Total Betti numbers of modules of finite projective dimension

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Abstract

The Buchsbaum-Eisenbud-Horrocks Conjecture predicts that the $i^{th}$ Betti number $\beta_i(M)$ of a nonzero module $M$ of finite length and finite projective dimension over a local ring $R$ of dimension $d$ should be at least $\binom{d}{i}$. It would follow from the validity of this conjecture that $\sum_i \beta_i(M) \geq 2^d$. We prove the latter inequality holds in a large number of cases and that, when $R$ is a complete intersection in which 2 is invertible, equality holds if and only if $M$ is isomorphic to the quotient of $R$ by a regular sequence of elements.

1. Introduction

We recall a long-standing conjecture (see [3, 1.4] and [6, Prob. 24]):

Conjecture (Buchsbaum-Eisenbud-Horrocks Conjecture). Let $R$ be a commutative Noetherian ring such that $\text{Spec}(R)$ is connected, and let $M$ be a nonzero, finitely generated $R$-module of finite projective dimension. For any finite projective resolution $0 \rightarrow P_d \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ of $M$, we have

$$\text{rank}_R(P_i) \geq \binom{c}{i},$$

where $c = \text{height}_R(\text{ann}_R(M))$, the height of the annihilator ideal of $M$.

The validity of the Buchsbaum-Eisenbud-Horrocks Conjecture would imply that the “total rank” of any projective resolution of $M$ is at least $2^c$. In this paper, we prove this latter inequality holds in a large number of cases:

Theorem 1. Assume $R$, $M$, and $P$ are as in the Buchsbaum-Eisenbud-Horrocks Conjecture and, in addition, that

Keywords: Betti numbers, Buchsbaum-Eisenbud-Horrocks Conjecture
AMS Classification: Primary: 13D02.
This work was partially supported by grant #318705 from the Simons Foundation.
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(1) \( R \) is locally a complete intersection and \( M \) is 2-torsion free, or
(2) \( R \) contains \( \mathbb{Z}/p \) as a subring for an odd prime \( p \).

Then \( \sum_i \text{rank}_R(P_i) \geq 2^c \), where \( c = \text{height}_R(\text{ann}_R(M)) \).

Theorem 2 below is the special case of Theorem 1 in which we assume \( R \) is a local ring and \( M \) has finite length. We record it as a separate theorem since Theorem 1 follows immediately from it and also because in the local situation we can say a bit more.

For a local ring \( R \) and a finitely generated \( R \)-module \( M \), let \( \beta_i(M) \) be the \( i \)th Betti number of \( R \), defined to be the rank of the \( i \)th free module in the minimal free resolution of \( M \).

**Theorem 2.** Assume \((R, \mathfrak{m}, k)\) is a local (Noetherian, commutative) ring of Krull dimension \( d \) and that \( M \) is a nonzero \( R \)-module of finite length and finite projective dimension. If either
(1) \( R \) is the quotient of a regular local ring by a regular sequence of elements and 2 is invertible in \( R \), or
(2) \( R \) contains \( \mathbb{Z}/p \) as a subring for an odd prime \( p \),
then \( \sum_i \beta_i(M) \geq 2^d \).

Moreover, if the assumptions in (1) hold and \( \sum_i \beta_i(M) = 2^d \), then \( M \) is isomorphic to the quotient of \( R \) by a regular sequence of \( d \) elements.

To see that Theorem 1 follows from Theorem 2, with the notation of the first theorem, let \( \mathfrak{p} \) be a minimal prime containing \( \text{ann}_R(M) \) of height \( c \). Then \( \dim(R_\mathfrak{p}) = c, M_\mathfrak{p} \) has finite length, and \( \beta_i(M_\mathfrak{p}) \leq \text{rank}_R(P_i) \) for all \( i \). Moreover, if \( M \) is 2-torsion free, then 2 \( \not\in \mathfrak{p} \) and hence is invertible in \( R_\mathfrak{p} \).

I thank Seth Lindokken, Michael Brown, Claudia Miller, Peder Thompson and Luchezar Avramov for useful conversations about this paper.

**2. Complete intersections of residual characteristic not 2**

In this section we prove part (1) of Theorem 2 and the assertion concerning when the equation \( \sum_i \beta_i(M) = 2^d \) holds; see Theorem 2.4 below.

For any local ring \((R, \mathfrak{m}, k)\), let \( \text{Perf}^{\mathfrak{fl}}(R) \) be the category of bounded complexes of finite rank free \( R \)-modules \( F \) such that \( H_i(F) \) has finite length for all \( i \), and define \( K^{\mathfrak{fl}}_0(R) \) to be the Grothendieck group of \( \text{Perf}^{\mathfrak{fl}}(R) \). Recall that \( K^{\mathfrak{fl}}_0(R) \) is generated by isomorphism classes of objects of \( \text{Perf}^{\mathfrak{fl}}(R) \), modulo relations coming from short exact sequences and quasi-isomorphisms.

Let \( \psi^2 : K^{\mathfrak{fl}}_0(R) \to K^{\mathfrak{fl}}_0(R) \) be the 2nd Adams operation, as defined by Gillet-Soulé [4]. Gillet-Soulé’s definition involves the Dold-Kan correspondence between complexes and simplicial modules, but if 2 is invertible in \( R \), then \( \psi^2 \) admits a simpler description: For \( F \in \text{Perf}^{\mathfrak{fl}}(R) \), let \( T^2(F) \) denote its second tensor power \( F \otimes_R F \) endowed with the action of the symmetric group \( \Sigma_2 = \langle \tau \rangle \).
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given by
\[ \tau \cdot (x \otimes y) = (-1)^{|x||y|} y \otimes x. \]
Since \( \frac{1}{2} \in R \), we have a direct sum decomposition \( T^2(F) = S^2(F) \oplus \Lambda^2(F) \),
where \( S^2(F) := \ker(\tau - \text{id}) \) and \( \Lambda^2(F) := \ker(\tau + \text{id}) \). By [1, 6.14] we have
\[ (2.1) \quad \psi^2[F] = [S^2(F)] - [\Lambda^2(F)] \in K^0(R). \]

Let \( \ell_R \) denote the length of an \( R \)-module, and write \( \chi : K^0(R) \to \mathbb{Z} \) for the Euler characteristic map: \( \chi([F]) = \sum_i (-1)^i \ell_R H_i(F) \).

**Proposition 2.2** (Gillet-Soulé; see [4, 7.1]). If \( R \) is a local complete intersection of dimension \( d \), then \( \chi \circ \psi^2 = 2^d \cdot \chi \).

**Definition 2.3.** A local ring \( (R, m, k) \) of dimension \( d \) such that 2 is invertible in \( R \) will be called a quasi-Roberts ring if there we have an equality of maps \( \chi \circ \psi^2 = 2^d \cdot \chi \).

**Theorem 2.4.** Let \( (R, m, k) \) be a local ring of dimension \( d \) such that 2 is invertible in \( R \). If \( R \) is a quasi-Roberts ring, then for any nonzero \( R \)-module \( M \) of finite length and finite projective dimension, we have \( \sum_i \beta_i(M) \geq 2^d \).

Moreover, if \( \sum_i \beta_i(M) = 2^d \), then \( M \cong R/(y_1, \ldots, y_d) \) for some regular sequence of elements \( y_1, \ldots, y_d \in m \).

**Proof.** Let \( F \) be the minimal free resolution of \( M \), so that \( \chi(F) = \ell_R(M) \) and \( \text{rank}_R(F_i) = \beta_i(M) \). Using (2.1) we get
\[ (2.5) \quad 2^d \cdot \ell_R(M) = \chi(\psi^2(F)) = \sum_i (-1)^i \ell_R H_i(S^2(F)) - \sum_j (-1)^j \ell_R H_j(\Lambda^2(F)) \]
\[ \leq \sum_{i \text{ even}} \ell_R H_i(S^2(F)) + \sum_{i \text{ odd}} \ell_R H_i(\Lambda^2(F)). \]

Since \( S^2(F) \) and \( \Lambda^2(F) \) are direct summands of \( F \otimes_R F \),
\[ (2.6) \quad \sum_{i \text{ even}} \ell_R H_i(S^2(F)) + \sum_{i \text{ odd}} \ell_R H_i(\Lambda^2(F)) \leq \sum_i \ell_R H_i(F \otimes_R F). \]

For each \( i \), \( H_i(F \otimes_R F) \cong H_i(F \otimes_R M) \) is a subquotient of \( F_i \otimes_R M \) and thus
\[ (2.7) \quad \ell_R H_i(F \otimes_R M) \leq \ell_R H_i(F \otimes_R F) = \text{rank}(F_i) \cdot \ell_R(M) = \beta_i(M) \cdot \ell_R(M). \]
Putting the inequalities (2.5), (2.6), and (2.7) together yields
\[ 2^d \cdot \ell_R(M) \leq \ell_R(M) \cdot \sum_i \beta_i(M), \]
and since \( \ell_R(M) > 0 \), we conclude \( \sum_i \beta_i(M) \geq 2^d \).

Now suppose \( \sum_i \beta_i(M) = 2^d \). Then the inequalities (2.5), (2.6), and (2.7) must actually be equalities, which means that \( H_i(S^2(F)) = 0 \) for all odd \( i \), \( H_j(\Lambda^2(F)) = 0 \) for all even \( j \), and \( F \otimes_R M \) has trivial differential. Since
$H_0(\Lambda^2(F)) \cong \Lambda^2(M)$ is the classical second exterior power, $M$ must be cyclic, i.e., of the form $R/I$ for some ideal $I$. Since $F \otimes_R R/I$ has trivial differential, $I/I^2 \cong \text{Tor}^1_R(R/I, R/I)$ is free as an $R/I$-module, and thus a result of Ferrand and Vasconcelos (see [2, 2.2.8]) gives that $I$ is generated by a regular sequence of elements. □

3. Rings of odd characteristic

In this section we prove part (2) of Theorem 2. The main idea is to replace the Euler characteristic $\chi$ occurring in the proof of part (1) with the Dutta multiplicity.

Definition 3.1. Assume $(R, m, k)$ is a complete local ring of dimension $d$ that contains $\mathbb{Z}/p$ as a subring for some prime $p$ and that $k$ is a perfect field. For $F \cdot \in \text{Perf}^{fl}(R)$, define

$$\chi_{\infty}(F) = \lim_{e \to \infty} \frac{\chi(\varphi^e F)}{p^d e},$$

where $\varphi^e$ denotes extension of scalars along the $e^{th}$ iterate of the Frobenius endomorphism of $R$. The limit is known to exist by, e.g., [9, 7.3.3].

Proof of Theorem 2 part (2). There is a faithfully flat map $(R, m, k) \to (R', m', k')$ of local rings such that $m \cdot R' = m'$, $R'$ is complete and $k'$ is algebraically closed; see [5, 0.10.3.1]. Letting $M' := M \otimes_R R'$, we have that $M'$ is a nonzero $R'$-module of finite length and finite projective dimension, $\beta^R_i(M') = \beta^R_i(M)$ for all $i$, and $\dim(R') = \dim(R)$. We may therefore assume $R$ is complete with algebraically closed residue field.

Let $F$ be the minimal free resolution of $M$. Since $R$ is complete with perfect residue field, a result of Roberts [9, 7.3.5] gives

$$\chi_{\infty}(F) > 0$$

and a result of Kurano-Roberts [7, 3.1] gives (using (2.1))

$$\chi_{\infty}(S^2(F)) - \chi_{\infty}(\Lambda^2(F)) = \chi_{\infty}(\psi^2(F)) = 2^d \cdot \chi_{\infty}(F).$$

For each $e \geq 0$, we have $\varphi^e S^2(F) \cong S^2(\varphi^e F)$ and $\varphi^e \Lambda^2(F) \cong \Lambda^2(\varphi^e F)$, and thus

$$\chi_{\infty}(S^2(F)) = \lim_{e \to \infty} \frac{1}{p^d e} \sum_i (-1)^i \ell_R H_i(S^2(\varphi^e F)),$$

$$\chi_{\infty}(\Lambda^2(F)) = \lim_{e \to \infty} \frac{1}{p^d e} \sum_i (-1)^i \ell_R H_i(\Lambda^2(\varphi^e F)).$$
As in the proof of Theorem 2.4, for a fixed $e$, we have
\[
\frac{1}{p^{de}} \sum_i (-1)^i \ell_R H_i(S^2(\varphi^e F)) - \frac{1}{p^{de}} \sum_i (-1)^i \ell_R H_i(A^2(\varphi^e F)) \\
\leq \sum_j \ell_R H_j(T^2(\varphi^e F)).
\]

By [8, 1.7], the complex $\varphi^e(F)$ is the minimal free resolution of the finite length module $\varphi^e(M)$ for each $e \geq 0$. As in the proof of Theorem 2.4, for each $i$, we have
\[
\ell_R H_i(T^2(\varphi^e F)) \leq \text{rank}(\varphi^e F_i) \cdot \ell_R(\varphi^e M) = \beta_i(M) \cdot \chi(\varphi^e F).
\]

We have proven that
\[
\frac{1}{p^{de}} \sum_i (-1)^i \ell_R H_i(\varphi^e S^2(F)) - \frac{1}{p^{de}} \sum_i (-1)^i \ell_R H_i(\varphi^e A^2(F)) \\
\leq \frac{1}{p^{de}} \chi(\varphi^e F) \cdot \sum_i \beta_i(M)
\]
holds for each $e \geq 0$. Taking limits and using (3.3) gives
\[
2^d \cdot \chi_\infty(F) \leq \chi_\infty(F) \cdot \sum \beta_i(M).
\]
Since $\chi_\infty(F) > 0$ by (3.2), we conclude $\sum \beta_i(M) \geq 2^d$. \qed

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(Received: April 7, 2017)
(Revised: May 22, 2017)

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