notice that if $i\alpha = (i+1)\alpha$ for any $1 \leq i \leq n-1$, then $h_i(\alpha)$ and $h_{i+1}(\alpha)$ are adjacent lines, and that if $i\alpha \neq (i+1)\alpha$ then $h_i(\alpha)$ lower than $h_{i+1}(\alpha)$. Since $\alpha \neq \theta$ there exists at least one $i \in X_{n-1}$ such that $h_i(\alpha)$ lower than $h_{i+1}(\alpha)$. Suppose that there exist $k$ many lower horizontal lines, say $h_{i_1}(\alpha), \ldots, h_{i_k}(\alpha)$ with $i_1 < \cdots < i_k < n$. Notice that $i_1\alpha = 1$. Then let $\alpha f = P_\alpha$ where

$$P_\alpha = (0,0) - \cdots - (i_1-1,0) - (i_1,0) - (i_1,1) - \cdots - (i_1, i_2\alpha - 1) - (i_2, i_2\alpha - 1) - \cdots - (i_2, i_3\alpha - 1) - (i_2, i_3\alpha - 1) - \cdots - (i_{k-1}, i_k\alpha - 1) - (i_{k-1}+1, i_k\alpha - 1) - \cdots - (i_{k-1}+1, i_k\alpha - 1) - (i_k, i_k\alpha - 1) - \cdots - (i_k, n\alpha - 1) - (i_k+1, n\alpha - 1) - \cdots - (n, n\alpha - 1) - (n, n\alpha) - \cdots - (n, n),$$

which is clearly a Dyck path.

Let $\alpha$ and $\beta$ be distinct two elements in $\mathcal{C}_n$. Then there exists at least one $2 \leq i \leq n$ such that $i\alpha \neq i\beta$, and so the horizontal lines $h_i(\alpha)$ and $h_i(\beta)$ are different. Thus $f$ is injective.

For any $P \in DP_n$, there are $n$ many horizontal lines of length $1$ in $P$, say $h_1, h_2, \ldots, h_n$ from left to right. Let $y_1, y_2, \ldots, y_n$ be the ordinates of the horizontal lines, respectively. Notice that $y_1 \leq y_2 \leq \cdots \leq y_n$ and that $y_i = 0, y_i \leq i-1$ for $2 \leq i \leq n$. Then we consider the transformation $\alpha : X_n \rightarrow X_n$ defined by $i\alpha = y_i + 1$ for each $i \in X_n$. Then it is clear that $\alpha \in \mathcal{C}_n$ and $\alpha f = P$, and so $f$ is onto. Therefore, $f$ is a bijection, as required. $\square$

For an example, let

$$\alpha = \left( \begin{array}{cccc} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 3 & 3 & 5 \end{array} \right) \in \mathcal{C}_5,$$

and let $f$ be the function defined in the proof of Proposition 2.1. Then the horizontal lines are

$$h_1(\alpha) = (0,0) - (1,0), \quad h_2(\alpha) = (1,0) - (2,0), \quad h_3(\alpha) = (2,2) - (3,2),$$

$$h_4(\alpha) = (3,2) - (4,2), \quad h_5(\alpha) = (4,4) - (5,4).$$

Moreover, the Dyck path associated with $\alpha$ is

$$\alpha f = (0,0) - (1,0) - (2,0) - (2,1) - (2,2) - (3,2) - (4,2) - (4,3) - (4,4) - (5,4) - (5,5).$$

3. Zero-divisors of $\mathcal{C}_n$

In this section, we find the left zero-divisors, right zero-divisors and two-sided zero-divisors of $\mathcal{C}_n$ and their numbers.

**Lemma 3.1.** For $n \geq 2$, let $L$ be the set of left zero-divisors and $R$ be the set of right zero-divisors of $\mathcal{C}_n$. Then we have

$$L = \{ \alpha \in \mathcal{C}_n \mid n\alpha < n \} \text{ and } R = \{ \alpha \in \mathcal{C}_n \mid 2\alpha = 1 \}.$$

**Proof.** Let $n \geq 2$, and let $\alpha \in \mathcal{C}_n$ such that $n\alpha < n$. If we consider the transformation $\alpha f$ which defined by

$$i\beta = \left\{ \begin{array}{ll} 1 & i \leq n\alpha \\ 2 & i > n\alpha, \end{array} \right.$$ 

then it is clear that $\beta \in \mathcal{C}_n^* = \mathcal{C}_n \setminus \{\theta\}$ and $\alpha\beta = \theta$.

Conversely, let $\alpha$ be a left zero-divisor of $\mathcal{C}_n$. Then there exists $\gamma \in \mathcal{C}_n^*$ such that $\alpha\gamma = \theta$. If we assume that $n\alpha = n$, then we have

$$n\gamma = (n\alpha)\gamma = n\theta = 1,$$
and so \( \gamma = \theta \), which is a contradiction. Therefore, the set of all the left-zero divisors of \( C_n \) is \( L \).

Let \( \alpha \in C_n \) such that \( 2\alpha = 1 \). If we consider the transformation which defined by

\[
i\lambda = \begin{cases} 
1 & i = 1 \\
2 & i \neq 1,
\end{cases}
\]

then it is clear that \( \lambda \in C_n^* = C_n \setminus \{\theta\} \) and \( \lambda\alpha = \theta \).

Conversely, let \( \alpha \) be a right zero-divisor of \( C_n \). Then there exists \( \mu \in C_n^* \) such that \( \mu\alpha = \theta \). If we assume that \( 2\alpha \neq 1 \), then we have \( 2\alpha = 2 \) and \( i\alpha \geq 2 \) for every \( 2 \leq i \leq n \). Since the equation \((i\mu)\alpha = i\theta = 1\), we must have \( i\mu = 1 \) for every \( 1 \leq i \leq n \), and so \( \mu = \theta \), which is a contradiction. Therefore, the set of all the right-zero divisors of \( C_n \) is \( R \). \( \square \)

**Lemma 3.2.** For \( n \geq 2 \) let \( L \) be the set of left zero-divisors and \( R \) be the set of right zero-divisors of \( C_n \). Then we have

\[
|L| = |R| = \frac{3}{n+1}\binom{2n-2}{n}.
\]

**Proof.** For \( n \geq 2 \) let \( A = \{\alpha \in C_n \mid n\alpha = n\} \). If we consider the function \( f : A \to C_{n-1} \) defined by \( \alpha f = \alpha|X_{n-1} \) for every \( \alpha \in A \), then it is clear that \( f \) is a bijection, and so \( |A| = |C_{n-1}| \). Then it follows from Lemma 3.1 that \( L = C_n \setminus A \), and so

\[
|L| = \frac{1}{n+1}\binom{2n}{n} - \frac{1}{n+1}\binom{2n-2}{n-1} = \frac{3}{n+1}\binom{2n-2}{n}.
\]

Let \( B = \{\alpha \in C_n \mid 2\alpha = 2\} \). For every \( \alpha \in B \) if we consider the transformation \( \hat{\alpha} : X_{n-1} \to X_{n-1} \) defined by \( i\hat{\alpha} = (i+1)\alpha - 1 \), then it is clear that \( \hat{\alpha} \in C_{n-1} \). Moreover, if we consider the function \( g : B \to C_{n-1} \) defined by \( \alpha g = \hat{\alpha} \) for every \( \alpha \in B \), then it is also clear that \( g \) is a bijection, and so \( |B| = |C_{n-1}| \). Similarly, it follows from Lemma 3.1 that \( |R| = \frac{3}{n+1}\binom{2n-2}{n} \), as required. \( \square \)

If \( T \) is the set of all two sided zero-divisors of \( C_n \), then it is clear that

\[
T = L \cap R = \{\alpha \in C_n \mid 2\alpha = 1 \text{ and } n\alpha < n\}.
\]

Thus, if \( n = 2 \) then \( T = \{\theta\} \), and if \( n \geq 3 \) then \( T \setminus \{\theta\} \neq \emptyset \).

**Lemma 3.3.** For \( n \geq 3 \) and \( T \) be the two sided zero-divisors set of \( C_n \). If \( n = 3 \) then \( |T| = 2 \) and if \( n \geq 4 \) then

\[
|T| = \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n}.
\]

**Proof.** For \( n = 3 \) it is clear that \( |T| = 2 \). For \( n \geq 4 \) let \( A = \{\alpha \in C_n \mid n\alpha = n\} \) and \( B = \{\alpha \in C_n \mid 2\alpha = 2\} \). Then since \( T = L \cap R \), it follows from Lemma 3.1 that

\[
T = (C_n \setminus A) \cap (C_n \setminus B) = C_n \setminus (A \cup B).
\]

Since \( A \cap B = \{\alpha \in C_n \mid 2\alpha = 2 \text{ and } n\alpha = n\} \), we similarly define a bijection from \( A \cap B \) to \( C_{n-2} \) so that the cardinality of \( A \cap B \) is Catalan number \( C_{n-2} \). Therefore,

\[
|T| = |C_n| - |A| - |B| + |A \cap B| = C_n - 2C_{n-1} + C_{n-2} = \frac{3}{n+1}\binom{2n-2}{n} - \frac{3}{n-3}\binom{2n-4}{n},
\]

as required. \( \square \)
4. Zero-divisor graph of $\mathbb{C}_n$

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an undirected graph, $V(\Gamma)$ denotes the vertex set of $\Gamma$ and $E(\Gamma)$ denotes the edge set of $\Gamma$. If $\Gamma$ contains no loops or multiple edges then $\Gamma$ is called a simple graph. In this section we shall assume that $\Gamma$ is a simple graph. For distinct two vertices $u, v \in V(\Gamma)$ if there exist distinct vertices $v_0, v_1, \ldots, v_n \in V(\Gamma)$ such that $v_0 = u$, $v_n = v$ and $v_{i-1} - v_i$ is an edge in $E(\Gamma)$ for each $1 \leq i \leq n$, then $u - v_1 - \cdots - v_{n-1} - v$ is called a path from $u$ to $v$ of length $n$ in $\Gamma$. For every distinct two vertices $u, v \in V(\Gamma)$ if there exits a path from $u$ to $v$, then $\Gamma$ is called a connected graph. Let $u, v \in V(\Gamma)$ and let $u, v$ be different vertices, then the length of the shortest path between $u$ and $v$ in $\Gamma$ is denoted by $d_\Gamma(u, v)$. The eccentricity of a vertex $v$ in a connected simple graph $\Gamma$ is denoted by $\text{ecc}(v)$ and

$$\text{ecc}(v) = \max\{d_\Gamma(u, v) \mid u \in V(\Gamma)\}.$$ 

The diameter $\text{diam}(\Gamma)$, the radius $\text{rad}(\Gamma)$ and the central vertex set $C(\Gamma)$ of $\Gamma$ defined by

\[
\begin{align*}
\text{diam}(\Gamma) &= \max\{\text{ecc}(v) \mid v \in V(\Gamma)\}, \\
\text{rad}(\Gamma) &= \min\{\text{ecc}(v) \mid v \in V(\Gamma)\} \quad \text{and} \\
C(\Gamma) &= \{v \in V(\Gamma) \mid \text{ecc}(v) = \text{rad}(\Gamma)\},
\end{align*}
\]

respectively. The degree of a vertex $v \in V(\Gamma)$ is denoted by $\text{deg}_\Gamma(v)$ and it is the number of adjacent vertices to $v$ in $\Gamma$. Moreover $\Delta(\Gamma)$ shows that the maximum degree and $\delta(\Gamma)$ shows that minimum degree among all the degrees in $\Gamma$.

Let $D$ be a non-empty subset of the vertex set $V(\Gamma)$ of $\Gamma$. For each vertices of $\Gamma$, if the vertex in $D$ or the vertex is adjacent to $D$ then $D$ is called a dominating set for $\Gamma$. The domination number of $\Gamma$ is

$$\min\{|D| \mid D \text{ is a dominating set of } \Gamma\}$$

and this number is denoted by $\gamma(\Gamma)$. In $\Gamma$ the length of shortest cycle is called girth of $\Gamma$ and it is denoted by $\text{gr}(\Gamma)$, moreover if $\Gamma$ does not contain any cycles, then its girth is defined to be infinity.

Let $C$ be the non-empty subset of $V(\Gamma)$, if $u$ and $v$ are adjacent vertices in $\Gamma$ for every $u, v \in C$, then $C$ is called a clique. Number of all the vertices in any maximal clique of $\Gamma$ is called clique number of $\Gamma$ and it is denoted by $\omega(\Gamma)$. If we colour all the vertices in $\Gamma$ with the rule of no two adjacent vertices have the same colour, then the minimum number of colours needed to colour of $\Gamma$ is called chromatic number of $\Gamma$, it is denoted by $\chi(\Gamma)$.

Let $V' \subseteq V(\Gamma)$. The (vertex) induced subgraph $\Gamma' = (V', E')$ is a subgraph of $\Gamma$ and its vertex set is $V'$, moreover its edge set consists of all of the edges in $E(\Gamma)$ that have both endpoints in $V'$. If $\chi(\Lambda) = \omega(\Lambda)$ for each induced subgraph $\Lambda$ of $\Gamma$, in this case $\Gamma$ is called a perfect graph. A chordal graph is a simple graph, it does not contain an induced cycle of length 4 or more. Thus in chordal graphs every induced cycle has exactly three vertices.

In this section we prove that $\Gamma(\mathbb{C}_n)$ is a connected graph and then we find the diameter, radius, girth, domination number, clique number and chromatic numbers and the degrees of all vertices of $\Gamma(\mathbb{C}_n)$ for $n \geq 4$. Moreover, we prove that $\Gamma(\mathbb{C}_n)$ is a chordal graph, and so a perfect graph for $n \geq 4$.

For $n \geq 2$ and $\alpha \in \mathbb{C}_n^\ast$, let $k_\alpha$ be the element of $X_{n-1}$ such that $k_\alpha \alpha = 1$ and $(k_\alpha + 1)\alpha \neq 1$, and moreover, let $t_\alpha = n\alpha$. For $n \geq 3$ and for $\alpha \in T^* = T \setminus \{\theta\}$ observe that

$$2 \leq k_\alpha, t_\alpha \leq n - 1.$$

**Lemma 4.1.** Let $n \geq 3$ and $\alpha, \beta \in T^*$. Then $\alpha \beta = \theta$ if and only if $t_\alpha \leq k_\beta$. In particular, $\alpha^2 = \theta$ if and only if $t_\alpha \leq k_\alpha$.

**Proof.** ($\Rightarrow$) For $n \geq 3$ and for $\alpha, \beta \in T^*$ let $\alpha \beta = \theta$. Assume that $t_\alpha > k_\beta$. Then $1 = n(\alpha \beta) = (n\alpha)\beta = t_\alpha \beta \geq (k_\beta + 1)\beta \geq 2$, which is a contradiction. Thus $t_\alpha \leq k_\beta$. 

($\Leftarrow$) For $n \geq 3$ and for $\alpha, \beta \in T^*$ let $t_\alpha = k_\beta$. Assume that $t_\alpha \neq k_\beta$. Then $1 = n(\alpha \beta) = (n\alpha)\beta = t_\alpha \beta \neq 2$, which is a contradiction. Thus $t_\alpha \leq k_\beta$. 


(⇐) For $n \geq 3$ and for $\alpha, \beta \in T^*$ let $t_\alpha \leq k_\beta$. Since $n(\alpha\beta) = (n\alpha)\beta = t_\alpha\beta = 1$, $\alpha\beta = \theta$, as required.

As a result, for $n \geq 3$ and for $\alpha, \beta \in T^*$, $\alpha\beta = \theta = \beta\alpha$ if and only if $t_\alpha \leq k_\beta$ and $t_\beta \leq k_\alpha$.

Since $\Gamma(\mathcal{E}_3)$ is a graph which has exactly only one vertex and no edge, from now on we only consider the case $n \geq 4$. Moreover, for convenience, we use $\Gamma$ instead of $\Gamma(\mathcal{E}_n)$. From Lemma 3.3 we have the following immediate corollary.

**Corollary 4.2.** For $n \geq 4$, $|V(\Gamma)| = \frac{3}{n+1}(\frac{2n-2}{n}) - \frac{3}{n-3}(\frac{2n-4}{n}) - 1$.

For $n \geq 4$ we fix the following zero-divisor of $\mathcal{E}_n$:

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 1 & \cdots & 1 & 2 \end{pmatrix}.$$  \hspace{1cm} (4.1)

**Lemma 4.3.** $\Gamma$ is a connected graph for each $n \geq 4$. In fact $\gamma(\Gamma) = 1$ for $n \geq 4$.

**Proof.** For $n \geq 4$ consider $\pi$ defined in (4.1). For each $\alpha \in T^* \setminus \{\pi\}$ it follows from Lemma 4.1 that $\alpha\pi = \theta = \pi\alpha$ since $t_\alpha \leq n - 1$ and $2 \leq k_\alpha$. Thus every element in $T^* \setminus \{\pi\}$ is adjacent to $\pi$ in $\Gamma$, so $\Gamma$ is connected and $\gamma(\Gamma) = 1$. \hspace{1cm} \Box

**Lemma 4.4.** $\text{diam}(\Gamma) = 2$ and $\text{rad}(\Gamma) = 1$ for $n \geq 4$. Moreover, $C(\Gamma) = \{\pi\}$.

**Proof.** For $n \geq 4$ if we consider $\pi$ defined in (4.1), then it is clear that $\text{diam}(\Gamma) \leq 2$ and $\text{rad}(\Gamma) = 1$. To show that $\text{diam}(\Gamma) = 2$ consider two elements $\alpha$ and $\beta$ in $T^* \setminus \{\pi\}$ such that $2 \leq k_\alpha \leq n - 2$ and $t_\beta = n - 1$. Now it follows from Lemma 4.1 that $\beta\alpha \neq \theta$, and so $\alpha$ and $\beta$ are not adjacent vertices in $\Gamma$. Thus $\text{ecc}(\alpha) = 2$, and so $\text{diam}(\Gamma) = 2$.

In addition consider two elements $\alpha$ and $\beta$ in $T^* \setminus \{\pi\}$ such that $3 \leq t_\alpha \leq n - 1$ and $k_\beta = 2$. Similarly, from Lemma 4.1, $\alpha\beta \neq \theta$, and so $\alpha$ and $\beta$ are not adjacent vertices in $\Gamma$. Thus $\text{ecc}(\alpha) = 2$, and so

$$C(\Gamma) = \{\alpha \in T^* \mid k_\alpha = n - 1 \text{ and } t_\alpha = 2\} = \{\pi\},$$

as required. \hspace{1cm} \Box

**Theorem 4.5.** $\text{gr}(\Gamma) = \begin{cases} \infty & \text{if } n = 4 \\ 3 & \text{if } n \geq 5. \end{cases}$

**Proof.** Since $\Gamma(\mathcal{E}_4)$ is isomorphic to the following graph

```
  o---o---o
  |   |   |
  o   o   o
```

$\text{gr}(\Gamma(\mathcal{E}_4)) = \infty$. For $n \geq 5$ if we consider $\alpha$ and $\beta$ in $T^* \setminus \{\pi\}$ such that $k_\alpha = 3$, $t_\alpha = 2$, $k_\beta = 2$ and $t_\beta = 3$. Notice that $\alpha \neq \pi$ since $n \geq 5$. Then it follows from Lemma 4.1 that $\alpha$ and $\beta$ are adjacent vertices in $\Gamma$. Thus $\pi - \alpha - \beta - \pi$ is a cycle of length 3 in $\Gamma$. Therefore, $\text{gr}(\Gamma) = 3$ since $\Gamma$ is simple. \hspace{1cm} \Box

**Theorem 4.6.** For $n \geq 4$ and for $\alpha \in T^* = V(\Gamma)$ let $k = k_\alpha$ and $t = t_\alpha$. Then

$$\text{deg}_\Gamma(\alpha) = \begin{cases} \binom{n-t+k-1}{k-1} - 1 & \text{if } k < t \\ \binom{n-t+k-1}{k-1} - 2 & \text{if } k = t \text{ or } k = t+1 \\ \binom{n-t+k-1}{k-1} - \binom{n-t+k-1}{k-t-2} - 2 & \text{if } k > t+1. \end{cases}$$
Proof. For $n \geq 4$ and for $\alpha \in T^*$ let $k = k_\alpha$ and $t = t_\alpha$. For any $\beta \in T^*$ suppose that $\alpha \beta = \theta = \beta \alpha$. It follows from Lemma 4.1 that $t_\beta \leq k$ and $t \leq k_\beta$. Let

$$A_\alpha = \{ \beta \in T^* \mid t_\beta \leq k \text{ and } t \leq k_\beta \}.$$ 

Thus if $\alpha^2 \neq \theta$, then the degree of $\alpha$ is $|A_\alpha|$, and if $\alpha^2 = \theta$, then the degree of $\alpha$ is $|A_\alpha| - 1$.

Consider the elements of $DP_n$, the set of all the Dyck paths from $(0, 0)$ to $(n, n)$, which have the following form

$$(0, 0) - (1, 0) - \cdots - (t, 0) - \cdots - (n, k - 1) - (n, k) - \cdots - (n, n),$$

and consider the function $f: \mathcal{E}_n \to DP_n$ defined in the proof of Proposition 2.1. Denote the set of all elements in $DP_n$ of the above form by $DP_{n, \alpha}$. For any $P \in DP_{n, \alpha}$ it is clear that $Pf^{-1} \in T$ since $2 \leq t \leq n - 1$ and $2 \leq k \leq n - 1$. If $Q \in DP_{n, \alpha}$ is the path

$$(0, 0) - (1, 0) - \cdots - (t, 0) - \cdots - (n, 0) - (n, 1) - \cdots - (n, k) - (n, k + 1) - \cdots - (n, n)$$

then $Qf^{-1} = \theta$, and so we have $|A_\alpha| = |DP_{n, \alpha}| - 1$.

Suppose that $k \leq t + 1$. Then the $NE$ lattice paths from $(t, 0)$ to $(n, k - 1)$ do not cross the diagonal, and so $|DP_{n, \alpha}| = (\frac{n + t + k - 1}{k - 1} - 1)$. If $k < t$ then $\alpha^2 \neq \theta$, and so $\text{deg}_{\Gamma}(\alpha) = (\frac{n + t + k - 1}{k - 1} - 1)$. If $k = t$ or $k = t + 1$ then $\alpha^2 = \theta$, and so $\text{deg}_{\Gamma}(\alpha) = (\frac{n + t + k - 1}{k - 1} - 2).

Suppose that $k > t + 1$. Then $\alpha^2 = \theta$ and some of the $NE$ lattice paths from $(t, 0)$ to $(n, k - 1)$ do cross the diagonal. Let us find the number of $NE$ lattice paths which crossing the diagonal. If we use the reflection principle, then $(n, k - 1)$ reflects to $(k - 2, n + 1)$ according to the line $x = y + 1$. Thus the number of those paths are equal to the number of all $NE$ lattice paths from $(t, 0)$ to $(k - 2, n + 1)$ is $(\frac{n + t + k - 1}{k - 1} - 1)$. Therefore, $\text{deg}_{\Gamma}(\alpha) = (\frac{n + t + k - 1}{k - 1} - 1) - (\frac{n + t + k - 1}{k - 1} - 2)$, as required.

For $n \geq 4$ if we consider $\pi$ defined in (4.1) and simplicity of $\Gamma$, then it is clear that $\Delta(\Gamma) = |T^*| - 1$. Moreover, if we consider $\alpha$ in $V(\Gamma)$ such that $t_\alpha = n - 1$ and $k_\alpha = 2$, then it follows from Theorem 4.6 that $\text{deg}_{\Gamma}(\alpha) = 1$. Thus we have the following immediate corollary.

Corollary 4.7. $\Delta(\Gamma) = \frac{3}{n+1}(\binom{2n-2}{n} - \binom{2n-4}{n}) - 2$ and $\delta(\Gamma) = 1$ for $n \geq 4$.

Theorem 4.8. For $n \geq 4$ $\Gamma$ is a chordal, and so a perfect graph.

Proof. For $n \geq 4$ assume that there exists an induced subgraph of $\Gamma$ which is an $m$-cycle with $m \geq 4$. Let $v_1 - v_2 - \cdots - v_m - v_1$ be an $m$-cycle in $\Gamma$ with $m \geq 4$. Let $k_i = k_{v_i}$ and $t_i = t_{v_i}$ for each $1 \leq i \leq m$. Moreover, let $k = \min\{k_i \mid 1 \leq i \leq m\}$. Without losing generality assume that $k = k_1$. Then $t_2 \leq k$ and $t_m \leq k$ since $v_1$ is adjacent to both $v_2$ and $v_m$. Since $t_2 \leq k \leq k_m$ and $t_m \leq k \leq k_2$, it follows that $v_2$ and $v_m$ are adjacent vertices, which is a contradiction.

It is well-known that every chordal graph is a perfect graph (see, for example [7,16]).

Lemma 4.9. For $n \geq 2$ let $A = \{ \alpha \in \mathcal{E}_n^+ \mid k_\alpha = k \text{ and } t_\alpha = t \}$. Then

$$|A| = \begin{cases} 1 & \text{if } t = 2 \\ (\frac{n - k + t - 3}{t - 2}) & \text{if } 2 < t \leq k + 1 \\ (\frac{n - k + t - 3}{t - 2}) - (\frac{n - k + t - 3}{t - k + 2}) & \text{if } t > k + 1. \end{cases}$$

Proof. For $n \geq 2$ let $A = \{ \alpha \in \mathcal{E}_n^+ \mid k_\alpha = k \text{ and } t_\alpha = t \}$. Consider the elements of $DP_n$, the set of Dyck paths from $(0, 0)$ to $(n, n)$, which have the following form

$$(0, 0) - \cdots - (k, 0) - (k, 1) - \cdots - (n - 1, t - 1) - (n, t - 1) - (n, t) - \cdots - (n, n).$$

Denote the set of all elements in $DP_n$ of the above form by $DP_{n,k,t}$. Thus it follows from Proposition 2.1 and its proof that $|A| = |DP_{n,k,t}|$. 


For $t = 2$ there is only one Dyck path, namely
\[(0, 0) - \cdots - (k, 0) - (k, 1) - \cdots - (n - 1, 1) - (n, 1) - \cdots - (n, n),\]
and so $|A| = 1$.

Suppose that $2 < t \leq k + 1$. Then it is clear that all the $NE$ lattice paths from $(k, 1)$ to $(n - 1, t - 1)$ do not cross the diagonal, and so $|A| = \binom{n - k + t - 3}{t - 2}$.

Suppose that $t > k + 1$. Then some of the $NE$ lattice paths from $(k, 1)$ to $(n - 1, t - 1)$ cross the diagonal. Let us find the number of $NE$ lattice paths which cross the diagonal. If we use the reflection principle, then $(n - 1, t - 1)$ reflects to $(t - 2, n)$ according to the line $y = x + 1$. Thus the number of those paths is equal to the number of all $NE$ lattice paths from $(k, 1)$ to $(t - 2, n)$ which is $\binom{n - k + t - 3}{t - k - 2}$. Therefore, $|A| = \binom{n - k + t - 3}{t - k - 2}$.

Notice that in Lemma 4.9 for $n \geq 3$ if $2 \leq k \leq n - 1$ and $2 \leq t \leq n - 1$, then $A \subseteq T^*$. Let $A(n, k, t) = \{\alpha \in C_n \mid k_\alpha = k$ and $t_\alpha = t\}$. Then we have a partition of $T^*$, namely
\[T^* = \bigcup_{k=2}^{n-1} \left( \bigcup_{t=2}^{n-1} A(n, k, t) \right)\]
for $n \geq 3$. Thus we have the following immediate corollary.

**Corollary 4.10.** For $n \geq 5$

\[
n - 2 + \sum_{r=0}^{n-4} \left( \sum_{k=2+r}^{n-1} \binom{n - k + r}{r + 1} \right) + \sum_{s=1}^{n-4} \left( \sum_{k=2}^{s+1} \left( \binom{n - k + s}{s + 1} - \binom{n - k + s}{s + 1 - k} \right) \right) = \frac{3}{n + 1} \left( \frac{2n - 2}{n} \right) - \frac{3}{n - 3} \left( \frac{2n - 4}{n} \right) - 1.
\]

**Lemma 4.11.** Let $K$ be a complete subgraph of $\Gamma$ and $V(K) = \{v_1, v_2, \ldots, v_m\}$. Let $k_i = k_{v_i}$ and $t_i = t_{v_i}$ for $1 \leq i \leq m$. Then there is at most one $1 \leq i \leq m$ such that $t_i > k_i$.

**Proof.** Assume that there are two distinct $i$ and $j$ such that $t_i > k_i$ and $t_j > k_j$. Without loss of generality suppose that $k_i \leq k_j$. Since $v_i$ and $v_j$ are adjacent vertices in $K$, we have $t_j \leq k_i$. Thus $t_j \leq k_i \leq k_j$, which is a contradiction. \qed

**Theorem 4.12.** For $n \geq 4$

\[
\chi(\Gamma) = \omega(\Gamma) = \max \left\{ \sum_{k=i}^{n-1} \sum_{t=2}^{n-1} |A(n, k, t)| \mid 2 \leq i \leq n - 1 \right\}.
\]

**Proof.** Let $n \geq 4$ and $K$ be a maximal complete subgraph of $\Gamma$ with the vertices set $V(K) = \{v_1, \ldots, v_m\}$, and let $k_i = k_{v_i}$, $t_i = t_{v_i}$ for $1 \leq i \leq m$. From Lemma 4.11, since $K$ is complete, without loss of generality either $t_1 > k_1$ and $t_i \leq k_i$ for all $2 \leq i \leq m$ or $t_i \leq k_i$ for all $1 \leq i \leq m$. Notice that it is possible $t_x = t_y$ or $k_x = k_y$ for some $1 \leq x \neq y \leq m$.

Suppose that $t_1 > k_1$ and $t_i \leq k_i$ for all $2 \leq i \leq m$. Let $C$ be the set of all the different numbers among $t_2, t_3, \ldots, t_m$ and let $D$ be the set of all the different numbers among $k_2, k_3, \ldots, k_m$. From Lemma 4.1, since $t_1 \leq k_1$ and $t_1 \leq k_i$ for every $2 \leq i \leq m$, \(\min(D) - \max(C) \geq 1\). Let $\min(D) = p$ (notice that $p \geq 2$), and let $\Omega$ be the induced subgraph of $\Gamma$ with vertex set $V(\Omega) = (V(K) \setminus \{v_1\}) \cup A(n, p, p)$. Then it is clear that $\Omega$ is a complete subgraph, and that $|V(\Omega)| \geq |V(K)|$. Moreover, it is also clear that $t_v \leq k_v$ for all $v$ in $\Omega$. Therefore, there exists at least one maximal complete subgraphs $\Gamma$ of $K$ such that $t_v \leq k_v$ for all $v$ in $K$.

Let $K$ be a maximal complete subgraph of $\Gamma$ with the vertices set $V(K) = \{v_1, \ldots, v_m\}$, and let $k_i = k_{v_i}$, $t_i = t_{v_i}$ and $t_i \leq k_i$ for $1 \leq i \leq m$. For any $1 \leq i \leq m$ since $K$ is a complete subgraph of $\Gamma$, $t_i \leq k_j$ for all $j \in \{1, \ldots, m\} \setminus \{i\}$, and so $t_i \leq k_j$ for all $1 \leq j \leq m$. Therefore, $\min(D) \geq \max(C)$ where $C$ is the set of all the different numbers.
among $t_1, t_2, \ldots, t_m$ and $D$ is the set of all the different numbers among $k_1, k_2, \ldots, k_m$. Now assume that $\min(D) \neq \max(C)$. Let $\max(C) = q$, and let $\Omega$ be the induced subgraph of $\Gamma$ with vertex set $V(\Omega) = V(K) \cup A(n, q, q)$. Then it is clear that $\Omega$ is a complete subgraph, and that $|V(\Omega)| > |V(K)|$ which is a contradiction. Thus $\min(D) = \max(C) = r$. From the maximality of the complete subgraph of $\Gamma$ we deduce that $C = \{2, 3, \ldots, r\}$ and $D = \{r, r+1, \ldots, n-1\}$. Thus, $V(K) = \bigcup\{A(n, k, t) \mid k \in D \text{ and } t \in C\}$, and so $|V(K)| = \sum_{k=r}^{n-1} \sum_{t=2}^{r} |A(n, k, t)|$. Therefore, $\omega(\Gamma) = \max \left\{ \sum_{k=i}^{n-1} \sum_{t=2}^{i} |A(n, k, t)| \mid 2 \leq i \leq n-1 \right\}$.

Since $\Gamma$ is perfect graph, $\chi(\Gamma) = \omega(\Gamma)$, as required.

**Example 4.13.** If $\Gamma = \Gamma(\mathbb{C}_6)$ then $|V(\Gamma)| = 61$, $\text{diam}(\Gamma) = 2$, $\text{rad}(\Gamma) = 1$, $\text{gr}(\Gamma) = 3$, $\gamma(\Gamma) = 1$, $\Delta(\Gamma) = 60$, $\delta(\Gamma) = 1$, $\chi(\Gamma) = \omega(\Gamma) = 9$. Moreover, $\text{deg}_\Gamma(\alpha) = 8$ where $\alpha = \left( \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 1 & 3 & 4 & 4 \end{array} \right) \in V(\Gamma)$.

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