ON SOME LIE GROUPS AS 5-DIMENSIONAL ALMOST CONTACT B-METRIC MANIFOLDS WITH THREE NATURAL CONNECTIONS

MIROSŁAWA IVANOVA AND HRISTO MANEV

ABSTRACT. Almost contact manifolds with B-metric are considered. There are studied three natural connections (i.e. linear connections preserving the structure tensors) determined by conditions for their torsions. These connections are investigated on a family of Lie groups considered as 5-dimensional almost contact B-metric manifolds.

INTRODUCTION

In differential geometry of manifolds with additional tensor structure, there are important the so-called natural connections, i.e. linear connections with respect to which the structure tensors of the manifolds are parallel.

The natural connections are studied in the geometry of manifolds with some additional structures: almost Hermitian structure [5], almost contact metric structure [1], almost complex structure with Norden metric [3].

The geometry of almost contact B-metric manifolds is the geometry of the structures \( \varphi, \xi, \eta, g \) and \( \tilde{g} \). For this geometry a natural connection \( D \) is parallel with respect to \( \varphi \) and \( g \) (consequently, \( \xi, \eta \) and \( \tilde{g} \)). By \( \mathcal{F}_0 \) is denoted the class of the considered manifolds with parallel structure \( \varphi \) with respect to the Levi-Civita connection \( \nabla \). The natural connections play the same role outside the class \( \mathcal{F}_0 \) such a role plays \( \nabla \) in \( \mathcal{F}_0 \).

Three natural connections (\( \varphi \)KT-connection, \( \varphi \)B-connection and \( \varphi \)-canonical connection) on almost contact B-metric manifolds are an object of special interest in this paper. The goal of the present work is the investigation of those three natural connections on a family of Lie groups as 5-dimensional almost contact B-metric manifolds belonging to a basic class.

The paper is organized as follows. In Sec. 1 we give necessary facts about considered manifolds. In Sec. 2 we consider an example on a family of Lie groups as 5-dimensional \( \mathcal{F}_7 \)-manifolds with a parallel torsion of the \( \varphi \)KT-connection. In Sec. 3 we describe the \( \varphi \)B-connection and the \( \varphi \)-canonical connection on these manifolds.

1. Almost contact manifolds with B-metric

Let \( (M, \varphi, \xi, \eta, g) \) be a \((2n + 1)\)-dimensional almost contact B-metric manifold. In details, \((\varphi, \xi, \eta)\) is an almost contact structure determined by a tensor field \( \varphi \) of

Key words and phrases. Almost contact manifold; B-metric; natural connection; \( \varphi \)KT-connection; \( \varphi \)B-connection; \( \varphi \)-canonical connection; totally real section; holomorphic section; Lie group; Lie algebra.
type (1,1), a vector field \( \xi \) and its dual 1-form \( \eta \) as follows:

\[
\varphi \xi = 0, \quad \varphi^2 = -\text{Id} + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,
\]

where \( \text{Id} \) is the identity. Moreover, \( g \) is a pseudo-Riemannian metric such that for any differentiable vector fields \( x, y \) on \( M \), it is valid: [4]

\[
g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y).
\]

Further, \( x, y, z, w \) will stand for arbitrary elements of \( \mathfrak{X}(M) \) or vectors of the tangent space at arbitrary point in \( M \).

Let us remark that the restriction of a B-metric on the contact distribution \( H = \ker(\eta) \) coincide with the corresponding Norden metric with respect to the almost complex structure (the restriction of \( \varphi \) on \( H \)) acting as an anti-isometry on the metric (the restriction of \( g \)) of \( H \). Thus, it is obtained a correlation between a \((2n+1)\)-dimensional almost contact B-metric manifold and a \(2n\)-dimensional almost complex manifold with Norden metric (or an \( n \)-dimensional complex Riemannian manifold).

The associated metric \( \tilde{g} \) of \( g \) on \( M \) given by \( \tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y) \) is a B-metric, too. The manifold \((M, \varphi, \xi, \eta, \tilde{g})\) is also an almost contact B-metric manifold. Both metrics \( g \) and \( \tilde{g} \) are indefinite of signature \((n+1, n)\).

The structure group of \((M, \varphi, \xi, \eta, g)\) is \( \mathcal{G} \times \mathcal{I} \), where \( \mathcal{I} \) is the identity on \( \text{span}(\xi) \) and \( \mathcal{G} = \mathcal{GL}(n; \mathbb{C}) \cap \mathcal{O}(n, n) \).

Let \( \nabla \) be the Levi-Civita connection of \( g \). The tensor \( F \) of type (0,3) on \( M \) is defined by \( F(x, y, z) = g((\nabla_x \varphi) y, z) \). It has the following properties:

\[
F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi).
\]

A classification of the almost contact B-metric manifolds is introduced in [4], where eleven basic classes \( \mathcal{F}_i \) \((i = 1, 2, \ldots, 11)\) of these manifolds are characterized with respect to \( F \). These basic classes intersect in the special class \( \mathcal{F}_0 \) determined by the condition \( F(x, y, z) = 0 \). Hence \( \mathcal{F}_0 \) is the class of almost contact B-metric manifolds with \( \nabla \)-parallel structures, i.e. \( \nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0 \).

Let \( g_{ij} \), \( i, j \in \{1, 2, \ldots, 2n+1\} \), are the components of the matrix of \( g \) with respect to a basis \( \{e_i\}_{i=1}^{2n^2} = \{e_1, e_2, \ldots, e_{2n+1}\} \) of the tangent space \( T_p M \) of \( M \) at an arbitrary point \( p \in M \), and \( g^{ij} \) – the components of the inverse matrix of \( (g_{ij}) \).

It is defined the square norm of \( \nabla \varphi \) by: [10]

\[
\|\nabla \varphi\|^2 = g^{ij} g^{k\ell} g(\nabla_{e_i} \varphi, e_k, \nabla_{e_j} \varphi, e_\ell).
\]

It is clear, the equality \( \|\nabla \varphi\|^2 = 0 \) is valid if \((M, \varphi, \xi, \eta, g)\) is a \( \mathcal{F}_0 \)-manifold, but the inverse implication is not always true. An almost contact B-metric manifold having a zero square norm of \( \nabla \varphi \) is called an isotropic-\( \mathcal{F}_0 \)-manifold [10].

The Nijenhuis tensor \( N \) of the almost contact structure is defined by \( N = [\varphi, \varphi] + d\eta \otimes \xi, \) where \( [\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2 [x, y] - \varphi [\varphi x, y] - \varphi [x, \varphi y] \) for \([x, y] = \nabla_x y - \nabla_y x \) and \( d\eta \) is the exterior derivative of \( \eta \). In [11], it is defined an associated Nijenhuis tensor \( \tilde{N} \) by \( \tilde{N} = \{\varphi, \varphi\} + (L_\xi g) \otimes \xi \), where \( \{\varphi, \varphi\}(x, y) = \varphi^2 \{x, y\} + \{\varphi x, \varphi y\} - \varphi \{\varphi x, y\} - \varphi \{x, \varphi y\} \) for \( \{x, y\} = \nabla_x y + \nabla_y x \) and \( L_\xi g \) is the Lie derivative with respect to \( \xi \) of the metric \( g \).
Hence, \( N \) and \( \hat{N} \) in terms of the covariant derivatives has the following form:
\[
N(x, y) = (\nabla_x \varphi) y - \varphi (\nabla_x y) + (\nabla_x \eta) y \cdot \xi
- (\nabla_y \varphi) x + \varphi (\nabla_y x) x - (\nabla_y \eta) x \cdot \xi,
\]
\[
\hat{N}(x, y) = (\nabla_x \varphi) y - \varphi (\nabla_x y) + (\nabla_x \eta) y \cdot \xi
+ (\nabla_y \varphi) x - \varphi (\nabla_y x) x + (\nabla_y \eta) x \cdot \xi.
\]

The corresponding tensors of type (0,3) on \((M, \varphi, \xi, \eta, g)\) are determined by
\[
N(x, y, z) = g(N(x, y), z) \quad \text{and} \quad \hat{N}(x, y, z) = g(\hat{N}(x, y), z).
\]

In [12], a tensor \( L \) of type (0,4) on \( M \) with properties
\[
L(x, y, z, w) = -L(y, x, z, w) = -L(x, y, w, z),
\]
\[
L(x, y, z, w) + L(y, z, x, w) + L(z, x, y, w) = 0
\]
is called a curvature-like tensor.

The curvature tensor \( R \) of \( \nabla \) is determined by \( R(x, y)z = \nabla_x \nabla_y z - \nabla_y \nabla_x z - \nabla_{[x, y]} z \). The corresponding tensor of \( R \) of type (0,4) is denoted by the same letter and is defined by the condition \( R(x, y, z, w) = g(R(x, y)z, w) \).

The Ricci tensor \( \rho \) and the scalar curvature \( \tau \) for \( R \) as well as their associated quantities are defined by the traces \( \rho(x, y) = g^{ij}R(e_i, x, y, e_j) \), \( \tau = g^{ij}\rho(e_i, e_j) \), \( \rho^*(x, y) = g^{ij}R(e_i, x, y, \varphi e_j) \) and \( \tau^* = g^{ij}\rho^*(e_i, e_j) \), respectively. In a similar way there are determined the Ricci tensor \( \rho(L) \), the scalar curvature \( \tau(L) \) and their associated quantities for any curvature-like tensor \( L \).

Let \( \alpha \) be a non-degenerate 2-plane (section) in the tangent space \( T_p M \), \( p \in M \). The special 2-planes with respect to the almost contact B-metric structure \((\varphi, \xi, \eta, g)\) are: totally real section if \( \alpha \) is orthogonal to its \( \varphi \)-image \( \varphi \alpha \) and \( \xi \), \( \varphi \)-holomorphic section if \( \alpha \) coincides with \( \varphi \alpha \) and \( \xi \)-section if \( \xi \) lies on \( \alpha \) [21].

The sectional curvature \( k(\alpha, p)(L) \) of \( \alpha \) with an arbitrary basis \( \{x, y\} \) at \( p \) regarding a curvature-like tensor \( L \) is defined by
\[
k(\alpha, p)(L) = \frac{L(x, y, y, x)}{g(x, x)g(y, y) - g(x, y)^2}.
\]

It is known, [10], a linear connection \( D \) is called a natural connection on the manifold \((M, \varphi, \xi, \eta, g)\) if the almost contact structure \((\varphi, \xi, \eta)\) and the B-metric \( g \) (consequently also \( \hat{g} \)) are parallel with respect to \( D \), i.e. \( D \varphi = D \xi = D \eta = D g = D \hat{g} = 0 \). In [15], it is proved that a linear connection \( D \) is natural on \((M, \varphi, \xi, \eta, g)\) if and only if \( D \varphi = D g \). A natural connection exists on any almost contact B-metric manifold and coincides with the Levi-Civita connection only on a \( \mathcal{F}_0 \)-manifold.

The torsion tensor \( T \) of \( D \) is determined by \( T(x, y) = Dx y - Dy x = [x, y] \) and the corresponding tensor of type (0,3) is defined by the condition \( T(x, y, z) = g(T(x, y), z) \).

Let \( Q \) be the potential tensor of \( D \) with respect to \( \nabla \) determined by
\[
D x y = \nabla x y + Q(x, y).
\]
The corresponding tensor of type (0,3) is defined as follows \( Q(x, y, z) = g(Q(x, y), z) \).

According to [10], a linear connection \( D \) is a natural connection on an almost contact B-metric manifold if and only if
\[
Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z),
\]
\[
Q(x, y, z) = -Q(x, z, y).
\]
A natural connection \( \tilde{D} \) on the manifold \((M, \varphi, \xi, \eta, g)\), which torsion tensor \( \tilde{T} \) is totally skew-symmetric (i.e. a 3-form), exists in the basic classes \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \), it is unique and is called the \( \varphi \)KT-connection \[10\]. Let us remark that this connection in the Hermitian geometry is known as the Bismut connection or the KT-connection.

In \[12\], it is introduced a natural connection \( \bar{D} \) on \((M, \varphi, \xi, \eta, g)\) in all basic classes by

\[
(1.5) \quad \bar{D}_x y = \nabla x y + \frac{1}{2} \left\{ (\nabla x \varphi) \varphi y + (\nabla x \eta) y \cdot \xi \right\} - \eta(y) \nabla x \xi.
\]

This connection is called a \( \varphi \)B-connection in \[13\] and it is studied for the main classes \( \mathcal{F}_1, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_{11} \) in Refs. \[12\], \[9\] and \[7\]. The \( \varphi \)B-connection is the odd-dimensional counterpart of the B-connection on the corresponding almost complex manifold with Norden metric, studied for the class \( \mathcal{W}_1 \) in \[2\].

In \[14\], a natural connection \( \bar{D} \) is called a \( \varphi \)-canonical connection on \((M, \varphi, \xi, \eta, g)\) if its torsion tensor \( \bar{T} \) satisfies the following identity:

\[
(1.6) \quad \bar{T}(x, y, z) = \bar{T}(x, z, y) - \bar{T}(x, \varphi y, \varphi z) + \bar{T}(x, \varphi z, \varphi y) = \\
\eta(x) \left\{ \bar{T}(\xi, y, z) - \bar{T}(\xi, z, y) - \bar{T}(\xi, \varphi y, \varphi z) + \bar{T}(\xi, \varphi z, \varphi y) \right\} \\
+ \eta(y) \left\{ \bar{T}(x, \xi, z) - \bar{T}(x, z, \xi) - \eta(x) \bar{T}(z, \xi, \xi) \right\} \\
- \eta(z) \left\{ \bar{T}(x, \xi, y) - \bar{T}(x, y, \xi) - \eta(x) \bar{T}(y, \xi, \xi) \right\}.
\]

In this work we pay attention on the case when the manifold \((M, \varphi, \xi, \eta, g)\) belongs to the class \( \mathcal{F}_7 \) from the mentioned classification. This basic class is characterized by the conditions: \[9\]

\[
\mathcal{F}_7 : \quad F(x, y, z) = F(x, y, \xi) \eta(z) + F(x, z, \xi) \eta(y), \\
F(x, y, \xi) = -F(y, x, \xi) = -F(\varphi x, \varphi y, \xi).
\]

The class \( \mathcal{F}_3 \oplus \mathcal{F}_7 \) is characterized by the condition \( \tilde{N} = 0 \). This is the only class of \((M, \varphi, \xi, \eta, g)\) where the \( \varphi \)KT-connection exists \[10\]. The basic classes \( \mathcal{F}_3 \) and \( \mathcal{F}_7 \) are the horizontal component and the vertical one of \( \mathcal{F}_3 \oplus \mathcal{F}_7 \), respectively. We are interested in \( \mathcal{F}_7 \) because the contact distribution \( \ker(\eta) \) of any \( \mathcal{F}_3 \)-manifold is an almost complex manifold with Norden metric belonging to the class \( \mathcal{W}_3 \) of quasi-Kähler manifolds with Norden metric which are well-studied in relation with their curvature properties and natural connections (see e.g. \[20\], \[16\], \[17\], \[13\], \[19\]). The mentioned topics on \( \mathcal{F}_7 \)-manifolds are studied in \[10\], \[11\] and \[13\]. In \[15\], it is proved that if \((M, \varphi, \xi, \eta, g)\) is an arbitrary manifold in \( \mathcal{F}_i, \ i \in \{ 3, 7 \} \), then \( \varphi \)B-connection \( \bar{D} \) is the average connection of the \( \varphi \)KT-connection \( \bar{D} \) and the \( \varphi \)-canonical connection \( \tilde{D} \), i.e. \( 2\bar{D} = \bar{D} + \tilde{D} \).

2. A Family of Lie Groups as 5-Dimensional \( \mathcal{F}_7 \)-Manifolds

Let us consider the example given in \[10\]. Let \( G \) be a 5-dimensional real connected Lie group and \( \mathfrak{g} \) be its Lie algebra. Let \( \{e_i\}_{i=1}^5 \) be a global basis of left-invariant vector fields of \( G \). An almost contact B-metric structure is introduced
by
\begin{equation}
\begin{aligned}
\varphi_1 &= e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2, \quad \varphi e_5 = 0; \\
\xi &= e_5; \quad \eta(e_1) = \eta(e_2) = \eta(e_3) = \eta(e_4) = 0, \quad \eta(e_5) = 1;
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
g(e_1, e_1) &= g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = g(e_5, e_5) = 1, \\
g(e_1, e_j) &= 0, \quad i \neq j; \quad i, j \in \{1, 2, 3, 4, 5\}; \\
[e_1, e_2] &= -[e_3, e_4] = -\lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + 2\mu_1 e_5, \\
[e_1, e_4] &= -[e_2, e_3] = -\lambda_3 e_1 - \lambda_4 e_2 - \lambda_1 e_3 - \lambda_2 e_4 + 2\mu_2 e_5.
\end{aligned}
\end{equation}

It is proved that \((G, \varphi, \xi, \eta, g)\) belongs to the class \(\mathcal{F}_7\). It is not an \(\mathcal{F}_0\)-manifold if and only if \((\mu_1, \mu_2) \neq (0, 0)\) holds \([10]\).

Actually, \(G\) is a family of manifolds determined by six real parameters \(\lambda_1, \ldots, \lambda_4, \mu_1, \mu_2\).

In \([10]\), there are obtained the components of the Levi-Civita connection \(\nabla\) and the \(\varphi\)KT-connection \(\tilde{D}\) on the manifold \((G, \varphi, \xi, \eta, g)\):

\begin{equation}
\begin{aligned}
\nabla_{e_1} e_1 &= -\nabla_{e_3} e_3 = \tilde{D}_{e_1} e_1 = -\tilde{D}_{e_3} e_3 = \lambda_1 e_2 - \lambda_3 e_4, \\
\nabla_{e_1} e_2 &= -\nabla_{e_4} e_4 = -\lambda_1 e_1 + \lambda_3 e_3 + \mu_1 e_5, \\
\nabla_{e_1} e_3 &= \nabla_{e_2} e_2 = -\lambda_3 e_1 - \lambda_1 e_3 + \mu_2 e_5, \\
\nabla_{e_1} e_4 &= \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = -\lambda_1 e_1 - \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + 2\mu_1 e_5, \\
\nabla_{e_1} e_5 &= \nabla_{e_5} e_5 = -\lambda_3 e_1 - \lambda_4 e_2 - \lambda_1 e_3 - \lambda_2 e_4 + 2\mu_2 e_5.
\end{aligned}
\end{equation}

The \(\varphi\)KT-connection \(\tilde{D}\) in \(\mathcal{F}_7\) is defined by: \([10]\)

\[\tilde{T}(x, y) = 2\{\eta(x)\nabla_y \xi - \eta(y)\nabla_x \xi + (\nabla_x \eta) y \cdot \xi\}.
\]

It is obtained that the basic non-zero components of the torsion of \(\tilde{D}\) are:

\[\begin{aligned}
\tilde{T}_{125} &= -\tilde{T}_{215} = -\tilde{T}_{345} = \tilde{T}_{335} = -2\mu_1, \\
\tilde{T}_{145} &= -\tilde{T}_{235} = \tilde{T}_{325} = -\tilde{T}_{415} = -2\mu_2.
\end{aligned}\]

For the square norm \(\|\tilde{T}\|^2\) we obtain

\begin{equation}
\|\tilde{T}\|^2 = g^{ip} g^{jq} g^{ks} \tilde{T}_{ipq} \tilde{T}_{ksj},
\end{equation}

\begin{equation}
\|\tilde{T}\|^2 = 16 (\mu_1^2 - \mu_2^2).
\end{equation}

For \(\nabla\) (respectively, \(\tilde{D}\)) there are computed the basic components \(R_{ijkl} = R(e_i, e_j, e_k, e_l)\) of the curvature tensor \(R\) (respectively, \(\tilde{R}\)), \(\rho_{jk} = \rho(e_j, e_k)\) of the
Ricci tensor $\rho$ (respectively, $\bar{\rho}$) and the values of the scalar curvature $\tau$ (respectively, $\bar{\tau}$) as follows (the remaining ones are obtained, according to (1.1) and (1.2)):

\[
R_{1212} = R_{3434} = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 3\mu_1^2,
\]
\[
R_{1234} = -(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2\mu_1^2 + \mu_2^2,
\]
\[
R_{1414} = R_{2323} = -(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 3\mu_2^2,
\]
\[
(2.5)
\]
\[
R_{1423} = (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + \mu_1^2 - 2\mu_2^2,
\]
\[
R_{1214} = -R_{1223} = R_{2334} = -R_{1434} = 2(\lambda_1\lambda_3 + \lambda_2\lambda_4) + 3\mu_1\mu_2,
\]
\[
R_{1324} = -(\mu_1^2 + \mu_2^2), \quad R_{1535} = R_{2545} = -2\mu_1\mu_2,
\]
\[
R_{1515} = R_{2525} = -R_{3535} = -R_{4545} = -\mu_1^2 + \mu_2^2;
\]
\[
(2.6)
\]
\[
\rho_{11} = \rho_{22} = -\rho_{33} = -\rho_{44} = -2(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2(\mu_1^2 - \mu_2^2),
\]
\[
\rho_{13} = \rho_{24} = -4(\lambda_1\lambda_3 + \lambda_2\lambda_4) - 4\mu_1\mu_2, \quad \rho_{55} = 4(\mu_1^2 - \mu_2^2);
\]
\[
(2.7)
\]
\[
\tau = -8(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 4(\mu_1^2 - \mu_2^2);
\]
\[
\bar{\tau} = -8(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 16(\mu_1^2 - \mu_2^2).
\]

In [10], it is established

**Proposition 2.1.** The following conditions are equivalent:

1. The manifold $(G, \varphi, \xi, \eta, g)$ is an isotropic-$F_0$-manifold;
2. The scalar curvatures for $\nabla$ and $\bar{\nabla}$ are equal;
3. The vectors $\nabla e_i, \xi$ $(i = 1, 2, 3, 4)$ are isotropic;
4. The equality $\mu_1 = \pm \mu_2$ is valid.

Moreover, the manifold $(G, \varphi, \xi, \eta, g)$ is Einsteinian if and only if the following conditions are valid: [10]

\[
\mu_1\mu_2 = -(\lambda_1\lambda_3 + \lambda_2\lambda_4), \quad \mu_1^2 - \mu_2^2 = -\frac{1}{3}(\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2).
\]

3. A FAMILY OF LIE GROUPS AS 5-DIMENSIONAL $F_7$-MANIFOLDS — FURTHER INVESTIGATIONS

In this section we continue the studying of the manifold $(G, \varphi, \xi, \eta, g)$. In [8], there is determined the form of the Nijenhuis tensor $N$ on $(M, \varphi, \xi, \eta, g) \in F_7$ as follows:

\[
N(x, y) = 4(\nabla_x \eta) y \cdot \xi.
\]

According to (2.1), (2.2) and (2.3), we obtain the following non-zero components $N_{ij} = N(e_i, e_j)$ of $N$:

\[
N_{12} = -N_{21} = -N_{34} = N_{43} = -4\mu_1\xi,
\]
\[
N_{14} = -N_{23} = N_{32} = -N_{41} = -4\mu_2\xi.
\]
For the square norm $\|N\|^2 = g^{ik}g^{js}g(N_{ij}, N_{ks})$ we obtain

\begin{equation}
\|N\|^2 = 64 \left( \mu_1^2 - \mu_2^2 \right).
\end{equation}

Using (2.4), (2.5), (2.6) and (2.8), we compute the components of the associated Ricci tensors for $\nabla$ and $\tilde{D}$, respectively. The non-zero components of $\rho^*$ and associated scalar curvatures are the following:

\begin{align*}
\rho_{11}^* &= \rho_{22}^* = -\rho_{33}^* = -\rho_{44}^* = -4 (\lambda_1 \lambda_3 + \lambda_2 \lambda_4) - 6 \mu_1 \mu_2, \\
\rho_{55}^* &= 4 \mu_1 \mu_2, \\
\rho_{13}^* &= \rho_{24}^* = \rho_{31}^* = \rho_{42}^* = 2 \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) + 3 \left( \mu_1^2 - \mu_2^2 \right), \\
\tau^* &= -16 (\lambda_1 \lambda_3 + \lambda_2 \lambda_4) - 20 \mu_1 \mu_2, \\
\tilde{\rho}_{11}^* &= \tilde{\rho}_{22}^* = -\tilde{\rho}_{33}^* = -\tilde{\rho}_{44}^* = -4 (\lambda_1 \lambda_3 + \lambda_2 \lambda_4) - 8 \mu_1 \mu_2, \\
\tilde{\rho}_{13}^* &= \tilde{\rho}_{24}^* = \tilde{\rho}_{31}^* = \tilde{\rho}_{42}^* = 2 \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) + 4 \left( \mu_1^2 - \mu_2^2 \right), \\
\tilde{\tau}^* &= -16 (\lambda_1 \lambda_3 + \lambda_2 \lambda_4) - 32 \mu_1 \mu_2.
\end{align*}

Bearing in mind (1.3), let $k_{ij}$ ($i \neq j$) be the sectional curvature for $\nabla$ (or, with respect to $R$) of the basic 2-plane $\alpha_{ij}$ with a basis $\{e_i, e_j\}$, where $e_i, e_j \in \{e_1, \ldots, e_5\}$. The basic 2-planes $\alpha_{ij}$ of $(G, \phi, \xi, \eta, g)$ are the following:

- totally real sections — $\alpha_{12}, \alpha_{14}, \alpha_{23}, \alpha_{34}$;
- $\phi$-holomorphic sections — $\alpha_{13}, \alpha_{24}$;
- $\xi$-sections — $\alpha_{51}, \alpha_{52}, \alpha_{53}, \alpha_{54}$.

Then, using (1.3), (2.2) and (2.5), we compute the sectional curvatures $k_{ij}$ for $\nabla$ and obtain:

\begin{align*}
k_{12} &= k_{34} = - \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) - 3 \mu_1^2, \\
k_{14} &= k_{23} = - \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) + 3 \mu_2^2; \\
k_{13} &= k_{24} = 0; \\
k_{51} &= k_{52} = k_{53} = k_{54} = \mu_1^2 - \mu_2^2.
\end{align*}

Analogously, from (1.3), (2.2) and (2.8) we calculate the sectional curvatures $\tilde{k}_{ij}$ for $\varphi$KT-connection $\tilde{D}$ and obtain:

\begin{align*}
\tilde{k}_{12} &= \tilde{k}_{34} = - \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) - 4 \mu_1^2, \\
\tilde{k}_{14} &= \tilde{k}_{23} = - \left( \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 \right) + 4 \mu_2^2; \\
\tilde{k}_{13} &= \tilde{k}_{24} = \tilde{k}_{51} = \tilde{k}_{52} = \tilde{k}_{53} = \tilde{k}_{54} = 0.
\end{align*}

Let us consider the $\varphi$B-connection $\tilde{D}$ on $(G, \varphi, \xi, \eta, g)$ defined by (1.4) and (1.5). Then, by (2.1) and (2.3) we compute its components as follows:

\begin{align*}
\tilde{D}_{e_1}e_1 &= -\tilde{D}_{e_2}e_3 = \lambda_1 e_2 - \lambda_3 e_4, & \tilde{D}_{e_3}e_1 &= -\mu_1 e_2 + \mu_2 e_4, \\
\tilde{D}_{e_1}e_2 &= -\tilde{D}_{e_3}e_4 = -\lambda_1 e_1 + \lambda_3 e_3, & \tilde{D}_{e_5}e_2 &= \mu_1 e_1 - \mu_2 e_3, \\
\tilde{D}_{e_1}e_3 &= \tilde{D}_{e_3}e_1 = \lambda_3 e_2 + \lambda_1 e_4, & \tilde{D}_{e_5}e_3 &= -\mu_2 e_2 - \mu_1 e_4, \\
\tilde{D}_{e_1}e_4 &= \tilde{D}_{e_3}e_2 = -\lambda_3 e_1 - \lambda_1 e_3, & \tilde{D}_{e_5}e_4 &= \mu_2 e_1 + \mu_1 e_3, \\
\tilde{D}_{e_2}e_1 &= -\tilde{D}_{e_4}e_3 = \lambda_2 e_2 - \lambda_4 e_4, & \tilde{D}_{e_5}e_5 &= 0, \quad i \in \{1, 2, 3, 4, 5\}. \\
\tilde{D}_{e_2}e_2 &= -\tilde{D}_{e_4}e_4 = -\lambda_2 e_1 + \lambda_4 e_3, & \tilde{D}_{e_5}e_1 &= \lambda_4 e_2 + \lambda_2 e_4, \\
\tilde{D}_{e_2}e_3 &= \tilde{D}_{e_4}e_1 = \lambda_4 e_2 + \lambda_2 e_4, & \tilde{D}_{e_5}e_3 &= -\lambda_4 e_1 - \lambda_2 e_3, \\
\tilde{D}_{e_2}e_4 &= \tilde{D}_{e_4}e_2 = -\lambda_4 e_1 - \lambda_2 e_3.
\end{align*}
Thus, we get that the basic non-zero components of the torsion of $\mathcal{D}$ are:

\[
\begin{align*}
\hat{T}_{125} &= -\hat{T}_{215} = 2\hat{T}_{251} = -\hat{T}_{345} \\
&= 2\hat{T}_{354} = \hat{T}_{435} = -2\hat{T}_{521} = -2\hat{T}_{534} = -2\mu_1, \\
\hat{T}_{145} &= -\hat{T}_{235} = 2\hat{T}_{253} = \hat{T}_{325} \\
&= -2\hat{T}_{415} = 2\hat{T}_{451} = -2\hat{T}_{523} = -2\hat{T}_{541} = -2\mu_2.
\end{align*}
\]

For the square norm $\|\hat{T}\|^2 = g^{ij}g^{kl}\hat{T}_{ijkl}\hat{T}_{pqrs}$ we obtain

\[
(3.10) \quad \|\hat{T}\|^2 = 20 (\mu_1^2 - \mu_2^2).
\]

According to (3.9), we obtain the following components $\hat{R}_{ijkl} = \hat{R}(e_i, e_j, e_k, e_l)$ of the curvature tensor $\hat{R}$ of $\mathcal{D}$ on the manifold (the remaining ones are obtained, according to (1.4) and (1.5)):

\[
\begin{align*}
\hat{R}_{1212} &= -\hat{R}_{1234} = -\hat{R}_{3412} = \hat{R}_{3434} \\
&= (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 2\mu_1^2, \\
\hat{R}_{1414} &= -\hat{R}_{1423} = -\hat{R}_{2314} = \hat{R}_{2323} \\
&= - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 2\mu_2^2, \\
\hat{R}_{1214} &= \hat{R}_{1412} = -\hat{R}_{1232} = -\hat{R}_{1432} = -\hat{R}_{2312} = \hat{R}_{2334} \\
&= -\hat{R}_{3414} = \hat{R}_{3432} = 2 (\lambda_1\lambda_3 + \lambda_2\lambda_4) + 2\mu_1\mu_2.
\end{align*}
\]

Using (2.1), (2.2) and (3.11), we compute the components of the Ricci tensor $\hat{\rho}$, the value of the scalar curvature $\hat{\rho}$ and their associated quantities for the $\varphi$-B-connection $\mathcal{D}$. The non-zero components of these tensors and the scalar curvatures are the following:

\[
\begin{align*}
\hat{\rho}_{11} &= \hat{\rho}_{22} = -\hat{\rho}_{33} = -\hat{\rho}_{44} = \hat{\rho}_{13} = \hat{\rho}_{14} = \hat{\rho}_{24} = \hat{\rho}_{34} = -\hat{\rho}_{42} \\
&= -2 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2 (\mu_1^2 - \mu_2^2), \\
\hat{\rho}_{13} &= \hat{\rho}_{24} = \hat{\rho}_{31} = \hat{\rho}_{42} = \hat{\rho}_{11} = \hat{\rho}_{22} = \hat{\rho}_{33} = \hat{\rho}_{44} \\
&= -4 (\lambda_1\lambda_3 + \lambda_2\lambda_4) - 4\mu_1\mu_2, \\
\hat{\tau} &= -8 (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 8 (\mu_1^2 - \mu_2^2), \\
\hat{\tau}^* &= -16 (\lambda_1\lambda_3 + \lambda_2\lambda_4) - 16\mu_1\mu_2.
\end{align*}
\]

Taking into account (1.3), (2.2) and (3.11), we obtain the following basic sectional curvatures $k_{ij}$ for the $\varphi$-B-connection:

\[
\begin{align*}
k_{12} &= k_{34} = - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) - 2\mu_1^2, \\
k_{14} &= k_{23} = - (\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2) + 2\mu_2^2; \\
k_{13} &= k_{24} = k_{51} = k_{52} = k_{53} = k_{54} = 0.
\end{align*}
\]

Let us consider the $\varphi$-canonical connection $\tilde{\mathcal{D}}$ on $(\mathcal{G}, \varphi, \xi, \eta, g)$ defined by (1.6). Then, by (2.4) and (2.3) we compute its components as follows:

\[
\begin{align*}
\tilde{\mathcal{D}}_{e_i} e_j &= \tilde{\mathcal{D}}_{e_i} e_j, \quad i, j \in \{1, 2, 3, 4\}, \\
\tilde{\mathcal{D}}_{e_i} e_5 &= \tilde{\mathcal{D}}_{e_i} e_5 = 0, \quad i \in \{1, 2, 3, 4, 5\}.
\end{align*}
\]
The basic non-zero components of the torsion of $\DD$ are:

$$
\begin{align*}
\tilde{T}_{125} &= -\tilde{T}_{215} = \tilde{T}_{251} = -\tilde{T}_{345} \\
&= \tilde{T}_{354} = \tilde{T}_{435} = -\tilde{T}_{521} = -\tilde{T}_{534} = -2\mu_1, \\
\tilde{T}_{145} &= -\tilde{T}_{235} = \tilde{T}_{253} = \tilde{T}_{325} \\
&= -\tilde{T}_{415} = \tilde{T}_{451} = -\tilde{T}_{523} = -2\mu_2.
\end{align*}
$$

For the square norm $\|\tilde{T}\|^2 = g^{ij}g^{pq}g^{rs}\tilde{T}_{ijk}\tilde{T}_{pqrs}$ we obtain

$$
\|\tilde{T}\|^2 = 32 (\mu_1^2 - \mu_2^2).
$$

Using (3.10) and (3.11), we obtain that the components $\tilde{R}_{ijkl}$ of the curvature tensor $\tilde{R}$ of $\tilde{D}$ and the components $\tilde{R}_{ijkl}$ of the curvature tensor $\tilde{R}$ of $\tilde{D}$ are equal on $(G, \varphi, \xi, \eta, g)$, i.e.

$$
\tilde{R}_{ijkl} = \tilde{R}_{ijkl}, \quad i, j, k, l \in \{1, \ldots, 5\}.
$$

Thus, it implies the following

**Proposition 3.1.** The curvature tensors of the $\varphi$B-connection and the $\varphi$-canonical connection are equal on $(G, \varphi, \xi, \eta, g)$.

**Proposition 3.2.** The following characteristics are valid for $(G, \varphi, \xi, \eta, g)$:

1. The $\varphi$-holomorphic sectional curvatures for $\nabla$, $\DD$, $\DD$ and $\DD$ are zero;
2. The $\xi$-sectional curvatures for $\DD$, $\DD$ and $\DD$ are zero;
3. The associated Ricci tensor of $\nabla$ is proportional to the metric $g$ if and only if the following identities hold:
   $$
   \lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = \mu_1^2 - \mu_2^2 = \mu_1\mu_2 + \frac{2}{3} (\lambda_1\lambda_3 + \lambda_2\lambda_4) = 0;
   $$
4. The manifold is scalar flat with respect to $\nabla$, $\DD$, $\DD$ and $\DD$ if and only if the identities $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = \mu_1^2 - \mu_2^2 = 0$ hold;
5. The manifold has vanishing associated scalar curvature with respect to $\nabla$ (respectively, $\DD$, $\DD$ and $\DD$) if and only if the identity
   $$
   \mu_1\mu_2 = \nu (\lambda_1\lambda_3 + \lambda_2\lambda_4)
   $$
   is valid for $\nu = -\frac{4}{3}$ (respectively, $\nu = -1$, $\nu = -\frac{1}{2}$ and $\nu = -1$);
6. The manifold has natural connections $\DD$, $\DD$ and $\DD$ coinciding with the Levi-Civita connection $\nabla$ if and only if $(G, \varphi, \xi, \eta, g)$ belongs to $\mathcal{F}_0$;
7. The manifold is flat with respect to $\nabla$ (respectively, $\DD$, $\DD$ and $\DD$) if and only if it belongs to $\mathcal{F}_0$ and the conditions $\lambda_1^2 + \lambda_2^2 - \lambda_3^2 - \lambda_4^2 = \lambda_1\lambda_3 + \lambda_2\lambda_4 = 0$ hold;
8. The manifold has vanishing $\rho$, i.e. it is Ricci-flat (respectively, $\rho^*$, $\tilde{\rho}$, $\rho^*$, $\rho^*$ vanish) if and only if $(G, \varphi, \xi, \eta, g)$ is flat, $R = 0$.

**Proof.** Equation (3.18) implies that the components of the Ricci tensor, the value of the scalar curvature (respectively, their associated quantities) and the basic sectional curvatures of the $\varphi$B-connection and $\varphi$-canonical connection are equal on $(G, \varphi, \xi, \eta, g)$.

The statements (1) and (2) are corollaries of (3.7), (3.8), (3.15) and (3.18).

By virtue of (3.3) and (2.2), we get that $\rho^*$ is proportional to $g$ if and only if the identities in (3) are valid.

Using (2.7), (2.10), (3.13) and (3.18), we obtain the statement (4).
From (3.4), (3.14), (3.6) and (3.18) we get immediately the statement (5). The statement (6) follow from (2.3), (3.9) and (3.16) because of $\mu_1 = \mu_2 = 0$. Equation (2.5) and (6) imply (7). The statement (8) follows from (2.6), (2.9), (3.3), (3.5), (3.12), (3.18) and (7). \hfill \Box

Proposition 3.3. The following conditions are equivalent:

(1) The manifold $(G, \varphi, \xi, \eta, g)$ is an isotropic-$\mathcal{F}_0$-manifold;
(2) The scalar curvatures for $\nabla$, $\dot{D}$, $\ddot{D}$ and $\dddot{D}$ are equal;
(3) The vectors $\nabla e_i \xi$ $(i = 1, 2, 3, 4)$ are isotropic;
(4) The Nijenhuis tensor $N$ is isotropic;
(5) The torsion tensors of $\dot{D}$, $\ddot{D}$ and $\dddot{D}$ are isotropic;
(6) The sectional curvatures of the $\xi$-sections with respect to $\nabla$ vanish;
(7) The equality $\mu_1 = \pm \mu_2$ is valid.

Proof. According to Proposition 2.1 the conditions (1), (3) and (7) are equivalent. The other conditions are equivalent of (7), bearing in mind (2.7), (2.10), (3.13), (3.18) for (2); (3.2) for (4); (2.4), (3.10), (3.17) for (5) and (3.7) for (6). \hfill \Box

References

[1] V. Alexiev and G. Ganchev, Canonical connection on a conformal almost contact metric manifolds, Ann. Univ. Sofia Fac. Math. Inform. 81, 1 (1987) 29–38.
[2] G. Ganchev, K. Gribachev and V. Mihova, B-connections and their conformal invariants on conformally Kaehler manifolds with B-metric, Publ. Inst. Math. Beograd (N.S.) 42, 56 (1987) 107–121.
[3] G. Ganchev and V. Mihova, Canonical connection and the canonical conformal group on an almost complex manifold with B-metric, Ann. Univ. Sofia Fac. Math. Inform. 81, (1987) 195–206.
[4] G. Ganchev, V. Mihova and K. Gribachev, Almost contact manifolds with B-metric, Math. Balkanica (N.S.) 7, (1993) 261–276.
[5] S. Kobayashi and K. Nomizu, Foundations of differential geometry, Intercis. Publ., New York 1, 2, 1963 (1969).
[6] M. Manev, Properties of curvature tensors on almost contact manifolds with B-metric, Proc. of Jubilee Sci. Session of Vasil Leveiski Higher Mil. School, Veliko Tarnovo 27, (1993) 221–227.
[7] M. Manev, Contactly conformal transformations of general type of almost contact manifolds with B-metric, Applications, Math. Balkanica (N.S.) 11, (1997) 347–357.
[8] M. Manev, On the conformal geometry of almost contact manifolds with B-metric, Ph.D. thesis, Flovdiv University, (1998).
[9] M. Manev, Almost contact B-metric hypersurfaces of Kaehlerian manifolds with B-metric, In: Perspectives of Complex analysis, Differential Geometry and Mathematical Physics, Eds. St. Dimiev and K. Sekigawa, World Sci. Publ., Singapore, 2001, pp. 159–170.
[10] M. Manev, A connection with totally skew-symmetric torsion on almost contact manifolds with B-metric, Int. J. Geom. Methods Mod. Phys. 9, 5 (2012).
[11] M. Manev, Curvature properties on some classes of almost contact manifolds with B-metric, C. R. Acad. Bulg. Sci. 65, 3 (2012) 283–290.
[12] M. Manev and K. Gribachev, Conformally invariant tensors on almost contact manifolds with B-metric, Serdica Math. J. 20, (1994) 133–147.
[13] M. Manev and M. Ivanova, A natural connection on some classes of almost contact manifolds with B-metric, C. R. Acad. Bulg. Sci. 65, 4 (2012) 429–436.
[14] M. Manev and M. Ivanova, Canonical-type connection on almost contact manifolds with B-metric, Ann. Global Anal. Geom. 43, 4 (2013) 397–408.
[15] M. Manev and M. Ivanova, A classification of the torsions on almost contact manifolds with B-metric, Cent. Eur. J. Math. 12, 10 (2014) 1416–1432.
[16] D. Mekerov, On the geometry of the B-connection on quasi-K"ahler manifolds with Norden metric, C. R. Acad. Bulg. Sci. 61, (2008) 1105–1110.
[17] D. Mekerov, A connection with skew-symmetric torsion and Kähler curvature tensor on quasi-Kähler manifolds with Norden metric, *C. R. Acad. Bulg. Sci.* **61**, (2008) 1249–1256.

[18] D. Mekerov, Canonical connection on quasi-Kähler manifolds with Norden metric, *J. Tech. Univ. Plovdiv Fundam. Sci. Appl. Ser. A Pure Appl. Math.* **14**, (2009) 73–86.

[19] D. Mekerov, On the geometry of the connection with totally skew-symmetric torsion on almost complex manifolds with Norden metric, *C. R. Acad. Bulgare Sci.* **63**, (2010), 19-28.

[20] D. Mekerov and M. Manev, On the geometry of quasi-Kähler manifolds with Norden metric, *Nihonkai Math. J.* **16**, 2 (2005) 89–93.

[21] G. Nakova and K. Gribachev, Submanifolds of some almost contact manifolds with B-metric with codimension two, *I, Math. Balkanica (N.S.)* **12**, 1-2 (1998) 93–108.

(Miroslava Ivanova) DEPARTMENT OF INFORMATICS AND MATHEMATICS, TRAKIA UNIVERSITY, STARA ZAGORA, 6000, BULGARIA, E-MAIL: mivanova@uni-sz.bg

(Hristo Manev) DEPARTMENT OF PHARMACEUTICAL SCIENCES, MEDICAL UNIVERSITY OF PLOVDIV, PLOVDIV, 4002, BULGARIA; DEPARTMENT OF ALGEBRA AND GEOMETRY, UNIVERSITY OF PLOVDIV, PLOVDIV, 4027, BULGARIA, E-MAIL: hmanev@uni-plovdiv.bg