Brane solitons of (1, 0) superconformal theories in six dimensions with hyper-multiplets

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Abstract
We solve the Killing spinor equations of six-dimensional (1, 0) superconformal theories which include hyper-multiplets in all cases. We show that the solutions preserve 1, 2, 3, 4 and 8 supersymmetries. We find models with self-dual string solitons which are smooth and supported by instantons with an arbitrary gauge group, and 3-brane solitons as expected from the M-brane intersection rules.

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1. Introduction

A consequence of AdS/CFT correspondence is that the field theory dual of M-theory on the AdS$_7 \times S^4$ background is a (2, 0) superconformal theory in six dimensions [1]. So far an action$^1$ for such a theory has not been constructed which is local and six-dimensional (6D) Lorentz covariant, though there have been suggestions [2–4] which either preserve a subset of the required symmetries or do not have a general gauge group because of the rigidity in the existence of Euclidean 3-Lie algebras [5, 6]. Within this context a (1, 0) superconformal theory was suggested in [7] and later modified in [8] to include hyper-multiplets. Although some of these models admit a local action [9] and are manifestly classically (1, 0)-supercovariant, they suffer from several pathologies which include the nonexistence of a ground state and possibly the presence of negative norm states. Nevertheless they exhibit some desirable features like classical superconformal invariance and have smooth string solitons supported by instantons [10], see also [11] for some mathematical aspects. These string solitons are in accordance

$^1$ Apart from the well-known problem with self-dual 3-forms, there may not be an action for such a theory but we shall consider superconformal theories with an action as a working hypothesis.
with the M-brane intersection rules [12, 13]. So these theories can be thought of belonging in the same universality class of theories as that which is dual to M-theory on AdS7 × S7, and possibly describe multiple M5-branes.

An exhaustive investigation of the solutions to the Killing spinor equations (KSEs) of the models described in [7] has been presented in [10]. In particular, the KSEs have been solved in all cases and the fractions of supersymmetry preserved by the solitons have been identified. Moreover a class of string solitons solutions have been found in some models which are smooth and are supported by instanton configurations.

In this paper, we shall extend the investigation of the KSEs to the models of [8] which include hyper-multiplets. The technique we use to solve the KSEs is based on spinorial geometry [14] as it has been adapted in [15] to investigate the solutions to the KSEs of 6D (1, 0) supergravity. Because of this, we shall not provide details of the calculation and the proof. Instead, we shall directly state the results for the hyperini KSEs, which are the KSEs associated with the hyper-multiplets, and refine some of the conditions that appear on the fields. The solution of the gaugini and tensorini KSEs, which are the KSEs associated to the vector and tensor multiplets, is the same as that given in [10] and so the analysis will not be repeated here. Then we shall use the solution of the gaugini and tensorini KSEs in [10] and that of the hyperini KSEs presented here to find new string and 3-brane solitons in some of the models of [8]. The new string solitons we find are supported by instantons of arbitrary gauge group, they are smooth at a generic point of the instaton moduli space and the string charge is related to the instanton number. The 3-brane solitons are supported by holomorphic maps of the hyper-scalars and they are complex curves in the hyper-multiplets target space which is a hyper-Kähler cone. The existence of such solitons are in agreement with expectations from the M-brane intersection rules.

This paper has been organized as follows. In section 2, we present the fields, couplings, field equations and Bianchi identities of (1, 0) superconformal theories with hyper-multiplets. In section 3, we give the solutions to the hyperini KSEs. In section 4, we construct new string and 3-brane solutions and in section 5, we give our conclusions.

2. (1, 0) superconformal theory and KSEs

2.1. Fields and KSEs

The (1, 0) superconformal models constructed in [7, 8] have vector, tensor and hyper-multiplets as well as appropriate higher form fields which appear in Stückelberg-type of couplings. The field content of the vector multiplets is \((A'_\mu, \lambda^r, Y^{ij r})\), where \(r\) labels the different vector multiplets and \(i, j = 1, 2\) are the \(Spt(1)\) R-symmetry indices, \(A'_\mu\) are 1-form gauge potentials, \(\lambda^r\) are symplectic Majorana–Weyl spinors and \(Y^{ij r}\) are auxiliary fields. The field content of the tensor multiplets is \((\phi^I, \chi^i I, B^I_{\mu\nu})\), where \(I\) labels the different tensor multiplets, \(\phi^I\) are scalars, \(\chi^i I\) are symplectic Majorana–Weyl spinors, of opposite chirality from those of the vector multiplets, and \(B^I_{\mu\nu}\) are the 2-form gauge potentials. The field content of the hyper-multiplets are \((q^a, \psi^a)\), where \(q^a\) are the ‘hyper-scalars’, which are maps from the spacetime to a hyper-Kähler cone\(^2\), and \(\psi^a\) are symplectic Majorana–Weyl spinors of the same chirality as \(\chi^i I\).

\(^2\) Supersymmetry requires that the hyper-scalars take values on a hyper-Kähler manifold. In addition superconformal symmetry requires that the hyper-Kähler manifold admits a homothetic motion associated with a potential making the hyper-Kähler manifold locally a hyper-Kähler cone.
The field strengths of the 1- and 2-form gauge potentials associated with the vector and tensor multiplets are
\[
{\mathcal{F}}_{\mu\nu}^{\tau} \equiv 2\partial_{[\mu}A_{\nu]}^{\tau} - f_{\mu\nu}^{\tau}A_{\rho}^{\tau} + h_{\mu\nu}^{\tau}B^{\tau},
\]
\[
{\mathcal{H}}_{\mu\nu\rho}^{\tau} \equiv 3D_{[\mu}B_{\nu\rho]}^{\tau} + 6d_{\mu}^{\tau}A_{[\nu}^{\tau}A_{\rho]}^{\tau} - 2f_{\mu\nu\rho}^{\tau}d_{\rho}^{\tau}A_{[\mu}^{\tau}A_{\nu]}^{\tau} + g^{\tau}C_{\mu\nu\rho},
\]
respectively, where \( f_{\mu\nu}^{\tau}, h_{\mu\nu}^{\tau}, g^{\tau} \) and \( d_{\mu}^{\tau} = d_{\mu}^{\tau} \) are coupling constants, and \( C_{\mu\nu\rho} \) are 3-form gauge potentials introduced via a St"uckelberg-type of coupling. In addition,
\[
D_{\mu}A^{\tau} \equiv \partial_{\mu}A^{\tau} + A^{\tau}_{\nu}(X_{\nu})^{\tau}, \quad D_{\mu}A^{\tau} \equiv \partial_{\mu}A^{\tau} + A^{\tau}_{\nu}(X_{\nu})^{\tau}A^{\tau},
\]
where \( X_{\nu} \) are given by
\[
(X_{\nu})^{\tau} = -f_{\nu}^{\tau} + d_{\nu}^{\tau}h_{\nu}^{\tau}, \quad (X_{\nu})^{\tau} = 2h_{\nu}^{\tau}d_{\nu}^{\tau} - g^{\tau}b_{\nu}^{\tau}.
\]
The various coupling satisfy a long list of restrictions required by gauge invariance and for these models to have an action which is given in [7], see also [10] for a summary, and it will not be repeated here. In particular, these models are described by an action provided there is a maximally split signature metric\(^3\) \( \eta_{IJ} \) such that
\[
g^{\tau} = \eta^{\tau\tau}h_{\tau\tau}, \quad d_{\nu}^{\tau} = \frac{1}{2}\eta^{\nu\tau}b_{\nu}^{\tau}.
\]
From now on, the indices \( I, J \) are raised and lowered with \( \eta \).

To couple hyper-multiplets to the above system [8], one assumes that the hyper-K"ahler cone, which is the target space of hyper-multiplet scalars, admits tri-holomorphic isometries generated by the vector fields \( X_{\nu}^{(m)} = X_{\nu}^{(m)} \partial_{\nu} \). Typically only some of the vector multiplets will be gauged. For this introduce the embedding tensor \( \theta_{m}^{\mu} \) and define
\[
A^{m} = A^{\mu}\theta_{m}^{\mu}, \quad \lambda^{m} = \lambda^{\nu}\theta_{m}^{\nu}, \quad Y_{ij}^{m} = Y_{ij}^{\alpha}\theta_{m}^{\alpha},
\]
where for consistency with the gauge transformations
\[
h_{\nu}^{\mu}\theta_{m}^{\nu} = 0, \quad f_{\nu}^{\tau}\theta_{m}^{\nu} = \theta_{m}^{\nu}\theta_{\nu}^{\rho}f_{\nu}^{\mu},
\]
and where \( [X_{\alpha}, X_{\beta}] = -f_{\alpha\beta}X_{\gamma}(m) \). The KSEs of the model, which are the vanishing conditions for the supersymmetry transformations of the fermions evaluated at the locus where all fermions vanish, are
\[
\delta \lambda^{\nu} = \frac{1}{2}C_{\nu}^{\mu\tau}Y^{\mu\nu}Y^{\nu} - \frac{1}{2}Y^{\nu}Y^{\nu}Y^{\nu} + \frac{1}{4}h_{\nu}^{\mu}h_{\nu}^{\nu} = 0,
\]
\[
\delta \lambda^{\nu} = \frac{1}{2}C_{\nu}^{\mu\tau}Y^{\mu\nu}Y^{\nu} + \frac{1}{2}D_{\mu}Y^{\mu\nu}Y^{\nu} = 0,
\]
\[
\delta Y_{ij}^{\alpha} = \frac{1}{2}D_{\mu}Y^{\mu\nu}Y^{\nu} = 0,
\]
where
\[
D_{\mu}Y^{\mu\nu} = \partial_{\mu}Y^{\mu\nu} - A^{m}X^{\alpha}_{\nu}(m).
\]
In addition, \( E_{\alpha}^{\mu} \) is the symplectic frame of the hyper-K"ahler cone, i.e., the hyper-K"ahler metric and hypercomplex structure are given as
\[
g_{\alpha\beta} = \epsilon_{ij}\epsilon_{ab}E_{\alpha}^{ia}E_{\beta}^{aj}, \quad (E_{\alpha}^{ia})_{\beta} = -i(\epsilon_{\gamma})_{\beta}E_{\alpha}^{ia}E_{\beta}^{ii},
\]
where \( \epsilon_{ij} \) and \( \epsilon_{ab} \) are the symplectic (fundamental) forms of \( Sp(1) \) and \( Sp(n) \), respectively, and \( \epsilon_{\gamma} \) are the Pauli matrices. In analogy with similar variations in 6D (1, 0) supergravity, we refer to these KSEs as the gaugini, tensorini and hyperini KSEs, respectively.

The Lagrangian for these theories consist of two parts. One part, \( \mathcal{L}_{VT} \), involves the vector and tensor multiplets, and the second part, \( \mathcal{L}_{H} \), contains the hyper-multiplets. These two parts are independently supersymmetric and the supersymmetry transformation of the vector multiplets used in the coupling of the hyper-multiplets in \( \mathcal{L}_{H} \) is obtained by contraction with the embedding tensor.

\(^3\) Since the metric is maximally split, the kinetic energy of some of the fields is negative which may lead to ghosts in the spectrum. This is an issue affecting this class of theories.
2.2. Field equations

The field equations of the system are

\[ D_\mu D^\mu \phi_I = - \frac{1}{2} d_{Irs} \left( F^r_{\mu \nu} F^{\mu \nu s} - 4 Y_{ij}^{(s)} Y^{ij} \right) - 3 d_{Irs} h^r_k \phi^l_k \phi^s_l, \]

\[ b_{Irs} Y^s_{ij} \phi^l_J = \frac{1}{2} b_{Irs} \epsilon_{\mu \nu \lambda \rho \sigma} \mathcal{H}_I^{(4) \lambda \rho \sigma}, \]

\[ g_{\alpha \beta} \nabla_\mu D^\mu q^\alpha_I = - Y_{ij} \delta_\alpha \mu^{ij}, \] (2.11)

where

\[ \lambda \text{ is a constant, and } \mu^{(m)} \tau, \]

\[ X^\beta_{(m)}(\alpha_t) \beta_\alpha = - \partial_\alpha \mu^{(m)} \tau, \]

\[ (\omega_\tau)_{\alpha \beta} = g_{\alpha \gamma} (\omega \tau)^{\gamma}_{\beta}, \] (2.13)

are the moment maps. Observe that generically the theory has a cubic scalar field interaction and so the potential term is not bounded from below. These field equations are also supplemented with the Bianchi identities

\[ D_\mu [F^\nu_{\mu \nu}] = \frac{1}{4} h^r_k \mathcal{H}_I^{(4) k}, \]

\[ D_\mu [\mathcal{H}^{(4)}_{\mu \nu \sigma \tau}] = \frac{1}{2} d_{r s} [F^r_{\mu \nu}] F^{s \rho \sigma \tau} + \frac{1}{8} \mathcal{H}^{(4) \mu \nu \sigma \tau}, \]

\[ D_\mu [\mathcal{H}^{(5)}_{\mu \nu \rho \sigma \tau}] = - 4 d_{Irs} f^{r s} \mathcal{H}^{(4)}_{\mu \nu \rho \sigma \tau} + \frac{1}{2} \partial_\mu \mathcal{H}^{(5)}_{\mu \nu \rho \sigma \tau} \]

(2.14)

where \( \mathcal{H}^{(4)}_{\mu \nu \rho \sigma \tau} \) is the field strength of the 3-form, and the duality relations

\[ \frac{1}{5!} \epsilon_{\mu \nu \rho \sigma \tau} \theta^m \mathcal{H}^{(5)}_{\mu \nu \rho \sigma \tau} = (X^r)_{ij} \phi^l D_\mu \phi^l + \frac{2}{\lambda} \theta^m X_{(m)} A^\mu q^\alpha, \] (2.15)

i.e., the 5-form field strength is dual to the hyper-scalars.

3. Solution hyperini KSEs

3.1. Spinorial geometry

The inclusion of the hyper-multiplets does not alter the form of the first two KSEs in (2.8).

Because of this, the solution of these equations described in [10], using the spinorial geometry technique of [14], still applies and the results will not be repeated here.

It remains to solve the hyperini KSEs in (2.8). This has a similar structure to that of the hyperini KSEs which has been solved in the context of 6D (1, 0) supergravity, see section 3 in [15]. Because of this we shall not give details of the derivation. Instead, we shall state the result of the spinorial geometry calculation and then re-express it in a way which allows for an improved interpretation of the conditions that arise. This is instrumental in the description of 3-brane solitons in some of the (1, 0) superconformal models in section 4.

The Killing spinors and their isotropy groups in Spin(5, 1) · Sp(1) that arise in the solution of all KSEs in (2.8) are summarized in table 1. The isotropy groups are either compact or non-compact and we shall use this to distinguish solutions with the same supersymmetry. Note that solutions with \( N = 3 \) supersymmetry arise only when hyper-multiplets are present.

A detailed analysis of how the Killing spinors in table 1 are selected is given in [15]. Here we shall use this selection and substitute it in the hyperini KSEs to find the conditions
Table 1. The first column gives the number of invariant spinors, the second column the associated isotropy groups and the third column representatives of the invariant spinors. The isotropy group of more than four linearly independent spinors is the identity.

| N | Isotropy groups | Killing spinors |
|---|---|---|
| 1 | $Sp(1) \cdot Sp(1) \rtimes \mathbb{H}$ | $1 + \epsilon_{1234}$ |
| 2 | $(U(1) \cdot Sp(1)) \rtimes \mathbb{H}$ | $1 + \epsilon_{1234}, i(1 - \epsilon_{1234})$ |
| 3 | $Sp(1) \rtimes \mathbb{H}$ | $1 + \epsilon_{1234}, i(1 - \epsilon_{1234}), \epsilon_1 - \epsilon_3$ |
| 4 | $Sp(1)$ | $1 + \epsilon_{1234}, i(1 - \epsilon_{1234}), \epsilon_1 - \epsilon_3, i(\epsilon_1 + \epsilon_3)$ |
| 2 | $Sp(1)$ | $1 + \epsilon_{1234}, \epsilon_1 + \epsilon_3$ |
| 4 | $U(1)$ | $1 + \epsilon_{1234}, i(1 - \epsilon_{1234}), \epsilon_1 + \epsilon_3 \epsilon_4, i(\epsilon_1 + \epsilon_3 \epsilon_4)$ |

that arise on the fields. For this note that in the context of spinorial geometry the lowering of the $Sp(1)$ indices on the supersymmetry parameter $\epsilon$ that appears in the hyperini KSEs is implemented by

$$\epsilon_1 = -\epsilon^2, \quad \epsilon_2 = \Gamma_{14}\epsilon^1,$$

(3.1)

where $(\epsilon^1, e^2, \epsilon^2)$ are the two Weyl spinors that we used to construct the symplectic-Majorana representation, and the gamma matrices $\Gamma_3, \Gamma_4$ are along two auxiliary directions which arise because we have identified the 6D symplectic-Majorana spinors with the $SU(2)$ invariant spinors of the positive chirality Majorana–Weyl representation of Spin(9, 1). For the spinor notation we use and other details see [10, 15].

3.2. $N = 1$

The Killing spinor can be chosen as $\epsilon = 1 + \epsilon_{1234}$ which is invariant under the subgroup $Sp(1) \cdot Sp(1) \rtimes \mathbb{H}$ of Spin(5, 1) \cdot Sp(1). In this case, $(\epsilon^1, e^2, \epsilon^2) = (1, x_{1234})$, and using the relations in (3.1) we have $(\epsilon_1) = (\epsilon_1, \epsilon_2) = (-\epsilon_{1234}, \epsilon_{34})$. Substituting these into the hyperini KSEs, we find

$$D_+ q^a E^{ia}_a = 0, \quad -D_1 q^a E^{1a}_a + D_2 q^a E^{2a}_a = 0, \quad D_3 q^a E^{3a}_a + D_4 q^a E^{4a}_a = 0,$$

(3.2)

where in the evaluation of the above conditions the 6D spacetime decomposes into two light-cone directions and four transverse directions which are written in terms of complex coordinates, e.g., the Minkowski spacetime metric is written as

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = 2 dx^+ dx^- + \delta_{mn} dx^m dx^n = 2(dx^+ dx^- + dx^1 dx^1 + dx^2 dx^2).$$

(3.3)

This choice of coordinates on the 6D spacetime arises naturally in the solution of the KSEs as it can be seen in (3.2), and later in (3.7) and (3.9), and the rest of the conditions that arise from the hyperini KSEs, as these are naturally expressed using complex structures on the transverse to the light-cone directions. The first condition simplifies to

$$D_+ q^a = 0,$$

(3.4)

i.e., the hyper-scalars are covariantly constant along one of the light-cone directions and so in the gauge $A_m^a = 0$ do not depend on $x^+$. The remaining two conditions in (3.2) can be written in a more covariant form as

$$(\tau^m)_{\rho}^a D_{m\rho} q^a E^{ia}_a = 0,$$

(3.5)

where $(\tau^m) = (i\sigma_r, 1_{2\times2}), m = 1, 2, 3, 4$ and $\sigma_r$ are the Pauli matrices, or equivalently in a coordinate basis as

$$(\tau^m)_{\rho}^a D_{m\rho} q^a = 0,$$

(3.6)

where $(\tau^m) = (I_r, 1_{4\times4r})$. 


3.3. $N = 2$ non-compact

From table 1, the Killing spinors can be chosen as $1 + \epsilon_{1234}$ and $i(1 - \epsilon_{1234})$. Using these in the hyperini KSEs, we find the following conditions

$$D_\alpha q^\alpha = 0, \quad D_1 q^\alpha E_1^\alpha = D_2 q^\alpha E_2^\alpha = D_3 q^\alpha E_3^\alpha = 0, \quad D_4 q^\alpha E_4^\alpha = D_5 q^\alpha E_5^\alpha = 0, \quad D_6 q^\alpha E_6^\alpha = D_7 q^\alpha E_7^\alpha = D_8 q^\alpha E_8^\alpha = 0. \quad (3.7)$$

The last two conditions in the above equation can be rewritten as a Cauchy–Riemann (CR) type of equation

$$D_m q^\alpha (I_3)^{ia}_{jb} E_i^{jb} = J^n m D_n q^\alpha E_i^{ia}, \quad (3.8)$$

where $(I_3)^{ia}_{jb} = (-i\sigma_3)^i_j \delta^a_b$ and $J^n_m = (i\delta^s_t, -i\delta^t_s)$. Therefore in the absence of gauge fields, the above condition becomes the CR equation and $q$ is a holomorphic map from the transverse space to the light-cone to the hyper-Kähler cone with respect to the pair of complex structures $\{J, I_3\}$.

3.4. $N = 2$ compact

The two Killing spinors can be chosen as $1 + \epsilon_{1234}$ and $\epsilon_{14} + \epsilon_{2345}$. The conditions imposed by the hyperini KSEs evaluated on $\epsilon_{15} + \epsilon_{2345}$ can be written

$$D_\alpha q^\alpha = 0, \quad -D_2 q^\alpha E_1^\alpha + D_4 q^\alpha E_2^\alpha = 0, \quad D_1 q^\alpha E_1^\alpha + D_5 q^\alpha E_5^\alpha = 0. \quad (3.9)$$

Combining these conditions with those associated with $1 + \epsilon_{1234}$, we find that the hyper-scalars satisfy

$$D_\alpha q^\alpha = 0, \quad \tau = -, +, 1, \quad (3.10)$$

and

$$D_\tau q^\alpha = -\epsilon_{\tau' \eta} (K_\tau)^\eta_{\rho} D_\rho q^\rho, \quad \tau', \eta = 2, 3, 4, \quad (3.11)$$

where we have made a $(3+3)$ (real) split of the spacetime and $(K_\tau)^{ia}_{jb} = -i(\sigma_{\tau-1})_j^i \delta^a_b$ with $\epsilon_{34} = 1$. Therefore the hyper-scalars are covariantly constant along the first three directions of the spacetime and obey (3.11) along the other three.

3.5. $N = 3$ non-compact

The three Killing spinors are $1 + \epsilon_{1234}, i(1 - \epsilon_{1234})$ and $\epsilon_{12} - \epsilon_{34}$. The conditions imposed by the hyperini KSEs on the hyper-scalars are

$$D_\alpha q^\alpha = 0, \quad D_1 q^\alpha E_1^\alpha = D_2 q^\alpha E_2^\alpha = D_3 q^\alpha E_3^\alpha = 0, \quad D_4 q^\alpha E_4^\alpha = D_5 q^\alpha E_5^\alpha = D_6 q^\alpha E_6^\alpha = 0, \quad (3.12)$$

As in the previous non-compact cases, the hyper-scalars are covariantly constant along one of the light-cone directions. The remaining conditions can be written as

$$(J_\tau)^{\nu}_{\rho} D_\nu q^\rho = (I_3)^{\nu}_{\rho} D_\rho q^\rho, \quad \tau = 1, 2, 3, \quad (3.13)$$

for an appropriate choice of a hypercomplex structure $J_\tau$ in the directions transverse to the light cone with $J_3 = J$. Therefore in the absence of gauge couplings, the hyper-scalars are quaternionic maps from the directions transverse to the light-cone to the hyper-Kähler cone.

4. The choice of complex structure on the hyper-Kähler cone depend on the choice of representatives for the Killing spinors. For a generic choice, the complex structure $I_3$ should be replaced with a linear combination of all three complex structures.

5. Clearly, the directions transverse to the light-cone can be identified with the quaternions $\mathbb{H}$. If the Obata curvature vanishes, then it is possible to introduce quaternionic coordinates on the hyper-Kähler cone. In such a case $q^\alpha$'s can be written as quaternions $q$ and (3.13) implies that $q = q(x, x^-), x \in \mathbb{H}$.
3.6. \( N = 4 \) non-compact

The four Killing spinors can be chosen as \( 1 + \epsilon_{1234}, i(1 - \epsilon_{1234}), \epsilon_{12} - \epsilon_{34} \) and \( i(\epsilon_{12} + \epsilon_{34}) \). The only non-vanishing component of the hyper-scalars is \( D_{-} q^{\alpha} \), i.e. in the absence of gauge fields the hyper-scalars depend only on the light-cone direction \( x^- \).

3.7. \( N = 4 \) compact

The four Killing spinors can be chosen as \( 1 + \epsilon_{1234}, i(1 - \epsilon_{1234}), \epsilon_{15} + \epsilon_{2345} \) and \( i(\epsilon_{15} - \epsilon_{2345}) \). The conditions imposed on the hyper-scalars from the hyperini KSEs are

\[
D_{r} q^{\alpha} = 0, \quad r = -, +, 1, \bar{1},
\]
\[
D_{2} q^{\alpha} E_{a}^{1} = D_{2} q^{\alpha} E_{a}^{2} = 0. \tag{3.14}
\]

Therefore there is a \((4+2)\) (real) split of the spacetime and the hyper-scalars are covariantly constant along the first four directions. In the remaining two directions, the hyper-scalars satisfy a CR type of equation

\[
J_{r} D_{r} q^{\alpha} = (I_{3})^{\alpha}_{\beta} D_{r} q^{\beta}, \tag{3.15}
\]

where \( J_{r} = (i\delta^{22}, -i\delta^{\bar{2}2}) \).

3.8. Maximal supersymmetry

All solutions of the hyperini KSEs with more than four Killing spinors are maximally supersymmetric, i.e., they preserve all eight supersymmetries. In addition, the hyperini KSEs imply that for maximally supersymmetric backgrounds the hyper-scalars are covariantly constants, i.e.,

\[
D_{\mu} q^{\alpha} = 0. \tag{3.16}
\]

This concludes the description of solutions of the hyperini KSEs.

4. Brane solitons

4.1. Self-dual string solitons

4.1.1. A class of models. A large class of models has been constructed in [7, 8] by considering a Lie algebra \( \mathfrak{g} \) and a representation \( \mathcal{R} \). The bosonic fields of the vector and tensor multiplets are chosen as

\[
A^{\prime} = (A^{m}, A^{A}), \quad Y^{\prime} = (Y^{m}, Y^{A}), \quad B^{\prime} = (B^{A}, B_{A}), \quad \phi^{\prime} = (\phi^{A}, \phi_{A}). \tag{4.1}
\]

i.e., \( A \) and \( Y \) take values in \( \mathfrak{g} \oplus \mathcal{R} \) while \( B \) and \( \phi \) take values in \( \mathcal{R} \oplus \mathcal{R}^{*} \). Moreover the non-vanishing couplings are chosen as

\[
\eta^{A}_{B} = \eta^{B}_{A} = \delta^{A}_{B}, \quad h^{B}_{A} = s^{B}_{A} = \delta^{B}_{A}, \quad f_{ma}^{B} = -\frac{1}{2} (T_{m})^{B}_{A}, \quad f_{mn}^{B},
\]
\[
d^{B}_{mA} = \frac{1}{2} b^{B}_{Am}, \quad \frac{1}{2} b^{B}_{ma} = \frac{1}{2} (T_{m})^{B}_{A}, \quad d_{AB} = d_{(ABC)} = b_{BCA},
\]
\[
d_{ABm} = d_{(AB)m} = \frac{1}{2} b_{ABm} = \frac{1}{2} b_{AmB}, \quad d_{ABm}, \quad b_{A(mn)} = 2d_{A(mn)}, \quad \theta_{m}^{n} = \delta_{m}^{n}. \tag{4.2}
\]

where \( T_{m} \) are the representation matrices of \( \mathfrak{g} \) in \( \mathcal{R} \). These solve all the constraints on the couplings imposed on these models provided that \( d_{mAB}, d_{mA} \) and \( d_{ABC} \) are invariant under the action of \( \mathfrak{g} \).
4.1.2. Self-dual string solitons from instantons. Motivated from the M-brane intersection rules, we shall seek self-dual string solitons in the class of models described in the previous section which preserve 1/2 of the supersymmetry. The relevant class of supersymmetric backgrounds for self-dual string solitons are those with four Killing spinors that have isotropy group $Sp(1) \ltimes H$ in table 1. The conditions on the field of the vector and tensor multiplets are given in [10] and in section (3.6) for the hyper-scalars. Similar solutions have been found in [10] for another class of models. The string soliton on a single M5-brane has been found in [16] and it is singular at the position of the brane.

To solve the supersymmetry conditions, Bianchi identities and field equations, suppose that the fields have support on 4-directions transverse to the light-cone coordinates $(x^+, x^-)$ which are smooth and the string charge is related to the instanton number, see also [18] for more details. In addition for generic values of instanton moduli space, all the string solutions are smooth and the string charge is related to the instanton number, see also [18] for more details.

In addition for generic values of instanton moduli space, all the string solutions are smooth and the string charge is related to the instanton number, see also [18] for more details.
4.2. 3-branes

Motivated from the M-brane intersection rules which state that two M5-branes intersect on a 3-brane, we shall describe a class of models which exhibit 3-brane solitons. These are those for which all the potentials vanish and the only active fields are those of the hyper-multiplets. Moreover, the hyper-multiplet scalars depend only on the two transverse directions of the 3-brane soliton. First to identify the models with 3-brane solitons suppose that the hyper-multiplets are not gauged, i.e., the embedding tensor $\theta = 0$. Moreover set all the fields apart from the hyper-multiplet scalars $q$ and $H^{(5)}$ equal to zero. The only non-trivial conditions that have to be satisfied to construct solutions are the field equations for $q$ and the hyperini KSEs.

To solve the hyperini KSEs, we shall take the case with four supersymmetries and compact isotropy group. The relevant equations are given in (3.14) and (3.15). From the KSEs, the hyper-multiplet scalars do not depend on four directions as expected for a 3-brane soliton and (3.15) is a CR equations implying that $q$ is a holomorphic curve into the hyper-Kähler cone. In addition, the field equation for the $q$’s is automatically satisfied.

A similar argument based on the supersymmetry conditions in section 3.3 leads to the existence of a string soliton preserving 1/4 of supersymmetry supported by a holomorphic surface embedded into the hyper-Kähler cone. Such solitons are expected to exist as they can be associated with a triple M5-brane intersection on a string. Similarly, the ‘quaternionic’ solitons of section 3.5 can be associated with M-brane intersections at angles.

5. Concluding remarks

Combining the results of [10] with those of this paper, we have solved the KSEs of the (1, 0) superconformal models of [7, 8] in all cases and identified the fractions of supersymmetry preserved. The theories have a large number of solitons. Here we have presented new self-dual string solutions and 3-brane solutions which are expected to exist because of the M-brane intersection rules. The former supported by instantons with arbitrary gauge group, they are smooth at a generic point of the instanton moduli space and the string charge is related to the instanton number. The latter are holomorphic curves from the six-dimensional spacetime into the hyper-Kähler cone of the hyper-multiplets.

Although it may appear that the techniques we have used to solve the KSEs apply to a particular class of models, this is not the case. In fact, the same method can be applied to solve the KSEs of any (1, 0) supersymmetric theory in six dimensions as it has been demonstrated in [15].

It is clear that there are many more supersymmetric solutions like the string solitons mentioned in the previous section associated with triple M5-brane intersections that preserve 1/4 of supersymmetry. This can be seen either by directly inspecting the solutions of the KSEs we have presented or by continuing to explore the analogy with the M-brane intersection rules. The KSEs that we have solved are the most general ones for models with (1, 0) supersymmetry in six dimensions. So, there is now a systematic way to find all solitons of these superconformal theories and to explore their applications.

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