TOEPLITZ-COMPOSITION $C^*$-ALGEBRAS FOR CERTAIN FINITE BLASCHKE PRODUCTS

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Abstract. Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. Then there exists a relation between the associated composition operator $C_R$ on the Hardy space and the $C^*$-algebra $O_R(J_R)$ associated with the complex dynamical system $(R^n)_n$ on the Julia set $J_R$. We study the $C^*$-algebra $TC_R$ generated by both the composition operator $C_R$ and the Toeplitz operator $T_z$ to show that the quotient algebra by the ideal of the compact operators is isomorphic to the $C^*$-algebra $O_R(J_R)$, which is simple and purely infinite.

1. Introduction

Let $D$ be the open unit disk in the complex plane and $H^2(D)$ be the Hardy (Hilbert) space of analytic functions whose power series have square-summable coefficients. For an analytic self-map $\varphi : D \to D$, the composition operator $C_\varphi : H^2(D) \to H^2(D)$ is defined by $C_\varphi(g) = g \circ \varphi$ for $g \in H^2(D)$ and is known to be a bounded operator by the Littlewood subordination theorem [14]. The study of composition operators on the Hardy space $H^2(D)$ gives a fruitful interplay between complex analysis and operator theory as shown, for example, in the books of Shapiro [29], Cowen and MacCluer [4] and Martínez-Avendaño and Rosenthal [15]. Since the work by Cowen [2], good representations of adjoints of composition operators have been investigated. Consult Cowen and Gallardo-Gutiérrez [3], Martín and Vukotić [17], Hammond, Moorhouse and Robbins [8], and Bourdon and Shapiro [1] to see recent achievements in adjoints of composition operators with rational symbols. In this paper, we only need an old result by McDonald in [18] for finite Blaschke products.

On the other hand, for a branched covering $\pi : M \to M$, Deaconu and Muhly [6] introduced a $C^*$-algebra $C^*(M, \pi)$ as the $C^*$-algebra of the $\tau$-discrete groupoid constructed by Renault [27]. In particular, they study rational functions $R$ on the Riemann sphere $\hat{\mathbb{C}}$. Iterations $(R^n)_n$ of $R$ by composition give complex dynamical systems. In [11] Kajiwara and the second-named author introduced slightly
different C*-algebras $\mathcal{O}_R(\hat{\mathbb{C}})$, $\mathcal{O}_R(J_R)$ and $\mathcal{O}_R(F_R)$, associated with the complex dynamical system $(R^n)_n$ on the Riemann sphere $\hat{\mathbb{C}}$, the Julia set $J_R$ and the Fatou set $F_R$ of $R$. The C*-algebra $\mathcal{O}_R(J_R)$ is defined as a Cuntz-Pimsner algebra of a Hilbert bimodule, called a C*-correspondence, $C(\text{graph } R|_{J_R})$ over $C(J_R)$. We regard the algebra $\mathcal{O}_R(J_R)$ as a certain analog of the crossed product $C(\Lambda_\Gamma) \rtimes \Gamma$ of $C(\Lambda_\Gamma)$ by a boundary action of a Kleinian group $\Gamma$ on the limit set $\Lambda_\Gamma$.

The aim of this paper is to show that there exists a relation between composition operators on the Hardy space and the C*-algebras $\mathcal{O}_R(J_R)$ associated with the complex dynamical systems $(R^n)_n$ on the Julia sets $J_R$.

We recall that the C*-algebra $\mathcal{T}$ generated by the Toeplitz operator $T_\varphi$ contains all continuous symbol Toeplitz operators, and its quotient by the ideal of the compact operators on $H^2(\mathbb{T})$ is isomorphic to the commutative C*-algebra $C(\mathbb{T})$ of all continuous functions on $\mathbb{T}$. For an analytic self-map $\varphi : \mathbb{D} \to \mathbb{D}$, we denote by $TC_\varphi$ the Toeplitz-composition C*-algebra generated by both the composition operator $C_\varphi$ and the Toeplitz operator $T_\varphi$. Its quotient algebra by the ideal $\mathcal{K}$ of the compact operators is denoted by $\mathcal{O}C_\varphi$. Recently Kriete, MacCluer and Moorhouse [12, 13] studied the Toeplitz-composition C*-algebra $TC_\varphi$ for a certain linear fractional self-map $\varphi$. They describe the quotient C*-algebra $\mathcal{O}C_\varphi$ concretely as a subalgebra of $C(\Lambda) \otimes M_2(\mathbb{C})$ for a compact space $\Lambda$. If $\varphi(z) = e^{-2\pi i \theta} z$ for some irrational number $\theta$, then the Toeplitz-composition C*-algebra $TC_\varphi$ is an extension of the irrational rotation algebra $A_\theta$ by $\mathcal{K}$ and studied by Park [25]. Jury [9, 10] investigated the C*-algebra generated by a group of composition operators with the symbols belonging to a non-elementary Fuchsian group $\Gamma$ to relate it with extensions of the crossed product $C(\mathbb{T}) \rtimes \Gamma$ by $\mathcal{K}$.

In this paper we study the class of finite Blaschke products $R$ of degree $n \geq 2$ with $R(0) = 0$. The boundary $\mathcal{T}$ of the open unit disk $\mathbb{D}$ is the Julia set $J_R$ of the Blaschke product $R$. We show that the quotient algebra $\mathcal{O}C_R$ of the Toeplitz-composition C*-algebra $TC_R$ by the ideal $\mathcal{K}$ is isomorphic to the C*-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system $(R^n)_n$, which is simple and purely infinite. We should remark that the notion of transfer operator considered by Exel in [7] is one of the keys to clarifying the above relation. In fact the corresponding operator of the composition operator in the quotient algebra is the implementing isometry operator.

The Toeplitz-composition C*-algebra depends on the analytic structure of the Hardy space by construction. The finite Blaschke product $R$ is not conjugate with $z^n$ by any M"{o}bius automorphism unless $R(z) = \lambda z^n$. But we can show that the quotient algebra $\mathcal{O}C_R$ is isomorphic to $\mathcal{O}C_{\varphi^n}$ as a corollary of our main theorem. This enables us to compute $K_0(\mathcal{O}C_R)$ and $K_1(\mathcal{O}C_R)$ easily.

### 2. Toeplitz-composition C*-algebras

Let $L^2(\mathbb{T})$ denote the square integrable measurable functions on $\mathbb{T}$ with respect to the normalized Lebesgue measure. The Hardy space $H^2(\mathbb{T})$ is the closed subspace of $L^2(\mathbb{T})$ consisting of the functions whose negative Fourier coefficients vanish. We put $H^\infty(\mathbb{T}) := H^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$.

The Hardy space $H^2(\mathbb{D})$ is the Hilbert space consisting of all analytic functions $g(z) = \sum_{k=0}^\infty c_k z^k$ on the open unit disk $\mathbb{D}$ such that $\sum_{k=0}^\infty |c_k|^2 < \infty$. The inner
product is given by
\[(g|h) = \sum_{k=0}^{\infty} c_k \overline{d_k}\]
for \(g(z) = \sum_{k=0}^{\infty} c_k z^k\) and \(h(z) = \sum_{k=0}^{\infty} d_k z^k\).
We identify \(H^2(D)\) with \(H^2(T)\) by a unitary \(U : H^2(D) \to H^2(T)\). We note that \(\tilde{g} = Ug\) is given as
\[\tilde{g}(e^{i\theta}) := \lim_{r \to 1^-} g(re^{i\theta}) \quad \text{a.e. } \theta\]
for \(g \in H^2(D)\) by Fatou’s theorem. Moreover the inverse \(\tilde{f} = U^*f\) is given as a Poisson integral
\[\tilde{f}(re^{i\theta}) := \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)f(e^{it})dt\]
for \(f \in H^2(T)\), where \(P_r\) is the Poisson kernel defined by
\[P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}, \quad 0 \leq r < 1, \quad 0 \leq \theta \leq 2\pi.\]

Let \(P_H : L^2(T) \to H^2(T) \subset L^2(T)\) be the projection. For \(a \in L^\infty(T)\), the Toeplitz operator \(T_a\) on \(H^2(T)\) is defined by \(T_af = P_Ha \cdot f\) for \(f \in H^2(T)\).

Let \(\varphi : D \to D\) be an analytic self-map. Then the composition operator \(C_\varphi\) on \(H^2(D)\) is defined by \(C_\varphi g = g \circ \varphi\) for \(g \in H^2(D)\). By the Littlewood subordination theorem, \(C_\varphi\) is always bounded.

We can regard Toeplitz operators and composition operators as acting on the same Hilbert space by the unitary \(U\) above. More precisely, we put \(\hat{T}_a = U^*T_aU\) and \(\hat{C}_\varphi = UC_\varphi U^*\). If \(\varphi\) is an inner function, then we know that \(\hat{g} \circ \hat{\varphi} = \hat{g} \circ \hat{\varphi}\) for \(g \in H^2(D)\) by Ryff [28, Theorem 2]. Therefore we may write \(C_\varphi f = f \circ \hat{\varphi}\) for \(f \in H^2(T)\).

**Definition.** For an analytic self-map \(\varphi : D \to D\), we denote by \(TC_\varphi\) the \(C^*\)-algebra generated by the Toeplitz operator \(\hat{T}_z\) and the composition operator \(C_\varphi\) on \(H^2(D)\). The \(C^*\)-algebra \(TC_\varphi\) is called the Toeplitz-composition \(C^*\)-algebra with symbol \(\varphi\). Since \(TC_\varphi\) contains the ideal \(K(H^2(D))\) of compact operators, we define a \(C^*\)-algebra \(OC_\varphi\) to be the quotient \(C^*\)-algebra \(TC_\varphi/K(H^2(D))\).

By the unitary \(U : H^2(D) \to H^2(T)\) above, we usually identify \(TC_\varphi\) with the \(C^*\)-algebra generated by \(T_z\) and \(C_\varphi\). We also use the same notation \(TC_\varphi\) and \(OC_\varphi\) for the operators on \(H^2(T)\). But we sometimes need to treat them carefully. Therefore we often use the notation \(\tilde{g} = Ug\) and \(\tilde{f} = U^*f\) to avoid confusion in the paper. Wise readers may neglect this troublesome notation.

3. \(C^*\)-algebras associated with complex dynamical systems

We recall the construction of Cuntz-Pimsner algebras [26]. Let \(A\) be a \(C^*\)-algebra and \(X\) be a Hilbert right \(A\)-module. We denote by \(L(X)\) the algebra of the adjointable bounded operators on \(X\). For \(\xi, \eta \in X\), the operator \(\theta_{\xi, \eta}(\zeta) = \xi(\eta^* \zeta)_A\) for \(\zeta \in X\). The closure of the linear span of these operators is denoted by \(K(X)\). We say that \(X\) is a Hilbert \(C^*\)-bimodule (or \(C^*\)-correspondence) over \(A\) if \(X\) is a Hilbert right \(A\)-module with a \(*\)-homomorphism \(\phi : A \to L(X)\). We always assume that \(X\) is full and \(\phi\) is injective. Let \(F(X) = \bigoplus_{n=0}^{\infty} X^k\times_n\) be
the full Fock module of $X$ with a convention $X^{\otimes 0} = A$. For $\xi \in X$, the creation operator $T_\xi \in L(F(X))$ is defined by

$$T_\xi(a) = \xi a \quad \text{and} \quad T_\xi(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n.$$ 

We define $i_{F(X)} : A \to L(F(X))$ by

$$i_{F(X)}(a)(b) = ab \quad \text{and} \quad i_{F(X)}(a)(\xi_1 \otimes \cdots \otimes \xi_n) = \phi(a)\xi_1 \otimes \cdots \otimes \xi_n$$

for $a, b \in A$. The Cuntz-Toeplitz algebra $T_X$ is the $C^*$-algebra acting on $F(X)$ generated by $i_{F(X)}(a)$ with $a \in A$ and $T_\xi$ with $\xi \in X$.

Let $j_K : K(X) \to T_X$ be the homomorphism defined by $j_K(\theta_{\xi, \eta}) = T_\xi T_\eta^*$. We consider the ideal $I_X := \phi^{-1}(K(X))$ of $A$. Let $\mathcal{J}_X$ be the ideal of $T_X$ generated by $\{i_{F(X)}(a) - (j_K \circ \phi)(a) : a \in I_X\}$. Then the Cuntz-Pimsner algebra $\mathcal{O}_X$ is defined as the quotient $T_X / \mathcal{J}_X$. Let $\pi : T_X \to \mathcal{O}_X$ be the quotient map. We set $S_\xi = \pi(T_\xi)$ and $i(a) = \pi(i_{F(X)}(a))$. Let $i_K : K(X) \to \mathcal{O}_X$ be the homomorphism defined by $i_K(\theta_{\xi, \eta}) = S_\xi S_\eta^*$. Then $\pi((j_K \circ \phi)(a)) = (i_K \circ \phi)(a)$ for $a \in I_X$.

The Cuntz-Pimsner algebra $\mathcal{O}_X$ is the universal $C^*$-algebra generated by $i(a)$ with $a \in A$ and $S_\xi$ with $\xi \in X$, satisfying the fact that $i(a)S_\xi = S_{\phi(a)\xi}$, $S_\xi(a) = S_{\xi a}$, $S_\xi S_\eta = i(\langle \xi | \eta \rangle)_A$ for $a \in A$, $\xi, \eta \in X$ and $i(a) = (i_K \circ \phi)(a)$ for $a \in I_X$. We usually identify $i(a)$ with $a$ in $A$. If $A$ is unital and $X$ has a finite basis $\{u_i\}_{i=1}^m$ in the sense that $\xi = \sum_{i=1}^n u_i \langle u_i | \xi \rangle_A$, then the last condition can be replaced by the fact that there exists a finite set $\{v_j\}_{j=1}^m \subset X$ such that $\sum_{j=1}^m S_{v_j} S_{v_j}^* = I$. Then $\{v_j\}_{j=1}^m$ becomes another finite basis of $X$ automatically.

Next we introduce the $C^*$-algebras associated with complex dynamical systems as in [11]. Let $R$ be a rational function of degree at least two. The sequence $(R \circ \cdots \circ R)_n$ of iterations of composition by $R$ gives a complex dynamical system on the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The Fatou set $F_R$ of $R$ is the maximal open subset of $\hat{\mathbb{C}}$ on which $(R \circ \cdots \circ R)_n$ is equicontinuous (or a normal family) and the Julia set $J_R$ of $R$ is the complement of the Fatou set in $\hat{\mathbb{C}}$. We denote by $e(z_0)$ the branch index of $R$ at $z_0$. Let $A = C(\hat{\mathbb{C}})$ and $X = C(\text{graph } R)$ be the set of continuous functions on $\hat{\mathbb{C}}$ and graph $R$ respectively, where $\text{graph } R = \{(x, y) \in \hat{\mathbb{C}}^2 : y = R(x)\}$ is the graph of $R$. Then $X$ is an $A$-$A$ bimodule by

$$(a : \xi, b)(x, y) = a(x)\xi(x, y)b(y), \quad a, b \in A, \ \xi \in X.$$ 

We define an $A$-valued inner product $\langle \cdot | \cdot \rangle_A$ on $X$ by

$$\langle \xi | \eta \rangle_A(y) = \sum_{x \in R^{-1}(y)} e(x)\overline{\xi(x, y)}\eta(x, y), \quad \xi, \eta \in X, \ y \in \hat{\mathbb{C}}.$$

Thanks to the branch index $e(x)$, the inner product above gives a continuous function and $X$ is a full Hilbert bimodule over $A$ without completion. The left action of $A$ is unital and faithful.

Since the Julia set $J_R$ is completely invariant under $R$, i.e., $R(J_R) = J_R = R^{-1}(J_R)$, we can consider the restriction $R|_{J_R} : J_R \to J_R$, which will often be denoted by the same letter $R$. Let graph $R|_{J_R} = \{(x, y) \in J_R \times J_R : y = R(x)\}$ be the graph of the restriction map $R|_{J_R}$ and $X_R = C(\text{graph } R|_{J_R})$. In the same way as above, $X_R$ is a full Hilbert bimodule over $C(J_R)$. 


**Definition.** The $C^*$-algebra $O_R(\hat{\mathbb{C}})$ on $\hat{\mathbb{C}}$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $X = C(\text{graph } R)$ over $A = C(\hat{\mathbb{C}})$. We also define the $C^*$-algebra $O_R(J_R)$ on the Julia set $J_R$ as the Cuntz-Pimsner algebra of the Hilbert bimodule $X_R = C(\text{graph } R|_{J_R})$ over $A = C(J_R)$.

**4. Transfer operators and adjoints of composition operators**

In the rest of this paper, we assume that $R$ is a finite Blaschke product of degree at least two with $R(0) = 0$; that is,

$$R(z) = \lambda z \prod_{k=1}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z} = \lambda \prod_{k=0}^{n-1} \frac{z - \lambda z_k}{1 - \overline{z_k} z}, \quad z \in \hat{\mathbb{C}},$$

where $n \geq 2$, $z_1, \ldots, z_{n-1} \in \mathbb{D}$, $|\lambda| = 1$ and $\lambda z_0 = 0$. Thus $R$ is a rational function with degree $\text{deg } R = n$. Since $R$ is an inner function, $R$ is an analytic self-map on $\mathbb{D}$. We consider the composition operator $C_R$ with symbol $R$. Since $R$ is inner and $R(0) = 0$, $C_R$ is an isometry by Nordgren [24]. We note the following fact:

$$\frac{z R'(z)}{R(z)} = 1 + \sum_{k=1}^{n-1} \frac{1 - |z_k|^2}{|z - z_k|^2} > 0, \quad z \in \mathbb{T}.$$ 

Thus $R$ has no branched points on $\mathbb{T}$ and the branch index $e(z) = 1$ for any $z \in \mathbb{T}$. Furthermore the Julia set $J_R$ of $R$ is $\mathbb{T}$ (see, for example, [19] pages 70-71), and it coincides with the boundary of the disk $\mathbb{D}$.

Let $A = C(\mathbb{T})$ and $h \in A$ be a positive invertible element. Set graph $R|_\mathbb{T} = \{(z, w) \in \mathbb{T}^2 ; w = R(z)\}$ and $X_{R,h} = C(\text{graph } R|_\mathbb{T})$. We define

$$(a \cdot \xi \cdot b)(z, w) = a(z)\xi(z, w)b(w), \quad (\xi | \eta)_{A,h}(w) = \sum_{z \in R^{-1}(w)} h(z)\xi(z, w)\eta(z, w)$$

for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. We see that $X_{R,h}$ is a pre-Hilbert $A$-$A$ bimodule whose left action is faithful. Since the Julia set $J_R = \mathbb{T}$ has no branched point, for the constant function $h = 1$, $X_{R,1}$ has a finite basis and coincides with the Hilbert $A$-$A$ bimodule $X_{R} = C(\text{graph } R|_\mathbb{T})$. Let $\{u_k\}_{k=1}^N$ be a basis of $X_R$. For a positive invertible element $h \in A$, put $v_k = h^{-1/2}u_k$. Then $\{v_k\}_{k=1}^N$ is a basis of $X_{R,h}$, and $X_{R,h}$ is also a Hilbert module without completion.

**Definition.** The $C^*$-algebra $O_{R,h}(\mathbb{T})$ is defined as the Cuntz-Pimsner algebra of the Hilbert bimodule $X_{R,h} = C(\text{graph } R|_\mathbb{T})$ over $A = C(\mathbb{T})$. The $C^*$-algebra $O_{R,h}(\mathbb{T})$ is the universal $C^*$-algebra generated by $\{\hat{S}_\xi ; \xi \in X_{R,h}\}$ and $A$ satisfying the following relations:

$$a \hat{S}_\xi = \hat{S}_a \xi, \quad \hat{S}_\xi b = \hat{S}_{\xi \cdot b}, \quad \hat{S}_\xi \hat{S}_\eta = (\xi | \eta)_{A,h}, \quad \sum_{k=1}^N \hat{S}_{v_k} \hat{S}^*_{v_k} = I$$

for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. The $C^*$-algebra $O_{R,h}(\mathbb{T})$ is in fact a topological quiver algebra in the sense of Muhly and Tomforde [21]. See also Muhly and Solel [20] Example 5.4.

We will use the symbol $a$ and $S_\xi$ to denote the generator of $O_R(J_R)$ for $a \in A$ and $\xi \in X_R$.
In the rest of this paper, we choose and fix $h \in A$ defined by
\[
h(z) = \frac{nR(z)}{zR'(z)} = \frac{n}{|R'(z)|}, \quad z \in \mathbb{T}.
\]
Then $h$ is positive and invertible by (1).

We need and collect several facts as lemmas to prove our main theorem. Some of them might be considered folklore.

**Lemma 1.** Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. Then the $C^*$-algebra $\mathcal{O}_{R,h}(\mathbb{T})$ is isomorphic to the $C^*$-algebra $\mathcal{O}_R(J_R)$ by an isomorphism $\Phi$ such that $\Phi(a) = a$ and $\Phi(S_\xi) = h^{1/2}S_\xi$ for $a \in A, \xi \in X_{R,h}$.

**Proof.** Let $V_\xi = h^{1/2}S_\xi$ for $\xi \in X_{R,h}$. Then we have that
\[
aV_\xi = V_a \xi, \quad V_\xi b = V_\xi b, \quad V_\xi V_\eta = (h^{1/2} \cdot \eta)A = (\xi \cdot \eta)A, h, \sum_{k=1}^N V_{\xi k}V_{\xi k}^* = I
\]
for $a, b \in A$ and $\xi, \eta \in X_{R,h}$. By universality, we have the desired isomorphism. □

**Definition.** For a function $f$ on $\mathbb{T}$, we define a function $\mathcal{L}_R(f)$ on $\mathbb{T}$ by
\[
\mathcal{L}_R(f)(w) = \frac{1}{n} \sum_{z \in \mathbb{T}^{-1}(w)} h(z)f(z) = \sum_{z \in \mathbb{T}^{-1}(w)} \frac{R(z)}{zR'(z)}f(z), \quad w \in \mathbb{T}.
\]

We mainly consider the restrictions of $\mathcal{L}_R$ to $A = C(\mathbb{T})$ and $H^2(\mathbb{T})$ and use the same notation if no confusion can arise.

The notion of transfer operator by Exel in [7] is one of the keys to clarifying our situation.

**Lemma 2.** Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. Let $A = C(\mathbb{T})$ and $\alpha : A \to A$ be the unital $^*$-endomorphism defined by $(\alpha(a))(z) = a(R(z))$ for $a \in A$ and $z \in \mathbb{T}$. Then the restriction of $\mathcal{L}_R$ to $A = C(\mathbb{T})$ is a transfer operator for the pair $(A, \alpha)$ in the sense of Exel; that is, $\mathcal{L}_R$ is a positive linear map such that
\[
\mathcal{L}_R(\alpha(ab)) = a\mathcal{L}_R(b) \quad \text{for} \quad a, b \in A.
\]
Moreover $\mathcal{L}_R$ satisfies the fact that $\mathcal{L}_R(1) = 1$.

**Proof.** The only non-trivial statement is to show that $\mathcal{L}_R(1) = 1$. But this is an easy calculation as follows: We fix $w \in \mathbb{T}$ and let $\alpha_1, \ldots, \alpha_n$ be exactly different solutions of $R(z) = w$. By the partial fraction decomposition, there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that
\[
\frac{R(z)/z}{R(z) - w} = \sum_{k=1}^n \frac{\lambda_k}{z - \alpha_k}.
\]
Multiplying by $z$ and letting $z \to \infty$, we have $\sum_{k=1}^n \lambda_k = 1$. Moreover multiplying by $z - \alpha_l$ and letting $z \to \alpha_l$, we get $\lambda_l = \frac{R(\alpha_l)}{\alpha_l R'(\alpha_l)}$. This shows that $\mathcal{L}_R(1) = 1$. See, for example, [5]. □

Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. J. N. McDonald [18] calculated the adjoint of $C_R$ and gave a formula. We follow some of his argument, but we also need a different formula to prove our main theorem. We claim that there exist $\theta_0 \in [0, 2\pi]$ and a strictly increasing continuously differentiable function $\psi : [\theta_0 - 2\pi, \theta_0] \to \mathbb{R}$ such that $\psi(\theta_0 - 2\pi) = 0$, $\psi(\theta_0) = 2n\pi$ and $R(e^{i\theta}) = e^{i\psi(\theta)}$. 

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Lemma 3. Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. Then $\mathcal{L}_R(H^2(\mathbb{T})) \subset H^2(\mathbb{T})$ and the restriction $\mathcal{L}_R : H^2(\mathbb{T}) \to H^2(\mathbb{T})$ is a bounded operator such that $\widetilde{C}_R^* = \mathcal{L}_R$.

Proof. Let $\psi$ be the map defined above. Set $t = \psi(\theta)$ and $\sigma_k(t) = \psi^{-1}(t + 2(k-1)\pi)$ for $1 \leq k \leq n$ and $0 \leq t \leq 2\pi$. If we differentiate $R(e^{i\sigma_k(t)}) = e^{it}$ with respect to $t$, then

$$\sigma'_k(t) = \frac{R(e^{i\sigma_k(t)})}{R'(e^{i\sigma_k(t)})e^{i\sigma_k(t)}}.$$ 

Therefore for $\xi_l(z) := z^l$ we have that

$$(\widetilde{C}_R \xi_l, f) = \frac{1}{2\pi} \int_0^{2\pi} R(e^{i\theta}) f(e^{i\theta}) d\theta = \frac{1}{2\pi} \int_{\theta_0-2\pi}^{\theta_0} R(e^{i\theta}) f(e^{i\theta}) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{it} f(e^{i\psi^{-1}(t)}(\psi^{-1}(t))^k dt = \frac{1}{2\pi} \sum_{k=1}^n \int_0^{2\pi} e^{it} f(e^{i\sigma_k(t)}) \sigma'_k(t) dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{it} \mathcal{L}_R(f)(e^{it}) dt = (\xi_l, \mathcal{L}_R(f))$$

for $f \in H^2(\mathbb{T})$ and $l \geq 0$. Thus $\widetilde{C}_R^* = \mathcal{L}_R$ and $\mathcal{L}_R$ is bounded. \qed

We need an appropriate basis of $H^2(\mathbb{T})$ for $R$. Let

$$e_l(z) = \begin{cases} \sqrt{1 - |\beta_0|^2}, & l = 0, \\ \frac{\alpha_l \sqrt{1 - |\beta_k|^2}}{1 - \beta_k z} \prod_{k=0}^{l-1} \frac{z - \beta_k}{1 - \beta_k z}, & l \geq 1, \end{cases}$$

where $\alpha_{k+1} = \lambda^k$, $\beta_{k+1} = z_k$ for $k \geq 0$ and $0 \leq l \leq n-1$. Since $\sum_{k=0}^\infty (1 - |\beta_k|) = \infty$, $\{e_k\}_{k=0}^\infty$ is an orthonormal basis of $H^2(\mathbb{T})$ as in Ninness, Hjalmarsson and Gustafsson [23 Theorem 2.1], and Ninness and Gustafsson [22 Theorem 1]. We write

$$Q_l(z) = \frac{\sqrt{1 - |z|^2}}{1 - \overline{z_l} z} \quad \text{and} \quad R_l(z) = \begin{cases} 1, & l = 0, \\ \prod_{k=0}^{l-1} \frac{z - z_k}{1 - z_k z}, & l \geq 1. \end{cases}$$

Thus $e_{kn+l} = Q_l R_l R_k$ for $k \geq 0$ and $0 \leq l \leq n-1$, where $R^k$ is the $k$-th power of $R$ with respect to pointwise multiplication.

Proposition 4. Let $R$ be a finite Blaschke product of degree at least two with $R(0) = 0$. Then for any $a \in C(\mathbb{T})$, we have

$$\widetilde{C}_R^* T_a \widetilde{C}_R = T_{\mathcal{L}_R(a)}.$$ 

Proof. We first examine the case that $a(z) = z^j$ for $j \geq 0$. Consider the $L^2$-expansion of $a$ by the basis $\{e_l\}_{l=0}^\infty$:

$$a = \sum_{l=0}^\infty c_l R^l + g,$$
where \( g \in (\text{Im} \tilde{C}_R)^\perp \cap H^2(\mathbb{T}) \). For \( \xi_m(z) = z^m \) with \( m \geq 0 \), we have
\[
(T_a \tilde{C}_R \xi_m)(z) = a(z) R(z)^m = \sum_{l=0}^{\infty} c_l R(z)^{l+m} + g(z) R(z)^m.
\]

It is clear that
\[
\text{Im} \tilde{C}_R = \overline{\text{span}} \{ e_{kn} \ ; \ k \geq 0 \}
\]
and
\[
(\text{Im} \tilde{C}_R)^\perp \cap H^2(\mathbb{T}) = \overline{\text{span}} \{ e_{kn+l} \ ; \ k \geq 0, \ 1 \leq l \leq n - 1 \},
\]
where \( \overline{\text{span}} \) means the closure of a linear span. Therefore \( g R^m \) is also in \((\text{Im} \tilde{C}_R)^\perp \cap H^2(\mathbb{T})\). Since \( \tilde{C}_R \) is an isometry, we have that
\[
(\tilde{C}_R^* T_a \tilde{C}_R \xi_m)(z) = \sum_{l=0}^{\infty} c_l z^{l+m}.
\]

On the other hand, \( \tilde{C}_R^* = L_R \) by Lemma 3, hence
\[
L_R (a) = \sum_{l=0}^{\infty} c_l L_R (R^l).
\]

By Lemma 2,
\[
L_R (R^l)(w) = \frac{1}{n} \sum_{z \in R^{-1}(w)} h(z) R(z)^l = (L_R(1)(w)) w^l = w^l.
\]
Thus
\[
L_R (a)(w) = \sum_{l=0}^{\infty} c_l w^l \quad \text{as an } L^2\text{-convergence.}
\]

Since \( L_R (a) \in H^\infty(\mathbb{T}) \), we have
\[
(T_{L_R(a)} \xi_m)(w) = (L_R(a) \xi_m)(w) = (M_{\xi_m} L_R(a))(w) = \sum_{l=0}^{\infty} c_l w^{l+m}
\]
for \( m \geq 0 \), where \( M_{\xi_m} \) is a multiplication operator by \( \xi_m \). Therefore we obtain that
\[
\tilde{C}_R^* T_a \tilde{C}_R = T_{L_R(a)}.
\]

For the remaining case where \( j \leq 0 \), the formula \( \tilde{C}_R^* T_a \tilde{C}_R = T_{L_R(a)} \) also holds because \( L_R \) is positive and \( T_a^* = T_\pi \).

**Lemma 5.** Let the notation be as above. Then
\[
\sum_{k=1}^{n} T_{Q_{k-1} R_{k-1}} \tilde{C}_R \tilde{C}_R^* (T_{Q_{k-1} R_{k-1}})^* = I.
\]

**Proof.** We have
\[
T_{Q_{k-1} R_{k-1}} \tilde{C}_R \xi_l = Q_{k-1} R_{k-1} R^l = e_{(n+(k-1))}
\]
for \( 1 \leq k \leq n \) and \( l \geq 0 \). Thus \( T_{Q_{k-1} R_{k-1}} \tilde{C}_R \) is an isometry, and the desired equality follows. \( \square \)

We can now state our main theorem.
Theorem 6. Let $R$ be a finite Blaschke product of degree at least two and let $R(0) = 0$; that is, 
\[ R(z) = \lambda z \prod_{k=1}^{n-1} \frac{z - z_k}{1 - \overline{z_k} z}, \]
where $n \geq 2$, $|\lambda| = 1$ and $|z_k| < 1$. Then the quotient $C^*$-algebra $\mathcal{O}_R$ of the Toeplitz-composition $C^*$-algebra $\mathcal{T}_R$ by the ideal of compact operators is isomorphic to the $C^*$-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system on the Julia set $J_R$ by an isomorphism $\Psi$ such that $\Psi(\pi(T_a)) = a$ for $a \in C(\mathbb{T})$ and $\Psi(\pi(C_R)) = \sqrt{\frac{R}{z}} S_1$, where $\pi$ is the canonical quotient map $\mathcal{T}_R$ to $\mathcal{O}_R$ and $1$ is the constant map in $X_R$ taking constant value $1$. Moreover the $C^*$-algebra $\mathcal{O}_R$ is simple and purely infinite.

Proof. For any $\xi \in X_{R,h}$ and $a \in A$, let $p(z) = \xi(z, R(z))$, $\rho(a) = \pi(T_a)$, $V_\xi = n^{1/2} \rho(p) \pi(C_R)$, we have 
\[ \rho(a) V_\xi = n^{1/2} \rho(a) \pi(C_R) = V_\alpha \xi. \]
Since, for $b \in A$
\[ \pi(C_R) \rho(b) = \pi(C_R T_b) = \pi(T_{b \circ R} C_R) = \rho(b \circ R) \pi(C_R), \]
we have that 
\[ V_\xi \rho(b) = n^{1/2} \rho(p) \pi(C_R) \rho(b) = n^{1/2} \rho(p) \rho(b \circ R) \pi(C_R) = n^{1/2} \rho(p(b \circ R)) \pi(C_R) = V_\xi \rho. \]
For any $\eta \in X_{R,h}$, define $q(z) = \eta(z, R(z))$. By Proposition \[ V_\xi V_\eta = n \pi(C_R^*) \rho(\overline{\eta}) \rho(\eta) \pi(C_R) = n \pi(C_R^*) \pi(T_{b \circ R}) = n \pi(T_{b \circ R(\xi)}) = \rho(n \mathcal{L}_R(\eta)) = \rho((\xi, a)_{A,h}). \]
Set $v_k(z, R(z)) = n^{-1/2} Q_{r-1}(z) R_{k-1}(z)$ for $1 \leq k \leq n$. By Lemma \[ \sum_{k=1}^{n} V_{v_k} V_{v_k}^* = \sum_{k=1}^{n} \pi(T_{Q_{r-1} R_{k-1}} C_R^* (T_{Q_{r-1} R_{k-1}})^*) = I. \]
By the universality and the simplicity of $\mathcal{O}_{R,h}(\mathbb{T})$, there exists an isomorphism $\Omega : \mathcal{O}_{R,h}(\mathbb{T}) \to \mathcal{O}_R$ such that $\Omega(S_\xi) = V_\xi$ and $\Omega(a) = \rho(a)$ for $\xi \in X_{R,h}$, $a \in A$. Let $\Phi$ be the map in Lemma \[ \Phi = \pi \circ \Omega^{-1}. \]
Then $\Phi$ is the desired isomorphism. Since it is proved in \[ \Phi \] that the $C^*$-algebra $\mathcal{O}_R(J_R)$ is simple and purely infinite, so is the $C^*$-algebra $\mathcal{O}_R$. The rest is now clear. \hfill $\Box$

Remark. It is important to notice that the element $\pi(C_R)$ of the composition operator in the quotient algebra corresponds exactly to the implementing isometry operator in Excel’s crossed product $A \times_{\alpha} \mathcal{L}_R \mathbb{N}$ in \[ \Phi \], which depends on the transfer operator $\mathcal{L}_R$. It follows directly from the fact that $\mathcal{O}_{R,h}(\mathbb{T})$ is naturally isomorphic to $A \times_{\alpha} \mathcal{L}_R \mathbb{N}$.

Example. Let $R(z) = z^n$ for $n \geq 2$. Then the Hilbert bimodule $X_R$ over $A = C(\mathbb{T})$ is isomorphic to $A^n$ as a right $A$-module. In fact, let $u_i(z, w) = \frac{1}{\sqrt{n}} z^{i-1}$ for $i = 1, \ldots, n$. Then $\{u_i | u_j\} = \delta_{i,j} I$ and $\{u_1, \ldots, u_n\}$ is a basis of $X_R$. Hence $S_i := S_{u_i}$, $i = 1, \ldots, n$ are generators of the Cuntz algebra $\mathcal{O}_n$. We see that $(z \cdot u_i)(z, R(z)) = u_{i+1}(z, R(z))$ for $i = 1, \ldots, n - 1$ and $(z \cdot u_n)(z, R(z)) = z^n = (u_1 \cdot z)(z, R(z))$. 

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and that the left multiplication by $z$ is a unitary $U$. Therefore the $C^*$-algebra $\mathcal{O}_R(J_R)$ associated with the complex dynamical system on the Julia set $J_R$ is the universal $C^*$-algebra generated by a unitary $U$ and $n$ isometries $S_1, ..., S_n$ satisfying $S_1S_1^* + \cdots + S_nS_n^* = I$, $US_i = S_{i+1}$ for $i = 1, ..., n-1$ and $US_n = S_1U$. In this way, the operator $S_1$ corresponds to the element $\pi(\hat{C}_R) \in \mathcal{O}_R$ of the composition operator $\hat{C}_R$ and the unitary $U$ corresponds to the element $\pi(T_z) \in \mathcal{O}_R$. Moreover we find that the commutation relation $U^nS_1 = S_1U$ in the $C^*$-algebra $\mathcal{O}_R(J_R)$ and the commutation relation $T_{R(z)}\hat{C}_R = \hat{C}_RT_z$ in the Toeplitz-composition $C^*$-algebra $\mathcal{T}_R$ are essentially the same.

The Toeplitz-composition $C^*$-algebra depends on the analytic structure of the Hardy space by the construction. The finite Blaschke product $R$ is not conjugate with $z^n$ by any Möbius automorphism unless $R(z) = \lambda z^n$. But we can show that the quotient algebra $\mathcal{O}_R$ is isomorphic to $\mathcal{O}_z$ as a corollary of our main theorem.

**Corollary 7.** Let $R$ be a finite Blaschke product of degree $n \geq 2$ with $R(0) = 0$. Then the quotient $C^*$-algebra $\mathcal{O}_R$ is isomorphic to the $C^*$-algebra $\mathcal{O}_z$. Moreover $K_0(\mathcal{O}_R) \cong \mathbb{Z} \oplus \mathbb{Z}/(n-1)\mathbb{Z}$ and $K_1(\mathcal{O}_R) \cong \mathbb{Z}$.

**Proof.** Let $\psi$ be the strictly increasing continuously differentiable function defined in the paragraph before Lemma 3. Since

$$\psi'(\theta) = \frac{e^{i\theta} R'(e^{i\theta})}{R(e^{i\theta})} = 1 + \sum_{k=1}^{n-1} \frac{1 - |z_k|^2}{|e^{i\theta} - z_k|^2} > 1,$$

$R|_T$ is an expanding map of degree $n$ and $R|_{T}$ is topologically conjugate to $z^n$ on $T$ by [30]. See a general condition given in Martin [16]. Therefore the statement follows from the above theorem and the results in [11].

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