Force fluctuations in granular materials

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Force fluctuations in granular materials are investigated. A continuum equation is derived starting from a discrete model proposed in the literature. The influence of boundary conditions is investigated. For periodic boundary conditions the average weight is found to increase linearly with depth while it saturates to a constant value for absorbing boundary conditions, which models the existence of walls. The scale dependencies of the saturation weight, the saturation depth and the average squared fluctuations are obtained. The analytical results are compared with previous works and with numerical simulations in one dimension.

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I. INTRODUCTION

A wide variety of technical processes involve the storage and transport of granular materials. On the other hand, granular materials have very unusual properties which have intrigued researchers in physics over recent years [1]. Of great importance is the characterization of stress fluctuations in dry granular materials. Optical measurements in two and three dimensional arrays of granular materials have shown that the stress in packed granular materials is not distributed uniformly inside the medium, but is concentrated along "chains". Further experiments dealing with force fluctuations in the bottom of relative small containers have shown an exponential distribution of vertical forces (weight) $w$. Another interesting phenomena associated with weight fluctuations is the problem of arching. Arching refers to the observation that the average weight at the bottom of a column of grains saturates at a value $W_s$ as the depth $t$ measured from the top of the column is increased, where $W_s$ shows large fluctuations when repeating this procedure with the same amount of grain. About one century ago Janssen [2] showed, using a simple argument, that $W(t) = W_s[1 - \exp(-t/t_s)]$, where $t_s$ is the saturation depth which scales as $t_s \sim L$, where $L$ is the linear horizontal size of the container.

More recently, Liu et al. [3] proposed a simple discrete model (q-model) which describe the essential features of force fluctuations in granular materials. For instance, numerical simulations of the model in $2 + 1$ dimensions give rise to an exponential distribution $P(v) \sim \exp(-\lambda v)$ for the normalized weight $v = w/t$, in agreement with experimental observations and numerical simulations of sphere packings [4,5]. Later Peralta-Fabi et al. [6] showed through numerical simulations in $1 + 1$ dimensions and mean field (MF) analysis that the same model, but with different boundary conditions, explains the process of arching, obtaining the Janssen law but with $W_s \sim L^2$ and $t_s \sim L^2$.

Generalizations of the q-model has been considered [7,8]. Some of them including a local slip condition which leads to anisotropies in the stress transmission [8] and introduces correlations [9]. These correlations, nevertheless, do not alter the asymptotic exponential decay of the normalized weight distribution [9]. On the other hand, other authors have introduced more realistic models which takes into account the stress fluctuations in the horizontal cross-section [10,11]. Some of their predictions are in agreement with those obtained for the q-model, such as the exponential asymptotic decay of the normalized weight distribution characteristic of the q-model.

We also count with some coarse-grained equations for the q-model [12]. Claudin et al. [13] obtained a diffusion equation for the weight with a multiplicative noise and derived some analytical results for the weight correlation function. However, their analysis was limited to pile configurations and silos with periodic boundary conditions. On the contrary, Peralta-Fabi et al. [6] the case when part of the weight is supported by the walls of the silo, which lead to a saturation of the average weight profile. Nevertheless, they neglected fluctuations in the local stress transmission.

In the present work we focus our attention in obtaining a continuum model to describe weight fluctuations and the process of arching in granular materials, for both periodic and wall boundary conditions. We take as starting point the q-model introduced by Liu et al. [3]. We obtain a coarse-grained continuum equation of this model which is similar to the Edwards-Wilkinson equation, where the grain weight acts as an external force and noise is added to consider fluctuations in shape and orientation. Boundary and initial conditions are imposed following the definition of the discrete model. Two particular cases are analyzed, periodic and absorbing boundaries. In the case of periodic boundary conditions the average weight increases linearly with depth and is independent of the lattice size, while in the case of absorbing boundary conditions it saturates for $t \gg t_s \sim L^2$ to $W_s \sim L^2$. Moreover, in both cases fluctuations around the average are of the order of $L^{\zeta}$, with $\zeta = (4 - d)/2$.

The paper is organized as follows. In section 1 we derive the coarse-grained equation for the q-model and solve it for the case of periodic and absorbing boundary conditions. In section II our results are compared...
with numerical simulations in one dimension, and with other works in the literature. Finally the summary and conclusions are given in section IV.

II. THE MODEL

A. Discrete model

Liu et al. [3] introduce a simple model (q-model) that describe some of the experimental observations in bead packs. The model assumes that the dominant physical mechanism leading to force fluctuations is the inhomogeneity of the packing, which causes an unequal distribution of weight on the beads supporting a given grain. Spatial correlations in these fractions and variations in the coordination numbers of the grains are ignored. Only the vertical components of the forces are considered explicitly.

Consider a \( d + 1 \) dimensional array of grains in which each layer has \( L^d \) beads. The total weight on a given bead is transmitted unevenly to \( N \) adjacent beads in the layer underneath. Specifically, a fraction \( q_{ij} \) of the total weight supported by the bead \( j \) in layer \( t \) is transmitted to bead \( i \) in layer \( t + 1 \). Thus, a site at depth \( t + 1 \) has weight \( w_i(t+1) \), due to its own weight \( w_0 \) and to weights of neighbors at depth \( t \), according to

\[
w_i(t + 1) = w_0 + \sum_j q_{ij}(t)w_j(t),
\]

where the sum runs over the \( N \) neighbors, in layer \( t \), of the site \( i \), in layer \( t + 1 \). The fractions \( q_{ij}(t) \) are taken to be random variables, independent except for the constraint

\[
\sum_i q_{ij}(t) = 1,
\]

where the sum runs over the \( N \) neighbors, in layer \( t + 1 \), of site \( j \), in layer \( t \).

To complete model definition, boundary and initial conditions must be specified. In the top layer \( (l = 0) \) the grains have no neighbors above and, therefore, they only support their own weight, i.e. \( w_i(0) = w_0 \). On the other hand, we can take different boundary conditions. We consider two cases, periodic and absorbing boundary conditions. Absorbing boundary conditions are more appropriate to describe the force fluctuations in silos where part of the weight is supported (“absorbed”) by the wall containing the grains. The absorbing boundary conditions are implemented by simply imposing that boundary sites have zero weight. If a site near the boundary transmit part of its weight to a boundary site (i.e. to the wall) this weight fraction is “dissipated”, while the boundary site never transmit weight to its neighbors because by definition its weight is zero. This boundary conditions are different but equivalent to the one used by Peralta-Fabi et al. [4]. On the other hand, for periodic boundary conditions we recover the original q-model [3].

B. Coarse-grained equation

To derive a coarse-grained continuum equation of eq. (1) let us introduce the new random fraction \( Q_{ij}(t) \), such that

\[
q_{ij}(t) = \frac{1}{N} + Q_{ij}(t),
\]

which following [3] satisfies the constraint

\[
\sum_i Q_{ij}(t) = 0.
\]

Substituting \( q_{ij}(t) \) in eq. (1) by eq. (3) and substrating \( w_i(t) \) to both sides it results that

\[
w_i(t+1) - w_i(t) = \frac{1}{N} \sum_j [w_j(t) - w_i(t)] + w_0 + \eta_i(t),
\]

where \( \eta_i(t) \) is a noise term associated with the random fractions \( Q_{ij}(t) \) and with weights at neighbor sites. In average the contribution of \( \eta_i(t) \) should be zero because of the constraint in eq. (4).

Eq. (6) can be coarse-grained to obtain a continuum equation for the effective \( w(\vec{x}, t) \). In the left hand side we have the discrete depth derivative, while the first term in the right hand side is the discrete Laplacian in the horizontal direction. After coarse-graining it result that

\[
\lambda \frac{\partial}{\partial l} w(\vec{x}, t) = \Gamma \nabla^2 w(\vec{x}, t) + w_0 + \eta(\vec{x}, t),
\]

where \( \lambda \) and \( \Gamma \) are coarse-grained coefficients. \( \lambda \sim a_\perp \) and \( \Gamma \sim a_\parallel^2/N \), where \( a_\perp \) and \( a_\parallel \) are characteristic lengths in the vertical and horizontal direction, respectively.

In \((1+1)\)-dimensions eq. (7) is actually very similar to the one derived in [12]. Claudin et al. [13] considered the explicit multiplicative nature of the noise. However, it is not clear how their analysis can be extended to \((2+1)\)-dimensions, in particular the precise form of the coarse-grained noise is in this case not clear for us. Instead, we assume that \( \eta(\vec{x}, t) \) is a Gaussian noise with zero mean and uncorrelated in space, with noise correlator

\[
\langle \eta(\vec{x}, t)\eta(\vec{x}', t') \rangle = \delta(\vec{x} - \vec{x}')\Delta(t - t').
\]

\( \Delta(t) \) is a monotonically decreasing even function. We do not know the precise form of \( \Delta(t) \), but we assume it decays to zero beyond depth \( t_c \). Depending on the value
of $t_c$ we can obtain different behaviors. For $t_c \sim a_L$, the noise will be also uncorrelated in the vertical direction, i.e. $\Delta(t) \sim \delta(t)$. On the contrary, if $t_c$ is much larger than any characteristic depth then $\Delta(t)$ may be considered constant. We expect the noise $\eta(x, t)$ to be strongly correlated in the vertical direction. According to eq. (6) the noise actually depends on weights at the different sizes, which should be strongly correlated from layer to layer. We have avoided the multiplicative nature of the noise but in compensation we must keep the correlations in the vertical direction. In any case we are going to consider different choices of $\Delta(t)$ and compare the results with numerical simulations.

C. Average weight and fluctuations

Eq. (3) can be interpreted as the equation of motion of an interface profile $w(x, t)$, where depth plays the role of time, under an external force $w_0$ and annealed noise $\eta(x, t)$. This observation is very important because it shows that force fluctuations in granular media can be described through a more general framework, that if interface dynamics, which has been extensively studied in the literature [13]. In this context, eq. (3) is known as the Edwards-Wilkinson (EW) equation after [14].

A central quantity of interest is the width $\Delta_w$ of the fluctuating "interface", given by $\Delta w^2(L, t) = \langle L \int_0^L \delta^d[w(x, t) - \langle w \rangle]^2 \rangle$, where $\langle w \rangle$ is the mean height of the interface (mean weight). In general the "surface roughness" $\Delta w$ has the following asymptotic behavior

$$\Delta w(L, t) = L^\zeta f(t/L^z),$$

where $\zeta$ and $z$ are the roughness and dynamic exponent, respectively. In the thermodynamic limit $L \to \infty$, being linear, eq. (3) is readily solved via Fourier methods. Direct integration shows that, if the noise is spatially and temporally uncorrelated, the roughness exponent is given by $\zeta = (2-d)/2$ [14].

In this section we investigate the solution of eq. (3) for different boundary conditions and noise correlators $\Delta(t)$. We are interested in the stationary solution and, therefore, the initial condition is irrelevant.

Let first analyze the case of periodic boundary conditions. In this case the solution can be written as

$$w(x, t) = \frac{w_0}{\lambda} t + y(x, t).$$

After substituting this expression in eq. (3) we obtain the following equation for $y(x, t)$

$$\lambda \partial_t y(x, t) = \Gamma \nabla^2 y(x, t) + \eta(x, t).$$

The solution of this equation clearly has zero average, i.e. $\langle y(x, t) \rangle = 0$, and therefore

$$\langle w(x, t) \rangle = \frac{w_0}{\lambda} t.$$  

The average weight thus increases linearly with $t$ and no saturation is observed.

On the contrary, for absorbing boundary conditions the solution cannot be proposed as in eq. (10). In this case weight dissipation at the boundary leads to a stationary state where the average weight saturates. In the language of interface dynamics this situation correspond with an elastic interface pinned at the boundary under an external force $w_0$. In this case is better to look for a solution of the form

$$w(x, t) = W(x) + y(x, t),$$

where $W(x)$ is the solution of the stationary problem

$$\Gamma \nabla^2 W(x) + w_0 = 0,$$  

with homogeneous boundary conditions ($W|_{\text{boundary}} = 0$), and $y(x, t)$ satisfies eq. (11) but with homogeneous boundary conditions.

Again, $y(x, t)$ has zero average and, therefore, the mean weight in the stationary state is $W(x)$. The solution of eq. (13) can be easily obtained for certain geometries. In $d = 1$ we can look for the solution in the interval $0 < x < L$ with $W(0) = W(L) = 0$, obtaining

$$W(x) = \frac{w_0 L^2}{2\Gamma} \left( \frac{x}{L} - \frac{x^2}{L^2} \right).$$

For $d = 2, d = 3$ and $d > 3$ (only the case $d = 2$ has a physical realization) we can look for the solution in a circle, sphere and hyper-sphere of radius $L$ such that $W = W(r)$, with $0 < r < L$ and the boundary condition $W(L) = 0$, where $r$ is the distance to the center. The solution is in this case given by

$$W(r) = \frac{w_0 L^2}{2d\Gamma} \left( 1 - \frac{r^2}{L^2} \right).$$

Thus, in any dimension, $L$ is the characteristic length of the system and the saturation weight scales as $L^2$.

Now we are going to analyze the fluctuations around the average, described by eq. (13) with the corresponding boundary conditions and noise correlator. A formal solution of eq. (13) is given by

$$y(x, t) = \int_0^t dt' \int d^d x' G_d(x, x', t-t'; L) \eta(x', t'),$$

where $G_d(x, x', t; L)$ is the Green function of corresponding boundary problem, which depends on the dimension of the horizontal space $d$ and on the linear size of the system $L$. The precise form of the Green function can be obtained only for some suitable geometries, however, in general it satisfies the scaling relation

$$G_d(x, x', t; L) = \frac{1}{L^d} g_{d} \left( \frac{x}{L}, \frac{x'}{L}, \frac{t}{L^2} \right),$$

3
where \( g_d(\vec{x}, \vec{y}, t) \) depends on the spatial dimension and boundary conditions and \( L \) is a characteristic linear dimension of the system. \( g_d \) can be obtained exactly, for instance, in one dimension and in a systems with radial symmetry.

Using equations (17), (18), and (8) we compute the "surface roughness" \( \Delta w \), obtaining the scaling relation in eq. (8) with \( z = 2 \) and a roughness exponent \( \zeta \) which depends on the choice of \( \Delta(t) \). If the noise is uncorrelated in the vertical direction \( t \) (\( t_c \to 0 \)) then

\[
\zeta = \frac{2 - d}{2},
\]

which corresponds with the EW universality class. On the contrary, if it is strongly correlated in the vertical direction (\( t_c \to \infty \)) then

\[
\zeta = \frac{4 - d}{2}.
\]

In both cases the exponent \( \zeta \) is independent of the choice of boundary conditions, and they are identical to the exponents obtained from the Fourier analysis for a \( L \to \infty \) system. Thus, while the average profile is strongly dependent on the boundary conditions, the "surface roughness" only depends on the noise correlator.

III. NUMERICAL SIMULATIONS AND DISCUSSION

To test results obtained in the previous section we have performed numerical simulations of the discrete model in one dimension \( (N = 2) \) with a uniform \( q \)-distribution. We have used \( w_0 = 1 \) and lattice sizes \( L = 50, 100, 200 \) and \( 400 \). Average were taken over 1000 realizations for the three smallest lattice sizes and over 100 realizations for the largest one. Different magnitudes associated with force fluctuations were computed.

Our analytical approach reveals that the existence of absorbing boundaries is necessary to obtain a saturation weight, otherwise the weight increases linearly with depth. In fig. 1 we have plotted the global average weight

\[
W(t) = \left\langle \frac{1}{L} \sum_{i=1}^{L} w_i(t) \right\rangle
\]

at depth \( t \), for periodic and absorbing boundary conditions. The agreement with our prediction becomes evident.

Moreover, in the case of absorbing boundary conditions, we have computed the average weight profile after saturation which in one dimension is given by eq. (17). The average weight after saturation as a function of lattice position \( x \) is shown in fig. 2 for different lattice sizes. The scaled variables \( w/L^2 \) vs. \( x/L \) have been used as suggested by eq. (13). All the curves collapse in a single plot showing that our choice of scaled variables is correct. Moreover, the scaled plot was fitted to the quadratic dependency \( P(x) = a + bx - cx^2 \), with \( a = 0.02 \pm 0.01 \), \( b = 3.91 \pm 0.1 \) and \( c = 3.92 \pm 0.1 \). These fitting parameters are in very good agreement with eq. (15) since they satisfy \( a \ll b \) and \( b \approx c \). We have thus obtained the correct profile for the saturation weight.

To investigate the transient region before saturation we have computed the global average weight as a function of depth. The result for different lattice sizes is shown in fig. 3. The scaled variables \( W(t)/L^2 \) vs. \( t/L^2 \) have been used. For small depths \( W(t) \) scales linearly with \( t \), as for periodic boundary conditions. In this region the force chains starting from bulk sites have not reached the boundary and, therefore, the weight dissipation at the boundary
is negligible. This transient behavior was previously observed in numerical simulations by Peralta-Fabi et al in one dimension [7].

The average squared fluctuations (interface roughness) \( \Delta W^2 = \langle (W - \langle W \rangle)^2 \rangle \) as a function of \( t \) are shown in fig. 4. Using the scaled variables \( \Delta W^2/L^2 \zeta \), with \( \zeta = 3/2 \), vs. \( t/L^2 \) we obtain a good data collapse. This scaled plot is very sensitive to the choice of \( \zeta \), a good data collapse is only obtained for \( \zeta = 1.50 \pm 0.05 \). If the noise is uncorrelated in the vertical direction then \( \zeta = 1/2 \), which is much smaller than our numerical estimate. On the contrary, if the noise is strongly correlated in the vertical direction then \( \zeta = 3/2 \), in very good agreement with our numerical estimate. When we avoid the multiplicative character of the noise we guess that, in compensation, strong correlations in the vertical direction should be considered. This supposition is now supported by numerical simulations.

The simplicity of the continuum model we have proposed in the previous section allowed us to obtain some analytical results. We have avoided the complicated form of the noise, associated with the fluctuations in the fractions \( Q_{ij} \), introducing an "annealed" noise with correlations in the vertical direction. Within this approximation we have obtained the average weight in the stationary state and characterized the fluctuations around the average, for both periodic and absorbing boundary conditions.

In earlier numerical simulations by Liu et al [8], they considered periodic boundary conditions and compute the distribution of normalized weight \( v = w/t \). They observe that the distribution of normalized weight \( P(v) \) was independent of \( t \) for large \( t \). This numerical result was later corroborated by Coppersmith et al using a MF theory for the discrete model [9]. According to our approach, for periodic boundary conditions the average weight is given by \( w_0 t/\lambda \) and, therefore, \( \langle v \rangle = w_0 / \lambda \). Since the average normalized weight is independent of \( t \) so should be its distribution, in agreement with the numerical simulations and MF theory of the discrete model.

The robustness of the asymptotic exponential decay of the normalized weight distribution \( P(v) \) is one of the main features of the \( q \)-model. We thus check if this behavior is still observed when absorbing boundary conditions are considered. Before going to the numerical results we should remember that in this case for \( t \gg t_s \) the average weight does not increases linearly with \( t \) but scales as \( L^2 \). Hence, the appropriate choice of normalized weight is in this case \( v = w/L^2 \).

With this remark, in fig. 4 we show the distribution of normalized weight for different lattice sizes. First we notice that \( P(v) \) is independent of lattice size, supporting our choice of normalized weight. Moreover, the resulting universal curve clearly show an exponential decay for large \( v \). However, for small \( v \) the data does not display the linear behavior predicted by the \( q \)-model for a uniform distribution [8]. This discrepancy for small \( v \) is just a consequence of the different boundary conditions.

On the other hand, in the case of absorbing boundary conditions our analytical approach is in agreement with previous MF theory by Peralta-Fabi et al [7] in one dimension. Peralta-Fabi et al analyzed the case \( Q_{ij} \equiv 0 \), obtaining the scaling dependencies \( W_s \sim L^2 \) and \( t_s \sim L^2 \). We have shown that these scaling relations are valid in larger dimensions and are independent of the noise. However, within the MF theory by Peralta-Fabi et al one cannot determine the scale dependency of the average squared fluctuations \( \Delta w^2 \), which constitutes one of our main results.

The scaling dependencies \( W_s \sim L^2 \) and \( t_s \sim L^2 \) are, nevertheless, in contradiction with the linear scaling obtained in classical theories [10], some generalizations of the \( q \)-model [11] and even in experimental observations [12]. However, this discrepancy cannot be attributed to our continuum analysis, which shows a very good agreement with the numerical simulations of the \( q \)-model.
IV. SUMMARY AND CONCLUSIONS

We have obtained a coarse-grained equation of the discrete model introduced by Liu et al to describe force fluctuations in granular media. The multiplicative nature of the noise has been considered assuming strong correlations in the vertical direction. The stationary and transient behavior were obtained analytically for periodic and absorbing boundary conditions.

In this way we have demonstrated that the existence of walls, modeled by the absorbing boundary conditions, are a necessary condition to obtain an stationary state were the average weight is independent of depth. We have also shown that the the scaling of the saturation weight \( W_s \sim L^2 \) and depth \( t_s \sim L^2 \) with lattice size are valid in any dimension and are independent of the noise, generalizing previous MF calculations in one dimension.

For the first time we have obtained the scale dependency of the average squared fluctuations. The comparison of this scale dependence with the one obtained in the numerical simulations support our guess about the existance of strong correlations of the noise in the vertical direction.

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![FIG. 5. Distribution of normalized weight \( v = w/L^2 \), after saturation \( t \gg t_s \), for the \( q \)-model with absorbing boundary conditions. The straight line is a fit to an exponential decay.](image_url)