Fully general device-independence for two-party cryptography and position verification

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Device-independent quantum cryptography allows security even if the devices used to execute the protocol are untrusted - whether this is due to unknown imperfections in the implementation, or because the adversary himself constructed them to subvert the security of the protocol. While device-independence has seen much attention in the domain of quantum key distribution, relatively little is known for general protocols. Here we introduce a new model for device-independence for two-party protocols and position verification in the noisy-storage model. For the first time, we show that such protocols are secure in the most general device-independent model in which the devices may have arbitrary memory, states and measurements. In our analysis, we make use of a slight modification of a beautiful new tool developed in [1] called “Entropy Accumulation Theorem”. What’s more, the protocols we analyze use only simple preparations and measurements, and can be realized using any experimental setup able to perform a CHSH Bell test. Specifically, security can be attained for any violation of the CHSH inequality, where a higher violation merely leads to a reduction in the amount of rounds required to execute the protocol.

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I. INTRODUCTION

Possibly the most well known application of quantum communication is quantum key distribution (QKD)\cite{2, 3}, which allows Alice and Bob to protect their communication from the prying eyes of an eavesdropper. Yet quantum communication offers advantages for the implementation of many other cryptographic tasks such as two-party cryptography, and position based cryptography \cite{4-9}. The goal of two-party cryptography is to enable two parties, Alice and Bob, to solve common problems without the need for mutual trust. Important examples of such problems include the realization of private database access, or secure identification. The goal of position verification on the other hand is to certify that a protocol participant really does reside at a claimed location in space, such as an agent who will only receive information if he is at a specific place.

Maybe not surprisingly, security for such tasks is more difficult to attain than for QKD, where Alice and Bob can collaborate to check on the eavesdropper. In the realm of two-party protocols, Alice and Bob themselves do not trust each other, and thus have no partner with whom to collaborate to check on the adversary’s behaviour. Such a lack of collaboration has dire consequences, and it has been shown that even using quantum communication, Alice and Bob cannot achieve security for two-party quantum protocols without making additional assumptions \cite{10–13}. Due to the practical importance of problems such as identification, however, one is generally willing to make some assumptions in order to achieve security. Classically, one often relies on computational assumptions, i.e., that solving a computational puzzle requires a larger amount of computing resources than are available to the adversary. Computational assumptions are highly relevant in practise, although many well known assumptions are known to become insecure once a quantum computer becomes available. Interestingly, however, it is also possible to make physically motivated assumptions, for example that the adversary’s ability to store information is limited. This line of assumptions was pioneered by Maurer \cite{14}, who considered an adversary who can only store a limited number of classical bits. This assumption is known as the bounded-storage model. While conceptually greatly appealing, classical storage is cheap and plentiful rendering this assumption less reliable. Most significantly, however, it was shown that any protocol in which the honest parties store $O(n)$ bits in order to run the protocol becomes insecure once the adversary can store $O(n^2)$ bits \cite{13}. Storing quantum information reliably, however, is an extremely difficult problem, which motivated the so-called bounded-quantum storage \cite{16, 17} or more generally noisy-storage model \cite{18, 19}. Importantly, the noisy-storage model admits protocols that require no quantum storage for the honest execution and that can be implemented in a manner similar to QKD using BB84 \cite{18, 20}, six-state \cite{21} or continuous variable \cite{22} encodings. Significantly, security can always be achieved as long as the number of qubits $n$ sent in the protocol is only slightly larger than the number of qubits $r$ that the adversary can store, that is, whenever $r \lesssim n - O(\log n)$ \cite{20}, which is essentially optimal. First implementations of bit commitment \cite{23} and oblivious transfer \cite{24} in the noisy-storage model demonstrated their technological feasibility. What makes storage assumptions especially appealing is that security remains preserved even if the adversary obtains a much better quantum storage device in the future, making this assumption fully future proof.

Implementing quantum protocols in practise is a challenging undertaking. Even for relatively simple protocols like QKD, many attacks on practical implementations have been found \cite{25, 30}. This motivates the study of device-independent quantum cryptography, in which security can in principle be obtained even if the quantum devices are entirely uncharacterized and possibly constructed by the adversary. Roughly speaking, device independent quantum cryptography models each device as a black box. Inside box, the quantum state and measurements can be arbitrary, and indeed the device may also have arbitrary memory of all previous rounds of a protocol. The only access to the device is by its classical input output behaviour. We can instruct it to perform a certain measurement as a classical instruction (input), and observe the resulting classical measurement outcome (output) \cite{31}. Yet, there are no guarantees that the desired measurement is actually being performed. Curiously, it turns out that in the quantum domain it is possible to obtain security purely by observing the input-output statistics of such black boxes. The first clues to this were present already in \cite{2}, where it was shown that the violation of a Bell \cite{32} inequality is intimately linked to cryptographic security. Mayers and Yao \cite{33, 34} went on to realise that indeed quantum systems can be self-tested: certain quantum properties can be verified purely by observing the classical input-output behaviour, which started the field of device-independent (DI) quantum cryptography. In DI cryptography instead of assuming that we know how the devices work, we test them during the protocol by using them to exhibit Bell nonlocality \cite{33}. DI cryptography has been one of the most active research topics within quantum cryptography, predominantly in the context of QKD \cite{36, 42} and randomness expansion or amplification \cite{42, 50}.

In contrast, DI security has seen relatively little attention for general quantum protocols. A device independent blind quantum computation scheme has been recently analyzed in \cite{51}. In \cite{52, 53}, the DI security of a protocol for imperfect coin flipping and weak bit commitments has been analysed. The spirit of these works differs form the approach pursued here, in that we work in the noisy-storage model which allows the implementation of essentially perfect two-party primitives under a realistic physical assumption. More recently, Adlam and Kent have proposed a DI relativistic bit commitment protocol \cite{54}, which allows security for a fixed amount of time under the assumption
that each party is split into space-like separated agents. We remark that in contrast to relativistic assumptions, we only require the noisy-storage assumption to be valid for a short waiting time during the execution of the protocol.

A. Model

Constructing DI protocols for more general quantum cryptographic tasks brings additional challenges that require both a careful modelling, as well as a different analysis than in QKD. The fact that Alice and Bob can check on Eve in QKD, leads to a model in which Eve first prepares all the devices, and Alice and Bob then produce key simply by giving inputs and collecting outputs. In particular, there is no quantum communication going back to Eve during the execution of the protocol. This is in sharp contrast to DI in the realm of two-party cryptography, where the correctness of the protocol requires that there is quantum communication going back to the adversary who prepared the devices. Before stating our results let us thus first explain how we model DI in this setting.

As in QKD, we will always assume that the dishonest party prepares the devices. For our protocols, it will turn out that indeed only Alice can be affected by faulty devices, and hence we will focus on the case of dishonest Bob. Figure 1 summarizes the model.

Assumptions 1 (Device-independent model).

1. Bob can prepare the following devices Alice will use: the main device, the source and the testing device. These devices can be fully malicious and exhibit arbitrary states and measurements, as well as correlations to the previous rounds. The main measurement device takes one bit $\Theta \in \{0, 1\}$ and a state as input, and outputs one bit $X' \in \{0, 1\}$ while the the testing device takes one trit $\hat{\Theta} \in \{0, 1, \perp\}$, one bit $T \in \{0, 1\}$ and a state as input, and outputs a trit $Y \in \{0, 1, \perp\}$. The source prepares the quantum states.

2. We assume that Alice’s measurement devices cannot send any information outside her lab. This assumption is also made in DI QKD, but we list it here explicitly.

3. We assume that Alice holds a device called switch to decide whether to test or to send the state to Bob. This switch - i.e. the decision whether to test or not - is trusted.

4. We also assume that Alice can isolate the source from the measurement devices, so that the source cannot receive any information from these devices. The source can only prepare and send the state to the switch. We remark that we prove security in a more general model in which there is no outside source, but the device of Alice first prepares the states of all rounds, and subsequently decides which rounds to measure or to pass to Bob. This is precisely analogous to DI QKD in which the state is prepared essentially ahead of time in the analysis. However, since we want protocols which require no memory to be implemented in practise, we use the source to explicitly model that the states which are generated do not carry information about the previous rounds to Bob. Indeed, we prove that any model in which there is no separated source and Alice’s device simply generates the quantum states sequentially is necessarily insecure.

5. All classical operations that Alice performs are trusted. This is assumed in all DI quantum protocols.

B. Protocols

We first prove security of a cryptographic building block known as Weak String Erasure [19] (see Figure 2). Weak string erasure is universal in that it can be used to solve any other two-party cryptographic problem. Using classical communication, we can easily obtain bit commitment, or given a quantum secure protocol for interactive hashing, oblivious transfer. In essence, weak string erasure uses quantum communication to generate a simple form of classical correlations which can then be turned into interesting tasks using classical communication. Our DI WSE protocol, Protocol 11 is a combination of the simple BB84 based protocol for WSE presented in [19], with a DI test using the CHSH inequality. We remark that our analysis is easily extended to a DI test using other Bell inequalities, including higher dimensional ones. However, we focus on CHSH since it can indeed be implemented using merely qubit systems, and reflects the qubit nature of the WSE protocol about which we like to gain DI confidence.

To state our results, we will take the simple case of a storage assumption in which Bob’s quantum memory has a maximum dimension $d$. Our analysis can easily be extended to the regime of noisy-quantum storage, as we will outline later. Specifically, we prove the following results.
Figure 1. DI model in round $i \in [n]$ of the protocol. The source creates a state $\rho_{A_iB_i} = \Phi_{A_iB_i}$ and sends it to the switch. The switch always sends the $A_i$ system to the main device, but the $B_i$ system either to the testing device (with probability $\mu$) or to Bob (with a probability $1 - \mu$). The source is isolated from the measurement devices (see assumptions II).

- Theorem (informal): Under the physical assumption that the adversary holds a noisy quantum storage Protocol 11 is secure with an min-entropy rate $\lambda \approx h - \mu$, where $\mu$ is the probability of testing, and $h$ is a function of the minimal number of CHSH games that have to be won during the execution of the protocol. See Theorem 12 for a precise statement.

- Theorem (informal): Under the physical assumption that the adversary holds “noisy entanglement” Protocol 18 is secure with $P_{\text{cheat}} \leq 2^{-\kappa n}$ being the probability that a cheater can trick the verifiers into falsely believing he is at the claimed position. $n$ is the number of rounds, and $\kappa > 0$ a constant which is depends on the minimal number of CHSH games that have to be won during the execution of the protocol. See section III B 1 for a precise statement.

Figure 2. Weak String Erasure (WSE). If both Alice and Bob are honest, then Alice holds a random bit string $X^n_i \in \{0, 1\}^n$ while Bob should get $(\mathcal{I}, X^\mathcal{I}_i)$ where $\mathcal{I} \subset \{1, \ldots, n\}$ is a random subset of indices and $X^\mathcal{I}_i$ is a substring of $X^n_i$ corresponding to the indices in $\mathcal{I}$. Security means that dishonest Alice cannot learn $\mathcal{I}$, and Bob has high min-entropy about $X$ - that is, while holding $X^\mathcal{I}_i$ it is difficult for him to guess the entire string $X$.

II. PRELIMINARIES

A. Notation

We denote $\mathcal{H}_A$ the Hilbert space of the system $A$ with dimension $|A|$ and $\mathcal{H}_{AB} := \mathcal{H}_A \otimes \mathcal{H}_B$ the Hilbert space of the composite system $A$ and $B$, with $\otimes$ the tensor product. By $\mathcal{L}(\mathcal{H}), \mathcal{S}_n(\mathcal{H}), \mathcal{P}(\mathcal{H})$ and $\mathcal{S}(\mathcal{H})$ we mean the set of linear, Hermitian, positive semidefinite and (quantum) density operators on $\mathcal{H}$, respectively. $\mathcal{S}(\mathcal{H}) \subseteq \mathcal{P}(\mathcal{H}) \subseteq \mathcal{S}_n(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$. For two operators $A, B \in \mathcal{S}_n(\mathcal{H})$, $A \geq B$ means $(A - B) \in \mathcal{P}(\mathcal{H})$. For $M \in \mathcal{L}(\mathcal{H})$, we denote $|M| := \sqrt{M^\dagger M}$, and the Schatten p-norm $\|M\|_p := \text{tr}(|M|^p)^{1/p}$ for $p \in [1, \infty]$. For $M \in \mathcal{P}(\mathcal{H})$, $\|M\|_\infty$ is the largest eigenvalue of $M$. For $M \in \mathcal{P}(\mathcal{H})$, $M^{-1}$ is the general inverse of $M$, meaning that the relation $MM^{-1} = M$ holds. If $\rho_{AB} \in \mathcal{S}(\mathcal{H}_{AB})$ then we denote $\rho_A := \text{tr}_B(\rho_{AB})$ and $\rho_B := \text{tr}_A(\rho_{AB})$ to be the respective reduced states. We use $[n]$ as a shorthand
for \( \{1, \ldots, n\} \). If we deal with a system composed of \( n \) subsystems \( A_1, \ldots, A_n \), we use \( A_i^j \) (\( i, j \in [n] : i \leq j \)) for \( A_i, \ldots, A_j \). We write \( X \in \mathcal{X} \) to indicate that a classical value \( X \) is chosen uniformly at random in a finite alphabet \( \mathcal{X} \).

We use \( \log(\cdot) \) to denote the logarithm in base two while \( \ln(\cdot) \) is the natural logarithm.

For a bit string \( x \in \{0,1\}^n \), \( |x| \) denotes its length \( n \) and the Hamming weight \( w_H(x) \) is the number of 1’s in \( x \). For a finite alphabet \( \mathcal{C} \), and for a random variable \( C_1^n \) on \( \mathcal{C}^n \), we write \( \text{freq}(C_1^n) \) for the distribution over \( \mathcal{C} \) defined by

\[
\text{freq}(C_1^n)(c) := \frac{|\{i \in [n] : C_i = c\}|}{n}, \text{ for any } c \in \mathcal{C}.
\]

CPTP maps denotes Completely Positive and Trace Preserving maps, and LOCC denotes the set of Local Operation and Classical Communication. POVM denotes a Positive-Operator Valued Measure which is a set of positive semi-definite operators \( \{P_x\} \) such that \( \sum_x P_x = \mathbb{1} \).

For two states \( \rho, \sigma \in \mathcal{S}(\mathcal{H}) \), we say that \( \rho \approx_c \sigma \) if the purified distance \([55, \text{Definition 3.8}]\) between \( \rho \) and \( \sigma \) is less than \( \epsilon \).

For classical-quantum states (or cq-states)

\[
\rho_{XA} := \sum_{x \in \mathcal{X}} p_x \cdot |x\rangle\langle x| \otimes \rho_{A|x},
\]

\( \{p_x\} \) is a probability distribution over the alphabet \( \mathcal{X} \) of \( X \). We define a cq-state \( \rho_{XA|\Omega} \) conditioned on an event \( \Omega \subset \mathcal{X} \) as,

\[
\rho_{XA|\Omega} := \frac{1}{p_{\Omega}} \sum_{x \in \Omega} p_x \cdot |x\rangle\langle x| \otimes \rho_{A|x}, \text{ where } p_{\Omega} := \sum_{x \in \Omega} p_x.
\]

To generalise result in the noisy storage model (in Weak String Erasure), we use \( E^{(1)}_C(\mathcal{E}) \) \([56, \text{Definition 10}]\) being the one shot entanglement cost to simulate a channel \( \mathcal{E} : \mathcal{L}(\mathcal{H}_A) \rightarrow \mathcal{L}(\mathcal{H}_B) \) using LOCC and preshared entanglement

\[
E^{(1)}_C(\mathcal{E}) := \min_{M,A} \{\log(M) : \forall \rho_A \in \mathcal{L}(\mathcal{H}_A), \Lambda(\rho_A \otimes \Psi^M_{AB}) = \mathcal{E}(\rho_A)\}
\]

where \( \Lambda \) is a LOCC with \( AA \rightarrow 0 \) (no output) on Alice’s side and \( B \rightarrow B \) on Bob’s side, and \( M \in \mathbb{N} \). Note that we require a single LOCC map to simulate the effect of the channel \( \mathcal{E} \) so \( \Lambda \) must be independent of \( \rho_A \).

Similarly to generalise our result to the noisy entanglement model (in PV) we will use \( E^{(1)}_{C,\text{LOCC}}(\rho_{AB}) \) being the one shot entanglement cost to create a bipartite state \( \rho_{AB} \) from a maximally entangled state using only local operations and classical communication. It is formally defined as

\[
E^{(1)}_{C,\text{LOCC}}(\rho) := \min_{M,A} \{\log(M) : \Lambda(\Psi^M_{AB}) = \rho_{AB}, \Lambda \in \text{LOCC}, M \in \mathbb{N}\},
\]

where \( \Psi^M_{AB} \) is a maximally entangled state of dimension \( M \)

\[
\Psi^M_{AB} := |\Psi^M_{AB}\rangle\langle \Psi^M_{AB}|, \quad |\Psi^M_{AB}\rangle := \frac{1}{\sqrt{M}} \sum_{i=1}^M |i\rangle |i\rangle.
\]

\[\text{B. Markov Condition}\]

All the statements in this work heavily rely on one condition, namely the Markov chain condition sometimes called Markov condition.

**Definition 2** (Markov condition). Let \( \rho_{ABC} \in \mathcal{S}(\mathcal{H}_{ABC}) \) be a quantum state. We say that \( \rho_{ABC} \) satisfies the Markov condition \( A \leftrightarrow B \leftrightarrow C \) if and only if

\[
I(A : C|B)_\rho = 0,
\]

where \( I(A : C|B)_\rho \) is the mutual information between \( A \) and \( C \) conditioned on \( B \) for the state \( \rho_{ABC} \).

This condition becomes trivial when \( A, B, \) and \( C \) are independent random variables. To know more about the Markov condition see \([1, \text{Section 2.2 & Appendix C}]\).
C. Entropies

All along this work we use the smooth entropies defined as,

**Definition 3.** Let $\rho_{AB} \in S(\mathcal{H}_{AB})$ be a quantum state, and $\epsilon \in [0, 1]$. The min- and max-entropies of $A$ conditioned on $B$ are defined as,

$$H_{\min}(A|B)_{\rho} := -\log \left( \inf_{\hat{\rho}_{AB}} \| \rho_{AB} \sigma_B^{-1/2} \|_2^2 \right)$$  \hspace{1cm} (7)

$$H_{\max}(A|B)_{\rho} := \log \left( \sup_{\sigma_B} \| \rho_{AB} \sigma_B^{-1/2} \|_1^2 \right)$$  \hspace{1cm} (8)

and their smooth version as

$$H_{\min}^\epsilon(A|B)_{\rho} := \sup_{\hat{\rho}_{AB}} H_{\min}(A|B)_{\hat{\rho}}$$  \hspace{1cm} (9)

$$H_{\max}^\epsilon(A|B)_{\rho} := \inf_{\hat{\rho}_{AB}} H_{\max}(A|B)_{\hat{\rho}},$$  \hspace{1cm} (10)

where the infimum and supremum over $\hat{\rho}_{AB}$ is taken in an $\epsilon$-ball (in terms of the purified distance) centered in $\rho_{AB}$ in the set of sub-normalised states $S_\epsilon(\mathcal{H}_{AB})$. Let $\sigma_B \in S(\mathcal{H}_B)$. Note that if $A$ is classical the optimisation can be restricted to be in the set of states $S(\mathcal{H}_{AB})$.

The min-entropy has a nice interpretation in terms of guessing probability.

**Property 4** (see [56]). Let $\rho_{XA}$ be a cq-state. We define the guessing probability $P_{\text{guess}}(X|A)$ as follows

$$P_{\text{guess}}(X|A) := \max_{\{P^x, x \in \mathcal{X}\}} \sum_{x \in \mathcal{X}} \text{tr} \left( |x\rangle \langle x | \otimes P^A \rho_{XA} \right),$$  \hspace{1cm} (11)

where the maximisation is done over all POVMs $\{P_x, x \in \mathcal{X}\}$ acting on $\mathcal{H}_A$. Then the min-entropy can be written as

$$H_{\min}(X|A)_{\rho} := -\log (P_{\text{guess}}(X|A)_{\rho}).$$  \hspace{1cm} (12)

In order to bound these min- and max-entropies we sometimes need to use the sandwiched relative Rényi entropies.

**Definition 5.** For a state $\rho_{AB} \in S(\mathcal{H}_{AB})$ and for $\alpha \in \mathbb{R} \cup [1, \infty]$ the sandwiched $\alpha$-Rényi entropy of $A$ conditioned on $B$ is defined as

$$H_\alpha(A|B)_{\rho} = -\frac{\alpha}{\alpha - 1} \log \left( \| \rho_{AB} \sigma_B^{-\alpha/2} \|_2^{\alpha} \right).$$  \hspace{1cm} (13)

D. The Entropy Accumulation Theorem (EAT)

The Entropy Accumulation Theorem (EAT) aims to relate the conditional $\epsilon$-min-entropy of a $n$-subsystem state to the Von Neumann entropy of each subsystem.

We recall here some of the useful definitions and theorems from [1], and prove a variant of the EAT which will be handier to use afterwards.

The first result we recall is the Corollary 3.4 from [1].

**Theorem 6** ([1], Corollary 3.4). Let $\rho_{R_0A_1B_1}$ be a state, and $\mathcal{M}$ be a CPTP map from $R_0$ to $A_2B_2$. Let us assume that the state $\rho_{A_1B_1A_2B_2} := (\mathcal{M} \otimes 1_{A_1B_1})(\rho_{R_0A_1B_1})$ satisfies the Markov condition $A_1 \leftrightarrow B_1 \leftrightarrow B_2$. Then we have,

$$\inf_{\omega} H_\alpha(A_2|B_2A_1B_1)_{\mathcal{M}(\omega)} \leq H_\alpha(A_1A_2|B_1B_2)_{\rho} - H_\alpha(A_1|B_1)_{\rho} \leq \sup_{\omega} H_\alpha(A_2|B_2A_1B_1)_{\mathcal{M}(\omega)}$$  \hspace{1cm} (14)

where the supremum and the infimum are taken over all states $\omega_{R_0A_1B_1} \in S(\mathcal{H}_{R_0A_1B_1})$.

Note that as for the remark 7 (below), if some registers are classical in $\rho_{R_0A_1B_1}$ then they can be taken classical in $\omega_{R_{0i}A_1B_1}$.

Let us now write some necessary definitions, which will permit us to state the main theorem (the EAT) of [1]. For $i \in [n]$ let $\mathcal{M}_i$ be a CPTP map from $R_{i-1}$ to $C_iA_iB_iR_i$ (where $C_i$ is a classical register) that takes any input $\sigma_{R_{i-1}R_i}$
and outputs $\sigma_{C,A,B_i,R_i} = (\mathcal{M}_i \otimes \mathbb{I}_R)(\sigma_{R_{i-1}})$. 

The EAT below applies to states on the form,

$$\rho_{C_1^i A_1^i B_1^i E} = (\text{tr}_{R_0} \circ \mathcal{M}_n \circ \cdots \circ \mathcal{M}_1 \otimes \mathbb{I}_E)(\rho_{R_0 E}),$$

where $\rho_{R_0 E} \in \mathcal{S}(\mathcal{H}_{R_0 E})$. Moreover we demand that the state satisfies the following Markov condition

$$\forall i \in [n], \quad A_i^{i-1} \leftrightarrow B_i^{i-1} \leftrightarrow B_i.$$  

Let $\mathcal{P}(C)$ be the set of distribution on the alphabet $C$ of the classical register $C_i$, and $R$ is a system isomorphic to $R_{i-1}$. For any $q \in \mathcal{P}(C)$ we define the set of states

$$\Sigma_i(q) := \{ \sigma_{C,A,B_i,R_i} = (\mathcal{M}_i \otimes \mathbb{I}_R)(\omega_{R_{i-1}}) : \omega \in \mathcal{S}(\mathcal{H}_{R_{i-1}} \otimes \mathcal{H}_R) \& \sigma_{C_i} = q \}$$

Definition 7. A real function $f$ on $\mathcal{P}(C)$ is called a \underline{min-tradeoff function} for a map $\mathcal{M}_i$ if

$$f(q) \leq \inf_{\sigma \in \Sigma_i(q)} H(A_i|B_i R),$$

and \underline{max-tradeoff function} for a map $\mathcal{N}_i$ if

$$f(q) \geq \sup_{\sigma \in \Sigma_i(q)} H(A_i|B_i R).$$

If $\Sigma_i(q) = \emptyset$ then the infimum is taken to be $+\infty$ and the supremum $-\infty$.

Remark 1 (From [1] Remark 4.3). We can impose some constraints on the sets $\Sigma_i(q)$ defined above. Indeed the system $R$ can be restricted to be isomorphic to $A_i^{i-1}B_i^{i-1}$. We can also impose that for any register in $\rho$ (as defined above) that is classical, we can take it to be classical for all states in $\Sigma_i(q)$.

Let us now state the EAT as in [1] before modifying it.

**Theorem 8** (EAT from [1] Theorem 4.4). Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be CPTP maps from $R_{i-1}$ to $C_i A_i B_i R_i$ (where $C_i$ is classical), and $\rho_{C_1^i A_1^i B_1^i E}$ be a state such that (15), the Markov conditions (16), and the property that the classical value $C_1^n$ can be obtained by measuring the marginal $\rho_{C_1^i B_1^i}$ hold. Let $h \in \mathbb{R}$, and $f$ be an affine \underline{min-tradeoff function} for $\mathcal{M}_1, \ldots, \mathcal{M}_n$, and let $\epsilon \in ]0,1[$. Then, for any event $\Omega \subset \mathbb{C}^n$ such that $f(\text{freq}(C_1^n)) \geq h$,

$$H_{\min}^c(A_i^n | B_i^n E)_{\rho_{\Omega}} \geq nh - v \sqrt{n},$$

where $\rho_{\Omega} = \rho_{C_1^i A_1^i B_1^i E | \Omega}$, and $v = 2(\log(1 + 2d_A) + ||\nabla f||_{\infty}) \sqrt{1 - 2 \log(\epsilon \cdot \rho_{\Omega})}$, where $d_A$ is the maximum dimension of the system $A_i$. In the same way we have,

$$H_{\max}^c(A_i^n | B_i^n E)_{\rho_{\Omega}} \leq nh + v \sqrt{n},$$

when we replace $f$ by an affine \underline{max-tradeoff function} $\tilde{f}$, such that the event $\Omega$ implies $h \geq \tilde{f}(\text{freq}(C_1^n))$.

Even though Theorem 8 is powerful, it is a bit inconvenient to apply for our applications. Indeed, to apply the theorem directly we need that the classical value $C_1^n$ can be obtained by measuring $A_i^n B_i^n$, which will not necessarily be true in our case. That is why we will state a variant of this theorem where we can circumvent this condition. Note that the two versions are in fact equivalent, but we state the modified version here in order to clarify the upcoming proofs.

**Theorem 9** (Modified EAT). Let $\mathcal{M}_1, \ldots, \mathcal{M}_n$ be CPTP maps from $R_{i-1}$ to $C_i A_i B_i R_i$ (where $C_i$ is classical), let $\rho_{C_1^i A_1^i B_1^i E}$ be a state such that (15) and the Markov conditions

$$\forall i \in [n], \quad A_i^{i-1}C_i^{i-1} \leftrightarrow B_i^{i-1} \leftrightarrow B_i$$

hold, let $h \in \mathbb{R}$, and $f$ be an affine \underline{min-tradeoff function} for $\mathcal{M}_1, \ldots, \mathcal{M}_n$, and let $\epsilon \in ]0,1[$. Then, for any event $\Omega \subset \mathbb{C}^n$ such that $f(\text{freq}(C_1^n)) \geq h$,

$$H_{\min}^c(A_i^n C_i^n | B_i^n E)_{\rho_{\Omega}} \geq nh - v \sqrt{n},$$

where $v = 2(\log(1 + 2d_A d_C) + ||\nabla f||_{\infty}) \sqrt{1 - 2 \log(\epsilon \cdot \rho_{\Omega})}$, where $d_A$ (respectively $d_C$) is the maximum dimension of the system $A_i$ (respectively $C_i$). In the same way we have,

$$H_{\max}^c(A_i^n C_i^n | B_i^n E)_{\rho_{\Omega}} \leq nh + v \sqrt{n},$$

when we replace $f$ by an affine \underline{max-tradeoff function} $\tilde{f}$, such that the event $\Omega$ implies $h \geq \tilde{f}(\text{freq}(C_1^n))$. 

Proof. Apply Theorem 8 with the following replacement:

\[ \forall i \in [n], A_i \rightarrow A_iC_i \]  
\[ \forall i \in [n], B_i \rightarrow B_i \]  
\[ E \rightarrow E \]  

(25a)  
(25b)  
(25c)

It is easy to see that \( C_1^n \) can be obtained by measuring \( A_1^nC_1^nB_1^n \) since \( C_1^n \) is a classical register contained in the register \( A_1^nC_1^nB_1^n \), and the Markov condition \( (22) \) holds by assumption of the theorem, which justify that we can apply the Theorem 8. Note that in the definitions of the tradeoff functions (Definition 7) we also need to do the replacements \( (25a) \) and \( (25b) \). \( \square \)

The main advantage of Theorem 9 over the original EAT is that now we drop the restriction that it must be possible to obtain \( C_1^n \) by measuring \( A_1^nB_1^n \).

### III. APPLICATIONS OF THE EAT TO 2PC AND PV

#### A. Weak String Erasure

1. The protocol

Weak String Erasure (WSE) is a two-party cryptographic primitive introduced in [19]. In [8] it was proven that the security of Position Verification can be derived from the security of WSE under some assumptions.

The goal of WSE is that at the end of the protocol, executed by the two parties Alice and Bob, Alice gets a random string \( X_1^n \) and Bob gets a random set \( I \subset [n] \) of indices and the corresponding substring \( X_I \) of \( X_1^n \) (see Figure 2). Moreover we want the protocol to be secure meaning that Alice is ignorant about \( I \), and that it is difficult for Bob to guess the entire string \( X_1^n \). For a formal definition of a \( (n, \lambda, \epsilon) \)-WSE scheme we use [19, Definition 3.1].

**Definition 10** ([19],Definition 3.1). An \( (n, \lambda, \epsilon) \)-Weak String Erasure scheme is a protocol between two parties, Alice and Bob, that satisfies the following properties:

**Correctness:** If Alice and Bob are honest, then the ideal state \( \sigma_{X_1^nIX_I} \) is defined such that

1. The joint distribution of the bit string \( X_1^n \) and the subset \( I \) is uniform:

\[
\sigma_{X_1^nIX_I} = \frac{1}{2^n} \otimes \frac{1}{2^n}.
\]

(26)

2. The joint state \( \rho_{AB} \) created by the real protocol is close to the ideal state:

\[
\rho_{AB} \approx \epsilon \sigma_{X_1^nIX_I},
\]

(27)

where \( (A, B) \) is identified with \( (X_1^n, IX_I) \).

**Security for Alice:** If Alice is honest then there exists an ideal state \( \sigma_{\hat{A}X_1^n} \) such that

1. The amount of information that Bob gets about \( X_1^n \) from \( \hat{B} \) is limited:

\[
H_{\min}(X_1^n|\hat{B})_\sigma \geq \lambda n.
\]

(28)

2. The joint state \( \rho_{AB} \) created by the real protocol is \( \epsilon \)-close to the ideal state:

\[
\rho_{AB} \approx \epsilon \sigma_{X_1^nB},
\]

(29)

where \( (A, \hat{B}) \) is identified with \( (X_1^n, \hat{B}) \).

**Security for Bob:** If Bob is honest, then there exists an ideal state \( \sigma_{\hat{A}X_1^nI} \) such that

1. The random variable \( I \) is independent of \( \hat{A}X_1^n \) and uniformly distributed over the set of subsets of \([n]\):

\[
\sigma_{\hat{A}X_1^nI} = \sigma_{\hat{A}X_1^n} \otimes \frac{1}{2^n}
\]

(30)
We present now the Protocol 11 that implements a Device Independent Weak String Erasure (DI WSE). The DI WSE protocol consists of a standard Weak String Erasure protocol in which some rounds, randomly chosen, are used to certify the devices. This certification is made by performing a CHSH test between two spatially separated devices on Alice’s side. The CHSH test allows to ensure security against a dishonest Bob with bounded quantum storage even in the paranoid scenario where Alice cannot trust her devices.

**Protocol 11** (Device Independent Weak String Erasure). When the two parties are honest, the protocol runs as follows (see figure 1)

1. For \( i \in [n] \) (where \( n \) is the total number of rounds in the protocol),
   
   (a) Alice uses her source to produce an EPR pair \( \rho_{A,B_i} = \Phi_{A,B_i} \) and sends it to her switch.
   
   (b) She chooses at random a bit \( T_i \in \{0,1\} \) (such that \( T_i \) has a probability \( \mu \) to be one), and inputs it in her switch. The switch sends the \( A \) part to the main device. If \( T_i = 1 \) then the switch sends \( \rho_{B_i} \) to Alice’s testing device and \( |\perp\rangle\langle\perp| \) to Bob. Else it sends \( \rho_{B_i} \) to Bob and \( |\perp\rangle\langle\perp| \) to the testing device. The register going from the switch to the testing device is called \( B_t \) and the one going to Bob is called \( B_t \).
   
   (c) Just after the switch has sent the \( B \) part to Bob or to the testing device, Alice inputs a randomly chosen bit \( \Theta_i \in R \{0,1\} \) and \( \Theta_i \in \{0,1,\perp\} \) in her main and testing devices respectively such that
   
   \[
   \begin{cases}
   \Theta_i \in R \{0,1\} & \text{if } T_i = 1 \\
   \Theta_i = \perp & \text{if } T_i = 0.
   \end{cases}
   \]
   
   The main device is supposed to measure in the standard basis for \( \Theta_i = 0 \) and in the Hadamard basis when \( \Theta_i = 1 \) while the testing device is supposed to measure in the basis \( \{\cos \frac{3\pi}{8}|0\rangle + \sin \frac{3\pi}{8}|1\rangle; \cos \frac{5\pi}{8}|0\rangle + \sin \frac{5\pi}{8}|1\rangle\} \) for \( \Theta_i = 1 \) and in \( \{\cos \frac{\pi}{8}|0\rangle + \sin \frac{\pi}{8}|1\rangle; \cos \frac{3\pi}{8}|0\rangle - \sin \frac{3\pi}{8}|1\rangle\} \) for \( \Theta_i = 0 \). When \( \Theta_i = \perp \) the testing device just outputs \( \perp \). To sum up the main device produces the outcome \( X'_i \in \{0,1\} \) and the testing device produces \( Y_i \in \{0,1,\perp\} \) where,
   
   \[
   \begin{cases}
   Y_i \in \{0,1\} & \text{if } T_i = 1 \\
   Y_i = \perp & \text{if } T_i = 0.
   \end{cases}
   \]
   
   (d) Alice classically computes a value \( C_i \in \{0,1,\perp\} \) as follows:
   
   \[
   \begin{cases}
   C_i = 1 & \text{if } X'_i \oplus Y_i = \Theta_i \cdot \Theta_i \\
   C_i = 0 & \text{else} \\
   C_i = \perp & \text{if } T_i = 0
   \end{cases}
   \]
   
   (e) After that, Alice classically post-processes the value of \( X'_i \): If \( T_i = 0 \) then she does nothing, \( X'_i \) stays unchanged, and if \( T_i = 1 \) then she changes \( X'_i \) and replaces its value by \( \perp \). Let’s call now the post processed bit \( X_i \in \{0,1,\perp\} \).
   
   (f) When he receives the state \( \rho_{B_t} \) from Alice, if \( T_i = 0 \) Bob measures it according to a randomly chosen basis \( \Theta_i \in R \{0,1\} \) and gets the output \( X_i \in \{0,1\} \). When \( T_i = 1 \) Bob just sets \( X_i = \perp \). Note that Bob measures either in the standard or the Hadamard bases. Then we can see that if \( \Theta_i = \Theta_i \) Alice and Bob measure the same outcome, i.e. \( X_i = X_i \).

2. Alice computes the fraction of CHSH games that are won:

\[
\omega := \frac{|\{i \in [n] : T_i = 1 \& C_i = 1\}|}{|\{i \in [n] : T_i = 1\}|}.
\]

Alice aborts the protocol if \( \omega < \delta \), for some fixed parameter \( \delta \in ]\frac{3}{4}, \frac{1}{2}\sqrt{2}]. \)
3. Bob counts how many test rounds there was,

$$\omega' := |\{i \in [n] : T_i = 1\}|.$$  

(32)

If $$\omega' > \mu n + \sqrt{n \ln(1/\lambda)} = n - \tilde{n}$$ Bob aborts the protocol.

4. Alice and Bob wait for a predetermined duration $$\Delta t$$ before Alice sends to Bob the list of bases $$\Theta^n_i$$ she has input in her main device. When Bob receives $$\Theta^n_i$$ he computes the set of indices $$\mathcal{I} := \{i \in [n] : \Theta_i = \Theta, T_i = 0\}$$, and erases all the bits of $$\hat{X}^n_1$$ whose index are not in $$\mathcal{I}$$. Bob finally has $$(\mathcal{I}, X^\prime_1)$$. Alice computes her bits string $$X^k_1$$ by erasing all the bits in $$\hat{X}^n_1$$ that are $$\perp$$.

We will assume here that if one of the two parties aborts the protocol, then this party will output the maximally mixed state (independently of all the previous communication). This permits to satisfy the security condition when a honest party aborts.

Let us now summarise our result in Theorem 12 whose proof is given later in Lemmas 13, 14 and 15.

**Theorem 12.** The Protocol 11 implements a Device Independent $$(k, \lambda, 2\epsilon)$$-WSE scheme with,

$$\tilde{n} \leq k \leq n,$$ where $$\tilde{n} := (1 - \mu)n - \frac{\sqrt{n \ln(1/\lambda)}}{2}$$ and $$n$$ is the number of rounds of the protocol.

The value of $$k$$ depends on how many test rounds there are during the protocol. The number $$\tilde{n}$$ is chosen such that for two honest parties the probability that one of them aborts is very low.

$$\begin{align*}
\lambda &= h - \mu - \frac{\tilde{v}(\epsilon)}{\sqrt{n}} + \frac{3 \log(1 - \sqrt{1 - (\epsilon/4)^2}) - \log(d)}{n}, \\
&\text{where } h := (1 - \mu)\frac{1 - (4\delta - 2)\sqrt{16\delta(1 - \delta) - 2}}{2},
\end{align*}$$

$$\begin{align*}
\epsilon &\in [0, 1], \text{ being a security parameter}, \\
\mu &\in [0, 1] \text{ is the testing probability}, \\
\delta &\in \left[\frac{3}{4}, \frac{1}{2} + \frac{1}{2\sqrt{2}}\right] \text{ is a threshold},
\end{align*}$$

where $$\tilde{v}(\epsilon) := 2(\log(19) + \|\nabla \tilde{f}\|_\infty)\sqrt{1 - 2\log(\epsilon^2) + 27\log(7) - \log(\epsilon^2(1 - \sqrt{1 - (\epsilon/4)^2}))}$$, $$\tilde{f}$$ is the affine min-tradeoff function defined by (22), and $$d$$ is an upper bound on the dimension (or the one shot entanglement cost) of a dishonest Bob’s quantum memory. Our proof are given against an adversary whose quantum memory is bounded in dimension by $$d$$. Nevertheless we can extend the result to a noisy storage model by using the arguments given in (2), where $$d$$ is now a bound on the one shot entanglement cost of the dishonest Bob’s quantum memory i.e. $$E^{(1)}_C(\mathcal{E}) \leq d$$, where $$\mathcal{E}$$ is the channel representing Bob’s memory.

**Proof.** Lemma 13 proves the $$2\epsilon$$-correctness of the protocol, Lemma 14 proves security for Bob, and Lemma 15 proves security for Alice.

The idea to extend the result to a noisy storage model is to use the definition of the one shot entanglement cost (3) of a channel (here the channel will be Bob’s memory) which tells us that any channel whose one shot entanglement cost is bounded by $$d$$ can be simulated by a bounded channel of dimension $$d$$ and LOCC. This means that any strategy using a memory whose entanglement cost is bounded by $$d$$ can be simulated by another strategy using a bounded memory (bounded in dimension by $$d$$), and then our security proofs also holds for the noisy storage model. 

2. Correctness

The following lemma proves that the Protocol 11 is $$2\epsilon$$-correct.

**Lemma 13.** The Protocol 11 is $$2\epsilon$$-correct meaning that if the protocol does not aborts then it is correct, and that the protocol aborts with a probability at most $$2\epsilon$$.

**Proof.** It is clear that for honest Alice and Bob, if they do not abort the protocol then Alice has a string $$X^k_1 \in \{0, 1\}^k$$ and Bob has $$(\mathcal{I}, X^\prime_1)$$ where $$\mathcal{I}$$ is a subset of $$[k]$$ and $$X^\prime_1$$ is the corresponding substring of $$X^k_1$$. Moreover if the protocol does not abort we have $$n \geq k \geq \tilde{n}$$. It is then sufficient to prove that the probability that Alice aborts is less than $$\epsilon$$ and that the probability that Bob aborts is less than $$\epsilon$$. 

For Alice: Remember that Alice aborts if $\omega < \delta$. Also if Alice and Bob are honest the state they will measure is $\Phi^{\otimes n}$ a tensor product of EPR pairs and all the measurement are the same for all rounds, so we can just multiply the probability for each testing rounds. Also each round has a probability $\mu$ to be a testing round. Then the probability that $w < \delta$ is given by

$$P_{\text{Alice aborts}} := \sum_{k=0}^{n} \binom{n}{k} \mu^k (1-\mu)^{n-k} \sum_{j=0}^{\delta k} \binom{k}{j} p^j (1-p)^{k-j}, \quad \text{where } p := \frac{1}{2} + \frac{1}{2\sqrt{2}}. \quad (33)$$

Using Hoeffding’s inequality we have

$$\sum_{j=0}^{\delta k} \binom{k}{j} p^j (1-p)^{k-j} \leq \exp(-2(p-\delta)^2 k) = (e^{-2(p-\delta)^2})^k, \quad (34)$$

and thus:

$$P_{\text{Alice aborts}} \leq \sum_{k=0}^{\infty} \binom{n}{k} \left( \mu e^{-2(p-\delta)^2} \right)^k (1-\mu)^{n-k} = \left( 1 - \mu \left( 1 - e^{-2(p-\delta)^2} \right) \right)^n, \quad (35)$$

where the last equality comes from the binomial theorem. So for $n \geq \frac{\ln(\epsilon)}{\ln(1-\mu(1-e^{-2(p-\delta)^2})))}$,

$$P_{\text{Alice aborts}} \leq \epsilon. \quad (36)$$

For Bob: Again if Alice and Bob are honest $T_i$ follows a Bernoulli distribution, and each round is independent from the others. Thus, because Bob aborts if $\omega' \geq \mu n + \sqrt{n \frac{\ln(1/\epsilon)}{2}}$, the probability that he aborts the protocol is given by

$$P_{\text{Bob aborts}} := \sum_{k=\mu n + \sqrt{n \frac{\ln(1/\epsilon)}{2}}}^{n} \binom{n}{k} \mu^k (1-\mu)^{n-k} \quad (37)$$

and using Hoeffding inequality we get

$$P_{\text{Bob aborts}} \leq \exp \left( -2 \left( \sqrt{\frac{\ln(1/\epsilon)}{2n}} \right)^2 n \right) = \epsilon. \quad (38)$$

Thus the probability that either Alice or Bob abort the protocol (for $n \geq \frac{\ln(\epsilon)}{\ln(1-\mu(1-e^{-2(p-\delta)^2})))}$ is less than $2\epsilon$ which ends the proof of the $2\epsilon$-correctness of protocol [11].

3. Security for Bob

We argue in this section that Protocol [11] is secure device independently for Bob.

**Lemma 14.** Protocol [11] is secure for a honest Bob.

**Proof.** It is very intuitive that Alice is ignorant about $I$. To see that note that the set $I$ that Bob has at the end of Protocol [11] is completely determined by $\Theta^\ast_1$ and $\hat{\Theta}^\ast_1$. More precisely the set of indices $I$ can be defined as being the set of indices for which the corresponding trits of the string $\Theta^\ast_1 \oplus \hat{\Theta}^\ast_1$ are zero (where $\oplus$ is the bit wise addition modulo 2, and as $\hat{\Theta}$, can be $\perp$ we take the convention that $\Theta \oplus \perp = \perp$). So as Alice knows the string $\Theta^\ast_1$, it is equivalent for her to guess $I$ or the value of $\Theta_i \in \{0, 1\}$ for all $i$ being a non-testing round (for the testing rounds she already knows that $\Theta_i = \perp$). But for the non-testing rounds Bob chooses $\Theta_i$ independently and uniformly at random.
in \( \{0, 1\} \), and he never sends any information to Alice, so the state between Alice and Bob (tracing out all irrelevant registers) must be

\[
\rho_{AI} = \rho_A \otimes \frac{I_I}{2^r}
\]

meaning that Alice gains no information about the set of indices \( I \).

To complete the proof we need to define state \( |11\rangle \) (for more details see Ref. [19]). For the case where Bob’s device perform projective measurements the proof follows straightforwardly as in Ref. [19]. However note that we do not want to make any assumption about Bob’s device, and we even allow that Alice prepares his device. Therefore in the most general scenario we may assume that Bob’s device perform imperfect measurements or general POVMs. This scenario can be modelled as a Weak String Erasure with Errors (WSEE) [58], and condition [39] guarantees security for this more general case as shown in Ref. [58].

Note that since we do not make any assumption on Bob’s device, this is a device independent security proof for a honest Bob (against Alice).

### 4. Security for Alice

In this section we want to show that the Protocol \([\text{11}]\) is secure for Alice. The proof is given in Lemma \([\text{15}]\).

When Bob is dishonest, he does not necessarily measure the system \( \hat{B}_1 \) he receives from Alice, instead he applies a general CPTP map \( E_{\hat{B}_1 \rightarrow \hat{B}_K} \) to it, where \( K \) is an arbitrary classical register, and \( \hat{B} \) is a quantum register whose dimension is bounded by \( d \) (bounded quantum memory). When Bob receives \( \Theta_1^p \) from Alice after the duration \( \Delta t \), Bob applies a CPTP map \( D_{\hat{B}_K \rightarrow \hat{X}_1^p} \) where is a classical register \( \hat{X}_1^p \) representing his guess for the entire string \( X_1^p \) of Alice.

We want to prove now that in the Protocol \([\text{11}]\) either the min-entropy \( H_{min}(X_1^k | \Theta_1^p T_1^p K \hat{B})_{\rho_{1\Omega}} \) is high or the protocol aborts with a high probability (at least \( 1 - \epsilon \)), where \( \Omega \) is the event corresponding to \( \omega \geq \delta \) or equivalently Alice does not abort the protocol in step\([\text{2}]\).

For that we first note that,

\[
H_{min}(X_1^k | \Theta_1^p T_1^p K \hat{B})_{\rho_{1\Omega}} = H_{min}(X_1^p | \Theta_1^p T_1^p K \hat{B})_{\rho_{1\Omega}} \geq H_{min}(X_1^p | \Theta_1^p T_1^p K)_{\rho_{1\Omega}} - \log(|\hat{B}|)
\]

The first equality coming form the fact that the only difference between \( X_1^p \) and \( X_1^p \), is that \( X_1^p \) contains some additional \( \bot \) values corresponding to the testing rounds determined by \( T_1^p \). Now we just need to bound \( H_{min}(X_1^i | \Theta_1^p T_1^p K)_{\rho_{1\Omega}} \). This basically means that we are interested in bounding the min-entropy for an purely classical adversary Bob. In other words we replace Bob’s maps by \( E_{\hat{B}_1 \rightarrow K} \) and \( D_{K \rightarrow \hat{X}_1^p} \), where now there is no quantum register at all. In the following we will only consider such a classical Bob.

We can think about Alice’s measurements as a composition of maps \( \mathcal{N}_1, \ldots, \mathcal{N}_n \) as described in figure \([\text{3}]\) and where the maps \( \mathcal{N}_i \) are described in the figure \([\text{4}]\). The global state before Bob makes his guess can then be written as \( \rho_{C_2^n X_1^n Y^n \Theta_1^n T_1^n K} = (\mathcal{R}_{R_0} \circ \mathcal{N}_n \circ \ldots \circ \mathcal{N}_1 \otimes \mathcal{I}_K)(\rho_{R_{6}K}) \), where \( R_0 := A^p B^p \) (see figure \([\text{3}]\). We prove in the appendix \([\text{A}]\) that the function \( f \) defined as,

\[
\forall q \in \mathbb{P}(\{0, 1, \bot\}), \quad f(q) := (1 - \mu) \frac{1 - (4p - 2) \sqrt{16p(1 - p) - 2}}{2} + p := \frac{q(1)}{1 - q(\bot)},
\]

is a convex min-tradeoff function for the maps \( \mathcal{N}_1, \ldots, \mathcal{N}_n \), and hence its differential function \( \hat{f} \) at the point \( q = (1 - (1 + \delta) \mu, \mu, \mu) \) (where \( \hat{f} \) is the transpose) is an affine min-tradeoff function.

**Lemma 15.** At the end of the Protocol \([\text{11}]\) we are in one of the following situations:

1. The min-entropy \( H_{min}(X_1^k | \Theta_1^p T_1^p K \hat{B})_{\rho_{1\Omega}} \) is bounded as follows:

\[
H_{min}(X_1^k | \Theta_1^p T_1^p K \hat{B})_{\rho_{1\Omega}} \geq (h - \mu)n - \bar{v}(\epsilon) \sqrt{n} + 3 \log \left( 1 - \sqrt{1 - (\epsilon/4)^2} \right) - \log(d)
\]

where \( h := (1 - \mu) \frac{1 - (4p - 2) \sqrt{16p(1 - p) - 2}}{2} \) and

\[
\bar{v}(\epsilon) := 2(\log(19) + \| \nabla \hat{f} \|_{\infty}) \sqrt{1 - 2 \log(\epsilon^2) + 2 \log(7)} \sqrt{-\log(\epsilon^2(1 - \sqrt{1 - (\epsilon/4)^2}))}
\]
Figure 3. Diagram representing Alice and Bob’s operations when Bob is dishonest. In this figure Alice produces all the \( n \) systems at the same time using the source \( S \), and then according to the random string \( T^n_1 \in \{0,1\}^n \) she chooses which ones she sends to Bob. Proving security for such a protocol also proves it for the Protocol 11 under the assumption 1. After she sent the information to Bob she can measure her system that we call \( R_0 := A^n_1 \bar{B}^n_1 \). The operations she makes are modeled by the maps \( N_1, \ldots, N_n \) described in more details in the figure 4. When Bob receives the system \( \hat{B}^n_1 \) he measures it, which gives him the classical value \( K \).

Figure 4. Description of the map \( N_i \). \( \Theta_i \in \{0,1\} \) represents the “basis” in which the device \( A_i \) measures its input to get the output \( X'_i \in \{0,1\} \) which will be post-processed afterwards into \( X_i \in \{0,1,\perp\} \). This device represents Alice’s main device. \( \bar{\Theta}_i \in \{0,1,\perp\} \) represents the “basis” in which the “testing” device \( T_i \) measures its input to get the output \( Y_i \in \{0,1,\perp\} \). If \( T_i = 0 \) we have \( \bar{\Theta}_i = \perp \) and then \( Y_i = \perp \); this corresponds to the non tested rounds. Else \( \Theta_i, Y_i \in \{0,1\} \). If \( T_i = 0 \) then \( C_i = \perp \), else \( C_i = 0 \) if \((\Theta_i, \bar{\Theta}_i, X_i, Y_i)\) do not verify the CHSH condition and \( C_i = 1 \) when they do.

2. Alice aborted the protocol with a probability at least \( 1 - \epsilon \).

Proof. First note that the state \( \rho_{C^n_1X^n_1Y^n_1\bar{\Theta}_i^1T^n_1K} \) clearly satisfies the following Markov condition,

\[
\forall i \in [n], \ X'_i | C^n_1 \leftrightarrow \Theta^n_1T^n_1K \leftrightarrow \Theta^n_{i+1}T^n_{i+1},
\]

since each bit of the strings \( \Theta^n_1 \) and \( T^n_1 \) are chosen independently at random. We can now apply Theorem 10 with the replacements

\[
\forall i \in [n], \ A_i \rightarrow X_i \quad (45) \\
\forall i \in [n], \ C_i \rightarrow C_i \quad (46) \\
\forall i \in [n], \ B_i \rightarrow \Theta_iT_i \quad (47) \\
E \rightarrow K \quad (48)
\]
and the affine min-tradeoff function $\tilde{f}$ to lower bound $H'_{\min}(X^n_i C^n_i | \Theta^n_i T^n_i K)_{p_\Omega}$:

$$H'_{\min}(X^n_i C^n_i | \Theta^n_i T^n_i K)_{p_\Omega} \geq nh - v\sqrt{n}, \quad (49)$$

where $h := (1 - \mu)(1 - (4\delta - 2)\sqrt{16\delta(1 - \delta)^2})$ and $v := 2(\log(19) + \|\nabla \tilde{f}\|_\infty)\sqrt{1 - 2\log(\epsilon \cdot p_\Omega)}$. It is easy to check that for the event $\Omega$ (which correspond to $\omega \geq \delta$) $\tilde{f}(\text{freq}(C^n_i)) \geq (1 - \mu)(1 - (4\delta - 2)\sqrt{16\delta(1 - \delta)^2}) =: h$ which justifies the inequality (49).

Then by using the chain rule for min-entropy from (53) Equation (6.57)) we have,

$$H'_{\min}(X^n_i | \Theta^n_i T^n_i K)_{p_\Omega} \geq H'_{\min}(X^n_i C^n_i | \Theta^n_i T^n_i K)_{p_\Omega} - H'_{\max}(C^n_i | X^n_i \Theta^n_i T^n_i K)_{p_\Omega} + 3\log(1 - \sqrt{1 - (\epsilon/4)^2}) \quad (50)$$

$$\geq H'_{\min}(X^n_i C^n_i | \Theta^n_i T^n_i K)_{p_\Omega} - H'_{\max}(C^n_i | T^n_i K)_{p_\Omega} + 3\log(1 - \sqrt{1 - (\epsilon/4)^2}). \quad (51)$$

It just remains us to upper bound $H'_{\max}(C^n_i | T^n_i K)_{p_\Omega}$ which is done in the appendix [2]

$$H'_{\max}(C^n_i | T^n_i K)_{p_\Omega} \leq \mu n + n(\alpha - 1)\log^2(7) + \frac{\alpha}{\alpha - 1} \log \left( \frac{1}{p_\Omega} \right) - \frac{\log(1 - \sqrt{1 - (\epsilon/4)^2})}{\alpha - 1} \quad (52)$$

Taking $\alpha = 1 + \sqrt{-\log(p_\Omega(1 - \sqrt{1 - (\epsilon/4)^2)})/n\log^2(7)}$ gives us,

$$H'_{\max}(C^n_i | T^n_i K)_{p_\Omega} \leq \mu n + 2\sqrt{n}\log(7) - \log(p_\Omega(1 - \sqrt{1 - (\epsilon/4)^2})). \quad (53)$$

Combining equations (49), (51) and (53) we get,

$$H'_{\min}(X^n_i | \Theta^n_i T^n_i K)_{p_\Omega} \geq (h - \mu)n - \tilde{v}\sqrt{n} + 3\log(1 - \sqrt{1 - (\epsilon/4)^2}) \quad (55)$$

where $\tilde{v} := 2(\log(19) + \|\nabla \tilde{f}\|_\infty)\sqrt{1 - 2\log(\epsilon \cdot p_\Omega)} + 2\log(7)\sqrt{-\log(p_\Omega(1 - \sqrt{1 - (\epsilon/4)^2}))}$.

Now if the probability $p_\Omega$ that the event $\Omega$ happens is such that $p_\Omega \geq \epsilon$ then the previous inequality becomes,

$$H'_{\min}(X^n_i | \Theta^n_i T^n_i K)_{p_\Omega} \geq (h - \mu)n - \tilde{v}(\epsilon)\sqrt{n} + 3\log(1 - \sqrt{1 - (\epsilon/4)^2}) \quad (56)$$

and by combining it with (51) we have,

$$H'_{\min}(X^n_i | \Theta^n_i T^n_i K \tilde{B})_{p_\Omega} \geq (h - \mu)n - \tilde{v}(\epsilon)\sqrt{n} + 3\log(1 - \sqrt{1 - (\epsilon/4)^2}) - \log(d) \quad (57)$$

where $\tilde{v}(\epsilon) := 2(\log(19) + \|\nabla \tilde{f}\|_\infty)\sqrt{1 - 2\log(\epsilon^2)} + 2\log(7)\sqrt{-\log(\epsilon^2(1 - \sqrt{1 - (\epsilon/4)^2}))}$. On the contrary if $p_\Omega < \epsilon$ this means (by definition of $\Omega$) that Alice aborts the protocol with probability at least $1 - \epsilon$ and we are in the second case of the theorem.

\section*{B. Position Verification (PV)}

\subsection*{1. The Protocol}

Position Verification (PV) is a protocol where one Prover (P) tries to convince two Verifiers (V1 and V2) that he stands at some claimed geographical position (Figure 5). In this paper we limit the analysis to the case of a one dimensional space.

It is known that for a purely classical PV no security is achievable [53] and for a quantum version it is shown that a fully powerful adversary can always cheat at the protocol [6, 7]. However we can have security under the realistic assumption that the adversary (composed of two malicious provers that we call M1 and M2) has a bounded amount of entanglement [8, 60]. Most of the previous security proofs rely on the fact that the state and the measurements done during the protocol by the Verifiers are known and trusted or are assumed to be IID. Here we will present a protocol that implements a secure PV even when the measurements are unknown and not trusted, meaning that the security still holds if the adversary prepares the Verifiers' devices.

In order to get an intuition on the problem, we start by informally defining the cheating probability for a Position Verification protocol.
Definition 16 (Informal). The probability $P_{\text{cheatPV}}$ that two dishonest provers $M1$ and $M2$ succeed to cheat at the protocol, meaning that they pass the protocol while none of them is at the claimed position, is defined as the maximum probability that they both guess $V1$’s bit string $X^n_1$. The maximum is taken over all possible strategies they can have. A strategy is compose of an entangled state $\rho_{M1M2}$ and two pairs of CPTP maps $(E_1, D_1)$ and $(E_2, D_2)$ (see Figure 6 in section III B 2).

Now we state the requirements for a secure PV protocol.

Definition 17. A PV protocol is called secure if either

- the protocol aborts with a probability $1 - p_\Omega$ exponentially close to one:

$$1 - p_\Omega \geq 1 - \epsilon_n,$$

where $\exists \alpha > 0 : \epsilon_n = 2^{-\alpha n}$ and $n$ is the number of rounds of the protocol.

- or the probability $P_{\text{cheatPV}}$ (see Definition 16) is such that

$$\exists \kappa > 0, P_{\text{cheatPV}} \leq 2^{-\kappa n}$$

for $n$ big enough, where $n$ is the number of rounds of the protocol.

We now present our Device-independent Position Verification protocol. The protocol consists of a standard PV protocol with additional randomly chosen test rounds, similar to the one introduced in Protocol 11, where the provers check the number of CHSH games that are won to prevent any malicious behavior of the devices.

Protocol 18. The honest execution of PV runs as follows (see Figure 5).

1. For $i \in [n], \ldots $
(a) The Verifier 1 (V1) uses his source to produce an EPR pair \( \rho_{V1,P} = \Phi_{V1,P} \) and sends it to his switch.

(b) • V1 chooses at random a bit \( T_i \in \{0, 1\} \) (such that \( T_i \) has a probability \( \mu \) to be one) and inputs it in his switch. The switch sends the \( V_1 \) part to the main device. If \( T_i = 1 \) then the switch sends \( p_{\rho_i} \) to V1’s testing device and \( |\perp\rangle\langle\perp| \) to the Prover P. Else it sends \( p_{\rho_i} \) to the Prover P and \( |\perp\rangle\langle\perp| \) to the testing device. The register that goes to V1’s testing device is called \( P_i \) and the one going to the Prover is called \( P_i \).

• At the same time V2 sends to the Prover the choice of basis \( \Theta_i \) in which V1 will measure his state \( \rho_{V1,i} \) with his main device.

(c) Just after the switch has sent the \( \{P_i\}_{i \in [n]} \) part to the Prover or to the testing device, V1 inputs the randomly chosen bits \( \Theta_i \in \{0, 1\} \) and \( \Theta_i \in \{0, 1, \perp\} \) in his main and testing devices respectively such that,

\[
\begin{align*}
\Theta_i &\in \{0, 1\} \quad \text{if} \ T_i = 1 \\
\Theta_i &\in \perp \quad \text{if} \ T_i = 0
\end{align*}
\]

The main device is supposed to make a measurement in the standard basis for \( \Theta_i = 0 \) and in the Hadamard basis for \( \Theta_i = 1 \), while the testing device is supposed to measure in the basis \( \{\cos \frac{\pi}{8}|0\rangle - \sin \frac{\pi}{8}|1\rangle : \cos \frac{\pi}{8}|0\rangle + \sin \frac{\pi}{8}|1\rangle\} \) for \( \Theta_i = 1 \) and in \( \{\cos \frac{\pi}{8}|0\rangle + \sin \frac{\pi}{8}|1\rangle : \cos \frac{\pi}{8}|0\rangle - \sin \frac{\pi}{8}|1\rangle\} \) for \( \Theta_i = 0 \). When \( \Theta_i = \perp \) the testing device just outputs \( \perp \). To sum up, the main device produces the outcome \( X'_i \in \{0, 1\} \) and the testing device produces \( Y_i \in \{0, 1, \perp\} \) where,

\[
\begin{align*}
Y_i &\in \{0, 1\} \quad \text{if} \ T_i = 1 \\
Y_i &\in \perp \quad \text{if} \ T_i = 0
\end{align*}
\]

(d) V1 classically computes a value \( C_i \in \{0, 1, \perp\} \) as follows:

\[
\begin{align*}
C_i = 1 &\quad \text{if} \ X'_i \oplus Y_i = \Theta_i \cdot \Theta_i \quad \text{if} \ T_i = 1 \\
C_i = 0 &\quad \text{else} \quad \text{if} \ T_i = 1 \\
C_i = \perp &\quad \text{if} \ T_i = 0
\end{align*}
\]

(e) After that, V1 classically post-processes the value of \( X'_i \): if \( T_i = 0 \) then he does nothing, \( X'_i \) stays unchanged, and if \( T_i = 1 \) then he changes \( X'_i \) and replaces its value by \( \perp \). Let’s call now the post processed bit \( X_i \in \{0, 1, \perp\} \).

2. Each time P receives a state \( \rho_{P_i} \) from V1 and the corresponding \( \Theta_i \) from V2, he uses the bit \( \Theta_i \) to measure the state in the same basis as V1 does with his main device (we adopt the convention here that measuring \( |\perp\rangle\langle\perp| \) outputs \( \perp \) independently of the basis \( \Theta_i \)). Just after he measures, P sends back his outcome measurement to both verifiers.

3. V1 computes the fraction

\[
\omega := \frac{|\{i \in [n] : T_i = 1 \& C_i = 1\}|}{|\{i \in [n] : T_i = 1\}|}.
\]

V1 aborts the protocol if \( \omega < \delta \), for some fixed parameter \( \delta \in \left[\frac{1}{3}, \frac{1}{2} + \frac{1}{2\sqrt{2}}\right] \). We call \( \Omega \) the event of not aborting.

4. • The two Verifiers check that the Prover replied within a duration \( \Delta t \).

• For each round \( i \) in which \( T_i = 0 \) they check whether the Prover’s outcome measurement agrees with V1’s one.

5. The Verifiers accept that the prover is at the claimed position if the two previous conditions are satisfied. We say that the prover passes the protocol.

As for WSE we will assume that when aborting V1 outputs a maximally mixed state for \( X'_i \).

It is easy to check that, conditioned on the fact that the protocol does not abort, an honest prover always passes the protocol. In the exact same way as WSE we can prove that for an honest implementation of Protocol [15] is \( \epsilon \)-correct, meaning that the protocol aborts with a probability less that \( \epsilon \) and when it does not abort a honest prover always passes the protocol.

Let us summarise our result on PV in the following theorem.
Theorem 19 (Informal). Under the Assumptions (below), the Protocol implements a secure Device-Independent Position Verification scheme. This means that the protocol is \( \varepsilon \)-correct and secure. Saying that it is \( \varepsilon \)-correct means that for a honest prover, the protocol abort with a probability less than \( \varepsilon \), and is correct if it does not abort. And we say that the protocol is secure if it satisfies all the requirements of Definition.

The \( \varepsilon \)-correctness of the protocol follows straightforwardly as we argued before. The security of the protocol will be proven in Lemma below.

Remark 2 (On Theorem 19). Note that even though Theorem 19 seems to be an asymptotic statement, our proofs can also give bounds for the finite regime, where an appropriate choice of values of \( \kappa \) and \( n \) can provide security with any desired security parameter. Here we only focus on a security proof for PV, but a finite regime analysis can be obtained from the parameters of WSE.

2. Security

![Figure 6. Strategy for the two cheaters M1 and M2. The choice of the state \( \rho_{M_1M_2} \) and of the two pairs of CPTP maps \( (\mathcal{E}_1, \mathcal{D}_1) \) and \( (\mathcal{E}_2, \mathcal{D}_2) \) constitute the strategy for M1 and M2. The maps \( \mathcal{D}_i, i \in [2] \) output classical register \( Y^n_1 \) and \( Z^n_1 \). In our model of attack M1 and M2 are not allowed to have quantum communication. Also as stated in Assumptions we will assume that the state \( \rho_{M_1M_2} \) has a local dimension bounded by \( d \). We say that the cheaters cheat at the protocol or succeed the attack if \( Y^n_1 = Z^n_1 = X^n_1 \).

A malicious prover M alone cannot cheat at the protocol, meaning that he cannot pass the protocol without being at the claimed position. Indeed, because a message cannot go faster than the speed of light, if he tries to cheat he will never be able to reply to both verifiers on time because he is further away (compared to the claimed position) from one of the two verifiers. However we can wonder what happens if there is several malicious provers trying to impersonate one honest prover at the claimed position. In one dimension we can without loss of generality assume that there is at most to malicious provers M1 and M2.

Our goal is to prove that Protocol is secure against two malicious provers (M1 and M2) trying to impersonate an honest prover at the claimed position (see Figure 6). To do so we will make use of the security result we have for WSE. Indeed, as proved in Ref., the security of WSE implies security for PV for the case where the two malicious provers do not have access to quantum channels. These assumptions are formalized below.
Assumptions 20.

1. The two malicious provers share a bipartite state $\rho_{M_1M_2}$ whose one shot entanglement cost is bounded:
   $$E^{(1)}_{C,\text{LOCC}}(\rho_{M_1M_2}) \leq d.$$ 

2. The two malicious prover do not have any quantum channel. However they can have an arbitrary classical channel. They can also have arbitrary large quantum and classical memory.

We are now ready to state the main result of this section. The Lemma 21 proves security for the Protocol 18 under the Assumptions 1, where V1 plays the role of Alice, and the Assumptions 20.

**Lemma 21.** The Protocol 18 is secure.

**Sketch of Proof.** As in the previous section if the probability $p_{\Omega}$ that V1 does not abort is such that $p_{\Omega} < \epsilon_n = 2^{-\alpha n}$, then the protocol aborts with a probability at least $1 - \epsilon_n = 1 - 2^{-\alpha n}$.

Now let’s consider the case where $p_{\Omega} \geq \epsilon_n$ and the protocol does not abort, then the state that Alice and Bob share is $\rho_{\Omega}$ i.e. the state conditioned on the event $\Omega$. In [8] it is proven that, under the Assumptions 20, $P_{\text{cheatPV}}$ can be linked to the probability $P_{\text{cheatWSE}} := 2^{-H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}}$ which a dishonest Bob cheats at an $(n, \lambda, \epsilon)$-WSE:

$$P_{\text{cheatPV}} \leq P_{\text{cheatWSE}} = 2^{-H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}},$$

(59)

where “Bob” refers to all quantum and classical information that Bob has at the end of WSE (before he makes a guess for $X^n_1$). In the previous section on Weak String Erasure we had a bound on the smooth min-entropy $H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}$ and thanks to Lemma 30 of Appendix 1 we can link the smooth min-entropy with the min-entropy as follows,

$$2^{-H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}} \leq \epsilon_n + 2^{-H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}},$$

(60)

Then using Lemma 31 (the lower bound on the smooth min-entropy [13] satisfies the conditions of Lemma 31 since $\alpha$ can be taken arbitrarily small), we have

$$\exists \kappa > 0 : 2^{-H_{\text{min}}(X^n_1|\text{Bob})_{\rho_{\Omega}}} \leq 2^{-\kappa n}$$

(61)

which proves the security for PV.

**IV. CONCLUSION**

Here we have considered Weak String Erasure and Position Verification in the noisy-storage model (or noisy-entanglement model). We have proved security of WSE and PV in the most paranoid scenario where the devices used to create or measure states are not trusted (device-independent scenario). This implies the device-independent security of any two-party cryptographic protocol that can be made from WSE and classical communications, and certain position-based cryptographic tasks.

Our proof techniques are based on the recently proven Entropy Accumulation Theorem [1].

We stress that the security of our protocols can be achieved for any violation of the CHSH inequality, under the cost of increasing the number of rounds required to execute the protocol. Moreover, for the maximal violation of the CHSH inequality our bound on the min-entropy is essentially tight (it grows like $\frac{n}{2}$, which is the amount achieved by the honest strategy). However we did not look at tightness of our bound for other values of the violation.

Our result constitutes the first device independent security proof on two-party cryptography and Position Verification where no IID assumption is made. Together with the fact that we prove security for any violation of the CHSH inequality, this opens the door to the experimental realization of many quantum cryptographic protocols in the most adversarial scenario.

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[1] F. Dupuis, O. Fawzi, and R. Renner, “Entropy accumulation,” arXiv:1607.01796, July 2016.
[2] A. K. Ekert, “Quantum cryptography based on bell’s theorem,” Phys. Rev. Lett., vol. 67, pp. 661–663, Aug 1991.

[3] C. H. Bennett and G. Brassard, “Quantum cryptography: Public key distribution and coin tossing,” Theoretical Computer Science, vol. 560, Part 1, pp. 7 – 11, 2014. Theoretical Aspects of Quantum Cryptography – celebrating 30 years of {BB84}.

[4] N. Chandran, V. Goyal, R. Moriarty, and R. Ostrovsky, “Position based cryptography,” Advances in Cryptology-CRYPTO, 2009.

[5] N. Chandran, S. Fehr, G. Gelles, V. Goyal, and R. Ostrovsky, “Position-based quantum cryptography,” arXiv:1005.1750 2010.

[6] H. Buhrman, N. Chandran, S. Fehr, R. Gelles, V. Goyal, R. Ostrovsky, and C. Schaffner, “Position-based quantum cryptography: Impossibility and constructions,” SIAM Journal on Computing, vol. 43, no. 1, pp. 150–178, 2014.

[7] S. Beigi and R. Koenig, “Simplified instantaneous non-local quantum computation with application to position-based cryptography,” arXiv:1101.1065 2011.

[8] J. Ribeiro and F. Grosshans, “A tight lower bound for the BB84-states quantum-position-verification protocol,” arXiv:1504.07117 2015.

[9] J. Ribeiro, L. Phuc Thinh, J. Kaniewski, J. Helsen, and S. Wehner, “Device-independence for two-party cryptography and position verification,” arXiv:1606.08750 June 2016.

[10] D. Mayers, “Unconditionally secure quantum bit commitment is impossible,” Phys. Rev. Lett., vol. 78, pp. 3414–3417, Apr 1997.

[11] H.-K. Lo and H. F. Chau, “Is quantum bit commitment really possible?,” Phys. Rev. Lett., vol. 78, pp. 3410–3413, Apr 1997.

[12] H.-K. Lo and H. F. Chau, “Why quantum bit commitment and ideal quantum coin tossing are impossible,” Physica D Nonlinear Phenomena, vol. 120, pp. 177–187, Sept. 1998.

[13] R. Colbeck, “Impossibility of secure two-party classical computation,” Phys. Rev. A, vol. 76, p. 062308, Dec 2007.

[14] U. Maurer, “Conditionally-perfect secrecy and a provably-secure-randomized cipher,” Journal of Cryptology, vol. 5, no. 1, pp. 53–66, 1992.

[15] C. Cachin and U. Maurer, “Unconditional security against memory-bounded adversaries,” in Advances in Cryptology — CRYPTO ’97, vol. 1294 of Lecture Notes in Computer Science, pp. 292–306, Springer-Verlag, Aug. 1997.

[16] I. B. Damgård, S. Fehr, L. Salvail, and C. Schaffner, “Cryptography in the bounded-quantum-storage model,” in IEEE Information Theory Workshop on Theory and Practice in Information-Theoretic Security, 2005, pp. 24–27, Oct 2005.

[17] I. B. Damgård, S. Fehr, L. Salvail, and C. Schaffner, Advances in Cryptology - CRYPTO 2007: 27th Annual International Cryptology Conference, Santa Barbara, Calif., USA, August 19-23, 2007. Proceedings, ch. Secure Identification and QKD in the Bounded-Quantum-Storage Model, pp. 342–359. Berlin, Heidelberg: Springer Berlin Heidelberg, 2007.

[18] S. Wehner, C. Schaffner, and B. M. Terhal, “Cryptography from noisy storage,” Phys. Rev. Lett., vol. 100, p. 220502, Jun 2008.

[19] R. König, S. Wehner, and J. Wullschleger, “Unconditional security from noisy quantum storage,” IEEE Transactions on Information Theory, vol. 58, pp. 1962–1984, March 2012.

[20] F. Dupuis, O. Fawzi, and S. Wehner, “Entanglement sampling and applications,” IEEE Transactions on Information Theory, vol. 61, pp. 1093–1112, Feb 2015.

[21] M. Berta, O. Fawzi, and S. Wehner, “Quantum to classical randomness extractors,” IEEE Transactions on Information Theory, vol. 60, pp. 1168–1192, Feb 2014.

[22] F. Furrer, C. Schaffner, and S. Wehner, “Continuous-Variable Protocols in the Noisy-Storage Model,” arXiv:1509.09123 Sept. 2015.

[23] N. H. Y. Ng, S. K. Joshi, C. Chen Ming, C. Kurtsiefer, and S. Wehner, “Experimental implementation of bit commitment in the noisy-storage model,” Nature Communications, vol. 3, no. 1326, 2012.

[24] C. Erven, N. Ng, N. Gigo, R. Laflamme, S. Wehner, and G. Weihs, “An experimental implementation of oblivious transfer in the noisy storage model,” Nature Communications, vol. 5, no. 3418, 2014.

[25] B. Qi, C.-H. F. Fung, H.-K. Lo, and X. Ma, “Time-shift attack in practical quantum cryptosystems,” Quantum Info. Comput., vol. 7, pp. 73–82, Jan. 2007.

[26] Y. Zhao, C.-H. F. Fung, B. Qi, C. Chen, and H.-K. Lo, “Quantum hacking: Experimental demonstration of time-shift attack against practical quantum-key-distribution systems,” Phys. Rev. A, vol. 78, p. 042333, Oct 2008.

[27] V. Makarov, A. Anisimov, and J. Skaar, “Effects of detector efficiency mismatch on security of quantum cryptosystems,” Phys. Rev. A, vol. 74, p. 022313, Aug 2006.

[28] N. Gisin, S. Fasel, B. Kraus, H. Zbinden, and G. Ribordy, “Trojan-horse attacks on quantum-key-distribution systems,” Phys. Rev. A, vol. 73, p. 022320, Feb 2006.

[29] C.-H. F. Fung, B. Qi, K. Tamaki, and H.-K. Lo, “Phase-remapping attack in practical quantum-key-distribution systems,” Phys. Rev. A, vol. 75, p. 032314, Mar 2007.

[30] S. Sajeed, I. Radchenko, S. Kaiser, J.-P. Bourgoin, A. Pappa, L. Monat, M. Legré, and V. Makarov, “Attacks exploiting deviation of mean photon number in quantum key distribution and coin tossing,” Phys. Rev. A, vol. 91, p. 032326, Mar 2015.

[31] It is assumed that the device produces no hidden output, such as an undetected radio signal, to the adversary. As such, the devices are unknown but well shielded from interactions outside the laboratory.

[32] J. S. Bell, “On the einstein podolsky rosen paradox,” Physics 1, 195-200, 1964.

[33] D. Mayers and A. Yao, “Quantum cryptography with imperfect apparatus,” in Proceedings of the 39th Annual Symposium on Foundations of Computer Science, FOCS ’88, (Washington, DC, USA), pp. 503–, IEEE Computer Society, 1988.

[34] D. Mayers and A. Yao, “Self testing quantum apparatus,” Quantum Info. Comput., vol. 4, pp. 273–286, July 2004.
Appendix A: min-tradeoff function

We prove here that the function $f$ defined as,

$$f(q) := \frac{1}{2} - \frac{(4p - 2)\sqrt{16p(1-p) - 2}}{2}, \quad \text{where } p := \frac{q(1)}{1 - q(\bot)},$$

is a convex min-tradeoff function for the maps $N_1, \ldots, N_n$ defined in Figure 4. The convexity is easy to check so we will focus on proving that $f$ is a min-tradeoff function for the maps $N_1, \ldots, N_n$. We prove it in Lemma 25.

Before we state the Lemma 25 we first need to define the effective anti-commutator [61] and the absolute effective anti-commutator [62] of two observables.

**Definition 22.** For a state $\omega_{AB}$ and two binary measurements acting on system $A$ associated to the observables $A_0$ and $A_1$ the effective anti-commutator is defined as,

$$\epsilon := \frac{1}{2} \text{tr}(\{A_0, A_1\} \omega_A),$$

and the absolute effective anti-commutator is

$$\epsilon_+ := \frac{1}{2} \text{tr}(\{|A_0, A_1\} \omega_A).$$

These two quantities are measures of how much (in)compatible two measurements are. If $\epsilon = 0$ or $\epsilon_+ = 0$ then the measurements associated to the observables $A_0$ and $A_1$ are maximally incompatible. When $|\epsilon| = 1$ or $|\epsilon_+| = 1$ then the measurement are compatible and there is no uncertainty on the measurement outcomes. They also have another nice property, namely they can be related to the probability of winning a CHSH game [61, 62].

**Property 23.** Let $\omega_{AB}$ be a bipartite state, $A_0, A_1$ two observables acting on the system $A$, and $B_0, B_1$ two observables acting on $B$. We define the CHSH value to be $S := \text{tr}(W \omega_{AB})$, where $W := A_0 \otimes B_0 + A_0 \otimes B_1 + A_1 \otimes B_0 - A_1 \otimes B_1$. The following inequalities hold

$$|\epsilon| \leq \epsilon_+ \leq \frac{S}{4} \sqrt{8 - S^2} :=: \zeta,$$

where $\epsilon$ and $\epsilon_+$ are respectively the effective anti-commutator and the absolute effective anti-commutator as defined above. The first inequality comes from the fact that for any hermitian operator $O$ and for any state $\omega$ we have $|\text{tr}(O \omega)| \leq \text{tr}(|O| \omega)$, the second inequality comes from the result of Ref. [62].

It is well known that the CHSH value is linked to the probability of success in a CHSH game [62] as:

$$p = \frac{1}{2} + \frac{S}{8},$$

where $p$ is the probability to win the CHSH game. Therefore we can express $\zeta$ in terms of $p$:

$$\zeta = (4p - 2)\sqrt{16p(1 - p) - 2}.$$

This property is important because it permits us to link the “(in)compatibility” of two measurements to the number of times we can win a CHSH game. In other words, by playing many CHSH games and counting the number of wins we can have an idea on how much (in)compatible two unknown measurements are. Note that because we do not use IID assumption, we cannot really estimate $\zeta$. However through the EAT we can use the number of wins of the CHSH game to prove security.

Before we prove the Lemma 25 we need to show a short technical lemma.

**Lemma 24.** Let $\omega_{AK}$ be a q-state

$$\omega_{AK} := \sum_k p_k \omega_A |k\rangle \otimes |k\rangle \langle k|_K,$$

and $A_0, A_1$ be two observables acting on the system $A$. If we measure the observables according to a random bit $\Theta \in_R \{0, 1\}$ and store the outcome in a register $X$, then the conditional Shannon entropy satisfies

$$H(X|\Theta K)_{\omega_{XAK}} \geq \frac{1 - \zeta}{2},$$

where $\zeta$ is defined in property 23 and characterizes how much the measurements associated with the observables $A_0, A_1$ are compatible.
Proof. We begin using the definition of the conditional Shannon entropy to write it as,

$$H(X|\Theta K) = \sum_k p_k H(X|\Theta K = k).$$  \hfill (A7)

Let’s define the effective anti-commutator conditioned on the value $K = k$ as $\epsilon_k := \frac{1}{2} \tr[\{A_0, A_1\} \omega_{A|k}]$. Using the main result from [61] we have,

$$H(X|\Theta K = k) \geq \frac{1}{2} h\left(\frac{1 + \sqrt{|\epsilon_k|}}{2}\right),$$  \hfill (A8)

where $h(x) := -x \log(x) - (1 - x) \log(1 - x)$. Unfortunately this lower bound is a concave function of $|\epsilon_k|$, and we will need convexity property, so let’s lower bound it by the largest possible convex function of $|\epsilon_k|$, namely

$$H(X|\Theta K = k) \geq \frac{1}{2} h\left(\frac{1 + \sqrt{|\epsilon_k|}}{2}\right) \geq \frac{1 - \epsilon_k}{2}. $$  \hfill (A9)

Using the last inequality in the equation (A7) we get,

$$H(X|\Theta K) \geq \sum_k p_k \frac{1 - |\epsilon_k|}{2} = \frac{1 - \sum_k p_k |\epsilon_k|}{2}. $$  \hfill (A10)

For any hermitian operator $O$ and state $\omega$ we have that $|\tr(O\omega)| \leq \tr(|O|\omega)$ which implies that

$$\sum_k p_k |\epsilon_k| = \frac{1}{2} \sum_k p_k \tr[\{A_0, A_1\} \omega_{A|k}] \leq \frac{1}{2} \sum_k p_k \tr[\{A_0, A_1\} |\omega_{A|k}|] = \frac{1}{2} \tr[\{A_0, A_1\} |\omega_A|] = \epsilon_{+} \leq \zeta,$$  \hfill (A11)

where the last inequality comes from property [23] and combined with equation (A10) concludes the proof. \hfill \Box

Lemma 25. For any of the maps $N_{i}$, $i \in [n]$ defined in Figure 4 we have:

$$f(q) \leq \inf_{\omega} H(X_i|\Theta_i T_i K)(N_{i} \otimes 1_K)(\omega_{RK}), $$  \hfill (A12)

where $f$ is defined in (A1) and the infimum is taken over all qc-states $\omega_{RK}$ such that $(N_{i} \otimes 1_K)(\omega_{RK})c_i = q$.  

Proof. First we use the fact that $X_i = \perp$ when $T_i = 1$ (this happens with probability $\mu$), which gives us,

$$H(X_i|\Theta_i T_i K)_{N_{i}}(\omega) = \mu H(X_i|\Theta_i T_i = 1K)_{N_{i}}(\omega) + (1 - \mu) H(X_i|\Theta_i T_i = 0K)_{N_{i}}(\omega) $$  \hfill (A13)

Note that for $T_i = 0$ we have $X_i = X'_i$, where $X'_i$ is Alice’s measurement outcome before she post-processes it (see step 1e of Protocol 11). Therefore we can say that

$$H(X_i|\Theta_i T_i = 0K)_{N_{i}}(\omega) = H(X'_i|\Theta_i T_i = 0K)_{N_{i}}(\omega) $$  \hfill (A15)

where $N'_i$ the map representing Alice operations before she post-processes her outputs. Also we know that the measurement made by her main device is independent from $T_i$, and then,

$$H(X'_i|\Theta_i K)_{N'_i}(\omega) = H(X'_i|\Theta_i K)_{N'_i}(\omega). $$  \hfill (A16)

Now we can use the Lemma 24 and we get,

$$H(X'_i|\Theta_i K)_{N'_i}(\omega) \geq \frac{1 - \zeta}{2}. $$  \hfill (A17)

Rewriting $\zeta$ in terms of the CHSH winning probability (see property 23) we have

$$H(X'_i|\Theta_i K)_{N'_i}(\omega) \geq \frac{1 - (4p - 2)\sqrt{16p(1 - p) - 2}}{2} $$  \hfill (A18)

where $p$ is the probability of winning a CHSH game. Note that the bit $C_i$ encodes whether the CHSH game is won, lost or not played, and then $p$ can be rewritten as $p = \frac{q(1)}{1 - q(\perp)}$ (which is just the winning probability conditioned on whether we effectively play the game), where $q = (q(0), q(1), q(\perp))^t$ (where $^t$ is the transpose) is the probability distribution on the alphabet of $C_i$, namely $\{0, 1, \perp\}$. Putting the equations (A14), (A15), (A16), and (A18) together finishes the proof. \hfill \Box
Appendix B: Upper bound on $H_{\max}$

In this section we want to upper bound $H_{\max}^{\epsilon/4}(C_1^n|T_1^nK)_{\rho|\Omega}$. To do so we first relate this max-entropy to $H_{\max}(C_1^n|T_1^nK)_{\rho}$ and then use Theorem 10 to upper bound the $1/\alpha$-entropy.

Let us now state some useful lemmas which permit us to relate $H_{\max}^{\epsilon/4}(C_1^n|T_1^nK)_{\rho|\Omega}$ with $H_{\max}(C_1^n|T_1^nK)_{\rho}$.

**Lemma 26** ([1], Lemma B.9). For any density operator $\omega$, $\alpha \in [1,2]$, and $\epsilon \in ]0,1[$,
\[
H_{\max}(A|B)_{\omega} \leq H_{\max}(A|B) + \frac{g(\epsilon)}{\alpha - 1},
\]
where $g(\epsilon) := -\log(1 - 1 - \epsilon^2)$.

We will also need to relate $H_{\max}(C_1^n|T_1^nK)_{\rho}$ with $H_{\max}(C_1^n|T_1^nK)_{\rho|\Omega}$.

**Lemma 27** ([1], Lemma B.5). For a quantum state $\omega_{AB}$ on the form $\omega_{AB} = \sum_x p_x \omega_{AB|x}$, where $\{p_x\}$ is a probability distribution on some alphabet $X$, we have for any $x \in X$ and $\alpha \in [1,2]$,
\[
H_{\max}(A|B)_{\omega|x} \leq H_{\max}(A|B) + \frac{\alpha}{\alpha - 1} \log \frac{1}{p_x}.
\]

Let us now find an upper bound on the max-entropy.

**Lemma 28.** For a state $\rho|\Omega$, specified by Protocol 11, the max-entropy $H_{\max}^{\epsilon/4}(C_1^n|T_1^nK)_{\rho|\Omega}$ satisfies
\[
H_{\max}^{\epsilon/4}(C_1^n|T_1^nK)_{\rho|\Omega} \leq \mu n + n(\alpha - 1) \log^2(7) + \frac{\alpha}{\alpha - 4} \log \left( \frac{1}{p_{\Omega}} \right) - \frac{\log(1 - \sqrt{1 - (\epsilon/4)^2})}{\alpha - 1},
\]
for $\alpha \in [1,2]$ and $\epsilon \in ]0,1[$, where $\Omega$ is the event corresponding to $\omega \geq \delta$ and $\mu$ is the probability of the round being a test round ($T_i = 1$) as described by Protocol 11.

**Proof.** Indeed, using Lemma 26 we get
\[
H_{\max}^{\epsilon/4}(C_1^n|T_1^nK)_{\rho|\Omega} \leq H_{\max}(C_1^n|T_1^nK)_{\rho|\Omega} - \frac{\log(1 - \sqrt{1 - (\epsilon/4)^2})}{\alpha - 1}.
\]

Now by taking $\rho = p_{\Omega} \cdot \rho|\Omega + (\rho - p_{\Omega} \cdot \rho|\Omega)$ and using Lemma 27 we can bound $H_{\max}(X_1^n|T_1^nK)_{\rho|\Omega}$ as follows,
\[
H_{\max}(X_1^n|T_1^nK)_{\rho|\Omega} \leq H_{\max}(C_1^n|T_1^nK)_{\rho} + \frac{\alpha}{\alpha - 1} \log \left( \frac{1}{p_{\Omega}} \right).
\]

Let us use the Theorem 11 $n$ times to upper bound $H_{\max}(X_1^n|T_1^nK)_{\rho}$. First we see that the Markov conditions
\[
\forall i \in [n], C_i^{i-1} \leftrightarrow T_i^{i-1}K \leftrightarrow T_i
\]
are clearly satisfied, so we can apply the theorem with the following substitution,
\[
\forall i \in [n], A_1 \rightarrow C_i^{i-1}
\]
\[
\forall i \in [n], A_2 \rightarrow C_i
\]
\[
\forall i \in [n], B_1 \rightarrow T_i^{i-1}K
\]
\[
\forall i \in [n], B_2 \rightarrow T_i
\]
and we get
\[
H_{\max}(C_1^n|T_1^nK)_{\rho} \leq \sum_{i=1}^{n} \sup_{\omega_i} H_{\max}(C_i|T_iK)_{\omega_i}
\]
\[
\leq \sum_{i=1}^{n} \sup_{\omega_i} H(C_i|T_iK)_{\omega_i} + n(\alpha - 1) \log^2(1 + 2d_C)
\]
where \(d_C = 3\) is the dimension of \(C_i\), and the second inequality comes from the Lemma 29 below which also imposes the constraint \(1 < \alpha < 1 + \frac{1}{\log(1 + 2d_A)}\). Because \(C_i = \perp\) when \(T_i = 0\) (which happens with a probability \(1 - \mu\)) we can say that \(H(C_i|T_i = 0K)_{\omega_i} = 0\), and conditioned on \(T_i = 1 C_i\) takes only two values. Then by decomposing the Shannon entropy we get the following bound,

\[
H(C_i|T,K)_{\omega_i} = \mu \cdot H(C_i|T = 1K)_{\omega_i} + (1 - \mu) \cdot H(C_i|T = 0K)_{\omega_i} \leq 1
\]

Combining equations (B4) (B5) (B12) and (B14) concludes the proof.

\[\square\]

Lemma 29 ([1], Lemma B.8). For any state \(\rho_{AB}\) and for \(1 < \alpha < 1 + \frac{1}{\log(1 + 2d_A)}\),

\[
H_{\frac{\mu}{2}}(A|B)_\rho < H(A|B)_\rho + (\alpha - 1) \log^2(1 + 2d_A),
\]

where \(d_A\) is the dimension of the system \(A\).

Appendix C: Attack against a fully sequential prepare and measure protocol

Here we show an attack where by preparing the state in some way, the adversary can break the non leakage assumption that no information is transmitted to the adversary. Note that this attack also works in QKD, and implicitly forces one to assume that the source is isolated from the measurement devices.

In a protocol where Alice produces a new state at each round with a non isolated source, there is a simple attack which permits Bob to guess Alice’s measurement outcomes \(X_i\) with probability almost one. Indeed as we allow Bob to create Alice’s devices (except for the switch which is trusted) Bob can integrate the source in the main device so that the main device is now allowed to produce the states. In this case (or equivalently if there is full communication between the source and the main device), for each round \(i \in [n]\):

1. Bob asks the main device to produce the state \(\rho_{A,B_i} = \Phi_{A_iB_i} \otimes |X_i^{i-1}\rangle_B^i \rangle_{B''}^i\) where \(\Phi_{A_iB_i}^i\) is an EPR pair, the register \(B_i = B_i' B_i''\), \(B_i''\) is a classical register encoding \(X_i^{i-1}\) (as defined in the Protocol [11] \(X_i^n\) is the string of all measurement outcomes for the rounds \(k \in [i-1]\) before Alice post-processes them).

2. He also asks his main device to execute the exact same measurements as the honest ones (described in Protocol [11] on the system \(A_i\), and he asks his testing device to proceed to the honest measurements on the system \(B_i'\).

3. • If the round \(i\) is tested, Bob receives \(|\perp\rangle \langle \perp|\) from Alice, so he sets the bit \(\hat{X}_i = \perp\) (because for the testing rounds \(j X_j = \perp\)).

• If the round is a non-testing round, Bob receives the system \(B_i = B_i' B_i''\). Then he reads the classical system \(B_i''\) where the string \(X_i^{i-1}\) is encoded so that he can set all the bits \(\hat{X}_j\) for the subset of the rounds \(j \in [i-1]\) that are not tested to \(X_j = X_j\) (indeed for the non-testing rounds \(k\) we have \(X_j = X_j\)). Then he stores the qubit \(B_i'\) in his one qubit memory he holds (erasing all previous quantum states that he received previously).

After the duration \(\Delta t\) Alice sends the list of bases \(\Theta_j^n\) to Bob. As soon as Bob receives it he measures the last qubit \(B_j'\) \((j \in [n])\) he has stored in the corresponding basis \(\Theta_j\), and because the state \(\rho_{A_jB_j'}\) is an EPR pair Bob’s measurement outcome \(\hat{X}_j\) for the round \(j\) is the same as Alice’s one for this round, so we can say that \(\hat{X}_j = X_j\).

In the end, if Alice does not abort the protocol, Bob holds a string \(X^n_1 = X^n_1\). Moreover since the measurements made by Alice’s devices during the protocol are the exact same as the honest ones, and because they are made on EPR pairs exactly as in the honest case, the protocol aborts with a probability less than \(\epsilon\) (see the correctness of the Protocol [11]). This means that Bob guesses Alice’s string \(X^n_1\) with a probability at least \(1 - \epsilon\) using only one qubit memory as resource.

Appendix D: Technical lemmas for Position Verification

We prove here two technical lemmas about smooth min-entropy and min-entropy.
Lemma 30. Let $\rho_{XA} \in \mathcal{S}(\mathcal{H}_{XA})$ be a cq-state on the form $\rho_{XA} = \sum_{x \in X} p_x |x\rangle \otimes \rho_{A|x}$, $X$ some alphabet, and $\epsilon \in [0, 1]$, then we have

$$2^{-H_{\min}(X|A)_{\rho}} - 2^{-H'_{\min}(X|A)_{\rho}} \leq \epsilon. \tag{D1}$$

Proof. First note that by definition of the smooth min-entropy there exists a state $\tilde{\rho}_{XA}$ that is $\epsilon$-close to $\rho_{XA}$ (in terms of purified distance) such that

$$H_{\min}(X|A)_{\tilde{\rho}} = H'_{\min}(X|A)_{\tilde{\rho}}. \tag{D2}$$

We will then proceed by contradiction and suppose that (D1) is not true meaning that there exists two cq-states $\sigma_{XA}, \tilde{\sigma}_{XA} \in \mathcal{S}(\mathcal{H}_{XA})$ such that $\sigma \approx_{\epsilon} \tilde{\sigma}$, $H_{\min}(X|A)_{\sigma} = H'_{\min}(X|A)_{\sigma}$ and

$$2^{-H_{\min}(X|A)_{\sigma}} - 2^{-H_{\min}(X|A)_{\sigma}} > \epsilon, \tag{D3}$$

and from that show that there exists a binary measurement which distinguishes $\sigma$ and $\tilde{\sigma}$ with a probability $P_{\text{dist}}(\sigma, \tilde{\sigma}) > \frac{1}{2}(1 + \epsilon)$ which is a contradiction with the fact that $\sigma \approx_{\epsilon} \tilde{\sigma}$. Indeed the purified distance upper bounds the trace distance $\Delta(\sigma, \tilde{\sigma})$, and $P_{\text{dist}}(\sigma, \tilde{\sigma}) = \frac{1}{2}(1 + \Delta(\sigma, \tilde{\sigma}))$, so as $\sigma \approx_{\epsilon} \tilde{\sigma}$ we must have $P_{\text{dist}}(\sigma, \tilde{\sigma}) \leq \frac{1}{2}(1 + \epsilon)$.

Let us show that such a measurement exists:

**First step:** Let us consider the POVM $\mathcal{F} := \{P^A_x, x \in X\}$ acting on $\mathcal{H}_A$, which maximizes (over the POVMs $\{P^A_x, x \in X\}$) $\sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes P^A_{X|A} \sigma_{XA})$ being the probability that one guesses the value of $X$ by measuring $\sigma_A$. In the same way we consider $\tilde{\mathcal{F}} := \{\tilde{P}^A_x, x \in X\}$ the POVM that maximizes $\sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes \tilde{P}^A_{X|A} \tilde{\sigma}_{XA})$. Note that

$$\sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes P^A_{X|A} \sigma_{XA}) - \sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes \tilde{P}^A_{X|A} \tilde{\sigma}_{XA})$$

$$= \max_{\{P^A_x, x \in X\}} \sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes P^A_{X|A} \sigma_{XA}) - \max_{\{\tilde{P}^A_x, x \in X\}} \sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes \tilde{P}^A_{X|A} \tilde{\sigma}_{XA})$$

$$= 2^{-H_{\min}(X|A)_{\sigma}} - 2^{-H_{\min}(X|A)_{\sigma}} > \epsilon. \tag{D6}$$

The first equality coming from the definitions of $\mathcal{F}$ and $\tilde{\mathcal{F}}$, the second from the operational interpretation of min-entropy (see property 4), and the final inequality comes from the hypothesis made in (D3).

**Second step:** We write now the maximum distinguishing probability between $\sigma$ and $\tilde{\sigma}$ as:

$$P_{\text{dist}}(\sigma, \tilde{\sigma}) := \max_{\{\hat{O}, 1 - \hat{O}\}} \left[ \frac{1}{2}(\text{tr}(\hat{O}\sigma) + \text{tr}((1 - \hat{O})\tilde{\sigma})) \right] \tag{D7}$$

$$\geq \max_{\{P_x, x \in X\}} \left[ \frac{1}{2}(1 + \text{tr}(O(\sigma - \tilde{\sigma}))) \right] \tag{D8}$$

$$= \frac{1}{2}(1 + \max_{\{P_x, x \in X\}} [\text{tr}(O\sigma) - \text{tr}(O\tilde{\sigma})]) \tag{D9}$$

$$\geq \frac{1}{2}(1 + \max_{\{P_x, x \in X\}} [\text{tr}(O\sigma)] - \max_{\{P_x, x \in X\}} [\text{tr}(O\tilde{\sigma})]), \tag{D10}$$

where the maximization in the first line is done over all binary POVMs $\{\hat{O}, 1 - \hat{O}\}$ acting on $\mathcal{H}_{AB}$, and the maximisation from the second to the last line is done over binary POVMs $\{O, 1 - O\}$ such that $O$ is on the form $O = \sum_{x \in X} |x\rangle \langle x| \otimes P^A_x$ for some POVM $\{P^A_x, x \in X\}$. The second line comes from linearity of the trace, the normalization of the state $\tilde{\sigma}$, and because we restrict the optimization to a smaller set of POVM. The last line comes from the fact that for any set $Z$ and any functions $f, g : Z \mapsto \mathbb{R}$ we have:

$$\left( \max_{z \in Z} \text{f}(z) - \max_{z \in Z} \text{g}(z) \geq 0 \right) \implies \left( \max_{z \in Z} \text{f}(z) - \text{g}(z) \geq 0 \right),$$

and the l.h.s. is satisfied because $H_{\min}(X|A)_{\sigma} = H'_{\min}(X|A)_{\sigma} \leq H_{\min}(X|A)_{\sigma}$.

By definition of $O$ we have:

$$\max_{\{P_x, x \in X\}} [\text{tr}(O\sigma)] - \max_{\{P_x, x \in X\}} [\text{tr}(O\tilde{\sigma})] \tag{D11}$$

$$= \max_{\{P^A_x, x \in X\}} \sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes P^A_x \sigma_{XA}) - \max_{\{P^A_x, x \in X\}} \sum_{x \in X} \text{tr}(|x\rangle \langle x| \otimes P^A_x \tilde{\sigma}_{XA}), \tag{D12}$$
and this is just the difference between two guessing probabilities, so it can be rewritten as

$$\max_{\{P_x, x \in X\}} \{ \text{tr}(O\sigma) \} - \max_{\{P_x, x \in X\}} \{ \text{tr}(O\tilde{\sigma}) \} = 2^{-H_{\text{min}}(X|A)_{\sigma}} - 2^{-H_{\text{min}}(X|A)_{\tilde{\sigma}}} > \epsilon,$$  \hspace{1cm} (D13)

where the inequality comes from the hypothesis (D3). Inserting the last expression into (D10) shows that the best guessing probability between $\sigma$ and $\tilde{\sigma}$ is $P_{\text{dist}}(\sigma, \tilde{\sigma}) > 2^{-\epsilon \max_{\{P_x, x \in X\}} \{ \text{tr}(O\sigma) \} = 2^{-\epsilon \min_{\{P_x, x \in X\}} \{ \text{tr}(O\sigma) \}}$, which contradicts that $\sigma \approx_{\epsilon} \tilde{\sigma}$ and proves the statement.

\[ \square \]

We will prove now that if the smooth min-entropy is lower bounded by a linear bound (in $n$) when $\epsilon$ decays exponentially with $n$.

**Lemma 31.** Let $\rho_{X^n|A}$ be a cq-state where $X^n_1$ is a classical register containing $n$ values and $A$ is an arbitrary quantum system. Then if

$$\exists \alpha, C > 0 : H_{\epsilon_n}^\alpha(X^n_1|A)_\rho \geq Cn$$  \hspace{1cm} (D14)

where $\epsilon_n = 2^{-\alpha n}$, then we have (for $n$ big enough)

$$\exists \kappa : H_{\min}(X^n_1|A)_\rho \geq \kappa n.$$  \hspace{1cm} (D15)

**Proof.** By using the previous Lemma 30 and we have,

$$2^{-H_{\min}(X^n_1|A)_\rho} \leq \epsilon_n + 2^{-H_{\min}^\alpha(X^n_1|A)_\rho}$$  \hspace{1cm} (D16)

and because $\epsilon_n = 2^{-\alpha n}$ and $H_{\min}^\alpha(X^n_1|A)_\rho \geq Cn$ we have

$$2^{-H_{\min}(X^n_1|A)_\rho} \leq 2^{-\alpha n} + 2^{-Cn} \leq 2^{-\min(\alpha, C)n+1}$$  \hspace{1cm} (D17)

which is equivalent to

$$H_{\min}(X^n_1|A)_\rho \geq \kappa n$$  \hspace{1cm} (D18)

for any $0 < \kappa < \min(\alpha, C)$ and $n$ big enough.

\[ \square \]