Wasserstein Distributionally Robust Estimation in High Dimensions: Performance Analysis and Optimal Hyperparameter Tuning

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Abstract

Wasserstein distributionally robust optimization has recently emerged as a powerful framework for robust estimation, enjoying good out-of-sample performance guarantees, well-understood regularization effects, and computationally tractable reformulations. In such framework, the estimator is obtained by minimizing the worst-case expected loss over all probability distributions which are close, in a Wasserstein sense, to the empirical distribution. In this paper, we propose a Wasserstein distributionally robust estimation framework to estimate an unknown parameter from noisy linear measurements, and we focus on the task of analyzing the squared error performance of such estimators. Our study is carried out in the modern high-dimensional proportional regime, where both the ambient dimension and the number of samples go to infinity at a proportional rate which encodes the under/over-parametrization of the problem. Under an isotropic Gaussian features assumption, we show that the squared error can be recovered as the solution of a convex-concave optimization problem which, surprisingly, involves at most four scalar variables. Importantly, the precise quantification of the squared error allows to accurately and efficiently compare different ambiguity radii and to understand the effect of the under/over-parametrization on the estimation error. We conclude the paper with a list of exciting research directions enabled by our results.

1 Introduction

We consider the problem of estimating an unknown parameter \( \theta_0 \in \Theta \subseteq \mathbb{R}^d \) which describes the relationship between two random variables \( X \) and \( Y \) through the noisy linear model \( Y = \theta_0^\top X + Z \). Here, \( (X,Y) \) is distributed according to an unknown probability distribution \( \mathbb{P} \) that is only indirectly observable through a set of \( n \) independent training samples \( \{(x_i,y_i)\}_{i=1}^n \), and \( Z \) represents the measurement noise distributed according to an unknown distribution \( \mathbb{P}_Z \). A popular solution to this problem is then to approximate \( \mathbb{P} \) using the \( n \) i.i.d. samples as \( \mathbb{P} \approx \hat{\mathbb{P}}_n = \frac{1}{n} \sum_{i=1}^n \delta_{(x_i,y_i)} \), leading to an estimate of the unknown \( \theta_0 \) via the following empirical risk minimization problem

\[
\min_{\theta \in \Theta} \mathbb{E}_{(X,Y) \sim \hat{\mathbb{P}}_n} \left[ L(Y - \theta^\top X) \right] = \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n L(y_i - \theta^\top x_i),
\]

for some loss function \( L : \mathbb{R} \to \mathbb{R}_{\geq 0} \). However, \( \hat{\mathbb{P}}_n \) invariably differs from \( \mathbb{P} \), and optimizing in view of \( \hat{\mathbb{P}}_n \) instead of \( \mathbb{P} \) may lead to estimators that display a poor performance on test data. For example,
it is well known that standard machine learning models trained in view of the empirical distribution \(\hat{P}_n\) can be easily fooled by adversarial examples, that is, test samples subject to seemingly negligible noise that cause these models to make a wrong prediction. Even worse, the decision problem at hand could suffer from a distribution shift, that is, the training data may originate from a distribution other than \(P\), under which estimators are evaluated.

A promising strategy to mitigate these issues would be to minimize the worst-case expected loss with respect to all distributions in some neighborhood of \(\hat{P}_n\). There is ample evidence that, for many natural choices of the neighborhood of \(\hat{P}_n\), this distributionally robust approach leads to tractable optimization models and provides a simple means to derive powerful generalization bounds [80, 67, 23, 123, 37, 46, 38, 103]. Specialized distributionally robust approaches may even enable generalization in the face of domain shifts [42, 126, 68, 37] or may make the training of deep neural networks more resilient against adversarial attacks [106, 128, 122, 65].

Motivated by this, in this paper we propose a distributionally robust approach to estimate the unknown parameter \(\theta_0\), which comes in the form of a minimax optimization problem where the maximization robustifies against perturbations (in the probability space) around the empirical distribution \(\hat{P}_n\). These perturbations are captured through an ambiguity set \(B_\varepsilon(\hat{P}_n)\), that is, an \(\varepsilon\)-neighborhood of \(\hat{P}_n\) with respect to an optimal transport discrepancy on the probability space \(\mathcal{P}(\mathbb{R}^{d+1})\). Then, the distributionally robust estimation (DRE) problem of interest becomes

\[
\min_{\theta \in \Theta} \sup_{Q \in B_\varepsilon(\hat{P}_n)} \mathbb{E}_{(X,Y) \sim Q} [L(Y - \theta^\top X)], \tag{1}
\]

Problem (1) can be viewed as a zero-sum game between a statistician, who chooses the estimator \(\theta\), and a fictitious adversary, often envisioned as ‘nature’, who chooses the distribution \(Q\) of \((X,Y)\). Throughout the paper, we denote an optimal solution of (1) by \(\hat{\theta}_{DRE}\). From now on, we define nature’s feasible set \(B_\varepsilon(\hat{P}_n) = \{Q \in \mathcal{P}(\mathbb{R}^{d+1}) : W_p(Q, \hat{P}_n) \leq \varepsilon\}\) using the optimal transport discrepancy

\[
W_p(Q, \hat{P}_n) = \inf_{\gamma \in \Gamma(Q, \hat{P}_n)} \mathbb{E}_{((X_1, Y_1), (X_2, Y_2)) \sim \gamma} [\|X_1 - X_2\|^p + \infty |Y_1 - Y_2|], \tag{2}
\]

where \(\|\cdot\|\) represents the standard Euclidean norm, and \(\Gamma(Q, \hat{P}_n)\) represents the set of all joint probability distributions of \((X_1, Y_1)\) and \((X_2, Y_2)\) with marginals \(Q\) and \(\hat{P}_n\), respectively. In particular, we will concentrate on the cases \(p = 1, 2\), which are the most important in applications. To avoid confusion, in what follows we will refer to \(W_1\) as the type-1 Wasserstein discrepancy and to \(W_2\) as the type-2 Wasserstein discrepancy. Notice that the underlying transportation cost \(\|x_1 - x_2\|^p + \infty |y_1 - y_2|\) on \(\mathbb{R}^{d+1}\) assigns infinite cost when \(y_1 \neq y_2\), restricting the distributional robustness to the input distribution, as will be later shown in Lemma 3.3. This covers the non-stationary environment known as covariate shift, which has been recently attracting significant attention in machine learning [17, 110].

This transportation cost has been shown to recover some very popular regularized estimators such as square-root LASSO (using \(W_2\)), as well as regularized logistic regression and support vector machines (using \(W_1\)) [102, 22]. However, for general loss functions, the distributional robustness introduces only an implicit regularization, without having, in general, an equivalent explicit regularized formulation; see [103] for more details on the implicit regularization effect and when it reduces to explicit regularization.

In the past few years, considerable effort has been put into understanding the statistical guarantees and the regularization effect of problem (1). We summarize the major developments in what follows.

- **Statistical guarantees.** Based on the decay rate of the ambiguity radius \(\varepsilon\) (with the number of training samples \(n\)), many guarantees have been established. First, if \(\varepsilon\) is chosen in the
order of $n^{-p/\max\{2,d\}}$, the concentration inequality proposed in [45, Theorem 2] shows that the underlying data-generating distribution $\mathbb{P}$ is contained in the ambiguity set $\mathbb{B}_\varepsilon(\hat{\mathbb{P}}_n)$ with high probability. This has been exploited in [80] to conclude that, for such choice of radius, for any $\theta$, the Wasserstein robust loss is an upper bound on the true loss with high probability. Secondly, through an asymptotic analysis, the works [23, 20] show that if $\varepsilon$ is chosen in the order of $n^{-p/2}$, then the Wasserstein ball contains at least one distribution (not necessarily equal to $\mathbb{P}$) for which there exists an optimal solution which is equal to $\arg\min_{\theta \in \Theta} \mathbb{E}(X,Y) \sim \mathbb{P} \left[L(Y - \theta^\top X)\right]$, with high probability. This result is valid asymptotically, namely, as $n \to \infty$, while keeping $d$ fixed. Thirdly, [46] shows that a non-asymptotic $n^{-p/2}$ rate for $\varepsilon$ guarantees that, up to a high-order residual, the true loss is upper bounded by the Wasserstein robust loss, uniformly for all $\theta$. Similarly, the analysis carried out in this paper will be in the setting where $\varepsilon$ satisfies the $n^{-p/2}$ decrease rate, i.e., $\varepsilon = \varepsilon_0/(np^{p/2})$, for some constant $\varepsilon_0$. Finally, statistical guarantees have also been proposed in [106, 68, 86] for the setting where the ambiguity radius remains fixed, i.e., it does not vary with the sample size.

- **Variation regularization effect.** For type-1 Wasserstein ambiguity sets and convex loss functions, [80] shows that the distributional robustness is equivalent to a Lipschitz norm regularization of the empirical loss. More generally, for the type-$p$ Wasserstein case, [47] shows that the inner supremum in the distributionally robust optimization problem (1) is closely related (and asymptotically equivalent, as $\varepsilon \to 0$, at the rate $n^{-p/2}$) to a variation regularized estimation problem, where the variation boils down to the Wasserstein ball contains at least one distribution (not necessarily equal to $\mathbb{P}$) for which there exists an optimal solution which is equal to $\arg\min_{\theta \in \Theta} \mathbb{E}(X,Y) \sim \mathbb{P} \left[L(Y - \theta^\top X)\right]$, with high probability. This result is valid asymptotically, namely, as $n \to \infty$, while keeping $d$ fixed. Thirdly, [46] shows that a non-asymptotic $n^{-p/2}$ rate for $\varepsilon$ guarantees that, up to a high-order residual, the true loss is upper bounded by the Wasserstein robust loss, uniformly for all $\theta$. Similarly, the analysis carried out in this paper will be in the setting where $\varepsilon$ satisfies the $n^{-p/2}$ decrease rate, i.e., $\varepsilon = \varepsilon_0/(np^{p/2})$, for some constant $\varepsilon_0$. Finally, statistical guarantees have also been proposed in [106, 68, 86] for the setting where the ambiguity radius remains fixed, i.e., it does not vary with the sample size.

Differently from the existing literature, the focus of this paper is on studying the performance of the distributionally robust estimator $\hat{\theta}_{DRE}$ in the high-dimensional regime, where $d$ is of the same order as (or possibly even larger than) $n$. Driven by the wide empirical success of deep neural networks, the high-dimensional regime has gained tremendous attention in the recent years [14, 4, 76, 7, 15]. In this regime, classical statistical asymptotic theory [57], where the sample size $n$ is taken to infinity, while the dimension $d$ is kept fixed, often fails to provide useful predictions, and standard methods generally break down [127]. For instance, it is known that for $d$ fixed or $d/n \to 0$ fast enough, least squares is optimal for Gaussian errors, whereas least absolute deviations is optimal for double-exponential errors. The work [13] shows that this is no longer true in the high-dimensional regime, with the answer depending, in general, on the under/over-parametrization parameter $\rho$, as well as the form of the error distribution. Another example is shown in the work [111] where, in the context of logistic regression, the authors show that the maximum-likelihood estimate is biased, and that its variability is far greater.

\footnote{The term $p$ in all the decrease rates appears since we did not employ the $1/p$ exponent in the definition of the optimal transport discrepancy (2).}
than classically estimated. Such observations, as well as many others, have sparked a lot of interest in the modern area of high-dimensional statistics, leading to numerous works trying to unravel and solve these surprising phenomena. Many of these works have focused on linear models, i.e., $y_i = \theta_0^T x_i + z_i$, for $i = 1, \ldots, n$, under the assumption of isotropic Gaussian features $x_i \sim \mathcal{N}(0, d^{-1}I_d)$, and have studied different aspects of the problem in the proportional asymptotic regime $d, n \to \infty$, with $d/n = \rho$; see [31, 40, 120] and the references therein.

This setting has been shown to be both theoretically rich and practically powerful in capturing many interesting phenomena observed in more complex models, as detailed in what follows. A direct connection between linear models and neural networks has been established in many works [58, 35, 5, 27, 52, 10]. One important example is constituted by the double descent behavior of neural networks [15], which has been theoretically analyzed through the lenses of linear regression [3, 52, 16, 9, 84, 76]. To the best of our knowledge, the idea of using Gaussian features can be traced back to the work [32] which studied the phase-transition of $\ell_1$-minimization in the compressed sensing problem. Remarkably, compared to other assumptions on the features, which allow the characterization of the error performance only up to loose constants [25], the isotropic Gaussian features assumption allowed the authors to obtain an asymptotically precise upper bound on the minimum number of measurements which guarantee, with probability 1, the exact recovery of a structured signal from noiseless linear measurements. Interestingly, for many problems, the i.i.d. Gaussian ensembles are known to enjoy a universality property, extending the validity of the results to broader classes of probability ensembles [12, 90, 120, 33, 1, 92, 55, 81].

In this paper, we undertake this setting and focus on studying the performance of the distributionally robust estimator $\hat{\theta}_{DRE}$ in the high-dimensional asymptotics, where $d, n \to \infty$, with $d/n = \rho \in (0, \infty)$. Practically, $\rho$ encodes the under-parametrization or the over-parametrization of the problem, corresponding to $\rho \in (0, 1]$ and $\rho \in (1, \infty)$, respectively. The structural constraints on $\theta_0$ are encoded in our analysis by assuming that $\theta_0$ is sampled from a probability distribution $\mathbb{P}_{\theta_0}$. We quantify the performance of $\hat{\theta}_{DRE}$ through the normalized squared error $\|\hat{\theta}_{DRE} - \theta_0\|^2/d$. Notice that the normalization $d^{-1}$ is necessary when $d \to \infty$, since an arbitrarily small error in each entry of the vector $\hat{\theta}_{DRE} - \theta_0$ will result in an infinite $\|\hat{\theta}_{DRE} - \theta_0\|$.

In the spirit of the works described above, the analysis throughout this paper will be done under the following assumptions, which are relatively standard in the high-dimensional statistics literature.

**Assumption 1.1.**

(i) *Isotropic Gaussian features*: the covariate vectors $x_i$, $i \in \{1, \ldots, n\}$, are i.i.d. $\mathcal{N}(0, d^{-1}I_d)$.

(ii) The true parameter $\theta_0$, the measurement noise $Z$, and the feature vectors $x_i$, $i \in \{1, \ldots, n\}$ are independent random variables.

(iii) The dimension of the problem and the number of measurements go to infinity at a fixed ratio, i.e., $d, n \to \infty$, with $d/n = \rho \in (0, \infty)$.

As highlighted previously, many results derived for Gaussian ensembles have been rigorously proven and/or empirically observed to enjoy a universality property, in the sense that they continue to hold true for a broad family of probability ensembles. In Section 5, we will validate numerically that the results proposed in this paper also demonstrate such universality. As a consequence, Assumption 1.1(i) introduces a convenient theoretical limitation, which has been shown in many cases to not affect the
practical validity of the results in more general scenarios. Moreover, our distributionally robust optimization formulation (1) naturally captures all the covariate probability distributions which are “close” to $\mathcal{N}(0, d^{-1}I_d)$, including more general probability ensembles, as well as features whose entries are correlated. Finally, Assumption 1.1(iii) is the standard setting in high-dimensional statistics (compared to the classical setting, where $n \to \infty$, and $d/n \to 0$).

These assumptions allow us to use the Convex Gaussian Minimax Theorem (CGMT), which is an extension of a classical Gaussian comparison inequality, that under convexity assumptions is shown in [119] to become tight. The CGMT will be one of the main technical ingredients in a sequence of steps, at the end of which we will show that the normalized squared error $\|\hat{\theta}_{DRE} - \theta_0\|^2/d$ converges in probability to a non-trivial deterministic limit (despite both $\hat{\theta}_{DRE}$ and $\theta_0$ being stochastic), which we recover as the solution of a convex-concave optimization problem which, surprisingly, involves at most four scalar variables. Both the under/over-parametrization parameter $\rho$ and the ambiguity radius $\varepsilon$ appear explicitly in the objective function of this optimization, while the loss function $L$ and the noise distribution $P_Z$ will appear implicitly through an expected Moreau envelope. Our results build upon the framework proposed in [120], which focused on convex-regularized estimation problems. In this paper, we extend their methodology and results to Wasserstein distributionally robust estimation problems.

The main contributions of this paper can be summarized as follows.

1) **Asymptotic error for type-1 Wasserstein DRE.** In the high-dimensional proportional regime, where both the ambient dimension and the number of samples go to infinity at a proportional rate which encodes the under/over-parametrization of the problem, i.e., $d, n \to \infty$, with $d/n = \rho \in (0, \infty)$, we show that the normalized squared error $\|\hat{\theta}_{DRE} - \theta_0\|^2/d$ converges in probability to a deterministic value which can be recovered from the optimal solution of a convex-concave minimax optimization problem that involves four scalar variables.

2) **Asymptotic error for type-2 Wasserstein DRE.** In the high-dimensional proportional regime, the normalized squared error $\|\hat{\theta}_{DRE} - \theta_0\|^2/d$ converges in probability to a deterministic value which can be recovered from the optimal solutions of two convex-concave minimax optimization problems that involve at most four scalar variables. Interestingly, the analysis reveals that in order to precisely recover this error, an upper bound on the over-parametrization of the problem $d/n$ should hold. We then adopt a distributionally regularized relaxation of the DRE problem, and show that the normalized squared error can be recovered from the minimizer of only one (simpler) convex-concave minimax problem, which involves only three scalar variables.

3) **Hyperparameter tuning in high-dimensional DRE problems.** We show how the precise quantification of the squared error opens the path to accurate comparisons between different ambiguity radii $\varepsilon$, and to providing an answer to the following fundamental questions: “What is the minimum estimation error achievable by $\hat{\theta}_{DRO}$?”, “How to optimally choose $\varepsilon$ in order to achieve the minimum estimation error?”, “How does the under/over-parametrization parameter $\rho$ affect the estimation error?”.

Aside from the contribution to the distributionally robust optimization community, we hope that this paper can bring the DRE formulation (1) to the attention of the high-dimensional statistics community, whose focus has been so far centered primarily around convex-regularized estimation problems. In particular, this paper complements and contributes to the rapidly expanding research stream which studies the precise high-dimensional asymptotics of estimators, and which has so far produced many
results in the context of phase transitions in $\ell_1$-minimization [34, 108, 109] and convex optimization problems [6], LASSO [11, 79] and generalized LASSO [91, 89, 118, 54], least squares [52] and convex-regularized least squares [26], estimation [31, 70] and convex-regularized estimation [63, 40, 116, 120], generalized linear estimation [48, 41], binary models [64, 115, 28, 82, 77], specifically SVM [56, 62] and logistic regression [24, 74, 98, 99], to name a few.

**Paper organization.** The paper is organized as follows. In Section 2 we introduce the asymptotic CGMT, and discuss how it will be employed in this paper. Then, in Section 3 we study the distributionally robust estimation problem, and prove that it can be brought into the form required by the CGMT. Subsequently, in Section 4 we present the main results of this paper, which quantify the distributionally robust estimation error in high-dimensions. All the proof are deferred to the appendix. Finally, in Section 5, we numerically validate the theoretical findings, and in Section 6 we discuss some exciting research directions enabled by the results in this paper.

**Notation.** Throughout the paper, $\| \cdot \|$ will denote the standard Euclidean norm. For a convex function $f : \mathbb{R} \to \mathbb{R}$, we denote by $\partial f(x)$ the subdifferential of $f$ at $x$, and by $f'_+(x)$ the quantity $\sup_{t \in \partial f(x)} |t|$. Moreover, we denote by $e_f : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}$ its Moreau envelope, defined as $e_f(x, \tau) = \min_{\xi \in \mathbb{R}} |x - \xi|^2/(2\tau) + f(\xi)$, and by Lip$(f)$ its Lipschitz constant. The expectation of a random variable $\zeta \sim \mathbb{P}$ are denoted by $\mathbb{E}_{\zeta \sim \mathbb{P}} [\zeta]$, and the probability of an event is denoted by $\Pr(\cdot)$. Given two probability distributions $\mathbb{P}$, $\mathbb{Q}$, we denote by $\mathbb{P} \otimes \mathbb{Q}$ their product distribution. Projection maps are denoted by $\pi$.

Finally, we use the notation $X_n \overset{D}{\to} c$ to denote that the sequence of random variables $\{X_n\}_{n \in \mathbb{N}}$ converges in probability to the constant $c$, i.e., for any $\epsilon > 0$, $\lim_{n \to \infty} \Pr(|X_n - c| > \epsilon) = 0$. In this case we say that $|X_n - c| \leq \epsilon$ holds with probability approaching 1 (henceforth abbreviated as w.p.a. 1).

## 2 Convex Gaussian Minimax Theorem

The key technical instrument employed in the high-dimensional analysis, where both the number of measurements and the dimension of the problem go to infinity, is the asymptotic version of the Convex Gaussian Minimax Theorem (CGMT), which we will present in Fact 2.1. A self-contained exposition of this result is presented in what follows.

Consider the general problem of analyzing the following primary optimization (PO) problem

$$\Phi(A) := \min_{w \in \mathcal{S}_w} \max_{u \in \mathcal{S}_u} u^\top Aw + \psi(w, u),$$

where $A$ is a random matrix, with entries i.i.d. standard normal $\mathcal{N}(0, 1)$, $\psi : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, and $\mathcal{S}_w \subset \mathbb{R}^d$, $\mathcal{S}_u \subset \mathbb{R}^n$ are compact sets. We denote by $w_\Phi(A)$ an optimal solution of (3).

Analyzing the (PO) problem is in general very challenging, due to the bilinear term which includes the random matrix $A$. The CGMT associates to the (PO) problem a simpler auxiliary optimization (AO) problem, from which we can infer properties of the optimal cost and solution of the original (PO) problem. The (AO) problem is a “decoupled” version of the (PO) problem, where the bilinear term $u^\top Aw$ is replaced by the two terms $\|w\|g^\top u$ and $\|u\|h^\top w$, as follows

$$\phi(g, h) := \min_{w \in \mathcal{S}_w} \max_{u \in \mathcal{S}_u} \|w\|g^\top u + \|u\|h^\top w + \psi(w, u),$$

where $g$ and $h$ are random vectors with entries i.i.d. standard normal $\mathcal{N}(0, 1)$, and $\| \cdot \|$ is the standard Euclidean norm. We denote by $w_\phi(g, h)$ an optimal solution of (4). Then, [119] shows that an extension
of Gordon’s Gaussian minimax theorem (GMT, [50]) guarantees that the following inequality between the optimal values of the (PO) and (AO) problems holds

\[ \Pr(\Phi(A) < c) \leq 2 \Pr(\phi(g, h) \leq c), \]

for all \( c \in \mathbb{R} \). Specifically, (5) says that whenever \( \Pr(\phi(g, h) \leq c) \) is close to zero, and therefore \( c \) is a high probability lower bound on \( \phi(g, h) \), we also have that \( c \) is a high probability lower bound on \( \Phi(A) \).

Moreover, [119] shows that under additional convexity assumptions, namely if \( \psi \) is also convex-concave on \( \mathcal{S}_w \times \mathcal{S}_u \), with \( \mathcal{S}_w \in \mathbb{R}^d \) and \( \mathcal{S}_u \in \mathbb{R}^n \) convex compact sets, whenever \( c \) is a high probability upper bound on \( \phi(g, h) \), we also have that \( c \) is a high probability upper bound on \( \Phi(A) \), i.e.,

\[ \Pr(\Phi(A) > c) \leq 2 \Pr(\phi(g, h) \geq c). \]

Combining inequalities (5) and (6) leads to the CGMT, which states that under the additional convexity assumptions, the (AO) problem tightly bounds the optimal value of the (PO) problem, in the sense that for all \( \mu \in \mathbb{R} \) and \( t > 0 \),

\[ \Pr(|\Phi(A) - \mu| > t) \leq 2 \Pr(|\phi(g, h) - \mu| \geq t). \]

The concentration inequality (7) shows that if the optimal value \( \phi(g, h) \) concentrates around \( \mu \) with high probability, and therefore \( \Pr(|\phi(g, h) - \mu| \geq t) \) is small, then \( \Phi(A) \) concentrates around \( \mu \) with high probability. In particular, (7) will be essential in the study of the asymptotic (as both \( d \) and \( n \) go to infinity) properties of the optimal solution \( w_\Phi(A) \) of the (PO) problem, based solely on the optimal value \( \phi(g, h) \) of the (AO) problem.

In the rest of this section, we focus on explaining how the concentration inequality (7) will be used in this paper. First of all, we are interested in quantifying the performance of the distributionally robust estimator through the normalized squared error \( \|\hat{\theta}_{DRE} - \theta_0\|^2/d \), therefore we introduce the following change of variables

\[ w := \frac{\theta - \theta_0}{\sqrt{d}}. \]

We use the notation \( \hat{w}_{DRE} \) to denote the value of \( w \) when the estimator \( \theta = \hat{\theta}_{DRE} \) is used in (8). Now, in order to prove that the norm of \( \hat{w}_{DRE} \) (which is random by nature) converges (in probability) to some deterministic value \( \alpha_* \), we will show that w.p.a. 1 as \( d, n \to \infty \), we have that for all \( \eta > 0 \),

\[ \hat{w}_{DRE} \in \mathcal{S}_\eta := \{ w \in \mathcal{S}_w : \|\|w\| - \alpha_*\| < \eta \}. \]

In Section 4, we will show that the value of \( \alpha_* \), which quantifies the performance of the distributionally robust estimator, can be recovered from the optimal solution of a convex-concave minimax problem which involves at most four scalars. While showing this, the CGMT will play a major role, as explained in what follows.

We will start by showing that problem (1) can be rewritten as a (PO) problem, in the form (3), with matrix \( A \) constructed using the measurement vectors \( x_i \). Then, following the reasoning presented thus far, we will formulate the (AO) problem associated to it. Assume now that the set \( \mathcal{S}_u \) takes the form \( \mathcal{S}_u = \{ u \in \mathbb{R}^n : \|u\| \leq K_\beta \} \). In Section 4, we will see that this is precisely the case here. Now, in order to arrive at the scalar convex-concave minimax problem, whose optimal solution is \( \alpha_* \), we will need to work with the following slight variation of the (AO) problem (4),

\[ \phi'(g, h) := \max_{0 \leq \beta \leq K_\beta} \min_{w \in \mathcal{S}_w} \max_{\|u\| \leq \beta} \|w\|^T u + \|u\|^T w + \psi(w, u). \]
Notice that the two problems (4) and (10) can be obtained easily from one-another by exchanging the minimization over \( w \) and the maximization over \( \beta \). However, despite the small difference, the two problems are generally not equivalent. This is due to the fact that, differently from the (PO) problem (3), the objective function of the (AO) problem (4) is not convex-concave in \((w, u)\). Indeed, due to the random vectors \( g \) and \( h \), the terms \( \|w\|g^\top u \) and \( \|u\|h^\top w \) could be, for example, concave in \( w \) and convex in \( u \) if \( g^\top u \) is negative and \( h^\top w \) is positive, respectively, prohibiting the use of standard tools (such as Sion’s minimax principle) to exchange the order of the minimization and maximization in (10).

Nonetheless, the tight relation between the (PO) problem (3) and the (AO) problem (4) shown by the CGMT, together with the convexity of the (PO) problem, can be used to asymptotically (as both \( d \) and \( n \) go to infinity) infer properties of the optimal solution \( w_\Phi(A) \) of the (PO) problem, based solely on the optimal value \( \phi(g, h) \) of the modified (AO) problem (10). The following result is instrumental in the high-dimensional error analysis presented in Section 4.

**Fact 2.1 (Asymptotic CGMT).** Let \( \mathcal{S} \) be an arbitrary open subset of \( \mathcal{S}_w \) and \( \mathcal{S}_c = \mathcal{S}_w / \mathcal{S} \). We denote by \( \phi_{\mathcal{S}_c}(g, h) \) the optimal value of the modified (AO) problem (10) when the minimization over \( w \) is restricted as \( w \in \mathcal{S}_c \). If there exist constants \( \overline{\phi} \) and \( \phi_{\mathcal{S}_c} \), with \( \phi < \overline{\phi} \), such that \( \phi(g, h) \xrightarrow{P} \overline{\phi} \) and \( \phi_{\mathcal{S}_c}(g, h) \xrightarrow{P} \phi_{\mathcal{S}_c} \) as \( d, n \to \infty \), then

\[
\lim_{d, n \to \infty} \Pr(w_\Phi(A) \in \mathcal{S}) = 1. \tag{11}
\]

The proof of this result can be found in [120, Lemma 7]. Armed with Fact 2.1, we can conclude the desired result \((w_{DRE} \xrightarrow{P} \alpha)\) with \( w_\Phi(A) = w_{DRE} \) and \( \mathcal{S} = \mathcal{S}_\eta \) (for all \( \eta > 0 \)). This will necessitate studying the limits \( \phi(g, h) \xrightarrow{P} \overline{\phi} \) and \( \phi_{\mathcal{S}_c}(g, h) \xrightarrow{P} \phi_{\mathcal{S}_c} \) for the modified (AO) problem, required by Fact 2.1, which will be the main technical challenge.

### 3 Wasserstein Distributionally Robust Estimation

In this section we study the DRE problem (1), and prove that it can be brought into the form of a (PO) problem, as required by the CGMT framework. While the the distributional robustness induced by \( W_1 \) corresponds to a Lipschitz regularization, making it possible to rely on results from the high-dimensional analysis of regularized estimators literature, the distributional robustness induced by \( W_2 \) does not generally have a regularized formulation. Nonetheless, we will present a strong duality result which will enable its representation as a (PO) problem and, with it, the high-dimensional analysis based on the asymptotic CGMT result presented in Fact 2.1.

At different points in the paper, we will make use of the following assumptions on the loss function \( L \) and the baseline distribution \( \mathbb{P} \). While reading the assumptions, recall that the two scenarios of interest in this paper are \( p = 1 \) and \( p = 2 \).

**Assumption 3.1.**

(i) The loss function \( L \) is proper, continuous, and convex.

(ii) The loss function \( L \) satisfies the growth rate \( L(u) \leq C(1 + |u|^p) \), for some \( C > 0 \).

(iii) The distribution \( \mathbb{P} \) has finite \( p \)th moment, i.e., \( \mathbb{E}_\mathbb{P} \|(X, Y)\|^p < \infty \).

(iv) For \( p = 2 \), the loss function \( L \) is \( M \)-smooth, for some \( M > 0 \), i.e., it is differentiable and has a Lipschitz continuous gradient, with Lipschitz constant \( M \).
These are essential and common assumptions in distributionally robust optimization. Specifically, Assumption 3.1(i) ensures that the inner maximization in (1) admits a strong dual, which we will present in Fact 3.6. Moreover, Assumptions 3.1(ii)-(iii) will ensure that the optimal value of (1) is finite (see Lemma 3.2). Finally, Assumption 3.1(iv) will be used for the type-2 Wasserstein case, in Lemma 3.8, to show that the dual formulation mentioned above is convex-concave minimax problem.

**Lemma 3.2** (Finite optimal value of DRE problem). Let Assumptions 3.1(ii)-(iii) be satisfied. Then the optimal value of (1) is finite.

Recall from Section 2 and Fact 2.1 that, after rewritesing the DRE problem (1) as a (PO) problem, and subsequently as an (AO) problem, we can study the optimal solution of the (PO) problem based solely on the optimal value of the (AO) problem. In this stream of reasoning, having a finite optimal value is crucial. The proof of this lemma can be easily recovered from [103, Proposition 2.5], and is therefore omitted. Before proceeding to the main results of this section, we present the following lemma, which sheds light on the distributional robustness introduced by the transportation cost in the definition of $B_\epsilon(\hat{P}_n)$.

**Lemma 3.3** (Ambiguity set). Let $Q$ be a distribution inside the ambiguity set $B_\epsilon(\hat{P}_n)$. Moreover, given $Q_1, Q_2 \in \mathcal{P}(\mathbb{R}^d)$, let $d_p(Q_1, Q_2) = \inf_{\gamma \in \Gamma(Q_1, Q_2)} \mathbb{E}_{(X_1, X_2) \sim \gamma} [\|X_1 - X_2\|^p]$. Then the marginals of $Q$ with respect to $X$ and $Y$ satisfy

\[
(\pi_Y)_\# Q = \frac{1}{n} \sum_{i=1}^n \delta_{y_i} \quad \text{and} \quad d_p(\pi_X)_\# Q, \frac{1}{n} \sum_{i=1}^n \delta_{x_i}) \leq \epsilon.
\]

Lemma 3.3 shows that the ambiguity set $B_\epsilon(\hat{P}_n)$ contains distributions $Q$ whose marginal distribution of $Y$ is equal to the empirical distribution $\frac{1}{n} \sum_{i=1}^n \delta_{y_i}$ and whose marginal distribution of $X$ can be both discrete or continuous, and whose support is not necessarily concentrated on the samples $\{x_i\}_{i=1}^n$.

In what follows, we will consider separately the two cases of distributional robustness induced by $W_1$ and $W_2$. Moreover, for $W_2$, we will also consider a distributionally regularized alternative to (1), which will be introduced and motivated in Section 3.3.

### 3.1 Type-1 Wasserstein DRE

We first concentrate on the type-1 Wasserstein case and recall the well-known fact that, for the type-1 Wasserstein discrepancy, the distributional robustness in problem (1) has an explicit Lipschitz regularization effect. For details, we refer to [102] for the case when the loss function is convex, and to [47] for nonconvex Lipschitz loss functions.

**Fact 3.4** (Regularized estimation for $p = 1$). Let Assumptions 3.1(i)-(ii) be satisfied for $p = 1$. Then the optimal value of (1) is equal to the optimal value of the following regularized estimation problem

\[
\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n L(y_i - \theta^\top x_i) + \epsilon \text{Lip}(L)\|\theta\|.
\]

The regularized estimation formulation of problem (1) presented in Fact 3.4 will constitute the starting point of the high-dimensional error analysis presented in Section 4.1. For illustration purposes, we conclude this section with the following example, which shows (informally) that problem (1) with $W_1$ can be rewritten as a (PO) problem.
Example 3.5 (DRE → (PO)). Consider the DRE problem (1), with type-1 Wasserstein discrepancy, which has the explicit regularized formulation (12). Using the fact that \( y_i = \theta_0^\top x_i + z_i \), and introducing the change of variables \( w = (\theta - \theta_0)/\sqrt{d} \), we obtain

\[
\min_{w \in \mathcal{W}} \frac{1}{n} \sum_{i=1}^{n} L(z_i - \sqrt{d} w^\top x_i) + \varepsilon \lip(L) \|\sqrt{d} w + \theta_0\|.
\]

with \( \mathcal{W} \) the feasible set of \( w \) obtained from \( \Theta \) and the change of variable. Now, introducing the convex conjugate of \( L \), we obtain

\[
\min_{w \in \mathcal{W}} \frac{1}{n} \sum_{i=1}^{n} \left( \sup_{u_i \in \mathbb{R}} u_i(z_i - \sqrt{d} w^\top x_i) - L^*(u_i) \right) + \varepsilon \lip(L) \|\sqrt{d} w + \theta_0\|,
\]

which can be rewritten as

\[
\min_{w \in \mathcal{W}} \sup_{u \in \mathbb{R}^n} -\frac{1}{n} u^\top (\sqrt{d} A) w + \frac{1}{n} u^\top z - \frac{1}{n} \sum_{i=1}^{n} L^*(u_i) + \varepsilon \lip(L) \|\sqrt{d} w + \theta_0\|.
\]

Assume now that we can restrict this optimization to \( w \in \mathcal{S}_w \) and \( u \in \mathcal{S}_u \), for some convex and compact sets \( \mathcal{S}_w, \mathcal{S}_u \), without changing its optimal value. Then, by defining \( \psi(w, u) = n^{-1} u^\top z - n^{-1} \sum_{i=1}^{n} \lambda_i(u_i) + \varepsilon \lip(L) \|\sqrt{d} w + \theta_0\| \), which can be easily seen to be convex-concave in \((w, u)\), and noticing that \( \sqrt{d} A \) has entries i.i.d. standard normal (since the feature vectors \( x_i \) are i.i.d. \( \mathcal{N}(0, d^{-1} I_d) \) by Assumption 1.1), we see that problem (1) can be rewritten in the form of the (PO) problem.

\( \square \)

3.2 Type-2 Wasserstein DRE

We now move to the type-2 Wasserstein case and proceed by presenting the minimax dual reformulation of problem (1). Similarly to Fact 3.4, this formulation will represent the starting point of the high-dimensional analysis conducted in Section 4.2.

Fact 3.6 (Minimax reformulation for \( p = 2 \)). Let Assumption 3.1(i) be satisfied for \( p = 2 \). Then the optimal value of (1) is equal to the optimal value of the following minimax problem

\[
\inf_{\theta \in \Theta, \lambda \geq 0} \sup_{u \in \mathbb{R}^n} \lambda \varepsilon + \frac{1}{n} \sum_{i=1}^{n} u_i(y_i - \theta^\top x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - L^*(u_i).
\]

(13)

The proof of this fact is a simple application of [103, Theorem 3.8(i)] and is therefore omitted. In the rest of Section 3.2, we will concentrate on the dual formulation (13) and study the conditions under which (13) is concave in \( u \), and the norm of its optimal solution \( u_* \) is in the order of \( \sqrt{n} \). In particular, the \( \sqrt{n} \)-growth rate of \( u_* \) will allow us to guarantee that, after the rescaling \( u \to u\sqrt{n} \), the variable \( u \) can be restricted without loss of generality to a convex compact set \( \mathcal{S}_u \). Recall from Section 2 that the concavity in \( u \), together with the convexity and compactness of \( \mathcal{S}_u \) are essential in the high-dimensional error analysis using the CGMT framework. It turns out that both properties hold as long as the ambiguity radius \( \varepsilon \) satisfies a mild upper-bound, which will be presented in Lemmas 3.8 and 3.9. These results build upon the work [21], and require the following assumption on the set \( \Theta \).

Assumption 3.7. The set \( \Theta \) is defined as \( \Theta := \{ \theta \in \mathbb{R}^d : \|\theta\| \leq R_\theta \sqrt{d} \} \).

Assumption 3.7 is essential for the application of the CGMT framework in the high-dimensional error analysis. Specifically, it will be used first in the proof of Lemma 3.9, while studying the growth rates of the optimal solutions.
Before proceeding, we need to introduce the following concepts used in Lemmas 3.8 and 3.9. Let \( L_n \) and \( T_n \) denote two constants satisfying
\[
0 < L_n \leq \mathbb{E}_{\hat{P}_n} \left[ L'(Y - \theta^T X)^2 \right] \leq T_n,
\]
(14)
for all \( \theta \in \Theta \). Recall from Assumption 3.1(iv) that \( L \) is differentiable, and therefore \( L' \) is well-defined. Notice that, due to the upper bound \( L(u) \leq C(1 + |u|^2) \) from Assumption (ii), the value \( T_n \) is finite. Moreover, excluding the trivial and practically not relevant case where the loss function \( L \) is \( \hat{P}_n \)-a.s. constant, resulting in \( L_n = 0 \), in general \( L_n \) is strictly positive.

In the following lemma, we show that an upper bound on the radius \( \varepsilon \) of the ambiguity set guarantees that (13) is a convex-concave minimax problem.

**Lemma 3.8** (Concavity in \( u \) of (13)). Let Assumptions 3.1(i) and (iv), and Assumption 3.7 be satisfied. Moreover, let \( \varepsilon = \varepsilon_0/n \), for some constant \( \varepsilon_0 \) satisfying \( \varepsilon_0 \leq \rho^{-1}M^{-2}R^{-2}L_n \). Then, problem (13) is equivalent to the following convex-concave minimax problem
\[
\inf_{\theta \in \Theta} \sup_{u \in \mathbb{R}^n} \lambda \varepsilon + \frac{1}{n} \sum_{i=1}^n u_i(y_i - \theta^T x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - L^*(u_i),
\]
with \( \Lambda = \{ \lambda \geq 0 : \lambda \geq MR\sqrt{d}\|\theta\|/2 \} \).

Moreover, in the next lemma we show that the same upper bound on \( \varepsilon \) guarantees that the norm of the optimal solution \( u_* \) in (13) can be upper bounded by \( K_u \sqrt{n} \), for some constant \( K_u > 0 \). Recall that the CGMT requires \( u \) to live in a compact set. This will be achieved in Theorem 4.5 using this upper bound and the rescaling \( u \to u\sqrt{n} \).

**Lemma 3.9** (Growth rate of \( u_* \) in (13)). Let Assumption 3.1(i) and Assumption 3.7 be satisfied. Moreover, let \( \varepsilon = \varepsilon_0/n \), for some constant \( \varepsilon_0 \) satisfying \( \varepsilon_0 \leq \rho^{-1}M^{-2}R^{-2}L_n \). Then the optimal \( u_* \) in (13) satisfies \( \|u_*\| \leq K_u \sqrt{n} \), for some constant \( K_u > 0 \).

### 3.3 Type-2 Wasserstein Distributional Regularization

The results presented in Section 3.2 will be used in Section 4.2 to analyze the estimation error in high-dimensions. As will be shown in the proof of Theorem 4.5, one of the main challenges in the analysis will be posed by the minimization over \( \lambda \) which appears in the dual formulation (13). This challenge will be one of the main reasons for the very complicated proof. To address this, in Section 4.3 we will show that by considering problem
\[
\min_{\theta \in \Theta} \sup_{Q \in \mathbb{P}(\mathbb{R}^{d+1})} \mathbb{E}_Q \left[ L(Y - \theta^T X) \right] - \lambda W_p(Q, \hat{P}_n),
\]
(15)
as an alternative to (1), a much simpler proof, together and a stronger result can be obtained. We will refer to problem (15) as distributionally regularized estimation, and we will denote its optimal solution by \( \hat{\theta}_{DReg} \). In what follows, we will mirror the results obtained in Section 3.2, i.e., the minimax reformulation, the concavity in \( u \), and the growth rate of \( u_* \), for problem (15). We start by presenting a strong dual reformulation of problem (15).

**Fact 3.10** (Minimax reformulation of (15)). Let Assumption 3.1(i) be satisfied for \( p = 2 \). Then the optimal value of (15) is equal to the optimal value of the following minimax problem
\[
\min_{\theta \in \Theta} \sup_{u \in \mathbb{R}^n} \frac{1}{n} \sum_{i=1}^n u_i(y_i - \theta^T x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - L^*(u_i).
\]
(16)
Similarly to Fact 3.6, the proof of this fact is a simple application of [103, Theorem 3.8(i)] and is therefore omitted. Next, we will show that when the regularization parameter \( \lambda \) satisfies a certain lower bound, the dual formulation (16) is a convex-concave minimax problem.

**Lemma 3.11** (Concavity in \( u \) of (16)). Let Assumptions 3.1 (i) and (iv), and Assumption 3.7 be satisfied. Moreover, let \( \lambda = d\lambda_0 \), for some constant \( \lambda_0 \) satisfying \( \lambda_0 \geq MR_\theta^2/2 \). Then, the objective function in (16) is concave in \( u \).

Before proceeding, we would like to highlight the connection between the upper bound on \( \varepsilon \) in Lemma 3.9 and the lower bound on \( \lambda \) in Lemma 3.12. In the proof of Lemma 3.9 we have shown that the upper bound \( \varepsilon \leq (\rho^{-1}M^{-2}R_\theta^2L_n)/n \) guarantees that \( \lambda \geq MR_\theta\sqrt{d}\|\theta\|/2 \). Therefore, the lower bound \( \lambda \geq MR_\theta^2d/2 \) corresponds to replacing \( \|\theta\| \) by its upper bound \( R_\theta\sqrt{d} \).

Finally, in the following lemma we show that when \( \lambda \) satisfies the same lower bound, the optimal solution \( u_* \) is in the order of \( \sqrt{n} \), allowing us to apply the CGMT after the re-scaling \( u \to u\sqrt{n} \).

**Lemma 3.12** (Growth rate of \( u_* \) in (16)). Let Assumption 3.1(i) and Assumption 3.7 be satisfied. Moreover, let \( \lambda = d\lambda_0 \), for some constant \( \lambda_0 \) satisfying \( \lambda_0 > MR_\theta^2/2 \). Then the optimal \( u_* \) in (16) satisfies \( \|u_*\| \leq K_\beta\sqrt{n} \), for some constant \( K_\beta > 0 \).

## 4 High-Dimensional Error Analysis

In this section, we present the main results of this paper, which quantify the performance of both the distributionally robust estimator \( \hat{\theta}_{DRE} \) and the distributionally regularized estimator \( \hat{\theta}_{DReg} \) in the high-dimensional regime. Building on the results of the previous section, namely the dual formulations presented in Facts 3.4, 3.6, and 3.10, as well as on the asymptotic CGMT presented in Fact 2.1, we will show that the quantities \( \|\hat{\theta}_{DRE} - \theta_0\|^2/d \) and \( \|\hat{\theta}_{DReg} - \theta_0\|^2/d \) converge in probability to two deterministic values which can be recovered from the solutions of two convex-concave minimax optimization problems that involve at most four scalar variables. The analysis carried out in what follows requires the following assumptions on the loss function \( L \) and the true parameter \( \theta_0 \).

**Assumption 4.1.**

(i) The loss function \( L \) is real-valued and satisfies \( L(0) = \min_{u \in \mathbb{R}} L(u) = 0 \).

(ii) The true parameter \( \theta_0 \) satisfies \( \|\theta_0\|^2/d \overset{P}{\to} \sigma_{\theta_0}^2 \), with \( \sigma_{\theta_0} > 0 \).

Assumption 4.1(i) is natural in estimation problems, while Assumption 4.1(ii) will be essential while studying the converge in probability of the quantities \( \|\hat{\theta}_{DRE} - \theta_0\|^2/d \) and \( \|\hat{\theta}_{DReg} - \theta_0\|^2/d \) in Theorems 4.3, 4.5 and 4.13.

### 4.1 Type-1 Wasserstein DRE

We start by focusing on the type-1 Wasserstein case. Recall from Fact 3.4 that, in this case, the DRE problem is equivalent to the regularized estimation problem (12). This regularized formulation turns out to be very convenient, since it allows us to rely on the results in [120] for the high-dimensional error analysis. Making use of these results requires the following additional mild assumptions on the loss function \( L \) and the noise \( Z \).
Assumption 4.2.

(i) Either there exists \( u \in \mathbb{R} \) at which \( L \) is not differentiable, or there exists an interval \( \mathcal{I} \subset \mathbb{R} \) where \( L \) is differentiable with a strictly increasing derivative.

(ii) The absolutely continuous part of the noise distribution \( \mathbb{P}_Z \) has a continuous Radon-Nikodym derivative.

(iii) The noise \( Z \) has nonzero variance.

Before proceeding, we need to introduce some notation, which will be later used in the statement of Theorem 4.3. We first define the expected Moreau envelope \( \mathcal{L} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) associated to \( L \) as

\[
\mathcal{L}(c, \tau) := \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} \left[ e_L(cG + Z; \tau) \right].
\]  

with \( G \) a standard normal random variable \( \mathcal{N}(0,1) \). Notice that the expectation is with respect to the joint distribution \( \mathcal{N}(0,1) \otimes \mathbb{P}_Z \) of the two independent random variables \( G \) and \( Z \).

The effects of the loss function \( L \) and the noise \( Z \) on the error \( \| \hat{\theta}_{DRE} - \theta_0 \|^2/d \) will only appear implicitly. In Theorem 4.3 we will see that the function \( \mathcal{L} \) quantifies their contribution. Moreover, we introduce the function \( \mathcal{G} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), defined as follows

\[
\mathcal{G}(c, \tau) := \begin{cases} 
\sqrt{(c^2 + \sigma_0^2)/\rho - \tau/(2\rho) - \sigma_0/\sqrt{\rho}} & \text{if } \sqrt{\rho\sqrt{c^2 + \sigma_0^2}} > \tau, \\
(c^2 + \sigma_0^2)/(2\tau) - \sigma_0/\sqrt{\rho} & \text{if } \sqrt{\rho\sqrt{c^2 + \sigma_0^2}} \leq \tau.
\end{cases}
\]

The effects of the ambiguity radius \( \varepsilon \) and the under/over-parametrization parameter \( \rho \) on the error \( \| \hat{\theta}_{DRE} - \theta_0 \|^2/d \) will appear explicitly. In Theorem 4.3 we will see that the function \( \mathcal{G} \), as well as other explicit terms, quantifies their contribution. We are now ready to state the main result on the high-dimensional error analysis for \( p = 1 \).

Theorem 4.3 (Performance of \( \hat{\theta}_{DRE} \) for \( p = 1 \)). Let Assumptions 1.1, 3.1, 4.1 and 4.2 be satisfied. Moreover, let \( \Theta = \mathbb{R}^d \) and \( \varepsilon = \varepsilon_0/\sqrt{n} \), for some constant \( \varepsilon_0 \). Then, as \( d, n \to \infty \) with \( d/n = \rho \), it holds in probability that

\[
\lim_{d \to \infty} \frac{\| \hat{\theta}_{DRE} - \theta_0 \|^2}{d} = \alpha_*^2,
\]

where \( \alpha_* \) is the unique solution\(^2\) of the convex-concave minimax scalar problem

\[
\min_{\alpha \geq 0, \tau_1 > 0, \beta \geq 0, \tau_2 > 0} \max_{\tau_1 > \tau_2 > 0} \frac{\beta \tau_1}{2} - \frac{\alpha \tau_2}{2} - \frac{\alpha^2}{2\tau_2} + \frac{1}{\rho} \mathcal{L} \left( \alpha, \frac{\tau_1}{\beta} \right) + \varepsilon_0 \text{Lip}(L) \mathcal{G} \left( \frac{\alpha \beta}{\tau_2}, \frac{\alpha \varepsilon_0 \text{Lip}(L)}{\tau_2} \right),
\]

with \( \mathcal{L}(\cdot, \cdot) \) and \( \mathcal{G}(\cdot, \cdot) \) defined in (17) and (18), respectively.

The Moreau envelope \( e_L(c, \tau) \) can be computed in closed-form for many important classes of estimators. One such example, which we will recall in Section 5, is the LAD estimator.

Example 4.4 (Moreau envelope for the LAD estimator). Consider the least absolute deviations (LAD) estimator, where the loss function is defined as \( L(u) = |u| \). Then, it can be easily shown that its Moreau envelope is equal to

\[
e_L(c, \tau) = \begin{cases} 
c^2/(2\tau) & \text{if } |c| \leq \tau \\
|c| - \tau/2 & \text{otherwise}.
\end{cases}
\]

\(^2\)Provided that the minimizer of (20) over \( \alpha \) is bounded.
4.2 Type-2 Wasserstein DRE

We now proceed to the type-2 Wasserstein case where, differently from the $W_1$ case, the distributional robustness does not generally have an equivalent regularized formulation, and therefore the results in [120] are not applicable anymore. However, building on the analysis and results presented in Section 3.2, we will show that the CGMT framework can be applied to study the high-dimensional performance of the distributionally robust estimator $\hat{\theta}_{DRE}$. This will be shown in Theorem 4.5.

Similarly to Section 4.1, we need to introduce some notation, which will be later used in the statement of Theorem 4.5. In the same spirit as equation (17), we define the expected Moreau envelope $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ associated to a convex function $f$ as

$$F(c, \tau) := \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} [e_f(cG + Z; \tau)],$$

where, as before, $G$ is a standard normal random variable $\mathcal{N}(0,1)$, and the expectation is with respect to the joint distribution $\mathcal{N}(0,1) \otimes \mathbb{P}_Z$ of the two independent random variables $G$ and $Z$. As in the case of $W_1$, the effects of the loss function $L$ and the noise $Z$ on the error $||\hat{\theta}_{DRE} - \theta_0||^2/d$ will only appear implicitly, this time through the expected Moreau envelope of a convex function $f$ which can be derived from the loss function $L$.

Secondly, we introduce the following two functions used in the statement of Theorem 4.5,

$$O_1(\alpha, \tau_1, \tau_2, \beta) := \frac{\beta \tau_1}{2} + \frac{\varepsilon_0 \beta \tau_2}{2} - \frac{\beta^2}{2M} + \mathcal{F}(\alpha, \tau_1/\beta) - \alpha \beta \sqrt{p} \left( \frac{\rho \varepsilon_0}{\tau_2} \sigma_{\theta_0}^2 + 1 + \frac{\sqrt{\varepsilon_0} \beta}{2\tau_2} \right) \left( \sigma_{\theta_0}^2 + \alpha^2 \right),$$

and

$$O_2(\alpha, \tau_1, \beta) := \inf_{\tau_2 > 0} \frac{\beta \tau_1}{2} + \frac{\rho \tau_2}{2} + \frac{\beta^2 \tau_2}{2q} - \frac{\beta^2}{2M} + \mathcal{F}(\alpha, \tau_1/\beta) - \alpha \sqrt{p} \left( \frac{p + \beta^2}{q} \right)^2 \frac{\rho \sigma_{\theta_0}^2}{\tau_2^2} + \beta^2 + \rho \left( p + \frac{\beta^2}{q} \right) \frac{\sigma_{\theta_0}^2 + \alpha^2}{2\tau_2},$$

with constants $p := (\varepsilon_0 \sqrt{p} MR_\theta)/2$ and $q := 2\sqrt{p} MR_\theta$, and $f$ defined below, in the statement of Theorem 4.5. Recall that $M$ is the parameter of the smoothness of $L$, defined in Assumption 3.1(iv), and $R_\theta$ is the bound on the norm of $\theta/\sqrt{d}$, defined in Assumption 3.7. We are now ready to state the main result on the high-dimensional error analysis.

**Theorem 4.5** (Performance of $\hat{\theta}_{DRE}$ for $p = 2$). Let Assumptions 1.1, 3.1, 3.7, and 4.1 be satisfied, and suppose that the loss function $L$ can be written as the Moreau envelope with parameter $M$, i.e., $L(\cdot) = e_f(\cdot, 1/M)$, of a convex function $f$ which satisfies

$$\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} [f(\alpha G + Z)] < \infty,$$

for all $\alpha \leq \sigma_{\theta_0}$. Moreover, let $R_\theta \geq 2\sigma_{\theta_0}$ and $\varepsilon = \varepsilon_0/n$, for some constant $\varepsilon_0$ satisfying $\varepsilon_0 \leq \rho^{-1} M^{-2} R_\theta^{-2} L_n$, and consider the following convex-concave minimax optimization problems

$$\mathcal{V}_1 := \inf_{\alpha \in [0, \sigma_{\theta_0}]} \sup_{\beta > B} O_1(\alpha, \tau_1, \tau_2, \beta), \quad \text{and} \quad \mathcal{V}_2 := \inf_{\alpha \in [0, \sigma_{\theta_0}]} \max_{\beta \leq B} O_2(\alpha, \tau_1, \beta),$$

with $O_1(\alpha, \tau_1, \tau_2, \beta)$ and $O_2(\alpha, \tau_1, \beta)$ defined above, and $B = \sqrt{\varepsilon_0 \rho} MR_\theta$. If the optimal solutions $\alpha_{*,1}$ and $\alpha_{*,2}$ attaining $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively, are strictly smaller than $\sigma_{\theta_0}$, then, as $d, n \to \infty$ with $d/n = \rho$, 

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it holds in probability that

\[
\lim_{d \to \infty} \frac{||\hat{\theta}_{DRE} - \theta_0||^2}{d} = \begin{cases} 
\alpha_{*,1}^2 & \text{if } \mathcal{V}_1 > \mathcal{V}_2 \\
\alpha_{*,2}^2 & \text{if } \mathcal{V}_2 > \mathcal{V}_1 \\
\max\{\alpha_{*,1}^2, \alpha_{*,2}^2\} & \text{if } \mathcal{V}_1 = \mathcal{V}_2.
\end{cases}
\]

(24)

**Remark 4.6.** Theorem 4.5 requires the optimal solutions of the two optimization problems to be strictly smaller than \(\sigma_{\theta_0}\), i.e., \(\alpha_{*,1}, \alpha_{*,2} < \sigma_{\theta_0}\). This condition is equivalent to imposing the following relative error bound

\[
\frac{||\hat{\theta}_{DRE} - \theta_0||}{||\theta_0||} < 1,
\]

which is practically desirable in any estimation problem.

**Remark 4.7.** In the proof of Theorem 4.5 we show that the objective function appearing in the definition of \(\mathcal{O}_2(\alpha, \tau_1, \beta)\) is convex in \(\tau_2\). Consequently, \(\mathcal{O}_2(\alpha, \tau_1, \beta)\) can be easily computed for any \((\alpha, \tau_1, \beta)\).

Many loss functions which have at-most-quadratic growth rate (as required by Assumption 3.1(ii)) can be written as the Moreau envelope with parameter \(M\) of some convex function \(f\) which satisfies the integrability condition \(\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} [f(\alpha G + Z)] < \infty\). In the following example we show that the Huber loss satisfies this requirement.

**Example 4.8** (Huber loss). Consider the Huber loss function with parameter \(\delta\), defined as

\[
L(u) = \begin{cases} 
\frac{u^2}{2} & \text{if } |u| \leq \delta \\
\delta |u| - \delta^2/2 & \text{otherwise.}
\end{cases}
\]

(25)

It can be immediately checked that \(M = 1\) in this case. It can be easily shown that its convex conjugate is equal to

\[
L^*(u) = \begin{cases} 
\frac{u^2}{2} & \text{if } |u| \leq \delta \\
\infty & \text{otherwise.}
\end{cases}
\]

In the proof of Theorem 4.5 we have seen that \(f\) can be recovered from \(L^*\) as follows

\[
f(u) = (L^*(\cdot) - 1/2(\cdot)^2)^*(u) = \sup_{v \in \mathbb{R}} uv - \begin{cases} 
0 & \text{if } |v| \leq \delta \\
\infty & \text{otherwise}
\end{cases}
= \delta |u|.
\]

Finally, notice that

\[
\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} [\delta |\alpha G + Z|] < \infty
\]

for all \(\alpha \in \mathbb{R}\) is a consequence of the finite first moment of \(G\) and \(Z\) (Assumption 3.1(iii) implies that \(Z\) has a finite second moment, and therefore also a finite first moment).

There are also loss functions which can not be written in such way. One important example is the squared loss, as explained in the following remark.
Remark 4.9. Consider the squared loss function $L(u) = u^2$. It can be easily checked that $L^*(v) = v^2/4$, and $M = 2$. Therefore, $f^* = L^*(\cdot) - (\cdot)^2/4 = 0$, and consequently,

$$f(u) = \sup_{v \in \mathbb{R}} uv = \begin{cases} 0 & \text{if } u = 0 \\ \infty & \text{otherwise,} \end{cases}$$

from which $\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} [f(\alpha G + Z)] = \infty$ follows.

However, for squared losses this requirement is not necessary, and Theorem 4.5 continues to hold if the expected Moreau envelope $\mathcal{F}(\alpha, \tau_1/\beta)$ is replaced by the term $\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} \left[\frac{\beta}{2 \tau_1} (\alpha G + Z)^2\right]$. This is explained in the next remark.

Remark 4.10. Recall from the proof of Theorem 4.5 that $f$ was introduced as a consequence of the decomposition $L^*(u) = u^2/(2M) + f^*(u)$ (which, in turn, was needed to preserve the concavity in $u$ of the objective function, in order to apply the CGMT). Whenever $L(u) = Mu^2/2$, we have that $L^*(u) = u^2/(2M)$, and therefore $f^*(u) = 0$ for all $u \in \mathbb{R}$. In this case, there is no need to introduce the function $f$, and the result of Theorem 4.5 continues to hold by simply replacing the expected Moreau envelope $\mathcal{F}(\alpha, \tau_1/\beta) = \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} [e_f(cG + Z; \tau)]$ with the term

$$\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} \left[\frac{\beta}{2 \tau_1} (\alpha G + Z)^2\right].$$

Furthermore, the minimization over $\tau_1$ can be now solved in closed-form as

$$\inf_{\tau_1 > 0} \frac{\beta \tau_1}{2} + \frac{\beta}{2 \tau_1} \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} \left[(\alpha G + Z)^2\right] = \beta \sqrt{\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{F}_z} [(\alpha G + Z)^2]} = \beta \sqrt{\alpha^2 + \sigma_Z^2},$$

where $\sigma_Z^2$ denotes the second moment of $Z$, i.e., $\sigma_Z^2 := \mathbb{E}_{\mathbb{F}_z} [Z^2]$, reducing the number of scalar variables from four to three in the two minimax problems (23).

Alternatively, it can be shown that for the particular case of squared losses some of the assumptions (such as $\varepsilon_0 \leq \rho^{-1} M^{-2} R_\theta^{-2} L_n$) can be dropped, and that the estimation error can be obtained from the solution of only one optimization problem. The reason for this is that when $L(u) = u^2$, the distributional robustness has the following equivalent regularized formulation [22],

$$\min_{\theta \in \Theta} \sup_{Q \in \mathcal{B}_a(\Pi_n)} \mathbb{E}_Q \left[(Y - \theta^\top X)^2\right] = \left(\min_{\theta \in \Theta} \sqrt{\mathbb{E}_{\mathbb{F}_n} [(Y - \theta^\top X)^2]} + \sqrt{\varepsilon} \|\theta\|\right)^2. \quad (26)$$

As a consequence, the estimation error can be recovered from Theorem 1 in [120], as summarized in the following proposition. Before doing so, we need to introduce some notation. We define the function $\mathcal{E} : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$,

$$\mathcal{E}(\varepsilon, \tau) := \begin{cases} \sqrt{(c^2 + \sigma_Z^2) - \tau/2} - \sigma_Z/\sqrt{\rho} & \text{if } \sqrt{c^2 + \sigma_Z^2} > \tau \\ (c^2 + \sigma_Z^2)/(2\tau) - \sigma_Z & \text{if } \sqrt{c^2 + \sigma_Z^2} \leq \tau, \end{cases}$$

which is used in the statement of the following proposition.

**Proposition 4.11 (Performance of $\hat{\theta}_{DRE}$ for $L(\cdot) = (\cdot)^2$).** Let Assumptions 1.1, 3.1, and 4.1 be satisfied for the loss function $L(u) = u^2$. Moreover, let $\Theta = \mathbb{R}^d$, $\sigma_Z^2 := \mathbb{E}_{\mathbb{F}_z} [Z^2]$, and $\varepsilon = \varepsilon_0/n$, for some constant $\varepsilon_0$. Then, as $d, n \rightarrow \infty$ with $d/n = \rho$, it holds in probability that

$$\lim_{d \rightarrow \infty} \frac{\|\hat{\theta}_{DRE} - \theta_0\|^2}{d} = \alpha_*^2,$$
provided that \( \alpha_* \) is the unique minimizer of the following convex-concave minimax scalar problem

\[
\min_{\alpha \geq 0, \tau_1 > 0} \max_{\beta \geq 0, \tau_2 > 0} \frac{\beta \tau_1}{2} - \frac{\alpha \tau_2}{2} \frac{\beta^2}{2\tau_2} + \frac{1}{\rho} E \left( \alpha, \frac{\tau_1}{\beta} \right) + \sqrt{\frac{\alpha}{\tau_2}} G \left( \frac{\alpha \beta}{\tau_2}, \frac{\alpha \sqrt{\varepsilon_0}}{\tau_2} \right),
\]

with \( E \) defined above, and \( G \) defined in (18).

**Remark 4.12** (Uniqueness of \( \alpha_* \)). Notice that, differently from Theorem 4.5, the uniqueness of the optimizer \( \alpha_* \) is not guaranteed in Proposition 4.11. When \( \alpha_* \) is not unique, and Proposition 4.11 can not be used to recover the estimation error, the adaptation of Theorem 4.5 for squared losses, proposed in Remark 4.10, can be used to recover the estimation error, provided that the minimizers of \( V_1 \) and \( V_2 \) in (23) are strictly smaller than \( \sigma_{\theta_0} \).

\[\square\]

### 4.3 Type-2 Wasserstein Distributional Regularization

In Theorem 4.5 we have shown that the estimation error in the distributionally robust setting with \( W_2 \) can be easily computed from the optimal solution of two scalar convex-concave minimax problems. Nonetheless, the result has an inconvenient limitation when compared to the results obtained for \( p = 1 \): the estimation error can be found only when the minimizers of \( V_1 \) and \( V_2 \) in (23) are strictly smaller than \( \sigma_{\theta_0} \). In what follows, we will show that by considering the distributionally regularized problem (15) this assumption can be dropped. Additionally, the estimation error can be recovered from the minimizer of only one (simpler) convex-concave minimax problem, which involves only three scalar variables. This is shown in the following theorem.

**Theorem 4.13** (Performance of \( \hat{\theta}_{\text{DR}2} \) for \( p = 2 \)). Let Assumptions 1.1, 3.1, 3.7, and 4.1 be satisfied, and suppose that the loss function \( L \) can be written as the Moreau envelope of a convex function \( f \) which satisfies

\[
E_{\mathcal{N}(0,1) \otimes \mathbb{P}_2} [f(\alpha G + Z)] < \infty,
\]

for all \( \alpha \geq 0 \). Moreover, let \( R_\theta \geq \sigma_{\theta_0} \) and \( \lambda = d \lambda_0 \), for some constant \( \lambda_0 \) satisfying \( \lambda_0 > MR_\theta^2/2 \). Then, as \( d, n \to \infty \) with \( d/n = \rho \), it holds in probability that

\[
\lim_{d \to \infty} \frac{||\hat{\theta}_{\text{DR}2} - \theta_0||^2}{d} = \begin{cases} 
\alpha_*^2 & \text{if } \alpha_* \in [0, R_\theta - \sigma_{\theta_0}] \\
\alpha_*^2 & \text{if } \alpha_* \in (R_\theta - \sigma_{\theta_0}, R_\theta + \sigma_{\theta_0}],
\end{cases}
\]

(28)

where \( \alpha_* \) is the unique solution of the following convex-concave minimax scalar problem

\[
\min_{0 \leq \alpha \leq R_\theta + \sigma_{\theta_0}} \max_{\tau_1 > 0} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \mathcal{F}(\alpha, \frac{\tau_1}{\beta}) + \frac{\beta^2}{4\lambda_0} \left( \sigma_{\theta_0}^2 + \alpha^2 \right) - \alpha \beta \sqrt{\rho + \frac{\beta^2\sigma_{\theta_0}^2}{4\lambda_0}}.
\]

**Remark 4.14**. As stated in Section 3.3, there is a close connection between the upper bound on \( \varepsilon_0 \) in Theorem 4.5 and the lower bound on \( \lambda_0 \) in Theorem 4.13. In Lemma 3.9 we have shown that the upper bound \( \varepsilon \leq (\rho^{-1} M^{-2} R_\theta^2 \mathbb{E}_n)/n \) guarantees that \( \lambda \geq MR_\theta \sqrt{d} ||\theta||/2 \). Therefore, the lower bound \( \lambda \geq MR_\theta^2 d/2 \) corresponds to replacing \( ||\theta|| \) by its upper bound \( R_\theta \sqrt{d} \).

Recall from Remark 4.9 that the squared loss \( L(u) = Mu^2/2 \) can not be written as the Moreau envelope with parameter \( M \) of some convex function \( f \) which satisfies the integrability condition \( E_{\mathcal{N}(0,1) \otimes \mathbb{P}_2} [f(\alpha G + Z)] < \infty \). Nonetheless, in the following corollary we show how Theorem 4.13 can be easily adapted to account for this case as well, resulting in a simpler convex-concave minimax problem, which involves only two scalar variables.
Corollary 4.15 (Performance of $\hat{\theta}_{DReg}$ for $L(\cdot) = (\cdot)^2$). Let Assumptions 1.1, 3.1, 3.7, and 4.1 be satisfied for the loss function $L(u) = u^2$. Moreover, let $R_0 \geq \sigma_{\theta_0}$, $\sigma_Z^2 := \mathbb{E}_Z[Z^2]$, and $\lambda = d\lambda_0$, for some constant $\lambda_0$ satisfying $\lambda_0 > R_0^2$. Then, as $d, n \to \infty$ with $d/n = \rho$, it holds in probability that

$$\lim_{d \to \infty} \frac{\|\hat{\theta}_{DReg} - \theta_0\|^2}{d} = \begin{cases} \alpha_*^2 & \text{if } \alpha_* \in [0, R_0 - \sigma_{\theta_0}] \\ \leq \alpha_*^2 & \text{if } \alpha_* \in (R_0 - \sigma_{\theta_0}, R_0 + \sigma_{\theta_0}], \end{cases}$$

where $\alpha_*$ is the unique solution of the following convex-concave minimax scalar problem

$$\min_{\theta_0 \leq \alpha \leq R_0 + \sigma_{\theta_0}} \max_{\beta \geq 0} \frac{\beta}{\beta \sigma^2 + \sigma_Z^2 - \frac{\beta^2}{2M} + \frac{\beta^2}{4\lambda_0} \left(\sigma_{\theta_0}^2 + \alpha^2\right) - \alpha \beta \sqrt{\rho + \frac{\beta^2\sigma_{\theta_0}^2}{4\lambda_0^2}}}{\sigma^2 + \sigma_Z^2 - \frac{\beta^2}{2M} + \frac{\beta^2}{4\lambda_0} \left(\sigma_{\theta_0}^2 + \alpha^2\right) - \alpha \beta \sqrt{\rho + \frac{\beta^2\sigma_{\theta_0}^2}{4\lambda_0^2}}},$$

(30)

5 Numerical Experiments

In this section we numerically validate the theoretical results presented so far in the context of hyperparameter tuning for regression problems in high dimensions. Specifically, we consider the problem of choosing the radius $\varepsilon$ which minimizes the normalized squared estimation error in the high-dimensional regime, where both the dimension of the problem $d$ and the number of samples $n$ are very large. In such regimes, numerical techniques such as cross-validation are computationally very demanding or even prohibitive. Instead, Theorems 4.3, 4.5, and 4.13 provide an easy-to-compute relationship between the normalized squared estimation error and the values of $\varepsilon$ and $\rho$, which appear explicitly in the objective functions of (20) and (23). Although these results were recovered in the asymptotic regime and for isotropic Gaussian features, in what follows we will show numerically that they are valid for broader classes of probability ensembles, already when $d, n \approx 500$.

5.1 Radius Tuning in High Dimensions

We focus on the problem of estimating an unknown parameter $\theta_0$ in $d$ dimensions from $n$ noisy linear measurements of the form $y_i = \theta_0^T x_i + z_i$. Throughout this entire section, we consider the case where the true parameter $\theta_0$ is sparse, with roughly 10% of its entries being non-zero. In our experiments, we encode such sparsity in a probabilistic fashion, by sampling the entries of $\theta_0$ according to the distribution $\mathbb{P}_{\theta_0} = 0.1\mathcal{N}(0,10) + 0.9 \delta_0$. From the definition of $\mathbb{P}_{\theta_0}$ we automatically recover, by the weak law of large numbers, that $\sigma_{\theta_0}^2 = 1$ (recall the definition of $\sigma_{\theta_0}^2$ from Assumption 4.1). Moreover, we assume that the noise values $z_i$ are i.i.d. according to the distribution $\mathbb{P}_Z = \mathcal{N}(0,0.1)$, resulting in $\sigma_Z^2 = 0.1$.

We start by considering the type-1 Wasserstein DRE problem with loss function $L(\cdot) = |\cdot|$. Notice that Assumptions 4.2 and 4.4 are trivially satisfied. Following Theorem 4.3, the asymptotic normalized squared error $\lim_{d \to \infty} \frac{\|\hat{\theta}_{DRE} - \theta_0\|^2}{d}$ is equal to $\alpha_*^2$, where $\alpha_*$ is the solution of the convex-concave minimax scalar problem (20). Since the Moreau envelope is equal to the Huber loss (see Example 4.4), we have that the expected Moreau envelope can be easily recovered in closed-form for our choice of $\mathbb{P}_Z$. Finally, we consider the case where $\rho = d/n = 0.8$. In Figure 1(a) we show in red the asymptotic normalized squared error $\alpha_*^2$ as a function of the radius $\varepsilon_0$ (recall that $\varepsilon = \varepsilon_0/\sqrt{n}$). We now investigate to what extent the asymptotic error $\alpha_*^2$ is informative in realistic scenarios, where the dimension $d$ might be large but nonetheless finite. For this, we show on the same plot the normalized squared error averaged over 5 independent realizations for dimensions $d = 500, 2000$, and 2500, while keeping fixed the value of $\rho$ at 0.8. It is important to notice that, although there is a slight discrepancy between the asymptotic and the three non-asymptotic error curves, the asymptotic curve captures the true behavior.
We now proceed to the type-2 Wasserstein DRE problem with loss function $L(\cdot) = (\cdot)^2$. In this case, the asymptotic normalized squared error $\alpha^2_\star$ can be recovered from Proposition 4.11 under less restrictive assumptions than Theorem 4.5. In Figure 1(b) we plot in red the value of $\alpha^2_\star$ as a function of the radius $\varepsilon_0$. Moreover, we plot the normalized squared error averaged over 5 independent realizations for dimensions $d = 500, 1000,$ and $2500$, while keeping fixed the value of $\rho$ at 0.8. Similarly to the type-1 Wasserstein case, the asymptotic error curve captures very well the true behavior of the error even when the dimension is as low as 500.

We now study the effect of the under/over-parametrization $\rho = d/n$ on the normalized squared estimation error. We consider again the type-2 Wasserstein DRE problem described above for the two values $\rho = 0.8$ (under-parametrized DRE problem) and $\rho = 1.2$ (over-parametrized DRE problem). In Figure 2(a) we plot both the asymptotic and the non-asymptotic (for $d = 2500$) error curves, which show that a smaller estimation error can be obtained in the case $\rho = 0.8$, i.e., where the number of samples is larger than the problem dimension. Moreover, we explore numerically the validity of our theoretical results beyond the assumption that the features are Gaussian. For this, we consider the cases where the entries of the vectors $x_i$ are i.i.d. Bernoulli or Poisson random variables with the same mean and variance as in Assumption 1.1(i). We consider the type-2 Wasserstein DRE problem described above, and we plot the results in Figure 2(b), where it can be seen that the theoretical results allow us to find the right tuning of the radius $\varepsilon_0$ even in cases where the features fail to be Gaussian.

Finally, we numerically validate the type-2 Wasserstein distributionally regularized estimator presented in Theorem 4.13. For simplicity, we focus on the case where $L(\cdot) = (\cdot)^2$, so that the asymptotic error can be computed using the scalar minimax optimization problem in Corollary 4.15. We consider the exact same scenario as in the previous experiments, with $\sigma^2_{\text{theta}} = 1$ and $\sigma^2_Z = 0.1$. However, differently than above, the radius $\varepsilon_0$ is replaced by the regularization parameter $\lambda_0$. Since the theoretical guarantees only hold for $\lambda_0 \geq R^2_{\theta}$, with $R_{\theta}$ defined in Assumption 3.7, we are incentivized to pick the value of $R_{\theta}$ as close as possible to the norm of the true parameter $\theta_0$. We thus pick $R_{\theta} = \sigma_{\theta_0}$.
acknowledging that this value will result in an optimal solution $\alpha_*$ to the scalar optimization problem \( \mathcal{O} \) that is only an upper bound on the true asymptotic normalized squared error. We illustrate this in Figure 3. Notice that, in this case, picking a value of $R_\theta$ larger than $\sigma_\theta$ will result in a lower bound on $\lambda_0$ which is much larger than 1. This, in turn, will cause the asymptotic curve to miss the value of $\lambda_0$ which attains the minimum error. As in the previous cases, it is important to highlight that, although we only obtain an upper bound on the true estimation error in high dimensions, the asymptotic curve captures the true behavior of the error even for dimensions as low as $d = 500$.

6 Conclusion and Future Work

In this paper, we have addressed the problem of quantifying the performance of Wasserstein distributionally robust estimators in high dimensions. To do so, we have adopted a modern high-dimensional statistical viewpoint, and we have shown that in the high-dimensional proportional regime, and un-
der the assumption of isotropic Gaussian features, the squared error of such estimators can be easily recovered from the solution of a convex-concave minimax problem which involves at most four scalar variables. We envision numerous interesting directions in which the results of this paper can be employed, extended, or serve as starting point for future work:

1. **Beyond Gaussian features:** The analysis presented in this paper relies on the assumption that the measurement matrix $A$ has entries i.i.d. $\mathcal{N}(0,1/d)$. One future direction is to extend these results to the case where $A$ is an isotropically random orthogonal matrix, i.e., a matrix sampled uniformly at random from the manifold of row-orthogonal matrices satisfying $AA^\top = I$. This class of measurement matrices is known to be practically relevant due to the fact that their condition number is 1, and they do not amplify the noise. In [117] the authors extended the CGMT framework to recover the squared error of LASSO, under the assumptions that $A$ is isotropically random orthogonal and $Z$ is Gaussian. Similar ideas could be employed to study the performance of Wasserstein distributionally robust estimators isotropically random orthogonal measurement matrices.

2. **Fundamental limits and optimal tuning of $L$ and $\varepsilon$:** The results presented in this paper open the path towards answering the following two fundamental question: “What is the minimum estimation error $\alpha_{\min}$ achievable by $\hat{\theta}_{DRO}$ as a function of $p$?” and “How to optimally choose $L$ and $\varepsilon$ in order to achieve $\alpha_{\min}$?” So far, these and similar type questions have been addressed in the context of unregularized estimators [13, 31, 2], ridge-regularized estimation [116], and convex-regularized least-squares for linear models over structured (e.g., sparse, low-rank) signals [26]. In particular, the methodology proposed in [116] builds upon the CGMT framework, and could be employed to provide an answer to the aforementioned questions in the context of Wasserstein distributionally robust estimators.

3. **Binary models:** The analysis and results presented in this paper could be extended to binary observation models, following similar ideas as in [98, 115]. In particular, the problem fundamental limits and optimal tuning of $L$ and $\varepsilon$ could also be studied for these models, similarly to what was done in [115] for Empirical Risk Minimization estimators.

4. **f-Divergence distributionally robust estimation:** The reasoning presented in this paper could be employed to recover the error of other distributionally robust estimation formulations, where the ambiguity set is constructed using a different statistical distance on the probability space. In particular, the case where the statistical distance is chosen to be an f-Divergence has recently received a lot of attention in Operations Research and Machine Learning, due to its computational tractability [18, 104, 87], variance regularization effect [66, 67, 36], and optimality in terms of out-of-sample disappointment [124, 112, 19]. Recovering the error of f-Divergence distributionally robust estimators would enable a direct comparison between the performances of these two very popular distributionally robust formulations.

5. **Fundamental trade-offs in distributional adversarial training:** Most machine learning models are known to be highly vulnerable to small adversarial perturbations to their inputs [113]. For example, in the context of image classification, even small perturbations of the image, which are imperceptible to a human, can lead to incorrect classification by these models [49, 83]. To solve this issue, many works have proposed adversarial training as a principled and effective technique
to improve robustness of machine learning models against adversarial perturbations (see [105] for a survey). Nonetheless, while adversarial training has been successful at improving the accuracy of the trained model on adversarially perturbed inputs (robust accuracy), often this success comes at the cost of decreasing the accuracy on natural unperturbed inputs (standard accuracy). Understanding the tradeoff between the standard and robust accuracies has been the subject of many recent works, which considered the adversarial setting with \( \ell_p \)-norm-bounded perturbations [73, 100, 121, 95, 131, 61, 60, 29, 114, 30, 78]. However, the \( \ell_p \)-norm can be a poor adversarial setting when dealing, for example, with images: two similar images for human perception are not necessarily close under the \( \ell_p \)-norm, and a slight translation or rotation of an image may change the \( \ell_p \)-distance drastically. Interestingly, these limitations are generally not present when the adversarial attacks are modeled using the Wasserstein distance, which has been shown to effectively capture the geometric information in the image space (see [129, 130] and the references therein). Consequently, the Wasserstein adversarial model, often referred to as distribution shifts, has recently received considerable attention [107, 106, 94, 75, 71]. Despite this, an understanding of the fundamental tradeoff between the standard and robust accuracies in the Wasserstein adversarial training is still lacking. The results presented in this paper could serve as a starting point for this problem. Specifically, we believe that this problem could be solved by employing the same reasoning as in [61], which characterized this fundamental tradeoff for squared losses and \( \ell_2 \)-norm-bounded adversarial attacks by: (i) analyzing the Pareto optimal points of the two dimensional region consisting of all the achievable (standard risk, robust risk) pairs; (ii) characterizing precisely the algorithmic standard and robust risks achieved by the \( \ell_2 \)-norm adversarially trained estimator (this has been done in the high-dimensional regime, where \( d, n \to \infty \) with \( d/n = \rho \), using the CGMT framework); (iii) showing that, as \( \rho \) decreases, the algorithmic tradeoff curve approaches the Pareto-optimal tradeoff curve. In the context of Wasserstein adversarial training, the Pareto-optimal tradeoff curve from step (i) has been characterized in [75] for the particular case of type-2 Wasserstein discrepancy and squared losses, where the distributional robustness simplifies to the norm-regularized formulation presented in equation (26).

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A Proofs for Section 3

Proof of Lemma 3.3. Notice that (2) reduces to

\[
W_p \left( \mathbb{Q}, \widehat{\mathbb{P}}_n \right) = \inf_{\gamma \in \Gamma(\mathbb{Q}, \widehat{\mathbb{P}}_n)} E_{\gamma} \left[ \left\| x_1 - x_2 \right\|_p^p + \infty \left| y_1 - y_2 \right| \right]
\]

\[
= \inf_{\gamma \in \Gamma(\mathbb{Q}, \widehat{\mathbb{P}}_n)} E_{\pi(x_1, x_2)} \# \gamma \left[ \left\| x_1 - x_2 \right\|_p^p + E_{\pi(y_1, y_2)} \# \gamma [\infty \left| y_1 - y_2 \right|] \right],
\]
where \((\pi(X_1, X_2))\#\gamma\) and \((\pi(Y_1, Y_2))\#\gamma\) are the marginal distributions of \((X_1, X_2)\) and \((Y_1, Y_2)\), respectively. From this we see that the marginal distributions \((\pi_Y)\#Q\) and \((\pi_Y)\#\hat{P}_n\) need to be equal, otherwise the term \(E(\pi(Y_1, Y_2))\#\gamma[\infty|y_1 - y_2|]\) would become infinity. As a consequence, the ambiguity set \(\mathbb{B}_\varepsilon(\hat{P}_n)\) only contains distributions \(Q\) whose marginal distribution of \(Y\) is \(n^{-1}\sum_{i=1}^n \delta_{y_i}\). In this case, the term \(E(\pi(Y_1, Y_2))\#\gamma[\infty|y_1 - y_2|]\) becomes zero (for the optimal coupling), and from the ambiguity set definition, we see that the marginal distribution of \(X\) is \(\varepsilon\) close to \(n^{-1}\sum_{i=1}^n \delta_{x_i}\), when evaluated with respect to \(d_\rho\) defined in the statement of the lemma. This concludes the proof. \(\square\)

**Proof of Lemma 3.8.** Notice first that problem (13) is always convex in \((\theta, \lambda)\). This follows from the fact that \(\|	heta\|^2/\lambda\) is the perspective function of \(\|	heta\|^2\), and therefore it is jointly convex in \((\theta, \lambda)\). In what follows, we will show that assumption \(\varepsilon_0 \leq \rho^{-1}M^{-2}R_\theta^{-2}L_n\) guarantees the concavity in each \(u_i\) of the function

\[
    u_i(y_i - \theta^\top x_i) + \frac{u_i^2}{4\lambda} \|	heta\|^2 - L^*(u_i).
\]

(31)

Now, from Assumption 3.1(iv) we know that \(L\) is \(M\)-smooth. As a consequence, its convex conjugate \(L^*\) is \(1/M\)-strongly convex, and therefore can be decomposed as follows

\[
    L^*(u_i) = \frac{u_i^2}{2M} + f(u_i),
\]

for some convex function \(f\). Introducing this into (31), we obtain

\[
    u_i(y_i - \theta^\top x_i) + \frac{u_i^2}{4\lambda} \|	heta\|^2 - \frac{u_i^2}{2M} - f(u_i),
\]

which can be seen to be concave in \(u_i\) whenever \(\lambda\) satisfies

\[
    \lambda \geq \frac{M\|	heta\|^2}{2}.
\]

(32)

We will now translate this lower bound on \(\lambda\) in the upper bound on the ambiguity radius presented in the statement of the lemma. To do so, we rely on the following result from [21] which bounds the optimal solution \(\lambda_*(\theta)\) as a function of \(\theta\)

\[
    \lambda_*(\theta) \geq \frac{1}{2\sqrt{\varepsilon}} \sqrt{L_n\|	heta\|}.
\]

(33)

Notice that the lower bound presented in (33) contains the additional \(\sqrt{\varepsilon}\) compared to the one presented in [21]. This is because of the scaling by \(\sqrt{\varepsilon}\) of \(\lambda\) used in [21].

Now, using the bound \(\|	heta\| \leq R_\theta\sqrt{d}\) from Assumption 3.13.7 in (32), and the lower bound (33), we can conclude concavity in \(u_i\) if

\[
    \frac{MR_\theta\sqrt{d}||\theta||}{2} \leq \frac{1}{2\sqrt{\varepsilon}} \sqrt{L_n\|	heta\|}
\]

or, equivalently, the ambiguity radius \(\varepsilon\) satisfies

\[
    \varepsilon \leq \frac{L_n}{dM^2R_\theta^2}
\]

from which we obtain the upper bound on \(\varepsilon_0\) using the fact that \(d/n = \rho\). \(\square\)
Proof of Lemma 3.9. We will recover the stated bound on the growth rate of \( u_* \) from the growth rates of \( \lambda_* \) and \( \theta \), and the relationship between these three variables. For a bound on the growth rate of \( \lambda_* \), we rely on the following result from [21], which upper bounds \( \lambda_* \) by a function of the norm of \( \theta \), as follows

\[
\lambda_* \leq \left( \frac{1}{2} MR_\theta \sqrt{d} + \sqrt{\overline{L}_n} \right) \|\theta\| = \left( \frac{1}{2} \sqrt{\rho} MR_\theta + \sqrt{\frac{\overline{L}_n}{n}} \right) \|\theta\| \sqrt{n}. \tag{34}
\]

Notice that, differently from the upper bound presented in [21], the multiplication by \( \sqrt{\varepsilon} \) is missing from the right-hand side. This is because of the scaling by \( \sqrt{\varepsilon} \) of \( \lambda \) used in [21].

We will now turn to (13) and find the precise relationship between \( u_* \), \( \lambda_* \), and \( \theta_* \). Due to Lemma 3.8, we know that the objective function in (13) is concave in \( u \). Moreover, it is clear that the objective function is convex in \( \theta \) and \( \lambda \). These, together with the upper bound (34), allow us to rely on Sion’s minimax principle to exchange the order of the minimization over \( \lambda \) and the maximization over \( u \) in (13) without changing the optimal value. We thus obtain

\[
\min_{\theta \in \Theta} \sup_{u \in \mathbb{R}^n} \inf_{\lambda \in \Lambda} \lambda_* \varepsilon + \frac{1}{n} \sum_{i=1}^n u_i(y_i - \theta^T x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - L^*(u_i). \tag{35}
\]

We can now solve the inner minimization over \( \lambda \), and recover the optimal solution

\[
\lambda_*(\theta, u_*) = \begin{cases} 
M R_\theta \sqrt{d} \|\theta\| / 2 & \text{if } \|\theta\| \|u_*\| / (2\sqrt{n\varepsilon}) \leq M R_\theta \sqrt{d} \|\theta\| / 2 \\
\|\theta\| \|u_*\| / (2\sqrt{n\varepsilon}) & \text{otherwise.}
\end{cases}
\]

In the first case, when \( \|\theta\| \|u_*\| / (2\sqrt{n\varepsilon}) \leq M R_\theta \sqrt{d} \|\theta\| / 2 \), we have that \( \|u_*\| \leq \sqrt{\varepsilon_0 d M R_\theta} = \sqrt{\varepsilon_0 \rho} M R_\theta \sqrt{n} \), which gives the desired bound on \( \|u_*\| \). In the second case, the bound on \( u_* \) can be obtained from the upper bound (34), as follows

\[
\lambda_*(\theta, u_*) = \frac{1}{(2\sqrt{n\varepsilon})} \|\theta\| \|u_*\| \leq \left( \frac{1}{2} \sqrt{\rho} M R_\theta + \sqrt{\frac{\overline{L}_n}{n}} \right) \|\theta\| \sqrt{n},
\]

from which we recover the upper bound on the norm of \( u_* \),

\[
\|u_*\| \leq 2\sqrt{\varepsilon_0} \left( \frac{1}{2} \sqrt{\rho} M R_\theta + \sqrt{\frac{\overline{L}_n}{n}} \right) \sqrt{n}.
\]

Finally, the desired result follows by noticing that \( \overline{L}_n \) has growth rate at most \( n \), due to Assumption (ii).

Proof of Lemma 3.11. We will prove the concavity in each \( u_i \) of the objective function in the inner supremum in (16), i.e.,

\[
\sup_{u_i \in \mathbb{R}^n} u_i(y_i - \theta^T x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - L^*(u_i). \tag{35}
\]

From Assumption 3.1(iv) we know that \( L \) is \( M \)-smooth. As a consequence, its convex conjugate \( L^* \) is \( 1/M \)-strongly convex, and therefore can be decomposed as follows

\[
L^*(u_i) = \frac{u_i^2}{2M} + f(u_i),
\]
for some convex function $f$. Introducing this into (31), we obtain

$$u_i(y_i - \theta^T x_i) + \frac{u_i^2}{4\lambda} \|\theta\|^2 - \frac{u_i^2}{2M} - f(u_i),$$

which can be seen to be concave in $u_i$ whenever $\lambda$ satisfies

$$\lambda \geq \frac{M\|\theta\|^2}{2}.$$

However, this follows automatically from the assumption that $\lambda \geq MR^2d/2$, since $\|\theta\| \leq R\theta\sqrt{d}$. $\square$

**Proof of Lemma 3.12.** Since this result will be used in the proof of Theorem 4.13, we will use the same formulation as there. Therefore, we introduce the change of variable $w = (\theta - \theta_0)/\sqrt{d}$, and express (16) in vector form, resulting in

$$\min_{w \in \mathcal{W}} \max_{u \in \mathbb{R}^n} -\frac{1}{n} u^T(\sqrt{d}A)w + \frac{1}{n} u^T z + \frac{1}{4\lambda n} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{1}{n} \sum_{i=1}^n L^*(u_i).$$

(36)

Here, $\mathcal{W}$ is the feasible set of $w$ obtained from $\Theta$ after the change of variable, $A \in \mathbb{R}^{n \times d}$ is a matrix whose rows are the vectors $x_i$, for $i = 1, \ldots, n$, and $z \in \mathbb{R}^n$ is the measurement noise vector with entries i.i.d. distributed according to $\mathcal{P}_2$.

We will prove that the constraint $u \in \mathbb{R}^n$ can be restricted, without loss of generality, to $\{u \in \mathbb{R}^n : \|u\| \leq K_\beta \sqrt{n}\}$, for some constant $K_\beta > 0$. For this, we will proceed in two steps. We will first show that for any $w \in \mathcal{W}$, the optimizer $u_{*,1}$ of

$$\max_{u \in \mathbb{R}^n} -\frac{1}{n} u^T(\sqrt{d}A)w + \frac{1}{n} u^T z + \frac{1}{4\lambda n} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{1}{n} \sum_{i=1}^n L^*(u_i)$$

(37)

is upper bounded by the optimizer $u_{*,2}$ of the following optimization problem

$$\max_{u \in \mathbb{R}^n} -\frac{1}{n} u^T(\sqrt{d}A)w + \frac{1}{n} u^T z + \frac{1}{4\lambda n} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{\|u\|^2}{2nM}.$$ 

(38)

Secondly, we will show that the optimizer $u_{*,2}$ satisfies $\|u_{*,2}\| \leq K_\beta \sqrt{n}$, for some $K_\beta > 0$.

The first step follows easily by noticing that the objective function in (37) is a difference of convex functions, with the subtracting convex function $\sum_{i=1}^n L^*(u_i)/n$ which satisfies the following two properties $\min_{u \in \mathcal{R}} L^*(u) = L^*(0) = 0$, and $L^*(v) \geq v^2/(2M) \geq 0$. In particular, the first property follows from the fact that $\min_{v \in \mathcal{R}} L(v) = L(0) = 0$, and the second property follows from the additional fact that $L^*$ is $1/M$-strongly convex. Consequently, due to these properties, we have that $u_{*,2} \geq u_{*,1}$.

We will now focus on the second step. Notice first that the assumptions $\lambda > MR^2d/2$ and $\|\theta\|^2 = \|\theta_0 + \sqrt{d}w\|^2 \leq R_d^2d$ ensure that the objective function in (38) is concave in $u$. Indeed, can be easily seen from the concavity in $u$ of

$$\frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{\|u\|^2}{2M}.$$

From the first order optimality conditions in (38), we have that

$$u_{*,2} = \left(\frac{\|\theta_0 + \sqrt{d}w\|^2}{2\lambda} - \frac{1}{M}\right)^{-1} \left(\sqrt{d}Aw - z\right).$$

(39)

Now, since $\lambda > MR^2d/2 \geq M\|\theta_0 + \sqrt{dw_\star}\|^2/2$, we have that there exists a constant $c_1$ such that

$$\left(\frac{\|\theta_0 + \sqrt{d}w\|^2}{2\lambda} - \frac{1}{M}\right)^{-1} < c_1.$$
Moreover, we have that there exist constants $c_2, c_3$ such that, w.p.a. 1, $\|A\| \leq c_2$ (since $A$ is a Gaussian matrix with entries i.i.d. $\mathcal{N}(0, 1/d)$), and $\|z\| \leq c_3 \sqrt{n}$ (due to the weak law of large numbers). Finally, using the fact that $w$ lives in the compact set $\mathcal{W}$, we have that there exists some constant $K_\beta$ such that $\|u_*\| \leq K_\beta \sqrt{n}$. This concludes the proof.

B Proofs for Section 4

Proof of Theorem 4.3. The proof is an application of the main results of [120] to the particular regularized estimation problem (12). In what follows, we explain why all the assumptions required by [120] hold, and why their result simplifies to (20) in our case.

First of all, notice that in (12) the loss function is separable, while the regularizer function is not. As a consequence, the minimax scalar problem (20) is obtained as a combination of Theorem 1 (general case) and Theorem 2 (separable case) from [120]. Therefore, we will verify that the loss function and the noise satisfy the assumptions of Theorem 2 and that the regularizer function satisfies the assumptions of Theorem 1.

We start by rewriting (12) in the following form

$$\min_{\theta \in \mathbb{R}^d} \frac{1}{d} \left( \sum_{i=1}^{n} L(y_i - \theta^T x_i) + \sqrt{n} \varepsilon_0 \text{Lip}(L) \|\theta\| \right).$$

The normalization factor $1/d$ guarantees that the optimal cost is of constant order. Moreover, the formulation (12) is now in the form required by [120] (see the proof of Theorem 1 for details), with loss function $\sum_{i=1}^{n} L(\theta^T x_i - y_i)$ and regularizer $\sqrt{n} \varepsilon_0 \text{Lip}(L) \|\theta\|$.

We first show that the loss function and noise satisfy the assumptions of Theorem 2 from [120]:

1. The loss function and noise are such that

$$\mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} \left[ L'(cG + Z)^2 \right] < \infty.$$  \hfill (41)

Proof: From Assumptions 3.1 (i)-(ii) we know that $L$ is continuous, convex and grows linearly at infinity, with slope at most $C$. Therefore, $L'_+$ is bounded and the quantity in (41) is finite. Alternatively, this can also be seen from Assumption (iv), which guarantees that $L'$ is continuous, and the fact that $L'$ is at most $C$ at infinity (due to Assumption 3.1(ii)). This assumption has been shown in [120] to guarantee that the quantity $e_L(cG + Z; \tau) - L(Z)$ under the expectation in (17) is absolutely integrable.

2. Either $\mathbb{E}_Z[Z^2] < \infty$, i.e., $Z$ has finite second moment, or $\sup_{u \in \mathbb{R}} |L(u)|/|u| < \infty$, i.e., $L$ grows at most linearly at infinity.

Proof: As discussed above, in our case the latter condition holds. Notice that from Assumption 3.1(iii), which states that the true distribution $\mathbb{P}$ has finite first moment, $Z$ necessarily has finite first moment, but the second moment might be infinite.

We now show that the objective function from Theorem 1 in [120] reduces to (20) in our case. For this, we focus on (40), and consider $\varepsilon_0 \text{Lip}(L)$ as the regularization parameter, and $\sqrt{n} \|\theta\|$ as the regularizer. Consequently, we need to study the convergence in probability of the quantity

$$\frac{1}{d} \left( e_{\sqrt{n} \|\cdot\|}(ch + \theta_0; \tau) - \sqrt{n} \|\theta_0\| \right)$$

(42)
with \( h \) a standard Gaussian vector, with entries \( \mathcal{N}(0, 1) \), and show that its limit is exactly the function \( G \) defined in (18). First notice that, due to Assumption 4.1(ii), we have

\[
\frac{1}{d} \sqrt{n} \|\theta_0\| \xrightarrow{P} \frac{\sigma_{\theta_0}}{\sqrt{\rho}}.
\]  
(43)

Now let’s concentrate on the first term, which involves the Moreau envelope of \( \sqrt{n} \cdot \). After some basic convex optimization steps, it can be seen that

\[
e_{\sqrt{n} \cdot}((ch + \theta_0); \tau) = \begin{cases} 
\sqrt{n} \|ch + \theta_0\| - n\tau/2 & \text{if } \|ch + \theta_0\| > \sqrt{n}\tau, \\
\|ch + \theta_0\|^2/(2\tau) & \text{if } \|ch + \theta_0\| \leq \sqrt{n}\tau.
\end{cases}
\]  
(44)

Finally, the convergence in probability to \( G(c, \tau) \) follows from

\[
\frac{\|ch + \theta_0\|^2}{d} \xrightarrow{P} c^2 + \sigma_{\theta_0}^2,
\]  
(45)

using Assumption 4.1(ii) and Lemma C.1.

We now show that \( G(c, \tau) \) satisfies the following two assumptions required by Theorem 1 in [120]:

1. \( \lim_{\tau \to 0^+} G(\tau, \tau) = 0. \)

   \textbf{Proof:} For \( \tau \to 0^+ \), we have \( \|\tau h + \theta_0\| > \sqrt{n}\tau \), and thus \( G(\tau, \tau) \) satisfies

\[
\lim_{\tau \to 0^+} G(\tau, \tau) = \lim_{\tau \to 0^+} \frac{\sqrt{\tau^2 + \sigma_{\theta_0}^2}}{\sqrt{\rho}} - \frac{\tau}{2\rho} - \frac{\sigma_{\theta_0}}{\sqrt{\rho}} = \frac{\sigma_{\theta_0}}{\sqrt{\rho}} - \frac{\sigma_{\theta_0}}{\sqrt{\rho}} = 0.
\]

2. \( \lim_{c \to +\infty} (c^2/(2\tau) - G(c, \tau)) = +\infty \) for all \( \tau > 0. \)

   \textbf{Proof:} For \( c \to +\infty \), we have \( \|ch + \theta_0\| > \sqrt{n}\tau \), and thus \( G(c, \tau) \) satisfies

\[
G(c, \tau) = \frac{c^2 + \sigma_{\theta_0}^2}{\sqrt{\rho}} - \frac{\tau}{2\rho} - \frac{\sigma_{\theta_0}}{\sqrt{\rho}},
\]

from which the desired limit follows immediately.

Finally, we show that for the particular case of regularized estimators of the form (12), the definition of the expected Moreau envelope \( \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} \left[ e_L(cG + Z; \tau) - L(Z) \right] \) used in [120] can be simplified to the expected Moreau envelope \( L(c, \tau) \) defined in (17), i.e., the term \(-L(Z)\) can be dropped. In [120], the term \(-L(Z)\) was added to account for the case when the noise \( Z \) has unbounded moments. In that case, the sequence

\[
\frac{1}{n} \sum_{i=1}^{n} e_L(\alpha G + Z, \frac{\tau_1}{\beta})
\]  
(46)

might not converge as \( d \to \infty \), while by subtracting \( L(Z) \) it is shown to converge. In our case, the convergence of (46) is automatically guaranteed by Assumption 3.1(ii) on the growth rate of the loss function \( L \), even though the noise \( Z \) might have unbounded second moment. Indeed, its limit (in probability) can be immediately recovered from the weak law of large numbers as

\[
\frac{1}{n} \sum_{i=1}^{n} e_L(\alpha G + Z, \frac{\tau_1}{\beta}) \xrightarrow{P} \mathbb{E}_{\mathcal{N}(0,1) \otimes \mathbb{P}_Z} \left[ e_L(\alpha G + Z, \frac{\tau_1}{\beta}) \right],
\]  
(47)
which is precisely the expected Moreau envelope. The quantity under the expectation in (47) is absolutely integrable, and therefore the expected Moreau envelope is well defined, as explained in what follows. For every \( \beta \geq 0 \) and \( \tau_1 > 0 \) we have
\[
\int_{\mathbb{R}} \epsilon_L(\alpha G + Z; \frac{\tau_1}{\beta}) = \min_{v \in \mathbb{R}} \beta \left( \frac{\tau_1}{2\tau_1} (\alpha G - v)^2 + L(v) \right)
\]
which is integrable due to the fact that both \( G \) and \( Z \) have finite first moment. In particular, the fact that \( Z \) has finite first moment is a direct consequence of Assumption 3.1(iii).

The result now follows from Theorem 1 and Theorem 2 in [120].

\[ \square \]

**Proof of Theorem 4.5.** Since the proof is long, we divide it into steps, which we briefly explain before jumping into the technical details.

**Step 1: We show that problem (1) can be equivalently rewritten as a (PO) problem.** The proof builds upon the dual formulation (13) presented in Fact 3.6. After introducing the change of variable \( w = (\theta - \theta_0)/\sqrt{\alpha} \) and expressing it in vector form, (13) becomes
\[
\min_{w \in \mathcal{W}, \lambda \geq 0} \max_{u \in \mathbb{R}^n} \lambda z - \frac{1}{n} \sum_{i=1}^{n} \theta_i \theta_i^\top \sum_{i=1}^{n} u_i^\top (\sqrt{\alpha} A) w + \frac{1}{n} \sum_{i=1}^{n} u_i^\top z + \frac{1}{4\lambda n} \sum_{i=1}^{n} \theta_i^2 + \sqrt{\alpha} \| u \|^2 - \frac{1}{n} \sum_{i=1}^{n} L^*(u_i) \tag{48}
\]
where \( \mathcal{W} \) is the feasible set of \( w \) obtained from \( \Theta \) after the change of variable, \( A \in \mathbb{R}^{n \times d} \) denotes the matrix whose rows are the vectors \( z_i \), for \( i = 1, \ldots, n \), and \( z \in \mathbb{R}^n \) is the measurement noise vector with entries i.i.d. distributed according to \( \mathbb{P}_Z \). Since we are interested only in the case where \( w \) satisfies \( \| w \| < \sigma_{\theta_0} \), instead of the constraint set \( \mathcal{W} \), we will work with a set of the form \( \mathcal{S}_w := \{ w \in \mathbb{R}^d : \| w \| \leq K_\alpha \} \), for some \( K_\alpha < \sigma_{\theta_0} \). This limitation on the magnitude of \( w \), which might seem arbitrary and not necessary at this point, will be shown to be fundamental later in the proof. Therefore, from now on we impose such a constraint on \( w \) (i.e., \( w \in \mathcal{S}_w \)). Notice that if \( K_\alpha < \sigma_{\theta_0} \), then the condition \( R_{\theta} \geq 2\sigma_{\theta_0} \) ensures that \( \mathcal{S}_w \subset \mathcal{W} \) w.p.a. 1 as \( d, n \to \infty \).

Due to Lemma 3.8, we know that \( \epsilon_0 \leq \rho^{-1} M^{-2} R_{\theta}^{-2} L_{\alpha} \) guarantees that \( \lambda \geq M R_{\theta} \sqrt{\alpha} \| \theta_0 + \sqrt{\alpha} d \| / 2 \). As a consequence, we define the set \( \Lambda := \{ \lambda \in \mathbb{R} : \lambda \geq M R_{\theta} \sqrt{\alpha} \| \theta_0 + \sqrt{\alpha} d \| / 2 \} \), and equivalently rewrite (49) as
\[
\min_{w \in \mathcal{S}_w, \lambda \in \Lambda} \max_{u \in \mathbb{R}^n} \lambda z - \frac{1}{n} \sum_{i=1}^{n} \theta_i \theta_i^\top \sum_{i=1}^{n} u_i^\top (\sqrt{\alpha} A) w + \frac{1}{n} \sum_{i=1}^{n} u_i^\top z + \frac{1}{4\lambda n} \sum_{i=1}^{n} \theta_i^2 + \sqrt{\alpha} \| u \|^2 - \frac{1}{n} \sum_{i=1}^{n} L^*(u_i) \tag{49}
\]
Moreover, from Lemma 3.9 we know that the optimal solution \( u^* \) is in the order of \( \sqrt{n} \). As a consequence, by rescaling the variable \( u \) as \( u \to u/\sqrt{n} \), and introducing the convex compact set \( \mathcal{S}_u := \{ u \in \mathbb{R}^n : \| u \| \leq K_\beta \} \), for some sufficiently large \( K_\beta > 0 \), we can equivalently rewrite (49) as
\[
\min_{w \in \mathcal{S}_w, \lambda \in \Lambda} \max_{u \in \mathcal{S}_u} \lambda z - \frac{1}{n} \sum_{i=1}^{n} \theta_i \theta_i^\top \sum_{i=1}^{n} u_i^\top (\sqrt{\alpha} A) w + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} u_i^\top z + \frac{1}{4\lambda n} \| \theta_0 + \sqrt{\alpha} d \|^2 \| u \|^2 - \frac{1}{n} \sum_{i=1}^{n} L^*(u_i/\sqrt{n}) \tag{50}
\]
We will now show that the assumption that \( L \) can be written as the Moreau envelope with parameter \( M \) of a convex function \( f \) is natural, and always satisfied for \( M \)-smooth functions (which is precisely our case, due to Assumption 3.1(iv)). Since \( L \) is \( M \)-smooth, we have that its convex conjugate \( L^* \) is \( 1/M \)-strongly convex, and therefore \( L^* \) can be written as
\[
L^*(\cdot) = \frac{1}{2M} (\cdot)^2 + f^*(\cdot), \tag{51}
\]
where $f^*$ is the conjugate of a convex function $f$. Notice that $f$ can be easily obtained (as a function of $L$) from (51) as $f = (L^* - 1/(2M)(\cdot)^2)^*$. Conversely, the loss function $L$ is nothing but the Moreau envelope of $f$, as can be seen from the following

$$L(\cdot) = \left( \frac{1}{2M}(\cdot)^2 + f^*(\cdot) \right)^* = \left( \frac{M}{2}(\cdot)^2 \ast_{\text{inf}} f(\cdot) \right)(\cdot) = e_f \left( \cdot, \frac{1}{M} \right),$$

where $\ast_{\text{inf}}$ denotes the infimal convolution operation. Since $L$ is continuous and convex (notice that $L$ is trivially proper due to Assumption 4.1(i)), with $\min_{\nu \in \mathbb{R}} L(\nu) = 0$, we have that $L^*$ is lower semicontinuous and convex, and therefore both $f$ and $f^*$ are lower semicontinuous and convex.

Using the decomposition (51), problem (50) can be rewritten as

$$\min_{u \in S_u, \lambda \in \Lambda} \max_{w \in S_w} \lambda e - \frac{1}{\sqrt{n}} u^T(\sqrt{d}A)w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{1}{2M} \|u\|^2 - \frac{1}{n} \sum_{i=1}^n f^*(u_i \sqrt{n}).$$

(52)

It is important now to highlight that the expression

$$\frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{1}{2M} \|u\|^2$$

is concave is $u$. This follows from the constraint $\lambda \in \Lambda$, which ensures that $\lambda \geq MR_\theta \sqrt{d}/2 \geq M\|\theta\|^2/2 = M\|\theta_0 + \sqrt{d}w\|^2/2$. As will be shown later, the decomposition (51) and the concavity in $u$ of the above-mentioned expression will allow us to use the CGMT.

For ease of notation, we introduce the function $F^*: \mathbb{R}^n \to \mathbb{R}$, defined as follows

$$F^*(v) := \sum_{i=1}^n f^*(v_i).$$

Since $f, f^*$ are lower semicontinuous and convex, we have that $F^*$ is lower semicontinuous, and convex, and therefore its convex conjugate $F = F^{**}$ can be easily recovered as follows

$$F(u) = \sup_{v \in \mathbb{R}^n} v^T u - F^*(v) = \sum_{i=1}^n \sup_{v_i \in \mathbb{R}} v_i u_i - f^*(v_i) = \sum_{i=1}^n f(u_i).$$

We now rewrite $F^*(u \sqrt{n})$ using its convex conjugate as follows

$$\min_{u \in S_u} \max_{\lambda \in \Lambda, s \in \mathbb{R}^n} \lambda e - \frac{1}{\sqrt{n}} u^T(\sqrt{d}A)w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2 - \frac{1}{2M} \|u\|^2 - \frac{1}{\sqrt{n}} s^T u + \frac{1}{n} F(s)$$

(53)

where we have used Sion’s minimax principle to exchange the minimization over $s$ with the maximization over $u$. In particular, this last step is possible only because the objective function in (53) is concave in $u$, due to the decomposition (51).

Since the objective function in (53) is convex-concave in $(\lambda, u)$ and $S_u$ is a convex compact set, we can use Sion’s minimax principle to exchange the minimization over $\lambda$ to obtain the equivalent optimization problem

$$\min_{u \in S_u} \max_{\lambda \in \Lambda, s \in \mathbb{R}^n} \lambda e - \frac{1}{\sqrt{n}} u^T(\sqrt{d}A)w + \frac{1}{\sqrt{n}} u^T z - \frac{1}{2M} \|u\|^2 - \frac{1}{\sqrt{n}} s^T u + \frac{1}{n} F(s) + \min_{\lambda \in \Lambda} \lambda e + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 \|u\|^2$$

(54)
Notice that the new objective function in (54) (which contains the minimization over $\lambda$) is convex in $(w, s)$ and concave in $u$. Indeed, the concavity in $u$ follows easily, since the objective function in (54) is the result of minimization of concave functions. Moreover, the convexity in $w$ follows from the fact that the objective function in (53) is jointly convex in $(w, \lambda)$ and partial minimization of jointly convex functions remains convex. Indeed, the joint convexity in $(w, \lambda)$ can be seen by noticing that $1/(4\lambda)\|\theta_0 + \sqrt{d}w\|u\|^2$ is nothing but a shifted version of the perspective function of $1/4\|\sqrt{d}w\|u\|^2$. Finally, the convexity in $s$ can be concluded from the convexity of $F$, and the joint convexity in $(w, s)$ follows easily since $w$ and $s$ are decoupled.

Using Assumption 1.1, we can see that $\sqrt{d}A$ has entries i.i.d. $\mathcal{N}(0, 1)$. Moreover, the objective function (54) is convex-concave in $(w, u)$, with $S_w, S_u$ convex compact sets. As a consequence, problem (54) is a (PO) problem, in the form (3).

**Step 2:** We apply CGMT and obtain two modified (AO) problems. Since (54) is a (PO) problem, we can associate to it the following (AO) problem on the optimal value of the modified (AO) problem.

We now proceed by eliminating the variable $\lambda$ from (56). The objective function is convex in $\lambda$, and its optimizer $\lambda_*$ can be easily computed as follows

$$
\lambda_*(w, u) = \begin{cases} 
\frac{\|\theta_0 + \sqrt{d}w\|u\|}{2\sqrt{\varepsilon}} & \text{if } \|\theta_0 + \sqrt{d}w\|u\|/(2\sqrt{\varepsilon}) > M R \sqrt{d}\|\theta_0 + \sqrt{d}w\|/2 \\
M R \sqrt{d}\|\theta_0 + \sqrt{d}w\|/2 & \text{otherwise},
\end{cases}
$$

or, equivalently,

$$
\lambda_*(w, u) = \begin{cases} 
\sqrt{\varepsilon}\|\theta_0 + \sqrt{d}w\|u\|/(2\sqrt{\varepsilon_0}) & \text{if } \|u\| > B := \sqrt{\varepsilon_0 R} M R \\
M R \sqrt{d}\|\theta_0 + \sqrt{d}w\|/2 & \text{otherwise},
\end{cases}
$$

where we have used the fact that $\varepsilon = \varepsilon_0/n$. 

As a consequence, by introducing the optimal solution \( \lambda_* \) into (56), we can equivalently rewrite problem (56) as

\[
\max \{ V_{d,1}, V_{d,2} \} \tag{58}
\]

with

\[
V_{d,1} := \max_{B < \beta \leq K_\beta} \min_{w \in S_w, ||u|| = \beta} \max_{s \in \mathbb{R}^n} \frac{1}{\sqrt{n}} ||w||g^T u - \frac{1}{\sqrt{n}} ||u||h^T w + \frac{1}{\sqrt{n}} u^T z - \frac{1}{2M} ||u||^2 - \frac{1}{\sqrt{n}} s^T u + \frac{1}{n} F(s) + \frac{\sqrt{\epsilon_0}}{\sqrt{n}} ||\theta_0 + \sqrt{d}w||u||, \tag{59}
\]

and

\[
V_{d,2} := \max_{0 \leq \beta \leq B} \min_{w \in S_w, ||u|| = \beta} \max_{s \in \mathbb{R}^n} \frac{1}{\sqrt{n}} ||w||g^T u - \frac{1}{\sqrt{n}} ||u||h^T w + \frac{1}{\sqrt{n}} u^T z - \frac{1}{2M} ||u||^2 - \frac{1}{\sqrt{n}} s^T u + \frac{1}{n} F(s) + \frac{\|\theta_0 + \sqrt{d}w\|}{\sqrt{n}} \left( p + \frac{||u||^2}{q} \right), \tag{60}
\]

where, for ease of notation, we have introduced the two constants \( p := (\epsilon_0 \sqrt{pM} R_0)/2 \) and \( q := 2\sqrt{p} M R_0 \).

In what follows, we will first simplify the two problems (59) and (60), by reducing them to scalar problems. Then, following the reasoning of Fact 2.1, we will study their asymptotic optimal values for \( d, n \to \infty \). We first focus on problem (59), but similar reasoning will be used subsequently for problem (60).

**Step 3.1:** We consider the first modified (AO) problem (59), and show that it can be reduced to a scalar problem, which involves only four scalar variables. We start by scalarizing problem (59) over the magnitude of \( u \), leading to the following problem

\[
\max_{B < \beta \leq K_\beta} \min_{w \in S_w} \frac{\beta}{\sqrt{n}} ||w||g + z - s|| - \frac{\beta}{\sqrt{n}} h^T w + \frac{\sqrt{\epsilon_0} \beta}{\sqrt{n}} \|\theta_0 + \sqrt{d}w\| - \frac{\beta^2}{2M} + \frac{1}{n} F(s). \tag{61}
\]

We now employ the square-root trick to rewrite the first term in the objective function of (61) as follows

\[
\frac{1}{\sqrt{n}} ||w||g + z - s|| = \inf_{\tau_1 > 0} \frac{\tau_1}{2} + \frac{1}{2n\tau_1} ||w||g + z - s||^2.
\]

Introducing this in (61) and re-organizing the terms gives

\[
\max_{B < \beta \leq K_\beta} \min_{w \in S_w, \tau_1 > 0} \frac{\beta}{2} \frac{\tau_1}{2} - \frac{\beta^2}{2M} + \frac{\beta}{2n\tau_1} \|w||g + z - s||^2 + \frac{1}{n} F(s) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\sqrt{\epsilon_0} \beta}{\sqrt{n}} \|\theta_0 + \sqrt{d}w\|,
\]

which allows us to introduce the Moreau envelope of \( F \), i.e.,

\[
\epsilon_F(\alpha g + z, \frac{\tau_1}{\beta}) := \min_{s \in \mathbb{R}^n} \frac{\beta}{2\tau_1} ||\alpha g + z - s||^2 + F(s), \tag{62}
\]

giving rise to

\[
\max_{B < \beta \leq K_\beta} \min_{w \in S_w, \tau_1 > 0} \frac{\beta}{2} \frac{\tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} \epsilon_F(||w||g + z, \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\sqrt{\epsilon_0} \beta}{\sqrt{n}} \|\theta_0 + \sqrt{d}w\|. \tag{63}
\]
We employ the square-root trick a second time and rewrite the last term in the objective function of problem (63) as
\[
\frac{1}{\sqrt{n}}\|\theta_0 + \sqrt{d}w\| = \inf_{\tau_2 > 0} \frac{\tau_2}{2} + \frac{1}{2n\tau_2} \|\theta_0 + \sqrt{d}w\|^2.
\]
leading to
\[
\max_{\begin{subarray}{l}B<\beta \leq K_\beta \\ \tau_1,\tau_2 > 0\end{subarray}} \min_{w \in S_w} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\|w\|g + z, \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^\top w + \frac{\sqrt{\epsilon_0} \beta \tau_2}{2} + \frac{\sqrt{\epsilon_0} \beta}{2n\tau_2} \|\theta_0 + \sqrt{d}w\|^2.
\]

We now proceed with the aim of scalarizing the problem over the magnitude of \( w \). Similarly to what was done for \( u \), we can separate the constraint \( w \in S_w \) into the two constraints \( 0 \leq \alpha \leq K_\alpha \) and \( \|w\| = \alpha \), and by expressing \( \|\theta_0 + \sqrt{d}w\|^2 \) as \( \|\theta_0\|^2 + d\|w\|^2 + 2\sqrt{d}\|\theta_0\|w \), we can now scalarize the problem over the magnitude of \( w \) and obtain
\[
\max_{\begin{subarray}{l}B<\beta \leq K_\beta \\ 0 \leq \alpha \leq K_\alpha \\ \tau_1,\tau_2 > 0\end{subarray}} \inf_{\begin{subarray}{l}w \in S_w \\ \alpha \in S_\alpha\end{subarray}} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) - \frac{\alpha \beta}{\sqrt{n}} \frac{\|\theta_0\| - h\|}{\tau_2} + \frac{\sqrt{\epsilon_0} \beta}{2n\tau_2} \left( \|\theta_0\|^2 + d\alpha^2 \right).
\]

We denote by \( O_{d,1}(\alpha, \tau_1, \tau_2, \beta) \) the objective function in problem (66), parametrized by the dimension \( d \), where the index 1 recalls the fact that we are focusing on the first problem (59).

**Step 3.2:** We show that \( O_{d,1} \) is continuous on its domain, jointly convex in \((\alpha, \tau_1, \tau_2)\), and concave in \( \beta \). Let’s first concentrate on the continuity property. Notice that we only need to show that the Moreau envelope \( e_F(\alpha g + z, \tau_1/\beta) \) is continuous, since all the other terms in the objective function are trivially continuous. Since \( F \) is lower semicontinuous and convex (recall that \( F(u) = \sum_{i=1}^n f(u_i) \)), the continuity of the Moreau envelope \( e_F(\alpha g + z, \tau_1/\beta) \) follows from [96, Theorem 2.26(b)].

Let’s now focus on the joint convexity in \((\alpha, \tau_1, \tau_2)\). Since \( \tau_1 \) and \( \tau_2 \) are decoupled, it will be enough to prove the joint convexity in \((\alpha, \tau_1)\) and \((\alpha, \tau_2)\). We first concentrate on the pair \((\alpha, \tau_1)\), and notice that it suffices to prove that the objective function on the right-hand side of (62) is jointly convex in \((\alpha, \tau_1, s)\). Then, after minimizing over \( s \in \mathbb{R}^n \), the Moreau envelope \( e_F(\alpha g + z, \tau_1/\beta) \) remains jointly convex in \((\alpha, \tau_1)\). But this is certainly the case since \( \|\alpha g - s\|^2 \) is jointly convex in \((\alpha, s)\), \( 1/\tau_1 \|\alpha g - s\|^2 \) is the perspective function of \( \|\alpha g - s\|^2 \), and therefore jointly convex in \((\alpha, \tau_1, s)\), and the shifted function \( 1/\tau_1 \|\alpha g + z - s\|^2 \) remains jointly convex.

We now concentrate on the pair \((\alpha, \tau_2)\). Proving the joint convexity will not be as straightforward, due primarily to the minus sign in front of the fifth term of \( O_{d,1} \). Moreover, due to the randomness of the objective function (coming from the random vector \( h \)), the joint convexity will be shown to hold true w.p.a. 1 as \( d \to \infty \). However, this is not restrictive for the asymptotic study of interest here. We start by considering the last two terms in the objective function \( O_{d,1} \) (notice that the linear term \( \sqrt{\epsilon_0} \beta \tau_2/2 \) does not affect the convexity). Since \( \beta, \epsilon_0, \rho, \|\theta_0\| \) have no influence on the convexity, and \( \beta > 0 \), we can simplify the part of the objective function containing the last two terms, which after some manipulations becomes
\[
-\alpha \left( \frac{1}{\tau_2} \frac{\theta_0}{\|\theta_0\|} - 
\frac{1}{\sqrt{\rho \epsilon_0} \|\theta_0\|} \frac{h}{\|\theta_0\|} \right) + \frac{1}{\tau_2} \left( \frac{\|\theta_0\|^2/d + \alpha^2}{2 \|\theta_0\|/\sqrt{d}} \right)
\]
or, equivalently,
\[
-\alpha \sqrt{\frac{1}{\tau_2^2} + \frac{\|h\|^2}{\rho \epsilon_0 \|\theta_0\|^2} - \frac{1}{\tau_2} \frac{2 \theta_0^\top h}{\sqrt{\rho \epsilon_0} \|\theta_0\|^2}} + \frac{1}{\tau_2} \left( \frac{\|\theta_0\|^2/d + \alpha^2}{2 \|\theta_0\|/\sqrt{d}} \right).
\]
Due to Assumption 4.1(ii) we know that \( \|\theta_0\| / \sqrt{d} \xrightarrow{P} \sigma_{\theta_0} \). In addition, \( \|h\| / \sqrt{d} \xrightarrow{P} 1 \) follows from the weak law of large numbers. As a consequence, the term \( \|h\|^2 / (\rho \varepsilon_0 \|\theta_0\|^2) \) converges in probability to \( 1 / (\rho \varepsilon_0 \sigma_{\theta_0}^2) \). Moreover, from Lemma C.1 we know that

\[
\frac{\theta_0^\top h}{d} \xrightarrow{P} 0,
\]

and therefore the term \( 2\theta_0^\top h / (\sqrt{\rho \varepsilon_0 \|\theta_0\|^2}) \) which multiplies \( 1/\tau_2 \) converges in probability to 0.

These facts are employed to prove that the Hessian (with respect to \( (\alpha, \tau_2) \)) of (67) is positive semidefinite w.p.a. 1 ad \( d \to \infty \). After some algebraic manipulations, it can be checked that the diagonal terms of the Hessian are positive for all \( \alpha \geq 0, \tau_2 > 0 \) w.p.a. 1 ad \( d \to \infty \). Moreover, it can be checked that the determinant of the Hessian is nonnegative (and therefore the Hessian is positive semidefinite) w.p.a. 1 as \( d \to \infty \) if and only if the following condition holds

\[
\alpha \leq \frac{\|\theta_0\|}{\sqrt{d}} \left( 1 + \frac{\tau_2^2}{\rho \varepsilon_0 \|\theta_0\|^2} - \frac{2\theta_0^\top h}{\sqrt{\rho \varepsilon_0 \|\theta_0\|^2}} \right). \tag{69}
\]

From (69), we recover the following sufficient condition,

\[
\alpha \leq \sigma_{\theta_0} - \delta, \tag{70}
\]

for some arbitrarily small \( \delta > 0 \), which guarantees that \( \mathcal{O}_{d,1} \) is jointly convex in \( (\alpha, \tau_2) \) w.p.a. 1 as \( d \to \infty \). As a consequence, from now on, we will work with \( K_\alpha := \sigma_{\theta_0} - \delta \).

Recalling the definition of \( \alpha \), this condition is equivalent to imposing the following relative error bound

\[
\left| \frac{\hat{\theta}_{DRE} - \theta_0}{\|\theta_0\|} \right| < 1,
\]

which is practically desirable in any estimation problem.

We will now prove the concavity in \( \beta \) of \( \mathcal{O}_{d,1} \). This follows easily from the concavity of the Moreau envelope \( e_F(\alpha g + z, \tau_1 / \beta) \) in \( \beta \) (as a minimization of affine functions), the concavity of the term \(-\beta^2 / (2M)\), and the linearity in \( \beta \) of all the other terms.

As a consequence of the continuity and convexity-concavity, we can apply Sion’s minimax principle to exchange the minimization and the maximization in (66) and obtain the following (equivalent w.p.a. 1) problem

\[
\inf_{0 \leq \alpha \leq \sigma_{\theta_0} - \delta} \left\{ \max_{\tau_1, \tau_2 > 0} \frac{\beta \tau_1}{2} + \frac{\sqrt{\varepsilon_0} \beta \tau_2}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) - \frac{\alpha \beta}{\sqrt{n}} \frac{\sqrt{\varepsilon_0}}{\tau_2} \|\theta_0\| - \frac{\sqrt{\varepsilon_0} \beta}{2n \tau_2} \left( \|\theta_0\|^2 + \alpha^2 \right) \right\}. \tag{71}
\]

**Step 3.3: We study the convergence in probability of \( \mathcal{O}_{d,1} \).** We have now arrived at the scalar formulation (71), which has the same optimal value as the modified (AO) problem (59) (under the additional constraint (70)), w.p.a. 1 as \( d \to \infty \). Therefore, following the reasoning presented in Fact 2.1, the next step is to study the convergence in probability of its optimal value. To do so, we start by studying the convergence in probability of its objective function \( \mathcal{O}_{d,1} \).

Notice first that the last two terms of \( \mathcal{O}_{d,1} \) converge in probability as follows.

\[
\frac{\alpha \beta}{\sqrt{n}} \frac{\sqrt{\varepsilon_0}}{\tau_2} \|\theta_0\| - \frac{\sqrt{\varepsilon_0} \beta}{2n \tau_2} \left( \|\theta_0\|^2 + \alpha^2 \right) \xrightarrow{P} \frac{\sqrt{\varepsilon_0} \beta}{2n \tau_2} \left( \sigma_{\theta_0}^2 + \alpha^2 \right), \tag{72}
\]

\[
\left( \|\theta_0\|^2 + \alpha^2 \right) \xrightarrow{P} \frac{\sqrt{\varepsilon_0} \beta}{2n \tau_2} \left( \sigma_{\theta_0}^2 + \alpha^2 \right), \tag{73}
\]

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where (72) can be recovered using (68) and the continuous mapping theorem (which states that continuous functions preserve limits in probability, when their arguments are sequences of random variables).

We will now study the convergence in probability of the the normalized Moreau envelope $e_F(\alpha g + z, \frac{\tau_1}{\beta})/n$. Since $F(u) := \sum_{i=1}^n f(u_i)$, (62) reduces to

$$
\frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) = \frac{1}{n} \sum_{i=1}^n \min_{s_i \in \mathbb{R}} \frac{\beta}{2\tau_1} (\alpha G - Z - s_i)^2 + f(s_i) = \frac{1}{n} \sum_{i=1}^n e_f(\alpha G + Z; \frac{\tau_1}{\beta}),
$$

and its limit (in probability) can be immediately recovered from the weak law of large numbers as

$$
\frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) \xrightarrow{P} \mathbb{E}_{\mathbb{N}(0,1) \otimes \mathbb{P}} \left[ e_f(\alpha G + Z; \frac{\tau_1}{\beta}) \right], \tag{74}
$$

which is precisely the expected Moreau envelope $F(\alpha, \frac{\tau_1}{\beta})$ defined in (21). The quantity under the expectation in (74) is absolutely integrable, and therefore the expected Moreau envelope $F$ is well defined, as explained in what follows. For every $\beta > B$ and $\tau_1 > 0$ we have

$$
\left| e_f(\alpha G + Z; \frac{\tau_1}{\beta}) \right| = \min_{v \in \mathbb{R}} \frac{\beta}{2\tau_1} (\alpha G + Z - v)^2 + f(v) \leq \frac{\beta}{2\tau_1} (\alpha G + Z)^2 + f(0) = \frac{\beta}{2\tau_1} (\alpha G + Z)^2
$$

which is integrable due to the fact that both $G$ and $Z$ have finite second moment. In particular, the fact that $Z$ has finite second moment follows directly from Assumption 3.1(iii). In the first and last equality we have used the fact that $f(0) = \min_{v \in \mathbb{R}} f(v) = 0$, which can be shown to hold true as follows. Since $L(0) = \min_{v \in \mathbb{R}} L(v) = 0$ (from Assumption 4.1(i)), we have that $L^*(0) = \min_{v \in \mathbb{R}} L^*(v) = 0$, and from $f^*(\cdot) = L^*(\cdot) - 1/(2M)(\cdot)^2$, with $L^*$ that is $1/M$-strongly convex, we have that $f^*(0) = \min_{v \in \mathbb{R}} f^*(v) = 0$, which results in the desired $f(0) = \min_{v \in \mathbb{R}} f(v) = 0$. Moreover, for $\tau_1 \to 0^+$,

$$
\lim_{\tau_1 \to 0^+} \left| e_f(\alpha G + Z; \frac{\tau_1}{\beta}) \right| = \lim_{\tau_1 \to 0^+} \min_{v \in \mathbb{R}} \frac{\beta}{2\tau_1} (\alpha G + Z - v)^2 + f(v) = f(\alpha G + Z),
$$

whose expectation is finite by assumption. In particular, the second equality follows from [96, Theorem 1.25].

Therefore, from (72)-(74), we have that $O_{d,1}(\alpha, \tau_1, \tau_2, \beta)$ converges in probability to the function

$$
O_1(\alpha, \tau_1, \tau_2, \beta) := \frac{\beta \tau_1}{2} + \frac{\varepsilon_0 \beta \tau_2}{2} - \frac{\beta^2}{2M} + F(\alpha, \tau_1/\beta) - \alpha \beta \sqrt{\rho} \sqrt{\frac{\rho \varepsilon_0}{\tau_2^2 \sigma_0^2 + 1} + \frac{\sqrt{\varepsilon_0} \beta \rho}{2\tau_2} \left( \sigma_0^2 + \alpha^2 \right)}. \tag{75}
$$

Notice that $O_1$ is precisely the first objective function of the minimax problem (23) from the statement of Theorem 4.5.

**Step 3.4:** We show that the asymptotic objective function $O_1$ is jointly convex in $(\alpha, \tau_1, \tau_2)$, jointly strictly convex in $(\alpha, \tau_2)$, and concave in $\beta$. Since $O_1$ is the pointwise limit (in probability, for each $(\alpha, \tau_1, \tau_2, \beta)$) of the sequence of objective functions $O_{d,1}$ which are convex-concave (w.p.a. 1 as $d \to \infty$), and convexity is preserved by pointwise limits (see Lemma C.2), we have that $O_1$ is jointly convex in $(\alpha, \tau_1, \tau_2)$ and concave in $\beta$. The convexity-concavity of $O_1$ can also be checked directly from (75), following similar lines as the proof of convexity-concavity of $O_1$. In fact, in what follows we will use such arguments to prove that $O_1$ is jointly strictly convex in $(\alpha, \tau_2)$, This, in turn, will help us later to the uniqueness of the optimal solution $\alpha_{*,1}$, which will be fundamental in the convergence analysis required by Fact 2.1.
For the strict convexity of $\mathcal{O}_1$ in $(\alpha, \tau_2)$ we can restrict attention only to the last two terms of $\mathcal{O}_1$. Once again, since $\beta, \rho, \varepsilon_0, \sigma_{\theta_0}$ have no influence on the convexity, we can simplify these terms, which after some manipulations become

$$-\alpha \sqrt{\frac{1}{\tau_2^2} + \frac{1}{\rho \varepsilon_0 \sigma_{\theta_0}^2}} + \frac{1}{\tau_2} \left( \frac{\sigma_{\theta_0}^2}{2 \sigma_{\theta_0}} + \alpha^2 \right)$$

(76)

Notice that (76) is nothing but the limit in probability of (67), for every $\alpha \geq 0, \tau_2 > 0$. We will prove the strict convexity in $(\alpha, \tau_2)$ of $\mathcal{O}_1$ by showing that the Hessian of $\mathcal{O}_1$ with respect to $(\alpha, \tau_2)$ of (76) is positive definite. After some algebraic manipulations, it can be checked that the diagonal terms of the Hessian are positive for all $\alpha \geq 0, \tau_2 > 0$. Moreover, it can be checked that the determinant of the Hessian is positive if and only if the following condition holds

$$\alpha < \sigma_{\theta_0} \sqrt{1 + \tau_2^2 \frac{1}{\rho \varepsilon_0 \sigma_{\theta_0}^2}}.$$  

(77)

As a consequence of this and (70), we have that the constraint $\alpha \leq \sigma_{\theta_0} - \delta$ also ensures the joint strict convexity in $(\alpha, \tau_2)$ of $\mathcal{O}_1$. We will now show that the strict convexity in $(\alpha, \tau_2)$ of $\mathcal{O}_1$ guarantees the uniqueness of the optimizer $\alpha_{*,1}$. With this aim in mind, consider the asymptotic minimax optimization problem

$$\inf_{0 \leq \alpha \leq \sigma_{\theta_0} - \delta} \max_{\tau_1, \tau_2 > 0} \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta).$$

(78)

**Step 3.5: We show that the asymptotic problem (78) has an unique minimizer $\alpha_{*,1}$.** In order to prove uniqueness of $\alpha_{*,1}$, we will show that $\inf_{\tau_1, \tau_2 > 0} \max_{B < \beta \leq K_\beta} \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta)$ is strictly convex in $\alpha$, for $\alpha > 0$. First, we consider the inner maximization in (78), i.e., $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2) := \max_{B < \beta \leq K_\beta} \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta)$. Since the function $\mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta)$ is continuous in $\beta$, we can extend it $\beta = B$, by setting $\lim_{\beta \rightarrow B} \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta) = \mathcal{O}_1(\alpha, \tau_1, \tau_2, B)$. Therefore, we have $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2) = \max_{B < \beta \leq K_\beta} \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta)$. Notice that $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2)$ is jointly convex in $(\alpha, \tau_1, \tau_2)$ since we have seen that $\mathcal{O}_1$ is jointly convex in $(\alpha, \tau_1, \tau_2)$, and pointwise supremum of convex functions is convex. We will now show that $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2)$ is jointly strictly convex in $(\alpha, \tau_2)$. From the strict convexity of $\mathcal{O}_1$ in $(\alpha, \tau_2)$ it follows that for any $\lambda \in (0, 1)$, $(\alpha^{(1)}, \tau_2^{(1)}) \neq (\alpha^{(2)}, \tau_2^{(2)})$, and $\beta$ we have that

$$\mathcal{O}_1(\lambda \alpha^{(1)} + (1 - \lambda) \alpha^{(2)}, \tau_1, \lambda \tau_2^{(1)} + (1 - \lambda) \tau_2^{(2)}) < \lambda \mathcal{O}_1(\alpha^{(1)}, \tau_1, \tau_2^{(1)}, \beta) + (1 - \lambda) \mathcal{O}_1(\alpha^{(2)}, \tau_1, \tau_2^{(2)}, \beta)$$

$$\leq \lambda \mathcal{O}_1^\beta(\alpha^{(1)}, \tau_1, \tau_2^{(1)}) + (1 - \lambda) \mathcal{O}_1^\beta(\alpha^{(2)}, \tau_1, \tau_2^{(2)}).$$

We can now take the supremum over $\beta \in [B, K_\beta]$ on the left-hand side, and since this will be attained (due to the continuity in $\beta$ of $\mathcal{O}_1$, and the compactness of $[B, K_\beta]$), we can conclude that

$$\mathcal{O}_1^\beta(\lambda \alpha^{(1)} + (1 - \lambda) \alpha^{(2)}, \tau_1, \lambda \tau_2^{(1)} + (1 - \lambda) \tau_2^{(2)}) < \lambda \mathcal{O}_1^\beta(\alpha^{(1)}, \tau_1, \tau_2^{(1)}) + (1 - \lambda) \mathcal{O}_1^\beta(\alpha^{(2)}, \tau_1, \tau_2^{(2)}).$$

This shows that $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2)$ is strictly convex in $(\alpha, \tau_2)$.

Secondly, we consider $\mathcal{O}^{\tau_1\beta}(\alpha, \tau_2) := \inf_{\tau_1 > 0} \mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2)$. Since $\mathcal{O}_1^\beta(\alpha, \tau_1, \tau_2)$ is jointly convex in $(\alpha, \tau_1, \tau_2)$, and partial minimization of convex functions is convex, we have that $\mathcal{O}^{\tau_1\beta}(\alpha, \tau_2)$ is convex in $(\alpha, \tau_2)$. Moreover, recall that the function $\mathcal{O}_1$ is decoupled in $\tau_1$ and $\tau_2$. As a consequence, taking the infimum over $\tau_1$ will not alter the joint strict convexity in $(\alpha, \tau_2)$. 

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Finally, we consider the function $O_1^{\tau_1, \beta}(\alpha) := \inf_{\tau_2 > 0} O_1^{\tau_1, \beta}(\alpha, \tau_2)$. Before proving that $O_1^{\tau_2, \beta}$ is strictly convex, we first need to show that for any $\alpha \leq \sigma_0 - \delta$, $\tau_1 > 0$, and $\beta > B$, the infimum is attained by some $\tau_{2*} > 0$. Indeed, this follows by noticing that

$$\lim_{\tau_2 \to 0^+} O_1(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to 0^+} \frac{\varepsilon_0 \beta \tau_2}{2} - \alpha \sqrt{\rho} \sqrt{\frac{\rho \varepsilon_0^2 \tau_2^2 \sigma_0^2 + 1}{\tau_2^2 \sigma_0^2 + \alpha^2}} = \lim_{\tau_2 \to 0^+} \frac{\varepsilon_0 \beta (\sigma_0 - \alpha)^2}{\tau_2} = \infty,$$

and that

$$\lim_{\tau_2 \to \infty} O_1(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to \infty} \frac{\varepsilon_0 \beta \tau_2}{2} - \alpha \sqrt{\rho} \sqrt{\frac{\rho \varepsilon_0^2 \tau_2^2 \sigma_0^2 + 1}{\tau_2^2 \sigma_0^2 + \alpha^2}} = \lim_{\tau_2 \to \infty} \frac{\varepsilon_0 \beta \tau_2}{2} - \alpha \sqrt{\rho} = \infty. \tag{79}$$

Consider now $\alpha(1), \alpha(2) > 0$, and choose $\gamma(1), \gamma(2)$ satisfying $O_1^{\tau_1, \beta}(\alpha(1), \gamma(1)) = O_1^{\tau_2, \beta}(\alpha(2))$ and $O_1^{\tau_1, \beta}(\alpha(2), \gamma(2)) = O_1^{\tau_2, \beta}(\alpha(1))$, respectively. Then, for any $\lambda \in (0, 1)$ we have

$$O_1^{\tau_2, \beta}(\lambda \alpha(1) + (1 - \lambda) \alpha(2)) \leq O_1^{\tau_1, \beta}(\lambda \alpha(1) + (1 - \lambda) \alpha(2), \gamma(1)) + (1 - \lambda) O_1^{\tau_1, \beta}(\alpha(2), \gamma(2)) \leq \lambda O_1^{\tau_1, \beta}(\alpha(1), \gamma(1)) + (1 - \lambda) O_1^{\tau_1, \beta}(\alpha(2), \gamma(2))$$

from which we conclude that $O_1^{\tau_2, \beta}(\alpha) = \inf_{\tau_1, \tau_2 > 0} \max_{B < \beta \leq K_\beta} O_1(\alpha, \tau_1, \tau_2, \beta)$ is strictly convex in $\alpha$, for $\alpha > 0$. As a consequence, the optimizer $\alpha_{*, 1}$ of (78) is unique.

**Step 3.6: We anticipate how the uniqueness of $\alpha_{*, 1}$, together with Fact 2.1, can be used to conclude the proof.** The uniqueness of $\alpha_{*, 1}$ is a key element in the convergence analysis required by Fact 2.1, as explained in what follows. We first define, for arbitrary $\eta > 0$, the sets $S_\eta := \{\alpha \in [0, \sigma_0 - \delta] : |\alpha - \alpha_{*, 1}| < \eta\}$, with $\alpha_{*, 1}$ the unique solution of (78), and $S_\eta^0 := [0, \sigma_0 - \delta] \setminus S_\eta$. Since $\alpha_{*, 1}$ is the unique solution of (78), we have that

$$\inf_{0 \leq \alpha \leq \sigma_0 - \delta} \max_{\tau_1, \tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta) = \inf_{\alpha \in S_\eta^0} \max_{\tau_1, \tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta) \tag{80}$$

Now, due to (80), if we prove that the optimal value of the (scalarized version of the) modified (AO) problem (71) satisfies

$$\inf_{0 \leq \alpha \leq \sigma_0 - \delta} \max_{\tau_1, \tau_2 > 0} O_{d, 1}(\alpha, \tau_1, \tau_2, \beta) \leq \inf_{\alpha \in S_\eta^0} \max_{\tau_1, \tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta), \tag{81}$$

and that, when additionally restricted to $\alpha \in S_\eta^0$, for arbitrary $\eta > 0$, it satisfies

$$\inf_{\alpha \in S_\eta^0} \max_{\tau_1, \tau_2 > 0} O_{d, 1}(\alpha, \tau_1, \tau_2, \beta) \leq \inf_{\alpha \in S_\eta^0} \max_{\tau_1, \tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta), \tag{82}$$

we can directly conclude the desired result (24) for the first case (59) from Fact 2.1. Therefore, in order to finish the first case (59), we only need to prove the two convergences in probability (81) and (82). It will be easier to prove the two convergences in probability in the following equivalent form (by Sion’s minimax theorem)

$$\min_{\alpha} \max_{B < \beta \leq K_\beta} \inf_{\tau_1, \tau_2 > 0} O_{d, 1}(\alpha, \tau_1, \tau_2, \beta) \to \min_{\alpha} \max_{B < \beta \leq K_\beta} \inf_{\tau_1, \tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta).$$

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The two convergences in probability (81) and (82) are a consequence of Lemma C.3, as explained in what follows. For convenience, we will drop the “w.p.a. 1 as \( d \to \infty \)” whenever we refer to the convexity of the functions \( O_{d,1} \) in the convergence analysis that follows, but the reader should remember that this is implicit. Moreover, we will drop the “under the condition \( \alpha \leq \sigma_\theta_0 - \delta \)” whenever we will say that \( O_{d,1}, O_1 \), and their partial minimizations are convex.

**Step 3.7:** We prove the two convergences in probability (81) and (82), and conclude the proof for the first modified (AO) problem (59). First, for fixed \((\alpha, \beta, \tau_1)\), \(\{O_{d,1}(\alpha, \tau_1, \cdot, \cdot)\}_{d \in \mathbb{N}}\) is a sequence of random real-valued convex functions, converging in probability (pointwise, for every \( \tau_2 > 0 \)) to the function \(O_1(\alpha, \tau_1, \cdot, \cdot)\). Moreover,

\[
\lim_{\tau_2 \to \infty} O_1(\alpha, \tau_1, \tau_2, \beta) = \infty,
\]
as shown in (79). As a consequence, we can apply statement (iii) of Lemma C.3 to conclude that

\[
\inf_{\tau_2 > 0} O_{d,1}(\alpha, \tau_1, \tau_2, \beta) \xrightarrow{P} \inf_{\tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \beta).
\]

We now define \(O_1^{\tau_2}(\alpha, \tau_1, \cdot, \cdot) := \inf_{\tau_2 > 0} O_1(\alpha, \tau_1, \tau_2, \cdot)\) and \(O_{d,1}^{\tau_2}(\alpha, \tau_1, \cdot, \cdot) := \inf_{\tau_2 > 0} O_{d,1}(\alpha, \tau_1, \tau_2, \cdot)\). For fixed \((\alpha, \beta)\), \(\{O_{d,1}^{\tau_2}(\alpha, \cdot, \cdot)\}_{d \in \mathbb{N}}\) is a sequence of random, real-valued convex functions (since they are defined as the partial minimization of jointly convex functions), converging in probability (pointwise, for every \( \tau_1 > 0 \)) to the function \(O_1^{\tau_2}(\cdot, \cdot, \cdot, \cdot)\). Moreover, since \(\tau_1\) and \(\tau_2\) are decoupled, we have

\[
\lim_{\tau_1 \to \infty} O_1^{\tau_2}(\alpha, \tau_1, \cdot, \cdot) = \infty \iff \lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} + \mathcal{F}(\alpha, \frac{\tau_1}{\beta}) = \infty.
\]

Notice now that

\[
\lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} + \mathcal{F}(\alpha, \frac{\tau_1}{\beta}) \geq \lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} = \infty,
\]
where the inequality follows from the fact that \(\mathcal{F}(\alpha, \tau_1/\beta) \geq 0\) (recall that \(f(v) \geq 0\) for all \(v \in \mathbb{R}\)). As a consequence, we can apply again statement (iii) of Lemma C.3 to conclude that

\[
\inf_{\tau_1 > 0} O_{d,1}^{\tau_2}(\alpha, \tau_1, \cdot, \cdot) \xrightarrow{P} \inf_{\tau_1 > 0} O_1^{\tau_2}(\alpha, \tau_1, \cdot, \cdot).
\]

We now define \(O_1^{\tau_1, \tau_2}(\alpha, \beta) := \inf_{\tau_1 > 0} O_1^{\tau_2}(\alpha, \tau_1, \cdot, \cdot)\) and \(O_{d,1}^{\tau_1, \tau_2}(\alpha, \beta) := \inf_{\tau_1 > 0} O_{d,1}(\alpha, \tau_1, \cdot, \cdot)\). For fixed \(\alpha\), \(\{O_{d,1}^{\tau_1, \tau_2}(\alpha, \cdot)\}_{d \in \mathbb{N}}\) is a sequence of random, real-valued concave functions (since they are defined as a minimization of concave functions), converging in probability (pointwise, for every \( \beta > B \)) to the function \(O_1^{\tau_1, \tau_2}(\cdot, \cdot)\). Therefore, we can apply statement (ii) of Lemma C.3 (for \(0 < \beta - B \leq K_\beta - B\)) to conclude that

\[
\sup_{B < \beta \leq K_\beta} O_{d,1}^{\tau_1, \tau_2}(\alpha, \beta) \xrightarrow{P} \sup_{B < \beta \leq K_\beta} O_1^{\tau_1, \tau_2}(\alpha, \beta).
\]  

(83)

We will now show that the constraint \(B < \beta \leq K_\beta\) can be relaxed to \(\beta > B\) without loss of generality. Recall that \(K_\beta\) was inserted for convenience, in equation (50), to be able to use the CGMT (which required \(u\) to live in a compact set). Now, notice that

\[
\lim_{\beta \to \infty} O_1^{\tau_1, \tau_2}(\alpha, \beta) \leq \lim_{\beta \to \infty} O_1^{\tau_1, \tau_2}(\alpha, \beta) = \lim_{\beta \to \infty} \frac{\beta \tau_1}{2} + \varepsilon_0 \frac{\beta \tau_2}{2} - \frac{\beta^2}{2M} + \mathcal{F}(\alpha, \frac{\tau_1}{\beta}) - \alpha \beta \sqrt{\frac{\varepsilon_0}{\tau_1^2} \sigma_{\theta_0}^2 + 1} + \frac{\sqrt{\varepsilon_0 \beta \rho}}{2\tau_2} (\sigma_{\theta_0}^2 + \alpha^2)
\]

\[
= \lim_{\beta \to \infty} -\frac{\beta^2}{2M} + O(\beta) = -\infty,
\]
for some constants \( \tilde{\tau}_1, \tilde{\tau}_1 > 0 \) (let’s say, for example, equal to 1). Here, \( O(\beta) \) encapsulates the remaining terms, which grow (at most) linearly in \( \beta \). The inequality above follows from the fact that \( O^{\tau_1, \tau_2}_1 \) is defined as a minimization over \( \tau_1, \tau_2 > 0 \). The assumption \( \mathbb{E}_{N(0,1) \otimes P} [f(\alpha G + Z)] < \infty \) is very important here, since it guarantees that

\[
\lim_{\beta \to \infty} \mathcal{F}(\alpha, \frac{\tilde{\tau}_1}{\beta}) = \lim_{\beta \to \infty} \mathbb{E}_{N(0,1) \otimes P} \left[ e_f(\alpha G + Z; \frac{\tilde{\tau}_1}{\beta}) \right] = \mathbb{E}_{N(0,1) \otimes P} \left[ \lim_{\beta \to \infty} e_f(\alpha G + Z; \frac{\tilde{\tau}_1}{\beta}) \right] = \mathbb{E}_{N(0,1) \otimes P} [f(\alpha G + Z)] < \infty,
\]

where the second equality follows from the monotone convergence theorem, and the last equality follows from \cite[Theorem 1.25]{96}.

Therefore, there exists some sufficiently large \( K_\beta \) such that

\[
\sup_{B < \beta \leq K_\beta} O^{\tau_1, \tau_2}_1(\alpha, \beta) = \sup_{\beta > B} O^{\tau_1, \tau_2}_1(\alpha, \beta),
\]

leading to

\[
\sup_{B < \beta \leq K_\beta} O^{\tau_1, \tau_2}_1(\alpha, \beta) \overset{P}{\to} \sup_{\beta > B} O^{\tau_1, \tau_2}_1(\alpha, \beta). \tag{84}
\]

Notice that the convergence (84) can also be proven using statement (iii) of Lemma C.3, since \( \lim_{\beta \to \infty} O^{\tau_1, \tau_2}_1(\alpha, \beta) = -\infty \).

We now define \( O^{\beta, \tau_1, \tau_2}_1(\alpha) := \sup_{\beta > B} O^{\tau_1, \tau_2}_1(\alpha, \beta) \) and \( O^{\beta, \tau_1, \tau_2}_1(\alpha) := \sup_{B < \beta \leq K_\beta} O^{\tau_1, \tau_2}_1(\alpha, \beta) \). Each of these functions is convex in \( \alpha \), since they were obtained by first minimizing over \( \tau_1, \tau_2 \) a jointly convex function in \( (\alpha, \tau_1, \tau_2) \), and then maximizing over \( \beta \) a convex function in \( \alpha \).

For the final step of the proof, we will distinguish between the two cases \( \alpha_{*,1} = 0 \) and \( \alpha_{*,1} > 0 \). This is due to the Convexity Lemma C.2, which stands at the core of the convergence analysis, and which requires the domain of the functions of interest to be open. We first consider the case \( \alpha_{*,1} = 0 \). Since (83) is valid for any \( \alpha \geq 0 \), it is in particular valid for \( \alpha = 0 \), and thus we have that

\[
O^{\beta, \tau_1, \tau_2}_1(0) \overset{P}{\to} O^{\beta, \tau_1, \tau_2}_1(0). \tag{85}
\]

Moreover, using statement (i) of Lemma C.3, for any \( K > 0 \), we have that

\[
\min_{K \leq \alpha \leq \sigma_0 - \delta} O^{\beta, \tau_1, \tau_2}_1(\alpha) \overset{P}{\to} \min_{K \leq \alpha \leq \sigma_0 - \delta} O^{\beta, \tau_1, \tau_2}_1(\alpha). \tag{86}
\]

The result for \( \alpha_{*,1} = 0 \) now follows from (85) and (86) and the fact that

\[
O^{\beta, \tau_1, \tau_2}_1(0) < \min_{K \leq \alpha \leq \sigma_0 - \delta} O^{\beta, \tau_1, \tau_2}_1(\alpha),
\]

for any \( K > 0 \), due to the uniqueness of \( \alpha_{*,1} \).

We now focus on the case \( \alpha_{*,1} > 0 \). We need to prove (81) and (82), which are equivalent to the following two convergences in probability

\[
\inf_{0 < \alpha \leq \sigma_0 - \delta} O^{\beta, \tau_1, \tau_2}_1(\alpha) \overset{P}{\to} \inf_{0 < \alpha \leq \sigma_0 - \delta} O^{\beta, \tau_1, \tau_2}_1(\alpha), \tag{87}
\]

and

\[
\inf_{\alpha \in S^c} O^{\beta, \tau_1, \tau_2}_1(\alpha) \overset{P}{\to} \inf_{\alpha \in S^c} O^{\beta, \tau_1, \tau_2}_1(\alpha), \tag{88}
\]

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respectively. Since \( \{O_{d,1}^{\beta,\tau_1,\tau_2}(\cdot)\}_{d \in \mathbb{N}} \) is a sequence of random, real-valued convex functions, converging in probability (pointwise, for every \( \alpha > 0 \)) to the function \( O_1^{\beta,\tau_1,\tau_2}(\cdot) \), (87) follows immediately from statement (ii) of Lemma C.3. Moreover, since \( S^c_\eta = (0, \alpha_{*,1} - \eta] \cup [\alpha_{*,1} + \eta, \sigma_0 - \delta] \), (88) follows from statement (i) of Lemma C.3, for the interval \([\alpha_{*,1} + \eta, \sigma_0 - \delta] \), and statement (ii) of the same lemma, for the interval \((0, \alpha_{*,1} - \eta] \).

This concludes the proof of the two convergences in probability (81) and (82), and with it, the analysis of the first case (59). We will now consider the second case (60), and proceed by first scalarizing for the interval \([\alpha_{*,1} + \eta, \sigma_0 - \delta] \). We will now consider the second case (60), and proceed by first scalarizing for the interval \([\alpha_{*,1} + \eta, \sigma_0 - \delta] \).

Step 4: We anticipate the results that will be obtained for the second modified (AO) problem (60), and briefly explain how everything can be used to conclude the proof of this theorem. Recall from the sequence of arguments (56)-(60) that the optimal value of the modified (AO) problem (56) is equal to \( \max \{V_{d,1}, V_{d,2}\} \), with \( V_{d,1} \) the optimal value of problem (59), and \( V_{d,2} \) the optimal value of problem (60). So far, we have shown that \( V_{d,1} \xrightarrow{p} V_1 \), with

\[
V_1 := \inf_{0 \leq \alpha \leq \sigma_0 - \delta} \max_{\tau_1, \tau_2 > 0} \{O_1(\alpha, \tau_1, \tau_2, \beta)\},
\]

and that the optimal solution \( \alpha_{*,1} \), which attains \( V_1 \), is unique. In what follows, we will show that similar things happen in the second case, i.e., \( V_{d,2} \xrightarrow{p} V_2 \), for some \( V_2 \) that will be defined later, and that the solution \( \alpha_{*,2} \), which attains \( V_2 \), is unique. The two converges \( V_{d,1} \xrightarrow{p} V_1 \) and \( V_{d,2} \xrightarrow{p} V_2 \) are very important because they allow us to conclude what is the asymptotic optimal value of the modified (AO) problem. Indeed, if \( V_1 > V_2 \), then we know that \( V_{d,1} > V_{d,2} \) w.p.a. 1 as \( d \to \infty \). In that case, we have that \( \max \{V_{d,1}, V_{d,2}\} = V_{d,1} \) asymptotically, and from Fact 2.1 and the analysis performed in (80)-(88), we can conclude the desired result, i.e., \( \|\hat{\theta}_{DRE} - \theta_0\|/\sqrt{d} \xrightarrow{p} \alpha_{*,1} \). Similarly, if \( V_2 > V_1 \), we can conclude that \( \|\hat{\theta}_{DRE} - \theta_0\|/\sqrt{d} \xrightarrow{p} \alpha_{*,2} \). Finally, if \( V_2 = V_1 \), we can not assess which of the two values \( V_{d,1} \) or \( V_{d,2} \) is larger, and therefore we can only conclude that \( \|\hat{\theta}_{DRE} - \theta_0\|/\sqrt{d} \leq \max\{\alpha_{*,1}, \alpha_{*,2}\} \) w.p.a. 1 as \( d \to \infty \).

Therefore, in order to finish the proof, we still need to prove the convergence \( V_{d,2} \xrightarrow{p} V_2 \), together with the uniqueness of \( \alpha_{*,2} \). We will do this in what follows, at a slightly faster pace when the steps are similar to the ones done for \( V_{d,1} \).

Step 5: We re-do all the steps for the second modified (AO) problem (60), following similar reasoning as above. Recall that \( V_{d,2} \) is the optimal value of the following optimization problem

\[
\max_{0 \leq \beta \leq B} \min_{s \in \mathbb{R}^n} \max_{u \in \mathbb{R}^n} \frac{1}{\sqrt{n}} \left\{ \|w\| g + z - s \right\}^\top u - \frac{1}{\sqrt{n}} \|w\| h^\top w - \frac{\|u\|^2}{2M} + \frac{1}{n} F(s) + \frac{1}{\sqrt{n}} \left( \|\theta_0 + \sqrt{d} w\| + \frac{1}{q} \|u\|^2 \right) \}
\]

with \( p := (\varepsilon_0 \sqrt{p} M R_g)/2 \) and \( q := 2\sqrt{p} R_g \).

We start by scalarizing over the magnitude of \( u \), leading to

\[
\max_{0 \leq \beta \leq B} \min_{s \in \mathbb{R}^n} \frac{\beta}{\sqrt{n}} \left\{ \|w\| g + z - s \right\} - \frac{1}{\sqrt{n}} \beta h^\top w - \frac{\beta^2}{2M} + \frac{1}{n} F(s) + \frac{1}{\sqrt{n}} \left( \|\theta_0 + \sqrt{d} w\| + \frac{1}{q} \beta^2 \right) \}
\]

\]

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and by rewriting the first term using the square-root trick, and introducing the Moreau envelope of $F$, we obtain
\[
\max_{0 \leq \beta \leq B} \inf_{\tau_1, \tau_2 > 0} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\|w\|g + z, \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\|\theta_0 + \sqrt{d} w\|}{\sqrt{n}} (p + \frac{\beta^2}{q}). \tag{89}
\]

From
\[
- \frac{\beta^2}{2M} + \frac{\|\theta_0 + \sqrt{d} w\|}{\sqrt{n}} \beta = - \frac{\beta^2}{2M} \left(1 - \frac{\|\theta_0 + \sqrt{d} w\|}{R_\theta \sqrt{d}}\right),
\]
and $\|\theta_0 + \sqrt{d} w\| \leq R_d \sqrt{d}$, we have that the objective function in (89) is concave in $\beta$ (recall that $e_F(\|w\|g + z, \tau_1/\beta)$ is concave in $\beta$).

We now employ again the square-root trick on the last term in (89), leading to
\[
\max_{0 \leq \beta \leq B} \inf_{\tau_1, \tau_2 > 0} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\|w\|g + z, \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\tau_2}{2} + \frac{\|\theta_0 + \sqrt{d} w\|^2}{2n\tau_2} \left(p + \frac{\beta^2}{q}\right). \tag{90}
\]

Notice that after introducing the square-root trick, the objective function in (90) is not concave in $\beta$ anymore. However, since the objective function in (89) was concave in $\beta$, we have that the function
\[
\inf_{\tau_2 > 0} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\|w\|g + z, \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\tau_2}{2} + \frac{\|\theta_0 + \sqrt{d} w\|^2}{2n\tau_2} \left(p + \frac{\beta^2}{q}\right) \tag{91}
\]
is concave in $\beta$.

We now separate the constraint $w \in S_w$ into the two constraints $0 \leq \alpha \leq K_\alpha$ and $\|w\| = \alpha$, and by expressing $\|\theta_0 + \sqrt{d} w\|^2$ as $\|\theta_0\|^2 + d\|w\|^2 + 2\sqrt{d}\theta_0^T w$, we can now scalarize the problem over the magnitude of $w$ and obtain
\[
\max_{0 \leq \beta \leq B} \inf_{0 \leq \alpha \leq K_\alpha} \inf_{\tau_1, \tau_2 > 0} \frac{\beta \tau_1}{2} + \frac{p\tau_2}{2} + \frac{\beta^2\tau_2}{2q} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) - \frac{\alpha}{\sqrt{n}} \sqrt{\beta} \left(p + \frac{\beta^2}{q}\right) \frac{\theta_0}{\tau_2} - \beta h \right\| + \\
+ \frac{\|\theta_0\|^2 + d\alpha^2}{2n\tau_2} \left(p + \frac{\beta^2}{q}\right). \tag{92}
\]

We denote by $O_{d,2}(\alpha, \tau_1, \tau_2, \beta)$ the objective function in (92), where the index 2 recalls the fact that we are in the second case. Before proceeding, we would like to highlight that the function
\[
\inf_{\tau_2 > 0} \frac{\beta \tau_1}{2} + \frac{p\tau_2}{2} + \frac{\beta^2\tau_2}{2q} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \frac{\tau_1}{\beta}) - \frac{\alpha}{\sqrt{n}} \sqrt{\beta} \left(p + \frac{\beta^2}{q}\right) \frac{\theta_0}{\tau_2} - \beta h \right\| + \\
+ \frac{\|\theta_0\|^2 + d\alpha^2}{2n\tau_2} \left(p + \frac{\beta^2}{q}\right) \tag{93}
\]
is concave in $\beta$, since it is the result of a minimization (over $\|w\| = \alpha$) of concave functions (recall that (91) is concave in $\beta$). Since we have concavity in $\beta$ only after the minimization over $\tau_2$, and ultimately we are interested in exchanging the minimization over $\alpha$ with the maximization over $\beta$ (in order to recover the optimal solution $\alpha^* \beta$), we define by $O_{d,2}^*(\alpha, \tau_1, \beta)$ the function in (93), and therefore, in this second case, we will be working with the optimization problem
\[
\max_{0 \leq \beta \leq B} \inf_{0 \leq \alpha \leq K_\alpha} O_{d,2}^*(\alpha, \tau_1, \beta). \tag{94}
\]
Therefore, we have that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is concave in \( \beta \). Furthermore, we will show that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is jointly convex in \( (\alpha, \tau_1) \), and continuous on its domain. Let’s first concentrate on the joint convexity in \( (\alpha, \tau_1) \). For this, we will first show that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \tau_2, \beta) \) is jointly convex in \( (\alpha, \tau_1, \tau_2) \), and therefore after the partial minimization over \( \tau_2 \), we will have that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is jointly convex in \( (\alpha, \tau_1) \). Since \( \tau_1 \) and \( \tau_2 \) are decoupled, it will be enough to prove the joint convexity in \( (\alpha, \tau_1) \) and \( (\alpha, \tau_2) \). The joint convexity in \( (\alpha, \tau_1) \) follows from the joint convexity in \( (\alpha, \tau_1) \) of the Moreau envelope \( e_F(\alpha g + z, \tau_1/\beta) \), which we have proven previously, in the first case. Moreover, the joint convexity in \( (\alpha, \tau_2) \) will also be shown to follow from the analysis performed in the first case. In particular, we will now show that the last two terms in \( \mathcal{O}_{d,2}(\alpha, \tau_1, \tau_2, \beta) \) can be rewritten in the form \( (67) \), and therefore the reasoning used in the first case can be re-used here. Since \( \beta, p, q, \rho, ||\theta_0|| \) have no influence on the convexity, and \( (p + \beta/q) > 0 \), we can simplify the last two terms in \( \mathcal{O}_{d,2}(\alpha, \tau_1, \tau_2, \beta) \), and obtain

\[
-\alpha \left| \frac{1}{\tau_2} - \frac{\beta}{\sqrt{\rho(p + \beta^2/q)} ||\theta_0||} \right| h + \frac{1}{\tau_2} \left( \frac{||\theta_0||^2/d + \alpha^2}{2 ||\theta_0||/\sqrt{d}} \right)
\]

or, equivalently,

\[
-\alpha \sqrt{\frac{1}{\tau_2} + \frac{\beta^2}{\rho(p + \beta^2/q)^2 ||\theta_0||^2} + \frac{2\beta}{\sqrt{\rho(p + \beta^2/q)} ||\theta_0||^2} \frac{\theta_0^T h}{2 ||\theta_0||/\sqrt{d}}}.
\]

Notice now that the only differences between \( (95) \) and \( (67) \) are the two coefficients \( \beta^2/(\rho(p + \beta^2/q)^2) \) instead of \( 1/(\rho \varepsilon_0) \), and \( 2\beta/\sqrt{\rho(p + \beta^2/q)} \) instead of \( 1/\sqrt{\rho \varepsilon_0} \). However, since \( 0 \leq \beta \leq B \), we have that the new coefficients are bounded, and therefore they do not affect in any way the convexity analysis performed for \( (67) \). As a consequence, following the same exact steps as in \( (67)-(70) \), we have that \( (95) \) is jointly convex in \( (\alpha, \tau_2) \) w.p.a. \( 1 \) as \( d \to \infty \) if and only if the following condition holds

\[
\alpha \leq \frac{||\theta_0||}{\sqrt{d}} \sqrt{1 + \frac{\beta^2}{\rho(p + \beta^2/q)^2 ||\theta_0||^2} - \frac{2\beta}{\sqrt{\rho(p + \beta^2/q)} ||\theta_0||^2} \frac{\theta_0^T h}{2 ||\theta_0||/\sqrt{d}}}.
\]

From \( (96) \), we recover the same sufficient condition as in the first case, i.e.,

\[
\alpha \leq \sigma_{\theta_0} - \delta,
\]

for some arbitrarily small \( \delta > 0 \), which guarantees that both \( \mathcal{O}_{d,1} \) and \( \mathcal{O}_{d,2} \) are jointly convex in \( (\alpha, \tau_2) \) w.p.a. \( 1 \) as \( d \to \infty \). As a consequence, from now on, we will work with \( K_\alpha := \sigma_{\theta_0} - \delta \). Similarly to the first case, in what follows we will drop the “w.p.a. \( 1 \) as \( d \to \infty \)” whenever we refer to the convexity of the functions \( \mathcal{O}_{d,2} \). Moreover, we will drop the “under the condition \( \alpha \leq \sigma_{\theta_0} - \delta \)” whenever we will say that \( \mathcal{O}_{d,2} \), and its partial minimizations, are convex.

We have just shown that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \tau_2, \beta) \) is jointly convex in \( (\alpha, \tau_1, \tau_2) \), and therefore, after the partial minimization over \( \tau_2 \), we have that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is jointly convex in \( (\alpha, \tau_1) \).

We will now prove that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is continuous on its domain. First notice that \( \mathcal{O}_{d,2}(\alpha, \tau_1, \tau_2, \beta) \) is continuous on its domain (the proof of this follows exactly the same lines as the proof of continuity for \( \mathcal{O}_{d,1}(\alpha, \tau_1, \tau_2, \beta) \)). Then, we will show that the minimization over \( \tau_2 > 0 \) can be equivalently rewritten as a minimization over \( \tau_2 \in \mathcal{T}_2 \), with \( \mathcal{T}_2 \) a compact set. Consequently, \( \mathcal{O}_{d,2}(\alpha, \tau_1, \beta) \) is a continuous function. Now, in order to show that we can restrict \( \tau_2 > 0 \) to \( \tau_2 \in \mathcal{T}_2 \) without loss of generality, we
will show that the optimal solution $\tau_2^*$ satisfies $0 < \tau_2^* < \infty$. The fact that $\tau_2^* > 0$ follows from

$$
\lim_{\tau_2 \to 0^+} O_{d,2}(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to 0^+} \frac{p \tau_2}{2} + \frac{\beta^2 \tau_2}{2q} - \frac{\alpha}{\sqrt{n}} \left\| \frac{\sqrt{\rho}}{p} \right\| \theta_0 \frac{2}{\tau_2} - \beta h + \frac{\left\| \theta_0 \right\|^2 + d\alpha^2}{2n\tau_2} \left( p + \frac{\beta^2}{q} \right) = \lim_{\tau_2 \to 0^+} \frac{\alpha}{\sqrt{n}} \left( p + \frac{\beta^2}{q} \right) \left( \frac{\theta_0}{\tau_2} + \frac{\left\| \theta_0 \right\|^2 + d\alpha^2}{2n\tau_2} \left( p + \frac{\beta^2}{q} \right) \right) \right) = \lim_{\tau_2 \to 0^+} \frac{\alpha}{\sqrt{n}} \left( p + \frac{\beta^2}{q} \right) \left( \frac{\theta_0}{\tau_2} - \alpha \right)^2,
$$

which is equal to $\infty$, w.p.a. 1 as $d \to \infty$, due to the condition (97). Moreover, $\tau_2^* < \infty$ follows from

$$
\lim_{\tau_2 \to \infty} O_{d,2}(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to \infty} \frac{p \tau_2}{2} + \frac{\beta^2 \tau_2}{2q} - \frac{\alpha}{\sqrt{n}} \left\| \frac{\sqrt{\rho}}{p} \right\| \theta_0 \frac{2}{\tau_2} - \beta h + \frac{\left\| \theta_0 \right\|^2 + d\alpha^2}{2n\tau_2} \left( p + \frac{\beta^2}{q} \right) = \lim_{\tau_2 \to \infty} \frac{\alpha}{\sqrt{n}} \left( p + \frac{\beta^2}{q} \right) \left( \frac{\theta_0}{\tau_2} - \alpha \right)^2,
$$

which is equal to $\infty$, w.p.a. 1 as $d \to \infty$, since $\|h\|/\sqrt{n} \xrightarrow{p} \sqrt{\rho}$.

As a consequence of the continuity and convexity-concavity, we can apply Sion’s minimax principle to exchange the minimization and the maximization in (94) and obtain

$$
\inf_{0 \leq \alpha, \tau_1 \geq 0 - \delta} \max_{0 \leq \beta \leq B} O_{d,2}^2(\alpha, \tau_1, \beta).
$$

We have now arrived at the scalar formulation (98), which has the same optimal value as the modified (AO) problem (60), w.p.a. 1 as $d \to \infty$. Therefore, following the same reasoning as in the first case, the next step is to study the convergence in probability of its optimal value, and to show that its optimal solution $\alpha_{*,2}$ is unique. The convergence in probability of its objective function $O_{d,2}^2(\alpha, \tau_1, \beta)$ will be done in two steps: first, we will study the convergence in probability of the function $O_{d,2}(\alpha, \tau_1, \tau_2, \beta)$, and secondly, we will employ statement (iii) of Lemma C.3 to study the convergence in probability of $O_{d,2}^2(\alpha, \tau_1, \beta)$. Following this, we will prove that $\alpha_{*,2}$ is unique.

Following similar arguments to the convergence in probability $O_{d,1}(\alpha, \tau_1, \tau_2, \beta) \xrightarrow{p} O_1(\alpha, \tau_1, \tau_2, \beta)$ previously treated, we have that for fixed $(\alpha, \tau_1, \tau_2, \beta)$ satisfying the constraints in (98), $O_{d,2}(\alpha, \tau_1, \tau_2, \beta)$ converges in probability to

$$
O_2(\alpha, \tau_1, \tau_2, \beta) := \frac{\beta \tau_1}{2} + \frac{p \tau_2}{2} + \frac{\beta^2 \tau_2}{2q} - \frac{\beta^2}{2M} + F(\alpha, \tau_1/\beta) - \alpha \sqrt{\rho} \left( p + \frac{\beta^2}{q} \right) \frac{\rho \sigma^2_{\theta_0}}{\tau_2} + \beta^2 + \rho \left( p + \frac{\beta^2}{q} \right) \frac{\sigma^2_{\theta_0} + \alpha^2}{2\tau_2}.
$$

(99)

Now, for fixed $(\alpha, \tau_1, \beta)$ satisfying the constraints in (98), \{O_{d,2}(\alpha, \tau_1, \tau_2, \beta)\}_{d \in \mathbb{N}} is a sequence of random real-valued convex functions, converging in probability (pointwise, for every $\tau_2 > 0$) to the function $O_2(\alpha, \tau_1, \cdot, \beta)$. Moreover,

$$
\lim_{\tau_2 \to \infty} O_2(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to \infty} \frac{p \tau_2}{2} + \frac{\beta^2 \tau_2}{2q} - \alpha \sqrt{\rho} \left( p + \frac{\beta^2}{q} \right) \frac{\rho \sigma^2_{\theta_0}}{\tau_2} + \beta^2 + \rho \left( p + \frac{\beta^2}{q} \right) \frac{\sigma^2_{\theta_0} + \alpha^2}{2\tau_2} = \lim_{\tau_2 \to \infty} \frac{\beta^2 \tau_2}{2q} - \alpha \beta \sqrt{\rho} = \infty.
$$

(100)
As a consequence, using statement (iii) of Lemma C.3, we have that

\[ \mathcal{O}_{d,2}^{\tau_2}(\alpha, \tau_1, \beta) \overset{P}{\to} \mathcal{O}_2^{\tau_2}(\alpha, \tau_1, \beta) := \inf_{\tau_2 > 0} \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) \]  

(101)

Notice that \( \mathcal{O}_2^{\tau_2} \) is precisely the second objective function of the minimax problem (23) from the statement of Theorem 4.5. Since \( \mathcal{O}_2^{\tau_2} \) is the pointwise limit (in probability, for each \((\alpha, \tau_1, \beta)\)) of the sequence of objective functions \( \mathcal{O}_{d,2}^{\tau_2} \) which are convex-concave, and convexity is preserved by pointwise limits, we have that \( \mathcal{O}_2^{\tau_2} \) is jointly convex in \((\alpha, \tau_1)\) and concave in \(\beta\).

We now consider the asymptotic minimax optimization problem

\[ \inf_{0 \leq \alpha \leq \sigma_{\theta_0} - \delta} \max_{\tau_1 > 0} \mathcal{O}_2^{\tau_2}(\alpha, \tau_1, \beta), \]  

(102)

and prove that its optimal solution \( \alpha_{*,2} \) is unique. To prove this, we will show that the function \( \inf_{\tau_1 > 0} \max_{0 \leq \beta \leq B} \mathcal{O}_2^{\tau_2}(\alpha, \tau_1, \beta) \) is strictly convex in \(\alpha\). As in the case of \( \mathcal{O}_1(\alpha, \tau_1, \tau_2, \beta) \), and following the same reasoning as in (76)-(77), we have that \( \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) \) is jointly strictly convex in \((\alpha, \tau_2)\) if the condition (97) (i.e. \( \alpha \leq \sigma_{\theta_0} - \delta \)) is satisfied. Indeed, the joint strict convexity in \((\alpha, \tau_2)\) depends only on the last two terms in (99), which after a simplification (over \(\beta, p, q, \rho, \sigma_{\theta_0} \) which have no influence on the convexity), can be rewritten as

\[ -\alpha \sqrt{\frac{1}{\tau_2} + \frac{\beta^2}{\rho(p + \beta^2/q)^2\sigma_{\theta_0}^2} + \frac{1}{\tau_2} \left( \frac{\alpha^2}{2\sigma_{\theta_0}} + \frac{\alpha^2}{2\sigma_{\theta_0}} \right)}, \]

from which we recover (similarly as before) the following necessary and sufficient condition for the joint strict convexity in \((\alpha, \tau_2)\),

\[ \alpha < \sigma_{\theta_0} \sqrt{1 + \frac{\tau_2^2}{\rho(p + \beta^2/q)^2\sigma_{\theta_0}^2}}, \]

which holds whenever condition (97) holds.

This can be used to prove that the optimal solution \( \alpha_{*,2} \) of problem (102) is unique, as we have previously seen for \( \mathcal{O}_1 \). For completeness, however, we will briefly remind the main steps of the proof of uniqueness. First, from

\[
\lim_{\tau_2 \to 0^+} \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) = \lim_{\tau_2 \to 0^+} \frac{p\tau_2}{2} + \frac{\beta^2\tau_2}{2q} - \alpha \sqrt{\left( \frac{p + \beta^2}{q} \right)^2 \frac{\rho\sigma_{\theta_0}^2}{\tau_2^2} + 1 + \beta^2 + \rho \left( \frac{p + \beta^2}{q} \right) \frac{\sigma_{\theta_0}^2}{\tau_2^2}} = \infty,
\]

and

\[ \lim_{\tau_2 \to \infty} \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) = \infty, \]

(which was shown in (100)), we have that the minimization of \( \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) \) over \( \tau_2 \) is attained by some \( 0 < \tau_2^* < \infty \). From this and the joint strict convexity in \((\alpha, \tau_2)\) of \( \mathcal{O}_2(\alpha, \tau_1, \tau_2, \beta) \), we have that the function \( \mathcal{O}_2^{\tau_2}(\alpha, \tau_1, \beta) \) is strictly convex in \(\alpha\). In particular, \( \mathcal{O}_2^{\tau_2}(\alpha, \tau_1, \beta) \) can be written as the sum of a strictly convex function in \(\alpha\) and a jointly convex function in \((\alpha, \tau_1)\). Then, after minimizing over \( \tau_1 > 0 \) (notice that, by Sion’s minimax principle, we can exchange the maximization over \(\beta\) with the minimization over \(\tau_1\) in (102)) we are left with a function that is strictly convex in \(\alpha\) (since it is the
sum of a strictly convex and a convex function). Finally, since the maximization over \( \beta \) is attained, we have that \( \max_{0 \leq \beta \leq B} \inf_{\tau_1 > 0} O_d^{2\star}(\alpha, \tau_1, \beta) \) is strictly convex in \( \alpha \), and therefore \( \alpha_{\star, 2} \) is unique.

Similarly to the first case, the uniqueness of \( \alpha_{\star, 1} \) is the key element in the convergence analysis required by Fact 2.1. Again, we define, for arbitrary \( \eta > 0 \), the sets \( S_{\eta} := \{ \alpha \in [0, \sigma_{\theta_0} - \delta] : |\alpha - \alpha_{\star, 2}| < \eta \} \), with \( \alpha_{\star, 2} \) the unique solution of (102), and \( S_{\eta}^c := [0, \sigma_{\theta_0} - \delta] \setminus S_{\eta} \). Since \( \alpha_{\star, 2} \) is the unique solution of (102), we have that

\[
\inf_{0 \leq \alpha \leq \sigma_{\theta_0} - \delta} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta) < \inf_{\alpha \in S_{\eta}^c \cap [0, \sigma_{\theta_0} - \delta]} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta) \quad (103)
\]

Now, in order to finish the proof for the second case, we still have to prove the following two convergences in probability (pointwise, for every \( \tau_1 > 0 \))

\[
\inf_{0 \leq \alpha \leq \sigma_{\theta_0} - \delta} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta) \overset{P}{\underset{\tau_1 \to 0}{\longrightarrow}} \inf_{\alpha \in S_{\eta}^c \cap [0, \sigma_{\theta_0} - \delta]} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta), \quad (104)
\]

and

\[
\inf_{\alpha \in S_{\eta}^c \cap [0, \sigma_{\theta_0} - \delta]} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta) \overset{P}{\underset{\tau_1 \to 0}{\longrightarrow}} \inf_{\alpha \in S_{\eta}^c \cap [0, \sigma_{\theta_0} - \delta]} \max_{0 \leq \beta \leq B} O_d^{2\star}(\alpha, \tau_1, \beta). \quad (105)
\]

Then, the result will simply follow from Fact 2.1. Again, it will be easier to prove the two convergences in probability in the following equivalent form (by Sion’s minimax theorem)

\[
\min_{\alpha} \max_{0 \leq \beta \leq B} \inf_{\tau_1 > 0} O_d^{2\star}(\alpha, \tau_1, \beta) \overset{P}{\underset{\tau_1 \to 0}{\longrightarrow}} \min_{\alpha} \max_{0 \leq \beta \leq B} \inf_{\tau_1 > 0} O_d^{2\star}(\alpha, \tau_1, \beta).
\]

For fixed \((\alpha, \beta)\), with \( \beta > 0 \), \( \{O_d^{2\star}(\alpha, \cdot, \beta)\}_{d \in \mathbb{N}} \) is a sequence of random, real-valued convex functions (since they are defined as the partial minimization of jointly convex functions), converging in probability (pointwise, for every \( \tau_1 > 0 \)) to the function \( O_d^{2\star}(\alpha, \cdot, \beta) \). Moreover, we have that

\[
\lim_{\tau_1 \to \infty} O_d^{2\star}(\alpha, \tau_1, \beta) = \infty.
\]

This follows, as in the first case, from

\[
\lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} + \mathcal{F}(\alpha, \frac{\tau_1}{\beta}) \geq \lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} = \infty.
\]

As a consequence, when \( \beta > 0 \), we can apply again statement (iii) of Lemma C.3 to conclude that

\[
\inf_{\tau_1 > 0} O_d^{2\star}(\alpha, \tau_1, \beta) \overset{P}{\underset{\tau_1 \to 0}{\longrightarrow}} \inf_{\tau_1 > 0} O_d^{2\star}(\alpha, \tau_1, \beta).
\]

Moreover, for \( \beta = 0 \) the objective functions \( O_d^{2\star} \) are reduced to

\[
O_d^{2\star}(\alpha, \tau_1, 0) = \frac{p r_{2\star}}{2} + \frac{pp}{2 r_{2\star}} \left( \frac{\|\theta_0\| - \alpha}{\sqrt{d}} \right)^2,
\]

and therefore the terms involving \( \tau_1 \) are cancelled. Here, \( r_{2\star} \) is the optimal solution attaining the infimum (which exists as previously explained).

We now define \( O_2^{T_1, r_2}(\alpha, \beta) := \inf_{\tau_1 > 0} O_2^{T_1, r_2}(\alpha, \tau_1, \beta) \) and \( O_d^{T_1, r_2}(\alpha, \beta) := \inf_{\tau_1 > 0} O_d^{T_1, r_2}(\alpha, \tau_1, \beta) \). For fixed \( \alpha \), \( \{O_d^{T_1, r_2}(\alpha, \cdot)\}_{d \in \mathbb{N}} \) is a sequence of random, real-valued concave functions (since they are defined as a minimization of concave functions), converging in probability (pointwise, for every \( 0 < \beta \leq B \)) to the function \( O_2^{T_1, r_2}(\alpha, \cdot) \). Therefore, we can apply statement (ii) of Lemma C.3 to conclude that

\[
\sup_{0 < \beta \leq B} O_d^{T_1, r_2}(\alpha, \beta) \overset{P}{\underset{0 < \beta \leq B}{\longrightarrow}} \sup_{0 < \beta \leq B} O_2^{T_1, r_2}(\alpha, \beta),
\]

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Moreover, for \( \beta = 0 \), it can be easily seen that
\[
\mathcal{O}_{d,2}^{\tau_2}(\alpha, \tau_1, 0) \overset{P}{\to} \mathcal{O}_{2}^{\tau_2}(\alpha, \tau_1, 0) = \frac{p\tau_{2*}}{2} + \frac{pp}{2\tau_{2*}} (\sigma_{00} - \alpha)^2,
\]
where \( \tau_{2*} \) is the optimal solution attaining the infimum (which exists as previously explained). As a consequence, we have that
\[
\sup_{0 \leq \beta \leq B} \mathcal{O}_{d,2}^{\tau_1, \tau_2}(\alpha, \beta) \overset{P}{\to} \sup_{0 \leq \beta \leq B} \mathcal{O}_{2}^{\tau_1, \tau_2}(\alpha, \beta),
\]

(106)

We have now arrived at the final step of the convergence proof, where again, due to the Convexity Lemma C.2, we will distinguish between the case \( \alpha_{*,2} = 0 \) and \( \alpha_{*,2} > 0 \). The proof follows exactly the same lines as in (85)-(88), which we report in what follows for completeness.

We first consider the case \( \alpha_{*,2} = 0 \). Since (106) is valid for any \( \alpha \geq 0 \), it is in particular valid for \( \alpha = 0 \), and thus we have that
\[
\mathcal{O}_{d,2}^{\beta, \tau_1, \tau_2}(0) \overset{P}{\to} \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(0).
\]

(107)

Moreover, using statement (i) of Lemma C.3, for any \( K > 0 \), we have that
\[
\min_{K \leq \alpha \leq \sigma_{00} - \delta} \mathcal{O}_{d,2}^{\beta, \tau_1, \tau_2}(\alpha) \overset{P}{\to} \min_{K \leq \alpha \leq \sigma_{00} - \delta} \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(\alpha).
\]

(108)

The result for \( \alpha_{*,1} = 0 \) now follows from (107) and (108) and the fact that
\[
\mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(0) < \min_{K \leq \alpha \leq \sigma_{00} - \delta} \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(\alpha),
\]
for any \( K > 0 \), due to the uniqueness of \( \alpha_{*,2} \).

We now focus on the case \( \alpha_{*,2} > 0 \). We need to prove (104) and (105), which are equivalent to the following two convergences in probability
\[
\inf_{0 < \alpha \leq \sigma_{00} - \delta} \mathcal{O}_{d,2}^{\beta, \tau_1, \tau_2}(\alpha) \overset{P}{\to} \inf_{0 < \alpha \leq \sigma_{00} - \delta} \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(\alpha),
\]

(109)

and
\[
\inf_{\alpha \in S_0} \mathcal{O}_{d,2}^{\beta, \tau_1, \tau_2}(\alpha) \overset{P}{\to} \inf_{\alpha \in S_0} \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(\alpha),
\]

(110)

respectively. Since \( \{\mathcal{O}_{d,2}^{\beta, \tau_1, \tau_2}(\cdot)\}_{d \in \mathbb{N}} \) is a sequence of random, real-valued convex functions, converging in probability (pointwise, for every \( \alpha > 0 \)) to the function \( \mathcal{O}_{2}^{\beta, \tau_1, \tau_2}(\cdot) \), (109) follows immediately from statement (ii) of Lemma C.3. Moreover, since \( S_0 = (0, \alpha_{*,2} - \eta] \cup [\alpha_{*,2} + \eta, \sigma_{00} - \delta] \), (110) follows from statement (i) of Lemma C.3, for the interval \([\alpha_{*,2} + \eta, \sigma_{00} - \delta] \), and statement (ii) of the same lemma, for the interval \((0, \alpha_{*,2} - \eta] \). This concludes the proof of the two convergences in probability (104) and (105), and with it, the analysis of the second case (60), and the proof.

\[\Box\]

Proof of Proposition 4.11. The proof follows similar lines as the proof of Theorem 4.3, and therefore we only provide a sketch. We start by rewriting the minimization on right-hand side of (26) in the following form
\[
\min_{\theta \in \mathbb{R}^d} \frac{p}{d} \left( \sqrt{n}||y - Ax|| + \sqrt{\epsilon_0} \sqrt{n}||\theta|| \right),
\]

(111)
which enables us to quantify the high-dimensional estimation error using Theorem 1 in [120]. Here, the loss function and the regularization function are both equal to $\sqrt{n} \cdot \| \cdot \|$. Therefore, similarly to the proof of Theorem 4.3, we can compute the Moreau envelope of $\sqrt{n} \cdot \| \cdot \|$ as

$$e_{\sqrt{n} \cdot \|} (c; \tau) = \begin{cases} \sqrt{n} \cdot \| c \| - n \tau / 2 & \text{if } \| c \| > \sqrt{n} \tau, \\ \| c \|^2 / (2 \tau) & \text{if } \| c \| \leq \sqrt{n} \tau, \end{cases}$$

from which the two limits in probability from Theorem 1 in [120] can be computed as follows

$$\frac{1}{n} \left( e_{\sqrt{n} \cdot \|} (cg + z; \tau) - \sqrt{n} \cdot \| z \| \right) \xrightarrow{P} \begin{cases} \sqrt{(c^2 + \sigma^2)} - \tau / 2 - \sigma / \sqrt{\rho} & \text{if } c^2 + \sigma^2 > \tau, \\ (c^2 + \sigma^2) / (2 \tau) - \sigma / \sqrt{\rho} & \text{if } c^2 + \sigma^2 \leq \tau, \end{cases}$$

which is equal to the function $E(c, \tau)$ defined above, and

$$\frac{1}{d} \left( e_{\sqrt{n} \cdot \|} (ch + \theta_0; \tau) - \sqrt{n} \cdot \| \theta_0 \| \right) \xrightarrow{P} \begin{cases} \sqrt{(c^2 + \sigma^2_{\theta_0}) / \rho - \tau / (2 \rho) - \sigma_{\theta_0} / \sqrt{\rho}} & \text{if } \sqrt{\rho} (c^2 + \sigma^2_{\theta_0}) > \tau, \\ (c^2 + \sigma^2_{\theta_0}) / (2 \tau) - \sigma_{\theta_0} / \sqrt{\rho} & \text{if } \sqrt{\rho} (c^2 + \sigma^2_{\theta_0}) \leq \tau. \end{cases}$$

which is equal to the function $G(c, \tau)$ defined in (18). The proofs of these convergences follow the same arguments as in Theorems 4.3 and 4.5, thus are omitted. Moreover, all the assumptions required by Theorem 1 in [120] can be easily shown to hold (as in the proof of Theorems 4.3, or as in Section V.E in [120]). This concludes the sketch of the proof.

**Proof of Theorem 4.13.** Similarly to Theorem 4.5, since the proof is quite long and intricate, we divide it into steps, which we briefly explain before jumping into the technical details.

**Step 1:** We show that problem (15) can be equivalently rewritten as a (PO) problem. The proof builds upon the dual formulation (16) presented in Fact 3.10. After introducing the change of variable $w = (\theta - \theta_0) / \sqrt{d}$ and expressing it in vector form, (16) becomes

$$\min_{w \in \mathcal{W}} \max_{u \in \mathbb{R}^n} -\frac{1}{n} w^T (\sqrt{d} A) w + \frac{1}{n} u^T z + \frac{1}{4n} \| \theta_0 + \sqrt{d} w \|^2 \| u \|^2 - \frac{1}{n} \sum_{i=1}^n L^*(u_i)$$

(112)

where $\mathcal{W}$ is the feasible set of $w$ obtained from $\Theta$ after the change of variable, $A \in \mathbb{R}^{n \times d}$ denotes the matrix whose rows are the vectors $x_i$, for $i = 1, \ldots, n$, and $z \in \mathbb{R}^n$ is the measurement noise vector with entries i.i.d. distributed according to $\mathbb{P}_Z$. Notice that, due to Assumption 3.7, the set $\mathcal{W}$ is convex and compact. Moreover, following the assumptions 3.7 and 4.1(ii), we have that $\mathcal{W} \subset \{ w \in \mathbb{R}^d : \| w \| \leq R_\theta + \sigma_{\theta_0} \}$ a.s. as $d \to \infty$.

From Lemma 3.12 we know that the optimal solution $u_*$ is in the order of $\sqrt{n}$. As a consequence, by rescaling the variable $u$ as $u \to u / \sqrt{n}$, and introducing the convex compact set $S_u := \{ u \in \mathbb{R}^n : \| u \| \leq K_\beta \}$, for some sufficiently large $K_\beta > 0$, we can equivalently rewrite (112) as

$$\min_{w \in \mathcal{W}} \max_{u \in S_u} -\frac{1}{\sqrt{n}} w^T (\sqrt{d} A) w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \| \theta_0 + \sqrt{d} w \|^2 \| u \|^2 - \frac{1}{n} \sum_{i=1}^n L^*(u_i \sqrt{n})$$

(113)

We will now show that the assumption that $L$ can be written as the Moreau envelope with parameter $M$ of a convex function $f$ is natural, and always satisfied by $M$-smooth functions (which is precisely our case, due to Assumption 3.1(iv)). Since $L$ is $M$-smooth, we have that its convex conjugate $L^*$ is $1/M$-strongly convex, and therefore $L^*$ can be written as

$$L^*(\cdot) = \frac{1}{2M} (\cdot)^2 + f^*(\cdot),$$

(114)
where \( f^* \) is the conjugate of a convex function \( f \). Notice that \( f \) can be easily obtained (as a function of \( L \)) from (114) as \( f = (L^* - 1/(2M)\cdot)^2)^* \). Conversely, the loss function \( L \) is nothing but the Moreau envelope of \( f \), as shown in what follows

\[
L(\cdot) = \left( \frac{1}{2M} \cdot^2 + f^*(\cdot) \right)^* (\cdot) = \left( \frac{M}{2} \cdot^2 \star_{\text{inf}} f(\cdot) \right) (\cdot) = e_f \left( \cdot, \frac{1}{M} \right),
\]

where \( \star_{\text{inf}} \) denotes the infimal convolution operation. Since \( L \) is continuous and convex (notice that \( L \) is trivially proper due to Assumption 4.1(i)), with \( \min_{v\in\mathbb{R}} L(v) = 0 \), we have that \( L^* \) is lower semicontinuous and convex, and therefore both \( f \) and \( f^* \) are lower semicontinuous and convex.

Using the decomposition (114), problem (113) can be rewritten as

\[
\min_{w\in\mathcal{W}} \max_{u\in\mathcal{S}_u} -\frac{1}{\sqrt{n}} u^\top (\sqrt{d}A)w + \frac{1}{\sqrt{n}} u^\top z + \frac{1}{4\lambda}\|\theta_0 + \sqrt{d}w\|^2\|u\|^2 - \frac{1}{2M}\|u\|^2 - \frac{1}{n} \sum_{i=1}^n f^*(u_i\sqrt{n}). \tag{115}
\]

It is important now to highlight that the expression

\[
\frac{1}{4\lambda}\|\theta_0 + \sqrt{d}w\|^2\|u\|^2 - \frac{1}{2M}\|u\|^2
\]

is concave in \( u \). This follows from the assumption that \( \lambda > MR_0^2d/2 \geq M\|\theta_0 + \sqrt{d}w\|^2/2 \). As will be shown later, the decomposition (114) and the concavity in \( u \) of the above-mentioned expression will allow us to use the CGMT.

For ease of notation, we introduce the function \( F^*: \mathbb{R}^n \to \mathbb{R} \), which is defined as

\[
F^*(u) := \sum_{i=1}^n f^*(u_i).
\]

Since \( f, f^* \) are lower semicontinuous and convex, and therefore its convex conjugate \( F = F^{**} \) can be easily recovered as follows

\[
F(u) = \sup_{v\in\mathbb{R}^n} v^\top u - F^*(v) = \sum_{i=1}^n \sup_{v_i\in\mathbb{R}} v_i u_i - f^*(v_i) = \sum_{i=1}^n f(u_i).
\]

We now rewrite \( F^*(u\sqrt{n}) \) using its convex conjugate as follows

\[
\min_{w\in\mathcal{W}} \max_{u\in\mathbb{R}^n} -\frac{1}{\sqrt{n}} u^\top (\sqrt{d}A)w + \frac{1}{\sqrt{n}} u^\top z + \frac{1}{4\lambda}\|\theta_0 + \sqrt{d}w\|^2\|u\|^2 - \frac{1}{2M}\|u\|^2 - \frac{1}{n} s^\top u + \frac{1}{n} F(s) \tag{116}
\]

where we have used Sion’s minimax principle to exchange the minimization over \( s \) with the maximization over \( u \). In particular, this last step is possible only because the objective function in (116) is concave in \( u \), due to the decomposition (114).

Notice that the new objective function in (116) is convex in \((w, s)\) and concave in \( u \). Indeed, the concavity in \( u \) holds, as explained above, from the assumption that \( \lambda > MR_0^2d/2 \geq M\|\theta_0 + \sqrt{d}w\|^2/2 \). Moreover, the convexity in \( w \) can be easily seen from the convexity in \( w \) of the term \( 1/(4\lambda)\|\theta_0 + \sqrt{d}w\|^2\|u\|^2 \) and the linearity of the other terms. Finally, the convexity in \( s \) can be concluded from the convexity of \( F \), and the joint convexity in \((w, s)\) follows easily since \( w \) and \( s \) are decoupled.

Using Assumption 1.1, we can see that \( \sqrt{d}A \) has entries i.i.d. \( \mathcal{N}(0,1) \). Moreover, the objective function (116) is convex-concave in \((w, u)\), with \( \mathcal{W}, \mathcal{S}_u \) convex compact sets. As a consequence, problem (116) is a (PO) problem, in the form (3).
Step 2: We apply CGMT and obtain the modified (AO) problem. Since (116) is a (PO) problem, we can associate to it the following (AO) problem
\[
\min_{w \in \mathbb{W}} \max_{x \in \mathbb{X}} \frac{1}{\sqrt{n}} \|w\| g^T u - \frac{1}{\sqrt{n}} \|u\| h^T w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 - \frac{1}{2M} \|u\|^2 - \frac{1}{\sqrt{n}} s^T u + \frac{1}{n} F(s),
\]
which is tightly related to the (PO) problem in the high-dimensional regime, as explained in Section 2. At this point, following the discussion presented in Section 2, we consider the following modified (AO) problem,(117)
\[
0 \leq \beta \leq K \beta, \quad \min_{w \in \mathbb{W}} \max_{x \in \mathbb{X}} \frac{1}{\sqrt{n}} \|w\| g^T u - \frac{1}{\sqrt{n}} \|u\| h^T w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d}w\|^2 - \frac{1}{2M} \|u\|^2
\]
where we have separated the maximization over \(u \in S_u = \{u \in \mathbb{R}^n : \|u\| \leq K\}\) into the maximization over the magnitude of \(u\), i.e., \(0 \leq \beta \leq K\), and the maximization over \(\|u\| = \beta\). Recall that, since the (AO) problem is not convex-concave (due to the random vectors \(g\) and \(h\)), the exchange of the minimization over \((w, s)\) and the maximization over \(\beta\) is not justified. Therefore the two problems (117) and (118) might not be equivalent. However, as seen in Fact 2.1, the modified (AO) problem (118) can be used to study the optimal solution of the (PO) problem (116) in the asymptotic regime, based solely on the optimal value of the modified (AO) problem. We will do so after simplifying the problem, by reducing it to a scalar one, as shown in what follows.

Step 3: We consider the modified (AO) problem (118), and show that it can be reduced to a scalar problem, which involves only three scalar variables. We start by scalarizing problem (118) over the magnitude of \(u\), leading to the following problem
\[
\max_{0 \leq \beta \leq K \beta} \min_{w \in \mathbb{W}} \max_{x \in \mathbb{X}} \frac{\beta}{\sqrt{n}} \|w\| g + z - s \| - \frac{\beta}{\sqrt{n}} h^T w + \frac{\beta^2}{4d\lambda_0} \|\theta_0 + \sqrt{d}w\|^2 - \frac{\beta^2}{2M} + \frac{1}{n} F(s),
\]
where we have also used the fact that \(\lambda = d\lambda_0\).

We now employ the square-root trick to rewrite the first term in the objective function of (119) as follows
\[
\frac{1}{\sqrt{n}} \|w\| g + z - s \| = \inf_{\tau_1 > 0} \frac{\tau_1}{2} + \frac{1}{2n\tau_1} \|w\| g + z - s \|^2.
\]
Introducing this in (119) and re-organizing the terms gives
\[
\max_{0 \leq \beta \leq K \beta} \min_{w \in \mathbb{W}} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{\beta}{2n\tau_1} \|w\| g + z - s \|^2 + \frac{1}{n} F(s) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\beta^2}{4d\lambda_0} \|\theta_0 + \sqrt{d}w\|^2,
\]
which allows us to introduce the Moreau envelope of \(F\), i.e.,
\[
\epsilon_F(\alpha g + z; \frac{\tau_1}{\beta}) := \min_{s \in \mathbb{R}^n} \frac{\beta}{2\tau_1} \|\alpha g + z - s\|^2 + F(s),
\]
giving rise to
\[
\max_{0 \leq \beta \leq K \beta} \min_{w \in \mathbb{W}} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} \epsilon_F(\|w\| g + z; \frac{\tau_1}{\beta}) - \frac{\beta}{\sqrt{n}} h^T w + \frac{\beta^2}{4d\lambda_0} \|\theta_0 + \sqrt{d}w\|^2.
\]
We now proceed with the aim of scalarizing the problem over the magnitude of \( w \) (similarly to what we have done for \( u \)). In order to scalarize the problem over the magnitude of \( w \), instead of the constraint set \( \mathcal{W} \), we would need to work with a set of the form \( \mathcal{S}_w := \{ w \in \mathbb{R}^d : \| w \| \leq K_\alpha \} \). Notice that if we choose \( K_\alpha := R_\theta + \sigma \theta_\alpha \), then we have \( \mathcal{W} \subset \mathcal{S}_w \) w.p.a. \( 1 \) as \( d \to \infty \). Therefore, from now on we impose such a constraint on \( w \) (i.e., \( w \in \mathcal{S}_w \)), and we postpone the discussion about the implications that this constraint has on the error \( \| \hat{\theta}_{DRE} - \theta_0 \| \) to the end of the proof.

Similarly to what was done for \( u \), we can now separate the constraint \( w \in \mathcal{S}_w \) into the two constraints \( 0 \leq \alpha \leq K_\alpha \) and \( \| w \| = \alpha \), and by expressing \( \| \theta_0 + \sqrt{d}w \| \) as \( \| \theta_0 \|^2 + \| w \|^2 + 2\sqrt{d} \theta_0 \cdot w \), we can now scalarize the problem over the magnitude of \( w \) and obtain

\[
\max_{0 \leq \beta \leq K_\beta} \inf_{0 \leq \alpha \leq K_\alpha, \tau_1 \geq 0} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \tau_1 \beta) - \alpha \beta \left\| \frac{\beta \theta_0}{2\lambda_0 \sqrt{d}} - \frac{h}{\sqrt{n}} \right\| + \frac{\beta^2}{4\lambda_0} \left( \| \theta_0 \|^2 + \alpha^2 \right).
\] (122)

We denote by \( \mathcal{O}_d(\alpha, \tau_1, \beta) \) the objective function in problem (122), where the index \( d \) recalls that \( \mathcal{O}_d \) is parametrized by the dimension \( d \).

**Step 4:** We show that \( \mathcal{O}_d \) is continuous on its domain, jointly convex in \((\alpha, \tau_1)\), and concave in \( \beta \). Let’s first concentrate on the continuity property. Notice that we only need to show that the Moreau envelope \( e_F(\alpha g + z, \tau_1 / \beta) \) is continuous, since all the other terms in the objective function are trivially continuous. Since \( F \) is lower semicontinuous and convex (recall that \( F(u) = \sum_{i=1}^n f(u_i) \)), the continuity of the Moreau envelope \( e_F(\alpha g + z, \tau_1 / \beta) \) follows from [96, Theorem 2.26(b)].

Let’s now focus on the joint convexity in \((\alpha, \tau_1)\). Notice that the last two terms in (122) are linear and quadratic in \( \alpha \), and therefore trivially convex. Moreover, if we prove that the objective function on the right-hand side of (120) is jointly convex in \((\alpha, \tau_1, s)\), then, after minimizing over \( s \in \mathbb{R}^n \), the Moreau envelope \( e_F(\alpha g + z, \tau_1 / \beta) \) remains jointly convex in \((\alpha, \tau_1)\). But this is certainly the case since \( \| \alpha g - s \|^2 \) is jointly convex in \((\alpha, s)\), \( 1 / \tau_1 \| \alpha g - s \|^2 \) is the perspective function of \( \| \alpha g - s \|^2 \), and therefore jointly convex in \((\alpha, \tau_1, s)\), and the shifted function \( 1 / \tau_1 \| \alpha g + z - s \|^2 \) remains jointly convex.

Finally, we will show that \( \mathcal{O}_d(\alpha, \tau_1, \beta) \) is concave in \( \beta \). Notice first that the objective function in (121) is concave in \( \beta \). Indeed, this follows from the concavity of the Moreau envelope \( e_F(\alpha g + z, \tau_1 / \beta) \) (since it is obtained from the minimization of affine functions in \( \beta \)), and the concavity of

\[
-\frac{\beta^2}{2M} + \frac{\beta^2}{4d \lambda_0} \| \theta_0 + \sqrt{d}w \|^2,
\]

(due to the assumption that \( \lambda_0 > MR_\theta^2 / 2 \)). Now, since \( \mathcal{O}_d(\alpha, \tau_1, \beta) \) is the result of a minimization over these concave functions, it is concave in \( \beta \).

As a consequence of the continuity and convexity-concavity, we can apply Sion’s minimax principle to exchange the minimization and the maximization in (122) to obtain

\[
\inf_{0 \leq \alpha \leq K_\alpha} \max_{0 \leq \beta \leq K_\beta} \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + \frac{1}{n} e_F(\alpha g + z, \tau_1 \beta) - \alpha \beta \left\| \frac{\beta \theta_0}{2\lambda_0 \sqrt{d}} - \frac{h}{\sqrt{n}} \right\| + \frac{\beta^2}{4\lambda_0} \left( \| \theta_0 \|^2 + \alpha^2 \right).
\] (123)

**Step 5:** We study the convergence in probability of \( \mathcal{O}_d \). We have now arrived at the scalar formulation (123), which has the same optimal value as the modified (AO) problem (118). Therefore, following the reasoning presented in Fact 2.1, the next step is to study the convergence in probability of its optimal value. To do so, we start by studying the convergence in probability of its objective function \( \mathcal{O}_d \).
Notice first that the last two terms of $\mathcal{O}_d$ converge in probability as follows.

\[
\alpha \beta \left\| \frac{\beta}{2\lambda_0} \frac{\theta_0}{\sqrt{d}} - \frac{h}{\sqrt{n}} \right\|_2 \overset{P}{\to} \alpha \beta \left\| \frac{\beta^2 \sigma_\hat{\theta}_0^2}{4\lambda_0^2} + \rho, \right.
\]

\[
\frac{\beta^2}{4\lambda_0} \left( \frac{||\theta_0||^2}{d} + \alpha^2 \right) \overset{P}{\to} \frac{\beta^2}{4\lambda_0} \left( \sigma_\hat{\theta}_0^2 + \alpha^2 \right),
\]

where (124) can be recovered using

\[
\frac{\theta_0^T h}{d} \overset{P}{\to} 0,
\]

(see Lemma C.1) and the continuous mapping theorem (which states that continuous functions preserve limits in probability, when their arguments are sequences of random variables).

We will now study the convergence in probability of the the normalized Moreau envelope $e_F(\alpha g + z, \tau_1 / \beta) / n$. Since $F(u) := \sum_{i=1}^n f(u_i)$, (120) reduces to

\[
\frac{1}{n} e_F(\alpha g + z, \tau_1 / \beta) = \frac{1}{n} \sum_{i=1}^n \min_{s_i \in \mathbb{R}} \frac{\beta}{2\tau_1} (\alpha G + Z - s_i)^2 + f(s_i) = \frac{1}{n} \sum_{i=1}^n e_F(\alpha G + Z; \tau_1 / \beta),
\]

and its limit (in probability) can be immediately recovered from the weak law of large numbers as

\[
\frac{1}{n} e_F(\alpha g + z, \tau_1 / \beta) \overset{P}{\to} \mathbb{E}_{\mathcal{N}(0,1)\otimes \mathbb{P}_Z} \left[ e_F(\alpha G + Z; \tau_1 / \beta) \right],
\]

(126)

which is precisely the expected Moreau envelope $F(\alpha, \tau_1 / \beta)$ defined in (21). The quantity under the expectation in (126) is absolutely integrable, and therefore the expected Moreau envelope $F$ is well defined, as explained in what follows. For every $\beta \geq 0$ and $\tau_1 > 0$ we have

\[
\left| e_F(\alpha G + Z; \tau_1 / \beta) \right| = \min_{v \in \mathbb{R}} \frac{\beta}{2\tau_1} (\alpha G + Z - v)^2 + f(v) \leq \frac{\beta}{2\tau_1} (\alpha G + Z)^2 + f(0) = \frac{\beta}{2\tau_1} (\alpha G + Z)^2
\]

which is integrable due to the fact that both $G$ and $Z$ have finite second moment. In the first and last equality we have used the fact that $f(0) = \min_{v \in \mathbb{R}} f(v) = 0$, which can be shown to hold true as follows. Since $L(0) = \min_{v \in \mathbb{R}} L(v) = 0$ (from Assumption 4.1(i)), we have that $L^*(0) = \min_{v \in \mathbb{R}} L^*(v) = 0$, and from $f^*(\cdot) = L^*(\cdot) - 1/(2M)(\cdot)^2$, with $L^*$ that is $1/M$-strongly convex, we have that $f^*(0) = \min_{v \in \mathbb{R}} f^*(v) = 0$, which results in the desired $f(0) = \min_{v \in \mathbb{R}} f(v) = 0$. Moreover, for $\tau_1 \to 0^+$,

\[
\lim_{\tau_1 \to 0^+} \left| e_f(\alpha G + Z; \tau_1 / \beta) \right| = \lim_{\tau_1 \to 0^+} \min_{\tau_1 \to 0^+} \frac{\beta}{2\tau_1} (\alpha G + Z - v)^2 + f(v) = f(\alpha G + Z),
\]

whose expectation is finite by assumption. In particular, the second equality follows from [96, Theorem 1.25].

Therefore, from (124)-(126), we have that $\mathcal{O}_d(\alpha, \tau_1, \beta)$ converges in probability to the function

\[
\mathcal{O}(\alpha, \tau_1, \beta) := \frac{\beta \tau_1}{2} - \frac{\beta^2}{2M} + F(\alpha, \tau_1 / \beta) - \alpha \beta \sqrt{\frac{\beta^2 \sigma_\hat{\theta}_0^2}{4\lambda_0^2}} + \rho + \frac{\beta^2}{4\lambda_0} \left( \sigma_\hat{\theta}_0^2 + \alpha^2 \right).
\]

(127)

Notice that $\mathcal{O}$ is precisely the objective function of the minimax problem (29) from the statement of Theorem 4.13. Since $\mathcal{O}$ is the pointwise limit (in probability, for each $(\alpha, \tau_1, \beta)$) of the sequence of objective functions $\{\mathcal{O}_d\}_{d \in \mathbb{N}}$, which are convex-concave, and convexity is preserved by pointwise limits (see Lemma C.2), we have that $\mathcal{O}$ is jointly convex in $(\alpha, \tau_1)$ and concave in $\beta$.

Consider the asymptotic minimax optimization problem

\[
\inf_{0 \leq \alpha \leq K_\alpha} \max_{0 \leq \beta \leq K_\beta} \mathcal{O}(\alpha, \tau_1, \beta).
\]

(128)
Step 6: We show that the asymptotic problem \((128)\) has a unique minimizer \(\alpha_\star\). As explained in Section 2, the uniqueness of the optimal solution \(\alpha_\star\) is fundamental in the convergence analysis required by Fact 2.1. In order to prove this, we will show that the function \(\max_{0 \leq \beta \leq K_\beta} \inf_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta)\) is strictly convex in \(\alpha\) (by Sion’s minimax theorem, we have that this function is equal to \(\inf_{\tau_1 > 0} \max_{0 \leq \beta \leq K_\beta} \mathcal{O}(\alpha, \tau_1, \beta)\)).

First, notice that \(\mathcal{O}(\alpha, \tau_1, \beta)\) can be written as the sum of a function which is jointly convex in \((\alpha, \tau_1)\) and a function which is strictly convex in \(\alpha\). Therefore, after minimizing over \(\tau_1\), we have that \(\mathcal{O}^{\tau_1}((\alpha, \beta) : = \inf_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta)\) is the sum of a convex function in \(\alpha\) (since partial minimization of jointly convex functions is convex) and the (same as before) strictly convex function in \(\alpha\). Now, from the strict convexity of \(\mathcal{O}^{\tau_1}\) in \(\alpha\) it follows that for any \(\lambda \in (0, 1)\), \(\alpha^{(1)} \neq \alpha^{(2)}\), and \(\beta\) we have that

\[
\mathcal{O}^{\tau_1}(\lambda \alpha^{(1)} + (1 - \lambda) \alpha^{(2)}, \beta) < \lambda \mathcal{O}^{\tau_1}(\alpha^{(1)}, \beta) + (1 - \lambda) \mathcal{O}^{\tau_1}(\alpha^{(2)}, \beta)
\]

\[
\leq \lambda \max_{0 \leq \beta \leq K_\beta} \mathcal{O}^{\tau_1}(\alpha^{(1)}) + (1 - \lambda) \max_{0 \leq \beta \leq K_\beta} \mathcal{O}^{\tau_1}(\alpha^{(2)}).
\]

We can now take the maximum over \(\beta \in [0, K_\beta]\) on the left-hand side, and since this will be attained (due to the upper-semicontinuity in \(\beta\) of \(\mathcal{O}^{\tau_1}\), and the compactness of \([0, K_\beta]\)), we can conclude that

\[
\max_{0 \leq \beta \leq K_\beta} \mathcal{O}^{\tau_1}(\lambda \alpha^{(1)} + (1 - \lambda) \alpha^{(2)}, \beta) < \lambda \max_{0 \leq \beta \leq K_\beta} \mathcal{O}^{\tau_1}(\alpha^{(1)}) + (1 - \lambda) \max_{0 \leq \beta \leq K_\beta} \mathcal{O}^{\tau_1}(\alpha^{(2)}).
\]

This shows that \(\max_{0 \leq \beta \leq K_\beta} \inf_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta)\) is strictly convex in \(\alpha\), and therefore the optimal solution \(\alpha_\star\) is unique.

Step 7: We anticipate how the uniqueness of \(\alpha_\star\), together with Fact 2.1, can be used to conclude the proof. The uniqueness of \(\alpha_\star\) is a key element in the convergence analysis required by Fact 2.1, as explained in what follows. We first define, for arbitrary \(\eta > 0\), the sets \(\mathcal{S}_\eta := \{\alpha \in [0, K_\alpha] : |\alpha - \alpha_\star| < \eta\}\), with \(\alpha_\star\) the unique solution of \((128)\) and \(K_\alpha := R_0 + \sigma_0\), and \(\mathcal{S}_\eta^c := \{0, K_\alpha\} \setminus \mathcal{S}_\eta\). Since \(\alpha_\star\) is the unique solution of \((128)\), we have that

\[
\inf_{0 \leq \alpha \leq K_\alpha} \max_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta) < \inf_{\alpha \in \mathcal{S}_\eta^c} \max_{0 \leq \beta \leq K_\beta} \mathcal{O}(\alpha, \tau_1, \beta)
\]

(129)

Now, due to (129), if we prove that the optimal value of the (scalarized version of the) modified (AO) problem \((118)\) satisfies

\[
\inf_{0 \leq \alpha \leq K_\alpha} \max_{\tau_1 > 0} \mathcal{O}_{\alpha}(\alpha, \tau_1, \beta) \xrightarrow{P} \inf_{0 \leq \alpha \leq K_\alpha} \max_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta),
\]

(130)

and that, when additionally restricted to \(\alpha \in \mathcal{S}_\eta^c\), for arbitrary \(\eta > 0\), it satisfies

\[
\inf_{\alpha \in \mathcal{S}_\eta^c} \max_{\tau_1 > 0} \mathcal{O}_{\alpha}(\alpha, \tau_1, \beta) \xrightarrow{P} \inf_{\alpha \in \mathcal{S}_\eta^c} \max_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta),
\]

(131)

we can directly conclude the desired result \((28)\) from Fact 2.1. Therefore, in order to finish the proof, we only need to prove the two convergences in probability \((130)\) and \((131)\). It will be easier to prove the two convergences in probability in the following equivalent form (by Sion’s minimax theorem)

\[
\min_{\alpha} \max_{0 \leq \beta \leq K_\beta} \inf_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta) \xrightarrow{P} \min_{\alpha} \max_{0 \leq \beta \leq K_\beta} \inf_{\tau_1 > 0} \mathcal{O}(\alpha, \tau_1, \beta).
\]
Moreover, since for $\beta = 0$ we have $O_d \equiv 0$, for all $d \in \mathbb{N}$, and $O \equiv 0$ (this follows easily from $f(0) = \min_{v \in \mathbb{R}} f(v) = 0$), in the remaining of the proof we restrict our attention, without loss of generality, to the nontrivial case $\beta > 0$.

The two convergences in probability (130) and (131) are a consequence of Lemma C.3, as explained in what follows.

**Step 8: We prove the two convergences in probability (130) and (131), and conclude the proof.** First, for fixed $(\alpha, \beta)$, $\{O_d(\alpha, \cdot, \beta)\}_{d \in \mathbb{N}}$ is a sequence of random real-valued convex functions, converging in probability (pointwise, for every $\tau_1 > 0$) to the function $O(\alpha, \cdot, \beta)$. Moreover,
\[
\lim_{\tau_1 \to \infty} O(\alpha, \tau_1, \beta) = \lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2} + F(\alpha, \tau_1/\beta) + \text{const.} \geq \lim_{\tau_1 \to \infty} \frac{\beta \tau_1}{2},
\]
which is equal to $+\infty$. In particular, the inequality follows from the fact that $F(\alpha, \tau_1/\beta) \geq 0$.

As a consequence, we can apply statement (iii) of Lemma C.3 to conclude that
\[
\inf_{\tau_1 > 0} O(\alpha, \tau_1, \beta) \overset{P}{\to} \inf_{\tau_1 > 0} O(\alpha, \tau_1, \beta).
\]

We now define $O_\tau(\alpha, \beta) := \inf_{\tau_1 > 0} O(\alpha, \tau_1, \beta)$ and $O_d^\tau(\alpha, \beta) := \inf_{\tau_1 > 0} O_d(\alpha, \tau_1, \beta)$. For fixed $\alpha$, $\{O_d^\tau(\alpha, \cdot)\}_{d \in \mathbb{N}}$ is a sequence of random, real-valued concave functions (since they are defined as a minimization of concave functions), converging in probability (pointwise, for every $\beta \geq 0$) to the function $O_\tau(\alpha, \cdot)$. Therefore, we can apply statement (ii) of Lemma C.3 to conclude that
\[
\sup_{0 < \beta \leq K_\beta} O_d^\tau(\alpha, \beta) \overset{P}{\to} \sup_{0 < \beta \leq K_\beta} O_\tau(\alpha, \beta).
\]

We will now show that the constraint $0 < \beta \leq K_\beta$ can be relaxed to $\beta > 0$, without loss of generality. Recall that $K_\beta$ was inserted for convenience, in equation (113), to be able to use the CGMT (which required $u$ to live in a compact set). Now, notice that
\[
\lim_{\beta \to \infty} O_\tau(\alpha, \beta) \leq \lim_{\beta \to \infty} O_d^\tau(\alpha, \beta)
\]
\[
= \lim_{\beta \to \infty} \frac{\beta \bar{\tau}_1}{2} - \frac{\beta^2}{2M} + F(\alpha, \bar{\tau}_1/\beta) - \alpha \beta \sqrt{\frac{\beta^2 \rho^2}{4\lambda_0^2} + \rho + \frac{\beta^2}{4\lambda_0^2} \left(\sigma_{\theta_0}^2 + \alpha^2\right)}
\]
\[
= \lim_{\beta \to \infty} \frac{\beta^2}{2M} - \frac{\alpha \sigma_{\theta_0}^2}{2\lambda_0} + \frac{\beta^2}{4\lambda_0} \left(\sigma_{\theta_0}^2 + \alpha^2\right) + O(\beta)
\]
\[
= \lim_{\beta \to \infty} \beta^2 \left(\frac{1}{2M} + \frac{(\sigma_{\theta_0}^2 - \alpha^2)}{4\lambda_0}\right) + O(\beta) = -\infty
\]
for some constant $\bar{\tau}_1$ (let’s say, for example, equal to 1). Here, $O(\beta)$ encapsulates the remaining terms, which grow (at most) linearly in $\beta$. The inequality above follows from the fact that $O_\tau^1$ is defined as a minimization over $\tau_1 > 0$, and the last equality follows from the fact that
\[
-\frac{1}{2M} + \frac{(\sigma_{\theta_0}^2 - \alpha^2)}{4\lambda_0} < 0,
\]
since $\lambda_0 > MR_{\theta}^2/2$, and $\alpha \leq K_\alpha = R_{\theta} + \sigma_{\theta_0}$. The assumption $E_{\mathcal{N}(0,1) \circ \mathbb{P}_Z} [f(\alpha G + Z)] < \infty$ is very important here, since it guarantees that
\[
\lim_{\beta \to \infty} F(\alpha, \bar{\tau}_1/\beta) = \lim_{\beta \to \infty} E_{\mathcal{N}(0,1) \circ \mathbb{P}_Z} \left[ef(\alpha G + Z; \bar{\tau}_1/\beta)\right] = E_{\mathcal{N}(0,1) \circ \mathbb{P}_Z} \left[\lim_{\beta \to \infty} ef(\alpha G + Z; \bar{\tau}_1/\beta)\right]
\]
\[
= E_{\mathcal{N}(0,1) \circ \mathbb{P}_Z} [f(\alpha G + Z)] < \infty.
\]

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Here, the second equality follows from the monotone convergence theorem, and the last equality follows from [96, Theorem 1.25].

Therefore, there exists some sufficiently large $K_\beta$ such that

$$
\sup_{\beta > 0} \mathcal{O}^\tau_1 (\alpha, \beta) = \sup_{0 < \beta \leq K_\beta} \mathcal{O}^\tau_1 (\alpha, \beta),
$$

leading to

$$
\sup_{0 < \beta \leq K_\beta} \mathcal{O}^\tau_d (\alpha, \beta) \overset{P}{\to} \sup_{\beta > 0} \mathcal{O}^\tau_1 (\alpha, \beta). \tag{133}
$$

Notice that the convergence (133) can also be proven using statement (iii) of Lemma C.3, since $\lim_{\beta \to \infty} \mathcal{O}^\tau_1 (\alpha, \beta) = -\infty$.

We now define $\mathcal{O}^{\beta, \tau_1} (\alpha) := \sup_{\beta > 0} \mathcal{O}^\tau_1 (\alpha, \beta)$ and $\mathcal{O}^{\beta, \tau_d} (\alpha) := \sup_{0 < \beta \leq K_\beta} \mathcal{O}^\tau_d (\alpha, \beta)$. Each of these functions is convex in $\alpha$, since they were obtained by first minimizing over $\tau_1$ a jointly convex function in $(\alpha, \tau_1)$, and then maximizing over $\beta$ a convex function in $\alpha$.

For the final step of the proof, we will distinguish between the two cases $\alpha_* = 0$ and $\alpha_* > 0$. This is due to the Convexity Lemma C.2, which stands at the core of the convergence analysis, and which requires the domain of the functions of interest to be open. We first consider the case $\alpha_* = 0$. Since (132) is valid for any $\alpha \geq 0$, it is in particular valid for $\alpha = 0$, and thus we have that

$$
\mathcal{O}^{\beta, \tau_1} (0) \overset{P}{\to} \mathcal{O}^{\beta, \tau_1} (0). \tag{134}
$$

Moreover, using statement (i) of Lemma C.3, for any $K > 0$, we have that

$$
\min_{K \leq \alpha \leq K_\alpha} \mathcal{O}^{\beta, \tau_1} (\alpha) \overset{P}{\to} \min_{K \leq \alpha \leq K_\alpha} \mathcal{O}^{\beta, \tau_1} (\alpha). \tag{135}
$$

The result for $\alpha_* = 0$ now follows from (134) and (135) and the fact that

$$
\mathcal{O}^{\beta, \tau_1} (0) < \min_{K \leq \alpha \leq K_\alpha} \mathcal{O}^{\beta, \tau_1} (\alpha),
$$

for any $K > 0$, due to the uniqueness of $\alpha_*$. We now focus on the case $\alpha_* > 0$. We need to prove (130) and (131), which are equivalent to the following two convergences in probability

$$
\inf_{0 < \alpha \leq K_\alpha} \mathcal{O}^{\beta, \tau_d} (\alpha) \overset{P}{\to} \inf_{0 < \alpha \leq K_\alpha} \mathcal{O}^{\beta, \tau_d} (\alpha), \tag{136}
$$

and

$$
\inf_{\alpha \in S^\eta_\alpha} \mathcal{O}^{\beta, \tau_d} (\alpha) \overset{P}{\to} \inf_{\alpha \in S^\eta} \mathcal{O}^{\beta, \tau_d} (\alpha), \tag{137}
$$

respectively. Since $\{\mathcal{O}^{\beta, \tau_d} (\cdot)\}_{d \in \mathbb{N}}$ is a sequence of random, real-valued convex functions, converging in probability (pointwise, for every $\alpha > 0$) to the function $\mathcal{O}^{\beta, \tau_d} (\cdot)$, (136) follows immediately from statement (ii) of Lemma C.3. Moreover, since $S^\eta_\alpha = (0, \alpha_* - \eta] \cup [\alpha_* + \eta, K_\alpha]$, (137) follows from statement (i) of Lemma C.3, for the interval $[\alpha_* + \eta, K_\alpha]$, and statement (ii) of the same lemma, for the interval $(0, \alpha_* - \eta]$. This concludes the proof of the two convergences in probability (130) and (131).

The last step of the proof is to understand the implications of relaxing the constraint $w \in W$ to $w \in S_w := \{w \in \mathbb{R}^d : \|w\| \leq K_\alpha := R_\theta + \sigma_\theta_0\}$. Recall that this was a necessary step for scalarizing the problem over the magnitude of $w$. Since $0 \in W$ and $W \subset S_w$, w.p.a. 1, we have that

$$
\lim_{d \to \infty} \frac{\|\hat{\theta}_d - \theta_0\|^2}{d} \leq \alpha_*,
$$
where $\alpha_*$ is the optimal solution of the relaxed problem (i.e., with constraint $w \in \mathcal{S}_w$). However, since 
\[ \{w \in \mathbb{R}^d : \|w\| \leq R_0 - \sigma_{\theta_0}\} \subset \mathcal{W} \cap \mathcal{S}_w, \] we have that if $\alpha_* \leq R_0 - \sigma_{\theta_0}$, then 
\[ \lim_{d \to \infty} \frac{\|\hat{\theta}_{\text{DRE}} - \theta_0\|^2}{d} = \alpha_. \]
This concludes the proof.

**Proof of Corollary 4.15.** The proof can be easily obtained following the same lines as Theorem 4.13, and therefore we only provide a sketch, highlighting only the main differences from the proof of Theorem 4.13.

After expressing the dual formulation (16) in vector form, introducing the the change of variable $w = (\theta - \theta_0)/\sqrt{d}$, and motivating the restriction of the variable $u$ to the compact set $\mathcal{S}_u$, we arrive at the minimax formulation (113),
\[ \min_{w \in \mathcal{S}_w} \max_{u \in \mathcal{S}_u} \frac{1}{\sqrt{n}} u^T (\sqrt{d} A) w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d} w\|^2 \|u\|^2 - \frac{1}{n} \sum_{i=1}^n L^*(u_i \sqrt{n}) \]

Now, since $L(u) = u^2$, we have that $L^*(u) = u^2/4$, and therefore $M = 2$. Consequently, this formulation can be rewritten as 
\[ \min_{w \in \mathcal{S}_w} \max_{u \in \mathcal{S}_u} \frac{1}{\sqrt{n}} u^T (\sqrt{d} A) w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d} w\|^2 \|u\|^2 - \frac{1}{2M} \|u\|^2. \]

Since this problem is convex-concave, and both $w$ and $u$ live in convex, compact sets, this is in the form of a (PO) problem, and we can now apply the CGMT, and arrive at the modified (AO) problem,
\[ \max_{0 \leq \beta \leq K_\beta} \min_{w \in \mathcal{S}_w} \max_{||u|| = \beta} \frac{1}{\sqrt{n}} \|w\| g^T u - \frac{1}{\sqrt{n}} \|u\| h^T w + \frac{1}{\sqrt{n}} u^T z + \frac{1}{4\lambda} \|\theta_0 + \sqrt{d} w\|^2 \|u\|^2 - \frac{1}{2M} \|u\|^2. \]

Now, as in the proof of Theorem 4.13, we can first scalarize over $u$ and subsequently over $w$,
\[ \max_{0 \leq \alpha \leq K_{\alpha}} \min_{0 \leq \beta \leq K_{\beta}} \frac{\beta}{\sqrt{n}} \|\alpha g + z\| - \frac{\beta^2}{2M} - \alpha \beta \left\| \frac{\beta}{2\lambda_0} \theta_0 \right\| \frac{h}{\sqrt{n}} + \frac{\beta^2}{4\lambda_0} \left( \frac{\|\theta_0\|^2}{d} + \alpha^2 \right), \]
and subsequently use Sion’s minimax principle to exchange the minimizing and the maximizing, and obtain
\[ \min_{0 \leq \alpha \leq K_{\alpha}} \max_{0 \leq \beta \leq K_{\beta}} \frac{\beta}{\sqrt{n}} \|\alpha g + z\| - \frac{\beta^2}{2M} - \alpha \beta \left\| \frac{\beta}{2\lambda_0} \theta_0 \right\| \frac{h}{\sqrt{n}} + \frac{\beta^2}{4\lambda_0} \left( \frac{\|\theta_0\|^2}{d} + \alpha^2 \right), \]
The optimal value of this minimax problem can be shown to converge in probability, as $d \to \infty$, to
\[ \beta \sqrt{\alpha^2 + \sigma_z^2} - \frac{\beta^2}{2M} - \alpha \beta \sqrt{\frac{\beta^2 \sigma_{\theta_0}^2}{4\lambda_0^2} + \rho^2 + \frac{\beta^2}{4\lambda_0} \left( \sigma_{\theta_0}^2 + \alpha^2 \right)}. \]
Then, following the same lines as in the proof of Theorem 4.13, the result can be concluded.

**C  Technical preliminary lemmas**

**Lemma C.1.** Let $h \in \mathbb{R}^d$ be a random vector with entries i.i.d. standard normal, and $\theta_0 \in \mathbb{R}^d$ be a random vector, independent from $h$, which satisfies $\|\theta_0\|^2/d \overset{P}{\to} \sigma_{\theta_0}^2$, for $\sigma_{\theta_0} > 0$. Then,
\[ \frac{\theta_0^T h}{d} \overset{P}{\to} 0. \]
Proof. First notice that $\mathbb{E}[\theta_0^T h] = 0$ due to the zero mean of $h$, and the fact that $\theta_0$ and $h$ are independent. Now let $\mathbb{V}(\theta_0^T h)$ denote the variance of $\theta_0^T h$. Using $\mathbb{E}[h_i^2] = 1$, $\mathbb{E}[h_i h_j] = 0$, for $i \neq j$, it can be easily checked that the variance of $\theta_0^T h$ satisfies $\mathbb{V}(\theta_0^T h) = \mathbb{E}[\|\theta_0\|^2]$. Consequently, using the assumption $\|\theta_0\|^2/d \overset{P}{\to} \sigma_0^2$, we have that $\mathbb{E}[\|\theta_0\|^2]/d \to \sigma_0^2$, and therefore $\mathbb{V}(\theta_0^T h)/(d^2) \to 0$. The result now follows easily from [39, Theorem 2.2.6].

Lemma C.2 (Convexity lemma). Let $\{O_d : \mathcal{X} \to \mathbb{R}\}_{d \in \mathbb{N}}$ be a sequence of random functions, defined on an open and convex subset $\mathcal{X}$ of $\mathbb{R}^d$, which are convex w.p.a. 1 as $d \to \infty$. Moreover, let $O : \mathcal{X} \to \mathbb{R}$ be a deterministic function for which $O_d(x) \overset{P}{\to} O(x)$, for all $x \in \mathcal{X}$. Then, $O$ is convex and the convergence is uniform over each compact subset $\mathcal{K}$ of $\mathcal{X}$, i.e.,

$$\sup_{x \in \mathcal{K}} |O_d(x) - O(x)| \overset{P}{\to} 0.$$ 

Proof. This is a slight generalization of the standard Convexity Lemma (see [93, Section 6]), which assumes that the functions $\{O_d\}_{d \in \mathbb{N}}$ are convex (instead of convex w.p.a. 1). A similar proof, which we include for completeness, can be used to conclude the result in this slightly more general case, as shown in what follows.

We start by showing that the function $O$ is convex. If $O$ was not convex, then there would exist $x_1, x_2 \in \mathcal{X}$ and $\lambda \in (0, 1)$ such that $O(\lambda x_1 + (1 - \lambda)x_2) \neq \lambda O(x_1) + (1 - \lambda)O(x_2)$, for some $\epsilon > 0$. Now, since $O_d$ converges in probability pointwise to $O$, we have that

$$|O_d(\lambda x_1 + (1 - \lambda)x_2) - \lambda O_d(x_1) - (1 - \lambda)O_d(x_2)| \leq \epsilon$$

holds w.p.a. 1. Therefore,

$$O_d(\lambda x_1 + (1 - \lambda)x_2) - \lambda O_d(x_1) - (1 - \lambda)O_d(x_2) \leq \frac{\epsilon}{2}$$

holds w.p.a. 1. However, this contradicts the fact that $\{O_d\}_{d \in \mathbb{N}}$ are convex w.p.a. 1. Consequently, $O$ is a convex function.

We now proceed to proving the uniform convergence on compact subsets. For this, it is enough to consider the case where $\mathcal{K}$ is a cube with edges parallel to the coordinate directions $e_1, \ldots, e_d$, since every compact subset can be covered by finitely many such cubes. Now, fix $\epsilon > 0$. Since $O$ is convex, we know that it is continuous on the interior of its domain. Moreover, from the Heine-Cantor theorem we know that every continuous function on a compact set is uniformly continuous. Therefore, we can partition $\mathcal{K}$ in cubes of side $\delta$ such that the $O$ varies by less than $\epsilon$ on each cube of side $2\delta$. Moreover, we expand $\mathcal{K}$ by adding another layer of these $\delta$-cubes around each side, and we call this new cube $\mathcal{K}_\delta$.

Notice that we may pick $\delta$ small enough such that $\mathcal{K}_\delta \subset \mathcal{X}$.

Now, let $V$ be the set of all vertices of all $\delta$-cubes in $\mathcal{K}_\delta$. Since $V$ is a finite set, the convergence in probability is uniform over $V$, i.e.,

$$\max_{x \in V} |O_d(x) - O(x)| \overset{P}{\to} 0. \quad (138)$$

Each $x \in \mathcal{K}$ will belong to a $\delta$-cube with vertices $\{x_i\}_{i \in I}$ in $\mathcal{K}_\delta$, where $I$ is a finite set of indices. Therefore, $x$ can be written as a convex combination of these vertices, i.e., $x = \sum_{i \in I} \lambda_ix_i$. Since $O_d$ is convex w.p.a. 1, we have that $O_d(x) \leq \sum_{i \in I} \lambda_i O_d(x_i)$ holds w.p.a. 1. Therefore,

$$O_d(x) \leq \max_{i \in I} O_d(x_i) \leq \max_{i \in I} |O_d(x_i) - O(x_i) + O(x_i) - O(x) + O(x)|$$

$$\leq \max_{i \in I} |O_d(x_i) - O(x_i)| + \max_{i \in I} |O(x_i) - O(x)| + O(x)$$

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holds w.p.a. 1. Now, from (138) and the fact that $\mathcal{O}$ varies by less than $\epsilon$ on the $\delta$-cube with vertices \( \{x_i\}_{i \in \mathcal{I}} \), we have that the following upper bound holds w.p.a. 1

\[
\sup_{x \in \mathcal{K}} \mathcal{O}_d(x) - \mathcal{O}(x) \leq 2\epsilon.
\]

We will now prove the lower bound. Each $x \in \mathcal{K}$ lies into a $\delta$-cube with a vertex $x_0$, and can be written as $x = x_0 + \sum_{i=1}^{d} \lambda_i e_i$, with $0 \leq \lambda_i < \delta$ for all $i = 1, \ldots, d$. We now define $x_i := x_0 - \lambda_i e_i$, and notice that $x_i \in \mathcal{V}$. Then, $\theta_0$ can be written as the convex combination $x_0 = \beta x + \sum_{i=1}^{d} \beta_i x_i$, with $\beta = \delta / (\delta + \sum_{j=1}^{d} \lambda_j)$ and $\beta_i = \lambda_i / (\delta + \sum_{j=1}^{d} \lambda_j)$. Notice that $\beta \geq 1/(d+1)$.

Since $\mathcal{O}_d$ is convex w.p.a. 1, we have that $\mathcal{O}_d(x_0) \leq \beta \mathcal{O}_d(x) + \sum_{i=1}^{d} \beta_i \mathcal{O}_d(x_i)$ holds w.p.a. 1. Therefore,

\[
\beta \mathcal{O}_d(x) \geq \mathcal{O}_d(x_0) - \sum_{i=1}^{d} \beta_i \mathcal{O}_d(x_i)
\]

\[
= (\mathcal{O}_d(x_0) - \mathcal{O}(x_0)) + \left(\mathcal{O}(x_0) - \sum_{i=1}^{d} \beta_i \mathcal{O}(x_i)\right) - \sum_{i=1}^{d} \beta_i (\mathcal{O}_d(x_i) - \mathcal{O}(x_i))
\]

\[
\geq (\mathcal{O}_d(x_0) - \mathcal{O}(x_0)) + \left(\mathcal{O}(x) - \epsilon - \sum_{i=1}^{d} \beta_i (\mathcal{O}(x_i) + \epsilon)\right) - \sum_{i=1}^{d} \beta_i (\mathcal{O}_d(x_i) - \mathcal{O}(x_i))
\]

holds w.p.a. 1. Now, from (138) and the fact that $\mathcal{O}$ is convex, we have w.p.a. 1 that

\[
\beta \mathcal{O}_d(x) \geq -\epsilon + \beta \mathcal{O}(x) - \epsilon - 2 \sum_{i=1}^{d} \beta_i \epsilon,
\]

and therefore

\[
\inf_{x \in \mathcal{K}} \mathcal{O}_d(x) - \mathcal{O}(x) \geq -\frac{4\epsilon}{\beta} \geq -4(d+1)\epsilon
\]

holds w.p.a. 1. Since $d$ is finite, this concludes the proof. \( \square \)

**Lemma C.3 (Convergence of convex w.p.a. 1 optimization problems).** Let $\{\mathcal{O}_d : (0, \infty) \to \mathbb{R}\}_{d \in \mathbb{N}}$ be a sequence of random functions, which are convex w.p.a. 1 as $d \to \infty$. Moreover, let $\mathcal{O} : (0, \infty) \to \mathbb{R}$ be a deterministic function for which $\mathcal{O}_d(x) \overset{P}{\to} \mathcal{O}(x)$, for all $x \in (0, \infty)$. Then,

(i) for any $K_1, K_2 > 0$,

\[
\min_{K_1 \leq x \leq K_2} \mathcal{O}_d(x) \overset{P}{\to} \min_{K_1 \leq x \leq K_2} \mathcal{O}(x).
\]

(ii) for any $K > 0$,

\[
\inf_{0 < x \leq K} \mathcal{O}_d(x) \overset{P}{\to} \inf_{0 < x \leq K} \mathcal{O}(x).
\]

(iii) Moreover, if $\lim_{x \to \infty} \mathcal{O}(x) = +\infty$,

\[
\inf_{x > 0} \mathcal{O}_d(x) \overset{P}{\to} \inf_{x > 0} \mathcal{O}(x).
\]

**Proof.** This is a generalization of [120, Lemma 10], which assumes that the functions $\{\mathcal{O}_d\}_{d \in \mathbb{N}}$ are convex (instead of convex w.p.a. 1). This result builds upon the uniform convergence in probability presented in the Convexity Lemma C.2.
(i) The proof of Assertion (i) is a simple application of the Convexity Lemma, as explained in what follows. Let \( x_* \) and \( \{x_*^{(d)}\}_{d \in \mathbb{N}} \) be the minimizers of \( \mathcal{O} \) and \( \{\mathcal{O}_d\}_{d \in \mathbb{N}} \) on \([K_1, K_2]\), respectively. Since \( \mathcal{O}_d \xrightarrow{P} \mathcal{O} \) pointwise on \((0, \infty)\), from the Convexity Lemma we know that \( \mathcal{O}_d \xrightarrow{P} \mathcal{O} \) uniformly on the compact set \([K_1, K_2]\), i.e., sup_{x \in [K_1, K_2]} |\mathcal{O}_d(x) - \mathcal{O}(x)| \leq \epsilon \) w.p.a. 1. In particular, this holds true for the points \( x_*, \{x_*^{(d)}\}_{d \in \mathbb{N}} \). Therefore,

\[-\epsilon \leq \mathcal{O}_d(x_*^{(d)}) - \mathcal{O}(x_*^{(d)}) \leq \min_{K_1 \leq x \leq K_2} \mathcal{O}_d(x) - \min_{K_1 \leq x \leq K_2} \mathcal{O}(x) \leq \mathcal{O}_d(x_*) - \mathcal{O}(x_*) \leq \epsilon,\]

w.p.a. 1, proving Assertion (i).

(ii) First notice that if \( \mathcal{O} \) is convex by Lemma C.2. Now, since \( \mathcal{O} \) is real-valued (on \((0, \infty)\)) and convex, we have that \( \inf_{0 < x \leq K} \mathcal{O}(x) > -\infty \). Moreover, since \( \{\mathcal{O}_d\}_{d \in \mathbb{N}} \) are real-valued (on \((0, \infty)\)) and convex w.p.a. 1, we have that \( \inf_{0 < x \leq K} \mathcal{O}_d(x) > -\infty \) w.p.a. 1. We first prove this statement for \( \mathcal{O} \). Suppose that \( \inf_{0 < x \leq K} \mathcal{O}(x) = -\infty \). Then, there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} > 0 \) such that \( \mathcal{O}(x_n) \to -\infty \) as \( x_n \to 0 \). Consequently, for any \( x \in (0, K) \) there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \in (0, 1) \), with \( \lambda_n \geq (K - x)/K > 0 \), satisfying \( x = \lambda_n x_n + (1 - \lambda_n)K \). From the convexity of \( \mathcal{O} \), we have that

\[
\mathcal{O}(x) = \lambda_n \mathcal{O}(x_n) + (1 - \lambda_n) \mathcal{O}(K),
\]

which, for \( x_n \to 0 \), implies that \( \mathcal{O}(x) = -\infty \) for all \( x \in (0, K) \), leading to a contradiction (since \( \mathcal{O} \) is assumed real-valued). This reasoning readily extends to the sequence \( \{\mathcal{O}_d\}_{d \in \mathbb{N}} \), for which we have that \( \inf_{0 < x \leq K} \mathcal{O}_d(x) > -\infty \) w.p.a. 1.

In order to prove the desired result we have to show that for all \( \epsilon > 0 \),

\[
|\inf_{0 < x \leq K} \mathcal{O}_d(x) - \inf_{0 < x \leq K} \mathcal{O}(x)| \leq \epsilon
\]

w.p.a. 1 as \( d \to \infty \). We start by proving the upper bound \( \inf_{0 < x \leq K} \mathcal{O}_d(x) - \inf_{0 < x \leq K} \mathcal{O}(x) \leq \epsilon \). Let \( \bar{x} > 0 \) be such that \( \mathcal{O}(\bar{x}) - \inf_{0 < x \leq K} \mathcal{O}(x) \leq \epsilon/2 \). From the pointwise convergence of \( \mathcal{O}_d \) to \( \mathcal{O} \) we have that \( |\mathcal{O}_d(\bar{x}) - \mathcal{O}(\bar{x})| \leq \epsilon/2 \) w.p.a. 1. Consequently,

\[
\inf_{0 < x \leq K} \mathcal{O}_d(x) - \inf_{0 < x \leq K} \mathcal{O}(x) \leq \mathcal{O}_d(\bar{x}) - \mathcal{O}(\bar{x}) + \frac{\epsilon}{2} \leq \epsilon,
\]

w.p.a. 1, showing that the upper bound holds.

Now, we focus on proving the lower bound \( \inf_{0 < x \leq K} \mathcal{O}_d(x) - \inf_{0 < x \leq K} \mathcal{O}(x) > -\epsilon \), and we will do this by considering two cases. First, we consider the case where all the infimums \( \inf_{0 < x \leq K} \mathcal{O}_d(x) \) are attained, and the minimizers are lower bounded by some \( x_0 > 0 \), and show that the result immediately follow from the Convexity Lemma C.2. Let \( \{x_*^{(d)}\}_{d \in \mathbb{N}} > 0 \) be the minimizers of \( \{\mathcal{O}_d\}_{d \in \mathbb{N}} \) on \((0, K]\), and suppose that they belong to \([x_0, K]\). Now, since \( \mathcal{O}_d \xrightarrow{P} \mathcal{O} \) pointwise on \((0, \infty)\), from the Convexity Lemma we know that \( \mathcal{O}_d \xrightarrow{P} \mathcal{O} \) uniformly on the compact set \([x_0, K]\), i.e., sup_{x \in [x_0, K]} |\mathcal{O}_d(x) - \mathcal{O}(x)| \leq \epsilon \) w.p.a. 1. In particular, this holds true for the points \( \{x_*^{(d)}\}_{d \in \mathbb{N}} \). Therefore,

\[
\inf_{0 < x \leq K} \mathcal{O}_d(x) - \inf_{0 < x \leq K} \mathcal{O}(x) \geq \mathcal{O}_d(x_*^{(d)}) - \mathcal{O}(x_*^{(d)}) \geq -\epsilon
\]

w.p.a. 1, which concludes the proof in this case.
Secondly, we consider the case where the infimums are attained in the limit of \( x \to 0 \). Let \( \bar{x}_1 > 0 \) be such that \( \mathcal{O}_d(\bar{x}_1) < \epsilon/3 \). Moreover, let \( \bar{x}_2 > \bar{x}_1 > \bar{x}_1^{(d)} \) be such that \( \mathcal{O}(\bar{x}_1) - \inf_{0 < x \leq K} \mathcal{O}(x) < \epsilon/3 \) and \( \mathcal{O}(\bar{x}_2) - \inf_{0 < x \leq K} \mathcal{O}(x) < \epsilon/3 \). Then, w.p.a., we have

\[
\inf_{0 < x \leq K} \mathcal{O}_d(x) > \mathcal{O}_d(\bar{x}_1) - \epsilon/3 \geq \frac{1 - \lambda_d}{\lambda_d} \mathcal{O}_d(\bar{x}_2) - \epsilon/3 \\
\geq \frac{1}{\lambda_d} \left( \mathcal{O}(\bar{x}_1) - \epsilon/3 \right) - \frac{1 - \lambda_d}{\lambda_d} \left( \mathcal{O}(\bar{x}_2) + \epsilon/3 \right) - \epsilon/3 \\
> \frac{1}{\lambda_d} \inf_{0 < x \leq K} \mathcal{O}(x) - 2\epsilon/3 - \frac{1 - \lambda_d}{\lambda_d} \left( \inf_{0 < x \leq K} \mathcal{O}(x) + 2\epsilon/3 \right) - \epsilon/3 \\
= \inf_{0 < x \leq K} \mathcal{O}(x) - \epsilon,
\]

proving the desired lower bound. In particular, the second inequality holds w.p.a., and follows from the pointwise convergence w.p.a. 1 of \( \mathcal{O}_d \), with \( \lambda_d > (\bar{x}_2 - \bar{x}_1)/\bar{x}_2 \), while the third inequality follows from the pointwise convergence of \( \mathcal{O}_d \) to \( \mathcal{O} \), and it holds w.p.a. 1. This concludes the proof.

(iii) Since \( \mathcal{O} \) is real-valued, there exists \( C > 0 \) such that \( \inf_{x > 0} \mathcal{O}(x) \leq C \). Moreover, since \( \lim_{x \to \infty} \mathcal{O}(x) = +\infty \), there exists \( K_1 > 0 \) such that \( \mathcal{O}(x) > C \) for all \( x > K_1 \). Therefore,

\[
\inf_{x > 0} \mathcal{O}(x) = \inf_{0 < x \leq K_1} \mathcal{O}(x).
\]

Now, fix \( \epsilon > 0 \), and let \( K_2 > K_1 \) be such that \( \mathcal{O}(K_2) > \mathcal{O}(K_1) + 3\epsilon \). Notice that this is possible since \( \lim_{x \to \infty} \mathcal{O}(x) = +\infty \). From the pointwise convergence in probability, we have that both \( |\mathcal{O}_d(K_1) - \mathcal{O}(K_1)| \leq \epsilon \) and \( |\mathcal{O}_d(K_2) - \mathcal{O}(K_2)| \leq \epsilon \) hold w.p.a. 1. Therefore, \( \mathcal{O}_d(K_1) < \mathcal{O}_d(K_2) \) holds w.p.a. 1. Now, using this and the fact that \( \mathcal{O}_d \) is convex w.p.a. 1, we can conclude that \( \mathcal{O}_d(x) > \mathcal{O}_d(K_2) \) w.p.a. 1 for all \( x > K_2 \). Consequently,

\[
\inf_{x > 0} \mathcal{O}_d(x) = \inf_{0 < x \leq K_2} \mathcal{O}_d(x).
\]

holds w.p.a. 1. Finally, letting \( K = \max\{K_1, K_2\} \), we have that

\[
\inf_{x > 0} \mathcal{O}_d(x) = \inf_{0 < x \leq K} \mathcal{O}_d(x) \overset{P}{\to} \inf_{0 < x \leq K} \mathcal{O}(x) = \inf_{x > 0} \mathcal{O}(x),
\]

where the convergence in probability follows from Assertion (ii). This concludes the proof. \( \square \)