RESOLVENT AND SPECTRAL MEASURE ON NON-TRAPPING ASYMPTOTICALLY HYPERBOLIC MANIFOLDS III: GLOBAL-IN-TIME STRICHARTZ ESTIMATES WITHOUT LOSS

XI CHEN

Abstract. Applying the spectral measure estimates obtained in [7], we establish global-in-time Strichartz estimates without loss via truncated / microlocalized dispersive estimates as well as energy estimates.

1. Introduction

This paper, following the author’s joint works [6] and [7] with Andrew Hassell, is the last in a series of papers concerning the analysis of the resolvent family and spectral measure for the Laplacian on non-trapping asymptotically hyperbolic manifolds. The present paper is devoted to the application of spectral measure to Schrödinger equations.

Consider the Cauchy problem of Schrödinger equation
\[ \begin{cases} i \frac{\partial}{\partial t} u + \Delta u = F(t, z) \\ u(0, z) = f(z) \end{cases} \]
on a $n+1$-dimensional asymptotically hyperbolic manifold $X$ (See Section 2 for the definition), provided $f$ is orthogonal to the eigenfunctions of $\Delta$.

The goal is to prove global-in-time Strichartz type estimates without loss
\[ \| u \|_{L^q(L^r(X))} \leq C \left( \| f \|_{L^2(X)} + \| F \|_{L^\tilde{q}(L^\tilde{r}(X))} \right) \]
for hyperbolic Schrödinger admissible pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$, namely
\[ \frac{2}{q} + \frac{n+1}{r} \geq \frac{n+1}{2}, \quad q \geq 2, \quad r > 2, \quad (q, r) \neq (2, \infty). \] (1.2)
One may note this admissible set is much wider than the set of sharp Schrödinger admissible pairs on Euclidean space, which satisfy
\[ \frac{2}{q} + \frac{n+1}{r} = \frac{n+1}{2}, \quad q, r \geq 2, \quad (q, r) \neq (2, \infty). \]
This striking phenomenon indeed results from the dispersion of hyperbolic geometry. Banica, Carles and Staffilani [3] first observed this while studying the radial solution of Schrödinger equations.

Key words and phrases. Asymptotically hyperbolic manifolds, spectral measure, dispersive estimates, Strichartz estimates.

By the standard Keel-Tao endpoint approach, it is sufficient to reduce to energy estimates
\[ \| e^{it\Delta} f \|_{L^2} \leq \| f \|_{L^2} \]
and dispersive estimates
\[ \| e^{i(t-s)\Delta} f \|_{\infty} \leq |t - s|^{-(n+1)/2} \| f \|_{L^1} \]
for Schrödinger propagator on Euclidean space. However, unlike on Euclidean space, dispersive estimates in above form is too strong to hold on (asymptotically) hyperbolic manifolds for two reasons. On the one hand, the conjugate points arising on the manifold might do harm to the global estimates. On the other hand, the spectral measure occurs with distinct powers of spectral parameter at high and low energies, which results in distinct powers of the decay of the propagator in time.
The geometric influences on the behaviour of solutions to PDEs always catch mathematicians’ eyes. The geometry of asymptotically hyperbolic manifolds has a few distinctive phenomena. It would be illuminating to see what happens on model spaces. Anker and Pierfelice [1] independently proved hyperbolic dispersive estimates would be orthogonal to the eigenfunctions of $\Delta$.

Ionescu and Staffilani [16] proved hyperbolic dispersive estimates

\begin{equation}
|\text{Ker} e^{it\Delta_{n+1}}| \leq C \left\{ \begin{array}{ll}
t^{-3/2}(1 + d(z,z'))e^{-nd(z,z')/2} & \text{if } t \geq 1 + d(z,z') \\
t^{-(n+1)/2}(1 + d(z,z'))^{n/2}e^{-nd(z,z')/2} & \text{if } t \leq 1 + d(z,z')
\end{array} \right.
\end{equation}

on real hyperbolic space $\mathbb{H}^{n+1}$. Similar results on convex co-compact hyperbolic manifolds are proved by Burq, Guillarmou and Hassell [5]. It is obvious that the distance factors in any compact subset of $\mathbb{H}^{n+1}$ reduce to Euclidean ones. Beyond that, the conformal metric creates an exponentially growing volume. Therefore the Schrödinger propagator gains an extra exponential decay factor to cancel it for integrability. Additionally, as a result of the discrepant spectral measure estimates on $\mathbb{H}^{n+1}$, the long time dispersion is at a lower speed than on Euclidean space, though the short time dispersions are the same with Euclidean case.

More generally, when we study an asymptotically hyperbolic manifold with conjugate points, there is no global expression of geodesic distance function. It does have an impact on the integral representation of the Schrödinger propagator. By functional calculus, Schrödinger propagator is determined by spectral measure. The kernel of the spectral measure on asymptotically hyperbolic manifolds is a Lagrangian distribution microlocally supported on the geodesic flow-out emanating from the spherical conormal bundle of the diagonal of the double space. Because of arising conjugate points, there is no global expression of the geodesic distance function. The desired global estimates thus fail. Alternatively, we have microlocalized spectral measure estimates, which still give dispersive estimates, though in a weak form, sufficient to have Strichartz inequalities. This technique comes from [12] and [14] for the spectral measure and the Schrödinger propagator on asymptotically conic (Euclidean) manifolds.

Our strategy is to establish the Strichartz estimates via energy estimates and dispersive estimates. To get these estimates, we shall apply the spectral measure results from [7] to hyperbolic dispersive estimates [13] in Proposition [6] and Proposition [8] and energy estimates in Proposition [5] for Schrödinger propagators on asymptotically hyperbolic manifolds to prove

**Theorem 1** (Strichartz estimates). Suppose $(X, g)$ is an $(n + 1)$-dimensional non-trapping asymptotically hyperbolic manifold with no resonance at the bottom of spectrum. For any admissible pairs $(q, r)$ and $(\tilde{q}', \tilde{r}')$ satisfying (1.2), we have inhomogeneous Strichartz inequality

\begin{equation}
\|u\|_{L^q_t L^r_x (\mathbb{R} \times X)} \leq C (\|f\|_{L^\tilde{q}'_t L^{\tilde{r}'}_x (\mathbb{R} \times X)} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x (\mathbb{R} \times X)}),
\end{equation}

provided $f$ is orthogonal to the eigenfunctions of $\Delta$.

The main idea of the proof is due to Keel and Tao [17]. But their method doesn’t exploit a distinctive phenomenon of hyperbolic type spaces, the spatial decay at infinity, which actually creates a wider admissible set than usual. On the other hand, we have a discrepancy of dispersive estimates unlike what happens on asymptotically conic manifolds, for which we thus have to split the time-space norm of the solution in temporal variables. In the case of asymptotically hyperbolic manifolds, we don’t have global dispersive estimates but the microlocalized ones. Therefore, it is vital to understand the relation between microlocalized spectral measure estimates and dispersive estimates. In all, our proof, certainly based on Keel-Tao endpoint approach, not only borrows the idea from Anker and Pierfelice [11], Ionescu and Staffilani [16] to split in time and make use of the spatial decay, but also follows Guillarmou and Hassell [11], Hassell and Zhang [14] to perform microlocalization for conjugate points.

The geometric microlocal techniques used for spectral measure do require the non-trapping condition and conformal compactness on the space. Bouclet [4] investigates the local-in-time homogeneous Strichartz estimates without loss on more general asymptotically hyperbolic manifolds without taking these advantages. The author constructed a parametrix for Schrödinger propagators. However, the issue here is that the error may be difficult to control as time goes to infinity. For the consideration of long time behaviour, one needs an exact spectral measure or propagator...
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(a function of spectral measure) in these estimates. Thanks to our previous works of spectral measure on non-trapping asymptotically hyperbolic manifolds, we gain global-in-time estimates without loss, which can be applied to well-posedness and scattering for semilinear Schrödinger equation as in [1].

Combining Strichartz estimates and dispersive estimates in this paper with our previous results of resolvent in [6], spectral measure with applications to restriction theorem and spectral multiplier in [7], we have elucidated the following diagram on non-trapping asymptotically hyperbolic manifolds.

\[
\begin{array}{ccc}
\text{L^p boundedness} & \downarrow & \text{Resolvent construction} \\
\text{of spectral multiplier} & & \text{near continuous spectrum} \\
\text{Strichartz estimate} & \leftarrow & \text{Restriction theorem} \\
\text{of Schrödinger equation} & \uparrow & \text{Pointwise estimates} \\
\text{Dispersive estimate} & \leftarrow & \text{for spectral measure} \\
\text{of Schrödinger propagator} & \uparrow & \text{for spectral measure} \\
\end{array}
\]

The paper is organized as follows. First of all, we shall review asymptotically hyperbolic manifolds and spectral measure. Based on the spectral measure estimates, we give microlocalized / truncated expressions of Schrödinger propagators. We then turn to the proof of L^2-energy estimates. Afterwards we establish dispersive estimates for single microlocalized / truncated propagators and refined dispersive estimates for double microlocalized / truncated propagators, which prove Strichartz estimates in the last section. The proof of Strichartz estimates in this paper are heavily influenced by the works on hyperbolic space due to Anker and Pierfelice [1], Ionescu and Staffilani [16] as well as the works on asymptotically conic manifolds due to Hassell and Zhang [14].

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2. Spectral measure on asymptotically hyperbolic manifolds

Conformally compact manifold X is an (n+1)-dimensional manifold with boundary ∂X, compact closure ˜X and endowed with a Riemannian metric g which extends smoothly to its closure. One can write

\[
g = \frac{dx^2}{x^2} + \frac{h(x,y,dy)}{x^2},
\]

where x is a boundary defining function, and h is a metric on the boundary but depending parametrically on x. Mazzeo [19] shows g is complete and its sectional curvature approaches −|dx|^2_{x^2g} as it approaches the boundary. In particular, g is said to be asymptotically hyperbolic if −|dx|^2_{x^2g} = −1.

Let Δ be the Laplacian, on (n+1)-dimensional non-trapping asymptotically hyperbolic manifold X. The continuous spectrum of Δ is contained in [n^2/4, ∞), whilst the point spectrum is contained in (0, n^2/4).

Mazzeo and Melrose [20] constructed the parametrix \((\Delta - \sigma(n - \sigma))^{-1}\) on asymptotically hyperbolic manifolds for fixed parameter and proved the resolvent has a meromorphic extension except at points \((n + 1)/2 - Z_+\). The resolvent they construct is a 0-pseudo differential operator plus a smooth function on the 0-blown up double space \(X \times_0 X\) ( or \(X^2\) for short), where the space \(X \times_0 X\) is obtained by blowing up the boundary of the diagonal \(\partial \text{diag} = \{(0, y, 0, y)\} \in X^2\).
From here on, we will work on $X_0^2$ instead of $X^2$ for the nice expression of the resolvent and the spectral measure. A very important feature of $X_0^2$ is that the front face is a bundle with fibres similar to hyperbolic spaces. Therefore the hyperbolic space is a good model for asymptotically hyperbolic manifolds. Apart from these, we also get following useful asymptotic expansion of the geodesic distance near the boundary of $X_0^2$

$$d(z, z') = -\log \rho_L - \log \rho_R + b(z, z'),$$

where $\rho_L$ and $\rho_R$ are boundary defining functions of the left and the right faces respectively, $b$ is a uniformly bounded function on $X_0^2$. See [7, Proposition 10]. In particular, $b(z, z')$ is smooth on $X_0^2$ in the case of asymptotically hyperbolic manifolds of Cartan-Hadamard type. This was observed on hyperbolic spaces and proved on asymptotically hyperbolic manifolds by Melrose, Sá Barreto and Vasy [21]. On general asymptotically hyperbolic manifolds, arising conjugate points ruin the smoothness of $b$ but we still get the boundedness of $b$. Consequently, we have the asymptotic

$$e^{-d(z, z')} \approx \rho_L \rho_R,$$

if $\rho_L \rho_R$ is small.

Additionally, Melrose, Sá Barreto and Vasy [21], Wang [23], and ourselves [6] constructed the semiclassical resolvent at high energy (near the infinity of the spectrum). Specifically, the high energy resolvent defined on $X_0^2$ is a 0-pseudo differential operator plus a Fourier integral operator microlocally supported on the union of the diagonal conormal bundle and its bicharacteristic flow-out. To avoid unnecessary technical details, we wouldn’t clearly state the result over parametrix construction but omit bulky theories about 0-calculus, blow-up, flow-out, Lagrangian distribution, intersecting Lagrangian and etc, for which we refer the readers to [20] [21] [6].

Based on the results of the resolvent, we study, via Stone’s formula

$$2\pi i dE_L(\lambda) = R_L(\lambda + i0) - R_L(\lambda - i0),$$

provided $\lambda$ is in the continuous spectrum of $L$ [7] the spectral measure on asymptotically hyperbolic manifolds in [7]. Since the spectral measure is defined on the continuous spectrum ($n^2/4, \infty$), we are in particular concerned about the asymptotic behaviour around two endpoints $n^2/4$ (low energy) and $\infty$ (high energy) respectively, as the intermediate values can be estimated in either way.

On the one hand, it is convenient to have the smoothness of resolvent at bottom of spectrum to gain the asymptotic of spectral measure. We call it no resonance at the bottom of the continuous spectrum. Intriguingly, it is still unknown that what geometric conditions amounts to the analyticity of the resolvent at the bottom of spectrum. However, there are some sufficiency results. For instance Guillarmou and Qing [13] shows that the largest real scattering pole of $(\Delta - \sigma(n - \sigma))^{-1}$ on an $n + 1$-dimensional conformally compact Einstein manifold $(X, g)$ is less than $n/2 - 1$ if and only if the conformal infinity of $(X, g)$ is of positive Yamabe type, where $n > 1$. With the hypotheses of analyticity at $n^2/4$, we [7] deduce, from the resolvent of Mazzeo and Melrose, that

$$dE_P(\lambda)(z, z') = \lambda\left((\rho_L \rho_R)^{n/2+i\lambda}a(\lambda, z, z') - (\rho_L \rho_R)^{n/2-i\lambda}a(-\lambda, z, z')\right)$$

when $\lambda < 1$.

\[1\] We use Greek letters $\lambda, \mu, \zeta$ to denote the phase variables on cotangent bundle, respectively bold Greek letters $\lambda, \mu, \zeta$ to denote spectral parameters.
where \( P = \sqrt{(\Delta - n \tau^2/4)} \) and \( a \in C^\infty([0,1] \times X^2_0) \). A quick corollary of this result is that
\[
|dE_P(\lambda)(z, z')| \leq C \lambda^2 (1 + d(z, z')) e^{-nd(z, z')/2}.
\]
One may note the spectral measure doesn’t vanish as rapidly as its counterparts on some other space do at low energy. For example, Guillarmou, Hassell and Sikora [12], Hassell and Zhang [13] showed on \( n + 1 \)-dimensional asymptotically Euclidean manifolds there is a pseudo differential operator partition of unity
\[
I = \sum_{i=1}^{N} Q_i,
\]
such that the microlocalized spectral measure \( Q_i dE_P(\lambda)Q'_i \) takes the form
\[
\lambda^n e^{i\lambda d(z, z')} a_+(\lambda, z, z') + \lambda^n e^{-i\lambda d(z, z')} a_-(\lambda, z, z'),
\]
where the derivatives of \( a_\pm \) obeys
\[
\left| \frac{d^n}{d\lambda^n} a_\pm(\lambda, z, z') \right| \leq C \lambda^{-n} (1 + \lambda d(z, z'))^{-n/2}.
\]
Apart from 3-dimensional space, the spectral measure on asymptotically hyperbolic manifolds is unable to provide such decay for the endpoint dispersive estimate. Nonetheless, the property (2.1) for large distance on asymptotically hyperbolic manifolds compensates the lack of decay by an exponential vanishing at spatial infinity.

On the other hand, the spectral measure at high energy shares a microlocal structure with corresponding resolvent. Suppose we have local coordinates \( \{(x, y_1, \ldots, y_n)\} \) near \( \partial X \) and local coordinates \( \{(z_1, \ldots, z_{n+1})\} \) away from \( \partial X \). The 0-cotangent bundle \( 0T^*X^0 \), introduced by Mazzeo and Melrose [20], is a vector bundle with sections
\[
\frac{\lambda}{x} dx + \frac{\mu_1}{x} dy_1 + \cdots + \frac{\mu_n}{x} dy_n \quad \text{near} \; \partial X
\]
\[
\zeta_1 \frac{dz_1}{x} + \cdots + \zeta_{n+1} \frac{dz_{n+1}}{x} \quad \text{away from} \; \partial X.
\]
Recall from [6] that the microlocal support (or wavefront set) of the high energy resolvent is the diagonal conormal bundle \( N^* \text{diag} \subset 0TX^2_0 \) and its bicharacteristic flow-out \( \Lambda \), which is contained in \( 0SX^0 \times 0SX^0 \), where
\[
0S^*X^0 = \{ |\zeta|^2 = 1 \text{ or } |\lambda|^2 + |\mu|^2 = 1 \} \subset 0TX^0.
\]
By Stone’s formula, the spectral measure is microlocally supported on \( \Lambda \), while the singularity at \( N^* \text{diag} \) cancels out by the subtraction between the outgoing resolvent and the incoming resolvent. Therefore the spectral measure is a Fourier integral operator associated with Lagrangian \( \Lambda \). Apart from the boundary behaviour, this Lagrangian structure on asymptotically hyperbolic manifolds is analogous with the case of asymptotically Euclidean. So we gain similar spectral measure at high energy, but better in the sense of the spatial exponential decay at boundary, which causes the wider set of admissible pairs in the Strichartz estimates. To avoid the arising conjugate points, we adopt the similar techniques of microlocalization as in [12] and [13].

To state the spectral measure estimates explicitly, let us recall the partition of unity on \( 0T^*X^0 \) in [7]. First of all, we take \( Q_0 \) microlocally supported away from the spherical bundle \( 0S^*X^0 \), say \( \{ |\zeta|^2 > 3/2 \land |\lambda|^2 + |\mu|^2 > 3/2 \} \), which contains the wavefront set of the spectral measure. On the other hand, we divide the interval \( -3/2, 3/2 \) into a union of intervals \( I_1, \ldots, I_{N_1} \) with overlapping interiors, and with diameter \( \leq \delta \), which is a sufficiently small number, whilst each \( I_i \) intersects only \( I_{i-1} \) and \( I_{i+1} \). We also take a small strip neighbourhood of the boundary such that the sectional curvature is negative; in the meantime, we divide the 0-cotangent bundle over this strip into a union of small slices \( B_{I_1}, \ldots, B_{I_{N_1}} \) and with diameter \( \leq \eta \), which is also sufficiently small, and have \( Q_{N_1+1}, \ldots, Q_{N_2} \) supported on them respectively. With this partition, we have the estimates for microlocalized spectral measure.
Proposition 2 ([7]). One can choose a pseudodifferential operator partition of unity

$$Id = \sum_{k=0}^{N_2} Q_k(\lambda),$$

where $Q_k$ for $k \neq 0$ is supported around the spherical bundle, such that $Q_k$ for any $k$ and $N(\lambda)$ are uniformly (L^2)-bounded over $\lambda$ and

$$Q_k(\lambda)dE_P(\lambda)Q_k(\lambda) = \lambda^ne^{i\lambda d(z,z')}a_+(\lambda) + \lambda^ne^{-i\lambda d(z,z')}a_-(\lambda) + O(\lambda^{-\infty}),$$

for large $\lambda$, where $a_\pm$ are the Lagrangian distributions defined on the forward and backward bicharacteristic flow respectively and satisfying

$$\frac{d^j}{d\lambda^j} g_\pm(\lambda) = \begin{cases} O\left(\lambda^{-j} (1 + \lambda d(z,z'))^{-n/2}\right), & \text{if } d(z,z') \text{ is small} \\ O\left(\lambda^{-n/2-j} e^{-nd(z,z'')/2}\right), & \text{if } d(z,z') \text{ is large} \end{cases}.$$ 

Moreover, the restriction theorem (in the sense of Stein and Tomas)

$$\|dE_P(\lambda)\|_{L^p \to L^{p'}} \leq C\lambda^{(n+1)(1/p-1/p')}$$

where $p \in [1, 2(n+2)/(n+4)]$,

at high energy on non-trapping asymptotically hyperbolic manifolds follows from above spectral measure estimates. It is well-known that Strichartz [22] insightfully points out the deep relationship between Strichartz estimates and restriction theorem. It motivates us to show the Strichartz estimates from these spectral measure estimates, which are sufficient to give restriction theorem.

3. Schrödinger Propagators via Spectral Measure

The spectral theorem of projection valued measure form for unbounded self-adjoint operators gives following expression of Schrödinger propagators $e^{it\Delta}$ via spectral measure,

$$e^{it\Delta} = e^{itn^2/4} \int_0^\infty e^{it\lambda^2} dE_P(\lambda).$$

We are motivated to employ the spectral measure estimates, including the microlocalized form at high energy (say $\lambda > 1$) together with the global form at low energy (say $\lambda < 1$), to estimate the Schrödinger propagator. As seen on Euclidean space or asymptotically conic manifolds, the spectral measure behaves uniformly on the full continuous spectrum, for example see [12]. However, comparing (2.2) and Proposition 2 one can see that the discrepancy in the order of $\lambda$ of the spectral measure in low energies and high energies on $n+1$-dimensional asymptotically hyperbolic manifolds for $n+1 > 3$. We have to split up the propagator to remedy the discrepancy.

One may pick two smooth bump functions $\chi_{\text{low}}$ supported in $[0, 2)$ and $\chi_{\text{\infty}}$ supported in $(1, \infty)$ such that $\chi_{\text{low}} + \chi_{\text{\infty}} = 1$ and split the propagator as

$$U(t) = \int_0^\infty e^{it\lambda^2} \chi_{\text{low}}(\lambda) dE_P(\lambda) + \int_0^\infty e^{it\lambda^2} \chi_{\text{\infty}}(\lambda) dE_P(\lambda).$$

In accordance with Proposition 2 we also will have to microlocalize the spectral measure of high energy part by a family of semiclassical pseudo differential operators $\{Q_k\}_0^N$ as follows

$$U_k = \int_0^\infty e^{it\lambda^2} \chi_{\text{\infty}}(\lambda) Q_k(\lambda) dE_P(\lambda).$$

In summary, we truncate and microlocalize the propagator and gain following decomposition

$$e^{it\Delta - itn^2/4} = U_{\text{low}} + \sum_{k=0}^N U_k$$

for short time. In the next two sections, we prove the energy estimates Proposition 3 and dispersive estimates in the second half of Proposition 5 and Proposition 8 for them respectively.

Thanks to Keel and Tao [17], Strichartz estimates reduce to the energy estimates and dispersive estimates for Schrödinger propagators. In our case, we have to cope with these estimates for the truncated and microlocalized propagators in the next three sections. As on hyperbolic space, a
discrepancy of dispersive estimates in time will arise, which results from the discrepancy of spectral measure estimates above.

We thus treat the long and short time estimates separately for Strichartz estimates. On the one hand, the temporal decay in the long time dispersive estimates for $e^{it\Delta}$ proved in the first half of Proposition 5 is not as fast as Euclidean space, but it doesn’t really affect Strichartz estimates. Since the low speed part (the long time part) is supported away from $t = 0$, the temporal integration is always convergent as long as a decay with order higher than 1 occurs. On the other hand, we apply the microlocalized dispersive estimates and energy estimates for $U_k$ and $U_{\text{low}}$ to the short time part by the standard Keel-Tao arguments. We remark that the exponential decay plays a key role to make the Strichartz estimates of this part hold for non-sharp admissible pairs, provided we have such exponential decay. Therefore we can prove the Strichartz estimates in Theorem 1, if Proposition 3, Proposition 5 and Proposition 8 hold.

4. Energy estimates for Schrödinger propagators at high energy

We shall prove the $L^2$-boundedness of microlocalized and truncated Schrödinger propagators. More precisely,

**Proposition 3** (Energy estimates). The propagator $e^{it\Delta}$, low energy truncated propagator $U_{\text{low}}$, microlocalized high energy truncated propagators $U_0$ and $U_k$ for $k = 1, 2, \ldots$ are all $L^2$-bounded.

**Proof.** The boundedness of $e^{it\Delta}$ and $U_{\text{low}}$ is clear. Since the entire cut off propagator at high energy is of course $L^2$-bounded, we can ignore the $k = 0$ term but only consider $U_k$ for $k = 1, 2, \ldots$.

Our main tool is almost orthogonality lemma established by Cotlar, Knapp and Stein, see for example [10, p. 620].

**Lemma 4** (Almost orthogonality). Let $\{T_j\}_{j \in \mathbb{Z}}$ be a family of bounded operators on Hilbert space $H$ obeying

$$\|T_j^* T_k\|_{H \rightarrow H} + \|T_j T_k^*\|_{H \rightarrow H} \leq \gamma(j - k)$$

for any $j, k \in \mathbb{Z}$, where the function $\gamma : \mathbb{Z} \rightarrow \mathbb{R}^+$ satisfies $\sum_{j \in \mathbb{Z}} \sqrt{\gamma(j)} < \infty$. Then linear operator $T$, the limit of $\sum_{|j| < N} T_j$ in the norm topology of $H$ as $N$ goes to infinity, is $H$-bounded.

First of all, the propagators are well-defined on $L^2$ if the integrand is supported on a compact subset of $(0, \infty)$ in $\lambda$ as the pseudodifferential operator would be $L^2$-bounded uniformly with respect to $\lambda$. We want to extend the well-definedness to entire positive half real line by almost orthogonality.

The strategy is to get a decomposition of the microlocalized propagator such that every term is an integral of a compactly supported function with respect to the microlocalized spectral measure, and then show the almost orthogonality of the decomposition required in Lemma 4.

First of all, we take the decomposition by a compactly supported smooth function $\psi \in C_\infty^\infty[1/2, 2]$ valued in $[0, 1]$ such that

$$\sum_j \psi \left( \frac{\lambda}{2^j} \right) = 1.$$ 

Then we define

$$U_{i,j}(t) = \int_0^\infty e^{it\lambda^2} \chi_\infty(\lambda) \psi \left( \frac{\lambda}{2^j} \right) Q_i(\lambda) \, dE_P(\lambda)$$

$$= -\int_0^\infty \frac{d}{d\lambda} \left( e^{it\lambda^2} \chi_\infty(\lambda) \psi \left( \frac{\lambda}{2^j} \right) Q_i(\lambda) \right) E_P(\lambda).$$

This proof is essentially due to Hassell and Zhang [13] in case of asymptotically Euclidean manifolds, as only minor modifications are needed here. But we give the detailed proof for the self-containedness of the paper.
and calculate as follows

\[ U_{i,j}(t)U_{i,k}^*(t) = \int \frac{d}{d\lambda} \left( e^{it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) E_P(\lambda) \times E_P(\mu) \frac{d}{d\mu} \left( e^{-it\mu^2} \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) Q_{i,\mu}(\mu) \right) d\lambda d\mu \]

\[ = \int \frac{d}{d\lambda} \left( e^{it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) E_P(\lambda) \times \left( e^{-it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}^*(\lambda) \right) d\lambda \]

\[ + \int \frac{d}{d\lambda} \left( e^{-it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) E_P(\mu) \times \left( e^{-it\mu^2} \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) Q_{i,\mu}(\mu) \right) d\lambda d\mu. \]

We then perform integration by parts and get

\[ U_{i,j}(t)U_{i,k}^*(t) = \int \frac{d}{d\lambda} \left( e^{it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) E_P(\lambda) \times \left( e^{-it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}^*(\lambda) \right) d\lambda \]

\[ + \int \frac{d}{d\lambda} \left( e^{-it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) E_P(\mu) \times \left( e^{-it\mu^2} \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) Q_{i,\mu}(\mu) \right) d\lambda d\mu. \]

As implied, \( U_{i,j}(t)U_{i,k}^*(t) \) is indeed \( t \)-independent. For the moment, we shall prove the \( L^2 \)-boundedness for all \( t \) via \( U_{i,j}(0)U_{i,k}^*(0) \), which equals

\[ \int \left( \int E_P(\lambda) \frac{d}{d\lambda} \left( \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) \right) d\lambda \int \left( \int E_P(\mu) \frac{d}{d\mu} \left( \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) Q_{i,\mu}(\mu) \right) \right) d\lambda d\mu. \]

We claim \( U_{i,j}(0)U_{i,k}^*(0) \) obeys the almost orthogonality estimate

\[ \|U_{i,j}(0)U_{i,k}^*(0)\|_{L^2 \to L^2} \leq C2^{-|t|-k}. \]

In light of the \( L^2 \)-boundedness of spectral projection, it suffices to prove

\[ \frac{d}{d\lambda} \left( \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) Q_{i,\lambda}(\lambda) \right) \frac{d}{d\mu} \left( \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) Q_{i,\mu}(\mu) \right) \leq C2^{-|t|-k}. \]

We denote the operators in the parentheses \( Q_{i,j}^*(\lambda) \) and \( Q_{i,k}(\mu) \) respectively. We write the product of the two as

\[ Q_{i,j}^*(\lambda)Q_{i,k}(\mu) = \lambda^{n+1} \mu^{n+1} \int \int e^{it\lambda^2} \chi_{\infty}(\lambda) \psi \left( \frac{\lambda}{2\lambda} \right) q_{i,j}(\lambda', \zeta, \lambda) \times e^{it\mu^2} \chi_{\infty}(\mu) \psi \left( \frac{\mu}{2\mu} \right) d\lambda d\mu. \]
away from $\partial X$ or

$$Q_{i,j}^*(\lambda)Q_{i,k}(\mu) = \lambda^{n+1}\mu^{n+1}\iint Q_{i,j}(\frac{x''}{\lambda^2} + (y'' - y''')\cdot x''') q_{i,j}(x'', y'', \lambda, \mu, \lambda) e^{i\mathbf{u}(x'' - x') + (y'' - y')\cdot \mathbf{u}'} dx'dy',$$

near $\partial X$. The second case is indeed the same with the first, as one can denote $(x, y)$ by $(z_1, \ldots, z_n)$ and $(\lambda, \mu)$ by $(i\zeta_1, \ldots, i\zeta_n)$. Furthermore, one may assume $j > k$, equivalent to $\lambda > \mu$, due to the symmetry. We insert a differential operator $ix''\zeta \cdot \partial_{x''}/(|\lambda||\zeta|^2)$, to which $e^{i\lambda(z'' - z')/\zeta}x''$ is invariant, and take integration by parts.

$$\lambda^{-n-1}\mu^{-n-1}Q_{i,j}^*(\lambda)Q_{i,k}(\mu) =$$

$$= \iint Q_{i,j}(\frac{x''}{\lambda^2} + (y'' - y''')\cdot x''') q_{i,j}(x'', y'', \lambda, \mu, \lambda) e^{i\mathbf{u}(x'' - x') + (y'' - y')\cdot \mathbf{u}'} dx'dy',$$

where $i \neq 0$, $Q_s$ is microlocally supported around the spherical bundle, namely, $|\zeta| \approx |\zeta'| \approx 1$. Therefore, using the $L^2$-boundedness of semiclassical pseudodifferential operators and noting $\lambda, \mu \geq 1$ on the support of the high energy cut-off function $\chi_\infty$, we deduce

$$|Q_{i,j}^*(\lambda)Q_{i,k}(\mu)| \leq C\frac{\mu + x''}{\lambda} \leq C\frac{\mu}{\lambda} \leq C2^{-|j-k|},$$

which proves the almost orthogonality for $t = 0$. Almost orthogonality lemma then gives that $\sum_{|j| \leq 1} U_{i,j}^*(0)$ strongly converges in $L^2$, that is,

$$\lim_{l \to \infty} \sup_{m \to \infty} \left\| \sum_{|j| \leq m} U_{i,j}^*(0)f \right\|_{L^2} = 0.$$

We now extend this conclusion to any $t$. Given $f \in L^2$, we want to have

$$\lim_{l \to \infty} \sup_{m \to \infty} \left\| \sum_{|j| \leq m} U_{i,j}^*(t)f \right\|_{L^2} = \lim_{l \to \infty} \sup_{m \to \infty} \sum_{|j| \leq m} \langle U_{i,j}(t)U_{i,j}^*(t)f, f \rangle = 0.$$
5. DISPERITIVE ESTIMATES FOR SCHRODINGER PROPAGATORS

**Proposition 5** (Dispersive estimates I). The long time dispersive estimates for the microlocalized Schrödinger propagators at high energy

\[
\left| \int_0^\infty e^{it\lambda^2} \chi_\infty(Q_k(\lambda)dE_P(\lambda)Q_k^*(\lambda))(z,z') \, d\lambda \right| \leq C|t|^{-n}e^{-nd(z,z')/2}
\]

hold, provided \( t > 1 + d(z,z') \). The low energy truncated propagator obeys

\[
\left| \int_0^\infty e^{it\lambda^2} \chi_{\text{low}}(\lambda) \chi_d(\lambda) dE_P(\lambda, z, z') \, d\lambda \right| \leq C|t|^{-3/2}(1 + d(z,z'))e^{-nd(z,z')/2}
\]

for all times. On the other hand, we have short time dispersive estimates for the high energy truncated propagator microlocalized near the diagonal

\[
\left| \int_0^\infty e^{it\lambda^2} \chi_\infty(Q_k(\lambda)dE_P(\lambda)Q_k^*(\lambda))(z,z') \, d\lambda \right| \leq C|t|^{-(n+1)/2}(1 + d(z,z'))^{n/2}e^{-nd(z,z')/2}
\]

provided \( t < 1 + d(z,z') \).

**Remark 6.** If we work on a manifold without conjugate points, this result will reduce to the dispersive estimates \([1,3]\) on hyperbolic space, where the microlocalization is needless. Moreover, for short time estimates, say \( t < 1 + d(z,z') \), we can combine (5.2) and (5.3).

**Proof of (5.1).** Let us look at the long time dispersion first. Because we want to use stationary phase estimates, we have to split the amplitude of the microlocalized propagator into functions compactly supported in \( \lambda \). To do so, we select a bump function \( \phi \in \mathcal{C}_c^\infty[1/2, 2] \) such that \( \sum_j \phi(2^{-j}\lambda) = 1 \) and let \( \phi_0(\lambda) = \sum_{j \leq 0} \phi(2^{-j}\lambda) \). Then the Schrödinger propagator is decomposed as \( I_0 + \sum_{j > 0} I_j \), which is

\[
I_0 = \int_0^\infty e^{it\lambda^2} \chi_\infty(\lambda)Q_k(\lambda)dE_P(\lambda)Q_k^*(\lambda)\phi_0(\lambda) \, d\lambda
\]

\[
I_j = \int_0^\infty e^{it\lambda^2} \chi_\infty(\lambda)Q_k(\lambda)dE_P(\lambda)Q_k^*(\lambda)\phi(2^{-j}\lambda) \, d\lambda.
\]

- **Case 1:** \( d(z,z') \leq 1 \)

As \( t \) goes to infinity, the phase function is \( \lambda^2 \) which is clearly non-degenerate at the stationary point \( \lambda = 0 \).

Noting 0 is not on the support of \( \chi_\infty \), we have \( I_0 = O(t^{-\infty})e^{-d(z,z')} \). On the other hand, noting the phase function of the \( I_j \) terms are non-stationary, we deduce

\[
\sum_{j > 0} |I_j| = \sum_{j > 0} \left| \int_0^\infty \left( \frac{1}{2it\lambda} \frac{d}{d\lambda} \right)^N (e^{it\lambda^2})dE_P(\lambda)\phi(2^{-j}\lambda) \, d\lambda \right|
\]

\[
\leq C \sum_{j > 0} t^{-N}e^{-nd(z,z')/2} \int_{2^{-j-1}}^{2^{-j+1}} \lambda^{n-2N} \, d\lambda \leq Ct^{-N}e^{-nd(z,z')/2}.
\]

Let \( N \) go to \( \infty \) to finish the proof of this case.

- **Case 2:** \( d(z,z') \geq 1 \)

Since \( d(z,z') \) goes to \( \infty \) as well as \( t \), the phase function consists of not only \( \lambda^2 \) but also some other term coming from the spectral measure. The outgoing and incoming parts spectral measure contribute the oscillatory terms \( e^{-it\lambda d(z,z')} \) and \( e^{it\lambda d(z,z')} \). So the new phase function will be \( t\lambda^2 \mp \lambda d(z,z') \). In the incoming case, such phase function isn’t stationary. Then we can select the bump function \( \phi \) as above and get compactly supported amplitudes. Noting the support of \( \phi_0 \)
isn’t intersected with $\chi_{\infty}$, we can obtain the dispersive estimates by running the same argument of non-stationary phase and integration by parts

$$
\sum_{j>0} |I_j| = \sum_{j>0} \left| \int_0^\infty \frac{1}{2it \lambda + id(z, z')} \frac{d}{d\lambda} \right|^N (e^{it\lambda^2 + id(z, z') \lambda}) \phi(2^{-j} \lambda) d\lambda 
$$

$$
\leq C \sum_{j>0} t^{-N} \int_{2^{j-1}}^{2^{j+1}} \lambda^{n-2N} e^{-\lambda d(z, z')/2} d\lambda \leq Ct^{-N} e^{-\lambda d(z, z')/2},
$$

for any large $N$.

On the other hand, the phase function $t\lambda^2 - \lambda d(z, z')$ is stationary at $\lambda = d(z, z')/(2t)$. Nonetheless $d(z, z')/(2t) < 1$ doesn’t lie on the support of $\chi_{\infty}$ either, we thus can prove the dispersive estimates by the same argument. The proof is now complete.

---

**Proof of [5.2].** It can be deduced from the results of the spectral measure at low energy. We make a change of variable and get

$$
U_{\text{low}} = \int_0^\infty e^{it\lambda^2} \chi_{\text{low}}(\lambda) d\lambda.
$$

We decompose the LHS as $I_0 + I_{\infty}$, where

$$
I_0 = t^{-1/2} \int_0^1 e^{it\lambda^2} \chi_{\text{low}}(t^{-1/2} \lambda) d\lambda.
$$

$$
I_{\infty} = t^{-1/2} \int_1^\infty e^{it\lambda^2} \chi_{\text{low}}(t^{-1/2} \lambda) d\lambda.
$$

We use [2.3] for low energies to estimate $I_0$ as follows

$$
|I_0| = t^{-1/2} \left| \int_0^1 e^{it\lambda^2} \chi_{\text{low}}(t^{-1/2} \lambda) d\lambda \right|
$$

$$
\leq t^{-1/2} \int_0^1 (t^{-1/2} \lambda)^2 (1 + d(z, z')) e^{-\lambda d(z, z')/2} d\lambda
$$

$$
\leq Ct^{-3/2} (1 + d(z, z')) e^{-\lambda d(z, z')/2}
$$

On the other hand, we invoke [2.2] for low energies and adopt an integration by parts argument for $I_{\infty}$

$$
I_{\infty} = t^{-1/2} \int_1^\infty \left( \frac{1}{2it \lambda d\lambda} \right)^2 (e^{it\lambda^2}) \chi_{\text{low}}(t^{-1/2} \lambda) d\lambda.
$$

$$
= \frac{t^{-1}}{4} \int_1^\infty \left( \frac{1}{\lambda d\lambda} \right)^2 (e^{it\lambda^2}) \chi_{\text{low}}(t^{-1/2} \lambda) \left( \rho_{PL} \rho_R \right)^{n/2}
$$

$$
\times \left( \left( \rho_{PL} \rho_R \right)^{it^{-1/2}} a(t^{-1/2} \lambda) - \left( \rho_{PL} \rho_R \right)^{-it^{-1/2}} a(-t^{-1/2} \lambda) \right) d\lambda.
$$

$$
= I_{\infty,1} + I_{\infty,2}
$$

where we write

$$
I_{\infty,1} = \frac{t^{-1} \left( \rho_{PL} \rho_R \right)^{n/2}}{4} \left( \frac{1}{\lambda d\lambda} \right) (e^{it\lambda^2})
$$

$$
\times \left( \left( \rho_{PL} \rho_R \right)^{it^{-1/2}} a(t^{-1/2} \lambda) - \left( \rho_{PL} \rho_R \right)^{-it^{-1/2}} a(-t^{-1/2} \lambda) \right) \bigg|_{\lambda=1}
$$

$$
I_{\infty,2} = \frac{t^{-3/2} \left( \rho_{PL} \rho_R \right)^{n/2}}{4} \int_1^\infty \left( \frac{1}{\lambda d\lambda} \right) (e^{it\lambda^2})
$$
Now it suffices to prove both (5.3) and (5.4). We begin with the proof of (5.3). If \( t < M \) provided \( M \) is sufficiently large,

\[
|I_{\infty, 1}| \leq t^{-1} e^{-ndz(z', z'')/2} \leq CM^{1/2} t^{-3/2} e^{-ndz(z', z'')/2}.
\]

On the other hand, if \( t > M \) (i.e. \( t^{-1/2} \) is very small), we then use the smoothness of \( a \) at 0 and obtain

\[
\left( (\rho L \rho R)^{-it^{-1/2}} a(t^{-1/2}) \right) - (\rho L \rho R)^{-it^{-1/2}} a(-t^{-1/2}) \leq Ct^{-1/2}.
\]

Consequently, we conclude that

\[
|I_{\infty, 1}| \leq Ct^{-3/2} e^{-ndz(z', z'')/2}.
\]

For \( I_{\infty, 2} \), by (2.1), we observe the part of the integrand contained in the parentheses is bounded by \( C(1 + d(z,z')) \). We take integration by parts one more time and get

\[
I_{\infty, 2} \leq Ct^{-3/2} (1 + d(z,z')) e^{-ndz(z', z'')/2} \left( C + \frac{1}{1} \int_{1}^{\infty} \lambda^{-2} d\lambda \right)
\]

which completes the proof.

**Proof of (5.3).** Because of the distinction between the long and short distance, we discuss the two cases separately. In particular, the exponential decay is negligible in case of short distance, as \( e^{-ndz(z', z'')/2} \) is bounded from below. The proof of (5.3) in case of small distance is the same with the proof of the dispersive estimates on asymptotically conic manifolds by Hassell and Zhang \[14\], as the spectral measure for small \( d(z,z') \) and large \( \lambda \) obeys the same estimates as on asymptotically conic manifolds. The idea for both long distance and short distance is to perform an appropriate dyadic decomposition over the value of the derivative of the phase function for an integration by parts argument. We only give the proof for long distance for the distinctive exponential decay in \( d(z,z') \).

First of all, we rescale the microlocalized high energy truncated propagator \( U_k \) as follows

\[
U_k = \int_{0}^{\infty} e^{it\lambda^2} \chi_\infty \left( \frac{Q_k(\lambda) dE_P(\lambda) Q_\lambda^*(\lambda)}{2} \right) (z,z') \, d\lambda
\]

providing \( t < 1 + d(z,z') \). Applying Proposition 2 for high energies, we write

\[
U_k = t^{-(n+1)/2} T_+ + t^{-(n+1)/2} T_-
\]

where

\[
T_+ = \int_{0}^{\infty} e^{i(\lambda^2 t^{-1/2} \lambda d(z,z'))} \lambda^{-n} a_+ (t^{-1/2} \lambda, z,z') \, d\lambda
\]

\[
T_- = \int_{0}^{\infty} e^{i(\lambda^2 - t^{-1/2} \lambda d(z,z'))} \lambda^{-n} a_- (t^{-1/2} \lambda, z,z') \, d\lambda
\]

with smooth function \( a_\pm (\lambda, z,z') \) on \([1, \infty) \times X_0^* \) obeying

\[
\left| \frac{d}{d\lambda} a_\pm (t^{-1/2} \lambda, z,z') \right| = O \left( t^{n/4} \lambda^{-n/2-j} e^{-ndz(z', z'')/2} \right) \quad \text{if} \quad d(z,z') \text{is large.}
\]

Now it suffices to prove both \( T_+ \) and \( T_- \) are bounded by \((1 + d(z,z'))^{n/2} e^{-ndz(z', z'')/2} \).
We decompose the $T_+$ term further into $\sum_{j \geq 0} T_{j,+}$, where

$$T_{j,+} = \int_0^\infty e^{i(\lambda^2 + t^{-1/2} \lambda d(z,z'))} \lambda^n a_+ (t^{-1/2} \lambda, z, z') \phi(2^{-j} \lambda) d\lambda \quad \text{for } j > 0$$

$$T_{0,+} = \int_0^\infty e^{i(\lambda^2 + t^{-1/2} \lambda d(z,z'))} \lambda^n a_+ (t^{-1/2} \lambda, z, z') (1 - \sum_{j > 0} \phi(2^{-j} \lambda)) d\lambda,$$

where we denote, by a partition of unity $\sum_j \phi(2^{-j} \lambda) = 1$ with $\phi \in C^\infty_0[1/2, 2]$. It is clear that $T_{0,+}$ is bounded by $(1 + d(z, z'))^{n/2} e^{-nd(z, z')/2}$ after a quick application of Proposition 2. On the other hand, for each $T_{j,+}$, the phase function of this oscillatory integral is actually non-stationary. One thus can insert a differential operator $N$ times leaving the exponential term invariant to take integration by parts

$$|T_{j,+}| = \left| \int_0^\infty \frac{1}{i(2\lambda + t^{-1/2} d(z,z'))} \frac{\partial}{\partial \lambda} e^{i(\lambda^2 + t^{-1/2} \lambda d(z,z'))} \lambda^n a_+ (t^{-1/2} \lambda, z, z') \phi(2^{-j} \lambda) d\lambda \right|$$

$$\leq C \int_{|\lambda| \sim 2} t^{n/4} e^{-nd(z,z')/2} \lambda^{n/2 - 2N} d\lambda$$

$$\leq C (1 + d(z, z'))^{n/2} e^{-nd(z, z')/2} \int_{|\lambda| \sim 2} \lambda^{n/2 - 2N} d\lambda.$$

The sum of $T_{j,+}$ is clearly convergent if we make $N$ sufficiently large.

For the term $T_{-}$, the phase function may be stationary, we have to make a subtler decomposition. One may rewrite the integral as $T_{-} = \sum_{k \geq 0} T_{k,-}$, where

$$T_{0,-} = \int_0^\infty e^{i\lambda^2 - it^{-1/2} \lambda d(z,z')} \lambda^n a_- (t^{-1/2} \lambda, z, z') \sum_{k \leq 0} \psi_k(\lambda) d\lambda$$

$$T_{k,-} = \int_0^\infty e^{i\lambda^2 - it^{-1/2} \lambda d(z,z')} \lambda^n a_- (t^{-1/2} \lambda, z, z') \psi_k(\lambda) d\lambda, \quad k > 0$$

$$\psi_k(\lambda) = \phi (2^{-k} |2\lambda - t^{-1/2} d(z, z')|).$$

If we plug the estimates for $a_-$ in $T_{0,-}$, we will have $T_{0,-}$ bounded by

$$(1 + d(z, z'))^{n/2} e^{-nd(z, z')/2} \int_0^\infty \frac{t^{n/4}}{(1 + d(z, z'))^{n/2} \lambda^{n/2}} \lambda^n \sum_{k \leq 0} \psi_k(\lambda) d\lambda.$$

The latter integral is convergent. In fact, if $t^{-1/2} d(z, z')$ is bounded, $\lambda$ will also be bounded, because of

$$\text{supp} \left( \sum_{k \leq 0} \psi_k \right) = \{|2\lambda - t^{-1/2} d(z, z')| \leq 2\}.$$  

Because of the large distance (large $d(z, z')$), the short time (small $t$), and the high energy (large $\lambda$), the fraction in the integrand is also bounded on the domain. So the $\lambda$-integration is convergent. If $t^{-1/2} d(z, z')$ is large, the restriction $|2\lambda - t^{-1/2} d(z, z')| \leq 2$ from the support of $\sum_{k \leq 0} \psi_k$ implies $\lambda \sim t^{-1/2} d(z, z')$. Consequently, for any value of $t^{-1/2} d(z, z')$, we have

$$\frac{t^{n/4}}{(1 + d(z, z'))^{n/2} \lambda^{n/2}} \lambda^n \leq C$$

Then the integral $T_{0,-}$ is bounded by

$$C \int_{|2\lambda - t^{-1/2} d(z, z')| < 2} \sum_{k \leq 0} \psi_k d\lambda \leq C.$$
For the $T_{k,-}$ terms, since $|2\lambda - t^{-1/2}d(z, z')| > 1$, namely the phase function is non-stationary, we can employ the integration by parts argument. We denote

$$L_- = \frac{1}{2\lambda - t^{-1/2}d(z, z')} \frac{d}{d\lambda}$$

and get following estimates for $\sum_{k>0} T_{k,-}$

$$\sum_{k>0} \int_0^\infty e^{i(\lambda^2 - t^{-1/2}d(z, z')\lambda)} \lambda^n a_-(t^{-1/2} \lambda, z, z') \psi_k(\lambda) d\lambda$$

$$= \sum_{k>0} \int_0^\infty L_N(\lambda) \lambda^n a_-(t^{-1/2} \lambda, z, z') \psi_k(\lambda) d\lambda$$

$$\leq C e^{-nd(z, z')/2} \int_0^\infty \lambda^{n/2-N} d\lambda$$

$$\leq C(1 + d(z, z'))^{n/2} e^{-nd(z, z')/2},$$

provided $N$ is large enough.

\[\square\]

Remark 7. A key point of this integration by parts argument in this proof is that we can have a non-stationary phase function in the oscillatory integral. To get the dispersive estimates for the propagator $U_i(t)U_j^*(s)$, we will get a non-stationary phase function and run this argument again.

6. Refined dispersive estimates for Schrödinger propagators

Because we employ Keel-Tao bilinear approach to prove Strichartz estimates, distinct double microlocalized spectral measure $Q_dE_PQ_k^*$ will confront us. Before stating distinct double microlocalized dispersive estimates, we have to understand all possible relations between two distinct microlocalizations.

This was discussed by Guillarmou and Hassell [11] for Sobolev estimates, which is closely related to Strichartz estimates. Let us review their notions of outgoing / incoming relations. Suppose $g'$ is the geodesic/bicharacteristic flow and $Q, Q'$ are two semiclassical pseudo differential operators of semiclasicical order 0 and differential order $-\infty$. We say $Q$ is not outgoing related to $Q'$ if the forward flowout $g'(WF_h(Q'))$ with $t \geq 0$ doesn’t meet $WF_h(Q)$, whilst $Q$ is not incoming related to $Q'$ if the backward flowout $g'(WF_h(Q'))$ with $t \leq 0$ doesn’t meet $WF_h(Q)$. It is useful to note $Q$ not incoming related to $Q'$ is equivalent to $Q'$ not outgoing related to $Q'$.

Proposition 8 (Dispersive estimates II). There is a refined pseudo differential operator partition of unity $Id = \sum_{k=0}^N Q_k$ such that

\[\int_0^\infty e^{i(t-s)\lambda^2} \chi_\infty (Q_j(\lambda)dE_P(\lambda)Q_k^*(\lambda))(z, z') d\lambda \leq C|t-s|^{-\infty} e^{-nd(z, z')/2} \quad \text{for } t > 1 + d(z, z') \]

\[\int_0^\infty e^{i(t-s)\lambda^2} \chi_\infty (Q_j(\lambda)dE_P(\lambda)Q_k^*(\lambda))(z, z') d\lambda \leq C|t-s|^{-(n+1)/2(1 + d(z, z'))^{n/2} e^{-nd(z, z')/2} \quad \text{for } t < 1 + d(z, z'), \]

hold for all $t \neq s$ if $WF_h(Q_j) \cap WF_h(Q_k) \neq \emptyset$, for $t < s$ if $Q_j$ is not outgoing related to $Q_k$, for $s < t$ if $Q_j$ is not incoming related to $Q_k$.

Before proving the dispersive estimates, we have to refine the microlocalization for spectral measure and categorize the double microlocalization $\{(Q_j, Q_k^*)\}_{j,k=1}^N$, where $Q_0$ is neglected as it is not on the semiclassical wavefront set of spectral measure.

Lemma 9. The microlocalization pair $(Q_j, Q_k^*)$ with $j, k \geq 1$ must obey one of the following relations:
Proof. \(\epsilon\) is compact and geodesically convex. It is true as long as \(Q\) is sufficiently small. Also recall the cubes \(Q_1, \ldots, Q_{N_j}\) supported on \(B_1, \ldots, B_{N_j}\), affiliated with \(I_1, \ldots, I_{N_j}\), over \(\{x \leq \epsilon\}\) and \(Q_{N_j+1}, \ldots, Q_{N_j}\) supported on \(B_{N_j+1}, \ldots, B_{N_j}\) over \(\{x > \epsilon\}\). Now we also assume \(\{x \geq 2\epsilon\}\) is compact and geodesically convex. It is true as long as \(\epsilon\) is sufficiently small.

Therefore there are following cases of microlocalization:

1. \((Q_j, Q^*_k)\) with \(B_j \cap B_k \neq \emptyset, 1 \leq j \leq N_1\) and \(1 \leq k \leq N_1\)

Since \(B_j\) is intersected with \(B_k\), then \(I_j\) is intersected with \(I_k\). One may prescribe \(k = j + 1\). Note both \(I_j\) and \(I_{j+1}\) are small subintervals in \([-3/2, 3/2]\) and they are contained in \(I_j \cup I_{j+1}\). Then one can find a slice \(B_{j,j+1}\) in \(\{x < \epsilon\} \cap \{\lambda \in I_j \cup I_{j+1}\}\). Since \(I_j \cup I_{j+1}\) is a small interval in \([-3/2, 3/2]\), we can find a pseudodifferential operator \(Q_{j,j+1}\) microlocally supported on \(B_{j,j+1}\) such that \(Q_{j,j+1}dE_tQ_{j,j+1}^*\) satisfies (6.3). So does \(Q_jdE_tQ^*_j\).

2. \((Q_j, Q^*_k)\) with \(B_j \cap B_k \neq \emptyset, N_1 \leq j \leq N_2\) and \(1 \leq k \leq N_2\)

Since the diameter of \(B_j \cup B_k\) is bounded by a very small number, \(B_j\) and \(B_k\) are contained in a very small cube \(B_{j,k}\). Then \(Q_{j,k}dE_tQ_{j,k}^*\), with \(Q_{j,k}\) microlocally supported on \(B_{j,k}\), satisfies (6.3). Consequently, so does \(Q_jdE_tQ^*_j\).

3. \((Q_j, Q^*_k)\) with \(B_j \cap B_k \neq \emptyset, 1 \leq j \leq N_1\) and \(1 \leq k \leq N_2\)

Since the diameter of \(B_k\) is very small in the sense of Sasaki distance, we can narrow the range of \(\lambda\) variable in \(I_j\) and the range of \(x\) variable in \(\{x < 2\epsilon\}\) such that both \(B_k\) and \(B_j\) are contained in a small slice \(B_{j,k}\) near the boundary with \(\{\lambda \in I_j\}\). Then we again find a pseudodifferential operator \(Q_{j,k}\) microlocally supported on \(B_{j,k}\) such that \(Q_{j,k}dE_tQ_{j,k}^*\) satisfies (6.3).

4. \((Q_j, Q^*_k)\) with \(B_j \cap B_k = \emptyset, 1 \leq j \leq N_1\) and \(1 \leq k \leq N_1\)

Recall from [21] or [8] that the 0-Hamilton vector field with Hamiltonian \(p = \lambda^2 + h(y, \lambda, \mu)\) on asymptotically hyperbolic manifold \(X\) is

\[
\frac{\partial p}{\partial \lambda} \frac{\partial}{\partial x} + x \frac{\partial p}{\partial \mu} \frac{\partial}{\partial y} - \bigg( \lambda \frac{\partial p}{\partial \mu} + x \frac{\partial p}{\partial x} \bigg) \frac{\partial}{\partial \lambda} + \bigg( \frac{\partial p}{\partial \lambda} \mu - \frac{\partial p}{\partial y} \bigg) \cdot \frac{\partial}{\partial \mu}.
\]

The variable \(\lambda\), along the geodesic, decreases down to \(-1\), in a small neighbourhood of the boundary.

Without loss of generality, one may assume \(\inf(I_j) > \sup(I_k)\). Take a geodesic \(\gamma(t)\) with \(\gamma(0) \in B_k\). If \(\gamma(t)\) stays in \(\{x < \epsilon\}\) for \(t \geq 0\), \(\gamma(t) : t \geq 0\) will be disjoint from \(B_j\), since \(\lambda\) is nonincreasing along the forward bicharacteristic near the boundary. On the other hand, if \(\gamma(t)\) goes beyond \(\{x < \epsilon\}\) at time \(t_2\), i.e. \(\gamma(t_2) \in \{x \geq \epsilon\}\) we have \(\lambda(0) > 0\), hence \(\inf(I_j) > \sup(I_k) > 0\). So we can find a maximal interval \((t_1, t_3)\) containing \(t_2\) on \(\{x \geq \epsilon\}\) such that \(\lambda(t) > 0\) for all \(t < t_1\) and \(\lambda(t) < 0\) for all \(t > t_3\), since \(\lambda\) is nonincreasing in \(\{x < \epsilon\}\). Consequently, \(\gamma\) is disjoint from \(B_j\) whenever \(t > 0\): when \(0 < t_2 < t_1\) i.e. \(\lambda < \lambda(0) < \inf(I_j)\); when \(t_2 < t < t_1\) i.e. \(\lambda < \lambda(0) < \inf(I_j)\).
(5) \((Q_j, Q_k^*)\) with \(B_j \cap B_k = \emptyset\), \(1 \leq j \leq N_1\) and \(N_1 \leq k \leq N_2\).

Take a geodesic \(\gamma\) with \(\gamma(0) \in B_j\). If \(\sup I_j < 0\), then \(x(t)\) is non-increasing namely \(\gamma(t)\) will stay in \(\{x < \epsilon\}\) for \(t > 0\) and be disjoint from \(B_k\). In the meantime, if \(\inf I_j > 0\), \(\gamma(t)\) will stay in \(\{x < \epsilon\}\) for \(t < 0\). If \(0 \in I_j\) and \(\lambda(t_0) = 0\), \(x(t)\) is nonincreasing for all \(t > t_0\), since \(\lambda\) is non-positive afterwards. So \(\gamma(t)\) will stay in \(\{x < \epsilon\}\) for all \(t > t_0\).

(6) \((Q_j, Q_k^*)\) with \(B_j \cap B_k = \emptyset\), \(N_1 \leq j \leq N_2\) and \(N_1 \leq k \leq N_2\).

Consider the function \((z, t) \to x(g^t(z))\). Since \(dx(g^t(z))/dt \neq 0\) locally in \(\{x > \epsilon/2\}\), we apply implicit function theorem and get an implicit function \(t(z)\). We can find a time \(t(z)\) such that \(x(g^t(z)) = \epsilon/2\). Therefore, for any compact set \(K \subset \{|\zeta| - 1| \leq \delta\} \cap \{x > \epsilon/2\}\), there is a \(T_+ > 0\) respectively \(T_- < 0\) such that \(g^t(K) \subset \{x < \epsilon/2\}\) for all \(t > T_+\) respectively \(t < T_-\). Assuming \(B_j\) and \(B_k\) are outgoing related and incoming related, we shall get a contradiction. Under this hypothesis, there exist two sequences of points \(\{z_l\} \subset B_j \cap K\) and \(\{z^*_l\} \subset B_j \cap K\) with two sequences of times \(\{t_l\}\) and \(\{t^*_l\}\) both going to \(B_j\) from \(B_k\) via the two geodesics respectively. Using the compactness of the interior, we can have some accumulation points of the two sequences in \(K\). Let the diameter of \(K\) shrink. The two sequences will be converging to the same point \(z \in \{x > \epsilon/2\} \cap \{|\zeta| - 1| \leq \delta\}\) and such that \(\lim_{l \to \infty} g^{t_l}(z_l) = \lim_{l \to \infty} g^{t^*_l}(z^*_l)\).

Since \(T_- < t < t' \leq T_+\), we can find accumulation points \(t\) and \(t'\) respectively. Then we have \(g^t(z) = g^{t'}(z)\). It gives a periodic geodesic which contradicts the non-trapping condition. Therefore we have either \(Q_j\) is not outgoing related to \(Q_k\) or \(Q_j\) is not incoming related to \(Q_k\).

Based on this refined microlocalization, we can prove the dispersive estimates for double microlocalized high energy truncated Schrödinger propagators.

\(\square\)

Proof of Proposition\(^3\) It is proved by the argument of Proposition\(^5\) with minor changes based on the classification of microlocalizations in Lemma\(^9\).

If \(Q_j, dE_P(\lambda) Q_k^*\) obeys \(\text{(6.3)}\), we can get desired dispersive estimates by repeating the proof of Proposition\(^5\).

If \(Q_j\) and \(Q_k\) are not outgoing related\(^4\) we claim the \(U_j(t) U^*_k(s)\) for \(t < s\) is a Fourier integral operator

\[
\int_0^\infty \int_{\mathbb{R}^N} e^{i(t-s)\lambda^2 + \lambda \phi(\omega, h, z', \theta)} d\omega d\lambda,
\]

provided \(\phi(z, z', \theta) < -\epsilon < 0\) is the phase function of the wavefront set \(\Lambda\) of the spectral measure. Since we can always write the propagator in this integral form, the only point we need to justify is that \(\phi(z, z', \theta) < -\epsilon < 0\).

Recall that the forward bicharacteristic flow-out is that the flow-out of the Hamilton vector field of the metric function. By standard theory of Lagrangian distributions, the phase function \(\phi\) can parametrize the forward flow-out in the following way that is \(\Lambda^+\) is locally furnished coordinates

\(\{(z, \phi^t_0)|\phi^t_0 = 0\}\).

Hence phase function \(\phi\) of forward bicharacteristic flow-out \(\Lambda^+\) satisfies

\[
\phi(z, z', \theta) = r(z, z') \geq d(z, z'), \quad \text{when } \phi_0 = 0,
\]

where \(r\) is the curve length along the bicharacteristic and \(d\) is the geodesic distance. Since \(Q_j\) is not outgoing related to \(Q_k\), i.e. the forward geodesic flow-out of \(WF_h Q_k^*\) doesn’t meet \(WF_h Q_j\), they are connected by the backward flow-out, namely

\[
\phi = -r(z, z') \leq -d(z, z') < 0.
\]

The not outgoing relation gives a constantly negative sign of the phase function \(\phi\) of microlocalized spectral measure \(Q_j(\lambda)dE_P(\lambda) Q_k^*(\lambda)\). Since \(t - s < 0\), the phase function of the propagator is

\(^4\)The proof is exactly the same in case \(Q_j\) and \(Q_k\) are not incoming related.
negative. So it allows us to play the integration by parts argument in Proposition 5 by the differential operator
\[ -i\frac{2\lambda - \phi / \sqrt{s - t}}{\partial \lambda} \]
to get the prove (6.2), instead of \(-i/(2\lambda - t^{-1/2}d(z, z'))\partial \lambda \) in the proof of (5.3). On the other hand, noting \(\phi \) and \(t - s\) have the same sign, namely the phase is non-stationary, we apply the rapid decay estimates, which really shows (6.1).

\[ \square \]

7. Strichartz estimates

We turn to proving Theorem 1.

First of all, we shall establish the Strichartz estimates
\[ \|u\|_{L^q(\mathbb{R}, L^r(X))} \leq C\|f\|_{L^2(X)} \]
for the homogeneous equations (i.e. \(F \equiv 0\)). Recall the low energy truncated propagator and high energy microlocalized propagators. The solution \(u\) of the homogeneous equation reads
\[ e^{itm^2/4}u(t, x) = \left( U_{\text{low}}(t) + \sum_{j=0}^{N} U_j(t) \right) f(z) \]
for \(0 < t < 1\), where
\[ U_{\text{low}}(t) = \int_0^\infty e^{it\lambda^2} \chi_{\text{low}}(\lambda) dE_P(\lambda) \quad \text{and} \quad U_j = \int_0^\infty e^{it\lambda^2} \chi_{\infty}(\lambda) Q_j(\lambda) dE_P(\lambda). \]

The Strichartz estimates for homogeneous equations
\[ \|e^{itP^2}f(z)\|_{L_t^q L_x^r} \leq C\|f\|_{L_x^2} \]
are equivalent to
\[ \left\| \int e^{-isP^2} G(s, z) ds \right\|_{L_t^2} \leq C\|G\|_{L_t^1 L_x^r}. \]

Noting the decomposition
\[ e^{t(-s)P^2} = U_{\text{low}}^*(s) + \sum_{j=0}^{N} U_j^*(s), \]
it suffices to show
\[ \left\| \int U_k^*(s)G(s, z) ds \right\|_{L_t^2} \leq C\|G\|_{L_t^1 L_x^r}, \]
where \(k \in \{0, 1, \ldots, N, \text{low}\}\). By \(TT^*\), it is equivalent to
\[ \left\| \int (U_k(t)U_k^*(s)F(s, z)) ds \right\|_{L_t^q L_x^r} \leq C\|F\|_{L_t^q L_x^r}. \]

One can split the left hand side by time. The long time part reduces to
\[ (7.1) \quad \left( \int \left\| \int_{|t-s| \geq 1} U_k(t)U_k^*(s)F(s, z) ds \right\|_{L_x^q} dt \right)^{1/q}, \]
in the meantime, the short time part reduces to
\[ (7.2) \quad \left( \int \left\| \int_{|t-s| \leq 1} U_k(t)U_k^*(s)F(s, z) ds \right\|_{L_x^q} dt \right)^{1/q}. \]

To estimate these integrals, we need following mapping properties of the propagators, which we shall prove in the last section,
Lemma 10 (Long times). Suppose \(|t - s| \geq 1\) and \(2 < r, \tilde{r} \leq \infty\). Then the following inequalities hold

\[
\|U_{low}(t)U_{low}^{*}(s)\|_{L^{r'}_{\tilde{r}'} \rightarrow L^{r}_{\tilde{r}}} \leq C|t - s|^{-3/2}
\]
\[
\|U_{j}(t)U_{k}^{*}(s)\|_{L^{r'}_{\tilde{r}'} \rightarrow L^{r}_{\tilde{r}}} \leq C|t - s|^{-3/2}
\]
where the last one only holds for either \(t - s > 1\) or \(s - t > 1\) if \(j \neq k\) and for both if \(j = k\).

Lemma 11 (Short times). Suppose \(0 < |t - s| < 1\) and \(2 < r, \tilde{r} \leq \infty\). Then the following inequalities hold

\[
\|U_{low}(t)U_{low}^{*}(s)\|_{L^{r'}_{\tilde{r}'} \rightarrow L^{r}_{\tilde{r}}} \leq C|t - s|^{-\max\{1/2 - 1/r, 1/2 - 1/\tilde{r}\}(n+1)}
\]
\[
\|U_{j}(t)U_{k}^{*}(s)\|_{L^{r'}_{\tilde{r}'} \rightarrow L^{r}_{\tilde{r}}} \leq C|t - s|^{-\max\{1/2 - 1/r, 1/2 - 1/\tilde{r}\}(n+1)}
\]
where the last one only holds for either \(0 < t - s < 1\) or \(0 < s - t < 1\) if \(j \neq k\) and for both if \(j = k\).

Assuming these lemmas for the moment, we now continue the proof of Strichartz estimates.

We insert (7.4) and (7.3) into (7.1) and get

\[
\left( \int \left( \int_{|t-s| \geq 1} \|U_{k}(t)U_{k}^{*}(s)F(s, z)\|_{L^{r}_{\tilde{r}}} \, ds \right)^{q} \, dt \right)^{1/q} \leq C \left( \int \left( \int_{|t-s| \geq 1} |t-s|^{-3/2} \|F(s, z)\|_{L^{r'}_{\tilde{r}'}} \, ds \right)^{q} \, dt \right)^{1/q}
\]
\[
\leq \|F(s, z)\|_{L^{r'}_{\tilde{r}'} L^{r}_{\tilde{r}}}
\]

We remark the kernel \(|t-s|^{-3/2} \chi_{|t-s| > 1}\) is integrable so it maps \(L^q(\mathbb{R})\) to \(L^q(\mathbb{R})\) for any \(q \geq 2\), where no admissibility is needed.

On the other hand, one can use the short time estimates (7.5) and (7.6). For \((q, r) \neq (2, 2(n+1)/(n-1))\), we invoke the admissibility condition (1.2) and Hardy-Littlewood-Sobolev inequality

\[
\left( \int \left( \int_{|t-s| \leq 1} \|U_{k}(t)U_{k}^{*}(s)F(s, z)\|_{L^{r}_{\tilde{r}}} \, ds \right)^{q} \, dt \right)^{1/q} \leq C \left( \int \left( \int_{|t-s| \leq 1} \frac{1}{|t-s|^{(1/2+1/r)(n+1)}} \|F(s, z)\|_{L^{r'}_{\tilde{r}'} \rightarrow L^{r}_{\tilde{r}}} \, ds \right)^{q} \, dt \right)^{1/q}
\]
\[
\leq C \|F(s, z)\|_{L^{r'}_{\tilde{r}'} L^{r}_{\tilde{r}}}
\]

Here the last inequality requires \(q < 2\), which is invalid for endpoints.

The short time endpoint estimates are proved via dispersive estimates and energy estimates by the standard Kato-Tao argument.

Next, following an argument from [14], we prove the inhomogeneous Strichartz estimates by the homogeneous estimates we have proved, that is

\[
\|e^{iP^{2}}f(z)\|_{L^{r}_{\tilde{r}}L^{r'}_{\tilde{r}'}} \leq C\|f\|_{L^{r}_{\tilde{r}}}
\]
provided \((q, r)\) satisfies (1.2). These estimates are equivalent to

\[
\left\| \int e^{i(t-s)P^{2}} F(s) \right\|_{L^{r}_{\tilde{r}}L^{r'}_{\tilde{r}'}} \leq C\|F\|_{L^{r'}_{\tilde{r}'} L^{r}_{\tilde{r}}}
\]
provided \((\tilde{q}, \tilde{r})\) also satisfies (1.2). By Duhamel’s formula, the desired inhomogeneous Strichartz estimates are equivalent to the retarded estimates

\[
\left\| \int_{s<t} e^{i(t-s)P^{2}} F(s) \right\|_{L^{r}_{\tilde{r}}L^{r'}_{\tilde{r}'}} \leq C\|F\|_{L^{r'}_{\tilde{r}'} L^{r}_{\tilde{r}}}
\]
For the non-endpoint case i.e. neither of $(q,r)$ and $(\tilde{q},\tilde{r})$ is $(2,2(n+1)/(n-1))$, the retarded estimates follow immediately from Christ-Kiselev lemma

**Lemma 12** ([8]). Let $X,Y$ be Banach spaces, let $I$ be a time interval, let $K \in C^0(I \times I)$ be a kernel taking values in the space bounded operators from $X$ to $Y$. Suppose that $1 \leq p < q \leq \infty$ and

$$
\left\| \int K(t,s)f(s)\,ds \right\|_{L^q(I \rightarrow Y)} \leq C \|f\|_{L^p(I \rightarrow X)}.
$$

Then one has

$$
\left\| \int_{\{s \leq t, s< t\}} K(t,s)f(s)\,ds \right\|_{L^q(I \rightarrow Y)} \leq C \|f\|_{L^p(I \rightarrow X)}.
$$

On the other hand, in order to establish the endpoint inhomogeneous estimates, we end up with following bilinear estimates

$$
\begin{align*}
\int_{s<t} (e^{i(t-s)F(s)},G(t)) \, ds dt &\leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x. \\
\int_{s<t} (U_{j}(t)U_{j}\ast_k(s)F(s),G(t)) \, ds dt &\leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x. \\
\int_{s<t} (U_{low}(t)U_{low}\ast_k(s)F(s),G(t)) \, ds dt &\leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x.
\end{align*}
$$

We want to use the standard Keel-Tao endpoint argument ([17, Section 7]). So we have to establish the dispersive estimates and energy estimates for these propagators. The energy estimates are proved in Proposition 3. The dispersive estimates for the dispersive estimates and energy estimates for these propagators. The energy estimates are

$$(7.8) \quad \int_{s<t} (U_{j}(t)U_{j}\ast_k(s)F(s),G(t)) \, ds dt \leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x.$$  

We only have the dispersive estimates for $U_{j}(t)U_{j}\ast_k(s)$ when $t < s$ in the case of $Q_{j}$ is not outgoing related to $Q_{k}$, though (7.8) is proved as above for other cases. Namely, what we can prove by the Keel-Tao argument when $Q_{j}$ is not outgoing related to $Q_{k}$ is that

$$
\int_{t<s} (U_{j}(t)U_{j}\ast_k(s)F(s),G(t)) \, ds dt \leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x.
$$

Nevertheless, noting that the homogeneous Strichartz estimates, by duality, implies

$$
\left\| \int (U_{j}(t)U_{j}\ast_k(s)F(s),G(t)) \, ds dt \right\|_{L^2_tL^{r}\prime}_x \leq C \|F\|_{L^2_tL^{q}\prime}_x \|G\|_{L^2_tL^{r}\prime}_x.
$$

So we still obtain (7.8).

8. Mapping properties of Schrödinger propagators

It remains to prove Lemma 10 and Lemma 11.

**Proof of (7.5) and (7.6).** The short time behaviour (7.5) and (7.6) come from the interpolation among

$$(8.1) \quad \|\cdot\|_{L^1 \rightarrow L^{r}} \leq Ct^{-(n+1)/2} \quad \text{for any } r > 2,$$

$$(8.2) \quad \|\cdot\|_{L^{r} \rightarrow L^{\infty}} \leq Ct^{-(n+1)/2} \quad \text{for any } r > 2,$$

$$(8.3) \quad \|\cdot\|_{L^2 \rightarrow L^2} \leq C.$$

The last one (8.3) is indeed Proposition 3, while we shall prove (8.1) and (8.2), via dispersive estimates, by a comparison argument with hyperbolic space. The identical argument is used to show the restriction theorem in [7].
Observing the RHS of the short time dispersive estimates in Proposition 3, let us consider the kernel
\[ K_t(z, z') = t^{-(n+1)/2}(1 + d(z, z'))^n e^{-nd(z, z')} \]
instead of the propagators. We claim, for \( r > 2 \),
\[ \|K_t\|_{L^1 \to L^r} = \sup_{z} \|K_t\|_{L^r_z} \leq t^{-(n+1)/2}. \]
Thinking of \( K_t \) as a function supported on \( X_0^2 \), we decompose it as
\[ K_t = K_t \cdot \chi_U + K_t \cdot (1 - \chi_U), \]
where \( U \) is a small neighbourhood of the front face.

The second part is proved by the fact that \( d(z, z') \) is comparable to \(-\log(xx')\), away from the front face. Then we have
\[
\sup_{z'} \|K_t\|_{L^r_z} = t^{-(n+1)/2} \sup_{z'} \left( \int (1 + d(z, z'))^{nr/2} e^{-nrd(z, z')} \right)^{1/r} \\
\leq Ct^{-(n+1)/2} \int (x) \left( -\log(xx') \right)^{nr} \frac{dx}{x^{n+1}} \\
\leq Ct^{-(n+1)/2}.
\]

On the other hand, consider the spectral measure restricted to \( U \) say \( K_{t, U}(z, z') = K_t \cdot \chi_U \). Before looking into the specific estimate, we shall compare this region with hyperbolic space \( \mathbb{H}^{n+1} \). To do so, one may further decompose the set \( U \) into subsets \( U_i \), where on each \( U_i \), we have \( x \leq \epsilon, x' \leq \epsilon \) and \( d(y, y_i), d(y', y_i) \leq \epsilon \) for some \( y_i \in \partial X \) (where the distance is measured with respect to the metric \( h(y, dy) \) on \( \partial X \)). Choose local coordinates \((x, y)\) on \( X \), centred at \((0, y_i)\) in \( \partial X \), spanning the set \( V_i = \{ x \leq \epsilon, d(y, y_i) \leq \epsilon \} \), and use these local coordinates to define a map \( \phi_i \) from \( V_i \) to a neighbourhood \( V'_i \) of \((0,0)\) in hyperbolic space \( \mathbb{H}^{n+1} \) using the upper half-space model (such that the map is the identity in the given coordinates). The map \( \phi_i \) induces a diffeomorphism \( \Phi_i \) from \( U_i \subset X_0^2 \) to a subset of \( \mathbb{H}^n \), the double space for \( \mathbb{H}^{n+1} \), covering the set \( x \leq \epsilon, x' \leq \epsilon, |y|, |y'| \leq \epsilon \) in this space. Clearly, this map identifies \( \rho_L \) and \( \rho_R \) on \( U_i \) with corresponding boundary defining functions for the left face and right face on \( \mathbb{H}^{n+1} \). We now reduce the kernel to
\[
\phi_i \circ K_{t, U_i} \circ \phi_i^{-1}
\]
as an integral operator on \( \mathbb{H}^{n+1} \). After linking the front face to the hyperbolic case, we now can reduce to the estimate to the hyperbolic case as follows.
\[
\sup_{z'} \|K_t\|_{L^r_z(V_i)} = \sup_{z'} \|\tilde{K}\|_{L^r_z(V'_i)} \leq C|t|^{-(n+1)/2},
\]
where \( \tilde{K} \) is mapped to \( \tilde{K} \) on hyperbolic space and the \( L^r \) norm of \( \tilde{K} \) on hyperbolic space.

\begin{proof}[Proof of (7.3) and (7.4)]
The long time behaviour results from the interpolation among
\[
\|\cdot\|_{L^1 \to L^r} \leq Ct^{-3/2} \quad \text{for any } r > 2, \\
\|\cdot\|_{L^r' \to L^\infty} \leq Ct^{-3/2} \quad \text{for any } r > 2, \\
\|\cdot\|_{L^r' \to L^r} \leq Ct^{-3/2} \quad \text{for any } r > 2,
\]
provided \( t > 1 \).

The proofs of (8.5) and (8.6) are exactly the same with (8.1) and (8.2).

The novelty in the proof of (8.7) is a non-trivial non-Euclidean ingredient called the Kunze-Stein phenomenon, which is named after Kunze and Stein [18]. Specifically, the Kunze-Stein phenomenon on the hyperbolic space \( \mathbb{H}^{n+1} \) at \((2,2)\) is expressed as
\[
\|f \ast F\|_{L^2(\mathbb{H}^{n+1})} \leq C\|f\|_{L^2(\mathbb{H}^{n+1})} \cdot \int_0^\infty |F(\rho)|(1 + \rho)e^{\rho/2}d\rho,
\]

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for any $f, F \in C_0(\mathbb{H}^{n+1})$, provided $F(\rho)$ is a radial function. See Cowling’s work [9] for a general result on semi-simple Lie groups. There is a generalized inequality
\begin{equation}
\|f \ast F\|_{L^r(\mathbb{H}^{n+1})} \leq C\|f\|_{L^r(\mathbb{H}^{n+1})} \left( \int_0^\infty |F(\rho)|^{r/2} (1 + \rho)^{n\rho/2}\,d\rho \right)^{2/r}, \quad \text{for } r \geq 2
\end{equation}

obtained by Anker and Pierfelice [2].

According to the long time dispersive estimates (5.1) (5.2) we consider a kernel $K_t(z, z') = t^{-3/2}(1 + d(z, z'))e^{-n d(z, z')/2}$ on $X^n_d$ and decompose it as

\[ K_t = K_t \cdot \chi_U + K_t \cdot (1 - \chi_U), \]

where $U$ is a small neighbourhood of the front face.

The part away the front face is proved like the short time case

\[ \|K_t\|_{L^r \to L^r} = \|K_t\|_{L^r(X^n_d \setminus U)} \leq t^{-3/2} \left( \int (1 + d(z, z'))^{nr/2} e^{-n d(z, z')} \,dg_z \,dg_{z'} \right)^{1/r} \leq Ct^{-3/2}. \]

For the part near the front face, we link the front face to hyperbolic space as in [8,4] and then have

\[ \left\| \int K \ast f \right\|_{L^r(V_t)} = C \left\| \int \tilde{K} \ast \tilde{f} \right\|_{L^r(V_t)} \leq Ct^{-3/2} \left\| f \right\|_{L^r(V_t)}, \]

by invoking (8.8), where $K$ and $f$ are mapped to $\tilde{K}$ and $\tilde{f}$ on hyperbolic space respectively.

\[ \square \]

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\[ ^5 \text{Note } t^{-(n+1)/2} < t^{-3/2} \text{ for the intermediate times } 1 < |t - s| < 1 + d(z, z'). \]
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Mathematical Sciences Institute
Australian National University
Canberra 0200, Australia

E-mail address: xi.chen@anu.edu.au