We use the phenomenological approach to study properties of space-time in the vicinity of the Schwarzschild black-hole singularity. Requiring finiteness of the Schwarzschild-like metrics we come to the notion of integrable singularity that is, in a sense, weaker than the conventional singularity and allows the (effective) matter to pass to the white-hole region. This leads to a possibility of generating a new universe there. Thanks to the gravitational field of the singularity, this universe is already born highly inflated ('singularity-induced inflation') before the ordinary inflation starts.

Keywords: Black holes; singularity; phenomenology.

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1. Introduction

Since General Relativity (GR) came into being, the scientific community has had various opportunities to verify that it gives a viable description of phenomena that include strong gravitational fields and relativistic velocities. Its experimental basis once consisting of the three classical GR effects (perihelion precession, deflection of light and red shift) has recently acquired one of its crucial contributions – the Cosmological Standard Model (CSM) of the visible Universe (see Ref. 1 and references therein). Extrapolating this CSM to the past leads to one of the main features of GR – singularities.

As is long known, the latter are a common place in general-relativistic solutions including the most physical ones, the above-mentioned Friedmann–Robertson–Walker universe and black holes which probably reside in central parts of many

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galaxies. Although one can think of space–times where a singularity appears 'all of a sudden', it is usually understood that some curvature invariants grow infinitely as one approaches the singular hypersurface (later on we deal with this type of singularity). However, one expects that the Einstein equations will somehow change near the singularity. For example, this change may be due to quantum effects. In cosmology this possibility was first studied in Refs. 10, 11, 12, 13 while for the case of black holes in Ref. 14.

Currently, the exact form of the quantum corrections remains unknown and the variety of suggested modifications complicates the studies. Under the circumstances it is natural to parametrize properties of gravitational field near the singularity in a way similar to how probable gravity modifications are parametrized by the notions of dark matter and dark energy in cosmology.

In this paper we phenomenologically model ultrahigh-curvature processes near the singularity by introducing a continuous effective mass distribution $m(r, t)$ such that $m(r, t) \rightarrow 0$ as $r \rightarrow 0$. Note that we do not discuss physical nature of the mass function $m(r, t)$ while we do study the consequences of the equivalent assumption that metrics potentials become finite near $r = 0$. We also assume that the mass function $m(r, t)$ has the property that it appears to be point-like at large distances.

It should be also emphasized that our assumption is milder than the usual requirement that a physically admissible space–time be completely free of singularities. As applied to the case of spherically symmetric geometries it leads to the notion of integrable singularity. It is still a singularity, but a softer one. The motivation for considering it is the following. Infinite tidal forces near singularity are that unattractive feature that makes it singularity in the physical sense (at least, this is true for the singularities that are believed to be formed in nature). From this point of view, it is quite remarkable that there may exist a singularity with infinite tidal forces only in some directions, so that they do not influence the material flow that supports the geometry. In this sense, the integrable singularity allows us to connect the interior of the black hole to a daughter universe that is born with $\mathbb{R} \times S^2$ symmetry. Although transversal tidal forces are infinite, they do not influence the flow that respects the space–time symmetry and passes the singularity to form the new universe. This situation is reminiscent of what happens in cosmological models with dust and generalized Chaplygin gas.

Also, though finite, the gravitational potential is deep, which enhances the initial volume of the new universe. Inflation as an intermediate stage in this universe is only needed to make it isotropic. The result suggests that the mechanism of generating new universes inside a black hole can work without special assumptions on finiteness of density or curvature like those used in Refs. 11, 20, 21, 22 or in bouncing models. Also note that it is crucial for this mechanism that the black-hole interior contain no static regions, so we do not consider here wormholes (see, for example, Refs. 24, 25) or solutions with an internal $R$-region.

In the next two sections we introduce the definition of integrable singularity and
analyze its properties. Then in Sect. 3 we present two toy models containing the one. The first of them gives the answer to the question of the source of the Schwarzschild geometry while the second is an example of a universe emerging from the black-hole interior. Finally, in Sect. 5 we discuss the effective-matter approach, the type of integrable singularity and the suggested scenario of generating new universes.

2. Integrable singularity

Let us consider a class of metrics that, first, have the Schwarzschild form in the region where quantum effects are negligible, second, inherit the global Killing t-vector from the vacuum solution and, third, contain finite quantities \( N(r) \) and \( \Phi(r) \):

\[
ds^2 = N^2(1 + 2\Phi) dt^2 - \frac{dr^2}{1 + 2\Phi} - r^2 d\Omega^2,
\]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) is the line element on a unit 2-sphere in angular coordinates \((\theta, \phi)\). More exactly, the functions \( N(r) \) and \( \Phi(r) \) are finite along with their two first derivatives everywhere including \( r = 0 \), and when \( r \) exceeds some characteristic positive value \( r_0 \)

\[
N(r \geq r_0) = 1, \quad \Phi(r \geq r_0) = -\frac{GM}{r},
\]

where \( M \) is the external mass of the black hole and \( G \) the gravitational constant.

The well-known coordinate divergence at the horizon \( \Phi = -1/2 \) which separates the \( R^- (\Phi > -1/2) \) and \( T^- (\Phi < -1/2) \) is treated in the usual way.\(^a\)

The surface \( r = r_0 \) is a sort of border where quantum processes start playing a significant role as we go inside the black hole. We will assume that \( M >> M_P \sim 10^{19} \text{ GeV} \) which ensures that the Hawking radiation does not affect our further consideration and that the surface \( r = r_0 \) resides under the horizon and, hence, is spacelike. Since we required the finiteness of the metric potentials even at \( r = 0 \), it is only logical that the r.h.s. of the Einstein equations \( G_{\mu\nu} = 8\pi G T_{\mu\nu} \) is non-zero somewhere in \( r < r_0 \). To model the situation we consider the effective energy-momentum tensor (EMT) that emerges in a triggered way:

\[
T_{\mu\nu} \neq 0 \quad \text{when} \quad 0 \leq r \leq r_0 < 2GM,
\]

\[
T_{\mu\nu} = 0 \quad \text{when} \quad r > r_0.
\]

Let us find out which properties of the effective matter support the general spherical space–time when the functions \( N \) and \( \Phi \) depend on both coordinates \( r \).

\(^a\) We use the signature \((+ - - -)\), the sign \( R^\alpha{}_{\beta\gamma\delta} = \partial_\gamma R^\alpha{}_{\beta\gamma\delta} - \ldots \) and set the speed of light \( c = 1 \).

\(^b\) Physically, it may be thought of as a phase transition caused by quantum-gravitational processes of vacuum polarization and matter creation in intensive variable gravitational field (though such an interpretation is valid only in the quasiclassical limit). One of the candidates to play the role of the 'temperature' is the squared Riemann tensor \( \mathcal{I} = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \) which in the Schwarzschild solution equals to \( 48(GM/r^3)^2 \). Then the characteristic value \( r_0 \) can be estimated from the physical dimension: \( \mathcal{I} \sim 1/l_P^2 \), where \( l_P \sim 10^{-33} \text{ cm} \) is the Planck length. To give an example, for a black hole with mass of the order of that of the Sun \( r_0 \sim 10^{12} l_P \).
and \( t \). Making use of eq. (1) one obtains from the GR equations:

\[
\Phi = -\frac{Gm}{r},
\]

where the finite mass function

\[
m = m(r,t) = 4\pi \int_0^T t r^2 dr = m_0 - 4\pi r^2 \int_T^t dt
\]

vanishes at \( r = 0 \) \((m(0,t) = 0)\) thanks to the finiteness condition applied on the potential \( \Phi \). The function \( m_0 = m_0(r) \) is determined by initial/boundary conditions.

We also conclude that for the function \( m(r,t) \) to be finite, \( T^r_r t^2 \) must be integrable at \( r = 0 \). Hereafter, by definition, the distribution of longitudinal pressure/energy density (in \( T/R \)-regions, respectively) over \( r \) integrable in this sense is called to possess an integrable singularity.

In our case the l.h.s. of Eq. (4) depends only on \( r \). Hence, \( T^r_r = 0 \) and the EMT compatible with metrics (1) is

\[
T^\mu_\nu = \text{diag}\left(-p, \varepsilon, -p_\perp, -p_\perp\right).
\]

Note that, since the EMT emerges only under the horizon, the \( (r^r)\)-component plays the role of energy density, because the coordinate \( r \) plays that of time. The rest of the Einstein equations read:

\[
\frac{N'}{N} = \frac{4\pi G r^2 (\varepsilon + p)}{2Gm - r}, \quad m(r) = -4\pi \int_0^r p r^2 dr
\]

\[
p_\perp = \frac{N'}{2N} \left( \frac{m}{4\pi r^2} - r\varepsilon \right) - \frac{(r^2 \varepsilon)'}{2r},
\]

where the prime stands for the derivative with respect to \( r \). Extending the integral in Eq. (7) up to \( r_0 \) one obtains the external mass

\[
M = -4\pi \int_0^{r_0} p r^2 dr.
\]

In order to integrate these equations it is necessary to specify the effective matter Lagrangian or its equation-of-state. Of course, their exact form can be found only in the complete theory of quantum gravity. However, in the phenomenological framework we can depict some generic features of the space–times under consideration already now.

3. Some general properties

**Theorem 3.1.** Let the potential \( \Phi(r = 0) = \Phi_0 \) and its second derivative be finite at \( r = 0 \), and its first derivative vanish no slower than \( r \) as \( r \to 0 \). Then tidal forces remain finite on world line of the matter flow and

\[
p = \frac{\Phi_0}{4\pi Gr^2} \quad \text{as} \quad r \to 0.
\]
Proof. In order to study the story of an extended body falling to \( r = 0 \) we follow Ref. [30]. Non-vanishing components of the Riemann tensor in a locally inertial reference frame \((\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi})\) are given by the formulae:

\[
R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \Phi'' , \quad R_{\hat{t}\hat{\theta}\hat{\theta}} = R_{\hat{t}\hat{\phi}\hat{\phi}} = \frac{\Phi'}{r} , \quad (11)
\]

\[
R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} = -\frac{2\Phi}{r^2} , \quad R_{\hat{r}\hat{\theta}\hat{\theta}} = R_{\hat{r}\hat{\phi}\hat{\phi}} = -\frac{\Phi'}{r} . \quad (12)
\]

The only non-zero component of the 4-velocity of the matter is \( u^r \). Therefore, the equation, which governs how fast two particles separated by the spatial vector \( \xi^a , \hat{a} = (\hat{t}, \hat{\theta}, \hat{\phi}) \), accelerate relative to each other, reads:

\[
\frac{D^2 \xi^a}{dt^2} = -R_{\hat{r}\hat{\alpha}\hat{\beta}} \xi^\beta . \quad (13)
\]

As one can see, under conditions of the theorem the right-hand side is finite. Moreover, since \( R_{\hat{\theta}\hat{\phi}\hat{\theta}\hat{\phi}} \) is the only component in \((11), (12)\) that diverges as \( r \to 0 \), the tidal forces remains finite on any radial geodesic \( u^\mu = (u^t, u^r, 0, 0) \).

Also, Eq. (4) yields

\[
\Phi' r + \Phi = 4\pi G p r^2 .
\]

Hence, \( p = \Phi_0 / 4\pi G r^2 + O(1) \) as \( r \to 0 \). Since \( p \) is supposed to comprise every correction to the Einstein tensor (see Introduction), this implies that all these corrections diverge as \( r \to 0 \).

In what follows it is convenient to extend the values of \( r \) to the real domain \((-\infty, +\infty)\). This choice allows one to describe the black- and white-hole regions in a unified manner. Indeed, a physical way to probe a space–time is to study test particles’ trajectories. For example, one can mark the surfaces \( r = \text{const} \) by launching a particle along the radius \( (d\theta = d\phi = 0) \) and measuring its consecutive positions. As usual, in the \( T \)-region of the black hole the particle moves towards smaller values of \( r \). Also it has no problem passing the hypersurface \( r = 0 \). Hence, as the particle passes to the \( T \)-region of the white hole, we can consider it moving on to smaller, but now negative, values of \( r \). Another advantage of this choice is that it fits the fact that \( r \) as time (as it is in the \( T \)-regions) must ‘tick’ in one direction.

We can now classify all possible models by their properties under the time inversion \( r \to -r \) with respect to \( r = 0 \) into reversible and irreversible ones. Reversible solutions \((1)\) are given by even functions of \( r \) whereas the second-type models are non-invariant under the inversion and described by asymmetric profiles. From geometrical point of view, models of different types depend differently on intrinsic and extrinsic curvature of the Schwarzschild-like space–time (see Appendix). From physical point of view, the mechanism that underlies the reversible solutions is the vacuum polarization as described by the quantum corrections (see Introduction). In the irreversible regime some transformation of gravitational degrees of freedom into material ones must take place.
We are about to give a toy model of each type. As soon as the external mass $M$ is positive, the longitudinal pressure is, on average, negative (see Eq. (9)), and for simplicity we choose it to be vacuum-like, $p = -\varepsilon$. Then $N = 1$ and the energy density is found from Eq. (8) which becomes
\[
\frac{d (\varepsilon r^2)}{r dr} = -2p_\perp .
\] (14)

4. Two toy models

4.1. Black-white hole

Let us consider a symmetric step for the profile of the transverse pressure $p_\perp$:
\[
p_\perp^{(A)} = p_0 \cdot \theta \left( r_0^2 - r^2 \right) .
\] (15)
Then $M \equiv 8\pi r_0^3 p_0 / 3$ is the black-hole mass, and integrating Eq. (14) with the initial condition $\varepsilon(r \geq r_0) = 0$ yields
\[
\varepsilon^{(A)} = p_\perp^{(A)} \left( \frac{r_0^2}{r^2} - 1 \right) ,
\] (16)
and the potential
\[
\Phi^{(A)} = \begin{cases} 
- \frac{3GM}{2r_0} \left( 1 - \frac{1}{3} \left( \frac{r}{r_0} \right)^2 \right) , & |r| \leq r_0 , \\
- \frac{GM}{|r|} , & |r| \geq r_0 .
\end{cases}
\] (17)

A model of this type can be referred to as a 'black-white hole', because the twin white hole is linked to its black counterpart by the matter flow. The spacetime geometry is seemingly sourced by the effective EMT. However, this EMT alone cannot sustain the black hole, because the latter resides in the absolute past with respect to the former. The actual source is located on the border between another black-and-white pair, which lies under the point of intersection of the horizons $r = 2GM$ (see Fig. 1). In other words, the Penrose diagram of the black-white hole is an infinite chain of elementary Penrose diagrams engaged by material regions that source the geometry. Let us emphasize that, presumably (see footnote [1]), $\Phi_0 = \min \Phi^{(A)} = - \frac{3GM}{2r_0} \ll -1$.

This also helps to clarify the question of the source of the Schwarzschild spacetime. Indeed, in the limit $r_0 \rightarrow M = \text{const}$ we obtain a black or white hole maximally extended onto the empty space with a delta-like source localized at $r = 0$:
\[
\varepsilon = -p = 2p_\perp = M \frac{\delta (r)}{2\pi r^2} ,
\] (18)
where $\delta(r) = \theta'(r)$ is one-dimensional delta-function.
4.2. Cosmology-like solution

For this example we consider an asymmetric step for the transverse pressure profile:
\[ p^{(B)}_\perp = p_0 \cdot \theta (rr_0 - r^2) - p_1 \cdot \theta (-r), \]  
(19)

where \( p_1 \) is a positive constant. Analogously to the previous section we obtain:
\[ \varepsilon^{(B)} = -p^{(B)}_\perp + p_0 \frac{r_0^2}{r^2} \theta (r_0 - r). \]  
(20)

\[ \Phi^{(B)} = \begin{cases} \frac{3GM}{2r_0} \left( 1 + \frac{p_1}{3p_0} \left( \frac{r}{r_0} \right)^2 \right), & r \leq 0, \\ \Phi^{(A)}, & r \geq 0. \end{cases} \]  
(21)

The nature of this solution becomes clearer if we introduce the proper time \( d\tau = \left| 1 + 2\Phi \right|^{-1/2} dr \). Then for \( \tau \in (0, \infty) \) Eqs. (20) (21) yield the solution asymptotically
approaching the de Sitter (see Fig. [2]):

\[ r = -g_0 \frac{\sinh(H_1 \tau)}{H_1}, \quad \varepsilon = \frac{3H_1^2}{8\pi G} \left( 1 + \frac{1 + g_0^{-2}}{3 \sinh^2(H_1 \tau)} \right), \quad (22) \]

\[ ds^2 = d\tau^2 - g_0^2 \left( \cosh^2(H_1 \tau) \, dt^2 + \frac{\sinh^2(H_1 \tau)}{H_1^2} \, d\Omega^2 \right), \]

where \( g_0^2 \equiv -2 \Phi_0 - 1 = \left( \frac{3GM}{r_0} - 1 \right) > 1/2 \) and the constant \( H_1 = (8\pi Gp_1/3)^{1/2} \) acquires any value independent of the external mass of the black hole. Therefore,
the $T$-region of the white hole may, in principle, incorporate an entire universe. Such a universe originating from inside a black hole can be dubbed \textit{astrogenic}. The space of the created universe is homogeneous and huge owing to two parameters from Eq. (22): the large prefactor $g_0 \sim \sqrt{GM/r_0} \sim r_0/l_P \sim (M/M_P)^{1/3} \gg 1$ and a constant $H_1$ responsible for inflation. While the former stems from the violent matter production in the deep gravitational potential of integrable singularity the latter is ordinary space inflation due to phase transition in already expanding effective matter. The toy example also shows that although the astrogenic universe is anisotropic in the beginning (with the symmetry of the Kantowski-Sachs model), it may become isotropic at late times: $-p^{(B)} = \varepsilon^{(B)} \to p_1 = -p^{(B)}_\perp$ as $\tau \to +\infty$.

5. Discussion

Until the theory of quantum gravity is established the phenomenological effective-matter approach can be very useful in the study of singularities. One, however, has to be careful of how to apply it. This we can learn from the analogy with classical hydrodynamics. Long before a theory of microscopic motion was developed many paradoxes of the ideal fluid hydrodynamics had been known to be solved by introducing a coefficient of viscosity.\textsuperscript{31} Note that in the macroscopic theory the value of the coefficient is found by measuring the fluid motions, that is, inferred from the l.h.s. of the Navier–Stokes equations. The same scheme should be adopted for the effective Einstein equations. Indeed, results obtained so far in the field of gravity seem to be more instructive about the properties of space-times rather than those of effective matter.

Since it is little known about the effective equation-of-state, it is tempting to consider cases with high symmetry like that of the de Sitter vacuum.\textsuperscript{32} The latter has become so popular in the literature that references are too numerous to cite. However, the above considerations indicate that in the black-hole singularity problem it is reasonable to require the $\mathbb{R} \times S^2$ symmetry of the EMT. Another argument follows from the well-known BLK regime\textsuperscript{33} that predicts a break of isotropy in the vacuum solution. Although it was argued\textsuperscript{34} that quantum effects may dump the anisotropy, until the exact physics under the Planck scale is known the anisotropic choice is nothing less natural than the de Sitter.

Moreover, the second model from the preceding section offers an advantage. It demonstrates that a newly born universe can be connected to the black-hole interior without assuming density and/or curvature being finite. Here it is worth clarifying what ‘connected’ means, which brings us to the question how to classify the integrable singularity. According to Ref. \textsuperscript{35} it is a strong singularity. Indeed, writing metrics \textsuperscript{11} locally as $ds^2 = \delta \tau^2 - \delta x^2 - \delta y^2 - \delta z^2$ we find that the comoving volume built on three spacelike vectors in the $T$-region (see Eq. (A.2))

\[\delta V = \delta x \delta y \delta z = N|1 + 2\Phi|^{1/2}\tau^2 \det \{\omega_{ij}\} dtdy^1dy^2 \to 0 \quad \text{as} \quad r \to 0.\]
Therefore, on one hand, an extended body falling towards the singularity will be destroyed by tidal forces. But notice that the volume (23) tends to zero because of the infinitesimal radius of the 2-sphere. It means that it is basically the transverse tidal forces that destroy the body whereas the longitudinal forces stay finite. This fact also follows from Eqs. (11)-(13), because in the extended body there always exist points with non-zero angular components of 4-velocity. However, it is more important that the flow that respects the space-time symmetry does not suffer from those forces and it is this flow that forms the new universe after the integrable singularity.

Here we would like to make a few comments on the suggested scenario of generating new universes. First, the model proposed is based on the notion of eternal black hole. However, its further generalization to the case of a black hole originating from collapse does not offer principal difficulties. Second, there is no need of generalizing it to the cases of rotating or charged black holes. This is thanks to the ‘mass inflation’ phenomenon: as it was shown in Ref. 36, an observer in the interior static region of a rotating (or charged) black hole would measure internal mass the factor \( \sim \sqrt{M/M_{\text{Pl}}} \) larger than \( M \). Because of the internal mass this large, the geometry of the interior would be similar to the Schwarzschild one. Third, thanks to the gain factor \( g_0 \gg 1 \) in Eq. (22) the volume of the new universe is already large when \( \tau \sim H_{\text{r}}^{-1} \), i.e. before the ordinary inflation enhances it. This effect can be referred to as ‘singularity-induced inflation’, because this kind of inflation does not require any specific field and is of pure gravitational nature. Finally, residual cylindrical anisotropy in the present-day data would indicate the astrogenic origin of our universe. Although some authors claim to have discovered the global anisotropy (see, for example, Refs. 37, 38), the current precision of cosmological observations is insufficient to state that, and future observations should clarify the situation.

It is also necessary to point out limitations of the phenomenological approach. The main of them is that there is ambiguity in determining the effective equation-of-state. Speaking of the longitudinal pressure, the vacuum-like equation-of-state used above seems to be fairly common, because it prevents intersection of layers in the more realistic case of collapse\(^\text{\ref{footnote}}\). As for the other pressure component, behind the phase transition scenario we have drafted (see footnote \text{\ref{footnote}}), of course, there must be a specific model to be developed. Also, processes like passing of an elementary particle through the integrable singularity are beyond the scope of the effective-matter approach. Nevertheless, within its framework we were able think of a situation where a singularity does not ‘spoil’ the space–time and, more than that, leads to a natural cosmogenesis scenario. The very possibility of it indicates that the requirement that a physical space–time must be completely free of singularities in the mathematical sense may be too strong indeed. It may be a pity that a star voyager cannot travel to the new universe, but it is still not the reason to say that nature is afraid of singularities.
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Appendix A. Intrinsic and extrinsic curvature of Schwarzschild-type space–times
In the spirit of the ADM formalism\textsuperscript{[20]} the Schwarzschild-type geometry \textsuperscript{[11]} can be represented as a (2+2)-split. To begin with, remind that the quantity \( r \) plays two roles. On one hand, it is the curvature radius of a 2-dimensional sphere. On the other hand, it is one of the coordinates. In order to discriminate one from the other, let us present the metrics of an arbitrary spherically symmetric 4D space–time split into a pair of 2D spaces\textsuperscript{[39, 40]}

\[
dX^2 = n_{IJ} dx^I dx^J \quad \text{(A.1)}
\]

and

\[
dY^2 = \gamma_{ij} dy^i dy^j \equiv r^2 \omega_{ij} dy^i dy^j , \quad \text{(A.2)}
\]

where the functions \( n_{IJ} \) and \( r \) depend on the variables \( x^I = (x^1, x^2) \in \mathbb{R}^2 \) and are independent of the internal 2D coordinates \( y^i \) of the closed homogeneous and isotropic 2-surface \( S^2 \) of the unit curvature \( d\Omega^2 = \omega_{ij} dy^i dy^j \), \( \gamma_{ij} \equiv r^2 \omega_{ij} \). If the coordinates are chosen to be angular, \( y^i = (\theta, \varphi) \), we have \( \omega_{ij} = diag(1, \sin^2 \theta) \) with \( \theta \in [0, \pi] \) and \( \varphi \in [0, 2\pi) \).

By choosing the four coordinates of the covering grid one can turn four non-diagonal components of the full metric tensor into zero \( g_{Ii} = 0 \) and write it in the orthogonal reference frame \( x^\mu = (x^I, y^i) \):

\[
dx^2 = g_{\mu\nu} dx^\mu dx^\nu = dX^2 - dY^2 , \quad \text{(A.3)}
\]

where \( g_{\mu\nu} = diag(n_{IJ}, -\gamma_{ij}) \) is the metrics of the spherically symmetric geometry in the orthogonal split 2 + 2. The energy–momentum tensor corresponding to \textsuperscript{[A.3]} is \( T_{\mu\nu} = diag(T_{IJ}, p_\perp \gamma_{ij}) \), where \( p_\perp \) is the transversal pressure.

At this point the metric potential \( r \) in Eq. \textsuperscript{[A.2]} can be introduced in the invariant manner as radius of the intrinsic curvature \( \rho \) of the closed \( Y \)-space, where

\[
R_{ij}^{(Y)} = \rho \gamma_{ij} \quad \text{and} \quad \rho \equiv \frac{1}{2} R^{(Y)} = r^{-2} \quad \text{(A.4)}
\]

are the Ricci tensor and scalar constructed from the metrics \( \gamma_{ij} \). By definition, the 2-space \( \gamma_{ij} \) and its intrinsic curvature \( \rho \) are invariant with respect to interchanging
and $-r$ while the extrinsic curvature of $Y$ depends on the sign of $r$ and determines the orientation and evolution of the surface $S^2$ in the space–time $(A.3)$:

$$K_{ij} \equiv \frac{1}{2} \gamma_{ij,I} = K_I \gamma_{ij}, \quad K_I \equiv \frac{1}{2} \gamma^{ij} K_{ij,I} = \frac{r_{,I}}{r},$$

where the comma in the subscript stands for the partial derivative with respect to $x^I$. This fact becomes obvious in the coordinates where one of the variables $x^I$ is identically equal to $r$.

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