ON THE SEDIMENTATION OF A DROPLET IN STOKES FLOW

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Abstract. This paper is dedicated to the analysis of the transport-Stokes equation which describes sedimentation of inertialess suspensions in a viscous flow at mesoscopic scaling. First we present a global existence and uniqueness result for $L^1 \cap L^\infty$ initial densities with finite first moment. Secondly, we consider the case where the initial data is the characteristic function of an axisymmetric bounded domain and investigate the regularity of its surface. Using spherical parametrisation, a hyperbolic equation for the evolution of the radius of the droplet is derived and we present a local existence and uniqueness result. Finally, we investigate the case where the initial shape of the droplet is spherical and show that the solution corresponds to the Hadamard and Rybczynski result. We present numerical simulations in the spherical case.

1. Introduction

In this paper, we consider the sedimentation of a cloud of rigid particles in a viscous fluid. At the mesoscopic scaling, it has been showed [7, 10] that the equation describing the dynamics is the transport-Stokes problem in the case where inertia of both fluid and particles is neglected:

$$\begin{cases}
\partial_t \rho + \text{div}((u + \kappa g)\rho) &=& 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
-\Delta u + \nabla p &=& 6\pi r_0 \kappa \rho g, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div } u &=& 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\rho(0, \cdot) &=& \rho_0, & \text{on } \mathbb{R}^3.
\end{cases}$$

Here, the function $\rho$ stands for the density of the particles, $(u, p)$ are the velocity and pressure of the fluid, $g$ is the gravity vector, $R = \frac{2}{N}$ is the radius of the particles where $N$ the (large) number of particles in the suspension and $\kappa g = \frac{2}{5} R^2 (\bar{\rho} - \rho) g$ represents the fall speed of one particle sedimenting under gravitational force. Note in particular that the source term in the Stokes equation corresponds to $6\pi r_0 \kappa \rho g = N^\frac{4}{3} \pi R^3 (\rho_p - \rho_f) g \rho = \phi (\rho_p - \rho_f) g \rho$ where $\phi$ is the solid volume fraction of the suspension in the case $|\text{supp } \rho| = 1$.

At the microscopic scaling, the motion and shape evolution of a blob has been studied

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Experimental and numerical investigations lead to the conclusion that a spherical cloud of particles slowly evolves to a torus. Precisely, the particles at top of the cloud leak away from the cluster and form a vertical tail. The decrease of the number of particles at the vertical axis of the cloud leads to the apparition of the toroidal form. Moreover, it has been observed that the unstable torus breaks into two secondary droplets which deform into tori themselves in a repeating cascade.

At the macroscopic scaling, Hadamard [4] and Rybczynski [13] considered independently a coupled Stokes-Stokes model describing sedimentation of liquid spherical drop in a viscous fluid assuming a uniform surface tension on the sphere. Using the Stokes stream function for axisymmetric flow, authors show that the spherical shape of the drop is preserved.

We are interested in investigating the mesoscopic model by considering the transport-Stokes equation (1) when the initial density of the cloud is the characteristic function of a bounded domain $B_0$. First we present a global existence and uniqueness result for the transport-Stokes equation for $L^1 \cap L^\infty$ initial densities with finite first moment. Secondly, we derive a hyperbolic equation describing the evolution of the surface of axisymmetric drop $B_0$ and present a local existence and uniqueness result. We investigate then the case where the initial drop $B_0$ is spherical and show that we recover the result of Hadamard and Rybczynski on both the transport-Stokes equation and the hyperbolic equation. Finally we propose a numerical scheme for solving the hyperbolic equation and present some numerical simulations for the spherical case.

### 1.1. Description of the main results.

Existence and uniqueness of (1) has been proved in [7] for regular initial data $\rho_0$. The first step of this study is to extend the result for less regular data allowing to tackle blob distribution. Note that, as explained in [7], if $(\rho, u)$ are solutions to equation (1), then

$$(\bar{\rho}(t, x), \bar{u}(t, x)) = (\rho(t, x + t\kappa g), u(t, x + t\kappa g)),$$

is solution to

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
-\Delta u + \nabla p = 6\pi r_0 \kappa g, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div } u = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\rho(0, \cdot) = \rho_0, & \text{on } \mathbb{R}^3.
\end{cases}$$

Since $6\pi r_0 \kappa g = -6\pi r_0 \kappa |g| e_3$, without loss of generality, we consider in this paper the following transport-Stokes problem:

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
-\Delta u + \nabla p = -\rho e_3, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\text{div } u = 0, & \text{on } \mathbb{R}^+ \times \mathbb{R}^3, \\
\rho(0, \cdot) = \rho_0, & \text{on } \mathbb{R}^3.
\end{cases}$$

where $e_3$ is the third vector of the standard basis in $\mathbb{R}^3$.

The first result is a proof of existence and uniqueness of solutions for the transport-Stokes problem.
Theorem 1.1. Let $\rho_0 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ a measure with finite first moment. There exits a unique couple $(\rho, u) \in L^\infty(0, T; L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)) \times L^\infty(0, T; W^{1,\infty}(\mathbb{R}^3))$ satisfying the transport-Stokes equation (2) for all $T \geq 0$. Moreover, for all $s \in [0, T]$ there exists a unique characteristic flow $X(\cdot, s, \cdot) \in L^\infty(0, T, W^{1,\infty}(\mathbb{R}^3))$

\[
\begin{cases}
\partial_t X(t, s, x) = u(s, X(t, s, x)), & \forall t, s \in [0, T], \\
X(s, s, x) = x, & \forall s \in [0, T],
\end{cases}
\]

For all $s, t \in [0, T]$ the diffeomorphism $X(s, t, \cdot)$ is measure preserving and we have

$$\rho(t, \cdot) = X(t, 0, \cdot) \# \rho_0.$$ 

This result ensures the well posedness of the transport-Stokes equation when the initial density is the characteristic function of a bounded domain $B_0 \subset \mathbb{R}^3$. The proof relies on stability estimates using the first Wasserstein distance $W_1$. Moreover, the regularity of the characteristic flow ensures that if $\rho_0 = 1_{B_0}$ then we have for all time $\rho(t, \cdot) = 1_{B_t}$ where $B_t$ is transported along the flow. Consequently, in the second part of this paper we focus on investigating the regularity of the surface of the drop $B_t$. We consider the case of initial axisymmetric domains $B_0$ (invariant under rotations around the vertical axis $e_3$) described using a spherical parametrization and a radius function $r_0$ depending only on $\theta \in [0, \pi]$

$$B_0 = \left\{ r_0(\theta) \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}. \tag{3}$$

The motivation of considering such domains is that the Stokes equation preserves the invariance and ensures that $B_t$ is axisymmetric. We set then $c(t) = (0, 0, c_3(t)) \in B_t$ the position at time $t$ of a reference point such that $c(0) = 0$ and write $B_t = c(t) + \bar{B}_t$ where

$$\bar{B}_t = \left\{ r(t, \theta) \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}. \tag{4}$$

Remark 1.1. The reference point $c$ is not necessarily the center of mass of the droplet $B_t$. The decomposition $B_t = c(t) + \bar{B}_t$ with $\bar{B}_t$ defined in (4) is valid as long as $c(t) \in B_t$.

Using the weak formulation of the transport-Stokes equation we derive a hyperbolic equation for the evolution of the radius $r$.

$$\begin{cases}
\partial_t r + \partial_\theta r A_1[r] = A_2[r], \\
r(0, \cdot) = r_0.
\end{cases} \tag{5}$$

The operators $A_1$ and $A_2$ are defined in (16) and (17) in Proposition 3.1. See also Appendix A for a summary of the formulas. These operators depend non linearly and non locally on the unknown $r$, they also depend on the reference point $c$. We emphasize that there is a coupling between the evolution of the radius $r$ and the motion of the reference center $c$. Precisely, the velocity of $c$ can be seen as a parameter in the model. In particular, if we
choose \( c \) to be transported along the flow we get \( c = c[r] = (0, 0, c[r]_3) \) with
\[
\begin{align*}
\dot{c}[r]_3(t) &= -\frac{1}{4} \int_0^\pi r^2(t, \hat{\theta}) \sin(\hat{\theta}) \left(1 - \frac{1}{2} \sin^2(\hat{\theta})\right) d\theta, \\
\Rightarrow c[r]_3(0) &= 0,
\end{align*}
\]
see Proposition 3.1. We present a local existence and uniqueness result of \((r, c)\) for Lipschitz functions \( r_0 \) such that
\[ |r|_* = \inf_{(0, \pi)} r(\theta) > 0. \]

**Theorem 1.2.** Let \( r_0 \in C^0[0, \pi] \) such that \( |r_0|_* > 0 \). There exists \( T > 0 \) and a unique \( r \in C(0, \pi; C^0[0, \pi]) \) satisfying the hyperbolic equation (5). Moreover, there exists a unique associated reference point \( c = c[r] \in C(0, \pi) \) satisfying (6).

**Remark 1.2.** The same result holds true if the motion of the center \( c \) is defined in another way. The only properties needed is a uniform bound on \( \dot{r} \) and a stability estimate with respect to \( r \) if \( c = c[r] \), see (34).

We finish the second part by investigating the spherical case. We first prove that, analogously to the Hadamard-Rybczynski result, the spherical shape is preserved in the transport-Stokes model, see Corollary 3.3. The proof relies on the property proven by Hadamard-Rybczynski which states that the normal component of the velocity of the fluid is constant on the surface of the sphere. This constant velocity is denoted \( v^* \) and corresponds to the velocity fall of the center of the droplet \( c^* \) and is given by formula (12). In particular we present direct computations showing the Hadamard-Rybczynski property, see Lemma 3.2.

Regarding the hyperbolic equation, we set \( r_0 = 1 \) and distinguish two cases. If we choose \( c = c^* \), then a straightforward computation shows that \( A_2[1] = 0 \) and hence \( r = 1 \) is solution of the hyperbolic equation and we recover the Hadamard-Rybczynski result. On the other hand, if \( \dot{c} \neq c^* \), we show that the solution \( r \) corresponds to a spherical parametrization of the Hadamard-Rybczynski sphere \( B(c^*, 1) \) as long as the reference center \( c \) belongs to \( B(c^*, 1) \), see Proposition 3.4. Moreover, in the case where \( \dot{c} \) is given by (6), explicit computations show that \( |c(t) - c^*(t)| \leq 1 \) for all \( t \geq 0 \) and \( \lim_{t \to \infty} |c(t) - c^*(t)| = 1 \), see Proposition 3.5. This ensures that \( c(t) \in B(c^*, 1) \) for all time and shows global existence of the solution of equations (5), (6).

We finish the paper by proposing a numerical scheme in order to investigate the spherical case \( r_0 = 1 \). First, we present numerical simulations for the solution of (5) with \( \dot{c} \) fixed as in (6) and recover numerically Hadamard-Rybczynski solution. Second we investigate a test case for which \( \dot{c} \neq c^* \) and is such that the center \( c(t) \) leaves the sphere \( B(c^*, 1) \) after \( t = 0.5 \). Numerical computations show the validity of Proposition (3.4) until \( t = 0.5 \) and we observe negative values of the radius \( r \) after \( t = 0.5 \). The last test case illustrates the steady state i.e. \( c = c^* \) for which \( r = 1 \) is solution for all time. We finish by a discussion on the approximation scheme and possible future investigations.

This paper is divided into three main sections, the first one is dedicated to the existence and uniqueness of the transport-equation (2). The second section concerns the derivation
and analysis of the hyperbolic equation. The last subsection of the second part is dedicated
to the discussion on the link between the spherical case and the Hadamard-Rybczynski
result. Eventually, in the last section, we present numerical results for the spherical case.

2. Existence and uniqueness of the transport-Stokes equation

In order to prove Theorem 1.1 we recall some existence, uniqueness and stability esti-
mates for Stokes and transport equations.

2.1. Reminder on the Steady Stokes and transport equations. Equation (2) is a
steady Stokes problem coupled with a transport equation. We recall here some properties
coming from the Stokes problem on $\mathbb{R}^3$ and the transport equations.

Proposition 2.1. Let $\eta \in L^\infty(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$, The unique velocity field $u$ solution to the
Stokes equation:

$$\begin{cases}
-\Delta u + \nabla p = \eta, & \text{on } \mathbb{R}^3 \\
\text{div}(u) = 0, & \text{on } \mathbb{R}^3,
\end{cases}$$

is given by the convolution of the source term $\eta$ with the Oseen tensor $\Phi$

$$\Phi(x) = \frac{1}{8\pi} \left( \frac{I_3}{|x|} + \frac{x \otimes x}{|x|^3} \right).$$

Moreover, $u \in W^{1,\infty}(\mathbb{R}^3)$ and there exists a positive constant independent of the data such
that:

$$\|u\|_\infty + \|\nabla u\|_\infty \leq C\|\eta\|_{L^1 \cap L^\infty}.$$  (8)

A proof can be found in [7, Lemma 3.18] in the case $\eta \in X_\beta$ where $X_\beta$ is defined in [7, Definition 2.5]. The proof is mainly the same when considering $\eta \in L^1 \cap L^\infty$. We recall
now a stability estimate using the first Wasserstein distance $W_1$ which is well defined for
measures with finite first moment. The following Proposition uses arguments similar to [6, Proposition 3] and [5, Theorem 3.1].

Proposition 2.2 (Steady-Stokes stability estimates). Let $\eta_1, \eta_2 \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and
denote by $u_1$ and $u_2$ the associated Stokes solution. For all compact subset $K \subset \mathbb{R}^3$ one
can show that there exists a constant depending on $K$ such that

$$\|u_1 - u_2\|_{L^1(K)} + \|\nabla u_1 - \nabla u_2\|_{L^1(K)} \leq C(K) W_1(\eta_1, \eta_2).$$

Moreover, given a density $\rho \in L^1 \cap L^\infty$, there exists a positive constant independent of the
data such that:

$$\int_{\mathbb{R}^3} |u_1(x) - u_2(x)|\rho(dx) \leq C\|\rho\|_{L^1 \cap L^\infty} W_1(\eta_1, \eta_2)$$  (9)

Since similar computations will be used thereafter, we present the proof of the former
Proposition.
Proof. According to [14, Theorem 1.5], there exists an optimal transport map $T$ such that $\eta_2 := T \# \eta_1$ and we have:

$$W_1(\eta_1, \eta_2) = \int_{\mathbb{R}^3} |T(y) - y| \eta_1(dy).$$

This yields:

$$\int_K |u_2(x) - u_1(x)| dx = \int_K \left| \int_{\mathbb{R}^3} \Phi(x - y)\eta_1(dy) - \int_{\mathbb{R}^3} \Phi(x - T(y))\eta_1(dy) \right| dx$$

$$\leq C \int_K \left| \int_{\mathbb{R}^3} \frac{|T(y) - y|}{\min(|x - y|^2, |x - T(y)|^2)} \eta_1(dy) \right| dx$$

$$\leq \int_{\mathbb{R}^3} \int_K \left( \frac{1}{|x - y|^2} + \frac{1}{|x - T(y)|^2} \right) dx |T(y) - y| \eta_1(dy)$$

$$\leq C(K)W_1(\eta_1, \eta_2).$$

The proof of the last formula (9) is analogous to the estimate above where we replace $C(K)$ by $\|\rho\|_{L^1 \cap L^\infty}$. □

Given a velocity field having the same regularity as above, we recall now an existence, uniqueness and stability estimates for the transport equations. The stability estimate presented below is analogous to [6, Proposition 3] which is adapted from [8].

**Proposition 2.3.** Let $u \in L^\infty(0,T; W^{1,\infty}(\mathbb{R}^3))$ and $\rho_0 \in L^1 \cap L^\infty$, for all $T > 0$ there exists a unique solution $\eta \in L^\infty(0,T; L^1 \cap L^\infty)$ to the transport equation (10)

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\rho(0, \cdot) = \rho_0.
\end{cases}$$

Moreover, given two velocity fields $u_i$, $i = 1, 2$, if we denote by $\rho_i$ the solution to the associated transport equation, we have for all $t \geq s \geq 0$:

$$W_1(\rho_1(t), \rho_2(t)) \leq \left( W_1(\rho_1(s), \rho_2(s)) + \int_s^t \int_{\mathbb{R}^3} |u_2(\tau, x) - u_1(\tau, x)| \rho_1(\tau, x) dx d\tau \right) e^{Q_2(t-s)},$$

where $Q_i := \|u_i\|_{L^\infty(0,T; W^{1,\infty})}$.

**Proof.** Classical transport theory ensures the existence and uniqueness. Precisely, the characteristic flow satisfying

$$\begin{cases}
\partial_t X(t,s,x) = u(s, X(t,s,x)), & \forall t, s \in [0,T], \\
X(s,s,x) = x, & \forall s \in [0,T],
\end{cases}$$

is well defined in the sense of Carathéodory since $u$ is $L^\infty$ in time and Lipschitz regarding the space variable. Moreover, the following formula holds true

$$\rho(t, \cdot) = X(t,s,\cdot) \# \rho(s, \cdot).$$
Now, consider two velocity fields \( u_i \in L^\infty(0, T; W^{1,\infty}) \) and denote by \( X_i \) its associated characteristic flow. For all \( x \neq y, i = 1, 2 \) we have:

\[
|X_i(t, s, x) - X_i(t, s, y)| \leq |x - y| + \int_s^t |u_i(\tau, X_i(\tau, s, x)) - u_i(\tau, X_i(\tau, s, y))|d\tau
\]

\[
\leq |x - y| + Q_i \int_s^t |X_i(\tau, s, x) - X_i(\tau, s, y)|d\tau,
\]

which yields, using Gronwall’s inequality, for all \( t \geq s \geq 0 \):

\[
\text{Lip}(X_i(s, t, \cdot)) \leq e^{Q_i(t-s)}.
\]

We recall that at time \( s \geq 0 \), according to [14, Theorem 1.5], one can choose an optimal mapping \( T_s \) such that \( \rho_2(s) = T_s \# \rho_1(s) \) and

\[
W_1(\rho_1(s), \rho_2(s)) := \int |T_s(y) - y|\rho_1(s, dy),
\]

on the other hand, thanks to the flows \( X_i \) we can construct a mapping \( T_t \) at time \( t \geq s \) such that \( \rho_2(t) = T_t \# \rho_1(t) \) defined by

\[
T_t := X_2(t, s, \cdot) \circ T_s \circ X_1(s, t, \cdot).
\]

According to the definition of the Wasserstein distance and formulas [13], [14] we have:

\[
W_1(\rho_1(t), \rho_2(t)) \leq \int |T_i(x) - x|\rho_1(t, dx)
\]

\[
= \int |T_i(X_1(t, s, y)) - X_1(t, s, y)|\rho_1(s, dy)
\]

\[
= \int |X_2(t, s, T_s(y)) - X_1(t, s, y)|\rho_1(s, dy)
\]

\[
\leq \text{Lip}(X_2(t, s, \cdot))W_1(\rho_1(s), \rho_2(s)) + \int |X_2(t, s, y) - X_1(t, s, y)|\rho_1(s, dy).
\]

Now we have:

\[
\int |X_2(t, s, y) - X_1(t, s, y)|\rho_1(s, dy)
\]

\[
\leq \int_s^t \int |u_2(\tau, X_2(\tau, s, y)) - u_1(\tau, X_1(\tau, s, y))|\rho_1(s, dy)d\tau
\]

\[
\leq Q_2 \int_s^t \int |X_2(\tau, s, y) - X_1(\tau, s, y)|\rho_1(s, dy)d\tau + \int_s^t \int |u_2(\tau, x) - u_1(\tau, x)|\rho_1(\tau, dx)d\tau.
\]

Gronwall’s inequality yields:

\[
\int_{\mathbb{R}^3} |X_2(t, s, y) - X_1(t, s, y)|\rho_1(s, dy) \leq \left( \int_s^t \int_{\mathbb{R}^3} |u_2(\tau, x) - u_1(\tau, x)|\rho_1(\tau, dx)d\tau \right) e^{Q_2(t-s)}.
\]
Finally we get

\[ W_1(\rho_1(t), \rho_2(t)) \leq \text{Lip}(X_2(t, s, \cdot))W_1(\rho_1(s), \rho_2(s)) \]
\[ + \left( \int_s^t \int_{\mathbb{R}^3} |u_2(\tau, x) - u_1(\tau, x)| \rho_1(\tau, dx) d\tau \right) e^{Q_2(t-s)}, \]

with \( \text{Lip}(X_2(s, t, \cdot)) \leq e^{Q_2(t-s)}. \)

\[ \square \]

2.2. proof of the existence and uniqueness result.

Proof of Theorem 1.1. Let \( T \geq 0 \) and \( \rho_0 \in L^\infty \cap L^1 \) a measure with finite first moment. We construct a sequence of solutions as follows: Given \( \rho^N \) we define \((u^N, \rho^{N+1})\) as the solution to the system:

\[
\begin{aligned}
\partial_t \rho^{N+1} + \text{div}(u^N \rho^{N+1}) &= 0, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
-\Delta u^N + \nabla p^N &= -\rho^N e_3, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
\text{div} u^N &= 0, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
\rho^{N+1}(0, \cdot) &= \rho_0, \quad \text{on } \mathbb{R}^3,
\end{aligned}
\]

here \( u^N \) is given by \( u^N = -\Phi * \rho^N e_3 \) and \( p^N \) its associated pressure. We choose \( \rho^0(t, \cdot) = \rho_0 \) as first step. Since \( \rho^N \) is transported by an incompressible fluid we have for all time \( t \in [0, T] \):

\[ \|\rho^N(t)\|_{L^1 \cap L^\infty} \leq \|\rho_0\|_{L^1 \cap L^\infty}. \]

Formula (8) from Proposition 2.1 yields

\[ \|u^N\|_{W^{1, \infty}} \leq C\|\rho^N\|_{L^1 \cap L^\infty}. \]

This shows that \( u^N \) is uniformly bounded in \( W^{1, \infty} \) and admits a weakly-* converging subsequence to a limit \( u \).

On the other hand, applying formula (11) from Proposition 2.3 together with formula (9) from Proposition 2.2 we have:

\[ W_1(\rho^{N+1}, \rho^N) \leq e^{Q_N t} \int_0^t \int_{\mathbb{R}^3} |u^N(\tau, x) - u^{N+1}(\tau, x)| \rho^N(t, dx) d\tau, \]
\[ \leq C e^{Q_N t} \|\rho^N\|_{L^1 \cap L^\infty} \int_0^t W(\rho^N(\tau), \rho^{N-1}(\tau)) d\tau, \]

with

\[ Q_N := \sup_{\tau \leq t} \text{Lip}(u^{N+1}(\tau, \cdot)) \leq \sup_{\tau \leq t} \|u^{N+1}(\tau, \cdot)\|_{W^{1, \infty}} \leq C \sup_{\tau \leq t} \|\rho^N\|_{L^1 \cap L^\infty} \leq C \|\rho_0\|_{L^1 \cap L^\infty}. \]

Hence

\[ ||W_1(\rho^{N+1}, \rho^N)||_{L^\infty[0,T]} \leq \left( e^{C\|\rho_0\|_{L^1 \cap L^\infty} T} \right)^N ||W_1(\rho^N, \rho^0)||_{L^\infty[0,T]}. \]

Note that, if we set \( X^N \) the characteristic flow associated to \( u^N \), we have

\[ \int |x| \rho^{N+1}(dx) = \int |X^N(t, 0, x)| \rho_0(dx) \leq \int |x| \rho_0(dx) + T \sup_{[0,T]} \|u^N(t, \cdot)\|_{\infty} \|\rho_0\|_1, \]
which ensures that the sequence \( (\rho^N)_N \subseteq \mathbb{N} \) is in the space of finite first moment measures. If we take \( T \) small enough, formula (15) shows that \( \rho^N \) is a Cauchy sequence in the (complete) space of \( L^\infty \) functions from \([0, T]\) in the complete space of finite first moment measures metrized by the Wasserstein distance \( W_1 \), see [15, Theorem 6.16]. Hence there exists a limit \( \rho \) such that:

\[
\| W_1(\rho^N, \rho) \|_{L^\infty[0, T]} \rightarrow 0.
\]

Recall that for all compact sets \( K \) we have for all \( M > N \geq 0 \)

\[
\| u^N - u^M \|_{L^\infty([0, T], L^1(K))} + \| \nabla u^N - \nabla u^M \|_{L^\infty([0, T], L^1(K))} \leq C(K) \| W_1(\rho^N, \rho^M) \|_{L^\infty([0, T])}.
\]

Hence, \( u^N_K \) and \( \nabla u^N_K \) are Cauchy sequences in \( L^\infty(0, T; L^1(K)) \) and admit a limit in \( L^\infty(0, T; W^{1,\infty}(K)) \). Finally \( u \in L^\infty(0, T; W^{1,\infty} \cap W^{1,1}_{\text{loc}}) \).

Thanks to the convergence, in the space of measure-valued functions, of \( \rho^N \) to \( \rho \) and the strong convergence of \( u^N \) towards \( u \) in \( L^\infty(0, T; W^{1,1}_{\text{loc}}) \) one can show that \((u, \rho)\) satisfies weakly the system:

\[
\begin{align*}
\partial_t \rho + \text{div}(u \rho) &= 0, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
-\Delta u + \nabla \rho &= -\rho e_3, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
\text{div } u &= 0, \quad \text{on } [0, T] \times \mathbb{R}^3, \\
\rho(0, \cdot) &= \rho_0, \quad \text{on } \mathbb{R}^3.
\end{align*}
\]

Moreover, if we assume that there exists two fixed-points \((u_i, \rho_i), i = 1, 2\), then estimate [15]

\[
\| W_1(\rho_1, \rho_2) \|_{L^\infty([0, T])} \leq CT\|\rho_1\| e^{C\|\rho_0\| T} \| W_1(\rho_1, \rho_2) \|_{L^\infty([0, T])},
\]

ensures uniqueness for \( T > 0 \) small enough. In order to show the global existence in time we need to show that the solutions \( \rho \) and \( u \) do not blow up in finite time and this is ensured by the following estimates:

\[
\begin{align*}
\| \rho(t) \|_{L^1 \cap L^\infty} &\leq \| \rho_0 \|_{L^1 \cap L^\infty}, \\
\| u(t) \|_{L^\infty} + \| \nabla u(t) \|_{L^\infty} &\leq C\| \rho(t) \|_{L^1 \cap L^\infty}.
\end{align*}
\]

\( \square \)

3. Analysis of the surface of the drop

3.1. Derivation of the hyperbolic equation. In this part we investigate the contour evolution in the case where the initial blob is axisymmetric. Using a spherical parametrization we set

\[
B_0 = \left\{ r_0(\theta) \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\phi) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}.
\]

and denote by \( B_t \) the domain at time \( t \). In order to use a spherical parametrization we set \((c(t) = (0, 0, c_3(t))) \) the position at time \( t \) of a reference point and write \( B_t = c(t) + \bar{B}_t \).
where
\[
B_t = \left\{ r(t, \theta) \begin{pmatrix} \cos(\phi) \sin(\theta) \\ \sin(\phi) \sin(\theta) \\ \cos(\theta) \end{pmatrix}, (\theta, \phi) \in [0, \pi] \times [0, 2\pi] \right\}.
\]

The velocity of the point \( c(t) \) can be chosen arbitrarily and in particular can be chosen such that \( c(t) \) is transported along the flow meaning that \( \dot{c} = u(t, c) \).

Using the convolution formula for the velocity field \( u \) together with the weak formulation of (2) we get

**Proposition 3.1.** \( r \) satisfies the following hyperbolic equation
\[
\left\{ \begin{array}{l}
\partial_t r + \partial_\theta r A_1[r] = A_2[r], \\
r(0, \cdot) = r_0.
\end{array} \right.
\]

In the case where the reference point \( c = (0, 0, c_3) \) is transported along the flow i.e. \( u(c) = \dot{c} \) we have \( c = c[r] = (0, 0, c[r]_3) \) and

\[
\left\{ \begin{array}{l}
\dot{c}[r]_3(t) = -\frac{1}{4} \int_0^\pi r^2(t, \bar{\theta}) \sin(\bar{\theta}) \left( 1 - \frac{1}{2} \sin^2(\bar{\theta}) \right) d\bar{\theta}, \\
c[r]_3(0) = 0,
\end{array} \right.
\]

The operators \( A_1[r] \) and \( A_2[r] \) are defined as follows

\[
A_1[r](t, \theta) := -\frac{1}{8\pi r(t, \theta)} \int_0^{2\pi} \int_0^\pi \frac{r(t, \bar{\theta}) \sin(\bar{\theta}) - \partial_\theta r(t, \bar{\theta}) \cos(\bar{\theta})}{\beta[r](t, \theta, \phi)} r(t, \bar{\theta}) \sin(\bar{\theta}) \left( r(t, \theta) \cos(\phi) - r(t, \bar{\theta}) \left\{ \cos(\bar{\theta}) \cos(\theta) \cos(\phi) + \sin(\bar{\theta}) \sin(\theta) \right\} \right) d\bar{\theta} d\phi + \frac{\dot{c}_3 \sin(\theta)}{r(t, \theta)}
\]

\[
A_2[r](t, \theta) := -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \frac{r(t, \bar{\theta}) \sin(\bar{\theta}) - \partial_\theta r(t, \bar{\theta}) \cos(\bar{\theta})}{\beta[r](t, \theta, \phi)} r(t, \bar{\theta}) \sin(\bar{\theta}) \left( -r(t, \bar{\theta}) \sin(\theta) \cos(\bar{\theta}) \cos(\phi) + r(t, \bar{\theta}) \cos(\theta) \sin(\bar{\theta}) \right) d\bar{\theta} d\phi - \dot{c}_3 \cos(\theta).
\]

\[
\beta[r](\theta, \bar{\theta}, \phi)^2 = r^2(\theta) + r^2(\bar{\theta}) - 2r(\theta) r(\bar{\theta}) (\sin(\theta) \sin(\bar{\theta}) \cos(\phi) + \cos(\theta) \cos(\bar{\theta})).
\]

**Remark 3.1.** The volume of the drop is conserved in time
\[
\int_0^\pi \partial_t r(t, \theta) r^2(t, \theta) \sin(\theta) d\theta = 0.
\]

**Proof of Proposition 3.1.** In what follows we drop the dependencies with respect to time since the operators \( A_1 \) and \( A_2 \) depend on \( t \) only through \( r(t, \cdot) \).
Using the change of variable \( x = c(t) + \bar{x} \in B_t, \bar{x} \in \tilde{B}_t \) the weak formulation of the transport equation writes
\[
\int_0^T \int_{\tilde{B}_t} \partial_t \psi + \nabla \psi \cdot (u(c(t) + \cdot) - \dot{c}) d\bar{x} = 0, \forall \psi \in C^\infty_c([0, T] \times \mathbb{R}^3),
\]
with \( \dot{c} = u(c) \). Since the flow preserves the rotational invariance, we define the spherical parametrization of \( \tilde{B}_t \) as follows:
\[
\tilde{B}_t = \{ z e(\theta, \phi), (\theta, \phi) \in [0, \pi] \times [0, 2\pi], 0 \leq z \leq r(t, \theta) \},
\]
where
\[
e(\theta, \phi) = \begin{pmatrix}
\cos(\phi) \sin(\theta) \\
\sin(\phi) \sin(\theta) \\
\cos(\theta).
\end{pmatrix}.
\]
Passing to the spherical parametrization in the weak formulation and doing an integration by parts we get for all \( \psi \) compactly supported in \( (0, T) \times \mathbb{R}^3 \)
\[
\int_0^T \int_{\tilde{B}_t} \partial_t \psi(t, \tilde{x}) d\tilde{x} = - \int_0^T \int_{[0, \pi] \times [0, 2\pi]} \psi(t, r(t, \theta)e(\theta, \phi)) \partial_r r(t, \theta) r^2(t, \theta) \sin(\theta) d\theta d\phi dt
\]
(19)

for the second term a direct integration by parts yields
\[
\int_0^T \int_{\tilde{B}_t} \nabla \psi(t, \tilde{x})(u(c(t) + \cdot) - \dot{c}) d\bar{x} = \int_0^T \int_{\partial \tilde{B}_t} \psi(u(c(t) + \cdot) - \dot{c}) \cdot n d\sigma dt
\]
(20)

where \( s \) is the surface element on \( \partial \tilde{B}_t \) such that the unit normal vector satisfies \( n = \frac{\tilde{\alpha}}{|\tilde{\alpha}|} \) and we have
\[
s(\theta, \phi) = s[r](\theta, \phi) = \partial_\theta \tilde{y} \times \partial_\phi \tilde{y} = r^2 \sin(\theta) e(\theta, \phi) - r'(\theta) r(\theta) \sin(\theta) \partial_\theta e(\theta, \phi).
\]
Gathering (19), (20) and (21) and dropping the dependencies with respect to \( (t, \theta) \) we get
\[
- \partial_r r + \left( u(c + re) - \dot{c} \right) \cdot e - \frac{\partial r}{r}(u(c + re) - \dot{c}) \cdot \partial_\theta e = 0.
\]
Hence we set
\[
A_1[r] = \frac{1}{r}(u(c + re) - \dot{c}) \cdot e, \quad A_2[r] = (u(c + re) - \dot{c}) \cdot e.
\]
We recall that for all \( x \in \mathbb{R}^3 \):
\[
u(x) = \frac{1}{8\pi} \int_{B_t} \left( -\frac{1}{|x-y|} e_3 - \frac{(x-y) \cdot e_3}{|x-y|^3} (x-y) \right),
\]
which can be reformulated using an integration by parts as follows

$$u(x) = -\frac{1}{8\pi} \int_{\partial B_t} \left( \frac{(x_3 - y_3)}{|x - y|} n(y) - \frac{(x - y) \cdot n(y)}{|x - y|} c_3 \right) d\sigma(y).$$

Using again the spherical parametrization of $\partial \tilde{B}_t$, we set $y = c + \tilde{y}$, where $\tilde{y} = r(t, \tilde{\theta})e(\tilde{\theta}, \tilde{\phi})$ and

$$x = c + \bar{x} = c + r(t, \theta)e(\theta, \phi) \in \partial B_t, \quad (\theta, \phi) \in [0, \pi] \times [0, 2\pi].$$

We recall that the velocity does not depend on the azimuth angle $\phi$ hence we can set $\phi = 0$. We define the operator $\mathcal{U}[r]$ as $\mathcal{U}[r](t, \theta) = u(c(t) + r(t, \theta)e(\theta, 0))$ and we have

$$\mathcal{U}[r](t, \theta) = u(c + r(t, \theta)e(\theta, 0)) = -\frac{1}{8\pi} \int_{[0, \pi] \times [0, 2\pi]} \left( \frac{(r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})) \cdot c_3}{|r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})|} s[r](\tilde{\theta}, \tilde{\phi}) \right.

$$

$$- \left. \frac{(r(t, \theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})) \cdot s[r](\tilde{\theta}, \tilde{\phi})}{|r(t, \theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})|} c_3 \right) d\tilde{\theta} d\tilde{\phi}. $$

We recall that $A_1$ and $A_2$ are given by

$$A_1[r] = \frac{1}{r}(\mathcal{U}[r] - c) \cdot \partial_\theta e, \quad A_2[r] = (\mathcal{U}[r] - c) \cdot e.$$ 

We first compute the components of the vector $\mathcal{U}[r]$. For sake of clarity we use the shortcut $\beta = |x - y| = |\bar{x} - \tilde{y}| = |r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})|$, and we have

$$\beta^2 = r^2(\theta) + r^2(\tilde{\theta}) - 2r(\theta)r(\tilde{\theta})\left( \cos(\tilde{\phi}) \sin(\theta) \sin(\tilde{\phi}) + \cos(\theta) \cos(\tilde{\phi}) \right).$$

This yields:

$$\mathcal{U}[r]_1 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \frac{r(\theta)\cos(\theta) - r(\tilde{\theta})\cos(\tilde{\theta})}{\beta} r(\tilde{\theta}) \sin(\tilde{\theta})$$

$$\times \left( r(\tilde{\theta}) \sin(\tilde{\theta}) - r'(\tilde{\theta}) \cos(\tilde{\theta}) \right) \cos(\tilde{\phi}) d\tilde{\theta} d\tilde{\phi},$$

$$\mathcal{U}[r]_2 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \frac{r(\theta)\cos(\theta) - r(\tilde{\theta})\cos(\tilde{\theta})}{\beta} r(\tilde{\theta}) \sin(\tilde{\theta})$$

$$\times \left( r(\tilde{\theta}) \sin(\tilde{\theta}) - r'(\tilde{\theta}) \cos(\tilde{\theta}) \right) \sin(\tilde{\phi}) d\tilde{\theta} d\tilde{\phi},$$

$$\mathcal{U}[r]_3 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \frac{r(\tilde{\theta}) \sin(\tilde{\theta}) - r'(\tilde{\theta}) \cos(\tilde{\theta})}{\beta} r(\tilde{\theta}) \sin(\tilde{\theta})$$

$$\times \left\{ - r(\theta) \sin(\theta) \cos(\tilde{\phi}) + r(\tilde{\theta}) \sin(\tilde{\theta}) \right\} d\tilde{\theta} d\tilde{\phi}.$$
Finally if we assume $u(c) = \dot{c}$ we get:

$$
\dot{c} = u(c) = -\frac{1}{8\pi} \int_{\partial B_r} \left( -\frac{\bar{y}_3}{|\bar{y}|} s + e_3 \frac{\bar{y} \cdot s}{|\bar{y}|} \right) d\sigma(\bar{y}),
$$

recall that $|\bar{y}| = r(\theta)$ and since $e \perp \partial \theta e$ we get:

$$
\bar{y} \cdot s = r(\theta)e(\theta, \phi) \cdot \left( r^2(\theta) \sin(\theta) e(\theta, \phi) - r'(\theta) r(\theta) \sin(\theta) \partial_\theta e(\theta, \phi) \right) = r^3(\theta) \sin(\theta).
$$

This yields:

$$
\dot{c}_1 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi -\cos(\theta) \left( r^2(\theta) \sin(\theta) \cos(\phi) \sin(\theta) - r'(\theta) r(\theta) \sin(\theta) \cos(\phi) \cos(\theta) \right) d\theta d\phi = 0,
$$

$$
\dot{c}_2 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi -\cos(\theta) \left( r^2(\theta) \sin(\phi) \sin(\theta) - r'(\theta) r(\theta) \sin(\theta) \sin(\phi) \cos(\theta) \right) d\theta d\phi = 0.
$$

$$
\dot{c}_3 = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^\pi \left( -\cos(\theta) \left( r^2(\theta) \sin(\theta) \cos(\theta) + r'(\theta) r(\theta) \sin^2(\theta) \right) \right)
+ r^2(\theta) \sin(\theta) d\theta d\phi,
$$

$$
= -\frac{1}{4} \int_0^\pi \left( r^2(\theta) \sin^3(\theta) - r'(\theta) r(\theta) \cos(\theta) \sin^2(\theta) \right) d\theta,
$$

$$
= -\frac{1}{4} \int_0^\pi \left( r^2(\theta) \sin^3(\theta) + \frac{1}{2} r^2(\theta) \left( -\sin^3(\theta) + 2 \cos^2(\theta) \sin(\theta) \right) \right) d\theta,
$$

$$
= -\frac{1}{4} \int_0^\pi \frac{1}{2} r^2(\theta) \left( -\sin^3(\theta) + 2 \sin(\theta) \right) d\theta,
$$

$$
= -\frac{1}{4} \int_0^\pi r^2(\theta) \sin(\theta) \left( 1 - \frac{1}{2} \sin^2(\theta) \right) d\theta < 0.
$$
We conclude by replacing formulas (26) and (31), (32) in (22). For the volume conservation, direct computations using (21) yield
\[
\int_0^\pi \partial_t r(t, \theta) r^2(t, \theta) \sin(\theta) d\theta \\
= \int_0^\pi A_2[r](t, \theta) r^2(t, \theta) \sin(\theta) - \partial_\theta r(t, \theta) A_1[r](t, \theta) r^2(t, \theta) \sin(\theta) d\theta \\
= \int_0^\theta r^2 \sin(\theta)(u(c + re(\theta, 0)) - \dot{c}) \cdot e(\theta, 0) - r \partial_\theta r \sin(\theta)(u(c + re(\theta, 0)) - \dot{c}) \cdot \partial_\theta e(\theta, 0) \\
= \int_0^\theta (u(c + re(\theta, 0)) - \dot{c}) \cdot s(\theta, 0) d\theta \\
= \frac{1}{2\pi} \int_0^{2\pi} \int_0^\theta u(c + re(\theta, \phi)) \cdot s(\theta, \phi) d\theta d\phi - \frac{1}{2\pi} \int_0^\pi \partial_\theta \left( \frac{1}{2} r^2(t, \theta) \sin^2(\theta) \right) d\theta \\
= \frac{1}{2\pi} \int_{\partial B} u(c + x) \cdot n(x) d\sigma(x) \\
= \frac{1}{2\pi} \int_{\partial B} u \cdot n = 0.
\]

\[\Box\]

3.2. Proof of the local existence and uniqueness Theorem 1.2. This section is devoted to the proof of local existence and uniqueness of a solution for equation (5). Given \( r \in C(0, \pi) \), we recall the definition of the following quantity
\[ |r|_* = \inf_{(0, \pi)} r(\theta). \]

Proof. The main idea is to apply a fixed-point argument. We recall that the operators \( A_1 \) and \( A_2 \) are defined using the velocity field \( u \) defined in (25). It is possible to formulate otherwise the velocity \( u \) using a spherical parametrization of the droplet \( B_t = \{ c(t) + ze(\theta, \bar{\phi}, \bar{\sigma}) \in (0, \pi) \times (0, 2\pi), 0 \leq z \leq r(\bar{\theta}) \} \). This yields the following formula for \( u \)
\[
(33) \quad \mathcal{U}[r](\theta) = \int_{(0, \pi) \times (0, 2\pi)} \int_0^{r(\bar{\theta})} \Phi(r(\bar{\theta})e(\theta, 0) - z e(\bar{\theta}, \bar{\phi})) z^2 \sin(\bar{\theta}) dz d\bar{\theta} d\bar{\phi},
\]
with \( \Phi \) the Oseen tensor, see (7). With this definition, the operator \( \mathcal{U}[r] \) satisfies the following estimates for \( r \in W^{1, \infty} \) such that \( |r|_* > 1 \)
\[
|\mathcal{U}[r](\theta)| \leq C \int_{(0, \pi) \times (0, 2\pi)} \int_0^{r(\bar{\theta})} \frac{z^2 dz}{|r(\bar{\theta})e(\theta, 0) - z e(\bar{\theta}, \bar{\phi})|} \sin(\bar{\theta}) d\bar{\theta} d\bar{\phi}, \\
\leq \frac{\|r\|_{1, \infty}^5}{\sqrt{|r|_*}} \int_0^\pi \frac{\sin(\bar{\theta}) d\bar{\theta}}{|e(\bar{\theta}, \bar{\phi}) - e(\theta, 0)|}
\]
where we used the fact that
\[
|e(\theta) - e(\bar{\theta}, \bar{\phi})|^2 = z^2 + r(\theta)^2 - 2zr(\theta)\sin(\bar{\theta}) - e(\bar{\theta}, \bar{\phi})
\]
\[
= (z - r(\theta))^2 + zr(\theta)\sin(\bar{\theta}) - e(\theta, 0)^2
\]
\[
\geq zr(\theta)\sin(\bar{\theta}) - e(\theta, 0)^2,
\]
we conclude using Lemma B.1. For the derivative of \(U[r]\) we use the shortcuts \(e = e(\theta, 0), \bar{e} = e(\bar{\theta}, \bar{\phi}), r = r(\theta), \bar{r} = r(\bar{\theta})\) and obtain after an integration on \(z\)
\[
|\partial_\theta U[r]| \leq C(|r(\theta)| + |\partial_\theta r(\theta)|) \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta)} \frac{z^2 dz}{|r(\theta)e(\theta, 0) - e(\bar{\theta}, \bar{\phi})|^2} \sin(\bar{\theta})d\bar{\theta}d\bar{\phi},
\]
\[
= C\|r\|_1,\infty \int_0^{2\pi} \int_0^\pi \int_0^{r(\theta)} \frac{z^2 dz}{(z - re \cdot \bar{e})^2 + r^2(1 - \bar{e} \cdot \bar{e})^2} \sin(\bar{\theta})d\bar{\theta}d\bar{\phi},
\]
\[
= C\|r\|_1,\infty \int_0^{2\pi} \int_0^\pi \left( r(\theta) + re \cdot \bar{e} \log \frac{|re - \bar{r}e|}{r} + r \left( \frac{2e \cdot \bar{e}^2 - 1}{\sqrt{1 - \bar{e} \cdot \bar{e}^2}} \left[ \arctan \left( \frac{r(\theta) - re \cdot \bar{e}}{r\sqrt{1 - \bar{e} \cdot \bar{e}^2}} \right) + \arctan \left( \frac{\bar{e} \cdot e}{\sqrt{1 - \bar{e} \cdot \bar{e}^2}} \right) \right] \right) \sin(\bar{\theta})d\bar{\theta}d\bar{\phi}
\]
\[
\leq C\|r\|^2_1,\infty \left( 1 + \frac{\|r\|_\infty}{|r|_*} \right) \left[ \int_0^{2\pi} \int_0^\pi \frac{\sin(\bar{\theta})d\bar{\theta}d\bar{\phi}}{|e(\bar{\theta}, \bar{\phi}) - e(\theta, 0)|} + \int_0^{2\pi} \int_0^\pi \frac{\sin(\bar{\theta})d\bar{\theta}d\bar{\phi}}{1 - e(\theta, 0) \cdot e(\bar{\theta}, \bar{\phi})^2} \right],
\]
where we used the fact that \(\log(z)\) is uniformly bounded and that \(|re - \bar{r}e| \geq |r|_*|e - \bar{e}|\).
We conclude using Lemma B.1. Let \(r_1, r_2 \in C(0, \pi), |r_1|_*, |r_2|_* > 0\), reproducing the same arguments as previously we have the following stability estimate
\[
|U[r_1](\theta) - U[r_2](\theta)|
\]
\[
\leq \int_{(0,\pi)\times(0,2\pi)} \int_{r_1(\theta)}^{r_2(\theta)} \frac{z^2 dz}{r_1 e - z\bar{e}} \sin(\bar{\theta})d\bar{\theta}d\bar{\phi}
\]
\[
+ \|r_1 - r_2\|_\infty \int_{(0,\pi)\times(0,2\pi)} \int_0^{r_2(\theta)} \left( \frac{z^2 dz}{r_1 e - z\bar{e}} + \frac{z^2 dz}{r e - z\bar{e}} \right) dzd\bar{\theta}d\bar{\phi}
\]
\[
\leq C\|r_1 - r_2\|_\infty \left( \frac{\|r_1\|^{3/2}_\infty + \|r_2\|^{3/2}_\infty}{|r_1|_*} \right) + (\|r_1\|_\infty + \|r_2\|_\infty)
\]
\[
\times \left[ 1 + (\|r_1\|_\infty + \|r_2\|_\infty) \left( \frac{1}{|r_1|_*} + \frac{1}{|r_2|_*} \right) \log \left( \frac{(\|r_1\|_\infty + \|r_2\|_\infty)^2}{|r_1|_*|r_2|_*} \right) \right].
\]
On the other hand since $c[r]$ is defined in (6) we get
\begin{equation}
|c[r]| \leq C\|r\|_\infty,
|c[r]_1 - c[r]_2| \leq C\|r_1 - r_2\|_\infty(\|r\|_\infty + \|r_2\|_\infty).
\end{equation}
Since $A_1[r], A_2[r]$ are defined in (20) using the estimates of $U[r]$ and $\dot{c}$ we obtain
\begin{equation}
\|A_1[r]\|_{1,\infty} \leq C\frac{1}{|r|_*} \left(1 + \|r\|_{1,\infty} \left(1 + \frac{1}{|r|_*}\right)\right)(\|U[r]\|_{1,\infty} + \|r\|_{2,\infty}^2),
\end{equation}
\begin{equation}
\|A_2[r]\|_{1,\infty} \leq C\left(\|U[r]\|_{1,\infty} + \|r\|_{2,\infty}^2\right),
\end{equation}
\begin{equation}
\|A_1[r] - A_1[r_2]\|_\infty \leq K\left(\frac{1}{|r_1|_*}, \frac{1}{|r_2|_*}, \|r_1\|_{1,\infty}, \|r_2\|_{1,\infty}\right)\|r_1 - r_2\|_\infty
\end{equation}
Now, given $r$, we introduce $\Theta[r]$ the characteristic flow of the transport equation (5)
\begin{equation}
\begin{cases}
\dot{\Theta}[r](t, s, \theta) = A_1[r](t, \Theta[r](t, s, \theta)), \\
\Theta[r](t, t, \theta) = \theta.
\end{cases}
\end{equation}
Thanks to the regularity of $A_1[r]$ the characteristic flow is well defined and in particular the characteristic curves do not intersect and satisfy
\[\Theta[r](t, s, \cdot) \circ \Theta[r](s, t, \cdot) = \text{id}.\]
In particular since $A_1[r](0) = A_1[r](\pi) = 0$ we have $\Theta[r](t, s, 0) = 0$ and $\Theta[r](t, s, \pi) = \pi$ for all $t, s$. Thanks to this properties, for a given $r$ the unique solution of the transport equation
\begin{equation}
\begin{cases}
\partial_t \tilde{r} + \partial_\theta \tilde{r} A_1[r] = A_2[r], \\
\tilde{r}(0, \cdot) = r_0,
\end{cases}
\end{equation}
satisfies
\[\frac{d}{dt} \tilde{r}(t, \Theta[r](t, 0, \theta)) = A_2[r](t, \Theta[r](t, 0, \theta)),\]
since the characteristic curves are well defined and do not intersect we have
\begin{equation}
\tilde{r}(t, \theta) = r_0(\Theta[r](0, t, \theta)) + \int_0^t A_2[r](s, \Theta[r](s, t, \theta)) ds.
\end{equation}
Hence we define the mapping $L : C(0, T; C^{0,1}(0, \pi)) \to C(0, T; C^{0,1}(0, \pi))$ which associates to each $r$ the solution $\tilde{r}$ of equation (38) defined by (39). Thanks to estimates (35), (36) and (37) the operator $L$ satisfies for all $r, r_1, r_2$ such that $\|r\|_{1,\infty} \leq \|r_0\|_{1,\infty} \lambda$ and $|r|_* \geq \beta |r_0|_*$
with $\beta < 1 < \lambda$
\[\|L[r](t, \cdot)\|_{1,\infty} \leq \|r_0\|_{1,\infty} + TC(\lambda, \beta, \|r_0\|_{1,\infty}, |r_0|_*),\]
\[|L[r](t, \cdot)|_* > |r_0|_* - TC(\lambda, \beta, \|r_0\|_{1,\infty}, |r_0|_*),\]
\[\|L[r_1](t, \cdot) - L[r_2](t, \cdot)\|_{\infty} \leq C(\lambda, \beta, \|r_0\|_{1,\infty}, |r_0|_* )T \|r_1(t, \cdot) - r_2(t, \cdot)\|_{\infty}.\]
If we define the sequence $(r^n)_{n \in \mathbb{N}}$ such that $r_0 = r_0$ and $r^{n+1} = L[r^n]$ i.e.
\begin{equation}
\begin{cases}
\partial_t r^{n+1} + \partial_\theta r^{n+1} A_1[r^{n}] = A_2[r^n], \\
r^{n+1}(0, \cdot) = r_0,
\end{cases}
\end{equation}
previous estimates ensure that, for $T$ small enough, $v^n$ converges (up to a subsequence) to some $\overline{r} \in C(0, T; C(0, \pi))$ satisfying equation \( [5] \). Moreover, we have $\overline{r} \in C(0, T; C^{0,1}(0, \pi))$ and $|\overline{r}|_* > 0$. Uniqueness of the fixed-point is ensured thanks to the former stability estimates. Eventually, we recover the existence and uniqueness of $c[r]$ thanks to \( [34] \). □

3.3. **Comparison with Hadamard-Rybczynski analysis.** In this section we compare our analysis to the result of Hadamard \[1\] and Rybczynski \[13\] who investigate the motion of a liquid spherical drop $B$ falling in a viscous fluid, see also \[1\] \[2\] \[3\]. The equations considered are Stokes equations on both fluid and drop domain. Denoting by $\bar{\rho}$, $\bar{\mu}$ (resp. $\rho$, $\mu$) the density and viscosity of the drop (resp. density and viscosity of the fluid). Authors show that $B_t = v^* t + B_0$ i.e. the spherical form of the droplet is preserved and the velocity fall of the droplet $v^*$ is given by

\begin{equation}
(41)
 v^* = \frac{2}{9} \frac{R^2}{\mu} (\bar{\rho} - \rho) \frac{\mu + \bar{\mu}}{\bar{\mu} + \frac{2}{3} \mu} g.
\end{equation}

In the case where we drop the coefficient $\frac{R^2}{\mu} (\bar{\rho} - \rho)$ and set $\mu = \bar{\mu} = 1$, we get

\begin{equation}
(42)
 v^* = -\frac{4}{15} e_3.
\end{equation}

In particular, it is shown that the velocity of both the exterior fluid $u$ and interior fluid $\bar{u}$ satisfy the following property

**Lemma 3.2** (Hadamard-Rybczynski). Let $u_0 = -\Phi \ast 1_{B_0} e_3$. $v^* = -\frac{4}{15} e_3$. We have

\((u_0 (e(\theta, 0)) - v^*) \cdot e(\theta, 0) = 0, \text{ for all } \theta \in [0, \pi].\)

We present below a proof relying on direct computations

**Proof.** Let $\theta \in [0, \pi]$ and $e(\theta, 0) \in \partial B(0, 1)$. We recall formula \( [24] \)

\( u(e(\theta, 0)) = -\frac{1}{8\pi} \int_{\partial B(0, 1)} \left( \frac{(e(\theta, 0) - y) \cdot e_3}{|e(\theta, 0) - y|} n(y) - \frac{e(\theta, 0) \cdot y - 1}{|e(\theta, 0) - y|} e_3 \right) d\sigma(y). \)

We set $Q(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$ the rotation matrix such that $e(\theta, 0) = Q(\theta)e_3$

with $e_3 = (0, 0, 1)$ and use the change of variable $y = Q(\theta)\omega, \omega \in \partial B(0, 1)$ such that

\( n(y) = y = Q(\theta)n(\omega) = Q(\theta)\omega \text{ and } d\sigma(y) = d\sigma(\omega). \)

We drop the dependencies with respect to $\theta$ and write

\(-8\pi u(e) = Q \left( \int_{\partial B(0, 1)} \frac{(Qe)_3 - (Q\omega)_3}{|e_3 - \omega|} \omega d\sigma(\omega) \right) - \int_{\partial B(0, 1)} \frac{(Qe)_3 \cdot (Q\omega) - 1}{|e_3 - \omega|} e_3 d\sigma(\omega) \)

\(- (Qe)_3 Q \left( \int_{\partial B(0, 1)} \frac{w}{|e_3 - \omega|} d\sigma(\omega) \right) - Q \left( \int_{\partial B(0, 1)} \frac{(Q\omega)_3}{|e_3 - \omega|} \omega d\sigma(\omega) \right) \)

\(- \left( \int_{\partial B(0, 1)} \frac{\omega_3}{|e_3 - \omega|} d\sigma(\omega) \right) e_3 + \left( \int_{\partial B(0, 1)} \frac{1}{|e_3 - \omega|} d\sigma(\omega) \right) e_3. \)
We have \((Q\omega)_3 = -\sin(\theta)\omega_1 + \cos(\theta)\omega_3\), direct computations yield
\[
\int_{\partial B(0,1)} \frac{w}{|e_3 - \omega|} d\sigma(\omega) = \frac{4\pi}{3} e_3, \quad \int_{\partial B(0,1)} \frac{1}{|e_3 - \omega|} d\sigma(\omega) = 4\pi
\]
\[
\int_{\partial B(0,1)} \frac{w_1}{|e_3 - \omega|} \omega d\sigma(\omega) = \frac{16}{15} \pi e_1, \quad \int_{\partial B(0,1)} \frac{w_3}{|e_3 - \omega|} \omega d\sigma(\omega) = \frac{14}{15} 2\pi e_3,
\]
where \(e_1 = (1, 0, 0)\) hence we get
\[
-8\pi u(e) = \cos(\theta) \frac{4\pi}{3} Q e_3 - Q \left(-\sin(\theta) \frac{16}{15} \pi e_1 + \cos(\theta) \frac{14}{15} 2\pi e_3\right)
\]
\[
-4\pi e_3 + 4\pi e_3
\]
\[
= -\cos(\theta) Q e_3 \frac{8\pi}{15} + \sin(\theta) \frac{16\pi}{15} Q e_1 + \frac{8\pi}{3} e_3,
\]
which yields the desired result. \(\square\)

3.3.1. **Comparison with the transport-Stokes equation.** The above Lemma yields directly the following result.

**Corollary 3.3.** The solution \((u, \rho)\) of the transport-Stokes equation \((2)\) in the case where \(\rho_0 = 1_{B_0}\) is given by
\[
(43) \quad u(t, x) = u_0(x - v^*t), \quad \rho(t, x) = \rho_0(x - v^*t),
\]
\[
(44) \quad u_0 = -\Phi * \rho_0 e_3, \quad \rho_0 = 1_{B(0,1)}.
\]
In other words, the drop \(B_t\) remains spherical for all time.

**Proof.** Indeed we have using \((43), (44)\)
\[
\partial_t \rho + \nabla \rho \cdot u = (\nabla \rho_0 \cdot (u_0 - v^*))_{(-v^*)}, \quad 0 = 0,
\]
we conclude using Lemma \([3.2]\) since \(\nabla \rho_0 = n s^1\) where \(s^1\) is the surface measure on the sphere and \(n\) the unit normal on the sphere. \(\square\)

3.3.2. **Comparison with the hyperbolic equation.** We are interested now in showing that the solution of the hyperbolic equation \((5)\) corresponds also to the Hadamard-Rybczynski solution \(i.e.\)
\[
c + \partial \tilde{B}_t = \partial B(v^*t, 0),
\]
with \(\partial \tilde{B}_t = \{r(t, \theta) e(\theta, \phi), (\theta, \phi) \in [0, \pi] \times [0, 2\pi]\}\. First, in the case where the reference point \(c\) corresponds to the center \(c^*(t) := v^*t\) given by Hadamard-Rybczynski, the result is straightforward since the source term \(A_2[r]\) of the hyperbolic equations becomes according to formula \(26)\)
\[
A_2[r](\theta) = (U[r] - \hat{c}^*) \cdot e(\theta, 0) = (U[r](\theta) - v^*) \cdot e(\theta, 0),
\]
which vanishes for \(r = 1\) since \(\theta \mapsto U[1](\theta)\) corresponds to the velocity \(\theta \mapsto u_0(e(\cdot, 0))\) introduced in Lemma \([3.2]\). This shows that \(r = 1\) is a solution to the hyperbolic equation
in the case \( c = c^* = v^*t \).

In the general case \( \dot{c} \neq v^* \), by symmetry, it is enough to show that
\[
|c(t) + r(t, \theta)e(\theta, 0) - c^*|^2 = 1 \text{ for all } \theta \in [0, \pi] \text{ and } t.
\]

Equivalently, we consider the function \( \bar{r} \) satisfying the above formula and show that it satisfies the hyperbolic equation. This is shown in the following Proposition.

**Proposition 3.4.** Let \( r_0 = 1 \) and \((r, c)\) the solution of (5) with \( \dot{c} \neq v^* \). Denote by \( T > 0 \) the maximal time of existence of the solution such that \(|c - c^*| \leq 1\) with \( c^* = v^*t = -\frac{4}{15}e_3 t \). Then \( r \) is given by
\[
r(t, \theta) = -(c - c^*)_3 \cos(\theta) + \sqrt{1 - (c - c^*)_3^2 \sin^2(\theta)}, \quad (t, \theta) \in [0, T] \times [0, \pi]
\]

and satisfies
\[
|c(t) + r(t, \theta)e(\theta, 0) - v^*t|^2 = 1 \text{ for all } \theta \in [0, \pi] \text{ and } t \leq T.
\]

In other words
\[
\partial B_t := c + \partial \bar{B}_t = \partial B(c^*, 1) \text{ on } [0, T].
\]

**Proof.** First, note that \(|c(t) + r(t, \theta)e(\theta, 0) - c^*|^2 = 1\) corresponds to
\[
(45) \quad r^2 + 2(c - c^*)_3 r \cos(\theta) + (c - c^*)_3^2 = 1 = 0,
\]

Computing the solutions of the quadratic equation (45) we denote by \( \bar{r} \) the solution which satisfies \( \bar{r}(0, \cdot) = 1 \) given by
\[
\bar{r}(t, \theta) = -(c - c^*)_3 \cos(\theta) + \sqrt{1 - (c - c^*)_3^2 \sin^2(\theta)},
\]

which is well defined provided that \(|c - c^*| \leq 1\). We aim to prove that \( \bar{r} \) satisfies the hyperbolic equation (5). We have
\[
\partial_t \bar{r}(t, \theta) = -(\dot{c} - c^*) \frac{\bar{r} \cos(\theta) + (c - c^*)_3}{\bar{r} + (c - c^*) \cos(\theta)}, \quad \partial_\theta \bar{r}(t, \theta) = \frac{\bar{r}(t, \theta) \sin(\theta)(c - c^*)_3}{\bar{r} + (c - c^*)_3 \cos(\theta)}.
\]

Direct computations using formula (26) yield
\[
(\partial_t \bar{r} + \partial_\theta \bar{r} A_1[\bar{r}]) (\bar{r} + (c - c^*)_3 \cos(\theta)) = -(\dot{c} - c^*) \bar{r} \cos(\theta) - (\dot{c} - c^*) (c - c^*)_3
\]
\[
+ (\mathcal{U}[\bar{r}] - \dot{c})_1 \cos(\theta) \sin(\theta)(c^*)_3 - (\mathcal{U}[\bar{r}] - \dot{c})_3 (c - c^*)_3 \sin^2(\theta).\]

Taking the difference between the two above formulas we obtain
\[
A_2[\bar{r}] (\bar{r} + (c - c^*)_3 \cos(\theta)) (\partial_t \bar{r} + \partial_\theta \bar{r} A_1[\bar{r}] - A_2[\bar{r}] = (c^* - \mathcal{U}[\bar{r}]) \cdot (\bar{r} e(\theta, 0) + c - c^*).
\]

It remains to prove that the right hand side in the above formula is equal to zero. Indeed the term \( \bar{r} + (c - c^*)_3 \cos(\theta) = \sqrt{1 - (c - c^*)_2 \sin^2(\theta)} \) in the above left hand side cannot
be identically null for all \(t\) and \(\theta \in [0, \pi]\) since we are in the case \(\dot{c} \neq v^*\). We recall the formula of \(\mathcal{U}[r]\) given in (25)

\[
\mathcal{U}[r](t, \theta) = -\frac{1}{8\pi} \int_{[0,\pi] \times [0,2\pi]} \left( \frac{(r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})) \cdot e_3}{r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})} s[r](\tilde{\theta}, \tilde{\phi}) \right. \\
\left. - \frac{(r(t, \theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})) \cdot s[r](\tilde{\theta}, \tilde{\phi}) e_3}{r(\theta)e(\theta, 0) - r(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi})} \right) d\tilde{\theta} d\tilde{\phi}.
\]

We recall that \(\bar{r}\) is such that \(|c + \bar{r}(t, \theta)e(\theta, 0) - c^*| = 1\). We claim that for all \(\theta \in [0, \pi]\) there exists \(\gamma \in [0, \pi]\) such that

\[c + \bar{r}(t, \theta)e(\theta, 0) - c^* = e(\gamma, 0),\]

and the mapping \(\gamma \mapsto \theta\) is bijective. Indeed, let \(\theta \in [0, \pi]\), we search for \(\gamma \in [0, \pi]\) satisfying

\[c + \bar{r}(t, \theta)e(\theta, 0) - c^* = e(\gamma, 0),\]

which yields

\[
\cos(\gamma) = (c - c_3^*) + \bar{r}(\theta) \cos(\theta), \quad \sin(\gamma) = \bar{r}(\theta) \sin(\theta).
\]

Note that \(\theta \mapsto \bar{r}(\theta) \cos(\theta)\) is monotone indeed

\[
\partial_\theta [\theta \mapsto \bar{r}(\theta) \cos(\theta)] = -\frac{r^2(\theta) \sin(\theta)}{\sqrt{1 - \sin^2(\theta)(c - c^*)^2}} \leq 0,
\]

moreover,

\[\left[\theta \mapsto (c - c_3^*) + \bar{r}(\theta) \cos(\theta)\right]_{\theta=0} = 1, \quad \left[\theta \mapsto (c - c_3^*) + \bar{r}(\theta) \cos(\theta)\right]_{\theta=\pi} = -1\]

hence we have \(\theta \mapsto (c - c_3^*) + \bar{r}(\theta) \cos(\theta)\) is \([-1, 1]\] and bijective. This ensures that \(\gamma \mapsto \theta\) is bijective and in particular we have \(\gamma = 0\) when \(\theta = 0\) and \(\gamma = \pi\) when \(\theta = \pi\), see Figure 3 for an illustration.

Consequently, we introduce the change of variable \(c + \bar{r}(\tilde{\theta})e(\tilde{\theta}, \tilde{\phi}) - c^* = e(\tilde{\gamma}, \tilde{\phi}) := \omega \in \partial B(0, 1)\) and we set \(x' = c + \bar{r}(\theta)e(\theta, 0) - c^* = e(\gamma, 0) \in \partial B(0, 1)\). Direct computations yield

\[
\gamma = \arccos((c - c^*)_3 + \bar{r}(\theta) \cos(\theta))
\]

\[
d\gamma = \frac{\bar{r}(\theta)}{\bar{r}(\theta) + (c - c^*)_3 \cos(\theta)} d\theta.
\]
\[ s[\bar{r}](\theta, \phi)d\theta = \frac{\bar{r}(\theta) + (c - c^*)_3 \cos(\theta)\bar{r}(\theta)}{\bar{r}(\theta)} s[\bar{r}](\theta, \phi)d\gamma \]
\[ = \frac{\bar{r}(\theta) + (c - c^*)_3 \cos(\theta)}{\bar{r}(\theta)} \bar{r}(\theta) \sin(\theta)(\bar{r}e(\theta, \phi) - \partial_\theta \bar{r} \partial_\theta e(\theta, \phi))d\gamma \]
\[ = \bar{r} \sin(\theta) \left( \begin{array}{c} \bar{r} \sin(\theta) \cos(\phi) \\ \bar{r} \sin(\theta) \sin(\phi) \\ \bar{r} \cos(\theta) + (c - c^*) \end{array} \right) d\gamma \]
\[ = \sin(\gamma)e(\gamma, \phi)d\gamma \]
\[ = s[1](\gamma, \phi)d\gamma \]

where we used the fact that \( \sin(\gamma) = \bar{r}(\theta) \sin(\theta) \) and \( \cos(\gamma) = \bar{r}(\theta) \cos(\theta) + (c - c^*) \). We get eventually

\[ U[\bar{r}](t, \theta) = -\frac{1}{8\pi} \int_{\partial B(0,1)} \left( \frac{(e(\gamma, 0) - \omega) \cdot e_3}{|e(\gamma, 0) - \omega|} n(\omega) - \frac{(e(\gamma, 0) - \omega) \cdot n(\omega)}{|e(\gamma, 0) - \omega|} e_3 \right) d\sigma(\omega) \]
\[ = U[1](\gamma), \]

using lemma 3.2 and the fact that \( U[1](\cdot) \) corresponds to \( u_0(e(\cdot, 0)) \) defined in Lemma 3.2 we have \( U[1](\gamma)e(\gamma, 0) = v^*e(\gamma, 0) \) which yields using the fact that \( c + r(\theta)e(\theta, 0) - c^* = e(\gamma, 0) \)

\[ U[\bar{r}](t, \theta) \cdot (c + r(\theta)e(\theta, 0) - c^*) = v^* \cdot (c + r(\theta)e(\theta, 0) - c^*), \]

which concludes the proof. \qed

Proposition 3.4 suggests that the existence time of the solution depends on the choice of \( \dot{c} \). We complete the analysis by showing that the choice for which \( c \) is transported along the flow \textit{i.e.} \( \dot{c} \) is given by (6) is such that \( |c - c^*| \leq 1 \) for all time.
Proposition 3.5. Let \( r_0 = 1 \) and \((r, c)\) the solution of \((5)\) and \((6)\). Then for all time \( t \geq 0 \) we have \( c(t) \leq c^*(t) \), \(|c(t) - c^*(t)| \leq 1 \) and

\[
\lim_{t \to \infty} c(t) - c^*(t) = -1
\]

Proof. We recall the formula for \( r \) given by Proposition 3.4

\[
r(t, \theta) = -(c - c^*)_3 \cos(\theta) + \sqrt{1 - (c - c^*_3)^2 \sin^2(\theta)},
\]

and we have

\[
r^2 = 1 - (c - c^*_3)^2 - 2(c - c^*_3) r \cos(\theta).
\]

This yields

\[
\dot{c}_3 - \dot{c}_3^* = -\frac{1}{4} \int_0^\pi r^2(t, \theta) \sin(\theta) \left(1 - \frac{1}{2} \sin^2(\theta) \right) d\theta - v_3^*
\]

\[
= -v_3^* - \frac{1}{4} (1 - (c - c^*_3)^2) \int_0^\pi \sin(\theta) \left(1 - \frac{1}{2} \sin^2(\theta) \right) d\theta
\]

\[
+ \frac{1}{2} (c - c^*_3) \int_0^\pi \cos^2(\theta) \sin(\theta) \left(1 - \frac{1}{2} \sin^2(\theta) \right) d\theta
\]

\[
+ \int_0^\pi \cos(\theta) \sin(\theta) \sqrt{1 - (c - c^*_3)^2 \sin^2(\theta)} d\theta
\]

where we used the fact that the last integral vanishes using the change of variable \( \theta' = \pi - \theta \).

We get

\[
\dot{c}_3 - \dot{c}_3^* = -\frac{1}{15} + \frac{1}{15} (c - c^*_3)^2.
\]

solving the ODE \( \dot{x} = -\frac{1}{15} + \frac{1}{15} x^2 \) with \( x(0) = 0 \) we obtain

\[
c(t) - c_3^*(t) = \frac{1 - e^{\frac{2t}{15}}}{e^{\frac{2t}{15}} + 1},
\]

this shows that \( c \leq c^* \), \(|c - c^*| \leq 1 \) for all time and in particular \( c - c^* \to -1 \) when \( t \to \infty \).

\[\square\]

4. Numerical simulations

We present in this section numerical simulations in the spherical case \emph{i.e.} \( r_0 = 1 \).

In what follows we set \( T > 0 \), we consider \( N, M, L \in \mathbb{N}^* \) and define

\[
(\Delta t, \Delta \theta, \Delta \phi) = \left( \frac{T}{N}, \frac{\pi}{M}, \frac{2\pi}{L} \right),
\]

we set for \( i = 0, \ldots, M, j = 0, \ldots, L, n = 0, \ldots, N \)

\[
\theta_i = \Delta \theta i, \quad \phi_j = \Delta \phi j, \quad t^n = \Delta tn.
\]
Let $\theta_i \in [0, \pi]$, $(t^n)_{1 \leq n \leq N}$ be a subdivision of $[0, T]$ and $(\phi)_{1 \leq j \leq L}$ a subdivision of $[0, 2\pi]$. We discretise the radius and the center by setting

$$r(t, \theta) \sim (r^n_{1 \leq n \leq N}), r_i^n = r(t^n, \theta_i), c(t) \sim (c^n)_{1 \leq n \leq N}.$$ 

We use the following classical upwind finite difference scheme for the hyperbolic equation. Given $(r^n_i)_{1 \leq n \leq N}$ we define $(r^{n+1}_i)_{1 \leq n \leq N}$ as

$$r^{n+1}_i = r^n_i - \frac{\Delta t}{\Delta \theta} A^{i,n}_1 \left\{ \begin{array}{ll} r^n_i - r^{n-1}_i & \text{if } A^n_1 \geq 0, \\ r^n_{i+1} - r^n_i & \text{if } A^n_1 \leq 0, \\ i = 2, \cdots, M - 1, \\ \end{array} \right. + \Delta t A^n_2, i = 2, \cdots, M - 1,$$

where $A^n_1 = A_1[r^n](t^n, \theta_i), A^n_2 = A_2[r^n](t^n, \theta_i)$ are computed by discretizing the integrals. For $i = 1, M$ we note that $A_1[r](t, 0) = A_1[r](t, \pi) = 0$ for all function $r$ and $t \geq 0$, hence we set

$$r^{n+1}_1 = r^n_1 + \Delta t A^n_2, r^{n+1}_M = r^n_M + \Delta t A^n_2.$$ 

For a fixed time $T > 0$, the following conditions ensure a uniform bound of $\max_{1 \leq n \leq N, 1 \leq i \leq M} |r^n_i|$

$$\max_{1 \leq n \leq N, 1 \leq i \leq M} |A^{i,n}_1| \frac{\Delta t}{\Delta \theta} < 1, \max_{1 \leq k \leq N, 1 \leq i \leq M} |A^{i,k}_2| \leq C.$$ 

For the evolution of the center we set $c \sim (c^n)_{1 \leq n \leq N}$ with $c^0 = 0$. We distinguish three test cases according to the choice of the velocity of the center $c$.

4.1 First test case. The first test case corresponds to the case where $\dot{c}$ is given by (6). We set $(\Delta t, M, L) = (10^{-2}, 100, 200)$. Figure 2 illustrates the droplet evolution on the time interval $[0, 24]$ using the upwind finite difference scheme (46). Precisely we present the vertical section of the droplet parametrized with $\theta \mapsto (r(\theta) \sin(\theta), r(\theta) \cos(\theta)), \theta \in [0, \pi].$

Table 1 gathers the following values for each $t = 0, 2.5, \cdots, 25$

- the distance $|c^n - c^*|$ between the discretized centers $c^*$ and $c$
- The errors $E^n$ defined by

$$E^n_1 = \max_i (|r^n_i - \bar{r}(t^n, \theta_i)|), \quad E^n_2 = \frac{1}{n} \sum_i (|r^n_i - \bar{r}(t^n, \theta_i)|),$$

where $\bar{r}$ is the exact solution given by Proposition 3.3

$$\bar{r}(t, \theta) = -(c - c*)_3 \cos(\theta) + \sqrt{1 - (c - c*)^2_3 \sin^2(\theta)}, \quad (t, \theta) \in [0, T] \times [0, \pi].$$

- the relative error for the volume conservation $V^n$ defined by discretizing the integral

$$\text{Vol}(t) := \int_0^\pi r^3(\theta) \sin(\theta) d\theta = \frac{4\pi}{3},$$

$$V^n = \left| \text{Vol}^n - \frac{4\pi}{3} \right| \frac{3}{4\pi}.$$
Numerical computations are in agreement with Proposition 3.4 in the sense that $r^n_i \sim \bar{r}(t^n, \theta_i)$ i.e. the numerical result corresponds to the Hadamard-Rybczynski sphere which can also be noticed on Figure 2.

**Figure 2.** First test case. Droplet evolution for $t = 0, 3, \cdots, 24$

| $t$ | 0   | 2.5 | 5   | 7.5 | 10  | 12.5 | 15  | 17.5 | 20  | 22.5 | 25  |
|-----|-----|-----|-----|-----|-----|------|-----|------|-----|------|-----|
| $|c - c^*|$ | 0.0007 | 0.166 | 0.322 | 0.462 | 0.58 | 0.68 | 0.76 | 0.82 | 0.865 | 0.898 | 0.921 |
| $E_1^n (\times 10^{-2})$ | 2.10$^{-5}$ | 0.04 | 0.17 | 0.40 | 0.77 | 1.29 | 2.02 | 3.02 | 4.37 | 6.16 | 8.55 |
| $E_2^n (\times 10^{-2})$ | 8.10$^{-6}$ | 0.011 | 0.043 | 0.095 | 0.17 | 0.25 | 0.36 | 0.54 | 0.85 | 1.33 | 2.01 |
| $V^n (\times 10^{-3})$ | 0.08 | 0.15 | 0.17 | 0.16 | 0.22 | 0.48 | 1.07 | 2.06 | 3.45 | 5.21 | 7.3 |

Table 1. First test case. Evolution of $|c - c^*|$, $E_1^n$, $E_2^n$ and $V^n$ for the upwind finite difference scheme (46)
We provide in Table 2 the results obtained using the following finite volume scheme

\begin{equation}
\begin{align*}
  r_i^{n+1} &= r_i^n - \frac{\Delta t}{\Delta \theta} \left( \mathcal{F}_{i+\frac{1}{2}} - \mathcal{F}_{i-\frac{1}{2}} \right) + \Delta t S^{i,n}, \quad i = 1, \ldots, M - 1,
\end{align*}
\end{equation}

based on the conservative formula

\begin{equation}
\begin{align*}
  \partial_t r + \partial_\theta(r A_1[r]) &= A_2[r] + r \partial_\theta A_1[r],
\end{align*}
\end{equation}

with

\begin{equation}
\begin{align*}
  S^{i,n} &= A_i^{i,n} + r_i^n \frac{A_1^{i+1,n} - A_1^{i-1,n}}{2 \Delta \theta}.
\end{align*}
\end{equation}

The flux is defined as follows

\begin{equation}
\begin{align*}
  \mathcal{F}_{i+\frac{1}{2}} &= A_{i+\frac{1}{2}} \left\{ \begin{array}{ll}
  r_i^n & \text{if} \quad A_{i+\frac{1}{2}} \geq 0, \\
  r_{i+1}^n & \text{if} \quad A_{i+\frac{1}{2}} \leq 0,
\end{array} \right.
\end{align*}
\end{equation}

Numerical computations show that $A_{i+1/2} \leq 0$ for all $i = 0, \ldots, M$ and all $n = 0, \ldots, N$. This means that the finite volume scheme can be rewritten as

\begin{equation}
\begin{align*}
  r_i^{n+1} &= r_i^n - \frac{\Delta t}{\Delta \theta} \left( \frac{A_1^{i,n} + A_1^{i+1,n}}{2} (r_{i+1}^n - r_i^n) \right) + \Delta t A_2^{i,n},
\end{align*}
\end{equation}

Figure 3 represents the error $E^n$ and the volume conservation $V_n$ for the two schemes with $M = 50$ and $M = 100$ on the same time interval $[0, 25]$ with $(\Delta t, L) = (0.01, 200)$. According to this comparison we consider only the upwind finite difference scheme for the two remaining test cases.

4.2. Second test case. The second test case is chosen such that $\dot{c} = \lambda \dot{c}^*$ with $\lambda > 1$. We have

\begin{equation}
\begin{align*}
  |c(t) - c^*(t)| &= t(\lambda - 1)|v^*| = t(\lambda - 1) \frac{4}{15},
\end{align*}
\end{equation}

if we set for instance $\lambda = \frac{12}{5}$, the time $\bar{t}$ for which we have $|c(\bar{t}) - c^*(\bar{t})| = 1$ is $\bar{t} = 0.5$. We present in Table 3 the errors computed thanks to the upwind finite difference scheme $(\Delta t, M, L) = (0.01, 100, 200)$. In this case, numerical computations show that after $t = 0.5$ we obtain negative values for the radius. This suggests that the maximal time of existence of the solution depends on the choice of $\dot{c}$.
4.3. **third test case.** We investigate the case where \( c = c^* \) using the upwind finite difference scheme \((46)\). In this case we recall that \( \bar{r} = 1 \) is a steady solution to the hyperbolic equation. We present in Table 4 the values of \( E_1^n, E_2^n, V^n \).

4.4. **Discussion on the approximation scheme.** In this last part we discuss the main difficulties encountered regarding the numerical solving of the hyperbolic equation. Several schemes have been tested in addition of the upwind finite difference scheme \((46)\) and the finite volume scheme \((48),(51)\). First, a Lax-Friedrichs scheme for the conservative
Table 4. Third test case. Evolution of $E^1_n$, $E^2_n$ and $V^n$.

| $t$ | $E^1_n \times 10^{-3}$ | $E^2_n \times 10^{-3}$ | $V^n \times 10^{-3}$ |
|-----|------------------------|------------------------|---------------------|
| 0   | 4.10^{-4}              | 2.10^{-4}              | 0.08                |
| 2.5 | 0.1                   | 0.06                   | 0.28                |
| 5   | 0.22                  | 0.16                   | 0.47                |
| 7.5 | 0.43                  | 0.3                    | 0.67                |
| 10  | 0.78                  | 0.49                   | 0.86                |
| 12.5| 1.24                  | 0.73                   | 1.06                |
| 15  | 1.82                  | 1.02                   | 1.25                |
| 17.5| 2.53                  | 1.36                   | 1.44                |
| 20  | 3.38                  | 1.76                   | 1.63                |
| 22.5| 4.41                  | 2.22                   | 1.82                |
| 25  | 5.65                  | 2.73                   | 2.02                |

formulation \((49), (48), (50)\) defined using the following fluxes

$$F_{i+\frac{1}{2}} = \frac{r_{i+1}A_{i+1,n} + r_{i}A_{i,n}}{2} - \frac{\Delta \theta}{2\Delta t} (r_{i+1} - r_{i}),$$

yields less accurate estimate than previous schemes from the first iterations \((t \in [0,5])\) on the first test case.

Secondly, a conservative formulation has been investigated for the hyperbolic equation which writes as follows

$$\partial_t r(t, \theta) + \partial_r G(r(t, \theta), \theta) = A_2[r] + F(r(t, \theta), \theta),$$

with \(\partial_r G(r, \theta) = A_1[r], F(r, \theta) = \partial_r G(r, \theta)\). An analogous Lax-Friedrichs scheme with a discretization of the additional source term has been implemented but yields less accurate results from the first iterations \((t \in [0,2.5])\) on the first test case.

A more precise investigation of an adapted scheme for the hyperbolic equation would be interesting. In particular one of the main purposes is to ensure the steady state approximation, the positivity and the volume conservation. Keeping in mind that one of the goals is to consider different initial shapes for the droplet such as ellipsoids which correspond to the following initial conditions for instance

$$r_0(\theta) = \frac{1}{\sqrt{1 - \frac{3}{4}\cos^2(\theta)}}, \quad r_0(\theta) = \frac{1}{\sqrt{1 - \frac{3}{4}\sin^2(\theta)}}, \quad \theta \in [0, \pi],$$

depending on the considered orientation of the ellipsoid.

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Appendix A. Summary of formulas for the operators $A_1[r]$, $A_2[r]$ and $U[r]$

$$e(\theta, 0) = \begin{pmatrix} \sin(\theta) \\ 0 \\ \cos(\theta) \end{pmatrix}$$

$$A_1[r](\theta) = \frac{1}{r(\theta)}(U[r](\theta) - \dot{c}) \cdot \partial_\theta e(\theta, 0), \quad A_2[r](\theta) = (U[r](\theta) - \dot{c}) \cdot e(\theta, 0)$$

$$U[r]_1(\theta) = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} K(\bar{\theta}, \theta, \phi) \left\{ r(\theta) \cos(\theta) - r(\bar{\theta}) \cos(\bar{\theta}) \right\} \cos(\bar{\phi}) d\bar{\theta} d\bar{\phi}$$

$$U[r]_2(\theta) = \frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} K(\bar{\theta}, \theta, \phi) \left\{ r(\theta) \cos(\theta) - r(\bar{\theta}) \cos(\bar{\theta}) \right\} \sin(\bar{\phi}) d\bar{\theta} d\bar{\phi}$$

$$U[r]_3(\theta) = \frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} K(\bar{\theta}, \theta, \phi) \left\{ -r(\theta) \sin(\theta) \cos(\bar{\phi}) + r(\bar{\theta}) \sin(\bar{\theta}) \right\} d\bar{\theta} d\bar{\phi}$$

$$K(\bar{\theta}, \theta, \phi) = \frac{r(\bar{\theta}) \sin(\bar{\theta}) - r'(\bar{\theta}) \cos(\bar{\theta})}{\beta[r](\bar{\theta}, \theta, \phi)} r(\bar{\theta}) \sin(\bar{\theta})$$

$$\beta^2[r](\bar{\theta}, \theta, \phi) = r^2(\theta) + r^2(\bar{\theta}) - 2r(\theta)r(\bar{\theta}) \left( \cos(\bar{\phi}) \sin(\theta) \sin(\bar{\theta}) + \cos(\theta) \cos(\bar{\theta}) \right)$$

$$U[r](\theta) \cdot e(\theta, 0) = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} K(\bar{\theta}, \theta, \phi) r(\bar{\theta}) \left( -\sin(\theta) \cos(\theta) \cos(\bar{\phi}) + \cos(\theta) \sin(\theta) \right) d\bar{\theta} d\bar{\phi}$$

$$U[r](\theta) \cdot \partial_\theta e(\theta, 0) = -\frac{1}{8\pi} \int_0^{2\pi} \int_0^{\pi} K(\bar{\theta}, \theta, \phi) \left( r(\theta) \cos(\bar{\phi}) - r(\theta) \left\{ \cos(\bar{\theta}) \cos(\theta) \cos(\bar{\phi}) + \sin(\bar{\theta}) \sin(\theta) \right\} \right) d\bar{\theta} d\bar{\phi}$$

Appendix B. Technical lemma

Lemma B.1. There exists a positive constant $C > 0$. satisfying

$$\sup_{\theta \in [0, \pi]} \left( \int_{[0, \pi] \times [0, 2\pi]} \frac{\sin(\theta)}{|e(\theta, \phi) - e(\theta, 0)|} d\theta d\phi + \int_{[0, \pi] \times [0, 2\pi]} \frac{\sin(\theta) d\phi d\theta}{\sqrt{1 - e(\theta, 0) \cdot e(\theta, \phi)^2}} \right) \leq C.$$
Proof. In fact we can show a stronger result. The idea is to note that
\[
\int_{[0, \pi] \times [0, 2\pi]} \sin(\bar{\theta}) d\bar{\theta} d\bar{\phi} = \int_{\partial B(0, 1)} d\sigma(y) \big| e(\theta, 0) - y \big|
\]

Let \( \theta \in [0, \pi] \). We set \( Q(\theta) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \) the rotation matrix such that \( e(\theta, 0) = Q(\theta) e_3 \) with \( e_3 = (0, 0, 1) \) and use the change of variable \( y = Q(\theta) \omega, \omega \in \partial B(0, 1) \) such that \( |Q(e_3 - \omega)| = |(e_3 - \omega)| \) and \( d\sigma(y) = d\sigma(\omega) \). This yields
\[
\int_{\partial B(0, 1)} d\sigma(y) = \int_{\partial B(0, 1)} d\sigma(y) = 4\pi.
\]

We apply the same idea for the second integral using the fact that \( e(\theta, 0) \cdot e(\bar{\theta}, \bar{\phi}) = Q(\theta) e_3 \cdot Q(\theta) \omega = e_3 \cdot \omega \). \( \square \)

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