Uncertainty-aware Three-phase Optimal Power Flow based on Data-driven Convexification

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Abstract—This paper presents a novel optimization framework of modeling three-phase optimal power flow that involves uncertainty. The proposed uncertainty-aware optimization (UaO) framework is: 1) a deterministic framework that is less complex than the existing frameworks with uncertainty, and 2) convex such that it admits polynomial-time algorithms and mature distributed optimization methods. To construct the UaO framework, a methodology of learning-based uncertainty-aware modeling with prediction errors of stochastic variables as the measurement of uncertainty and a theory of data-driven convexification are proposed. Theoretically, the UaO framework is applicable for modeling general optimization problems under uncertainty.

Index Terms—Convex relaxation, data-driven, optimization under uncertainty, three-phase power flow.

I. INTRODUCTION

Optimization technologies have been widely used in many decision-making processes in the operation, control, and planning of power systems, such as optimal power flow (OPF). However, the increasing uncertainty introduced by distributed energy resources (DER) makes it extremely hard for operators to make accurate optimal decisions ahead of real time. There mainly exist three types of frameworks for modeling power system optimization problems that involve uncertainty: 1) stochastic framework, 2) robust framework, and 3) chance-constrained framework [1]. Unfortunately, these frameworks are rather computationally expensive for large-scale, highly-nonconvex problems. As a result, a large portion of existing works investigate proper assumptions to simplify these frameworks for power system applications. In contrast, based on regression analysis [2], this paper develops a novel uncertainty-aware optimization (UaO) framework using a new measurement of uncertainty that considers the prediction errors of stochastic variables (see Section III for more details).

Convex optimization [3] has applications in a broad range of disciplines including power system engineering, mainly because: 1) many classes of convex optimization problems are computationally tractable as they admit polynomial-time algorithms; and, 2) it plays a fundamental role in the theories of both distributed optimization and bi-level optimization. The general idea is to relax the nonconvex constraints into convex ones. However, the solutions of the resulting convex problem may be infeasible to the original nonconvex problem due to the nature of relaxations, which now becomes one of the bottlenecks of this technology. To mitigate the infeasibility issue, this paper proposes a data-driven approach to construct convex relaxations with stronger tightness and lower complexity (see Subsection II-B for more details). The resulting convex relaxation is applied to convexify the developed UoA framework. The paper demonstrates the UoA framework on a three-phase optimal power flow (3pOPF) problem with uncertainty introduced by distributed energy resources and uncontrollable loads. The 3pOPF is balanced for transmission networks while unbalanced for distribution networks. It is worth noting that, theoretically, the proposed methods can be applied to general optimization problems under uncertainty.

II. THEORY OF DATA-DRIVEN CONVEXIFICATION

A. Three-Phase Power Flow Equations

In an OPF problem, the objective function is generally convex or linear. Thus, we focus on the main nonconvex constraints, i.e. the power flow (PF) equations, which are also considered as the mathematical model of power networks. Let \( \mathcal{N} \) and \( \Phi \) denote the sets of buses and phases respectively. For each \( i, j \in \mathcal{N} \) and \( \phi, \phi' \in \Phi \), the compact formulation of three-phase PF equations [4] is given as

\[
\begin{align*}
\sum_{j \in \mathcal{N}} \sum_{\phi' \in \Phi} (G_{ij}^{\phi\phi'} e_{ij}^\phi - B_{ij}^{\phi\phi'} f_{ij}^\phi) + f_i^\phi \sum_{j \in \mathcal{N}} \sum_{\phi' \in \Phi} (B_{ij}^{\phi\phi'} e_{ij}^\phi + G_{ij}^{\phi\phi'} f_{ij}^\phi) &= p_i^\phi + p_{S,i}^\phi, \quad (1a) \\
-f_i^\phi \sum_{j \in \mathcal{N}} \sum_{\phi' \in \Phi} (G_{ij}^{\phi\phi'} e_{ij}^\phi - B_{ij}^{\phi\phi'} f_{ij}^\phi) - e_i^\phi \sum_{\phi' \in \Phi} [G_{ij}^{\phi\phi'} (e_{ij}^\phi - e_{ij}^{\phi'}) - B_{ij}^{\phi\phi'} (f_{ij}^\phi - f_{ij}^{\phi'})] + f_i^\phi \sum_{\phi' \in \Phi} [B_{ij}^{\phi\phi'} (e_{ij}^\phi - e_{ij}^{\phi'}) + G_{ij}^{\phi\phi'} (f_{ij}^\phi - f_{ij}^{\phi'})] &= q_i^\phi + q_{S,i}^\phi, \quad (1b) \\
\sum_{\phi' \in \Phi} [G_{ij}^{\phi\phi'} (e_{ij}^\phi - e_{ij}^{\phi'}) - B_{ij}^{\phi\phi'} (f_{ij}^\phi - f_{ij}^{\phi'})] - e_i^\phi \sum_{\phi' \in \Phi} [B_{ij}^{\phi\phi'} (e_{ij}^\phi - e_{ij}^{\phi'}) + G_{ij}^{\phi\phi'} (f_{ij}^\phi - f_{ij}^{\phi'})] &= p_{ij}^\phi, \quad (1c) \\
(e_i^\phi)^2 + (f_i^\phi)^2 &= v_i, \quad (1e)
\end{align*}
\]

where \( p_{S,i}^\phi \) and \( q_{S,i}^\phi \) denote the stochastic components of active and reactive power injections at each bus. Each of the quadratic equations in (1) can be compactly formulated as

\[
g(x) = x^T A x = y = z + u
\]

where \( z \) and \( u \) denote the deterministic and stochastic components of the power injections respectively. Further define a

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set $\Omega = \{(x, y)| P^k \leq p^0_i + p^0_{i,j} \leq \bar{p}^0_i, q^0_i \leq q^0_i + q^0_{i,j} \leq \bar{q}^0_i, (\pi^0_j)^2 + (\pi^0_{i,j})^2 \leq \bar{\pi}_{i,j}, \text{and } \bar{u}_i \leq \bar{v}_i \leq \bar{r}_i, \forall i \in \mathcal{N} \text{ and } \phi \in \Phi\}$, then the feasible set of $3\phi$PF $\Pi$ is $\Psi = \{(x, y) \in \Omega \cap \mathcal{N} \text{ and } \phi \in \Phi\}$. Note that $\Omega$ is convex while $\Psi$ is not. Moreover, the three-phase DistFlow model of radial networks is also a nonconvex quadratic system that can be represented in the form of $\Pi$. That means the proposed methods can be directly applied to DistFlow-based $3\phi$OPF.

B. Data-Driven Convex Relaxation

In this subsection, a methodology of data-driven convex relaxation is established and applied to construct a tight convex quadratic relaxation of $3\phi$PF equations $\Pi$. For the sake of simplicity, we start from a deterministic case, namely $u = 0$. Let $\mathcal{D}$ denotes a historical data set where the $k$th data point $D(k) = (x(k), y(k))$, we have $\mathcal{D} \subset \Psi$. The following regression algorithm is proposed to train $\mathcal{D}$ to obtain a positive semi-definite (PSD) matrix $P$ and a complementary vector $B$ and scalar $c$ for each quadratic equations in $\Pi$ (i.e., $\Pi$):

$$\min_{P,B,c} \frac{1}{|\mathcal{D}|} \sum_k t(k) \quad (3a)$$

s.t. $(x(k))^T P x(k) + B^T x(k) + c - y(k) = m(k) \leq 0 \quad (3b)$

$$\begin{bmatrix} 1 & m(k) & t(k) \\ \end{bmatrix}, \quad P \succeq 0 \quad (3c)$$

where $k = 1, \ldots, |\mathcal{D}|$. The dimensions of $P$'s, $B$'s, and $c$'s are consistent with the dimensions of the corresponding quadratic equations in $\Pi$. Note that the $A$ matrix in (1e) is already PSD. Therefore, we don’t need to train a $P$ for (1e). The optimization model (3) is a standard semidefinite programming problem which can be effectively and globally solved by mature solvers like MOSEK, GUROBI, and CPLEX.

Define a quadratic convex set:

$$\Theta = \{(x, y) \in \Omega | x^T P x + B^T x + c \leq y, \forall i \in \mathcal{N} \text{ and } \phi \in \Phi\},$$

we have the following theorem.

**Theorem of Data-driven Convex Relaxation.** The set $\Theta$ is a convex relaxation of the feasible set $\Psi$ of the original three-phase AC power flow $\Pi$ if:

a) the PSD matrices $P$'s, vectors $B$'s, and scalars $c$'s are obtained by training $\mathcal{D}$ using the regression algorithm $\Pi$.

b) $\mathcal{D}$ contains all extreme points $\Pi$ of $\Psi$.

Proof: Constraint $\Pi$ guarantees that each $D(k) \in \mathcal{D}$ satisfies

$(x(k))^T P x(k) + B^T x(k) + c \leq y(k)$,

which implies $\mathcal{D} \subset \Theta$. Therefore, $\Theta$ is a convex quadratic relaxation of $\mathcal{D}$ since $P$'s are PSD.

All extreme points of a feasible set are linearly independent according to the definition $\Pi$. Suppose $\psi$ is an arbitrary point in $\Psi$, there must exist a vector of extreme points $X = [\theta_1, \theta_2, \ldots, \theta_l]^T$ of $\Psi$ and a vector of multipliers $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_l]^T$ that satisfy

$$\psi = \alpha^T X,$$

where $0 \leq \alpha_i \leq 1 (i = 1, \ldots, l)$, $\sum_i \alpha_i = 1$, and $l$ equals to the dimension of the $(x, y)$-space. Since all $\theta_i \in \mathcal{D} \subset \Theta$ according to condition b), then $\psi \in \Theta$ due to the convexity of $\Theta$. Therefore, $\Psi \subset \Theta$ as $\psi$ is an arbitrary point in $\Psi$, which means $\Theta$ is convex relaxation of $\Psi$.

**Remark.** Condition b) in the theorem of data-driven convex relaxation is not easy to strictly satisfy. However, under the concept of Big Data, it is reasonable to assume that the data set $\mathcal{D}$ is big enough to represent the original feasible set $\Psi$, which implies $\Theta$ is highly close to a strictly convex relaxation of $\Psi$. Moreover, regression (3) is a convex optimization problem that can be globally solved, which implies that $\Theta$ is the tightest quadratic convex relaxation of $\Psi$.

III. UNCERTAINTY-AWARE OPTIMIZATION FRAMEWORK

A. Novel Measurement of Uncertainty

In the stochastic and chance-constrained frameworks, uncertainty is measured by probability distributions, while it is captured by a deterministic sets under the robust framework $\Pi$. Actually, the impact of uncertainty on power system operation is illustrated in Fig. 1. Suppose $u$ is the actual generation of stochastic resources at moment $t$ and its system response is $x$ as in Fig. 1(a). The operation decision is generally made in advance (e.g. 5 minutes, 1 hour, or even a day before) based on the forecast $\tilde{u}$ of $u$ as in Fig. 1(b). One can consider that the uncertainty originates from the prediction error $|\tilde{u} - u|$, since the resulting model output error $|\tilde{x} - x|$ may lead to failures in ahead-of-real-time decision-making of power systems. In fact, a “good prediction” and a “bad prediction” may have totally different impacts on the operation, control, and planning of power systems. However, none of the existing optimization frameworks considers the information of forecast in the measurement of uncertainty, which increases the conservativeness. In this research, we use the prediction error $|\tilde{u} - u|$ as the new measurement of uncertainty which will be incorporated into the following machine learning process.

B. Uncertainty-aware Modeling

Researchers from the field of renewable generation and load forecast attempts to reduce the prediction errors $|\tilde{u} - u|$, while this research focuses on reducing the model output errors $|\tilde{x} - x|$ in the process of system modeling given the information of $\tilde{u}$, for which the key is to properly learn a PSD matrix $P$ and complementary vector $B$ and scalar $c$ for each quadratic...
equation in (1). In order to do so, we first design a historical data set \( \tilde{D} \) that contains information of forecast errors and its \( k \)-th data point \( \tilde{D}^{(k)} = (\tilde{u}^{(k)}, u^{(k)}, z^{(k)}, x^{(k)}) \), where the historical operating point \( (x^{(k)}, u^{(k)} + z^{(k)}) \) is a solution of (2). Then, the regression model (3) is modified as

\[
\min_{P, b} \quad \text{s.t.} \quad (3c) \quad \text{and} \quad (4a)
\]

\[
(x^{(k)})^T P x^{(k)} + B^T x^{(k)} + c - u^{(k)} - z^{(k)} \leq 0
\]

\[
(x^{(k)})^T P x^{(k)} + B^T x^{(k)} + c - \tilde{u}^{(k)} - z^{(k)} = m_k
\]

to train the new data set \( \tilde{D} \). With the \( P, B, c \) inferred by regression (4), the following equation (5) is defined as an uncertainty-aware model (UaM) of \( \tilde{\phi} \). It relies on the predictions rather than the actual values (i.e. perfect predictions) of the stochastic parameters:

\[
h(x) = x^T P x + B^T x + c = \tilde{y} = z + \tilde{u}
\]

Regression process (3) aims at fitting a convex mapping function between the actual system response \( x \) and the actual input \( u \) of the stochastic variables. In contrast, regression (4) infers the convex mapping function between \( x \) and \( \tilde{u} \) (the aforesaid prediction of \( u \)). For a future case \( (x^{(D+1)}, y^{(D+1)}) \), a forecast \( \tilde{u}^{(D+1)} \) is used in the ahead-of-real-time decision-making since the actual value \( u^{(D+1)} \) of the stochastic parameters is not available at that moment. With \( \tilde{u}^{(D+1)} \) as input, the UaM (5) provides a close prediction \( \tilde{x}^{(D+1)} \) to the actual system response \( x^{(D+1)} \), as illustrated in Fig. 1(c). Constraint (4b) guarantees that \( \Theta \) with parameters inferred by (3) is a convex quadratic relaxation for the projection of \( \tilde{D} \) on \( (x, y) \)-space.

### C. Uncertainty-aware \( \tilde{\phi} \)OPF

With the typical objective that minimizes generation costs, the uncertainty-aware \( \tilde{\phi} \)OPF can be compactly formulated as

\[
\min_z \left\{ \sum_{i \in G} \sum_{\phi \in \Phi} C(z^{\phi}_i) : (x, z + \tilde{u}) \in \Theta \right\}
\]

where \( G \) denotes the generator set, and \( P \)'s, \( B \)'s, \( c \)'s in \( \Theta \) are inferred by training \( \tilde{D} \) using regression (4). A typical objective of OPF in distribution systems is to minimize the active power from transmission grids. It is worth noting that (6) is a deterministic optimization problem which is much less-complex than the existing robust, stochastic, and chance-constrained optimization frameworks.

### IV. Numerical Experiment

Due to page limit, we only present the preliminary studies on three scenarios of two small balanced networks, i.e. a 5-bus and a 57-bus systems. More comprehensive numerical studies based on real-world data will be presented in future publications to compare the proposed methods with existing convex relaxations and uncertainty-involved optimization frameworks. The three scenarios compared in this experiment are: 1) the original ACOPF with perfect predictions \( u \) of stochastic power injections which is used as the reference; 2) the original ACOPF with inaccurate predictions \( \tilde{u} \) which simulates the actual practice; and 3) the proposed UaO framework with inaccurate predictions \( \tilde{u} \).

In the first step of generating the training data sets \( \tilde{D} \), a set of 5000 load profiles \( y = z + u \) for each test system are randomly produced. Then, voltage profile \( x \) for each load profile of each system is obtained by solving PF (1). Finally, the prediction \( \tilde{u} \) is randomly generated based on the corresponding \( u \) assuming that the maximum forecast error is \( \pm 30\% \).\footnote{\textcopyright 2019 IEEE. Personal use is permitted, but republication/redistribution requires IEEE permission. See http://www.ieee.org/publications_standards/publications/rights/index.html for more information.} The UaO models (6) are obtained by training data sets \( \tilde{D} \) of the two systems, respectively, using regression (4). The three scenarios mentioned above are compared on 50 load cases for each test system. The average errors of objective values defined in in Table I are used to quantify the performance, where \( C_k \) denotes the optimal cost of the \( k \)-th scenario in the \( i \)-th load case. It can be observed that, with inaccurate forecast, the original ACOPF (scenario 2) produces inaccurate solutions. Nevertheless, the UaO (scenario 3) can provide better solutions than the original ACOPF model does since the forecast errors are taken into account in the data-driven modeling process. Although the UaO framework is still not able to produce a strictly accurate solution, fortunately, it can be improved by training a larger, better data set due to its learning-based nature.

**TABLE I**

|   | Average Errors of Optimal Costs |
|---|---------------------------------|
|   | \( E_1 = \frac{1}{50} \sum_{i=1}^{50} \frac{|C^{(i)} - C_1^{(i)}|}{C_1^{(i)}} \) | \( E_2 = \frac{1}{50} \sum_{i=1}^{50} \frac{|C^{(i)} - C_1^{(i)}|}{C_1^{(i)}} \) |
| 5-bus | 20.21% | 3.72% |
| 57-bus | 21.43% | 1.66% |

### V. Conclusion and Future Work

This paper presents a preliminary study on a novel UaO framework which shows that the UaO framework can effectively mitigate the impacts of uncertainty in solving \( \tilde{\phi} \)OPF. Our future work mainly consists of two aspects: first, explore advanced machine learning technologies, such as ensemble learning [2], to improve the efficiency of UaO framework; second, apply to UaO framework to model other power system optimization problems other than OPF problems.

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