FOLD MAPS WITH SINGULAR VALUE SETS HAVING NO SELF-INTERSECTIONS AND HOMOLOGICAL PROPERTIES OF THEIR REEB SPACES

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Abstract. In this paper, as a fundamental study on the theory of Morse functions and their higher dimensional versions or fold maps, and applications to geometric theory of manifolds, we study algebraic and differential topological properties of fold maps such that the sets of all the singular values (the singular value sets) have no self-intersections and their source manifolds. As a specific case, round fold maps are defined as fold maps whose singular value sets are concentric spheres and they were introduced in 2012–14 and have been systematically studied by the author. In the present paper, we concentrate on fold maps with singular value sets having no self-intersections; especially on ones such that the inverse images of regular values are disjoint unions of spheres. The author previously studied homology and homotopy groups and the homeomorphism and diffeomorphism types of manifolds admitting such round fold maps and here, we do such works for such fold maps which are not always round. For example, we show flexibility of (co)homology groups of Reeb spaces of such maps, which are essential tools in studying manifolds by using generic maps.

1. Introduction and fundamental notation and terminologies

1.1. Historical backgrounds. Morse functions and fold maps, which are regarded as higher dimensional versions of Morse functions, play important roles in studying smooth manifolds by using generic maps since related studies were started by Thom ([18]) and Whitney ([20]).

Let $m$ and $n$ be integers satisfying $m \geq n \geq 1$. A fold map from an $m$-dimensional closed smooth manifold into an $n$-dimensional smooth manifold without boundary is defined as a smooth map such that each singular point is of the form

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^m x_k^2)$$

2010 Mathematics Subject Classification. Primary: 57R45. Secondary: 57N15.

Key words and phrases. Singularities of differentiable maps; singular sets, fold maps. Differential topology.
for an integer $0 \leq i \leq \frac{m-n+1}{2}$ (the integer $i$ is uniquely determined and it is called the index of the singular point). We easily have the following.

**Proposition 1.**

1. The set consisting of all the singular points (the singular set) of the map is a closed smooth submanifold of dimension $n - 1$ of the source manifold.
2. The restriction map to the singular set is a smooth immersion of codimension 1.

As a branch of the theory of fold maps and its applications to studies of smooth manifolds, algebraic and differential topological properties of fold maps of several classes such as special generic maps and round fold maps and manifolds admitting such maps have been studied since 1990.

A special generic map is defined as a fold map such that the indices of singular points are always 0. A Morse function on a homotopy sphere with just two singular points is regarded as a simplest special generic map; every smooth homotopy sphere of dimension $k \neq 4$ and the 4-dimensional standard sphere $S^4$ admits such a function and conversely, a manifold admitting such a function is a homotopy sphere (see [9] and [12] and see also [11]). Moreover, we easily obtain a special generic map from any standard sphere of dimension $k_1 \geq 2$ into the $k_2$-dimensional Euclidean space $\mathbb{R}^{k_2}$ by a natural projection under the assumption that $k_1 \geq k_2 \geq 1$ holds. On the other hand, it was shown that a homotopy sphere of dimension $k_1$ admitting a special generic map into $\mathbb{R}^{k_2}$ is diffeomorphic to the standard sphere $S^{k_1}$ under the assumption that $1 \leq k_1 - k_2 \leq 3$ holds (see [12] and [13]). In addition, in [12] and [14], the diffeomorphism types of manifolds admitting special generic maps into $\mathbb{R}^2$ and $\mathbb{R}^3$ are completely or partially classified. A Morse function and its singular points tells us homology groups and some information on homotopy of the source manifold and a special generic map and its singular points often tells us more; the homeomorphism and diffeomorphism type of the source manifold.

Such interesting properties of special generic maps imply that considering appropriate classes of (stable) fold maps and studying algebraic and differential topological properties of such maps and manifolds admitting them is essential in the theory of fold maps. Motivated by this, round fold maps were introduced in 2012–2014 by the author in [4].

**Definition 1.** A round fold map is a fold map into an Euclidean space of dimension larger than 1 satisfying the following three.

1. The singular set is a disjoint union of standard spheres.
2. The restriction map to the singular set is an embedding.
3. The set consisting of all the singular values (the singular value set) of the map is a disjoint union of spheres embedded concentrically.

Some special generic maps are round fold maps. The standard sphere of dimension $m > 1$ admits such a map into $\mathbb{R}^n$ under the assumption that $m \geq$
$n \geq 2$ holds; we can construct the map by a natural projection as mentioned and the singular set is connected. Furthermore, in [12], Saeki constructed such a map into the plane on every homotopy sphere whose dimension is larger than 1 and not 4.

Homology and homotopy groups and more precisely, the homeomorphism and diffeomorphism types of manifolds admitting round fold maps were studied in [3], [4], [5] and [6] under appropriate conditions by the author. For example, homology and homotopy groups of manifolds admitting round fold maps such that the inverse images of regular values are disjoint unions of spheres were studied in [4] and [5]. In addition, the following has been shown through explicit constructions of fold maps in [3], [4] or [6].

**Proposition 2.** Let $m$ and $n$ be integers satisfying $m > n \geq 2$.

1. The total space of a smooth bundle over $S^n$ whose fiber is diffeomorphic to an almost-sphere admits a round fold map into $\mathbb{R}^n$ satisfying the following conditions.
   (a) The singular set consists of 2 connected components.
   (b) The inverse image of a regular value in the connected component of the regular value set, which is an open disc, is a disjoint union of two copies of the mentioned almost-sphere and these almost-spheres are both fibers of the bundle.

2. A smooth manifold represented as a connected sum of $l$ smooth manifolds regarded as the total spaces of smooth bundles over $S^n$ whose fibers are diffeomorphic to $S^{m-n}$ admits a round fold map into $\mathbb{R}^n$ satisfying the following conditions.
   (a) The inverse image of each regular value is a disjoint union of standard spheres and regarded as a fiber of a smooth bundle above.
   (b) The number of connected components of the inverse image of a regular value in the connected component of the regular value set, which is an open disc, is $l$.

Note that explicit constructions help us to know precise structures of manifolds such as topological properties of submanifolds obtained as inverse images of regular values and that they will help us to study manifolds by the theory of fold maps easy to handle, but constructions are often difficult except cases such as special generic maps and some round fold maps mentioned here.

1.2. **Main works of this paper and fundamental notation and terminologies.** In this paper, as a wider class of fold maps, we study algebraic and differential topological properties of fold maps with singular value sets having no self-intersections which are not always round and manifolds admitting such maps. Especially, we investigate such fold maps the inverse images of regular values of which are disjoint unions of spheres. We study structures of such maps and (co)homology groups and more precise information of manifolds
admitting such maps. In these studies, we introduce operations consisting of surgeries and construct several new explicit fold maps.

First, we review the Reeb space of a smooth map, which is defined as the space consisting of all the connected components of the inverse images of the map and is a fundamental tool in studying the manifold.

Second, we introduce a \((\text{normal})\) S-bubbling operation to a fold map with the singular value set having no self-intersections such that the inverse images of regular values are disjoint unions of spheres. Through [7] and [8], Kobayashi introduced a bubbling surgery as a surgery operation to obtain explicit stable fold maps and he succeeded in constructing explicit stable (fold) maps and S-bubbling operations are regarded as extensions of some bubbling surgeries. We apply S-bubbling operations to fold maps having singular value sets with no self-intersections such that the inverse images of regular values are disjoint unions of spheres and we obtain new maps satisfying similar conditions. We mainly study Reeb spaces of maps obtained by finite iterations of (normal) bubbling operations starting from a round fold map whose singular set is connected. As main theorems, through Theorems 1-4 and 6, we see that Euler numbers and (co)homology groups of the resulting Reeb spaces are flexible. In other word, for infinitely many integers, we obtain maps and Reeb spaces so that the Euler numbers are the given numbers and we see that there are many types of homology groups of the resulting Reeb spaces: in fact if we only consider round fold maps such that the inverse images of regular values are disjoint unions of almost-spheres, then (co)homology groups of Reeb spaces are not so flexible; this fact is also pointed out in Example 2. We also see some restrictions on (co)homology types of the resulting Reeb spaces in some statements of Theorem 2, Theorem 5, Example 4 and Remark 2 under various constraints.

Next, we explain isomorphisms between (co)homology and homotopy groups of manifolds admitting stable fold maps such that the inverse images of regular values are disjoint unions of spheres and those groups of their Reeb spaces, which were (partially) discussed in [15], [4] and [5]. By combining obtained results on (co)homology groups of Reeb spaces and these relations, we observe that (co)homology groups of manifolds admitting fold maps discussed here are flexible.

In this paper, for a smooth map \(c\), we define the \(\text{singular set of } c\) as the set consisting of all the singular points of \(c\) as in the presentation of the fundamental properties of fold maps before or Proposition 1 and denote it by \(S(c)\). For the smooth map \(c\), we call \(c(S(c))\) the \(\text{singular value set of } c\) as in the presentation of the definition of a round fold map or Definition 1 before. We call \(\mathbb{R}^n - c(S(c))\) the \(\text{regular value set of } c\).

We also note on (homotopy) spheres. In this paper, an \(\text{almost-sphere of dimension } k\) means a homotopy sphere given by gluing two \(k\)-dimensional standard closed discs together by a diffeomorphism between the boundaries.
We often use terminologies on (fiber) bundles in this paper (see also [17]). For a topological space $X$, an $X$-bundle is a bundle whose fiber is $X$. A bundle whose structure group is $G$ is said to be a trivial bundle if it is equivalent to the product bundle as a bundle whose structure group is $G$. A linear bundle is a smooth bundle whose fiber is a standard disc or a standard sphere and whose structure group consists of linear transformations on the fiber.

Throughout this paper, we assume that $M$ is a closed smooth manifold of dimension $m$, that $N$ is a smooth manifold of dimension $n$ without boundary, that $f : M \to N$ is a smooth map and that $m \geq n \geq 1$. Manifolds are of class $C^\infty$ and maps between manifolds are also of class $C^\infty$ unless otherwise stated in the proceeding sections. In addition, the structure groups of bundles such that the fibers are (smooth) manifolds are assumed to be (subgroups of) diffeomorphism groups.

2. Reeb spaces and (normal) S-bubbling operations

2.1. Definitions and fundamental properties of Reeb spaces and normal S-bubbling operations.

Definition 2. Let $X$ and $Y$ be topological spaces. For $p_1, p_2 \in X$ and for a map $c : X \to Y$, we define as $p_1 \sim_c p_2$ if and only if $p_1$ and $p_2$ are in a same connected component of $c^{-1}(p)$ for some $p \in Y$. $\sim_c$ is an equivalence relation.

We denote the quotient space $X/\sim_c$ by $W_c$ and call $W_c$ the Reeb space of $c$.

We denote the induced quotient map from $X$ into $W_c$ by $q_c$. We can define $\bar{c} : W_c \to Y$ uniquely so that the relation $c = \bar{c} \circ q_c$ holds.

For a (stable) fold map $c$, the Reeb space $W_c$ is regarded as a polyhedron. For example, for a Morse function, the Reeb space is a graph and for a special generic map, the Reeb space is regarded as a smooth manifold (see section 2 of [13]).

Definition 3. Let $f : M \to \mathbb{R}^n$ be a fold map. Let $P$ be a connected component of the regular value set $\mathbb{R}^n - f(S(f))$. Let $S$ be a connected and orientable closed submanifold of $P$ and $N(S)$, $N(S)_i$ and $N(S)_o$ be small closed tubular neighborhoods of $S$ in $P$ such that $N(S)_i \subset N(S) \subset N(S)_o$ holds. Furthermore, we can naturally regard $N(S)_o$ as a linear bundle whose fiber is an $(m - n + 1)$-dimensional disc of radius 1 and $N(S)_i$ and $N(S)$ are subbundles of the bundle $N(S)$ whose fibers are $(m - n + 1)$-dimensional discs of radii $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Let $f^{-1}(N(S)_o)$ have a connected component $Q$ such that $f|_Q$ makes $Q$ a bundle over $N(S)_o$ and that the fiber of this bundle is an almost-sphere.

Let us assume that there exists an $m$-dimensional closed manifold $M'$ and a fold map $f' : M' \to \mathbb{R}^n$ satisfying the following.
(1) $M - \text{Int}Q$ is a compact submanifold (with non-empty boundary) of $M'$ of dimension $m$.

(2) $f|_{M - \text{Int}Q} = f'|_{M - \text{Int}Q}$ holds.

(3) $f'(S(f'))$ is the disjoint union of $f(S(f))$ and $\partial N(S)$.

(4) $f'|_{(M' - (M - Q)) \cap f^{-1}(N(S)_1)}$ makes $(M' - (M - Q)) \cap f^{-1}(N(S)_1)$ the disjoint union of two bundles over $N(S)$ whose fibers are both almost-spheres.

These assumptions enable us to consider the procedure of constructing $f'$ from $f$ and we call it a normal $S$-bubbling operation to $f$ and, $Q_0 := f^{-1}(S) \cap q_f(Q)$, which is homeomorphic to $S$, the generating manifold of the normal $S$-bubbling operation.

Example 1. Let $f : M \to \mathbb{R}^n$ be a fold map. Let $P$ be a connected component of the set $\mathbb{R}^n - f(S(f))$. Let $S$ be a connected and orientable closed submanifold of $P$ such that there exists a connected component $Q_0$ of $f^{-1}(S)$ and that $f|_{Q_0} : Q_0 \to S$ makes $Q_0$ a trivial bundle over $S$; for example, let $S$ be in the interior of an open ball in the interior of $P$.

Then, by a normal $S$-bubbling operation to $f$ such that the generating manifold is $Q_0$, we can obtain a new fold map $f' : M' \to \mathbb{R}^n$ satisfying the following conditions where we abuse notation in Definition 3. We call this operation a trivial normal $S$-bubbling operation.

(1) $f|_{M - \text{Int}Q} = f'|_{M - \text{Int}Q}$.

(2) $f'|_{f'^{-1}(Q_0)}$ gives a disjoint union of two trivial bundles over $N(S)$ whose fibers are both almost-spheres.

(3) There exists a connected component of $f'^{-1}(N_o(S) - \text{Int}N_i(S))$ such that the restriction map of $f'$ to the component is regarded as the product of a Morse function with just one singular point on a manifold PL homeomorphic to the $(m - n + 1)$-dimensional standard sphere with the interior of the disjoint union of smoothly embedded three $(m - n + 1)$-dimensional standard closed discs and $\text{id}_{\partial N(S)}$.

Moreover, let the normal bundle or tubular neighborhood of $S$ be trivial. $N_o(S)$ is represented by $S \times D^{n - \text{dim}S}$ and $S$ is regarded as $S \times \{0\} \subset S \times D^{n - \text{dim}S}$. For example, let $S$ be the standard sphere embedded as an unknot in the interior of an open ball in the interior of $P$. Furthermore, let the restriction of $f'$ to $f'^{-1}(N_o(S))$ is regarded as the product of a surjective map $f'|_{f'^{-1}(D^{n - \text{dim}S})} : f'^{-1}(D^{n - \text{dim}S}) \to D^{n - \text{dim}S}$ where $D^{n - \text{dim}S}$ is a fiber of the trivial bundle $N_o(S)$ and $\text{id}_{S}$. Then we call the previous operation a strongly trivial normal $S$-bubbling operation.

**Proposition 3.** Let $f : M \to \mathbb{R}^n$ be a fold map such that $f|_{S(f)}$ is an embedding and that the inverse images of regular values are disjoint unions of
almost-spheres. Let $f': M \to \mathbb{R}^n$ be a fold map obtained by a normal S-bubbling operation to $f$ and let $S$ be the generating manifold of the normal S-bubbling operation and of dimension $k < n$. Then for any PID $R$, we have $H_i(W_{f'}; R) \cong H_i(W_f; R) \oplus (H_{i-(n-k)}(S; R))$.

We prove Proposition 3 in the proof of Proposition 4 later; Proposition 3 follows immediately from Proposition 4.

We have the following as a corollary.

**Corollary 1.** In the situation of Proposition 3, if $H_j(S; R)$ is free for any $j$ and $H_i(W_f; R)$ is free, then $H_i(W_{f'}; R)$ is also free.

In this paper, for a topological space $X$, we denote the Euler number of $X$ by $\chi(X)$. This is another corollary to Proposition 3

**Corollary 2.** In the situation of Proposition 3, we have the formula $\chi(W_{f'}) = \chi(W_f) + (-1)^{n-k} \chi(S)$.

2.2. Flexibility of the Euler numbers and homology modules (groups) of Reeb spaces of maps obtained by finite iterations of normal S-bubbling operations. First, applying Corollary 2, we have the following.

**Theorem 1.** Let $f : M \to \mathbb{R}^n$ be a fold map such that $f|_{S(f)}$ is an embedding and that the Euler number of $W_f$ is 1. For example, consider a Morse function with two singular points on a homotopy sphere or a round fold map whose singular set is connected (on a homotopy sphere).

1. Let $n$ be even (odd). If we perform a finite iteration of normal S-bubbling operations such that the Euler numbers of generating manifolds are not negative starting from $f$, then the resulting Euler numbers of the resulting Reeb spaces are not smaller (resp. larger) than 1. Moreover, every positive integer not smaller (resp. larger) than 1 is realized as the resulting Reeb space of an iteration of normal S-bubbling operations.

2. Let $n = 1$ (2). If we perform a finite iteration of normal S-bubbling operations, then the resulting Euler number of the resulting Reeb space is not larger (resp. smaller) than 1.

3. Let $n \geq 3$. In this situation, every integer is realized as the Euler number of the resulting Reeb space of an iteration of normal S-bubbling operations.

**Proof.** The former part of the first statement immediately follows from Corollary 2. The latter part is easily obtained by applying a finite iteration of normal S-bubbling operations such that the generating manifolds are points (the Euler number of point is $1 > 0$) starting from $f$ and Corollary 2.

We easily have the second statement for $n = 1$ by the fact that the generating manifold is always a point and by Corollary 2. For $n = 2$, we have the
result by the fact that the generating manifold is always a point or a circle (the Euler class of a circle is 0) and by Corollary 2.

We prove the last statement.

Let \( n \) be even. In this case, it is sufficient to show that for any integer \( l \) smaller than 1, by a finite iteration of normal S-bubbling operations starting from \( f \), we obtain a fold map such that the resulting Euler number of the resulting Reeb space is \( l \). We can take a closed and connected orientable surface of genus \( g \geq 2 \), whose Euler number is \( 2 - 2g \), so that \( 1 + (2 - 2g) = 2g - 1 \) is not larger than 1. We can also take \( l - (2g - 1) \) points which are not in the previous surface. By the finite iteration of normal S-bubbling operations whose generating manifolds are this surfaces and these points, we obtain a fold map such that the resulting Euler number of the resulting Reeb space is \( l \).

Let \( n \) be odd. \( \square \)

**Theorem 2.** Let \( R \) be a PID. Let \( f : M \to \mathbb{R}^n \) be a round fold map whose singular set is connected or more generally, a fold map such that \( f|_{S(f)} \) is an embedding, that the singular set is connected and that for the image \( f(M) \), \( H_*(f(M); R) \) and \( H_*(D^n; R) \) are isomorphic.

1. For any integer \( 0 \leq j \leq n \), we define \( G_j \) as a free finitely generated module over \( R \) so that \( G_0 \) is \( R \), that \( G_n \) is not a trivial module, and the sum \( \sum_{j=1}^{n-1} \text{rank } G_j \) of the ranks of \( G_j \) is not larger than the rank of \( G_n \). Then, by a finite iteration of normal S-bubbling operations starting from \( f \), we obtain a fold map \( f' \) such that the map obtained by the restriction to its singular set is an embedding and \( H_j(W_{f'}; R) \) is isomorphic to \( G_j \).

2. Let \( f' \) be a fold map obtained by a finite iteration of normal S-bubbling operations starting from \( f \). Then, \( H_0(W_{f'}; R) \) is \( R \) and \( H_n(W_{f'}; R) \) is finitely generated and free.

Suppose that for a positive integer \( k \) and for the Reeb space \( W_{f'} \), \( H_j(W_{f'}; R) \) is zero for \( 0 < j < k \) and \( H_k(W_{f'}; R) \) is not zero, then \( H_k(W_{f'}; R) \) is finitely generated and free and its rank is not larger than that of \( H_0(W_{f'}; R) \).

3. Let \( f' \) be a fold map obtained by a finite iteration of normal bubbling operations starting from \( f \). Then, \( H_{n-1}(W_{f'}; R) \) is finitely generated and free.

**Proof.** We prove the first statement. Set \( g_j := \text{rank} G_j \). We can choose a disjoint union of \( g_j \) copies of the \((n-j)\)-dimensional standard spheres for \( 0 < j \leq n \) smoothly embedded in an open disc in the interior \( \text{Int} f(M) \) of the image and \( g_n - \sum_{j=1}^{n-1} g_j \) points in \( \text{Int} f(M) \). We apply normal S-bubbling operations such that the generating manifolds are inverse images of spheres or points in the previous disjoint union by the maps from the Reeb spaces to
as in Definition 2 one after another starting from the given map \( f \); in fact we can apply strongly trivial normal S-bubbling operations. Thus, by virtue of Proposition 3, we can obtain a desired map. If we choose a disjoint union of \( g_j \) \((n - j)\)-dimensional manifolds whose homology types with coefficient rings \( R \) are isomorphic to that of an \((n - j)\)-dimensional sphere for \( 0 < j < n \) smoothly embedded in the interior \( \text{Int} f(M) \) and \( g_n - \sum_{j=1}^{n-1} g_j \) points there so that we can demonstrate a finite iteration of normal S-bubbling operations, then we always obtain a desired map.

We prove the second statement. \( H_0(W_{f'}; R) \) is \( R \) and \( H_n(W_{f'}; R) \) is finitely generated and free; by Proposition 3, this easily follows and the rank of \( H_n(W_{f'}; R) \) equals the number of the time of normal S-bubbling operations we need to obtain \( f' \). By Proposition 3, the maximum of the dimensions of the generating manifolds of these normal S-bubbling operations must be \( m - k \).

It also follows that \( H_k(W_{f'}; R) \) is free and that its rank and the number of the time of normal S-bubbling operations with generating manifolds of dimension \( m - k \) coincide.

We prove the last statement. Let \( f_1 : M_1 \to \mathbb{R}^n \) be a fold map such that \( f_1|_{S(f_1)} \) is an embedding and that the inverse images of regular values are disjoint unions of almost-spheres. Let \( f_2 : M_2 \to \mathbb{R}^n \) be a fold map obtained by a normal S-bubbling operation to \( f_1 \) whose generating manifold is \( S \) and of dimension \( k \). By Proposition 3, we have \( H_{n-1}(W_{f_2}; R) \cong H_{n-1}(W_{f_1}; R) \oplus (H_{k-1}(S; R) \otimes R) \). Note that \( H_{k-1}(S; R) \) is free; if \( k \) is 1 or 2, then \( S \) is, by the assumption, orientable and \( H_{k-1}(S; R) \) is finitely generated and free, and if \( k \) is larger than 2, then \( H^1(S; R) \) is finitely generated and free and by virtue of the Poincare duality theorem together with the assumption that \( S \) is orientable, so is \( H_{k-1}(S; R) \).

This completes the proof. □

Remark 1. By universal coefficient theorem, in the situation of Theorem 2 (1), \( H_j(W_{f'}; R) \) is isomorphic to \( H^j(W_{f'}; R) \) for \( 0 \leq j \leq n \) and in the situation of (2), \( H_j(W_{f'}; R) \) is isomorphic to \( H^j(W_{f'}; R) \) for \( 0 \leq j \leq k \) and \( j = n \).

Example 2. We can say that in [3] and [4], maps in Proposition 1 have been obtained by finite iterations of strongly trivial normal S-bubbling operations whose generating manifolds are points. Note that the Reeb spaces are simple homotopy equivalent to the bouquet of \( l - 1 \) copies of \( S^n \). These are specific cases of Theorem 2 (1).

In [8], Kobayashi invented a bubbling surgery to a stable (fold) map and we can also say that maps in Proposition 1 have been obtained by finite iterations of bubbling surgeries.

More precisely, a bubbling surgery is a surgery operation to construct a new stable fold map from a given stable fold map so that the singular value set of the new map is the disjoint union of that of the given map and a sphere
Corollary 3. Let $\Box$ normal S-bubbling operation. Let $\Box$ vanishes and which we can embed into $\mathbb{R}^{n-1}$. Then, by a normal S-bubbling operation to $f$, we have a fold map $f' : M' \to \mathbb{R}^n$ satisfying the following condition.

$$H_j(W_f'; R) \cong \begin{cases} R(j = 0) \\ H_{j-1}(S; R)(l + 1 \leq j \leq k - 1) \\ H_{j-1}(S; R) \oplus H_{j-k}(S; R)(k \leq j \leq n - 1) \\ R(k = n) \\ \{0\}(\text{otherwise}) \end{cases}$$

Proof. We can embed $S$ into $\mathbb{R}^{n-1}$ and there exists a manifold $S'$ regarded as $S$ whose Euler class vanishes and which we can embed into $\mathbb{R}^n$. For any PID $R$, we have

$$H_j(S'; R) \cong \begin{cases} H_j(S; R)(0 \leq j \leq k - l - 2) \\ H_j(S; R) \oplus H_{j-k+l+1}(S; R)(k - l - 1 \leq j \leq n - l - 2) \\ H_j(S; R)(j = n - l - 1) \end{cases}.$$

By virtue of Proposition 3, by a normal S-bubbling operation to $f$ such that the generating manifold is $f^{-1}(S')$, we obtain a desired map $f$; for example, take $S$ in an open ball in the interior of the image $f(M)$ and perform a trivial normal S-bubbling operation. \hfill \square

As a specific case, we have the following.

Corollary 3. Let $S$ be a closed and connected orientable manifold whose dimension is not smaller than 1 and not larger than $\frac{n}{2}$. Set $l$ be an integer satisfying $0 \leq l \leq n - 2 \dim S$ and if $\dim S = 1$ holds or $S$ is a circle, then assume that $0 \leq l < n - 2 \dim S$ holds.

Let $R$ be a PID. Let $f : M \to \mathbb{R}^n$ be a round fold map whose singular set is connected or more generally, a fold map such that $f|_{S(f)}$ is an embedding, that the singular set is connected and that for the image $f(M)$, $H_*(f(M); R)$ and $H_*(D^n; R)$ are isomorphic.

Then, by a normal S-bubbling operation to $f$, we have a fold map $f' : M' \to \mathbb{R}^n$ satisfying

bounding a closed disc whose interior does not contain singular values and that the outside of a small neighborhood of the disc does not change.
\[ H_j(W_f; R) \cong \begin{cases} 
R(j = 0) 
H_{j-l-1}(S; R)(l + 1 \leq j \leq n - \dim S - 1) 
H_{j-l-1}(S; R) \oplus H_{j-n+1}(S; R)(n - \dim S \leq j \leq n - 1) 
R(k = n) 
\{0\} \text{(otherwise)} 
\end{cases} \]

**Proof.** We apply Whitney embedding theorem, which states that any closed manifold \( X \) is embedded into \( \mathbb{R}^{2 \dim X} \). By virtue of this theorem, in the situation of Theorem 2, we can set \( n - k := \dim S \) and we have \( l < l + 1 < k < n \). Thus, we have the result.

Example 3. Let \( G \) be any finitely generated commutative group. There exists a closed and connected orientable 3-dimensional manifold \( S \) satisfying \( H_1(S; \mathbb{Z}) = G \). We can embed \( S \) into \( \mathbb{R}^n \) where \( n \geq 5 \) holds \((19)\). For such \( n \), we can apply Proposition 3 to a fold map \( f : M \to \mathbb{R}^n \) as in Theorem 1, Theorem 2 or Corollary 1 and for the resulting map \( f' : M' \to \mathbb{R}^n \), \( H_j(W_f; R) \) is \( \{0\} \) for \( 0 < j \leq n - 4 \) and \( H_{n-j}(W_{f'}; R) \) is \( R \) and \( H_{n-2}(W_f; R) \) is \( G \). By choosing suitable integers \( k \geq 1 \) and \( l \geq 0 \), we can also apply Theorem 2 or, for \( n \geq 6 \), Corollary 1. In this case, for the map \( f' \), \( H_{l+1}(W_{f'}; \mathbb{Z}) \) is \( \mathbb{Z} \) and \( H_{l+2}(W_{f''}; \mathbb{Z}) \) is \( G \) or \( G \oplus \mathbb{Z} \). Note that the group \( H_{l+2}(W_f; \mathbb{Z}) \) may not be free.

Related to this example. We show another result on flexibility of homology groups of Reeb spaces. Let \( n \) be an integer larger than 6 and let \( f : M \to \mathbb{R}^n \) be a fold map as in Theorem 1, 2 or Corollary 1. Let \( j \) be an integer satisfying \( 0 \leq j \leq n - 7 \) and let \( G_j \) be a finite commutative group. Let \( \{g_j\}_{j=0}^{n-7} \) be a sequence of integers satisfying the following.

- all the integers are not smaller than \(-1\).
- If \( g_j = -1 \), then \( G_j \) is a trivial group.

There exists a 3-dimensional closed and connected orientable manifold satisfying \( H_1(S_j; \mathbb{Z}) \cong G_j \), \( H_2(S_j; \mathbb{Z}) \cong \{0\} \) and \( H_0(S_j; \mathbb{Z}) \cong H_3(S_j; \mathbb{Z}) = \mathbb{Z} \) (for example consider connected sums of Lens spaces). If \( g_j > -1 \) holds, then we can choose the family of disjoint submanifolds in \( \mathbb{R}^n \) satisfying the following two.

1. The family includes a manifold diffeomorphic to \( S_j \times S^{n-j-4} \).
2. The number of manifolds whose dimensions are \( \dim(S_j \times S^{n-j-4}) \) is \( g_j + 1 \) and except the manifold diffeomorphic to \( S_j \times S^{n-j-4} \), homology types of these manifolds with coefficient rings \( \mathbb{Z} \) are same as that of a sphere.

For example, take each submanifold in an open ball in the interior of the image \( f(M) \). By demonstrating a finite iteration of normal S-bubbling operations (for example, trivial normal S-bubbling operations) such that the generating
manifolds are inverse images of these obtained manifolds by the natural maps from the Reeb spaces into $\mathbb{R}^n$ one after another and applying Proposition 3, we obtain a map $f' : M' \to \mathbb{R}^n$ as in the following theorem.

**Theorem 4.** Let $m$ and $n$ be positive integers satisfying $m > n \geq 7$. Let $f : M \to \mathbb{R}^n$ be a fold map as in Theorem 2, 3 or Corollary 3.

For integers $0 \leq j \leq n - 7$, we introduce integers $g_j$ and finite commutative groups $G_j$ satisfying the following two.

1. $g_j \geq -1$.
2. If $g_j = -1$, then $G_j$ is a trivial group.

By a finite iteration of (trivial) normal $S$-bubbling operations such that the generating manifold of each operation is a product of a 3-dimensional manifold and a standard sphere starting from $f$, we obtain a map $f' : M' \to \mathbb{R}^n$ satisfying the following:

$$H_i(W_{f'}; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z} & (i = 0) \\
\mathbb{Z}^{g_0} & (i = 1) \\
\mathbb{Z}^{g_{i-1}+1} \oplus G_{i-2} & (i = 2, 3) \\
\oplus_{j=0}^{n-7} G_j & (i = n - 2) \\
\{0\} & (i = n - 1) \\
\mathbb{Z}^{(\sum_{j=0}^{n-7} g_j) + n - 6} & (i = n) \\
\mathbb{Z}^{g_{i-1}+2} \oplus G_{i-2} & (4 \leq i \leq n - 6, g_{i-4} > -1) \\
\mathbb{Z}^{g_{i-1}+1} \oplus G_{i-2} & (4 \leq i \leq n - 6, g_{i-4} = -1) \\
\mathbb{Z} \oplus G_{n-7} & (4 \leq i = n - 5, g_{n-9} > -1) \\
G_{n-7} & (4 \leq i = n - 5, g_{n-9} = -1) \\
\mathbb{Z} & (4 \leq i = n - 4, g_{n-8} > -1) \\
\{0\} & (4 \leq i = n - 4, g_{n-8} = -1) \\
\oplus_{j \in \{j \mid g_j > -1\}} H_{n-j-4}(S_j \times S^{n-j-4}; \mathbb{Z}) & (i = n - 3) 
\end{cases}$$

We add precise presentations on the homology groups which are important in knowing the types of homology groups. We have

$$H_1(W_{f'}; \mathbb{Z}) \cong \begin{cases} 
H_0(S_0 \times S^{n-4}; \mathbb{Z}) \oplus \mathbb{Z}^{g_0} & (g_0 \neq -1) \\
\{0\} & (g_0 = -1) 
\end{cases}$$

$$H_2(W_{f'}; \mathbb{Z}) \cong \begin{cases} 
H_0(S_1 \times S^{n-5}; \mathbb{Z}) \oplus \mathbb{Z}^{g_1} \oplus H_1(S_0 \times S^{n-4}; \mathbb{Z}) & (g_0 > -1, g_1 > -1) \\
H_0(S_1 \times S^{n-5}; \mathbb{Z}) \oplus \mathbb{Z}^{g_1} & (g_0 = -1, g_1 > -1) \\
\mathbb{Z}^{g_1} \oplus H_1(S_0 \times S^{n-4}; \mathbb{Z}) & (g_0 > -1, g_1 = -1) \\
\mathbb{Z}^{g_1} & (g_0 = -1, g_1 = -1) 
\end{cases}$$
HOMOLOGICAL PROPERTIES OF THE REEB SPACES OF CERTAIN FOLD MAPS

\[ H_3(W_f; \mathbb{Z}) \cong \begin{cases} 
H_0(S_2 \times S^{n-6}; \mathbb{Z}) \oplus \mathbb{Z}^{g_2} \oplus H_1(S_1 \times S^{n-3}; \mathbb{Z})(g_1 > -1, g_2 > -1) \\
H_0(S_2 \times S^{n-6}; \mathbb{Z}) \oplus \mathbb{Z}^{g_2}(g_1 = -1, g_2 > -1) \\
H_1(S_1 \times S^{n-3}; \mathbb{Z})(g_1 > -1, g_2 = -1) \\
\{0\}(g_1 = -1, g_2 = -1)
\end{cases}. \]

Let \( H_j = H_{1+(n-j-4)}(S_j \times S^{n-j-4}; \mathbb{Z}) \cong G_j \) for \( g_j > -1 \) and let \( H_j = G_j \cong \{0\} \) for \( g_j = -1 \). We have

\[ H_{n-2} \cong \oplus_{j=0}^{n-7} H_j \cong \oplus_{j=0}^{n-7} G_j \].

\( 2 + (n - j - 4) = n - j - 2 \) and the \((n - j - 2)\)-th homology group of each generating manifold vanishes. These facts give us \( H_{n-1}(W_f; \mathbb{Z}) \cong \{0\} \).

\( H_n(W_f; \mathbb{Z}) \) is finitely generated free group and its rank equals the number of times of normal S-bubbling operations. For \( 4 \leq i \leq n - 6 \), we have the following:

\[ H_i(W_f; \mathbb{Z}) \cong \begin{cases} 
H_0(S_{i-1} \times S^{n-i-3}; \mathbb{Z}) \oplus \mathbb{Z}^{g_{i-1}} \oplus H_2(S_{i-4} \times S^{n-i}; \mathbb{Z}) \oplus G_{i-2} \\
\cong \mathbb{Z}^{g_{i-1}+2} \oplus G_{i-2}(g_{i-1} > -1, g_{i-4} > -1) \\
H_3(S_{i-4} \times S^{n-i}; \mathbb{Z}) \oplus G_{i-2}
\end{cases}. \]

For \( 4 \leq i = n - 5 \), we have the following:

\[ H_i(W_f; \mathbb{Z}) \cong \begin{cases} 
H_3(S_{n-9} \times S^5; \mathbb{Z}) \oplus H_1(S_{n-7} \times S^3; \mathbb{Z}) \cong \\
\mathbb{Z} \oplus G_{n-7}(g_{n-9} > -1, g_{n-7} > -1) \\
H_1(S_{n-7} \times S^3; \mathbb{Z}) \cong G_{n-7}(g_{n-9} = -1, g_{n-7} > -1) \\
H_3(S_{n-9} \times S^5; \mathbb{Z}) \cong \mathbb{Z} \oplus G_{n-7}(g_{n-9} > -1, g_{n-7} = -1) \\
\{0\} \cong G_{n-7}(g_{n-9} = -1, g_{n-7} = -1)
\end{cases}. \]

For \( 4 \leq i = n - 4 \), we have the following:

\[ H_i(W_f; \mathbb{Z}) \cong \begin{cases} 
H_3(S_{n-8} \times S^4; \mathbb{Z}) \cong \mathbb{Z}(g_{n-8} > -1) \\
\{0\}(g_{n-8} = -1)
\end{cases}. \]

Last, we have the following: \( H_{n-3}(W_f; \mathbb{Z}) \) is represented as the direct sum of all the groups \( H_{n-j-4}(S_j \times S^{n-j-4}; \mathbb{Z}) \) for \( j \) such that \( g_j > -1 \).

We can find homology types of Reeb spaces of maps we cannot obtain by the constructions of maps.
Proof. For any homology group \( d \) then \( W \) is not larger than \( n \). Then by the iteration of normal S-bubbling operations starting from \( f \), \( S \) is an infinite group. Furthermore, assume that there exists an integer \( d \) satisfying the following.

1. For an integer \( d \), the group \( H_d(W; \mathbb{Z}) \) is non-trivial and finite.
2. For any non-negative integer \( d \) smaller than \( d \), the group \( H_{d^2+d}(W; \mathbb{Z}) \) is non-trivial and finite.

Then \( d \) is not larger than \( d \).

Example 4. We consider the iteration of normal S-bubbling operations in the proof of Theorem 4. In this case, we can take \( d \) = 1 in the situation of Theorem 5.

Now we extend normal S-bubbling operations to obtain other maps and \( \mathbb{R}^n \) spaces.

Definition 4. In Definition 3, let \( S \) be the bouquet of finite connected and orientable closed submanifolds whose dimensions are smaller than \( n \) of \( P \) and
$N(S)$, $N(S)_i$ and $N(S)_o$ be small regular neighborhoods of $S$ in $P$ such that $N(S)_i \subset N(S) \subset N(S)_o$ holds and that these three are isotopic as regular neighborhoods. By a similar way, we define a similar operation and call the operation an $S$-bubbling operation to $f$. We call $Q_0 := f^{-1}(S) \cap q_f(Q)$, which is homeomorphic to $S$, the generating polyhedron of the $S$-bubbling operation.

We can define a trivial $S$-bubbling operation similarly as in Example 1.

**Proposition 4.** Let $f : M \to \mathbb{R}^n$ be a fold map such that $f|_{S(f)}$ is an embedding and that the inverse images of regular values are disjoint unions of almost-spheres. Let $f' : M \to \mathbb{R}^n$ be a fold map obtained by an $S$-bubbling operation to $f$. Let $S$ be the generating polyhedron of the $S$-bubbling operation. Let $k$ be a positive integer and $S$ be represented as the bouquet of submanifolds $S_j$ where $j$ is an integer satisfying $1 \leq j \leq k$. any integer $0 \leq i < n$, we have

$$H_i(W_{f'}; R) \cong H_i(W_f; R) \oplus \bigoplus_{j=1}^{k}(H_{i-(n-\dim S_j)}(S_j; R))$$
and we also have $H_n(W_{f'}; R) \cong H_n(W_f; R) \oplus R$.

**Proof.** Let $N(S)$ be a small regular neighborhood of $S$ in $W_f$ as in Definition 4. The space $W_{f'}$ is obtained by attaching $q_f'(q_f^{-1}(N(S)))$ to $W_f$ on $N(S) \subset W_f$. We have the following exact sequence.

$$\cdots \to H_i(N(S); R) \to H_i(W_f; R) \oplus H_i(q_f'(q_f^{-1}(N(S))); R) \to H_{i-1}(N(S); R) \to H_{i-1}(W_f; R) \oplus H_{i-1}(q_f'(q_f^{-1}(N(S))); R) \to \cdots$$

$N(S)$ is regarded as a boundary connected sum of the manifolds $N(S_j)$ and each $N(S_j)$ is regarded as the total space of a bundle over $S_j$ whose fiber is an $(n-\dim S_j)$-dimensional standard closed disc.

$q_f'(q_f^{-1}(N(S)))$ is regarded as a connected sum of $n$-dimensional orientable closed and connected manifolds $Q_j$ satisfying the following two where $j$ is an integer satisfying $1 \leq j \leq k$.

- $Q_j$ is regarded as the total space of a linear $S^{n-\dim S_j}$-bundle over $S_j$.
- The bundle $N(S_j)$ over $S_j$ is a subbundle of the bundle $Q_j$ whose fiber is $D^{n-\dim S_j} \subset S^{n-\dim S_j}$.

$H_i(Q_j; R)$ is of the form $\bigoplus_{i'=0}^{i} H_{i'}(N(S_j); R) \otimes H_{i-i'}(S^{n-\dim S_j}; R)$. The homomorphism from $H_i(N(S_j); R)$ into $H_i(Q_j; R)$ or $\bigoplus_{i'=0}^{i} H_{i'}(N(S_j); R) \otimes H_{i-i'}(S^{n-\dim S_j}; R)$ induced from the natural inclusion is injective and of the form $x \mapsto x \otimes 1 \in H_i(N(S_j); R) \otimes H_0(S^{n-\dim S_j}; R)$.

Since $q_f'(q_f^{-1}(N(S)))$ is regarded as a connected sum of manifolds $Q_j$, for any integer $0 < i < n$, $H_i(q_f'(q_f^{-1}(N(S))); R)$ is isomorphic to $\bigoplus_{j=1}^{k} H_i(Q_j; R)$ and $H_0(q_f'(q_f^{-1}(N(S))); R) \cong H_n(q_f'(q_f^{-1}(N(S))); R) \cong R$ holds.

By virtue of these observations, for any integer $0 < i < n$, we have

$$H_i(W_{f'}; R) \cong H_i(W_f; R) \oplus \bigoplus_{j=1}^{k}(H_{i-(n-\dim S_j)}(S_j; R))$$
We also have $H_0(W_f;R) \cong H_0(W_f;R)$ and $H_n(W_f;R) \cong H_n(W_f;R) \oplus R$. Note that the first result holds for $i = 0$ since $H_{-(n-\dim S_j)}(S_j;R)$ is zero for $i = 0$.

**Corollary 4.** In the situation of Proposition 4, if $H_{j_1}(S_{j_2};R)$ is free for any $j_1$ and $j_2$ and $H_i(W_f;R)$ is free, then $H_i(W_f;R)$ is also free.

By applying S-bubbling operation, we have results similar to Theorem 2. However, some statements are a bit different. For example, we have the following.

**Theorem 6.** Let $R$ be a PID. For any integer $0 \leq j \leq n$, we define $G_j$ as a torsion-free finitely generated module over $R$ so that $G_0$ is $R$ and that $G_n$ is not zero. Then, by a finite iteration of normal S-bubbling operations to a map $f$ as in Theorem 2, Theorem 3, Theorem 4 or Corollary 3, we obtain a fold map $f'$ such that the map obtained by the restriction to its singular set is an embedding and $H_j(W_f;R)$ is isomorphic to $G_j$.

**Proof.** Set $g_j = \text{rank} G_j$. We can choose $g_n \geq 1$ disjoint polyhedra in an open disc in the interior $\text{Int} f(M)$ of the image such that the following hold.

1. Each polyhedron is a bouquet of spheres whose dimensions are not smaller than 1 and not larger than $n - 1$ or a point.
2. The number of $k$-dimensional spheres used in all of these bouquets is $g_{n-k}$.

We can apply (trivial) S-bubbling operations such that the generating polyhedra are inverse images of connected components in the previous disjoint polyhedra by the maps from the Reeb spaces to $\mathbb{R}^n$ as in Definition 2 one after another starting from the given map $f$. By Proposition 3, we have a desired map.

Furthermore, if we choose $g_n \geq 1$ disjoint polyhedra in the interior $\text{Int} f(M)$ of the image such that the following two hold and that we can demonstrate a finite iteration of S-bubbling operations similarly, then we also have a desired map.

1. Each polyhedron is a bouquet of manifolds whose dimensions are not smaller than 1 and not larger than $n - 1$ or a point.
2. The number of $k$-dimensional manifolds used in all of these bouquets is $g_{n-k}$ and their homology types with coefficient rings $R$ are same as that of $k$-dimensional sphere.

**Remark 2.** In the situation of Theorem 6, sometimes we can not obtain a fold map $f'$ by any finite iteration of normal S-bubbling operations to the original map $f$.

For example, set $g_n = 1$. If we can obtain the map $f'$, then the time of a normal S-bubbling operation we need must be 1. Poincare duality theorem
restricts the homology types of the generating manifolds of normal S-bubbling operations and this fact and Proposition 3 restrict the homology type of the Reeb space $W_f$.

Last not only (algebraic) topological properties of Reeb spaces of these fold maps, we consider the source manifolds of these fold maps.

A simple fold map is a fold map such that the map $q_f|_{S(f)}: S(f) \subset M \to W_f$ is injective. A fold map such that the map $f|_{S(f)}$ is an embedding is a simple fold map. The following proposition includes propositions in [15] and [5].

**Proposition 5.** Let $m$ and $n$ be integers satisfying $m > n \geq 1$. Let $M$ be a closed and connected orientable manifold of dimension $m$. If $m - n = 1$ holds, then we also assume that $M$ is orientable.

Then, for a simple fold map $f : M \to \mathbb{R}^n$ such that inverse images of regular values are always disjoint unions of almost-spheres and that indices of singular points are always 0 or 1, two induced homomorphisms $q_f^* : \pi_j(M) \to \pi_j(W_f)$, $q_f^* : H_j(M; R) \to H_j(W_f; R)$, and $q_f^* : H^j(W_f; R) \to H^j(M; R)$ are isomorphisms for $0 \leq j \leq m - n - 1$ and for any ring $R$.

Note that isomorphisms between the homology groups or modules are not discussed in these papers. However, it is easy to know that the induced maps give mentioned isomorphisms.

Fold maps obtained in Theorems 1-4 and 6 satisfy the assumption of Proposition 5 (if $M$ is orientable in the $m - n = 1$ case). This means that there are many types of (co)homology groups of manifolds.

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