Harmonic Analysis on the Infinite Symmetric Group

Sergei Kerov, Grigori Olshanski, and Anatoly Vershik

Abstract. The infinite symmetric group $S(\infty)$, whose elements are finite permutations of $\{1, 2, 3, \ldots\}$, is a model example of a “big” group. By virtue of an old result of Murray–von Neumann, the one-sided regular representation of $S(\infty)$ in the Hilbert space $\ell^2(S(\infty))$ generates a type II_1 von Neumann factor while the two-sided regular representation is irreducible. This shows that the conventional scheme of harmonic analysis is not applicable to $S(\infty)$: for the former representation, decomposition into irreducibles is highly non-unique, and for the latter representation, there is no need of any decomposition at all. We start with constructing a compactification $\mathcal{S} \supset S(\infty)$, which we call the space of virtual permutations. Although $\mathcal{S}$ is no longer a group, it still admits a natural two-sided action of $S(\infty)$. Thus, $\mathcal{S}$ is a $G$-space, where $G$ stands for the product of two copies of $S(\infty)$. On $\mathcal{S}$, there exists a unique $G$-invariant probability measure $\mu_1$, which has to be viewed as a “true” Haar measure for $S(\infty)$. More generally, we include $\mu_1$ into a family $\{\mu_t : t > 0\}$ of distinguished $G$-quasiinvariant probability measures on virtual permutations. By making use of these measures, we construct a family $\{T_z : z \in \mathbb{C}\}$ of unitary representations of $G$, called generalized regular representations (each representation $T_z$ with $z \neq 0$ can be realized in the Hilbert space $L^2(\mathcal{S}, \mu_t)$, where $t = |z|^2$). As $|z| \to \infty$, the generalized regular representations $T_z$ approach, in a suitable sense, the “naive” two-sided regular representation of the group $G$ in the space $\ell^2(S(\infty))$. In contrast with the latter representation, the generalized regular representations $T_z$ are highly reducible and have a rich structure. We prove that any $T_z$ admits a (unique) decomposition into a multiplicity free continuous integral of irreducible representations of $G$. For any two distinct (and not conjugate) complex numbers $z_1, z_2$, the spectral types of the representations $T_{z_1}$ and $T_{z_2}$ are shown to be disjoint. In the case $z \in \mathbb{Z}$, a complete description of the spectral type is obtained. Further work on the case $z \in \mathbb{C} \setminus \mathbb{Z}$ reveals a remarkable link with stochastic point processes and random matrix theory.
0. Introduction

0.1. The infinite symmetric group: characters, factor representations, spherical representations. The present paper deals with harmonic analysis on the infinite symmetric group $S(\infty)$ (the group of finite permutations of the infinite set $\{1, 2, \ldots \}$). This group belongs to the class of big groups,\(^1\) such as the infinite dimensional classical groups, diffeomorphism and current groups. We present here the first example of harmonic analysis on a big group. A short exposition of our results was published in [KOV]; here we provide the detailed statements and proofs.

The most well–studied groups in representation theory — compact, abelian, and reductive Lie groups — are tame. Irreducible unitary representations of tame groups can be classified, and the basic problem of harmonic analysis consists in decomposing interesting reducible representations (e.g., the regular representation) into irreducible components.

The group $S(\infty)$ is wild, not tame. The wild groups do not admit any sensible classification of irreducible representations, and reducible representations can be decomposed into irreducibles in essentially different ways. Unlike tame groups, the wild groups have factor representations of von Neumann types II and III. All those facts imply that the representation theory of wild groups should be based on the principles distinct from those used for tame groups.\(^2\)

The possibility of developing such a theory was first demonstrated by E. Thoma [Tho1] exactly for the example of $S(\infty)$. In that paper Thoma discovered that the finite type factor representations of the group $S(\infty)$ admit an explicit classification. More precisely, those representations are labelled by the pairs $\omega = (\alpha; \beta)$ of nonincreasing sequences of nonnegative numbers

$$\alpha = (\alpha_1 \geq \alpha_2 \geq \ldots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \ldots \geq 0)$$

subject to the extra condition

$$\sum_{k=1}^{\infty} \alpha_k + \sum_{k=1}^{\infty} \beta_k \leq 1.$$

---

\(^1\)This term, introduced in Vershik [Ver], is somewhat vague but expressive and convenient. About representations of various big groups, see, e.g., Ismagilov [Ism], Neretin [Ner1], Olshanski [Ol14], Strătilă–Voiculescu [SV].

\(^2\)For general facts about tame and wild groups, see, e.g., Dixmier [Dix], Kirillov [Kir]. Note that tame groups are also called type I groups. About factor representations, see, e.g., Dixmier [Dix], Thoma [Tho2].
The set of such pairs forms an infinite-dimensional simplex $\Omega$ which we call the
Thoma simplex.\footnote{For more detail on Thoma’s theorem, see §9.6 below.}

For an arbitrary unitary representation of a countable group, generating a von
Neumann algebra of finite type, a decomposition into a direct integral of factor
representations always exists and is essentially unique. Therefore, the main problem
of harmonic analysis on $S(\infty)$ may consist in the actual decomposition of most
notable representations of finite type. This agrees with an old idea (especially
developed by Pukanszky) that as “elementary representations” for a wild group,
one should consider not irreducible representations but factor representations of an
appropriate class.

Instead of decompositions of representations one can speak of decompositions of
their characters. Denote by $X$ the set of positive definite functions $\chi$ on the
group $S(\infty)$, constant on conjugacy classes and normalized at the identity permutation
by the condition $\chi(e) = 1$. In the topology of pointwise convergence, the space
$X$ is a convex compact set. It is known that the extreme points of the set $X$ are
exactly the characters of factor representations of finite type. Moreover, the space
$X$ is a Choquet simplex, that is, any point $\chi \in X$ can be uniquely represented as a
continual convex combination of extreme characters.

However, as was shown by one of the authors (\cite{Ol2}, \cite{Ol3}), one can de
velop another approach to representation theory of the group $S(\infty)$, which makes it
possible to avoid factor representations. Recall that, by definition, the group $S(\infty)$
consists of finite permutations of the set $\{1, 2, \ldots\}$, fixing all but finitely many
points. Consider now the group $\overline{S(\infty)}$ of all bijections of the set $\{1, 2, \ldots\}$ onto
itself. Taking the subsets

$$S_m(\infty) = \{g \in S(\infty) : g(k) = k, k = 1, \ldots, m\},$$

where $m = 1, 2, \ldots,$ as a fundamental system of neighborhoods of identity we turn
$S(\infty)$ into a topological group.

The subgroup $S(\infty)$ is dense in $\overline{S(\infty)}$. Furthermore, denote by $\overline{G}$ the group of
pairs $(g_1, g_2) \in \overline{S(\infty)} \times \overline{S(\infty)}$, such that $g_1^{-1}g_2 \in S(\infty)$, and identify $\overline{S(\infty)}$ with
the diagonal subgroup $\overline{K}$ in $\overline{G}$. We introduce a topology in $\overline{G}$ by proclaiming
the subgroup $\overline{K}$ open. The group $\overline{S(\infty)} \times S(\infty)$ is dense in $\overline{G}$. The groups $\overline{K}$ and $\overline{G}$
are totally disconnected and not locally compact.

Remarkably enough, the topological group $\overline{G}$ is tame and its irreducible represen-
tations can be completely described, see Olshanski \cite{Ol3}, Okounkov \cite{Ok1}, \cite{Ok2}.
In the present paper we are only interested in spherical irreducible representations
of the group $\overline{G}$ (that is, representations containing a $\overline{K}$–invariant vector). It is
known that $(\overline{G}, \overline{K})$ is a Gelfand pair: in any irreducible spherical representation
there is only one, up to a scalar factor, $\overline{K}$–invariant vector $\xi$.

There exists a natural bijection $T \leftrightarrow \pi$ between irreducible spherical representa-
tions $T$ of the pair $(\overline{G}, \overline{K})$ and factor representations $\pi$ of finite type of the group
$S(\infty)$. Specifically, $\pi$ is the restriction of $T$ to the subgroup $S(\infty) \times \{e\} \subset \overline{G}$. The
character $\chi(s) = Tr \pi(s)$ of the representation $\pi$ (here “Tr” denotes the normalized
trace on the factor) is related to the spherical function $\varphi(g) = (T(g)\xi, \xi)$ by the
simple formula

$$\varphi(g_1, g_2) = \chi(g_1^{-1}g_2), \quad (g_1, g_2) \in \overline{G}.$$
Therefore, we can think of finite factor representations of the group $S(\infty)$ as of irreducible spherical representations of the pair $(\mathcal{G}, \mathcal{K})$, thus returning to the conventional setup of representation theory of tame groups.

0.2. The generalized regular representations $T_z$ and the problem of harmonic analysis. We shall now describe the representations of the group $G$ whose decomposition is the purpose of the present paper. The choice of those representations is itself a nontrivial problem. Traditionally, the object of harmonic analysis for a Gelfand pair $(G, K)$ is the decomposition of the natural representation of $G$ in the space $L^2(G/K)$ (for instance, for Riemannian symmetric spaces $G/K$, the decomposition problem was studied in classical works of Harish–Chandra and Gindikin–Karpelevich). In our situation the space $G/K \cong S(\infty)$ is discrete, so that the Hilbert space $L^2(G/K) = \ell^2(S(\infty))$ makes sense. However, the corresponding representation turns out to be irreducible so that there is no need of any further decomposition. This irreducibility effect is equivalent to the following fact (which is probably better known): the one–sided regular representation of the group $S(\infty)$ in the space $\ell^2(S(\infty))$ generates a type $\mathrm{II}_1$ factor.\(^4\)

Instead of the homogeneous space $G/K = S(\infty)$ we introduce a compact space $\mathcal{S}$ containing $S(\infty)$ as a dense subset. More precisely, the space $\mathcal{S}$ is defined as a projective limit of finite sets,

$$S(1) \leftarrow S(2) \leftarrow \ldots \leftarrow S(n) \leftarrow \ldots$$

Here $S(n)$ is the set of all permutations of $n$ objects, and the projections are specified in §1 below. The points of $\mathcal{S}$ are called virtual permutations. Unlike $S(\infty)$, the space $\mathcal{S}$ is not a group. However, it admits a canonical action of the group $G$, which is sufficient for our purposes. In particular, we have a two–sided action of the group $S(\infty)$ on $\mathcal{S}$.

On the space $\mathcal{S} \supset S(\infty)$ of virtual permutations there is a (unique) probability measure invariant with respect to the action of the group $\mathcal{G}$. This measure, which we denote as $\mu_1$, is much more interesting and useful than the counting (Haar) measure on $S(\infty)$. More generally, the measure $\mu_t$, $t > 0$, can be included into a one–parameter family of probability measures $\mu_t$, $t > 0$. All those measures are invariant with respect to the subgroup $K$, and quasiinvariant with respect to $\mathcal{G}$.\(^5\)

By applying a standard general construction of producing unitary representations from a group action with quasiinvariant measure, we arrive at a family of representations $T_z$ of the group $\mathcal{G}$. Here $z$ ranges over the set $\mathbb{C} \setminus \{0\}$, and the representation $T_z$ acts in the space $L^2(\mathcal{S}, \mu_t)$ where $t = |z|^2$. One can also define two more unitary representations $T_0$, $T_\infty$ in such a way that the resulting family is continuously parametrized by the points of the Riemann sphere $\mathbb{C} \cup \{\infty\}$. The representation $T_\infty$ coincides with the above mentioned two–sided regular representation in the space $\ell^2(S(\infty))$. We regard the family $\{T_z\}$ as a deformation of $T_\infty$.

\(^4\)More generally, for any discrete group whose conjugacy classes, except $\{e\}$, are infinite, the one–sided regular representation is a type $\mathrm{II}_1$ factor representation, while the two–sided regular representation is irreducible. See Murray–von Neumann [MvN, chapter 5], Naimark [Nai, chapter VII, §38.5].

\(^5\)The idea of extending the group space (of a big group) in order to obtain measures with good transformation properties comes from the measure theory on infinite–dimensional linear spaces and is rather old, see, e.g., Gelfand–Vilenkin [GV, chapter IV].
and we call $T_z$ the generalized regular representations.\(^6\)

The only irreducible representation in the family $\{T_z : z \in \mathbb{C} \cup \{\infty\}\}$ is $T_{\infty}$. All the representations $T_z$, $z \in \mathbb{C}$, are reducible, and their structure is rather complicated. We consider the explicit decomposition of those representations as the main problem of harmonic analysis for the group $S(\infty)$.

Each representation $T_z$ can also be realized as an inductive limit of the two–sided regular representations of finite groups $S(n) \times S(n)$,

$$\text{Reg}^1 \rightarrow \text{Reg}^2 \rightarrow \ldots \rightarrow \text{Reg}^n \rightarrow \ldots$$

(*)

with very special embeddings $\text{Reg}^n \rightarrow \text{Reg}^{n+1}$ depending on $z$. The existence of a finite dimensional approximation allows one to employ the powerful combinatorial and probabilistic techniques used in the theory of approximately finite–dimensional ($AF$–) algebras. In case of the group $S(\infty)$, these techniques, based on combinatorics of Young diagrams and the theory of symmetric functions, were developed in [VK2], [VK3].

0.3. Main results of the paper. These are as follows:

1. The representations $T_z$ and $T_{\bar{z}}$ are equivalent, and we describe an intertwining operator realizing their equivalence.

2. For any $z \in \mathbb{C}$, the representation $T_z$ can be decomposed into a multiplicity free direct integral of irreducible spherical representations. So, the equivalence class of $T_z$ is completely described by an equivalence class of measures on the Thoma simplex $\Omega$. We will refer to the latter equivalence class as to the spectral type of $T_z$.

3. The spectral types of two representations $T_{z_1}$ and $T_{z_2}$ are disjoint (mutually singular) whenever $z_1$ and $z_2$ are not equal or conjugate to each other. This means that there exist two disjoint Borel subsets in $\Omega$ supporting the measures from the spectral types of $T_{z_1}$ and $T_{z_2}$, respectively. Equivalently, there is no intertwining operator between $T_{z_1}$ and $T_{z_2}$.

4. The spectral type of $T_z$ substantially depends on whether $z$ is an integer or not. In the present paper we focus on the case when $z \in \mathbb{Z}$. Then we are able to describe the spectral type quite explicitly. Namely, let $\Omega(p,q)$ denote the subset of those pairs $(\alpha, \beta) \in \Omega$ for which $\alpha_1 + \cdots + \alpha_p + \beta_1 + \cdots + \beta_q = 1$ (consequently, all the remaining coordinates $\alpha_i$ and $\beta_j$ vanish); notice that $\Omega(p,q)$ is a $(p+q-1)$–dimensional face of the Thoma simplex $\Omega$. Our result says that when $z = 0, \pm 1, \pm 2, \ldots$, the spectral type of $T_z$ is determined by the union of Lebesgue measures on the finite–dimensional faces $\Omega(p,q)$ with $p - q = z$ (which agrees with the result on disjointness of spectral types stated above). We also show that if $z \in \mathbb{C} \setminus \mathbb{Z}$, then the spectral type of $T_z$ is concentrated “inside” the simplex $\Omega$. That is, all faces $\Omega(p,q)$ are null sets with respect to the spectral type of $T_z$.

0.4. The case $z \in \mathbb{C} \setminus \mathbb{Z}$. This case is studied in detail in a series of papers by Borodin and Olshanski, see [O15], [Bor2], [BO3], the expository papers [BO1], [O16], and references therein. When $z \in \mathbb{C} \setminus \mathbb{Z}$, there exists a distinguished probability measure in the spectral type of $T_z$ (in the present paper it is denoted as $\sigma_z$). The key idea is to assign to $(\Omega, \sigma_z)$ a stochastic point process, and to study its correlation functions. It turns out that the point processes thus obtained are close to those

\(^6\)Notice a certain similarity between our family $\{T_z\}$ and Neretin’s deformation of the natural representation on $L^2(\mathbb{G}/\mathbb{K})$, where $\mathbb{G}/\mathbb{K}$ is a Riemannian symmetric space, see Neretin [Ner3].
arising in random matrix theory. The link with random matrix theory seems to be especially interesting and promising.

The results of Borodin and Olshanski were obtained as a continuation of the project started in [KOV]. Together with the present work they provide a description of the spectral types for all representations $T_z$.

0.5. Organization of the paper.

In §1, we start with the definition of the canonical projection $p_n : S(n) \to S(n-1)$. Using the projections $p_n$ we define the space $\mathcal{S} = \bigcup_n S(n)$ of virtual permutations. Then we describe four different concrete realizations of that space. In one of them, $\mathcal{S}$ turns into the product of an infinite sequence of finite sets. Finally, we show that $\mathcal{S}$ is a $G$–space, where $G = S(\infty) \times S(\infty)$, and we introduce an additive $\mathbb{Z}$–valued 1–cocycle for the action of $G$ on $\mathcal{S}$.

In §2, we study the family $\{\mu_t\}_{t>0}$ of probability measures on the space $\mathcal{S}$. This is a deformation of the unique $G$–invariant probability measure (in our notation, $\mu_1$). The measures $\mu_t$ are central, i.e., invariant with respect to the diagonal subgroup $K \subset G$. We show that they turn into product measures in one of the realizations of $\mathcal{S}$. Moreover, they are essentially the only central measures with this property. Applying Kakutani’s theorem we show that the measures $\mu_t$ are pairwise disjoint. Then we show that any $\mu_t$ is quasiinvariant with respect to the action of the group $G$, i.e., for any $g \in G$, the shift $\mu_t^g$ of $\mu_t$ by $g$ is a measure in the equivalence class of $\mu_t$. We also calculate the Radon–Nikodym derivative $\mu_t^g/\mu_t$, which we will need later.

In §3, we construct the representations $T_z$ in two different ways. First, we realize $T_z$, where $z \in \mathbb{C}^+$, in the Hilbert space $L^2(\mathcal{S}, \mu_t)$, where $t = |z|^2$. Here we use a multiplicative 1–cocycle $\mathcal{S} \times G \to \mathbb{C}^*$ which depends on $z$ and is defined via the additive cocycle from §2. Second, we realize the same representation $T_z$ as the inductive limit corresponding to a chain of embeddings (*), see above. Using the latter realization, it is easy to complete the family $\{T_z\}$ by two limit representations, $T_0$ and $T_\infty$. Finally, we present a transparent interpretation of the embeddings in (*).

In §4, we define and study a distinguished matrix coefficient of $T_z$. The representation $T_z$ has a distinguished $K$–invariant vector $\xi_0$: in the first realization, this is the identity function $f_0 \equiv 1$ on the space $\mathcal{S}$. Though the space of $K$–invariant vectors is infinite dimensional, $\xi_0$ is the only $K$–invariant vector which is seen at once: constructing other examples of $K$–invariant vectors is already a nontrivial task (we do this in section 6). To the vector $\xi_0$ one associates a spherical function on $G$, $\varphi_z(g_1, g_2) = (T(g_1, g_2)\xi_0, \xi_0)$, and a character $\chi_z(s) = \varphi(s, e)$ of the group $S(\infty)$. We do not dispose of a simple expression for the values of the function $\chi_z$ on conjugacy classes of the group $S(\infty)$; instead of this, we find a very nice formula for the coefficients $M_z(\lambda)$ in the expansions

$$\chi_z|_{S(n)} = \sum_{\lambda \in \mathcal{Y}_n} M_z(\lambda) \tilde{\chi}^{\lambda}, \quad n = 1, 2, \ldots, \quad (**),$$

where $\mathcal{Y}_n$ denotes the set of Young diagrams with $n$ boxes, and $\tilde{\chi}^{\lambda}$ is the normalized irreducible character of $S(n)$, indexed by $\lambda$. In other words, we get the “Fourier coefficients” of $\chi_z$ (see Theorem 4.2.1).
This explicit formula for $M_z(\lambda)$ plays a key role in the present paper.\footnote{For alternative derivations of the formula and generalizations, see Kerov [Ker3], Borodin’s appendix in Olshanski [Ol5], Borodin–Olshanski [BO2].} We derive from it the following results:

First, we prove that $\xi_0$ is a cyclic vector if (and only if) $z \in \mathbb{C} \setminus \mathbb{Z}$. This means that for nonintegral values of $z$, the spectral type of $T_z$ is entirely determined by the decomposition of the matrix coefficient associated with $\xi_0$.

Second, we prove the equivalence $T_z \sim T_{\overline{z}}$, which is not evident from the construction of the representations. We also exhibit an isometric operator intertwining $T_z$ and $T_{\overline{z}}$.

Finally, following the general philosophy of [VK2], [VK3], we assign to the family $M_z = \{M_z^{(n)}\}$ the so-called transition probabilities $p_z(\lambda, \nu)$. Given $\lambda \in \mathbb{Y}_n$, the numbers $p_z(\lambda, \nu)$ form a probability distribution on the set of those diagrams $\nu$ that can be obtained from $\lambda$ by adding a single box.

In §5, we deal with the realization of $T_z$ as an inductive limit of regular representations of the finite groups $S(n) \times S(n)$. To any such inductive limit we assign a “transition function” defined on the edges of the Young graph. This leads to a description of the commutant of the representation. We explicitly calculate the transition function for the representations $T_z$. Using this, we get, for integral values of the parameter $z$, a decomposition of $T_z$ into a direct sum of subrepresentations $T_{pq}$ which we call the blocks of $T_z$.

In §6, we get a convenient realization of the subspace $V_z$ of $K$–invariant vectors in the Hilbert space of $T_z$. In terms of this realization we construct, for any $z \in \mathbb{Z}$, a $K$–invariant vector in each block of $T_z$, and we calculate the spectral decomposition of the corresponding matrix coefficient.

In §7, we prove that all the vectors constructed in section 6 are cyclic vectors in the corresponding blocks. Together with the results of section 6 this provides us with a complete description of the spectral type of the representation $T_z$ in case of integral $z$. The argument of this section turns out to be rather long. At the end we give an example illustrating the origin of one of the difficulties.

In §8, we deal with arbitrary values of the parameter $z$. Here we prove that the spectral types of the representations $T_z$ are pairwise disjoint (except the equivalence $T_z \sim T_{\overline{z}}$).

In §9 (Appendix), we collected a number of necessary definitions and facts of general nature.

**0.6. Concluding remarks.** The measures $\mu_t$ involved in the construction of the representations $T_z$ are very interesting objects in their own right. Each measure $\mu_t$ can be written as a projective limit $\varprojlim \mu^n_t$, as $n \to \infty$, where $\mu^n_t$ is a remarkable probability measure on the finite symmetric group $S(n)$. These measures $\mu^n_t$ were first discovered in the context of population genetics and were considered in many subsequent works (see, e.g., the encyclopedic article Tavaré–Ewens [TE] and references therein). We will call the measures $\mu_t$ the Ewens measures. The probability space $(\mathcal{G}, \mu_t)$ is closely related to the Chinese Restaurant Process construction, see Aldous [Ald], Pitman [Pit, §3.1].

There is a similarity between the spectral decomposition of the characters $\chi_z$ attached to the representations $T_z$, and the decomposition of the measures $\mu_t$ into $K$–invariant ergodic components. In a certain sense, these two problems are dual to each other: the latter refers to the “group level” while the former refers to the
“group dual level”. Moreover, there exists a general scheme unifying both problems and providing an interpolation between them, see Kerov [Ker3], Kerov–Okounkov–Olshanski [KOO], Borodin–Olshanski [BO2]. The decomposition of the measures $\mu_t$ is governed by the Poisson–Dirichlet distributions, see Kingman [Kin1], [Kin2], and also Olshanski [Ol5].

There is a deep analogy between the infinite symmetric group $S(\infty)$ and the infinite–dimensional unitary group $U(\infty)$. This analogy becomes apparent when one compares the description of characters of both groups, given in the fundamental papers Thoma [Tho1] and Voiculescu [Voi]. The theory of harmonic analysis for $S(\infty)$, as developed in the present paper and the papers of Borodin and Olshanski mentioned above, also has a counterpart for the group $U(\infty)$, see Olshanski [Ol6] and Borodin–Olshanski [BO5].

In particular, the counterparts of the Ewens measures $\mu_t$ are the so–called Hua–Pickrell measures, see Borodin–Olshanski [BO4]. More generally, similar measures can be associated to all 10 infinite series of classical Riemannian symmetric spaces of compact type, see Neretin [Ner2]. A pioneer work in this direction is that of Pickrell [Pic]; our construction of the space $\mathcal{S}$ of virtual permutations was largely influenced by that paper.

As was first observed by Borodin, the expression (**) for the characters $\chi_z$ can be analytically continued to provide a complementary series of characters. In the papers of Borodin and Olshanski, the characters of the complementary series are considered together with the characters $\chi_z$, the latter being viewed as the principal series (for a justification of this terminology, see Okounkov [Ok3]; note that there also exists a degenerate series). A natural question is whether these series of characters (and the corresponding representations) exhaust all “reasonable” objects of harmonic analysis for $S(\infty)$. In this direction little is known. Using the idea of Borodin [Bor], Rozhkovskaya [Rozh] obtained an elegant combinatorial characterization of the characters $\chi_z$ and their analytic continuation. Kerov [Ker1] considered the so–called Ewens–Pitman measures generalizing the Ewens measures $\mu_t$, as possible candidates for an extension of the basic construction of the representations $T_z$. It would be interesting to pursue further the study of this question.

§1. THE SPACE OF VIRTUAL PERMUTATIONS

1.1. Canonical projections. Let $S(n)$ be the group of permutations of the finite set $\{1, \ldots, n\}$, the symmetric group of degree $n$. We identify $S(n)$ with the subgroup of permutations $s \in S(n+1)$ preserving the last element $n+1$, i.e., $s(n+1) = n+1$. The inductive limit of groups $S(n)$ with respect to these embeddings (i.e., the union of these groups) will be denoted as $S(\infty) = \varinjlim S(n)$. The elements of $S(\infty)$ are finite permutations of the set $\{1, 2, \ldots\}$, fixing all but finitely many natural numbers. We call $S(\infty)$ the infinite symmetric group.

Given a permutation $\tilde{s} \in S(n+1)$, $n = 1, 2, \ldots$, we define its derivative permutation (we borrow the term from dynamical systems theory, see [CFS, chapter 1, §5]) $s = \tilde{s}' \in S(n)$ as follows

$$s(i) = \begin{cases} 
\tilde{s}(i), & \text{if } \tilde{s}(i) \leq n, \\
\tilde{s}(n+1), & \text{if } \tilde{s}(i) = n+1,
\end{cases}$$

where $i = 1, \ldots, n$. The map $\tilde{s} \mapsto s$, denoted $p_{n,n+1}$, will be referred to as the canonical projection of $S(n+1)$ onto $S(n)$. Here is an alternative description of the
canonical projection in terms of the cycle structure of permutations. Depending on the position of the element \( n+1 \) in the cycles of the permutation \( \tilde{s} \) we distinguish between two cases: \( n+1 \) belongs to a cycle \((\ldots \to i \to n+1 \to j \to \ldots)\) of length \( \geq 2 \), or \( n+1 \) is a fixed point of \( \tilde{s} \). In the former case we remove \( n+1 \) out of its cycle, i.e., we replace this cycle with \((\ldots \to i \to j \to \ldots)\). In the latter case \( \tilde{s} \) already belongs to the subgroup \( S(n) \subset S(n+1) \), and we set \( s = \tilde{s} \).

It is clear from the definition that the preimage of a permutation \( s \in S(n) \) with respect to the canonical projection contains \( n+1 \) permutations in \( S(n+1) \). In fact, in order to obtain a permutation \( \tilde{s} \in p_{n,n+1}^{-1}(s) \) one should insert \( n+1 \) in a cycle of \( s \) right before one of the elements \( j = 1, \ldots, n \), or take \( n+1 \) as a new 1–cycle. Note that in the latter case \( \tilde{s} = s \).

Yet another useful description of the canonical projection employs the following simple operation on graphs. It will be convenient to identify permutations with bipartite graphs. We associate with a permutation \( s \in S(n) \) a graph with the vertex set \( \{1, \ldots, n; 1', \ldots, n'\} \). Its edges are couples of the form \((i, j')\), where \( s(i) = j \). The projection \( p_{n,n+1} : \tilde{s} \mapsto s \) can now be described as follows. Take the graph of \( \tilde{s} \), and add an extra edge connecting the vertices \((n+1) \) and \((n+1)\).

Then the graph of the derivative permutation \( s \) arises if one takes for the edges the paths connecting the vertices in \( 1, \ldots, n \) with the vertices in \( 1', \ldots, n' \).

Note that the group \( S(n) \) acts by left and right multiplications on both \( S(n) \) and \( S(n+1) \).

**Proposition 1.1.1.** The canonical projection \( p_{n,n+1} \) is equivariant with respect to two–sided action of the group \( S(n) \). If \( n \geq 4 \), this is the only map \( S(n+1) \to S(n) \) with this property.

**Proof.** The first claim is immediate from the description of the canonical projection in terms of bipartite graphs, and the obvious graphical interpretation of the product of permutations. Assume now that the a map \( p : S(n+1) \to S(n) \) is equivariant. Then, for each permutation \( s \in S(n) \), we have \( s^{-1}p(e)s = p(s^{-1}es) = p(e) \), where \( e \) is the identity permutation. Since \( e \) is the only central element in \( S(n) \) for \( n \geq 3 \), this implies \( p(e) = e \). By the same token, \( s^{-1}p((n, n+1))s = p((n, n+1)) \) for every permutation \( s \in S(n-1) \). If \( n \geq 4 \), then \( e \) is the only element of \( S(n) \) commuting with all permutations in \( S(n-1) \) (for \( n = 2, 3 \) the transposition \((12)\) also shares this property). Therefore, \( n \geq 4 \) implies \( p((n, n+1)) = e \). Since the group \( S(n+1) \) is made of just two double \( S(n) \)–cosets (the group \( S(n) \) itself and the class containing \((n, n+1)\)), it follows that \( p = p_{n,n+1} \). \( \square \)

**Remark 1.1.2.** As it is clear from the proof, there are non–canonical projections \( p : S(n+1) \to S(n) \) for \( n = 2, 3 \). They are determined by the equalities \( p((23)) = (12) \) and \( p((34)) = (12) \) respectively. Note that there are lots of maps \( S(n+1) \to S(n) \) which are equivariant with respect to a one–sided (left or right) action of the group \( S(n) \).

1.2. *Virtual permutations.* Consider the sequence

\[
S(1) \leftarrow \cdots \leftarrow S(n) \leftarrow S(n+1) \leftarrow \ldots
\]

of canonical projections, and let

\[
\mathcal{S} = \lim_{\to} S(n)
\]
denote the projective limit of the sets $S(n)$. By definition, the elements of $\mathcal{G}$ are arbitrary sequences $x = (x_n \in S(n))$, such that $p_{n,n+1}(x_{n+1}) = x_n$ for all $n = 1, 2, \ldots$. The set $\mathcal{G}$ is a closed subset of the compact space of all sequences $(x_n)$, therefore it is a compact space itself. Let $p_n : \mathcal{G} \to S(n)$ denote the natural projection, $n = 1, 2, \ldots$. The group $S(\infty)$ can be identified with a subset of $\mathcal{G}$ via the map $s \mapsto x = (x_n)$, $x_n = s$ for sufficiently large $n$. In other words, $S(\infty) \subset \mathcal{G}$ consists of the stable sequences $(x_n)$.

The subset $S(\infty)$ is dense in $\mathcal{G}$, because $p_n(S(\infty)) = p_n(\mathcal{G}) = S(n)$ for any $n$. Hence, the space $\mathcal{G}$ is a compactification of the discrete space $S(\infty)$. The elements of $\mathcal{G}$ will be called virtual permutations of the set $\{1, 2, \ldots\}$.

The definition of the space $\mathcal{G}$ does not change if we remove from the projective limit any finite number of first terms, i.e., if we start from $S(n)$ instead of $S(1)$, where $n$ is chosen arbitrarily. This simple observation will be tacitly used in what follows.

In particular, it implies that changing the canonical projections $p_{n,n+1}$ for $n = 2, 3$ by noncanonical ones (see Remark 1.1.2) does not affect the construction of the space $\mathcal{G}$.

1.3. Realizations of the space of virtual permutations. We shall use four concrete realizations of the space $\mathcal{G}$ of virtual permutations:

(1) as of the infinite product $\prod_{n=1}^{\infty} \{0, 1, \ldots, n - 1\}$;
(2) as of the space of growing trees;
(3) as of the space of decreasing maps of the set $\{0, 1, \ldots\}$ in itself;
(4) as of the space of cyclic structures on the set $\{1, 2, \ldots\}$.

All the constructions use appropriate realizations of finite sets $S(n)$, and the canonical projections preserve the specific structures of those sets. In this sense, the above realizations are natural.

Proposition 1.3.1. There exists a natural homeomorphism $x = (x_n) \mapsto i = (i_n)$ between the space $\mathcal{G}$ and the infinite product

$$I = I_1 \times I_2 \times \ldots \quad \text{where } I_n = \{0, 1, \ldots, n - 1\}.$$

Proof. Given an element $x = (x_n) \in \mathcal{G}$, we define the sequence $i = (i_n) \in I$ as follows. Set $i_1 = 0$. For every $n = 1, 2, \ldots$ the coordinate $i_{n+1}$ encodes the relation of $s = x_n$ and $\tilde{s} = x_{n+1}$. Specifically, $i_{n+1} = 0$ means that $s = \tilde{s}$, and $i_{n+1} = j \in \{1, \ldots, n\}$ means that the element $n + 1$ is inserted in a cycle of $s$ immediately before $j$. One can easily check that this correspondence is indeed a homeomorphism $\mathcal{G} \to I$, where $I$ is equipped with the product topology. $\Box$

Likewise, we get a bijection $S(n) \to I_1 \times \cdots \times I_n$. The vector $i(x) = (i_1, i_2, \ldots, i_n)$ is called the code of a permutation $x \in S(n)$. We shall apply this terminology to virtual permutations, too.

We define a finite increasing tree as a rooted labelled tree with $n + 1$ vertices, labelled by the numbers $0, 1, \ldots, n$ in such a way that the root has label 0 and the labels increase along every path leading off the root (cf. [Sta], §1.3). Countable increasing trees are defined in a similar way.

Proposition 1.3.2. There exists a natural bijection between virtual permutations and countable increasing trees.
Proof. Let $\tau$ be a countable increasing tree. Removing vertices with the labels $n+1, n+2, \ldots$ we obtain a finite increasing tree which we denote as $\tau_n$. Thus, a tree $\tau$ can be considered as the union of a chain $\tau_1 \subset \tau_2 \subset \ldots$ of finite increasing trees, where $\tau_i$ has $i+1$ vertices. Given a tree $\tau_n$, there are $n+1$ options for the next tree $\tau_{n+1}$, which are naturally indexed by the numbers $0, 1, \ldots, n$. In fact, we can join the vertex $(n+1)$ to every one of the vertices $0, \ldots, n$. Therefore, each countable increasing tree $\tau$ is determined by the sequence $i = (i_n) \in I$. One can easily check that that the map $\tau \mapsto (i_n)$ provides a bijection onto $I$. From Proposition 1.3.1 we get a bijection between countable increasing trees, and virtual permutations. The above construction is quite similar to that of the bijection between permutations in $S(n)$, and increasing trees with $n + 1$ vertices, cf. [Sta], §1.3. □

Note that the bijection $\mathcal{S} \to I$ identifies finite permutations $s \in S(\infty) \subset \mathcal{S}$ with the sequences $i = (i_n)$ with only finitely many nonzero coordinates. There is also an obvious interpretation in terms of increasing trees.

The first two realizations of virtual permutations are actually very close to each other. The third realization is a version of the first one, too. We say that a map $\varphi$ of the set $\{0, 1, \ldots\}$ into itself is decreasing if $\varphi(0) = 0$ and $\varphi(n) < n$ for all $n > 0$. Setting $\varphi(n) = i_n$ for $n \geq 1$ we obtain a bijective correspondence between decreasing maps and the points of $I$.

In the next subsection we describe the forth realization of virtual permutations; its nature is rather different from the first three.

1.4. Cyclic structures. Let $J$ be an arbitrary set. We define a cyclic order on $J$ as a family of subsets $[i, j)$, called arcs, and labelled by ordered pairs $i$, $j$ of distinct points in $J$. We assume that the following four axioms hold:

1. for each pair $i, j$ (where $i \neq j$) the arcs $[i, j), [j, i]$ do not intersect, and their union is $J$;
2. for each arc $[i, j)$, $i \in [i, j)$ (and hence $j \notin [i, j)$);
3. if $i, j, k$ are pairwise distinct, then one of the arcs $[i, j)$, $[i, k)$ is a strict subset of the other;
4. if the arc $[i, j)$ is a strict subset of $[i, k)$, then $[j, k) = [i, k) \setminus [i, j)$.

If $|J| = 1$, there is only one cyclic structure on $J$ with the empty family of arcs.

Note that cyclic orders on a finite set $J$ are in a bijective correspondence with cyclic permutations of the elements of $J$. In fact, given a cyclic permutation $s$ of the set $J$, we define the corresponding cyclic order as follows: for each pair $i \neq j$ the arc $[i, j)$ consists of the points $i, s(i), \ldots, s^{m-1}(i)$ where $m$ is the least number with $s^m(i) = j$. On the contrary, for infinite sets there is no natural relation between cyclic orders and permutations.

More generally, we define a cyclic structure on a set $J$ as a partition of $J$ into nonempty disjoint subsets (called cycles) with a specified cyclic order on each cycle. Let $CS(J)$ denote the set of all cyclic structures on $J$. If $J$ is finite, there is a natural bijection between $CS(J)$ and the set of all permutations of $J$. Specifically, we associate with a permutation $s$ its cycle partition with the cyclic order on each cycle as defined above.

For every subset $J' \subset J$ there is a natural projection $CS(J) \to CS(J')$. Moreover, if $J$ is a union of an increasing sequence of subsets $(J_n)$, then $CS(J)$ can be naturally identified with the projective limit of the sets $CS(J_n)$. This observation leads to the following result.
Proposition 1.4.1. There exists a natural bijection between $\mathfrak{S}$ and the set $CS(\{1, 2, \ldots \})$ of cyclic structures on $\{1, 2, \ldots \}$.

Note that the bijection of Proposition 1.4.1 identifies $p_n : \mathfrak{S} \to S(n)$ with the projection $CS(\{1, 2, \ldots \}) \to CS(\{1, \ldots , n\}) = S(n)$.

We define the cycles of a virtual permutation $x \in \mathfrak{S}$ as the cycles of the corresponding cycle structure on the set $\{1, 2, \ldots \}$.

In the second realization of $\mathfrak{S}$ (see §1.3), the cycles of $x$ correspond to those subtrees of the rooted tree $\tau$ associated with $x$ which are one edge apart from the root of $\tau$.

In the third realization of $\mathfrak{S}$, the cycles correspond to the orbits of the associated map $\varphi$ of the set $\{1, 2, \ldots \}$ to itself. Here, by definition, two numbers $i, j$ belong to the same $\varphi$–orbit if there exist $k, l$, such that $\varphi^k(i) = \varphi^l(j)$.

1.5. The groups $G$ and $K$, and their action on the space $\mathfrak{S}$. Set $G = S(\infty) \times S(\infty)$. We call $G$ the infinite bisymmetric group. Let $K = \text{diag}(S(\infty))$ denote the diagonal subgroup $\{(s, s) : s \in S(\infty)\}$ in $G$. We shall also use parallel notation

$$G(n) = S(n) \times S(n), \quad K(n) = \text{diag}(S(n)) \subset G(n).$$

Clearly, the group $K$ is isomorphic to $S(\infty)$. We consider $S(\infty)$ as a right homogeneous space $K \backslash G$, where the action of $G$ is defined as follows

$$s \cdot g = g_2^{-1} s g_1, \quad s \in S(\infty), \quad g = (g_1, g_2) \in G = S(\infty) \times S(\infty).$$

Since the canonical projections $p_{n,n+1}$ are equivariant, this action can be naturally extended to the action $\mathfrak{S} \times G \to \mathfrak{S}$ defined as $x \cdot g = y$, where

$$y_n = x_n \cdot g$$

for all $n$ large enough.

Specifically, this equality holds whenever $n$ is so large that $g \in G$ already lies in $G(n)$.

Note that $G$ acts on $\mathfrak{S}$ by homeomorphisms. Thus, this is the (only) continuous extension of the action of $G$ on $K \backslash G$ to the space $\mathfrak{S}$.

Unfortunately, in all our realizations the description of this action of $G$ is rather awkward. Only the action of the subgroup $K$ can be easily described in terms of the forth realization of $\mathfrak{S}$. Indeed, the action of $K$ on $\mathfrak{S}$ is the only continuous extension of its action on $S(\infty)$ by conjugations. Under the identification of the space $\mathfrak{S}$ with the space of cyclic structures on $\{1, 2, \ldots \}$, this turns into the natural action of $K$ (as a group of permutations of $\{1, 2, \ldots \}$) on the set $CS(\{1, 2, \ldots \})$.

1.6. The fundamental cocycle. Given $s \in S(n)$, denote by $[s]$ the number of cycles in $s$. It is important for what follows that the difference $[x \cdot g] - [x]$ can be defined correctly for all $x \in \mathfrak{S}$ and $g \in G$.

Proposition 1.6.1. (i) There exists an integer valued function $c(x, g)$ on $\mathfrak{S} \times G$, uniquely defined by the following property: if $n$ is large enough, so that $g \in S(n) \times S(n)$, then

$$c(x, g) = [p_n(x \cdot g)] - [p_n(x)] = [p_n(x) \cdot g] - [p_n(x)].$$
(ii) This function is an additive cocycle:

\[ c(x, g_1 g_2) = c(x, g_1) + c(x \cdot g_1, g_2). \]

(iii) If \( g \in K \), then \( c(\cdot, g) \equiv 0 \).

Proof. (i) It suffices to prove the claim for \( g \) of the form \((s, e)\) or \((e, s)\), where \( s \) is a transposition \((ij) \in S(\infty)\). In this case (i) can be easily checked directly. In fact, let \( 1 \leq i < j \leq n \). If \( i, j \) are both in the same cycle of \( x \in S(n) \), then the multiplication by \((ij)\) from the left or from the right splits this cycle into two; otherwise the two cycles of \( x \) containing the elements \( i \) and \( j \) merge into a single cycle of the product \( x \cdot (ij) \) or \((ij) \cdot x \). On the other hand, if \( x = p_{n,n+1}(\bar{x}) \), where \( \bar{x} \in S(n+1) \), then \( i, j \) belong to one and the same cycle of \( x \) if and only if they belong to one and the same cycle of \( \bar{x} \). Therefore, \( |p_n(x) \cdot (ij)| - |p_n(x)| = [(ij) \cdot p_n(x)] - |p_n(x)| = \pm 1 \), and this number does not change when \( n \) is replaced by \( n + 1 \).

The claim (ii) follows from (i), and (iii) is obvious. □

We call \( c(x, g) \) the fundamental cocycle of the dynamical system \((\mathcal{G}, G)\).

Remark 1.6.2. Let \( \overline{G} \supset G \) and \( \overline{K} \supset K \) be topological groups as defined above in §0.1. One can prove that the action of \( G \) on \( \mathcal{G} \) can be extended to a continuous action \( \mathcal{G} \times \overline{G} \to \mathcal{G} \). In particular, the subgroup \( \overline{K} \subset \overline{G} \) also acts on \( \mathcal{G} \). Moreover, all claims of Proposition 1.6.1 hold when \( G \) and \( K \) are replaced by \( \overline{G} \) and \( \overline{K} \), respectively.

2. Quasiinvariant measures

2.1. The \( G \)–invariant measure on \( \mathcal{G} \). Recall that \( \mathcal{G} = \varprojlim S(n) \) is the space of virtual permutations, and that we have defined an action of the bisymmetric group \( G = S(\infty) \times S(\infty) \) on \( \mathcal{G} \). In what follows, all measures are Borel measures.

Proposition 2.1.1. There exists a unique \( G \)–invariant probability measure \( \mu_1 \) on \( \mathcal{G} \).

Proof. Let \( \mu^n_1 \) denote the normalized Haar measure on \( S(n) \). Clearly, \( \mu^n_1 \) coincides with the image of \( \mu^{n+1}_1 \) under the canonical projection \( p_{n,n+1} : S(n + 1) \to S(n) \). As \( n \to \infty \), the projective limit measure

\[ \mu_1 = \varprojlim \mu^n_1 \]

on \( \mathcal{G} \) is well defined. It is \( G \)–invariant, because the measures \( \mu^k_1 \), \( k \geq n \), are \( G(n) \)–invariant for all \( n = 1, 2, \ldots \).

Conversely, let \( \mu \) be a \( G \)–invariant probability measure on \( \mathcal{G} \). For any \( n \), the push–forward of \( \mu \) under the projection \( p_n : \mathcal{G} \to S(n) \) should coincide with \( \mu^n_1 \), the only \( G(n) \)–invariant probability measure on \( S(n) \). Therefore, \( \mu = \mu_1 \). □

Note that the assumption of \( G \)–invariance in Proposition 2.1.1 can be replaced by a weaker assumption of the invariance under the subgroups \( S(\infty) \times \{ e \} \subset G \) or \( \{ e \} \times S(\infty) \subset G \).

We think of \( \mu_1 \) as of a substitute of Haar measure for the group \( S(\infty) \).
2.2. Ewens measures $\mu_t$. We shall now include the measure $\mu_1$ into a one-parameter family of probability measures $\{\mu_t\}_{t>0}$ on $\mathcal{S}$.

For $t > 0$ and arbitrary $n = 1, 2, \ldots$ we define a measure $\mu^n_t$ on $S(n)$ by the formula

$$\mu^n_t(\{x\}) = \frac{t^{[x]}}{t(t+1) \ldots (t+n-1)}, \quad x \in S(n).$$

Here, as in §1.6, $[x]$ stands for the number of cycles in $x$. The measure $\mu^n_t$ is a probability distribution, as it follows from a well-known identity (see Stanley [Sta, Proposition 1.3.4])

$$\sum_{k=1}^{n} c(n, k) t^k = t(t+1) \ldots (t+n-1).$$

Here $c(n, k) = \#\{x \in S(n): [x] = k\}$ is the absolute value of the Stirling number of the first kind. Another proof of this fact follows from Proposition 2.2.1.

Proof. Let $i(x) = (i_1, \ldots, i_n) \in I_1 \times \ldots \times I_n$ be the code of a permutation $x \in S(n)$, and let $l$ denotes the number of 0’s among the coordinates $i_1, \ldots, i_n$. By the very definition,

$$\mu^n_t(i(x)) = \frac{t^l}{t(t+1) \ldots (t+n-1)}.$$

On the other hand, a coordinate $i_k$ vanishes if and only if the element $k$ creates a new cycle of the permutation $p_k(x)$, hence the number of cycles $[x]$ in $x$ coincides with the number $l$ of zeros in the vector $i(x)$. This concludes the proof. 

Corollary 2.2.2. Given $t > 0$, the canonical projections $p_{n-1,n}$ preserve the measures $\mu^n_t$, hence the measure

$$\mu_t = \varprojlim \mu^n_t$$

on $\mathcal{S}$ is correctly defined. Under the identification $\mathcal{S} \cong I$ of §1.3, the measure $\mu_t$ looks as the product measure

$$\mu_t = \bar{\mu}_t^1 \times \bar{\mu}_t^2 \times \ldots$$

Proof. Indeed, the canonical projection $p_{n-1,n}$ corresponds to deleting the last entry $i_n$ of the code $(i_1, i_2, \ldots, i_n) \in I_1 \times \ldots \times I_n$. This immediately implies the both claims. 

The measures $\mu^n_t$ on the groups $S(n)$ are known as Ewens measures (see §0.6). We will use the same name for the measures $\mu_t$ on the space $\mathcal{S}$, which are built from the measures $\mu^n_t$. 

14
Proposition 2.2.3. For any \( t > 0 \), the Ewens measure \( \mu_t \) is invariant under the action of the group \( K \) on \( \mathfrak{S} \).

Proof. Indeed, it suffices to prove that for any \( n \), the measure \( \mu_t^n \) on \( S(n) \) is invariant with respect to the action of the subgroup \( K(n) \subset K \), isomorphic to \( S(n) \). The action under question is simply the action of \( S(n) \) on itself by conjugations. Since the measure \( \mu_t^n \) on \( S(n) \) has constant weights on conjugacy classes, it is invariant. \( \square \)

Since the measures \( \mu_t \), \( 0 < t < \infty \), live on a compact space, it is natural to ask for their limits as \( t \) goes to \( 0 \) or to \( +\infty \).

Proposition 2.2.4. There exist weak limits

\[
\lim_{t \to 0} \mu_t = \mu_0, \quad \lim_{t \to +\infty} \mu_t = \mu_\infty.
\]

Here \( \mu_0 \) is supported by the subset of virtual permutations with a single cycle while \( \mu_\infty \) is the Dirac measure at the point \( e \in S(\infty) \subset \mathfrak{S} \). Under the identification \( \mathfrak{S} \cong I \), both \( \mu_0 \) and \( \mu_\infty \) become product measures.

Proof. Let us deal with the realization \( \mathfrak{S} \cong I \). Then it suffices to examine the limit behavior of the measure \( \bar{\mu}_t^n \) on \( I_n = \{0, \ldots, n-1\} \), where \( n \) is fixed and \( t \) goes to \( 0 \) or \( \infty \).

When \( t \to 0 \), the limit exists and is the measure \( \bar{\mu}_0^n \) such that

\[
\bar{\mu}_0^n(0) = 0, \quad \bar{\mu}_0^n(1) = \cdots = \bar{\mu}_0^n(n-1) = \frac{1}{n-1}.
\]

This means that the finite product measure \( \bar{\mu}_0^1 \times \cdots \times \bar{\mu}_0^n \), being transferred to \( S(n) \), lives on maximal cycles in \( S(n) \). Therefore, the infinite product measure \( \bar{\mu}_0^1 \times \bar{\mu}_0^2 \times \cdots \) on \( I \), being transferred to \( \mathfrak{S} \), lives on the virtual permutations with a single cycle in the sense of §1.4.

When \( t \to \infty \), the measure \( \bar{\mu}_t^n \) tends to the Dirac measure at \( 0 \in I_n \), which we denote as \( \bar{\mu}_\infty^n \). This means that the measure on \( \mathfrak{S} \) corresponding to the infinite product \( \bar{\mu}_\infty^1 \times \bar{\mu}_\infty^2 \times \cdots \) is simply the Dirac measure at the point \( x \) such that \( i(x) = (0,0,\ldots) \). This point is just \( e \). \( \square \)

2.3. \( K \)-invariant product measures on \( I \).

We can characterize the family of measures \( \mu_t \) as follows.

Proposition 2.3.1. The measures \( \mu_t \), \( 0 \leq t \leq \infty \), are precisely those probability measures on \( \mathfrak{S} \) that are both product measures (with respect to the identification \( \mathfrak{S} \cong I \) of subsection 1.3) and invariant under \( K \).

Proof. Let \( \mu \) be a \( K \)-invariant product measure on \( \mathfrak{S} \). Clearly, \( \mu = \lim \mu^n \) with \( \mu^n = p_n(\mu) \). We have to show that \( \mu \) coincides with one of the measures \( \mu_t \), \( 0 \leq t \leq \infty \).

Consider the measure \( \mu_t^2 \) on \( S(2) \). It coincides with some \( \mu_0^2 \), \( 0 \leq t \leq \infty \), and the parameter \( t \) is determined uniquely. In fact, \( \mu_0^2 \) is the Dirac measure at \( e \in S(2) \), \( \mu_0^2 \) is the Dirac measure at the involution \( (1,2) \in S(2) \), and the measure \( \mu_t^2 \) with \( 0 < t < \infty \) has weights \( \frac{1}{1+t} \) and \( \frac{1}{1-t} \) at the elements \( (1,2) \) and \( e = (1)(2) \) respectively. We shall prove by induction in \( n \geq 2 \) that \( \mu^n = \mu_t^n \) for all \( n \).
We start by considering the induction step \( n \to n + 1 \) in the degenerated cases of \( t = 0 \) and \( t = \infty \). Here we need not the assumption that \( \mu \) is a product measure. In the first case we assume that \( \mu^n \) is the uniform distribution on \( n \)-cycles in \( S(n) \). The preimage of \( n \)-cycles in \( S(n + 1) \) under the canonical projection is the set \( A \cup B \subset S(n + 1) \) where \( A \) consists of all \( n \)-cycles in \( S(n) \subset S(n + 1) \), and \( B \) is the set of all \((n + 1)\)-cycles. Each of the two sets is a diag \( S(n) \)-orbit, and \( B \) (but not \( A \)) is also a diag \( S(n + 1) \)-orbit. Since \( \mu^{n+1} \) is diag \( S(n + 1) \)-invariant, it is supported by \( B \) alone and uniform on \( B \).

The case \( t = \infty \) is similar. We assume that \( \mu^n \) is supported by the point \( \{e\} \). Then the measure \( \mu^{n+1} \) is supported by \( A \cup B \subset S(n + 1) \) where \( A \) consists of transpositions \((n + 1, j), 1 \leq j \leq n \), and \( B = \{e\} \). Since \( A \) is not \( S(n + 1) \)-invariant, \( \mu^{n+1} \) is supported by \( B \).

Let us now assume that \( \mu^n = \mu^t_t \) where \( t \neq 0, \infty \). We have to show that \( \mu^{n+1} = \mu^{t+1}_t \). We shall write an arbitrary permutation \( x \in S(n + 1) \) as a pair \( \{y, j\} \in S(n) \times I_{n+1} \) where \( y = p_{n,n+1}(x) \) and

\[
\begin{align*}
    j = \begin{cases} 
        0 & \text{if } x(n + 1) = n + 1; \\
        x(n + 1) & \text{if } x(n + 1) \neq n + 1.
    \end{cases}
\end{align*}
\]

Since \( \mu^{n+1} \) is a product measure,

\[
\mu^{n+1}(\{x\}) = \frac{t^{[y]} \nu(j)}{t(t + 1) \ldots (t + n - 1)}, \tag{2-3-1}
\]

where \( \nu(0), \ldots, \nu(n) \) are the weights of a probability measure on \( I_{n+1} \).

Applying a conjugation by appropriate permutation in \( S(n) \subset S(n + 1) \), one can replace any \( j \neq 0 \) with any other \( j' \neq 0 \) leaving the cycle structure of \( y \) intact. It follows that \( \nu(1) = \ldots = \nu(n) \), and we only have to find out the relation of \( \nu(0) \) and \( \nu(1) = \cdots = \nu(n) \). To this end, choose a permutation \( y \in S(n) \) such that the cycle containing 1 is of length \( \geq 2 \), i.e., \( y(1) = j \neq 1 \). Then

\[
(1, n + 1) \cdot \{y, 0\} \cdot (1, n + 1) = \{y', j\}.
\]

Here \( y' \) is obtained from \( y \) by removing \( j \) from its cycle and forming an additional trivial cycle \( (j) \). An important point is that the initial cycle containing \( j \) does not disappear completely. It follows that \([y'] = [y] + 1\). Since, by the invariance assumption, \( \mu^{n+1}([y, 0]) = \mu^{n+1}([y', j]) \), we obtain from (2-3-1)

\[
\nu(0) = t \nu(j) = t \nu(1).
\]

Since \( \nu(0) + n \nu(1) = 1 \), we obtain

\[
\nu(0) = \frac{t}{t + n}, \quad \nu(1) = \frac{1}{t + n},
\]

and the desired identity \( \mu^{n+1} = \mu^{t+1}_t \) follows. \( \Box \)

2.4. Disjointness of the measures \( \mu_t \).
Proposition 2.4.1. The measures $\mu_t$, $0 \leq t \leq \infty$, are disjoint (mutually singular).

Proof. We can replace the measures $\mu_t$ on $\mathcal{S}$ with the corresponding product measures

$$
\bar{\mu}_t = \prod_{n=1}^{\infty} \bar{\mu}_t^n
$$
on $I = I_1 \times I_2 \times \ldots$. Note that $\bar{\mu}_\infty$ is the Dirac measure at the point $i = (0, 0, \ldots) \in I$, and $\bar{\mu}_0$ is supported by the sequences $i = (i_1, i_2, \ldots)$ with nonzero coordinates: $i_n \neq 0$ for $n \geq 2$. Obviously, the measures $\bar{\mu}_0$, $\bar{\mu}_\infty$ and $\bar{\mu}_t$ are pairwise disjoint, for every $t \in (0, \infty)$.

Assume now that $s, t \in (0, \infty)$ and $s \neq t$. We shall show that $\mu_s$ and $\mu_t$ are disjoint, $\mu_s \perp \mu_t$, by applying the well–known Kakutani criterion [Kak]. To this end we check that the infinite product

$$
\prod_{n=1}^{\infty} a_n
$$
diverges. Set $u = \sqrt{st}$; then

$$
a_n = \sum_{i=0}^{n-1} \sqrt{\mu_s^n(i) \mu_t^n(i)},
$$
denotes. Set $u = \sqrt{st}$; then

$$
a_n = \frac{u + n - 1}{\sqrt{(s + n - 1)(t + n - 1)}} = \frac{1 + \frac{u}{n-1}}{\sqrt{(1 + \frac{s}{n-1})(1 + \frac{t}{n-1})}}
= 1 + \frac{u - \frac{1}{2}(s+t)}{n-1} + O\left(\frac{1}{(n-1)^2}\right).
$$

Now note that

$$
u - \frac{1}{2}(s+t) = \sqrt{st} - \frac{1}{2}(s+t) \neq 0,$$
since $s \neq t$. Therefore, the product $\prod_{n=1}^{\infty} a_n$ is indeed divergent. □

2.5. Quasiinvariance.

Proposition 2.5.1. Each of the measures $\mu_t$, $0 < t < \infty$ is quasiinvariant with respect to the action of $G$ on the space $\mathcal{S}$. More precisely,

$$
\frac{\mu_t(dx \cdot g)}{\mu_t(dx)} = t^{c(x,g)}; \quad x \in \mathcal{S}, \; g \in G \quad (2.5-1)
$$

where $c(x,g)$ is the fundamental additive cocycle of subsection 1.6. The measures $\mu_0$ and $\mu_\infty$ are not quasiinvariant.

Proof. It suffices to check that

$$
\mu_t(V \cdot g) = \int_V t^{c(x,g)} \mu_t(dx), \quad g \in G \quad (2.5-2)
$$
for every Borel subset $V \subseteq \mathcal{S}$. This would imply (2.5-1), hence the quasiinvariance of $\mu_t$. 

17
Fix $g \in G$ and choose $m$ so large that $g \in G(m)$. For arbitrary $n \geq m$ and $y \in S(n)$, let $V_n(y) \subset \mathfrak{G}$ denote the preimage of the point $y \in S(n)$ under the canonical projection $p_n: \mathfrak{G} \to S(n)$; this is a cylinder set. It suffices to check (2-5-2) for $V = V_n(y)$.

Note that $V_n(y) \cdot g = V_n(y \cdot g)$ and $\mu_t(V_n(y)) = \mu^n_t(\{y\})$, hence

$$\mu_t(V_n(y) \cdot g) = \mu^n_t(\{y \cdot g\}).$$

On the other hand,

$$c(x, g) = [p_n(x \cdot g)] - [p_n(x)] = [y \cdot g] - [y]; \quad x \in V(y),$$

so that

$$t^{c(x,y)} = t^{[y \cdot g] - [y]}, \quad x \in V_n(y).$$

The equation (2-5-2) takes the form

$$\mu^n_t(\{y \cdot g\}) = t^{[y \cdot g] - [y]} \mu^n_t(\{y\}),$$

which is immediate from the definition of the measure $\mu^n_t$. \hfill \square

Using a well-known trick from ergodic theory, one can replace quasiinvariant measures $\mu_t$ by invariant, though infinite, measures. In order to do this, consider the space $\tilde{\mathfrak{G}} = \mathfrak{G} \times \mathbb{Z}$ and define an action of $G$ on $\tilde{\mathfrak{G}}$ as

$$(x, k) \cdot g = (x \cdot g, k + c(x, g)), \quad x \in \mathfrak{G}, \quad k \in \mathbb{Z}, \quad g \in G. \quad (2-5-3)$$

Define the infinite measure $\nu_t$ on $\mathbb{Z}$ as

$$\nu_t(\{k\}) = t^{-k}, \quad k \in \mathbb{Z},$$

and introduce the infinite measure $\tilde{\mu}_t = \mu_t \times \nu_t$ on $\tilde{\mathfrak{G}}$.

**Proposition 2.5.2.** For every $0 < t < \infty$, the measure $\tilde{\mu}_t$ on the space $\tilde{\mathfrak{G}}$ is invariant with respect to the action (2-5-3) of the group $G$.

**Proof.** This is immediate from (2-5-1) and the definition of the measure $\nu_t$. \hfill \square

This construction will be used below, see §3.1.

**Remark 2.5.3.** One can prove that for every $t > 0$, the action of each of the subgroups $S(\infty) \times \{e\}$ and $\{e\} \times S(\infty)$ on the space $\mathfrak{G}$ with the measure $\mu_t$ is ergodic and topologically minimal (the latter means that every orbit is dense in $\mathfrak{G}$). These claims will not be used in the sequel.

### 3. Generalized regular representations

In this Section we introduce a family $\{T_z\}$ of unitary representations of the group $G = S(\infty) \times S(\infty)$ parameterized by points $z \in \mathbb{C} \cup \{\infty\}$ of the Riemann sphere. First we assume $z \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. 

18
3.1. The representations $T_z$. We shall always assume below that the parameters $t > 0$ and $z \in \mathbb{C}^*$ are related as $t = z \bar{z}$. By virtue of Proposition 2.5.1,

$$
\frac{\mu_t(dx \cdot g)}{\mu_t(dx)} = t^{c(x,g)} = |z^{c(x,g)}|^2, \quad x \in \mathcal{G}, \quad g \in G.
$$

Recall that $c(x,g)$ is an additive cocycle, so that $z^{c(x,g)}$ is a multiplicative one. Therefore, the following formula allows one to define a unitary representation $T_z$ of the group $G$ in the Hilbert space $H = L^2(\mathcal{G}, \mu_t)$,

$$(T_z(g)f)(x) = f(x \cdot g) \cdot z^{c(x,g)}, \quad g \in G, \quad x \in \mathcal{G}, \quad f \in H.$$ 

Note two cases when the multiplier $z^{c(x,g)}$ is equal to 1:

1) if $z = 1$ (then the measure $\mu_t = \mu_1$ is invariant);
2) if $z$ is arbitrary but $g \in K \subset G$ (since $c(x, \cdot) \equiv 0$ on the subgroup $K$).

By the reasons to be made clear later on we call representations $T_z$ the generalized regular representations of the group $G$.

**Remark 3.1.1.** The above construction is nothing but a specialization of a well-known general construction. Indeed, to any triple $(X, \mathcal{G}, \mu)$, where $\mathcal{G}$ is a group acting on the right on a space $X$ with a quasiinvariant measure $\mu$, one associates a one–parameter family of unitary representations acting in $L^2(X, \mu)$ according to the formula

$$(T(g)f)(x) = f(x \cdot g) \left(\frac{\mu(dx \cdot g)}{\mu(dx)}\right)^{\frac{1+s}{2}}, \quad g \in \mathcal{G}, \quad x \in X, \quad f \in L^2(X, \mu),$$

where $s \in \mathbb{R}$ is a parameter. When $(X, \mathcal{G}, \mu) = (\mathcal{G}, G, \mu_t)$, we obtain

$$
\left(\frac{\mu(dx \cdot g)}{\mu(dx)}\right)^{\frac{1+s}{2}} = \left(t^{c(x,g)}\right)^{\frac{1+s}{2}} = z^{c(x,g)},
$$

where $z = t^{\frac{1+s}{2}}$. Hence, we return to the definition of $T_z$.

One can introduce representations $T_z$ in a slightly different way, using the action

$$
g : (x, k) \mapsto (x, k) \cdot g = (x, g, k + c(x, g))
$$

of the group $G$ on the space $\mathcal{G} = \mathcal{G} \times \mathbb{Z}$ with the infinite invariant measure $\tilde{\mu}_t = \mu_t \times \nu_t$, see Proposition 2.5.2. There is a natural unitary representation $\tilde{T}_t$ of the group $G$ in the Hilbert space $L^2(\mathcal{G}, \tilde{\mu}_t)$,

$$(\tilde{T}_t(g)f)(x, k) = f((x, k) \cdot g), \quad g \in G.$$

Let $\mathbb{T}$ denote the unit circle $\{\xi \in \mathbb{C} : |\xi| = 1\}$. 

Proposition 3.1.2. For any $t > 0$, the representation $\tilde{T}_t$ of the group $G$ is unitary equivalent to the direct integral

$$\int_{\xi \in \mathbb{T}} T_{\xi \sqrt{t}} \, d\theta, \quad \xi = e^{2\pi i \theta},$$

of the generalized regular representations $T_z$ with $|z|^2 = t$.

Proof. Define a unitary representation of the group $\mathbb{Z}$ acting in the space of $\tilde{T}_t$ and commuting with the latter representation:

$$(\tilde{T}(l) f)(x, k) = f(x, k + l) t^{-1/2}, \quad l \in \mathbb{Z}.$$  

We claim that the decomposition in question is determined by this commuting representation of $\mathbb{Z}$. To see this, we shall pass to a slightly different realization of $\tilde{T}_t$.

Consider the Hilbert space $L^2(\mathbb{S} \times \mathbb{T}, \mu_t \times d\theta)$, where $d\theta$ is the normalized Lebesgue measure on the circle $\mathbb{T}$ (we again write $\xi = e^{2\pi i \theta}$). In this space, we introduce two commuting unitary representations of the groups $G$ and $\mathbb{Z}$, as follows

$$(\hat{T}_t(g) \hat{f})(x, \xi) = \hat{f}(x \cdot g, \xi) (\xi \sqrt{t})^{c(x, g)}, \quad g \in G,$$

$$(\hat{T}(l) \hat{f})(x, \xi) = \hat{f}(x, \xi) \xi^l, \quad l \in \mathbb{Z},$$

where $\hat{f}$ ranges over $L^2(\mathbb{S} \times \mathbb{T}, \mu_t \times d\theta)$.

Clearly, the representation $\hat{T}_t$ admits the required decomposition, which is determined by the action of $\mathbb{Z}$. Hence, it suffices to check that the representation $\hat{T}_t \times \hat{T}$ in the space $L^2(\mathbb{S} \times \mathbb{T}, \mu_t \times d\theta)$ is unitary equivalent to the representation $\hat{T}_t \times \hat{T}$ of the group $G \times \mathbb{Z}$ in the space $L^2(\mathbb{S} \times \mathbb{T}, \mu_t \times d\theta)$.

The desired unitary equivalence is provided by the transform $F : f \mapsto \hat{f}$,

$$\hat{f}(x, \xi) = \sum_{k=-\infty}^{\infty} f(x, k) t^{-k/2} \xi^{-k},$$

which is, in essence, the Fourier transform with respect to the second argument.

Clearly, $F$ is an isometry. Let us check that it intertwines both representations of $G$:

$$F(\hat{T}_t(g)f)(x, \xi) = \sum_{k=-\infty}^{\infty} (\hat{T}_t(g) f)(x, k) t^{-k/2} \xi^{-k} =$$

$$= \sum_{k=-\infty}^{\infty} f(x \cdot g, k + c(x, g)) (\xi \sqrt{t})^{-k} =$$

$$= \sum_{j=-\infty}^{\infty} f(x \cdot g, j) (\xi \sqrt{t})^{-j+c(x, g)} =$$

$$= \hat{T}_t(F(f))(x, \xi).$$

The intertwining property for the action of the group $\mathbb{Z}$ can be checked in a similar way.

$\square$
3.2. Admissibility. The definition of admissible representations is given in §9.9.

Proposition 3.2.1. All the representations $T_z$, $z \in \mathbb{C}^*$, are admissible representations of the pair $(G, K)$.

Proof. Given $n = 1, 2, \ldots$, consider the canonical projection $p_n : \mathcal{S} \to S(n)$. A function $F \circ p_n$, where $F$ is any function on $S(n)$, will be called a cylinder function of level $n$ on the space $\mathcal{S}$. We denote the space of such functions by $\text{Cyl}^n(\mathcal{S})$, and we call

$$\text{Cyl}(\mathcal{S}) = \bigcup_{n \geq 1} \text{Cyl}^n(\mathcal{S})$$

the space of cylinder functions on $\mathcal{S}$. Clearly,

$$\text{Cyl}(\mathcal{S}) \subset \mathcal{H} = L^2(\mathcal{S}, \mu)$$

for every $t > 0$.

Note that the canonical projection $S(n) \to S(m)$ is invariant with respect to conjugations with the elements of the subgroup $S_m(n) \subset S(n)$, for all $m < n$. It follows that

$$p_m(x \cdot u) = p_m(x), \quad x \in \mathcal{S}, \quad u \in K_m.$$  

Since the factor $z^{t(x, u)}$ is trivial on the group $K \subset G$, we derive that

$$\text{Cyl}^m(\mathcal{S}) \subseteq \mathcal{H}_m.$$  

The space $\text{Cyl}(\mathcal{S})$ is clearly dense in $\mathcal{H} = L^2(\mathcal{S}, \mu)$, hence $\mathcal{H}_\infty$ is also dense in $\mathcal{H}$, and the representation $T_z$ is admissible.  \hfill \square

Remark 3.2.2. The space $\text{Cyl}^n(\mathcal{S})$ is strictly smaller than $\mathcal{H}_m$. Indeed, the former space is finite–dimensional for any $m$, while the latter space (as we shall see later on) has infinite dimension even for $m = 0$.

Remark 3.2.3. According to a general result (see §9.9), Proposition 3.2.1 implies that the representations $T_z$ can be continued to the topological group $\mathfrak{G}$. This can be verified directly by making use of Remark 1.6.2.

3.3. Approximations by regular representations. For every $n = 1, 2, \ldots$, we denote by $H^n$ the finite dimensional space $L^2(S(n))$ defined by the normalized Haar measure $\mu^n$ on $S(n)$, and by $\text{Reg}^n$ the two–sided regular representation of the group $G(n) = S(n) \times S(n)$ in this space:

$$(\text{Reg}^n(g) f)(x) = f(g_2^{-1} x g_1)$$

where

$$g = (g_1, g_2) \in G(n), \quad x \in S(n), \quad f \in H^n.$$  

We shall show that every generalized regular representation $T_z$ can be obtained as an inductive limit of the representations $\text{Reg}^n$ determined by an appropriate family of isometric embeddings $L^2_n : H^n \to H^{n+1}$, $n = 1, 2, \ldots$, depending on $z \in \mathbb{C}^*$.

We define the operators $L^z_n : H^n \to H^{n+1}$ as follows: if $f \in H^n$ and $x \in S(n+1)$,

$$(L^z_n f)(x) = \begin{cases} 
  z \sqrt{\frac{n+1}{t+1}} f(x) & \text{if } x \in S(n) \subset S(n+1); \\
  \sqrt{\frac{n+1}{t+1}} f(p_n(x)) & \text{if } x \in S(n+1) \setminus S(n).
\end{cases} \quad (3.3-1)$$

Here and below we assume that $t = z \bar{z}$. 

21
Proposition 3.3.1. For any $z \in \mathbb{C}^*$ the operator $L^n_z$ provides an isometric embedding $H^n \rightarrow H^{n+1}$ which intertwines the $G(n)$–representations $\text{Reg}^n$ and $\text{Reg}^{n+1}$ $\upharpoonright_{G(n)}$.

Let $T'_z$ denote the inductive limit of the representations $\text{Reg}^n$ with respect to the embeddings

$$H^1 \xrightarrow{L^1_z} H^2 \xrightarrow{L^2_z} H^3 \xrightarrow{L^3_z} \ldots .$$

Then the representations $T'_z$ and $T_z$ are equivalent.

Proof. Note that for every $n = 1, 2, \ldots$ the subspace $\text{Cyl}^n \subset \mathcal{H} = L^2(\mathcal{G}, \mu_t)$ of cylinder functions of level $n$ is invariant with respect to the operators $T_z(g)$ where $g \in G(n)$. This follows from the definition of the representation $T_z$, and the fact that for all $g \in G(n)$ the function $x \mapsto c(x, g)$ is a cylinder function of level $n$:

$$c(x, g) = [p_n(x) \cdot g] - [p_n(x)], \quad x \in \mathcal{G}, \quad g \in G(n).$$

Since the image of the measure $\mu_t$ with respect to the canonical projection $p_n: \mathcal{G} \rightarrow S(n)$ coincides with $\mu^n_t$, we can identify the Hilbert spaces $\text{Cyl}^n \subset \mathcal{H}$ and $L^2(S(n), \mu^n_t)$. The operators $T_z(g) \mid_{\text{Cyl}^n}$, where $g \in G(n)$, take the form

$$(T_z(g) f)(x) = f(x \cdot g) z^{[x \cdot g] - [x]} \quad (3-3-2)$$

(here $x \in S(n)$, $f \in L^2(S(n), \mu^n_t)$.)

Define a function $F^n_z$ on the group $S(n)$ by the formula

$$F^n_z(x) = \left( \frac{n!}{t(t+1) \ldots (t+n-1)} \right)^{1/2} z^x, \quad x \in S(n). \quad (3-3-3)$$

Then (3-3-2) can be written in the form

$$(T_z(g) f)(x) = f(x \cdot g) \frac{F^n_z(x \cdot g)}{F^n_z(x)} .$$

Note that the function $|F^n_z(x)|^2$ coincides with the density of the measure $\mu^n_t$ with respect to the Haar measure $\mu_t^n$. It follows that the operator of multiplication by the function $F^n_z$ defines an isometry

$$\text{Cyl}^n \rightarrow L^2(S(n), \mu_t^n)$$

intertwining the representations $T_z \mid_{\text{Cyl}^n}$ and $\text{Reg}^n$ of the group $G(n)$.

Consider now the commutative diagram

$$L^2(S(n), \mu^n_t) = \mathbb{C}^n \xrightarrow{\mathbb{C}^{n+1}} \mathbb{C}^{n+1} = L^2(S(n+1), \mu_t^{n+1})$$

$$\downarrow \quad \downarrow$$

$$H^n \xrightarrow{T^n_z} H^{n+1}$$

where the top arrow denotes the natural embedding (lifting of functions via the projection $p_{n,n+1}$), the vertical arrows correspond to multiplication by $F^n_z$ and
of the term “generalized regular representation”.

As standard two–sided regular representation of \( C \cup \{\infty\} \),

we conclude that \( L_z \) is a family of representations \( T_z \).

Proposition 3.4.1. 

(i) For every \( n = 1, 2, \ldots \), the isometry \( L_z^n : H^n \to H^{n+1} \) admits a continuous continuation, with respect to the parameter \( z \in \mathbb{C}^* \), to the points \( z = 0 \) and \( z = \infty \) of the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). Therefore, the definition of the inductive limit representation \( T_z^\prime \) also makes sense for the values \( z = 0 \) and \( z = \infty \).

(ii) The representation \( T_\infty \) is equivalent to the natural two–sided regular representation of the group \( G = S(\infty) \times S(\infty) \) on the Hilbert space \( l^2(S(\infty)) \).

Proof. (i) Since \( t = z \bar{z} \), (3-3-1) implies that the limits

\[
L_0^n = \lim_{z \to 0} L_z^n \quad \text{and} \quad L_\infty^n = \lim_{|z| \to \infty} L_z^n
\]
do exist, and have the form

\[
(L_0^n f)(x) = \begin{cases} 
0, & \text{if } x \in S(n) \subset S(n+1), \\
\sqrt{\frac{n+1}{n}} f(p_n(x)), & \text{if } x \in S(n+1) \setminus S(n), 
\end{cases}
\]

\[
(L_\infty^n f)(x) = \begin{cases} 
\sqrt{n+1} f(p_n(x)), & \text{if } x \in S(n) \subset S(n+1), \\
0, & \text{if } x \in S(n+1) \setminus S(n). 
\end{cases}
\]

By continuity, \( L_0^n \) and \( L_\infty^n \) determine isometric embeddings \( H^n \to H^{n+1} \) commuting with the action of the group \( G(n) \). Thus, we can use them to construct inductive limits \( T_0^\prime \), \( T_\infty^\prime \) of the two–sided regular representations of the groups \( S(n) \times S(n) \).

(ii) For every \( n = 1, 2, \ldots \), consider the map \( l^2(S(n)) \to H^n \) defined as multiplication by the scalar \( \sqrt{n!} \). This is an isometry, since the counting measure on \( S(n) \) equals \( n! \mu_0^n \). Under identification of both spaces by this map, the embedding \( L_\infty^n \) turns into the natural embedding \( l^2(S(n)) \to l^2(S(n+1)) \). This completes the proof. □

Using the identification \( T_0 = T_\infty \) we now may extend the definition of representations \( T_z \) to the values \( z = 0 \) and \( z = \infty \) of the parameter \( z \). In this way we obtain a family of representations \( T_z \) parametrized by the points of the Riemann sphere \( \mathbb{C} \cup \{\infty\} \). We have shown that this family forms a continuous deformation of the standard two–sided regular representation of \( G \) in \( l^2(S(\infty)) \). This is a justification of the term “generalized regular representation”.
3.5. A construction of $T_z$ via representations $T_0$ and $T_∞$. Let us discuss now the formula for the isometric embeddings $L^n_z : H^n \to H^{n+1}$. We derived this formula from the initial definition of the representations $T_z$ with $z \in \mathbb{C}^*$. Then, taking a limit transition in the formula, we completed the construction of the representations at the points $z = 0$ and $z = \infty$.

Here we aim to show that these two steps can be realized in opposite order. We start with the definition (3-4-1) of the embeddings $L^n_0$ and $L^n_∞$, and then pass to the general operators $L^n_z$. First, we have to check that (3-4-1) indeed defines isometric embeddings, equivariant with respect to $G(n)$. For $L^n_∞$, this immediately follows from the basic property of the canonical projection. As for $L^n_0$, we observe that up to a scalar multiple, $L^n_0$ can be defined as lifting along the fibers of the canonical projection $p_{n,n+1} : S(n+1) \to S(n)$, omitting the natural section $S(n) \to S(n+1)$. This implies equivariance. To prove the isometry property, we use the fact that the fiber over any point of $S(n)$ consists just of $n$ points, except a single point belonging to the section.

Next, we note that the spaces $L^n_0(H^n)$ and $L^n_∞(H^n)$ are mutually orthogonal subspaces of $H^{n+1}$: the functions in the second space are supported by the subgroup $S(n) \subset S(n+1)$, and those in the first space are supported by $S(n+1) \setminus S(n)$.

Comparing (3-3-1) and (3-4-1) we see that for $z \in \mathbb{C}^*$, $L^n_z$ is a linear combination of $L^n_0$ and $L^n_∞$:

$$L^n_z = \frac{\sqrt{n}}{\sqrt{t+n}} L^n_0 + \frac{z}{\sqrt{t+n}} L^n_∞. \quad (3-5-1)$$

Moreover, the coefficients of $L^n_0$ and $L^n_∞$ satisfy the relation

$$\left| \frac{\sqrt{n}}{\sqrt{t+n}} \right|^2 + \left| \frac{z}{\sqrt{t+n}} \right|^2 = \frac{n + z\bar{z}}{t + n} = 1.$$  

It follows at once that the operator $L^n_z$ defined by the formula (3-5-1) is a $G(n)$–equivariant isometry.

Formula (3-5-1) looks very simple and natural. This is an argument in favor of “naturalness” of the representations $T_z$.

3.6. Connection between $T_z$ and $T_{-z}$. For any $s \in S(∞)$, the number $\text{inv}(s)$ of inversions in $s$ is finite. Let

$$\text{sgn}(s) = (-1)^{\text{inv}(s)} = \pm 1.$$  

Then $\text{sgn} : S(∞) \to \{±1\}$ is a (unique) nontrivial one–dimensional representation of the group $S(∞)$.

Proposition 3.6.1. For every $z \in \mathbb{C} \cup \{∞\}$, $T_{-z}$ is equivalent to $T_z \times (\text{sgn} \times \text{sgn})$.

Proof. Given $x \in \mathcal{G}$ and $g = (g_1, g_2) \in G$, let $n$ be so large that $g \in G(n)$. Then

$$c(x, g) = [g_2^{-1} p_n(x) g_1] - [p_n(x)].$$

It follows that $c(x, g) \in 2\mathbb{Z}$ if the permutation $g_1 g_2^{-1}$ is even, and $c(x, g) \in 2\mathbb{Z} + 1$ if $g_1 g_2^{-1}$ is odd. Using the initial definition of $T_z$ (for $z \neq 0, ∞$) we derive that

$$T_{-z}(g) = \text{sgn}(g_1 g_2^{-1}) T_z(g).$$

24
When \( z = 0, \infty \), Proposition 3.6.1 claims the equivalence
\[
T_0 \cong T_0 \otimes (\text{sgn} \times \text{sgn}), \quad T_\infty \cong T_\infty \otimes (\text{sgn} \times \text{sgn}).
\]
Such an equivalence is indeed provided by the operator of multiplication by the function \( \text{sgn}(\cdot) \) (we use the realization of the representations as inductive limits, see subsection 3.4). \( \square \)

4. The distinguished spherical function

4.1. The distinguished vector and the coherent system \( M_z \) (case \( z \neq 0, \infty \)). Assume \( z \in \mathbb{C} \setminus \{0\} \) and consider the generalized regular representation \( T_z \).

According to the initial construction of \( T_z \) in Hilbert space \( \mathcal{H} = L^2(\mathcal{S}, \mu_t) \) (see §3.1), \( T_z \) comes with a distinguished vector \( \xi_0 \): this vector is simply the function \( f_0 \equiv 1 \) on the space \( \mathcal{S} \). Clearly, \( \xi_0 \) is \( K \)-invariant and has norm 1.

In the inductive limit realization of \( T_z \) as described in §3.3, the same vector \( \xi_0 \) is represented by the functions \( F_n^z \in L^2(S(n), \mu_1^n) \) defined in (3-3-3).

Note that the whole space of \( K \)-invariant vectors in \( \mathcal{H} \) is infinite-dimensional. However, explicitly constructing invariant vectors other than the distinguished one is a nontrivial task.

Let \( \varphi_z \) denote the spherical function on the group \( G \) corresponding to the distinguished vector \( \xi_0 \), let \( \chi_z \) be the related character of the group \( S(\infty) \), and let \( M_z \) be the corresponding coherent system. We will derive a nice expression for \( M_z \).

Recall a standard notation related to Young diagrams. For a particular box \( b \in \lambda \) with coordinates \((i, j)\), the number \( c(b) = j - i \) and

\( c(b) \) is called the content of \( b \).

**Theorem 4.1.1.** Let \( z \in \mathbb{C}^* \) and \( t = z \bar{z} \). Consider the coherent system \( M_z = \{M_z^{(n)}\} \) as defined above. For any Young diagram \( \lambda \vdash n \),
\[
M_z(\lambda) = \frac{\prod_{b \in \lambda} |z + c(b)|^2}{t(t+1) \ldots (t+n-1) \dim^2 \lambda \ n!} \tag{4-1-1}
\]

where we abbreviate
\[
M_z(\lambda) = M_z^{(n)}(\lambda), \quad \lambda \in \mathbb{Y}_n.
\]

**Proof.** It will be convenient to identify \( T_z \) with the inductive limit \( T'_z \) of regular representations \( \text{Reg}^n \). Recall that in this realization the representation space is defined as the Hilbert completion \( H \) of the inductive limit of finite dimensional Hilbert spaces \( H^n = L^2(S(n), \mu_1^n) \). The distinguished vector \( f_0 \) belongs to \( H^1 \), hence to all of \( H^n \). As an element of \( H^n \) it coincides with the function \( F_z^n \) introduced in §3. Therefore, for \( s \in S(n) \)
\[
\chi_z|_{S(n)}(s) = \langle \text{Reg}^n(s, e) F_z^n, F_z^n \rangle = \frac{1}{n!} \sum_{s_1 \in S(n)} F_z^n(s_1) F_z^n(s_1) \overline{F_z^n(s_1)}.
\]

25
This can be rewritten as

\[ \chi_z|_{S(n)} = (F^n_z)^* \ast F^n_z, \]

where \( f^*(s) = \overline{f(s)} \) denotes the standard involution on the group algebra \( \mathbb{C}[S(n)] \), and \( \ast \) is the convolution product taken with respect to normalized Haar measure \( \mu^n_1 \).

Note that \( F^n_z \) is a central function on \( S(n) \), hence it can be decomposed as a sum of characters \( \chi^\lambda \),

\[ F^n_z = \sum_{\lambda \vdash n} a(\lambda) \chi^\lambda, \quad (4-1-2) \]

where \( a(\lambda) \) are appropriate complex coefficients. By virtue of the orthogonality relations,

\[ (\chi^\lambda)^* \ast \chi^\mu = \delta_{\lambda\mu} \frac{\chi^\lambda}{\dim \chi^\lambda}, \lambda, \mu \vdash n, \]

hence

\[ M_z(\lambda) = |a(\lambda)|^2. \quad (4-1-3) \]

Recall that

\[ F^n_z(x) = \left( \frac{n!}{t(t+1)\ldots(t+n-1)} \right)^{1/2} z^{[x]}, \quad (4-1-4) \]

where \([x]\) denotes the number of cycles of a permutation \( x \in S(n) \), see §3.3. We are interested in the decomposition of the central function \( z^{[x]} \).

**Lemma 4.1.2.** Given \( n = 1, 2, \ldots \) and \( z \in \mathbb{C}^* \), the decomposition of the central function \( x \mapsto z^{[x]} \) on the group \( S(n) \) along the characters \( \chi^\lambda, \lambda \vdash n \), can be written in the form

\[ z^{[x]} = \sum_{\lambda \vdash n} \prod_{b \in \lambda} (z + c(b)) \cdot \frac{\dim \lambda}{n!} \chi^\lambda(x) = \]

\[ = \sum_{\lambda \vdash n} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \chi^\lambda(x). \quad (4-1-5) \]

Keeping together (4-1-2), (4-1-3), (4-1-4), (4-1-5) we get the desired formula (4-1-1). Thus, it remains to prove the lemma.

**Proof of the lemma.** We switch from central functions on the group \( S(n) \) to symmetric functions. This is done using the classical characteristic map “ch” establishing a bijection between central functions on \( S(n) \) and homogeneous symmetric functions of degree \( n \), see [Mac, 1.7]. It is well known that \( \text{ch}(\chi^\lambda) = s_\lambda \), hence we have to prove the formula

\[ \text{ch}(z^{[x]}) = \sum_{\lambda \vdash n} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_\lambda, \quad (4-1-6) \]

where \( s_\lambda \) are the Schur functions. Let us recall the definition of \( \text{ch} \). If \( F \) is a central function on \( S(n) \), \( \rho = (1^{k_1} 2^{k_2} \ldots) \) is a partition of \( n \), and \( x_\rho \in S(n) \) is a permutation of cycle type \( \rho \), then

\[ \text{ch} F = \sum_{\rho \vdash n} z^{-1}_\rho F(x_\rho) p_\rho. \]
Here \( z_\rho \) is the order of the centralizer of \( x_\rho \),
\[
z_\rho = \frac{n!}{1^{k_1} 2^{k_2} \ldots k_1! k_2! \ldots}
\]
and \( p_\rho = p_1^{k_1} p_2^{k_2} \ldots \) are the monomials in the power sums \( p_1, p_2 \ldots \). Note that \([x_\rho] = k_1 + k_2 + \ldots\) and \( k_1 + 2k_2 + 3k_3 + \ldots = n\).
Denote by \( y_1, y_2, \ldots \) a sequence of formal variables of symmetric functions, and let \( u \) be still another formal variable. One can write
\[
1 + \sum_{n \geq 1} \text{ch}(z^{[1]}) u^n = \sum_{(k_1, k_2, \ldots)} \frac{z^{k_1+k_2+\ldots} u^{k_1+2k_2+\ldots}}{1^{k_1} 2^{k_2} \ldots k_1! k_2! \ldots p_1^{k_1} p_2^{k_2} \ldots} = \prod_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{z^k p_k u^n k^n}{n^k k!} = \exp z \left( \frac{u p_1}{1} + \frac{u^2 p_2}{2} + \ldots \right) = \exp z \left( \frac{u y_1}{1} + \frac{(u y_1)^2}{2} + \ldots \right) = \exp \left( -z \sum_{i=1}^{\infty} \ln(1 - uy_i) \right) = \prod_{i=1}^{\infty} (1 - uy_i)^{-z}.
\]
The formula (4-1-6) takes the form
\[
\prod_{i=1}^{\infty} (1 - uy_i)^{-z} = \sum_{\lambda} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_\lambda(uy_1, uy_2, \ldots),
\]
where \( \lambda \) in the right hand side runs over all Young diagrams. Replacing \( uy_i \) with \( y_i \), we arrive at the identity
\[
\prod_{i=1}^{\infty} (1 - y_i)^{-z} = \sum_{\lambda} \prod_{b \in \lambda} \frac{z + c(b)}{h(b)} \cdot s_\lambda(y_1, y_2, \ldots). \tag{4-1-7}
\]
Recall that the coefficients of Schur functions in the right hand side are the polynomials in \( z \), hence it suffices to prove (4-1-7) for \( z = N = 1, 2, \ldots \).

It is well known ([Mac, I.3, Example 4]) that
\[
\prod_{b \in \lambda} \frac{N + c(b)}{h(b)} = s_\lambda(1, \ldots, 1)_{N}
\]
(this is the dimension of the irreducible representation of the group \( GL(\mathbb{C}) \) with the highest weight \((\lambda_1, \ldots, \lambda_N)\) if \( \lambda_{N+1} = \lambda_{N+2} = \ldots = 0 \), and 0 otherwise). The formula (4-1-3) takes the form
\[
\prod_{i=1}^{\infty} (1 - y_i)^{-N} = \sum_{\lambda} s_\lambda(1, \ldots, 1)_{N} s_\lambda(y_1, y_2, \ldots).
\]
This is a specialization of a more general identity ([Mac, Ch. I, (4.3)])
\[
\prod_{j=1}^{\infty} \prod_{i=1}^{\infty} (1 - u_j y_i)^{-1} = \sum_{\lambda} s_\lambda(u_1, u_2, \ldots) s_\lambda(y_1, y_2, \ldots)
\]
where we put \( u_1 = \ldots = u_N = 1 \) and \( u_{N+1} = u_{N+2} = \ldots = 0 \). This completes the proof of Lemma 4.1.2 and Theorem 4.1.1. □

27
4.2. The limit coherent systems $M_0$ and $M_\infty$. In Theorem 4.1.1 we did not consider the parameter values $z = 0$ and $z = \infty$. However, one can see from (4-1-1) that there exist the limits

$$M_0 = \lim_{z \to 0} M_z, \quad M_\infty = \lim_{z \to \infty} M_z,$$

which are also coherent systems on the Young lattice. The coherent system $M_0$ is supported by hook diagrams only (see Proposition 4.3.1 (iii) below), and

$$M_\infty(\lambda) = \frac{\dim^2 \lambda}{n!}, \quad \lambda \in \mathbb{Y}_n, \quad (4-2-1)$$

is the so-called Plancherel measure.

According to Proposition 9.5.1 these limiting coherent systems give rise to certain characters $\chi_0$ and $\chi_\infty$ of the group $S(\infty)$, and to certain spherical functions $\varphi_0$ and $\varphi_\infty$.

**Proposition 4.2.1.** The functions $\varphi_0$, $\varphi_\infty$ are spherical functions of the representations $T_0 = T'_0$ and $T_\infty = T'_\infty$, respectively. That is, they coincide with matrix coefficients of certain $K$–invariant vectors of the representations in question.

**Proof.** In order to see this, examine the behavior of the distinguished vector $\xi_0$ (recall that as an element of the space $H^n$ it coincides with the function $F^n_x$) as long as $z \to 0$ or $z \to \infty$.

Set $z = r\zeta$ where $r > 0$ and $\zeta$ is a point of the unit circle $S^1 \subset \mathbb{C}^*$. If $\zeta$ is fixed, the limits

$$F^n_x = \lim_{r \to 0} F^n_{r\zeta}, \quad F^n_x = \lim_{r \to \infty} F^n_{r\zeta}, \quad n = 1, 2, \ldots$$

exist and have the form

$$F^n_0(x) = \begin{cases} \sqrt{n} \zeta & \text{if } [x] = 1, \text{ i.e., if } x \text{ is a cycle in } S(n) \text{ of maximal length } n, \\ 0 & \text{otherwise}; \end{cases}$$

$$F^n_\infty(x) = \begin{cases} \sqrt{n!} \zeta^n & \text{if } x = e, \\ 0 & \text{if } x \neq e. \end{cases}$$

In these expressions, $\zeta$ enters as a scalar factor only. Hence the corresponding spherical functions do not depend on the choice of $\zeta$. Clearly, they coincide with $\varphi_0$ and $\varphi_\infty$, respectively. \square

Thus, our definition of the distinguished vector $\xi_0$ (§4.1) can be extended to the limit cases $z = 0$ and $z = \infty$ — at least, up to an unessential scalar factor. Note that the representation $T_\infty$ can be realized in the Hilbert space $\ell^2(S(\infty))$, and then $\xi_0$ can be identified with the delta function at $e \in S(\infty)$.

Note also that $\varphi_\infty$ is simply the characteristic function of the subgroup $K \subset G$, and $\chi_\infty$ is the delta function at the identity element of $S(\infty)$.

We proceed to analysis of the formula (4-1-1).

4.3. Support of $M_z$. Here we consider the coherent system $M_z$ with arbitrary $z \in \mathbb{C} \cup \{\infty\}$. The definition of the support of a coherent system is given in §9.4.
Proposition 4.3.1. (i) If \( z \notin \mathbb{Z} \), then \( \text{supp}(M_z) \) is the whole set \( \mathcal{Y} \).

(ii) If \( \lambda \) is a nonzero integer, \( \lambda = k \) or \( \lambda = -k \), where \( k = 1, 2, \ldots \), then \( \text{supp}(M_z) \) consists of those Young diagrams \( \lambda \) that have no more than \( k \) rows, or, respectively, no more than \( k \) columns.

(iii) If \( z = 0 \), then \( \text{supp}(M_z) \) is the set of hook diagrams, i.e., Young diagrams contained inside the union of the first row and the first column.

Proof. (i) If \( z \notin \mathbb{Z} \cup \{ \infty \} \), then the numerator in (4-1-1) does not vanish for all \( \lambda \), hence \( M_z(\lambda) \neq 0 \). If \( z = \infty \), then it follows from (4-2-1) that \( M_\infty(\lambda) \neq 0 \).

(ii) Assume that \( z = k \). Then the zeros in the numerator of (4-1-1) correspond to the boxes \( b = (i,j) \in \lambda \) such that \( c(b) = j - i = -k \). These boxes lie on a diagonal of \( \lambda \) passing through the box \((k+1,1)\) in the first column. The lack of such boxes in \( \lambda \) is clearly equivalent to the fact that \( \lambda \) contains \( k \) or less rows. In a similar fashion, if \( z = -k \) then \( -k + c(b) \neq 0 \) for all \( b \in \lambda \) if and only if \( \lambda \) contains no more than \( k \) columns.

(iii) Assume that \( z = 0 \) and consider the limit \( M_0 = \lim_{z \to 0} M_z \). When \( z \to 0 \), the zero factor \( t = z\bar{z} \) in the denominator of (4-3-1) cancels with the factor in the numerator corresponding to the box \( b = (1,1) \) (every nonempty diagram \( \lambda \neq \emptyset \) contains this box). Other zero factors in the numerator correspond to the boxes \((2,2),(3,3),\ldots\) on the main diagonal. The absence of such boxes just means that \( \lambda \) is a hook. \( \square \)

If a hook diagram \( \lambda \) has arm length \( a = \lambda_1 - 1 \) and leg length \( l = \lambda'_1 - 1 \), then

\[
M_0(\lambda) = \frac{|z-l|^2 \ldots |z-1|^2 |z+1|^2 \ldots |z+a|^2}{(t+1)(t+2)(t+3)\ldots(t+n-1)} \frac{\dim^2 \lambda}{n!}.
\]

4.4. When the distinguished vector \( \xi_0 \) is cyclic.

Proposition 4.4.1. Assume \( z \notin \mathbb{Z} \), then the distinguished vector \( \xi_0 \) is a cyclic vector of the representation \( T_z \).

Proof. When \( z = \infty \), this is evident from the realization in the space \( \ell^2(S(\infty)) \).

Assume now \( z \in \mathbb{C} \setminus \mathbb{Z} \).

Since \( T_z \) is the inductive limit of regular representations \( \text{Reg}^n \) and \( \xi_0 \) belongs to the spaces \( H^n \) of all those representations, it suffices to check that \( \xi_0 \) is cyclic in \( \text{Reg}^n \) for all \( n \). Recall that

\[
\text{Reg}^n \cong \bigoplus_{\lambda^+}^n (\pi^\lambda \times \pi^\lambda).
\]

Each of the irreducible representations \( \pi^\lambda \times \pi^\lambda = \pi^\lambda \times (\pi^\lambda)^* \) of the group \( G(n) = S(n) \times S(n) \) is spherical with respect to diagonal subgroup \( K(n) \), and the corresponding spherical vector is the function \( \chi^\lambda \). Since \( \xi_0 = F^n_z \in H^n \) is a \( K(n) \)-invariant vector, too, it is cyclic if and only if all of its coefficients in the decomposition in functions \( \chi^\lambda \) are nonzero. But \( M_z(\lambda) \) is the square modulus of the coefficient of \( \chi^\lambda \), hence the claim follows from Proposition 4.3.1(i). \( \square \)

We shall see below that \( \xi_0 \) is not cyclic in \( T_z \) if \( z \in \mathbb{Z} \).

4.5. The equivalence of representations \( T_z \) and \( T_{\bar{z}} \).
**Proposition 4.5.1.** The representations $T_z$ and $\bar{T}_z$ are unitarily equivalent for all $z \in \mathbb{C} \cup \{\infty\}$.

**Proof.** If $z \in \mathbb{R}$ or $z = \infty$, there is nothing to prove. Hence, we may assume that $z \notin \mathbb{R} \cup \{\infty\}$, in particular, $z \notin \mathbb{Z}$. By virtue of Proposition 4.6.1, it suffices to check that $\varphi_z = \varphi_{\bar{z}}$ which is equivalent to $\chi_z = \chi_{\bar{z}}$ and to $M_z = M_{\bar{z}}$. But this last formula follows directly from (4-1-1). $\square$

Note that it is not immediate from the definition that $T_z$ and $\bar{T}_z$ are equivalent.

**Proposition 4.5.2.** Assume that $z \in \mathbb{C} \setminus \mathbb{Z}$.

(i) There exists a unique operator $A_z$ intertwining representation $T_z$ and $\bar{T}_z$, and identifying their distinguished spherical vectors.

(ii) Realize the representations $T_z, \bar{T}_z$ as inductive limits of representations $\text{Reg}_n$. Then the operator $A_z$ preserves the subspaces $H^n$, hence determines an operator $A_{n,z}: H^n \to H^n$ commuting with the representation $\text{Reg}_n$, for all $n = 1, 2, \ldots$.

(iii) The operator $A_{n,z}$ on the space $H^n = L^2(S(n), \mu^n)$ is the convolution operator with the central function

$$\Theta_{n,z} = \sum_{\lambda \vdash n} \left( \prod_{b \in \lambda} \frac{\bar{z} + c(b)}{z + c(b)} \right) \dim \lambda \cdot \chi^\lambda.$$ 

**Proof.**

(i) Follows from the fact that the distinguished vectors are cyclic (for $z \notin \mathbb{Z}$), and from the coincidence of the corresponding spherical functions $\varphi_z, \varphi_{\bar{z}}$.

(ii) Follows from (i) and the fact that the distinguished vector (for $z \notin \mathbb{Z}$) is a $G(n)$–cyclic vector in the representation $\text{Reg}_n$, for all $n = 1, 2, \ldots$.

(iii) We have to find the operator in $L^2(S(n), \mu^n)$ that commutes with the regular representation $\text{Reg}_n$, and transforms the function

$$F^n_z = C \sum_{\lambda \vdash n} \left( \prod_{b \in \lambda} \frac{\bar{z} + c(b)}{h(b)} \right) \chi^\lambda$$

into

$$F^n_{\bar{z}} = C \sum_{\lambda \vdash n} \left( \prod_{b \in \lambda} \frac{\bar{\bar{z}} + c(b)}{h(b)} \right) \chi^\lambda,$$

with the same factor $C \neq 0$ (we do not need its precise form at the moment). Every operator commuting with $\text{Reg}_n$ is a convolution operator with some central function

$$\Theta = \sum_{\lambda \vdash n} \theta(\lambda) \chi^\lambda.$$

Note that the convolution operator with the function $\dim \lambda \cdot \chi^\lambda$ is the projection onto the irreducible component $\pi^\lambda \times \pi^\lambda$ of the representation $\text{Reg}_n$. It follows that $\Theta = \Theta_{n,z}$. $\square$

**4.6. Reducibility of representations $T_z$.** For $z \in \mathbb{C} \cup \{\infty\}$, let $\bar{T}_z$ denote the subrepresentation in $T_z$ realized in the cyclic span of the distinguished vector. In other words, $\bar{T}_z$ is the cyclic unitary representation of the group $G$ generated by the positive definite function $\varphi_z$. If $z \notin \mathbb{Z}$, then $\bar{T}_z$ coincides with $T_z$ by Proposition
4.4.1; we shall see below that for \( z \in \mathbb{Z} \) it is a proper subrepresentation of \( T_z \). Note that
\[
\varphi_1(g) \equiv 1, \quad \varphi_{-1}(g) \equiv \text{sgn}(g_1g_2^{-1}), \quad g = (g_1, g_2) \in G,
\]
so that for \( z = \pm 1 \) our representation \( \tilde{T}_z \) is one-dimensional (more precisely, trivial for \( z = 1 \) and equivalent to \( \text{sgn} \times \text{sgn} \) for \( z = -1 \)). Moreover, for \( z = \infty \) the representation \( \tilde{T}_\infty \) is irreducible, since the two-sided regular representation \( T_\infty \) of the group \( G = S \times S \) in \( P^2(S) \) is irreducible. We shall show now that in all other cases our cyclic representation \( \tilde{T}_z \) (hence the entire representation \( T_z \)) is reducible.

**Proposition 4.6.1.** For every \( z \in \mathbb{C} \setminus \{\pm 1\} \) the cyclic representation \( \tilde{T}_z \subseteq T_z \) generated by the distinguished vector is reducible.

**Proof.** Let \( \chi_z \) denote the character corresponding to the spherical function \( \varphi_z \). Irreducibility of the representation \( T_z \) would imply that \( \chi_z \) is an extreme character and hence coincides with a certain character \( \chi^{\alpha\beta} \) from the Thoma list (see the Appendix). In this case we would also have the equality \( M_z = M^{(\alpha,\beta)} \) of the corresponding coherent systems on the Young lattice. We shall compare the values of \( M_z \) and \( M^{(\alpha,\beta)} \) on one-row diagrams \( \lambda = (n) \), \( n = 1, 2, \ldots \) and derive that the equality only holds for \( z = \pm 1, \infty \).

Indeed, it follows from (4-3-1) that the generating function for \( M_z((n)) \), \( z \neq 0, \infty \), is
\[
1 + \sum_{n \geq 1} M_z((n)) w^n = 2F_1(z, \bar{z}; z\bar{z}; w),
\]
where \( w \) is a parameter. On the other hand, the Thoma formula implies that
\[
1 + \sum_{n \geq 1} M^{\alpha\beta}((n)) w^n = e^{\gamma w} \prod_i \frac{1 + \beta_i w}{1 - \alpha_i w}, \quad \gamma = 1 - \sum_{k=1}^{\infty} (\alpha_k + \beta_k).
\]
Hence, we are led to study the possibility of the equality
\[
2F_1(z, \bar{z}; z\bar{z}; w) = e^{\gamma w} \prod_i \frac{1 + \beta_i w}{1 - \alpha_i w}.
\]
This formula would also imply that
\[
2F_1(-z, -\bar{z}; z\bar{z}; w) = e^{\gamma w} \prod_i \frac{1 + \alpha_i w}{1 - \beta_i w},
\]
since the tensor multiplication by the nontrivial one-dimensional representation \( \text{sgn} \times \text{sgn} \) switches \( z \) to \( -z \) (by Proposition 3.6.1) and replaces \( \chi^{\alpha\beta} \) with \( \chi^{\beta\alpha} \).

It is now easy to see that the equalities (4-8-1), (4-6-2) are only possible if \( z = \pm 1 \). In fact, the hypergeometric series converges absolutely in the open disk \( |w| < 1 \), hence the right-hand side of (4-6-1), (4-6-2) cannot have poles in this disk. Recall that all the Thoma parameters are positive, so that there are no cancellations between numerators and denominators. Therefore, \( \alpha_1 = 1 \) or \( \beta_1 = 1 \) or \( \gamma = 1 \), and all other parameters vanish. The first case corresponds to \( z = 1 \), the second one to \( z = -1 \), and the last one corresponds to \( z = \infty \) and cannot occur for a finite \( z \).

It remains to consider the case \( z = 0 \). In the limit \( z \to 0 \) the generating function takes the form \( 1 + \sum_{n \geq 1} \frac{1}{n} w^n \) and the equalities (4-6-1), (4-6-2) cannot hold in this case, too. \( \square \)
4.7. Transition probabilities. Recall from Proposition 4.5.1 that in case \( z \notin \mathbb{Z} \) the support of \( M_z \) is the entire Young graph. If \( z = \pm k \) where \( k = 1, 2, \ldots \), the support is made of the Young diagrams with \( k \) or less rows (columns). If \( z = 0 \), then \( M_z \) is supported by the hook diagrams.

Let \( p_n(\lambda, \nu) \) denote the transition probabilities of the coherent system \( M_z \), see \( \S \) 9. These quantities are defined for any \( \lambda \in \text{supp}(M_z) \).

**Proposition 4.7.1.** Let \( \lambda \in \mathcal{Y}_n, \nu \in \mathcal{Y}_{n+1}, \lambda \neq \nu \). We have

\[
p_\infty(\lambda, \nu) = \frac{\dim \nu}{(n + 1) \dim \lambda},
p_z(\lambda, \nu) = \frac{|z + c_{\lambda\nu}|^2}{z \bar{z} + n} p_\infty(\lambda, \nu), \quad \lambda \in \mathcal{Y}_n \cap \text{supp}(M_z), \quad z \neq \infty
\]

where \( c_{\lambda\nu} = c(\nu \setminus \lambda) \) is the content of the box \( \nu \setminus \lambda \).

**Proof.** Follows immediately from (4-1-1) and the definition of transition probabilities. \( \square \)

5. The commutant and block decomposition

5.1. Simplicity of spectrum. Let \( T_z, z \in \mathbb{C} \cup \{\infty\} \) be a generalized regular representation. We shall work with the realization of \( T_z \) as inductive limit of the two-sided regular representation \( \text{Reg}^n \) of the group \( G(n) = S(n) \times S(n) \). As before, we denote the space of the representation \( \text{Reg}^n \) by \( H^n \). The representation \( T_z \) acts in the Hilbert completion \( H \) of the space \( \bigcup_{n \geq 1} H^n \), where the maps \( L^z_n : H^n \to H^{n+1} \) (depending on \( z \)) were introduced in \( \S 3 \). It will be important that \( L^z_n \) depends continuously on the parameter \( z \) ranging over the Riemann sphere. Recall that \( \text{Reg}^n \) is the direct sum of irreducible representations \( \pi^\lambda \times \pi^\lambda, \lambda \in \mathcal{Y}_n \) of the group \( G(n) \). There are no multiple components in this decomposition.

Denote by \( P_n : H \to H^n \) the orthogonal projection onto \( H^n \), by \( H(\lambda) \subset H^n \) the space of the representation \( \pi^\lambda \times \pi^\lambda \), and by \( P(\lambda) : H \to H(\lambda) \) the orthogonal projection onto \( H(\lambda) \). Note that the projectors \( P(\lambda), \lambda \in \mathcal{Y}_n \), are pairwise orthogonal, and their sum equals \( P_n \).

Let \( \mathcal{A} \) be the commutant of \( T_z \), i.e., the algebra of all bounded operators in \( H \) commuting with the representation \( T_z \). We know that for \( z \notin \mathbb{Z} \) the representation \( T_z \) admits a cyclic \( K \)-invariant vector (the distinguished vector). On the other hand, \( (G, K) \) is a Gelfand pair [Ol3, \S 1]. It follows that for \( z \notin \mathbb{Z} \), the algebra \( \mathcal{A} \) is isomorphic to the commutant of a commutative operator \( * \)-algebra admitting a cyclic vector, whence \( \mathcal{A} \) is commutative. We shall presently give another proof of this fact, applicable for all \( z \).

**Proposition 5.1.1.** For every \( z \in \mathbb{C} \cup \{\infty\} \) the commutant \( \mathcal{A} \) of the representation \( T_z \) is a commutative algebra.

**Proof.** We have to prove that \( AB = BA \) for arbitrary \( A, B \in \mathcal{A} \). For every \( n \), the operators \( P_n A P_n \) and \( P_n B P_n \) viewed as operators in the space \( H^n \) commute with the representation \( \text{Reg}^n \). Since \( \text{Reg}^n \) multiplicity free, its commutant is commutative. Therefore,

\[
P_n A P_n B P_n = P_n B P_n A P_n \quad \text{for all } n = 1, 2, \ldots \quad (5-1-1)
\]
Since $H^n \subset H$ form an increasing chain of subspaces and their union is dense in $H$, the projectors $P_n$ converge to 1 strongly as $n \to \infty$. Moreover, the multiplication operation is continuous in the strong operator topology on every operator ball. Since the norms of all operators in (5-1-1) do not exceed the maximum of the numbers $1, \|A\|, \|B\|$, we can pass to the limit in (5-1-1), which gives $AB = BA$. □

**Corollary 5.1.2.** For every $z \in \mathbb{C}$, the representation $T_z$ is decomposable in a multiplicity free direct integral of admissible irreducible representations of the group $G$.

*Proof.* The existence of a decomposition into a multiplicity free integral of irreducible representations follows from the fact that the group $G$ is countable and the commutant is commutative. The admissibility is easily checked as in [Ol1, Theorem 3.6]. □

### 5.2. The transition function of $T_z$

In order to simplify the notation, we identify the isometric embedding $L^n_z: H^n \to H^{n+1}$ with the partially isometric operator $L^n_z P_n$ in the space $H$. For every $\lambda \in \mathcal{Y}_n$, $\nu \in \mathcal{Y}_{n+1}$ (where $\lambda \triangleright \nu$) fix an isometric embedding $E(\lambda, \nu): H(\lambda) \to H(\nu)$ commuting with the action of the group $G(n)$. The choice of $E(\lambda, \nu)$ is unique, up to a complex factor of modulus 1. We identify $E(\lambda, \nu)$ with the partially isometric operator $E(\lambda, \nu) P(\lambda)$ acting in the whole space $H$.

For each pair of Young diagrams $\lambda \in \mathcal{Y}_n$, $\nu \in \mathcal{Y}_{n+1}$ consider the operator $P(\nu) L^n_z P(\lambda)$. This operator intertwines representations $\pi^\lambda \times \pi^\lambda$ and $(\pi^\nu \times \pi^\nu)|_{G(n)}$, hence is 0 unless $\lambda \triangleright \nu$. In the latter case it is proportional to $E(\lambda, \nu)$:

$$P(\nu) L^n_z P(\lambda) = \alpha_z(\lambda, \nu) E(\lambda, \nu), \quad \lambda \triangleright \nu.$$ 

Set

$$\tilde{p}_z(\lambda, \nu) = |\alpha_z(\lambda, \nu)|^2, \quad \lambda \triangleright \nu.$$ 

Clearly, this function does not depend on the choice of $E(\lambda, \nu)$. It is also clear that for any $\xi \in H(\lambda)$

$$||P(\nu) \xi||^2 = \begin{cases} \tilde{p}_z(\lambda, \nu) ||\xi||^2, & \text{if } \lambda \triangleright \nu \\ 0, & \text{otherwise.} \end{cases}$$

Since the projections $P(\nu), \nu \in \mathcal{Y}_{n+1}$ are pairwise orthogonal and sum up to $P_{n+1}$, it follows that

$$\sum_{\nu \triangleright \lambda} \tilde{p}_z(\lambda, \nu) = 1 \quad \forall \lambda \in \mathcal{Y}_n, \quad n = 1, 2, \ldots.$$

We shall call $\tilde{p}_z(\lambda, \nu)$ the *transition function of the representation $T_z$*. In Theorem 5.5.1 below we show that it coincides with the transition probabilities of the coherent system $M_z$.

Let us emphasize that the transition function $\tilde{p}_z(\lambda, \nu)$ is defined on the edges of the graph $\mathcal{Y} \setminus \{\varnothing\}$, not the whole Young graph. That is, we do not attempt to define the value of this function when $\lambda$ is the empty diagram $\varnothing$ and $\nu$ is the one–box diagram.

33
5.3. The commutant in terms of the transition function. We shall presently show that the transition function of the representation $T_z$ determines its commutant completely.

Let $\tilde{A}$ denote the space of all bounded complex functions $A(\lambda)$ on the set of vertices of the graph $\mathbb{Y} \setminus \{\emptyset\}$ satisfying the condition

$$A(\lambda) = \sum_{\nu \searrow \lambda} \tilde{p}_z(\lambda, \nu) A(\nu), \quad \text{for all } \lambda \in \mathbb{Y}, \lambda \neq \emptyset,$$

where $\nu \searrow \lambda$ means $\lambda \not\succ \nu$. We consider $\tilde{A}$ as a Banach space with the norm $||A|| = \sup_{\lambda} |A(\lambda)|$.

**Proposition 5.3.1.** The commutant $\mathcal{A}$ of the generalized regular representation $T_z$ considered as a Banach space with the ordinary operator norm is naturally isometric to the space $\tilde{A}$.

**Proof.** For every operator $A \in \mathcal{A}$ and every $n = 1, 2, \ldots$ we have

$$P_n A P_n = \sum_{\lambda, \mu \in \mathbb{Y}_n} P(\lambda) A P(\mu).$$

But $P(\lambda) A P(\mu)$ intertwines the representations $\pi^\mu \times \pi^\mu$ and $\pi^\lambda \times \pi^\lambda$, hence can be nonzero only if $\lambda = \mu$. In this case the operator $P(\lambda) A P(\lambda)$ has to be proportional to $P(\lambda)$. Denoting the coefficient by $A(\lambda)$ we obtain

$$P_n A P_n = \sum_{\lambda \in \mathbb{Y}_n} A(\lambda) P(\lambda).$$

It is clear that

$$\|P_n A P_n\| = \sup_{\lambda \in \mathbb{Y}_n} |A(\lambda)|$$

which implies that

$$\|A\| = \sup_n \|P_n A P_n\| = \sup_{\lambda \in \mathbb{Y}_n} |A(\lambda)| = \|A(\cdot)\|,$$

where $\|A(\cdot)\|$ denotes the sup–norm of the function $A(\lambda)$.

Let us check now that for any $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$,

$$P(\lambda) P(\nu) P(\lambda) = \begin{cases} \tilde{p}_z(\lambda, \nu) P(\lambda), & \text{if } \lambda \not\succ \nu, \\ 0, & \text{otherwise.} \end{cases}$$

Indeed, the operator $P(\nu)$ commutes with the action of $G(n) \subset G(n + 1)$, hence the operator $P(\lambda) P(\nu) P(\lambda)$ which commutes with the irreducible representation $\pi^\lambda \times \pi^\lambda$ of the group $G(n)$ should be proportional to $P(\lambda)$. In order to find the coefficient we remark that for every $\xi \in H(\lambda)$ we have $P(\lambda) \xi = \xi$, so that

$$\langle P(\lambda) P(\nu) P(\lambda) \xi, \xi \rangle = \langle P(\nu) \xi, \xi \rangle = \langle P(\nu) P(\nu) \xi, \xi \rangle =$$

$$= \begin{cases} \tilde{p}_z(\lambda, \nu) \langle \xi, \xi \rangle, & \text{if } \lambda \not\succ \nu, \\ 0, & \text{otherwise.} \end{cases}$$
by the definition of the transition function. On the other hand, \((P(\lambda), \xi, \xi) = (\xi, \xi)\), so that the above coefficient equals \(\tilde{p}_z(\lambda, \nu)\) if \(\lambda \not\supset \nu\), and 0 otherwise.

Note now that \(P(\lambda)P(\nu)P(\mu) = 0\) if \(\lambda, \mu \in \mathbb{Y}_n, \mu \neq \lambda\). Therefore,

\[
P_n P(\nu) P_n = \sum_{\lambda \not\supset \nu} \tilde{p}_z(\lambda, \nu)P(\lambda) \quad \forall \nu \in \mathbb{Y}_{n+1}.
\]

Setting

\[
A_n = P_n A P_n, \quad n = 1, 2, \ldots
\]

we see that the property (5-3-1) of the function \(A(\cdot)\) simply means that

\[
A_n = P_n A_{n+1} P_n, \quad n = 1, 2, \ldots
\]

Thus, we have constructed above an isometric embedding of \(A\) into the space \(\tilde{\mathcal{A}}\). In the opposite direction, we shall show that every function \(A(\cdot) \in \tilde{\mathcal{A}}\) stems from some operator \(A \in \mathcal{A}\). To this end we set

\[
A_n = \sum_{\lambda \in \mathbb{Y}_n} A(\lambda) P(\lambda), \quad n = 1, 2, \ldots
\]

The condition (5-3-1) then implies that

\[
A_n = P_n A_{n+1} P_n, \quad n = 1, 2, \ldots
\]

and the condition \(\|A(\cdot)\| < \infty\) implies

\[
\sup_n \|A_n\| = \|A(\cdot)\| < \infty.
\]

It follows that there exists a bounded operator

\[
A = \lim_{n \to \infty} A_n
\]

where \(\lim\) denotes the limit in the weak operator topology. Since \(A_n\) commutes with the action of the group \(G(n)\), the operator \(A\) belongs to the commutant. It is clear that \(A_n\) coincides with \(P_n A P_n\) for all \(n\), hence our function \(A(\cdot)\) corresponds to this very operator. 

\[\blacksquare\]

5.4. The multiplication in the space \(\tilde{\mathcal{A}}\). Let us denote by \((A \circ B)(\cdot)\) the multiplication operation of functions \(A(\cdot), B(\cdot)\) in \(\tilde{\mathcal{A}}\) corresponding to the operator multiplication in \(\mathcal{A}\). Unfortunately there is no simple formula for this operation. Nevertheless, it can be described in terms of the transition function using an appropriate limit procedure.

It will be convenient to extend the definition of the transition function \(\tilde{p}_z(\lambda, \nu)\) to all pairs of Young diagrams \(\lambda, \nu\). Given \(\lambda \in \mathbb{Y}_n, \nu \in \mathbb{Y}_{N}\), we denote by \(T(\lambda, \nu)\) the set of paths

\[
\tau = (\tau_n \nearrow \tau_{n+1} \nearrow \cdots \nearrow \tau_N), \quad \tau_n = \lambda, \quad \tau_N = \nu.
\]
from $\lambda$ to $\nu$ in the Young graph (such paths are commonly called \textit{skew Young tableaux} of shape $\nu/\lambda$). Let

$$\bar{p}_z(\tau) = \prod_{k=n+1}^{N} \bar{p}_z(\tau_{k-1}, \tau_k)$$

denote the transition probability along the path $\tau$, and let

$$\bar{p}_z(\lambda, \nu) = \sum_{\tau \in \mathcal{T}(\lambda, \nu)} \bar{p}_z(\tau)$$

be the total probability of the transition from $\lambda$ to $\nu$. If $\lambda$ is not contained in $\nu$, then the set $\mathcal{T}(\lambda, \nu)$ is empty and $\bar{p}_z(\lambda, \nu) = 0$. If $\lambda \ncong \nu$, then the definition of $\bar{p}_z(\lambda, \nu)$ does not change.

Let $C(\mathbb{Y}_n)$ denote the algebra of functions on the finite set $\mathbb{Y}_n$, with pointwise multiplication. Given $n < N$, define a linear map

$$\alpha_{N,n} : C(\mathbb{Y}_N) \to C(\mathbb{Y}_n)$$

as follows: if $A(\cdot) \in C(\mathbb{Y}_N)$, then

$$(\alpha_{N,n}(A)) (\lambda) = \sum_{\nu \in \mathbb{Y}_N} \bar{p}_z(\lambda, \nu) A(\nu).$$

\textbf{Proposition 5.4.1.} Assume that $A(\cdot), B(\cdot) \in \tilde{A}$, and let

$$A_N(\cdot) = A(\cdot) |_{\mathbb{Y}_N}, \quad B_N(\cdot) = B(\cdot) |_{\mathbb{Y}_N}$$

denote their restrictions to $\mathbb{Y}_N$. Then

$$(A \circ B)(\lambda) = \lim_{N \to \infty} \alpha_{N,n}(A_N(\cdot) B_N(\cdot))(\lambda), \quad \lambda \in \mathbb{Y}_n, \quad (5-4-1)$$

\textbf{Proof.} Denote by $A, B \in \mathcal{A}$ the operators corresponding to $A(\cdot), B(\cdot)$ and set $A_N = P_N A P_N$, $B_N = P_N B P_N$. Then

$$A_N B_N = \sum_{\nu \in \mathbb{Y}_N} A_N(\nu) B_N(\nu) P(\nu),$$

hence

$$P(\lambda) A_N B_N P(\lambda) = \alpha_{N,n}(A_N(\cdot) B_N(\cdot))(\lambda)$$

for all $\lambda \in \mathbb{Y}_n$. On the other hand, $A_N B_N$ converges strongly to $AB$ as $N \to \infty$, so that the left hand side converges strongly to

$$P(\lambda) A B P(\lambda) = (A \circ B)(\lambda) P(\lambda),$$

which proves (5-4-1). $\Box$

\textbf{5.5. The identity of the functions $p_z$ and $\bar{p}_z$.}
Theorem 5.5.1. The transition function of the representation $T_z$, $z \in \mathbb{C} \cup \{\infty\}$, is given by the same expression as the transition probabilities of the distribution $M_z$, i.e.,

$$\tilde{p}_z(\lambda, \nu) = \frac{|z + c_{\lambda\nu}|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n + 1) \dim \lambda}, \quad (5-5-1)$$

where $\lambda \in \mathbb{Y}_n$, $\nu \in \mathbb{Y}_{n+1}$, $\lambda \not> \nu$ and $c_{\lambda\nu} = c(\nu \setminus \lambda)$.

Remark 5.5.2. If $z \in \mathbb{Z}$, the function $p_z(\lambda, \nu)$ is formally defined on the edges of the proper subgraph $\text{supp}(M_z) \subset \mathbb{Y}$ only, whereas the function $\tilde{p}_z(\lambda, \nu)$ is always defined on the whole graph $\mathbb{Y}$. However, the expression for $p_z(\lambda, \nu)$ given in §4.7 makes sense for all couples $\lambda \not> \nu$. This makes it possible to say that both functions coincide even for $z \in \mathbb{Z}$.

Proof. a) Let us introduce some notation which will also be used in the sequel. Let $\xi_{\lambda}, \lambda \in \mathbb{Y}_n$, be the vector in $H(\lambda) \subset H^\mathbb{Y} \subset H$ corresponding to $\chi_{\lambda}$ under the identification $H^\mathbb{Y} \cong L^2(S(n), \mu_n^1)$. It has unit length and is $K(n)$–invariant. The distinguished vector of $T_z$ will be denoted as $\xi_0$ as before. As a vector in $L^2(S(n), \mu_n^1)$ it is given by the function

$$F^n_z(x) = \left(\frac{n!}{t(t+1)\ldots(t+n-1)}\right)^{1/2} z^{[x]}, \quad x \in S(n),$$

and can be decomposed as

$$\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_{\lambda},$$

where

$$a_z(\lambda) = \left(\frac{1}{t(t+1)\ldots(t+n-1)}\right)^{1/2} \prod_{b \in \lambda} (z + c(b)) \cdot \frac{\dim \lambda}{\sqrt{n!}}.$$

b) We now show that, for every $\nu \in \mathbb{Y}_{n+1}$, the vector $\xi_{\nu}$ can be decomposed as

$$\xi_{\nu} = \sum_{\lambda \not> \nu} \xi_{\lambda\nu},$$

where the vectors $\xi_{\lambda\nu}$ are pairwise orthogonal, $K(n)$–invariant,

$$(\xi_{\lambda\nu}, \xi_{\lambda\nu}) = \frac{\dim \lambda}{\dim \nu},$$

and $\xi_{\lambda\nu}$ generates, under the action of $G(n)$, the representation $\pi^\lambda \times \pi^\lambda$.

Indeed, let $H(\pi')$ be the space of an irreducible representation $\pi'$ of the group $S(n+1)$, and let $\text{End} \ H(\pi')$ be the algebra of operators on this space. Endow $\text{End} \ H(\pi')$ with the inner product

$$(A, B) = \frac{\text{tr}(AB^*)}{\dim \nu}$$

and define an action of the group $G(n+1)$ on $\text{End} \ H(\pi')$ by

$$g \cdot A = \pi'(g_1) A \pi'(g_2)^{-1}, \quad A \in \text{End} \ H(\pi'), \quad g = (g_1, g_2) \in G(n+1).$$
It is convenient to identify the vector spaces $\text{End} H(\pi^\nu)$ and $H(\nu) \subset H^{n+1} = L^2(S(n+1), \mu_1^{n+1})$ as follows: to an operator $A \in \text{End} H(\pi^\nu)$ we assign the function $\hat{A}(s) = \text{tr}(A\pi^\nu(s^{-1}))$. The map $A \mapsto \hat{A}$ preserves the inner product and commutes with the action of the group $G(n+1)$.

Let $1_\nu$ denote the identity operator in the space $H(\pi^\nu)$. Its image under the correspondence $A \mapsto \hat{A}$ coincides with the vector $\xi_\nu$. Further, for any $\lambda \nrightarrow \nu$, let $1_{\lambda\nu} \in \text{End} H(\pi^\nu)$ denote the orthogonal projection onto the subspace of vectors that transform according to the representation $\pi^\lambda$ under the action of $S(n) \subset S(n+1)$.

Define $\xi_{\lambda\nu}$ as the image of the operator $1_{\lambda\nu}$ under the correspondence $A \mapsto \hat{A}$. Clearly, the vectors $\xi_{\lambda\nu}$ satisfy all the required properties.

c) Now let us remark that it suffices to prove (5-5-1) when $z \notin \mathbb{Z}$. In fact, the right–hand side of (5-5-1) is continuous in the parameter $z$ ranging over the Riemann sphere. It also follows from the definition (5-3-1) of $\hat{p}_z(\lambda, \nu)$ that this function is continuous in $z$, since so is the map $L_z^n$.

The assumption $z \notin \mathbb{Z}$ implies that $a_z(\lambda) \neq 0$ for all $\lambda$, which will be used in the computation below.

d) Equating the decompositions of $\xi_0$ in $\xi_\lambda$, $\lambda \in \mathbb{Y}_n$, and in $\xi_\nu$, $\nu \in \mathbb{Y}_{n+1}$, we conclude that

$$
\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_\lambda = \sum_{\nu \in \mathbb{Y}_{n+1}} a_z(\nu) \xi_\nu.
$$

Substituting the decomposition $\xi_\nu = \sum_{\lambda \nrightarrow \nu} \xi_{\lambda\nu}$, we arrive at

$$
\sum_{\lambda \in \mathbb{Y}_n} a_z(\lambda) \xi_\lambda = \sum_{\lambda \in \mathbb{Y}_n} \sum_{\nu \nrightarrow \lambda} a_z(\nu) \xi_{\lambda\nu}.
$$

Comparing the components in both sides that transform according to a given irreducible representation of $G(n) \subset G(n+1)$ we see that

$$
a_z(\lambda) \xi_\lambda = \sum_{\nu \nrightarrow \lambda} a_z(\nu) \xi_{\lambda\nu} \quad \text{for any } \lambda \in \mathbb{Y}_n.
$$

This implies that

$$
P_{\nu} \xi_\lambda = \frac{a_z(\nu)}{a_z(\lambda)} \xi_{\lambda\nu}, \quad \nu \nrightarrow \lambda,
$$

whence

$$
\hat{p}_z(\lambda, \nu) = \left( \frac{a_z(\nu)}{a_z(\lambda)} \right)^2 \|\xi_{\lambda\nu}\|^2 = \left( \frac{a_z(\nu)}{a_z(\lambda)} \right)^2 \frac{\dim \lambda}{\dim \nu}
$$

by the definition of the transition function. This last equation, along with the explicit formula for the coefficients $a_z(\cdot)$, implies the desired formula for $\hat{p}_z(\lambda, \nu)$. $\square$

### 5.6. The subgraphs $\mathbb{Y}(p, q)$ and levels of Young diagrams.

Given a couple $(p, q)$ of nonnegative integers, we define a subset $\mathbb{Y}(p, q) \subset \mathbb{Y}$ as follows

$$
\begin{align*}
\mathbb{Y}(p, q) &= \{ \lambda \in \mathbb{Y} \mid (p, q) \in \lambda, (p + 1, q + 1) \notin \lambda \}, \quad p, q \geq 1 \\
\mathbb{Y}(p, 0) &= \{ \lambda \in \mathbb{Y} \mid \lambda_{p+1} = \lambda_{p+2} = \cdots = 0 \}, \quad p \geq 1 \\
\mathbb{Y}(0, q) &= \{ \lambda \in \mathbb{Y} \mid (\lambda')_{q+1} = (\lambda')_{q+2} = \cdots = 0 \}, \quad q \geq 1 \\
\mathbb{Y}(0, 0) &= \{ \emptyset \}
\end{align*}
$$
In other words, the set $\mathcal{Y}(p, q)$, where $p, q \geq 1$, consists of all diagrams containing the rectangle of shape $p \times q$ but not the box $(p + 1, q + 1)$. The set $\mathcal{Y}(p, 0)$ consists of all diagrams with at most $p$ rows, and the set $\mathcal{Y}(0, q)$ consists of all diagrams with at most $q$ columns.

Each $\mathcal{Y}(p, q)$ may be viewed as a connected subgraph of the Young graph.

**Proposition 5.6.1.** Fix an arbitrary integer $k$ and remove from the Young graph all edges $\lambda \not\supset \nu$ such that the content of the box $\nu \setminus \lambda$ equals $-k$. Then we obtain a subgraph in $\mathcal{Y}$ whose connected components are exactly the $\mathcal{Y}(p, q)$’s with $p - q = k$.

**Proof.** This follows from the three claims which are readily checked. First, the sets $\mathcal{Y}(p, q)$ with fixed $p - q = k$ form a partition of the set of all Young diagrams. Second, if $\lambda \not\supset \nu$ and $c(\nu \setminus \lambda) \neq k$ then $\lambda$ and $\nu$ belong to one and the same part $\mathcal{Y}(p, q)$ of that partition. Third, if $\lambda \not\supset \nu$ and $c(\nu \setminus \lambda) = k$ then $\lambda$ and $\nu$ belong to different parts: specifically, if $\lambda \in \mathcal{Y}(p, q)$ then $\nu \in \mathcal{Y}(p + 1, q + 1)$. □

For an arbitrary $k \in \mathbb{Z}$, we define the $k$–level of a Young diagram $\lambda$ as follows

$$\text{lev}_k(\lambda) = \#\{b \in \lambda | c(b) = -k\}.$$ 

In other words, $\text{lev}_0(\lambda)$ equals the length of the main diagonal in $\lambda$, and $\text{lev}_k(\lambda)$ is the number of boxes on the diagonal shifted (with respect to the main diagonal) $k$ boxes downwards if $k \geq 0$, and $|k|$ boxes upwards, if $k \leq 0$.

**Proposition 5.6.2.** Fix an arbitrary integer $k$. The partition of the set $\mathcal{Y}$ into disjoint union of the sets $\mathcal{Y}(p, q)$ with $p - q = k$ coincides with the partition according the value of the $k$–level. Specifically, a given part $\mathcal{Y}(p, q)$ with $p - q = k$ is exactly the set of diagrams with $\text{lev}_k(\cdot) = l$, where $l = \min(p, q)$.

**Proof.** This is evident. □

### 5.7. The decomposition into blocks.

The knowledge of the transition function $\tilde{p}_z(\lambda, \nu)$, and the results of Propositions 5.3.1 and 5.4.1 lead us to a preliminary decomposition of representations $T_z$ for $z \in \mathbb{Z}$.

Define a function $A_{pq}(\cdot)$ on nonempty diagrams as follows. If $p, q = 1, 2, \ldots$ then this is the characteristic function of the set $\mathcal{Y}(p, q)$. If $p = 1, 2, \ldots$ and $q = 0$ then this is the characteristic function of $\mathcal{Y}(p, 0) \setminus \{\emptyset\}$. Similarly, if $p = 0$ and $q = 1, 2, \ldots$ then this is the characteristic function of $\mathcal{Y}(0, q) \setminus \{\emptyset\}$.

**Theorem 5.7.1.** Fix $k \in \mathbb{Z}$ and let $(p, q)$ be a couple of nonnegative integers, not vanishing simultaneously, and such that $p - q = k$.

(i) The function $A_{pq}(\cdot)$ satisfies the condition (5-3-1) involving the transition function $\tilde{p}_k(\lambda, \nu)$. Therefore, it determines an operator $A_{pq}$ in the commutant $\mathcal{A}$ of the representation $T_k$.

(ii) Any operator $A_{pq}$ is an orthogonal projection onto a subspace $H_{pq} \subset H$. The subspaces $H_{pq}$ are pairwise orthogonal, and their direct sum is the whole $H$. Thus, they determine a decomposition of the representation $T_k$ into a direct sum of subrepresentations,

$$T_k = \bigoplus_{p - q = k} T_{pq}, \quad (p, q) \in \mathbb{Z}_+^2 \setminus \{(0, 0)\}.$$
(iii) Denote by $\text{Reg}_{pq}^n$, where $n$ is large enough, the subrepresentation of the regular representation $\text{Reg}^n$, which is the union of the components $\pi^\lambda \times \pi^\lambda$ such that $\lambda \in \mathbb{Y}(p, q)$. Let $H_{pq}^n$ be the corresponding subspace in $H^n$. Then the isometric embedding $L_k^n : H^n \to H^{n+1}$ maps $H_{pq}^n$ into $H_{pq}^{n+1}$, and the representation $T_{pq} \subset T_k$ coincides with the inductive limit of the representations $\text{Reg}_{pq}^n \subset \text{Reg}^n$ as $n \to \infty$.

Proof. (i) For $\lambda \in \mathbb{Y}_n \cap \mathbb{Y}(p, q)$ we have a chain of identities

$$A_{pq}(\lambda) = 1 = \sum_{\nu \uparrow \lambda} \tilde{p}_k(\lambda, \nu) = \sum_{\nu \uparrow \lambda, \nu \in \mathbb{Y}(p, q)} \tilde{p}_k(\lambda, \nu) = \sum_{\nu \uparrow \lambda} \tilde{p}_k(\lambda, \nu)A_{pq}(\nu).$$

Therefore, the function $A_{pq}(\cdot)$ satisfies the condition (5-3-1). Since this function is bounded, it determines, according to Proposition 5.3.1, an operator $A_{pq}$ in the commutant $A$ of the representation $T_k$.

(ii) From Proposition 5.3.1 and the definition of the function $A_{pq}(\cdot)$ it follows that for all $n = 1, 2, \ldots$

$$P_nA_{pq}P_n = \sum_{\lambda \in \mathbb{Y}_n \cap \mathbb{Y}(p, q)} P(\lambda). \quad (5-6-2)$$

Since $P_nA_{pq}P_n$ is an orthoprojector for any $n$, so is the operator $A_{pq}$. A similar argument shows that the projectors $A_{pq}$ are pairwise orthogonal and sum up to the identity operator. Note that here we could also use Proposition 5.4.1.

(iii) Since all three operators $P_n, A_{pq}, P_nA_{pq}P_n$ are orthoprojectors, it follows that $P_n$ and $A_{pq}$ commute. Along with (5-6-2) this implies that the operator $P_nA_{pq} = A_{pq}P_n$ projects $H^n$ onto the invariant subspace $H_{pq}^n$. Since $P_{n+1}$ majorizes $P_n$, we see that $H_{pq}^n$ is a subspace of $H_{pq}^{n+1}$. ∎

We shall call the subrepresentations $T_{pq} \subset T_k$ the blocks of the representation $T_k$. We shall show below that the blocks are themselves reducible, and find their decomposition into a direct integral of irreducible representations. The only exceptions are the one-dimensional blocks $T_{10} \subset T_1$ and $T_{01} \subset T_{-1}$: each one is generated by the distinguished spherical vector. The block $T_{10}$ is the identity representation, and $T_{01}$ is $\text{sgn} \times \text{sgn}$.

5.8. Another approach to decomposition into blocks. It is worth noting that the decomposition of the representation $T_k, k \in \mathbb{Z}$, into blocks can be obtained in a different way – using the operator $A_z$ intertwining representations $T_z$ and $\overline{T_z}$.

Recall that the operator $A_z$ was described in §4.7 as inductive limit, as $n \to \infty$, of certain operators $A_{n,z} : H^n \to H^n$ commuting with the representation $\text{Reg}^n$. In other words, $A_z$ preserves $H^n$ for all $n = 1, 2, \ldots$, and the restriction of $A_z$ to $H^n$ is $A_{n,z}$. The operator $A_{n,z}$ acts in the space $H^n = L^2(S(n), \mu^n)$ as the operator of convolution with the central function

$$\Theta_{n,z} = \sum_{\lambda \vdash n} \left( \prod_{\lambda \in \lambda} \frac{\bar{z} + c(h)}{z + c(h)} \right) \dim \lambda \cdot \chi^\lambda$$

on the group $S(n)$ (see Proposition 4.7.1).

This function is correctly defined for all nonintegral values of the parameter $z$. If $z \in \mathbb{R} \setminus \mathbb{Z}$, then the function $\Theta_{n,z}$ is just $\delta_z = \sum_{\lambda \vdash n} \dim \lambda \cdot \chi^\lambda$, the delta function
at the identity element of the group \( S(n) \). Therefore, the associated operator is simply the identity operator in \( H^n \). Let us see what happens to \( \Theta_{n,z} \) when the parameter \( z \in \mathbb{C} \setminus \mathbb{R} \) approaches an integer point \( k \in \mathbb{Z} \). It turns out that there exists a nontrivial limit of \( \Theta_{n,z} \), depending this time on the direction in which \( z \) approaches \( k \).

In order to see this, set \( z = k + \varepsilon w \), where \( w \neq 0 \) is a fixed complex number and \( \varepsilon \) is a real number going to 0. Set \( u = \bar{w} / w \). Then one can easily see that

\[
\lim_{\varepsilon \to 0} \Theta_{n,z} = \sum_{\lambda \vdash n} u^{\text{lev}_n(\lambda)} \dim \lambda \cdot \chi^\lambda.
\]

Denote by \( A_{n,k}(u) \) the convolution operator with the latter function. Clearly, for each fixed \( u, |u| = 1 \), the sequence of operators \( (A_{n,k}(u)), n = 1, 2, \ldots \), is consistent with the embeddings \( L^p_k : H^n \to H^{n+1} \) and hence determines an operator \( A_k(u) \) in the space \( H \) of the representation \( T_k \), commuting with this representation.

On the other hand,

\[
A_k(u) |_{H_{pq}} = u^l \cdot 1, \quad l = \min(p, q),
\]

which implies that

\[
A_k(u) = \sum_{p-q=k} u^l A_{pq},
\]

i.e., \( A_k(u) \) can be considered as a generating function of the projectors \( A_{pq} \). One can derive from this fact another proof of Theorem 5.6.1, not relying on Proposition 5.3.1.

6. The invariant vectors

Let us outline the contents of the present Section.

We start with arbitrary \( z \) and describe a convenient realization of the space \( V_z \subset H(T_z) \) formed by \( K \)-invariant vectors. Specifically, we show that \( V_z \) is isomorphic to the space \( \mathcal{F}_z \) of functions on \( Y \) that satisfy two conditions: a harmonicity type condition and a Hardy type condition. In these terms, if \( z \) is an integer, the splitting of the space \( V_z \) induced by the block decomposition \( T_z = \bigoplus_{p-q=z} T_{pq} \) takes especially nice form.

Then we focus on the case when \( z \) is an integer. We prove two main results: Theorems 6.2.1 and 6.2.2. In Theorem 6.2.1 we construct, for any block \( T_{pq} \), a certain \( K \)-invariant vector \( v_{pq} \). In Theorem 6.2.2 we compute the spectral decomposition of the corresponding spherical function: we show that the spectral measure lives on a finite dimensional face \( \Omega(p, q) \) of the simplex \( \Omega \).

Later on in §7 we shall show that, for any couple \( (p, q) \), the vector \( v_{pq} \) is a cyclic vector in \( T_{pq} \). This allows us to completely understand the spectral decomposition of \( T_z \) at the integer points \( z \).

6.1. The space \( \mathcal{F}_z \). Recall that the vectors \( \xi_\lambda \) were introduced in §5.5 (see the proof of Theorem 5.5.1, part a)).

**Proposition 6.1.1.** For any \( n = 1, 2, \ldots \) and any \( \lambda \in \mathcal{Y}_n, \nu \in \mathcal{Y}_{n+1} \),

\[
(\xi_\lambda, \xi_\nu) = \begin{cases} 
(z + c_{\lambda\nu}) / \sqrt{(t+n)(n+1)}, & \text{if } \lambda \nleq \nu \\
0, & \text{otherwise},
\end{cases}
\]

41
where, as before, \( c_{\lambda \nu} \) is the content of the box \( \nu / \lambda \), \( t = |z|^2 \).

**Proof.** The inner product \((\xi_\lambda, \xi_\nu)\) is a continuous function in \( z \), hence it suffices to prove the formula under the assumption \( z \not\in \mathbb{Z} \). In this case \((\xi_0, \xi_\lambda) \neq 0 \) for all \( \lambda \). Indeed, this follows from the equality \((\xi_0, \xi_\lambda) = a_z(\lambda)\) and the explicit expression for \( a_z(\lambda) \), see the proof of Theorem 5.5.1, part a).

Denote by \( Q_\lambda \) the projection operator from the Hilbert space \( H \) of the representation \( T_z \) to the subspace of all vectors that transform, under the action of the subgroup \( G(n) \), according to the representation \( \pi^\lambda \otimes \pi^\lambda \) (that is, the range of \( Q_\lambda \) is the isotypical component of \( \pi^\lambda \otimes \pi^\lambda \) in \( T_z |_{G(n)} \)). Notice that \( P(\lambda) \leq Q_\lambda \) and \( Q_\lambda \xi_\lambda = \xi_\lambda \). It follows from the proof of Theorem 5.5.1, part b), that
\[
(Q_\lambda \xi_\nu, Q_\lambda \xi_\nu) = \begin{cases} \dim \lambda / \dim \nu, & \text{if } \lambda \succ \nu \\ 0, & \text{otherwise.} \end{cases}
\]

In particular, if the condition \( \lambda \succ \nu \) does not hold then \( Q_\lambda \xi_\nu = 0 \) and \((\xi_\lambda, \xi_\nu) = 0\).

Consider the decompositions
\[
\xi_0 = \sum_{\lambda \in \mathbb{Y}_n} (\xi_0, \xi_\lambda) \xi_\lambda = \sum_{\nu \in \mathbb{Y}_{n+1}} (\xi_0, \xi_\nu) \xi_\nu.
\]

Applying the operator \( Q_\lambda \) we derive
\[
Q_\lambda \xi_0 = (\xi_0, \xi_\lambda) \xi_\lambda = \sum_{\nu \succ \lambda} (\xi_0, \xi_\nu) Q_\lambda \xi_\nu.
\]

Fix a diagram \( \nu \) such that \( \nu \succ \lambda \). Taking the inner product with \( \xi_\nu \) we obtain
\[
(\xi_0, \xi_\lambda) (\xi_\lambda, \xi_\nu) = (\xi_0, \xi_\nu) (Q_\lambda \xi_\nu, \xi_\nu).
\]

We have used here the relation \((Q_\lambda \xi_\nu, \xi_\nu) = 0 \) for any \( \nu' \in \mathbb{Y}_{n+1} \setminus \{\nu\} \), which in turn follows from the fact that \( Q_\lambda H(\nu) \subset H(\nu) \) for any \( \nu \in \mathbb{Y}_{n+1} \).

Now remark that
\[
(Q_\lambda \xi_\nu, \xi_\nu) = (Q_\lambda \xi_\nu, Q_\lambda \xi_\nu) = \frac{\dim \lambda}{\dim \nu},
\]
and hence
\[
(\xi_0, \xi_\lambda) (\xi_\lambda, \xi_\nu) = (\xi_0, \xi_\nu) \frac{\dim \lambda}{\dim \nu}.
\]

Since \((\xi_0, \xi_\lambda) \neq 0 \) this implies
\[
(\xi_\lambda, \xi_\nu) = \frac{(\xi_0, \xi_\nu)}{(\xi_0, \xi_\lambda)} \frac{\dim \lambda}{\dim \nu} = \frac{a_z(\nu)}{a_z(\lambda)} \frac{\dim \lambda}{\dim \nu}.
\]

Substituting the explicit expression for \( a_z(\cdot) \) (see the proof of Theorem 5.5.1 part a)) concludes the proof. \( \Box \)
Definition 6.1.2. Denote by $F_z$ the space of complex–valued functions $f(\lambda)$ on the vertices $\lambda \neq \emptyset$ of the Young graph, satisfying the following two conditions.

(i) **Pseudoharmonicity**: for any $\lambda \in \mathbb{Y}_n$, $n = 1, 2, \ldots$,

$$f(\lambda) = \sum_{\nu \downarrow \lambda} f(\nu) (\xi_\nu, \xi_\lambda) = \sum_{\nu \downarrow \lambda} f(\nu) \frac{\bar{z} + c_{\lambda \nu}}{\sqrt{(t + n)(n + 1)}}$$

(we have used here the formula of Proposition 6.1.1).\(^8\)

(ii) **Hardy type condition**: for any $n = 1, 2, \ldots$

$$\|f\|^2 := \sup_n \sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2 < \infty.$$ 

It is worth noting that for any $f$ satisfying the pseudoharmonicity condition, the sum $\sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2$ does not decrease, as $n \to \infty$ (this follows from the proof of Proposition 6.1.3 below). This shows that we could equally well define the norm $\|f\|$ by the formula

$$\|f\|^2 = \lim_{n \to \infty} \sum_{\lambda \in \mathbb{Y}_n} |f(\lambda)|^2.$$ 

Proposition 6.1.3. Let $V_z$ be the subspace of $K$–invariant vectors of the representation $T_z$. Then the map

$$v \mapsto f_v, \quad f_v(\lambda) = (v, \xi_\lambda)$$

provides a linear isomorphism $V_z \to F_z$, preserving the norm.

**Proof.** Recall that by $P_n$ we denote the orthogonal projection from $H = H(T_z)$ onto $H^n$. Denote by $V^n$ the subspace of $K(n)$–invariant vectors in $H^n$. Note that $V^n$ is not contained in $V_z$, but $P_n V_z \subseteq V^n$ and $P_n V^{n+1} \subseteq V^n$. The vectors $\xi_\lambda$, where $\lambda \in \mathbb{Y}_n$, form an orthonormal basis in $V^n$. Given two vectors

$$v_n = \sum_{\lambda \in \mathbb{Y}_n} a(\lambda) \xi_\lambda \in V^n \quad \text{and} \quad v_{n+1} = \sum_{\nu \in \mathbb{Y}_{n+1}} b(\nu) \xi_\nu \in V^{n+1},$$

we have

$$v_n = P_n v_{n+1} \iff a(\lambda) = \sum_{\nu \downarrow \lambda} b(\nu) (\xi_\nu, \xi_\lambda) \quad \forall \lambda \in \mathbb{Y}_n.$$ 

Note that these relations imply that

$$\sum_{\lambda \in \mathbb{Y}_n} |a(\lambda)|^2 = \|v_n\|^2 \leq \|v_{n+1}\|^2 = \sum_{\nu \in \mathbb{Y}_{n+1}} |b(\lambda)|^2.$$ 

Assume now that $v \in V_z$ and $v_n = P_n v$ for $n = 1, 2, \ldots$ Then

$$f_v(\lambda) = (v, \xi_\lambda) = (v_n, \xi_\lambda), \quad n = |\lambda|.$$ 

\(^8\)Cf. the definition of harmonic functions on $\mathbb{Y}$, see the end of §9.3
The above argument shows that the function \( f_v \) satisfies the pseudoharmonicity condition and
\[
\|f_v\|^2 := \sup_n \|v_n\|^2 = \lim_{n \to \infty} \|v_n\|^2 = \|v\|^2 < \infty,
\]
hence \( f_v \in \mathcal{F}_z \). Therefore, the function \( v \mapsto f_v \) provides an isometric embedding \( V_z \to \mathcal{F}_z \).

Let us show now that the map is surjective. Given an arbitrary \( f \in \mathcal{F}_z \), we set
\[
v_n = \sum_{\lambda \in \mathcal{Y}_n} f(\lambda) \xi_\lambda, \quad n = 1, 2, \ldots.
\]
Then \( v_n \in V^n \), \( v_n = P_n v_{n+1} \) and
\[
\lim_{n \to \infty} \|v_n\|^2 = \sup_n \|v_n\|^2 = \|f\|^2.
\]
Therefore, the vectors \( v_n \) converge to a vector \( v \in H(T_z) \). For every \( m \) the vector \( v \) is \( K(m) \)-invariant, because the vectors \( v_n \) possess this property for all \( n \geq m \).
Hence, \( v \in V_z \) and it follows that \( f = f_v \).  

6.2. Two theorems. We assume from this point to the end of §6 that \( z \) is an integer and we write \( z = k \). Recall that in §5 we have introduced invariant subspaces \( H(T_{pq}) \subset H(T_k) \) (with the indices \( p, q \) subject to the condition \( p - q = k \)), called the blocks. Denote by \( V_{pq} = V \cap H(T_{pq}) \) the subspace of all \( K \)-invariant vectors in the block \( H(T_{pq}) \). Let \( \mathcal{F}_{pq} \) be the subspace of \( \mathcal{F}_k \) formed by the functions supported by \( \mathcal{Y}(p, q) \subset \mathcal{Y} \). The decomposition
\[
\mathcal{F}_k = \bigoplus_{p - q = k} \mathcal{F}_{pq}
\]
is parallel to
\[
V_k = \bigoplus_{p - q = k} V_{pq},
\]
and the latter corresponds to the decomposition
\[
H(T_k) = \bigoplus_{p - q = k} H(T_{pq}),
\]
of §5.

Our goal is to produce a certain vector \( v_{pq} \in V_{pq} \), which will be described in terms of the realization \( V_{pq} = \mathcal{F}_{pq} \). The following encoding of a Young diagram \( \lambda \in \mathcal{Y}(p, q) \) will be convenient. Recall that \( \lambda \) belongs to \( \mathcal{Y}(p, q) \) if and only if \( \lambda \) contains the rectangle of shape \( p \times q \) (\( p \) rows and \( q \) columns) but not the box \( (p + 1, q + 1) \). One can represent such a diagram as union of three parts: the \( p \times q \)-rectangle, a diagram \( \lambda^+ \) to its right, and a diagram below the rectangle. The transpose of the latter diagram will be denoted by \( \lambda^- \). Formally:
\[
\lambda^+ = (\lambda_1^+, \ldots, \lambda_p^+), \quad \text{where } \lambda_i^+ = \lambda_i - q, \quad 1 \leq i \leq p;
\]
\[
\lambda^- = (\lambda_1^-, \ldots, \lambda_q^-), \quad \text{where } \lambda_j^- = (\lambda_j') - p, \quad 1 \leq j \leq q.
\]

If \( q = 0 \) then \( \lambda^- = \emptyset \) and \( \lambda = \lambda^+ \). Likewise, if \( p = 0 \) then \( \lambda^+ = \emptyset \) and \( \lambda = (\lambda^-)' \).

If both \( p, q \) are nonzero then the correspondence \( \lambda \leftrightarrow (\lambda^+, \lambda^-) \) establishes a bijection between the sets \( \mathcal{Y}(p, q) \) and \( \mathcal{Y}(p, 0) \times \mathcal{Y}(q, 0) \).

Below we set \( n = |\lambda| \).
Theorem 6.2.1. Let $z = k$ be an integer and let $T_{pq}$ be an arbitrary block of the representation $T_k$ (recall that $p, q$ are nonnegative integers such that $p - q = k$ and $(p, q) \neq (0, 0)$ if $k = 0$).

In the space $H(T_{pq})$ there is a $K$-invariant vector $v_{pq}$, such that the corresponding function $f_{pq}(\lambda) = f_{v_{pq}}(\lambda)$ in the space $F_{pq} \subset F_z$ of functions on the graph $Y(p, q)$ has the following form

$$f_{pq}(\lambda) = (-1)^{|\lambda|} \frac{\sqrt{(p - q)^2 + n - 1)! n!}}{(p^2 + q^2 - pq + n - 1)!} \prod_{1 \leq i < j \leq p} (\lambda_i^+ - \lambda_j^+ + j - i) \prod_{1 \leq r < s \leq q} (\lambda_r^+ - \lambda_s^+ + s - r),$$

where $n = |\lambda|$.

In the next theorem we use the concept of spectral measure; it is explained in §9.7. We also need the finite–dimensional faces $\Omega(p, q) \subset \Omega$, which are defined in §9.6 ($\Omega(p, q)$ is a simplex of dimension $p + q - 1$).

Theorem 6.2.2. Let $v_{pq}$ be the vector defined in Theorem 6.2.1, let

$$\varphi_{pq}(g) = \frac{(T_z(g)v_{pq}, v_{pq})}{\|v_{pq}\|^2} = \frac{(T_{pq}(g)v_{pq}, v_{pq})}{\|v_{pq}\|^2}, \quad g \in G,$$

be the corresponding spherical function on the group $G$, and let $\sigma_{pq}$ be the spectral measure on $\Omega$, corresponding to $\varphi_{pq}$.

The measure $\sigma_{pq}$ is supported by $\Omega(p, q) \subset \Omega$ and has the density

$$\|v_{pq}\|^{-2} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2$$

with respect to Lebesgue measure on $\Omega(p, q)$.

In order to prove these two theorems, we need a few lemmas.

6.3. Preliminary Lemmas.

Lemma 6.3.1. Let $p = 1, 2, \ldots$ be fixed. The function $g_p(\lambda)$ on the graph $Y(p, 0)$, determined by

$$g_p(\lambda) = \prod_{1 \leq i < j \leq p} (\lambda_i - \lambda_j + j - i),$$

satisfies the relation

$$(p^2 + |\lambda|) g_p(\lambda) = \sum_{\nu \backslash \lambda} (p + c_{\lambda\nu}) g_p(\nu).$$

Proof. By setting

$$l_i = \lambda_i + p - i \quad (i = 1, \ldots, p), \quad |\lambda| = l_1 + \ldots + l_p,$$

$$V(l_1, \ldots, l_p) = \prod_{1 \leq i < j \leq p} (l_i - l_j),$$
we transform the relation to the form
\[
\left( p^2 - \frac{p(p-1)}{2} + |l| \right) V(l_1, \ldots, l_p) = \sum_{i=1}^{p} (l_i + 1) V(l_1, \ldots, l_i + 1, \ldots, l_p).
\]
The sum extends over all indices \( i = 1, \ldots, p \), not just those with \( l_{i-1} > l_i + 1 \), because in case of \( l_{i-1} = l_i + 1 \) the term \( V(l_1, \ldots, l_i + 1, \ldots, l_p) \) vanishes.

Let us check the latter relation. The right hand side, being a skew symmetric polynomial in \( l_1, \ldots, l_p \), is divisible by \( V(l_1, \ldots, l_p) \). Since its highest term is \( |l| V(l_1, \ldots, l_p) \), the right hand side has the form
\[
(|l| + \text{const}) V(l_1, \ldots, l_p).
\]
In order to determine the constant, we specialize the identity
\[
\sum_{i=1}^{p} (l_i + 1) V(l_1, \ldots, l_i + 1, \ldots, l_p) = (|l| + \text{const}) V(l_1, \ldots, l_p).
\]
at
\[(l_1, l_2, \ldots, l_p) = (p-1, p-2, \ldots, 0),\]
Then we obtain
\[
p V(p, p-2, p-3, \ldots, 0) = \left( \frac{p(p-1)}{2} + \text{const} \right) V(p-1, p-2, \ldots, 0),
\]
whence
\[
\text{const} = p^2 - \frac{p(p-1)}{2} = \frac{p(p+1)}{2},
\]
and we are done. □

**Remark 6.3.2.** One can suggest another proof of Lemma 6.3.1. It is seemingly more round about, but better explains the origin of the result. The idea is that in case of \( p = k \), \( q = 0 \) we already dispose of an example of a pseudoharmonic function on the Young lattice \( \mathbb{Y}(p, 0) \): the function
\[
\tilde{g}_p(\lambda) := (\xi_0, \xi_\lambda) = \left( \frac{1}{(p^2)^n} \right)^{1/2} \frac{\dim \lambda}{\sqrt{n!}} \prod_{b \in \lambda} (p + c(b)), \quad n = |\lambda|
\]
(cf. the proof of Theorem 5.5.1, part a); in this setup \( t = p^2 \). Since
\[
\frac{\dim \lambda}{n!} = \frac{V(l_1, \ldots, l_p)}{l_1! \ldots l_p!}
\]
and
\[
\prod_{b \in \lambda} (p + c(b)) = \frac{l_1! \ldots l_p!}{(p-1)! \ldots 0!},
\]
we obtain that
\[
\tilde{g}_p(\lambda) = \frac{1}{(p-1)! \ldots 0!} \left( \frac{n!}{(p^2)^n} \right)^{1/2} g_p(\lambda),
\]
and the required relation for \( g_p(\lambda) \) follows from the pseudoharmonicity condition for \( \tilde{g}_p(\lambda) \). □

Using Lemma 6.3.1 and the fact that the graph \( \mathcal{Y}(p, q) \) is isomorphic to the product \( \mathcal{Y}(p, 0) \times \mathcal{Y}(q, 0) \) we will produce a pseudoharmonic function on \( \mathcal{Y}(p, q) \).

Let \( (x_n)_{n \geq pq} \) be a sequence of positive real numbers satisfying the recurrence relation

\[
\frac{x_{n+1}}{x_n} = \frac{n + p^2 + q^2 - pq}{\sqrt{(n + (p - q)^2)(n + 1)}}.
\]

For instance, one can set

\[
x_n = \frac{\Gamma(p^2 + q^2 - pq + n)}{\sqrt{\Gamma((p - q)^2 + n)\Gamma(n + 1)}}.
\] (6-3-1)

Below we use the correspondence \( \lambda \mapsto (\lambda^+, \lambda^-) \) introduced in §6.2.

**Lemma 6.3.3.** Let \( p, q \) be nonnegative integers, \( (p, q) \neq (0, 0) \). The function \( f_{pq}(\lambda) \) on the graph \( \mathcal{Y}(p, q) \) determined by the formula

\[
f_{pq}(\lambda) = \frac{(-1)^{|\lambda^-|}}{x_n} \prod_{1 \leq i < j \leq p} (\lambda_i^+ - \lambda_j^+ + j - i) \prod_{1 \leq r < s \leq p} (\lambda_r^- - \lambda_s^- + s - r),
\]

satisfies the pseudoharmonicity condition with the parameter \( k = p - q \), i.e.,

\[
f_{pq}(\lambda) = \sum_{\nu \succ \lambda} \frac{p - q + c_{\lambda\nu}}{\sqrt{((p - q)^2 + n)(n + 1)}} f_{pq}(\nu)
\]

for any \( \lambda \in \mathcal{Y}(p, q) \cap \mathcal{Y}_n \).

In this formula we assume that \( \nu \) belongs to \( \mathcal{Y}(p, q) \). However, the formula remains true without this assumption, because if \( \lambda \rhd \nu, \lambda \in \mathcal{Y}(p, q) \), and \( \nu \notin \mathcal{Y}(p, q) \) then the factor \( p - q + c_{\lambda\nu} \) vanishes.

**Proof.** Assume first that both \( p \) and \( q \) are positive. In the notation of Lemma 6.3.1,

\[
f_{pq}(\lambda) = \frac{(-1)^{|\lambda^-|}}{x_n} g_p(\lambda^+) g_q(\lambda^-).
\]

Let \( \nu \in \mathcal{Y}(p, q) \) denote a Young diagram with \( n + 1 \) boxes. The condition \( \lambda \rhd \nu \) implies one of the following two conditions:

(i) \( \lambda^+ \rhd \nu^+ \), \( \lambda^- = \nu^- \);

(ii) \( \lambda^+ = \nu^+ \), \( \lambda^- \rhd \nu^- \).

Note that

\[
c_{\lambda\nu} = \begin{cases} 
q + c_{\lambda^+\nu^+}, & \text{in case (i)}, \\
-p - c_{\lambda^-\nu^-}, & \text{in case (ii)}.
\end{cases}
\]

Therefore,

\[
p - q + c_{\lambda\nu} = \begin{cases} 
p + c_{\lambda^+\nu^+}, & \text{in case (i)}, \\
-(q + c_{\lambda^-\nu^-}), & \text{in case (ii)}.
\end{cases}
\]
It follows that
\[
\sum_{\nu \gtrsim \lambda} (p - q + c_{\lambda \nu}) f_{pq}(\nu) =
\]
\[
= \sum_{\nu^+ \gtrsim \lambda^+} \frac{(p + c_{\lambda^+ \nu^+}) g_p(\nu^+) g_q(\lambda^-)}{x_{n+1}}
\]
\[
+ \sum_{\nu^- \gtrsim \lambda^-} \frac{(q + c_{\lambda^- \nu^-}) g_p(\lambda^+) g_q(\nu^-)}{x_{n+1}}.
\]

In the first sum $|\nu^-| = |\lambda^-|$, and in the second one $|\nu^-| = |\lambda^-| + 1$. Hence, the sign in both formulas is $(-1)^{|\lambda^-|}$.

Applying the Lemma 6.3.1, we derive that the last expression equals
\[
\sum_{\nu \gtrsim \lambda} (p - q + c_{\lambda \nu}) f_{pq}(\nu) = \frac{x_n}{x_{n+1}} (p^2 + q^2 + n - pq) f_{pq}(\lambda).
\]

We have shown that
\[
\sum_{\nu \gtrsim \lambda} (p - q + c_{\lambda \nu}) f_{pq}(\nu) = \frac{x_n}{x_{n+1}} (p^2 + q^2 + n - pq) f_{pq}(\lambda).
\]

Dividing both sides by $\sqrt{(n + (p - q)^2)(n + 1)}$ we obtain that the coefficient of $f_{pq}(\lambda)$ in the right–hand side equals
\[
\frac{x_n}{x_{n+1}} \frac{p^2 + q^2 + n - pq}{\sqrt{(n + (p - q)^2)(n + 1)}},
\]
which is 1 by definition of the numbers $x_n$. The lemma follows.

If $q = 0$ or $p = 0$, our relation is simply equivalent to Lemma 6.3.1. □

**Lemma 6.3.4.** The function $f_{pq}(\lambda)$ introduced in Lemma 6.3.3 satisfies the Hardy type condition of Definition 6.1.2. Hence, $f_{pq} \in F_{pq}$.

**Proof.** In order to simplify the notation, set $a = \lambda_1^+, b = \lambda^-_1$, i.e.,
\[
a_1 = \lambda_1^+, \ldots, a_p = \lambda_p^+; \quad b_1 = \lambda_1^-, \ldots, b_q = \lambda_q^-.
\]

When $\lambda$ ranges over $Y(p, q) \cap \mathbb{Y}_n$, the couple $(a, b)$ ranges over the subset in $\mathbb{Z}_+^{p+q}$ determined by the conditions
\[
a_1 \geq \ldots \geq a_p, \quad b_1 \geq \ldots \geq b_q, \quad \sum a_i + \sum b_j = n - pq.
\]

We now have
\[
\sum_{\lambda \in Y(p, q)} |f_{pq}(\lambda)|^2 = \frac{1}{x_n} \sum_{a, b} \prod_{1 \leq i < j \leq p} (a_i - a_j + j - i)^2 \prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)^2.
\]

48
Assuming here $n > pq$, we set

$$\bar{\iota}_n(\lambda) = (\alpha; \beta) = (\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q),$$

where

$$\alpha_1 = \frac{a_1}{n - pq}, \ldots, \alpha_p = \frac{a_p}{n - pq}; \quad \beta_1 = \frac{b_1}{n - pq}, \ldots, \beta_q = \frac{b_q}{n - pq}.$$

The vector $(\alpha; \beta)$ belongs to the lattice $\frac{1}{n - pq} \mathbb{Z}^{p+q}$ and satisfies the conditions

$$\alpha_1 \geq \cdots \geq \alpha_p, \quad \beta_1 \geq \cdots \geq \beta_q, \quad \sum \alpha_i + \sum \beta_j = 1.$$

In terms of $(\alpha; \beta)$, the expression for $\sum |f_{pq}(\lambda)|^2$ can be written as

$$\frac{n^{p(p-1)+q(q-1)}}{x_n^2} \sum_{(\alpha; \beta)} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2 \left(1 + O\left(\frac{1}{n}\right)\right).$$

summed over all $(\alpha; \beta)$ subject to the conditions stated above, that is, over the set

$$\Omega(p, q) \cap \left(\frac{1}{n - pq} \mathbb{Z}^{p+q}\right),$$

where $\Omega(p, q)$ is the simplex introduced in §9.6.

From the well-known formula

$$\frac{\Gamma(n + \text{const})}{\Gamma(n)} \sim n^{\text{const}}, \quad n \to \infty,$$

we get

$$x_n \sim n^{p^2+q^2-pq \frac{(p-q)^2+1}{2}} = n^{\frac{p^2+q^2-1}{2}},$$

hence the coefficient in front of the sum has the asymptotics (as $n \to \infty$)

$$n^{p(p-1)+q(q-1)-(p^2+q^2-1)} = n^{-(p+q-1)}.$$

But $p + q - 1$ is exactly the dimension of $\Omega(p, q)$, hence we have

$$\lim_{n \to \infty} \sum_{\lambda \in Y(p, q) \cap Y_n} |f_{pq}(\lambda)|^2 = \int_{\omega=(\alpha; \beta) \in \Omega(p, q)} \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq p} (\beta_r - \beta_s)^2 d\omega,$$

where $d\omega$ is Lebesgue measure on $\Omega(p, q)$. Since the integral in the right–hand side is finite, we conclude that the Hardy type condition is satisfied. $\square$
6.4. Proof of Theorems 6.2.1 and 6.2.2. The definition of the function \( f_{pq}(\lambda) \) in Theorem 6.2.1 is identical with that given in Lemma 6.3.3. By virtue of Lemmas 6.3.3 and 6.3.4, the function \( f_{pq} \) belongs to \( \mathcal{F}_{pq} \). Therefore, it determines a \( K \)-invariant vector \( v_{pq} \in H(T_{pq}) \), which concludes the proof of the theorem.

Let us turn to Theorem 6.2.2. Compare two probability measures on \( \mathcal{Y}_n \).

The first measure, denoted as \( M^{(n)}_{pq} \), comes from the coherent system corresponding to the spherical function \( \phi_{pq} \) introduced in the statement of Theorem 6.2.2, \( M^{(n)}_{pq}(\lambda) = \|Q_{\lambda}v_{pq}\|_2^2 = \|v_{pq}\|_2^2, \lambda \in \mathcal{Y}_n \).

By the general theory (see Theorem 9.7.3), the spectral measure \( \sigma_{pq} \) is the weak limit of the measures \( \iota_n(M^{(n)}_{pq}) \), where \( \iota_n: \mathcal{Y}_n \to \Omega \) is the embedding defined just before Theorem 9.7.3.

The second measure, which we denote as \( \overline{M}^{(n)}_{pq} \), has the form \( \overline{M}^{(n)}_{pq}(\lambda) = \|Q_{\lambda}v^{(n)}_{pq}\|_2^2 = \frac{f_{pq}(\lambda)}{\|v^{(n)}_{pq}\|_2^2}, \lambda \in \mathcal{Y}_n, \) where \( v^{(n)}_{pq} = P_nv_{pq} \) is the projection of \( v_{pq} \) onto the subspace \( H^n \), and the projection \( Q_{\lambda} \) was defined in §6.1. We know that \( \overline{M}^{(n)}_{pq} \) is concentrated on the subset \( \mathcal{Y}_n(p, q) = \mathcal{Y}(p, q) \cap \mathcal{Y}_n \).

Below we use the standard norm on signed Borel measures: given such a measure \( \mu \), its norm \( \|\mu\| \) is defined as the variance of \( \mu_+ + \mu_- \), where \( \mu = \mu_+ - \mu_- \) stands for the Jordan decomposition of \( \mu \).

Lemma 6.4.1. We have

\[
\|M_{pq}^{(n)} - \overline{M}_{pq}^{(n)}\| \to 0, \quad n \to \infty.
\]

Proof. Indeed, as is well known, for any two probability Borel measures \( \mu_1, \mu_2 \), defined on one and the same Borel space,

\[
\|\mu_1 - \mu_2\| \leq 2 \sup_X |\mu_1(X) - \mu_2(X)|,
\]

where the supremum is taken over arbitrary Borel subsets. Let us apply this inequality to \( \mu_1 = M_{pq}^{(n)}, \mu_2 = \overline{M}_{pq}^{(n)} \). To simplify the notation, set

\[
\xi_1 = \|v_{pq}\|^{-1} v_{pq}, \quad \xi_2 = \|v^{(n)}_{pq}\|^{-1} v^{(n)}_{pq}
\]

and, for any subset \( X \subset \mathcal{Y}_n \), set

\[
Q_X = \sum_{\lambda \in X} Q_{\lambda}.
\]

The operators \( Q_{\lambda} \), with \( \lambda \) ranging over \( \mathcal{Y}_n \), are pairwise orthogonal projectors whose sum equal 1, whence \( \|Q_X\| \leq 1 \) for any \( X \). It follows

\[
\|M_{pq}^{(n)} - \overline{M}_{pq}^{(n)}\| \leq 2 \sup_{X < \mathcal{Y}_n} |(Q_X \xi_1, \xi_1) - (Q_X \xi_2, \xi_2)| \leq 4\|\xi_1 - \xi_2\|.
\]

50
But $\|\xi_1 - \xi_2\| \to 0$ as $n \to \infty$, because $\|v_{pq} - v^{(n)}_{pq}\| \to 0$. This completes the argument. □

On the other hand, it follows from the proof of Lemma 6.3.4 that the measures $\tilde{\iota}_n(M^{(n)}_{pq})$ on $\Omega(p,q)$ weakly converge to a probability measure, which is absolutely continuous with respect to Lebesgue measure on $\Omega(p,q)$ and whose density is exactly as required in the statement of the theorem.

**Lemma 6.4.2.** The measures $\iota_n(M^{(n)}_{pq})$ on $\Omega$ weakly converge, as $n \to \infty$, to the same limit measure concentrated on $\Omega(p,q)$.

**Proof.** Indeed, let us compare the two embeddings, $\tilde{\iota}_n : Y_n(p,q) \to \Omega(p,q) \subset \Omega$ and $\iota_n : Y_n(p,q) \to \Omega$.

For any $\lambda \in Y_n(p,q)$, we can write (viewing $\tilde{\iota}_n(\lambda)$ as an element of $\Omega$)

\[ \tilde{\iota}_n(\lambda) = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots), \]

where $\alpha_{p+1} = \alpha_{p+2} = \cdots = \beta_{q+1} = \beta_{q+2} = \cdots = 0$, and similarly

\[ \iota_n(\lambda) = (\alpha_1, \alpha_2, \ldots; \beta_1, \beta_2, \ldots), \]

where $\alpha_{d+1} = \alpha_{d+2} = \cdots = \beta_{d+1} = \beta_{d+2} = \cdots = 0$, $d = \max(p,q)$.

Further, from the definition of $\tilde{\iota}_n(\lambda)$ and $\iota_n(\lambda)$ it follows that for any fixed $i, j$,

\[ |\tilde{\alpha}_i - \alpha_i| \leq \frac{\text{const}}{n}, \quad |\tilde{\beta}_j - \beta_j| \leq \frac{\text{const}}{n} \]

where the constant does not depend on $\lambda$. Consequently, if $F$ is an arbitrary bounded continuous function on $\mathbb{R}^\infty \times \mathbb{R}^\infty$ depending on finitely many coordinates only, then the result of coupling between $f$ and $\tilde{\iota}_n(M^{(n)}_{pq}) - \iota_n(M^{(n)}_{pq})$ is $O(1/n)$. Since $\Omega$ is a compact subset of $\mathbb{R}^\infty \times \mathbb{R}^\infty$, the same is true for any continuous test function $F$ on $\Omega$, which completes the proof. □

By Lemma 6.4.1,

\[ \|\tilde{\iota}_n(M^{(n)}_{pq}) - \iota_n(M^{(n)}_{pq})\| \to 0, \quad n \to \infty, \]

so that the measures $\iota_n(M^{(n)}_{pq})$ must have the same limit as the measures $\iota_n(M^{(n)}_{pq})$. Together with Lemma 6.4.2 this completes the proof of Theorem 6.2.2.

**7. The commutant of representation $T_{pq}$**

**7.1. The results.** Recall that if $z$ is an integer, $z = k \in \mathbb{Z}$, then the representation $T_z = T_k$ splits into a direct sum of subrepresentations called the blocks,

\[ T_k \sim \bigoplus_{p-q=k} T_{pq} \]

where $p, q$ are assumed to be nonnegative integers, and $(p,q) \neq (0,0)$ if $k = 0$ (see §5.7).

The main goal of this Section is to prove the following result.
Theorem 7.1.1. Let $z = k$ be an integer and let $T_{pq}$ be an arbitrary block of the representation $T_k$. Then the vector $v_{pq} \in H(T_{pq})$ constructed in §6 is a cyclic vector of $T_{pq}$.

Along with the Theorem 6.2.2 this implies our main result on representations $T_k$ at integer points $z = k$:

Theorem 7.1.2. The block $T_{pq} \subset T_z$, where $z = k$ is an integer and $p - q = k$, is equivalent to the direct integral of irreducible spheric representations labelled by the points $\omega$ of the finite dimensional face $\Omega(p, q)$ of the Thoma simplex, with respect to Lebesgue measure $d\omega$.

Corollary 7.1.3. The representations $T_{pq}$, where $p, q \in \mathbb{Z}$, are pairwise disjoint. In particular, the representations $T_k$, $k \in \mathbb{Z}$, are pairwise disjoint.

7.2. The commutant $A_{pq}$. Recall that the representation $T_{pq}$ is an inductive limit of finite dimensional representations of the groups $G(n)$ in the spaces $H^n \cap H(T_{pq}) \subset \text{Reg}^n$. Since

$$(v_{pq}, \xi_\lambda) = f_{pq}(\lambda) \neq 0 \quad \text{for } \lambda \in \mathbb{Y}(p, q),$$

the projection $v_{pq}^{(n)}$ of the vector $v_{pq}$ onto subspace $H^n \cap H(T_{pq})$ is a cyclic vector, for every $n$. If one of the numbers $p, q$ vanishes, we have $v_{pq}^{(n)} = v_{pq}$ and hence $v_{pq}$ is obviously a cyclic vector. But if $v_{pq}^{(n)} \neq v_{pq}$, the fact that each vector $v_{pq}^{(n)}$ is cyclic in the corresponding representation of the subgroup $G(n)$ does not formally imply that the limiting vector is cyclic, too. See subsection 7.8. for a counterexample.

In order to prove that the vector $v_{pq}$ is cyclic we study the commutant of $T_{pq}$, making use of Proposition 5.3.1 and Theorem 5.5.1. We shall show that the matrix element $(\cdot, v_{pq})$ provides a faithful state on the commutant, which implies the cyclicity immediately. Unfortunately, the proof turns out to be rather long.

Fix a block $T_{pq} \subset T_k$, where $k = p - q$, and assume that $p \geq 1, q \geq 1$ (if one of the numbers $p, q$ vanishes, the claim of the Theorem 7.1.1 is trivial).

Set $\mathbb{Y}_n(p, q) = \mathbb{Y}_n \cap \mathbb{Y}(p, q)$; this is the $n$th level of the graph $\mathbb{Y}(p, q)$. It is not empty, starting with $n = pq > 0$. The graph $\mathbb{Y}(p, q)$ can be identified with the direct product of the graphs $\mathbb{Y}(p, 0)$ and $\mathbb{Y}(0, q)$.

Denote by $A_{pq}$ the commutant of the representation $T_{pq}$. According to Proposition 5.3.1, there is a linear isomorphism $A_{pq} \to \tilde{\mathcal{A}}_{pq}$, where $\tilde{\mathcal{A}}_{pq}$ is the Banach space of complex-valued bounded functions $A(\lambda)$ on $\mathbb{Y}(p, q)$, satisfying the condition

$$A(\lambda) = \sum_{\nu \leq \lambda} \tilde{p}_z(\lambda, \nu) A(\nu), \quad \lambda \in \mathbb{Y}(p, q),$$

with the norm

$$\|A\| = \sup_{\lambda} |A(\lambda)|.$$

According to Theorem 5.5.1, the function $\tilde{p}_z(\lambda, \nu)$ has the form

$$\tilde{p}_z(\lambda, \nu) = p_z(\lambda, \nu) = \frac{|z + c_{\lambda, \nu}|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n + 1) \dim \lambda}.$$

It will be important for us that $p_z(\lambda, \nu) \neq 0$ for all couples $\lambda \not\succ \nu$ in the graph $\mathbb{Y}(p, q)$. 

52
Denote by $\mathcal{A}_{pq}^+$ the cone of nonnegative Hermitian operators in $\mathcal{A}_{pq}$. Its image with respect to the isomorphism $\mathcal{A}_{pq} \to \tilde{\mathcal{A}}_{pq}$ is contained in the cone $\tilde{\mathcal{A}}_{pq}^+ \subset \tilde{\mathcal{A}}_{pq}$ generated by the functions $A \in \tilde{\mathcal{A}}_{pq}$ with nonnegative values. One can show that the image actually coincides with $\tilde{\mathcal{A}}_{pq}^+$, but we do not need this fact.

7.3. A faithful state on the algebra $\mathcal{A}_{pq}$. Denote by $\lambda_{\min}$ the rectangular diagram of size $p \times q$; this is the only vertex of the graph $Y(p, q)$ at the level $pq$.

Lemma 7.3.1. The linear functional
\[ \eta : A \mapsto A(\lambda_{\min}), \quad A \in \tilde{\mathcal{A}}_{pq} \]
provides a faithful trace on the algebra $\mathcal{A}_{pq}$.

Proof. Clearly, the functional $\eta$ equals 1 at the function $A(\lambda) \equiv 1$ (which determines the identity of $\mathcal{A}_{pq}$), has norm 1 and is nonnegative on the cone $\mathcal{A}_{pq}^+$. Hence, it defines a state on the algebra $\mathcal{A}_{pq}$. It remains to prove that this state is faithful.

To do this we shall show that $\eta(A) > 0$ for every non-zero function $A(\lambda)$ in $\tilde{\mathcal{A}}_{pq}^+$. Define a “weight function” $\mathcal{M}_{pq}(\lambda)$ on the graph $Y(p, q)$ by the recurrence relation
\[ \mathcal{M}_{pq}^{(n)}(\nu) = \sum_{\lambda, \lambda \not\rightarrow \nu} \mathcal{M}_{pq}(\lambda) p_z(\lambda, \nu), \quad |\nu| > pq \]
and the initial condition
\[ \mathcal{M}_{pq}(\lambda_{\min}) = 1. \]

Denote by $\mathcal{M}^{(n)}_{pq}$ the measure on $Y_n(p, q)$, $n \geq pq$, with the weights $\mathcal{M}_{pq}(\lambda)$ at the vertices $\lambda \in Y_n(p, q)$. Since $p_z(\lambda, \nu)$ has the property
\[ \sum_{\nu \not\rightarrow \lambda} p_z(\lambda, \nu) = 1, \]
all $\mathcal{M}^{(n)}_{pq}$ are probability measures.

By virtue of the main relation for the functions $A(\cdot) \in \tilde{\mathcal{A}}_{pq}$, the expression
\[ \eta_n(A) = \langle \mathcal{M}^{(n)}_{pq}, A \rangle = \sum_{\lambda \in Y_n(p, q)} \mathcal{M}_{pq}(\lambda) A(\lambda), \quad n \geq pq, \]
does not depend on $n$. Hence, it coincides with $\eta_{pq}(A) = \eta(A)$.

Since $p_z(\lambda, \nu) > 0$, all the weights $\mathcal{M}_{pq}(\lambda)$ are strictly positive. Let now $A(\cdot)$ be a non-zero function from $\tilde{\mathcal{A}}_{pq}$. Then there exists $\lambda \in Y(p, q)$, such that $A(\lambda) > 0$. If $n = |\lambda|$, then $\langle \mathcal{M}^{(n)}_{pq}, A \rangle > 0$, hence we conclude that $\eta(A) = \eta_n(A) > 0$. □

7.4. Evaluation of $\mathcal{M}_{pq}$. As in §6 above, we associate with a diagram $\lambda \in Y(p, q)$ a couple of diagrams $(\lambda^+, \lambda^-)$, and in order to simplify the notation we set $a_i = \lambda^+_i$, $b_r = \lambda^-_r$ for $1 \leq i \leq p$, $1 \leq r \leq q$.  

53
Lemma 7.4.1. The “weight function” $\mathcal{M}_{pq}(\lambda)$ introduced in the proof of Lemma 7.2.1 can be given by the explicit formula

$$\mathcal{M}_{pq}(\lambda) = \frac{C(p, q)}{(n - pq + 1)p^2 + q^2 - pq - 1} \times \prod_{1 \leq i < j \leq p} (a_i - a_j + j - i)^2 \prod_{1 \leq r < s \leq q} (b_r - b_s + s - r)^2 \prod_{i, r} (a_i + b_r + p - i + q - r + 1),$$

where $n = |\lambda|$ and

$$C(p, q) = \frac{(p^2 + q^2 - pq - 1)! \prod_i (p - i + q - r + 1)}{(\prod_i (p - i)! \prod_r (q - r)!)^2}.$$

Proof. Recall that $\mathcal{M}_{pq}(\lambda)$ satisfies the recurrence relation

$$\mathcal{M}_{pq}(\nu) = \sum_{\lambda \vdash \lambda \nu} \frac{(p - q + c_\lambda)^2 \dim \lambda}{(p - q)^2 + n \cdot (n + 1) \dim \lambda} \mathcal{M}_{pq}(\lambda),$$

where $|\nu| = n + 1 > pq$. Set in this formula

$$\mathcal{M}_{pq}(\lambda) = \prod_{b \in \lambda \setminus \lambda_{\min}} \frac{(p - q + c(b))^2 \dim \lambda}{(p - q)^2 + n - 1)!} \mathcal{M}'_{pq}(\lambda),$$

with a new unknown function $\mathcal{M}'_{pq}(\lambda)$, $n = |\lambda|$. Then for $\mathcal{M}'_{pq}(\lambda)$ we get the recurrence relation

$$\mathcal{M}'_{pq}(\nu) = \sum_{\lambda \vdash \lambda \nu} \mathcal{M}'_{pq}(\lambda).$$

Its solution has the form

$$\mathcal{M}'_{pq}(\lambda) = \text{const} \cdot \text{Dim} \lambda,$$

where Dim $\lambda$ stands for the number of paths, in the graph $\mathcal{Y}(p, q)$, going from $\lambda_{\min}$ to $\lambda$ (the dimension function of the graph $\mathcal{Y}(p, q)$).

Since the graph $\mathcal{Y}(p, q)$ is isomorphic to the product of the graphs $\mathcal{Y}(p, 0) \times \mathcal{Y}(q, 0)$ (with the empty diagrams $\emptyset$ added to $\mathcal{Y}(p, 0)$ and to $\mathcal{Y}(0, q)$), we get

$$\text{Dim} \lambda = \frac{(n - pq)!}{|\lambda^+||\lambda^-|!} \dim \lambda^+ \dim \lambda^-.$$

It follows that

$$\mathcal{M}_{pq}(\lambda) = \text{const} \prod_{b \in \lambda \setminus \lambda_{\min}} \frac{(p - q + c(b))^2 \dim \lambda}{(p - q)^2 + n - 1)!} \frac{\dim \lambda^+ \dim \lambda^-}{|\lambda^+||\lambda^-|!} (n - pq)!,$$

Substitute here the explicit expressions:

$$\prod_{b \in \lambda \setminus \lambda_{\min}} (p - q + c(b))^2 = \left( \prod_{i=1}^{p} \frac{(a_i + p - i)!}{(p - i)!} \frac{q}{r=1} \frac{(b_r + q - r)!}{(q - r)!} \right)^2.$$
Lemma 7.5.1. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that

\[
\sum_{\lambda \in \mathcal{Y}_n(p,q) : a_p + b_q \leq \delta n} \mathcal{M}_{pq}(\lambda) \leq \varepsilon,
\]

for all \( n \) large enough.

Proof will be given in §7.7.

Lemma 7.5.2. The function

\[
\frac{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2}{\prod_{i,r}(\alpha_i + \beta_r)}
\]

on the simplex \( \Omega(p,q) \) is integrable with respect to Lebesgue measure \( d\omega \).

Proof will be given in §7.8.

Consider now the embedding \( \iota_n : \mathcal{Y}_n(p,q) \rightarrow \Omega(p,q) \) introduced in §6.3. Let \( \widehat{\mathcal{M}}_{pq}^{(n)} = \iota_n(\mathcal{M}_{pq}^{(n)}) \) be the push–forward of the probability measure \( \mathcal{M}_{pq}^{(n)} \); this is a probability measure on \( \Omega(p,q) \).

Lemma 7.5.3. As \( n \rightarrow \infty \), the measures \( \widehat{\mathcal{M}}_{pq}^{(n)} \) on \( \Omega(p,q) \) weakly converge to a certain probability measure \( \widehat{\mathcal{M}}_{pq}^{(\infty)} \). The measure \( \widehat{\mathcal{M}}_{pq}^{(\infty)} \) is absolutely continuous with respect to Lebesgue measure \( d\omega \) on \( \Omega(p,q) \), and has the density

\[
C(p,q) \frac{\prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j)^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s)^2}{\prod_{i,r}(\alpha_i + \beta_r)},
\]

where the constant is the same as in Lemma 7.4.1.
Proof. Let $\lambda \in \mathcal{Y}_n(p,q)$ and $\alpha;\beta = \bar{\iota}_n(\lambda)$. It follows from the expression for $M_{pq}(\lambda)$ in Lemma 7.4.1 that

$$
M_{pq}(\lambda) = C(p,q) \cdot n^{-(p+q-1)} \cdot (1 + O(1/n)) \times \prod_{1 \leq i < j \leq p} (\alpha_i - \alpha_j + O(1/n))^2 \prod_{1 \leq r < s \leq q} (\beta_r - \beta_s + O(1/n))^2 \times \prod_{i,r}(\alpha_i + \beta_r + O(1/n)).
$$

Given $\delta > 0$, we split the simplex $\Omega(p,q)$ into the union of two subsets,

$$
\Omega(p,q) = \Omega_{\geq \delta}(p,q) \cup \Omega_{< \delta}(p,q),
$$

determined by the conditions $\alpha_p + \beta_q \geq \delta$ and $\alpha_p + \beta_q < \delta$, respectively. On the set $\Omega_{\geq \delta}(p,q)$ the denominator of the expression for $M_{pq}(\lambda)$ is bounded from below, uniformly in $n$. This implies the weak convergence of measures

$$
\lim_{n \to \infty} \left. \tilde{M}_{pq}^{(n)} \right|_{\Omega_{\geq \delta}(p,q)} \to \left. \tilde{M}_{pq}^{(\infty)} \right|_{\Omega_{\geq \delta}(p,q)}.
$$

By Lemma 7.5.1, the total mass of the set $\Omega_{< \delta}(p,q)$ with respect to $\tilde{M}_{pq}^{(n)}$ can be made arbitrarily small (uniformly in $n$), provided that $\delta$ is chosen sufficiently small, and $n$ is large enough. By Lemma 7.5.2, the mass of the set $\Omega_{< \delta}(p,q)$ with respect to $\tilde{M}_{pq}^{(\infty)}$ also tends to 0, together with $\delta$. This implies that $\tilde{M}_{pq}^{(n)}$ weakly converges to $\tilde{M}_{pq}^{(\infty)}$ on the entire simplex $\Omega(p,q)$. □

Lemma 7.5.4. The state

$$
A \mapsto \frac{(Av_{pq}, v_{pq})}{(v_{pq}, v_{pq})}
$$

on the algebra $A_{pq}$ is faithful.

Proof. We have to prove that if $A \in A_{pq}^+, A \neq 0$, then $(Av_{pq}, v_{pq}) > 0$. Set

$$
A^{(n)} = P_n AP_n, \quad v_{pq}^{(n)} = P_n v_{pq}.
$$

If $n \to \infty$, the operator $P_n AP_n$ converges to $A$ strongly, hence

$$
(Av_{pq}, v_{pq}) = \lim_{n \to \infty} (P_n AP_n v_{pq}, v_{pq}) = \lim_{n \to \infty} (A^{(n)} v_{pq}^{(n)}, v_{pq}^{(n)}).
$$

According to Proposition 5.3.1,

$$
A^{(n)} = \sum_{\lambda \in \mathcal{Y}_n(p,q)} A(\lambda) P(\lambda),
$$

where $A(\lambda)$ is the function in $\tilde{A}_{pq}^+$, corresponding to the operator $A$. On the other hand,

$$
v_{pq}^{(n)} = \sum_{\lambda \in \mathcal{Y}_n(p,q)} f_{pq}(\lambda) \xi_{\lambda}
$$
by definition of the vector $v_{pq}$. Therefore,

$$\langle A^{(n)} v_{pq}^{(n)}, v_{pq}^{(n)} \rangle = \sum_{\lambda \in \mathcal{Y}_n(p,q)} |f_{pq}(\lambda)|^2 A(\lambda) = \sum_{\lambda \in \mathcal{Y}_n(p,q)} A(\lambda) \mathfrak{M}_{pq}(\lambda) \frac{|f_{pq}(\lambda)|^2}{M_{pq}(\lambda)}. $$

We have explicit expressions for both $f_{pq}(\lambda)$ and $\mathfrak{M}_{pq}(\lambda)$, see Theorem 6.2.1 and Lemma 7.4.1. These expressions imply that

$$|f_{pq}(\lambda)|^2 \frac{1}{M_{pq}(\lambda)} = c_{pq} \prod_{i,r} (a_i + b_r + p - i + q - r + 1),$$

where

$$c_{pq} = \frac{n - pq + 1}{x_n^2 \cdot C(p, q)},$$

(we have employed here an equivalent expression for $f_{pq}(\lambda)$, see Lemma 6.3.3). Since

$$x_n \sim n^{(p^2+q^2-1)/2},$$

we have

$$c_{pq} \sim \frac{1}{C(p, q) n^{pq}},$$

whence

$$|f_{pq}(\lambda)|^2 \frac{1}{M_{pq}(\lambda)} = \left( \text{const} + o(1) \right) \prod_{i,r} \frac{a_i + b_r + p - i + q - r + 1}{n} = \text{const} \prod_{i,r} (\alpha_i + \beta_r) + o(1),$$

where we assume that $(\alpha; \beta) = \tilde{i}_n(\lambda)$.

Let $\mathfrak{N}_{pq}^{(n)}$ denote the measure on $\mathcal{Y}_n(p,q)$ defined by

$$\mathfrak{N}_{pq}^{(n)}(\lambda) = A(\lambda) \mathfrak{M}_{pq}(\lambda), \quad \lambda \in \mathcal{Y}_n(p,q).$$

In the notation introduced in the proof of Lemma 7.3.1,

$$\sum_{\lambda \in \mathcal{Y}_n(p,q)} A(\lambda) \mathfrak{M}_{pq}(\lambda) = \eta_n(A).$$

We have noted there that $\eta_n(A)$ does not depend on $n$, and is strictly positive. Therefore, the mass of the set $\mathcal{Y}_n(p,q)$ with respect to the measure $\mathfrak{N}_{pq}^{(n)}$ is strictly positive and does not depend on $n$. Consequently, passing to an appropriate subsequence of indices $n_1 < n_2 < \ldots$, we may conclude that the measures $\mathfrak{N}_{pq}^{(\infty)} := \tilde{i}_n(\mathfrak{N}_{pq}^{(n)})$ converge weakly on $\Omega(p,q)$ to a certain nonzero measure $\mathfrak{N}_{pq}^{(\infty)}$.

On the other hand, for the weights of the measures $\mathfrak{N}_{pq}^{(n)}$ and $\mathfrak{M}_{pq}^{(n)}$ there is an estimate

$$\mathfrak{N}_{pq}^{(n)}(\lambda) = A(\lambda) \frac{\mathfrak{M}_{pq}^{(n)}(\lambda)}{\mathfrak{M}_{pq}(\lambda)} \leq \|A\| \mathfrak{M}_{pq}^{(n)}(\lambda),$$
hence
\[ \hat{\mathcal{M}}_{pq}^{(n)} \leq \| A \| \hat{\mathcal{M}}_{pq}^{(\infty)}. \]

By Lemma 7.5.3, the measures \( \hat{\mathcal{M}}_{pq}^{(n)} \) converge weakly to a measure \( \hat{\mathcal{M}}_{pq}^{(\infty)} \) which is absolutely continuous with respect to the Lebesgue measure \( d\omega \) on the simplex \( \Omega(p, q) \). It follows that
\[ \hat{\mathcal{N}}_{pq}^{(\infty)} \leq \| A \| \hat{\mathcal{M}}_{pq}^{(\infty)}, \]
which implies that \( \hat{\mathcal{N}}_{pq}^{(\infty)} \) is absolutely continuous, too.

Return now to the quantity \( (A_{pq}, v_{pq}) \). We have shown that it can be represented as a limit
\[ (A_{pq}, v_{pq}) = \lim_{n \to \infty} (A_{pq}^{(n)}, v_{pq}^{(n)}) = \lim_{n \to \infty} \text{const} \left\langle \hat{\mathcal{N}}_{pq}^{(n)}, \left( \prod_{i,r} (\alpha_i + \beta_r) + o(1) \right) \right\rangle, \]
where \( \text{const} > 0 \). Hence,
\[ (A_{pq}, v_{pq}) = \text{const} \left\langle \hat{\mathcal{N}}_{pq}^{(\infty)}, \prod_{i,r} (\alpha_i + \beta_r) \right\rangle. \]

Since the measure \( \hat{\mathcal{N}}_{pq}^{(\infty)} \) is absolutely continuous and nonzero, and the function \( \prod (\alpha_i + \beta_r) \) on the simplex \( \Omega(p, q) \) is continuous and positive almost everywhere, the result is a strictly positive number. \( \square \)

7.6. The Cauchy determinant. We shall need a generalization of the classical Cauchy identity
\[ \det \begin{bmatrix} 1 \\ x_i + y_j \end{bmatrix}_{i,j=1}^{m} = \frac{V(x) V(y)}{\prod_{i,j} (x_i + y_j)}, \]
where
\[ V(x) = \prod_{1 \leq i < j \leq m} (x_i - x_j), \]
to the case when the numbers of \( x_i \)'s and \( y_j \)'s are not necessarily equal.

Set
\[ x = (x_1, \ldots, x_p), \quad y = (y_1, \ldots, y_q), \]
and assume (to be definite) that \( p \leq q \). We shall denote by the symbol \( y = y' \sqcup y'' \) an arbitrary decomposition of the set of variables \( y = (y_1, \ldots, y_q) \) into disjoint subsets of cardinalities \( q - p \) and \( p \) respectively:
\[ y' = (y_1', \ldots, y_{q-p}'), \quad y'' = (y_1'', \ldots, y_p''), \]
where
\[ i_1 < \ldots < i_{q-p}, \quad j_1 < \ldots < j_p, \quad \{i_1, \ldots, i_{q-p}\} \cup \{j_1, \ldots, j_p\} = \{1, \ldots, q\}. \]

Set \( \text{sgn}(y', y'') = \pm 1 \), where the sign plus/minus is taken depending on parity/imparity of the number of inversions in the permutation \( (i_1, \ldots, i_{q-p}, j_1, \ldots, j_p) \) of the numbers \( 1, \ldots, q \).
Lemma 7.6.5. The following formula generalizes the Cauchy determinant:
\[
\frac{V(x_1, \ldots, x_p) V(y_1, \ldots, y_q)}{\prod_{i,j} (x_i + y_j)} = \sum_{y'\cup y''=y} \text{sgn}(y', y'') V(y'_1, \ldots, y'_{q-p}) \det \left[ \frac{1}{x_i + y_j'} \right]_{i,j=1}^p.
\]

Proof. One can easily derive this identity from the classical Cauchy identity by replacing the variables \(x_1, \ldots, x_p\) with \(x_1', \ldots, x_p'\), adding a new group of variables \(x_1', \ldots, x_q'_{p-1}\), writing down the Cauchy identity in terms of the variables \((x_1', \ldots, \varepsilon x_1'_{q-p}; x_1'', \ldots, x_p'')\) and \(y_1, \ldots, y_q\), and then applying the Laplace rule while \(\varepsilon\) goes to 0. □

Another proof. The argument given below is similar to the well–known proof of the classical Cauchy identity.

Denote the right–hand side of the identity in question by \(A(x, y)\). Then the product \(A(x, y)\Pi(x_i + y_j)\) is a polynomial. It suffices to check the following three claims:

(i) \(A(x, y)\) is skew symmetric, separately in \(x\) and in \(y\).
(ii) \(\deg A(x, y) = \deg V(x) + \deg V(y) - \deg \Pi(x_i + y_j)\).

(iii) Let us order the variables as \(x_1 > \cdots > x_p > y_1 > \cdots > y_q\) and consider the corresponding order on the monomials; then the leading term in the expansion of \(A(x, y)\Pi(x_i + y_j)\) is the same as that for \(V(x) V(y)\).

Now we have:

(i) The skew symmetry with respect to \(x\) is evident. Let us show that \(A(x, y)\) changes the sign upon the elementary transposition \(y_j \leftrightarrow y_{j+1}\), where \(j = 1, \ldots, q-1\). If the variables \(y_j, y_{j+1}\) enter the same group, \(y'\) or \(y''\), then the corresponding term is already skew symmetric. Consider now the terms for which \(y_j\) and \(y_{j+1}\) belong to different groups. Those terms split into pairs: in each pair the terms are switched by the transposition \(y_j \leftrightarrow y_{j+1}\) and the corresponding signs are opposite. Thus, the skew symmetry follows.

(ii) This is trivial.

(iii) It is readily verified that the leading monomial comes from the partition \(y' = (y_1, \ldots, y_{q-p}), y'' = (y_{q-p+1}, \ldots, y_q)\). □

7.7. Proof of Lemma 7.5.1. Without loss of generality we may assume that \(p \leq q\). Set
\[
x_i = a_i + p - i, \quad 1 \leq i \leq p; \quad y_j = b_j + q - j, \quad 1 \leq j \leq q.
\]

Taking into account the explicit formula for \(\mathfrak{M}_{pq}(\lambda)\) (Lemma 7.4.1) we have to prove the estimate
\[
\frac{1}{n^{pq} q^{q^2-1}} \sum_{x,y} V(x)^2 V(y) = O(\delta),
\]
summed over the integer vectors \(x \in \mathbb{Z}_+^p, y \in \mathbb{Z}_+^q\) satisfying the conditions
\[
x_1 \geq \cdots \geq x_p \geq 0, \quad y_1 \geq \cdots \geq y_q \geq 0, \quad x_p + y_q \leq \delta n
\]
\[
x_1 + \cdots + x_p + y_1 + \cdots + y_q = n + \text{const}, \quad (7-6-1)
\]
where “const” is an integer not depending on \( n \).

Apply the identity of Lemma 7.6.1. Since all variables are of order \( O(n) \), we have

\[
V(x) V(y) V(y') = O \left( n^{\frac{1}{2}(p(p-1)+q(q-1)+(q-p)(q-p-1))} \right) = O \left( n^{p^2-pq+q^2-q} \right).
\]

Using this and expanding the determinant in the right hand side of the identity in question, we reduce the problem to the following bound:

\[
\frac{1}{n^{q-1}} \sum_{x,y} \frac{1}{(x_1 + y''_{\sigma(1)} + 1) \ldots (x_p + y''_{\sigma(p)} + 1)} = O(\delta),
\]

where a splitting of variables \( y' \sqcup y'' = y \) is fixed, \( \sigma \) is a fixed permutation of the indices \( 1, \ldots, p \), and summation is again taken under the conditions (7-6-1).

There are three possibilities:

(i) \( y'' \) contains \( y_q \), and \( y''_{\sigma(p)} = y_q \);
(ii) \( y'' \) contains \( y_q \), but \( y''_{\sigma(p)} \neq y_q \);
(iii) \( y'' \) does not contain \( y_q \).

The case (iii) reduces to that of (i), since replacing \( y''_{\sigma(p)} \) with \( y_q < y''_{\sigma(p)} \) only increases the sum.

The case (ii) can also be reduced to (i). Indeed, there exists \( i < p \) such that \( y''_{\sigma(i)} = y_q \). We can now change \( \sigma \) by switching \( y''_{\sigma(i)} \) and \( y''_{\sigma(p)} \). This can be done by virtue of the inequality

\[
\frac{1}{(A_1 + B_2)(A_2 + B_1)} < \frac{1}{(A_1 + B_1)(A_2 + B_2)},
\]

(correct for \( A_1 > A_2 > 0, B_1 > B_2 > 0 \)) which we apply to

\[
A_1 = x_i + \frac{1}{2}, \quad A_2 = x_p + \frac{1}{2}, \quad B_1 = y''_{\sigma(p)} + \frac{1}{2}, \quad B_2 = y''_{\sigma(i)} + \frac{1}{2}.
\]

Hence, we are left with the case (i), where \( y''_{\sigma(p)} = y_q \). Let us relax the system of restrictions (7-6-1) by removing from it the inequalities \( x_1 > \ldots > x_p, y_1 > \ldots > y_q \) (but we still assume that all variables are nonnegative integers). This will result in a bigger amount of arrays \( x, y \), hence the sum will increase, too.

Set

\[
w_1 = x_1 + y''_{\sigma(1)}, \quad \ldots, \quad w_{p-1} = x_{p-1} + y''_{\sigma(p-1)}, \quad w_p = x_p + y''_{\sigma(p)} = x_p + y_p.
\]

Note that for every \( w_i \) there are precisely \( w_i + 1 \) ways to represent it as a sum of two nonnegative terms. Therefore, our bound reduces to the following one: the number of vectors \((y'_1, \ldots, y'_{q-p}, w_1, \ldots, w_p) \in \mathbb{Z}_+^q \) such that

\[
y'_1 + \cdots + y'_{q-p} + w_1 + \cdots + w_p = n + \text{const}, \quad w_p \leq \delta n
\]

is of order \( O(\delta)n^{q-1} \). This bound is readily verified.
7.8. Proof of Lemma 7.5.2. We shall prove the analogous claim for the same density on the larger simplex $\Omega(p, q)$, obtained by removing the restrictions $\alpha_1 \geq \ldots \geq \alpha_p$, $\beta_1 \geq \ldots \geq \beta_q$. In other words, $\Omega(p, q)$ is just the standard $(p + q - 1)$-dimensional simplex. Without loss of generality we may assume that $p \leq q$. By the Cauchy type identity of Lemma 7.6.5 and taking into account the trivial bounds $\alpha_i \leq 1$, $\beta_j \leq 1$ we are reduced to the following claim:
\[
\int \frac{dw}{(\alpha_1 + \beta_1) \ldots (\alpha_p + \beta_p)} < \infty,
\]
where $dw$ is the Lebesgue measure on the standard $(p + q - 1)$-dimensional simplex $\sum \alpha_i + \sum \beta_i = 1$, $\alpha_i \geq 0$, $\beta_j \geq 0$, the domain of integration. Project our simplex onto a $(q - 1)$-dimensional simplex by applying the map
\[
(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q) \mapsto (\alpha_1 + \beta_1, \ldots, \alpha_p + \beta_p; \beta_{p+1}, \ldots, \beta_q).
\]
Under this projection, the push–forward of the measure with density
\[
\frac{1}{(\alpha_1 + \beta_1) \ldots (\alpha_p + \beta_p)}
\]
is simply the Lebesgue measure (this is a particular case of a more general fact on the behavior of Dirichlet measures under projections of simplices), which completes the proof of the lemma.

This completes the proofs of Theorems 7.1.1, 7.1.2.

7.9. Example of a noncyclic vector with cyclic projections. One can think that the proof of the cyclicity of the vectors $v_{pq}$ given above is a little bit too involved. Surely, one would try to simplify it. But the claim is not entirely trivial, and the goal of the present addendum is show that by an example.

We shall provide a representation $T = \lim T_n$ of a group $G = \lim G_n$, and a vector $\xi \in H(T)$, such that, for each $n$, the projection $\chi_n$ of the vector $\xi$ onto the subspace $H(T_n)$ is cyclic for $T_n$, though the vector $\xi$ itself is not.

Set $G_n = \mathbb{Z}_2^n$, so that $G = \bigoplus_1^\infty \mathbb{Z}_2$. The dual group to $G_n$ is again $\mathbb{Z}_2^n$, while the dual group to $G$ is a compact group, $\hat{G} = \prod_1^\infty \mathbb{Z}_2$. With arbitrary finite Borel measure $\sigma$ on $\hat{G}$ one can associate a unitary representation $T$ of the group $G$ acting in the Hilbert space $H = L^2(\hat{G}, \sigma)$ by the formula
\[
(T(g))f(\chi) = \chi(g)f(\chi), \quad g \in G, \quad \chi \in \hat{G}, \quad f \in H.
\]
Denote by $H_n \subset H$ the subspace of functions $f(\chi)$ depending on $\chi_n := \chi|_{G_n}$ only, and by $T_n$ the natural representation of the group $G_n$ in $H_n$. The representation $T$ coincides with $\lim T_n$. The space $H_n$ can be identified with $L^2(\mathbb{Z}_2^n, \sigma^{(n)})$, where $\sigma^{(n)}$ is the image of the measure $\sigma$ under the projection $\hat{G} \rightarrow \mathbb{Z}_2^n$ (taking of the first $n$ coordinates). If $\xi = f(\cdot) \in H$, then the vector $\chi_n \in H_n = L^2(\mathbb{Z}_2^n, \sigma^{(n)})$ can be obtained by integrating $f$ along the fibers of the projection $G \rightarrow \mathbb{Z}_2^n$.

For $p \in (0, 1)$ denote by $\sigma_p$ the Bernoulli measure on $\hat{G} = \prod_1^\infty \mathbb{Z}_2 = \prod_1^\infty \{0, 1\}$ with the weights $p$ and $1 - p$ at the points 0 and 1 respectively. By virtue of the law of large numbers, $\sigma_p$ is supported by the set $X_p \subset \prod_1^\infty \{0, 1\}$ formed by 0–1 sequences with the limiting frequency of 0’s equal to $p$. 

61
Take two distinct numbers \( p, p' \in (0, 1) \) and set \( \sigma = \sigma_p + \sigma_{p'} \). The measure \( \sigma \) is supported by the union of two disjoint sets \( X_p, X_{p'} \) each of which has measure 1. Take for \( \xi \) the characteristic function of the set \( X_p \). Clearly, \( \xi \) is not cyclic, since its cyclic span is a proper subspace generated by the functions in \( L^2(\hat{G}, \sigma) \) supported by \( X_p \).

On the other hand, the vector \( \xi_n \), as a function on \( \mathbb{Z}_2^n \), coincides with the Radon–Nikodym derivative \( \sigma_p^{(n)}/(\sigma_p^{(n)} + \sigma_{p'}^{(n)}) \), hence is a strictly positive function. Therefore, \( \xi_n \) is a cyclic vector for \( T_n \) for any \( n \).

8. Disjointness

8.1. The results. Our aim is to prove the following result

**Theorem 8.1.1.** If \( z \) ranges over the upper half-plane \( \Im z \geq 0 \), then the representations \( T_z \) are pairwise disjoint.

The assumption \( \Im z \geq 0 \) is introduced because \( T_z \sim T_{\bar{z}} \). The definition of disjoint representations is given in §9...

Let \( \sigma_z \) the spectral measure of the character \( \chi_z \) (§4.1). We shall deduce Theorem 8.1.1 from the following result.

**Theorem 8.1.2.** Assume \( z \) ranges over the set \( \{ z \in \mathbb{C} \setminus \mathbb{Z}, \Im z \geq 0 \} \).

(i) The measures \( \sigma_z \) are pairwise disjoint.

(ii) Each of the faces \( \Omega(p, q) \subset \Omega \) is a null set with respect to \( \sigma_z \).

Derivation of Theorem 8.1.1 from Theorem 8.1.2. Let \( z_1, z_2 \) be two distinct complex numbers from the upper half-plane \( \Im z \geq 0 \). We have to prove that \( T_{z_1} \) and \( T_{z_2} \) are disjoint. Assume first that both \( z_1 \) and \( z_2 \) are not integers. We know that if \( z \in \mathbb{C} \setminus \mathbb{Z} \) then the distinguished spherical vector \( \xi_0 \) is a cyclic vector; hence the measure \( \sigma_z \) determines the representation \( T_z \) entirely. According to claim (i) of Theorem 8.1.2, the measures \( \sigma_{z-1} \) and \( \sigma_{z_2} \) are disjoint; therefore the representations \( T_{z_1} \) and \( T_{z_2} \) are disjoint, too. Assume now that \( z_1 \in \mathbb{C} \setminus \mathbb{Z} \) while \( z_2 \in \mathbb{Z} \). According to Theorem 7.1.2, the representation \( T_{z_2} \) decomposes into a direct sum of representations labelled by the faces \( \Omega(p, q) \), \( p - q = z_2 \). By virtue of claim (ii) of Theorem 8.1.2, \( T_{z_1} \) and \( T_{z_2} \) are disjoint. Finally, when \( z_1, z_2 \) are two distinct integers, the disjointness of the representations was pointed out in Corollary 7.1.3. \( \square \)

Now we proceed to the proof of Theorem 8.1.2.

8.2. Reduction to central measures. There is a one-to-one correspondence \( \sigma \leftrightarrow M \) between probability measures \( \sigma \) on the Thoma simplex \( \Omega \) and central probability measures \( \hat{M} \) on the path space \( \mathcal{T} \), see §9...

**Lemma 8.2.1.** Let \( \sigma_1 \) and \( \sigma_2 \) be two probability measures on \( \Omega \), and let \( \hat{M}_1 \) and \( \hat{M}_2 \) be the corresponding central measures on \( \mathcal{T} \). Then \( \sigma_1, \sigma_2 \) are disjoint if and only if \( \hat{M}_1 \) and \( \hat{M}_2 \) are disjoint.

**Proof.** First, introduce a notation. Given two finite (not necessarily normalized) Borel measures \( \nu_1, \nu_2 \) on a Borel space, let us denote by \( \nu_1 \wedge \nu_2 \) their greatest lower bound. Its existence can be verified as follows. Let \( f_1 \) and \( f_2 \) be the Radon–Nikodym derivatives of \( \nu_1 \) and \( \nu_2 \) with respect to \( \nu_1 + \nu_2 \), then we set \( \nu_1 \wedge \nu_2 = \min(f_1, f_2)(\nu_1 + \nu_2) \). Observe that \( \nu_1 \) and \( \nu_2 \) are disjoint if and only if \( \nu_1 \wedge \nu_2 = 0 \).
Next, observe that the correspondence $\sigma \leftrightarrow \tilde{M}$ can be extended to not necessarily normalized measures.

Now we can proceed to the proof. In one direction the implication is trivial. Namely, if $\sigma_1$ and $\sigma_2$ are not disjoint, then $\sigma_1 \land \sigma_2$ is a nonzero measure. Let $\tilde{M}$ be the corresponding central measure. From the integral representation of Theorem 9.7.2 it follows that $\tilde{M} \leq \tilde{M}_1$ and $\tilde{M} \leq \tilde{M}_2$, so that $\tilde{M}_1 \land \tilde{M}_2 \neq 0$, whence $\tilde{M}_1$ and $\tilde{M}_2$ are not disjoint.

In the opposite direction, assume that $\tilde{M}_1$ and $\tilde{M}_2$ are not disjoint, so that $\tilde{M}_1 \land \tilde{M}_2$ is nonzero. We claim that $\tilde{M}_1 \land \tilde{M}_2$ is a central measure. Indeed, this follows from the characterization of central measures as invariant measures with respect to a countable group action, as explained in Proposition 9.4.1. Now, let $\sigma$ be the measure on $\Omega$ corresponding to $\tilde{M}_1 \land \tilde{M}_2$. It is a nonzero measure. Next, since $\tilde{M}_1 \land \tilde{M}_2 \leq \tilde{M}_1$ and $\tilde{M}_1 \land \tilde{M}_2 \leq \tilde{M}_2$, we also have $\sigma \leq \sigma_1$, $\sigma \leq \sigma_2$. (Indeed, this claim can be restated as follows: if $\tilde{M}$, $\tilde{M}'$ are two central probability measures such that $\tilde{M} \leq \text{const } \tilde{M}'$ then the same inequality holds for the corresponding spectral measures on $\Omega$, and the latter claim can be checked by applying Theorem 9.7.3 or Theorem 9.7.4.) Therefore, $\sigma_1$ and $\sigma_2$ are not disjoint. □

8.3. Proof of claim (ii) of Theorem 8.1.2. Let $p, q \in \mathbb{Z}_+$ be not equal to 0 simultaneously. Denote by $\Gamma(p, q)$ the set

$$\Gamma(p, q) = \{(i, j) \mid 1 \leq i \leq p, \quad j = 1, 2, \ldots \} \cup \{(i, j) \mid 1 \leq j \leq q, \quad i = 1, 2, \ldots \},$$

(a “fat hook”) and by $\mathcal{T}(p, q)$ the set of paths $\tau = (\tau_n) \in \mathcal{T}$ such that $\tau_n \subset \Gamma(p, q)$ for all $n$.

**Lemma 8.3.1.** If a measure $\sigma$ is supported by $\Omega(p, q) \subset \Omega$, then the corresponding central measure $\tilde{M}$ is supported by $\mathcal{T}(p, q) \subset \mathcal{T}$.

**Proof.** Let $(M^{(n)})$ be the coherent system corresponding to $\sigma$. We have (see §9...)

$$M^{(n)}(\lambda) = \dim \lambda \int_{\omega \in \Omega(p, q)} s_{\lambda}(\omega) \sigma(d\omega),$$

where $s_{\lambda}(\omega)$ is the supersymmetric Schur polynomial $s_{\lambda}(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q)$. It is known (Macdonald, [Mac, Example I.5.23 (a)]) that this polynomial vanishes unless $\lambda$ belongs to $\Gamma(p, q)$. Thus, for each $n$, $M^{(n)}$ vanishes outside $\Gamma(p, q) \cap \mathcal{V}_n$. This is equivalent to the statement of the lemma. □

**Lemma 8.3.2.** Let $z \in \mathbb{C} \setminus \mathbb{Z}$ and $p, q \in \mathbb{Z}_+$ be fixed. We assume that $p + q > 0$ thus excluding the case $p = q = 0$. Let $p_z(\lambda, \nu)$ be the coherent system $M_z$.

There exists $\varepsilon > 0$ depending on $z$, $p$, $q$ only, with the following property. If $\lambda \subset \Gamma(p, q)$ is an arbitrary Young diagram such that the set $\nu = \lambda \cup \{(p + 1, q + 1)\}$ is also a diagram, then

$$p_z(\lambda, \nu) \geq \varepsilon/n, \quad n = |\lambda|.$$ 

**Proof.** Since the content of the box $\nu/\lambda$ is $q - p$, we have

$$p_z(\lambda, \nu) = \frac{|z + q - p|^2}{|z|^2 + n} \cdot \frac{\dim \nu}{(n + 1) \dim \lambda}.$$
The first factor can be easily estimated from below: since \( z \notin \mathbb{Z} \), there exists \( \varepsilon_1 > 0 \) depending on \( z \) only, such that
\[
\frac{|z + q - p|^2}{|z|^2 + n} \geq \frac{\varepsilon_1}{n}.
\]
Now consider the second factor. It follows from the hook formula, that
\[
\dim \nu \left( \dim \lambda \right) = \prod h(b) - 1,
\]
where the product is taken over the boxes \( b \in \lambda \) such that either the arm or the leg of \( b \) contains the box \((p + 1, q + 1)\). There are exactly \( p + q \) such boxes \( b \), namely
\[
(p + 1, j), \quad 1 \leq j \leq q; \quad (i, q + 1), \quad 1 \leq i \leq p.
\]
Therefore, there is a product of \( p + q \) factors of the form \( k/(k+1) \), where \( k = 1, 2, \ldots \). Each of the factors is greater or equal to \( 1/2 \), and the entire product is not less than \( 2^{-(p+q)} \). This provides the required estimate. □

Lemma 8.3.3. Let \( z \in \mathbb{C} \setminus \mathbb{Z} \) and let \( \tilde{M}_z \) be the central measure on \( T \) corresponding to the measure \( \sigma_z \). We have \( \tilde{M}_z(T(p, q)) = 0 \) for all \( p, q \).

Proof. Denote by \( T'(p, q) \) the set of those paths \( \tau \in T(p, q) \) that are not contained the smaller sets \( T(p - 1, q) \) and \( T(p, q - 1) \). It suffices to prove that \( T'(p, q) \) has measure 0 with respect to \( \tilde{M}_z \).

Let \( \mu \) be an arbitrary diagram in \( \Gamma(p, q) \) that contains those two boxes, set \( m = |\mu| \), and denote by \( T(p, q; \mu) \) the set of paths \( \tau \in T(p, q) \) with \( \tau_m = \mu \). For any path \( \tau = (\tau_n) \in T'(p, q) \) there exists a number \( n \) such that the diagram \( \tau_n \) contains the boxes \((p + 1, q)\) and \((p, q + 1)\). Consequently the set \( T'(p, q) \) is the countable sum of the sets of the form \( T(p, q; \mu) \). Thus, it remains to prove that each set \( T(p, q; \mu) \) has measure 0.

It will be convenient to look at the measure \( \tilde{M}_z \) as describing a Markov process with the transition function \( p_z(\lambda, \nu) \). Set
\[
p_n = \text{Prob}\{\tau_{n+1} \mid \mu \subseteq \tau_n \subset \Gamma(p, q)\}.
\]
The measure of the set \( T(p, q; \mu) \) coincides with the probability of the event \( \tau_m = \mu \), multiplied by the product of the conditional probabilities \( \prod_{n \geq m} p_n \).

By Lemma 8.3.2, we have
\[
p_n \leq 1 - \frac{\varepsilon}{n}
\]
so that \( \prod_{n \geq m} p_n = 0 \). □

Proof of claim (ii) of Theorem 8.1.2. Let \( \sigma \) be the restriction of the measure \( \sigma_z \) to \( \Omega(p, q) \), and let us show that \( \sigma = 0 \). Let \( \tilde{M} \) be the central measure corresponding to \( \sigma \). According to Lemma 8.2.1, \( \tilde{M} \) is supported by \( T(p, q) \). On the other hand, it follows from Lemma 8.3.3 that \( T(p, q) \) is a null set for \( \tilde{M}_z \). Since \( \tilde{M} \leq \tilde{M}_z \), we conclude that \( \tilde{M} = 0 \), hence \( \sigma = 0 \). □
8.4. Proof of claim (i) of Theorem 8.1.2. Recall that if \( z \in \mathbb{C} \setminus \mathbb{Z} \), then the measure \( M_z^{(n)} \) has nonzero weights \( M_z^{(n)}(\lambda) \) for all \( \lambda \in \mathbb{Y}_n \), and we have an explicit formula for \( M_z^{(n)}(\lambda) \), see Theorem 4.1.1.

Fix two distinct numbers \( z_1, z_2 \) in the upper half–plane \( \Im z \geq 0 \), which are not integers. We have to prove that the spectral measures \( \sigma_{z_1} \) and \( \sigma_{z_2} \) are disjoint. By virtue of Lemma 8.2.1, it suffices to prove that the corresponding central measures \( \tilde{M}_{z_1} \) and \( \tilde{M}_{z_2} \) are disjoint. To simplify the notation, we set \( \tilde{M}_1 = \tilde{M}_{z_1}, \tilde{M}_2 = \tilde{M}_{z_2} \).

We also denote by \((M_1^{(n)})\) and \((M_2^{(n)})\) the corresponding coherent systems.

Introduce a sequence \( \varphi_n(\tau) \) of functions on \( T \),

\[
\varphi_n(\tau) = \frac{M_2^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n)}, \quad n = 1, 2, \ldots, \quad \tau = (\tau_n) \in T.
\]

Let \( X \) be the set of paths \( \tau \in T \) such that the sequence \( \varphi_n(\tau)_{n \geq 1} \) converges, as \( n \to \infty \), to a finite nonzero limit. This is a Borel subset of \( T \).

**Lemma 8.4.1.** We have \( \tilde{M}_1(X) = \tilde{M}_2(X) = 0 \).

**Proof.** We shall show that \( X \) is contained in the union of the sets \( T(p, q) \), so that the claim will follow from Lemma 8.3.3.

Denote by \( c_k(\tau) \) the content of the \( k \)th box \( \tau_k \setminus \tau_{k-1} \). By virtue of the explicit formula for \( M_z^{(n)} \) (Theorem 4.1.1), we get

\[
\varphi_n(\tau) = \prod_{k=1}^{n} \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 \frac{|z_1|^2 + k - 1}{|z_2|^2 + k - 1}.
\]

Therefore, \( X \) consists of those paths \( \tau \) for which the infinite product

\[
\varphi(\tau) = \prod_{k=1}^{\infty} \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 \frac{|z_1|^2 + k - 1}{|z_2|^2 + k - 1}
\]

converges. In particular, the \( k \)th factor in the product should go to 1. Since the second fraction in right–hand side converges to 1, as \( k \to \infty \), we conclude that

\[
\lim_{k \to \infty} \left| \frac{z_2 + c_k(\tau)}{z_1 + c_k(\tau)} \right|^2 = 1, \quad \tau \in X.
\]

It follows from our assumptions on \( z_1, z_2 \) that the equality \( |z_2 + c|^2 = |z_1 + c|^2 \) may hold for at most one real number \( c \). Indeed, this equation on \( c \) describes the set of points \( c \) that are equidistant from \( -z_1 \) and \( -z_2 \). Since \( z_1 \neq z_2 \), this set is a line in the complex plane \( \mathbb{C} \), which cannot coincide with the real axis \( \mathbb{R} \), because \( z_1, z_2 \) are both in the upper half–plane. Thus, the line is either parallel to \( \mathbb{R} \) (then there is no real \( c \) at all) or intersects \( \mathbb{R} \) at a single point.

Now, we fix an arbitrary integer \( c \) such that

\[
\left| \frac{z_2 + c}{z_1 + c} \right|^2 \neq 1.
\]

For any \( \tau \in X \), the existence of the limit above implies that there is only a finite number of integers \( k \) such that \( c_k(\tau) = c \). This means that any path \( \tau \in X \) may contain only a finite number of boxes \((p, q)\) on the diagonal \( q - p = c \). Therefore, \( \tau \) is contained in some subset of type \( T(p, q) \), which concludes the proof. \( \square \)
Lemma 8.4.2. Let $\tilde{A}$ and $\tilde{B}$ be two central probability measures on $\mathcal{T}$ and $(A^{(n)})$, $(B^{(n)})$ be the corresponding coherent systems. Assume $A \leq \text{const} B$ and let $f(\tau)$ denote the Radon–Nikodym derivative of $\tilde{A}$ with respect to $\tilde{B}$. Assume further that $B^{(n)}(\lambda) \neq 0$ for all $n$ and all $\lambda \in \mathcal{Y}_n$. Then

$$\lim_{n \to \infty} \frac{A^{(n)}(\tau_n)}{B^{(n)}(\tau_n)} = f(\tau)$$

for almost all paths $\tau = (\tau_n) \in \mathcal{T}$ with respect to $\tilde{B}$.

Proof. Let $\mathcal{T}^{[n]}$ denote the set of finite paths in $\mathcal{Y}$ going from $\emptyset$ to a vertex in $\mathcal{Y}_n$. There is a natural projection $\mathcal{T} \to \mathcal{T}^{[n]}$ assigning to a path $\tau$ its finite part $\tau^{[n]} = (\tau_0, \ldots, \tau_n)$. Notice that the infinite path space $\mathcal{T}$ can be identified with the projective limit space $\varprojlim \mathcal{T}^{[n]}$.

Denote by $\Sigma^{[n]}$ the finite algebra of cylinder subsets with the bases in $\mathcal{T}^{[n]}$. The algebras $\Sigma^{[n]}$ form an increasing family, and the union $\Sigma = \bigcup \Sigma^{[n]}$ coincides with the algebra of all Borel sets with respect to the topology of $\mathcal{T}$.

Consider the probability space $(\mathcal{T}, \Sigma, \tilde{B})$. The function $f$ is bounded and $\tilde{B}$–measurable. Hence, by the martingale theorem (cf., e.g., Shiryaev [Shir, Ch. VII, Section 4, Theorem 3])

$$\lim_{n \to \infty} \mathbb{E}(f \mid \Sigma^{[n]}) = f.$$ 

almost everywhere.

On the other hand, let $A^{[n]}$ and $B^{[n]}$ be the push–forwards of the measures $\tilde{A}$ and $\tilde{B}$ taken with respect to the projection $\mathcal{T} \to \mathcal{T}^{[n]}$. The conditional expectation $\mathbb{E}(f \mid \Sigma^{[n]})$ is nothing but the function

$$f_n(\tau) = \frac{A^{[n]}(\tau^{[n]})}{B^{[n]}(\tau^{[n]})}$$

Since $\tilde{A}$ is a central measure, we have

$$A^{[n]}(\tau^{[n]}) = \frac{1}{\dim \tau_n} A^{(n)}(\tau_n),$$

and similarly

$$B^{[n]}(\tau^{[n]}) = \frac{1}{\dim \tau_n} B^{(n)}(\tau_n),$$

It follows

$$f_n(\tau) = \frac{A^{(n)}(\tau_n)}{B^{(n)}(\tau_n)},$$

and the proof is completed. □

Proof of claim (i) of Theorem 8.1.2. We have to show that the measures $\tilde{M}_1$ and $\tilde{M}_2$ are disjoint. Let $\tilde{A} = \tilde{M}_1$, $\tilde{B} = (\tilde{M}_1 + \tilde{M}_2)/2$. Then $\tilde{A} \leq 2\tilde{B}$ and hence there exists the Radon–Nikodym derivative of $\tilde{A}$ with respect to $\tilde{B}$. Denote it by $f(\tau)$. We have $0 \leq f(\tau) \leq 2$. The measures $\tilde{M}_1$ and $\tilde{M}_2$ are disjoint if and only if $f(\tau)$ takes only two values 0 and 1, almost surely with respect to the measure $\tilde{B}$. 66
On the other hand, by virtue of Lemma 8.4.2 above, \( f(\tau) \) is \( \tilde{B} \)-almost surely the limit of the functions \( f_n(\tau) \). Let \( Y \) be the set of those paths \( \tau \) for which the limit of \( f_n(\tau) \) exists and is distinct from 0 and 2. Observe that

\[
f_n(\tau) = \frac{A(n)(\tau_n)}{B(n)(\tau_n)} = 2 \frac{M_1^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n) + M_2^{(n)}(\tau_n)} = 2 \left( 1 + \frac{M_2^{(n)}(\tau_n)}{M_1^{(n)}(\tau_n)} \right)^{-1} = \frac{2}{1 + \varphi_n(\tau)},
\]

in the notation introduced before Lemma 8.4.1. Consequently, \( Y \) coincides with the set of those paths \( \tau \) for which \( \varphi_n(\tau) \) has a finite nonzero limit, that is, \( Y = X \). But \( \tilde{M}_1(X) = \tilde{M}_2(X) = 0 \) by virtue of Lemma 8.4.1. Hence, \( \tilde{B}(Y) = B(X) = 0 \), so that \( f(\tau) \) is 0 or 2 almost surely with respect to \( \tilde{B} \). \( \square \)

The proof of Theorem 8.1.2 is completed.

9. Appendix

9.1. Young diagrams and representations of finite symmetric groups. We identify partitions with Young diagrams, and denote by \( \mathbb{Y}_n \) the set of Young diagrams with \( n \) boxes. Given a Young diagram \( \lambda \), we denote by \( |\lambda| \) the number of its boxes, and by \( \lambda' \) the transposed diagram.

Recall that \( \mathbb{Y}_n \) is a natural set of labels for irreducible representations of the finite symmetric group \( S(n) \). Given \( \lambda \in \mathbb{Y}_n \), we denote by \( \pi^\lambda \) the corresponding irreducible representation of \( S(n) \) and by \( \chi^\lambda \) its character. Let

\[
\dim \lambda = \dim \pi^\lambda = \chi^\lambda(e).
\]

The quantity \( \dim \lambda \) is called the dimension of \( \lambda \). There are several explicit formulas for the dimension. For example, the hook formula says

\[
\dim \lambda = \frac{n!}{\prod_{b \in \lambda} h(b)}, \quad \lambda \vdash n.
\]

Here the symbol \( b \in \lambda \) means that \( b \) is a box of \( \lambda \), and if \((i, j)\) are its coordinates than \( h(b) \), the hook length of \( b \), is defined by

\[
h(b) = \lambda_i + \lambda'_j - i - j + 1.
\]

Given two Young diagrams \( \lambda \) and \( \mu \), we write \( \mu \succ \lambda \) if \( \mu \subset \lambda \) and \( |\lambda| = |\mu| + 1 \), that is, \( \lambda \) is obtained from \( \mu \) by adding a box.

The classical Young branching rule says that for any \( \lambda \in \mathbb{Y}_n \)

\[
\pi^\lambda|_{S(n-1)} \sim \sum_{\mu \in \mathbb{Y}_{n-1} : \mu \succ \lambda} \pi^\mu.
\]

This implies

\[
\chi^\lambda|_{S(n-1)} = \sum_{\mu \in \mathbb{Y}_{n-1} : \mu \succ \lambda} \chi^\mu.
\]
9.2. The Young graph and cotransition probabilities. Let $\mathcal{Y}$ be the set of all Young diagrams: the disjoint union of the sets $\mathcal{Y}_n$, where $n = 0, 1, \ldots$ (we agree that $\mathcal{Y}_0$ consists of a single element, the empty diagram $\emptyset$). We view $\mathcal{Y}$ as the set of vertices of a graph, called the Young graph and denoted also by $\mathcal{Y}$. By definition, the edges of the Young graph are arbitrary couples $\mu \succcurlyeq \lambda$. The Young graph is a convenient way to encode the Young branching rule.

For $\mu \in \mathcal{Y}_{n-1}$ and $\lambda \in \mathcal{Y}_n$ set

$$q(\mu, \lambda) = \begin{cases} \dim \mu / \dim \lambda, & \text{if } \mu \succcurlyeq \lambda \\ 0, & \text{otherwise.} \end{cases}$$

By convention, $q(\emptyset, \lambda) = 1$ for the single element $\lambda \in \mathcal{Y}_1$ (the one–box diagram).

By the Young branching rule,

$$\sum_{\mu \in \mathcal{Y}_{n-1}, \mu \succcurlyeq \lambda} \dim \mu = \dim \lambda,$$

so that

$$\sum_{\mu \in \mathcal{Y}_{n-1}, \lambda} q(\mu, \lambda) = 1.$$

The numbers $q(\mu, \lambda)$ are called the cotransition probabilities (see Kerov [Ker2], [Ker4]). They constitute a probability distribution for any fixed $\lambda$ — the cotransition distribution.

9.3. Coherent systems of distributions on the Young graph. Let $\Delta_n$ be the set of probability distributions on the finite set $\mathcal{Y}_n$. This is a finite–dimensional simplex whose vertices are Dirac measures $\delta_\lambda$ with $\lambda \in \mathcal{Y}_n$.

For any $n = 1, 2, \ldots$, define an affine map $\Delta_n \to \Delta_{n-1}$ by

$$\delta_\lambda \to \sum_{\mu \in \mathcal{Y}_{n-1}} q(\mu, \lambda) \delta_\mu.$$

Let

$$\Delta = \lim_{\leftarrow} \Delta_n$$

be the projective limit of the simplices taken with respect to these maps.

By the very definition, an element of $\Delta$ is a sequence $M = \{M^{(n)}\}_{n=0,1,\ldots}$ such that $M^{(n)}$ is a probability distribution on $\mathcal{Y}_n$ and any two measures $M^{(n-1)}, M^{(n)}$ with consecutive indices fulfill the coherency relation

$$M^{(n-1)}(\mu) = \sum_{\lambda \in \mathcal{Y}_n} q(\mu, \lambda) M^{(n)}(\lambda), \quad \forall \mu \in \mathcal{Y}_{n-1}.$$
• Third, $M(\emptyset) = 1$.

Indeed, the only point to be checked is that if $M$ satisfies the above conditions then its restriction $M^{(n)} = M|_{\mathcal{Y}_n}$ is a probability distribution on $\mathcal{Y}_n$, that is, $\sum_{\lambda} M^{(n)}(\lambda) = 1$. But this is readily proved by induction on $n$.

A function $f(\lambda)$ on $\mathcal{Y}$ is called harmonic if it satisfies the relation

$$f(\lambda) = \sum_{\nu \searrow \lambda} f(\nu) \quad \forall \lambda \in \mathcal{Y}.$$  

The coherency relation for a function $M(\lambda)$ is equivalent to the harmonicity relation for the function $f(\lambda) = M(\lambda)/\dim \lambda$.

For further details, see Vershik–Kerov [VK2], Kerov [Ker2], [Ker4].

9.4. Central measures on paths and transition probabilities. By a path in the Young graph we mean a sequence of vertices

$$\tau = (\tau_n \nearrow \tau_{n+1} \nearrow \ldots), \quad \tau_i \in \mathcal{Y}_i,$$

which may be finite or infinite. Let $\mathcal{T}$ be the set of all infinite paths starting at $\emptyset$. This is a subset of the infinite product set $\prod_{n=0}^{\infty} \mathcal{Y}_n$. We endow $\mathcal{T}$ with the induced topology. Then it becomes a compact totally disconnected topological space. Given a probability measure on the path space $\mathcal{T}$, we may speak about random paths.

To any coherent system $M$ on $\mathcal{Y}$ we assign a probability measure $\widetilde{M}$ on $\mathcal{T}$ with cylinder probabilities defined as follows. Let $\lambda \in \mathcal{Y}_n$ be an arbitrary vertex and $\tau$ be an arbitrary finite path going from $\emptyset$ to $\lambda$, then the probability that the $\widetilde{M}$–random path goes along $\tau$ (up to $\lambda$) equals $M(\lambda)/\dim \lambda$.

The measure $\widetilde{M}$ is central in the sense that the cylinder probabilities depend only on the final vertices $\lambda$ but not on the paths $\tau_n$ chosen. Conversely, any central probability measure on the path space comes from a (unique) coherent system.

There is a useful characterization of central measures as invariant measures with respect to a countable group of transformations of $\mathcal{T}$. This group is defined as follows. First, for each $n$ we let $\mathcal{G}(n)$ be the group of the transformations $g : \mathcal{T} \to \mathcal{T}$ such that for any path $\tau = (\tau_n) \in \mathcal{T}$, we have $\tau_m = (g(\tau))_m$ for all $m \geq n$. Clearly, this is a finite group and we have $\mathcal{G}(n) \subset \mathcal{G}(n+1)$. Next, we define the group $\mathcal{G}$ as the union of the groups $\mathcal{G}(n)$.

**Proposition 9.4.1.** A measure on $\mathcal{T}$ is central if and only if it is invariant under the action of $\mathcal{G}$.

Define the support of a coherent system $M$ as the subset

$$\text{supp}(M) = \{ \lambda \in \mathcal{Y} : M(\lambda) \neq 0 \} \subset \mathcal{Y}.$$  

The measure $\widetilde{M}$ is concentrated on the subspace of paths entirely contained in $\text{supp}(M)$. We may view $\widetilde{M}$ as a Markov chain on the state set $\text{supp}(M)$, with the transition probabilities

$$p(\lambda, \nu) = \text{Prob}\{\tau_{n+1} = \nu \mid \tau_n = \lambda\}, \quad \lambda \in \mathcal{Y}_n, \quad \nu \in \mathcal{Y}_{n+1},$$

where $\tau = (\tau_n)$ is the random path. The transition probabilities $p(\lambda, \nu)$ are unambiguously defined for all $\lambda \in \text{supp}(M)$ by

$$p(\lambda, \nu) = \frac{M(\nu)}{M(\lambda)} \cdot \frac{\dim \lambda}{\dim \nu}, \quad \lambda \in \text{supp}(M).$$
The system of transition probabilities uniquely determines the initial central measure, so that distinct central measures have distinct transition probabilities. On the other hand, all central measures have one and the same system of cotransition probabilities, which are nothing but the quantities \( q(\mu, \lambda) \) introduced in §9.2. That is, we have

\[
q(\mu, \lambda) = \text{Prob}\{\tau_{n-1} = \mu \mid \tau_n = \lambda\}, \quad \mu \in \mathcal{Y}_{n-1}, \quad \lambda \in \mathcal{Y}_n.
\]

For further details, see Vershik–Kerov [VK2], Kerov [Ker2], [Ker4].

9.5. Characters of the group \( S(\infty) \). Recall that we have defined the infinite symmetric group \( S(\infty) \) as the inductive limit of the finite symmetric group \( S(n) \) as \( n \to \infty \). A function \( \chi : S(\infty) \to \mathbb{C} \) will be called a character if it is central (i.e. constant on conjugacy classes), positive definite, and normalized at the unity (i.e. \( \chi(e) = 1 \)). The set of all characters of \( S(\infty) \) will be denoted by \( \mathcal{X} \).

**Proposition 9.5.1.** There is a natural bijective correspondence

\[
\mathcal{X} \ni \chi \leftrightarrow M \in \Delta
\]

between characters of \( S(\infty) \) and coherent systems of probability measures on the Young graph.

**Proof.** Let \( \chi \in \mathcal{X} \). For any \( n \), set \( \chi_n = \chi|_{S(n)} \). This is a central, positive definite, normalized function on \( S(n) \). As readily verified, such functions are exactly the convex combinations of normalized irreducible characters \( \chi^\lambda / \dim \lambda \). Thus,

\[
\chi_n = \sum_{\lambda \in \mathcal{Y}_n} M^{(n)}(\lambda) \frac{\chi^\lambda}{\dim \lambda}
\]

with certain nonnegative coefficients \( M^{(n)}(\lambda) \),

\[
\sum_{\lambda \in \mathcal{Y}_n} M^{(n)}(\lambda) = 1.
\]

These coefficients may be viewed as the Fourier coefficients of the function \( \chi_n \). They form a probability distribution on \( \mathcal{Y}_n \); denote it by \( M^{(n)} \). Let us check that the distributions \( M^{(n)} \) obey the coherency relation. Indeed, by virtue of the Young branching rule, this is simply equivalent to saying that the function \( \chi_n-1 \) coincides with the restriction of \( \chi_n \) to \( S(n-1) \). Thus, starting from a character \( \chi \) we obtain a coherent system \( M = \{M^{(n)}\} \).

Conversely, let \( M \) be a coherent system. Then, for any \( n \), we may define a function \( \chi_n \) on \( S(n) \) as above. These functions are pairwise compatible and hence define a function \( \chi \) on the group \( S(\infty) \). It is readily verified that \( \chi \) is a character. \( \Box \)

Both \( \mathcal{X} \) and \( \mathcal{Y} \) are certain sets of functions (on \( S(\infty) \) and \( \mathcal{Y} \), respectively). These sets are convex, and the bijection \( \mathcal{X} \leftrightarrow \mathcal{Y} \) is an isomorphism of convex sets.

Further, both \( \mathcal{X} \) and \( \mathcal{Y} \) are compact topological spaces with respect to the topology of pointwise convergence, and our bijection is a homeomorphism with respect to this topology.

70
9.6. Thoma's theorem. Let \( \text{Ex}X \) denote the set of extreme points of the convex set \( X \). Elements of \( \text{Ex}X \) will be called extremal characters of the group \( S(\infty) \).

The first examples of extremal characters are as follows. Let \( \Omega(p, q) \subset \mathbb{R}^p \times \mathbb{R}^q \) be the set of couples \((\alpha, \beta)\), where \( \alpha = (\alpha_1 \geq \cdots \geq \alpha_p \geq 0) \) and \( \beta = (\beta_1 \geq \cdots \geq \beta_q \geq 0) \) be two collections of numbers such that

\[
\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j = 1.
\]

Here one of the numbers \( p, q \) may be zero (then the corresponding collection \( \alpha \) or \( \beta \) disappears).

For \((\alpha, \beta) \in \Omega(p, q)\) and \( k = 1, 2, \ldots \), set

\[
\tilde{p}_k(\alpha, \beta) = \sum_{i=1}^{p} \alpha_i^k + (-1)^{k-1} \sum_{j=1}^{q} \beta_j^k.
\]

Notice that \( \tilde{p}_1(\alpha, \beta) \equiv 1 \).

Given \( s \in S(\infty) \), we denote by \( m_k(s) \) the number of \( k \)-cycles in \( s \). Since \( s \) is a finite permutation, we have

\[
m_1(s) = \infty, \quad m_k(s) < \infty \text{ for } k \geq 2, \quad m_k(s) = 0 \text{ for } k \text{ large enough}.
\]

In this notation, we define a function on \( S(\infty) \) by

\[
\chi^{(\alpha, \beta)}(s) = \prod_{k=1}^{\infty} (\tilde{p}_k(\alpha, \beta))^{m_k(s)} = \prod_{k=2}^{\infty} (\tilde{p}_k(\alpha, \beta))^{m_k(s)}, \quad s \in S(\infty),
\]

where we agree that \( 1^\infty = 1 \) and \( 0^0 = 1 \). Any such function turns out to be an extremal character: this claim is a particular case of a more general result stated below.

If \( p = 1 \) and \( q = 0 \) (i.e., \( \alpha_1 = 1 \) and all other parameters disappear) then we get the trivial character, which equals 1 identically. If \( p = 0 \) and \( q = 1 \) then we get the alternate character \( \text{sgn}(s) = \pm 1 \), where the plus–minus sign is chosen according to the parity of the permutation. More generally, we have

\[
\chi^{(\alpha, \beta)} \cdot \text{sgn} = \chi^{(\beta, \alpha)}.
\]

Let \( \mathbb{R}^\infty \) denote the direct product of countably many copies of \( \mathbb{R} \). We equip \( \mathbb{R}^\infty \) with the product topology. Let \( \Omega \) be the subset of \( \mathbb{R}^\infty \times \mathbb{R}^\infty \) formed by couples \( \alpha \in \mathbb{R}^\infty, \beta \in \mathbb{R}^\infty \) such that

\[
\alpha = (\alpha_1 \geq \alpha_2 \geq \cdots \geq 0), \quad \beta = (\beta_1 \geq \beta_2 \geq \cdots \geq 0), \quad \sum_{i=1}^{\infty} \alpha_i + \sum_{j=1}^{\infty} \beta_j \leq 1.
\]

We call \( \Omega \) the Thoma simplex. As affine coordinates of the simplex one can take the numbers

\[
\alpha_1 - \alpha_2, \ldots, \alpha_{p-1} - \alpha_p, \alpha_p, \beta_1 - \beta_2, \ldots, \beta_{q-1} - \beta_q, \beta_q
\]
but we will not use these coordinates. We equip \( \Omega \) with topology induced from that of the space \( \mathbb{R}^\infty \times \mathbb{R}^\infty \). It is readily seen that \( \Omega \) is a compact space. Clearly, each set \( \Omega(p, q) \) may be viewed as a subset of \( \Omega \) (this is one of finite-dimensional faces of \( \Omega \)).

Notice that the union of the simplices \( \Omega(p, q) \) is dense in \( \Omega \). For instance, the point \( (0, 0) = (\alpha \equiv 0, \beta \equiv 0) \in \Omega \) can be approximated by points of the simplices \( \Omega(p, q) \) as \( p \to \infty \),

\[
(0, 0) = \lim_{p \to \infty} \left( \frac{1}{p}, \ldots, \frac{1}{p}, 0 \right).
\]

Now we extend by continuity the definition of the functions \( \chi^{(\alpha, \beta)} \) given above. First, for any \( k = 2, 3, \ldots \) we define the function \( \tilde{p}_k \) on \( \Omega \) as follows. If \( \omega = (\alpha, \beta) \in \Omega \) then

\[
\tilde{p}_k(\omega) = \tilde{p}_k(\alpha, \beta) = \sum_{i=1}^{\infty} \alpha_i^k + (-1)^{k-1} \sum_{j=1}^{\infty} \beta_j^k.
\]

Note that \( \tilde{p}_k \) is a continuous function on \( \Omega \). It should be emphasized that the condition \( k \geq 2 \) is necessary here: the similar expression with \( k = 1 \) (that is, the sum of all coordinates) is not continuous.

Next, for any \( \omega = (\alpha, \beta) \in \Omega \) we set

\[
\chi^{(\omega)}(s) = \chi^{(\alpha, \beta)}(s) = \prod_{k=2}^{\infty} \tilde{p}_k(\alpha, \beta)^{m_k(s)}, \quad s \in S(\infty),
\]

**Theorem 9.6.1 (Thoma’s theorem).** The functions \( \chi^{(\omega)} \) are precisely the extremal characters of the group \( S(\infty) \).

That is, for any \( \omega \in \Omega \) the function \( \chi^{(\omega)} \) is an extremal character, each extremal character is obtained in this way, and different points \( \omega \in \Omega \) define different characters.

In particular, the character \( \chi^{(0,0)} \) is the delta function at \( e \in S(\infty) \). It corresponds to the biregular representation defined in §1.3.

Notice that the set \( \mathcal{X} \) carries a natural topology – that of pointwise convergence on the group. Endow the subset \( \text{Ex} \mathcal{X} \subset \mathcal{X} \) with the induced topology. Then the correspondence \( \text{Ex} \mathcal{X} \leftrightarrow \Omega \) given by Thoma’s theorem becomes a homeomorphism of topological spaces.

This implies, in particular, that characters \( \chi^{(\omega)} \) with parameters \( \omega \in \cup_{p,q} \Omega(p, q) \) are dense in the whole set \( \text{Ex} \mathcal{X} \) with respect to the topology of pointwise convergence on the group \( S(\infty) \).

**Comments to Thoma’s theorem.** 1. The original proof of Thoma was given in his paper [Tho1] published in 1964. Thoma first proved (Satz 1 in [Tho1]) that a character \( \chi \) is extremal if and only if it is a multiplicative class function, that is,

\[
\chi(s) = \prod_{k=2}^{\infty} p_k^{m_k(s)}, \quad s \in S(\infty),
\]

with certain real numbers \( p_2, p_3, \ldots \). This reduced the problem to the following one: find all sequences \( (p_2, p_3, \ldots) \) such that the expression above is a positive
definite function on the group $S(\infty)$. An equivalent condition on $(p_2, p_3, \ldots)$ is as follows: let $h_1, h_2, \ldots$ be defined by

$$1 + h_1u + h_2u^2 + h_3u^3 + \cdots = \exp \left( u + \sum_{k=2}^{\infty} \frac{p_k u^k}{k} \right)$$

and set $h_0 = 1, h_{-1} = h_{-2} = \cdots = 0$; then

$$\det_{1 \leq i, j \leq \ell} [h_{\lambda_i, -i-j}] \geq 0 \quad \text{for any } \lambda \in \mathbb{Y}.$$

Then Thoma succeeded to prove that the sequences $(h_1 = 1, h_2, h_3, \ldots)$ with this property are exactly those given by the formula

$$1 + h_1u + h_2u^2 + h_3u^3 + \cdots = e^\gamma u \prod_{i=1}^{\infty} \frac{1 + \beta_i u}{1 - \alpha_i u},$$

where

$$(\alpha, \beta) \in \Omega, \quad \gamma = 1 - \sum_{i=1}^{\infty} (\alpha_i + \beta_i),$$

which implies the theorem. Actually, this result is equivalent to Edrei’s classification (1952) of one-sided, totally positive sequences in the sense of Schoenberg, see Edrei [Edr]. Since the paper [Tho1] contains no reference to Schoenberg or Edrei, one may conclude that Thoma was unaware of their work.

2. Quite a different proof of Thoma’s theorem was given by Vershik and Kerov ([VK1], [VK2], 1981). Instead of function-theoretic arguments of Edrei and Thoma, Vershik and Kerov used an asymptotic method (whose general idea was suggested by Vershik’s paper [Ver]): approximation of extremal characters of $S(\infty)$ by irreducible normalized characters $\chi_\lambda/\dim \lambda$ of finite groups $S(n)$, as $n \to \infty$. The asymptotic method explains the origin of Thoma’s parameters $\alpha_i, \beta_i$: they arise as limits of normalized Frobenius coordinates of the growing diagram $\lambda$.

3. An important combinatorial lemma, stated in [VK2] without proof (see [VK2, §5, Lemma 1]), was proved in Kerov–Olshanski [KO]. A particular case of it (which is sufficient for completing the proof of Thoma’s theorem) was proved in Wassermann’s thesis ([Was]). For more detail, see also Okounkov–Olshanski [OkOl, §8], Olshanski–Regev–Vershik [ORV].

4. Okounkov’s work [Ok1, Ok2] provides one more approach to Thoma’s theorem. In particular, Okounkov showed that a crucial step in Thoma’s proof can be replaced by a simple representation-theoretic argument.

5. Finally, the paper Kerov–Okounkov–Olshanski [KOO] contains a far generalization of Thoma’s theorem obtained by the asymptotic method of [VK2].

9.7. Spectral decomposition of characters.

Theorem 9.7.1. (i) For any character $\chi \in X$, there exists a probability measure $\sigma$ on the Thoma simplex $\Omega$ such that

$$\chi(s) = \int_{\Omega} \chi^{(\omega)}(s) \sigma(d\omega), \quad s \in S(\infty).$$
Such a measure is unique.

Conversely, for any probability measure $\sigma$ on $\Omega$, the function $\chi$ defined by the above formula is a character of $S(\infty)$.

Thus, $\mathcal{X}$ is isomorphic, as a convex set, to the set of all probability measures on the compact space $\Omega$.

We call this integral representation the spectral decomposition of a character. The measure $\sigma$ will be called the spectral measure of $\chi$. If $\chi$ is extremal, i.e., $\chi = \chi(\omega)\lambda$, then its spectral measure reduces to the Dirac mass at $\omega$.

Theorem 9.7.1 admits an equivalent formulation in terms of coherent systems. To state it we need to extend the above definition of the functions $\tilde{p}_k(\omega)$ to arbitrary symmetric functions. Let $\Lambda$ be the algebra of symmetric functions (see Macdonald [Mac]). The power–sums $p_k$ are algebraically independent generators of $\Lambda$, so that the assignment $p_k \mapsto \tilde{p}_k$ can be extended to a homomorphism of the algebra $\Lambda$ into the algebra $C(\Omega)$ of continuous functions on the Thoma simplex $\Omega$. Given $f \in \Lambda$, we denote by $\tilde{f}$ its image in $C(\Omega)$. In particular, we apply this to the Schur functions $s_\lambda$: the corresponding functions $\tilde{s}_\lambda(\omega)$ are called the extended Schur functions (see Vershik–Kerov [VK3, §6], Kerov–Okounkov–Olshanski [KOO, Appendix]). Notice that the restriction of $\tilde{s}_\lambda$ to $\Omega(p,q) \subset \Omega$ is nothing but the supersymmetric Schur polynomial $s_\lambda(\alpha_1, \ldots, \alpha_p; -\beta_1, \ldots, -\beta_q)$ (see, e.g., Berele–Regev [BR], Macdonald [Mac, Example I.3.23]; in Macdonald’s notation, this is $s_\lambda(\alpha_1, \ldots, \alpha_p; -\beta_1, \ldots, -\beta_q)$).

Theorem 9.7.2. Let $\chi$ be a character of $S(\infty)$, $M = (M^{(n)})$ be the corresponding coherent system, and $\sigma$ be the spectral measure of $\chi$. For any $n = 0, 1, 2, \ldots$ and any $\lambda \in \mathcal{Y}_n$ we have

$$M^{(n)}(\lambda) = \int_\Omega \dim \lambda \cdot \tilde{s}_\lambda(\omega) \sigma(d\omega).$$

In particular, the coherent system corresponding to an extremal character $\chi(\omega)$ has the form $M^{(n)}(\lambda) = \dim \lambda \cdot \tilde{s}_\lambda(\omega)$.

For a proof of Theorem 9.7.2, see Kerov–Okounkov–Olshanski [KOO] (that paper actually contained a more general result).

The claim of the theorem is similar to the Poisson integral representation of harmonic functions (see Kerov [Ker2], [Ker4], Kerov–Okounkov–Olshanski [KOO]). Notice that the role of the Poisson kernel is played here by the function $(\lambda, \omega) \mapsto \tilde{s}_\lambda(\omega)$.

Theorems 9.7.1 and 9.7.2 involve the claim that a certain convex set (that of characters or coherent systems, or yet equivalently, that of central measures) is a Choquet simplex. That is, each point of the convex set in question is uniquely representable by a probability measure on the subset of extreme points. This fact does not rely on the specific nature of the group $S(\infty)$ or the Young graph, and can be derived from some very general theorems. See, e.g., Diaconis–Freedman [DS, section 4], Olshanski [Ol7, §9].

The next result can be viewed as a kind of Fatou’s theorem on boundary values of harmonic functions. To state it we need to introduce some important notation and definitions.
Recall the definition of the Frobenius coordinates of a nonempty diagram \( \lambda \in \mathbb{Y} \): these are the integers \( p_1 > \cdots > p_d \geq 0 \), \( q_1 > \cdots > q_d \geq 0 \), where \( d \) is the number of boxes on the main diagonal of \( \lambda \) and

\[
p_i = \lambda_i - i, \quad q_i = \lambda'_i - i, \quad i = 1, \ldots, d.
\]

Any collection of integers \( p_1 > \cdots > p_d \geq 0 \), \( q_1 > \cdots > q_d \geq 0 \) corresponds to a Young diagram. The transposition \( \lambda \mapsto \lambda' \) corresponds to interchanging \( p_i \leftrightarrow q_i \).

We also need the so called modified Frobenius coordinates of a diagram \( \lambda \), which are defined by

\[
\tilde{p}_i = p_i + \frac{1}{2}, \quad \tilde{q}_i = q_i + \frac{1}{2}, \quad i = 1, \ldots, d.
\]

It is convenient to agree that \( \tilde{p}_i = \tilde{q}_i = 0 \), \( i > d \).

Notice that

\[
\sum_{i=1}^{\infty} (\tilde{p}_i + \tilde{q}_i) = |\lambda|.
\]

For each \( n \) we define an embedding \( \iota_n : \mathbb{Y}_n \to \Omega \) by

\[
\iota_n(\lambda) = (\alpha, \beta), \quad \alpha_i = \frac{\tilde{p}_i}{n}, \quad \beta_i = \frac{\tilde{q}_i}{n}, \quad i = 1, 2, \ldots, \lambda \in \mathbb{Y}_n.
\]

Notice that the union of the finite sets \( \iota_n(\mathbb{Y}_n) \) is dense in \( \Omega \).

**Theorem 9.7.3.** Let \( \chi \) be a character of \( S(\infty) \), \( \sigma \) be its spectral measure, and \( M = (M^{(n)}) \) be the coherent system of distributions corresponding to \( \chi \). Further, let \( \iota_n(M^{(n)}) \) be the push–forward of the measure \( M^{(n)} \) under the embedding \( \iota_n : \mathbb{Y}_n \to \Omega \).

As \( n \to \infty \), the measures \( \iota_n(M^{(n)}) \) converge to \( \sigma \) in the weak topology of measures on the compact space \( \Omega \).

For a proof, see Kerov–Okounkov–Olshanski [KOO].

Finally, we shall state a related result concerning central measures. An infinite path \( \tau = (\tau_n) \in \mathcal{T} \) is called regular if the points \( \iota_n(\tau_n) \in \Omega \) converge to a limit as \( n \to \infty \); then the limit point is called the end of the path. Let \( \mathcal{T}' \) be the set of regular paths; this is a Borel subset of \( \mathcal{T} \). Assigning to a regular path its end we get a Borel map \( \mathcal{T}' \to \Omega \).

**Theorem 9.7.4.** Any central probability measure \( \tilde{M} \) is supported by \( \mathcal{T}' \). The push-forward of \( \tilde{M} \) under the map \( \mathcal{T}' \to \Omega \) coincides with the spectral measure \( \sigma \) of the character \( \chi \leftrightarrow \tilde{M} \).

The proof is similar to (and actually simpler than) the proof of Theorem 10.2 in Olshanski [Olsh7].

**9.8. Spherical representations and spherical functions.** Let \( G \) be the bisymmetric group \( S(\infty) \times S(\infty) \) and \( K \) be the diagonal subgroup in \( G \), canonically isomorphic to \( S(\infty) \).

Assume we are given a unitary representation \( T \) of the group \( G \) in a Hilbert space \( \mathcal{H} \). A vector \( \xi \in \mathcal{H} \) is said to be a cyclic vector if the linear span of the vectors
$T(g)\xi$, where $g$ ranges over $G$, is dense in $\mathcal{H}$. Suppose that $\xi$ is cyclic, invariant under the action of the subgroup $K$, and $\|\xi\| = 1$. In such a case we say that the couple $(T, \xi)$ is a \textit{spherical representation}. We will call $\xi$ the \textit{spherical vector}.

Denote by $\Phi$ the set of all functions on $G$ that are positive definite, $K$–invariant, and normalized at the unity. If $(T, \xi)$ is a spherical representation of $(G, K)$, then the matrix coefficient corresponding to the spherical vector,

$$\varphi(g) = (T(g)\xi, \xi), \quad g \in G,$$

is an element of $\Phi$. We call $\varphi$ the \textit{spherical function} of $(T, \xi)$. The couple $(T, \xi)$ is uniquely (up to a natural equivalence) reconstructed from its spherical function, by use of the Gelfand–Naimark construction. Moreover, any $\varphi \in \Phi$ comes from a certain $(T, \xi)$. Thus, there is a one–to–one correspondence between functions $\varphi \in \Phi$ and (equivalence classes of) spherical representations $(T, \xi)$.

Assume $T$ is an irreducible unitary representation of $G$. Then the space of its $K$–invariant vectors has dimension 0 or 1 (indeed, this follows from the fact that $(G, K)$ is a Gelfand pair, see Olshanski [Ol3, §1]). Thus, if $T$ possesses a nonzero $K$–invariant vector $\xi$ then $\xi$ is unique, within a scalar multiple. Observe that $\xi$ is automatically cyclic, because any nonzero vector in an irreducible representation is cyclic. Thus, assuming $\|\xi\| = 1$, we see that $(T, \xi)$ is a spherical representation. The only lack of uniqueness in the choice of $\xi$ is reduced to multiplying $\xi$ by a complex scalar of absolute value 1, which does not affect the spherical function $\varphi(g) = (T(g)\xi, \xi)$. Notice that $\varphi$ is an extreme point of the convex set $\Phi$. Conversely, if $\varphi \in \Phi$ is extreme then the corresponding spherical representation is irreducible.

\textbf{Proposition 9.8.1.} There is a natural bijective correspondence $\chi \leftrightarrow \varphi$ between characters $\chi \in \mathcal{X}$ and spherical functions $\varphi \in \Phi$.

\textit{Proof.} Indeed, the relation between $\chi$ and $\varphi$ has the form

$$\varphi(g_1, g_2) = \chi(g_1 g_2^{-1}), \quad \chi(s) = \varphi(s, e),$$

where $g_1, g_2, s$ are elements of $S(\infty)$. Clearly, the normalization $\chi(e) = 1$ is equivalent to the normalization $\varphi(e) = 1$. It is readily verified that $\chi$ is constant on conjugacy classes if and only if $\varphi$ is constant on double cosets. Next, let $\{g_i\} = \{(g_{i_1}, g_{i_2})\}$ be a finite collection of element of the group $G$, and let $s_i = g_{i_1} g_{i_2}^{-1}$ be the corresponding elements in $S(\infty)$. Remark that $g_j^{-1} g_i$ lies in the same double coset modulo $K$ as $(s_j^{-1} s_i, e)$. It follows that $\varphi(g_j^{-1} g_i) = \chi(s_j^{-1} s_i)$, so that $\varphi$ is positive definite if and only if $\chi$ is. Thus, $\chi \leftrightarrow \varphi$ is indeed a bijective correspondence between the two sets. \hfill $\square$

Clearly, the bijection $\chi \leftrightarrow \varphi$ is an isomorphism of convex set. Therefore, irreducible spherical representations of $(G, K)$ are parametrized by extremal characters, and finally by points $\omega \in \Omega$.

More generally, combining Proposition 9.8.1 with the description of characters given in Theorem 9.7.1 we obtain a general description of spherical representations $(T, \xi)$. Specifically, any such $(T, \xi)$ is determined by a probability measure $\sigma$ on $\Omega$. It is worth noting that, as long we are dealing with reducible spherical representations, a given $T$ may well possess a lot of $K$–invariant cyclic vectors. If $\xi$ is replaced by another spherical vector $\xi'$ then $\sigma$ is replaced by an equivalent probability measure.
σ′. Thus, the equivalence class of T is determined by the equivalence class of σ. This equivalence class of measures on Ω will be called the spectral type of T.

Using the abstract machinery of direct integrals of Hilbert spaces one can show that any spherical representation (T, ξ) can be decomposed into a multiplicity free direct integral of irreducible spherical representations (T^{(ω)}, ξ^{ω}). This decomposition, which may be called the spectral decomposition of (T, ξ), is unique, and it is governed by the spectral measure σ:

\[ T = \int_ω T^{(ω)}σ(dω), \quad ξ = \int_ω ξ^{(ω)}σ(dω). \]

Assume (T′, ξ′) is another spherical representation and σ′ is the corresponding spectral measure. The measures σ and σ′ are said to be disjoint if they are singular with respect to each other (then there exist two disjoint Borel sets supporting them). The representations T and T′ are said to be disjoint if they have no equivalent nonzero subrepresentations. Disjointness of σ and σ′ is equivalent to disjointness of T and T′.

9.9. Admissible representations. Spherical representations enter a wider class of representations that will be defined now.

For m ≤ n, let S_m(n) ⊆ S(n) denote the subgroup fixing the points 1, 2, ..., m. Set

\[ S_m(∞) = \bigcup_{n \geq m} S_m(n) \subset S(∞) \]

and denote by K_m ⊂ K the corresponding subgroup of K ≅ S(∞). Recall that by G(m) we denote the subgroup S(m) × S(m) in the bi–symmetric group G. An important fact is that the subgroups K_m and G(m) commute to each other.

Given a unitary representation T of the group G in a Hilbert space H, let H_m = H^{K_m} be the subspace of K_m–invariant vectors, and set

\[ H_∞ = \bigcup_m H_m. \]

We remark that H_∞ is a G–invariant (algebraic) subspace in H. Indeed, for any m, H_m is invariant under G(m), because G(m) and K_m commute. Since G is the union of G(m)’s, it follows that H_∞ is invariant under G. Thus, the closure of H_∞ is an invariant subspace of the representation T.

We say that T is an admissible representation of the pair (G, K) if the subspace H_∞ as defined above is dense in H.

For more detail about this definition, see [Ol3] and also [Ol2], [Ol4]. Not all representations of G are admissible, for it may well happen that H_∞ is reduced to {0}. If T is irreducible then either it is admissible or H_∞ = {0}.

Any spherical representation (T, ξ) is admissible. Indeed, the spherical vector ξ belongs to the subspace H_∞. Therefore, all vectors T(g)ξ, where g ∈ G, are also in H_∞. Since these vectors generate a dense algebraic subspace, H_∞ is dense in H, so that T is admissible.

As shown in [Ol3], admissible representations are exactly those unitary representations that can be continuously extended to the topological group \( \overline{G} \supset G \). Thus, admissible representations of (G, K) are in essence the same as continuous...
unitary representations of the group $G$. However, for technical reasons, admissible representations are more convenient to deal with than representations of $G$.

Any admissible representation is a type I representation, i.e., the von Neumann algebra generated by it is of type I ([Ol3, Theorem 4.1]). This means that inside the class of admissible representation there is no pathologies occurring for general representations of non–tame groups. Irreducible admissible representations admit a complete classification (Okounkov [Ok1], [Ok2]), their explicit realization is described in [Ol3, §5].

References

[Ald] D. J. Aldous, *Exchangeability and related topics*, In: Springer Lecture Notes in Math. 1117 (1985), pp. 2–199.
[BR] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. 64 (1987), 118–175.
[Bor1] A. M. Borodin, *Multiplicative central measures on the Schur graph*, In: Representation theory, dynamical systems, combinatorial and algorithmic methods II (A. M. Vershik, ed.), Zapiski Nauchnyh Seminarov POMI 240, Nauka, St. Petersburg, 1997, pp. 44–52 (Russian); English transl. in J. Math. Sci. (New York) 96 (1999), no. 5, 3472–3477.
[Bor2] A. Borodin, *Harmonic analysis on the infinite symmetric group and the Whittaker kernel*, Algebra and Analysis 12 (2000), no. 5, 28–63 (Russian); English translation: St. Petersburg Math. J. 12 (2001), no. 5, 733–759.
[BO1] A. Borodin and G. Olshanski, *Point processes and the infinite symmetric group*, Math. Research Lett. 5 (1998), 799–816; [arXiv:math/9810015].
[BO2] A. Borodin and G. Olshanski, *Harmonic functions on multiplicative graphs and interpolation polynomials*, Electronic J. Comb. 7 (2000), paper #R28; [arXiv:math/9912124].
[BO3] A. Borodin and G. Olshanski, *Distributions on partitions, point processes, and hypergeometric kernel*, Commun. Math. Phys. 211 (2000), 335–356; [arXiv:math/9904013].
[BO4] A. Borodin and G. Olshanski, *Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes*, Ann. Math., to appear; [arxiv:math/0109194].
The boundary of Young graph with Jack edge multiplicities, Intern. Math. Res. Notices (1998), no. 4, 173–199.

S. Kerov, G. Olshanski, Polynomial functions on the set of Young diagrams, Comptes Rendus Acad. Sci. Paris Sér. I 319 (1994), 121–126.

S. Kerov, G. Olshanski, A. Vershik, Harmonic analysis on the infinite symmetric group. A deformation of the regular representation, Comptes Rendus Acad. Sci. Paris, Sér. I 316 (1993), 773–778.

J. F. C. Kingman, The population structure associated with the Ewens sampling formula, Theoret. Population Biology 11 (1977), 274–283.

J. F. C. Kingman, Poisson processes, Oxford University Press, 1993.

A. A. Kirillov, Elements of the theory of representations, Grundlehren der mathematischen Wissenschaften 220, Springer, Berlin-Heidelberg-New York, 1976.

I. G. Macdonald, Symmetric functions and Hall polynomials, 2nd edition, Oxford University Press, 1995.

F. J. Murray and J. von Neumann, On rings of operators IV, Ann. Math. 44 (1943), 716–808.

M. A. Naimark, Normed rings, translated from the first Russian edition. Groningen, The Netherlands, Wolters-Noordhoff Publishing, 1970.

Yu. A. Neretin, Categories of symmetries and infinite–dimensional groups, London Math. Soc. Monographs (New Series) 16, Oxford Univ. Press, 1996; Russian edition: URSS, Moscow, 1998.

Yu. A. Neretin, Hua type integrals over unitary groups and over projective limits of unitary groups, Duke Math. J. 114 (2002), 239–266. arXiv:math-ph/0010014

Yu. A. Neretin, Plancherel formula for Berezin deformation of $L^2$ on Riemannian symmetric space, J. Funct. Anal. 189 (2002), no. 2, 336–408. arXiv:math/9910229

A. Yu. Okounkov, Thoma’s theorem and representations of infinite bisymmetric group, Funct. Anal. Appl. 28 (1994), no. 2, 101–107.

A. Yu. Okounkov, On representations of the infinite symmetric group, Representation Theory, Dynamical Systems, Combinatorial and Algorithmic Methods II (A. M. Vershik, ed.), Zap. Nauchn. Semin. POMI, vol. 240, 1997, pp. 167–229 (Russian); English translation: St. Petersburg Math. J.

A. Yu. Okounkov, $SL(2)$ and $z$–measures, In: Random matrix models and their applications (P. M. Bleher and A. R. Its, eds). Mathematical Sciences Research Institute Publications 40, Cambridge Univ. Press, 2001, 407–420.

A. Okounkov and G. Olshanski, Shifted Schur functions, Algebra i Analiz 9 (1997), no. 2, 73–146 (Russian); English translation: St. Petersburg Math. J. 9 (1998), no. 2, 239–300.

G. Olshanski, Unitary representations of the infinite–dimensional classical groups $U(p, \infty)$, $SO(p, \infty)$, $Sp(p, \infty)$ and the corresponding motion groups, Funktion. Anal. Prilozhen. 12 (1978), no. 3, 20–44 (Russian); English translation: Funct. Anal. Appl. 12 (1979), 185–195.

G. Olshanski, Unitary representations of infinite–dimensional pairs $(G, K)$ and the formalism of $R$. Howe, Soviet Math. Doklady 27 (1983), no. 2, 290–294.

G. Olshanski, Unitary representations of $(G, K)$–pairs connected with the infinite symmetric group $S(\infty)$, Algebra i Analiz 1 (1989), no. 4, 178–209 (Russian); English translation: Leningrad Math. J. 1 (1990), 983–1014.

G. Olshanski, Unitary representations of infinite–dimensional pairs $(G, K)$ and the formalism of $R$. Howe, In: Representation of Lie Groups and Related Topics (A. Vershik and D. Zhelobenko, eds.), Advanced Studies in Contemporary Math. 7, Gordon and Breach Science Publishers, New York etc., 1999, pp. 269–463.

G. Olshanski, Point processes related to the infinite symmetric group, In: The orbit method in geometry and physics: in honor of A. A. Kirillov (Ch. Duval, L. Guieu, V. Ovsienko, eds.), Progress in Mathematics 213, Birkhäuser, 2003, pp. 349–393; arXiv: math/9804085

G. Olshanski, An introduction to harmonic analysis on the infinite symmetric group, In: Asymptotic Combinatorics with Applications to Mathematical Physics (A. Vershik, ed.). Springer Lecture Notes in Math. 1815, 2003; arXiv:math/0311369

G. Olshanski, The problem of harmonic analysis on the infinite–dimensional unitary group, J. Funct. Anal. 205 (2003), 464–524; arXiv:math/0109193.
ORV: G. Olshanski, A. Regev, and A. Vershik, Frobenius–Schur functions, In: Studies in Memory of Issai Schur (A. Joseph, A. Melnikov, R. Rentschler, eds), Progress in Mathematics 210, Birkhäuser, 2003, pp. 251–300. arXiv:math/0110077.

Pick: D. Pickrell, Measures on infinite dimensional Grassmann manifold, J. Func. Anal. 70 (1987), 323–356.

Pit: J. Pitman, Combinatorial stochastic processes, Lecture Notes for St. Flour Summer School, July 2002, available via http://stat-www.berkeley.edu/users/pitman/.

Rozh: N. A. Rozhkovskaya, Multiplicative distributions on Young graph, Representation theory, dynamical systems, combinatorial and algorithmic methods II (A. M. Vershik, ed.), Zapiski Nauchnykh Seminarov POMI 240, Nauka, St. Petersburg, 1997, pp. 246-257 (Russian); English translation: J. Math. Sci. (New York) 96 (1999), no. 5, 3600–3608.

Shir: A. Shiryaev, Probability, Springer-Verlag, New York, 1996.

Sta: R. P. Stanley, Enumerative combinatorics, Wadsworth, Inc., 1986.

SV: S. Strătilă and D. Voiculescu, Representations of AF–algebras and of the group $U(\infty)$, Springer Lecture Notes in Math. 486, 1975.

TE: S. Tavaré, W. J. Ewens, The Ewens Sampling Formula, In: Encyclopedia of Statistical Sciences (S. Kotz, C. B. Read, D. L. Banks, Eds.) Vol. 2, Wiley, New York. 1998, pp. 230-234.

Tho1: E. Thoma, Die unzerlegbaren, positive-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe, Math. Zeitschr. 85 (1964), 40–61.

Tho2: E. Thoma, Characters of infinite groups, In: Operator algebras and group representations (Gr. Arsene, S. Strătilă, A. Verona, and D. Voiculescu, Eds.), vol. 2, Pitman, 1984, pp. 23–32.

Ver: A. Vershik, Description of invariant measures for the actions of some infinite-dimensional groups, Soviet Math. Doklady 15 (1974), 1396–1400.

VK1: A. M. Vershik and S. V. Kerov, Characters and factor representations of the infinite symmetric group, Doklady AN SSSR 257 (1981), 1037–1040 (Russian); English translation in Soviet Math. Doklady 23 (1981), 389–392.

VK2: A. M. Vershik and S. V. Kerov, Asymptotic theory of characters of the symmetric group, Funct. Anal. Appl. 15 (1981), no. 4, 246–255.

VK3: A. M. Vershik and S. V. Kerov, The Grothendieck group of the infinite symmetric group and symmetric functions with the elements of the $K_0$-functor theory of AF-algebras, Representation of Lie groups and related topics (A. M. Vershik and D. P. Zhelobenko, eds.), Adv. Stud. Contemp. Math. 7, Gordon and Breach, 1990, pp. 36–114.

Voi: D. Voiculescu, Représentations factorielles de type $II_1$ de $U(\infty)$, J. Math. Pures et Appl. 55 (1976), 1–20.

Was: A. J. Wassermann, Automorphic actions of compact groups on operator algebras, Thesis, University of Pennsylvania (1981).