Moment Conditions for Convergence of Particle Filters with Unbounded Importance Weights

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Abstract

In this paper, we derive moment conditions for particle filter importance weights, which ensure the mean square and almost sure convergence of particle filter estimates even when the importance weights are unbounded. The result extends the previously derived conditions by not requiring the boundedness of weights, but only finite second or fourth order moments. We show that the boundedness of the second order moments of the weights implies the convergence of the estimates bounded functions in the mean square sense, and the $L^4$ convergence as well as the almost sure convergence are assured by the boundedness of the fourth order moments of the weights. We also present an example class of models and importance distributions where the moment conditions hold, but the boundedness does not. The unboundedness in these models is caused by isolated singularities in the weights which still leave the weight moments bounded. We show by using simulated data that the particle filter for this kind of model also performs well in practice.

Index Terms

Particle filter, convergence, unbounded importance weights, moment condition.

I. INTRODUCTION

Particle filters are sequential Monte Carlo based methods for numerical solving Bayesian filtering problems by forming a weighted set of Monte Carlo samples $\{(x_i^t, \bar{w}_i^t) : i = 1, \ldots, N\}$ (see, e.g., [1], [2]) and by approximating the filtering probability measure as a linear combination of delta measures located at the particles $x_i^t$ with the weights given by $\bar{w}_i^t$.
In probabilistic form, the Bayesian estimation problem can be expressed as state inference in a state space model

\[ x_t \sim f_t(x_t \mid x_{t-1}), \]
\[ y_t \sim g_t(y_t \mid x_t), \]

where \( x_t \in \mathbb{R}^n \) is the state of the system, \( y_t \in \mathbb{R}^m \) is the measurement, \( f_t(x_t \mid x_{t-1}) \) is the transition probability density modeling the dynamics of the system, and \( g_t(y_t \mid x_t) \) is the conditional probability density of measurements modeling the distribution of measurements. In applications, the densities are usually with respect to the Lebesgue measure or counting measure, but other reference measures are possible as well.

An important feature of any particle filter algorithm is that it should converge to the correct distribution as the number of particles tends to infinity \( N \to \infty \). This property of particle filters is well studied and there exists a number of convergence results for particle filters (see, e.g., [1], [3]–[16] and references therein). However, the effect of importance distribution on the convergence is less studied and it is typical either to assume that the dynamic model is used as the importance distribution, leading to so called bootstrap filter, or that the unnormalized importance weights are point-wise bounded. Although in central limit theorem type analysis of particle filters this point-wise boundedness is not always assumed (see, e.g., [7]), it is a standard assumption in \( L^p \)-type analysis of particle filters [1], [4], [6]. In this paper, we extend the proofs for the mean-square-convergence and empirical measures convergence in [4] to the case of potentially unbounded importance weights and thus enlarge the class of state space models in which particle filters are ensured to converge.

II. PARTICLE FILTERING

Particle filters are related to the Bayesian filtering problem, which refers to the construction of the filtering probability density function \( p(x_t \mid y_{1:t}) \). The construction of \( p(x_t \mid y_{1:t}) \) is done recursively by Bayesian filtering equations (see, e.g., [2]). Let \( \phi \) be a bounded test function, \( \pi_{t|t-1} \) the measure corresponding to the probability density \( p(x_t \mid y_{1:t-1}) \), and \( \pi_{t|t} \) the measure corresponding to the density \( p(x_t \mid y_{1:t}) \). Then the Bayesian filtering equations can be written as

\[ (\pi_{t|t-1}, \phi) = (\pi_{t-1|t-1}, f_t \phi), \]
\[ (\pi_{t|t}, \phi) = \frac{(\pi_{t|t-1}, \phi g_t)}{(\pi_{t|t-1}, g_t)}, \]
where \((\rho, \phi) \triangleq \int \phi \, d\rho\), and \(f(\rho | x) \triangleq \int f(z | x) \phi(z) \, d\nu_f(z)\), where \(\nu_f\) is the reference measure used for the density \(f\). In this article, we also denote \(\rho_t \triangleq \pi_{t|t-1}\), and define an importance distribution for particle filters as \(\tilde{\rho}_t \triangleq \int \pi_{t-1|t-1}(dx_{t-1}) \Xi_t(y_t, x_{t-1}, \cdot)\), where \(\Xi_t\) is a transition kernel (cf. [4]).

Due to intractability of (2), for most state space models, we usually need to approximate them. A particle filter for approximating the solutions of (2) is given in Algorithm 1. The convergence results for particle filters, found in [4], are given for the posterior distribution of the whole trajectory, with assumption that the importance weights are bounded. In this paper we do not consider the whole trajectory, but instead we provide the convergence results for the filtering posterior distribution of the current state. For simplicity, we do not consider the Markov chain Monte Carlo (MCMC) step as in [4], because it is not often used in practical particle filters. Our contribution is to prove the convergence of particle filters such that we do not restrict the importance weights to be bounded.

Algorithm 1  Particle filter

- At \(t = 0\), for \(i = 1, \ldots, N\), sample \(x_{0}^{(i)} \sim \pi_{0|0}(dx_{0})\)
- At \(t \geq 1\),
  1) For \(i = 1, \ldots, N\), sample \(\tilde{x}_{t}^{(i)} \sim \Xi_t(y_t, x_{t-1}^{(i)}, dx_t)\).
  2) Evaluate the normalized importance weights \(\tilde{w}_t^{(i)}\) for \(i = 1, \ldots, N\) as
    \[
    \tilde{w}_t^{(i)} \propto g_t\left(y_t, \tilde{x}_t^{(i)}\right) h_t\left(y_t, \tilde{x}_t^{(i)}\right),
    \]
    where \(h_t(y_t, \cdot) = d\rho_t/d\tilde{\rho}_t\). Define
    \[
    \tilde{\rho}_t^N(dx_t) \triangleq \frac{1}{N} \sum_{i=1}^{N} \delta_{\tilde{x}_t^{(i)}}(dx_t)
    \]
    and
    \[
    \tilde{\pi}_t^N(dx_t) \triangleq \frac{1}{N} \sum_{i=1}^{N} \tilde{w}_t^{(i)} \delta_{\tilde{x}_t^{(i)}}(dx_t).
    \]
  3) Do resampling to obtain new samples \(x_t^{(i)}\) for \(i = 1, \ldots, N\) and define \(\pi_t^N(dx_t)\) as the associated empirical measure.
  4) \(t \leftarrow t + 1\)

III. CONVERGENCE OF MEAN SQUARE ERROR

In this section, the convergence of particle filters is analyzed by computing the bound for the mean square error. This is done by proving the following main convergence theorem which shows that the mean square error introduced by sampling is of the order \(1/N\).
Theorem III.1. For all \( t \geq 0 \), there exist a constant \( c_t \) such that, for any bounded function \( \phi \)

\[
E \left[ \left| \left( \pi_t^N, \phi \right) - \left( \pi_t, \phi \right) \right|^2 \right] \leq c_t \frac{\|\phi\|_2^2}{N},
\]

(4)

where \( \|\phi\| \triangleq \sup_{x \in \mathbb{R}^n} |\phi(x)| \).

In order to establish a similar bound, Crisan and Doucet [4], among others, used the following assumptions.

Assumption III.2. The measurement model \( g_t \) is bounded.

Assumption III.3. The resampling procedure satisfies (see, e.g., [4], [6] for the sufficient conditions for this):

\[
E \left[ \left| \left( \pi_t^N, \phi \right) - \left( \bar{\pi}_t^N, \phi \right) \right|^2 \right] \leq C_t \frac{\|\phi\|_2^2}{N}.
\]

(5)

In previous convergence proofs an additional assumptions has been that the unnormalized importance weights are point-wise bounded. However, here, to cope with potentially unbounded importance weights, we use the aforementioned assumptions together with the following assumption.

Assumption III.4. Let \( w_t(x_t, x_{t-1}) = g_t(y_t, x_t) h_t(y_t, x_t) \) be the unnormalized importance weight function, then \( E[(w_t(x_t, x_{t-1}))^2 | x_{t-1}] \leq C_w < \infty \), with the expectation taken over \( \Xi_t \).

Using Assumptions III.2, III.3 and III.4 we thus prove Theorem III.1 for the case where the importance weights are not necessarily (point-wise) bounded. In the following we use the notation \( \pi_t \triangleq \pi_{\{t\}} \).

Additionally, \( F_{t-1} \) denotes the \( \sigma \)-field generated by the particles \( \{x_{t-1}^{(i)}\}_{i=1}^N \).

Proof. At the initial step, \( t = 0 \), because the \( N \)-particles from the prior distribution \( (\pi_0^N) \) are assumed to be independent and identically distributed variables, we have

\[
E \left[ \left| \left( \pi_0^N, \phi \right) - \left( \pi_0, \phi \right) \right|^2 \right] \leq c_0 \frac{\|\phi\|_2^2}{N}.
\]

(6)

For \( t \geq 1 \), the result follows by proving Lemmas III.5 and III.6 below. \( \square \)

Lemma III.5. Assume that for any bounded function \( \phi \):

\[
E \left[ \left| \left( \pi_{t-1}^N, \phi \right) - \left( \pi_{t-1}, \phi \right) \right|^2 \right] \leq c_{t-1} \frac{\|\phi\|_2^2}{N}.
\]

(7)

Then after steps 1 and 2 of Algorithm [7] we have

\[
E \left[ \left| \left( \pi_t^N, \phi \right) - \left( \pi_t, \phi \right) \right|^2 \right] \leq \tilde{c}_t \frac{\|\phi\|_2^2}{N}.
\]

(8)
Proof. We have

$$E \left[ (\tilde{\rho}_t^N, \phi) \mid F_{t-1} \right] = (\pi_{t-1}^N, f \phi g),$$

(9)

and we know that

$$(\tilde{\pi}_t^N, \phi) - (\pi_t, \phi) = \left( \frac{\tilde{\rho}_t^N, \phi g h_t}{\tilde{\rho}_t^N, g h_t} \right) - \left( \frac{\tilde{\rho}_t, \phi g h_t}{\tilde{\rho}_t, g h_t} \right).$$

(10)

Let $w_t = g_t h_t$, then

$$(\bar{\pi}_t^N, \phi) - (\pi_t, \phi) = (\bar{\pi}_t^N, \phi) \left( \tilde{\rho}_t, w_t \right) - \left( \tilde{\rho}_t, w_t \right)$$

(11)

Consider $(\tilde{\rho}_t^N, \phi w_t) - (\tilde{\rho}_t, \phi w_t) = \Pi_1 + \Pi_2$, where

$$\Pi_1 = (\tilde{\rho}_t^N, \phi w_t) - E[(\tilde{\rho}_t^N, \phi w_t)],$$

(12)

$$\Pi_2 = E[(\tilde{\rho}_t^N, \phi w_t)] - (\tilde{\rho}_t, \phi w_t).$$

(13)

We need to evaluate $E[|\Pi_1|^2]$ and $E[|\Pi_2|^2]$. Using the boundedness of $\phi$ and Assumption III.4, we get

$$E \left[ |\Pi_1|^2 \mid F_{t-1} \right] = E \left[ (\tilde{\rho}_t^N, \phi w_t)^2 \mid F_{t-1} \right] - (\pi_{t-1}^N, \Xi_t \phi w_t)^2$$

$$\leq \frac{1}{N} E \left[ \frac{1}{N} \left( \sum_{i=1}^{N} \phi(x_t(i)) w_t(x_t(i), x_{t-1}(i)) \right)^2 \mid F_{t-1} \right]$$

$$\leq \frac{\|\phi\|^2}{N^2} \sum_{i=1}^{N} E \left[ (w_t(x_t(i), x_{t-1}(i)))^2 \mid F_{t-1} \right] \leq C_w \frac{\|\phi\|^2}{N}. \quad (14)$$

For the second part, using (7), we get

$$E \left[ |\Pi_2|^2 \mid F_{t-1} \right]$$

$$= E \left[ \left| (\pi_{t-1}, f \phi g) - (\pi_{t-1}, f \phi g) \right|^2 \mid F_{t-1} \right]$$

$$\leq c_{t-1} \frac{\|g\|^2 \|\phi\|^2}{N} = \tilde{c}_1 \frac{\|\phi\|^2}{N}. \quad (15)$$

Using Minkowski inequality, we combine (14) and (15) to get

$$E \left[ \left| (\tilde{\rho}_t^N, \phi w_t) - (\tilde{\rho}_t, \phi w_t) \right|^2 \right] \leq \tilde{c}_2 \frac{\|\phi\|^2}{N}. \quad (16)$$
which with $\phi = 1$, then implies $E\left[ \left| \left( \tilde{\rho}_t^N, w_t^N \right) - \left( \tilde{\rho}_t, w_t \right) \right|^2 \right] \leq \bar{c}_2^2/N$. Using these results and by introducing the Minkowski inequality to (11) we get
\[
\left( E\left[ \left| \left( \pi_t^N, \phi \right) - \left( \pi_t, \phi \right) \right|^2 \right] \right)^{1/2} \leq \left( \frac{\left( \pi_t^N, \phi \right)}{(\tilde{\rho}_t, w_t)} \right) \left( \bar{c}_2 \frac{1}{N} \right)^{1/2} + \left( \frac{1}{(\tilde{\rho}_t, w_t)} \right) \left( \bar{c}_2 \frac{\|\phi\|^2}{N} \right)^{1/2} \leq \left( \frac{\|\phi\|}{(\tilde{\rho}_t, w_t)} \right) \sqrt{\bar{c}_2} \frac{1}{N^{1/2}} + \left( \frac{1}{(\tilde{\rho}_t, w_t)} \right) \sqrt{\bar{c}_2} \frac{\|\phi\|}{N^{1/2}} = \sqrt{\bar{c}_2} \frac{\|\phi\|}{N^{1/2}},
\]
which completes the proof of Lemma III.5.

Lemma III.6. Assume that Lemma III.5 holds. Then
\[
E\left[ \left| \left( \pi_t^N, \phi \right) - \left( \pi_t, \phi \right) \right|^2 \right] \leq c_t \frac{\|\phi\|^2}{N},
\]
(17)

Proof. We split $(\pi_t^N, \phi) - (\pi_t, \phi)$ as
\[
(\pi_t^N, \phi) - (\pi_t, \phi) = (\pi_t^N, \phi) - (\pi_t^N, \phi) + (\pi_t^N, \phi) - (\pi_t, \phi).
\]
Then, using Assumption III.3 and Lemma III.5 we have
\[
\left( E\left[ \left| \left( \pi_t^N, \phi \right) - \left( \pi_t, \phi \right) \right|^2 \right] \right)^{1/2} \leq \left( E\left[ \left| \left( \pi_t^N, \phi \right) - (\pi_t^N, \phi) \right|^2 \right] \right)^{1/2} + \left( E\left[ \left| (\pi_t^N, \phi) - (\pi_t, \phi) \right|^2 \right] \right)^{1/2} \leq \sqrt{C_t} \frac{\|\phi\|}{N^{1/2}} + \sqrt{\bar{c}_2} \frac{\|\phi\|}{N^{1/2}} = \sqrt{\bar{c}_2} \frac{\|\phi\|}{N^{1/2}},
\]
where the result follows.

IV. CONVERGENCE OF EMPIRICAL MEASURES

In this section, we show the convergence of the empirical measure, which means that almost surely, $\lim_{N \to \infty} \pi_t^N = \pi_t$. To guarantee the convergence of the empirical measure, we impose the following assumptions.

Assumption IV.1. The measurement model $g_t$ is bounded.

Assumption IV.2. The resampling procedure satisfies the condition (cf. [4], [6]:
\[
E\left[ \left| \left( \pi_t^N, \phi \right) - (\pi_t^N, \phi) \right|^4 \right] \leq C_1 \frac{\|\phi\|^4}{N^2},
\]
(18)
Assumption IV.3. \( E[(w_t(x_t, x_{t-1}))^4 \mid x_{t-1}] < \infty \).

Given the assumptions, we get the following theorem.

**Theorem IV.4.** For all \( t \geq 0 \) we have, almost surely,

\[
\lim_{N \to \infty} \pi^N_t = \pi_t.
\]

**Proof.** The result follows by using the Markov inequality and Borel-Cantelli argument provided that we can show that for all \( t \geq 0 \), for some constants \( c_t \), we have

\[
E\left[\left| (\pi^N_t, \phi) - (\pi_t, \phi) \right|^4 \right] \leq c_t \frac{\|\phi\|^4}{N^2}.
\]  

(19)

Certainly, this is true for \( t = 0 \) and the cases for \( t \geq 1 \) result follows from Lemmas IV.5 and IV.6 below.

**Lemma IV.5.** Assume that we have

\[
E\left[\left| (\pi^N_{t-1}, \phi) - (\pi_{t-1}, \phi) \right|^4 \right] \leq c_{t-1} \frac{\|\phi\|^4}{N^2}.
\]  

(20)

Then, after steps 1 and 2 of Algorithm I we have

\[
E\left[\left| (\pi^N_t, \phi) - (\pi_t, \phi) \right|^4 \right] \leq \tilde{c} \frac{\|\phi\|^4}{N^2}.
\]  

(21)

**Proof.** We have defined \( (\pi^N_t, \phi) - (\pi_t, \phi) \) in Equation (11). From the equation, consider

\[
(\tilde{\rho}^N_t, \phi w_t) - (\bar{\rho}_t, \phi w_t) = \left[ (\tilde{\rho}^N_t, \phi w_t) - E[(\tilde{\rho}^N_t, \phi w_t) \mid F_{t-1}] \right]
\]

\[
+ \left[ E[(\tilde{\rho}^N_t, \phi w_t) \mid F_{t-1}] - (\bar{\rho}_t, \phi w_t) \right].
\]  

(22)

Using Lemmas 7.1 and 7.2 from [11], we get

\[
E\left[\left| (\tilde{\rho}^N_t, \phi w_t) - E[(\tilde{\rho}^N_t, \phi w_t) \mid F_{t-1}] \right|^4 \mid F_{t-1}\right]
\]

\[
\leq \frac{16\|\phi\|^4}{N^4} \left( \sum_{i=1}^{N} E\left[|w_t(x_t^{(i)}, x_{t-1}^{(i)})|^4 \mid F_{t-1}\right] \right)
\]

\[
+ \frac{16\|\phi\|^4}{N^4} \left( \sum_{i=1}^{N} E\left[|w_t(x_t^{(i)}, x_{t-1}^{(i)})|^2 \mid F_{t-1}\right] \right)^2
\]

\[
\leq \frac{16\|\phi\|^4}{N^3} \left( \sum_{i=1}^{N} C^4_w + \left( \sum_{i=1}^{N} C^2_w \right)^2 \right) = \bar{c}_1 \frac{\|\phi\|^4}{N^2}.
\]  

(23)
Proceeding as in (15) we get that
\[ E \left[ \left| \mathbb{E}[\hat{\rho}_t^N, \phi w_t] - (\hat{\rho}_t, \phi w_t) \right|^4 \right] \leq \tilde{c}_2 \frac{\|\phi\|^4}{N^2}. \] (24)

The result follows by combining (23) and (24) with Minkowski’s inequality.

**Lemma IV.6.** Assume that Lemma IV.5 holds. Then
\[ E \left[ \left| \left( \pi_t^N, \phi \right) - (\pi_t, \phi) \right|^4 \right] \leq c_t \frac{\|\phi\|^4}{N^2}. \] (25)

**Proof.** Analogous to the proof of Lemma III.6.

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**V. Analytical and Numerical Example**

Assume that we have a Cox process, where the a priori dynamics of the state can be modeled as a reflected Brownian motion \( x(\tau) \triangleq q^{1/2} |B(\tau)| \), where \( B(\tau) \) is a standard Brownian motion, and the measurements are Poisson distributed with an intensity parameter \( \lambda(\tau) = cx(\tau) \), where \( c > 0 \) is a constant, and the measurements are obtained at discrete times \( t \in \{1, 2, 3, 4, \ldots \} \). The model can now be formulated as a discrete-time model for the measurement times:
\[
\begin{align*}
  f(x_t | x_{t-1}) &= \frac{1}{\sqrt{2\pi q}} \left[ e^{-\frac{(x_t - x_{t-1})^2}{2q}} + e^{-\frac{(x_t + x_{t-1})^2}{2q}} \right], \\
  g(y_t | x_t) &= \frac{(c x_t)^{y_t} \exp(-c x_t)}{y_t!},
\end{align*}
\]
where \( f(x_t | x_{t-1}) \) is a density with respect to the Lebesgue measure and \( g(y_t | x_t) \) with respect to the counting measure.

Let us now select a Gamma distribution with constant parameters \( \alpha, \beta > 0 \) as the importance distribution for a particle filter. Thus the importance sampling density (w.r.t. the Lebesgue measure) is
\[ \pi(x_t) = \frac{\beta^\alpha}{\Gamma(\alpha)} x_t^{\alpha-1} \exp(-\beta x_t). \] (26)

At some point of time we eventually reach a zero measurement \( y_t = 0 \). Let us assume that \( \alpha > 1 \). What now happens, is that the importance weight \( w(x_t, x_{t-1}) \) becomes unbounded at origin, because \( \pi(0) = 0 \), but the numerator is nonzero. Thus according to the classical result for particle filters [1], [4], [6] the particle filter is not guaranteed to converge in mean square or empirical measure.

Let us now select some \( \alpha \in (1, \infty) \setminus \mathbb{Z} \) and \( \beta \in (0, 4c/3) \). It is now easy to show that even when \( y_t = 0 \), we have for \( p = 2, 4 \):
\[ \int_0^\infty (w(x_t, x_{t-1}))^p \pi(x_t) \, dx_t < \infty \] (27)
and hence according to the present theory the particle filter converges in mean squared sense and in empirical measure.

Because this model is single-dimensional, we can use numerical integration (naive Riemann sum in this case) to approximate the filtering solution in a dense grid. The result of applying the grid filter to a simulated process with $c = 1/2$, $q = 1/10$, $x_0 = |\xi|$, where $\xi$ is unit Gaussian, is shown in Figure 1 on the left. The right hand side of Figure 1 shows the result of a particle filter with 10000 particles and with the importance distribution parameters $\alpha = 1.5$ and $\beta = 0.5$. For visualization the number of particles is reduced to 100 per time step. As can be seen, the result is well in line with the grid based
result. Figure 2 shows the filtering distribution approximations at step \( t = 11 \), where \( y_{11} = 0 \) and hence the importance weight is unbounded. The particle filter result is well in line with the grid based result despite the unboundedness of the weight.

VI. Conclusion and Discussion

In this paper, we have derived moment conditions for importance weights of particle filters, which ensure the mean square and empirical convergence of the particle filter even in the case when the importance weights are not point-wise bounded. We have also provided an example of a model and a particle filter for which the present theory guarantees the mean square and almost sure convergences although the previously developed particle filter theory does not. We have also shown by a numerical simulation that the particle filter performs well in practice.

The numerical example showed an example situation when the weight moments can be bounded when the weights are not point-wise bounded. This happened, because the weight was unbounded only at an isolated point (origin in this case). We can conclude that the same phenomenon is possible whenever there are isolated singularities in the weights caused by, for example, isolated nulls in the importance distribution. The present theory covers this kind of cases whereas the classical theory does not. Another advantage of the present moment conditions is that when the importance distribution is constructed indirectly (as in, e.g., [17]), the weight moment condition can be easier to check than the point-wise boundedness.

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