COMPUTING THE ADDITIVE STRUCTURE OF INDECOMPOSABLE MODULES OVER DEDEKIND-LIKE RINGS USING GRÖBNER BASES.

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Abstract. We introduce a general constructive method to find a $p$-basis (and the Ulm invariants) of a finite Abelian $p$-group $M$. This algorithm is based on Gröbner bases theory. We apply this method to determine the additive structure of indecomposable modules over the following Dedekind-like rings: $\mathbb{Z}C_p$, where $C_p$ is the cyclic group of order a prime $p$, and the $p-$pullback $\{\mathbb{Z} \to \mathbb{Z}_p \leftarrow \mathbb{Z}\}$ of $\mathbb{Z} \oplus \mathbb{Z}$.

1. Introduction

Let $R$ be an algebra. Finding the additive structure of an $R$-module as an Abelian group associated to a representation is a classical problem solved in a similar way to obtaining the Jordan canonical form of a matrix over a field, see [7, Chapter III] and [2, Chapter 12, §2]. This information is used, for example, to determine the matrices associated to the group representation. This is accomplished by finding a $p$-basis for the torsion part of the group that permits a unique matrix representation for this Abelian finite $p$-group. In 1949, Szekeres started the classification and matrix description of modules over $\mathbb{Z}_pC_p$. Since then it has been studied in detail, see [1, 4, 12, 13, 14, 15, 17]. In [12, 13], Levy studied these modules in the more general context of modules over a pullback of two Dedekind rings with a common field, which he called Dedekind-like rings.

Until now, the simplest way to find the additive structure of an $R$-module consists in writing the relations as a matrix with entries in $\mathbb{Z}$, performing elementary transformations over an Euclidean domain (like $\mathbb{Z}$), and using the division algorithm to write the matrix in a canonical form, see [7 Theorem 16.8]. This approach becomes rather difficult when the generating set is not minimal and there are several relations among the generators. In here, we present a different method that has the advantage of producing different group presentations by writing the relations as polynomials and changing the term orders used to reduce them. Furthermore, we show how to use this procedure to find a good $p$-basis which gives the Ulm invariants [9] of $M$ and also the type of $M$.

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The main contribution of this paper is a constructive method to find a $p$-basis (and the Ulm invariants) of a finite Abelian $p$-group $M$ from a given presentation of $M$ encoding the action of $p$. The algorithm is obtained by noting that there are some invariant properties between the order of elements in an Abelian group and the basis elements of certain toric ideals. To accomplish this, we use several tools from Gröbner bases and chain-modules. Furthermore, this method can be used in general for modules over algebras on $\mathbb{Z}$ and $\mathbb{Z}/p^n$. 

Let $M$ be a finitely generated Abelian group. We assume that $M$ is also finitely presented, that is, $M = \langle C, R \rangle$, where $C$ is a nonminimal finite generating set, and $R$ is a finite set of relations among the elements of $C$, see [11]. For example

$$M = \langle c_1, \ldots, c_n \mid \sum_{j=1}^{q} a_{ij} c_j = 0 \text{ with } a_{ij} \in \mathbb{Z}, \text{ for all } i = 1, \ldots, n \text{ and } j = 1, \ldots, q \rangle.$$

We want to find the torsion-free rank of $M$ and the Ulm invariants of the $p_j$-Sylow subgroups of $M$. This is an old problem, the new aspects in this work are: (1) we use Gröbner bases to solve the problem, and (2) using the notation and classification introduced by Levy, we apply this method to determine the additive structure of indecomposable modules over certain Dedekind-like rings. In this case, the algorithm computes a $p$-basis for the torsion part of the group.

This paper is organized as follows: in Section 2, we introduce toric ideals associated to finitely generated Abelian groups. In Section 3, we give a description of the reduced Gröbner basis of a toric ideal associated to a finitely generated Abelian $p$-group. As a consequence, in Section 4, we obtain an algorithm to compute the $p$-basis and the type of any finite Abelian $p$-group. As an application of this algorithm, in Section 5, we show how to obtain the additive structure of any indecomposable module over $\mathbb{Z}C_p$, where $C_p$ is the cyclic group of order a prime $p$ and over the $p$--pullback $\{\mathbb{Z} \rightarrow \mathbb{Z}_p \leftarrow \mathbb{Z}\}$ of $\mathbb{Z} \oplus \mathbb{Z}$.

2. Gröbner bases associated to finitely generated Abelian groups

We start by reviewing some concepts in finitely generated Abelian group theory.

**Definition 2.1** (type). If $p_1 < \cdots < p_r$ and $M$ is a finitely generated Abelian group, such that

$$M \cong \mathbb{Z}^{s_0} \oplus (\mathbb{Z}_{p_1}^{s_{i_1}} \oplus \cdots \oplus \mathbb{Z}_{p_1}^{s_{i_1^r}}) \oplus \cdots \oplus (\mathbb{Z}_{p_r}^{s_{i_r}} \oplus \cdots \oplus \mathbb{Z}_{p_r}^{s_{i_r^r}}),$$

as an Abelian group, then the type of $M$ is

$$\mathcal{t}(M) = (s_0, s_{i_1}, \ldots, s_{i_1^r}, s_{i_2}, \ldots, s_{i_2^r}, \ldots, s_{i_r}, \ldots, s_{i_r^r}) \in \mathbb{Z}^{n_1 + \cdots + n_r + 1}.$$

The number $s_0$ is the torsion-free rank of $M$, the number $n_i$ is the torsion rank of $M_i$, and the sequence of numbers $s_{i_1}, \ldots, s_{i_1^r}$, are the Ulm invariants of the $p_i$-Sylow subgroup $M_i = \mathbb{Z}_{p_i}^{s_{i_1}} \oplus \cdots \oplus \mathbb{Z}_{p_i}^{s_{i_1^r}}$ of $M$.

**Definition 2.2** ($p$-basis). If $M$ is a $p$-group, for some prime number $p$, a set $B = \{b_1, \ldots, b_d\} \subseteq M$ is called a $p$-basis of $M$ if $M \cong \mathbb{Z}_p \langle b_1 \rangle \oplus \cdots \oplus \langle b_d \rangle$. A set $B$ is a $p$-basis of $M$ if and only if, for all $m \in M$ the sum $m = \sum_{i=1}^{d} l_i b_i$ is unique, where $0 \leq l_i \leq \text{ord}(b_i)$ and $\text{ord}(b_i)$ is the order of $b_i$ in the group $M$, see [12].
Let \( M = \mathbb{Z}^q \bigoplus_{i=1}^q M_i \) be a finitely generated Abelian group, where \( M_i \) is the \( p_i \)-Sylow subgroup of \( M \) with \( p_i \)-rank equal to \( d_i \). Consider a nonminimal generating set \( C_i \) of each \( M_i \), such that, \( C_i \cap C_j = \emptyset \) for all \( i \neq j \), and a generating set \( C_0 \) of \( \mathbb{Z}^q \). If \( C = \bigcup_{i=0}^q C_i = \{c_1, \ldots, c_q\} \), where \( q \geq \sum_{i=1}^q d_i + s \), then \( \langle C \rangle = M \). Consider the semigroup homomorphism

\[
\gamma : \mathbb{N}^q \rightarrow M, \quad v = (v_1, \ldots, v_q) \mapsto \sum_{i=1}^q v_ic_i.
\]

Let \( k \) be an infinite field. The previous map lifts to the following short exact sequence

\[
0 \rightarrow \text{Ker}(\gamma) \rightarrow k[x] \xrightarrow{\tilde{\gamma}} k[M] \rightarrow 0,
\]

where \( k[x] = k[x_1, \ldots, x_q] \cong k[\mathbb{N}^q] \) is the polynomial ring in \( q \) indeterminates over \( k \). The monomials in \( k[x] \) are denoted by \( x^a = x_1^{a_1} \cdots x_q^{a_q} \), where \( a = (a_1, \ldots, a_q) \in \mathbb{N}^q \). On the other hand, if \( d = \sum_{i=1}^q d_i \), we have the following isomorphism

\[
k[M] \cong k[t] = k[t_{1d}, \ldots, t_{1d+1}, \ldots, t_{qdn+1}, \ldots, t_{qdn+s}] / \langle t_{1d+1} - 1, \ldots, t_{qdn+1} - 1 \rangle,
\]

where \( k_i \) is the order of the corresponding element in the external direct sum of \( M \). Furthermore, in this external direct sum, the element \( c_i \in C \) can be expressed as a tuple \( c_i = (c_{i1}, \ldots, c_{id}, c_{id+1}, \ldots, c_{id+s}) \). So, we have a homomorphism of semigroup algebras

\[
\tilde{\gamma} : k[x] \rightarrow k[t], \quad x_i \mapsto t^{c_i} = t_{1d}^{c_{i1}} \cdots t_{d+1}^{c_{id}} \cdots t_{d+s}^{c_{id+s}}.
\]

We denote the kernel of \( \tilde{\gamma} \) by \( I_C \). We will show how to obtain a minimal generating set that is a \( p \)-basis of \( M \), from a certain Gröbner basis of this ideal. In the following, we assume that we have a term order \( \prec \) defined in \( k[x] \). Then every nonzero polynomial \( f \in k[x] \) has a unique initial monomial, denoted \( \text{in}_\prec(f) \). Observe that for any \( v = (v_1, \ldots, v_q) \in \mathbb{Z}^q \), we can write \( v = v^+ - v^- \), where \( v^+ = (v_1^+, \ldots, v_q^+) \) and \( v^- = (v_1^-, \ldots, v_q^-) \) are nonnegative integer tuples. Denote by \( \text{Ker}(\gamma) \) the subgroup of \( \mathbb{Z}^q \) consisting of all elements \( v \) such that \( \gamma(v^+) = \gamma(v^-) \). Let

\[
P(\text{Ker}(\gamma)) = \{x^{v^+} - x^{v^-} | v \in \text{Ker}(\gamma)\}.
\]

The following lemma follows immediately from [16] Lemma 4.1.

**Lemma 2.3.** The ideal \( I_C \) is generated as a \( k \)-vector space by the set \( P(\text{Ker}(\gamma)) \).

Recall that \( C \) is a nonminimal finite generating set of \( M \). We assume that there exists a finite set of defining relations \( R \) for \( C \) in \( M \). We use the notation \( v^+ \) and \( v^- \) to write the relations as \( \sum_{i=1}^q v_i^+ c_i = \sum_{i=1}^q v_i^- c_i \). These relations induce a subset of vectors in \( \mathbb{Z}^q \) and a subset of polynomials in \( I_C \)

\[
\mathcal{R} = \{v \in \mathbb{Z}^q : \sum_{i=1}^q v_i^+ c_i = \sum_{i=1}^q v_i^- c_i \text{ is in } \mathcal{R} \} \subset \mathbb{Z}^q,
\]

\[
P(\mathcal{R}) = \{P(v) = x^{v^+} - x^{v^-} | v \in \mathcal{R}\} \subset I_C.
\]

Let \( G_{\mathcal{R}} \) denote the reduced Gröbner basis of the ideal generated by \( P(\mathcal{R}) \), with respect to the order \( \prec \). Similarly, this Gröbner basis induces the set \( \mathcal{R}_{\mathcal{R}} \) of tuples
\[ \mathcal{R}_G = \{ v \in \mathbb{Z}^q \mid x^{v^+} - x^{v^-} \in \mathcal{G}_R \} \]

and the set \( \mathcal{R}_G \) of relations

\[ R_G = \{ \sum_{i=1}^{q} v_i^+ c_i = \sum_{i=1}^{q} v_i^- c_i \text{ such that } v \in \mathcal{R}_G \} \].

**Proposition 2.4.** Let \( v_k \in \mathcal{R} \) with \( in_<(P(v_k)) = x^{v_k^+} \) for \( k = 1, 2 \). Also let \( w_1, w_2 \in \mathbb{N}^q \) such that \( w_1 + v_1^+ = w_2 + v_2^+ \). If \( P(v) = x^{w_1} P(v_1) - x^{w_2} P(v_2) \) then \( v = v_1 - v_2 \).

**Proof.** We have \( P(v) = x^{w_1} P(v_1) - x^{w_2} P(v_2) = x^{w_2+v_2^-} - x^{w_1+v_1^-} \). But

\[ \gamma(w_1 + v_1^+) = \gamma(w_2 + v_2^+) = \gamma(w_1 + v_1^-) = \gamma(w_2 + v_2^-), \]

and \( v = (w_2+v_2^-)-(w_1+v_1^-) = (w_2+v_2^-)-(w_1+v_1^-)+(w_1+v_1^+)-(w_2+v_2^+)=v_1-v_2. \)

\[ \Box \]

**Theorem 2.5.** The set \( \mathcal{R}_G \) is a set of relations for \( C \) in \( M \).

**Proof.** First observe that, if \( v \in \text{Ker}(\gamma) \), then \( \sum_{i=1}^{q} v_i c_i = 0 \). So \( v \in \mathcal{R} \) and since \( \langle \mathcal{R} \rangle \subset \text{Ker}(\gamma) \), then the set \( \mathcal{R} \) generates the subgroup \( \text{Ker}(\gamma) \). This implies that the ideal \( I_C \) is generated by \( P(<\mathcal{R}> \rangle) \). Next, we use the Buchberger algorithm to obtain the reduced Gröbner basis of \( I_C \) from \( P(<\mathcal{R}> \rangle) \).

By Proposition 2.4, we have that the \( S \)-polynomial \( S(P(u_1), P(u_2)) = P(v) \) satisfies \( v = u_1 - u_2 \in \text{Ker}(\gamma) \). Let \( S \) be the set of all nonzero \( S \)-polynomials obtained in the Buchberger algorithm and let \( S = \{ v \in \mathbb{Z}^q \mid P(v) \in S \} \). Clearly \( \langle \mathcal{R} \rangle = \langle \mathcal{R} \rangle \cup S \). We denote by \( \mathcal{R}' = \mathcal{R} \cup S \). Now, we reduce the set of polynomials in \( P(<\mathcal{R}> \rangle) \). Suppose \( in_<(P(v)) = x^{v^+} \) divides \( in_<(P(v_1)) = x^{v_1^+} \), with \( v, v_1 \in \mathcal{R}' \).

There exists \( w \in \mathbb{N}^q \) such that \( w + v_1^+ = v_1^- \) and \( P(v_2) = P(v_1) - x^w P(v) \). Hence \( v_2 = v_1 - v \in \mathcal{R}' \) by Proposition 2.4. Then \( \langle \mathcal{R}_G \rangle = \langle \mathcal{R} \rangle \). This proves our claim. \( \Box \)

### 3. The reduced presentation of \( M \)

Given a generating set \( C \) of a finite Abelian \( p \)-group \( M \), one can obtain a set of relations by studying the action of \( p \) over the elements in \( C \). In the last section, we saw that any Gröbner basis of \( I_C \) gives a set of relations for \( M \). In this section, we describe a particular Gröbner basis that gives a \( p \)-basis of \( M \). We assume that the elements of \( C \) have orders \( \text{ord}(c_1) \geq \cdots \geq \text{ord}(c_q) \). Consider the following chain of subgroups of \( M \)

\[ \langle c_1 \rangle \subseteq \langle c_1, c_2 \rangle \subseteq \cdots \subseteq \langle c_1, c_2, \ldots, c_s \rangle \cdots \subseteq \langle C \rangle = M. \]

For \( s \geq 2 \), let \( r_s = \min \{ k \mid p^k c_s \in \langle c_1, c_2, \ldots, c_{s-1} \rangle \} \). There are two possibilities, either \( r_s < \text{ord}(c_s) \), or \( r_s = \text{ord}(c_s) \), in this case, \( p^{r_s} c_s = 0 \in \langle c_1, c_2, \ldots, c_s \rangle \).

Thus, we have the following set of relations

\[ R_p = \{ p^{r_s} c_1 = 0, \ p^{r_s} c_s = p^{r_s} \sum_{i<s} a_{st} c_t, \text{ where } a_{st} \in \mathbb{Z}, \text{ for } 2 \leq s \leq q \} \].

**Proposition 3.1.** The relations \( R_p \) together with the set \( C \) is a presentation of the \( p \)-group \( M \).
Proof. Suppose \( \sum_{t=1}^{q} c_t = 0 \). Dividing \( c_t \) by \( p^{r_t} \), we obtain \( c_t = s_t p^{r_t} + s'_t \) with \( 0 \leq s'_t < p^{r_t} \). Therefore, \( s_t p^{r_t} c_t + \sum_{t=1}^{q-1} c_t = 0 \). Suppose that \( s'_t \neq 0 \). Using the relation \( p^{r_t} c_t = \sum_{t=1}^{q} s'_t c_t + \sum_{t=1}^{q-1} c_t = 0 \). If \( \gcd(s'_t, p) = 1 \), then \( c_t = \sum_{t=q-1}^{q} c_t = \sum_{t=1}^{q-1} c_t = 0 \). If \( \gcd(s'_t, p) = p \), let \( r' \) be the maximum number such that \( p^{r'} \) divides \( s'_t \). Then \( s'_t = -\sum_{t=1}^{q-1} \ell'_t c_t \). But this is impossible, because \( p^{r'} \) is the minimum with this condition. Thus \( \gcd(s'_t, p) = 1 \).

Repetition of this argument shows that \( \sum_{t=1}^{q} \ell_t c_t = \ell'_1 c_1 = 0 \) with \( \gcd(\ell'_1, p) = 1 \). But this implies that \( c_1 = \ldots = c_q = 0 \) which is impossible. Then \( s'_t = 0 \) and the relation \( \sum_{t=1}^{q} \ell_t c_t = 0 \) is a linear combination of the relations in \( R \).

\[ \text{(3.2)} \]

**Proposition 3.2.** Let \( \prec \) be the lexicographic ordering with \( x_1 \prec x_2 \prec \cdots \prec x_q \). Then, the reduced Gröbner basis of \( I_C \) with respect to \( \prec \) equals

\[ \mathcal{G}_p = \{ x_1^{p^{r_1}} - 1, x_2^{p^{r_2}} - x_2^{a_2 p^{r_2}}, \ldots, x_q^{p^{r_q}} - \prod_{t=1}^{q-1} x_t^{a_t p^{r_t}} \} \]

Proof. Observe that \( \mathcal{G}_p = P(R_p) \). Thus, by Theorem 2.1, \( \mathcal{G}_p \) generates \( I_C \). Furthermore, \( \mathcal{G}_p \) is a reduced Gröbner basis since \( \gcd(\in_{\prec}(p_1), \in_{\prec}(p_2)) = 1 \), for any \( p_1 \) and \( p_2 \) in \( R_p \). This forces all S-polynomials to be zero modulo \( \mathcal{G}_p \), see [6]. \( \square \)

Let \( pR \) be the following set of relations for \( C \) in \( M \)

\[ \text{(3.1)} \]

\[ pR = \{ pc_q = 0, \quad pc_j = \sum_{t \geq j} a_{jt} c_t, \quad \text{for all } 1 \leq j \leq q-1, \quad \text{with } 0 \leq a_{jt} \in \mathbb{Z} \} \]

These relations can be used to find the order of any element in \( M \), since \( pM \) is the Frattini subgroup of \( M \), [6]. On the other hand, from the action of \( p \), we can find the minimal number of generators of \( M \), that is, the \( p \)-rank of \( M \) by the Burnside Basis Theorem for finite groups \( (M/pM) \), [6]. Let \( d \) be the \( p \)-rank of \( M \). For each \( t \geq 2 \), let \( D_t \) be the set

\[ D_t = \{ c_t = \sum_{j=1}^{t-1} a_{ts} c_j \mid 0 \leq a_{ts} \leq \text{ord}(c_j) \} \]

If \( b_t \) is the element of maximal order in \( D_t \), then \( p^t = \text{ord}(b_t) \). Therefore, we have the following set of relations, denoted by \( R_{p\text{basis}} \)

\[ \{ p^{r_t} c_t = 0, \quad p^{r_t} c_t = p^{r_t} \sum_{j < t} a_{jt} c_s \text{ for } 2 \leq t \leq d, \quad \text{and } c_t = \sum_{j < d} a_{jt} c_j \text{ for } d < t \leq q \} \]

It is clear that \( M = \langle b_1, b_2, \ldots, b_d \rangle \), so \( R_{p\text{basis}} \) is a set of relations for \( C \) in \( M \). As a corollary of Proposition 3.2, we have

**Corollary 3.3.** Let \( \prec \) be the lexicographic ordering with \( x_1 \prec x_2 \prec \cdots \prec x_q \). Then, the reduced Gröbner basis of \( I_C \) with respect to \( \prec \) equals

\[ \mathcal{G}_{p\text{basis}} = \{ x_1^{p^{r_1}} - 1, \ldots, x_d^{p^d} - \prod_{t=1}^{d-1} x_t^{a_{tt} p^d}, x_{d+1} - \prod_{t=1}^{d} x_t^{a_{tt} d+1}, \ldots, x_q - \prod_{t=1}^{d} x_t^{a_{tt} q} \} \]
Note that $G_{\text{basis}}$ is just a refinement of $G_p$ obtained by setting some of the $p^v$ equal to 1. The next theorem is the key to our algorithm. It says that the generating set obtained from $G_{\text{basis}}$ is actually a $p$-basis of $M$.

**Theorem 3.4.** The set $\mathcal{C} = \{c_1, \ldots, c_d - \sum_{i=1}^{d-1} a_i b_i \}$ is a $p$-basis of $M$.

**Proof.** We have seen that $M = \langle b_1, b_2, \ldots, b_d \rangle = \langle \mathcal{C} \rangle$. Now, we will prove that the sum $\langle b_1 \rangle + \cdots + \langle b_d \rangle$ is actually a direct sum. If $y \in \langle b_1 \rangle \cap \langle b_1, \ldots, b_{t-1} \rangle$. Then $y = \alpha_1 b_1 + \sum_{i=1}^{t-1} \alpha_i b_i$. Thus, $\alpha t b_i = \sum_{j=1}^{t-1} \alpha_j c_j$. The argument preceding this theorem says that $\alpha_i \geq p^{r_i}$, so $\alpha_i = \alpha_i' p^{r_i} + \beta_i$, with $0 \leq \beta_i < p^{r_i}$. Thus $\beta_i c_i = \sum_{j=1}^{t-1} \alpha_j c_j$ which implies $\beta_i = 0$ and $y = \alpha_i' p^{r_i} b_i = 0$. So $M = \bigoplus_{t=1}^{d} \langle b_t \rangle$. □

We can summarize the above results as follows.

**Remark 3.5.**

1. Given a presentation of a finite Abelian $p$-group $M = \langle C, R \rangle$, there exists a term ordering such that the reduced Gröbner basis of the toric ideal $I_C$ gives a $p$-basis for $M$.

2. Given a homomorphism $\overline{\gamma}$ as in (3.1). The presentations for the corresponding finite Abelian group $M$ can be obtained from Gröbner bases of the toric ideal $\text{Ker}(\overline{\gamma})$.

### 4. The $p$-basis Algorithm and the Additive Structure of $M$

Corollary 3.3 gives an explicit description of a reduced Gröbner basis for $I_C$. Moreover, Theorem 3.4 shows that the corresponding set of generators is a $p$-basis of $M$. Nevertheless, we obtained this Gröbner basis from a very special set of relations whose definition was nonconstructive, namely $R_{\text{basis}}$. In particular, this set of relations specified the ordering on the indeterminates for the specific lexicographic order needed in Corollary 3.3. In this section, we put all these results together to compute the invariants of a finite Abelian $p$-group $M$ from a particular presentation.

Let $pR$ be the finite presentation of $M$ introduced in (3.1), that is, assume that the action of $p$ in a generating set $C$ is known. Following Remark 3.5, we need to find an ordering of the indeterminates, such that, the Gröbner basis with respect to the corresponding lexicographic order has the form (3.2). Note that there might be several such orderings. In the last section, we saw that if $\text{ord}(c_i) < \text{ord}(c_j)$ then $x_j < x_i$. We also need to break ties among the elements in $C$ with the same order in the group.

In practice, one first break ties arbitrarily. If the Gröbner basis has the required form, we are done. Otherwise, there is an element in the Gröbner basis of the form $x_j^q - x_i \prod_{i=1, i \neq j}^{q-1} x_i^{a_j v_i}$, with $\text{ord}(c_j) = \text{ord}(c_i)$. In this case, we need to invert the order of $x_j$ to $x_i$. This process eventually terminates, moreover; it effectively gives the desired Gröbner basis since the $p$-basis itself always exists. The output of the algorithm consists on the $p$-basis and the Ulm invariants of $M$, that is, the type $t(M)$.

**Algorithm 4.1.** Input: $C$, $pR$.

(A1) Write the relations in $pR$ as polynomials in $k[x]$ as follows: $x_q^p - 1$, and $x_j^p - \prod_{i \neq j}^{q-1} x_i^{a_j v_i}$, for $1 \leq j \leq q$.

(A2) Find the order of all $c_j$ by computing all the univariate polynomials in the ideal $I$ generated by the polynomials obtained in (A1).
(A3) Find an ordering of the indeterminates, such that, the reduced lexicographic Gröbner basis \( G_p \) of \( I \) has the form \( \{x_1^\alpha, \ldots, x_n^\alpha\} \).

(A4) Let \( d \) be the number of polynomials in \( G_p \) such that the initial term has exponent \( > 1 \). If \( p^{r_j} \) \( > 1 \) and \( x_j^{p^{r_j}} - \prod_{l=1}^{j-1} x_l^{a_l p^{r_l}} \in G_p \), then add \( b_j \) to the \( p \)-basis, where \( b_j \) is the following element of order \( p^{r_j} \):

\[
b_j = c_j - \sum_{i=1}^{j-1} a_j c_i.
\]

(A5) To compute the type of \( M \), let \( s_r \) be the number of elements with the same order \( p^r \). Then \( \ell(M) = (s_1, \ldots, s_n) \).

Output: \( B = \{b_1, \ldots, b_d\} \) and \( M \cong (\mathbb{Z}_p)^{s_1} \oplus \cdots \oplus (\mathbb{Z}_p)^{s_n} \).

Example 4.2. Let \( M = \langle c_1, c_2, c_3, c_4, c_5, c_6, c_7, c_8 \rangle \) be a 5-group, with the following relations:

\[
\begin{align*}
5c_1 - c_4 - 4c_5 - 2c_6 - 3c_7 &= 0, \\
5c_2 - 4c_5 - 2c_6 - 2c_7 &= 0, \\
5c_3 - 4c_7 &= 0, \\
5c_4 &= 0, \\
5c_5 &= 0, \\
5c_6 &= 0, \\
5c_7 &= 0, \\
5c_8 &= 0.
\end{align*}
\]

The corresponding polynomials are

\[
\{x_1^5 - x_8 x_5^2 x_6^2 x_7^3, x_2^5 - x_6 x_7^2, x_3^5 - x_4^4, x_4^5 - 1, x_5^5 - 1, x_6^5 - 1, x_7^5 - 1, x_8^5 - 1\}.
\]

The reduced lexicographic Gröbner basis equals

\[
\{x_1^{25} - 1, \ x_2^{25} - 1, \ x_3^{25} - 1, \ x_4^{25} - 1, \ x_5^{25} - 1, \\
\ x_6 - x_3 x_2^{20}, \ x_7 - x_3^{20}, \ x_8 - x_5 x_3^{10} x_2^{10} x_1\}.
\]

In this case, \( d = 5 \). So, the Gröbner basis gives the following information

\[
\begin{align*}
25c_1 &= 0, & 25c_2 &= 0, & 25c_3 &= 0, & 5c_4 &= 0, & 5c_5 &= 0, \\
c_6 &= 15c_3 + 20c_2, & c_7 &= 20c_3, & c_8 &= c_5 + 10c_3 + 10c_2 + 5c_1.
\end{align*}
\]

Hence, the \( p \)-basis is equal to \( B = \{c_1, c_2, c_3, c_4, c_5\} \), \( M \cong \mathbb{Z}_5^2 \oplus \mathbb{Z}_2^{13} \), and \( \ell(M) = (2, 3) \).

The classical way to solve this problem, using matrix transformations over an Euclidean domain, appears in [2]. We remark that it is possible to perform the second step in the algorithm because by definition, \( I_C \) is a zero-dimensional ideal. Moreover, each univariate polynomial in \( I_C \) has the form \( x_j^{\text{ord}(c_j)} - 1 \).

5. Indecomposable modules over Dedekind-like rings

Let \( R_1 \) and \( R_2 \) be two rings. Let \( R \) be the pullback ring of the rings \( R_i \) over a common ring \( \overline{R} \), that is, \( R = \{R_1 \rightarrow \overline{R} \leftarrow R_2\} \). In [12], L. Levy studied the separated representation of an \( R \)-module \( M \). In [13], he described the indecomposable \( R \)-modules when \( R_1 \) and \( R_2 \) are Dedekind domains and \( \overline{R} \) is a field \( k \) (\( R \) is called a Dedekind-like ring). In particular, he studied modules over two rings: \( \mathbb{Z} C_p \), where \( C_p \) is the cyclic group of order a prime number \( p \), and the \( p \)-pullback \( \{\mathbb{Z} \rightarrow \mathbb{Z}_p \leftarrow \mathbb{Z}\} \) of \( \mathbb{Z} \oplus \mathbb{Z} \).

An \( R \)-module \( S \) is separated if it is an \( R \)-submodule of a direct sum \( S_1 \oplus S_2 \), where each \( S_i \) is an \( R_i \)-module. A separated representation of an \( R \)-module \( M \) is
an $R$-module epimorphism $\phi : S \twoheadrightarrow M$, such that, $S$ is a separated $R$-module and if $\phi$ admits a factorization $\phi : S \twoheadrightarrow S' \twoheadrightarrow M$ with $S'$ also a separated $R$-module, then $f$ must be one to one. Let $P_i = \ker(R_i \rightarrow k)$, then $P = \{P_1 \rightarrow 0 \leftarrow P_2\}$ is an ideal of $R$. We call an $R$-module $M$ $P$-mixed, if each torsion element $m$ is annihilated by some power of $P$. The separated modules $S = \{S_1, \ldots, S_m\}$ satisfying one of the following two conditions: (1) $S_i \cong$ nonzero ideal of $R_i$, or (2) $S_i = R_i/P_i^e$ form the basic building blocks for all finitely generated, $P$-mixed $R$-modules. If $S$ is a building block, then $S$ has exactly one submodule which has the form $\{X \rightarrow 0 \leftarrow 0\}$ and is $R$-isomorphic to $k$ (left $k$ of $S$). Similarly, $S$ has a right $k$ of $S$.

**Definition 5.1.** *(Deleted Cycle and Block Cycle Indecomposables)*

(a) Let $S^{(1)}, \ldots, S^{(m)}$ be a sequence of basic building blocks, such that,

\[
\begin{array}{cccccc}
S^{(1)} & \rightarrow & S_{21} & S^{(i)} & \rightarrow & S_{2i} \\
\downarrow & & \downarrow & & \downarrow & \\
S_{11} & \rightarrow & k & S_{1i} & \rightarrow & k
\end{array}
\]

and suppose that for $1 \leq i \leq m$, $S^{(i)}$ has a right $k$ and $S^{(i+1)}$ has a left $k$. A deleted cycle indecomposable $M$ is the direct sum $S = \bigoplus_{i=1}^{m} S^{(i)}$ modulo a relation which identifies the right $k$ of $S^{(i)}$ with the left $k$ of $S^{(i+1)}$, that is, first choose $p_{j} \in P_{j} - P_{j}^{2}$ for $j = 1, 2$, then make the following identification

\[
p_{2}^{d(2,i)} - 1 s_{2,i} = -p_{1}^{d(1,i+1)} - 1 s_{1,i+1}, \text{ where } s_{j,i} \in S_{ji}, \sigma_{ji}(s_{ji}) = \mathbb{T},
\]

for $j = 1, 2; 1 \leq i \leq m - 1$, and $d(j, i)$ the length of $S_{ji}$. In other words, it is the direct sum $S$ modulo

\[
\{p_{2}^{d(2,i)} - 1 s_{2,1} + p_{1}^{d(1,2)} - 1 s_{1,2}, \ldots, p_{2}^{d(2,m-1)} - 1 s_{2,m-1} + p_{1}^{d(1,m)} - 1 s_{1,m}\}.
\]

(b) Let $S^{(1)}, \ldots, S^{(m)}$ be a sequence of basic building blocks

\[
\begin{array}{cccccc}
S^{(1)} & \rightarrow & S_{21} & S^{(i)} & \rightarrow & S_{2i} \\
\downarrow & & \downarrow & & \downarrow & \\
S_{11} & \rightarrow & k & S_{1i} & \rightarrow & k
\end{array}
\]

each with a left and a right $k$. Write $m = \frac{m}{m}$, where $m$ is the unique smallest positive integer, such that, for all $i$, $S^{(i)} \cong S^{(i+m)}$. Let $f(z) = \lambda_0 + \lambda_1 z + \cdots + \lambda_l z^l - 1 + z^l$ be a power of an irreducible polynomial in $k[z]$. A block cycle indecomposable $M$ is a deleted cycle indecomposable modulo the following relation

\[
-p_{2}^{d(2,i)} - 1 s_{2,m} = \sum_{j=0}^{l-1} \lambda_{j} p_{1}^{d(1,j)} - 1 s_{1,(jm+1)},
\]

which identifies the right $k$ of $S^{(m)}$ with a one-dimensional subspace of $S_{1,1} \oplus S_{1,2m+1} \oplus S_{1,(2m+1)} \oplus \cdots$. In other words, it is the direct sum $S$ modulo

\[
\{p_{2}^{d(2,1)} - 1 s_{2,1} + p_{1}^{d(1,2)} - 1 s_{1,2}, \ldots, p_{2}^{d(2,m-1)} - 1 s_{2,m-1} + p_{1}^{d(1,m)} - 1 s_{1,m},
\]

\[
p_{2}^{d(2,m)} - 1 s_{2,m} + \sum_{j=0}^{l-1} \lambda_{j} p_{1}^{d(1,j)} - 1 s_{1,(jm+1)}\}.
\]
As a consequence, if $M$ is a deleted cycle then $1 \leq d(2, i) \neq \infty$, for $1 \leq i \leq m - 1$ and $1 \leq d(1, i) \neq \infty$, for $2 \leq i \leq m$. But the length of either one of $S_{11}$ or $S_{2n}$ may be infinite. If $M$ is a block cycle, then $1 \leq d(j, i) \neq \infty$ for $1 \leq i \leq m$ and $j = 1, 2$.

**Remark 5.2.** The indecomposable, finitely generated, $P$-mixed modules are deleted cycle indecomposables and block cycle indecomposables. Every separated $R$-module is a direct sum of basic building blocks. Moreover, basic building blocks are always indecomposable $R$-modules, see [12].

5.1. **Additive descriptions.** Using Algorithm [4.1] we describe the additive structure of the indecomposable $R$-modules when $R$ is one of the following rings: $\mathbb{Z}C_p$, or the $p$-pullback of $\mathbb{Z} \oplus \mathbb{Z}$, $\{ \mathbb{Z} \to \mathbb{Z}_p \leftarrow \mathbb{Z} \}$. In these two cases the concept of $P$-mixed coincides with $p$-mixed.

The ring $\mathbb{Z}C_p$:

$$
\begin{array}{ccc}
\mathbb{Z}C_p & \longrightarrow & \mathbb{Z}[\zeta] \\
\downarrow & & \downarrow \nu_2 \\
\mathbb{Z} & \overset{\nu_1}{\longrightarrow} & \mathbb{Z}_p
\end{array}
$$

Let $\zeta$ be a primitive $p$th root of unity, and let $x$ be a generator of $C_p$. Then $\mathbb{Z}C_p \cong \{ \mathbb{Z} \overset{\nu_1}{\longrightarrow} \mathbb{Z}_p \leftarrow \mathbb{Z}[\zeta] \}$, where the isomorphism is given by $x \longrightarrow (1 \longrightarrow \mathbb{T} \leftarrow \zeta)$. The action of $p_1$ and $p_2$ in $\Lambda = \mathbb{Z}C_p$ is given by the following formulas:

$$
p_1 = x^{p-1} + x^{p-2} + \cdots + x + 1 = \{ p \to 0 \leftarrow \zeta^{p-1} + \zeta^{p-2} + \cdots + \zeta + 1 \} \quad \text{and}
$$

$$
p_2 = x - 1 = \{ 0 \to 0 \leftarrow \zeta - 1 \}, \quad p = (p, p) = p_1 + p_2^{p-1} \sigma(p_2), \quad p_1p_2 = 0.
$$

where $\sigma(p_2)$ is a polynomial in $p_2$, with degree less or equal than $p-1$, which exists because the sum equals $p$. So, every element $m$ of a $\mathbb{Z}C_p$-module $M = \langle a_1, \ldots, a_n \rangle$ is a linear combination of these generators and the elements resulting from the action of $p_1$ and $p_2$ over them.

**Example 5.3.** Let $\Lambda = \mathbb{Z}C_3$ and $M = \langle a \rangle_{\mathbb{Z}C_3}$ be a deleted cycle indecomposable with $d(1) = d(2) = 3$, and $3 = p_1 + 2p_2^2$. We need to compute the action of $p$ in $\Lambda$ over the generator $a$ to obtain a generating set for $M$ over $\mathbb{Z}$. This is the classical way to begin this problem in Abelian group theory. Thus,

$$
3a = p_1a + 2p_2^2a, \quad 3p_1a = p_1^2a, \quad 3p_2a = p_2^3a, \quad 3p_2^2a = 0.
$$

The generating set is $C = \{ a, p_2a, p_1a, p_2^2a, p_1^2a \}$, the corresponding ideal is generated by the binomials

$$
\{ x_1^3 - x_3x_4, \ x_3 - x_5, \ x_5^3 - 1, \ x_2^3 - x_4, \ x_4^3 - 1 \}.
$$

The order of each element in $C$ is $\{ 27, 9, 9, 3, 3 \}$. The reduced Gröbner basis is equal to

$$
\{ x_1^{27} - 1, \ x_2^9 - 1, \ x_3 - x_2x_1, \ x_4 - x_2^3, \ x_5 - x_1^9 \}.
$$

So, Algorithm [4.1] outputs

$$
B = \{ a, \ p_2a \}, \quad M \cong \mathbb{Z}_9 \oplus \mathbb{Z}_{27}, \quad \sharp(M) = (0, 0, 1, 1).
$$

The extra zero in the type means that the torsion-free rank equals 0. Therefore, the action of $\Lambda$ does not change if we consider $M$ as a module over $\mathbb{Z}_{27}C_3$. 
Example 5.4. Let $\Lambda = \mathbb{Z}C_3$ and $M = \langle a \rangle_{\mathbb{Z}C_3}$ be a block cycle indecomposable with $d(1) = 4$, $d(2) = 4$, and $f(z) = z - 2$. The action of $p = 3$ is given by $3 = p_1 + 2p_2^2$. We also have the relation $p_1^2a = 2p_2^2a$. The action of $p = 3$ over $a$ is given by

$$3a = p_1a + 2p_2^2a, \quad 3p_1a = p_1^2a, \quad 3p_2a = p_1^3a, \quad 3p_3a = 0, \quad 3p_2a = 2p_3^2a, \quad 3p_2a = 0.$$

The generating set is $C = \{a, p_1a, p_2a, p_1^2a, p_2^2a, p_1^3a, p_3^2a\}$. In this case, the corresponding toric ideal is generated by

$$\{x_1^3 - x_2x_3^2, x_2^3 - x_4, x_4^3 - x_6, x_6^3 - 1, x_3^3 - x_7^2, x_7^3 - 1, x_5^3 - 1, x_6 - x_7^2\}.$$

The order of each element in $C$ is $\{81, 27, 9, 9, 3, 3, 3\}$. The reduced Gröbner basis is equal to

$$\{x_1^{81} - 1, x_2^3 - x_9^1, x_3^3 - x_7^27, x_4 - x_9^1, x_5 - x_2x_1^{78}, x_6 - x_1^{27}, x_7 - x_5^{84}\}.$$

Hence, Algorithm 4.4 outputs

$$B = \{a, p_1a - 3a, p_2a - 9a\}, \quad M \cong \mathbb{Z}_4 \oplus \mathbb{Z}_3, \quad \mathcal{I}(M) = (0, 2, 0, 0, 1).$$

The $p$-pullback ring of $Z \oplus Z$: The $p$-pullback of $Z \oplus Z$ is the subring $\Lambda = \{Z \rightarrow Z_p \leftarrow Z\}$ of $Z \oplus Z$. In this case, let $p_1 = (p, 0)$ and $p_2 = (0, p)$. Then $p = (p, p) = p_1 + p_2$.

Example 5.5. Consider the pullback ring $\Lambda = \{Z \rightarrow Z_3 \leftarrow Z\}$ and a deleted cycle indecomposable module $M = \langle a_1, a_2 \rangle_{\Lambda}$, with $d(1) = 3$, $d(1, 2) = 3$, $d(2, 1) = 3$, and $-4p_3^2a_1 = p_4^2a_2$. Note that the order of these elements is 3, since they are in the socle $M[p]$ of $M$; thus, the last relation is $2p_2^2a_1 = p_1^2a_2$. Also $p = p_1 + p_2$. Therefore, the generators are

$$C = \{a_1, a_2, p_1a_1, p_2a_1, p_1a_2, p_2a_2, p_1^2a_1, p_2^2a_2, p_1^3a_2, p_2^3a_2\}.$$

Besides the previous relation $2p_2^2a_1 = p_1^2a_2$, the relations obtained from the action of $p$ are

$$3a_1 = p_1a_1 + p_2a_1, \quad 3p_1a_1 = p_1^2a_1, \quad 3p_2a_1 = p_2^2a_1, \quad 3p_1a_1 = 0, \quad 3p_2a_1 = 0,$$

$$3a_2 = p_1a_2 + p_2a_2, \quad 3p_1a_2 = p_1^2a_2, \quad 3p_2a_2 = p_2^2a_2, \quad 3p_1a_2 = 0, \quad 3p_2a_2 = 0.$$

The toric ideal is generated by

$$\{x_1^3 - x_3x_4, x_3^3 - x_7, x_4^3 - x_8, x_7^3 - 1, x_8^3 - 1, x_9^3 - x_7^2, x_7^3 - 1, x_8^3 - 1, x_9^3 - x_8^2\}.$$

The order of each element in $C$ is $\{27, 27, 9, 9, 9, 3, 3, 3, 3\}$. The Gröbner basis is equal to

$$\{x_1^{27} - 1, x_2^{27} - 1, x_3^3 - 1, x_4 - x_3^2x_1^3, x_5^3 - x_3^2x_1^1, x_6 - x_5^2x_3^2x_2^1, x_7 - x_3^2, x_8 - x_3^2x_1^1, x_9 - x_3^2x_1^1, x_10 - x_3^2x_2^1\}.$$

Using the algorithm, we obtain the $p$-basis

$$B = \{a_1, a_2, p_1a_1, p_1a_2 - p_2a_1 - 6a_1\}, \quad M \cong \mathbb{Z}_3 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_{27^2}, \quad \mathcal{I}(M) = (0, 1, 1, 2).$$

Let $M$ be an indecomposable $R$-module and let $S = \bigoplus_{i=1}^{m} S^{(i)}$ be the separated representation of $M$. If $m = 1$, then $S = \{S_1 \rightarrow k \leftarrow S_2\} = \langle a \rangle$ is a basic building block, and length($S_j$) $= d(j) \neq \infty$, for $j = 1, 2$, because $M$ is $\mathbb{Z}_p^m$-free. Thus, the subset $A = \{a, p_1a, \ldots, p_1^{d(1)-1}a, p_2a, \ldots, p_2^{d(2)-1}a\}$ generates $S$ as an Abelian
The next theorem shows how to use Algorithm 4.1 to obtain the type and a \( p \)-basis of any basic building block with torsion part.

**Theorem 5.6.** Let \( S = \{S_1 \to k \leftarrow S_2\} = \langle a \rangle \) be a basic building block. Then

(i) If \( d(1) \cdot d(2) < \infty \), then Algorithm 4.1 gives a \( p \)-basis for \( S \) using the presentation \( S = \langle A, pA \rangle \).

(ii) If \( d(j) = \infty \) for exactly one \( j \), one can obtain a basis for \( S \), by adding to the input of Algorithm 4.1 the number \( \exp(t(S)) + 2 \), where \( \exp(t(S)) \) denotes the exponent of the torsion subgroup of \( (S, +) \).

(iii) If both lengths are infinite then the Abelian group \( (S, +) \) is torsion free. In this case, the rank is \( p \), and \( \{a, p_1a, \ldots, p^{p-1}_2a\} \) is a \( p \)-basis for \( R = ZC_p \).

\( R = \{Z \to Z_p \leftarrow Z\} \) is the \( p \)-pullback of \( Z \oplus Z \), then either \( \{a, p_1a\} \) or \( \{a, p_2a\} \) is a \( p \)-basis for \( S \).

**Proof.** In part (iii), the case \( R = ZC_p \) is a direct consequence of [13, Application 1.10] and the case \( R = \{Z \to Z_p \leftarrow Z\} \) is trivial. If \( d(1) < \infty \) and \( d(2) < \infty \), then \( S \) is an \( R_p \)-module. So, \( \langle A, pA \rangle \) is a presentation of \( S \). Hence, applying Algorithm 4.1 we obtain a \( p \)-basis. If \( d(1) = \infty \) or \( d(2) = \infty \), we change the infinite length for \( \exp(t(S)) + 2 \). After this, we can apply Algorithm 4.1 to get a \( p \)-basis. Using the proof of Theorem 11.6 in [13], we can recover the basis for \( S \). If \( R = ZC_p \) and \( d(2) = \infty \), there are \( p - 1 \) elements of order \( \exp(t(S)) + 2 \) in the basis, by [13, Application 1.10]. These elements have infinite order and the remaining elements in the basis form the \( p \)-basis for the torsion part. If \( d(1) = \infty \), then there is one element with infinite order in the basis. If \( R \) is the \( p \)-pullback of \( Z \oplus Z \), we have one element of infinite order in the basis. \( \square \)

Theorem 5.7 describes how to find the additive structure, in general, for any indecomposable \( R \)-module after computing the \( p \)-height of the elements that connect the building blocks in \( M \). Let \( d_{ji} \) denote \( d(j, i) \). Also let \( d_i = p^{d_{2i} - 1} s_{2i} = -p^{d_{1i} - 1} s_{1i + 1} \) for \( 1 \leq i \leq m - 1 \). If \( d_{11} \neq \infty \), let \( d_0 = p^{d_{11} - 1} s_{11} \), and if \( d_{2m} \neq \infty \), let \( d_m = p^{d_{2m} - 1} s_{2m} \). The \( p \)-height of the element \( d_i = p^{k-1} d' \) is \( h_p(d_i) = k - 1 \) in the Abelian group \( t(M) \). Let \( \ell_a \) be the number of elements \( d_i \) such that \( h_p(d_i) = \alpha - 1 \), see [9].

Let \( S = \bigoplus_i S^{(i)} \) be a separated representation of an indecomposable \( R \)-module \( M \). If \( M \) is block cyclic, then we consider the separated module \( S' = \bigoplus_i S^{(i)} \) such that \( d'_{ji} = d_{ji} - 1 \) for all \( (j, i) \). If \( M \) is deleted cyclic, then we consider the separated module \( S' = \bigoplus_i S^{(i)} \) such that \( d'_{ji} = d_{ji} - 1 \) for \( (j, i) \neq (1, 1) \) and \( (j, i) \neq (2, m) \).

**Theorem 5.7.** Let \( M \) be an indecomposable \( R \)-module, and let \( S = \bigoplus_{i=1}^m S^{(i)} \) be the separated representation of \( M \). Then

\[
\ell(M) = \sum_{i=1}^m \ell(S^{(i)}) + (0, \ell_1 - \ell_2, \ldots, \ell_{n-1} - \ell_n, \ell_n),
\]

where \( n = \exp(t(M)) \).

**Proof.** First, suppose that \( M = \langle a_1, \ldots, a_m \rangle \) is a deleted cycle indecomposable \( R \)-module. If \( m = 1 \), the theorem is obviously true. Suppose the result is proved for \( m - 1 \). Then consider \( M/\langle d_1 \rangle = S^{(1)} \oplus M' \), where \( M' = \langle a_2, \ldots, a_m \rangle/\langle d_1 \rangle \). Since \( M' \) is generated by \( m - 1 \) elements, we can apply induction. By [4, Corollary 3.3],
if \( h_p(d_1) = \beta \) then

\[
\mathfrak{f}(M) = \mathfrak{f}(M/(d_1)) + v(\beta) = \mathfrak{f}(S^{(1)}) + \sum_{k=2}^{n} \mathfrak{f}(S^{(k)}) + (0, \ell'_1 - \ell'_2, \ldots, \ell'_{n-1} - \ell'_n, \ell'_n) + v(\beta),
\]

where \( \ell'_n \) is the number of elements \( d'_k \) such that \( h_p(d'_k) = \alpha - 1 \) in \( M' \). The vector \( v(\beta) = (v_0, v_1, \ldots, v_n) \in \mathbb{Z}^{n+1} \) satisfies \( v_{\beta - 1} = -1, v_\beta = 1 \) and \( v_i = 0 \) otherwise. It is clear that \( h_p(d'_k) = h_p(d_k) \) for \( k \geq 2 \). So our claim holds.

Now suppose \( M \) is a block cyclic indecomposable module. Observe that the result holds for the module \( M' = M/(d_0) \), since \( M' \) is a deleted cyclic module. Hence our claim holds by [4, Corollary 3.3]. □

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