Log-normal distribution in growing systems with weighted multiplicative interactions

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Many-body stochastic processes with weighted multiplicative interactions are investigated analytically and numerically. An interaction rate between particles with quantities \(x, y\) is controlled by a homogeneous symmetric kernel \(K(x, y) \propto x^\alpha y^\beta\) with a weight parameter \(w\). When \(w < 0\), a method of moment inequalities is used to derive log-normal type tails in probability distribution functions. The variance of log-normal distributions is expressed in terms of the weight \(w\) and interaction parameters. When interactions are weak and a growth rate of systems is small, in particular, the variance is in proportion to the growth rate. This behavior is totally different from that of one-body stochastic processes, where the variance is independent of the growth rate. At \(w > 0\), Monte Carlo simulations show that the processes end up with a winner-take-all state.

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A number of probability distribution functions (PDF) with fat tails of log-normal type have been observed in a wide variety of processes including not only natural but also social and economic phenomena\(^1\),\(^2\),\(^3\). When we conduct a keyword search on "log-normal", we find thousands of literatures in various fields, say, size distributions of small grains\(^4\),\(^5\),\(^6\),\(^7\), aerosols\(^8\),\(^9\), clouds\(^10\),\(^11\),\(^12\), foams\(^13\), galaxies\(^14\),\(^15\),\(^16\), the abundance of species\(^17\),\(^18\),\(^19\), the number of employees in manufacturing plants\(^20\), the prices of insurance claims\(^21\),\(^22\), and the size of farms\(^23\), and so on. In spite of accumulations of a vast amount of data, understandings of log-normal distributions are still quite poor compared with those of power-law distributions. For example, the concepts of self-organized criticality\(^24\),\(^25\),\(^26\) and scale-free network\(^27\),\(^28\) are useful in explaining power-law distributions. For log-normal distributions, in contrast, there exist no corresponding explanations. It is well-known that in systems of a single degree of freedom, a multiplicative stochastic process exhibits a log-normal distribution\(^29\),\(^30\). In most of actual processes, however, interactions play a significant role and systems must be treated as those with multiple degrees of freedom. In addition, experiments and simulations give only circumstantial evidences. So, there is a real need for analytical derivation of log-normal distributions in many-body systems. To our knowledge, however, the theory of log-normal distributions has not been developed successfully in many-body processes which is irreducible to one-body problem.

In recent years, stochastic processes with multiplicative interactions have been attracted considerable attention\(^30\),\(^31\),\(^32\),\(^33\). Although the processes are simple extensions of the one-body multiplicative stochastic process to many-body processes, it does not exhibit a log-normal distribution but a power law at the tail of PDF\(^33\). In this Letter, we investigate many-body stochastic processes with weighted multiplicative interactions analytically. In consequence, we succeed in deriving a log-normal type tail. Furthermore, we find that when interactions are weak and a growth rate of systems is small, the variance of log-normal distribution is in proportion to the growth rate. This behavior is totally different from that of one-body systems, where the variance is independent of the growth rate.

We consider a system of \(N\) particles with positive quantities \(x_i(> 0)\) \((i = 1,\ldots,N)\). At each time step, the system evolves with a binary interaction between particles labeled by \(i\) and \(j\) \((i \neq j)\), and two quantities \(x_i, x_j\) are transformed into \(x'_i, x'_j\) by the rule

\[
x'_i = \alpha x_i + \beta x_j, \quad x'_j = \beta x_i + \alpha x_j, \tag{1}
\]

where \(\alpha, \beta > 0\) are positive interaction parameters. In the limit \(N \to \infty\), the processes are described by the master equation

\[
\frac{\partial f(z, t)}{\partial t} = \int_0^\infty dx \int_0^\infty dy f(x, t) f(y, t) K(x, y; t) \times \left[ \frac{1}{2} \left[ \delta(z - (\alpha x + \beta y)) + \delta(z - (\beta x + \alpha y)) \right] - \delta(z - x) - \delta(z - y) \right], \tag{2}
\]

where \(f(x, t)\) is a PDF of the quantities and \(K(x, y)\) is a kernel representing interaction rates. It has been already reported that when a kernel \(K\) is constant, in other words, the interaction takes place between randomly chosen particles, the PDF has a power-law tail\(^33\). The exponent of the tail is a continuous function of parameters \(\alpha, \beta\) and is calculated analytically via a transcendental equation. In many cases, however, the kernel \(K\) depends on quantities \(x\) and \(y\). A typical example is the case where a quantity \(x\) represents a mass \(M\) or size \(R\) of particles. Generally, a transport (diffusion) coefficient and then an interaction (reaction) rate depend
on $M$ and $R$. So, it is natural to think what happens in the processes with non-constant kernels. To answer this question, we address the case with a homogeneous symmetric kernel

$$K(x, y; t) = \frac{x^w y^w}{(m_w(t))^2},$$

(3)

where $w$ is a weight parameter and $m_w$ is a $w-$th order moment defined by $m_w(t) = \int_0^\infty x^w f(x, t) dx$. Hereafter, we deal with the case $\alpha > 1$ or $\beta > 1$. In this case, the total sum of the quantities increases and the system grows as a whole. When $w$ is positive, interactions between particles with large quantities are accelerated. Thus, the distribution is considered to become wider than the power-law. At $w < 0$, the PDF is expected to have a narrower tail. Firstly, the case $w < 0$ is investigated by considering an asymptotic moment function of PDF. A log-normal law. At $w > 0$, the system grows and does not have a steady state. Then, we deal with the case $p > 1$. The existence of winner-take-all type distribution is elucidated.

As mentioned above, we treat the case where the system grows and does not have a steady state. Then, we attempt to find a scaling solution of Eq. (2) and assume the relations

$$\gamma = \xi e^{\gamma t}, \quad f(z, t) = e^{-\gamma t} \Psi(\xi),$$

(4)

where $\gamma$ is a scaling parameter representing a growth rate of systems. Substituting Eq. (4) into Eq. (2), we have a scaled form of the master equation,

$$p \gamma \mu_p = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \Psi(\xi_1) \Psi(\xi_2) \frac{(\xi_1)^w (\xi_2)^w}{(\mu_w)^2} \times [ (\alpha \xi_1 + \beta \xi_2)^p - (\xi_1)^p ], \quad (p \in \mathbb{R}^+),$$

(5)

where $\mu_p$ is a scaled $p-$th moment. The parameter $\gamma$ is defined by Eq. (5) at $p = 1$,

$$\gamma = (\alpha + \beta - 1) \frac{\mu_1 + w}{\mu_1 \mu_w}.$$  

(6)

Note that Eq. (5) is symmetric with respect to interaction parameters $\alpha$ and $\beta$. Without loss of generality, therefore, we put $\alpha \geq \beta$. To proceed further, we follow the method of moment inequalities which is used to study inelastic hard sphere models by Bobylev et al. Applying lemma 2 to Eq. (5), we obtain inequalities

$$\frac{1}{(\mu_w)^2} \sum_{k=1}^{k_p} \left( \frac{p}{k} \right)^k \left( \frac{\beta}{\alpha} \right)^{p-k} \frac{\mu_k w \mu_{p-k} w}{\mu_{p+w} w} \leq \frac{1}{(\mu_w)^2} \sum_{k=1}^{k_p} \left( \frac{p}{k} \right)^k \left( \frac{\beta}{\alpha} \right)^{p-k} \frac{\mu_k w \mu_{p-k} w}{\mu_{p+w} w},$$

(7)

where $k_p$ denotes the integer part of $(p+1)/2$ for $p > 1$. It should be noted that when $p$ is an integer $n \in \mathbb{N}$, inequalities (7) are reduced to the equality

$$n(\alpha + \beta - 1) \frac{\mu_n}{\mu_1} = (\alpha^n + \beta^n - 1) \frac{\mu_1^w \mu_{n+w}}{\mu_{n+w+1}} + \sum_{i=1}^{n-1} \left( \frac{n}{i} \right)^i \beta^n \frac{\mu_i^w \mu_{n-i+w} \mu_{n+i+w}}{\mu_{n+w+1}}.$$  

(8)

The goal of this analysis is to estimate the asymptotic form of moment function $\mu_p$ ($p \to \infty$) when $w < 0$. Here, we assume the asymptotic form of $p-$th moment

$$\mu_p = \exp\left( a p^2 + b p \ln p + c p \right),$$

(9)

where $a$, $b$, and $c$ are unknown constants. It is found that Eq. (9) satisfies inequalities

$$0 \leq \left( \frac{p}{k} \right)^\frac{\mu_k + w \mu_p - k + w}{\mu_p} \leq \hat{C} \exp \left( -\tilde{A} p + \tilde{B} \ln p \right),$$

(10)

for $1 \leq k \leq k_p$, where $\tilde{A}, \tilde{B}, \hat{C} > 0$ are certain positive constants. It follows that in Eq. (10), l.h.s. of the first inequality and r.h.s. of the second inequality vanish in the limit $p \to \infty$. Hence, the following recurrence equation is fulfilled in the $p-$asymptotic region

$$p(\alpha + \beta - 1) \mu_p = (\alpha^p + \beta^p - 1) \frac{\mu_1^w \mu_{p+w}}{\mu_{p+w+1}}, \quad (p \to \infty).$$

(11)

Substitution of Eq. (9) into Eq. (11) leads to

$$a = \frac{\ln \alpha}{2|w|}, \quad b = \frac{1}{|w|},$$

$$c = \frac{1}{|w|} \ln \left( \frac{1 + \delta_{\alpha, \beta} \mu_1}{(\alpha + \beta - 1) \mu_{w+1}} \right) + \frac{\ln \alpha}{2}, \quad \frac{1}{|w|},$$

(12)

where $\delta_{\alpha, \beta}$ is the Kronecker delta. Consequently, the asymptotic solution of moment functions are determined self-consistently. We compare this result with data of Monte Carlo(MC) simulations and numerical solutions of the recurrence equations (8) and (11). Values obtained by MC simulations are used for the initial values of the recurrence equations. Notice that Eq. (8) and (11) provide exact results asymptotically. Two typical results at $w = -1$ and $-0.5$ are illustrated in Figs. 1. In the large $p$ region, $\ln(\mu_p)$ behaves as a quadratic function with a curvature consistent with $a$ in Eq. (12). MC results agree well at small $p$. Deviation at large $p$ comes from the finiteness of $N$.

Next, let us consider the shape of PDF. Inverse Mellin transform of the asymptotic moment function Eq. (9) with (12)

$$\Psi(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\mu_{s-1} \xi^{-s}}{\mu_{s-1} \xi^{-s}} ds,$$

(13)
FIG. 1: Semi-logarithmic plots of the scaled $p$-th moment $\mu_p$ versus $p^2$ at a weight parameter $w = -1$ (a) and $-0.5$ (b). Interaction parameters are $\alpha = 1.43589$ and $\beta = 0.1$. Data of MC simulations are illustrated in solid lines. In MC simulations, the number of particles is $N = 10^7$ and the number of interactions is $T = 200 \times N$. Solutions of the recurrence Eq. (8) and (11) are plotted in square dots ((a) only) and triangular dots, respectively. The results of Eq. (9) with parameters (12) are shown in dashed lines.

It is immediately found that the leading order term of Eq. (9) is transformed into a log-normal distribution.

$$\Psi(\xi) \simeq \frac{1}{\sqrt{4a\pi\xi}} \exp \left( -\frac{(\ln \xi)^2}{4a} \right), \quad (\xi \gg 1).$$  \hfill (14)

It is also checked by MC simulations that the PDF is log-normally distributed. It should be emphasized that the variance of the log-normal distribution is determined by $a = \ln(\alpha)/2|w|$ and is independent of the smaller interaction parameter $\beta < \alpha$. In many actual phenomena, in addition, interactions are weak and a growth rate is small. In this case ($0 < \beta \ll \gamma < 1$), Eq. (9) and (12) give

$$a \simeq \frac{1}{2|w|} \frac{\mu_1\mu_w}{\mu_{1+w}} \gamma \propto \gamma.$$  \hfill (15)

This dependence of the variance $a$ on the growth rate $\gamma$ differs qualitatively from that of one-body processes, where $a$ is independent of $\gamma$. This shows that the log-normal distributions of many-body processes belong to a totally different universality class from those of one-body processes.

The sign of the weight parameter $w$ is essential to the tail of PDF. When $w = 0$, Eq. (11) reduces to a transcendental equation $(\alpha + \beta - 1)p = \alpha^p + \beta^p - 1$, which coincides with the result in \cite{33}. When $w > 0$, it is suggested by MC simulations that any scaling solution does not hold. This is confirmed by the fact that the growth rate $\gamma$ is not kept constant and $m_1(t)$ diverges drastically after a critical time $t_c$ as shown in Fig. 2(a). Instead, in the long time limit, the processes end up with a winner-take-all state where two particular particles are selected almost invariably and gain almost all of the total sum of quantities as illustrated in Fig. 2(b). It becomes evident that the power-law distribution ($w = 0$) emerges at

FIG. 2: (a) Semi-logarithmic plot of temporal evolution of the first order moment $m_1(t)$. The inset shows the divergence of $m_1(t)$. (b) Double logarithmic plot of temporal evolution of the PDF. ($\alpha = 1.0998749$, $\beta = 0.005$, and $w = 0.5$.) In MC simulations, the number of particles is $N = 10^7$ and unit of time is defined by the number of interaction steps $N/2$. 

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the boundary between the log-normal tail \((w < 0)\) and the winner-take-all state \((w > 0)\). This situation is similar to those of self-organized criticality and scale-free network\(^3\), that is, criticality appears at the edge of chaos\(^3\), \(^3\).

As far as we know, this work gives the first analytical derivation of log-normal type distribution in many-body processes which is irreducible to one-body problem. It can be concluded that the following two conditions are essential for the emergence of log-normal distributions: (i) systems grow with multiplicative interactions, (ii) interactions between particles with smaller quantities are accelerated. In this Letter, we have treated only the case with a kernel given by Eq. \((3)\). Recently, we have examined the processes with a generic kernel \(K(x, y)\) and obtained essentially the same results. The details will be reported elsewhere\(^7\).

The processes are rather simple and expected to become a prototype of certain log-normal type distributions in nature. For example, production methods of fine particles are categorized into two types: breaking-down and building-up. The former may be described by one-body processes as the zeroth approximation. However, the latter must be treated as many-body processes because interactions between particles are essential for the building-up process in general. This suggests that log-normal distributions by building-up methods belong to the universality class of many-body processes with \(a \propto \gamma^0 \) while those by breaking-down methods are in the class of one-body processes with \(a \sim \gamma^0 = \text{const.} \). The difference is expected to be verifiable experimentally.

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