Quantum interference of polarized electrons in the presence of magnetic strings

Silviu Olariu
Institute of Physics and Nuclear Engineering, Heavy-Ion Physics Department
76900 Magurele, P.O. Box MG-6, Bucharest, Romania

Abstract

The interaction of polarized electrons with bare and shielded magnetic strings is studied with the aid of the Dirac equation. It is found that the difference between the amplitudes for the scattering by bare and shielded strings of incident wave packets of width $\delta$ and impact parameter $d$ is proportional to $\exp(-d^2/2\delta^2)$.

PACS numbers: 03.65.Bz
The basic requirement of an Aharonov-Bohm experiment [1] is the separation between the region of space accessible to the interfering electron and the region of space containing the magnetic flux. In electron interference experiments this separation is obtained by placing suitable shields in the path of the incident electron beam. In experiments with Josephson junctions or mesoscopic rings the electrons are prevented from leaving the conductors by surface potential barriers. In theoretical models the Aharonov-Bohm effect is demonstrated by showing the persistence of finite, flux-dependent phase shifts in the presence of barriers of arbitrary height or thickness.

Percoco and Villalba [2] have studied the eigenfunctions of the Dirac equation in the presence of a magnetic string shielded by a cylindrical barrier and have found that for a barrier of finite radius it is not possible that all four components of the Dirac wave function should vanish on the surface of this cylinder. Therefore, when we represent the state of the electron by solutions of the Dirac equation, the separation between the region accessible to the electron and the region of the magnetic flux has to be achieved by a barrier of finite height and sufficient thickness.

A flux of $\frac{h}{2e} = 2.07 \cdot 10^{-15} \text{Tm}^2$ which would produce a shift of a half-fringe in an electron interference pattern would be obtained from a 2 T magnetic field from a tube of magnetic flux having a radius $r_{h/2e} = 1.81 \cdot 10^{-8} \text{m}$, and the wave vector of an electron having an energy of 1 keV is $\tilde{k}_1 = 1.62 \cdot 10^{11} \text{m}^{-1}$, so that $\tilde{k}_1 r_{h/2e} = 2.9 \cdot 10^3 \gg 1$. Nevertheless, a concept often used in theoretical models is that of magnetic string, which is a tube of arbitrarily small radius carrying a certain finite amount of magnetic flux $F$. The interaction of a particle of charge $q$ with the flux $F$ is measured by the coupling constant $\alpha = qF/2\pi\hbar$, in Système International (SI) units. For an electron we have $q = -e < 0$.

If the state of the particle of charge $q$ is represented by the Schrödinger equation, thereby neglecting the electron spin, the eigenfunctions in the presence of a magnetic string are all vanishing on the string line, so that the inclusion of additional shielding barriers is not necessary in this case. A finite amount of momentum is exchanged, however, between the electron and the string at the surface of the string [3]. The eigenfunctions of an electron endowed with spin interacting with a magnetic string may in some cases differ significantly from the eigenfunctions of a spinless electron interacting with that magnetic string. Hagen [4] and Olariu [5] have found that the representation of the electron states by solutions of the Dirac equation involves an eigenfunction which becomes infinite on the
magnetic string. However, while [4] obtained divergent terms in the pair of principal components of the four-component Dirac wave function, in [3] the divergent terms were proportional to $\hbar k/Mc$, becoming vanishingly small in the limit $\hbar k/Mc \to 0$, where $\hbar k$ is the momentum and $M$ the mass of the incident electron.

The Dirac eigenfunctions of an electron in the presence of a shielded magnetic string are determined in this work by a limiting process in which the radius $r_0$ of the tube of magnetic flux tends to zero and the product $kR_0$ tends to zero while the radius $R_0$ of the shielding barrier remains finite. It is shown that the scattering of polarized electrons by a shielded magnetic string can be represented, in the limit $\hbar k/Mc \to 0$, by the Aharonov-Bohm wave function for spinless electrons. The scattering of a plane wave by a shielded magnetic string is then compared with the scattering of a plane wave by a bare magnetic string, and it is shown that the difference between the scattering by bare and shielded strings persists in the non-relativistic limit. By comparing further the scattering of wave packets by bare and shielded magnetic strings it is found that the difference of the corresponding scattering amplitudes is proportional to the probability density in the region of the string. This probability is made as small as possible in real Aharonov-Bohm experiments. Thus, the scattering by a shielded magnetic string and the scattering by a bare magnetic string are really different only when there is a significant probability density in the region of the magnetic flux, i.e. when there is a direct interaction between the magnetic moment of the electron and the magnetic field.

The separation between the region of space accessible to the incident electron and the region of the magnetic flux is a basic requirement of an Aharonov-Bohm experiment. The concept of shielded magnetic string has been introduced [3],[5] by assuming that the tube of magnetic flux of radius $r_0$ is surrounded by a cylindrical potential barrier of height $U$, $E - Mc^2 < U < E + Mc^2$, and radius $R_0 > r_0$, where $E$ is the electron energy, including rest mass. The eigenfunctions for a shielded magnetig string can then be determined for fixed $R_0$ in the limit $r_0 \to 0$ and $kR_0 \to 0$, the latter being equivalent to $k \to 0$. Thus, in addition to the vector potential of the tube of magnetic flux, of radius $r_0$, which in cylindrical coordinates is

\[
\begin{align*}
A_x^{(r_0)} &= 0, \\
A_y^{(r_0)} &= 0,
\end{align*}
\]
\[ A_{\theta}^{(r_0)} = \begin{cases} F/2\pi r & \text{for } r > r_0, \\ Fr/2\pi r_0^2 & \text{for } r < r_0, \end{cases} \]

we assume that there is a static electric potential given by
\[ q\phi = \begin{cases} U & \text{for } r < R_0, \\ 0 & \text{for } r > R_0, \end{cases} \]

where \( q \) is the charge of the incident particle and \( r \) the distance to the axis of the tube of flux.

The eigenstates of the particle of charge \( q \) in the presence of the potentials represented in eqs. (1) and (2) have the form
\[
\begin{pmatrix} \psi_{11}^{(R_0)} \\ \psi_{12}^{(R_0)} \end{pmatrix} = \begin{pmatrix} \chi_{11}^{(R_0)}(r) e^{il\theta} \\ \chi_{12}^{(R_0)}(r) e^{i(l+1)\theta} \end{pmatrix} e^{-iEt/h}, \quad \begin{pmatrix} \psi_{33}^{(R_0)} \\ \psi_{44}^{(R_0)} \end{pmatrix} = \begin{pmatrix} \chi_{33}^{(R_0)}(r) e^{il\theta} \\ \chi_{44}^{(R_0)}(r) e^{i(l+1)\theta} \end{pmatrix} e^{-iEt/h},
\]

where \( l \) is an integer. The radial parts are determined in the standard representation [1], [6] and in SI units, from the relations
\[
\frac{d^2\chi_{11}^{(R_0)}}{dr^2} + \frac{1}{r} \frac{d\chi_{11}^{(R_0)}}{dr} + \frac{[E - q\phi(r)]^2 - M^2 c^4}{h^2 c^2} - \frac{1}{r^2} \left( l - \frac{q}{\hbar} r A_{\theta}^{(r_0)}(r) \right)^2 + \frac{q}{\hbar} B(r) \chi_{11}^{(R_0)} = 0, \quad (4)
\]
\[
\frac{d^2\chi_{12}^{(R_0)}}{dr^2} + \frac{1}{r} \frac{d\chi_{12}^{(R_0)}}{dr} + \frac{[E - q\phi(r)]^2 - M^2 c^4}{h^2 c^2} - \frac{1}{r^2} \left( l + 1 - \frac{q}{\hbar} r A_{\theta}^{(r_0)}(r) \right)^2 + \frac{q}{\hbar} B(r) \chi_{12}^{(R_0)} = 0, \quad (5)
\]
\[
\chi_{33}^{(R_0)} = -\frac{i\hbar c}{E + Mc^2 - q\phi(r)} \left( \frac{d}{dr} + \frac{l + 1}{r} - \frac{q}{\hbar} A_{\theta}^{(r_0)}(r) \right) \chi_{12}^{(R_0)}, \quad (6)
\]
\[
\chi_{44}^{(R_0)} = -\frac{i\hbar c}{E + Mc^2 - q\phi(r)} \left( \frac{d}{dr} - \frac{l}{r} + \frac{q}{\hbar} A_{\theta}^{(r_0)}(r) \right) \chi_{11}^{(R_0)}, \quad (7)
\]

where \( B(r) \) is the magnetic field represented by the vector potential (1).

In the region of the magnetic flux, \( r < r_0 \), the radial eigenfunctions are determined by the condition of regularity at \( r=0 \), and are
\[
\begin{align*}
\chi_{11}^{(R_0)} &= a_{11} B_{11}^{(R_0)} (k_{1r})^{|l|} \exp(-\alpha r^2/2r_0^2) F \left( \frac{|l| + 1 - l}{2} - \frac{\kappa_{1r}^2 r_0^2}{4\alpha} \right) |l + 1| \left| \frac{\alpha r^2}{r_0^2} \right|, \\
\chi_{12}^{(R_0)} &= a_{12} B_{12}^{(R_0)} (k_{2r})^{|l+1|} \exp(-\alpha r^2/2r_0^2) F \left( \frac{|l + 1| - l}{2} - \frac{\kappa_{2r}^2 r_0^2}{4\alpha} \right) |l + 1| + 1 \left| \frac{\alpha r^2}{r_0^2} \right|, 
\end{align*}
\]

where \( a_{11}, a_{12}, B_{11}^{(R_0)}, B_{12}^{(R_0)} \) are constant coefficients, \( B = F/p\pi r_0^2 \),
\[
\kappa_{1r}^2 = \kappa^2 + \frac{qB}{\hbar}, \quad \kappa_{2r}^2 = \kappa^2 - \frac{qB}{\hbar}, \quad (10)
\]
\[-\kappa^2 = \frac{(E - U)^2 - M^2 c^4}{\hbar^2 c^2} < 0, \quad (11)\]

and \(F(a|c|z)\) is the confluent hypergeometric function \[.\]

The eigenfunctions for a bare magnetic string will be obtained first as the eigenfunctions for a tube of magnetic flux of radius \(r_0\), in the limit \(r_0 \to 0\). The radial solutions \(\chi_{11}, \chi_{12}\) of the wave equations for the flux region \(r < r_0\) can be obtained from eqs. (8), (9) by putting \(U = 0\) in eq. (11). In the field-free region \(r > r_0\), the radial solutions are of the form

\[
\chi_{11} = a_{11} \left( J_{|l-\alpha|}(kr) + A_{11} H^{(1)}_{|l-\alpha|}(kr) \right) \quad \text{for} \quad r > r_0, \quad (12)
\]

\[
\chi_{12} = a_{12} \left( J_{|l+1-\alpha|}(kr) + A_{12} H^{(1)}_{|l+1-\alpha|}(kr) \right) \quad \text{for} \quad r > r_0, \quad (13)
\]

where

\[
k^2 = \frac{E^2 - M^2 c^4}{\hbar^2 c^2}. \quad (14)
\]

The coefficients \(A_{11}, A_{12}\) can be determined from the conditions of continuity at \(r = r_0\) of the four components of the wave function, and have the expressions

\[
A_{11} = \frac{-J'_{|l-\alpha|}(kr_0) - \Lambda_1(k_1/k)J_{|l-\alpha|}(kr_0)}{H^{(1)'}_{|l-\alpha|}(kr_0) - \Lambda_1(k_1/k)H^{(1)}_{|l-\alpha|}(kr_0)}, \quad (15)
\]

\[
A_{12} = \frac{-J'_{|l+1-\alpha|}(kr_0) - \Lambda_2(k_2/k)J_{|l+1-\alpha|}(kr_0)}{H^{(1)'}_{|l+1-\alpha|}(kr_0) - \Lambda_2(k_2/k)H^{(1)}_{|l+1-\alpha|}(kr_0)}, \quad (16)
\]

where \(J'_\nu(\zeta) = dJ_\nu(\zeta)/d\zeta, \ H^{(1)'}_\nu(\zeta) = dH^{(1)}_\nu(\zeta)/d\zeta; \ \Lambda_1 = (d\chi_{11}/k_1dr)/\chi_{11}, \ \Lambda_2 = (d\chi_{12}/k_2dr)/\chi_{12}\) are the logarithmic derivatives at \(r = r_0\) for the region \(r < r_0\); and \(k_1^2 = k^2 + qB/\hbar, \ k_2^2 = k^2 - qB/\hbar\).

As can be seen from eqs. (8), (9) the logarithmic derivative \(\Lambda_1\) is, for \(l \geq 0\) and in the limit \(r_0 \to 0, \ \Lambda_1 \to (|l| - \alpha)/k_1r_0\). For \(kr_0 \ll 1\), the leading term of the denominator in eq. (15) is proportional to

\[
H^{(1)'}_{|l-\alpha|}(kr_0) - \Lambda_1(k_1/k)H^{(1)}_{|l-\alpha|}(kr_0) \sim |l - \alpha| + |l| - \alpha. \quad (17)
\]

When \(kr_0 \ll 1\), the coefficients are \(A_{11} \sim (kr_0)^{2|l-\alpha|}\), becoming vanishingly small unless

\[
|l - \alpha| + |l| - \alpha = 0, \ |l - \alpha| < 1, \quad (18)
\]

when the ratio in eq. (15) tends to a finite limit. This equation has solutions only when \(\alpha > 0\), the solution being \(l = \lfloor \alpha \rfloor\), where \(\lfloor \alpha \rfloor\) is the integer part of \(\alpha\), i.e. the nearest integer to the left of \(\alpha\), so that
that the magnetic moment is oriented in the -z direction, as the flux cancellation occurs for the \( H_{\nu}^{(1)} = (i/\sin(\pi \nu)) \left( e^{-i\pi \nu J_\nu} - J_{-\nu} \right) \) if \( A_{[\alpha]1} \to i \sin(\pi \alpha) e^{i\pi \alpha} \), so that \( \chi_{[\alpha]1} \sim J_{-\alpha+\alpha}(kr) \), instead of the usual \( J_{\alpha-\alpha}(kr) \). For \( l < 0 \), \( \Lambda_1 \) remains of the order of \( 1/k_1 r_0 \), but the leading term of the denominator in eq. (15) is different from 0 for all values of \( l \) in this range, and the coefficients \( A_{l1} \sim (kr_0)^{2|l-\alpha|} \), are becoming vanishingly small as \( kr_0 \ll 1 \). The logarithmic derivative \( \Lambda_2 \) is, for \( l \leq -1 \) and in the limit \( r_0 \to 0 \), \( \Lambda_2 \to (|l+1|+\alpha)/k_2 r_0 \). For \( kr_0 \ll 1 \), the leading term of the denominator in eq. (16) is proportional to

\[
H_{[l+1-\alpha]}^{(1)}(kr_0) - \Lambda_2(k_2/k) H_{[l+1-\alpha]}^{(1)}(kr_0) \sim |l+1-\alpha| + |l+1| + \alpha,
\]

and when \( kr_0 \ll 1 \), the coefficients are \( A_{l2} \sim (kr_0)^{2|l+1-\alpha|} \), becoming vanishingly small, unless

\[
|l+1-\alpha| + |l+1| + \alpha = 0, \quad |l+1-\alpha| < 1,
\]

when the ratio in eq. (16) tends to a finite limit. This equation has solutions only when \( \alpha < 0 \), the solution being again \( l = [\alpha] \). It can be shown that \( A_{[\alpha]2} \to -i \sin(\pi \alpha) e^{-i\pi \alpha} \), so that \( \chi_{[\alpha]2} \sim J_{\alpha-[\alpha]-1}(kr) \), instead of the usual \( J_{[\alpha]+1-\alpha}(kr) \). For \( l > -1 \), \( \Lambda_2 \) remains of the order of \( 1/k_2 r_0 \), but the leading term of the denominator in eq. (16) is different from 0 for all values of \( l \) in this range, and the coefficients are \( A_{l2} \sim (kr_0)^{2l+1-\alpha} \), becoming vanishingly small as \( kr_0 \ll 1 \). Thus, in the limit \( r_0 \to 0 \) the radial eigenfunctions for the bare string are, in the region \( r > r_0 \) and for \( \alpha > 0 \),

\[
\chi_{l1} \sim J_{[l-\alpha]}(kr), \quad l \neq [\alpha]; \quad \chi_{[\alpha]1} \sim J_{-\alpha+[\alpha]}(kr) \quad \text{for } \alpha > 0,
\]

\[
\chi_{l2} \sim J_{[l+1-\alpha]}(kr) \quad \text{for } \alpha > 0,
\]

and, for \( \alpha < 0 \),

\[
\chi_{l1} \sim J_{[l-\alpha]}(kr) \quad \text{for } \alpha < 0,
\]

\[
\chi_{l2} \sim J_{[l+1-\alpha]}(kr), \quad l \neq [\alpha]; \quad \chi_{[\alpha]2} \sim J_{\alpha-[\alpha]-1}(kr) \quad \text{for } \alpha < 0.
\]

It can be checked that the cancellations occur when the magnetic moment of the electron and the magnetic flux have the same orientation. For example, in the case of an electron of charge \( q = -e < 0 \) a positive \( \alpha \) means a negative \( F \), i.e. a magnetic flux oriented in the -z direction. For \( \alpha > 0 \) the cancellation occurs for the \( \chi_{[\alpha]1} \) component, which represents a spin oriented in the +z direction, so that the magnetic moment is oriented in the -z direction, as the flux \( F \).
The solutions for a shielding barrier of fixed radius \( R_0 \) enclosing a flux tube of radius \( r_0 \to 0 \) will be discussed further. In the shielding region \( r_0 < r < R_0 \) the radial eigenfunctions are superpositions of Bessel functions of imaginary argument \( \kappa r \),

\[
\chi^{(R_0)}_{il} = a_{i1} \left( C^{(R_0)}_{il} H^{(1)}_{|l-\alpha|}(i\kappa r) + D^{(R_0)}_{il} H^{(2)}_{|l-\alpha|}(i\kappa r) \right) \quad \text{for } r_0 < r < R_0, \\
\chi^{(R_0)}_{l2} = a_{i2} \left( C^{(R_0)}_{l2} H^{(1)}_{|l+1-\alpha|}(i\kappa r) + D^{(R_0)}_{l2} H^{(2)}_{|l+1-\alpha|}(i\kappa r) \right) \quad \text{for } r_0 < r < R_0,
\]

where \( C^{(R_0)}_{il}, D^{(R_0)}_{il}, C^{(R_0)}_{l2}, D^{(R_0)}_{l2} \) are constant coefficients, and \( H^{(1)}_{\nu}, H^{(2)}_{\nu} \) are the Hankel functions of the first and second kind \( \mathcal{H} \). It can be shown as previously that in the limit \( r_0 \to 0 \) the radial eigenfunctions in the region \( r_0 < r < R_0 \) are, for \( \alpha > 0 \),

\[
\chi^{(R_0)}_{i1} \sim J_{|l-\alpha|}(i\kappa r), \quad l \neq [\alpha]; \quad \chi^{(R_0)}_{i1} \sim J_{-\alpha+[\alpha]}(i\kappa r) \quad \text{for } \alpha > 0,
\]

\[
\chi^{(R_0)}_{i2} \sim J_{|l+1-\alpha|}(i\kappa r) \quad \text{for } \alpha > 0,
\]

and, for \( \alpha < 0 \),

\[
\chi^{(R_0)}_{i1} \sim J_{|l-\alpha|}(i\kappa r) \quad \text{for } \alpha < 0,
\]

\[
\chi^{(R_0)}_{i2} \sim J_{|l+1-\alpha|}(i\kappa r), \quad l \neq [\alpha]; \quad \chi^{(R_0)}_{i2} \sim J_{\alpha-[\alpha]-1}(i\kappa r) \quad \text{for } \alpha < 0.
\]

As can be seen from eqs. (21)-(24) and (27)-(30), the presence of the barrier changes the argument of the Bessel solutions from a real number in the case of the bare string into an imaginary number in the case of the shielded string, but does not change the divergent character of one of the principal components of the eigenfunction for \( l = [\alpha] \) at \( r = r_0 \), when \( r_0 \to 0 \). However, if \( \kappa R_0 \gg 1 \), all the eigenfunctions in eqs. (27), (28) or (29), (30) become for \( r < R_0 \) and \( r \approx R_0 \) proportional to \( \exp(\kappa r)/r^{1/2} \). Therefore, the eigenfunctions are multiplied in the region \( r < R_0 \) by coefficients of the order of \( \exp(-\kappa R_0) \), which become vanishingly small for \( \kappa R_0 \gg 1 \). For example, if we would choose, in a gedanken experiment, \( U = Mc^2 \), then \( \kappa \approx Mc/h \), and we would have to fulfil the conditions \( McR_0/h \gg 1 \) (effective shielding) and \( kR_0 \ll 1 \) (thin flux tube). For \( k = 1.62 \cdot 10^{11} \text{ m}^{-1} \), which corresponds to a kinetic energy \( h^2k^2/2M=1 \text{ keV} \), and \( h/Mc = 3.86 \cdot 10^{-13} \text{ m} \), we could take \( R_0 = 10^{-12} \text{ m} \), so that \( kR_0=0.16,McR_0/h=2.59, \) and \( \exp(-2McR_0/h) = 5.6 \cdot 10^{-3} \).

In the region \( r > R_0 \) the radial principal components \( \chi^{(R_0)}_{i1}, \chi^{(R_0)}_{i2} \) have the form

\[
\chi^{(R_0)}_{i1} = a_{i1} \left( J_{|l-\alpha|}(kr) + A^{(R_0)}_{i1} H^{(1)}_{|l-\alpha|}(kr) \right) \quad \text{for } r > R_0,
\]

\[
\chi^{(R_0)}_{i2} = a_{i2} \left( C^{(R_0)}_{i2} H^{(1)}_{|l+1-\alpha|}(kr) + D^{(R_0)}_{i2} H^{(2)}_{|l+1-\alpha|}(kr) \right) \quad \text{for } r > R_0,
\]

where \( C^{(R_0)}_{i1}, D^{(R_0)}_{i1}, C^{(R_0)}_{i2}, D^{(R_0)}_{i2} \) are constant coefficients, and \( H^{(1)}_{\nu}, H^{(2)}_{\nu} \) are the Hankel functions of the first and second kind \( \mathcal{H} \). It can be shown as previously that in the limit \( r_0 \to 0 \) the radial eigenfunctions in the region \( r_0 < r < R_0 \) are, for \( \alpha > 0 \),

\[
\chi^{(R_0)}_{i1} \sim J_{|l-\alpha|}(i\kappa r), \quad l \neq [\alpha]; \quad \chi^{(R_0)}_{i1} \sim J_{-\alpha+[\alpha]}(i\kappa r) \quad \text{for } \alpha > 0,
\]

\[
\chi^{(R_0)}_{i2} \sim J_{|l+1-\alpha|}(i\kappa r) \quad \text{for } \alpha > 0,
\]

and, for \( \alpha < 0 \),

\[
\chi^{(R_0)}_{i1} \sim J_{|l-\alpha|}(i\kappa r) \quad \text{for } \alpha < 0,
\]

\[
\chi^{(R_0)}_{i2} \sim J_{|l+1-\alpha|}(i\kappa r), \quad l \neq [\alpha]; \quad \chi^{(R_0)}_{i2} \sim J_{\alpha-[\alpha]-1}(i\kappa r) \quad \text{for } \alpha < 0.
\]
\[ \chi_{12}^{(R_0)} = a_{12} \left( J_{|l+1-\alpha|}(kr) + A^{(R_0)}_{12} H_{|l+1-\alpha|}^{(1)}(kr) \right) \text{ for } r > R_0, \]  

The coefficients \( A^{(R_0)}_{11}, A^{(R_0)}_{12} \) appearing in the expression of the eigenfunctions for \( r > R_0 \), eqs. (31),(32), can be determined from the condition of continuity at \( r = R_0 \) of the four components \( \chi_{11}^{(R_0)}, \chi_{12}^{(R_0)}, \chi_{13}^{(R_0)}, \chi_{14}^{(R_0)} \). The expressions of \( A^{(R_0)}_{11}, A^{(R_0)}_{12} \) are

\[
A^{(R_0)}_{11} = -\frac{f^{(R_0)}_{11} (kR_0) - (\kappa/k) f^{(R_0)}_{11} (kR_0)}{H_{|l-\alpha|}^{(1)}(kR_0) - (\kappa/k) f^{(R_0)}_{11} H_{|l-\alpha|}^{(1)}(kR_0)},
\]

\[
A^{(R_0)}_{12} = -\frac{f^{(R_0)}_{12} (kR_0) - (\kappa/k) f^{(R_0)}_{12} (kR_0)}{H_{|l+1-\alpha|}^{(1)}(kR_0) - (\kappa/k) f^{(R_0)}_{12} H_{|l+1-\alpha|}^{(1)}(kR_0)},
\]

where

\[
f^{(R_0)}_{11} = \frac{E + Mc^2}{E + Mc^2 - U} \Lambda_{11}^{(R_0)} - \frac{U}{E + Mc^2 - U} \frac{l - \alpha}{\kappa R_0},
\]

\[
f^{(R_0)}_{12} = \frac{E + Mc^2}{E + Mc^2 - U} \Lambda_{12}^{(R_0)} + \frac{U}{E + Mc^2 - U} \frac{l + 1 - \alpha}{\kappa R_0}.
\]

The quantities \( \Lambda_{11}^{(R_0)}, \Lambda_{12}^{(R_0)} \) appearing in eqs. (35), (36) are the logarithmic derivatives of the radial eigenfunctions written in eqs. (27),(28) or (29),(30), evaluated at \( r = R_0 \),

\[
\Lambda_{11}^{(R_0)} = \frac{d \chi_{11}^{(R_0)}/kdr}{\chi_{11}^{(R_0)}}, \Lambda_{12}^{(R_0)} = \frac{d \chi_{12}^{(R_0)}/kdr}{\chi_{12}^{(R_0)}} \text{ for } r \to R_0, r < R_0.
\]

Due to the exponential dependence of the eigenfunctions in the case \( \kappa R_0 \gg 1 \), the quantities defined in eq. (37) converge to 1, \( \Lambda_{11}^{(R_0)} \to 1, \Lambda_{12}^{(R_0)} \to 1 \). The denominators in eqs. (33), (34) then are

\[
H_{|l-\alpha|}^{(1)}(kR_0) - (\kappa/k) f^{(R_0)}_{11} H_{|l-\alpha|}^{(1)}(kR_0) \sim \frac{1}{E + Mc^2 - U} \left[ (E + Mc^2) \left( -\frac{|l - \alpha|}{kR_0} - \frac{\kappa}{k} \right) + U \frac{|l - \alpha| + l - \alpha}{kR_0} \right],
\]

\[
H_{|l+1-\alpha|}^{(1)}(kR_0) - (\kappa/k) f^{(R_0)}_{12} H_{|l+1-\alpha|}^{(1)}(kR_0) \sim \frac{1}{E + Mc^2 - U} \left[ (E + Mc^2) \left( -\frac{|l + 1 - \alpha|}{kR_0} - \frac{\kappa}{k} \right) + U \frac{|l + 1 - \alpha| - (l + 1 - \alpha)}{kR_0} \right],
\]

and are different from 0 for all values of \( l \). From the fact that, for \( kR_0 \ll 1 \), \( J_{|l-\alpha|}(kR_0) \sim (kR_0)^{|l-\alpha|} \), \( H_{|l-\alpha|}(kR_0) \sim (kR_0)^{-|l-\alpha|} \), it follows that

\[
A^{(R_0)}_{11} \sim (kR_0)^{2|l-\alpha|}, \quad A^{(R_0)}_{12} \sim (kR_0)^{2|l+1-\alpha|},
\]

so that the coefficients \( A^{(R_0)}_{11}, A^{(R_0)}_{12} \) are vanishing in the limit \( kR_0 \to 0 \). Then from eqs. (6),(7),(31),(32) we obtain the Dirac eigenfunctions of an electron in the presence of a \textit{shielded} magnetic string as

\[
\chi_{11}^{(sh)} = a_{11} J_{|l-\alpha|}(kr)
\]
\[
\chi_{l2}^{(sh)} = a_{l2} J_{|l+1|-\alpha}(kr)
\]  
\[
\chi_{l3}^{(sh)} = \frac{i\hbar c a_{l2}}{E + Mc^2} \left( \frac{d}{dr} + \frac{l + 1 - \alpha}{r} \right) J_{|l+1|-\alpha}(kr),
\]
\[
\chi_{l4}^{(sh)} = -\frac{i\hbar c a_{l1}}{E + Mc^2} \left( \frac{d}{dr} - \frac{l - \alpha}{r} \right) J_{|l|-\alpha}(kr),
\]
for \( r > R_0, R_0 \to 0 \). These eigenfunctions have been first written in eqs. (16) of ref. [5], but the simplified mathematical deduction presented there is not applicable for the eigenfunctions of angular momentum \( l = [\alpha] \).

For a given \( \alpha \), the ensemble of eigenfunctions for the shielded magnetic string, \( U \neq 0 \), and the ensemble of eigenfunctions for the bare magnetic string, \( U=0 \), differ only by the eigenfunction of angular momentum \( l = [\alpha] \), the eigenfunctions for all other values of \( l \) being identical. If \( 0 < \alpha < 1 \), the radial components of the eigenfunction \( l=0 \) for the bare string are
\[
\chi_{01}^{(b)} = b_{01} J_{-\alpha}(kr),
\]
\[
\chi_{02}^{(b)} = b_{02} J_{1-\alpha}(kr),
\]
\[
\chi_{03}^{(b)} = -\frac{i\hbar c k b_{02}}{E + Mc^2} J_{-\alpha}(kr),
\]
\[
\chi_{04}^{(b)} = \frac{i\hbar c k b_{01}}{E + Mc^2} J_{1-\alpha}(kr),
\]
For \( 0 < \alpha < 1 \), the radial components of the eigenfunction \( l=0 \) for the shielded string are
\[
\chi_{01}^{(sh)} = a_{01} J_{\alpha}(kr),
\]
\[
\chi_{02}^{(sh)} = a_{02} J_{1-\alpha}(kr),
\]
\[
\chi_{03}^{(sh)} = -\frac{i\hbar c k a_{02}}{E + Mc^2} J_{-\alpha}(kr),
\]
\[
\chi_{04}^{(sh)} = -\frac{i\hbar c k a_{01}}{E + Mc^2} J_{1-\alpha}(kr).
\]

The definitions of the parameters \( \alpha \) in [4] and [5] differ by sign. The choice \( \alpha = qF/2\pi\hbar \) is similar to standard electromagnetic relations like \( \mathbf{F} = q\mathbf{E} \).
Although it is not possible that all four components of the Dirac wave function should vanish for finite $kR_0$, it can be seen from eqs. (41)-(44),(49)-(52) that the vanishing of the four components is possible in the limit $kR_0 \ll 1$, except for $l = [\alpha]$ when the $\chi_{[\alpha]3}$ and $\chi_{[\alpha]4}$ components are divergent at $r = 0$.

The scattering of a plane wave by a shielded magnetic string will now be compared with the scattering of a plane wave by a bare magnetic string. As shown in [5], the scattering of a plane wave by a shielded magnetic string can be represented, for $0 < \alpha < 1$, by the Dirac wave function

$$
\Psi_{\alpha}^{(sh)} = \begin{pmatrix}
    a_1 \\
    a_2 \\
    -\frac{\hbar c a_2}{E + M c^2} e^{-iEt/\hbar} \\
    -\frac{\hbar c a_3}{E + M c^2}
\end{pmatrix} \psi_{\alpha}^{(sh)}(r, \theta) e^{-iEt/\hbar} + \begin{pmatrix}
    0 \\
    0 \\
    -\frac{\hbar c a_2}{E + M c^2} \exp(i\pi \alpha/2) \sin(\pi \alpha) H_\alpha^{(1)}(kr) \\
    \frac{\hbar c a_3}{E + M c^2} \exp(-i\pi \alpha/2) \sin(\pi \alpha) H_1^{(1)}(kr) e^{i\theta}
\end{pmatrix} e^{-iEt/\hbar}, \tag{53}
$$

where

$$
\psi_{\alpha}^{(sh)}(r, \theta) = \sum_{l=-\infty}^{\infty} e^{-i\pi |l-\alpha|} J_{|l-\alpha|}(kr) e^{il\theta} \tag{54}
$$

is the Aharonov-Bohm wave function for the scattering of spinless electrons [1]. Unlike $\psi_{\alpha}^{(sh)}$, which is vanishing as $r \to 0$, the second term on the right-hand side of eq. (53) is divergent at $r=0$. This term vanishes however in the limit $\hbar k/Mc \to 0$. The asymptotic expansion of the wave function, eq. (53), is

$$
\Psi_{\alpha}^{(sh)} = \begin{pmatrix}
    a_1 \\
    a_2 \\
    -\frac{\hbar c a_2}{E + M c^2} e^{-iEt/\hbar} \\
    -\frac{\hbar c a_3}{E + M c^2}
\end{pmatrix} e^{-ikr \cos \theta + i\alpha \theta - iEt/\hbar} - \begin{pmatrix}
    a_1 e^{i\theta/2} \\
    a_2 e^{i\theta/2} \\
    \frac{\hbar c a_2}{E + M c^2} e^{-i\theta/2} \\
    \frac{\hbar c a_3}{E + M c^2} e^{3i\theta/2}
\end{pmatrix} \frac{\sin(\pi \alpha)}{\cos(\theta/2)} e^{ikr + i\pi/4 - iEt/\hbar} \frac{1}{(2\pi kr)^{1/2}}. \tag{55}
$$

The scattering of a plane wave by a bare magnetic string can be represented, for $0 < \alpha < 1$, by the Dirac wave function

$$
\Psi_{\alpha}^{(b)} = \Psi_{\alpha}^{(sh)} + \begin{pmatrix}
    ia_1 e^{i\pi \alpha/2} \sin(\pi \alpha) H_\alpha^{(1)} \\
    0 \\
    0 \\
    \frac{\hbar c a_3}{E + M c^2} e^{i\pi \alpha/2} \sin(\pi \alpha) H_1^{(1)} e^{i\theta}
\end{pmatrix} e^{-iEt/\hbar}. \tag{56}
$$

The expression of the first component of the wave function in eq. (56) is

$$
\psi_{\alpha}^{(b)}(r, \theta) = \sum_{l=-\infty, l \neq 0}^{\infty} e^{-i\pi |l-\alpha|} J_{|l-\alpha|}(kr) e^{il\theta} + e^{i\pi \alpha} J_{-\alpha}(kr). \tag{57}
$$
The asymptotic expansion of the wave function, eq. (56), is

\[
\Psi^{(b)}_\alpha = \begin{pmatrix}
a_1 \\
a_2 \\
\frac{-\hbar c a_2}{E + Mc^2} \\
\frac{-\hbar c a_1}{E + Mc^2}
\end{pmatrix} e^{-i k r \cos \theta + i \alpha \theta - i Et / \hbar} + \begin{pmatrix}
a_1 e^{-i \theta / 2} \\
-a_2 e^{i \theta / 2} \\
\frac{-\hbar c a_2}{E + Mc^2} e^{-i \theta / 2} \\
\frac{\hbar c a_1}{E + Mc^2} e^{i \theta / 2}
\end{pmatrix} \frac{\sin(\pi \alpha)}{\cos(\theta / 2)} \frac{e^{ikr + i\pi / 4 - iEt / \hbar}}{(2\pi kr)^{1/2}}.
\]

(58)

According to eq. (53), the scattering of a plane wave by a shielded magnetic string can be represented in the limit \(\hbar k/Mc \to 0\) with the aid of the Aharonov-Bohm wave function for spinless electrons, eq. (54). On the other hand, according to eq. (56), there is a difference between the scattering of polarized electrons by bare and shielded strings, and the difference persists in the non-relativistic limit.

Recently, Moroz [9] remarked that the divergent terms in the eigenfunction of an electron endowed with spin in the presence of a bare magnetic string are also obtained by using the Pauli equation for an electron gyromagnetic ratio \(g=2\), while the divergent terms disappear if the anomalous part \(g_m - 2 = 0.0023\) of the gyromagnetic ratio, due to radiative corrections, is included in the analysis. Therefore, for magnetic fluxes which are odd multiples of \(h/2e\), the modifications to an eigenfunction described by Hagen [4] will be observed only when \(g_m - 2 < kr_0 \ll 1\), i.e. when the tube of flux of radius \(r_0\) can be regarded as a string, and at the same time the tube is not so thin that the anomalous magnetic moment wipes out the effect. As mentioned previously, the radius of a tube of flux of \(h/2e\) obtained with a 2 T magnetic field is \(r_{h/2e} = 1.81 \cdot 10^{-8}\) m, so that if we choose \(kr_{h/2e} = 0.2\), then the required kinetic energy of the incident electron would be \(4.65 \cdot 10^{-6}\) eV, which is rather low for an electron interference experiment.

An alternative approach by which the wave function for an electron interference experiment can be made very small in the region of the tube of magnetic flux is to represent the state of an incident electron by a wave packet [10]. The propagation of this wave packet can be studied with the aid of a Green’s function [3], [11]. If relativistic corrections are neglected, the state of the incident electron is described by the two principal components of the Dirac wave function, and for a tube of magnetic flux oriented in the z direction these principal components are solutions of decoupled equations of the Schrödinger type. Then for each of the two principal components, the Green’s function can be obtained as a sum of products of eigenfunctions. It was found that the eigenfunctions for a bare magnetic string and for a shielded magnetic string differ only for \(l = [\alpha]\). Thus, if \(\alpha > 0\), the wave
function $J_{\alpha-|\alpha|}(kr) \exp(i[\alpha] \theta)$ would be replaced by $J_{[\alpha]-|\alpha|}(kr) \exp(i[\alpha] \theta)$. The difference between the Green’s function for the bare magnetic string $G_{\alpha}^{(b)}$ and the conventional Green’s function $G_{\alpha}$ is, in the non-relativistic limit,

$$G_{\alpha}^{(b)} - G_{\alpha} = \frac{1}{2\pi} \int_0^\infty k dk \left( J_{[\alpha]-|\alpha|}(kr) J_{[\alpha]-|\alpha|}(kr') - J_{[\alpha]-|\alpha|}(kr) J_{[\alpha]-|\alpha|}(kr') \right) e^{-i\hbar k^2 t/2M + i[\alpha](\theta - \theta')} .$$

The result of the integration is [12]

$$G_{\alpha}^{(b)} - G_{\alpha} = -\frac{iM}{2\pi \hbar t} \left[ e^{\frac{i}{\hbar} r \alpha -(\alpha) -\alpha} \left( \frac{M r r'}{\hbar t} \right) - e^{-\frac{i}{\hbar} (\alpha-(\alpha))} \left( \frac{M r r'}{\hbar t} \right)\right] \times \exp \left( \frac{iM}{2\hbar t}(r^2 + r'^2) + i[\alpha](\theta - \theta') \right) ,$$

which can also be written as

$$G_{\alpha}^{(b)} - G_{\alpha} = \left( \frac{M}{2\pi \hbar t} \sin(\pi(\alpha - [\alpha])) e^{\frac{i}{\hbar} (\alpha -[\alpha])} H_{\alpha}^{(1)} \left( \frac{M r r'}{\hbar t} \right) \exp \left( \frac{iM}{2\hbar t}(r^2 + r'^2) + i[\alpha](\theta - \theta') - \frac{i\pi}{4} \right) \right) .$$

For $M r r'/\hbar t \gg 1$ the difference $G_{\alpha}^{(b)} - G_{\alpha}$ is

$$G_{\alpha}^{(b)} - G_{\alpha} = \left( \frac{M}{2\pi \hbar^3 t r r'} \right)^{1/2} \sin(\pi(\alpha - [\alpha])) \exp \left[ \frac{iM}{2\hbar t}(r^2 + r'^2 + 2rr') + i[\alpha](\theta - \theta') - \frac{i\pi}{4} \right] .$$

The initial wave function of a packet of width $\delta$, centered at the time $t=0$ at the point $\rho_0, \theta_0$ and of incident momentum $\hbar k$ oriented in the $-x$ direction can be represented as [3]

$$\Psi_{\delta,\alpha}(r', \theta', 0) = \frac{1}{\pi^{1/2} \delta} \exp \left[ i\alpha \theta' - ikr' \left( 1 - \frac{\theta'^2}{2} \right) - \frac{r'^2 + \rho_0^2 - 2r' \rho_0 \left[ 1 - (\theta' - \theta_0)^2 / 2 \right]}{2\delta^2} \right] .$$

The plane-wave contribution $-ikr' \cos \theta'$ could be expanded about $\theta' = 0$ because $|\theta_0| \ll 1, \delta/\rho_0 \ll 1$. The difference at the point of polar coordinates $r, \theta$ and at the time $t$ of the wave functions representing the scattering by the bare string and by the shielded string can be obtained as

$$\Delta(r, \theta, t) = \int \left( G_{\alpha}^{(b)} - G_{\alpha} \right) \Psi_{\delta,\alpha} \, dr' \, d\theta' .$$

For $k\rho_0 \gg 1, kr \gg 1, \rho_0 \ll k\delta^2, r \ll k\delta^2$, the difference $\Delta$ is

$$\Delta(r, \theta, t) = \frac{\epsilon^{\pi/4}}{2\pi^{1/2} \pi \delta} \sin(\pi(\alpha - [\alpha])) \exp\left[ ikr + i[\alpha] \theta \right] \left( kr \right)^{1/2} \exp \left[ -\frac{i\hbar k^2 t}{2M} - \frac{\rho_0^2 \theta_0^2}{2\delta^2} - \frac{\left( r + \rho_0 - \frac{\hbar k t}{M} - \frac{\rho_0 \theta_0^2}{2} \right)^2}{2\delta^2} \right] .$$

where $\alpha > 0$. For times $t$ such that $t > \rho_0(1 - \theta_0^2/2)M/\hbar k$, the difference $\Delta$ of the wave functions describing the scattering of the incident packet by a bare string and by a shielded string is proportional to $\exp(-\rho_0^2 \theta_0^2/2\delta^2)$, or $\exp(-d^2/2\delta^2)$, where $d = \rho_0 \theta_0$ is the impact parameter of the incident wave packet. As was shown in [13], in the case of the scattering of a wave packet by a magnetic string the whole scattering amplitude is proportional to the exponentially small factor $\exp(-d^2/2\delta^2)$, and
we have just seen that the replacement of $J_{α−[α]}$ by $J_{[α]−α}$ does not change this fact.

The scattering by a shielded magnetic string and the scattering by a bare magnetic string are really different only when there is a significant probability density in the region of the magnetic flux, i.e. when there is a direct interaction between the magnetic moment of the electron and the magnetic field. This is basically due to the fact that the interaction with a magnetic dipole moment is proportional to the magnetic field, and for a shielded string the magnetic field is equal to zero in the region where there is a non-vanishing probability density. Actually, as shown recently [13], the Aharonov-Bohm effect is relevant not so much for the problem of the description of the electromagnetic continuum by field strengths or electromagnetic potentials, but rather it demonstrates the global character of the states in quantum mechanics.

ACKNOWLEDGMENT

This work has been supported by a research grant from the Romanian Academy of Sciences.
References

[1] Y. Aharonov and D. Bohm, Phys. Rev. 115 (1959) 485.

[2] U. Percoco and V. M. Villalba, Phys. Lett. A 140 (1989) 105.

[3] S. Olariu and I. I. Popescu, Rev. Mod. Phys. 57 (1985) 339.

[4] C. R. Hagen, Phys. Rev. Lett. 64 (1990) 503.

[5] S. Olariu, Phys. Lett. A 144 (1990) 287.

[6] V. Berestetski, E. Lifchitz and L. Pitayevski, Théorie quantique relativiste, Première partie, in L. Landau and E. Lifchitz, Physique théorique, Tome IV (Mir, Moscou, 1972), pp. 95, 100.

[7] Ph. M. Morse and H. Feshbach, Methods of theoretical physics (McGraw-Hill, New York, 1953) p. 604.

[8] Ref. [7], pp. 623-624.

[9] A. Moroz, Phys. Rev. A 53 (1996) 669.

[10] S. Olariu and I. I. Popescu, Phys. Rev. D 27 (1983) 383.

[11] M. Kretzschmar, Z. Physik 185 (1965) 84.

[12] G. N. Watson, A treatise on the theory of Bessel functions (Cambridge University Press, Second Edition, 1948), pp. 77 and 395.

[13] S. Olariu, Phys. Rev. A, accepted for publication.