Existence of mild solutions for Riemann-Liouville fractional differential equations with nonlocal conditions *†

Zhan-Dong Mei ‡ Ji-Gen Peng §

Abstract

In this paper, we are concerned with the mild solutions of Riemann-Liouville fractional differential equations with nonlocal conditions in Banach space. We use Banach contraction principle to prove the existence and uniqueness. Moreover, we derive the existence by using Krasnoselkii’s theorem. An illustrative example is presented.

Key words: Riemann-Liouville fractional differential equations; fractional resolvent; mild solution.

*This work was supported by the Natural Science Foundation of China (Grant No. 11301412 and 11131006), and the Fundamental Research Funds for the Central Universities (Grant No. 2012jdhz52)
†2010 Mathematics Subject Classification. Primary: 34A08; Secondary: 47D06.
‡Corresponding author, School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China; Email: zhdmei@mail.xjtu.edu.cn
§School of Mathematics and Statistics, Xi’an Jiaotong University, Xi’an 710049, China; Email: jg-peng@mail.xjtu.edu.cn
1 Introduction

Consider inhomogeneous abstract Riemann-Liouville fractional differential equations described by

\[
\begin{aligned}
D_t^\alpha u(t) &= Au(t) + f(t, t^{1-\alpha} u(t), (Ku)(t)), \quad t \in (0, T], \\
\tilde{u}(0) &= x + g(u),
\end{aligned}
\]

where \(0 < \alpha \leq 1\), \(u(\cdot)\) is the state, \(A : D(A) \subset X \to X\) is a closed and densely defined linear operator, \((X, \| \cdot \|)\) is a Banach space, \(D(A)\) is the domain of \(A\) endowed with the graph norm \(\| \cdot \|_{D(A)} = \| \cdot \| + \| A \cdot \|\), \(D_t^\alpha\) is the \(\alpha\)-order Riemann-Liouville fractional derivative operator, \((Ku)(t) = \int_0^t r(t, s)u(s)ds\), \(f : [0, T] \times X \times X \to X\), \(\tilde{u}(0) = \lim_{t \to 0^+} \Gamma(\alpha) t^{1-\alpha} u(t)\), \(x \in X\), \(g\) is a function from a certain function space on \(X\) to \(X\).

Fractional differential equations have received increasing attention because the behavior of many physical systems, such as fluid flows, electrical networks, viscoelasticity, chemical physics, electron-analytical chemistry, biology, control theory, can be properly described by using the fractional order system theory etc. (see [7, 15, 21, 22, 25]). Fractional derivatives appear in the theory of fractional differential equations; they describe the property of memory and heredity of materials, and it is the major advantage of fractional derivatives compared with integer order derivatives. Many of the references on fractional differential equations were focused on the existence and/or uniqueness of solutions for fractional differential equations [3, 4, 7, 8, 13].

The nonlocal Cauchy problem, an initial problem for the corresponding equations with nonlocal initial data, was first studied by Byszewski [6]. Such problem has better effects than the normal Cauchy problem with the classical initial data because nonlocal condition can be applied in physics with better effect in applications than the classical initial condition since nonlocal conditions are usually more precise for physical measurements than the classical initial condition (cf., e.g., [1, 6, 9, 10, 19, 20, 23, 28] and references therein). Very recently, the existence
and uniqueness of solutions of Caputo fractional abstract differential equations with a nonlocal initial condition were discussed by some references (cf., e.g., Anguraj e.t. [1], Balachandran e.t. [2], Li, e.t. [16], Zhou, e.t. [29]). N’Guerekata [24] studied the mild solutions of fractional differential equations with nonlocal conditions related to Riemann-Liouville derivative, which results in singularity at zero. However, Li, Peng and Gao [18] pointed out that the definition of the mild solution in [24] is incorrect and the similar situation can be found in [12]. Motivated by this, in this paper, we will use fractional resolvent developed by Li and Peng [17] and introduce a new norm to study the existence and uniqueness of equation (1.1).

The arrangement of this paper is as follows. Sec. 2 is to introduce some related preliminaries. In Sec. 3, Banach contraction principle is used to prove the existence and uniqueness and Krasnoselkii’s theorem is used to derive the existence of the mild solutions of (1.1).

2 Preliminaries

Let \((X, \| \cdot \|)\) be a Banach space. For \(q \geq 1\), \(L^q((0, T); X)\) denotes the space of all \(X\)-valued functions \(u : (0, T) \to X\) with the norm \(\|u\|_{L^q((0, T); X)} = \left( \int_0^T \|u(t)\|^q dt \right)^{\frac{1}{q}}\). Denote by \(C_{1-\alpha}([0, T], X)\) all the functions such that \(t \mapsto t^{1-\alpha}u(t)\) is continuous on \([0, T]\) with the norm \(\|u\|_{C_{1-\alpha}([0, T]; X)} = \sup_{t \in [0, T]} \|t^{1-\alpha}u(t)\|\). Obviously, \(L^q((0, T); X)\) and \(C_{1-\alpha}([0, T], X)\) are Banach spaces. Let \(n \in \mathbb{N}\), \(1 \leq q < \infty\). Let \(I = (0, T)\), or \(I = [0, T]\), or \(I = (0, \infty)\). The Sobolev spaces \(W^{n,p}(I; X)\) is defined as follows ([4 Appendix]):

\[
W^{n,p}(I; X) = \{ u | \exists \varphi \in L^p(I; X) : u(t) = \sum_{k=0}^{n-1} c_k \frac{t^k}{k!} + \frac{t^{n-1}}{(n-1)!} * \varphi(t), \ t \in I \}.
\]

In this case, we have \(\varphi(t) = u^{(n)}(t), \ c_k = u^{(k)}(0)\).
For the convenience of the readers, we shall introduce some definitions and some fundamental properties of fractional calculus theory, which can be found in [11, 15, 25, 27].

**Definition 2.1** For any $u \in L^1((0, T); X)$, the $\alpha$-order Riemann-Liouville fractional integral of $u$ is defined by

$$J^\alpha_t u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau. \tag{2.1}$$

We denote $J^0_t u(t) = u(t)$. Obviously, the fractional integral operators $\{J^\alpha_t\}_{\alpha \geq 0}$ satisfies the semigroup property $J^\alpha_t J^\beta_t = J^{\alpha+\beta}_t$, $\alpha, \beta \geq 0$.

**Definition 2.2** Let $\alpha \in (0, 1)$. The $\alpha$-order Riemann-Liouville fractional derivative of $u$ is defined by

$$D^\alpha_t u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\sigma)^{-\alpha} u(\sigma) d\sigma.$$  

**Definition 2.3** [17] Let $0 < \alpha < 1$. A family $\{T(t)\}_{t > 0}$ of bounded linear operators on Banach space $X$ is called an $\alpha$-order fractional resolvent if it satisfies the following assumptions:

(P1) for any $x \in X$, $T(\cdot)x \in C((0, \infty), X)$, and

$$\lim_{t \to 0^+} \Gamma(\alpha) t^{1-\alpha} T(t)x = x \text{ for all } x \in X; \tag{2.2}$$

(P2) $T(s)T(t) = T(t)T(s)$ for all $t, s \geq 0$;

(P3) for all $t, s > 0$, there holds

$$T(t)J^\alpha_s T(s) - J^\alpha_t T(t)T(s) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} J^\alpha_s T(s) - \frac{s^{\alpha-1}}{\Gamma(\alpha)} J^\alpha_t T(t), \tag{2.3}$$

where $J^\alpha_t$ is $\alpha$-order Riemann-Liouville fractional integral operator.

The generator $A$ of fractional resolvent $\{T(t)\}_{t > 0}$ is defined by

$$D(A) = \{x \in X : \text{the limit } \lim_{t \to 0^+} \frac{t^{1-\alpha} T(t)x - \frac{x}{t^\alpha}}{t^\alpha} \text{ exists}\}.$$
and

\[ Ax = \lim_{t \to 0^+} \frac{t^{1-\alpha}T(t)x - \frac{x}{\Gamma(\alpha)}}{t^\alpha}. \]

**Theorem 2.4** [17] Let \( \{T(t)\}_{t>0} \) be a fractional resolvent and \( A \) its generator. Then, we have

(a) For any \( x \in X \), \( \int_0^t \frac{(t-s)\alpha-1}{\Gamma(\alpha)} T(s)xds \in D(A) \) and

\[ T(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x + A \int_0^t \frac{(t-s)\alpha-1}{\Gamma(\alpha)} T(s)xds, \ t > 0; \] (2.4)

(b) For any \( x \in D(A) \),

\[ T(t)x = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x + \int_0^t \frac{(t-s)\alpha-1}{\Gamma(\alpha)} T(s)Axds, \ t > 0; \] (2.5)

**Theorem 2.5** [13](Krasnoselskii) Let \( B \) be a closed convex and nonempty subset of a Banach space \( X \). Let \( A \) and \( B \) be two operators such that

(i) \( Au + Bv \in B \) whenever \( u, v \in B \);

(ii) \( A \) is a contraction mapping;

(iii) \( B \) is compact and continuous.

Then there exists \( z \in B \) such that \( z = Az + Bz \).

### 3 Existence of mild solution

In this section, we shall prove the existence and uniqueness of the mild solution of (1.1). To begin with, we introduce the notion mild solution.

**Definition 3.1** A function \( u \in C((0,T], X) \) is called a mild solution of equation (1.1), if \( J_1^\alpha u(t) \in D(A), \ t \in (0,T], \) and there holds

\[ u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} [x - g(u)] + AJ_1^\alpha u(t) + J_1^\alpha f(t,t^{1-\alpha}u(t), (Ku)(t)), \ t \in (0,T], \] (3.1)
Lemma 3.2 Suppose that \( A \) generates a fractional resolvent \( \{ S(t) \}_{t>0} \). Let \( f \in C([0,T] \times X \times X) \). If \( u \in C((0,T], X) \) is a mild solution of system (1.1), then
\[
    u(t) = S(t)[x - g(u)] + \int_0^t S(t-s)f(s, s^{1-\alpha}u(s), (Ku)(s)d\sigma)ds;
\]
(3.2)
conversely, if \( u \in C((0,T], X) \) satisfies (3.2), then \( u \) is a mild solution of system (1.1).

Proof. Assume that \( u \in C((0,T], X) \) is a mild solution of (1.1). Then, \( g_\alpha(t) \ast u(t) = J_t^\alpha u(t) \in D(A) \). By (2.4), it follows that
\[
    \frac{t^{\alpha-1}}{\Gamma(\alpha)} * u(t) = \left( S(t) - A g_\alpha(t) * S(t) \right) * u(t)
\]
\[
    = S(t) * u(t) - A S(t) * g_\alpha(t) * u(t)
\]
\[
    = S(t) * u(t) - S(t) * A \left( g_\alpha(t) * u(t) \right)
\]
\[
    = S(t) * (u(t) - A J_t^\alpha u(t))
\]
\[
    = S(t) * \left( g_\alpha(t)x - g_\alpha(t) * f(t) \right)
\]
\[
    = \frac{t^{\alpha-1}}{\Gamma(\alpha)} * \left( S(t)[x - g(u)] + \int_0^t S(t-s)f(s, s^{1-\alpha}u(s), (Ku)(s)d\sigma)ds \right).
\]
(3.3)
By Titchmarsh’s theorem, we have
\[
    u(t) = S(t)[x - g(u)] + \int_0^t S(t-s)f(s, s^{1-\alpha}u(s), (Ku)(s)d\sigma)ds.
\]
Assume that \( u \in C((0,T], X) \) satisfies (3.2). Then, by (a) of Theorem 2.4 it follows that
\[
    J_t^\alpha u(t) = J_t^\alpha S(t)[x - g(u)] + g_\alpha(t) * S(t) * f(t, t^{1-\alpha}u(t), (Ku)(t)) \in D(A)
\]
and

\[ \text{and} \]

\[ AJ_i^\alpha u(t) = \left( S(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) [x - g(u)] + \left( S(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \right) * f \left( t, t^{1-\alpha} u(t), (Ku)(t) \right) \]

\[ = S(t)(x - g(u)) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} (x - g(u)) + S(t) * f \left( t, t^{1-\alpha} u(t), (Ku)(t) \right) \]

\[ - \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f \left( t, t^{1-\alpha} u(t), (Ku)(t) \right) \]

\[ = u(t) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} (x - g(u)) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} * f \left( t, t^{1-\alpha} u(t), (Ku)(t) \right). \]

The proof is therefore completed. \( \blacksquare \)

**Lemma 3.3** Assume that \( x, y > 0 \) and \( 0 < \gamma < 1 \). Then

\[ |x^\gamma - y^\gamma| \leq |x - y|^\gamma. \hspace{1cm} (3.4) \]

**Proof.** Assume that \( x > y \). Then (3.4) is equivalent to

\[ \left( \frac{x}{y} \right)^\gamma - 1 \leq \left( \frac{x}{y} - 1 \right)^\gamma. \hspace{1cm} (3.5) \]

Define function \( g(z) := (z + 1)^\gamma - z^\gamma - 1, \ z > 0. \) We can easily obtain that the derivative of \( g \) at each \( z > 0 \) satisfies that \( g'(z) = \gamma(z + 1)^{\gamma-1} - \gamma z^{\gamma-1} < 0. \) This means that \( g(z) \) is monotone-decreasing function. So we have

\[ g \left( \frac{x}{y} - 1 \right) = \left( \frac{x}{y} \right)^\gamma - \left( \frac{x}{y} - 1 \right)^\gamma - 1 \leq g(0) = 0, \]

that is, (3.5) holds. The proof is therefore completed. \( \blacksquare \)

Denote \( M = \max_{t \in [0,T]} \| t^{1-\alpha} S(t) \|, \ N = \max_{0 \leq s \leq t \leq T} r(t, s), \ P = \max_{0 \leq s \leq t \leq T} m(t, s). \)

In order to derive our main results, the following hypothesis are introduced:

\( (H_1) \) There exists two constants \( \alpha_1, \alpha_2 \in (0, \alpha) \) and real-valued functions \( m_1(t) \in L^{\frac{1}{\alpha_1}}([0, T], R), \)

\( m_2(t) \in L^{\frac{1}{\alpha_2}}([0, T], R) \) such that

\[ \| f(t, u_1, u_2) - f(t, v_1, v_2) \| \leq m_1(t) \| u_1 - v_1 \| + m_2(t) \| u_2 - v_2 \|. \]
There exists a constant $\alpha_3 \in (0, \alpha)$ and real-valued function $h(t) \in L^{\frac{1}{\alpha}}([0, T], R)$ such that

$$\|f(t, u_1, v_1)\| \leq h(t), \ t \in [0, T], \ u_1, v_1 \in X.$$ 

There exists a constant $b$ such that

$$\|g(u) - g(v)\| \leq b\|u - v\|_\ast, \ u, v \in C_{1-\alpha}([0, T], X).$$

We denote $M_1 = \|m_1\|_{L^{\frac{1}{\alpha}}([0, T], R)}$, $M_2 = \|m_2\|_{L^{\frac{1}{\alpha}}([0, T], R)}$ and $H = \|h\|_{L^{\frac{1}{\alpha}}([0, T], R)}$.

**Theorem 3.4** Assume that $(H_1) - (H_3)$ hold. If

$$\Omega := Mb + \frac{M M_1 T^{1-\alpha_1}}{(\frac{\alpha}{1-\alpha_1})^{1-\alpha_1}} + \frac{M M_2 NT^{\alpha_2+1-\alpha_2}}{\alpha(\frac{\alpha}{1-\alpha_2})^{1-\alpha_2}} < 1,$$

then system (1.1) has a unique solution.

**Proof.** Consider the following operator:

$$(Nu)(t) = S(t)[x - g(u)] + \int_0^t S(t - s)f(s, s^{1-\alpha}u(s), (Ku)(s))ds, \ u \in C_{1-\alpha}([0, T], X), \ t > 0. \quad (3.6)$$
We shall first verify that $\mathcal{N} : C_{1-\alpha}([0,T],X) \to C_{1-\alpha}([0,T],X)$. Let $u \in C_{1-\alpha}([0,T],X)$, $t, \delta > 0$, $t + \delta \leq T$. We compute

$$\left\| t^{1-\alpha}(\mathcal{N}u)(t) - \frac{1}{\Gamma(\alpha)}[x - g(u)] \right\|$$

$$\leq \left\| t^{1-\alpha}S(t) - \frac{1}{\Gamma(\alpha)}[x - g(u)] \right\| + \left\| t^{1-\alpha} \int_{0}^{t} S(t-s) f(s, s^{1-\alpha}u(s), (Ku)(s)) ds \right\|$$

$$\leq \left\| t^{1-\alpha}S(t) - \frac{1}{\Gamma(\alpha)}[x - g(u)] \right\| + t^{1-\alpha} \int_{0}^{t} (t-s)^{\alpha-1} \| (t-s)^{1-\alpha}S(t-s) \| \| h(s) \| ds$$

$$\leq \left\| t^{1-\alpha}S(t) - \frac{1}{\Gamma(\alpha)}[x - g(u)] \right\| + Mt^{1-\alpha} \left( \int_{0}^{t} (t-s)^{\alpha-1} ds \right)^{\frac{\alpha-1}{\alpha}} \left( \int_{0}^{t} \| h(s) \|_{L^\alpha} ds \right)^{\alpha}$$

$$= \left\| t^{1-\alpha}S(t) - \frac{1}{\Gamma(\alpha)}[x - g(u)] \right\| + \frac{Mt^{1-\alpha}}{\left( \frac{\alpha-1}{1-\alpha} \right)^{1-\alpha} \| h \|_{L^\alpha}}.$$  \hspace{1cm} (3.7)

Since $\lim_{t \to 0^+} \Gamma(\alpha) t^{1-\alpha} S(t)[x - g(u)] = x - g(u)$, the inequality (3.7) implies that the limit

$$\lim_{t \to 0^+} t^{1-\alpha}(\mathcal{N}u)(t)$$

exists and

$$\lim_{t \to 0^+} t^{1-\alpha}(\mathcal{N}u)(t) = \frac{1}{\Gamma(\alpha)}[x - g(u)].$$ \hspace{1cm} (3.8)
Denote $p_s(t) = t^{1-\alpha}(t-s)^{\alpha-1}$ and $q_s(t) = (t-s)^{1-\alpha}S(t-s)$. Using Lemma 3.3, we deduce the following inequality,

$$\| (t+\delta)^{1-\alpha}S(t+\delta-s) - t^{1-\alpha}S(t-s) \|$$

$$= \| p_s(t+\delta)q_s(t+\delta) - p_s(t)q_s(t) \|$$

$$\leq \| p_s(t+\delta)q_s(t+\delta) - p_s(t+\delta)q_s(t) \| + \| p_s(t+\delta)q_s(t) - p_s(t)q_s(t) \|$$

$$\leq |p_s(t+\delta)| \| q_s(t+\delta) - q_s(t) \| + M|p_s(t+\delta) - p_s(t)|$$

$$\leq |p_s(t+\delta)| \| q_s(t+\delta) - q_s(t) \| + M|t+\delta|^{1-\alpha}|t+\delta-s|^{1-\alpha} - |t^{1-\alpha}(t-s)^{1-\alpha} - t^{1-\alpha}(t-s)^{1-\alpha}|$$

$$+ M|t+\delta|^{1-\alpha}|t+\delta-s|^{1-\alpha} - (t-s)^{1-\alpha}|$$

$$\leq |p_s(t+\delta)| \| q_s(t+\delta) - q_s(t) \| + M|t+\delta|^{1-\alpha} |t+\delta-s|^{1-\alpha} - (t-s)^{1-\alpha}|$$

$$+ M|t-s|^{1-\alpha} |t+\delta|^{1-\alpha} - t^{1-\alpha}|$$

$$\leq \sup_{0 \leq s \leq T-\delta} \| q_s(t+\delta) - q_s(t) \| |(t+\delta)^{1-\alpha}(t+\delta-s)^{\alpha-1} + M|t+\delta|^{1-\alpha} \delta^{1-\alpha} + M|t-s|^{1-\alpha} \delta^{1-\alpha}|$$

$$\leq \sup_{0 \leq s \leq T-\delta} \| q_s(t+\delta) - q_s(t) \| T^{1-\alpha}(t+\delta-s)^{\alpha-1} + 2MT^{1-\alpha} \delta^{1-\alpha}.$$ (3.9)
Combining inequality (3.9) and Lemma 3.3, we have
\[
\left\| (t + \delta)^{1-\alpha} \int_0^{t+\delta} S(t + \delta - s)f(s, s^{1-\alpha}u(s), (Ku)(s)) \, ds \right\|
\]
\[
- t^{1-\alpha} \int_0^t S(t - s)f(s, s^{1-\alpha}u(s), (Ku)(s)) \, ds \right\|
\]
\[
\leq \int_0^t \left\| (t + \delta)^{1-\alpha} S(t + \delta - s) - t^{1-\alpha} S(t - s) \right\| h(s) \, ds + (t + \delta)^{1-\alpha} \int_t^{t+\delta} \left\| S(t + \delta - s) \right\| h(s) \, ds
\]
\[
\leq \sup_{0 \leq s \leq t \leq T - \delta} \| q_s(t + \delta) - q_s(t) \| T^{1-\alpha} \int_0^t (t + \delta - s)^{\alpha-1} h(t) \, dt
\]
\[
+ 2MT^{1-\alpha} \delta^{1-\alpha} \int_0^t h(s) \, ds + MT^{1-\alpha} \int_t^{t+\delta} (t + \delta - s)^{\alpha-1} h(s) \, ds
\]
\[
\leq \frac{T^{1-\alpha} H \left[ (t + \delta)^{1-\alpha} - \delta^{1-\alpha} \right]}{\left( \frac{a-\alpha}{1-\alpha} \right)^{1-\alpha}} \sup_{0 \leq s \leq t T - \delta} \| q_s(t + \delta) - q_s(t) \| \leq \frac{HT^{1-\alpha} \delta^{1-\alpha}}{\left( \frac{a-\alpha}{1-\alpha} \right)^{1-\alpha}} + 2MT^{1-\alpha} \delta^{1-\alpha} HT^{1-\alpha} \delta^{a-\alpha}.
\]
(3.10)

Since the function \( t \mapsto t^{1-\alpha} S(t) \) is uniformly continuous over \([0, T]\), we have that
\[
\lim_{\delta \to 0^+} \sup_{0 \leq s \leq t \leq T - \delta} \| q_s(t + \delta) - q_s(t) \| = 0.
\]
Let \( \delta \to 0^+ \), the right side of inequality (3.10) tends to zero. We obtain that the function
\[
t^{1-\alpha} \int_0^t S(t - s)f(s, s^{1-\alpha}u(s), (Ku)(s)) \, ds
\]
is continuous on \([0, T]\). Observe that the function \( t \mapsto t^{1-\alpha}(x - g(u)) \) is continuous on \([0, T]\). The combination of (3.3) and (3.10) implies that \( t^{1-\alpha} \mathcal{J}(t) \) is continuous on \([0, T]\), which implies
that $\mathcal{N}u \in C_{1-\alpha}([0, T], X)$.

Next, we shall prove that the operator $\mathcal{N} : C_{1-\alpha}([0, T], X) \to C_{1-\alpha}([0, T], X)$ is contraction mapping on $C_{1-\alpha}([0, T], X)$. Let $u, v \in C_{1-\alpha}([0, T], X)$. We compute

$$
\|\mathcal{N}u - \mathcal{N}v\|_s \\
\leq \max_{t \in [0, T]} \|t^{1-\alpha}S(t)\| \|g(u) - g(v)\| \\
+ \max_{t \in [0, T]} t^{1-\alpha} \int_0^t \|S(t-s)\| f(s, s^{1-\alpha}u(s), (Ku)(s)) - f(s, s^{1-\alpha}v(s), (Kv)(s)) \|ds \\
\leq Mb\|u - v\|_s + M \max_{t \in [0, T]} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1}m_1(s) \|s^{1-\alpha}(u(s) - v(s))\|ds \\
+ MN \max_{t \in [0, T]} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1}m_2(s) \int_0^s \|k(s, \sigma)(u(\sigma) - v(\sigma))\|d\sigma ds \\
\leq Mb\|u - v\|_s + MM_1 \max_{t \in [0, T]} t^{1-\alpha} \left( \int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha}} ds \right)^{1-\alpha_1} \|u - v\|_s \\
+ \frac{MN}{\alpha} \max_{t \in [0, T]} t^{1-\alpha} \int_0^t (t-s)^{\alpha-1}m_2(s) s^{\alpha} ds \|u - v\|_s \\
\leq Mb\|u - v\|_s + MM_1 \max_{t \in [0, T]} t^{1-\alpha} \left( \int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha}} ds \right)^{1-\alpha_1} \|u - v\|_s \\
+ \frac{MN T^{\alpha}}{\alpha} \max_{t \in [0, T]} t^{1-\alpha} \left( \int_0^t (t-s)^{\frac{\alpha-1}{1-\alpha}} ds \right)^{1-\alpha_2} \|u - v\|_s \\
\leq Mb\|u - v\|_s + MM_1 T^{1-\alpha} \left( \frac{\alpha-1}{1-\alpha_1} \right)^{1-\alpha_1} \|u - v\|_s + \frac{MM_2 NT^{\alpha+1-\alpha_2}}{\alpha \left( \frac{\alpha-1}{1-\alpha_2} \right)^{1-\alpha_2}} \|u - v\|_s.
$$

So we have

$$
\|\mathcal{N}u - \mathcal{N}v\|_s \leq \Omega \|u - v\|_s.
$$
By Banach contraction principle, we can obtain that $\mathcal{N}$ has an unique fixed point which is just the solution of system (1.1). □

**Theorem 3.5** Assume that $A$ generates a fractional resolvent and $H_1$-$H_3$ hold. If $Mb < 1$ and there exists an $r > 0$ such that

$$M\left[\|x\| + \max_{u \in B_r} \|g(u)\|\right] + \frac{MT^{1-\alpha_3}}{\left(\frac{\alpha_1-1}{1-\alpha_3}\right)^{1-\alpha_3}}\|h\|_{L^{\alpha_3}} \leq r.$$  

Then system (1.7) has at least one solution.

**Proof.** We consider the operator $\mathcal{N} : C_{1-\alpha}([0,T], X) \to C_{1-\alpha}([0,T], X)$ defined by (3.6).

From the proof of the above theorem, we know $\mathcal{N}$ is well defined. We divide $\mathcal{N}$ into two operators

1. $$(Au)(t) := S(t)[x - g(u)],$$
2. $$(Bu)(t) := \int_0^t S(t-s)f(s, s^{1-\alpha}u(s), (Ku)(s))ds.$$  

Let $u, v \in B_r$. Assume that $t \in (0,T]$. By the inequality (3.7), we have

$$\|t^{1-\alpha}(Au)(t) - t^{1-\alpha}(Bv)(t)\| \leq \|t^{1-\alpha}S(t)\|\|x\| + \|g(u)\| + \left\|t^{1-\alpha} \int_0^t S(t-s)f(s, s^{1-\alpha}v(s), (Kv)(s))ds\right\|$$

$$\leq M\left[\|x\| + \max_{u \in B_r} \|g(u)\|\right] + \frac{MT^{1-\alpha_3}}{\left(\frac{\alpha_1-1}{1-\alpha_3}\right)^{1-\alpha_3}}\|h\|_{L^{\alpha_3}} \leq r,$$

which implies that (i) of Theorem 2.5 holds.

For any $u, v \in B_r$, we have that

$$\|t^{1-\alpha}(Au)(t) - t^{1-\alpha}(Av)(t)\| = \|t^{1-\alpha}S(t)g(u) - t^{1-\alpha}S(t)g(v)\|$$

$$\leq Mb\|u - v\|_*.$$  

The assumption $Mb < 1$ implies that (ii) of Theorem 2.5 holds.
Assume that \( u_n, u \in B_r, n = 1, 2, \cdots, u_n \rightarrow u \) in the norm of \( B_r \). From the proof of Theorem 3.4, we derive that
\[
\| t^{1-\alpha}(Bu_n) - t^{1-\alpha}(Bu) \| \leq \frac{MM_1T^{1-\alpha}}{(1-\alpha)} \| u_n - u \|_\ast + \frac{MM_2NT^{\alpha+1-\alpha_2}}{\alpha(1-\alpha_2)} \| u_n - u \|_\ast.
\]

Then, \( B \) is continuous. The combination of inequality (3.7) and inequality (3.10) implies that \( B \) is uniformly bounded and equicontinuous. By the Arzela-Ascoli’s theorem, \( B \) is compact. This means that (iii) of Theorem 2.5 holds. The proof is completed directly by Theorem 2.5.

**Example 3.6** As an application, we consider the following partial differential equations with Dirichlet boundary conditions.

\[
\begin{align*}
D_t^\alpha u(t, x) &= k^2 \frac{\partial^2}{\partial x^2} u(t, x) + \frac{\mu_1 t^{1-\alpha}}{1+1-\alpha|u(t,x)|} + \frac{\mu_2}{1+|\int_0^t e^{t-s}u(s,x)ds|}, \\
u(t,0) &= u(t,1) = 0, \\
\lim_{t \to 0^+} J^{1-\alpha} t u(t,x) &= p(x) + g(u(\cdot, x)).
\end{align*}
\]  

(3.11)

In order to write the system (3.11) as the abstract form of system (1.1), we take

- \( X = L^2(0,\pi) \);
- \( A = k^2 \frac{\partial^2}{\partial x^2} \) with domain \( D(A) = \{ g \in W^{2,2}(0,1) : g(0) = g(1) = 0 \} \);
- \( r(t,s) = e^{t-s}, 0 \leq s \leq t \leq 1 \);
- \( f : I \times X \times X \to X \) defined by
  \[
  f(t, w_1(\cdot), w_2(\cdot)) = \frac{\mu_1 |w_1(\cdot)|}{1 + |w_1(\cdot)|} + \frac{\mu_2}{1 + |w_2(\cdot)|}, \quad t > 0, \ w_1(\cdot), w_2(\cdot) \in X.
  \]

Observe that \( A \) is closed, densely defined and has eigenvalues \( \lambda_n = -k^2n^2\pi^2 \) with eigenfunctions \( \{ \sin(nx) \}_{n \in \mathbb{N}} \). Moreover, we can obtain \( \rho(A) = \mathbb{C}/\{ \sin(kn\pi x) \}_{n \in \mathbb{N}} \). For \( g(x) = \sum_{n=1}^\infty g_n \sin(kn\pi x) \), we define the family \( \{ S(t) \}_{t \geq 0} \) by
\[
(S(t)g)(x) = \sum_{n=1}^\infty t^{1-\alpha} E_{\alpha,\alpha}(-k^2n^2\pi^2 t^\alpha)g_n \sin(kn\pi x).
\]
By [17] Example 3.1, it follows that \( \{S(t)\}_{t>0} \) generates a fractional resolvent. By [26], it follows that the function \( E_{\alpha,\alpha}(-\cdot) \) is complete monotonicity thereby monotone nonincreasing function over \((0, +\infty)\). Since \( t \mapsto E_{\alpha,\alpha}(-t) \) is continuous on \([0, +\infty)\), function \( E_{\alpha,\alpha}(-\cdot) \) is nonincreasing on \([0, +\infty)\). We compute

\[
\|t^{1-\alpha}S(t)\|^2 = \sup_{g \in L^2(0,1)} \|t^{1-\alpha}T(t)g\|^2 \\
= \sup_{g \in L^2(0,1), \|g\|=1} \int_0^1 |(T(t)g)(x)|^2 \, dx \\
= \sup_{g \in L^2(0,1), \|g\|=1} \sum_{n=1}^{\infty} \left( E_{\alpha,\alpha}(-k^2n^2\pi^2t^{\alpha}) \right)^2 (g_n)^2 \\
\leq \left( E_{\alpha,\alpha}(-k^2\pi^2t^{\alpha}) \right)^2 \sup_{g \in L^2(0,1), \|g\|=1} \sum_{n=1}^{\infty} (g_n)^2 \\
\leq \frac{1}{\Gamma(\alpha)^2}.
\]

This means that \( M \leq \frac{1}{\Gamma(\alpha)} \).

We compute

\[
\|f(t, w_1(\cdot), w_2(\cdot)) - f(t, v_1(\cdot), v_2(\cdot))\| \\
\leq \mu_1 \left\| \frac{|w_1(\cdot)|}{1 + |w_1(\cdot)|} - \frac{|v_1(\cdot)|}{1 + |v_1(\cdot)|} \right\| + \mu_2 \left\| \frac{1}{1 + |w_2(\cdot)|} - \frac{1}{1 + |v_2(\cdot)|} \right\| \\
\leq \mu_1 \|w_1(\cdot) - v_1(\cdot)\| + \mu_2 \|w_2(\cdot) - v_2(\cdot)\|.
\]

(I) In the case that \( g(u(\cdot, x)) = \sum_{i=1}^{l} a_i u(t_i, x) \), we have

\[
\|\sum_{i=1}^{l} a_i u(t_i, \cdot) - \sum_{i=1}^{l} a_i v(t_i, \cdot)\| \leq \sum_{i=1}^{l} a_i t_i^{\alpha-1} \|t_i^{1-\alpha}u(t_i, \cdot) - t_i^{1-\alpha}v(t_i, \cdot)\| \\
\leq \sum_{i=1}^{l} a_i t_i^{\alpha-1} \|u - v\|_*;
\]

by Theorem 3.34 if there exist \( \alpha_1, \alpha_2 \in (0, \alpha) \) such that

\[
\frac{\mu_1}{\Gamma(\alpha)} \left( \frac{\alpha - \alpha_1}{1 - \alpha_1} \right)^{1-\alpha_1} + \frac{\sum_{i=1}^{l} a_i t_i^{\alpha-1}}{\Gamma(\alpha)} + \frac{\mu_2 e}{\Gamma(\alpha + 1)} \left( \frac{\alpha - \alpha_2}{1 - \alpha_2} \right)^{1-\alpha_2} < 1,
\]

15
then system (1.1) has a unique solution.

(II) In the case that

\[ g(u(\cdot, x)) = \sum_{i=1}^{l} a_i |u(t_i, x)| / (1 + \sum_{i=1}^{l} a_i |u(t_i, x)|), \]

we have

\[ \|g(u(\cdot, \cdot)) - g(v(\cdot, \cdot))\| \leq \left\| \sum_{i=1}^{l} a_i |u(t_i, \cdot)| - \sum_{i=1}^{l} a_i |v(t_i, \cdot)| \right\| \]

\[ \leq \sum_{i=1}^{l} a_i t_i^{\alpha - 1} \|t_i^{1-\alpha} u(t_i, \cdot) - t_i^{1-\alpha} v(t_i, \cdot)\| \]

\[ \leq \sum_{i=1}^{l} a_i t_i^{\alpha - 1} \|u - v\|_*; \]

by Theorem 3.5 if

\[ \frac{\mu_1}{\Gamma(\alpha) \left( \frac{q-\alpha}{q-\alpha_1} \right)^{1-\alpha_1} } < 1, \]

then system (1.1) has at least one solution.

References

[1] A. Anguraj, P. Karthikeyan, G.M. N’Guerekata, Nonlocal Cauchy problem for some fractional abstract differential equations in Banach spaces, Commun. Math. Anal., 6(1) (2009) 31-35.

[2] K. Balachandran, J.Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, Nonlinear Anal., 71 (2009) 4471-4475.

[3] E. Bazhlekov, Fractional Evolution Equations in Banach Spaces. University Press Facilities, Eindhoven University of Technology, 2001.
[4] E. Bazhlekova, Existence and uniqueness results for a fractional evolution equation in Hilbert space, Fract. Calc. Appl. Anal., 15(2) (2012) 232-243.

[5] H. Brezis, Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert. Math. Studies 5, North-Holland, Amsterdam, 1973.

[6] L. Byszewski, V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, Applicable Anal., 40(1) (1991) 11-19.

[7] S.D. Eidelman, A.N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations, 199 (2004) 211-255.

[8] H.X. Fan, J. Mu, Initial value problem for fractional evolution equations, Adv. Difference Equations, 2012, 2012:49 http://www.advancesindifferenceequations.com/content/2012/1/49.

[9] X. Fu, K. Ezzinbi, Existence of solutions of a semilinear functional-differential evolution equations with nonlocal conditions, Nonlinear Anal. 54 (2003) 215-227.

[10] E. P. Gatsori, Controllability results for nondensely defined evolution differential inclusions with nonlocal conditions, J. Math. Anal. Appl., 297(1) (2004) 194-211.

[11] R. Hilfer, Fractional time evolution, in Applications of Fractional Calculus in Physics, R. Hilfer, ed., World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000, pp. 87-130.

[12] L. Hu, Y. Ren, R. Sakthivel, Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays, Semigroup Forum., 79 (2009) 507-514.
[13] R.W. Ibrahim, Existence and uniqueness of holomorphic solutions for fractional Cauchy problem, J. Math. Anal. Appl., 380 (2011) 232-240.

[14] M.A. Krasnosel’skii, Topological Methods in the Theory of Nonlinear Integral Equations. Pergamon, Elmsford (1964).

[15] V. Lakshmikantham, S. Leela, Theory of fractional dynamic systems, Cambridge Academic Publishers, Cambridge (2009).

[16] F. Li, J. Liang, H.K. Xu, Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions, J. Math. Anal. Appl., 391 (2012) 510-525.

[17] K.X. Li, J.G. Peng, Fractional resolvents and fractional evolution equations, Appl Math Lett., 25 (2012) 808-812.

[18] K.X. Li, J.G. Peng, On some properties of the of the $\alpha$-Exponential function, Integral Transforms Spec. Funct., In press.

[19] J. Liang, J.H. Liu, T.J. Xiao, Nonlocal impulsive problems for nonlinear differential equations in Banach spaces, Math. Comput. Model., 49(3-4) (2009) 798-804.

[20] Y. Lin, J. Liu, Semilinear integrodifferential equations with nonlocal Cauchy problem, Nonlinear Anal., 26 (1996) 1023-1033.

[21] M.M. Meerschaert, E. Nane, P. Vellaisamy, Fractional Cauchy Problems on bounded domains, Ann. Anal., 37 (2009) 979-1007.

[22] R. Metzler and J. Klafter, The random walk’s guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 339 (2000) 1-77.
[23] G.M. N’guerekata, Existence and uniqueness of an integral solution to some Cauchy problem with nonlocal conditions, Differential and Difference Equations and Applications, pp 843-849, Hindawi Publ, Corp, New York, 2006.

[24] G.M. N’guerekata, A Cauchy problem for some fractional abstract differential equation with nonlocal condition, Nonlinear Anal., 70 (2009) 1873-1876.

[25] I. Podlubny, Fractional Differential Equations, Academic Press, New Yourk, 1999.

[26] W.R. Schneider, Completely monotone generalized Mittag-Leffler functions, Expositiones Mathematicae 14 (1996) 3-16.

[27] H.M. Srivastava, Ž. Tomovski, Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel, Appl. Math. Comput., 211 (2009) 198-210.

[28] T.J. Xiao, J. Liang, Existence of classical solutions to nonautonomous nonlocal parabolic problems, Nonlinear Anal., 63 (2003) 225-232.

[29] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Anal RWA., 11 (2010) 4465-4475.