On mean and/or variance mixtures of normal distributions

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Abstract

Parametric distributions are an important part of statistics. There is now a voluminous literature on different fascinating formulations of flexible distributions. We present a selective and brief overview of a small subset of these distributions, focusing on those that are obtained by scaling the mean and/or covariance matrix of the (multivariate) normal distribution with some scaling variable(s). Namely, we consider the families of mean mixture, variance mixture, and mean-variance mixture of normal distributions. Its basic properties, some notable special/limiting cases, and parameter estimation methods are also described.

1 Introduction

The normal distribution plays a central role in statistical modelling and data analysis, but real data rarely follow this classical distribution. The quest for more flexible distributions has led to an ever-growing development in the literature of parametric distribution. In the past two decades or so, intense interest has been in the area of skew or asymmetric distributions; see, for example, the book edited by Genton (2004), the monograph by Azzalini and Capitanio (2014), and the papers by Azzalini (2005), Arellano-Valle and Azzalini (2006) and Adcock and Azzalini (2021) for recent accounts of the literature on skew distributions. Many of these formulations belong to the class of skew-symmetric distributions, which is a generalization of the classical skew normal (SN) distribution by Azzalini and Dalla Valle (1996). This SN distribution can be characterized as a mean mixture of normal (MMN) distribution, where the mean of a normal random variable is scaled by a truncated normal random variable (Negarestani et al., 2019). Another related and extensively studied family of distributions that can render asymmetric distributional shapes is the mean-variance mixture of normal (MVMN) distribution. Introduced by Barndorff-Nielsen et al. (1982), the MVMM distribution is obtained by scaling both mean and variance of a normal random variable with the same (positive scalar) random variable.

This paper presents a brief overview of flexible distributions that arise from scaling either/both the mean and variance of a normal random variable. For simplicity, we focus on the case of a univariate scaling variable. Apart from the aforementioned MMN and MVMN families, a third family called variance mixture of normal (VMN) distributions
can be defined by scaling only the variance of a normal random variable. Although VMN does not produce asymmetric distributions (at least not in the case of a scalar scaling variable), we include this family in this paper for completeness.

Following conventional notation, a $p$-dimensional random vector $Y$ is said to follow a (multivariate) normal distribution, denoted by $Y \sim N_p(\mu, \Sigma)$, if its density is given by

$$
\phi_p(y; \mu, \Sigma) = (2\pi)^{-\frac{p}{2}}|\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(y-\mu)^\top \Sigma^{-1} (y-\mu)},
$$

where $\mu$ is a $p \times 1$ vector of location parameters and $\Sigma$ is a $p \times p$ positive definite symmetric matrix of scale parameters. The mean and variance of $Y$ are $E(Y) = \mu$ and $\text{cov}(Y) = \Sigma$, respectively. The vector $Y$ can be expressed as a location-scale variant of a standard normal random variable, that is,

$$
Y = \mu + \Sigma^{1/2} Z,
$$

where $Z \sim N_p(0, I_p)$, $0$ is a vector of zeros, and $I_p$ is the $p \times p$ identity matrix. By ‘scaling’ or ‘mixing’ $Y$, we mean that $\mu$ is mixed with $W$ and/or $\Sigma$ is weighted by $\sqrt{W}$, where $W$ is a positive random variable independent of $Z$. We consider each of these cases in Sections 2 to 4. By adopting a range of different distributions for $W$, a wide variety of non-normal distributions can be constructed.

## 2 Variance mixture of normal distributions

Variance mixture, or scale mixture, of normal (VMN) distributions refers to the family of distributions generated by scaling the variance matrix $\Sigma$ in (1) with a (scalar) positive scaling variable $W$. More formally, it refers to distributions with the following stochastic representation,

$$
Y = \mu + \sqrt{W} \Sigma^{1/2} Z,
$$

where $Z \sim N_p(0, I_p)$ and $W$ are independent. Let the density of $W$ be denoted by $h(w; \theta)$, where $\theta$ is the vector of parameters associated with $W$. It follows that the density is in the form of an integral given by

$$
f(y; \mu, \Sigma, \theta) = \int_0^\infty \phi_p(y; \mu, W\Sigma) h(w; \theta) dw.
$$

A similar expression to (4) above can be given in the case where $W$ has a discrete distribution; see, for example, equation (3) of Lee and McLachlan (2013). As can be observed from (4), the family of VMN distributions have constant mean but variable scale depending on $W$. This allows the VM distributions to have lighter or heavier tails than the normal distribution and thus are suitable for modelling data with tails thickness that deviate from the normal. However, this distribution in the unimodal family remain symmetric in shape.

### 2.1 Properties

The moments of VMN distributions can be readily obtained from (3). For example, the first and second moments of $Y$ are given by, respectively, $E(Y) = \mu$ and $\text{cov}(Y) = E(W)\Sigma$. Further, the moment generating function (mgf) of $Y$ can be expressed as

$$
M_Y(t) = e^{t^\top \mu} M_W \left( \frac{1}{2} t^\top \Sigma t \right),
$$
where $M_W(\cdot)$ denotes the mgf of $W$.

Some nice properties of the normal distribution remain valid for VMN distributions, including closure under affine transformation, marginalization, and conditioning. Let $Y \sim VMN_q(\mu, \Sigma; h(w; \theta))$ denotes $Y$ having the density (3). Let also $A$ be a $q \times p$ matrix of full row rank and $a$ be a $q$-dimensional vector. Then the affine transformation $AY + a$ still has a VMN distribution with density

$$AY + a \sim VMN_q(A\mu + b, A\Sigma^\top A; h(w; \theta)).$$

Furthermore, if $X \sim VMN_q(\mu^*, \Sigma^*; h(w; \theta))$ is independent of $Y$, then the linear combination $AY + X$ has density given by

$$AY + X \sim VMN_q(A\mu + \mu^*, A\Sigma^\top A + \Sigma^*; h(w; \theta)).$$

Suppose $Y$ can be partitioned as $Y^\top = (Y_1^\top, Y_2^\top)$ with respective dimensions $p_1$ and $p_2$ where $p_1 + p_2 = p$. Accordingly, let $\mu^\top = (\mu_1^\top, \mu_2^\top)$ and $\Sigma$ be partitioned into four block matrices $\Sigma_{11}, \Sigma_{12}, \Sigma_{21}$ and $\Sigma_{22}$. Then the marginal density of $Y_1$ is $VMN_{p_1}(\mu_1, \Sigma_{11}; h(w; \theta))$ and the conditional density of $Y_2 | Y_1 = y_1$ is $VMN_{p_2}(\mu_{1,2} + \Sigma_{21}^{-1}(y_2 - \mu_2), \Sigma_{22})$, where $\mu_{1,2} = \mu_1 + \Sigma_{11}^{-1}(y_1 - \mu_1)$ and $\Sigma_{11} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$.

### 2.2 Special cases

The family of VMN distribution encompasses many well-known distributions, including the $t$, Cauchy, symmetric generalized hyperbolic, and logistic distributions. The slash, Pearson type VII, contaminated normal, and exponential power distributions can also be represented as a VMN distribution; see also [Andrews and Mallows (1974)] and [Lee and McLachlan (2019)] for some other special cases of VMN distributions.

The $(p \times p)$-dimensional $t$-distribution can be obtained by letting $W \sim IG\left(\frac{p}{2}, \frac{\nu}{2}\right)$ in (3), where $IG(\cdot)$ denotes the inverse gamma distribution and $\nu$ is a scalar parameter commonly known as the degrees of freedom. This tuning parameter regulates the thickness of the tails of the $t$-distribution, allowing it to model heavier tails than the normal distribution. The Cauchy and normal distributions are special/limiting cases of the $t$-distribution (by letting $\nu = 1$ and $\nu \to \infty$, respectively).

The (symmetric) generalized hyperbolic distribution is another important special case of the VMN distribution. It arises when $W$ follows a generalized inverse Gaussian (GIG) distribution, which includes the IG distribution as a special case. Thus, the above mentioned $t$-distribution and its nested cases are also members of the symmetric generalized hyperbolic distribution.

### 2.3 Parameter estimation

From (3), a VMN distribution can be expressed in a hierarchical form given by $Y|W = w \sim N_p(\mu, w\Sigma)$ and (with a slight abuse of notation) $W \sim h(w; \theta)$. This facilitates maximum likelihood estimation of the model parameters via the Expectation-Maximization (EM) algorithm [Dempster et al. (1977)]. Technical details can be found in many reports, for example, [Lange and Sinsheimer (1993)].

### 3 Mean-mixture of normal distributions

Rather than weighting $\Sigma$ with $W$, the mean-mixture (or location-mixture) of normal (MMN) distribution [Negarestani et al. (2019)] is obtained by mixing $\mu$ with $W$. Note
that, in general, \( W \) need not be a positive random variable in the case MMN distribution. More formally, the MMN distribution arises from the stochastic expression

\[
Y = \mu + W\delta + \Sigma^{\frac{1}{2}}Z,
\]

where \( \delta \) is \( p \times 1 \) vector of shape parameters. The density \( f(Y) \) is asymmetric if \( W \) has an asymmetric distribution. In this case, \( \delta \) may be interpreted as a vector of skewness parameters. A prominent example is the (positively) truncated normal or half-normal distribution, that is, \( W \sim TN(0,1;\mathbb{R}^+) \). This leads to the classical characterization of the skew normal (SN) distribution proposed by Azzalini and Dalla Valle (1996). It should be noted that while the MMN distribution includes the SN distribution as a special case, some other commonly used skew-elliptical distributions such as the skew-

Concerning the marginal and conditional distributions of MMN random variables, let \( Y_1, \mu, \) and \( \Sigma \) be partitioned as in Section 2.1. Similarly, partition \( \lambda \) into \( \lambda^\top = (\lambda_1^\top, \lambda_2^\top) \). Then the marginal density of \( Y_1 \) is \( MMN_p(\mu_1, \Sigma_{11}, \lambda_1; h(w; \theta)) \) and the conditional density of \( Y_1 | Y_2 = y_2 \) is \( MMN_p(\mu_{1,2}, \Sigma_{11,2}, \lambda_{1,2}; h(w; \theta)) \), where \( \lambda_{1,2} = \lambda_1 - \Sigma_{12} \Sigma_{22}^{-1} \lambda_2 \), and \( \mu_{1,2} \) and \( \Sigma_{11,2} \) are defined in Section 2.1.
3.2 Special cases

As mentioned previously, taking \( W \sim TN(0,1;\mathbb{R}^+) \) leads to the classical SN density given by

\[
f(y; \mu, \Sigma, \delta) = 2 \phi_p(y; \mu, \Omega) \Phi_1(\delta^\top \Omega^{-1}(y - \mu); 0, 1 - \delta^\top \Omega^{-1} \delta),
\]

where \( \Omega = \Sigma + \delta \delta^\top \) and \( \Phi_1(\cdot; \mu, \sigma^2) \) denotes the corresponding distribution function of \( \phi_1(\cdot; \mu, \sigma^2) \). When \( \delta = 0 \), the SN distribution reduces to the (multivariate) normal distribution.

Another special case of the MMN distribution were presented in Negarestani et al. (2019). Taking \( W \) to have a standard exponential distribution, that is \( W \sim \text{exp}(1) \), leads to the MMN exponential (MMNE) distribution. It can be shown that the density is given by

\[
f(y; \mu, \Sigma, \delta) = \sqrt{2\pi} e^{\frac{\beta^2}{2}} \Phi_p(y; \mu, \Sigma) \Phi_1(\beta),
\]

where \( \alpha^2 = \delta^\top \Sigma^{-1} \delta \) and \( \beta = \alpha^{-1} [\delta^\top \Sigma^{-1}(y - \mu) - 1] \). For further details and properties of the MMNE distribution, the reader is referred to Section 8.1 in Negarestani et al. (2019).

3.3 Parameter estimation

Utilizing the hierarchical representation (10), EM algorithm can be implemented to provide maximum likelihood estimates of the model parameters. Although the technical details for this EM algorithm are not given in Negarestani et al. (2019), it is analogous to the univariate case presented in Section 4 of the above reference.

4 Mean-variance mixture of normal distributions

The mean-variance mixture of normal (MVMN) distribution, sometimes called the location-scale mixture of normal distribution, is a generalization of the VMN distribution described in Section 2. Compared to (3), the scaling variable \( W \) is now also mixed with \( \mu \) like in the case of the MMN distribution. The MVN distribution has the following stochastic representation

\[
Y = \mu + W \delta + \sqrt{W} \Sigma^{\frac{1}{2}} Z.
\]

In this case, both the location and scale of the distribution vary with \( W \). Moreover, \( W \) is a positive random variable and hence the MVN distribution is asymmetric when \( \delta \neq 0 \). It is important to note that while the MVMN distribution reduces to the VMN distribution when \( \delta = 0 \), the MMN distribution described in Section 3 is not a special case of the MVMN distribution.

Following the definition (10), the density of \( p \)-dimensional MVMN distribution can be expressed as

\[
f(y; \mu, \Sigma, \delta; h(w; \theta)) = \int_0^\infty \phi_p(y; \mu + w\delta, w\Sigma) h(w; \theta) dw.
\]

The notation \( Y \sim MVMN_p(\mu, \Sigma, \delta; h(w; \theta)) \) will be used when \( Y \) has density in the form of (17). Analogous to the VMN and MMN distributions, the MMN distribution can be conveniently expressed in a hierarchical form given by

\[
Y|W = w \sim N_p(\mu + w\delta, w\Sigma) \perp W \sim h(w; \theta).
\]
4.1 Properties

Some basic properties of the MVMN distribution have been studied in Barndorff-Nielsen et al. (1982), among other works. The moments of $Y \sim \text{MVMN}_p(\mu, \Sigma; h(w; \theta))$ can be derived directly from (16). Specifically, the first two moments of $Y$ are given by $E(Y) = \mu + E(W)\delta$ and $\text{cov}(Y) = \text{var}(W)\delta\delta^\top + E(W)\Sigma$, respectively. Further, the mgf of $Y$ is given by

$$M_Y(t) = e^{t^\top \mu M_W \left( t^\top \delta + \frac{1}{2} t^\top \Sigma t \right)}. \quad (19)$$

As can be expected, the MVMN distribution shares certain nice properties with the VMN distribution such as closure under linear transformation and marginalization. Let $A$ be a $q \times p$ matrix of full row rank, and $a$ be a $q$-dimensional vector. Then the affine transformation $AY + a$ remains a MVMN distribution with density

$$AY + a \sim \text{MVMN}_q(A\mu + b, A\Sigma^\top A, A\delta; h(w; \theta)). \quad (20)$$

Similar to the MN distribution, a linear combination of a MVMN and a normal random variable remains a MVMN random variable. If $X \sim N_q(\mu^*, \Sigma^*)$ is independent of $Y$, then the linear combination $AY + X$ has density given by

$$AY + X \sim \text{MVMN}_q(A\mu + \mu^*, A\Sigma^\top A + \Sigma^*, A\delta; h(w; \theta)). \quad (21)$$

Marginal distributions and conditional distributions of MVMN random variables can also be derived. Let $Y$, $\mu$, $\Sigma$, and $\delta$ be partitioned as in Section 3.1. Then the marginal density of $Y_1$ is $\text{MVMN}_{p_1}(\mu_1, \Sigma_{11}, \lambda_1; h(w; \theta))$ and the conditional density of $Y_1|Y_2 = y_2$ is $\text{MVMN}_{p_1}(\mu_{12}, \Sigma_{112}, \lambda_{12}; h(w; \theta))$, where $\lambda_{12}$, $\mu_{12}$, and $\Sigma_{112}$ are defined in Section 3.1.

4.2 Special cases

Perhaps the most well-known special case of the MVMN distribution is the generalized hyperbolic (GH) distribution, which is widely applied in finance and other fields. This distribution is obtained by letting $W \sim GIG(\psi, \chi, \lambda)$, yielding the following density (McNeil et al., 2005).

$$f(y; \mu, \Sigma, \delta, \psi, \chi, \lambda) = \frac{\psi^{\frac{\chi}{2}} K_\lambda(-\frac{\chi}{2})}{(2\pi)^{\frac{p}{2}} |\Sigma|^2} \left( \psi + d_\delta (\chi + d_y) \right)^\frac{\chi - \frac{p}{2}}{2}, \quad (22)$$

where $d_\delta = \delta^\top \Sigma^{-1} \delta$, $d_y = (y - \mu)^\top \Sigma^{-1} (y - \mu)$, and $K_\lambda(\cdot)$ denotes the modified Bessel function of the third kind with index $\lambda$. The GH distribution, as the name suggests, contains the symmetric GH distribution mentioned in Section 2.2 and an asymmetric version of some of its members. However, it cannot obtain the SN distribution as a special/limiting case. Other noteworthy special cases of the GH distribution include the normal inverse Gaussian, variance gamma, and asymmetric Laplace distributions. The GH distribution and its properties have been well studied in the literature; see, for example, Iversen (1999) and Deng and Yao (2018).

Two other less well-known MVMN distributions were recently considered by Pourmousa et al. (2013) and Naderi et al. (2018). The former presented a MVN of Birnbaum-Saunders (MVNBS) distribution, where $W$ has a Birnbaum-Saunders distribution with shape parameter $\alpha$ and scale parameter $\beta$. In the second reference, the authors assumed $W$ follows a Lindley distribution, which is a mixture of exp($\alpha$) and gamma(2, $\alpha$) distributions. This leads to the so-called MVN Lindley (MVNL) distribution.
4.3 Parameter estimation

The EM algorithm can be employed to estimate the parameters of the MVMN distribution. For special cases of MVMN distribution such as the GH, MVNBS, and MVNL distributions, explicit expressions for the implementation of the EM algorithm can be found in Browne and McNicholas (2015), Pourmousa et al. (2015), and Naderi et al. (2018), respectively.

5 Conclusions

A concise description of three generalizations of the (multivariate) normal distribution has been presented. These families of flexible distributions arise by mixing the mean and/or weighting the variance matrix of a normal random variable. Two of these families, namely the variance mixture (VMN) and mean-variance mixture of normal (MVMN) distributions have a relatively long history in the literature, whereas the third family (mean-mixture of normal (MMN) distribution) were introduced more recently. Each of these families has their own merits and limits. We have presented their basic properties, some important special/limiting cases, and references for parameter estimation procedures. Some further versions and/or generalizations of MVMN would be of interest for future investigation; for example, a scale mixture of MMN distributions (as suggested by Negarestani et al. (2019)) and a MVMN distribution where different mixing variables can be used for the mean and variance of the normal random variable.

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