NONSYMMETRIC ROGERS-RAMANUJAN SUMS AND THICK DEMAZURE MODULES

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Abstract. We consider expansions of products of theta-functions associated with arbitrary root systems in terms of nonsymmetric Macdonald polynomials at $t = \infty$ divided by their norms. The latter are identified with the graded characters of Demazure slices, some canonical quotients of thick (upper) level-one Demazure modules, directly related to recent theory of generalized (nonsymmetric) global Weyl modules. The symmetric Rogers-Ramanujan-type series considered by Cherednik-Feigin were expected to have some interpretation of this kind; the nonsymmetric setting appeared necessary to achieve this. As an application, the coefficients of the nonsymmetric Rogers-Ramanujan series provide formulas for the multiplicities of the expansions of tensor products of level-one Kac-Moody representations in terms of Demazure slices.

Key words: Rogers-Ramanujan series; Hecke algebras; nonsymmetric Macdonald polynomials; Kac-Moody algebras; Demazure modules; global Weyl modules.

Contents

1. Introduction 4
  1.1. Major objectives 4
    1.1.1. Rogers-Ramanujan summations 4
    1.1.2. Demazure modules 4
    1.1.3. Nonsymmetric setting 5
  1.2. Main results 6
    1.2.1. $E$– polynomials 6
| Section | Title                                           | Page |
|---------|------------------------------------------------|------|
| 1.2.2  | Rogers-Ramanujan sums                           | 7    |
| 1.2.3  | Demazure slices                                 | 8    |
| 2      | **Affine roots systems and DAHA**              | 9    |
| 2.1    | **Affine Weyl groups**                          | 9    |
| 2.1.1  | Extended Weyl groups                            | 10   |
| 2.1.2  | The length on $\hat{W}$                        | 10   |
| 2.1.3  | Elements $\pi_b, u_b$                           | 11   |
| 2.1.4  | Affine Weyl chamber                             | 12   |
| 2.2    | Partial ordering on $P$                         | 12   |
| 2.2.1  | The definition                                 | 12   |
| 2.2.2  | Bruhat ordering etc                             | 13   |
| 2.3    | **Polynomial representation**                  | 13   |
| 2.3.1  | $Y$-operators                                   | 14   |
| 2.3.2  | The $\mu$-function                              | 15   |
| 2.4    | $E$ -polynomials                                | 16   |
| 2.4.1  | Using $Y$-operators                             | 16   |
| 2.4.2  | Standard identities                             | 17   |
| 3      | **Theta-products via $E$ -polynomials**         | 17   |
| 3.1    | **Mehta-Macdonald identities**                 | 17   |
| 3.1.1  | Basic notations                                 | 18   |
| 3.1.2  | The key expansions                              | 19   |
| 3.2    | Using $E$ -dag polynomials                      | 20   |
| 3.2.1  | The numbers $n_c(b)$                            | 20   |
| 3.2.2  | An example for $A_2$                            | 22   |
| 3.2.3  | Proof of monomiality                            | 22   |
| 3.3    | Three major expansions                          | 23   |
| 3.3.1  | Non-spherical formulas                          | 23   |
| 3.3.2  | The limit at zero                               | 24   |
| 3.3.3  | The remaining cases                             | 25   |
| 3.4    | **Iteration formulas**                         | 26   |
| 3.4.1  | Some remarks                                    | 26   |
| 3.4.2  | Using intertwiners                              | 28   |
1. Introduction

Generalizing [ChFB], we expand the products of (standard) theta-functions associated with arbitrary root systems in terms of nonsymmetric Macdonald polynomials at $t = \infty$ divided by their norms. The latter are identified with the graded characters of Demazure slices, canonical quotients of the level-one thick (upper) Demazure modules, which are directly related to recent theory of generalized global Weyl modules; see [FKM, FMO, KL] and references there. We conjecture and partially prove (in the “mixed case”) that the expansions above, the nonsymmetric Rogers-Ramanujan series, match the decomposition of tensor products of level-one integrable representations of Kac-Moody algebras in terms of Demazure slices. This provides formulas for the corresponding multiplicities, which are generally difficult to obtain representation-theoretically.

1.1. Major objectives. Let us briefly overview the problems we approach and (partially) settle and the major techniques that are used. The identification of nonsymmetric Macdonald polynomials at $t = \infty$ divided by their norms with the graded characters of Demazure slices is the key; interestingly, the DAHA-based expansion of a (single) theta-function is used in our proof. Applications of this fact to tensor products of arbitrary integrable Kac-Moody modules will require further efforts. We only consider the tensor products of level-one modules.

1.1.1. Rogers-Ramanujan summations. The expansions of products of standard theta-functions associated with an arbitrary (simple) root system in terms of $q$–Hermite polynomials from [ChFB] generalize the celebrated Rogers-Ramanujan summations and its various multi-rank extensions. There are connections with [An2, War, GOW] and a vast literature on the Rogers-Ramanujan identities, the representation theory of Kac-Moody algebras, and related mathematical physics, including the so-called Nahm conjecture; see [VZ, ChFB, CGZ]. Here the $q$–Hermite polynomials, the symmetric Macdonald polynomials at $t = 0$, were sufficient. The first usage of the $q$–Hermite polynomials in this context is actually due to Rogers.

1.1.2. Demazure modules. When we take a single copy of the theta-function, the expansion in [ChFB] was expected to be related to the filtrations of level-one representations of affine Lie algebra in terms of
thick Demazure modules, as in [Kas1], but the representation-theoretical tools for making this a theorem were developed only recently.

Thick Demazure modules are infinite dimensional in general, as opposed to the thin (lower) Demazure modules that are always finite-dimensional; see e.g. [Kum]. It is well known that thin Demazure modules of level one and nonsymmetric Macdonald polynomials at $t = 0$ are closely related through the local Weyl modules of (twisted) current algebras [San, Ion, CL, FL, FK]. Moreover, the symmetric $q$–Hermite polynomials divided by their norms were interpreted as the global Weyl modules [LNS, FMS], with some technical reservations. Furthermore, the filtration of parabolic Verma modules in terms of the global Weyl modules was provided in [CI], which is important to us. The latter filtration was generalized to any integrable highest weight modules of affine Lie algebra in [KL]. Namely, the existence of the filtration of any such modules in terms of global Weyl modules was proven there.

This theory combined with [FKM, FMO, OS] is essentially sufficient to connect thick Demazure modules with global and generalized global Weyl modules, at least for the types $ADE$. We use this approach and develop a systematic theory of Demazure-Joseph functor [Jos] in the thick case. One of the applications is a connection between thin and thick Demazure modules; also, any twisted root systems can be considered. From the nil-DAHA perspective, the Demazure-Joseph functor is closely related to the DAHA intertwiners from [ChO], which is one of the key advantages of the usage of nonsymmetric Macdonald polynomials vs. the symmetric ones.

1.1.3. Nonsymmetric setting. In order to fully employ these and other techniques, it is necessary to consider all Demazure modules, not only those stable under the action of the classical part of the affine Lie algebra (i.e. in the case of dominant weights). This corresponds to the passage from the symmetric to nonsymmetric Macdonald polynomials in DAHA theory. Here the limits $t \to 0$ and $t \to \infty$ result in very different families of nonsymmetric polynomials. The nonsymmetric $q$–Hermite polynomials, called $E\bar{E}$–polynomials in this paper, correspond to $t \to 0$; they are significantly simpler than the $E\bar{t}$–polynomials corresponding to $t \to \infty$. A direct relation is only in the direction from $E\bar{t}$ to $E\bar{E}$. However $E\bar{t}$–polynomials are dual to $E\bar{E}$–polynomials with the respect to some natural inner product; this is important to us.
The goal of this paper is to present, examine and apply three explicit expansion formulas for the product of standard theta-functions in terms of $E^-$ and $E^\dagger-$polynomials (divided by their norms). All three coincide with the corresponding expansion from [ChFB] upon the symmetrization; the case when we expand in terms of (nonsymmetric) $E^-$polynomials is actually close to the (symmetric) one from [ChFB].

A general problem is actually to expand products of the theta-functions multiplied by any $E^-$ or $E^\dagger-$polynomials. Such formulas of course require nonsymmetric theory. These $E^-$factors must be omitted (unless for minuscule weights) if one wants to obtain Rogers-Ramanujan-type series of $PSL(2, \mathbb{Z})-$modular type, as in [VZ, ChFB, CGZ]; their presence destroys the modularity. When they are absent, we therefore expand the same products of theta-functions as in [ChFB] and our formulas are (non-trivial) partitions of those considered there. The main point of this paper is that (even without the $E^-$factors) such partitions have deep representation-theoretic meaning. For instance, the DAHA intertwiners, which require the nonsymmetric setting, are closely connected with the Demazure-Joseph functors [Jos].

1.2. Main results.

1.2.1. $E^-$ polynomials. Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a simple root system, $\{\alpha_i\}$ simple roots, $W$ the Weyl group, $P, Q$ the weight and root lattices. Let $b^-$ denote the antidominant element (i.e. that from $P_-$) that is $W-$conjugate to $b \in P$. The normalization of the standard form in $\mathbb{R}^n$ is $(\alpha, \alpha) = 2$ for short roots; the corresponding affine system is twisted:

$$\tilde{R} = \{[\alpha, \nu_{\alpha}j] \mid \alpha \in R, j \in \mathbb{Z}, \nu_{\alpha} = (\alpha, \alpha)/2\}.$$ 

See Section 2.4 for the definition of the nonsymmetric Macdonald polynomials $E_b(X; q, t)$ ($b \in P$). By $^*$, we denote the standard DAHA anti-involution sending $X_b \mapsto X_b^{-1}, q \mapsto q^{-1}, t \mapsto t^{-1}$. These polynomials are in terms of pairwise commutative variables $X_c$ ($c \in P$); $q, t = \{t_{|\alpha|}\}$ are their parameters. Let

$$\overline{E}_b \overset{\text{def}}{=} E_b(X; q, t \to 0), \quad E^\dagger_0 = E^\ast_0(t \to 0), \quad k^0_b$$

be the limit of the norm of $E_b(X, q, t)$ as $t \to 0$, a very explicit product of certain $(1 - q^j)$. Note that the polynomials $E^\dagger_0$ naturally appear in our expansions, not the dag-polynomials $E^\dagger_0 \overset{\text{def}}{=} E^\ast_0(t \to \infty).$
We have a standard theta-function $\theta$ for $R$: $\theta(X) \overset{\text{def}}{=} \sum_{b \in P} q^{(b,b)/2} X_b$, and its special normalization $\hat{\theta} \overset{\text{def}}{=} \theta/\langle \theta \mu_0(t \to 0) \rangle$. See (3.8) for the formula for the constant term $\langle \theta \mu_0 \rangle$ of $\theta$ as $t \to 0$; $\mu_0 = \mu/\langle \mu \rangle$ for the DAHA $\mu$-function. Under this normalization, $\hat{\theta}$ can be identified with the graded character of a level-one integrable representation $L$ of the twisted affinization $\hat{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$ corresponding to the root system $R$.

1.2.2. Rogers-Ramanujan sums. The following particular case of formula (3.48) (the “mixed formula” from Theorem 3.4) is the key.

**Theorem 1.1.** For any $c \in P$, $p \in \mathbb{N}$, and $b = \{b_k \in P \mid 1 \leq k \leq p \}$,

$$E_{c^\dagger} \hat{\theta}^p = \sum_b C_b q^{(c_\alpha^P - b_\alpha^P)^2 + (b_\alpha^P - b_\alpha^P)^2 + \cdots + (b_{p-1}^P - b_{p-1}^P)^2}/2 \frac{E_{by_p}^{\dagger*}}{h_{by_p}^0},$$

where $C_b$ is some power of $q$ depending on $b$, whose definition requires the theory of $E^{\dagger*}$-polynomials.

Setting $c = 0$, we arrive at an expansion of $\hat{\theta}^p$ in terms of $E^{\dagger*}$-polynomials divided by their norms. With a reservation about the range of $b_k$ (which is $P$, not $P_-$) and the $q$-factors $C$, this expansion is quite similar to that from [ChFB]. Actually, it can be reduced to the formula there (in the absence of $c$) using some theory of $E^{\dagger}$-polynomials.

To understand this series from representation-theoretic perspective, we use the associated graded pieces of thick Demazure filtrations of level-one integrable representations of $\hat{\mathfrak{g}}$, called in this paper the Demazure slices.

The Rogers-Ramanujan theory is of course not only about the summations; the product formulas (if they exist) are very important. We expect interesting representation-theoretic applications here. Let us also mention the identities connecting different expansion of the (same) products of theta-function. The latter are described in [ChD] for arbitrary $t$ in the symmetric setting; the passage to the non-symmetric sums is straightforward. This is closely related to the topological vertex and DAHA-Jones polynomials of Hopf links. Another perspective is a generalization of this paper to the spinor $q$–Whittaker global function from [ChO]. Affine Hall functions from [Ch6] must be mentioned too, especially Section 2.5 there (devoted to the Kac-Moody limit).
1.2.3. Demazure slices. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra corresponding to $R$, $\hat{\mathfrak{g}}$ its twisted affinization; $L$ a level-one integrable representation of $\hat{\mathfrak{g}}$ (sometimes called basic).

**Theorem 1.2.** The $E^\dagger$–polynomials divided by their norms are precisely the graded characters of Demazure slices.

Theorem 1.2 is (homologically) dual to the results of [San, Ion], though the $E^\dagger$–polynomials are significantly more involved than the $\overline{E}$–polynomials. It is also a “nonsymmetric” analogue of that in [CI] and is closely related to [KL]. Using Theorem 1.2, we derive the following theorem.

**Theorem 1.3.** Let $p$ be a positive integer. For a level one thin Demazure module $D_b$ associated to $b \in P$, the module $D_b^\vee \otimes L^\otimes p$ admits a filtration by the Demazure slices (as constituents).

This is essentially our representation-theoretic interpretation of Theorem 1.1. Combining Theorem 1.3 and Theorem 1.1, we obtain (quite non-trivial) formulas for the multiplicities of the Demazure slices in $D_b^\vee \otimes L^\otimes p$. Here obviously the “numerical” nil-DAHA approach is very reasonable for obtaining explicit formulas vs. direct representation-theoretic calculations; this was already demonstrated in [ChFB].

Let us provide some details. We first prove the symmetric analogue of Theorem 1.2, which generalizes [KL] (the ADE case) to the twisted affinizations. Then we follow [FKM] to obtain that Demazure slices have graded characters (homologically) dual to those of thin Demazure modules. Using this, Theorem 1.2 follows from the identification of the proper Ext–pairing and the pairing from the theory of Macdonald polynomials. Papers [ChO, Kas3, Kat1] are used here. Finally, the machinery of Demazure-Joseph functors gives Theorem 1.3.

The end of the paper contains conjectures concerning the interpretation of the remaining cases of the “numerical” theta-function expansions. Let us mention here that Section 5 is more compressed and technically involved than the rest of the paper. We provide sufficient references, but it is mostly aimed at specialists in affine Lie algebras.

We note that one of the key application in [ChFB] was that Rogers-Ramanujan-type expansions there almost automatically satisfy the level-rank duality, which is generally involved representation-theoretically. It will be interesting to see which kind of level-rank duality the nonsymmetric theory presented in this paper has.
2. Affine roots systems and DAHA

Let $R = \{ \alpha \} \subset \mathbb{R}^n$ be a root system of type $A, B, ..., F, G$ with respect to a euclidean form $(z, z')$ on $\mathbb{R}^n \ni z, z'$, $W$ the Weyl group generated by the reflections $s_\alpha$, $R_+$ the set of positive roots corresponding to fixed simple roots $\alpha_1, ..., \alpha_n$, $\Gamma$ the Dynkin diagram with $\{ \alpha_i, 1 \leq i \leq n \}$ as the vertices, $\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha$, $R^\vee = \{ \alpha^\vee = 2\alpha/(\alpha, \alpha) \}$.

The root lattice and the weight lattice are:

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i \subset P = \oplus_{i=1}^n \mathbb{Z}\omega_i,$$

where $\{ \omega_i \}$ are fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$ for the simple coroots $\alpha_j^\vee$. Replacing $\mathbb{Z}$ by $\mathbb{Z}_+ = \{ k \in \mathbb{Z}, \pm k \geq 0 \}$ we obtain $Q_+, P_+$.

Here and further see [B, Ch4].

The form will be normalized by the condition $(\alpha, \alpha) = 2$ for the short roots in this paper. Thus $\nu_\alpha \overset{\text{def}}{=} (\alpha, \alpha)/2$ can be either 1, $\{ 1, 2 \}$, or $\{ 1, 3 \}$. We set $\nu_R \overset{\text{def}}{=} \{ \nu_\alpha | \alpha \in R \}$. This normalization leads to the inclusions $Q \subset Q^\vee, P \subset P^\vee$, where $P^\vee$ is defined to be generated by the fundamental coweights $\{ \omega_i^\vee \}$ dual to $\{ \alpha_i \}$.

We note that $Q^\vee = P$ for $C_n$ ($n \geq 2$), $P \subset Q^\vee$ for $B_{2n}$ and $P \cap Q^\vee = Q$ for $B_{2n+1}$; the index $[Q^\vee : P]$ is $2^{n-1}$ for any $B_n$ (in the sense of lattices).

2.1. Affine Weyl groups. The vectors $\vec{\alpha} = [\alpha, \nu_{\alpha j}] \in \mathbb{R}^n \times \mathbb{R} \subset \mathbb{R}^{n+1}$ for $\alpha \in R, j \in \mathbb{Z}$ form the affine root system $\tilde{R} \supset R$; this is the so-called twisted case.

The vectors $z \in \mathbb{R}^n$ are identified with $[z, 0]$. We add $\alpha_0 \overset{\text{def}}{=} [-\vartheta, 1]$ to the simple roots for the maximal short root $\vartheta \in R_+$. It is also the maximal positive coroot because of the choice of normalization. The (dual) Coxeter number is then $h = (\rho, \vartheta) + 1$. The corresponding set $\tilde{R}_+$ of positive roots equals $R_+ \cup \{ [\alpha, \nu_{\alpha j}], \alpha \in R, j > 0 \}$.

We complete the Dynkin diagram $\Gamma$ of $R$ by $\alpha_0$ (by $-\vartheta$, to be more exact); this is called the affine Dynkin diagram $\tilde{\Gamma}$.

The set of the indices of the images of $\alpha_0$ by all the diagram automorphisms of $\tilde{\Gamma}$ will be denoted by $O$; $O = \{ 0 \}$ for $E_8, F_4, G_2$. Let $O' \overset{\text{def}}{=} \{ r \in O, r \neq 0 \}$. The elements $\omega_r$ for $r \in O'$ are the so-called minuscule weights: $(\omega_r, \alpha^\vee) \leq 1$ for $\alpha \in R_+$ (here $(\omega_r, \vartheta) \leq 1$ is sufficient).
2.1.1. Extended Weyl groups. Given $\tilde{\alpha} = [\alpha, \nu, \tilde{\alpha}] \in \tilde{R}$, $b \in P$, the corresponding reflection in $\mathbb{R}^{n+1}$ is defined by the formula

$$(2.1) \quad s_{\tilde{\alpha}}(z) = \tilde{z} - (z, \tilde{\alpha})\tilde{\alpha}, \quad b'(z) = [z, \zeta - (z, b)],$$

where $\tilde{z} = [z, \zeta] \in \mathbb{R}^{n+1}$.

The affine Weyl group $\tilde{W}$ is generated by all $s_{\tilde{\alpha}}$ (we write $\tilde{W} = \langle s_{\tilde{\alpha}}, \tilde{\alpha} \in \tilde{R}_+ \rangle$). One can take the simple reflections $s_i = s_{\alpha_i}$ ($0 \leq i \leq n$) as its generators and introduce the corresponding notion of the length. This group is the semidirect product $W \ltimes Q'$ of its subgroups $W = \langle s_\alpha, \alpha \in \tilde{R}_+ \rangle$ and $Q' = \{a', a \in Q\}$, where

$$(2.2) \quad \alpha' = s_{\alpha}s_{[\alpha, \nu, \tilde{\alpha}]} = s_{[\alpha, -\nu, \tilde{\alpha}]}s_{\alpha} \quad \text{for} \ \alpha \in R.$$

The extended Weyl group $\tilde{W}$ generated by $W$ and $P'$ (instead of $Q'$) is isomorphic to $W \ltimes P'$:

$$(2.3) \quad (wb')(\langle z, \zeta \rangle) = [w(z), \zeta - (z, b)] \quad \text{for} \ w \in W, b \in P.$$

From now on, $b$ and $b'$, $P$ and $P'$ will be identified.

2.1.2. The length on $\tilde{W}$. Given $b \in P_+$, let $w_0^b$ be the longest element in the subgroup $W_0^b \subset W$ of the elements preserving $b$. This subgroup is generated by simple reflections. We set

$$(2.4) \quad u_b = w_0w_0^b \in W, \quad \pi_b = b(u_0)^{-1} \in \tilde{W}, \quad u_i = u_{\omega_i}, \quad \pi_i = \pi_{\omega_i},$$

where $w_0$ is the longest element in $W, 1 \leq i \leq n$.

The elements $\pi_r \overset{\text{def}}{=} \pi_{\omega_r}, r \in O'$ and $\pi_0 = \text{id}$ leave $\tilde{\Gamma}$ invariant and form a group denoted by $\Pi$, which is isomorphic to $P/Q$ by the natural projection $\{\omega_r \mapsto \pi_r\}$. As to $\{u_r\}$, they preserve the set $\{\vartheta, \alpha_i, i > 0\}$. The relations $\pi_r(\alpha_0) = \alpha_r = (u_r)^{-1}(\vartheta)$ distinguish the indices $r \in O'$. Moreover,

$$(2.5) \quad \tilde{W} = \Pi \ltimes \tilde{W}, \quad \text{where} \quad \pi_r \pi_i \pi_r^{-1} = \pi_r(\alpha_i) = \alpha_j, \quad 0 \leq j \leq n.$$

Setting $w = \pi_r u \in \tilde{W}$, $\pi_r \in \Pi$, $u \in \tilde{W}$, the length $l(w)$ is by definition the length of the reduced decomposition $u = s_i \ldots s_{i_2}s_{i_1}$ in terms of the simple reflections $s_i, 0 \leq i \leq n$. The number of $s_i$ in this decomposition such that $\nu_i = \nu$ is denoted by $l_{\nu}(w)$. 
The length can be also defined as the cardinality $|\lambda(w)|$ of the $\lambda$-sets:

\begin{align}
\lambda(w) & \overset{\text{def}}{=} \tilde{R}_+ \cap w^{-1}(\tilde{R}_-) = \{ \tilde{\alpha} \in \tilde{R}_+ , \ w(\tilde{\alpha}) \in \tilde{R}_- \} , \ w \in \widehat{W} ; \\
\lambda(w) & \overset{\text{def}}{=} \{ \tilde{\alpha} \in \lambda(w) , \ \nu(\tilde{\alpha}) = \nu \}. 
\end{align}

See, e.g. \cite{(2.7),(2.10),(2.9)} \cite{B, Hu} and also \cite{Ch4}.

2.1.3. Elements $\pi_b, u_b$. Extending the definition of $\pi_b, u_b$ from $b \in P_+$ to any $b \in P$, we have the following proposition.

**Proposition 2.1.** Given $b \in P$, there exists a unique decomposition $b = \pi_b u_b$, $u_b \in W$ satisfying one of the following equivalent conditions:

\begin{enumerate}
\item $l(\pi_b) + l(u_b) = l(b)$ and $l(u_b)$ is the greatest possible,
\item $\lambda(\pi_b) \cap R = \emptyset$.
\end{enumerate}

The latter condition implies that $l(\pi_b) + l(w) = l(\pi_b w)$ for any $w \in W$. Besides, the relation $u_b(b) \overset{\text{def}}{=} b_- \in P_- = -P_+$ holds, which, in its turn, determines $u_b$ uniquely if one of the following equivalent conditions is imposed:

\begin{enumerate}
\item[(iii)] $l(u_b)$ is the smallest possible,
\item[(iv)] if $\alpha \in \lambda(u_b)$ then $(\alpha, b) \neq 0$.
\end{enumerate}

\hspace{1cm} $\square$

We will need the following explicit description of the sets $\lambda(b)$. For $\tilde{\alpha} = [\alpha, \nu_\alpha] \in \tilde{R}_+$, one has:

\begin{align}
\lambda(b) & = \{ \tilde{\alpha} , \ (b, \alpha^\vee) > j \geq 0 \text{ if } \alpha \in R_+ , \ (b, \alpha^\vee) \geq j > 0 \text{ if } \alpha \in R_- \} , \\
\lambda(\pi_b) & = \{ \tilde{\alpha} , \ \alpha \in R_- , \ (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+ , \ (b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_- \} , \\
\lambda(\pi_b)^{-1} & = \{ \tilde{\alpha} \in \tilde{R}_+ , \ -(b, \alpha^\vee) > j \geq 0 \}, \\
\lambda(u_b) & = \{ \alpha \in \tilde{R}_+ , \ (b, \alpha^\vee) > 0 \}.
\end{align}

For instance, $l(b) = l(b_-) = -2(\rho^\vee, b_-)$ for $2\rho^\vee = \sum_{\alpha > 0} \alpha^\vee$.

There is an important interpretation of the length and the elements $\pi_b, u_b$ in terms of the following affine action of $\widehat{W}$ on $z \in \mathbb{R}^n$:

\begin{align}
(wb)(z) & = w(b + z), \ w \in \widehat{W}, b \in P , \\
s_{\tilde{\alpha}}(z) & = z - ((z, \alpha^\vee) + j)\alpha , \ \tilde{\alpha} = [\alpha, \nu_\alpha] \in \tilde{R}.
\end{align}
For instance, \( (bw)(0) = b \) for any \( w \in W \). The relation to the above action is given in terms of the affine pairing \( ([z, l], z' + d) \overset{\text{def}}{=} (z, z') + l \):
\[
(\hat{w}([z, l]), w([z']) + d) = ([z, l], z' + d) \quad \text{for} \quad \hat{w} \in \hat{W},
\]
where we treat \((, + d)\) formally (one can add \( d \) to \( \mathbb{R}^{n+1} \) and extend \((, )\) correspondingly; compare with Sect. 4.1.1).

2.1.4. **Affine Weyl chamber.** Introducing the basic affine Weyl chamber
\[
\mathcal{C}^a = \bigcap_{i=0}^{n} \mathcal{L}_{\alpha_i}, \quad \mathcal{L}_{[\alpha, j]} = \{ z \in \mathbb{R}^n, (z, \alpha) > 0 \},
\]
we come to another interpretation of the \( \lambda \)-sets:
\[
\lambda_{\nu}(w) = \{ \tilde{\alpha} \in \tilde{R}_+, \mathcal{C}^a \not\subset w(\mathcal{L}_{\tilde{\alpha}}), \nu_{\alpha} = \nu \}. \tag{2.14}
\]
Equivalently, taking a vector \( \xi \in \mathcal{C}^a \),
\[
\lambda(w) = \{ \tilde{\alpha} \in \tilde{R} \mid (\tilde{\alpha}^\vee, \xi + d) > 0 > (\tilde{\alpha}^\vee, \xi' + d) \}, \tag{2.15}
\]
for \( \xi' \in \tilde{w}^{-1}((\mathcal{C}^a)) \). Geometrically, \( \Pi \) is the group of all elements of \( \hat{W} \) preserving \( \mathcal{C}^a \) with respect to the affine action. Similarly, the elements \( \pi_b^{-1} \) for \( b \in P \) are exactly those sending \( \mathcal{C}^a \) to the basic nonaffine Weyl chamber \( \mathcal{C} \overset{\text{def}}{=} \{ z \in \mathbb{R}^n, (z, \alpha_i) > 0 \text{ for } i > 0 \} \).

2.2. **Partial ordering on \( P \).**

2.2.1. **The definition.** It is necessary in the theory of nonsymmetric polynomials; see [Op, Ma]. This ordering was also used in [Ch1] for calculating the coefficients of \( Y \)-operators. The definition is as follows:
\[
b \ll c, \ c \gg b \quad \text{for} \quad b, c \in P \quad \text{if} \quad c - b \in Q_+ \quad \text{and} \ b \neq c \tag{2.16}
\]
\[
b \prec c, \ c \succ b \quad \text{if} \quad b_\prec \ll c_\prec \quad \text{or} \quad \{b_\prec = c_\prec \quad \text{and} \ b \ll c \}. \tag{2.17}
\]
Recall that \( b_\prec = c_\prec \) means that \( b, c \) belong to the same \( W \)-orbit. We write \( <, > \) respectively if \( b \) can coincide with \( c \).

The following sets
\[
\sigma(b) \overset{\text{def}}{=} \{ c \in P, c \succeq b \}, \quad \sigma_+(b) \overset{\text{def}}{=} \{ c \in P, c \succeq b \},
\]
\[
\sigma(b) \overset{\text{def}}{=} \sigma(b_\prec), \quad \sigma_+(b) \overset{\text{def}}{=} \sigma_+(b_\prec) = \{ c \in P, c \gg b \_ \}. \tag{2.18}
\]
are convex. By convex, we mean that if \( c, d = c + r\alpha \in \sigma \) for \( \alpha \in R_+, r \in \mathbb{Z}_+ \), then
\[
\{ c, c + \alpha, \ldots, c + (r - 1)\alpha, d \} \subset \sigma.
\] (2.19)

2.2.2. Bruhat ordering etc. We will use the standard Bruhat ordering. Given \( w \in \hat{W} \), the standard Bruhat set \( B(w) \) is formed by \( u \) obtained by striking out any number of \( \{ s_j \} \) from a reduced decomposition of \( w \in \hat{W} \). The notation is \( u \leq w \). The set \( B(w) \) does not depend on the choice of the reduced decompositions.

**Proposition 2.2.** (i) Let \( c = u((0)), b = w((0)) \) and \( u \in B(w) \). Then \( c \geq b \) and \( b - c \) is a linear combination of the non-affine components of the corresponding roots from \( \lambda(w^{-1}) \). Also, \( c = b \) if and only if \( u \) is obtained by striking out \( s_j \) from \( v \in W \) such that \( w = \pi b v \) and this product is reduced, i.e. \( \ell(w) = \ell(\pi b) + \ell(v) \).

(ii) Letting \( b = s_i((c)) \) for \( 0 \leq i \leq n \), if the element \( s_i\pi c \) belongs to \( \{ \pi a, a \in P \} \) then it equals \( \pi b \). It happens if and only if \( (\alpha_i, c + d) \neq 0 \).

More precisely, the following three conditions are equivalent:
\[
\{ c \succ b \} \iff \{ (\alpha_i, c + d) > 0 \} \iff \{ s_i\pi c = \pi b, \ell(\pi b) = \ell(\pi c) + 1 \}.
\] (2.20)
The latter relation implies that \( \lambda(\pi b) = \pi c^{-1}(\alpha_i) \cap \lambda(\pi c) \).

The following lemma is Lemma 1.7 from [Ch5]; it extends part (ii) to the case \( (\alpha_i, c + d) = 0 \).

**Lemma 2.3.** The condition \( (\alpha_i, c + d) = 0 \) for \( 0 \leq i \leq n \), equivalently, the condition \( (\alpha_i, b + d) = 0 \) for \( b = s_i((c)) \), implies that \( u_c(\alpha_i) = \alpha_j \) for \( i > 0 \) or \( u_c(-\vartheta) = \alpha_j \) for \( i = 0 \) for a proper index \( j > 0 \). Given \( c \) and \( i \geq 0 \), the existence of such \( \alpha_j \) and the equality \( (\alpha_j, c_-) = 0 \) are equivalent to \( (\alpha_i, c + d) = 0 \). □

2.3. Polynomial representation. For the variables \( X_1, \ldots, X_n \), let
\[
X_{\tilde{b}} = \prod_{i=1}^{n} X_i^{l_i} q^k \quad \text{if} \quad \tilde{b} = [b, k], \quad w(X_{\tilde{b}}) = X_{w(\tilde{b})},
\] (2.21)

where \( b = \sum_{i=1}^{n} l_i \omega_i \in P \), \( k \in \mathbb{Q} \), \( w \in \hat{W} \).

For instance, \( X_0 \overset{\text{def}}{=} X_{\omega_0} = qX_{\omega_0}^{-1} \). We will set \( (\tilde{b}, \tilde{c}) = (b, c) \), i.e. we ignore the affine extensions in this pairing.
Note that $\pi^{-1}_r$ is $\pi_{r^*}$ and $u^{-1}_r$ is $u_{r^*}$ for $r^* \in O$, where the reflection $\{\cdot\}^*$ is induced by an involution $\iota$ of the nonaffine Dynkin diagram $\Gamma$.

By $e$, we will denote the least natural number such that $e(P, P)/2 = \mathbb{Z}$. Thus $e = 4$ for $D_{2k}$, $e = 2$ for $B_{2k}$ and $C_k$, otherwise $e = 2|\Pi|$.

2.3.1. $Y$-operators. For $\tilde{\alpha} = [\alpha, \nu_{\alpha^j}] \in \tilde{\Delta}$, $0 \leq i \leq n$, we set

$$t_{\tilde{\alpha}} = t_{\alpha} = t_{\nu_{\alpha^j}} = t_i.$$

The Demazure-Lusztig operators are as follows:

$$T_i = t_i^{1/2} s_i + (t_i^{1/2} - t_i^{-1/2})(X_{\alpha_i} - 1)^{-1}(s_i - 1), \quad 0 \leq i \leq n.$$

They obviously preserve $\mathbb{Z}[q][X_b, b \in P]$. We note that only the formula for $T_0$ involves $q$:

$$t_0^{1/2} s_0 + (t_0^{1/2} - t_0^{-1/2})(X_{\alpha_0} - 1)^{-1}(s_0 - 1),$$

where $X_0 = qX_0^{-1}$, $s_0(X_b) = X_b X_0^{-\varphi(b\varrho)} q^{(b\varrho)}$, $\alpha_0 = [-\varrho, 1]$.

We will also need $\pi_r$ ($r \in O$); they act via the general formula $w(X_b) = X_{w(b)}$ for $w \in \tilde{W}$.

Given $w \in \tilde{W}$, $r \in O$, the product

$$T_{\pi_r w} \overset{\text{def}}{=} \pi_r \prod_{k=1}^l T_{i_k}, \quad \text{where} \quad w = \prod_{k=1}^l s_{i_k}, l = l(\tilde{w}),$$

does not depend on the choice of the reduced decomposition (because $T_i$ satisfy the same “braid” relations as $s_i$ do). Moreover,

$$T_v T_w = T_{vw}$$

whenever $l(vw) = l(v) + l(w)$ for $v, w \in \tilde{W}$.

In particular, we arrive at the pairwise commutative elements

$$Y_b \overset{\text{def}}{=} \prod_{i=1}^n Y_i^{b_i} \quad \text{if} \quad b = \sum_{i=1}^n l_i \omega_i \in P, \quad Y_i \overset{\text{def}}{=} T_{\omega_i}, b \in P.$$

The action of $T_i$, $\pi_r$ ($r \in O$) and $X_b$, considered to the operators of multiplication by $X_b$ (see (2.21)), induces a representation of DAHA, the abstract algebra generated by these operators. It is called the polynomial representation; the notation is

$$\mathcal{V} \overset{\text{def}}{=} \mathbb{Z}[q^{\pm 1/(2e)}, q^{\pm 1/2}][X_b, b \in P].$$

It is a faithful DAHA-module if $q$ is not a root of unity, so we can skip the definition of DAHA in this paper.
2.3.2. The $\mu$-function. The following exponential notation for $t_\alpha$ in terms of parameters (or complex numbers) $k_\nu$ will be convenient in quite a few formulas. For $\tilde{\alpha} = [\alpha, \nu_\alpha] \in \tilde{R}$, $0 \leq i \leq n$, $k_\alpha \overset{\text{def}}{=} k_{\nu_\alpha}$, we set

$$
(2.27) \quad t_\tilde{\alpha} = t_\alpha = t_{\nu_\alpha} = q^{k_\nu_\alpha}, \quad q_\alpha = q^{\nu_\alpha}, \quad \rho_k \overset{\text{def}}{=} (1/2) \sum_{\alpha > 0} k_\alpha \alpha.
$$

For instance, by $X_\alpha(q^{\rho_k}) = q^{(\rho_k, \alpha)}$, we mean $\prod_{\nu \in \nu_R} t_\nu^{(\rho_k, \alpha)/\nu}$, where $\alpha \in R$. This product contains only integral powers of $t_\text{sh}_{\alpha}$ and $t_\text{sh}_{\alpha}$ ($t_\alpha$ for short and long roots). Note that $(\rho_k, \alpha^\vee) = k_\alpha = k_{\alpha^\vee}$ for $i > 0$.

The truncated theta-function, which is the key in the definition of the inner product of the polynomial DAHA representation (and in the theory of nonsymmetric Macdonald polynomials) is as follows:

$$
(2.28) \quad \mu(X; q, t) = \prod_{\alpha \in \nu_R} \prod_{j=0}^{\infty} \frac{(1 - X_\alpha q^j_\alpha)(1 - X_{\alpha^{-1}} q^{-j+1}_\alpha)}{(1 - X_\alpha t_\alpha q^j_\alpha)(1 - X_{\alpha^{-1}} t_\alpha q^{-j+1}_\alpha)}.
$$

We will consider $\mu$ as a Laurent series with coefficients in the ring $\mathbb{Q}[t_\nu][[q]]$. The constant term of a Laurent series $f(X)$ will be denoted by $\langle f \rangle$ through the paper. One has:

$$
(2.29) \quad \langle \mu \rangle = \prod_{\alpha \in \nu_R} \prod_{i=1}^{\infty} \frac{(1 - q^{(\rho_k, \alpha) + \nu_\alpha})^2}{(1 - t_\alpha q^{(\rho_k, \alpha) + \nu_\alpha})(1 - t_\alpha^{-1} q^{(\rho_k, \alpha) + \nu_\alpha})}.
$$

Using that $q^{(z, \alpha^\vee)} = q^{(z, \alpha^\vee)}$, we can set here $q^{(\rho_k, \alpha) + \nu_\alpha} = q^{(\rho_k, \alpha^\vee) + i}$. This formula is equivalent to the Macdonald constant term conjecture.

Let $\mu_\circ \overset{\text{def}}{=} \mu / \langle \mu \rangle$. The coefficients of the Laurent series $\mu_\circ$ are from the field of rationals $\mathbb{Q}(q, t) \overset{\text{def}}{=} \mathbb{Q}(q_\nu, t_\nu)$, where $\nu \in \nu_R$. We set

$$
(2.30) \quad \langle f, g \rangle \overset{\text{def}}{=} \langle f \, g^* \mu_\circ \rangle = \langle g, f \rangle^* \quad \text{for} \quad f, g \in \mathbb{Q}(q^{1/2}, t^{1/2})[X],
$$

where $X_b^* = X_{-b}$, $(t_\nu^u)^* = t_\nu^{-u}$, $(q_\nu^u)^* = q^{-u}$ for $u \in \mathbb{Q}$.

Note that $\mu_\circ^* = \mu_\circ$. Also $\langle A(f), g \rangle = \langle f, A^{-1}(g) \rangle$, for the DAHA generators $A = X_i$ ($1 \leq i \leq n$), $\pi_r$ ($r \in O$), $T_i$ ($i \geq 0$) and therefore for any $A = T_w^v Y_b$. So $\mathcal{V}$ is formally a $*$-unitary representation of DAHA.
We will mostly need this pairing in the limit $t_{\nu} \to 0$. For later reference, let $\pi \overset{\text{def}}{=} \mu(t_{\nu} \to 0, \nu \in \nu_R)$. Then

$$
(2.31) \quad \pi = \prod_{\alpha \in R_+} \prod_{j=0}^{\infty} (1 - X_{\alpha} q^j_{\alpha})(1 - X_{\alpha}^{-1} q^{j+1}_{\alpha}),
$$

$$
(2.32) \quad \langle \pi \rangle = n \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{1}{1 - q^j_i}, \quad \text{where} \quad q_i = q^{\nu_i}, \; \nu_i = \nu_{\alpha_i} = \frac{(\alpha_i, \alpha_i)}{2}.
$$

2.4. $E$-polynomials. There are two equivalent definitions of the non-symmetric Macdonald polynomials, denoted by $E_b = E_b(X; q, t)$ for $b \in P$. They belong to $\mathbb{Q}(q,t)[X_a, a \in P]$ and, using the pairing $\langle , \rangle$, can be introduced by means of the conditions

$$
(2.33) \quad E_b - X_b \in \Sigma_+(b) \overset{\text{def}}{=} \oplus_{c>b} \mathbb{Q}(q,t) X_c, \; \langle E_b, X_c \rangle = 0 \quad \text{for} \; P \ni c > b.
$$

They are well defined because the pairing is nondegenerate (for generic $q, t$) and form a basis in $\mathbb{Q}(q,t)[P]$.

This definition is due to Macdonald (for $k_{\text{sht}} = k_{\text{ling}} \in \mathbb{Z}_+$), who extended Opdam’s nonsymmetric polynomials introduced in the differential case in [Op] (Opdam mentions Heckman’s unpublished lectures in [Op]). The general (reduced) case was considered in [Ch2].

2.4.1. Using $Y$-operators. Another approach to $E$–polynomials is based on the $Y$–operators. We continue using the same notation $X, Y, T$ for these operators acting in the polynomial representation.

**Proposition 2.4.** The polynomials $\{E_b, b \in P\}$ are unique (up to proportionality) eigenfunctions of the operators $\{L_f \overset{\text{def}}{=} f(Y_1, \ldots, Y_n)\}$, where $f \in \mathbb{Q}[X]$, acting in $\mathbb{Q}_q,t[X]$ :

$$
(2.34) \quad L_f(E_b) = f(q^{-b_2}) E_b \quad \text{for} \quad b_2 \overset{\text{def}}{=} b - u_b^{-1}(\rho_k),
$$

$$
(2.35) \quad X_a(q^b) = q^{(a,b)}, \quad \text{where} \quad a, b \in P, \; u_b = \pi_b^{-1} b,
$$

$u_b$ is from Proposition 2.1, $b_2 = \pi_b((-\rho_k))$.

The coefficients of the Macdonald polynomials are rational functions in terms of $q, t_{\nu}$ (here either approach can be used). Note that $b_2 = b - \rho_k$ for $b \in P_-$ and $b_2 = b + \rho_k$ for generic $b \in P_+$ (such that $(b, \alpha_i) > 0$ for $i = 1, \ldots, n$).
2.4.2. **Standard identities.** We will need the following evaluation and norm formulas. One has:

\[
E_b(q^{-\rho_k}) = q^{(\rho_k, b_-)} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left( \frac{1-q^2_{\alpha} t_\alpha X_\alpha(q^{\rho_k})}{1-q^2_{\alpha} X_\alpha(q^{\rho_k})} \right),
\]

(2.36)

\[
\lambda'(\pi_b) = \{ [\alpha, j] \mid [-\alpha, \nu_\alpha j] \in \lambda(\pi_b) \}.
\]

(2.37)

Explicitly (see (2.9)),

\[
\lambda'(\pi_b) = \{ [\alpha, j] \mid \alpha \in R_+, -(b_-, \alpha^\vee) > j > 0 \text{ if } u_b^{-1}(\alpha) \in R_-,
\]

\[
-(b_-, \alpha^\vee) \geq j > 0 \text{ if } u_b^{-1}(\alpha) \in R_+ \}.
\]

(2.38)

Formula (2.36) is the nonsymmetric version of the Macdonald evaluation conjecture from [Ch2]. The norm-formula is as follows:

\[
\langle E_b, E_c \rangle = \delta_{bc} \prod_{[\alpha, j] \in \lambda'(\pi_b)} \left( \frac{1-q^2_{\alpha} t_\alpha^{-1} X_\alpha(q^{\rho_k})}{1-q^2_{\alpha} X_\alpha(q^{\rho_k})} \right) \left( \frac{1-q^2_{\alpha} t_\alpha X_\alpha(q^{\rho_k})}{1-q^2_{\alpha} X_\alpha(q^{\rho_k})} \right).
\]

(2.39)

For later reference, let

\[
g_b(q, t) \overset{\text{def}}{=} E_b(q^{-\rho_k}), \quad h_b(q, t) \overset{\text{def}}{=} \langle E_b, E_b \rangle \text{ for } b \in P.
\]

Assuming that \( t_\nu \to 0 \) for all \( \nu \in \nu_R \) (we set \( t \to 0 \)),

\[
\lim_{t \to 0} q^{-(\rho_k, b_-)} g_b = 1, \quad h_b^0 \overset{\text{def}}{=} \lim_{t \to 0} h_b = \prod_{[\alpha, j] \in \lambda'(\pi_b)} (1-q^2_{\alpha} t_\alpha).
\]

(2.40)

\[
i.e. \text{ the last product is over } [-\alpha, \nu_\alpha j] \in \lambda(\pi_b) \text{ with simple } \alpha. \text{ Also:}
\]

\[
\langle E_b E^*_c \mu \theta(t \to 0) \rangle = \delta_{bc} h_b^0 \text{ for } E_b = E_b(t \to 0), E^*_c = E^*_c(t \to 0).
\]

(2.42)

See formula (3.42) from [ChO].

3. **Theta-products via \( E \)-polynomials**

3.1. **Mehta-Macdonald identities.**
3.1.1. **Basic notations.** Our approach is based on the difference Mehta-Macdonald formulas for *standard theta-functions*. Given a root system $R$ as above and the corresponding $P, Q$, they generally depend on the choice of a character $v: \Pi = P/Q \to \mathbb{C}^*$. The group of such characters will be denoted by $\Pi'$; the trivial character will be $1'$. Let

$$\zeta_v(X_a) = v(a)X_aT_wY_b \text{ for } a \in P.$$  

(3.1)

For a character $v \in \Pi'$, we set

$$\theta_v(X) \overset{\text{def}}{=} \sum_{b \in P} v(b)q^{(b,b)/2}X_b = \zeta_v(\theta), \quad \text{where } \theta \overset{\text{def}}{=} \theta_{V'}.$$  

(3.2)

The characters $v$ play here the role of the classical *theta-characteristics*. Definition (3.2) is directly related to that from [ChFB], though we used a somewhat different setting there. Namely, theta-functions were introduced using the partial summations in the series for $\theta$, where the images of $b$ were taken from some subsets $\varpi \subset \Pi$. Using $v$ instead of $\varpi$, we obtain two different basis in the same space (of theta-functions). The usage of $\varpi$ has some advantages for the $\text{PSL}(2, \mathbb{Z})$--modularity and when the *string functions* are considered [KP]. One of the most important observations in [ChFB] was that the Rogers-Ramanujan formulas give almost immediate justification of the *level-rank duality* for related string functions. See also Lemma 3.1 from [ChD] concerning the “topological DAHA-vertex”.

We will use Theorem 5.1 from [Ch3] (the first formula) and Theorem 3.4.5 from [Ch4]. Enhancing them by the characters $v \in \Pi'$ follows Theorem 3.2 from [ChD], where (3.5) below was proven; the justification of (3.6) is quite similar.

We will need non-symmetric spherical polynomials $E_b \overset{\text{def}}{=} E_b/E_b(q^{-\rho_k})$ for $b \in P$. See [Ch4], and formula (6.30) from [Ch5]. Using these polynomials and the definition of $b_{\sharp}$ from (2.34), the *duality theorem* states that

$$E_b(q^{-2}) = E_c(q^{b_{\sharp}}), \text{ where } b_{\sharp} = b - u_b^{-1}(\rho_k), \ b, c \in P.$$  

(3.3)

We will also use that $\mu_{\circ} = \mu_0$, where $\mu_{\circ}$ is understood as a Laurent series $1 + \ldots$ in terms of $X_b$ with *rational* $q, t$--coefficients. This readily give the relations:

$$\langle E_b, E_c \rangle = \langle E_{b_{\sharp}}, E_{c_{\sharp}} \rangle = \delta_{bc}\langle E_b, E_b \rangle = \delta_{bc}(E_{b_{\sharp}}, E_{b_{\sharp}})^* \text{ for } b, c \in P.$$  

(3.4)

One can see the $*$--invariance of $\langle E_b, E_{b_{\sharp}} \rangle$ directly from (2.39).
3.1.2. The key expansions. Recall that $X_b^* = X_{\bar{b}}^{-1} = X_{-\bar{b}}$, for $b \in P$, $q^* = q^{-1}$, $t^* = t^{-1}$ and $u_b(b) = b_{-} \in P_{-}$. The following formulas are the key for us.

**Theorem 3.1.** For $b, c \in P$,

\begin{align}
(3.5) \quad v(b + c)\langle E_b E_c \theta_{c} \mu_o \rangle &= q^{b^2/2 + c^2/2 - (b_{-} + c_{-} \rho_k)} E_c(q^{b_{-}}) \langle \theta \mu_o \rangle, \\
(3.6) \quad v(b - c)\langle E_b E_c^* \theta_{c} \mu_o \rangle &= q^{b^2/2 + c^2/2 - (b_{-} + c_{-} \rho_k)} E_c^*(q^{b_{-}}) \langle \theta \mu_o \rangle.
\end{align}

Here the coefficients of the Laurent series for $\mu_o$ are naturally expanded in terms of positive powers of $q$ and the proportionality factor is a $q$-generalization of the Mehta-Macdonald integral:

\begin{align}
(3.7) \quad \langle \theta \mu_o \rangle &= \prod_{\alpha \in R_+} \prod_{j=1}^{\infty} \left( \frac{1 - t^{-1}q^{(\rho_k, \alpha') + j}_a}{1 - q^{(\rho_k, \alpha') + j}_a} \right).
\end{align}

For later reference, let us make $t \to 0$ in (3.7):

\begin{align}
(3.8) \quad \langle \theta \mu_o \rangle = \langle \mu_o \rangle^{-1} = \prod_{i=1}^{n} \prod_{j=1}^{\infty} (1 - q^j_i) \text{ when } t \to 0.
\end{align}

Switching here (and below) to

\begin{align}
(3.9) \quad \hat{\theta}_v \overset{\text{def}}{=} \theta_v / \langle \theta \mu_o \rangle,
\end{align}

the formulas above can be interpreted as the following expansions:

\begin{align}
(3.10) \quad E_c \hat{\theta}_v &= \sum_{b \in P} \frac{\langle E_b E_c \hat{\theta}_v \mu_o \rangle}{\langle E_b, E_b \rangle} E_b^* = \sum_{b \in P} \frac{q^{b^2/2 + c^2/2 - (b_{-} + c_{-} \rho_k)}}{v(b + c)\langle E_b, E_b \rangle} E_b(q^{c_{-}}) E_b^*, \\
(3.11) \quad E_c^* \hat{\theta}_v &= \sum_{b \in P} \frac{\langle E_b E_c^* \hat{\theta}_v \mu_o \rangle}{\langle E_b, E_b \rangle} E_b^* = \sum_{b \in P} \frac{q^{b^2/2 + c^2/2 - (b_{-} + c_{-} \rho_k)}}{v(b - c)\langle E_b, E_b \rangle} E_b^*(q^{c_{-}}) E_b^*, \\
(3.12) \quad E_c \hat{\theta}_v &= \sum_{b \in P} \frac{\langle E_b^* E_c \hat{\theta}_v \mu_o \rangle}{\langle E_b, E_b \rangle} E_b = \sum_{b \in P} \frac{q^{b^2/2 + c^2/2 - (b_{-} + c_{-} \rho_k)}}{v(c - b)\langle E_b, E_b \rangle} E_b^*(q^{c_{-}}) E_b.
\end{align}

Here (3.11) and (3.12) follow from (3.6), where we use the duality. Formula (3.12) in the symmetric variant was the starting point for paper [ChFB].

Proposition 3.6 and Theorem 3.7 from [ChD] contain a formal theory of iterations of these relation. They were stated there for symmetric
Macdonald polynomials. Change the summations from $P_+$ to $P$, replace $-c - \rho_k$ for $c \in P_+$ by $e_\tau$, and use $\frac{b^2}{2} + \frac{c^2}{2} - (b_- + c_- + \rho_k)$ instead of $\frac{b^2}{2} + \frac{c^2}{2} + (b + c, \rho_k)$ to transfer the iteration formulas in [ChD] to the $E$–polynomials.

The formulas for iterations with generic $t$ are quite involved because there are no explicit formulas for $E^*_b(q^\pm)$. However these quantities become some relatively simple $q$–monomials in the limit $t \to 0$ and also when $t = 1$. Let us discuss a little the case $t = 1$, called “free theory”. One has: $\mu_0 = 1, \langle \mu \rangle = 1, \mathcal{E}_b = E_b = X_b$ for any $b \in P$, and $\mathcal{E}_b(q^\pm) = q^{(b,c)}$. The expansion above reads $X_c \theta_v = \sum_{b \in P} q^{n_c(b)} X_b \mod \Sigma_+(c)$, where $n_c(b) \in \mathbb{Z}_-$, $c \in P$.

3.2. Using $E$-dag polynomials. The two limiting cases $t \to 0$ and $t \to \infty$ of the DAHA theory are of significant importance:

(3.13) $\overline{E}_b(X; q) = E_b(X; q, t \to 0), \overline{E}_b^\dagger(X; q) = E_b(X; q, t \to \infty), b \in P$.

These polynomials are well defined; see [ChO]. We will simply call them $E$-dag and $E$-bar polynomials. The former are nonsymmetric generalizations of the $q$–Hermite polynomials, which coincide with the level-one Demazure characters in the twisted setting [San, Ion]. The latter are more recent; they were studied in [ChO]. The coefficients of $E^\dagger_b$, which are from $\mathbb{Z}[q^{-1}]$, were conjectured there to be in $\mathbb{Z}_+[q^{-1}]$. Furthermore, it was conjectured that

(3.14) $E^\dagger_c = \sum_{b \in W(c)} q^{n_c(b)} X_b \mod \Sigma_+(c)$, where $n_c(b) \in \mathbb{Z}_-$, $c \in P$,

where we take only $b$ in the summation such that $X_b$ is present in $E^\dagger_c$.

The first conjecture was (generalized and) verified in [OS]. The presentation (3.14) was checked in [ChFE] for $c = c_-$ and verified in [NNS] in the simply-laced case. It was mentioned in [ChFE] that it formally follows from the case $c = c_-$; we will provide the proof below.

3.2.1. The numbers $n_c(b)$. By constriction, $n_c(c) = 0$ and $q^{n_c(c_-)} = 0$ unless $c = c_-$. Let us use formula (3.52) from [ChO].

We assume in the next calculation that $(c, \alpha_i) = (c_-, u_c(\alpha_i)) < 0$ for $1 \leq i \leq n$. Then $u_c(\alpha_i) \in R_+$. Also, $\pi_c = s_i \pi_{s_i(c)}$ is reduced
\( (l(\pi_c) = l(s_i) + l(\pi_{s_i(c)})) \) and, equivalently, the product \( u_{s_i(c)} = u_c s_i \) is reduced. One has:

\[
\left( \begin{array}{ll}
E_{s_i(c)}^+ &= \begin{cases}
(1 - q^{(c, \alpha_i)})^{-1}(T_i^\dagger)(E_c^\dagger) & \text{if } u_c(\alpha_i) \text{ is simple}, \\
(T_i^\dagger)(E_c^\dagger) & \text{otherwise},
\end{cases} \\
(T_i^\dagger)' &= T_i^\dagger - 1 = \frac{X_{\alpha_i}}{X_{\alpha_i} - 1} (s_i - 1), \quad T_i^\dagger \overset{\text{def}}{=} t_i^{-1/2}T_i(t \to \infty).
\end{array} \right.
\]

Recall that \( q^{(b, \alpha_i)} = q_i^{(b, \alpha_i)}, q_i = q^{(\alpha_i, \alpha_i)/2} \). For any \( b \in P \),

\[
(T_i^\dagger)' (X_b) \mod \Sigma_+(b) = \begin{cases}
X_{s_i(b)}, & \text{if } (b, \alpha_i) < 0 \\
-X_b, & \text{if } (b, \alpha_i) > 0 \\
0, & \text{if } (b, \alpha_i) = 0.
\end{cases}
\]

Therefore:

\[
(T_i^\dagger)' (E_c^\dagger) \mod \Sigma_+(b) = \sum_{b \in W(c)} (q^{n_c(s_i(b))} - q^{n_c(b)}) X_b,
\]

\[
q^{n_{s_i(c)}(b)} = \begin{cases}
0, & \text{unless } (b, \alpha_i) > 0, \\
\frac{q^{n_c(s_i(b))} - q^{n_c(b)}}{1 - q^{(c, \alpha_i)}}, & \text{if } u_c(\alpha_i) \text{ is simple}, \\
q^{n_c(s_i(b))} - q^{n_c(b)}, & \text{otherwise},
\end{cases}
\]

where we impose \( (b, \alpha_i) > 0 \) in the latter two equalities; \( q^{n_c(b)} \) may be 0 for such \( b \). The role of simplicity \( u_c(\alpha_i) \) is as follows:

\[
(3.20) \quad \text{provided } (b, \alpha_i) > 0, \ u_c(\alpha_i) \text{ is simple } \iff h_{s_i(c)}^0 \neq h_{s_i(c)}^0.
\]

Employing the monomiality claim:

\[
q^{n_{s_i(c)}(b)} = \begin{cases}
0, & \text{if } (b, \alpha_i) \leq 0 \text{ or } q^{n_c(s_i(b))} = q^{n_c(b)}, \\
q^{n_c(s_i(b))}, & \text{if } (b, \alpha_i) > 0 \text{ and } q^{n_c(s_i(b))} \neq q^{n_c(b)}.
\end{cases}
\]

Moreover, in the latter case, \( q^{n_c(b)} = q^{n_c(s_i(b)) + (c, \alpha_i)} \) for simple \( u_c(\alpha_i) \) and \( q^{n_c(b)} = 0 \) if \( u_c(\alpha_i) \) is not simple; this root is positive due to \( (c, \alpha_i) < 0 \).

Using relations (3.21), we can obtain all \( q^{n_c(b)} \) for any \( c \in W(c_-) \ni b \) assuming that they are known for \( c = c_- \); see [ChFE] for the latter case. Indeed, any element from the orbit \( W(c) \) can be obtained from \( c = c_- \) by consecutively applying proper \( s_i \) under the negativity condition above: \( c' = s_i(c), c'' = s_{\nu'} c' \) for \( (c', \alpha_{\nu'}) < 0 \), and so on.
3.2.2. An example for \( A_2 \). It makes some sense to provide a simple example (the simplest beyond \( A_1 \)). For \( A_2 \), let \( c_\ominus = -\rho = -\omega_1 - \omega_2 \), \( s_i = s_1 \), \( s_\varphi = s_2 \). Thus \( c = -\omega_1 - \omega_2 \), \( c' = s_1(c) = \omega_1 - 2\omega_2 \), \( c'' = s_2(c') = 2\omega_2 - \omega_1 \). Recall that \( X_i = X_{\omega_i} \). The first polynomial we provide below is \( E_\varphi^c \). It is followed by \( \simeq \) and the corresponding \( \sum_{b \in W(c_\ominus)} q^{n_s_i(c)(b)} X_b \) calculated via formula (3.21) for \( c, s_i \). Note that it does coincide with the contribution of \( X_b \) for \( b \in W(c') \) in \( E_\varphi^c \), which is provided next. Similarly, \( \sum_b q^{n_{s_i(c')(c')}}X_b \) from (3.21) for \( c', s_\varphi \) is given after \( \simeq \), which coincides with the corresponding portion of \( E_\varphi^c \). The latter is the last polynomial we provide.

\[
\frac{1}{X_1 X_2} + \frac{X_2}{q X_1} + \frac{1}{q} + \frac{2}{q} \frac{X_1}{q X_2} + \frac{X_1}{q^2 X_2} + \frac{X_1 X_2}{q} + \frac{X_2^2}{q^2 X_1} \simeq X_1 + \frac{X_1^2}{X_2} + \frac{X_2}{X_2} + \frac{1}{X_1} \frac{X_1^2}{X_2} + \frac{X_1 X_2}{q} + \frac{X_2}{X_1} + \frac{X_1 X_2}{q} + 1.
\]

3.2.3. Proof of monomiality. The monomiality claim (3.14) was convenient to use when checking (3.21). This is not necessary; the monomiality can be actually deduced from the same argument and the case \( c = c_\ominus, c' = s_i(c) \).

Proposition 3.2. Let \( \zeta_c(b) \) be the coefficient of \( X_b \) for \( b \in W(c) \) in \( E_\varphi^c \); we set \( \zeta_{c_\ominus}(b) = \zeta_{c_\ominus}(b) \). Then given \( c_\ominus \), \( \zeta_{c_\ominus}(b) \) is either 0 or coincides with \( \zeta_{c_\ominus}(u_\varphi(b)) \). More exactly, if \( c' = s_i(c) \) and \( (\alpha_i, c) < 0 \), then relations (3.21) hold and one can proceed by induction. In particular, we arrive at the monomiality statement from (3.14).

Proof. We argue by induction with respect to \( l = l(u_\varphi) \) following Proposition 7.4 from [ChFE] and the next mini-section there “On embeddings of dag-polynomials”. They provide the expansion (3.21) for \( c = c_\ominus \) and the monomiality for \( c' = s_i(c_\ominus) \), i.e. the induction step \( l = 1 \). We will also use Corollary 3.6 from [ChO], which establishes that \( \zeta_c(b) \) belong to \( \mathbb{Z}[q^{-1}] \). To simplify the reasoning we use the positivity of \( \zeta_c(b) \) from [OS] (which can be actually avoided here).

For any \( c \), let \( c' = s_i(c) \). In the case of simple \( u_\varphi(\alpha_i) \), we have \( u_\varphi(\alpha_i) = \alpha_j \) for \( 1 \leq j \leq n \). Then either both terms in \( \zeta_c(s_i(b)) - \zeta_c(b) \) vanish or neither of them. Indeed, if only one is zero, then it would
contradict to \( \zeta_c'(b) \in \mathbb{Z}[q^{-1}] \) for \( c' = s_i(c) \). If both vanish then \( \zeta_c'(b) = 0 \). If both are nonzero, the we obtain by the induction claim for \( c \):

\[
\zeta_c'(b) = \frac{\zeta_c(s_i(b)) - \zeta_c(b)}{1 - q^{(c, \alpha_i)}} = \frac{\zeta_-(u_es_i(b)) - \zeta_-(u_e(b))}{1 - q^{(c, \alpha_i)}}
\]

\[
= \frac{\zeta_-(s_ju_e(b)) - \zeta_-(u_e(b))}{1 - q^{(c, \alpha_j)}},
\]

where we use \( u_es_i = s_ju_e \).

Here the product \( u_es_i \) is reduced by construction and \( (u_e(b), \alpha_j) = (b, \alpha_i) > 0 \); so we can use step \( l = 1 \) (for \( c' = s_j(c_-) \)).

If \( u_e(\alpha_i) \) is not simple, then we need to check that \( \zeta_-(u_e(b)) \) from \( \zeta_c'(b) = \zeta_c(s_i(b)) - \zeta_c(b) = \zeta_-(u_es_i(b)) - \zeta_-(u_e(b)) \) is 0 unless these two terms coincide. Note that there is no denominator now. This follows from some (minor) development of the method from [ChFE] for \( c = c_- \). Alternatively, we can simply use for this step that \( \zeta_c'(b) \in \mathbb{Z}_+[q^{-1}] \) from [OS]; the presence of nonzero \( -\zeta_c(b) = -\zeta_-(u_e(b)) \) readily contradicts the positivity proven there. This gives the required. \( \square \)

### 3.3. Three major expansions.

#### 3.3.1. Non-spherical formulas

In the theory of \( E^\dagger \)-polynomials it is convenient to use the following (obvious) identity:

\[
E^\dagger_b(X; q) = E^*_b(X^{-1}; q^{-1}, t \to 0), \ b \in P; \ \text{see} \ (2.30).
\]

Let us consider the limit \( t \to 0 \) in formulas (3.10,3.11,3.12). First, we need to restate them in terms of \( E \)-polynomials using \( g_b(q,t) \) from (2.40) and employing the duality in the middle formula:

\[
\frac{E_c \tilde{\theta}_v}{g_c} = \sum_{b \in P} \frac{q^{v^2 + c^2} - (b, +c, -\rho_k)}{v(b+c)\langle E_b, E_b \rangle} g_b g_b^* E_b(q^{v^2}) E_b^*,
\]

\[
\frac{E_c^* \tilde{\theta}_v}{g_c^*} = \sum_{b \in P} \frac{q^{v^2 + c^2} - (b, +c, -\rho_k)}{v(b-c)\langle E_b, E_b \rangle} g_b g_b^* E_b^*(q^{v^2}) E_b^*,
\]

\[
\frac{E_c \tilde{\theta}_v}{g_c} = \sum_{b \in P} \frac{q^{v^2 + c^2} - (b, +c, -\rho_k)}{v(c-b)\langle E_b, E_b \rangle} g_b g_b^* E_b^*(q^{v^2}) E_b.
\]
3.3.2. The limit at zero. Now let \( t \to 0 \) (i.e. \( t_\nu \to 0 \) for all \( \nu \)). We will use (2.41) and \( h_b^0 = \lim_{t \to 0} h_b \). The last formula is somewhat simpler to analyze. Note the cancelation of \( g_b \)-factors; also, \( 1/g_c \) on the l.h.s. cancels \( q^{-c - \rho_k} \) on the r.h.s. One obtains:

\[
E_c \hat{\theta}_v = \sum_{b \in P} \frac{q^{\frac{b^2}{2} + \frac{c^2}{2}}}{v(c-b)h_b^0} \left( \lim_{t \to 0} q^{-(b - \rho_k)} E_b^* (q^{\nu t}) \right) E_b.
\]

Using that \( c_t = u_c^{-1}(c_\nu - \rho_k) \), the monomials \( X_{-a} \) from \( E_b^* \) that do not vanish in the limit \( t \to 0 \) upon the evaluation at \( q^{\nu t} \) are for \( a \in W(b) \) and must satisfy

\[
(b_\nu, \rho_k) = (a, u_c^{-1}(\rho_k)) = (u_c(a), \rho_k)
\]

for generic \( k \) (before the limit). This gives the relation \( a = u_c^{-1}(b_\nu) \).

Let us comment on our usage of \( q \) and \( k \) here. We assume that \( 0 < q_\alpha < 1 \) and that \( k_\nu \to +\infty \) (they are generic real numbers); then \( t_\nu \to 0 \), which is exactly what we need here. Thus \( X_{-a} \) from \( E_b^* \) contributes to the limit (3.26) only if

\[
\lim_{k_\nu \to \infty} q^{-(b - \rho_k)} X_{-a}(q^{\nu t}) \neq 0,
\]

and we arrive at the relation above.

Now let us use the expansion

\[
E_b^*(X; q, t \to 0) = E^t(X^{-1}; q^{-1}) = \sum_{a \in W(b)} X_{-a} q^{-n_b(a)} \mod \Sigma_+(b),
\]

where \( n_b(a) \) are from (3.14). One has:

\[
\lim_{t \to 0} q^{-(b - \rho_k)} E_b^* (q^{\nu t}) = q^{-n_b(a)} X_{-a}(q^{u_c^{-1}(c_-)})
\]

\[
= q^{-n_b(a)} q^{-(a,u_c^{-1}(c_-))} = q^{-n_b(a) - (b_\nu, c_-)},
\]

\[
\frac{q^{\frac{b^2}{2} + \frac{c^2}{2}}}{v(c-b)h_b^0} \left( \lim_{t \to 0} q^{-(b - \rho_k)} E_b^* (q^{\nu t}) \right) = \frac{q^{\frac{(c_- - b_\nu)^2}{2} - n_b(a)}}{v(c_- - b_\nu)h_b^0}.
\]

As above, we take only \( a \) such that \( X_a \) is present in \( E_b^* \); we technically set \( q^{-n_b(a)} = 0 \) otherwise. Also we can obviously replace here \( v(c) \) by \( v(c_-) \). Thus (3.26) becomes in the limit:

\[
E_c \hat{\theta}_v = \sum_{b \in P} \frac{q^{\frac{(c_- - b_\nu)^2}{2} - n_b(u_c^{-1}(b_-))}}{v(c_- - b_\nu)h_b^0} E_b.
\]
Assuming that $c = c_-$ we come to the setting of [ChFB]. Indeed, $u_c = \text{id}$ and $q^{-n_b(u_c^{-1}(b_-))} = 0$ unless $b = b_-$. For $b = b_\in P_-$, the polynomials $E_{b_-}$ coincide with the corresponding symmetric bar-polynomials, and therefore the resulting decomposition is exactly as it was in [ChFB]. Interestingly, this decomposition is quite non-trivial when $c \not\in P_-; \text{the numbers } n_b(c) \text{ are involved.}

3.3.3. The remaining cases. Actually, the calculation is very similar in the other two cases. Let us proceed with (3.24). After the cancelations of $g$ as above, now $q^{-\rho_k(c_-)}$ replaces $q^{-\rho_k(b_-)}$ on the r.h. To proceed, let us set $E^*_c \overset{\text{def}}{=} E^*_c(X^{-1}; q^{-1})$. Then

$$
E^*_c \overset{\hat{\theta}_v}{=} \sum_{b \in P} \frac{q^{\frac{b_+^2}{2}}}{v(b-c)h_b} \left( \lim_{t \to 0} q^{-(c_-) \rho_k} E^*_c(q^{(c_-)} b) \right) E^*_b.
$$

(3.30)

The roles of $b$ and $c$ within $\lim(\cdots)$ are now exactly the opposite to those above. Therefore the monomial $X^a$ from $E^*_c$ contributes to the limit if and only if $a = u_b^{-1}(c_-)$ and the final formula becomes:

$$
E^*_c \overset{\hat{\theta}_v}{=} \sum_{b \in P} \frac{q^{\frac{(b_- - c_-)^2}{2}}}{v(b-c)h_b} E^*_b.
$$

(3.31)

The analysis of the remaining (first) formula is also quite close to that for (3.12). The cancelation of the $g$–factors is exactly the same. The only change is that the polynomials $E_b$ replace $E^*_b$ there. Proposition 3.1 of [ChO] states that

$$
E_b(X; q, t \to 0) = \sum_{a \in W(b)} \varsigma_b(a) X_a \mod \Sigma_+(b),
$$

(3.32)

where $\varsigma_b(a) = 1$ if $u_a \geq u_b$ in the sense of the Bruhat order and 0 otherwise. This replaces more involved (3.27). For the (unique) $X_a$ in $E_b$ (instead of $X_a$ in $E^*_b$) contributing to the limit one now has:

$$(b_-, \rho_k) = -(a, u_c^{-1}(\rho_k)) = -(u_c(a), \rho_k).$$

Therefore, $a = u_c^{-1} w_0(b_-),$ where we use that $-w_0(\rho_k) = \rho_k$, and

$$
\lim_{t \to 0} q^{-(b_- - \rho_k)} E_b(q^{\rho_k}) = X_a(q^{u_c^{-1}(c_-)}) = q^{(a, u_c^{-1}(c_-))} = q^{-(b_- c_-)}
$$

(3.33)
for \( c' \overset{\text{def}}{=} -w_0(c) \). Note that \( u_{c'} = w_0u_cw_0 \) and \( (c_-)^t = (c^t)_- \). Finally:

\[
E_c^t \widehat{\theta}_v = \sum_{b \in P} \zeta_b(w_0^{-1}(b_-)) \frac{q^{(b_--c_-)^2}}{v(b_-c_-)h_b^0} E_b^{1*},
\]

where we changed \( c \) to \( c^t \) in the second formula.

As an example, let us consider the case \( c \in P_- \). Then all \( b \in P \) will occur in this summation because \( u_c = \text{id} \).

3.4. Iteration formulas. Let us summarize the analysis above in the following theorem.

**Theorem 3.3.** Let us fix \( v \in \Pi' = \text{Hom}(\Pi, \mathbb{C}^*) \). We use the function \( n_c(b) \) from (3.14); recall that we set \( q^{-\alpha_0(a)} = 0 \) if \( X_a \) is not present in \( E_b^\dagger \). Also, \( \zeta_0(a) = 1 \) if \( u_a \geq u_b \) in the sense of the Bruhat order in \( W \) and 0 otherwise, \( c' = -w_0(c) \) and \( h_b^0 = \prod_{i \in \alpha \cap \check{\beta}} (1-q_i^2) \) for \( b, c \in P \); see (2.41) and (2.38). One has:

\[
E_c^t \widehat{\theta}_v = \sum_{b \in P} \zeta_b(w_0u_c^{-1}(b_-)) \frac{q^{(b_--c_-)^2}}{v(b_-c_-)h_b^0} E_b^{1*},
\]

\[
E_c^{1*} \widehat{\theta}_v = \sum_{b \in P} \frac{q^{(b_--c_-)^2} \cdot n_c(u_b^{-1}(c_-))}{v(b_-c_-)h_b^0} E_b^{1*},
\]

\[
E_c \widehat{\theta}_v = \sum_{b \in P} \frac{q^{(c_--b_-)^2} \cdot n_b(u_c^{-1}(b_-))}{v(c_-b_-)h_b^0} E_b.
\]

In particular for \( c = 0 \) (notice \( b \in P_- \) in the second sum below):

\[
\widehat{\theta}_v = \sum_{b \in P} \frac{q^{b_2}}{v(b_-)h_b^0} E_b^{1*} = \sum_{b \in P_+} \frac{q^{b_2}}{v(-b_-)h_b^0} E_b.
\]

3.4.1. Some remarks. \( (i) \) Formula (3.37) for \( v = 1' \) is actually not new. It can be deduced from formula (4.43) from [ChO]. The notation \( \overline{\gamma} \) there becomes \( \theta \) in this paper, and \( a_{b,c} \) used in (3.43) is the coefficient of \( X_{-u_{c'}^{-1}(b_-)} \) in \( E_b^v(t \to 0) = E_b^{1*} \); see Proposition 4.3 from [ChO].
\( (ii) \) Let us discuss some symmetries of the expansions above. First of all, (3.36) formally follows from (3.37). Indeed, the coefficient of \( E^*_b/h^0_b \) in this expansion equals \( \langle E^*_b \hat{\vartheta}_b E_b \hat{\varpi}_c \rangle \). The coefficient of \( \overline{E}_c/h^0_c \) in the r.h.s. of (3.37) is given by the same expression: \( \langle \overline{E}_b \hat{\vartheta}_b E^*_c \hat{\varpi}_c \rangle \).

Similarly, the coefficient of \( E^*_b/h^0_b \) in (3.35) is \( \langle \overline{E}_c \hat{\vartheta}_b E_b \hat{\varpi}_c \rangle \), which is symmetric under \( c^i \leftrightarrow b \). This readily gives a general relation:

\[
\varsigma_b(u_c^{-1} w_0(b_-)) = \varsigma_c(u_b^{-1} w_0(c_-)), \quad b, c \in P.
\]

Let us outline a direct justification of the latter relation. Setting \( u_c^{-1} w_0(b_-) = a = u_a^{-1}(b_-) \), Proposition 3.4 from [ChO] states that \( X_a \) occurs in \( \overline{E}_b \) (always with the coefficient 1) if and only if \( u_b \leq u_a \) in for the Bruhat order in \( W \). Equivalently, \( u_b \leq w_0 u_c \) due to the minimality of \( u_b \) modulo the centralizer of \( b_- \) on the left. Generally, if \( w' = w_0 \) and \( l(u') < l(u) \) for \( u' = u s_a \), which corresponds to deleting one simple reflection from the reduced decomposition of \( u \), then \( u'(v)^{-1} = w_0 \) for \( v = v s_a \), \( l(v') > l(v) \), and therefore \( v' > v \). Thus \( u_b \leq w_0 u_c \Leftrightarrow w_0 u_b \geq u_c \), which gives the required.

\( (iii) \) The second expansion from (3.38), namely the expansion in terms of \( \overline{E}_b \), is a special case of (3.37). The first formula there, which is for \( \hat{\vartheta}_b \) via \( E_b^\dagger \), formally follows from the second due to the identity:

\[
(3.39) \quad \sum_{b \in W(c)} \frac{q^2}{h^{2b}_b} E_b^{\dagger} \overline{E}_b = \frac{q^2}{h^{2b}_b} \overline{E}_b \quad \text{for} \quad b_\epsilon = (\neg w_0(b))_\epsilon = -w_0(b_-),
\]

where \( q^{b^2/2} = q^{b_\epsilon^2/2} \) and therefore both \( q \)-powers can be omitted. Equivalently, \( P^\dagger_{b_-} = \sum_{b \in W(c)} E_b^{\dagger} (h^{b_\epsilon}_b/h^{b}_b)^* \) for the corresponding symmetric Macdonald polynomial \( P^\dagger_{b_-} = P_{b_-}(t \to \infty) \). As an example, let us provide the latter identity for \( A_1 \). Setting \( X = X_w \) for \( n \geq 0 \):

\[
E_{-n}^\dagger + (1 - q^{-n}) E_{n}^\dagger = P_{-n}^\dagger, \quad \text{where} \quad E_n = E_{nw_1}; \quad \text{letting} \quad n = 2:
\]

\[
(X^{-2} + X^2 + \frac{1+q}{q^2}) + (1 - \frac{1}{q^2})(X^2+1) = P_{-2}^\dagger - X^2 + X^{-2} + \frac{1+q}{q}.
\]

Formula (3.39) results from (2.57) or general formula (4.19) from Proposition 4.2 in [ChO]: \( E_{w(b)/h(w(b))}^* = t^{l(w)/2} \hat{\varpi}_w(E_b^*/h_b) \) for \( b = b_- \), in the notation there. We obtain that the l.h.s. of (3.39) is \( W \)-invariant and therefore proportional to \( P_b^* = P_{t(b)} \) in the limit \( t \to 0; \)
the coefficient of proportionality is simple. One can also make $t \to \infty$ in formula (3.3.15) from [Ch4], which expresses $P_{b_-}$ in terms of $E_{w(b_-)}$.

3.4.2. Using intertwiners. It is instructional to check that applying the intertwiners to (3.37) results in relations for $q^{\alpha_0(b)}$ obtained above. Let $\zeta_c(b) = q^{\alpha_0(b)}$ be the coefficient of $X_b$ in $E_c^b$. Then for $v = 1'$,

$$
(3.40) \quad \bar{E}_c \hat{\theta} = \sum_{b \in P} q^{(c-b-1)^2/2} \zeta_b^*(u_c^{-1}(b_-)) E_b/h_b^0.
$$

The first relation in (3.18) gives that for $(b, \alpha_i) < 0$,

$$
(3.41) \quad \{\zeta_{s_i(b)}(u_c^{-1}(b_-)) \neq 0\} \Rightarrow \{0 < (u_c^{-1}(b_-), \alpha_i) = (b_-, u_c(\alpha_i))\} \Rightarrow \{u_c(\alpha_i) \in -R_+\} \iff \{(c, \alpha_i) > 0\}.
$$

Recall that the equality $(c, \alpha_i) = 0$ here is equivalent to $u_c(\alpha_i) = \alpha_j$ for simple $\alpha_j$ such that $(\alpha_j, c_-) = 0$, which contradicts to $(b_-, u_c(\alpha_i)) > 0$. See Proposition 2.2 and Lemma 2.3.

Following Section 4.4 from [ChO],

$$
(3.42) \quad \bar{T}_i'(E_b) = \begin{cases} 
E_{s_i(b)} & \text{if } (b, \alpha_i) > 0, \\
E_b & \text{if } (b, \alpha_i) \leq 0,
\end{cases}
$$

where $\bar{T}_i = 1 + \bar{T}_i = 1 + (X_{s_i} - 1)^{-1}(s_i - 1) (1 \leq i \leq n)$. Explicitly:

$$
(3.43) \quad \bar{T}_i'(X_b) = \begin{cases} 
X_{s_i(b)} + X_b, & \text{if } (b, \alpha_i) > 0, \\
X_b, & \text{if } (b, \alpha_i) = 0, \\
0, & \text{if } (b, \alpha_i) < 0.
\end{cases}
$$

Applying $\bar{T}_i'$ to (3.40):

$$
(3.44) \quad \bar{T}_i'(\bar{E}_c \hat{\theta}) = \bar{T}_i'(\bar{E}_c) \hat{\theta}
$$

$$
= \sum_{(b, \alpha_i) > 0} q^{(c-b-1)^2/2} \zeta_b^*(u_c^{-1}(b_-)) E_{s_i(b)}/h_b^0 + \sum_{(b, \alpha_i) \leq 0} q^{(c-b-1)^2/2} \zeta_b^*(u_c^{-1}(b_-)) E_b/h_b^0.
$$

When $(c, \alpha_i) \leq 0$, we arrive at the identities:

$$
\frac{\zeta_b^*(u_c^{-1}(b_-))}{h_b^0} + \frac{\zeta_{s_i(b)}^*(u_c^{-1}(b_-))}{h_{s_i(b)}^0} = \frac{\zeta_{s_i(b)}^*(u_c^{-1}(b_-))}{h_{s_i(b)}^0} \text{ if } (b, \alpha_i) > 0.
$$

Since $(s_i(b), \alpha_i) < 0$ and $(c, \alpha_i) \leq 0$, the relation $\zeta_b(u_c^{-1}(b_-)) = 0$ follows from (3.41).
Now let \((c, \alpha_i) > 0\). Then \(u_c = u_{s_i(c)}s_i\) (the product is reduced) and we obtain:

\[
\frac{\zeta^*_b(u_c^{-1}(b_-))}{h^0_b} + \frac{\zeta^*_{s_i(b)}(u_c^{-1}(b_-))}{h^0_{s_i(b)}} = \frac{\zeta^*_{s_i(b)}(s_iu_c^{-1}(b_-))}{h^0_{s_i(b)}} \quad \text{if} \quad (b, \alpha_i) > 0.
\]

Collecting the terms with \(s_i(b)\) in the r.h.s., we obtain the last two relations from (3.18).

3.4.3. Main Theorem. Let us now iterate the formulas from Theorem 3.3. We set \(v = \{v_1, \ldots, v_p\} \in \Pi', \tilde{\theta}_\psi = \tilde{\theta}_{v_1} \cdots \tilde{\theta}_{v_p}\), and \(b = \{b_k \in P, 1 \leq k \leq p\}\). We will use the following system of notations:

\[
m_c(b) \overset{\text{def}}{=} -n_c(u_b^{-1}(c_-)) = -n_c(u_b^{-1}u_c(c)),
\]

\[
x_b(c) \overset{\text{def}}{=} \zeta_0(u_b^{-1}w_0(b_-)) = \zeta_0(u_b^{-1}w_0u_b(b)),
\]

\[
(\alpha_i^\vee, b)^* \overset{\text{def}}{=} -(\alpha_i^\vee, b_-) \quad \text{when} \quad u_b^{-1}(\alpha_i) \in \mathbb{R}_+,
\]

and \((\alpha_i^\vee, b)^* \overset{\text{def}}{=} -(\alpha_i^\vee, b_-) - 1 \quad \text{otherwise}.
\]

Note that \((\alpha_i^\vee, b)^* \geq 0\); it is needed in the norms in the denominators, where we use (2.41) and (2.42):

\[
\langle E_bE_c^{\dagger}E_{\Psi} \rangle = \delta_{bc}h^0_b = \delta_{bc} \prod_{j=1}^{(\alpha_i^\vee, b)^*} (1 - q_i^j);
\]

see (2.9). We will also set \(b^- = b_-\) to improve the visibility of the formulas below.

Recall that \(q^{-m_c(b)}\) is the coefficient of \(X_{ub^{-1}(c_-)}\) in \(E_c^{\dagger}\). The monomiality claim is that \(n_c(b) \in \mathbb{Z}_+\) (unless this coefficient is 0). Switching to \(m_c(b) = -n_c(b)\) is quite natural here. We set \(m_c(b) = +\infty\) and \(q^{+\infty} = 0\) if such \(X\) is not present in \(E_c^{\dagger}\). Since \(m_c(b)\) is a non-negative integer otherwise, the sum of \(m\)-terms is \(\infty\) if and only if at least one of the terms is \(\infty\) (+\(\infty\) to be exact).
Theorem 3.4. For an arbitrary sequence \( \mathbf{v} \) and \( c \in P \),

\[
E_{c}^\ast \hat{\theta}_{\mathbf{v}} = \sum_{\mathbf{b}} q^{((c_{1} - b_{1})^{2} + (c_{2} - b_{2})^{2} + \cdots + (c_{p} - b_{p})^{2})/2} \prod_{i=1}^{n} \prod_{k=1}^{p-1} (\hat{\alpha}_{i}^{\prime}, b_{k}) (1 - q_{i}^{\prime}) 
\times v_{1}(b_{1} - c) v_{2}(b_{2} - b_{1}) \cdots v_{p}(b_{p} - b_{p-1}) E_{b_{p}}^\ast \xi_{b_{p+1}}^{\ast} \theta_{c_{r+1}} \hat{\theta}_{\mathbf{v}}
\]

\[
(3.46)
\]

\[
\overline{E}_{c} \hat{\theta}_{\mathbf{v}} = \sum_{\mathbf{b}} q^{((c_{1} - b_{1})^{2} + (c_{2} - b_{2})^{2} + \cdots + (c_{r} - b_{r})^{2})/2} \prod_{i=1}^{n} \prod_{k=1}^{p-1} (\hat{\alpha}_{i}^{\prime}, b_{k}) (1 - q_{i}^{\prime}) 
\times v_{1}(c - b_{1}) v_{2}(b_{1} - b_{2}) \cdots v_{r}(b_{r-1} - b_{r}) v_{r+1}(b_{r} + b_{r+1}) 
\times q_{m_{r+1}(b_{r+2}) + m_{r+2}(b_{r+3}) + \cdots + m_{p-1}(b_{p})} \prod_{i=1}^{n} \prod_{k=1}^{p-1} (\hat{\alpha}_{i}^{\prime}, b_{k}) (1 - q_{i}^{\prime}) 
\times v_{r+2}(b_{r+2} - b_{r+1}) v_{r+3}(b_{r+3} - b_{r+2}) \cdots v_{p}(b_{p} - b_{p-1}) \hat{\theta}_{\mathbf{v}}
\]

\[
(3.47)
\]

Recall that in these formulas and those below any \( b_{k}, c \) inside \( \mathbf{v}( ) \) can be replaced by \( b_{k}^{\ast}, c^{\ast} \). We obtain that the coefficient of \( E_{b_{p}} \) in the first or the second expansion depends only on \( b_{p} \), the unordered set \( \mathbf{v} \) and initial \( c \). The same holds for the following mixed expansion:

\[
\overline{E}_{c} \hat{\theta}_{\mathbf{v}} = \sum_{\mathbf{b}} q^{((c_{1} - b_{1})^{2} + (c_{2} - b_{2})^{2} + \cdots + (c_{p} - b_{p})^{2})/2 + (c_{r} b_{r+1})} \prod_{i=1}^{n} \prod_{k=1}^{p-1} (\hat{\alpha}_{i}^{\prime}, b_{k}) (1 - q_{i}^{\prime}) 
\times v_{1}(c - b_{1}) v_{2}(b_{1} - b_{2}) \cdots v_{r}(b_{r-1} - b_{r}) v_{r+1}(b_{r} + b_{r+1}) 
\times q_{m_{r+1}(b_{r+2}) + m_{r+2}(b_{r+3}) + \cdots + m_{p-1}(b_{p})} \prod_{i=1}^{n} \prod_{k=1}^{p-1} (\hat{\alpha}_{i}^{\prime}, b_{k}) (1 - q_{i}^{\prime}) 
\times v_{r+2}(b_{r+2} - b_{r+1}) v_{r+3}(b_{r+3} - b_{r+2}) \cdots v_{p}(b_{p} - b_{p-1}) \hat{\theta}_{\mathbf{v}}
\]

\[
(3.48)
\]

where \( 0 \leq r \leq p - 1 \). In this expansion, we switch from \( E \) to \( E_{c}^\ast \) at place \( r + 1 \) using (3.35).

\[
\square
\]

3.4.4. Comments. The following lemma is important to understand how far the summations above are from those over \( P_{-} \). It readily results from the implication \( \{ b \in P_{-}, m_{c}(b) \neq \infty \} \Rightarrow \{ c \in P_{-}, m_{c}(b) = 0 \} \). Here we use that \( u_{b} = \text{id} \) if and only if \( b \in P_{-} \). We will set \( c = b_{0} \) for the sake of uniformity.

Lemma 3.5. Assuming that \( b_{r} \in P_{-} \) in a nonzero term (a product) from the summation in (3.46), all previous \( b_{s} \) \((0 \leq s \leq r)\) must be then from \( P_{-} \) in this product and \( m_{b_{s-1}}(b_{s}) = 0 \) for \( 1 \leq s' \leq r \). Similarly,
if some \( b_c \) belongs to \( P_- \) in a nonzero product in \((3.47)\), then \( b_s \in P_- \) for \( s \geq r \) and \( m_{b_s}(s+1) = 0 \) for \( r \leq s \leq p - 1 \). \( \square \)

Let us focus on \((3.46)\). We set
\[
\Xi_{c,a}^{\alpha_{P_\infty}} \overset{\text{def}}{=} \langle E_c^{\dagger \ast} \hat{\theta}_{\Xi} E_a \Pi_0 \rangle,
\]
which is the coefficient of \( E_c^{\dagger \ast} / h_0^0 \) in \((3.46)\).

Note that a somewhat different definition of theta-functions was used in \([ChFB]\). Namely, we set for a collection \( \varpi = \{ \varpi_k \subset \Pi, 1 \leq k \leq p \} \):
\[
\theta_{\varpi}(X) \overset{\text{def}}{=} \sum_{b \in \varpi + Q} q^{(b,b)/2} X_b, \quad \hat{\theta}_{\varpi} \overset{\text{def}}{=} \prod_{k=1}^{p} \hat{\theta}_{\varpi_k} \left( \langle \theta_{\varpi_k} \rangle^p \right).
\]

Then the following modification of \((3.46)\) is necessary:
\[
E_c^{\dagger \ast} \hat{\theta}_{\varpi} = \sum_{b} q^{(c-b_1^2+(b_1-b_2)^2+\ldots+(b_{p-1}-b_p)^2)/2} \prod_{i=1}^{n} \prod_{k=1}^{p-1} \prod_{j=1}^{r} (1 - q_i^j)^{-m_c(b_1)+m_1(b_2)+\ldots+m_{p-1}(b_p)} E_c^{\dagger \ast} / h_{b_p}^0,
\]

where \( c-b_1 \in \varpi_1 + Q \), \( b_1-b_2 \in \varpi_2 + Q \), \ldots, \( b_{p-1}-b_p \in \varpi_p + Q \).

Accordingly, we must switch from \( \Xi_{c,a}^{\alpha_{P_\infty}} \) in \((3.49)\) to
\[
\Xi_{c,a}^{\alpha_{P_{\infty}}} \overset{\text{def}}{=} \langle E_c^{\dagger \ast} \hat{\theta}_{\varpi} E_a \Pi_0 \rangle.
\]

Let us now discuss Theorem 2.3 from \([ChFB]\), which addresses the \( PSL(2,\mathbb{Z}) \)-modularity, adjusting it to the \( E^{\dagger} \)-expansions from the theorem. The modularity (generally) occurs for \( c=0 \) and minuscule \( a \).

**Corollary 3.6.** We assume that \( v_k \) from \( v \) are trivial in \((P \cap Q^\dagger) / Q\).

Then \( \Xi_{0,a}^{\alpha_{P_\infty}} \overset{\text{def}}{=} \langle \hat{\theta}_{\alpha_{P_\infty}} E_a \Pi_0 \rangle \) is a modular function of weight 0 for \( a = 0 \) or for minuscule \(-a \in P_+ \) with respect to some congruence subgroup of \( PSL(2,\mathbb{Z}) \). Due to Lemma 3.5, only \( b_k \in P_- \) contribute to these summations. Namely, setting \( b' = \{ b_1, \ldots, b_{p-1} \} \in P_{p-1}^\ast ;
\]
\[
\Xi_{0,a}^{\alpha_{P_\infty}} = \sum_{b'} q^{(b_1^2+(b_1-b_2)^2+\ldots+(b_{p-2}-b_{p-1})^2+(b_{p-1}-a)^2)/2} \prod_{i=1}^{n} \prod_{k=1}^{p-1} \prod_{j=1}^{r} (1 - q_i^j)^{-m_c(b_1)+m_1(b_2)+\ldots+m_{p-1}(b_p)} \times \prod_{j=1}^{p-1} \prod_{i=1}^{n} \prod_{k=1}^{p-1} \prod_{j=1}^{r} (1 - q_i^j)^{-m_c(b_1)+m_1(b_2)+\ldots+m_{p-1}(b_p)} v_1(b_1) v_2(b_2-b_1) \ldots v_{p-1}(b_{p-1}-b_{p-2}) v_p(a-b_{p-1}) \cdot
\]
Using $\hat{\theta}_c$, the corresponding $\Xi_{p,a}^{c,a}$ for $c = 0$ reduce to modular Rogers-Ramanujan type sums from Theorem 2.3 in [ChFB] upon the restrictions $\varpi_k = \varpi_k + Q'/Q$ ($1 \leq k \leq p$).

Proof. We use relation (3.45) and then follow [ChFB], mainly Section 2.3 there. See also [An1, An2, VZ, War, Za] and more recent [CGZ, GOW].

3.4.5. The case of $A_1$. We set $X = X_{\omega_1}, t = t_{sht}$, and denote $n\omega_1$ for $n \in \mathbb{Z}$ simply by $n$; thus $s(n) = -n$ in this notation for $s = s_1$. One has: $q^{(n\omega_1)^2/2} = q^{n^2/4}$ and $h_n^0 = \prod_{j=1}^{[n]/2}(1-q^j)$, where $|0|' = 0$, $| - n'| = |n| = |n + 1'|$ if $n \geq 0$. Then $u_{n+1} = s$, $u_{-n} = \text{id}$ and the coefficient of $X^n$ in $E_n^n$ is $q^{-n}$ for $n \geq 0$. Therefore for $b, c \in \mathbb{Z}$,

$$ m_c(b) = -n_c(b^{-1}(-|c|)) = \begin{cases} 0 & \text{for } c \leq 0, b \leq 0 \text{ or } c > 0, b > 0, \\ |c| & \text{for } c \leq 0, b > 0, \\ \infty & \text{for } c > 0, b \leq 0. \end{cases} $$

Recall that we set $q^\infty = 0$. Finally, the characters $v$ for $A_1$ are $(\pm 1)^n$.

Let $n' = \{n_1, \ldots, n_{p-1}\}$, $n_0 = c$, $n_p = a$, and $\tilde{n}' = \{c, n', a\}$. Then given any $v = \{v_1, \ldots, v_p\}$ and $a, c \in \mathbb{Z}$, we arrive at:

$$ \Xi_{p,v}^{c,a} = \sum_{n'} q^{((|c|+|n_1|)^2+|n_1|+|n_2|^2+\ldots+|n_{p-2}|+|n_{p-1}|)^2+(|n_{p-1}|+|a|)^2/4} \prod_{k=1}^{p-1} \prod_{j=1}^{[n]/2}(1-q^j) \prod_{k=1}^{n'/2}(1-q^j) q^{m_c(n_1)+m_a(n_2)+\ldots+m_a(n_p-1)+m_{a-1}(a)} \times \frac{v_1(n_1-c) v_2(n_2-n_1) \cdots v_{p-1}(n_{p-1}-n_{p-2}) v_p(a-n_{p-1})}{v_1(n_1-c) v_2(n_2-n_1) \cdots v_{p-1}(n_{p-1}-n_{p-2}) v_p(a-n_{p-1})}. $$

Note that for $a \leq 0$, the terms are nonzero if and only if the set $\{n_0, \ldots, n_{p-1}\}$ is from $P_-$ and $c \leq 0$, which matches Corollary 3.6. Similarly, if $c > 0$, then $\{n_1, \ldots, n_p\}$ must be all from $1 + \mathbb{Z}_+$. Generally, nonzero terms are exactly for $\tilde{n}$ such that $\tilde{n} = n^- [r] \cup n^+[r + 1]$, where

$$ n^- [r] = \{n_0, \ldots, n_r\} \subset -\mathbb{Z}_+, \quad n^+[r + 1] = \{n_{r+1}, \ldots, n_p\} \subset 1 + \mathbb{Z}_+ $$

for $-1 \leq r \leq p$. I.e. the sequence $\tilde{n}$ must have no single transition from strictly positive $n_k$ to non-negative $n_{k+1}$. We will call the corresponding $n'$ good and set $\eta(\tilde{n}) = 0$ unless $c \leq 0$ and $a > 0$; in the latter case let
\[ \eta(\tilde{n}) = |n_r|. \] Using this analysis:

\[
(3.55) \quad \Xi_{c,a}^{c,a} = \sum_{\text{good } n'} q^{\left( (|c|-|n_1|)^2 + (|n_1|-|n_2|)^2 + \ldots + (|n_{p-2}| - |n_{p-1}|)^2 + (|n_{p-1}| - |a|)^2 \right)/4} \\
\times \prod_{k=1}^{p-1} \prod_{j=1}^{n_k} (1 - q^j) \\
\times q^{\eta(\tilde{n})} \\
\times v_1(n_1 - c) v_2(n_2 - n_1) \cdots v_{p-1}(n_{p-1} - n_{p-2}) v_{p}(a - n_{p-1}).
\]

4. Demazure slices

In this section, we create representation-theoretical tool to interpret formula (3.38) from Theorem 3.3, which is actually the key in there.

We use the results from [FKM] and [KL], generalize them and provide some new proofs. Our approach is based on the associated graded of the filtration of integrable highest weight modules by its thin (usual) and thick (upper) Demazure submodules.

Following [Kas1, AKT, Kat2], we compare the associate graded of the thick Demazure filtration in a level one integrable modules of an affine Lie algebra \( \mathfrak{g} \) with the Weyl modules of (twisted) current algebras. Due to [FL, FMS] and the expansion formula from [ChFB], this construction results in symmetric Macdonald polynomials at \( t = 0 \), which is Theorem 4.7. Concerning the second one, we need a proper version of the Demazure-Joseph functor from [Jos] together with Section 4 from [FKM]. This connects the thick associated graded with the non-symmetric Macdonald polynomials at \( t = \infty \) (Corollary 4.20); the Ext-interpretation of the pairing (3.45) between the Macdonald polynomials at \( t = 0 \) and \( t = \infty \) is used here.

List of basic modules and functors. For the convenience of readers, we provide a list of basic modules and functors to be used in Sections 4 and 5, with the links to corresponding subsections.

The elements \( u, w, \ldots \) will be now from \( \hat{W} \) (not from \( W \) as above); \([w]\) stands for the image of \( w \in \hat{W} \) in \( \Pi \). Through this part of the paper, \( M^\vee \) denotes the restricted dual of a \( \tilde{\mathfrak{g}} \)-semisimple module \( M \) with finite-dimensional weight spaces; for finite-dimensional \( M \), usual dual \( M^* \) is sufficient.

Note that \( \text{gr}^w L \) in the table below is a \( \tilde{\mathfrak{g}}^- \)-module, while \( \text{Gr}^b L \) is generally a \( \mathfrak{g}_{\leq 0} \)-module.
\[ L = L(\Lambda_{[w]}), \quad L(\Lambda_{[b]}) \quad \text{level-one integrable modules: 4.1.1} \]

\[ L^w, L^b = L^{\pi_b} \subset L(\Lambda_{[b]}) \quad \text{thick (upper) Demazure modules: 4.1.1} \]

\[ \text{gr}^w L \ (w \in \widehat{W}), \quad \text{Gr}^b L \quad \text{associated graded (pieces) of } L: 4.1.2 \]

\[ \mathbb{D}_b = \text{gr}^\pi_b L \text{ for } b \in P \quad \text{modules with } \text{gch} = q^{\frac{k^2}{2} w_0(E^+_b)} \cdot h^0_b: 4.1.2 \]

\[ \mathcal{D}_i(i \geq 0), \quad \mathcal{D}_w(w \in \widehat{W}) \quad \text{Demazure-Joseph functors: 4.2.1} \]

\[ \mathbb{W}_b(\subset \text{Gr}^{b^+} L), \quad b \in P \quad \text{generalized global Weyl modules: 4.2.2} \]

\[ D^\gamma_b(\subset L(\Lambda_{[b]})), \quad b \in P \quad \text{dual of thin Demazure modules: 4.2.3} \]

\[ W_b \ (\text{covered by } \mathbb{W}_b) \quad \text{generalized local Weyl modules: 5.1.3} \]

\[ L(b + k\Lambda_0) \text{ for } b \in P_+ \quad \text{level-}k \text{ integrable modules: 5.2.1}. \]

4.1. Demazure slices.

4.1.1. Thick Demazure modules. We identify the cosets in \( \widehat{W}/W \) with their minimal length representatives in \( \widehat{W} \). Namely, each \( w \in \widehat{W} \) can be uniquely represented in the form \( w = bu \) for \( b \in P, u \in W \). Then \( \pi_b \) is such a minimal representative of \( w \) in the notation from Proposition 2.1. Following (2.5), we set from now on:

\[ \widehat{W} \ni w = bu \mapsto [w] = \omega_r \quad \text{for } r \in O \text{ such that } b - \omega_r \in Q, \]

i.e. the images of \( w \) and minuscule \( \omega_r \) (or 0) coincide in \( \widehat{W}/\widehat{W} = \Pi \). To simplify the notations, we will use \( w \) till the end of the paper without “hat” for the elements in \( \widehat{W} \). Also, \( \alpha \) will be generally affine roots, unless in \( \tilde{\alpha} = [\alpha, j\nu_\alpha] \). Let \( \tilde{\mathfrak{g}} \) be the affine Kac-Moody algebra over \( \mathbb{C} \) with \( \tilde{\mathfrak{h}} \) as its set of real roots and the degree operator \( d \), which corresponds to \( d \) from (2.13). Its Cartan subalgebra will be denoted by \( \tilde{\mathfrak{n}} \). See [Kac], Chapter 7 and 8 here and below.

For each \( \alpha \in \tilde{\mathfrak{h}} \), the corresponding root space will be denoted by \( \tilde{\mathfrak{g}}_\alpha \). We also set \( \mathfrak{sl}(2)_\alpha \overset{\text{def}}{=} \tilde{\mathfrak{g}}_\alpha \oplus [\tilde{\mathfrak{g}}_\alpha, \tilde{\mathfrak{g}}_{-\alpha}] \oplus \tilde{\mathfrak{g}}_{-\alpha} \), that is a Lie subalgebra of \( \tilde{\mathfrak{g}} \) isomorphic to \( \mathfrak{sl}(2) \). For \( 0 \leq i \leq n \), we also set \( \mathfrak{sl}(2)_i \overset{\text{def}}{=} \mathfrak{sl}(2)_{\alpha_i} \). The triangular decomposition of \( \tilde{\mathfrak{g}} \) is: \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{n}} \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^- \), where the set of
$\tilde{h}$–weights of $\tilde{n}$ is $\tilde{R}_+$ completed by the positive imaginary roots. We also set: $\tilde{b}^+ \overset{\text{def}}{=} \tilde{h} \oplus \tilde{n}$, $\tilde{b}^- \overset{\text{def}}{=} \tilde{h} \oplus \tilde{n}^-$. 

The degree operator $d$ is normalized so that the non-positive degree part $\tilde{g}_{\leq 0}$ of $\tilde{g}$ contains $\tilde{b}^-$; let $\tilde{g}'_{\leq 0} \overset{\text{def}}{=} [\tilde{g}_{\leq 0}, \tilde{g}_{\leq 0}]$. The $d$-zero-part of $\tilde{g}'_{\leq 0}$, denoted by $\tilde{g}_0$, is the simple Lie algebra corresponding to $\tilde{R}$. One has:

$$g = n \oplus h \oplus n^-, \quad \text{where} \quad n = \bigoplus_{\alpha \in R_+} \tilde{g}_\alpha, \quad h = \tilde{h} \cap g.$$ 

We will also use $b^- \overset{\text{def}}{=} \tilde{h} \oplus n^-$ and $g_0 \overset{\text{def}}{=} g + \tilde{h}$. The latter is the $d$-zero-part of $\tilde{g}$.

The affine roots $\alpha^\vee = \alpha/\nu_\alpha$ will be considered in this and the next sections as coroots, i.e. the corresponding elements of $h$. Accordingly, $\tilde{\alpha} = [\alpha, j\nu_\alpha] \in \tilde{R}$ and $\tilde{\alpha}^\vee = \tilde{\alpha}/\nu_\alpha$ will be interpreted as

$$\tilde{\alpha} = \alpha + j\nu_\alpha \delta, \quad \tilde{\alpha}^* = \alpha^\vee + jK \in \tilde{\mathfrak{h}} \cap [\tilde{g}, \tilde{g}],$$

where $K$ is the standard central element of $\tilde{g}$, $\delta \in \tilde{\mathfrak{h}}^*$ is the (positive) primitive imaginary root.

The basic level-one fundamental weight $\Lambda_0 \in \tilde{\mathfrak{h}}^*$ is defined as follows:

$$\Lambda_0(\alpha^*_+) = \delta_{00}, \quad \text{equivalently,} \quad \Lambda_0(K) = 1, \Lambda_0(\mathfrak{h}) = 0.$$ 

Here and further we will frequently use the identification $\Lambda(\tilde{\alpha}^*) = \tilde{\alpha}^*(\Lambda)$. Also, note that $\delta(K) = 0$ and

$$\tilde{\alpha}^*(\beta + z\delta) = \alpha^\vee(\beta) \quad \text{for} \quad \tilde{\alpha} = [\alpha, j\nu_\alpha] \in \tilde{R}, \quad \beta \in R, \quad z \in \mathbb{C}.$$

For each $w \in \hat{W}$, we set $\Lambda_w \overset{\text{def}}{=} w(\Lambda_0) \in \tilde{\mathfrak{h}}^*$. One has:

$$\Lambda_w = w((0)) + \Lambda_0 \mod \mathbb{Q}\delta \quad \text{for the affine action from (2.12)}.$$ 

For $[w]$ from (4.1), $\Lambda_{[w]}$ is a level-one fundamental weight of $\tilde{g}$. The exact formula from [Kac] is:

$$\Lambda_{[w]} = c + \Lambda_0 - \frac{c^2}{2} \delta \in \Lambda_{[w]} + Q + \mathbb{Z}\delta.$$ 

Thus we have $c = \omega_r$ when $w = [w] = \omega_r$ for some $r \in O$.

Let $v_{[w]}$ be a unique up to proportionality $\tilde{n}$–fixed vector of the corresponding integrable irreducible level-one $\tilde{g}$–module denoted by $L(\Lambda_{[w]})$. For each $w \in \hat{W}$, there exists a unique (up to a scalar) vector $v_w \in L(\Lambda_{[w]})$ of $\mathfrak{h}$–weight $\Lambda_w$. One has:

$$\tilde{g}_\alpha v_w = \{0\} \quad \text{for any} \quad \alpha \in w(\tilde{R}_+).$$
The thick (upper) Demazure modules $L^w$ are defined as follows:

$$L^w \overset{\text{def}}{=} U(\tilde{\mathfrak{g}}^-)v_w \subset L(\Lambda_{[w]}),$$

where $w \in \hat{W}$; they are $\tilde{\mathfrak{h}}$–modules.

By (4.3), the set of vectors $\{v_w\}_{w \in \hat{W}}$ is in bijection with $P$ via the map $w \mapsto w(0)$. We restrict it to the elements of the form $\pi_b \in \hat{W}$; recall that $\pi_b$ map to $b$ under this map. Finally we set:

$$L^b \overset{\text{def}}{=} L^{\pi_b}, \quad v^b \overset{\text{def}}{=} v_{\pi_b} \in L(\Lambda_{[\pi_b]}),$$

where $b \in P$.

Any thick Demazure modules are of the form $L^b$ for proper $b \in P$.

By the triangular decomposition, each $L^w$ is a $\tilde{\mathfrak{h}}$–semisimple module. We have the Bruhat order $\leq$ on $\hat{W}$ defined as $v \leq w$ if and only if $v \in B(w)$; see Propositions 2.1, 2.2. Recall that if $W(b) = W(c) \subset P$ for $b, c \in P$, then $b \ll c$ by the partial order from (2.16) if and only if $u_b < u_c$ for the Bruhat order.

**Lemma 4.1.** Let $w \in \hat{W}$. For each $\alpha \in \hat{R}_+$, we have

$$v_{s_\alpha w} \in \begin{cases} \tilde{\mathfrak{g}}_\alpha^{-*}(\Lambda_w)v_w & (\alpha^*\Lambda_w > 0) \\ \mathbb{C}v_w & (\alpha^*\Lambda_w = 0) \\ \tilde{\mathfrak{g}}_\alpha^{-*}(\Lambda_w)v_w & (\alpha^*\Lambda_w < 0). \end{cases}$$

Here and further $\tilde{\mathfrak{g}}_\alpha^m$ ($m \geq 1$) is the $m$-th multiplicative power of $\tilde{\mathfrak{g}}_\alpha$ in the universal enveloping algebra $U(\tilde{\mathfrak{g}})$. In particular, the spaces in the right-hand side are always non-zero and one-dimensional.

**Proof.** Since $L(\Lambda_{[w]})$ is an integrable $\tilde{\mathfrak{g}}$–module, the $\hat{\mathfrak{g}}$–eigenvalue spaces for any weights from $\tilde{W}(\Lambda_{[w]})$ are of dimension one. Also, if $v_w$ is non-zero, then so is $v_{s_\alpha w}$; consider the action of $\mathfrak{sl}(2)_\alpha$. This gives the required. \(\square\)

**Corollary 4.2.** For each $w, u \in \hat{W}$, we have $w \leq u$ if and only if $L^u \subset L^w$.

**Proof.** We first prove the “only if” part of the corollary. By [BB, Section 2.2], there exist $w \leq x < u$ such that $u^{-1}x$ is a reflection and $\ell(x) = \ell(u) - 1$. Here and below we use the Bruhat order; see also [Hu] here and below. Then Lemma 4.1 implies that $L^u \subset L^x$; continuing we obtain the “only if” part.

The “if” part is as follows. We assume $L^u \subset L^w$ and need to prove that $u \geq w$. The inclusion $L^u \subset L^w$ gives that $v_u \in L^w$. Then we
use that the module $L^w$ is stable with respect to the action of $\mathfrak{sl}(2)_i$ corresponding to $\alpha_i$, where $0 \leq i \leq n$, assuming that $s_i w > w$. The relation (4.4) is applied and the PBW theorem.

Let us provide some details. One has that $v_{s_iu}, v_u$ belong to a single $\mathfrak{sl}(2)_i$–string by Lemma 4.1. Therefore $v_u \in L^w$ implies that $v_{s_iu}, v_u$ belong to a single $\mathfrak{sl}(2)_i$–string in $L^w$ and that

$$v_{s_iu}, v_u \in \begin{cases} L^{s_iw} & (s_iw < w) \\ L^w & (s_iw > w). \end{cases}$$

Now let us assume that $w \not\leq u$ and $v_u \in L^w$ and prove that this is impossible. Here $u$ can be taken minimal satisfying these two conditions for the Bruhat order on $\hat{W}$. One has either $\ell(u) = 0$ or $\ell(u) > 0$. In the latter case, there exists $0 \leq i \leq n$ such that $\ell(s_iu) = \ell(u) - 1$.

If $\ell(u) = 0$, then $v_u$ has to be the highest weight vector of $L(\Lambda_{\lbrack u\rbrack})$. Hence, $v_u \in L^w$ implies $L^w = L^u$ and therefore $w = u$. Thus, this case cannot occur due to the assumption $w \not\leq u$.

If $\ell(s_iu) = \ell(u) - 1$, then $s_iu < u$. The minimality of $u$ gives that $s_iu > \min\{w, s_iw\}$ for any $0 \leq i \leq n$. There are two possibilities here: $s_iw > w$ or $s_iw < w$. If $s_iw > w$, then $u > s_iu > w$, which contradicts to $w \not\leq u$. If $s_iw < w$, then $s_iu > s_iw$ and again $u > w$ due to [BB, Proposition 2.2.7] (see also [Hu]).

Finally, we conclude that there is no pair $u, w \in \hat{W}$ such that $w \not\leq u$ and $v_u \in L^w$, which proves the “if” part.

4.1.2. Filtrations of $L(\Lambda_{\lbrack b\rbrack})$. The quantum analogue of $L^w$ admits a global base in the sense of Kashiwara [Kas1, Section 4, Proposition 4.1]. The Littelmann path model [Lit] and the interpretation of its initial and final directions from [AKT, Theorem 6.23] results in the following theorem. We note that [Kas1] and [Kat2, Corollary 2.18] give its independent proof.

**Theorem 4.3.** For each $S \subset \hat{W}$, there exists $S' \subset \hat{W}$ such that

$$\bigcap_{w \in S} L^w = \sum_{u \in S'} L^u.$$

For each $w \in \hat{W}$, let $\text{gr}^w L \overset{\text{def}}{=} L^w / \sum_{u > w} L^u$, where the quotient is well-defined due to Corollary 4.2.

Let us use Proposition 2.2, (i). See also [Ma2, (2.7.5) and (2.7.11)]. One has that $u \geq w$ implies $u(\lbrack 0\rbrack) \preceq w(\lbrack 0\rbrack)$ for $u, w \in \hat{W}$. For each
$b \in P_+$, we set

$$\text{Gr}^b L \overset{\text{def}}{=} \left( L^b + \sum_{c \prec b} L^c \right) / \sum_{c \prec b} L^c .$$

We will denote the image of $v_b$ in $\text{Gr}^b L$ by the same letter. The $\mathfrak{h}$–weight of $v_b$ is $b$; see (4.3).

**Corollary 4.4.** For every $u, w \in \widehat{W}$, the vector subspace

$$\left( L^u \cap L^w \right) / \sum_{x > w} L^x \subset \text{gr}^w L$$

is either $\text{gr}^w L$ or $\{0\}$.

**Proof.** By Theorem 4.3, we have

(4.5) $$L^u \cap L^w = \sum_{x \in S} L^x ,$$

where $S \subset \widehat{W}$. Corollary 4.2 gives that $x \geq w$ for each $x \in S$ and, moreover, $x > w$ when $u \not\leq w$.

The space $L^u \cap L^w$ belongs to $\sum_{x > w} L^x$ when $u \not\leq w$, so the quotient is $\{0\}$ in this case. Otherwise, $L^w \subset L^u$ and the quotient is $\text{gr}^w L$ as stated.

Let us introduce the $\tilde{\mathfrak{h}}$–modules $D_b \overset{\text{def}}{=} \text{gr}^{\pi_b} L$, where the latter are $\text{gr}^w L$ above for $w = \pi_b$. By Proposition 2.1:

$$\{\text{gr}^w L\}_{w \in \widehat{W}} = \{D_b\}_{b \in P} .$$

**Corollary 4.5.** For every $b \in P_+$, the vector space $\text{Gr}^b L$ admits a $\tilde{\mathfrak{g}}_{\leq 0}$–action. Moreover, there is a natural finite filtration of $\text{Gr}^b L$ by $\{D_c\}_{c \in W(0)}$, where every $D_c$ occurs exactly once. We will say that $\text{Gr}^b L$ is filtered by $\{D_c\}_{c \in W(0)}$ with multiplicities one.

**Proof.** We have $n v_b = 0$ by construction. In particular, $U(\mathfrak{g}) v_b$ is a finite-dimensional $\mathfrak{g}$–module with the highest weight $b$. Therefore, the PBW theorem implies that the $\tilde{\mathfrak{h}}$–action on $L^b$ extends to the $\tilde{\mathfrak{g}}_{\leq 0}$–action. Then we use Corollary 4.4, which gives that the $\tilde{\mathfrak{h}}$–module $\text{Gr}^b L$ has a natural finite filtration by $\{\text{gr}^w L\}_{w \in \widehat{W}}$. Since $w \in \widehat{W}$ such that $\text{gr}^w L$ appears in this filtration must satisfy $w \geq \pi_b$ and $w \not\geq \pi_c$ for every $b \succ c \in P_+$, we obtain the required. □
4.1.3. Characters of Demazure slices. Recall that the $U(\tilde{g}_{\leq 0})$–module $Gr^bL$ has a cyclic vector $v_b$ ($b \in P_+$) of $\mathfrak{h}$–weight $b$.

**Proposition 4.6.** Let $b \in P_+$. Consider the cyclic $U(\tilde{g}_{\leq 0})$–module $W'_b$ generated by the cyclic vector $v$ subject to the following relations:

1. $Hv = b(H)v$ for each $H \in \mathfrak{h}$;
2. $\tilde{g}_\alpha v = 0$ for each $\alpha \in \tilde{R} \cap (R_+ + Z_{\leq 0}\delta)$;
3. $\tilde{g}_{\alpha}^{h^*} v = 0$ when $\alpha = [\beta, 0]$ or $[\beta, -\nu_\beta]$ for some $\beta \in R_+$.

Then, $W'_b$ maps surjectively onto $Gr^bL$ as $U(\tilde{g}_{\leq 0})$–modules.

**Proof.** Setting, $q(b + \Lambda_0 + m\delta) \overset{\text{def}}{=} \frac{\nu^2}{2} - m$, the weights $\Lambda_b$ for $b \in P$ satisfy the following (hyperbolic) equation: $q(b + \Lambda_0 + m\delta) = 0$. Moreover, they are exactly solutions of this equations from all $\tilde{h}$–weights of $\bigoplus_{\pi \in \Pi} L(\Lambda_\pi)$, which generally satisfy the inequality $q(b + \Lambda_0 + m\delta) \leq 0$ (i.e. are inside the corresponding hyperboloid).

Thanks to Lemma 4.1 and the construction of $Gr^bL$, the only $\tilde{h}$–weights $\Lambda$ of $Gr^bL$ satisfying $q(\Lambda) = 0$ are $\Lambda_{wb}$ for $w \in W$. One has $q < 0$ for all the other weights in $Gr^bL$. Hence, the cyclic vector $v = v_b \in Gr^bL$ satisfies the conditions (1), (3) above and the following modification of (2):

2'. $\tilde{g}_\alpha v = 0$ when $\alpha = [\beta, 0], [\beta, -\nu_\beta]$ for some $\beta \in R_+$.

Therefore, it suffices to check that condition (2) results from (1), (3) and (2') to prove (1, 2, 3) from the proposition.

Upon the restriction to $\mathfrak{sl}(2)_\alpha \otimes \mathbb{C}[z]$-calculation, we obtain that

\[(4.6) \tilde{g}_{\alpha + n\nu_\alpha \delta} v = 0 \quad n \leq 0\]

for each $\alpha \in R_+$; see e.g. [FMO]. Therefore,

$$\tilde{g}_\alpha v = 0 \quad \text{for each} \quad \alpha \in \tilde{R} \cap (R_+ + Z_{\leq 0}\delta),$$

which gives the required. The surjectivity $W'_b \to Gr^bL$ readily follows. \hfill $\square$

Next, we use that the $\tilde{g}_{\leq 0}$–action in $W'_b$ can be naturally extended to the $\tilde{g}_{\leq 0}$–action by the formula $Kv = 0$. The grading of the cyclic vector $v$ is 0; we put $d v = 0$.

Let $\mathfrak{B}$ be the category of finitely generated $U(\tilde{g}^-)$–modules with semisimple $\mathfrak{h}$–action and such that every weight space is finite dimensional with its weight in $P \oplus \mathbb{Z}\Lambda_0 \oplus \frac{1}{2}\mathbb{Z}\delta \subset \tilde{h}^*$. As in Section 2.3, here $e$ is the minimal positive integer satisfying $e(P, P)/2 \subset \mathbb{Z}$. 

For each $M \in \mathfrak{B}$, we set (formally):

$$
\text{gch } M \overset{\text{def}}{=} \sum_{c-m\delta \in P \oplus \mathbb{Z}\delta} q^m X_c \cdot \dim \text{Hom}_{\mathfrak{B}\subset\mathfrak{q}}(\mathbb{C}c-m\delta, M).
$$

We put $f \leq g$ for two polynomials $f, g \in \mathbb{Z}[q^{\pm 1/e}][X_b, b \in P]$ for $X_b$ from (2.21) if this inequality holds coefficient-wise, i.e. for all pairs of corresponding (integer) coefficients of the monomials $q^m X_b$ ($m \in \mathbb{Z}, b \in P$) in $f$ and $g$.

**Theorem 4.7.** For each $b \in P_+$ and $W'_b$ from Proposition 4.6, we have

$$
q^{-\frac{b^2}{2}} \text{gch } \text{Gr}^b L = \text{gch } W'_b = \frac{E_b}{h^0_{b-}}.
$$

**Proof.** We use [FL, Definition 2], [CFS, 3.6], and [FMS, (3.3)]. Proposition 4.6 implies that $W'_b$ is a quotient of the global Weyl modules there. Then [CI, Proposition 4.3] implies the inequality

$$
(4.7) \quad \text{gch } W'_b \leq \frac{E_b}{h^0_{b-}}.
$$

Since $v_b$ has $d$-degree $-\frac{b^2}{2}$ in $L(\Lambda_{[b]})$, (4.7) results in

$$
\text{gch } L(\Lambda_{[b]}) = \sum_{c \in (b+Q) \cap P_+} \text{gch } \text{Gr}^b L
\leq \sum_{c \in (b+Q) \cap P_+} q^{\frac{c^2}{2}} \text{gch } W'_c \leq \sum_{c \in (b+Q) \cap P_+} \frac{q^{\frac{c^2}{2}} E_{c-}}{h^0_{c-}}.
$$

Here we employ Proposition 4.6 and (4.7). Using now (3.38),

$$
\text{gch } L(\Lambda_{[b]}) = \sum_{c \in (b+Q) \cap P_+} \frac{q^{\frac{c^2}{2}} E_{c-}}{h^0_{c-}}.
$$

Therefore the inequality in (4.7) is actually an equality.

\[\square\]

4.2. Demazure slices and $E^\dagger_b$ ($b \in P$).
4.2.1. Demazure-Joseph functors. The main reference here is [Jos]. It is for semi-simple Lie algebras, but the construction there can be extended to our (affine, twisted) case. This is what we are going to do now.

For each \( b \in P \) and \( k \in \mathbb{Z}, m \in (1/e)\mathbb{Z}, \) let

\[
M(b + k\Lambda_0 + m\delta) \overset{\text{def}}{=} U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{g}}^+)} \mathbb{C}_{b + k\Lambda_0 + m\delta},
\]

where \( \mathbb{C}_{b + k\Lambda_0 + m\delta} \) is the natural \( \tilde{\mathfrak{b}}^- \)–module for the \( \tilde{\mathfrak{h}} \)–weight \( b + k\Lambda_0 + m\delta. \) This is the usual definition of Verma modules of \( \tilde{\mathfrak{g}}. \)

**Proposition 4.8 ([CG]).** For any \( b \in P, k \in \mathbb{Z}, m \in (1/e)\mathbb{Z}, \) the Verma module \( M(b + k\Lambda_0 + m\delta) \) viewed as a \( \tilde{\mathfrak{b}}^- \)–module, is the projective cover of \( \mathbb{C}_{b + k\Lambda_0 + m\delta} \) in \( \mathfrak{B}. \)

For each \( 0 \leq i \leq n, \) we set \( p_i^- \overset{\text{def}}{=} \tilde{\mathfrak{g}}_{\alpha_i} \oplus \tilde{\mathfrak{b}}^- \). A \( p_i^- \)–module is said to be \( \mathfrak{sl}(2) \)–integrable if it is \( \tilde{\mathfrak{h}} \)–semisimple and is a direct sum of finite-dimensional modules of \( \mathfrak{sl}(2). \)

Let us introduce some general terminology. For an abstract Lie algebra \( \mathfrak{L} \) and its finite-dimensional reductive Lie subalgebra \( \mathfrak{r}, U(\mathfrak{r}) \)–semisimple \( U(\mathfrak{L}) \)–modules will be called \( (\mathfrak{L}, \mathfrak{r}) \)–modules.

Given \( 0 \leq i \leq n \) and a \( (\tilde{\mathfrak{b}}^-, \tilde{\mathfrak{h}}) \)–module \( M, \) consider the \( U(\tilde{\mathfrak{b}}^-) \)–module \( \mathcal{D}_i(M) \) obtained as the maximal \( \mathfrak{sl}(2) \)–integrable quotient of \( U(p_i^-) \otimes_{U(\tilde{\mathfrak{b}}^-)} M. \) It is straightforward to see that \( \mathcal{D}_i(M) \) is \( \tilde{\mathfrak{h}} \)–semisimple. The correspondence \( M \mapsto \mathcal{D}_i(M) \) gives rise to a functor, which is usually called the Demazure-Joseph functor.

**Theorem 4.9 ([Jos]).** The functors \( \{\mathcal{D}_i\}_{0 \leq i \leq n} \) satisfy the following.

1. Each \( \mathcal{D}_i \) is right exact.
2. For \( i, j \in I \) such that \( (s_i s_j)^m = 1, \) one has

\[
\mathcal{D}_i \mathcal{D}_j \cdots \cong \mathcal{D}_j \mathcal{D}_i \cdots.
\]

3. There is a natural morphism \( \text{Id} \rightarrow \mathcal{D}_i. \)
4. For a \( \mathfrak{sl}(2) \)–integrable \( p_i^- \)–module \( M: \mathcal{D}_i(M) \cong M. \) In particular, \( \mathcal{D}_i^2 \cong \mathcal{D}_i. \)
5. For a \( \mathfrak{sl}(2) \)–integrable \( p_i^- \)–module \( M \) and a \( \tilde{\mathfrak{b}}^- \)–module \( L, \)

\[
\mathcal{D}_i(L \otimes M) \cong \mathcal{D}_i(L) \otimes M.
\]
By Theorem 4.9, we can consider the left derived functor $L D_i$ in the category of $(\tilde{b}^-, \tilde{h})$–modules.

For each $w \in \tilde{W}$ with a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$, let

$$D_w \overset{\text{def}}{=} D_{i_1} \circ D_{i_2} \circ \cdots \circ D_{i_\ell}.$$ 

Thanks to the same theorem, $D_w$ does not depend on the particular choice of the reduced expression of $w$.

For $M$ with finite-dimensional $\tilde{h}$–weight spaces, we denote by $M^\vee$ its restricted dual (i.e. the direct sum of the duals of the weight spaces). Clearly, $M^\vee$ is again a $(\tilde{b}^-, \tilde{h})$–module with finite-dimensional weight spaces, and we have $M \cong (M^\vee)^\vee$. We set $D^\sharp_i \overset{\text{def}}{=} \vee \circ D_i \circ \vee$.

Theorem 4.10 ([FKM] Proposition 5.7). For any two $\tilde{h}$–semisimple $U(\tilde{b}^-)$–modules $M$ and $N$ and for $0 \leq i \leq n$, one has:

$$\text{Ext}_{(\tilde{b}^-, \tilde{h})}^p (L D_i (M), N) \cong \text{Ext}_{(\tilde{b}^-, \tilde{h})}^p (M, L R D^\sharp_i (N)),$$

where $\text{Ext}_{(\tilde{b}^-, \tilde{h})}^p (\cdot, \cdot)$ is the relative extension (see e.g. [Kum, III]). 

Recall for the following Theorem and below, that $\prec$ is the orderings defined in (2.17) and $<$ is the Bruhat order from Proposition 2.2.

Theorem 4.11. Let $w \in \tilde{W}$ and $0 \leq i \leq n$. If $s_i w < w$, then $D_i(L_w) = L_{s_i w}$. If $s_i w > w$, then one has: $D_i(L_w) = L_w$. Moreover, $\mathbb{L} L^\prec D_i (L_w) = \{0\}$.

Proof. Using quantum group, this result follows from its analog of [Kas1, Proposition 3.3.4]; see also Theorem 4.3 above. In the geometric approach from [Kat2, Theorem 2.15], on can obtain this claim as a corollary of [KS, Lemma 3.1 and Proposition 3.2].

4.2.2. More on Demazure-Joseph functors. For any $b \in P_+$, let $\mathbb{W}_c$ be the image of $L_c$ in $\text{Gr}^h L$, where $c \in W(b)$. It is a $\tilde{b}^-\text{-module}$. By Corollary 4.2 and Corollary 4.5, $\mathbb{W}_c \cong D_c$ as $\tilde{b}^-\text{-modules}$ when $c = b_-$. Also, $\mathbb{W}_c \cong \mathbb{W}_c' \otimes \mathbb{C}_{\Lambda_0 - \Delta b}^+$ when $c = b_+$; combine here Theorem 4.7 and Proposition 4.6.
Lemma 4.12. For each $b \in P_+$, the module $\mathcal{W}_b$ admits a finite filtration by $\mathbb{D}_c$ (as constituents) with $c \in W(b)$ such that each of them appears exactly once; we say that $\mathcal{W}_b$ is filtered by $\{\mathbb{D}_c\}_{c \in W(b)}$ with multiplicities one.

Proof. Apply Corollary 4.5 to $\mathcal{W}_c \cong \text{Gr}^b L$. \hfill \Box

Proposition 4.13. Let $c \in P$ and $0 < i \leq n$. Then

$$\mathcal{D}_i(\mathcal{W}_c) = \begin{cases} \mathbb{W}_{s_i(c)} & (s_i(c) \geq c), \\ \mathbb{W}_c & (s_i(c) \not\geq c). \end{cases}$$

Moreover, $\mathbb{L}^\leq \mathcal{D}_i(\mathcal{W}_c) = \{0\}$.

Proof. By the $q$–invariance, $\mathcal{D}_i(\text{Gr}^b L) \cong \text{Gr}^b L$ and $\mathbb{L}^\leq \mathcal{D}_i(\text{Gr}^b L) = \{0\}$ for every $b \in P_+$. Then we use that by construction,

$$\mathcal{W}_c \cong (L^c + M)/M,$$

where $M \subset L^c$ is the sum of $L^b$ such that $b \prec c$ and $b \in P_+$. Applying $\mathcal{D}_i$ to the short exact sequence

$$0 \to M \to (L^c + M) \to \mathcal{W}_c \to 0,$$

and utilizing Theorem 4.11, we obtain the exact sequence

$$0 \to \mathbb{L}^\leq \mathcal{D}_i(\mathcal{W}_c) \to M \to (L^{c'} + M) \to \mathcal{D}_i(\mathcal{W}_c) \to 0,$$

where $c' = s_i(c)$ if $s_i(c) \geq c$, and $c' = c$ if $s_i(c) < c$. Therefore $\mathbb{L}^\leq \mathcal{D}_i(\mathcal{W}_c) = \{0\}$ and either $\mathcal{D}_i(\mathcal{W}_c) \cong \mathcal{W}_c$ (for $s_i(c) \leq c$) or $\mathcal{W}_{s_i c}$ (for $s_i(c) \geq c$) as required. \hfill \Box

Corollary 4.14. For $b \in P_-$, one has: $\mathcal{D}_{w_0}(\mathbb{D}_b) \cong \mathcal{W}_{w_0(b)}$. \hfill \Box

Corollary 4.15. Let $b \in P$ and $0 \leq i \leq n$. If $s_i(b) \succ b$, then we have a short exact sequence of $\mathcal{B}^-$–modules:

$$0 \to \mathbb{D}_b \to \mathcal{D}_i(\mathbb{D}_b) \to \mathbb{D}_{s_i(b)} \to 0.$$

If $s_i(b) \preceq b$, then $\mathcal{D}_i(\mathbb{D}_b) = \{0\}$. Moreover, $\mathbb{L}^\leq \mathcal{D}_i(\mathbb{D}_b) = \{0\}$ in each of these two cases.

Proof. Using Corollary 4.5, $\mathbb{D}_b \cong L^b/\sum_{u \prec_0 b} L^u$. Let $S \overset{\text{def}}{=} \{u \in \hat{W} \mid u \not\leq \pi_b, s_i u \not\leq \pi_b\}$ and $M \overset{\text{def}}{=} \sum_{u \in S} L^u$. It is straightforward to see that $\mathcal{D}_i(M) \cong M$. Also, if $u \not\leq \pi_b$ and $s_i \pi_b < \pi_b$, then we have
4.2.3. Orthogonality relations. For a \( \mathbb{Z} \)-graded vector space \( V = \bigoplus_{i \in \mathbb{Z}} V_i \), we define

\[
\text{end}(V) \overset{\text{def}}{=} \bigoplus_{j \in \mathbb{Z}} \prod_{i \in \mathbb{Z}} \text{Hom}(V_i, V_{i+j}) \subset \text{End}(V),
\]

which is a ring. If \( V \) has a \( \tilde{b}^- \)-module structure, then the \( d \)-grading can be used here as the \( \mathbb{Z} \)-grading. Accordingly, we define

\[
\text{end}_{\tilde{b}^-}(V) \subset \text{end}(V),
\]

the subring of all endomorphisms commuting with the action of \( \tilde{b}^- \).

Each \( \mathbb{W}_b \) is cyclically generated by a \( \tilde{b}^- \)-eigenvector. For instance, all elements of \( \text{end}_{\tilde{b}^-}(\mathbb{W}_b) \) commute with \( g \) assuming that \( \mathbb{W}_b \) is \( g \)-invariant, which occurs for \( b \in P_+ \).

For each \( b \in P \), the (restricted) dual of the level one integrable representation \( L(\Lambda_{[b]})^\vee \) has a unique extremal weight vector \( v_{\pi b} \) (up to a scalar) that is dual to \( v_b = v_{\pi b} \in L(\Lambda_{[b]}) \). The \( \tilde{b}^- \)-weight of \( v_{\pi b}^* \) is \( -\Lambda_{\pi b} \).

We define the usual (thin) Demazure module for \( b \in P \) to be

\[
D_b \overset{\text{def}}{=} U(\tilde{b}^-)v_{\pi b}^* \subset L(\Lambda_{[b]})^\vee.
\]

These modules are referred to as Demazure modules in [Kum].

**Proposition 4.16 ([San, Ion, CI]).** For any \( b \in P \) and the reduced decomposition \( b = s_{i_1} s_{i_2} \cdots s_{i_\ell} \pi \) in \( \widehat{W} \) with \( \pi \in \Pi \) (the reduced decomposition of any \( bw \) for \( w \in W \) can be taken here), we have an isomorphism of \( \tilde{b}^- \)-modules

\[
D_b \cong \mathcal{R}_{i_1} \circ \cdots \circ \mathcal{R}_{i_\ell} (\mathbb{C}_{-\pi \Lambda_0}).
\]

Moreover, we have the followings:

1. \( \text{gch } D_b = q^{-b^2/2}w_b\overline{E_b} \) (called below the character equality);
(2) for \( b \in P_+ \), the module \( \mathcal{W}_b \) admits a decreasing separable filtration as \( \mathfrak{g}_{\leq 0} \)-modules whose associated graded is isomorphic to a direct sum of \( D_{-b} \otimes \mathbb{C}_{2\Lambda_0} \) (i.e. \( \mathcal{W}_b \) is a self-extension of \( D_{-b} \otimes \mathbb{C}_{2\Lambda_0} \)). In addition, \( \text{end}_{\mathfrak{g}_{-}}(\mathcal{W}_b) \) is a polynomial ring.

Proof. The character equality is from \([\text{Ion}]\). The second claim follows from \([\text{CI}], \text{Corollary } 2.10\). □

Proposition 4.17 ([FKM] Appendix A). For \( M \in \mathfrak{B} \) and any \( b \in P \),

\[
\langle w_0(\mathcal{E}_b(\mu/\langle \mu \rangle)) \text{gch} M \rangle = q^{-\frac{b^2}{2}} \sum_{p \geq 0, m, k \in \mathbb{Q}} (-1)^p q^{-m} \dim_{\mathbb{C}} \text{Ext}^p_{(b^-_b)}(M \otimes_{\mathbb{C}} \mathbb{C}_{m\delta + k\Lambda_0}, D_c^\vee),
\]

where \( \langle \cdot \rangle \) denotes the constant term. Recall that \( D_c^\vee \) is the restricted dual (coinciding with the full dual since \( \dim_{\mathbb{C}} D_b < \infty \)).

Proof. This claim is essentially from \([\text{FKM}]\) for the ADE systems. The same approach is applicable to arbitrary twisted \( \tilde{R} \). Namely, we use (2.31) and (2.32); they result in \( \text{gch} n^- = w_0\mu_\circ(t \to 0) \). Then Proposition 4.13 and Corollary 4.15 are applied to extend the proof of Theorem 5.12 and Proposition 5.9 in \([\text{FKM}]\). We also use \([\text{CI}], \text{Theorem } 4.4\). □

Corollary 4.19. For any \( b, c \in P \), one has:

\[
\langle (\text{gch} D_b) w_0(\mathcal{E}_c \mu/\langle \mu \rangle) \rangle = \delta_{b,c}.
\]

Proof. Combine Proposition 4.17 and Theorem 4.18. □
Corollary 4.20 (Demazure slices and \( E^\dagger \)-polynomials). For any \( b \in P \),
\[
gch D_b = q^2 \frac{w_0(E^\dagger_{b^*})}{h^\dagger_0}.
\]

Proof. One has: \( gch D_b \leq gch W_{b^*} \) due to Theorem 4.7 and Corollary 4.5. The orthogonality relations from (2.42) with the polynomials \( \{ E_b \}_{b \in P} \), provided by Corollary 4.19 for \( gch D_b \), and the comparison of the coefficients of the leading monomials give the required.

We mention that all \( \mathfrak{h} \)-weights of \( W_{b^*} \) belong to \( \sigma^{-}(b) \) from (2.18) by [CI, Proposition 4.3]. Therefore \( gch D_b \in \sum_{b^* \leq c} \mathbb{Q}(q) X_c \). The inequality for \( c \) here is weaker than \( b \leq c \) in the definition/construction of the \( E \)-polynomials, but using this fact is not necessary anyway in our approach (the orthogonality relations with \( E_b \) are sufficient).

\[
\text{Corollary 4.21.} \text{ For } b \in P, \text{ the module } D_b \text{ is projective in the category of } \mathfrak{h} \text{-semisimple } U(\mathfrak{b}^-) \text{-modules with the weights } c \text{ satisfying } c \geq b.
\]

Proof. We use Theorem 4.18 and that \( X_c \) occur in \( E_b \) only for \( c \geq b \). Thus \( \text{Ext}^{>0}_{(\mathfrak{b}^-)}(D_b \otimes \mathbb{C} \mathbb{C}_{mb+k\Lambda_0}, \mathbb{C}_{b'}) = \{0\} \) for every \( b \leq b' \) and \( m, k \in \mathbb{Q} \); here the vanishing property holds of course for any \( m, k \in \mathbb{C} \).

5. Filtrations of tensor products

In this section, the identification of the characters of Demazure slices with \( E \)-dag polynomials will be used to address Theorem 3.4. We employ Theorem 5.2, which is related to similar results in [Kas3, Kat1] in the ADE case (it establishes a connection with the intertwiners from [ChO]). The vanishing theorem from [KL] and, its variant, Corollary 5.7 are also important to us; see also Theorems 5.9 and 5.11. These facts are essentially sufficient to interpret the existence of the expansion from (3.48) representation-theoretically. At the end, we discuss the remaining theta-function expansions from Theorem 3.4. The exposition is compressed in this section, with quite a few references to prior works and some details omitted (especially if the corresponding results are known in the ADE case). Recall that we provided the list of main modules under consideration in the beginning of Section 4.

5.1. The \( \mathbb{W}_c \)-modules.
5.1.1. Demazure operators. Recall that we have the Demazure operators:

\[ T^+_i : \mathbb{C}((q^{1/e}))[X_b] \ni f \mapsto \frac{f - X_{\alpha_i}s_i(f)}{1 - X_{\alpha_i}}, \]

where \( X_{\alpha_0} = q^{-1}X_{-\vartheta}, \) for each \( 0 \leq i \leq n \) defined in (3.16), where \( T_i \) is from (2.22).

**Lemma 5.1.** We assume that \( M \in \mathfrak{B} \) has only finitely many distinct \( \mathfrak{h} \)-weights and the central element \( K \in \tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}} \) acts trivially (is zero) in \( M \). Then \( \text{gch} \mathcal{D}_i(M) = T^+_i(\text{gch} M) \) for each \( 0 \leq i \leq n \).

**Proof.** We can follow here [Jos, 5.4 Lemma], using that our \( \mathfrak{h} \)-weights are bounded by the assumption. \( \square \)

5.1.2. Level-zero theory. Here, the action of \( K \) is assumed to be trivial. The Lie algebra \( \tilde{\mathfrak{g}}_{\leq 0} \) has one-dimensional representations \( \mathbb{C}k\Lambda_0 \) and \( \mathbb{C}m\delta \) for \( k \in \mathbb{Z}, m \in (1/e)\mathbb{Z} \) for \( e \) as above: \( e(P, P)/2 = \mathbb{Z} \). Tensoring with such one-dimensional modules will be called character twist. The trivial (zero) \( K \)-action can be always achieved by an appropriate character twist.

The following is an extension of the corresponding result from [Kat1].

**Theorem 5.2.** For each \( b \in P \) and \( 0 \leq i \leq n \), we have

\[
\text{gch} \mathbb{W}_{s'_i(b)} = \begin{cases} 
q^{-\delta \omega^i(v(b))}T^+_i(\text{gch} \mathbb{W}_b) & (\alpha^*_i(b) \leq 0), \\
\text{gch} \mathbb{W}_b & (\alpha^*_i(b) > 0), 
\end{cases}
\]

where \( s'_i = s_i \) (\( i \neq 0 \)) and \( s'_0 = s_\vartheta \) (\( i = 0 \)). Also, \( \mathbb{L}^< \mathcal{D}_i(\mathbb{W}_b) = \{0\} \) for every \( 0 \leq i \leq n \).

**Proof.** The case \( i \neq 0 \) is Proposition 4.13. Hence, assume \( i = 0 \); we will omit some details in the proof below.

When \( \alpha^*_0(b) \geq 0 \), it suffices to show that \( \mathbb{W}_b \) is an integrable \( \mathfrak{p}^-_0 \)-module; see Theorem 4.9. To establish this, we will check that the \( \mathfrak{b}^- \)-action on \( \mathbb{W}_b \) can be enhanced to a \( \mathfrak{p}^- \)-action.

For every \( b \in P \), there is an embedding \( \mathbb{W}_b \subset \mathbb{W}_{b+} \) by definition of \( \mathbb{W}_b \). The set of \( \mathfrak{h} \)-weights of \( \mathbb{W}_b \) is contained in the convex hull of \( W(b) = W(b_+) \); see Theorem 4.7. In particular, \( \alpha^*_0(b) \geq 0 \) implies that

\[
\mathbb{g}_{-\alpha_0}^{\alpha^*_0(b)+1}v_b = \{0\}, \quad \text{for the } \mathfrak{b}^-\text{-cyclic vector } v_b \in \mathbb{W}_b.
\]

By Proposition 4.6 and Theorem 4.7, we obtain that \( v_b \) satisfies the relations from Proposition 4.6 upon the application of \((w_0u_b)^{-1}\) to the
roots there. Therefore \( \alpha_0^*(b) \geq 0 \) gives that
\[
\mathfrak{g}_{-\alpha_0}^\alpha \nu_b \neq \{0\}. \tag{5.2}
\]

Next, \( \nu_{s\varphi(b)} \) has the same \( \mathfrak{h} \)--weight as that of \( \mathfrak{g}_{-\alpha_0}^\alpha \nu_b \), but its \( \mathfrak{d} \)--degree is different by \( \vartheta^\vee(b) \). Thus Theorem 5.3 below implies that the isomorphism \( \mathbb{C} \nu_{s\varphi(b)} \cong \mathfrak{g}_{-\alpha_0}^\alpha \nu_b \) induces the following embedding of \( \mathfrak{b}^- \)--modules:
\[
\mathbb{W}_{s\varphi(b)} \hookrightarrow \mathbb{W}_b.
\]
Moreover, we can equip \( \sum_{k=0}^{\alpha_0^*(b)} \mathfrak{g}_{-\alpha_0}^k \nu_b \) with a structure of \( \mathfrak{sl}(2)_0 \)--module; see (5.2), and (5.1). Since \( \nu_b \) is a cyclic vector of \( \mathbb{W}_b \), the PBW theorem implies that \( \mathbb{W}_b \) admits an integrable \( \mathfrak{p}_0^- \)--action. In particular,
\[
\mathcal{D}_0(\mathbb{W}_b) \cong \mathbb{W}_b, \quad \mathbb{L}^{<0} \mathcal{D}_0(\mathbb{W}_b) \cong \{0\}
\]
whenever \( \alpha_0^*(b) \geq 0 \) by Theorem 4.9. Therefore, the case \( \alpha_0^*(b) \geq 0 \) of the assertion is proved.

Let us now consider the case \( \alpha_0^*(b) \leq 0 \). We have now:
\[
\mathbb{W}_b \hookrightarrow \mathbb{W}_{s\varphi(b)},
\]
where the latter space is an integrable \( \mathfrak{p}_0^- \)--module with finitely many distinct \( \mathfrak{h} \)--weights. This induces the following homomorphism of \( \mathfrak{b}^- \)--modules:
\[
\mathcal{D}_0(\mathbb{W}_b) \xrightarrow{\eta} \mathbb{W}_{s\varphi(b)}, \quad \text{where } \mathbb{W}_{s\varphi(b)} \ni \nu_{s\varphi(b)} \in \eta(\mathcal{D}_0(\mathbb{W}_b));
\]
use (5.1). Therefore, \( \eta \) is surjective and
\[
T_0^\dagger (\text{gch } \mathbb{W}_b) \geq q^{-\vartheta^\vee(b)} \text{gch } \mathbb{W}_{s\varphi(b)} \tag{5.3}
\]
by \([\text{Kat1}, \text{Lemma 4.4}]\).

Employing now Lemma 4.12 and Corollary 4.4, we obtain that \( \mathbb{W}_c \) \((c \in \mathcal{P}) \) has a filtration by \( \mathbb{D}_{c'} \) (as always, for appropriate character twists) for \( c' \in \mathcal{W}(c) \) such that \( c' \leq c \); these modules occur (as constituents) with multiplicities one. By Corollary 4.20, the inequality (5.3) is equivalent to
\[
\sum_{c \in \mathcal{W}_b} \zeta_c(b) T_0^\dagger \left( \frac{w_0 E_{c'}^{c'} h_c^0}{h_0^c} \right) \geq \sum_{c \in \mathcal{W}_b} q^{-\vartheta^\vee(b)} \zeta_{s\varphi(b)}(s\varphi(b)) \frac{w_0 E_{c'}^{c'} h_c^0}{h_0^c}; \tag{5.4}
\]
see Theorem 3.3. Thus (5.4) is in fact an equality, and therefore (5.3) is an equality too. We obtain that \( \mathcal{D}_0(\mathbb{W}_b) \cong \mathbb{W}_{s\varphi(b)} \) up to a character.
twist and $L^{<0}\mathcal{D}_b(\mathcal{W}_b) = \{0\}$ in the case $\alpha_1^0(b) \leq 0$. Here we use again [Kat1, Lemma 4.4]. □

5.1.3. The ring $\text{end}(\mathcal{W}_c)$.

**Theorem 5.3** ([FL], [FMS], [CI] Corollary 2.4). Let $c \in P_+$. The ring $\text{end}(\mathcal{W}_c)$ is isomorphic to a graded polynomial ring, namely, to:

$$\bigotimes_{i=1}^n \mathbb{C}[X_{i,1}, \ldots, X_{i,m_i}]^{S_{m_i}}, \text{ where } m_i = \alpha_i^\vee(c_+), \deg X_{i,j} = 1,$$

$S_m$ are the symmetric groups. Moreover, $\text{end}(\mathcal{W}_c)$ is isomorphic to $\text{Hom}_h(\mathcal{C}_b, \mathcal{W}_c)$ for every $b \in W(c)$. □

Using the results from the previous section, we can extend this theorem to any $c \in P$.

**Corollary 5.4.** For any $c \in P$, the ring $\text{end}(\mathcal{W}_c)$ does not depend on $c \in W(b)$ and therefore it is isomorphic to the graded polynomial ring from Theorem 5.3.

Proof. The case $c = c_+$ is Theorem 5.3. This ring is isomorphic to $\text{Hom}_h(\mathcal{C}_c, \mathcal{W}_c) \subset \mathcal{W}_c$ for such $c$.

Now let us consider an arbitrary $c \in P$. We will proceed by induction. Namely, we assume this claim for $c$ and prove it for $s_i(c)$ for $0 \leq i \leq n$ provided that either the corresponding length increases for $i \neq 0$, or that the corresponding length decreases for $i = 0$.

For any $0 \leq i \leq n$, $\mathcal{D}_i$ is a functor and therefore induces the following homomorphism of algebras:

(5.5) $\text{end}(\mathcal{W}_c) \rightarrow \text{end}(\mathcal{D}_i(\mathcal{W}_c)) = \text{end}(\mathcal{W}_{s_i(c)})$,

where $s_i(c) \gg c$ for $i \neq 0$ and $s_0(c) \ll c$ if $i = 0$; see (2.16).

Moreover, Theorem 4.9 gives a homomorphism of $\mathfrak{h}^-$--modules $\mathcal{W}_c \rightarrow \mathcal{D}_i(\mathcal{W}_c)$, which is an inclusion. The latter follows directly from the definition of $\mathcal{W}_c$ for $i \neq 0$ and results from Theorem 5.2 when $i = 0$.

For every $0 \leq i \leq n$ and $c \in W(b)$, the reduction to $\mathfrak{sl}(2)_i$ gives that:

(5.6) $\mathcal{W}_c \supset \text{Hom}_h(\mathcal{C}_c, \mathcal{W}_c) \cong \text{Hom}_h(\mathcal{C}_{s_i(c)}, \mathcal{W}_{s_i(c)}) \subset \mathcal{W}_{s_i(c)}$, with a possible character twist of $\mathcal{W}_{s_i(c)}$ when $i = 0$. These are maps of linear spaces.

Since $\mathcal{W}_c$ is cyclic, $\text{end}(\mathcal{W}_c) \subset \text{Hom}_h(\mathcal{C}_c, \mathcal{W}_c)$ for every $c \in P$. Let us assume that

$$\text{end}(\mathcal{W}_c) \cong \text{Hom}_h(\mathcal{C}_c, \mathcal{W}_c)$$

and $\mathcal{D}_i(\mathcal{W}_c) \cong \mathcal{W}_{s_i(c)}$. 

Then (5.6) gives that
\[ \text{end}(W_c) \cong \text{Hom}_0(C_c, W_c) \cong \text{Hom}_0(C_{s_i(c)}, W_{s_i(c)}) \supset \text{end}(W_{s_i(c)}). \]
Therefore the image of the map from (5.5) is the whole eigenspace for the $\mathfrak{h}$–weight $s_i(c)$ in $W_{s_i(c)}$, and we conclude that
\[ \text{end}(W_c) \cong \text{Hom}_0(C_{s_i(c)}, W_{s_i(c)}) \cong \text{end}(W_{s_i(c)}). \]

The assumption we used here holds for $c \in P$ if it holds for $s_i(c) \gg c$ for some $i \neq 0$ or if it holds for $s_{\vartheta}(c) \ll c$. Hence, we can employ the induction, starting with $c = c_+$.

5.1.4. Introducing $W$-modules. Let $W_c \overset{\text{def}}{=} (C_0 \otimes \text{end}(W_c) W_c \otimes C_{-\Lambda_0}$ for $c \in P$; the action of $K$ is trivial (zero) in this module.

**Corollary 5.5.** Similar to Theorem 5.2,
\[ \text{gch } W_{s_i(b)} = \begin{cases} q^{-\delta_0 \vartheta \gamma(b)} T_i^\dagger \text{ (gch } W_b) & (\alpha_i^*(b) \leq 0) \\ \text{gch } W_b & (\alpha_i^*(b) > 0), \end{cases} \]
for any $0 \leq i \leq n$ and $b \in P$. Moreover, $L^{<0} \mathcal{D}(W_b) = \{0\}$.

**Proof.** We use that $W_{s_i(b)}$ and $W_b$ are free modules over the polynomial ring $\text{end}(W_{b_+})$. The *Koszul resolution* of $C$ considered as an $\text{end}(W_{b_+})$–module therefore results in $\mathfrak{g}_{\leq 0}$–module resolutions of $W_{s_i(b)}$ and $W_b$ by some complexes whose terms are direct sums of $W_{s_i(b)}$ and $W_b$, respectively. Then we apply $\mathcal{D}_i$ to these resolutions and deduce the claim from Theorem 5.2.

**Proposition 5.1.** For any $b \in P_+$, we have an isomorphism $W_b \cong D_{-b}$ as $\mathfrak{g}_{\leq 0}$–modules, which may require a character twist.

**Proof.** This follows from [CI, Theorem 2.7]. See also Theorem 4.16.

5.2. Vanishing theorems.

5.2.1. General results. Let $L(b + k\Lambda_0)$ be the *integrable highest weight* $\mathfrak{g}$–module for the highest weight $b + k\Lambda_0 \in \mathfrak{h}^*$. Here $b \in P_+$, $k \in \mathbb{Z}_{\geq 0}$ and we assume that $\alpha_i^*(b + k\Lambda_0) \in \mathbb{Z}_{\geq 0}$, equivalently, $(\vartheta \gamma, b) \in \mathbb{Z}_{\leq k}$; otherwise it does not exist.
**Theorem 5.6 ([KL]).** For any \(b, b' \in P_+\), \(k \in \mathbb{Z}_{>0}\), and \(k' \in \mathbb{Q}\) such that \((\vartheta^\vee, b) \leq k\), we have

\[
\text{Ext}^p_{(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})}(L(b + k\Lambda_0), W_{b'}^\vee \otimes_{\mathbb{C}} C_{k'\Lambda_0 + m\delta}) = \{0\} \quad \text{for} \quad p > 0,
\]

where \(\text{Ext}^p_{(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})}(\cdot, \cdot)\) are defined for the relative Lie algebra cohomology; see [Kum, III].

**Proof.** The corresponding claim from [KL] can be extended to the twisted case following Section 2.1. We omit details. \(\square\)

**Corollary 5.7.** Let \(b, b' \in P_+, k \in \mathbb{N}\) and \(m, k' \in \mathbb{Q}\) provided the inequality \((\vartheta^\vee, b) \leq k\). Then

\[
\text{Ext}^p_{(\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})}(L(b + k\Lambda_0), D_{b'}^\vee \otimes_{\mathbb{C}} C_{k'\Lambda_0 + m\delta}) = \{0\} \quad \text{for} \quad p > 0.
\]

**Proof.** We know that \(W_c \cong D_c\) for \(c \in P_+\) by Lemma 5.1. Thus it suffices to see that the higher \(\text{Ext}\) in the category of \((\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{h}})\)-modules are the same as those for the action of \((\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{h}})\). This can be seen from the Hochschild-Serre spectral sequence ([Kum, E.12]). Indeed,

\[
\text{Hom}_{(\mathfrak{g}, \mathfrak{h})}(M, N) = \bigoplus_{p \in \mathbb{Z}} \text{Ext}^p_{(\mathfrak{g}, \mathfrak{h})}(M, N) \cong \bigoplus_{p \in \mathbb{Z}} \text{Ext}^p_{(\mathfrak{b}, \mathfrak{h})}(M, N)
\]

for every \(\mathfrak{g}\)-modules \(M\) and \(N\) that are direct sums of finite-dimensional \(\mathfrak{g}\)-modules. This finiteness property holds, in particular, for the minimal projective resolution of \(L(b + k\Lambda_0)\) in the category of \(\widetilde{\mathfrak{h}}\)-semisimple \(\widetilde{\mathfrak{g}}_{\leq 0}\)-modules. \(\square\)

### 5.2.2. Filtrations by \(\mathbb{W}, \mathbb{D}\)-modules.

**Theorem 5.8.** Let \(M\) be a \((\mathfrak{b}^-, \mathfrak{h})\)-module, whose set of \(\mathfrak{h}\)-weights is contained in \(\bigcup_{j=1}^m (\Lambda_j + \sum_{i=0}^n \mathbb{Z}_{\leq 0} \alpha_i)\) for a finite subset \(\{\Lambda_j\}_{j=1}^m \subset \mathfrak{h}^*\).

(i) A module \(M\) admits a decreasing separable filtration such that its associated graded components are of the form \(\{\mathbb{W}_b\}_{b \in P_+}\), possibly with character twists, if and only if it is obtained as the restriction of some \((\mathfrak{g}_{\leq 0}, \mathfrak{g} + \mathfrak{h})\)-module and, additionally, the following holds for any \(c \in P_-\) and \(m, k \in \mathbb{Q}\):

\[
\text{Ext}^1_{(\mathfrak{g}_{\leq 0}, \mathfrak{g} + \mathfrak{h})}(M \otimes_{\mathbb{C}} C_{mk\Lambda_0}, D_c^\vee) = \{0\}.
\]

(ii) Similarly, \(M\) admits a decreasing separable filtration such that its associated graded components are of the form \(\{\mathbb{D}_b\}_{b \in P}\), possibly with...
character twists, if and only if for any \(c \in P\) and \(m, k \in \mathbb{Q}\):

\[
\text{Ext}^1_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (M \otimes \mathbb{C}_{m\delta + k\Lambda_0}, D_c^\vee) = \{0\}.
\]

For (i) or (ii), the \(\text{Ext}^p\)–spaces vanish in all positive degrees \(p\).

Proof. We will begin with the “only if” part. For (ii), it follows from Theorem 4.18 and a (repeated) usage of the long exact sequences. To prove the “only if” part in the case of (i), one needs an analogue of this theorem for the modules \(\mathbb{W}_{-c}\) and \(D_b^\vee\) when \(b, c \in P_-\). Let us outline the proof in this case.

For \(b \in P_-\), the module \(D_b\) is \(\mathfrak{g}\)–invariant. Hence, \(D_i (D_b) \cong D_b\) and \(L_{>0} D_i (D_b) = \{0\}\) for each \(1 \leq i \leq n\). Thus, \(\mathbb{W}_{w_0}(D_b) = D_b\). Thanks to Theorem 4.10 and Corollary 4.14:

\[
\{0\} = \text{Ext}^>_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (\mathbb{C}_c \otimes \mathbb{C}_{m\delta}, D_b^\vee) \cong \text{Ext}^>_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (\mathbb{W}_{w_0(c)} \otimes \mathbb{C}_{m\delta}, D_b^\vee)
\]

for any \(b, c \in P_-\). Then we use the long exact sequences, which gives the “only if” part of (i).

Let us come to the “if” part. We will consider only (ii) (the first case is similar). Let \(c \in P\) be a maximal element with respect to \(\preceq\) such that

\[
\bigoplus_{m, k \in \mathbb{Q}} \text{Hom}_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (M, D_c^\vee \otimes \mathbb{C}_{m\delta + k\Lambda_0}) \neq \{0\}.
\]

Using Proposition 4.16 and Section 2.4, every \(\mathfrak{h}\)–weight of \(D_c^\vee\) satisfies \(\succeq c\), and \(c\) occurs with multiplicity one. Recall that for any \(c' \in P\), there exists a unique simple \(\tilde{\mathfrak{b}}^-\)–submodule of \(D_c^\vee\) and it is isomorphic to \(\mathbb{C}_{c'}\). Therefore the maximality of \(c\) results in

\[
\bigoplus_{m, k \in \mathbb{C}} \text{Hom}_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (M, \text{coker}(\mathbb{C}_c \rightarrow D_c^\vee) \otimes \mathbb{C}_{m\delta + k\Lambda_0}) = \{0\}.
\]

Let \(M_1\) be the maximal quotient (a \(\tilde{\mathfrak{b}}^-\)–module) of \(M\) such that all its \(\mathfrak{h}\)–weights \(b\) satisfy \(b \succeq c\). By our maximality assumption, every simple quotient of \(M_1\) is isomorphic to \(\mathbb{C}_{c'}\). Hence, Corollary 4.21 results in a surjective \(\mathbb{C}_{\leq 0}\)–homomorphism \(\phi: (\mathbb{D}_c)^{\oplus r} \rightarrow M_1\) for some \(r \in \mathbb{Z}_{>0} \cup \{\infty\}\). We can further assume that every direct summand of \((\mathbb{D}_c)^{\oplus r}\) maps non-trivially to \(M_1\), changing \(r\) if necessary.

If a simple quotient of \(\ker \phi\) contains \(\mathbb{C}_{c'}\), then \(c' \geq c\) and

\[
\text{Ext}^1_{(\tilde{\mathfrak{b}}-\tilde{\mathfrak{h}})} (M \otimes \mathbb{C}_{m\delta + k\Lambda_0}, \mathbb{C}_{c'}) \neq \{0\} \quad \text{for some} \quad m, k \in \mathbb{Q}.
\]
Using that $\text{Ext}^{>0}$ are vanishing and the long exact sequences associated with the short exact sequences for subquotients of $\{D^\vee_b\}_{b \geq c}$.

\[
\text{Ext}_{(b-,\tilde{b})}^p(M \otimes \mathbb{C}_{m\delta+k\Lambda_0},\mathbb{C}_{c'}) = \{0\} \text{ for every } c' \geq c, p > 0, m, k \in \mathbb{Q}.
\]

We use here (5.7). This contradicts (5.8). Therefore,

\[
\text{Ext}_{(b-,\tilde{b})}^1(M,\mathbb{C}_{c'} \otimes \mathbb{C}_{m\delta+k\Lambda_0}) = \{0\} \text{ for every } c' \geq c, m, k \in \mathbb{Q},
\]

and we obtain an isomorphism $(D_{c'})^\oplus \cong M_1$. Then we replace $M$ with $\ker (M \rightarrow M_1)$ and proceed by induction; note that our condition on weights is stable under taking subquotients. This concludes the “if part” for (ii). The last assertion follows from Theorem 4.18 for both, (i) and (ii). □

Theorem 5.8 is actually a level-one version of the (non-affine) van der Kallen’s criterion [vdK]. The proof we suggest here is applicable only to level one, since we cannot generalize Corollary 4.21 to any levels.

5.2.3. The passage to $W$–modules. We will use the results above for $W$–modules from Section 5.1.4.

**Theorem 5.9.** Let $\Lambda$ be a level-one dominant integral weight. Then for any $b, c \in P_+, m \in \mathbb{Q}$ and $p > 0$,

\[
(5.9) \quad \text{Ext}_{(b-,\tilde{b})}^p(W_b \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}_{\Lambda_0+m\delta}) = \{0\}.
\]

**Proof.** Since $W_b, D_b, L(\Lambda), W_c^\vee, D_b^\vee$ are indecomposable $\tilde{b}^-$–modules, the $\mathfrak{h}$–weights of each of them coincide in $P/Q$. Therefore $\text{Hom}$ and $\text{Ext}$ vanish if the weights in the first and the second $\text{Ext}^p$–component from (5.9) have different images in $P/Q$. Thus the right-hand side there is $\{0\}$ when $b + c + \Lambda - \Lambda_0 \notin Q + \mathbb{Z}\delta$.

For any $b, c \in P_+$ and $p \geq 0$ ($p = 0$ is allowed), one has:

\[
\text{Ext}_{(b-,\tilde{b})}^p(W_b \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}_{\Lambda_0+m\delta})
\]

\[
\cong \text{Ext}_{(b-,\tilde{b})}^p(W_b \otimes \mathbb{C}_{-\Lambda_0} \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}_{m\delta}).
\]

Since $L(\Lambda)$ and $\mathbb{C}_{m\delta}$ are integrable $\tilde{g}$–modules, this gives that

\[
\text{Ext}_{(b-,\tilde{b})}^p(\mathcal{O}_i(W_b \otimes \mathbb{C}_{-\Lambda_0}) \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}_{m\delta})
\]

\[
\cong \text{Ext}_{(b-,\tilde{b})}^p(W_b \otimes \mathbb{C}_{-\Lambda_0} \otimes L(\Lambda), (\mathcal{O}_i(W_c))^\vee \otimes \mathbb{C}_{m\delta})
\]

for any $0 \leq i \leq n$ and $p \geq 0$. We used Theorem 4.9 and Theorem 4.10.
Let $\Lambda' \overset{\text{def}}{=} \Lambda_{[-b]}$ and $v^*$ be the lowest weight vector of $L(\Lambda')^\vee$. Then we have an embedding $W_b \otimes \mathbb{C}_{-\Lambda_0} \subset L(\Lambda')^\vee$ of $\mathfrak{g}_{\leq 0}$-modules, possibly up to a character twist; see Lemma 5.1. Let us use Proposition 4.16. We need a reduced decomposition $\pi_{w_0(b)} = s_{i_1} s_{i_2} \cdots s_{i_\ell} \pi$, where $i_1, \ldots, i_\ell \in [0, n]$, $\pi \in \Pi$. Then:

(5.10) $\quad W_b \otimes \mathbb{C}_{-\Lambda_0} \cong \mathcal{D}_{i_1} \circ \mathcal{D}_{i_2} \circ \cdots \circ \mathcal{D}_{i_{\ell}}(\mathbb{C} v^*)$.

Since $\mathcal{D}_{i}(\mathbb{C} v^*) \cong \mathbb{C} v^*$ when $\alpha_i^*(\Lambda') = 0$, we can replace $\pi_{w_0(b)}$ here with the maximal length representative $x \in \hat{W}$ in the double coset $W \pi_{w_0(b)} W$. The relation (5.10) will hold.

Obviously $x^{-1}$ is also a maximal length representative of the corresponding element in $W \backslash \hat{W}/W$. Thus Corollary 5.5 implies that there exists $m' \in \mathbb{Z}$ such that

(5.11) $\quad \mathcal{D}_{\pi x^{-1}}(W_c) = \mathcal{D}_{i_1} \circ \mathcal{D}_{i_{\ell-1}} \circ \cdots \circ \mathcal{D}_{i_1}(W_c) \cong W_c \otimes \mathbb{C} m' \delta$.

Let us use now that $\pi \in \Pi$ above induces an automorphism of $\tilde{\mathfrak{g}}$ preserving $\mathfrak{h}$ and $\mathfrak{h}$ and that $\pi \Lambda_0 = \Lambda_{[-b]}$. There exists $m'' \in \mathbb{Q}$ and some $d \in P$ such that

$$\text{Ext}^p_{(b^{-}, \tilde{\mathfrak{h}})}(W_b \otimes \mathbb{C}_{-\Lambda_0} \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C} m_d) \cong \text{Ext}^p_{(b^{-}, \tilde{\mathfrak{h}})}(\mathcal{D}_{x}(\mathbb{C}_{-\Lambda_0}) \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C} m_d) \cong \text{Ext}^p_{(b^{-}, \tilde{\mathfrak{h}})}(\mathbb{C}_{-\Lambda'} \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}(m-m') \delta).$$

Recall that $b, c \in P_+$, $m \in \mathbb{Q}$ are arbitrary and $m'$ is determined by (5.11).

Let us use now that $\pi \in \Pi$ above induces an automorphism of $\tilde{\mathfrak{g}}$ preserving $\mathfrak{h}$ and $\mathfrak{h}$ and that $\pi \Lambda_0 = \Lambda_{[-b]}$. There exists $m'' \in \mathbb{Q}$ and some $d \in P$ such that

$$\text{Ext}^p_{(b^{-}, \tilde{\mathfrak{h}})}(\mathbb{C}_{-\Lambda'} \otimes L(\Lambda), W_c^\vee \otimes \mathbb{C}(m-m') \delta) \cong \text{Ext}^p_{(b^{-}, \tilde{\mathfrak{h}})}(\mathbb{C}_{-\Lambda_0} \otimes L(\Lambda''), D_d^\vee \otimes \mathbb{C}_{-\Lambda_0 + m''} \delta)$$

for the standard (thin) Demazure module $D_d^\vee \subset L(\pi^{-1} \Lambda_{[-c]})$. Here $\Lambda'' \overset{\text{def}}{=} \pi^{-1} \Lambda$ is a level-one dominant weight and there is an explicit (quadratic) formula for $m''$ in terms of $\Lambda''$, initial (level-one dominant) $\Lambda$ and $\Lambda'$ above. We used the fact that the definition of Demazure modules is stable under the diagram automorphisms.
Next, we use that $D_i(L(\Lambda''')) \cong L(\Lambda''')$ for any $0 \leq i \leq n$. One has:

$$\text{Ext}^p_{(\tilde{b},\tilde{c})}(L(\Lambda'''), D_d^\vee \otimes \mathbb{C}_{m''\delta}) \cong \text{Ext}^p_{(\tilde{b},\tilde{c})}(L(\Lambda'''), D_d^\vee \otimes \mathbb{C}_{m''\delta})$$

$$\cong \text{Ext}^p_{(\tilde{b},\tilde{c})}(L(\Lambda'''), D_d^\vee \otimes \mathbb{C}_{m''\delta})$$

$$\cong \text{Ext}^p_{(\tilde{b},\tilde{c})}(L(\Lambda'''), D_d^\vee \otimes \mathbb{C}_{m''\delta})$$

$$= \text{Ext}^p_{(\tilde{b},\tilde{c})}(L(\Lambda'''), D_d^\vee \otimes \mathbb{C}_{m''\delta}).$$

Finally, the usage of Corollary 5.7 completes the proof. Here, as with the proof of the previous theorem, we omit some technical details. □

5.3. **Theta-products via $D_b$.**

5.3.1. **Theta-products via $W_c$.** The following stratification is an application of the theory above.

**Corollary 5.10.** Let $v = (\pi_1, \ldots, \pi_p)$ be a sequence of elements of $\Pi$. For each $b \in P_+$, the tensor product

$$W_b \otimes L(\pi_1 \Lambda_0) \otimes \cdots \otimes L(\pi_p \Lambda_0)$$

admits a decreasing separable filtration by $\{W_c\}_{c \in P_+}$ (as constituents), possibly with character twists. Moreover, the multiplicities in this filtration are given by formula (3.47) in Theorem 3.4 for $c = c_-$. 

**Proof.** We first prove the existence of the filtration by induction on $p$. In the case $p = 1$, we apply Theorem 5.9; the vanishing of Ext follows from Theorem 5.8. Generally, the existence of such a filtration by $\{W_c\}_{c \in P_+}$ implies the existence of the filtration by modules $\{W_c\}_{c \in P_+}$. Therefore, we can go from $p = k - 1$ to $p = k$ by taking the associated graded. This gives the existence claim.

The multiplicity claim for this filtration follows then from formula (3.47) for $c = c_-$ because the graded characters of $\{W_c\}_{c \in P_+}$ are linearly independent. □

5.3.2. **More on vanishing Ext.**

**Theorem 5.11.** Let $\Lambda_1, \ldots, \Lambda_s$ be a sequence level-one dominant integral weight. We set

$$L(\vec{\Lambda}) \overset{\text{def}}{=} L(\Lambda_1) \otimes \cdots \otimes L(\Lambda_s).$$
For any $b, c \in P$, $m \in \mathbb{Q}$ and $p > 0$, we have

\[(5.12) \quad \operatorname{Ext}^p_{(\tilde{b}, \tilde{b})} (D_b \otimes L(\tilde{\Lambda}), D_c^\vee \otimes \mathbb{C}^{(s-2)A_0 + m\delta}) = \{0\}.\]

Proof. Let $\Lambda \overset{\text{def}}{=} \sum_{j=1}^s \Lambda_j$ and $\Lambda = \omega + s\Lambda_0 \mod \mathbb{Z}d$ for proper $\omega \in P_+$. By Corollary 5.10, we know that $D_{-b} \otimes L(\tilde{\Lambda})$ admits a decreasing separable filtration by $\{W_c\}_{c \in P_+}$, where $b \in P_\geq$. Using Theorem 3.3 and formula (3.35), we conclude that $W_c$ occurs in $D_b \otimes L(\tilde{\Lambda})$ if and only if $c \in -b + \omega + Q \mod \mathbb{Z}d$.

Lemma 4.12 gives that each $W_c$ admits a filtration by $\{D_b\}$ with $b \in W(c)$ and that every such module occurs with multiplicity one. The character twists can be needed here; actually one (common) character twist serves all $b \in W(c)$.

Now let us use that the $\tilde{b}^-$-submodule

$$D_{b^+} \otimes L(\tilde{\Lambda}) \subset D_{b^-} \otimes L(\tilde{\Lambda})$$

generates the whole $D_{b^-} \otimes L(\tilde{\Lambda})$ as a $g$-module, which is Corollary 4.14. By the PBW theorem and the fact that $D_{b^+} \otimes L(\tilde{\Lambda})$ is $\tilde{b}^-$-invariant, we obtain that $D_{b^-} \otimes L(\tilde{\Lambda})$ generates $D_{b^-} \otimes L(\tilde{\Lambda})$ as a $n$-module.

Let $b' \in P_\geq$ be such that $W_{b'}$ appears in $D_{b^-} \otimes L(\tilde{\Lambda})$ for the filtration of Corollary 5.10; as always in this section, this is up to a suitable character twist. Then there exists a pair of $\tilde{g}_{\leq 0}$-submodules $M' \subset M \subset D_{b^-} \otimes L(\tilde{\Lambda})$ such that (a) $M'$ and $M$ admit compatible decreasing separable filtrations by $\{W_c\}_{c \in P_+}$, (b) $M/M' \cong W_c$, and (c) $M$ is generated by $(D_{b^+} \otimes L(\tilde{\Lambda})) \cap M$ as a $n$-module. This results in the existence of an irreducible $(g + \tilde{h})$-submodule $V \subset M$ generating $W_c \cong M/M'$ as a $\tilde{g}_{\leq 0}$-module, where every $\tilde{h}$-eigenvector of $V$ of the weight $c'$ from $W(c)$ generates $W_{c'}$ in $W_c \cong M/M'$.

Generally, a $\tilde{b}^-$-invariant subspace $U \subset W_c$ ($c \in P_\geq$) generates $W_c$ with respect to the action $n$ if it contains the largest $\tilde{h}$-submodule $U_c$ of $W_c$ that is a direct sum of some copies of $\mathbb{C}_{c^-}$. We use here (3.14).

By (the proof of) Corollary 5.4, $U_c \subset W_{c^-}$ for such $U_c$. The vector space $U_c$ contains the cyclic vector of $W_{c^-}$ and generates the whole $W_{c^-}$ considered as a $\tilde{b}^-$-module. We conclude that every $\tilde{h}$-submodule $U_c^+ \subset M$ such that $(U_c^+ + M')/M' \cong U_c$ generates $W_{c^-} \cong \mathbb{D}_{c^-}$ upon the passage to $W_c \cong M/M'$. 
Let us now use that \( M \) is generated by \( (D_{b+} \otimes L(\tilde{\Lambda})) \cap M \) as a \( n \)-module. We can choose \( U^+_c \subset (D_{b+} \otimes L(\tilde{\Lambda})) \cap M \). This implies that

\[
\left( (D_{b+} \otimes L(\tilde{\Lambda})) \cap M \right) / \left( (D_{b+} \otimes L(\tilde{\Lambda})) \cap M' \right) \supset \mathbb{D}_{c+}.
\]

As a consequence, if \( \mathcal{W}_c \) occurs in a filtration of \( D_{b-} \otimes L(\tilde{\Lambda}) \) (upon a suitable character twist) from Corollary 5.10, then \( \mathbb{D}_{c-} \) occurs in \( D_{b+} \otimes L(\tilde{\Lambda}) \) for the induced filtration of \( D_{b-} \otimes L(\tilde{\Lambda}) \) (with the same character twist). This correspondence preserves the multiplicities.

For \( s = 1 \) case, the above count of multiplicities results in the inequality:

\[
(5.13) \quad q^{b^2/2} \text{gch } D_{b+} \otimes L(\tilde{\Lambda}) \geq \sum_{c \in (-b+w+Q) \cap P} q^{(b-\cdot c-)^2/2} \cdot q^{c^2/2} \text{gch } \mathbb{D}_c,
\]

where \( q^{(b-\cdot c-)^2/2} \) is due to Theorem 5.10 and formula (3.29) for \( c = c_- \).

The equality holds here if and only if \( D_{b+} \otimes L(\tilde{\Lambda}) \) has a filtration by \( \{ \mathbb{D}_c \}_{c \in P_-} \). However we know that (5.13) is actually an equality due to (3.35), which gives the required.

This argument remains essentially unchanged for any \( s > 1 \). The left-hand side of (5.13) will then correspond to (3.47) with \( c, b \in P_- \). See Theorem 4.16, Corollary 4.20, and (3.48).

Thus \( D_{b+} \otimes L(\tilde{\Lambda}) \) has a required filtration by \( \{ \mathbb{D}_c \}_{c \in P_-} \) for every \( s > 0 \) and we can use Corollary 4.15 and Theorem 5.8 to establish that \( \mathcal{D}(D_{b+} \otimes L(\tilde{\Lambda})) \) has a filtration by \( \{ \mathbb{D}_c \}_{c \in P} \). This concludes the proof of the theorem. \( \square \)

The following application is one of the main results of this half of the paper; it finally interprets formula (3.48) representation-theoretically.

**Theorem 5.12.** Let \( b \in P \). For each sequence \( v = (\pi_1, \ldots, \pi_p) \) of elements in \( \Pi \), the tensor product

\[
D_b \otimes L(\pi_1\Lambda_0) \otimes \cdots \otimes L(\pi_p\Lambda_0)
\]

admits a decreasing separable filtration by \( \{ \mathbb{D}_c \}_{c \in P} \) with possible character twists. The multiplicities in this filtration are given by Theorem 3.4, namely by the \( r = 0 \) case of formula (3.48).

**Proof.** In view of Theorem 5.11, the existence of a filtration is a direct consequence of Theorem 5.8. For its character, use (3.48) and Corollary 4.20. \( \square \)
5.3.3. Some perspectives.

**Conjecture 5.13.** For any $c \in P$, the module $D_c$ is free over the following graded polynomial ring $R_c$, similar to that in Theorem 5.3:

$$R_c \defeq \bigotimes_{i=1}^{n} C[X_{i,1}, \ldots, X_{i,m_i}]^{\mathfrak{m}_i}, \quad \text{where} \quad m_i = (\alpha_i^\vee, c)^\mathfrak{m}, \quad \deg X_{i,j} = 1.$$  

See Section 3.4.3 for the notation $(\cdot, \cdot)^\mathfrak{m}$. Moreover, we conjecture the existence of a maximal (universal) cyclic $\tilde{\mathfrak{b}}^-\mathfrak{g}$-modules $\hat{D}_c$ such that

1. the module $D_c$ maps surjectively onto $\hat{D}_c$ (as $\tilde{\mathfrak{b}}^-\mathfrak{g}$-modules);

2. $\text{end}_{\tilde{\mathfrak{b}}^-}(D_c) \cong R_c$ and $D_c$ is free as an $R_c$-module;

3. and also, $h_c^0 \cdot \text{gch} D_c = \text{gch} \hat{D}_c$ for $h_c^0$ from (2.41).

For such $D_c$, we conjecture that $\hat{D}_c \defeq C_0 \otimes_{R_c} \hat{D}_c$ (local Demazure slices) satisfy the following orthogonality relations:

$$\dim C \text{Ext}^p_{(\tilde{\mathfrak{b}}^- \tilde{\mathfrak{h}})}(D_b \otimes \mathbb{C}_{m_0}, \hat{D}_c^\vee) = \delta_{p,0} \delta_{b,c} \delta_{m_0}.$$

Conjecture 5.13 for $\mathfrak{g}$ of type ADE were essentially proved in [FKM]. It yields an analogue of Theorem 5.8 for $\{D_b\}_b$ (instead of $\{D_b\}_b$).

Taking them into account, our “explanation” of the remaining two decompositions in Theorem 3.4 (but not the exact formulas for the coefficients there) is as follows:

**Conjecture 5.14.** Let $\Lambda$ be a level-one fundamental weight. Then for any $b, c \in P$, $m \in \mathbb{Q}$ and $p > 0$, we have:

$$\text{Ext}^p_{(\tilde{\mathfrak{b}}^- \tilde{\mathfrak{h}})}(D_b \otimes L(\Lambda), \hat{D}_c^\vee \otimes \mathbb{C}_{\Lambda_0+m_0}) = \{0\},$$

$$\text{Ext}^p_{(\tilde{\mathfrak{b}}^- \tilde{\mathfrak{h}})}(\hat{D}_b \otimes L(\Lambda), D_c^\vee \otimes \mathbb{C}_{\Lambda_0+m_0}) = \{0\}.$$

The equalities here are equivalent to each other. The first corresponds to the decomposition in (3.47), the second interprets (3.46). We follow the proof of Corollary 5.10 and use the relations:

$$\text{gch} D_b^\vee = q^{-b^2/2} w_0(\mathcal{T}_{b^*}), \quad \text{gch} \hat{D}_b = q^{b^2/2} w_0(E_{b^*}^t), \quad \text{where} \quad b \in P.$$

See Proposition 4.16, Corollary 4.20, and Conjecture 5.13.

5.3.4. Conclusion. As we discussed in the Introduction, the nonsymmetric Rogers-Ramanujan-type identifies from our paper can be seen (upon some simplifications) as partitions of the symmetric ones, so they are formally not “brand new”. However they are an important developments, since powerful DAHA tools can be used to study them,
which are non-existent in the symmetric theory. The DAHA technique of intertwiners is a major example, which is exactly the bridge between the $E, E^\dagger$–polynomials and Demazure modules.

In contrast to general $E$–polynomials (for any $q, t$), the coefficients of $E_b, E_b^\dagger$ are positive integers, which alone is very remarkable. This also holds for the coefficients of expansions of the products of theta-functions in terms of these polynomials. The interpretation of such integrality and positivity via the theory of level-one thick Demazure modules (in the twisted setting) is what we do in this paper.

The restriction to the level-one Demazure modules requires an explanation. This case is exceptional in Kac-Moody theory; the corresponding (basic) integrable representations and Demazure modules have many unique features. The characters of the level-one Demazure modules (thin and thick) have important applications in the classical finite-dimensional Lie theory; in a sense, they are “no worse” than the characters of finite-dimensional simple Lie algebras.

Our paper is mostly based on the following special feature of the Demazure level-one modules: they constitute a remarkable families of orthogonal polynomials. More exactly, we use very much the orthogonality relation between the limits $t \to 0$ and $t \to \infty$, interpreted as some homological $Ext$–duality in the (based on prior works). This is the key for our identification of the characters of Demazure slices with $E^\dagger$–polynomials divided by their norms.

The connection between the limits $t \to 0$ and $t \to \infty$, is quite non-trivial in the nonsymmetric theory vs. the symmetric theory. Algebraically (combinatorially), the $E$–polynomials ($t = 0$) can be obtained from the $E^\dagger$–polynomials ($t = \infty$), but the latter are significantly more involved than the former. The Kac-Moody interpretation shed light on this asymmetric behavior; let us touch it upon.

The level-one Kac-Moody modules are naturally unions (inductive limits) of the usual (thin) Demazure modules. On the other hand, the natural associate graded of the level-one Kac-Moody modules are direct sums of Demazure slices (quotients of thick Demazure modules). For instance, generally there are no natural embeddings between neighboring (in the Bruhat sense) Demazure slices in contrast to usual (thin) Demazure modules.

At the level of graded characters of level-one integrable modules, which are essentially theta-functions, these two fundamental relations
become as follows. The theta-function associated with a given root system is: (a) the limit of $E$-polynomials and (b) a certain Rogers-Ramanujan sum of $E^\dagger$--polynomials divided by their norms.

The nonsymmetric Rogers-Ramanujan sums are dual-purpose in our paper. First, we use them to prove the coincidence of the characters of the Demazure slices with the $E^\dagger$--polynomials divided by their norm. This was expected in [ChFB], but the representation-theoretical tools at that time were insufficient to approach this problem. Second, we obtain that the multiplicities of the Demazure slices in tensor products of level-one Kac-Moody modules can be found via the DAHA-based “numerical” machinery. Let us comment on the latter.

The existence of such a decomposition of tensor products of level-one Kac-Moody modules in terms of Demazure slices is quite a theorem, even without exact formulas for the multiplicities. The latter are involved (unless in the case of a single level-one Kac-Moody module); they are remarkable string functions in some cases. Potentially, we can extend our approach (the expansion via Demazure slices combined with the DAHA formulas) to the decomposition of any integrable Kac-Moody modules, but this is beyond this paper.

To conclude, we think that this paper (a) significantly extends the usage of nonsymmetric Macdonald polynomials in the theory of (at least) integrable representations of affine Lie algebra, (b) clarifies the fundamental (but somewhat mysterious) role of the level-one Demazure characters in this theory and beyond, (c) interprets the positivity phenomenon in the theory of non-symmetric Macdonald polynomials upon $t \to \infty$, which opens a road to their categorification and that of the corresponding $q$–Whittaker function.

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