Burgers’ equation in 2D SU(N) YM.

H. Neuberger

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855, U.S.A

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Abstract

It is shown that the logarithmic derivative of the characteristic polynomial of a Wilson loop in two dimensional pure Yang Mills theory with gauge group SU(N) exactly satisfies Burgers’ equation, with viscosity given by $1/(2N)$. The Wilson loop does not intersect itself and Euclidean space-time is assumed flat and infinite. This result explicitly confirms in two dimensional YM the recent interpretation by Blaizot and Nowak of the field theoretic short to long scale transition as an onset of turbulent behavior.
1 Introduction.

Recent numerical work provides evidence that Wilson loops in $SU(N)$ gauge theory in two, three and four dimensions exhibit an infinite $N$ phase transition as they are dilated from a small size to a large one; in the course of this dilation the eigenvalue distribution of the untraced Wilson loop unitary matrix expands from a small arc on the unit circle to encompassing the entire unit circle [1, 2]. An analogous effect takes place in the two dimensional principal chiral model for $SU(N)$ [3].

The universality class of this transition is that of a random multiplicative ensemble of unitary matrices. The transition was discovered by Durhuus and Olesen [4] (DO) when they solved the Migdal-Makeenko [5] loop equations in two dimensional planar QCD. The associated multiplicative random matrix ensemble [6] can be axiomatized in the language of noncommutative probability [7]. It provides a generalization of the familiar law of large numbers. The essential feature making a difference is that one case is commutative and the other not. Various recent insights into the DO transition [8, 9, 10] point to possibly deeper interpretations of the transition.

In this note, motivated by a recent paper by Blaizot and Nowak [10], I present an exact map from the average characteristic polynomial associated with a Wilson loop to Burgers’ equation. This extends to finite $N$ the original work of DO at $N = \infty$, where the inviscid Burger’s equation plays a central role. The main observation is that all finite $N$ effects are exactly represented by reinstating a finite viscosity in Burgers’ equation, given by $\frac{1}{N}$. Positive $N$ gives positive viscosity, so the equation knows at least that $N$ should not be negative. I suspect that integral $N$’s are identified as special by a Mittag-Leffler [11] representation of the solution, stemming from a product representation of
the average characteristic polynomial, and depending also on the initial condition.

In addition to making the insight of [10] particularly transparent, I hope that this result would also aid future efforts to exploit large \( N \) universality in dimensions higher than two for obtaining analytical quantitative estimates of the ratio between a scale describing perturbative phenomena and the scale of confinement. This was the original motivation for seeking to establish numerically large \( N \) phase transitions in Wilson loops [1].

2 Characteristic polynomial.

An \( N \times N \) simple unitary Wilson loop matrix \( W \), defined on a curve that does not self intersect, with \( \tau \) denoting the dimensionless area in units of the ’t Hooft gauge coupling, has the following probability distribution:

\[
P_N(W, \tau)dW = \sum_R d_R \chi_R(W)e^{-\tau C_2(R)}dW \tag{1}
\]

The sum is over all irreducible representations \( R \) with character \( \chi_R(W) \) and second order Casimir \( C_2(R) \). \( dW \) is the Haar measure. Normalization conventions are standard [2] and \( \tau \geq 0 \). We introduce the average characteristic polynomial

\[
Q_N(z, \tau) = \langle \det(z - W) \rangle_{P_N(\tau)} \tag{2}
\]

One can think about \( Q_N(z, \tau) \) as the generating function for the \( \langle \chi_R(W) \rangle \) with totally antisymmetric \( R \). Simple manipulations [2] produce an integral representation:

\[
Q_N(z, \tau) = \sqrt{\frac{N\tau}{2\pi}} \int_{-\infty}^{\infty} du e^{-\frac{N}{2} \tau u^2} \left[ z - e^{-\tau(u+1/2)} \right]^N \tag{3}
\]

It is more convenient to study

\[
q_N(y, \tau) = (-1)^N e^{-\frac{N\tau}{2}} e^{-\frac{N\tau}{2} y^2} Q_N(-e^y, \tau) \tag{4}
\]

where, for the time being, \( y \) is kept real. \( q_N(y, t) \) is even in \( y \) and this is the main reason for extracting the exponential factor from \( Q_N \). Changing the integration variable \( u \) to \( x = y + \tau(u + 1/2) \) gives:

\[
q_N(y, \tau) = \sqrt{\frac{N}{2\pi \tau}} \int_{-\infty}^{\infty} dx e^{-\frac{N}{\tau}(y-x)^2} e^{N\log(2\cosh(x/2))} \tag{5}
\]

3 Derivation of equation.

It is now a trivial matter to observe that

\[
\frac{\partial q_N}{\partial \tau} = \frac{1}{2N} \frac{\partial^2 q_N}{\partial y^2} \tag{6}
\]
with initial condition
\[
\lim_{\tau \to 0} \left[ q_N(y, \tau) \right] = (2 \cosh(y/2))^N
\]  
(7)

The initial condition is a consequence of
\[
P_N(W, 0) = \delta(W, 1) \quad \text{with} \quad \int dW \delta(W, W_0) f(W) = f(W_0)
\]  
(8)

for any \( W_0 \in SU(N) \). This equation can be also directly derived from the polynomial formula of \( Q_N \), without going to the integral representation. This heat equation is related to Burgers’ equation (for example, see [12], problem 12(a), p. 214) by
\[
\phi_N(y, \tau) = -\frac{1}{N} \frac{\partial \log q_N(y, \tau)}{\partial y}
\]  
(9)

Burgers’ equation and the initial condition are
\[
\frac{\partial \phi_N}{\partial \tau} + \phi_N \frac{\partial \phi_N}{\partial y} = \frac{1}{2N} \frac{\partial^2 \phi_N}{\partial y^2}, \quad \phi_N(y, 0) = -\frac{1}{2} \tanh \frac{y}{2}
\]  
(10)

At \( N = \infty \), \( N \) drops out of the equation giving the inviscid limit:
\[
\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial y} = 0
\]  
(11)

The initial condition is \( N \) independent so we can drop the \( N \) subscript on \( \phi \) at \( N = \infty \). So long as \( \phi \) is uniquely defined, this is the point-wise \( N = \infty \) limit of \( \phi_N \).

The equation can be solved by the method of characteristics (for example, see [12], p. 16.) for an arbitrary initial condition
\[
\phi(y, 0) = h(y)
\]  
(12)

The solution is given implicitly by
\[
\phi(y, \tau) = h(y - \tau \phi(y, \tau))
\]  
(13)

This equation is known to produce a shock at a time \( \tau^* > 0 \) which is the first time at which multiple solutions become available. \( \tau^* \) is the smallest positive value satisfying
\[
\tau^* = -\frac{1}{(dh/dy)(y^*)} \quad \text{with} \quad (d^2 h/dy^2)(y^*) = 0
\]  
(14)

We are interested only in solutions odd in \( y \); hence, assuming \( h(y) \) to be smooth near \( y = 0 \) we expand:
\[
h(y) = ay + by^3 + cy^5 + \ldots
\]  
(15)

This implies that \( y^* = 0 \) and therefore
\[
\tau^* = -\frac{1}{a}
\]  
(16)
A shock will form if $a < 0$. In the case of $N = \infty$ 2D YM we have
\[ h(y) = -\frac{1}{2} \tanh \frac{y}{2} = -y/4 + y^3/48 - .... \] (17)
Therefore, the critical area corresponds to
\[ \tau^* = 4, \] (18)
the well known critical value \[4, 6\].

Universality can be invoked now in a sense that applies to the nonlinear equation producing a generic shock \[13, 14\]. This means taking the simplest polynomial $h(y)$ capable of producing shocks:
\[ h(y) = ay + by^3 \] (19)
with $a < 0$, $b > 0$. The $y$ location of the shock is at the origin, $y = y^* = 0$. Extending $h$ and $y$ to the complex plane provides a geometric view of this universality in terms of the structure of the evolving Riemann surface $y(\phi, \tau)$ parameterized by $\tau \geq 0$. One can also take $\tau$ into the complex plane.

4 Large $N = \text{small viscosity}$.

Making the viscosity nonzero is a singular perturbation which eliminates the shock and has the same effect as making $N$ finite. Large $N$ universality will hold in the vicinity of the critical area and corresponds to universal behavior in the vicinity of the would-be shock for small viscosities, which is the simplest dissipative regularization of the shock.

The important new insight is that the large $N$ transition is equivalent to a movable singularity, determined by the initial condition, rather than by the evolution rule. Thus, the simplest initial condition producing a shock will also lead to a universal small viscosity smoothing of the shock.

Running the derivation backwards, with the minimal initial condition
\[ h(y) = -y/4 + y^3/48 \] (20)
produces an integral representation on which a double scaling limit can be taken directly, exactly reproducing the limit used in matching to the large $N$ transitions in higher dimensions than two in \[1, 2\]. The critical exponents $\mu = 1/2, 3/4$ associated with the scalings $N^\mu$ that need to be taken \[2\] are identical to those found in defining the small viscosity limit \[10\]. The associated integral, studied in detail in \[2\] (see \[17\] for a plot), is related to Pearcey’s integral by a contour change, as indicated in \[10\].

The particular initial condition \[19\] has been analyzed in great detail in \[16\].

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1 The shock can be regulated also dispersively, in which case we could use a third derivative on the right hand side of the inviscid Burgers’ equation, producing the KdV equation. If there were a symmetry restricting to a Hamiltonian partial differential equations, this might have been the equation defining the universality class.

2 Something similar happens in the context of models consisting of one or several large matrices, where Painlevé equations enter (see for example \[5\] and \[15\]).
5 Higher critical points.

We have become accustomed to expect higher critical points, of reduced degrees of stability, to accompany a basic large $N$ critical point. Looking at (15) it seems plausible that setting $b = 0$ and making $c > 0$ would produce a critical point of one degree of stability less. Obviously, if this is true, a whole hierarchy will be generated, by initial conditions of the form $ay + by^{2m-1}$ with integer $m \geq 1$. If one is not worried about the convergence of the associated universal integrals and one is also willing to give up the $y \rightarrow -y$ parity symmetry, also higher critical points with half integer $m$ could be studied, at least as formal originators of asymptotic series.

It would be intriguing if parent models existed with physical symmetries that selected one of these higher critical points. More work on this is left for the future.

6 Product representation.

It certainly is true that

$$Q_N(z, \tau) = \langle \det(z - W) \rangle_{P_N(\tau)} = \prod_{i=1}^{N}(z - z_i(\tau))$$

(21)

One may view the $z_i(\tau)$ as certain averages of the eigenvalues of $W$, but not as usually defined:

$$\det(z - W) = \prod_{i=1}^{N}(z - \hat{z}_i(W)), \quad \hat{z}_i^{av}(\tau) = \langle \hat{z}_i(W) \rangle_{P_N(\tau)}$$

(22)

By large $N$ factorization, keeping away from $N = \infty$ critical points, one probably can identify identically ordered $\hat{z}_i^{av}(\tau)$’s and $z_i(\tau)$’s. I suspect that this stays true also in the double scaling limit. If this suspicion is validated, we shall obtain a new method to identify, using numerical simulations, the location and nature of the large $N$ transition in dimensions higher than two.

There is a finite number of $z_i$’s but the change of variables $z = -e^y$ produces an infinite number of pre-images in $y$-plane, once we take $y$ complex. In [2] it was proved that $|z_i(\tau)| = 1$ for all $i = 1, ..., N$ (see [17] for a plot). This means that all zeros in $y$ are on the imaginary axis. This remains true in the universal case, as pointed out in [16], on the basis of an old theorem [18]. Therefore the location of the zeros on the imaginary axis seems generic and well matched with the unitary character of the matrix $W$. For $q_N(y, \tau)$, the infinite product representation needs to be defined using well known methods of complex analysis (see for example [11]).

Consider now the equation governing the movement of the zeros $z_i(\tau)$ as $\tau$ varies in the YM case. Zeros of $q_N$ become poles of $\phi_N$; setting convergence issues aside (often, they do not present a problem), I now sketch a derivation of the equations governing the motion of the poles. The technique is quite well known.

The initial condition in the 2D YM case can be written as a sum over poles:

$$\phi_N(y, 0) = -2y \sum_{k=1}^{\infty} \frac{1}{((2k-1)\pi)^{2} + y^2}$$
\[ -\sum_{y}^{\infty} \left[ \frac{1}{y+(2k-1)\pi i} + \frac{1}{y-(2k-1)\pi i} \right] = -\sum_{\text{odd } k} \frac{1}{y+k\pi i} \]  

(23)

This leads to an ansatz of Calogero type:

\[ \phi_{N}(y, \tau) = -\frac{1}{N} \sum_{j} \frac{1}{y - y_{j}(\tau)} \]  

(24)

where the \(y_{j}(\tau)\) are complex in general, but, based on \[2\], actually purely imaginary. Plugging into the heat equation we get, denoting \(dy_{j}/d\tau = \dot{y}_{j}\)

\[ \sum_{j} \dot{y}_{j}(\tau) = -\frac{1}{2N} \sum_{j \neq i} \left( \frac{1}{y - y_{i}(\tau)} \cdot \frac{1}{y - y_{j}(\tau)} \right) \]  

(25)

Using

\[ \frac{1}{y - y_{i}(\tau)} \cdot \frac{1}{y - y_{j}(\tau)} = \left( \frac{1}{y - y_{i}(\tau)} - \frac{1}{y - y_{j}(\tau)} \right) \cdot \frac{1}{y_{i}(\tau) - y_{j}(\tau)} \]  

(26)

we get

\[ \dot{y}_{j}(\tau) = \frac{1}{N} \sum_{\{k; k \neq j\}} \frac{1}{y_{k}(\tau) - y_{j}(\tau)} \]  

(27)

The initial conditions is

\[ y_{j}(0) = (2j - 1)\pi i \]  

(28)

Each \(y_{j}(0)\) is \(N\)-fold degenerate, making the initial condition singular. For good initial conditions, this system determines the motion of the poles of \(\phi_{N}(y, \tau)\). Initially purely imaginary \(y_{j}(\tau)\) will stay imaginary as \(\tau\) varies. With our specific initial condition, the \(z_{i}(\tau) = -e^{y_{i}(\tau)}\) ought to spread out equally round the unit circle as \(\tau \to \infty\). I hope to work out more details about the motion of the poles some time in the future, drawing from the work in \[16\].

7 Large \(\tau\) behavior.

The regularization of the shock provides a smooth connection between small and large loops. In two dimensions Burgers’ equation provides an exact renormalization group type of equation allowing the evaluation of \(\phi_{N}(y, \tau)\) when \(\tau \to \infty\), given \(\phi_{N}(y, \tau)\) in the limit \(\tau \to 0\). The approach to the limit \(\tau \to \infty\) gives the dimensionless string tension associated with the dimensionless area \(\tau\). Here we only show how the correct \(\phi_{N}(y, \tau = \infty)\) is obtained. It is clear that \(Q_{N}(z, \tau = \infty) = z^{N} - 1\). This simply says that at infinite \(\tau\) all \(\langle W^{m} \rangle\) terms, for any \(m > 0\), can be replaced by zero.

Using \[14\], we conclude that the large \(\tau\) behavior is given by:

\[ \lim_{\tau \to \infty} \left( e^{\frac{N}{2} \pi} q_{N}(y, \tau) \right) = 2(-1)^{N} \sinh \frac{Ny}{2} \]  

(29)

We now wish to recover the ensuing \(\phi_{N}(y, \tau = \infty)\) from Burgers’ equation. The route is again in reverse of our derivation: First go to the heat equation, then get the
integral representation in order to incorporate the initial condition. Finally, in order to get the asymptotic behavior for large $\tau$, change variables in the integral representation, arriving at:

\[
\frac{1}{N} \partial_y \log q_N(y, \tau) = \frac{\int du e^{-\frac{Nu^2}{2}} \sinh((u\sqrt{\tau}+y)/2) (2 \cosh((u\sqrt{\tau}+y)/2))^{N-1}}{\int du e^{-\frac{Nu^2}{2}} (2 \cosh((u\sqrt{\tau}+y)/2))^N}
\]  

(30)

For large $\tau$, one of the two exponents making up each hyperbolic function dominates, depending on the sign of $u$:

\[
\lim_{\tau \to \infty} \left( \frac{1}{N} \partial_y \log q_N(y, \tau) \right) = \frac{1}{2} \lim_{\tau \to \infty} \left( \frac{\int du e^{-\frac{Nu^2}{2}} \varepsilon(u)e^{N[\varepsilon(u)(u\sqrt{\tau}+y)/2]}}{\int du e^{-\frac{Nu^2}{2}} e^{N[\varepsilon(u)(u\sqrt{\tau}+y)/2]}} \right)
\]

(31)

Here, $\varepsilon(u)$ is the sign function. The above equation implies that

\[
\phi_N(y, \infty) = \lim_{\tau \to \infty} \left( -\frac{1}{N} \partial_y \log q_N(y, \tau) \right) = -\frac{1}{2} \tanh \frac{Ny}{2}
\]

(32)

This is the expected result, showing that the roots $y_j(\infty)$ are non-degenerate.

At infinite $N$, the hyperbolic tangent becomes a sign function. In an electrostatic picture it is obvious that the above result holds if the poles of $\phi(y, \tau)$ are uniformly spaced and dense on the circle $|e^y| = 1$: Viewing the poles as charges, the jump $\varepsilon(y)$ comes from crossing the line charge at $z = -1$ as $y$ goes through zero along the real axis [2]. That the solution has this limiting behavior is essential for confinement, which would be indicated by the leading correction to the above result being exponentially small in $\tau$.

Note that $\tau$ was taken to infinity at finite $N$; the final result admits a subsequent infinite $N$ limit. Had we taken $N \to \infty$ first, we could have interpreted the shock, appearing first at $\tau = 4$, as a jump between two extremal solutions of the implicit equation defining the solution for $\tau < 4$. With the wrong initial conditions this jump might not grow to the full size required for consistency with confinement; thus, the transition in itself is insufficient to guarantee confinement. If we want to add the input that there is confinement we need to put a constraint on the initial condition. The mere onset of turbulent behavior does not guarantee that “maximal” turbulent behavior (total chaos in a statistical sense) will develop at asymptotic times. Again, I leave details for further work.

8 Discussion.

The primary objective of this paper was the derivation of (10) as an exact equation holding in two dimensional Yang Mills theory with gauge group $SU(N)$ defined on the
infinite Euclidean plane. A surprising simplicity in the area dependence of the average characteristic polynomial of simple Wilson loops was found. Nevertheless, the essential feature of the existence of a large $N$ phase transition is captured by this observable. In this respect the average characteristic polynomial of the Wilson loop is superior to traces of the Wilson loop in some fixed representation. As explained in [2] this observable has other advantages, in dimensions three and four.

The simple and exact finite $N$ relation to Burgers’ equation presented above seems to provide opportunity for progress in different directions, as emphasized in the course of this paper. The secondary objective of the paper was to present enough observations to convince the reader that there are many interesting issues left to explore. Last, but not least, the insight of Blaizot and Nowak [10] seems very promising both on the qualitative and on the quantitative level.

The shock at $\tau = 4$ is reminiscent of the possibility that instantons at infinite $N$ might herald, as $\tau \to 4^-$, a jump in certain particularly sensitive quantities in 4D YM [19].

It should also be mentioned that workers in lattice field theory [20] have shown numerically that in four dimensions the trace $2 \cos \theta$ of a Wilson loop for $SU(2)$ seems to evolve with the area as if $\theta$ were diffusing on the $SU(2)$ group manifold where the eigenvalues of $W$ are $e^{\pm i\theta}$. For $N = 2$ there is no essential distinction between the characteristic polynomial and any other gauge invariant observable related to the matrix $W$.

9 Added note.

Blaizot and Nowak [21] have independently identified the viscosity as $\frac{1}{N}$.

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