Multiphoton blockade in the two-photon Jaynes-Cummings model

Fen Zou,† Xiao-Ya Zhang,† Xun-Wei Xu,‡ Jin-Feng Huang,§,* and Jie-Qiao Liao,†

†Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China
‡Department of Applied Physics, East China Jiaotong University, Nanchang 330013, China

(Dated: November 12, 2019)

We study multiphoton blockade and photon-induced tunneling effects in the two-photon Jaynes-Cummings model, where a single-mode cavity field and a two-level atom are coupled via a nondipolar two-photon interaction. We consider both the cavity-field-driving and atom-driving cases, and find that the single-photon blockade and photon-induced tunneling effects can be observed when the cavity mode is driven, while the two-photon blockade effect appears when the atom is driven. For the atom-driving case (the two-photon physical transition process), we present a new criteria of the correlation functions for the multiphoton blockade effect. Specifically, we show that the quantum interference effect can enhance the conventional photon blockade in the cavity-field-driving case. Our results are confirmed by analytically and numerically calculating the second- and third-order correlation functions of the cavity-field mode. Our work has potential applications in quantum information processing and paves the way for the study of multiphoton quantum coherent devices.

I. INTRODUCTION

The photon blockade (PB) effect [1], as a typical photon correlation phenomenon, not only has significance in the study of the fundamentals of quantum optics, but also possesses wide application potential in modern quantum devices and quantum information science. So far, there exist two kinds of PB: the conventional photon blockade (CPB) and unconventional photon blockade (UPB), which are based on different physical mechanisms. The former is caused by the nonlinearity in the energy spectrum, while the latter is induced by the quantum interference effect between different transition channels. In CPB, the capture of a single photon in a nonlinear system blocks the excitation of the second and subsequent photons. Thus, a sequence of single photons can be generated and such systems can be implemented as single-photon source devices. In this sense, PB can change a classical light field into a nonclassical light field. In general, the signatures of PB can be observed from photon antibunching and sub-Poissonian photon-number statistics.

In recent years, great advances have been made in the topic of PB. On one hand, the CPB effect has been theoretically investigated in a variety of quantum systems, e.g., cavity quantum electrodynamics (QED) systems [2–15], circuit-QED systems [16–18], the Kerr-type nonlinear cavities [1, 19–22], optomechanical systems [23–29], and other systems [30–34]. The CPB effect has also been experimentally demonstrated with a single atom trapped in an optical cavity [3], a quantum dot in a photonic crystal cavity [4], and a single superconducting artificial atom coupled to a microwave transmission-line resonator [16, 17]. On the other hand, the UPB effect has been theoretically studied in the coupled Kerr-type nonlinear cavities [35–41], cavity-QED systems [42–44], coupled optomechanical systems [45–47], and other systems [48–52]. The UPB effect has also been experimentally demonstrated in an optical microcavity coupled to a single semiconductor quantum dot [53] and in a superconducting circuit consisting of two coupled resonators [54].

Previously, the studies on PB are mainly aimed at the single-photon blockade (1PB). Most recently, the two-photon blockade (2PB) has been experimentally [55] and theoretically [56–67] investigated in various configurations. The 2PB means that the resonance absorption of two photons in a nonlinear system will suppress the transmission of the subsequent photons. Such systems with 2PB can be used for two-photon source devices. In addition, the photon-induced tunneling (PIT) with photon bunching has also been explored in a photonic crystal cavity coupled to a quantum dot [4, 68–70], optomechanical systems [71], and other systems [63, 67], i.e., the absorption of the first photon favors that of the second or subsequent photons. PIT has been observed experimentally in Refs. [4, 68, 70].

Based on the physical picture of multiphoton blockade, a nature question is: what is the influence of the multiphoton physical transition processes on the multiphoton blockade effect? To study this question, in this work we propose to study the multiphoton blockade effect in the two-photon Jaynes-Cummings (JC) model [72–76], which describes the nondipolar two-photon interaction of a single bosonic mode with a two-level system. This model has become an interesting and important research topic in quantum optics and quantum information sciences [77–82]. The strong nonlinearity induced by the nondipolar interaction gives rise to many important quantum effects at the level of few photons. We will consider the PB effects in this system by driving either the cavity field or the atom. When the cavity mode is
II. MODEL

We consider a two-photon JC model [Fig. 1(a)], which is composed of a single-mode cavity field coupled to a two-level atom via a nondipolar two-photon physical interaction [82]. The Hamiltonian of the two-photon JC model reads ($\hbar = 1$)

$$\hat{H}_{2pJC} = \omega_c \hat{a}^\dagger \hat{a} + \omega_0 \hat{\sigma}_+ \hat{\sigma}_- + J \hat{\sigma}_- \hat{\sigma}_+ \hat{a} \hat{a}^\dagger,$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the creation and annihilation operators of the single-mode cavity field with the resonance frequency $\omega_c$. The operators $\hat{\sigma}_+ = |e\rangle \langle g|$ and $\hat{\sigma}_- = |g\rangle \langle e|$ are the raising and lowering operators of the two-level atom with an energy separation $\omega_0$ between the excited state $|e\rangle$ and ground state $|g\rangle$. The last term in Eq. (1) denotes the nondipolar two-photon-process interaction between the cavity field and the two-level atom with the coupling strength $J$. Note that the two-photon JC model can be implemented with superconducting circuits [82].

It is generally believed that the CPB effect is caused by the anharmonicity of the eigenenergy spectrum. To study the PB effect in the two-photon JC model, in the following we calculate its eigensystem and analyze its energy spectrum. In the two-photon JC model, the weighted excitation number operator $\hat{N} = 2\hat{\sigma}_+ \hat{\sigma}_- + \hat{a} \hat{a}^\dagger$ is a conserved quantity due to the commutative relation $[\hat{N}, \hat{H}_{2pJC}] = 0$.

The subspace corresponding to the weighted excitation number $N = 0, 1, 2, 3, \ldots, n, \ldots$ is spanned over the basis states $\{|g,0\rangle, \{|g,1\rangle, \{|g,2\rangle, \langle e, 0\rangle, \{|g,3\rangle, \langle e, 1\rangle, \ldots, |g,n\rangle, \langle e, n\rangle, \langle e, n-2\rangle, \ldots\}$, where $|n\rangle (n = 0, 1, 2, \ldots)$ denotes the number states of the cavity-field mode.

In the zero-excitation subspace, the eigensystem can be obtained as $\hat{H}_{2pJC} |\varepsilon_0\rangle = \varepsilon_0 |\varepsilon_0\rangle$ with the eigenstate $|\varepsilon_0\rangle = |g,0\rangle$ and the eigenvalue $\varepsilon_0 = 0$. In the one-excitation subspace, the eigensystem can be written as $\hat{H}_{2pJC} |\varepsilon_1\rangle = \varepsilon_1 |\varepsilon_1\rangle$ with the eigenstate $|\varepsilon_1\rangle = |g,1\rangle$ and the eigenvalue $\varepsilon_1 = \omega_c$. In the n-excitation $(n \geq 2)$ subspace, the eigensystem can be expressed as $\hat{H}_{2pJC} |\varepsilon_{n\pm}\rangle = \varepsilon_{n\pm} |\varepsilon_{n\pm}\rangle$, where the eigenvalues and eigenstates are, respectively, defined by

$$\varepsilon_{n\pm} = 2(n-1)\omega_c + \omega_0 \pm \sqrt{(2\omega_c - \omega_0)^2 + 4n(n-1)J^2} \frac{2}{2},$$

and

$$|\varepsilon_{n+}\rangle = C_{g,n}^\dagger |g,n\rangle + C_{e,n-2}^\dagger |e, n-2\rangle, \quad (3a)$$

$$|\varepsilon_{n-}\rangle = C_{g,n}^\dagger |g,n\rangle + C_{e,n-2}^\dagger |e, n-2\rangle. \quad (3b)$$

The superposition coefficients in Eq. (3) are given by

$$C_{g,n}^\dagger = C_{e,n-2} = \cos \theta_n, \quad (4a)$$

$$C_{g,n}^\dagger = -C_{e,n-2} = \sin \theta_n, \quad (4b)$$

with the mixing angle $\theta_n$ defined by

$$\tan(2\theta_n) = \frac{2\sqrt{n(n-1)J}}{2\omega_c - \omega_0} \quad (5)$$

The rest of this paper is organized as follows. In Sec. II, we introduce the two-photon JC model. In Sec. III, we present the criteria of the nPB and PB effects. In Secs. IV and V, we study the photon blockade effects in the cavity-field-driving and atom-driving cases, respectively. Conclusions will be given in Sec. VI.
In the resonant case $\omega_0 = 2\omega_c$, the eigenvalues and eigenstates of the system are reduced as $\varepsilon_{n\pm} = n\omega_c \pm \sqrt{n(n-1)}J$ and $|\varepsilon_{n\pm}\rangle = (|g, n\rangle + |e, n-2\rangle)/\sqrt{2}$, respectively.

For studying the 1PB and 2PB effects, we consider the weak-driving case in which the Hilbert space of the cavity field can be truncated up to $n = 4$. Figure 1(b) shows the eigenenergy spectrum of the Hamiltonian $H_{2pJC}$ versus the atomic frequency $\omega_0$ in units of the cavity-field frequency $\omega_c$ in the subspace associated with zero, one, two, three, and four photons for $J/\omega_c = 0.08$. Obviously, the eigenenergy spectrum of the system is anharmonic in the vicinity of the resonance point ($\omega_0 \approx 2\omega_c$), which means that the PB effect is more evident around the resonance point. In Fig. 1(c), we show the eigenenergy spectrum of the Hamiltonian $H_{2pJC}$ in the resonant case $\omega_0 = 2\omega_c$.

Below, we study the 1PB and 2PB effects in this system by driving either the cavity, $\hat{H}_d = \Omega (\hat{a} e^{-i\omega_t t} + \hat{a}^\dagger e^{i\omega_t t})$, or the atom, $H'_d = \Omega_L (\hat{\sigma}_+ e^{-i\omega_t t} + \hat{\sigma}_- e^{i\omega_t t})$. Here $\Omega$ ($\Omega_L$) and $\omega_d$ ($\omega_c$) are the driving strength and driving frequency of the cavity field (atom), respectively. When the driving frequency $\omega_d$ matches the eigenenergy separation $\omega_c$ between the first excited state $|\varepsilon_1\rangle$ and the ground state $|\varepsilon_0\rangle$, the single-photon transition ($|\varepsilon_0\rangle \rightarrow |\varepsilon_1\rangle$) becomes resonant, but the subsequent transitions ($|\varepsilon_1\rangle \rightarrow |\varepsilon_{1\pm}\rangle$) induced by the second photon are blocked due to the anharmonicity of the eigenenergy spectrum. This indicates that the 1PB effect can occur in this system. Similarly, when the driving frequency $2\omega_d$ ($\omega_L$) matches the energy-level differences $2\omega_c \pm \sqrt{2}J$ between $|\varepsilon_{2\pm}\rangle$ and $|\varepsilon_0\rangle$, the two-photon transitions ($|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$) become resonant, while the subsequent transitions ($|\varepsilon_{2\pm}\rangle \rightarrow |\varepsilon_{3\pm}\rangle$) are blocked, i.e., the 2PB effect can be observed in this system.

In the weak-driving case, the transition amplitude between two states is proportional to the ratio of the transition matrix element over the transition detuning. The transition behavior in this model induced by the driving terms can be analyzed by calculating the transition amplitudes between the involved energy levels in the eigenstate representation. In Figs. 1(d) and 1(e), we show the transition matrix elements between different energy levels in the resonant case $\omega_0 = 2\omega_c$ when the cavity field and the atom are driven, respectively. In the following we will discuss the details of the corresponding transition matrix elements in these two cases.

When driving the cavity mode, the transitions can occur between different energy levels in the neighboring subspaces. For the transitions $|\varepsilon_0\rangle \rightarrow |\varepsilon_1\rangle$ and $|\varepsilon_1\rangle \rightarrow |\varepsilon_{2\pm}\rangle$, the transition matrix elements are, respectively, $T_{|\varepsilon_0\rangle \rightarrow \varepsilon_1\rangle} = \Omega (\varepsilon_1 a_0^\dagger \varepsilon_0 = \Omega$ and $T_{|\varepsilon_1\rangle \rightarrow \varepsilon_{2\pm}\rangle} = \Omega (\varepsilon_2 a_1^\dagger \varepsilon_{2\pm} = \Omega$, with the corresponding energy separations $\omega_c$ and $\omega_c \pm 2\sqrt{2}J$. For the transitions from the $n$-excitation ($n \geq 2$) to the $(n+1)$-excitation subspaces, the transition matrix elements can be obtained by [55]

\[
T_{|\varepsilon_{n+}\rangle \rightarrow |\varepsilon_{(n+1)\pm}\rangle} = \Omega (\varepsilon_{(n+1)\pm} a_1^\dagger \varepsilon_{n+}) = \\
= \Omega (\sqrt{n+1} \pm \sqrt{n-1})/2, \quad (6a)
\]

\[
T_{|\varepsilon_{n-}\rangle \rightarrow |\varepsilon_{(n+1)\pm}\rangle} = \Omega (\varepsilon_{(n+1)\pm} a_1^\dagger \varepsilon_{n-}) = \\
= \Omega (\sqrt{n+1} \mp \sqrt{n-1})/2, \quad (6b)
\]

where the eigenstates of the system are $|\varepsilon_{n\pm}\rangle = (|g, n\rangle + |e, n-2\rangle)/\sqrt{2}$ [Fig. 1(d)]. The corresponding energy separations between these states are

\[
\varepsilon_{(n+1)\pm} - \varepsilon_{n+} = \omega_c \pm \sqrt{(n+1)J} - \sqrt{n(n-1)J}, \quad (7a)
\]

\[
\varepsilon_{(n+1)\pm} - \varepsilon_{n-} = \omega_c \pm \sqrt{(n+1)J} + \sqrt{n(n-1)J}. \quad (7b)
\]

Based on the energy separations, we can calculate the driving detunings in the cavity-field-driving case.

In the atom-driving case, the transitions from the $n$-excitation ($n \geq 0$) subspace to the $(n+2)$-excitation subspaces can occur, while the transitions between different energy levels of the neighboring subspaces are forbidden. Hence, the 1PB effect cannot be observed because the transition $|\varepsilon_0\rangle \rightarrow |\varepsilon_1\rangle$ is forbidden, i.e., $T_{|\varepsilon_0\rangle \rightarrow |\varepsilon_1\rangle} = \Omega_L (\varepsilon_1 a_0^\dagger |\varepsilon_0\rangle = 0$. For the transitions $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$, the corresponding transition amplitudes can be obtained by calculating the transition matrix elements $T_{|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle} = \Omega_L (\varepsilon_{2\pm} a_0^\dagger |\varepsilon_0\rangle = \pm \Omega_L/\sqrt{2}$. The energy separations between the state $|\varepsilon_0\rangle$ and states $|\varepsilon_{2\pm}\rangle$ are $2\omega_c \pm \sqrt{2}J$. For the transitions from the $n$-excitation ($n \geq 2$) subspace to the $(n+2)$-excitation subspace, the corresponding transition amplitudes can be obtained by

\[
T_{|\varepsilon_{n+}\rangle \rightarrow |\varepsilon_{(n+2)\pm}\rangle} = \Omega_L (\varepsilon_{(n+2)\pm} a_0^\dagger |\varepsilon_{n+}\rangle = \pm \Omega_L/2, \quad (8a)
\]

\[
T_{|\varepsilon_{n-}\rangle \rightarrow |\varepsilon_{(n+2)\pm}\rangle} = \Omega_L (\varepsilon_{(n+2)\pm} a_0^\dagger |\varepsilon_{n-}\rangle = \pm \Omega_L/2, \quad (8b)
\]

which indicate that all transitions have equal transition amplitudes $\Omega_L/2$ [Fig. 1(e)]. The corresponding energy separations are

\[
\varepsilon_{(n+2)\pm} - \varepsilon_{n+} = 2\omega_c \pm \sqrt{(n+2)(n+1)J} - \sqrt{n(n-1)J}, \quad (9a)
\]

\[
\varepsilon_{(n+2)\pm} - \varepsilon_{n-} = 2\omega_c \pm \sqrt{(n+2)(n+1)J} + \sqrt{n(n-1)J}. \quad (9b)
\]

Similar to the cavity-field-driving case, the driving detuning in the atom-driving case can be analyzed by comparing the energy separation between these states and the resonance frequency of the driving light. In our following discussions, we will analyze the locations of the peaks and dips in these correlation functions by calculating the resonance conditions for single- and multi-photon transitions.

III. CRITERIA OF THE nPB AND PIT EFFECTS

The physical picture of the $n$PB and PIT effects can be explained by analyzing the photon-number distribution.
$P_n \equiv \langle n \rangle \langle n \rangle$ and the equal-time $n$th-order correlation function $g^{(n)}(0) \equiv \langle \hat{a}^n \hat{a}^n \rangle / \langle \hat{n} \rangle^n$, with $\hat{n} = \hat{a}^\dagger \hat{a}$ being the photon number operator. For an ideal nPB, the absorption of the first $n$ photons blocks the entrance of the subsequent photons. Therefore, the photon-number distributions corresponding to a perfect nPB satisfy [55]

$$P_m = 0, \quad \text{for } m > n, \quad (10a)$$

$$P_n \neq 0, \quad (10b)$$

with the normalization condition $\sum_{n=0}^{\infty} P_n = 1$. The condition of this perfect nPB is hard to achieve in experiments. In order to observe the nPB effect, Hamsen et al. [55] proposed two criteria. The first criterion is based on a comparison between the photon-number distributions and the Poisson distributions of a coherent state. In this case, the criterion is defined by

$$P_m < P_m', \quad \text{for } m > n, \quad (11a)$$

$$P_n \geq P_n', \quad (11b)$$

where $P_n$ are the Poisson distributions defined by

$$P_n = \frac{\langle \hat{n} \rangle^n}{n!} \exp(-\langle \hat{n} \rangle), \quad (12)$$

with $\langle \hat{n} \rangle$ being the average photon number. Equation (11) indicates that the probability of $n$ photons is enhanced and the probabilities of other photon numbers ($> n$) are suppressed for the nPB effect. The other criterion is based on the equal-time $n$th-order correlation function $g^{(n)}(0)$. In the case of weak driving, the mean photon number is very small, i.e., $\langle \hat{n} \rangle \ll 1$ The criteria of the correlation functions for the nPB effect are [55, 63]

$$g^{(n)}(0) \geq 1, \quad (13a)$$

$$g^{(n+1)}(0) < 1, \quad (13b)$$

which means the $n$th-order super-Poissonian photon statistics or Poisson photon statistics, and the $(n + 1)$th-order sub-Poissonian photon statistics. For instance, the correlation functions $g^{(2)}(0) \geq 1$ and $g^{(3)}(0) < 1$ are satisfied for the 2PB effect. The correlation function $g^{(2)}(0) \ll 1$ is a signature of the 1PB effect.

On the other hand, for PIT, the absorption of the first photon favors that of the second or subsequent photons, so the PIT effect is usually characterized by the super-Poissonian photon statistics. Obviously, the process of PIT is inverse to the PB. Therefore, we refer to PIT if the $n$th-order correlation functions $g^{(n)}(0) > 1$ $(n = 2, 3)$ are satisfied in the weak-driving case [63]. Note that the criteria of PIT have been analyzed more detailed in Refs. [4, 67–71].

It should be mentioned that the criteria of the nPB in Eq. (13) and PIT are mainly used for the single-photon physical transition process. In the two-photon JC model, the single-photon physical transition process occurs when the cavity field is driven, while the two-photon physical transition processes, namely the creation or annihilation of two photons, happen when driving the atom. Hence, we propose that the criteria of the correlation functions for the nPB effect in the two-photon physical transition process should be

$$g^{(n)}(0) \geq 1, \quad (14a)$$

$$g^{(n+1)}(0) < 1, \quad (14b)$$

$$g^{(n+2)}(0) < 1. \quad (14c)$$

For instance, in the atom-driving case, the correlation functions $g^{(2)}(0) \geq 1$ and $g^{(n)}(0) < 1$ $(n = 3, 4)$ are satisfied for the 2PB effect, and the PIT effect can be characterized by the conditions of $g^{(n)}(0) > 1$ $(n = 2, 3, 4)$.

### IV. PB IN THE CAVITY-FIELD-DRIVING CASE

In this section, we study the PB effect by analytically and numerically calculating the second- and third-order correlation functions of the cavity mode in the cavity-field-driving case.

#### A. Analytical results

When the cavity field is continuously driven by a monochromatic weak field, the driving Hamiltonian is described by

$$\hat{H}_d = \Omega (\hat{a}\hat{e}^{-i\omega_dt} + \hat{a}^\dagger e^{i\omega_dt}), \quad (15)$$

where $\Omega$ and $\omega_d$ are the driving strength and driving frequency, respectively. Then the total Hamiltonian of the system becomes

$$\hat{H}_{\text{sys}} = \hat{H}_{2\text{pJC}} + \hat{H}_d. \quad (16)$$

In a rotating frame defined by the unitary operator $\exp[-i\omega_d(\hat{a}\hat{a}^\dagger + \hat{\sigma}_z)t]$, the Hamiltonian of the system becomes

$$\hat{H}_{\text{sys}}^{(f)} = \hat{H}_{2\text{pJC}}^{(f)} + \Omega(\hat{a}\hat{a}^\dagger + \hat{\sigma}_z), \quad (17)$$

with

$$\hat{H}_{2\text{pJC}}^{(f)} = \Delta_c \hat{a}^\dagger \hat{a} + \Delta_0 \hat{\sigma}_+ \hat{\sigma}_- + J(\hat{a} \hat{\sigma}_+ + \hat{\sigma}_- \hat{a}^\dagger), \quad (18)$$

where $\Delta_c = \omega_c - \omega_d$ $(\Delta_0 = \omega_0 - 2\omega_d)$ is the detuning of the cavity-field (atomic) frequency with respect to the driving frequency. In the low-excitation subspace, the Hamiltonians $\hat{H}_{2\text{pJC}}^{(f)}$ and $\hat{H}_{2\text{pJC}}^{(0)}$ have the same eigenstates, but the eigenvalues should be replaced by $\varepsilon_0 = 0$, $\varepsilon_1 = \Delta_c$, and $\varepsilon_{2\pm} = [\Delta_0 + \Delta_c \pm \sqrt{(2\Delta_c - \Delta_0)^2 + 8J^2}]^2 / 2$.

To include the influence of the dissipations of the cavity field and the atom on the PB effect, we phenomenologically add the imaginary dissipation terms into Hamiltonian (17) as follows

$$\hat{H}_{\text{eff}} = (\Delta_c - i\kappa/2)\hat{a}^\dagger \hat{a} + (\Delta_0 - i\gamma/2)\hat{\sigma}_+ \hat{\sigma}_- + J(\hat{a} \hat{\sigma}_+ + \hat{\sigma}_- \hat{a}^\dagger) + \Omega(\hat{a}^\dagger + \hat{a}), \quad (19)$$
where we have assumed that the cavity field and the atom are connected with two individual vacuum reservoirs, with \( \kappa \) and \( \gamma \) being the corresponding decay rates.

In the weak-driving regime (\( \Omega \ll \kappa \)), we truncate the Hilbert space of the cavity field up to \( n = 3 \). In this subspace, a general state of the system can be written as

\[
|\psi(t)\rangle = C_{g0}(t)|g,0\rangle + C_{g1}(t)|g,1\rangle + C_{g2}(t)|g,2\rangle + C_{e0}(t)|e,0\rangle + C_{g3}(t)|g,3\rangle + C_{e1}(t)|e,1\rangle,
\]

where the coefficients \( C_{s,j}(t) \) (\( s = g, e \) and \( j = 0, 1, 2, 3 \)) are the probability amplitudes. Based on the Schrödinger equation \( i|\psi(t)\rangle = H_{\text{eff}}|\psi(t)\rangle \), we obtain the equations of motion for these probability amplitudes as

\[
i\dot{C}_{g0} = \Omega C_{g1},
\]

\[
i\dot{C}_{g1} = \Omega C_{g0} + \left( \Delta_c - i\frac{\kappa}{2} \right) C_{g1} + \sqrt{2}\Omega C_{g2},
\]

\[
i\dot{C}_{g2} = \sqrt{2}\Omega C_{g1} + (2\Delta_c - i\kappa)C_{g2} + \sqrt{2}JC_{e0} + \sqrt{3}\Omega C_{g3},
\]

\[
i\dot{C}_{e0} = \sqrt{2}JC_{g2} + \left( \Delta_0 - i\frac{\gamma}{2} \right) C_{e0} + \Omega C_{e1},
\]

\[
i\dot{C}_{g3} = \sqrt{3}\Omega C_{g2} + 3 \left( \Delta_c - i\frac{\kappa}{2} \right) C_{g3} + \sqrt{6}JC_{e1},
\]

\[
i\dot{C}_{e1} = \Omega C_{e0} + \sqrt{6}JC_{g3} + \left( \Delta_c + \Delta_0 - i\frac{\kappa + \gamma}{2} \right) C_{e1}.
\]

Equation (21) can be approximately solved by using a perturbation method. Under the weak-driving condition (\( \Omega \ll \kappa \)), we have the approximate scales \( C_{g0} \sim 1, C_{g1} \sim \Omega/\kappa, \{C_{g2}, C_{e0}\} \sim \Omega^2/\kappa^2 \), and \( \{C_{g3}, C_{e1}\} \sim \Omega^3/\kappa^3 \), i.e., \( C_{g0} \gg C_{g1} \gg \{C_{g2}, C_{e0}\} \gg \{C_{g3}, C_{e1}\} \). To approximately solve Eq. (21), we discard the higher-order terms in the equations of motion for the lower-order variables. Then the equations of motion for these probability amplitudes become

\[
i\dot{C}_{g0} \approx 0,
\]

\[
i\dot{C}_{g1} \approx \Omega C_{g0} + \left( \Delta_c - i\frac{\kappa}{2} \right) C_{g1},
\]

\[
i\dot{C}_{g2} \approx \sqrt{2}\Omega C_{g1} + (2\Delta_c - i\kappa)C_{g2} + \sqrt{2}JC_{e0},
\]

\[
i\dot{C}_{e0} \approx \sqrt{2}JC_{g2} + \left( \Delta_0 - i\frac{\gamma}{2} \right) C_{e0},
\]

\[
i\dot{C}_{g3} \approx \sqrt{3}\Omega C_{g2} + 3 \left( \Delta_c - i\frac{\kappa}{2} \right) C_{g3} + \sqrt{6}JC_{e1},
\]

\[
i\dot{C}_{e1} \approx \Omega C_{e0} + \sqrt{6}JC_{g3} + \left( \Delta_c + \Delta_0 - i\frac{\kappa + \gamma}{2} \right) C_{e1}.
\]

We assume \( C_{g0}(0) = 1 \), then the steady-state solution of Eq. (22) can be obtained by setting \( \partial C_{s,j}/\partial t = 0 \) as

\[
C_{g0} = 1,
\]

\[
C_{g1} = -\frac{2\Omega}{2\Delta_c - i\kappa},
\]

\[
C_{g2} = 2\sqrt{2}(\gamma + 2i\Delta_0)\Omega^2 W^{-1},
\]

\[
C_{e0} = 8J\Omega^2 W^{-1},
\]

\[
C_{g3} = -\frac{4\sqrt{6}[J^2 - (\gamma + 2i\Delta_0)V][\Omega^3]}{2W[8J^2 + (2\Delta_c + \kappa)V]},
\]

\[
C_{e1} = -\frac{i16JV\Omega^3}{W[8J^2 + (2\Delta_c + \kappa)V]},
\]

where we introduce the variables

\[
W = (2\Delta_c - i\kappa)[4J^2 + (\gamma + 2i\Delta_0)(2\Delta_c + \kappa)],
\]

\[
V = \gamma + 2i(\Delta_0 + \Delta_c) + \kappa.
\]

Based on Eq. (23), we obtain the steady state of the system, then the equal-time second- and third-order correlation functions can be expressed as

\[
g^{(2)}(0) \equiv \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^2} = \frac{2P_2 + 6P_3}{(P_1 + 2P_2 + 3P_3)^2} \approx \frac{2P_2}{P_1^2},
\]

and

\[
g^{(3)}(0) \equiv \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} \rangle}{\langle \hat{a}^\dagger \hat{a} \rangle^3} = \frac{6P_3}{(P_1 + 2P_2 + 3P_3)^3} \approx \frac{6P_3}{P_1^3},
\]

where the photon-number distributions are given by

\[
P_0 = \langle |C_{g,0}|^2 + |C_{e,0}|^2 \rangle / N,
\]

\[
P_1 = \langle |C_{g,1}|^2 + |C_{e,1}|^2 \rangle / N,
\]

\[
P_2 = \langle |C_{g,2}|^2 / N, \rangle
\]

\[
P_3 = \langle |C_{g,3}|^2 / N, \rangle
\]

with the normalization constant

\[
N = |C_{g,0}|^2 + |C_{g,1}|^2 + |C_{g,2}|^2 + |C_{e,0}|^2 + |C_{g,3}|^2 + |C_{e,1}|^2.
\]

For the weak-driving case, this normalization constant can be omitted because of \( N \approx 1 \).

The condition \( g^{(2)}(0) < 1 \) [\( g^{(2)}(0) > 1 \)] indicates the sub-Poissonian (super-Poissonian) photon statistics. In particular, the 1PB effect can be observed when the two-photon probability is significantly suppressed in the system. It follows from the relation \( g^{(2)}(0) \approx 2P_2/P_1^2 \) that the correlation function \( g^{(2)}(0) \ll 1 \) is a signature of the 1PB effect. In this low-excitation subspace, a case corresponding to a perfect photon blockade is \( C_{g2} = 0 \), which means that there are no two-photon probability in the cavity. The parameter condition for this perfect 1PB effect can be obtained as

\[
\Delta_0 = 0, \quad \gamma = 0.
\]

In the bare-state representation, there is only a path leading to the two-photon excitation \( |g,0\rangle \xrightarrow{\Omega} |g,1\rangle \xrightarrow{\Omega} \cdots \).
of the two different paths ($|\epsilon_0\rangle \rightarrow |\epsilon_{2\pm}\rangle$) [34]. The detailed analysis of the destructive quantum interference will be given in the next section.

In the 2PB case, the resonant absorption of two photons will suppress the absorption of the third or subsequent photons. Hence, the 2PB effect is characterized by the correlation functions $g^{(2)}(0)$ and $g^{(3)}(0)$

\begin{align}
g^{(2)}(0) & \geq 1, \\
g^{(3)}(0) & < 1,
\end{align}

which imply the second-order super-Poissonian photon statistics or Poisson photon statistics, and the third-order sub-Poissonian photon statistics.

### B. Numerical results

In order to confirm our analytical results, we numerically calculate the equal-time second- and third-order correlation functions of the cavity mode. Numerical computations were performed using the Python package QuTiP [83, 84]. We assume that the cavity and the two-level atom are connected with two individual vacuum baths. Then the dynamics of the system is governed by the quantum master equation

$$\frac{d\hat{\rho}(t)}{dt} = i[\hat{\rho}(t), \hat{H}_{\text{sys}}] + \frac{\kappa}{2}[2\hat{\sigma}\hat{\rho}(t)\hat{\sigma}^\dagger - \hat{\sigma}^\dagger\hat{\sigma}\hat{\rho}(t) - \hat{\rho}(t)\hat{\sigma}^\dagger\hat{\sigma}],$$

where $\hat{\rho}(t)$ is the density matrix of the system, $\hat{H}_{\text{sys}}$ is the Hamiltonian defined in Eq. (17), and $\kappa$ ($\gamma$) is the decay rate of the cavity (atom).

By numerically solving Eq. (31), we can get the steady-state density operator $\hat{\rho}_{ss}$ of the system, and then the photon-number distributions $P_{n=0,1,2,3} = \text{Tr}[n|\hat{\rho}_{ss}\rangle\langle n|$ can be calculated. The second- and third-order correlation functions can also be obtained by $g^{(2)}(0) = \text{Tr}((\hat{a}^2)^2\hat{\rho}_{ss})/\langle \text{Tr}(\hat{a}\hat{a}^\dagger)^2 \rangle_0^2$ and $g^{(3)}(0) = \text{Tr}((\hat{a}^3)^2\hat{\rho}_{ss})/\langle \text{Tr}(\hat{a}\hat{a}^\dagger)^3 \rangle_0^2$, respectively.

For studying the PB effect in this model, we consider both the resonant ($\omega_0 = 2\omega_c$) and non-resonant ($\omega_0 \neq 2\omega_c$) cases. In Fig. 2(a), we plot the photon-number distributions $P_{n=0,1,2,3}$ as functions of the cavity-field driving frequency $\omega_d/\omega_c$ in the resonant case $\omega_0 = 2\omega_c$. The colored solid curves are plotted based on the analytical results given in Eq. (27). We see in Fig. 2(a) that the relations $P_0 \approx 1$ and $P_0 > P_1 > P_2 > P_3$ in the weak-driving case. In addition, there is a peak located at $\omega_d/\omega_c = 1$ in the curve of the single-photon probability $P_1$ (green solid curve), while there are a dip and two peaks in the curve of the two-photon probability $P_2$ (red solid curve), with the locations $\omega_d/\omega_c = 1$ and $1 \pm J/\sqrt{2}\omega_c$, respectively. By analyzing the energy spectrum of this system, we find that the locations of these peaks in the curves of $P_1$ and $P_2$ are determined by the single- and two-photon resonance transitions $|\epsilon_0\rangle \rightarrow |\epsilon_1\rangle$ and $|\epsilon_0\rangle \rightarrow |\epsilon_{2\pm}\rangle$, respectively. To be clearer, we mark these peaks in the curves of $P_1$ and $P_2$ as $p_{0.1}$ and $p_{0.2\pm}$. In the curve of the three-photon probability $P_3$ (yellow solid curve), we see that there are five peaks located at $\omega_d/\omega_c = 1, 1 \pm J/(\sqrt{2}\omega_c)$, and $1 \pm \sqrt{6}J/(3\omega_c)$, respectively. The locations of the two main peaks $p_{0.3\pm}$ are determined by the three-photon resonance transitions $|\epsilon_0\rangle \rightarrow |\epsilon_{3\pm}\rangle$, while the rest three peaks are induced by the single- and two-photon resonance transitions, and hence the locations of the three peaks are the same as those of the three peaks in the curves of $P_1$ and $P_2$. The dip in the curve of $P_2$ can be explained by the destructive quantum interference between the two different paths ($|\epsilon_0\rangle \rightarrow |\epsilon_{2\pm}\rangle$) of the two-photon excitation. Thus, we mark this dip as $d_{\text{int,2}}$. To prove this point, in the
following we will present a detailed analysis on the influence of quantum interference effect in the eigenstate representation on the photon-number distributions.

In the eigenstate representation, a general pure state of the system in the low-excitation subspace can be expressed as

$$\Psi(t) = D_0(t)|\epsilon_0\rangle + D_1(t)|\epsilon_1\rangle + D_{-1}(t)|\epsilon_{-1}\rangle
+ D_2(t)|\epsilon_2\rangle + D_{-2}(t)|\epsilon_{-2}\rangle. \quad (3)$$

According to the Schrödinger equation $i|\dot{\Psi}(t)\rangle = \hat{H}_{\text{eff}}|\Psi(t)\rangle$, we can obtain the equations of motion for these probability amplitudes $D_i(t) (i = 0, 1)$ and $D_{-i}(t) (j = 2, 3$ and $s = \pm)$ (see the Appendix). The steady-state solutions of these probability amplitudes can be obtained using the perturbation method. The zero-photon (one-photon) probability can be expressed as $P_0 \approx |D_0|^2$ ($P_1 \approx |D_1|^2$) due to $C_{g0} \gg C_{c0}$ ($C_{g1} \gg C_{c1}$). The two- and three-photon probabilities can also be obtained as

$$P_2 = |D_2 - C_{g2}[^{-1}]|^2 + |D_2 + C_{g2}[^{+}]|^2 + D_2^* - D_2^* C_{g2}[^{-}] C_{g2}[^{+}]$$
$$+ D_2^* + D_2^* C_{g2}[^{+}] C_{g2}[^{-}], \quad (33a)$$

$$P_3 = |D_3 - C_{g3}[^{-}]|^2 + |D_3 + C_{g3}[^{+}]|^2 + D_3^* - D_3^* C_{g3}[^{-}] C_{g3}[^{+}]$$
$$+ D_3^* + D_3^* C_{g3}[^{+}] C_{g3}[^{-}], \quad (33b)$$

where the first two terms in Eq. (33a) are the two-photon probability of the non-quantum-interference contribution, the rest terms (cross terms) are induced by quantum interference between the two different paths of the two-photon excitation. To confirm the quantum interference effect, we show the non-quantum-interference part (gray dotted curve) of the two-photon probability $P_2$ as a reference in Fig. 2(a). Here we see that the two peaks in the curve of $P_2$ have an excellent agreement with those of the non-quantum-interference result, while this dip $d_{0.2}$ in the curve of $P_2$ becomes a peak in the non-quantum-interference result. Therefore, the dip in the curve of $P_2$ can be explained based on the destructive quantum interference between the two different paths ($|\epsilon_0\rangle \rightarrow |\epsilon_{\pm}\rangle$).

To seek an optimal cavity-field driving frequency of the 1PB, in Fig. 2(b) we plot the second-order correlation function $g^{(2)}(0)$ versus the cavity-field driving frequency $\omega_4/\omega_c$. Here the red solid (blue dotted) curve represents the numerical (analytical) results, while the gray dotted curve is the analytical result of the non-quantum-interference part. From Fig. 2(b), we see that the analytical result has an excellent agreement with the numerical result, and that the two peaks of the non-quantum-interference result can also match well the analytical and numerical results, but the dip cannot. It is shown that the non-quantum-interference result can predict the location of the optimal driving frequency, but the exact value of $g^{(2)}(0)$ cannot be obtained. In addition, we find that the locations of the dip $d_{0.1}$ and the two peaks $p_{0.2}$ in the curve of $g^{(2)}(0)$ correspond to single- and two-photon resonance transitions, respectively. In the single-photon resonance case, we can observe the 1PB effect, i.e., $g^{(2)}(0) \ll 1$. In the two-photon resonance case, we see that $g^{(2)}(0) > 1$. To further investigate the 2PB effect, we show the third-order correlation function $g^{(3)}(0)$ versus the cavity-field driving frequency $\omega_4/\omega_c$ in Fig. 2(c). According to the expression of $g^{(3)}(0) \approx 6P_3/P_1^3$, we find that the locations of the dip $d_{0.1}$ and the four peaks ($p_{0.2}$ and $p_{0.3}$) in the curve of $g^{(3)}(0)$ correspond to the single-, two-, and three-photon resonance transitions, respectively. In particular, the correlation functions exhibit $g^{(2)}(0) > 1$ and $g^{(3)}(0) > 1$ at $1 \pm J/(\sqrt{2}\omega_c)$, which is a signature of PIT in the two-photon resonance case.

Figure 3(a) displays $P_{n=0,1,2,3}$ as functions of $\omega_4/\omega_c$ in the non-resonant case $\omega_0/\omega_c = 1.4$. Here we can see that there is a peak $p_{0.1}$ in the curve of $P_1$ located at...
ω_d/ω_c = 1, i.e., the driving frequency corresponding to single-photon resonance transition. In addition, there are three peaks in the curve of \( P_2 \) located at \( \omega_d/\omega_c = 1 \) and \( 3.4 \pm \sqrt{0.36 + 8(\epsilon/\omega_c)^2}/4 \), respectively. The locations of the two main peaks \( p_{0,2\pm} \) correspond to the two-photon resonance transitions \( |\epsilon_0\rangle \rightarrow |\epsilon_{2\pm}\rangle \) because the driving frequency \( 2\omega_d/\omega_c \) matches the energy-level differences \( 3.4 \pm \sqrt{0.36 + 8(\epsilon/\omega_c)^2}/2 \) between \( |\epsilon_{2\pm}\rangle \) and \( |\epsilon_0\rangle \). The peak \( p_{0,1} \) is induced by the single-photon resonance transition. We point out that there is a dip \( d_{\text{int},2} \) in the curve of \( P_2 \), which disappears in the non-quantum-interference result (gray dotted curve). Here, the location of the dip in the curve of \( P_2 \) is different from that of the peak in the curve of \( P_1 \), different from the results in the resonant case \( \omega_0 = 2\omega_c \). In the curve of \( P_3 \) there are five peaks located at \( \omega_d/\omega_c = 1, 3.4 \pm \sqrt{0.36 + 8(\epsilon/\omega_c)^2}/4, \) and \( 5.4 \pm \sqrt{0.36 + 24(\epsilon/\omega_c)^2}/6 \), respectively. The locations of these peaks \( (p_{0,1}, p_{0,2\pm}, \) and \( p_{0,3\pm} \) match with those of the single-, two-, and three-photon resonance transitions. Moreover, the two dips \( d_{\text{int},3} \) in the curve of \( P_3 \) are induced by destructive quantum interference between the two different transition paths \( (|\epsilon_0\rangle \rightarrow |\epsilon_{3\pm}\rangle) \) of the three-photon excitation, which can be confirmed by comparing the analytical result with the non-quantum-interference result (gray dotted curve).

In the non-resonant case \( \omega_0 = 1.4\omega_c \), we analyze the optimal cavity-field driving frequency of 1PB by showing \( g^{(2)}(0) \) as a function of \( \omega_d/\omega_c \) in Fig. 3(b). It follows from the relation \( g^{(2)}(0) \approx 2P_2/P_1^2 \) that there are two dips in the curve of \( g^{(2)}(0) \) which is a signature of the 1PB effect. One of the two dips \( d_{\text{int},1} \) corresponds to the single-photon resonance transition, the other dip \( d_{\text{int},2} \) is caused by destructive quantum interference between the two different paths \( |\epsilon_0\rangle \rightarrow |\epsilon_{2\pm}\rangle \). To further explain the quantum interference effect, we show the analytical result of non-quantum-interference part (gray dotted curve). We find
that the dip $d_{\text{int},2}$ caused by quantum interference effect disappears in the non-quantum-interference result. In Fig. 3(c), $g^{(3)}(0)$ is plotted as a function of $\omega_d/\omega_c$. We find that the locations of the four peaks ($p_{0,2,3}$ and $p_{0,3,2}$) in the curve of $g^{(3)}(0)$ correspond to two- and three-photon resonance transitions, respectively. While the three dips ($d_{0,1}$ and two $d_{\text{int},3}$) in the curve of $g^{(3)}(0)$ are caused by the single-photon resonance transition and the destructive quantum interference between the two different paths $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$. Moreover, the relations $g^{(2)}(0) > 1$ and $g^{(3)}(0) > 1$ indicates that PIT can be observed in the two-photon resonance case.

With regard to the analysis of the non-resonant case, we only consider a particular case in Fig. 3. A more comprehensive analysis of the non-resonant case is shown in Fig. 4(a), in which we show $\log_{10} g^{(2)}(0)$ as a function of $\omega_d/\omega_c$ and $\omega_0/\omega_c$. It is clear that the optimal parameter conditions to observe the 1PB effect are $\omega_d/\omega_c = 1$ and $\omega_0/\omega_d = 2$, respectively, i.e., $\Delta_c = 0$ and $\Delta_0 = 0$. The condition $\Delta_c = 0$ can be explained based on the single-photon resonance transition $|\varepsilon_0\rangle \rightarrow |\varepsilon_1\rangle$, and the condition $\Delta_0 = 0$ can be interpreted by $C_{g2} = 0$ corresponding to destructive quantum interference between the two different paths $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$. The white dotted curves (red areas) represent the two-photon resonance transitions $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$. We observe that $g^{(2)}(0) > 1$ in the two-photon resonance case. In order to further investigate PIT in the non-resonant case, $\log_{10} g^{(3)}(0)$ is plotted as a function of $\omega_d/\omega_c$ and $\omega_0/\omega_c$ in Fig. 4(b). It can be seen that $g^{(3)}(0) < 1$ in the single-photon resonance case $\omega_d/\omega_c = 1$. The white (black) dotted curves represent the two-photon (three-photon) resonance transitions. The green areas in regimes I and II between the white and black dotted curves are induced by destructive quantum interference between the two different paths $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$ of the three-photon excitation. Obviously, we see that $g^{(3)}(0) > 1$ under the two-photon resonance transitions. Therefore, PIT can be observed in the two-photon resonance case because of $g^{(2)}(0) > 1$ and $g^{(3)}(0) > 1$. To see more clearly, the correlation functions $g^{(2)}(0)$ (blue solid curves) and $g^{(3)}(0)$ (red dashed curves) are plotted versus $\omega_d/\omega_c$ at different values of $\omega_0/\omega_c$ in Figs. 4(c-e). To be clearer, we mark the locations of these dips and peaks in the curves of $g^{(2)}(0)$ and $g^{(3)}(0)$ with the resonance transitions and the destructive quantum interference. Obviously, we observe that $g^{(2)}(0) < 1$ at the locations of the single-photon resonance transition and the destructive quantum interference between the two different paths $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$, respectively. We also find that $g^{(2)}(0) > 1$ and $g^{(3)}(0) > 1$ in the two-photon resonance case. This implies that the 1PB effect and PIT can occur by driving the cavity, while the 2PB effect cannot occur in this case. In addition, the numerical results indicate that the dips $d_{\text{int},3}$ induced by the destructive quantum interference between the two different paths $|\varepsilon_0\rangle \rightarrow |\varepsilon_{2\pm}\rangle$ disappear in a range around $\omega_0/\omega_c = 2$ (roughly from 1.4 to 2.6).

Our results can also be confirmed by comparing the photon-number distributions $P_n=0,1,2,3$ and the Poisson distributions $P_n=0,1,2,3$. In Fig. 5(a) we plot $P_n=0,1,2,3$ (colored solid curves) and $P_n=0,1,2,3$ (colored dash-dotted curves) as functions of $\omega_d/\omega_c$ in the resonant case $\omega_0/\omega_c = 2$. Figure 5(b) is a zoomed-in plot of $P_1$ and $P_3$ versus $\omega_d/\omega_c$. At the two-photon resonance transitions ($\omega_d/\omega_c = 1 \pm J/\sqrt{2\omega_c}$), the single-photon probability is suppressed because of $P_1 < P_3$, while the two- and three-photon probabilities are enhanced due to $P_2 > P_1$ and $P_3 > P_2$. This means that PIT occurs by driving the cavity in the two-photon resonance case. To further illustrate PIT, we show the relative deviations of the photon-number distribution to the standard Poisson distribution with the same mean photon number at $\omega_d/\omega_c = 1 \pm J/\sqrt{2\omega_c}$ in Fig. 5(c). Here we can see that the relative population grows as the photon number increases, which is another signature of PIT.

We proceed to investigate the influence of the coupling strength $J/\omega_c$ and the cavity-field decay rate $\kappa/\omega_c$ on the 1PB effect. In Figs. 6(a) and 6(b), we plot the second-order correlation function $g^{(2)}(0)$ as a function of $J/\omega_c$ at different values of $\Delta_c/\omega_c$ ($\Delta_0 = 0$) and $\Delta_0/\omega_c$ ($\Delta_c = 0$), respectively. Here we see $g^{(2)}(0) \ll 1$, which means that the 1PB effect can be observed. We also see that $g^{(2)}(0)$ decreases monotonically as $J/\omega_c$ increases. This implies...
that the 1PB effect is more obvious with the increase of the coupling strength. The reason is that the stronger coupling will cause larger energy nonharmonicity, and makes it more difficult to induce the multiphoton excitation. In addition, we find that the 1PB effect becomes weak as the detunings $(\Delta_c$ and $\Delta_0$) increase.

The second-order correlation function $g^{(2)}(0)$ is plotted as a function of $\kappa/\omega_c$ at different values of $\Delta_c/\omega_c$ for $\Delta_0 = 0$ in Fig. 7(a) and at different values of $\Delta_0/\omega_c$ for $\Delta_c = 0$ in Fig. 7(b). Clearly, we find that $g^{(2)}(0) \ll 1$ in the optimal parameter conditions, namely, $\Delta_0 = 0$ and $\Delta_c = 0$. This implies that the 1PB effect can be observed. In addition, the second-order correlation function $g^{(2)}(0)$ increases monotonically with the increasing of $\kappa/\omega_c$, which means that the cavity-field decay rate attenuates the 1PB effect. Similarly, we find that the 1PB effect becomes weak with the increase of the detunings $\Delta_c$ and $\Delta_0$.

V. PB IN THE ATOM-DRIVING CASE

In this section, we study PB effect in the atom-driving case by numerically calculating the second- and third-order correlation functions of the cavity-field mode.

![Image](image-url)

**FIG. 6.** (Color online) The correlation function $g^{(2)}(0)$ as a function of $J/\omega_c$ at different values of (a) $\Delta_c/\omega_c$ for $\Delta_0 = 0$ and (b) $\Delta_0/\omega_c$ for $\Delta_c = 0$. Other parameters used are given by $\kappa/\omega_c = \gamma/\omega_c = 0.001$ and $\Omega/\kappa = 0.2$.

**FIG. 7.** (Color online) The correlation function $g^{(2)}(0)$ as a function of $\kappa/\omega_c$ at different values of (a) $\Delta_c/\omega_c$ for $\Delta_0 = 0$ and (b) $\Delta_0/\omega_c$ for $\Delta_c = 0$. Other parameters used are given by $J/\omega_c = 0.08$, $\gamma/\omega_c = 0.001$, and $\Omega/\kappa = 0.2$.

A. Theoretical analysis

When a monochromatic weak driving field is applied to the atom, the driving Hamiltonian is described by

$$\hat{H}_d = \Omega_L(\hat{\sigma}_+ e^{-i\omega_L t} + \hat{\sigma}_- e^{i\omega_L t}),$$

(34)

where $\Omega_L$ and $\omega_L$ are the driving strength and driving frequency, respectively. In this case, the total Hamiltonian of the system reads

$$\hat{H}_{\text{sys}} = \hat{H}_{2\text{pIC}} + \hat{H}_d.$$  

(35)

In a rotating frame defined by the unitary operator $\exp[-i\omega_L t(\hat{a}^\dagger \hat{a} + \hat{\sigma}_z)/2]$, the Hamiltonian of the system becomes

$$\hat{H}'_{\text{sys}} = \Delta'_c \hat{a}^\dagger \hat{a} + \Delta'_0 \hat{\sigma}_+ \hat{\sigma}_- + J(\hat{a}^\dagger^2 \hat{\sigma}_+ + \hat{\sigma}_+ \hat{a}^2) + \Omega_L(\hat{\sigma}_+ + \hat{\sigma}_-),$$

(36)

where $\Delta'_c = \omega_c - \omega_L/2$ ($\Delta'_0 = \omega_0 - \omega_L$) is the detuning of the cavity-field (atomic) frequency with respect to the driving frequency.

By numerically solving quantum master equation (31) under the replacement of $\hat{H}_{\text{sys}}^{(1)} \rightarrow \hat{H}'_{\text{sys}}$, the steady-state density operator $\hat{\rho}_{\text{ss}}$ of the system can be obtained and then we can calculate the photon-number distributions $P_n = \text{Tr}[|n\rangle \langle n| \hat{\rho}_{\text{ss}}]$ in the cavity. Similarly, the equal-time second- and third-order correlation functions...
can be obtained as
\[ g^{(2)}(0) = \frac{\text{Tr}(\hat{a}^\dagger \hat{a}^2 \hat{\rho}_{ss}^{\text{eq}})}{[\text{Tr}(\hat{a}^\dagger \hat{a} \hat{\rho}_{ss}^{\text{eq}})]^2}, \]
\[ g^{(3)}(0) = \frac{\text{Tr}(\hat{a}^\dagger \hat{a}^3 \hat{\rho}_{ss}^{\text{eq}})}{[\text{Tr}(\hat{a}^\dagger \hat{a}^2 \hat{\rho}_{ss}^{\text{eq}})]^3}. \]  

By analyzing these correlation functions, we can study the PB effect and PIT for the cavity photons.

**B. Numerical results**

When the atom is driven, the 1PB effect cannot be observed due to the transition \( |\varepsilon_0\rangle \xrightarrow{\Omega_L} |\varepsilon_1\rangle \) is forbidden. In order to prove it, we show \( \log_{10} g^{(2)}(0) \) as a function of \( \omega_L/\omega_c \) and \( \omega_0/\omega_c \) in Fig. 8(a). The white dotted curves represent the two-photon resonance transitions \( |\varepsilon_0\rangle \xrightarrow{\Omega_L} |\varepsilon_\pm\rangle \). Clearly, we observe that \( g^{(2)}(0) > 1 \) in the entire parameter area which implies the 1PB effect cannot appear by driving the atom. Different from the cavity-field-driving case, two photons can be produced when driving the atom. To further study the 2PB effect, \( \log_{10} g^{(4)}(0) \) is plotted as a function of \( \omega_L/\omega_c \) and \( \omega_0/\omega_c \) in Fig. 8(b). The white (black) dotted curves correspond to the two-photon (four-photon) resonance transitions, namely, \( |\varepsilon_0\rangle \xrightarrow{\Omega_L} |\varepsilon_\pm\rangle \) (\( |\varepsilon_0\rangle \xrightarrow{\Omega_L} |\varepsilon_\pm\rangle \)). At the two-photon resonance transitions, the correlation function \( g^{(4)}(0) < 1 \) for some parameters. This means that the 2PB can be observed by driving the atom in the two-photon resonance case, i.e., \( g^{(2)}(0) > 1 \) and \( g^{(4)}(0) < 1 \). To see more clearly, in Figs. 8(c-e) the correlation functions \( g^{(2)}(0) \) (blue solid curves), \( g^{(3)}(0) \) (red dashed curves), and \( g^{(4)}(0) \) (green dash-dotted curves)
are plotted versus $\omega_L/\omega_c$ at different values of $\omega_0/\omega_c$. Here we find that the locations of these dips $d_{0,\pm}$ in the curves of $g^{(2)}(0)$ correspond to two-photon resonance transitions, while the locations of these dips $d_{0,\pm}$ and peaks $(p_{1,3\pm}$ and $p_{0,4\pm})$ in the curves of $g^{(3)}(0)$ correspond to two-, three, and four-photon resonance transitions, respectively. In the curves of $g^{(4)}(0)$, the locations of these dips $d_{0,\pm}$ and peaks $p_{0,4\pm}$ correspond to two- and four-photon resonance transitions, respectively. From Figs. 8(c-e), we find that the 2PB effect can occur in the grey areas due to $g^{(2)}(0) > 1$ and $g^{(n)}(0) < 1$ ($n = 3, 4$). The yellow area of Fig. 8(c) corresponds to PIT due to $g^{(n)}(0) > 1$ ($n = 2, 3, 4$). It is noteworthy that the blue area of Fig. 8(d) indicates the enhanced two- and four-photon correlations $|g^{(n)}(0)| > 1$ ($n = 2, 4$) and the suppressed three-photon correlation $g^{(3)}(0) < 1$.

The 2PB effect can also be confirmed by comparing the photon-number distributions $P_n^{(a,b,c)}$ and the Poisson distributions $P_n^{(0,1,2,3)}$. In Fig. 9(a) we plot $P_n^{(a,b,c)}$ (colored solid curves) and $P_n^{(0,1,2,3)}$ (colored dash-dotted curves) as functions of the atomic driving frequency $\omega_L/\omega_c$ in the resonant case $\omega_0/\omega_c = 2$. Figure 9(b) is a zoomed-in plot of $P_n^{(a,b,c)}$ and $P_n^{(0,1,2,3)}$ versus $\omega_L/\omega_c$. We see in Fig. 9(a) that there are two peaks $p_{0,2\pm}$ in the curve of $P_1$ (green solid curve) located at $\omega_L = 2 \omega_c \pm \sqrt{2}J$, which corresponds to the population of the $|e_0\rangle$ induced through the Raman processes $|e_0\rangle \overset{\Omega_L}{\longrightarrow} |e_{2\pm}\rangle \overset{\kappa}{\longrightarrow} |e_1\rangle$. The physical processes involve the transitions $|e_0\rangle \rightarrow |e_{2\pm}\rangle$ at the atomic driving frequency $\omega_L = 2 \omega_c \pm \sqrt{2}J$, and the decay process $|e_{2\pm}\rangle \rightarrow |e_1\rangle$. In the curve of $P_2$ (red solid curve), we see that there are two peaks $p_{0,2\pm}$ located at $\omega_L = 2 \omega_c \pm \sqrt{2}J$, i.e., the two-photon resonance transitions. In the curve of $P_3$ (yellow solid curve), we see that there are six peaks $p_{0,4\pm}$, $p_{1,3\pm}$ and $p_{1,3\pm}$ located at $\omega_L = 2 \omega_c \pm \sqrt{2}J$, $2 \omega_c \pm \sqrt{3}J$, and $2 \omega_c \pm \sqrt{6}J$, respectively. The two peaks $p_{0,4\pm}$ are induced by the processes $|e_0\rangle \overset{\Omega_2}{\rightarrow} |e_{2\pm}\rangle \overset{\kappa}{\longrightarrow} |e_{3\pm}\rangle$, and the two peaks $p_{1,3\pm}$ are induced by the processes $|e_0\rangle \overset{\Omega_2}{\rightarrow} |e_{2\pm}\rangle \overset{\kappa}{\longrightarrow} |e_1\rangle \overset{\Omega_2}{\rightarrow} |e_{3\pm}\rangle$. Similarly, we mark the locations of these peaks in the photon-number distributions $P_n^{(a,b,c)}$. At the locations of the two-photon resonance transitions ($\omega_L = 2 \omega_c \pm \sqrt{2}J$), we see that the single- and three-photon probabilities are suppressed due to $P_1 < P_3$ and $P_1 < P_3$, while the two-photon probability is enhanced because of $P_2 > P_3$. This indicates that the 2PB effect can be observed by driving the atom. To further illustrate the 2PB effect, in Fig. 9(c) we display the relative deviations of the photon-number distribution to the standard Poisson distribution with the same mean photon number at $\omega_L = 2 \omega_c \pm \sqrt{2}J$. We observe that only the value of the two-photon relative population is greater than 0, i.e., $P_2 > P_3$, which implies that the 2PB effect can appear in the atom-driving case.
that $g^{(2)}(0)$ and $g^{(3)}(0)$ increase monotonically with the increasing of $\kappa/\omega_c$, which implies that the cavity-field decay rate weakens the 2PB effect. Furthermore, it can be seen that the 2PB effect disappears when $\kappa/\omega_c > 0.005$ in the resonant case $\omega_0/\omega_c = 2$ [Fig. 11(b)]. This is because $g^{(3)}(0) > 1$ when $\kappa/\omega_c > 0.005$ at $\omega_0/\omega_c = 2$.

VI. CONCLUSIONS

In conclusion, we have studied the multiphoton blockade and PIT effects of the two-photon JC model in both the cavity-field-driving and atom-driving cases. We have obtained the analytical results of the correlation functions by perturbatively solving the equations of motion for these probability amplitudes. These analytical results are confirmed by numerically solving the quantum master equation including both the cavity-field and the atomic dissipations in the truncated Hilbert space. We have found that the 1PB and PIT effects can be observed in this system when the cavity mode is driven, while the 2PB cannot occur. In particular, we have shown that the 1PB effect can be enhanced by the destructive quantum interference effect between the two different paths in the non-resonant case. Furthermore, we have found that the 2PB effect can be induced by driving the atom, while the 1PB effect cannot be observed due to the single-photon transition is forbidden in this case. Our results will pave the way for the study of multiphoton quantum correlation and multiphoton quantum coherent devices.

ACKNOWLEDGMENTS

J.-Q.L would like to thank Dr. S. Felicetti for helpful discussions. J.-Q.L. is supported in part by National Natural Science Foundation of China ( Grants No. 11822501, No. 11774087, and No. 11935006), Natural Science Foundation of Hunan Province, China (Grant No. 2017JJ1021), and Hunan Science and Technology Plan Project (Grant No. 2017XK2018). J.-F.H. is supported in part by the National Natural Science Foundation of China (Grant No. 11505055) and Scientific Research Fund of Hunan Provincial Education Department (Grant No. 18A007).

Appendix: Derivation of the probability amplitudes in the eigenstate representation

In the low-excitation subspace, a general state of the system can be expressed in the eigenstate representation as

$$|\Psi(t)\rangle = D_0(t)|\varepsilon_0\rangle + D_1(t)|\varepsilon_1\rangle + D_2_-(t)|\varepsilon_2_-angle + D_2_+(t)|\varepsilon_2_+angle + D_3_-(t)|\varepsilon_3_-angle + D_3_+(t)|\varepsilon_3_+\rangle,$$

where the coefficients $D_i(t)$ ($i = 0, 1$) and $D_{js}(t)$ ($j = 2, 3$ and $s = \pm$) are the probability amplitudes. Based on the Schrödinger equation $i|\dot{\Psi}(t)\rangle = \hat{H}_{\text{eff}}|\Psi(t)\rangle$, the equations of motion for these probability amplitudes can be obtained...
as

\[ i\dot{D}_0(t) = \varepsilon_0 D_0(t) + \Omega D_1(t), \]
\[ i\dot{D}_1(t) = \left( \varepsilon_1 - i \frac{\kappa}{2} \right) D_1(t) + \Omega \left[ D_0(t) + \sqrt{2} C_{g2}^{[+]} D_{2+}(t) + \sqrt{2} C_{g2}^{[-]} D_{2-}(t) \right], \]
\[ i\dot{D}_{2-}(t) = \left( \varepsilon_2 - i \frac{\gamma_1}{2} \right) C_{g2}^{[+]} \left( \varepsilon_2 - i \frac{\gamma_1}{2} \right) C_{g2}^{[-]} \right) D_{2-}(t) - i \left( \kappa C_{g2}^{[+]} C_{g2}^{[-]} + \frac{\gamma}{2} C_{c0}^{[+]} C_{c0}^{[-]} \right) D_{2+}(t) + \sqrt{2} \Omega C_{g2}^{[-]} D_1(t) \]
\[ + \Omega \left[ \left( \sqrt{3} C_{g2}^{[+] C_{g2}^{[-]} + C_{g0}^{[+] C_{g0}^{[-]} \right) D_{3-}(t) + \left( \sqrt{3} C_{g2}^{[+] C_{g0}^{[+]} C_{c1}^{[-]} \right) D_{3+}(t) \right), \]
\[ i\dot{D}_{2+}(t) = \left( \varepsilon_2 + i \frac{\gamma_1}{2} \right) C_{g2}^{[+]} \left( \varepsilon_2 + i \frac{\gamma_1}{2} \right) C_{g2}^{[-]} \right) D_{2+}(t) - i \left( \kappa C_{g2}^{[+]} C_{g2}^{[-]} + \frac{\gamma}{2} C_{c0}^{[+]} C_{c0}^{[-]} \right) D_{2-}(t) + \sqrt{2} \Omega C_{g2}^{[+] D_1(t) \]
\[ + \Omega \left[ \left( \sqrt{3} C_{g2}^{[+] C_{g2}^{[+]} C_{g0}^{[+]} C_{c1}^{[-]} \right) D_{3+}(t) + \left( \sqrt{3} C_{g2}^{[+] C_{g0}^{[+]} C_{c1}^{[-]} \right) D_{3-}(t) \right), \]
\[ i\dot{D}_{3-}(t) = \left[ \varepsilon_3 - i \frac{\kappa}{2} \left( \frac{\gamma_2}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_2}{2} \right) C_{c1}^{[+] C_{c1}^{[-]} \right) D_{3-}(t) - i \left[ \frac{\kappa}{2} \right] \left( 3 C_{g3}^{[+] C_{g3}^{[-]} + C_{c1}^{[+] C_{c1}^{[-]} \right) + \frac{\gamma}{2} C_{c1}^{[+] C_{c1}^{[-]} \right) D_{3+}(t) \]
\[ + \Omega \left[ \left( \sqrt{3} C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_3}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_3}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} \right) D_{3-}(t) \right), \]
\[ i\dot{D}_{3+}(t) = \left[ \varepsilon_3 + i \frac{\kappa}{2} \left( \frac{\gamma_2}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_2}{2} \right) C_{c1}^{[+] C_{c1}^{[-]} \right) D_{3+}(t) - i \left[ \frac{\kappa}{2} \right] \left( 3 C_{g3}^{[+] C_{g3}^{[-]} + C_{c1}^{[+] C_{c1}^{[-]} \right) + \frac{\gamma}{2} C_{c1}^{[+] C_{c1}^{[-]} \right) D_{3-}(t) \]
\[ + \Omega \left[ \left( \sqrt{3} C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_3}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} + \sqrt{2} \left( \frac{\gamma_3}{2} \right) C_{g3}^{[+] C_{g3}^{[-]} C_{g0}^{[+]} C_{c1}^{[-]} \right) D_{3-}(t) \right). \]
\]

(A.2)

In the weak-driving case, Eq. (A.2) can be solved approximately by using the perturbation method, namely discarding the higher-order terms in the equations of motion for the lower-order variables. We assume \(D_0 = 1\), then the steady-state solution of Eq. (A.2) can be approximately obtained by setting \(\partial D_i/\partial t = 0\) and \(\partial D_{3\pm}/\partial t = 0\) as

\[
D_0 = 1, \quad D_1 = -2\Omega/(2\varepsilon_1 - i\kappa), \quad D_{2-} = 2\sqrt{2}\left(2iC_{g2}^{[+] C_{c0}^{[+]} C_{g2}^{[-]} \gamma + C_{c0}^{[+]} C_{g2}^{[-]} \gamma \right) \Omega^2/M(2\varepsilon_1 - i\kappa), \quad D_{2+} = 2\sqrt{2}\left(2iC_{g2}^{[+] C_{c0}^{[+]} C_{g2}^{[-]} \gamma + C_{c0}^{[+]} C_{g2}^{[-]} \gamma \right) \Omega^2/M(2\varepsilon_1 - i\kappa), \quad \]

(A.3)

where we introduce the coefficient

\[
M = i \left[ \left( C_{c0}^{[+]} C_{g2}^{[+]} \gamma + 2\kappa C_{c0}^{[+]} C_{g2}^{[+]} \gamma \right) \varepsilon_2 + \left( C_{c0}^{[+]} C_{g2}^{[+]} \gamma + 2\kappa C_{c0}^{[+]} C_{g2}^{[+]} \gamma \right) \varepsilon_2 + 2\varepsilon_2 - \varepsilon_2 \right] + \left( C_{g2}^{[+] C_{c0}^{[+]} C_{g2}^{[-]} \gamma + C_{c0}^{[+]} C_{g2}^{[-]} \gamma \right) \gamma. \quad \]

(A.4)

Note that the two expressions of the steady-state solution of \(D_{3\pm}\) are too complex to be shown here.

[1] A. Imamoglu, H. Schmidt, G. Woods, and M. Deutsch, Strongly Interacting Photons in a Nonlinear Cavity, Phys. Rev. Lett. 79, 1467 (1997).
[2] L. Tian and H. J. Carmichael, Quantum trajectory simulations of two-state behavior in an optical cavity containing one atom, Phys. Rev. A 46, R6801(R) (1992).
[3] K. M. Birnbaum, A. Boca, R. Miller, A. D. Boozer, T. E. Northup, and H. J. Kimble, Photon blockade in an optical cavity with one trapped atom, Nature (London) 436, 87 (2005).
[4] A. Faraon, I. Fushman, D. Englund, N. Stoltz, P. Petroff, and J. Vučković, Coherent generation of non-classical light on a chip via photon-induced tunnelling and blockade, Nat. Phys. 4, 859 (2008).
[5] A. Faraon, A. Majumdar, and J. Vučković, Generation of nonclassical states of light via photon blockade in optical nanocavities, Phys. Rev. A 81, 033838 (2010).
[6] A. Reinhard, T. Volz, M. Winger, A. Badolato, K. J. Hennelly, E. L. Hu, and A. Imamoglu, Strongly correlated photons on a chip, Nat. Photonics 6, 93 (2012).
[7] A. Ridolfo, M. Leib, S. Savasta, and M. J. Hartmann, Photon Blockade in the Ultrastrong Coupling Regime, Phys. Rev. Lett. 109, 193602 (2012).
[8] T. Peyronel, O. Firstenberg, Q.-Y. Liang, S. Hofferberth,
A. V. Gorshkov, T. Pohl, M. D. Lukin, and V. Vuletić, Quantum nonlinear optics with single photons enabled by strongly interacting atoms, *Nature (London)* **488**, 57 (2012).

[9] M. Bajcsy, A. Majumdar, A. Rundquist, and J. Vučković, Photon blockade with a four-level quantum emitter coupled to a photonic-crystal nanocavity, *New J. Phys.* **15**, 025014 (2013).

[10] K. Müller, A. Rundquist, K. A. Fischer, T. Sarmiento, K. G. Lagoudakis, Y. A. Kelaita, C. Sánchez Muñoz, E. del Valle, F. P. Laussy, and J. Vučković, Coherent Generation of Nonclassical Light on Chip via Detuned Photon Blockade, *Phys. Rev. Lett.* **114**, 233601 (2015).

[11] M. Radulaski, K. A. Fischer, K. G. Lagoudakis, J. L. Zhang, and J. Vučković, Photon blockade in two-emitter-cavity systems, *Phys. Rev. A* **96**, 011801(R) (2017).

[12] C. Wang, Y.-L. Liu, R. Wu, and Y.-x. Liu, Phase-modulated photon antibunching in a two-level system coupled to two cavities, *Phys. Rev. A* **96**, 013818 (2017).

[13] Y. F. Han, C. J. Zhu, X. S. Huang, and Y. P. Yang, Electromagnetic control and improvement of nonclassicality in a strongly coupled single-atom cavity-QED system, *Phys. Rev. A* **98**, 033828 (2018).

[14] R. Trivedi, M. Radulaski, K. A. Fischer, S. Fan, and J. Vučković, Photon Blockade in Weakly Driven Cavity Quantum Electrodynamics Systems with Many Emitters, *Phys. Rev. Lett.* **122**, 243602 (2019).

[15] K. Hou, C. J. Zhu, Y. P. Yang, and G. S. Agarwal, Interfering pathways for photon blockade in cavity QED with one and two qubits, arXiv:1907.05997.

[16] A. J. Hoffman, S. J. Srinivasan, S. Schmidt, L. Spietz, K. Müller, A. Rundquist, K. A. Fischer, T. Sarmiento, C. Lang, D. Bozyigit, C. Eichler, L. Steffen, J. M. Fink, K. Hou, C. J. Zhu, Y. P. Yang, and G. S. Agarwal, Interfering pathways for photon blockade in cavity QED with one and two qubits, arXiv:1907.05997.

[17] A. J. Hoffman, S. J. Srinivasan, S. Schmidt, L. Spietz, J. Aumentado, H. E. Türeci, and A. A. Houck, Dispersive Photon Blockade in a Superconducting Circuit, *Phys. Rev. Lett.* **107**, 053602 (2011).

[18] C. Lang, D. Bozyigit, C. Eichler, L. Steffen, J. M. Fink, A. A. Abdumalikov, M. Baur, S. Filipp, M. P. da Silva, A. Blais, and A. Wallraff, Observation of Resonant Photon Blockade at Microwave Frequencies Using Correlation Function Measurements, *Phys. Rev. Lett.* **106**, 243601 (2011).

[19] Y.-x. Liu, X.-W. Xu, A. Miranowicz, and F. Nori, From blockade to transparency: Controllable photon transmission through a circuit-QED system, *Phys. Rev. A* **89**, 043818 (2014).

[20] W. Leonišk and R. Tanaš, Possibility of producing the one-photon state in a kicked cavity with a nonlinear Kerr medium, *Phys. Rev. A* **49**, R20 (1994).

[21] S. Ferretti, L. C. Andreani, H. E. Türeci, and D. Gerace, Photon correlations in a two-site nonlinear cavity system under coherent drive and dissipation, *Phys. Rev. A* **82**, 013841 (2010).

[22] J.-Q. Liao and C. K. Law, Correlated two-photon transport in a one-dimensional waveguide side-coupled to a nonlinear cavity, *Phys. Rev. A* **82**, 053836 (2010).

[23] S. Ghosh and T. C. H. Liew, Dynamical Blockade in a Single-Mode Bosonic System, *Phys. Rev. Lett.* **123**, 013602 (2019).

[24] P. Rabl, Photon Blockade Effect in Optomechanical Systems, *Phys. Rev. Lett.* **107**, 063601 (2011).

[25] J.-Q. Liao and F. Nori, Photon blockade in quadratically coupled optomechanical systems, *Phys. Rev. A* **88**, 023853 (2013).

[26] J.-Q. Liao and C. K. Law, Correlated two-photon scattering in cavity optomechanics, *Phys. Rev. A* **87**, 043809 (2013).

[27] P. Kómár, S. D. Bennett, K. Stannigel, S. J. M. Habraken, P. Rabl, P. Zoller, and M. D. Lukin, Single-photon nonlinearities in two-mode optomechanics, *Phys. Rev. A* **87**, 013839 (2013).

[28] H. Wang, X. Gu, Y.-x. Liu, A. Miranowicz, and F. Nori, Tunable photon blockade in a hybrid system consisting of an optomechanical device coupled to a two-level system, *Phys. Rev. A* **92**, 033806 (2015).

[29] G.-L. Zhu, X.-Y. Lü, L.-L. Wan, T.-S. Yin, Q. Bin, and Y. Wu, Controllable nonlinearity in a dual-coupling optomechanical system under a weak-coupling regime, *Phys. Rev. A* **97**, 033830 (2018).

[30] F. Zou, L.-B. Fan, J.-F. Huang, and J.-Q. Liao, Enhancement of few-photon optomechanical effects with cross-Kerr nonlinearity, *Phys. Rev. A* **99**, 043837 (2019).

[31] H. Zheng, D. J. Gauthier, and H. U. Baranger, Cavity-Free Photon Blockade Induced by Many-Body Bound States, *Phys. Rev. Lett.* **107**, 233601 (2011).

[32] A. Majumdar and D. Gerace, Single-photon blockade in doubly resonant nanocavities with second-order nonlinearity, *Phys. Rev. B* **87**, 235319 (2013).

[33] J.-F. Huang, J.-Q. Liao, and C. P. Sun, Photon blockade induced by atoms with Rydberg coupling, *Phys. Rev. A* **87**, 023822 (2013).

[34] S. Zeytinoğlu and A. İmamoğlu, Interaction-induced photon blockade using an atomically thin mirror embedded in a microcavity, *Phys. Rev. A* **89**, 051801 (2018).

[35] F. Zou, D.-G. Lai, and J.-Q. Liao, Photon blockade effect in a coupled cavity system, arXiv:1803.06642.

[36] T. C. H. Liew and V. Savona, Single Photons from Coupled Quantum Modes, *Phys. Rev. Lett.* **104**, 183601 (2010).

[37] M. Bamba, A. İmamoğlu, I. Carusotto, and C. Ciuti, Origin of strong photon antibunching in weakly nonlinear photonic molecules, *Phys. Rev. A* **83**, 021802(R) (2011).

[38] X.-W. Xu and Y. Li, Tunable photon statistics in weakly nonlinear photonic molecules, *Phys. Rev. A* **90**, 043822 (2014).

[39] H. Z. Shen, Y. H. Zhou, and X. X. Yi, Tunable photon blockade in coupled semiconductor cavities, *Phys. Rev. A* **91**, 063808 (2015).

[40] H. Flayac and V. Savona, Unconventional photon blockade, *Phys. Rev. A* **96**, 053810 (2017).

[41] Y. Wang, G.-Q. Zhang, and W.-L. You, Photon blockade with cross-Kerr nonlinearity in superconducting circuits, *Laser Phys. Lett.* **15**, 105201 (2018).

[42] A. Ryu, D. Rosser, A. Saxena, T. Fryett, and A. Majumdar, Strong photon antibunching in weakly nonlinear two-dimensional exciton-polaritons, *Phys. Rev. B* **97**, 235307 (2018).

[43] W. Zhang, Z. Yu, Y. Liu, and Y. Peng, Optimal photon antibunching in a quantum–dot–bimodal-cavity system, *Phys. Rev. A* **89**, 043832 (2014).

[44] J. Tang, W. Geng, and X. Xu, Quantum Interference Induced Photon Blockade in a Coupled Single Quantum Dot-Cavity System, *Sci. Rep.* **5**, 9252 (2015).

[45] J.-Q. Liao, G.-Z. Wang, Y.-x. Liu, and F. Nori, Mode coupling and photon antibunching in a bimodal cavity containing a dipole quantum emitter, *Phys. Rev. A* **93**, 013856 (2016).

[46] X.-W. Xu and Y.-J. Li, Antibunching photons in a cavity coupled to an optomechanical system, *J. Phys. B: At.
B. Sarma and A. K. Sarma, Unconventional photon blockade in three-mode optomechanics, Phys. Rev. A 98, 013826 (2018).

B. Li, R. Huang, X. Xu, A. Miranowicz, and H. Jing, Nonreciprocal unconventional photon blockade in a spinning optomechanical system, Photonics Res. 7, 630 (2019).

M.-A. Lemonde, N. Didier, and A. A. Clerk, Antibunching and unconventional photon blockade with Gaussian squeezed states, Phys. Rev. A 90, 063824 (2014).

D. Gerace and V. Savona, Unconventional photon blockade in doubly resonant microcavities with second-order nonlinearity, Phys. Rev. A 89, 031803(R) (2014).

Y. H. Zhou, H. Z. Shen, and X. X. Yi, Strong photon antibunching with weak second-order nonlinearity under dissipation and coherent driving, Opt. Express 24, 17332 (2016).

B. Sarma and A. K. Sarma, Quantum-interference-assisted photon blockade in a cavity via parametric interactions, Phys. Rev. A 96, 053827 (2017).

H. J. Snijders, J. A. Frey, H. Norman, H. Flayac, V. Savona, A. C. Gossard, J. E. Bowers, M. P. van Exter, D. Bouwmeester, and W. Löffler, Observation of the Unconventional Photon Blockade, Phys. Rev. Lett. 121, 043601 (2018).

C. Vaneph, A. Morvan, G. Aiello, M. Féchant, M. Aprili, J. Gabelli, and J. Estève, Observation of the Unconventional Photon Blockade in the Microwave Domain, Phys. Rev. Lett. 121, 043602 (2018).

C. Hanssen, K. N. Tolazzi, T. Wilk, and G. Rempe, Two-Photon Blockade in an Atom-Driven Cavity QED System, Phys. Rev. Lett. 118, 133604 (2017).

S. S. Shamailov, A. S. Parks, M. J. Collett, and H. J. Carmichael, Multi-photon blockade and dressing of the dressed states, Opt. Commun. 283, 766 (2010).

A. Miranowicz, M. Paprzycka, Y.-x. Liu, J. Bajer, and F. Nori, Two-photon and three-photon blockades in driven nonlinear systems, Phys. Rev. A 87, 023809 (2013).

A. Miranowicz, J. Bajer, M. Paprzycka, Y.-x. Liu, A. M. Zagoskin, and F. Nori, State-dependent photon blockade via quantum-reservoir engineering, Phys. Rev. A 90, 033831 (2014).

G. H. Hovsepyan, A. R. Shahinyan, and G. Y. Kryuchkyan, Multiphoton blockades in pulsed regimes beyond stationary limits, Phys. Rev. A 90, 013839 (2014).

H. J. Carmichael, Breakdown of Photon Blockade: A Dissipative Quantum Phase Transition in Zero Dimensions, Phys. Rev. X 5, 031028 (2015).

W.-W. Deng, G.-X. Li, and H. Qin, Enhancement of the two-photon blockade in a strong-coupling qubit-cavity system, Phys. Rev. A 91, 043831 (2015).

C. J. Zhu, Y. P. Yang, and G. S. Agarwal, Collective multiphoton blockade in cavity quantum electrodynamics, Phys. Rev. A 95, 063842 (2017).

R. Huang, A. Miranowicz, J.-Q. Liao, F. Nori, and H. Jing, Nonreciprocal Photon Blockade, Phys. Rev. Lett. 121, 153601 (2018).

Q. Bin, X.-Y. Lü, S.-W. Bin, and Y. Wu, Two-photon blockade in a cascaded cavity-quantum-electrodynamics system, Phys. Rev. A 98, 043858 (2018).

J. Z. Lin, K. Hou, C. J. Zhu, and Y. P. Yang, Manipulation and improvement of multiphoton blockade in a cavity-QED system with two cascade three-level atoms, Phys. Rev. A 99, 053850 (2019).

C. J. Villas-Boas and D. Z. Rossatto, Multiphoton Jaynes-Cummings Model: Arbitrary Rotations in Fock Space and Quantum Filters, Phys. Rev. Lett. 122, 123604 (2019).

A. Majumdar, M. Bajcsy, A. Rundquist, and J. Vučković, Loss-Enabled Sub-Poissonian Light Generation in a Bimodal Nanocavity, Phys. Rev. Lett. 108, 183601 (2012).

A. Majumdar, M. Bajcsy, and J. Vučković, Probing the ladder of dressed states and nonclassical light generation in quantum-dot–cavity QED, Phys. Rev. A 85, 041801 (2012).

A. Rundquist, M. Bajcsy, A. Majumdar, T. Sarmiento, K. Fischer, K. G. Lagoudakis, S. Buckley, A. Y. Pigott, and J. Vučković, Nonclassical higher-order photon correlations with a quantum dot strongly coupled to a photonic-crystal nanocavity, Phys. Rev. A 90, 023846 (2014).

X.-W. Xu, Y.-J. Li, and Y.-x. Liu, Photon-induced tunneling in optomechanical systems, Phys. Rev. A 87, 025803 (2013).

C. V. Sukumar and B. Buck, Multi-phonon generalisation of the Jaynes-Cummings model, Phys. Lett. A 83, 211 (1981).

S. Singh, Field statistics in some generalized Jaynes-Cummings models, Phys. Rev. A 25, 3206 (1982).

C. C. Gerry, Two-photon Jaynes-Cummings model interacting with the squeezed vacuum, Phys. Rev. A 37, 2683 (1988).

J.-s. Peng and G.-x. Li, Influence of the virtual-photon processes on the squeezing of light in the two-photon Jaynes-Cummings model, Phys. Rev. A 47, 3167 (1993).

K. Ng, C. Lo, and K. Liu, Exact eigenstates of the two-photon Jaynes-Cummings model with the counter-rotating term, Eur. Phys. J. D 6, 119 (1999).

I. Travénc, Solvability of the two-photon Rabi Hamiltonian, Phys. Rev. A 85, 043805 (2012).

Q.-H. Chen, C. Wang, S. He, T. Liu, and K.-L. Wang, Exact solvability of the quantum Rabi model using Bogoliubov operators, Phys. Rev. A 86, 023822 (2012).

Y.-L. Dong, S.-Q. Zhu, and W.-L. You, Quantum-state transmission in a cavity array via two-photon exchange, Phys. Rev. A 85, 023833 (2012).

S. Felicetti, J. S. Pedernales, I. L. Egusquiza, G. Romero, L. Lamata, D. Braak, and E. Solano, Spectral collapse via two-phonon interactions in trapped ions, Phys. Rev. A 92, 033817 (2015).

R. Puebla, M.-J. Hwang, J. Casanova, and M. B. Plenio, Protected ultrastrong coupling regime of the two-photon quantum Rabi model with trapped ions, Phys. Rev. A 95, 063844 (2017).

S. Felicetti, D. Z. Rossatto, E. Rico, E. Solano, and P. Fern-Díaz, Two-photon quantum Rabi model with superconducting circuits, Phys. Rev. A 97, 013851 (2018).

J. R. Johansson, P. D. Nation, and F. Nori, QuTiP: An
open-source Python framework for the dynamics of open quantum systems, *Comput. Phys. Commun.* **183**, 1760 (2012).

[84] J. R. Johansson, P. D. Nation, and F. Nori, QuTiP 2: A Python framework for the dynamics of open quantum systems, *Comput. Phys. Commun.* **184**, 1234 (2013).