A two-species competition model with mixed dispersal and free boundaries in time-periodic environment

Qiaoling Chen
School of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, PR China
School of Science, Xi’an Polytechnic University, Xi’an 710048, PR China

Fengquan Li
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, PR China

Sanyi Tang
School of Mathematics and Information Science, Shaanxi Normal University, Xi’an 710062, PR China

Feng Wang
School of Mathematics and Statistics, Xidian University, Xi’an 710071, PR China

Abstract. This paper is concerned with a Lotka-Volterra type competition model with free boundaries in time-periodic environment. One species is assumed to adopt nonlocal dispersal and the other one adopt mixed dispersal, which is a combination of both random dispersal and nonlocal dispersal. We show that this free boundary problem with more general growth functions admits a unique solution defined for all time. A spreading-vanishing dichotomy is obtained and criteria for spreading and vanishing are provided.

Keywords: Competition model; Free boundary; Mixed dispersal; Time-periodic environment; Spreading-vanishing dichotomy.

AMS subject classifications (2000): 35K57, 35K61, 35R35, 92D25.

1 Introduction

In this paper, we study the dynamical behavior of the solution \((u(t, x), v(t, x), g(t), h(t))\) to the following Lotka-Volterra type competition model with mixed dispersal and free boundaries in
time-periodic environment

\[
\begin{aligned}
\partial_t u &= d_1 \left( \int_{g(t)}^1 J(x-y) u(t,y) dy - u \right) + u(a(t) - u - b(t)v), \quad t > 0, \ g(t) < x < h(t), \\
\partial_t v &= d_2 \left[ \tau \partial_y^2 v + (1 - \tau) \left( \int_{g(t)}^1 J(x-y) v(t,y) dy - v \right) \right] \\
&\quad + v(c(t) - v - d(t)u), \quad t > 0, \ g(t) < x < h(t), \\
&\quad u(t,g(t)) = u(t,h(t)) = v(t,g(t)) = v(t,h(t)) = 0, \quad t \geq 0, \\
h'(t) &= -\mu v_x(t,h(t)) + \rho_1 \int_{h(t)}^1 J(x-y) u(t,y) dy dx \\
&\quad + \rho_2 \int_{g(t)}^1 J(x-y) v(t,y) dy dx, \quad t \geq 0, \\
g'(t) &= -\mu u_x(t,g(t)) - \rho_1 \int_{g(t)}^1 J(x-y) u(t,y) dy dx \\
&\quad - \rho_2 \int_{g(t)}^1 J(x-y) v(t,y) dy dx, \quad t \geq 0, \\
u(0,0) &= u_0(x), \quad v(0,0) = v_0(x), \quad |x| \leq h_0, \\
h(0) &= -g(0) = h_0.
\end{aligned}
\]  

(1.1)

Here \(u(t,x)\) and \(v(t,x)\) represent the population densities of two competing species; the positive constants \(d_1, d_2\) are dispersal rates of \(u,v\) and the constant \(0 < \tau \leq 1\) measures the fraction of individuals adopting random dispersal; \(h_0, \mu\) and \(\rho_i\) \((i = 1, 2)\) are positive constants; the kernel function \(J : \mathbb{R} \to \mathbb{R}\) satisfies that

(J) \quad \(J\) is Lipschitz continuous, \(J(x) \geq 0, J(0) > 0, \int_\mathbb{R} J(x) dx = 1, J\) is symmetric and \(\sup_\mathbb{R} J < \infty;\)

\(a(t), c(t)\) represent the intrinsic growth rates of species, \(b(t), d(t)\) represent competition between species and they satisfy that

\(a(t), b(t), c(t), d(t)\) are positive \(T\)-periodic functions and

\(a, b \in C([0,T]), c, d \in C^\alpha([0,T])\) for \(0 < \alpha < 1;\)

the initial functions \(u_0\) and \(v_0\) satisfy

\[
\begin{aligned}
u_0 &\in C^{1\alpha}([-h_0,h_0]), \quad u_0(\pm h_0) = 0, \quad u_0 > 0 \quad \text{in} \ (-h_0,h_0), \\
v_0 &\in C^{1\alpha}([-h_0,h_0]), \quad v_0(\pm h_0) = 0, \quad v_0 > 0 \quad \text{in} \ (-h_0,h_0),
\end{aligned}
\]

(1.2)

where \(C^{1\alpha}([-h_0,h_0])\) is defined as the Lipschitz continuous function space.

Ecologically, problem (1.1) describes the dynamical process of two competing species which spread and invade to new environment with daily or seasonal changes via the same free boundaries. All the individuals in the population \(u\) adopt nonlocal dispersal, while in the population \(v\) a fraction of individuals adopt nonlocal dispersal and the remaining fraction assumes random dispersal. The latter strategy is called mixed dispersal, which was first proposed by Kao et al.\cite{24}. We assume that the spreading fronts expand at a speed that is proportional to the outward flux of the population of the two species at the front, which give rise to the free boundary conditions in (1.1). Problem (1.1) is a variation of the following two species competition system studied in \cite{24}:

\[
\begin{aligned}
\partial_t u &= d_1 \left( \int_{\mathbb{R}^N} J(x-y) u(t,y) dy - u \right) + u(a(x) - u - v), \quad t > 0, \ x \in \mathbb{R}^N, \\
\partial_t v &= d_2 \left[ \tau \partial_y^2 v + (1 - \tau) \left( \int_{\mathbb{R}^N} J(x-y) v(t,y) dy - v \right) \right] + v(a(x) - u - v), \quad t > 0, \ x \in \mathbb{R}^N.
\end{aligned}
\]

They investigated how the mixed dispersal affects the invasion of a single species and how the mixed dispersal strategies will evolve in spatially periodic but temporally constant environment.
If $\tau = 0$ and $a(t), b(t), c(t), d(t)$ are constants, (1.1) reduces to a two species nonlocal diffusion system with free boundaries studied by Du et al. [17]. They proved the model has a unique global solution, established a spreading-vanishing dichotomy and obtained criteria for spreading and vanishing. Moreover, for the weak competition case they determined the long-time asymptotic limit of the solution when spreading happens. If $\tau = 1$ and $a(t), b(t), c(t), d(t)$ are constants, (1.1) becomes a free boundary problem of ecological model with nonlocal and local diffusions considered in [31]. They also obtained well-posedness of solutions and spreading-vanishing results. Moreover, Cao et al. [4] recently considered a nonlocal diffusion Lotka-Volterra type competition model with free boundary in [10], and similar results including the existence and uniqueness of global solutions were obtained in [3], from which one can see that the nonlocal diffusion brings many essential difficulties in analysis.

In the absence of the species $v$ (i.e. $v \equiv 0$) and $a(t)$ is a constant, (1.1) reduces to the following nonlocal diffusion model with free boundaries

$$
\begin{aligned}
\partial_t u &= d_1 \left( \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - u \right) + u(a-u), \quad t > 0, \quad g(t) < x < h(t), \\
\partial_t v &= d_2 \left( \int_{g(t)}^{h(t)} J(x-y)v(t,y)dy - v \right) + v(b-v), \quad t > 0, \quad g(t) < y < h(t), \\
\partial_t v &= d_3 \left( \int_{g(t)}^{h(t)} J(x-y)v(t,y)dy - v \right) + v(c-v), \quad t > 0, \quad g(t) < y < h(t), \\
\partial_t u &= d_4 \left( \int_{g(t)}^{h(t)} J(x-y)u(t,y)dy - u \right) + u(a-u), \quad t > 0, \quad g(t) < x < h(t), \\
\partial_t v &= d_5 \left( \int_{g(t)}^{h(t)} J(x-y)v(t,y)dy - v \right) + v(b-v), \quad t > 0, \quad g(t) < y < h(t), \\
\partial_t v &= d_6 \left( \int_{g(t)}^{h(t)} J(x-y)v(t,y)dy - v \right) + v(c-v), \quad t > 0, \quad g(t) < y < h(t),
\end{aligned}
$$

which has been studied in [3]. Problem (1.3) is a nature extension of the local diffusion model with free boundary in [10], and similar results including the existence and uniqueness of global solutions for more general growth function $f(t, x, u)$ and the spreading-vanishing results in the homogeneous environment were obtained in [3], from which one can see that the nonlocal diffusion brings many essential difficulties in analysis.

Since the work of Du and Lin [10], the local diffusion models with free boundary(ies) have been studied extensively. For example, the model in [10] has been extended to other situations of single species model such as in higher dimensional space, heterogeneous environment, time-periodic environment, or with other boundary conditions, general nonlinear term, advection term, we refer the readers to [2, 6, 9, 11, 13–15, 19, 23, 25, 27, 28, 33, 39, 41] and references therein. Moreover, two-species Lotka-Volterra type competition problems and predator-prey problems with free boundary(ies) have also been considered in the homogeneous environment or heterogeneous time-periodic environment, e.g., [7, 12, 16, 20, 21, 23, 24, 30, 32, 35, 38, 40]. The epidemic models with free boundary(ies) have also been considered in [5, 18, 20].

The aim of this paper is to study the well-posedness and long-time behaviors of solutions to problem (1.1). We first investigate the existence and uniqueness of solutions to (1.1) with more general growth functions. To achieve it, we shall establish the maximum principle for linear parabolic equations with mixed dispersal, and prove that the nonlinear parabolic equations with mixed dispersal (see (2.5)) admit a unique positive solution under the assumption that $g'(t), h'(t)$ and $u(t, x)$ are only continuous functions by approximation method, which plays an important role in the process of using the fixed point theorem. Then we establish a spreading-vanishing dichotomy and criteria for spreading and vanishing. To discuss the spreading and vanishing, we need to consider the existence and properties of principle eigenvalue of time-periodic parabolic-type eigenvalue problems with random/mixed dispersal. Since the intrinsic growth rates $a(t)$ and $c(t)$
are independent of spatial variable, we can transform the parabolic-type eigenvalue problems into elliptic-type eigenvalue problems. This transformation is also used in discussing the asymptotic behavior of solution (see Theorem 4.1). Finally, it is worth mentioning that, due to the effect of mixed dispersal, in Theorem 4.4 we only prove the vanishing result for two cases, but whether the other situations still hold true is unknown, we leave it for future research.

The rest of the paper is organized as follows. In Section 2, we establish the global existence and uniqueness of solutions to problem (1.1) with more general growth functions. The comparison principle in the moving domain and the discussions on eigenvalue problems are given in Section 3. In Section 4, we investigate spreading and vanishing of species.

**Notations.** Throughout the paper, we denote $\Omega_{T_0}^{g,h} = (0, T_0] \times (g(t), h(t))$, $D_{T_0} = (0, T_0] \times (-1, 1)$ and $\alpha_T = \frac{1}{T} \int_0^T a(t) dt$. Under the transform $x(t, z) = \frac{(h(t) - g(t))z + h(t) + g(t)}{2}$, we always denote $\hat{f}(t) = f(t, x(t, z)) = f\left(\frac{(h(t) - g(t))z + h(t) + g(t)}{2}\right)$.

## 2 Well-posedness

In this section, we give the global well-posedness of solutions to problem (1.1) with more general growth functions. More precisely, we consider the following free boundary problem

\[
\begin{align*}
\partial_t u &= d_1 \left( \int_{g(t)}^{h(t)} J(x,y)u(t,y)dy - u \right) + f_1(t, x, u, v), \quad t > 0, \quad g(t) < x < h(t), \\
\partial_t v &= d_2 \left[ \partial_x^2 v + (1 - \tau) \left( \int_{g(t)}^{h(t)} J(x,y)v(t,y)dy - v \right) \right] + f_2(t, x, u, v), \quad t > 0, \quad g(t) < x < h(t), \\
u(t, g(t)) &= \hat{u}(t, h(t)) = v(t, g(t)) = v(t, h(t)) = 0, \quad t \geq 0, \\
h'(t) &= -\mu v_x(t, h(t)) + \rho_1 \int_{g(t)}^{h(t)} J(x,y)u(t,y)dydx \\
&\quad + \rho_2 \int_{h(t)}^{\infty} J(x,y)v(t,y)dydx, \quad t \geq 0, \\
g'(t) &= -\mu v_x(t, g(t)) - \rho_1 \int_{g(t)}^{h(t)} J(x,y)u(t,y)dydx \\
&\quad - \rho_2 \int_{-\infty}^{g(t)} J(x,y)v(t,y)dydx, \quad t \geq 0, \\
u(0, x) &= u_0(x), v(0, x) = v_0(x), \quad |x| \leq h_0, \\
h(0) &= -g(0) = h_0,
\end{align*}
\]

(2.1)

where the growth terms $f_i(t, x, u, v)$ ($i = 1, 2$) satisfy the following assumptions:

**F1** $f_1(t, x, 0, v), f_2(t, x, u, 0) \equiv 0$, and there exists a constant $K > 0$ such that $f_1(t, x, u, v) < 0$ for all $u > K$, $v \geq 0$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, and $f_2(t, x, u, v) < 0$ for all $u \geq 0$, $\nu > K$ and $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$;

**F2** For any given $T, l, K_1, K_2 > 0$, there exists a constant $L = L(T, l, K_1, K_2)$ such that

\[\|f_2(\cdot, x, u, v)\|_{C^*(0, T)} \leq L\]

for all $x \in [-l, l], u \in [0, K_1]$ and $v \in [0, K_2]$;

**F3** For any $K_1, K_2 > 0$, there exists a constant $L^* = L^*(K_1, K_2) > 0$ such that

\[|f_i(t, x, u, v) - f_i(t, y, u, v)| \leq L^*|x - y|\]

for all $u \in [0, K_1], v \in [0, K_2]$ and all $(t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$;
(f4) \( f_i(t, x, u, v) \) is locally Lipschitz in \( u, v \in \mathbb{R}^+ \) uniformly for \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \), i.e., for any \( K_1, K_2 > 0 \), there exists a constant \( \hat{L} = \hat{L}(K_1, K_2) > 0 \) such that

\[
|f_i(t, x, u_1, v_1) - f_i(t, x, u_2, v_2)| \leq \hat{L}(|u_1 - u_2| + |v_1 - v_2|)
\]

for all \( u_1, u_2 \in [0, K_1] \), \( v_1, v_2 \in [0, K_2] \) and all \( (t, x) \in \mathbb{R}^+ \times \mathbb{R} \).

It is easy to check that the growth functions in (1.1) satisfy the conditions (f1) – (f4). The main result of this section is stated in the following theorem.

**Theorem 2.1.** Assume that (J) and (f1)-(f4) hold. For any given \((u_0, v_0)\) satisfying (1.2), the problem (2.1) admits a unique global solution \((u, v, g, h)\) defined on \([0, T_0]\) for any \(0 < T_0 < \infty\) and

\[
(u, v, g, h) \in C^{1,1-}(\overline{\Omega}_{T_0}^{g, h}) \times C^{1+\frac{\alpha}{2}+\alpha}(\Omega_{T_0}^{g, h}) \times [C^{1+\frac{\alpha}{2}}([0, T_0])]^2,
\]

\[
0 < u \leq K_1 := \max\{\|u_0\|_{\infty}, K\}, \quad 0 < v \leq K_2 := \max\{\|v_0\|_{\infty}, K\}, \quad \forall (t, x) \in \Omega_{T_0}^{g, h},
\]

\[
0 < -v_x(t, h(t)), \quad v_x(t, g(t)) \leq K_3 := 2K_2 \max\left\{\sqrt{\frac{L + d_2(1 - \tau)}{2d_2}}, \frac{4\|v_0\|_{\infty} + 1}{3K_2}\right\}, \quad 0 < t \leq T_0,
\]

where \(C^{1,1-}(\overline{\Omega}_{T_0}^{g, h})\) denotes the class of functions that are \(C^1\) in \(t\) and Lipschitz continuous in \(x\), and \(\hat{L} = \hat{L}(K_1, K_2)\) is the Lipschitz constant defined in (f4).

To prove Theorem 2.1, we first establish the maximum principle for linear parabolic equations with mixed dispersal. For some \(h_0, T_0\), we define

\[
\mathbb{H}_{T_0}^{h_0} := \{h \in C^1([0, T_0]) : h(0) = h_0, \quad 0 < h'(t) \leq R(t)\},
\]

\[
\mathbb{G}_{T_0}^{h_0} := \{g \in C^1([0, T_0]) : \quad g \in \mathbb{H}_{T_0}^{h_0}\}
\]

with

\[
R(t) := \mu K_3 + 2(h_0\rho_1 K_1 + h_0\rho_2 K_2 + \mu K_3)e^{(\rho_1 K_1 + \rho_2 K_2)t}.
\]

**Lemma 2.1.** (Maximum Principle) Assume that (J) holds and \((g, h) \in \mathbb{G}_{T_0}^{h_0} \times \mathbb{H}_{T_0}^{h_0}\). If \(v(t, x) \in C^{1,2}(\Omega_{T_0}^{g, h}) \cap C(\overline{\Omega}_{T_0}^{g, h})\) satisfies, for some \(c \in L^\infty(\Omega_{T_0}^{g, h})\),

\[
\begin{aligned}
\partial_t v &\geq d_2 \left[\tau \partial_x^2 v + (1 - \tau) \int_{g(t)}^{h(t)} J(x - y)v(t, y)dy - v\right] + c(t, x)v, \quad (t, x) \in \Omega_{T_0}^{g, h}, \\
v(t, g(t)) &\geq 0, \quad v(t, h(t)) \geq 0, \quad t \in (0, T_0], \\
v(0, x) &\geq 0, \quad x \in [-h_0, h_0],
\end{aligned}
\]

then \(v(t, x) \geq 0\) for all \((t, x) \in \overline{\Omega}_{T_0}^{g, h}\). Moreover, if \(v(0, x) \neq 0\) in \([-h_0, h_0]\), then \(v(t, x) > 0\) in \(\Omega_{T_0}^{g, h}\).

**Proof.** (i) Let \(\omega(t, x) = e^{-kt}v(t, x)\), where \(k > 0\) is a constant chosen large enough such that

\[
- k + c(t, x) < 0 \quad \text{for all } (t, x) \in \Omega_{T_0}^{g, h}.
\]

Then

\[
\partial_t \omega \geq d_2 \left[\tau \partial_x^2 \omega + (1 - \tau) \int_{g(t)}^{h(t)} J(x - y)\omega(t, y)dy\right] + [-k - d_2(1 - \tau) + c(t, x)]\omega.
\]
We are now in a position to prove that $\omega \geq 0$ in $\overline{\Omega_{T_0}^g}$. Suppose that $\omega_{\inf} = \inf_{(t,x) \in \overline{\Omega_{T_0}^g}} \omega(t,x) < 0$. By (2.3), $\omega \geq 0$ on the parabolic boundary of $\overline{\Omega_{T_0}^g}$, and hence there exists $(t_*, x_*) \in \Omega_{T_0}^g$ such that $\omega_{\inf} = \omega(t_*, x_*) < 0$. Since $\partial_t \omega(t_*, x_*) \leq 0$, $\partial^2_t \omega(t_*, x_*) \geq 0$, then

$$
\partial_t \omega(t_*, x_*) \geq d_{2} \left[ \tau \partial^2_x \omega(t_*, x_*) + (1 - \tau) \int_{\partial \Omega} h(t_*) J(x_{*} - y) \omega(t_*, y) dy \right] + [-k - d_2(1 - \tau) + c(t_*, x_*)] \omega(t_*, x_*)
$$

$$
\geq d_2 \tau \partial^2_x \omega(t_*, x_*) + d_2(1 - \tau) \omega_{\inf} \int_{\Omega} J(x_{*} - y) dy + [-k - d_2(1 - \tau) + c(t_*, x_*)] \omega_{\inf}
$$

$$
= d_2 \tau \partial^2_x \omega(t_*, x_*) + [-k + c(t_*, x_*)] \omega_{\inf}.
$$

Since $[-k + c(t_*, x_*)] \omega_{\inf} < 0$, we can get a contradiction. Thus, $\omega(t,x) \geq 0$ in $\overline{\Omega_{T_0}^g}$, which implies that

$$
v(t,x) \geq 0 \text{ for all } (t,x) \in \overline{\Omega_{T_0}^g}.
$$

(ii) Now assume that $v(0,x) \neq 0$ in $[-h_0,h_0]$. By (2.4) and the fact $J(x) \geq 0$, we have

$$
\partial_t v \geq d_2 \left[ \tau \partial^2_x v + (1 - \tau) \left( \int_{\partial \Omega} h(t) J(x_{*} - y) v(t, y) dy - v \right) \right] + c(t,x)v
$$

$$
\geq d_2 \tau \partial^2_x v - (1 - \tau) v + c(t,x)v
$$

$$
= d_2 \tau \partial^2_x v + [c(t,x) - d_2(1 - \tau)]v.
$$

Define the transform

$$
x(t,z) = \frac{(h(t) - g(t))z + h(t) + g(t)}{2}, \quad \text{that is, } z(t,x) = \frac{2x - g(t) - h(t)}{h(t) - g(t)},
$$

and let $\tilde{v}(t,z) = v(t,x(t,z))$ and $\tilde{c}(t,z) = c(t,x(t,z))$, then $\tilde{v}(t,z)$ satisfies

$$
\begin{cases}
\partial_t \tilde{v} \geq d_2 \tau \xi(t) \partial^2_z \tilde{v} + \eta(t,z) \partial_z \tilde{v} + [\tilde{c}(t,z) - d_2(1 - \tau)]\tilde{v}, & (t,z) \in D_{T_0}, \\
\tilde{v}(t,-1) \geq 0, \quad \tilde{v}(t,1) \geq 0, & t \in [0,T_0], \\
\tilde{v}(0,z) = v(0,h_0z) \geq 0, & z \in [-1,1],
\end{cases}
$$

where

$$
\xi(t) = \frac{4}{(h(t) - g(t))^2}, \quad \eta(t,z) = \frac{h'(t) + g'(t)}{h(t) - g(t)} + \frac{(h'(t) - g'(t))z}{h(t) - g(t)}.
$$

By the classical maximum principle for parabolic equation, we know

$$
\tilde{v}(t,z) > 0, \quad \forall (t,z) \in D_{T_0}.
$$

Thus, $v(t,x) > 0$ in $\Omega_{T_0}^g$. This completes the proof.

According to Lemma 2.1, we can derive the following comparison principle.
Lemma 2.2. (Comparison principle) Suppose that \((J)\) holds, \((g, h) \in \mathbb{G}^{h_{0}}_{T_0} \times \mathbb{H}^{h_{0}}_{T_0}\) and \(f(t, x, u, v)\) satisfies (f4). Let \(v_1(t, x), v_2(t, x) \in C^{1,2}(\Omega^{g,h}_{T_0}) \cap C(\overline{\Omega}^{g,h}_{T_0})\) satisfy

\[
\begin{aligned}
&\partial_t v_1 - d_2 \left[ \tau \partial^2_v v_1 + (1 - \tau) \left( f^h(t) J(x - y) v_1(t, y) dy - v_1 \right) \right] - f(t, x, u, v_1) \\
&\geq \partial_t v_2 - d_2 \left[ \tau \partial^2_v v_2 + (1 - \tau) \left( f^h(t) J(x - y) v_2(t, y) dy - v_2 \right) \right] - f(t, x, u, v_2), \\
&v_1(t, x) \geq v_2(t, x), \quad t \in (0, T_0], \quad x = g(t) \text{ or } x = h(t), \\
&v_1(0, x) \geq v_2(0, x), \quad x \in [-h_0, h_0].
\end{aligned}
\]

If \(u(t, x) \in [0, c_1], v_1(t, x), v_2(t, x) \in [0, c_2] \) in \(\Omega^{g,h}_{T_0}\) for some constants \(c_1, c_2 > 0\), then we have

\[
v_1(t, x) \geq v_2(t, x), \quad \forall (t, x) \in \Omega^{g,h}_{T_0}.
\]

If we further assume that \(v_1(0, x) \neq v_2(0, x)\) for \(x \in [-h_0, h_0]\), then

\[
v_1(t, x) > v_2(t, x), \quad \forall (t, x) \in \Omega^{g,h}_{T_0}.
\]

Proof. Let \(w = v_1 - v_2\), then we have

\[
\begin{aligned}
&\partial_t w - d_2 \left[ \tau \partial^2_v w + (1 - \tau) \left( f^h(t) J(x - y) w(t, y) dy - w \right) \right] \\
&\geq f(t, x, u, v_1) - f(t, x, u, v_2), \quad (t, x) \in \Omega^{g,h}_{T_0}, \\
w(t, x) \geq 0, \quad t \in (0, T_0], \quad x = g(t) \text{ or } x = h(t), \\
w(0, x) \geq 0, \quad x \in [-h_0, h_0].
\end{aligned}
\]

Since \(f(t, x, u, v)\) satisfies (f4), we have

\[
f(t, x, u, v_1) - f(t, x, u, v_2) = \int_{v_2}^{v_1} \frac{\partial f(t, x, u, \eta)}{\partial \eta} d\eta = \int_{0}^{1} \frac{\partial f(t, x, u, \eta)}{\partial \eta} \bigg|_{\eta = v_2 + s(v_1 - v_2)} ds \cdot (v_1 - v_2)
\]

\[
= c(t, x) w.
\]

Denote \(\hat{L}(c_1, c_2)\) by the Lipschitz constant of \(f\) for \((t, x, u, v) \in \mathbb{R}^+ \times \mathbb{R} \times [0, c_1] \times [0, c_2]\), then \(\|c\|_{L^\infty} \leq \hat{L}(c_1, c_2)\). By applying Lemma 2.1, we can get the desired results. \( \square \)

Next, by applying the classical upper and lower method we shall prove that nonlinear parabolic equations with mixed dispersal (see (2.5)) admit a unique positive classical solution under the assumption that \(g'(t), h'(t)\) and \(u(t, x)\) are Hölder continuous. For some \(h_0, T_0 > 0\), we define

\[
\begin{aligned}
\mathbb{G}^{h_{0}}_{T_0} := \{ h \in C^{1+\frac{\alpha}{2}}([0, T_0]) : h(0) = h_0, \quad 0 < h'(t) \leq R(t) \}, \\
\mathbb{H}^{h_{0}}_{T_0} := \{ g \in C^{1+\frac{\alpha}{2}}([0, T_0]) : -g \in \mathbb{G}^{h_{0}}_{T_0} \}.
\end{aligned}
\]

Lemma 2.3. Suppose that \((J)\) holds, \((g, h) \in \mathbb{G}^{h_{0}}_{T_0} \times \mathbb{H}^{h_{0}}_{T_0}\), \(u \in C^{1+\frac{\alpha}{2}}(\overline{\Omega}^{g,h}_{T_0})\), \(f_2\) satisfies (f1)-(f4) and \(v_0\) satisfies (1.2). Then for any \(T_0 > 0\), the following problem

\[
\begin{aligned}
\partial_t v &= d_2 \left[ \tau \partial^2_v v + (1 - \tau) \left( f^h(t) J(x - y) v(t, y) dy - v \right) \right] + f_2(t, x, u, v), \quad (t, x) \in \Omega^{g,h}_{T_0}, \\
v(t, g(t)) &= v(t, h(t)) = 0, \quad t \in (0, T_0], \\
v(0, x) &= v_0(x), \quad x \in [-h_0, h_0]
\end{aligned}
\]

(2.5)
admits a unique solution \( v(t, x) \in C^{1+\frac{\alpha}{2},2+\alpha}(\Omega_{T_0}^{g,h}) \). Moreover, \( v(t, x) \) satisfies
\[
\begin{align*}
0 < v(t, x) &\leq K_2 \quad \text{for } (t, x) \in \Omega_{T_0}^{g,h}, \\
0 < -v_x(t, h(t)), v_x(t, g(t)) &\leq K_3 \quad \text{for } t \in (0, T_0].
\end{align*}
\] (2.6)

**Proof.** We mainly adopt the classical upper and lower solutions method. Since the mixed dispersal is considered, we give the details of the proof. A function \( \tilde{v} \) is called an upper solution of (2.5) if \( \tilde{v} \in C^{1,2}(\Omega_{T_0}^{g,h}) \cap C(\overline{\Omega}_{T_0}^{g,h}) \) satisfies
\[
\begin{align*}
\partial_t \tilde{v} &\geq d_2 \left[ \tau \partial_x^2 \tilde{v} + (1-\tau) \left( f_{g(t)}^{h(t)} J(x-y) \tilde{v}(t, y) dy - \tilde{v} \right) \right] + f_2(t, x, \tilde{v}), \quad (t, x) \in \Omega_{T_0}^{g,h}, \\
\tilde{v}(t, g(t)) &\geq 0, \quad \tilde{v}(t, h(t)) \geq 0, \quad t \in (0, T_0], \\
\tilde{v}(0, x) &\geq v_0(x), \quad x \in [-h_0, h_0],
\end{align*}
\]
and a function \( \underline{v} \) is called a lower solution of (2.5) if reversing all the above inequalities.

**Step 1.** We claim that, if \( \tilde{v}, \underline{v} \) are respectively nonnegative upper and lower solutions of (2.5), then (2.5) has a unique solution \( v(t, x) \) satisfying
\[
v(t, x) \leq \underline{v}(t, x) \leq \tilde{v}(t, x), \quad \forall (t, x) \in \overline{\Omega}_{T_0}^{g,h}.
\]

Indeed, since \( u \in C^{\frac{\alpha}{2},\alpha}(\Omega_{T_0}^{g,h}) \) and \( \tilde{v}, \underline{v} \in C(\overline{\Omega}_{T_0}^{g,h}) \), there exists a constant \( M > 0 \) such that \( 0 \leq u, \tilde{v}, \underline{v} \leq M \) for \( (t, x) \in \overline{\Omega}_{T_0}^{g,h} \). By (14), we have, for some constant \( k > d_2(1-\tau) \),
\[
|f_2(t, x, u, \xi(1)) - f_2(t, x, u, \xi(2))| \leq [k - d_2(1-\tau)]|\xi(1) - \xi(2)| \quad \text{for any } (t, x) \in \overline{\Omega}_{T_0}^{g,h} \text{ and } u, \xi(1), \xi(2) \in [0, M].
\]

For any \( \vartheta \in C(\overline{\Omega}_{T_0}^{g,h}) \) satisfying \( \vartheta \in [0, M] \), we define a mapping \( \Phi \) by \( v = \Phi \vartheta \), where \( v \in C^{\frac{\alpha}{2}+1,\alpha}(\overline{\Omega}_{T_0}^{g,h}) \) is the unique solution of
\[
\begin{align*}
\partial_t v - d_2 \tau \partial_x^2 v + k v = f_2(t, x, \vartheta) + k \vartheta, \quad (t, x) \in \Omega_{T_0}^{g,h}, \\
v(t, g(t)) = v(t, h(t)) = 0, \quad t \in (0, T_0], \\
v(0, x) = v_0(x), \quad x \in [-h_0, h_0].
\end{align*}
\] (2.7)

The existence and uniqueness of \( v \in C^{\frac{\alpha}{2}+1+\alpha}(\overline{\Omega}_{T_0}^{g,h}) \) is guaranteed by the \( L^p \) theory for linear parabolic equation and the Sobolev imbedding theorem. More precisely, let \( \bar{v}(t, z) = v(t, x(t, z)), \bar{\vartheta}(t, z) = \vartheta(t, x(t, z)) \) and \( \bar{f}_2(t, z, \bar{v}, \bar{\vartheta}) = f_2(t, x(t, z), \bar{v}, \bar{\vartheta}) \), then (2.7) becomes
\[
\begin{align*}
\partial_t \bar{v} - d_2 \tau \partial_z^2 \bar{v} - \eta(t, z) \partial_z \bar{v} + k \bar{v} = f_2(t, z, \bar{v}, \bar{\vartheta}) + k \bar{\vartheta} + \bar{f}_2(t, z, \bar{v}, \bar{\vartheta}), \quad (t, z) \in D_{T_0}, \\
\bar{v}(t, -1) = \bar{v}(t, 1) = 0, \quad t \in (0, T_0], \\
\bar{v}(0, z) = v_0(0, z), \quad z \in [-1, 1].
\end{align*}
\] (2.8)

Note that the right hand of the equation in (2.8) is continuous in \( D_{T_0} \) and then belongs to \( L^p(D_{T_0}) \) with any \( p > 3 \). \( \xi(t) \in C([0,T_0]) \) with \( \|\xi\|_{L^\infty([0,T_0])} \leq \frac{1}{h_0} \) and \( \|\eta\|_{L^\infty([0,T_0])} \leq \frac{2R(T_0)}{h_0} \). Applying the \( L^p \) theory to (2.8) and the Sobolev imbedding theorem, we can obtain a unique solution \( \bar{v} \in W^{1,2}_{p}(D_{T_0}) \rightarrow C^{\frac{\alpha}{2}+1+\alpha}(D_{T_0}) \), and then get a unique solution \( v \in C^{\frac{\alpha}{2}+1,\alpha}(\overline{\Omega}_{T_0}^{g,h}) \) to (2.8).
We shall show that \( \Phi \) is monotone in the sense that if any \( \vartheta_1, \vartheta_2 \in C(\Omega_{T_0}^{g,h}) \) satisfy \( 0 \leq \vartheta_1, \vartheta_2 \leq M \) and \( \vartheta_2 \geq \vartheta_1 \), then \( \Phi \vartheta_2 \geq \Phi \vartheta_1 \). To see that, let \( w = \Phi \vartheta_2 - \Phi \vartheta_1 \), then \( w \) satisfies

\[
\begin{align*}
\partial_t w - d_2 \tau \partial_x^2 w + kw &= d_2 (1 - \tau) \left( J(x-y)(\vartheta_2(t,y) - \vartheta_1(t,y))dy - (\vartheta_2 - \vartheta_1) \right) \\
&\quad + f_2(t, x, u, \vartheta_2) - f_2(t, x, u, \vartheta_1) + k(\vartheta_2 - \vartheta_1), \quad (t, x) \in \Omega_{T_0}^{g,h}, \\
w(t, g(t)) &= w(t, h(t)) = 0, \quad t \in (0, T_0], \\
w(0, x) &= 0, \quad x \in [-h_0, h_0].
\end{align*}
\]

Since the equation in (2.9) satisfies similar as the proof of Lemma 2.1 (ii), we can get \( \hat{w}(t, z) = w(t, x(t, z)) \geq 0 \) in \( \Omega_{T_0}^{g,h} \) by the maximum principle for linear parabolic equation. Thus, we have \( w(t, x) \geq 0 \) in \( \Omega_{T_0}^{g,h} \) and then \( \Phi \vartheta_2 \geq \Phi \vartheta_1 \).

Next, we shall show that \( \Phi \vartheta \leq \vartheta \) provided that \( \vartheta \) is an upper solution. In fact, let \( v = \Phi \vartheta \), then

\[
\begin{align*}
\partial_t (\vartheta - v) - d_2 \tau \partial_x^2 (\vartheta - v) + k(\vartheta - v) &= 0, \quad (t, x) \in \Omega_{T_0}^{g,h}, \\
(\vartheta - v)(t, g(t)) &\leq 0, \quad (\vartheta - v)(t, h(t)) \geq 0, \quad t \in (0, T_0], \\
(\vartheta - v)(0, x) &\geq 0, \quad x \in [-h_0, h_0].
\end{align*}
\]

Similar as above, we have \( \vartheta - v \geq 0 \) in \( \Omega_{T_0}^{g,h} \), i.e., \( \Phi \vartheta \leq \vartheta \). Similarly, we can also prove that \( \Phi \vartheta \geq \vartheta \) provided that \( \vartheta \) is a lower solution.

We then construct two sequences \( \{v^{(n)}\} \) and \( \{w^{(n)}\} \) as follows

\[
v^{(1)} = \Phi \vartheta, \quad v^{(2)} = \Phi v^{(1)}, \quad \ldots, \quad v^{(n)} = \Phi v^{(n-1)}, \quad \ldots, \\
w^{(1)} = \Phi \bar{v}, \quad w^{(2)} = \Phi w^{(1)}, \quad \ldots, \quad w^{(n)} = \Phi w^{(n-1)}, \quad \ldots
\]

Thus,

\[
\underline{v} \leq w^{(1)} \leq w^{(2)} \leq \cdots \leq w^{(n)} \leq v^{(n)} \leq \cdots \leq v^{(2)} \leq v^{(1)} \leq \bar{v}.
\]

We conclude that the pointwise limits

\[
w^*(t, x) = \lim_{n \to \infty} w^{(n)}(t, x), \quad v^*(t, x) = \lim_{n \to \infty} v^{(n)}(t, x)
\]

exist at each point in \( \Omega_{T_0}^{g,h} \) and

\[
\underline{v}(t, x) \leq w^*(t, x) \leq v^*(t, x) \leq \bar{v}(t, x) \quad \text{in} \quad \Omega_{T_0}^{g,h}.
\]
We first prove that $\Phi$ is continuous. For any $\theta_1, \theta_2 \in D$, we still define $w = \Phi \theta_2 - \Phi \theta_1$, then $w$ satisfies (2.9). Let $\bar{w}(t, z) = w(t, x(t, z))$, $\hat{u}(t, z) = u(t, x(t, z))$, $\hat{\theta}_i(t, z) = \theta_i(t, x(t, z))$ and $\hat{f}_2(t, z, \hat{u}, \hat{\theta}_i) = f_2(t, x(t, z), \hat{u}, \hat{\theta}_i)$ ($i = 1, 2$), then (2.9) becomes

$$
\begin{align*}
\frac{\partial \bar{w}}{\partial t} - d_2 \tau \xi (t) \frac{\partial^2 \bar{w}}{\partial z^2} - \eta(t, z) \frac{\partial \bar{w}}{\partial z} + k \bar{w} = d_2 (1 - \tau) \left( J \left( \frac{\beta(t) - g(t)}{2} \right) \frac{1}{J} \int_0^1 (\hat{\theta}_2(t, s) - \hat{\theta}_1(t, s)) ds \right) \\
+ \hat{f}_2(t, z, \hat{u}, \hat{\theta}_2) - \hat{f}_2(t, z, \hat{u}, \hat{\theta}_1) + k(\hat{\theta}_2 - \hat{\theta}_1), \quad (t, z) \in D_{T_0},
\end{align*}
$$

(2.11)

Applying the $L^p$ theory to (2.11), for any $p > 1$,

$$
\left\| \bar{w} \right\|_{W^1_p(D_{T_0})} \leq C_1 \left\| \frac{\beta(t) - g(t)}{2} \frac{1}{J} \int_0^1 (\hat{\theta}_2(t, s) - \hat{\theta}_1(t, s)) ds \right\|_{L^p(D_{T_0})} + \left\| \hat{f}_2(t, z, \hat{u}, \hat{\theta}_2) - \hat{f}_2(t, z, \hat{u}, \hat{\theta}_1) \right\|_{L^p(D_{T_0})} + \left\| \hat{\theta}_2 - \hat{\theta}_1 \right\|_{L^p(D_{T_0})}
$$

where we have used the estimates

$$
\left\| \frac{\beta(t) - g(t)}{2} \frac{1}{J} \int_0^1 (\hat{\theta}_2(t, s) - \hat{\theta}_1(t, s)) ds \right\|_{L^p(D_{T_0})} \leq \left\| \hat{\theta}_2 - \hat{\theta}_1 \right\|_{C(T_{T_0})}
$$

and

$$
\left\| \hat{f}_2(t, z, \hat{u}, \hat{\theta}_2) - \hat{f}_2(t, z, \hat{u}, \hat{\theta}_1) \right\|_{L^p(D_{T_0})} = \left\| f_2(t, x(t, z), \hat{u}, \hat{\theta}_2) - f_2(t, x(t, z), \hat{u}, \hat{\theta}_1) \right\|_{L^p(D_{T_0})} \leq \left\| \hat{\theta}_2 - \hat{\theta}_1 \right\|_{C(T_{T_0})} ||f||_{C^0(R^2)} (2T_0)^{\frac{p}{2}}.
$$

By the Sobolev imbedding theorem, we have

$$
\left\| \bar{w} \right\|_{C(T_{T_0})} \leq \left\| \bar{w} \right\|_{C^q_p(\Omega_{T_0}^g)} \leq C_4 \left\| \bar{w} \right\|_{W^1_p(D_{T_0})} \leq C_4 \left\| \hat{\theta}_2 - \hat{\theta}_1 \right\|_{C(T_{T_0})},
$$

which is equivalent to

$$
\left\| w \right\|_{C(\Omega_{T_0}^{\beta, g})} \leq \left\| w \right\|_{C^q_p(\Omega_{T_0}^{\beta, g})} \leq C_5 \left\| w \right\|_{W^1_p(\Omega_{T_0}^{g})} \leq C_6 \left\| \hat{\theta}_2 - \hat{\theta}_1 \right\|_{C(\Omega_{T_0}^{\beta, g})}. \tag{2.10}
$$

Thus, $\Phi : D \rightarrow C(\Omega_{T_0}^{\beta, g})$ is continuous.

Similar as above, we can show that, for any given constant $M_1 > 0$, there exists a constant $M_2 > 0$ independent of $\theta$ such that $\left\| \Phi \theta \right\|_{C^q_p(\Omega_{T_0}^{\beta, g})} \leq M_2$ for any $\theta$ satisfying $\left\| \theta \right\|_{C(\Omega_{T_0}^{\beta, g})} \leq M_1$, which implies that $\Phi : D \rightarrow C(\Omega_{T_0}^{\beta, g})$ is a compact operator. Thus, from the fact $\left\| v^{(n)} \right\|_{C(\Omega_{T_0}^{\beta, g})} \leq M$ we know $\{v^{(n)}\} = \{\Phi v^{(n-1)}\}$ has a convergent subsequence in $C(\Omega_{T_0}^{\beta, g})$. By the monotonicity of $v^{(n)}$ in $n$, we have $v^{(n)} \rightarrow v^*$ in $C(\Omega_{T_0}^{\beta, g})$. Therefore, $v^* = \Phi v^*$, which means $v^* \in C^{1+\frac{\beta}{2}, 1+\alpha}(\Omega_{T_0}^{g})$ is a solution of (2.5), and then $\tilde{v}^* \in C^{1+\frac{\beta}{2}, 1+\alpha}(\Omega_{T_0}^g)$ is a solution of (2.8) with $\bar{v}, \tilde{\theta}$ replaced by $\tilde{v}^*$. Since $h, g \in C^{1+\frac{\beta}{2}, 1+\alpha}(\Omega_{T_0}^g)$, we have $\xi \in C^{\frac{\beta}{2}}([0, T_0])$ and $\eta \in C^{\frac{\beta}{2}-\alpha}([0, T_0])$. Moreover, by the
assumption of $f_2$, we know $f_2 \in C^{\hat{\alpha},\alpha}(\overline{D_{T_0}})$. By the Lipschitz continuity of $J$, we deduce

\[
\begin{align*}
&\left|\frac{h(t_1) - g(t_1)}{2} \int_{t_1}^{t_2} J \left(\frac{h(t_1) - g(t_2)}{2}(z_1 - s)\right) \tilde{v}^*(t_1, s) ds - \frac{h(t_2) - g(t_2)}{2} \int_{t_1}^{t_2} J \left(\frac{h(t_2) - g(t_2)}{2}(z_2 - s)\right) \tilde{v}^*(t_2, s) ds\right| \\
\leq& \left|\frac{h(t_1) - g(t_1)}{2} \int_{t_1}^{t_2} J \left(\frac{h(t_1) - g(t_1)}{2}(z_1 - s)\right) \tilde{v}^*(t_1, s) ds\right| \\
+& \left|\int_{t_1}^{t_2} J \left(\frac{h(t_2) - g(t_1)}{2}(z_1 - s)\right) \tilde{v}^*(t_1, s) ds\right| \\
<& \int_{t_1}^{t_2} J \left(\frac{h(t_2) - g(t_1)}{2}(z_1 - s)\right) \tilde{v}^*(t_1, s) ds \\
+& \left|\int_{t_1}^{t_2} J \left(\frac{h(t_2) - g(t_2)}{2}(z_2 - s)\right) \tilde{v}^*(t_2, s) ds\right| \\
\leq& C \int_{t_1}^{t_2} \left|\frac{h(t_1) - g(t_1)}{2}(z_1 - s) - \frac{h(t_2) - g(t_2)}{2}(z_2 - s)\right| ds \\
+& C \int_{t_1}^{t_2} \left|\frac{h(t_2) - g(t_1)}{2}(z_1 - s)\right| ds + C \int_{t_1}^{t_2} \left|\frac{h(t_2) - g(t_2)}{2}(z_2 - s)\right| ds \\
\leq& C(|z_1 - z_2| + |t_1 - t_2| + \int_{t_1}^{t_2} |\tilde{v}^*(t_1, s) - \tilde{v}^*(t_2, s)| ds) \\
\leq& C(|z_1 - z_2|^{1/2} + |t_1 - t_2|^{1/2}),
\end{align*}
\]

which means that \(\frac{h(t) - g(t)}{2} \int_{t_1}^{t_2} J \left(\frac{h(t) - g(t)}{2}(z - s)\right) \tilde{v}^* ds \in C^{\hat{\alpha},\alpha}(\overline{D_{T_0}})\). Applying the Schauder regularity theory to (2.8) with \(\tilde{v}, \tilde{\varphi}\) replaced by \(\tilde{v}^*, \varphi^*\), we can deduce that \(\tilde{v}^* \in C^{1+\hat{\alpha},2+\alpha}(D_{T_0})\), and then \(v^* \in C^{1+\hat{\alpha},2+\alpha}(\Omega_{T_0}^h)\) is a classical solution to (2.5). Similarly, we can prove \(w^*\) is also a classical solution of (2.5).

Now we prove the uniqueness of solution in \([\tilde{v}, \tilde{v}^*]\). In (2.10), we have obtained \(w^* \leq v^*\). By Lemma 2.2, we also get \(w^* \geq v^*\). Thus, \(w^* = v^*\). If \(v(t, x)\) is a solution of (2.5) and satisfies \(v \leq v \leq \tilde{v}\), then \(v = \Phi v\). From Step 1, we know

\[
w_n = \Phi^n v_n \leq \Phi^n v = v \leq \Phi^n \tilde{v} = v_n.
\]

Since \(\lim_{n \to \infty} w_n = w^* = v^* = \lim_{n \to \infty} v_n\), we have

\[
w^*(t, x) = v(t, x) = v^*(t, x).
\]

**Step 2.** It is easy to check that \(\underline{v} = 0\) and \(\bar{v} = K_2\) are lower and upper solutions of (2.5), respectively. Then there exists a unique solution \(v\) satisfying \(0 < v \leq K_2\). Note that \(f_2(t, x, u, v)\) satisfies the assumption (f4). Lemma 2.2 implies that \(v\) is unique solution of (2.5).

We define

\[
\Omega := \{ (t, x) : 0 < t \leq T_0, \ h(t) - M^{-1} < x < h(t) \}
\]

and construct an auxiliary function

\[
\psi(t, x) = K_2 [2M(h(t) - x) - M^2(h(t) - x)^2].
\]

We will choose \(M\) such that \(\psi(t, x) \geq v(t, x)\) holds over \(\Omega\).

Direct calculations show that, for \((t, x) \in \Omega\),

\[
\partial_t \psi = 2K_2 Mh'(t)(1 - M(h(t) - x)) \geq 0,
\]

\[
- \partial_{xx} \psi = 2K_2 M^2, \ f_2(t, x, u, v) \leq \dot{L} u.
\]
It follows that
\[
\partial_t \psi - d_2 \left[ \tau \partial_{xx} \psi + (1 - \tau) \left( \int_{\gamma(t)}^{h(t)} f(x-y, \psi(t,y))dy - \psi \right) \right] \\
\geq 2d_2 \tau K_2 M^2 - d_2(1 - \tau)K_2 \int_{\gamma(t)}^{h(t)} f(x-y)dy \\
\geq 2d_2 \tau K_2 M^2 - d_2(1 - \tau)K_2 \geq LK_2 \\
\geq \hat{L}v \geq \partial_t v - d_2 \left[ \tau \partial_{xx} v + (1 - \tau) \left( \int_{\gamma(t)}^{h(t)} f(x-y,v(t,y))dy - v \right) \right] \quad \text{in } \Omega,
\]
if \( M^2 \geq \frac{L+d_2(1-\tau)}{2d_2 \tau} \). On the other hand,
\[
\psi(t, h(t) - M^{-1}) = K_2 \geq v(t, h(t) - M^{-1}), \quad \psi(t, h(t)) = 0 = v(t, h(t)).
\]
Choosing
\[
M := \max \left\{ \sqrt{\frac{L+d_2(1-\tau)}{2d_2 \tau}}, \frac{4\|u_0\|_{C^{1,1}([-h_0, h_0])}}{3K_2} \right\},
\]
we can prove that \( v_0(x) \leq \psi(0, x) \) for \( x \in [h_0 - M^{-1}, h_0] \). Then we can apply Lemma 2.1 to \( \psi - v \) over \( \Omega \) to deduce that
\[
v(t,x) \leq \psi(t,x) \quad \text{for } (t,x) \in \Omega.
\]
It then follows that
\[
v_x(t,h(t)) \geq -2K_2 M.
\]
Moreover, since \( v(t, h(t)) = 0 \) and \( v > 0 \) in \( \Omega^g_{T_0}, \) we have \( v_x(t, h(t)) < 0 \). The estimates for \( v_x(t, g(t)) \) can be similarly obtained.

Now, by approximation method we get the unique strong solution of (2.5) provided that \( g'(t), h'(t) \) and \( u(t, x) \) are only continuous functions, which plays an important role in the proof of Lemma 2.5 later.

**Lemma 2.4.** Suppose that \( (J) \) holds, \( (g,h) \in G_{U_0}^{\Phi_0} \times H_{T_0}^{h_0}, u \in C(\Omega_{T_0}^g), f_2 \) satisfies \( (f1)-(f4) \) and \( v_0 \) satisfies (1.2). Then the problem (2.5) admits a unique solution \( v \in W^{1,2}_p(\Omega_{T_0}^g) \cap C^{\frac{1+\alpha}{p}}(\Omega_{T_0}^g) \) with any \( p > 3 \). Moreover, \( v \) satisfies (2.6).

**Proof.** Step 1. (Uniqueness) Let \( \tilde{v}(t, z) = v(t, x(t,z)), \tilde{f}(t, z, \tilde{u}, \tilde{v}) = f(t, x(t,z), \tilde{u}(t, x(t,z)), v(t, x(t,z))), \) then the problem becomes

\[
\begin{aligned}
\partial_t \tilde{v} &= d_2 \tau \xi(t) \partial_z^2 \tilde{v} + \eta(t,z) \partial_z \tilde{v} \\
&+ d_2 (1 - \tau) \left( \frac{h(t)}{2} \int_1^{h(t)} J(s) ds \right) \tilde{v}(t,s) ds - \tilde{v} + \tilde{f}_2(t, z, \tilde{u}, \tilde{v}), \quad (t,z) \in D_{T_0}, \\
\tilde{v}(t,-1) &= \tilde{v}(t,1) = 0, \quad t \in (0, T_0], \\
\tilde{v}(0, z) &= v_0(h_0 z), \quad z \in [-1, 1].
\end{aligned}
\]

(2.13)

Assume that \( v_i(t, x) \in W^{1,2}_p(\Omega_{T_0}^g) \cap C^{\frac{1+\alpha}{p}}(\Omega_{T_0}^g), i = 1, 2, \) are two solutions of (2.5), then \( \tilde{v}_i(t, z) = v_i(t, x(t,z)) \in W^{1,2}_p(D_{T_0}) \cap C^{\frac{1+\alpha}{p}}(D_{T_0}) \) are two solutions of (2.13). Let \( \tilde{w} = \tilde{v}_1 - \tilde{v}_2, \)
then $\tilde{w}$ satisfies
\[
\begin{cases}
\partial_{t}\tilde{w} = d_{2}\tau\xi(t)\partial_{x}\tilde{w} + \eta(t,z)\partial_{z}\tilde{w} + d_{2}(1-\tau)\left(\frac{h(t)-g(t)}{2}\int_{-1}^{1}J\left(\frac{h(t)-g(t)}{2}(z-s)\right)\tilde{w}(t,s)ds - \tilde{w}\right) \\
+ \hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{1}) - \hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{2}), \quad (t,z) \in D_{T_{0}},
\end{cases}
\]
\[
\tilde{w}(t,-1) = \tilde{w}(t,1) = 0, \quad t \in (0,T_{0}],
\]
\[
\tilde{w}(0,z) = 0, \quad z \in [-1,1].
\]

(2.14)

Multiplying the equation in (2.14) by $\tilde{w}\chi_{[0,t]}$, where $\chi_{[0,t]}$ is the characteristic function in $[0,t]$ with any $0 < t \leq T_{0}$, and then integrating over $(0,T_{0}] \times [-1,1]$ gives
\[
\frac{1}{2}\int_{-1}^{1}\tilde{w}^{2}(t,z)dz \leq -d_{2}\tau\int_{0}^{T_{0}}\int_{-1}^{1}\xi(t)(\partial_{x}\tilde{w})^{2}dzdt + f_{2}(t)\int_{0}^{T_{0}}\int_{-1}^{1}\eta(t,z)\tilde{w}\partial_{x}\tilde{w}dzdt
\]
\[
+ d_{2}(1-\tau)\int_{0}^{T_{0}}\int_{-1}^{1}\left(\frac{h(t)-g(t)}{2}\int_{-1}^{1}J\left(\frac{h(t)-g(t)}{2}(z-s)\right)\tilde{w}(t,s)ds - \tilde{w}(t,z)\right)\tilde{w}(t,z)dzdt
\]
\[
+ \int_{0}^{T_{0}}\int_{-1}^{1}[\hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{1}) - \hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{2})]\tilde{w}(t,z)dzdt.
\]

By the Young’s inequality with $0 < \varepsilon < \frac{4d_{2}\tau}{(\chi_{[0,t]}-g(t))^{2}}$,
\[
\int_{0}^{T_{0}}\int_{-1}^{1}\eta(t,z)\tilde{w}\partial_{x}\tilde{w}dzdt \leq \varepsilon \int_{0}^{T_{0}}\int_{-1}^{1}(\partial_{x}\tilde{w})^{2}dzdt + C(\varepsilon)\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt.
\]

By the continuity of $J$ and Hölder inequality,
\[
d_{2}(1-\tau)\int_{0}^{T_{0}}\int_{-1}^{1}\left(\frac{h(t)-g(t)}{2}\int_{-1}^{1}J\left(\frac{h(t)-g(t)}{2}(z-s)\right)\tilde{w}(t,s)ds - \tilde{w}(t,z)\right)\tilde{w}(t,z)dzdt
\]
\[
\leq d_{2}(1-\tau)C_{1}\int_{0}^{T_{0}}\int_{-1}^{1}|\tilde{w}(t,z)|^{2}dzdt - d_{2}(1-\tau)\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt
\]
\[
\leq d_{2}(1-\tau)C_{1}\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt.
\]

By the Lipschitz continuity of $f_{2}$ with respect to $\tilde{v}$,
\[
\int_{0}^{T_{0}}\int_{-1}^{1}[\hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{1}) - \hat{f}_{2}(t,z,\tilde{u},\tilde{v}_{2})]\tilde{w}(t,z)dzdt \leq L\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt.
\]

Combining the above estimates, we have
\[
\int_{-1}^{1}\tilde{w}^{2}(t,z)dz \leq C\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt.
\]

By the Gronwall’s inequality, we know $\int_{0}^{T_{0}}\int_{-1}^{1}\tilde{w}^{2}dzdt = 0$, which implies that $\tilde{w} = 0$, a.e. in $(0,t] \times [-1,1]$. Since $t \in (0,T_{0}]$ is arbitrary and $\tilde{w} \in C(\overline{D}_{T_{0}})$, we can obtain $\tilde{w} = 0$ for all $(t,z)$ in $[0,T_{0}] \times [-1,1]$, which implies the uniqueness of solution.

Step 2. (Existence) For any $(g,h) \in G^{h_{0},h_{n}}_{T_{0}} \times \mathbb{H}^{h_{0},h_{n}}_{T_{0}}$, we can find some sequences $(g_{n},h_{n}) \in G^{h_{0},h_{n}}_{T_{0}} \times \mathbb{H}^{h_{0},h_{n}}_{T_{0}}$ such that $g_{n} \to g$ and $h_{n} \to h$ in $C^{1}([0,T_{0}])$. Moreover, for every $u(t,x) \in C^{1}(\overline{D}_{T_{0}})$, we can obtain $\tilde{u}(t,z) = u(t,z(t,z)) \in C(\overline{D}_{T_{0}})$ and find some sequence $\tilde{u}_{n} \in C^{1}(\overline{D}_{T_{0}})$ such that $\tilde{u}_{n} \to \tilde{u}$ in $C(\overline{D}_{T_{0}})$. Taking $u_{n}(t,x) = \tilde{u}_{n}(t,\frac{2x-g_{n}(t)}{h_{n}(t)-g_{n}(t)})$, we know $u_{n} \in C^{1}(\overline{D}_{T_{0}})$.

Consider the approximate problem
\[
\begin{cases}
\partial_{t}v = d_{2}\left[\tau\partial_{x}^{2}v + (1-\tau)\left(\frac{h_{n}(t)}{g_{n}(t)}J(x-y)v(t,y)dy - v\right)\right] + f_{2}(t,x,u_{n},v), \quad (t,x) \in \Omega^{g_{n},h_{n}}_{T_{0}},
\end{cases}
\]
\[
v(t,g_{n}(t)) = v(t,h_{n}(t)) = 0, \quad t \in (0,T_{0}],
\]
\[
v(0,x) = v_{0}(x), \quad x \in [-h_{0},h_{0}].
\]

(2.15)
By Lemma 2.3, we know (2.15) has a unique classical solution $v_n \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_{T_0}^{g_n, h_n})$, and satisfies

$$0 < v_n \leq K_2 \quad \text{for } (t, x) \in \Omega_{T_0}^{g_n, h_n},$$

$$0 < -\partial_x v_n(t, h_n(t)), \partial_x v_n(t, g_n(t)) \leq K_3 \quad \text{for } t \in (0, T_0].$$

Let $\tilde{v}_n(t, z) = v_n(t, x_n(t, z))$ and $\tilde{f}_2(t, z, \tilde{u}_n, \tilde{v}_n) = f_2(t, x_n(t, z), u_n(t, x_n(t, z)), v_n(t, x_n(t, z)))$ with

$$x_n(t, z) = \frac{(h_n(t) - g_n(t))z + h_n(t) + g_n(t)}{2},$$

then $\tilde{v}_n(t, z) \in C^{1+\frac{\alpha}{2}, 2+\alpha}(D_{T_0})$ is the unique solution of

$$\begin{cases}
\partial_t \tilde{v}_n = d_2 \xi(t) \partial_z^2 \tilde{v}_n + \eta_n(t, z) \partial_z \tilde{v}_n + d_2 (1 - \tau) \left(\frac{h_n(t) - g_n(t)}{2} \int_{-1}^1 J(\frac{h_n(t) - g_n(t)}{2}(z - s))\tilde{v}_n(t, s)ds - \tilde{v}_n\right) \\
+ \tilde{f}_2(t, z, \tilde{u}_n, \tilde{v}_n), \quad (t, z) \in D_{T_0},
\end{cases}
$$

$$\tilde{v}_n(t, -1) = \tilde{v}_n(t, 1) = 0, \quad t \in (0, T_0],$$

$$\tilde{v}_n(0, z) = v_0(h_0 z), \quad z \in [-1, 1],$$

(2.16)

and satisfies

$$0 < \tilde{v}_n \leq K_2 \quad \text{in } D_{T_0},$$

$$0 < -\frac{2}{h_n(t) - g_n(t)} \partial_z \tilde{v}_n(t, 1), \frac{2}{h_n(t) - g_n(t)} \partial_z \tilde{v}_n(t, -1) \leq K_3 \quad \text{for } t \in (0, T_0].$$

(2.17)

Let

$$g(t, z) := d_2 (1 - \tau) \left(\frac{h_n(t) - g_n(t)}{2} \int_{-1}^1 J(\frac{h_n(t) - g_n(t)}{2}(z - s))\tilde{v}_n(t, s)ds\right) + \tilde{f}_2(t, z, \tilde{u}_n, \tilde{v}_n),$$

we know $g \in L^\infty(D_{T_0})$. Applying the $L^p$ theory for linear parabolic equations to (2.16), we have the solution $\tilde{v}_n$ satisfies

$$\|\tilde{v}_n\|_{W^{1, 2}_p(D_{T_0})} \leq C,$$

where $C$ is independent of $n$. By the weak compactness of the bounded set in $W^{1, 2}_p(D_{T_0})$ and $W^{1, 1}_p(D_{T_0})$ and the compactly imbedding theorem ($W^{1, 1}_p(D_{T_0}) \hookrightarrow L^p(D_{T_0})$), there exists a subsequence, still denoted by $\{\tilde{v}_n\}$, such that $\tilde{v}_n \rightarrow \tilde{v}$ in $W^{1, 2}_p(D_{T_0}) \cap W^{1, 1}_p(D_{T_0})$, $\partial_x \tilde{v}_n \rightarrow \partial_x \tilde{v}$ in $L^p(D_{T_0})$ and $\tilde{v}_n \rightarrow \tilde{v}$ in $L^p(D_{T_0})$, which implies that $\tilde{v} \in W^{1, 2}_p(D_{T_0}) \cap W^{1, 1}_p(D_{T_0})$ is the strong solution of (2.13). By the Sobolev imbedding theorem, $\tilde{v} \in C^{1+\frac{\alpha}{2}, 1+\alpha}(\overline{D_{T_0}})$.

Note that $\tilde{v}_n$ satisfies (2.17). From the fact $\partial_x \tilde{v}_n \rightarrow \partial_x \tilde{v}$, $\tilde{v}_n \rightarrow \tilde{v}$ in $L^p(D_{T_0})$ (then a.e. in $D_{T_0}$) and $\tilde{v} \in C^{1+\frac{\alpha}{2}, 1+\alpha}(\overline{D_{T_0}})$, we have $0 < \tilde{v} \leq K_2$ in $D_{T_0}$ and $0 < -\frac{2}{h_n(t) - g_n(t)} \partial_z \tilde{v}(t, 1), \frac{2}{h_n(t) - g_n(t)} \partial_z \tilde{v}(t, -1) \leq K_3$ for $t \in (0, T_0]$. Then $v(t, x) = \tilde{v}(t, z(t, x))$ satisfies (2.6), which completes the proof. \hfill \Box

In the following lemma, we prove the well-posedness for (2.1) with any fixed $(g, h) \in \mathcal{G}_{T_0}^{h_0} \times \mathbb{R}_{T_0}^{h_0}$ by the fixed point theorem. Denote

$$X_{T_0}^1 := \left\{ u \in C(\overline{D_{T_0}^{g, h}}) : 0 \leq u \leq K_1, u(0, x) = u_0(x), u(t, g(t)) = u(t, h(t)) = 0 \right\},$$

$$X_{T_0}^2 := \left\{ v \in C(\overline{D_{T_0}^{g, h}}) : 0 \leq v \leq K_2, v(0, x) = v_0(x), v(t, g(t)) = v(t, h(t)) = 0 \right\},$$

$$X_{T_0}^{1, 2} := X_{T_0}^1 \times X_{T_0}^2.$$
Lemma 2.5. For any $T_0 > 0$ and $(g, h) \in \mathbb{C}^{h_0}_{T_0} \times \mathbb{H}^{h_0}_{T_0}$, the problem

\[
\begin{aligned}
\partial_t u &= d_1 \left( \int_{g(t)}^{h(t)} J(x-y) u(t,y) dy - u \right) + f_1(t, x, u, v), \quad (t, x) \in \Omega^{g,h}_{T_0}, \\
\partial_t v &= d_2 \left[ \tau \partial_z^2 v + (1 - \tau) \left( \int_{g(t)}^{h(t)} J(x-y) v(t,y) dy - v \right) \right] + f_2(t, x, u, v), \quad (t, x) \in \Omega^{g,h}_{T_0}, \\
u(t, g(t)) &= u(t, h(t)) = v(t, g(t)) = v(t, h(t)) = 0, \quad t \in [0, T_0], \\
u(0, x) &= u_0(x), v(0, x) = v_0(x), \quad x \in [-h_0, h_0]
\end{aligned}
\]

(2.18)

admits a unique solution $(u, v) \in \mathcal{X}^{g,h}_{T_0}$, and $(u, v)$ satisfy

\[
0 < u \leq K_1, 0 < v \leq K_2 \quad \text{in} \quad \Omega^{g,h}_{T_0},
\]

\[
0 < -v_x(t, h(t)), v_x(t, g(t)) \leq K_3 \quad \text{in} \quad (0, T_0].
\]

Moreover, $v \in W^{1,2}_p(\Omega^{g,h}_{T_0}) \cap \mathcal{C}^{2,1+\alpha}(\overline{\Omega}^{g,h}_{T_0})$ with any $p > 3$.

**Proof.** For $u^* \in \mathcal{X}^{s}_s$ with $0 < s \leq T_0$, from Lemma 2.4 we know that the initial-boundary value problem (2.5) with $(u, T_0)$ replaced by $(u^*, s)$ admits a unique solution $v \in \mathcal{X}^{s}_s$. For such $v \in \mathcal{X}^{s}_s$, we consider

\[
\begin{aligned}
\partial_t u &= d_1 \left( \int_{g(t)}^{h(t)} J(x-y) u(t,y) dy - u \right) + f_1(t, x, u, v), \quad (t, x) \in \Omega^{g,h}_{T_0}, \\
u(t, g(t)) &= u(t, h(t)) = 0, \quad t \in [0, T_0], \\
u(0, x) &= u_0(x), \quad x \in [-h_0, h_0].
\end{aligned}
\]

By Lemma 2.3 in [3], it admits a unique solution $u \in \mathcal{X}^{1}_s$. We define a mapping $\mathcal{F}_s : \mathcal{X}^{s}_s \rightarrow \mathcal{X}^{1}_s$ by $\mathcal{F}_s u^* = u$. If $\mathcal{F}_s u^* = u^*$, then $(u^*, v)$ solves (2.18) with $T_0$ replaced by $s$.

Next, we shall prove that $\mathcal{F}_s$ has a fixed point in $\mathcal{X}^{1}_s$ provided that $s$ is small enough. For $i = 1, 2$, we assume $u^*_i \in \mathcal{X}^{1}_s$, $u_i = \mathcal{F}_s u^*_i$, and $v_i$ be the unique solution of (2.5) with $(u, T_0)$ replaced by $(u^*_i, s)$. Denote $\theta^* = u^*_1 - u^*_2$, $\theta = u_1 - u_2$ and $w = v_1 - v_2$. Note that $w$ satisfies

\[
\begin{aligned}
\partial_t w &= d_2 \left[ \tau \partial_z^2 w + (1 - \tau) \left( \int_{g(t)}^{h(t)} J(x-y) w(t,y) dy - w \right) \right] + a_0(t, x) w + b_0(t, x) \theta^*, \quad (t, x) \in \Omega^{g,h}_{s}, \\
w(t, g(t)) &= w(t, h(t)) = 0, \quad t \in [0, s], \\
w(0, x) &= 0, \quad x \in [-h_0, h_0],
\end{aligned}
\]

where

\[
\begin{aligned}
a_0(t, x) &= \int_0^1 f_{2,v}(t, x, u_1^*, v_2 + (v_1 - v_2) \tau) d\tau, \\
b_0(t, x) &= \int_0^1 f_{2,u}(t, x, u_2^* + (u_1^* - u_2^*) \tau, v_2) d\tau.
\end{aligned}
\]

Let $\tilde{\theta}(t, z) = \theta^*(t, x(t, z))$, $\tilde{\theta}(t, z) = w(t, x(t, z))$, $\tilde{a}_0(t, z) = a_0(t, x(t, z))$, $\tilde{b}_0(t, z) = b_0(t, x(t, z))$. It is easy to see that $\tilde{w}$ satisfies

\[
\begin{aligned}
\partial_t \tilde{w} &= d_2 \tau(t) \partial_z^2 \tilde{w} + \eta(t, z) \partial_z \tilde{w} + [\tilde{a}_0(t, z) - d_2(1 - \tau)] \tilde{w} \\
&+ d_2(1 - \tau) \left( h(t) - g(t) \right) \int_0^s J(\frac{h(t) - g(t)}{z - s})(z - s) \tilde{w}(t, z) ds + \tilde{b}_0(t, z) \tilde{\theta}^*, \quad (t, z) \in D_s, \\
\tilde{w}(t, -1) &= \tilde{w}(t, 1) = 0, \quad t \in [0, s], \\
\tilde{w}(0, z) &= 0, \quad z \in [-1, 1].
\end{aligned}
\]
By the $L^p$ theory for linear parabolic equation, we have
\[
\|\tilde{w}\|_{W^{1,2}_p(D_s)} \leq C \left( \left\| \frac{h(t)-g(t)}{2} \int_0^1 J \left( \frac{h(t)-g(t)}{2} (z-s) \right) \tilde{w}(t,s) ds \right\|_{L^p(D_s)} + \| \tilde{\theta} \|_{L^p(D_s)} \right)
\]
\[
\leq C(\|\tilde{w}\|_{C(\overline{D_s})} \left\| \int_0^{(z+1)/x_0} J(y) dy \right\|_{L^p(D_s)} + \| \tilde{\theta} \|_{L^p(D_s)}).
\]
From the proof of Theorem 1.1 in [34], we know the Hölder semi-norm $[\tilde{w}]_{C^{\frac{1}{s}}(\overline{D_s})} \leq C'\|\tilde{w}\|_{W^{1,2}_p(D_s)}$, where $C'$ is independent of $\frac{1}{s}$. Thus,
\[
|\tilde{w}(t,z)| = |\tilde{w}(t,z) - \tilde{w}(0,z)| \leq [\tilde{w}]_{C^{\frac{1}{s}}(\overline{D_s})} \left\| \tilde{w} \right\|_{W^{1,2}_p(D_s)} \frac{\tilde{\theta}}{\tilde{\theta}},
\]
which implies that
\[
\|\tilde{w}\|_{C(\overline{D_s})} \leq C'\|\tilde{w}\|_{W^{1,2}_p(D_s)} \frac{\tilde{\theta}}{\tilde{\theta}}, \tag{2.20}
\]
Choosing $s$ small such that $CC'(2s)^{\frac{1}{2}} \frac{\tilde{\theta}}{\tilde{\theta}} < \frac{1}{2}$, we have
\[
\|\tilde{w}\|_{W^{1,2}_p(D_s)} \leq 2C\|\tilde{\theta}\|_{L^p(D_s)} \leq 2C(2s)^{\frac{1}{2}} \|\tilde{\theta}\|_{C(\overline{D_s})} = 2C(2s)^{\frac{1}{2}} \|\theta\|_{C(\overline{D_s})}.
\]
Similar to the proof of Lemma 2.3 (Step 3) in [31], we can choose $s$ small enough such that
\[
\|\theta\|_{C(\overline{D_s})} \leq \frac{1}{2} \|\theta\|_{C(\overline{D_s})}.
\]
By the contraction mapping theorem, we know that $F_s$ has a unique fixed point $u \in X_1^I$.

**Proof of Theorem 2.1.** By Lemma 2.5, for any $T_0 > 0$ and $(g, h) \in G_{T_0}^{h_0} \times H_{T_0}^{h_0}$, we can find a unique $(u, v) \in X_{T_0}^{g,h}$ that solves (2.18), and (2.19) holds. For $0 < t \leq T_0$, define the mapping $G(g, h) = (\tilde{g}, \tilde{h})$ by

\[
\tilde{h}(t) = h_0 - \mu \int_0^t v_x(\tau, h(\tau)) d\tau + \rho_1 \int_0^t \int_{h_0(\tau)}^{\infty} J(x-y)u(\tau, x) dy dx d\tau + \rho_2 \int_0^t \int_{h_0(\tau)}^{\infty} J(x-y)v(\tau, x) dy dx d\tau,
\]
\[
\tilde{g}(t) = -h_0 - \mu \int_0^t v_x(\tau, g(\tau)) d\tau - \rho_1 \int_0^t \int_{g(\tau)}^{\infty} J(x-y)u(\tau, x) dy dx d\tau - \rho_2 \int_0^t \int_{g(\tau)}^{\infty} J(x-y)v(\tau, x) dy dx d\tau.
\]
To prove this theorem, we will show that if $T_0$ is sufficiently small, then $G$ maps a suitable closed subset $\Sigma_{T_0}$ of $G_{T_0}^{h_0} \times H_{T_0}^{h_0}$ into itself and is a contraction mapping.

**Step 1.** There exists a closed subset $\Sigma_\tau \subset G_{T_0}^{h_0} \times H_{T_0}^{h_0}$ such that $G(\Sigma_\tau) \subset \Sigma_\tau$.

Let $(g, h) \in G_{T_0}^{h_0} \times H_{T_0}^{h_0}$. The definitions of $\tilde{h}(t)$ and $\tilde{g}(t)$ indicate that they belong to $C^1([0, T_0])$ and for $0 < t \leq T_0$,

\[
\tilde{h}'(t) = -\mu v_x(t, h(t)) + \rho_1 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t, x) dy dx + \rho_2 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)v(t, x) dy dx,
\]
\[
\tilde{g}'(t) = -\mu v_x(t, g(t)) - \rho_1 \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)u(t, x) dy dx - \rho_2 \int_{g(t)}^{h(t)} \int_{-\infty}^{g(t)} J(x-y)v(t, x) dy dx.
\]
It follows that
\[
[\hat{h}(t) - \tilde{g}(t)]' = -\mu[v_x(t, h(t)) - v_x(t, g(t))] + \rho_1 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y) u(t, x) dy dx + \rho_2 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y) v(t, x) dy dx.
\]
(2.21)

Note that from (J) we know there exist constants $\tilde{\epsilon} \in (0, \frac{\rho_1}{8\mu K^3})$ and $\eta_0 > 0$ such that $J(x - y) > \eta_0$ if $|x - y| \leq \tilde{\epsilon}$. Take
\[
0 < \varepsilon_0 < \min \left\{ \epsilon, \frac{8\mu K^3}{\rho_1 K_1 + \rho_2 K_2} \right\}, \quad M_1 = 2h_0 + \frac{\varepsilon_0}{4}, \quad 0 < T_1 \leq \frac{\varepsilon_0}{4[2\mu K^3 + (\rho_1 K_1 + \rho_2 K_2)M_1]}
\]
such that $h(T_1) - g(T_1) \leq M_1$. Estimating the right hand of (2.21), we have
\[
[\hat{h}(t) - \tilde{g}(t)]' \leq 2\mu K^3 + \rho_1 K_1 [h(T_1) - g(T_1)] + \rho_2 K_2 [h(T_1) - g(T_1)] \leq 2\mu K^3 + (\rho_1 K_1 + \rho_2 K_2)M_1.
\]
This implies
\[
\hat{h}(t) - \tilde{g}(t) \leq 2h_0 + t[2\mu K^3 + (\rho_1 K_1 + \rho_2 K_2)M_1] \leq M_1, \quad t \in [0, T_1].
\]
Similarly, we can show that
\[
\hat{h}(t), \quad -\tilde{g}(t) \leq \mu K^3 + (\rho_1 K_1 + \rho_2 K_2)M_1 =: \tilde{R} \leq R(t), \quad t \in [0, T_1].
\]
It is easy to check that
\[
h(t) \in [h_0, h_0 + \varepsilon_0^2], \quad g(t) \in [-h_0 - \varepsilon_0^2, -h_0], \quad t \in [0, T_1].
\]
Similar to (2.15) in [31], we can prove that, for $t \in (0, T_1]$,
\[
\hat{h}(t) \geq \rho_1 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y) u(t, x) dy dx \geq \frac{1}{4} \varepsilon_0 \eta_0 \rho_1 \int_{h_0}^{h_0 + \frac{\varepsilon_0}{4}} J(x - y) u_0(x) dx =: \rho_1 c_0
\]
and
\[
\tilde{g}(t) \leq -\rho_1 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y) u(t, x) dy dx \leq -\frac{1}{4} \varepsilon_0 \eta_0 \rho_1 \int_{-h_0}^{-h_0 + \frac{\varepsilon_0}{4}} J(0) u_0(x) dx =: -\rho_1 \tilde{c}_0.
\]
We now define, for $\tau \in (0, T_1]$,
\[
\Sigma_\tau = \{(g, h) \in G_{g_0}^{h_0} \times H_{x_0}^{h_0} : \rho_1 c_0 \leq \hat{h}(t) \leq \tilde{g}(t) \leq \rho_1 \tilde{c}_0, h(\tau) - g(\tau) \leq M_1\}.
\]
Our analysis above shows that
\[
\mathcal{G}(\Sigma_\tau) \subset \Sigma_\tau \quad \text{for } \tau \in (0, T_1].
\]

Step 2. $\mathcal{G}$ is contraction mapping on $\Sigma_\tau$ for sufficiently small $\tau > 0$.

For $(g_i, h_i) \in \Sigma_\tau$ with $0 < \tau \leq \min\{T_1, 1\}$, let $\mathcal{G}(g_i, h_i) = (\tilde{g}_i, \tilde{h}_i) \quad (i = 1, 2)$, $g = g_1 - g_2, h = h_1 - h_2, \tilde{g} = \tilde{g}_1 - \tilde{g}_2, \tilde{h} = \tilde{h}_1 - \tilde{h}_2, u = u_1 - u_2$ and $v = v_1 - v_2$, where $(u_i, v_i) \in X_{T}^{g_i, h_i} (i = 1, 2)$ are solutions of (2.18) with $(g(t), h(t))$ replaced by $(g_i(t), h_i(t))$. By Lemma 2.5, $v_i \in W^{1,2}_{p}^{(\Omega_\tau^{h_i})} \cap C_{1+\alpha}^{1,1+\alpha}(\Omega_\tau^{h_i})$ with any $p > 3$. Make the zero extension of $u_i, v_i$ in $([0, \tau] \times \mathbb{R}) \setminus \Omega_\tau^{h_i}$. It is easy to see that
\[
|\hat{h}'(t)| \leq \mu |\partial_x v_1(t, h_1(t)) - \partial_x v_2(t, h_2(t))|
\]
\[
+ \rho_1 \left| \int_{g_1(t)}^{h_1(t)} \int_{h_1(t)}^{\infty} J(x - y) u_1(t, x) dy dx - \int_{g_2(t)}^{h_2(t)} \int_{h_2(t)}^{\infty} J(x - y) u_2(t, x) dy dx \right|
\]
\[
+ \rho_2 \left| \int_{g_1(t)}^{h_1(t)} \int_{h_1(t)}^{\infty} J(x - y) v_1(t, x) dy dx - \int_{g_2(t)}^{h_2(t)} \int_{h_2(t)}^{\infty} J(x - y) v_2(t, x) dy dx \right|
\]
\[
=: E_1 + E_2 + E_3.
\]
We first estimate $E_1$. It follows from (2.18) that, for $i = 1, 2$,
\[
\begin{cases}
\partial_i v_i = d_2 \left[ \tau \partial^2_{xx} v_i + (1 - \tau) \left( \int_{D_1}^1 J(x - y)v_i(t,y)dy - v_i \right) \right] + f_2(t,x,u_i,v_i) , \quad (t,x) \in \Omega_1^2, \\
v_i(t, g_i(t)) = v_i(t, h_i(t)) = 0 , \quad t \in (0, \tau), \\
v_i(0,x) = v_0(x) , \quad x \in [-h_0,h_0]. 
\end{cases}
\]

(2.22)

Let $\tilde{u}_i(t,z) = u_i(t, x_i(t,z))$ and $\tilde{v}_i(t,z) = v_i(t, x_i(t,z))$ with $x_i(t,z) = \frac{1}{4}[(h_i(t) - g_i(t))z + h_i(t) + g_i(t)]$ $(i = 1, 2)$, then (2.22) becomes into
\[
\begin{cases}
\partial_i \tilde{v}_i - d_2 \tau \xi_i(t) \partial^2_z \tilde{v}_i - \eta_i(t,z) \partial_z \tilde{v}_i = d_2(1 - \tau) \left( \int_{1}^1 J \frac{(h_i(t) - g_i(t))}{2}(z - s) \tilde{v}_i ds - \tilde{v}_i \right) \\
+ f_2(t,x_i(t,z), \tilde{u}_i, \tilde{v}_i) , \quad (t,z) \in D_\tau, \\
\tilde{v}_i(t, -1) = \tilde{v}_i(t, 1) = 0 , \quad t \in (0, \tau), \\
\tilde{v}_i(0,z) = v_0(h_0 z) , \quad z \in [-1,1],
\end{cases}
\]

(2.23)

where $\xi_i$ and $\eta_i$ are defined as $\xi$ and $\eta$ with $(g,h)$ replaced by $(g_i,h_i)$. By Lemma 2.4, we know the unique solution $\tilde{v}_i \in W^{1,2}_p(\Omega_1^{2,h}) \cap C^{\frac{n-1+\alpha}{2},\alpha}(\Omega_1^{2,h})$ satisfies $0 < \tilde{v}_i \leq K_2$, then the right hand of the equation in (2.23) is bounded. Applying the $L^p$ theory, we know $\|\tilde{v}_i\|_{W^{1,2}_p(D_\tau)} \leq C$. From the proof of Theorem 1.1 in [13, 14], we have $|\partial_2 \tilde{v}_i|_{C^{\frac{n-1+\alpha}{2},\alpha}(\Omega_1^{2,h})} \leq C' \|\tilde{v}_i\|_{W^{1,2}_p(D_\tau)}$, where $C'$ is independent of $\tilde{v}_i$. Then we can deduce that $\|\partial_2 \tilde{v}_i\|_{C(D_\tau)} \leq h_0 \|\tilde{v}_i\|_{C([-h_0,h_0])} + \tau \tilde{v}_i \|\partial_2 \tilde{v}_i\|_{C^{\frac{n-1+\alpha}{2},\alpha}(\Omega_1^{2,h})} \leq h_0 \|	ilde{v}_i\|_{C([-h_0,h_0])} + CC'.

Let $\tilde{v} = \tilde{v}_1 - \tilde{v}_2$, $\tilde{u} = \tilde{u}_1 - \tilde{u}_2$, we have
\[
\begin{cases}
\partial_i \tilde{v} - d_2 \tau \xi_i(t) \partial^2_z \tilde{v} - \eta_i(t,z) \partial_z \tilde{v} = d_2(1 - \tau) \left( \int_{1}^1 J \frac{(h_i(t) - g_i(t))}{2}(z - s) \tilde{v} ds - \tilde{v} \right) - a_1(t,z) \tilde{v} - b_1(t,z) \tilde{u} + c_1(t,z) \tilde{u} + d_1(t,z) \tilde{u}_2(t,z) , \quad (t,z) \in D_\tau, \\
\tilde{v}(t, -1) = \tilde{v}(t, 1) = 0 , \quad t \in (0, \tau), \\
\tilde{v}(0,z) = 0 , \quad z \in [-1,1],
\end{cases}
\]

where
\[
\begin{align*}
a_1(t,z) &= \int_{1}^1 \partial_i f_2(t,x_1(t,z), \tilde{u}_1, \tilde{v}_2 + \theta \tilde{v}) d \theta , \\
b_1(t,z) &= f_2(t,x_1(t,z), \tilde{u}_1, \tilde{v}_2) - f_2(t,x_2(t,z), \tilde{u}_1, \tilde{v}_2) , \\
c_1(t,z) &= \int_{1}^1 \partial_i f_2(t,x_2(t,z), \tilde{u}_2 + \theta \tilde{u}, \tilde{v}_2) d \theta , \\
d_1(t,z) &= \frac{h_1(t) - g_1(t)}{2} \int_{1}^1 J \frac{(h_i(t) - g_i(t))}{2}(z - s) \tilde{v}_1 ds \\
&- \frac{h_2(t) - g_2(t)}{2} \int_{1}^1 J \frac{(h_i(t) - g_i(t))}{2}(z - s) \tilde{v}_2 ds.
\end{align*}
\]

Similar to (2.12), by the Lipschitz continuity of $J$ and the boundness of $h_i(t), g_i(t)$ and $\tilde{v}_i$,
\[
|d_1(t,z)| \leq C \left( \int_{1}^1 |\frac{(h_i - g_i)}{2}(z - s)\tilde{v}_1| ds + \int_{1}^1 |\frac{(h_i - g_i)}{2}\tilde{v}_1| ds + \int_{1}^1 |\frac{(h_i - g_i)}{2}\tilde{v}_2| ds \right).
\]
Note that
\[ \| \xi_1 - \xi_2 \|_{L^\infty((0,\tau))} \leq \frac{h_0 + \frac{\varepsilon_0}{4}}{h_0^4} \| g, h \|_{C([0,\tau])}, \quad \| \eta_1 - \eta_2 \|_{L^\infty(D_\tau)} \leq \frac{\bar{R} + h_0 + \frac{\varepsilon_0}{4}}{h_0^2} \| g, h \|_{C^1([0,\tau])}, \]
\[ \| a_1, c_1 \|_{L^\infty(D_\tau)} \leq \hat{L}, \quad \| b_1 \|_{L^\infty(D_\tau)} \leq L^*. \]

Applying the \( L^p \) theory, we get
\[ \| \tilde{v} \|_{W^{1,2}(D_\tau)} \leq C(\| g, h \|_{C^1([0,\tau])} + \| \tilde{u} \|_{C(\overline{D_\tau})}) + C_1 \| \tilde{v} \|_{C(\overline{D_\tau})}(2\tau)^{\frac{1}{2}}. \]

From (2.20), we know \( \| \tilde{v} \|_{C(\overline{D_\tau})} \leq C'(\| g, h \|_{C^1([0,\tau])} + \| \tilde{u} \|_{C(\overline{D_\tau})}). \)

Similar as the proof of Lemma 2.6 in [31], we can prove that, for \( \tau \) small enough,
\[ \| \tilde{u} \|_{C(\overline{D_\tau})} \leq C(\| u \|_{C(\overline{\Omega^*_\tau})} + \| g, h \|_{C([0,\tau])}), \]
where \( \Omega^*_\tau := \Omega^1_{\tau, h_1} \cup \Omega^2_{\tau, h_2}. \) By the similar arguments in the proof of Lemma 2.5 (Steps 1-3) in [31] and Theorem 2.1 (Step 2) in [17], we can also get
\[ E_1 \leq C\tau \tilde{\varphi}(\| g, h \|_{C^1([0,\tau])} + \| u \|_{C(\overline{D_\tau})}), \]
\[ E_2 \leq C(\| u \|_{C(\overline{D_\tau})} + \tau \| g, h \|_{C^1([0,\tau])}), \]
\[ E_3 \leq C(\| v \|_{C(\overline{D_\tau})} + \tau \| g, h \|_{C^1([0,\tau])}), \]
\[ \| u \|_{C(\overline{D_\tau})} \leq C\tau \| g, h \|_{C^1([0,\tau])}. \]

Thus, for small \( \tau > 0 \), we have
\[ \| \tilde{g}, \tilde{h} \|_{C^1([0,\tau])} \leq \frac{1}{2} \| g, h \|_{C^1([0,\tau])}, \]
which implies that \( \mathcal{G} \) is a contraction map on \( \sum_\tau. \)

The rest of the proof can be obtained by using similar arguments as that of Theorem 2.1 in [17, 31], here we omit the details. \( \square \)

## 3 Comparison principle and some eigenvalue problems

In this section, we first give a comparison principle for (1.1), and then investigate the existence and properties of principle eigenvalue of some eigenvalue problems. These results will play an important role in later sections.

### 3.1 The comparison principle

In this subsection, we discuss the comparison principle for (1.1).
Lemma 3.1. (The Comparison Principle) Suppose that $T_0 \in (0, \infty), \bar{g}, \bar{h} \in C^1([0, T_0]), \bar{u} \in C(T_{T_0}^{\bar{g}}), \bar{v} \in C^1, 2(\Omega_{T_0}^{\bar{g}}) \cap C(T_{T_0}^{\bar{h}})$, and $(\bar{u}, \bar{v}, \bar{g}, \bar{h})$ satisfy

\[
\begin{cases}
\partial_t \bar{u} \geq d_1 \left( f_{\bar{g}(t)}^{\bar{h}(t)} J(x-y) \bar{u}(t,y)dy - \bar{u} \right) + \bar{u}(a(t) - \bar{u}), & (t, x) \in \Omega_{\bar{T}_0}^{\bar{g}}, \\
\partial_t \bar{v} \geq d_2 \left[ \tau \bar{v}^2 \bar{v} + (1 - \tau) \left( f_{\bar{g}(t)}^{\bar{h}(t)} J(x-y) \bar{v}(t,y)dy - \bar{v} \right) \right] + \bar{v}(c(t) - \bar{v}), & (t, x) \in \Omega_{\bar{T}_0}^{\bar{h}}, \\
\bar{u}(t, \bar{g}(t)) \geq 0, \bar{u}(t, \bar{h}(t)) \geq 0, & 0 < t \leq T_0, \\
\bar{v}(t, \bar{g}(t)) = 0, \bar{v}(t, \bar{h}(t)) = 0, & 0 < t \leq T_0, \\
\bar{h}'(t) \geq -\mu \bar{v}_x(t, \bar{h}(t)) + \rho_1 \int_{\bar{g}(t)}^{\bar{h}(t)} J(x-y) \bar{u}(t,x)dydx + \rho_2 \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{h}(t)}^{\infty} J(x-y) \bar{v}(t,x)dydx, & 0 < t \leq T_0, \\
\bar{g}'(t) \leq -\mu \bar{v}_x(t, \bar{g}(t)) - \rho_1 \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{g}(t)}^{\infty} J(x-y) \bar{u}(t,x)dydx - \rho_2 \int_{\bar{g}(t)}^{\bar{h}(t)} \int_{\bar{g}(t)}^{\infty} J(x-y) \bar{v}(t,x)dydx, & 0 < t \leq T_0, \\
\bar{u}(0, x) \geq u_0(x), \bar{v}(0, x) \geq v_0(x), & |x| \leq h_0, \\
\bar{h}(0) \geq h_0, \bar{g}(0) \leq -h_0.
\end{cases}
\]

(3.1)

Let $(u, v, g, h)$ be the unique solution of (1.1), then

\[ g(t) \geq \bar{g}(t), \quad h(t) \leq \bar{h}(t) \quad \text{in} \quad (0, T_0], \quad u(t, x) \leq \bar{u}(t, x), \quad v(t, x) \leq \bar{v}(t, x) \quad \text{for} \quad (t, x) \in \Omega_{T_0}^{\bar{g}}. \]

**Proof.** Thanks to Lemma 2.2 in [4] and Lemma 2.1, one sees that $\bar{u}, \bar{v} > 0$ for $(t, x) \in \Omega_{\bar{T}_0}^{\bar{g}}$.

We first consider the case $\bar{h}(0) > h_0, \bar{g}(0) < -h_0$. Then $\bar{h}(t) > h(t), \bar{g}(t) < g(t)$ hold true for small $t > 0$. We claim that $\bar{h}(t) > h(t), \bar{g}(t) < g(t)$ for all $t \in (0, T_0]$. In fact, if this is not true, there exists $t_1 \leq T_0$ such that

\[ \bar{h}(t) > h(t), \quad \bar{g}(t) < g(t) \quad \text{for} \quad t \in (0, t_1) \quad \text{and} \quad [\bar{h}(t_1) - h(t_1)][\bar{g}(t_1) - g(t_1)] = 0. \]

Without loss of generality, we may assume that

\[ \bar{g}(t_1) \leq g(t_1) \quad \text{and} \quad \bar{h}(t_1) = h(t_1). \]

Thus, $\bar{h}'(t_1) \leq h'(t_1)$. Since $\bar{v}(0, x) \geq v_0(x)$ for $x \in [-h_0, h_0], \bar{v}(t, g(t)) \geq 0 = v(t, g(t))$ and $\bar{v}(t, h(t)) \geq 0 = v(t, h(t))$ for $t \in (0, t_1]$, by applying Lemma 2.2, we have $\bar{v} > v$ in $\Omega_{t_1}^{\bar{g}}$. Moreover, by the fact that $\bar{v}(t_1, h(t_1)) = v(t_1, h(t_1)) = 0 = v(t_1, h(t_1))$, we deduce that $\bar{v}(t_1, h(t_1)) < v_x(t_1, h(t_1))$. Similarly, using Lemma 2.2 in [4], we can obtain $\bar{u} > u$ in $\Omega_{t_1}^{\bar{h}}$. It follows that

\[
\bar{h}'(t_1)
\]

\[ \geq -\mu \bar{v}_x(t_1, \bar{h}(t_1)) + \rho_1 \int_{\bar{g}(t_1)}^{\bar{h}(t_1)} \int_{\bar{h}(t_1)}^{\infty} J(x-y) \bar{u}(t,x)dydx + \rho_2 \int_{\bar{g}(t_1)}^{\bar{h}(t_1)} \int_{\bar{h}(t_1)}^{\infty} J(x-y) \bar{v}(t,x)dydx \\
\geq -\mu \bar{v}_x(t_1, h(t_1)) + \rho_1 \int_{\bar{g}(t_1)}^{h(t_1)} \int_{h(t_1)}^{\infty} J(x-y) \bar{u}(t,x)dydx + \rho_2 \int_{\bar{g}(t_1)}^{h(t_1)} \int_{h(t_1)}^{\infty} J(x-y) \bar{v}(t,x)dydx \\
> -\mu v_x(t_1, h(t_1)) + \rho_1 \int_{\bar{g}(t_1)}^{h(t_1)} \int_{h(t_1)}^{\infty} J(x-y) u(t,x)dydx + \rho_2 \int_{\bar{g}(t_1)}^{h(t_1)} \int_{h(t_1)}^{\infty} J(x-y) v(t,x)dydx \\
= \bar{h}'(t_1),
\]
which is a contradiction. Hence, \( h(t) < \bar{h}(t), \ g(t) > \bar{g}(t) \) for all \( t \in (0, T_0] \), and \( \bar{u}(t, x) > u(t, x) \), \( \bar{v}(t, x) > v(t, x) \) in \( \Omega_{T_0}^{d, h} \).

For the general case that \( \bar{h}(0) \geq h_0, \ \bar{g}(0) \leq -h_0 \), we can adopt the same method as the proof for Remark 3.1.

\[\square\]

**Remark 3.1.** From the proof of Lemma 3.1, we can see that the conditions \( \bar{v}(t, \bar{g}(t)) = 0, \bar{v}(t, \bar{h}(t)) = 0 \) are necessary in deriving the contradiction from the relationship between \( \bar{h}'(t) \) and \( h'(t) \). If \( \tau = 0 \), as considered in [17], then the expressions of \( h'(t), g'(t) \) in (1.1) and \( \bar{h}'(t), \bar{g}'(t) \) in (3.1) do not include the terms \( -\mu v_x(t, h(t)), -\mu v_x(t, g(t)) \) and \( -\mu \bar{v}_x(t, \bar{h}(t)), -\mu \bar{v}_x(t, \bar{g}(t)) \), respectively, in such case the conditions \( \bar{v}(t, \bar{g}(t)) = 0, \bar{v}(t, \bar{h}(t)) = 0 \) can be weaken into \( \bar{v}(t, \bar{g}(t)) \geq 0, \bar{v}(t, \bar{h}(t)) \geq 0 \).

### 3.2 Some eigenvalue problems

In this subsection, we mainly study some eigenvalue problems and analyze the properties of their principle eigenvalue. Hereafter, we always assume \( \Omega \) be a bounded, connected open interval in \( \mathbb{R} \) and \( |\Omega| \) be its length.

Consider the following operator

\[-(L_{\Omega} + a(t))[\phi](t, x) = \phi_t(t, x) - d_1 \int_{\Omega} J(x-y)\phi(t, y)dy - \phi(t, x) - a(t)\phi(t, x), \quad (t, x) \in \mathbb{R} \times \overline{\Omega},\]

(3.2)

where \( a \in C_T(\mathbb{R}) := \{ a \in C(\mathbb{R}) : a(t + T) = a(t) > 0, \forall t \in \mathbb{R} \} \). For convenience, we define the space \( \mathcal{X}_\Omega, \mathcal{X}_\Omega^+, \mathcal{X}_\Omega^{++} \) as follows:

\[
\mathcal{X}_\Omega = \{ \phi \in C^{1,0}(\mathbb{R} \times \overline{\Omega}) : \phi(t + T, x) = \phi(t, x), \ (t, x) \in \mathbb{R} \times \overline{\Omega} \},
\]

\[
\mathcal{X}_\Omega^+ = \{ \phi \in \mathcal{X}_\Omega : \phi(t, x) \geq 0, \ (t, x) \in \mathbb{R} \times \overline{\Omega} \},
\]

\[
\mathcal{X}_\Omega^{++} = \{ \phi \in \mathcal{X}_\Omega : \phi(t, x) > 0, \ (t, x) \in \mathbb{R} \times \overline{\Omega} \},
\]

where \( C^{1,0}(\mathbb{R} \times \overline{\Omega}) \) denotes the class of functions that are \( C^1 \) in \( t \) and continuous in \( x \).

We define

\[
\lambda_1(-(L_{\Omega} + a(t))) = \inf \left\{ \Re \lambda : \lambda \in \sigma(-(L_{\Omega} + a(t))) \right\},
\]

where \( \sigma(-(L_{\Omega} + a(t))) \) is the spectrum of \( -(L_{\Omega} + a(t)) \). By Theorem A (1) in [29], we know that \( \lambda_1(-(L_{\Omega} + a(t))) \) is the principle eigenvalue of \( -(L_{\Omega} + a(t)) \), which means that there exists an eigenfunction \( \phi \in \mathcal{X}_\Omega^{++} \) such that

\[-(L_{\Omega} + a(t))[\phi](t, x) = \lambda_1(-(L_{\Omega} + a(t)))\phi.\]

**Lemma 3.2.** (see Theorem B in [29]) Assume that \( J \) satisfies (J) and \( a \in C_T(\mathbb{R}) \). Let \( u(t, x; u_0) \) be a solution of

\[
\begin{aligned}
&u_t = d_1 \int_{\Omega} J(x-y)u(t, y)dy - u(t, x) + a(t)u(t, x), \quad t > 0, x \in \overline{\Omega}, \\
&u(0, x) = u_0(x), \quad x \in \overline{\Omega},
\end{aligned}
\]

21
where \( u_0 \in C(\Omega) \) is non-negative and not identically zero. The following statements hold:

(i) If \( \lambda_1(- (L_\Omega + a(t))) < 0 \), then the equation

\[
 u_t = d_1 \left[ \int_\Omega J(x-y)u(t,y)dy - u(t,x) \right] + u(a(t) - u), \quad t \in \mathbb{R}, x \in \Omega \tag{3.3}
\]

admits a unique solution \( u^* \in X^+_\Omega \), and there holds

\[
 \|u(t, \cdot; u_0) - u^*(t, \cdot)\|_{C(\overline{\Omega})} \to 0 \quad \text{as} \quad t \to \infty,
\]

(ii) If \( \lambda_1(- (L_\Omega + a(t))) > 0 \), then the equation (3.3) admits no solution in \( X^+_\Omega \setminus \{0\} \) and there holds

\[
 \|u(t, \cdot; u_0)\|_{C(\overline{\Omega})} \to 0 \quad \text{as} \quad t \to \infty.
\]

**Remark 3.2.** For the case \( \lambda_1(- (L_\Omega + a(t))) = 0 \), (3.3) has been shown in \[29\] to admit no solution in \( X^+_\Omega \setminus \{0\} \), but the global dynamics is not provided. Since \( a(t) \) is independent of spatial variable, we can also get \( \|u(t, \cdot; u_0)\|_{C(\overline{\Omega})} \to 0 \), more details can be seen in the proof of Theorem 4.1.

In what follows, we present some further properties of \( \lambda_1 \).

**Lemma 3.3.** Let \( J \) satisfies (J) and \( a \in C_T(\mathbb{R}) \). Then

(i) \( \lambda_1(- (L_\Omega + a(t))) \) is strictly decreasing and continuous in \( \|\Omega\| \);

(ii) \( \lim_{|\Omega| \to +\infty} \lambda_1(- (L_\Omega + a(t))) = -a_T \), where \( a_T = \frac{1}{T} \int_0^T a(t)dt \);

(iii) \( \lim_{|\Omega| \to 0} \lambda_1(- (L_\Omega + a(t))) = d_1 - a_T \).

**Proof.** Let \( \phi \in X^+_\Omega \) be an eigenfunction of \( -(L_\Omega + a(t)) \) associated with the principle eigenvalue \( \lambda_1(- (L_\Omega + a(t))) \). We define

\[
 \psi(t,x) = e^{- \int_0^t (a(s) - a_T)ds} \phi(t,x), \quad \forall (t,x) \in \mathbb{R} \times \overline{\Omega}.
\]

It is easy to check that \( \psi \in X^+_\Omega \).

Multiplying the equation \( -(L_\Omega + a(t))\phi = \lambda_1(- (L_\Omega + a(t))\phi \) by the function \( t \mapsto e^{- \int_0^t (a(s) - a_T)ds} \), we have

\[
 -\psi_t(t,x) + d_1 \left[ \int_\Omega J(x-y)\psi(t,y)dy - \psi(t,x) \right] + a_T \psi(t,x) + \lambda_1(- (L_\Omega + a(t))\psi(t,x) = 0
\]

for \( (t,x) \in \mathbb{R} \times \overline{\Omega} \). Taking \( \psi_T(x) = \frac{1}{T} \int_0^T \psi(t,x)dt \) for \( x \in \overline{\Omega} \), and integrating the above equation over \([0, T]\) with respect to \( t \), we have

\[
 d_1 \left[ \int_\Omega J(x-y)\psi_T(y)dy - \psi_T(x) \right] + a_T \psi_T(x) + \lambda_1(- (L_\Omega + a(t))\psi_T(x) = 0, \quad x \in \overline{\Omega}.
\]

That is, \( -\lambda_1(- (L_\Omega + a(t))) \) is the principle eigenvalue of the following nonlocal operator \( L_\Omega + a_T : C(\overline{\Omega}) \to C(\overline{\Omega}) \) defined by

\[
 (L_\Omega + a_T)[\omega](x) := d_1 \left[ \int_\Omega J(x-y)\omega(y)dy - \omega(x) \right] + a_T \omega(x)
\]
with an eigenfunction $\psi_T \in X_{\Omega}^{++}$. Denote by $\lambda_1(\mathcal{L}_\Omega + a_T)$ the principle eigenvalue of $\mathcal{L}_\Omega + a_T$, then we have

$$-\lambda_1(-(\mathcal{L}_\Omega + a(t))) = \lambda_1(\mathcal{L}_\Omega + a_T).$$

(3.5)

Without loss of generality, we assume that $\Omega = (l_1,l_2)$. According to Proposition 3.4 in [3], we know the following results hold:

(i) $\lambda_1(\mathcal{L}_\Omega + a_T)$ is strictly increasing and continuous in $|\Omega| = l_2 - l_1$;
(ii) $\lim_{t_1 \to -\infty} \lambda_1(\mathcal{L}_\Omega + a_T) = a_T$;
(iii) $\lim_{t_1 \to 0} \lambda_1(\mathcal{L}_\Omega + a_T) = a_T - d_1$.

Combining the above conclusions and (3.5), we can get the desired results. \qed

Now, we consider another periodic-parabolic eigenvalue problem

$$\left\{ \begin{array}{l}
-(\mathcal{L}_\Omega + c(t))[\varphi](t,x) \\
= \varphi_t - d_2[\tau\varphi_{xx} + (1-\tau)(\int_\Omega J(x-y)\varphi(t,y)dy - \varphi(t,x))] - c(t)\varphi = \lambda\varphi, \quad \text{in } [0,T] \times \Omega, \\
\varphi(t,x) = 0, \quad \text{on } [0,T] \times \partial\Omega, \\
\varphi(0,x) = \varphi(T,x), \quad \text{in } \Omega.
\end{array} \right.$$

(3.6)

Define a linear nonlocal operator $\mathcal{K}$ on $C([0,T];L^p(\Omega))$ ($\forall p \geq 1$) by

$$(\mathcal{K}\varphi)(t,x) := \int_\Omega J(x-y)\varphi(t,y)dy - \varphi(t,x).$$

For any given $0 < \tau \leq 1$, we can check that $\{A(t) : 0 \leq t \leq T\} := \{-d_2[\tau\varphi_{xx} + (1-\tau)\mathcal{K}] - c(t)I : 0 \leq t \leq T\}$ satisfy the hypotheses (11.5) in [22]. As showed in Section II.14 of [22], based on the Krein-Rutman theorem, we can prove that (3.6) admits a principle eigenvalue $\lambda_1(-(\mathcal{L}_\Omega + c(t)))$ with principle eigenfunction $\varphi$.

For later applications, we give the following lemma.

Lemma 3.4. Let $J$ satisfies (J) and $c \in C_T(\mathbb{R})$. Then

(i) $\lambda_1(-(\mathcal{L}_\Omega + c(t)))$ is a strictly decreasing continuous function in $|\Omega|$ and $\lambda_1(-(\mathcal{L}_\Omega + c(t))) = 0$ has a unique root $|\Omega| = h^*$;
(ii) if $\lambda_1(-(\mathcal{L}_\Omega + c(t))) < 0$, then the problem

$$\left\{ \begin{array}{l}
\varphi_t - d_2[\tau\varphi_{xx} + (1-\tau)(\int_\Omega J(x-y)\varphi(t,y)dy - \varphi(t,x))] = \varphi(c(t) - \varphi), \quad \text{in } (0,\infty) \times \Omega, \\
\varphi(t,x) = 0, \quad \text{on } (0,\infty) \times \partial\Omega.
\end{array} \right.$$

admits a unique positive $T$-periodic solution $\varphi^*$, and $\varphi^*$ is globally asymptotically stable.

Proof. (i) Let $\varphi$ be an eigenfunction of (3.6) associated with the principle eigenvalue $\lambda_1(-(\mathcal{L}_\Omega + c(t)))$. Define

$$\psi(t,x) = e^{-\int_0^t(c(s) - c_T)ds}\varphi(t,x), \quad \forall (t,x) \in \mathbb{R} \times \overline{\Omega}.$$

23
Similar as the proof of Lemma 3.3, $\lambda_1(-\tilde{L}_\Omega + c(t))$ is the principal eigenvalue of the following elliptic-type problem

$$
\begin{align*}
-\tilde{L}_\Omega + c(t)[\omega] &= -d_2[\tau \omega_{xx} + (1 - \tau)(\int_\Omega J(x - y)\omega(y)dy - \omega(x))] - c_T \omega = \lambda \omega, & \text{in } \Omega, \\
\omega(x) &= 0, & \text{on } \partial \Omega
\end{align*}
$$

(3.7)

with an eigenfunction $\omega(x) = \frac{1}{\sqrt{\pi}} \int_0^T \psi(t, x)dt$. Denote by $\lambda_1(-\tilde{L}_\Omega + c(t))$ the principle eigenvalue of (3.7), then we have

$$
\lambda_1(-\tilde{L}_\Omega + c(t)) = \lambda_1(-\tilde{L}_\Omega + c_T)).
$$

(3.8)

The continuity of $\lambda_1(-\tilde{L}_\Omega + c_T))$ with respect to $|\Omega|$ can be obtained by using a simple re-scaling argument of the spatial variable $x$. Note that $\lambda_1(-\tilde{L}_\Omega + c_T))$ can be expressed in a variational formulation

$$
\lambda_1(-\tilde{L}_\Omega + c_T)) = \inf_{0 \not= \omega \in H^1_0(\Omega)} \frac{d_2 \int_\Omega \omega_y^2(x)dx - d_2(1 - \tau) \int_\Omega \int_\Omega J(x - y)\omega(y)\omega(x)dydx}{\int_\Omega \omega^2(x)dx} + [d_2(1 - \tau) - c_T].
$$

By the zero extension of principle eigenfunction, we can get the monotonicity of $\lambda_1(\tilde{L}_\Omega + c_T)$ from the variational formulation of principle eigenvalue.

Next, we prove that $\lambda_1(-\tilde{L}_\Omega + c(t))) = 0$ has a unique root. Without loss of generality, we may assume that $\Omega = (0, l)$. Since

$$
\int_0^l \int_0^l J(x - y)\omega(y)\omega(x)dydx \leq \int_0^l \int_0^l J(x - y)\omega^2(y) + \omega^2(x) 2\,dydx \leq \int_0^l \omega^2(x)dx,
$$

we have

$$
\lambda_1(-\tilde{L}_{(0, l)} + c_T)) \geq \inf_{0 \not= \omega \in H^1_0((0, l))} \frac{d_2 \int_0^l \omega_y^2(x)dx}{\int_0^l \omega^2(x)dx} - c_T.
$$

By the fact that

$$
\inf_{0 \not= \omega \in H^1_0((0, l))} \frac{\int_0^l \omega_y^2(x)dx}{\int_0^l \omega^2(x)dx} = \frac{\pi^2}{4l^2},
$$

we know

$$
\lim_{l \to 0} \lambda_1(-\tilde{L}_{(0, l)} + c_T)) = +\infty
$$

(3.9)

and

$$
\lim_{l \to +\infty} \lambda_1(-\tilde{L}_{(0, l)} + c_T)) \geq -c_T.
$$

(3.10)

On the other hand, by (J), for any fixed $0 < \varepsilon < 1$, there exists $L = L(\varepsilon) > 0$ such that

$$
\int_{-L}^{L} J(x)dx > 1 - \varepsilon.
$$

For any large $l > 3L$, we choose the test function $\varphi_\varepsilon(x)$ defined as follows

$$
\varphi_\varepsilon(x) = \begin{cases} 
\frac{x}{\varepsilon}, & x \in [0, \varepsilon], \\
1, & x \in [\varepsilon, l - \varepsilon], \\
\frac{l - x}{\varepsilon}, & x \in [l - \varepsilon, l].
\end{cases}
$$
It is easy to check that $\varphi_\varepsilon \in H^1((0,l))$ and satisfies $\int_0^l \varphi^2_\varepsilon(x)dx = l - \frac{4}{3}\varepsilon$ and $\int_0^l (\partial_x \varphi_\varepsilon)^2(x)dx = \frac{2}{\varepsilon}$. Thus,

$$\lambda_1(-(\tilde{\mathcal{L}}_{(0,l)} + c_T)) \leq \frac{d_2 \int_0^l (\partial_x \varphi_\varepsilon)^2(x)dx - d_2 (1 - \tau) \int_0^l \int_0^l J(x-y)\varphi_\varepsilon(y)\varphi_\varepsilon(x)dydx}{\int_0^l \varphi^2_\varepsilon(x)dx} + [d_2 (1 - \tau) - c_T]$$

$$\leq \frac{2d_2 \varepsilon}{\varepsilon} - d_2 (1 - \tau) \int_{l+\varepsilon}^{l-L-\varepsilon} \int_{-\varepsilon}^{l-\varepsilon} J(x-y)dydx}{l - \frac{4}{3}\varepsilon} + [d_2 (1 - \tau) - c_T]$$

$$\leq \frac{2d_2 \varepsilon}{\varepsilon} - d_2 (1 - \tau) \int_{l+\varepsilon}^{l-L-\varepsilon} \int_{-\varepsilon}^{L} J(x)dxdx}{l - \frac{4}{3}\varepsilon} + [d_2 (1 - \tau) - c_T]$$

$$\leq \frac{2d_2 \varepsilon}{\varepsilon} - d_2 (1 - \tau)(l - 2L - 2\varepsilon)(1 - \varepsilon) + [d_2 (1 - \tau) - c_T]$$

$$\rightarrow -d_2 (1 - \tau)(1 - \varepsilon) + [d_2 (1 - \tau) - c_T]$$

as $l \rightarrow +\infty$.

Since $\varepsilon$ is arbitrary, it follows that

$$\limsup_{l \rightarrow +\infty} \lambda_1(-(\tilde{\mathcal{L}}_{(0,l)} + c_T)) \leq -c_T,$$

which together with (3.10) imply that

$$\lim_{l \rightarrow +\infty} \lambda_1(-(\tilde{\mathcal{L}}_{(0,l)} + c_T)) = -c_T.$$  

(3.11)

From (3.8), (3.9) and (3.11), we know that $\lambda_1(-(\tilde{\mathcal{L}}_{(0,l)} + c(t))) = 0$ has a unique root.

(ii) the proof is similar as that of Theorem 28.1 in [22], we omit the details. \hfill \square

4 Spreading and vanishing for problem (1.1)

In this section, we investigate the dynamics of problem (1.1), including the spreading-vanishing dichotomy and some sufficient conditions for spreading and vanishing. In view of (2.2), we see that the free boundaries $h(t), -g(t)$ are strictly increasing functions with respect to time $t$. Thus, $h_\infty := \lim_{t \rightarrow +\infty} h(t)$ and $g_\infty := \lim_{t \rightarrow +\infty} g(t)$ are well-defined. Clearly, $h_\infty, -g_\infty \leq +\infty$.

By similar argument as the proof of Proposition 3.1 in [32] with minor modifications, we have the following result.

**Lemma 4.1.** Let $d$, $\mu$ and $h^0$ be positive constants and $C \in \mathbb{R}$. Assume that $\varphi_0 \in C^2([-h^0, h^0])$ satisfies $\varphi_0(-h^0) = \varphi_0(h^0) = 0$ and $\varphi_0 > 0$ in $(-h^0, h^0)$. Let $(g, h) \in [C^{1+\frac{\alpha}{2}}[0, \infty])^2$, $\varphi \in C^{1+\frac{\alpha}{2}}((0,\infty) \times (g(t), h(t)))$ for some $\alpha \in (0, 1)$ and satisfy $g(t) < 0, h(t) > 0, \varphi(t, x) > 0$ for all $t \geq 0$ and $g(t) < x < h(t)$. We further suppose that $\lim_{t \rightarrow +\infty} g(t) > -\infty, \lim_{t \rightarrow +\infty} h(t) < +\infty, \lim_{t \rightarrow +\infty} g'(t) = \lim_{t \rightarrow +\infty} h'(t) = 0$ and there exists a constant $K > 0$ such that $\|\varphi\|_{C^{1+\frac{\alpha}{2}}[g(t), h(t)]} \leq K$.
for \( t > 1 \). If \((\varphi, g, h)\) satisfies

\[
\begin{cases}
\varphi_t - d\varphi_{xx} \geq C\varphi, & t > 0, \ g(t) < x < h(t), \\
\varphi = 0, & t \geq 0, \ x = g(t) \text{ or } x = h(t), \\
g'(t) \leq -\mu\varphi_x(t, g(t)), \ h'(t) \geq -\mu\varphi_x(t, h(t)), & t > 0, \\
g(0) = -h^0, \ h(0) = h^0, \\
\varphi(0, x) = \varphi_0(x), & -h^0 < x < h^0,
\end{cases}
\]

then \( \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} \varphi(t, x) = 0. \)

The next lemma provides an estimate for \( v \). The proof is a simple modification of that for Lemma 4.2 in [31], so we omit it here.

**Lemma 4.2.** Let \((u, v, g, h)\) be the unique global solution of (1.1) and \( h_\infty - g_\infty < \infty \). Then there exists \( C > 0 \) such that

\[
\|v\|_{C^{1/2,1}([h_\infty, h_\infty])} \leq C,
\]

where \( D_\infty := [0, \infty) \times [g(t), h(t)] \) (4.1) and hence

\[
\|v_x(t, g(t))\|_{C^1(\mathbb{R}_+)} + \|v_x(t, h(t))\|_{C^1(\mathbb{R}_+)} \leq C. \tag{4.2}
\]

**Lemma 4.3.** If \( h_\infty - g_\infty < \infty \), then \( \lim_{t \to \infty} g'(t) = \lim_{t \to \infty} h'(t) = 0. \)

**Proof.** It is easy to see that \( -\infty < g_\infty < h_\infty < \infty \). From (2.2), we can deduce that \( g'(t) \) and \( h'(t) \) defined in (1.1) are bounded. Let

\[
\varphi_1(t) = v_x(t, h(t)), \quad \varphi_2(t) = \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)dydx, \quad \varphi_3(t) = \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)v(t, x)dydx.
\]

By (4.2), we get \( |\varphi_1(t) - \varphi_1(s)| \leq C_1|t - s|^{3/2} \) for any \( t, s > 0 \). For \( \varphi_2 \), assume \( t > s \), we have \( h(t) > h(s), \ g(t) < g(s) \) and then

\[
\varphi_2(t) - \varphi_2(s)
\]

\[
= \int_{g(t)}^{h(t)} \int_{h(s)}^{\infty} J(x - y)u(t, x)dydx - \int_{g(s)}^{h(s)} \int_{h(s)}^{\infty} J(x - y)u(s, x)dydx
\]

\[
= \int_{g(s)}^{h(s)} \int_{h(t)}^{\infty} J(x - y)[u(t, x) - u(s, x)]dydx + \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)dydx
\]

\[
+ \int_{h(s)}^{h(t)} \int_{h(t)}^{\infty} J(x - y)u(t, x)dydx - \int_{g(s)}^{h(s)} \int_{h(t)}^{\infty} J(x - y)u(s, x)dydx
\]

\[
\leq \|\partial_t u\|_{L^\infty(D_\infty)} (t - s)(h(s) - g(s)) + \|u\|_{L^\infty(D_\infty)} (g(s) - g(t)) + 2\|u\|_{L^\infty(D_\infty)} (h(t) - h(s))
\]

\[
\leq C_2(t - s),
\]

where \( \|\partial_t u\|_{L^\infty(D_\infty)} \) is obtained by the first equation in (1.1) and the bound of \( u \). Thus,

\[
|\varphi_2(t) - \varphi_2(s)| \leq C_2|t - s|.
\]
Proof. Since $h$ we claim that

$$\varphi(t) - \varphi(s) \leq C_3 |t - s|.$$ 

Therefore, $h'(t) = -\mu \varphi_1 + \rho_1 \varphi_2 + \rho_2 \varphi_3$ is uniformly continuous in $[0, \infty)$. From $\lim_{t \to \infty} h(t) = h_\infty < \infty$, we know $\lim_{t \to \infty} h'(t) = 0$. Similarly, we can show $\lim_{t \to \infty} g'(t) = 0$. \hfill \Box

Theorem 4.1. If $h_\infty - g_\infty < \infty$, then the solution $(u, v, g, h)$ of (1.1) satisfies

$$\lim_{t \to \infty} \|u(t, \cdot)\|_{C([g(t), h(t)])} = \lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0.$$ 

Proof. Since $J \geq 0$ and $v > 0$, from the second equation in (1.1), there exists a constant $C > 0$ such that

$$\partial_t v - d_2 \partial_x^2 v \geq Cv.$$ 

According to Lemma 4.1, we get

$$\lim_{t \to \infty} \|v(t, \cdot)\|_{C([g(t), h(t)])} = 0.$$ 

We claim that

$$\lambda_1(-(L_{(g_\infty, h_\infty)} + a(t))) \geq 0,$$

where $-(L_{(g_\infty, h_\infty)} + a(t))$ is defined in (3.2). Assume on the contrary that $\lambda_1(-(L_{(g_\infty, h_\infty)} + a(t))) < 0$. For convenience, for any $\varepsilon > 0$ we define $h_\infty^{\pm \varepsilon} := h_\infty \pm \varepsilon$, $g_\infty^{\pm \varepsilon} := g_\infty \pm \varepsilon$. Thus, there exists $\varepsilon_1 > 0$ such that $\lambda_1(-(L_{(g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon})} + a(t) - b(t)) > 0$ for all $\varepsilon \in (0, \varepsilon_1)$. For such $\varepsilon > 0$, we can find $T_\varepsilon > 0$ such that, for $t > T_\varepsilon$,

$$h(t) > h_\infty^{\pm \varepsilon}, \ g(t) < g_\infty^{\pm \varepsilon}, \ \|v(t, \cdot)\|_{C([g(t), h(t)])} < \varepsilon.$$ 

Then $u$ satisfies

$$\begin{cases}
u_t \geq d_1 \int_{g_\infty^{\pm \varepsilon}}^{h_\infty^{\pm \varepsilon}} J(x - y) u(t, y) dy - d_1 u + u(a(t) - u - b(t)) \varepsilon, & t > T_\varepsilon, \ x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}], \\ u(T_\varepsilon, x) = u(T_\varepsilon, x), & x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}]. \end{cases}$$

Consider the following problem

$$\begin{cases}
\phi_t = d_1 \int_{g_\infty^{\pm \varepsilon}}^{h_\infty^{\pm \varepsilon}} J(x - y) \phi(t, y) dy - d_1 \phi + \phi(a(t) - \phi - b(t)) \varepsilon, & t > T_\varepsilon, \ x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}], \\ \phi(T_\varepsilon, x) = u(T_\varepsilon, x), & x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}]. \end{cases} \tag{4.4}$$

Since $\lambda_1(-(L_{(g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon})} + a(t) - b(t)) \varepsilon) < 0$, by Lemma 3.2 (i) we know that the solution $\phi_\varepsilon(t, x)$ of problem (4.4) converges to $\phi_\varepsilon^*(t, x)$ uniformly in $[g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}]$ as $t \to \infty$, where $\phi_\varepsilon^*(t, x) \in X^{+\varepsilon}$ is the unique periodic solution of

$$\phi_t = d_1 \int_{g_\infty^{\pm \varepsilon}}^{h_\infty^{\pm \varepsilon}} J(x - y) \phi(t, y) dy - d_1 \phi + \phi(a(t) - \phi - b(t)) \varepsilon, \ t \in \mathbb{R}, \ x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}].$$

By Lemma 2.2 in and a simple comparison argument, we get

$$u(t, x) \geq \phi_\varepsilon(t, x), \ \forall \ t > T_\varepsilon, \ x \in [g_\infty^{\pm \varepsilon}, h_\infty^{\pm \varepsilon}].$$

27
Hence, there exist two constants $\bar{T}_\varepsilon > T_\varepsilon$ and $C > 0$ such that
\[ u(t, x) \geq \frac{1}{2} \Phi^\varepsilon(t, x) \geq C > 0, \quad \forall t > \bar{T}_\varepsilon, \; x \in [g^\varepsilon, h^{-\varepsilon}] \]

It follows that, for $0 < \varepsilon < \min\{\varepsilon_1, \frac{\varepsilon}{2}\}$ and $t > \bar{T}_\varepsilon$,
\[
h'(t) \geq \rho_1 \int_{g(t)}^{h(t)} \int_{h(t)}^{\infty} J(x-y)u(t, x)dydx \geq \rho_1 \int_{g^\varepsilon}^{h^\varepsilon} \int_{h^\varepsilon}^{\infty} J(x-y)u(t, x)dydx \\
\geq \rho_1 \int_{h^\varepsilon}^{h^{-\varepsilon}} \int_{h^\varepsilon}^{\infty} \frac{1}{2} \Phi^\varepsilon(t, x)dx \geq \rho_1 \int_{h^{-\varepsilon}}^{h^\varepsilon} \delta_0 C dydx > 0,
\]
which implies that $h_\infty = \infty$. It is a contradiction and then (4.3) holds.

Let $\bar{u}$ be the unique solution of
\[
\left\{ \begin{array}{ll}
\bar{u}_t = d_1 \int_{g^\infty}^{h^\infty} J(x-y)\bar{u}(t, y)dy - d_1 \bar{u} + \bar{u}(a(t) - \bar{u}), & t > 0, x \in [g^\infty, h^\infty], \\
\bar{u}(0, x) = u_0(x), & x \in [-h_0, h_0];
\end{array} \right. \\
\]
\[ \bar{u}(0, x) = 0, \quad x \in [g^\infty, h^\infty] \setminus [-h_0, h_0]. \]

Now we prove that $\lim_{t \to \infty} \bar{u}(t, x) = 0$ uniformly in $[g^\infty, h^\infty]$. Since (4.3) holds, we divide the discussion into two cases:

(i) For the case $\lambda_1(-(L_{(g^\infty, h^\infty)} + a(t))) > 0$, applying Lemma 3.2 (ii) we can get the desired result.

(ii) For the case $\lambda_1(-(L_{(g^\infty, h^\infty)} + a(t))) = 0$, we define
\[ w(t, x) = e^{-\int_0^t [a(s) - a_T]ds}\bar{u}(t, x), \]
then $w(t, x)$ satisfies
\[
\left\{ \begin{array}{ll}
w_t = d_1 \int_{g^\infty}^{h^\infty} J(x-y)w(t, y)dy - d_1 w + w(a_T - e^{\int_0^t [a(s) - a_T]ds}w), & t > 0, x \in [g^\infty, h^\infty], \\
w(0, x) = u_0(x), & x \in [-h_0, h_0];
\end{array} \right. \\
w(0, x) = 0, \quad x \in [g^\infty, h^\infty] \setminus [-h_0, h_0].
\]

For any $t > 0$, we can write $t = nT + \tau$ with $\tau \in [0, T)$, and then
\[ e^{\int_0^t [a(s) - a_T]ds} = e^{\int_0^n [a(s) - a_T]ds} = e^{\int_0^n [a(s) - a_T]ds}, \]
which together with the continuity of $a(t)$ imply that $M_1 \leq e^{\int_0^n [a(s) - a_T]ds} \leq M_2$ for some positive constants $M_1$ and $M_2$. By the comparison principle, we know $w(t, x) \leq \tilde{w}(t, x)$ with $\tilde{w}(t, x)$ be the unique solution of
\[
\left\{ \begin{array}{ll}
\tilde{w}_t = d_1 \int_{g^\infty}^{h^\infty} J(x-y)\tilde{w}(t, y)dy - d_1 \tilde{w} + \tilde{w}(a_T - M_1), & t > 0, x \in [g^\infty, h^\infty], \\
\tilde{w}(0, x) = u_0(x), & x \in [-h_0, h_0];
\end{array} \right. \\
\tilde{w}(0, x) = 0, \quad x \in [g^\infty, h^\infty] \setminus [-h_0, h_0].
\]

Recall that in (3.5) we have $\lambda_1(\mathcal{L}_{(g^\infty, h^\infty)} + a_T) = -\lambda_1(-(L_{(g^\infty, h^\infty)} + a(t)))$, where $\mathcal{L}_{(g^\infty, h^\infty)} + a_T$ is defined in (3.4). By Proposition 3.5 in [8] (see also [1, 8]), we know that $\lim_{t \to \infty} \tilde{w}(t, x) = 0$ uniformly in $[g^\infty, h^\infty]$. Thus, $w(t, x)$ and $\bar{u}(t, x) = e^{\int_0^t [a(s) - a_T]ds}w(t, x)$ converge to 0 uniformly in $[g^\infty, h^\infty]$ as $t \to +\infty$, which completes the proof.

On the other hand, it is easy to know that
\[
\left\{ \begin{array}{ll}
\bar{u}_t \geq d_1 \int_{g(t)}^{h(t)} J(x-y)\bar{u}(t, y)dy - d_1 \bar{u} + \bar{u}(a(t) - \bar{u}), & t > 0, x \in (g(t), h(t)), \\
\bar{u}(t, g(t)) \geq 0, \; \bar{u}(t, h(t)) \geq 0, \\
\bar{u}(0, x) = u_0(x), & x \in [-h_0, h_0].
\end{array} \right.
\]
By the comparison principle, we know \( u(t, x) \leq \bar{u}(t, x) \) for any \( t > 0 \) and \( x \in [g(t), h(t)] \). Thus,
\[
\lim_{t \to \infty} ||u(t, \cdot)||_{C([g(t), h(t)])} = 0.
\]

From Lemma 4.1, we can obtain the following spreading-vanishing dichotomy.

**Corollary 4.1.** (Spreading-vanishing dichotomy) Let \((u, v, g, h)\) be the unique solution of (1.1). Then, the following alternative holds:

Either (i) spreading: \( \lim_{t \to \infty}(h(t) - g(t)) = \infty \), or (ii) vanishing: \( \lim_{t \to \infty}(g(t), h(t)) = (g_{\infty}, h_{\infty}) \)

is a finite interval and \( \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} u(t, x) = \lim_{t \to \infty} \max_{g(t) \leq x \leq h(t)} v(t, x) = 0 \).

In what follows, we will provide some sufficient conditions for spreading and vanishing.

**Theorem 4.2.** If \( h_{\infty} - g_{\infty} < \infty \), then

\[
h_{\infty} - g_{\infty} \leq h^*,
\]

where \( |\Omega| = h^* \) is the unique root of \( \lambda_1(-(\tilde{L}_\Omega + c(t))) = 0 \) with \(-(\tilde{L}_\Omega + c(t))\) defined in (3.6).

**Proof.** Recall that in Lemma 4.4 we have showed that \( h_{\infty} - g_{\infty} < \infty \) implies

\[
\lim_{t \to \infty} ||u(t, \cdot)||_{C([g(t), h(t)])} = \lim_{t \to \infty} ||v(t, \cdot)||_{C([g(t), h(t)])} = 0.
\]

(4.5)

Assume on the contrary that \( h_{\infty} - g_{\infty} > h^* \). Then there exists \( 0 < \varepsilon < 1 \) and \( T \gg 1 \) such that

\[
\begin{align*}
h^\varepsilon_{-} - g^\varepsilon_{+} &= h_{\infty} - g_{\infty} - 2\varepsilon > h^*_\varepsilon, \\
g(T) &< g^\varepsilon_{+}, \quad h(T) > h^\varepsilon_{-}, \\
0 &\leq u(t, x) < \varepsilon, \quad \forall t, x \in [g^\varepsilon_{-}, h^\varepsilon_{+}],
\end{align*}
\]

where \( |\Omega| = h^*_\varepsilon \) is the unique root of \( \lambda_1(-(\tilde{L}_\Omega + c(t) - d(t)\varepsilon)) = 0 \). Then \( v \) satisfies

\[
\begin{aligned}
v_t &\geq d_2 \left[ \tau v_{xx} + (1 - \tau) \left( \int_{g^\varepsilon_{-}}^{h^\varepsilon_{-}} J(x-y)v(t,y)dy - v \right) + v(c(t) - d(t)\varepsilon - v), \\
&\quad t > T, x \in (g^\varepsilon_{+}, h^\varepsilon_{-}),
\right. \\
v(t, g^\varepsilon_{+}) &> 0, \quad v(t, h^\varepsilon_{-}) > 0, \quad t \geq T,
\end{aligned}
\]

\[
\begin{aligned}
v(t, x) &> 0, \quad t \geq T, \quad x \in (g^\varepsilon_{+}, h^\varepsilon_{-}).
\end{aligned}
\]

Let \( \psi \) be the unique positive solution of

\[
\begin{aligned}
\psi_t &= d_2 \left[ \tau \psi_{xx} + (1 - \tau) \left( \int_{g^\varepsilon_{-}}^{h^\varepsilon_{-}} J(x-y)\psi(t,y)dy - \psi \right) + \psi(c(t) - d(t)\varepsilon - \psi), \\
&\quad t > T, x \in (g^\varepsilon_{+}, h^\varepsilon_{-}),
\right. \\
\psi(t, g^\varepsilon_{+}) &= 0, \quad \psi(t, h^\varepsilon_{-}) = 0, \quad t \geq T,
\end{aligned}
\]

\[
\begin{aligned}
\psi(T, x) &= v(T, x), \quad x \in (g^\varepsilon_{+}, h^\varepsilon_{-}).
\end{aligned}
\]

By Lemma 2.2, we have

\[
\psi(t, x) \leq v(t, x), \quad t \geq T, x \in [g^\varepsilon_{+}, h^\varepsilon_{-}].
\]
Since $h_\infty - g^{+}_\infty = h_\infty - g_\infty - 2\varepsilon > h^*_\infty$, we have $\lambda_1(-\tilde{L}(g^{+}_\infty, h_\infty)) < 0$, and then Lemma 3.4 implies that $\psi(t + nT, x) \to \omega(t, x)$ as $n \to \infty$ uniformly in the compact subset of $(g^{+}_\infty, h_\infty)$, where $\omega(t, x)$ is the unique positive periodic solution of

$$
\begin{align*}
\omega_t &= d_2 \left[ \tau \omega_{xx} + (1 - \tau) \left( j^{h^{-}_\infty}_{g^{+}_\infty} J(x - y) \omega(t, y) dy - \omega \right) \right] + \omega(c(t) - d(t)\varepsilon - \omega), \\
& \quad \quad \quad t \in [0, T], x \in (g^{+}_\infty, h_\infty), \\
\omega(t, g^{+}_\infty) &= 0, \quad \omega(t, h_\infty) = 0, \quad t \in [0, T], \\
\omega(0, x) &= \omega(T, x), \quad x \in (g^{+}_\infty, h_\infty).
\end{align*}
$$

Therefore, $\liminf_{n \to \infty} \nu(t + nT, x) \geq \lim_{n \to \infty} \psi(t + nT, x) = \omega(t, x) > 0$ for all $x \in (g^{+}_\infty, h_\infty)$, which is a contradiction to (4.5). This completes the proof.

\[ \square \]

**Corollary 4.2.** If $h_0 \geq \frac{1}{2}h^*$, then spreading occurs, that is, $h_\infty - g_\infty = +\infty$.

If $a_T \geq d_1$, then Lemma 3.3 implies that $\lambda_1(-(L_\Omega + a(t))) < 0$ for all $l := |\Omega| > 0$. Thus, the vanishing can not happen by the proof of Theorem 4.1, which means that $h_\infty - g_\infty = +\infty$ always holds.

**Theorem 4.3.** If $a_T \geq d_1$, then spreading always happens.

On the other hand, if $a_T < d_1$, then Lemma 3.3 implies that $\lambda_1(-(L_\Omega + a(t))) > 0$ for $0 < |\Omega| < 1$, and $\lambda_1(-(L_\Omega + a(t))) < 0$ for $|\Omega| > 1$. Since $\lambda_1(-(L_\Omega + a(t)))$ is strictly decreasing in $|\Omega|$, there exists a $l^* > 0$ such that $\lambda_1(-(L_\Omega + a(t))) = 0$ for $|\Omega| = l^*$, $\lambda_1(-(L_\Omega + a(t))) > 0$ for $|\Omega| < l^*$ and $\lambda_1(-(L_\Omega + a(t))) < 0$ for $|\Omega| > l^*$. From the proof of (4.3), we know that if $h_\infty - g_\infty < +\infty$ then $h_\infty - g_\infty \leq l^*$. Therefore, if $h_0 \geq \frac{l^*}{2}$ then we have $h_\infty - g_\infty = +\infty$.

**Theorem 4.4.** Assume $a_T < d_1$ and $h_0 < \frac{1}{2}\min\{h^*, l^*\}$. If one of the following conditions is satisfied:

(i) $\tau = 1$, (ii) $J(x)$ is equal to a positive constant on $[-2h_0 - 2\delta_0, 2h_0 + 2\delta_0]$ for some small constant $\delta_0 > 0$,

then there exists $\Lambda_0 > 0$ such that $h_\infty - g_\infty < +\infty$ when $\mu + \rho_1 + \rho_2 \leq \Lambda_0$.

**Proof.** Since $\lambda_1(-(L_{(h_0, h_0)} + a(t))) > 0$, we can choose $h_0 < h_1 < \frac{l^*}{2}$ such that $\lambda := \lambda_1(-(L_{(h_1, h_1)} + a(t))) > 0$.

Let $\bar{u}$ be the unique solution of

$$
\begin{align*}
\bar{u}_t &= d_1 \int_{-h_1}^{h_1} J(x - y) \bar{u}(t, y) dy - d_1 \bar{u} + a(t) \bar{u}, \quad t > 0, x \in [-h_1, h_1], \\
\bar{u}(0, x) &= u_0(x), \quad |x| \leq h_0; \quad \bar{u}(0, x) = 0, \quad h_0 < |x| \leq h_1.
\end{align*}
$$

And let $\varphi(t, x)$ be the corresponding eigenfunction associated with $\lambda$ and satisfies $\|\varphi\|_{L^\infty([0, T] \times [-h_1, h_1])} = 1$, that is,

$$
- (L_{(h_1, h_1)} + a(t)) [\varphi] = \lambda \varphi.
$$
Let $\omega(t, x) = Ce^{-\frac{\alpha}{2}t} \varphi(t, x)$ for some $C > 0$, it is easy to check that

$$
\omega_t - d_1 \int_{-h_1}^{h_1} J(x - y) \omega(t, y) dy + d_1 \omega - a(t) \omega
= Ce^{-\frac{\alpha}{2}} \left( \varphi_t - d_1 \int_{-h_1}^{h_1} J(x - y) \varphi(t, y) dy + d_1 \varphi - a(t) \varphi - \frac{\lambda}{2} \varphi \right)
= \frac{1}{2} \lambda Ce^{-\frac{\alpha}{2}} \varphi(t, x) > 0,
$$

for all $t > 0$ and $x \in [-h_1, h_1]$. Choosing $C > 0$ large such that $\omega(0, x) = C \varphi(0, x) > u_0(x)$ on $[-h_1, h_1]$. Applying Lemma 3.3 in [3], we have

$$
\bar{u}(t, x) \leq \omega(t, x) = Ce^{-\frac{\alpha}{2}t} \varphi(t, x) \leq Ce^{-\frac{\lambda}{2}},
$$

for all $t > 0$ and $x \in [-h_1, h_1]$.

On the other hand, since $h_0 < \frac{h_1}{2}$, we can choose a constant $h_2$ satisfying $h_0 < h_2 < \min\{\frac{h_1}{2}, h_1, h_0 + \delta_0\}$ such that $\lambda_1(-\left(\frac{d}{2}L_{(-h_2, h_2)} + c(t)\right)) > 0$. Let $\psi(t, x)$ be the corresponding normalized eigenfunction of $-\left(\frac{d}{2}L_{(-h_2, h_2)} + c(t)\right)$ associated with $\lambda_1(-\left(\frac{d}{2}L_{(-h_2, h_2)} + c(t)\right))$. Note that $\psi_x(t, h_2) < 0, \psi_x(t, -h_2) > 0$ in $[0, T]$. We claim that there exists a constant $\alpha > 0$ such that

$$
-x \psi_x(t, x) \leq \alpha \psi(t, x), \forall (t, x) \in [0, T] \times [-h_2, h_2].
$$

In fact, since $\pm h_2 \psi_x(t, \pm h_2) < 0$, by the continuity of $x \psi_x(t, x)$, we have $x \psi_x(t, x) < 0$ on some interval $[0, T] \times [-h_2, -h_2 + \delta_1] \cup [h_2 - \delta_2, h_2] \subset [0, T] \times [-h_2, h_2]$. Moreover, from the positivity and continuity of $\psi(t, x)$, we know there exists a constant $m > 0$ such that $\psi(t, x) \geq m$ for $(t, x) \in [0, T] \times [-h_2 + \delta_1, h_2 - \delta_2]$. Applying the continuity of $x \psi_x(t, x)$ again, we can choose a constant $\alpha > 0$ large enough such that $x \psi_x(t, x) \leq \alpha m \leq \alpha \psi(t, x)$ on $[0, T] \times [-h_2 + \delta_1, h_2 - \delta_2]$. This shows the claim is true.

Now we define

$$
s(t) = h_2 \varsigma(t), \varsigma(t) = 1 - \frac{\delta}{2} - \frac{\delta}{2} e^{-\sigma t}, \bar{v}(t, x) = ke^{-\sigma t} \psi(\xi(t), \eta(t, x)), \forall (t, x) \in [0, \infty) \times [-s(t), s(t)]
$$

with

$$
\xi(t) = \int_0^t \frac{1}{\varsigma^2(\theta)} d\theta, \eta(t, x) = \frac{h_2}{s(t)} x = \frac{x}{\varsigma(t)},
$$

where $k > 0, \sigma > 0, 0 < \delta < 1 - \frac{h_0}{h_2}$ are positive constants to be determined later. Then $\bar{v}(t, x)$
satisfies
\[
\ddot{v}_t(t, x) - d_2[\tau \ddot{v}_{xx} + (1 - \tau)(\int_{-s(t)}^{s(t)} J(x - y) \ddot{v}(t, y) dy - \ddot{v}(t, x))] - \ddot{v}(t, x)(c(t) - \ddot{v}(t, x)) \\
= ke^{-\sigma t}[\sigma \psi(\xi, \eta) - \frac{\zeta'(t)}{\zeta(t)} \eta \psi(\xi, \eta) \\
+ \frac{d_2(1 - \tau)}{c(t)} \int_{-h_2}^{h_2} J(\eta - \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta} - \zeta(t) \int_{-h_2}^{h_2} J(\zeta(t) \eta - \zeta(t) \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta}] \\
\geq ke^{-\sigma t}[\sigma \alpha + \frac{d_2(1 - \tau)}{c(t)} \int_{-h_2}^{h_2} J(\eta - \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta} - \zeta(t) \int_{-h_2}^{h_2} J(\zeta(t) \eta - \zeta(t) \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta})].
\]

(i) If $\tau = 1$, by the fact $\zeta(t) \to 1$ as $\delta \to 0$, we can choose $0 < \sigma, \delta \ll 1$ such that, for $(t, x) \in [0, \infty) \times (-s(t), s(t))$,
\[
\ddot{v}(t, x) - d_2[\tau v_{xx} + (1 - \tau)(\int_{-s(t)}^{s(t)} J(x - y)v(t, y) dy - v(t, x))] - v(t, x)(c(t) - \ddot{v}(t, x)) \\
\geq ke^{-\sigma t}[\sigma \alpha + \frac{1}{c(t)} \lambda_1 + \frac{1}{c(t)}(\xi - c(t))] \psi(\xi, \eta) \\
> 0.
\]

(ii) If $J(x)$ is equal to a positive constant $K$ for $x \in [-2h_2, 2h_2] \subset [-2h_0 - 2\delta_0, 2h_0 + 2\delta_0]$, then
\[
\int_{-h_2}^{h_2} J(\eta - \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta} - \zeta(t) \int_{-h_2}^{h_2} J(\zeta(t) \eta - \zeta(t) \bar{\eta}) \psi(\xi, \bar{\eta}) d\bar{\eta} \\
= \frac{1}{c(t)} \int_{-h_2}^{h_2} K \psi(\xi, \bar{\eta}) d\bar{\eta} - \zeta(t) \int_{-h_2}^{h_2} K \psi(\xi, \bar{\eta}) d\bar{\eta} \\
= K \left(\frac{1}{c(t)} - \zeta(t)\right) \int_{-h_2}^{h_2} \psi(\xi, \bar{\eta}) d\bar{\eta} > 0.
\]
By the fact $\zeta(t) \to 1$ as $\delta \to 0$, we can also choose $0 < \sigma, \delta \ll 1$ such that
\[
\ddot{v}(t, x) - d_2[\tau v_{xx} + (1 - \tau)(\int_{-s(t)}^{s(t)} J(x - y)v(t, y) dy - v(t, x))] - v(t, x)(c(t) - \ddot{v}(t, x)) > 0
\]
for $(t, x) \in [0, \infty) \times (-s(t), s(t))$.

Moreover, choosing $k$ large enough such that $\ddot{v}(0, x) \leq v_0(x)$ for $x \in [-h_0, h_0]$. Since $s(t) < h_2 < h_1$, we know $\ddot{u}$ satisfies
\[
\ddot{u}_t \geq d_1 \int_{-s(t)}^{s(t)} J(x - y) \ddot{u}(t, y) dy - d_1 \ddot{u} + \ddot{u}(a(t) - \ddot{u}), \quad t > 0, x \in (-s(t), s(t)).
\]
Similarly, we can get 
\[ -\ddot{v}_x(t, s(t)) = -\frac{k}{\zeta(t)}e^{-\sigma t}\psi_0(\xi(t), h_2) \leq \frac{k}{1-\delta}e^{-\sigma t}\|\psi\|_{C^1([0,T] \times [-h_2, h_2])}, \]
\[ \int_{-s(t)}^{s(t)} \int_{s(t)}^{\infty} J(x-y)\ddot{v}(t,x)dydx \leq 2kh_2e^{-\sigma t}, \]
\[ \int_{-s(t)}^{s(t)} \int_{s(t)}^{\infty} J(x-y)\dddot{u}(t,x)dydx \leq 2Ch_2 e^{-\frac{\sigma t}{2}}. \]

Since \( 0 < \sigma \ll 1 \), we may further assume that \( \sigma < \frac{\delta}{2} \). Suppose that 
\[ 0 < \mu + \rho_1 + \rho_2 \leq \frac{h_2 \delta \sigma}{2A} \quad \text{with} \quad A := \max \left\{ \frac{k}{1-\delta}\|\psi\|_{C^1([0,T] \times [-h_2, h_2])}, 2kh_2, 2Ch_2 \right\}, \]
we have 
\[ s'(t) = \frac{1}{2}h_2 \delta \sigma e^{-\sigma t} \geq \frac{k}{1-\delta}e^{-\sigma t}\|\psi\|_{C^1([0,T] \times [-h_2, h_2])} + 2kh_2 \rho_2 e^{-\sigma t} + 2Ch_2 \rho_1 e^{-\sigma t} \]
\[ \geq \frac{k}{1-\delta}e^{-\sigma t}\|\psi\|_{C^1([0,T] \times [-h_2, h_2])} + 2kh_2 \rho_2 e^{-\sigma t} + 2Ch_2 \rho_1 e^{-\frac{\sigma t}{2}} \]
\[ \geq -\mu \ddot{v}_x(t, s(t)) + \rho_1 \int_{-s(t)}^{s(t)} \int_{s(t)}^{\infty} J(x-y)\ddot{u}(t,x)dydx + \rho_2 \int_{-s(t)}^{s(t)} \int_{s(t)}^{\infty} J(x-y)\dddot{u}(t,x)dydx. \]

Similarly, we can get 
\[ -s'(t) \]
\[ \leq -\mu \dddot{u}_x(t, -s(t)) - \rho_1 \int_{-s(t)}^{s(t)} \int_{-\infty}^{-s(t)} J(x-y)\ddot{u}(t,x)dydx - \rho_2 \int_{-s(t)}^{s(t)} \int_{-\infty}^{-s(t)} J(x-y)\dddot{u}(t,x)dydx. \]

This shows that \((\ddot{u}, \dddot{v}, -s(t), s(t))\) is an upper solution of (1.1). Applying Lemma 3.1, we have 
\[ h(t) \leq s(t) \] and \( g(t) \geq -s(t) \), which implies that \( h_\infty - g_\infty \leq 2h_2 < +\infty. \]

**Remark 4.1.** To ensure (4.6) holds, we choose 
\[ s(t) = h_2 \zeta(t) = h_2(1 - \frac{\delta}{2} - \frac{\delta}{2} e^{-\sigma t}) \] with \( \zeta(t) < 1 \), which is slightly different from \( [10, 13] \). Here we only prove the vanishing result for two cases, but whether the other situations still hold true is unknown, we leave it for future research.

**Theorem 4.5.** Assume \( a_T < d_1 \).

(i) If \( h_0 \geq \frac{1}{2}\min\{h^*, l^*\} \), then spreading always occurs;

(ii) If \( h_0 < \frac{1}{2}\min\{h^*, l^*\} \), and one of the following conditions is satisfied:

\( (ii.1) \) \( \tau = 1 \), \( (ii.2) \) \( J(x) \) is equal to a positive constant on \([-2h_0 - 2\delta_0, 2h_0 + 2\delta_0] \) for some small constant \( \delta_0 > 0 \),

then there exist \( \Lambda^* > \Lambda_* > 0 \) such that \( h_\infty - g_\infty < +\infty \) when \( \mu + \rho_1 + \rho_2 \leq \Lambda_* \) and \( h_\infty - g_\infty = +\infty \) when \( \mu + \rho_1 + \rho_2 \geq \Lambda^* \).

**Proof.** (1) If \( h_0 \geq \frac{1}{2}h^* \), then Corollary 4.2 implies that the spreading always occurs. For the case \( h_0 \geq \frac{1}{2}l^* \), if the vanishing happens, then \((g_\infty, h_\infty)\) is a finite interval and its length strictly larger than \( 2h_0 \geq l^* \). Thus, \( \lambda_1(-(L_{(g_\infty, h_\infty)} + a(t))) < 0 \), which is a contraction to (4.3).
(2) From (2.2), we can deduce that
\[ h'(t) > -\mu v_x(t, h(t)), \quad h'(t) > \rho_1 \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J(x-y)u(t,x)dydx, \quad t \geq 0, \]
\[ g'(t) < -\mu v_x(t, g(t)), \quad g'(t) < -\rho_1 \int_{g(t)}^{h(t)} \int_{-\infty}^{\infty} J(x-y)u(t,x)dydx, \quad t \geq 0. \]
Since \( u, v \) are positive and bounded, we know that \( \int_{h(t)}^{g(t)} \int_{-\infty}^{\infty} J(x-y)u(t,x)dydx > 0 \) and there exists a constant \( C > 0 \) such that \( f_1 \geq Cu \) and \( f_2 \geq Cv \). For any given constant \( H > \frac{1}{2} \min\{h^*, l^*\} \), by Lemmas 5.1-5.2 in [31], there exist \( \mu^0 \) and \( \rho^0_1 \) such that \( g_\infty - h_\infty \geq 2H \) for any \( \mu \geq \mu^0 \) or \( \rho_1 \geq \rho^0_1 \). Taking \( \Lambda^0 = \mu^0 + \rho^0_1 \), we know \( g_\infty - h_\infty = +\infty \) for \( \mu + \rho_1 \geq \Lambda^0 \). Applying the continuity method, we can get the desired results. ∎

Combining Theorems 4.2-4.5 and Corollary 4.2, we immediately obtain the following criteria for spreading and vanishing.

**Corollary 4.3.** (Criteria for spreading and vanishing) Let \((u, v, g, h)\) be the unique solution of (1.1), \(|\Omega| = h^*\) and \(|\Omega| = l^*\) be the unique root of \( \lambda_1(-(L_\Omega + c(t))) = 0 \) and \( \lambda_1(-(L_\Omega + a(t))) = 0 \), respectively.

(i) If one of the following conditions is satisfied:

(i.1) \( a_T \geq d_1 \), (i.2) \( h_0 > \frac{1}{2} h^* \), (i.3) \( a_T < d_1 \) and \( h_0 > \frac{1}{2} l^* \),
then spreading happens.

(ii) If \( a_T < d_1 \), \( h_0 < \frac{1}{2} \min\{h^*, l^*\} \) and one of the following conditions is satisfied:

(ii.1) \( \tau = 1 \), (ii.2) \( J(x) \) is equal to a positive constant on \([-2h_0 - 2\delta_0, 2h_0 + 2\delta_0]\) for some small constant \( \delta_0 > 0 \),
then there exist \( \Lambda^* > \Lambda_* > 0 \) such that vanishing happens when \( \mu + \rho_1 + \rho_2 \leq \Lambda_* \) and spreading happens when \( \mu + \rho_1 + \rho_2 \geq \Lambda^* \).

**Acknowledgments**

Chen’s work was supported by NSFC (No:11801432). Li’s work was supported by NSFC (No:11571057). Tang’s work was supported by NSFC (No:61772017). Wang’s work was supported by NSFC (No:11801429) and the Natural Science Basic Research Plan in Shaanxi Province of China (No:2019JQ-136).

**References**

[1] P. Bates, G. Zhao, Existence, uniqueness and stability of the stationary solution to a nonlocal evolution equation arising in population dispersal, J. Math. Anal. Appl. 332 (2007) 428–440.

[2] G. Bunting, Y. Du, K. Krakowski, Spreading speed revisited: analysis of a free boundary model, Netw. Heterog. Media 7 (2012) 583–603.

[3] J.F. Cao, Y. Du, F. Li, W.T. Li, The dynamics of a Fisher-KPP nonlocal diffusion model with free boundaries, J. Funct. Anal. 277 (2019) 2772–2814.
[4] J.F. Cao, W.T. Li, J. Wang, A Lotka-Volterra competition model with nonlocal diffusion and free boundaries, arXiv:1905.09584.

[5] J.F. Cao, W.T. Li, J. Wang, F.Y. Yang, A free boundary problem of a diffusive SIRS model with nonlinear incidence, Z. Angew. Math. Phys. 68 (2017) 68:39.

[6] Q.L. Chen, F.Q. Li, F. Wang, A diffusive logistic problem with a free boundary in time-periodic environment: favorable habitat or unfavorable habitat, Discrete Contin. Dyn. Syst. B 21 (2016) 13–35.

[7] Q.L. Chen, F.Q. Li, F. Wang, A reaction-diffusion-advection competition model with two free boundaries in heterogeneous time-periodic environment, IMA J. Appl. Math. 82 (2017) 445–470.

[8] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, J. Differential Equations 249 (2010) 2921–2953.

[9] W.W. Ding, R. Peng, L. Wei, The diffusive logistic model with a free boundary in a heterogeneous time-periodic environment, J. Differential Equations 263 (2017) 2736–2779.

[10] Y.H. Du, Z.G. Lin, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary, SIAM J. Math. Anal. 42 (2010) 377–405.

[11] Y.H. Du, Z.M. Guo, Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary II, J. Differential Equations 250 (2011) 4336–4366.

[12] Y.H. Du, Z.G. Lin, The diffusive competition model with a free boundary: invasion of a superior or inferior competitor, Discrete Contin. Dyn. Syst. B 19 (2014) 3105–3132.

[13] Y.H. Du, Z.M. Guo, R. Peng, A diffusive logistic model with a free boundary in time-periodic environment, J. Funct. Anal. 265 (2013) 2089–2142.

[14] Y.H. Du, B.D. Lou, Spreading and vanishing in nonlinear diffusion problems with free boundaries, J. Eur. Math. Soc. 17 (2015) 2673–2724.

[15] Y.H. Du, H. Matsuzawa, M.L. Zhou, Sharp estimate of the spreading speed determined by nonlinear free boundary problems, SIAM J. Math. Anal. 46 (1)(2014) 375–396.

[16] Y. Du, M. Wang, M. Zhou, Semi-wave and spreading speed for the diffusive competition model with a free boundary, J. Math. Pures. Appl. 107 (2017) 253–287.

[17] Y. Du, M. Wang, M. Zhao, Two species nonlocal diffusion systems with free boundaries, arXiv:1907.04542.

[18] J. Ge, K. Kim, Z. Lin, H. Zhu, A SIS reaction-diffusion-advection model in a low-risk and high-risk domain, J. Differential Equations 259 (2015) 5486–5509.

[19] H. Gu, Z.G. Lin, B.D. Lou, Different asymptotic spreading speeds induced by advection in a diffusion problem with free boundaries, Proc. Amer. Math. Soc. 143 (2015) 1109–1117.

[20] J.S. Guo, C.H. Wu, On a free boundary problem for a two-species weak competition system, J. Dynam. Differential Equations 24 (2012) 873–895.
[21] J.S. Guo, C.H. Wu, Dynamics for a two-species competition-diffusion model with two free boundaries, Nonlinearity 28 (2015) 1–27.

[22] P. Hess, Periodic-parabolic boundary value problems and positivity, Pitman Res. Notes Math., vol. 247, Longman Sci. Tech., Harlow, 1991.

[23] Y. Kaneko, Spreading and vanishing behaviors for radially symmetric solutions of free boundary problems for reaction-diffusion equations, Nonlinear Anal.: RWA. 18 (2014) 121–140.

[24] C.Y. Kao, Y. Lou, W.X. Shen, Evolution of mixed dispersal in periodic environments, Discrete Contin. Dyn. Syst. B 17 (2012) 2047–2072.

[25] C.X. Lei, Z.G. Lin, Q.Y. Zhang, The spreading front of invasive species in favorable habitat or unfavorable habitat, J. Differential Equations 257 (2014) 145–166.

[26] Z. Lin, H. Zhu, Spatial spreading model and dynamics of West Nile virus in birds and mosquitoes with free boundary, J. Math. Biol. 75 (2017), 1381–1409.

[27] X. Liu, B.D. Lou, On a reaction-diffusion equation with Robin and free boundary conditions, J. Differential Equations 259 (2015) 423–453.

[28] J.L. Ren, D.D.Zhu, On a reaction-advection-diffusion equation with double free boundaries and nth-order Fisher non-linearity, IMA J. Appl. Math. 84 (2019) 197–227.

[29] Z.W. Shen, H. Vo, Nonlocal dispersal equations in time-periodic media: Principal spectral theory, limiting properties and long-time dynamics, J. Differential Equations 267 (2019) 1423–1466.

[30] C.R. Tian, S.G. Ruan, On an advection-reaction-diffusion competition system with double free boundaries modeling invasion and competition of Aedes albopictus and Aedes aegypti mosquitoes, J. Differential Equations 265 (2018) 4016–4051.

[31] J. Wang, M. Wang, Free boundary problems of ecological models with nonlocal and local diffusions, arXiv:1812.11643

[32] M.X. Wang, On some free boundary problems of the Lotka-Volterra type prey-predator model, J. Differential Equations 256 (2014) 3365–3394.

[33] M.X. Wang, A diffusive logistic equation with a free boundary and sign-changing coefficient in time-periodic environment, J. Funct. Anal. 270 (2016) 483–508.

[34] M.X. Wang, Existence and uniqueness of solutions of free boundary problems in heterogeneous environments, Discrete Contin. Dyn. Syst. B 24 (2019) 415–421.

[35] M.X. Wang, Y. Zhang, The time-periodic diffusive competition models with a free boundary and sign-changing growth rates, Z. Angew. Math. Phys. 67 (2016) 132.

[36] M.X. Wang, Y. Zhang, Dynamics for a diffusive prey-predator model with different free boundaries, J. Differential Equations 264 (2018) 3527–3558.

[37] J. Wang, L. Zhang, Invasion by an inferior or superior competitor: A diffusive competition model with a free boundary in a heterogeneous environment, J. Math. Anal. Appl. 423 (2015) 377–398.
[38] C.H. Wu, The minimal habitat size for spreading in a weak competition system with two free boundaries, J. Differential Equations 259 (2015) 873–897.

[39] C.H. Wu, Biased movement and the ideal free distribution in some free boundary problems, J. Differential Equations 265 (2018) 4251–4282.

[40] L. Zhou, S. Zhang, Z.H. Liu, An evolutional free-boundary problem of a reaction-diffusion-advection system, Proc. Roy. Soc. Edinb. Sect. A 147 (2017) 615–648.

[41] P. Zhou, D.M. Xiao, The diffusive logistic model with a free boundary in heterogeneous environment, J. Differential Equations 256 (2014) 1927–1954.