ON MAXIMAL GREEN SEQUENCE FOR QUIVERS ARISING FROM WEIGHTED PROJECTIVE LINES

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ABSTRACT. We investigate the existence and non-existence of maximal green sequences for quivers arising from weighted projective lines. Let $Q$ be the Gabriel quiver of the endomorphism algebra of a basic cluster-tilting object in the cluster category $\mathcal{C}_X$ of a weighted projective line $X$. It is proved that there exists a quiver $Q'$ in the mutation equivalence class $\text{Mut}(Q)$ of $Q$ such that $Q'$ admits a maximal green sequence. Furthermore, there is a quiver in $\text{Mut}(Q)$ which does not admit a maximal green sequence if and only if $X$ is of wild type.

1. INTRODUCTION

Maximal green sequences were introduced by Keller [14] in the study of refined Donaldson-Thomas invariants for quivers and implicitly by Gaiotto et al. in [9]. They are certain sequences of quiver mutations satisfying a certain combinatorial condition. It is known that not all quivers have maximal green sequences, but they do exist for important classes of quivers. We refer to the survey [16] for examples and recent progress. It is an open question to determine whether a given quiver admits a maximal green sequence or not.

The existence of maximal green sequences yields quantum dilogarithm identities in the associated quantum torus and provides explicit formulas for Kontsevich-Soibelman’s refined Donaldson-Thomas invariants (cf. [14]). It also has important applications in the theory of cluster algebras. In particular, Gross et al. [11] proved that the Fock-Goncharov conjecture about the existence of a canonical basis for cluster algebras holds when a cluster algebra has a quiver with a maximal green sequence and the cluster algebra is equal to its upper cluster algebra. It is also a sufficient condition for the existence of a generic basis in certain upper cluster algebras [22].

Cluster-tilting theory of hereditary abelian categories produces a large class of important quivers, which we denote it by $Q_{ct}$. Let $K$ be an algebraically closed field and $\mathcal{H}$ a hereditary abelian category over $K$ with tilting objects. The cluster category $\mathcal{C}(\mathcal{H})$ [2] is defined as the orbit category of the bounded derived category $\mathbb{D}^b(\mathcal{H})$ with respect to the auto-equivalence $\tau^{-1} \circ \Sigma$, where $\tau$ is the Auslander-Reiten translation and $\Sigma$ is the suspension functor of $\mathbb{D}^b(\mathcal{H})$, respectively. The cluster category $\mathcal{C}(\mathcal{H})$ is a 2-Calabi-Yau triangulated category with cluster-tilting objects (cf. [13]). For each basic cluster-tilting object $T \in \mathcal{C}(\mathcal{H})$, we denote by $Q_T$ the Gabriel quiver of the endomorphism algebra $\text{End}_{\mathcal{C}(\mathcal{H})}(T)$. Then $Q_{ct}$ consists of quivers which
are isomorphic to $Q_T$ for some basic cluster-tilting object $T$ and hereditary abelian category $\mathcal{H}$.

According to Happel’s classification theorem [12], each connected hereditary abelian $K$-category with tilting objects is either derived equivalent to the path algebra $KQ$ of a finite acyclic quiver $Q$ or to the category $\text{coh } \mathbb{X}$ of coherent sheaves over a weighted projective line in the sense of Geigle-Lenzing [10]. Therefore, $\mathcal{Q}_{\text{ct}}$ can be written as the union of two subclasses: $\mathcal{Q}_{\text{pa}}$ consists of quivers arising from path algebras and $\mathcal{Q}_{\text{wpl}}$ consists of quivers arising from weighted projective lines. Quivers in $\mathcal{Q}_{\text{pa}}$ and their associated cluster algebras were well-studied. It is natural to investigate the quiver in $\mathcal{Q}_{\text{wpl}}$ and their associated cluster algebras. The aim of this note is to study the existence and non-existence of maximal green sequences for quivers in $\mathcal{Q}_{\text{wpl}}$. Our main result is an existence and non-existence theorem (cf. Theorem 4.3) for quivers arising from weighted projective lines. Surprisingly, the existence and non-existence theorem is compatible with the classification of weighted projective lines.

The paper is structured as follows. In Section 2, we recall the definitions of quiver mutation and maximal green sequence. Quivers of finite mutation type are also discussed. In Section 3, we collect basic properties for weighted projective lines. It is proved that a quiver arising from a weighted projective line $\mathbb{X}$ is of finite mutation type if and only if $\mathbb{X}$ is not of wild type (Proposition 3.7). In Section 4, we present the proof of the main result (Theorem 4.3).

**Conventions.** Let $m \geq n$ be positive integers. For an integer matrix $B \in M_{m \times n}(\mathbb{Z})$, we refer to the submatrix formed by the first $n$ rows of $B$ the principal part of $B$ and the submatrix formed by the last $m - n$ rows the coefficient part.

For any integer vectors $\alpha = [a_1, \ldots, a_n]^T$, $\beta = [b_1, \ldots, b_n]^T \in \mathbb{Z}^n$, we denote by $\alpha \leq \beta$ if $a_i \leq b_i$ for $1 \leq i \leq n$. This endows a partial order on $\mathbb{Z}^n$. For $b \in \mathbb{Z}$, let $\text{sgn}(b)$ be $1$, $0$, or $-1$, depending on whether $b$ is positive, zero, or negative.

2. **Preliminaries**

2.1. **Quivers and mutation.** A quiver is an oriented graph, i.e., a quadruple $Q = (Q_0, Q_1, s, t)$ formed by a set of vertices $Q_0$, a set of arrows $Q_1$ and two maps $s$ and $t$ from $Q_1$ to $Q_0$ which send an arrow $\alpha$ respectively to its source $s(\alpha)$ and its target $t(\alpha)$. An arrow whose source and target coincide is a loop; a 2-cycle is a pair of distinct arrows $\alpha$ and $\beta$ such that $s(\alpha) = t(\beta)$ and $t(\alpha) = s(\beta)$. By convention, in the sequel, by a quiver we always mean a finite quiver without loops nor 2-cycles. An ice quiver is a pair $(Q, F)$, where $Q$ is a quiver and $F$ is a subset of $Q_0$ called frozen vertices, such that there are no arrows between frozen vertices. The non-frozen vertices of $(Q, F)$ are mutable vertices. The mutable part of $(Q, F)$ is the full subquiver of $(Q, F)$ consisting of mutable vertices.

**Definition 2.1.** Let $(Q, F)$ be an ice quiver and $k$ a mutable vertex. The mutation $\mu_k(Q, F)$ of $(Q, F)$ at vertex $k$ is the ice quiver obtained from $(Q, F)$ as follows:

- for each subquiver $i \xrightarrow{\beta} k \xrightarrow{\alpha} j$, we add a new arrow $[\alpha \beta] : i \to j$;
- we reverse all arrows with source or target $k$;
we remove the arrows in a maximal set of pairwise disjoint 2-cycles and any arrows that created between frozen vertices.

When \( F = \emptyset \), we also write \( \mu_k(Q) \) for \( \mu_k(Q, \emptyset) \).

Let \((Q, F)\) be an ice quiver with non frozen vertices \( \{1, \ldots, n\} \) and frozen vertices \( \{n+1, \ldots, m\} \). Up to an isomorphism fixing the vertices, such an ice quiver is given by an \( m \times n \) integer matrix \( B(Q, F) \) whose coefficient \( b_{ij} \) is the difference between the number of arrows from \( j \) to \( i \) and the number of arrows from \( i \) to \( j \). In particular, the principal part of \( B(Q, F) \) is skew-symmetric. Conversely, each \( m \times n \) integer matrix \( B \) with skew-symmetric principal part comes from an ice quiver. Let \( B(Q, F) = (b_{ij}) \) be the associated matrix of \((Q, F)\). For any mutable vertex \( k \), we denote by \( \mu_k(B(Q, F)) = (b_{ij}') \) the matrix associated to the ice quiver \( \mu_k(Q, F) \), then

\[
b_{ij}' = \begin{cases} 
-b_{ij} & \text{if } i = k \text{ or } j = k; \\
b_{ij} + \text{sgn}(b_{ik}) \max(0, b_{ik}b_{kj}) & \text{else.}
\end{cases}
\]

This is the matrix mutation rule introduced by Fomin and Zelevinsky [7]. It is clear that \( \mu_k(B(Q, F)) = B(\mu_k(Q, F)) \).

Mutation at a fixed vertex is an involution. Two ice quivers are mutation-equivalent if they are linked by a finite sequence of mutations. We will denote by \( \text{Mut}(Q, F) \) the set of all quivers that can be obtained from \((Q, F)\) by a finite sequence of mutations. We write \( \text{Mut}(Q) := \text{Mut}(Q, \emptyset) \).

2.2. Maximal green sequence.

**Definition 2.2.** Let \( Q \) be a quiver. The framed quiver \( \hat{Q} \) of \( Q \) is the ice quiver \((\hat{Q}, Q_0^*)\) such that:

\[
Q_0^* = \{i^* \mid i \in Q_0\}, \quad \hat{Q}_0 = Q_0 \sqcup Q_0^*, \quad \hat{Q}_1 = Q_1 \sqcup \{i \rightarrow i^* \mid i \in Q_0\}.
\]

The coframed quiver \( \hat{Q} \) is the ice quiver \((\hat{Q}, Q_0^*)\) such that:

\[
Q_0^* = \{i^* \mid i \in Q_0\}, \quad \hat{Q}_0 = Q_0 \sqcup Q_0^*, \quad \hat{Q}_1 = Q_1 \sqcup \{i \leftarrow i^* \mid i \in Q_0\}.
\]

**Definition 2.3.** Let \( R \in \text{Mut}(\hat{Q}, Q_0^*) \). A mutable vertex \( k \in R_0 \) is green if \( \{j^* \in Q_0^* \mid \exists j^* \rightarrow k \in R_1\} = \emptyset \). It is red if \( \{j^* \in Q_0^* \mid \exists j^* \leftarrow k \in R_1\} = \emptyset \).

We have the following sign-coherence property.

**Theorem 2.4.** [5, Theorem 1.7] Every mutable vertex of \( R \in \text{Mut}(\hat{Q}, Q_0^*) \) is either green or red.

**Remark 2.5.** A non-zero integer vector \( c \in \mathbb{Z}^n \) is sign-coherent if \( c \leq 0 \) or \( 0 \leq c \). Let \( Q \) be a quiver with vertex set \( \{1, \ldots, n\} \). For \( R \in \text{Mut}(\hat{Q}, Q_0^*) \), recall that \( B(R, Q_0^*) \) is the associated \( 2n \times n \) integer matrix. Theorem 2.4 can be restated as follows: each column vector of the coefficient part of \( B(R, Q_0^*) \) is sign-coherent.
**Definition 2.6.** A green sequence for a quiver $Q$ is a sequence $i = (i_1, \ldots, i_l)$ of vertices of $Q$ such that for any $1 \leq k \leq l$, the vertex $i_k$ is green in $\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*)$. The green sequence $i$ is maximal if every mutable vertex in $\mu_{i_l} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*)$ is red. We will simply denote the composition $\mu_{i_1} \circ \cdots \circ \mu_{i_l}$ by $\mu_i$. A green-to-red sequence is a sequence $i$ of vertices of $Q$ such that every mutable vertex in $\mu_i(\hat{Q}, Q_0^*)$ is red.

**Proposition 2.7.** [1, Proposition 2.10] Suppose that $Q$ admits a green-to-red sequence $i$. Then there is a unique isomorphism $\mu_i(\hat{Q}, Q_0^*) \rightarrow Q$ fixing the frozen vertices and sending a non frozen vertex $i$ to $\sigma(i)$ for a unique permutation $\sigma$ of the vertices of $Q$.

**Remark 2.8.** By definition and Proposition 2.7, it is known that a sequence $i$ is a green-to-red sequence of $Q$ if and only if the coefficient part of the matrix $B(\mu_i(\hat{Q}, Q_0^*)) = \mu_i(B(\hat{Q}, Q_0^*))$ is a permutation of $-I_n$. A sequence $i = (i_1, \ldots, i_l)$ is a maximal green sequence if and only if

- the $i_k$-th column vector of the coefficient part of $B(\mu_{i_{k-1}} \circ \cdots \circ \mu_{i_1}(\hat{Q}, Q_0^*))$ is positive for $1 \leq k \leq l$;
- the coefficient part of the matrix $B(\mu_i(\hat{Q}, Q_0^*)) = \mu_i(B(\hat{Q}, Q_0^*))$ is a permutation of $-I_n$.

By definition, all maximal green sequences are green-to-red sequences. There are quivers for which a maximal green sequence does not exist, but a green-to-red sequence does. Furthermore, there are quivers for which no green-to-red sequence exists.

**Example 2.9.** Let $a$, $b$, $c$ be non negative integers, denote by $Q_{a,b,c}$ the quiver with three vertices $1, 2, 3$ and $a$ arrows from $1$ to $2$, $b$ arrows from $2$ to $3$ and $c$ arrows from $3$ to $1$. It is known that $Q_{2,2,2}$ does not admit a green-to-red sequence. Furthermore, Muller [19, Theorem 12] proved that $Q_{a,b,c}$ does not admit maximal green sequences whenever $a, b, c \geq 2$.

**Lemma 2.10.** [19, Corollary 19] If a quiver $Q$ admits a green-to-red sequence, then any quiver mutation-equivalent to $Q$ also admits a green-to-red sequence.

Muller [19] also proved that the property of having a maximal green sequence is not invariant under mutation. The following is useful to show the non-existence of maximal green sequence for a given quiver.

**Lemma 2.11.** [19, Theorem 9 and 17] If a quiver $Q$ admits a green-to-red sequence (resp. maximal green sequence), then any full subquiver of $Q$ also admits a green-to-red sequence (resp. maximal green sequence). In particular, if $Q$ has a full subquiver $Q_{a,b,c}$ with $a, b, c \geq 2$, then $Q$ does not admit a maximal green sequence.

**Definition 2.12.** Let $Q$ be a quiver and $Q', Q''$ full subquivers. We say that $Q$ is a triangular extension of $Q'$ by $Q''$ if the set of vertices of $Q$ is the disjoint union of the sets of vertices of $Q'$ and $Q''$ and there are no arrows from vertices of $Q''$ to vertices of $Q'$. The following result was proved in [4, Theorem 4.5] using Lemma 2.11.
Lemma 2.13. [4, Theorem 4.5] If \( Q \) is a triangular extension of \( Q' \) and \( Q'' \), then \( Q \) has a maximal green sequence if and only if \( Q' \) and \( Q'' \) have maximal green sequences.

2.3. Tropical dualities between \( e \)-vectors and \( g \)-vectors. Let \( Q \) be a quiver with vertex set \( \{1, 2, \ldots, n\} \). Denote by \( T_n \) the \( n \)-regular tree whose edges are labeled by the numbers \( 1, \ldots, n \) such that the \( n \) edges emanating from each vertex have different labels. We write \( t \xrightarrow{k} t' \) to indicate that vertices \( t \) and \( t' \) are linked by an edge labeled by \( k \).

A quiver pattern of \( (\hat{Q}, Q_0') \) is an assignment of an ice quiver \( R_t \in \text{Mut}(\hat{Q}, Q_0'^*) \) to each vertex \( t \in T_n \) such that

1. there is a vertex \( t_0 \in T_n \) such that \( R_{t_0} = (\hat{Q}, Q_0'^*) \);
2. if \( t \xrightarrow{k} t' \), then \( R_{t'} = \mu_k(R_t) \).

Clearly, a quiver pattern of \( (\hat{Q}, Q_0'^*) \) is uniquely determined by assigning \( (\hat{Q}, Q_0'^*) \) to the vertex \( t_0 \in T_n \) and \( t_0 \) is called the root vertex of the quiver pattern.

We relabel the vertex \( i^* \) as \( n + i \) for each \( i \in Q_0' \) and fix a quiver pattern of \( (\hat{Q}, Q_0'^*) \). In particular, for each vertex \( t \in T_n \), we have a \( 2n \times n \) integer matrix \( B(R_t) := (b_{ij,t}) \). The coefficient part \( C_t \) of \( B(R_t) \) is the \( C \)-matrix at \( t \). Its column vectors are \( e \)-vectors.

Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{Z}^n \). For \( 1 \leq j \leq n \), denote by \( \beta_j \) the \( j \)th column of the principal part of \( B(R_{t_0}) \). For each vertex \( t \in T_n \), we also assign an integer matrix \( G_t := (g_{1,t}, \ldots, g_{n,t}) \) by the following recursion:

1. for any \( 1 \leq i \leq n \), \( g_{i,t_0} = e_i \);
2. suppose that \( G_t \) is defined and let \( t \xrightarrow{k} t' \) be an edge of \( T_n \), then

\[
\begin{align*}
g_{i,t'} &= \begin{cases} 
g_{i,t} & i \neq k; \\
-g_{k,t} + \sum_{j=1}^n [b_{jk,t} + g_{j,t'}] - \sum_{j=1}^n [b_{(n+j)k,t} + \beta_j] & i = k.
\end{cases}
\end{align*}
\]

We call \( G_t \) the \( G \)-matrix at \( t \) and its column vectors are \( g \)-vectors.

Proposition 2.14. [5, Theorem 1.7] For each vertex \( t \in T_n \), every row vector of \( G_t \) is sign-coherent.

The following is known as the tropical duality between \( e \)-vectors and \( g \)-vectors (cf. [20, 15, 21]).

Theorem 2.15. [20, Theorem 4.1] For each vertex \( t \in T_n \), we have

\[
G_t^T C_t = I_n.
\]

2.4. Finite mutation type. A quiver \( Q \) is of finite mutation type if \( \text{Mut}(Q) \) is a finite set. Quivers of finite mutation type have been classified in [6]. Here, we only recall the following.

Lemma 2.16. (1) Every quiver with two vertices is of finite mutation type.

(2) If \( Q \) is acyclic with at least three vertices, then \( Q \) is of finite mutation type if and only if \( Q \) is of Dynkin type or extended Dynkin type.
Each quiver in Figure 2.1 is of finite mutation type.

Proof. The statement (1) is obvious, (2) is proved by [3, Theorem 3.6]. For (3), one can verify the finiteness by the MutationApp of Keller [17] directly (cf. also [6, Theorem 6.1]).

Lemma 2.17. Let $2 \leq c \leq b \leq a$ be integers. If $a \geq 3$, then the quiver $Q_{a,b,c}$ is not of finite mutation type.

Proof. For a non-negative integer $t$, we set $Q^t := (\mu_2 \mu_1)^t(Q_{(a,b,c)})$ and a pair of integers $(b_t, c_t)$ by the following recursion:

$$b_0 = b, c_0 = c, b_t = ac_{t-1} - b_{t-1}, c_t = ab_t - c_{t-1}.$$ 

We claim that $0 < b_1 < c_1 \cdots < b_t < c_t < \cdots$. Indeed, it is straightforward to see that $0 < b_1 < c_1$. For $t \geq 2$, we have

$$b_t - c_{t-1} = ac_{t-1} - b_{t-1} - c_{t-1} = (a-1)c_{t-1} - b_{t-1} \geq 2c_{t-1} - b_{t-1} > 0$$

and

$$c_t - b_t = (a-1)b_t - c_{t-1} > 0$$

by induction. As a consequence, $Q^t = Q_{(a,b_t,c_t)}$ and $Q_{(a,b,c)}$ is not of finite mutation type. □

3. Quivers arising from Weighted projective lines

3.1. Weighted projective lines. Fix a positive integer $t \geq 2$. A weighted projective line $\mathbb{X} = \mathbb{X}(p, \lambda)$ over $K$ is given by a weight sequence $p = (p_1, \ldots, p_t)$ of positive integers, and a parameter sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of pairwise distinct points of the projective line $\mathbb{P}_1(K)$. Let $\mathbb{L}$ be the rank one abelian group generated by $\bar{x}_1, \ldots, \bar{x}_t$ with the relations

$$p_1 \bar{x}_1 = p_2 \bar{x}_2 = \cdots = p_t \bar{x}_t =: \bar{c},$$

where the element $\bar{c}$ is called the canonical element of $\mathbb{L}$. Denote by

$$\bar{\omega} := (t-2)\bar{c} - \sum_{i=1}^{t} \bar{x}_i \in \mathbb{L},$$

Figure 2.1. Quivers of tubular type $D_4^{(1,1)}$, $E_6^{(1,1)}$, $E_7^{(1,1)}$ and $E_8^{(1,1)}$.
EXISTENCE/NON-EXISTENCE OF MGS

which is called the dualizing element of \( \mathbb{L} \). Each element \( \vec{x} \in \mathbb{L} \) can be uniquely written into the normal form

\[
\vec{x} = \sum_{i=1}^{t} l_i \vec{x}_i + l \vec{c}, \text{ where } 0 \leq l_i < p_i \text{ and } l \in \mathbb{Z}.
\]

Let \( \vec{x} = \sum_{i=1}^{t} l_i \vec{x}_i + l \vec{c} \) and \( \vec{y} = \sum_{i=1}^{t} m_i \vec{x}_i + m \vec{c} \in \mathbb{L} \) be in normal form, denote by \( \vec{x} \leq \vec{y} \) if \( l_i \leq m_i \) for \( i = 1, \ldots, t \) and \( l \leq m \). This defines a partial order on \( \mathbb{L} \). It is known that each \( \vec{x} \in \mathbb{L} \) satisfies exactly one of the two possibilities:

\[
0 \leq \vec{x} \text{ or } \vec{x} \leq \vec{c} + \vec{\omega}.
\]

3.2. The category \( \text{coh} \, X \) of coherent sheaves. Let

\[
S := S(\mathbf{p}, \lambda) = K[X_1, \cdots, X_t]/I
\]

be the quotient of the polynomial ring \( K[X_1, \cdots, X_t] \) by the ideal \( I \) generated by \( f_i = X_i^{p_i} - X_i^2 + \lambda_i X_i^1 \) for \( 3 \leq i \leq t \). The algebra \( S \) is \( \mathbb{L} \)-graded by setting \( \deg X_i = \vec{x}_i \) for \( i = 1, \ldots, t \) and we have the decomposition of \( S \) into \( K \)-subspace

\[
S = \bigoplus_{\vec{x} \in \mathbb{L}} S_{\vec{x}}.
\]

The category \( \text{coh} \, X \) of coherent sheaves over \( X \) is defined to be the quotient category

\[
\text{coh} \, X := \text{mod}^L S / \text{mod}^L_0 S,
\]

where \( \text{mod}^L S \) is the category of finitely generated \( \mathbb{L} \)-graded \( S \)-modules, while \( \text{mod}^L_0 S \) is the Serre subcategory of \( \mathbb{L} \)-graded \( S \)-modules of finite length. For each sheaf \( E \) and \( \vec{x} \in \mathbb{L} \), denote by \( E(\vec{x}) \) the grading shift of \( E \) with \( \vec{x} \). The free module \( S \) gives the structure sheaf \( O \), and each line bundle is given by the grading shift \( O(\vec{x}) \) for a unique element \( \vec{x} \in \mathbb{L} \). Moreover, we have

\[
\text{Hom}_X(O(\vec{x}), O(\vec{y})) = S_{\vec{y} - \vec{x}} \text{ for any } \vec{x}, \vec{y} \in \mathbb{L}.
\]

In [10], Geigle and Lenzing proved that \( \text{coh} \, X \) is a connected hereditary abelian category with tilting objects and has Serre duality of the form

\[
\mathbb{D} \text{ Ext}^1_X(E, F) = \text{Hom}_X(F, E(\vec{\omega}))
\]

for all \( E, F \in \text{coh} \, X \). In particular, \( \text{coh} \, X \) admits almost split sequences with the Auslander-Reiten translation \( \tau \) given by the grading shift with \( \vec{\omega} \). Recall that an object \( T \in \text{coh} \, X \) is a tilting object if \( \text{Ext}^1_X(T, T) = 0 \) and for \( X \in \text{coh} \, X \) with \( \text{Hom}_X(T, X) = 0 = \text{Ext}^1_X(T, X) \), we have that \( X = 0 \).

Denote by \( \text{vect} \, X \) the full subcategory of \( \text{coh} \, X \) consisting of vector bundles, i.e. torsion-free sheaves, and by \( \text{coh}_0 \, X \) the full subcategory consisting of sheaves of finite length, i.e. torsion sheaves. Each coherent sheaf is the direct sum of a vector bundle and a finite length sheaf. Each vector bundle has a finite filtration by line bundles and there is no nonzero
morphism from \( \text{coh}_0 \mathcal{X} \) to \( \text{vect} \mathcal{X} \). We remark that \( \text{coh} \mathcal{X} \) does not contain nonzero projective objects. Denote by

\[
p_\lambda : \mathbb{P}_1(k) \to \mathbb{N}, \quad p_\lambda(\mu) = \begin{cases} p_i & \text{if } \mu = \lambda_i \text{ for some } i, \\ 1 & \text{else}. \end{cases}
\]

the weight function associated with \( \mathcal{X} \).

**Proposition 3.1.** [10, Proposition 2.5] The category \( \text{coh}_0 \mathcal{X} \) is an exact abelian, uniserial subcategory of \( \text{coh} \mathcal{X} \) which is stable under Auslander-Reiten translation. The components of the Auslander-Reiten quiver of \( \text{coh}_0 \mathcal{X} \) form a family of pairwise orthogonal standard tubes \((T_\mu)_{\mu \in \mathbb{P}_1(k)}\), where each tube \( T_\mu \) has rank \( p_\lambda(\mu) \).

For \( \lambda_i \) with weight \( p_i \geq 2 \), there is exactly one simple object \( S_i \) in \( T_{\lambda_i} \) satisfying \( \text{Hom}_\mathcal{X}(O, S_i) \neq 0 \). Moreover, there exists a sequence of exceptional objects and epimorphisms

\[
S_i^{[p_i-1]} \to S_i^{[p_i-2]} \to \cdots \to S_i^{[1]} = S_i,
\]

where \( S_i^{[j]} \) has length \( j \) and top \( S_i \).

The following is well-known (cf. [10]).

**Proposition 3.2.** Both

\[
T_{\text{can}}(\mathcal{X}) := \bigoplus_{0 \leq \lambda \leq \beta} \mathcal{O}(\lambda) \quad \text{and} \quad T_{\text{sq}}(\mathcal{X}) := \mathcal{O} \oplus \mathcal{O}(\beta) \oplus \bigoplus_{i=1}^t \bigoplus_{k=1}^{p_i-1} (\bigoplus_{i=1}^{p_i-1} S_i^{[p_i-k]})
\]

are tilting objects of \( \text{coh} \mathcal{X} \).

### 3.3. Quivers associated with \( T_{\text{can}}(\mathcal{X}) \) and \( T_{\text{sq}}(\mathcal{X}) \)

Denote by \( \mathcal{D}^b(\text{coh} \mathcal{X}) \) the bounded derived category of \( \text{coh} \mathcal{X} \) with suspension functor \( \Sigma \). Let \( \tau : \mathcal{D}^b(\text{coh} \mathcal{X}) \to \mathcal{D}^b(\text{coh} \mathcal{X}) \) be the Auslander-Reiten (AR) translation functor, which restricts to the AR translation of \( \text{coh} \mathcal{X} \).

**Definition 3.3.** The cluster category \( \mathcal{C}_\mathcal{X} \) associated with \( \mathcal{X} \) is defined as the orbit category \( \mathcal{D}^b(\text{coh} \mathcal{X})/\langle \tau^{-1} \circ \Sigma \rangle \); it has the same objects as \( \mathcal{D}^b(\text{coh} \mathcal{X}) \), morphism spaces are given by \( \bigoplus_{\tau \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\text{coh} \mathcal{X})}(X, (\tau^{-1} \circ \Sigma)^i Y) \) with obvious composition.

The cluster category \( \mathcal{C}_\mathcal{X} \) admits a canonical triangle structure such that the projection \( \pi_\mathcal{X} : \mathcal{D}^b(\text{coh} \mathcal{X}) \to \mathcal{C}_\mathcal{X} \) is a triangle functor (cf. [13]). The suspension functor \( \Sigma \) (resp. the AR translation \( \tau \)) of \( \mathcal{D}^b(\text{coh} \mathcal{X}) \) induces the suspension functor (resp. the AR translation) of \( \mathcal{C}_\mathcal{X} \), which will be denoted by \( \Sigma \) (resp. \( \tau \)) as well. It was shown in [13] that \( \mathcal{C}_\mathcal{X} \) is a 2-Calabi-Yau triangulated category, i.e., for any \( X, Y \in \mathcal{C}_\mathcal{X} \), we have bifunctorially isomorphisms

\[
\text{Hom}_{\mathcal{C}_\mathcal{X}}(X, \Sigma^2 Y) \cong \mathbb{D} \text{Hom}_{\mathcal{C}_\mathcal{X}}(Y, X).
\]

By the 2-Calabi-Yau property, we clearly have \( \tau = \Sigma \) in \( \mathcal{C}_\mathcal{X} \).

**Definition 3.4.** An object \( T \in \mathcal{C}_\mathcal{X} \) is a cluster-tilting object if \( \text{Ext}_{\mathcal{C}_\mathcal{X}}^1(T, T) = 0 \) and \( \text{Ext}_{\mathcal{C}_\mathcal{X}}^1(T, X) = 0 \) implies that \( X \in \text{add} T \), where \( \text{add} T \) is the full subcategory of \( \mathcal{C}_\mathcal{X} \) consisting of direct summands of direct sum of finite copies of \( T \).
Since \( \text{coh} \, \mathcal{X} \) has no nonzero projective objects, the composition of the embedding of \( \text{coh} \, \mathcal{X} \) into \( D^b(\text{coh} \, \mathcal{X}) \) with the projection functor \( \pi_X \) yields a bijection between the set of isomorphism classes of indecomposable objects of \( \text{coh} \, \mathcal{X} \) and the set of isomorphism classes of indecomposable objects of \( C_X \). We may identify the objects of \( \text{coh} \, \mathcal{X} \) with the ones of \( C_X \) by the bijection.

**Lemma 3.5.** [2, Section 3] An object \( T \in \text{coh} \, \mathcal{X} \) is a tilting object if and only if \( T \) is a cluster-tilting object of \( C_X \).

In particular, \( T_{\text{can}}(\mathcal{X}) \) and \( T_{\text{sq}}(\mathcal{X}) \) are basic cluster-tilting objects of \( C_X \). We denote by \( Q_{T_{\text{can}}(\mathcal{X})} \) (resp. \( Q_{T_{\text{sq}}(\mathcal{X})} \)) the Gabriel quiver of the endomorphism algebra \( \text{End}_{C_X} (T_{\text{can}}(\mathcal{X})) \) (resp. \( \text{End}_{C_X} (T_{\text{sq}}(\mathcal{X})) \)). The quivers have been listed in Figure 3.2 and Figure 3.3 respectively. We remark that the relation of the corresponding algebra is quite complicated in general and we do not need in the sequel.

**Figure 3.2.** Quiver \( Q_{T_{\text{sq}}(\mathcal{X})} \) with weight sequence \((p_1, \ldots, p_t)\).

**Figure 3.3.** Quiver \( Q_{T_{\text{can}}(\mathcal{X})} \) with weight sequence \((p_1, \ldots, p_t)\), where the label \( t-2 \) means that there are \( t-2 \) arrows from \( \mathcal{O}(\vec{c}) \) to \( \mathcal{O} \).

The following is a direct consequence of [8, Theorem 1.2].

**Proposition 3.6.** Let \( T \) be a basic cluster-tilting object of \( C_X \) and \( Q_T \) the Gabriel quiver of the endomorphism algebra of \( T \). Then

1. \( Q_T \) is mutation-equivalent to \( Q_{T_{\text{can}}(\mathcal{X})} \). In particular, the quiver \( Q_{T_{\text{can}}(\mathcal{X})} \) is mutation-equivalent to \( Q_{T_{\text{sq}}(\mathcal{X})} \).
2. \( Q_T \) admits a green-to-red sequence.
3.4. The classification. Denote by \( p = \operatorname{lcm}(p_1, \ldots, p_t) \) the least common multiple of \( p_1, \ldots, p_t \). The genus \( g_X \) of \( X \) is defined as

\[
g_X = 1 + \frac{1}{2}((t - 2)p - \sum_{i=1}^{t} \frac{p}{p_i}).
\]

A weighted projective line of genus \( g_X < 1(g_X = 1, \text{ resp. } g_X > 1) \) is of domestic (tubular, resp. wild) type. The domestic types are, up to permutation, \((1, p)\) with \( p \geq 1 \), \((p, q)\) with \( p, q \geq 2 \), \((2, 2, n)\) with \( n \geq 2 \), \((2, 3, 3)\), \((2, 3, 4)\) and \((2, 3, 5)\), whereas the tubular types are, up to permutation, \((2, 2, 2)\), \((3, 3, 3)\), \((2, 4, 4)\) and \((2, 3, 6)\). It is worth pointing out that a weighted projective line of domestic type is derived equivalent to a finite dimensional hereditary algebra of tame type.

**Proposition 3.7.** Let \( X \) be a weighted projective line. The quiver \( Q_{T_n(X)} \) is of finite mutation type if and only if \( X \) is of domestic type or of tubular type.

**Proof.** The “if” part follows from Lemma 2.16. More precisely, if \( X \) is of domestic type, then \( \text{coh} \ X \) is derived equivalent to a finite dimensional hereditary algebra of tame type. As a consequence, the quiver \( Q_{T_n(X)} \) is mutation-equivalent to an acyclic quiver of extended Dynkin type. If \( X \) is of tubular type, then the quiver \( Q_{T_n(X)} \) is as in Figure 2.1.

For the “only if” part, it suffices to prove that \( Q_{T_n(X)} \) is not of finite mutation type provided that \( X \) is of wild type. According to Lemma 2.17, it suffices to show that there is a quiver \( Q \) in \( \text{Mut}(Q_{T_n(X)}) \) such that \( Q \) admits a full subquiver \( Q_{a,b,c} \) for some \( 2 \leq c \leq b \leq a \) and \( a \geq 3 \).

Let \( X \) be a wild weighted projective line. According to the classification of weighted projective lines, the quiver \( Q_{T_n(X)} \) admits one of the following quivers as a subquiver

1. \( Q_{T_n(X')} \) with weight sequence \((2, 3, 7)\);
2. \( Q_{T_n(X')} \) with weight sequence \((2, 4, 5)\);
3. \( Q_{T_n(X')} \) with weight sequence \((3, 3, 4)\);
4. \( Q_{T_n(X')} \) with weight sequence \((2, 2, 2, 3)\);
5. \( Q_{T_n(X')} \) with weight sequence \((2, 2, 2, 2)\).

Let \( p \) be one of the weight sequences in (1)-(5). According to Proposition 3.6, there is a quiver \( Q_{p} \) in \( \text{Mut}(Q_{T_n(X)}) \) such that \( Q_{p} \) admits \( Q_{T_n(X')} \) as a full subquiver, where \( X' \) has the weight sequence \( p \). It suffices to show that there is a quiver in \( \text{Mut}(Q_{T_n(X')}) \) which admits a subquiver \( Q_{a,b,c} \) for \( 2 \leq c \leq b \leq a \) and \( 3 \leq a \). Let us label the vertices of \( Q_{T_n(X')} \) as in Figure 3.3. For \( p = (2, 3, 7) \), let

\[
i = (O, O(6\bar{x}_3), O(\bar{c}), O(2\bar{x}_3), O(\bar{x}_3), O(2\bar{x}_2), O(6\bar{x}_3), O(5\bar{x}_3), O(\bar{x}_2), O(2\bar{x}_3), O(3\bar{x}_3), O(2\bar{x}_2), O(\bar{x}_3), O(\bar{c})).
\]

For \( p = (2, 4, 5) \), let

\[
i = (O, O(\bar{c}), O(\bar{x}_3), O(2\bar{x}_3), O(3\bar{x}_3), O(3\bar{x}_2), O(\bar{c}), O(3\bar{x}_3), O(4\bar{x}_3), O(\bar{x}_2), O(2\bar{x}_3)).
\]
For \( \mathbf{p} = (3, 3, 4) \), let \( i = (\mathcal{O}, \mathcal{O}(\bar{x}_1), \mathcal{O}(\bar{x}_2), \mathcal{O}(\bar{x}_3), \mathcal{O}(\bar{c}), \mathcal{O}) \). For \( \mathbf{p} = (2, 2, 2, 3) \), let \( i = (\mathcal{O}(\bar{c}), \mathcal{O}) \). It is straightforward to check that \( Q_{2,2,3} \) is a subquiver of \( \mu_i(Q_{\text{tw}(X)}) \) in each case. Finally, for \( \mathbf{p} = (2, 2, 2, 2) \), denote by \( i = (\mathcal{O}(\bar{c}), \mathcal{O}) \). We find that \( Q_{2,3,5} \) is a subquiver of \( \mu_i(\mathcal{O}(\bar{c})(Q_{\text{tw}(X)})) \) in this case. This completes the proof.

4. The Existence and Non-Existence of Maximal Green Sequence

This section is devoted to proving the main result of this note. We begin with the hyperbolic case. Recall that a weighted projective line \( X \) with weight sequence \((p_1, \ldots, p_t)\) is of hyperbolic type if \( p_1 = p_2 = \cdots = p_t = 2 \).

Let \( X \) be of hyperbolic type. We denote by \( Q_t \) the quiver \( Q_{\text{tw}(X)} \) in this case and relabel the vertices of \( Q_t \) as in Figure 4.4. We will always identify \( Q_t \) with a full subquiver of \( Q_{t+1} \) such that the vertex \( t+1 \) is the unique vertex which does not belong to \( Q_t \).

**Lemma 4.1.** Let \( i_t \) be a maximal green sequence of \( Q_t \). Denote by \( \bullet \rightarrow \circ \) the unique multiple arrows in \( \mu_{i_t}(Q_t) \). If \( i_{t+1} := (i_t, t+1, \circ, \bullet) \) is a maximal green sequence of \( Q_{t+1} \), then \( \circ \rightarrow t+1 \) is the unique multiple arrows in \( \mu_{i_{t+1}}(Q_{t+1}) \) and \( i_{t+2} := (i_{t+1}, t+2, t+1, \circ) \) is a maximal green sequence of \( Q_{t+2} \).

**Proof.** We apply \( \mu_{i_t} \) to the quiver \( Q_{t+2} \). Since \( i_t \) is a sequence of vertices of \( Q_t \), \( \mu_{i_t}(Q_t) \) is a full subquiver of \( \mu_{i_t}(Q_{t+2}) \). In particular, the vertex set of \( \mu_{i_t}(Q_t) \) is a subset of the vertex set of \( \mu_{i_t}(Q_{t+2}) \). Since \( \mu_{i_t}(Q_t) \cong Q_t \), we will denote the vertex set of \( \mu_{i_t}(Q_t) \) by \( \{\bullet, \circ, 1, \ldots, t\} \) and the vertex set of \( \mu_{i_t}(Q_{t+2}) \) by \( \{\bullet, \circ, 1, \ldots, t, t+1, t+2\} \).

Let \( \hat{B} = (b_{ij}) \in M_{2(t+4)}(\mathbb{Z}) \) be the skew-symmetric matrix associated to the framed quiver \( \hat{Q}_{t+2} \) and \( \hat{B}^\circ \) the submatrix of \( \hat{B} \) consisting of the first \( t+4 \) columns. We index the columns of \( \hat{B}^\circ \) by \( \bullet, \circ, 1, \ldots, t+2 \).

**Claim 1:** The principal part of \( \mu_{i_t}(\hat{B}^\circ) \) is

\[
\begin{bmatrix}
0 & -2 & 1 & \cdots & 1 & 1 & 1 \\
2 & 0 & -1 & \cdots & -1 & -1 & -1 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & & & \ddots & & & \vdots \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
\end{bmatrix} \in M_{t+4}(\mathbb{Z}).
\]
The proof of this claim will be separated into three steps. Here we work on the quivers.

**Step 1:** The full subquiver of $\mu_{4t}(Q_{t+1})$ consisting of vertices $\bullet$, $\circ$, $t + 1$ and $t + 2$ has the following form:

Suppose that there are $a$ arrows from vertex $t + 1$ to vertex $\bullet$ and $b$ arrows from vertex $\circ$ to vertex $t + 1$ and denote by $Q(a, b)$ the full subquiver consisting of vertices $\bullet$, $\circ$ and $t + 1$. Since there is a symmetry between $t + 1$ and $t + 2$ in $Q_{t+2}$, it suffices to prove that $a = b = 1$.

Denote by $B = \begin{bmatrix} 0 & -2 & a \\ 2 & 0 & -b \\ -a & b & 0 \end{bmatrix}$ the associated skew-symmetric matrix of $Q(a, b)$. Denote by

$$c_{12} = -2 + \text{sgn}(a)[ab]_+, \quad c_{13} = a - \text{sgn}(c_{12})[bc_{12}]_+, \quad c_{23} = -b + \text{sgn}(c_{12})[-c_{12}c_{13}]_+.$$  

By Fomin-Zelevinsky’s matrix mutation formula, we obtain

$$C := \mu_s(\mu_o(\mu_{t+1}(B))) = \begin{bmatrix} 0 & c_{12} & c_{13} \\ -c_{12} & 0 & c_{23} \\ -c_{13} & -c_{23} & 0 \end{bmatrix}.$$  

Note that the associated skew-symmetric matrix of $\mu_s(\mu_o(\mu_{t+1}(Q(a, b))))$ is the matrix $C$. By the assumption that $i_{t+1}$ is a maximal green sequence of $Q_{t+1}$, we have $\mu_{i_{t+1}}(Q_{t+1}) \cong Q_{t+1}$. In particular, the quiver $\mu_s(\mu_o(\mu_{t+1}(Q(a, b))))$ is a full subquiver of $Q_{t+1}$ via the isomorphism $\mu_{i_{t+1}}(Q_{t+1}) \cong Q_{t+1}$. The remaining proof is a discussion of the values of $a$ and $b$, from which we can deduce that $a = 1 = b$. We will denote by $Q(C)$ the associated quiver of the skew-symmetric matrix $C$.

**Case 1:** $a < 0$, $b > 0$. A direct computation shows that $c_{12} = -2$, $c_{13} = a$ and $c_{23} = -b$. Consequently, the associate quiver $Q(C)$ is not a full subquiver of $Q_{t+1}$.

**Case 2:** $a > 0$, $b < 0$. We have $c_{12} = -2$, $c_{13} = a - 2b \geq 3$, which implies that the associated quiver $Q(C)$ is not a full subquiver of $Q_{t+1}$.

**Case 3:** $a \leq 0$, $b \leq 0$. We have $c_{12} = -2 - ab \leq -2$. Since $Q(C)$ is a full subquiver of $Q_{t+1}$, we have $c_{12} = -2$. Hence $ab = 0$, i.e., $a = 0$ or $b = 0$. In each case, one can show that $Q(C)$ is not a full subquiver of $Q_{t+1}$.

**Case 4:** $a \geq 0$, $b \geq 0$. Similar to the Case 3, we obtain $-2 \leq c_{12} = -2 + ab \leq 2$. In particular, $0 \leq ab \leq 4$. A direct computation shows that $a = 1 = b$ is the unique value such that $Q(C)$ is a full subquiver of $Q_{t+1}$. This completes the proof for the statement in **Step 1**.

As a direct consequence of the statement of **Step 1**, the quiver $Q(C)$ has the form as in Figure 4.5. Consequently, $\circ \Rightarrow t + 1$ is the unique multiple arrows in $\mu_{i_{t+1}}(Q_{t+1})$.  


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Step 2: There are no arrows between vertex $k \in \{1, \ldots, t\}$ and vertex $t + 1$ in the quiver $\mu_i(Q_{t+2})$.

Without loss of generality, we may assume that there are $a$ arrows from vertex $i$ to vertex $t + 1$ and we consider the full subquiver consisting of vertices $\bullet, \circ, t + 1, i$:

By applying the mutation sequence $t + 1, \circ, \bullet$ to the above quiver, we obtain

By the assumption that $i_{t+1}$ is a maximal green sequence of $Q_{t+1}$, we know that the quiver $Q(4)$ is a full subquiver of $\mu_{i_{t+1}}(Q_{t+1})$ and we conclude that $a = 0$. This completes the proof of the statement in Step 2.

Step 3: There are no arrows between vertex $k \in \{1, \ldots, t + 1\}$ and vertex $t + 2$ in the quiver $\mu_i(Q_{t+2})$.

Note that there is a symmetry between vertex $t + 1$ and $t + 2$ in the quiver $Q_{t+2}$ and the mutation sequence $i_k$ does not involve the vertices $t + 1$ and $t + 2$. Then the statement of Step 3 follows the statement of Step 2 directly.

Now Claim 1 is a direct consequence of the statements in Step 1, 2, 3.

Claim 2: Up to a permutation of the rows associated to $\bullet^*, \circ^*, 1^*, \ldots, t^*$, the coefficient part $C_{i_k}$ of $\mu_{i_k}(\hat{B}^o)$ has the following form:

$$
\begin{bmatrix}
-I_{t+2} & X \\
0 & I_2
\end{bmatrix}
$$

where $X \in M_{(t+2) \times 2}(\mathbb{Z})$ with non-negative entries.

We fix a quiver pattern of the framed quiver $\tilde{Q}_{t+2}$ of $Q_{t+2}$ by assigning $\tilde{Q}_{t+2}$ to the root vertex $t_0 \in T_{t+4}$. Each sequence $i$ of vertices of $Q_{t+2}$ induces a path of $T_{t+4}$ with starting point $t_0$. We denote by the ending point $s_i$ and denote by $G_i := G_{s_i}$ (resp. $C_i$) the $G$-matrix (resp. $C$-matrix) at $s_i$. 
Since the sequence $i_t$ does not involves the vertices $t+1$ and $t+2$. It follows that $G_{i_t} = \begin{bmatrix} A & 0 \\ Y & I_2 \end{bmatrix}$, where $A \in M_{t+2}(\mathbb{Z})$ is invertible and $Y \in M_{2 \times (t+2)}(\mathbb{Z})$ with non negative entries. By the tropical dualities between $G$-matrices and $C$-matrices (2.15), we have

$$C_{i_t} = G_{i_t}^{-T} = \begin{bmatrix} A^{-T} & -A^{-T}Y^T \\ 0 & I_2 \end{bmatrix}.$$ 

Since $i_t$ is a maximal green sequence, it follows that $A^{-T}$ is a permutation of $-I_{t+2}$. On the other hand, the entries of $-A^{-T}Y^T$ are non negative by the sign-coherence of $c$-vectors. This finishes the proof of Claim 2.

According to Claim 1 and Claim 2, up to permutation of indices, we may assume that

$$\mu_{i_t}(\hat{B}^o) = \begin{bmatrix} \bullet & \circ & 1 & \cdots & t & t+1 & t+2 \\ 0 & -2 & 1 & \cdots & 1 & 1 & 1 \\ 2 & 0 & -1 & \cdots & -1 & -1 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & 0 & \cdots & a_\bullet & a_\circ & a_\bullet \\ 0 & -1 & 0 & \cdots & a_0 & a_0 & a_0 \\ 0 & 0 & -1 & \cdots & a_1 & a_1 & a_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_t & a_t \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

where $\bullet, \circ, 1, \cdots, t+2$ are (relabelled) vertices of $\mu_{i_t}(Q_{t+2})$, $a_\bullet, a_0, a_1, \ldots, a_t$ are non negative integers.

By Fomin-Zelevinsky’s mutation rule, we obtain $\mu_{i_{t+1}}(\hat{B}^o)$ as in Figure 4.6. Note that $i_{t+1} = (i_t, t+1, \circ, \bullet)$ is a maximal green sequence of $Q_{t+1}$. It follows that the submatrix formed by the first $t+3$ row indices and the first $t+3$ column indices of the coefficient part of $\mu_{i_{t+1}}(\hat{B}^o)$ is a permutation of $-I_{t+1}$. Hence we have

$$a_\bullet = 1, a_0 = 1, a_1 = 0, \ldots, a_t = 0.$$ 

Finally, we apply the mutation sequence $\circ, t+1, t+2$ to the matrix $\mu_{i_{t+1}}(\hat{B}^o)$, we compute the matrix $\mu_{i_{t+2}}(\hat{B}^o)$ as in Figure 4.7. Note that the coefficient part of the matrix $\mu_{i_{t+2}}(\hat{B}^o)$ is a permutation of $-I_{t+4}$. According to Remark 2.8, we conclude that $i_{t+2}$ is a maximal green sequence of $Q_{t+2}$. This completes the proof of the lemma.

\[\Box\]

**Proposition 4.2.** Assume that $t \geq 3$. The quiver $Q_{T_w(\infty)}$ admits a maximal green sequence.

**Proof.** According to Lemma 2.13, it suffices to show that the quiver $Q_t$ admits a maximal green sequence for $t \geq 3$. We label the vertices of $Q_t$ as in Figure 4.4. It is straightforward
\[ \mu_{t+1}(\hat{B}^\circ) = \mu_\bullet \circ \mu_0 \circ \mu_{t+1}(\hat{B}^\circ) \]

\[
\begin{bmatrix}
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & 1 & \cdots & 1 & -2 & 1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-1 & 2 & -1 & \cdots & -1 & 0 & -1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
1 - a_\bullet & -1 & 0 & \cdots & 0 & 0 & a_\bullet \\
1 - a_0 & 0 & 0 & \cdots & 0 & -1 & a_0 \\
-a_1 & 0 & -1 & \cdots & 0 & 0 & a_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
-a_1 & 0 & 0 & \cdots & -1 & 0 & a_1 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\end{bmatrix}
\]

**Figure 4.6.** The matrix \( \mu_{t+1}(\hat{B}^\circ) \).

to check that \( i_3 = (\circ, 1, 2, \circ, *, 3, 2, 1, *, \circ) \) is a maximal green sequence for \( Q_3 \) and \( * \Rightarrow 3 \) is the unique multiple arrows of \( \mu_{i_3}(Q_3) \). Furthermore, \( i_4 := (i_3, 4, 3, *) \) is a maximal green sequence of \( Q_4 \). Now the result follows from Lemma 4.1.

**Theorem 4.3.** Let \( \mathcal{X} \) be a weighted projective line.

1. There is a quiver \( Q' \) in \( \text{Mut}(Q_{\text{Im}}(\mathcal{X})) \) such that \( Q' \) admits a maximal green sequence.
2. There is a quiver \( Q'' \) in \( \text{Mut}(Q_{\text{Im}}(\mathcal{X})) \) such that \( Q'' \) does not admit a maximal green sequence if and only if \( \mathcal{X} \) is of wild type.

**Proof.** If \( t = 2 \), then \( Q_{\text{Im}}(\mathcal{X}) \) is an acyclic quiver. Hence \( Q_{\text{Im}}(\mathcal{X}) \) admits a maximal green sequence (cf. [1] for instance). Now assume that \( t \geq 3 \). According to Proposition 3.6, we know that \( Q_{\text{Im}}(\mathcal{X}) \) belongs to \( \text{Mut}(Q_{\text{Im}}(\mathcal{X})) \). Consequently, the first statement follows from Proposition 4.2 directly.

The “only if” part of (2) follows from the main result of [18]. Namely, let us assume that \( \mathcal{X} \) is not of wild type, then \( Q_{\text{Im}}(\mathcal{X}) \) is of finite mutation type by Proposition 3.7. According to the main result of [18], each quiver in \( \text{Mut}(Q_{\text{Im}}(\mathcal{X})) \) admits a maximal green sequence. To prove the “if” part, we use Lemma 2.11. Let \( \mathcal{X} \) be a weighted projective line of wild type. Similar to the proof of the “only if” of Proposition 3.7, we conclude that there is a quiver \( Q \) in \( \text{Mut}(Q_{\text{Im}}(\mathcal{X})) \) such that \( Q_{a,b,c} \) is a full subquiver of \( Q \), where \( 2 \leq c \leq b \leq a \) and \( 3 \leq a \). Consequently, \( Q \) does not admit a maximal green sequence by Lemma 2.11.

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\[ \mu_{t+1}(\widehat{B}^o) = \]

\[
\begin{pmatrix}
\cdot & o & 1 & \cdots & t & t+1 & t+2 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & 1 & \cdots & 1 & -2 & 1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
-1 & 2 & -1 & \cdots & -1 & 0 & -1 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
0 & -1 & 0 & \cdots & 0 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
\mu_{t+2}(\widehat{B}^o) = \]

\[
\begin{pmatrix}
\cdot & o & 1 & \cdots & t & t+1 & t+2 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & -2 \\
-1 & -1 & -1 & \cdots & -1 & 2 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
0 & 0 & -1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -1 & 0 \\
-1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\end{pmatrix}
\]

**Figure 4.7.** The matrix \( \mu_{t+2}(\widehat{B}^o) \).

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