CORRECTIONS TO THE PAGELS-STOKAR FORMULA

FOR $f_\pi$ *

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Abstract

Within the composite operator formalism we derive a formula for the pion decay constant $f_\pi$, as defined directly from the residue at the pion pole of the meson propagator, rather than from the matrix element of the axial current. The calculation is performed under some simplifying assumptions, and we verify the complete consistency with soft-pion results, in particular with the Adler-Dashen relation. The formula one obtains for (the pole-defined) $f_\pi^2$ differs from the previous Pagels-Stokar expression by an additive term, and it still provides $f_\pi^2$ in terms of the quark self-energy. We make some numerical estimates leading to $(30 \div 40)\%$ deviation for $f_\pi^2$ with respect to the Pagels-Stokar formula.
1. Introduction

In quantum chromodynamics the pion, as well-known, is a pseudo Nambu-Goldstone boson associated with the spontaneous breaking of chiral symmetry. Its decay constant $f_\pi$ plays a key dynamical role in the chiral symmetry breaking mechanism of QCD, and analogous quantities appear in other theories which use similar paradigms, such as electroweak symmetry breaking through a new strong sector. In 1979 Pagels and Stokar proposed an approximate expression for the calculation of $f_\pi$ [1]. Their derivation used a sum rule due to Jackiw and Johnson [2], plus additional assumptions within the so-called dynamical perturbation theory, and allowed for an approximate expression for $f_\pi$, as defined from the matrix element of the axial current, in terms of the quark self-energy. The formula is of great utility in QCD and also in theories derived from the old technicolour concept.

There has been a vast literature on the Pagels-Stokar (PS) formula with the general conclusion that it leads to a sensible result for $f_\pi$, within uncertainties not easily controllable in view of the theoretical approximations present in the derivation [3]. The Pagels-Stokar expression for the pion decay constant is

\[
 f_{PS}^2 = \frac{d(\mathbf{r})}{(2\pi)^2} \int_0^\infty dk^2 \frac{\Sigma_0^2(k^2) - \frac{1}{2} k^2 \Sigma_0(k^2) \frac{d\Sigma_0(k^2)}{dk^2}}{\left[k^2 + \Sigma_0^2(k^2)\right]^2} \tag{1}
\]

where $d(\mathbf{r})$ is the dimension of the quark colour representation ($d(\mathbf{r}) = 3$ in QCD) and $\Sigma_0$ is the dynamical quark self-energy in the chiral limit.

We shall present below a new formula for $f_\pi$, which shares with the PS formula the advantage of only depending on the self-energy $\Sigma_0$, and that we derive within the composite operator formalism developed in ref. [4] and as modified in ref. [5]. Within such schemes one could think of two different calculations of the pion decay constant. One can evaluate the coupling of the pseudo-Goldstone to the axial-vector current. Alternatively, one can directly evaluate the residue at the pion pole of the meson propagator. These procedures correspond to different definitions, but only the second one agrees with the current-algebra and soft-pion results [6] (in particular the Adler-Dashen relation). We shall follow the second procedure, that is calculating the residue at the pion pole. Within the composite operator formalism, for a first approximate understanding, we discuss here the so-called “rigid case”, in which the presumably small logarithmic corrections coming from the renormalization-group analysis are neglected. For the asymptotic behaviour of the self-energy function we choose the one dictated by the operator product expansion (OPE) [3] up to a logarithm which we neglect. The motivation for such approximate study
is mainly that in this way it is possible to derive analytically the complete expression for the effective action at two fermion loops. Furthermore a phenomenological analysis we have previously performed of the pseudoscalar masses [5] in this contest did not show any major inadequacy of such a treatment. Our expression for \( f_\pi^2 \) is

\[
f_\pi^2 = \frac{d(\xi)}{(2\pi)^2} \int_0^\infty dk^2 \left[ k^2 \frac{\Sigma_0^2(k^2)}{[k^2 + \Sigma_0^2(k^2)]^2} \right. \\
+ k^6 \left( \frac{d\Sigma_0(k^2)}{dk^2} \right)^2 - k^4 \Sigma_0^2(k^2) \left( \frac{d\Sigma_0(k^2)}{dk^2} \right)^2 - k^4 \Sigma_0(k^2) \frac{d\Sigma_0(k^2)}{dk^2} \right] \\
\left. \frac{2[k^2 + \Sigma_0^2(k^2)]^2}{2} \right] \\
\text{Eq. (2)}
\]

The first term in Eq. (2) is \( f_{PS}^2 \) of Eq. (1). By writing \( f_\pi^2 = f_{PS}^2 (1 + \delta^2) \), to get an evaluation of \( \delta \) we can go back to two different Ansatz for \( \Sigma_0 \) we had used in the past to study low energy QCD [3, 4]. We find for \( \delta^2 \) values such as 0.35 and 0.37, suggesting that the correction to the Pagels-Stokar expression is presumably large (30 ÷ 40)% and presumably rather insensitive to the modification of the self-energy shape, provided the ultra-violet behaviour in \( k^2 \) is roughly maintained.

2. The effective action

We start from the effective Euclidean action for the composite operator formalism

\[
\Gamma(\Sigma) = -\text{Tr} \ln \left[ S_0^{-1} + \frac{\delta \Gamma_2}{\delta S} \right] + \text{Tr} \left[ \frac{\delta \Gamma_2}{\delta S} S - \Gamma_2(S) \right] + \text{counterterms} \quad \text{(3)}
\]

where \( S_0^{-1} = (i\hat{p} - m) \), \( m \) is the bare quark mass matrix, \( \Gamma_2(S) \) is the sum of all two-particle irreducible vacuum diagrams with fermionic propagator \( S \) and \( \Sigma = -\delta \Gamma_2/\delta S \). Eq. (3) is the modification of the effective action of Cornwall, Jackiw and Tomboulis [4] which was introduced in ref. [4] to account for the correct stability properties of the theory. At two-loop level \( \Gamma_2 = \frac{1}{2} \text{Tr}(S \Delta S) \), where \( \Delta \) is the gauge boson propagator, so that \( \Sigma = -\Delta S \), \( \text{Tr} [\delta \Gamma_2/\delta S] = 2\Gamma_2 \), and one can rewrite Eq. (3) in terms of \( \Sigma \)

\[
\Gamma(\Sigma) = -\text{Tr} \ln \left[ S_0^{-1} - \Sigma \right] + \Gamma_2(\Sigma) + \text{counterterms} \]

\[
= -\text{Tr} \ln \left[ S_0^{-1} - \Sigma \right] + \frac{1}{2} \text{Tr} \left( \Sigma \Sigma^{-1} \Sigma \right) + \text{counterterms} \quad \text{(4)}
\]

Here the variable \( \Sigma \) plays the role of a dynamical variable. At the minimum of the functional action, that is when the Schwinger-Dyson equation is satisfied, \( \Sigma \) is nothing
but the fermion self-energy. A parametrization for $\Sigma$, employed in [5], was

$$\Sigma = (s + i\gamma_5 p)f(k) \equiv \Sigma_s + i\gamma_5 \Sigma_p$$

with a suitable Ansatz for $f(k)$, and with $s$ and $p$ scalar and pseudoscalar constant fields respectively.

3. The effective potential

The effective potential one obtains from Eq. (4) (see ref. [5]) is

$$V = \frac{\Gamma}{\Omega} = -\frac{8\pi^2 d(\mathcal{L})}{3C_2(\mathcal{L})g^2} \int \frac{d^4k}{(2\pi)^4} \text{tr} [\Sigma_s \Box_k \Sigma_s + \Sigma_p \Box_k \Sigma_p] - d(\mathcal{L}) \text{Tr} \ln \left[ ik - (m + \Sigma_s) - i\gamma_5 \Sigma_p \right] + \delta Z \text{tr}(ms)$$

where $C_2(\mathcal{L})$ is the quadratic Casimir of the fermion colour representation (for $SU(3)_c$ $C_2 = 4/3$) and $\Sigma_s = \lambda_\alpha s_\alpha f(k)/\sqrt{2}$, $\Sigma_p = \lambda_\alpha p_\alpha f(k)/\sqrt{2}$, $m = \lambda_\alpha m_\alpha/\sqrt{2}$ ($\alpha = 0, \ldots, 8$, $\lambda_0 = \sqrt{2}/3$, $\lambda_i = $ Gell-Mann matrices, $i = 1, \ldots, 8$). Furthermore $\delta Z$ has a divergent piece to compensate the leading divergence proportional to $\text{tr}(ms)$ in the logarithmic term. For a quark of mass $m$ the effective potential is

$$V(s, p, m) = -d(\mathcal{L})c \int \frac{d^4k}{(2\pi)^4} \left[ \Sigma_s \Box_k \Sigma_s + \Sigma_p \Box_k \Sigma_p \right] - 2d(\mathcal{L}) \int \frac{d^4k}{(2\pi)^4} \text{ln} \left[ k^2 + (m + \Sigma_s)^2 + \Sigma_p^2 \right] + \delta Z ms$$

where we have defined $c = 2\pi^2/g^2$. In ref. [5] after fixing $\delta Z$ so as to cancel the leading divergence proportional to $ms$ in the logarithm, we had imposed the normalization condition

$$\lim_{m \to 0} \frac{1}{m} \left. \frac{\partial V}{\partial \langle \bar{\psi}\psi \rangle} \right|_{\text{extr}} = 1$$

or, with $\langle \bar{\psi}\psi \rangle = (d(\mathcal{L})M^3/2\pi^2) c\bar{s}$

$$\lim_{m \to 0} \frac{1}{m} \left. \frac{\partial V}{\partial s} \right|_{\text{extr}} = \frac{d(\mathcal{L})M^3}{2\pi^2} c$$

where $M$ is a momentum scale for the self energy and $\bar{s}$ is the extremum of the effective potential. The extrema of the effective potential in the massless case $m = 0$ depend only on $c$. Therefore Eq. (9) becomes, in this case, an equation for $c$ and $M$ is left undetermined. This is nothing but the usual dimensional transmutation. The numerical values for $c$ and $s_0$, the minimum of the effective potential in the massless case, are obtained once one has fixed the Ansatz for $f(k)$. 
With $\Sigma_0(k) = s_0 f(k)$ we shall write in general
\[
\delta Z = d(\mathcal{L}) \left[ \frac{M^3}{2\pi^2} + \frac{4}{s_0} \int \frac{d^4 k}{(2\pi)^4} \frac{\Sigma_0 s}{k^2 + \Sigma_0^2} \right]
\]
(10)

The gap equation, from $\frac{dV}{ds} = 0$, is
\[
\frac{d(\mathcal{L})}{s} \left[ -2c \int \frac{d^4 k}{(2\pi)^4} \Sigma_s \Sigma_s - 4 \int \frac{d^4 k}{(2\pi)^4} \frac{(m + \Sigma_s)\Sigma_s}{k^2 + (m + \Sigma_s)^2} \right] + m\delta Z = 0
\]
(11)

where $\Sigma_s = \bar{s}f(k)$ and $\bar{s}$ is the value at the minimum in the presence of the bare mass.

Let us now turn to the effective action. The fields $s$ and $p$ depend in this case on the space coordinates and we shall use the Weyl symmetrization prescription
\[
\Sigma = (s + i\gamma_5 p)f(k) \to \frac{1}{2} [s(x) + i\gamma_5 p(x), f(k)]
\]
(12)

We are interested in oscillations around the minimum of the effective potential, so we introduce
\[
\chi(x) = s(x) - \bar{s}, \quad \pi(x) = p(x) - \bar{p} \equiv p(x)
\]
\[
v(x) = \chi(x) + i\gamma_5 \pi(x)
\]
\[
\bar{S}(k) = i\hat{k} - (m + \Sigma_s(k))
\]
(13)

The $Tr \ln$ term in Eq. (4), which we denote as $\Gamma_{log}$, becomes
\[
\Gamma_{log} = -Tr \ln \left[ i\hat{k} - m - \Sigma \right] = -Tr \ln \left[ \bar{S}^{-1} - \frac{1}{2} [v(x), f(k)] \right]
\]
(14)

As we are interested in the 2-points function, we expand to second order in $v(x)$ and after some calculation we obtain for the Fourier transform of $\Gamma_{log}$ in Eq. (14)
\[
\Gamma_{log} = d(\mathcal{L}) \left\{ -2\Omega \int \frac{d^4 k}{(2\pi)^4} Tr \ln \left[ k^2 + (m + \Sigma_s(k))^2 \right] + \int \frac{d^4 k}{(2\pi)^4} f(k) Tr \left[ \bar{S}(k)\chi(0) \right] 
\right.
\]
\[
\left. + \frac{1}{2} \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} Tr \left[ \bar{S}(k)i\gamma_5 V(k, k + q)\pi(-q)\bar{S}(k + q)i\gamma_5 V(k + q, k)\pi(q) \right] + (\text{pseudoscalar} \leftrightarrow \text{scalar}, i\gamma_5 \leftrightarrow 1) \right\} + \cdots
\]
(15)

where
\[
V(k_1, k_2) = \frac{1}{2} [f(k_1) + f(k_2)]
\]
(16)
For \( \Gamma_2 \) we obtain (working in Landau’s gauge)

\[
\Gamma_2 = -d(r) c \Omega \int \frac{d^4k}{(2\pi)^4} \text{tr} \left[ \Sigma_s \Box_k \Sigma_s \right] \\
-2d(r)c \left( \int \frac{d^4k}{(2\pi)^4} f(k) \Box_k f(k) \right) \text{tr} \left( \bar{s} \chi(0) \right) \\
+ \int \frac{d^4q}{(2\pi)^4} \text{tr} \left\{ -d(r) c \pi(-q) \left( \int \frac{d^4k}{(2\pi)^4} f(k) \Box_k f(k) \right) \pi(q) \right\} \\
+(\text{pseudoscalar} \leftrightarrow \text{scalar})
\]

(17)

For the counterterm one has

\[
\Gamma_{ct} = \text{tr} \left[ ms \Omega \delta Z + \text{tr} \left( m \chi(0) \right) \delta Z \right]
\]

(18)

4. The improved expression for \( f_\pi \)

We note that each term in the effective action consists of a constant, a linear and a quadratic term in the fields. The constant term gives back the original potential Eq. (6) at the minimum. Such a term controls the normalization. The linear term vanishes by virtue of the gap equation, Eq. (11). The quadratic term stands up for the effective action up to the second order in the fields. In space-time coordinates

\[
\Gamma = \int d^4x d^4y \int \frac{d^4q}{(2\pi)^4} e^{-iq(x-y)} \pi_\alpha(x) \cdot \\
\text{tr} \left\{ -d(r) c \frac{\lambda_\alpha}{\sqrt{2}} \int \frac{d^4k}{(2\pi)^4} f(k) \Box_k f(k) \frac{\lambda_\beta}{\sqrt{2}} + \\
+ \frac{1}{2} d(r) \int \frac{d^4k}{(2\pi)^4} \left[ \bar{S}(k) i\gamma_5 \frac{\lambda_\alpha}{\sqrt{2}} V(k, k + q) S(k + q) i\gamma_5 \frac{\lambda_\beta}{\sqrt{2}} V(k + q, k) \right] \right\} \\
\cdot \pi_\beta(y) + (\text{pseudoscalar} \leftrightarrow \text{scalar}, i\gamma_5 \leftrightarrow 1) + \cdots
\]

(19)

From

\[
G^{-1}_{\alpha\beta}(x - y) = \frac{\delta^2 \Gamma}{\delta \pi_\alpha(x) \delta \pi_\beta(y)}
\]

(20)

one finds for the Fourier transform of \( G^{-1}_{\alpha\beta}(x - y) \)

\[
G^{-1}_{\alpha\beta}(q^2) = \text{tr} \left\{ -2d(r) c \frac{\lambda_\alpha}{\sqrt{2}} \int \frac{d^4k}{(2\pi)^4} f(k) \Box_k f(k) \frac{\lambda_\beta}{\sqrt{2}} + \\
+ d(r) \int \frac{d^4k}{(2\pi)^4} \left[ \bar{S}(k) i\gamma_5 \frac{\lambda_\alpha}{\sqrt{2}} V(k, k + q) S(k + q) i\gamma_5 \frac{\lambda_\beta}{\sqrt{2}} V(k + q, k) \right] \right\}
\]

(21)
By using the gap equation to eliminate $c$ we get, for a quark of mass $m$
\[
    G_{a\beta}^{-1}(q^2) = d(x) \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ \bar{S}(k)i\gamma_5 V(k, k + q)S(k + q)i\gamma_5 V(k + q, k) \right] + 4 \frac{d(x)}{s^2} \int \frac{d^4 k}{(2\pi)^4} \left( \frac{m + \Sigma_s \bar{\Sigma}_s}{k^2 + (m + \Sigma_s)^2} - \frac{m}{s} \delta Z \right)
\]

We can eliminate $\delta Z$ by using the normalization condition, see Eq. (10), and the relation $\langle \bar{\psi}\psi \rangle = (d(x)M^3/2\pi^2)cs$ to obtain
\[
    G_{a\beta}^{-1}(q^2) = -\frac{m}{s^2} \langle \bar{\psi}\psi \rangle + d(x) \left\{ \frac{4m}{s^2} \int \frac{d^4 k}{(2\pi)^4} \frac{\Sigma_0(k)}{k^2 + \Sigma_0^2(k)} + \frac{4}{s^2} \int \frac{d^4 k}{(2\pi)^4} \frac{(m + \Sigma_s) \bar{\Sigma}_s}{k^2 + (m + \Sigma_s)^2} + \int \frac{d^4 k}{(2\pi)^4} \text{tr} \left[ \bar{S}(k)i\gamma_5 V(k, k + q)S(k + q)i\gamma_5 V(k + q, k) \right] \right\}
\]

Note that the second and third terms in Eq. (23) regularize each other in the ultraviolet. In the limit of small quark masses, expanding in $q^\mu$, we find
\[
    G_{a\beta}^{-1}(q^2) \equiv \left( \frac{F}{\sqrt{2}s_0} \right)^2 \left( q^2 - \frac{2m}{F^2} \langle \bar{\psi}\psi \rangle_0 \right)
\]

with
\[
    F^2 = \frac{d(x)}{(2\pi)^2} \int_0^\infty dk^2 \left[ k^2 \frac{\Sigma_0^2(k^2) - \frac{1}{2}k^2 \Sigma_0(k^2) \frac{d\Sigma_0(k^2)}{dk^2}}{[k^2 + \Sigma_0^2(k^2)]^2} + k^6 \left( \frac{d\Sigma_0(k^2)}{dk^2} \right)^2 - k^4 \Sigma_0^2(k^2) \left( \frac{d\Sigma_0(k^2)}{dk^2} \right)^2 - k^4 \Sigma_0(k^2) \frac{d\Sigma_0(k^2)}{dk^2} \right]^{\frac{1}{2}}
\]

In Minkowski metrics the propagator (24) has a pole at $q^2 = m_\pi^2 = -2m \langle \bar{\psi}\psi \rangle_0/F^2$, with residue $(\sqrt{2}s_0/F)^2$, where $\langle \bar{\psi}\psi \rangle_0 = (d(x)M^3/2\pi^2)cs_0$. The Adler-Dashen relation (which follows from the symmetries and current algebra) requires the identification $f_\pi^2 = F^2$, so that $q^2 = m_\pi^2 = -2m \langle \bar{\psi}\psi \rangle_0/f_\pi^2$. The rescaling factor $b$ relating the canonical field $\varphi_\pi$ (with unit residue at the pole) to the field $\pi$, $\varphi_\pi = b\pi$, is then $b = -f_\pi/\sqrt{2}s_0$, as indeed follows from current algebra and soft pions theorem (see ref. 5). Comparison of (25) with the Pagels-Stokar formula (1) leads to our new formula (2) of the introduction.

To get a numerical insight into the problem we use the dynamical calculations in ref. 4, 7, where overall fits to low energy QCD were made on the basis of two alternative Ansatz for $\Sigma(k) = sf(k)$: a smooth Ansatz $f(k) = M^3/(M^2 + k^2)$ for which
the relevant parameters took values $c = 0.554$, $s_0 = -4.06$, and a step-function Ansatz $f(k) = M[\theta(M^2 - k^2) + (M^2/k^2)\theta(k^2 - M^2)]$ for which one found $c = 0.32$, $s_0 = -2.69$. Our new expression for $f_\pi$ has $f_\pi^2 = f^2_{PS}(1 + \delta^2)$, where $\delta$ follows from Eq. (25). We find $\delta^2 = 0.347$ in the case of the smooth Ansatz, and $\delta^2 = 0.376$ for the step-function Ansatz.

We want to remark that in the massless limit we consider, the correction $\delta$ depends only on $s_0$ (or $c$) and the shape of $f(k)$, but not on $M$. In particular, because of the fact that the relevant contribution to the chiral symmetry breaking phenomenon comes from relatively short-distance effects, the corrections will depend mainly on the ultraviolet behaviour of the self-energy. That is, the correction does not depend on the fit we have made to low energy QCD.

It therefore seems that: (i) the corrections are relevant with respect to the old Pagels-Stokar formula; (ii) the corrections do not seem to vary in a sensible way when varying the Ansatz for the self energy, at least within the Ansatz we have used.

It is obvious that the next step one should take is to see whether the neglected corrections, which we know must be there, can modify the results. However, due to previous experience from the study of the pseudoscalar masses \[3\], we would not expect substantial changes in the overall picture of dynamical symmetry breaking.

Finally we may stress that the new formula (2) we have obtained for $f_\pi$, within the approximations made, does not require additional inputs beyond those already present in the Pagels-Stokar formula.

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