Integral dimension of a noetherian ring

Caijun Zhou

Department of Mathematics, Shanghai Normal University,
Shanghai, 200234, China

1 Introduction

Throughout this paper we always use $R$ to denote a commutative noetherian ring with an identity. The main purpose of the paper is to introduce a new notion, integral dimension, for noetherian rings.

Let $J$ be an ideal of $R$. An element $x \in J$ is said to be integral over $J$, if there is an equation of the form

$$x^n + a_1x^{n-1} + \cdots + a_n = 0$$

where $a_i \in J^i$ for $1 \leq i \leq n$. It is not difficult to show that the set of all elements which are integral over $J$ is an ideal of $R$. We will use $\overline{J}$ to denote this ideal, and call it the integral closure of the ideal $J$. An ideal $I$ with $I \subseteq \overline{J}$ will be said to be integral over $J$.

One of the main interests on $\overline{J}$ is when $\overline{J^n}$ is contained in a power of $J$.

Let us recall several remarkable results established in the past years. Rees [Re] showed that if $R$ is an analytically unramified local ring and $J$ an ideal of $R$, then there is an integer $k$, depending on $J$, such that $\overline{J^n} \subseteq J^{n-k}$ for $n \geq k$.

Let $R$ be the ring of convergent power series in $n$ variables $Z_1, \ldots, Z_n$ over the field of complex numbers $\mathbb{C}$. Let $f \in R$ be a non unit element and $J = (\partial f/\partial Z_1, \ldots, \partial f/\partial Z_n)$ be the Jacobian ideal of $f$. It can be proved [cf. HS] that $f$ is integral over $J$. Briançon and Skoda [BS] showed by analytic method that $f^n \in J$, and this is known as the Briançon-Skoda theorem.

Lipman and Sathaye [LS] extended the theorem of Briançon-Skoda to arbitrary regular rings by purely algebraic method. They proved if $R$ is a regular ring of dimension $d > 0$, then for all ideals $J$ of $R$, $\overline{J^n} \subseteq J^{n-d+1}$ for $n \geq d - 1$.

Later on, Huneke [Hu, Theorem 4.13] proved a far reaching result for a broad class of noetherian rings. For example, he obtained that, for any reduced excellent local ring $R$, there exists an integer $k \geq 0$, such that for all ideals $J$ of $R$, $\overline{J^n} \subseteq J^{n-k}$ for $n \geq k$. The result of Huneke is known as the uniform Briançon-Skoda-Huneke theorem. Recently, Andersson and Wulcan [AW] presented global effective versions of the Briançon-Skoda-Huneke theorem by analytic method.

In this paper, instead of considering the integral closure of the power ideal $J^n$, we will study the power of the integral closure of an ideal $\overline{J^n}$. We prove in
the next section for any ideal $J$ of $R$,

$$J^n \subseteq \overline{J} \subseteq \overline{J}^n.$$ 

Hence the comparison of $\overline{J}^n$ with the power of $J$ may be easier than that of $\overline{J}$. Motivated by this, we introduce the following notion of integral dimension for noetherian rings.

**Definition 1.1** Let $R$ be a noetherian ring. If there exists an integer $k \geq 0$ such that for all ideals $J$ of $R$ and for $n \geq k$

$$\overline{J}^n \subseteq J^{n-k},$$

then we define the integral dimension of $R$, denoted by $i(R)$, to be the least integer $k$ such that (1.1) holds for all $J$ and all $n \geq k$. If such $k$ does not exist, then we set $i(R) = \infty$.

It is obvious that $i(R) = k < \infty$ if and only if $k$ is the least integer such that for every pair of ideals $J \subseteq I$, where $I$ is integral over $J$,

$$I^n \subseteq J^{n-k}$$

for $n \geq k$. Moreover, we have $Q^{k+1} = 0$ for the nilradical $Q$ of $R$ because it is integral over the zero ideal. Clearly, if $R$ is an artinian ring, then $i(R) \leq l(R)$, where $l(R)$ denotes the length of $R$.

It follows easily from the Lipman-Sathaye theorem, $i(R) \leq d - 1$ for every regular ring of dimension $d > 0$. The above mentioned result of Huneke implies that if $R$ is a reduced excellent local ring, then $i(R) = \infty$.

In this paper, we will prove some basic facts about the behavior of $i(R)$ under several operations on $R$. The results show that $i(R)$ behaves well under operations such as localization and completion.

We will give a lower bound for $i(R)$ in terms of the dimension $d$ of $R$. Explicitly, we will show $i(R) \geq d - 1$. In particular, it follows from Lipman-Sathaye theorem, if $R$ is a regular ring of dimension $d > 0$, then $i(R) = d - 1$. We can prove a noetherian ring $R$ with $i(R) = 0$ if and only if $R$ is a regular ring of dimension $d \leq 1$.

The main goal of the paper is to show that $i(R) < \infty$ for a large class of noetherian rings. Recall that a $S$-algebra $R$ is said to be essentially of finite type over a ring $S$, if $R$ is a localization of a finitely generated $S$-algebra. Our main result of the paper states as follows:

**Theorem 3.6** Let $R$ be a noetherian ring of finite dimension. If $R$ satisfies one of the following conditions, then $i(R) < \infty$.

(i) $R$ is essentially of finite type over a local ring $S$.
(ii) $R$ is a ring of characteristic $p$, and $R$ is module finite over $R^p$.
(iii) $R$ is essentially of finite type over the ring of integer numbers $\mathbb{Z}$.

In particular, one can conclude from Theorem 3.6, $i(R) < \infty$ for any noetherian local ring $R$. It implies that the notion of the integral dimension becomes
a well-defined notion and has concrete meaning in the class of local rings. We do not know whether \( i(R) \) is finite for every noetherian ring \( R \). It seems very difficult to give an answer to the question in general. However, we can prove (see, Corollary 2.11) that \( \dim(R) < \infty \) if \( i(R) < \infty \). We make the following conjecture.

**Conjecture 1.2** Let \( R \) be a noetherian ring of finite dimension. Then \( i(R) < \infty \).

Huneke [Hu] observed that the uniform Briançon-Skoda property plays an important role in the study of the uniform Artin-Rees property of a noetherian ring. As pointed out in Theorem 3.2, the finiteness property of integral dimension is also useful in proving a ring with such uniform Artin-Rees property. The proof of our main result Theorem 3.6 depends heavily on the uniform Artin-Rees theorem of Huneke [Hu]. The main technique of Huneke came from the paper of Lipman and Sathaye [LS] in characteristic 0 and the theory of tight closure of [HH1] [HH2] in characteristic \( p \). It can not be applied directly to proving a result more general than Theorem 3.6.

Even for the local case, it would be very interesting to reveal more about the relationship between \( i(R) \) and \( i(R/aR) \) for an element \( a \) in \( R \) with \( \dim(R/aR) = \dim(R) - 1 \). We guess that they may be related to each other by something like the multiplicities, \( e(R) \) and \( e(R/(a)) \), of the maximal ideals of \( R \) and \( R/(a) \). The answer is not known even for \( i(R) \) and \( i(R[X]) \), where \( R[X] \) is the polynomial ring of \( R \) in the variable \( X \). Hence there are a lot of questions about integral dimension remained for further studying.

## 2 Basic properties

In this section, we will recall some basic facts about the integral closure of an ideal, and then presents some properties of the integral dimension \( i(R) \). Some of the results are known, one can find in [cf. HS], we will prove them for convenience of readers.

Let \( Q \) be the nilradical of \( R \) and \( R_{\text{red}} \) be the ring of \( R \) modulo \( Q \). Since there is an integer \( n \) such that \( Q^n = 0 \), it is easy to see that \( Q \) is contained in \( \mathcal{J} \) for every ideal \( J \) of \( R \), and an element \( x \) is integral over \( J \) if and only if \( \bar{x} \), the image of \( x \) in \( R_{\text{red}} \), is integral over \( JR_{\text{red}} \). Thus we have:

**Proposition 2.1** Let \( J \) be an ideal of \( R \). Then \( \overline{JR_{\text{red}}} = \mathcal{J}R_{\text{red}} \).

Integral dependence behaves well under localization:

**Proposition 2.2** Let \( J \) be an ideal of \( R \). Then for any multiplicatively closed subset \( T \) of \( R \), \( T^{-1}\mathcal{J} = \mathcal{T}^{-1}\mathcal{J} \).
Proof. Clearly, \( T^{-1} \mathcal{J} \subseteq \overline{T^{-1} \mathcal{J}} \) by the definition of integral dependence. Let \( a \in T^{-1} \mathcal{J} \). By the definition of integral dependence again, there is an equation of the form
\[
a^n + a_1 a^{n-1} + \cdots + a_n = 0
\]
in the ring \( T^{-1} \mathcal{R} \), where \( a_i \in T^{-1} \mathcal{J}_i \) for \( 1 \leq i \leq n \). Choose an element \( b \in T \) such that \( ba \in \mathcal{R} \) and \( ba_i \in \mathcal{J}_i \) for \( 1 \leq i \leq n \). Hence we have
\[
(ba)^n + a_1 (ba)^{n-1} + \cdots + a_n = 0
\]
is an equation with coefficients in \( \mathcal{R} \), and there is an element \( b' \in T \) such that
\[
b'(ba)^n + a_1 b' (ba)^{n-1} + \cdots + a_n = 0
\]
in \( \mathcal{R} \). It shows that
\[
(b'a)^n + a_1 b' (b'a)^{n-1} + \cdots + (b'a)_n = 0
\]
is the desired equation which implies \( b'a \) is integral over \( \mathcal{J} \). So \( a \in T^{-1} \mathcal{J} \), and \( \overline{T^{-1} \mathcal{J}} \subseteq \overline{T^{-1} \mathcal{J}} \). This proves the proposition.

Corollary 2.3 Let \( T \) be any multiplicatively closed subset of \( \mathcal{R} \). Then \( \iota(T^{-1} \mathcal{R}) \leq \iota(\mathcal{R}) \).

Proof. If \( \iota(\mathcal{R}) = \infty \), there is nothing to prove. Now, assume that \( \iota(\mathcal{R}) = k \) is finite. For any ideal \( \mathcal{K} \) of \( T^{-1} \mathcal{R} \), there exists an ideal \( \mathcal{J} \) of \( \mathcal{R} \) such that \( \mathcal{K} = T^{-1} \mathcal{J} \). By Proposition 2.2, \( \mathcal{K} = T^{-1} \mathcal{J} \). Note that \( \mathcal{J}^n \subseteq \mathcal{J}^{n-k} \) for \( n \geq k \). Thus, for \( n \geq k \), we have
\[
\overline{\mathcal{K}}^n = (T^{-1} \mathcal{J})^n = T^{-1} \mathcal{J}^n \subseteq T^{-1} \mathcal{J}^{n-k} = (T^{-1} \mathcal{J})^{n-k} = \mathcal{K}^{n-k}
\]
and it follows that \( \iota(T^{-1} \mathcal{R}) \leq k \).

It is well-known the dimension of \( \mathcal{R} \) can be computed as follows:
\[
\dim(\mathcal{R}) = \sup \{ \dim(\mathcal{R}_m) \mid m \text{ a maximal ideal of } \mathcal{R} \},
\]
where \( \mathcal{R}_m \) denotes the localization of \( \mathcal{R} \) at the maximal ideal \( m \). Similarly, we have the following computation for integral dimension.

Corollary 2.4 For a noetherian ring \( \mathcal{R} \), \( \iota(\mathcal{R}) = \sup \{ \iota(\mathcal{R}_m) \mid m \text{ a maximal ideal of } \mathcal{R} \} \).

Proof. Let us set
\[
t = \sup \{ \iota(\mathcal{R}_m) \mid m \text{ a maximal ideal of } \mathcal{R} \}.
\]

By Corollary 2.3, we have \( \iota(\mathcal{R}) \geq t \). On the other hand, if \( t = \infty \), then \( \iota(\mathcal{R}) = \infty \) and thus \( \iota(\mathcal{R}) = t \) holds in this case.

Now, assume that \( t < \infty \). For any ideal \( \mathcal{I} \) of \( \mathcal{R} \), we have \( \iota(\mathcal{R}_m) \leq t \) and
\[
(\mathcal{I} \mathcal{R}_m)^n \subseteq (\mathcal{I} \mathcal{R}_m)^{n-t}
\]
for all \( n \geq t \).
for all for $n \geq t$ and for any maximal ideal $m$ of $R$. By Proposition 2.2, $\mathcal{T}R_m = \mathcal{T}^nR_m$. It implies
\[
\mathcal{T}^n R_m \subseteq I^{n-t} R_m
\]
for all maximal ideals $m$. Hence $\mathcal{T}^n \subseteq I^{n-t}$ for $n \geq t$, and consequently $i(R) \leq t$. This proves the corollary.

Now, we recall a useful tool, the reduction of an ideal, for studying the integral closure of an ideal. An ideal $J \subseteq I$ is said to be a reduction of the ideal $I$, if there exists an integer $k$ such that $I^{k+1} = JI^k$. One important result about reduction is the following proposition.

**Proposition 2.5** Let $R$ be a noetherian ring and $J \subseteq I$ a pair of ideals of $R$. Then the following conditions are equivalent.

(i) $J$ is a reduction of $I$.

(ii) $I$ is integral over $J$.

*Proof.* (i) $\Rightarrow$ (ii) By assumption, there is an integer $k$ such that $I^{k+1} = JI^k$. As $R$ is a noetherian ring, the ideal $I^k$ is finitely generated. Set $I^k = (a_1, a_2, \ldots, a_n)$. For $a \in I$ and $i(1 \leq i \leq n)$, write $aa_i = \sum_{j=1}^n a_{ij}a_j$ for some $a_{ij} \in J$. Let $A = (a_{ij})$ be the matrix $(\delta_{ij}a - a_{ij})$, where $\delta_{ij}$ is the Kronecker delta function. By Cramer’s Rule, $\det(A)I^k = 0$. In particular, $\det(A)a_k = 0$, and an expansion of $\det(A)a_k = 0$ yields the desired equation of integral dependence of $a$ over $J$. Hence $I$ is integral over $J$.

(ii) $\Rightarrow$ (i) Suppose that $I$ is integral over $J$. Write $I = (b_1, b_2, \ldots, b_s)$. For each $i, 1 \leq i \leq s$, there is an equation of the form
\[
b_i^{n_i} + c_{i1}b_i^{n_i-1} + \cdots + c_{i_m_i} = 0
\]
where $c_{ij} \in I^j$ for $1 \leq j \leq n_i$. Thus it implies $b_i^{n_i} \in JI^{n_i-1}$. Put $k = \sum_{j=1}^s n_j + 1$. It is clear that if $r_1 + r_2 + \cdots + r_s = k$, every element of the form $b_1^{r_1}b_2^{r_2} \cdots b_s^{r_s}$ lies in $JI^{k-1}$, and this proves that $I^k = JI^{k-1}$.

As an easy consequence of the proposition, we have the following result mentioned in the last section.

**Corollary 2.6** Let $J$ be an ideal of $R$. Then for any integer $n$, $J^n \subseteq \mathcal{J}^n \subseteq \mathcal{J}$.  

*Proof.* It follows from Proposition 2.5, $J$ is a reduction of $\mathcal{J}$. Thus there is an integer $k$ such that $\mathcal{J}^{k+1} = J\mathcal{J}^k$. For an integer $n$, we have $(\mathcal{J})^{k+1} = J^n(\mathcal{J})^k$. By Proposition 2.5 again, $\mathcal{J}^n$ is integral over $J^n$, and consequently $J^n \subseteq \mathcal{J}^n \subseteq \mathcal{J}$.

Let $\varphi : R \rightarrow S$ be a morphism of noetherian rings. The property of persistence of integral closure states that $\varphi(\mathcal{J}) \subseteq \mathcal{J}$ for any ideal $J$ of $R$. This follows as by applying $\varphi$ to an equation of integral dependence of an element $a$ over $J$ to obtain an equation of integral dependence of $\varphi(a)$ over the ideal $\mathcal{J}$. 

Proposition 2.7 Let \( \varphi : R \to S \) be a faithfully flat morphism of noetherian rings. Then \( i(R) \leq i(S) \).

Proof. If \( i(S) = \infty \), there is nothing to prove. Now, we assume \( i(S) = k \) is finite. Let \( J \) be an ideal of \( R \). Since \( i(S) = k \), it shows \( \varphi(J^n) \subseteq \varphi(J^{n-k}) \) for \( n \geq k \). By persistence of integral closure, we have \( \varphi(J) \subseteq \varphi(J) \). So \( \varphi(J^n) \subseteq \varphi(J^{n-k}) \). It implies \( \varphi(J^n) \subseteq \varphi(J^{n-k}) \). Note that \( \varphi \) is a faithfully flat morphism of noetherian rings, we conclude \( J^n \subseteq J^{n-k} \) for \( n \geq k \), and this proves \( i(R) \leq i(S) \).

If \( R \) is a local ring with the unique maximal ideal \( m \), we use \( \hat{R} \) to denote the \( m \)-adic completion of \( R \). Since \( \hat{R} \) is faithfully flat over \( R \), \( i(R) \leq i(\hat{R}) \) by Proposition 2.7. We prove that this inequality is indeed an equality.

Proposition 2.8 Let \((R, m)\) be a noetherian local ring. Then \( i(R) = i(\hat{R}) \).

Proof. It suffices to prove \( i(\hat{R}) \leq i(R) \) by Proposition 2.7. If \( i(R) = \infty \), there is nothing to prove. In fact, such case cannot happen, we will prove in the next section, \( i(R) \) is always a finite number. Now, put \( i(R) = k \), and naturally regard \( R \) as a sub-ring of \( \hat{R} \). Let \( K = (b_1, b_2, \cdots, b_r) \) be an arbitrary ideal of \( \hat{R} \).

If \( K \) is a \( mR \)-primary ideal, then there exists an integer \( t \) such that \( m^t \hat{R} \subseteq K \). By the definition of completion of a ring, we can choose elements \( a_i \in R \) such that \( b_i - a_i \) lies in \( m^{t+1}R \) for \( 1 \leq i \leq r \). Let \( J \) be the ideal of \( R \) which is generated by \( a_1, a_2, \cdots, a_r \). Clearly, \( K \subseteq J \hat{R} + mK \). So it follows that \( K = J \hat{R} \) by Nakayama lemma. Since the integral closure \( \hat{K} \) of \( K \) is also a \( m \hat{R} \)-primary ideal, there is an ideal \( I \) of \( R \), replacing \( I \) by \( I + J \), we may assume \( J \subseteq I \), such that \( \hat{K} = I \hat{R} \). By Proposition 2.5, \( J \hat{R} \) is a reduction of \( I \hat{R} \), and there is an integer \( n_0 \) such that \( I^{n+1} \hat{R} = JI^{n_0} \hat{R} \). As \( \hat{R} \) is faithfully flat over \( R \), we obtain \( I^{n+1} = JI^{n_0} \). Hence for \( n \geq k \), \( I^n \subseteq J^{n-k} \), and consequently

\[
\hat{K}^n = I^n \hat{R} \subseteq J^{n-k} \hat{R} = (J \hat{R})^{n-k} = K^{n-k}.
\]

For an arbitrary ideal \( K \) of \( \hat{R} \), we have for \( n \geq k \) and all positive integer \( i \)

\[
\overline{K^n} \subseteq K + m^i R^n \subseteq (K + m^i \hat{R})^{n-k} \subseteq K^{n-k} + m^i \hat{R}
\]

by the case we have just proved. So \( \overline{K^n} \subseteq K^{n-k} \) for \( n \geq k \) by Nakayama lemma again. This proves \( i(\hat{R}) \leq i(R) \), and the proof of the proposition is complete.

Now, we turn to giving a lower bound for \( i(R) \). We will show that if \( R \) is of finite dimension \( d > 0 \), then \( i(R) \) is at least \( d - 1 \). Let us begin with recalling Monomial Conjecture of Hochster which asserts that given any system of parameters \( x_1, x_2, \cdots, x_d \) of a \( d \)-dimensional local ring \( R \), then for all \( n > 1 \)

\[
(x_1 x_2 \cdots x_d)^n \notin (x_1^{n+1}, x_2^{n+1}, \cdots, x_d^{n+1}).
\]

Hochster proved the conjecture in the equicharacteristic case \([Ho1]\) \([Ho2]\). In mixed cases, the conjecture remains undetermined, despite much effort. However, Hochster \([Ho1]\) pointed out, for any system of parameters \( x_1, x_2, \cdots, x_d \) of \( R \), there exists an integer \( t_0 \), such that for all \( t \geq t_0, x_1^t, x_2^t, \cdots, x_d^t \) satisfying Monomial Conjecture. In particular, we have:
Lemma 2.9 If $R$ is a noetherian local ring of dimension $d > 0$, then there exists a system of parameters $x_1, x_2, \ldots, x_d$ of $R$ such that for all $n > 0$

$$(x_1 x_2 \cdots x_d)^n \notin (x_1^{n+1}, x_2^{n+1}, \ldots, x_d^{n+1}).$$

By means of Lemma 2.9, we can prove the following interesting result.

Proposition 2.10 Let $R$ be a noetherian ring dimension $d > 0$. Then $i(R) \geq d - 1$.

Proof. Choose a maximal ideal $\mathfrak{m}$ of $R$ such that $\text{ht}(\mathfrak{m}) = d$. It is clear, $\dim(R_\mathfrak{m}) = d$. By Corollary 2.3, $i(R) \geq i(R_\mathfrak{m})$. Replacing $R$ by $R_\mathfrak{m}$, we may assume $R$ is a local ring with the maximal ideal $\mathfrak{m}$.

If $d = 1$, it follows from the definition of integral dimension that $i(R) \geq 0$, and thus the conclusion is true in this case.

Now we assume $d \geq 2$. By Lemma 2.9, there is a system of parameters $x_1, x_2, \ldots, x_d$ of $R$ such that the element $(x_1 x_2 \cdots x_d)^n$ is not contained in the ideal $(x_1^{n+1}, x_2^{n+1}, \ldots, x_d^{n+1})$ for any $n > 0$.

Let us consider the two ideals $I = (x_1^d, x_1^{d-1}x_d, x_2^d, x_2^{d-1}x_d, \ldots, x_d^{d-1}, x_d, x_d^d)$ and $J = (x_1^d, x_2^d, \ldots, x_d^d)$ of $R$. Since for each $i \ (1 \leq i \leq d - 1)$

$$(x_i^{d-1} x_d)^d = (x_i^d)^{d-1} x_d^d \in (x_i^d, x_d^d)(x_i^{d-1} x_d, x_d^d)^{d-1},$$

it yields that

$$(x_i^d, x_i^{d-1} x_d, x_d^d)^d = (x_i^d, x_d^d)(x_i^{d-1} x_d, x_d^d)^{d-1}.$$

By Proposition 2.5, $x_i^{d-1} x_d$ is integral over the ideal $(x_i^d, x_d^d)$, and thus $x_i^{d-1} x_d$ is integral over $J$ for $i \ (1 \leq i \leq d - 1)$. It shows $I$ is generated by elements which are integral over $J$, and thus $I$ is integral over $J$.

Suppose that $i(R) < d - 1$. We must have $I^{d-1} \subseteq J$ by the definition of integral dimension. In particular, it implies the multiplication of the following $d - 1$ elements

$x_1^{d-1} x_d, \ldots, x_d^{d-1} x_d$

lies in $J$, i.e.

$$(x_1 x_2 \cdots x_d)^{d-1} \in (x_1^d, x_2^d, \ldots, x_d^d),$$

and this contradicts the choices of $x_1, x_2, \ldots, x_d$. Therefore $i(R) \geq d - 1$, and this proves the conclusion.

As an immediate consequence of Corollary 2.4 and Proposition 2.10, we have:

Corollary 2.11 Let $R$ be a noetherian ring. If $i(R) < \infty$, then $\dim(R) < \infty$.

Motivated by Proposition 2.10, we will say a noetherian local ring $R$ of dimension $d$ is of minimal integral dimension if $i(R) = d - 1$. It follows from the Lipman-Sathaye theorem [LS], $i(R) \leq d - 1$ for every regular ring of dimension $d > 0$. So by Proposition 2.10, we have:

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Corollary 2.12 \( \text{Let } R \text{ be a regular local ring of dimension } d > 0. \text{ Then } R \text{ is of minimal integral dimension, i.e. } i(R) = d - 1. \)

In the rest of the section, we consider the local ring \( R \) with \( i(R) = 0. \) Clearly every ideal of \( R \) is integrally closed for such rings. By Proposition 2.10, we have \( \dim R \leq 1. \) If \( R \) is an artinian local ring, then every ideal \( I \) of \( R \) with \( I \neq R \) is integral over the zero ideal 0. So \( I = 0, \) and \( R \) must be a field. For \( \dim R = 1, \) we have

Proposition 2.13 \( \text{Let } (R, \mathfrak{m}) \text{ be a noetherian local ring with } i(R) = 0. \text{ If } R \text{ is not a field, then } R \text{ is a one dimensional regular local ring.} \)

Proof. As we point out as above, under the assumption of \( R, \) \( R \) must be a local ring of dimension 1. Since every ideal of \( R \) is integrally closed, it follows that \( R \) is reduced because the nilradical of \( R \) is integral over the zero ideal.

Now, we show that \( R \) is a domain. It suffices to prove that \( R \) has only one minimal prime ideal. Let \( P_1, P_2, \ldots, P_r \) are all the minimal prime ideals of \( R. \)

If \( r \geq 2, \) we can choose elements \( a, b \) such that

\[
a \in P_1, a \notin \cap_{i=2}^{r} P_i \text{ and } b \in \cap_{i=2}^{r} P_i, b \notin P_1.\]

It is obvious \( ab = 0, (a + b) \notin P_1 \) and \( a^2 - (a + b)a = 0. \) It shows \( a \) is integral over the ideal \((a + b)R, \) and thus \( a \in (a + b)R \) because of \( (a + b)R = (a + b)R. \)

Write \( a = (a + b)c \) for some \( c \in R. \) If \( c \notin \mathfrak{m}, \) then \( c \) is a unit of \( R. \) So

\[
a + b = c^{-1} a \in P_1,\]

and this contradicts the choice of \( a + b. \) Hence \( c \in \mathfrak{m}, \) and it implies \( a(1-c) = bc. \) Thus

\[
a \in \cap_{i=2}^{r} P_i\]

because \( 1 - c \) is a unit of \( R, \) which also contradicts the choice of \( a. \) Hence \( r = 1 \) and \( R \) is a domain.

Let \( Q(R) \) be the field of the fractions of \( R. \) In order to prove that \( R \) is a regular local ring, it suffices to show that \( R \) is integrally closed in \( Q(R) \) [Ma].

Let \( \frac{a}{b} \) be an element of \( Q(R) \) such that \( \frac{a}{b} \) is integral over \( R, \) where \( a, b \in R. \)

Hence there exists an equation of the form

\[
\left(\frac{a}{b}\right)^n + a_1\left(\frac{a}{b}\right)^{n-1} + \cdots + a_n = 0
\]

where \( a_i \in R \) for \( 1 \leq i \leq n. \) It yields that

\[
a^n + a_1 ba^{n-1} + \cdots + b^n a_n = 0.
\]

It implies \( a \) is integral over the ideal \( bR. \) Since \( bR \) is integrally closed, it shows \( a \in bR, \) and thus \( \frac{a}{b} \in R. \) Therefore \( R \) is a integrally closed domain, and the proof of the proposition is complete.
The result of Proposition 2.13 means that a one dimensional noetherian local ring with minimal integral dimension is a regular local ring. It would be very interesting to study the properties of local rings with nonzero minimal integral dimension. We conjecture:

**Conjecture 2.14** Let \((R, m)\) be a reduced local ring with minimal integral dimension \(i(R) > 0\). Then \(R\) is a regular local ring.

### 3 Main result

In this section, we will present several sufficient conditions for a ring \(R\) having finite integral dimension. Let us begin with recalling the notion of uniform Artin-Rees property studied by Huneke [Hu].

Let \(R\) be a noetherian ring, we say that \(R\) has the uniform Artin-Rees property, if for every pair \(N \subseteq M\) of finitely generated \(R\)-modules, there exists a number \(k\), depending on \(N, M\), such that for all ideal \(I\) of \(R\) and all \(n \geq k\)

\[ I^n M \cap N \subseteq I^{n-k} N. \]

First of all, we observe that if \(R\) has the uniform Artin-Rees property, then we can reduce the question when \(i(R)\) is finite to the reduced case when \(i(R_{\text{red}}) < \infty\).

**Proposition 3.1** Let \(R\) be a noetherian ring. Then

(i) \(i(R_{\text{red}}) \leq i(R)\).

(ii) If \(i(R_{\text{red}}) < \infty\), then \(i(R) < \infty\), provided that \(R\) has the uniform Artin-Rees property.

**Proof.** (i) If \(i(R) = \infty\), we must have \(i(R_{\text{red}}) \leq \infty\) by (i), and the conclusion holds in this case.

Now we assume \(i(R)\) is a finite number and put \(i(R) = k\). For any ideal \(K\) of \(R_{\text{red}}\), there is an ideal \(I\) with \(Q \subseteq I\) such that \(K = IR_{\text{red}}\). Note that \(I^n \subseteq I^{n-k}\) for \(n \geq k\). By Proposition 2.1, \(K = T R_{\text{red}}\). So

\[ K^n = T^n R_{\text{red}} \subseteq I^{n-k} R_{\text{red}} = K^{n-k} \]

for \(n \geq k\), this shows \(i(R_{\text{red}}) \leq k\), and the proof of (i) is complete.

(ii) Let \(Q\) be the nilradical of \(R\). If \(Q = 0\), then \(R = R_{\text{red}}\), and there is nothing to prove. Now, we assume \(Q \neq 0\). Let \(n_0\) be the positive integer such that \(Q^{n_0} \neq 0\) and \(Q^{n_0+1} = 0\). Since \(R\) has the uniform Artin-Rees property, for each \(i(1 \leq i \leq n_0)\), there exists integer \(k_i > 0\) such that for all ideals \(I\) of \(R\) and all \(n \geq k_i\)

\[ I^n \cap Q^i \subseteq I^{n-k_i} Q^i. \] (3.1)

Put \(k = i(R_{\text{red}})\). By Proposition 2.1, for any ideal \(I\) of \(R\), \(T R_{\text{red}} = T R_{\text{red}}\).

It yields

\[ T^n R_{\text{red}} \subseteq I^{n-k} R_{\text{red}} \]
for $n \geq k$. Equivalently, we have for $n \geq k$

$$T^n \subseteq I^{n-k} + Q. \quad (3.2)$$

Since $I^{n-k} \subseteq T^{n-k}$, it shows

$$T^n \subseteq I^{n-k} + T^{n-k} \cap Q.$$

Hence if $n \geq k + k_1$, it follows from (3.1)

$$T^n \subseteq I^{n-k} + T^{n-k-k_1}Q,$$ 

and then by (3.2), we have for $n \geq n - 2k - k_1$

$$T^n \subseteq I^{n-k} + (I^{n-2k-k_1} + Q)Q \subseteq I^{n-2k-k_1} + Q^2. \quad (3.3)$$

In the following, we will use induction on $i$ to prove

$$T^n \subseteq I^{n-i-k_1-\cdots-k_{i-1}} + Q^i. \quad (3.4)$$

for $i (2 \leq i \leq n_0 + 1)$ and for all $n \geq ik + k_1 + \cdots + k_{i-1}$.

Clearly, the conclusion holds for $i = 2$ by (3.3). Suppose that

$$T^n \subseteq I^{n-jk-k_1-\cdots-k_{j-1}} + Q^j.$$ 

for some $j$ with $(2 \leq j < n_0)$ and for all $n \geq jk + k_1 + \cdots + k_{j-1}$. Then we have

$$T^n \subseteq I^{n-jk-k_1-\cdots-k_{j-1}} + T^{n-jk-k_1-\cdots-k_{j-1} - k_j} \cap Q^j.$$ 

Hence by (3.1), we have for $n \geq jk + k_1 + \cdots + k_{j-1} + k_j$

$$T^n \subseteq I^{n-jk-k_1-\cdots-k_{j-1}} + T^{n-jk-k_1-\cdots-k_{j-1} - k_j} \cap Q^j.$$ 

It follows from (3.2) that for $n \geq (j+1)k + k_1 + \cdots + k_{j-1} + k_j$

$$T^n \subseteq I^{n-(j+1)k-k_1-\cdots-k_j} + Q^{j+1}.$$ 

Hence by induction, (3.4) holds for all $i (2 \leq i \leq n_0 + 1)$. Note that $Q^{n_0+1} = 0$.

Thus we have shown

$$T^n \subseteq I^{n-(n_0+1)k-k_1-\cdots-k_{n_0}}.$$ 

for $n \geq (n_0+1)k + k_1 + \cdots + k_{n_0}$ and for all ideals $I$ of $R$. Therefore $i(R) < \infty$, and this ends the proof of (ii).

To give a sufficient condition for a ring $R$ having the uniform Artin-Rees property, Huneke [Hu] studied two ideals $T(R)$ and $CM(R)$ of $R$.

For an integer $k$, set $T_k(R) = \bigcap_{I,n} (I^{n-k} : T^n)$, where the intersection is taken over all $n$ and all ideals $I$ of $R$. We define $T(R) = \bigcup_k T_k(R)$. It is clear if $k$
increases, the ideals $T_k$ also increase. Hence $T(R)$ is an ideal of $R$. If $T(R) = R$, we say that $R$ has the uniform Briançon-Skoda property. Clearly, $R$ has the uniform Briançon-Skoda property if and only if there exists an integer $k \geq 0$ such that for $n \geq k$ and all ideals $I$ of $R$

$$T^n \subseteq I^{n-k}.$$  

By Corollary 2.6, one can see easily $i(R) < \infty$ if $R$ has the uniform Briançon-Skoda property.

We need the following notion of standard complex of free modules to describe the definition of the ideal $CM(R)$. Recalling that a complex of finitely generated free $R$-modules

$$0 \rightarrow F_n \xrightarrow{f_n} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{f_1} F_0$$

is said to satisfy the standard condition on rank if $\operatorname{rank}(f_n) = \operatorname{rank}(F_n)$, and $\operatorname{rank}(f_{i+1}) + \operatorname{rank}(f_i) = \operatorname{rank}(F_i)$ for $1 \leq i < n$. Here we think $f_i$ as given by a matrix $A_i$, and the rank of $f_i$ is the determinantal rank of $A_i$. Let $I(f_i)$ be the ideal generated by the rank-size minors of $A_i$. We say that $F_\ast$ satisfies the standard condition on height if $\operatorname{ht}(I(f_i)) \geq i$ for all $i$. Note that the height an ideal $I$ is $\infty$ if $I = R$. We define $CM(R)$ to be the ideal generated by all elements $a \in R$ such that for all complexes $F_\ast$ of finitely generated free $R$-modules satisfying the standard conditions on height and rank, $xH_i(F_\ast) = 0$ for all $i \geq 1$.

As an easy consequence of Buchsbaum-Eisenbud criterion theorem for exactness of a free complex [BE], it follows that $CM(R) = R$ if and only if $R$ is a Cohen-Macaulay ring. One of the important questions concerning $CM(R)$ is when $CM(R)$ is not contained in any minimal prime ideal of $R$. This question is closely related to the existence of uniform local cohomological annihilators of $R$ which was introduced by the author [Zh1].

Recall that an element $a \in R$ is said to be a uniform local cohomological annihilator of $R$, if

(i) $a$ is not contained in any minimal prime ideal of $R$,

(ii) For every maximal ideal $\mathfrak{m}$, $a$ kills the $i$-th local cohomology module $H^i_\mathfrak{m}(R)$ for $i < \operatorname{ht}(\mathfrak{m})$.

It is known [cf. Zh2] that $CM(R)$ is not contained in any minimal prime ideal of $R$ if and only if $R$ has a uniform local cohomological annihilator. Moreover, Zhou proved a noetherian domain of finite dimension has a uniform local cohomological annihilator, provided $R$ is a homomorphic image of a Cohen-Macaulay ring [Zh1] or $R$ is an excellent ring [Zh2].

By means of $T(R/P)$ and $CM(R/P)$ for every prime ideal $P$ of $R$, Huneke [Hu] proved an important criterion for rings having the uniform Artin-Rees property. The same proof of Huneke works for the following criterion, which replaces the condition $T(R/P) \neq 0$ of [Hu, Theorem 3.4] by $i(R/P) < \infty$.

**Theorem 3.2** Let $R$ be a noetherian ring with infinite residue fields. Assume that for all nonzero prime ideals $P$ of $R$ the following conditions hold:
(i) $i(R/P) < \infty$.
(ii) $CM(R/P) \neq 0$.

Then $R$ has the uniform Artin-Rees property.

Combining Theorem 3.2 with the mentioned results of the author [Zh1] [Zh2], we have:

**Corollary 3.3** Let $R$ be a finite dimensional noetherian ring with $i(R/P) < \infty$ for any prime ideal of $R$. If $R$ satisfies one of the following conditions:

(i) $R$ is an excellent ring.
(ii) $R$ is a quotient ring of a Cohen-Macaulay ring of finite dimension.

Then $R$ has the uniform Artin-Rees property.

The following remarkable result is proved by Huneke [Hu, Theorem 4.12], which gives three sufficient conditions for rings having the uniform Artin-Rees property. In particular, the theorem asserts that every local ring has the uniform Artin-Rees property.

**Theorem 3.4** Let $R$ be a noetherian ring of finite dimension. $R$ has the uniform Artin-Rees property if $R$ satisfies one of the following conditions:

(i) $R$ is essentially of finite type over a local ring.
(ii) $R$ is a ring of characteristic $p > 0$, and $R$ is module finite over $R^p$.
(iii) $R$ is essentially of finite type over the ring of integer numbers $\mathbb{Z}$.

The another remarkable result of Huneke [Hu] is the following uniform Briançon-Skoda theorem [Hu, Theorem 4.13].

**Theorem 3.5** Let $R$ be a reduced noetherian ring of finite dimension. $R$ has the uniform Briançon-Skoda property if $R$ satisfies one of the following conditions:

(i) $R$ is essentially of finite type over an excellent local ring.
(ii) $R$ is a ring of characteristic $p > 0$, and $R$ is module finite over $R^p$.
(iii) $R$ is essentially of finite type over the ring of integer numbers $\mathbb{Z}$.

Now, we turn to the main result of the paper. By means of Proposition 3.1, Theorem 3.4 and Theorem 3.5, we can prove the following main result, which is a weaker generalization of Theorem 3.5.

**Theorem 3.6** Let $R$ be a noetherian ring of finite dimension. Then $i(R) < \infty$ if $R$ satisfies one of the following conditions:

(i) $R$ is essentially of finite type over a noetherian local ring.
(ii) $R$ is a ring of characteristic $p$, and $R$ is module finite over $R^p$.
(iii) $R$ is essentially of finite type over the ring of integer numbers $\mathbb{Z}$.

**Proof.** (i) By Corollary 2.3, we may assume $R$ is of finite type over a local ring $(S, \mathfrak{m})$. Let $\hat{S}$ be the completion of $S$ in the $\mathfrak{m}$-adic topology. It is known $\hat{S}$ is faithfully flat over $S$, and thus $B = R \otimes_S \hat{S}$ is of finite type and faithfully flat over $\hat{S}$. From Proposition 2.7, it suffices to prove $i(B) < \infty$. Replacing $R$ by $B$, and $S$ by $\hat{S}$, we may assume $R$ is of finite type over an excellent ring $S$. 


Write $R = S[X_1, X_2, \cdots, X_r]/I$, where $I$ is an ideal of the polynomial ring $S[X_1, X_2, \cdots, X_r]$. Let $P_1, P_2, \cdots, P_t$ be all the minimal prime ideals of $I$. Set $p_i = P_i \cap S$ for $1 \leq i \leq t$. It is not difficult to see that

$$R_{red} = S[X_1, X_2, \cdots, X_r]/P_1 \cap P_2 \cap \cdots \cap P_t$$

is of finite type over the reduced excellent ring $S/p_1 \cap p_2 \cap \cdots \cap p_t$. By Theorem 3.5, $R_{red}$ has the uniform Briançon-Skoda property. In particular $i(R_{red}) < \infty$.

Now from Theorem 3.4 and Proposition 3.1, we conclude $i(R) < \infty$, and this proves (i).

(ii) First, we show that $R_{red}$ is module finite over $(R_{red})^p$. In fact, let $P_1, P_2, \cdots, P_r$ are all the minimal prime ideals of $R$. It is easy to see $R^p/(P_i \cap R^p)$ is naturally isomorphic to $(R/P_i)^p$ and $P_i \cap R^p = P_i^p$ is a prime ideal of $R^p$. Hence the nilpotent ideal of $R^p$

$$I = P_1 \cap P_2 \cap \cdots \cap P_r \cap R^p = P_1^p \cap P_2^p \cap \cdots \cap P_r^p$$

is an intersection of prime ideals of $R^p$. Conversely the nilradical ideal $J$ is clearly contained in $I$, so $I$ is the nilradical ideal of $R^p$. It is easy to see $P_i^p$ are all the distinct minimal prime ideals of $R^p$. Moreover, there is a natural isomorphism between the rings $R^p/I$ and $(R_{red})^p$. So if $R$ is module finite over $R^p$, then $R_{red}$ is module finite over $R^p/I$, and thus is module finite over $(R_{red})^p$.

Now from Theorem 3.5, $R_{red}$ has the uniform Briançon-Skoda property. In particular $i(R_{red}) < \infty$. Now from Theorem 3.4 and Proposition 3.1, we conclude $i(R) < \infty$, and this ends the proof of (ii).

(iii) Clearly, $R_{red}$ is also essentially of finite type over the ring of integer numbers $\mathbb{Z}$. Hence from Theorem 3.5 $i(R_{red}) < \infty$. The conclusion follows from Theorem 3.4 and Proposition 3.1, and this ends the proof of the theorem.

An immediate consequence of Theorem 3.5 is the following finiteness property of $i(R)$ for any noetherian local ring $R$.

**Corollary 3.7** Let $R$ be a noetherian local ring. Then $i(R) < \infty$.

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