$L^p$-ESTIMATES OF EXTENSIONS OF HOLOMORPHIC FUNCTIONS DEFINED ON A NON-REDUCED SUBVARIETY

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Abstract. Let $D$ be a strictly pseudoconvex domain in $\mathbb{C}^N$ and $X$ a pure-dimensional non-reduced subvariety that behaves well at $\partial D$. We provide $L^p$-estimates of extensions of holomorphic functions defined on $X$.

1. Introduction

Let $D$ be a pseudoconvex domain in $\mathbb{C}^N$ and let $X$ be a smooth submanifold of dimension $n$. For any holomorphic function $\phi$ on $X$ there is a holomorphic extension $\Phi$ to $D$. The celebrated Ohsawa-Takegoshi theorem, [19], provides very precise weighted $L^2$-estimates of such extensions. This theorem, and various variants, have played a decisive role in complex and algebraic geometry during the last decades, see., e.g., [20]. There are also quite recent extension results, see, e.g., [14] and [16], obtained by $L^2$-methods, in certain cases when $X$ is not reduced.

In case $D$ is strictly pseudoconvex there are $L^p$- and $H^p$-estimates of extensions from smooth submanifolds, based on integral representation, see [2, 15, 17]. Notably is that if $D$ is strictly pseudoconvex, and $X$ behaves reasonably at $\partial D$, then any bounded holomorphic function on $X$ admits a bounded extension. In [1] there are estimates of extensions from non-smooth hypersurfaces. These results are based on integral formulas for representing the extensions or for solving $\bar{\partial}$-equations in $D$.

Let $i: X \rightarrow D$ be a non-reduced subspace of pure dimension $n$ of a pseudoconvex domain $D$. That is, we have a coherent ideal sheaf $\mathcal{J} \rightarrow D$ of pure dimension $n$ so that the sheaf $\mathcal{O}_X$ of holomorphic functions on $X$, the structure sheaf, is isomorphic to $\mathcal{O}_D/\mathcal{J}$. We thus have a natural mapping $i^*: \mathcal{O}_D \rightarrow \mathcal{O}_X$, and we say that $\Phi$ is an extension of a function $\phi$ on $X$, or that $\Phi$ interpolates $\phi$, if $i^*\Phi = \phi$. In [6] we introduced a pointwise coordinate invariant norm $|\phi|_X$ of holomorphic functions $\phi$ on $X$. In this paper we will only consider $X$ such that the underlying reduced space $Z$, i.e., the zero set of $\mathcal{J}$, is smooth. In this case the norm $|\phi|_X$ is well-defined on compact subsets up to multiplicative constants. Recall that a holomorphic differential operator $L$ in $D$ is Noetherian with respect to $\mathcal{J}$ if $L\Psi$ vanishes on $Z$ as soon as $\Psi$ is in $\mathcal{J}$. Such an $L$ induces an intrinsic mapping $\mathcal{L}: \mathcal{O}_X \rightarrow \mathcal{O}_Z$ that we also call a Noetherian operator. In [6] we introduced a locally finitely generated coordinate invariant $\mathcal{O}_D$-sheaf $\mathcal{N}_X$ of Noetherian operators such that $\phi = 0$ if and only if $\mathcal{L}\phi = 0$ for all $\mathcal{L}$ in $\mathcal{N}_X$. We defined the pointwise norm locally as

$$|\phi(z)|_X = \sum_j |\mathcal{L}_j\phi(z)|,$$

\(1.1\)

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where $\mathcal{L}_f$ is finite set of generators of $\mathcal{N}_X$. For a precise description of $\mathcal{N}_X$, see Section 2. Notice that $|\phi(z)|_X = 0$ in an open set if and only if $\phi$ vanishes identically there. Roughly speaking $| \cdot |_X$ is the smallest invariant norm with this property, see Remark 2.3.

By means of $| \cdot |_X$ we can define $L^p$-norms of $\phi$ in $\mathcal{O}_X$. It is then natural to look for $L^p$-estimates of extensions of holomorphic functions on $X$. In this paper we present a couple of such results when $D$ is strictly pseudoconvex. We do not look for the most general possible statements but our aim is to point out some new ideas. In order not to conceal them with technicalities we make some additional assumptions on the behaviour of $X$ at the boundary of $D$. Here is our main result.

**Theorem 1.1.** Let $D \subset \subset \Omega \subset \mathbb{C}^N$ be a strictly pseudoconvex domain with smooth boundary, and let $i : X \to \Omega \subset \mathbb{C}^N$ be a non-reduced subspace of pure dimension $n$ such that $Z = X_{\text{red}}$ is smooth and intersects $\partial D$ transversally. Assume that $\mathcal{O}_X$ is Cohen-Macaulay at each point on $Z \cap \partial D$. Let $\kappa = N - n$. Assume that $1 \leq p < \infty$ and that $r > -1$. Let $\delta(z) = \text{dist}(z, \partial D)$ be the distance to the boundary. Each holomorphic function $\phi$ in $\mathcal{O}(X \cap D)$ admits a holomorphic extension $\Phi \in \mathcal{O}(D)$ such that

$$
(1.2) \quad \int_D \delta^r |\Phi|^pdV_D \leq C_{r,p}^p \int_{Z \cap D} \delta^{r+\kappa} |\phi|^p dV_Z,
$$

provided that the right hand side is finite.

Here $dV_D$ and $dV_Z$ denote some volume forms on $D$ and $Z$, respectively. Since $X$ is defined in $\Omega$, the $L^p$-norms are well-defined up to multiplicative constants.

The transversally condition means that if $\rho$ is a defining function for $D$ and $(\zeta, \eta)$ are local coordinates such that $Z = \{ \eta = 0 \}$, then $\partial \rho \wedge d\eta_1 \wedge \ldots \wedge d\eta_\kappa$ is non-vanishing on $\partial D \cap Z$. In particular, $D \cap Z$ is a strictly pseudoconvex domain in $Z$ with smooth boundary.

Assume that $D \subset \mathbb{C}^{n+\kappa}$ is the unit ball, $Z = \{ \tau = 0 \}$ and $X = Z$ is reduced. If $\phi(\zeta)$ is holomorphic on $Z \cap D$ and $\Phi(\zeta, \eta) = \phi(\zeta)$ is the trivial extension to the entire ball, and $\delta(\zeta, \tau) = 1 - |\zeta|^2 - |\tau|^2$, then

$$
\int_D \delta^r |\Phi|^pdV_D = c_{r,\kappa} \int_{Z \cap D} \delta^{r+\kappa} |\phi|^p dV_Z,
$$

where $c_{r,\kappa} = \pi^\kappa / (r + 1) \cdots (r + \kappa)$. It follows that the estimate (1.2) is sharp up to the constant $C_{r,p}$ when $X$ is reduced. In the non-reduced case it is not, as we will see in our second result.

Assume that $Z$ is a smooth hypersurface in $\Omega$ defined by the function $f$ in $\Omega$, i.e., $Z = Z(f)$ and $df \neq 0$ on $Z$, let $\mathcal{J} = \{ f^{M+1} \}$ and let $\mathcal{O}_X = \mathcal{O}_\Omega / \mathcal{J}$. It turns out that then $\mathcal{N}_X$ is generated by all differential operators of order at most $M$, so that

$$
|\phi|_X = \sum_{k=0}^M \sum_{|\beta| = k} |\partial^\beta \phi|_Z.
$$

**Theorem 1.2.** Let $D \subset \subset \Omega$ be as in Theorem 1.1. Assume that $Z$ is a smooth hypersurface in $\Omega$ that intersect $\partial D$ transversally. Assume that that $Z$ is defined by the function $f$ and let $\mathcal{O}_X = \mathcal{O}_D / \langle f^{M+1} \rangle$. Moreover, assume that $1 \leq p < \infty$, $r > -1$, and let $\delta$ be the distance to the boundary. Each function $\phi$ on $\mathcal{O}(D \cap X)$
has an extension $\Phi \in \mathcal{O}(D)$ such that

$$
\int_D \delta^r |\Phi|^p dV_D \leq C_{r,p} \sum_{k=0}^M \int_{Z \cap D} \delta^{r+1+k/2} |\partial^\beta \phi|^p dV_Z,
$$

provided that the right hand side is finite.

Thus the requirement is less restrictive for higher derivatives of $\phi$.

The extension $\Phi$ in Theorems 1.1 and 1.2 is obtained by an integral formula, that in turn is constructed by means of the residue currents in [9] and the division-interpolation formulas in [4]. A main novelty is the technique to carry out the estimates in terms of the norm in [6].

When $D$ is a ball the extension formula is explicitly given in terms of the residue current associated with $X$. In the general case the analogously constructed formula does not provide a holomorphic extension, so it has to be slightly modified by a technique inspired by a classical idea of Kerzman-Stein and Ligocka, see, e.g., [21].

To this end we have to construct a linear solution operator for the $\bar{\partial}$-equation for $\bar{\partial}$-closed smooth $(0,1)$-forms in $E$ for a quite arbitrary ideal sheaf $\mathcal{J}$, Theorem 8.1.

In Sections 2 to 5 we recall the definition of the norm $| \cdot |_X$, the residue currents associated with $X$, and we make the construction of interpolation-division formulas in strictly pseudoconvex domains. In the remaining sections we prove Theorem 1.1, Theorem 1.2, and Theorem 8.1.

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2. THE POINTWISE NORM ON $X$

Let $\Omega \subset \mathbb{C}^N$ be an open pseudoconvex domain, let $Z$ be a submanifold of dimension $n < N$, and let $\kappa = N - n$. The $\mathcal{O}_\Omega$-sheaf of Coleff-Herrera currents was introduced by Björk, see [13], $\mathcal{CH}^Z_\Omega$ consists of all $\bar{\partial}$-closed $(N,\kappa)$-currents in $\Omega$ with support on $Z$ that are annihilated by $\mathcal{J}_Z$, i.e., by all $\bar{\partial}h$ where $h$ is in $\mathcal{J}_Z$. It is well-known that $\mathcal{CH}^Z_\Omega$ is coherent. Notice that if $\mathcal{J} \subset \mathcal{O}_\Omega$ is an ideal sheaf with zero set $Z$, then $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega)$ is the subsheaf of $\mu$ in $\mathcal{CH}^Z_\Omega$ that are annihilated by $\mathcal{J}$.

Remark 2.1. If $Z$ is not smooth, then $\mathcal{CH}^Z_\Omega$ is defined in the same way, but one need an additional regularity condition at $Z_{\text{sing}}$, see, [13] or, e.g., [7] Section 2.1. □

Consider the embedding $i: X \to \Omega \subset \mathbb{C}^N$. Locally, in say $U \subset \Omega$, we have have coordinates $(\zeta, \tau) = (\zeta_1, \ldots, \zeta_n, \tau_1, \ldots, \tau_n)$ so that $Z \cap U = \{ \tau = 0 \}$. Then the mapping $\pi: U \to Z \cap U$, $(\zeta, \tau) \mapsto \zeta$ is a submersion, and locally any submersion appears in this way.

If $\mu$ is a section of $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega)$ in $U$, then

$$
(2.1) \quad \pi_* (\phi \mu) =: \mathcal{L} \phi \ dz
$$

defines a holomorphic differential operator $\mathcal{L}: \mathcal{O}(X \cap U) \to \mathcal{O}(Z \cap U)$. Following [6] we define $\mathcal{N}_X$ as the set all such local operators $\mathcal{L}$ obtained from some $\mu$ in $\text{Hom}(\mathcal{O}_\Omega/\mathcal{J}, \mathcal{CH}^Z_\Omega)$ and local submersion. It follows from (2.1) that if $\xi$ is in $\mathcal{O}_Z$, then $\xi \mathcal{L} \phi = \mathcal{L}(\pi^* \xi \phi)$. Thus $\mathcal{N}_X$ is a left $\mathcal{O}_Z$-module. It is in fact coherent, in particular it is locally finitely generated.
Example 2.2. Assume that we have a local embedding and local coordinates \((\zeta, \tau)\) as above in \(\Omega\). Let \(M = (M_1, \ldots, M_\kappa)\) be a tuple of non-negative integers and consider the ideal sheaf

\[
\mathcal{I} = \left\langle \tau_1^{M_1+1}, \ldots, \tau_\kappa^{M_\kappa+1} \right\rangle.
\]

Let \(X\) be the analytic space with structure sheaf \(\mathcal{O}_X = \mathcal{O}_\Omega/\mathcal{I}\). Consider the tensor product of currents

\[
\hat{\mu} = \partial \frac{d\tau_1}{\tau_1^{M_1+1}} \wedge \ldots \wedge \partial \frac{d\tau_\kappa}{\tau_\kappa^{M_\kappa+1}},
\]

where \(d\tau_j/\tau_j^{M_j+1}\) is the principal value current. We recall that if \(\varphi = \varphi_0(\zeta, \tau)d\zeta \wedge d\bar{\zeta}\) is a test form, then

\[
\hat{\varphi}_0 = \mathcal{O}_\Omega \text{ with } \varphi = \varphi_0(\zeta, \tau)d\zeta \wedge d\bar{\zeta},
\]

where \(\mathcal{O}_\Omega\) and \(\mathcal{O}_X\) are holomorphic functions.

If \(\Psi(\zeta, \tau)\) is any representative in \(\Omega\) for \(\varphi\), then it follows from \([6, (4.22)]\), that

\[
\hat{\varphi}_0 = \mathcal{O}_\Omega \text{ with } \varphi = \varphi_0(\zeta, \tau)d\zeta \wedge d\bar{\zeta},
\]

where \(\mathcal{O}_\Omega\) and \(\mathcal{O}_X\) are holomorphic functions.

Assume that we have a local embedding and local coordinates \((\zeta, \tau)\) as above. Assume furthermore that \(\mathcal{O}_X\) is Cohen-Macaulay. Then one can find monomials \(1, \tau^{\alpha_1}, \ldots, \tau^{\alpha_{\nu-1}}\) such that each \(\hat{\phi}\) in \(\mathcal{O}_X\) has a unique representative

\[
\hat{\phi} = \hat{\phi}_0(z) \otimes 1 + \cdots + \hat{\phi}_{\nu-1}(z) \otimes \tau^{\alpha_{\nu-1}}.
\]
where \( \hat{\varphi}_j \) are in \( \mathcal{O}_Z \), see, e.g., [7, Corollary 3.3]. In this way \( \mathcal{O}_X \) becomes a free \( \mathcal{O}_Z \)-module. By [6, Theorem 4.1 (iii)],

\[
(2.10) \sum |\hat{\varphi}_j(z)| \leq C|\varphi|_X
\]

and in fact \( |\cdot|_X \) is the smallest norm such that (2.10) holds for any choice of coordinates and of monomial basis. \( \Box \)

3. Residue currents associated with a free resolution

If \( \mathcal{J} \) is coherent ideal sheaf in \( \Omega \), then we can find a free resolution

\[
(3.1) 0 \rightarrow \mathcal{O}(E_\nu) \xrightarrow{f_\nu} \mathcal{O}(E_{\nu-1}) \cdots \xrightarrow{f_1} \mathcal{O} \rightarrow \mathcal{O}/\mathcal{J} \rightarrow 0
\]

of \( \mathcal{O}/\mathcal{J} \) in a slightly smaller pseudoconvex domain that we for simplicity denote by \( \Omega \) as well. If the (trivial) vector bundles \( E_k \) are equipped with Hermitian metrics we say that (3.1) is a hermitian resolution. For each Hermitian resolution there are, [9], associated residue currents

\[
R = \sum_{k=\kappa}^{\nu} R_k, \quad U = \sum_{\ell,k} U_{k,\ell},
\]

where \( R_k \) are currents of bidegree \((0,k)\) with support on \( Z := Z(\mathcal{J}) \) that take values in \( \text{Hom}(E_0, E_k) \simeq E_k \), and \( U_{k,\ell} \) are \((0,k-\ell)\)-currents that are smooth outside \( Z \) and take values in \( \text{Hom}(E_\ell, E_k) \).

Remark 3.1. The currents \( R \) and \( U \) are defined even if (3.1) is just a pointwise generically exact complex. In general then \( R \) has components \( R_{k,\ell} \) with values in \( \text{Hom}(E_\ell, E_k) \) even for \( \ell \geq 1 \). \( \Box \)

If \( \mathcal{J} \) is Cohen-Macaulay, then one can choose (3.1) so that \( \nu = \kappa \). In that case the components of \( R = R_k \) are in \( \text{Hom}(\mathcal{O}/\mathcal{J}, \mathcal{C}H^2_{\Omega}) \). If we only assume that \( \mathcal{O}_X \) has pure dimension, then we may have components \( R_k \) for \( k \leq N-1 \), see, e.g., [8, 7]. They can be written, [7, Lemma 6.2],

\[
(3.2) R_k \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_N = a_k \mu,
\]

where \( \mu \) is in \( \text{Hom}(\mathcal{O}/\mathcal{J}, \mathcal{C}H^2_{\Omega}) \) with values in a trivial bundle \( F \) and \( a_k \) are currents in \( \Omega \) that take values in \( \text{Hom}(F, E_k) \). Moreover, \( a_k \) are smooth outside a Zariski closed set \( W \subset \Omega \) such that \( Z \setminus W \) is the set of all Cohen Macaulay points on \( Z \). In particular \( Z \cap W \) has positive codimension on \( Z \). The currents \( a_k \) are almost semi-meromorphic in the terminology from [8, 11]. For us the important point is that

\[
(3.3) a_k \mu = \lim_{\epsilon \to 0} \chi(|f|^2/\epsilon) a_k \mu,
\]

if \( f \) is a holomorphic tuple with zero set \( W \) and \( \chi \) is a smooth function on \([0, 1)\) that is 1 for \( t > 1/2 \) and 0 for \( t > 1 \). Notice that for each \( \epsilon, \chi(|f|^2/\epsilon) a_k \mu \) is the product of a current and a smooth form.
4. Integral representation of holomorphic functions

Following [3] we recall a formalism to generate representation formulas for holomorphic functions. Let $z$ be a fixed point in $\Omega$, let $\delta_{z-z^*}$ be contraction with the vector field

$$2\pi i \sum_{j=1}^{N} (\zeta_j - z_j) \frac{\partial}{\partial \zeta_j}$$

in $\Omega$ and let $\nabla_{\zeta-z^*} = \delta_{z-z^*} - \bar{\partial}$. We say that a current $g = g_{0,0} + \cdots + g_{n,n}$, where $g_{k,k}$ has bidegree $(k,k)$, is a weight with respect to $z$, if $\nabla_{\zeta-z} g = 0$, $g$ is smooth in a neighborhood of $z$, and $g_{0,0}$ is 1 when $\zeta = z$. Notice that if $g$ and $g'$ are weights, one of which is smooth, then $g' \wedge g$ is again a weight. The basic observation is that if $g$ is a weight (with respect to $z$) with compact support in $\Omega$, then

$$\phi(z) = \int g \phi$$

if $\phi$ is holomorphic in $\Omega$, [3] Proposition 3.1].

If $\Omega$ is pseudoconvex and $D \subset \subset \Omega$, then, see [4] Example 1], we can find a weight $g$, with respect to $z \in \overline{D}$, with compact support in $\Omega$, such that $g$ depends holomorphically on $z \in \overline{D}$. If $D$ and $\Omega$ are balls with center at 0 $\in \Omega$, then we can take

$$g = \chi - \bar{\partial} \chi \wedge \frac{\sigma}{\nabla_{\zeta-z} \sigma} = \chi - \bar{\partial} \chi \wedge \sum_{\ell=1}^{N} \frac{1}{(2\pi i)^{\ell}} \frac{\zeta \cdot d\zeta \wedge (d\zeta \cdot \bar{d\zeta})^{\ell-1}}{(|\zeta|^2 - \zeta \cdot z)^{\ell}},$$

where $\chi$ that is 1 in a neighborhood of $\overline{D}$, with compact support in $\Omega$, and

$$\sigma = \frac{1}{2\pi i} \frac{\zeta \cdot d\zeta}{|\zeta|^2 - \zeta \cdot z}.$$ 

4.1. Division-interpolation formulas. Let $(E,f)$ be a Hermitian resolution in $\Omega$ of $\mathcal{O}/\mathcal{J}$ as in Section 3. In order to construct division-interpolation formulas with respect to $(E,f)$, in [4] was introduced the notion of an associated family $H = (H_{k}^{\ell})$ of Hefer morphisms. The $H_{k}^{\ell}$ are holomorphic $(k - \ell)$-forms with values in $\text{Hom}(E_{\zeta,k}, E_{z,\ell})$ that are connected in the following way: To begin with, $H_{k}^{\ell} = 0$ if $k - \ell < 0$, and $H_{k}^{k}$ is equal to $I_{E_{k}}$ when $\zeta = z$. In general,

$$\delta_{z-z^*} H_{k+1}^{\ell+1} = H_{k}^{\ell} f_{k+1}(\zeta) - f_{\ell+1}(z) H_{k+1}^{\ell+1}. \tag{4.2}$$

If $R$ and $U$ are the associated currents in Section 3 then

$$HR = \sum_k H_{k}^{0} R_k, \quad H^{1} U = \sum_k H_{k}^{1} U_{k},$$

are scalar-valued currents, cf. Remark 3.1. It turns out that

$$g' = f_{1}(z) H^{1} U + HR$$

is a weight with respect to $z$ for each $z \in \Omega \backslash Z$. If $g$ is a smooth weight with respect to $z \in \overline{D} \subset \Omega$, depending holomorphically on $z$, with compact support in $\Omega$ and $\Psi$ is holomorphic in $\Omega$, then by (4.1),

$$\Psi(z) = \int g' \wedge g \Psi = f_{1}(z) \int_{\zeta} H^{1} U \wedge g \Psi + \int_{\zeta} HR \wedge g \Psi \tag{4.3}$$

for $z \in D \backslash Z$. Since the right hand side has a holomorphic extension across $Z$, actually [4,3] holds for all $z$ in $D$ by continuity.
Now assume that $\phi$ is a section of $\mathcal{E}/\mathcal{J}$ in $\Omega$. Since $\Omega$ is pseudoconvex there is some holomorphic extension $\Psi$ of $\phi$ to $\Omega$. Since $R$ annihilates $\mathcal{J}$, the current $R\phi := R\Psi$ is independent of the extension and thus intrinsic. Since $f_1(z)$ is in $\mathcal{J}$, we conclude from (4.3) that

$$\Phi(z) = \int HR \wedge g\phi$$

is a holomorphic function in $D$ that extends $\phi$. In order to obtain interesting estimates however, we must replace $g$ by a weight with support on $D$.

For future reference notice that if $g$ only depends smoothly on $z \in D$, then (4.4) is a smooth function in $D$ such that $\Phi - \phi$ is in $E(J)$, where $E$ is the sheaf of smooth functions.

5. Integral formulas in strictly pseudoconvex domains

The material in this section is basically well-known but we need it for the construction of our formula. Assume that $D \subset \subset \Omega \subset \mathbb{C}^N$ is strictly pseudoconvex with smooth boundary. We can assume that $D = \{ \rho < 0 \}$ where $\rho$ is strictly plurisubharmonic in $\Omega$. If $D$ is the ball we can take $\rho = |\zeta|^2 - 1$. If $D$ is strictly convex, then $\delta_{\zeta - z} \partial \rho$ is holomorphic in $z \in D$ and if $\rho$ is strictly convex, then

$$2\text{Re} \delta_{\zeta - z} \partial \rho \geq \rho(\zeta) - \rho(z) + c|\zeta - z|^2$$

for some constant $c > 0$. If

$$v(\zeta, z) := \delta_{\zeta - z} \partial \rho - \rho(\zeta) = -\rho(\zeta) - \sum_j \frac{\partial \rho}{\partial \zeta_j}(\zeta_j - \zeta_j),$$

because of the strict convexity, therefore

$$2\text{Re} v(\zeta, z) \geq -\rho(z) - \rho(\zeta) + c|\zeta - z|^2,$$

and moreover,

$$d(\text{Im} v)|_{\zeta = z} = d^c\rho(z)/4\pi.$$

Altogether it follows that if $z$ (or $\zeta$) is a fixed point $p$ on $\partial D$, then the level sets of $|v(\zeta, z)|$ are non-isotropic so-called Koranyi balls around $p$. More precisely, if $x_1 = -\rho(\zeta)$ and $x_2 = \text{Im} v(\zeta, z)$, and $x_3, \ldots, x_{2N}$ are chosen so that $x_1, \ldots, x_{2N}$ is a local (non-holomorphic) coordinate system at $p$ such that $x(p) = 0$, and $y$ are the corresponding coordinates for $z$, then

$$|v(\zeta, z)| \sim x_1 + y_1 + |x_2 - y_2| + \sum_{j=3}^{2N} (x_j - y_j)^2 + O(|x - y|^3).$$

One can make a similar construction of $v$ if $D$ is strictly pseudoconvex, see Remark 5.2. If $D$ is the ball and $\rho = |\zeta|^2 - 1$, then

$$v(\zeta, z) = 1 - \bar{\zeta} \cdot z$$

which is anti-holomorphic in $\zeta$. In general, unfortunately, $\partial_{\zeta} v$ will only vanish to first order on the diagonal. For our formula we need such a function $v$ that is (essentially) anti-holomorphic in $\zeta$ so we must elaborate the construction.
5.1. Definition of $v$ in the general case. First assume that $\rho(z)$ is strictly plurisubharmonic and real-analytic. Then close to the diagonal we choose $v(\zeta, z)$ so that $v(\zeta, z)$ is the (unique) holomorphic extension of $-\rho(z)$ from the the totally real subspace $\{ \zeta = \bar{z} \}$ of $\Omega_c \times \Omega_r$. Then $v(z, \zeta) = v(\zeta, z)$ and $v$ is anti-holomorphic in $\zeta$. We can represent $v$ by the power series

$$v(\zeta, z) = -\sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha \rho}{\partial \zeta^\alpha}(\zeta - \zeta)^\alpha.$$  

We claim that

$$2\Re v = -\rho(\zeta) - \rho(z) + L\rho(\zeta) + O(|\zeta - z|^3),$$

where $L\rho(\zeta)$ is the Levi form in the Taylor expansion of $\rho$ at $\zeta$. In fact, from (5.5) we have, using the notation $\rho_j = \partial \rho / \partial \zeta^j (\zeta)$ etc and $\eta_j = z_j - \zeta_j$,

$$2\Re v = -2\rho(\zeta) - 2\Re \sum_j \rho_j \eta_j - \Re \sum_{jk} \rho_{jk} \eta_j \eta_k + O(|\eta|^3) =$$

$$-\rho(\zeta) + L\rho(\zeta) - \left( \rho(\zeta) + 2\Re \sum_j \rho_j (\zeta) \eta_j + \Re \sum_{jk} \rho_{jk} \eta_j \eta_k + L\rho(\zeta) \right) + O(|\eta|^3) =$$

$$-\rho(\zeta) + L\rho(\zeta) - \rho(\zeta) + O(|\eta|^3).$$

Since $\rho$ is strictly plurisubharmonic it follows from (5.10) that (5.2) holds, and since also (5.3) holds, the level sets of $|v|$ are the Koryani balls discussed above and (5.4) holds. From (5.5) it is easy to find a $(1, 0)$-form $q$, depending holomorphically on $z$, such that

$$v = \delta_{\zeta - z} q - \rho(\zeta).$$

We now turn to the case when $\rho$ is just smooth. Let $\chi$ be a smooth function on $[0, \infty)$ that is 1 when $t < 1/2$ and 0 when $t > 1$. Then the series

$$v(\zeta, z) = -\sum_{\alpha} \frac{1}{\alpha!} \frac{\partial^\alpha \rho}{\partial \zeta^\alpha}(\zeta - \zeta^\alpha \chi(c_\alpha |z - \zeta|^2)$$

converges and defines a smooth function close to the diagonal if $c_\kappa$ tends to infinity fast enough (depending on the ultra-differentiable class of $\rho$). The function $v(\zeta, z)$ is a so-called almost holomorphic extension of $-\rho(\zeta)$, and $v$ is therefore almost anti-holomorphic in $\zeta$ in the sense that

$$\partial_{\bar{\zeta}} v = O(|\zeta - z|^\infty).$$

Again one can find $q$ such that (5.7) holds. Moreover, $v(z, \zeta) - v(\zeta, z) = O(|\zeta - z|^\infty)$ but this property is not used in this paper.

We extend $v$ to $\Omega \times \Omega$ by patching with $|\zeta - z|^2$, that is, if $\eta = \zeta - z$ we let

$$\tilde{v} = \chi(|\eta|^2) v + (1 - \chi(|\eta|^2)) \eta^2, \quad \tilde{q} = \chi(|\eta|^2) q + (1 - \chi(|\eta|^2)) \partial |\eta|^2.$$ 

so that $\tilde{v} = \delta_{\zeta - z} \tilde{q} - \rho(\zeta)$. In what follows, for simplicity, we write $v$ and $q$ even for the extensions.
5.2. The weight \( g^\alpha \). Let \( \alpha \) be any complex number. We claim that for each fixed \( z \in D \),
\[
(5.10) \quad g^\alpha = \left( \frac{1 + \nabla_{\zeta - z} \bar{q}}{\rho} \right)^{-\alpha + 1} = \left( \frac{\bar{v}}{\rho} + \bar{\partial}_\rho \right)^{-\alpha + 1}
\]
is a weight with respect to \( z \). In fact, the scalar term within the second brackets has positive real part in view of (5.2), and hence \( g^\alpha \) is well-defined by elementary functional calculus, see [3], and \( \nabla_{\zeta - z} \bar{q} = 0 \) since \( \nabla_{\xi - z} \bar{z} = 0 \). It is also clear that \( g^\alpha_{0,0} = 1 \) when \( \zeta = z \). Thus the claim holds.

A simple computation gives that
\[
(5.11) \quad g^\alpha = \sum_{k=0}^{N} c_{k,\alpha} \frac{(-\rho)^{\alpha} \beta_k}{\bar{v}^{\alpha + k + 1}},
\]
where \( c_{\alpha,k} \) are constants and \( \beta_k \) are \((k,k)\)-forms that are smooth in \( \Omega \). If \( \Re \alpha \) is positive, then \( g^\alpha \) vanishes on the boundary of \( D \) for each fixed \( z \in D \).

In case \( D \) is the ball, this weight depends holomorphically on \( z \in D \). In general however it only depends smoothly on \( z \); however, (5.9) holds, which is crucial in the proofs of Theorems 1.1 and 1.2.

**Remark 5.1.** For a given \( X \) in Theorem 1.1 or Theorem 1.2 it is in fact enough for our proofs to choose \( \nu \) such that
\[
\partial_\nu v = \partial'(|\zeta - z|^{\nu})
\]
for a large enough \( \nu \). Such a \( v \) is obtained by restricting the sum (5.8) to \( |\alpha| \leq \nu + 1 \); then of course the factors \( c(v^{|\zeta - z|^2}) \) are not needed. \( \square \)

**Remark 5.2.** If \( D \) is strictly pseudoconvex, due to Fornaess embedding theorem, one can find an embedding \( \psi: \Omega \to \Omega' \) into a higher dimensional domain, and a strictly convex subset \( D' \subset \Omega' \) so that \( \psi(\Omega) \) intersect \( \partial D' \) transversally, and such that \( D = \psi^{-1} D' \). If \( \rho' \) is strictly convex in \( \Omega' \) and defines \( D' \), then \( \rho = \psi^* \rho' \) is strictly plurisubharmonic in \( \Omega \) and defines \( D \). One can then define \( v \) in \( D \times D' \) as the pullback of \( v' \) in \( D' \times D' \). Then clearly \( v \) will depend holomorphically on \( z \).

Moreover, there is a \((1,0)\)-form \( q \) in \( D \), depending holomorphically on \( z \), such that \( v = \delta_{\zeta - z} \bar{q} - \rho(\zeta) \). This function plays the same role as the function \( v \) in the strictly convex case described above. In the rare case when one can choose \( D' \) to be a ball, \( v \) is anti-holomorphic in \( \zeta \). \( \square \)

6. Proof of Theorem 1.1 when \( D \) is the ball

Let us first assume that our \( v(\zeta, z) \) is defined in \( \Omega \times \Omega \), holomorphic in \( z \) and antiholomorphic in \( \zeta \), as in the case with the ball. Let us also assume that \( X \) is defined and Cohen-Macaulay in \( \Omega \). Then we can assume that our Hermitian resolution \((E, f)\) has length \( \kappa = N - n \), and hence \( HR = H^0 R_\kappa \) has bidegree \((\kappa, \kappa)\). Recall, cf. (5.11), that for fixed \( z \in D \), the weight \( g^\alpha \) vanishes to order \( \alpha \) at \( \partial D \). If we define it as 0 outside \( D \), it is therefore of class \( C^{\alpha - 1} \).

**Lemma 6.1.** If \( \alpha \) is large enough and \( \phi \) is holomorphic in a neighborhood of \( X \cap \overline{D} \), then
\[
(6.1) \quad \Phi(z) = \int_D HR \wedge \frac{(-\rho)^\alpha \beta_\alpha}{\bar{v}^{\alpha + 1}} \phi
\]
is a holomorphic extension of \( \phi \) to \( D \).
Proof. Assume that \( \alpha \) is larger than the order of the currents \( U \) and \( R \). Notice that the function that is \((-\rho)^\alpha\) in \( D \) and 0 outside \( D \) is in \( C^{\alpha-1} \). For each fixed \( z \in D \), therefore \( g^\alpha \), defined as 0 outside \( D \), is a weight in \( \Omega \) of class \( C^{\alpha-1} \). Thus \((4.3)\) holds with \( g = g^\alpha \). As in Section 4.1 we conclude that \((4.4)\), that is, \((6.1)\), is a holomorphic extension of \( \phi \) to \( D \). \( \square \)

We shall now make an a priori estimate of \( \Phi \) in terms of \( \phi \). If either \(|\zeta - z| \geq \epsilon\) or \( \zeta \) is far from \( \partial D \), then \(|v|\) is strictly positive in view of \((5.4)\).

By a suitable partition of unity we therefore have to estimate the \( L^p \) norm of a finite number of terms

\[
(6.2) \quad \int_D HR \wedge (-\rho)^\alpha \frac{\alpha}{v^{n+\alpha+1}} \beta(\zeta,z)\phi, 
\]

where \( \beta \) is smooth with compact support in a small neighborhood \( U \) of a point \( p \in \partial D \cap Z \), plus some terms with no singularity at all.

Let us consider a term \((6.2)\). Let us change notation and replace \( \zeta \) by coordinates \((\zeta,\tau)\) in \( U \) such that \( Z \cap U = \{ \tau = 0 \} \). Let \( \mu \) be one of the components of \( R = R_\kappa \).

We may assume that

\[
\mu = \gamma(\zeta,\tau) \frac{d\tau}{\tau^{M+1}}
\]
as in \((2.7)\), cf. \((2.2)\) and \((2.4)\). Let us incorporate \( H \) in \( \beta \). Integrating with respect to \( \eta \), that is, taking the push-forward \( \pi^* \), where \( \pi \) is the projection \((\zeta,\tau) \mapsto \zeta \), we get, see \((2.3)\),

\[
\int_{\zeta \in Z \cap D} \frac{1}{v^{n+\alpha+1}} (M^m) \partial^M_{\tau} (-\rho)^\alpha \beta \partial^m(\gamma \phi).
\]

Thus we get a sum of terms of the form

\[
\int_{\zeta \in \Omega} \frac{1}{v^{n+\alpha+1}} (-\rho)^{a-\ell} \beta \partial^m(\gamma \phi),
\]

where \( \ell \leq |M - m| \) and \( \beta \) is smooth. Since \( \rho \) is a defining function we may assume that \( \partial \rho \) is nonzero in \( U \). If

\[
T = \frac{1}{|\partial \rho|^2} \sum_j \frac{\partial \rho}{\partial \zeta_j} \frac{\partial}{\partial \zeta_j},
\]

then \( T \rho = 1 \) and hence

\[
(-\rho)^{a-\ell} \beta = \beta' T (-\rho)^{a-\ell+1},
\]

where \( \beta' = \beta / (\ell - \alpha - 1) \). If \( T' \) is the formal adjoint of \( T \), again using that \( v \) is anti-holomorphic in \( \zeta \), we get

\[
\int_{\zeta \in \Omega} \frac{1}{v^{n+\alpha+1}} (-\rho)^{a-\ell+1} T' (\beta' \partial^m_\zeta(\gamma \phi)).
\]

Repeating this procedure \( \ell \) times we get a sum of terms

\[
(6.3) \quad A(z) = \int_{\zeta \in \Omega} \frac{1}{v^{n+\alpha+1}} (-\rho)^{a-\ell} \beta \partial^m_\zeta(\gamma \phi),
\]
where $a$ is a multiindex such that $|a| \leq |M - m|$ and $\beta$ is smooth. It follows from (2.6) and (2.8) that
\[ |\partial^\alpha \partial^\beta \phi| \lesssim |\phi|_X. \]
Assume that $r > -1$. Provided that $\alpha$ is large enough, from (6.5) in Lemma 6.3 below we have
\[ \int \delta^\alpha |A| \lesssim \int (\rho)^{N+n+r} |\phi|_X. \]
Summing up all terms we get the desired a priori estimate (1.2) in case $p = \infty$. Below we have
\[ \int \rho \leq \rho \leq \rho \leq \rho \leq \rho \]
Assume that $p > 0$. Below we have
\[ \rho \leq \rho \leq \rho \leq \rho \leq \rho \leq \rho \]
by (6.6) if $r > \rho > -1$. An application of (6.5), then gives (1.2) for $p = 2$. First notice that
\[ |A|^2 \lesssim \int_{Z \cap D} \frac{\delta^\alpha}{\rho^{a+\alpha+1}} \int_{Z \cap D} \frac{\delta^{\alpha+\beta}}{\rho^{n+\alpha+1}} |\phi|^2_X \lesssim \delta(z)^{-\epsilon} \int_{Z \cap D} \frac{\delta^{\alpha+\beta}}{\rho^{n+\alpha+1}} |\phi|^2_X \]
by (6.5) if $r - \epsilon > -1$. An application of (6.5), then gives (1.2) for $p = 2$.

If $\phi$ is just defined in $X \cap D$ we apply the same construction and argument to the slightly smaller strictly pseudoconvex domains $D_\epsilon = \{ \rho < -\epsilon \}$. It is not hard to see that the same computation works in $D_\epsilon$, with estimates that are uniform in $\epsilon$. By Lemma 6.4 we thus get $\Phi_\epsilon$ in $D_\epsilon$ that interpolate $\phi$ in $D_\epsilon \cap X$ and such that
\[ \int_{D_\epsilon} \delta^\alpha |\Phi_\epsilon|^p dV \leq C_{r,p} \int_{Z \cap D_\epsilon} \delta^{\alpha+r} |\phi|^p_X dV, \]
where $C_{r,p}$ is uniform in $\epsilon$. If the right hand side of (1.2) is finite, then furthermore
\[ \Phi(z) = \int_D \frac{HR}{\rho^{n+\alpha+1}} \phi = \lim_{\epsilon \to 0} \int_{D_\epsilon} \frac{HR}{\rho^{n+\alpha+1}} \phi \]
eq 0 for all $\mu \in \mathcal{H}(\Theta_X^1 J, \mathcal{C}^0_{\Theta_X})$ on compact subsets of $D$, this must hold for $\Phi$ as well. Thus the image of $\Phi$ in $\Theta_X$ is $\phi$, that is, $\Phi$ is an extension of $\phi$. Hence Theorem 1.1 is proved in case $D$ is a ball and $\Theta_X$ is Cohen-Macaulay.

We will now point out how to estimate (6.1) if $\Theta_X$ has non-Cohen-Macaulay points in $D$. Then, cf. Section 4
\[ HR = H^0 \phi R_k + \cdots + H^0 \phi R_{N-1}. \]
Recall the representations (3.2). Since $\Theta_X$ is Cohen-Macaulay at points on $\partial D \cap Z$, $a_k$ are smooth there and hence we can proceed in the same way as before at such points.

Let neighborhood $U \subset \subset Z \cap D$ be a small neighborhood of a point on $Z \cap D$ and let us choose coordinates $(\zeta, \tau)$ in $U$ as before. Then we have, cf. (3.2) and (2.7), that
\[ H^0 k R_k = a_k \mu = a_k \gamma \partial \frac{d\tau}{\tau^{M+1}}. \]
Thus we get terms like
\[ \int_{(\zeta, \tau) \in D} R \wedge \beta \phi = \int_{(\zeta, \tau) \in D} a_k \beta \partial \frac{d\tau}{\tau^{M+1}} \wedge \gamma \phi, \]
where $\beta$ is smooth and has compact support in $U$. Integrating with respect to $\tau$, that is, applying $\pi_*$, we get by Lemma 6.2 a sum of terms like

$$\int_{\zeta \in Z \cap D} b_m(\cdot, z) \partial^{m}_\tau (\gamma \phi)$$

for $m \leq M$, where $b_m(\zeta, z)$ are currents with compact support in $U$ that depend holomorphically on $z$ in $D$. By usual Cauchy estimates, (6.4) is controlled by the $L^p$-norm of $\partial^m_\tau (\gamma \phi)$ over $U$. In view of (2.6) and (2.7) we get the same a priori estimate as before. Thus Theorem 1.1 is fully proved in the case when $D$ is the ball, except for the following lemma.

**Lemma 6.2.** With the notation in the proof, let $a = \beta \alpha_k$, and $\psi = \gamma \phi$. Then

$$\pi_*(\bar{\partial} \frac{d\tau}{\tau^{M+1}} a \phi) = \sum_{m \leq M} b_m \partial^m_\tau \psi|_{\tau = 0},$$

where $b_m$ are currents on $U$ with compact support in $U$. If, in addition, $\beta$ depends holomorphically on a parameter $z$, then also $b_m$ will do.

**Proof.** Recall from Section 3 that

$$a \bar{\partial} (d\tau/\tau^{M+1}) = \lim_{\epsilon \rightarrow 0} \chi(|f|^2/\epsilon) d\bar{\partial} (d\tau/\tau^{M+1}),$$

where $f$ is a holomorphic tuple with zero set $W$. It follows that $\tau^{M'} a \bar{\partial} (d\tau/\tau^{M+1}) = 0$ if $\tau^{M'}$ is in the ideal $<\tau^{M+1}>$, that is, if $M'_j \geq M_j + 1$ for some $j$. Since $\psi$ is holomorphic we have

$$\psi(\zeta, \tau) = \sum_{m \leq M} \psi_m(\zeta) \tau^m + \cdots$$

where $\cdots$ are terms in $<\tau^{M+1}>$. It follows that

$$a \psi \bar{\partial} (d\tau/\tau^{M+1}) = \sum_{m \leq M} \psi_m(\zeta) a \bar{\partial} (d\tau/\tau^{M-m+1}),$$

and hence

$$\pi_*(a \psi \bar{\partial} (d\tau/\tau^{M+1})) = \sum_{m \leq M} \psi_m(\zeta) \pi_*(a \bar{\partial} (d\tau/\tau^{M-m+1})).$$

Now the lemma follows, since the last factor depends holomorphically on $z$. $\square$

**Lemma 6.3.** With the notation above we have, for $s > -1$ and $b > 0$, we have the estimates

$$\int_{z \in D} \frac{\delta(z)^s dV(z)}{|v|^{n+1+s+b}} \lesssim \frac{1}{\delta(z)^b}$$

and

$$\int_{Z \cap D} \frac{\delta(z)^s dV(z)}{|v|^{n+1+s+b}} \lesssim \frac{1}{\delta(z)^b}.$$

This lemma is well-known and follows in a standard way from the local representation (5.4) of $|v|$.

**Remark 6.4.** There is a somewhat different way to construct holomorphic extensions from $X$, which is, e.g., used in [2]. Let $(E, f)$ be a Hermitian resolution of $\mathcal{O}_D\mathcal{J}$ as before and let $\nabla_f = f - \bar{\partial}$, cf. [4, 9]. The associated currents $U$ and $R$ are related by the formula $\nabla_f U^0 = I - R$, that is, $f_{k+1} U_{k+1} - \bar{\partial} U_k^0 = I - R_k$, $k = 0, 1, \ldots$. If
\( \phi \in \mathcal{O}(X \cap D) \), then \( R\phi \) is well-defined. By solving a sequence of \( \bar{\partial} \)-equations in \( D \) one can find a current \( V = V_1 + V_2 + \cdots + V_N \) such that \( f_{k+1}V_{k+1} - \partial V_k = -R_k\phi \), \( k \geq 1 \). We claim that \( \Phi = f_1V_1 \) is a holomorphic extension in \( D \) of \( \phi \). Since one can solve \( \bar{\partial} \) with estimates one get estimates of \( \Phi \). However, except in the case \( X \) when is reduced, we cannot see how to obtain Theorems 1.1 or 1.2 with this approach.

In case \( \kappa = 1 \) there is just one step in this procedure so that if \( K \) is a solution operator for \( \bar{\partial} \) in \( D \), then \( \Phi = f_1K(R_1\phi) \) is a holomorphic extension of \( \phi \).

Let us sketch a proof of the claim: Let \( \varphi \) be any holomorphic extension of \( \phi \) to \( D \). Then

\[
\nabla_f U^0\varphi = (I - R)\varphi = \varphi - R\phi.
\]

Furthermore, \( \nabla_f V = \Phi - R\phi \). Hence \( \nabla_f(V - U^0\varphi) = \varphi - \Phi \). By solving another sequence of \( \bar{\partial} \)-equations one can find a holomorphic \( w \) such that \( \varphi - \Phi = f_1w \). This precisely means that \( \varphi - \Phi \in \mathcal{J} \).

\section{Proof of Theorem 1.1 in the general case}

As described in Section 5 in the general case we get a similar kernel \( v(\zeta, z) = \delta_{\zeta-z}q - \rho(\zeta) \) but instead of being holomorphic in \( z \) and anti-holomorphic in \( \zeta \) we have \( \bar{\partial}_z v = 0 \) close to the diagonal \( \Delta \) and the property (5.9), respectively. Notice, cf. (5.8), for future reference that we can choose \( g \) so that \( \bar{\partial}_z q = 0 \) close to \( \Delta \).

In this section we let the \( \bar{\partial} \) in \( \nabla_{\zeta-z} = \delta_{\zeta-z} - \bar{\partial} \) act on both \( z \) and \( \zeta \). Thus also anti-holomorphic differentials with respect to \( z \) will occur in \( g^a \), cf. (5.10).

\[
g := (f_1(z)H^1U + HR) \wedge g^a.
\]

However, we only have holomorphic differentials with respect to \( \zeta \). Then still \( \nabla_{\zeta-z} g^a = 0 \) and \( \nabla_{\zeta-z} g = 0 \).

Let \( \Omega \) be a neighborhood of \( \overline{D} \) and assume that \( \phi \) is defined in \( \Omega \cap X \). Moreover, let \( \Psi \) be a holomorphic extension to \( \Omega \). Then

\[
\Psi(z) = \int_{\zeta} g^0_{N,N} \Psi, \quad z \in D,
\]

where upper and lower indices denote bidegree in \( z \) and \( \zeta \), respectively. Hence (the \((0,0)\)-component of \( z \))

\[
\varphi(z) := \int_{\zeta \in D} HR \wedge g^a(\zeta, z)\phi(\zeta)
\]

is a smooth function in \( D \) that interpolates \( \phi \) in the sense that \( \varphi - \phi \) is in \( \mathcal{E}^{0,0} \mathcal{J} \), cf. Section 1.1.

We shall now modify the kernel in (7.2) so that it produces a holomorphic extension. To this end we invoke a result that should be of independent interest. We formulate and prove a somewhat more general version in Section 8.

**Proposition 7.1.** Assume that \( \bar{D} \subseteq \bar{\mathcal{D}} \) are pseuodconvex neighborhoods of \( \overline{D} \). There is a linear operator \( T : \mathcal{E}^{0,1}(\bar{D}) \cap \ker \bar{\partial} \to \mathcal{E}^{0,0}(\bar{D}) \) such that \( \bar{\partial}T\xi = \xi \) in \( \bar{D} \) and furthermore \( T\xi \in \mathcal{E}^{0,0}(\mathcal{J}(\bar{D})) \) if \( \xi \in \mathcal{E}^{0,1}(\mathcal{J}(\bar{D})) \).

Here \( \xi \in \mathcal{E}^{0,1}(\mathcal{J}(\bar{D})) \) means that \( \xi \) is a smooth \((0,1)\)-form in \( \bar{D} \) such that locally \( \xi \) has a representation \( \xi = \xi_1\eta_1 + \cdots + \xi_p\eta_p \), where \( \xi_j \) are smooth \((0,1)\)-forms and \( \nu_j \) are functions in \( \mathcal{J} \).
Recall from Section 5 that $\overline{\partial}_z g = 0$ and $\overline{\partial}_z v = 0$ in a set $W = \{|z - z| < \epsilon\}$. It follows from (5.2) that there is a pseudovex neighborhood $\widetilde{D}$ of $D$ such that $\overline{\partial}_z g^\alpha$ is smooth in $D_\overline{\zeta} \times \overline{D}_z$. It follows that also $\overline{\partial}_z g$ is smooth in $\widetilde{D}$ for $\zeta \in D$. Since $\nabla_{\zeta - z} g = 0$, the component $g_{N,N}$ of $g$ of total bidegree $(N,N)$ is $\overline{\partial}_z$-closed, and hence
\begin{equation}
\overline{\partial}_z g_{N,N}^0 + \overline{\partial}_z g_{N,N-1}^0 = 0
\end{equation}
in $D \times \overline{D}$. Since $\overline{\partial}_z g = 0$ in $W$, no anti-holomorphic differentials with respect to $z$ can occur in $g^\alpha$, cf. (5.10) there, and hence $g_{N,N-1}^0 = 0$ in $W \cap D \times \overline{D}$.

Notice that $\overline{\partial}_z (HR \wedge g_\alpha) = HR \wedge \overline{\partial}_z g_\alpha$. We now define
\begin{equation}
\mathcal{A}(\zeta, z) = T(H(\zeta, t)R(\zeta) \wedge \overline{\partial}_tg_\alpha(\zeta, t))(z), \quad \zeta \in D, \, z \in \overline{D}.
\end{equation}
Then clearly
\begin{equation}
HR \wedge g_\alpha(\zeta, z) - \mathcal{A}(\zeta, z)
\end{equation}
is holomorphic in $z \in D$. Thus
\begin{equation}
\Phi(z) := \int_{\zeta \in D} (HR \wedge g_\alpha(\zeta, z) - \mathcal{A}(\zeta, z)) \phi
\end{equation}
is holomorphic in $D$. We claim that it indeed is an extension of $\phi$.

**Proof of the claim.** As noticed above $g_{N,N-1}^0$ vanishes in $W$. Hence it is it is smooth in $D$ and vanishes to high order at the boundary. Since $\Psi$ is holomorphic thus
\begin{equation}
\int_{\zeta \in D} \overline{\partial}_z g_{N,N-1}^0 \Psi = 0
\end{equation}
by Stokes’ theorem. In view of (7.3), cf. (7.1), we therefore have
\begin{equation}
\int_{\zeta \in D} HR \wedge \overline{\partial}_tg_\alpha \phi = - \int_{\zeta \in D} f_1(t)H^1U \wedge \overline{\partial}_tg_\alpha \Phi.
\end{equation}
Applying $T$ we get
\begin{equation}
\int_{\zeta \in D} \mathcal{A}(\zeta, z) \phi(\zeta) = T \left( \int_{\zeta \in D} HR \wedge \overline{\partial}_tg_\alpha \phi \right) = - T \left( \int_{\zeta \in D} (f_1(t)H^1U \wedge \overline{\partial}_tg_\alpha \Psi) \right).
\end{equation}
In fact, the change of order of $T$ and integration with respect to $\zeta \in D$ is legitimate since the currents $U$ and $R$, as well as $(-\rho(\zeta))^\tau$ go outside and what is left are forms depending on $t$ that are smooth in $\overline{D}$. Since
\begin{equation}
\int_{\zeta \in D} f_1(t)H^1U \wedge \overline{\partial}_tg_\alpha \Phi
\end{equation}
is in $\mathcal{E}^{0,1}J(\overline{D})$ and $\overline{\partial}_t$-closed, it follows from Proposition 7.1 that
\begin{equation}
T \left( \int_{\zeta \in D} f_1(t)H^1U \wedge \overline{\partial}_tg_\alpha \Psi \right)
\end{equation}
is in $\mathcal{E}^{0,0}J(\overline{D})$ with respect to $z$. We conclude that (7.7) is in $\mathcal{E}^{0,0}J(\overline{D})$. Thus $\Phi - \phi$ is in $\mathcal{E}^{0,0}J(D)$, and since $\Phi$ is holomorphic, therefore $\Phi - \phi$ is in $J$, see Lemma 7.2. Thus the claim is proved. □
Now the proof of Theorem 1.1, that is, estimating the extension $\Phi$, is concluded in the essentially same way as for the case with the ball in Section 6. Since $A$ has no singularities at the diagonal the second term in the definition (7.5) of $\Phi$ offers no problems at all. The first term is handled as in the proof of Theorem 1.1. In fact, close to a point $\partial D \cap Z$ the same arguments as before work. Each time a holomorphic derivative falls on $v$ we get $O(|\zeta - z|)$ which cancels the singularity in view of (5.2). In a neighborhood of a (possibly non-Cohen Macaulay) point in $D \cap Z$ one proceed precisely as in the the proof of Theorem 1.1.

**Lemma 7.2.** If $\Phi$ is holomorphic and in $E_{0,0} J$, then it is in $J$. More explicitly, if $\eta_1, \ldots, \eta_v$ generate $J$, $\Phi = a_1 \eta_1 + \cdots + a_v \eta_v$ for some smooth functions $a_j$ and $\Phi$ is holomorphic, then one can choose holomorphic such $a_j$. This must be well-known but we include a short proof. Notice that the components of $f_1$ generate $J$.

*Proof.* It is a local statement. Since $\Phi$ is holomorphic we can choose a local representation formula

$$\Phi(z) = \int_\zeta (f_1(z)H^1 U + HR) \wedge g \Phi,$$

where the kernel depends holomorphically on $z$. The hypothesis implies that $\Phi$ annihilates $R$, and hence the conclusion follows. $\Box$

8. The $\bar{\partial}$-equation for forms in $\mathcal{E} J$

In this section we let $J$ be a quite arbitrary ideal sheaf of dimension $n$ in a pseudoconvex domain $\Omega \subset \mathbb{C}^N$. For simplicity we assume that the underlying reduced space $Z$ is smooth.

**Theorem 8.1.** Let $J$ be an ideal sheaf in a pseudoconvex domain $\Omega \subset \mathbb{C}^N$ such that $Z = X_{\text{red}}$ has pure dimension $n$, and let $\Omega' \subset \Omega$. There is a linear operator $T: \mathcal{E}^{0,1}(\Omega) \cap \ker \bar{\partial} \to \mathcal{E}^{0,0}(\Omega')$, such that $\bar{\partial} T \xi = \xi$ in $\Omega'$ and furthermore $T \xi \in \mathcal{E}^{0,0} J(\Omega')$ if $\xi \in \mathcal{E}^{0,1} J(\Omega)$.

*Proof.* In a possibly slightly smaller pseudoconvex domain, that we denote by $\Omega$ as well, we can choose a Hermitian free resolution (3.1) of $\mathcal{O}_\Omega / J$. Let $U$ and $R$ be that associated currents and let $H$ be a Hefer morphism associated (3.1). Moreover, let $g$ be a smooth weight with respect to $U \in \Omega'$ with compact support in $\Omega$, cf. Section 4. We also assume that $g$ depends holomorphically on $z$. Furthermore, let $B$ be the component of the full Bochner-Martinelli form, see [3, Section 2], that only has holomorphic differentials with respect to $\zeta$. It follows from [3, Section 7.4], see also [8, 7], that if $v$ is a smooth $(0,1)$-form in $\Omega$, then

$$v(z) = \int_\zeta (f_1(z)H^1 U + HR) \wedge g \wedge B \wedge \bar{\partial} v + \int_\zeta (f_1(z)H^1 U + HR) \wedge g v$$

for $z \in \Omega'$. In fact, one can choose regularizations $U^\varepsilon$ and $R^\varepsilon$ of $U$ and $R$, respectively, so that

$$g^\varepsilon = f_1(z)H^1 U^\varepsilon + HR^\varepsilon$$

are smooth weights, and then

$$v = \int_\zeta g^\varepsilon \wedge g \wedge B \wedge \bar{\partial} v + \int_\zeta g^\varepsilon \wedge g v$$
holds for $\epsilon > 0$, see, Remark [3] and, e.g., [8]. Now
\[
g' \rightarrow g' := f_1(z)H^1U + HR
\]
as currents when $\epsilon \to 0$. Notice that $g' \wedge B$ is a tensor product of currents and hence well-defined in $\Omega \times \Omega$, and that $g' \wedge B \to g' \wedge B$. Thus (8.1) follows from (8.2).

Let $\psi$ be a $\bar{\partial}$-closed smooth $(0,1)$-form in $\Omega$ and let $v$ be a (smooth) solution to $\bar{\partial}v = \psi$ in $\Omega$. Since the second term in (8.1) is holomorphic, it follows that
\[
\text{(8.3)}
T\psi := \int_\zeta (f_1(z)H^1U + HR) \wedge g \wedge B \wedge \psi
\]
is a solution to $\bar{\partial}u = \psi$ in $\Omega'$. Since two solutions only differ by a holomorphic function it is clear that $T\psi$ is smooth. This is also seen directly, noticing that
\[
\text{(8.4)}
T\psi = v - \int_\zeta (f_1(z)H^1U + HR) \wedge gv.
\]

Now assume that, in addition, $\psi \in \mathfrak{e}^{0,1}J$. Then $R\psi = 0$ and thus the second term in (8.3), i.e., $HR \wedge g \wedge B \wedge \psi$, vanishes since it is a tensor product times a smooth form. Thus
\[
T\psi(z) = f_1(z) \int_\zeta H^1U \wedge g \wedge B \wedge \psi =: f_1(z)b(z).
\]
However, we do not know that $b$ is smooth; in fact it is (probably) not in general, and hence we cannot conclude directly that $u \in \mathfrak{e}^{0,0}J$. Notice for instance that $1 = f(1/f)$ although $1$ is not in $\langle f \rangle$. To prove that $u$ is indeed in $u \in \mathfrak{e}^{0,0}J$ we will first use the following lemma.

**Lemma 8.2.** If $\psi \in \mathfrak{e}^{0,1}J$, $\bar{\partial}\psi = 0$ and $u = T\psi$, then $Ru = 0$.

Since $u$ is smooth, $Ru$ is well-defined.

**Proof.** Let $R_z$ denote $R$ depending on $z$. First notice that $R_z \wedge U$ is a well-defined current in $\Omega_\zeta \times \Omega_\zeta$ since it is a tensor product. Moreover, $B$ is an almost semmeromorphic form and therefore, cf. (3.3),
\[
R_z \wedge H^1U \wedge B := \lim_{\epsilon \to 0} R_z \wedge H^1U \wedge B^\epsilon
\]
is a well-defined current, where $B^\epsilon = \chi(|\zeta - z|^2/\epsilon)B$. See also [8 7 11].

Since $u$ is smooth and $R_z^\epsilon \to R_z$ we have that $R_z^\epsilon u \to R_zu$. Moreover,
\[
R_z^\epsilon u = \int_\zeta R_z^\epsilon \wedge f_1(z)H^1U \wedge B \wedge g\psi.
\]
We claim that
\[
\text{(8.5)}
W_k = \lim_{\epsilon \to 0} R_{z,k}^\epsilon \wedge H^1U \wedge B - R_{z,k} \wedge H^1U \wedge B = 0, \quad k = 0, 1, \ldots.
\]

The proof of this claim relies on the fact that all currents involved are pseudommeromorphic and that such currents fulfills the dimension principle: If $\mu$ is pseudommeromorphic, has bidegree $(*, \ell)$, and support on a subvariety of codimension strictly larger than $\ell$, then $\mu$ must vanish. See [10 8].

**Proof of the claim.** Since $R_{z,k} \wedge U$ is a tensor product, $R_{z,k}^\epsilon \wedge U \to R_{z,k} \wedge U$. Since $B$ is smooth outside the diagonal $\Delta$, therefore $W_k = 0$ there. That is, $W_k$ has support on $\Delta$. 

Recall that $H^1U$ is a sum of currents of bidegree $(*,*)$ in $\zeta$ so that $H^1U \wedge B$ is a sum of currents of bidegree at most $(N,N-1)$. Thus $W_k$ has bidegree at most $(N,N-1+k)$. Since $R_k$ has support on $Z$ we have that $W_k$ has support on $\Delta \cap \Omega \times Z$ which we can think of as $Z \subset \Delta \subset \Omega \times \Omega$, and hence it has codimension $N + \kappa$ in $\Omega \times \Omega$. By the dimension principle we conclude that $W_k = 0$ if $k \leq \kappa$.

Next we use the fact that outside a Zariski closed set $Z_1 \subset Z$ with codimension at least 1 in $Z$ there is a smooth form $\alpha_1$ such that $R_{\kappa+1} = \alpha_1 R_\kappa$, see, \cite{9}. Outside $Z_1$ thus $W_{\kappa+1} = \alpha_1 W_\kappa = 0$. Thus $W_{\kappa+1}$ has anti-holomorphic degree at most $N-1+\kappa+1$ and support on $Z_1 \subset \Delta \subset \Omega \times \Omega$. Again by the dimension principle it must vanish. In general, there are Zariski closed sets $Z_\ell \subset Z$ of codimension at least $\ell$ in $Z$, and smooth forms $\alpha_\ell$ outside $Z_\ell \subset Z$ such that $R_{\kappa+\ell+1} = \alpha_\ell R_{\kappa+\ell}$ there. The claim now follows by finite induction. □

From the claim we conclude that

$$R_z T\psi(z) = \int_\zeta R_z f_1(z) H^1U \wedge g \wedge B \wedge \psi = \lim_{\ell \to 0} \int_\zeta R_z f_1(z) H^1U \wedge g \wedge B^* \wedge \psi = 0$$

since $R_z f_1(z) = 0$ and $R_z f_1(z) H^1U$ is a tensor product times the smooth form $H$. Thus the lemma is proved. □

We can now conclude the proof of Theorem 8.1. Since $\partial u = \psi$, that is,

$$\partial u / \partial \bar{\zeta} = \psi_j, \quad j = 1, \ldots, N,$$

where each $\psi_j$ is in $\mathcal{E}^{0,0} J$, we conclude that

$$(\partial^\alpha u / \partial \bar{\zeta}^\alpha u) R = 0$$

for all $\alpha \geq 0$. It now follows from \cite{9} Theorem 5.1 that $u$ is in $\mathcal{E}^{0,0} J$. □

Remark 8.3. If $f$ is a holomorphic tuple that vanishes on $Z$ and $\chi(t)$ is as before then one can take $U^\epsilon = \chi(|f|^2/\epsilon) U$ and then define $R^\epsilon$ so that $\nabla_j U^{\epsilon,0} = I - R^\epsilon$, cf. Remark 6.4. Notice that $R\epsilon_k$ are non-vanishing for all $k \geq 0$. □

9. Proof of Theorem 1.2

If $J = \langle f^{M+1} \rangle$, then we have the simple resolution

$$0 \to \mathcal{O}(E_1) f^{M+1} \to \mathcal{O}(E_0) \to \mathcal{O}/J \to 0,$$

where $E_1$ and $E_0$ are trivial line bundles. Moreover,

$$U = \frac{1}{f^{M+1}}, \quad R = R_1 = \bar{\partial} \frac{1}{f^{M+1}},$$

and if $h$ is a holomorphic $(1,0)$-form in $\Omega$ for each $z \in \Omega$ such that $\delta_{\zeta-z} h = f - f(z)$, then

$$H = \sum_{k=0}^M f(\zeta)^{M-k} f(z)^k h$$

is a Hefer form for $f^{M+1}$, that is,

$$\delta_{\zeta-z} H = f(\zeta)^{M+1} - f(z)^{M+1}.$$

Thus

$$HR = H \bar{\partial} \frac{1}{f^{M+1}} = \sum_{k=0}^M f^k(z) h \wedge \bar{\partial} \frac{1}{f^{k+1}}.$$
Let us first assume that we are in the ball so that \( v(\zeta, z) \) is holomorphic in \( z \) and anti-holomorphic in \( \zeta \). Then we get our extension
\[
\Phi(z) = \int_{D \cap X} \sum_{k=0}^{M} f^k(z) \overline{\partial} \frac{1}{f^{k+1}} \wedge h \wedge g^\alpha \phi
\]
for a suitably large \( \alpha \). Arguing precisely as in Section 6, cf. (6.3), we see that
\[
\Phi(z) = \int_{D \cap Z} \sum_{k=0}^{M} f^k(z) \left( -\frac{\partial}{\partial \phi} \right)^\alpha \frac{1}{v^\alpha + n + 1} \beta_k \sum_{|\beta|=k} \partial^\beta \phi
\]
where \( \beta_k \) are smooth forms. If \( \zeta \in Z \), then \( f(z) = f(z) - f(\zeta) = O(|\zeta - z|) \) and hence \( |f(z)| \leq \sqrt{|v|} \). Using the same estimates as in Section 6 now Theorem 1.2 follows in the case with the ball. Combining with the arguments in Section 7 the general case follows.

Remark 9.1. It is reasonable to believe that it is possible to get a similar sharpening of Theorem 1.1 for instance, if \( Z \) has higher codimension and \( J \) is a jet ideal \( J_{M+1} \).

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