Benders Decomposition for a Class of Mathematical Programs with Constraints on Dual Variables

Geunyeong Byeon · Pascal Van Hentenryck

Abstract Interdependent systems with mutual feedback are best represented as a multi-level mathematical programming, in which the leader decisions determine the follower operations, which subsequently affect the leader operations. For instance, a recent paper [3] formulated the optimization of electricity and natural gas systems with physical and economic couplings as a tri-level program that incorporates some constraints cutting off first-level solutions (e.g., unit commitment decisions) based on the dual solutions of the third-level problem (e.g., natural gas prices) to ensure economic viability. This tri-level program can be reformulated as a single-level Mixed-Integer Second-Order Cone Program (MISOCP), which is equivalent to a “standard” MISOCP for a joint electricity-gas system with additional constraints linking the first-level variables and the dual variables of the inner-continuous problem.

This paper studies how to apply Benders decomposition to this class of mathematical programs. Since a traditional Benders decomposition results in computationally difficult subproblems, the paper proposes a dedicated Benders decomposition where the subproblem is further decomposed into two more tractable subproblems. The paper also shows that traditional acceleration techniques, such as the normalization of Benders feasibility cuts, can be adapted to this setting. Experimental results on a gas-aware unit commitment for coupled electricity and gas networks demonstrate the computational benefits of the approach compared to a state-of-the-art mathematical programming solver and an advanced Benders method with acceleration schemes.

Keywords Benders decomposition · Strong duality · Multi-level programming · Mixed-Integer Conic Linear Programming

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1 Introduction

Motivation This paper is motivated by interdependent systems with mutual feedback that can be represented as multi-level mathematical programs in which the leader decisions determine the follower operations, which subsequently affect the leader operations. This is the case for instance of the electricity and gas systems in the United States. The increased interdependency between these systems has introduced a wider range of risks—both physical and economic—to the power system. For example, during the 2014 polar vortex event in the Northeastern United States area, the record high natural gas prices invalidated the reliability and efficiency of the day-ahead unit commitment decisions, which in turn led spikes in electricity prices. This historical event highlighted a crucial need for power-system operators who must be aware of the physical and economic feedback from the gas system when committing their generators.

To mitigate these risks, a recent paper [3] introduced the unit commitment problem with Gas Network Awareness (UCGNA), a tri-level mathematical program where the first and second levels determine how to commit and dispatch electric power generating units and the third level decides how to operate the gas network given the natural gas demands of committed gas-fueled generators that are determined in the first and second levels. The economic feedback from the gas network, i.e., the natural gas zonal prices, is given by the dual solution $\mathbf{y}$ of the third-level optimization and the first-level optimization is subject to constraints over both $\mathbf{y}$ and commitment decisions $\mathbf{z}$ in order to ensure the robustness of the unit commitment decisions against the economic feedback from the gas system. The paper [3] showed that the tri-level problem can be reformulated as a single-level MISOCP by utilizing strong duality on the third-level problem and an approximation of lexicographic optimization to merge the second and third level problems. Interestingly, the resulting single-level program can be expressed as a simple MISOCP that optimizes the joint electricity and natural gas problem to which a constraint on the dual variables of its inner-continuous problem has been added.

Problem Formulation Given some matrices $\mathbf{A} \in \mathbb{R}^{k \times m}$, $\mathbf{B} \in \mathbb{R}^{k \times n}$, a proper cone $\mathcal{K} \subseteq \mathbb{R}^m$, and some vectors $\mathbf{b} \in \mathbb{R}^k$, $\mathbf{c} \in \mathbb{R}^m$, $\mathbf{h} \in \mathbb{R}^n$, consider the following mixed-integer conic linear programming:

$$\begin{align*}
\min & \quad \mathbf{c}^T \mathbf{x} + \mathbf{h}^T \mathbf{z} \\
\text{s.t.} & \quad \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{z} \geq \mathbf{b}, \\
& \quad \mathbf{x} \in \mathcal{K}, \quad \mathbf{z} \in \mathbb{B}^n,
\end{align*}$$

(1)

where $\mathbf{x}$ and $\mathbf{z}$ respectively represent $m$-dimensional continuous and $n$-dimensional binary variables.

Consider the inner continuous problem of Problem (1), described in Problem (2) where the binary variables are fixed as some $\bar{\mathbf{z}} \in \mathbb{B}^n$, and its dual, described in Problem (3) where $\mathcal{K}^*$ denotes the dual cone of $\mathcal{K}$:
Mathematical programs with constraints on dual variables 3

\[
\begin{align*}
\min & \quad c^T x \\
\text{s.t.} & \quad Ax \geq b - B\bar{z}, \quad x \in \mathcal{X}.
\end{align*}
\tag{2}
\]

\[
\begin{align*}
\max & \quad y^T (b - B\bar{z}) \\
\text{s.t.} & \quad y^T A \preceq \chi^*, \quad c^T y \\
& \quad y \geq 0.
\end{align*}
\tag{3a}
\]

\[
\begin{align*}
\max & \quad y^T (b - B\bar{z}) \\
\text{s.t.} & \quad y^T A \preceq \chi^*, \quad c^T y \\
& \quad y \geq 0.
\end{align*}
\tag{3b}
\]

\[
\begin{align*}
\max & \quad y^T (b - B\bar{z}) \\
\text{s.t.} & \quad y^T A \preceq \chi^*, \quad c^T y \\
& \quad y \geq 0.
\end{align*}
\tag{3c}
\]

The target application desires to exclude some binary solution \(\bar{z}\) of Problem (1) if its subsequent dual solution \(\bar{y}\) does not satisfy some requirements, i.e., it adds a constraint linking variables \(y\) of Problem (3) and variables \(z\) of Problem (1) that can be expressed as

\[
Ey \leq a + Mz,
\tag{4}
\]

for some matrices \(E \in \mathbb{R}^{l \times k}\) and \(M \in \mathbb{R}^{l \times n}\) and some vector \(a \in \mathbb{R}^l\).

This paper also makes the following assumption

\((A0):\) Strong duality holds for the inner-continuous problem.

This problem class is called Mixed-Integer Conic Programming with Additional Constraints on the Dual Variables of its inner continuous problem (MIPAD) in this paper. For notational simplicity, this paper focuses on the mixed-integer programming subclass of MIPAD, i.e., \(\mathcal{X} = \mathbb{R}^m\). The results however can be easily generalized to mixed-integer conic programming that satisfies (A0).

**Contributions** The structure of the considered problems, together with assumption (A0), make them amenable to Benders decomposition. Unfortunately, the Benders dual subproblem, even when enhanced with Benders cut normalization [5] and an in-out approach [1], is computationally and numerically challenging for the target applications that are concerned with large electricity and gas networks.

To address these challenges, this paper proposes a dedicated Benders method for MIPAD that decomposes the Benders dual subproblem further into two independent and simpler problems: the dual problem and the primal problem with some additional variables and constraints. Moreover, the paper shows how to generalize this novel decomposition to produce normalized Benders feasibility cuts using a Lagrangian dual formulation of the normalized problem and a subgradient adaption of Newton’s method.

The proposed approach was evaluated on instances of the UCGNA problem [3]. They show that the novel decomposition provides significant benefits compared to a state-of-the-art mathematical programming solvers and a standard Benders decomposition enhanced with the acceleration schemes (i.e., normalization and in-out approach). The benefits stem from reduced computational requirements at each iteration and increased numerical stability.

The rest of the paper is organized as follows. Section 2 discusses the general modeling framework for MIPAD. Section 3 proposes the dedicated Benders method for MIPAD. Section 4 shows how to generalize the approach to produce normalized feasibility cuts and Section 5 presents the in-out approach adopted in this paper. Section 6 presents detailed computational results demonstrating the benefits of the proposed approach and Section 7 concludes the paper.
2 A Modeling Framework for MIPAD

This section proposes a general modeling framework for MIPAD. It assumes that \( \mathcal{X} = \mathbb{R}^m \) and

(A1): Problem (1) is feasible and bounded.

(A1) is a reasonable assumption since otherwise there is no purpose in adding constraints on dual variables. If Problem (1) is unbounded for some \( \bar{z} \), then Problem (2) is unbounded, which implies that Problem (3) is infeasible for any binary \( z \). Therefore, no additional constraints on dual variables can be satisfied. If Problem (1) is infeasible, then the problem remains infeasible when constraints are added. Note also that (A1) implies that Problem (3) is feasible.

By the strong duality, the MIPAD problem can be expressed as follows:

\[
\min_{z \in \mathbb{R}^n} h^T z + f(z) \tag{5a}
\]

where

\[
f(z) = \min \, c^T x \tag{6a}
\]

s.t. \( Ax \geq b - Bz, \quad x \geq 0, \tag{6b} \)

\( c^T x \leq y^T (b - Bz), \tag{6c} \)

\( y^T A \leq c^T, \quad y \geq 0, \tag{6d} \)

\( Ey \leq a + Mz. \tag{6e} \)

Constraints (6b) and (6d) respectively ensure primal and dual feasibility, Constraint (6c) captures strong duality, and Constraints (6e) represents the additional constraints on dual variables. Problem (6) contains a bilinear term of \( y^T Bz \) in Constraint (6c). Assuming that \( y \) has an upper bound of \( \bar{y} \), this term can be linearized: First introduce an additional vector of variables \( s \in \mathbb{R}^{kn} \) and constraints

\[
s_{(i-1)\times n+j} = B_{ij} y_i z_j, \quad \forall i = 1, \ldots, k, \quad j = 1, \ldots, n \tag{7}
\]

to represent \( y^T Bz \) as \( s^T 1 \). Then use a McCormick transformation to replace Equation (7) by a set of linear constraints of the form

\[
G y + K s \geq e + H z, \quad s \geq 0,
\]

for some matrices \( G, K, H \), and some vector \( e \). Then, \( f(z) \) can be obtained by solving the following problem:

\[
f(z) = \min \, c^T x \tag{8a}
\]

s.t. \( Ax + Bz \geq b, \tag{8b} \)

\( -c^T x + b^T y - s^T 1 \geq 0, \tag{8c} \)

\( -A^T y \geq -c^T, \tag{8d} \)

\( Mz - Ey \geq -a, \tag{8e} \)

\( -Hz + G y + K s \geq e, \tag{8f} \)

\( x \geq 0, \quad y \geq 0, \quad s \geq 0. \tag{8g} \)
In the following, (MIP) denotes the resulting mixed-integer linear programming, i.e., Problem (5) where \( f(z) \) is defined by Problem (8).

### 3 A Dedicated Benders Decomposition Method for MIPAD

This section proposes a novel decomposition method for the Benders subproblem arising in MIPAD. Benders Decomposition (BD) is defined by a Relaxed Master Problem (RMP) and a Benders SubProblem (BSP). The RMP generates a guess \( \bar{z} \in \mathbb{B}^n \) for the binary variables \( z \) and its initial version is simply

\[
\min_{z \in \mathbb{B}^n} h^T z + t
\]

s.t. \( t \in \mathbb{R} \).  

(9)

For a guess \( \bar{z} \), the BSP is defined by the dual of Problem (8):

\[
\max \quad y^T (b - B\bar{z}) - \left[ c^T x + u^T (a + M\bar{z}) - v^T (e + H\bar{z}) \right]
\]

s.t. \( y^T A \leq c^T (w + 1) \),

\( Ax + E^T u - G^T v \geq bw \),

\( K^T v \leq 1w \),

\( y, w, x, u, v \geq 0 \).

(10a)

(10b)

(10c)

(10d)

(10e)

where \( y, w, x, u, \) and \( v \) represent the dual variable associated with Constraints (8b), (8c), (8d), (8e), and (8f) respectively. If Problem (10) is unbounded, then Problem (8) is infeasible and BD adds a feasibility cut to the RMP using an unbounded ray of Problem (10). If Problem (10) has a finite optimal value, which means \( \bar{z} \) is feasible to (MIP), then BD adds an optimality cut to the RMP. BD iteratively solves the updated RMP (Problem (9)) and the BSP (Problem (10)) until the value of \( t \) in the optimal RMP solution and the optimal BSP solution agree.

Unfortunately, for the applications considered in this paper, Problem (10) is highly complex since it has constraints from the primal problem (e.g., (10c) and (10d)), and from the dual problem (e.g., (10b)) which are linked by variable \( w \). The main contribution of this paper is to show that Problem (10) does not need to be solved as a whole. Rather, the Benders cuts of Problem (10) can be obtained by solving two more tractable problems, i.e., the dual problem (3) and the primal problem with some additional variables.

**Theorem 1** Problem (10) can be solved by solving Problem (3) and the problem

\[
\min \quad c^T x + u^T (a + M\bar{z}) - v^T (e + H\bar{z})
\]

s.t. \( Ax + E^T u - G^T v \geq bw \),

\( K^T v \leq 1w \),

\( x, u, v \geq 0 \).

(11a)

(11b)

(11c)

(11d)
Remark 1 The feasibility of Problem (11) is also guaranteed by (A1). Consider the dual of the restriction of Problem (11) where $u$ is fixed as 0:

\[
\begin{align*}
\max & \quad y^T b - s^T 1 \\
\text{s.t.} & \quad y^T A \leq c \\
& \quad Gy + Ks \geq e + H\bar{z}, \\
& \quad x, s \geq 0,
\end{align*}
\]

where $y$ and $s$ are dual variables associated with Constraints (11b) and (11c) respectively. For any binary $\bar{z}$, the McCormick relaxation (i.e., Constraint (12c)) is exact and Problem (12) becomes equivalent to Problem (3). If Problem (11) is infeasible, then Problem (3) must be unbounded for any binary $\bar{z}$, which contradicts (A1).

Proof of Theorem 1 The proof strategy is to show that there is a surjective mapping from the possible outcomes of Problems (3) and (11) to those of Problem (10), which implies that Problem (10) is completely determined by Problems (3) and (11).

Let $U_i$ and $F_i$ respectively denote the unbounded and finite outcome of Problem $i$ for $i \in \{3, 10, 11\}$. The combination of all possible outcomes of Problems (3) and (11) is given by

\[
A = \{U_3, (F_3, U_1), (F_3, F_1), (F_3, F_2)_1, (F_3, F_2)_2\}
\]

where the case of $(F_3, F_2)_k$ is divided into two disjoint cases with some additional condition, denoted by $(F_3, F_2)_k$ for $k = 1, 2$. Likewise, the possible outcomes of Problem (10) can be expressed as $B = \{U_{10}, F_{10}\}$. The proof gives a surjective mapping $g : A \rightarrow B$, showing the solution of Problem (10) can be obtained from the solutions of Problems (3) and (11).

1. Outcome $U_3$: Let $\hat{y}$ be the unbounded ray of Problem (3). Then $(y, x, u, v, w) = (\hat{y}, 0, 0, 0, 0)$ is an unbounded ray of Problem (10) and Problem (10) is unbounded.

2. Outcome $F_3$: Let $\mathcal{O}_1$ denote the optimal objective value of Problem (3) and $\hat{y}$ denote its optimal solution.

   (a) Outcome $U_1$: Let $(\hat{x}, \hat{u}, \hat{v})$ be the unbounded ray of Problem (11). Then $(y, x, u, v, w) = (0, \hat{x}, \hat{u}, \hat{v}, 0)$ is an unbounded ray of Problem (10) and Problem (10) is unbounded.

   (b) Outcome $F_1$: Let $\mathcal{O}_2$ denote the optimal objective value of Problem (11) and $(\hat{x}, \hat{u}, \hat{v})$ denote its optimal solution.

      i. Outcome $(F_1, F_1)$ with $\mathcal{O}_1 > \mathcal{O}_2$: $(y, x, u, v, w) = (\hat{y}, \hat{x}, \hat{u}, \hat{v}, 1)$ is an unbounded ray of Problem (10). For any $\alpha > 0$,

\[
(y, 0, 0, 0, 0) + \alpha(\hat{y}, \hat{x}, \hat{u}, \hat{v}, 1)
\]

is a feasible solution to Problem (10) and has objective value of $\hat{y}^T (b - B\bar{z}) + \alpha(\mathcal{O}_1 - \mathcal{O}_2)$, which increases as $\alpha$ increases. Hence Problem (10) is unbounded.
ii. Outcome \((F_0, F_1)\) with \(\mathcal{O}_1 \leq \mathcal{O}_2\): The proof is by a case analysis over two versions of Problem \((10)\) in which \(w = 0\) and \(w > 0\). When \(w = 0\), Problem \((10)\) can be decomposed into Problem \((3)\) and

\[
\begin{align*}
\min & \quad c^T x + u^T (a + M\bar{z}) - v^T (e + H\bar{z}) \\
\text{s.t.} & \quad Ax + E^T u - G^T v \geq 0, \\
& \quad K^T v \leq 0, \\
& \quad x, u, v \geq 0,
\end{align*}
\]

Problem \((13)\) is either unbounded or zero at optimality, since it has a trivial solution with all variables at zeros. Therefore, its optimum must be zero since otherwise Problem \((11)\) is unbounded. Hence, \((y, x, u, v, w) = (\bar{y}, 0, 0, 0, 0)\) is the optimal solution to Problem \((10)\). When \(w > 0\), by stating \((y', x', u', v', w') = (\frac{y}{\bar{w}}, \frac{x}{\bar{w}}, \frac{u}{\bar{w}}, \frac{v}{\bar{w}}, 1)\), Problem \((10)\) becomes separable into Problem \((3)\) and Problem \((11)\) and its optimal objective value becomes

\[
y^T (b - B\bar{z}) + w \left\{ y^T (b - B\bar{z}) - \left[c^T \bar{x} + u^T (a + M\bar{z}) - v^T (e + H\bar{z})\right]\right\}
\]

\[
= y^T (b - B\bar{z}) + w (\mathcal{O}_1 - \mathcal{O}_2).
\]

Note that, if \(\mathcal{O}_1 < \mathcal{O}_2\), the solution of Problem \((10)\) with a strictly positive value of \(w\) is dominated by the solution with \(w = 0\). Otherwise, \(\mathcal{O}_1 = \mathcal{O}_2\), the two solutions have the same optimal objective value, and \((y, x, u, v, w) = (\bar{y}, 0, 0, 0, 0)\) is the optimal solution to Problem \((10)\).

Theorem \((11)\) implies that there is no need to deal directly with the dual constraints of Problem \((8)\). Instead, it is sufficient to solve Problem \((2)\) (the Benders subproblem when there is no additional constraints on the dual variables) and Problem \((11)\), the modified primal problem.

Let \(\mathcal{J}_1\) and \(\mathcal{R}_1\) be the set of all extreme points and rays of Problem \((3)\) and \(\mathcal{J}_2\) and \(\mathcal{R}_2\) be the set of all extreme points and rays of Problem \((11)\), respectively. With \(\mathcal{J}_1, \mathcal{J}_2, \mathcal{R}_1,\) and \(\mathcal{R}_2\), MIP is equivalent to the following problem:

\[
\begin{align*}
\min & \quad h^T z + t \\
\text{s.t.} & \quad t \geq y^T (b - Bz), & & \forall y \in \mathcal{J}_1, \\
& \quad 0 \geq \tilde{y}^T (b - B\bar{z}), & & \forall \tilde{y} \in \mathcal{R}_1, \\
& \quad c^T \bar{x} + u^T (a + M\bar{z}) - v^T (e + Hz) \geq 0, & & \forall (\bar{x}, \hat{u}, \hat{v}) \in \mathcal{R}_2, \\
& \quad c^T \bar{x} + u^T (a + M\bar{z}) - v^T (e + Hz) \geq \tilde{y}^T (b - B\bar{z}), & & \forall (\tilde{y}, \hat{u}, \hat{v}) \in \mathcal{J}_1 \times \mathcal{J}_2.
\end{align*}
\]

Observe that Equation \((14d)\) is valid for all combinations of extreme points of Problems \((3)\) and \((11)\), since it enforces \(O_1(z) \leq O_2(z)\), where \(O_1\) and \(O_2\) are the objective functions of Problems \((3)\) and \((11)\). Similarly, Equation \((14e)\) is valid for all extreme
Algorithm 1: The Benders Separation Algorithm.

begin
Input: \( z \in \mathbb{B}^{n} \)
Solve Problem (3);
if \( O_{1} = \infty \) with an unbounded ray \( \hat{y} \in \mathbb{R}^{1} \) then
Add the feasibility cut \( \hat{y}^{T} (b - Bz) \leq 0 \) to the RMP;
else
Obtain its optimal dual solution \( \hat{y} \in J_{1} \);
Solve Problem (11);
if \( O_{2} = -\infty \) with an unbounded ray \( (\hat{x}, \hat{u}, \hat{v}) \in \mathbb{R}^{2} \) then
Add the feasibility cut \( c^{T} \hat{x} + \hat{u}^{T} (a + Mz) - \hat{v}^{T} (e + Hz) \geq 0 \) to the RMP;
else
Obtain its optimal solution \( (\hat{x}, \hat{u}, \hat{v}) \in J_{2} \);
if \( O_{1} > O_{2} \) then
Add the feasibility cut \( \hat{y}^{T} (b - Bz) - c^{T} \hat{x} - \hat{u}^{T} (a + Mz) + \hat{v}^{T} (e + Hz) \leq 0 \) to the RMP;
else
Add the optimality cut \( t \geq \hat{y}^{T} (b - Bz) \) to the RMP; Update the best primal bound with the obtained feasible solution;
end

Let \( C_{1}, C_{2}, C_{3}, C_{4} \) denote the set of all constraints in Equation (14a), (14b), (14c), and (14d) respectively. At each iteration, the RMP is a relaxation of Problem (14) with a subset of the constraints, i.e., \( \tilde{C}_{1} \subseteq C_{1}, \tilde{C}_{2} \subseteq C_{2}, \tilde{C}_{3} \subseteq C_{3}, \) and \( \tilde{C}_{4} \subseteq C_{4} \). The Benders separation routine at each iteration for an optimal solution \( \bar{z} \) of the RMP is given by Algorithm 1 instead of by solving Problem (10) and produces a violated constraints in \( C_{i} \setminus \tilde{C}_{i} \), for some \( i \in \{1, \cdots, 4\} \).

4 Normalizing Benders Feasibility Cuts

Fischetti et al. [5] have shown that normalizing the ray used in Benders feasibility cuts can improve the performance of Benders decomposition. The main contribution of this section is to show that the Benders subproblem decomposition can be generalized to produce a normalized ray.

When Problem (10) is unbounded, the problem at hand consists in solving (10) with an additional normalization constraint, i.e.,

\[
\begin{align*}
\text{max} & \quad y^{T} (b - B\bar{z}) - \left[ c^{T} x + u^{T} (a + M\bar{z}) - v^{T} (e + H\bar{z}) \right] \\
\text{s.t.} & \quad y^{T} A \leq c^{T} w, \\
& \quad Ax + E^{T} u - G^{T} v \geq bw, \\
& \quad K^{T} v \leq 1w, \\
& \quad \| (y, x, u, v, w) \|_{1} = 1, \\
& \quad y, w, x, u, v \geq 0.
\end{align*}
\]
Note that, in Problem (15), the constants have been set to zero, since the goal is to find a ray. In particular, the right-hand side of Equation (15b) has become $c^Tw$ instead of $c^T(w + 1)$. The proof of Theorem 1 showed that Problem (10) has three different types of extreme unbounded rays:

(i) $(\hat{y}, 0, 0, 0)$ for $\hat{y} \in \mathcal{R}_1$, when Problem (3) is unbounded.
(ii) $(0, \hat{x}, \hat{u}, \hat{v})$ for $(\hat{x}, \hat{u}, \hat{v}) \in \mathcal{R}_2$, when Problem (3) has a finite optimal objective value and Problem (11) is unbounded.
(iii) $(\hat{y}, \hat{x}, \hat{u}, \hat{v}, 1)$ for $(\hat{y}, \hat{x}, \hat{u}, \hat{v}) \in \mathcal{F}_1 \times \mathcal{F}_2$, when both Problem (3) and Problem (11) have finite optimal values and $\partial \Omega > \partial O_2$.

Cases (i) and (ii) are simple: It suffices to solve Problem (3) and Problem (11) with the additional constraint of $\|y\|_1 = 1$ and $\|(x, u, v)\|_1 = 1$ respectively. Case of (iii) is more difficult and requires to find a normalized ray $\hat{r} = (\hat{y}, \hat{x}, \hat{u}, \hat{v}, \hat{w})$ that maximizes Equation (15g) while satisfying $\|\hat{r}\|_1 = 1$ and $\hat{w} > 0$. Note that $(\hat{y}, \hat{x}, \hat{u}, \hat{v}, 1)/\|y, \hat{x}, \hat{u}, \hat{v}, 1\|_1$ is a feasible solution to Problem (15). Hence, Problem (15) is feasible and bounded.

Consider the Lagrangian relaxation of Problem (15) with $w > 0$ that penalizes the violation of Constraint (15c) with some $\lambda \in \mathbb{R}$:

$$\lambda + \sup_{w > 0} y^T(b - Bz - \lambda 1) - [(c + \lambda 1)^T x + u^T(a + Mz + \lambda 1) - v^T(e + Hz - \lambda 1)] - \lambda w$$

$$\text{s.t. } y^T A \leq c^Tw, \quad (16a)$$
$$Ax + E^T u - G^T v \geq bw, \quad (16b)$$
$$K^T v \leq lw, \quad (16c)$$
$$y, x, u, v \geq 0, w > 0, \quad (16d)$$

For $w > 0$, using $(y', x', u', v', w') = (\frac{y}{w}, \frac{x}{w}, \frac{u}{w}, \frac{v}{w}, 1)$, Problem (15) becomes

$$\min_{\lambda \in \mathbb{R}} \left\{ \lambda + \sup_{w > 0} \left\{ w(t^1(\lambda) - t^2(\lambda) - \lambda) \right\} \right\}, \quad (17)$$

where

$$t^1(\lambda) = \max_{w > 0} y^T(b - Bz - \lambda 1), \quad (18a)$$
$$\text{s.t. } (3b), \quad (18b)$$

and

$$t^2(\lambda) = \min (c + \lambda 1)^T x + u^T(a + Mz + \lambda 1) - v^T(e + Hz - \lambda 1) \quad (19a)$$
$$\text{s.t. } (11b), (11c), (11d). \quad (19b)$$

Define $t(\lambda) := t^1(\lambda) - t^2(\lambda) - \lambda$. If $t(\lambda) < 0$, the optimal objective value of the inner optimization problem of Problem (17) approaches zero as $w$ converges to
0. If \( t(\hat{\lambda}) > 0 \) then Problem (17) is unbounded. Therefore, Problem (15) becomes equivalent to the following problem:

\[
\min_{\lambda \in \mathbb{R}} \{ \lambda : t(\lambda) \leq 0 \}.
\]  

Note that \( t(\hat{\lambda}) \) is non-increasing in \( \lambda \) and \( t(0) = \theta_1 - \theta_2 > 0 \). Therefore, the optimal solution \( \hat{\lambda}^* \) of Problem (20) is the solution of \( t(\hat{\lambda}) = 0 \). Since \( t(\hat{\lambda}) \) is a convex piecewise linear function of \( \lambda \), Problem (20) can be solved by Newton’s method, using subgradients (instead of gradients) as shown in Algorithm 2. At each iteration, \(-\hat{y}_k, \hat{u}_k, \hat{v}_k, 1\)^T \( I \), where \( \hat{y}_k \) and \( (\hat{u}_k, \hat{v}_k, \hat{v}_k) \) are the solutions of \( t^1(\lambda_k) \) and \( t^2(\lambda_k) \) respectively, is a subgradient of \( t \) at \( \lambda_k \) and is denoted by \( \delta t(\lambda_k) \). Observe that Problems (18) and (19) are the counterparts to Problem (3) and (11), demonstrating that the subproblem decomposition carries over to the decomposition.

5 An In-Out Approach

Ben-Ameur et al. [11] proposed an acceleration scheme (the in-out method) for general cutting-plane algorithms. The method carefully chooses the separation point, rather than using the solution obtained from the RMP. The method considers two points: a feasible point \( z_{in} \) to Problem (14) and the optimal solution \( z_{out} \) of the RMP. It uses a convex combination of these two points when generating the separating cut, i.e., it solves Problem (10) with \( \bar{z} = \lambda z_{in} + (1 - \lambda)z_{out} \) for some \( \lambda \in (0, 1) \).

Fischetti et al. [4] applied the in-out approach with an additional perturbation to solve facility location problems:

\[
\bar{z} = \lambda z_{in} + (1 - \lambda)z_{out} + \epsilon I,
\]  

for some \( \lambda \in (0, 1) \) and \( \epsilon > 0 \), and showed a computational improvement.

This paper also employs the in-out approach equipped with some perturbation as Fischetti et al. [4]. It periodically finds \( z_{in} \) in a heuristic manner and chooses the separation point according to Equation (21). The implementation starts with \( \lambda = 0.5 \) and \( \epsilon = 10^{-6} \) and decrease \( \lambda \) by half if the BD halts (i.e., it does not improve the optimality gap for more than 3 consecutive iterations). If the algorithm halts and \( \lambda \) is smaller than \( 10^{-5} \), \( \epsilon \) is set to 0. After 3 more consecutive iterations without a lower bound improvement, the algorithm returns to the original BD. Whenever a new best incumbent solution is found, the in-out approach is applied again with this new feasible point.
6 Performance Analysis of the Dedicated Benders Method

This section studies the performance of the decomposition approach (Section 3) and the benefits of the acceleration schemes explained in Sections 4 and 5. All algorithms were implemented with the C++/Gurobi 8.0.1 interface and executed on an Intel Core i5 PC at 2.7 GHz with 8 GB of RAM. Each run has a wall-time limit of 1 hour.

6.1 Test Instances

The evaluation is performed on the instances of the UCGNA problem [3]. The instances are based on the gas-grid test system, which is representative of the natural gas and electric power systems in the Northeastern United States [2]. There are 42 different instances, each of which constructed by uniformly increasing the demand of each system by some percentage; \(\eta_p\) denotes the stress level imposed on the power system which takes values from \{1, 1.3, 1.6\} and \(\eta_g\) denotes the stress level of the gas system that has values of \{1, 1.1, \ldots, 2.2, 2.3\}. For example, \((\eta_p, \eta_g) = (1.3, 2.3)\) means the demands of the power and natural gas systems are increased uniformly by 30% and 130% respectively. Before we experiment the solution approaches on the instances of the UCGNA problem, we apply some preprocessing step from [3] which eliminates invalid bids with regard to a lower bound on natural gas zonal prices. Detailed description of the instances and the preprocessing step can be found in [3].

6.2 Computational Performance

This section compares three different solution approaches for MIPAD:

- D: our dedicated Benders method (Section 3) with the acceleration schemes (Sections 4 and 5);
- G: a state-of-the-art solver (Gurobi 8.0.1);
- B: the standard Benders method with the acceleration schemes (Sections 4 and 5).

The implementation of D is sequential, although Problems (3) and (11) can be solved independently. All solution approaches use the same values for the Gurobi parameters, i.e., the default values except NumericFocus set at 3, DualReductions at 0, ScaleFlag at 0, BarQCPConvTol at 1e-7, and Aggregate at 0 for more rigorous attempts to detect and manage numerical issues.

Tables 1-3 report the computation times and optimality gaps of the three solution methods. The symbol † indicates that a method reaches the time limit and the symbol ‡ that the method did not find any incumbent solution.

The results for \(\eta_p = 1\) are summarized in Table 1. D timed out for two instances, G reached the time limit for 5 instances, and B timed out for all the instances. For the two instances with \(\eta_p = 1.8, 1.9\), where all methods time out, D found incumbent solutions within optimality gaps of 1.8% and 1.3% and B found solutions with gaps of 6.7% and 10.6%. On the other hand, G did not find any incumbent solution. For easy instances that both D and G found optimal solutions within two minutes, G is faster than D by a factor of 2 in average.
Table 1: Computational Performance Comparison ($\eta_p = 1$).

| Instance | D   | G   | B   |
|----------|-----|-----|-----|
|          | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) |
| 1        | 25.42 | 0.0  | 15.28 | 0.0  | † 6.8   |
| 1.1      | 25.91 | 0.0  | 23.24 | 0.0  | † 4.3   |
| 1.2      | 25.86 | 0.0  | 14.78 | 0.0  | † 2.2   |
| 1.3      | 29.33 | 0.0  | 31.17 | 0.0  | † 4.4   |
| 1.4      | 26.60 | 0.0  | 6.76  | 0.0  | † 2.6   |
| 1.5      | 25.80 | 0.0  | 13.24 | 0.0  | † 6.2   |
| 1.6      | 27.04 | 0.0  | 33.56 | 0.0  | † 3.1   |
| 1.7      | 100.82| 0.0  | 22.78 | 0.0  | † 4.5   |
| 1.8      | †    | 1.3  | †    | †    | † 10.6  |
| 1.9      | †    | 1.3  | †    | ‡    | † 10.8  |
| 2.0      | 67.13 | 0.0  | †    | 3.2  | † 20.0  |
| 2.1      | 1091.88| 0.0 | †    | 3.2  | † 20.0  |
| 2.2      | 566.94| 0.0  | †    | 3.6  | † 19.1  |
| 2.3      | 31.52 | 0.0  | 15.94 | 0.0  | † 8.4   |

Table 2: Computational Performance Comparison ($\eta_p = 1.3$).

| Instance | D   | G   | B   |
|----------|-----|-----|-----|
|          | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) |
| 1        | 31.01 | 0.0  | 4.37  | 0.0  | † 1.9   |
| 1.1      | 28.93 | 0.0  | 3.20  | 0.0  | † 2.8   |
| 1.2      | 30.87 | 0.0  | 3.28  | 0.0  | † 2.9   |
| 1.3      | 48.22 | 0.0  | 2.93  | 0.0  | † 3.3   |
| 1.4      | 32.69 | 0.0  | 12.07 | 0.0  | † 3.8   |
| 1.5      | 44.13 | 0.0  | 23.89 | 0.0  | † 2.2   |
| 1.6      | †    | 0.3  | †    | †    | † 4.1   |
| 1.7      | †    | 3.5  | †    | †    | † 11.0  |
| 1.8      | †    | 3.2  | †    | †    | † 10.9  |
| 1.9      | †    | 3.3  | †    | †    | † 17.4  |
| 2        | †    | 4.2  | †    | 19.9 | † 14.9  |
| 2.1      | †    | 4.3  | †    | †    | † 9.7   |
| 2.2      | †    | 4.0  | †    | †    | † 14.8  |
| 2.3      | 43.23 | 0.0  | 10.43| 0.0  | † 5.7   |

For instances with $\eta_p = 1.3$, reported in Table 2, D and G timed out for 7 instances and B reached the time limit for all the instances. For the 7 instances with $\eta_g = 1.6, \cdots, 2.2$, where all methods reached the time limit, D found incumbent solutions within 4.3% of optimality and B found worse solutions. On the other hand, G did not find any incumbent solution except the two instances with $\eta_g = 1.6$ and 2. For easy instances that both D and G found optimal solutions within two minutes, G is faster than D by a factor of around 7 in average.

Instances with $\eta_p = 1.6$ display similar behaviors. While B failed to find optimal solutions for all the instances, D and G found optimal solutions for 7 instances. For the hard instances where all methods timed out, D found incumbent solutions with optimality gaps less than 7.5%, B found worse solutions, and G failed to find any
incumbent solution. For the instances where both D and G found optimal solutions, G is faster than D.

To compare the computational performance of D and G more precisely, Figure 1 visualizes the performance of D and G for all the instances. Figure 1a reports the computation times of D and G, Figure 1b displays the optimality gaps of the two methods for all the instances, and the reference lines (in red) serve to delineate when a method is faster than the other. For Figure 1b, the axes are in logarithmic scale and a 100% optimality gap is assigned to instances with no incumbent. The figure indicates that, although D is slower than G for some easy instances (the points at the bottom left corner of Figure 1a), it has notable benefits for hard instances (the points in the upper left side of Figures 1a and 1b).
6.3 Benefits of the Acceleration Schemes

This section studies the benefits of the acceleration schemes by comparing the performance of the dedicated Benders method with different combinations of acceleration schemes applied. It uses $D(n_k, i_k)$ to denote the dedicated Benders method with acceleration schemes ($n_k$, $i_k$) where

- $n_k$: $k = 1$ if the normalization scheme is applied; $k = 0$ otherwise;
- $i_k$: $k = 1$ if the in-out approach is applied; $k = 0$ otherwise.

Tables 4-6 summarize the computational performance of the dedicated Benders methods with the four combinations of acceleration schemes.

Table 4 displays the computation times and optimality gaps for instances with $\eta_p = 1$. Without the in-out approach, $D(n_1, i_0)$ and $D(n_0, i_0)$ timed out for all instances. Although both $D(n_1, i_0)$ and $D(n_0, i_0)$ reach the time limit for all instances, the normalization scheme does improve optimality gaps. On the other hand, with the in-out approach, $D(n_0, i_1)$ solved 10 instances within 100 seconds. However, $D(n_0, i_1)$ still cannot solve the two instances with $\eta_p = 2.1, 2.2$. The slight increase in computation time of $D(n_1, i_1)$ for some instances, compared to $D(n_0, i_1)$, is due to the additional computation time required to find a normalized ray.

The results for instances with $\eta_p = 1.3$ are reported in Table 5. Again, without the in-out approach, $D(n_1, i_0)$ and $D(n_0, i_0)$ timed out for all instances, but $D(n_1, i_0)$ has significant improvement in optimality gaps for some instances. With the in-out approach, $D(n_0, i_1)$ solved 7 instances within 150 seconds and so did $D(n_1, i_1)$. The normalization scheme does have some computational benefits, as $D(n_1, i_1)$ has smaller optimality gaps than $D(n_0, i_1)$ for the remaining 7 instances except one instance with $\eta_p = 2.2$. Moreover, for some hard instances where $D(n_0, i_1)$ reached the time limit, $D(n_1, i_0)$ has smaller optimality gaps (i.e., $\eta_p = 1.7, \cdots, 2$).

The acceleration schemes display similar behaviors for instances with $\eta_p = 1.6$. Without the in-out approach, $D(n_0, i_0)$ timed out for all instances, while $D(n_1, i_0)$...
Table 5: Benefits of the Acceleration Schemes ($\eta_p = 1.3$).

| $\eta_g$ | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) |
|---|---|---|---|---|---|---|---|---|
| 1  | 31.01 | 0.00 | 30.83 | 0.00 | † | 63.96 | † | 63.78 |
| 1.1 | 28.93 | 0.00 | 27.83 | 0.00 | † | 54.30 | † | 63.93 |
| 1.2 | 30.87 | 0.00 | 143.36 | 0.00 | † | 60.95 | † | 63.65 |
| 1.3 | 48.22 | 0.00 | 52.89 | 0.00 | † | 56.01 | † | 64.09 |
| 1.4 | 32.69 | 0.00 | 31.04 | 0.00 | † | 51.67 | † | 64.85 |
| 1.5 | 44.13 | 0.00 | 44.98 | 0.00 | † | 53.98 | † | 64.80 |
| 1.6 | † | 0.31 | † | 1.08 | † | 1.94 | † | 65.07 |
| 1.7 | † | 3.53 | † | 5.34 | † | 3.42 | † | 65.99 |
| 1.8 | † | 3.15 | † | 4.01 | † | 3.73 | † | 65.92 |
| 1.9 | † | 3.26 | † | 8.28 | † | 7.97 | † | 66.22 |
| 2  | † | 4.24 | † | 4.59 | † | 4.51 | † | 64.58 |
| 2.1 | † | 4.27 | † | 4.12 | † | 4.29 | † | 63.36 |
| 2.2 | † | 4.03 | † | 4.07 | † | 4.08 | † | 64.46 |
| 2.3 | 43.23 | 0.00 | 48.06 | 0.00 | † | 14.51 | † | 62.93 |

Table 6: Benefits of the Acceleration Schemes ($\eta_p = 1.6$).

| $\eta_g$ | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) | Time (s) | Gap (%) |
|---|---|---|---|---|---|---|---|---|
| 1  | 43.51 | 0.00 | 44.01 | 0.00 | † | 45.17 | † | 69.59 |
| 1.1 | 27.88 | 0.00 | 26.88 | 0.00 | † | 59.44 | † | 69.33 |
| 1.2 | 26.63 | 0.00 | 26.84 | 0.00 | † | 14.54 | † | 69.51 |
| 1.3 | 22.19 | 0.00 | 30.55 | 0.00 | † | 34.22 | † | 69.81 |
| 1.4 | 29.75 | 0.00 | 30.51 | 0.00 | † | 6.91 | † | 69.95 |
| 1.5 | 330.88 | 0.00 | 208.22 | 0.00 | † | 2.58 | † | 71.69 |
| 1.6 | † | 2.10 | † | 2.09 | † | 2.13 | † | 71.43 |
| 1.7 | † | 2.05 | † | 3.84 | † | 2.11 | † | 71.73 |
| 1.8 | † | 6.16 | † | 7.80 | † | 6.68 | † | 71.86 |
| 1.9 | † | 7.43 | † | 7.62 | † | 7.49 | † | 71.80 |
| 2  | † | 3.75 | † | 3.81 | † | 3.77 | † | 67.66 |
| 2.1 | † | 5.04 | † | 5.15 | † | 5.05 | † | 68.12 |
| 2.2 | † | 5.01 | † | 5.15 | † | 5.01 | † | 67.27 |
| 2.3 | 12.44 | 0.00 | 13.75 | 0.00 | 73.32 | 0.00 | † | 67.84 |

solves one instance to optimality and has significant improvements in optimality gaps. With the in-out approach, both $D(n_0, i_1)$ and $D(n_1, i_1)$ solve 7 instances within 350 seconds, and $D(n_1, i_0)$ has smaller optimality gaps for the unsolved instances. Again, for some hard instances for which $D(n_0, i_1)$ reached the time limit, $D(n_1, i_0)$ has smaller optimality gaps (i.e., $\eta_g = 1.7, \ldots, 2.2$).

6.4 Benefits of the Decomposition Method

Section 6.2 indicated that the decomposition method has significant benefits for solving MIPAD. The decomposition method not only shortens computation times re-
quired for solving the dual of the inner-continuous problem, but also allows us to address the numerical issues of MIPAD.

Figure 2 displays the average computation time for generating a Benders cut, where the error bars represent the standard deviation. In average, the cut generation time of \( D \) is faster than \( B \) by a factor of 3.94. Since the subproblems that \( D \) solves to generate cuts (i.e., Problems (3) and (11)) can be solved independently, an implementation in parallel computing would improve the computation time even further.

Moreover, the decomposition method deals better with numerical issues arising from the complex inner-continuous problem of MIPAD. Figure 3 displays the convergence behavior of \( D \) and \( B \) for two instances, \((\eta_p, \eta_g) = (1, 1.2), (1.6, 1.8)\). For instance \((\eta_p, \eta_g) = (1, 1.2)\) (i.e., Figure 3a and Figure 3b), \( D \) closes the gap in 30 seconds, but \( B \) does not improve its lower bound even if it finds a good incumbent solution early. For instance \((\eta_p, \eta_g) = (1.6, 1.8)\) (i.e., Figure 3c and Figure 3d), although both \( D \) and \( B \) timed out, \( B \) improves its lower bound much slower than \( D \). This behavior of \( B \) is explained by the fact that it suffers from numerical issues when solving Problem (10); it sometimes terminates with an optimal solution even if there exists an unbounded ray. This incorrect evaluation of the first-stage variable leads to ineffective cut generation and a slower convergence rate. On the other hand, the decomposition method effectively decomposes Problem (10) into two more stable and smaller problems, which addresses the numerical issues effectively.

7 Conclusion

This paper was motivated by multi-level mathematical programs where solutions of the inner-most problem impose constraints on the binary variables of the first-level problem. These mathematical programs arise naturally when connecting interdependent physical infrastructures operated independently. Through strong duality and McCormick reformulations, these mathematical programs can be rewritten as Mixed-Integer Conic Linear Programming with Additional Constraints on Dual Variables (MIPAD).
This paper proposed a dedicated Benders decomposition algorithm to solve MI-PAD, recognizing that the Benders subproblem is not necessarily easy to solve for large MIPADs. The dedicated approach decomposes the Benders subproblem into two more tractable problems: the dual problem and the primal problem with some additional variables and constraints. In addition, the paper shows how to adapt existing acceleration schemes to this decomposition. In particular, it shows how to normalize Benders feasibility cuts using a Newton’s (subgradient) method and how to carefully choose the separation points using the in-out approach [1].

The resulting Benders method significantly improves the performance of a standard Benders method and outperforms a state-of-the-art mathematical-programming solvers for hard instances. The experimental results showed the benefits of acceleration schemes—normalizing feasibility rays and the in-out approach—and demonstrated that decomposing the Benders subproblem not only shortens the computation time for generating Benders cuts but also addresses the numerical issues arising when solving complex Benders subproblems.
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