Piecewise Interference and Stability of Branched Flow

Bo Li

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Dated: September 3, 2014)

The defining feature of chaos is its hypersensitivity to small perturbations. However, we report a stability of branched flow against large perturbations where the classical trajectories are chaotic, showing that strong perturbations are largely ignored by the quantum dynamics. The origin of this stability is accounted for by the piecewise nature of the interference, which is largely ignored by the traditional theory of scattering. Incorporating it into our theory, we introduce the notion of piecewise classical stable paths (PCSPs). Our theory shall have implications for many different systems, from electron transport in nanostructures, light propagation in nonhomogeneous photonic structures to freak wave formations in oceans.

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$t_{k+1}$. Nonetheless, $\vec{p}_{k+1}$ is not necessarily the initial momentum at $t_{k+1}$. Instead, the trajectory from $t_{k+1}$ to $t_{k+2}$ can restart with any momentum $\vec{p}_{k+1}$. This difference is better illustrated with a figure, as in Fig. 1(a). Whereas a classical trajectory connecting A and C in Fig. 1(a) is uniquely specified by the initial momentum $\vec{p}_A$ at A, both the initial momenta at A($\vec{p}_A$) and B($\vec{p}_B$) and the position of B($\vec{r}_B$) are needed for a PCT. To find the most stable classical trajectory, one needs to optimize with regards to only $\vec{p}_A$. However, in order to find the most stable PCT, one needs to optimize with regards to $\vec{p}_A$, $\vec{p}_B$ and $\vec{r}_B$. Therefore, the most stable PCT is at least as stable as the most stable classical trajectory. In chaotic systems, all classical trajectories are unstable in the long range, where the long range is defined as large compared with the characteristic length leading to the chaos. On the contrary, PCT has the ability to adjust its momentum at any point to follow the most stable paths. As we add more intermediate points between A and C, the prefactor $\prod_k |\frac{\partial \vec{q}_{k+1}}{\partial \vec{p}_{k}}|^\frac{1}{2}$ for an optimized PCT is exponentially larger than that of the most stable classical trajectory. Consequently, the long-range quantum dynamics is dictated by the least chaotic PCTs, which we name piecewise classical stable paths (PCSPs).

It is interesting to ask how PCSPs arise in such systems and the answer lies in the prefactor, which appears as a result of the stationary phase approximation used to derive VVG. This prefactor summarizes the effect of the constructive interference of neighboring in-phase Feynman Paths and the interference is piecewise. PCSPs arise when the electrons follow a path that consists of many short paths that are continuously boosted by piecewise constructive interference. Consider the simplest case in Figure 1, where the black path is part of one PCSP consisting of two short classical trajectories, one from i to m($pa_{i,m}$) and the other from m to f($pa_{m,f}$). In this case, $|\frac{\partial \vec{q}_{i,m}}{\partial \vec{p}_{i}}|^{-\frac{1}{2}}$ includes the contributions from all the red paths that are in phase with $pa_{i,m}$, while $|\frac{\partial \vec{q}_{m,f}}{\partial \vec{p}_{m}}|^{-\frac{1}{2}}$ includes the contributions from all the blue paths that are in phase with $pa_{m,f}$. The numbers of blue and red paths are intentionally chosen to be different to emphasize that the interference is piecewise. In classical mechanics, each classical trajectory corresponds to a point in the classical phase space spanned by ($\vec{p}$, $\vec{q}$) at any given time. On the contrary, any quantum initial conditions must occupy a region with a finite volume in the phase space due to the uncertainty principle. Quantum dynamics can be modeled based on classical dynamics by assuming each classical trajectory carries a phase. Classical chaos implies that points starting close in phase space will separate from each other exponentially fast in the long range, which is due to the fact that any small deviation in initial conditions will be exponentially magnified over time. As a result, most classical trajectories will make negligible contributions due to their fluctuating phases. However, in the short range, certain regions in phase space, called stable regions, will separate from each other relatively slower compared to the others purely by chance. Points in these stable regions will carry similar phases and will contribute largely to the quantum propagator due to constructive interference. Since quantum dynamics can restart with any momentum at any time, paths consisting of a series of piecewise paths from stable regions will have dominant effect in the long range since by construction, they will have the smallest Lyapunov...
One property of such PCSPs that differs significantly from classical trajectories is its stability against perturbations. Unlike classical chaotic trajectories which are exponentially unstable to small perturbations, PCSPs can actually tolerate a moderate amount of perturbations. In Fig.1c, the black path starting at o ending at e is one PCSP in the absence of perturbations. After perturbations are added to the systems, the red path is one possible way how such PCSP can get recovered and the green and blue paths are two other possible paths that could help recover the original black path. The reason for such recovery lies in PCSPs’ ability to adjust its momentum at any point. Once the perturbed path intersects the original path in the coordinate space, the original path can get recovered. This is very different from classical trajectories. Two classical trajectories with different momentum can intersect in coordinate space, but they will never be merged into the same path due to the difference in momentum. As a matter of fact, two such classical trajectories will separate from each other exponentially fast over time after intersection if the system is chaotic.

To make this difference more concrete, we provide an example where long range stability was not expected from existing theories, but should exist due to the arguments above. We consider the electron flow through a random potential in two dimensions. This system has been intensively studied in the context of Anderson Localization[11] and Universal Conductance Fluctuations[12]. Here, we consider a different regime where the random potential is weakly correlated and chaotic scattering results in the so-called "branched flow"[13]. Previously, it has been shown that classical trajectories combined with semiclassical initial conditions are enough to explain all the observed effects related to branched flow[15][17] and the branched flow pattern simulated using this approach bears close resemblance to the quantum flow pattern[13][15][19]. However, classical trajectories are exponentially unstable to perturbations and the prevailing theory on branching would imply that the branched flow pattern should be sensitive to perturbations. We numerically simulate both the quantum flow patterns and the branched flow pattern using classical trajectories with semiclassical conditions after an area of strong perturbations is introduced into the systems. More information about our numerical methods can be found in the supplementary material. The results are shown in Fig.1. All parameters(Fermi energy $E_F$, wavelength $\lambda_F$, donor to two-dimensional electron gases(2DEGs) distance, sample mobility etc.) are chosen to match that in a previous experiment[14]. The random potential has a standard deviation of 8%$E_F$ and the same correlation length is superimposed on the original random potential. As shown in Fig.1c and Fig.1d, all the branches involving only classical trajectories are destroyed in the long range(long in terms of the correlation length of the random potential). However, for the quantum simulations in Fig.1a and Fig.1b, the branches are only distorted inside and near the region of perturbations and recover themselves in the long range. This observation shall not be confused with that of a previous experiment[14], in which case, classical theory suffices to explain the observed stability[15][20].

In the following, we show how PCSPs can explain this stability. As shown in [22], one can reproduce the experimental flow pattern by carefully constructing an initial wavepacket and propagating it through the scattering region. For such a wavepacket $\Psi(\vec{q}, t_f)$, it is given by

$$
\Psi(\vec{q}, t_f) = \int d\vec{q}_0 G(\vec{q}, t_f; \vec{q}_0, t_0) \Psi(\vec{q}_0, t_0)
$$

Since electrons flow along narrow branches, $\Psi(\vec{q}, t_f)$ can be written as

$$
\Psi(\vec{q}, t_f) = \sum_t \Psi_t(\vec{q}) + \Psi_r(\vec{q})
$$

where each $\Psi_t(\vec{q})$ is a compact wavepacket corresponding to a branch and $\Psi_r(\vec{q})$ is a small residue. Time reversal symmetry implies that

$$
\Psi^*(\vec{q}, t_f) = e^{-iH(t_f-t_0)/\hbar}\Psi^*(\vec{q}, t_f)
$$

$$
= \sum_l e^{-iH(t_f-t_l)/\hbar}\Psi^*_t(\vec{q}) + e^{-iH(t_f-t_0)/\hbar}\Psi^*_r(\vec{q})
$$

(6)

$\Psi^*_t(\vec{q})$ is nothing but a compact wavepacket and $e^{-iH(t_f-t_0)/\hbar}\Psi^*_t(\vec{q})$ is the resulting wavepacket after $t_f - t_0$, which should consist of a new set of compact wavepackets corresponding to the resulting branches that can be written as

$$
e^{-iH(t_f-t_0)/\hbar}\Psi^*_t(\vec{q}) = \sum_{l,s} \Psi^*_{t,l,s}(\vec{q}) + \Psi^*_{r,l}(\vec{q})
$$

(7)

Plugging (7) into (6),

$$
\Psi^*(\vec{q}, t_0) = \sum_{l,s} \Psi^*_{t,l,s}(\vec{q}) + \sum_l \Psi^*_{r,l}(\vec{q}) + e^{-iH(t_f-t_0)/\hbar}\Psi^*_r(\vec{q})
$$

(8)

$\Psi^*(\vec{q}, t_0)$ consists of one or several compact wavepacket at typical experimental temperature[22]. If $\Psi^*_{t,l,s}(\vec{q})$ barely overlaps with $\Psi^*(\vec{q}, t_0)$ in space, its contribution has to be cancelled by other wavepackets and (8) can be further simplified to

$$
\Psi(\vec{q}, t_0) = \sum_{l,s} \Psi_{t,l,s}(\vec{q}) + \Psi_{r,l}(\vec{q})
$$

(9)

where the summation is over wavepackets that spatially overlap with $\Psi(\vec{q}, t_0)$. 
Every PCSP is reversible, so the major contribution to each $\Psi_{l,f}(\vec{q})$ comes from $\Psi_{l,l',p}(\vec{q})$, which, by construction, occupies a more localized region in space than $\Psi(\vec{q},t_0)$.

We now consider how the perturbed region influences each $\Psi_{l,l',p}(\vec{q})$. First of all, we need to remember that the disorder is weak, resulting in only small angle scattering. If we consider the wavepacket $\Psi_{0,\text{pert}}(\vec{q})$ after passing through the perturbed region, it will spatially overlap largely with the one $\Psi(\vec{q},t_0)$ in the absence of perturbations. This can be inferred from our quantum simulations in Fig.1a and Fig.1b and the reason is that small angle scattering are not effective in reducing spatial overlap. However, it can reduces the coherence between the two wavepackets through phase randomization. One can model this phase randomization by changing the momentum of each $\Psi_{l,l',p}(\vec{q})$. That is, we assume $\Psi_{l,l',p}(\vec{q}) \approx f_0(\vec{q} - \vec{q}_l, 0)e^{i\vec{p}_f \cdot (\vec{q} - \vec{q}_f, 0)}$ and the perturbed region changes it to $\Psi_{l,l',p}(\vec{q}) \approx f_0, p(\vec{q} - \vec{q}_l, 0)e^{i\vec{p}_f \cdot (\vec{q} - \vec{q}_f, 0)}$, where both $f_0(\vec{q} - \vec{q}_l, 0)$ and $f_0, p(\vec{q} - \vec{q}_l, 0)$ are real and have large spatial overlap. In the absence of perturbations, assume that the resulting wavepacket at $t_f$ is $\Psi_l(\vec{q}) \approx f_f(\vec{q} - \vec{q}_l, f)e^{i\vec{p}_f \cdot (\vec{q} - \vec{q}_f, f)}$, with $f_f(\vec{q} - \vec{q}_l, f)$ real. The effect of the perturbed region is to change the initial momentum of each wavepacket corresponding to a branch. In free space, a change in the initial momentum will drift wavepackets apart linearly fast with time. However, with PCSPs, the two wavepackets are actually pulled towards each other. To see the reason, we consider the following integral that measures the probability that the perturbed wavepacket will lead to the same original branch.

\[
< \Psi_l | e^{-iHt/\hbar} | \Psi_{l,l',p} >= \int dq_0 dq_f \Psi_{l,l',p}(\vec{q}_0) \Psi^\dagger_l(\vec{q}_f) \times \frac{1}{(2\pi \hbar)^N} \prod_k \left| \frac{\partial \vec{q}_{k+1}}{\partial \vec{p}_k} \right|^{-\frac{i}{2}} \frac{e^{iS_j(\vec{q}_{k+1}, \vec{q}_0, t_{k+1}, f)/\hbar}}{\sqrt{vy_{j,k,k+1}}} (10)
\]

This is by no means a simple integral. Luckily, the prefactor $\prod_k \left| \frac{\partial \vec{q}_{k+1}}{\partial \vec{p}_k} \right|^{-\frac{i}{2}}$ ensures that we only need to consider the neighborhood of PCSPs when $\vec{q}_l, 0$ and $\vec{q}_l, f$ are far apart. Moreover, the integrand is oscillatory, so we focus on stationary phase points, the conditions for which are [23]

\[
\frac{\partial S_j}{\partial \vec{q}_0} + \vec{p}_o = 0 \quad \frac{\partial S_j}{\partial \vec{q}_f} - \vec{p}_f = 0
\]

For PCSPs, Equation (11) is equivalent to

\[
\sum_{k=1}^{N-1} (\vec{p}_k - \vec{p}_f) \frac{\partial \vec{q}_k}{\partial \vec{q}_0} + (\vec{p}_o - \vec{p}_0) = 0 \quad \sum_{k=1}^{N-1} (\vec{p}_k - \vec{p}_f) \frac{\partial \vec{q}_k}{\partial \vec{q}_f} + (\vec{p}_f - \vec{p}_0) = 0
\]

(12)

In the case of a single classical trajectory connecting $\vec{q}_f$ and $\vec{q}_0$, (12) yields $\vec{p}_f = \vec{p}_f$ and $\vec{p}_o = \vec{p}_0$, which is the classical limit as $\hbar \to 0$.

In order to accommodate a moderate change in $\vec{p}_o$, we can make small adjustments to each $\vec{p}_k$ such that (12) still holds. Mathematically, it is given by

\[
\delta \vec{p}_o = \sum_{k=0}^{N-1} \left( \delta \vec{p}_k - \sum_{i>k} \frac{\partial \vec{q}_i}{\partial \vec{p}_k} \delta \vec{q}_i - (\vec{p}_f - \vec{p}_0) + \frac{\partial^2 \vec{q}_f}{\partial \vec{q}_0 \partial \vec{p}_k} \delta \vec{p}_k \right)
\]

(13)

The above argument shows the existence of stationary phase points in the vicinity of PCSPs if $|\vec{q}_f - \vec{q}_0|$ is long compared to the correlation length of the random potential causing the chaos. This implies that even if the two wavepackets have different initial momenta, they can still end up in the same branch. In the example provided above, the simulated region is about seventy correlation lengths, which is considered to be long range in this case.

In summary, we've shown that interference is of piecewise nature in open quantum systems where the classical dynamics is chaotic. This piecewise nature gives rise to a
long range stability in branched flow that challenges the prevailing interpretation of branching. Our theory is an extension to the traditional theory of scattering and the result on the long range stability of branched flow can be tested experimentally in either 2DEGs or photonic systems [24]. Moreover, our theory shall have implications for the wave dynamics in many other open system where the classical ray dynamics is chaotic.

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* Electronic address: bliu@physics.harvard.edu

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