We propose how to explicitly introduce the Jaffe-Witten mass gap into the quantum Yang-Mills theory. Through the full gluon propagator it is defined as such nonperturbative scale that when it formally goes to zero, then the perturbative phase survives in the theory only. The close link between mass gap and strong infrared singularities which are due to dominated in the QCD vacuum self-interaction of massless gluons is also discussed. This interaction leads thus to the zero momentum modes enhancement effect in the QCD NP vacuum. Using theory of distributions, we argue that strong infrared singularities can be put under control. A new, intrinsically nonperturbative phase in QCD is established.

1 Introduction

The system of quantum, dynamical equations of motion, the so-called Schwinger-Dyson (SD) equations for lower (propagators) and higher (vertices and kernels) Green's functions [1] can serve as an adequate and effective tool for the nonperturbative (NP) approach to QCD. However, to say today that QCD is the NP theory is almost tautology. The problem is how to define it exactly since we surely know that QCD has a perturbative (PT) phase as well because of asymptotic freedom (AF) [1]. In order to exactly define NP QCD, let us first start from the Yang-Mills (YM) sector. The two-point Green’s function, describing the full gluon propagator, is

\[ iD_{\mu\nu}(q) = \left\{ T_{\mu\nu}(q) d(-q^2, \xi) + \xi L_{\mu\nu}(q) \right\} \frac{1}{q^2}. \]  

(1)

Here \( \xi \) is a gauge fixing parameter and \( T_{\mu\nu}(q) = g_{\mu\nu} - (q_{\mu}q_{\nu}/q^2) = g_{\mu\nu} - L_{\mu\nu}(q) \). For its free (tree level) counterpart see Eq. (8) below. The solutions of the corresponding SD equation for the full gluon propagator (1) are supposed to reflect the complexity of the quantum structure of the QCD true ground state. It is a highly nonlinear system of four-dimensional integrals containing many different, unknown in general propagators, vertices and kernels, so there is no hope for exact solution(s). In any case, however, the solutions of this equation can be distinguished by their behavior in the deep infrared (DIR) limit, describing thus many (several) different types of quantum excitations and fluctuations of gluon field configurations in the QCD NP vacuum. Evidently, not all of them can reflect its real structure.
The realistic ultraviolet (UV) limit of these solutions is uniquely determined since we know solutions of the renormgroup equations because of AF. At the same time, the solutions of the renormgroup equations in the DIR region is not known. That is why in this case one has to rely on the formal asymptotics of the full gluon propagator as they follow from its expression (1). The DIR asymptotics of the full gluon propagator can be generally classified into the two different types: singular which means the IR enhanced (IRE) or smooth which means the IR finite (IRF) or even the IR vanishing (IRV) gluon propagators. However, the smooth behavior of the full gluon propagator is possible only in one exceptional gauge - the Landau gauge ($\xi = 0$), i.e., it is a gauge artefact. We will not therefore discuss it in what follows, so we are left with the IRE one only.

Let us now emphasize that any deviation in the behavior of the full gluon propagator in the DIR domain from the free one automatically assumes its dependence on a scale parameter (at least one) in general different from QCD asymptotic scale parameter $\Lambda_{QCD}$. It can be considered as responsible for the NP dynamics (in the IR region) in the QCD vacuum. If QCD itself is a confining theory, then such a characteristic scale is very likely to exist. In what follows, let us denote it, say, $\Delta$ (see below).

2 Jaffe-Witten mass gap

**Truly NP phase.** The phenomenon of ”dimensional transmutation” [1] only supports our general conclusion that QCD exhibits a mass determining the scale of NP dynamics in its ground state. In terms of the full gluon propagator it can be exactly defined as follows:

$$d^{TNP}(-q^2, \Delta^2) = d(-q^2, \Delta^2) - d(-q^2, \Delta^2 = 0),$$

where we introduce explicit dependence on the above-mentioned NP mass scale parameter $\Delta$. This subtraction can be also considered as the definition of the truly NP (TNP) part of the full gluon propagator since when the NP scale parameter goes formally to zero then the TNP part vanishes. The definition (2) also explains the difference between the TNP part $d^{TNP}(-q^2)$ and the full gluon propagator $d(-q^2)$ which is NP itself. On the other hand, the subtraction (2) is equivalent to

$$d^{TNP}(-q^2, \Delta^2) = \Delta^2 f(-q^2, \Delta^2),$$

where the function $f(q^2, \Delta^2)$ is of the corresponding dimension and it has a finite limit as $\Delta^2 \to 0$. Thus in the initially massless quantum YM theory the characteristic mass appears explicitly. In Ref. [2] Jaffe and Witten (JW)
have argued that quantum YM fields should exhibit characteristic mass scale
the so-called mass gap $\Delta$ in order to explain why the nuclear force is strong
but short-range. The definitions (2) and (3) is precisely our proposal how the
JW mass gap could be explicitly introduced into the quantum YM theory. In
other words, by defining TNP QCD, we identify the NP scale with their mass
gap. Let us note further that the limit $\Delta^2 \to 0$ is usually equivalent to the UV
limit $-q^2 \to \infty$, so it is almost obvious that because of AF one can identify
\begin{equation}
    d(-q^2, \Delta^2 = 0) \equiv d^{PT}(-q^2).
\end{equation}
Thus the relation (4) is our definition of the PT phase in QCD, then the
relation (2) becomes
\begin{equation}
    d(-q^2, \Delta^2) = d^{TNP}(-q^2, \Delta^2) + d^{PT}(-q^2).
\end{equation}
Let us underline that this is an exact relation (the gluon momentum runs
over the whole range $[0, \infty)$), so at this stage there is no approximation made.
Due to the inevitable identification (4), one can say that we define the NP
scale parameter in the way that when it formally goes to zero (i.e., when the
TNP part vanishes, Eq. (3)), then the PT phase only survives in the full gluon
propagator. This allows one to clearly separate the JW mass gap from all other
mass scale parameters which may be present in the full gluon propagator. It
is worth emphasizing that in the realistic models of the full gluon propagator
its TNP part usually coincides with its DIR asymptotics determining thus the
strong intrinsic influence of the IR properties of the theory on its NP dynamics.

Substituting the exact decomposition (5) into the full gluon propagator
(1), one obtains
\begin{equation}
    D_{\mu\nu}(q, \Delta) = D_{\mu\nu}^{TNP}(q, \Delta) + D_{\mu\nu}^{PT}(q),
\end{equation}
where
\begin{align}
    D_{\mu\nu}^{TNP}(q, \Delta) &= -iT_{\mu\nu}(q)d^{TNP}(-q^2, \Delta^2)\frac{1}{q^2}, \\
    D_{\mu\nu}^0(q) &= -i\{T_{\mu\nu}(q) + \xi L_{\mu\nu}(q)\}\frac{1}{q^2}, \\
    D_{\mu\nu}^{PT}(q) &= -i\{T_{\mu\nu}(q)d^{PT}(-q^2) + \xi L_{\mu\nu}(q)\}\frac{1}{q^2}.
\end{align}
For further purposes the explicit expression for the free gluon propagator
$D_{\mu\nu}^0(q)$ is also given. Thus the exact decomposition (6) has a few remark-
able features. First of all, the dependence of the full gluon propagator on the

JW mass gap $\Delta$ is exactly placed in its TNP part. Secondly, the explicit gauge dependence of the full gluon propagator is exactly shifted from its TNP part to its PT part (for reasons to proceed in this way, see discussion below). Thus we clearly separate the NP phase from the PT one in QCD.

**Intrinsically NP phase.** Up to this moment we have dealt only with exact decompositions and definitions, i.e., only algebraic manipulations have been done. It is the time now to introduce nontrivial dynamics into this scheme. Evidently, the only place where it can be done is, of course, the DIR region, i.e., by saying something nontrivial about the DIR asymptotics of the full gluon propagator one can additionally distinguish between its TNP and PT parts. Fortunately, we have an exact criterion for establishing the DIR structure of the full gluon propagator. The free gluon propagator (8) has an exact ($1/q^2$) singularity in the DIR limit since it is defined in the whole range. The PT part, presented by Eq. (9) also in the whole range, possesses the same property. This means that the free gluon singularity ($1/q^2$) at $q^2 \to 0$ is an exact separation line between the TNP and PT parts in the full gluon propagator (6). It is worth reemphasizing that decomposition (6) is exact, i.e., the gluon momentum runs over the whole range in both terms in its right hand side. The PT phase in QCD can be defined as one which in the IR is singular as the free gluon propagator. Its existence in QCD is important from conceptual point of view and it is determined by AF.

Contrary to the PT part, the TNP part of the full gluon propagator (7) should have then singularities in the DIR domain stronger than ($1/q^2$). Of course, such strong singularities can be only of dynamical origin. The only dynamical mechanism in QCD which can produce such severe singularities in the vacuum is self-interaction of massless gluons in the DIR domain. Precisely this self-interaction in the UV limit leads to AF. Thus one comes to the inevitable conclusion that the TNP part of the full gluon propagator should have strong IR singularities different from those of the PT part. In what follows this type of singularities will be called as NP IR singularities. Thus they should be summarized by the full gluon propagator and effectively correctly described by its TNP part in the DIR domain. In this way the TNP phase becomes intrinsically NP (INP) one. So the definition of the INP YM quantum theory consists of the two conditions.

I). The first necessary condition defines the TNP part of the full gluon propagator (7), on account of the definitions (2) and (3).

II). The second sufficient condition specifies the existence and structure of more stronger than ($1/q^2$) IR singularities in the QCD NP vacuum.

The longitudinal part of the full gluon propagator is exactly singular in the IR as ($1/q^2$), and therefore it does not make any sense to decompose the
gauge fixing parameter $\xi$ similar to Eq. (2), though formally it is possible to do, of course. Thus the INP part of the full gluon propagator is manifestly gauge invariant and only transfer (physical) degrees of freedom of the gauge bosons are important for the NP dynamics in QCD.

3 ZMME quantum model of the QCD ground state

The quantum structure of the QCD NP vacuum is dominated by such types of excitations and fluctuations of gluon field configurations there which are due to self-interaction of massless gluons since precisely this interaction is the main quantum, dynamical effect in QCD. In the UV region it implies AF. In the DIR region it becomes strongly singular and thus to be responsible for the zero momentum modes enhancement (ZMME) effect in the QCD NP vacuum. In the quantum YM theory all the NP IR singularities should be absorbed into the full gluon propagator, in its INP part. Thus the ZMME (or simply zero modes enhancement (ZME) since we work always in the momentum space) quantum, dynamical model of the QCD true ground state is based on the existence and importance of such kind of the NP excitations and fluctuations of gluon field configurations there which are due to self-interaction of massless gluons only (without explicitly involving some extra degrees of freedom). They are to be summarized by the INP part of the full gluon propagator and are to be effectively correctly described by its behavior in the DIR domain.

In general, all the Green’s functions in QCD are generalized functions, i.e., they are distributions. Especially this is true for the severe NP IR singularities due to self-interaction of massless gluons in the QCD vacuum. Not loosing generality, the severe NP IR singularities can be analytically taken into account in terms of the TNP gluon form factor in Eq. (7) with a Euclidean signature ($-q^2 \to q^2$) as follows:

$$d^{INP}(q^2, \Delta^2) = (\Delta^2)^{-\lambda-1}(q^2)^{\lambda} \times f(q^2),$$

where obviously we include $1/q^2$ from Eq. (7) into the exponent $\lambda$ which in general is arbitrary (any complex number with $Re\lambda < 0$, see below). The function $f(q^2)$ is a dimensionless one which is regular at zero and otherwise remaining arbitrary, but preserving AF in the UV limit. This is nothing else but the analytical formulation of the second sufficient condition of the existence of the INP phase in the YM theory. That is why we replaced superscript “TNP” by “INP” in Eq. (10). Since we are particularly interested in the DIR region, the arbitrary function $f(q^2)$ should be also expanded around zero in the form of the Taylor series in powers of $q^2$. As a result, we will be left with finite sum of power terms with exponent decreasing by one starting from $-\lambda$. All
other remaining terms from the Taylor expansion starting from the term having already the PT IR singularity should be shifted to the PT part of the full gluon propagator.

The distribution theory (DT) [3] tells us that the distribution \((q^2)_{\lambda}\) will have a simple pole at points \(\lambda = -(n/2) - k, \ (k = 0, 1, 2, 3...),\) where \(n\) denotes the number of dimensions in Euclidean space \((q^2 = q_0^2 + q_1^2 + q_2^2 + ... + q_{n-1}^2)\).

In order to actually define the system of SD equations in the DIR domain, it is convenient to introduce the IR regularization parameter \(\epsilon\), defined as \(D = n + 2\epsilon, \ \epsilon \rightarrow 0^+\) within a gauge invariant dimensional regularization (DR) method [1]. As a result, all the Green’s functions should be regularized with respect to \(\epsilon\) which is to be set to zero at the end of computations. The structure of the NP IR singularities is then determined (when \(n\) is even number) as follows [3,4]:

\[
(q^2)_{\lambda} = \frac{C^{(k)}_{-1}}{\lambda + (D/2) + k} + \text{finite terms},
\]

where the residue is

\[
C^{(k)}_{-1} = \frac{\pi^{n/2}}{2^{2k}k!\Gamma((n/2) + k)} \times L^n\delta^n(q)
\]

with \(L = (\partial^2/\partial q_0^2) + (\partial^2/\partial q_1^2) + ... + (\partial^2/\partial q_{n-1}^2)\). In terms of \(\epsilon\) from Eq. (11), one has

\[
(q^2)^{-\frac{D}{2} - k + \epsilon} = \frac{1}{\epsilon} C^{(k)}_{-1} + \text{finite terms},
\]

where we can put \(D = n\) now. This means that the order of singularity does not depend on \(\lambda\), i.e., it is always a simple pole, \((1/\epsilon)\). Thus the wide-spread opinion that severe IR singularities (stronger than \((1/q^2)\)) in QCD NP vacuum cannot be controlled is not justified (for details see Refs. [3,4]).

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