Truncations of Random Unitary Matrices Drawn from Hua-Pickrell Distribution

Zhaofeng Lin\textsuperscript{1} · Yanqi Qiu\textsuperscript{2,3} · Kai Wang\textsuperscript{4}\textsuperscript{in}

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Abstract
Let $U$ be a random unitary matrix drawn from the Hua-Pickrell distribution $\mu_{U(n+m)}^{(\delta)}$ on the unitary group $U(n+m)$. We show that the eigenvalues of the truncated unitary matrix $[U_{i,j}]_{1 \leq i,j \leq n}$ form a determinantal point process $\mathcal{X}_{n}^{(m,\delta)}$ on the unit disc $\mathbb{D}$ for any $\delta \in \mathbb{C}$ satisfying $\text{Re} \, \delta > -1/2$. We also prove that the limiting point process taken by $n \to \infty$ of the determinantal point process $\mathcal{X}_{n}^{(m,\delta)}$ is always $\mathcal{X}^{[m]}$, independent of $\delta$. Here $\mathcal{X}^{[m]}$ is the determinantal point process on $\mathbb{D}$ with weighted Bergman kernel

$$K^{[m]}(z, w) = \frac{1}{(1 - z \bar{w})^{m+1}}$$

with respect to the reference measure $d\mu^{[m]}(z) = \frac{m}{\pi} (1 - |z|)^{m-1} d\sigma(z)$, where $d\sigma(z)$ is the Lebesgue measure on $\mathbb{D}$.

Keywords Determinantal point process · Hua-Pickrell measure · Truncated unitary matrix · Limiting point process · Weighted Bergman kernel

Mathematics Subject Classification Primary 60G55; Secondary 46E22 · 30B20 · 30H20

In Memory of Professor Jörg Eschmeier
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✉ Kai Wang
kwang@fudan.edu.cn

Extended author information available on the last page of the article
1 Introduction

Determinantal point processes, which arise in quantum physics, have been studied extensively after being initiated by Macchi [28] in the seventies. They have been used to model fermions in quantum mechanics, eigenvalues and singular values distribution of random matrices, zero sets of random analytic functions, and many other objects in representation theory and combinatorics. All of them can be described by probabilistic models that give the likelihood as matrix determinants of kernel functions. We refer the reader to [3, 23, 38–40] for further background of determinantal point processes.

1.1 Truncations of Haar Unitary Matrices

By the theorem established by Macchi [28] and Soshnikov [39], as well as Shirai and Takahashi [40], one knows how to determine whether a reproducing kernel function $K(\cdot, \cdot)$ yields a determinantal point process. However, it is still a hard problem to illustrate a concrete determinantal point process for a given kernel function, even for the classical weighted Bergman kernel.

Let $X[m]$ be the determinantal point process on the unit disc $\mathbb{D}$ with weighted Bergman kernel

$$K[m](z, w) = \frac{1}{(1 - zw)^{m+1}}$$

with respect to the reference measure

$$d\mu[m](z) = \frac{m}{\pi} (1 - |z|^2)^{m-1} d\sigma(z),$$

where $d\sigma(z)$ is the usual Lebesgue measure. In a breakthrough work [38], Peres and Virág established a concrete model for $X[1]$ through the random analytic function theory. Namely, the zeros of the random analytic function

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

where $a_k, k \geq 0$, are i.i.d. standard complex Gaussian random variables, form a determinantal point process on $\mathbb{D}$ with Bergman kernel

$$K[1] = \frac{1}{(1 - zw)^2}$$

with respect to the normalized Lebesgue measure $d\mu[1](z) = \frac{1}{\pi} d\sigma(z)$.

Krishnapur [23, 25] extended the result of Peres and Virág to general positive integer $m$, which is exactly the determinantal point process $X[m]$. In detail, let $G_k, k \geq 0$, be i.i.d. $m \times m$ matrices with i.i.d. standard complex Gaussian entries. Then the zeros of the random analytic function

$$F(z) = \det(G_0 + G_1 z + G_2 z^2 + \ldots)$$

(1.1)
form a determinantal point process on $D$ with the weighted Bergman kernel $K_{[m]}$ with respect to the reference measure $\mu_{[m]}$.

The truncated unitary matrix theory plays an important role in Krishnapur’s work. Let $U$ be a random unitary matrix drawn from the Haar distribution $\mu_{U(n+m)}$ on unitary group $U(n+m)$ for some fixed positive integers $n, m$. Zyczkowski and Sommers [42] showed that the eigenvalues distribution $X_{[m]}$ of the truncated unitary matrix $[U_{i,j}]_{1 \leq i, j \leq n}$ form a determinantal point process on the unit disc $D$ with kernel

$$K_{[m]}(z, w) = \sum_{k=0}^{n-1} \frac{(m+1)(m+2)\ldots(m+k)}{k!} (zw)^k$$

with respect to the reference measure $d\mu_{[m]}$. Krishnapur obtained the distribution of the zeros of the random analytic function $F$ in (1.1) by establishing a remarkable link between his model and Zyczkowski and Sommers’ result.

1.2 Hua-Pickrell Measure on Unitary Group

Let $\mu_{U(N)}$ be the Haar measure on the unitary group $U(N)$ for $N \in \mathbb{N}_+$; see e.g. [17]. The celebrated theorem by Dyson and Weyl in [8–11] state that the eigenvalues of unitary matrix drawn from the Haar distribution $\mu_{U(N)}$ on $U(N)$ form a determinantal point process, called circular unitary ensemble, on the unit circle $T$ with kernel

$$K_N(e^{i\theta}, e^{i\phi}) = \sum_{k=0}^{N-1} e^{ik(\theta - \phi)}$$

with respect to the normalized Lebesgue measure $\frac{1}{2\pi} d\theta$. We refer the reader to Pereira’s short note [34] for a quick survey for more Hermitian random matrix model, and Hedenmalm and Wennman’s recent work [21] for non-Hermitian cases.

For any fixed $\delta \in \mathbb{C}$ with $\text{Re} \, \delta > -1/2$, the Hua-Pickrell measure $\mu_{U(N)}^{(\delta)}$ on $U(N)$ is defined by

$$E_{\mu_{U(N)}^{(\delta)}}[f(U)] = \frac{E_{\mu_{U(N)}}[|\det(I - U)^{\delta}|^2 f(U)]}{E_{\mu_{U(N)}}[|\det(I - U)^{\delta}|^2]}$$

for any continuous function $f \in C(U(N))$. In other words, the Hua-Pickrell measure $\mu_{U(N)}^{(\delta)}$ is a probability measure on $U(N)$ satisfying

$$d\mu_{U(N)}^{(\delta)}(U) \propto |\det(I - U)^{\delta}|^2 d\mu_{U(N)}(U).$$

When $\delta = 0$, $\mu_{U(N)}^{(\delta)}$ is just the Haar measure $\mu_{U(N)}$. Here, the notation $\mu \propto \nu$ for two measure $\mu$, $\nu$ means that there exists a constant $c$ such that $\mu = c\nu$.

The Hua-Pickrell measure has already been studied on the finite dimensional unitary group by Hua [18]. And results about the infinite dimensional case were given by
Pickrell [35, 36]. It has been widely studied in recent years; see, e.g. [2, 4, 6, 14, 30, 32].

Similarly, the eigenvalues of unitary matrix drawn from the Hua-Pickrell distribution $\nu^\delta_U(N)$ on $U(N)$ also form a determinantal point process on the unit circle $\mathbb{T}$, in which the kernel $K^\delta_N$ involves Gauss’s hypergeometric functions [2, 6].

1.3 Main Results

We consider the point process for eigenvalues of the truncated unitary matrix drawn from the Hua-Pickrell distribution $\nu^\delta_U(n + m)$ on unitary group $U(n + m)$, where $n, m \in \mathbb{N}_+$ and $\delta \in \mathbb{C}$ satisfying $\text{Re} \delta > -1/2$. Let

$$d\nu^{(m, \delta)}(z) = |(1 - z)\delta|^2 (1 - |z|^2)^{m-1} d\sigma(z)$$

be the reference measure. Denote $\{P_k^{(m, \delta)}\}_{k=0}^\infty$ the family of orthonormal polynomials obtained by applying Gram-Schmidt orthogonalization procedure to $\{z^k\}_{k=0}^\infty$ in $L^2(\mathbb{D}, \nu^{(m, \delta)})$.

**Theorem 1.1** Let $U$ be a random unitary matrix drawn from the Hua-Pickrell distribution $\nu^\delta_U(n + m)$ on unitary group $U(n + m)$. Then the eigenvalues of the truncated unitary matrix $[U_{i,j}]_{1 \leq i, j \leq n}$ form a determinantal point process $\mathcal{X}^{(m, \delta)}_n$ on the unit disc $\mathbb{D}$ with kernel

$$K^{(m, \delta)}(z, w) = \sum_{k=0}^{n-1} P_k^{(m, \delta)}(z) \overline{P_k^{(m, \delta)}(w)}$$

with respect to the reference measure $\nu^{(m, \delta)}$.

Clearly, the determinantal point process $\mathcal{X}^{(m, \delta)}_n$ is equal to the determinantal point process $\mathcal{X}^{[m]}_n$ only if $\delta = 0$. However, we next show that the limiting point process $\mathcal{X}^{[m]}$ is always $\mathcal{X}^{[m]}$, independent of the parameter $\delta$.

**Theorem 1.2** For any $\delta \in \mathbb{C}$ satisfying $\text{Re} \delta > -1/2$, the limiting point process taken by $n \to \infty$ of the determinantal point process $\mathcal{X}^{(m, \delta)}_n$ is always $\mathcal{X}^{[m]}$, independent of $\delta$. Here $\mathcal{X}^{[m]}$ is the determinantal point process on the unit disc $\mathbb{D}$ with weighted Bergman kernel

$$K^{[m]}(z, w) = \frac{1}{(1 - zw)^{m+1}}$$

with respect to the reference measure $d\nu^{[m]}(z) = \frac{m}{\pi} (1 - |z|)^{m-1} d\sigma(z)$.

2 Preliminaries

Let us recall some notations and definitions of determinantal point processes and reproducing kernels.
2.1 Determinantal Point Process

Let $E$ be a locally compact Polish space, $B_0(E)$ the collection of all pre-compact Borel subsets of $E$. Denote $\text{Conf}(E)$ the space of all locally finite configurations over $E$, that is,

$$\text{Conf}(E) = \{ \xi = \sum_i \delta_{x_i} \mid \forall i, x_i \in E \text{ and } \xi(\Delta) < \infty \text{ for all } \Delta \in B_0(E) \}.$$ 

Consider the vague topology on $\text{Conf}(E)$, the weakest topology on $\text{Conf}(E)$ such that for any $f \in C_c(E)$, the map $\text{Conf}(E) \ni \xi \mapsto \int_E f \, d\xi$ is continuous. Here $C_c(E)$ is the space of all continuous functions on $E$ with compact support. The configuration space $\text{Conf}(E)$ equipped with the vague topology is also a Polish space. The Borel $\sigma$-algebra $\mathcal{F}$ on $\text{Conf}(E)$ is generated by the cylinder sets $C_{\Delta}^n = \{ \xi \in \text{Conf}(E) \mid \xi(\Delta) = n \}$, where $n \in \mathbb{N} = \{0, 1, 2, \ldots \}$ and $\Delta \in B_0(E)$. By definition, a point process on $E$ is a random variable $X : (\Omega, \mathcal{F}(\Omega), \mathbb{P}) \to (\text{Conf}(E), \mathcal{F})$,

where $(\Omega, \mathcal{F}(\Omega), \mathbb{P})$ is any probability space. For more details, we refer the reader to [12, 13, 15, 23, 26, 27].

A point process $\mathcal{X}$ is called simple if it almost surely assigns at most measure one to singletons. In the simple case, $\mathcal{X}$ can be identified with a random discrete subset of $E$. And for any $\Delta \in B_0(E)$, $\mathcal{X}(\Delta)$ represents the number of points that fall in $\Delta$.

Let $\mu$ be a reference Radon measure on $E$ and $K : E \times E \to \mathbb{C}$ be a measurable function. A simple point process $\mathcal{X}$ is called determinantal on $E$ associated to the kernel $K$ with respect to the reference measure $\mu$, if for every $n \in \mathbb{N}_+$ and any family of mutually disjoint subsets $\Delta_1, \Delta_2, \ldots, \Delta_m \in B_0(E)$, $m \geq 1$, $n_k \geq 1$, $1 \leq k \leq m$,

$$n_1 + n_2 + \ldots + n_m = n,$$

$$\mathbb{E} \left[ \prod_{k=1}^m \frac{\mathcal{X}(\Delta_k)!}{(\mathcal{X}(\Delta_k) - n_k)!} \right] = \int_{\Delta_1 \times \cdots \times \Delta_m} \det[K(x_i, x_j)]_{1 \leq i, j \leq n} \, d\mu(x_1) \cdots d\mu(x_n).$$

(2.2)

2.2 Orthonormal Polynomials

From now on, we always assume that $E \subset \mathbb{C}$ is a domain and $\mu$ be the reference Radon measure on $E$. Suppose that $\{\varphi_k\}_{k=0}^{n-1}$ is a finite orthonormal set in $L^2(E, \mu)$. Denote

$$K(x, y) = \sum_{k=0}^{n-1} \varphi_k(x)\overline{\varphi_k(y)}.$$ 

Then there exists a unique determinantal point process on $E$ with kernel $K$ with respect to the reference measure $\mu$. And the number of points in this determinantal point process is equal to $n$, almost surely.
Consider a random vector in $E^n \subset \mathbb{C}^n$ with density

$$\frac{1}{n!} \det \left[ K(x_i, x_j) \right]_{1 \leq i, j \leq n} \prod_{k=1}^{n} d\mu(x_k).$$

Erase the labels and regard it as a point process on $E$ with $n$ points, then it implies a determinantal point process on $E$ with kernel $K$ with respect to the reference measure $\mu$. The proof of these results can be found in [22, 23].

Let $\{P_k\}_{k=0}^\infty$ be a family of orthonormal polynomials obtained by applying Gram-Schmidt orthogonalization procedure to $\{x^k\}_{k=0}^\infty$ in $L^2(E, \mu)$. For any $n \in \mathbb{N}^+$, set

$$K_n(x, y) = \sum_{k=0}^{n-1} P_k(x) \overline{P_k(y)}. \quad (2.3)$$

It follows from the Gram-Schmidt orthogonalization procedure and the definition of Vandermonde determinant that

$$\det \left[ K_n(x_i, x_j) \right]_{1 \leq i, j \leq n} = \prod_{k=0}^{n-1} \text{lc}^2(P_k) \prod_{1 \leq i < j \leq n} |x_i - x_j|^2,$$

where $\text{lc}(P)$ denotes the leading coefficient of polynomial $P$.

Now consider a random vector in $E^n \subset \mathbb{C}^n$ whose density is proportional to

$$\prod_{1 \leq i < j \leq n} |x_i - x_j|^2 \prod_{k=1}^{n} d\mu(x_k). \quad (2.4)$$

It follows that the point process on $E \subset \mathbb{C}$ induced by erasing the labels is determinantal with kernel $K_n$ with respect to the reference measure $\mu$. The ensemble (2.4), called the orthogonal polynomial ensemble, has been extensively studied in random matrix models; see, e.g. [7–10, 16, 23, 29, 31] and references therein.

### 2.3 Reproducing Kernel

When the kernel function $K$ appearing in the formula (2.2) is Hermitian, it is a reproducing kernel for some Hilbert space.

Recall that a reproducing kernel Hilbert space $\mathcal{H}$ on $E$ is a Hilbert space endowed with some measurable functions satisfying that for every $x \in E$, the linear evaluation functional $ev_x : \mathcal{H} \to \mathbb{C}$ defined by $ev_x(f) = f(x)$ is bounded. Since every bounded linear functional is given by the inner product with a unique vector in $\mathcal{H}$, we know that for every $x \in E$, there is a unique vector $k_x \in \mathcal{H}$ such that $f(x) = \langle f, k_x \rangle_{\mathcal{H}}$ for every $f \in \mathcal{H}$. The function $k_x$ is called the reproducing kernel of the point $x$. The
two-variable function $K_{\mathcal{H}} : E \times E \to \mathbb{C}$ defined by
\[ K_{\mathcal{H}}(x, y) = k_y(x) \]
is called the reproducing kernel of $\mathcal{H}$. We refer the reader to [1, 37] for more properties and details of reproducing kernels.

It is well known that the weighted Bergman space
\[ L^2_a(\mathbb{D}, \mu^{[m]}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and } \int_{\mathbb{D}} |f(z)|^2 d\mu^{[m]}(z) < \infty \right\} \]
is a reproducing kernel Hilbert space with reproducing kernel
\[ K^{[m]}(z, w) = \frac{1}{(1 - z\overline{w})^{m+1}}, \]
which is called the weighted Bergman kernel. The linear subspace $\text{span}\{z^k\}_{k=0}^{\infty}$ is dense in $L^2_a(\mathbb{D}, \mu^{[m]})$, and the orthonormal polynomials have the following form:
\[ P_k^{[m]}(z) = \sqrt{(m+1)(m+2)\ldots(m+k)} \frac{z^k}{k!}, \quad k \geq 0. \]

We refer the reader to [20, 41] for a comprehensive overview.

Suppose that the reproducing kernel Hilbert space $\mathcal{H}$ is a subspace of $L^2(E, \mu)$ with reproducing kernel $K_{\mathcal{H}}(\cdot, \cdot)$. If we assume moreover that the reproducing kernel of $\mathcal{H}$ is locally of trace class, then by a theorem obtained by Macchi [28] and Soshnikov [39], as well as Shirai and Takahashi [40], there exists a unique determinantal point process $\mathcal{X}_{\mathcal{H}}$ on $E$ with kernel $K_{\mathcal{H}}(\cdot, \cdot)$ with respect to the reference measure $\mu$.

In fact, the kernel given in (2.3) is the reproducing kernel of the finite dimensional reproducing kernel Hilbert space
\[ \mathcal{H}_n = \text{span}\{x^k\}_{k=0}^{n-1} \subset L^2(E, \mu). \]
Furthermore, the associated determinantal point process $\mathcal{X}_{\mathcal{H}_n}$ is exactly determined by the orthogonal polynomial ensemble (2.4).

Next we consider an infinite dimensional reproducing kernel Hilbert space $\mathcal{H} \subset L^2(E, \mu)$, and suppose that $\text{span}\{x^k\}_{k=0}^{\infty}$ is dense in $\mathcal{H}$. Then the reproducing kernel of $\mathcal{H}$ has the following form:
\[ K_{\mathcal{H}}(x, y) = \sum_{k=0}^{\infty} P_k(x) \overline{P_k(y)}, \quad (2.5) \]
where $\{P_k\}_{k=0}^{\infty}$ is the family of orthonormal polynomials obtained by applying Gram-Schmidt orthogonalization procedure to $\{x^k\}_{k=0}^{\infty}$ in $L^2(E, \mu)$.

Hence the determinantal point process $\mathcal{X}_{\mathcal{H}}$ with kernel (2.5) can be seen as the limiting point process taken by $n \to \infty$ of the determinantal point process $\mathcal{X}_{\mathcal{H}_n}$ with
kernel (2.3). For more concrete models, more connections between determinantal point processes and reproducing kernels; see, e.g. [5, 23–25, 33].

3 Proof of Theorem 1.1

In this section, we are going to establish the point process of eigenvalues of the truncated unitary matrix drawn from the Hua-Pickrell distribution \( \mu_\delta \).

Let \( \mathcal{T}(n, m) \) be the product measurable space

\[
\mathcal{T}(n, m) = U(m) \times \mathcal{V}(n) \times (S^{2m-1})^n \times \mathbb{C}^n,
\]

where \( \mathcal{V}(n) \) is the submanifold of \( n \times n \) unitary matrices with non-negative diagonal elements and \( S^{2m-1} \) is the \((2m - 1)\)-dimensional unit sphere. There exists a product probability measure \( \mathcal{T}(n, m) \) on \( \mathcal{T}(n, m) \) defined by

\[
d\mathcal{T}(n, m)(W, V, \omega_1, \ldots, \omega_n, z_1, \ldots, z_n) = C_{(n, m)} \prod_{k=1}^n (1 - |z_k|^2)^{m-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \ d\mu(U(m)) \ d\nu(W) \prod_{k=1}^n \ d\Theta(\omega_k) \prod_{k=1}^n \ d\sigma(z_k),
\]

where \( C_{(n, m)} \) is the normalization constant, \( d\nu \) is the restriction of Haar measure \( d\mu(U) \) on the submanifold \( \mathcal{V}(n) \) and \( d\Theta \) is the usual Lebesgue measure on \( S^{2m-1} \).

Let \( \mathcal{P} : \mathcal{T}(n, m) \to \mathbb{C}^n \) be the projection map. The following decomposition for the Haar measure \( \mu_{U(n+m)} \) on unitary group \( U(n + m) \) may be well-known. Since the lack of references, we present a full proof for completeness.

Lemma 3.1 There exists a measurable transformation

\[
T : (U(n + m), \mu_{U(n+m)}) \to (\mathcal{T}(n, m), \mathcal{T}(n, m)),
\]

which possesses the following two properties:

(i) for \( \mu_{U(n+m)} \) a.e. \( U \in U(n + m) \), the set \( \{\mathcal{P}T(U) \} : 1 \leq i \leq n \} \) is equal to the set \( \lambda([U_{i,j}]_{1 \leq i, j \leq n}) \) of eigenvalues of the matrix \( [U_{i,j}]_{1 \leq i, j \leq n} \);

(ii) for any integrable function \( \varphi \in L^1(\mathcal{T}(n, m), \mathcal{T}(n, m)) \),

\[
\int_{U(n+m)} \varphi \circ T \ d\mu_{U(n+m)} = \int_{\mathcal{T}(n, m)} \varphi \ d\mathcal{T}(n, m).
\]

Proof of Lemma 3.1 We will prove Lemma 3.1 by the following five steps.

Step I For a matrix \( M \in GL(n + m, \mathbb{C}) \), partition it as

\[
M = \begin{bmatrix} X & C \\ B^* & A \end{bmatrix},
\]

(3.6)

where \( X \) has size \( n \times n \). By omitting a lower dimensional submanifold of \( GL(n + m, \mathbb{C}) \), whose Lebesgue measure is zero, we may assume that the eigenvalues of \( X \)
are mutually distinct. Then we can put the eigenvalues of \(X\) in lexicographical order by \(z_1 < z_2 < \cdots < z_n\).

The submatrix \(X\) can be uniquely written as Schur decomposition

\[
X = V(Z + T)V^*,
\]

where \(V\) is unitary with \(V_{i,i} \geq 0, 1 \leq i \leq n\), \(T\) is strictly upper triangular, \(Z\) is the diagonal matrix \(\text{diag}(z_1, z_2, \ldots, z_n)\); see e.g. [19]. Therefore, there is a unique matrix decomposition of \(M\) as follow:

\[
M = \begin{bmatrix}
V & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
Z + T & S \\
R^* & A
\end{bmatrix}
\begin{bmatrix}
V^* & 0 \\
0 & I
\end{bmatrix},
\]

where \(R = V^*B\) and \(S = V^*C\).

Write \(R = [R_1^T, R_2^T, \ldots, R_n^T]^T\), where each \(R_k\) is a \(m\)-tuple row vector. For \(1 \leq k \leq n\), let \(R_k = [R_1^T, R_2^T, \ldots, R_k^T]^T\) denote the submatrix consisting of the first \(k\) rows of \(R\), and \(\omega_k\) be the submatrix consisting of the first \(k\) rows and columns of matrix \(Z + T\). In particular, \(R_1 = R_1, R_n = R\), and \(\omega_1 = z_1, \omega_n = Z + T\). For \(1 \leq k \leq n\), since \(I + (R_k^*\omega_k^{-1})(R_k^*\omega_k^{-1})^*\) is a positive definite matrix, the positive definite square root \([I + (R_k^*\omega_k^{-1})(R_k^*\omega_k^{-1})^*]^{1/2}\) makes sense. Denote

\[
\begin{align*}
\tilde{R}_1 &= R_1, \\
\tilde{R}_{k+1} &= R_{k+1}[I + (R_k^*\omega_k^{-1})(R_k^*\omega_k^{-1})^*]^{1/2}, & k &= 1, 2, \ldots, n-1, \\
\tilde{A} &= [I + (R_n^*\omega_n^{-1})(R_n^*\omega_n^{-1})^*]^{1/2}A.
\end{align*}
\]

For \(k = 1, 2, \ldots, n\), write \(\tilde{R}_k = r_k\omega_k\) in the spherical coordinate system, where the \(m\)-tuple row vector \(\omega_k\) lies in the \((2m-1)\)-dimensional unit sphere \(S^{2m-1}\) and \(r_k\) is a non-negative number. Let \(A = WD^{1/2}\) be the matrix’s polar decomposition of \(A\), i.e., \(W\) lies in the \(m \times m\) unitary group \(U(m)\) and \(D^{1/2}\) is the positive definite square root of a positive definite matrix \(D\).

Let us introduce some notations. We will use the notations

\[
d\sigma(z_{[n]}) = \bigwedge_{1 \leq k \leq n} d\sigma(z_k),
\]

\[
d\Theta(\omega_{[n]}) = \bigwedge_{1 \leq k \leq n} d\Theta(\omega_k),
\]

and

\[
r_{[n]}dr_{[n]} = \bigwedge_{1 \leq k \leq n} r_k dr_k.
\]

And for any \(u \times v\) matrix \(Y\), denote

\[
|dY_{[u \times [v]}|^2 = \bigwedge_{1 \leq i \leq u, 1 \leq j \leq v} |dY_{i,j}|^2.
\]
and
\[
dY_{[u] \times [v]} = \bigwedge_{1 \leq i \leq u, 1 \leq j \leq v} dY_{i,j}.
\]

In addition, when \( u = v \), denote
\[
|dY_{[u] \times [u]}, <|^2 = \bigwedge_{1 \leq i < j \leq u} |dY_{i,j}|^2,
\]
and
\[
dY_{[u] \times [u], \neq} = \bigwedge_{1 \leq i, j \leq u, i \neq j} dY_{i,j}.
\]

We have the following measure decomposition for Lebesgue measure on the general linear group \( GL(n + m, \mathbb{C}) \).

**Fact 3.2** In the above notations, the Lebesgue measure on \( GL(n + m, \mathbb{C}) \) has the following decomposition:

\[
|dM_{[n+m] \times [n+m]}|^2 \propto \frac{f(D) \prod_{k=1}^n r_k^{2m-2} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}{\det[I + (R_n^* \circ f_n^{-1})(R_n^* \circ f_n^{-1})^*] \prod_{k=1}^{n-1} \det[I + (R_k^* \circ f_k^{-1})(R_k^* \circ f_k^{-1})^*]} \\
\cdot \sigma(z_{[n]}) \wedge d\Theta(\omega_{[n]}) \wedge d\nu_n(V) \wedge \mu_{U(m)}(W) \\
\cdot |dT_{[n] \times [n]}, <|^2 \wedge |dS_{[n] \times [n]}|^2 \wedge |dD_{[m] \times [m]}|^2 \wedge |dA_{[m] \times [m]}|^2,
\]

where \( f \) is a smooth function of \( D \).

**Proof of Fact 3.2**: It follows from the partitioned matrix form (3.6) that
\[
|dM_{[n+m] \times [n+m]}|^2 = |dX_{[n] \times [n]}|^2 \wedge |dC_{[n] \times [m]}|^2 \wedge |dB_{[n] \times [m]}|^2 \wedge |dA_{[m] \times [m]}|^2.
\]

Noticing that \( R = V^* B, S = V^* C \) and \( V \) is unitary, we have
\[
|dB_{[n] \times [m]}|^2 = |dR_{[n] \times [m]}|^2,
\]
and
\[
|dC_{[n] \times [m]}|^2 = |dS_{[n] \times [m]}|^2.
\]

Therefore,
\[
|dM_{[n+m] \times [n+m]}|^2 = |dX_{[n] \times [n]}|^2 \wedge |dS_{[n] \times [m]}|^2 \wedge |dR_{[n] \times [m]}|^2 \wedge |dA_{[m] \times [m]}|^2.
\]

Applying the Ginibre’s measure decomposition in Page 105 in [23] to the matrix’s Schur decomposition (3.7), we have
\[
|dX_{[n] \times [n]}|^2 \propto \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \sigma(z_{[n]}) \wedge d\nu_n(V) \wedge |dT_{[n] \times [n]}, <|^2.
\]
We next turn to the computation of $|dR_{[n] \times [m]}|^2$ and $|dA_{[m] \times [m]}|^2$. The formulas (3.9) implies that

$$|dR_{[n] \times [m]}|^2 = \prod_{k=1}^{n-1} \det[I + (R_k^* \omega_k^{-1})(R_k \omega_k^{-1})^*]^{-1} |d\tilde{R}_{[n] \times [m]}|^2,$$

and

$$|dA_{[m] \times [m]}|^2 = \det[I + (R_n^* \omega_n^{-1})(R_n \omega_n^{-1})^*]^{-m} |d\tilde{A}_{[m] \times [m]}|^2.$$

Based on the spherical coordinate systems $\tilde{R}_k = r_k \omega_k$, $k = 1, 2, \ldots, n$, we have

$$|d\tilde{R}_{[n] \times [m]}|^2 = \prod_{k=1}^{n} r_k^{2m-2} r_{[n]} dr_{[n]} \wedge d\Theta(\omega_{[n]}),$$

and

$$|dR_{[n] \times [m]}|^2 = \frac{\prod_{k=1}^{n} r_k^{2m-2} r_{[n]} dr_{[n]} \wedge d\Theta(\omega_{[n]})}{\prod_{k=1}^{n-1} \det[I + (R_k^* \omega_k^{-1})(R_k \omega_k^{-1})^*]}.$$

(3.12)

Moreover, by Lemma 6.6.1 in [23], the matrix’s polar decomposition $A = W D^{1/2}$ yields that

$$|dA_{[m] \times [m]}|^2 = f(D) dD_{[m] \times [m]} \wedge_{1 \leq i, j \leq m} \Omega_i, j(W),$$

where $f$ is a smooth function of $D$ and $\Omega(W) = W^* dW$ is a matrix-valued one form. It also follows from the statement in Page 101 of [23] that

$$d\mu_{U(m)}(W) \propto \wedge_{1 \leq i, j \leq m} \Omega_i, j(W).$$

This concludes that

$$|dA_{[m] \times [m]}|^2 \propto \frac{f(D) dD_{[m] \times [m]} \wedge d\mu_{U(m)}(W)}{\det[I + (R_n^* \omega_n^{-1})(R_n \omega_n^{-1})^*]^{-m}}.$$ (3.13)

Combining (3.11), (3.12), (3.13) with (3.10), we obtain Fact 3.2.

**Step II** For a matrix $M \in GL(n+m)$, write $M = U P^{1/2}$ in the polar decomposition, where $U$ lies in the unitary group $U(n+m)$ and $P^{1/2}$ is the positive definite square root of a positive definite matrix $P$. We denote a normal matrix $Q$ by

$$Q = \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} P \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix},$$ (3.14)

and partition it as

$$Q = \begin{bmatrix} Q^{(1)} & Q^{(2)} \\ Q^{(3)} & Q^{(4)} \end{bmatrix},$$ (3.15)

where $Q^{(1)}$ has size $n \times n$. Here $V$ is the matrix appeared in the matrix’s Schur decomposition (3.7).
By further studying for Fact 3.2, we have the following measure decomposition for Lebesgue measure on the general linear group $\text{GL}(n + m, \mathbb{C})$.

**Fact 3.3** In the above notations, the Lebesgue measure on $\text{GL}(n + m, \mathbb{C})$ has the following decomposition:

$$
|dM_{[n+m] \times [n+m]}|^2 \propto \frac{f(D) \prod_{k=1}^{n} r_k^{2m-2} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}{\det(\omega_n^* \omega_n + R_n R_n^*) \prod_{k=1}^{n-1} \det(\omega_n^* \omega_k + R_k R_k^*)} 
\times d\sigma(z_{[n]}) \land d\Theta(\omega_{[n]}) \land d\nu_n(V) \land d\mu_{U(m)}(W) 
\land dQ_{[n] \times [n], \neq}^1 \land dQ_{[n] \times [m]}^{2} \land dQ_{[m] \times [m]}^{3} \land dD_{[m] \times [m]} \land r_{[n]} dr_{[n]}.
$$

Proof of Fact 3.3: It is easy to verify that

$$
\begin{bmatrix}
(Z + T)^* & R \\
S^* & A^*
\end{bmatrix}
\begin{bmatrix}
Z + T \\
S \\
R^* \\
A
\end{bmatrix} = Q.
$$

This implies that

$$(Z + T)^* S + RA = Q^2. \tag{3.16}$$

Noticing that

$$A = \left[ I + (R_n^* \omega_n^{-1})(R_n^* \omega_n^{-1})^* \right]^{-1/2} WD 1/2, \tag{3.17}$$

we have

$$S = \omega_n^{*-1} \left( Q^2 - R_n \left[ I + (R_n^* \omega_n^{-1})(R_n^* \omega_n^{-1})^* \right]^{-1/2} WD 1/2 \right). \tag{3.18}$$

Based on (3.9), one can prove by induction that $R_n$ is depended on $Z, T, r_1, r_2, \ldots, r_n, \omega_1, \omega_2, \ldots, \omega_n,$ and $\omega_n$ is depended on $Z, T$. This implies that

$$dS_{[n] \times [m]} = \frac{1}{(\det \omega_n^*)^m} dQ_{[n] \times [m]}^2 + \cdots,$$

where $\cdots$ consists of many terms involving $dz_k, dk, d\omega_k$, $1 \leq k \leq n$, and $dD_{i,j}, dW_{i,j}$, $1 \leq i, j \leq m$, as well as $dT_{i,j}$, $1 \leq i < j \leq n$. Noting that $Q^2 = Q^3$, we get

$$|dS_{[n] \times [m]}|^2 = \frac{1}{(\det \omega_n^*)^m} dQ_{[n] \times [m]}^2 \land dQ_{[m] \times [n]}^3 + \cdots. \tag{3.19}$$

For $1 \leq k \leq n - 1$, denote $k$-tuple column vectors

$$t_{k+1} = [T_{k,k+1}, T_{k+1,k+1}, \ldots, T_{k,k+1}]^T,$$

and

$$q_{k+1} = [Q_{1,k+1}^{(1)}, Q_{2,k+1}^{(1)}, \ldots, Q_{k,k+1}^{(1)}]^T. \tag{3.20}$$
It follows from (3.16) that

$$A_k^* t_{k+1} + R_k R_{k+1}^* = q_{k+1}. \tag{3.21}$$

By the fact that

$$R_{k+1} = r_{k+1} \omega_{k+1} \left[ I + (R_k^* A_k^{-1})(R_k^* A_k^{-1})^* \right]^{-1/2}, \tag{3.22}$$

we get

$$t_{k+1} = A_k^{-1} \left( q_{k+1} - r_{k+1} R_k \left[ I + (R_k^* A_k^{-1})(R_k^* A_k^{-1})^* \right]^{-1/2} \omega_{k+1} \right). \tag{3.22}$$

Using the fact that $R_k$ is depended on $z_1, z_2, \ldots, z_k, r_1, r_2, \ldots, r_k, \omega_1, \omega_2, \ldots, \omega_k$ and $T_{i,j}, 1 \leq i < j \leq k$, and $A_k$ is depended on $z_1, z_2, \ldots, z_k, T_{i,j}$ again, we obtain that

$$\wedge 1 \leq i \leq k dT_{i,k+1} = \frac{1}{\det A_k} \wedge 1 \leq i \leq k dQ_{i,k+1}^{(1)} + [\cdots],$$

where $[\cdots]$ consists of many terms involving $dz_j, 1 \leq j \leq k$, and $dr_j, d\omega_j, 1 \leq j \leq k + 1$, as well as $dT_{i,j}, 1 \leq i < j \leq k$. Noting that $Q^{(1)*} = Q^{(1)}$, we get

$$\wedge 1 \leq i \leq k |dT_{i,k+1}|^2 = \frac{1}{\det (A_k^* A_k)} \wedge 1 \leq i \leq k dQ_{i,k+1}^{(1)} \wedge 1 \leq j \leq k dQ_{k+1,i}^{(1)} + [\cdots]. \tag{3.23}$$

A direct computation shows that for any $1 \leq k \leq n$,

$$\det (A_k^* A_k) \det \left[ I + (R_k^* A_k^{-1})(R_k^* A_k^{-1})^* \right] = \det (A_k^* A_k) \det \left[ I + R_k^* (A_k^* A_k)^{-1} R_k \right] = \det (A_k^* A_k) \det \left[ I + (A_k^* A_k)^{-1} R_k R_k^* \right] = \det (A_k^* A_k + R_k R_k^*).$$

Combing it with (3.19), (3.23) and Fact 3.2, we obtain Fact 3.3.

**Step III** Denote

$$G = Q^{(2)*} (A_n^* A_n)^{-1} R_n \left[ I + (R_n^* A_n^{-1})(R_n^* A_n^{-1})^* \right]^{-1/2} W. \tag{3.24}$$

and for $1 \leq k \leq n - 1$,

$$\alpha_k = \text{Re} \left( \omega_{k+1} \left[ I + (R_k^* A_k^{-1})(R_k^* A_k^{-1})^* \right]^{-1/2} R_k^* (A_k^* A_k)^{-1} q_{k+1} \right). \tag{3.25}$$

By further studying for Fact 3.3, we have the following measure decomposition for Lebesgue measure on the general linear group $GL(n + m, \mathbb{C})$. 
Fact 3.4 In the above notations, the Lebesgue measure on \( GL(n + m, \mathbb{C}) \) has the following decomposition:

\[
|dM_{[n+m] \times [n+m]}|^2 \propto \frac{f(D)g(G, D)\prod_{k=1}^{\alpha} r_{k}^{2m-2} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}{\det(\omega_n^* \omega_n + R_n R_n^*)^m \prod_{k=1}^{\alpha-1} \det(\omega_k^* \omega_k + R_k R_k^*) \prod_{k=1}^{n-1} (1 - \alpha_k r_{k+1}^{-1})} \cdot d\sigma(z[n]) \wedge d\Theta(\omega[n]) \wedge d\nu_k(V) \wedge d\mu(U(n))W \wedge dQ_{[n+m] \times [n+m]},
\]

where \( g \) is a smooth function of \((G, D)\).

Proof of Fact 3.4: It follows from (3.16) that

\[
S^* S + A^* A = Q^{(4)}.
\]

Based on (3.17) and (3.18), we have

\[
D - GD^{1/2} - D^{1/2}G^* + Q^{(2)*} (\omega_n^* \omega_n)^{-1} Q^{(2)} = Q^{(4)}. \tag{3.26}
\]

This implies that

\[
dD_{[m] \times [m]} = g(G, D) \wedge dQ^{(4)}_{[m] \times [m]} + \cdots, \tag{3.27}
\]

where \( g \) is a smooth function of \((G, D)\). Here \( \cdots \) consists of many terms involving \( dz_k, d\omega_k, 1 \leq k \leq n, dW_{i,j}, 1 \leq i, j \leq m, \) and \( dQ^{(2)}_{i,j}, 1 \leq i \leq n, 1 \leq j \leq m, dQ^{(1)}_{i,j}, 1 \leq i, j \leq n, i \neq j, \) as well as \( dr_k, 1 \leq k \leq n \).

It follows from (3.16) that

\[
\begin{cases}
|z_1|^2 + R_1 R_1^* = Q^{(1)}_{1,1}, \\
\sum_{k=1}^{n} |z_{k+1}|^2 + R_{k+1} R_{k+1}^* = Q^{(1)}_{k+1,k+1}, \quad k = 1, 2, \ldots, n - 1.
\end{cases}
\]

Based on (3.21) and (3.22), we have

\[
\begin{align*}
r_1^2 &= Q^{(1)}_{1,1} - |z_1|^2, \\
r_{k+1}^2 - 2\alpha_k r_{k+1} + q_{k+1}^* (\omega_k^* \omega_k)^{-1} q_{k+1} &= Q^{(1)}_{k+1,k+1} - |z_{k+1}|^2, \quad k = 1, 2, \ldots, n - 1. \tag{3.28}
\end{align*}
\]

This implies that

\[
\begin{align*}
2r_1 dr_1 &= dQ^{(1)}_{1,1} - \bar{z}_1 dz_1 - z_1 d\bar{z}_1, \\
2r_{k+1} dr_{k+1} &= (1 - \alpha_k r_{k+1}^{-1})^{-1} dQ^{(1)}_{k+1,k+1} + \cdots, \quad k = 1, 2, \ldots, n - 1. \tag{3.29}
\end{align*}
\]

Here \( \cdots \) consists of many terms involving \( dz_j, 1 \leq j \leq k, d\omega_j, 1 \leq j \leq k + 1, dQ^{(1)}_{i,j}, 1 \leq i, j \leq k + 1, i \neq j, \) as well as \( dr_j, 1 \leq j \leq k \).

Combing Fact 3.3 with (3.27) and (3.29), we obtain Fact 3.4.

Step IV By further studying for Fact 3.4, we have the following measure decomposition for Lebesgue measure on the general linear group \( GL(n + m, \mathbb{C}) \).
Fact 3.5 In the above notations, the Lebesgue measure on $GL(n + m, \mathbb{C})$ has the following decomposition:

$$|dM_{[n+m] \times [n+m]}|^2 \propto \frac{f(D)g(G, D) \prod_{k=1}^{n} \prod_{i \leq j \leq n} |z_i - z_j|^2}{\det(\omega_n^* \omega_n + R_n R_n^*) \prod_{k=1}^{n} (1 - \alpha_k r_{k+1}^{-1})} \cdot d\sigma(z_{[n]}) \wedge d\Theta(\omega_{[n]}) \wedge d\nu_n(V) \wedge d\mu_{U(n)}(W) \wedge dP_{[n+m] \times [n+m]}.$$

Proof of Fact 3.5: Based on (3.14), we have

$$dQ_{[n+m] \times [n+m]} = dP_{[n+m] \times [n+m]} + \ldots, \quad (3.30)$$

where $\ldots$ consists of many terms involving $dV_{i,j}$, $1 \leq i, j \leq n$. Notice that the dimension of the submanifold consisted by $n \times n$ unitary matrices with non-negative diagonal elements is $n^2 - n$. It follows from Claim 6.3.1 in [23] that for any $1 \leq i, j \leq n$,

$$d\nu_n(V) \wedge dV_{i,j} = 0.$$

Hence putting (3.30) into Fact 3.4, we obtain Fact 3.5.

Step V We now turn to consider the Haar measure $\mu_{U(n+m)}$ on the unitary group $U(n + m)$. We have the following measure decomposition.

Fact 3.6 In the above notations, the Haar measure $\mu_{U(n+m)}$ on $U(n + m)$ has the following decomposition:

$$d\mu_{U(n+m)}(U) \propto \prod_{k=1}^{n} (1 - \alpha_k r_{k}^{-1}) \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 d\mu_{U(n)}(W) d\nu_n(V) \prod_{k=1}^{n} d\Theta(\omega_k) \prod_{k=1}^{n} d\sigma(z_k).$$

Proof of Fact 3.6: The matrix’s polar decomposition $M = UP^{1/2}$ yields that

$$|dM_{[n+m] \times [n+m]}|^2 = h(P) d\mu_{U(n+m)}(U) \wedge dP_{[n+m] \times [n+m]}, \quad (3.31)$$

where $h$ is a smooth function of $P$.

Based on the polar decomposition $M = UP^{1/2}$, the unitary group $U(n + m)$ is the submanifold of the general linear group $GL(n + m, \mathbb{C})$ defined by the equation $P = I_{n+m}$.

By Fact 6.6.3 in [23], combining (3.31) with Fact 3.5, we obtain

$$h(I_{n+m}) d\mu_{U(n+m)}(U) \propto \frac{f(I_m)g(0_{m \times m}, I_m) \prod_{k=1}^{n} (1 - \alpha_k r_{k}^{-1}) \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}{\det(\omega_n^* \omega_n + R_n R_n^*) \prod_{k=1}^{n} (1 - \alpha_k r_{k+1}^{-1})} \cdot d\sigma(z_{[n]}) \wedge d\Theta(\omega_{[n]}) \wedge d\nu_n(V) \wedge d\mu_{U(m)}(W),$$
that is
\[
d\mu_{U(n+m)}(U) \propto \frac{\prod_{k=1}^{n} (1 - |z_k|^2)^{m-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2}{\det(A_n^*A_n^* + R_n R_n^*)^m \prod_{k=1}^{n+1} \det(A_k^*A_k + R_k R_k^*)} \cdot d\mu_{U(m)}(W) dv_n(V) \prod_{k=1}^{n} d\Theta(\omega_k) \prod_{k=1}^{n} d\sigma(z_k).
\]

Here, we use the fact \( Q = I_{n+m}, G = 0_{m \times m} \) and \( \alpha_k = 0, k = 1, 2, \ldots, n-1, \)
\( D = I_m \) and \( r_k^2 = 1 - |z_k|^2, k = 1, 2, \ldots, n, \) which are implied by \( P = I_{n+m}. \)

For any \( 1 \leq k \leq n, \) since \( Q = I_{n+m}, \) it follows from (3.16) that
\[
A_k^*A_k + R_k R_k^* = I_k.
\]

Therefore,
\[
\det(A_k^*A_k + R_k R_k^*) = 1,
\]
which implies Fact 3.6.

In fact, the above steps gives a measurable transformation
\[
T : (U(n + m), \mu_{U(n+m)}) \to (\mathcal{T}(n, m), \mathcal{T}(n,m)) : U \mapsto (W, V, \omega_1, \ldots, \omega_n, z_1, \ldots, z_n).
\]

Since \( z_1, \ldots, z_n \) are eigenvalues of the matrix \([U_{i,j}]_{1 \leq i, j \leq n}, \) we obtain the first assertion of Lemma 3.1. And the second assertion of Lemma 3.1 means that \( T \) is a measure preserving transformation, which follows from Fact 3.6.

This completes the proof of Lemma 3.1. \( \square \)

We now turn to the Hua-Pickrell measure \( \mu_{U(n+m)}^{(\delta)} \) on the unitary group \( U(n + m), \)
where \( \text{Re}\ \delta > -1/2. \) Let us start with the following Lemma.

**Lemma 3.7** For \( \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \in U(n + m), \) suppose the \( n \times n \) matrix \( M_1 \) satisfies that \( M_1 \) and \( I - M_1 \) are invertible. Then the \( m \times m \) matrix
\[
[I - M_3^*(I - M_1)^{-1}M_1^{*-1}M_3][I + (M_3^*M_1^{-1})(M_3^*M_1^{-1})^*]^{-1/2} \in U(m).
\]

**Proof of Lemma 3.7** In order to prove that
\[
N = [I - M_3^*(I - M_1)^{-1}M_1^{*-1}M_3][I + (M_3^*M_1^{-1})(M_3^*M_1^{-1})^*]^{-1/2} \in U(m),
\]
we shall show that
\[
N^*N = [I + (M_3^*M_1^{-1})(M_3^*M_1^{-1})^*]^{-1/2}[I - M_3^*(I - M_1)^{-1}M_1^{*-1}M_3]^*
\]
\[
\cdot [I - M_3^*(I - M_1)^{-1}M_1^{*-1}M_3][I + (M_3^*M_1^{-1})(M_3^*M_1^{-1})^*]^{-1/2} = I.
\]
So that it is enough to prove that
\[
[I - M_2^* (I - M_1)^{-1} M_1^{* -1} M_3]^* [I - M_2^* (I - M_1)^{-1} M_1^{* -1} M_3] = I + (M_2^* M_1^{-1}) (M_2^* M_1^{-1})^*.
\]
Since \[
\begin{bmatrix} M_1 & M_2 \\ M_2^* & M_4 \end{bmatrix}
\]
is unitary, we have
\[
M_1^* M_1 + M_3 M_3^* = I.
\]
Therefore,
\[
\text{LHS} = \left[ I - M_2^* M_1^{-1} (I - M_1^*)^{-1} M_3 \right] \left[ I - M_2^* (I - M_1)^{-1} M_1^{* -1} M_3 \right]
\]
\[
= I + M_2^* M_1^{-1} (I - M_1^*)^{-1} M_3 M_2^* (I - M_1)^{-1} M_1^{* -1} M_3
\]
\[
- M_2^* M_1^{-1} (I - M_1^*)^{-1} M_3 - M_2^* (I - M_1)^{-1} M_1^{* -1} M_3
\]
\[
= I + M_2^* M_1^{-1} (I - M_1^*)^{-1} \left[ M_2^* M_3^* - M_1^* (I - M_1) - (I - M_1^*) M_1 \right] (I - M_1)^{-1} M_1^{* -1} M_3
\]
\[
= I + M_2^* M_1^{-1} (I - M_1^*)^{-1} \left[ M_2^* M_3^* + 2M_2^* M_1 - M_1^* - M_1 \right] (I - M_1)^{-1} M_1^{* -1} M_3
\]
\[
= I + M_2^* M_1^{-1} (I - M_1^*)^{-1} \left[ (I - M_1^*) (I - M_1) \right] (I - M_1)^{-1} M_1^{* -1} M_3
\]
\[
= I.
\]
This completes the proof of Lemma 3.7. \qed

With the help of Lemma 3.7, we have the following lemma about the Hua-Pickrell measure \( \mu_{U(n+m)}^{(\delta)} \) in the case of \( \text{Re} \delta > -1/2 \).

**Lemma 3.8** There exists a probability measure \( T_{(n,m)}^{(\delta)} \) on \( \mathcal{F}(n, m) \) and a measurable transformation
\[
T^{(\delta)} : (U(n + m), \mu_{U(n+m)}^{(\delta)}) \rightarrow (\mathcal{F}(n, m), T_{(n,m)}^{(\delta)}),
\]
which possess the following properties:
(i) there is an \( m \times m \) unitary matrix \( H(z_1, \ldots, z_n, \omega_1, \ldots, \omega_n) \in U(m) \) related to \( z_1, \ldots, z_n \) and \( \omega_1, \ldots, \omega_n \), such that
\[
dT_{(n,m)}^{(\delta)}(W, V, \omega_1, \ldots, \omega_n, z_1, \ldots, z_n)
\]
\[
= C_{(n,m)}^{(\delta)} \left| \det (I - H(z_1, \ldots, z_n, \omega_1, \ldots, \omega_n) W) \right|^{\delta} \prod_{k=1}^{n} \left| 1 - z_k \right|^{\delta / 2}
\]
\[
\cdot \prod_{k=1}^{n} (1 - |z_k|^2)^{m-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 d\mu_{U(m)}(W) d\nu_n(V) \prod_{k=1}^{n} d\Theta(\omega_k) \prod_{k=1}^{n} d\sigma(z_k),
\]
where \( C_{(n,m)}^{(\delta)} \) is the normalization constant;
ii) for $\mu_{U(n+m)}^{(\delta)}$ a.e. $U \in U(n+m)$, the set \{$(PT^{(\delta)}(U))_i : 1 \leq i \leq n$\} is equal to the set $\lambda([U_{i,j}]_{1 \leq i,j \leq n})$ of eigenvalues of the matrix $[U_{i,j}]_{1 \leq i,j \leq n}$.

(iii) for any integrable function $\varphi \in L^1(\mathcal{T}(n,m), \mathcal{T}^{(\delta)}_{(n,m)})$, 

$$
\int_{U(n+m)} \varphi \circ T^{(\delta)} d\mu_{U(n+m)}^{(\delta)} = \int_{\mathcal{T}(n,m)} \varphi d\mathcal{T}^{(\delta)}_{(n,m)}.
$$

**Proof of Lemma 3.8** We will divide the proof of Lemma 3.8 into two steps.

**Step I** We have the following measure decomposition for the Hua-Pickrell measure $\mu_{U(n+m)}^{(\delta)}$.

**Fact 3.9** In the notations of the proof of Lemma 3.1, the Hua-Pickrell measure $\mu_{U(n+m)}^{(\delta)}$ on $U(n+m)$ has the following decomposition:

$$
d\mu_{U(n+m)}^{(\delta)}(U) \propto \left| \det \left( I - [I - R_n^* (I - H_n)^{-1} H_n^{-1} R_n] [I + (R_n^* H_n^{-1}) (R_n^* H_n^{-1})^*]^{-1/2} W \right) \right|^2
\cdot \prod_{k=1}^n \left| (1 - z_k) \right|^2 \prod_{k=1}^n \left| (z_k)^2 \right|^{-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2
\cdot d\mu_{U(n)}(W) \nu_n(V) \prod_{k=1}^n d\Theta(\omega_k) \prod_{k=1}^n d\sigma(z_k).
$$

Proof of Fact 3.9: Recall that

$$
d\mu_{U(N)}^{(\delta)}(U) \propto \left| \det (I - U)^{\delta} \right|^2 d\mu_{U(N)}(U). \quad (3.32)
$$

Using (3.8), we have

$$
\det(I - U) = \det \left( I - \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Z + T & S \\ R^* & A \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} \right)
= \det \left( I - (Z + T) - S \\ -R^* \right) \det \left( I - A \right)
= \det \left( I - (Z + T) \right) \det \left( I - A - R_n^* [I - (Z + T)]^{-1} S \right)
= \det \left( I - A - R_n^* (I - H_n)^{-1} S \right) \prod_{k=1}^n (1 - z_k).
$$

Noting that $Q^{(2)} = 0_{n \times m}$ and $D = I_m$, and by (3.17) and (3.18), we get

$$
\det \left( I - A - R_n^* (I - H_n)^{-1} S \right)
= \det \left( I - [I - R_n^* (I - H_n)^{-1} H_n^{-1} R_n] [I + (R_n^* H_n^{-1}) (R_n^* H_n^{-1})^*]^{-1/2} W \right).
$$
It follows that

\[
|\det(I - U)^\delta|^2 \\
= |\det \left( I - \left[ I - R_n^*(I - \mathbb{A}_n)^{-1}\mathbb{A}_n^{-1} R_n \right] \left[ I + (R_n^*\mathbb{A}_n^{-1})(R_n^*\mathbb{A}_n^{-1})^* \right]^{-1/2} W \right) |^2 \\
\cdot \prod_{k=1}^n |(1 - z_k)^\delta|^2.
\]

(3.33)

Combining (3.32), (3.33) with Fact 3.6, we obtain Fact 3.9.

**Step II** Denote the \( m \times m \) matrix

\[
H(z_1, z_2, \ldots, z_n, \omega_1, \omega_2, \ldots, \omega_n) \\
= \left[ I - R_n^*(I - \mathbb{A}_n)^{-1}\mathbb{A}_n^{-1} R_n \right] \left[ I + (R_n^*\mathbb{A}_n^{-1})(R_n^*\mathbb{A}_n^{-1})^* \right]^{-1/2}.
\]

Since that \( \begin{bmatrix} Z + TS \\ R^* A \end{bmatrix} = \begin{bmatrix} V^* & 0 \\ 0 & I \end{bmatrix} U \begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} \) is a unitary matrix, Lemma 3.7 implies that \( H(z_1, z_2, \ldots, z_n, \omega_1, \omega_2, \ldots, \omega_n) \in U(m) \) immediately.

This completes the proof of Lemma 3.8. \( \square \)

We now turn to prove Theorem 1.1.

**Proof of Theorem 1.1** Let \( U \) be a random unitary matrix drawn from the Hua-Pickrell distribution \( \mu_{\text{H}(U,m+m)}^{(\delta)} \) on \( U(n + m) \), where \( \text{Re} \delta > -1/2 \). We are going to derive the density of eigenvalues of the truncated unitary matrix \([U_{i,j}]_{1 \leq i,j \leq n}\), i.e. the joint density of vector \((z_1, z_2, \ldots, z_n)\) in uniform random order.

By Fact 3.9, all we need is to calculate the integral

\[
\int_{U(m) \times (S^{2m-1})^n} \left| \det \left[ I - H(z_1, \ldots, z_n, \omega_1, \ldots, \omega_n) W \right] \right|^2 d\mu_{U(m)}(W) \prod_{k=1}^n d\Theta(\omega_k),
\]

where the \( m \times m \) unitary matrix \( H(z_1, z_2, \ldots, z_n, \omega_1, \omega_2, \ldots, \omega_n) \in U(m) \) related to \( z_1, z_2, \ldots, z_n \) and \( \omega_1, \omega_2, \ldots, \omega_n \).

Because the Haar measure on \( U(m) \) is invariant under left multiplication by unitary matrices, the integral

\[
\int_{U(m)} \left| \det \left[ I - H(z_1, \ldots, z_n, \omega_1, \ldots, \omega_n) W \right] \right|^2 d\mu_{U(m)}(W)
\]

is a constant, independent of \( z_1, z_2, \ldots, z_n, \omega_1, \omega_2, \ldots, \omega_n \). And then the integral (3.34) is independent of \( z_1, z_2, \ldots, z_n \).

Therefore, we conclude from Fact 3.9 that the joint density of vector \((z_1, z_2, \ldots, z_n)\) is proportional to

\[
\prod_{k=1}^n |(1 - z_k)^\delta|^2 \prod_{k=1}^n (1 - |z_k|^2)^{m-1} \prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \prod_{k=1}^n d\sigma(z_k).
\]
Writing that
\[
d\mu^{(m,\delta)}(z) = |(1 - z)^\delta|^2(1 - |z|^2)^{m-1}d\sigma(z),
\]
we can restate that the joint density of the vector \((z_1, z_2, \ldots, z_n)\) is proportional to
\[
\prod_{1 \leq i < j \leq n} |z_i - z_j|^2 \prod_{k=1}^n d\mu^{(m,\delta)}(z_k).
\]
This is an orthogonal polynomial ensemble by (2.4).

It follows from (3.35) that the eigenvalues of the truncated unitary matrix
\[
[U_{i,j}]_{1 \leq i, j \leq n}
\]
form a determinantal point process \( \mathcal{P}_n^{(m,\delta)} \) on the unit disc \( \mathbb{D} \) with the kernel
\[
K_n^{(m,\delta)}(z, w) = \sum_{k=0}^{n-1} P_k^{(m,\delta)}(z) P_k^{(m,\delta)}(w)
\]
with respect to the reference measure \( \mu^{(m,\delta)} \). Here \( \{P_k^{(m,\delta)}\}_{k=0}^{\infty} \) is the family of orthonormal polynomials obtained by applying Gram-Schmidt orthogonalization procedure to \( \{z^k\}_{k=0}^{\infty} \) in \( L^2(\mathbb{D}, \mu^{(m,\delta)}) \).

This completes the proof of Theorem 1.1. \( \Box \)

4 Proof of Theorem 1.2

This section is devoted to the proof of Theorem 1.2. Let us denote the reproducing kernel Hilbert space
\[
L^2_a(\mathbb{D}, \mu^{(m,\delta)}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and } \int_\mathbb{D} |f(z)|^2 d\mu^{(m,\delta)}(z) < \infty \right\}.
\]

Lemma 4.1 The linear subspace \( \text{span}\{z^k : k = 0, 1, 2, \ldots\} \) is dense in \( L^2_a(\mathbb{D}, \mu^{(m,\delta)}) \).

Proof Let \( L^2_a(\mathbb{D}, \mu^{(m)}) \) be the standard weighted Bergman space
\[
L^2_a(\mathbb{D}, \mu^{(m)}) = \left\{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is holomorphic and } \int_\mathbb{D} |f(z)|^2 d\mu^{(m)}(z) < \infty \right\},
\]
where the measure \( d\mu^{(m)}(z) = (1 - |z|^2)^{m-1}d\sigma(z) \). It suffices to show that the linear subspace \( \text{span}\{(1 - z)^\delta z^k : k = 0, 1, 2, \ldots\} \) is dense in \( L^2_a(\mathbb{D}, \mu^{(m)}) \), or
\[
1 \in \text{span}L^2_a(\mathbb{D}, \mu^{(m)}) \{(1 - z)^\delta z^k : k = 0, 1, 2, \ldots\}.
\]

In the case \( \text{Re} \delta \geq 0 \), one sees that \( \sup_{z \in \mathbb{D}} |(1 - z)^\delta| < \infty \). For any \( 0 < r < 1 \), since there exists a sequence of polynomials \( \{p_k^{(r)}\} \) which converges to \( \frac{1}{(1-rz)^\delta} \) uniformly
on $\overline{\mathbb{D}}$, we have
\[
\left(\frac{1 - z}{1 - rz}\right)^\delta \in \text{span}^{L^2_{\overline{\mathbb{D}}}(\mu^{(m)})\{(1 - z)^k : k = 0, 1, 2, \ldots}\}.
\]

Note $|\frac{1 - z}{1 - rz}| \leq |\frac{1 - z}{1 - r}| + |\frac{z}{1 - rz}| \leq 2$. By Lebesgue’s dominated convergence theorem, we have
\[
\lim_{r \to 1} -1 \int_{\mathbb{D}} \left|\frac{1 - z}{1 - rz}\right|\delta - 1\right|^2 d\mu^{(m)}(z) = 0.
\]
This implies that (4.36).

In the case $-1/2 < \text{Re} \delta < 0$, $(1 - z)^{-\delta}$ is a holomorphic function in the unit disc $\mathbb{D}$ and continuous to the unit circle $\mathbb{T}$. Hence by Mergelyan’s theorem, there is a sequence of polynomials $\{q_k\}$ such that $q_k$ converges to $(1 - z)^{-\delta}$ uniformly on $\mathbb{D}$.

Because of
\[
\int_{\mathbb{D}} \left|1 - z\right|^\delta q_k(z) - 1\right|^2 d\mu^{(m)}(z) = \int_{\mathbb{D}} \left|q_k(z) - (1 - z)^{-\delta}\right|^2 (1 - z)^\delta |d\mu^{(m)}(z)
\leq \left\|q_k(z) - (1 - z)^{-\delta}\right\|^2 \|\mu^{(m)}(\mathbb{D})\),
\]
we also obtain (4.36).

This completes the proof of Lemma 4.1.

Recall that the correlation kernel of the determinantal point process $\mathcal{P}^{(m, \delta)}_n$ established in Theorem 1.1 is
\[
K_n^{(m, \delta)}(z, w) = \sum_{k=0}^{n-1} P_k^{(m, \delta)}(z) P_k^{(m, \delta)}(w),
\]
where $\{P_k^{(m, \delta)}\}_{k=0}^{\infty}$ is the family of orthonormal polynomials obtained by applying Gram-Schmidt orthogonalization procedure to $\{z^k\}_{k=0}^{\infty}$ in $L^2(\mathbb{D}, \mu^{(m, \delta)})$. By Lemma 4.1, the limiting kernel taken by $n \to \infty$ of $K_n^{(m, \delta)}$ is the reproducing kernel of $L^2_a(\mathbb{D}, \mu^{(m, \delta)})$.

It is easy to verify $L^2_a(\mathbb{D}, \mu^{(m, \delta)}) = (1 - z)^{-\delta} L^2_a(\mathbb{D}, \mu^{(m)})$. Thus by [37, Proposition 5.20], we obtain the following lemma.

**Lemma 4.2** The reproducing kernel of $L^2_a(\mathbb{D}, \mu^{(m, \delta)})$ is
\[
K^{(m, \delta)}(z, w) = \frac{m}{\pi (1 - z)^{\delta}(1 - z\overline{w})^{m+1}(1 - \overline{w})^{\delta}}.
\]

It follows from Lemma 4.2 that the limiting point process $\mathcal{P}^{(m, \delta)}_n$, taken by $n \to \infty$ of $\mathcal{P}^{(m, \delta)}_n$, is the determinantal point process on $\mathbb{D}$ with kernel $K^{(m, \delta)}$ with respect to the reference measure $\mu^{(m, \delta)}$. 

Recall that the determinantal point process $\mathcal{X}^{[m]}$ introduced in subsection 1.1, which is a determinantal point process on the unit disc $\mathbb{D}$ with weighted Bergman kernel
\[
K^{[m]}(z, w) = \frac{1}{(1 - z\overline{w})^{m+1}}
\]
with respect to the reference measure $d\mu^{[m]}(z) = \frac{m}{\pi} (1 - |z|)^{m-1} d\sigma(z)$. It is worth noticing that for any $n \in \mathbb{N}_+$,
\[
\det \left[ K^{(m, \delta)}(z_i, z_j) \right]_{1 \leq i, j \leq n} \prod_{k=1}^{n} d\mu^{(m, \delta)}(z_k) = \det \left[ K^{[m]}(z_i, z_j) \right]_{1 \leq i, j \leq n} \prod_{k=1}^{n} d\mu^{[m]}(z_k).
\]

Hence by definition (2.2), for any $\delta \in \mathbb{C}$ satisfying $\text{Re} \, \delta > -1/2$, the determinantal point process $\mathcal{X}^{(m, \delta)}$ is always equal to $\mathcal{X}^{[m]}$, independent of $\delta$.

This completes the proof of Theorem 1.2.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors state that there is no conflict of interest.

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Authors and Affiliations

Zhaofeng Lin¹ · Yanqi Qiu²,³ · Kai Wang⁴

Zhaofeng Lin
zflin18@fudan.edu.cn

Yanqi Qiu
yanqi.qiu@hotmail.com

1 Shanghai Center for Mathematical Sciences, Fudan University, Shanghai 200438, China
2 School of Mathematics and Statistics, Wuhan University, Wuhan 430072, Hubei, China
3 Institute of Mathematics & Hua Loo-Keng Key Laboratory of Mathematics, AMSS, CAS, Beijing 100190, China
4 School of Mathematical Sciences, Fudan University, Shanghai 200433, China