STABLE SPHERICAL VARIETIES AND THEIR MODULI

VALERY ALEXEEV AND MICHEL BRION

Abstract. We introduce a notion of stable spherical variety which includes the spherical varieties under a reductive group $G$ and their flat equivariant degenerations. Given any projective space $\mathbb{P}$ where $G$ acts linearly, we construct a moduli space for stable spherical varieties over $\mathbb{P}$, that is, pairs $(X, f)$, where $X$ is a stable spherical variety and $f : X \rightarrow \mathbb{P}$ is a finite equivariant morphism. This space is projective, and its irreducible components are rational. It generalizes the moduli space of pairs $(\mathbb{P}, D)$, where $X$ is a stable toric variety and $D$ is an effective ample Cartier divisor on $X$ which contains no orbit. The equivariant automorphism group of $\mathbb{P}$ acts on our moduli space; the spherical varieties over $\mathbb{P}$ and their stable limits form only finitely many orbits. A variant of this moduli space gives another view to the compactifications of quotients of thin Schubert cells constructed by Kapranov and Lafforgue.

Contents

0. Introduction 2
1. Varieties with reductive group action 4
1.1. Preliminaries 4
1.2. Polarized varieties 6
1.3. Spherical varieties 9
1.4. Examples 12
2. Stable spherical varieties 14
2.1. Affine SSV’s 14
2.2. Polarized SSV’s 19
2.3. Structure 20
2.4. Examples 23
3. Families 26
3.1. Families of affine SSV’s 26
3.2. Families of polarized SSV’s 30
3.3. Families of stable spherical pairs 32
4. Moduli 35
4.1. Existence of a quasiprojective moduli scheme 35
4.2. Projectivity 38
4.3. Group actions 40
4.4. Examples 43
5. Generalizations 44

Date: May 31, 2005.
0. Introduction

The starting point of this work is the construction and study of a moduli space for stable toric pairs in [Al02], and its generalization to stable reductive pairs in [AB04a, AB04b]. Both spaces, as well as the space of semiabelic pairs of [Al02], parametrize certain pairs \((X,D)\), where \(X\) is a projective variety (possibly reducible, but not too singular), and \(D\) is an effective ample Cartier divisor on \(X\). In addition, an algebraic group \(G\) acts on \(X\) with finitely many orbits, none of them being contained in \(D\); in the two first cases, the invertible sheaf \(\mathcal{O}_X(D)\) is \(G\)-linearized.

In both cases, the first step in the construction of the moduli space is to classify the “stable \(G\)-varieties” \(X\) and their orbit structure, in terms of combinatorial invariants. The stable toric varieties turn out to be unions of toric varieties glued along torus-invariant subvarieties, and hence the well-known classification of toric varieties may be applied. In the case of stable reductive varieties (which includes the \(G \times G\)-equivariant compactifications of a reductive group \(G\) and their flat equivariant degenerations), the classification may be reduced to that of stable toric varieties with additional symmetries, by considering the closures of appropriate torus orbits.

Toric varieties and their non-commutative analogs, reductive varieties, are examples of spherical varieties. These may be defined as the normal projective varieties where our reductive group \(G\) acts with finitely many orbits, so that this finiteness is preserved under any equivariant modification. In the present work, we introduce a notion of stable spherical variety and we show the existence of a projective moduli space of stable pairs in this setting.

Actually, we find it more convenient to consider pairs \((X,f)\), where \(X\) is a stable spherical variety and \(f : X \to \mathbb{P} = \mathbb{P}(V)\) is a finite equivariant morphism to the projectivization of an arbitrary \(G\)-module \(V\). We then say that \(X\) is a stable spherical variety over \(\mathbb{P}\). Clearly, for any pair \((X,D)\), the translates \(gD, g \in G\), span a base-point-free linear system; hence they define a pair \((X,f)\). It follows that the moduli spaces of stable pairs \((X,D)\) are, in fact, special cases of those of stable pairs \((X,f)\). For example, in the case when \(G\) is a torus, the space of pairs \((X,D)\) of [Al02] with linearized \(\mathcal{O}_X(D)\) corresponds to a multiplicity-free module \(V\). (We note, however, that [Al02] also deals with the case when the sheaf \(\mathcal{O}_X(D)\) is not linearized.)

For the bulk of the paper we work with a connected reductive group \(G\) over an algebraically closed field \(k\) of characteristic zero. As we will see in Section 5, this assumption can be weakened in several ways: \(k\) need not
be algebraically closed, and when $G$ is a split torus one can work over an arbitrary base scheme, for example Spec $\mathbb{Z}$. Section 5 also contains another generalization: to stable varieties over a closed $G$-invariant subscheme $Z$ of $\mathbb{P}(V)$. In the case when $Z$ is a grassmanian, our moduli space is related to the compactifications of quotients of thin Schubert cells constructed by Kapranov and Lafforgue, as we explain in Example 5.3.4.

Our construction goes along similar lines as in [Al02, AB04a, AB04b], but an essential difference is that the combinatorial classification of spherical varieties is unknown in general (although there are many partial results, and a promising program; this will be discussed in more detail below). So we systematically resort to qualitative arguments.

A key ingredient is the finiteness of spherical varieties over a fixed projective space $\mathbb{P}$, up to equivariant automorphisms of $\mathbb{P}$ (Theorem 1.3.4). This result is deduced from the finiteness of spherical orbit types in a fixed $G$-variety, proved in [AB05, Section 3]; this follows, in turn, from a vanishing theorem of Knop [Kn94] that implies the local rigidity of a class of spherical varieties.

These finiteness results, proved by non-effective geometric arguments, should rather be deduced from the classification (in progress) of spherical varieties. The latter has been established by Luna and Vust for varieties having a prescribed open $G$-orbit, see [LV83] and also [Kn91]. Such spherical embeddings are described in terms of combinatorial objects called colored fans, which also determine the orbit structure; they generalize the fans of toric geometry.

The next aim is to classify the spherical homogeneous spaces. This was achieved by Wassermann in [Wa96] for spaces of rank at most 2, and then by Luna for groups $G$ of type $A$, see [Lu01]. The combinatorial invariants introduced there make sense, in fact, for any reductive group $G$, and they allowed Bravi and Pezzini to extend Luna’s result to groups of type $D$ [BP04]. But these developments do not yet suffice to establish, in full generality, the finiteness results that we need.

Here our main combinatorial invariant is the moment polytope of a polarized spherical variety $(X, L)$, a rational convex polytope which governs the representations of $G$ in the spaces of sections of powers of the ample line bundle $L$. In Corollary 1.3.6 we obtain a fundamental boundedness property:

For any rational convex polytope $Q$, there are only finitely many isomorphism classes of polarized spherical varieties with moment polytope $Q$, and given any bounded set $K$, there are only finitely many moment polytopes contained in $K$.

In the toric case, the moment polytopes are precisely the integral convex polytopes, and their faces correspond to toric subvarieties. But for a non-abelian group $G$, our understanding of moment polytopes is very incomplete: their characterization is an open problem, and their faces correspond only to certain Borel-invariant subvarieties, see [GS05].
Another open problem is the combinatorial classification of spherical varieties over a fixed projective space $\mathbb{P}$. In our moduli space, they form finitely many orbits under the natural action of the equivariant automorphism group of $\mathbb{P}$. The orbit closures may be described in terms of the Luna–Vust classification of spherical embeddings; this will be developed elsewhere. In all examples that we know of, there is a unique open orbit. Equivalently, all spherical varieties over $\mathbb{P}$ with prescribed weight group and moment polytope are degenerations of a unique one.

This is closely related to the Delzant conjecture in symplectic geometry (see [De90] and also [Wo96]), which asserts that any compact multiplicity-free Hamiltonian manifold is uniquely determined by its moment polytope and principal isotropy subgroup. It suggests that the nonsingular spherical varieties over $\mathbb{P}$ are classified by their weight group and moment polytope.

In this work, we have endeavoured to make the exposition self-contained, in particular, independent of the classification of spherical embeddings. Thus, we have provided complete proofs of some statements (Propositions 2.1.8, 3.1.2 and 3.1.4) which generalize the corresponding results in [AB04a, AB04b], but where the original proofs relied on the classification. Examples at the end of Sections 1, 2, and 4 discuss stable toric and reductive varieties in detail and exhibit some new features of stable spherical varieties, which form their natural and definitive generalization.

Acknowledgments 1. The first author would like to thank the École Normale Supérieure for its hospitality; his research was also partially supported by NSF under DMS-0401795.

1. Varieties with reductive group action

1.1. Preliminaries. In this section, we fix the notation and gather preliminary results concerning algebraic groups and varieties. We use [Ha77] as a general reference for algebraic geometry, and [PoVi94], [Gr97] for algebraic transformation groups.

The ground field $k$ is algebraically closed, of characteristic zero. By a variety, we mean a connected separated reduced scheme of finite type over $k$; by a subvariety, we mean a closed subvariety.

A variety $X$ equipped with an algebraic action of a linear algebraic group $G$ is called a $G$-variety; an equivariant morphism between $G$-varieties is called a $G$-morphism.

We will only consider $G$-quasiprojective varieties, i.e., those admitting an ample $G$-linearized invertible sheaf. A quasiprojective $G$-variety $X$ is $G$-quasiprojective whenever $X$ is normal (by [Su74]), or when $G$ is finite.

Throughout this paper, we denote by $G$ a connected reductive algebraic group; we choose a Borel subgroup $B$ of $G$, and a maximal torus $T$ of $B$. Then $B = TU$, where $U$ denotes the unipotent part of $B$. We denote by $W = N_G(T)/T$ the Weyl group of $(G,T)$, and by $B^- = TU^-$ the unique Borel subgroup of $G$ such that $B^- \cap B = T$. The weight group of $G$ is the
character group $X(T)$, identified with the character group of $B$ and denoted by $\Lambda$.

By a $G$-module, we mean a rational, possibly infinite-dimensional $G$-module. Since $G$ is reductive, any $G$-module is semi-simple. Moreover, any simple $G$-module $V$ is finite-dimensional and contains a unique line of $B$-eigenvectors; the corresponding weight $\lambda \in \Lambda$ determines $V$ uniquely. This yields a bijection $\lambda \mapsto V(\lambda)$ from the set $\Lambda^+$ of dominant weights, to the set of isomorphism classes of simple $G$-modules.

We may view $\Lambda$ as a lattice in the real vector space $\Lambda_R := \Lambda \otimes \mathbb{Z} \mathbb{R}$; then $\Lambda^+$ is the intersection of $\Lambda$ with the positive Weyl chamber $\Lambda^+_R := \mathbb{R}_{>0} \Lambda^+$.

Given a $G$-module $V$, the weight set $\Lambda^+(V)$ is the set of dominant weights of simple submodules of $V$; the weight cone $C(V) := \mathbb{R}_{>0} \Lambda^+(V)$ is the subcone of $\Lambda^+_R$ generated by $\Lambda^+(V)$. We say that $V$ is multiplicity-free if it is the direct sum of pairwise non-isomorphic simple $G$-modules. Then $V$ is uniquely determined by its weight set.

Let $X$ be an affine $G$-variety. Then the affine ring $R := H^0(X, \mathcal{O}_X)$ is a $G$-module; this defines the weight set $\Lambda^+(X) := \Lambda^+(R)$ and the weight cone $C(X) := C(R)$. Both are discrete invariants of the $G$-variety $X$, that satisfy the following basic properties (see [Sj98] for the proofs).

**Lemma 1.1.1.**

(i) If $X$ is irreducible, then $\Lambda^+(X)$ is a finitely generated submonoid of $\Lambda^+$. Thus, $C(X)$ is a rational polyhedral convex cone.

(ii) For an arbitrary $X$ and a $G$-subvariety $Y$ holds $\Lambda^+(Y) \subseteq \Lambda^+(X)$, whence $C(Y) \subseteq C(X)$. Moreover, $\Lambda^+(X)$ is the union of the weight monoids $\Lambda^+(Y)$, where $Y$ runs over the irreducible components of $X$.

(iii) Let $X'$ be an affine $G$-variety and $f : X' \to X$ a finite surjective $G$-morphism. Then $\Lambda^+(X') \subseteq \Lambda^+(X') \subseteq \frac{1}{N} \Lambda^+(X)$ for some positive integer $N$. Thus, $C(X') = C(X)$.

The affine $G$-variety $X$ is multiplicity-free, if its affine ring is a multiplicity-free $G$-module. For an irreducible $G$-variety, the multiplicity-freeness is equivalent to the existence of a dense $B$-orbit, and also to the finiteness of the number of $B$-orbits. In particular, every affine multiplicity-free $G$-variety $X$ contains only finitely many $G$-orbits. Also, recall the following result (see, e.g., [AB05] Lemma 1.3] for the easy proof):

**Lemma 1.1.2.** The $G$-automorphism group of any affine multiplicity-free $G$-variety is a diagonalizable linear algebraic group.

Next let $X$ be an irreducible (possibly non-affine) $G$-variety. Then $G$, and hence $B$, acts on the function field $k(X)$. The set of weights of $B$-eigenvectors in $k(X)$ is a subgroup of $\Lambda$ that we denote by $\Lambda(X)$ and call the weight group of $X$; this is a birational invariant of the $G$-variety $X$. The abelian group $\Lambda(X)$ is free of finite rank: the rank of $X$, denoted by $\text{rk}(X)$.

If $X$ is affine, then $\Lambda^+(X) \subseteq \Lambda(X) \cap C(X)$, and $\Lambda^+(X)$ generates the group $\Lambda(X)$. Thus, the rank of $X$ is the dimension of the cone $C(X)$.
For an arbitrary irreducible $G$-variety $X$, we record the following result, a consequence of Lemma 1.1.1 together with the local structure of $G$-varieties [Kn93, Section 2.2].

**Lemma 1.1.3.** (i) Let $Y$ be an irreducible $G$-subvariety of $X$. Then $\Lambda(Y)$ is a subgroup of $\Lambda(X)$. In particular, $\text{rk}(Y) \leq \text{rk}(X)$.

(ii) Let $X'$ be an irreducible $G$-variety and $f : X' \to X$ a finite surjective $G$-morphism. Then $\Lambda(X')$ contains $\Lambda(X)$ as a subgroup of finite index. In particular, $\text{rk}(X') = \text{rk}(X)$.

1.2. **Polarized varieties.** We now adapt classical notions related to polarized varieties (see, e.g., [Vi95]) to our equivariant setting.

**Definition 1.2.1.** A polarized $G$-variety is a pair $(X, L)$, where $X$ is a projective $G$-variety and $L$ is an ample $G$-linearized invertible sheaf on $X$.

A $G$-morphism $\varphi : (X', L') \to (X, L)$ of polarized $G$-varieties is a pair $(f : X' \to X, \gamma : L' \to f^*L)$, where $f$ is a $G$-morphism and $\gamma$ is an isomorphism of $G$-linearized sheaves.

For any such pair $(f, \gamma)$, the morphism $f$ is finite, since $f^*L$ is ample. Also, note that $\gamma$ is uniquely determined by $f$ up to scalar multiplication, since $H^0(X', \mathcal{O}_{X'}) = k$ (as $X'$ is reduced and connected).

Thus, the $G$-automorphisms of $(X, L)$ are exactly the pairs $(f, z f^*)$, where $f$ is a $G$-automorphism of $X$ that fixes the isomorphism class $[L]$ of the $G$-linearized invertible sheaf $L$, and $z$ is a non-zero scalar. This yields an isomorphism

$$\text{Aut}^G(X, L) \simeq \mathbb{G}_m \times \{f \in \text{Aut}^G(X) \mid f^*[L] = [L]\}.$$  

Moreover, the $G$-isomorphism classes of polarized varieties are the equivalence classes of pairs $(X, [L])$ under isomorphisms of $G$-varieties.

To any polarized $G$-variety $(X, L)$ is associated its section ring

$$R(X, L) := \bigoplus_{n=0}^{\infty} H^0(X, L^n),$$

where $L^n$ denotes the $n$-th tensor power of $L$. This is a finitely generated graded algebra on which $G$ acts by automorphisms, via its natural action on every $H^0(X, L^n)$. In other words, $R(X, L)$ carries an action of the group

$$\tilde{G} := \mathbb{G}_m \times G,$$

where the multiplicative group $\mathbb{G}_m$ acts with weight $n$ on $H^0(X, L^n)$. Note that $\tilde{G}$ is a connected reductive group with Borel subgroup $\tilde{B} := \mathbb{G}_m \times B$, maximal torus $\tilde{T} := \mathbb{G}_m \times T$, weight group $\tilde{\Lambda} := \mathbb{Z} \times \Lambda$, and set of dominant weights $\tilde{\Lambda}^+ := \mathbb{Z} \times \Lambda^+$. Moreover,

$$\tilde{X} := \text{Spec } R(X, L)$$

is an affine $\tilde{G}$-variety, the affine cone over $X$. This variety has a special closed point 0, associated with the maximal homogeneous ideal of $R(X, L)$.
and fixed by $\tilde{G}$. As is well-known (and follows from [Ra70, Proposition I.2.3]), the natural map
\[
\pi : \tilde{X} \setminus \{0\} \to X = \text{Proj} \ R(X, L)
\]
is a principal $\mathbb{G}_m$-bundle, and there are isomorphisms of $G$-linearized sheaves $\mathcal{O}_X(n) \simeq L^n$ for all $n \in \mathbb{Z}$.

Also, note that $X$ is normal (resp. semi-normal) if and only if $\tilde{X}$ is (see [AB04b, Lemma 2.1] for the semi-normality).

Given another polarized $G$-variety $(X', L')$ with affine cone $\tilde{X}'$, one easily sees that the $G$-morphisms $(X', L') \to (X, L)$ are in bijective correspondence with the finite $\tilde{G}$-morphisms $\tilde{X}' \to \tilde{X}$. In particular,
\[
(1.2.1) \quad \text{Aut}_G(X, L) \simeq \text{Aut}_G(\tilde{X}).
\]
On the other hand, any $G$-subvariety $Y$ of $X$ yields a polarized $G$-variety $(Y, L) := (Y, L|_Y)$, and hence a finite morphism $\tilde{Y} \to \tilde{X}$ which is injective, but need not be a closed immersion.

Next we define the weight set $\tilde{\Lambda}^+(X, L) := \tilde{\Lambda}^+(\tilde{X})$, the weight cone $\tilde{C}(X, L) := C(\tilde{X})$, and the moment set
\[
Q(X, L) := \tilde{C}(X, L) \cap (\{1\} \times \tilde{\Lambda}_+^R) \subset \tilde{\Lambda}_R = \mathbb{R} \times \Lambda_R.
\]
Then $\tilde{C}(X, L) = \text{Cone} Q(X, L)$ (the cone over the moment set). Moreover, $Q(X, L)$ is a finite union of rational convex polytopes associated with the irreducible components of $X$ (as follows from Lemma 1.1.1 see [Sj99] for details). We identify $Q(X, L)$ with its projection to $\tilde{\Lambda}_R^+$.

If $X$ is irreducible, then $Q(X, L)$ is a rational convex polytope in $\tilde{\Lambda}_R^+$, the moment polytope. Moreover, the first projection $\tilde{\Lambda} \to \mathbb{Z}$ yields an exact sequence
\[
0 \to \Lambda(X) \to \tilde{\Lambda}(X, L) \to \mathbb{Z} \to 0,
\]
where $\tilde{\Lambda}(X, L) := \tilde{\Lambda}(\tilde{X})$ is the weight group of $(X, L)$. The real vector space $\Lambda(X)_R$ is spanned by the differences of the points of $Q(X, L)$ (for these results, see [Sj99]). In particular, $\text{dim} Q(X, L) = \text{rk}(X)$.

Clearly, the weight set, weight cone, and moment set of the pair $(X, L)$ depend only on its $G$-isomorphism class. Also, note the following consequence of Lemmas 1.1.1(iii) and 1.1.3(ii):

**Lemma 1.2.2.** Let $(f, \gamma) : (X', L') \to (X, L)$ be a morphism of polarized $G$-varieties, where $f$ is surjective. Then $Q(X', L') = Q(X, L)$.

We say that a polarized $G$-variety $(X, L)$ is multiplicity-free, if every $G$-module $H^0(X, L^n)$ is multiplicity-free (as is easy to see, it suffices to check the multiplicity-freeness of $H^0(X, L^n)$ for $n \gg 0$). Equivalently, the affine cone $\tilde{X}$ is a multiplicity-free $\tilde{G}$-variety. Then (1.2.1) and Lemma 1.1.2 yield:

**Lemma 1.2.3.** The $G$-automorphism group of any multiplicity-free polarized $G$-variety is a diagonalizable linear algebraic group.
Returning to general polarized varieties, the simplest examples are, of course, the pairs \((\mathbb{P}(V), \mathcal{O}(1))\), where \(V\) is a finite-dimensional \(G\)-module with projectivization \(\mathbb{P}(V) := \text{Proj} \text{Sym}(V^*)\), and \(\mathcal{O}(1)\) is equipped with its natural \(G\)-linearization. Then \(R(\mathbb{P}(V), \mathcal{O}(1)) = \text{Sym}(V^*)\). We now introduce a stronger notion of polarization by prescribing a \(G\)-morphism to some \((\mathbb{P}(V), \mathcal{O}(1))\).

**Definition 1.2.4.** Let \(V\) be a finite-dimensional \(G\)-module. A **\(G\)-variety over** \(\mathbb{P}(V)\) is a pair \((X, f, \gamma)\), where \(X\) is a projective \(G\)-variety and \(f : X \to \mathbb{P}(V)\) is a **finite \(G\)-morphism**.

A **\(G\)-morphism** from another pair \((X', f')\) to \((X, f)\) is a \(G\)-morphism \(\varphi : X' \to X\) such that \(f' \circ \varphi = f\).

The \(G\)-varieties \((X, f)\) over \(\mathbb{P}(V)\) are in bijective correspondence with the polarized \(G\)-varieties \((X, L)\) equipped with a \(G\)-module map

\[
\gamma : V^* = H^0(\mathbb{P}(V), \mathcal{O}(1)) \to H^0(X, L)
\]

such that the image of \(\gamma\) is base-point-free. Namely, one associates to \((X, f)\) the sheaf \(L := f^*\mathcal{O}(1)\) and the map \(\gamma := f^*\). This also yields a finite \(\tilde{G}\)-morphism

\[
\tilde{f} : \tilde{X} \to \mathbb{A}(V),
\]

where \(\tilde{X}\) is the affine cone over \((X, L)\), and \(\mathbb{A}(V)\) denotes the affine space \(\text{Spec} \text{Sym}(V^*)\). The group \(\tilde{G} = \mathbb{G}_m \times G\) acts on \(\mathbb{A}(V)\) via the scalar action of \(\mathbb{G}_m\) and the given action of \(G\).

Clearly, there are only finitely many \(G\)-morphisms \(\varphi : (X', f') \to (X, f)\) between any two prescribed \(G\)-varieties over \(\mathbb{P}(V)\). Every such morphism defines a (special) \(G\)-morphism \((\varphi, \text{id}) : (X', L') \to (X, L)\) of polarized varieties.

The \(G\)-automorphism group \(\text{Aut}^G \mathbb{P}(V)\) acts on the set of \(G\)-varieties \((X, f)\) over \(\mathbb{P}(V)\), and the isomorphism class \([L]\) of the \(G\)-linearized sheaf \(L = f^*\mathcal{O}(1)\) depends only on the orbit. In the multiplicity-free case, this observation may be refined as follows:

**Lemma 1.2.5.** Given a finite-dimensional \(G\)-module \(V\), there is a bijection between:

- the \(\text{Aut}^G \mathbb{P}(V)\)-orbits of multiplicity-free \(G\)-varieties \((X, f)\) over \(\mathbb{P}(V)\), and
- the triples \((X, [L], F)\), where \((X, L)\) is multiplicity-free and \(F\) is a subset of the weight set of \(V^*\) such that \(L\) is generated by a submodule of global sections with weight set \(F\).

**Proof.** Consider a pair \((X, f)\) and the associated map \(f^* : V^* \to H^0(X, L)\). Since the \(G\)-module \(H^0(X, L)\) is multiplicity-free, the image of \(f^*\) is uniquely determined by its weight set \(F\). Moreover, the triple \((X, [L], F)\) only depends on the \(\text{Aut}^G \mathbb{P}(V)\)-orbit of \((X, f)\).

Conversely, given a triple \((X, [L], F)\), let \(H^0(X, L)_F\) be the unique \(G\)-submodule of \(H^0(X, L)\) with weight set \(F\). Our assumptions on weight
sets imply the existence of a surjective $G$-module map $\gamma : V^* \to H^0(X, L)_F$. Moreover, any two such maps are conjugate under an element of $\text{GL}(V^*)^G \cong \text{GL}(V)^G$. Finally, any isomorphism of $G$-linearized sheaves $L \to L'$ yields a $G$-module isomorphism $H^0(X, L') \to H^0(X, L)$ which maps isomorphically $H^0(X, L')_F$ to $H^0(X, L)_F$. Thus, the corresponding maps $\gamma, \gamma'$ are still conjugate under $\text{GL}(V^*)^G$. □

The structure of $\text{Aut}^G \mathbb{P}(V)$ is easily described: since $G$ is connected, one obtains an exact sequence

$$1 \to \mathbb{G}_m \to \text{GL}(V)^G \to \text{Aut}^G \mathbb{P}(V) \to 1,$$

where $\text{GL}(V)^G$ denotes the group of linear $G$-automorphisms of $\mathbb{A}(V)$, and $\mathbb{G}_m$ acts on $\mathbb{A}(V)$ by scalar multiplication. Further, we have a canonical isomorphism of $G$-modules

$$(1.2.3) \quad V \cong \bigoplus_{\lambda \in F} E(\lambda) \otimes V(\lambda),$$

where $F$ is a finite subset of $\Lambda^+$, and each $E(\lambda) = \text{Hom}^G(V(\lambda), V)$ is a non-zero vector space of finite dimension. Here, of course, $G$ acts on every summand $E(\lambda) \otimes V(\lambda)$ via its action on $V(\lambda)$. Then

$$(1.2.4) \quad \text{GL}(V)^G \cong \prod_{\lambda \in F} \text{GL}(E(\lambda)),$$

where each factor $\text{GL}(E(\lambda))$ acts on $V$ via its natural action on $E(\lambda)$.

1.3. Spherical varieties. Recall that an affine irreducible $G$-variety $X$ is spherical if it is normal and multiplicity-free. Likewise, a polarized variety $(X, L)$ is spherical if $X$ is normal and $(X, L)$ is multiplicity-free; equivalently, the affine cone $\tilde{X}$ is a spherical $\tilde{G}$-variety. We then say for brevity that $(X, L)$ is a PSV.

We now gather some fundamental facts on polarized spherical varieties.

**Proposition 1.3.1.** Let $(X, L)$ be a PSV with weight group $\Gamma$ and moment polytope $Q$, and let $Y$ be an irreducible $G$-subvariety. Then:

(i) $(Y, L)$ is a PSV.

(ii) $\tilde{\Lambda}(Y, L)$ is a direct summand of $\Gamma$. Thus, $\Lambda(Y)$ is a direct summand of $\Lambda(X)$.

(iii) $Q(Y, L)$ is a face of $Q(X, L)$, which determines $Y$ uniquely.

(iv) The restriction map $H^0(X, L^n) \to H^0(Y, L^n)$ is surjective for all $n \geq 0$.

(v) As $G$-modules,

$$H^0(X, L^n) \cong \bigoplus_{\lambda \in \Lambda^+, (m, \lambda) \in \Gamma \cap nQ} V(\lambda).$$
Equivalentlly, as $\tilde{G}$-modules,
\[ R(X, L) \simeq \bigoplus_{\tilde{\lambda} \in \Gamma \cap \text{Cone}(Q)} V(\tilde{\lambda}). \]

(vi) $L$ is globally generated.
(vii) $H^i(X, L^n) = 0$ for all $i \geq 1$, $n \geq 0$.

Proof. The natural map $f : \tilde{Y} \to \tilde{X}$ between affine cones is finite and injective. Thus, $f(\tilde{Y})$ is an irreducible $\tilde{G}$-subvariety of the affine spherical variety $\tilde{X}$. Hence $f(\tilde{Y})$ is normal (e.g., by [ABU11, Lemma 2.2]). It follows that $f$ is an isomorphism; this implies (i) and (iv).

Likewise, (ii), (iii) and (v) follow from their affine analogs proved for example in [loc. cit.].

For (vi), if $L$ is not globally generated, then its base locus contains a closed $G$-orbit $Y$. Since $L$ is ample and $Y$ is a flag variety, then $H^0(Y, L) \neq 0$ which contradicts (iv).

Finally, (vii) is well-known, see, e.g., [ABH13, Corollary 5.8].

Lemma 1.3.2. Let $X$ be a spherical $G$-variety, $Y$ an irreducible $G$-variety, and $f : X \to Y$ a finite surjective $G$-morphism. Then:

(i) $f^{-1}(Z)$ is irreducible for every irreducible $G$-subvariety $Z$ of $Y$. Equivalently, the preimage of every $G$-orbit is a unique $G$-orbit.

(ii) The pair $(X, f)$ is uniquely determined (up to $G$-isomorphism over $Y$) by the data of $Y$ and the weight group $\Lambda(X)$. In particular, if $\Lambda(X) = \Lambda(Y)$, then $f$ is the normalization map.

Proof. (i) is a direct consequence of Proposition 1.3.1 (iii) combined with Lemma 1.2.2.

(ii) First we consider the case where $Y = G/I$ is a unique $G$-orbit. Then we may write $X = G/H$, where $H$ is a subgroup of finite index of $I$. Thus, the connected component $I^0$ satisfies $I^0 \subseteq H \subseteq I$. Now, by [BPS7] Section 5.2, we may choose $B$ and $T$ so that $BI^0$ is open in $G$, and $I = I^0(T \cap I)$. Thus, $H = I^0(T \cap H)$. Moreover, $\Lambda(X) = \chi(T/T \cap H)$ as a subgroup of $\Lambda = \chi(T)$, by [BPS7] Section 2.9). So $\Lambda(X)$ determines uniquely the subgroup $T \cap H$ of $T$ which, in turn, determines uniquely $H$.

In the general case, let $G/I$ be the open orbit in $Y$. Since $f$ is finite, $f^{-1}(G/I) = G/H$, where $H$ is as above. Moreover, $X$ is the normalization of $Y$ in the function field $k(G/H)$.

Our final preliminary result concerns the $G$-automorphism group of a spherical variety $X$. This group acts on the function field $k(X)$ and preserves each subset $k(X)^{(B)}(\lambda)$ of $B$-eigenvectors with prescribed weight $\lambda \in \Lambda(X)$. Since $X$ contains a dense $B$-orbit, such an eigenvector is determined by its weight up to multiplication by a non-zero scalar. This defines a character $\chi_\lambda : \text{Aut}^G(X) \to \mathbb{G}_m$ and, in turn, a homomorphism
\[ \iota = \iota_X : \text{Aut}^G(X) \to \text{Hom}(\Lambda(X), \mathbb{G}_m), \quad \varphi \mapsto (\lambda \mapsto \chi_\lambda(\varphi)), \]
where the target is the torus with character group $\Lambda(X)$. Now [Kn96, Theorems 5.1, 5.5] yields the following

**Lemma 1.3.3.** Let $X$ be a spherical variety. Then every irreducible $G$-subvariety $Y$ is invariant under $\text{Aut}^G(X)$, and the diagram

$$
\begin{array}{ccc}
\text{Aut}^G(X) & \xrightarrow{\iota_X} & \text{Hom}(\Lambda(X), \mathbb{G}_m) \\
\downarrow \text{res}_Y & & \downarrow \text{res}_{\Lambda(Y)} \\
\text{Aut}^G(Y) & \xrightarrow{\iota_Y} & \text{Hom}(\Lambda(Y), \mathbb{G}_m)
\end{array}
$$

commutes. Moreover, $\iota_X$ and $\iota_Y$ are injective.

This lemma is easily checked directly if $X$ is affine, see, e.g. [AB05, Lemma 1.3]. By (1.2.1), it also applies to $\text{Aut}^G(X,L)$, where $(X,L)$ is any polarized spherical variety.

We now come to a key finiteness result.

**Theorem 1.3.4.** Let $V$ be a finite-dimensional $G$-module. Then there are only finitely many orbits of spherical varieties over $\mathbb{P}(V)$, for the action of $\text{Aut}^G \mathbb{P}(V)$.

**Proof.** Consider such a variety $(X, f)$ and the associated finite $\hat{G}$-morphism $\hat{f} : \hat{X} \to V$ of (1.2.2). By [AB05, Corollary 3.3], there are only finitely many possibilities for the image $\hat{f}(\hat{X})$, up to the action of $\text{GL}(V)^G$. So we may fix $\hat{Y} := \hat{f}(\hat{X})$.

By Lemma 1.1.3, $\hat{\Lambda}(\hat{X})$ contains $\hat{\Lambda}(\hat{Y})$ as a subgroup of finite index. Since the torsion subgroup of the quotient $\hat{\Lambda}/\hat{\Lambda}(\hat{Y})$ is finite, there are only finitely many possibilities for the weight group $\hat{\Lambda}(\hat{X})$. Moreover, by Lemma 1.3.2, $\hat{Y}$ and the latter group determine uniquely the pair $(\hat{X}, \hat{f})$, and hence $(X, f)$. $\square$

Together with Lemma 1.2.5 this implies readily the following:

**Corollary 1.3.5.** Let $V$ be a finite-dimensional $G$-module. Then there are only finitely many $G$-isomorphism classes of PSV’s $(X, L)$, where $L$ is generated by a $G$-module of global sections which is a quotient of $V$.

This implies, in turn, another finiteness result:

**Corollary 1.3.6.** Let $K$ be a bounded subset of $\Lambda_{\mathbb{R}}$. Then there are only finitely many $G$-isomorphism classes of PSV’s with moment polytope contained in $K$.

**Proof.** Let $(X, L)$ be a PSV such that $Q(X, L) \subseteq K$. By Proposition 1.3.1, $L$ is generated by a $G$-module of global sections which is a quotient of the $G$-module $\bigoplus_{\lambda \in \Lambda^* \cap K} \mathcal{V}(\lambda)^*$. So the statement follows from Corollary 1.3.5. $\square$

In particular, any prescribed bounded set contains only finitely many moment polytopes of PSV’s. This boundedness property is not obvious, since the polytopes under consideration may have non-integral vertices; see, e.g., Examples 1.4(2), 1.4(3).
1.4. Examples. We now consider three natural subclasses of the class of polarized spherical varieties, beginning with the simplest one:

1) Toric varieties.

Here $G = T$ is a torus with character group $\Lambda$. The spherical varieties for $T$ are just the normal varieties where $T$ acts with a dense orbit, i.e., the toric varieties for a quotient of $T$.

As is well-known, the assignment $X \mapsto (\Lambda(X), C(X))$ yields a bijective correspondence from the affine toric varieties for a quotient of $T$, to the pairs $(\Gamma, C)$, where $\Gamma$ is a subgroup of $\Lambda$, and $C$ is a rational polyhedral convex cone in $\Gamma_\mathbb{R}$ with non-empty interior. Moreover, $\text{Aut}^T(X) = \text{Hom}(\Gamma, \mathbb{G}_m)$.

Likewise, the assignment $(X, L) \mapsto (\Lambda(X, L), Q(X, L))$ yields a bijective correspondence from the polarized toric varieties for a quotient of $T$, to the pairs $(\Gamma, Q)$, where $Q$ is a convex lattice polytope in $\{1\} \times \Lambda_\mathbb{R}$, and the subgroup $\Gamma$ of $\Lambda$ contains the subgroup generated by the vertices of $Q$ as a subgroup of finite index. Moreover, $\text{Aut}^T(X, L) = \text{Hom}(\Gamma, \mathbb{G}_m)$.

Finally, given a finite-dimensional $T$-module $V$, the toric varieties over $\mathbb{P}(V)$ correspond to those pairs $(\Gamma, Q)$ where the vertices of $Q$ are weights of $V^*$, i.e., opposites of weights of $V$. Equivalently, the moment polytopes of toric varieties over $\mathbb{P}(V)$ are exactly the convex hulls of opposites of weights of $V$.

2) Reductive varieties.

Here the acting group is $G \times G$ with Borel subgroup $B^- \times B$, maximal torus $T \times T$, weight group $\Lambda \times \Lambda$, and set of dominant weights $(-\Lambda^+) \times \Lambda^+$. The affine reductive varieties of [AB04a] are the affine spherical $G \times G$-varieties whose weight group is a direct summand of the anti-diagonal $\{(-\lambda, \lambda) \mid \lambda \in \Lambda\} \simeq \Lambda$. These varieties are classified by $W$-admissible cones, i.e., rational polyhedral convex cones $\sigma$ in $\Lambda_\mathbb{R}$ such that

1. the relative interior $\sigma^0$ meets $\Lambda_\mathbb{R}^+$, and
2. the distinct $w\sigma^0$ ($w \in W$) are disjoint.

The weight lattice of the reductive toric variety $X_\sigma$ is the group generated by $\sigma \cap \Lambda$, and its weight cone is $\sigma \cap \Lambda_\mathbb{R}^+$.

Likewise, the polarized reductive varieties of [AB04d] are the PSV’s for $G \times G$ such that their affine cone is a reductive variety. These varieties are classified by $W$-admissible polytopes, i.e., convex lattice polytopes $\delta$ in $\Lambda_\mathbb{R}$ satisfying the above conditions. The moment polytope of the polarized reductive variety $(X_\delta, L_\delta)$ is the intersection of $\delta$ with the positive Weyl chamber.

An example of a $W$-admissible polytope is the convex hull of the Weyl group orbit of a dominant weight $\lambda$. The vertices of the corresponding moment polytope are $\lambda$ and its projections on the faces of the positive chamber. This yields many examples of moment polytopes with non-integral vertices. Specifically, if $G = \text{SL}(n+1)$ with fundamental weights $\omega_1, \omega_2, \ldots, \omega_n$, and $\lambda = \omega_1$, then the vertices are $\omega_1, \frac{1}{2}\omega_2, \ldots, \frac{1}{n}\omega_n$. More complicated examples
exist in which the moment polytope of a spherical variety is not obtained by intersecting a $W$-invariant integral convex polytope with the positive chamber.

Also, note that the data of the weight lattice and weight cone (or moment polytope) do not suffice to distinguish reductive varieties. This happens, e.g., for $G = SL(2)$ and the subvarieties of $\mathbb{P}(M_2 \oplus k)$ defined by the equations $ad - bc = z^2$, resp. $ad - bc = 0$. Here $M_2$ denotes the space of $2 \times 2$ matrices with coefficients $a, b, c, d$, where $G \times G$ acts by left and right multiplication; and $k$ denotes the trivial $G \times G$-module with coordinate $z$.

3) Spherical varieties for $SL(2)$.

Let $G = SL(2)$ with Borel subgroup $B$ of upper triangular matrices, and maximal torus $T$ of diagonal matrices. We identify $T$ to $G_m$ via $t \mapsto \text{diag}(t, t^{-1})$. This identifies $\Lambda$ to $\mathbb{Z}$, and $\Lambda^+$ to the set $\mathbb{N}$ of non-negative integers. Each simple $G$-module $V(n)$ may be realized as the space of homogeneous polynomials of degree $n$ in two variables $x, y$, where $G$ acts by linear change of variables.

The classification of the non-trivial projective spherical $G$-varieties $X$ together with their weight group $\Gamma = \Lambda(X)$ is easy; the results are as follows.

(i) $X$ is the projective line $\mathbb{P}^1 = G/B$. Here $\Gamma$ is trivial.

(ii) $X$ is a rational ruled surface $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$ ($e \geq 1$), where $G$ acts via its natural action on $\mathbb{P}^1$ and on the linearized sheaf $\mathcal{O}_{\mathbb{P}^1}(e)$. Then $X$ contains a dense $G$-orbit with isotropy group $U_e$, the semi-direct product of $U$ with the subgroup of $e$-th roots of unity in $T$. The complement of this orbit consists of two closed orbits $C_+, C_-$, sections of the fibration $\mathbb{F}_e \to \mathbb{P}^1$. The self-intersection of $C_\pm$ is $\pm e$. Here $\Gamma = e\mathbb{Z}$.

(iii) $X$ is a normal projective surface $S_e$ obtained from $\mathbb{F}_e$ by contracting the negative section $C_-$. Then $X$ still contains a dense orbit with isotropy group $U_e$; its complement is the disjoint union of a fixed point and the positive section $C_+$. Again, $\Gamma = e\mathbb{Z}$.

(iv) $X = \mathbb{P}^1 \times \mathbb{P}^1$ where $G$ acts diagonally. The orbits are the diagonal and its complement, a dense orbit isomorphic to $G/T$. Here $\Gamma = 2\mathbb{Z}$.

(v) $X = \mathbb{P}^2$ where $G$ acts by the projectivization of its linear action on the quadratic forms in two variables. The orbits are the conic of degenerate forms and its complement, a dense orbit isomorphic to $G/N_G(T)$. Here $\Gamma = 4\mathbb{Z}$.

Note that $S_1 \simeq \mathbb{P}^2$ as abstract varieties, but not as $G$-varieties (since the orbit structures are different).

We now describe the ample invertible sheaves $L$ on these varieties $X$ and the corresponding moment polytopes $Q = Q(X, L)$ and weight groups $\tilde{\Gamma} = \tilde{\Lambda}(X, L) \subseteq \mathbb{Z} \times \Lambda = \mathbb{Z}^2$ (recall that any invertible sheaf on a normal $G$-variety admits a unique $G$-linearization, since $G$ is semi-simple and simply connected).
(i) $L = \mathcal{O}_p(n)$, where $n \geq 1$. Then $Q$ is just the point $n$. We may realize $X$ as the orbit $G \cdot [x^n]$ in $\mathbb{P}(V(n))$; then $L$ is the restriction of $\mathcal{O}(1)$, and $\tilde{\Gamma} = \mathbb{Z}(1,n)$.

(ii) The degrees of the restriction of $L$ to the sections $C_\pm$ are positive integers $n_\pm$. One checks that $L$ is uniquely determined by these integers, which satisfy $n_- < n_+$ and $n^+ - n^-$ is divisible by $e$; we write $L = \mathcal{O}(n_-, n_+)$. Then the moment polytope is the interval $[n_-, n_+]$. We may realize $X$ as the closure of the orbit $G \cdot [x^n \pm x^{n_+}]$ in $\mathbb{P}(V(n_-) \oplus V(n_+))$; then $L$ is the restriction of $\mathcal{O}(1)$. The group $\tilde{\Gamma}$ is generated by $(1, n_+), (0, e)$.

(iii) Likewise, $L$ is uniquely determined by the degree $n$ of its restriction to $C_+$, a positive integer; we write $L = \mathcal{O}(n)$. Then $Q = [0, n]$. Moreover, $X$ is the closure of the orbit $G \cdot [1 \pm x^n]$ in $\mathbb{P}(V(0) \oplus V(n))$, and $L$ is the restriction of $\mathcal{O}(1)$. The group $\tilde{\Gamma}$ is generated by $(1, n)$ and $(0, e)$.

(iv) $L = \mathcal{O}_p(m, n)$ with obvious notation, where $m, n \geq 1$. Then $Q = [m - n, m + n]$. We may realize $X$ as the closure of the orbit $G \cdot [x^m \otimes y^n]$ in $\mathbb{P}(V(m) \otimes V(n))$; then $L$ is the restriction of $\mathcal{O}(1)$. The group $\tilde{\Gamma}$ is generated by $(1, m + n)$ and $(0, 2)$.

(v) $L = \mathcal{O}_p^2(n)$, where $n \geq 1$. Then $Q = [0, 2n]$. We may realize $X$ as the closure of the orbit $G \cdot [x^n y^n]$ in $\mathbb{P}(V(2n))$; then $L$ is the restriction of $\mathcal{O}(1)$. The group $\tilde{\Gamma}$ is generated by $(1, 2n)$ and $(0, 4)$.

Note that $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(m, n))$ may be realized as a spherical variety over $\mathbb{P}(V(m + n))$, via the multiplication map $V(m) \otimes V(n) \to V(m + n)$. The corresponding morphism $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}(V(m + n))$ is injective with non-normal image if $m \neq n$, resp. of degree 2 with smooth image if $m = n$.

In case (iv), both vertices of the moment polytope lie in $\Gamma$, but the vertex $|m - n|$ does not arise from any $G$-subvariety. On the other hand, in case (v) the vertex 0 is not in $\tilde{\Gamma}$ for odd $n$.

Also, note that the PSV’s $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$ and $(S_2, \mathcal{O}(2))$ have the same weight lattice and moment polytope, but are non-isomorphic; similarly for $(\mathbb{P}^2, \mathcal{O}(2))$ and $(S_4, \mathcal{O}(4))$. In both cases, the nonsingular variety can be $G$-equivariantly degenerated to the singular one.

Finally, given a finite-dimensional $G$-module $V$ with weight set $F \subset \mathbb{N}$, the moment polytopes of spherical varieties over $\mathbb{P}(V)$ are the intervals with endpoints in $F$ and also the intervals $[n - 2p, n]$, where $n \in F, p \in \mathbb{N}$, and $2p \leq n$.

2. Stable spherical varieties

2.1. Affine SSV’s. In this section, we begin by formulating a general definition of stable spherical varieties. Then we obtain several characterizations and properties of these varieties, in the affine case.

We fix a subgroup $\Gamma$ of the weight group $\Lambda$.

**Definition 2.1.1.** A *stable spherical variety with weights in $\Gamma$* is a $G$-variety $X$ satisfying the following conditions:

...
We then say for brevity that $X$ is a SSV.

**Remark 2.1.2.** This definition should be compared with that of a stable toric variety under a torus $T$ (in the sense of [Al02, Definition 1.1.5]), i.e., a semi-normal variety where $T$ acts with finitely many orbits and connected isotropy groups. Our conditions (i) and (ii) are direct analogs of their toric versions, while (iii) is a more hidden analog of the connectedness of isotropy groups. This will be developed in Lemma 2.1.5.

By Proposition 1.3.1, any spherical variety is a SSV. Conversely, any SSV is obtained by gluing spherical $G$-varieties along $G$-subvarieties:

**Proposition 2.1.3.** Let $X$ be a SSV and denote by $Q(X)$ the finite set of $G$-orbit closures in $X$, partially ordered by inclusion. Then every such orbit closure $Y$ is a spherical variety. Moreover, the natural map

$$f : \lim_{\to} Y \to X$$

is an isomorphism.

**Proof.** Let $\nu : Y' \to Y$ be the normalization, $O$ a $G$-orbit in $Y$, and $Z := \overline{O}$ its closure. Put $Z' := \nu^{-1}(Z)$, then $Z'$ is irreducible by Lemma 1.3.2. Further, $\Lambda(Z) \subseteq \Lambda(Z')$ as a subgroup of finite index, by Lemma 1.1.1 (ii). On the other hand, $\Lambda(Z)$ is a direct summand of $\Gamma$, and Lemma 1.1.1 (i) yields

$$\Lambda(Z') \subseteq \Lambda(Y') = \Lambda(Y) \subseteq \Gamma.$$  

It follows that $\Lambda(Z') = \Lambda(Z)$. By Lemma 1.3.2 again, this implies that $\nu$ restricts to an isomorphism $\nu^{-1}(O) \to O$. Since this holds for all orbits, $\nu$ is bijective. Moreover, since $Z'$ is normal by Proposition 1.3.1, the restriction $\nu|_{Z'}$ is the normalization. This yields a map

$$f' : \lim_{\to} Y' \to X,$$

which is clearly finite and bijective. (Since the morphisms $Z' \to Y'$ are closed immersions, the limits in the categories of schemes and of affine schemes coincide). As $X$ is semi-normal, $f'$ is an isomorphism; this implies both statements. \qed

**Remark 2.1.4.** One shows by similar arguments that the $G$-varieties obtained as direct limits of a finite poset of spherical varieties (with arrows being closed immersions) are characterized by the conditions (i) and (ii) of Definition 2.1.3 together with

(iii)' For any $G$-orbit closures $Z \subseteq Y$, the weight group $\Lambda(Z)$ is a direct summand of $\Lambda(Y)$. 

However, there exist $G$-varieties which satisfy (i), (ii), (iii)', but such that (iii) fails for any subgroup $\Gamma$ of $\Lambda$ (for example, take $G = \mathbb{G}_m$ and let $X$ be the union of two affine lines where $G$ acts linearly with weights 2, 3).

Until the end of this section, we will only consider affine SSV’s. We begin by relating them to stable toric varieties. For this, recall that each affine $G$-variety $X = \text{Spec } R$ defines an affine $T$-variety

$$X//U := \text{Spec } R^U,$$

where $R^U$ denotes the ($T$-stable) subalgebra of $U$-invariants in $R$. Clearly, $X$ and $X//U$ have the same weight set. In fact, many properties of $X$ can be read off $X//U$. This is developed in [Po87, §6] and [Gr97, Chapter 3]; we will freely use the results there.

**Lemma 2.1.5.** The following conditions are equivalent for an affine $G$-variety $X$:

(i) $X$ is a SSV with weights in $\Gamma$.

(ii) The action of $T = \text{Hom}(\Lambda, \mathbb{G}_m)$ on $X//U$ factors through an action of the quotient torus $T_\Gamma := \text{Hom}(\Gamma, \mathbb{G}_m)$ for which $X//U$ is a stable toric variety.

*Proof.* (i) $\Rightarrow$ (ii) By [AB04a, Lemma 2.1], $X//U$ is a semi-normal $T$-variety. Its irreducible components are the $Y//U$, where $Y$ runs over the irreducible components of $X$. Since $Y$ is normal by Proposition 2.1.3 each $Y//U$ is an affine toric variety. And since the weight group $\Lambda(Y)$ is a direct summand of $\Gamma$, the $T$-action on $Y//U$ factors through a $T_\Gamma$-action having a dense orbit with connected isotropy group: the subtorus of $T_\Gamma$ with character group $\Gamma/\Lambda(Y)$. It follows that any $T_\Gamma$-isotropy group in $Y//U$ is connected. This means that $X//U$ is a $T_\Gamma$-stable toric variety.

(ii) $\Rightarrow$ (i) By [AB04a, Lemma 2.1] again, $X$ is semi-normal. Moreover, by [Al02, Section 2.3], $Y//U$ is an affine $T_\Gamma$-toric variety for any irreducible component $Y$ of $X$. Since $Y$ is normal by Proposition 2.1.3 each $Y//U$ is an affine toric variety. And since the weight group $\Lambda(Y)$ is a direct summand of $\Gamma$, the $T$-action on $Y//U$ factors through a $T_\Gamma$-action having a dense orbit with connected isotropy group: the subtorus of $T_\Gamma$ with character group $\Gamma/\Lambda(Y)$. It follows that any $T_\Gamma$-isotropy group in $Y//U$ is connected. This means that $X//U$ is a $T_\Gamma$-stable toric variety.

Clearly, any affine SSV is multiplicity-bounded, i.e., the multiplicities of simple $G$-modules in its affine ring are uniformly bounded. Those affine SSV’s that are multiplicity-free admit a useful characterization, which follows from Lemma 2.1.5 and [AB04a, Lemma 2.3].

**Lemma 2.1.6.** For an affine multiplicity-free variety $X$, the following conditions are equivalent:

(i) $X$ is a SSV with weights in $\Gamma$.

(ii) The weight set $\Lambda^+(X)$ is saturated in $\Gamma$, i.e., $\Lambda^+(X) = C(X) \cap \Gamma$. 

From this lemma, we now deduce important invariance properties of $G$-subvarieties of SSV's:

**Lemma 2.1.7.** Let $X$ be an affine multiplicity-free SSV. Then:

(i) Any $G$-subvariety $Y$ is a multiplicity-free SSV, invariant under $\text{Aut}^G(X)$.

(ii) For any two $G$-subvarieties $Y, Z$, the scheme-theoretic intersection $Y \cap Z$ is non-empty, connected and reduced.

**Proof.** By Lemma 2.1.5, we may assume that $G = T = T_{\Gamma}$. Then $\Gamma = \Lambda$, and $X$ is a stable toric variety.

(i) By Lemma 2.1.6, to show that $Y$ is a SSV, it suffices to check that $C(Y) \cap \Lambda \subseteq \Lambda^+(Y)$. Given $\lambda \in C(Y) \cap \Lambda$, we may find a positive integer $n$ such that $n\lambda \in \Lambda^+(Y)$. Thus, $\lambda \in \Lambda^+(X)$ since the latter is saturated in $\Gamma$. Let $R$ (resp. $S$) be the affine ring of $X$ (resp. $Y$), and $I_Y$ the ideal of $Y$ in $R$. By the exact sequence of $G$-modules

$$0 \to I_Y \to R \to S \to 0$$

and multiplicity-freeness, $\Lambda^+(X) = \Lambda^+(R)$ is the disjoint union of $\Lambda^+(Y) = \Lambda^+(S)$ and $\Lambda^+(I_Y)$. Since $\Lambda^+(I_Y)$ (as $n\lambda \notin \Lambda^+(I_Y)$), it follows that $\lambda \in \Lambda^+(Y)$ as desired.

To show that $Y$ is invariant under $G$-automorphisms of $X$, we note that the irreducible components of $X$ are affine toric varieties with pairwise distinct weight cones (by multiplicity-freeness again): these components are pairwise non-isomorphic. Thus, each component is invariant under $\text{Aut}^G(X)$. Together with Lemma 1.3.3 this completes the proof.

(ii) Every $G$-invariant regular function on $X$ is constant, since $X$ is multiplicity-free. Thus, $X$ contains a unique closed $G$-orbit. It follows that $Y \cap Z$ is non-empty and connected.

To show the reducedness, consider a $T$-eigenvector $f \in R_{\lambda}$ and a positive integer $n$ such that $f^n \in I_{Y \cap Z} = I_Y + I_Z$. Since $f^n \in R_{n\lambda}$ and this weight space is a line, then $f^n \in I_Y$ or $f^n \in I_Z$. Thus, $f \in I_Y$ or $f \in I_Z$, so that $f \in I_{Y \cap Z}$. \hfill $\square$

Finally, we obtain a simple sufficient condition for a stable spherical variety to be Cohen–Macaulay:

**Proposition 2.1.8.** Let $X$ be an affine multiplicity-free SSV. If the weight cone $C(X)$ is convex, then $X$ is Cohen–Macaulay.

**Proof.** We adapt the argument of AB04b, Lemma 5.15. By Po87 (see also Gr97, AB05 Section 2), $X$ admits a flat degeneration to an affine $G$-variety $X_0$ which is “horospherical”, i.e., $X_0 = G \cdot X_0^U$; the base of this degeneration may be taken to be the affine line. Moreover, the affine rings of $X$ and $X_0$ are isomorphic as $G$-modules, so that $X_0$ is a multiplicity-free SSV as well. Since the property of being Cohen–Macaulay is open for fibers of this degeneration, we may assume that $X$ itself is horospherical.
Now let $Y := X^{U^-}$, so that $X = G \cdot Y$. By [AB05, Lemma 2.4], the composed map $Y \to X \to X//U$ is a $T$-isomorphism. Together with Lemma 2.1.6 it follows that $Y$ is an affine stable toric variety for $T$, and

\[ \Lambda^+(Y) = \Lambda^+(X) = \Gamma \cap C(X). \]

Thus, $Y$ is Cohen-Macaulay by [A02, Theorem 2.3.19]. Moreover, the dualizing module $H^0(Y, \omega_Y)$, regarded as a $T$-module, has weight set $\Gamma \cap C(X)^0$, where $C(X)^0$ denotes the relative interior of the cone $C(X)$ (see [AB04b, p. 405–406]).

Let $P$ be the set of all $g \in G$ such that $g \cdot Y = Y$; this is a parabolic subgroup of $G$ containing $B^-$. Moreover, any $\lambda \in \Lambda^+(X)$ extends to a character of $P$; thus, the $B^-$-action on $Y$ (via the quotient $B^- \to B^-/U^- = T$) extends to an action on $P$. In fact, $P$ is the largest parabolic subgroup containing $B^-$ such that every $\lambda \in \Lambda^+(X)$ extends to a character of $P$. Thus, any such $\lambda$ yields a $G$-linearized invertible sheaf $\mathcal{L}(\lambda)$ on $G/P$, which is globally generated. Moreover, $\mathcal{L}(\lambda)$ is ample for any $\lambda \in \Gamma \cap C(X)^0$.

Let $X'$ denote the fiber product $G \times^P Y$. In other words, $X'$ is a $G$-variety equipped with a $G$-morphism

\[ p : X' \to G/P \]

such that the fiber at $P$ is the $P$-variety $Y$. Then $p$ is a locally trivial fibration; thus, $X'$ is Cohen–Macaulay. We also have a proper, surjective $G$-morphism

\[ \pi : X' \to X, \quad (g, y)P \mapsto g \cdot y. \]

By a lemma of Kempf, to show that $X$ is Cohen–Macaulay, it suffices to check that $\pi_*(\mathcal{O}_{X'}) = \mathcal{O}_X$, $R^i\pi_*(\mathcal{O}_{X'}) = 0$ for all $i \geq 1$, and $R^i\pi_*(\omega_{X'}) = 0$ for all $i \geq 1$, where $\omega_{X'}$ denotes the dualizing sheaf of $X'$. Since $X$ is affine, this is equivalent to $H^0(X', \mathcal{O}_{X'}) = H^0(X, \mathcal{O}_X)$, $H^i(X', \mathcal{O}_{X'}) = 0$ for all $i \geq 1$, and $H^i(X', \omega_{X'}) = 0$ for all $i \geq 1$.

Since the morphism $p$ is affine and $p_*\mathcal{O}_{X'} \simeq \bigoplus_{\lambda \in \Gamma \cap C(X)} \mathcal{L}(\lambda)$, we obtain

\[ H^i(X', \mathcal{O}_{X'}) \simeq \bigoplus_{\lambda \in \Gamma \cap C(X)} H^i(G/P, \mathcal{L}(\lambda)) \text{ for all } i \geq 0. \]

Thus, the desired assertions on $H^i(X', \mathcal{O}_{X'})$ follow from the Borel-Weil theorem. To compute $H^i(X', \omega_{X'})$, we use the isomorphism of $G$-linearized sheaves $\omega_{X'} \simeq (p^*\omega_{G/P}) \otimes \omega_p$, where $\omega_p$ denotes the relative dualizing sheaf. Thus, $p_*\omega_{X'} \simeq \omega_{G/P} \otimes p_*\omega_p$. Moreover, $p_*\omega_p$ is the $G$-linearized sheaf on $G/P$ associated with the $P$-module $H^0(Y, \omega_Y) \simeq \bigoplus_{\lambda \in \Gamma \cap C(X)^0} \mathcal{L}(\lambda)$. Thus,

\[ H^i(X', \omega_{X'}) \simeq \bigoplus_{\lambda \in \Gamma \cap C(X)^0} H^i(G/P, \omega_{G/P} \otimes \mathcal{L}(\lambda)) \]

and the latter vanishes for all $i \geq 1$, by the Kodaira vanishing theorem. □

**Remark 2.1.9.** By this argument, an affine multiplicity-free SSV is Cohen–Macaulay whenever its weight cone $C$ satisfies the following conditions:
(1) $C$ is homeomorphic to a convex cone.
(2) $C$ is a finite union of convex cones $C_i$ such that the smallest face of
the positive chamber containing $C_i$ is independent of $i$.

2.2. Polarized SSV’s. In this section, we adapt the definition of stable
spherical varieties to the setting of polarized varieties, and we generalize
results of Sections 1.3 and 2.1 to these polarized SSV’s.

We fix a subgroup $\Gamma$ of $\tilde{\Lambda}$.

Definition 2.2.1. A polarized stable spherical variety with weights in $\Gamma$ is
a polarized $G$-variety $(X, L)$ satisfying the following conditions:
(i) $X$ is semi-normal.
(ii) $X$ contains only finitely many $G$-orbits, and these are spherical.
(iii) $\tilde{\Lambda}(Y, L)$ is contained in $\Gamma$ as a direct summand, for any $G$-orbit closure $Y$.

We then say for brevity that $(X, L)$ is a PSSV.

Now [AB04b, Lemma 2.1] and Lemma 2.1.3 imply readily the following:

Lemma 2.2.2. A polarized $G$-variety $(X, L)$ is a PSSV for $G$ if and only
if its affine cone is a SSV for $\tilde{G}$. Then

$$(X, L) = \lim_{\to \infty} (Y, L),$$

where $Q(X)$ denotes the poset of $G$-orbit closures in $X$.

We may now generalize the results of Propositions 1.3.1, 2.1.7 and 2.1.8 to
the polarized setting. From now on we only consider multiplicity-free SSV’s.
Indeed, we are mainly interested in spherical varieties and their stable limits,
which are all multiplicity-free. The general SSVs can be studied along the
lines of [Al02, Section 2].

Proposition 2.2.3. Let $(X, L)$ be a PSSV with weights in $\Gamma$ and moment
set $Q$, and let $Y$ be a $G$-subvariety of $X$. Then:
(i) $(Y, L)$ is a PSSV, invariant under $\text{Aut}^G(X, L)$.
(ii) The scheme-theoretic intersection $Y \cap Z$ is reduced, for any $G$-subvariety
$Z$ of $X$.
(iii) The restriction map $H^0(X, L^n) \to H^0(Y, L^n)$ is surjective for all $n \geq 0$.
(iv) As $G$-modules,

$$H^0(X, L^n) \simeq \bigoplus_{\lambda \in \Lambda^+} V(\lambda).$$

Equivalently, as $\tilde{G}$-modules,

$$R(X, L) \simeq \bigoplus_{\tilde{\lambda} \in \Gamma \cap \text{Cone}(Q)} V(\tilde{\lambda}).$$

(v) $L$ is globally generated.
(vi) \( H^i(X, L^n) = 0 \) for all \( i, n \geq 1 \).

(vii) If \( Q \) is convex, then the affine cone over \((X, L)\) is Cohen–Macaulay. In particular, \( X \) is Cohen–Macaulay.

**Proof.** (i) and (ii) follow from Lemmas 2.1.7 and 2.2.2.

(iii) Consider the finite injective map \( f : \tilde{Y} \to \tilde{X} \). By Lemma 2.1.7 again, \( f(\tilde{Y}) \) is semi-normal. Thus, \( f \) is a closed immersion. In other words, the restriction map \( R(X, L) \to R(Y, L) \) is surjective.

(iv) is a consequence of Lemma 2.1.6.

(v) follows from (iii) as in the proof of Lemma 1.3.1.

(vi) Let \( Y \) be an irreducible component of \( X \), and \( Z \) the union of all the other irreducible components. Consider the Mayer–Vietoris exact sequence

\[
0 \to \mathcal{O}_X \to \mathcal{O}_Y \oplus \mathcal{O}_Z \to \mathcal{O}_{Y \cap Z} \to 0.
\]

Note that \((Y, L)\) is a PSV, and the connected components of \( Z, Y \cap Z \) (equipped with the restriction of \( L \)) are multiplicity-free PSV’s, strictly smaller than \( X \). By Lemma 1.3.1 and induction, this yields an exact sequence

\[
0 \to H^0(X, L^n) \to H^0(Y, L^n) \oplus H^0(Z, L^n) \to H^0(Y \cap Z, L^n) \to H^1(X, L^n) \to 0
\]

and also \( H^i(X, L^n) = 0 \) for all \( i \geq 2, n \geq 1 \). To complete the proof, it suffices to check that the restriction map

\[
H^0(Y, L^n) \to H^0(Y \cap Z, L^n)
\]

is surjective for \( n \geq 1 \). Since \( Y \cap Z \) may be disconnected, the statement (iii) does not apply readily. We circumvent this by introducing

\[
R(Y \cap Z, L) := k \oplus \bigoplus_{n=1}^{\infty} H^0(Y \cap Z, L^n)
\]

(a reduced, finitely generated algebra) and

\[
\tilde{Y} \cap Z := \text{Spec } R(Y \cap Z, L).
\]

Then we obtain a finite injective map \( f : \tilde{Y} \cap Z \to \tilde{Y} \), with image a (connected) \( \tilde{G} \)-subvariety. By Lemma 2.1.7 again, \( f \) is a closed immersion; this is equivalent to the desired surjectivity.

(vii) follows from Proposition 2.1.8. □

2.3. **Structure.** In this section, we obtain a partial generalization of the classifications of stable toric varieties (in [Al02]) and of stable reductive varieties (in [AB04a, AB04b]), to the setting of stable spherical varieties. We only treat the polarized case, the affine case being entirely similar, and we still assume multiplicity-freeness throughout. The classification of SSV’s over a fixed projective space will be given in Lemma 4.3.3.
Lemma 2.3.1. Let \((X, L)\) be a PSSV with poset \(Q\) of \(G\)-orbit closures, and moment set \(Q\). Then:

(i) \(Q = \bigcup_{Y \in Q} Q(Y, L)\).

(ii) \(Y \cap Z \in Q\) for any \(Y, Z\) in \(Q\). Moreover, \(Q(Y \cap Z, L)\) is the unique common face to \(Q(Y, L)\) and \(Q(Z, L)\).

(iii) \(Z \subseteq Y\) if and only if \(Q(Z, L)\) is a face of \(Q(Y, L)\). In particular, \(Q(Y, L)\) determines \(Y\) uniquely.

Proof. Recall that \(Q\) is the union of the moment polytopes of the irreducible components of \(X\). Moreover, \(Q(Z, L)\) is a face of \(Q(Y, L)\) whenever \(Z \subseteq Y\).

This shows (i) and one implication of (iii). For the converse implication, if \(Q(Z, L) \subseteq Q(Y, L)\), then the same inclusion holds for the weight sets of the affine cones \(\tilde{Z}, \tilde{Y}\). By multiplicity-freeness, the opposite inclusion holds for the ideals of \(\tilde{Z}, \tilde{Y}\) in \(R(X, L)\). Thus, \(\tilde{Z} \subseteq \tilde{Y}\) and \(Z \subseteq Y\).

(ii) By (the proof of) Proposition 2.2.3 (vi), the restriction maps \(R(Y, L) \to R(Y \cap Z, L), R(Z, L) \to R(Y \cap Z, L)\) are both surjective. Thus, the Mayer–Vietoris sequence

\[ 0 \to R(Y \cup Z, L) \to R(Y, L) \oplus R(Z, L) \to R(Y \cap Z, L) \to 0 \]

is exact. Moreover, \(R(Y \cup Z, L), R(Y, L), R(Z, L), R(Y \cap Z, L)\) are all multiplicity-free \(\tilde{G}\)-modules, and the corresponding weight sets are saturated in \(\Gamma\). It follows that \(\tilde{C}(Y \cap Z, L) = \tilde{C}(Y, L) \cap \tilde{C}(Z, L)\), so that

\[ Q(Y \cap Z, L) = Q(Y, L) \cap Q(Z, L). \]

In particular, \(Q(Y \cap Z, L)\) is convex. On the other hand, \(Q(Y \cap Z, L)\) is a union of faces of \(Q(Y, L)\) corresponding to the irreducible components of \(Y \cap Z\), by Proposition 1.3.1. Thus, \(Q(Y \cap Z, L)\) is a unique face of \(Q(Y, L)\), and \(Y \cap Z\) is irreducible. \(\square\)

With the preceding assumptions and notation, the moment polytopes \(Q(Y, L), Y \in Q\), form a partial subdivision of \(Q\) by rational convex polytopes. The characterization of those subdivisions that arise from PSSV’s is an open problem; some examples are considered in 2.4.

We now describe those PSSV’s \((X, L)\) that have prescribed building blocks \((Y, L)\), where \(Y \in Q\). Recall from Proposition 2.1.3 and Lemma 2.2.2 that

\[ (X, L) \simeq \varinjlim_{Y \in Q} (Y, L_Y), \]

where \(L_Y\) denotes the restriction of \(L\) to \(Y\), and the directed system is defined by the inclusions

\[ i_{Z_Y} : (Z, L_Z) \to (Y, L_Y) \quad (Z \subset Y). \]

We introduce twists of this directed system, as follows. Let \(t = (t_{Z_Y})_{Z \subset Y}\), where the \(t_{Z_Y} \in \text{Aut}^G(Z, L_Z)\) satisfy

\[ t_{Y_3Y_1} = t_{Y_2Y_1} |_{Y_3} \circ t_{Y_3Y_2}. \]
whenever $Y_3 \subset Y_2 \subset Y_1$ (recall that the group $\text{Aut}^G(Y_2, L_{Y_2})$ is abelian and leaves $(Y_3, L_{Y_3})$ invariant, see Lemma 1.3.3). This defines a twisted direct system

$$i_{ZY} \circ t_{ZY} : (Z, L_Z) \to (Y, L_Y).$$

We denote its direct limit by $(X_t, L_t)$.

Note that the twists $t$ are precisely the 1-cocycles of the complex of abelian groups $C^*(Q, \text{Aut})$:

$$\prod_{Y_1} \text{Aut}^G(Y_1, L_{Y_1}) \to \prod_{Y_2 \subset Y_1} \text{Aut}^G(Y_2, L_{Y_2}) \to \prod_{Y_3 \subset Y_2 \subset Y_1} \text{Aut}^G(Y_3, L_{Y_3}) \to \cdots$$

with the usual differentials. This easy yields

**Lemma 2.3.2.** For any 1-cocycle $t \in Z^1(Q, \text{Aut})$, the limit $(X_t, L_t)$ is a PSSV having the same building blocks $(Y, L_Y)$, $Y \in Q$, as $(X, L)$. Moreover, $\text{Aut}^G(X_t, L_t) = H^0(Q, \text{Aut})$, and the isomorphism classes of the polarized $G$-varieties $(X_t, L_t)$ are parametrized by the first cohomology group $H^1(Q, \text{Aut})$ (a diagonalizable linear algebraic group).

**Remark 2.3.3.** The complex $C^*(Q, \text{Aut})$ admits an interpretation in terms of sheaf cohomology on the poset $Q$. Indeed, consider $Q$ as a topological space, where the open subsets are the decreasing subsets, i.e., the unions of subsets

$$Q \leq Y := \{Z \in Q \mid Z \subseteq Y \}.$$

Then a sheaf $\mathcal{F}$ of abelian groups on $Q$ is given by a collection of abelian groups $\mathcal{F}(Y)$, where $Y \in Q$ (the stalks of $\mathcal{F}$), together with restriction maps $\mathcal{F}(Y) \to \mathcal{F}(Z)$, where $Z \subset Y$, which are compatible in an obvious sense. This yields a complex $C^*(Q, \mathcal{F})$ which sheafifies to a resolution of $\mathcal{F}$ by flabby sheaves, analogous to Godement’s canonical resolution. Thus, the $H^i(Q, \text{Aut})$ are the cohomology groups of the sheaf $\text{Aut} : Y \to \text{Aut}^G(Y, L_Y)$.

Observe that $H^i(Q \leq Y, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ and any $i \geq 1$ (since the functor $\mathcal{F} \mapsto H^0(Q \leq Y, \mathcal{F}) = \mathcal{F}(Y)$ is exact). Moreover, by Lemma 2.3.1 (ii), the intersection of any family of basic open subsets $Q \leq Y$ is a basic open subset. Thus, the cohomology groups of $\mathcal{F}$ are those of the Cech complex associated with the covering $Q = \bigsqcup_i Q \leq Y_i$, where the $Y_i$ are the irreducible components of $X$:

$$\bigoplus_i \mathcal{F}(Y_i) \to \bigoplus_{i < j} \mathcal{F}(Y_{ij}) \to \bigoplus_{i < j < k} \mathcal{F}(Y_{ijk}) \to \cdots$$

where the $Y_{ij} = Y_i \cap Y_j$ are the double intersections, $Y_{ijk} = Y_i \cap Y_j \cap Y_k$ are the triples intersections, etc.

Also, note that the group $H^1(Q, \text{Aut})$ may be infinite. In fact, this already happens for stable toric surfaces, see [Al02, Example 2.2.10] and also Example 2.4(1).

Together with Lemma 1.2.5, it follows that there are infinitely many orbits of projective stable toric surfaces over some $\mathbb{P}(V)$, for the action of
Aut\(^G\) \(\mathbb{P}(V)\). In other words, Theorem [1.3.4] does not extend to all stable spherical varieties; however, it holds for SSV’s which contain a unique closed \(G\)-orbit (see Example 2.4(5)).

For arbitrary SSV’s, we obtain a weaker finiteness statement:

**Lemma 2.3.4.** Let \(X\) be a \(G\)-subvariety of \(\mathbb{P}(V)\), where \(V\) is a finite-dimensional \(G\)-module. Then there are only finitely many SSV’s \((X', f)\) over \(\mathbb{P}(V)\) such that \(f(X') = X\).

**Proof.** Let \(Y'\) be a \(G\)-orbit closure in \(X'\), with image \(Y\) in \(X\). Then there are only finitely many possibilities for \((Y', f|_{Y'})\), by Lemma [1.3.2]. Moreover, the assignment \(Y' \mapsto Y\) is a bijection from the poset of \(G\)-orbit closures in \(X'\) to the poset \(Q\) of \(G\)-orbit closures in \(X\), by Lemmas [1.3.2] and [2.3.1]. So we may fix the building blocks \((Y', L_{Y'})\) of the pairs \((X', f)\) over \((X, L)\), where \(L := O(1)\) and \(L' := f^* L\).

Fix such a pair \((X', L')\), so that an arbitrary pair may be written as \((X'_t, L'_t)\), where \(t \in Z^1(Q, \text{Aut})\). Let \(t = (t_{Z', Y'})\), then

\[
t_{Z', Y'} \in \text{Aut}^G(Z', L') \subseteq \text{Hom}(\tilde{\Lambda}(Z', L'), \mathbb{G}_m).
\]

Moreover, the restriction of \(t_{Z', Y'}\) to the subgroup \(\Lambda(Z, L)\) of \(\tilde{\Lambda}(Z', L')\) is constant, since \((X', L')\) and \((X'_t, L'_t)\) map to the same pair \((X, L)\). Thus, \(t_{Z', Y'}\) belongs to a finite subgroup of \(\text{Aut}^G(Z', L')\). So there are only finitely many possibilities for \(t\). \(\square\)

### 2.4. Examples.

We illustrate the notions and results of Section 2 in the cases considered in 1.4, and in two additional cases.

1) **Stable toric varieties.**

Let \(G = T\), then the stable spherical varieties with weights in \(\Lambda\) are exactly the stable toric varieties of \([A102]\) (for the quotient \(T_\Gamma\) of \(T\)), as follows from Lemma [2.1.5]. The moment polytopes of orbit closures of a multiplicity-free polarized stable toric variety form a subdivision of the moment set \(Q\) into convex lattice polytopes. Any such subdivision arises from a multiplicity-free polarized stable toric variety, by \([A102\text{ Section 2}]\).

In the case where \(Q\) is the subdivision of a triangle given in \([A102\text{ Example 2.2.10}]\), one obtains \(H^1(Q, \text{Aut}) \simeq \mathbb{G}_m\). This yields infinitely many stable toric surfaces which share the same moment set, but are pairwise non-isomorphic.

2) **Stable reductive varieties.**

With the notation of Example 1.4(2), the stable reductive varieties are exactly the SSV’s for \(G \times G\) with weights in the anti-diagonal. They are classified by complexes of \(W\)-admissible cones in the affine case, resp. polytopes in the polarized case; see \([AB04a\text{ Section 5}]\) and \([AB04b\text{ Section 2}]\). Again, there are non-isomorphic multiplicity-free stable spherical varieties having the same weight set and the same subdivision of their moment polytope.

3) **PSSV’s for SL(2).**
We consider projective multiplicity-free SSV’s for the group $G = \text{SL}(2)$. By Example 1.4(3) and Lemma 2.3.1, these varieties are exactly the chains

$$X = X_1 \cup X_2 \cup \cdots \cup X_r$$

of spherical $G$-varieties, such that:

- $X_2, \ldots, X_r$ are all isomorphic to the same rational ruled surface $F_e$,
- $X_1$ is isomorphic to $F_e$, or to $S_e$, or to $\mathbb{P}^1 \times \mathbb{P}^1$ (if $e = 2$), or to $\mathbb{P}^2$ (if $e = 4$), for $i = 2, \ldots, r - 1$, the positive section of $X_i$ is identified to the negative section of $X_{i+1}$,
- likewise, the positive section of $X_1$ (in the case of $F_e$ or $S_e$) or its closed orbit (in the other cases) is identified to the negative section of $X_2$.

(Such identifications are unique, since the $G$-automorphism group of the projective line is trivial).

Moreover, any ample, $G$-linearized line bundle $L$ on $X$ is uniquely determined by the degrees of its restrictions to the various sections. These degrees form a strictly increasing sequence $(n_0, n_1, \ldots, n_r)$ in the case of $F_e$ or $S_e$, resp. $(n_1, \ldots, n_r)$ in the other cases. The moment set $Q = Q(X,L)$ is the concatenation of the moment intervals of the $X_i$. So $Q = [n_0, n_r]$ in the case of $F_e$ or $S_e$; $Q = [n_1 - 2p, n_r]$ for some integer $p$ such that $0 \leq p \leq \frac{n_1}{2}$ in the case of $\mathbb{P}^1 \times \mathbb{P}^1$; and $Q = [0, n_r]$ in the case of $\mathbb{P}^2$.

4) Some PSSV’s for $SL(2) \times SL(2)$.

Let $G = SL(2) \times SL(2)$. We construct a multiplicity-free PSSV $(X,f)$ over some projective space $\mathbb{P}(V)$, such that $X$ is the union of two spherical varieties $X_1, X_2$ glued along a spherical subvariety $X_{12}$; the corresponding moment polytopes $Q_1, Q_2, Q_{12}$ form a subdivision of a convex polytope $Q$, but there exists no spherical variety over $\mathbb{P}(V)$ with moment polytope $Q$.

For this, we adapt an unpublished example of D. Luna who showed that the moduli space of affine spherical varieties with a prescribed weight monoid (defined in [AB05, Section 1.4]) may have several irreducible components.

The weight group of $G$ is $\mathbb{Z}^2$ and the subset of dominant weights is $\mathbb{N}^2$. Each simple $G$-module $V(n,n')$ is the space of polynomials in the variables $x,y,x',y'$ which are homogeneous of degree $n$ in $x,y$, and of degree $n'$ in $x',y'$; then $x^n x'^{n'}$ is a highest weight vector. Consider the $G$-module

$$V := V(2,0) \oplus V(4,2).$$

Then $\mathbb{P}(V)$ has two closed $G$-orbits, $G \cdot [x^2]$ and $G \cdot [x^4 x'^2]$.

Let $x_{12} := [x^2 + x^4 x'^2]$ and $X_{12} := G \cdot x_{12}$. Then $X_{12}$ is a spherical subvariety of $\mathbb{P}(V)$. Let $L_{12}$ be the restriction of $O(1)$ to $X_{12}$, then the weight group $\Lambda(X_{12}, L_{12}) \subset \mathbb{Z}^3$ is generated by $(1,2,0)$ and $(1,4,2)$. Moreover, the moment polytope $Q_{12}$ is the segment $[(2,0),(4,2)]$. Any $G$-subvariety of $\mathbb{P}(V)$ which contains both closed orbits must also contain $X_{12}$.

Next let $x_1 := [xy + x^4 x'^2] \in \mathbb{P}(V)$ and let $X_1$ be the normalization of $G \cdot x_1$ (one can show that the latter orbit closure is non-normal). Then the $G$-isotropy group of $x_1$ is the semi-direct product of the additive group
\((x \mapsto x, y \mapsto y, x' \mapsto x', y' \mapsto y' + ux')\) with the diagonalizable group \((x \mapsto tx, y \mapsto t^{-1}y, x' \mapsto \varepsilon t^{-2}x', y' \mapsto \varepsilon t^2y')\), where \(\varepsilon^2 = 1\). It follows that \(X_1\) is a spherical variety.

Let \(L_1\) be the pull-back of \(O(1)\) to \(X_1\). Then, by restricting to the open orbit \(G \cdot x_1\), one checks that the weight group \(\Lambda(X_1, L_1)\) is generated by the three vectors \((1, 0, 0), (1, 2, 0), (1, 4, 2)\); moreover, the weight cone \(\tilde{C}(X_1, L_1)\) is contained in the cone spanned by these vectors. It follows that the moment polytope \(Q_1 := Q(X_1, L_1) \subset \mathbb{Z}^2\) is the triangle with vertices \((0, 0), (2, 0), (4, 2)\). Indeed, we just saw that \(Q_1\) is contained in this triangle. On the other hand, \(\mathbb{C} \cdot x_1\) contains the \(G\)-subvarieties \(X_{12}\) and \(Y_1 := \mathbb{C} \cdot [xy]\). Thus, \(Q_1\) contains as faces the moment polytopes of \(X_{12}\) and \(Y_1\), i.e., the segments \([(2, 0), (4, 2)]\) and \([(0, 0), (2, 0)]\).

Next let \(x_2 := [x^2 + x^3x'y'] \in \mathbb{P}(V)\) and let \(H_2 \subseteq G\) be the subgroup \((x \mapsto \varepsilon x, y \mapsto \varepsilon y + ux, x' \mapsto t^2x', y' \mapsto t^2y')\), where \(\varepsilon^2 = 1\). Then \(H_2\) is a subgroup of index 2 of the \(G\)-isotropy group of \(x_2\). Denote by \(X_2\) the normalization of \(\mathbb{C} \cdot x_2\) in the function field \(k(G/H_2)\), then \(X_2\) is a spherical variety.

Let \(L_2\) be the pull-back of \(O(1)\) to \(X_2\). One checks that \(\Lambda(X_2, L_2) = \Lambda(X_1, L_1)\). Moreover, the moment polytope \(Q_2 := Q(X_2, L_2) \subset \mathbb{Z}^2\) is the triangle with vertices \((2, 0), (4, 2), (4, 0)\). Indeed, \(\mathbb{C} \cdot x_2\) contains both \(X_{12}\) and \(Y_2 := \mathbb{C} \cdot [x^3x'y']\). Thus, the segments \([(2, 0), (4, 2)]\) and \([(4, 2), (4, 0)]\) are faces of \(Q_2\). Since this convex polytope consists of dominant weights, this leaves no other choice.

By the description of their weight groups and moment polytopes, \((X_1, L_1)\) and \((X_2, L_2)\) contain both \((X_{12}, L_{12})\). So we may glue them to a multiplicity-free PSSV \((X, L)\) with moment polytope \(Q := Q_1 \cup Q_2\), the triangle with vertices \((0, 0), (4, 0), (4, 2)\). The corresponding partial subdivision \(Q\) consists of the triangles \(Q_1, Q_2\), the segments \([(0, 0), (2, 0)], [(2, 0), (4, 2)], [(4, 2), (4, 0)]\), and the points \((2, 0), (4, 2)\).

We claim that there exists no spherical variety over \(\mathbb{P}(V)\) with moment polytope \(Q\). In other words, \(Q\) is not the moment polytope of any irreducible multiplicity-free \(G\)-subvariety \(X \subseteq \mathbb{P}(V)\). Indeed, if \(X\) contains both closed \(G\)-orbits, then it contains \(X_{12}\); this contradicts the fact that \(Q_{12}\) is not a face of \(Q\). So \(X\) contains a unique closed orbit. Hence one of the projections \(\mathbb{P}(V) \rightarrow \mathbb{P}(V(2, 0)), \mathbb{P}(V) \rightarrow \mathbb{P}(V(4, 2))\) restricts to a finite morphism \(X \rightarrow \mathbb{P}(V(2, 0))\) or \(X \rightarrow \mathbb{P}(V(4, 2))\). The former case is excluded, since \(\text{rk}(X) = 2\). Thus, we reduce to the case where \(X \subseteq \mathbb{P}(V(4, 2))\). Now \(X\) is multiplicity-free of dimension 4 and contains semi-stable points, as \(Q\) contains the origin. By inspection, one obtains the unique possibility \(X = \mathbb{C} \cdot [x^2y^2x'y']\). But then the moment polytope contains \((0, 2)\), a contradiction.

One checks that \(\text{Aut}^G(X_{12}, L_{12}) = \mathbb{G}_m \times \mathbb{G}_m\) and likewise for \(X_1, X_3\). It follows that \(H^0(Q, \text{Aut}) = \mathbb{G}_m \times \mathbb{G}_m\) and \(H^1(Q, \text{Aut})\) is trivial. In
other words, $\text{Aut}^G(X, L) = \mathbb{G}_m \times \mathbb{G}_m$, and any PSSV with building blocks $(X_1, L_1), (X_2, L_2), (X_{12}, L_{12})$ is isomorphic to $(X, L)$.

5) **Simple PSSV’s.**

Recall that a $G$-variety is said to be *simple* if it contains a unique closed $G$-orbit. Let $(X, L)$ be a simple multiplicity-free PSSV with closed orbit $Z$. Then the $G$-module $H^0(Z, L)$ is simple, and hence lifts to a unique simple submodule of $H^0(X, L)$. Clearly, the latter submodule is base-point-free and invariant under $\text{Aut}^G(X, L)$. Thus, there exist a unique dominant weight $\lambda$ and a finite $G$-morphism $f : X \to \mathbb{P}(V(\lambda))$ such that $L = f^*\mathcal{O}(1)$. Moreover, $\text{Aut}^G(X, L)$ is a finite extension of $\mathbb{G}_m$. Conversely, any PSSV over the projectivization of a simple $G$-module is also simple, as follows, e.g., from Lemma 1.3.2.

Given $\lambda \in \Lambda^+$, we claim that *there are only finitely many multiplicity-free PSSV’s $(X, f)$ over $\mathbb{P}(V(\lambda))$. Indeed, there are only finitely many spherical varieties over $\mathbb{P}(V(\lambda))$ by Theorem 1.3.3, so that we may fix the building blocks $(Y, L_Y)$ of $(X, L)$. Then each $\text{Aut}^G(Y, L_Y)$ is a finite extension of $\mathbb{G}_m$. Moreover, $H^1(Q, \mathbb{G}_m)$ is trivial, since the poset $Q$ admits a unique minimal element. It follows that $H^1(Q, \text{Aut})$ is finite. By Lemma 2.3.2, this implies our claim.*

### 3. Families

3.1. **Families of affine SSV’s.** From now on we consider families of varieties over schemes; by a scheme, we mean a Noetherian scheme over our base field $k$ (still assumed to be algebraically closed, of characteristic zero). Morphisms (resp. products) are understood to be morphisms (resp. products) over $k$.

Recall from [AB04a, Section 7] that a *family of affine $G$-varieties over a scheme $S$ is a morphism $\pi : \mathcal{X} \to S$ satisfying the following conditions:

(i) $\mathcal{X}$ is a scheme equipped with an action of the constant group scheme $G \times S \to S$.

(ii) $\pi$ is flat, affine, and of finite type, with geometrically connected and reduced fibers.

A *$G$-morphism* of families over the same scheme $S$ is an equivariant morphism over $S$. The family $\pi : \mathcal{X} \to S$ is *trivial* if it is $G$-isomorphic to $X \times S$ equipped with the projection to $S$, where $X$ is some affine $G$-variety.

Given a family of affine $G$-varieties $\pi : \mathcal{X} \to S$ and a point $s \in S$, the geometric fiber $\mathcal{X}_s$ is an affine $G(k(\bar{s}))$-variety, where $k(\bar{s})$ denotes an algebraic closure of the residue field $k(s)$. Moreover,

$$\mathcal{R} := \pi_* (\mathcal{O}_\mathcal{X})$$

is a flat $\mathcal{O}_S$-algebra equipped with a compatible action of $G$, which is rational by [MPK94, I.1]; we say that $\mathcal{R}$ is a $\mathcal{O}_S$-$G$-algebra. This yields an
isomorphism of $\mathcal{O}_S$-modules
\begin{equation}
\mathcal{R} \simeq \bigoplus_{\lambda \in \Lambda^+} \mathcal{F}_\lambda \otimes V(\lambda),
\end{equation}
where each $\mathcal{F}_\lambda$ is a flat $\mathcal{O}_S$-module. Moreover, each $\mathcal{F}_\lambda$ is a coherent sheaf of modules over the invariant subalgebra, $\mathcal{R}^G \simeq \mathcal{F}_0$.

Thus, if $\mathcal{R}^G$ is finitely generated over $\mathcal{O}_S$ (e.g., if each geometric fiber contains only finitely many orbits), then each $\mathcal{F}_\lambda$ is a locally free $\mathcal{O}_S$-module, since we assume $S$ to be Noetherian. The rank of $\mathcal{F}_\lambda$ at any $s \in S$ is the multiplicity of the simple $G(k(\bar{s}))$-module $k(\bar{s}) \otimes V(\lambda)$ in the affine ring of $X_{\bar{s}}$.

Taking $U$-invariants in (3.1.5) yields an isomorphism of $\mathcal{O}_S$-algebras
\begin{equation}
\mathcal{R}^U \simeq \bigoplus_{\lambda \in \Lambda^+} \mathcal{F}_\lambda.
\end{equation}

**Definition 3.1.1.** Given a subgroup $\Gamma$ of $\Lambda$, a **family of affine SSV’s with weights in $\Gamma$** is a family of affine $G$-varieties $\pi : X \to S$ such that each geometric fiber is a stable spherical variety with weights in $\Gamma$.

If $S$ is connected and some geometric fiber $X = X_{\bar{s}}$ is multiplicity-free (e.g., a spherical variety over $k(\bar{s})$), then the $\mathcal{O}_S$-module $\mathcal{F}_\lambda$ is invertible whenever $\lambda \in \Lambda^+(X)$, and $\mathcal{F}_\lambda = 0$ otherwise. Thus, all the geometric fibers are multiplicity-free SSV’s with the same weight set.

We now obtain an important isotriviality result for families of spherical varieties.

**Proposition 3.1.2.** Let $\pi : X \to S$ be a family of affine spherical varieties over an excellent integral scheme. Then there exist a non-empty open subscheme $S_0$ of $S$ and a finite surjective morphism $\varphi : S' \to S_0$ such that the pull-back family $\pi' : X \times_S S' \to S'$ is trivial.

**Proof.** By [AB04a, Lemma 7.4], the family $\pi//U : X//U := \text{Spec}_{\mathcal{O}_S} \mathcal{R}^U \to S$ is locally trivial, with fiber an affine toric variety $Y$. Replacing $S$ with an open subset, we may assume that there is an isomorphism $X//U \to Y \times S$ over $S$. This yields a morphism $\varphi : S \to M_Y$, where $M_Y$ denotes the moduli scheme of affine spherical varieties of type $Y$ defined in [AB05, Section 1.3]; then $X$ is the pull-back of the universal family of $M_Y$. Recall from [AB05, Corollary 3.4] that $M_Y$ has an action of $T$ with finitely many orbits; these are in bijection with the isomorphism classes of affine spherical varieties $X$ such that $X//U \simeq Y$. So, by shrinking $S$ again, we may assume that the image of $\varphi$ is contained in a unique $T$-orbit $O$. After a finite surjective base change, we may assume that $\varphi : S \to O$ lifts to a morphism $\psi : S \to T$. Since the $T$-action on $M_Y$ lifts to an action on the universal family by [AB05, Section 2.1], it follows that $\pi$ is trivial.

Next we study **one-parameter degenerations** of affine spherical varieties, that is, families of affine $G$-varieties $\pi : X \to S$, where $S$ is a regular integral
scheme of dimension 1, and the geometric generic fiber $X_\eta$ is a spherical variety.

Here is a construction of such families. Let $\tilde{X}$ be an affine spherical $\tilde{G}$-variety (recall that $\tilde{G} = G_m \times G$) equipped with a surjective $\tilde{G}$-morphism $f : \tilde{X} \to \mathbb{A}^1$, where $\tilde{G}$ acts on $\mathbb{A}^1$ via the trivial action of $G$ and the scalar action of $G_m$. Then $f$ restricts to a trivial family over $\mathbb{A}^1 \setminus \{0\} \simeq \mathbb{G}_m$, with fiber some affine spherical $G$-variety $X$.

Lemma 3.1.3. With the preceding notation, let $\tilde{C} \subseteq \mathbb{R} \times \Lambda_{\mathbb{R}}$ (resp. $C \subseteq \Lambda_{\mathbb{R}}$) be the weight cone of $\tilde{X}$ (resp. $X$).

(i) There exists a unique function $h : C \to \mathbb{R}$ such that

\begin{equation}
\tilde{C} = \{(t, \lambda) \in \mathbb{R} \times C \mid h(\lambda) \leq t\}.
\end{equation}

Moreover, $h$ is lower convex, piecewise linear, and takes rational values at all rational points of $C$.

(ii) The special fiber $f^{-1}(0)$ is reduced if and only if $h$ takes integral values at all points of $\Lambda^+(X) = C \cap \Lambda(X)$. Then $f^{-1}(0)$ is a stable spherical variety, so that $f$ is a one-parameter degeneration.

(iii) There exists a positive integer $N$ such that the pull-back family under the morphism $\mathbb{A}^1 \to \mathbb{A}^1$, $z' \mapsto z'^N = z$, has reduced fibers.

Proof. (i) Let $R$ (resp. $\tilde{R}$) be the affine ring of $X$ (resp. $\tilde{X}$). Then $\tilde{R} \subseteq R[z, z^{-1}]$, as $\tilde{X} \setminus f^{-1}(0) \simeq X \times \mathbb{G}_m$. So $\tilde{C} \subseteq \mathbb{R} \times C$. Moreover, $(1, 0) \in \tilde{C}$ since $f \in \tilde{R}$, but $(-1, 0) \notin \tilde{C}$ since $f$ is not invertible in $\tilde{R}$. Since $\tilde{C}$ is a closed convex cone, it may be written as (3.1.7) for a unique convex function $h$. The piecewise linearity and rationality of $h$ follow from the fact that $\tilde{C}$ is a rational polyhedral cone.

(ii) Assume that there exists $\lambda \in \Lambda^+(X)$ such that $h(\lambda) \notin \mathbb{Z}$. Let $n$ be the smallest integer such that $n > h(\lambda)$, and let $f_\lambda \in R$ be a $B$-eigenvector of weight $\lambda$. Then $\tilde{f}_\lambda := z^n f_\lambda$ is in $\tilde{R}$, but not in $z\tilde{R}$. Now let $N$ be a positive integer such that $Nh(\lambda) \in \mathbb{Z}$. Then $\tilde{f}_\lambda^N := z^{Nh(\lambda)} f_\lambda^N$ is in $z\tilde{R}$, since $Nn \geq Nh(\lambda) + 1$. So the algebra $\tilde{R}/z\tilde{R}$ is not reduced.

Conversely, if $h$ takes integral values at all points of $\Lambda^+(X)$, then the preceding argument shows that no $B$-eigenvector in $\tilde{R}/z\tilde{R}$ is nilpotent. It follows that $\tilde{R}/z\tilde{R}$ is reduced. On the other hand, $\tilde{R}/z\tilde{R}$ is multiplicity-free. By Lemma 2.1.6 this implies that the special fiber is a SSV.

(iii) The pull-back under consideration is a similar family, but where $h$ is replaced with $Nh$. Since $h$ is rational, and linear on each cone of a finite subdivision of $C$ into rational polyhedral convex cones, we may choose $N$ so that $Nh$ takes integral values at all points of $\Lambda^+(X)$. Then the assertion follows from (ii).

A one-parameter degeneration as in Lemma 3.1.3 will be called standard. We now show that every one-parameter degeneration becomes standard after appropriate base changes.
Proposition 3.1.4. Let $\pi : X \to S = \text{Spec} A$ be a one-parameter degeneration of affine spherical varieties, where $A$ is a discrete valuation ring. Then, after a finite surjective base change, $\pi$ is isomorphic to the pull-back of a standard family $p : \tilde{X} \to k^1$.

Proof. We adapt the argument of [AB04a, Proposition 7.13]. Let $\eta = \text{Spec} K$ be the generic point of $S$, where $K$ is the fraction field of $A$, and let $s = \text{Spec}(A/zA)$ be the closed point, where $z$ is a generator of the maximal ideal of $A$. Applying Proposition 3.1.2, we may assume (after a finite surjective base change) that $X_\eta = X \times \{\eta\}$, where $X = \text{Spec} R$ is an affine spherical variety. Then $X_\eta = \text{Spec} R_\eta$, where

$$R_\eta = K \otimes R \simeq \bigoplus_{\lambda \in \Lambda^+(X)} K \otimes V(\lambda).$$

Moreover, $X = \text{Spec} R$, where $R = \bigoplus_{\lambda \in \Lambda^+(X)} F_\lambda \otimes V(\lambda)$, and each $F_\lambda$ is an invertible $R$-submodule of $K$. Thus,

$$R = \bigoplus_{\lambda \in \Lambda^+(X)} z^{h(\lambda)} R \otimes V(\lambda)$$

for some function $h : \Lambda^+(X) \to \mathbb{Z}$.

We claim that the subspace

$$\tilde{R} := \bigoplus_{\lambda \in \Lambda^+(X)} z^{h(\lambda)} k[z] \otimes V(\lambda) \subseteq R$$

is in fact a subalgebra. Indeed, consider three weights $\lambda, \mu, \nu$ in $\Lambda^+(X)$, such that $V(\nu)$ occurs in the decomposition of the product $V(\lambda) \cdot V(\mu)$ in $R$. Then the product $z^{h(\lambda)} V(\lambda) \cdot z^{h(\mu)} V(\mu)$ (in $R$) contains $z^{h(\lambda)+h(\mu)} V(\nu)$, so that $h(\nu) \leq h(\lambda) + h(\mu)$. This implies our claim.

Clearly, $\tilde{R}$ contains the polynomial ring $k[z]$, and

$$k(s) \otimes \tilde{R}/z\tilde{R} \simeq R/zR,$$

where $k(s) = A/zA$. Since the algebra $R/zR$ is the affine ring of the special fiber, it is finitely generated over $k(s)$. It follows that $\tilde{R}/z\tilde{R}$, and hence $\tilde{R}$, is finitely generated as well. Moreover, $\tilde{R}/z\tilde{R}$ is reduced, since $R/zR$ is. By considering powers of $B$-eigenvectors as in the proof of Lemma 3.1.3, this implies the equality $h(n\lambda) = nh(\lambda)$ for any positive integer $n$ and any $\lambda \in \Lambda^+(X)$. So $\tilde{X} := \text{Spec} \tilde{R}$ is an irreducible multiplicity-free $\tilde{G}$-variety with a saturated weight set, i.e., a spherical $\tilde{G}$-variety. Moreover, the morphism $z : \tilde{X} \to A^1$ is a standard degeneration. Finally, the multiplication map yields an isomorphism of $A$-algebras

$$A \otimes_{k[z]} \tilde{R} \simeq R,$$

which completes the proof. □
3.2. **Families of polarized SSV’s.** A family of polarized $G$-varieties over $S$ is a pair $(\pi : X \to S, L)$ satisfying the following conditions.

(i) $X$ is a scheme equipped with an action of the constant group scheme $\tilde{G} \times S$ over $S$.

(ii) $\pi$ is flat, proper, with geometrically connected and reduced fibers.

(iii) $L$ is a $\pi$-ample, $G$-linearized invertible sheaf on $X$.

We then put

$$R(X, L) := \bigoplus_{n=0}^{\infty} \pi^*(L^n).$$

This is a sheaf of $O_S$-$\tilde{G}$-algebras, the (relative) section ring of $(X, L)$.

**Definition 3.2.1.** Given a subgroup $\Gamma$ of $\tilde{\Lambda}$, a family of PSSV’s with weights in $\Gamma$ is a family of polarized $G$-varieties such that every geometric fiber is a PSSV with weights in $\Gamma$.

From now on we will only consider families of multiplicity-free PSSV’s $\pi : X \to S$, with weights in a prescribed subgroup $\Gamma$ of $\tilde{\Lambda}$. By Proposition 2.2.3 $H^1(X, L^n) = 0$ for all geometric points $\bar{s}$ and for all $n \geq 1$. Thus, every $O_S$-module $\pi_*(L^n)$ is locally free and satisfies $\pi_*(L^n) \otimes_{O_S} k(\bar{s}) = H^0(X, L^n)$ for all $\bar{s}$, by the theorem on cohomology and base change [Ha77, Chapter III, Theorem 12.11]. This yields isomorphisms of $O_S$-$G$-modules

$$\pi_*(L^n) \cong \bigoplus_{\lambda \in \tilde{\Lambda}^+} F_{n,\lambda} \otimes V(\lambda),$$

where each non-zero $F_{n,\lambda}$ is an invertible $O_S$-module.

In other words, the morphism

$$\tilde{\pi} : \tilde{X} := \text{Spec}_{O_S} R(X, L) \to S$$

is a family of affine multiplicity-free SSV’s (for $\tilde{G}$) with $X = \text{Proj}_{O_S} R(X, L)$ and $L = O_X(1)$. Moreover,

$$R(X, L) \cong \bigoplus_{\lambda \in \tilde{\Lambda}^+} F_{\lambda} \otimes V(\lambda), \quad R(X, L)^U \cong \bigoplus_{\lambda \in \tilde{\Lambda}} F_{\lambda},$$

and $R(X, L) \otimes_{O_S} k(\bar{s}) = R(X, L_{\bar{s}})$ for any geometric point $\bar{s}$. The sheaves $F_{\lambda}$ are equipped with multiplication maps

$$m_{\lambda, \mu} : F_{\lambda} \otimes_{O_S} F_{\mu} \to F_{\lambda+\mu}.$$

In particular, this yields $N$-th power maps

$$F_{\lambda}^N \to F_{N\lambda},$$

which are isomorphisms whenever $F_{\lambda}$ is non-zero, since the fibers of $\pi$ are reduced.

Thus, if $S$ is connected then all the geometric fibers have the same moment set $Q$; we say that the family is of type $Q$. 
Next we study the families of SSV’s over the projectivization of a fixed $G$-module $V$.

**Definition 3.2.2.** A family of SSV’s over $\mathbb{P}(V)$ is a pair $(\pi : \mathcal{X} \to S, f : \mathcal{X} \to \mathbb{P}(V))$, where $(\mathcal{X}, \mathcal{L} := f^*\mathcal{O}(1))$ is a family of PSSV’s.

Then the product morphism $f \times \pi : \mathcal{X} \to \mathbb{P}(V) \times S$ is finite, so that every geometric fiber of $\pi$ is a SSV over $\mathbb{P}(V)$. We also say that $\mathcal{X}$ is a SSV over $\mathbb{P}(V) \times S$.

A morphism between two SSV’s over $\mathbb{P}(V) \times S$ is a $G$-morphism of schemes over $\mathbb{P}(V) \times S$.

We now obtain two boundedness results for these families, which will play an essential role in the construction of the moduli space.

**Lemma 3.2.3.** Given a finite-dimensional $G$-module $V$, there exists a finite collection $\mathcal{Q}$ of rational convex polytopes in $\Lambda^+_R$ such that the following properties hold for any SSV $\mathcal{X}$ over $\mathbb{P}(V) \times S$:

(i) The moment polytope of any $G$-subvariety of any geometric fiber $X_s$ is a union of polytopes in $\mathcal{Q}$.

(ii) For any $Q \in \mathcal{Q}$ and $\check{\lambda}, \check{\mu}$ in $\Gamma \cap \text{Cone}(Q)$, the map $m_{\check{\lambda}, \check{\mu}}$ of (3.2.11) is an isomorphism.

**Proof.** By Theorem 1.3.4, the set of moment polytopes of spherical varieties over $\mathbb{P}(V)$ is finite. Therefore, we may choose a finite common subdivision $\mathcal{Q}$ of all these polytopes by rational convex polytopes. Clearly, this subdivision satisfies (i).

Let $s \in S$ and put $X := X_s$, $L := L_s$. Then, by multiplicity-freeness, the multiplication map $R(X, L)^U_{\check{\lambda}} \otimes R(X, L)^U_{\check{\mu}} \to R(X, L)^U_{\check{\lambda} + \check{\mu}}$ is an isomorphism whenever both $\check{\lambda}, \check{\mu}$ belong to $\Gamma \cap \text{Cone}Q(Y, L)$, where $Y$ is an irreducible component of $X$. It follows that $\mathcal{Q}$ satisfies (ii). \qed

Any family of SSV’s $f \times \pi : \mathcal{X} \to \mathbb{P}(V) \times S$ defines a morphism of $O_S\tilde{G}$-algebras

$$f^* : O_S \otimes \text{Sym}(V^*) \to \mathcal{R}(\mathcal{X}, f^*\mathcal{O}(1)) := \mathcal{R},$$

which makes $\mathcal{R}$ a finite module over $O_S \otimes \text{Sym}(V^*)$. It follows that $\mathcal{R}^U$ is a finite module over $O_S \otimes \text{Sym}(V^*)^U$. For every $\check{\lambda} \in \check{\Lambda}^+$, we denote by

(3.2.11) $$f^*_\check{\lambda} : O_S \otimes \text{Sym}(V^*)^U_{\check{\lambda}} \to \mathcal{R}^U_{\check{\lambda}} = \mathcal{F}_{\check{\lambda}}$$

the restriction of $f^*$ to the component of weight $\check{\lambda}$.

**Lemma 3.2.4.** Let $\mathcal{Q}$ be as in Lemma 3.2.3. Then:

(i) There exists a finite subset $F$ of $\check{\Lambda}^+$ such that the monoid $\Gamma \cap \text{Cone}(Q)$ is generated by a subset of $F$, for any $Q \in \mathcal{Q}$.

(ii) For any SSV $\mathcal{X}$ over $\mathbb{P}(V) \times S$, the $O_S$-algebra $\mathcal{R}^U$ is generated by the $\mathcal{F}_{\check{\lambda}}$, where $\check{\lambda} \in F$. Moreover, there exists a positive integer $N$ (depending only on $V$) such that the map $f^*_N$ of (3.2.11) is surjective for any $\check{\lambda} \in \check{\Lambda}^+$. 

Proof. By Gordan’s lemma, every monoid $\Gamma \cap \text{Cone}(Q)$, $Q \in \mathbb{Q}$, is finitely generated. Thus, we may choose a finite subset $F \subseteq \lambda^+$ containing generators of all these monoids. Then the sheaves $\mathcal{F}_\lambda$, $\lambda \in F$, generate the algebra $\mathcal{R}^U$ by Lemma 3.2.3 (ii).

To complete the proof, by Lemma 3.2.3 again, it suffices to show the existence for any $\tilde{\lambda}$ of $N = N(\tilde{\lambda})$ such that $f^*_N$ is surjective. Let

$$R^U_{(\tilde{\lambda})} := \bigoplus_{n=0}^{\infty} \mathcal{R}(\mathcal{X}, \mathcal{L})^U_{n\tilde{\lambda}} = \bigoplus_{n=0}^{\infty} \mathcal{F}_{n\tilde{\lambda}}.$$  

This is a finitely generated, graded $O_S$-subalgebra of $\mathcal{R}^U$; we may assume that $R^U_{(\tilde{\lambda})} \neq O_S$. Let $N_0$ be the smallest positive integer such that $F_{N_0\tilde{\lambda}} \neq 0$. By saturation, $F_{n\tilde{\lambda}} \neq 0$ if and only if $N_0$ divides $n$. It follows that

$$\mathcal{R}^U_{(N\tilde{\lambda})} \simeq \text{Sym}_{O_S}(\mathcal{F}_{N\tilde{\lambda}})$$

whenever $N$ is a multiple of $N_0$. In particular, $\mathcal{R}^U_{(N\tilde{\lambda})}$ is locally isomorphic to the polynomial ring $O_S[z]$, where $z$ is a variable of degree $N$.

Define likewise the finitely generated, graded algebra

$$\text{Sym}(V^*_n)^U_{(\tilde{\lambda})} = \bigoplus_{n=0}^{\infty} \text{Sym}(V^*_n)^U_{n\tilde{\lambda}}.$$  

Then there exists a positive integer $N_1$ such that the algebra $\text{Sym}(V^*_n)^U_{(N\tilde{\lambda})}$ is generated by its subspace of degree $N$, whenever $N$ is a multiple of $N_1$.

On the other hand, by weight considerations, $\mathcal{R}^U_{(N\tilde{\lambda})}$ is a finite module over $O_S \otimes \text{Sym}(V^*_n)^U_{(N\tilde{\lambda})}$ for any $N$. It follows easily that the desired surjectivity holds for any $N$ divisible by $N_0$ and $N_1$. $\blacksquare$

**Remark 3.2.5.** By similar arguments, one obtains the existence of a positive integer $N'$ (depending only on $V$) such that the map

$$f^*_n : O_S \otimes \text{Sym}^n(V^*) \rightarrow \mathcal{R}_n$$

is surjective for any SSV $f \times \pi : \mathcal{X} \rightarrow \mathbb{P}(V) \times S$ and for any multiple $n$ of $N'$. In particular, $f^* \mathcal{O}(N')$ is very ample relatively to $\pi$. Since $f^* \mathcal{O}(1)$ is $\pi$-globally generated, it follows that $f^* \mathcal{O}(n)$ is $\pi$-very ample for any $n \geq N'$.

### 3.3. Families of stable spherical pairs.

In this section, we introduce the notion of stable spherical pairs, and we reduce their classification to that of stable spherical varieties over the projective spaces of certain $G$-modules.

We still consider families of *multiplicity-free* PSSV’s with weights in a prescribed group $\Gamma$.

**Definition 3.3.1.** A family of stable spherical pairs (in short, SSP’s) over a scheme $S$ consists of a family $(\pi : \mathcal{X} \rightarrow S, \mathcal{L})$ of PSSV’s, together with a section $\sigma \in H^0(\mathcal{X}, \mathcal{L})$ satisfying the following condition:
For any geometric point \( \bar{s} \), the pull-back of \( \sigma \) to the geometric fiber \( \mathcal{X}_s \) does not vanish on any orbit of \( G(k(\bar{s})) \).

Then the divisor of zeroes \( D := (\sigma)_0 \) is an effective, \( \pi \)-ample Cartier divisor on \( \mathcal{X} \). Since \( \mathcal{L} = \mathcal{O}_\mathcal{X}(D) \), the pair \( (\mathcal{X}, D) \) encodes our triple \((\pi : \mathcal{X} \to S, \mathcal{L}, \sigma)\).

A morphism from another pair \((\pi' : \mathcal{X}' \to S, D')\) to \((\pi : \mathcal{X} \to S, D)\) is a \( G \)-morphism \( \varphi : \mathcal{X}' \to \mathcal{X} \) over \( S \) such that \( D' = \varphi^*(D) \).

We now associate with any family of SSP’s \((\pi : \mathcal{X} \to S, D)\) of a fixed type \( Q \), a family of SSV’s over the projectivization of a \( G \)-module \( V_Q \) depending only on \( Q \). By \((3.3.12)\), we have a canonical isomorphism of \( \mathcal{O}_S \)-modules

\[
\pi_*(\mathcal{L}) = \bigoplus_{\lambda \in F} \mathcal{F}_{1,\lambda} \otimes V(\lambda),
\]

where \( F := \Gamma \cap Q \) is a finite set of dominant weights. Then

\[
\sigma \in H^0(\mathcal{X}, \mathcal{L}) = \bigoplus_{\lambda \in F} H^0(S, \mathcal{F}_{(1,\lambda)}) \otimes V(\lambda).
\]

Write accordingly \( \sigma = \sum_{\lambda \in F} f_\lambda \otimes x_\lambda \). This yields a linear map

\[
\gamma : \bigoplus_{\lambda \in F} \text{End} V(\lambda) \to H^0(\mathcal{X}, \mathcal{L}), \quad \sum u_\lambda \mapsto \sum f_\lambda \otimes u_\lambda(x_\lambda).
\]

In particular, \( \gamma(\sum_{\lambda \in F} \text{id}_\lambda) = \sigma \) with obvious notation. Also, \( \gamma \) is \( G \)-equivariant, where \( G \) acts on each \( \text{End} V(\lambda) \cong V(\lambda)^* \otimes V(\lambda) \) via its action on \( V(\lambda) \). Put

\[
(3.3.12) \quad V_Q := \bigoplus_{\lambda \in F} (\text{End} V(\lambda))^* = \bigoplus_{\lambda \in F} V(\lambda)^* \otimes V(\lambda)^*,
\]

regarded as a \( G \)-module via the action on the spaces \( V(\lambda)^* \). Then we have a morphism of \( G \)-modules

\[
\gamma : V_Q^* \to H^0(\mathcal{X}, \mathcal{L}).
\]

**Proposition 3.3.2.** Let \((\pi : \mathcal{X} \to S, D)\) be a family of SSP’s of type \( Q \). Then the above morphism \( \gamma \) yields a \( G \)-morphism \( f : \mathcal{X} \to \mathbb{P}(V_Q) \) such that \( \mathcal{L} = f^* \mathcal{O}(1) \) and \( \sigma = f^*(\sum_{\lambda \in F} \text{id}_\lambda) \). In particular, \( \mathcal{X} \) is an SSV over \( \mathbb{P}(V_Q) \times S \).

This defines a bijective correspondence from families of SSP’s of type \( Q \) over \( S \), to SSV’s of the same type over \( \mathbb{P}(V_Q) \times S \); this correspondence preserves morphisms.

**Proof.** Let \( \bar{s} \) be a geometric point of \( S \). Then, by assumption, the translates \( gD_{\bar{s}}, g \in G(k(\bar{s})) \), have no common zero on the geometric fiber \( \mathcal{X}_s \). Since \( G(k) \) is dense in \( G(k(\bar{s})) \), the same holds for the translates \( gD_{\bar{s}}, g \in G(k) \). Let \( g_\lambda \) denote the image of \( g \) in \( \text{GL}(V(\lambda)) \subset \text{End} V(\lambda) \). Then

\[
\gamma(\sum g_\lambda) = \sum g_\lambda \sigma_\lambda = g \sigma.
\]
Thus, the subspace $\gamma(V_Q^\ast)$ of $H^0(\mathcal{X}, \mathcal{L})$ is base-point-free in each geometric fiber. It follows that $f : \mathcal{X} \to \mathbb{P}(V_Q)$ is well-defined, and $\mathcal{L} = f^*\mathcal{O}(1)$. Clearly, $f^*(\sum_{\lambda \in F} \text{id}_\lambda) = \sigma$.

Conversely, let $(\pi : \mathcal{X} \to S, f : \mathcal{X} \to \mathbb{P}(V_Q))$ be a family of SSV’s over $\mathbb{P}(V_Q)$. Let $\sigma := f^*(\sum_{\lambda \in F} \text{id}_\lambda)$; this is a global section of $\mathcal{L} := f^*\mathcal{O}(1)$. We show that $\sigma$ does not vanish identically on any $G(k(\bar{s})$-orbit $Y$ in a geometric fiber $X = \mathcal{X}_{\bar{s}}$. We may assume that $Y$ is closed, and (to simplify notation) $k(\bar{s}) = k$. Then $Y \cong G/P$, where $P$ is a parabolic subgroup of $G$ containing $B$. By assumption, we have a finite morphism $f_{\bar{s}} : X \to \mathbb{P}(\bigoplus_{\lambda \in F} V(\lambda) \otimes V(\lambda)^\ast)$.

By highest weight theory, there exists a unique $\lambda = \lambda(Y) \in F$ such that $f_{\bar{s}}$ restricts to an embedding $Y \to \mathbb{P}(V(\lambda) \otimes V(\lambda)^\ast)$, $gP \mapsto [v \otimes gv_{\lambda^\ast}]$, where $v \in V(\lambda)$ is non-zero, and $v_{\lambda^\ast} \in V(\lambda)^\ast$ is a highest weight vector. Moreover, $\sigma(v \otimes gv_{\lambda^\ast}) = \langle v, gv_{\lambda^\ast} \rangle$ is non-zero for some $g \in G$, since the translates $gv_{\lambda^\ast}$ span $V(\lambda)^\ast$.

This establishes the desired correspondence, which is clearly functorial. □

For example, if $G = T$ is a torus, then the families of stable toric pairs of type $Q$ may be identified with the families of stable toric varieties over $\mathbb{P}(V_Q)$, where $V_Q := \bigoplus_{\lambda \in \Gamma \cap Q} k^{-\lambda}$. Here $k_\lambda$ denotes the line $k$ on which $G$ acts with weight $\lambda$.

**Remark 3.3.3.** We mention the following connection with the singularities of pairs, generalizing results of [AB04b, Section 5] which mainly concern stable reductive varieties.

Recall that a spherical variety $X$ has two kinds of group boundaries: $\partial_G X$, the codimension one part of the complement of the open $G$-orbit, and $\partial_B X$, the complement of the open $B$-orbit minus $\partial_G X$. A canonical divisor for $X$ is

\[(3.3.13) \quad K_X = -\Delta_G - \Delta_B,\]

where $\Delta_G$ is the divisor $\partial_G X$ with reduced structure, and $\Delta_B$ is a unique effective divisor with support $\partial_B X$.

Next consider a multiplicity-free stable spherical variety $X$ with convex moment set. Then $X$ is Cohen–Macaulay by Proposition 2.2.3 and one easily shows that $X$ has only simple crossings in codimension one. It follows that (3.3.13) still holds, where $\Delta_G$ denotes the sum of all irreducible $G$-invariant divisors which are not contained in the double locus, and $\Delta_B$ denotes the sum of the $\Delta_B$’s of these components.

In [AB04b Theorem 5.3] we proved that for a spherical variety $X$ the pair $(X, \Delta_G + |\Delta_B|)$ has log canonical singularities. (Here $|\Delta_B|$ means that one
has to pick a general element of this linear system.) And an easy extension of [AB04b, Theorems 5.9, 5.12] gives the following:

**Proposition 3.3.4.** Let \((X, D)\) be a multiplicity-free SSP whose moment set \(Q\) is convex. Then for \(0 < \varepsilon \ll 1\), the pair \((X, \Delta_G + |\Delta_B| + \varepsilon D)\) has semi-log canonical singularities (resp. log canonical if \(X\) is irreducible).

4. **Moduli**

4.1. **Existence of a quasiprojective moduli scheme.** We fix a subgroup \(\Gamma\) of \(\tilde{\Lambda}\) and a finite-dimensional \(G\)-module \(V\). Consider the contravariant functor

\[ M = M_{\Gamma, \mathbb{P}(V)} \]

from schemes to sets, that assigns to any scheme \(S\) the set of isomorphism classes of multiplicity-free SSV’s over \(\mathbb{P}(V) \times S\) with weights in \(\Gamma\).

**Theorem 4.1.1.** The functor \(M\) is coarsely represented by a quasiprojective scheme \(M_{\Gamma, \mathbb{P}(V)}\).

**Proof.** Choose a finite collection \(Q\) of polytopes, a finite subset \(F\) of \(\tilde{\Lambda}^+\), and a positive integer \(N\) satisfying the statements of Lemma 3.2.4. For simplicity, we begin with the case where \(N = 1\). Then, for any family \(f \times \pi : \mathcal{X} \to \mathbb{P}(V) \times S\) with section ring \(\mathcal{R} := \mathcal{R}(\mathcal{X}, f^*\mathcal{O}(1))\), the maps \(f_\lambda^*\) of (3.2.11) are all surjective. Thus, the map

\[ f^* : \mathcal{O}_S \otimes \text{Sym}(V^*) \to \mathcal{R} \]

is surjective as well. In other words, the morphism

\[ \tilde{f} \times \tilde{\pi} : \tilde{\mathcal{X}} \to \Lambda(V) \times S \]

is a closed immersion, where \(\tilde{\mathcal{X}} = \text{Spec}_{\mathcal{O}_S} \mathcal{R}\). The image of this morphism yields a family of \(\tilde{G}\)-subvarieties of \(\Lambda(V)\), parametrized by \(S\). The corresponding Hilbert function

\[ h : \tilde{\Lambda}^+ \to \mathbb{N} \]

(in the sense of [AB05, Definition 1.4]) is given by: \(h(\tilde{\lambda}) = 1\) if there exists \(Q \in Q\) such that \(\tilde{\lambda} \in \Gamma \cap \text{Cone}(Q)\), and \(h(\tilde{\lambda}) = 0\) otherwise.

Recall from [AB05] that the families of \(\tilde{G}\)-subschemes of \(\Lambda(V)\) with Hilbert function \(h\) admit a fine moduli space: the invariant Hilbert scheme

\[ H := \text{Hilb}_{\tilde{G}}^{\tilde{\lambda}}(V), \]

a quasi-projective scheme. Moreover, the families of \(\tilde{G}\)-subvarieties of \(\Lambda(V)\) with Hilbert function \(h\) are parametrized by an open subscheme \(H'\) of \(H\), the locus where the fibers of the universal family are reduced; see [AB04b, p. 265]. So we obtain a morphism \(S \to H'\) such that \(\tilde{\mathcal{X}}\) is isomorphic to the pull-back of the universal family. Conversely, the universal family over \(H'\) is a family of stable spherical \(\tilde{G}\)-subvarieties of \(\Lambda(V)\) (by Lemma 2.1.6),
and hence a family of $SSV$’s over $\mathbb{P}(V)$. Thus, $\mathcal{M}$ is represented by the quasi-projective scheme $H'$.

We now turn to the general case, where $N$ is arbitrary. A little problem we need to overcome is that some of our generating sections of $\mathcal{R}$ provided by Lemma 3.2.4 have weights $\tilde{\lambda}_i = (n_i, \lambda_i)$ of degrees $n_i > 1$, and moreover only their $N$-th power may belong to the image of $\mathcal{O}_S \otimes \text{Sym}(V^*)$. We solve this by taking $N$-th roots, thus making extra choices, and then dividing by these choices. As a result, the moduli space is only coarse and not fine.

The $N$-th roots live on a finite abelian Galois covering of the base scheme $S$. Specifically, write $F = \{\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m\}$ and $\tilde{\lambda}_i = (n_i, \lambda_i)$ for $i = 1, \ldots, m$, where $n_i$ is a positive integer and $\lambda_i$ is a dominant weight. Choose a basis $(\varphi_{ij})_{j \in J_i}$ of the finite-dimensional vector space $\text{Sym}(V^*)^{\otimes \tilde{\lambda}_i}$.

Consider again a family $f \times \pi : X \to \mathbb{P}(V) \times S$ with section ring $\mathcal{R}$, and the associated maps $f^*_{\tilde{\lambda}_i}$ of (3.2.11). By Lemma 3.2.4, each map $f^*_{\tilde{\lambda}_i}$ is surjective. Let $\sigma_{ij} := f^*_{\tilde{\lambda}_i}(\varphi_{ij}) \in H^0(S, F_{N\tilde{\lambda}_i})$, then each invertible $\mathcal{O}_S$-module $F_{N\tilde{\lambda}_i}$ is generated by its global sections $\sigma_{ij}$, $j \in J_i$.

Let $S'$ be the scheme obtained from $S$ by taking the $N$-th roots of these sections for all $i,j$ (see [EV92, §3]) and let $A$ be the corresponding product of the groups $\mu_N$ of $N$-th roots of unity. Then the finite abelian group $A$ acts on $S'$ with a flat quotient map $p : S' \to S$. Moreover, every invertible $\mathcal{O}_{S'}$-module $F'_{N\tilde{\lambda}_i}$ is equipped with sections $\sigma'_{ij}$ such that $(\sigma'_{ij})^N = \sigma_{ij}$. Since $(F'_{\tilde{\lambda}_i})^N$ is identified with $F_{N\tilde{\lambda}_i}$ via (3.2.11), it follows that the $\mathcal{O}_{S'}$-module $F'_{\tilde{\lambda}_i}$ is generated by the $\sigma'_{ij}$, $j \in J_i$. So, by Lemma 3.2.4 again, the $\mathcal{O}_{S'}$-algebra

$$\mathcal{R}' := \mathcal{R} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'} \simeq \bigoplus_{\tilde{\lambda}_i \in \tilde{\Lambda}^+} F'_{\tilde{\lambda}_i} \otimes V(\tilde{\lambda}_i)$$

is generated by its subspaces $\sigma'_{ij} \otimes V(\tilde{\lambda}_i)$. This defines a $\tilde{G}$-module map

$$\gamma : V \oplus \bigoplus_{i,j \in J_i} V(\tilde{\lambda}_i) \to \mathcal{R}'$$

where the restriction $V \to \mathcal{R}'$ is the composed map $V \to H^0(S, \mathcal{R}) \to \mathcal{R}'$. The image of $\gamma$ generates the $\mathcal{O}_{S'}$-algebra $\mathcal{R}'$. Let

$$V' := V^* \oplus \bigoplus_{i,j} V(\tilde{\lambda}_i)^*$$

and $\tilde{\Lambda}' := \text{Spec}_{\mathcal{O}_{S'}} \mathcal{R}'$.

Then $\gamma$ yields a closed $\tilde{G}$-embedding

$$\tilde{\Lambda}' \hookrightarrow \Lambda(V') \times S$$
over $S'$. Moreover, the group $A$ acts linearly on $\mathbb{A}(V')$ via the trivial action on $V$ and the scalar action of each factor $\mu_N$ on the corresponding factor $V(\bar{\lambda}_i)$; this action commutes with the $\tilde{G}$-action. The structure map $\tilde{X}' \to S'$ is $A$-equivariant, the projection $\tilde{X}' \to \mathbb{A}(V') \times S'$ is finite, and the quotient $\tilde{X}'/A$ is $\tilde{G}$-isomorphic to $\tilde{X}$ (since $(\mathcal{R}')^A = (\mathcal{O}_{\mathcal{S}'(\mathcal{R})})^A \simeq \mathcal{R})$.

Let $h : \bar{\lambda}^+ \to \mathbb{N}$ be as in \eqref{eq:lambda_bar}. Consider the invariant Hilbert scheme $H := \text{Hilb}_{\mathbb{C}}^{\tilde{G}}(V')$, and its locus $H'$ parametrizing those families $\tilde{X}' = \text{Spec}_{\mathcal{S}',(\mathcal{R}')}$ having reduced fibers and such that the maps

$$ f^*_{\mathcal{R}'} : \mathcal{O}_{\mathcal{S}'} \otimes \text{Sym}((V')^*)^U_{\mathbb{N}} \to \mathcal{R}'^U_{\mathbb{N}} $$

are surjective for $i = 1, \ldots, m$. Clearly, $H'$ is an open subscheme, invariant under the $A$-action on $H$ (induced by the linear $A$-action on $\mathbb{A}(V')$ that commutes with the $\tilde{G}$-action). Since $H$ is quasi-projective, the quotient $H'/A$ is quasi-projective as well.

We now show that $H'/A$ coarsely represents $\mathcal{M}$. Indeed, we just saw that every family of SSV’s over $\mathbb{P}(V) \times S$ is the pullback of the universal family over $H'$ under a functorially defined $A$-equivariant morphism $S' \to H', S = S'/A$. This gives a morphism $\phi : \mathcal{M} \to \text{Hom}(\ast, H'/A)$ to the categorical quotient $H'/A$, and $\phi$ is universal among such morphisms.

The universal family over $H'$ is a family of affine SSV’s for $\tilde{G}$ (by Lemma \ref{lem:universal}) which is finite over $\mathbb{A}(V) \times H'$ (as follows from Lemma \ref{lem:finite}). Thus, each closed point of $H'/A$ defines a SSV over $\mathbb{P}(V)$, which is unique up to $G$-isomorphism. Hence, $H'/A$ satisfies the conditions in the definition of the coarse moduli space of $\mathcal{M}$. \hfill \Box

**Remarks 4.1.2.** 1) Consider the subfunctors $\mathcal{M}_{\mathbb{P}(V), Q}$ of $\mathcal{M}_{\mathbb{P}(V), V}$, obtained by prescribing the moment set $Q$. Clearly, each $\mathcal{M}_{\mathbb{P}(V), Q}$ is an open and closed subfunctor of $\mathcal{M}_{\mathbb{P}(V), V}$. This yields moduli schemes $M_{\mathbb{P}(V), Q}$ which are unions of connected components of $M_{\mathbb{P}(V), V}$.

2) Likewise, fix a subdivision $Q$ of $Q$ and consider the subfunctor $\mathcal{M}_{\mathbb{P}(V), Q}$ where the geometric fibers have all moment set $Q$ and subdivision $Q$. Then $M_{\mathbb{P}(V), Q}$ is a locally closed subfunctor. Indeed, it is given by the conditions that the map $m_{\bar{\lambda}, \bar{\mu}}$ of \eqref{eq:moment_map} is an isomorphism whenever $\bar{\lambda}, \bar{\mu}$ belong to the same cone over a polytope in $Q$, and is zero otherwise. Further, it suffices to check these conditions for finitely many pairs $(\bar{\lambda}, \bar{\mu})$, by Lemmas \ref{lem:finite} and \ref{lem:finite}.

3) It is easy to check that any morphism of SSV’s over $\mathbb{P}(V) \times S$ of the same type $Q$ is an isomorphism, and the automorphism group functor of any such SSV is representable by a finite group scheme. Together with the argument of Theorem \ref{thm:galois} it follows that the families of SSV’s over $\mathbb{P}(V)$ with weights in $\Gamma$ and type $Q$ form an algebraic stack of finite type in the sense of \cite{LaMT00} Chapitre 4, with coarse moduli space $M_{\mathbb{P}(V), Q}$. In fact, this stack is proper, as will follow from Proposition \ref{prop:proper}.
4.2. Projectivity. In this section, we show that the schemes $M_{\Gamma, \mathbb{P}(V)}$ of Theorem 4.1.1 are projective. Since we know that they are quasiprojective, it suffices to check that the valuative criterion of properness is satisfied. This is the content of the following result.

Proposition 4.2.1. Let $A$ is a discrete valuation ring and let $S = \text{Spec } A$ with generic point $\eta$. Then every SSV $\mathcal{X}_\eta$ over $\mathbb{P}(V) \times \eta$ extends to a unique SSV $\mathcal{X}$ over $\mathbb{P}(V) \times S$, possibly after a finite surjective base change $S' \to S$.

Proof. Let $s = \text{Spec}(A/\mathfrak{z}A)$ be the closed point of $S$, where $\mathfrak{z}$ is a generator of the maximal ideal of $A$; let $K = k(\eta)$ be the fraction field of $A$. Consider the section ring $\mathcal{R}_\eta$ of $\mathcal{X}_\eta$; this is a $K$-algebra, finite over $K \otimes \text{Sym}(V^*)$. The desired extension $\mathcal{X}$ corresponds to an $A$-algebra $\mathcal{R}$ of $\mathcal{R}_\eta$, finite over $A \otimes \text{Sym}(V^*)$ and such that $\mathcal{R}_s = \mathcal{R}/\mathfrak{z}\mathcal{R}$ is reduced.

First we consider the case where the geometric generic fiber $\mathcal{X}_\eta$ is a spherical variety. Then $\mathcal{R}$ is an integral domain having the same quotient field as $\mathcal{R}_\eta$. Further, since $\mathcal{R}/\mathfrak{z}\mathcal{R}$ satisfies $(R_0)$ and $(S_1)$, then $\mathcal{R}$ satisfies $(R_1)$ and $(S_2)$. Thus, $\mathcal{R}$ is normal by Serre’s criterion: so $\mathcal{R}$ is the integral closure of $A[V]$ in $\mathcal{R}_\eta$. This shows the uniqueness of the extension $\mathcal{X}$.

For the existence, we consider of course the integral closure $\mathcal{R}$ of $A[V]$ in $\mathcal{R}_\eta$. This $A$-algebra is clearly finite over $A[V]$ and flat over $A$, and hence multiplicity-free. It remains to show that $\mathcal{R}/\mathfrak{z}\mathcal{R}$ (the section ring of the special fiber) is reduced, possibly after a finite surjective base change. For this, we may assume (by Proposition 3.1.2) that

$$\mathcal{R}_\eta = K \otimes R \simeq \bigoplus_{\lambda \in \Lambda^+(\mathcal{X})} K \otimes V(\lambda),$$

where $R$ is the affine ring of an affine spherical $\tilde{G}$-variety $X$. As in the proof of Proposition 3.1.4, it follows that $\mathcal{R}$ is obtained by base change from a standard family $z : \tilde{X} \to \mathbb{A}^1$. The special fiber of this family may be non-reduced, but it becomes reduced after a finite base change $z = z' \bar{N}$, by Lemma 3.1.3.

We now show that our extension $\mathcal{X}$ is compatible with $G$-subvarieties. Let $\mathcal{Y}_\eta$ be an irreducible $G$-subvariety of $\mathcal{X}_\eta$. After a finite extension, we may assume that $\mathcal{Y}_\eta$ is obtained by base change from a $G$-subvariety $\mathcal{Y}_0$ of $\mathcal{X}_\eta$. Let $\mathcal{Y}$ be the closure of $\mathcal{Y}_0$ in $\mathcal{X}$. Then $\mathcal{Y}$ is flat over $S$ and finite over $\mathbb{P}(V) \times S$. Moreover, its special fiber $\mathcal{Y}_s$ is a closed $G$-subscheme of $\mathcal{X}_s$, and the weight set of its affine cone satisfies

$$\tilde{\Lambda}^+(\mathcal{Y}_s) = \tilde{\Lambda}^+(\mathcal{Y}_0) = \tilde{\Lambda}^+(\mathcal{Y}).$$

Thus, $\tilde{\Lambda}^+(\mathcal{Y}_s)$ is the intersection of $\tilde{\Lambda}^+(\mathcal{X})$ with a face of the corresponding weight cone. By multiplicity-freeness (as in the proof of Lemma 2.1.7), it follows that $\mathcal{Y}_s$ is reduced. So $\mathcal{Y}_s$ is reduced as well, and $\mathcal{Y}$ yields the desired extension.

Finally, we treat the general case, where $\mathcal{X}_\eta$ may have several irreducible components. After a finite extension, we may assume that all the irreducible
G-subvarieties of $X_\eta$ are defined over $K$. By Propositions 2.1.3 and 2.2.2, this yields affine spherical $\tilde{G}$-varieties $\tilde{Y}_\eta$ such that $\tilde{X}_\eta = \varprojlim \tilde{Y}_\eta$. By the first step of the proof, after a further finite base change, each $\tilde{Y}_\eta$ extends uniquely to a family $\tilde{Y}$, finite over $V \times S$. Since these extensions are compatible with $\tilde{G}$-subvarieties, they form a directed system of SSV’s. The direct limit of this system is the desired extension. □

Next we consider the moduli scheme

$$M := M_{\Gamma,P(V),Q},$$

where $Q$ is assumed to be convex. Then any SSV $(X, f)$ of type $Q$ is equidimensional, as follows, e.g., from Proposition 2.2.3 (vii). Clearly, the associated cycle $f^*[X]$ in $P(V)$ depends only on the isomorphism class of $(X, f)$, and its dimension and degree depend only on the combinatorial data $\Gamma, Q$. This defines a cycle map from the underlying set of $M$ to some Chow variety, Chow $P(V)$ (see [Ko96, Chapter 1] for background on Chow varieties and related issues of semi-normality).

We now show that this cycle map, properly defined, is a finite morphism. (This generalizes a result of [AB04b, Section 4.4] regarding stable reductive varieties, but the exposition there is inaccurate.)

**Proposition 4.2.2.** With the preceding notation, the cycle map extends to a finite morphism $\gamma : M^{sn} \to \text{Chow } P(V)$, where $M^{sn}$ denotes the semi-normalization of $M$.

**Proof.** To show the existence of $\gamma$, we construct a kind of “universal family” of cycles over $M^{sn}$. For this, we follow the notation as in the proof of Theorem 1.1. The subgroup $G_m = G_m \times \{1\}$ of $\tilde{G} = G_m \times G$ acts linearly on $A(V')$ with positive weights. Let $P(V')$ be the associated weighted projective space, which may differ from the projectivization of $V'$, but contains the usual projective space $P(V)$. Then $X' = \text{Proj}_{G_m} \mathcal{O}'$ is a $G$-subscheme of $P(V') \times H'$, finite over $P(V) \times H'$ via the projection $P(V') - \to P(V)$. The corresponding diagonal embedding $X' \hookrightarrow P(V') \times P(V) \times H'$ is $A$-equivariant. Thus, it yields an $A$-morphism

$$H' \to \text{Hilb}(P(V') \times P(V)).$$

Together with [Ko96, Theorem 6.3, Proposition 7.2.3], this yields in turn an $A$-morphism

$$H'^{sn} \to \text{Chow}(P(V') \times P(V))$$

which sends any point of $H'^{sn}$ to the associated cycle of the corresponding subscheme of $P(V') \times P(V)$. Further, the projection $P(V') \times P(V) \to P(V)$ defines a morphism $\text{Chow}(P(V') \times P(V)) \to \text{Chow } P(V)$, by [Ko96, Theorem 6.8]. So we obtain an $A$-invariant morphism $H'^{sn} \to \text{Chow } P(V)$, that is, a morphism

$$H'^{sn}/A \to \text{Chow } P(V).$$
But $H^m/A$ is semi-normal (e.g., by \([\text{AB04a}, \text{Lemma 2.1}])$, and hence isomorphic to $H^m$. This yields the desired morphism $\gamma$.

To show that $\gamma$ is finite, it suffices to check that the cycle map has finite fibers, as $M$ is proper. Consider a PSSV $(X, f)$ of type $Q$ over $\mathbb{P}(V)$. Then $f(X)$ is uniquely determined by the cycle $\gamma(X)$. By Lemma \([2.3.4])$ it follows that there are only finitely many possibilities for $(X, f)$.

4.3. \textbf{Group actions.} The group $GL(V)^G$ acts on the scheme $M_{\Gamma,F(V)}$ via the natural action of its quotient $Aut^G \mathbb{P}(V)$. Recall the isomorphisms $V \simeq \bigoplus_{\lambda \in F} E(\lambda) \otimes V(\lambda)$ and $GL(V)^G \simeq \prod_{\lambda \in F} GL(E(\lambda))$. In particular, if the $G$-module $V$ is multiplicity-free, then

$$V \simeq \bigoplus_{\lambda \in F} V(\lambda) := V_F,$$

and $GL(V)^G$ is the torus $G_m^F$ (product of copies of $G_m$ indexed by $F$). We then put

$$M_{\Gamma,F} := M_{\Gamma,P(V)}.$$

We now describe the isotropy group $Stab_{G_m^F}(\xi)$ of a closed point $\xi \in M_{\Gamma,F}$. Let $(X, f)$ be a representative of $\xi$. The $G$-module spanned by the image of the associated morphism $\tilde{f} : \tilde{X} \to V_F$ depends only on $\xi$; let $F(\xi) \subseteq F$ be its weight set. Then one easily checks that

$$Stab_{G_m^F}(\xi) \simeq G_m^F/F(\xi) \times Aut^G(X,L)/Aut^G(X,f),$$

where $L = f^*\mathcal{O}(1)$.

Returning to an arbitrary $G$-module $V$ with weight set $F$, we will establish a bijective correspondence between $GL(V)^G$-orbits in $M_{\Gamma,F(V)}$ and $G_m^{F(\xi)}$-orbits in $M_{\Gamma,F}$. For this, choose lines $\ell(\lambda)$ in $E(\lambda)$ for all $\lambda \in F$. This yields an injective $G$-module map $V_F \hookrightarrow V$ and an injective homomorphism $G_m^{F} \hookrightarrow GL(V)^G$, that we regard both as inclusions. We also regard $M_{\Gamma,F}$ as a closed subscheme of $M_{\Gamma,F(V)}$; this subscheme is invariant under the subgroup $P_F$ of $GL(V)^G$ that stabilizes all the lines $\ell(\lambda)$. Note that $P_F$ is a parabolic subgroup of $GL(V)^G$ containing $G_m^F$ as a direct factor, and

$$GL(V)^G/P_F \simeq \prod_{\lambda \in F} \mathbb{P}(E(\lambda)).$$

Consider again a closed point $\xi \in M_{\Gamma,P(V)}$ with representative $(X, f)$, and the associated morphism $\tilde{f} : \tilde{X} \to V$. Since $\tilde{X}$ is multiplicity-free and the image $\tilde{f}(\tilde{X})$ only depends on $\xi$, the span of this image decomposes uniquely as

$$\bigoplus_{\lambda \in F(\xi)} \ell(\lambda, \xi) \otimes V(\lambda),$$

where every $\ell(\lambda, \xi)$ is a line in $E(\lambda)$, and $F(\xi)$ is a subset of $F$. In fact, $F(\xi)$ only depends on the $GL(V)^G$-orbit of $\xi$. This implies readily the following statement:
Lemma 4.3.1. Let $\Omega$ be a $\text{GL}(V)^G$-orbit in $M_{\Gamma,F}(V)$. Then $\Omega$ meets $M_{\Gamma,F}$ along a unique $\mathbb{G}^F_m$-orbit, $\Omega_F$. Moreover, $\Omega$ is a homogeneous bundle with fiber $\Omega_F$ over $\prod_{\lambda \in F(\xi)} \mathbb{P}(E(\lambda))$, where $\xi$ is any point of $\Omega_F$.

By Theorem 1.3.3, $M_{\Gamma,F}(V)$ contains only finitely many orbits of spherical varieties over $\mathbb{P}(V)$. Let $M_{\Gamma,F}^{\text{main}}(V)$ (the main part of $M_{\Gamma,F}(V)$) denote the union of the closures of these orbits in $M_{\Gamma,F}(V)$ and define $M_{\Gamma,F}^{\text{main}}$ similarly. Then $M_{\Gamma,F}^{\text{main}}(V) = \text{GL}(V)^G M_{\Gamma,F}^{\text{main}}$, by Lemma 4.3.1. Since any orbit closure of a torus contains only finitely many orbits, this yields:

Theorem 4.3.2. The main part $M_{\Gamma,F}^{\text{main}}(V)$ contains only finitely many orbits of $\text{Aut}^G \mathbb{P}(V)$.

Moreover, by Theorem 1.3.4, only finitely many subgroups of $\hat{A}$ arise as weight groups of spherical varieties over $\mathbb{P}(V)$. It follows that there are only finitely many stable limits of such spherical varieties, up to isomorphism and action of $\text{Aut}^G \mathbb{P}(V)$.

In contrast, the full moduli space $M_{\Gamma,F}(V)$ may contain infinitely many orbits of $\text{Aut}^G \mathbb{P}(V)$, as shown by Example 2.4(1). However, the number of closed orbits is always finite. Indeed, by Lemma 4.3.1 it suffices to check that $M_{T,F}$ contains only finitely many fixed points of $\mathbb{G}^F_m$. Consider such a fixed point $\xi$ with representative $f : X \to \mathbb{P}(V_F)$. Each irreducible component of $f(X)$ is a multiplicity-free subvariety of $\mathbb{P}(V_F)$, invariant under $\mathbb{G}^F_m$. By [A05, Corollary 3.3], there are only finitely many such varieties. So the same holds for $(X,f)$, by Lemma 2.3.4.

Rather than considering the orbit structure of $M_{\Gamma,F}(V)$, we will introduce a coarser stratification, still invariant under $\text{Aut}^G \mathbb{P}(V)$. For this, we first classify the SSV’s $(X,f)$ over $\mathbb{P}(V)$ having the same building blocks: the irreducible $G$-subvarieties $Y$, their line bundles $L_Y$, and their weight sets $F_Y$.

Lemma 4.3.3. The set of SSV’s $f : X \to \mathbb{P}(V)$ with the given building blocks is fibered over a product of projective spaces $\mathbb{P}(E(\lambda))$, into principal homogeneous spaces under a diagonalizable group. Moreover, the automorphism group of $f : X \to \mathbb{P}(V)$ is also diagonalizable.

The groups, which we construct explicitly in the proof, are analogs of the groups $H^i(\check{M}^*)$, $i = 0,1$ of [A02]. In the case where $G$ is a torus, $\check{M}^*$ is a complex of tori and hence its cohomology groups are diagonalizable groups. For an arbitrary $G$ and multiplicity-free $V$, the groups $\check{M}^*$ are diagonalizable, so that the $H^i(\check{M}^*)$ are diagonalizable as well. In the general case where $V$ is arbitrary, we get a fibration into diagonalizable groups over a product of projective spaces, and this bears close resemblance to the semiabelian case of [A02].

Proof. Let $Y_i$ denote the irreducible components of $X$, $Y_{ij} = Y_i \cap Y_j$ the double intersections, $Y_{ijk}$ the triple intersections, etc. For each $(Y,L_Y)$ we
have the group $\text{Aut}^G(Y, L_Y)$, which is diagonalizable by Lemma 1.2.3 acting on the set $\hat{\text{Fun}}(Y, L_Y, F_Y)$ of morphisms of polarized varieties $(Y, L_Y) → (\mathbb{P}(V), O(1))$ such that the image of $V^* → H^0(Y, L_Y)$ has weight set $F_Y$.

We first consider the case where $V = V_F$ is multiplicity-free. Then each $\hat{\text{Fun}}(Y, L_Y, F_Y) = G^F_{m_Y}$ is also a group and the action is described by a homomorphism $\phi_{Y,L} : \text{Aut}^G(Y, L) → \hat{\text{Fun}}(Y, L_Y, F_Y)$ so that $a.f = \phi_i(a)f$.

Let $\hat{\text{M}}^*$ denote the cone of the homomorphism

$$C^*(\phi) : C^*(\text{Aut}) → C^*(\hat{\text{Fun}}).$$

Explicitly:

$$\hat{\text{M}}^0 = C^0(\text{Aut}) = \oplus_i \text{Aut}^G(Y_i, L_i),$$

$$\hat{\text{M}}^1 = C^1(\text{Aut}) \oplus C^0(\hat{\text{Fun}}) = \oplus_{i<j} \text{Aut}^G(Y_{ij}, L_{ij}) \oplus \hat{\text{Fun}}(Y_i, L_i, F_i),$$

$$\hat{\text{M}}^2 = C^2(\text{Aut}) \oplus C^1(\hat{\text{Fun}}) = \oplus_{i<j<k} \text{Aut}^G(Y_{ijk}, L_{ijk}) \oplus \hat{\text{Fun}}(Y_{ijk}, L_{ijk}, F_{ijk})$$

and the differentials $d^i : \hat{\text{M}}^i → \hat{\text{M}}^{i+1}$ are of the form $(d^i_{\text{Aut}} \times \phi_{ij}^{-1}, d^i_{\hat{\text{Fun}}}^{-1}).$

Fix one variety $f : X → \mathbb{P}(V)$ with the given building blocks. Then any other variety $(X', f')$ over $\mathbb{P}(V)$ with the same blocks differs from $(X, f)$ by an element of $Z^1(\hat{\text{M}}) = \ker(\hat{\text{M}}^1 → \hat{\text{M}}^2)$. Indeed, $Z^1(\hat{\text{M}})$ describes all other ways to glue the $(Y_i, L_i)$ together compatible on the triple intersections, and the maps $f_i$ twisted correspondingly. The group $B^1(\hat{\text{M}}) = \text{im}(\hat{\text{M}}^0 → \hat{\text{M}}^1)$ describes the effect of changing each $(Y_i, L_i)$ by an automorphism on the gluing and maps $f_i$.

Hence, $X'$ differs from $X$ by an element of $H^1(\hat{\text{M}})$, and the automorphism group is $H^0(\hat{\text{M}})$. Note that all $\hat{\text{M}}^i$ are diagonalizable groups and so are the cohomology groups.

Now, consider the general case. Then the isomorphism classes of PSSV’s over $\mathbb{P}(V)$ are still classified by the first cohomology set $H^1(\hat{\text{M}})$ of a complex of sets $\hat{\text{M}}^*$ properly understood: as the collection of pairs $(a_{ij}, f_i)$ coinciding on intersections, i.e., $(a_{ij})$ is a cocycle and

$$a_{ij}.f_i|_{X_{ij}} = f_j|_{X_{ij}},$$

modulo the action of collections $(a_i)$ on $(a_{ij}, f_i)$. The automorphism group is still $H^0(\hat{\text{M}})$.

Any map $f : X → \mathbb{P}(V)$ defines a surjective homomorphism

$$V^* = \bigoplus_{\lambda \in \mathcal{E}} E(\lambda)^* \otimes V(\lambda)^* → H^0(X, L)_{F(X)} = \bigoplus_{\lambda \in F(X)} V(\lambda)$$

which is a point of $\prod_{\lambda \in F(X)} \mathbb{P}(E(\lambda))$. If one point appears for some $X$ then so do all others. And for a fixed point $P$, the first cohomology set is a principal homogeneous space under $H^1(\hat{\text{M}}^*)$, by the multiplicity-free case. So we are done.
Theorem 4.3.4. The moduli space $M_{\Gamma,P(V)}$ has a natural stratification by locally closed subsets, invariant under $\text{Aut}^G P(V)$. Each subset is fibered over a product of projective spaces, with fibers being principal homogeneous spaces over a diagonalizable group.

In particular, each irreducible component of $M_{\Gamma,P(V)}$ is a rational variety.

Proof. Let $A$ be the set of possible moment sets $Q$, subdivisions $Q$ of $Q$ into moment polytopes of spherical varieties, and weight sets $F'$. Then $A$ is a finite set and by Remark 4.1.2(2) the corresponding strata $M_{\alpha}, \alpha \in A$, are locally closed subschemes of $M_{\Gamma,P(V)}$.

Next, consider a finer stratification in which, in addition to the above data, the isomorphism classes of the irreducible components $(Y_i, L_i)$ are fixed as well. We claim that these finer strata are locally closed subsets. Indeed, let $f \times \pi : (\mathcal{X}, \mathcal{L}) \to P(V) \times S$ be a family of SSV’s over $P(V)$ with base $S$, and consider the sheaf of $O_S$-algebras $\mathcal{R} = \mathcal{R}(\mathcal{X}, \mathcal{L})$. The identities $m_{\tilde{\lambda}, \tilde{\mu}} = 0$ in the definition of $M_{\alpha}$ imply that for each polytope $Q_i \in Q$, sending to zero the $\tilde{\lambda}$-components outside the cone over $Q_i$ gives a quotient algebra $\mathcal{R}_i$ of $\mathcal{R}$, and hence a closed subfamily of PSV’s $(Y_i, L_i)$.

The subsets of $S$ where the fibers $(Y_i, L_i)$ are isomorphic are locally closed, since they correspond to unions of orbits in the moduli space of spherical subvarieties of an affine space (cf. the proof of Theorem 1.3.4). Finally, Lemma 4.3.3 gives the stated structure of these locally closed subsets. □

Remark 4.3.5. The image of the cycle map $\gamma : M^{sn} \to \text{Chow} P(V)$ is contained in the multiplicity-free part $\text{Chow}^{mf} P(V)$, consisting of positive combinations of irreducible multiplicity-free $G$-subvarieties. This closed subset of $\text{Chow} P(V)$ has a natural, but easier, stratification according to the cycle decomposition and the isomorphism classes of irreducible components. In homological language, each stratum corresponds to the global sections of a sheaf describing the components. The map $\gamma$ is compatible with the stratifications. But in general it is neither injective nor surjective.

In the case where $G$ is a torus, our cycle map is a version of the Chow morphism from the toric Hilbert scheme to the toric Chow variety, studied in [HS05, Section 5].

Finally, we briefly discuss the moduli space of stable spherical pairs of type $Q$, i.e., $M_{\Gamma, Q} := M_{\Gamma,P(V_Q), Q}$, where $V_Q = \bigoplus_{\lambda \in \Gamma \cap Q} V(\lambda)^* \otimes V(\lambda)$. Then $E(\lambda) = V(\lambda)^*$, so that a natural choice for the line $\ell(\lambda)$ is the highest weight line in $V(\lambda)^*$. The corresponding subspace $M_{\Gamma, F}$ of $M_{\Gamma, Q}$ consists of the isomorphism classes of those $(X, D)$ such that the divisor $D$ is stable under the action of $U$.

4.4. Examples. We reconsider the examples discussed in Section 2.4, from the viewpoint of moduli spaces.

1) By Proposition 3.3.2, the moduli space of stable toric pairs of type $Q$ is $M_{\Gamma, F}$, where $F = -\Gamma \cap Q$. This space is described in [Al02, Section 2]. Its
main part $M_{\Gamma,P}^{\text{main}}$ is irreducible. The normalization of its image under the cycle map $\gamma$ of Section 4.2 is the toric variety associated with the secondary polytope of the pair $(Q, \Gamma \cap Q)$, see [GKZ94, Chapter 8].

More generally, one easily checks that $M_{\Gamma,P(V),Q}^{\text{main}}$ is irreducible, if $G = T$ is a torus. This space parametrizes the normalizations of those $T$-orbit closures in $P(V)$ having moment polytope $Q$, and their limits as stable toric varieties over $P(V)$.

2) The moduli space of stable reductive pairs is described in [AB04b, 4.5]. The main tool is a bijective correspondence between stable reductive varieties and stable toric varieties with a compatible action of $W$, see [loc.cit., Theorem 2.8]. For stable reductive varieties, one checks that the main part of each $M_{\Gamma,P(V),Q}^{\text{main}}$ is irreducible.

3) Likewise, if $G = \text{SL}(2)$ then each $M_{\Gamma,P(V),Q}^{\text{main}}$ turns out to be irreducible.

4) Let $G$, $\Gamma$, $V$, $Q$ and $(X,f)$ be as in Example 2.4(4) and let $\xi$ be the corresponding point of the space $M_{\Gamma,P(V),Q}$. Then the connected component of $\xi$ in this space does not meet the main part (parametrizing the spherical varieties and their stable limits). This yields an example of a connected component of our moduli space that contains no irreducible variety. By the results of the present section, this still holds if $V = V(2,0) \oplus V(4,2)$ is replaced with a direct sum of any number of copies of $V(2,0)$, $V(4,2)$.

5) In the case where $V = V(\lambda)$ is a simple $G$-module, our moduli space $M_{\Gamma,P(V)}$ consists of finitely many points, by Example 2.4(5). Thus, if $V = E(\lambda) \otimes V(\lambda)$ is an isotypical $G$-module, then $M_{\Gamma,P(V)}$ consists of finitely many $GL(E(\lambda))$-orbits, all of them being closed.

5. Generalizations

5.1. Split tori. In the case when $G = G_m^r$ is a split torus, the existence of a projective moduli space and most of the other constructions of this paper can be extended to an arbitrary base scheme, including Spec $\mathbb{Z}$. We point out the changes that have to be made:

For the construction of the moduli space $M_{\Gamma,P(V)}$ we used the $\tilde{G}$-Hilbert scheme $\text{Hilb}\tilde{G}$ constructed in [AB05] over an algebraically closed field of characteristic zero. This has to be replaced with the multigraded Hilbert scheme, which was constructed in [HS05] in full generality over any base. (Similarly to the classical Grothendieck Hilbert scheme, even the Noetherian condition may be removed but then one has to work with locally free instead of flat families).

Quite generally, an action of a split torus $G_m^r$, resp. a diagonalizable group $G$, on an affine scheme Spec $R$ is the same as a grading of $R$ by $\mathbb{Z}^r$, resp. the dual group $\hat{G}$, a finitely generated abelian group, see f.e. [SGA3]. The arguments in the proof of Theorem 4.1.1 are statements about graded algebras, therefore they apply over a general base.
In particular, the scheme $S'$ obtained by extracting the $N$-th root of a section $s \in \Gamma(S, F^N)$ is $\text{Spec}_{O_S} A$, where $A$ is the graded algebra $\bigoplus_{i=0}^{N-1} F^{-i}$ with the multiplication given by $s \in \text{Hom}(F^{-N}, O_S) = \text{Hom}(O_S, \mathcal{F}^N)$, and $S$ is the quotient of $S'$ by the split diagonalizable group $\mu_N$. Similarly, the group $A$ appearing in the proof is a direct product of several copies of $\mu_N$.

5.2. Non-algebraically closed fields. We chose to work over an algebraically closed field of characteristic zero, to ease understanding and to avoid introducing cumbersome notation. However, all the results and arguments may be adapted readily to an arbitrary field $k$ of characteristic zero, and to a split reductive group $G$.

Moreover, the main statements 4.1.1, 4.2.1 about existence of projective coarse space hold for a nonsplit group $G$ as well. A family $X \rightarrow S$ of, say affine, $G$-varieties is by definition multiplicity-free for our purposes, if all the geometric fibers $X_s$ are multiplicity-free. For constructing the moduli space, one can work on an étale cover $\bigcup S_i$ of $S$, and there exist such covers with $G \times S S_i$ split over $S_i$. Similarly, the main theorem is easily generalized to the case of non-split tori over $Z$.

In the non-split setting, we note that the statements about stratifications of the moduli space have to significantly modified, as some strata glue together. Also, rationality Theorem 4.3.4 no longer holds.

5.3. Stable spherical varieties over a closed subscheme of $\mathbb{P}(V)$. We still fix a subgroup $\Gamma$ of $\tilde{\Lambda}$, and a finite-dimensional $G$-module $V$. Let $Z$ be a $G$-invariant closed subscheme of $\mathbb{P}(V)$. (We could consider an arbitrary closed subscheme $Z$ but this would not add any flexibility, since it can be replaced by its unique maximal $G$-invariant closed subscheme.)

**Lemma 5.3.1.** The subfunctor $\mathcal{M}_{\Gamma,Z}$ of stable spherical varieties $f : X \rightarrow \mathbb{P}(V)$ with weights in $\Gamma$ that factor through $Z$ is a closed subfunctor of $\mathcal{M}_{\Gamma, \mathbb{P}(V)}$.

**Proof.** Let $I(Z) \subset \text{Sym}(V^*)$ be the homogeneous ideal of $Z$. Then $Z$ is the zero subscheme of the homogeneous component $I(Z)_n$ for some $n$ (for example, for any $n \gg 0$). Choose such an $n$ and consider a family $f \times \pi : X \rightarrow \mathbb{P}(V) \times S$; put $\mathcal{L} := \pi^* \mathcal{O}(1)$, so that $\pi_*(\mathcal{L}^n)$ is a locally free $O_S$-module by Lemma 2.2.3 vi. The map $I(Z)_n \rightarrow H^0(X, \mathcal{L}^n)$ yields a morphism of $O_S$-modules $\varphi : O_S \otimes I(Z)_n \rightarrow \pi_*(\mathcal{L}^n)$. The zero subscheme of $\varphi$ is the locus of $S$ where our family factors through $Z$. This is a closed subscheme $S_Z \subseteq S$, whose formation commutes with base change (since this holds for $\pi_*(\mathcal{L}^n)$, see [Har77, Proposition III.9.3]).

**Corollary 5.3.2.** The functor $\mathcal{M}_{\Gamma,Z}$ has a coarse moduli space which is a projective scheme.

Next we obtain an interpretation of the main part of the moduli space $\mathcal{M}_{\Gamma,Z}$, in the case of a torus $T$. Then $Z$ is the disjoint union of strata $Z_S$, where $S$ denotes a finite subset of $\Lambda$, and $Z_S$ consists of those points
Further, \( \frac{Z}{\mathbb{T}} \) is isomorphic to \( \frac{\mathbb{M}}{\mathbb{Q}} = \text{Conv} \). Proof. The polarized toric variety associated with any point of \( \frac{Z}{\mathbb{T}} \) is locally closed and \( \mathbb{T} \)-invariant, and the \( \mathbb{T} \)-action on \( \mathbb{M} \) factors through a free action of a quotient torus \( \mathbb{T}_S \). This stratification is a refinement of the one by thin Schubert-like cells considered in [Hu95].

We now make a very simple observation. A point of the quotient \( \frac{Z}{\mathbb{T}} = \frac{Z}{\mathbb{T}_S} \) is the same as the image of a \( \mathbb{T} \)-morphism \( f : \mathbb{T}_S \to \mathbb{Z} \), and the same as the closure \( \overline{f(\mathbb{T}_S)} \subseteq \mathbb{Z} \), which is a possibly non-normal toric variety. Further, \( f \) can be extended in a unique way to a finite birational morphism \( f : \mathbb{X} \to \mathbb{Z} \) from the polarized \( \mathbb{T}_S \)-toric variety \( (\mathbb{X}, \mathbb{L}) \) with moment polytope \( Q = \text{Conv} \mathbb{S} \). Hence, a point of \( \frac{Z}{\mathbb{T}_S} \) gives a point of the moduli space \( \mathbb{M} = M_{\mathbb{A}, \mathbb{Z}, \mathbb{S}} \), where \( \mathbb{A} \) is the character group of \( \mathbb{T}_S \), i.e., the subgroup of \( \mathbb{A} \) generated by differences of elements of \( \mathbb{S} \). This observation may be refined as follows:

**Lemma 5.3.3.** The open stratum \( \mathbb{M}_{\text{irr}} \) of \( \mathbb{M} \) parametrizing irreducible varieties is isomorphic to \( \mathbb{Z}/\mathbb{T}_S \). Thus, the main part of \( \mathbb{M} \) is a compactification of \( \mathbb{Z}/\mathbb{T}_S \).

Proof. The polarized toric variety associated with any point of \( \mathbb{M}_{\text{irr}} \) is \( (\mathbb{X}, \mathbb{L}) \), and the corresponding finite map to \( \mathbb{Z} \) restricts to an immersion on the open orbit \( \mathbb{X}_0 \simeq \mathbb{T}_S \). Thus, each automorphism group \( \text{Aut}^\mathbb{T}(\mathbb{X}, f) \) is trivial, and \( \mathbb{M}_{\text{irr}} \) is a fine moduli space: it admits a universal family \( \mathbb{X} \to \mathbb{M}_{\text{irr}} \). Let \( \mathbb{X}_0 \) be the union of all \( \mathbb{T}_S \)-orbits of maximal dimension. Then we have a morphism \( \mathbb{X}_0 \to \mathbb{Z} \) and therefore a morphism \( \mathbb{M}_{\text{irr}} = \mathbb{X}_0/\mathbb{T}_S \to \mathbb{Z}/\mathbb{T}_S \).

We now construct the inverse of this morphism. Consider the family of tori \( \mathbb{Z}_S \to \mathbb{Z}/\mathbb{T}_S \); it can be completed to a family of toric varieties in a unique way as

\[
\pi : \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X} \to \mathbb{Z}/\mathbb{T}_S,
\]

where \( \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X} \) denotes the quotient of \( \mathbb{Z}_S \times \mathbb{X} \) by the diagonal \( \mathbb{T}_S \)-action. We claim that there exists a \( \mathbb{T} \)-morphism

\[
f : \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X} \to \mathbb{Z}
\]

such that the product map \( \pi \times f : \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X} \to \mathbb{Z}/\mathbb{T}_S \times \mathbb{Z} \) is finite, and the restriction of \( f \) to \( \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X}_0 \simeq \mathbb{Z}_S \) is the inclusion of \( \mathbb{Z}_S \) into \( \mathbb{Z} \).

To check this, consider the invertible sheaf \( \mathcal{O}(1) \otimes \mathbb{L} \) on \( \mathbb{Z}_S \times \mathbb{X} \). This sheaf is linearized for the diagonal action of \( \mathbb{T}_S \), and hence descends to an invertible sheaf \( \mathcal{L} \) on \( \mathbb{Z}_S \times \mathbb{X} \). Moreover, for any \( \lambda \in \mathbb{S} \), we have a space of global sections \( V^*_\lambda \) of \( \mathcal{O}(1) \), and a line of global sections \( k_\lambda \) of \( \mathcal{L} \), which are eigenspaces of \( \mathbb{T}_S \) of opposite weights. Thus, we obtain a space of global sections \( V^*_\lambda \otimes k_\lambda \) of \( \mathcal{L} \). By the definition of \( \mathbb{Z}_S \), the direct sum of these subspaces (over all \( \lambda \in \mathbb{S} \)) is base-point-free and hence yields a morphism

\[
f : \mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X} \to \mathbb{P}
\left( \bigoplus_{\lambda \in \mathbb{S}} V^*_\lambda \otimes k_\lambda \right) \subseteq \mathbb{P}(\mathbb{V}).
\]

One easily checks that the map \( \pi \times f \) is finite, and \( f(\mathbb{Z}_S \times_{\mathbb{T}_S} \mathbb{X}_0) \subseteq \mathbb{Z}_S \). Thus, the image of \( f \) is contained in \( \mathbb{Z} \); this completes the proof of the claim.
By this claim, we obtain a family of $T_S$-toric varieties over $\mathbb{Z}$ with base $Z_S/T_S$, and hence a morphism $Z_S/T_S \to M^{irr}$ which is the desired inverse. □

Example 5.3.4. As an application, we give another view to the compactified spaces constructed by Lafforgue in [Laf03]. In this example, one takes $Z$ to be the grassmannian $Gr$ embedded into a projective space $\mathbb{P}(V)$ by Plücker coordinates. Special to this case is the fact that a $T$-orbit in $Gr$ corresponds to a very particular “matroid” polytope $Q$ (called in [Laf03] “entier”), and the weight set $S$ is the full set of lattice points in $Q$. (These polytopes have a number of nice properties, for example their integral points generate the lattices they lie in.) Thus, the compactification of $Z_S/T$ lies already in the open part of the moduli space of stable toric varieties over $\mathbb{P}(V)$ corresponding to subdivisions of $(Q,S)$ into matroid polytopes.

Now let us explain this in more detail. Fix $r \in \mathbb{N}$ and a free graded $\mathbb{Z}$-module $E = E_0 \oplus E_1 \oplus \cdots \oplus E_n$. The grassmanian $Gr = Gr^{r,E}$ parameterizing locally free quotients of $E$ of corank $r$ is a smooth projective scheme over $\mathbb{Z}$. It is contained in $\mathbb{P}(V)$, where $V = \Lambda^r E = \bigoplus_{i=(i_0,\ldots,i_n), \sum i_\alpha = r} \Lambda^{i_\alpha} E_\alpha$. Further, $Gr$ has a stratification by locally closed strata, usually called thin Schubert cells

$$Gr_\Delta = Gr^{r,E}_\Delta = \{ L \subseteq E \mid \dim(L \cap E_I) = d_I \}$$

for some sets of integers $\Delta = (d_I)$ labeled by subsets $I$ of $\{0,1,\ldots,n\}$ satisfying in particular $d_\emptyset = 0$, $d_{\{0,1,\ldots,n\}} = r$ and $d_I + d_J \leq d_{I \cup J} + d_{I \cap J}$.

Let $T = G_m^{n+1}$, acting on $E$ in the obvious way, so that the quotient $T = T/\text{diag}(G_m)$ acts on $Gr$ and leaves each $Gr_\Delta$ invariant. Note here that if $E$ is multiplicity-free, i.e. all rank $E_\alpha = 1$, then so is $V$. Let $S^{r,E}$ be the weight set of the action of $T$ on $V$, i.e.,

$$S^{r,E} = \{(i_0,\ldots,i_n) \in \mathbb{Z}^{n+1} \mid 0 \leq i_\alpha \leq \text{rank} E_\alpha, \sum i_\alpha = r \}$$

and $S^{r,E}_\mathbb{R} = \text{Conv} S^{r,E}$ be the corresponding polytope. Further, let

$$S^{r,E}_d = \{ (i_0,\ldots,i_n) \in S^{r,E} \mid \sum_{\alpha \in I} i_\alpha \geq d_I \}, \quad S^{r,E}_d\mathbb{R} = \text{Conv}(S^{r,E}_d)$$

We put $S = S^{r,E}_\mathbb{R}$ and $Q = S^{r,E}_d\mathbb{R}$ for simplicity. By [Laf03 Prop.1.1,1.5], $S$ is the common weight set of all elements of $Gr_\Delta \subseteq \mathbb{P}(V)$, and $S = Q \cap \mathbb{Z}^{n+1}$. Thus $Gr_\Delta = Gr_S$ with the preceding notation. Let

$$\overline{Gr}_\Delta = \overline{Gr}^{r,E}_\Delta = Gr^{r,E}_\Delta /T.$$

The main aim of [Laf03] is the construction of compactifications $\overline{Gr}^{r,E}_\Delta$ of the schemes $\overline{Gr}^{r,E}_\Delta$. The case where all rank $E_\alpha = 1$ and with generic $d$
was considered by Kapranov [Ka93] who used the Chow variety. Another paper [HKT05] on the multiplicity-free case uses the toric Hilbert scheme and gives additional moduli interpretation for the compactification; see also [Al04]. We will compare Lafforgue’s compactifications with those obtained from Lemma 5.3.3 that we denote by $M_{\mu,Gr}$.

**Lemma 5.3.5.** There exists a finite morphism $\Omega_{r,E} \rightarrow M_{\mu,Gr}$ which restricts to the identity on $Gr_{r,E}$. 

**Proof.** The construction of $\Omega_{r,E}$ is as follows. One first constructs a projective morphism of toric varieties $\tilde{A}_S \rightarrow A_S$, which is flat with geometrically reduced fibers, such that the generic fiber is a toric variety for the polytope $Q$: see [Laf03, Prop.4.3(i)]. It immediately follows that this is a family of stable toric varieties.

Recall that the secondary polytope $\Sigma(Q,S)$ is a lattice polytope whose faces are in bijection with convex (same as coherent, or regular) subdivisions of $Q$ with vertices in $S$, see [GKZ94]. Let us denote by $B_S$ the projective toric variety corresponding to $\Sigma(Q,S)$. One observes that $A_S$ is just the open subset of $B_S$ corresponding to some special convex subdivisions of $(Q,S)$, into matroid polytopes $(Q_i,S_i)$ such that $S = \cup S_i$.

Next, one constructs:

1. ([Laf03 Thm.2.4]) A certain scheme, call it $A_{S,E}$, with a morphism $A_{S,E} \rightarrow \prod_{i \in S} \mathbb{P}(\Lambda E_i)$ such that every fiber is isomorphic to $A^S$.

2. ([Laf03 Sec.4.3]) A family $\pi: \tilde{A}_{S,E} \rightarrow A_{S,E}$ which is isomorphic to $A^S \rightarrow A^S$, fiber-wise over $\prod_{i \in S} \mathbb{P}(\Lambda E_i)$, and a morphism $\tilde{A}_{S,E} \rightarrow \mathbb{P}(V)$ giving a finite morphism $\tilde{A}_{S,E} \rightarrow A_{S,E} \times \mathbb{P}(V)$.

3. The closed subscheme $\Omega_{r,E}$ of $A_{S,E}$ corresponding to the subfamily mapping to the grassmanian $Gr \subset \mathbb{P}(V)$, same as in our Lemma 5.3.1.

Hence, we have a family of stable toric varieties over $\mathbb{P}(V)$ parameterized by $A_{S,E}$ and a family over $Gr$ parameterized by $\Omega_{r,E}$, obtained from the previous one by a base change. This gives classifying morphisms $f_1: A_{S,E} \rightarrow M_{\mu,\mathbb{P}(V)}$ and $f_2: \Omega_{r,E} \rightarrow M_{\mu,Gr}$. To prove that $f_2$ is finite it is sufficient to show that $f_1$ is finite to an open subscheme of $M_{\mu,\mathbb{P}(V)}$.

Let $M^0$ be the open part of $M_{\mu,\mathbb{P}(V)}$ parameterizing varieties over $\mathbb{P}(V)$ with the full weight set $S$. By the mentioned fact about moment polytopes of $T$-orbits in $Gr$, the image of $f_2$ lies in $M^0$. In Section 4.3 we showed that $M^0$ has a fibration over $\prod_{i \in S} \mathbb{P}(\Lambda E_i)$ similar to that above for $A_{S,E}$, and every geometric fiber is isomorphic to the moduli space $M_{\mu,\mathbb{P}(V_S)}$ for the multiplicity-free module $V_S = \bigoplus_{i \in S} \mathbb{Z}$.

Therefore, the morphism $f_1$ is finite to an open subscheme of $M_{\mu,\mathbb{P}(V)}$ if and only if the morphism from the toric variety $A^S$ to an open subscheme of $M_{\mu,\mathbb{P}(V_S)}$ is finite for the multiplicity-free module $V_S$. (Thus, we reduced
the problem to the case when \( V \) is multiplicity-free without assuming that \( E \) is.) We now prove the latter finiteness statement.

The “Chow toric variety”, or “Chow toric scheme” \( C^S \) is defined as the inverse limit of the set of toric varieties corresponding to the fibers of the polytope map from the simplex \( \sigma_S \) with vertex set \( S \) to \( Q \), see [KSZ91]. The secondary variety \( B^S \) is the normalization of the main irreducible component of \( C^S \). One also has a finite “Chow morphism” from the multigraded Hilbert scheme to \( C^S \) ([HS05, Sec.5]); and by our construction of \( M_{d,F(V_S)} \) through the multigraded Hilbert scheme, a finite morphism \( M_{d,F(V_S)} \to C^S \) (cf. also [A02, 2.11.11]). Thus, \( B^S \) is the normalization of the main components of \( C^S \) and of \( M_{d,F(V_S)} \), and the statement follows. □

We expect that \( M_{d,F(E)} \to M_{d,Gr} \) is in fact a closed embedding. Note finally that in the multiplicity-free case, the space \( M_{d,F(V)} \) is the same as the moduli space of stable toric pairs by Section 3.3. The morphism \( M_{d,Gr} \to M_{d,F(V)} \) in this case has an interpretation as the toric analog of the extended Torelli map \( \overline{M}_g \to \overline{A}_g \), see [A04].

References

[A02] V. Alexeev, Complete moduli in the presence of semiabelian group action, Ann. of Math. (2) 155 (2002), 611–708.

[A04] V. Alexeev, Compactified Jacobians and Torelli map, Publ. RIMS, Kyoto Univ. 40 (2004), 1241–1265.

[AB04a] V. Alexeev and M. Brion, Stable reductive varieties I: Affine varieties, Invent. math. 157 (2004), 227–274.

[AB04b] V. Alexeev and M. Brion, Stable reductive varieties II: projective case, Adv. Math. 184 (2004), 380–408.

[AB05] V. Alexeev and M. Brion, Moduli of affine schemes with reductive group action, J. Algebraic Geom. 14 (2005), 83–117.

[BP04] P. Bravi and G. Pezzini, Wonderful varieties of type D, Preprint (2004), arXiv:math.RT/0410472.

[BP87] M. Brion and F. Pauer, Valuations des espaces homogènes sphériques, Comment. Math. Helv. 62 (1987), 265–285.

[De90] T. Delzant, Classification des actions hamiltoniennes complètement intégrables de rang deux, Ann. Global Anal. Geom. 8 (1990), 87–112.

[EV92] H. Esnault and E. Viehweg, Lectures on vanishing theorems, DMV Seminar Band 20, Birkhäuser, 1992.

[GKZ94] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants, and multidimensional determinants, Birkhäuser, 1994.

[Gr97] F. Grosshans, Algebraic Homogeneous Spaces and Invariant Theory, Lecture Notes in Math. 1673, Springer–Verlag, 1997.

[GS05] V. Guillemin and R. Sjamaar, Convexity theorems for varieties invariant under a Borel subgroup, Preprint (2005), arXiv:math.SG/0504537.

[Ha77] R. Hartshorne, Algebraic Geometry, Graduate Texts in Math. 52, Springer–Verlag, 1977.

[HKT05] P. Hacking, S. Keel, and J. Tevelev, Compactification of the moduli space of hyperplane arrangements, Preprint (2005), arXiv:math.AG/0501227.

[HS05] M. Haiman and B. Sturmfels, Multigraded Hilbert schemes, J. Algebraic Geom. 13 (2004), 725–769.
[Hu95] Y. Hu, (W − R)-matroids and thin Schubert-like cells attached to algebraic torus actions, Proc. Amer. Math. Soc. 123 (1995), no. 9, 2607–2617.

[Ka93] M. M. Kapranov, Chow quotients of Grassmannians. I, I. M. Gelfand Seminar, Adv. Soviet Math. 16, Amer. Math. Soc., Providence, RI, 1993, 29–110.

[KSZ91] M. M. Kapranov, B. Sturmfels, and A. V. Zelevinsky, Quotients of toric varieties, Math. Ann. 290 (1991), no. 4, 643–655.

[Kn91] F. Knop, The Luna–Vust theory of spherical embeddings, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991.

[Kn93] F. Knop, Über Bewertungen, welche unter einer reduktiven Gruppe invariant sind, Math. Ann. 295 (1993), 333–363.

[Kn94] F. Knop, A Harish-Chandra homomorphism for reductive group actions, Ann. of Math. 140 (1994), 253–288.

[Kn96] F. Knop, Automorphisms, root systems, and compactifications of homogeneous varieties, J. Amer. Math. Soc. 9 (1996), 153–174.

[Ko96] J. Kollár, Rational curves on algebraic varieties, Ergeb. der Math. (3) 32, Springer–Verlag, 1996.

[Laf03] L. Lafforgue, Chirurgie des grassmanniennes, CRM Monograph Series, vol. 19, American Mathematical Society, Providence, RI, 2003.

[LaMB00] G. Laumon and L. Moret–Bailly, Champs algébriques, Ergeb. der Math. (3) 39, Springer–Verlag, 2000.

[Lu01] D. Luna, Variétés sphériques de type A, Publ. Math. Inst. Hautes Etudes Sci. 94 (2001), 161–226.

[LV83] D. Luna and Th. Vust, Plongements d’espaces homogènes, Comment. Math. Helv. 58 (1983), 186–245.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, Geometric Invariant theory, third enlarged edition, Ergeb. der Math. 34, Springer–Verlag, 1994.

[Po87] V. L. Popov, Constructions of the actions of reductive algebraic groups, Math. USSR Sbornik 58 (1987), 311–335.

[PoVi94] V. L. Popov and E. B. Vinberg, Invariant theory, in: Algebraic Geometry IV, Encyclopaedia of Mathematical Sciences 55, Springer–Verlag, 1994.

[Ray70] M. Raynaud, Faisceaux amples sur les schémas en groupes et les espaces homogènes, Lecture Notes in Math. 119, Springer–Verlag, 1970.

[SGA3] M. Demazure, A. Grothendieck, et al., SGA3. Schémas en groupes, Lecture Notes in Math. 152, Springer–Verlag, 1970.

[Sj98] R. Sjamaar, Convexity properties of the moment mapping re-examined, Adv. Math. 138 (1998), 46–91.

[Su74] H. Sumihiro, Equivariant completion, J. Math. Kyoto Univ. 14 (1974), 1–28.

[Vi95] E. Viehweg, Quasi-projective moduli for polarized manifolds, Ergeb. der Math. (3) 30, Springer–Verlag, 1995.

[Wa96] B. Wasserman, Wonderful varieties of rank two, Transform. Groups 1 (1996), 375–403.

[Wo96] C. Woodward, The classification of transversal multiplicity-free group actions, Ann. Global Anal. Geom. 14 (1996), 3–42.

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
E-mail address: valery@math.uga.edu

Institut Fourier, B. P. 74, 38402 Saint-Martin d’Hères Cedex, France
E-mail address: Michel.Brion@ujf-grenoble.fr