Some applications of parameterized Picard–Vessiot theory

C. Mitschi

Abstract. This is an expository article describing some applications of parameterized Picard–Vessiot theory. This Galois theory for parameterized linear differential equations was Cassidy and Singer’s contribution to an earlier volume dedicated to the memory of Andrey Bolibrukh. The main results we present here were obtained for families of ordinary differential equations with parameterized regular singularities in joint work with Singer. They include parametric versions of Schlesinger’s theorem and of the weak Riemann–Hilbert problem as well as an algebraic characterization of a special type of monodromy evolving deformations illustrated by the classical Darboux–Halphen equation. Some of these results have recently been applied by different authors to solve the inverse problem of parameterized Picard–Vessiot theory, and were also generalized to irregular singularities. We sketch some of these results by other authors. The paper includes a brief history of the Darboux–Halphen equation as well as an appendix on differentially closed fields.

Keywords: complex linear ordinary differential equations, differential Galois theory, parameterized Picard–Vessiot theory, monodromy, isomonodromy, monodromy evolving deformations, Darboux–Halphen equation, inverse problems.

§ 1. Parameterized Picard–Vessiot theory

The classical Picard–Vessiot theory or differential Galois theory (PV-theory for brevity) associates with any linear differential system

$$\partial Y = AY,$$  \hspace{1cm} (1)

where the entries of the square matrix $A$ belong to a differential field $k$ of characteristic zero with derivation $\partial$ and algebraically closed field of constants, the so-called Picard–Vessiot extension of $k$. This is a differential field extension of $k$ generated by the entries of a fundamental solution. It has no new constants and its derivation is given by (1). The Picard–Vessiot extension is unique up to differential $k$-isomorphisms, and its group of differential $k$-automorphisms is called the Picard–Vessiot group (or differential Galois group). This is a linear algebraic group.
which reflects many properties of the equation, such as solubility, reducibility and the existence of algebraic solutions.

In the special volume [1] dedicated to Andrey Bolibrukh, Cassidy and Singer developed a parameterized Picard–Vessiot theory (PPV-theory for short), based on seminal work by Cassidy, Kolchin and Landesman. In PPV-theory, the differential base field \( k \) is endowed with a set of commuting derivations \( \Delta = \{ \partial_0, \partial_1, \ldots, \partial_m \} \). It is also called a \( \Lambda \)-field. As in PV-theory, one wants to associate with a (square) differential system

\[
\partial_0 Y = AY
\]  

(2)

with coefficients in \( k \) a unique parameterized Picard–Vessiot extension, that is, a \( \Delta \)-differential field extension of \( k \) generated by the entries of a fundamental solution of (2) (generated as a field extension by these entries and their \( \Delta \)-derivatives of any order) with no new \( \partial_0 \)-constants. Here \( \Delta \) denotes any set of derivations extending those of \( k \) in a field extension. The parameterized Picard–Vessiot group of a PPV-extension is its group of \( \Delta \)-differential \( k \)-automorphisms, with the usual expected properties, such as a parameterized version of the ‘Galois correspondence’. The following example ([1], p. 118) shows that some assumptions are needed to meet these requirements.

**Example 1.1.** Consider the scalar differential equation

\[
\frac{dy}{dx} = \frac{t}{x} y.
\]

(3)

For every fixed \( t \in \mathbb{C} \) we can apply the classical PV-theory over the differential fields \( \mathbb{C}(x) \) or \( \mathbb{Q}(x) \). An easy calculation shows that the PV-group of (3) over \( \mathbb{C}(x) \) (resp. \( \mathbb{Q}(x) \)) is \( \mathbb{C}^* \) (resp. \( \mathbb{Q}^* \)) when \( t \not\in \mathbb{Q} \), and a cyclic subgroup (of roots of unity) otherwise.

We now regard (3) as a parameterized family over the differential field \( k = \mathbb{C}(x, t) \) of rational functions of \( x \) and \( t \) with derivations \( \{ \frac{d}{dx}, \frac{d}{dt} \} \). Its PPV-extension is

\[
K = \mathbb{C}(x, t, x^t, \log x).
\]

We claim that the corresponding PPV-group \( G \) is equal to \( \mathbb{C}^* \) and that the element \( \log x \) of \( K \) is invariant under \( G \). (If the Galois correspondence held for \( G \), then the subfield \( K^G \) of all elements of \( K \) that are invariant under \( G \) would coincide with the base field \( k \).) Indeed, since every automorphism \( \sigma \in G \) is uniquely determined by the elements \( \sigma(x^t) \) and \( \sigma(\log x) \) and commutes with both derivations, it is of the form

\[
\sigma(x^t) = a_\sigma x^t, \quad \sigma(\log x) = \log x + c_\sigma,
\]

where \( c_\sigma \) is the logarithmic derivative of \( a_\sigma \), and the coefficients \( a_\sigma \in \mathbb{C}^* \), \( c_\sigma \in \mathbb{C} \) depend only on \( t \). An easy calculation shows that

\[
G = \{ a \in \mathbb{C}(t)^*, \; a''a - a'^2 = 0 \} = \mathbb{C}^*.
\]

where \( a' \), \( a'' \) are the first and second derivatives with respect to \( t \), and that \( G = \mathbb{C}^* \) since \( a \) is a rational function of \( t \). In particular, it follows that \( c_\sigma = 0 \) for all \( \sigma \in G \), whence \( \sigma(\log x) = \log x \) for all \( \sigma \in G \).
To have $K^G = k$ in Example 1.1, the group $G$ must contain non-constant elements. Assuming that the field $k_0 = k[\partial_0]$ of $\partial_0$-constants is differentially closed (see the appendix), Cassidy and Singer ([1], p. 116) proved that for every equation (2) there is a unique PPV-extension of $k$ and that its PPV-group is a linear differential algebraic group over $k_0$, that is, a subgroup of $GL(n, k_0)$ defined by differential polynomial equations. In other words, this group is closed in the Kolchin topology, whose elementary closed sets are the zero sets of $\{\partial_1, \ldots, \partial_n\}$-differential polynomials. For more facts about differential algebraic groups we refer to Cassidy [2], who first introduced these objects, and to [3], [4]. Then we have a Galois correspondence between closed differential subgroups of the PPV-group and intermediate $\Delta$-differential extensions of $k$ in the PPV-extension.

Note that since $k_0$ is assumed to be differentially closed, it is in particular algebraically closed, and the usual PV-theory holds for the equation (2). The PPV-group, which is Kolchin-closed in $GL(n, k_0)$, is not closed in general in the (weaker) Zariski topology, and its Zariski closure is precisely the PV-group.

In what follows we consider only those families of differential equations whose coefficients are complex-analytic functions depending analytically on complex parameters. In the parametric case we first need to clarify the notion of a regular singular point.

§ 2. Parameterized singular points

Consider a family of linear differential equations

$$\frac{\partial Y}{\partial x} = A(x, t)Y \tag{4}$$

parameterized by $t$, where $A \in gl_n(\mathcal{O}_U(\{x-\alpha(t)\}))$ depends analytically on $x$ and $t$, as explained in Notation 2.1.

In what follows we use the words ‘system’ or ‘equation’ indifferently for a matrix equation, that is, a system of equations.

Notation 2.1. Let $\mathcal{U} \subset \mathbb{C}^r$ be an open connected subset containing 0. We write $\mathcal{O}_{\mathcal{U}}$ for the ring of analytic functions of the multi-variable $t$ on $\mathcal{U}$. The elements $\alpha \in \mathcal{O}_{\mathcal{U}}$ with $\alpha(0) = 0$ can be thought of as moving singularities near 0. Let $\mathcal{O}_{\mathcal{U}}((x-\alpha(t)))$ be the ring of formal Laurent series of the form

$$f(x, t) = \sum_{i \geq m} a_i(t)(x-\alpha(t))^i$$

with coefficients in $\mathcal{O}_{\mathcal{U}}$, where $m \in \mathbb{Z}$ is independent of $t$, and let $\mathcal{O}_{\mathcal{U}}(\{x-\alpha(t)\})$ be the ring of those series $f(x, t) \in \mathcal{O}_{\mathcal{U}}((x-\alpha(t)))$ which for every fixed $t \in \mathcal{U}$ converge in the punctured disc $0 < |x-\alpha(t)| < R_t$ for some $R_t > 0$. Note that for every compact neighbourhood $\mathcal{N} \subset \mathcal{U}$ of 0 one can choose $R_t$ to be independent of $t$ for $t \in \mathcal{N}$.

Under these assumptions and notation, we can expand the matrix $A$ in (4) as

$$A(x, t) = \frac{A_{-m}(t)}{(x-\alpha(t))^m} + \frac{A_{-m+1}(t)}{(x-\alpha(t))^{m-1}} + \cdots = \sum_{i \geq -m} (x-\alpha(t))^i A_i(t),$$

where $A_i(t) \in gl_n(\mathcal{O}_{\mathcal{U}})$ for all $i \geq -m$, and $m \in \mathbb{N}$ is independent of $t$. 
Definition 2.2. Two parametric equations
\[ \frac{\partial Y}{\partial x} = AY, \quad \frac{\partial Y}{\partial x} = BY \]
with matrices \( A, B \in \text{gl}_n(\mathcal{O}_U(\{x - \alpha(t)\})) \) are said to be equivalent if the following equality holds for some invertible matrix \( P \in \text{GL}_n(\mathcal{O}_U(\{x - \alpha(t)\})) \):
\[ B = \frac{\partial P}{\partial x} P^{-1} + PAP^{-1}. \]

Definition 2.3. With notation as before, we say that
(1) an equation (4) has simple singular points near 0 if \( m = 1 \) and \( A_{-1} \neq 0 \) as an element of \( \text{gl}_n(\mathcal{O}_U) \).
(2) an equation (4) has parameterized regular singular points near 0 (notation: prs\(_0\)) if it is equivalent to an equation with simple singular points near 0.

Example 2.4. Let
\[ A = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix} \frac{1}{(x-t)^2} + \begin{pmatrix} t & 0 \\ 0 & t-2 \end{pmatrix} \frac{1}{x-t}, \quad B = \begin{pmatrix} t-1 & 0 \\ 0 & t-1 \end{pmatrix} \frac{1}{x-t}. \]
These equations are equivalent via
\[ P = \begin{pmatrix} \frac{1}{x-t} & -1 \\ 0 & \frac{1}{x-t} \end{pmatrix}. \]
Since the second equation has simple singular points near 0, the first equation has parameterized regular singular points near 0.

As in the non-parameterized case, solutions of an equation (4) with parameterized regular singular points near 0 have ‘uniformly’ moderate growth as \( x \) tends to \( \alpha(t) \) and \( t \) tends to 0 (see [5], Corollary 2.6).

Proposition 2.5. Assume that an equation (4) has regular singular points near 0. Then there is an open connected subset \( \mathcal{U}' \) of \( \mathcal{U} \) with the following properties.
1) Equation (4) has a solution \( Y \) of the form
\[ Y(x,t) = \left( \sum_{i \geq i_0} (x-\alpha(t))^i Q_i(t) \right) (x-\alpha(t))^\tilde{A}(t) \quad (5) \]
with \( \tilde{A} \in \text{gl}_n(\mathcal{O}_{\mathcal{U}'}(\{x - \alpha(t)\})) \) and \( Q_i \in \text{gl}_n(\mathcal{O}_{\mathcal{U}'}(\{x - \alpha(t)\})) \) for all \( i \geq i_0 \).
2) For every \( r \)-tuple \( (m_1, \ldots, m_r) \) of non-negative integers there is an integer \( N \) such that for every fixed \( t \in \mathcal{U}' \) and any sector \( S_t \) with vertex \( \alpha(t) \) and opening less than \( 2\pi \) in the complex plane, we have
\[ \lim_{x \to \alpha(t)} \left( x - \alpha(t) \right)^N \frac{\partial^{m_1+\cdots+m_r} Y(x,t)}{\partial^{m_1}t_1 \cdots \partial^{m_r}t_r} = 0. \]
Solutions of parameterized differential equations with \textit{irregular} singular points were studied by Babbitt and Varadarajan in [6], by Schäfke in [7] and more recently by Dreyfus in [8]. Assuming that 0 is a (non-moving) irregular singular point, these authors gave conditions on the exponential part of a formal solution in its usual form

\[ \hat{Y}(z) = \hat{H}(z)z^J e^Q \]

ensuring that the coefficients of the formal series \( \hat{H}(z) \) depend analytically on the multi-parameter \( t \).

\section{3. PPV-theory and monodromy}

From the beginning of Picard–Vessiot theory in the nineteenth century, monodromy has been closely related to the ‘group of transformations’ (now called the Picard–Vessiot group) of a linear differential equation. More information about the history of the monodromy group and the Picard–Vessiot group can be found in [9] and [10].

\subsection{3.1. The classical Picard–Vessiot theory and monodromy.}

In classical Picard–Vessiot theory it is commonly admitted that ‘the monodromy matrices belong to the differential Galois group’. This is true for differential equations (1) over the base field \( \mathbb{C}(x) \), but does not hold over \( \overline{\mathbb{Q}}(x) \). Moreover, when the equation (1) has only regular singular points, Schlesinger’s theorem (see [11], §§ 159, 160, [12], Theorem 5.8) tells us that the monodromy matrices generate a Zariski-dense subgroup of the differential Galois group over \( \mathbb{C}(x) \). For example, the equation

\[ \frac{dy}{dx} = \frac{t}{x} y \]

(see Example 1.1) with a constant non-zero complex number \( t \) has two regular singular points, 0 and \( \infty \). The monodromy ‘matrices’ with respect to the solution \( x^t \) (under a fixed choice of \( \log x \)) at the points 0 and \( \infty \) are the scalars \( m_0 = e^{2\pi i t} \) and \( m_\infty = e^{-2\pi i t} \). It is easy to see that the Zariski closure in \( \mathbb{C}^* \) of the subgroup generated by \( m_0 \) (or \( m_\infty \)) is the PV-group over \( \mathbb{C}(x) \) as given above (it is equal to either \( \mathbb{C}^* \) or a finite cyclic group).

What happens over the differential field \( \overline{\mathbb{Q}}(x) \) if \( t \in \overline{\mathbb{Q}} \)? In this case the monodromy scalars \( e^{\pm 2\pi i t} \) may be transcendental and hence do not belong to the PV-group, which is a subgroup of \( \overline{\mathbb{Q}}^* \). But the results described above show that the PV-group is defined by the same equation in \( \mathbb{C}^* \) or \( \overline{\mathbb{Q}}^* \) for \( t \in \mathbb{C}^* \) or \( t \in \overline{\mathbb{Q}}^* \) respectively. The following example also illustrates the importance of the base field.

\begin{example}

The equation

\[ \frac{dY}{dx} = \begin{pmatrix} 1/x & 1 \\ 0 & 0 \end{pmatrix} Y \]

has two regular singular points (one Fuchsian at 0, one at \( \infty \)). The monodromy matrix of this equation with respect to the fundamental solution

\[ \begin{pmatrix} x & x \log x \\ 0 & 1 \end{pmatrix} \]
at the point 0 is equal to
\[ M = \begin{pmatrix} 1 & 2\pi i \\ 0 & 1 \end{pmatrix}. \]

If we consider the equation over \( \overline{\mathbb{Q}}(x) \), then \( M \) does not belong to the PV-group over \( \overline{\mathbb{Q}}(x) \) since it has a transcendental entry.

To adjust Schlesinger’s theorem to this situation, we use the following result (see [5], Proposition 3.1 and Corollary 3.2).

**Proposition 3.2.** Let \( C_0 \subset C_1 \) be algebraically closed fields. Consider the differential fields \( k_0 = C_0(x) \) and \( k_1 = C_1(x) \), where \( c' = 0 \) for all \( c \in C_1 \) and \( x' = 1 \), and the differential equation
\[ Y' = AY, \tag{6} \]
where \( A \in \text{gl}_n(k_0) \). If \( G(C_0) \subset \text{GL}_n(C_0) \) is the PV-group over \( k_0 \) of the equation (6) with respect to some fundamental solution, where \( G \) is a linear algebraic group defined over \( C_0 \), then \( G(C_1) \) is the PV-group of (6) over \( k_1 \) with respect to some fundamental solution.

Thus, in Example 3.1 we easily see that the PV-group over \( \overline{\mathbb{Q}}(x) \) is
\[ G = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \lambda \in \overline{\mathbb{Q}} \right\}, \]
and the PV-group over \( \mathbb{C}(x) \) is the group of \( \mathbb{C} \)-points of \( G \):
\[ G(\mathbb{C}) = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}, \lambda \in \mathbb{C} \right\}. \]

The monodromy matrices do belong to the PV-group after extending scalars.

**Corollary 3.3.** Consider an equation (6) with \( A \in \text{gl}_n(C_0(x)) \), where \( C_0 \) is an algebraically closed subfield of \( \mathbb{C} \). Assuming that 0 is a non-singular point, we take it for the base point of \( \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S) \), where \( S \) is the set of all singular points of (6) on \( \mathbb{P}^1(\mathbb{C}) \). Let \( G(C_0) \) be the PV-group of (6) over \( C_0(x) \), where \( G \) is a linear algebraic group defined over \( C_0 \). If \( C_1 \) is any algebraically closed subfield of \( \mathbb{C} \) containing \( C_0 \) and the entries of the monodromy matrices, then the monodromy matrices are elements of the PV-group \( G(C_1) \) of (6) over \( C_1(x) \).

### 3.2. Monodromy matrices in the PPV-group.

In PPV-theory too, the equation may have coefficients in some differentially closed field to which the entries of the parameterized monodromy matrices do not belong.

In [5] we proved an analogue of Proposition 3.2 for parameterized Picard–Vessiot extensions. Consider equations of the form
\[ \partial_x Y = A(x,t)Y, \tag{7} \]
where \( A(x,t) \in \text{gl}_n(\mathcal{O}_U(x)) \) and \( t = (t_1, \ldots, t_r) \in U \) for some domain \( U \subset \mathbb{C}^r \). Denoting the differentiations with respect to \( x, t_1, \ldots, t_r \) by \( \partial_x, \partial_{t_1}, \ldots, \partial_{t_r} \) respectively, we put \( \Delta = \{ \partial_x, \partial_{t_1}, \ldots, \partial_{t_r} \} \) and \( \Delta_t = \{ \partial_{t_1}, \ldots, \partial_{t_r} \} \).
Let $C$ be a $\Delta_t$-differentially closed extension of some field of functions that are analytic on some domain in $\mathbb{C}^r$ and, for every $i$, let $\partial_i$ be the derivation extending $\partial_t$. We consider the $\Delta$-differential field structure on $k = C(x)$ given by $\partial_x(x) = 1$, $\partial_i(x) = 0$ for each $i$ and $\partial_x(c) = 0$ for all $c \in C$, and we assume that $A \in \text{gl}_n(k)$.

**Proposition 3.4.** Take $C_0 \subset C_1$ to be differentially closed $\Delta_t$-fields as $C$ above, inducing $\Delta$-field structures on $k_0 = C_0(x)$ and $k_1 = C_1(x)$. Let

$$\partial_x Y = AY$$

be a differential equation with $A \in \text{gl}_n(k_0)$. If $G(C_0) \subset \text{GL}_n(C_0)$ is the PPV-group over $k_0$ of the equation (8) with respect to some fundamental solution, where $G$ is a linear differential algebraic group over the differential $\Delta_t$-field $C_0$, then $G(C_1)$ is the PPV-group of (8) over $k_1$ with respect to some fundamental solution.

Let us define the parameterized monodromy matrices, which belong to the PPV-group in the same sense as in the non-parameterized case (that is, after extending the base field).

Let $D$ be an open subset of $\mathbb{P}^1(\mathbb{C})$ with $0 \in D$. Assume that $\mathbb{P}^1(\mathbb{C}) \setminus D$ is the union of $m$ disjoint discs $D_i$ such that for every $t \in U$ the equation (7) has a unique singular point in $D_i$. Let $\gamma_i, i = 1, \ldots, m$, be elementary loops generating $\pi_1(D_i, 0)$. We fix a fundamental solution $Z_0$ of (7) in a neighbourhood of 0 and, for every fixed $t \in U$, consider the monodromy matrices of (7) with respect to this solution and the loops $\gamma_i$. These matrices, which depend on $t$, are by definition the parameterized monodromy matrices of the equation (7).

As in the non-parameterized case, to prove that the monodromy matrices belong to the PPV-group, one has to perform ‘analytic continuation’ of some expression $P(Z_0)$ in the entries of $Z_0$, where $P$ is a differential polynomial with coefficients in $C_0(x)$ over a differentially closed $\Delta_t$-field $C_0$ containing $\mathbb{C}$. Indeed, the following result of Seidenberg [13], [14] enables us to regard $P(Z_0)$ as an analytic function.

**Theorem 3.5** (Seidenberg). Let $\mathbb{Q} \subset K \subset K_1$ be finitely generated differential extensions of the field $\mathbb{Q}$ of rational numbers. Assume that $K$ consists of meromorphic functions on some domain $\Omega \subset \mathbb{C}^r$. Then $K_1$ is isomorphic to a field $F$ of functions that are meromorphic on some domain $\Omega_1 \subset \Omega$ such that $K|\Omega_1 \subset F$.

This leads to the expected analogue of Corollary 3.3.

**Theorem 3.6.** Assume in equation (7) that $A \in \text{gl}_n(C_0(x))$, where $C_0$ is any differentially closed $\Delta_t$-field containing $\mathbb{C}$, and let $C_1$ be any differentially closed $\Delta_t$-field containing $C_0$ and the entries of the parameterized monodromy matrices of the equation (7) with respect to a fundamental solution of (7). Then the parameterized monodromy matrices belong to $G(C_1)$, where $G$ is the PPV-group of (7) over the $\Delta$-field $C_0(x)$.

### 3.3. A parameterized version of Schlesinger’s theorem.

Consider a family of equations

$$\frac{\partial Y}{\partial x} = A(x, t)Y,$$

(9)
where the entries of \( A \) are rational in \( x \) and analytic in \( t \) on some open subset \( U \) of \( \mathbb{C}^r \). As above, let \( D \) be an open subset of \( \mathbb{P}^1(\mathbb{C}) \) with \( 0 \in D \). Assume that \( \mathbb{P}^1(\mathbb{C}) \setminus D \) is the union of \( m \) disjoint discs \( D_i \) such that for every \( t \in U \) the equation (9) has a unique singular point \( \alpha_i(t) \) in each \( D_i \) and no singular points otherwise. Let \( \gamma_i, i = 1, \ldots, m, \) be elementary loops generating \( \pi_1(\mathcal{D}, 0) \). Locally at \( 0 \) we can fix a fundamental solution \( Z_0 \) which is analytic on \( \mathcal{V} \times U \), where \( \mathcal{V} \) is a neighbourhood of \( 0 \) in \( D \subset \mathbb{C} \) and \( U \) is a neighbourhood of \( 0 \) in \( \mathbb{C}^r \). As above, we put \( \Delta = \{ \partial_x, \partial_{t_1}, \ldots, \partial_{t_r} \} \) and \( \Delta_t = \{ \partial_{t_1}, \ldots, \partial_{t_r} \} \).

In [5] we proved the following parameterized analogue of Schlesinger’s theorem.

**Theorem 3.7.** Under the notation and assumptions as above, suppose that the equation (9) has only parameterized regular singular points near each point \( \alpha_i(0), i = 1, \ldots, m \). Let \( k \) be a differentially closed \( \Delta_t \)-field containing the \( x \)-coefficients of the entries of \( A \), the singularities \( \alpha_i(t) \) of (9) and the entries of the parameterized monodromy matrices with respect to \( Z_0 \). Then the parameterized monodromy matrices generate a Kolchin-dense subgroup of \( G(k) \), where \( G \) is the PPV-group of (9) over \( k(x) \).

**Proof.** By the Galois correspondence in PPV-theory, it suffices to show that every element of the PPV-extension \( k(x)(Z_0) \) (\( \Delta \)-differentially generated by the fundamental solution \( Z_0 \)) which is invariant under the action of the parameterized monodromy matrices, belongs to the base field \( k(x) \). We fix such an element \( f \in k(x)(Z_0) \) which is invariant under all parameterized monodromy matrices. Let \( \mathcal{F}_0 \) be the differential \( \Delta_t \)-subfield of \( k \) generated over \( \mathbb{Q} \) by the \( x \)-coefficients of \( A \), the singular points \( \alpha_i(t) \) and the entries of the parameterized monodromy matrices with respect to elementary loops around the points \( \alpha_i(t) \). Further let \( \mathcal{F}_1 \) be any \( \Delta_t \)-subfield of \( k \) containing \( \mathcal{F}_0 \) such that \( f \in \mathcal{F}_1(x)(Z_0) \). By Seidenberg’s theorem (Theorem 3.5) we can regard \( f \) as a meromorphic function on a suitable domain of the \( (x,t) \)-space. For every fixed \( t \), the function \( f \) is invariant under the monodromy matrices and, by Proposition 2.5, has moderate growth at each singular point. Therefore it is indeed a rational function of \( x \) for each fixed \( t \) (cf. [55], proof of Theorem 2.28). As in the non-parameterized case, since \( f \) is single-valued, it has an isolated pole at each singular point of the equation (see [15], Preparation Theorem 18.2, p. 118). To show that it is globally a rational function of \( x \), we use the following lemma inspired by a result of Palais [16]. □

**Lemma 3.8.** Let \( \mathcal{F} \) be a \( \Delta \)-field of functions that are meromorphic on \( \mathcal{V} \times \mathcal{U} \), where \( \mathcal{V} \subset \mathbb{C} \) and \( \mathcal{U} \subset \mathbb{C}^r \) are connected open sets, and let

\[
\mathcal{C}_x = \{ u \in \mathcal{F} \mid \partial_x u = 0 \}.
\]

Furthermore assume that \( x \in \mathcal{F} \). Let \( f \in \mathcal{F} \) be such that \( f(x,t) \in \mathbb{C}(x) \) for each \( t \in \mathcal{U} \). Then for some \( m \in \mathbb{N} \) there are \( a_0, \ldots, a_m, b_0, \ldots, b_m \in \mathcal{C}_x \) such that

\[
f(x,t) = \frac{\sum_{i=0}^{m} a_i x^i}{\sum_{i=0}^{m} b_i x^i}.
\]
§ 4. PPV-characterization of isomonodromy

We recall that classical differential Galois theory, or PV-theory, extends easily and naturally to differential fields with several derivations. More precisely, let $k$ be a $\Delta$-differential field with derivations $\Delta = \{\partial_0, \partial_1, \ldots, \partial_r\}$. Consider a system of linear equations
\[
\begin{align*}
\partial_0 Y &= A_0 Y, \\
\partial_1 Y &= A_1 Y, \\
\vdots \\
\partial_r Y &= A_r Y,
\end{align*}
\]
(10)
where $A_0, A_1, \ldots, A_r \in \text{gl}(n, k)$.

If the subfield $C$ of $\Delta$-constants of $k$ is algebraically closed, then for every system (10) there is a unique PV-extension $K$ of $k$, that is, a $\Delta$-differential extension of $k$ generated by the entries of a fundamental solution of (10) with no new $\Delta$-constants. The corresponding PV-group of differential $k$-automorphisms of $K$ is a linear algebraic group $G \subset \text{GL}(n, C)$, which is unique up to a differential isomorphism and satisfies the Galois correspondence.

4.1. Integrable systems. The notion of integrability has a nice interpretation in terms of PPV-theory. Integrability over abstract differential fields has the same definition as over fields of analytic functions (see [1]).

Definition 4.1. Let the notation be as above.

(1) A differential system (10) is integrable if
\[
\partial_i A_j - \partial_j A_i = [A_i, A_j]
\]
for all $i, j$ with $0 \leq i, j \leq r$, where $[\cdot, \cdot]$ stands for the Lie bracket.

(2) An equation
\[
\partial_0 Y = AY, \quad A \in \text{gl}(n, k),
\]
is completely integrable if it can be completed to a system (10) with $A_0 = A$.

For completely integrable equations, PV-theory and PPV-theory are close (see [1], Lemma 9.9).

Lemma 4.2. With notation as above, assume that the field $k_0$ of $\partial_0$-constants of $k$ is $\Delta$-differentially closed, and let
\[
\partial_0 Y = AY, \quad A \in \text{gl}(n, k),
\]
(11)
be a completely integrable system, completable into an integrable system (10) as above. Then every PV-extension of $k$ for (10) is a PPV-extension of $k$ for (11).

The proof of the lemma relies on the fact that every differentially closed field is a fortiori algebraically closed, and that the field of constants of an algebraically closed differential field is itself algebraically closed. This lemma was used by Cassidy and Singer to give the following PPV-characterization of integrability ([1], Proposition 3.9).
Proposition 4.3 (Cassidy–Singer). With notation as above, assume that $k_0$ is differentially closed and let $C \subset k_0$ be the subfield of $\Delta$-constants of $k$.

(1) Equation $\text{(11)}$ is completely integrable if and only if its PPV-group over $k$ is conjugate in $\text{GL}(n,k_0)$ to the group $G(C)$ of all $C$-points of some linear algebraic group defined over $C$.

(2) In particular, the conditions of part (1) hold for $A \in \text{gl}(C)$.

4.2. Isomonodromy. Let us again consider the case of differential fields containing analytic functions. As in §3.3, we consider a parameterized system

$$\partial_x Y = A(x,t)Y,$$

where the entries of $A$ are analytic on $D \times U$ for some open polydisc $U \subset \mathbb{C}^r$ containing 0 and some open subset $D$ of $\mathbb{P}^1(\mathbb{C})$ containing 0, and the group $\pi_1(D,0)$ is generated by elementary loops $\gamma_1, \ldots, \gamma_m$. More precisely, we assume that $\mathbb{P}^1(\mathbb{C}) \setminus D$ is the union of $m$ disjoint discs $D_i$ and that for every $t \in U$ the equation $\text{(12)}$ has a unique singular point $\alpha_i(t)$ in each $D_i$ and no singular points otherwise.

Definition 4.4. Equation $\text{(12)}$ is said to be isomonodromic on $D \times U$ if there are constant matrices $M_1, \ldots, M_m \in \text{GL}(n,\mathbb{C})$ such that for every fixed $t \in U$ there is a local fundamental solution $Y_t$ of $\text{(12)}$ at 0 such that the analytic continuation $Y_t^{\gamma_i}$ of $Y_t$ along $\gamma_i$ satisfies

$$Y_t^{\gamma_i} = Y_t M_i$$

for $i = 1, \ldots, m$.

Note that $Y_t$ need not be analytic in $t$. Nevertheless, following a proof by Bolibrukh in the Fuchsian case (see [17]), one can show the existence of a solution $Y_t$ which is analytic in $t$. In particular, this argument uses the fact that $U$ is a Stein variety, on which any topologically trivial analytic bundle is analytically trivial (see [18]).

Here is a useful criterion for isomonodromy.

Theorem 4.5 (Sibuya [19]). Consider an equation $\text{(12)}$ under the same notation and assumptions as above.

(1) The equation $\text{(12)}$ is isomonodromic on $D \times U$ if and only if it is completely integrable, that is, forms a part of an integrable system

$$\begin{align*}
\partial_0 Y &= A_0 Y, \\
\partial_1 Y &= A_1 Y, \\
\cdots \\
\partial_r Y &= A_r Y
\end{align*}$$

with $A_0 = A$ and $A_i$ analytic on $D \times U$ for all $i$.

(2) Assume that $\text{(12)}$ is isomonodromic and has only parameterized regular singular points and $A$ is rational in $x$. Then all matrices $A_i$ are rational in $x$.

Cassidy and Singer [1] give an algebraic criterion for isomonodromy using PPV-theory. As above, let $\Delta = \{\partial_x, \partial_{t_1}, \ldots, \partial_{t_r}\}$ and $\Delta_t = \{\partial_{t_1}, \ldots, \partial_{t_r}\}$ be the tuples of partial differentiations with respect to $x$ and to the components of the multi-parameter $t$. 
Theorem 4.6 (Cassidy–Singer). As above, consider the equation
\[ \partial_x Y = A(x, t)Y, \]
where \( A \) has entries analytic on \( D \times U \) and rational in \( x \), with parameterized regular singular points only, one in each disc \( D_i \). Let \( k = C_0(x) \), where \( C_0 \) is the differential closure of the \( \Delta_t \)-field generated over \( \mathbb{C}(t_1, \ldots, t_r) \) by the \( x \)-coefficients of the entries (which are rational functions of \( x \)) of \( A \). The equation is isomonodromic if and only if its PPV-group over \( k \) is conjugate in \( \text{GL}(n, C_0) \) to a linear algebraic subgroup of \( \text{GL}(n, \mathbb{C}) \).

The proof of this theorem relies on Sibuya’s criterion and Proposition 4.3.

§ 5. Projective isomonodromy

As above, consider a parameterized equation
\[ \partial_x Y = A(x, t)Y \tag{13} \]
on \( D \times U \) with \( m \) isolated singular points, one in each disc \( D_i \), where \( D = \mathbb{P}^1(\mathbb{C}) \setminus \bigcup_{i=1}^m D_i \). We are now interested in a special case of the so-called monodromy evolving deformations, which was studied by Chakravarty and Ablowitz [20] and Ohyama [21], [22] in the classical case of the Darboux–Halphen equation.

Definition 5.1. Equation (13) is projectively isomonodromic if there are constant matrices \( \Gamma_1, \ldots, \Gamma_m \in \text{GL}(n, \mathbb{C}) \) and analytic functions \( c_1, \ldots, c_m \in \mathcal{O}_U \) such that for every fixed \( t \in U \) one can find a local fundamental solution \( Y_t \) of (13) at 0 such that the parameterized monodromy matrix of (13) with respect to the loop \( \gamma_i \) is equal to \( c_i(t)\Gamma_i \) for each \( i \).

As in the isomonodromic case, the solution \( Y_t \) need not be analytic in \( t \). In [23] we mimicked Bolibrukh’s proof to show the existence of a particular solution that is analytic in \( t \). We need such a solution to interpret the projective isomonodromy algebraically in terms of PPV-theory.

In the special case of a Fuchsian parameterized equation
\[ \partial_x Y = \sum_{i=1}^m \frac{A_i(t)}{x - \alpha_i(t)}Y, \tag{14} \]
the projective isomonodromy is related to isomonodromy in a natural way (see [23]).

Proposition 5.2. Equation (14) is projectively isomonodromic if and only if for every \( i \) we can write
\[ A_i(t) = B_i(t) + b_i(t)I, \]
where the scalars \( b_i \) and all entries of the matrices \( B_i \) are analytic on \( U \) and the equation
\[ \partial_x Y = \sum_{i=1}^m \frac{B_i(t)}{x - \alpha_i(t)}Y \]
is isomonodromic.
For general equations (13) with parameterized regular singularities we give in [23] an algebraic characterization of projective isomonodromy in terms of a PPV-group.

**Theorem 5.3.** With notation as before, consider a parameterized equation

\[ \partial_x Y = A(x, t)Y, \]  

where the entries of A are analytic on \( D \times U \) and rational in x. Assume that this equation has parameterized regular singular points only, one in each disc \( D_i \). Put \( k = k_0(x) \), where \( k_0 \) is the differential closure of the \( \Delta_t \)-field generated over \( \mathbb{C}(t_1, \ldots, t_r) \) by the x-coefficients of the rational functions that are entries of A. The equation (15) is projectively isomonodromic if and only if its PPV-group over \( k \) is conjugate in \( \text{GL}(n, k_0) \) to a subgroup of the group

\[ \text{GL}(n, \mathbb{C}) \cdot k_0 I \subset \text{GL}(n, k_0), \]

where \( k_0 I \) is the subgroup of scalar matrices in \( \text{GL}(n, k_0) \).

Combining topological arguments in the Kolchin topology and the Zariski topology and using Schur’s lemma, we get a corollary of this result for absolutely irreducible equations over \( k \), that is, equations that are irreducible over any finite extension of \( k \). We recall that an equation is said to be irreducible if the corresponding differential polynomial is irreducible (has no factors of strictly smaller order) or, equivalently, if its differential Galois group acts irreducibly on its solution space in any Picard–Vessiot extension.

**Corollary 5.4.** Let \( A, k_0 \) and \( k \) be as in Theorem 5.3. An absolutely irreducible equation (15) is projectively isomonodromic if and only if the commutator subgroup \((G, G)\) of its PPV-group \( G \) is conjugate in \( \text{GL}(n, k_0) \) to a subgroup of \( \text{GL}(n, \mathbb{C}) \).

§ 6. The Darboux–Halphen equation

The results of the previous section are well illustrated by the Darboux–Halphen equation. This equation describes projective isomonodromy in the same way as Schlesinger’s equation accounts for the isomonodromy (of Schlesinger type) of parameterized Fuchsian systems. The Darboux–Halphen V equation

\[ \begin{align*}
\omega_1' &= \omega_2 \omega_3 - \omega_1 (\omega_2 + \omega_3) + \phi^2, \\
\omega_2' &= \omega_3 \omega_1 - \omega_2 (\omega_3 + \omega_1) + \theta^2, \\
\omega_3' &= \omega_1 \omega_2 - \omega_3 (\omega_1 + \omega_2) - \theta \phi, \\
\phi' &= \omega_1 (\theta - \phi) - \omega_3 (\theta + \phi), \\
\theta' &= -\omega_2 (\theta - \phi) - \omega_3 (\theta + \phi)
\end{align*} \]  

occurs in physics as a reduction of the self-dual Yang–Mills equation (SDYM). When \( \theta = \phi \), the equation (DH V) is equivalent to Einstein’s self-dual vacuum equations. When \( \theta = \phi = 0 \), it coincides with Halphen’s original equation (H II) solving a geometry problem of Darboux about orthogonal surfaces.

In contrast with other SDYM reductions (such as the Painlevé equations), the equation (DH V) does not possess the Painlevé property since it has a boundary of movable essential singularities. It is therefore not likely to describe isomonodromy.
6.1. History of the DH equation. Halphen’s equation (H II) goes back to Darboux’s work [24], [25] on orthogonal systems of surfaces. Darboux’s original problem was the following.

Problem 1. What condition on a given pair \((\mathcal{F}_1, \mathcal{F}_2)\) of orthogonal families of surfaces in \(\mathbb{R}^3\) implies that there exists a family \(\mathcal{F}_3\) such that \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\) is a tri-orthogonal system of pairwise-orthogonal families?

In [24] Darboux gave a necessary and sufficient condition on \((\mathcal{F}_1, \mathcal{F}_2)\) to solve the problem: the intersection of any surfaces \(S_1 \in \mathcal{F}_1\) and \(S_2 \in \mathcal{F}_2\) must be a curvature line of both \(\mathcal{F}_1\) and \(\mathcal{F}_2\). The necessity of this condition was already known as Dupin’s theorem (1813).

Problem 2. What condition on the parameter \(u = \varphi(x, y, z)\) of a one-parameter family \(\mathcal{F}\) of surfaces in \(\mathbb{R}^3\) implies that \(\mathcal{F}\) belongs to a tri-orthogonal system \((\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)\) of pairwise-orthogonal families?

In [25] Darboux found and solved a third-order partial differential equation satisfied by \(u\) and, using previous work by Bonnet and Cayley, obtained the general solution from a particular family of ruled helicoidal surfaces. Cartan [26] later used his exterior differential calculus to prove that Problem 2 has a solution. He also generalized the problem, replacing orthogonality by any prescribed angle and considering \(p\) pairwise-orthogonal families of hypersurfaces in the \(p\)-dimensional space.

Darboux stated yet another problem on orthogonal surfaces.

Problem 3. Given two families \(\mathcal{F}_1\) and \(\mathcal{F}_2\) consisting each of parallel surfaces, does there exist a family \(\mathcal{F}\) orthogonal to both \(\mathcal{F}_1\) and \(\mathcal{F}_2\)?

It is an easy exercise to prove that any solution consists either of planes or of ruled quadrics. If \(\mathcal{F}\) consists of quadrics with a centre, then they have simultaneously reducible equations

\[
\frac{x^2}{a(u)} + \frac{y^2}{b(u)} + \frac{z^2}{c(u)} = 1,
\]

which depend on the parameter \(u = \varphi(x, y, z)\) of \(\mathcal{F}\). One can show that \(\mathcal{F}\) solves Problem 3 if and only if \(a, b, c\) satisfy the Darboux equations

\[
a(b' + c') = b(c' + a') = c(a' + b'),
\]

where \(a', b', c'\) are the derivatives with respect to \(u\). Darboux could not solve the problem though: “These equations do not seem to be integrable by known procedures” (Darboux, 1878).

He gave up on this part of the problem and restricted his study to centreless quadrics. He solved the particular case of the family \(\mathcal{F}\) of paraboloids

\[
\frac{y^2}{\alpha + u} + \frac{z^2}{\alpha - u} = 2x + \alpha \log u
\]

and claimed that some surfaces of revolution solve the problem as well.
In 1881 Halphen [27], [28] completely solved Darboux’s second problem in the following form:

\[
\begin{align*}
\omega'_1 + \omega'_2 &= \omega_1\omega_2, \\
\omega'_2 + \omega'_3 &= \omega_2\omega_3, \\
\omega'_3 + \omega'_1 &= \omega_3\omega_1,
\end{align*}
\]

known as the Halphen I equation, and solved the more general QHDS (quadratic homogeneous differential system) of the form

\[
\begin{align*}
\omega'_1 &= a_1\omega_1^2 + (\lambda - a_1)(\omega_1\omega_2 + \omega_3\omega_1 - \omega_2\omega_3), \\
\omega'_2 &= a_2\omega_2^2 + (\lambda - a_2)(\omega_2\omega_3 + \omega_1\omega_2 - \omega_3\omega_1), \\
\omega'_3 &= a_3\omega_3^2 + (\lambda - a_3)(\omega_3\omega_1 + \omega_2\omega_3 - \omega_1\omega_2),
\end{align*}
\]

known as the Halphen II equation, by means of hypergeometric functions. He considered even more general QHDSs of the form

\[
\{\omega'_r = \psi_r(\omega_1, \ldots, \omega_l)\}_{r=1,\ldots,l}
\]

with quadratic forms \(\psi_r\) under some additional symmetry conditions. Special examples of such QHDSs are the equation (DH V) above and its particular form (H II), which we consider now.

### 6.2. Application of PPV-theory to the Darboux–Halphen equation.

As shown in [21], the equation (H II) is equivalent to the system

\[
x'_i = Q_i(x_1, x_2, x_3), \quad i = 1, 2, 3,
\]

where \(Q_i(x_1, x_2, x_3) = x_i^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2\) for some constants \(a, b, c\).

The equation (H II) is in fact the integrability condition of the Lax pair

\[
\begin{align*}
\frac{\partial Y}{\partial x} &= \left(\frac{\mu I}{(x - x_1)(x - x_2)(x - x_3)} + \sum_{i=1}^{3} \frac{\lambda_i C}{x - x_i}\right) Y, \\
\frac{\partial Y}{\partial t} &= \left(\nu I + \sum_{i=1}^{3} \lambda_i x_i C\right) Y - Q(x) \frac{\partial Y}{\partial x},
\end{align*}
\]

where

\[
Q(x) = x^2 + a(x_1 - x_2)^2 + b(x_2 - x_3)^2 + c(x_3 - x_1)^2
\]

and \(x_i = x_i(t)\) are the parameterized (simple) singularities. Moreover, \(C\) is a constant traceless \(2 \times 2\) matrix, \(I\) is the identity matrix, \(\mu \neq 0\) and the \(\lambda_i\) are constants such that \(\lambda_1 + \lambda_2 + \lambda_3 = 0\) (hence there is no singular point at \(\infty\)), and the function \(\nu\) is a solution of the equation

\[
\frac{\partial \nu}{\partial x} = -\mu \frac{x + x_1 + x_2 + x_3}{(x - x_1)(x - x_2)(x - x_3)}.
\]

Since the solutions of the last equation are not rational in \(x\), Sibuya’s criterion yields that the equation (16) is not isomonodromic. To describe the monodromy
of (16), we fix a fundamental solution $Y$ of the Lax pair at some point $x_0$ lying outside the fixed discs $D_i$ with centres $x_i(t)$, for all $i$. Note that $Y$ must be analytic in both $x$ and $t$. A computation shows that the parameterized monodromy matrix of (16) with respect to $Y$ and $x_i(t)$ is

$$M_i(t) = \exp\left\{-2\pi \sqrt{-1} \mu \int_{t_0}^{t} \beta_i(t) \, dt\right\} \exp\left\{2\pi \sqrt{-1} L_i(t_0)\right\},$$

where $L_i(t)$ is an analytic function of $t$ such that, for some fundamental solution $Y_0$ of (16) in the neighbourhood of a given non-singular point $x_0$, the analytic extension of $Y_0$ to a neighbourhood of $x_i(t)$ is

$$Y(t, x) = Y_i(t, x - x_i(t))(x - x_i(t))^{L_i(t)}$$

with a single-valued function $Y_i$. The coefficients $\beta_i(t)$ are given by the formula

$$\frac{x + \sum_{i=1}^{3} x_i}{\prod_{i=1}^{3}(x - x_i(t))} = \sum_{i=1}^{3} \frac{\beta_i(t)}{x - x_i(t)}.$$

For each $i$, the monodromy matrix is of the form

$$M_i(t) = c_i(t)M_i(t_0),$$

where

$$c_i(t) = \exp\left\{-2\pi \sqrt{-1} \mu \int_{t_0}^{t} \beta_i(t) \, dt\right\}, \quad M_i(t_0) = \exp\left\{2\pi \sqrt{-1} L_i(t_0)\right\}.$$

Hence the equation (16) is projectively isomonodromic. Moreover, it is an example of a Fuchsian projectively isomonodromic equation. Proposition 5.2 applies to this equation since it can be written in the form

$$\frac{\partial Y}{\partial x} = \left(\sum_{i=1}^{3} \frac{A_i(t)}{(x - x_i)}\right)Y,$$

where

$$A_i(t) = B_i(t) + b_i(t)I, \quad B_i(t) = \lambda_i C, \quad b_i(t) = \frac{\mu}{\prod_{j \neq i}(x_i - x_j)},$$

and where the equation

$$\frac{\partial Y}{\partial x} = \left(\sum_{i=1}^{3} \frac{\lambda_i C}{(x - x_i)}\right)Y$$

is clearly isomonodromic.
7. Inverse problems

7.1. A parameterized version of the weak Riemann–Hilbert problem.

In [5] we adapted Bolibrukh’s techniques and construction of holomorphic bundles (see [29]–[34]) to give a parameterized version of the weak Riemann–Hilbert problem.

**Theorem 7.1.** Let $S = \{a_1, \ldots, a_s\}$ be a finite subset of $\mathbb{P}^1(\mathbb{C})$, $D$ an open polydisc in $\mathbb{C}^r$, $\gamma_1, \ldots, \gamma_s$ generators of $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus S; a_0)$ for some fixed base point $a_0 \in \mathbb{P}^1(\mathbb{C}) \setminus S$, and $M_i : D \to \text{GL}_n(\mathbb{C})$, $i = 1, \ldots, s$, analytic maps with $M_1 \cdot \ldots \cdot M_s = I_n$. Then there is a parameterized linear differential system

$$\partial_x Y = A(x, t)Y$$

with matrix $A \in \text{gl}_n(\mathcal{O}_{D'}(x))$ for some open polydisc $D' \subset D$, with only regular singular points, all in $S$, such that for some parameterized fundamental solution, the parameterized monodromy matrix along each $\gamma_i$ is equal to $M_i$. Furthermore, for every fixed $a_i \in \{a_1, \ldots, a_s\}$ the entries of $A$ may be chosen to have at worst simple poles at all points $a_j \neq a_i$.

The proof here, as in the non-parameterized case, relies on a parameterized version of the Birkhoff–Grothendieck theorem (see [35], Proposition 4.1, [36], Theorem 2 and [37], Theorem A.1).

7.2. The inverse problem of PPV-theory. Again in analogy with the non-parameterized case, we deduced in [5] the following consequence of the parameterized versions of Schlesinger’s theorem (see Theorem 3.7) and the theorem on the solubility of the weak Riemann–Hilbert problem (see Theorem 7.1). As above, let $t = (t_1, \ldots, t_r)$ be a multi-parameter, and let $\Delta_t = \{\partial_{t_1}, \ldots, \partial_{t_r}\}$ be the corresponding partial derivations. We consider the differential field $k = k_0(x)$, where $k_0$ is a differentially closed $\Delta_t$-field containing $\mathbb{C}(t_1, \ldots, t_r)$, and endow $k$ with the derivations $\Delta = \{\partial_x, \partial_{t_1}, \ldots, \partial_{t_r}\}$.

**Theorem 7.2.** Let $G$ be a linear $\Delta_t$-differential algebraic group defined over $k_0$. Assume that $G(k_0)$ contains a finitely generated Kolchin-dense subgroup $H$. Then $G(k_0)$ is the PPV-group of some PPV-extension of $k = k_0(x)$.

The condition in Theorem 7.2 that $G(k_0)$ contains a finitely generated Kolchin-dense subgroup $H$ actually characterizes those linear differential algebraic groups over $k_0$ which are PPV-groups. The necessity of this condition was proved by Dreyfus [8] as a consequence of his parameterized version of the Ramis density theorem (see, for example, [12], Theorem 8.10, and [55], Theorem 2.47). The Ramis theorem says that the local (at 0) differential Galois group over $\mathbb{C}(\{x\})$ of a linear differential system of order $n$ is equal to the Zariski closure in $\text{GL}(n, \mathbb{C})$ of a subgroup finitely generated by the so-called formal monodromy, Stokes matrices and exponential torus, together also called the generalized monodromy data, which generalize to irregular singularities the notion of monodromy matrices for regular singularities.

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1On March 31, 1998, Andrey Bolibrukh introduced his weekly graduate course in Strasbourg with the poetic injunction “Récitons les fruits de ce théorème”, about the classical Birkhoff–Grothendieck theorem.
Moreover, it can be proved that the (global) differential Galois group over \( \mathbb{C}(x) \) of a linear differential system is equal to the Zariski closure of the subgroup generated by the finitely many ‘local’ differential Galois groups just mentioned, which can be simultaneously embedded as subgroups in the global PV-group. Dreyfus [8] defined a parameterized version of the generalized monodromy data and gives a parameterized version of this theorem, which in turn gives the converse result of Theorem 7.2.

In the non-parametric case, the solution of the differential Galois inverse problem over \( \mathbb{C}(x) \) by Tretkoff and Tretkoff [38] uses the fact, proved by the same authors, that every linear algebraic group over an algebraically closed field of characteristic zero is the Zariski closure of some finitely generated subgroup. However, the latter does not hold for linear differential algebraic groups. This can in particular be seen in the case of the additive group \( \mathbb{G}_a(k_0) \) (using the same notation as above for the differential field \( k_0 \)), which has the striking property that the Kolchin closure of any of its finitely generated subgroups is a proper subgroup of \( \mathbb{G}_a(k_0) \) (see [5]). It was also shown in [39] and [1] that neither \( \mathbb{G}_a(k_0) \) nor \( \mathbb{G}_m(k_0) \) is the PPV-group of a PPV-extension of \( k_0(x) \). Using Corollary 7.2, Singer [40] proved the following result.

**Theorem 7.3.** With notation as above, a linear algebraic group \( G \) defined over \( k_0 \) is the PPV-group of some PPV-extension of \( k_0(x) \) if and only if the identity component of \( G \) has no quotient isomorphic to \( \mathbb{G}_a(k_0) \) or \( \mathbb{G}_m(k_0) \).

More recently Minchenko, Ovchinnikov and Singer [41] gave a characterization of those unipotent linear differential algebraic groups that can be realized as PPV-groups.

**Theorem 7.4** (Minchenko, Ovchinnikov, Singer). A unipotent linear differential algebraic group \( G \) over \( k_0 \) is the Kolchin closure of a finitely generated subgroup if and only if it has differential type 0.

Here the condition ‘\( G \) has differential type 0’ means that the so-called ‘differential dimension’ of \( G \) is finite. The latter is defined as the transcendence degree over \( k_0 \) of the ‘differential function field’ \( k_0\langle G^0 \rangle \), where \( G^0 \) is the identity component of \( G \). If \( G \subset \text{GL}(n,k_0) \), then \( k_0\langle G^0 \rangle \) is the quotient field of the differential coordinate ring \( R/I \) of the group \( G^0 \). More precisely, \( R/I \) is the quotient of the ring of differential polynomials \( k_0\{y_1,1,\ldots,y_{n,n}\} \) in \( n^2 \) differential indeterminates (differential with respect to \( \Delta_1 \)) by the differential ideal \( I \) of those differential polynomials that vanish on \( G^0 \).

The same authors gave in [42] a characterization of those reductive linear differential algebraic groups that can occur as PPV-groups over \( k_0(x) \). In both [41] and [42] the authors gave algorithms to determine whether the PPV-group is of the relevant type and to compute this group if it is.

**§ 8. Appendix**

Let \( (K,\partial) \) be an ordinary differential field, and let \( K\{X\} \) be the differential ring of differential polynomials in one differential variable. By definition, \( K\{X\} \) is the ring \( K[X_0,X_1,\ldots,X_n,\ldots] \) of polynomials in the indeterminates \( X_0,X_1,\ldots,X_n,\ldots \), with the derivation \( \partial \) extended by the formula \( \partial X_i = X_{i+1} \).
for all \( i \geq 0 \). In \( K\{X\} \) one writes \( X \) for \( X_0 \), \( X' \) for \( X_1 \) and \( X^{(i)} := \partial^{(i)} X = X_i \) for each \( i \). The order \( o(f) \) of an element \( f \in K\{X\} \) is defined as the least integer \( n \) such that \( f \in K[X_0, X_1, \ldots, X_n] \) (if \( f \notin K \)) or by the formula \( o(f) = -1 \) (if \( f \in K \)). The basic facts and model-theoretic properties of the theory DCF of differentially closed fields are given, for example, in [43]–[47].

The following definition is close to the definition of algebraic closedness. It is due to Blum [48], who simplified an earlier definition introduced by Robinson [49].

**Definition 8.1** (Blum). The differential field \((K, \partial)\) is said to be **differentially closed** if for any \( f, g \in K\{X\} \) with \( f \notin K \) and \( o(g) < o(f) \) there is an \( a \in K \) such that \( f(a) = 0 \) and \( g(a) \neq 0 \).

This definition is well illustrated by Example 1.1 above:

\[
\frac{dy}{dx} = \frac{t}{x^3}.
\]

Let us show that over \( K(x) \), where \( K \) is a differentially closed field containing \( \mathbb{C}(t) \), the obstruction (described in this example) to Galois correspondence vanishes. We recall that the PPV-extension of this equation over \( K(x) \) is \( K(x, x^t, \log x) \) and that all elements \( \sigma \) of the PPV-group are of the form

\[
\sigma(x^t) = a_{\sigma} x^t, \quad \sigma(\log x) = \log x + c_{\sigma},
\]

where the coefficients \( a_{\sigma} \in K^* \) satisfy the equalities

\[
a_{\sigma}'' a_{\sigma} - a_{\sigma}'^2 = 0, \quad c_{\sigma} = \frac{a_{\sigma}'}{a_{\sigma}}
\]

and \( a_{\sigma}', a_{\sigma}'' \) are derivatives with respect to the differentiation extending \( d/dt \).

To avoid the situation when \( \log x \) is invariant under the PPV-group (in which case the invariant field of the PPV-group does not coincide with the base field \( K(x) \)), we need at least one element \( \sigma \) with \( \sigma(\log x) \neq \log x \). The coefficient \( a_{\sigma} \in K^* \) must satisfy

\[
a_{\sigma}'' a_{\sigma} - a_{\sigma}'^2 = 0, \quad \frac{a_{\sigma}'}{a_{\sigma}} \neq 0.
\]

Since \( K \) is differentially closed, such an element exists by Definition 8.1 applied to \( f(X) = X'' X - X'^2 \) and \( g(X) = X' \).

The definition of general (non-ordinary) differentially closed fields is due to Kolchin [50], who called them ‘constrainedly closed’. For ordinary differential fields this definition is equivalent to Definition 8.1.

**Definition 8.2** (Kolchin). Let \( K \) be a differential \( \Delta \)-field endowed with a finite set \( \Delta \) of commuting derivations on \( K \). The \( \Delta \)-field \( K \) is **differentially closed** if it has no proper constrained extensions.

The definition of constrained extensions is the following.

**Definition 8.3.** Let \( K \) be a differential \( \Delta \)-field. A differential extension \( L \) of \( K \) is said to be **constrained** if for every family of elements \( (\eta_1, \ldots, \eta_s) \) of \( L \) there is a \( \Delta \)-differential polynomial \( P \in K\{y_1, \ldots, y_s\} \) such that \( P(\eta_1, \ldots, \eta_s) \neq 0 \) whereas \( P(\zeta_1, \ldots, \zeta_s) = 0 \) for every non-generic differential specialization \( (\zeta_1, \ldots, \zeta_s) \) of \( (\eta_1, \ldots, \eta_s) \) over \( K \).
In Kolchin’s terminology, a differential specialization $\zeta = (\zeta_1, \ldots, \zeta_s)$ of a tuple $\eta = (\eta_1, \ldots, \eta_s)$ in some extension of $K$ is said to be generic if the defining ideals of $\zeta$ and $\eta$ in $K\{y_1, \ldots, y_s\}$ are the same. We refer to Kolchin’s original work for details about these notions (see [3], [50], [51]).

**Definition 8.4.** Let $K$ be a $\Delta$-field. A differential closure of $K$ is a differentially closed $\Delta$-extension of $K$ which can be embedded in any differentially closed differential extension of $K$.

**Theorem 8.5.** Every differential field $K$ has a unique differential closure.

This result was proved by Morley [52], Blum [48], Shelah [53] and Kolchin [50]. However, unlike the algebraic closure, the differential closure fails to be minimal, even in characteristic 0. Although it had been conjectured by some authors to be minimal (see [54]), Kolchin, Rosenlicht and Shelah proved independently that it is not. In particular, Shelah [53] proved that the ordinary differential closure $\tilde{Q}$ of $Q$ is not minimal by exhibiting an infinite strictly decreasing sequence of differentially closed intermediate differential extensions of $Q$ in $\tilde{Q}$.

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Claude Mitschi
Institut de Recherche Mathématique Avancée,
Université de Strasbourg, France
E-mail: mitschi@math.unistra.fr

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