Simultaneous Estimation of Dimension, States and Measurements: Computation of representative density matrices and POVMs

Cyril Stark

Institute for Theoretical Physics, ETH Zürich, 8093 Zürich, Switzerland

This investigation continues a program aiming at obtaining effective quantum models to describe measurement statistics. In [Stark, arXiv:1209.5737 (2012)], we have described how the Gram matrix \( G \) associated to the prepared states and the performed POVM elements can be estimated via convex relaxation of a rank minimization problem. This Gram matrix \( G \) determines the density matrices and the POVM elements uniquely up to simultaneous rotations (with respect to the Hilbert-Schmidt inner product) in the vector space of Hermitian matrices. However, when the description of the experiment needs to be connected to textbook quantum mechanics, explicit expressions for the states and the POVM elements in terms of positive semidefinite matrices are required. In this paper, we describe a heuristic algorithm that takes the state-measurement Gram matrix \( G \) as input and searches explicit realizations of \( G \) in terms of quantum mechanically valid density matrices and POVM elements.

Assume you have built up an experiment that allows you to prepare a quantum system in different states, and subsequently perform different measurements on this system. The catch is that you do not know exactly what states are actually prepared, and what the measurement devices truly measure. But maybe you are willing to assume that the experiment can be repeated such that one can gather statistics. Then, the question is what can be deduced from this knowledge. We have previously introduced a general program for simultaneous estimation of states and measurements. In [1] we described how the Gram matrix \( G \), associated to the prepared states and the POVM elements can be estimated via convex relaxation of a rank minimization problem. We showed that \( G \) is not always uniquely determined by the acquired measurement data. In [2] we introduced a method to test whether or not specific assumptions on the Gram matrix can guarantee its uniqueness. The Gram matrix \( G \) only determines the density matrices and the POVM elements uniquely up to simultaneous rotations (with respect to the Hilbert-Schmidt inner product) in the vector space of Hermitian matrices. If we compute them via standard factorizations of \( G \), we typically do not get matrices which are positive semidefinite as we normally would expect of proper density operators and POVM elements. In this paper we complete the proof of principle of the simultaneous estimation of dimension, states and measurements, in terms of a heuristic algorithm that delivers a collection of positive and normalized density operators, in terms of a heuristic algorithm that delivers a collection of positive semidefinite matrices.

In Section [I] we introduce the general setting, define the general task more precisely, and clarify its role in the general program outlined in [1] and [2]. In Section [II] we introduce the general program for simultaneous estimation of states and measurements. In [3] we described how the Gram matrix \( G \) associated to the prepared states and the POVM elements can be estimated via convex relaxation of a rank minimization problem. This Gram matrix \( G \) determines the density matrices in terms of the prepared states and the performed POVM elements. We showed that \( G \) is not always uniquely determined by the acquired measurement data. In [4] we introduced a method to test whether or not specific assumptions on the Gram matrix can guarantee its uniqueness. The Gram matrix \( G \) only determines the density matrices and the POVM elements uniquely up to simultaneous rotations (with respect to the Hilbert-Schmidt inner product) in the vector space of Hermitian matrices. If we compute them via standard factorizations of \( G \), we typically do not get matrices which are positive semidefinite as we normally would expect of proper density operators and POVM elements. In this paper we complete the proof of principle of the simultaneous estimation of dimension, states and measurements, in terms of a heuristic algorithm that delivers a collection of positive and normalized density operators. We introduce the general setting, define the general task more precisely, and clarify its role in the general program outlined in [1] and [2].

This paper is organized as follows. In Section [I] we introduce the general setting, define the general task more precisely, and clarify its role in the general program outlined in [1] and [2]. In Section [II] we introduce the general program for simultaneous estimation of states and measurements. In Section [III] we introduce the general program for simultaneous estimation of states and measurements. In this paper, we describe a heuristic algorithm that takes the state-measurement Gram matrix \( G \) as input and searches explicit realizations of \( G \) in terms of quantum mechanically valid density matrices and POVM elements.

I. SETTING

We imagine that we are able to prepare \( W \) different states \( \rho_w \) and to measure \( V \) choices of POVMs \( (E_{vk})_{vk} \) \((k\) enumerates the different outcomes). To keep the notation simple, we assume that the number of outcomes, \( K \), is the same for all measurements. Both the density matrices and the POVMs are unknown a priori even though you might have an educated guess what they might look like. Performing independent repetitions of each of the measurements on each of the states, we can determine frequency distributions \( f_{w,vk} \) for each pairing of a state \( 'w' \) with a measurement \( 'v' \). Here, \( f_{w,vk} \) denotes the frequency for measuring \( 'k' \) in case we have prepared state \( 'w' \) and measured POVM \( 'v' \). In the asymptotic limit (i.e., the number of measurement repetitions goes to infinity), \( f_{w,vk} \) is equal to its associated probability \( p_{w,vk} \). Consequently,

\[
f_{w,vk} = \text{tr}(E_{vk}\rho_w) = \sum_{i,j=1}^{d} (E_{vk})_{ij} (\rho_w)_{ij} = (\vec{E}_{vk})^T \vec{\rho}_w.
\]

Here, we have created the vectors \( \vec{E}_{vk} \) and \( \vec{\rho}_w \) in \( \mathbb{C}^{d^2} \) out of the matrices \( E_{vk}, \rho_w \) by stacking all the matrix-columns on top of each other. When collecting all the
to the task of finding proper states and POVMs. We cannot exclude that its application may require modifications of the suggested parameter values in the algorithm. If the algorithm does not converge quickly enough to a desired realization, the evolution of the computation needs to be analyzed to adjust these parameters. This applies mainly to the parameters dictating the switching from one phase of the algorithm (REGULAR, SELECTION_OF_FASTEST, and PARTIAL; see below) to another phase.

II. THE ALGORITHM

Here we present the an algorithm to solve (1). The main idea is to iteratively solve sequences of convex optimization problems. To this end we first observe that although non-convex, problem (1) has a lot of structure. When the first factor is replaced by a constant matrix, $P^T \mapsto P_0^T$, we arrive at a family of independent least squares problems, since

$$
\|P_0^T P - G\|_2^2 = \sum_j \|P_0^T P(:,j) - G(:,j)\|_2^2. \tag{5}
$$

Here, $P(:,j)$ and $G(:,j)$ refer to the $j$-th columns of $P$ and $G$ respectively. Each summand of the RHS of (5) corresponds to a convex optimization problem, which thus can be solved efficiently and reliably. This observation leads to Algorithm 1, see Listing 2 for the description of the routine ‘REGULAR’. There, $\text{mat}(\cdot)$ refers to the inverse of the transition $\rho \mapsto \bar{\rho}$.

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**Algorithm 1** Draft of SESAM_realizations

**Require:** Gram matrix $G$, Hilbert space dimension $d$, error threshold
1: Initialize $P$ (cf. Eq. (2)) with random density matrices and POVM elements.
2: $\text{error} := \infty$
3: while $\text{error} \geq \text{threshold}$ do
4: for $\text{ind} = 1:\text{size}(P,2)$ do
5: call REGULAR
6: end for
7: $\text{error} = \|P^T P - G\|_2/\|G\|_2$
8: end while
9: Density matrices and POVM elements correspond to columns of $P$; cf. Eq. (2).

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**Listing 2 REGULAR**

Determine $P(:,\text{ind})$ via

$$
\begin{align*}
\text{argmin}_\bar{\rho} & \quad \|P^T \bar{\rho} - G_{:,\text{ind}}\|_2 \\
\text{subject to} & \quad \text{mat}(\bar{\rho}) \text{ positive semidefinite}
\end{align*}
$$

By explicit tests of Algorithm 1 on simple models, one observes that it generally appears to work well on single qubits, or when we remove the constraints that $\text{mat}(P):
must be positive semidefinite for all \( j \). However, for estimations in \((d > 2)\)-dimensional quantum systems, the Algorithm \( \text{PARTIAL} \) appears to easily get stuck away from a global minimum. We suspect that this is connected to the somewhat ‘edgy’ nature of the cone of positive matrices over \( \mathbb{C}^d \) \((d > 2)\).

The observation that the vectors \( \vec{v} \) in subroutine \text{REGULAR} get stuck reflects that the demands imposed on \( \vec{v} \) are too restrictive if \( \text{mat}(\vec{v}) \) is cornered in an unfortunate part of the cone of positive semidefinite matrices. This brings us to the introduction of subroutine \text{PARTIAL} (see Algorithm 3) line 10 and Listing 4.

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**Algorithm 3 SESAM_realizations**

Require: Gram matrix \( G \), Hilbert space dimension \( d \), error threshold

1. Initialize \( P \) (cf. Eq. (2)) with random density matrices and POVM elements.
2. mode := ‘PARTIAL’
3. error := \( \infty \)
4. Use \( G \) to compute euclidean distances between columns of \( P \). Store all relative distances within a matrix \( D_{rel} \):
   \[ D_{rel}(i, j) := \sqrt{G_{ii} - 2G_{ij} + G_{jj}} \]
5. while error \( \geq \) threshold do
6. for ind = 1:size(P,2) do
7. old_vec := P(:,ind)
8. switch mode do
9. case ‘PARTIAL’
10. break
11. end
12. case ‘SELECTION_OF_FASTEST’
13. call SELECTION_OF_FASTEST
14. break
15. case ‘REGULAR’
16. call REGULAR
17. break
18. end
19. end for
20. call initialization_of_switching_parameters
21. call switching_decision
22. end while
23. Density matrices and POVM elements correspond to columns of \( P \); cf. Eq. (2).

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**Listing 4 PARTIAL**

Use \( D_{rel} \) (cf. line 1 in Algorithm 3) to compute the index set \( \text{ind}_{\text{NH}} \) containing the indices of the \( d^2 \) columns of \( P \) which are closest to \( P(:, \text{ind}) \). Then, determine \( P(:, \text{ind}) \) via

\[
\begin{align*}
\arg\min_{\vec{v}} & \quad \| P^T \vec{v} \|_{\text{ind}_{\text{NH}}} - G_{\text{ind}_{\text{NH}}, \text{ind}} \|_2 \\
\text{subject to} & \quad \text{mat}(\vec{v}) \text{ positive semidefinite}
\end{align*}
\]

The purpose of running \text{PARTIAL} is to remove some of the demands specified in terms of \( G(:, j) \) in the objective function in \text{REGULAR}. More precisely, \text{PARTIAL} does not try to align the dynamic \( P \)-column \( \vec{v} \) as well as possible with respect to all states and measurements. Instead it only tries to align \( \vec{v} \) as well as possible with the vectors of its local neighborhood. Thus, \text{PARTIAL} aims at restoring local neighborhoods while \text{REGULAR} tries to directly arrange the whole family of states and measurements. In Listing 4 we chose the minimal choice \( d^2 \) for the size of the local neighborhood. If we selected fewer vectors, the result of each execution of \text{PARTIAL} would not be unique, since the vectors \( P(:, j) \) live in a \( d^2 \)-dimensional space (recall that the matrices \( \text{mat}(P(:, j)) \) are forced to be Hermitian).

To run \text{PARTIAL}, we need to determine the local neighborhood of each column \( P(:, j) \) by selecting the \( d^2 \) nearest neighbors (with respect to Hilbert-Schmidt distance) of \( P(:, j) \). The necessary knowledge about all the pairwise relative distances \( D_{rel}(i, j) \) between columns \( i \) and \( j \) of \( P \) can be computed directly via the Gram matrix \( G \):

\[
D_{rel}(i, j)^2 = \langle P(:, i) - P(:, j), P(:, i) - P(:, j) \rangle - G_{ii} + 2G_{ij} + G_{jj}.
\]

This calculation is performed on line 4 of Algorithm 3. Looking at Algorithm 3 we recognize that—for the moment we ignore the routine \text{SELECTION_OF_FASTEST}—in some iterations of the while-loop we run \text{REGULAR} and in some other iterations we run \text{PARTIAL}. Thus, we need to prescribe which routine to select. This is specified in Listing 4 but we postpone the explanation of the switching criteria for later.

In summary we have arrived at an Algorithm that switches back and forth between \text{REGULAR} and \text{PARTIAL}. \text{PARTIAL} tries to restore local neighborhoods while \text{REGULAR} tries to find the global, complete configuration. \text{REGULAR} succeeds if it starts with an initial condition that is close to a solution. This might be achieved via an educated guess. However, even if \text{PARTIAL} manages to arrange all the vectors such that supposedly near vectors are near, we have not necessarily constructed a good starting point for \text{REGULAR}. This is due to the complicated geometry of the cone of positive semidefinite matrices in case of Hilbert space dimensions \( d > 2 \). The edged shape of the cone of positive semidefinite matrices makes it necessary to rotate the complete collection of states-measurement vectors as a whole so that the edges of the contour of the states-measurement configuration fits into the edges of the cone of positive semidefinite matrices. Since we only allow one vector to move in each iteration (that is the precondition for using convex optimization in each step) this becomes a tiresome business because the dynamic vector is held back by all the static vectors due to the interactions described by \( G(:, j) \). The dynamics of the whole states-measurement configuration thus resembles the movement of a worm: each segment of the worm corresponds to one column \( P(:, j) \). We need the worm to travel from \( A \) to \( B \). Thus, each segment of the worm has to travel from \( A \) to \( B \). In each time step, only one segment of the worm is dynamic and wants to reach \( B \) but its movement is highly constrained by its bonding to the static segments. Although this analogy is admittedly a little far fetched, it neverthe-
less provides an intuition for the dynamics of the bunch of state-measurement vectors. Speeding up the motion of the states-measurements worm is the main purpose of the remaining routine.

The routine $\text{SELECTION\_OF\_FASTEST}$, see Listing 5, takes into account that some worm segments move faster than others, and it operates by ignoring the slowest moving segments. Thus, instead of letting the worm being slowed down by its small segments, we ignore the slow segments. Consequently, during $\text{SELECTION\_OF\_FASTEST}$, the evolution of the columns $P(:, j)$ only tries to align the dynamic vector with respect to the fast moving components of the state-measurement bunch.

### Listing 5 SELECTION\_OF\_FASTEST

Use shifts (cf. line 2 in Listing 6) to determine the index set $\text{ind}_{\text{fast}}$ containing the indices of the $d^2$ columns of $P$ which have moved the fastest in the preceding round. Then, determine $P(:, \text{ind})$ via

$$\arg\min_{\vec{G}} \|P^T \vec{G}_{\text{ind}_{\text{fast}}} - G_{\text{ind}_{\text{fast}}, \text{ind}}\|_2$$

subject to $\text{mat} (\vec{G})$ positive semidefinite

### III. SWITCHING SUBROUTINES

In the previous section we described the main parts of the algorithm. Here we describe the criteria for the switching between these subroutines. This is specified in Listing 5. To detect the right moment to switch from one routine to the other, we require some basic monitoring of the computation. The monitoring in Listing 5 records a history of how fast the individual vectors move (→ shifts), a history of the temporary error $\|P^T P - G\|_2 / ||G||_2$ (→ error/history), and the current trend of the error (→ trend). Moreover, we store the quantities $\text{zenith}$ and $\text{counter}_{\text{no\_zenith}}$. The variable $\text{zenith}$ stores the largest error that has been encountered until the last switching. The quantity $\text{counter}_{\text{no\_zenith}}$ counts for how many iterations the procedure has not updated the value of $\text{zenith}$.

Listing 7 initializes some parameters necessary for switching decision. We have chosen random initializations to break periodicities (we observed that in some situations the complete states-measurement vectors undergoes periodic movements).

We have chosen to switch between the different routines according to the following sequential ordering:

1. $\text{REGULAR} \quad \rightarrow \quad \text{PARTIAL}$
2. $\text{PARTIAL} \quad \downarrow \quad \text{SELECTION\_OF\_FASTEST}$
3. $\text{SELECTION\_OF\_FASTEST} \quad \rightarrow \quad \text{REGULAR}$

1. We decide to leave $\text{REGULAR}$ whenever staying in this phase does not lead to meaningful improvements.

The variable trend is updated in Listing 6. The threshold $\text{switch}_{\text{REG\_TO\_PARTIAL}}$ is randomly updated in Listing 7.

2. This transition is enforced under two circumstances. In the first case, we switch whenever $\|\text{shifts}(:, :)^{-1}, \text{size}(P, 2) + 1, :\|_\infty \leq 0.002$, i.e., we switch whenever PARTIAL gets stuck. The second case occurs when the procedure has not encountered a new zenith in $\text{no\_zenith}_{\text{threshold-many}}$ iterations. This rule is motivated as follows: Recall that we start PARTIAL after REGULAR got stuck in a local minimum (eventually just a plateau). We apply PARTIAL with the intention that it should lift the states-measurement configuration out of the local minimum. To do so, we have to cross an ‘error-barrier’. If the procedure has not reached a new zenith in $\text{no\_zenith}_{\text{threshold-many}}$ iterations, then we take this as an indication that the states-measurement configuration may have crossed a barrier, and we could check whether or not the bottom of the current valley coincides with the global minimum $\|P^T P - G\|_2 = 0$.

3. Before exploring the valley with REGULAR, we first apply $\text{SELECTION\_OF\_FASTEST}$, since we do anticipate the ‘worm-like’ dynamics described in the previous section. This phase is left when it yields no more improvements. Usually, this happens rather quickly because typically, $\text{SELECTION\_OF\_FASTEST}$ leads to chaotic movements soon after its initiation.

### IV. EXAMPLES

The goal of this section is to present our experiences when running the proposed algorithm in different situations. We are considering 2- to 4-dimensional systems with different levels of randomness in the states and the POVM elements. We cover the following three cases:

- **Scenario ‘pure’**. We are sampling pure states and projective measurements uniformly from the Haar measure.

- **Scenario ‘partly mixed’**. First, we uniformly sample $\eta_w$ from $[\eta_{lb}, \eta_{ub}]$ for each state ‘$w$’, and $\mu_{vk}$ from $[\mu_{lb}, \mu_{ub}]$ for each POVM element ‘$(vk)$’. We chose
\begin{verbatim}
Listing 7 initialization of switching parameters
1: Throw a random coin; prob(‘heads’) = 0.7
2: if coin=‘heads’ then
3:   switch_FAST_to_REG = -0.08
4: else
5:   switch_FAST_to_REG = -0.05
6: end if
7: Throw a random coin; prob(‘heads’) = 0.7
8: if coin=‘heads’ then
9:   switch_REG_to_PARTIAL = -0.01
10: else
11:   switch_REG_to_PARTIAL = -0.02
12: end if
13: Throw a random coin; prob(‘heads’) = 0.7
14: if coin=‘heads’ then
15:   no_zenith_threshold = 3 * size(P, 2)
16: else
17:   no_zenith_threshold = 7 * size(P, 2)
18: end if
\end{verbatim}

\begin{verbatim}
Listing 8 switching decision
1: switch mode do
2:   case ‘PARTIAL’
3:     if ||shifts(end-2*size(P, 2)+1:end)||_\infty \leq 0.002 then
4:       mode = ‘SELECTION_OF_FASTEST’
5:       zenith = -1
6:       break
7:     end if
8:     if counter_for_no_zenith > no_zenith_threshold then
9:       mode = ‘SELECTION_OF_FASTEST’
10:      zenith = -1
11:     break
12:   end if
13:   case ‘SELECTION_OF_FASTEST’
14:     if trend \geq switch_FAST_to_REG then
15:       mode = ‘REGULAR’
16:       zenith = -1
17:       break
18:     end if
19:   case ‘REGULAR’
20:     if trend \geq switch_REG_to_PARTIAL then
21:       mode = ‘PARTIAL’
22:       zenith = -1
23:       break
24:     end if
\end{verbatim}

\( \eta_{hh} = \mu_{hh} = 0.6 \) and \( \eta_{ab} = \mu_{ab} = 0.8 \). To generate the states, we sample separately for each state a unitary \( U_w \) from the Haar measure, and define

\[ \rho_w := U_w \text{diag}(\eta_{w, 1} - \eta_{w, 0}, 0, 0, 0) U_w^*. \]

To generate the POVM elements, we set \( P_{vk} \), such that

\[ (P_{vk})_{ij} := \mu_{vk} \delta_{ij} \delta_{ik}. \]

Then, for \( k = 1, \ldots, (K - 1) \),

\[ E_{vk} := U_{vk} P_{vk} U_{vk}^*. \]

and

\[ E_{vK} := \mathbb{I} - \sum_{k=1}^{K-1} E_{vk}. \]

Here, all the unitary matrices are again sampled according to the Haar measure.

- **Scenario ‘purified’**. All POVM effects on \( \mathcal{H} \) associated to the Kn possible outcomes are of the form \( \{ |k|\mathcal{U}|a\rangle \}_{k=1}^{K} \) where \( \mathcal{U} \) is a unitary on \( \mathcal{H} \otimes \mathcal{H}_{\text{anc}} \), \( \mathcal{H}_{\text{anc}} \) a \( K \)-dimensional ancilla system, \( \{ |k\rangle \}_{k=1}^{K} \) is an orthonormal basis in \( \mathcal{H}_{\text{anc}} \), and \( |a\rangle \) is an arbitrary state in \( \mathcal{H}_{\text{anc}} \). Moreover, for all choices of \( \mathcal{U} \), the definitions

\[ M_k := \langle k|\mathcal{U}|a\rangle \]

yield valid effects and therefore, \( E_k := M_k^* M_k \) defines a POVM. In the present scenario ‘purified’, we construct the POVM ‘\( \mathcal{U} \)’ by sampling \( U_w \) on \( \mathcal{H} \otimes \mathcal{H}_{\text{anc}} \) with respect to the Haar measure, by defining \( M_{vk} := \langle k\mathcal{U}|1\rangle \), and by setting \( E_{vk} := M_{vk}^* M_{vk} \). To sample the states, we set

\[ \rho_{w}^{(0)} := \text{diag}(p_{u1}, \ldots, p_{ud}) \]

with \( p_{wn} \) sampled uniformly from \([0, 1] \). Then,

\[ \rho_w := \frac{U_w \rho_{w}^{(0)} U_w^*}{\text{tr}(\rho_{w}^{(0)})} \]

with \( U_w \) sampled from the Haar measure.

We have run each of the scenarios ‘pure’, ‘partly mixed’, and ‘purified’ 100 times for 2-, 3-, and 4-dimensional systems. The stopping criteria was

\[ \max_{ij} \{ \text{tr}(\rho_i E_j) - G_{ij} \} \leq 10^{-2}, \forall i, j. \quad (7) \]

Each run converged successfully. However, we observe that some cases required considerably more iterations (i.e., calls of the subroutines PARTIAL, SELECTION_OF_FASTEST, and REGULAR) than others. This is visible in Fig. [1] to Fig. [2]. Table [1] summarizes our findings. Calculations have been performed using CVX [12, 13] calling SeDuMi [14].

\section{Conclusion}

We have introduced a heuristic algorithm to generate proper density operators and POVMs from the knowledge of the Gram matrix of a collection of a priori unknown states and measurements. In other words, based only on the knowledge of the (Hilbert-Schmidt) inner products between states and POVM elements, between states and states, and between POVM elements and POVM elements, this technique delivers proper density
TABLE I: Numerical findings

| Scenario  | d | W | V | K | successes | failures |
|-----------|---|---|---|---|-----------|----------|
| 'pure'    | 2 | 8 | 4 | 2 | 100       | 0        |
| 'partly mixed' | 2 | 8 | 4 | 2 | 100       | 0        |
| 'purified' | 3 | 27| 9 | 3 | 100       | 0        |
| 'pure'    | 3 | 27| 9 | 3 | 100       | 0        |
| 'partly mixed' | 3 | 27| 9 | 3 | 100       | 0        |
| 'purified' | 4 | 56| 9 | 4 | 100       | 0        |
| 'pure'    | 4 | 56| 9 | 4 | 100       | 0        |
| 'partly mixed' | 4 | 56| 9 | 4 | 100       | 0        |
| 'purified' | 4 | 56| 9 | 4 | 100       | 0        |

operators and POVM elements, i.e., operators that are positive semidefinite and satisfy the standard normalization conditions. We do not know to what extent this computation of operators in the positive cone, consistent with the acquired data, is a computationally hard problem.

This paper concludes a proof of principle that it is possible to simultaneously estimate the dimension, states, and measurements in the asymptotic regime. This program was introduced in [1], where we showed how to estimate the Gram matrix $G$ associated to the involved states and POVM elements. In [2] we focused on the freedom in choosing $G$, and introduced a method to test whether or not specific assumptions on the Gram matrix can guarantee its uniqueness. By the extraction of states and POVMs presented here, we have thus completed the task of simultaneous estimation of dimension, states and POVMs.

However, several questions remain concerning all the steps of the estimation procedure, ranging from scenarios involving non-asymptotic sampling (leading to statistical fluctuations in the Gram matrix $G$) to efficient, stable, and practically applicable numerical procedures.

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FIG. 1: Scenario ‘pure’, \((d, W, V, K) = (2, 8, 4, 2)\).

FIG. 2: Scenario ‘partly mixed’, \((d, W, V, K) = (2, 8, 4, 2)\).

FIG. 3: Scenario ‘purified’, \((d, W, V, K) = (2, 8, 4, 2)\).
FIG. 4: Scenario ‘pure’, \((d, W, V, K) = (3, 27, 9, 3)\).

FIG. 5: Scenario ‘partly mixed’, \((d, W, V, K) = (3, 27, 9, 3)\).

FIG. 6: Scenario ‘purified’, \((d, W, V, K) = (3, 27, 9, 3)\).
FIG. 7: Scenario ‘pure’, \((d, W, V, K) = (4, 56, 9, 4)\).

FIG. 8: Scenario ‘partly mixed’, \((d, W, V, K) = (4, 56, 9, 4)\).

FIG. 9: Scenario ‘purified’, \((d, W, V, K) = (4, 56, 9, 4)\).