New Coalescences for the Painlevé Equations

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Abstract

The Painlevé equations are here connected to other classes of equations with the Painlevé Property (Ince’s equations) by the same degeneracy procedure that connects the Painlevé equations (coalescence). These Ince’s equations here are also connected among themselves like in the traditional Painlevé’s coalescence cascade. Such degeneracy is considered also for the special equations, symmetric equations and Bäcklund transformations.

1 Introduction

At the end of the 19th century, Painlevé and his collaborators worked on the challenge of finding all possible rational second-order ordinary differential equations free of movable critical points (Painlevé Property). By a laborious effort on calculations and classification, such work resulted in 50 classes of equations that comprehend all rational second-order equations with the Painlevé Property up to Möbius Transformations. Among these 50, 44 could be linearized or solved by known functions, but 6 could not, therefore they defined new functions, the so-called Painlevé Transcendents.

Such a list of the 50 equations can be found in its full presentation on the classical Ince’s book of differential equations [8].

Since then, Painlevé equations have been continuously found in the most diverse areas of mathematical physics and especially for integrable models. Painlevé equations appearing as reductions for integrable PDEs is the core of the ARS conjecture [1], and are reductions for Dressing Chain of the Schrödinger operator [16]. Together with them, elliptic functions (here, some of Ince’s equations) play a major role in the theory of solitons for the traveling wave reduction of integrable models [16][5].

Gambier [6] described that we can reduce the list of 50 equations to only 24 by identifying that the other 26 belong to orbits of these 24 classes of equivalence by birational transformations [9].

Gambier in his paper [6] lists the 24 equations in a table, such that its correspondence with the usual list of 50 from Ince [8] is as follows:

| Gambier | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | ... |
|---------|---|---|---|---|---|---|---|---|---|----|----|----|-----|
| Ince (n) | 1 | 2 | 3 | 4 | P_I | 6 | 5 | 7 | 8 | 9 | P_{II} | 11 | 12 | P_{III} |
| ... | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | ... |
| ... | 14 | 24 | 27 | 29 | 30 | 31 | P_{IV} | 32 | 37 | 38 | 39 | P_{V} | 49 | 50 | P_{VI} |

Since Ince is the main reference for them, I will refer to the equations as $I_n$, where the index $n$ is their number on such list. The Painlevé equations will be kept as $P_n$ since this is the usual notation.

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Composing these 24, we have the 6 Painlevé equations \((P_I, ..., P_V)\), 6 autonomous equations solvable by Elliptic functions \((I_3, I_8, I_{12}, I_{30}, I_{38}, I_{49})\), 7 absent of parameters \((I_1, I_2, I_7, I_{11}, I_9, I_{32}, I_{37})\), and 5 with arbitrary functions \((I_5, I_6, I_{14}, I_{24}, I_{27})\).

P. Painlevé himself saw himself that it was possible to connect six Painlevé equations noticing that one can transform their variables and parameters artificially introducing a parameter (usually called \(\epsilon\)) in a so specific way that the limiting procedure of \(\epsilon \to 0\) turns an equation into some other. The \(P_{VI}\) equation is considered a “master” equation for the other five since it degenerates into them [11].

The Painlevé equations are non-linear differential equations with their critical points being just poles, they also have parameters that allow one to construct an infinite chain of solutions for each set of parameters through Bäcklund Transformations [10]. Since the degeneracy cascade mentioned above change the nature of the poles by “coalescing” them at each step, it is commonly known as coalescence cascade. The coalescence also coalesces the parameters as is seen in section (7) and in [14].

The goal of this paper is to present a full degeneracy cascade connecting not only the usual 6 Painlevé equations, which is known but also the other 13 equations. The 5 equations with arbitrary functions, cannot be generated by this limiting procedure like the others, therefore will not be considered. Since the word “coalescence” brings an idea of quantities coalescing and this may not be the case in some situations here, I will keep it only for the Painlevé and autonomous equations, and calling by “degeneracy”, which is more general, the other results presented here.

2 The traditional coalescence cascade

The original coalescence cascade for Painlevé equations was discovered by Painlevé himself [11] and is here reproduced in order to extend it in the following sections.

As a matter of notation, we will use upper case letters for the equations that degenerate into the other and lower case letters for the resulting equation. The equations will always be composed by a function \(w(z)\) with the appropriate parameters \(a, b, c, d, e\). \(\epsilon\) will be the parameter to be taken to zero.

The cascade will be as follows

\[ P_{VI} \rightarrow P_{V} \rightarrow P_{IV} \rightarrow P_{II} \rightarrow P_{I} \]

We start by the \(P_{VI}\) equation

\[
P_{VI} : \quad W'' = \frac{1}{2} \left( \frac{1}{W - Z} + \frac{1}{W} + \frac{1}{W - 1} \right) (W')^2 - \left( \frac{1}{W - Z} + \frac{1}{Z} + \frac{1}{Z - 1} \right) W' + \frac{W(W - 1)(W - Z)}{Z^2(Z - 1)^2} \left( A + \frac{BZ}{W^2} + \frac{C(Z - 1)}{(W - 1)^2} + \frac{D(Z - 1)Z}{(W - Z)^2} \right) \]

that with the transformation

\[ W(Z) = w(z), \quad Z = 1 + \epsilon z, \quad A = a, \quad B = b, \quad C = \frac{c}{\epsilon} - \frac{d}{\epsilon^2}, \quad D = \frac{d}{\epsilon} \]

followed by \(\epsilon \to 0\), becomes the \(P_{V}\) equation:

\[
P_{V} : \quad w'' = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) (w')^2 - \frac{w'}{z} + \frac{(w - 1)^2}{z^2} \left( aw + \frac{b}{w} \right) + \frac{cw}{z} + \frac{d(w + 1)w}{w - 1} \]

The \(P_{V}\) equation [11] above can degenerate to both \(P_{IV}\) and \(P_{II}\) with different transformations.
The $P_V$ equation (4) with the transformations
\[ W(Z) = \frac{\epsilon w(z)}{\sqrt{2}}, \quad Z = 1 + \sqrt{2} \epsilon z, \quad A = \frac{1}{2 \epsilon^3}, \quad B = \frac{b}{4}, \quad C = -\frac{1}{\epsilon^4}, \quad D = \frac{a}{\epsilon^2} - \frac{1}{2 \epsilon^4} \]
followed by $\epsilon \to 0$, becomes the $P_{IV}$ equation:
\[ P_{IV}: \quad w'' = \frac{3w^3}{2} + \frac{(w')^2}{2w} + 4w^2 z + 2w (z^2 - a) + \frac{b}{w} \]  

The $P_V$ equation (4) with the transformations:
\[ W(Z) = 1 + \epsilon zw(z), \quad Z = z^2, \quad A = \frac{a}{4 \epsilon} + \frac{c}{8 \epsilon^2}, \quad B = -\frac{c}{8 \epsilon^2}, \quad C = \frac{eb}{4}, \quad D = \frac{de^2}{8} \]
followed by $\epsilon \to 0$, becomes the $P_{III}$ equation:
\[ P_{III}: \quad w'' = \left( \frac{w'}{w} - \frac{aw^2 + b}{z} \right) + cw^3 + \frac{d}{w} \]  

The $P_{IV}$ equation (6) with the transformations
\[ W(Z) = 2^{2/3} \frac{w(z)}{\epsilon} + \frac{1}{\epsilon^3}, \quad Z = 2^{-2/3} \epsilon z - \frac{1}{\epsilon^3}, \quad A = -2a - \frac{1}{2 \epsilon^6}, \quad B = -\frac{1}{2 \epsilon^{12}} \]
followed by $\epsilon \to 0$, becomes the $P_{II}$ equation:
\[ P_{II}: \quad w'' = 2w^3 + wz + a \]  

The $P_{III}$ equation (8) with the transformations
\[ W(Z) = 1 + 2 \epsilon w(z), \quad Z = 1 + \epsilon^2 z, \quad A = -\frac{1}{2 \epsilon^6}, \quad B = \frac{1}{2 \epsilon^6} + \frac{2a}{\epsilon^3}, \quad C = -D = \frac{1}{4 \epsilon^6} \]
followed by $\epsilon \to 0$, becomes the $P_{II}$ equation (10).

The $P_{II}$ equation (10) with the transformations
\[ W(Z) = \epsilon w(z) + \frac{1}{\epsilon^3}, \quad Z = \epsilon^2 z - \frac{6}{\epsilon^10}, \quad A = \frac{4}{\epsilon^{15}} \]
followed by $\epsilon \to 0$, becomes the $P_I$ equation:
\[ P_I: \quad w'' = 6w^2 + z \]  

3 Autonomous Painlevé equations

These autonomous equations are not solvable by Painlevé transcendentst, but by Elliptic functions. They are named like that due to their direct connection with each of the Painlevé equations both in their form as in degeneracy.

These limits are present explicitly here, which will connect the following equations:
So, the $P_{VI}$ equation (1) under the transformations:

$$W(z) = w(z), \quad Z = a + \epsilon z, \quad A = \frac{(a - 1)^2}{e^2}, \quad B = \frac{ae(a - 1)}{e^2}, \quad C = \frac{a^2d(a - 1)}{e^2}, \quad D = \frac{ae(a - 1)}{e^2}$$

followed by $\epsilon \to 0$, becomes the $I_{49}$ equation:

$$I_{49}: \quad w'' = \frac{1}{2} \left( 1 \frac{1}{w - a} + \frac{1}{w} + \frac{1}{w - 1} \right) (w')^2 + w(w - 1)(w - a) \left( b + \frac{c}{w^2} + \frac{d}{(w - 1)^2} + \frac{e}{(w - a)^2} \right)$$

The $P_{V}$ equation (4) under the transformations:

$$W(z) = w(z), \quad Z = \epsilon z, \quad A = \frac{a}{e^2}, \quad B = \frac{b}{e^2}, \quad C = \frac{2c - d}{2e^2}, \quad D = \frac{d}{2e^2}$$

followed by $\epsilon \to 0$, becomes the $I_{38}$ equation:

$$I_{38}: \quad w'' = \left( \frac{1}{2w} + \frac{1}{w - 1} \right) (w')^2 + (w - 1)w \left( a(w - 1) + \frac{b(w - 1)}{w^2} + \frac{c}{w - 1} + \frac{d}{(w - 1)^2} \right)$$

The $P_{IV}$ equation (6) under the transformations:

$$W(z) = \frac{w(z)}{\epsilon}, \quad Z = \epsilon z + \frac{a}{\epsilon}, \quad A = \frac{a^2 - b}{e^2}, \quad B = \frac{c}{e^2}, \quad C = a, \quad D = \frac{d}{e^3}$$

followed by $\epsilon \to 0$, becomes the $I_{30}$ equation:

$$I_{30}: \quad w'' = \frac{3w^3}{2} + \frac{(w')^2}{2w} + 4aw^2 + 2bw + \frac{c}{w}$$

The $P_{III}$ equation (8) under the transformations:

$$W(z) = \frac{w(z)}{\epsilon}, \quad Z = \epsilon z, \quad A = \frac{b}{e^3}, \quad B = \frac{c}{e^3}, \quad C = a, \quad D = \frac{d}{e^3}$$

followed by $\epsilon \to 0$, becomes the $I_{12}$ equation:

$$I_{12}: \quad w'' = \frac{(w')^2}{w} + aw^3 + bw^2 + c + \frac{d}{w}$$

The $P_{II}$ equation (10) under the transformations:

$$W(z) = \frac{w(z)}{\epsilon}, \quad Z = \epsilon z + \frac{a}{2e^4}$$

followed by $\epsilon \to 0$, becomes the $I_{8}$ equation:

$$I_{8}: \quad w'' = 2w^3 + aw + b$$

The $P_{I}$ equation (13) under the transformations:

$$W(z) = \frac{w(z)}{e^2}, \quad Z = \epsilon z + \frac{1}{2e^4}$$

followed by $\epsilon \to 0$, becomes the $I_{3}$ equation:

$$I_{3}: \quad w'' = 6w^2 + \frac{1}{2}$$
4 Coalescence cascade between the autonomous Painlevé

Now continuing the same reasoning as Painlevé, we are also able to connect the autonomous equations in a cascade in the same fashion as the original Painlevé coalescence cascade.

In fact, the transformations for the variables \( w \) and \( z \) are very similar in both cases. The cascade will follow the same lines as the Painlevé ones:

\[
I_{49} \rightarrow I_{38} \rightarrow I_{30} \rightarrow I_8 \rightarrow I_3
\]

So, we start by the \( I_{49} \) equation (15), that under the transformations:

\[
W(Z) = \frac{w(z)}{w'(z)}, \quad Z = \epsilon z,
\]

\[
A = \frac{a}{\epsilon}, \quad B = \frac{4\epsilon(a - b \epsilon^2 + b) + d(\epsilon - 1)}{\epsilon^2 (\epsilon^3 - 3\epsilon + 2)}, \quad C = \frac{b}{\epsilon}, \quad D = \frac{a}{(1 - \epsilon^2)^2\epsilon},
\]

\[
E = \frac{\alpha \epsilon (\epsilon^3 - 4\epsilon^2 + 4) - \epsilon(\epsilon - 1)^2(b(\epsilon^3 - 8\epsilon^2 + 12\epsilon - 4) - c(\epsilon - 2)) - d(\epsilon - 1)^4}{(\epsilon - 2)(\epsilon - 1)^2\epsilon^4}
\]

followed by \( \epsilon \to 0 \), becomes the \( I_{38} \) equation (17).

The \( I_{38} \) equation (17) under the transformations:

\[
W(Z) = \sqrt{2} w(z), \quad Z = \epsilon z, \quad A = \frac{1}{\epsilon^4}, \quad B = C = 2,\quad D = 4\epsilon^2 + 2\epsilon^4 - 2\epsilon^6
\]

followed by \( \epsilon \to 0 \), becomes the \( I_{30} \) equation (19).

The \( I_{30} \) equation (19) under the transformations:

\[
W(Z) = 1 + \epsilon w(z), \quad Z = z, \quad A = \frac{a}{2\epsilon^2}, \quad B = \frac{b}{2} + \epsilon(2\epsilon - c), \quad C = \epsilon(2\epsilon - c), \quad D = d\epsilon^2
\]

followed by \( \epsilon \to 0 \), becomes the \( I_{12} \) equation (21).

The \( I_{12} \) equation (21) under the transformations:

\[
W(Z) = \frac{w(z)}{\epsilon} + \frac{1}{\epsilon^3}, \quad Z = \epsilon z - \frac{1}{\epsilon^3}, \quad A = -\frac{1}{\epsilon^3} - \frac{a\epsilon}{12}, \quad B = \frac{3 + a\epsilon^4}{2\epsilon^6}, \quad C = -\frac{4a\epsilon^4 - 6be^6 + 3}{6\epsilon^{12}}
\]

followed by \( \epsilon \to 0 \), becomes the \( I_8 \) equation (23).

The \( I_8 \) equation (23) under the transformations:

\[
W(Z) = \epsilon w(z) + \frac{1}{\epsilon^5}, \quad Z = \epsilon^2 z - \frac{6}{\epsilon^{10}}, \quad A = -\frac{6}{\epsilon^{10}}, \quad B = \frac{8 + \epsilon^{12}}{2\epsilon^{15}}
\]

followed by \( \epsilon \to 0 \), becomes the \( I_3 \) equation (25).
5 Degeneracy to parameterless equations

The parameterless equations have a very simple form, such that they in some cases are just the corresponding Painlevé equation with all parameters set to zero. For all the cases the transformation for both Painlevé and corresponding autonomous Painlevé is the same, with the only exception being \( P_{VI} \) and \( I_{49} \) to \( I_{32} \). Also, if some parameter is not specified, it will be naturally deleted by the transformation.

\[
\begin{align*}
P_{VI} & \quad I_{32} \\
P_{V}, I_{38} & \quad I_{37} \\
P_{IV}, I_{30} & \quad I_{29} \\
P_{I}, I_{12} & \quad I_{11} \\
P_{I}, I_{8} & \quad I_{7} \\
P_{I}, I_{3} & \quad I_{2} \\
\end{align*}
\]

The \( P_{VI} \) equation (1) under the transformations

\[ W(Z) = \epsilon w(z), \quad Z = \frac{e^z}{\epsilon}, \quad B = -\frac{\epsilon^2}{2}, \quad C = \epsilon \]  \hspace{1cm} (32)

(here \( e^z \) is the exponential of \( z \)) followed by \( \epsilon \to 0 \) becomes the \( I_{32} \) equation:

\[ I_{32} : \quad w'' = \frac{w'^2 - 1}{2w} \]  \hspace{1cm} (33)

The \( I_{49} \) equation (15) under the transformations

\[ W(Z) = \epsilon w(z), \quad Z = \frac{z}{\epsilon}, \quad A = a, \quad B = \frac{\epsilon^2}{a}, \quad C = -\frac{\epsilon^4}{2a}, \quad D = \frac{\epsilon^2}{a}, \quad E = -2a\epsilon^2 \]  \hspace{1cm} (34)

followed by \( \epsilon \to 0 \) becomes the \( I_{32} \) equation (33).

The \( P_{V} \) (4) and \( I_{38} \) (17) under the transformations

\[ W(Z) = \epsilon w(z), \quad Z = \epsilon \ln(z) \]  \hspace{1cm} (35)

followed by \( \epsilon \to 0 \) becomes the \( I_{37} \) equation:

\[ I_{37} : \quad w'' = \left( \frac{1}{2w} + \frac{1}{w-1} \right) w'^2 - \frac{w'}{z} \]  \hspace{1cm} (36)

The \( P_{IV} \) (6) and \( I_{30} \) (19) under the transformations

\[ W(Z) = \frac{w(z)}{\epsilon}, \quad Z = \epsilon z \]  \hspace{1cm} (37)

followed by \( \epsilon \to 0 \) becomes the \( I_{29} \) equation:

\[ I_{29} : \quad w'' = \frac{w'^2}{2w} + \frac{3w^3}{2} \]  \hspace{1cm} (38)

The \( P_{III} \) (8) and \( I_{12} \) (21) under the transformations

\[ W(Z) = w(z), \quad Z = \epsilon z \]  \hspace{1cm} (39)

followed by \( \epsilon \to 0 \) becomes the \( I_{11} \) equation:

\[ I_{11} : \quad w'' = \frac{w'^2}{w} \]  \hspace{1cm} (40)

The \( P_{II} \) (10) and \( I_{8} \) (23) under the transformations

\[ W(Z) = \frac{w(z)}{\epsilon}, \quad Z = \epsilon z \]  \hspace{1cm} (41)
followed by $\epsilon \to 0$ becomes the $I_7$ equation:

$$I_7 : \quad w'' = 2w^3$$ (42)

The $P_I$ (13) and $I_3$ (25) under the transformations

$$W(Z) = \frac{w(z)}{\epsilon^2}, \quad Z = \epsilon z$$ (43)

followed by $\epsilon \to 0$ becomes the $I_2$ equation:

$$I_2 : \quad w'' = 6w^2$$ (44)

The $P_I$ (13) and $I_3$ (25) under the transformations

$$W(Z) = \epsilon w(z), \quad Z = \epsilon z$$ (45)

followed by $\epsilon \to 0$ becomes the $I_1$ equation:

$$I_1 : \quad w'' = 0$$ (46)

6 Riccati equations

Painlevé equations are known to have Riccati equations as special solutions, such being classical special functions.

These special functions also have a coalescence relation among them and it was fully exposed at [15], so in this discussion I will only focus on the coalescence limits leading to the corresponding autonomous equations of these special functions, using exactly the same limits as in section (3).

I also present these special cases of Painlevé equations in a different form, showing that in all cases we are able to naturally obtain the conditions for the Riccati equations.

Here I also follow the generalization used by [13], where the parameters $\theta_0, \theta_1, \theta_2$ are signal choices, i.e., they are $\pm 1$.

We begin by the most complex case, the $P_{VI}$ equation (11), which with the special choice of parameters:

$$a = \frac{\alpha^2}{2}, \quad b = -\frac{1}{2}(\beta - \theta_0(\gamma \theta_1 \theta_2 + 1))^2, \quad c = \frac{1}{2}(\alpha \theta_1 + \gamma)^2, \quad d = \frac{1}{2}(1 - \beta^2)$$ (47)

can be written as:

$$(P_{VI}) : \quad F'(z) = F(z)^2 \left( \frac{z}{2w(z)} - \frac{w(z)}{2} \right) + F(z) \left( -\frac{\theta_0(-\beta + \gamma \theta_0 \theta_1 \theta_2 + \theta_0)}{(z-1)w(z)} - \frac{\alpha \theta_2 w(z) + 2z - 1}{(z-1)z} \right)$$ (48)

where

$$F(z) = \frac{1}{(w(z) - 1)(w(z) - z)} \left( w'(z) - \left( \frac{\alpha \theta_2 w(z)^2}{(z-1)z} + \frac{\theta_0(\beta - \theta_0(\gamma \theta_1 \theta_2 + 1))}{z-1} \right) + \frac{-\alpha \gamma \theta_1 + \beta \gamma \theta_0 \theta_1 \theta_2 - \gamma^2 + zw(z)((\alpha - \beta)(\alpha + \beta) + \alpha \gamma \theta_1 - \alpha \theta_2 + \beta \gamma \theta_0 \theta_1 \theta_2 + \beta \theta_0 - \gamma \theta_1 \theta_2)}{(z-1)z(-\alpha \theta_2 + \beta \theta_0 - \gamma \theta_1 \theta_2)} \right)$$ (49)
Notice here for this notation that the \( F(z) = 0 \) is also a Riccati equation, and in fact that is the one treated extensively in literature as the special case for \( PV I \).

\( F(z) = 0 \) can be linearized with a Cole-Hopf transformation:

\[
 w(z) = \frac{-z(z-1)w'(z)}{\alpha \theta_2 u(z)}
\]

having its solutions given in terms of Gauss-Hypergeometric functions.

The coalescence reduction of \( F(z) = 0 \) with the same limits as of \( PV I \) yields:

\[
 w'(z) = \sqrt{2b \theta_2 w(z)^2 + \theta_0 \sqrt{-2ac}} - \frac{w(z)(a(ab + b + c + d) + c + c)}{\sqrt{2 \left( a \sqrt{b \theta_2 - \theta_0 \sqrt{-ac}} \right)}}
\]

For completeness of notation, I will express here the others Painlevé equations in the same form as above, but this time I will just make some notation changes, as for \( a \) and \( d \) in (47) and will NOT impose conditions on them \textit{a priori}, like those upon \( b \) and \( c \) in (47).

The equation \( PV (4) \) can be rewritten without lose of generality with parameters

\[
 a = \frac{\alpha^2}{2}, \quad b = -\frac{\beta^2}{2}, \quad \delta = -\frac{1}{2}
\]

by

\[
 (PV) : \quad F'(z) = F(z)^2(1 - w(z)) + F(z) \left( -\frac{\alpha \theta_2 w(z)}{z} + \frac{\theta_1}{w(z) - 1} + \frac{\alpha \theta_2 - 1}{z} \right) + \frac{\alpha \theta_1 \theta_2 - \beta \theta_0 \theta_1 + c - \theta_1}{2z(w(z) - 1)}
\]

with

\[
 F(z) = \frac{1}{2(w(z) - 1)w(z)} \left( w'(z) - \left( w(z) \left( \theta_1 - \frac{\alpha \theta_2 + \beta \theta_0}{z} \right) + \frac{\alpha \theta_2 w(z)^2}{z} + \frac{\beta \theta_0}{z} \right) \right)
\]

Now if we set \( F(z) = 0 \), (52) is solvable by Confluent Hypergeometric functions (or Whittaker functions) and (51) imposes the condition on the parameters:

\[
 c = \theta_1 (-\alpha \theta_2 + \beta \theta_0 + 1)
\]

We can see here that expressing \( PV \) as (51) give us automatically the conditions on the parameters required for it to be a special function.

The coalescence reduction of \( F(z) = 0 \) with the same limits as of \( PV \) \( \rightarrow I_{38} \) yields:

\[
 w' = a \theta_2 w^2 - w (a \theta_2 + b \theta_0) + b \theta_0
\]

The equation \( P IV (6) \) can be rewritten without lose of generality with parameter

\[
 b = -\frac{\beta^2}{2}
\]

by

\[
 (P IV) : \quad F'(z) = -F(z)^2 - 2F(z) (\theta_1 w(z) + \theta_1 z) + \frac{1}{2} (-2a - \beta \theta_0 \theta_1 - 2\theta_1)
\]

with

\[
 F(z) = \frac{1}{2w(z)} \left( w'(z) - \beta \theta_0 - \theta_1 w(z)^2 - 2\theta_1 zw(z) \right)
\]
Now if we set $F(z) = 0$, (55) is solvable by Hermite functions and (54) imposes the condition on the parameter:

$$a = \frac{1}{2} (-\beta \theta_0 - 2\theta_1)$$

The coalescence reduction of $F(z) = 0$ with the same limits as of $P_{IV} \to I_{30}$ (18) yields:

$$w' = \theta_1 w^2 + 2a\theta_1 w + \sqrt{-2c\theta_0}$$

The equation $P_{III}$ (8) can be rewritten without lose of generality with parameter

$$c = 1, \quad d = -1$$

by

$$(P_{III}) : \quad F'(z) = F(z) \left( \frac{\theta_0}{w(z)} - \theta_1 w(z) - \frac{1}{z} \right) + \frac{a\theta_1 \theta_0 + b - 2\theta_0}{z w(z)}$$

with

$$F(z) = \frac{1}{w(z)} \left( w'(z) - \frac{(a\theta_1 - 1) w(z)}{z} - \theta_0 - \theta_1 w(z)^2 \right)$$

Now if we set $F(z) = 0$, (58) is solvable by Bessel functions and (57) imposes the condition on the parameter:

$$b = 2\theta_0 - a\theta_1 \theta_0$$

The coalescence reduction of $F(z) = 0$ with the same limits as of $P_{III} \to I_{12}$ (20) yields:

$$w' = \theta_1 w^2 + b\theta_1 w$$

The equation $P_{II}$ (10) can be rewritten without lose of generality by

$$(P_{II}) : \quad \theta_0 F'(z) = -2w(z)F(z) + a + \frac{\theta_0}{2}$$

with

$$F(z) = \theta_0 w'(z) - w(z)^2 - \frac{z}{2}$$

Now if we set $F(z) = 0$, (61) is solvable by Airy functions and (60) imposes the condition on the parameter:

$$a = -\frac{\theta_0}{2}$$

The coalescence reduction of $F(z) = 0$ with the same limits as of $P_{II} \to I_8$ (22) yields:

$$w' = \theta_0 w^2 + \frac{a\theta_0}{2}$$

One can immediately observe that all of these autonomous limits of special functions have the general form

$$w' = c_2 w^2 + c_1 w + c_0$$

and also that the number of parameters decrease from higher equations to lower ones, therefore the degeneracy cascade here is trivial, only needing sometimes to do a shift by a constant like $w \to w + C$, and then match the parameters.

The linearization of such general form via Cole-Hopf transformation yields:

$$u'' = K_1 u' + K_2 u \implies u(z) = c_1 e^{\frac{1}{2}z(K_1 - \sqrt{K_1^2 + 4K_2})} + c_2 e^{\frac{1}{2}z(K_1 + \sqrt{K_1^2 + 4K_2})}.$$ 

The form (63) was expected, since all the autonomous Painlevé have solutions in terms of Jacobi Elliptic Functions and (63) is the corresponding Riccati form for them.
7 Coalescence for the symmetric $P_{IV}$ and $P_{V}$

In this section I present some of the previous results for $P_{IV}$ and $P_{V}$ but in the framework of their symmetric equations.

I show how the traditional coalescence appears as additional terms in the equations of motion and how the coalescence to autonomous Painlevé appears as a constraint ($\sigma = 0$) on the parameters.

The idea behind it was first noticed by [16], in the context of dressing chains, however the link with Ince’s equations, and coalescence was not considered.

7.1 $P_{IV}$

7.1.1 Traditional symmetric $P_{IV}$

We start with the symmetric $P_{IV}$ [10]:

\[
\begin{align*}
    f_0' &= f_0 (f_1 - f_2) + \alpha_0, \\
    f_1' &= f_1 (f_2 - f_0) + \alpha_1, \\
    f_2' &= f_2 (f_0 - f_1) + \alpha_2,
\end{align*}
\]

(64)

where $f_i = f_i(z)$ and $' = d/dz$.

By summing these equations we get

\[
f_0' + f_1' + f_2' = \alpha_0 + \alpha_1 + \alpha_2
\]

Defining $\sigma := \alpha_0 + \alpha_1 + \alpha_2$ and making one integration, we get

\[
f_0 + f_1 + f_2 = \sigma z + \chi
\]

(65)

Setting the integration constant $\chi = 0$ and eliminating $f_2 = \sigma z - f_0 - f_1$ from (64) we obtain:

\[
\begin{align*}
    f_0'(z) &= f_0 (-\sigma z + f_0 + 2f_1) + \alpha_0, \\
    f_1'(z) &= f_1 (\sigma z - 2f_0 - f_1) + \alpha_1,
\end{align*}
\]

(66)

while the third equation in (64) can be obtained by summing the above two equations.

Traditionally, either $\sigma$ is set to 1 or it can be absorbed by the following transformations [2]:

\[
\begin{align*}
    f_i(z) &\rightarrow f_i(z) + 1\epsilon, \\
    z &\rightarrow z + 2\epsilon\sigma^2, \\
    \alpha_0 &\rightarrow \epsilon\alpha_0 - \frac{1}{\epsilon^2}, \\
    \alpha_1 &\rightarrow \epsilon\alpha_1 + \frac{1}{\epsilon^2}, \\
    \alpha_2 &\rightarrow \epsilon\alpha_2.
\end{align*}
\]

(69)

7.1.2 Degeneracies on the symmetric $P_{IV}$

Here we formulate coalescence in the setting of the symmetric $P_{IV}$ equations through the following transformations [2]:

\[
\begin{align*}
    f_i(z) &\rightarrow f_i(z) + \frac{1}{\epsilon}, \\
    z &\rightarrow z + \frac{2}{\sigma\epsilon^2}, \\
    \alpha_0 &\rightarrow \epsilon\alpha_0 - \frac{1}{\epsilon^2}, \\
    \alpha_1 &\rightarrow \epsilon\alpha_1 + \frac{1}{\epsilon^2}, \\
    \alpha_2 &\rightarrow \epsilon\alpha_2.
\end{align*}
\]

(69)
Applying the above transformation to the first order equations (64) yields:

\[ f'_0(z) = f_0 (f_1 - f_2) + \frac{f_1 - f_2}{\epsilon} + \epsilon \alpha_0 - \frac{1}{\epsilon^2} \]
\[ f'_1(z) = f_1 (f_2 - f_0) + \frac{f_2 - f_0}{\epsilon} + \epsilon \alpha_1 + \frac{1}{\epsilon^2} \]
\[ f'_2(z) = f_2 (f_0 - f_1) + \frac{f_0 - f_1}{\epsilon} + \epsilon \alpha_2 \]  

(70)

Summing the equations above and making one integration, we arrive at

\[ f_0 + f_1 + f_2 = \epsilon \sigma z + \left(\frac{\epsilon - \frac{1}{\epsilon}}{\epsilon}\right) \]  

(71)

that term in parenthesis is a suitably chosen integration constant corresponding to \( \chi \) in (65).

Eliminating \( f_2 \) we get:

\[ f'_0(z) = \epsilon \left(\alpha_0 - \sigma z f_0 + \xi f_0\right) + \frac{2f_0 + 2f_1}{\epsilon} + f_0^2 + 2f_1 f_0 - \sigma z + \xi, \]
\[ f'_1(z) = \epsilon \left(\alpha_1 + \sigma z f_1 - \xi f_1\right) + \frac{-2f_0 - 2f_1}{\epsilon} - f_1^2 - 2f_0 f_1 + \sigma z - \xi. \]  

(72)

By eliminating \( f_1 \) from (72) we obtain a second order ODE for \( f_0(z) \) depending on \( \sigma, \xi \) and \( \epsilon \), which admits different solutions for different limits of these parameters.

- Such an equation is the traditional \( P_{IIV} \) (68) by absorbing \( \xi \) into \( z \), followed by (67) and absorbing \( \epsilon \) from \( f_0 \) and \( \alpha_i \) (inverse of (69));
- if we take the limit \( \sigma \to 0 \) and absorbing \( \epsilon \) from \( f_0 \) and \( \alpha_i \) (inverse of (69)), we get the \( I_{30} \) equation:

\[ f''_0(z) = \frac{f_0^3}{2f_0} + \frac{3}{2} f_0^3 + f_0^2 \left(2 \xi \epsilon - \frac{4}{\epsilon}\right) + f_0(z) \left(\alpha_0 + 2\alpha_1 - 2z + \frac{\xi^2 \epsilon^2}{2} + \frac{2}{\epsilon^2}\right) - \frac{\alpha_0^2}{2f_0} \]  

(73)

Instead of the previous steps’ limits, one keeps \( \sigma \) and takes \( \epsilon \to 0 \) yielding:

\[ f''_0(z) = 2f_0^3 - 2(\sigma z - \xi) f_0 + 2\alpha_0 + 2\alpha_1 - \sigma \]  

(74)

which is

- the \( P_{II} \) equation (10) by either absorbing \( \xi \) into \( z \), or taking \( \xi = 0 \);
- the \( I_8 \) equation (23) by taking \( \sigma = 0 \);
- the same \( I_8 \) if one had taken first the limit \( \sigma \to 0 \) and then \( \epsilon \to 0 \).

One can see that the use of the \( \epsilon \sigma \) parameter here turns this procedure equivalent to (9) by the relations (67).

This whole procedure can be visualized as:

\[ P_{IV} \xrightarrow{(\epsilon, \sigma, \xi)} P_{IV}(\epsilon, \sigma, \xi) \xrightarrow{\epsilon \to 0} P_{II}(\sigma, \xi) \]
\[ I_{30}(\epsilon, \xi) \xrightarrow{\epsilon \to 0} I_8 \]

where \( P_{IV} \) is (68); \( (\epsilon, \sigma, \xi) \) is (69) with (71); \( P_{IV}(\epsilon, \sigma, \xi) \) is the second order ODE from (72); \( \epsilon \) is (69) without the \( z \) transformation; and the leftmost \( I_{30}(\epsilon, \xi) \) is the equation (73). 

11
7.2 \(P_V\)

7.2.1 The symmetric \(P_V\)

Here the traditional symmetric \(P_V\) is described as in the literature and also the coalescence to \(I_{38}\) is presented.

The symmetric equations for \(P_V\), as described by Noumi [10], correspond to the system of differential equations:

\[
\begin{align*}
zf'_{0} & = \left(\frac{\sigma}{2} - \alpha_2\right) f_0 + \alpha_0 f_2 + (f_1 - f_3) f_2 f_0 \\
zf'_{1} & = \left(\frac{\sigma}{2} - \alpha_3\right) f_1 + \alpha_1 f_3 + (f_2 - f_0) f_3 f_1 \\
zf'_{2} & = \left(\frac{\sigma}{2} - \alpha_0\right) f_2 + \alpha_2 f_0 + (f_3 - f_1) f_0 f_2 \\
zf'_{3} & = \left(\frac{\sigma}{2} - \alpha_1\right) f_3 + \alpha_3 f_1 + (f_0 - f_2) f_1 f_3
\end{align*}
\]

(75)

where \(\sigma := \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3\).

The following change of variables is equivalent to rescale \(\sigma \to 1\):

\[
f_i(z) = \sqrt{\sigma} \tilde{f}_i(x), \quad z = (\sigma x)^{1/\sigma}, \quad \alpha_i = \sigma \tilde{\alpha}_i, \quad i = 0, 1, 2, 3.
\]

(76)

By summing the first and third (resp. second and fourth) equations in (75) we get, respectively:

\[
\begin{align*}
zf'_{0} + zf'_{2} & = \frac{\sigma}{2} (f_0 + f_2) \\
zf'_{1} + zf'_{3} & = \frac{\sigma}{2} (f_1 + f_3)
\end{align*}
\]

(77)

we are able to perform 2 integrations, therefore obtaining 2 integration constants, \(\epsilon_0\) and \(\epsilon_1\):

\[
\begin{align*}
f_0 + f_2 & = \epsilon_0 z^{\sigma/2} (f_0 + f_2) \\
f_1 + f_3 & = \epsilon_1 z^{\sigma/2} (f_1 + f_3)
\end{align*}
\]

(78)

(79)

\(\epsilon_0\) and \(\epsilon_1\) are normally set to 1, but for reasons explained ahead, they will be kept here.

By eliminating \(f_2\) and \(f_3\) in (75) using the above relations, we get for \(f_0\):

\[
\begin{align*}
zf'_{0} & = \alpha_0 \left(\epsilon_0 z^{\sigma/2} - f_0\right) + f_0 \left( f_1 \left(\epsilon_0 z^{\sigma/2} - f_0\right) - \left(\epsilon_0 z^{\sigma/2} - f_0\right) \left(\epsilon_1 z^{\sigma/2} - f_1\right) \right) + f_0 \left(\frac{\sigma}{2} - \alpha_2\right) \\
zf'_{1} & = \alpha_1 \left(\epsilon_1 z^{\sigma/2} - f_1\right) + f_1 \left( f_1 \left(\epsilon_0 z^{\sigma/2} - f_0\right) \left(\epsilon_1 z^{\sigma/2} - f_1\right) - f_0 \left(\epsilon_1 z^{\sigma/2} - f_1\right) \right) + f_1 \left(\frac{\sigma}{2} - \alpha_3\right)
\end{align*}
\]

(80)

(81)

we are able to solve the above equations either for \(f_0\) or for \(f_1\), yielding second order differential equations. For simplicity, only \(f_0\) will be shown hereinafter, and it gets the form:

\[
f''_0 = \frac{f_{10}^2}{2} \left( \frac{1}{f_0} - \frac{1}{(\epsilon_0 z^{\sigma/2} - f_0)} \right) - \frac{f'_0}{z} + ... \]

(82)

such form of equation suggests it is a Painleve equation, and we should perform the following change of variables, as described in Ince’s book:

\[
f_0(z) = \frac{\epsilon_0 z^{\sigma/2}}{1 - g_0(z)},
\]

(83)
and in order to eliminate some powers of $\sigma$, we also perform the transformation:

$$z = x^{1/\sigma}.$$  \hspace{1cm} (84)

Such change of variables reveals us it is a $P_V$ equation with parameters, respectively:

$$g_0: \quad a = \frac{\alpha_0^2}{2\sigma^2}, \quad b = -\frac{\alpha_2^2}{2\sigma^2}, \quad c = \frac{(\alpha_3 - \alpha_1)\epsilon_0\epsilon_1}{\sigma^2}, \quad d = -\frac{\epsilon_0^2\epsilon_1^2}{2\sigma^2}$$  \hspace{1cm} (85)

one can notice that the parameters above are related by an index shift.

The case for $\sigma = 0$ is also immediately noticeable, since it appears as a singularity in step (84).

In such case we have to go back to the $g_0$ equation still in the $z$ variable and then set $\sigma \to 0$ followed by the transformation

$$z = e^x$$

This yields the $I_{38}$ equation \[ with parameters, respectively:

$$g_0: \quad \alpha_0 = -\sqrt{2}\sqrt{\alpha}, \quad \alpha_1 = \frac{1}{4} \left(2\sqrt{2} \left(\sqrt{\alpha} + i\sqrt{b}\right) - \frac{2c}{\epsilon_0\epsilon_1} - \epsilon_0\epsilon_1\right), \quad \alpha_2 = -i\sqrt{2}\sqrt{\alpha},
\alpha_3 = \frac{1}{4} \left(2\sqrt{2} \left(\sqrt{\alpha} + i\sqrt{b}\right) + \frac{2c}{\epsilon_0\epsilon_1} + \epsilon_0\epsilon_1\right), \quad d = -\epsilon_0^2\epsilon_1$$  \hspace{1cm} (86)

### 7.2.2 Coalescence to $P_{III}$

An effective and very symmetrical way of obtaining the coalescence from $P_V$ to $P_{III}$ is by adding terms $\beta_0(f_1 + f_3)$ and $\beta_1(f_0 + f_2)$, such that:

$$z f_0' = \left(\frac{\sigma}{2} - \alpha_2\right) f_0 + \alpha_0 f_2 + (f_1 - f_3) f_2 f_0 - \beta_0 (f_1 + f_3)$$
$$z f_1' = \left(\frac{\sigma}{2} - \alpha_3\right) f_1 + \alpha_1 f_3 + (f_2 - f_0) f_3 f_1 + \beta_1 (f_0 + f_2)$$
$$z f_2' = \left(\frac{\sigma}{2} - \alpha_0\right) f_2 + \alpha_2 f_0 + (f_1 - f_3) f_0 f_2 + \beta_0 (f_1 + f_3)$$
$$z f_3' = \left(\frac{\sigma}{2} - \alpha_1\right) f_3 + \alpha_3 f_1 + (f_0 - f_2) f_1 f_3 - \beta_1 (f_0 + f_2)$$  \hspace{1cm} (87)

such approach was first developed in [4].

We repeat the same steps as in the traditional case up to \[; namely, eliminating $f_2$ and $f_3$ from the system, solving it for $f_0''$ and $f_1''$, making the variable change \[ followed by \[, we end up with a $P_V$ equation with parameters:

$$g_0: \quad a = \frac{(\alpha_0\epsilon_0 + \epsilon_1\beta_0)^2}{2\sigma^2}, \quad b = -\frac{(\alpha_2\epsilon_0 - \epsilon_1\beta_0)^2}{2\sigma^2}, \quad c = \frac{\epsilon_0 ((\alpha_3 - \alpha_1)\epsilon_1 + 2\epsilon_0\beta_1)}{\sigma^2}, \quad d = -\frac{\epsilon_0^2\beta_1^2}{2\sigma^2}$$  \hspace{1cm} (88)

That is, the addition of $\epsilon_i$ terms is equivalent to:

$$\alpha_0 \to \alpha_0\epsilon_0 + \beta_0\epsilon_1, \quad \alpha_1 \to \alpha_1\epsilon_1 - \beta_1\epsilon_0, \quad \alpha_2 \to \alpha_2\epsilon_0 - \beta_0\epsilon_1, \quad \alpha_3 \to \alpha_3\epsilon_1 + \beta_1\epsilon_0.$$

Now, if we instead of making the transformations \[ when we arrive at the second-order equation with the form \[, we take $\epsilon_0 \to 0$ (respectively $\epsilon_1 \to 0$ for the $f_1$ equation), we arrive at the equation:

$$f_0'' = \frac{f_0^2}{f_0} - \frac{f_0'}{f_0} + \epsilon_1 z^{\sigma-2} \left(f_0^3 - \frac{\beta_0^2}{f_0}\right) + \epsilon_1 z^{\sigma-2} \left(\beta_0 (-\alpha_1 - \alpha_3 + \sigma) + (\alpha_3 - \alpha_1) f_0 (z)^2\right)$$  \hspace{1cm} (89)
that can have the powers of $\sigma$ eliminated by the transformation

$$z = x^{2/\sigma}$$

yielding the traditional $P_{III}$ equation (8) with parameters:

$$f_0 : \quad a = \frac{4(\alpha_3 - \alpha_1)\epsilon_1}{\sigma^2}, \quad b = \frac{4\epsilon_1\beta_0(\alpha_0 + \alpha_2)}{\sigma^2}, \quad c = \frac{4\epsilon_1^2}{\sigma^2}, \quad d = -\frac{4\epsilon_1^2\beta_0^2}{\sigma^2}, \quad (90)$$

as usual, we notice that the point $\sigma = 0$ is a singularity, therefore setting $\sigma = 0$ on (89), followed the variable change $z = e^x$, it becomes $I_{12}$ (21) with parameters:

$$f_0 : \quad a = \epsilon_1^2, \quad b = \epsilon_1(\alpha_3 - \alpha_1), \quad c = -\epsilon_1\beta_0(\alpha_1 + \alpha_3), \quad d = -\epsilon_1^2\beta_0^2 \quad (91)$$

The complementary case, of $f_{0zz}$ with $\epsilon_1 = 0$ is trivial since this limit is not a singularity, therefore it is just $P_V$ (or $I_{38}$ when $\sigma = 0$) equation.

If we translate that process of going from $P_V$ to $P_{III}$ in literature’s language, that is, calling $\epsilon_0 = \epsilon$ and making the appropriate relabeling of the other parameters, this process is exactly like the old-fashioned coalescence limit (7).

Equations $I_{38}(g_0(x))$ and $I_{12}(f_0(x))$ are also connected by the transformations already described.

This whole procedure can be visualized as:

$$P_V(g_0(z); \sigma, \beta_i) \xrightarrow{f_0 \leftrightarrow g_0} P_V(f_0(z); \sigma, \beta_i) \xrightarrow{\epsilon_0 \to 0} P_{III}(f_0(z))$$

$$I_{38}(g_0(z)) \xrightarrow{f_0 \leftrightarrow g_0} I_{38}(f_0(z)) \xrightarrow{\epsilon_0 \to 0} I_{12}(f_0(z))$$

$$I_{38}(g_0(x)) \xrightarrow{\epsilon_i \parallel \beta_i} I_{12}(f_0(x))$$

where $P_V(f_0(z); \sigma, \beta_i)$ is the equation with the form (82) originated from the system (7); $f_0 \leftrightarrow g_0$ is the transformation (83); and since specifically $I_{38}(g_0(x))$ and $I_{12}(f_0(x))$ that have the form described in literature they feature here too.

### 7.3 The Bäcklund Transformations for $\sigma = 0$

The theory for the behavior of the affine Weyl groups of the Bäcklund Transformations for the Painlevé equations under coalescence is well described in [14] and can be seen to some extent here too by noticing how the parameters combine to become the resulting ones eliminating singularities.

The result of taking $\sigma \to 0$ on Bäcklund transformations is different from what happens in coalescence, there the group structure shrinks and degenerates into a subgroup at each step of coalescence; here the group structure is still present, but totally spoiled due to an extra relation that does not allow the generations of an infinite chain of solutions, as will be seen.

Since the conclusion here is simple we take as a case study the symmetric $P_{IV}$, which is invariant by the Bäcklund transformations [10]:

$$\begin{array}{c|ccc|ccc} & \alpha_0 & \alpha_1 & \alpha_2 & f_0 & f_1 & f_2 \\ \hline s_0 & -\alpha_0 & \alpha_1 + \alpha_0 & \alpha_2 + \alpha_0 & f_0 & f_1 + \frac{\alpha_1}{f_2} & f_2 - \frac{\alpha_2}{f_2} \\ s_1 & \alpha_0 + \alpha_1 & -\alpha_1 & \alpha_2 + \alpha_1 & f_0 - \frac{\alpha_1}{f_1} & f_1 & f_2 + \frac{\alpha_1}{f_1} \\ s_2 & \alpha_0 + \alpha_2 & \alpha_1 + \alpha_2 & -\alpha_2 & f_0 + \frac{\alpha_2}{f_2} & f_1 - \frac{\alpha_2}{f_2} & f_2 \end{array} \quad (92)$$
\[(s_i)^2 = 1, \quad s_isjisj = sjsisj\]

The above table should be read as \(s_0(\alpha_0) = -\alpha_0, \quad s_0(\alpha_1) = \alpha_1 + \alpha_2\) and so on, meaning that applying \(s_i\) keeps system (44) invariant.

This implies that if one has a solution (obtained by any means) for \(f_i(z)\) with constants \(\{\alpha_0, \alpha_1, \alpha_2\}\) for \(P_{IV}\) [15], we are immediately able to obtain a new solution with the new set of constants, and by combining the transformations \(s_i\), it is easy to see that every time the constants combine into \(\alpha_0 + \alpha_1 + \alpha_2 = \sigma \neq 0\) such procedure can be continued indefinitely always summing \(\sigma\) after some steps, creating an infinite chain of solutions for \(P_{IV}\). For more details the author refers to [10].

One notices that the \(\sigma\)-parameter plays no evident role here, so using
\[\alpha_2 = -\alpha_0 - \alpha_1\]
and after plugging it in the transformations above, they become:

|     | \(\alpha_0\) | \(\alpha_1\) |
|-----|--------------|--------------|
| \(s_0\) | \(-\alpha_0\) | \(\alpha_1 + \alpha_0\) |
| \(s_1\) | \(\alpha_0 + \alpha_1\) | \(-\alpha_1\) |
| \(s_2\) | \(-\alpha_1\) | \(-\alpha_0\) |

\[(93)\]

one can easily see that the transformations now are not independent anymore:
\[s_isjisj = s_k, \quad \text{or} \quad s_isjisj = s_ksisj \quad t \neq j \neq k\]
\[(94)\]
since relation \((s_i)^2 = 1\) and \(s_isjisj = sjsisj\) are still valid, we see that the other possibility of combination of \(s_i\) also degenerates to this simpler case:
\[s_isjisjisj = s_ksisj = s_j\]
\[(95)\]

therefore all possible combinations of constants are restricted to \(\pm \alpha_i\) and \(\pm \alpha_i \pm \alpha_j\).

Since one cannot generate a chain of solutions, one neither can create a chain of rational solutions nor associate a polynomial for them, like the Hermite polynomials that would be associated with the rational solutions of \(P_{IV}\) [7], in this example. Since it applies to \(I_{12}\) (and to the others autonomous equations), this is the reason such a chain of rational solutions does not appear in Jacobi Elliptic functions.

The generalization of (94) to higher \(N\) can be written as:
\[\left(s_{i+1} \ldots s_{i+N-1}\right)s_{i+N} = s_i\left(s_{i+1} \ldots s_{i+N-1}\right)\]
\[(96)\]
and similar observations apply.

### 8 Conclusion

Hybrid Painlevé equations have been a goal of several researchers, e.g. [9, 12], and to expand equations with their possible degeneracy parameters allows one to obtain a hybrid equation with both a Painlevé equation and an elliptic equation, for example. This was done in [4, 3] and [2] and here the complete framework for such equations was provided since the basic recipe there was to first find the coalescence in the framework of symmetric equations and then extend the parameter space to all possible constants of integration the system provides.

The coalescence cascades were seen here to be preserved for multiple properties and reductions, like for autonomous equations, Riccati equations, symmetric equations, and Bäcklund transformations.

The limit to autonomous equations here explored \((\sigma \to 0)\) is a bridge from Painlevé equations to any simpler ODE by this systematic procedure of coalescence, and can be applied to a multitude of properties of these equations, even possibly for numerical algorithms.
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References

[1] Ablowitz M J, Ramani A, Segur H 1978 Nuovo Cimento 23 333, 1980 J. Math. Phys. 21 715
[2] Alves V C C, Aratyn H, Gomes J F and Zimerman A H 2020 J. Phys. A: Math. Theor. 53 445202
[3] Alves V C C, Aratyn H, Gomes J F and Zimerman A H 2019 J. Phys.: Conf. Ser. 1194 012003
[4] Aratyn H, Gomes J F, Ruy D V and Zimerman A H 2016 J. Phys. A: Math. Theor. 49 045201
[5] Conte R M and Musette M 2008 The Painlevé Handbook (Springer Netherlands)
[6] Gambier B 1910 Acta Math. 33 1
[7] Gromak V I, Laine I, Shimomura S 2002 Painlevé Differential Equations in the Complex Plane (de Gruyter Studies in Mathematics) Volume 28
[8] Ince E L 1956 Ordinary Differential Equations (New York: Dover)
[9] Kudryashov N 2001 J. Phys. A: Math. Gen. 35 93
[10] Noumi M and Yamada Y, Higher order Painlevé equations of type $\mathcal{A}_1^{(1)}$ (arXiv:9808003)
[11] Painlevé P 1906 C. R. Acad. Sc. Paris 143 p.1111
[12] Rogers C 2017 J. of Nonlin. Math. Ph. 24:2 239
[13] Smith J 2016 Painleve Equations and Orthogonal Polynomials (https://kar.kent.ac.uk/id/eprint/54758)
[14] Suzuki M, Tahara N, Takano K 2004 J. Math. Soc. Japan 56 no.4 1221
[15] Tamizhmani K M et al 1998 J. Phys. A: Math. Gen. 31 5799
[16] Veselov A P, Shabat A B 1993 Funktsional. Anal. i Priložen 27:2 1