SOME REMARKS ON A RESULT OF JENSEN AND TILTING MODULES 
FOR $\text{SL}_3(k)$ AND $q\text{-GL}_3(k)$

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Abstract. This paper reviews a result of Jensen on characters of some tilting modules for $\text{SL}_3(k)$, where $k$ has characteristic at least five and fills in some gaps in the proof of this result. We then apply the result to finding some decomposition numbers for three part partitions for the symmetric group and the Hecke algebra. We review what is known for characteristic two and three. The quantum case is also considered: analogous results hold for the mixed quantum group where $q$ is an $l$th root of unity with $l$ at least three and thus also hold for the associated Hecke algebra.

Introduction

In this paper, we look at the main result of Jensen [12], the published version of the main result in Jensen’s PhD thesis, supervised by H. H. Andersen. When we were looking at the proof of this result more closely, we realised that there were some unfortunate holes in proof of this result. This paper is an attempt to fill these holes and thus establish the validity of his results. We also consider applications of the main result to the symmetric group and show that analogous results hold for the quantum group and the associated Hecke algebra.

Our proof does not differ significantly from that in [12]. In fact, it uses the same techniques and principles and we do not claim to have any new insights nor to offer anything other than a corrected version of Jensen’s proof, with further applications of the results. Usually, the original author would offer corrections but this author has since left mathematics and the likelihood of an author correction seems slim.

Finding the characters of the tilting modules for $\text{SL}_n(k)$, the special linear group of $n$ by $n$ matrices, where $n$ is a positive integer and $k$ and algebraically closed field of characteristic $p$ is equivalent to finding the characters of the tilting modules for $\text{GL}_n(k)$ (the general linear group) which in turn is equivalent to finding decomposition numbers for the symmetric group. The main result of [12] is a description of the characters of the tilting modules for $\text{SL}_3(k)$ for $p \geq 5$ and when the highest weight lies on the edge of the dominant region and lies in the second $p^2$ alcove away from the origin. Thus we may deduce the decomposition numbers for the symmetric group provided our partitions have at most three parts and whose difference between the first and second part is at most $2p^2$ (approximately). Actually, Jensen does goes past the 2nd $p^2$ alcove slightly and so we may deduce decomposition numbers up to $2p^2 + 2p - 2$. 
We then consider $p = 2$ and $3$ where the information flows the other way — we use the known decomposition numbers for the symmetric group to deduce various tilting modules.

Finally, we consider what may be said about the mixed quantum case and deduce that analogous results hold there also.

1. Notation

We first review the basic concepts and most of the notation that we will be using. The reader is referred to [10] and [14] for further information. This material is also in [11] where it is presented from the group schemes point of view.

Throughout this paper $k$ will be an algebraically closed field of characteristic $p$, $p$ is usually greater than or equal to 5. Let $G = \text{SL}_3(k)$. We take $T$ to be the diagonal matrices in $G$ and $B$ to be the lower triangular matrices. We let $W$ be the Weyl group of $G$ which is isomorphic to the symmetric group on three letters.

We will write $\text{mod}(G)$ for the category of finite dimensional rational $G$-modules. Most $G$-modules considered in this paper will belong to this category. Let $X(T) = X = \mathbb{Z}^2$ be the weight lattice for $G$ and $Y(T) = Y = \mathbb{Z}^2$ the dual weights. The natural pairing $\langle -, - \rangle : X \times Y \to \mathbb{Z}$ is bilinear and induces an isomorphism $Y \cong \text{Hom}_\mathbb{Z}(X, \mathbb{Z})$. We take $R$ to be the roots of $G$. For each $\gamma \in R$ we take $\gamma^\vee \in Y$ to be the coroot of $\gamma$. We set $\alpha = (2, -1)$ and $\beta = (-1, 2)$, with $\alpha, \beta \in R$. Then $R^+ = \{\alpha, \beta, \alpha + \beta\}$ are the positive roots and $S = \{\alpha, \beta\}$ are the simple roots. We also have $\rho = \alpha + \beta = (1, 1)$, which is also half the sum of the positive roots.

We have a partial order on $X$ defined by $\mu \leq \lambda \Leftrightarrow \lambda - \mu \in \mathbb{N}S$. A weight $\lambda$ is dominant if $\langle \lambda, \gamma^\vee \rangle \geq 0$ for all $\gamma \in S$ and we let $X^+$ be the set of dominant weights.

Take $\lambda \in X^+$ and let $k_\lambda$ be the one-dimensional module for $B$ which has weight $\lambda$. We define the induced module, $\nabla(\lambda) = \text{Ind}_B^G(k_\lambda)$. This module has formal character given by Weyl’s character formula and has simple socle $L(\lambda)$, the irreducible $G$-module of highest weight $\lambda$. Any finite dimensional, rational irreducible $G$-module is isomorphic to $L(\lambda)$ for a unique $\lambda \in X^+$. We will denote the socle of a module $M$ by $\text{soc}(M)$.

We may use the transpose matrix map to define an antiautomorphism on $G$. From this morphism we may define $^\circ$, a contravariant dual. It does not change a module’s character, hence it fixes the irreducible modules. We define the Weyl module, to be $\Delta(\lambda) = \nabla(\lambda)^\circ$. Thus $\Delta(\lambda)$ has simple head $L(\lambda)$.

We say that a $G$-module has a good filtration if it has a filtration whose sections are isomorphic to induced modules and we say it has a Weyl filtration if it the sections are isomorphic to Weyl modules. A tilting module is a module with both a good filtration and a Weyl filtration. For each $\lambda \in X^+$ there is a unique indecomposable tilting module $T(\lambda)$, with $[T(\lambda) : L(\lambda)] = 1$ where the square brackets denote the composition multiplicity of $L(\lambda)$ in $T(\lambda)$. A tilting module can
be decomposed as a sum of these indecomposable ones. Note that tilting modules are self-dual $T^\circ(\lambda) \cong T(\lambda)$ and hence that their socles must be isomorphic to their heads.

We return to considering the weight lattice $X$ for $G$. There are also the affine reflections $s_{\gamma, mp}$ for $\gamma$ a positive root and $m \in \mathbb{Z}$ which act on $X$ as $s_{\gamma, mp}(\lambda) = \lambda - ((\lambda, \gamma^\vee) - mp)\gamma$. These generate the affine Weyl group $W_p$. We mostly use the dot action of $W_p$ on $X$ which is the usual action of $W_p$, with the origin shifted to $-\rho$. So we have $w \cdot \lambda = w(\lambda + \rho) - \rho$. Each reflection in $W_p$ defines a hyperplane in $X$. A \textit{facet} for $W_p$ is a non-empty set of the form

$$F = \{ \lambda \in X \otimes \mathbb{R} \mid (\lambda + \rho, \gamma^\vee) = n_\gamma p \ \forall \gamma \in R^+_p(F), \quad (n_\gamma - 1)p < (\lambda + \rho, \gamma^\vee) < n_\gamma p \ \forall \gamma \in R^+_1(F) \}$$

for suitable $n_\gamma \in \mathbb{Z}$ and for a disjoint decomposition $R^+ = R^+_0(F) \cup R^+_1(F)$.

The \textit{closure} $\bar{F}$ of $F$ is

$$\bar{F} = \{ \lambda \in X \otimes \mathbb{R} \mid (\lambda + \rho, \gamma^\vee) = n_\gamma p \ \forall \gamma \in R^+_0(F), \quad (n_\gamma - 1)p \leq (\lambda + \rho, \gamma^\vee) \leq n_\gamma p \ \forall \gamma \in R^+_1(F) \}$$

The \textit{lower closure} of $F$ is

$$\{ \lambda \in X \otimes \mathbb{R} \mid (\lambda + \rho, \gamma^\vee) = n_\gamma p \ \forall \gamma \in R^+_0(F), \quad (n_\gamma - 1)p \leq (\lambda + \rho, \gamma^\vee) < n_\gamma p \ \forall \gamma \in R^+_1(F) \}$$

A facet $F$ is an \textit{alcove} if $R^+_0(F) = \emptyset$, (or equivalently $F$ is open in $X \otimes \mathbb{R}$). If $F$ is an alcove for $W_p$ then its closure $\bar{F} \cap X$ is a fundamental domain for $W_p$ acting on $X$. The group $W_p$ permutes the alcoves simply transitively. We set $C = \{ \lambda \in X \otimes \mathbb{R} \mid 0 < (\lambda + \rho, \gamma^\vee) < p \ \forall \gamma \in R^+ \}$ and call $C$ the \textit{fundamental alcove}.

A facet $F$ is a \textit{wall} if there exists a unique $\beta \in R^+$ with $(\lambda + \rho, \beta^\vee) = mp$ for some $m \in \mathbb{Z}$ and for all $\lambda \in F$.

We will also consider the group $W_{p^2}$, which is generated by $s_{\alpha, 0}$, $s_{\beta, 0}$ and $s_{\alpha, p^2}$. We may also define $p^2$-alcoves and walls using the hyperplanes associated with $W_{p^2}$.

We say that $\lambda$ and $\mu$ are \textit{linked} if they belong to the same $W_p$ orbit on $X$ (under the dot action). If two irreducible modules $L(\lambda)$ and $L(\mu)$ are in the same $G$ block then $\lambda$ and $\mu$ are linked.

We will extensively use \textit{translation functors}.

**Definition 1.1.** Given weights $\lambda, \mu$ in the closure of some alcove $F$, there is a unique dominant weight $\nu$ in $W(\mu - \lambda)$. We define the \textit{translation functor} $T^\mu_\lambda$ from $\lambda$ to $\mu$ on a module $V$ by $T^\mu_\lambda V = \text{pr}_\mu (L(\nu) \otimes \text{pr}_\lambda V)$, where $\text{pr}_\tau V$ is the largest submodule of $V$ all of whose composition factors have highest weights in $W_{p^2} \tau$.

The properties that we require are summarised in [12]. In particular we note the following: translates of tilting modules are also tilting modules. It is this principle that the proof of the main result is based on. The real question becomes how to decompose the translates into their indecomposable summands. Usually we translate away from the origin, then the indecomposable tilting
module of interest is the unique indecomposable summand with highest weight in the dominance order.

One result used [12, Proposition 4.1(ii)] is that the character of a tilting module for \( \text{SL}_3(k) \) is also the sum of characters of tilting modules for \( q\text{-GL}_3(C) \), the quantum group in characteristic zero with \( q \) a \( p \)th root of unity, \( r \geq 1, r \in \mathbb{Z} \). Such “quantum character considerations” will prove crucial in proving the indecomposability of tilting modules for \( \text{SL}_3(k) \). The character of such a tilting module must simultaneously be the character of a (decomposable) tilting module for \( \sqrt[p]{1}\text{-GL}_3(C) \), \( \sqrt[2]{p}\text{-GL}_3(C) \), \( \sqrt[3]{p}\text{-GL}_3(C) \) and so on. As the characters of the tilting modules for \( q\text{-GL}_3(C) \) are known, this gives a lower bound for the possible character of a tilting module for \( \text{SL}_3(k) \).

2. The Base Case

We now assume that \( G = \text{SL}_3(k) \) although much of what we say in this section generalises in an appropriate way to the more general case. Throughout this paper we will draw diagrams of the weight space for \( G \). We will usually have large triangles for the \( p^2 \)-alcoves, and smaller ones for the usual alcoves. We will also draw short lines for walls. We will almost always label \( -\rho \), which is the bottom corner of the fundamental alcove.

When drawing such pictures we are usually drawing the “character” for a tilting module. This means entering the highest weights with multiplicities of the induced modules appearing in a good filtration of the tilting module. It is easy to apply translation functors to such diagrams, using the following proposition.

**Proposition 2.1** (Janzten [11, proposition 7.13]). Let \( \mu, \lambda \in \overset{\circ}{C} \) and \( w \in W_p \) with \( w \cdot \mu \in X^+ \), then \( T^\lambda \nabla(w \cdot \mu) \) has a good filtration. Moreover the factors are \( \nabla(ww_1 \cdot \lambda) \) with \( w_1 \in \text{Stab}_{W_p}(\mu) \) and \( ww_1 \cdot \lambda \in X^+ \). Each different \( ww_1 \cdot \lambda \) occurs exactly once.

We refer to weights which lie on only one hyperplane as wall weights and those which lie on more than one (and hence three as this is \( \text{SL}_3(k) \)) as Steinberg weights. When visualising the weight space we will refer to the subset of dominant weights as the dominant region. Weights which lie close to the edge of this dominant region and which do not have a Steinberg weight lying in the lower closure of the facet containing them are referred to as just dominant weights.

The characters of the indecomposable tilting modules in the bottom \( p^2 \) alcove are well known. They are either the translate of a near-by simple module with highest weight a Steinberg weight, or they can be deduced by decomposing translates (necessary for tilting modules whose highest weight is just dominant.) In all cases the characters coincide with the characters of the indecomposable tilting modules for an associated quantum group in characteristic zero with \( q \) a \( p \)th root of unity.

Once we have a starting set of characters we can produce more indecomposable tilting modules using translation.
Lemma 2.2. Suppose $\lambda$ is not a Steinberg weight and is not just dominant. Let $\sigma$ be the unique Steinberg weight lying in the lower closure of the facet containing $\lambda$. Then

$$T(\lambda) \cong T_\sigma T(\sigma).$$

Proof. This presumably is well known, but can be deduced by generalising the argument in [12, proposition 4.2] or by using Donkin’s tilting tensor product formula [5, proposition 2.1] and the known information about the injective $G_1$-hulls of simple modules for $SL_3(k)$. □

The question really becomes then: What are the characters of the tilting modules whose highest weight is just dominant?

3. The Inductive Step

Throughout this section we will assume that $p \geq 5$. We need $p \geq 3$ in order to apply Andersen’s sum formula [2], and the “inductive step” for $p = 3$ is only needed for two modules, neither of which are difficult to decompose and do not display the generic behaviour seen in the inductive step.

The problem in Jensen’s proof lies in his inductive step ([12, sections 5.3, 5.4]), namely, he doesn’t explicitly state the embedding as part of the inductive hypothesis, and consequently perhaps, this embedding is not proved properly as part of the induction. In particular, summands are removed from the embedding without comment nor justification. We thus consider this part of the proof in greater detail, and take the base case of his induction as given, but take a much loser look at the “minimal embeddings”. We consider the wall case rather than the alcove case as there are less weights involved, thus cutting down the notated weights. The wall version and alcove versions are equivalent by translation and [12, proposition 4.2]. The alcove case may be obtained by translating the wall case off the wall, an indecomposable tiling module remains indecomposable, but the number of weights involved doubles.

We have the following picture (figure 1) as our base case for the induction. This is the wall version of the base case of Jensen’s induction [12, figure 1 (d)], see also figure 5 (a) in the same reference. We have a minimal embedding

$$T(\lambda) \hookrightarrow T(c) \oplus T(d).$$

It is worth clarifying what such a minimal embedding really means. It turns out that the modules of the right hand side of the above equation both have simple socle. It also turns out that they are injective for an associated generalised Schur algebra for a suitable value of the degree of this Schur algebra. Details may be found in [3]. Thus, for a suitable truncation of mod$G$ the module $T(c) \oplus T(d)$ is the injective hull of $T(\lambda)$. Indeed, in the proof of the following theorem, we could truncate mod($G$) to those modules whose composition factors have highest weight in the bottom 4 $p^2$-alcoves. In this category, any indecomposable tilting module whose highest weight is not just
Figure 1. Diagram showing highest weights of the induced modules appearing in a good filtration for $T(\lambda)$. The large triangles are $p^2$-alcoves, the small short lines are $p$-walls and the filled-in circles are the weights in the filtration. The only weight with multiplicity two has a second circle around it.

dominant and is not in the bottom $p^2$-alcove is injective. Since the tilting modules are self-dual, the injective hull of $T(\lambda)$ is also its projective cover. We do not use this fact in the sequel however.

The following theorem differs from that in Jensen in that the embeddings proved as part of the induction are not the ones he used. We can recover the embeddings he used which are in fact minimal, and do so in Proposition 3.2 but we could not see how to get them as part of the proof of Theorem 3.1 but rather as a corollary of it.

**Theorem 3.1.** We have the following cases:

Case (a) Suppose $\lambda$ is just dominant and lies on an $\alpha$-wall then $T(\lambda)$ has character as depicted in figure 2 (a) and has an embedding:

$$T(\lambda) \hookrightarrow T(a) \oplus T(b) \oplus T(c) \oplus T(d)$$

Case (b) Suppose $\mu$ is just dominant and lies on a horizontal wall ($\rho$-wall) then $T(\mu)$ has character as depicted in figure 2 (b) and has an embedding:

$$T(\mu) \hookrightarrow T(e) \oplus T(f) \oplus T(g) \oplus T(h)$$

**Proof.** We prove the result by induction taking the base case depicted in figure 1 as read.

For this part compare with [12, section 5.3]. We assume that we have a tilting module $T(\lambda)$ as depicted in figure 2 (a). We now translate $T(\lambda)$ to the wall containing $\mu$. We get a module $M_1$ which is tilting, but not necessarily indecomposable, which is depicted in figure 3.
Some remarks on a result of Jensen and tilting modules for $\text{SL}_3(k)$ and $\gamma$-$\text{GL}_3(k)$

We translate $M_1$ again to the next wall and call this tilting module $M_2$. This module is depicted in figure 4 (a). Now it is clear that the tilting module with highest weight $j$ may be removed by $\text{SL}_2$ considerations. The dotted line in figure 4 (a) and (b) is the line parallel to $\alpha$ which goes through $\mu$. Restricting to the corresponding Levi subgroup, which is isomorphic to $\text{SL}_2(k)$ and using the results of [6, section 4.2], the multiplicities of the weights along this line are the same as the multiplicities in $\text{SL}_2(k)$. In particular, $\nabla(j)$ is not a component of $T(\eta)$. As $j$ is the next highest weight in $M_2$, this means that $T(j)$ is in fact a direct summand of $M_2$.

We now want to remove two copies of $T(i)$ from $M_2$ if possible. We can do this using Andersen’s sum formula [2]. The value we get for the sum formula for $M_2$ with the summand $T(j)$ removed, is zero and so $T(i)$ occurs with multiplicity two in $M_2$. If we translate back again to $\mu$ then we see that $T(h)$ occurs with multiplicity two in $M_1$. (We can do this by applying [12, lemma 4.7] to the translates of $M_2$ and $M_1$ into the alcoves and noting that translating off a wall into an alcove preserves the number of indecomposable components.)

We thus have that $M_1$ has $T(h) \oplus T(h)$ as a direct summand and $M_2$ has $T(i) \oplus T(i) \oplus T(j)$ as a direct summand. We can therefore write $M_2$ as $Q \oplus T(i) \oplus T(i) \oplus T(j)$. We will return to decompose $Q$ further later, but we claim that $Q = T(\eta) \oplus T(v)$.

Figure 2. Diagram showing highest weights of the induced modules appearing in a good filtration for $T(\lambda)$ and $T(\mu)$. The large triangles are $p^2$-alcoves, the small short lines are $p$-walls and the filled-in circles are the weights in the filtration. The only weight with multiplicity two has a second circle around it. Diagram (a) is the wall version of [12, figure 2(f)] and diagram (b) is the wall version of [12, figure 2(e)].
Figure 3. Diagram showing highest weights of the induced modules appearing in a good filtration of $M_1 := T_\mu T(\lambda)$. The large triangles are $p^2$-alcoves, the small short lines are $p$-walls and the filled-in circles are the weights in the filtration. Multiplicities are indicated by extra concentric circles.

Now the module we get when we remove $T(h) \oplus T(h)$ from $M_1$ is tilting and has the character depicted in figure 2(a). Now a tilting module with such a character must be indecomposable by quantum character considerations. Thus we must have $M_1 = T(\mu) \oplus T(h) \oplus T(h)$ and we have shown that $T(\mu)$ has the desired character. We now consider what the socle of $T(\mu)$ can be.

Now since $T(\mu)$ is self-dual and has a good filtration, we must have

\[ \text{soc} T(\mu) \subseteq \left( \bigoplus (\text{hd} \nabla(\nu))^{(T(\mu), \nabla(\nu))} \right) \bigcap \left( \bigoplus (\text{soc} \nabla(\nu))^{(T(\mu), \nabla(\nu))} \right). \]

We also know that the head of $\nabla(\mu)$ ( = $L(l)$) must appear in the socle of $T(\mu)$ (as it is self-dual) and the socle of the “bottom” $\nabla$ in a good filtration must be in the socle. Thus

\[ L(o) \oplus L(l) \subseteq \text{soc} T(\mu) \subseteq L(h) \oplus L(l) \oplus L(m) \oplus L(n) \oplus L(o), \]

where we use the labelling of figure 3 for the weights.

We now translate the previous embedding. So

\[ T_\mu T(\lambda) \hookrightarrow T_\lambda (T(a) \oplus T(h) \oplus T(c) \oplus T(d)) \]

Thus

\[ T(\mu) \oplus T(h) \oplus T(h) \hookrightarrow T(g) \oplus T(h) \oplus T(h) \oplus T(e) \oplus T(f) \oplus T(h) \oplus T(r) \oplus T(s) \]
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where h and r coincide for the very first application of the inductive step, but are generically different. Hence

\[ T(\mu) \hookrightarrow T(g) \oplus T(e) \oplus T(f) \oplus T(h) \oplus T(r) \oplus T(s) \]

The socles (in order) of the tilting modules on the right hand side are: \( L(o), L(l), L(h), L(n), L(t) \) and \( L(u) \). Thus by intersecting this list and equation (1) we get that

\[ \text{soc } T(\mu) \subseteq L(h) \oplus L(l) \oplus L(n) \oplus L(o) \]

and

\[ T(\mu) \hookrightarrow T(g) \oplus T(e) \oplus T(f) \oplus T(h) \]

(2)

In [12, section 5.3], it is rightly claimed that when we translate the minimal embedding (Jensen’s claimed embedding) of \( T(\lambda) \) to \( T(\mu) \) that we get an embedding

\[ T(\mu) \oplus T(h) \oplus T(h) \hookrightarrow T(e) \oplus T(f) \oplus T(g) \oplus T(h) \oplus T(h) \oplus T(h) \oplus T(r) \]

(corresponding to the \( \theta, \pi, \tau, \alpha \) and \( \psi \) of [12, figure 6] respectively). After removal of two copies of \( T(h) \) (= \( T(\alpha) \)) and \( T(r) \), whose socle does not coincide with the highest weight of \( \nabla \) appearing

Figure 4. (a) Diagram showing highest weights of the induced modules appearing in a good filtration of \( M_2 := T_\eta \mu M_1 \). (b) Diagram showing the module \( Q \) which is \( M_2 \) with \( T(j) \) removed and \( T(i) \) removed twice. In both diagrams, the large triangles are \( p^2 \)-alcoves, the small short lines are \( p \)-walls and the filled-in circles are the weights in the filtration. Multiplicities are indicated by extra concentric circles.
in \( T(\mu) \) we get an embedding

\[
T(\mu) \hookrightarrow T(e) \oplus T(f) \oplus T(g) \oplus T(h)
\]

not \( T(e) \oplus T(f) \oplus T(g) \) as claimed in [12]. There is no justification given for the removal of the extra \( T(h) \). In fact, we have to work harder to remove this extra summand in the next proposition.

We take the embedding in equation (2) and translate it to \( \eta \). We get

\[
T_\mu^\eta T(\mu) = Q \oplus T(j) \hookrightarrow T_\mu^\eta (T(e) \oplus T(f) \oplus T(g) \oplus T(h))
\]

\[= T(w) \oplus T(j) \oplus T(i) \oplus T(x) \oplus T(y) \oplus T(z).\]

Note that \( T_\mu^\eta T(h) = T(i) \). So

\[
Q \hookrightarrow T(w) \oplus T(j) \oplus T(i) \oplus T(x) \oplus T(y) \oplus T(z) \tag{3}
\]

using the labels depicted in figure 4 (b), (corresponding to the \( \eta, \mu, \gamma, \nu, \iota \) and \( \kappa \) of [12, figure 6] respectively). Note that \( j \) may coincide with \( x \) although generically it doesn’t. This embedding is in contrast to the claimed embedding obtained by Jensen in [12, section 5.4]. He gets

\[
Q \hookrightarrow T(\eta) \oplus T(\iota) \oplus T(\kappa) \oplus T(\lambda).
\]

Firstly, the \( \lambda \) should be a \( \nu \). This is not a serious error as \( T(\lambda) \) embeds into \( T(\nu) \). The extra copy of \( T(\mu) \) is missing — possibly due to thinking that the translate of \( T(\pi) \) is \( T(\mu) \) rather than \( T(\mu) \oplus T(\mu) \). He doesn’t have a \( T(\gamma) \) as this comes from the extra \( T(\alpha) \). The remark about the multiplicities and referring to \( \lambda_1 \) (surely not relevant to the induction?) doesn’t make sense, as they already have multiplicity one.

We continue with our own argument by noting that \( T(v) \) embeds in \( T(x) \) so our obtained embedding for \( Q \) is still consistent with our claim that \( Q = T(\eta) \oplus T(v) \).

Now the socle of \( T(i) \) is not the head of any \( \nabla(Q) \) appearing in a good filtration of \( Q \) and so the socle of \( Q \) cannot contain the socle of \( T(i) \). Thus we can remove \( T(i) \) from the right hand side of the previous equation to get,

\[
Q \hookrightarrow T(w) \oplus T(j) \oplus T(x) \oplus T(y) \oplus T(z) =: N
\]

where we have defined \( N \) to be the module given by the right hand side of this equation. This module is depicted in figure 5. (The module \( N \) is analogous to the module \( Q \) in [12, section 5.4] but with an extra \( T(\mu) \) and \( T(\nu) \) in place of \( T(\lambda) \).)

We have \( (N : \nabla(v)) = (Q : \nabla(n)) = 2 \) and so \( \text{Hom}_G(\Delta(v), Q) \cong \text{Hom}_G(\Delta(v), N) \cong k^2 \). We may now repeat the argument of this part of the proof of [12] to show that \( T(v) \) is a summand of \( Q \).

When we remove \( T(v) \) from \( Q \) we get a module which is indecomposable by consideration of quantum characters and we so get a tilting module, which must be \( T(\eta) \), of the desired character.
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Figure 5. Diagram showing the module $N$. In the diagram, the large triangles are $p^2$-alcoves, the small short lines are $p$-walls and the filled-in circles are the highest weights in a good filtration of the module. Multiplicities are indicated by extra concentric circles.

Since $\text{soc} T(v) = \text{soc} T(x)$ and this is distinct from the socles of $T(j)$, $T(w)$, $T(y)$ and $T(z)$ we also get that

$$T(\eta) \hookrightarrow T(w) \oplus T(j) \oplus T(y) \oplus T(z)$$

and this is the desired embedding.

Proposition 3.2. The following maps are minimal embeddings.

$$T(\mu) \hookrightarrow T(e) \oplus T(f) \oplus T(g)$$

and

$$T(\lambda) \hookrightarrow T(a) \oplus T(b) \oplus T(c).$$

Proof. Consider equation (2). It remains to show that $T(h)$ is not needed for this embedding. This is equivalent to showing that the socle of $T(h)$ which is $L(n)$ is not in the socle of $T(\mu)$.

Now $L(n)$ is in the socle of $T(\mu)$ if the socle of $\nabla(n)$ which appears in $T(\mu)$ moves down into the socle of $T(h)$. Since the unique indecomposable extension of $\nabla(o)$ by $\nabla(n)$ has simple socle, this is equivalent to saying that this extension does not embed into $T(\mu)$. Since this extension is isomorphic to $T_{\eta}^{\mu} \nabla(v)$ (see figure 4 for $v$), this is the same as saying that $\nabla(v)$ does not embed in $T_{\eta}^{\mu}T(\mu) = Q \oplus T(j)$. (Recall that $Q$ is the tilting module in figure 4 (b).) But we have shown that $\nabla(v)$ embeds in $Q$ and hence that $L(n)$ is not in the socle of $T(\mu)$. 
Now note that this embedding must be minimal. Firstly we must have $T(e) \oplus T(g)$ on the right hand side as the socle of $T(e) \oplus T(g)$ must be contained in the socle of $T(\mu)$ using equation (1). Also the good filtration of $T(e) \oplus T(g)$ does not contain $\nabla(a')$ which is in $T(\mu)$ and so characters tell us that $T(e) \oplus T(g)$ cannot give us the minimal embedding for $T(\mu)$.

The previous proof showed that $T(\lambda)$ embeds into $T(a) \oplus T(b) \oplus T(c) \oplus T(d)$. We wish to show that $T(d)$ is not required for this embedding. I.e. that the socle of $T(d)$ which $L(b)$ is not in the socle of $T(\lambda)$. Now if $L(b)$ were in the socle of $T(\lambda)$ then we must have

$$T^\mu L(b) \hookrightarrow T^\mu T(\lambda).$$

Now $T^\mu L(b) = L(u)$ which isn’t in the socle of $T^\mu T(\lambda)$ and so $L(b)$ cannot be in the socle of $T(\lambda)$.

Also note that this embedding must be minimal as both the socles of $T(a)$ and $T(b)$ are the weights of “bottom” $\nabla$’s in a good filtration of $T(\lambda)$ and the socle of $T(c)$ is the head of $\nabla(\lambda)$ and so must also be in the socle of $T(\lambda)$.

4. The next $p^2$ alcove.

What happens when $\mu$ or $\lambda$ lies on a $p^2$ wall? We may then proceed as Jensen did to produce pictures of the characters for various tilting modules, so we do not reproduce his results here. We unfortunately get to the same impasse as Jensen in that we cannot prove that his picture in [12, figure 7] is the character of a tilting module. However we have verified the results in [12, figure 3]. So is the following picture the character of an indecomposable tilting module?

![Figure 6. Conjectured character of an indecomposable tilting module](image)

5. Multiplicities for $\text{GL}_n(k)$

In this section we give the filtration multiplicities calculated so far in weight form rather than diagram form and convert them to weights for $\text{GL}_n(k)$. These are easily calculated using the action
of the affine Weyl group and we just summarise the results in the tables below. Since filtration multiplicities for tilting modules are the same as decomposition numbers for the symmetric group, this also gives some three-part decomposition numbers, using the formula \((T(\lambda) : \nabla(\mu)) = [S^\mu : D^\lambda]\). Here \(S^\mu\) is a Specht module and \(D^\lambda\) is a simple module for the symmetric group. See [7] (classical case) and [6, (5) section 4.4] (quantum case) for details.

Let \(a, b \in \mathbb{N}\) with \(0 \leq a \leq p - 2\) and \(0 \leq b \leq p - 2\), and where \(p\) is the characteristic of \(k\). For \(p \geq 5\) we have the following non-zero multiplicities for \((T(\lambda) : \nabla(\mu))\), all other multiplicities are zero.

| \(\mu \setminus \lambda\) | \((p^2 + pa + p - 1 + b, b, 0)\) |
|--------------------------|------------------------------|
| \((p^2 + pa + p - 1 + b, b, 0)\) | 1 |
| \((p^2 + pa + b - 1, p - 1, b + 1)\) | 1 |
| \((p^2 + b - 1, pa + p + b, 0)\) | 1 |
| \((p^2 + b - 2, pa + b, b + 1)\) | 2 |
| \((p^2 + b - 1, pa + p - 1, b + 1)\) | 1 |
| \((p^2 - 2, pa + p + b, b + 1)\) | 1 |
| \((p^2 + b - 1, pa + b, p)\) | 1 |

with the convention that if a weight above is not dominant (i.e. does not have \(\mu_1 \geq \mu_2 \geq \mu_3\) then the multiplicity is zero.

For \(a, b \in \mathbb{N}\) with \(1 \leq a \leq p - 2\) and \(0 \leq b \leq p - 2\). For \(p \geq 5\) we have the following non-zero multiplicities for \((T(\lambda) : \nabla(\mu))\), all other multiplicities are zero.

| \(\mu \setminus \lambda\) | \((p^2 + pa + p - 2, p - 2 + b, 0)\) |
|--------------------------|------------------------------|
| \((p^2 + pa + p - 2, p - 2 + b, 0)\) | 1 |
| \((p^2 + pa + p - 3, p - 1, 0)\) | 1 |
| \((p^2 + pa - 2, p - 1, p - b - 1)\) | 1 |
| \((p^2 + p - 2, pa + p - b - 2, 0)\) | 1 |
| \((p^2 + p - b - 3, pa + p - 1, 0)\) | 1 |
| \((p^2 + p - 2, pa - 1, p - b - 1)\) | 1 |
| \((p^2 - 2, pa + p + b - 1, p - b - 1)\) | 1 |
| \((p^2 + p - b - 3, pa - 1, p)\) | 1 |
| \((p^2 - 2, pa + p - b - 2, p)\) | 1 |

again, with the convention that if a weight above is not dominant then the multiplicity is zero.

The alcove version is as follows: Let \(a, r, s \in \mathbb{N}\) with \(2 \leq a \leq p - 2\) and \(0 \leq r + s \leq p - 3\). For \(p \geq 5\) we have the following non-zero multiplicities for \((T(\lambda) : \nabla(\mu))\), all other multiplicities
are zero.

| $\mu$ \ $\lambda$ | $(p^2 + pa + r + s, s, 0)$ | $(p^2 + pa - p + s - 1, p + r + s + 1, 0)$ |
|-------------------|-----------------------------|---------------------------------------------|
| $\mu_1$          | $(p^2 + pa + r + s, s, 0)$  | 1                                           |
| $\mu_2$          | $(p^2 + pa + s - 1, r + s + 1, 0)$ | 1                                           |
| $\mu_3$          | $(p^2 + pa - p + r + s, p + s, 0)$ | 0                                           |
| $\mu_4$          | $(p^2 + pa - 2, r + s + 1, s + 1)$ | 0                                           |
| $\mu_5$          | $(p^2 + pa - p + r + s, p - 1, s + 1)$ | 1                                           |
| $\mu_6$          | $(p^2 + pa - p + s - 1, p - 1, r + s + 2)$ | 1                                           |
| $\mu_7$          | $(p^2 + pa - p - 2, p + s, r + s + 2)$ | 0                                           |
| $\mu_8$          | $(p^2 + p + s - 1, pa - p + r + s + 1, 0)$ | 0                                           |
| $\mu_9$          | $(p^2 + r + s, pa + s, 0)$ | 1                                           |
| $\mu_{10}$       | $(p^2 + p - 2, pa - p + r + s + 1, s + 1)$ | 1                                           |
| $\mu_{11}$       | $(p^2 + p + s - 1, pa - p - 1, r + s + 2)$ | 0                                           |
| $\mu_{12}$       | $(p^2 + s - 1, pa + r + s + 1, 0)$ | 1                                           |
| $\mu_{13}$       | $(p^2 + r + s, pa - 1, s + 1)$ | 2                                           |
| $\mu_{14}$       | $(p^2 + p - 2, pa - p + s, r + s + 2)$ | 1                                           |
| $\mu_{15}$       | $(p^2 - 2, pa + r + s + 1, s + 1)$ | 1                                           |
| $\mu_{16}$       | $(p^2 + s + 1, pa - 1, r + s + 1)$ | 2                                           |
| $\mu_{17}$       | $(p^2 + r + s, pa - p + s, p)$ | 1                                           |
| $\mu_{18}$       | $(p^2 - 2, pa + s, r + s + 2)$ | 1                                           |
| $\mu_{19}$       | $(p^2 + s - 1, pa - p + r + s + 1, p)$ | 1                                           |
| $\mu_{20}$       | $(p^2 + r + s, pa - p - 1, p + s + 1)$ | 0                                           |
| $\mu_{21}$       | $(p^2 - 2, pa - p + r + s + 1, p + s + 1)$ | 0                                           |

Again, with the convention that if a weight above is not dominant then the multiplicity is zero. The $\text{SL}_3(k)$ picture of the weights $\mu_i$ is depicted in figure 7.

6. Small primes

All the results in the previous sections were for $p \geq 5$. In this section we review some of what is known for $p = 2$ or $p = 3$. For $p \geq 5$ it is easier to calculate the characters of the tilting modules than it is to calculate decomposition numbers for the symmetric group. For small primes the information often flows the other way. Thus in this section we convert the known decomposition numbers for the symmetric group into character diagrams for the tilting modules for $p = 2$.

Explicit decomposition matrices are known for the symmetric group for $p = 2$ up to $n = 18$ and for $p = 3$ up to $n = 17$. These decomposition matrices were found by Jürgen Müller and have been implemented in Gap4, [8].
Figure 7. Diagram indicating the $\text{SL}_3(k)$ weight corresponding to $\mu_i$. In the diagram, the large triangles are $p^2$-alcoves, the small triangles are $p$-alcoves, and these are labelled by the subscript $i$. The dashed lines are lines parallel to $\alpha$ and are meant as an aid to determine which $W_p$ element is used to map one weight to another.

For the prime 3 the results known for 3 part partitions are essentially the same as the results obtained for $p \geq 5$. We can actually push the 3 part decomposition numbers further, up to $n = 22$. We may take the tilting module $T(17,0)$ and continue translating, as in the $p \geq 5$ case. The translates remain indecomposable and so we may obtain the tilting modules (on the edge) up to $n = 22 = 2 \cdot 9 + 2 \cdot 3 - 2$ as in the $p \geq 5$ case.

For prime 2 more interesting things happen — partly because 18 is bigger that $2^4$! Thus the prime two case gives a hint at what may happen for larger primes once we get past the next $p^2$ wall. We include pictures of all the edge cases for prime two up to the $\text{SL}_3$ weight $(18,0)$. (The non-edge cases may be found as for the other primes by translating the appropriate tilting module off a Steinberg weight.) In these pictures we have drawn the $2$-hyperplanes as dashed lines, the $2^2$-hyperplanes as solid lines, the $2^3$-hyperplanes as thicker solid lines and so on. Multiplicities higher than 3 are indicated with numbers rather than numerous concentric circles.

The first 8 pictures may be thought of as degenerate versions of the (wall version) of the $p \geq 5$ result. Once we have highest weight $(10,0)$, however, the patterns change.
Some Remarks on a Result of Jensen and Tilting Modules for $\text{SL}_3(k)$ and $q$-$\text{GL}_3(k)$

Close study of these pictures do reveal some patterns. For instance, the “top” half of the diagram for $T(16,0)$ is the same as all of $T(8,0)$. These type of repetitions are to be expected, as the corresponding Schur algebra often has a quotient which is isomorphic to a smaller Schur algebra. (The quotient result is implicit in [13] and more explicit in [9].) Unfortunately, perhaps two is still too small to give insight into the generic case. Especially given its divergence after $T(10,0)$.

We also see that quantum character arguments are not always enough to show that a module is indecomposable. If we consider the character for $T(15,0)$ we see that this tilting module is not predicted to be indecomposable using a quantum character argument rather character arguments predict that $T(4,1)$ is a summand of this module.

7. The Quantum Case

In this section we consider what we can say about the quantum case. Here we use the Dipper-Donkin quantum group $q$-$\text{GL}_3(k)$ defined in [4]. The interesting case is the so called “mixed case”, the one where $q$ is a primitive $l$th root of unity and $k$ has characteristic $p$. We necessarily have $p > l$. We may generalise our argument from $\text{SL}_3(k)$ to $q$-$\text{GL}_3$ by replacing all occurrences of $p^2$ with $pl$ (and $p^3$ by $p^2l$ and so on). To do this we need to assume that there are alcove weights, so that Andersen’s sum formula is valid. Thus, we need to assume that $l \geq 3$ and hence that $p \geq 5$.

The other key ingredient is the result that the character of a quantum tilting module in the mixed case should also be the sum of characters of tilting modules in characteristic zero with $q$ now a $l$th, $pl$th, $p^2l$th etc, root of unity. This generalisation of the “classical” result is not so straightforward. The proof works the same way, in that we need a more general ring structure on the tilting module which can then be specialised to different rings. Now [1, Section 5.3] shows that a mixed tilting module lifts to the local ring $\mathbb{Z}[v, v^{-1}]_m$ with $m$ being the kernel of the specialisation to an $l$th root in a characteristic $p$ field. Hence the character of a mixed tilting module is a sum of tilting characters over any field which is an algebra over this local ring. This includes for instance the complex numbers made into such an algebra by specialising $v$ to a primitive $l$th root of 1. To get the character result we need $q$ to be a $p^rl$th root of unity. Now if $n = p^rl$ then the cyclotomic polynomial $\Phi_n$ specialises to 0 when $v$ is specialised to a primitive $l$th root of 1 in a characteristic $p$ field. This shows that the complex numbers are in fact an algebra over the relevant local ring. Hence the method indicated does in fact generalise the result of [12, Proposition 4.1(ii)] to the mixed quantum case.

Equipped with these results and all the usual translation theory etc. for $q$-$\text{GL}_3(k)$ we may now obtain the analogous “pictures” of the tilting modules. We need to be a bit careful — really we must work with the analogue of $\text{GL}_3(k)$ and not $\text{SL}_3(k)$. Of course, the connection between $\text{GL}_3(k)$ and $\text{SL}_3(k)$ is in this case really a matter of tensoring with the appropriate power of the determinant module. We could equally well work with the weights for $\text{GL}_3(k)$ and apply translation functors in exactly the same way as for $\text{SL}_3(k)$. In other words, we can be confident that the quantum
analogue of the decomposition numbers in the tables in section 5 are also true with $p^2$ replaced by $pl$ and $p$ replaced by $l$ where $l$ is at least 3 and $p$ is at least 5. These decomposition numbers are thus also decomposition numbers for the corresponding Hecke algebra using the Schur functor.

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