A Construction of Binary Linear Codes from Boolean Functions✩

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Abstract

Boolean functions have important applications in cryptography and coding theory. Two famous classes of binary codes derived from Boolean functions are the Reed-Muller codes and Kerdock codes. In the past two decades, a lot of progress on the study of applications of Boolean functions in coding theory has been made. Two generic constructions of binary linear codes with Boolean functions have been well investigated in the literature. The objective of this paper is twofold. The first is to provide a survey on recent results, and the other is to propose open problems on one of the two generic constructions of binary linear codes with Boolean functions. These open problems are expected to stimulate further research on binary linear codes from Boolean functions.

Keywords: Almost bent functions, bent functions, difference sets, linear codes, semibent functions, o-polynomials.

1. Introduction

Let $p$ be a prime and let $q = p^m$ for some positive integer $m$. An $[n, k, d]$ code $C$ over $\mathbb{F}_p$ is a $k$-dimensional subspace of $\mathbb{F}_p^n$ with minimum (Hamming) distance $d$. Let $A_i$ denote the number of codewords with Hamming weight $i$ in a code $C$ of length $n$. The weight enumerator of $C$ is defined by $1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$. The sequence $(1, A_1, A_2, \cdots, A_n)$ is called the weight distribution of the code $C$. A code $C$ is said to be a $t$-weight code if the number of nonzero $A_i$ in the sequence $(A_1, A_2, \cdots, A_n)$ is equal to $t$.

Boolean functions are functions from $\mathbb{F}_{2^m}$ or $\mathbb{F}_2^m$ to $\mathbb{F}_2$. They are important building blocks for certain types of stream ciphers, and can also be employed to construct binary codes. Two famous families of binary codes are the Reed-Muller codes [64, 60] and Kerdock codes [10, 11, 47]. In the literature two generic constructions of binary linear codes from Boolean functions have been well investigated. A lot of progress on the study of one of the two constructions has been made in the past decade. The objective of this paper is twofold. The first one is to provide a survey on recent development on this construction, and the other is to propose open problems on this generic constructions of binary linear codes with Boolean functions. These open problems are expected to stimulate further research on binary linear codes from Boolean functions.

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2. Mathematical foundations

2.1. Difference sets

For convenience later, we define the difference function of a subset \(D\) of an abelian group \((A, +)\) as

\[
\text{diff}_D(x) = |D \cap (D + x)|,
\]

where \(D + x = \{y + x : y \in D\}\).

A subset \(D\) of size \(k\) in an abelian group \((A, +)\) with order \(v\) is called a \((v, k, \lambda)\) difference set in \((A, +)\) if the difference function \(\text{diff}_D(x) = \lambda\) for every nonzero \(x \in A\). A difference set \(D\) in \((A, +)\) is called cyclic if the abelian group \(A\) is so.

Difference sets could be employed to construct linear codes in different ways. The reader is referred to \([27, 28]\) for detailed information. Some of the codes presented in this survey paper are also defined by difference sets.

2.2. Group characters in \(\text{GF}(q)\)

An additive character of \(\text{GF}(q)\) is a nonzero function \(\chi\) from \(\text{GF}(q)\) to the set of nonzero complex numbers such that \(\chi(x + y) = \chi(x)\chi(y)\) for any pair \((x, y) \in \text{GF}(q)^2\). For each \(b \in \text{GF}(q)\), the function

\[
\chi_b(c) = e^{\text{Tr}(bc)} \quad \text{for all } c \in \text{GF}(q)
\]

defines an additive character of \(\text{GF}(q)\), where and whereafter \(e_p = e^{2\pi \sqrt{-1}/p}\) is a primitive \(p\)th root of unity and \(\text{Tr}\) is the absolute trace function. When \(b = 0\), \(\chi_0(c) = 1\) for all \(c \in \text{GF}(q)\), and is called the trivial additive character of \(\text{GF}(q)\). The character \(\chi_1\) in \([24]\) is called the canonical additive character of \(\text{GF}(q)\). It is known that every additive character of \(\text{GF}(q)\) can be written as \(\chi_b(x) = \chi_1(bx)\) \([51\text{, Theorem 5.7]}\).

2.3. Special types of polynomials over \(\text{GF}(q)\)

It is well known that every function from \(\text{GF}(q)\) to \(\text{GF}(q)\) can be expressed as a polynomial over \(\text{GF}(q)\). A polynomial \(f \in \text{GF}(q)[x]\) is called a permutation polynomial if the associated polynomial function \(f : a \mapsto f(a)\) from \(\text{GF}(q)\) to \(\text{GF}(q)\) is a permutation of \(\text{GF}(q)\).

Dickson polynomials of the first kind over \(\text{GF}(q)\) are defined by

\[
D_h(x, a) = \sum_{i=0}^{\lfloor h/2 \rfloor} \frac{h - i}{i} (-a)^i x^{h - 2i},
\]

where \(a \in \text{GF}(q)\) and \(h\) is called the order of the polynomial. Some of the linear codes that will be presented in this paper are defined by Dickson permutation polynomials of order 5 over \(\text{GF}(2^m)\).

A polynomial \(f \in \text{GF}(q)[x]\) is said to be \(e\)-to-1 if the equation \(f(x) = b\) over \(\text{GF}(q)\) has either \(e\) solutions \(x \in \text{GF}(q)\) or no solution for every \(b \in \text{GF}(q)\), where \(e \geq 1\) is an integer, and \(e\) divides \(q\). By definition, permutation polynomials are 1-to-1. In this survey paper, we need \(e\)-to-1 polynomials over \(\text{GF}(2^m)\) for the construction of binary linear codes.
2.4. Boolean functions and their expressions

A function $f$ from $\text{GF}(2^m)$ or $\text{GF}(2^m)$ to $\text{GF}(2)$ is called a Boolean function. A function $f$ from $\text{GF}(2^m)$ to $\text{GF}(2)$ is called linear if $f(x + y) = f(x) + f(y)$ for all $(x, y) \in \text{GF}(2^m)^2$. A function $f$ from $\text{GF}(2^m)$ to $\text{GF}(2)$ is called affine if $f$ or $f - 1$ is linear.

The Walsh transform of $f : \text{GF}(2^m) \rightarrow \text{GF}(2)$ is defined by

$$\hat{f}(w) = \sum_{x \in \text{GF}(2^m)} (-1)^{f(x) + \text{Tr}(wx)}$$

where $w \in \text{GF}(2^m)$. The Walsh spectrum of $f$ is the following multiset \{ $\hat{f}(w) : w \in \text{GF}(2^m)$ \}.

Let $f$ be a Boolean function from $\text{GF}(2^m)$ to $\text{GF}(2)$. The support of $f$ is defined to be

$$D_f = \{ x \in \text{GF}(2^m) : f(x) = 1 \} \subseteq \text{GF}(2^m).$$

Clearly, $f \mapsto D_f$ is a one-to-one correspondence between the set of Boolean functions from $\text{GF}(2^m)$ to $\text{GF}(2)$ and the power set of $\text{GF}(2^m)$.

3. The first generic construction of linear codes from functions

Let $f$ be any polynomial from $\text{GF}(q)$ to $\text{GF}(q)$, where $q = p^m$. A code over $\text{GF}(p)$ is defined by

$$C(f) = \{ c = (\text{Tr}(af(x) + bx))_{x \in \text{GF}(q)} : a \in \text{GF}(q), b \in \text{GF}(q) \},$$

where $\text{Tr}$ is the absolute trace function. Its length is $q$, and its dimension is at most $2m$ and is equal to $2m$ in many cases. The dual of $C(f)$ has dimension at least $q - 2m$.

Let $f$ be any polynomial from $\text{GF}(q)$ to $\text{GF}(q)$ such that $f(0) = 0$. A code over $\text{GF}(p)$ is defined by

$$C^*(f) = \{ c = (\text{Tr}(af(x) + bx))_{x \in \text{GF}(q^*)} : a \in \text{GF}(q), b \in \text{GF}(q) \}.$$  

Its length is $q - 1$, and its dimension is at most $2m$ and is equal to $2m$ in many cases. The dual of $C^*(f)$ has dimension at least $q - 1 - 2m$.

This is a generic construction of linear codes, which has a long history and its importance is supported by Delsarte’s Theorem [24]. It gives a coding-theory characterisation of APN monomials, almost bent functions, and semibent functions (see, for examples, [13], [8] and [44]) when $q = 2$. We will not deal with this construction in this paper.

4. The second generic construction of linear codes from functions

In this section, we present the second generic construction of linear codes over $\text{GF}(p)$ with any subset $D$ of $\text{GF}(p^m)$, and introduce basic results about the linear codes. In Section 5, we will consider specific families of binary linear codes from Boolean functions obtained with this generic construction.
4.1. The description of the construction of linear codes

Let $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{GF}(q)$, where again $q = p^m$. Recall that $\text{Tr}$ denotes the trace function from $\mathbb{GF}(q)$ onto $\mathbb{GF}(p)$ throughout this paper. We define a linear code of length $n$ over $\mathbb{GF}(p)$ by

$$C_D = \{ \langle \text{Tr}(x d_1), \text{Tr}(x d_2), \ldots, \text{Tr}(x d_n) \rangle : x \in \mathbb{GF}(q) \},$$

and call $D$ the defining set of this code $C_D$. By definition, the dimension of the code $C_D$ is at most $m$.

This construction is generic in the sense that many classes of known codes could be produced by selecting the defining set $D \subseteq \mathbb{GF}(q)$ properly. This construction technique was employed in [28], [29], [30], [33] and other papers for obtaining linear codes with a few weights. If the set $D$ is properly chosen, the code $C_D$ may have good or optimal parameters. Otherwise, the code $C_D$ could have bad parameters.

4.2. The weights in the linear codes $C_D$

It is convenient to define for each $x \in \mathbb{GF}(q)$,

$$c_x = \langle \text{Tr}(x d_1), \text{Tr}(x d_2), \ldots, \text{Tr}(x d_n) \rangle.$$  (7)

The Hamming weight $\text{wt}(c_x)$ of $c_x$ is $n - N_x(0)$, where

$$N_x(0) = |\{1 \leq i \leq n : \text{Tr}(x d_i) = 0\}|$$

for each $x \in \mathbb{GF}(q)$.

It is easily seen that for any $D = \{d_1, d_2, \ldots, d_n\} \subseteq \mathbb{GF}(q)$ we have

$$pN_x(0) = \sum_{i=1}^{n} \sum_{y \in \mathbb{GF}(p)} e^{2\pi i \text{Tr}(x d_i)/p} = \sum_{i=1}^{n} \sum_{y \in \mathbb{GF}(p)} \chi_1(y x d_i) = n + \sum_{y \in \mathbb{GF}(p)^*} \chi_1(y x D)$$

where $\chi_1$ is the canonical additive character of $\mathbb{GF}(q)$, $aD$ denotes the set $\{ad : d \in D\}$, and $\chi_1(S) := \sum_{x \in S} \chi_1(x)$ for any subset $S$ of $\mathbb{GF}(q)$. Hence,

$$\text{wt}(c_x) = n - N_x(0) = \frac{(p - 1)n - \sum_{y \in \mathbb{GF}(p)^*} \chi_1(y x D)}{p}. $$

4.3. Differences between the first and second generic constructions

The second generic construction of this section is different from the first generic construction of Section 3 in the following aspects:

- While the length of the codes in the first generic construction in Section 3 is either $q$ or $q - 1$, that of the codes in the second generic construction could be any integer between 1 and $q$, depending on the underlying defining set $D$.
- While the dimension of the codes in the first construction in Section 3 is usually $2m$, that of the codes in the second construction is usually $m$ and is at most $m$. 

4
5. Binary codes from the preimage $f^{-1}(b)$ of Boolean functions $f$

Let $f$ be a function from $\mathbb{GF}(2^m)$ to $\mathbb{GF}(2)$, and let $D$ be any subset of the preimage $f^{-1}(b)$ for any $b \in \mathbb{GF}(2)$. In general, it is very hard to determine the parameters of the code $C_D$. Recall the support $D_f$ of $f$ defined in [5]. Let $n_f = |D_f|$. In this section, we deal with the binary code $C_{D_f}$ with length $n_f$ and dimension at most $m$, and will focus on the weight distribution of the code $C_{D_f}$ for several classes of Boolean functions $f$.

The following theorem plays a major role in this section whose proof can be found in [28].

**Theorem 1.** Let $f$ be a function from $\mathbb{GF}(2^m)$ to $\mathbb{GF}(2)$, and let $D_f$ be the support of $f$. If $2n_f + \hat{f}(w) \neq 0$ for all $w \in \mathbb{GF}(2^m)^*$, then $C_{D_f}$ is a binary linear code with length $n_f$ and dimension $m$, and its weight distribution is given by the following multiset:

$$\left\{ \left\{ \frac{2n_f + \hat{f}(w)}{4} : w \in \mathbb{GF}(2^m)^* \right\} \right\} \cup \{0\}.$$  \hspace{1cm} (9)

Theorem 1 establishes a connection between the set of Boolean functions $f$ such that $2n_f + \hat{f}(w) \neq 0$ for all $w \in \mathbb{GF}(2^m)^*$ and a class of binary linear codes. The determination of the weight distribution of the binary linear code $C_{D_f}$ is equivalent to that of the Walsh spectrum of the Boolean function $f$ satisfying $2n_f + \hat{f}(w) \neq 0$ for all $w \in \mathbb{GF}(2^m)^*$. When the Boolean function $f$ is selected properly, the code $C_{D_f}$ has only a few weights and may have good parameters. We will demonstrate this in the remainder of this section.

We point out that Theorem 1 can be generalized into the following whose proof is the same as that of Theorem 1.

**Theorem 2.** Let $f$ be a function from $\mathbb{GF}(2^m)$ to $\mathbb{GF}(2)$, and let $D_f$ be the support of $f$. Let $e_w$ denote the multiplicity of the element $\frac{2n_f + \hat{f}(w)}{4}$ and $e$ the multiplicity of 0 in the following multiset of (9). Then $C_{D_f}$ is a binary linear code with length $n_f$ and dimension $m - \log_2 e$, and the weight distribution of the code is given by

$$\frac{2n_f + \hat{f}(w)}{4} \text{ with frequency } \frac{e_w}{e}$$

for all $\frac{2n_f + \hat{f}(w)\cdot e}{4}$ in the multiset of (9).

5.1. Linear codes from bent functions

A function from $\mathbb{GF}(2^m)$ to $\mathbb{GF}(2)$ is called bent if $|\hat{f}(w)| = 2^{m/2}$ for every $w \in \mathbb{GF}(2^m)$. Bent functions exist only for even $m$ [6].

It is well known that a function $f$ from $\mathbb{GF}(2^m)$ to $\mathbb{GF}(2)$ is bent if and only if $D_f$ is a difference set in $(\mathbb{GF}(2^m), +)$ with the following parameters

$$(2^m, 2^{m-1} \pm 2^{(m-2)/2}, 2^{m-2} \pm 2^{(m-2)/2}).$$ \hspace{1cm} (10)

Let $f$ be bent. Then by definition $\hat{f}(0) = \pm 2^{m/2}$. It then follows that

$$n_f = |D_f| = 2^{m-1} \pm 2^{(m-2)/2}$$ \hspace{1cm} (11)

As a corollary of Theorem 1 we have the following [28].
Corollary 3. Let \( f \) be a Boolean function from \( \text{GF}(2^m) \) to \( \text{GF}(2) \) with \( f(0) = 0 \), where \( m \geq 4 \) and is even. Then \( C_D \) is an \( [n_f, m, (n_f - 2^{(m-2)/2})/2] \) two-weight binary code with the weight distribution in Table 1, where \( n_f \) is defined in (12), if and only if \( f \) is bent.

There are many constructions of bent functions and thus Hadamard difference sets. We refer the reader to [4], [58], [59], the book chapter [12] and the references therein for details. Any bent function can be plugged into Corollary 3 to obtain a two-weight binary linear code.

The construction of binary codes with bent functions above can be generalized as follows.

Theorem 4. Let \( D \) be a \( (2^m, n, \lambda) \) difference sets in \( \text{GF}(2^m) \). Then \( C_D \) is a two-weight binary code with parameters \([n, m]\) and weight enumerator

\[
1 + \frac{(2^m - 1)\sqrt{n - \lambda - n} - \frac{n-\lambda}{4}}{2\sqrt{n - \lambda}} + \frac{(2^m - 1)\sqrt{n - \lambda + n} + \frac{n-\lambda}{4}}{2\sqrt{n - \lambda}}.
\]

5.2. Linear codes from semibent functions

Let \( m \) be odd. Then there is no bent Boolean function on \( \text{GF}(2^m) \). A function \( f \) from \( \text{GF}(2^m) \) to \( \text{GF}(2) \) is called semibent if \( \hat{f}(w) \in \{0, \pm 2^{(m+1)/2}\} \) for every \( w \in \text{GF}(2^m) \).

Let \( f \) be a semibent function from \( \text{GF}(2^m) \) to \( \text{GF}(2) \). It then follows from the definition of semibent functions that

\[
n_f = |D_f| = \begin{cases} 
2^{m-1} - 2^{(m-1)/2} & \text{if } \hat{f}(0) = 2^{(m+1)/2}, \\
2^{m-1} + 2^{(m-1)/2} & \text{if } \hat{f}(0) = -2^{(m+1)/2}, \\
2^{m-1} & \text{if } \hat{f}(0) = 0.
\end{cases} \tag{12}
\]

Corollary 5. Let \( f \) be a Boolean function from \( \text{GF}(2^m) \) to \( \text{GF}(2) \) with \( f(0) = 0 \), where \( m \) is odd. Then \( C_D \) is an \( [n_f, m, (n_f - 2^{(m-1)/2})/2] \) three-weight binary code with the weight distribution in Table 2, where \( n_f \) is defined in (12), if and only if \( f \) is semibent.

### Table 1: The weight distribution of the codes of Corollary 3

| Weight \( w \) | Multiplicity \( A_w \) |
|----------------|-----------------|
| 0              | 1               |
| \( \frac{n_f - 2^{(m-1)/2}}{2} \) | \( 2^{m-1} - n_f2^{-(m+1)/2} \) |
| \( \frac{n_f + 2^{(m-1)/2}}{2} \) | \( 2^{m-1} + n_f2^{-(m-1)/2} \) |

### Table 2: The weight distribution of the codes of Corollary 5

| Weight \( w \) | Multiplicity \( A_w \) |
|----------------|-----------------|
| 0              | 1               |
| \( \frac{n_f - 2^{(m-1)/2}}{2} \) | \( n_f(2^m - n_f)2^{-m} - n_f2^{-(m+1)/2} \) |
| \( \frac{n_f + 2^{(m-1)/2}}{2} \) | \( 2^m - 1 - n_f(2^m - n_f)2^{-(m-1)/2} \) |
| \( \frac{n_f - 2^{(m-1)/2}}{2} \) | \( n_f(2^m - n_f)2^{-m} + n_f2^{-(m+1)/2} \) |
There are a lot of constructions of semibent functions from GF\(2^m\) to GF(2). We refer the reader to [16, 21, 37, 54, 55, 57] for detailed constructions. All semibent functions can be plugged into Corollary 5 to obtain three-weight binary linear codes.

5.3. Linear codes from almost bent functions

For any function \(g\) from GF\(2^m\) to GF\(2^m\), we define
\[
\lambda_g(a, b) = \sum_{x \in \text{GF}(2^m)} (-1)^{\text{Tr}(ag(x) + bx)}, \ a, b \in \text{GF}(2^m).
\]
A function \(g\) from GF\(2^m\) to GF\(2^m\) is called almost bent (AB) if \(\lambda_g(a, b) = 0\), or \(\pm 2^{(m+1)/2}\) for every pair \((a, b)\) with \(a \neq 0\). By definition, almost bent functions over GF\(2^m\) exist only for odd \(m\). Specific almost bent functions are available in [5, 12].

By definition, \(\lambda_g(1, 0) \in \{0, \pm 2^{(m+1)/2}\}\) for any almost bent function \(g\) on GF\(2^m\). It is straightforward to deduce the following lemma.

**Lemma 6.** For any almost bent function \(g\) from GF\(2^m\) to GF\(2^m\), define \(f = \text{Tr}(g)\). Then we have
\[
n_f = |D_{\text{Tr}(g)}| = \begin{cases} 
2^{m-1} + 2^{(m-1)/2} & \text{if } \lambda_g(1, 0) = -2^{(m+1)/2}, \\
2^{m-1} - 2^{(m-1)/2} & \text{if } \lambda_g(1, 0) = 2^{(m+1)/2}, \\
2^{m-1} - 1 & \text{if } \lambda_g(1, 0) = 0.
\end{cases}
\]

As a corollary of Theorem 11 we have the following [28].

**Corollary 7.** Let \(g\) be an almost bent function from GF\(2^m\) to GF\(2^m\) with \(\text{Tr}(g(0)) = 0\), where \(m\) is odd. Define \(f = \text{Tr}(g)\). Then \(C_f\) is an \([n_f, m, (n_f - 2^{(m-1)/2})/2]\) three-weight binary code with the weight distribution in Table 2 where \(n_f\) is given in (13).

The following is a list of almost bent functions \(g(x) = x^d\) on GF\(2^m\) for odd \(m\):

1. \(d = 2^h + 1\), where \(\gcd(m, h) = 1\) is odd [44].
2. \(d = 2^h - 2^h + 1\), where \(h \geq 2\) and \(\gcd(m, h) = 1\) is odd [46].
3. \(d = 2^{(m-1)/2} + 3\) [46].
4. \(d = 2^{(m-1)/2} + 2^{(m-1)/4} - 1\), where \(m \equiv 1 \pmod{4}\) [44, 45].
5. \(d = 2^{(m-1)/2} + 2^{(3m-1)/4} - 1\), where \(m \equiv 3 \pmod{4}\) [44, 45].

This list of almost bent monomials \(g(x)\) are permutation polynomials on GF\(2^m\). Hence, the length of the code \(C_f\) is equal to \(2^{m-1}\), and the weight distribution of the code is given in Table 11.

5.4. Linear codes from quadratic Boolean functions

Let
\[
f(x) = \text{Tr}_{2^m/2} \left( \sum_{i=0}^{[m/2]} f_i x^{2^i + 1} \right)
\]
be a quadratic Boolean function from GF\(2^m\) to GF(2), where \(f_i \in GF(2^m)\). The rank of \(f\), denoted by \(r_f\), is defined to be the codimension of the GF(2)-vector space
\[
V_f = \{ x \in GF(2^m) : f(x + z) - f(x) - f(z) = 0 \ \forall \ z \in GF(2^m) \}.
\]
Table 3: The Walsh spectrum of quadratic Boolean functions

| $f(w)$ | the number of $w$'s |
|--------|---------------------|
| 0      | $2^m - 2r_f$       |
| $2^{-m-r_f/2}$ | $2^{r_f - 1} + 2^{(r_f - 2)/2}$ |
| $-2^{m-r_f/2}$ | $2^{r_f - 1} - 2^{(r_f - 2)/2}$ |

The Walsh spectrum of $f$ is known [9] and given in Table 3.

Let $D_f$ be the support of $f$. By definition, we have

$$n_f = |D_f| = 2^{m-1} - \frac{\hat{f}(0)}{2} = \begin{cases} 2^{m-1} & \text{if } \hat{f}(0) = 0, \\ 2^{m-1} - 2^{m-1-r_f/2} & \text{if } \hat{f}(0) = 2^{m-r_f/2}, \\ 2^{m-1} + 2^{m-1-r_f/2} & \text{if } \hat{f}(0) = -2^{m-r_f/2}. \end{cases}$$ (15)

The following theorem then follows from Theorem 1 and Table 3.

**Theorem 8.** Let $f$ be a quadratic Boolean function of the form in (14) such that $r_f > 2$. Then $C_{D_f}$ is a binary code with length $n_f$ given in (15), dimension $m$, and the weight distribution in Table 4 where

$$(\varepsilon_1, \varepsilon_2, \varepsilon_3) = \begin{cases} (1, 0, 0) & \text{if } \hat{f}(0) = 0, \\ (0, 1, 0) & \text{if } \hat{f}(0) = 2^{m-1-r_f/2}, \\ (0, 0, 1) & \text{if } \hat{f}(0) = -2^{m-1-r_f/2}. \end{cases}$$ (16)

Table 4: The weight distribution of the code $C_{D_f}$ in Theorem 8

| Weight $w$ | $A_w$ |
|------------|-------|
| $0$        | 1     |
| $\frac{n_f}{2}$ | $2^{m-2r_f - \varepsilon_1}$ |
| $\frac{n_f+2^{m-1-r_f/2}}{2}$ | $2^{r_f - 1} + 2^{(r_f - 2)/2} - \varepsilon_2$ |
| $\frac{n_f-2^{m-1-r_f/2}}{2}$ | $2^{r_f - 1} - 2^{(r_f - 2)/2} - \varepsilon_3$ |

Note that the code $C_{D_f}$ in Theorem 8 defined by any quadratic Boolean function $f$ is different from any subcode of the second-order Reed-muller code, due to the difference in their lengths. The weight distributions of the two codes are also different.

5.5. Some binary codes $C_{D_f}$ with three weights

**Theorem 9.** Let $m \geq 4$ be even. Then the code $C_{D_f}$ has parameters $[2^{m-1}, m, 2^{m-2} - 2^{(m-2)/2}]$ and the weight distribution of Table 6 where $e = 2$, for $f(x) = \Tr(x^e)$ for the following $d$:

1. $d = 2^h + 1$, where $\gcd(m,h)$ is odd and $1 \leq h \leq m/2$ [44],
2. $d = 2^{2h} - 2^h + 1$, where $\gcd(m,h)$ is odd and $1 \leq h \leq m/2$ [46],
3. $d = 2^{m/2} + 2^{(m+2)/4} + 1$, where $m \equiv 2 \pmod{4}$ [23].
Table 5: Boolean functions with three-valued Walsh spectrum

| $f(w)$ | the number of $w$'s |
|--------|---------------------|
| 0      | $2^m - 2^{m-e}$    |
| $2^{(m+2)/2}$ | $2^m - 1 + 2^{(m-e-2)/2}$ |
| $-2^{(m+2)/2}$ | $2^{m-e-1} - 2^{(m-e-2)/2}$ |

4. $d = 2^{(m+2)/2} + 3$, where $m \equiv 2 \pmod{4}$ [23].

**Proof.** It can be verified that $\gcd(d,2^m-1) = 1$ for all the $d$ listed above. Hence $n_f = |D_f| = 2^{m-1}$. The Walsh spectrum of the functions $f$ above is given in Table 5 according to the references given in this theorem. The desired conclusions on the parameters and the weight distribution of the code $C_{D_f}$ then follow from Theorem 1.

Table 6: The weight distribution of some three-weight codes

| Weight $w$ | Multiplicity $A_w$ |
|------------|--------------------|
| 0          | $2^m - 2^{m-e} - 1$ |
| $2^{(m+2)/2} + 2^{(m+4)/2}$ | $2^{m-e-1} + 2^{(m-e-2)/2}$ |
| $2^{(m+2)/2} - 2^{(m+4)/2}$ | $2^{m-e-1} - 2^{(m-e-2)/2}$ |

5.6. Binary codes $C_{D_f}$ with four weights

Table 7: Boolean functions with four-valued Walsh spectrum: Case I

| $f(w)$ | the number of $w$'s |
|--------|---------------------|
| $-2^{m/2}$ | $(2^m - 2^{m/2})/3$ |
| 0      | $2^{m-1} - 2^{(m-2)/2}$ |
| $2^{m/2}$ | $2^{m/2}$ |
| $2^{(m+2)/2}$ | $(2^{m-1} - 2^{(m-2)/2})/3$ |

The code $C_{D_f}$ has four weights and its weight distribution is known when $f(x) = \text{Tr}(x^d)$ and $d$ is given in the following list.

- When $d = 2^{(m+2)/2} + 1$ and $m \equiv 0 \pmod{4}$, the code $C_{D_f}$ has length $2^{m-1}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem 1 and Table 7 where $h = 1$ [61].
- When $d = 2^{(m+2)/2} - 1$ and $m \equiv 2 \pmod{4}$, the code $C_{D_f}$ has length $2^{m-1} - 2^{m/2}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem 1 and Table 9 [61]. Note that in this case, $\gcd(d,2^m-1) = 3$. 

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Table 8: Boolean functions with four-valued Walsh spectrum: Case 2

| $f(w)$ | the number of $w$'s |
|--------|---------------------|
| $-2^{m/2}$ | $2^{m-1} - 2^{(3m-4)/4}$ |
| $0$ | $2^{3m/4} - 2^m / 4$ |
| $2^{m/2}$ | $2^{m-1} - 2^{(3m-4)/4}$ |
| $2^{3m/4}$ | $2^m / 4$ |

Table 9: Boolean functions with four-valued Walsh spectrum: Case 3

| $f(w)$ | the number of $w$'s |
|--------|---------------------|
| $-2^{m/2}$ | $(2^m - 2^{m/2} - 2) / 3$ |
| $0$ | $2^{m-1} - 2^{(m-2)/2} + 2$ |
| $2^{m/2}$ | $2^{m/2} - 2$ |
| $2^{(m+2)/2}$ | $(2^{m-1} - 2^{(m-2)/2} + 2) / 3$ |

- When $d = (2^{m/2} + 1)(2^{m/4} - 1) + 2$ and $m \equiv 0 \pmod{4}$, the code $C_{D_f}$ has length $2^{m-1}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem [1] and Table [8].
- When $d = \frac{2^{(m+2)/2} - 1}{2-1}$ and $m \equiv 0 \pmod{4}$, where $1 \leq h < m$ and $\gcd(h, m) = 1$, the code $C_{D_f}$ has length $2^{m-1}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem [1] and Table [7] where $h = 1$ [33].
- When $d = \frac{2^{(m+2)/2} - 1}{2-1}$ and $m \equiv 2 \pmod{4}$, where $1 \leq h < m$ and $\gcd(h, m) = 1$, the code $C_{D_f}$ has length $2^{m-1} - 2^{m/2}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem [1] and Table [7] [61]. Note that in this case, $\gcd(d, 2^{m-1}) = 3$.
- When $d = \frac{2^m + 2h+1 - 2^{m/2+1}}{2^h - 1}$, where $2h$ divides $m/2$ and $m \equiv 0 \pmod{4}$, the code $C_{D_f}$ has length $2^{m-1}$ and dimension $m$, and the weight distribution of $C_{D_f}$ is deduced from Theorem [1] and Table [7] [43].
- When $d = (2^{m/2} - 1)s + 1$ with $s = 2^h (2^h \pm 1)^{-1} \pmod{2^{m/2} + 1}$, where $e_2(h) < e_2(m/2)$ and $e_2(h)$ denotes the highest power of 2 dividing $h$, the parameters and the weight distribution of the code $C_{D_f}$ can be deduced from Theorem [1] and the results in [36].
- Let $d$ be any integer such that $1 \leq d \leq 2^{m} - 2$ and $d(2^{\ell} + 1) \equiv 2^h \pmod{2^{m} - 1}$ for some positive integers $\ell$ and $h$. Then the parameters and the weight distribution of the code $C_{D_f}$ can be deduced from Theorem [1] and the results in [42].

All these cases of $d$ above are derived from the cross-correlation of a binary maximum-length sequence with its $d$-decimation version.

5.7. Other binary codes $C_{D_f}$ with at most five weights

The code $C_{D_f}$ has at most five weights for the following $f$:
Theorem 10. Let $d$ be a prime such that $2$ is a primitive root modulo $r^m$. Let $q = 2^{\phi(r^m)}$, where $\phi$ is the Euler function. Define

$$D = \left\{ x \in GF(q) : \text{Tr} \left( x^{\frac{q^m}{2}} \right) = 0 \right\}.$$  

(17)

The following theorem was proved in [69].

**Theorem 10.** Let $2^m \geq 9$ and let $D$ be defined in (17). Then the set $C_D$ of (6) is a binary code with length $(q-1)(r^m - r + 1)/r^m$, dimension $(r-1)r^{m-1}$ and the weight distribution in Table 10.

### 5.9. Binary codes from Boolean functions whose supports are relative difference sets

Let $(A, +)$ be an abelian group of order $m \ell$ and $(N, +)$ a subgroup of $A$ of order $\ell$. A $n$-subset $D$ of $A$ is called an $(m, \ell, n, \lambda)$ relative difference set, if the multiset $\{ \{ d_1 - d_2 : d_1, d_2 \in D, d_1 \neq d_2 \} \}$ does not contain all elements in $N$, but every element in $A \setminus N$ exactly $\lambda$ times.

It well known in combinatorics that $|\chi(D)|^2 \in \{ n, n - \lambda \ell \}$ for any nontrivial group character $\chi$. Hence, any relative difference set $D$ in $(GF(2^m), +)$ defines a binary code $C_D$ with at most the following four weights:

$$\frac{n \pm \sqrt{n}}{2}, \quad \frac{n \pm \sqrt{n - \lambda \ell}}{2}.$$  

Obviously, $D$ is the support of a Boolean function on $GF(2^m)$.
6. Binary codes from the images of certain functions on GF($2^m$)

Let $f(x)$ be a function from GF($2^m$) to GF($2^m$). We define

$$D(f) = \{ f(x) : x \in \text{GF}(2^m) \} \quad \text{and} \quad D(f)^* = \{ f(x) : x \in \text{GF}(2^m) \} \setminus \{0\}.$$ 

In this section, we consider the code $C_{D(f)}$. In general, it is difficult to determine the length $n_f := |D(f)|$ of this code, not to mention its weight distribution. However, in certain special cases, the parameters and the weight distribution of $C_{D(f)}$ can be settled.

If $0 \notin D(f)$, then the two codes $C_{D(f)}$ and $C_{D(f)}^*$ are the same. Otherwise, the length of the code $C_{D(f)}^*$ is one less than that of the code $C_{D(f)}$, but the two codes have the same weight distribution. Thus, we will not give information about the codes $C_{D(f)}^*$ in this section.

Let $D$ be any subset of GF($2^m$). The characteristic function, denoted by $f_D(x)$, of $D$ is defined by

$$f_D(x) = \begin{cases} 1 & \text{if } x \in D, \\ 0 & \text{otherwise}. \end{cases}$$

Hence, the Boolean function $f_D$ has support $D$. Thus, the code $C_{D(f)}$ is in fact defined by the support of the characteristic function (Boolean function) of the set $D(f)$. Therefore, the construction method of this section is actually equivalent to that of Section 5.

6.1. The codes $C_{D(f)}$ from o-polynomials on GF($2^m$)

A permutation polynomial $f$ on GF($2^m$) is called an o-polynomial if $f(0) = 0$, and for each $s \in \text{GF}(2^m)$,

$$f_s(x) = (f(x + s) + f(s))x^{2^m-2}$$

is also a permutation polynomial. O-polynomial can be used to construct hyperovals in finite geometry.

In the original definition of o-polynomials, it is required that $f(1) = 1$. However, this is not essential, as one can always normalise $f(x)$ by using $f(1)^{-1}f(x)$ due to that $f(1) \neq 0$.

In this section, we consider binary codes $C_{D(f)}$, where $f$ is defined by an o-polynomial in some way.

6.1.1. O-polynomials and their binary codes $C_{D(f_s)}$

For any permutation polynomial $f(x)$ over GF($2^m$), we define $T(x) = xf(x^{2^m-2})$, and use $f^{-1}$ to denote the compositional inverse of $f$, i.e., $f^{-1}(f(x)) = x$ for all $x \in \text{GF}(2^m)$.

The following two theorems introduce basic properties of o-polynomials whose proofs can be found in references about hyperovals.

**Theorem 11.** Let $f$ be an o-polynomial on GF($2^m$). Then the following statements hold:

1. $f^{-1}$ is also an o-polynomial;
2. $f(x^{2^j})x^{2^m-1}$ is also an o-polynomial for any $1 \leq j \leq m - 1$;
3. $T$ is also an o-polynomial; and
4. $f(x + 1) + f(1)$ is also an o-polynomial.
Theorem 12. Let $x^k$ be an o-polynomial on $\text{GF}(2^m)$. Then every polynomial in \[
\left\{x^k, x^{1-k}, x^{1-k}+x, x^{1-k}+x^{m-1}, x^{1-k}+x^{m-1}+x\right\}
\] is also an o-polynomial, where $1/k$ denotes the multiplicative inverse of $k$ modulo $2^m - 1$.

The following property of o-polynomials plays an important role in our construction of binary linear codes with o-polynomials.

Theorem 13. A polynomial $f$ from $\text{GF}(2^m)$ to $\text{GF}(2^m)$ with $f(0) = 0$ is an o-polynomial if and only if $f_u := f(x) + ux$ is 2-to-1 for every $u \in \text{GF}(2^m)^*$. Let $f$ be any o-polynomial over $\text{GF}(2^m)$. Define $f_u(x) = f(x) + ux$ where $u \in \text{GF}(2^m)^*$. It follows from Theorem 13 that $f_u$ is 2-to-1 for every $u \in \text{GF}(2^m)^*$. In the rest of Section 6.1, we consider the codes $C_{D(f_u)}$ defined by o-polynomials. By Theorem 13 the o-polynomial property of $f$ guarantees that the length of the code $C_{D(f_u)}$ is equal to $2^m - 1$ for any $u \in \text{GF}(2^m)^*$. The dimension of $C_{D(f_u)}$ usually equals $m$, but may be less than $m$. The minimum weight and the weight distribution of $C_{D(f_u)}$ cannot be determined by the o-polynomial property alone, and differ from case to case.

6.1.2. Binary codes from the translation o-polynomials

The translation o-polynomials are described in the following theorem [66].

Theorem 14. $\text{Trans}(x) = x^{2^h}$ is an o-polynomial on $\text{GF}(2^m)$, where $\gcd(h, m) = 1$.

The following is a list of known properties of translation o-polynomials.

1. $\text{Trans}^{-1}(x) = x^{2^m-h}$ and
2. $\text{Trans}(x) = x f(x^{2^m-2}) = x^{2^m-2^m-h}$.

The proof of the following theorem is straightforward.

Theorem 15. Let $f(x) = x^{2^h}$, where $\gcd(h, m) = 1$. Then for any $u \in \text{GF}(2^m)^*$, the code $C_{D(f_u)}$ has parameters $[2^{m-1}, m-1, 2^{m-2}]$ and is a one-weight code.

The codes $C_{D(f_u)}$ in Theorem 15 have the same parameters as a subcode of the first order binary Reed-Muller code.

6.1.3. Binary codes from the Segre and Glynn o-polynomials

The following theorem describes a class of o-polynomials, which are an extension of the original Segre o-polynomials.

Theorem 16. Let $m$ be odd. Then $\text{Segre}_a(x) = x^6 + ax^4 + a^2x^2$ is an o-polynomial on $\text{GF}(2^m)$ for every $a \in \text{GF}(2^m)$.

Proof. The conclusion follows from $\text{Segre}_a(x) = (x + \sqrt{a})^6 + \sqrt{a}^3$. 

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We have the following remarks on this family of o-polynomials.

1. Segre_{0}(x) = x^6 is the original Segre o-polynomial [40, 67]. So this is an extended family.
2. Segre_{1}(x) = xD_3(x, a) = a^2D_5(x^5, a^2x^2) where D_5(x, a) = x^5 + ax^4 + a^2x, which is the Dickson polynomial of the first kind of order 5.
3. Segre_{i}(x) = D_5(x^{2^m-2}, a) = a^2x^{2^m-2} + ax^{2^m-4} + x^{2^m-6}.
4. Segre_{j}(x) = (x + \sqrt{a}) \frac{5x^{2^m-1} - 2}{2} + \sqrt{a}.

**Theorem 17** ([32]). Let m be odd. Then

$$\text{Segre}_{1}^{-1}(x) = \left( D_{13}^{\ast}(x, 1) \right)^{2^m - 2}.$$  \hspace{1cm} (19)

Glynn discovered two families of o-polynomials [40]. The first is described as follows.

**Theorem 18.** Let m be odd. Then Glynn_{i}(x) = x^{3 \times 2^{(m+1)/2} + 4} is an o-polynomial.

**Conjecture 19.** Let m \geq 3 be odd, and let f(x) = x^{3 \times 2^{(m+1)/2} + 4} be the Glynn o-polynomial. When m \in \{5, 7\}, C_{D(f_i)} is a [2^{m-1}, m] code with the weight distribution of Table [11] When m \geq 9, C_{D(f_i)} is a [2^{m-1}, m] code with five weights.

An extension of the second family of o-polynomials discovered by Glynn is documented in the following theorem.

**Theorem 20** ([32]). Let m be odd. Then

$$\text{Glynn}_{i}(x) = \left\{ \begin{array}{ll} x^{4(m+1)/2 + 2^{(3m+1)/4}} + ax^{2(m+1)/2} + (ax)^{2^{3m+1}/4} & \text{if } m \equiv 1 \pmod{4}, \\
 x^{4(m+1)/2 + 2^{(3m+1)/4}} + ax^{2(m+1)/2} + (ax)^{2^{3m+1}/4} & \text{if } m \equiv 3 \pmod{4}. \end{array} \right.$$ is an o-polynomial for all a \in GF(q).

**Proof.** Let m \equiv 1 \pmod{4}. Then

$$\text{Glynn}_{i}(x) = (x + a(m-1)/4) \left( x^{2(m+1)/2 + 2^{(3m+1)/4}} + a^{2(m+1)/2 + 2^{(3m+1)/4}} \right).$$

The desired conclusion for the case m \equiv 1 \pmod{4} can be similarly proved. \hspace{1cm} \square

Note that Glynn_{i}(x) is the original Glynn o-polynomial. So this is an extended family. For some applications, the extended family may be useful.

For certain quadratic Boolean functions f, the code C_{D(f_i)} has good parameters and its weight distribution is known. The following result is an extended version of a result proved in [28].

**Theorem 21.** Let p = 2^i + 2^j and define

$$f\alpha(x) = x^p + u, u \in GF(2^m)^\times.$$ If f\alpha(x) is 2-to-1 on GF(2^m) and gcd(2^k + 1, 2^m - 1) = 1, where \kappa = j - i, then the binary code C_{D(f_i)} has parameters [2^{m-1}, m, 2^{m-2} - 2^{(m-3)/2}] and the weight distribution of Table [7] for any u \in GF(2^m)^\times.

The following p satisfies the conditions of Theorem 21.
Table 11: The weight distribution of the codes of Theorem 21

| Weight $w$ | Multiplicity $A_w$ |
|------------|-------------------|
| $2^{m-2} - 2(m-3)/2$ | $2^{m-2} + 2(m-3)/2$ |
| $2^{m-2}$ | $2^{m-1} - 1$ |
| $2^{m-2} + 2(m-3)/2$ | $2^{m-2} - 2(m-3)/2$ |

- $\rho = 6$ (Segre case).
- $\rho = 2^6 + 2^5$ with $\sigma = (m + 1)/2$ and $4\pi \equiv 1 \mod m$ (Glynn I case).

**Theorem 22.** Let $f(x) = xD_5(x, a)$, where $a \in \text{GF}(2^m)$, and let $m$ be odd. Then the code $C_{D(f)}$ has parameters $[2^{m-1}, m]$ and the weight distribution of Table 11 for any $u \in \text{GF}(2^m)^*$. 

**Proof.** It is easily verified that $f(x) = (x + \sqrt{a})^6 + \sqrt{a}^3$. The desired conclusions then follow from Theorem 21 in the Segre case, i.e., $\rho = 6$. \hfill $\square$

**Theorem 23.** Let $f(x) = x^{(5 \times 2^{m-1} - 2)/3}$, and let $m$ be odd. Then code $C_{D(f)}$ has parameters $[2^{m-1}, m]$ and the weight distribution of Table 11 for any $u \in \text{GF}(2^m)$.

**Proof.** Note that the multiplicative inverse of 6 modulo $2^m - 1$ is equal to $(5 \times 2^{m-1} - 2)/3$. The desired conclusion then follows from Theorem 21. \hfill $\square$

6.1.4. Binary codes from the Cherowitzo o-polynomials

The following describes another conjectured class of o-polynomials.

**Conjecture 24 ([32]).** Let $m$ be odd and $e = (m + 1)/2$. Then

$$\text{Cherowitzo}_a(x) = x^{2^e} + ax^{2^e+2} + a^{2^e+2}x^{3 \times 2^e+4}$$

is an o-polynomial on $\text{GF}(2^m)$ for every $a \in \text{GF}(2^m)$.

We have the following remarks on this family.

1. Cherowitzo$_1(x)$ is the original Cherowitzo o-polynomial [17, 18]. So this is an extended family.
2. No proof of the o-polynomial property is known in the literature.
3. Cherowitzo$_a(x) = x^{2^e} - 2^e^2 + ax^{2^e-2} + a^{2^e-2}x^{2^e-3 \times 2^e-4}$.
4. Carlet and Mesnager showed that Cherowitzo$_1^{-1}(x) = x(x^{2^e+1} + x^3 + x)^{2^e-1-1}$.

We can prove the following.

**Theorem 25 ([32]).**

$$\text{Cherowitzo}_a^{-1}(x) = x(ax^{2^e+1} + a^{2^e}x^3 + x)^{2^e-1-1}.$$ 

**Theorem 26 ([32]).**

$$\text{Cherowitzo}_a = (ax^{2^e-2} + a^{2^e}x^{2^e-4} + x^{2^e-2})^{2^e-1-1}.$$
Conjecture 27. Let $m$ be odd, and let
\[ f(x) = b^{2^{(m+1)/2} + 2^{(m+1)/2}} x^{2^{(m+1)/2} + b^{2^{(m+1)/2} + 4} x^{2^{(m+1)/2} + 2} + x^{3^{2^{(m+1)/2} + 4}}, \]
where $b \in \text{GF}(2^m)$. If $m \in \{5, 7\}$, $C_{\text{D}(f_u)}$ is a $[2^{m-1}, m]$ code with at most five weights for every $u \in \text{GF}(2^m)^*$. If $m \geq 9$, $C_{\text{D}(f_u)}$ is a five-weight code with length $2^{m-1}$ and dimension $m$ for every $u \in \text{GF}(2^m)^*$.

6.1.5. Binary codes from the Payne o-polynomials

The following documents a conjectured family of o-trinomials.

Conjecture 28 ([32]). Let $m$ be odd. Then $\text{Payne}_a(x) = x^5 + ax^3 + a^2 x^{\frac{1}{3}}$ is an o-polynomial on $\text{GF}(2^m)$ for every $a \in \text{GF}(2^m)$.

We have the following remarks on this family.
1. $\text{Payne}_1(x)$ is the original Payne o-polynomial [62]. So this is an extended family.
2. $\text{Payne}_a(x) = x \text{D}_5(x^\frac{1}{3}, a)$.
3. $\text{Payne}_a(x) = a^{2^m - 3} \text{Payne}_{a^{2^m - 2}}(x)$.
4. Note that
\[ \frac{1}{6} = \frac{5 \times 2^{m-1} - 2}{3}. \]

We have then
\[ \text{Payne}_a(x) = x^{2^{m-1} + 2} + ax^{2^m - 1} + a^2 x^{2 	imes 2^{m-1} - 2}. \]

Theorem 29 ([32]). Let $m$ be odd. Then
\[ \text{Payne}_1^{-1}(x) = \left( D_{\frac{3 	imes 2^{m-2}}{2}}(x, 1) \right)^6 \]
and $\text{Payne}_1(x)$ are an o-polynomial.

Proof. Note that the multiplicative inverse of 5 modulo is $\frac{3 \times 2^{m-2}}{2}$. The conclusion then follows from the definition of the Payne polynomial and the fact that
\[ D_5(x, 1)^{-1} = D_{\frac{3 	imes 2^{m-2}}{2}}(x, 1). \]

Conjecture 30 ([32]). Let $m$ be odd, and let
\[ f(x) = x^5 + bx^3 + b^2 x^{\frac{1}{2}} = D_5 \left( x^{\frac{5 \times 2^{m-1} - 2}{3}}, b \right), \]
where $b \in \text{GF}(2^m)$. If $m \geq 7$, $C_{\text{D}(f_u)}$ is a three-weight or five-weight code with length $2^{m-1}$ and dimension $m$ for all $u \in \text{GF}(2^m)^*$. 

6.1.6. Binary codes from the Subiaco o-polynomials

The Subiaco o-polynomials are given in the following theorem [19].

**Theorem 31.** Define

\[
\text{Subiaco}_a(x) = ((a^2(x^4 + x) + a^2(1 + a + a^2)(x^3 + x^2))(x^4 + a^2x^2 + 1)^{2m-2} + x^{2m-1},
\]

where Tr(1/a) = 1 and d \not\in \text{GF}(4) if m \equiv 2 \mod 4. Then Subiaco_a(x) is an o-polynomial on GF(2^m).

As a corollary of Theorem 31, we have the following.

**Corollary 32.** Let m be odd. Then

\[
\text{Subiaco}_1(x) = (x + x^2 + x^3 + x^4)(x^4 + x^2 + 1)^{2m-2} + x^{2m-1}
\]

is an o-polynomial over GF(2^m).

Experimental data shows that the binary codes \( C_{D(f)} \) from the Subiaco o-polynomials have many weights and have smaller minimum weights compared with binary codes from other o-polynomials described in the previous subsections. Hence, the binary code \( C_{D(f)} \) from an o-polynomial could be very good and bad, depending on the specific o-polynomial.

6.2. Binary codes \( C_{D(f)} \) from functions on GF(2^m) of the form \( f(x) = F(x) + F(x + 1) + 1 \)

A function \( F(x) \) over GF(2^m) is called *almost perfect nonlinear (APN)* if

\[
\max_{a \in \text{GF}(2^m)} \max_{b \in \text{GF}(2^m)} |\{ x \in \text{GF}(2^m) : F(x + a) - F(x) = b \}| = 2.
\]

Let \( F \) be any function on GF(2^m). Define

\[
f(x) = F(x) + F(x + 1) + 1.
\]

For certain APN functions \( F(x) \) over GF(2^m), it is known that \( f \) is 2-to-1.

**Conjecture 33.** Let \( F(x) = x^{2^{m-1}+1} \) and \( m \) be odd. It is known that \( F \) is both APN and AB. If \( m \in \{5,7\}, C_{D(f)} \) is a three-weight code with length \( 2m-1 \) and dimension \( m \). If \( m \geq 9 \), \( C_{D(f)} \) is a five-weight code with length \( 2m-1 \) and dimension \( m \).

**Conjecture 34.** Let \( F(x) = x^{2^h-2^h+1} \), and let gcd\((h,m) = 1\). It is known that \( F \) is both APN and AB.

When \( h = 1, \) \( C_{D(f)} \) is a \( [2m-1, m-1, 2m-2] \) one-weight code.

When \( h \geq 2 \) and \( m \) is odd, \( C_{D(f)} \) is a three-weight or five-weight code with length \( 2m-1 \) and dimension \( m \).

In particular, when \( h = 3 \) and \( m \) is odd, \( C_{D(f)} \) is a three-weight code with length \( 2m-1 \) and dimension \( m \) for every odd \( m \geq 5 \) and \( m \not\equiv 0 \) (mod 3). In this case, \( d = 57 \) and the weight distribution of the code \( C_{D(f)} \) is given in Table [17].

It is known that \( f(x) = x^{2^h-2^h+1} + (x + 1)^{2^h-2^h+1} \) is 2'-to-1, where \( s = \text{gcd}(h,m) \) [25].
Theorem 35. Let $F(x) = x^{2^h+1}$, and let $\gcd(h,m) = 1$. Then $C_{D(f)}$ is a one-weight code with parameters $[2^{m-1}, m-1, 2^{m-2}]$.

Proof. The proof is straightforward and omitted, as $f(x)$ is linear.

We have the following comments on other APN monomials.

a) Let $F(x) = x^{2^m-2}$. Then $C_{D(f)}$ is a binary code with length $2^{m-1}$ and dimension $m$, and has at most $m$ weights. The weights are determined by the Kloosterman sums.

b) For the Niho function $F(x) = x^{2^{(m-1)/2} + 2^{(m-1)/4} - 1}$, where $m \equiv 1 \pmod{4}$, the code $C_{D(f)}$ has length $2^{m-1}$ and dimension $m$, but many weights.

c) For the Niho function $F(x) = x^{2^{(m-1)/2} + 2^{(m-1)/4} - 1}$, where $m \equiv 3 \pmod{4}$, the code $C_{D(f)}$ has length $2^{m-1}$ and dimension $m$, but many weights.

It would be extremely difficult to determine the weight distribution of the code $C_{D(f)}$ for these three classes of APN monomials.

6.3. Binary linear codes from some trinomials

A lot of constructions of cyclic difference sets in $(\text{GF}(2^m))^*$ with the Singer parameters $(2^m - 1, 2^{m-1}, 2^{m-2})$ or $(2^m - 1, 2^{m-1} - 1, 2^{m-2} - 1)$ are proposed in the literature [26, 27]. These difference sets can certainly be plugged into the second generic construction of this paper and obtain binary linear codes with good parameters. But determining the parameters of the binary linear codes may be difficult in general.

There are also a number of conjectured cyclic difference sets in $(\text{GF}(2^m))^*$ with Singer parameters (see Chapter 4 of [27]).

Conjecture 36. [27, Chapter 4] For any $f \in \text{GF}(2^m)[x]$, we define

$$D(f)^* = \{ f(x) : x \in \text{GF}(2^m) \} \setminus \{ 0 \}.$$ Let $m \geq 5$ be odd. Then $D(f)^*$ is a difference set in $(\text{GF}(2^m))^*$ with Singer parameters $(2^m - 1, 2^{m-1}, 2^{m-2})$ for the following trinomials $f \in \text{GF}(2^m)[x]$:

- a) $f(x) = x^{2^{m-1}-17} + x^{2^{(m+19)/3}+1}$.
- b) $f(x) = x^{2^{m-2}2^{-1}-1} + x^{2^{m-2}2^{-4}4^2} + x$.
- c) $f(x) = x^{2^{m-3}3} + x^{2^{2(m+1)/2}+2(m+1)/4} + x$.
- d) $f(x) = x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x$.
- e) $f(x) = x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x$.
- f) $f(x) = x^{2^{m-2}2^{(m+1)/2}2^{(m-1)/2}2^{-1}+1} + x^{2^{m-2}2^{(m+1)/2}2^{(m-1)/2}2^{-1}+1} + x$.
- g) $f(x) = x^{2^{m-3}2^{(m+1)/2}2^{-1}+1} + x^{2^{(m+1)/2}+2(m-1)/2^{-2}2^{-1}+1} + x$.
- h) $f(x) = x^{2^{m-2}2^{-1}+1} + x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x$.
- i) $f(x) = x^{2^{m-2}2^{-1}+1} + x^{2^{m-2}2^{(m+1)/2}2^{-1}+1} + x$.
- j) $f(x) = x^{2^{m-2}2^{(m-1)/2}2^{-1}+1} + x^{2^{m-2}2^{(m+1)/2}2^{-1}+1} + x$.
- k) $f(x) = x^{2^{m-2}2^{(m+1)/2}2^{-1}+1} + x^{2^{m-2}2^{(m+1)/2}2^{-1}+1} + x$. 

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For the linear codes $C_{D(f)^*}$ of the conjectured difference sets $D(f)^*$ in Conjecture 36, we have the following conjectured parameters.

**Conjecture 37.** Let $m \geq 5$ and let $D(f)^*$ be defined as in Conjecture 36. Then for every $f$ given in Conjecture 36, the binary linear code $C_{D(f)^*}$ has parameters $[2^{m-1}, m, 2^{m-2} - 2^{(m-3)/2}]$ and weight enumerator

$$1 + (2^{m-2} - 2^{(m-3)/2})x^{2^{m-2} - 2^{(m-3)/2}} + (2^{m-1} - 1)x^{2^{m-2}} + (2^{m-2} + 2^{(m-3)/2})x^{2^{m-2} + 2^{(m-3)/2}}.$$ 

The dual code of $C_{D(f)^*}$ has parameters $[2^{m-1}, 2^{m-1} - m, 3]$.

Conjecture 37 describes binary three-weight codes for the case that $m$ is odd. The next one is about binary three-weight codes for the case that $m$ is even.

**Conjecture 38.** Let $f(x) = x + x^{2^{m-2}/2} + x^{2^{m-2}/2 - 1} \in \text{GF}(2^m)[x]$, where $m \equiv 2 \mod 4$ and $m \geq 6$. Define

$$D(f) = \{ f(x) : x \in \text{GF}(2^m) \}.$$ 

Then the binary code $C_{D(f)}$ has parameters $[2^{m-1}, m, 2^{m-2} - 2^{(m-4)/2}]$ and weight enumerator

$$1 + (2^{m-2} - 2^{(m-4)/2} + 3 \times 2^{m-2} - 1)x^{2^{m-2}} + (2^{m-3} - 2^{(m-4)/2})x^{2^{m-2} + 2^{(m-4)/2}}.$$ 

It was conjectured in [27, Chapter 4] that $D(f)^*$ is a difference set in $(\text{GF}(2^m)^*, \times)$ with the parameters $(2^m - 1, 2^{m-1} - 1, 2^{m-2} - 1)$.

**Table 12:** The weight distribution of the codes of Conjecture 38

| Weight $w$ | Multiplicity $A_w$ |
|------------|-------------------|
| $2^{m-2} - 2^{(m-2)/2}$ | $2^{(m-2)/2}$ |
| $2^{m-2} - 2^{(m-4)/2}$ | $2^{m-1} - 2^m/2$ |
| $2^{m-2}$ | $2^{m/2} + 2^{(m-2)/2} - 1$ |
| $2^{m-2} + 2^{(m-4)/2}$ | $2^{m-1} - 2^m/2$ |

Binary four-weight codes may also be produced with difference sets in $(\text{GF}(2^m)^*, \times)$ as follows.

**Conjecture 39.** Let $f(x) = x + x^2 + x^{2^{m-2} - 2^{m/2} + 1} \in \text{GF}(2^m)[x]$, where $m \geq 4$ and $m$ is even. Define

$$D(f) = \{ f(x) : x \in \text{GF}(2^m) \}.$$ 

Then the binary linear code $C_{D(f)}$ has parameters $[2^{m-1}, m, 2^{m-2} - 2^{(m-4)/2}]$ and the weight distribution of Table 12. It was conjectured in [27, Chapter 4] that $D(f)^*$ is a difference set in $(\text{GF}(2^m)^*, \times)$ with parameters $(2^m - 1, 2^{m-1} - 1, 2^{m-2} - 1)$.

The following is another list of conjectured cyclic difference sets in $(\text{GF}(2^m)^*, \times)$ with Singer parameters (see Chapter 4 of [27]).
Conjecture 40. [27, Chapter 4] For any $f \in \text{GF}(2^m)[x]$, we define

$$D(f) = \{ f(x^2 + 1) : x \in \text{GF}(2^m) \}.$$  

Let $m \geq 4$. Then $D(f)^*$ is a difference set in $(\text{GF}(2^m)^*, \times)$ with Singer parameters $(2^m - 1, 2^{m-1} - 1, 2^{m-2} - 1)$ for the following polynomials $f \in \text{GF}(2^m)[x]$:

1. $f(x) = x + x^{2^{(m+1)/2}} + x^{2^m - 2^{(m+1)/2} + 1}$, where $m$ is odd.
2. $f(x) = x + x^{2^{(m+1)/3}} + x^{2^{m+1} - 1}/3$, where $m$ is odd.
3. $f(x) = x + x^{2^{(m+2)/2}} + x^{2^m - 2^{m/2} + 1}$, where $m$ is even.

For the linear codes $C_{D(f)^*}$ of the conjectured difference sets $D(f)^*$ in Conjecture 40, we have the following conjectured parameters.

Conjecture 41. Let $m \geq 4$ and let $D(f)$ be defined as in Conjecture 40. Then for every $f$ given in Conjecture 40, the binary linear code $C_{D(f)^*}$ has parameters $[2^{m-1} - 1, m - 1, 2^{m-2}]$ and is a one-weight code.

To determine the weight distribution of the code $C_{D(f)^*}$ or $C_{D(f)}^c$ of the conjectured difference sets listed in this section, one does not have to prove the difference set property of the set $D(f)$ or $D(f)^*$.

7. An expansion of the binary codes

Table 13: The weight distribution of the codes of Theorem 42

| Weight $w$ | Multiplicity $A_w$ |
|------------|-------------------|
| 0          | 1                 |
| $2^{m-2} + 2^{(m-3)/2}$ | $2^{m-1} + 2^{(m-1)/2}$ |
| $2^{m-2}$  | $2^{m-1} - 2$     |
| $2^{m-2} + 2^{(m-3)/2}$ | $2^{m-1} - 2^{(m-1)/2}$ |
| $2^{m-1}$  | 1                 |

Let $\mathbf{1}$ denote the all-one vector, that is, $(1, 1, \ldots, 1)$, of any length. The complement of any vector $\mathbf{c} \in \text{GF}(2)^n$ is defined to be $\mathbf{c} + \mathbf{1}$. For any binary code $C$, we define

$$\overline{C} = C \cup \{ \mathbf{c} + 1 : \mathbf{c} \in C \}.$$  

Then $\overline{C}$ is a binary linear code, which has the same length as $C$. For most of the binary codes $C$ presented in this paper, the dimension of $\overline{C}$ is one more than that of $C$. In many cases, the weight distribution of $\overline{C}$ can be deduced from that of $C$. As an example, we have the following.

Theorem 42. Let $m$ be odd and let $C$ be any binary linear code with parameters $[2^{m-1}, m]$ and the weight distribution of Table 17. Then $\overline{C}$ is binary linear code with parameters $[2^{m-1}, m + 1]$ and the weight distribution of Table 13.

Example 43. When $m = 5$, the code $\overline{C}$ of Theorem 42 has parameters $[16, 6, 6]$ and is optimal. When $m = 7$, the code $\overline{C}$ of Theorem 42 has parameters $[64, 8, 28]$ and is almost optimal.
8. Concluding remarks

In this paper, we surveyed binary linear codes from Boolean functions and functions on GF(2^m) obtained from the second generic construction. Our focus was on such binary linear codes with at most five weights. Many one-weight codes, two-weight codes, three-weight codes, four-weight codes are presented in this paper. Some of them are optimal and some are almost optimal. The codes are also quite interesting in the sense that they may have applications in secret sharing [1, 14, 71] and authentication codes [31]. The parameters of some of the binary codes are different from those in [3, 6, 22, 20, 38, 49, 50, 70, and 72].

A number of conjectures were presented in this paper as open problems. All the conjectures on difference sets, o-polynomials and the corresponding binary codes were confirmed for sufficiently many integers m by Magma. The reader is warmly invited to attack these open problems.

Finally, we make it clear that this is by no means a survey of all binary linear codes from Boolean functions, but a survey of binary linear codes from Boolean functions from the second generic construction described in Section 4.

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