Principal bundle groupoids, their gauge group and their nerve.

Alfonso Garmendia*†       Sylvie Paycha‡

March 9, 2023

Abstract

We consider groupoids in the category of principal bundles, which we call principal bundles (PB) groupoids. Inspired by work by Th. Nikolaus and K. Waldorf, we generalise bundle gerbes over manifolds to bundle gerbes over groupoids and discuss a functorial correspondence between PB groupoids and bundle gerbes over groupoids. From a PB groupoid over a fibre product groupoid, we build a bundle gerbe over another fibre product groupoid. Conversely, from a bundle gerbe over a Lie groupoid, we build a PB groupoid. It has a trivial base and from any PB groupoid with trivial base, we build a bundle gerbe over a Lie groupoid. In that case, the resulting bundle gerbe is isomorphic as a groupoid to a partial quotient of the PB groupoid. We describe the nerves of PB groupoids and their partial quotients, which are simplicial objects in the category of principal bundles. Applying this construction enables us to define the inner transformation group of the nerve of a partial quotient groupoid and to describe the transformations of the corresponding bundle gerbe.

Contents

1 Principal bundle groupoids
   1.1 Review of Lie 2-groups ........................................ 4
   1.2 From Lie 2-group actions to principal bundle groupoids .... 5
   1.3 The partial quotient groupoid of a PB groupoid .......... 6
   1.4 Groupoid principal bundles arising from groupoid actions .. 8
   1.5 $G(1)$-principal 2-bundles in terms of principal bundle groupoids .................................................. 8
2 A functorial correspondence between PB groupoids and bundle gerbes 12
   2.1 Bundle gerbes over Lie groupoids .............................. 12
   2.2 From a PB groupoid over a fiber product groupoid to a bundle gerbe ............................................... 14
   2.3 From a bundle gerbe over a groupoid to a base trivial PB groupoid ............................................... 16
   2.4 From a base trivial PB groupoid to a bundle gerbe over a groupoid ............................................... 17
   2.5 Conclusions ....................................................... 19
3 Nerve of a principal bundle groupoid 20
   3.1 Nerve of a small category ....................................... 20
   3.2 The nerve of a Lie 2-group ...................................... 20
   3.3 The nerve of a PB groupoid and of its partial quotient groupoid .................................................. 21

*Departament de Matemàtiques, Universitat Politècnica de Catalunya (UPC), Barcelona, Spain
†Centre de Recerca Matemàtica, Barcelona, Spain Email: agarmendia@crm.cat
‡Institut für Mathematik, University of Potsdam, Potsdam, Germany Email: paycha@math.uni-potsdam.de
Preamble

We choose to work with left group actions for principal bundles. The main reason for this choice is aesthetic since left actions combine nicely with the groupoid composition, taking an arrow from right to left in contrast with [NW13], where they are taken from left to right. Nevertheless, our conventions for bundle gerbes over Lie 2-groups coincides with those of [NW13].

Introduction

This is an exploratory paper prompted by the wish to investigate higher analogs of gauge groupoids. Our interest for this class of groupoids stems from the fact that they correspond to the Lie groupoids that host direct connections studied in [AFBP22].

We consider principal bundles groupoids, namely groupoids in the category of principal bundles. Principal bundles groupoids are of the form $P(1) \to M(1)$, with $P(1) \rightrightarrows P$ and $M(1) \rightrightarrows M$, two Lie groupoids, with a structure group given by a Lie group groupoid $G(1) = H \rtimes G \rightrightarrows G$ (Definition 1.15). Their structure group, also called a Lie 2-group, can also be viewed as a crossed module $H \rtimes G$ built from the action of a Lie group $G$ on another Lie group $H$ and a homomorphism $d : H \to G$.

Our definition differs from other known notions of principal bundles that involve the action of a groupoid, such as principal groupoid bundles arising in [MM03], principal bundles over groupoids in [LGTX07], or principal 2-bundles considered in [NW13] (these last ones can nevertheless be viewed as principal bundle groupoids, see Proposition 1.32), hence the terminology “principal bundle groupoids” we adopt here. The main differences lie in the fact that

- we allow $M(1)$ to be any groupoid over $M$, not only the identity groupoid,
- we allow $G(1)$ to be any Lie 2-group, generalising other approaches that only allow Lie groups,
- the action we consider is not a groupoid action on a manifold but a Lie 2-group action on a groupoid.

We consider the partial quotient by the action on $P(1)$ of the identity bisection of $H \rtimes G$, i.e. by $\{e\} \rtimes G$, and use the shorthand notation $P(1) / G$ (Def. 1.21.1) justified by:

$$P(1) / G := (P(1) / \{e\} \rtimes G) \rightrightarrows M.$$

Partial quotients $P(1) / G$ are relevant for the study of Lie 2-group bundle gerbes over a Lie groupoid $Y(1) \rightrightarrows Y$. These generalise the classical notion of Lie 2-group bundle gerbe over a smooth manifold [NW13 Definition 5.1.1]. Specialising to bundle gerbes over fibre bundle Lie groupoids
$Y^{[2]} \Rightarrow Y$ yields back the classical Lie 2-group bundle gerbe over the orbit space $M \cong Y/Y^{[2]}$. From a principal Lie 2-group bundle groupoid $P^{(1)} \Rightarrow P$ over a fibre bundle groupoid $Y^{[2]} \Rightarrow Y$, there is a bundle gerbe over the fibre bundle groupoid $P^{[2]} \Rightarrow P$, which yields a Lie 2-group bundle gerbe over the quotient space $P/P^{[2]} \cong M$. We also build a Lie 2-group bundle gerbe over $Y^{(1)}$ from a Lie 2-group principal bundle groupoid $P^{(1)} \Rightarrow P$ over the Lie groupoid $Y^{(1)} \Rightarrow Y$, with trivial base, namely such that $P \cong Y \times G$. In both cases, the resulting bundle gerbes are Morita equivalent to the partial quotient $P^{(1)}/G$. When $P^{(1)} \Rightarrow P$ has trivial base, the resulting bundle gerbe is actually isomorphic to $P^{(1)}/G$ as a groupoid.

Finally, we discuss the inner transformations of a principal $H \rtimes G$-bundle groupoid $P^{(1)} \rightarrow M^{(1)}$ as well as inner transformations of its partial quotient $P^{(1)}/G$, then generalizing them to the $k$-nerves of the corresponding objects. In particular, this applies to inner transformations of the bundle gerbe built from a principal Lie 2-group bundle groupoid $P^{(1)} \Rightarrow P$ with trivial base.

Let us briefly describe the structure of this note.

- In Section 1, we review some known facts on Lie 2-groups and define principal bundle groupoids, relating them to affine concepts in the literature, such as in [NW13], [LGTX07] and [MM03]. Proposition 1.32 describes a principal 2-bundle on a manifold $M$ as defined in [NW13] as a principal bundle groupoid over a Lie groupoid which is Morita equivalent to $M$.

- Section 2 gives a functorial correspondence, in part inspired by [NW13], between certain classes of PB groupoids and bundle gerbes over groupoids (see Definition 2.4). From a bundle gerbe $B \rightarrow Y^{(1)}$ over a groupoid $Y^{(1)} \rightarrow Y$, we build a trivial base groupoid $P^{(1)} \Rightarrow P$ (Proposition 2.13) which can in turn be sent to a bundle gerbe $B \cong P^{(1)}/G$ (Proposition 2.11). In Theorem 2.8 starting from a PB groupoid over $Y^{[2]} = Y \times_M Y$ we build a bundle gerbe over $P \times_M P$: these two groupoids are Morita equivalent to the identity groupoid over the manifold $M$.

- In Section 3 we describe the nerve $N^*(P^{(1)})$ of a Principal bundle groupoid $P^{(1)}$ as a simplicial set of principal bundles. Where, $G^{(k)} \hookrightarrow P^{(k)} \rightarrow M^{(k)}$ (Theorem 3.3). We further interpret the nerve $N^*(P^{(1)})$ of $P^{(1)}$ as a principal $G$-bundle over the nerve $N^*(P^{(1)}/G)$ of the quotient groupoid (Proposition 3.4).

- Section 4 is dedicated to inner automorphisms of PB groupoids and their nerves. Let us denote by $\text{Mor}(P^{(1)})$, the morphisms of principal bundle groupoid over the identity on its base $M^{(1)}$. We get an isomorphism for nerves which makes clear that any morphism is an automorphism (see (4.0.2)):

$$C_{H \rtimes G}^{\infty}(P^*, H^* \rtimes G) \cong \text{Mor}(P^*) = \text{Aut}(P^*).$$

The previous results lead to a natural definition (Definition 4.1) of the set $\text{Aut}(P^*/G)$ of morphisms of the nerve of the partial quotient groupoid $P^{(1)}/G$ in terms of the group $\text{Aut}(P^*)$ and by Theorem 4.2 we have the following group isomorphism:

$$\text{Aut}(P^*/G) \cong C_{H \rtimes G}^{\infty}(P^*, H^*).$$

Then, specialising to case of the (trivial base) PB groupoid $P^{(1)} = \Psi(B)$ built from a bundle gerbe $B$ as in Proposition 2.11 In this case and $B \cong P^{(1)}/G$ which yields a decrion of the automorphism group of a bundle gerbe (Corollary 4.3):

$$\text{Aut}(B^*) = \text{Aut}(P^*/G) \cong C_{H \rtimes G}^{\infty}(P^*, H^*).$$
We end this paper with an example, specialising the above to the pair groupoids $P^{(1)} := P^{(1)}(P)$, resp. $M^{(1)} := P^{(1)}(M)$ of a principal bundle $G \rightrightarrows P \to M$. In that case $H = G$ and the partial quotient $P^{(1)}/G = G^{(1)}(P)$ is the gauge groupoid of $P$, leading to Corollary 4.4 which shows the group isomorphism:

$$\text{Aut}(G^*(P)) \simeq C_{G*G}^\infty(P^*(P), G^*),$$

and a canonical embeddings between the various groups as in Theorem 4.5:

$$\text{Aut}_M(P) \hookrightarrow \text{Aut}(P^{(1)}(P)) \hookrightarrow \text{Aut}(P^*(P)) \simeq C_{G*G}^\infty(P^*(P), G^* \rtimes G).$$

With PB groupoids at hand, the next step would be to equip them with (infinitesimal) connections, which is the object of ongoing work by the first author.

Acknowledgments

We are very grateful to Sara Azzali for many fruitful discussions at various stages of the paper. We were also inspired by prior conversations on higher gauge theory with Alessandra Frabetti, whom we would also like to thank.

This work is supported by the Spanish State Research Agency, through the Severo Ochoa and María de Maeztu Program for Centers and Units of Excellence in R&D (CEX2020-001084-M)

1 Principal bundle groupoids

In this section, after some necessary prerequisites, we define principal bundle groupoids.

Let us recall that a groupoid is a small category $\mathcal{M}$ in the category SETS of sets such that every arrow is invertible. This means that the arrows $M^{(1)} := \text{Arr(\mathcal{M})}$ (also called morphisms for a big category) and the points $M := \text{Ob(\mathcal{M})}$ (also called objects for a big category) are both sets and the correspondences:

- source and target $s : M^{(1)} \to M$ and $t : M^{(1)} \to M,$
- composition $\circ : M^{(1)} \times_1 M^{(1)} \to M^{(1)},$
- identity and inverse $e : M \to M^{(1)}$ and $(\cdot)^{-1} : M^{(1)} \to M^{(1)},$

are all maps (morphisms in SETS).

A Lie groupoid is a groupoid $\mathcal{M}$ in the category of manifolds. That means that $M^{(1)}, M$ are manifolds and $(s, t, \circ, e, (\cdot)^{-1})$ are smooth maps (the condition of $s, t$ being submersion is required to make sense for $\circ$ being smooth). In the rest of this paper we will denote groupoids as $M^{(1)} \Rightarrow M,$ the double arrows stands for the source and target maps.

In this section, we introduce groupoids in the category of principal bundles, which we call principal bundle groupoids or equivalently, for a Lie 2-group $G^{(1)},$ a principal $G^{(1)}$-bundle groupoid or a principal bundle groupoid with structure group $G^{(1)}.$
1.1 Review of Lie 2-groups

Definition 1.1. A Lie 2-group is a groupoid in the category of Lie groups. This is a small category whose space of arrows \( G^{(1)} \) and space of points \( G \) are both Lie groups and whose groupoid structure maps \((s, t, o, e, (-)^{-1})\) are Lie group morphisms (homomorphisms).

Example 1.2. Any Lie group \( G_0 \) is a Lie 2-group where \( G^{(1)} = G_0, G = G_0 \), the group structure is given by \( G_0 \) and the groupoid structure is given by: \( s = t = e = (-)^{-1} = o = I_G \). The tangent space \( TG \) is also a Lie 2-group (more details in Example 1.10).

Since the category of Lie groups is included in the category of smooth manifolds, any Lie 2-group is a Lie groupoid.

Definition 1.3. A (smooth) crossed module over the category of groups is a pair of Lie groups \((H, G)\) together with an action (by homomorphisms) \( C: G \ni g \mapsto Cg \in \text{Aut}(H) \) of \( G \) on \( H \) and a map \( d: H \rightarrow G \) which defines an action (by diffeomorphisms) of \( H \) on \( G \) by \((h, g) \mapsto d(h) \cdot g\), with compatibility conditions between the two maps \( C \) and \( d\):

\[
\begin{align*}
\bullet \ d(Cgh) &= g d(h) g^{-1}, \\
\bullet \ C_{d(h)} h' &= hh'h^{-1},
\end{align*}
\]

for all \( g \in G \) and \( h, h' \in H \).

Remark 1.4. As we shall see, \( d \) defines a groupoid structure, \( C \) a group structure.

There is a categorical equivalence between (smooth) crossed modules over groups—a notion initially introduced by Whitehead in [Whi46, Whi49]—and (Lie) 2-groups, also called (Lie) group groupoids, see [BS76], [BH11], [BL04]. Let us recall this equivalence in terms of semi-direct products of groups.

Proposition 1.5. (Lie 2-group of a crossed module) Let \((H, G, C, d)\) be a (smooth) crossed module over the category of groups.

1. \( H \times G \) is the set of arrows of a Lie groupoid over \( G \) with source and target given by \( t(h, g) := d(h)g, s(h, g) := g \) and the groupoid composition defined as:

\[
\text{Groupoid Str:} \quad (h_2, t(h_1, g_1)) \circ (h_1, g_1) := (h_2 h_1, g_1).
\]

2. The groupoid \( H \times G \Rightarrow G \), as in item 1. above, is a Lie 2-group with the semi-direct product:

\[
\text{Group Str:} \quad (h_2, g_2) \cdot (h_1, g_1) := (h_2 C_{g_2} h_1, g_2 g_1),
\]

as the product on the arrows and the natural product in \( G \) on the points.

Proposition 1.6. (Crossed module of a Lie 2-group) Let \( G^{(1)} \Rightarrow G \) be a Lie 2-group and \( H := \ker(s) \).

- The group \( G \) acts on \( H \) via homomorphisms \( Cgh := e(g)h e(g)^{-1} \in H \) for all \( g \in G \);
- \((H, G, C, d := t|_H)\) is a (smooth) crossed module in the category of groups.
Proof. Note that $s(C_s h) = s(e(g)) \cdot s(h) \cdot s(e(g))^{-1} = s(e(g)) \cdot s(e(g))^{-1} = 1$ which implies that $C_s(h)$ lies in $H = \text{Ker}(s)$.

Also, clearly, we have $t(C_s h) = e(g)t(h)e(g)^{-1}$ and

$$(C_{t(h)} h', t(h)) = (e, t(h)) \cdot (h', e) = \left((h, e) \circ (h^{-1}, t(h))\right) \cdot (h'h, e) \circ (e, e)$$

$$= (h'h, e) \circ (h^{-1}, t(h)) = (h'h^{-1}, t(h))$$

so that Conditions (1.3.1) are verified.

□

**Proposition 1.7.** (see e.g. [BL04] and references therein) Let $G^{(1)} \Rightarrow G$ be a Lie 2-group, $(H, G, C, t_{|H})$ the crossed module of Prop. 1.6 and $H \times G \Rightarrow G$ the Lie 2-group of Prop. 1.5. Then the map:

$$\varphi: H \triangleright G \to G^{(1)}; (h, g) \mapsto h e(g)$$

is a Lie 2-group isomorphism (a group and Lie groupoid isomorphism).

**Remark 1.8.** The isomorphism $\varphi$ is given by a known construction. If $s: G^{(1)} \to G$ is a homomorphism with splitting $e: G \to G^{(1)}$ there is an isomorphism $\varphi: \ker(s) \triangleright G \to G^{(1)}$.

Using the above identifications, we write any Lie 2-group as $H \rtimes C G \Rightarrow G$ where $G, H$ are Lie groups. In the sequel, for the sake of simplicity, we shall often drop the subscript $C$.

**Example 1.9.** Let $H$ be any normal subgroup of a Lie group $G$. The conjugate action $C: g \mapsto (C_g: h \mapsto g h g^{-1})$ of $G$ on $H$ and the inclusion map $d: H \to G$ define a crossed module and therefore a Lie 2-group $H \triangleright G \Rightarrow G$. When $H = G$ one gets a Lie 2-group $G \rtimes G \Rightarrow G$, which is isomorphic to the pair groupoid of $G$.

**Example 1.10.** Let $G$ be a Lie group and $V$ a vector space ($V$ is also a Lie group with the sum) carrying a representation $C: G \to \text{Aut}(V)$ of $G$. The action combined with the map $d: V \to G$, which sends $v$ to $e_G$ defines a crossed module and hence a Lie 2-group $V \rtimes C G \Rightarrow G$.

The above applied to $V = \mathfrak{g}$ and $C$ the adjoint action $\text{Ad}$ of $G$ on $\mathfrak{g}$ equips $\mathfrak{g} \rtimes \text{Ad} G$ with a Lie 2-group structure. Since the tangent bundle $T G$ of any Lie group $G$ is isomorphic to $\mathfrak{g} \rtimes \text{Ad} G$, $T G$ inherits a Lie 2-group structure.

**Example 1.11.** The action of $O(2k)$ on the Clifford algebra $\text{Cliff}(2k)$ gives an action of $O(2k)$ in $\text{Pin}(2k)$ by homeomorphisms. The mentioned action and the covering projection $d: \text{Pin}(2k) \to O(2k)$ defines a crossed module and therefore a Lie 2-group $\text{Pin}(2k) \rtimes O(2k) \Rightarrow O(2k)$.

### 1.2 From Lie 2-group actions to principal bundle groupoids

In this paragraph we define principal Lie 2-group bundle groupoids which involve group actions.

Let us now recall the action of a Lie 2-group on another Lie groupoid.

**Definition 1.12.** A Lie 2-group action of a Lie 2-group $(H \rtimes G) \Rightarrow G$ on a Lie groupoid $P^{(1)} \Rightarrow P$ consists of Lie group actions of $H \rtimes G$ on the manifold $P^{(1)}$ and of $G$ on the manifold $P$ such that the action map

$$\begin{array}{ccc}
(H \rtimes G) \times P^{(1)} & \longrightarrow & P^{(1)} \\
\downarrow \circ \downarrow \circ \downarrow \circ \downarrow & & \downarrow \circ \downarrow \circ \downarrow \circ \downarrow \\
G \times P & \longrightarrow & P
\end{array}$$

is a Lie groupoid map from the cartesian product groupoid $(H \rtimes G) \times P^{(1)}$ to the Lie groupoid $P^{(1)}$. 
Example 1.13. When \( H = \{ e_H \} \), the group action of the Lie 2-group \( H \rtimes G \cong G \) on \( P(1) \) boils down to the mere group action of \( G \).

Just as not every Lie group \( G \) acting on a manifold \( P \) yields a quotient manifold \( P/G \), the action of a Lie 2-group \( H \rtimes G \) on a Lie groupoid \( P(1) \) does not always give rise to a Lie groupoid structure on the quotient \( P(1)/\!\!/H \rtimes G \). Yet, as in the case of Lie group actions on manifolds, it does under a freeness and properness assumption on the action.

Proposition 1.14. [Gar19] §5.2 and references therein] Consider a free and proper right action of a Lie 2-group \( (H \rtimes G) \triangleright \! \! \! \! \! \! \leftarrow G \) on a Lie groupoid \( P(1) \triangleright \! \! \! \! \! \! \leftarrow P \). Define the equivalence relations

\[
R := \{(p, gp) \in P \times P \mid p \in P \text{ and } g \in G\} \text{ on } P,
\]

\[
R^{(1)} := \{ (\xi, (h, g) \xi) \in P^{(1)} \times P^{(1)} \mid \xi \in P^{(1)} \text{ and } (h, g) \in H \rtimes G \} \text{ on } P^{(1)}.
\]

Then \( M^{(1)} := P^{(1)}/R^{(1)} \) and \( M := P/R \) are manifolds, and \( M^{(1)} \triangleright \! \! \! \! \! \! \leftarrow M \) acquires a canonical Lie groupoid structure making the quotient map a Lie groupoid fibration.

Let us recall that a Lie groupoid fibration is a Lie groupoid morphism \( \pi^{(1)} : P^{(1)} \to M^{(1)} \) such that the map \( \pi_{(1)} : P^{(1)} \to M^{(1)} \times_P P ; \phi \mapsto (\pi^{(1)}(\phi), s(\phi)) \) is a surjective submersion. This means that given an arrow \( \gamma \in M^{(1)} \) and a point \( p \in P \), over the source of \( \gamma \), there is a way to lift smoothly \( \gamma \) to an arrow in \( P^{(1)} \) starting in \( p \). The interested reader can find further details in [Mac05], see [CZ13] for a specific instance of the above proposition.

On the grounds of Proposition 1.14, we set the following definition, which compares with [CLS14] §2.8. Categorical principal bundles| in the category of Lie groupoids.

Definition 1.15. A principal \( (H \rtimes G) \)-bundle groupoid \( (P^{(1)} \triangleright \! \! \! \! \! \! \leftarrow P) \) over a Lie groupoid \( M^{(1)} \triangleright \! \! \! \! \! \! \leftarrow M \) consists of a free and proper group action of the Lie 2-group \( (H \rtimes G) \triangleright \! \! \! \! \! \! \leftarrow G \) on a Lie groupoid \( P^{(1)} \triangleright \! \! \! \! \! \! \leftarrow P \) whose quotient is isomorphic to \( M^{(1)} \triangleright \! \! \! \! \! \! \leftarrow M \), corresponding to the following diagramme:

\[
\begin{array}{ccc}
(H \rtimes G) \times P^{(1)} & \xrightarrow{(\pi_{(1)}, s \times s)} & P^{(1)} \\
\downarrow \rho & & \downarrow s \\
G \times P & \xrightarrow{s} & P \\
\end{array}
\]

\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
target equal to the projection \((TP \to P)\) and the composition equal to the addition. The following formula gives a Lie 2-group action of \(\mathfrak{g} \rtimes \text{Ad} G \Rightarrow G\) on \(TP\) (as a Lie groupoid):

\[
(w, g) \cdot v = a(w, gp) + dL_g(v) \quad \text{for} \quad v \in T_pP, \; w \in \mathfrak{g} \; \text{and} \; g \in G.
\]

The principal \(G\)-bundle \(P \to M\) therefore induces a principal bundle groupoid

\[(\mathfrak{g} \rtimes \text{Ad} G) \curvearrowright TP \to TM.\]

**Definition 1.19.** A (strict) morphism of \(H \rtimes G\)-principal bundle groupoids

\[f: \left(P^{(1)}_1 \to M^{(1)}_1\right) \longrightarrow \left(P^{(1)}_2 \to M^{(1)}_2\right)\]

consists of two groupoid morphisms \(f_P: P^{(1)}_1 \to P^{(1)}_2\) and \(f_M: M^{(1)}_1 \to M^{(1)}_2\) such that \(f_P\) is \((H \rtimes G)\)-equivariant and \(f_M\) is \((H \rtimes G)\)-invariant.

A principal bundle groupoid \(G^{(1)} \curvearrowright P^{(1)} \to M^{(1)}\) is trivial if there is an isomorphism of groupoids \(P^{(1)} \cong G^{(1)} \times M^{(1)}\) where the r.h.s is the cartesian product groupoid.

**Remark 1.20.** Strict morphisms defined as above extend to ones from a principal \((H_1 \rtimes G_1)\)-bundle groupoid to a principal \((H_2 \rtimes G_2)\)-bundle groupoid above a Lie 2-group morphism \(H_1 \rtimes G_1 \longrightarrow H_2 \rtimes G_2\), but we shall only refer to the case when \(H_1 \rtimes G_i = H \rtimes G, i = 1, 2\) with the identity Lie 2-group morphism, hence the above definition.

### 1.3 The partial quotient groupoid of a PB groupoid

We now consider the following groupoid:

**Definition 1.21.** Let \((H \rtimes G) \Rightarrow G\) be a Lie 2-group and \(P^{(1)} \Rightarrow P\) be a \((H \rtimes G)\)-principal bundle groupoid over \(M^{(1)} \Rightarrow M\). We define the partial quotient groupoid of \(P^{(1)}\) as the quotient by the identity bisection action of \(H \rtimes G\) i.e. \(P^{(1)}/(\{e_H\} \rtimes G)\) which we will denote simply by:

\[P^{(1)}/G \Rightarrow (P/G \cong M)\]

**Proposition 1.22.** The canonical projection map \(Q: P^{(1)} \longrightarrow (P^{(1)}/G)\) is a principal \(G\)-bundle and a principal \(G\)-bundle groupoid.

**Proof.** This follows from Proposition [1.14] with \(H = \{e_H\}\). The fact that it is a principal \(G\)-bundle follows from the fact that for \(H = \{e_H\}\), the group action of \(H \rtimes G\) amounts to the group action of \(G\) (see Example [1.13]).

**Example 1.23.** Given a principal \(G\)-bundle \(\pi_0: P \to M\), let \(P^{(1)} = P \times P\) be the pair groupoid of \(P\). The partial quotient groupoid of the principal bundle groupoid in Example [1.17] is called the gauge groupoid \(G(P) \Rightarrow M\) of \(P \to M\) ([Mac05, Example 1.1.15] and references therein).

### 1.4 Groupoid principal bundles arising from groupoid actions

In the previous paragraphs, we considered the group action of a Lie 2-group. Instead, one can consider its groupoid action leading to principal bundles stemming from the action of a Lie groupoid on a manifold [Hae84, §2.2], [MM03, §5.7], see also [NW13, Definition 2.2.1], and Definition [1.25] below. This approach is useful since principal bundle groupoids relate to bundle gerbes, see Section 2 below.
Definition 1.24. A left (resp. right) \textbf{action of a Lie groupoid} $G^{(1)} \xrightarrow{t_s} G$ on a manifold $B$ is given by a surjective submersion $\rho: B \to G$ called anchor map, and a smooth map

$$\star: G^{(1)} \times_{\rho} B \to B \quad \text{(resp. } \star: B \times_{t} G^{(1)} \to B)$$

called the action, as in the following diagram (resp. for the left action):

\[
\begin{array}{c}
G^{(1)} \\ \downarrow t \quad \quad \quad \quad \quad \downarrow s \\
B \\
\downarrow \rho \\
G
\end{array}
\]

such that, for any $b$ in $B$, any composable $\gamma_2, \gamma_1$ in $G^{(1)}$, we have

1. $\rho(\gamma_1 \star b) = t(\gamma_1)$ (resp. $\rho(b \star \gamma_1) = s(\gamma_1)$)
2. $\gamma_2 \star (\gamma_1 \star b) = (\gamma_2 \circ \gamma_1) \star b$ (resp. $(b \star \gamma_1) \star \gamma_2 = b \star (\gamma_1 \circ \gamma_2)$).

$B$ is also called a left (resp. right) $G^{(1)}$-space.

In an effort to clarify notation, this article uses ‘$\star$’ to denote groupoid actions whereas ‘·’ denotes group actions.

Let us recall the notion of principal groupoid bundle of [NW13]:

Definition 1.25. [Hae84, §2.2], see also [NW13, Definition 2.2.1] Let $G^{(1)} \Rightarrow G$ be a Lie groupoid and $M$ be a smooth manifold. A \textbf{principal $G^{(1)}$-bundle} (left (resp. right) bundle) on $M$ consists of a groupoid action on a smooth manifold $B$

$$\star: G^{(1)} \times_{\rho} B \to B \quad \text{(resp. } \star: B \times_{t} G^{(1)} \to B)$$

with anchor map $\rho: B \to G$, together with a surjective submersion $\Pi: B \to M$, as illustrated by the following diagram:

\[
\begin{array}{c}
G^{(1)} \\ \downarrow t \quad \quad \quad \quad \quad \downarrow s \\
B \\
\downarrow \rho \\
M
\end{array}
\]

such that:

- The quotient of $B$ by the action of $\star$ is a manifold $B/G^{(1)}$.
- $\Pi$ is $\star$-invariant and its induced map from $B/G^{(1)}$ to $M$ is a diffeomorphism.

Remark 1.26. In [MM03, §5.7], [Hae84], see also [Ros04, Definition 2.1], [Iva01, Definition 3.1], the authors call principal groupoid bundle a principal ($G^{(1)} \Rightarrow G$)-bundle. We prefer the latter terminology to avoid confusion with principal bundle groupoids as in Definition 1.15.

Remark 1.27. The equation $g \cdot p = p \cdot (g^{-1})$ transforms a right action into a left action and vice-versa. So, without loss of generality, we can use principal $G^{(1)}$-bundles induced by left or right actions.
Principal bundle groupoids as defined in Definition 1.15, which involve the group action of a Lie 2-group, can be viewed as a smooth version of principal categorical bundles discussed in [CLS14].

Yet they differ from other notions of principal bundles involving the action of a groupoid. The main differences lies in the fact that:

- we allow \( M^{(1)} \) to be any groupoid over \( M \), not only the groupoid of identities. In [NW13, Definition 6.1.5.], the authors define **principal 2-bundles** with a manifold as the base in contrast with our principal bundle groupoids whose base space is any groupoid as in Definition 1.21.

- We allow \( G^{(1)} \) to be any Lie 2-group, generalising other approaches that only allow Lie groups. In [LGTX07, Definition 2.2], the authors define **principal bundles over groupoids** from a free and proper Lie 2-group action of the identity Lie 2-group \( G \Rightarrow G \) in a groupoid \( P^{(1)} \) with quotient a Lie groupoid \( M^{(1)} \). Our definition is a generalisation which allows \( G \) to be any Lie 2-group. Note that we get a principal bundle over a groupoid when considering the partial quotient groupoid of a principal bundle groupoid as in Proposition 1.22.

The two definitions of principal bundle over a groupoid in [LGTX07] on the one hand and in [MM03] on the other, relate by generalised morphisms of Lie groupoids.

**Definition 1.28.** [LGTX07, Definition 2.8] A **generalised groupoid homomorphism** from \( G^{(1)} \to G \) to \( H^{(1)} \to H \) is given by a manifold \( P \), two smooth maps \( G \xleftarrow{\pi} P \xrightarrow{\rho} H \), a left action of \( G^{(1)} \) w.r.t. to \( \rho \), a right action of \( H^{(1)} \) with respect to \( \pi \), such that the two actions commute, and \( P/H^{(1)} \cong G \).

\[
\begin{array}{c}
G^{(1)} \xleftarrow{s} P \xrightarrow{t} H^{(1)} \\
\downarrow \pi \quad \downarrow \rho \\
G \quad H
\end{array}
\]

Let \( H \) be a Lie group. A principal \( H \)-bundle over a groupoid \( G^{(1)} \Rightarrow G \) as in [LGTX07, Definition 2.2] (the authors consider right actions) corresponds to a generalised morphism from \( G^{(1)} \) to \( H \Rightarrow \{*\} \) (there is an equivalence of categories [LGTX07 Proposition 2.11]).

### 1.5 \( G^{(1)} \)-principal 2-bundles in terms of principal bundle groupoids

We now consider a right group action of a Lie 2-group instead of the above groupoid action. We shall compare Definition 1.15 with the subsequent definition.

We first recall the notion of weak equivalence.

**Definition 1.29.** [NW13, Theorem 2.3.13] Let \( P^{(1)} \Rightarrow P \) and \( M^{(1)} \Rightarrow M \) be two Lie groupoids. A smooth functor \( \varphi^{(1)}: P^{(1)} \to M^{(1)} \) is a **weak equivalence** if:

- \( \varphi^{(1)} \) is full and faithful, in other words, for any \( p, q \in P \), the map \( \pi: P^{(1)}(p, q) \to M^{(1)}(\varphi(p), \varphi(q)) \) is bijective.

- \( \varphi^{(1)} \) is essentially surjective, in other words, the map \( M^{(1)} \times_{\varphi} P \to M; (\alpha, p) \mapsto t(\alpha) \) is a surjective submersion.
Lemma 1.30. Given a fibration $\pi^{(1)} : P^{(1)} \to Y^{(1)}$ and a surjective morphism $\varphi^{(1)} : Y^{(1)} \to M$, the inclusion $P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_M P^{(1)}$ is a weak equivalence if and only if $\varphi^{(1)}$ is a weak equivalence.

Proof. We need to prove that the canonical injection $\iota : P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_M P^{(1)}$ is full, faithful and essentially surjective, if and only if $\varphi^{(1)} : Y^{(1)} \to M$ is also full, faithful and essentially surjective.

- If the canonical inclusion $\iota : P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_M P^{(1)}$ is full, faithful and essentially surjective, then $\varphi^{(1)} : Y^{(1)} \to M$ is also full, faithful and essentially surjective. This can be read off the following commutative diagram, which shows that the bottom projection is not full, faithful and essentially surjective then the top inclusion cannot be:

  $\begin{array}{ccc}
P^{(1)} \times_{Y^{(1)}} P^{(1)} & \xrightarrow{\iota} & P^{(1)} \times_M P^{(1)} \\
\downarrow & & \downarrow \\
Y^{(1)} & \xrightarrow{\varphi^{(1)}} & M
\end{array}$

- Assuming that $\varphi^{(1)} : Y^{(1)} \to M$ is full, faithful and essentially surjective, we now prove that $\iota : P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_M P^{(1)}$ is full and faithful. For any $(\alpha, \beta) \in P^{(1)} \times_M P^{(1)}$ there is only one $\gamma \in Y^{(1)}$ with $\varphi^{(1)}(\gamma) = \varphi^{(1)}(\pi^{(1)}(\alpha)) = \varphi^{(1)}(\pi^{(1)}(\beta))$ which means that $\pi^{(1)}(\alpha) = \pi^{(1)}(\beta) = \gamma$ and $(\alpha, \beta) \in P^{(1)} \times_{Y^{(1)}} P^{(1)}$.

Let us check that the map $\iota : P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_M P^{(1)}$ is also essentially surjective. The map $\iota : P^{(1)} \times_{Y^{(1)}} P^{(1)} \to P^{(1)} \times_P P$ is a surjective submersion by definition. To prove the statement, we now need to show that around any $(\alpha, \beta) \in P^{(1)} \times_M P^{(1)}$, there is a smooth choice $\gamma_{\alpha\beta} \in P^{(1)}$ such that $(\alpha \circ \gamma, \beta) \in P^{(1)} \times_M P^{(1)}$ and $s(\alpha \circ \gamma, \beta) \in P \times_P P$. Since $\varphi(s(\pi^{(1)}(\alpha))) = \varphi(s(\pi^{(1)}(\beta))$ and $\varphi^{(1)}$ is full and faithful, there is only one smooth choice $\omega_{\alpha\beta} \in Y^{(1)}$ from $q := s(\pi^{(1)}(\beta))$ to $p := s(\pi^{(1)}(\alpha))$. The desired $\gamma_{\alpha\beta}$ is any arrow in $P^{(1)}$ with target $t(\alpha)$ such that $\pi^{(1)}(\gamma_{\alpha\beta}) = \omega_{\alpha\beta}$, this smooth choice exists since $\pi^{(1)}$ is a fibration.

Let us recall the notion of principal groupoid 2-bundle of [NW13]:

Definition 1.31. [NW13, Definition 6.1.5.] Let $M$ be a smooth manifold and $(G \ltimes H)$ a Lie 2-group. A principal $(G \ltimes H)$- 2-bundle over $M$ is defined from a right Lie 2-group action

$\star : P^{(1)} \times (G \ltimes H) \to P^{(1)}$

$(\phi, (g, h)) \mapsto \phi \star (g, h),$

on another Lie groupoid $P^{(1)} \rightrightarrows P$, with a groupoid morphism $P^{(1)} \to M$, where $M \rightrightarrows M$ is the identity groupoid over a manifold $M$, for which the morphism:

$P^{(1)} \times (G \ltimes H) \xrightarrow{pr_1 \times q} P^{(1)} \times_M P^{(1)}$

$\downarrow pr \times G \downarrow sp \times G \downarrow pr \times P \downarrow sp \times P$

$P \times G \xrightarrow{pr_1 \times a} P \times_M P$

is a weak equivalence.
Principal 2-bundles are, in a non trivial way, principal bundle groupoids, as it is shown in the subsequent proposition.

**Proposition 1.32.** Let \( G^{(1)} \rightrightarrows G \) be a Lie 2-group. A principal \( G^{(1)} \)-2-bundle on a manifold \( M \) as in Definition [1.37] is a principal \( G^{(1)} \)-bundle groupoid (in the sense of Definition [1.35]) over a Lie groupoid \( Y^{(1)} \rightrightarrows Y \), which is Morita equivalent to the manifold \( M \).

**Proof.** With the notations of Definition [1.31] let \( P^{(1)} \rightrightarrows M \) be a \( G^{(1)} \)-principal 2-bundle. We want to define a principal \( G^{(1)} \)-bundle groupoid \((P^{(1)} \rightrightarrows P)\) over a Lie groupoid \( Y^{(1)} \rightrightarrows Y \), which is Morita equivalent to \( M \) by Example [A.3].

We set \( Y^{(1)} = P^{(1)} / G^{(1)} \) in which case \( G^{(1)} \times P^{(1)} \cong P^{(1)} \times_{Y^{(1)}} P^{(1)} \) and the quotient map \( \pi^{(1)} : P^{(1)} \rightrightarrows Y^{(1)} \) is a fibration. Because the morphism \( P^{(1)} \rightrightarrows M \) is \( G^{(1)} \)-invariant, there exists a morphism \( \varphi : Y^{(1)} \rightrightarrows M \). By Lemma [1.30] the inclusion \( P^{(1)} \times_{Y^{(1)}} P^{(1)} \rightrightarrows P^{(1)} \times M P^{(1)} \) is a weak equivalence iff \( \varphi^{(1)} : Y^{(1)} \rightrightarrows M \) is a weak equivalence. By Lemma [A.5] \( Y^{(1)} \) is Morita equivalent to \( M \) (see Example [A.4]).

\[ \square \]

## 2 A functorial correspondence between PB groupoids and bundle gerbes

In this section, we generalise the classical notion of Lie 2-group bundle gerbe over a smooth manifold [NW13, Definition 5.1.1] to that of a Lie 2-group bundle gerbe over a Lie groupoid \( Y^{(1)} \rightrightarrows Y \). Specialising to bundle gerbes over fibre bundle Lie groupoids \( Y^{[2]} \rightrightarrows Y \) yields back the classical Lie 2-group bundle gerbe over the orbit space \( M = Y / Y^{[2]} \). From a principal Lie 2-group bundle groupoid \( P^{(1)} \rightrightarrows P \) over a fibre bundle groupoid \( Y^{[2]} \rightrightarrows Y \), we build a bundle gerbe over the fibre bundle groupoid \( P^{[2]} \rightrightarrows P \), which yields a bundle gerbe over the quotient space \( P^{[2]} / P \). From a trivial base Lie 2-group principal bundle groupoid \( P^{(1)} \rightrightarrows P \) over a Lie groupoid \( Y^{(1)} \rightrightarrows Y \), we build a Lie 2-group bundle gerbe over \( Y^{(1)} \).

Let us first recall the definition of a fibre bundle groupoid.

**Definition 2.1.** [Mac05] To a surjective submersion \( \pi : Y \rightrightarrows M \) between two smooth manifolds \( Y \) and \( M \), we associate the fiber product groupoid of \( \pi \), whose set of points is \( Y \) and whose set of arrows is given by:

\[
y^{[2]} := Y \times_M Y = \{(y_2, y_1) \in Y^2 : \pi(y_2) = \pi(y_1)\}.
\]

for any \((y_3, y_2), (y_2, y_1) \in Y^{[2]}\) there is

\[
\begin{align*}
s(y_2, y_1) &:= y_1, \\
t(y_2, y_1) &:= y_2, \\
(y_3, y_2) &\circ (y_2, y_1) := (y_3, y_1), \\
e_{y_1} &:= (y_1, y_1) \in Y^{[2]} \quad \text{and} \quad (y_2, y_1)^{-1} := (y_1, y_2).
\end{align*}
\]

### 2.1 Bundle gerbes over Lie groupoids

**Definition 2.2.** [NW13, Definition 5.1.1] Given a Lie 2-group \( H \rtimes G \), a \((H \rtimes G)\)-bundle gerbe over a smooth manifold \( M \) consists of

- A Lie groupoid \( B \rightrightarrows Y \) with composition \( \circ_B : B \times_B B \rightrightarrows B \);
• A surjective submersion $\pi: Y \rightarrow M$;

• A $(H \ltimes G \rightrightarrows G)$ groupoid action $\star: (H \ltimes G) \times Y \times B \rightarrow B$ with anchor $\rho: B \rightarrow G$.

Such that:

1. The image of the map $t \times s: B \rightarrow Y \times Y$ is the submanifold $Y^{(2)} = Y \times_M Y$. In other words, the map $\pi$ gives a diffeomorphism between the orbit space $Y/B$ and $M$, where the orbit space $Y/B$ is the quotient of $Y$ by the equivalence relation: $y_1 \sim y_2 \iff \exists b \in B, s(b) = y_1 \land t(b) = y_2$.

2. The induced map $\Pi: B \rightarrow Y^{(2)}$ is a submersion and an $(H \ltimes G \rightrightarrows G)$-principal bundle with respect to the groupoid action, as the following diagram suggests:

\[ H \ltimes G \quad \xrightarrow{\star} \quad B \xrightarrow{\Pi} Y^{(2)} \]

\[ \xymatrix{ G \\ B \ar[u]^s \ar[d]_t \ar[r]^\rho & Y \ar[u]_s \ar[d]^t \ar[r]^\pi & M } \]

3. The composition $\circ_\mu$ is $(H \ltimes G)$-equivariant, i.e.:

\[ ((h_2, g_2) \star b_2) \circ_\mu ((h_1, g_1) \star b_1) = ((h_2, g_2) \cdot (h_1, g_1)) \star (b_2 \circ_\mu b_2), \]

for $(h_2, g_2), (h_1, g_1) \in H \ltimes G$ and composable elements $b_2, b_1 \in B$.

**Remark 2.3.** The above definition corresponds to an interpretation of the bundle gerbe product $\mu$ in [NW13, Definition 5.1.1] in terms of a groupoid composition $\circ_\mu$, in viewing $B$ as a groupoid over $Y$. The existence of an identity and the invertibility in $B$ result from [NW13, Lemma 5.2.5].

We generalise the above definition to bundle gerbes over groupoids.

**Definition 2.4.** Given a Lie 2-group $H \ltimes G$ and a groupoid $Y^{(1)} \rightrightarrows Y$, an $(H \ltimes G)$-bundle gerbe over a Lie groupoid $Y^{(1)} \rightrightarrows Y$ consists of

• A Lie groupoid $B \rightrightarrows Y$, with composition $\circ_\mu: B \times_B B \rightarrow B$;

• A groupoid morphism $\Pi: B \rightarrow Y^{(1)}$ over the identity of $Y$;

• A $(H \ltimes G \rightrightarrows G)$-groupoid action $\star: (H \ltimes G) \times B \rightarrow B$ and anchor map $\rho: B \rightarrow G$,

such that:

1. $\Pi: B \rightarrow Y^{(1)}$ is a $(H \ltimes G \rightrightarrows G)$-principal bundle.

2. The composition $\circ_\mu$ of $B$ is $(H \ltimes G)$-equivariant i.e.

\[ ((h_2, g_2) \star b_2) \circ_\mu ((h_1, g_1) \star b_1) = ((h_2, g_2) \cdot (h_1, g_1)) \star (b_2 \circ_\mu b_2), \]

for all $(h_2, g_2), (h_1, g_1) \in H \ltimes G$ and $b_2, b_1 \in B$,

as suggested by the following diagram:

\[ H \ltimes G \quad \xrightarrow{\star} \quad B \xrightarrow{\Pi} Y^{(1)} \]

\[ \xymatrix{ G \\ B \ar[u]^s \ar[d]_t \ar[r]^\rho & Y \ar[u]_s \ar[d]^t } \]
**Example 2.5.** Given a submersion $\pi: Y \to M$ and taking $Y^{(1)} = Y^{[2]} := Y \times_\pi Y$ to be the tensor product groupoid over $Y$, gives back a bundle gerbes over the orbit space $M = Y/Y^{[2]}$.

**Definition 2.6.** A (strict) morphism of $H \rtimes G$-bundle gerbes

$$f: \left( B_1 \to Y_1^{(1)} \right) \to \left( B_2 \to Y_2^{(1)} \right)$$

consists of two groupoid morphisms $f_B: B_1 \to B_2$ and $f_Y: Y_1^{(1)} \to Y_2^{(1)}$ such that $f_B$ is $(H \rtimes G)$-equivariant and $f_Y$ is $(H \rtimes G)$-invariant (as groupoid actions).

**Remark 2.7.** Later we shall see how a morphism of bundle gerbes arises from a morphism of principal bundle groupoids.

### 2.2 From a PB groupoid over a fiber product groupoid to a bundle gerbe

We build a functor

$$\Phi: \text{PB groupoids over fibre product groupoids} \to \text{Bundle gerbes over fiber product groupoids}$$

**Proposition 2.8.** Let $H \rtimes G$ be a Lie 2-group, $\pi: Y \to M$ be a submersion between the manifolds $Y$ and $M$. Given a principal $(H \rtimes G)$-bundle groupoid $(P^{(1)} \to P)$ over the fiber product groupoid $(Y^{[2]} \to Y)$ (see Definition 1.15)

$$
\begin{array}{c}
(H \rtimes G) \times P^{(1)} \\
\downarrow t \times s \downarrow \downarrow t \downarrow s \\
G \times P \\
\downarrow s \downarrow t \\
P \to Y
\end{array}
$$

the following data defines a bundle gerbe $\Phi(P^{(1)}):= G \times P^{(1)}$ over $P^{[2]} = P \times_M P$:

$$
\begin{array}{c}
H \rtimes G \\
\downarrow t \\
G \times P^{(1)} \\
\downarrow s \\
\Pi \Rightarrow P \times_M \rho \\
\downarrow t \downarrow s \\
P \to Y \\
\downarrow \pi \\
M
\end{array}
$$

(2.8.1)

where for any $(g, \phi)$ in $G \times P^{(1)}$:

$$
\Pi(g, \phi) := (g \cdot t(\phi), s(\phi)) , \quad \rho(g, \phi) := g ,
$$

$$(h, g) \star (g, \phi) := \left( d(h)g , (C_{g^1}h_{g^1}^{-1}, e) \cdot \phi \right)$$

using the notations of Definition 1.3

$$(g_2, \phi_2) \circ (g_1, \phi_1) := \left( g_2 \cdot g_1 \cdot (e, g_1^{-1} \cdot \phi_2) \circ \phi_1 \right),$$

for $(h, g) \in H \rtimes G, \phi \in P^{(1)}$ and composable $(g_2, \phi_2), (g_1, \phi_1) \in G \times P^{(1)}$ (i.e. $s(\phi_2)) = g_1 \cdot t(\phi_1)$).

**Remark 2.9.** Starting from a PB groupoid over $Y^{[2]}$ we get a bundle gerbe over $P \times_M P$: these two groupoids are Morita equivalent to the identity groupoid over $M$ (see Example A.4).

Proposition 2.8 is proven in [NW13]. We give here a proof that allow us to generalize it, as Proposition 2.13.
Proof. • ∗ is a groupoid action with anchor ρ. It is free because $H \times G$ acts freely in $P^{(1)}$. It is proper since $\Pi$ is a submersion and invariant under this action.

• $\circ_\mu$ is associative and smooth because $\cdot$ and $\circ$ are associative and smooth.

• Let us prove that $\Pi$ is a surjective submersion, which uses the fact that $Y^{[2]}$ is a fibre product groupoid:
It follows from Definition 1.15 that the map $Q^{(1)} : P^{(1)} \to Y^{[2]}$, where $Y^{[2]}$ is viewed as a groupoid and $Q^{(1)}(\phi) = (Q(t(\phi)), Q(s(\phi)))$, is a fibration, which implies that the map

$$Q^{(1)} : P^{(1)} \to Y^{(1)} \times_Q P; \phi \mapsto (Q^{(1)}(\phi), s(\phi))$$

is a surjective submersion. We want to show that $\Pi : G \times P^{(1)} \to P \times_M P$ is a surjective submersion.

Hence, for any $(q, p) \in P \times_M P$ and $(V, U) \in T_{(p,q)}(P \times_M P) = T_p P \times_T M T_q P$ there is $\phi \in P^{(1)}$ and $W \in T_\phi P^{(1)}$ such that:

$$s(\phi) = p, \quad \text{and} \quad Q^{(1)}(\phi) = (Q(q), Q(p)) \in Y \times_M Y.$$

d$s(W) = X$ and $dQ^{(1)}(W) = (dQ(V), dQ(U)) \in T_{(Q(q), Q(p))}(Y \times_M Y)$.

This implies that:

$\Pi$ is surjective: $Q(q) = Q(t(\phi))$ so there is $g \in G$ such that $\Pi(g, \phi) = (gt(\phi), s(\phi)) = (q, p)$.

$\Pi$ is a submersion: $dQ(Y) = dQ(dt(W))$ so there is $v \in T_g G$ such that $d\Pi(v, W) = (V, U)$.

• Let us prove that $\circ_\mu$ is $(H \times G)$-equivariant:
For any composable elements $(g_2, \phi_2), (g_1, \phi_1) \in G \times P^{(1)}$ and $h_2, h_1 \in H$, using the notations $C : G \to \text{Aut}(H)$ and $d : H \to G$ of Definition 1.13, let us set:

• $g := d(h_2) g_2 \cdot d(h_1) g_1$,

• $(h, g) := (C_{g_1^{-1} d(h_1^{-1}) g_2^{-1}} h_2^{-1}, g_1^{-1} d(h_1^{-1}) g_1)$.

The elements $(h, g)$ and $(C_{g_1^{-1} h_1^{-1}} e)$ are composable in $H \times G$ since by (1.3.1) we have

$$t(C_{g_1^{-1} h_1^{-1}} e) = d(C_{g_1^{-1} h_1^{-1}} e) = g_1^{-1} d(h_1^{-1}) g_1 = s(h, g)$$

and:

$$(h, g) \circ (C_{g_1^{-1} h_1^{-1}} e) = (h C_{g_1^{-1} h_1^{-1}} e) = (C_{g_1^{-1} d(h_1^{-1}) g_2^{-1} h_2^{-1}} C_{g_1^{-1} h_1^{-1}} e) = (C_{g_1^{-1}} (h_1^{-1} C_{g_2^{-1} h_2^{-1}}) h_1) C_{g_1^{-1} h_1^{-1}} e = (C_{g_1^{-1}} (h_1^{-1} C_{g_2^{-1} h_2^{-1}}), e).$$

Moreover, on the one hand we have:

$$((h_2, g_2) \star (g_2, \phi_2)) \circ_\mu ((h_1, g_1) \star (g_1, \phi_1))$$

$$= (d(h_2) g_2, (C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_2) \circ_\mu (d(h_1) g_1, (C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_1)$$

$$= (g, ((e, g_1^{-1} d(h_1^{-1}) e) \cdot \phi_2) \circ ((C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_1)$$

$$= (g, C_{g_1^{-1} d(h_1^{-1}) g_2^{-1} h_2^{-1}} g_1^{-1} d(h_1^{-1}) \cdot \phi_2) \circ ((C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_1)$$

$$= (g, C_{g_1^{-1} d(h_1^{-1}) g_2^{-1} h_2^{-1}} g_1^{-1} \cdot \phi_2) \circ ((C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_1)$$

$$= (g, (h, g) \circ (e, g_1^{-1} \cdot \phi_2) \circ ((C_{g_1^{-1} h_1^{-1}} e) \cdot \phi_1).$$
On the other hand,

\[ ((h_2, g_2) \cdot (h_1, g_1)) \circ ((g_2, \phi_2) \circ \mu (g_1, \phi_1)) \]

\[ = (h_2 \cdot C_{g_2 \cdot h_1, g_2 \cdot g_1} \circ (g_2 \cdot g_1, ((e, g_1^{-1}) \cdot \phi_2) \circ \phi_1) \]

\[ = (d(h_2) \cdot d(C_{g_2 \cdot h_1}) \circ (g_2 \cdot g_1), ((C_{g_2 \cdot h_1})^{-1}(h_2 \cdot C_{g_2 \cdot h_1}^{-1}, e) \cdot ((e, g_1^{-1}) \cdot \phi_2) \circ \phi_1) \]

\[ = (d(h_2) \cdot g_2 \cdot (h_1) \cdot g_1, ((C_{g_2 \cdot h_1})^{-1}(h_2 \cdot C_{g_2 \cdot h_1}^{-1}, e) \cdot ((e, g_1^{-1}) \cdot \phi_2) \circ \phi_1) \]

\[ = (g, (h, g) \circ (C_{g^{-1}h_1^{-1}, e}) \circ ((g, e^{-1}) \cdot \phi_2) \circ \phi_1) \].

The two expressions coincide since \cdot is a Lie 2-group action:

\[ ((h, g) \cdot ((g, e^{-1}) \cdot \phi_2)) \circ ((C_{g^{-1}h_1^{-1}, e}) \cdot \phi_1) = ((h, g) \circ (C_{g^{-1}h_1^{-1}, e}) \cdot ((g, e^{-1}) \cdot \phi_2) \circ \phi_1), \]

proving that

\[ ((h_2, g_2) \star (g_2, \phi_2)) \circ ((h_1, g_1) \star (g_1, \phi_1)) = ((h_2, g_2) \cdot (h_1, g_1)) \star ((g_2, \phi_2) \circ \mu (g_1, \phi_1)). \]

This ends the proof. \(\square\)

The above proposition gives rise to a functor \(\Phi\) from \((H \ltimes G)\)-groupoids to \((H \ltimes G)\)-bundle gerbes by sending a morphism \(f\) of \((H \ltimes G)\)-groupoids to the morphism \(\text{Id}_G \times f\) of the corresponding bundle gerbes.

### 2.3 From a bundle gerbe over a groupoid to a base trivial PB groupoid

**Definition 2.10.** A trivial base \((H \rtimes G)\)-principal bundle groupoid is a principal bundle groupoid such that the 0 level is a trivial principal bundle i.e., \(P = G \times Y \rightarrow Y\), where as before, the source and target of the Lie 2-group \(G(1) = H \rtimes G \Rightarrow G\) built from a map \(d := t|_H : H \rightarrow G\), are given by \(s_{G(1)}(h, g) := g, t_{G(1)}(h, g) := d(h)g\) (see Proposition [1.5]).

We build a functor

\[ \Psi : \text{Bundle gerbes} \rightarrow \text{base trivial PB groupoids}. \]

**Proposition 2.11.** Given a bundle gerbe \(B\) over \(Y(1)\) described by the diagram (2.4.1), there is a base trivial PB groupoid given by:

\[
\begin{align*}
H \rtimes G & \quad \xrightarrow{\quad} \quad G \times B \\
\downarrow t & \quad \quad \quad \quad \downarrow s \\
G & \quad \xrightarrow{\quad} \quad G \times Y \\
\downarrow t & \quad \quad \quad \downarrow s \\
Y & \quad \xrightarrow{\quad} \quad Y
\end{align*}
\]

where:

- For all \((k_1, b_1) \in G \times B\) we have: \(s(k_1, b_1) = (k_1, s(\Pi(b_1)))\) and \(t(k_1, b_1) = (k_1, t(\Pi(b_1)))\).
• for \((k_2, b_2), (k_1, b_1) \in (G \times B)_s \times (G \times B)\) then:

\[(k_2, b_2) \circ (k_1, b_1) = (k_1, (b_2 \circ_\mu b_1)).\]

• for \((h, g) \in H \rtimes G\) and \((k_1, b_1) \in G \times B\) we have the Lie 2-Group action:

\[(h, g) \cdot (k_1, b_1) = \left( g k_1, \left( C_{\rho(b_1)k_1^{-1}g^{-1}}h^{-1}, \rho(b_1) \right) \star b_1 \right).\]

• The identity morphism of an object \((g, y)\) is \((g, 1_B(y))\), where \(1_B\) is the unit element of \(B\).

• The inverse of a morphism \((g, b)\) is \((g \rho(b)^{-1}, i_B(b))\), where \(i_B\) is the inverse map of \(B\).

Proof. The proof follows the construction of principal 2-bundles from bundle gerbes carried out in [NW13 §7.2.], compare the definition of the groupoid composition and the Lie 2-group action with [NW13 Formula (7.2.1)]. □

This yields a functor \(\Psi\) which sends an \((H \rtimes G)\)-bundle gerbe morphism \(f\) to the \((H \rtimes G)\)-groupoid morphism \(\text{Id}_G \times f\).

Remark 2.11. Note that the image of \(\Psi\) restricted to the bundle gerbes over fiber product groupoids \(Y^{[2]}\), is a subset of the principal bundle groupoids over fiber product groupoids with trivial base.

### 2.4 From a base trivial PB groupoid to a bundle gerbe over a groupoid

On the grounds of Proposition 2.11, we specialise to the class of trivial base principal bundle groupoids. The functor \(\Phi\) built above induces a functor:

\[\Xi : \text{trivial base } (H \rtimes G)\text{- PB groupoid over } Y^{(1)} \to (H \rtimes G)\text{-bundle gerbes over } Y^{(1)}\]

Let \(H \rtimes G \leadsto P^{(1)} \to Y^{(1)}\) be any base trivial PB groupoid. Let \(\text{pr}_G : P = G \times Y \to G\) be the canonical projection onto the first component and consider the map

\[s_G := \text{pr}_G \circ s : P^{(1)} \to G.\]

By the implicit function theorem, the preimage of the neutral element \(e_G \in G\) given by \(B := \{\xi \in P^{(1)} : s_G(\xi) = e_G\}\) is a submanifold of \(P^{(1)}\). Recall that \(G^{(1)} = H \rtimes G\) acts on \(P^{(1)}\) over the trivial \(G\)-action on \(P = G \times Y\). We have the following diffeomorphism:

\[
\varphi : P^{(1)} \to G \times B \\
\xi \mapsto \left(s_G(\xi), s_G(\xi)^{-1} \xi\right),
\]

(2.12.1)

(where \(g \xi = (e, g) \cdot \xi\)), corresponding to the diagramme:

\[
\begin{array}{ccc}
G \times B & \xrightarrow{\varphi} & G \times Y \\
\downarrow & & \downarrow \\
H \rtimes G & \xrightarrow{i} & P^{(1)} \xrightarrow{t} Y^{(1)}.
\end{array}
\]
In the particular case $Y^{(1)} = Y^{[2]}$, the construction of the previous section yields an $(H \rtimes G)$-bundle gerbe $\Psi(p^{(1)}) \cong G \times G \times B$ over $P^{[2]} \cong G \times G \times Y^{[2]}$. We want to interpret $B$ as a bundle gerbe over $Y^{[2]}$. Going back to the case $p^{(1)} \to Y^{(1)}$, let us first note that the following diagram commutes:

\[
\begin{array}{ccc}
G \times B & \xleftarrow{\varphi} & p^{(1)} \\
\downarrow{\text{pr}_B} & & \downarrow{\pi} \\
B & \xrightarrow{\Pi} & Y^{(1)}
\end{array}
\]

Since $\pi$ is a surjective submersion, so is $\Pi := (\pi \circ \varphi^{-1})|B$ a surjective submersion.

To prove that $\Pi : B \to Y^{(1)}$ is an $(H \rtimes G)$-bundle gerbe, we follow the construction in [NW13 §7.1], where the authors build a bundle gerbe from a PB groupoid, as well as the construction in [NW13 §7.2], where the authors build a PB groupoid from a bundle gerbe, see in particular Formula (7.2.1).

**Proposition 2.13.** Let $\pi : p^{(1)} \to Y^{(1)}$ be a trivial-base $H \rtimes G$-PB groupoid with $p^{(1)} \Rightarrow P$ such that $P \cong G \times Y$. The submanifold $\Xi(p^{(1)}) := B = \text{Ker}(\mathfrak{s}_G)$ is a $(H \rtimes G)$-bundle gerbe $H \rtimes G \cong B \times Y^{(1)}$ over the groupoid $Y^{(1)}$ with:

- **projection** $\Pi := (\pi \circ \varphi^{-1})_B : B \to Y^{(1)}$,
- **anchor** $\rho = (t_G)^{-1} := (-)^{-1} \circ \text{pr}_G \circ (t|_B) : B \to G; b \mapsto (\text{pr}_G(t(b)))^{-1}$,
- **the** $H \rtimes G \Rightarrow G$ **action given in terms of the groupoid action**:
  \[
  (h, \rho(b)) \star b = (C_{\rho(b)}^{-1}h^{-1}, e) \cdot b,
  \]
  where $b$ is viewed as an element of $p^{(1)}$.
- **bundle gerbe product in terms of the groupoid composition** $\circ_{p^{(1)}}$:
  \[
  \circ_{\mu} : B_{\text{an}} \times_{\Pi} B_{\gamma_1} \to B \\
  (b_2, b_1) \mapsto ((e, \rho(b_1)^{-1}) \cdot b_2) \circ p^{(1)} b_1 ,
  \]
  where $b_2$ is viewed as an element of $p^{(1)}$.

**Remark 2.14.** Note that the above formula for the action $\star$, resp. the bundle gerbe product $\circ_{\mu}$ follows from setting $\phi = b$, $g = \rho(b)$, resp. $g_1 = \rho(b_1)$ in the formulas for the action, resp. the bundle gerbe product of Proposition 2.8 which we recall here for convenience:

\[
(h, g) \star (g, \phi) := (d(h)g, (C_{g^{-1}h^{-1}}, e) \cdot \phi),
\]

resp.

\[
(g_2, \phi_2) \circ_{\mu} (g_1, \phi_1) := (g_2 \cdot g_1, (e, g_1^{-1} \cdot \phi_2) \circ p^{(1)} \phi_1),
\]

and then applying the projection $\text{pr}_B$. 

18
Proof. We know that \( \Pi := (\pi \circ \varphi^{-1})|B \) is a surjective submersion. The map \( \varphi \) is a \( H \rtimes G \)-equivariant isomorphism where \( H \rtimes G \) acts on \( G \times B \) by a trivial action on \( G \) and the groupoid action on \( B \) and \( B \to Y^{(1)} \) is a \( H \rtimes G \)-principal bundle.

As a consequence of the above remark, the proof of the properties of the various operators is similar to that of Proposition 2.8:

- \( \ast \) is a Lie groupoid action since \( \cdot \) is a smooth action. It acts freely since \( \cdot \) acts freely.
- \( \circ_\mu \) is smooth and associative since \( \cdot \) and \( \circ \) are smooth and associative.
- The bundle gerbe product is \( (H \rtimes G) \)-equivariant following the same argument used in the proof of Proposition 2.8.

\( \square \)

This gives a functor \( \Xi \) from \( (H \rtimes G) \)-PG groupoids to \( (H \rtimes G) \)-bundle gerbes by restriction of the PG groupoid morphisms to \( B = \text{Ker}(s_G) \).

2.5 Conclusions

Let \( \Phi \) be the correspondence of Proposition 2.8, \( \Psi \) be the correspondence of Proposition 2.11 and \( \Xi \) be the correspondence of Proposition 2.13. For a fixed Lie 2-group \( H \rtimes G \), the correspondences \( \Phi, \Psi \) and \( \Xi \) are functors between the (strict)-categories of \( H \rtimes G \)-principal bundle groupoids and \( H \rtimes G \)-bundle gerbes.

The correspondences \( \Xi \) and \( \Psi \) are functors between the categories of base trivial principal bundles with strict morphisms and gerbes with (strict) morphisms. Moreover, they are inverses of each other.

For a fixed Lie 2-group \( H \rtimes G \), the correspondences \( \Psi \) and \( \Phi \) are functors between the (strict)-categories of \( H \rtimes G \)-principal bundle groupoids over fiber product groupoids and \( H \rtimes G \)-bundle gerbes over fiber product groupoids. Although \( \Psi \) and \( \Phi \) are not inverses of each other, in [NW13] it is shown that they give an equivalence of categories, when seen as 2-categories with weak equivalences and anafunctors.

The following statement follows from the above constructions summarised in the diagrams (2.8.1) and (2.13.1).

Proposition 2.15. Let \( H \rtimes G \) be a Lie 2-group and \( P^{(1)} \to Y^{(1)} \) a principal \( (H \rtimes G) \)-principal bundle groupoid. The bundle gerbes \( \Phi(P^{(1)}) \) and \( \Xi(P^{(1)}) \) are equivalent to the partial quotient \( P^{(1)}/_{|G} \) of Definition 1.21. To show this we use Definition A.2.

1. If \( Y^{(1)} = Y^{[2]} \) then the bundle gerbe \( B := \Phi(P^{(1)}) = G \times P^{(1)} \) is isomorphic (as a groupoid) to the pullback by \( Q : P \to Y \) of the partial quotient \( P^{(1)}/_{|G} \); this is

\[
B \cong P_{G \times (P^{(1)})_{s \times Q} P},
\]

with the isomorphism given by \( (g, \gamma) \mapsto (g t(\gamma), [\gamma], s(\gamma)) \) for any \( g \in G \) and \( \gamma \in P^{(1)} \).

2. If \( P \cong G \times Y \), so if \( P^{(1)} \to Y^{(1)} \) is a trivial base PB groupoid, the bundle gerbe \( B' := \Xi(P^{(1)}) = P^{(1)}|_{\text{Ker}(s_G)} \) is isomorphic to the partial quotient \( P^{(1)}/_{|G} \).
3 Nerve of a principal bundle groupoid

This section discusses higher analogs of the principal bundle groupoids introduced in the previous one. We first review some concepts used to build these higher analogs.

3.1 Nerve of a small category

Let \( \Delta \) denote the category whose objects are finite ordered sets \([n] = \{0, 1, 2, \cdots, n\}\) and whose morphisms are order-preserving functions \([m] \to [n]\). One defines the injection \(d_i: [n - 1] \to [n]\) omitting \(i \in [n]\) and the surjection \(e_i: [n] \to [n - 1]\) repeating \(i \in [n - 1]\).

A simplicial set in a category \(\mathcal{C}\) is a contravariant functor \(X: \Delta \to \mathcal{C}\) (equivalently, a covariant functor \(X: \Delta^{op} \to \mathcal{C}\)). An element \(x_k \in X^{(k)} := X([k])\) is called an \(k\)-simplex. A morphism of simplicial objects in \(\mathcal{C}\) is a natural transformation of such functors.

Concretely, a simplicial object in the category \(\mathcal{C}\) amounts to a sequence \(\mathcal{C}^{(k)}, k \in \mathbb{Z}_{\geq 0}\) of objects in \(\mathcal{C}\) called simplices and a collection of morphisms called face maps \(d_k: \mathcal{C}^{(k)} \to \mathcal{C}^{(k-1)}\) and degeneracy maps \(e_k: \mathcal{C}^{(k-1)} \to \mathcal{C}^{(k)}\) which obey the following conditions (see e.g. [Fri12])

\[
\begin{align*}
    d_i d_j &= d_{j-1} d_i & \text{for } i < j, \\
    d_i e_j &= e_j d_{j-1} & \text{for } i > j + 1, \\
    e_i e_j &= e_{j+1} e_i & \text{for } i > j, \\
    d_j e_j &= d_{j+1} e_j = \text{Id}, \\
    d_i e_j &= s_{j-1} d_i & \text{for } i < j.
\end{align*}
\]

Let \(\mathcal{C}\) be a category. The nerve \(N^\bullet(M^{(1)}) = M^\bullet\) of a small category \(M^{(1)}\) in \(\mathcal{C}\) is a simplicial set (in the category of sets). The 0-simplices \(M^{(0)}\) are given by the points in \(M\), the 1-simplices by the arrows \(M^{(1)}\), its \(k\)-simplices are the sets of \(k\) arrows morphisms in \(M^{(1)}\), i.e. they are the sets

\[
M^{(k)} = N^k(M^{(1)}) := M^{(1)} \times_{\cdots \times} M^{(1)} = \left\{(\alpha_1, \cdots, \alpha_k) \in (M^{(1)})^k : \alpha_i \text{ is composable with } \alpha_{i+1}\right\}.
\]

For \(k = 1\), \(d_0\), resp. \(d_1\) is the source, resp. the target map from the arrows to the points of the category. For \(k \geq 2\), the two outer face maps \(d_0\), respectively \(d_{k+1}\), are defined by forgetting the first, respectively the last morphism in such a sequence. The \(k\) inner face maps \(d_j, j = 1, \cdots, k\) are given by composing the \(j\)-th morphism with the \(j+1\)-st morphism in the sequence. The degeneracy maps \(e_j, j = 0, \cdots, k + 1\) are given by inserting an identity morphism on \(x_j\).

Remark 3.1. The nerve alone contains all the information needed to reconstruct the small category \(M^{(1)}\). If the big category \(C\) is closed by fiber products then \(M^\bullet\) is a simplicial set in \(C\).

Nerves of groupoids, which we shall consider in this paper, are in one to one correspondence with Kan complexes. These are simplicial sets \(X\) with the property that, for \(0 < k \leq n\), any morphism of simplicial sets \(\Delta^n_k \to X\), where \(\Delta^n_k\) is obtained by removing the interior of \(\Delta^n\) and that of the faces \(d_k \Delta^n\), can be extended to simplicial morphism \(\Delta^n \to X\).

3.2 The nerve of a Lie 2-group

Associated to any Lie 2-group \(G^{(1)} \ni G\), there is a simplicial manifold \(N^\bullet(G^{(1)}) = G^\bullet\). It is a simplicial set in the category of Lie groups i.e., the \(k\)-th nerve \(N^k(G) = G^{(k)} = G^{(1)} \times_{\cdots \times} G^{(1)}\) is endowed with the structure of a Lie group via the coordinate-wise multiplication, and the faces and degeneracy maps are Lie group morphisms. The nerve of a Lie-2 group \(G^{(1)}\) is a simplicial set on the category of Lie groups.
We shall give a few properties of the nerve of a Lie 2-group which even if elementary will be useful for the sequel.

The Lie 2-group \( G^{(1)} = H \rtimes G \) defines other two distinct simplicial sets in the category of Lie groups:

1. The family of Lie groups given by \( \{ H \rtimes \left( \cdots \rtimes \left( H \rtimes G \right) \right) \}_{k \geq 0} \) together with face and degeneracy maps given by:
   - \( \delta_i^k(h_k, \ldots, h_1, g) = (h_k, \ldots, h_{i+1}, h_i, \ldots, h_1, g) \in H \rtimes \left( \cdots \rtimes \left( H \rtimes G \right) \right) \),
   - \( e_j^k(h_k, \ldots, h_1, g) = (h_k, \ldots, h_{j+1}, e_M h_j, \ldots, h_1, g) \in H \rtimes \left( \cdots \rtimes \left( H \rtimes G \right) \right) \),
   for all \( 0 < i < k \) and \( 0 \leq j \leq k \).

2. The family of Lie groups given by \( \{ H^k \rtimes G \}_{k \geq 0} \) together with:
   - Face maps: \( \delta_i^k(h_k, \ldots, h_1, g) = (h_k, \ldots, h_{i+1}, h_i, \ldots, h_1, g) \in H^{k-1} \rtimes G \),
   - Degeneracy maps: \( e_j^k(h_k, \ldots, h_1, g) = (h_k, \ldots, h_j, h_j, \ldots, h_1, g) \in H^{k+1} \rtimes G \),
   for all \( 0 < i < k \) and \( 0 \leq j \leq k \).

These three simplicial sets in the category of Lie groups are isomorphic.

**Proposition 3.2.** The following maps define an isomorphism of simplicial sets in the category of Lie groups for the Lie-2 group \( G^{(1)} \equiv H \rtimes G \).

\[
G^{(k)} \to H \rtimes \left( \cdots \rtimes \left( H \rtimes G \right) \right) \quad (3.2.1)
\]

\[
(h_k, \cdots, h_1, g) \times \cdots \times (h_1, g) \mapsto (h_k, \ldots, h_1, g).
\]

\[
H \rtimes \left( \cdots \rtimes \left( H \rtimes G \right) \right) \to H^k \rtimes G \quad (3.2.2)
\]

\[
(h_k, \ldots, h_1, g) \mapsto (h_k, \ldots, h_2, h_1, g) := (h_k \cdots h_1, \ldots, h_2 h_1, h_1, g).
\]

### 3.3 The nerve of a PB groupoid and of its partial quotient groupoid

The following statement is motivated by Propositions 1.14 and 1.16.

**Theorem 3.3.** The nerve of a principal bundle groupoid is a collection of principal bundles given by the nerves of the groupoids:

\[
\left[ P^{(1)} \overset{G^{(1)}}{\longrightarrow} M^{(1)} \right] \sim \left[ P^* \overset{G^*}{\longrightarrow} M^* \right].
\]

Moreover, a principal \((G^{(1)} \Rightarrow G)\)-bundle groupoid \((P^{(1)} \Rightarrow P)\) over \((M^{(1)} \Rightarrow M)\) defines a simplicial set in the \((1)\)-principal bundle category, given by \(G^*\) acting on \(P^*\) with quotient space being \(M^*\).
Proof. By functoriality, this follows from the fact that the action map \(G^{(1)} \times P^{(1)} \to P^{(1)} \to M^{(1)}\) is a Lie groupoid morphism. Nonetheless, we give a pedestrian proof of the statement.

**Simplicial set:** We note that the action map extends to a map of simplicial sets:

\[
(G^{(1)} \times P^{(1)})^{(k)} \to P^{(k)} \to M^{(k)} \quad \forall k \in \mathbb{N},
\]

Moreover, \((G^{(1)} \times P^{(1)})^{(k)} \cong G^{(k)} \times P^{(k)}\), which yields a map of simplicial sets:

\[
G^{(k)} \times P^{(k)} \to P^{(k)} \to M^{(k)} \quad \forall k \in \mathbb{N}.
\]

**Simplicial set on Principal bundles:** Since the action map \(G^{(1)} \times P^{(1)} \to P^{(1)}\) is a group action, the maps given by the cartesian product \((G^{(1)})^{k} \times (P^{(1)})^{k} \to (P^{(1)})^{k}\) are also group actions. From which it follows that the restriction to the submanifolds \(G^{(k)} \times P^{(k)} \to P^{(k)}\) are also group actions \(\forall k \in \mathbb{N}\).

Moreover, \((G^{(1)})^{k} \times (P^{(1)})^{k} \to (P^{(1)})^{k} \to (M^{(1)})^{k}\) are principal bundles and hence so do the following submanifolds build principal bundles:

\[
G^{(k)} \times P^{(k)} \to P^{(k)} \to M^{(k)}, \quad \forall k \in \mathbb{N}.
\]

For a groupoid \(\mathcal{A} \in \{G^{(1)}, P^{(1)}, M^{(1)}\}\), let \(d^{\mathcal{A}}_{i} : \mathcal{A}(k) \to \mathcal{A}(k-1)\) and \(e^{\mathcal{A}}_{i} : \mathcal{A}(k-1) \to \mathcal{A}(k)\) denote the face and degeneracy maps respectively. The maps \(d^{G}_{i} \times d^{P}_{i} : G^{(k)} \times P^{(k)} \hookrightarrow G^{(k-1)} \times P^{(k-1)}\) and \(e^{G}_{i} \times e^{P}_{i} : G^{(k-1)} \times P^{(k-1)} \hookrightarrow G^{(k)} \times P^{(k)}\) are the face and degeneracy maps of the cartesian product groupoid \(P^{(1)} \times G^{(1)}\). Consider the following diagrammes here combined in one sole diagramme, each of which involves the face maps \(d_{i}\), resp. the degeneracy maps \(e_{i}\):

\[
\begin{align*}
G^{(k)} \times P^{(k)} \quad &\xrightarrow{d^{G}_{i} \times d^{P}_{i}} \quad P^{(k)} \quad \xrightarrow{d^{P}_{i}} \quad M^{(k)} \\
G^{(k-1)} \times P^{(k-1)} \quad &\xrightarrow{e^{G}_{i} \times e^{P}_{i}} \quad P^{(k-1)} \quad \xrightarrow{e^{P}_{i}} \quad M^{(k-1)}
\end{align*}
\]

They commute in so far as the the face maps \(d_{i}\), resp. the degeneracy maps \(e_{i}\), commute with the horizontal ones since the horizontal arrows are simplicial sets morphisms coming from a groupoid morphism. Therefore, the maps \((d^{G}_{i}, d^{P}_{i}, d^{M}_{i})\) and \((e^{G}_{i}, e^{P}_{i}, e^{M}_{i})\) are principal bundle morphisms. \(\square\)

The subsequent statement follows from Proposition [1.22] combined with Theorem [3.3].

**Proposition 3.4.** With the notations of Proposition [1.22] the projection \(N^{*}(Q) : N^{*}(P^{(1)}) \to N^{*}(P^{(1)}/G)\) induced by \(Q\) is a simplicial set of principal \(G\)-bundles and \(N^{*}(Q)\) induces the following isomorphisms for any \(k \in \mathbb{N}\):

\[
\tilde{N}^{k}(Q) : \left(N^{k}(P^{(1)})\right)_{/G} \xrightarrow{\cong} N^{k}\left(P^{(1)}/G\right) \quad \forall \phi_{1}, \ldots, \phi_{k} \quad \mapsto \quad ([\phi_{1}], [\phi_{2}], \ldots, [\phi_{k}]).
\]  

(3.4.1)

Moreover, the face maps on the nerve of \(P^{(1)}\) give rise to face maps \(\partial_{i}^{k}\) for any \(k \in \mathbb{N}\):

\[
\partial_{i}^{k} : N^{k}\left(P^{(1)}\right)_{/G} \quad \to \quad N^{k-1}\left(P^{(1)}\right)_{/G} \quad \forall \phi_{1}, \ldots, \phi_{k} \quad \mapsto \quad [\phi_{1}, \ldots, \phi_{i} \circ \phi_{i+1}, \ldots, \phi_{k+1}].
\]  

(3.4.2)
Denoting by $\delta_i^k$ the face maps of the groupoid $G^{(k)}$, the subsequent diagramme commutes:

$$
\begin{align*}
\delta_i^k & : N^k(P^{(1)}/G) \longrightarrow N^{k-1}(P^{(1)}/G) \\
\delta_i^k & : N^k(Q) \longrightarrow N^{k-1}(Q) \\
\delta_i^k & : N^k(P^{(1)})/G \longrightarrow N^{k-1}(P^{(1)})/G
\end{align*}
$$

We apply the above constructions to the pair groupoid of a principal bundle.

**Example 3.5.** Let $P \rightarrow M$ be a principal $G$-bundle and $P^{(1)} = T^{(1)}(P), M^{(1)} = T^{(1)}(M)$ be the corresponding pair groupoids. We saw that in this case, $P^{(1)}/G$ is the gauge groupoid $G(P)$.

The isomorphisms of Proposition 3.4 are the following maps:

$$
\tilde{N}^k(Q) : \tilde{T}^{(k)}(P)/G \xrightarrow{\cong} G^{(k)}(P)
$$

$$
[p] := [p_1, \cdots, p_{k+1}] \mapsto ([p_1, p_2], [p_2, p_3], \cdots, [p_k, p_{k+1}]). \tag{3.5.1}
$$

Moreover, the face maps on the nerve of the pair groupoid $\tilde{T}(P)$ give rise to face maps $\tilde{\delta}_i^k$:

$$
\tilde{\delta}_i^k : \tilde{T}^{(k)}(P)/G \longrightarrow \tilde{T}^{(k)}(P)/G
$$

$$
[p_1, \cdots, p_{k+1}] \mapsto [p_1, \cdots, \tilde{p}_i, \cdots, p_{k+1}]. \tag{3.5.2}
$$

Denoting by $\delta_i^k$ the face maps of the partial quotient groupoid $G^{(k)}(P)$, the subsequent diagramme commutes:

$$
\begin{align*}
\delta_i^k & : \tilde{T}^{(k)}(P)/G \longrightarrow \tilde{T}^{(k-1)}(P)/G \\
Q_k & \downarrow \quad Q_{k-1} \\
G^{(k)}(P) & \longrightarrow \tilde{G}^{(k)}(P)
\end{align*}
$$

## 4 Inner transformations of a PB groupoid

In this section we investigate gauge transformations of principal bundle groupoids and specialise to the case of the pair groupoid of a principal bundle manifold.

With the notations of Definition 1.15 let $P^{(1)} \Rightarrow P$ be a $H \rtimes G$-principal bundle groupoid over a Lie groupoid $M^{(1)} \Rightarrow M$.

### 4.1 Inner transformations of a PB groupoid

Let us recall that the gauge transformations of the principal $G$-bundle $\pi : P \rightarrow M$ is the group of principal bundle morphisms over the identity in $M$, i.e.

$$
\text{Mor}(P) := \{ \phi \in C^\infty(P, P) : \pi(p) = \pi(\phi(p)) ; \phi(g p) = g \phi(p) \ \forall g \in G , p \in P \}.
$$

This is a group because there is a well known isomorphism:

$$
\text{Mor}(P) \simeq C^\infty_G(P, G),
$$
where the multiplication in $\text{Mor}(P)$ is the composition of morphisms; the set on the r.h.s. are $G$-equivariant smooth maps from $P$ to $G$ with the group multiplication, and $G$ acts by the left action on $P$ and by the (left) adjoint action on $G$.

Here and in what follows $\text{Aut}(P)$ stands for invertible elements in $\text{Mor}(P)$; the mention of a group in the lower index such as $G$ in $C^\infty_G$, refers to equivariance maps with respect to the action of the group $G$. A priori $\text{Aut}(P) \subset \text{Mor}(P)$, but the isomorphism $\text{Mor}(P) \simeq C^\infty_G(P, G)$ leads to the conclusion $\text{Aut}(P) = \text{Mor}(P)$. Therefore, in the rest of this paper $\text{Aut}(\cdot)$ and $\text{Mor}(\cdot)$ are interchangeable.

Propositions 3.3 and 3.2 give rise to a simplicial set of principal bundles whose $k$-nerves are principal $H^k \rtimes G$-bundle groupoids $P^{(k)} \to M^{(k)}, k \in \mathbb{N}$ and bijections:

$$\Psi_k : \text{Aut}(P^{(k)}) \xrightarrow{\simeq} C^\infty_G(P^{(k)}, H^k \rtimes G),$$

(4.0.1)

which in turn give rise to group isomorphisms. In a compact form, this reads

$$\text{Aut}(P^\bullet) \cong C^\infty_{H^\bullet \rtimes G}(P^\bullet, H^\bullet \rtimes G).$$

(4.0.2)

and we refer to $\text{Aut}(P^\bullet)$ as the group of inner transformations of the nerve of the principal bundle groupoid $P^{(1)}$, or for short the inner transformation group of $P^\bullet$.

### 4.2 Inner transformations of the partial quotient of a PB groupoid

Given a PB-groupoid $H \rtimes G \rightrightarrows P^{(1)} \rtimes \pi^{(1)} M^{(1)}$, let $P^{(1)}/G$ be the partial quotient (as in 3.4.1) and let us abuse notation to call the map given by the original PB-groupoid $\pi^{(1)} : P^{(1)}/G \to M^{(1)}$ again. This is not a PB-groupoid, in general there is no Lie 2-group acting on $P^{(1)}/G$. Let us denote following sets and maps for every $k \in \mathbb{N}$:

$$C^\infty(P^{(k)}/G, P^{(k)}/G)^{M^{(k)}} := \left\{ \varphi \in C^\infty(P^{(k)}/G, P^{(k)}/G) : \varphi \circ \pi^{(k)} = \varphi \right\},$$

$$\Pi_k : \text{Aut}(P^{(k)}) \longrightarrow C^\infty(P^{(k)}/G, P^{(k)}/G)^{M^{(k)}}$$

$$(\vec{\phi} \mapsto \varphi(\vec{\phi}) = (\varphi_0(\vec{\phi}), \ldots, \varphi_{k+1}(\vec{\phi}))) \longmapsto \left([\vec{\phi}] \mapsto [\varphi(\vec{\phi})] = \left[ (\varphi_0(\vec{\phi}), \ldots, \varphi_{k+1}(\vec{\phi})) \right] \right),$$

$$\Gamma_k : C^\infty(P^{(k)}, H^k \rtimes G) \longrightarrow C^\infty(P^{(k)}, H^k)$$

$$(h_k, \ldots, h_1, g_0) \longmapsto \left( C_{g_0} h_k, \ldots, C_{g_0} h_1 \right).$$

(4.0.3)

The maps $\Gamma_k$ clearly preserves equivariant properties w.r.to the partial action of $H^k \rtimes G$ on $P^{(k)}$ and the adjoint action of $H^k \rtimes G$ on $H^k$ by the adjoint action. Therefore, let us consider the maps

$$\Xi_k : C^\infty(P^{(k)}, H^k) \longrightarrow C^\infty(P^{(k)}/G, P^{(k)}/G)^{M^{(k)}}$$

$$(\vec{\phi} \mapsto (h_k(\vec{\phi}), \ldots, h_1(\vec{\phi}))) \longmapsto \left([\vec{\phi}] \mapsto \left[ (h_k(\vec{\phi}), \ldots, h_1(\vec{\phi}), 1 \cdot \vec{\phi}) \right] \right),$$

(4.0.4)

which can easily seen to be injective.

The following diagramme commutes:
Correspondingly, inner transformations of $P^k/G$ form a group with the following group isomorphism:

$$
\text{Aut}(P^k/G) \simeq \text{Aut}(P^k)/\sim_{\Pi_k} \simeq C_{H^k \rtimes G}^\infty(P^k, H^k).
$$

Correspondingly, inner transformations of $P^*/G$ form a group with the following group isomorphism:

$$
\text{Aut}(P^*/G) = \text{Aut}(P^*)/\sim_{\Pi_k} \simeq C_{H^* \rtimes G}^\infty(P^*, H^*).
$$

We now consider a bundle gerbe $B^{(1)}$ and, as before, denote by $\Psi$ the functor of Proposition 2.11. Then $p^{(1)} = \Psi(B^{(1)})$ is a base trivial PB groupoid, $B = \ker(s_G) \xrightarrow{\pi^{(1)}} M^{(1)}$ as in Prop 2.13; also $B^{(k)} \simeq P^k/G$ by Proposition 2.15. Since the functor $\Psi$ of Proposition 2.11 and the functor $\Xi$ of Proposition 2.13 are inverses of each other we get that

$$
\text{Aut}(B^{(1)}) := \left\{ \varphi \in C_{H^* \rtimes G}^\infty(B^{(1)}, B^{(1)}) : \varphi \circ \pi^{(1)} = \varphi \right\} \simeq \text{Aut}(p^{(1)})|_{B^{(1)}} \simeq \text{Aut}(P^{(1)}/G)
$$

so that the following statement is a straightforward consequence of Theorem 4.2.

**Corollary 4.3.** For any $k \in \mathbb{N}$, there is an isomorphism:

$$
\text{Aut}(B^{(k)}) = \text{Mor}(B^{(k)}) \simeq \text{Aut}(P^k/G) \simeq C_{H^k \rtimes G}^\infty(P^k, H^k).
$$

Correspondingly, inner transformations of the nerve of $B^*$ form a group with the following group isomorphism:

$$
\text{Aut}(B^*) \simeq \text{Aut}(P^*/G) \simeq C_{H^* \rtimes G}^\infty(P^*, H^*).
$$
4.3 The case of the pair groupoid of a principal bundle

Specialising Proposition [1.22] to the pair groupoid \( P^{(1)} = \mathcal{P}^{(1)}(P) \), we consider the canonical projection of the principal bundle \( G \curvearrowright P \to M \),

\[
Q: P^{(1)} \longrightarrow \mathcal{G}(P) := P^{(1)}/_G
\]
on to the quotient partial quotient groupoid \( \mathcal{G}(P) \) by the diagonal action. Applying Theorem [4.2] (with \( H = G \)) we get the following corollary.

**Corollary 4.4.** Gauge transformations of the nerve of the partial quotient groupoid \( \mathcal{G}(P) \) of a principal \( G \)-bundle \( P \to M \) are invertible and form a group

\[
\text{Aut}(\mathcal{G}^\bullet(P)) \cong C^0_{G\times_\mathcal{G}(\mathcal{P}^\bullet(P),G^\bullet)}.
\] (4.4.1)

Moreover, a groupoid morphism \( \tilde{f}: \mathcal{P}^{(1)}(P) \to \mathcal{P}^{(1)}(P) \) induces a map \( N^\bullet(\tilde{f}): \mathcal{P}^\bullet(P) \to \mathcal{P}^\bullet(P) \). Also, a principal bundle morphism \( f \in \text{Aut}(P) \) induces a groupoid morphism

\[
\tilde{f}: \mathcal{P}^{(1)}(P) \rightarrow \mathcal{P}^{(1)}(P)
\]

\[
(p,q) \mapsto (f(p), f(q)).
\]

The corresponding nerve map \( N^k(\tilde{f}): \mathcal{P}^\bullet(P) \to \mathcal{P}^\bullet(P) \) reads:

\[
N^k(\tilde{f})(p_0, \ldots, p_k) = (f(p_0), \ldots, f(p_k)).
\]

**Theorem 4.5.** Given a principal \( G \)-bundle \( P \to M \), we have the following canonical injections of gauge group transformations:

\[
\text{Aut}(P) \hookrightarrow \text{Aut}(\mathcal{P}^{(1)}(P)) \xrightarrow{N^\bullet} \text{Aut}(\mathcal{P}^\bullet(P)) \cong C^0_{G\times_\mathcal{G}(\mathcal{P}^\bullet(P),G^\bullet \times G)}.
\] (4.5.1)

A Appendix: Morita equivalence

In order to interpret \( H \times G \rightrightarrows G \)-principal 2-bundles in terms of principal bundle groupoids, we need the notion of Morita equivalence [Mac05], [MM03], [LGTX07]. Here are three equivalent definitions.

The following corresponds to a generalised isomorphism in the sense of Definition [1.28].

**Definition A.1.** Let \( G^{(1)} \rightrightarrows G \) and \( H^{(1)} \rightrightarrows H \) be two Lie groupoids. A **bitorsor** is a manifold \( X \) and surjective submersions \( G \xrightarrow{\rho} X \xrightarrow{\sigma} H \), such that

1. \( X \xrightarrow{\rho} G \) is a principal left \( G^{(1)} \)-bundle over \( H \), so \( X/H^{(1)} \cong G \),
2. \( X \xrightarrow{\sigma} H \) is a principal right \( H^{(1)} \)-bundle over \( G \), so \( X/G^{(1)} \cong H \),
3. The \( G^{(1)} \)- and \( H^{(1)} \)-actions on \( X \) commute,
and the following diagram commutes:

\[
\begin{array}{ccc}
G^{(1)} & \star & H^{(1)} \\
\downarrow s & & \downarrow s \\
G & \leftarrow & H
\end{array}
\]

**Definition A.2.** Two Lie groupoids $G^{(1)} \Rightarrow G$ and $H^{(1)} \Rightarrow H$ are **Morita equivalent** if there is a manifold $P$ with surjective submersions $G \leftarrow P \rightarrow H$, such that $\rho^{-1}G^{(1)} \cong \sigma^{-1}H^{(1)}$ as Lie groupoids over $P$.

**Definition A.3.** Two Lie groupoids $G^{(1)} \Rightarrow G$ and $H^{(1)} \Rightarrow H$ are **weak equivalent** if there is a groupoid manifold $P^{(1)} \Rightarrow P$ with weak equivalences $G^{(1)} \leftarrow P^{(1)} \rightarrow H^{(1)}$ (see Definition 1.29 for the notion of weak equivalence of groupoids).

**Example A.4.** Let $\pi : Y \rightarrow M$ be a surjective submersion, and $M^{(1)} \Rightarrow M$ be a Lie groupoid. The pull-back groupoid $\pi^{-1}M^{(1)} \Rightarrow Y$ is Morita equivalent, and weak equivalent to $M^{(1)} \Rightarrow M$, with $\rho = \text{Id}$ and $\sigma = \pi$. In particular, when $M^{(1)} = M$, then $\pi^{-1}M^{(1)} \cong Y^{[2]}$ is Morita equivalent and weak equivalent to $M$.

One easily checks that Morita equivalence and weak equivalence defines an equivalence relation. Another important property is its stability under generalised isomorphisms (Definition 1.28).

**Lemma A.5.** Two Lie groupoids $G^{(1)} \Rightarrow G$ and $H^{(1)} \Rightarrow H$ are Morita equivalent if and only if they are weak equivalent or if and only if there exists a bitorsor.

This was proven in [MM03] and [LGTX07] along the lines briefly sketched here: A bitorsor is a Morita equivalence since the actions give an isomorphism of the pullback groupoids. A Morita equivalence gives a weak equivalence by means of the pullback groupoid in the middle. From a weak equivalence one can build a bitorsor. For further details, see [Gar19].
References

[AFBP22] S Azzali, A Frabetti, Y Boutaib, and S Paycha. Groupoids with direct connections and their jet prolongations. 
https://www.math.uni-potsdam.de/~paycha/paycha/Unpublished_notes_May_2022.pdf, 2022.

[BH11] J Baez and J Huerta. An invitation to higher gauge theory. General Relativity and Gravitation, 43:2335–2392, 2011.

[BL04] J Baez and A Lauda. Higher-dimensional algebra v: 2-groups. Theory Appl. Categ., 12(423–491), 2004.

[BS76] R Brown and Ch Spencer. G-groupoids, crossed modules and the fundamental groupoid of a topological group. Indag. Math., 38(4):296–302, 1976.

[CLS14] S Chatterjee, A Lahiri, and A Sengupta. Path space connections and categorical geometry. Journal of Geometry and Physics, 75, 2014.

[CZ13] A Cattaneo and M Zambon. A supergeometric approach to Poisson reduction. Comm. Math. Phys., 318(3):675–716, 2013.

[Fri12] G Friedman. Survey article: An elementary illustrated introduction to simplicial sets. Rocky Mountain J. Math, 42:353–423, 2012.

[Gar19] A Garmendia. Groupoids and singular foliations. arXiv:2107.10502 PhD thesis, 2019.

[GZ21] A Garmendia and M Zambon. Quotients of singular foliations and Lie 2-group actions. J. Noncommut. Geom. 15, no. 4, pp. 1251–1283, 2021.

[Hae84] A Haefliger. Groupoides d'holonomie et classifiants. Structures transverses des feuilletages. Astérisque, 116:70–97, 1984.

[Iva01] Gh Ivan. Principal fibre bundles with structural groupoids. Balkan Journal of Geometry and Its Applications, 6:39–48, 2001.

[LGTX07] C Laurent-Gengoux, J Tu, and P Xu. Chern–Weil map for principal bundles over groupoids. Mathematische Zeitschrift, 225:451–491, 2007.

[Mac05] K Mackenzie. General theory of Lie groupoids and Lie algebroids. London Mathematical Society lecture note series. Cambridge University Press, 213, 2005.

[MM03] I Moerdijk and J Mrcun. Introduction to foliations and Lie groupoids. Cambridge Studies in Advanced Mathematics, Cambridge University Press, 91, 2003.

[NW13] Th Nikolaus and K Waldorf. Four equivalent versions of non-abelian gerbes. Pacific J. Math., 264:355–420, 2013.

[Ros04] A Rossi. Principal bundles with groupoid structure: local vs. global theory and non abelian Cech cohomology. https://arxiv.org/abs/math/0404449, 2004.
[Whi46] J Whitehead. Note on a previous paper entitled: On adding relations to homotopy group. *Ann. Math.*, 47:806–810, 1946.

[Whi49] J. Whitehead. Combinatorial homotopy II. *Bull. Amer. Math. Soc.*, 55:453–496, 1949.