Principal Poincaré Pontryagin function associated to some families of Morse real polynomials

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Abstract
It is known that the principal Poincaré Pontryagin function is generically an Abelian integral. We give a sufficient condition on monodromy to ensure that it is also an Abelian integral in non-generic cases.

In non-generic cases it is an iterated integral. Uribe (2006 J. Dyn. Control. Syst. 12 109–34, 2009 J. Diff. Eqns 246 1313–41) gives in a special case a precise description of the principal Poincaré Pontryagin function, an iterated integral of length at most 2, involving logarithmic functions with only 1 ramification at a point at infinity. We extend this result to some non-isomonodromic families of real Morse polynomials.

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(Some figures may appear in colour only in the online journal)

1. Introduction
Throughout the paper \(F\) denotes a Morse polynomial \(F(x, y) : \mathbb{C}^2 \to \mathbb{C}\) with real coefficients, of degree \(d \geq 3\) and with \(d\) distinct real points at infinity. It always has \((d-1)^2\) critical points but in non-generic cases it has fewer than \((d-1)^2\) critical values. The one-form \(dF\) defines a foliation of \(\mathbb{C}^2\). We consider a continuous family of ovals \(\delta(t)\) in regular fibres \(F = t\) for \(t\) in some open interval and a transverse section to these ovals, parametrized by \(t\). Let \(\omega(x, y)\) be
a real polynomial one-form. To a one-parameter foliation defined by the perturbation \(dF + \varepsilon \omega\) for a small parameter \(\varepsilon\) is associated the displacement map which is the difference of the first return map and the identity. It is analytic with respect to \(\varepsilon\). The family of ovals is destroyed if and only if the expansion with respect to \(\varepsilon\) of the displacement map is not identically 0. To find a bound of the number of zeros depending on the degrees of the polynomial \(F\) and the one-form \(\omega\) of the first non-zero coefficient of its expansion in \(\varepsilon\) is the infinitesimal 16th Hilbert problem. This first non-zero coefficient is called the generating function in [11] and the first non-zero Melnikov function in [4]. Following [9] we will call it the principal Poincaré Pontryagin function.

It is proved in [11] that the principal Poincaré Pontryagin function is an iterated integral of polynomial one-forms (see [6] for iterated integrals) and that its length depends on topological properties of the Milnor fibration associated with the non-perturbed polynomial \(F\). If some singular fibre is non-reduced or non-connected one has to compute iterated integrals of one-forms polynomial with respect to the variables \(x, y\) but rational in \(F\), that is one-forms \(\eta(x, y)\) such that for some polynomial in 1 variable \(\tau, \tau(F)\eta(x, y)\) is polynomial (see [5]). We give an explicit computation in section 2.

An upper bound of the number of zeros of the principal Poincaré Pontryagin function is given in [4] (see also references herein). The principal Poincaré Pontryagin function satisfies a Picard–Fuchs system (see [4,9]). Clearly its dimension grows with the length of the iterated integral since the derivative of an iterated integral of length \(L\) involves iterated integrals of length lower than \(L\), see [4,9,8]. Hence the number of zeros also increases with the length of the principal Poincaré Pontryagin function as an iterated integral, see [4]. That motivates to bound from above this length.

The essential tool is the monodromy of the fibration. Generically, that is if all \((d - 1)^2\) critical values are distinct, the monodromy acts transitively on the homology with complex coefficients of regular fibres \(F^{-1}(t)\) and the principal Poincaré Pontryagin function is up to the sign the Abelian integral \(\int_{\delta(t)} \omega\). In non-generic cases the principal Poincaré Pontryagin function may also be an Abelian integral, the integral on the oval \(\delta(t)\) of polynomial one-form algorithmically computed from \(\omega\), see [13,14]. It is shown in [11] that the principal Poincaré Pontryagin function is well defined on the free homotopy class of the oval \(\delta(t)\) and not on its homology class. Hence one has to compute the action of the monodromy on the free homotopy class of \(\delta(t)\) as described in [11]. We give an example in section 2 and using results of [11] and [9] we show in proposition 3 that if the orbit of some oval generates a codimension 1 subspace of the homology then the principal Poincaré Pontryagin function is an Abelian integral.

If it is not an Abelian integral, the simplest case is the one where it is a length 2 iterated integral of one-forms polynomial with respect to \(x, y\) but may be rational with respect to \(F\), for example if the 0-level of \(F\) is a triangle, see [12], or more generally if \(F\) is the product of \(d\) linear factors, \(F = \ell_1 \cdots \ell_d\), satisfying the following hypothesis, see [18, 19].

**Hypothesis.** The \(d\) points at infinity \(\ell_k = 0\) are distinct, all critical points are Morse points, and the 0-level is the only critical level containing more than 1 critical point. The \(d(d - 1)/2\) intersection points of the line \(\ell_k = 0\) are real.

These properties ensure that the 0-level of \(F\) is what A’Campo calls a divide in [1,2], that is a union of immersions of \([0, 1]\) into some disc without self-tangencies, transversal to each other and to the boundary of the disc. We keep this terminology.

**Definition 1.** A polynomial \(F = \ell_1 \cdots \ell_d\) is a generic divide in lines if it satisfies the above hypothesis.
In [19] the second author proves that for generic divides in lines the principal Poincaré Pontryagin function is an iterated integral of length at most 2. The proof uses monodromy properties of divides. The divide shows all the homology of regular fibres $F^{-1}(t)$ and also allows to one compute the monodromy. Since our hypothesis remains true after a small perturbation it is natural to hope that the principal Poincaré Pontryagin function remains a length 2 iterated integral after a small perturbation. In section 3 we give two examples of one-parameter small perturbations of generic divides in lines and we check that the principal Poincaré Pontryagin function is still of length at most 2. Therefore, we note that the fibration defined by perturbed polynomial has more monodromy operators than the fibration defined by the generic divide in lines.

In section 4 we generalize examples of section 3 by introducing the following.

**Definition 2.** A simple connecting family is a continuous family of Morse polynomials $F_\lambda, \lambda \in [0, 1]$, $F_1 = F$ such that all $F_\lambda, \lambda \in [0, 1]$ are isomonodromic, satisfy the above hypothesis and $F_0$ is a generic divide in lines.

Two fibrations are isomonodromic if their monodromy groups are conjugated, we give a precise description of isomonodromy in section 4. Note that a connected family is not necessarily isomonodromic because $F_0$ may be more degenerated than $F_1$ if it has less critical values than $F_1$, hence monodromy groups have not the same number of generators and that is precisely the interesting case.

**Notation.** For regular $t$ we will denote by $H^1_{c}(t)$ the $\mathbb{C}$-vector space defined by the homology of the compactification of the fibre $F_\lambda = t, \lambda \in [0, 1]$ with coefficients in $\mathbb{C}$.

**Theorem A.** Let $F_\lambda$ be a simple connecting family, $\lambda \in [0, 1]$. Then the $\mathbb{C}$-vector space generated by the orbit of some oval $\delta(t)$ of the fibre $F_1 = t$ contains $H^1_{c}(t)$.

It is proved in [19] that if $F = \ell_1 \cdot \cdot \cdot \ell_d$ is a generic divide in lines the principal Poincaré Pontryagin function is generically not an Abelian integral and that it is of length at most 2. The polynomial one-forms generate a module of Abelian integral on $\mathbb{C}[t]$ where the module structure is given by $\int_{\delta(t)} (Homega) = t \int_{\delta(t)} omega$. As we have seen above we have to consider a $\mathbb{C}(t)$ vector space of Abelian or iterated integrals. In this case of generic divide in lines one has to use length 2 iterated integrals, that is integrals of multivalued one-forms $\Psi_1 \eta$ along each regular fibre $F = t$ and $\eta$ is a polynomial one-form. Then the principal Poincaré Pontryagin function belongs to the $\mathbb{C}(t)$ vector space of integrals along $\delta(t)$ of polynomial one-forms and one-forms such as $\Psi_k \eta$, thus are of length at most 2. We generalize this result in section 5.

**Theorem B.** If $F_\lambda(x, y)$ is a simple connecting family of Hamiltonians, then for $\lambda \neq 0$ there exist $v \leq d$ functions $\Psi_1, \ldots, \Psi_v$ such that each function $\Psi_k$ has logarithmic ramifications at some infinity points of the fibres $F_{\lambda}^{-1}(t)$ for regular values $t$ and is univalued out of the infinity points, and the principal Poincaré Pontryagin function is an element of the $\mathbb{C}(t)$ vector space generated by Abelian integrals on $\delta(t)$ and integrals along $\delta(t)$ of one-forms such as $\Psi_k \eta$ where $1 \leq k \leq v$ and $\eta$ is a polynomial one-form.

Thus the principal Poincaré Pontryagin function is of length at most 2.

2. **Codimension 1 case**

As usually we suppose that $F$ is a generic at infinity real Morse polynomial.
**Notation.** Let $t$ be a regular value of $F$. We denote as $\text{Orb}(\delta(t))$ the $\mathbb{C}$-vector subspace $H_1(t)$ generated by the orbit of the oval $\delta(t)$ considered as a cycle in regular fibres of $F$ under the monodromy action.

**Proposition 3.** If for regular $t$ the codimension of $\text{Orb}(\delta(t))$ is 1 in the $\mathbb{C}$-vector space $H_1(t)$ then the principal Poincaré Pontryagin function is an Abelian integral.

**Proof.** We denote by $r$ the dimension of $\text{Orb}(\delta(t))$, so that $r + 1$ is the Milnor number of the fibration defined by $F$, and we denote by $\delta_1(t), \ldots, \delta_r(t)$ a basis of $\text{Orb}(\delta(t))$ for regular $t$. We complete it to some basis $\delta_1(t), \ldots, \delta_r(t), \sigma(t)$ of $H_1(t)$, for regular $t$. The cycle $\sigma(t)$ can be chosen in such a way that it is invariant under the monodromy action thus it is a residual cycle at infinity. We use the monodromy representation of the principal Poincaré Pontryagin function as defined in [9, 11]. The same letters $\delta_1(t), \sigma(t)$ now denote loops with some base point $p(t)$ or even free loops. Let $S$ be the family of loops $\delta_1, \ldots, \delta_r$. We construct a family of free loops $\hat{S} = \{hsh^{-1}, h \in \Pi_1(F^{-1}(t), p(t)), s \in S\}$ and finally the main geometric object $H_1$$\delta(t)$$[\Pi_1(F^{-1}(t), p(t)), \hat{S}]$. This means that the elements of $H_1\delta(t)$ can be uniquely written as $\delta_1^p \cdot \ldots \cdot \delta_r^p \cdot \sigma^0$, since we have supposed that $\text{Orb}(\delta(t))$ does not contain $\sigma$. Thus there is a natural injection from $H_1\delta(t)$ into $H_1(F^{-1}(t), \mathbb{Z})$. Moreover, the group $H_1\delta(t)$ is finitely generated and from [9] it has moderate growth. It only remains to use the main result of [11]. □

This can be applied to the symmetric figure eight of [13, 14] or to the above mentioned example of a perturbation of the Hamiltonian triangle. If $\text{Orb}(\delta(t))$ contains all the homology but a two-dimensional space generated by 2 residual cycles at infinity, again we denote by $r$ the dimension of $\text{Orb}(\delta(t))$ for regular $t$, now $r + 2$ is the Milnor number. We complete with two cycles $\sigma_1, \sigma_2$. Then the elements of $H_1\delta(t)$ are written as, for instance, $\delta_1^{\beta_1} \cdot \ldots \cdot \delta_r^{\beta_r} \cdot \sigma_1^{\gamma_1} \cdot \sigma_2^{\gamma_2} \cdot \beta_1 + \beta_2 + \gamma_1 + \gamma_2 = 0$ and to any such free loop there corresponds the same cycle $\alpha_1 \delta_1 + \ldots + \alpha_r \delta_r$. Clearly this map is no more injective, the principal Poincaré Pontryagin function is an iterated integral on brackets and it is no more an Abelian integral, at least in the general case, see [12, 18].

We now assume that $F$ is a product of two irreducible polynomials $F_a, F_b$ and that the 0-level $F = 0$ is a divide in the real plane and all the homology of regular fibres $F^{-1}(t)$ is seen on this divide. This is a very particular case, as we will prove.

**Lemma 4.** For any oval $\delta(t)$ the orbit $\text{Orb}(\delta(t))$ does not contain all the homology of the fibre $F = t$.

**Proof.** The one-form $F_a dF_a$ is relatively cohomologous to $F \frac{dF_a}{F_a}$. Hence its integral is not identically 0 on residual cycles around infinity points $F_a = 0$. So this one-form is not algebraically relatively exact.

Nevertheless $\int_{\delta(t)} F_a dF_a = t \int_{\delta(t)} \frac{dF_a}{F_a} = 0$ since the cycle $\delta(t)$, which is vanishing at some critical value, does not turn around any point at infinity of fibres $F^{-1}(t)$. □

**Lemma 5.** If the perturbation is $\omega = \alpha F_b dF_a + \beta F_a dF_b$ for some reals $\alpha, \beta$, then the family of ovals is not destroyed.

**Proof.** First note that $\omega = F d \left( \ln(F_a^\delta, F_b^\delta) \right) = F \, dg$ with $g = \ln(F_a^\delta, F_b^\delta)$. The value 0 is a critical one, hence ovals are in some fibre $F = t, t \neq 0$. The only branching points of the logarithm are 0 and $\infty$. It is clear that the ovals $\delta(t)$ for regular $t$ do not turn around any branch.
of \( F = 0 \), so the function \( g \) is univalued along \( \delta(t) \). The perturbed polynomial one-form is \( dF + \varepsilon F dg = d(F + \varepsilon Fg) - \varepsilon g dF \). Since \( g \) is univalued along \( \delta(t) \), \( \int_{\delta(t)} d(F + \varepsilon Fg) \equiv 0 \) and of course \( \int_{\delta(t)} -\varepsilon g dF \equiv 0 \), so that \( \int_{\delta(t)} dF + \varepsilon \omega \equiv 0 \). □

This is in fact a Darboux integrable case, see [16]. The following lemma shows that this case is very particular.

**Lemma 6.** The homology is seen on the divide \( FaFb = 0 \) only if the 2 factors are of degree at most 2 and the degree 2 factors have no real point at infinity. Then the vector space \( \text{Orb}(\delta(t)) \) contains all the homology \( Hc^1 \) and its codimension is 1.

**Proof.** We will use a formula given without proof in [1]. For completeness we give a proof based on the computation of the Euler characteristic since the divide provides a decomposition of the disc. On the divide we see double points and regions. We denote by \( k \) the number of saddle points and \( K \) the number of compact closed regions of the divide \( F = 0 \). We suppose that \( r_\infty \leq d \) points at infinity of level curves of \( F \) are real. Hence this divide has \( 2r_\infty \) branches going to infinity. Hence

\[
1 = K + 2r_\infty - \left( \frac{4k}{2} + 3 \frac{2r_\infty}{2} \right) + k + 2r_\infty = K - k + r_\infty.
\]

From this formula we get that the homology is seen on the divide if

\[
(d - 1)^2 = k + K = 1 + 2k - r_\infty.
\]

We denote by \( d_a \) the degree of \( Fa \), by \( d_b \) the degree of \( Fb \). Using \( k \leq d_ad_b \) we obtain

\[
0 \geq -r_\infty \geq d_a(d_a - 2) + d_b(d_b - 2).
\]

The first result follows since both degrees \( d_a, d_b \) are at least 1. Divides are drawn in figure 1.

One can check that in the case of a line and an ellipse, \( \dim(Hc^1(t)) = 4 \) and \( \dim(\text{Orb}(\delta(t))) = 3 \). In the case of two ellipses, the Milnor number is 9 and \( \dim(\text{Orb}(\delta(t))) = 8 \). The cycle which is not in \( \text{Orb}(\delta(t)) \) has 0 as intersection number with any cycle vanishing cycle at a centre critical value, thus it is a residual cycle at infinity. □
3. Perturbations of a product of 3 or 4 linear factors in general position

The polynomial $F$ is a generic divide in lines of degree 3 or 4. We choose some oval $\delta(t)$ for regular $t$ and we denote by $\text{Orb}(\delta(t))$ the vector space generated by the orbit of this oval under the monodromy action.

If degree($F$)=3 then for regular $t$, $\text{Orb}(\delta(t))$ is a two-dimensional $\mathbb{C}$-vector space and contains the homology of the compactification of regular fibres, that is $H_1^c(t)$. It is complementary to the $\mathbb{C}$-vector space generated by two residual cycles at infinity. The coordinates can be chosen in such a way that $F(x, y) = xy(x+y-1)$ and $F^\epsilon = (xy+\epsilon)(x+y-1)$ (figure 2). The 0-level and the critical points of $F$ and $F^\epsilon$ are shown in figure 2. The principal Poincaré Pontryagin function is not an Abelian integral, see [12, 18].

The monodromy group of the Hamiltonian Triangle has only two generators: the monodromy around the critical value 0 and the monodromy around the centre type critical value. Since all singular points are of Morse type, there remains three saddles and one centre after perturbation if $\epsilon$ is sufficiently small but one saddle lies on a non-zero critical level, and now the monodromy group has three generators. Up to reparametrization the family $F^\epsilon$ is a simple connecting family, and $F_0$ is strictly more degenerated than $F^\epsilon$, $\epsilon \neq 0$. One can check that the codimension of $\text{Orb}(\delta(t))$ is 1 in the perturbed case. Then the principal Poincaré Pontryagin function is computed using only one multivalued function $\varphi$ defined by

$$d\varphi = F^\epsilon \frac{x+y-1}{x+y} = (xy+\epsilon)dx + dy$$

and it is an Abelian integral.

The following perturbation of the generic divide in four lines is very similar. Let $F_0 = xy \left( x + \frac{1}{2}y - 1 \right) \left( \frac{1}{2}x + y + 1 \right)$. The level $F = 0$ contains four lines in general position intersecting at six saddle points, the homology of regular fibres is a nine-dimensional $\mathbb{C}$-vector space, it is generated by three vanishing cycles at centre type singular values and the six cycles vanishing at 0, each one surrounding one saddle point. The only critical values are three critical values of centre type and 0, hence the monodromy group has only four generators. The orbit of any oval generates a codimension three vector space in $H_1^c(t)$. Hence one needs three functions $\varphi_1, \varphi_2, \varphi_3$ to perform the computation of the principal Poincaré Pontryagin function (see [19] for details) and this principal Poincaré Pontryagin function is generically not an Abelian integral.
Figure 3. Singular curves in the phase portrait of the perturbation \( F_\varepsilon = (xy + \varepsilon) \left( x + \frac{1}{2} y - 1 \right) \left( \frac{1}{2} x + y + 1 \right), \varepsilon > 0 \) of four straight lines. This divide is included in the union of two critical levels.

Let now

\[ F_\varepsilon = (xy + \varepsilon) \left( x + \frac{1}{2} y - 1 \right) \left( \frac{1}{2} x + y + 1 \right), \varepsilon > 0. \]

In this perturbed situation, the critical level 0 contains five saddle points, and the other four critical levels contain each one critical point, as can be seen in figure 3. There are now five critical values. Again up to reparametrization the family \( F_\varepsilon \) is a simple connecting family, and \( F_0 \) is strictly more degenerated than \( F_\varepsilon, \varepsilon \neq 0 \).

The orbit of any oval contains \( H_1^c(t) \) and its codimension is 2. Thus one needs now only two functions to compute the principal Poincaré Pontryagin function. The principal Poincaré Pontryagin function is generically not an Abelian integral, at least if the degree of the perturbative one-form is sufficiently great as shown in [19]. We can define \( \varphi_1, \varphi_2 \) as relative primitives of polynomial one-forms \( F_\varepsilon d \left( x + \frac{1}{2} y - 1 \right) \) and \( F_\varepsilon d \left( \frac{1}{2} x + y + 1 \right) \).

We can use an additional parameter to break one more connexion:

\[ F_{\varepsilon, \varepsilon'} = (xy + \varepsilon) \left( \left( x + \frac{1}{2} y - 1 \right) \left( \frac{1}{2} x + y + 1 \right) + \varepsilon' \right), \varepsilon' \ll \varepsilon \ll 1. \]

Now the codimension of \( \text{Orb}(\delta(t)) \) is 1 and the principal Poincaré Pontryagin function is an Abelian integral. We can put \( F_{\varepsilon, \varepsilon'} \) into a chain of two simple connecting families, what we could call a connecting family. Theorem A remains true for (non-necessary simple) connecting families. Note that \( F_{\varepsilon, \varepsilon'} \) is not a two-parameter family since we have \( \varepsilon' \ll \varepsilon \ll 1 \).

4. Proof of theorem A

Here we prove that if \( F \) is in a simple connecting family \( F_\lambda \), then the vector space \( \text{Orb}(\delta(t)) \) contains all the homology \( H_1^c(t) \) for any oval and regular values \( t \). Therefore, we prove that the monodromy group has in some sense more generators in the Milnor fibration defined by \( F \) than in the Milnor fibration defined by the divide in lines \( F_0 \).
Notation. We denote by $\Sigma_\lambda$ the set of critical values of $F_\lambda$.

Lemma 7. Let $F_\lambda, \lambda \in [0, 1]$ be a simple connecting family of polynomials. The map $(x, y, \lambda) \to (F_\lambda(x, y), \lambda)$ defines a fibration from $\{(x, y, \lambda)\} = \mathbb{C}^2$ to the subset $\mathbb{C}^2 \setminus \{\lambda \times \Sigma_\lambda\}$ of $\{(\lambda, t)\} = \mathbb{C}^2$.

Proof. We consider $\lambda$ as a complex parameter. From Ehresmann fibration theorem (see [20]) this mapping defines a fibration with basis the complement of the set where its rank is not maximal. So the basis of this fibration is the complementary of $\{(c, \lambda)\}$ such that $c$ is a critical value of $F_\lambda$. Hence the basis of this fibration is $\mathbb{C}^2 \setminus \{\lambda \times \Sigma_\lambda\}$. \hfill $\square$

For simplicity we will project the fibres into $\{(x, y)\} = \mathbb{C}^2$ and denote by $F^{-1}_\lambda(t)$ any regular fibre, a Riemann surface in $\mathbb{C}^2$. Now again we restrict to $\lambda \in [0, 1]$ where the deformation is isomonodromic. That means the following. From preceding lemma, if $\lambda$ and $\mu$ are two near values of the parameter, there is a local connection sending $\Sigma_\lambda$ to $\Sigma_\mu$, a regular fibre $F^{-1}_\lambda(t)$ to some regular fibre $F^{-1}_\mu(t)$, and $H_{1,\lambda}(t)$ onto $H_{1,\mu}(t)$, where $H_{1,\lambda}(t)$ denotes the $\mathbb{C}$-homology of the fibre $F^{-1}_\lambda(t)$. Since $F_\lambda$ and $F_\mu$ are isomonodromic they have the same number of critical values and the connection establishes a one–one correspondence from $\Sigma_\lambda$ onto $\Sigma_\mu$. Moreover, this connection commutes with the monodromy. Namely assume that some critical value $c_\lambda$ of $F_\lambda$ is sent to $c_\mu$ of $F_\mu$. Then for any cycle $\delta_\lambda(t_\lambda)$ we get the same result if we first let act the monodromy around $c_\lambda$ and then the connection, or if we first transport the cycle into $H_{1,\mu}(t_\mu)$ and the let act the monodromy around $C_\mu$.

Now the polynomial $F_0$ is not isomonodromic with $F_\lambda$ but since all critical points are of Morse type they vary continuously with respect to $\lambda$. Hence we have the following.

Lemma 8. The limit of critical values of $F_\lambda$ when $\lambda$ goes to 0 is one of the critical values of $F_0$.

Denote as $c_j^\lambda$ the critical values of $F_\lambda$, which go to a critical value of $F_0$ of centre type. Denote as $z_j^\lambda$ the critical values of $F_\lambda$, which go to 0 when $\lambda$ goes to 0.

Lemma 9. The critical levels $c_j^\lambda$ contain only 1 critical point.

Proof. This is true for $\lambda = 0$. Since it is an open property it is true for $\lambda$ near 0. By isomonodromy for $\lambda \in [0, 1]$ it is true for $\lambda \in [0, 1]$. \hfill $\square$

The homology of the regular fibres $F_\lambda = t$ varies continuously. The vanishing cycles around critical values $c_j^\lambda$ will be denoted by $\delta_j^\lambda$. If a cycle in the Milnor fibration defined by $F_\lambda$, $\lambda \in [0, 1]$ vanishes at some $z_j^\lambda$ then when $\lambda \to 0$ it goes to a vanishing cycle of the divide in lines $F_0$, more precisely a cycle vanishing at 0, which shrinks at a double point, intersection of two lines $\ell_m = 0$ and $\ell_n = 0$. We will denote it by $\gamma_{m,n}^j$. The homology at infinity contains all residual cycles around $\ell_\mu = 0$, $n = 1, \ldots, d$. Moreover, we suppose that points at infinity are fixed for $\lambda \in [0, 1]$. With the same notations as above, the connection of lemma 7 sends the homology at infinity of $F^{-1}_\lambda(t_\lambda)$ onto the homology at infinity of $F^{-1}_\mu(t_\mu)$.

The polynomial $F_0 = \ell_1 \cdots \ell_d$ is a generic divide in lines, thus the critical level $F_0(x, y) = 0$ contains $d(d - 1)/2$ saddle points. Since $F_0$ is a Morse polynomial there are $(d - 1)^2$ critical points, hence $K = (d - 1)(d - 2)/2$ critical points of centre type and from genericity hypothesis all these critical points lie on distinct non-zero critical levels. The monodromy operators of the degenerated polynomial $F_0$ are the monodromy around 0 and the $K$ monodromy operators around critical values of centre type, $c_j^0$. The monodromy operators of $F_\lambda$, $\lambda \in [0, 1]$ are the monodromy around $K$ critical values $c_j^\lambda$ and monodromy operators.
around each critical value \( z_{\lambda}^j \). We will only use the monodromy generated by a loop turning once counterclockwise around all critical values \( z_{\lambda}^j \) and only around them, and by \( K \) loops turning once counterclockwise around 1 of the critical values \( c_{\lambda}^j \).

**Definition 10.** The subgroup of the monodromy generated by the monodromy operators around each \( c_{\lambda}^j \) and by a loop turning once clockwise around all critical values \( z_{\lambda}^j \) will be called the sub-monodromy.

This sub-monodromy of \( F_{\lambda} \) has \( 1 + K \) generators, exactly as many generators as the monodromy of the Milnor fibration defined by \( F_0 \). Choose some oval \( \delta(t_1) \) in the homology of one regular fibre \( F_1 = t_1 \). It varies continuously with \( \lambda \), thus it is in a family denoted by \( \delta_{\lambda}(t_\lambda) \). Its limit when \( \lambda \) goes to 0 is one of the ovals of \( F_0 = t_0 \) where \([t_0, t_1]\) is a path in \( \mathbb{C} \) such that \( t_\lambda \) is regular for \( F_{\lambda}, \lambda \in [0, 1] \). Since everything depends continuously on \( \lambda \), the orbit under the action of the sub-monodromy of \( \delta(t) \) also varies continuously. Thus the dimension of the vector space generated by this orbit is constant.

**Notation.** The dimension of \( \text{Orb}(\delta_{\lambda}(t)) \) is denoted by \( r_{\lambda} \).

**Lemma 11.** The dimension \( r_{\lambda} \) is at least the dimension in the degenerated case: \( r_{\lambda} \geq r_0 \).

**Proof.** The vector space \( \text{Orb}(\delta_{\lambda}(t_\lambda)) \) contains the vector space generated by the action of the sub-monodromy. □

When \( \lambda \) goes to 0, the limit of \( \text{Orb}(\delta_{\lambda}(t_\lambda)) \) is a vector space containing \( \text{Orb}(\delta_0(t_0)) \) as a subspace. We know from [19] that \( \text{Orb}(\delta_0(t_0)) \) contains all homology of the compactification of regular fibres. Moreover, the homology at infinity does not depend on \( \lambda \). That finishes the proof of theorem A.

5. Proof of theorem B

If \( \int_{\delta(t)} \omega \) is not identically 0 then it is an Abelian integral and it is the principal Poincaré–Pontryagin function and we are done. If \( \int_{\delta(t)} \omega \equiv 0 \) we have to compute further and first we construct a convenient basis of the relative cohomology for some regular \( t \).

**Lemma 12.** If the orbit of some cycle \( \delta(t) \) is not the whole homology, then there exist polynomial one-forms such that their integral on \( \delta(t) \) is identically 0 and their integral on cycles of the complementary of \( \text{Orb}(\delta(t)) \) is not 0.

**Proof.** We denote by \( r \) the dimension of \( \text{Orb}(\delta(t)) \), \( v = (d - 1)^2 - r \). The Petrov module of the integrals of polynomial one-forms on \( \delta \) has dimension \( r \), see [10]. It contains the integrals on \( \delta(t) \) of polynomial one-forms \( \omega_1, \ldots, \omega_v \), free as elements of a \( \mathbb{C}(t) \)-vector space. This family can be completed with polynomial one-forms to a \( \mathbb{C}(t) \)-basis of the relative cohomology. From the dimension of the Petrov module of Abelian integrals on \( \delta \), one can construct a basis of the relative cohomology of polynomial one-forms \( \omega_1, \ldots, \omega_v, \psi_1, \ldots, \psi_v \) in such a way that the basis is such that \( \int_{\delta(t)} \psi_k \equiv 0 \) if \( k = 1, \ldots, v \). □

**Remark.** Since this is a basis of the relative cohomology the integrals of the one-forms \( \psi_1, \ldots, \psi_v \) on cycles of a complementary of the orbit \( \text{Orb}(\delta(t)) \) are free in the \( \mathbb{C}(t) \)-vector space of Abelian integrals, hence also in the \( \mathbb{C}[t] \)-module of Abelian integrals.
Let us use generalized Françoise’s algorithm, see [7, 8] with multivalued functions. If $M_1(t) = -\int_{\delta(t)} \omega = 0$ then there exist polynomials $\alpha_k(F), k = 1, \ldots, v$ such that the one-form $\omega = \sum_{k=1}^v \alpha_k(F)\psi_k$ has integral 0 on all cycles of $F = t$, that is this form is topologically relatively exact. From [5] we know that there exists a polynomial $T$ in $F$ such that $T(F) (\omega - \sum_{k=1}^v \alpha_k(F)\psi_k)$ is algebraically relatively exact, that is there exist polynomials $Q, R$ in $x, y$ such that

$$T(F) \left( \omega - \sum_{k=1}^v \alpha_k(F)\psi_k \right) = Q(x, y) \, dF + dR(x, y).$$

This polynomial is called torsion in [5] and it depends on reducible or non-connected fibres $F = t$. This yields

$$\omega = \sum_{k=1}^v \alpha_k(F)\psi_k + \frac{dR(x, y)}{T(F)} + \frac{Q(x, y)}{T(F)} \, dF.$$ 

Now $\frac{dR(x, y)}{T(F)} = d \left( \frac{R(x, y)}{T(F)} \right) + T'(F) \frac{R(x, y)}{T'(F)} \, dF$ where $T'$ denotes the usual derivative of the polynomial $T$ with respect to $F$.

We define a primitive of any $\psi_k$, $k = 1, \ldots, v$ as follows and we will denote it by $\Psi_k$. We choose a base point in some regular fibre $F = t_0$, this allows one to compute $\Psi_k$ restricted to this fibre. It is multivalued since the one-form $\psi_k$ is not algebraically relatively exact, but it is univalued along the fiber $\delta(t_0)$ and along all cycles of $\text{Orb}(\delta(t_0))$. Then we fix a section transversal to our family of regular fibres. This allows one to compute $\Psi_k$ on fibres $F = t$ for $t$ in our family of regular values. For any $t$, the function $\Psi_k$ is not univalued but it is univalued along all cycles of $\text{Orb}(\delta(t))$. Note that

$$\alpha_k(F)\psi_k = d(\alpha_k(F)\Psi_k) - \Psi_k \, d(F\alpha_k(F)), \quad k = 1, \ldots, v.$$

Finally there exist functions $f_1, g_1$ which are polynomials in $x, y, \Psi_1, \ldots, \Psi_v$ and rational in $F$ such that

$$\omega = g_1(F, x, y, \Psi_1, \ldots, \Psi_v) \, dF + df_1(F, x, y, \Psi_1, \ldots, \Psi_v).$$

And we can go to the next step of the algorithm and compute $M_2(t) = \int_{\delta(t)} g_1 \omega$ that is a length-2 iterated integral. This function lies in the $C(t)$-vector space generated by Abelian integrals and integrals such as $\int_{\delta(t)} \Psi_k \Psi_j$.

**Notation.** We denote by $I_{k,j} = \int_{\delta(t)} \Psi_k \Psi_j, 1 \leq k < v, 1 \leq j \leq v$. If we compute these integrals on the fibres of $F_k$ we will denote them by $I_{k,k}^i(t)$ or $I_{k,j}^i(t)$, and $\psi_k, k = 1, \ldots, v$ the one-forms with relative primitives $\Psi_k$.

Indeed since $\Psi_k \psi_j + \Psi_k \psi_j = d(\Psi_k \Psi_j)$, integrals $I_{k,j}^i(t)$ and $I_{j,k}^i(t)$ are opposite. So we use $I_{k,j}^i(t), 1 \leq k < j < r$. Recall that from the definition of length-2 iterated integrals, see [6] $I_{k,j}^i(t) = \int_{\delta(t)} \Psi_k \Psi_j$, that is we integrate a multivalued one-form. This makes sense only on paths. So we have to suppose that there is some base point on $F = t$ and that any cycle is represented as a path. By abuse we denote again as $\delta(t)$ this path. If we change the base point the primitive $\Psi_k$ may become $\Psi_k + C_k$ for some constant $C_k$. We know from [11] that the principal Poincaré Pontryagin function is base point independent. Indeed after some change of the base point the length-2 integral $I_{k,j}^i(t)$ becomes $\int_{\delta(t)} (\Psi_k + C_k) \Psi_j$. Its variation is $\int_{\delta(t)} C_k \Psi_j$ which is identically 0 by construction. Thus our integrals $I_{k,j}^i(t)$ are base point independent. It was shown in [12] that they are not Abelian. We show a little more in the following essential lemma.
Lemma 13. The integrals of length 2 are free as elements of the $\mathbb{C}(t)$-vector space generated by Abelian integrals and the $I_{k,j}(t)$.

Proof. Here we consider $F = F_1$ as an element of a connecting family $F_{\lambda}$. Recall that for $\lambda = 0$, that is for the divide in lines, we have used one-forms $\varphi_k = F \frac{dt}{d\tau_k}$, $k = 1, \ldots, d - 1$. We can suppose that the complementary to $\text{Orb}(\delta_i(t))$ contains precisely the residual cycles dual to one-forms $\varphi_k = F \frac{dt}{d\tau_k}$, $k = 1, \ldots, v$.

Furthermore, the polynomials $F_\lambda$ have same points at infinity, hence same degree $d$ terms. Hence we can choose one-forms $\psi_k$ in such a way that for $t$ going to $\infty$ we have $\psi_k \sim \varphi_k$ for $k = 1, \ldots, v$. This yields $I_{\delta_i}(t_\lambda) \sim I_{\delta_i}(t_0)$ when $t$ goes to $\infty$.

It was proved in [19] that the non-Abelian integrals $I^0_{k,j}(t_0)$ are free as elements of a $\mathbb{C}(t)$-vector space, for $1 \leq k < j \leq d - 1$, hence for $1 \leq k < j \leq v$. Now we want to know what happens if for some polynomials $\alpha^\lambda_{k,j}(F)$,

$$\sum_{k,j} \alpha^\lambda_{k,j}(t) I^\lambda_{k,j}(t) = 0.$$  

If these polynomials are not identically 0 they have higher degree terms. This is the dominating term of preceding combination for $t$ going to $\infty$. Since we have supposed that points at infinity of the polynomial $F_{\lambda}$ and one-forms $\psi_k$ are independent of $\lambda$, then this dominating term does not depend on $\lambda$. But its limit when $\lambda$ goes to 0 is 0 because the integrals $I_{k,j,0}(t)$ are free as elements of a $\mathbb{C}(t)$-vector space. Thus this dominating term is 0 and this is a contradiction. □

Lemma 14. The iterated integrals of length 3 such as $\int_{\delta(t)} \psi_k \psi_j \psi_m$ are not base point independent.

Proof. Again we have to choose a base point and to integrate along paths lying in the fibre $F = t$ and starting at this base point and we denote as $\int_{\delta(t)} \psi_k \psi_j \psi_m$ an iterated integral along some path representing the cycle $\delta(t)$. If we denote by $\gamma(p)$ the piece of $\delta$ starting at the base point and going to the point $p$ of $\delta$ then $\int_{\delta(t)} \psi_k \psi_j \psi_m$ it computed as the integral along $\delta$ of the multivalued one-form which takes the value $\left(\int_{\gamma(p)} \psi_k \psi_j \psi_m(p)\right)$ when $p$ varies along $\delta$.

The length 2 iterated integral $\int_{\delta(t)} \psi_k \psi_j$ is computed as the integral on the path $\gamma(p)$ of the multivalued one-form $\psi_k \psi_j$. If we move the base point then $\psi_k$ may become $\psi_k + C_k$ for some constant $C_k$, the integral $\int_{\delta(t)} \psi_k \psi_j \psi_m$ becomes $\int_{\delta(t)} \psi_k \psi_j \psi_m + C_k I_{j,m}(t)$. Its variation is non-zero since this integral $I_{j,m}(t)$ is non-zero. □

The following result finishes the proof.

Corollary 15. The only base point independent one-forms are the Abelian integrals and $I_{j,m}(t)$.

It was proved in [18] that generically the principal Poincaré Pontryagin function of order 2, $M_2(t)$, is not an Abelian integral if the degree of the perturbative one-form is at least 5 and the Hamiltonian $F_0$ is of degree 3.

6. Conclusion and perspectives

Clearly the final goal is to apply these results to the infinitesimal 16th Hilbert Problem. Before that a first generalization would be to use chains of simple connecting families. Namely we can use a simple connecting family connecting $F$ to a more degenerated polynomial which
could be less degenerated than a generic divide in lines. The product of these paths in the manifold of Morse polynomials of degree \( d \) with fixed \( d \) real points at infinity is a connecting family. Proof of theorem A shows that the orbit of some oval in \( F = t \) for generic \( t \) contains all the homology \( H^c_1(t) \). Theorem B can be generalized to such families.

Next one could relax hypothesis, for instance allow to points at infinity to move but keep generic at infinity polynomials along all the connecting family. This would allow complex points at infinity, which is natural since the monodromy works in \( \mathbb{C}^2 \) and thus the fact that points at infinity are real or not is irrelevant. Therefore, one has to adapt proof of lemma 13.

We conjecture that as soon as \( F \) is a Morse polynomial of degree \( d \) with \( d \) distinct points at infinity such that at least \( d(d + 1)/2 \) critical levels contain only one critical point then the principal Poincaré–Pontryagin function is an iterated integral of length at most 2. This conjecture is based on the stratification of polynomials given by the Zariski–Tarskii theorem (or Chevalley theorem), see [3, 17]. Indeed any Morse polynomial \( F \) lies in some stratum. If there is a generic divide in lines in the boundary of this stratum then there exists a simple connecting family from \( F \) to the generic divide in lines and we are done. Else we can use the following technic indicated to us by Maxim Kazarian who solves similar questions in [15].

We suppose that \( F \) is a real polynomial of degree \( d \) with \( d \) real points at infinity, that all critical points are of Morse type, that at most \( d/2 \) critical points lie on the level \( F = 0 \) and that other critical levels contain only one critical point. We are going to prove that such polynomials can be put in a simple connecting family.

Therefore, we choose \( d \) lines \( \ell_1, \cdots, \ell_d \) such that the algebraic curve \( \ell_1 \cdots \ell_d = 0 \) contains all \( d/2 \) or \( (d - 1)/2 \) critical points of the 0-level of \( F \) and both polynomials \( F \) and \( F_0 = \ell_1 \cdots \ell_d \) have the same points at infinity. Consider the family of polynomials

\[
F_\lambda = \lambda F + (1 - \lambda) F_0, \quad F_1 = F, \quad F_0 = \ell_1 \cdots \ell_d.
\]

All polynomials \( F_\lambda, \lambda \in [0, 1] \) have the same points at infinity. Moreover, all \( d/2 \) or \( (d - 1)/2 \) critical points of the critical level \( F = 0 \) are critical points of \( F_\lambda, \lambda \in [0, 1] \).

If all critical points of \( F_\lambda \) are of Morse type and all non-zero critical levels of \( F_\lambda, \lambda \in [0, 1] \) contain only one critical point, we have constructed a simple connecting family and the result is proved. Else it means that for some isolated values of \( \lambda \) either one critical point is not Morse or one non-zero critical level contains more than one critical point. By Zariski–Tarski theorem each of these two conditions define an algebraic set in \( \mathbb{C}^{d-1}(x, y) \), vector space of polynomials of degree at most \( d \) with fixed points at infinity. Thus it can be avoided by following a path in \( \mathbb{C}^{d-1}(x, y) \) instead of \( \mathbb{R}^{d-1}(x, y) \). Again we have constructed a simple connecting family and the result is proved.

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