SOME PROBLEMS IN DIFFERENTIATION

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1. INTRODUCTION

In this note we discuss three problems.

Problem 1: What is the $n$th derivative of an inverse function?

Problem 2: What is the $n$th derivative of a function given parametrically?

Problem 3: What is the $n$th derivative of an implicit function?

Let us be more precise. Suppose $x = f(t)$ and $y = g(t)$ for nice enough functions $f$ and $g$. What is $d^n y / dx^n$ in terms of the derivatives of $f$ and $g$? Or suppose that an equation $F(x, y) = 0$ defines $y$ as an implicit function of $x$, and that $F$ is nice enough to have equality of all the relevant mixed partial derivatives. What is $d^n y / dx^n$ in terms of the partial derivatives of $F$? These are problems 2 and 3 respectively.

Note that problem 1 is the special case $g(t) = t$ of problem 2, and the special case $F(x, y) = f(y) - x$ of problem 3, so it does not need a separate treatment. Moreover, I have discussed it before in [5], which contains my rediscovery of Sylvester’s solution [8]. Problem 1 was apparently solved first by Murphy [6].

Sylvester pointed out in a postscript to [8] that his argument extends without great difficulty to problem 2, but he did not give the details. A survey of the problem was given later by Gallop [4], who had read [8] but was not sympathetic to it. The following treatment is, I believe, close to what Sylvester had in mind.

We begin by working out the first several derivatives. Evidently we have

$$\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = g'(t) (f'(t))^{-1}.$$  

Then

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{1}{f'(t)} \left\{ g''(t) [f'(t)]^{-1} - g'(t) [f'(t)]^{-2} f''(t) \right\}$$

$$= g''(t) [f'(t)]^{-2} - g'(t) f''(t) [f'(t)]^{-3}$$

and

$$\frac{d^3 y}{dx^3} = g'''(t) [f'(t)]^{-3} - 3g''(t) f''(t) [f'(t)]^{-4}$$

$$- g'(t) f'''(t) [f'(t)]^{-4} + 3g'(t) f''(t) [f'(t)]^{-2} f'(t) [f'(t)]^{-5}.$$  

To illustrate the combinatorics behind these formulas we take (1.1). Besides the order $n$ of the derivative (here $n = 3$) there is another crucial parameter that we will call $k$. Each term of (1.1) represents a partition of \{1, 2, \ldots, k + 3\} into $k + 1$ blocks, where $k + 3$ is the exponent of $[f'(t)]^{-1}$ and the sign of each term

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is \((-1)^k\). The term \(g''(t)[f'(t)]^{-3}\), with \(k = 0\), represents \(\{1, 2, 3\}\). The term
\(-3g''(t)f'''(t)[f'(t)]^{-4}\), with \(k = 1\), represents the three partitions \(\{1, 2\}\), \(\{3, 4\}\) and
\(\{13\}, \{24\}\) and \(\{14\}, \{23\}\), where the block containing 1 and one other element

\[
\text{corresponds to } g''(t) \text{ and the other block to } f'''(t). \text{ The block containing 1 always}
\text{corresponds to and has the same size as the order of the } g \text{ derivative, so the other}
\text{ } k = 1 \text{ term } -g''(t)f'''(t)[f'(t)]^{-4} \text{ corresponds to } \{1, \{3, 4\}. \text{ The remaining term}
\text{ } 3g''(t)[f'''(t)]^2[f'(t)]^{-5} \text{ has } k = 2 \text{ and represents the partitions } \{1, \{2, 3\}, \{4, 5\}\text{ and}
\(\{1\}, \{2, 4\}, \{3, 5\}\) and \(\{1\}, \{2, 5\}, \{3, 4\}.\)

Note that there is never an } f'(t) \text{ term in the numerators of these formulas; only
higher derivatives of } f \text{ can occur. This means that the corresponding partitions

\text{cannot have any singleton blocks besides } \{1\}, \text{ which in turn explains why } k \text{ can’t exceed 2 in this example: if } k = 3, \text{ we would have to put } \{1, 2, 3, 4, 5, 6\} \text{ into 4
blocks without putting any of } \{2, 3, 4, 5, 6\} \text{ in a singleton, which is impossible. }

Let us try to predict } d^4y/dx^4 \text{ from this point of view. At the same time we
illustrate a definition we shall need for the general case. We have to consider partitions of } \{1, 2, \ldots, k + 4\} \text{ with } k + 1 \text{ blocks, where only 1 can be in a singleton.}

If } k = 0 \text{ this means four elements and one block. The only such partition is}
\(\{1, 2, 3, 4\}, \text{ which gives us the term } g''''(t)[f'(t)]^{-4}. \text{ If we set } P_{4,0}(t) = g''''(t) \text{ then}
this term is } [f'(t)]^{-4} P_{4,0}(t).

When } k = 1 \text{ we have five elements and two blocks, so the block sizes are either
one and four or two and three. With one and four we can only have } \{1\}, \{2, 3, 4, 5\}, \text{ which corresponds to } -g''(t)f'''(t)[f'(t)]^{-5}. \text{ With two and three there are two
possibilities: 1 is in a block of size two, or 1 is in a block of size three. In the first
case there are 4 other elements that could be together with 1, so these partitions give
the term } -4g''(t)f'''(t)[f'(t)]^{-5}. \text{ In the second case there are } \binom{4}{2} = 6 \text{ ways to choose
the block containing 1, so these partitions give the term } -6g''''(t)f''''(t)[f'(t)]^{-5}. \text{ If we set } P_{4,1}(t) = g''''(t)f''''(t) + 4g''(t)f'''(t) + 6g'''(t)f''(t), \text{ then these three terms
together are } -[f'(t)]^{-5} P_{4,1}(t).

If } k = 2 \text{ we have six elements and three blocks. Since there can only be one single-
ton, the block sizes must be either 3-2-1 or 2-2-2. In the former case we have the
block } \{1\}, \text{ and we complete the partition by choosing the block of size two in any of
\(\binom{5}{2} = 10 \text{ ways. Therefore these partitions correspond to } 10g''(t)f'''(t)f''''(t)[f'(t)]^{-6}. \text{ If all three
blocks have size two, then we pick an element to put with 1, and then pick one to put with the smallest remaining element, and the partition is determined, so there are } 5 \cdot 3 = 15 \text{ partitions in this case and the corresponding term is}
15g''(t)[f''''(t)]^2[f'(t)]^{-6}. \text{ If we set } P_{4,2}(t) = 10g''(t)f'''(t)f''''(t) + 15g''(t)[f''''(t)]^2
\text{ then these two terms together are } [f'(t)]^{-6} P_{4,2}(t).

If } k = 3 \text{ we have seven elements and four blocks. This forces at least one
singleton, but there can’t be more than one, so the only possibility is to have the
block } \{1\} \text{ and the other six elements in three blocks of size two. As above there are 15 such partitions, so this gives the term } -15g''(t)[f''''(t)]^2[f'(t)]^{-7}. \text{ or, setting
} P_{4,3}(t) = 15g''(t)[f''''(t)]^3, \text{ the term } -[f'(t)]^{-7} P_{4,3}(t). \text{ It is also clear from this
example that we would get an impossible number of singleton blocks if } k > 3, \text{ so we define } P_{4,k}(t) = 0 \text{ in this case.}
Adding all these terms together we have

\[ \frac{d^4y}{dx^4} = \sum_{k=0}^{3} (-1)^k [f'(t)]^{-4-k} P_{4,k}(t), \]

where we could just as well leave the upper limit of the sum unrestricted. When written out broadly this says

\[ \frac{d^4y}{dx^4} = g'''(t) [f'(t)]^{-4} - 6g''(t)f''(t) [f'(t)]^{-5} + 15g''(t)f'''(t) [f'(t)]^{-6} \]
\[ \quad - 4g''(t)f''(t) [f'(t)]^{-5} + 10g'(t)f''''(t) [f'(t)]^{-6} \]
\[ \quad - g'(t)f'''(t) [f'(t)]^{-5} - 15g'(t) [f''''(t)]^3 [f'(t)]^{-7} \]

which is right.

To state the general result we make the definition illustrated above.

**Definition 1.** Given two functions \( f(t) \) and \( g(t) \) with at least \( n \) derivatives, define \( P_{n,k}(t) \) as the sum over all the partitions of \{1, 2, ..., \( k + n \)\} with \( k + 1 \) blocks where only the block containing 1 can be a singleton, and as above each partition corresponds to a product of derivatives, where the size of the block containing 1 is the order of the derivative of \( g \) and the sizes of the other blocks are orders of derivatives of \( f \).

We note some extreme cases. If \( k = 0 \) we have only the partition \{1, 2, ..., \( n \)\}, so \( P_{n,0}(t) = g^{(n)}(t) \) for \( n \geq 1 \). At the other end, if \( k = n - 1 \) we have partitions of \{1, 2, ..., 2n - 1\} into \( n \) blocks, where only the element 1 can be in a singleton. Therefore all the other blocks must be doubletons, and there are \( 1 \cdot 3 \cdot 5 \cdot \cdots (2n - 3) \) ways to choose them, so

\[ P_{n,n-1}(t) = 1 \cdot 3 \cdot 5 \cdot \cdots (2n - 3)g'(t) [f''''(t)]^{n-1} \quad \text{for } n \geq 1. \]

If \( k > n - 1 \) then we have an impossible number of singletons, so \( P_{n,k}(t) = 0 \) if \( k > n - 1 \), or for that matter if \( k < 0 \). It is also not hard to see that

\[ P_{n,n-2}(t) = 1 \cdot 3 \cdot 5 \cdot \cdots (2n - 3)g''(t) [f'''(t)]^{n-2} \]
\[ \quad + \binom{2n - 3}{3} 1 \cdot 3 \cdot 5 \cdot \cdots (2n - 7)g'(t) [f''''(t)]^{n-3} f'''(t) \quad \text{for } n \geq 2. \]

We need the following recurrence for \( P_{n,k}(t) \).

\[ (1.2) \quad P_{n+1,k}(t) = P_{n,k}'(t) + (n + k - 1)f''(t)P_{n,k-1}(t). \]

We can obtain partitions of \{1, 2, ..., \( k + n + 1 \)\} with \( k + 1 \) blocks from a partition of \{1, 2, ..., \( k + n \)\} with \( k + 1 \) blocks by adding the element \( k + n + 1 \) to each existing block in turn. This corresponds perfectly to how \( d/dt \) acts on each term of the sum, increasing the order of one derivative by one while leaving the others alone, and doing this for each factor in turn. The only problem is that we cannot obtain all of the partitions of the desired type this way: we are missing the ones where \( k + n + 1 \) is in a doubleton with one of \{2, 3, ..., \( k + n \)\}, because those elements could not have been in singleton blocks. These partitions come from the term \((n + k - 1)f''(t)P_{n,k-1}(t)\), for we know that \( P_{n,k-1}(t) \) corresponds to partitions of \{1, 2, ..., \( k + n - 1 \)\} with \( k \) blocks. We can add \{\( k + n, k + n + 1 \)\} as a doubleton to each of these, and then we can switch \( k + n \) with any of \{2, 3, ..., \( k + n - 1 \)\}
to get the remaining ones. This proves (1.2), and now we can prove the following formula.

**Theorem 1.** If \( x = f(t) \) and \( y = g(t) \), where \( f \) and \( g \) have at least \( n \) derivatives, then

\[
\frac{d^n y}{dx^n} = \sum_{k=0}^{n-1} (-1)^k [f'(t)]^{n-k} P_{n,k}(t).
\]

We have checked the cases \( n = 1, 2, 3, 4 \) of this. Suppose it holds for \( n \), and take \( d/dx \) of the right side of (1.3) by taking \( d/dt \) and dividing by \( dx/dt = f'(t) \). It is convenient to leave the sum unrestricted, as we may since \( P_{n,k}(t) = 0 \) if \( k < 0 \) or \( k > n - 1 \). We get

\[
\frac{1}{f'(t)} \left[ \sum_j (-1)^j (-n-j) [f'(t)]^{-n-j-1} f''(t) P_{n,j}(t) + \sum_j (-1)^j [f'(t)]^{-n-j} P'_{n,j}(t) \right] = \sum_j (-1)^{j+1} (n+j) [f'(t)]^{-n-(j+1)} f''(t) P_{n,j}(t) + \sum_j (-1)^j [f'(t)]^{-n-1-j} P'_{n,j}(t) = \sum_k (-1)^k (n+k-1) [f'(t)]^{-n-1-k} f''(t) P_{n,k-1}(t) + \sum_k (-1)^k [f'(t)]^{-n-1-k} P'_{n,k}(t).
\]

Because of (1.2) we know that this is

\[
\sum_k (-1)^k [f'(t)]^{-n-1-k} P_{n+1,k}(t).
\]

Therefore (1.3) holds for \( n + 1 \) if it holds for \( n \), and hence Theorem 1 is true.

Next we take up problem 3. It has been treated in two papers by Comtet [1], [3], whose excellent book [2] also has references (p. 153) to some older literature. More recently it has been discussed in [9] and [7]. The forthcoming paper by Shaul Zemel [10] gives a valuable survey of the various approaches, including my own below, and a new method. I am putting this version of my paper on the arXiv at his request.

If \( F(x, y) = 0 \) then we have

\[
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0,
\]

so

\[
\frac{dy}{dx} = -\frac{\partial F}{\partial y} \frac{dx}{dy},
\]

or, as we will prefer to write,

\[
\frac{dy}{dx} = -F_x F_y^{-1}.
\]

Differentiating this with respect to \( x \) we have

\[
\frac{d^2 y}{dx^2} = -\frac{\partial}{\partial x} F_x F_y^{-1} - \frac{dy}{dx} \frac{\partial}{\partial y} F_x F_y^{-1} = -F_{xx} F_y^{-1} + F_x F_y^{-2} F_{xy} + F_x F_y^{-1} (F_{xy} F_y^{-1} - F_x F_y^{-2} F_{yy}) = -F_{xx} F_y^{-1} + 2 F_x F_{xy} F_y^{-2} - F_x F_{yy} F_y^{-3}.
\]
As before, we want to associate a family of set partitions to these formulas. The general result we are after has the form

\[
\frac{d^n y}{dx^n} = \sum_{k \geq 1} (-1)^k F_y^{-k} I_{n,k},
\]

where \( I_{n,k} \) has still to be explained. We have the examples \( I_{2,1} = F_{xx}, I_{2,2} = 2F_x F_{xy}, I_{2,3} = F_x^2 F_{yy}, \) and \( I_{2,k} = 0 \) for \( k > 3 \). In general \( I_{n,k} \) will be a sum over partitions of \( \{1, 2, \ldots, n+k-1\} \) with \( k \) blocks. Let us call \( \{1, 2, \ldots, n\} \) the small elements, and \( \{n+1, \ldots, n+k-1\} \) the large elements. The small elements will correspond to \( x \) derivatives and the large elements to \( y \) derivatives, and the partitions are restricted only in that the large elements can’t be in singleton blocks. When \( k = 1 \) all the elements are small and must be together, giving us the term \( F_{xx} \) when \( n = 2 \) and in general the term \( \partial^n y / \partial x^n \). When \( n = 2 = k \) we have partitions of \( \{1, 2, 3\} \) into two blocks, where 3 is large and hence can’t be in a singleton. There are two such partitions, \( \{1\}, \{2, 3\} \) and \( \{1, 3\}, \{2\} \), and with 1 and 2 corresponding to \( x \) and 3 to \( y \) these give the term \( 2F_x F_{xy} \). When \( n = 2 \) and \( k = 3 \) we have partitions of \( \{1, 2, 3, 4\} \) with three blocks, where 3 and 4 are large. There is only one such partition, \( \{1\}, \{2\}, \{3, 4\} \), and it corresponds to \( F_x^2 F_{yy} \). If \( n = 2 \) and \( k = 4 \) we would have to put \( \{1, 2, 3, 4, 5\} \) into 4 blocks without putting 3, 4, or 5 into a singleton, which is impossible, and similarly there are no more partitions with \( n = 2 \) and a larger \( k \).

Let us work out \( d^3 y / dx^3 \) from this point of view; we have \( n = 3 \) and various values of \( k \). As before, when \( k = 1 \) we have only the term \( I_{3,1} = F_{xxx} \). When \( k = 2 \) we have partitions of \( \{1, 2, 3, 4\} \) with two blocks, where 4 can’t be in a singleton. All three partitions with two blocks of size two are admissible, and they give us the term \( 3F_{xx} F_{xy} \). Of the four partitions with block sizes one and three we can only use those with 4 in a triplet, and they give us the term \( 3F_x F_{xx} F_{xy} \). Therefore \( I_{3,2} = 3F_{xx} F_{xy} + 3F_x F_{xx} F_{xy} \).

When \( n = 3 \) and \( k = 3 \) we have partitions of \( \{1, 2, 3, 4, 5\} \) with three blocks, where 4 and 5 are large. The block sizes must be either 3-1-1 or 2-2-1. In the former case the singletons must be two of \( \{1, 2, 3\} \), so there are only three possibilities, which together give the term \( 3F_x^2 F_{yy} \). In the latter case there are two subcases. If the two large elements 4 and 5 are together in a doubleton, the partition is determined by which of the three small elements is alone, and we get the term \( 3F_x F_{xx} F_{xy} \). If 4 and 5 are in different doubletons then there are six possibilities, and we get the term \( 6F_x F_{xy} \). Therefore \( I_{3,3} = 3F_x^2 F_{yy} + 3F_x F_{xx} F_{xy} + 6F_x F_{xy} \).

When \( n = 3 \) and \( k = 4 \) we have partitions of \( \{1, 2, 3, 4, 5, 6\} \) with four blocks, where 4, 5, 6 are large. The block sizes must be either 3-1-1-1 or 2-2-1-1. There is only one admissible partition in the former case, namely \( \{1\}, \{2\}, \{3\}, \{4, 5, 6\} \), which corresponds to the term \( F_x^3 F_{yy} \). In the latter case we can choose one of 1, 2, 3 to be in a doubleton with one of 4, 5, 6, and after that the partition is determined (since the other two small elements must be the singletons), so there are nine possibilities, which together contribute the term \( 9F_x^3 F_{yy} \). Hence \( I_{3,4} = F_x^3 F_{yy} + 9F_x F_{xy} F_{yy} \).

When \( n = 3 \) and \( k = 5 \) we have partitions of \( \{1, 2, 3, 4, 5, 6, 7\} \) with five blocks, where 4, 5, 6, 7 are large. With this constraint we cannot have block sizes 3-1-1-1-1, so we have only to consider the case 2-2-1-1-1. The singletons must be 1, 2, 3, and there are three such partitions, which collectively give us \( 3F_x^3 F_{yy} = I_{3,5} \).
In general there are no partitions of the desired type when \( k > 2n - 1 \), so we have all of them for \( n = 3 \). If \( k = 2n \) then we would have to put \( 3n - 1 \) elements into \( 2n \) blocks, which forces at least \( n + 1 \) singletons; but only the \( n \) small elements can be in singletons, so this is impossible. If \( k > 2n \) then even more elements have to be in singletons, which is even more impossible. Therefore we define \( I_{n,k} = 0 \) if \( k > 2n - 1 \).

The recurrence we will need for \( I_{n,k} \) is rather complicated.

\[
I_{n+1,k} = \frac{\partial}{\partial x} I_{n,k} + F_x \frac{\partial}{\partial y} I_{n,k-1} + (k-1)F_{xy} I_{n,k-1} + (k-2)F_x F_{yy} I_{n,k-2}.
\]

The left side is a sum over partitions of \( \{1, 2, \ldots, n + k\} \) with \( k \) blocks, where \( \{1, 2, \ldots, n + 1\} \) are small and \( \{n + 2, \ldots, n + k\} \) are large, and we have to argue that the right side also represents this class of partitions. As before, a derivative adds an element to an existing block while leaving the others alone—here an \( x \) derivative adds a small element and a \( y \) derivative a large one.

We interpret the first term \( \frac{\partial}{\partial x} I_{n,k} \) as adding the element 1 to an existing block in a partition of \( \{1, 2, \ldots, n + k - 1\} \) with \( k \) blocks of the desired type and relabeling all the other elements up one; thus we have one more small element and the same number of large elements, as desired. Note that with this operation 1 can’t be in a singleton block, nor can it be in a doubleton with a large element, but it gives all the partitions of the desired type except for these restrictions.

The second term \( F_x \frac{\partial}{\partial y} I_{n,k-1} \) should be an operation on partitions of \( \{1, 2, \ldots, n + k - 2\} \) with \( k - 1 \) blocks where \( \{1, 2, \ldots, n\} \) are small. We think of it as adding \( \{1\} \) as a singleton block, relabeling all the other elements up one, and adding \( n + k \) to an existing block. This gives the correct numbers of small and large elements, but in addition to putting 1 in a singleton it also means that \( n + k \) is not in a doubleton with another large element.

The third term \( (k-1)F_{xy} I_{n,k-1} \) is an operation on the same partitions as the second term. We think of it as first adding \( \{1, n + k\} \) as a doubleton and relabeling all the other elements up one, then switching \( n + k \) with each of the \( k - 2 \) relabeled large elements \( \{n + 2, \ldots, n + k - 1\} \) in turn to create \( k - 2 \) more partitions of the desired type. These are the missing ones from the first term, having 1 in a doubleton with a large element.

Finally, the last term \( (k-2)F_x F_{yy} I_{n,k-2} \) should be an operation on partitions of \( \{1, 2, \ldots, n + k - 3\} \) of the desired type with \( k - 2 \) blocks where \( \{1, 2, \ldots, n\} \) are small. Here we first add \( \{1\} \) as a singleton block, relabel the other elements up one, and add the doubleton \( \{n + k - 1, n + k\} \). Then we switch \( n + k - 1 \) with each of the \( k - 3 \) relabeled large elements \( \{n + 2, \ldots, n + k - 2\} \) in turn to create \( k - 3 \) more partitions of the desired type. These are the missing ones from the second term, where 1 is in a singleton and \( n + k \) is in a doubleton with another large element. This proves (1.4).

**Theorem 2.** If \( F(x,y) = 0 \) for a sufficiently nice function \( F \), then

\[
\frac{d^n y}{dx^n} = \sum_{k=1}^{2n-1} (-1)^k F_y^{-k} I_{n,k}
\]

with \( I_{n,k} \) as defined above.
In the proof it is convenient to leave the sum unrestricted, as we may since \( I_{n,k} = 0 \) if \( k < 1 \) or if \( k > 2n - 1 \). On the right side of (1.5) we calculate the derivative with respect to \( x \) as
\[
\frac{\partial}{\partial x} + \frac{dy}{dx} \frac{\partial}{\partial y} = \frac{\partial}{\partial x} \frac{F_y}{F_x} \frac{\partial}{\partial y}
\]
Applying \( \frac{\partial}{\partial x} \) to the right side of (1.5) we get
\[
\sum_k (-1)^k (-k)F_y^{-k-1}F_{xy}I_{n,k} + \sum_k (-1)^k F_y^{-k} \frac{\partial}{\partial x} I_{n,k}
\]
\[
= \sum_j (-1)^{j+1} jF_y^{-(j+1)}F_{xy}I_{n,j} + \sum_j (-1)^{j} F_y^{-j} \frac{\partial}{\partial x} I_{n,j}
\]
\[
= \sum_k (-1)^k (k-1)F_y^{-k}F_{xy}I_{n,k-1} + \sum_k (-1)^k F_y^{-k} \frac{\partial}{\partial x} I_{n,k}.
\]
Applying \( \frac{F_x}{F_y} \frac{\partial}{\partial y} \) to the right side of (1.5) we get
\[
\frac{F_x}{F_y} \sum_k (-1)^k (-k)F_y^{-k-1}F_{yy}I_{n,k} - \frac{F_x}{F_y} \sum_k (-1)^k F_y^{-k} \frac{\partial}{\partial y} I_{n,k}
\]
\[
= \sum_j (-1)^{j+2} jF_y^{-(j+1)}F_x F_{yy}I_{n,j} + \sum_j (-1)^{j+1} F_y^{-j+1} F_x \frac{\partial}{\partial y} I_{n,j}
\]
\[
= \sum_k (-1)^k (k - 2)F_y^{-k}F_x F_{yy}I_{n,k-2} + \sum_k (-1)^k F_y^{-k} F_x \frac{\partial}{\partial y} I_{n,k-1}.
\]
Therefore, the derivative with respect to \( x \) of the right side of (1.5) is
\[
\sum_k (-1)^k F_y^{-k} \left[ (k - 1)F_{xy}I_{n,k-1} + \frac{\partial}{\partial x} I_{n,k} + (k - 2)F_x F_{yy}I_{n,k-2} + F_x \frac{\partial}{\partial y} I_{n,k-1} \right]
\]
which is
\[
\sum_k (-1)^k F_y^{-k} I_{n+1,k}
\]
by (1.3). Therefore, (1.5) holds for \( n+1 \) if it holds for \( n \), so Theorem 2 is true.

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