JKLMR conjecture and Batyrev construction

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Received 21 December 2018
Accepted for publication 15 February 2019
Published 22 March 2019

Abstract. We study a mirror interpretation of the relation between the exact partition functions of $N = (2, 2)$ gauged linear sigma-models (GLSM) on $S^2$ and Kähler potentials on the moduli spaces of the CY manifolds proposed by Jockers et al. We use the Batyrev mirror construction for establishing the explicit relation between GLSM and the corresponding mirror family of the Calabi–Yau manifolds, defined as hypersurfaces in weighted projective spaces. We demonstrate how to do this by the explicit calculation in the case of the quintic threefold and its mirror.

Keywords: algebraic structures of integrable models, conformal field theory

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As it was shown in [1] in order to obtain the space-time supersymmetric effective theory in four dimensions it is necessary to compactify the superstring theory on Calabi–Yau (CY) manifolds. Equivalently, it means that the compact sector of the theory has to be described by a $N = 2$ superconformal field theory with the central charge $c = 9$ [2]. The dynamics of the massless sector of this theory is governed by Kähler potentials of the moduli spaces of CY manifold. Therefore for studying the low energy dynamics one has to compute the CY moduli space geometry.

Taking into account this aim we study the mirror version of the recently discovered by Jockers et al [3] relation (JKLMR conjecture) between the Kähler potentials on CY moduli space and the exact partition functions of $N = (2, 2)$ gauged linear sigma-models (GLSM) on $S^5$ [5, 6]. We use the Batyrev mirror construction [12] for fixing the explicit correspondence between GLSM with its gauge group, the set of the chiral superfields and the corresponding mirror family of the Calabi–Yau manifolds $X$, defined as hypersurfaces in weighted projective spaces $\mathbb{P}^d_{k_1, k_2, k_3, k_4, k_5}$. The key point of our approach is as follows. Let a CY hypersurface $X$ be given by zeros of the superpotential

$$W_X(x_1, x_2, x_3, x_4, x_5 | \psi_1, \ldots, \psi_h) = \sum_{a=1}^{h+5} C_a \prod_{i=1}^5 x_i^{v_{ai}}, \quad \sum_{i=1}^5 k_i v_{ai} = d \quad \text{and} \quad d = \sum k_i.$$  

(1)

Here $h$ is equal to the Hodge number $h_{21}$. The coefficients $C_a$ are some functions of the complex structure moduli $\psi_1, \ldots, \psi_h$.

In the Batyrev approach to mirror symmetry the set of the exponents $v_{ai}$ corresponds to the lattice points of the reflexive polytope $\Delta_X$. They are coordinates of the vectors $\vec{V}_a \in \mathbb{R}^5$, that is $v_{ai} = (\vec{V}_a)_i$. These vectors $\vec{V}_a$ being five-dimensional are subject to the linear relations

$$\sum_{a=1}^{h+5} Q_{al} \vec{V}_a = 0, \quad l = 1, \ldots, h,$$  

(2)

where the $Q_{al}$ is a set of integer numbers which corresponds to the integral basis in the linear relations between the exponents of the monomials in $W_X$. That is, every linear relation between $\vec{V}_a$ can be expressed as a sum of the relations (2) with integral coefficients. This condition implies that if $\sum_{l \leq h} Q_{al} m_l \in \mathbb{Z}$ for all $a$, then $m_l \in \mathbb{Z}$ for all $l$.

These vectors $\vec{V}_a$ are the ‘edges’ of the fan that defines the toric manifold $\mathbb{C}^N // (\mathbb{C}^*)^h$, where $N = h + 5$. The symbol $\mathbb{C}^N // (\mathbb{C}^*)^h$ denotes the quotient $\mathbb{C}^N / (\mathbb{C}^*)^h$ where $Z$ is a certain invariant subset. This toric manifold can be provided with a set of projective coordinates $y_1, \ldots, y_N \in \mathbb{C}^N$, which are subject to the identification
\( (y_1, \ldots, y_N) \sim (\lambda^{Q_{1l}}y_1, \ldots, \lambda^{Q_{Nl}}y_N), \quad l = 1, \ldots, h. \) (3)

Then the Calabi–Yau manifold \( Y \), which is the mirror of \( X \), is realized as a subspace in the toric manifold \( \mathbb{C}^N/(\mathbb{C}^*)^h \) defined by the critical locus of a polynomial \( W_Y(y_1, \ldots, y_N) \) which is weighted homogeneous with respect to all \( h \) dilations

\[ W_Y(\lambda^{Q_{1l}}y_1, \ldots, \lambda^{Q_{Nl}}y_N) = W_Y(y_1, \ldots, y_N), \quad l = 1, \ldots, h. \] (4)

The monomials

\[ z_j = \prod_{a=1}^{N} y_a^{v_{aj}}, \]

invariant with respect to the group action, can be taken as natural coordinates \((z_1, \ldots, z_{N-h})\) on the toric manifold \( \mathbb{C}^N/(\mathbb{C}^*)^h \).

Thus \( W_Y \) can be written as their linear combination

\[ W_Y(y_1, \ldots, y_N) = \sum_{j=1}^{5} C_j \prod_{a=1}^{N} y_a^{v_{aj}}. \] (5)

The manifold \( Y \) given by the equation \( W_Y = 0 \) corresponds to the reflexive polytope \( \Delta_Y \) that is polar dual to the polytope \( \Delta_X \). Note that the matrix of the exponents in (5) is the transposed of the exponents matrix in (1)\(^6\).

On the other hand, it was shown by Witten [9] that Calabi–Yau manifolds of such type arise as supersymmetric vacua manifolds in two-dimensional \( N = (2,2) \) GLSM and the weights \( Q_{al} \) are just the charges of \( N \) chiral superfields \( (\Phi_1, \ldots, \Phi_N) \) under the gauge groups \( U(1)_l \) where the subscript indicates the particular \( U(1) \)-component of the gauge group. Thus, knowing the family of CY manifolds \( X \) defined as hypersurfaces in a weighted projective space by a family of polynomials \( W_Y \), we can find the corresponding \( N = (2,2) \) gauged linear sigma-model. We can also find the set of its integer charges \( \{ Q_{al} \} \) as such solution of the equation (2) which gives the integral basis of the linear relations between the set \( \tilde{v}_a \).

Here we shortly recollect the connection [9] between the supersymmetric vacua in GLSM and the hypersurfaces in toric manifolds. Consider the \( U(1)^{h} = \prod_{i=1}^{h} U(1)_i \) \( N = (2,2) \) gauge model with \( h \) vector superfields and \( N \) chiral matter superfields \( (\Phi_1, \ldots, \Phi_N) \) whose charges under \( U(1)_i \) are denoted by \( (Q_{il}, \ldots, Q_{Ni}) \). The Lagrangian of the model [9] also depends on the coupling constants \( (e_1, \ldots, e_h) \) and on Fayet–Iliopoulos (FI) parameters \( r_l, \ l = 1, \ldots, k \), the theta angles and the superpotential \( W_Y \).

The potential term for the scalar fields in this Lagrangian is

\[ U(\phi) = \sum_{l=1}^{h} \frac{e_l^2}{2} \left( \sum_{a=1}^{N} Q_{al}|\phi_a|^2 - r_l \right)^2 + \frac{1}{4} \sum_{a=1}^{N} \left| \frac{\partial W_Y}{\partial \phi_a} \right|^2, \] (6)

where \( \phi_1, \ldots, \phi_N \) are the scalar components of the chiral superfields \( \Phi_1, \ldots, \Phi_N \). The supersymmetric ground states of the theory are parametrized by the minima of (6) modulo gauge equivalences. For \( r_l > 0 \) this space of vacua defines a manifold

\(^6\) This connection reminds the Berglund–Hubsch mirror construction [7, 8].
\[ Y_r = \left\{ (\phi_1, \ldots, \phi_N) \mid \sum_{a=1}^{N} Q_{al} |\phi_a|^2 = r_l, \ l = 1, \ldots, h, \ \frac{\partial W_Y}{\partial \phi_a} = 0 \right\} / U(1)^h. \] (7)

If we send all coupling constants \( e_l \) to infinity all massive modes decouple and the model effectively reduces to the \( N = (2, 2) \) non-linear sigma model with the target space \( Y_r \).

It can be shown that the critical locus of \( W_Y \) in the toric manifold \( \mathbb{C}^N/\mathbb{C}^* \) and \( Y_r \) are equivalent as symplectic manifolds [11]. This construction establishes a one-to-one correspondence between the hypersurfaces \( Y \) in toric manifolds and the GLSM.

It was shown in [5, 6] that GLSM can be put on the two-sphere while preserving the \( N = (2, 2) \) supersymmetry. The \( N = (2, 2) \) supersymmetry allows one to compute the partition function of this theory exactly, using the supersymmetric localization technique. It was done in the papers [5, 6]. The result is given by the Mellin–Barnes type integral

\[ Z_Y = \sum_{m \in \mathbb{Z}} \prod_{l=1}^{h} e^{-i\theta_l m_l} \int_{C_l} \ldots \int_{C_h} \frac{d\tau_l}{(2\pi i)} e^{4\pi i \tau_l} \prod_{a=1}^{N} \frac{\Gamma \left( \frac{q_a/2 + \sum_{l=1}^{h} Q_{al}(\tau_l - m_l/2)}{2} \right)}{\Gamma \left( 1 - \frac{q_a}{2} - \sum_{l=1}^{h} Q_{al}(\tau_l + m_l/2) \right)}, \] (8)

where the contours \( C_l \) go along the imaginary axis. The precise choice of the contours \( C_l \) depends on the R-charges \( q_a \) (see [5, 6]). The partition function (8) does not depend either on the coupling constants \( (e_1, \ldots, e_h) \) or on the details of the polynomial \( W_Y \). The independence of the partition function on the coupling constants means that we can send \( e_l \) to infinity from the very beginning, as it was done by Witten in [9]. Thus, effectively \( Z_Y \) picks up only the contribution of the massless fields and gives the exact partition function of the non-linear sigma-model.

It was conjectured by Jockers \textit{et al} in [3] that the exact expression for the sphere partition function (8) coincides with the \( \mathbb{C}^N/\mathbb{C}^* \) on the quantum Kähler moduli space for the Calabi–Yau manifold \( Y \). This conjecture was verified in [3] for a few examples (see a physical proof in [4]). The problem with this check is a lack of a simple definition of the function \( K_Y \). Therefore, instead of doing this, we adopt a different approach based on mirror symmetry and on the Batyrev approach to it. The mirror symmetry implies an equality

\[ K_Y^X = K_X^Y, \quad e^{-K_X^Y} = -i \int_X \Omega \wedge \bar{\Omega}, \] (9)

where \( K_X^Y \) is the Kähler potential for complex structure moduli of the mirror manifold \( X \). The last potential has a transparent definition in terms of integral of a uniquely (up to a function) defined holomorphic form \( \Omega \) on \( X \). The manifold \( X \) is canonically related to the manifold \( Y \) by the mirror map [12] realized as a hypersurface in toric manifold. In these terms the conjecture [3] can be reformulated [10] as

\[ \int_X \Omega \wedge \bar{\Omega} = iZ_Y. \] (10)

We will call this the mirror version of JKLMR conjecture.

https://doi.org/10.1088/1742-5468/ab081a
Here we give a direct check of the conjecture for the Quintic threefold. In this case the CY $X$ is defined as a hypersurface in the projective space $\mathbb{P}^4$, that is a set of five complex coordinates $(x_1, \ldots, x_5)$ identified by

$$(x_1, x_2, x_3, x_4, x_5) \sim (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4, \lambda x_5).$$

(11)

The hypersurface $X \in \mathbb{P}^4$ is given by zeros of the superpotential

$$ W(x_1, x_2, x_3, x_4, x_5|\psi_1, \ldots, \psi_h) = \sum_{i=1}^{5} x_i^5 + \sum_{l=1}^{101} \psi_l e_l(x),$$

(12)

where

$$ e_l(x) = \prod_{i=1}^{5} x_i^{s_{li}}, \quad \text{with} \quad 0 \leq s_{li} \leq 3, \quad h = 101 \quad \text{and} \quad \sum_{i=1}^{5} s_{li} = 5. $$

The parameters $\psi_l$ represent deformations of the complex structure of the manifold $X$. The manifold $Y$, which is mirror to $X$, can be realized as a subspace in the toric manifold defined as the quotient $\mathbb{C}^{h+5}/(\mathbb{C}^*)^h$. The action of the group $(\mathbb{C}^*)^h$ is given by the matrix $Q_{al}$, as was explained above. Batyrev’s construction tells us how to find this matrix.

Namely, the vectors of exponents $\tilde{v}_{ai}$ in (12)

$$ \tilde{v}_{ai} = \begin{cases} 5\delta_{a,i}, & 1 \leq a \leq 5, \\ s_{a-5,i}, & 6 \leq a \leq h + 5. \end{cases} $$

(13)

The vectors of exponents $\tilde{v}_{ai}$ are simply connected with vectors $v_{ai}$ which span the one-dimensional part of the fan of the manifold $\mathbb{C}^{h+5}/(\mathbb{C}^*)^h$. Namely,

$$ v_{ai} = \tilde{v}_{ai} - 1. $$

As it was explained above one can find the charges $Q_{al}$ of the chiral superfields in the corresponding GLSM from the linear dependencies between these vectors

$$ \sum_{a=1}^{106} Q_{al} \cdot v_{ai} = 0. $$

(14)

We can use in (14) the matrix $\tilde{v}_{ai}$ instead of $v_{ai}$ using the additional property of the charges $Q_{al}$

$$ \sum_{a=1}^{106} Q_{al} = 0. $$

(15)

The convenient choice of solutions for this equation is

$$ Q_{al} = \begin{cases} s_{la}, & 1 \leq a \leq 5, \\ -5\delta_{a-5,l}, & 6 \leq a \leq 106. \end{cases} $$

These relations are manifestations of the fact that any monomial can be expressed as a product of powers of $x_i$. 

https://doi.org/10.1088/1742-5468/ab081a
We note, however, that this choice of \( \tilde{Q}_{al} \) does not give the integral basis \( Q_{al} \) in the lattice of all possible linear relations among \( \{v_i\}_{i\leq 106} \). That is, not all integral linear relations between \( v_{al} \) can be obtained as sums of \( \tilde{Q}_{al} \) with integer coefficients.

For instance, consider \( s_2 = (3,2,0,0,0) \), \( s_3 = (2,3,0,0,0) \), which correspond to monomials \( x_1^3x_2^2 \) and \( x_1^2x_2^3 \). Then we see, that \( Q_{a2} + Q_{a3} = (5,5,0,0,0,−5,−5,0,\ldots,0) \) is a linear relation between the vectors \( v_{al} \). This relation is divisible by 5 and the resulting relation \( (1,1,0,0,0,−1,−1,0,\ldots,0) \) cannot be obtained from integral linear combinations of \( \tilde{Q}_{al} \).\(^7\)

However, the choice of \( \tilde{Q}_{al} \) instead of \( Q_{al} \) is very convenient in computations. Although \( \tilde{Q}_{al} \) does not give the correct charge matrix for our GLSM, it still can be used with one remark.

In the formula (8) \( m_l \in \mathbb{Z} \), which is a consequence of the quantization condition for magnetic fluxes [5]. The actual integral basis of linear relations between \( v_{al} \) is given by the matrix \( Q_{al} = \tilde{Q}_{ak}B_l^k \) for an invertible \( 101 \times 101 \) matrix \( B_l^k \). In the formula (8) \( m_l \) run through all possible integers.

If we plug in the charge matrix \( Q_{al} = \tilde{Q}_{ak}B_l^k \) in the formula (8), then we can define \( \tilde{m}_l := m_k(B^{-1})^k_l, \tilde{r}_l = r_k(B^{-1})^k_l \), the similar definition for \( \tilde{\theta}_l := \theta_kB_l^k \) and substitute the expressions into (8) to get rid of the matrix \( B \).

In the obtained formula the set of all possible \( \tilde{m}_l \) is not all integers, but can be described as a set of all rational numbers such that for all \( a \)

\[
\sum_{l \leq 101} \tilde{Q}_{al}\tilde{m}_l \in \mathbb{Z}.
\]

This follows from the remark after the formula (2). Below for simplicity we will omit the sign ‘tilde’ in new notations \( \tilde{m}_l, \tilde{V} \) and \( \tilde{Q}_{al} \).

We see from (15) that \( \sum_{a=1}^{106} Q_{al} = 0 \). Therefore, as explained in [9] the FI parameters and the theta parameters do not run with RG flow and the full \( U(1) \) axial symmetry is unbroken. The FI and theta parameters \( r_l \) and \( \theta_l \) remain arbitrary parameters of the quantum (not only classical) theory.

The superpotential \( W_Y \) can be written in terms of the invariant coordinates as it is presented in the formula (5). It is convenient to introduce the new notations for the 106 chiral matter fields \( (\Phi_1, \ldots, \Phi_{106}) \) and their scalar components, whose charges under \( U(1)_l \) are given in (15), as follows

\[
\Phi_a = \begin{cases} 
S_a, & 1 \leq a \leq 5, \\
P_{a-5}, & 6 \leq a \leq 106.
\end{cases}
\]

The field \( P_1 \) corresponds to \( v_{b1} \), whose components \( v_{b1} = 1 \) play a distinguished role. The superpotential \( W_Y \) in the considered case can be written as

\[
W_Y = P_1 \cdot G(S_1, \ldots, S_5; P_2, \ldots, P_{101}).
\]

Being a bit sloppy, we also assign the R-charges to the fields as \( q_{P_1} = 2, q_{P_l} = 0, l > 1 \) and \( q_{S_i} = 0 \) such that the R-charge of the potential \( W_Y \) is equal to 2. The potential

\(^7\) This means that the gauge group \( U(1)^{101} \) acting on the chiral fields has a discrete subgroup (consisting of certain fifth roots of unity) which acts trivially on all the chiral fields, which is not the case for an actual theory we are building. In other words, our choice of the charge matrix gives a bigger gauge group.
term for the scalar fields in the Lagrangian, which is given by (6) above, will then take the form

\[ U(\phi) = \sum_{l=1}^{101} \left( \sum_{a=1}^{5} s_{ia} |S_a|^2 - 5|P_l|^2 - r_l \right)^2 + \frac{1}{4} |G(S_1, \ldots, S_5; P_2, \ldots, P_{101})|^2 \\
+ \frac{1}{4} |P_1|^2 \sum_{a=1}^{5} \left| \frac{\partial G}{\partial S_a} \right|^2 + \frac{1}{4} \left| \sum_{l=2}^{101} \frac{\partial G}{\partial P_l} \right|^2. \]  

(19)

Depending on the values of FI parameters in the theory the different phases are expected to occur \([9, 11]\). In a suitable region of \(r_l\) the vacuum manifold is a set of \((S_1, P_l)\) obeying the equations

\[ \sum_{a=1}^{5} s_{ia} |S_a|^2 - 5|P_l|^2 - r_l = 0, \quad G(S_1, \ldots, S_5; P_2, \ldots, P_{101}) = 0, \quad P_1 = 0 \]  

(20)
divided by the gauge group action, that is the manifold \(Y\).

The exact partition function \(Z_Y\) can be written in the form

\[ Z_Y = \sum_{m_l \in V} \int_{C_1} \cdots \int_{C_{101}} \prod_{l=1}^{101} \frac{d\tau_l}{(2\pi i)} \left( z_l^{-\tau_l + \frac{m_l}{2}} z_l^{m_l} \right) \times \frac{\Gamma(1 - 5(\tau_l - \frac{m_l}{2}))}{\Gamma(5(\tau_l + \frac{m_l}{2}))} \prod_{a=1}^{5} \frac{\Gamma(\sum_l s_{ia}(\tau_l - \frac{m_l}{2}))}{\Gamma(1 - \sum_l s_{ia}(\tau_l + \frac{m_l}{2}))} \prod_{l=2}^{101} \frac{\Gamma(-5(\tau_l - \frac{m_l}{2}))}{\Gamma(1 + 5(\tau_l + \frac{m_l}{2}))}, \]  

(21)

where

\[ z_l = e^{-(2\pi r_l + i\theta_l)}. \]  

(22)

The contours \(C_l\) in (21) go slightly to the left of the imaginary axes: \(\tau_l = -\epsilon + it_l\). The set \(V\) is defined as set of all possible \(m_l\) such that \(\sum_{l \leq 101} Q_{al} m_l \in \mathbb{Z}\). We consider the expansion of this integral for large values of \(|z_l|\), that is for \(r_l \ll 0\). For \(r_l < 0\) each contour \(C_l\) can be closed to the right half-plane picking up the poles at

\[ 5 \left( \tau_l - \frac{m_l}{2} \right) - 1 = p_l, \quad 5 \left( \tau_l - \frac{m_l}{2} \right) = p_l; \quad p_l = 1, 2, \ldots, p_l = 0, 1, \ldots \text{ such that } p_l + 5m_l > 0. \]  

(23)

It is convenient to introduce the notation \(\tilde{p}_l = p_l + 5m_l\). Then the partition function can be rewritten as

\[ Z = \pi^{-5} \sum_{p_l > 0, p_l \in \mathbb{Z}} \sum_{\tilde{p}_l \in \Sigma_p} \prod_l \frac{(-1)^{p_l}}{p_l!} z_l^{\tilde{p}_l} \frac{p_l}{5} \prod_{l=1}^{5} \Gamma \left( \frac{h}{5} \sum_{l=1}^{h} s_{il} p_l \right) \Gamma \left( \frac{1}{5} \sum_{l=1}^{h} s_{il} \tilde{p}_l \right) \frac{\sin \left( \frac{\pi}{5} \sum_{l=1}^{h} s_{il} \tilde{p}_l \right)}{\Gamma \left( \frac{1}{5} \sum_{l=1}^{h} s_{il} \tilde{p}_l \right)}, \]  

(24)

where the \(\Sigma_p\) is a set of all \(\{\tilde{p}_l\}\) such that \(\{\tilde{p}_l\}\) that \(\sum Q_{al}(\tilde{p}_l - p_l) / 5 = \sum Q_{al} m_l \in \mathbb{Z}\) as follows from (16). Using the explicit expression for \(Q_{al}\) the latter condition can be rewritten as \(\tilde{p}_l \in \mathbb{Z}\) and \(\sum s_{il}(\tilde{p}_l - p_l) \in 5\mathbb{Z}\). Also noting that each term in (24), such that \(\sum_{l=1}^{101} s_{il} \tilde{p}_l = 0 \text{ mod } 5\), vanishes, we conclude that the sum in (24) effectively goes over the sets.
\[ S_\mu = \left\{ p_l, \bar{p}_l : \sum_{l=1}^{101} s_{il}p_l = \sum_{l=1}^{101} s_{il}\bar{p}_l \equiv \mu_i \pmod{5}, 1 \leq \mu_i \leq 4 \right\}. \]  

Finally, using the identity
\[
\prod_{i=1}^{5} \sin \left( \frac{\pi}{5} \sum_{i=1}^{h} s_{il}\bar{p}_l \right) = (-1)^{|\mu|} \prod_{i=1}^{5} \sin \left( \frac{\pi \mu_i}{5} \right) \prod_{l=1}^{h} (-1)^{\bar{p}_l},
\]
we find that
\[
Z = \sum_{\mu} (-1)^{|\mu|} \prod_{i=1}^{5} \frac{\Gamma \left( \frac{\mu_i}{5} \right)}{\Gamma \left( 1 - \frac{\mu_i}{5} \right)} \left| \sigma_\mu(z) \right|^2,
\]
where
\[
\sigma_\mu(z) = \sum_{n_i \geq 0} \prod_{i=1}^{5} \frac{\Gamma \left( \frac{\mu_i}{5} + n_i \right)}{\Gamma \left( \frac{\mu_i}{5} \right)} \sum_{p \in S_\mu, l=1}^{101} (-1)^{p_l} z_l^p \frac{\bar{p}_l}{p_l!}.
\]

The expression for the partition function \( Z \) in (27) and (28) coincides with the expression for
\[
e^{-\kappa \mathcal{F}} = -i \int_X \Omega \wedge \bar{\Omega},
\]
obtained by the direct computation in [13]. For this identification we have to assume that
\[
z_l = -\psi_l^{-5}.
\]
This relation is nothing but the mirror map for the considered case.

Actually, in the region \( r_l < 0 \), where the formula (24) was obtained, the theory is expected to describe the Landau–Ginzburg phase [9, 11]. On the other hand, in order to get a metric on the Kähler moduli space of the manifold \( Y \) which is mirror to the Quintic one need to perform an analytic continuation to other region of \( r_l \)'s. We note that the integral formula (21) together with the equality (29) serve as a tool for this analytic continuation.

In this note we have shown how, starting from the superpotential \( W(x) \) which defines a CY manifold \( X \), to construct \( N = (2, 2) \) gauged linear sigma model whose vacuum manifold \( Y \) is the mirror of \( X \). To do this we use Batyrev mirror construction for fixing the explicit correspondence between GLSM and the corresponding mirror Calabi–Yau model. As an example of this approach we have verified the conjecture by Jockers et al [3] for the quintic threefold. This paper is intended to explain our check of the conjecture which we sketched for the class of CY manifolds of Fermat type in [15] using the results from [14].

**Acknowledgments**

We are grateful to V Batyrev, F Benini, G Bonelli, D Gepner, A Gerhardus, H Jockers, S Parkhomenko, P Putrov, A Tanzini and F Quevedo for the interesting discussions.
and useful comments. We are also thankful to A Gerhardus for pointing out a mistake in the manuscript. AB is grateful to Weizmann Institute for the kind hospitality. This work was done in Landau Institute and has been supported by the Russian Science Foundation under the grant 18-12-00439.

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