Black Holes and Thermodynamics

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Abstract

We review the remarkable relationship between the laws of black hole mechanics and the ordinary laws of thermodynamics. It is emphasized that – in analogy with the laws of thermodynamics – the validity the laws of black hole mechanics does not appear to depend upon the details of the underlying dynamical theory (i.e., upon the particular field equations of general relativity). It also is emphasized that a number of unresolved issues arise in “ordinary thermodynamics” in the context of general relativity. Thus, a deeper understanding of the relationship between black holes and thermodynamics may provide us with an opportunity not only to gain a better understanding of the nature of black holes in quantum gravity, but also to better understand some aspects of the fundamental nature of thermodynamics itself.
1 Introduction

Undoubtedly, one of the most remarkable developments in theoretical physics to have occurred during the past twenty five years was the discovery of a close relationship between certain laws of black hole physics and the ordinary laws of thermodynamics. It appears that these laws of “black hole mechanics” and the laws of thermodynamics are two major pieces of a puzzle that fit together so perfectly that there can be little doubt that this “fit” is of deep significance. The existence of this close relationship between these laws may provide us with a key to our understanding of the fundamental nature of black holes in a quantum theory of gravity, as well as to our understanding of some aspects of the nature of thermodynamics itself. The aim of this article is to review the nature of the relationship between the black hole and thermodynamics laws. Although some notable progress has been made, many mysteries remain.

It was first pointed out by Bekenstein [1] that a close relationship might exist between certain laws satisfied by black holes in classical general relativity and the ordinary laws of thermodynamics. The area theorem of classical general relativity [2] states that the area, $A$, of a black hole can never decrease in any process

$$\Delta A \geq 0.$$  \hspace{1cm} (1)

Bekenstein noted that this result is closely analogous to the statement of ordinary second law of thermodynamics: The total entropy, $S$, of a closed system never decreases in any process

$$\Delta S \geq 0.$$  \hspace{1cm} (2)

Indeed, Bekenstein proposed that the area of a black hole (times a constant of order unity in Planck units) should be interpreted as its physical entropy.

A short time later, the analogy between certain laws of black hole physics in classical general relativity and the laws of thermodynamics was developed systematically by Bardeen, Carter, and Hawking [3]. They proved that in general relativity, the surface gravity, $\kappa$, of a stationary black hole (defined by eq. (19) below) must be constant over the event horizon of the black hole. They noted that this result is analogous to the zeroth law of thermodynamics, which states that the temperature, $T$, must be uniform over a body in thermal equilibrium. Finally, Bardeen, Carter, and Hawking proved the “first
law of black hole mechanics”. In the vacuum case, this law states that the differences in mass, \( M \), area, \( A \), and angular momentum, \( J \) of two nearby stationary black holes must be related by

\[
\delta M = \frac{1}{8\pi} \kappa \delta A + \Omega \delta J,
\]

where \( \Omega \) denotes the angular velocity of the event horizon (defined by eq. (18) below). Additional terms may appear on the right side of eq. (3) when matter fields are present. They noted that this law is closely analogous to the ordinary first law of thermodynamics, which states that differences in energy, \( E \), entropy, and other state parameters of two nearby thermal equilibrium states of a system is given by

\[
\delta E = T \delta S + \text{“work terms”}.
\]

If we compare the zeroth, first, and second laws of ordinary thermodynamics with the corresponding laws of black hole mechanics, we see that the analogous quantities are, respectively, \( E \leftrightarrow M \), \( T \leftrightarrow \alpha \kappa \), and \( S \leftrightarrow A/8\pi \alpha \), where \( \alpha \) is a undetermined constant. Even in the context of classical general relativity, a hint that this relationship might be of physical significance arises from the fact that \( E \) and \( M \) represent the same physical quantity, namely the total energy of the system. However, in classical general relativity, the physical temperature of a black hole is absolute zero, so there can be no physical relationship between \( T \) and \( \kappa \). Consequently, it also would be inconsistent to assume a physical relationship between \( S \) and \( A \). For this reason, at the time the paper of Bardeen, Carter, and Hawking appeared, most researchers (with the notable exception of Bekenstein) viewed the analogy between the black hole and thermodynamical laws as a mathematical curiosity, devoid of any physical significance.

That view changed dramatically with Hawking’s discovery [4] that, due to quantum particle creation effects, a black hole radiates to infinity all species of particles with a perfect black body spectrum, at temperature (in units with \( G = c = \hbar = k = 1 \))

\[
T = \frac{\kappa}{2\pi}.
\]

Thus, \( \kappa/2\pi \) truly is the \textit{physical} temperature of a black hole, not merely a quantity playing a role mathematically analogous to temperature in the
laws of black hole mechanics. This left little doubt that the laws of black hole mechanics must correspond physically to the laws of thermodynamics applied to a system consisting of a black hole. As will be discussed further below, it also left little doubt that \( A/4 \) must represent the physical entropy of a black hole in general relativity.

Thus, Hawking's calculation of particle creation effectively gave a resoundingly positive answer to the question of whether there exists any physical significance to mathematical the relationship between the laws of black hole mechanics and the laws of thermodynamics. This conclusion is particularly intriguing, since, as I shall review in sections II and III, the the derivations of the laws of black hole mechanics are so different in nature from those of the laws of thermodynamics that it is hard to see how it is possible that these laws could really be “the same”. As I will discuss further in section IV, this conclusion also raises a number of new questions and issues, most prominently that of providing a physical explanation for the origin of black hole entropy.

2 The nature of the laws of ordinary thermodynamics

The ordinary laws of thermodynamics are not believed to be fundamental laws in their own right, but rather to be laws which arise from the fundamental “microscopic dynamics” of a sufficiently complicated system when one passes to a macroscopic description of it. The great power and utility of the laws of thermodynamics stems mainly from the fact that the basic form of the laws does not depend upon the details of the underlying “microscopic dynamics” of particular systems and, thus, the laws have a “universal” validity – at least for a very wide class of systems.

The analysis of how the laws of ordinary thermodynamics arise applies to a classical or quantum system with a large number of degrees of freedom, whose time evolution is governed by Hamiltonian dynamics. It is important to emphasize that it is crucial here, at the outset, that there be a well defined notion of “time translations” (to which the Hamiltonian is conjugate), and that the Hamiltonian, \( H \), (and, thus, the dynamics) be invariant under these time translations. It then follows that the total energy, \( E \), of the system (i.e.,
the value of $H$) is conserved.

We now shall focus on the case of a classical dynamical system. By the previous remark, it follows that each dynamical orbit in phase space is confined to its “energy shell” $\Sigma_E$, i.e., the hypersurface in phase space defined by the equation $H(x) = E = \text{constant}$. The crucial assumption needed for the applicability of thermodynamical laws to such a system is that “generic” orbits in phase space behave “ergodically” in the sense that they come arbitrarily close to all points of $\Sigma_E$, spending “equal times in equal volumes”; equivalently, the total energy of the system is the only nontrivial constant of motion for generic orbits. (By a slight modification of these arguments, the laws also can accommodate the presence of a small number (compared with the number of degrees of freedom) of additional constants of motion – such as the angular momentum of a rotationally invariant system.) Of course, even when such ergodic behavior occurs, it would take a dynamical orbit an infinite amount of time to completely “sample” $\Sigma_E$. The degree of “sampling” which is actually needed for the applicability of thermodynamics depends upon what observable (or collection of observables) of the dynamical system is being measured – or, in more common terminology, the amount of “coarse graining” of phase space that one does. For a “fine-grained” observable (corresponding to measuring detailed information about the “microscopic degrees of freedom” of the system), the “sampling” of the energy shell must be extremely good, and the amount of time needed for this “sampling” will be correspondingly long, thereby making the laws of thermodynamics inapplicable or irrelevant for the system. However, for the types of “macroscopic, coarse-grained” observables, $\mathcal{O}$, usually considered for systems with a huge number of degrees of freedom, the “sampling” need only be quite modest and the required “sampling time” (= the timescale for the system to “reach thermal equilibrium”) is correspondingly short. One may then get considerable predictive power from the laws of thermodynamics about the values of $\mathcal{O}$ which one would expect to observe for the system under various physical conditions.

The statistical entropy, $S_{\mathcal{O}}$, of a classical dynamical system relative to a macroscopic, coarse-grained observable (or collection of observables), $\mathcal{O}$, is defined to be the observable whose value at point $x$ on $\Sigma_E$ is the logarithm of the volume of the region of the energy shell at which $\mathcal{O}$ takes the same
value as it does at $x$, i.e.,
\[
S_\mathcal{O}(x) = \ln[\text{vol}(\mathcal{R}_x)]
\] (6)
where
\[
\mathcal{R}_x = \{y \in \Sigma_E | \mathcal{O}(y) = \mathcal{O}(x)\}.
\] (7)

For the types of coarse-grained observables $\mathcal{O}$ which are normally considered, the largest region, $\mathcal{R}_\text{max}$, of the form (7) will have a volume nearly equal to that of the entire energy shell $\Sigma_E$. If dynamical orbits sample $\Sigma_E$ and spend “equal times in equal volumes”, then we would expect $S_\mathcal{O}$ to increase in value until it reaches its maximum possible value, namely, $\ln[\text{vol}(\mathcal{R}_\text{max})] \simeq \ln[\text{vol}(\Sigma_E)]$. Subsequently, $S_\mathcal{O}$ should remain at that value for an extremely long time. During this extremely long period, the value of $\mathcal{O}$ remains unchanged, so no change would be perceived in the system, and the system would be said to have achieved “thermal equilibrium”.

The *thermodynamic entropy*, $S$, of the system is defined by
\[
S = \ln[\text{vol}(\Sigma_E)]
\] (8)
Unlike $S_\mathcal{O}$, $S$ is not an observable on phase space but rather a function on a low dimensional *thermodynamic state space* comprised by the total energy, $E$, of the system, and any parameters (such as, for example, an external magnetic field) appearing in the Hamiltonian which one might contemplate varying, together with any additional constants of motion for the system (such as, the total angular momentum for a rotationally invariant system). These variables characterizing thermodynamic state space are usually referred to as state parameters. The temperature, $T$, is defined by
\[
\frac{1}{T} = \frac{\partial S}{\partial E},
\] (9)
where the remaining state parameters are held fixed in taking this partial derivative. Like $S$, the temperature, $T$, is a function on thermodynamic state space, not an observable on phase space.

It follows from the above discussion that when a system is in thermal equilibrium, its statistical entropy, $S_\mathcal{O}$, equals its thermodynamic entropy, $S$. Similarly suppose a system consists of weakly interacting subsystems, so that each subsystem can be viewed as an isolated system in its own right. Suppose
that $O$ is comprised by a collection of observables, $O_i$, for each subsystem, and suppose, in addition, that each subsystem (viewed as an isolated system) is in thermal equilibrium – although the entire system need not be in thermal equilibrium. Then $S_O$ will equal the sum of the thermodynamic entropies, $S_i$, of the subsystems

$$S_O = \sum_i S_i.$$  \hfill (10)

We now are in a position to explain the origin of the laws of thermodynamics. As argued above, if $S_O$ is less than its maximum possible value (for the given value of $E$ and the other state parameters), we should observe it to increase until “thermal equilibrium” is reached. In particular, in the case where the total system consists of subsystems and these subsystems are individually in thermal equilibrium at both the beginning and the end of some process (but not necessarily at the intermediate stages), we should have

$$\sum_i (S_i)_{1} \geq \sum_i (S_i)_{0},$$  \hfill (11)

where $(S_i)_{0}$ and $(S_i)_{1}$ denote, respectively, the initial and final thermodynamic entropies of the $i$th subsystem. This accounts for the second law of thermodynamics, eq. (3).

It should be noted that the time asymmetry present in this formulation of the second law arises from the implicit assumption that, commonly, $S_O$ is initially below its maximum possible value. (Otherwise, a more relevant formulation of the second law would merely state that only very rarely and/or briefly would we expect to observe $S_O$ to fluctuate below its maximum possible value.) The fact that we do commonly observe systems with $S_O$ below its maximum possible value shows that the present state of our universe is very “special”.

The zeroth law of thermodynamics is an immediate consequence of the fact that if the subsystems appearing in eq. (11) are at different temperatures, then $S_O$ can be increased by transferring energy from a subsystem of high temperature to a subsystem of low temperature. (This fact follows immediately from the definition, eq. (4), of $T$.) Thus, for a thermal equilibrium state – where, by definition, $S_O$ achieves its maximum value – it is necessary that $T$ be uniform.

Finally, since $S$ is a function on thermodynamic state space, its gradient
can be written as

\[ dS = \frac{1}{T} dE + \sum_j X_j d\alpha_j \quad (12) \]

where \( \alpha_j \) denotes the state parameters other than \( E \), and \( X_j \equiv \partial S / \partial \alpha_j \) (where \( E \) and the state parameters other than \( \alpha_j \) are held fixed in taking this partial derivative). Using Liouville’s theorem, one can argue that \( S \) should be constant when sufficiently slow changes are made to parameters appearing in the Hamiltonian. This fact gives \( TX_j \) the interpretation of being the “generalized force” conjugate to \( \alpha_j \) (at least for the case where \( \alpha_j \) is a parameter appearing in the Hamiltonian), and it gives \( TX_j d\alpha_j \) the interpretation of being a “work term”. This accounts for the first law of thermodynamics, eq. (4).

Thus far, our discussion has been restricted to the case of a classical dynamical system. However, as discussed in more detail in [5], a completely parallel analysis can be for a quantum system. In this analysis, the classical coarse-grained observable \( \mathcal{O} \) on phase space is replaced by a self-adjoint operator \( \hat{\mathcal{O}} \) acting on the Hilbert space of states with energy between \( E \) and \( E + \Delta E \). The spectral decomposition of \( \hat{\mathcal{O}} \) takes the form

\[ \hat{\mathcal{O}} = \sum \lambda_m \hat{P}_m \quad (13) \]

where the \( \{ \hat{P}_m \} \) are a family of orthogonal projection operators, and it is assumed – in correspondence with the assumptions made about coarse-graining in the classical case – that the degeneracy subspaces of \( \hat{\mathcal{O}} \) are large. The statistical entropy, \( \hat{S}_\mathcal{O} \) is then defined to be the quantum observable

\[ \hat{S}_\mathcal{O} = \sum \ln(d_m) \hat{P}_m \quad (14) \]

where \( d_m \) is the dimension of the \( m \)th degeneracy subspace of \( \lambda_m \), i.e.,

\[ d_m = \text{tr}(\hat{P}_m). \quad (15) \]

Again, it is assumed that the maximum value of \( d_m \) is essentially the dimension of the entire Hilbert space of states of energy between \( E \) and \( E + \Delta E \).

The corresponding definition of the thermodynamic entropy, \( S \), of a quantum system is

\[ S = \ln n \quad (16) \]
where $n$ denotes the dimension of the Hilbert space of states between $E$ and $E + \Delta E$, i.e., $n$ is proportional to the density of quantum states per unit energy. Again, $S$ is not a quantum observable, but rather a function on thermodynamic state space. Arguments for the zeroth, first, and second laws of thermodynamics then can be made in parallel with the classical case.

It should be noted that, thus far, I have made no mention of the third law of thermodynamics. In fact, there are two completely independent statements which are referred to as the “third law”. The first statement consists of the rather vague claim that it is physically impossible to achieve $T = 0$ for a (sub)system. To the extent that it is true, I would view this claim as essentially a consequence of the second law, since it always is highly entropically favorable to take energy away from a subsystem at finite temperature and add that energy to a subsystem whose temperature is very nearly 0. The second statement, usually referred to as “Nernst’s theorem”, consists of the claim that $S \to 0$ as $T \to 0$. This claim is blatantly false in classical physics – it fails even for a classical ideal gas – but it holds for many quantum systems (in particular, for boson and fermion ideal gases). Clearly, the “Nernst theorem” is actually a claim about the behavior of the density of states, $n(E)$, as the total energy of the system goes to its minimum possible value. Indeed, more precisely, as explained in section 9.4 of [6], it should be viewed as a statement about the extrapolation to minimum energy of the continuum approximation to $n(E)$. Elsewhere, I shall give some examples of quantum systems which violate the “Nernst theorem” [7] (see also Section IV below). Thus, while the “Nernst theorem” holds empirically for systems studied in the laboratory, I do not view it as a fundamental aspect of thermodynamics. In particular, I do not feel that the well known failure of the analog of the “Nernst theorem” to hold for black hole mechanics – where there exist black holes of finite area with $\kappa = 0$ – should be viewed as indicative of any breakdown of the relationship between thermodynamics and black hole physics.

The above discussion explains the nature and origin of the laws of thermodynamics for “ordinary” classical and quantum systems. However, as I shall now briefly describe, when general relativity is taken into account, a number of new issues and puzzles arise.

In the first place, it should be noted that general relativity is a field theory and, as such, ascribes infinitely many degrees of freedom to the spacetime metric/gravitational field. If these degrees of freedom are treated classi-
cally, no sensible thermodynamics should be possible. Indeed, this situation also arises for the electromagnetic field, where a treatment of the statistical physics of a classical electromagnetic field in a box yields the Rayleigh-Jeans distribution and its associated “ultraviolet catastrophe”. As is well known, this difficulty is cured by treating the electromagnetic field as a quantum field. I see no reason not to believe that similar difficulties and cures will occur for the gravitational field. However, it should be emphasized at the outset that one should not expect any thermodynamic laws to arise from a statistical physics treatment of classical general relativity; a quantum treatment of the degrees of freedom of the gravitational field should be essential.

A much more perplexing issue arises from the fact that, as emphasized above, the arguments for the validity of thermodynamics for “ordinary systems” are based upon the presence of a well defined notion of “time translations”, which are symmetries of the dynamics. Such a structure is present when one considers dynamics on a background spacetime whose metric possesses a suitable one-parameter group of isometries, and when the Hamiltonian is invariant under these isometries. However, such a structure is absent in general relativity, where no background metric is present. Furthermore, when the degrees of freedom of the gravitational field are excited, one would not expect the dynamical spacetime metric to possess a time translation symmetry. The absence of any “rigid” time translation structure in general relativity can be viewed as being responsible for making notions like the “energy density of the gravitational field” ill defined in general relativity. Notions like the “entropy density of the gravitational field” are not likely to fare any better. It may still be possible to use structures like asymptotic time translations to define the notion of the total entropy of an (asymptotically flat) isolated system. (As is well known, total energy can be defined for such systems.) However, for a closed universe, it seems unlikely that any meaningful notion will exist for the “total entropy of the universe” (including gravitational entropy). If so, it is far from clear how the second law of thermodynamics is to be formulated for a closed universe. This issue appears worthy of further exploration.

Another important issue that arises in the context of general relativity involves ergodic behavior. As discussed above, ordinary thermodynamics is predicated on the assumption that generic dynamical orbits “sample” the entire energy shell, spending “equal times in equal volumes”. However, gross violations of such ergodic behavior occur in classical general relativity on
account of the irreversible tendency for gravitational collapse to produce singularities – from which one cannot then evolve back to “uncollapsed” states. Interestingly, however, there are strong hints that ergodic behavior could be restored in quantum gravity. In particular, the quantum phenomenon of black hole evaporation (see Section IV below) provides a means of evolving from a collapsed state back to an uncollapsed configuration.

Finally, as noted above, the fact that we commonly observe the increase of entropy shows that the present state of the universe is very “special”. As has been emphasized by Penrose [8], the “specialness” of the present state of the universe traces back, ultimately, to extremely special initial conditions for the universe at the “big bang”. This specialness of the initial state of the universe should have some explanation in a complete, fundamental theory. However, at present, it remains a matter of speculation as to what this explanation might be.

The above comments already give a clear indication that there are deep and fundamental issues lying at the interface of gravitation and thermodynamics. As we shall see in the next two sections, the theory of black holes gives rise to very significant further relationships between gravitation and thermodynamics.

3 The nature of the laws of classical black hole mechanics

As previously indicated in the Introduction, it appears overwhelmingly likely that the laws of classical black hole mechanics must arise, in a fundamental quantum theory of gravity, as the classical limit of the laws of thermodynamics applied to a system comprised by a black hole. However, as we shall see in this section, the present derivations of the laws of classical black hole mechanics could hardly look more different from the arguments for the corresponding laws of thermodynamics, as given in the previous section. Nevertheless, as I shall emphasize here, the derivations of the laws of black hole mechanics appear to share at least one important feature of the thermodynamic arguments: There appears to be a “universality” to the laws of black hole mechanics in that the basic form of the laws appears to be independent of the details of the precise Lagrangian or Hamiltonian of the underlying
theory of gravity — in a manner analogous to the “universality” of the form of the laws of ordinary thermodynamics.

In this section, we will consider theories of gravity which are much more general than general relativity, but we shall restrict attention to geometric theories, wherein spacetime is represented by a pair \((M, g_{ab})\), where \(M\) is a manifold and \(g_{ab}\) is a metric of Lorentzian signature. Other matter fields also may be present on spacetime. For definiteness, I will assume that \(M\) is 4-dimensional, but all results below generalize straightforwardly to any dimension \(n \geq 2\). For our discussion of the first and second laws of black hole mechanics, it will be assumed, in addition, that the field equations of the theory have been obtained from a diffeomorphism covariant Lagrangian.

In physical terms, a black hole in a spacetime, \((M, g_{ab})\), is a region where gravity is so strong that nothing can escape. In order to make this notion precise, one must have in mind a region of spacetime to which one can contemplate escaping. For an asymptotically flat spacetime (representing an isolated system), the asymptotic portion of the spacetime “near infinity” is such a region. The black hole region, \(B\), of an asymptotically flat spacetime, \((M, g_{ab})\), is defined as

\[
B \equiv M - I^- (I^+),
\]

where \(I^+\) denotes future null infinity and \(I^-\) denotes the chronological past. The event horizon, \(\mathcal{H}\), of a black hole is defined to be the boundary of \(B\).

If an asymptotically flat spacetime \((M, g_{ab})\) contains a black hole \(B\), then \(B\) is said to be stationary if there exists a one-parameter group of isometries on \((M, g_{ab})\) generated by a Killing field \(t^a\) which is unit timelike at infinity. The black hole is said to be static if it is stationary and if, in addition, \(t^a\) is hypersurface orthogonal — in which case there exists a discrete “time reflection” isometry about any of the orthogonal hypersurfaces. The black hole is said to be axisymmetric if there exists a one parameter group of isometries which correspond to rotations at infinity. A stationary, axisymmetric black hole is said to possess the “\(t - \phi\) orthogonality property” if the 2-planes spanned by \(t^a\) and the rotational Killing field \(\phi^a\) are orthogonal to a family of 2-dimensional surfaces. In this case, there exists a discrete “\(t - \phi\)” reflection isometry about any of these orthogonal 2-dimensional surfaces.

For a black hole which is static or is stationary-axisymmetric with the \(t - \phi\) orthogonality property, it can be shown [3] that there exists a Killing
field $\xi^a$ of the form

$$\xi^a = t^a + \Omega \phi^a$$

which is normal to the event horizon, $\mathcal{H}$. The constant $\Omega$ defined by eq. (18) is called the \textit{angular velocity of the horizon}. (For a static black hole, we have $\Omega = 0$.) A null surface whose null generators coincide with the orbits of a one-parameter group of isometries is called a \textit{Killing horizon}, so the above result states that the event horizon of any black hole which is static or is stationary-axisymmetric with the $t - \phi$ orthogonality property must always be a Killing horizon. A stronger result holds in general relativity, where, under some additional assumptions, it can be shown that the event horizon of any stationary black hole must be a Killing horizon \cite{10}. From this result, it also follows that in general relativity, a stationary black hole must be nonrotating (from which staticity follows \cite{1}, \cite{2}) or axisymmetric (though not necessarily with the $t - \phi$ orthogonality property).

Now, let $\mathcal{K}$ be any Killing horizon (not necessarily required to be the event horizon, $\mathcal{H}$, of a black hole), with normal Killing field $\xi^a$. Since $\nabla^a (\xi^b \xi_b)$ also is normal to $\mathcal{K}$, these vectors must be proportional at every point on $\mathcal{K}$. Hence, there exists a function, $\kappa$, on $\mathcal{K}$, known as the \textit{surface gravity} of $\mathcal{K}$, which is defined by the equation

$$\nabla^a (\xi^b \xi_b) = -2\kappa \xi^a$$

It follows immediately that $\kappa$ must be constant along each null geodesic generator of $\mathcal{K}$, but, in general, $\kappa$ can vary from generator to generator. It is not difficult to show (see, e.g., \cite{13}) that

$$\kappa = \lim (V_a)$$

where $a$ is the magnitude of the acceleration of the orbits of $\xi^a$ in the region off of $\mathcal{K}$ where they are timelike, $V \equiv (-\xi^a \xi_a)^{1/2}$ is the “redshift factor” of $\xi^a$, and the limit as one approaches $\mathcal{K}$ is taken. Equation (20) motivates the terminology “surface gravity”. Note that the surface gravity of a black hole is defined only when it is “in equilibrium” (i.e., stationary), analogous to the fact that the temperature of a (sub)system in ordinary thermodynamics is defined only for thermal equilibrium states.

In the context of an arbitrary metric theory of gravity, the zeroth law of black hole mechanics may now be stated as the following theorem \cite{9}, \cite{14}.
For any black hole which is static or is stationary-axisymmetric with the \( t - \phi \) orthogonality property, the surface gravity \( \kappa \), must be constant over its event horizon \( \mathcal{H} \). The key ingredient in the proof of this theorem is the identity

\[
\xi_\alpha \nabla_\beta \kappa = -\frac{1}{4} \epsilon_{\alpha\beta\gamma\delta} \nabla^\gamma \omega^\delta
\] (21)

which holds on an arbitrary Killing horizon, where \( \omega_\alpha \equiv \epsilon_{\alpha\beta\gamma\delta} \xi^\beta \nabla^\gamma \xi^\delta \) denotes the twist of the Killing field \( \xi^\alpha \). For a static black hole, we have \( \omega_\alpha = 0 \), and the constancy of \( \kappa \) on \( \mathcal{H} \) follows immediately. Further arguments similarly establish the constancy of \( \kappa \) for a stationary-axisymmetric black hole with the \( t - \phi \) orthogonality property. It should be emphasized that this result is “purely geometrical”, and involves no use of any field equations.

A stronger version of the zeroth law holds in general relativity. There it can be shown that if Einstein’s equation holds with the matter stress-energy tensor satisfying the dominant energy condition, then \( \kappa \) must be constant on any Killing horizon. In particular, one need not make the additional hypothesis that the \( t - \phi \) orthogonality property holds.

An important consequence of the zeroth law is that if \( \kappa \neq 0 \), then in the “maximally extended” spacetime representing the black hole, the event horizon, \( \mathcal{H} \), comprises a branch of a “bifurcate Killing horizon”. (A precise statement and proof of this result can be found in [14]. Here, a bifurcate Killing horizon is comprised by two Killing horizons, \( \mathcal{H}_A \) and \( \mathcal{H}_B \), which intersect on a spacelike 2-surface, \( \mathcal{C} \), known as the bifurcation surface.) As stated above, the event horizon of any black hole which is static or is stationary-axisymmetric with the \( t - \phi \) orthogonality property, necessarily is a Killing horizon and necessarily satisfies the zeroth law. Thus, the study of such black holes divides into two cases: “degenerate” black holes (for which, by definition, \( \kappa = 0 \)), and black holes with bifurcate horizons. Again, this result is “purely geometrical” – involving no use of any field equations – and, thus, it holds in any metric theory of gravity.

We turn, now, to the consideration of first law of black hole mechanics. For this analysis, it will be assumed that the field equations of the theory arise from a diffeomorphism covariant Lagrangian 4-form, \( \mathbf{L} \), of the general structure

\[
\mathbf{L} = \mathbf{L}(g_{ab}, R_{abcd}, \nabla_a R_{bde}, \ldots; \psi, \nabla_a \psi, \ldots)
\] (22)

where \( \nabla_a \) denotes the derivative operator associated with \( g_{ab}, R_{abcd} \) denotes
the Riemann curvature tensor of $g_{ab}$, and $\psi$ denotes the collection of all matter fields of the theory (with indices suppressed). An arbitrary (but finite) number of derivatives of $R_{abcd}$ and $\psi$ are permitted to appear in $L$. Here and below we use boldface letters to denote differential forms and we will suppress their indices. We also shall denote the complete collection of dynamical fields, $(g_{ab}, \psi)$ by $\phi$ (thereby suppressing the indices of $g_{ab}$ as well). Our treatment will follow closely that given in [15]; much of the mathematical machinery we shall use also has been extensively employed in analyses of symmetries and conservation laws of Lagrangian systems (see, e.g., [16] and references cited therein).

The Euler Lagrange equations of motion, $E = 0$, are obtained by writing the variation of the Lagrangian in the form

$$\delta L = E(\phi)\delta \phi + d\theta(\phi, \delta \phi).$$

where no derivatives of $\delta \phi$ appear in the first term on the right side. Usually, the manipulations yielding eq.(23) are performed under an integral sign, in which case the second term on the right side becomes a “boundary term”, which normally is discarded. In our case, however, our interest will be in the mathematical structure to the theory provided by $\theta$. Indeed, the precise form of $\theta$ and the auxiliary structures derived from it will play a crucial role in our analysis, whereas the precise form of the field equations, $E = 0$, will not be of interest here.

The sympletic current 3-form on spacetime, $\omega$ – which is a local function of a field configuration, $\phi$, and two linearized perturbations, $\delta_1 \phi$ and $\delta_2 \phi$ off of $\phi$ – is obtained by taking an antisymmetrized variation of $\theta$

$$\omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_2 \theta(\phi, \delta_1 \phi) - \delta_1 \theta(\phi, \delta_2 \phi)$$

The (pre-)symplectic form, $\Omega$ – which is a map taking field configurations, $\phi$, together with a pairs of linearized perturbations off of $\phi$, into the real numbers – is obtained by integrating $\omega$ over a Cauchy surface, $\Sigma$

$$\Omega(\phi, \delta_1 \phi, \delta_2 \phi) = \int_{\Sigma} \omega$$

(This integral is independent of choice of Cauchy surface when $\delta_1 \phi$ and $\delta_2 \phi$ satisfy the linearized field equations.) The (pre-)symplectic form, $\Omega$, provides the structure needed to define the phase space of the theory [17]. It also
provides the structure needed to define the notion of a Hamiltonian, $H$, conjugate to an arbitrary vector field, $\eta^a$, on spacetime. $H$ is a function on phase space satisfying the property that about any solution, $\phi$, the variation of $H$ satisfies

$$\delta H = \Omega(\phi, \delta \phi, L_{\eta} \phi),$$

(26)

where $L_{\eta}$ denotes the Lie derivative with respect to the vector field $\eta^a$. (Equation (26) can be put in the more familiar form of Hamilton’s equations of motion by solving it for $L_{\eta} \phi$, thus expressing the “time derivative” of $\phi$ in terms of functional derivatives of $H$.)

On account of the diffeomorphism covariance of $L$, the infinitesimal diffeomorphism generated by an arbitrary vector field, $\eta^a$, is a local symmetry of the theory. Hence, there is an associated, conserved Noether current 3-form, $j$, defined by

$$j = \theta(\phi, L_{\eta} \phi) - \eta \cdot L$$

(27)

where the “$\cdot$” denotes the contraction of the vector field $\eta^a$ into the first index of the differential form $L$. One can show \cite{18} that $j$ always can be written in the form

$$j = dQ + \eta^a C_a,$$

(28)

where $C_a = 0$ when the equations of motion hold, i.e., $C_a$ corresponds to “constraints” of the theory. Equation (28) defines the Noether charge 2-form $Q$, which is unique up to

$$Q \rightarrow Q + \eta \cdot X(\phi) + Y(\phi, L_{\eta} \phi) + dZ(\phi, \eta).$$

(29)

where $X$, $Y$, and $Z$ are arbitrary forms which are locally constructed from the fields appearing in their arguments (and with $Y$ being linear in $L_{\eta} \phi$ and $Z$ being linear in $\eta$). Here the term $\eta \cdot X$ arises from the ambiguity $L \rightarrow L + dX$ in the choice of Lagrangian, the term $Y(\phi, L_{\eta} \phi)$ arises from the ambiguity $\theta \rightarrow \theta + dY$ in eq. (23), and the term $dZ$ arises directly from eq. (28).

The first law of black hole mechanics is a direct consequence of the variational identity

$$\delta j = \omega(\phi, \delta \phi, L_{\eta} \phi) + d(\eta \cdot \theta),$$

(30)

$\footnote{As discussed in \cite{17}, it will, in general, be necessary to choose $\eta^a$ to be “field dependent”, in order that it “project” to phase space.}$
which follows directly from eqs. (23), (24) and (27) above. One immediate consequence of this identity, together with eq. (26), is that if a Hamiltonian, \( H \), conjugate to \( \eta^a \) exists, it must satisfy

\[
\delta H = \int_\Sigma [\delta j - d(\eta \cdot \theta)]
\]

(31)

From eq. (28) it follows, in addition, that “on shell” – i.e., when the equations of motion hold and, hence, \( C_a = 0 \) – we have

\[
\delta H = \int_\Sigma d[\delta Q - \eta \cdot \theta],
\]

(32)

so, “on shell”, \( H \) is given purely by “surface terms”. In the case of an asymptotically flat spacetime, the surface term, \( H_\infty \), arising from infinity has the interpretation of being the total “canonical energy” (conjugate to \( \eta^a \)) of the spacetime.

Now, let \( \phi \) be any solution to the field equations \( E = 0 \) with a Killing field \( \xi^a \), and let \( \delta \phi \) be any solution to the linearized field equations off \( \phi \) (not necessarily satisfying \( \mathcal{L}_\xi \delta \phi = 0 \)). It follows immediately from eq. (30) (with \( \eta^a = \xi^a \)) together with the variation of eq. (28) that

\[
d[\delta Q - \xi \cdot \theta] = 0
\]

(33)

We apply this equation to a spacetime containing a black hole with bifurcate Killing horizon, with \( \xi^a \) taken to be the Killing field \( \xi^a \) normal to the horizon, \( \mathcal{H} \). (As mentioned above, the assumption of a bifurcate Killing horizon involves no loss of generality if the zeroth law holds and \( \kappa \neq 0 \).) We integrate this equation over a hypersurface, \( \Sigma \), which extends from the bifurcation surface, \( \mathcal{C} \), of the black hole to infinity. The result is

\[
\delta H_\infty = \delta \int_\mathcal{C} Q
\]

(34)

where the fact that \( \xi^a = 0 \) on \( \mathcal{C} \) has been used.

We now evaluate the surface terms appearing on each side of eq. (34). As noted above, \( H_\infty \) has the interpretation of being the canonical energy conjugate to \( \xi^a \). For \( \xi^a \) of the form \( \alpha \mathcal{I} \), we have (see [14])

\[
\delta H_\infty = \delta M - \Omega \delta J + ...
\]

(35)
where the “...” denotes possible additional contributions from long range matter fields. On the other hand, it is possible to explicitly compute $\mathbf{Q}$, and thereby show that \[ \delta \int_C \mathbf{Q} = \frac{\kappa}{2\pi} \delta S_{\text{bh}}, \] \[ (36) \] where \[ S_{\text{bh}} \equiv -2\pi \int_C \frac{\delta L}{\delta R_{abcd}} n_{ab} n_{cd}. \] \[ (37) \] Here $n_{ab}$ is the binormal to $C$ (normalized so that $n_{ab} n^{ab} = -2$), $L$ is the Lagrangian (now viewed as a scalar density rather than a 4-form), and the functional derivative is taken by formally viewing the Riemann tensor as a field which is independent of the metric in eq.(22). Combining eqs.(34), (35), and (36), we obtain \[ \delta M = \frac{\kappa}{2\pi} \delta S_{\text{bh}} + \Omega \delta J + ..., \] \[ (38) \] which is the desired first law of black hole mechanics. Indeed, this result is actually stronger than the form of the first law stated in the Introduction, since eq.(38) holds for non-stationary perturbations of the black hole, not merely for perturbations to other stationary black hole states.

For the case of vacuum general relativity, where $L = R \sqrt{-g}$, a simple calculation yields \[ S_{\text{bh}} = A/4. \] \[ (39) \] However, if one considers theories with non-minimally-coupled matter or “higher derivative” theories of gravity, additional curvature contributions will appear in the formula for $S_{\text{bh}}$. Nevertheless, as eq. (37) explicitly shows, in all cases, $S_{\text{bh}}$ is given by an integral of a “local, geometrical expression” over the black hole horizon.

The above analysis also contains a strong hint that the second law of black hole mechanics may hold in a wide class of theories. Consider a stationary black hole with bifurcate Killing horizon, but now let us normalize the Killing field, $\xi^a$, normal to the horizon by the local condition that $\nabla_a \xi_b = n_{ab}$ on $C$ (or, equivalently, $\nabla_a \xi_b \nabla^a \xi^b = -2$ on $\mathcal{H}$), rather than by the asymptotic behavior \[ (18) \] of $\xi^a$ at infinity. With this new normalization, eq.(33) (with the $\delta$’s removed) becomes \[ S_{\text{bh}} = 2\pi \int_C \mathbf{Q}[\xi^a]. \] \[ (40) \]
It is easy to show that for a stationary black hole, this equation continues to hold when \( C \) is replaced by an arbitrary cross-section, \( \sigma \), of \( \mathcal{H} \).

Now, consider a process in which an initially stationary black hole evolves through a non-stationary era, and then “settles down” to another stationary final state. Let \( \xi^a \) be any vector field which coincides with the Killing field normal to \( \mathcal{H} \) (with the above, new normalization) in the two stationary regimes. Then, by eqs.(40) and (28) (together with \( C_a = 0 \)), we obtain

\[
\Delta S_{bh} = 2\pi \int_{\sigma_1} \mathcal{Q}[\xi^a] - 2\pi \int_{\sigma_0} \mathcal{Q}[\xi^a] = 2\pi \int_{\mathcal{H}} j[\xi^a],
\]

where \( \sigma_0 \) and \( \sigma_1 \) are, respectively, cross-sections of \( \mathcal{H} \) in the initial and final stationary regimes. Equation (41) states that the change in black hole entropy is proportional to the net flux of Noether current (conjugate to \( \xi^a \)) through \( \mathcal{H} \). Now, in many circumstances, the Noether current conjugate to a suitable time translation can be interpreted as the 4-density of energy-momentum. Thus, eq.(41) suggests that the second law of black hole mechanics, \( \Delta S_{bh} \geq 0 \), may hold in all theories which have suitable positive energy properties (such as, perhaps, a positive “Bondi energy flux” at null infinity). However, I have not, as yet obtained any general results along these lines. Nevertheless, it has long been known that the second law holds in general relativity [2], provided that the matter present in spacetime satisfies the following positive energy property (known as the “null energy condition”): for any null vector \( k^a \), the matter stress-energy tensor, \( T_{ab} \), satisfies \( T_{ab} k^a k^b \geq 0 \).

### 4 Quantum black hole thermodynamics

In the previous section, we used a purely classical treatment of gravity and matter fields to derive analogs of the laws of thermodynamics for black holes. However, as already noted in the Introduction, in classical physics, these laws of black hole mechanics cannot correspond physically to the laws of thermodynamics. It is only when quantum effects are taken into account that these subjects appear to merge.

The key result establishing a physical connection between the laws of black hole mechanics and the laws of thermodynamics is, of course, the thermal particle creation effect discovered by Hawking [3]. This result is derived
in the context of “semiclassical gravity”, where the effects of gravitation are still represented by a classical spacetime \((M, g_{ab})\), but matter fields are now treated as quantum fields propagating in this classical spacetime. In its most general form, this result may be stated as follows (see [19] for further discussion): Consider a black hole formed by gravitational collapse, which “settles down” to a stationary final state. By the zeroth law of black hole mechanics, the surface gravity, \(\kappa\), of this stationary black hole final state will be constant over its event horizon. Consider a quantum field propagating in this background spacetime, which is initially in any (non-singular) state. Then, at asymptotically late times, particles of this field will be radiated to infinity as though the black hole were a perfect black body at the Hawking temperature, eq. (5). Thus, a stationary black hole truly is a state of thermal equilibrium, and \(\kappa/2\pi\) truly is the physical temperature of a black hole. It should be noted that this result relies only on the analysis of quantum fields in the region exterior to the black hole. In particular, the details of the gravitational field equations play no role, and the result holds in any metric theory of gravity obeying the zeroth law.

The physical connection between the laws of black hole mechanics and the laws of thermodynamics is further cemented by the following considerations. If we take into account the “back reaction” of the quantum field on the black hole (i.e., if the gravitational field equations are used self-consistently, taking account of the gravitational effects of the quantum field), then it is clear that if energy is conserved in the full theory, an isolated black hole must lose mass in order to compensate for the energy radiated to infinity in the particle creation process. As a black hole thereby “evaporates”, \(S_{bh}\) will decrease, in violation of the second law of black hole mechanics. (Note that in general relativity, this can occur because the stress-energy tensor of quantum matter does not satisfy the null energy condition – even for matter for which this condition holds classically – in violation of one of the hypotheses of the area theorem.) On the other hand, there is a serious difficulty with the ordinary second law of thermodynamics when black holes are present: One can simply take some ordinary matter and drop it into a black hole, where, classically at least, it will disappear into a spacetime singularity. In this latter process, one loses the entropy initially present in the matter, but no compensating gain of ordinary entropy occurs, so the total entropy, \(S\), decreases. Note, however, that in the black hole evaporation process, although \(S_{bh}\) decreases, there is significant amount of ordinary entropy generated outside the black hole.
hole due to particle creation. Similarly, when ordinary matter (with positive energy) is dropped into a black hole, although $S$ decreases, by the first law of black hole mechanics, there will necessarily be an increase in $S_{bh}$.

The above considerations motivated the following proposal [1], [20]. Although the second law of black hole mechanics breaks down when quantum processes are considered, and the ordinary second law breaks down when black holes are present, perhaps the following law, known as the generalized second law always holds: In any process, the total generalized entropy never decreases

$$\Delta S' \geq 0,$$

where the generalized entropy, $S'$, is defined by

$$S' \equiv S + S_{bh}. \quad (43)$$

A number of analyses [21], [22], [23], [24] have given strong support to the generalized second law. Although these analyses have been carried out in the context of general relativity, the arguments for the validity of the generalized second law should be applicable to a general theory of gravity, provided, of course, that the second law of black hole mechanics holds in the classical theory.

The generalized entropy (43) and the generalized second law (42) have obvious interpretations: Presumably, for a system containing a black hole, $S'$ is nothing more than the “true total entropy” of the complete system, and (42) is then nothing more than the “ordinary second law” for this system. If so, then $S_{bh}$ truly is the physical entropy of a black hole.

Although I believe that the above considerations make a compelling case for the merger of the laws of black hole mechanics with the laws of thermodynamics, there remain many puzzling aspects to this merger. One such puzzle has to do with the existence of a “thermal atmosphere” around a black hole. It is crucial to the arguments for the validity of the generalized second law (see, in particular, [21]) that near the black hole, all fields are in thermal equilibrium with respect to the notion of time translations defined by the horizon Killing field $\xi^a$ (see eq. (18) above). For an observer following an orbit of $\xi^a$ just outside the black hole, the locally measured temperature (of all species of matter) is

$$T = \frac{\kappa}{2\pi V},$$

(44)
where $V = (-\xi^a \xi_a)^{1/2}$. Note that, in view of eq.(20) above, we see that $T \to a/2\pi$ as the black hole horizon, $\mathcal{H}$, is approached. Thus, in this limit eq.(14) corresponds to the flat spacetime Unruh effect [25].

Since $T \to \infty$ as the horizon is approached, this “thermal atmosphere” has enormous entropy. Indeed, if no cut-off is introduced, the entropy of the thermal atmosphere is divergent. However, inertial observers do not “see” this thermal atmosphere, and would attribute the physical effects produced by the thermal atmosphere to other causes, like radiation reaction effects [21]. In particular, with respect to a notion of “time translations” which would be naturally defined by inertial observers who freely fall into the black hole, the entropy of quantum fields outside of a black hole should be negligible.

Thus, it is not entirely clear what the quantity “$S$” appearing in eq.(43) is supposed to represent for matter near, but outside of, a black hole. Does $S$ include contributions from the thermal atmosphere? If so, $S$ is divergent unless a cutoff is introduced – although changes in $S$ (which is all that is needed for the formulation of the generalized second law) could still be well defined and finite. If not, what happens to the entropy in a box of ordinary thermal matter as it is slowly lowered toward the black hole? By the time it reaches its “floating point” [21], its contents are indistinguishable from the thermal atmosphere, so has its entropy disappeared? These questions provide good illustrations of some of the puzzles which arise when one attempts to consider thermodynamics in the framework of general relativity, as previously discussed at the end of Section II.

However, undoubtedly the most significant puzzle in the relationship between black holes and thermodynamics concerns the physical origin of the entropy, $S_{\text{bh}}$, of a black hole. Can the origin of $S_{\text{bh}}$ be understood in essentially the same manner as in the thermodynamics of conventional systems (as suggested by the apparently perfect merger of black hole mechanics with thermodynamics), or is there some entirely new phenomena at work here (as suggested by the radical differences in the present derivations of the laws

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2If a cut-off at the Planck scale is introduced, the entropy of the thermal atmosphere agrees, in order of magnitude, with the black hole entropy (39). There have been a number of attempts (see [24] for further discussion) to attribute the entropy of a black hole to this thermal atmosphere, or to “quantum hair” outside the black hole. However, if any meaning can be given to the notion of where the entropy of a black hole “resides”, it seems much more plausible to me that it resides in the “deep interior” of the black hole (corresponding to the classical spacetime singularity).
of black hole mechanics and the laws of thermodynamics)? As already re-
marked near the end of Section II, a classical treatment (as in Section III), or
even a semiclassical treatment (as above in this Section), cannot be adequate
to analyze this issue; undoubtedly, a fully quantum treatment of all of the
degrees of freedom of the gravitational field will be required.

Our present understanding of quantum gravity is quite rudimentary. Most
approaches used to formulate the quantum theory of fields propagating in a
Minkowski background spacetime either are inapplicable or run into severe
difficulties when one attempts to apply them to the formulation of a quantum
theory of the spacetime metric itself. However, several approaches have been
developed which have some appealing features and which hold out some hope
of overcoming these difficulties. The most extensively developed of these
approaches is string theory, and, very recently, some remarkable results have
been obtained in the context of string theory relevant to understanding the
origin of black hole entropy. I will very briefly describe some of the key
results here, referring the reader to the contribution of Horowitz [26] for
further details and discussion.

In the context of string theory, one can consider a “low energy limit” in
which the “massive modes” of the string are neglected. In this limit, string
theory should reduce to a 10-dimensional supergravity theory. If one treats
this supergravity theory as a classical theory involving a spacetime metric,
g_{\alpha\beta}, and other classical fields, one can find solutions describing black ho-
les. These classical black holes can be interpreted as providing a description of
states in string theory which is applicable at “low energies”.

On the other hand, one also can consider a “weak coupling” limit of string
theory, wherein the states are treated perturbatively about a background,
flat spacetime. In this limit, the dynamical degrees of freedom of the theory
are described by perturbative string states together with certain soliton-like
configurations, known as “D-branes”. In the weak coupling limit, there is
no literal notion of a black hole, just as there is no notion of a black hole in
linearized general relativity. Nevertheless, certain weak coupling states
comprised of “D-branes” can be identified with certain black hole solutions
of the low energy limit of the theory by a correspondence of their energy and
charges.

Now, the “weak coupling” states are, in essence, ordinary quantum dy-
amical degrees of freedom in a flat background spacetime. The ordinary
entropy, S, of these states can be computed using eq. (16) of Section II. The
corresponding classical black hole states of the “low energy limit” of the theory have an entropy, $S_{\text{bh}}$, given by eq. (39) of Section III. The remarkable results referred to above are the following: For certain classes of extremal ($\kappa = 0$) [27] and nearly extremal [28] black holes, the “ordinary entropy”, (16), of the weak coupling D-brane states agrees exactly with the entropy, (39), of the corresponding classical black hole states occurring in the low energy limit of the theory. Since these entropies have a nontrivial functional dependence on energy and charges, it is hard to imagine that this agreement could be the result of a random coincidence. Furthermore, for low energy scattering, the absorption/emission coefficients (“gray body factors”) of the corresponding D-brane and black hole configurations also agree [29]. In particular, since the temperatures of the corresponding D-brane and black hole configurations agree (by virtue of the equality of entropies), it follows that the “Hawking radiation” from the corresponding D-brane and black hole configurations also agree, at least at low energies.

Since the entropy of the D-brane configurations arises entirely from conventional “state counting”, the above results strongly suggest that the physical origin of black hole entropy should be essentially the same as for conventional thermodynamic systems. However, undoubtedly, a much more complete understanding of the nature of black holes in quantum gravity when one is neither in a “weak coupling” nor “low energy” limit will be required before a definitive answer to this question can be given.

As I have tried to summarize in this article, a great deal of progress has been made in our understanding of black holes and thermodynamics, but many unresolved issues remain. I believe that the attainment of a deeper understanding of the nature of the relationship between black holes and thermodynamics is our most promising route toward an understanding of the fundamental nature of both quantum gravity and thermodynamics.

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