Homogeneous spherically symmetric bodies with a nonminimal coupling between curvature and matter: the choice of the Lagrangian density for matter

Orfeu Bertolami*†
Departamento de Física e Astronomia, Faculdade de Ciências,
Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Jorge Paramos‡
Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico,
Universidade Técnica de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

(Dated: May 7, 2014)

PACS numbers: 04.20.Fy, 04.80.Cc, 97.10.Cv

I. INTRODUCTION

Despite its great experimental success (see e.g. Refs. [1, 2]), General Relativity (GR) is not the most encompassing way to couple matter with curvature. Indeed, matter and curvature can be coupled, for instance, in a nonminimal way [3], a fact that can have a bearing on the dark matter [3–5] and dark energy [6–8] problems. This putative nonminimal coupling impacts the well-known energy conditions [9] and can give rise to wormhole and time machines [10]. In Ref. [11], several phenomenological aspects of the dynamics of perfect fluids nonminimally coupled to curvature were addressed — in particular, the scenario of an axisymmetric dust distribution with constant density.

Another interesting issue that arises in the context of gravity theories with a nonminimal coupling between curvature and matter is the fact that the Lagrangian degeneracy in the description of a perfect fluid, encountered in GR [12, 13] is lifted [14]: indeed, since this quantity explicitly appears in the modified equations of motion, two Lagrangian densities leading to the same energy-momentum tensor have different dynamical implications, whereas in GR they are physically indistinguishable.

In what follows we shall examine the role that two possible descriptions of a perfect fluid (i.e. two different choices of the Lagrangian density) have on the structure of a spherically symmetric gravitational body.

This does not contradict the previous work arguing that a more suitable choice for the Lagrangian density of a perfect fluid is $\mathcal{L} = -\rho$ [14] — but aims to show that, under adequate circumstances, the adopted Lagrangian density is not all that crucial in determining the observable implications of the nonminimal coupling between matter and curvature.

II. THE MODEL

One considers a model that exhibits a non-minimal coupling between geometry and matter, as expressed in the action functional [3],

$$ S = \int [\kappa f_1(R) + f_2(R)\mathcal{L}] \sqrt{-g} d^4x \quad .$$

where $f_i(R)$ $(i = 1, 2)$ are arbitrary functions of the scalar curvature, $R$, $g$ is the determinant of the metric and $\kappa = c^4/16\pi G$.

Variation with respect to the metric yields the modified field equations,

$$(\kappa F_1 + F_2 \mathcal{L}) G_{\mu\nu} = \frac{1}{2} f_2 T_{\mu\nu} + \Delta_{\mu\nu} (\kappa F_1 + F_2 \mathcal{L}) + (2) \frac{1}{2} g_{\mu\nu} [\kappa (f_1 - F_1 R) - F_2 R \mathcal{L}] \quad ,$$

with $F_i \equiv df_i/dR$ and $\Delta_{\mu\nu} = \nabla_\mu \nabla_\nu - g_{\mu\nu} \Box$. As expected, GR is recovered by setting $f_1(R) = R$ and $f_2(R) = 1$.

The trace of Eq. (2) reads

$$(\kappa F_1 + F_2 \mathcal{L}) R = \frac{1}{2} f_2 T - 3 \Box (\kappa F_1 + F_2 \mathcal{L}) + 2\kappa f_1 \quad .$$

Resorting to the Bianchi identities, one concludes that the energy-momentum tensor of matter may not be (covariantly) conserved, since

$$\nabla_\mu T^{\mu\nu} = \frac{F_2}{f_2} (g^{\mu\nu} \mathcal{L} - T^{\mu\nu}) \nabla_\mu R \quad ,$$

can be non-vanishing.
Using the energy-momentum tensor of a perfect fluid,

\[ T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu} \rightarrow T = 3p - \rho , \]

with \( u^\mu u_\nu = -1 \) and \( u^\mu \nabla_\nu u_\mu = 0 \), together with the projection operator

\[ P_{\mu\nu} = u_\mu u_\nu + g_{\mu\nu} \rightarrow P_{\mu\nu}u^\mu = 0 , \]

one gets

\[ P_{\mu\beta} \nabla_\alpha [(\rho + p)u^\alpha u^\beta + pg^{\alpha\beta}] = \]

\[ (\rho + p)u^\alpha \nabla_\alpha u_\mu + P^{\alpha\beta}_\mu \nabla_\alpha p = (\rho + p)a_\mu + P^{\beta\mu}p_\beta , \]

with \( a_\mu = a^\alpha \nabla_\alpha u_\mu \) and, for \( F_2 \neq 0 \),

\[ P_{\mu\beta} \left( g^{\alpha\beta} L - T^{\alpha\beta} \right) \nabla_\alpha R = (u_\mu u^\alpha + \delta^\alpha_\mu)(L - p)R_\alpha , \]

so that Eq. (4) yields

\[ (\rho + p)a_\mu = -P^{\mu\beta}_\beta p_\beta + \frac{F_2}{f_2} \left( u_\mu u^\alpha + \delta^\alpha_\mu \right)(L - p)R_\alpha . \]

If \( F_2(L - p) \neq 0 \), this equation shows that the energy-momentum tensor is not covariantly conserved. As will be shown in subsequent sections, this possibility will play a crucial role in the dynamical behaviour of the pressure inside a homogeneous spherical body.

### III. STATIONARY CASE

Imposing spherical symmetry and stationarity, one adopts the line element

\[ ds^2 = -e^{2\phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\Omega^2 , \]

from which follows that \( u^\alpha = -(e^{-\phi}, 0) \),

\[ P^{00} = (u^0)^2 + g^{00} = 0 , \quad P^{rr} = g^{rr} = e^{-2\lambda} , \]

and

\[ a_\mu = u^\beta \nabla_\beta u_\mu = \Gamma^{0}_{0\mu} = \delta^\alpha_\mu \phi'(r) , \]

where the prime denotes differentiation with respect to the radial coordinate \( r \). With the above, Eq. (9) becomes

\[ (\rho + p)\phi'(r) = -\phi'(r) + \frac{F_2}{f_2}(L - p)R'(r) . \]

One now introduces the mass function \( m(r) \) through

\[ e^{-2\lambda} = 1 - \frac{m(r)}{8\pi K}\]

which allows the rewriting of the Einstein tensor components as

\[ G_{00} = e^{2\lambda} \frac{m(r)}{8\pi K} , \]

\[ G_{rr} = \frac{1}{r} \left( 2\phi' - \frac{m(r)}{r[8\pi K - m(r)]} \right) , \]

and the energy-momentum components

\[ T_{00} = \rho e^{2\phi} , \quad T_{rr} = pe^{2\lambda} , \quad T_{\theta\theta} = pr^2 , \]

with trace \( T = 3p - \rho \). One also has

\[ \Delta_{00} f_i = e^{2(\phi - \lambda)} \left[ f_{ii}'' + \left( \frac{2}{r} - \lambda' \right) f_i' \right] , \]

\[ \Delta_{rr} f_i = - \left( \frac{2}{r} + \phi' \right) f_i' , \]

\[ \Delta_{\theta\theta} f_i = - e^{-2\lambda} r^2 \left[ f_{ii}'' + \left( \frac{1}{r} + \phi' - \lambda' \right) f_i' \right] , \]

\[ \Box f_i = e^{-2\lambda} \left[ f_{ii}'' + \left( \frac{2}{r} + \phi' - \lambda' \right) f_i' \right] . \]

With the above, the \( 0 - 0 \) component of Eq. (2) becomes

\[ m' \left( \frac{2}{r} + \frac{d}{dr} \right) (\kappa F_1 + F_2 L) = \]

\[ \left[ 2 (8\pi K r - m) \frac{d^2}{dr^2} + \left( 32 \pi K - \frac{3m}{r} \right) \frac{d}{dr} \right] (\kappa F_1 + F_2 L) + 8\pi K [\kappa (F_1 R - f_i) + f_2 p + F_2 R L] , \]

while the \( r - r \) component reads

\[ \phi' \left[ \kappa F_1 + F_2 L + \frac{r d}{2 dr} (\kappa F_1 + F_2 L) \right] = \]

\[ - \frac{d}{dr} (\kappa F_1 + F_2 L) + \frac{m}{2r(8\pi K r - m)} (\kappa F_1 + F_2 L) + \]

\[ \frac{2\pi K r^2}{8\pi K r - m} [\kappa (f_1 - F_1 R) + f_2 p - F_2 R L] . \]

The trace of the equations of motion, Eq. (3), becomes

\[ (\kappa F_1 + F_2 L) R \frac{1}{2} f_2 (3p - \rho) + 2\kappa f_1 \]

\[ - 3 \left( 1 - \frac{m}{8\pi K} \right) \times \]

\[ \left[ \frac{d^2}{dr^2} + \left( \phi' + \frac{3}{2r} + \frac{1}{2} \frac{8\pi K - m'}{8\pi K r - m} \right) \frac{d}{dr} \right] (\kappa F_1 + F_2 L) . \]
One thus obtains three differential equations for four unknowns, $m(r)$, $\phi(r)$, $\rho(r)$ and $p(r)$. Solving these requires an additional equation, namely an equation of state (EOS) relating the pressure with the energy density, $p = p(\rho)$. In the following sections one assumes instead a homogeneous density, which allows for a considerable simplification of the dynamical behaviour of the aforementioned system — thus highlighting the impact of a nonminimal coupling between matter and curvature and the choice of the Lagrangian density for matter.

A. Homogeneous sphere

In order to isolate the effect of the non-minimal coupling, one considers the linear form for the curvature term $f_1(R) = R$. One studies the impact of the nonminimal coupling on a homogeneous sphere, $\rho = \rho_0$. Considering the issue of how to properly choose the Lagrangian density of a perfect fluid, one writes $\mathcal{L} = -\alpha \rho = -\alpha \beta_0 \kappa^2$, with $\alpha = 1$ or $\alpha = -\omega(r)$ and $\beta_0 \equiv \rho / \kappa^2$ — the former is the EOS parameter, defined through $\omega(r) = p(r)/\rho$. With the above, Eq. (13) becomes

\[
(1 + \omega)\rho_0 \phi' = -\omega' \rho_0 - \frac{F_2}{f_2}(\alpha + \omega)\rho_0 R' \rightarrow 
\]
\[
\frac{\omega'}{1 + \omega} = \phi' + \frac{F_2}{f_2} \frac{\alpha + \omega}{1 + \omega} R'. 
\]

Noticing that the combination

\[
\gamma \equiv \frac{\alpha + \omega}{1 + \omega} = \left\{ \begin{array}{ll}
0 & , \quad \mathcal{L} = p \rightarrow \alpha = -\omega \\
1 & , \quad \mathcal{L} = -\rho \rightarrow \alpha = 1
\end{array} \right. , \quad (22)
\]

acts as a “binary” variable, the above equation is integrable,

\[
\frac{\omega'}{1 + \omega} = \phi' + \gamma \frac{F_2}{f_2} R' \rightarrow 
\]
\[
\omega = A e^{-\phi} \frac{F_2}{f_2} - 1 .
\]

If one considers a non-relativistic dust distribution with vanishing pressure, $\omega = 0$, and assumes that $\mathcal{L} = -\rho \rightarrow \gamma = 1$, the above can be recast as

\[
f_2^\gamma \propto \frac{1}{\sqrt{-\beta_0}} ,
\]

a relation previously found in Ref. [11]. One aims here to further explore the insight gained from that study, allowing for a non-vanishing pressure and the two possible choices of Lagrangian density already discussed.

Together with Eqs. (18), (19) and (20) and the definition of the scalar curvature $R$, one has a closed set of equations for $\omega$, $\phi$ and $m$.

\[
m' \left( \frac{2}{r} + \frac{d}{dr} \right) \left( F_2 \alpha - \frac{1}{\beta_0 \kappa} \right) = \quad (25)
\]
\[
8 \pi \kappa R \left( F_2 R \alpha - f_2 \right) + 
\]
\[
\left[ 2 (8 \pi \kappa - m) \frac{d^2}{dr^2} + \left( \frac{32 \pi \kappa - 3 m}{r} \right) \frac{d}{dr} \right] (F_2 \alpha) ,
\]

\[
\left( \frac{1}{\beta_0 \kappa} + F_2 \alpha \right) R = \frac{1}{2} f_2 (1 - \omega) - 3 \left( 1 - \frac{m}{8 \pi \kappa \kappa} \right) \times 
\]
\[
\left[ \frac{d^2}{dr^2} + \left( \phi' + \frac{3}{2 r} + \frac{1}{2} \frac{8 \pi \kappa - m}{8 \pi \kappa \kappa - m} \right) \frac{d}{dr} \right] (F_2 \alpha) ,
\]

\[
\omega = A e^{-\phi} \frac{F_2}{f_2} - 1 ,
\]

with the scalar curvature given by

\[
R = \frac{\sqrt{1 - \frac{m}{8 \pi \kappa}}}{2 \pi} \times \left( 4 - \frac{4}{\sqrt{1 - \frac{m}{8 \pi \kappa}}} \right) + \quad (26)
\]
\[
r \left[ \frac{1}{2} \frac{m - m r}{2 \pi \kappa} + \phi' \left( 4 + \phi' \right) + 2 \phi'' r \right] .
\]

IV. LINEAR COUPLING

One considers now a linear coupling between curvature and matter

\[
f_2(R) = 1 + \beta_2 \frac{R}{\kappa} ,
\]

as this yields a more tractable problem that allows for the direct extraction of relevant consequences of the non-minimal coupling; one also defines the dimensionless parameter $\epsilon \equiv \beta_0 \beta_2$. As expected, one finds that GR is recovered if either the coupling between matter and curvature vanishes, $\beta_2 = 0$, or if there is no matter, $\rho \sim \beta_0 = 0$.

In Ref. [15], it was found that a linear coupling between curvature and matter of the form of Eq. (27) is compatible with Starobinsky inflation and able to drive post-inflationary preheating if $10^{10} < \beta_2 < 10^{14}$. Considering an upper bound for the density of $\rho_N = 10^{18}$ kg/m$^3$, the typical density of a neutron star’s core, one gets

\[
\epsilon < 10^{-62} ,
\]

indicating that the nonminimal coupling is highly perturbative.

The dimensionless functions are introduced below,

\[
\varrho \equiv \frac{R}{8 \pi \kappa \beta_0} , \quad \mu \equiv \sqrt{\frac{\beta_0}{8 \pi \kappa}} m ,
\]

written in terms of the dimensionless variable
so that Eq. (25) becomes

\[ x \equiv \sqrt{8\pi\kappa\beta_0 r} \]  

(30)

so that Eq. (25) becomes

\[
\mu' \left(\frac{2}{x} + \frac{d}{dx}\right) \left(\alpha - \frac{1}{\epsilon}\right) = -x \left(\phi(1 - \alpha) + \frac{1}{8\pi\epsilon}\right) \]  

(31)

\[ + \left[ 2 \left(\frac{1}{x} + \frac{\epsilon}{\lambda}ight) + \frac{d^2}{dx^2} + \frac{1}{4 - \mu} \frac{d}{dx} \right] \alpha , \]

\[
\left(\frac{1}{\epsilon} + 1 + 3\omega - \frac{1}{2}\right) \phi = \frac{1 - 3\omega}{16\pi} - 3 \left(1 - \frac{\mu}{x}\right) \times \]

\[ + \frac{d^2}{dx^2} + \left(\phi' + \frac{1}{2} \frac{4x - 3\mu - \mu'x}{x - \mu} \right) \frac{d}{dx} \alpha , \]

\[
\phi' = \frac{c\alpha' + \frac{1}{x - \mu} \left[ \frac{x^2}{4} \left(\frac{\omega}{8\pi} + (\alpha + \omega)\epsilon\phi + \frac{\mu}{2x} (1 - \alpha\epsilon)\right) \right]}{1 - \epsilon (\alpha + \frac{1}{2}\alpha'x)} , \]

\[
\omega = A_e^{-\phi} \left(1 + 8\pi\epsilon\phi\right) - 1 , \]

with the prime now denoting a derivative with respect to \( x \).

The integration constant \( A \) may be determined from the

\[ A = \sqrt{48\pi(1 - \epsilon) - x^2} \]

(35)

Defining

\[ y = \sqrt{1 - \frac{x^2}{48\pi(1 - \epsilon)}} , \quad y_1 = y(x_1) \]

(36)

one then has

\[
\omega = \frac{(y_1 - y)(1 + 2\epsilon)}{(1 + 2\epsilon)y - 3y_1} . \]

(37)

The central pressure is given by

\[ \omega_c \equiv \omega(x = 0) = \frac{|y_1 - 1|(1 + 2\epsilon)}{(1 + 2\epsilon) - 3y_1} , \]

(38)

and collapse is inevitable if it diverges, \( \omega_c \to \infty \), leading to

\[ y_1 = \frac{1 + 2\epsilon}{3} \to \]

(39)

\[ x_1^2 = \frac{64\pi}{3} (1 - \epsilon)^2 (2 + \epsilon) \to \]

\[ r_1^2 = \frac{8}{3} (1 - \epsilon)^2 (2 + \epsilon) \]

(40)

As expected, the standard result of GR,

\[ r_1 = \frac{16}{3\kappa\beta_0} = \frac{16\kappa}{3\rho} \to \frac{GM}{r_1c^2} = \frac{4}{9} , \]

is obtained by setting \( \epsilon = 0 \).

From Eq. (32), one gets

\[ \phi' = \frac{\omega'}{1 + \omega} - \frac{8\pi\epsilon}{1 + 8\pi\epsilon} \frac{\alpha + \omega}{1 + \omega} \phi' \to \]

(34)

\[ \frac{x}{48\pi(1 - \epsilon) - x^2} = \frac{1}{2} \frac{1 - \epsilon}{(1 + \omega)(1 + 3\omega + 2\epsilon)} \to \]

\[ \frac{x}{48\pi(1 - \epsilon) - x^2} \to \frac{1 + 3\omega + 2\epsilon}{1 + \omega} . \]

Using Eq. (13), one obtains

\[ g_{00} = -e^{2\phi} = -A \left[ (1 + 8\pi\epsilon\phi)(1 + \omega)\right]^{-2} \]

(41)

and

\[ -\frac{B}{4} \left(3 - \frac{1 + 2\epsilon}{1 + \epsilon} \frac{y}{y_1}\right)^2 . \]

Continuity with the Schwarzschild exterior metric

V. \emph{\( L = -\rho \) CASE}

If one considers that \( L = -\rho \) is the Lagrangian density of a perfect fluid, then the scenario of a homogeneous sphere naturally yields a very simplified set of equations: since both \( F_2 \) and \( L \) are constants, the additional terms found in Eq. (31) involving spatial derivatives of the latter vanish.

Substituting \( \alpha = 1 \) into Eq. (31), one gets

\[ \mu' = \frac{x^2}{16\pi(1 - \epsilon)} \to \mu = \frac{x^3}{48\pi(1 - \epsilon)} \]  

(32)

\[
\left(\frac{1}{\epsilon} + 1 + 3\omega\right) \phi = \frac{1 - 3\omega}{16\pi} \epsilon\phi \]

\[
\phi' = \frac{x^2}{4} \left[ \frac{\omega}{8\pi} + (\alpha + \omega)\epsilon\phi + \frac{\mu}{2x} (1 - \alpha\epsilon) \right] \]

\[
\omega = A_e^{-\phi} \left(1 + 8\pi\epsilon\phi\right) - 1 \]

The second equation above may be used to get

\[ [2 + \epsilon(1 + 3\omega)] \phi' = -3\omega' \left(\frac{1}{8\pi} + \epsilon\phi\right) \]  

(33)

Using Eq. (13), one obtains

\[ \phi' = -\frac{x}{1 + \omega} - \frac{8\pi\epsilon}{1 + 8\pi\epsilon} \frac{\alpha + \omega}{1 + \omega} \phi' \to \]

(34)

\[ \frac{x}{48\pi(1 - \epsilon) - x^2} = -2 \frac{1}{(1 + \omega)(1 + 3\omega + 2\epsilon)} \omega' \to \]

\[ \frac{\sqrt{48\pi(1 - \epsilon) - x^2}}{A \frac{1 + 3\omega + 2\epsilon}{1 + \omega}} . \]
\[ ds^2_+ = - \left(1 - \frac{M}{8\pi\kappa r} \right) dt^2 + \frac{1}{1 - \frac{M}{8\pi\kappa r}} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \]

at \( y = y_1 \) implies that \( m(r_1) = M \) and

\[ B = 4y_1^2 \left( \frac{1 + \epsilon}{2 + \epsilon} \right)^2 \]

so that

\[ g_{00} = - \left[ \frac{3(1 + \epsilon)y_1 - (1 + 2\epsilon)y}{2 + \epsilon} \right]^2 \]

From the above set of results, one finds that the strength of the nonminimal coupling must obey \( \epsilon < 1 \), so that all quantities are well defined (in particular, so that the dimensionless coordinate \( y \) is real and the sign of the 00 component of the metric is correct). Given the stringent bound, Eq. (28), this requirement is automatically fulfilled.

If one relaxes the compatibility with the preheating scenario posited in Ref. [15], then a dominant, negative nonminimal coupling, \( \epsilon \rightarrow -\infty \) is not precluded: from Eqs. (32), (37), (41) and (45), one sees that taking this limiting case yields

\[ \mu \sim - \frac{x^3}{48\pi\epsilon} \sim 0 \]
\[ \omega \sim \frac{x_1^2 - x^2}{96\pi\epsilon} \sim 0 \]
\[ \varrho \sim \frac{1}{8\pi\epsilon} \sim 0 \]
\[ g_{00} \sim -1 + 3x_1^2 - 2x^2 \sim -1 \]

Thus, one finds that a dominant negative nonminimal coupling effectively masks the presence of a spherical body; in order to prevent this unphysical result, the former must be perturbative, \( \epsilon \ll 1 \).

**VI. \( \mathcal{L} = p \) CASE**

If one instead considers that \( \mathcal{L} = p \) is the suitable Lagrangian density to describe a perfect fluid, then a more involved set of equations is expected, since in this scenario the spatial derivative terms found in Eq. (31) do not vanish.

Substituting \( \alpha = -\omega \) into Eq. (31), one gets

\[ x \left[ \varrho (1 + \omega) + \frac{1}{8\pi\epsilon} \right] = \]

\[ 2(\mu - x) \omega'' + \left( \frac{3\mu}{x} + \mu' - 4 \right) \omega' + \frac{2}{x} \left( \frac{1}{\epsilon} + \omega \right) \mu' \]
\[ \left( \frac{1}{\epsilon} + \frac{\omega - 1}{2} \right) \varrho = \frac{1 - 3\omega}{16\pi\epsilon} \]
\[ 3 \left( 1 - \frac{\mu}{x} \right) \left[ \omega'' + \left( \phi' + \frac{1}{2} \frac{4x - 3\mu - \mu'x}{x - \mu} \omega \right) \phi' \right] = \]
\[ -\epsilon \omega' + \frac{1}{x - \mu} \left[ \frac{\omega''}{2(1 + \epsilon)} + \omega' \left( 1 + \epsilon \omega \right) \right] \]
\[ \omega = Ae^{-\phi} - 1 \]

Using the definition of the scalar curvature and the last of the above equations, one can write

\[ \varrho = \frac{4x - 3\mu}{x^2} \frac{\omega'}{1 + \omega} + \frac{\mu'}{x} \left( \frac{2}{x} - \frac{\omega'}{1 + \omega} \right) + \]
\[ \frac{2}{x(\mu - x)} \frac{2(\omega')^2 - \omega''(1 + \omega)}{(1 + \omega)^2} \]

so that Eqs. (50)-(53) can be reduced to

\[ \frac{x}{16\pi} + \frac{\mu'}{x} (\epsilon - 1) = \]
\[ 2(\mu - x) \epsilon \omega'' + \left( \frac{3\mu}{x} + \mu' - 4 \right) \epsilon \omega' + 2x - \frac{\mu}{1 + \omega} \epsilon (\omega')^2 \]

and

\[ 2(1 - \epsilon) \omega' + \epsilon \omega (\omega')^2 = \]
\[ \frac{1}{\mu - x} \left[ \frac{x^2 \omega}{16\pi} + \frac{\mu}{x} (1 + \epsilon \omega) \right] \]

Solving for \( \mu \), one gets

\[ \mu = \frac{x^2}{16\pi} \left( 1 + \omega \right) \omega + 32\pi (1 - \epsilon) \omega' + 16\pi \epsilon \omega (\omega')^2 \]

\[ \frac{2}{16\pi} 2(1 - \epsilon) x^2 \omega' + \epsilon x^2 (\omega')^2 - (1 + \omega)(1 + \epsilon \omega) \]

Considering that the boundary of the spherical body is signaled by a vanishing pressure, \( \omega(x_1) = 0 \), one has

\[ \mu(x_1) = \frac{x_1^2}{16\pi} \left( 32\pi (1 - \epsilon) \omega' + 16\pi \epsilon x_1 (\omega')^2 \right) \]

\[ \frac{2}{16\pi} 2(1 - \epsilon) x_1 \omega' + \epsilon x_1^2 (\omega')^2 - 1 \]

**A. Constant solution**

One notes that a constant pressure solution is available: indeed, setting \( \omega = \text{const.} \), Eqs. (50)-(53) yield the solution

\[ \omega = \frac{1}{2\epsilon - 3} \]
\[ \mu = \frac{x^3}{48\pi (1-\epsilon)} , \]
\[ \theta = \frac{1}{8\pi (1-\epsilon)} , \]
\[ \phi = \frac{1}{2} \log \left( 1 - \frac{x^2}{48\pi (1-\epsilon)} \right) . \]

where \( \phi \) has been normalized following the previous procedure to match the metric at the boundary of the spherical body.

Notice that this solution cannot simultaneously yield a positive mass \( \mu > 0 \) (\( \epsilon < 1 \)) and pressure \( \omega \geq 0 \) (if \( \epsilon > 3/2 \)). Furthermore, if the effect of the nonminimal coupling is perturbative, \( \epsilon \sim 0 \), a spherical body with positive mass and curvature is obtained, but with negative pressure, \( \omega \approx -1/3 \).

Conversely, a dominant nonminimal coupling \( |\epsilon| \gg 1 \) leads to a Minkowski space with \( \omega \sim \mu \sim \rho \sim 0 \) and \( g_{\mu\nu} \sim \eta_{\mu\nu} \), as found in the previous section if \( \mathcal{L} = -\rho \).

Again, this is unreasonable and thus implies that the effect of the nonminimal coupling should be perturbative, as supported by Ref. [15].

B. Numerical solution

One may substitute Eq. (57) into Eq. (55) and solve the ensuing second-order differential equation for \( \omega \). To do so, one ascertains the typical order of magnitude of \( x_1 \), assuming a perturbative nonminimal coupling,

\[ x_1 = \sqrt{8\pi \kappa \beta_0 r_1} = \sqrt{\frac{8\pi \rho}{\kappa} r_1} \sim \sqrt{\frac{96\pi GM}{r_1}} , \quad (60) \]

which, considering that the classical upper bound \( G M / r_1 c^2 \lesssim 4/9 \) remains approximately valid if the nonminimal coupling is perturbative, that is

\[ x_1 \lesssim \sqrt{\frac{2\pi}{3}} \sim 10 . \quad (61) \]

Figs. 1 and 2 show the numerical solution of Eqs. (55) and (57) for different values of the coupling strength \( \epsilon \); boundary conditions \( \omega(x_1) = 0 \) and \( \omega'(0) = 0 \) are assumed, for \( x_1 = 10 \), the upper bound obtained above. The relative deviations \( \delta\mu/\mu \equiv 1 - \mu/\mu_{GR} \) and \( \delta\omega/\omega \equiv 1 - \omega/\omega_{GR} \) with respect to their GR counterparts \( \mu_{GR} \) and \( \omega_{GR} \) are shown, with latter being defined as

\[ \mu_{GR} = \frac{x^3}{48\pi} , \quad \omega_{GR} = \frac{1 - \frac{x^2}{48\pi}}{1 - \frac{x^2}{3 - 3 \frac{x^2}{48\pi}}} . \quad (62) \]

Since the bound Eq. (28) for the latter indicates that it is almost vanishing, much higher values for \( \epsilon \) are shown, in order to better illustrate the effect of the nonminimal coupling.

A numerical analysis does not yield an analytical expression for the deviation of the Schwarzschild mass \( M \) due to the effect of the nonminimal coupling; since the latter is perturbative, one expects that the latter yields a linear correction to the GR value \( M_{GR} = (4\pi/3)r_1^3\rho; \) this is confirmed in Fig. (3), where the relative deviation is plotted together with a linear fit that allows one to estimate that

\[ 1 - \frac{\mu(x_1)}{\mu_{GR}(x_1)} \sim \frac{\delta M}{M_{GR}} \sim 0.723\xi . \quad (63) \]

VII. DISCUSSION AND OUTLOOK

In this work, we have computed the effect of a linear coupling between matter and curvature on a spherical body with homogeneous density, for the choices of Lagrangian density \( \mathcal{L} = -\rho \) and \( \mathcal{L} = p \). In doing so, it
complements two previous studies: one on the analogous effect on the Sun, modelled as a polytrope with polytropic index \( n \sim 3 \) [16], and on the modification of the collapse of a homogeneous spherical body due to a linear nonminimal coupling between curvature and matter [17]. Although the ensuing dynamical equations ruling the inner structure of such body are widely different, our results show that both formulations imply that the nonminimal coupling should be perturbative, \( f_2(R) \sim 1 \) (consistent with previous studies): the converse would imply that the mass \( M \) of the spherical body (as inferred from the Schwarzschild metric probed by an external observer) would be negative or, in the extreme case of a very large, negative nonminimal coupling, vanish altogether — and thus lead to an external Minkowski spacetime, allowing for the masking of very large central masses.

Furthermore, since the widely different dynamical behaviour found in Eqs. (32) and (55)-(57) is naturally suppressed by a perturbative nonminimal coupling, our study shows that the effect of the latter on the mass \( M \) is rather similar for both choices of Lagrangian densities,

\[
M = \frac{4\pi}{3} \rho r_1^3 \sim \frac{4\pi}{3} \rho r_1^3 (1 + \epsilon), \quad \mathcal{L} = -\rho \ ,
\]

\[
M \sim \frac{4\pi}{3} \rho r_1^3 (1 + 0.723\epsilon), \quad \mathcal{L} = \rho .
\]

Returning to the main motivation of this work, that is on the choice of the Lagrangian density in a nonminimally coupled model, one sees that it does not have a strong impact on the relevant observables: the results here presented indicate that in a stationary and perturbative regime, the selected form for \( \mathcal{L} \) does not affect greatly the impact on the structure of a spherical body. This is contrasting with respect with what occurs in a more dynamical context such as a gravitational collapse — which, although beginning in a perturbative regime, inevitably evolves towards more extreme scenarios, with widely different consequences depending on the choices of \( \mathcal{L} \) [17].

This criterion allows us to reduce the degeneracy between different choices of nonminimal coupling and Lagrangian densities: future studies aiming at testing the nonminimal coupling should focus on perturbative, stationary scenarios. Conversely, we argue that if a nonminimal coupling is assumed, the best environment to test what is the form of the Lagrangian density is found in time-evolving phenomena, where the effect of the nonminimal coupling eventually surfaces from its initial perturbative nature.

[1] C. M. Will, Living Rev. Rel. 9, 3 (2006).
[2] O. Bertolami and J. Páramos, “The experimental status of Special and General Relativity”, to appear in Handbook of Spacetime, Springer, Berlin (2013); arXiv:1212.2177 [gr-qc].
[3] O. Bertolami, C. G. Böhmer, T. Harko and F. S. N. Lobo, Phys. Rev. D 75, 104016 (2007).
[4] O. Bertolami and J. Páramos, JCAP 03, 009 (2010).
[5] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D 86, 044034 (2012).
[6] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D 81, 104046 (2010).
[7] O. Bertolami and J. Páramos, Phys. Rev. D 84, 064022 (2011).
[8] O. Bertolami, R. March and J. Páramos, in preparation.
[9] O. Bertolami and M. C. Sequeira, Phys. Rev. D 79, 104010 (2009).
[10] O. Bertolami and R. Z. Ferreira, Phys. Rev. D 85, 104050 (2012).
[11] O. Bertolami and A Martins, Phys. Rev. D 85, 024012 (2012).
[12] B. F. Schutz, Phys. Rev. D 2, 2762 (1970).
[13] J. D. Brown, Class. Quantum Gravity 10, 1579 (1993).
[14] O. Bertolami, F. S. N. Lobo and J. Páramos, Phys. Rev. D 78, 064036 (2008).
[15] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D 83, 044010 (2011).
[16] O. Bertolami and J. Páramos, Phys. Rev. D 77, 084018 (2008).
[17] J. Páramos and C. Bastos, Phys. Rev. D86, 103007 (2012).