DEFORMED OSCILLATOR ALGEBRAS AND HIGHER-SPIN GAUGE INTERACTIONS OF MATTER FIELDS IN 2+1 DIMENSIONS

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Abstract

We formulate a non-linear system of equations which describe higher-spin gauge interactions of massive matter fields in 2+1 dimensional space-time and explain some properties of the deformed oscillator algebra which underlies this formulation. In particular we show that the parameter of mass $M$ of matter fields is related to the deformation parameter in this algebra.

1 Introduction

Dmitrij Vassilievich Volkov was a brilliant scientist who had made a great contribution to theoretical physics and created a remarkable scientific school in Kharkov. His most famous results are related to the creation of supersymmetric theories. The original approach invented by Volkov and collaborators was based on the application of invariant connection forms \[1\]. A further development of these geometric ideas in the modern field theory was extremely fruitful. In this talk we argue that a proper generalization of the Volkov’s ideas leads to a universal method of description of relativistic dynamics in terms of certain zero-curvature conditions supplemented with appropriate constraints. We will illustrate this by considering an example of matter fields interacting through higher-spin (HS) gauge fields in 2+1 dimensions.

2 Higher-Spin Symmetries in 2+1 Dimensions and Deformed Oscillator Algebras

HS algebras in $d$ space-time dimensions are certain infinite-dimensional extensions of space-time symmetry algebras $s_d$ \[\mathfrak{s}\] \[\mathfrak{s}\], which act on appropriate physical fields. HS
symmetries can be gauged by virtue of introducing appropriate HS gauge fields. In 2+1 dimensions, HS gauge fields do not propagate rather mediating interactions of matter sources analogously to the case of the gravitational field in 2+1 dimensions [4, 5]. This is a greatly simplifying property compared to the HS dynamics in four and higher dimensions. HS symmetries in 2+1 dimensions are still non-trivial as well as HS matter multiplets, i.e. the multiplets of fields on which the HS symmetries are realized. They are however very simple: ordinary scalar and spinor fields of an arbitrary mass. The analysis of HS interactions of relatively simple lower dimensional models sheds some light on general properties of HS models.

It is most useful to start with the space-time symmetries of (anti) - de Sitter type \( s_d = o(d - 1, 2) \) analyzing a possibility of taking a flat limit afterwards. The case of \( d = 3 \) is special because \( s_3 = o(1, 2) \oplus o(1, 2) = sp(2) \oplus sp(2) \) is not simple. Originally it was conjectured [6] that a 3d HS algebra is the direct sum of two Heisenberg-Weyl algebras (more precisely, of their Lie supercommutator superalgebras), each constructed from the ordinary oscillators \([y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}\). Because 3d HS gauge fields are not propagating one can write the Chern-Simons action for the pure gauge HS system,

\[
S = \int_M \text{str}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A) \quad \text{with the gauge fields} \quad A \quad \text{taking values in the HS algebra. In [7, 8, 9] it was shown that there exists a one-parametric class of infinite-dimensional algebras which we denote } \text{hs}(2; \nu) \quad \text{all containing } sp(2) \text{ as a subalgebra. This allows one to define a class of HS algebras } g = \text{hs}(2; \nu) \oplus \text{hs}(2; \nu). \text{ The supertrace operation was defined in [9] where also a useful realization of the supersymmetric extension of } \text{hs}(2; \nu) \text{ was given, based on a certain deformed oscillator algebra. Since this construction will be used below and also gets interesting applications in a number of different physical problems let us explain its properties in somewhat more details.}

Consider an associative algebra \( Aq(2; \nu) \) with a general element of the form

\[
f(q, K) = \sum_{n=0}^{\infty} \sum_{A=0,1} \frac{1}{n!} f^{A\alpha_1...\alpha_n}(K)^A q_{\alpha_1} \cdots q_{\alpha_n},
\]

under condition that the coefficients \( f^{A\alpha_1...\alpha_n} \) are symmetric with respect to the indices \( \alpha_j = 1, 2 \) and that the generating elements \( q_\alpha \) satisfy the relations

\[
[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu K), \quad K q_\alpha = -q_\alpha K, \quad K^2 = 1,
\]

where \( \nu \) is an arbitrary parameter. In other words, \( Aq(2; \nu) \) is the enveloping algebra for the relations (2), the deformed oscillator algebra.

An important property of this algebra is that for all \( \nu \) the bilinears

\[
T_{\alpha\beta} = \frac{1}{4i}\{q_\alpha, q_\beta\}
\]

have \( sp(2) \) commutation relations and rotate \( q_\alpha \) as a \( sp(2) \) vector

\[
[T_{\alpha\beta}, T_{\gamma\eta}] = (\epsilon_{\alpha\gamma} T_{\beta\eta} + \epsilon_{\beta\gamma} T_{\alpha\eta} + \epsilon_{\alpha\eta} T_{\beta\gamma} + \epsilon_{\beta\eta} T_{\alpha\gamma}),
\]

\[
[T_{\alpha\beta}, q_\gamma] = \epsilon_{\alpha\gamma} q_\beta + \epsilon_{\beta\gamma} q_\alpha.
\]

1 The indices \( \alpha, \beta, \gamma = 1, 2 \) are treated as spinor indices in 2+1 dimensions. These are lowered and raised with the aid of the symplectic form \( \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \epsilon_{12} = \epsilon^{12} = 1, A^\alpha = \epsilon^{\alpha\beta} A_\beta, A_\alpha = A^\beta \epsilon_{\beta\alpha}. \)
The deformed oscillators described above have a long history and were originally discovered by Wigner [10] who addressed a question whether it is possible to modify the commutation relations for the normal oscillators $a^{\pm}$ in such a way that the basic commutation relations $[H, a^{\pm}] = \pm a^{\pm}, H = \frac{1}{2}\{a^+, a^-\}$ remain valid. By analyzing this problem in the Fock-type space Wigner found a one-parametric deformation of the standard commutation relations which corresponds to a particular realization of the commutation relations (2) with the identification $a^+ = q_1, a^- = \frac{1}{2}i q_2, H = T_{01}$ and $K = (-1)^N$ where $N$ is the particle number operator. These commutation relations were discussed later by various authors in particular in the context of parastatistics (see e.g. [11]).

According to (3) and (5) the $sp(2)$ symmetry generated by $T_{\alpha\beta}$ extends to $osp(1,2)$ supersymmetry by identifying the supergenerators with $q_\alpha$. In fact, as shown in [12], one can start from the $osp(1,2)$ algebra to derive the deformed oscillator commutation relations. Since this construction is instructive in many respects we reproduce it here.

One starts with the (super)generators $T_{\alpha\beta}$ and $q_\alpha$ which by definition of $osp(1,2)$ satisfy the commutation relations (3)-(5). Since $\alpha$ and $\beta$ take only two values one can write

$$[q_\alpha, q_\beta] = 2i\epsilon_{\alpha\beta}(1 + Q),$$

where $Q$ is some new “operator” while the unit term is singled out for convenience. Inserting this back into (3) with the substitution of (3) and completing the commutations one observes that (3) is true if and only if $Q$ anticommutes with $q_\alpha$,

$$Qq_\alpha = -q_\alpha Q.\quad (7)$$

The relation (3) does not add anything new since it is a consequence of (3) and (3). As a result we arrive [12] at the following important

**Corollary:** The enveloping algebra of $osp(1,2)$, $U(osp(1,2))$, is isomorphic to the enveloping algebra of the deformed oscillator relations (3) and (3).

In other words, the associative algebra with the generating elements $q_\alpha$ and $Q$ subject to the relations (3) and (3) is the same as the associative algebra with the generating elements $q_\alpha$ and $T_{\alpha\beta}$ subject to the $osp(1,2)$ commutation relations (3)-(3).

Computing the quadratic Casimir operator of $osp(1,2)$

$$C_2 = -\frac{1}{2} T_{\alpha\beta} T^{\alpha\beta} - \frac{i}{4} q_\alpha q_\alpha.$$

one easily derives using (3) that

$$C_2 = -\frac{1}{4}(1 - Q^2).\quad (9)$$

Let us now consider the factor algebra of $U(osp(1,2))$ over its ideal $I_{(C_2 + \frac{1}{4}(1 - \nu^2))}$ generated by the element $(C_2 + \frac{1}{4}(1 - \nu^2))$ where $\nu$ is an arbitrary number. In other words we assume that every element of $U(osp(1,2))$ which is of the form $(C_2 + \frac{1}{4}(1 - \nu^2)) a, \forall a \in U(osp(1,2))$ is equivalent to zero. This factorization can be achieved in terms of the deformed oscillators (3), (3) by setting [3]

$$Q = \nu K, \quad K^2 = 1 \quad Kq_\alpha = -q_\alpha K.\quad (10)$$

\[\text{The point } \nu = 0 \text{ is special since one can consider a case(s) with } Q^2 = 0, Q \neq 0.\]
Thus, it is shown \cite{2} that the algebra $Aq(2, \nu)$ introduced in \cite{1} is isomorphic to $U(osp(1, 2))/I_{(C_2+\frac{1}{2}(1-\nu^2))}$. This fact has a number of simple but important consequences. For example, any representation of the superalgebra $osp(1, 2)$ with $C_2 = -\frac{1}{4}(1 - \nu^2)$ forms a representation of $Aq(2, \nu)$ ($\nu \neq 0$) and vice versa (for all $\nu$ including $\nu = 0$). In particular this is the case for finite-dimensional representations corresponding to the values $\nu = 2l + 1, l \in \mathbb{Z}$ with $C_2 = l(l+1)$. This fact has been used in \cite{13} for the construction of the generalized Toda field theories interpolating between ordinary finite-component Toda field theories. Let us note that the even subalgebra of $Aq$ with the generators $A_{q, \nu}^{fg}$ such that $str(A_{q, \nu}^{fg}) = \delta_{fg}$ forms a representation of $Aq(2, \nu)$ ($\nu \neq 0$) and vice versa (for all $\nu$ including $\nu = 0$). In particular this is the case for finite-dimensional representations corresponding to the values $\nu = 2l + 1, l \in \mathbb{Z}$ with $C_2 = l(l+1)$. This fact has been used in \cite{13} for the construction of the generalized Toda field theories interpolating between ordinary finite-component Toda field theories. Let us note that the even subalgebra of $Aq(2, \nu)$ spanned by the elements of the form (\ref{1}) with $f(q, K) = f(-q, K)$ decomposes into a direct sum of two subalgebras $Aq^{\pm}(2, \nu)$ spanned by the elements $P_{\pm} f(q, K)$ with $f(-q, K) = f(q, K)$, $P_{\pm} = \frac{1}{2}(1 \pm K)$. These algebras can be shown to be isomorphic to the factor algebras $U(osp(2))/I_{(C_2+\frac{3+2\nu-\nu^2}{4})}$ where $C_2 = -\frac{1}{2}T_{\alpha\beta}T^{\alpha\beta}$ is the quadratic Casimir operator of $sp(2)$ and can be interpreted as (infinite-dimensional) algebras interpolating between the ordinary finite-dimensional matrix algebras. Such interpretation of $U(osp(2))/I_{(C_2-\nu)}$ was given by Feigin in \cite{14}.

A very important property of $Aq(2; \nu)$ is that it admits \cite{1} a uniquely defined supertrace operation

$$str(f) = f^0 - \nu f^1,$$

such that $str(fg) = (-1)^{\nu_f \nu_g} str(gf)$, $\forall f, g$ having a definite parity, $f(-q, K) = (-1)^{\nu_f} f(q, K)$ (i.e. $str(1) = 1$, $str(K) = -\nu$ while all higher monomials of $q_\alpha$ in \cite{1} do not contribute under the supertrace). This supertrace reduces \cite{13} to the ordinary supertrace of finite-dimensional algebras for the special values of the parameter $\nu = 2l + 1$ which correspond to the values of the Casimir operator related to finite-dimensional representations of $osp(1, 2)$ ($sp(2)$ in the bosonic case). This property allows one to handle the algebras $Aq(2; \nu)$ very much the same way as ordinary finite-dimensional (super)matrix algebras. What happens for special values of $\nu = 2l + 1$ is that $Aq(2; \nu)$ acquires ideals $I_l$ such that $Aq(2; \nu)/I_l$ amounts to appropriate (super)matrix algebras. These ideals were described in \cite{1} as null vectors of the invariant bilinear form $str(ab)$, $a, b \in Aq(2; \nu)$.

The identification of $Aq(2; \nu)$ with $U(osp(1, 2))/I_{(C_2+\frac{1}{4}(1-\nu^2))}$ makes transparent such properties of the deformed oscillator algebra as relationship of the representations of $Aq(2; \nu)$ with those of $osp(1, 2)$ (including its finite-dimensional representations for special values of $\nu = 2l + 1, \forall l \in \mathbb{Z}$) and $N = 1$ supersymmetry (as inner $osp(1, 2)$ automorphisms). A more interesting property \cite{12} is that $Aq(2; \nu)$ admits $N = 2$ supersymmetry $osp(2, 2)$ with the generators

$$T_{\alpha\beta} = \frac{1}{4i} \{ q_\alpha, q_\beta \}, \quad Q_\alpha = q_\alpha, \quad S_\alpha = q_\alpha K, \quad J = K + \nu.$$

These properties find interesting applications (see, e.g., \cite{13} and references therein).

As we demonstrate below the deformed oscillator algebras serve as a main tool for the description of the d=3 HS dynamics. The reason is that they are related to the enveloping algebras of the space-time symmetries and allow us to formulate a non-linear dynamics with explicit local Lorentz symmetry. In its turn, the analysis of the HS dynamics presented below is interesting in the context of the deformed oscillator algebra itself because algebraically it reduces to the construction of its embedding into a direct product of two ordinary Heisenberg-Weyl (i.e. oscillator) algebras equipped with certain twist operators.

Coming back to the HS problem in 2+1 dimensions we note that to describe a doubling of the elementary algebras in $g = hs(2; \nu) \oplus hs(2; \nu)$ it suffices to introduce an additional
central involutive generating element $\psi$: $[\psi, q] = 0$, $[\psi, K] = 0$, $\psi^2 = 1$. The two simple subalgebras of $g$ are singled out by the projection operators $\Pi_\pm = \frac{1}{2} (1 \pm \psi)$. The full set of HS gauge fields in 2+1 dimensions, the gauge fields of $g$, thus is

$$A(q, K, \psi|x) = dx^\nu \sum_{n=0}^{\infty} \sum_{B=0,1} \frac{1}{2n!} (\omega^B_{\alpha_1 \ldots \alpha_n}(x) + \psi h^B_{\alpha_1 \ldots \alpha_n}(x)) (K)^B q_{\alpha_1} \ldots q_{\alpha_n}. \quad (13)$$

The field strengths and gauge transformation laws are defined in the usual way

$$R(q, K, \psi|x) = dA(q, K, \psi|x) + A(q, K, \psi|x) \wedge A(q, K, \psi|x), \quad (14)$$
$$\delta A(q, K, \psi|x) = d\epsilon(q, K, \psi|x) + [A(q, K, \psi|x), \epsilon(q, K, \psi|x)], \quad (15)$$

where $d = dx^\nu \frac{\partial}{\partial x^\nu}$. The gravitational fields

$$A^{gr} = \frac{1}{4i} (\omega^{\alpha\beta} + h^{\alpha\beta} \psi) q_\alpha q_\beta \quad (16)$$

take values in the subalgebra $sp(2) \oplus sp(2)$. The pure gauge Chern-Simons HS action reduces to the Witten gravity action $[5]$ in the spin 2 sector and to the Blencowé’s HS action $[6]$ in the case of $\nu = 0$.

3 Unfolded equations

In this report which is based on a recent papers $[16, 17]$ we answer to the question how to introduce interactions of HS gauge fields with propagating matter fields at the level of equations of motion using an approach which we call “unfolded formulation” $[18]$. It consists of reformulation of dynamical equations in a form of certain zero-curvature conditions and covariant constancy conditions

$$d\omega + \omega \wedge \omega = 0, \quad dB^A + \omega^i t_i A_B B^B = 0, \quad (17)$$

supplemented with some gauge invariant constraints

$$\chi(B) = 0 \quad (18)$$

which do not contain space-time derivatives. Here $\omega(x) = dx^\nu \omega^i_{\nu}(x) T_i$ is a gauge field taking values in some Lie superalgebra $l$ ($T_i \in l$), and $B^A(x)$ is a set of 0-forms which take values in the representation space of some representation $(t_i)^B_A$ of $l$.

An interesting property of this form of equations is that their dynamical content is hidden in the constraints $(18)$. Indeed, locally one can integrate out $(17)$ explicitly as $\omega = g(x) dg^{-1}(x), B(x) = t_{g(x)}(B_0)$ where $g(x)$ is an arbitrary invertible element while $B_0$ is an arbitrary $x$-independent representation element and $t_{g(x)}$ is the exponential of the representation $t$ of $l$. Since the constraints $\chi(B)$ are gauge invariant one is left with the only condition $\chi(B_0) = 0$. Let $g(x_0) = I$ for some point of space-time $x_0$. Then $B_0 = B(x_0)$.

Such a formulation can in principle be applied to an arbitrary dynamical system provided that a representation $t$ is infinite-dimensional. Being based on zero-curvature conditions it has deep similarities with the original approach by Volkov and collaborators $[1]$. A crucial feature of our approach is that the set of 0-forms $B$ has to be reach enough
to describe all space-time derivatives of the dynamical fields while the constraints (18) effectively impose all restrictions on the space-time derivatives required by the dynamical equations under consideration. Given solution of (18) one knows all derivatives of the dynamical fields and can therefore reconstruct these fields by analyticity in some neighborhood of $x_0$. The specificity of the HS dynamics which makes such an approach adequate is that HS symmetries mix all orders of derivatives which therefore have to be contained in a representation $t$ of HS symmetries.

Let us illustrate this by the example of a scalar field $\phi$ obeying the massless Klein-Gordon equation $\Box \phi = 0$ in a flat space-time of an arbitrary dimension $d$. Here $l$ is identified with the Poincaré algebra $\text{iso}(d - 1, 1)$ which gives rise to the gauge fields $\omega_\nu = (h_\nu^a, \omega_\nu^{ab})$ ($a, b = 0 - (d - 1)$). The zero curvature conditions of $\text{iso}(d - 1, 1)$, $R_{\nu\mu}^a = 0$ and $R_{\nu\mu}^{ab} = 0$, imply that the vierbein $h_\nu^a$ and Lorentz connection $\omega_\nu^{ab}$ describe the flat geometry. Fixing the local Poincaré gauge transformations one can set

$$h_\nu^a = \delta^a_\nu, \quad \omega_\nu^{ab} = 0. \quad (19)$$

To describe dynamics of a spin zero massless field $\phi(x)$ let us introduce an infinite collection of 0-forms $\phi_{a_1...a_n}(x)$ which are totally symmetric traceless tensors

$$\eta^{bc} \phi_{bca_3...a_n} = 0, \quad (20)$$

where $\eta^{bc}$ is the flat Minkowski metrics. The “unfolded” version of the Klein-Gordon equation has a form of the following infinite chain of equations

$$\partial_\nu \phi_{a_1...a_n}(x) = h_\nu^b \phi_{b a_1...a_n}(x), \quad (21)$$

where we have replaced the Lorentz covariant derivative by the ordinary flat derivative $\partial_\nu$ using the gauge condition (19). The tracelessness condition (20) is a specific realization of the constraints (18) while the system of equations (21) is a particular example of the equations (17). It is easy to see that this system is formally consistent.

To show that the system (21) is equivalent to the free massless field equation $\Box \phi(x) = 0$ let us identify the scalar field $\phi(x)$ with the $n = 0$ member of the tower of 0-forms $\phi_{a_1...a_n}$. Then the first two equations (21) read $\partial_\nu \phi = \phi_\nu$ and $\partial_\nu \phi = \phi_\nu$, respectively. The former tells us that $\phi_\nu$ is a first derivative of $\phi$. The latter implies that $\phi_\nu$ is a second derivative of $\phi$. However, because of the tracelessness condition (20) it imposes the Klein-Gordon equation $\Box \phi = 0$. It is easy to see that all other equations in (21) express highest tensors in terms of the higher-order derivatives $\phi_{\nu_1...\nu_n} = \partial_{\nu_1}...\partial_{\nu_n} \phi$ and impose no additional conditions on $\phi$. The tracelessness conditions are all satisfied once the Klein-Gordon equation is true.

## 4 Free fields in 2+1 AdS space

Let us now confine ourselves to the 2+1 dimensional case and generalize the above analysis of the scalar field dynamics to the AdS geometry. The gauge fields of the AdS algebra $\text{o}(2, 2) \sim \text{sp}(2) \oplus \text{sp}(2)$ are identified with the gravitational fields, $A_\nu = (\lambda h_{\nu a\beta}; \omega_{\nu a\beta})$. The zero-curvature conditions $R_{\nu\mu} = 0$ for the AdS algebra in its orthogonal realization take a form

$$R_{\nu\mu, ab} = \lambda^2 (h_{\nu a} h_{\mu b} - h_{\nu b} h_{\mu a}), \quad R_{\nu\mu, a} = 0 \quad (22)$$
(ν, µ...; a, b... = 0 − 2), where $R_{νμ,ab}$ and $R_{νμ,a}$ are the Riemann and torsion tensors, respectively. From (22) one concludes that the zero curvature equations for the algebra $o(2, 2)$ on a 3d manifold do indeed describe the AdS space provided that $h_{ν}^{α}$ is identified with a dreibein and is invertible.

It is an important property of the 3d geometry that one can resolve the tracelessness conditions (20) by using the formalism of two-component spinors: a totally symmetric traceless tensor $φ_{a1...an}$ is equivalent to a totally symmetric multispinor $C_{α1...2n}$. Let us now address the question what is a general form of the equations analogous to (21) such that their integrability conditions reduce to (22). The result is that, up to a freedom in field redefinitions, these are equations of the form [18]

$$DC_{α1...2n} = h^{βγ}C_{α1...2n,βγ} + 2n(2n-1)e(2n, λ, M)h_{[α1α2}C_{α3...2n]}_α ,$$

(23)

where $D$ is the Lorentz covariant derivative, $DB_α ≡ dB_α + ω_α βB_β$, and

$$e(l, λ, M) = \frac{1}{4}λ^2 - \frac{1}{2}M^2 \frac{M^2}{l^2 - 1} \quad (l ≥ 2).$$

(24)

One can see that the freedom in an arbitrary parameter $M$ is just the freedom of the relativistic field equations in the parameter of mass.

Thus the equations (23) describe a scalar field of an arbitrary mass in 2+1 dimensions. Now let us show how these equations can be generated with the aid of the generalized oscillators (2). To this end we introduce the generating function

$$C(q_α, K|x) = \sum_{n=0}^{∞} \sum_{A=0,1} \frac{1}{n!}C_{α1...an}(x)(K)^Aq^{α1}...q^{αn}.$$

(25)

The relevant equations acquire then the following simple form

$$DC(q_α, K|x) = \frac{1}{4i}\{h^{αβ}q_αq_β, C(q, K|x)\}$$

(26)

(from now on we use the dimensionless units with a unit AdS radius, $λ = 1$).

To see that the integrability conditions for (20) reduce to the zero-curvature conditions for $sp(2) ⊕ sp(2)$ one observes that there is an automorphism of the AdS algebra which changes a sign of the AdS translations. This automorphism allows one to introduce a “twisted representation” of the AdS algebra with the anticommutator instead of commutator in the translational part of the AdS algebra. This twisted representation just leads to the covariant constancy equations (26).

Since the terms in (24) which depend on the background gravitational fields only contain even combinations of the oscillators $q_α$ the full system of equations decomposes into four independent subsystems which can be singled out by virtue of the projection operators $P_± = \frac{1}{2}(1 ± K)$ either in the boson or in the fermion sectors (even (odd) functions $C(q_α, K|x)$ of $q_α$ describe bosons (fermions)). The explicit calculation which involves some reorderings of $q_α$ and rescalings of fields then shows that the irreducible boson subsystems projected out by $P_±$ indeed reduce to the equations of motion of the form (23) for a massive scalar field of mass $M^2 = \frac{1}{2}ν(ν + 2)$. Remarkably, the same equations in the fermion sector describe spin $\frac{1}{2}$ fermion fields of the mass $M^2 = \frac{1}{2}ν^2$.

An important achievement of the reformulation of the free field equations in the form (26) is that this form suggests that the global HS symmetry algebra realized on the matter...
fields of mass $M(\nu)$ is $g = hs(2; \nu) \oplus hs(2; \nu)$ with the gauge fields $[13]$. To simplify the formulation it is convenient to introduce two Clifford variables $\{\psi_i, \psi_j\} = 2\delta_{ij} \ (i, j = 1, 2)$ instead of $\psi$. One then introduces the full set of HS gauge fields as $W_\nu(q_\alpha, K, \psi_{1,2}|x)$ and realizes the gravitational fields as

$$W_\nu^{\gamma r} = \frac{1}{4i}(\omega_\nu^{\alpha \beta} + h_\nu^{\alpha \beta} \psi_1)q_\alpha q_\beta.$$  

(27)

The generating function for 0-forms is

$$C(q, K, \psi_{1,2}|x) = C^{\text{mat}}(q, K, \psi_1|x)\psi_2 + C^{\text{aux}}(q, K, \psi_1|x).$$  

(28)

Now let us consider the zero curvature equations

$$0 = R \equiv dW(q, K, \psi|x) + W(q, K, \psi|x) \wedge W(q, K, \psi|x)$$

(29)

along with the covariant constancy conditions in the adjoint representation of the HS algebra

$$0 = dC(q, K, \psi|x) + W(q, K, \psi|x)C(q, K, \psi|x) - C(q, K, \psi|x)W(q, K, \psi|x).$$  

(30)

Due to the factor of $\psi_2$ in front of $C^{\text{mat}}$ the equations for $C^{\text{mat}}$ turn out to be equivalent to the equations (20) in the gauge in which only the gravitational part (27) of the vacuum HS gauge fields is non-vanishing. The fields $C^{\text{aux}}$ can be shown [13] to be of a topological type so that each irreducible subsystem in this sector can describe at most a finite number of degrees of freedom and trivializes in a topologically trivial situation. Thus the effect of introducing a second Clifford element consists of addition of some topological fields.

5 Non-linear dynamics

To describe non-linear HS dynamics of matter fields in 2+1 dimensions we start with a system of equations which is very close to that introduced in [19] for a particular case of massless matter fields. We introduce three types of the generating functions $dx_\nu W_\nu(z_\alpha, y_\beta, K, \psi_1|x), s_\gamma(z_\alpha, y_\beta, K, \psi_1|x)$ and $B(z_\alpha, y_\beta, K, \psi_1|x)$ which depend on the space-time variables $x^\mu$ and auxiliary variables $(z_\alpha, y_\beta, K, \psi_1)$ such that the two Clifford elements $\psi_1$ commute to all other variables, while the bosonic spinor variables $z_\alpha$ and $y_\beta$ commute to each other but anticommute with $K$, i.e. $\{K, z_\alpha\} = \{K, y_\alpha\} = 0$, $K^2 = 1$. Their physical content is as follows: $dx_\nu W_\nu$ is the generating function for HS gauge fields, $B$ contains physical matter degrees of freedom along with some auxiliary variables, and $s_\gamma$ is entirely auxiliary variable which allows one to formulate the full system of equations in a compact form.

This formulation is based on the following star-product law which endows the space of functions $f(z, y)$ with a structure of associative algebra

$$(f \ast g)(z, y) = (2\pi)^{-2}\int d^2 ud^2 v f(z + u, y + u) \times g(z - v, y + v) \exp i(u_\alpha v^\alpha).$$  

(31)

This product law provides a particular symbol realization of the Heisenberg–Weyl algebra. In particular one finds that $[y_\alpha, y_\beta]_* = -[z_\alpha, z_\beta]_* = 2i\epsilon_{\alpha \beta}$. The full system of equations has the form:

$$dW + W \ast W = 0, \hspace{0.5cm} ds_\alpha + W_\ast s_\alpha - s_\alpha \ast W = 0, \hspace{0.5cm} dB + W \ast B - B \ast W = 0.$$  

(32)
\[ \tilde{s}_\alpha \ast s_\beta - \tilde{s}_\beta \ast s_\alpha = -2i\epsilon_{\alpha\beta}(1 + \kappa \ast B), \quad \tilde{B} \ast s_\alpha - s_\alpha \ast B = 0, \quad (33) \]

where
\[ \tilde{a}(z, y, K, \psi_1|x) = a(z, y, -K, \psi_1|x) \quad \forall a \quad (34) \]

and \( \kappa = K \exp i(z_\alpha y^\alpha) \) is a central element of the algebra which has vanishing star commutators with \( y_\alpha, z_\alpha, K \) and \( \psi_1 \).

The system of equations (32), (33) is explicitly invariant under the general coordinate transformations and the HS gauge transformations of the form
\[ \delta W = d\epsilon + W \ast \epsilon - \epsilon \ast W, \quad \delta B = B \ast \epsilon - \epsilon \ast B, \quad \delta s_\alpha = s_\alpha \ast \epsilon - \tilde{\epsilon} \ast s_\alpha. \quad (35) \]

To elucidate its physical content one has to analyze this system perturbatively near some vacuum solution. In the massless case the appropriate vacuum solution \([19]\) is
\[ B_0 = 0, \quad s_{0\alpha} = z_\alpha, \quad W_0 = \omega(y, K, \psi_{1,2}) \quad (36) \]

with the vacuum gauge field \( \omega \) satisfying the zero curvature condition \( d\omega + \omega \ast \wedge \omega = 0 \).

It can be shown along the lines of \([19]\) that the system of equations (32), (33) expanded near this vacuum solution properly describes dynamics of massless matter fields on the free field level and beyond.

The main result of this report consists \([17]\) of the observation that the same system (32), (33) expanded near another vacuum solution describes dynamics of matter fields with an arbitrary mass. This is a solution with
\[ B_0 = \nu, \quad (37) \]

where \( \nu \) is an arbitrary constant. For a constant field \( B_0 \) only the first of the equations (33) remains non-trivial. Remarkably it turns out to be possible to find its explicit solution
\[ s_{0\alpha} = z_\alpha + \nu(z_\alpha - y_\alpha) \int_0^1 dt e^{it(z_\beta y^\beta)} K \quad (38) \]

(it is not too difficult to check that (38) satisfies (33) by a direct substitution). Now let us turn to the equations (32). The third of these equations is trivially satisfied. The second one reads
\[ \tilde{W} \ast s_{0\alpha} - s_{0\alpha} \ast W = 0, \quad (39) \]

where we have taken into account that \( ds_{0\alpha} = 0 \). Eq. (39) is a complicated integral equation. The key observation however is that it admits the following two particular solutions: \( W_0 = q_\alpha \ (\alpha = 1, 2) \),
\[ q_\alpha = y_\alpha + \nu K(z_\alpha - y_\alpha) \int_0^1 dt (1 - t)e^{itz_\beta y^\beta}. \quad (40) \]

Taking into account that \( \ast \)-product is associative it allows us to describe a general solution of (33) as an arbitrary element \( W_0 = \omega(q_\alpha, K, \psi_{1,2}|x) \) whose arguments are treated as some non-commutative elements of the star-product algebra.

To make contact with the previous consideration it remains to check by explicit computation that the elements \( q_\alpha \) indeed obey (2). Thus, the vacuum solution with a constant
field (37) leads automatically to the deformed oscillator algebra with the deformed oscillators realized as some functions of \( z, y \) and \( K \), i.e. as elements of the tensor product of two Heisenberg-Weyl algebras (equipped with the operator \( K \)). Finally, it remains to observe that the first of the equations (32) reduces to the zero curvature equation which describes the AdS background space. Since, as argued in the section 4, \( \nu \) governs the parameter of mass of matter fields we arrive at the conclusion that a particular value of the parameter of mass is determined by a vacuum value of the field \( B \).

Next one can analyze the full system of equations perturbatively by inserting the expansions of the form:

\[
W = W_0 + W_1 + \ldots, \quad B = B_0 + B_1 + \ldots, \quad s_\alpha = s_{0\alpha} + s_{1\alpha} + \ldots
\]

In particular one can derive in the lowest orders that

\[
B_1(z, y, K, \psi | x) = C(q, K, \psi | x),
\]

\[
W_1(z, y, K, \psi | x) = \omega(q, K, \psi | x) + \Delta W_1(C), \quad s_{1\alpha} = s_{1\alpha}(C),
\]

where \( s_{1\alpha}(C) \) and \( \Delta W_1(C) \) are some functionals of the field \( C \) which remains arbitrary and has to be identified with generating function (28). Inserting this back into (32) one obtains the free field equations for \( C \) in the linearized approximation from the third equation and the equations of the form \( d\omega + \omega \star \wedge \omega - J(\omega, C^2) = 0 \) from the first one where \( J(\omega, C^2) \) is expected to describe HS currents (including the gravitational and spin-1 ones).

### 6 Lorentz Covariance

After it is argued that the system (32), (33) describes properly HS dynamics in 2+1 dimensions let us explain what a physical principle fixes a particular form of these equations. Remarkably, this is a simple and physically important requirement that local Lorentz symmetry should be a particular symmetry of the equations.

Let us consider the following element of the algebra

\[
L^\text{tot}_{\alpha\beta} = \frac{i}{4} (\{z_\alpha, z_\beta\}_* - \{y_\alpha, y_\beta\}_*).
\]

Infinitesimal local Lorentz transformations with a parameter \( \eta^{\alpha\beta} \) are generated as \( \delta Q(z, y) = [Q, \eta^{\alpha\beta} L^\text{tot}_{\alpha\beta}]_* \). Indeed these generators rotate properly the elementary spinor generating elements of the algebra, \( \delta z_\alpha = \eta_{\alpha\beta} z_\beta, \quad \delta y_\alpha = \eta_{\alpha\beta} y_\beta \), and therefore induce principal \( sp(2) \) transformations (automorphisms) of the whole algebra.

Although the argument above proves explicit local Lorentz invariance of the system of equations (32), (33), this symmetry is spontaneously broken due to the first of the constraints (33). Indeed, since the right hand side of this constraint has a non-vanishing vacuum value, \( s_\alpha \) itself must have a non-vanishing vacuum value (38). The question therefore is whether there exists another local Lorentz symmetry which rotates properly spinor indices of the dynamical fields leaving invariant a vacuum solution. In fact the existence of such a Lorentz symmetry in all orders in interactions is a highly non-trivial property which fixes the constraints (33).

The point is that according to the analysis of the previous section after the constraints (33) are solved the generating functions for matter fields and gauge fields are described by arbitrary functions of only one spinor variable \( q_\alpha, C(q, K, \psi | x) \) and \( \omega(q, K, \psi | x) \), respectively. In the linearized approximation the Lorentz generators which rotate properly \( q_\alpha \)
are \( L_{\alpha\beta} = \frac{1}{4i} \{ q_{\alpha}, q_{\beta} \} \). Thus what we need is a proper generalization of these generators, to all orders in interactions. The constraints (33) indeed guarantee that such Lorentz generators \( l_{\alpha\beta} \) can be constructed.

To see this we first observe that the constraints (33) give a particular realization of the deformed oscillator algebra (2). To this end it is convenient to introduce a new auxiliary generating element \( \rho \) which has the properties

\[
\{ \rho, K \} = 0, \quad \rho^2 = 1.
\]

(44)

Let us introduce a new variable \( t_{\alpha} = \rho s_{\alpha} \). A role of \( \rho \) is that it compensates the twiddle operation (34) in (32) and (33) so that the constraints (33) take the form

\[
t_{\alpha} \ast t_{\beta} - t_{\beta} \ast t_{\alpha} = -2i\epsilon_{\alpha\beta}(1 + \kappa \ast B), \quad B \ast t_{\alpha} - t_{\alpha} \ast B = 0, \quad \kappa \ast t_{\alpha} + t_{\alpha} \ast \kappa = 0,
\]

(45)

where \( \kappa = K \exp(i z_{\alpha} y^{\alpha}) \) is a central element of the original algebra which now anticommutes with \( t_{\alpha} \) due to (44). To make contact with (2) one identifies \( t_{\alpha} \), \( B \) and \( \kappa \) with \( iq_{\alpha}, \nu \), and \( K \), respectively. As a consequence of the general property (5) one concludes that the elements

\[
M_{\alpha\beta} = \frac{i}{4} \{ t_{\alpha}, t_{\beta} \} \ast = \frac{i}{4} (s_{\alpha} \ast s_{\beta} + s_{\beta} \ast s_{\alpha})
\]

(46)

obey the Lorentz commutation relations and rotate properly \( t_{\alpha} \). Now one can come back to the original \( \rho \)-independent variables arriving at the relations

\[
M_{\alpha\beta} \ast s_{\gamma} - s_{\gamma} \ast M_{\alpha\beta} = \epsilon_{\alpha\gamma}s_{\beta} + \epsilon_{\beta\gamma}s_{\alpha}.
\]

(47)

Let us now introduce the generators

\[
l_{\alpha\beta} = L_{\alpha\beta}^{tot} - M_{\alpha\beta}.
\]

(48)

From (33) it follows that

\[
\delta B = [B, \eta^\alpha_{\alpha\beta} l_{\alpha\beta}] \ast = [B, \eta^\alpha_{\alpha\beta} L_{\alpha\beta}^{tot}] \ast,
\]

(49)

i.e. \( l_{\alpha\beta} \) rotate properly physical fields like \( C(q, K, \psi \mid x) \) in all orders in interactions. Assuming that \( L_{\alpha\beta}^{tot} \) rotates properly \( s_{\alpha} \) one concludes that

\[
\delta s_{\alpha} = s_{\alpha} \ast \eta^{\gamma\beta}_{\gamma\beta} l_{\gamma\beta} - \eta^{\gamma\beta}_{\gamma\beta} l_{\gamma\beta} \ast s_{\alpha} = \frac{\delta s_{\alpha}}{\delta B} \delta B
\]

(50)

and that the gauge transformations induced by \( l_{\alpha\beta} \) satisfy \( sp(2) \) commutation relations. Also

\[
\delta W = D(\eta^\alpha_{\alpha\beta} l_{\alpha\beta}) = d(\eta^\alpha_{\alpha\beta} l_{\alpha\beta}) + [W, \eta^\alpha_{\alpha\beta} L_{\alpha\beta}^{tot}]
\]

(51)

because \( d(L_{\alpha\beta}^{tot}) = 0 \) while \( D(M_{\alpha\beta}) = 0 \) as a consequence of the second of the equations (32). From (51) one concludes that the gauge field for a true local Lorentz symmetry is

\[
W_{L} = \omega^\alpha_{\alpha\beta} l_{\alpha\beta}
\]

(52)

while all other gauge fields are rotated properly under the Lorentz transformations.

Let us emphasize that the above analysis guarantees Lorentz symmetry in all orders in interactions. Thus, it is the Lorentz symmetry principle which fixes a form of the equations and enforces appearance of the deformed oscillator algebra in the HS problem.

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3 Note that the vacuum solution \( t_{\alpha} = \rho s_{\alpha} \) (38) therefore again describes an embedding of the deformed oscillator algebra into a (equipped) direct product of the two Heisenberg-Weyl algebras.

4 There is some gauge ambiguity in the generic solution of the first of the constraints (33) for \( s_{\alpha} \) in terms of \( B \). The assumption above is true when \( s_{\alpha} \) is reconstructed entirely in terms of \( B \) without introducing any external constant spinors beyond those in the vacuum solution \( s_{0\alpha} \).
7 Concluding remarks

The proposed formulation of HS interactions admits interpretation of the parameter of mass as a module of the space of vacuum solutions, i.e. the same equations describe HS interactions of massive multiplets with different masses depending on a chosen vacuum solution. As a result different global HS symmetries of the linearized matter multiplets are different stability subgroups of the full HS symmetry, which leave invariant vacuum solutions. These global HS symmetries turn out to be pairwise non-isomorphic for different values of the parameter of mass. It is worth mentioning that the model under consideration (eq.(1)) possesses \( N = 2 \) supersymmetry \( osp(2; 2) \) with the generators \( (12) \). The constraints have a form of the deformed oscillator algebra as a consequence of the requirement that the equations of motion of matter fields interacting with HS fields must possess local Lorentz symmetry which is guaranteed by the properties \( (11) \) and \( (3) \).

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