Moving NRQCD

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I discuss the derivation and applications of Moving NRQCD (MNRQCD), a generalization of NRQCD which allows the treatment of heavy quarks moving with a finite velocity. This formalism is vital in reducing discretization errors in calculations of large recoil decays, such as $B \rightarrow K^{*} + \gamma$ or the Isgur-Wise function at large $w$.

1. INTRODUCTION

Many of the most interesting experimental probes of the Standard Model involve exclusive weak decays of $B$ mesons. Because the $B$ is much heavier than charmed and strange mesons, these decay products have large recoil momenta. This presents a problem for lattice calculations of these processes, since (naively) the spatial lattice spacing must be fine enough to control discretization errors of order $(a p_{\text{recoil}})$. For example, in the process $B \rightarrow K^{*} + \gamma$, the $K^{*}$ is kinematically constrained to have a momentum of about 2.5 GeV relative to the $B$; this means that an inverse lattice spacing of 10 GeV would be required just to reduce $(a p_{\text{recoil}})$ to about 25%.

A closely related problem has already been dealt with in lattice studies of $B$ physics, namely that the total energy of the $b$ quark is much larger than its (dynamically important) recoil momentum and kinetic energy within the meson. Several solutions have been found; the NRQCD and FNAL approaches \[1,2\] are the most widely used. In this talk I will discuss extending the NRQCD method to heavy quarks moving with finite velocity; this allows much of the recoil momentum to be removed from the calculation before discretization.

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The idea is that “removing the rest mass” in NRQCD is equivalent to shifting the 4-momentum of the quark, $P_{b}$, by an amount proportional to the time 4-vector, $\hat{t}$, while MNRQCD shifts the $b$ quark’s 4-momentum by a multiple of an arbitrary time-like 4-vector (preferably chosen to be the 4-velocity, $\hat{U}_{B}$, of the $B$ meson):

NRQCD : $P_{b} \rightarrow p_{b} + m_{b}\hat{t}$,

MNRQCD : $P_{b} \rightarrow p_{b} + m_{b}\hat{U}_{B}$,

where $p_{b}$ is the shifted 4-momentum of the quark, which is relevant for lattice discretization errors, and $m_{b}$ is an arbitrarily chosen energy shift parameter (usually chosen non-relativistically so that the kinetic and static masses of the physical state are equal)\[2\]. If $\hat{U}_{B} = \hat{t}$ the two methods are identical, but when $\hat{U}_{B} \neq \hat{t}$ the MNRQCD shift reduces the large spatial components of $P_{b}$.

Some work on this method has appeared previously. In the same way that MNRQCD is a generalization of NRQCD, the “moving static” formalism of Mandula and Ogilvie \[3\] (MO) generalizes the static theory, which in turn is the infinite mass limit of NRQCD. Continuum HQET derivations are usually done in a Lorentz-covariant manner; the terms in the continuum limit of the MNRQCD action at some order in $1/M$ should agree (up to field redefinitions) with the corresponding HQET action. Finally, Hashimoto and Matsufuru \[4\] (HM) have derived the $O(p^{2}/M)$ terms in the MNRQCD action and done numerical studies at low recoil; Both MO and HM have also studied non-perturbative renormalization of the input 4-velocity in the moving static and MNRQCD actions, respectively. In this talk, I discuss the kinematics of specific high-recoil decays and the expected improvement in discretization errors from using MNRQCD. I sketch a method of deriving the tree-level MNRQCD action by means:

Note that $|\hat{t}|^{2} = |\hat{U}_{B}|^{2} = -1$ even when working with the Wick-rotated Euclidean theory.

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ing the Foldy-Wouthuysen-Tani (FWT) transformation and use this method to write down all \( \mathcal{O}(1/M) \) terms in continuum MNRQCD.

2. DECAY KINEMATICS

The most illustrative decay to consider is \( B \to K^* + \gamma \). Because it is a two particle decay, the recoil 3-momentum of the \( K^* \) (in the rest frame of the \( B \)) is fixed to be 2.5GeV, which corresponds to an energy of 2.6GeV and a relativistic \( \Gamma \) factor of 3.0. The energy scale \( \mu_{HAD} \) which controls discretization errors in Clover light hadron spectroscopy is a few hundred MeV; inverse lattice spacings of 500-1000MeV give accuracies of a few percent. In the \( B \)'s rest frame, however, \( \mu_{K^*} \) will be controlled by \( p_{K^*} \), the \( K^* \)'s 4-momentum. This scale is about an order of magnitude larger, so inverse lattice spacings of 5-10GeV might be required to achieve few percent accuracies.

In contrast, consider the rest frame of the \( K^* \). If we make this the rest frame of our lattice discretization, then \( \mu_{K^*} \) will be the usual few hundred MeV, but the \( B \) will be boosted to \( \Gamma = 3.0 \); \( B \) meson discretization errors will dominate. There are two sorts of discretization errors we need to consider: “boosted muck” and “moving mass”. The boosted muck errors arise from Lorentz contraction; the brown muck cloud around the \( b \) quark will be Lorentz contracted in the direction of motion when the \( B \) meson is moving. This should result in discretization errors with a typical scale of \( \Gamma_B^2 \mu_{muck} \), where \( \mu_{muck} \approx \mu_{HAD} \) is the \( \mu \) for a \( B \) meson at rest.

The moving mass errors are the same ones we encountered when boosting the \( K^* \); a particle of mass \( m \) has spatial momentum \( |p| = m|v|\Gamma = m\sqrt{T^2 - 1} \) when moving with velocity \( v \) and boost factor \( \Gamma \). After performing the MN-RQCD shift of the 4-momentum in Eq. (3), the relevant mass is \( \Lambda_B = m_B - m_b \), for which most NRQCD calculations obtain a value of about 1GeV. The moving mass discretization scale in this case then is \( \Lambda_B \sqrt{T^2 - 1} \approx 3 \)GeV, i.e. roughly the same as for the \( K^* \) in the original frame. This is due to the numerical accident that \( m_B^2 \approx \Lambda_B \); \( v \) and \( \Gamma \) don’t care which meson is at rest so equal “mass” mesons have equal momenta.

The problem in both the above frames is that the discretization scales of the two mesons are very different. An optimal frame is one in which the two mesons are equally well discretized. Since \( m_K^* \approx \Lambda_B \), this means that the two mesons should have roughly equal boost factors, i.e. \( \Gamma_B = \Gamma_K^* = \sqrt{(1 + \Gamma_{tot})/2} \approx \sqrt{2} \), where \( \Gamma_B \) and \( \Gamma_K^* \) are the meson boosts relative to the lattice frame and \( \Gamma_{tot} \), their boost relative to each other, has been take to be about 3. This reduction in \( \Gamma \) by a factor of 2 leads to an even bigger improvement in the spatial momentum, which depends upon \( \Gamma \) like \( \sqrt{\Gamma^2 - 1} \). The spatial momenta of the mesons in this frame are \( p_K^* \approx p_B \approx 1 \)GeV, to be compared with the previous values of 2.5-3GeV. The boosted muck errors in this frame are less than twice \( \mu_{HAD} \); moving mass errors should dominate. Note that this is a worst-case analysis; the moving mass errors of the \( B \) meson might be further reduced by using a different choice of \( m_b \).

The other example I will consider is \( B \to D^* + e + \nu \). Since this is a three particle decay, the \( D^* \)'s recoil momentum can range between 0 and 2.3GeV. The high recoil region gives much cleaner experimental results for \( e^+e^- \) colliders running at the \( \Upsilon(4S) \), while most of the theoretical work has been done near zero recoil. If we work in the rest frame of the \( B \), then at maximum recoil the \( D^* \) has 3-momentum, energy, and \( \Gamma \) of 2.3GeV, 3.0GeV, and 1.1, respectively. If we use MNRQCD, however, we can subtract from the 4-momenta of both the \( B \) and \( D^* \) mesons. Since \( \Lambda \) should be the same for the two mesons (i.e. about 1GeV), again we want to choose the frame where they have equal boost factors (which will be about 1.025). This implies a moving mass momentum of 220MeV, which is about the same size as the (not very) boosted muck scale. This means that, for a given lattice spacing, these runs should have the same discretization errors as light spectroscopy calculations. Even if we decide not to use MNRQCD for the charm quark, the \( D^* \)'s spatial momentum will still only be about 500MeV.

3. DERIVATION

NRQCD is a (non-renormalizable) effective field theory which reproduces the QCD heavy
quark action. As such, one should perform the normal matching procedure to adjust the coefficients in the action. When one is working at tree-level, however, a (much simpler) classical derivation can be used; this is just the FWT transformation. The FWT transformation consists of going to a Dirac basis which diagonalizes $\gamma_0$ and then, order by order in $p/m$, block diagonalizing the fermion kernel into non-interacting quark and anti-quark sectors. This is equivalent to requiring that $\gamma_0$ commute with the fermion kernel. The 4-momentum is then shifted by rescaling the fermion fields by factors of $\exp(m_b \gamma_0 t)$.

To derive the tree-level MNRQCD action, one just needs to rewrite the NRQCD derivation in covariant language. This is done by writing the 4-vector components of 4-vectors as their dot product.

The FWT transformation consists of the following procedure to adjust the coefficients in the action. When one is working at tree-level, one should perform the normal matching procedure to adjust the coefficients of $m_b^2$. This is achieved by writing the Fermi-Walker-Transport (FWT) transformation so that the fermion kernel commutes with $\Gamma = \Gamma^{\mu\nu}/U_{\mu\nu}$ where $\Gamma = \Gamma^{\mu\nu}/U_{\mu\nu}$ is the velocity of the particle and $i$, $j$ run over spatial directions. This is a Lorentz contraction term; when $|v| = 1$ this term projects out the transverse momenta.

The other subtlety involves the other $O(1/M)$ term in the NRQCD action: $\sigma \cdot B/2m_0$. The transcribed expression is in terms of $\tilde{\sigma}_{\mu\nu}$, the commutator of the moving frame $\gamma$ matrices. The problem is that we need to write this term using $\sigma_{\mu\nu}$ (the commutator of the rest frame $\gamma$'s) without changing the spinor basis. This is achieved by boosting only the vector indices. One finds:

$$\tilde{\sigma} \cdot \tilde{B} \rightarrow \frac{\Gamma}{1 + \Gamma} (\sigma \cdot v) \left( \frac{v \cdot \mathbf{B}}{\mathbf{E}} \right),$$

where $\Gamma = \Gamma_U$. The first term is just the moving $B$ field written in terms of rest frame $E$ and $B$, while the second term again completes a transpose projector when $|v| = 1$ (the $E$ term is missing because $v \cdot E \times v$ vanishes). The full $O(1/M)$ continuum MNRQCD action is then

$$S = \tilde{Q} [D_i + H_{MN}] Q$$

and

$$H_{MN} = i \mathbf{v} \cdot \mathbf{D} - \frac{D_i \left( \delta^{ij} - v^i v^j \right) D_j}{2m_0}$$

where

$$\sigma_i \left( \delta^{ij} - v^i v^j \right) (\mathbf{B} + \mathbf{v} \times \mathbf{E})_j.$$