The off-resonant dielectronic recombination in a collision of an electron with a heavy hydrogen-like ion

Vladimir A. Yerokhin\textsuperscript{1,2,3} and Andrey Surzhykov\textsuperscript{1,2}

\textsuperscript{1}Institute of Physics, University of Heidelberg, Philosophenweg 12, D-69120 Heidelberg, Germany
\textsuperscript{2}Gesellschaft für Schwerionenforschung, Planckstraße 1, D-64291 Darmstadt, Germany
\textsuperscript{3}Center for Advanced Studies, St. Petersburg State Polytechnical University, Polytekhnicheskaya 29, St. Petersburg 195251, Russia

The recombination of an electron with an (initially) hydrogen-like ion is investigated. The effect of the electron-electron interaction is treated rigorously to the first order in the parameter $1/Z$ and within the screening-potential approximation to higher orders in $1/Z$, with $Z$ being the nuclear charge number. The two-electron correction contains the dielectronic-recombination part, which contributes to the process not only under the resonance condition for the projectile energy but also in the regions far from resonances. The mechanism of the off-resonant dielectronic recombination is studied in detail.

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I. INTRODUCTION

One of the main processes occurring in collisions of a highly charged ion with an electron is radiative recombination (RR), in which the electron is captured from the continuum into a bound state with emission of a photon. In the case when the ion initially possesses one or several electrons, the electron capture can proceed also via dielectronic recombination (DR), in which the energy excess goes to the excitation of the second electron, which then returns to the ground state via a radiative decay. DR is a resonant process and is usually studied under the condition that the excess energy is very close to the excitation energy of the second electron. In this case, DR is the dominant recombination channel, whereas RR is responsible for a nonresonant background. Outside of the resonance region, RR is the dominant process.

In the zeroth approximation, RR and DR are often considered as two independent recombination channels, which can be calculated separately and combined additively \cite{1}. More accurate calculations include the effects of quantum interference between DR and RR \cite{2,4}. Generally speaking, at the level of precision where effects of the electron-electron interaction come into play, RR and DR cannot be meaningfully separated. Outside of the resonance region, RR is the dominant process.

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The accuracy of experimental investigations of the RR process with heavy highly charged ions has gradually increased during the past years \cite{2,4}, reaching the level on which the electron-electron interaction effect can be clearly identified \cite{5}. A disagreement with the one-electron theory observed in the state-selective study of RR into hydrogen-like uranium \cite{8} calls for an accurate theoretical description of the electron-electron interaction effect.

Most of the previous calculations of the RR process into heavy few-electron ions accounted for the electron correlation by means of the Dirac-Fock method \cite{8,10}, disregarding the off-resonant DR mechanism. Evidences that the omitted contribution might be significant were reported in Refs. \cite{11,12}, where a part of the off-resonant DR (involving photon emission from a core electron) was studied. It was claimed that, for many-electron systems, this mechanism can significantly influence the RR process, yielding an order-of-magnitude enhancement in some specific cases.

In the present investigation we perform an \textit{ab initio} calculation of the electron-electron interaction effect on RR into a heavy hydrogen-like ion in the non-resonant region of energies of the incoming electron. A particular emphasis will be made on the contribution of the off-resonant DR, as this effect has not been carefully studied before. A similar study of RR into a helium-like uranium have been reported previously in Ref. \cite{13}.

Relativistic units ($\hbar = c = 1$) are used in this paper.

II. GENERAL APPROACH

We consider RR of an electron with an (initially) hydrogen-like ion. The initial state consists of the incident electron with the asymptotic momentum $p$, the energy $\varepsilon = \sqrt{p^2 + m^2}$, and the spin projection $\mu_s = \pm 1/2$ and the bound (core) electron in the state $a'$ with the relativistic angular quantum number $\kappa_{a'}$, and the momentum projection $\mu_{a'}$. In the final state, there is the two-electron bound state with the angular momentum $J$ and the projection $M$ and the outgoing photon with the momentum $k$, and the energy $\omega = |k|$. The wave function of the final two-electron state is

$$|JM\rangle = N \sum_{\mu_a \mu_v} C_{Ja \mu_a, J_{av}, \nu, \mu_{a'}} |\kappa_{a'}| \langle \kappa_{a} \mu_{a'} | \mu_a \mu_v \rangle ,$$

where $a$ and $v$ stand for the core and the valence electron, respectively, and $N = 1/\sqrt{2}$ for the equivalent electrons.
and $N = 1$ otherwise. The core electron state is not changed in the process, thus $\kappa_a = \kappa_{a'}. $ The wave function $\hat{\psi}$ is not antisymmetrized since we choose to perform antisymmetrization explicitly for the amplitude.

General formulas are conveniently written in the center-of-mass frame, which practically coincides with the rest system of the ion. The direction of the $z$ axis of the coordinate system is chosen to be the direction of the emitted photon.

In the following, we will assume that the fine-structure levels with different $J$'s are not resolved in the experiment (as is the case for the experiments conducted so far).

### A. Zeroth order

To the zeroth order, we neglect the electron-electron interaction. The core electron does not participate in the process and the transition amplitude is written as

$$\tau_{\mu_s, \mu_v}^{(0)} J M = N \sum_{\mu_a \mu_v} C^{JM}_{\mu_a \mu_v} \delta_{\mu_a \mu_v} \langle \mu_s | \hat{\psi}^{(0)} | \mu_v \rangle,$$

(2)

where $\tau_{\mu_s, \mu_v}^{(0)}$ is the amplitude for the recombinations with the bare nucleus. It reads

$$\tau_{\mu_s, \mu_v}^{(0)} = \langle v | \alpha \cdot \hat{u}^* e^{-i k \cdot r} | p \rangle,$$

(3)

where $|v\rangle \equiv |\kappa_v \mu_v\rangle$ denotes the bound state, $|p\rangle \equiv |\mu_a \rangle$ is the Dirac continuum-state wave function with a definite asymptotic momentum, and $\hat{u}$ is the unit polarization vector of the emitted photon. After summation over the final states and averaging over the initial states, the differential cross section of the process is written as

$$\frac{d\sigma^{(0)}}{d\Omega} = \frac{1}{2j_a + 1} \frac{\alpha \omega m}{4\beta^2 \varepsilon^2} \sum_{\mu_s \mu_v} |\tau_{\mu_s, \mu_v}^{(0)} J M|^2,$$

(4)

$$= N^2 \frac{\alpha \omega m}{4\beta^2 \varepsilon^2} \sum_{\mu_s \mu_v} |\tau_{\mu_s, \mu_v}^{(0)}|^2,$$

where $\beta = \sqrt{1 - m^2 / \varepsilon^2}$. Because of the summation over the initial and final states, the cross section does not depend on the polarization of the emitted photon, which can be fixed arbitrary. The formula (4) differs from the corresponding expression for the RR into the bare nucleus (1) only by a factor of $N^2 (= 1/2$ for the recombinations into the ground state and 1 otherwise).

The energy of the emitted photon in Eq. (4) is fixed by the energy conservation condition $\omega = \varepsilon - \varepsilon_i$ or, more generally, $\omega = \varepsilon - m + \varepsilon_i$, where $\varepsilon_i$ is the ionization energy of the atom in the final state.

### B. Electron-electron interaction

For a heavy few-electron ion, the electron-electron interaction can be effectively accounted for by a perturbative expansion in the parameter $1/Z$. The first-order correction is induced by a single virtual-photon exchange between the electrons, shown diagrammatically in Fig. 1.

The corresponding correction to the differential cross section can be written as

$$\frac{d\sigma^{(1)}}{d\Omega} = \frac{d\sigma^{(0)}}{d\Omega} + \frac{1}{2j_a + 1} \frac{\alpha \omega m}{4\beta^2 \varepsilon^2} \sum_{\mu_s \mu_v} 2 \text{Re} \left[ \tau_{\mu_s, \mu_v}^{(0)*} \tau_{\mu_s, \mu_v}^{(1)} J M \right],$$

(5)

where $\tau_{\mu_s, \mu_v}^{(1)}$ is the first-order correction to the amplitude and $\delta_\omega$ is induced by the change of the energy of the emitted photon (because of the shift of the energy of the final state due to the presence of the second electron),

$$\delta_\omega \frac{d\sigma^{(0)}}{d\Omega} = \frac{d\sigma^{(0)}}{d\Omega} \bigg|_{\omega = \omega^{(0)} + \delta_\omega} - \frac{d\sigma^{(0)}}{d\Omega} \bigg|_{\omega = \omega^{(0)}}.$$

(6)

We note that Eq. (5) assumes that the perturbative regime $(|\tau^{(1)}| \ll |\tau^{(0)}|)$ takes place.

Since the fine-structure sublevels of the final state are not resolved in the experiment, the dependence on $J$ and $M$ can be eliminated already in the general formulas. To achieve this, we write the correction to the amplitude as

$$\tau_{\mu_s, \mu_v}^{(1)} J M = N \sum_{\mu_a} C^{JM}_{\mu_a \mu_v} \tau_{\mu_s, \mu_a, \mu_v}^{(1)} J M,$$

(7)

where $\tau_{\mu_s, \mu_a, \mu_v}^{(1)}$ does not depend on $J$ and $M$. Inserting this formula and Eq. (2) into Eq. (5) and performing summations, we obtain

$$\frac{d\sigma^{(1)}}{d\Omega} = \delta_\omega \frac{d\sigma^{(0)}}{d\Omega} + \frac{N^2}{2j_a + 1} \frac{\alpha \omega m}{4\beta^2 \varepsilon^2} \sum_{\mu_s \mu_v} 2 \text{Re} \left[ \tau_{\mu_s, \mu_v}^{(0)*} \tau_{\mu_s, \mu_v}^{(1)} \right].$$

(8)

Here,

$$\tau_{\mu_s, \mu_v}^{(1)} \equiv \sum_{\mu_a} \tau_{\mu_s, \mu_a, \mu_v}^{(1)}.$$

(9)

is the amplitude of the recombinations with a closed-shell atom. So, we obtain that, in the situation when the fine-structure levels are not resolved in the experiment, formulas for the RR with a hydrogen-like ion differ from those for the RR with a helium-like ion only by a prefactor of $N^2/(2j_a + 1)$.

General expressions for the one-photon exchange correction to the RR of an electron with a heavy ion were derived in Ref. [12]. (For the closed-shell ions, such derivation was reported also in Ref. [13]). The correction to the transition amplitude consists of 8 terms corresponding to the 8 diagrams on Fig. 1

$$\tau_{\mu_s, \mu_v}^{(1)} = \sum_{i=1}^{8} \tau_{\mu_s, \mu_v}^{(1,i)}.$$

(10)
The individual contributions for each diagram are given by

\[
\tau^{(1,1)} = \sum_{\mu_a, n \neq v} \frac{\langle va|I(0)|na\rangle \langle n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|p\rangle}{\varepsilon_v - \varepsilon_n},
\]

(11)

\[
\tau^{(1,2)} = \sum_{\mu_a, n} \frac{\langle v\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|pa\rangle}{\varepsilon - \varepsilon_n(1 - i0)} ,
\]

(12)

\[
\tau^{(1,3)} = \sum_{\mu_a, n} \frac{\langle va|I(\varepsilon - \varepsilon_v)|mn\rangle \langle n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|a\rangle}{\varepsilon_a + \varepsilon_v - \varepsilon_n(1 - i0)},
\]

(13)

\[
\tau^{(1,4)} = \sum_{\mu_a, n} \frac{\langle a|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|n\rangle \langle vn|I(\varepsilon - \varepsilon_v)|pa\rangle}{\varepsilon_a - \varepsilon_v + \varepsilon - \varepsilon_n(1 - i0)},
\]

(14)

\[
\tau^{(1,5)} = - \sum_{\mu_a, n \neq v} \frac{\langle va|I(\varepsilon_v - \varepsilon_a)|na\rangle \langle n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|p\rangle}{\varepsilon_v - \varepsilon_n} - \frac{1}{2} \sum_{\mu_a, \mu_{a'}} \langle va'|I'(\varepsilon_v - \varepsilon_a)|v'\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|p\rangle,
\]

(15)

\[
\tau^{(1,6)} = - \sum_{\mu_a, n} \frac{\langle v\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|ap\rangle}{\varepsilon - \varepsilon_n(1 - i0)},
\]

(16)

\[
\tau^{(1,7)} = - \sum_{\mu_a, n} \frac{\langle av|I(\varepsilon - \varepsilon_a)|mn\rangle \langle n|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|a\rangle}{\varepsilon_a + \varepsilon_v - \varepsilon_n(1 - i0)},
\]

(17)

\[
\tau^{(1,8)} = - \sum_{\mu_a, n} \frac{\langle a|\alpha \cdot \hat{u}^* e^{-i\mathbf{k} \cdot \mathbf{r}}|n\rangle \langle vn|I(\varepsilon_a - \varepsilon_v)|ap\rangle}{\varepsilon_a - \varepsilon_v + \varepsilon - \varepsilon_n(1 - i0)}.
\]

(18)

Here, \(I(\Delta)\) is the operator of the electron-electron interaction,

\[
I(\Delta) = e^2 \alpha_{\mu} \alpha_{\nu} \, D^{\mu\nu}(\Delta, x_{12}),
\]

(19)

where \(D^{\mu\nu}\) is the photon propagator. In the Feynman gauge, the operator \(I\) takes the form

\[
I(\Delta) = \frac{\alpha}{4\pi} \frac{1 - \alpha_1 \cdot \alpha_2}{x_{12}} e^{i\Delta |x_{12}|}.
\]

(20)

The summations over \(n\) in Eqs. (11)-(18) extend over the complete spectrum of the Dirac equation. The second term on the right-hand-side of Eq. (15) corresponds to the \(n = v\) contribution excluded from the summation in the first term. The prime on the operator \(I\) denotes the derivative with respect to the energy argument. The state \(v'\) is the \(n = v\) state with the angular momentum projection \(\mu_{v'}\). The small imaginary addition to the intermediate-state energies in the energy denominators fixes the position of the energy argument of the electron propagator \(G(\mathcal{E})\) with respect to the branch cuts for \(|\mathcal{E}| > m\).
We now turn to the physical interpretation of individual diagrams in Fig. I. The first two graphs represent the effect of the screening of the nuclear charge by the core electron. The corresponding corrections \([\tau^{(1.1)}] and \([\tau^{(1.2)}]\) can be regarded as the first-order perturbations of the zeroth-order amplitude \([4]\) by the screening potential of the core electron,

\[
V_{\text{scr}}(x) = \alpha \int_0^\infty dy y^2 \frac{1}{\max(x, y)} \left[ g_a^2(y) + f_a^2(y) \right], \tag{21}
\]

where \(g_a\) and \(f_a\) are the upper and the lower radial components of the core electron state. The screening effect can easily be accounted for to all orders in \(1/Z\) by evaluating the zeroth-order amplitude for an electron in a combination of the nuclear and the screening potentials. Such treatment is exactly equivalent to the frozen-core Dirac-Fock method (as the core in our case contains one electron only).

The contribution of diagram (5) in Fig. II can be interpreted to represent the electron correlation on the bound-electron wave function (also known as the “relaxation” effect). It can be partly included by the standard many-body techniques like many-body perturbation theory or the multiconfiguration Dirac-Fock method.

The contribution of diagrams (4) and (8) in Fig. II contain resonant parts that become prominent when the projectile energy approaches the region where \(\varepsilon_v \approx \varepsilon_a \pm \varepsilon_n > m\), with \(\varepsilon_n\) being a Dirac bound-state energy. When the resonance condition is fulfilled, the core electron gets excited into a higher-lying bound state, which corresponds to the standard resonant DR mechanism. In that case, the electron propagator can be replaced by a contribution of the single state responsible for the resonance (the so-called “resonance” approximation), thus greatly simplifying the problem. In the region far from the resonance, however, the core electron gets “excited” in all possible virtual states of the energy spectrum, so that the usage of the full Dirac propagator becomes essential in the description of this process.

The diagrams (3), (6) and (7) in Fig. I correspond to other processes with participation of the core electron, in which the full energy spectrum of virtual states is probed. We will refer to the contribution of all the diagrams (3), (4), (6), (7), and (8) as the (off-resonant) DR correction. So, in the present work, the term DR is used to refer to the recombination with an assistance of the second electron, rather than only to the resonant part of this process, as is customary. It should be noted that the separation of the total two-electron effect in several parts is to a large extent artificial (e.g., the DR correction defined in this way is not gauge invariant). Its main justification is that the screening and correlation parts are easily accounted for by standard methods, whereas the DR part is not. The sum of all two-electron contributions, however, is gauge invariant and derived rigorously within QED.

So, we represent the total RR cross section as a sum of four terms,

\[
\sigma = \sigma^{(0)} + \sigma_{\text{scr}} + \sigma_{\text{corr}}^{(1)} + \sigma_{\text{DR}}^{(1)}, \tag{22}
\]

where \(\sigma^{(0)}\) is the zeroth-order cross section, \(\sigma_{\text{scr}}\) is the correction induced by the screening potential \(V_{\text{scr}}\) included to all orders, \(\sigma_{\text{corr}}^{(1)}\) is the correlation correction induced by \(\tau^{(1,5)}\), and \(\sigma_{\text{DR}}^{(1)}\) is the off-resonant DR contribution induced by \(\tau^{(1,3)}\), \(\tau^{(1,4)}\), \(\tau^{(1,6)}\), \(\tau^{(1,7)}\), and \(\tau^{(1,8)}\). The screening correction is calculated with the “correct” energy of the emitted photon and thus includes the \(\delta_\omega\) correction in Eq. (5). We assume that the projectile energy is far enough from the resonance condition to ensure that the perturbative regime is valid.

III. NUMERICAL EVALUATION

The integration over angular variables in the general formulas of the previous section can be performed by means of the standard Racah algebra, as illustrated in Ref. [13]. The resulting formulas for the zeroth-order transition amplitude and for the first-order corrections are given in Appendix. Performing our calculations, we found several sign mistakes in Ref. [13]. Namely, the contributions of Eqs. (A.3) and (A.4) were accounted for with the opposite sign in that work. Moreover, the incorrect sign was present in the first term of Eq. (A.6) in the case of the capture into the \(2p_1/2\) state.

The zeroth-order cross section \(\sigma^{(0)}\) and the screening correction \(\sigma_{\text{scr}}\) were evaluated according to Eqs. (I) and (A.2). The radial bound and continuum wave functions were obtained by solving the Dirac equation with an extended-nucleus Coulomb potential and the screening potential of the core electron, by using the RADIAL package by Salvat et al. [16].

The calculation of the first-order corrections \(\sigma_{\text{corr}}^{(1)}\) and \(\sigma_{\text{DR}}^{(1)}\) was more complicated, due to a larger number of radial integrations and the summations over the complete spectrum of the Dirac equation. In the evaluation of the \(\tau^{(1,5)}\) correction, we employed the dual kinetically balanced B-spline basis set [17] to represent the Dirac spectrum. In most of other cases, we used the analytical representation of the radial Dirac Coulomb Green function in terms of Whittaker functions [18]. For simplicity, we used the point-nucleus Green function, since the effect of the finite nuclear size turned out to be negligibly small. In the evaluation of the \(\tau^{(1,4)}\) and \(\tau^{(1,8)}\) corrections, we used the finite basis set when the energy argument of the Green function was smaller than the electron rest mass \(\mathcal{E} < m\), and the exact Green function, otherwise. The Dirac Coulomb Green function with \(\mathcal{E} > m\) is a complex-valued function and a care must be taken in order to choose the appropriate branch of it. The sign of the imaginary part of the Green function is fixed by the sign of the small imaginary addition in the energy denominators of Eqs. (12)-(17) and discussed in detail in Ref. [13].
A problem emerges in the numerical evaluation of the radial integrals when they contain, apart from the Bessel function, two continuum-state wave functions. In this case, the integrand is a rapidly oscillating function that falls off very slowly at large radial distances. In our case, such situation arises only in the evaluation of the $\tau^{(1,2)}$ correction for projectile energies $\varepsilon > m - \varepsilon_a + \varepsilon_e$. [The problem appears also for the $\tau^{(1,2)}$ correction if it is evaluated perturbatively but not if it is evaluated to all orders.]

Our scheme of evaluation of radial integrals was as follows. First, we introduce the parameter $R_1$ that represents the distance at which all bound-state wave functions become negligibly small. (Typically, $R_1 = 80/Z$ a.u.) At the distances $r > R_1$, all radial integrals with bound-state wave functions reach their asymptotic values, so that the problem reduces to the evaluation of one-dimensional integrals of the form

$$\int_{R_1}^{\infty} dr r^2 j_i(\omega r) f^\dagger(r) \Phi_\infty^\dagger(r),$$

where $j_i$ is a spherical Bessel function, $f^\dagger$ is a radial component of the continuum-state Dirac wave function and $\Phi_\infty^\dagger$ is the irregular solution of the Dirac equation (originating from the Green function). To evaluate these integrals, we introduce a small regulator parameter $\alpha > 0$ and multiply the integrand by $\exp(-\alpha r)$. The regularized integrals are cut off at large distances by a parameter $R_2 \propto 1/\alpha$ and evaluated numerically with Gauss-Legendre quadratures. The typical value of the regulator was $\alpha = 10^{-3}$. We checked that decreasing the regulator by a factor of 10 does not influence our numerical results significantly.

IV. RESULTS AND DISCUSSION

The calculational results for the total cross section of the RR of an electron with an (initially) hydrogen-like uranium are presented in Table I for the capture into the $(1s)^2, 1s2s, 1s2p_{1/2},$ and $1s2p_{3/2}$ states. $\sigma^{(1)}$ is the zeroth-order cross section. It is calculated with the energy of the emitted photon that includes all known one-electron corrections to the energy of the final state, i.e., $\omega = \varepsilon - m - \varepsilon_{io,H}$, where $\varepsilon_{io,H}$ is the ionization energy of the hydrogen-like ion. $\sigma_{\text{DR}}^{(1)}$ is the correction due to the screening of the nuclear charge by the core electron. It was obtained by re-evaluating the zeroth-order cross section with the wave functions calculated in the presence of the screening potential. The energy of the emitted photon includes all known corrections to the energy of the final state [10], i.e., $\omega = \varepsilon - m - \varepsilon_{io}$, where $\varepsilon_{io}$ is the ionization energy of the helium-like ion. $\sigma_{\text{corr}}^{(1)}$ is the correlation correction induced by $\tau^{(1,5)}$, $\sigma_{\text{DR}}^{(1)}$ is the off-resonant DR correction induced by $\tau^{(1,3)}, \tau^{(1,4)}, \tau^{(1,6)}, \tau^{(1,7)},$ and $\tau^{(1,8)}$. For the recombination into the excited states, $\sigma_{\text{DR}}^{(1)}$ contains a series of the DR resonance peaks in the region of projectile energies $E = 110 - 190$ MeV/u. The behaviour of $\sigma_{\text{DR}}^{(1)}$ in the vicinity of the peaks is shown in Fig. 2. In the case of recombination into the ground state, $\sigma_{\text{DR}}^{(1)}$ does not have any resonances.

Our calculation shows that the effect of the screening of the nuclear charge generally grows for larger projectile energies and the capture into higher excited states, approaching the limit of the complete screening (i.e., the case of the capture by a bare nucleus with the $Z = 1$ charge). The effect of the off-resonant DR mechanism is the strongest for the capture into the ground state and for low projectile energies. In this case, the DR contribution is of the similar size as the contribution of the screening effect. We conclude that for the capture into the ground state, the electron-electron interaction needs to be accounted for rigorously and with inclusion of the off-resonant DR mechanism. Results obtained by an effective one-electron theory or by standard many-body approaches such as the Dirac-Fock method provide only an order-of-magnitude estimate of the two-electron effect in this case. However, for the recombination into the excited states and the projectile energy beyond the DR resonance threshold, the DR correction is much smaller than the screening contribution and can be neglected for most practical purposes. For the projectile energies below the threshold, the off-resonant DR mechanism can be important in the vicinity of the peaks, even at relatively large distances from the region of resonance.

In order to illustrate the dependence of the effects studied on the nuclear charge number $Z$, Table II presents the calculational results for the recombination into the $(1s)^2$ and $1s2s$ states of the initially hydrogen-like tin ($Z = 50$). We observe that the relative contribution of the screening effect is roughly proportional to $1/Z$, as could be expected. It is remarkable that the electron correlation correction, which plays only a minor role for uranium, becomes important for tin in the case of capture into the ground state. The relative contribution of the off-resonant DR mechanism is slightly larger for tin than for uranium, but, in comparison to the screening effect, the DR correction becomes somewhat less significant for lighter ions.

In Fig. 3 we present the results for the differential cross section for the case of the capture into the ground state of uranium, for two values of the projectile energy $E = 50$ and 300 MeV/u, which are typical for the ESR storage ring at GSI. The differential cross section is calculated in the laboratory frame, in which the initially free electron is at rest. We observe that the screening and the DR contributions have different dependence on the observation angle. For the zero angle, they are of the opposite sign and significantly cancel each other, whereas for larger angles these two effects amplify each other.

One of the motivations of the present study was a deviation from predictions of one-electron theory reported in the experimental investigation of RR into a hydrogen-like uranium at very small projectile velocities [8]. An effect of about 10% was observed in the experiment, whereas
a much smaller contribution on the level of 1-2% was expected from theory [20].

Our \textit{ab initio} calculation demonstrates that the electron-electron interaction affects the RR cross section on the level of about 2% for the projectile energies of several MeV/u, which agrees with previous estimates. For smaller projectile energies, the cross section is well described by the asymptotic behaviour \( E \sigma(E) = \text{const} \), and the relative values of all corrections stay constant. So, our calculation cannot explain the large two-electron effect observed in Ref. [8]. We note, however, that the quantities actually measured in this experiment were not the cross sections but the recombination rates. A consistent interpretation of the experimental results requires a careful consideration of the recombination rates under the experimental conditions. Such a calculation in under way and will be reported elsewhere.

V. SUMMARY

We have performed an investigation of the radiative recombination of an electron with an (initially) hydrogen-like ion. The electron-electron interaction was treated rigorously to the first order in the parameter \( 1/Z \) and within the screening-potential approximation to the higher orders in \( 1/Z \). The contribution of the off-resonant dielectronic recombination was studied in detail. It was demonstrated that this mechanism contributes significantly to the total effect of the electron-electron interaction in the case of recombination into the ground state. For the recombination into the excited states, it is significant in the vicinity of the resonance peaks but becomes small for the projectile energies beyond the resonant dielectronic-recombination threshold.

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Appendix: Calculational formulas

The spherical-wave expansion of the Dirac wave function of an incident electron with a fixed asymptotic momentum is [14]

\[
|p_{\mu}\rangle = 4\pi \sum_{\kappa \mu} i^l e^{i\Delta \kappa} C_{\mu^I\mu^J}^{\mu^I}\ Y^*_{\mu^I}(\hat{p}) \ |\kappa \mu\rangle, \quad (A.1)
\]

where \( j = |\kappa| - 1/2, \ l = |\kappa + 1/2| - 1/2, \ \Delta \kappa \) is the phase shift, and \( |\kappa \mu\rangle \) is the continuum Dirac wave function with the relativistic angular quantum number \( \kappa \) and the angular momentum projection \( \mu \), normalized on the energy scale. After the integration over the angular variables (see Ref. [13] for details), the result for the zeroth-order amplitude is given by

\[
\tau^{(0)}_{\mu^I, \mu^J} (\hat{p}) = 4\pi \sum_\kappa i^l e^{i\Delta \kappa} C_{\mu^I \mu^J}^{\mu^I} \ Y^*_{\mu^I}(\hat{p}) \sum_{JL} i^{-1-L} \times \sqrt{2L+1} C_{01\lambda}^{JM}(\hat{p}) P_{JL}(\omega, \mu), \quad (A.2)
\]

where the radial integrals \( P_{JL} \) are defined as

\[
P_{JL}(\omega, ab) = \int_0^\infty dx x^2 J_L(\omega x) \left[ g_b(x) f_\alpha(x) S_{JL}(\kappa_b, -\kappa_a) - f_b(x) g_\alpha(x) S_{JL}(-\kappa_b, \kappa_a) \right]. \quad (A.3)
\]

The angular coefficients \( S_{JL}(\kappa_1, \kappa_2) \) are given, e.g., by Eqs. (C7)-(C9) of Ref. [21]. The momentum projections \( \mu, M, \) and \( m_J \) in Eq. (A.2) are fixed by the selection rules of Clebsch-Gordan coefficients. \( \lambda = \pm 1 \) corresponds to...
FIG. 3: Individual two-electron contributions to the differential cross section of RR into the ground state of the initially hydrogen-like uranium, for two values of the projectile energy, $E = 50$ MeV/u (left column) and $300$ MeV/u (right column), as a function of the observation angle $\theta$. The upper graphs represent the absolute contributions to the cross section in barns/sr and the lower graphs, the relative magnitude of the corrections, in units per cent of the zeroth-order cross section $d\sigma^{(0)}$. The dash-dotted line (blue on-line) corresponds to the screening part; the dotted line (green on-line), to the correlation correction; the dashed line (red on-line), to the DR correction; and the solid line (black on-line), to the total two-electron effect.

The one-photon exchange corrections to the transition amplitude $\tau^{(1,i)}(1,i)$ can be expressed in the form similar to that for the zeroth-order amplitude, with the radial integrals $P_{JL}$ substituted by their generalizations $\mathcal{F}_{JL}^{(1,i)}$. The results for the functions $\mathcal{F}_{JL}^{(1,i)}$ are

$$\mathcal{F}_{JL}^{(1,3)} = \alpha \sum_{n} \frac{P_{JL}(\omega, na)}{\varepsilon_a + \varepsilon_v - \varepsilon - \varepsilon_n} \times \frac{(-1)^{J+J_n-J_a}}{2J+1} R_J(\varepsilon - \varepsilon_v, na \varepsilon) , \quad (A.4)$$

$$\mathcal{F}_{JL}^{(1,4)} = \alpha \sum_{n} \frac{P_{JL}(\omega, an)}{\varepsilon_a - \varepsilon_v + \varepsilon - \varepsilon_n} \times \frac{(-1)^{J+J_a-J_n}}{2J+1} R_J(\varepsilon - \varepsilon_v, na \varepsilon) , \quad (A.5)$$

$$\mathcal{F}_{JL}^{(1,5)} = \alpha \sum_{n \neq v} \frac{P_{JL}(\omega, n\varepsilon)}{\varepsilon_v - \varepsilon_n} \sum_{L_0} \frac{(-1)^{J_a+j_v+L_0}}{2j_v + 1} R_{L_0}(\varepsilon_v - \varepsilon_a, av \varepsilon)$$

$$\quad \times \frac{\alpha^2}{2} \sum_{L_0} \frac{(-1)^{J_v+j_a+L_0}}{2j_a + 1} R'_{L_0}(\varepsilon_a - \varepsilon_v, av \varepsilon) , \quad (A.6)$$

$$\mathcal{F}_{JL}^{(1,6)} = \alpha \sum_{n \neq v} \frac{P_{JL}(\omega, vn)}{\varepsilon_v - \varepsilon_n} \sum_{L_0} \frac{(-1)^{J_a+j_v+L_0}}{2j_v + 1} R_{L_0}(\varepsilon_v - \varepsilon_a, av \varepsilon)$$

$$\quad \times \frac{\alpha^2}{2} \sum_{L_0} \frac{(-1)^{J_v+j_a+L_0}}{2j_a + 1} R'_{L_0}(\varepsilon_a - \varepsilon_v, av \varepsilon) , \quad (A.7)$$

$$\mathcal{F}_{JL}^{(1,7)} = \alpha \sum_{n} \frac{P_{JL}(\omega, na)}{\varepsilon_a + \varepsilon_v - \varepsilon - \varepsilon_n} \sum_{L_0} (-1)^{J-a-j_n} J$$

$$\quad \times \left\{ \frac{j_v}{j_a} \frac{j_j}{j_n} \frac{J}{L_0} \right\} R_{L_0}(\varepsilon - \varepsilon_a, av \varepsilon) , \quad (A.8)$$
TABLE I: The total cross section of the radiative recombination of an electron into the \((1s)^2\), \(1s2s\), \(1s2p_{1/2}\) and \(1s2p_{3/2}\) states of the initially hydrogen-like uranium, for different values of the projectile energy \(E\). \(\sigma^{(0)}\) is the zeroth-order cross section. \(\sigma_{\text{scr}}\) is the screening correction, \(\sigma_{\text{cor}}^{(1)}\) is the correction due to the electron correlation on the bound electron state, and \(\sigma_{\text{dir}}^{(1)}\) is the correction due to the dielectronic recombination. All corrections are given in units per cent of \(\sigma^{(0)}\).

| \(E\) [MeV/u] | \(\sigma^{(0)}\) [barn] | \(\sigma_{\text{scr}}\) [%] | \(\sigma_{\text{cor}}^{(1)}\) [%] | \(\sigma_{\text{dir}}^{(1)}\) [%] | \(\sigma^{(0)}\) [barn] | \(\sigma_{\text{scr}}\) [%] | \(\sigma_{\text{cor}}^{(1)}\) [%] | \(\sigma_{\text{dir}}^{(1)}\) [%] |
|--------------|-------------------|------------------|------------------|------------------|-------------------|------------------|------------------|------------------|
| \(1\) | \(1.588 \times 10^4\) | \(-0.850\) | \(0.138\) | \(-0.703\) | \(5.080 \times 10^3\) | \(-1.997\) | \(-0.232\) | \(0.387\) |
| \(2\) | \(7.917 \times 10^3\) | \(-0.851\) | \(0.140\) | \(-0.702\) | \(2.539 \times 10^3\) | \(-1.981\) | \(-0.228\) | \(0.390\) |
| \(5\) | \(3.142 \times 10^3\) | \(-0.852\) | \(0.146\) | \(-0.702\) | \(1.013 \times 10^3\) | \(-1.936\) | \(-0.217\) | \(0.396\) |
| \(10\) | \(1.550 \times 10^3\) | \(-0.854\) | \(0.156\) | \(-0.701\) | \(5.040 \times 10^2\) | \(-1.874\) | \(-0.201\) | \(0.408\) |
| \(25\) | \(5.967 \times 10^2\) | \(-0.864\) | \(0.182\) | \(-0.696\) | \(1.968 \times 10^2\) | \(-1.755\) | \(-0.164\) | \(0.444\) |
| \(50\) | \(2.806 \times 10^2\) | \(-0.888\) | \(0.217\) | \(-0.683\) | \(9.315 \times 10^1\) | \(-1.673\) | \(-0.124\) | \(0.524\) |
| \(75\) | \(1.765 \times 10^2\) | \(-0.917\) | \(0.245\) | \(-0.668\) | \(5.841 \times 10^1\) | \(-1.652\) | \(-0.101\) | \(0.669\) |
| \(100\) | \(1.254 \times 10^2\) | \(-0.949\) | \(0.268\) | \(-0.650\) | \(4.118 \times 10^1\) | \(-1.655\) | \(-0.086\) | \(1.141\) |
| \(125\) | \(9.524 \times 10^1\) | \(-0.981\) | \(0.286\) | \(-0.634\) | \(3.101 \times 10^1\) | \(-1.670\) | \(-0.076\) | \(-6.124\) |
| \(150\) | \(7.559 \times 10^1\) | \(-1.013\) | \(0.302\) | \(-0.617\) | \(2.438 \times 10^1\) | \(-1.690\) | \(-0.069\) | \(1.635\) |
| \(175\) | \(6.188 \times 10^1\) | \(-1.043\) | \(0.314\) | \(-0.601\) | \(1.977 \times 10^1\) | \(-1.712\) | \(-0.065\) | \(0.443\) |
| \(200\) | \(5.184 \times 10^1\) | \(-1.072\) | \(0.325\) | \(-0.587\) | \(1.642 \times 10^1\) | \(-1.735\) | \(-0.062\) | \(-0.044\) |
| \(250\) | \(3.827 \times 10^1\) | \(-1.125\) | \(0.341\) | \(-0.562\) | \(1.192 \times 10^1\) | \(-1.778\) | \(-0.060\) | \(-0.034\) |
| \(300\) | \(2.968 \times 10^1\) | \(-1.170\) | \(0.353\) | \(-0.542\) | \(9.102\) | \(-1.817\) | \(-0.060\) | \(-0.017\) |
| \(400\) | \(1.966 \times 10^1\) | \(-1.242\) | \(0.369\) | \(-0.515\) | \(5.879\) | \(-1.880\) | \(-0.063\) | \(0.015\) |
| \(500\) | \(1.419 \times 10^1\) | \(-1.292\) | \(0.378\) | \(-0.499\) | \(4.159\) | \(-1.926\) | \(-0.068\) | \(0.034\) |
| \(600\) | \(1.084 \times 10^1\) | \(-1.327\) | \(0.383\) | \(-0.492\) | \(3.127\) | \(-1.958\) | \(-0.074\) | \(0.043\) |
| \(700\) | \(8.621\) | \(-1.350\) | \(0.387\) | \(-0.495\) | \(2.456\) | \(-1.980\) | \(-0.079\) | \(0.045\) |

\[
\mathcal{F}_{jL}^{(1,8)} = \alpha \sum_n \frac{P_{jL}(\omega, an)}{\varepsilon_a - \varepsilon_v + \varepsilon - \varepsilon_n} \sum_{l=0} (-1)^{j_a-j_n+j} \sum_{j_\nu} \frac{\sum_{j_a}^j}{j_\nu} R_{L_0}(\varepsilon_v - \varepsilon_a, vna\varepsilon), \quad (A.9)
\]

where \(R_L\) is the relativistic generalization of the Slater integral (see Appendix C of Ref. [21]). The prime of \(R_L\) in Eq. (A.6) denotes the derivative with respect to the energy argument, \(R_L'(\varepsilon, abcd) = d/(d\varepsilon) R_L(\omega, abcd)|_{\omega=\varepsilon}.

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TABLE II: The total cross section of the radiative recombination of an electron into the (1s)² and 1s2s states of the initially hydrogen-like tin (Z = 50), for different values of the projectile energy E. Notations are the same as in Table I.

| E [MeV/u] | σ⁽⁰⁾ | σ⁽¹⁾ | σ⁽²⁾ | σ⁽¹⁾ | σ⁽²⁾ | σ⁽¹⁾ |
|-----------|------|------|------|------|------|------|
|           | [barn] | [%]  | [%]  | [%]  | [barn] | [%]  | [%]  |
|           | (1s)² state | 1s2s state |
| 1         | 5.036 × 10⁻³ | -1.373 | 0.399 | -0.981 | 1.528 × 10⁻³ | -3.407 | -0.271 | 0.405 |
| 2         | 2.492 × 10⁻³ | -1.370 | 0.419 | -0.982 | 7.598 × 10⁻² | -3.345 | -0.244 | 0.399 |
| 5         | 9.669 × 10⁻² | -1.362 | 0.473 | -0.983 | 2.976 × 10⁻² | -3.218 | -0.174 | 0.388 |
| 10        | 4.603 × 10⁻² | -1.356 | 0.552 | -0.984 | 1.425 × 10⁻² | -3.116 | -0.090 | 0.386 |
| 25        | 1.665 × 10⁻² | -1.364 | 0.721 | -0.970 | 4.931 × 10⁻¹ | -3.071 | 0.046 | 0.532 |
| 50        | 6.568 × 10⁻¹ | -1.408 | 0.873 | -0.929 | 1.965 × 10⁻¹ | -3.133 | 0.129 | 0.222 |
| 75        | 3.684 × 10⁻¹ | -1.455 | 0.947 | -0.887 | 1.078 × 10⁻¹ | -3.187 | 0.152 | 0.174 |
| 100       | 2.375 × 10⁻¹ | -1.494 | 0.985 | -0.852 | 6.835 | -3.223 | 0.153 | 0.133 |
| 125       | 1.661 × 10⁻¹ | -1.526 | 1.003 | -0.813 | 4.718 | -3.247 | 0.144 | 0.098 |
| 150       | 1.226 × 10⁻¹ | -1.552 | 1.010 | -0.804 | 3.447 | -3.265 | 0.131 | 0.069 |
| 175       | 0.941 | -1.575 | 1.011 | -0.796 | 2.626 | -3.277 | 0.117 | 0.043 |
| 200       | 0.746 | -1.593 | 1.009 | -0.761 | 2.064 | -3.286 | 0.102 | 0.022 |
| 250       | 0.4996 | -1.624 | 0.999 | -0.740 | 1.367 | -3.300 | 0.073 | -0.011 |
| 300       | 0.3575 | -1.647 | 0.986 | -0.677 | 0.970 | -3.308 | 0.047 | -0.035 |
| 400       | 0.2088 | -1.680 | 0.959 | -0.618 | 0.559 | -3.317 | 0.003 | -0.063 |
| 500       | 0.1372 | -1.699 | 0.935 | -0.601 | 0.364 | -3.319 | -0.031 | -0.080 |
| 600       | 0.0974 | -1.711 | 0.914 | -0.581 | 0.257 | -3.316 | -0.059 | -0.089 |
| 700       | 0.0732 | -1.716 | 0.896 | -0.554 | 0.192 | -3.310 | -0.083 | -0.094 |

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