Recent resurgence of interest on the Schwinger effect, the non-perturbative production of electron-positron pairs from vacuum in an external electric field, has yielded new insights into this peculiar yet unobserved prediction of QED \cite{1, 2}. Going beyond uniform field approximation, computations of vacuum decay rate show that mean number of produced particle pairs depends nontrivially on the shape, cycle structure and polarization of the external pulse \cite{3–7}. While such investigations predominantly deal with the time-dependent electric fields, laboratory fields are usually composed of Gaussian or X-ray beams, which have spatio-temporal profile and include magnetic fields as well. This makes the formulation of vacuum decay in multidimensional electromagnetic fields essential for the fully realistic treatment of the problem.

The technical challenge is to compute the imaginary part of QED effective action, $\text{Im} \Gamma[A_\mu]$, which requires the knowledge of vacuum persistence amplitude in the background gauge field, $A_\mu(x)$. The standard approach is to use the S-matrix formalism, which relates $\text{Im} \Gamma[A_\mu]$ to the WKB coefficients of the vacuum state by a Bogoliubov transformation\cite{8, 9}. While this method works perfectly well for one dimensional external fields, its extension to higher dimensional, non-separable backgrounds is exceedingly difficult. One way to circumvent this difficulty is the inverse scattering approach, where one starts with a plausible ansatz for the Dirac equation to establish a physical gauge configuration\cite{10}. In conjunction with the standard scattering methods, works aiming at generalization of quantum kinetic equation to 1+1 dimensions can be found in\cite{11}. Another but closely related subtlety in higher dimensions is the existence of conserved quantities. Apparent translational symmetry of the Hamiltonian in one dimensional backgrounds makes the quantization of the created particle states via conserved momentum (or energy) straightforward. In the multidimensional setting, identification of particle content within the framework of Bogoliubov type transformations remains elusive. Evidently, these problems also appear in semiclassical WKB analysis, which uses analytic continuation of WKB solutions in the complex domain\cite{12, 13}.

Here, we give semiclassical treatment of the multidimensional vacuum pair production by using the worldline formulation of QED\cite{14, 16}. The worldline approach could be considered more advantageous with respect to conventional WKB methods for two reasons. First reason is that in the worldline language $\Gamma[A_\mu]$ is represented by a path integral over closed trajectories in spacetime, thus the formalism admits a natural multidimensional description. Secondly, no specific choice of ansatz and Bogoliubov transformation are needed; calculation of $\text{Im} \Gamma[A_\mu]$ is relegated to finding periodic, tunneling trajectories, which are referred as worldline instantons. Basic formalism is not just relevant to Schwinger effect, but it also has the potential to deal with multidimensional tunnelling problems in general.

Main ingredient of the analysis is the QED analog of Gutzwiller trace formula for the effective action\cite{17, 18}. Upon performing saddle point approximation to path integral, $\Gamma[A_\mu]$ can be written as a sum over closed orbits $(g_{\mu\nu} = (+, −, −, −), \hbar = c = 1)$:

$$\Gamma[A_\mu] \approx -\frac{i}{2} \sum_p e^{-iW[T_p]−iM_p\pi/2} \sqrt{\text{det}(1−\mathbf{M})} \text{tr} \left[ e^{\frac{i}{\hbar} \int_0^{T_p} \sigma_{\mu\nu} F^{\mu\nu} \text{d}u} \right]$$

$$W[T_p] = \int_0^{T_p} p\dot{x}^\mu \text{d}u, \quad \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu]$$

The Monodromy matrix $\mathbf{M}$ comes from the saddle point expansion and corresponds to the density of the trajectories which start at the same point with different initial momenta. For each periodic trajectory labelled by $p$, the Morse index, $M_p$ is given by the number of negative eigenvalues of the determinant, whereas the spinor term contributes a $+\sim$ sign. The determinant can analytically be obtained when equations of motion separate, and the trajectory admits a simple form. Here, we will
work with non-separable cases so we focus our attention to Hamilton’s characteristic function, $W[T]$. Semiclassical value of $W[T]$ is given by the classical tunneling trajectories, which are imaginary proper time solutions of the classical force equation:

$$\ddot{x} = -ie\mu F^{\mu\nu}(x)\dot{x}_\nu, \quad u \to iu \quad (2)$$

Periodic solutions of (2) also exist for real $u$ [19]. Composite worldlines which are made of $x(u)$ and $x(iu)$ are responsible for possible interference effects, but their contribution to pair production rate is controlled by the tunneling segments $x(iu)$. For this reason, we are interested only on imaginary proper time solutions, as they are expected to yield a pretty accurate estimate in the non-perturbative domain. In the following, we argue that worldline trajectories are essentially multi-periodic in spatio-temporal backgrounds. Starting with a simple model, we discuss the implications of bounded motion in the multidimensional setting, and move onto a more realistic scenario, which incorporates the magnetic field.

![FIG. 1. Worldline trajectories in the background field (3).](image)

Temporal parameters are fixed as $E_0 = 0.1 m$, $\omega = 0.1 m$. Spatial parameters are chosen as $E_{0'} = 0.0 m$ (top-left), $E_{0} = 0.01 m$, $k = 0.001 m$ (top-right) $E_{0} = 0.1 m$, $k = 0.01 m$ (bottom-left), and $E_{0'} = 0.03 m$, $k = 0.01 m$ (bottom-right).

We first consider the combination of two Sauter pulses. Total electric field along $x_3$ is given as:

$$A_{x_0}(x_3) = \frac{E_0}{\omega} \tanh \omega x_0, \quad A_{x_3}(x_3) = \frac{E_0'}{k} \tanh kx_3$$

$$E_{x_1}(x_0, x_3) = E_0 \sech^2 \omega x_0 + E_0' \sech^2 kx_3 \quad (3)$$

where $k$ and $\omega$ represent spatial and temporal width respectively. In the following we keep the temporal parameters fixed, and vary $E_{0'}$ and $k$ to illustrate the effect of spatial inhomogeneity on the closed orbits. As observed in [19], in the purely time dependent case ($E_0' = 0$) tunneling trajectories are periodic with the classical period being $T$, and are quantized by the canonical momentum $p_3$. Introduction of spatial inhomogeneity causes periodic trajectories to have a second oscillation period $T_2$, which envelopes the oscillations with a smaller period, $T_1$ (Fig. 1). These trajectories are mainly quasi-periodic and form invariant tori in phase space. Given the time component of the above field is dominant ($E_0' < E_0$), such bounded trajectories usually persist when $k < \omega$. Our key observation is that in the constant spatial field limit: $k \to 0$, and also in the limit: $E_0' \to 0$, $T_2$ increases and effectively goes to infinity, leaving a single period $T_1 \equiv T$. This indicates finite values of $T_1$ and $T_2$ genuinely depend on the interplay between the temporal and spatial adiabaticity parameters of the external field. This applies not just to this particular case but holds general validity. Few other examples include pulse-shaped fields such as:

$$E_{x_1} = E_0 e^{-k^2 x_3^2 - \omega^2 x_0^2}, \quad E_{x_1} = E_0 e^{-k^2 x_3^2} \sech \omega x_0^2$$

$$E_{x_3} = E_0 e^{-\omega^2 x_0^2} + E_0' e^{-k^2 x_3^2}$$

Note that Hamiltonian, $\mathcal{H} = 1/2 (p_\mu - e/c A_\mu(x))^2$, for the external field in (3) has no longer translational symmetry along $x_3$. On the other hand, existence of quasi-periodic motion tells us that system possesses dynamical (hidden) symmetry. This is because Poincare sections of the quasi periodic orbits form closed curves in the reduced phase space, and imply the existence of an isolating integral of motion $\mathcal{C}$, in addition to $\mathcal{H}$. One may hope in this case the corresponding quantum system possesses simultaneous eigenstates of constants of motion by the virtue of Liouville integrability. Quantum tunneling via multi-periodic motion can then be interpreted as the creation of particle states quantized by $\mathcal{H}$ and $\mathcal{C}$. Although such interpretation could be valid, it is usually not possible to represent $\text{Im} \Gamma[A]$ as a sum over tunneling amplitudes labelled by the good quantum numbers. This stems from the fact that $\mathcal{C}$ is generally an intimate admixture of phase space variables in multidimensions. The power of trace formula is that effective action and so the decay rate can semiclassically be evaluated by using the periods of motion, even though the system does not admit separability.

The amplitude for the vacuum decay is governed by the characteristic function, whose evaluation on multi-periodic solutions must be handled with care. Direct approach would involve computing $W[T_2]$, where $T$ simply gets replaced by $T_2$. This however leads to an undesirable limiting behavior for the tunneling amplitude. To explain we reconsider the toy model above. For a weak spatial inhomogeneity ($E_0' \ll E_0$) we have the second period extended over a very large proper-timescale compared to $T_1$, so we have $T_1 \ll T_2$. In this regime, the smaller period almost coincides with the period of purely time-dependent background, whose characteristic function is $W[T]$. On the other hand, the evaluation of $W[T_2]$ yields $\sim T_2/T_1 W[T]$. Thus the corresponding vacuum decay rate is many orders of magnitude smaller than $e^{-|W[T]|}$, which gives the decay rate for the time-dependent field. Such a huge difference in tunneling probabilities is quite unexpected for a weak spatial perturbation. In fact, in
the limit: \( E_0' \to 0, T_2 \) gets infinitely large and so \( W[T_2] \)
will get infinitely large. This basically leads to vanishing pair production probability, which is clearly in contradiction with the purely time dependent case. To remedy this situation tunneling treatment of the trajectories should take both periods into account. To incorporate the second period into picture, we rewrite the effective action:

\[
\Gamma[A] = -\frac{i}{2} \int_0^\infty \frac{dT_1}{T_1} \text{tr} \left[ \frac{i T_1}{2 T_2} (m^2 c^2 + \partial^2) T_2 \right] \\
\approx -\frac{i}{2} \int_0^\infty e^{-iW[T_1, T_2]} \text{tr} \left[ \frac{i T_1}{2 T_2} \sigma \sqrt{m^2} du \right],
\]

where we have fixed the upper limit of the ordering parameter \( u \) as \( T_2 \). The trace is taken over the classical closed trajectories, for which \( T_1/T_2 \) is treated as a constant factor. With this in mind and making the change of variable \( (T_1 \to T_2) \) in the integration, saddle point approximation can be carried out in the usual way. Above transformation is very similar to gauge fixing of effective approximation can be carried out in the usual way. Above field introduces a source term \( \Phi \) which basically represents an exponentially decaying plane-wave with frequency \( \omega \) and the wavenumber \( k \). In conjunction with this, we consider the external field:

\[
A_{x_1}(x_0, x_3) = -E_0 e^{-\frac{x_3^2}{2 \sigma^2}} \left( -\frac{x_3^2}{2} \right) (e^{-i k x_3} f(x_0) + c.c), \\
E_{z_1}(x_0, x_3) = E_0 e^{-\frac{x_3^2}{2 \sigma^2}} \left( \frac{x_3}{2} \right) \cos (\omega x_0 - k x_3), \\
B_{z_2}(x_0, x_3) = E_0 e^{-\frac{x_3^2}{2 \sigma^2}} \left( -\frac{x_3^2}{2} \right) (e^{-i k x_3} g(x_0) + c.c), \\
f(x_0) = \sqrt{\pi \tau} \text{Erf} \left( \frac{x_0 - i \tau}{\sqrt{2 \tau \omega}} \right), \quad g(x_3) = \frac{x_3}{\sigma^2} + i k
\]

which basically represents an exponentially decaying plane-wave with frequency \( \omega \) and the wavenumber \( k \) (Fig. 2). The inverse temporal and spatial width are respectively given by \( \tau \) and \( \sigma \). Above field introduces a source current along \( x_1 \). This current may in principle affect the tunneling amplitude, because it contributes to the evolution of the momentum operator, \( d\xi_1/du \). However, the effect of the source term remains negligible, and classical solutions are expected to dominate tunneling, as long as the external field extends over a distance larger than the Compton wavelength [21].

The existence of invariant tori now depends on the relative magnitude of both \( \sigma \) and \( k \) with respect to temporal parameters. Quasi-periodic trajectories generally occur in the parameter region where \( \sigma \) is of same order of magnitude as \( \tau \) and \( k < \omega \). Here, we look for the closed orbits when the electric field is maximum, so we fix the initial positions as \( x_0(0) = x_3(0) = 0 \), and vary the conserved momentum \( \dot{x}_1(0) = ip_1 \), and the initial velocity \( \dot{x}_3(0) = i p_3 \). The remaining initial condition for \( \dot{x}_0(0) = ip_0 \) is fixed by the constraint: \( \dot{x}_0^2(u) - \dot{x}_1^2(u) - \dot{x}_3^2(u) = -m^2 \), where \( m \) is the electron mass. In order to show the effect of spatial inhomogeneity on tunneling, we specify the values of \( E_0, \tau \) and \( \omega \), in accordance with [20]. In terms of the normalized mass \( (m = 1) \) we fix the field parameters as: \( E_0 = 0.1 m, \tau = 100 m^{-1}, \omega = 0.03 m \). We choose spatial frequency to be \( k = 0.01 m \) and \( \sigma = 100 m^{-1} \). Multi-periodic orbits are
located by making use of a search algorithm that scans through the region: \( p_1 \in (0, m), p_3 \in (0, m) \), until the trajectory closes on itself within an accuracy of \( 10^{-6} \).

The greater accuracy makes the location of the orbits more precise but, from a practical point of view, increasing the accuracy further does not have appreciable effect on the tunneling rate. Figure 4 shows the locations of the closed orbits on the momentum plane. Multi-periodic trajectories are not isolated, but form a family and lie on what we would like to call a saddle curve in analogy with the saddle points of WKB solutions. Along the curves closed trajectories grow in amplitude and get steeper until the invariant tori breaks. For every closed trajectory on the first curve we have \( T_2/T_1 = 12 \), on the second, period doubling occurs and ratio becomes 23, and on the third curve it becomes 11. For \( T_2/T_1 = a/b \), the set of co-prime integers, \((a, b)\) fixes the shape and the number of self intersections for the orbit, and therefore it characterizes the orbit topology. Consequently, (1) is considered as a topological sum, where each saddle curve characterized the orbit topology [22].

Before we finally evaluate \( W[T_1, T_2] \), we look into the spinor term. Straightforward calculation of the spinor trace for the field configuration \([1] \) gives \((u \rightarrow iu)\)

\[
\text{tr} \left[ e^{-\frac{T_2}{2}\int_0^{T_2} \frac{1}{2} \sigma_{\mu\nu} F_{\mu\nu} du} \right] = \cos \left( \frac{1}{2} \sqrt{\mathcal{E}^2 - \mathcal{B}^2} \right) = T_1/T_2 \int_0^{T_2} E_{x_1} \, du, \quad \mathcal{B} = T_1/T_2 \int_0^{T_2} B_{x_2} \, du \quad (6)
\]

Upon integration along the orbits, the argument of cosine for a single closure yields \( \pi \). Spinor trace essentially gives a multiplicative phase factor which can be absorbed into the prefactor. This further simplifies imaginary part of the effective action to:

\[
\text{Im} \Gamma[A_\mu] \approx \sum_p e^{-W[T_1, T_2]}, \quad W[T_1, T_2] > 0 \quad (7)
\]

Having obtained the closed orbits, decay rate can now be determined upon evaluation of the characteristic function. For convenience, we use the configuration space where \( W[T_1, T_2] \) reads

\[
W[T_1, T_2] = T_1/T_2 \int_0^{T_2} (\dot{x}_0^2 - \dot{x}_3^2) \, du. \quad (8)
\]

Figure 4 shows vacuum decay probability density \( e^{-W[T_1, T_2]} \) on the saddle curves. Decay rate is higher on the first curve, which contains relatively low momenta orbits. Comparing results with the purely time dependent counterpart of (3), we see that pair production rate for the overlapping range of canonical momentum drops down by an order of magnitude [5]. This result is consistent with the qualitative picture of virtual particles; the external electric field along \( x_1 \) separates the electron-positron pairs, whereas the magnetic field applied in the perpendicular direction tries to bring the virtual pairs together. It is worth noting that tunneling probability increases along all the curves with increasing momenta. If we consider the region where quasi-periodic motion occurs, this behavior suggests that decay rate is maximized on the boundary of invariant tori. It would be of particular interest to see whether this remains true for other field configurations.

Worldline analysis presented here relies on the existence of quasi-periodic motion. In this respect, integrability of the Hamiltonian plays an important role in the pair production process. Given that \( \hat{\mathcal{H}} \) is integrable, the use of multi-periodic trajectories in quantum tunnelling naturally leads to the quantization of the fermionic modes by \( \hat{\mathcal{H}} \) and \( \hat{\mathcal{C}} \). For the external field given in (5), multi-periodic trajectories form a one parameter family but it is useful to keep in mind that there may be field configurations, where tunneling occurs via isolated trajectories. As a final remark on the use of transformation in (4), we would like to point out that averaging methods similar in spirit have been used in atomic molecular physics to obtain the spectrum of multidimensional bound systems. For instance, the evaluation of characteristic function by using the caustics or the Poincare sections of quasi-periodic trajectories has successfully generated the energy eigenvalues of multidimensional systems, such as 2-d coupled harmonic oscillators [23]. Here in our approach averaging arises naturally, with only requiring the periods of motion.

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