The correlation functions of the \((D_4, A_6)\) conformal model

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Abstract

In this work, we exploit the operator content of the \((D_4, A_6)\) conformal algebra. By constructing a \(Z_2\)-invariants fusion rules of a chosen subalgebra and by resolving the bootstrap equations consistent with these rules, we determine the structure constants of the subalgebra.

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1 Introduction

The biggest advantage in studying two dimensional conformal field theories, is the presence of an infinite amount of symmetries. This fact is very helpful to compute the correlators of the fields present in the theory.

The first approach proposed to reach this goal is the bootstrap approach [1]. Using the coulomb gas formulation, Dotsenko and Fateev have solved, in [2], the bootstrap equations and determined the conformal blocks as well as the structure constants of the operator algebras associated with the minimal models. These models are shown later to form the A-series of the A-D-E classification obtained in [3]. They involve only spinless fields.

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In the case of the models of the D-series, involving also spin fields, many approaches have been proposed for the determination of the structure constants and the correlators (see for example [4], [5], [6], [7] and [8]).

In [5] the authors have determined the structure constants appearing in the three point correlation functions for the particular case of the $(A_4, D_4)$-model or what they call the $D_5$-model. They have used the well known coulomb gas technique adapted to the cases where the correlators involve spin fields. Their results depend on the choice of the $Z_2$-symmetry used to separate the two copies of certain fields present in the theory. Among the four different possibilities, they took only one by imposing the consistency of the results with the bootstrap equations. Their choice has led to the vanishing of one of the structure constants. This fact was not predicted by the non-chiral $Z_2$ invariant fusion rules, but at the same time it does not disagree with a non-zero $Z_3$-invariant 3-pt function of extended fields [10]. The implementation of such discrete symmetries in non diagonal models is discussed in [11].

In our present work, we apply exactly the same method, as in [5], to determine the structure constants of the $(D_4, A_6)$-model. The only difference is that we consider a different $Z_2$-transformation. Our results are consistent with the bootstrap equations and also with the non-chiral $Z_2$ invariant fusion rules.

The paper is organised as follows. In section 2, we start with the operator content of our model. After that, we write the fusion rules and exhibit the different choices for the $Z_2$-symmetries. In section 3, we derive the desired $Z_2$-invariant fusion rules of our operator algebra. In section 4 and 5, we give the essential of the method used to determine the structure constants. In section 6, we apply it to the $(D_4, A_6)$-model and give the conclusion.

2 The conformal $(D_4, A_6)$ Model

The partition functions of unitary conformal field theories defined on the torus were classified in three infinite series: the A-D-E-series [3]. In a generic case, for a modular invariant model corresponding to the minimal one $\mathcal{M}(p = m + 1, p' = m)$ with

$$c = 1 - \frac{6}{m(m + 1)} < 1, \quad m = 2, 3, 4..., \quad (1)$$

the partition function is written as a sesquilinear form in the characters $\chi_h$ ($\chi_{h'}$) of the representations of the left (right) Virasoro algebra generated by the primary fields $\phi_h(z)$ ($\phi_{h'}(\bar{z})$)
\[ Z = \sum_{h,\overline{h} \in E(p,p')} N_{h,\overline{h}} \chi_h \chi_{\overline{h}}, \tag{2} \]

where \( E(p,p') \) is the set of the conformal weights given by the Kac formula

\[ h = h_{r,s} = \frac{[r(m+1) - sm]^2 - 1}{4m(m+1)}, \]
\[ 1 \leq r \leq m-1 ; \quad 1 \leq s \leq m \tag{3} \]

and having the symmetry property

\[ h_{r,s} = h_{p'-r,p-s}. \]

The nonnegative integers \( N_{h,\overline{h}} \) denote the multiplicities of occurrence of the corresponding left-right representation modules (the identity operator \( \Phi_{(1,1),(1,1)} \) being nondegenerate, makes \( N_{h_{1,1},\overline{h}_{1,1}} = N_{0,0} = 1 \).

The characters \( \chi_h \) and \( \chi_{\overline{h}} \) are shown to form a unitary representation of the modular group \( PSL(2,\mathbb{Z}) \). So if we note \( N \) the matrix whose elements are the integers \( N_{h,\overline{h}} \), the problem of finding all possible modular invariant forms of equation (2) is reduced to the problem of finding all the matrices \( N \) commuting with those representing the generators \( S \) and \( T \) of the modular group (in [11] one can find the explicit elements of the matrices \( S \) and \( T \)).

For a given minimal model with \( m \geq 5 \) there is always at least two solutions to this problem. A trivial one with

\[ N_{h,\overline{h}} = \delta_{h,\overline{h}}, \]

corresponds to the diagonal modular invariants models of the A-series

\[ Z_{(A_{p'-1},A_{p'-1})} = \sum_{(r,s) \in E(p,p')} |\chi_{r,s}|^2, \tag{4} \]

here \( \chi_{r,s} = \chi_{h_{r,s}} \cdot \)

The second solution (less trivial) gives the modular invariants models of the D-series. Those associated with the minimal models with \( p' = m = 2(2n+1) \) (to which belongs our model) are

\[ Z_{(D_{p'/2+1},A_{p-1})} = \frac{1}{4} \sum_{r \text{ odd}} \sum_{(r,s) \in E(p,p')} |\chi_{r,s} + \chi_{p'-r,s}|^2. \tag{5} \]

Then for the model with \( m = 6 \), associated to the minimal one \( \mathcal{M}(7,6) \) we have

\[ Z_{(D_4,A_6)} = \sum_{s=1,2,3} |\chi_{1,s} + \chi_{5,s}|^2 + 2 |\chi_{3,s}|^2. \tag{6} \]
The operator content of this model, resumed in table 1, can be read off directly from the last equation. We note that not all the fields of the $\mathcal{M}(7,6)$ minimal model are present, the existence of spinless as well as spin left-right combinations and finally the presence of two copies for each of the spinless combinations $\Phi(3,s|3,s) = \phi(3,s) \otimes \bar{\phi}(3,s)$ for $s = 1, 2, 3$. A third solution exist for some exceptional minimal models, it gives the modular invariants of the $E$-series (see [10] for a complete review on the conformal field theory).

### Table 1: The operator content of the $D_6$ model

| $\Phi(r,s),(r',s')$ | $(h, h)$ | $s = h - h$ | $N_{h,h}$ |
|---------------------|------------------|------------------|------------|
| $\Phi(1,1),(1,1)$   | $(0, 0)$         | 0                | 1          |
| $\Phi(3,1),(3,1)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 2          |
| $\Phi(3,2),(3,2)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 2          |
| $\Phi(5,1),(5,1)$   | $(5, 5)$         | 0                | 1          |
| $\Phi(1,5),(1,5)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 1          |
| $\Phi(1,4),(1,4)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 1          |
| $\Phi(1,3),(1,3)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 1          |
| $\Phi(1,2),(1,2)$   | $(\frac{1}{2}, \frac{1}{2})$ | 0                | 1          |
| $\Phi(5,1),(1,1)$   | $(5, 0)$         | 5                | 1          |
| $\Phi(1,4),(1,3)$   | $(\frac{1}{2}, \frac{1}{2})$ | 1                | 1          |
| $\Phi(1,3),(1,4)$   | $(\frac{1}{2}, \frac{1}{2})$ | -1              | 1          |
| $\Phi(1,5),(1,2)$   | $(\frac{1}{2}, \frac{1}{2})$ | 3                | 1          |
| $\Phi(1,2),(1,5)$   | $(\frac{1}{2}, \frac{1}{2})$ | -3              | 1          |

3 The fusion rules

The set of the primaries and their descendants, of a given minimal model, form a closed algebra with respect to the fusion operation. This is in general an operation among the operators forming representations of chiral algebras inherited from the operator product algebra. In our case and when considering only the "chiral part" of the conformal theory, i.e only the holomorphic part (or in the same manner only the anti-holomorphic part) the fusion between two conformal families is written
\[ \left[ \phi_{r,s} \right] \times \left[ \phi_{r',s'} \right] = \mathcal{N}_{(r,s),(r',s')}^{(r'',s'')} \left[ \phi_{r'',s''} \right] \] (7)

where the fusion coefficients \( \{ \mathcal{N}_{i,j}^k \} \) take the values 1 or 0 to indicate if at least one member of the conformal family \( \left[ \phi_{r,s} \right] \) is present in the fusion of the two families \( \left[ \phi_{r,s} \right] \) and \( \left[ \phi_{r',s'} \right] \). This looks similar to the decomposition of the usual tensor product of representations in terms of the irrep. There is a formula, due to Verlinde [9], expressing these coefficients for the minimal models in terms of the elements of the unitary matrix \( S \) of the modular group

\[ \mathcal{N}_{(r,s),(r',s')}^{(r'',s'')} = \sum_{(k,l) \in E(p,p')} S_{(r,s),(k,l)} S_{(r',s'),(k,l)} S_{(k,l),(r'',s'')} \] (8)

with

\[ S_{(r,s),(r',s')} = 2 \sqrt{\frac{2}{pp'}} (\frac{1}{pp'} + sr' + rs') \sin(\pi \frac{p}{pp'} r r') \sin(\pi \frac{p'}{p} s s') \] (9)

On the operator algebra, the fusion operation is realized by the short distance expansion of the operators product

\[ \Phi_{h_1,h_1}(z, \bar{z}) \Phi_{h_2,h_2}(0,0) = \sum_{h,\bar{h}} \tilde{C}_{(h_1,\bar{h}_1),(h_2,\bar{h}_2),(h,\bar{h})} z^{-h_1+h_2-h} \bar{z}^{-\bar{h}_1+\bar{h}_2-\bar{h}} \Phi_{h,\bar{h}}(0,0) [1 + O(z, \bar{z})] \] (10)

where the contribution of the descendants is contained in \( O(z, \bar{z}) \). The coefficients appearing in the expansion are the structure constants of the operator product algebra. They are of very interest because as we will see later they appear also in the 3-point correlation functions.

For our model, and if we restrict ourselves to the subalgebra containing the operators \( \Phi_{1,1),(1,1) , \Phi_{1,5),(5,1) , \Phi_{5,1),(1,1) , \Phi_{3,1),(3,1) \) and \( \Phi_{5,1),(5,1) \), we have for the chiral fusion rules, i.e. the fusion rules between the holomorphic parts only or the antiholomorphic ones

\[ \left[ \phi_{3,1} \right] \times \left[ \phi_{3,1} \right] = \left[ \phi_{1,1} \right] + \left[ \phi_{3,1} \right] + \left[ \phi_{5,1} \right] \]
\[ \left[ \phi_{5,1} \right] \times \left[ \phi_{5,1} \right] = \left[ \phi_{1,1} \right] \]
\[ \left[ \phi_{3,1} \right] \times \left[ \phi_{5,1} \right] = \left[ \phi_{3,1} \right] \] (11)

Now combining both the left and right part we obtain the non-chiral fusion rules

\[ \left[ \Phi_{3,1),(3,1) \right] \times \left[ \Phi_{3,1),(3,1) \right] = \left[ \Phi_{1,1),(1,1) \right] + \left[ \Phi_{1,1),(5,1) \right] + \left[ \Phi_{5,1),(1,1) \right] \]
It's now easy to verify that the fusion rules. Let's note $\Phi$ the fields will be fixed by imposing a consistency condition on the fusion parities to the two copies. The action of this transformation on the rest of the present in our subalgebra, one introduces a $\mathbb{Z}_2$-symmetry which gives opposite parities to the two copies. The action of this transformation on the rest of the fields will be fixed by imposing a consistency condition on the fusion rules. Let’s note $\Phi^+$ and $\Phi^-$ the two copies and take the transformation

$$\Phi^\pm \rightarrow \pm \Phi^\pm$$

(13)

It’s now easy to verify that the fusion

$$[\Phi(5,1),(1,1)] \times [\Phi(1,1),(5,1)] = [\Phi(5,1),(5,1)]$$

is preserved under the four following distinct transformations:

$$T_1 \quad \Phi(5,1),(5,1) \rightarrow +\Phi(5,1),(1,1) \quad ; \quad \Phi(1,1),(5,1) \rightarrow +\Phi(1,1),(5,1) \quad ;$$
$$\Phi(5,1),(1,1) \rightarrow +\Phi(5,1),(1,1)$$

$$T_2 \quad \Phi(5,1),(5,1) \rightarrow +\Phi(5,1),(5,1) \quad ; \quad \Phi(1,1),(5,1) \rightarrow -\Phi(1,1),(5,1) \quad ;$$
$$\Phi(5,1),(1,1) \rightarrow -\Phi(5,1),(1,1)$$

$$T_3 \quad \Phi(5,1),(5,1) \rightarrow -\Phi(5,1),(5,1) \quad ; \quad \Phi(1,1),(5,1) \rightarrow -\Phi(1,1),(5,1) \quad ;$$
$$\Phi(5,1),(1,1) \rightarrow +\Phi(5,1),(1,1)$$

$$T_4 \quad \Phi(5,1),(5,1) \rightarrow -\Phi(5,1),(5,1) \quad ; \quad \Phi(1,1),(5,1) \rightarrow +\Phi(1,1),(5,1) \quad ;$$
$$\Phi(5,1),(1,1) \rightarrow -\Phi(5,1),(1,1)$$

One can then choose four different $\mathbb{Z}_2$-symmetries. They lead to four different operator algebras, i.e. the fusion rules are contrained differently in each case. But as we will see later, only the results for the structure constants
obtained from the possibility $T_2$ are in total concordance with the fusion rules and the bootstrap equations. This transformation was already used in $[3]$ for the $D_{odd}$ models. This choice is different from that taken in $[5]$, where the possibility $T_1$ was preferred.

With the possibility $T_2$ the fusion rules (12) become:

\[
[\Phi^\pm] \times [\Phi^\pm] = [\Phi^{(1,1),(1,1)}] + [\Phi^{(5,1),(5,1)}] + [\Phi^+] \\
[\Phi^+] \times [\Phi^-] = [\Phi^-] + [\Phi^{(1,1),(5,1)}] + [\Phi^{(5,1),(1,1)}] \\
[\Phi^{(5,1),(5,1)}] \times [\Phi^{(5,1),(5,1)}] = [\Phi^{(1,1),(1,1)}] \\
[\Phi^\pm] \times [\Phi^{(5,1),(5,1)}] = [\Phi^\pm] \\
[\Phi^{(5,1),(1,1)}] \times [\Phi^{(5,1),(1,1)}] = [\Phi^{(1,1),(1,1)}] \\
[\Phi^{(5,1),(1,1)}] \times [\Phi^{(1,1),(5,1)}] = [\Phi^{(5,1),(1,1)}] \\
[\Phi^\pm] \times [\Phi^{(5,1),(1,1)}] = [\Phi^+] \\
[\Phi^\pm] \times [\Phi^{(1,1),(5,1)}] = [\Phi^+] \\
[\Phi^{(1,1),(5,1)}] \times [\Phi^{(5,1),(5,1)}] = [\Phi^{(1,1),(1,1)}] \\
[\Phi^{(5,1),(5,1)}] \times [\Phi^{(1,1),(5,1)}] = [\Phi^{(5,1),(1,1)}] \\
[\Phi^{(5,1),(1,1)}] \times [\Phi^{(5,1),(1,1)}] = [\Phi^{(1,1),(1,1)}] \\
\]

(14)

4 The correlation functions and the structure constants

In a 2-dimensionnal conformal field theory, the form of the two and 3-point correlation functions are fixed only from symmetry considerations. Considering only primary fields, and for a particular normalisation, the two and three point functions are written as

\[
\langle \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \Phi_{h_2,\bar{h}_2}(z_2, \bar{z}_2) \rangle = (-1)^{s_1} \delta_{h_1, h_2} \delta_{\bar{h}_1, \bar{h}_2} \frac{\delta_{z_1 \bar{z}_1}}{z_{12}^{2h_1} \bar{z}_{12}^{2\bar{h}_1}} 
\]

(15)

where $s_1 = h_1 - \bar{h}_1$ is the spin of the field $\Phi_{h_1,\bar{h}_1}$, and

\[
\langle \Phi_{h_1,\bar{h}_1}(z_1, \bar{z}_1) \Phi_{h_2,\bar{h}_2}(z_2, \bar{z}_2) \Phi_{h_3,\bar{h}_3}(z_3, \bar{z}_3) \rangle = \frac{\delta_{\bar{h}_1, \bar{h}_2} \delta_{\bar{h}_1 \bar{h}_3} \delta_{\bar{h}_2, \bar{h}_3} \delta_{z_{12}^{h_1-h_2} z_{23}^{h_2-h_3} z_{13}^{h_3-h_1} \bar{z}_{12}^{h_1-h_2} \bar{z}_{23}^{h_2-h_3} \bar{z}_{13}^{h_3-h_1}}{C_{123}} 
\]

(16)

with $z_{ij} = z_i - z_j$. In the limit $z \to \infty$ (short distance) the 3-point correlation behaves like
This means that the product $\Phi_{h_1, h_1} \times \Phi_{h_2, h_2}$ contains the field $\Phi_{h_3, h_3}$ with strength $\tilde{C}_{123}$. As already mentioned the constants $C_{ijk}$ of the 3-point correlation function are then related to the structure constants of the operator product algebra by

$$C_{123} = (-1)^{s_i} \tilde{C}_{123}$$

The 4-point correlation functions are fixed up to an undermined factor depending on the variable $z = \frac{z_{12}}{z_{13} z_{24}}$

$$G(z_1, ..., z_4) = \langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle = f(z, \bar{z}) \prod_{i<j} z_i^{-h_i} z_j^{-h_j} \bar{z}_{ij}^{\bar{h}_i + \bar{h}_j + \bar{h}/3}$$

where $h = \sum_{i=1}^4 h_i$ and $\bar{h} = \sum_{i=1}^4 \bar{h}_i$.

Using the operator product expansion (10), one can write $G(z_1, ..., z_4)$ in terms of the structure constants and because of the fact that it is not obvious for which pairs of fields we should compute the operator product first (the duality or the crossing symmetry), one obtains strong constraints on the structure constants: the bootstrap equations. In the $s$-channel ($z_{12}, z_{34} \to 0$, or $z \to 0$) one obtains

$$G(z_1, ..., z_4) = \sum_m \frac{(-1)^{s_m} \tilde{C}_{12m} \tilde{C}_{34m} \left[ 1 + O(z_{12}, \bar{z}_{34}) \right]}{z_{12}^{h_1 + h_2 - h_m} z_{34}^{h_3 + h_4 - h_m} z_{13}^{2h_m} z_{24}^{2h_m}}$$

and in the $t$-channel ($z_{14}, z_{23} \to 0$, or $(1 - z) \to 0$)

$$G(z_1, ..., z_4) = \sum_m \frac{(-1)^{s_m} \tilde{C}_{14m} \tilde{C}_{23m} \left[ 1 + O(z_{14}, \bar{z}_{23}) \right]}{z_{14}^{h_1 + h_4 - h_m} z_{23}^{h_3 + h_2 - h_m} z_{13}^{2h_m} z_{24}^{2h_m}}$$

On the other hand as one can always factorize the contribution of each primary into an holomorphic part and an anti-holomorphic one, the 4-point
correlation function can be written in the factorized forms

\[ G(z_1, \ldots, z_4) = \sum_{k,l} A_{kl} F_k(z_1, \ldots, z_4) \overline{F}_l(z_1, \ldots, z_4) \]  

(21)

where the \( F_i \) and \( \overline{F}_i \) are the conformal blocks. In the case where all the fields present in the correlation are symmetric left-right combination (i.e are spinless), the coefficients \( A_{kl} \) are diagonals. This is always the case in the models of the A-series but not for those of the D-series because of the presence of spin fields.

5 The conformal blocks and their integral representation

To determine the structure constants for the \((D_4, A_6)\) model, we use the method first used by Dotsenko in [2] for the models of the A-series and later adapted by McCabe for the \(D_5\) model in [5] and [6]. It consists at first in finding the integral representation of the conformal blocks in the coulomb gas formulation and to write their approximate forms as well as that of the associated 4-point correlation function (21) in both the s- and t- channel. After that one can compare the obtained expressions for \( G(z_1, \ldots, z_4) \) with those given by the equations (19) and (20) respectively and extract the structure constants.

In this section we try to reproduce, with very few details and using the same notation as in [2], the most important steps in the determination of the integral representation of the conformal blocks for the special case of the 4-point correlations containing the field \( \Phi_{(3,1),(3,1)} \) of the form

\[ G(z_1, \ldots, z_4) = \left\langle \Phi_{(m,n),(\overline{m},\overline{n})} \Phi_{(3,1),(3,1)} \Phi_{(3,1),(3,1)} \Phi_{(m,n),(\overline{m},\overline{n})} \right\rangle \]  

(22)

In the coulomb gas approach, the conformal blocks are written as

\[ F_k(z_i) = \left\langle \prod_{i=1}^{3} V_{\alpha_{rs}}(z_i) V_{-\alpha_{rs}+2\alpha_0}(z_4) Q^N Q^M \right\rangle \]  

(23)

where the \( V_{\alpha_{rs}}(z) \) is a vertex operator of charge \( \alpha_{rs} \)

\[ \alpha_{rs} = -\frac{1}{2}(r-1)\alpha_+ + \frac{1}{2}(s-1)\alpha_- \]

and conformal weight

\[ h_{rs} = \alpha_{rs}(\alpha_{rs} - 2\alpha_0) \]

\[ 2\alpha_0 = \alpha_+ + \alpha_- \quad ; \quad \alpha_+\alpha_- = -1 \]
\[ \alpha_+ = \sqrt{\frac{m + 1}{m}} = \frac{\sqrt{7}}{6} \quad ; \quad \alpha_- = -\sqrt{\frac{m}{m + 1}} = -\sqrt{\frac{6}{7}} \]

The operators \( Q_{\pm} \) are the screen operators, they are integrals over closed contour of the vertex \( V_{\alpha_{\pm}} \) of conformal dimension 1

\[ Q_{\pm} = \oint \! dv V_{\alpha_{\pm}} (u) \]

and their exponents \( N \) and \( M \) are fixed by the neutrality condition

\[ \sum_{i=1}^{3} \alpha_{r_{1i}} - \alpha_{r_{4i}} + 2\alpha_0 + N\alpha_+ + M\alpha_- = 0 \]

For the chosen 4-point correlation (22), the last equation is solved for \( N = 0 \) and \( M = 2 \). Noting that

\[ \langle \prod_{i=1}^{K} V_{\alpha_i}(z_i) \rangle = \begin{cases} \prod_{1<j}^{K} (z_i - z_j)^{2\alpha_i\alpha_j} & \text{if } \sum_{i=1}^{K} \alpha_i = 2\alpha_0 \\ 0 & \text{otherwise} \end{cases} \]

the conformal block is then written as

\[ F_k(z_i) = (z_{12} z_{13})^{2\alpha_{mn}\alpha_{31}} (z_{32})^{2\alpha_{31}'} (z_{24} z_{34})^{-2\alpha_{31}'(\alpha_{mn}-\alpha_+ - \alpha_-)} (z_{14})^{-2\alpha_{mn}(\alpha_{mn}-\alpha_+ - \alpha_-)} \]
\[ \times \oint_{C_1(k)} dv_1 \oint_{C_2(k)} dv_2 \left[ (v_1 - z_1) (v_2 - z_1) \right]^{2\alpha_-\alpha_{mn}} \]
\[ \left[ (v_1 - z_2) (v_1 - z_3) (v_2 - z_2) (v_2 - z_3) \right]^{2\alpha_-\alpha_{31}} \]
\[ \left[ (v_1 - z_4) (v_2 - z_4) \right]^{2\alpha_-\alpha_{mn}(\alpha_+ - \alpha_-)} (v_1 - v_2)^{2\alpha_-} \] (24)

A simpler form of this equation is given by

\[ F_k(z_i) = \prod_{i=1}^{4} \left( \frac{d\omega}{dz_i} \right)^{h_i} F_k(0, z, 1, \infty) \]

\[ = (z_{14})^{-2h_{mn}} \left( \frac{z_{14}}{z_{31} z_{24}} \right)^{2h_{31}} z^{2\alpha_{mn}\alpha_{31}} (1 - z)^{2\alpha_{31}'} \]
\[ \times I_k(2\alpha_{mn}\alpha_-, 2\alpha_{31}\alpha_-, 2\alpha_{31}\alpha_-, 2\alpha_-^2, z) \] (25)

with

\[ I_k(a, b, \tilde{C}, g, z) = \oint_{\tilde{C}_1(k)} dv_1 \oint_{\tilde{C}_2(k)} dv_2 f(v_1, v_2; a, b, \tilde{C}, g, z) \] (26)
To obtain (25) one perform the global conformal transformation

\[ z \rightarrow \omega(z) = \frac{z_{34}}{z_{31}} \left( \frac{z - z_1}{z - z_4} \right) \]

which fixes \( z_1, z_3 \) and \( z_4 \) to 0, 1 and \( \infty \) respectively, while \( z_2 \) is transformed into \( z = \frac{z_{12} z_{34}}{z_{13} z_{24}} \).

There is in principle many possibilities for the choice of the integration contours appearing in (25). But because of the fact that the integrand has branch cuts at 0, 1, \( z \) and \( \infty \), it is shown that there is only three independent integration contours (see [2] for more details) leading to three different blocks defined with the following integrals

\[
I_1(a, b, c, g, z) = \int_0^z dv_1 \int_0^{v_1} dv_2 f(v_1, v_2; a, b, c, g, z) \quad (28a)
\]
\[
I_2(a, b, c, g, z) = \int_1^\infty dv_1 \int_1^{v_1} dv_2 f(v_1, v_2; a, b, c, g, z) \quad (28b)
\]
\[
I_3(a, b, c, g, z) = \int_1^\infty dv_1 \int_0^z dv_2 f(v_1, v_2; a, b, c, g, z) \quad (28c)
\]

In the \( s \)-channel (\( z \rightarrow 0 \)) they behave like

\[
I_1(a, b, c, g, z) \simeq N_1 [1 + o(z)] \quad (29a)
\]
\[
I_2(a, b, c, g, z) \simeq N_2 z^{1+a+c} [1 + o(z)] \quad (29b)
\]
\[
I_3(a, b, c, g, z) \simeq N_3 z^{2(1+a+c)+g} [1 + o(z)] \quad (29c)
\]

Now one can write the 4-point correlation function of interest (22) in the \( s \)-channel and compare it with the corresponding one given by (19) to obtain some equations involving the desired structure constants. To do the same in the \( t \)-channel one uses the monodromy properties of the integrals \( I_k \) to transform them into functions of \( (1 - z) \) with the monodromy matrices

\[ I_k(a, b, c, g, z) = \sum_j \gamma_{kj} I_j(b, a, c, g, 1 - z) \]

\([\gamma]\) is the monodromy matrix. It’s general expression as well that of the \( N_k \) are given in [2].
In the particular case where the correlation function studied contains an operator having a null vector at the level one, like \( \Phi_{(1,1), (m,m)}(z, \bar{z}) \) for example, we have the additional constraint

\[
\partial_z \Phi_{(1,1), (m,m)}(z, \bar{z}) = 0
\]

This equation with the usual ones, obtained from the global conformal invariance, permit to obtain a very simple forms of the conformal blocks [5].

6 Application

As an application of the previous algorithm, in this section, explicit values of some structure constants of the subalgebra will be calculated.

6.1 The structure constant \( \tilde{C}_{(51|11), (11|51), (51|51)} \):

It is obtained by considering the following correlation function

\[
G_1 = \langle \Phi_{(11|51)}(1) \Phi_{(51|11)}(2) \Phi_{(51|11)}(3) \Phi_{(11|51)}(4) \rangle \tag{30}
\]

From the bootstrap equations (19) and (20) one can write in the \( s \)-channel :

\[
G_1 = \left( \tilde{C}_{(51|11), (11|51), (51|51)} \right)^2 \left[ 1 + \mathcal{O}(z, \bar{z}) \right] \left( z_{24} \bar{z}_{24} \right)^{10} \tag{31}
\]

and in the \( t \)-channel :

\[
G_1 = \left[ 1 + \mathcal{O}(1 - z, 1 - \bar{z}) \right] \left( z_{23} \bar{z}_{41} \right)^{10} \tag{32}
\]

On the other hand from the integral representation, one finds only one block and can write

\[
G_1 = \frac{A}{\left( z_{23} \bar{z}_{41} \right)^{10}} \tag{33}
\]

when \( z \to 0 \) the last expression behaves like

\[
G_1 = \frac{A \left[ 1 + \mathcal{O}(z, \bar{z}) \right]}{\left( z_{24} \bar{z}_{24} \right)^{10}} \tag{34}
\]

and for \( (1 - z) \to 0 \)
\[ G_1 = \frac{A}{(z_{23} z_{41})^{10}} \] (35)

Comparing the leading terms from the corresponding expressions one find :

\[ (\tilde{C}_{(51|11),(11|51),(51|51)})^2 = 1 \] (36)

6.2 The structure constant \( \tilde{C}_{(51|11)+-} \):

Here we use the function

\[ G_2 = \langle \Phi_{(51|11)}(1)\Phi^+(2)\Phi^+(3)\Phi_{(51|11)}(4) \rangle \] (37)

The bootstrap equations give, in the \( s \)–channel :

\[ G_2 = \frac{- \left( \tilde{C}_{(51|11)+-} \right)^2 [1 + \mathcal{O}(z, \overline{z})]}{(z_{12} z_{34})^5 (z_{24} \overline{z}_{24})^{\frac{5}{3}}} \] (38)

and in the \( t \)–channel :

\[ G_2 = \frac{- [1 + \mathcal{O}(1 - z, 1 - \overline{z})]}{(z_{41})^{10} (z_{23} \overline{z}_{23})^{\frac{5}{3}}} \] (39)

The development in terms of the conformal blocks is given by

\[ G_2 = \sum_{k=1}^{3} - A_k \frac{z^{-5} (1 - z)^{\frac{5}{3}}}{z_{14}^{\frac{5}{3}} (z_{23} z_{31} z_{24})^{\frac{5}{3}}} I_k \left( 5, \frac{-7}{3}, \frac{-7}{3}, \frac{7}{3}; z \right) \] (40)

The blocks \( \tilde{F}_l(z_1, ..., z_4) \) are determined using the fact that the correlation \( G_2 \) involves null state vectors at level one. From the equations (29a), (29b) and (29c) for the behaviour of the \( I_k \)'s when \( z \to 0 \), we have

\[ G_2 = \frac{-z^{-5} [1 + \mathcal{O}(z, \overline{z})]}{z_{14}^{\frac{5}{3}} (z_{23} z_{31} z_{24})^{\frac{5}{3}}} \left( A_1 N_1 + A_2 N_2 z^{\frac{14}{3}} + A_3 N_3 z^{\frac{22}{3}} \right) \] (41)

and a comparison of the both expressions in the \( s \)–channel gives :

\[ - \left( \tilde{C}_{(51|11)+-} \right)^2 = -A_1 N_1 \quad ; \quad A_2 = A_3 = 0 \] (42)
while the same thing when $1 - z \to 0$ yields to

$$G_2 = \frac{-(1 - z)^{\frac{5}{3}} [1 + \mathcal{O}(1 - z, 1 - \bar{z})]}{z_{14} (z_{23} z_{31} z_{24})^{\frac{2}{3}}} \times \{A_1 \gamma_{11} I_1 \left(\frac{-7}{3}, 5, -\frac{7}{3}, \frac{7}{3}; 1 - z \right) + A_1 \gamma_{12} I_2 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right) + A_1 \gamma_{13} I_3 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right)\}$$

$$= -\frac{1}{2} A_1 N_3 (1 - z)^{\frac{5}{3}} \left[1 + \mathcal{O}(1 - z, 1 - \bar{z})\right] \left(\frac{z_{23} z_{31} z_{24}}{z_{14}}\right)^{\frac{2}{3}} \times \{A_1 \gamma_{11} I_1 \left(\frac{-7}{3}, 5, -\frac{7}{3}, \frac{7}{3}; 1 - z \right) + A_1 \gamma_{12} I_2 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right) + A_1 \gamma_{13} I_3 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right)\}$$

$$= -\frac{1}{2} A_1 N_3 (1 - z)^{\frac{5}{3}} \left[1 + \mathcal{O}(1 - z, 1 - \bar{z})\right] \left(\frac{z_{23} z_{31} z_{24}}{z_{14}}\right)^{\frac{2}{3}} \times \{A_1 \gamma_{11} I_1 \left(\frac{-7}{3}, 5, -\frac{7}{3}, \frac{7}{3}; 1 - z \right) + A_1 \gamma_{12} I_2 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right) + A_1 \gamma_{13} I_3 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right)\}$$

$$= -\frac{1}{2} A_1 N_3 (1 - z)^{\frac{5}{3}} \left[1 + \mathcal{O}(1 - z, 1 - \bar{z})\right] \left(\frac{z_{23} z_{31} z_{24}}{z_{14}}\right)^{\frac{2}{3}} \times \{A_1 \gamma_{11} I_1 \left(\frac{-7}{3}, 5, -\frac{7}{3}, \frac{7}{3}; 1 - z \right) + A_1 \gamma_{12} I_2 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right) + A_1 \gamma_{13} I_3 \left(\frac{-7}{3}, 5, -\frac{7}{3}, -\frac{7}{3}; 1 - z \right)\}$$

Finally from the comparison one obtains

$$\left(\tilde{C}_{(51|11)}^{(++)}\right)^2 = 2 \frac{N_1 (5, -\frac{7}{3}, -\frac{7}{3}, \frac{7}{3})}{N_3 (\frac{7}{3}, 5, -\frac{7}{3}, \frac{7}{3})}$$

### 6.3 The structure constant \(\tilde{C}_{+++}\):

Considering the function

$$G_3 = \langle \Phi^+(1) \Phi^+(2) \Phi^+(3) \Phi^+(4) \rangle$$

we have in the \(s\)–channel :

$$G_3 = \left\{ \frac{1}{|z_{12} z_{34}|^{\frac{14}{3}}} + \frac{\left(\tilde{C}_{+++}\right)^2}{\left|z_{12} z_{34}\right|^{\frac{2}{3}} \left(z_{24} z_{24}\right)^{\frac{8}{3}}} + \frac{\left(\tilde{C}_{++(51|51)}\right)^2}{\left|z_{12} z_{34}\right|^{\frac{14}{3}} \left(z_{24} z_{24}\right)^{10}} \right\} \left[1 + \mathcal{O}(z, \bar{z})\right]$$

and in the \(t\)–channel :

$$G_3 = \left\{ \frac{1}{|z_{41} z_{23}|^{\frac{14}{3}}} + \frac{\left(\tilde{C}_{+++}\right)^2}{\left|z_{41} z_{23}\right|^{\frac{2}{3}} \left(z_{13} z_{13}\right)^{\frac{8}{3}}} + \frac{\left(\tilde{C}_{++(51|51)}\right)^2}{\left|z_{41} z_{23}\right|^{\frac{14}{3}} \left(z_{31} z_{31}\right)^{10}} \right\} \left[1 + \mathcal{O}(1 - z, 1 - \bar{z})\right]$$

The factorized form is

$$G_3 = \sum_{k,l=1}^{3} A_{kl} |z (1 - z)|^{\frac{14}{3}} f_k \left(\frac{-7}{3}, -\frac{7}{3}, -\frac{7}{3}, -\frac{7}{3}; z \right) \times \bar{f}_l \left(\frac{-7}{3}, -\frac{7}{3}, \frac{7}{3}, \frac{7}{3}; \bar{z} \right)$$
The comparison between its behavior when \( z \to 0 \) and \( (47) \) gives

\[
A_{33} N_3^2 = 1 \quad (50)
\]
\[
A_{11} N_1^2 = \left( \tilde{C}_{+++} \right)^2 \quad (51)
\]
\[
A_{22} N_2^2 = \left( \tilde{C}_{++} \right)^2 \quad (52)
\]

while that when \( (1 - z) \to 0 \) leads to

\[
1 = \frac{N_2^2}{4} (A_{11} + A_{22} + A_{33}) \quad (53)
\]
\[
\left( \tilde{C}_{+++} \right)^2 = N_2^2 (A_{11} + A_{33}) \quad (54)
\]
\[
\left( \tilde{C}_{++(51|51)} \right)^2 = \frac{N_1^2}{4} (A_{11} + A_{22} + A_{33}) \quad (55)
\]

Resolving the two systems of equations, we find:

\[
\left( \tilde{C}_{+++} \right)^2 = 8 \frac{N_2^2 \left( \frac{-7}{3}, \frac{-7}{3}, \frac{-7}{3}, \frac{7}{3} \right)}{N_1^2 \left( \frac{-7}{3}, \frac{2}{3}, \frac{-7}{3}, \frac{7}{3} \right)} \frac{N_3^2 \left( 5, \frac{-7}{3}, \frac{-7}{3}, \frac{7}{3} \right)}{N_4^2 \left( \frac{-7}{3}, 5, \frac{-7}{3}, \frac{7}{3} \right)} \quad (56)
\]

It is noted here that \( G_3 \) can be calculated as a correlator of the A-series, however we calculated it with non diagonal blocks form and we observed that only non vanishing elements of \( A_{ij} \) are those of the diagonal, this fact is not trivial when considering \( G_3 \) with different action of \( Z_2 \).

A similar calculus with \( \langle \Phi^-(1)\Phi^-(2)\Phi^-(3)\Phi^-(4) \rangle \) gives,

\[
\left( \tilde{C}_{---} \right)^2 = \left( \tilde{C}_{+++} \right)^2
\]

6.4 The sign of the structure constants:

6.4.1 sign of \( \tilde{C}_{+-} \):

Let consider the correlation:

\[
G_4 = \langle \Phi^+(1)\Phi^+(2)\Phi^-(3)\Phi^-(4) \rangle
\]

In the s- and t-channel we have respectively

\[
G_4 = \left\{ \frac{1}{|z_{12}z_{34}|^{12}} + \frac{\tilde{C}_{+++}\tilde{C}_{---}}{|z_{12}z_{34}|^{12} (z_{24} \bar{z}_{24})^{12}} + \frac{\tilde{C}_{++(51|51)}\tilde{C}_{--(51|51)}}{|z_{12}z_{34}|^{12} (z_{24} \bar{z}_{24})^{10}} \right\} \left[ 1 + O(z, \bar{z}) \right]
\]

(57)
\[ G_4 = \left\{ \frac{(\tilde{C}_{+\cdots})^2}{|z_{41}z_{23}|^{\frac{4}{3}} (z_{13}\bar{z}_{13})^{\frac{2}{3}}} - \frac{(\tilde{C}_{+-(51|11)})^2}{(z_{41}z_{23})^{\frac{2}{3}} (z_{13}\bar{z}_{13})^{\frac{1}{3}}} - \frac{(\tilde{C}_{+-(11|51)})^2}{(z_{41}z_{23})^{\frac{2}{3}} (z_{13}\bar{z}_{13})^{\frac{1}{3}}} \right\} \left[ 1 + O(1 - z, 1 - \bar{z}) \right] \]

A comparison with (49) in the s-channel leads to,

\[ A_{22}N_2^2 = 1 \quad (59) \]
\[ A_{31}N_3 = \tilde{C}_{++\cdot} \tilde{C}_{--\cdot} \]
\[ A_{13}N_1N_3 = \tilde{C}_{++(51|51)} \tilde{C}_{--(51|51)} \]

where,

\[ A_{13} = A_{31} \]

and in the t-channel to,

\[ \left( A_{13} + \frac{A_{22}}{2} \right) \frac{N_3^2}{2} = \left( \tilde{C}_{+-} \right)^2 \]
\[ A_{13}N_2^2 = - \left( \tilde{C}_{+-(51|11)} \right)^2 \]
\[ \left( A_{13} + \frac{A_{22}}{2} \right) \frac{N_1^2}{2} = - \left( \tilde{C}_{+-(11|51)} \right)^2 \]

The second equation in (60) gives \( A_{13} < 0 \), and the second one of (59) leads to

\[ \tilde{C}_{--\cdot} = -\tilde{C}_{++\cdot} \]

Finally, we note from the results obtained from \( G_4 \) that if we choose \( \tilde{C}_{++\cdot} \) to be positive (as all the structure constants of the A-series) \( \tilde{C}_{--\cdot} \) will be of negative sign.

### 6.4.2 Sign of \( \tilde{C}_{--(51|51)} \)

From the correlation function

\[ G_5 = \langle \Phi^- (1) \Phi^- (2) \Phi^+ (3) \Phi_{(51|51)} (4) \rangle \]

we have

\[ \tilde{C}_{--\cdot} \tilde{C}_{++(51|51)} = \tilde{C}_{--\cdot} \tilde{C}_{--(51|51)} \]
and if we take $\tilde{C}_{++(51|51)}$ to be positive we obtain

$$\tilde{C}_{++(51|51)} = \tilde{C}_{-(51|51)}$$

(63)

It is then noted here that the calculus of the conformal blocks and the corresponding monodromy matrices for the correlation of type $G_5$ are obtained in the same way as was done for the correlation $^{(22)}$.

### 6.4.3 Sign of $\tilde{C}_{(51|11),(11|51),(51|51)}$

Consider the correlation

$$G_5 = \langle \Phi_{(51|11)}(1)\Phi^+(2)\Phi^+(3)\Phi_{(11|51)}(4) \rangle$$

(64)

the comparison gives

$$-\tilde{C}_{(51|11)}+-\tilde{C}_{(11|51)}+- = \tilde{C}_{(51|11),(11|51),(51|51)} \tilde{C}_{++(51|51)}$$

(65)

The constants structure in l.h.s of the relation above can be chosen to be positives (this not alter the form of the two points correlation). This gives $\tilde{C}_{(51|11),(11|51),(51|51)}$

$$\tilde{C}_{(51|11),(11|51),(51|51)} = -1$$

The values for the remaining structure constants are of course obtained in the same manner. They are resumed in the table 2.

| The structure Constants | Value |
|-------------------------|-------|
| $\tilde{C}_{+-}$        | $-\frac{25 \sqrt{6} \Gamma \left(\frac{5}{3}\right)}{18 \pi^3/2 \Gamma \left(\frac{5}{3}\right)^3}$ |
| $\tilde{C}_{++}$        | $\frac{25 \sqrt{6} \Gamma \left(\frac{5}{3}\right)}{18 \pi^3/2 \Gamma \left(\frac{5}{3}\right)^3}$ |
| $C_{(51|11)++} = C_{(11|51)++}$ | $\frac{2 \sqrt{1173}}{81}$ |
| $C_{++(51|51)} = C_{-(51|51)}$ | $\frac{1564}{2187}$ |
| $C_{(51|11),(11|51),(51|51)}$ | $-1$ |

Table 2: The structure constants
7 Summary

To conclude this work, we would like to notice that the use of the transformation $T_2$ as the $Z_2$-symmetry in our case doesn’t give any inconsistencies with the non chiral fusion rules and the bootstrap equations. We also notice that the relative signs of the structure constants we obtained are in concordance with those obtained in [1].

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