TENSOR ALGEBRAS AND DISPLACEMENT STRUCTURE.
IV. INARIANT KERNELS

T. BANKS, T. CONSTANTINESCU, AND NERMINE EL-SISSI

ABSTRACT. In this paper we investigate the class of invariant positive definite kernels
on the free semigroup on $N$ generators. We provide a combinatorial description of
the positivity of the kernel in terms of Dyck paths and then we find a displacement
equation that encodes the invariance property of the kernel.

1. Introduction

In the previous parts of this paper, [3], [7], there were considered algebraic and
asymptotic properties of orthogonal polynomials in several variables associated with
a certain class of positive definite kernels on the free semigroup on $N$ generators.
These kernels were naturally associated with the Cuntz-Toeplitz defining relations,
$X_iX_j = \delta_{i,j}1, \ i, j = 1, \ldots, N$, but they are quite sparse (a lot of zero entries), which
makes their structure to be quite simple (see [7] for more details).

In this note we consider a more general class of positive definite kernels, which are
invariant under the action of the free semigroup on itself by concatenation, and our
main goal is to find combinatorial descriptions of the positive definiteness and of the
invariance property.

In answering the first question we establish a connection with the combinatorics of
Dyck paths. The invariance is then encoded into a displacement equation, and this
allows the use of the tools of the displacement structure theory.

The paper is organized as follows. In Section 2 we review some material on orthog-
onal polynomials and introduce the moment kernel of a q-positive functional on the
algebra of polynomials in several noncommuting variables. Then we describe the main
result about positive definite kernels and Dyck paths in Theorem 2.2. Two simple ap-
lications are given to the counting of paths in marine seismology and to the structure
of the Markov product introduced in [5]. Finally we discuss the connection between
orthogonal polynomials and displacement equations in our setting. The main result
here is given by Theorem 2.4. In Section 3 we introduce several examples of positive
definite invariant kernels. First we deal with a kernel involved in the dilation theory for
arbitrary families of contractions and calculate the orthogonal polynomials associated
with this kernel in Theorem 3.1. Then we show in Theorem 3.3 that the invariant
kernels are precisely the moment kernels associated with q-positive functionals on the
algebra of polynomials in $N$ noncommuting isometric variables. Finally, we show that
free products give many examples of invariant kernels. In Section 4 we prove the main
result about the displacement equation satisfied by an invariant kernel.
2. Preliminaries

Here we describe our setting, so that the paper can be read independently of [3] and [4]. We review material on orthogonal polynomials, positive definite kernels, and displacement structure. We also establish the description of positive definite kernels in terms of the combinatorics of Dyck paths.

2.1. Moment kernels and orthogonal polynomials. We introduce a class of positive definite kernels associated with some linear functionals on algebras of polynomials. Let \( \mathcal{P}_N \) be the algebra of polynomials in \( N \) noncommuting variables \( X_1, \ldots, X_N \) with complex coefficients. Each element \( P \in \mathcal{P}_N \) can be uniquely written in the form

\[
P = \sum_{\sigma \in \mathbb{P}_N} c_{\sigma} X_{\sigma},
\]

with \( c_{\sigma} \neq 0 \) for finitely many \( \sigma \)'s, where \( \mathbb{P}_N \) denotes the unital free semigroup on \( N \) generators \( 1, \ldots, N \) and with lexicographic order \( \preceq \). Also \( X_{\sigma} = X_{i_1} \ldots X_{i_k} \) for \( \sigma = i_1 \ldots i_k \in \mathbb{P}_N \). Instead of \( \mathbb{P}_N^+ \) we use the standard notation \( \mathbb{N}_0 \).

We can view \( \mathcal{P}_N \) as the free product of \( N \) copies of \( \mathcal{P}_1 \):

\[
\mathcal{P}_N = \mathcal{P}_1 \ast \ldots \ast \mathcal{P}_1 = \mathbb{C} \oplus \left( \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq i_2, \ldots, i_n-1 \neq i_n} \mathcal{P}_1^0 \otimes \ldots \otimes \mathcal{P}_1^0 \right),
\]

where \( \mathcal{P}_1^0 \) is the set of polynomials in the variable \( X_1 \), \( i = 1, \ldots, N \), without constant term. We also notice that \( \mathcal{P}_N \) is isomorphic with the tensor algebra over \( \mathbb{C}^N \), which is defined by the algebraic direct sum

\[
\mathcal{T}(\mathbb{C}^N) = \bigoplus_{k \geq 0} (\mathbb{C}^N)^{\otimes k},
\]

where \( (\mathbb{C}^N)^{\otimes k} \) denotes the \( k \)-fold tensor product of \( \mathbb{C}^N \) with itself. If \( \{e_1, \ldots, e_N\} \) is the standard basis of \( \mathbb{C}^N \), then the set

\[
\{1\} \cup \{e_{i_1} \otimes \ldots \otimes e_{i_k} \mid 1 \leq i_1, \ldots, i_k \leq N, k \geq 1\}
\]

is a basis of \( \mathcal{T}(\mathbb{C}^N) \). For \( \sigma = i_1 \ldots i_k \) we write \( e_\sigma \) instead of \( e_{i_1} \otimes \ldots \otimes e_{i_k} \), and the mapping \( X_\sigma \rightarrow e_\sigma, \sigma \in \mathbb{P}_N^+ \), extends to an isomorphism from \( \mathcal{P}_N \) to \( \mathcal{T}(\mathbb{C}^N) \), hence \( \mathcal{P}_N \cong \mathcal{T}(\mathbb{C}^N) \).

There is a natural involution on \( \mathcal{P}_{2N} \) introduced as follows:

\[
X_k^+ = X_{N+k}, \quad k = 1, \ldots, N,
\]

\[
X_l^+ = X_{l-N}, \quad l = N+1, \ldots, 2N;
\]

on monomials,

\[
(X_{i_1} \ldots X_{i_k})^+ = X_{i_k}^+ \ldots X_{i_1}^+,
\]

and finally, if \( Q = \sum_{\sigma \in \mathbb{P}_{2N}^+} c_{\sigma} X_{\sigma} \), then \( Q^+ = \sum_{\sigma \in \mathbb{P}_{2N}^+} \sigma_\sigma X_{\sigma}^+ \).

We say that \( \mathcal{A} \subset \mathcal{P}_{2N} \) is symmetric with respect to this involution if \( P \in \mathcal{A} \) implies \( cP^+ \in \mathcal{A} \) for some \( c \in \mathbb{C} - \{0\} \). Then the quotient of \( \mathcal{P}_{2N} \) by the two-sided ideal generated by \( \mathcal{A} \) is an associative algebra \( \mathcal{R}(\mathcal{A}) \). Letting \( \pi = \pi_\mathcal{A} : \mathcal{P}_{2N} \rightarrow \mathcal{R}(\mathcal{A}) \) denote the quotient map then the formula

\[
(2.1) \quad \pi(P)^+ = \pi(P^+)
\]
gives a well-defined involution on $\mathcal{R}(A)$. Thus $\mathcal{R}(A)$ is a unital $*$-algebra and $A$ is called the set of defining relations. A linear functional $\phi$ on $\mathcal{R}(A)$ is called $q$-positive if $\phi(\pi(P)^+\pi(P)) \geq 0$ for all $P \in \mathcal{P}_N$. The index set $G(A) \subset \mathbb{F}_N^+$ of $A$ is chosen as follows: if $\alpha \in G(A)$, choose the next element in $G(A)$ to be the least $\beta \in \mathbb{F}_N^+$ with the property that the elements $\pi(X_\alpha)$, $\alpha' \leq \alpha$, and $\pi(X_\beta)$ are linearly independent. We will avoid the degenerate situation in which $\pi(1) = 0$; if we do so, then $\emptyset \in G(A)$. Define $F_\alpha = \pi(X_\alpha)$ for $\alpha \in G(A)$. The moments of $\phi$ are the complex numbers
\[
s_{\alpha,\beta} = \phi(F_\alpha^+F_\beta), \quad \alpha, \beta \in G(A),
\]
and the moment kernel is defined by $K_\phi(\alpha, \beta) = s_{\alpha,\beta}$, $\alpha, \beta \in G(A)$. Since $\phi$ is $q$-positive on $\mathcal{R}(A)$, $K_\phi$ is a positive definite kernel on $G(A)$. However, $K_\phi$ does not determine $\phi$ uniquely.

In [3] and [7] the focus was on moment kernels associated with $q$-positive functionals on $\mathcal{R}(\mathcal{A}^N_{CT})$, where $\mathcal{A}^N_{CT} = \{1 - X_k^+X_k \mid k = 1, \ldots, N\} \cup \{X_k^+X_l, k, l = 1, \ldots, N, k \neq l\}$. The relations $X_k^+X_l = 0$, $k, l = 1, \ldots, N$, make the moment kernel to be sparse. In this paper we analyze the moment kernels of $q$-positive functionals on $\mathcal{R}(\mathcal{A}^N_O)$, where $\mathcal{A}^N_O = \{1 - X_k^+X_k \mid k = 1, \ldots, N\}$. We have that $G(\mathcal{A}^N_O) = \mathbb{F}_N^+$.

The orthonormal polynomials associated with a strictly $q$-positive functional on $\mathcal{R}(A)$ (that is, $\phi(\pi(P)^+\pi(P)) > 0$ for $\pi(P) \neq 0$) are introduced by the Gram-Schmidt procedure applied to the family $\{\pi(X_\alpha)\}_{\alpha \in G(A)}$ of linearly independent elements in the Hilbert space $\mathcal{H}_\phi$ associated with $\phi$ by the Gelfand-Naimark-Segal construction. Thus, the orthonormal polynomials are
\[
\varphi_\alpha = \sum_{\beta \leq \alpha} a_{\alpha,\beta} \pi(X_\beta), \quad a_{\alpha,\alpha} > 0.
\]
The polynomials $\varphi_\alpha$ are uniquely determined by the condition $a_{\alpha,\alpha} > 0$ and the orthonormality property
\[
\phi(\varphi_\beta^+\varphi_\alpha) = \delta_{\alpha,\beta}, \quad \alpha, \beta \in G(A).
\]

2.2. Positive definite kernels and Dyck paths. We will use several times a certain structure (and parametrization) of positive definite kernels on $\mathbb{N}_0$. For sake of completeness we briefly describe this structure here, while the details can be found in [6]. Also we discuss the connection with the combinatorics of Dyck paths.

For a contraction $\gamma \in \mathcal{L}(\mathcal{H}, \mathcal{H'})$, that is, a linear bounded operator between the Hilbert spaces $\mathcal{H}$ and $\mathcal{H'}$ with $\|\gamma\| \leq 1$, we define the defect operator $d_\gamma = (I - \gamma^*\gamma)^{1/2}$, and the corresponding defect space $\mathcal{D}_\gamma$, the closure of the range of $d_\gamma$. The Julia operator associated with $\gamma$ is defined by
\[
J(\gamma) = \left[ \begin{array}{cc} \gamma & d_\gamma \\ d_\gamma^* & -\gamma^* \end{array} \right];
\]
the Julia operator is unitary from $\mathcal{H} \oplus \mathcal{D}_\gamma$ onto $\mathcal{H'} \oplus \mathcal{D}_\gamma$. This construction can be extended to certain families of contractions as follows. Let $\{\gamma_{k,j}\}_{0 \leq k \leq j}$ be a family of contractions satisfying the compatibility conditions: $(i)$ $\gamma_{k,k} = 0$ for all $k \geq 0$ and $(ii)$...
\(\gamma_{k,j} \in \mathcal{L}(\mathcal{D}_{\gamma_{k+1,j}}, \mathcal{D}_{\gamma_{k,j-1}})\). Then the unitary operators \(U_{k,j}\) are recursively defined by:

\[
U_{k,k} = I_1 \quad \text{and for } k < j,
\]

\[
U_{k,j} = (J(\gamma_{k,k+1} \oplus I_{j-k-1}) (I_1 \oplus J(\gamma_{k,k+2} \oplus I_{j-k-2}) \ldots (I_{j-k-1} \oplus J(\gamma_{j,j})))
\]

\[
\times (U_{k+1,j} \oplus I_1);
\]

each \(U_{k,j}\) is a \((j - k + 1) \times (j - k + 1)\) block matrix and \(I_l\) denotes the identity \(l \times l\) block matrix.

\[
\begin{align*}
\text{Figure 1. Transmission line for } K(0, 3)
\end{align*}
\]

**Theorem 2.1.** Let \(K\) be a positive definite kernel on the set \(\mathbb{N}_0\) with values in a Hilbert space \(\mathcal{H}\). Then there is a uniquely determined family of contractions satisfying the compatibility conditions (i) and (ii), and such that

\[
(2.5) \quad K(l, m) = K(l, l)^{1/2} (P_{\mathcal{H}} U_{l,m}/\mathcal{H}) K(m, m)^{1/2}, \quad l \leq m,
\]

where \(P_{\mathcal{H}}\) denotes the orthogonal projection on the space \(\mathcal{H}\).

For a proof see \([6]\). We shall say that \(\{\gamma_{k,j}\}\) is the family of parameters associated with the kernel \(K\). Occasionally we write \(\gamma_{k,j}(K)\) in order to underline the dependence on \(K\). It is very useful to realize the above formula by a so-called time varying transmission line; for \(K(0, 3)\) this is illustrated in Figure 1 (for simplicity, assume \(K(l, l) = I\) for all \(l\)). Thus, if the identity operator \(I\) is the input at \(A\), then at \(B\) we read off the expression of \(K(0, 3)\) in terms of the parameters \(\gamma_{01}, \gamma_{02}, \gamma_{03}, \gamma_{12}, \gamma_{13}, \gamma_{23}\) and their defects. Likewise, if the input at \(C\) is the identity operator, then the output at \(B\) is now the expression of \(K(0, 2)\) (for more details see \([6]\)).

It was noticed in \([3]\) that there is a simple connection between transmission lines as in Figure 1 and Dyck (or Catalan) paths. We assume, again for simplicity, that \(K(l, m) \in \mathbb{C}\) for all \(l, m\) and that \(K(l, l) = 1\) for all \(l\). A Dyck path of length \(2k\) is a path in the positive quadrant of the lattice \(\mathbb{Z}^2\) which starts at \((0, 0)\), ends at \((2k, 0)\), and consists of rise steps \(\uparrow\) and fall steps \(\downarrow\) (see Figure 2). For more information on Dyck paths and their combinatorics, see \([13]\).
Let $\mathcal{D}_k$ be the set of Dyck paths of length $2k$ and let $\mathcal{A}_k$ be the set of points $(l, q)$, $q > 0$, with the property that there exists $p \in \mathcal{D}_k$ with $(l, q) \in p$. It is seen that

$$A_k = \{(j + i, j - i) \mid 0 \leq i < j \leq k\}.$$  

Also, we notice that if $p \in \mathcal{D}_k$ and $x = (l, q) \in p$, then there are only four types of behaviour of $p$ about $x$: (I) a rise step followed by a fall step; (II) a fall step followed by a rise step; (III) two consecutive rise steps; (IV) two consecutive fall steps (see Figure 3).

![Figure 2. A Dyck path of length 8](image)

Consequently, for each pair $i, j$ with $0 \leq i < j \leq k$ we define the function $a_{i,j} : \mathcal{D}_k \rightarrow \mathbb{C}$,

$$a_{i,j}(p) = \begin{cases} 1 & \text{if } x = (j + i, j - i) \notin p; \\ \gamma_{i,j} & \text{if } x = (j + i, j - i) \in p \text{ and (I) holds;} \\ \gamma_{i,j} & \text{if } x = (j + i, j - i) \in p \text{ and (II) holds;} \\ d_{i,j} & \text{if } x = (j + i, j - i) \in p \text{ and either (III) or (IV) holds.} \end{cases}$$

Let $p$ be a Dyck path in $\mathcal{D}_k$ such that $(2l, 0) \in p$. The restriction of $p$ from $(2l, 0)$ to $(2k, 0)$ is called a Dyck subpath starting at $(2l, 0)$ in $\mathcal{D}_k$ and denote by $\mathcal{D}_k^l$ the set of all these subpaths. There is a bijection between $\mathcal{D}_k^l$ and $\mathcal{D}_{k-l}$ so that the number of elements in $\mathcal{D}_k^l$ is given by the Catalan number $C_{k-l} = \frac{1}{k - l + 1} \binom{2(k - l)}{k - l}$; also, $\mathcal{D}_k^0 = \mathcal{D}_k$. If $q \in \mathcal{D}_k^l$ then there could be many Dick paths whose restrictions at $(2l, 0)$ coincide with $q$. However, we notice that if $p_1$ and $p_2$ are two such Dych paths, then $a_{i,j}(p_1) = a_{i,j}(p_2)$ for $j + i > 2l$. We will write $a_{i,j}(q)$ in order to denote this common value.

Now we can rewrite (2.5) as a cumulant type formula. In fact, we can establish a certain connection with free cumulants (see [15], [12]), which will be explored elsewhere.
**Theorem 2.2.** Let $K$ be a positive definite kernel on the set $\mathbb{N}_0$ with scalar values and $K(l, l) = 1$ for all $l$. Then, for $l < m$,

$$
(2.6) \quad K(l, m) = \sum_{q \in D_m} \prod_{l \leq i < j \leq m} a_{i,j}(q).
$$

**Proof.** Formula (2.6) is a direct consequence of Theorem 2.1 and the straightforward way in which we identify paths in a transmission line with Dyck paths. $\square$

Formula (2.6) looks quite intriguing. There is a well-established connection between continued fraction expansions and combinatorics of Dyck paths, see for instance [10], still (2.6) comes from the only requirement that the kernel is positive. A brief application of this result concerns the counting of paths in marine seismology. One has a layered medium with a perfect reflection at the 0-interface (see Figure 4).

![Figure 4](image)

**Figure 4.** A trajectory through a layered medium in 4 units of time

A unit impulse strikes at $A$, at time zero, and it propagates downwards through the medium. At each interface, the impulse is partially reflected and partially transmitted to the next layer. It is a consequence of Theorem 2.2 that the number of possible paths the impulse can take in order to return back to the 0-interface in $2n$ units of time, is precisely given by the Catalan number $C_n$.

We conclude this subsection with an application of the transmission line interpretation of Theorem 2.1 which provides a simple, conceptual proof of a result in [5]. Thus, let $A_1, A_2$ be two sets such that $A_1 \cap A_2 = \{a\}$ and $K_1, K_2$ be positive definite kernels on $A_1$, respectively $A_2$, such that $K_1(a, a) = K_2(a, a)$. The Markov product of the kernels $K_1$ and $K_2$ is a hermitian kernel $K$ on $A_1 \cup A_2$ defined in [5] by the rules:

1. $K|_{A_j \times A_j} = K_j$, \quad $j = 1, 2$;
2. $K(a_1, a_2) = K_1(a_1, a)K_2(a, a_2)$, \quad $a_1 \in A_1, a_2 \in A_2$;
3. $K(a_1, a_2) = K(a_2, a_1)^*$.

For our purpose we can restrict to the case of finite sets $A_1, A_2$, $A_1 = \{0, \ldots, n\}$, $A_2 = \{-m, \ldots, 0\}$. Let $\{\gamma_{k,j}(K_1)\}_{-m \leq k \leq j \leq 0}$ and $\{\gamma_{k,j}(K_2)\}_{0 \leq k \leq j \leq n}$ be the parameters associated with $K_1$, respectively $K_2$. The fact that the Markov product is positive
definite was proved in [4]. In addition, we provide here the structure of its associated parameters.

**Theorem 2.3.** The Markov product of two positive definite kernels $K_1$ and $K_2$ is a positive definite kernel with parameters $\{\gamma_{k,j}\}_{-m \leq k \leq j \leq n}$ given by:

$$
\gamma_{k,j} = \begin{cases} 
\gamma_{k,j}(K_1) & \text{if } -m \leq k \leq j \leq 0; \\
\gamma_{k,j}(K_2) & \text{if } 0 \leq k \leq j \leq n; \\
0 & \text{otherwise.}
\end{cases}
$$

**Proof.** The transmission line of $K(a_1, a_2)$, $a_1 \in A_1$, $a_2 \in A_2$, looks like in Figure 5.

![Figure 5. Transmission line for Markov products](attachment:figure5.png)

The Julia operator of 0 is $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and therefore the central block of Julia operators of 0 acts like a barrier. There is only one place for the signal to propagate from the left to right and that is the upmost wire. What comes through that wire is exactly $K_2(a, a_2)$. The transmission line to the right of $C$ will produce $K_1(a_1, a)$ and all together will get the product $K_2(a, a_2)K_1(a_1, a)$. □

2.3. **Displacement structure and orthogonal polynomials.** The displacement structure of a family $\{R(t)\}$ of matrices is encoded by an equation of the form

$$R(t) - F(t)R(t+1)F(t)^* = G(t)J(t)G^*(t),$$

where $F(t), G(t)$ are the so-called generators and $J(t)$ is a signature matrix (usually, $J(t) = I_p \oplus -I_q$, for some fixed $p, q$). The main feature in the use of displacement structure is that under suitable conditions on generators, the Gaussian elimination for $R(t)$ can be performed at the level of generators. This leads to faster algorithms for factorization of $R(t)$ and to useful lattice structures associated with these matrices.

There is a remarkable connection between orthogonal polynomials on the unit circle (when the moment kernel is Toeplitz) and displacement structure, as described in [10]. For our purpose it is convenient to obtain a similar connection in our more general setting. We discuss in details the following situation (with the notation introduced in Subsection 2.1): $N = 1$ and $\mathcal{A} = \emptyset$, so that $\mathcal{R}(\mathcal{A}) = \mathcal{P}_2$ and $G(\mathcal{A}) = \mathbb{N}_0$. The moment kernel of a q-positive functional $\phi$ on $\mathcal{P}_2$ is $K(n, m) = \phi((X_1^n)^+X_1^m)$, $n, m \in \mathbb{N}_0$, and there is no additional restriction on $K_\phi$ other then being positive definite. So, in a
certain sense, this is the most general possible situation. Next assume \( \phi \) is strictly \( q \)-positive (we say in this case that the moment kernel is strictly positive definite). It was showed in \([7]\) that the orthonormal polynomials associated with \( \phi \) obey the recurrence relation:

\begin{align}
\varphi_0(X_1, l) &= \varphi_0^\sharp(X_1, l) = s_{l,l}^{-1/2}, \quad l \in \mathbb{N}_0, \\
\varphi_n(X_1, l) &= \frac{1}{d_{l,n+l}} \left( X_1 \varphi_{n-1}(X_1, l + 1) - \gamma_{l,n+l} \varphi_{n-1}^\sharp(X_1, l) \right), \quad n \geq 1, l \in \mathbb{N}_0, \\
\varphi_n^\sharp(X_1, l) &= \frac{1}{d_{l,n+l}} \left( -\gamma_{l,n+l} X_1 \varphi_{n-1}(X_1, l + 1) + \varphi_{n-1}^\sharp(X_1, l) \right),
\end{align}

where \( \varphi_n(X_1) = \varphi_n(X_1, 0) \) and \( \{\gamma_{k,j}\} \) is the family of parameters associated with the moment kernel \( K_\phi \).

We now describe the displacement structure of the kernel \( K_\phi \). For each \( n \geq 0 \) we introduce the following elements (the generators of the relevant displacement equations): the \((n+1) \times (n+1)\) matrix

\[
F_n(t) = \begin{bmatrix} 0 & & 0 \\ 1 & 0 & & 0 \\ & 1 & \ddots & \ddots \\ & & 0 & 1 & 0 \\ 1 & & & & \end{bmatrix}, \quad t \in \mathbb{N}_0,
\]

and the \(2 \times 2\) matrix

\[
J(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad t \in \mathbb{N}_0;
\]

then for \( t \in \mathbb{N}_0 \), we introduce the \((n+1) \times 2\) matrix

\[
G_n(t) = s_{t,t}^{-1/2} \begin{bmatrix} s_{t,t} & 0 \\ s_{t,t+1} & s_{t,t+1} \\ \vdots & \vdots \\ s_{t,t+n} & s_{t,t+n} \end{bmatrix}.
\]

It was showed in \([8]\) that the displacement equation

\[
R_n(t) - F_n(t)R_n(t+1)F_n(t)^* = G_n(t)J(t)G_n(t)^*, \quad t \in \mathbb{N}_0,
\]

has a unique solution given by \( R_n(t) = [s_{k,j}]_{t \leq k, j \leq t+n} \). For this reason we say that the kernel \( K_\phi \) has displacement structure.

If \( K_\phi \) is a Toeplitz kernel then there is a strong connection between the displacement structure of its inverse and the orthogonal polynomials associated with \( \phi \). In the general case there is a trade-off. Certainly, the more general formulae are somewhat obscured by the necessary use of additional indices. On the other hand, the general case reveals some features obscured by the additional symmetries of the Toeplitz case.
The orthonormal polynomials of $\varphi$ have the expansion

$$\varphi_n(x_1, l) = \sum_{k=0}^{n} a_{n,k}^l x_1^k,$$

with $(a_{n,n}^l)^{-1} = s_{1/2}^{l+n+l} \prod_{k=1}^{n} d_{l+n-k,l+n}$, and similarly, the polynomials $\varphi_n^\sharp$ have the expansion

$$\varphi_n^\sharp(x_1, l) = \sum_{k=0}^{n} b_{n,k}^l x_1^k,$$

with $(b_{n,0}^l)^{-1} = s_{1/2}^{l+n+l} \prod_{k=1}^{n} d_{l+k}$. It follows from the proof of Theorem 3.2 in [7] that

$$R_n(t) \begin{bmatrix} a_{n,0}^t \\ \vdots \\ a_{n,n-1}^t \\ a_{n,n}^t \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ (a_{n,n}^t)^{-1} \end{bmatrix}$$

(2.11)

and

$$R_n(t) \begin{bmatrix} b_{n,0}^t \\ \vdots \\ b_{n,n}^t \end{bmatrix} = \begin{bmatrix} (b_{n,n}^t)^{-1} \\ \vdots \\ 0 \end{bmatrix}.$$  

(2.12)

We then define

$$H_n(t) = \begin{bmatrix} a_{n,0}^{t+1} & \cdots & a_{n,n-1}^{t+1} & a_{n,n}^{t+1} \\ \alpha_{n,0}^t & \cdots & \alpha_{n,n-1}^t & \beta_{n,n}^t \end{bmatrix},$$

and obtain the main result of this subsection.

**Theorem 2.4.** The family $\{R_n(t)^{-1}\}_{t \in \mathbb{N}_0}$ is the solution of the displacement equation

$$R_n(t+1)^{-1} - F_n(t)^* R_n(t)^{-1} F_n(t) = H_n(t)^* J(t) H_n(t), \quad t \in \mathbb{N}_0.$$

**Proof.** We define $K_n(t) = \begin{bmatrix} 0 & 0 \\ 0 & -s_{1/2}^{t+1} b_{n,0}^t \end{bmatrix}, \quad t \in \mathbb{N}_0$, and we have to show that

$$F_n(t) R_n(t+1) H_n(t)^* + G_n(t) J(t) K_n(t)^* = 0$$

(2.13)

and

$$H_n(t) R_n(t+1) H_n(t)^* + K_n(t) J(t) K_n(t)^* = J(t).$$

(2.14)

From (2.12) we deduce that

$$b_{n,0}^t \begin{bmatrix} s_{t+1,t+1} \\ \vdots \\ s_{t,t+n} \end{bmatrix} + \begin{bmatrix} s_{t+1,t+1} & \cdots & s_{t+1,t+n} \\ \vdots & \ddots & \vdots \\ s_{t+1,t+n} & \cdots & s_{t+n,t+n} \end{bmatrix} b_{n,1}^t = 0$$
which implies
\[
R_n(t+1) \begin{bmatrix}
    b_{n,1}' \\
    \vdots \\
    b_{n,n}' \\
    0
\end{bmatrix} = \begin{bmatrix}
    -b_{n,0}'s_{t,t+1} \\
    \vdots \\
    -b_{n,0}'s_{t,n+1} \\
    \end{bmatrix},
\]

where * denotes an entry whose actual value does not play any role here. Therefore, using the previous relation and (2.11), we deduce
\[
R_n(t+1)H_n(t)^* = \begin{bmatrix}
    0 & -b_{n,0}'s_{t,t+1} \\
    \vdots & \vdots \\
    0 & -b_{n,0}'s_{t,n+1} \\
    (a_{n,n}'^t)^{-1} & * \\
\end{bmatrix}
\]
and
\[
F_n(t)R_n(t+1)H_n(t)^* + G_n(t)J(t)K_n(t)^* = 0.
\]

In order to obtain (2.14) we calculate:
\[
H_n(t)R_n(t+1)H_n(t)^* = \begin{bmatrix}
    0 & 0 & \vdots & 0 \\
    0 & s_{t,t} & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & s_{t,t} & \vdots & \vdots \\
\end{bmatrix} + s_{t,t}^{-1/2} \begin{bmatrix}
    s_{t,t} & 0 & \vdots & 0 \\
    \vdots & s_{t,t} & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    \vdots & \vdots & \vdots & s_{t,t} \\
\end{bmatrix} \begin{bmatrix}
    1 & 0 & \vdots & 0 \\
    0 & 1 & \vdots & \vdots \\
    \vdots & \vdots & \ddots & \vdots \\
    -s_{t,t}^{-1/2}b_{n,0}' & \vdots & \vdots & 1 \\
\end{bmatrix}
\]

\[
= 0.
\]

The fact that the matrix \(H_n(t)R_n(t+1)H_n(t)^*\) is selfadjoint makes the north-east corner (the * entry) of the above matrix equal to 0. Also, formula (2.11) implies that
\[
\frac{b_{n,1}'s_{t,t+1} + \ldots + b_{n,n}'s_{t,n+1}}{10} = \left(b_{n,0}'\right)^{-1} - s_{t,t}b_{n,0}'.
\]
In conclusion,
\[ H_n(t)R_n(t + 1)H_n(t) + K_n(t)J(t)K_n(t) \]
\[
= \begin{bmatrix} 1 & 0 \\ 0 & -b_{t,0}^{-1} - s_t b_{n,0}^t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & s_t b_{n,0}^t \end{bmatrix}
\]
\[
= J(t).
\]

From (2.13) and (2.14) we deduce that
\[
F_n(t)G_n(t) + F_n(t)K_n(t)\]
and a Schur complement argument implies
\[
\text{In particular, we deduce that}
\]
\[
K_n(t)\text{ be a Hilbert space. A positive definite kernel}
\]
\[ F \text{ (under the action of } K \text{ and an isometric representation } U \text{ implies})
\]
\[ F \text{ gives a family of contractions } \}
\[
\{ F_{n,t} \}
\]
conditions: (3.1)
\[
\gamma_n F_{n,t} \text{ is total in } K \text{ and an isometric representation } U \text{ of } F_n^+ \text{ on } K \text{ such that}
\]
\[
3. \text{ INVARIANT KERNELS}
\]
In this section we introduce invariant kernels and provide several examples. Let \( H \) be a Hilbert space. A positive definite kernel \( K : F_N^+ \times F_N^+ \rightarrow \mathcal{L}(H) \) is called invariant (under the action of \( F_N^+ \) on itself by concatenation) if
\[
K(t,\tau) = K(t,\sigma'), \quad \tau,\sigma,\sigma' \in F_N^+.
\]
The invariant Kolmogorov decomposition theorem, [11], provides a certain structure of an invariant kernel. Thus, we can define a Hilbert space \( \mathcal{K} \), an operator \( V \in \mathcal{L}(H,\mathcal{K}) \), and an isometric representation \( U \) of \( F_N^+ \) on \( \mathcal{K} \) such that
\[
K(\sigma,\tau) = V^*U(\sigma)^*U(\tau)V, \quad \sigma,\tau \in F_N^+,
\]
and the set \( \{ U(\sigma V h \mid \sigma \in F_N^+, h \in H) \} \) is total in \( \mathcal{K} \). An application of Theorem 2.1 gives a family of contractions \( \{ \gamma_{\sigma,\tau} \mid \sigma,\tau \in F_N^+, \sigma \leq \tau \} \) satisfying the compatibility conditions: (i) \( \gamma_{\sigma,\sigma} = 0 \) for \( \sigma \in F_N^+ \) and (ii) \( \gamma_{\sigma,\tau} \in \mathcal{L}(D_{\gamma_{\sigma+1,\tau}},D_{\gamma_{\sigma,\tau-1}}) \), where \( \tau - 1 \) denotes the predecessor of \( \tau \) with respect to the lexicographic order on \( F_N^+ \) and \( \sigma + 1 \) denotes the successor of \( \sigma \). Using Theorem 1.6.1 in [6], we can describe \( V \) and \( U \) in (3.2) in terms of the parameters \( \gamma_{\sigma,\tau} \). However, at this stage it is not clear how to translate the invariance property of \( K \) into an invariance property of the parameters \( \gamma_{\sigma,\tau} \). In order to deal with this issue we first discuss several examples.
3.1. **Families of contractions.** Let \( T_1, \ldots, T_N \) be given contractions on the Hilbert space \( \mathcal{H} \). For \( \sigma = i_1^{k_1} \ldots i_n^{k_n}, \ i_j \neq i_{j+1}, \ j = 1, \ldots, n-1, \ k_j \in \mathbb{Z} - \{0\} \), a reduced word in the free group \( \mathbb{F}_N \) on \( N \) generators we define the contraction

\[
T_\sigma = T_{i_1}^{[k_1]} \ldots T_{i_n}^{[k_n]},
\]

where

\[
T^{[k]} = \begin{cases} 
T^k & k \geq 0; \\
T^{*-k} & k < 0.
\end{cases}
\]

Then the kernel \( K(\sigma, \tau) = T_{\sigma^{-1}\tau} \) is positive definite on \( \mathbb{F}_N \) \( [5, 1] \). It is also invariant, in the sense that (3.1) holds for \( \tau, \sigma, \sigma' \in \mathbb{F}_N \). Its restriction to \( \mathbb{F}_N^+ \), denoted \( K^+ \), is an invariant kernel on \( \mathbb{F}_N^+ \). If we try to calculate the parameters of \( K^+ \) (with respect to the lexicographic order), we notice that their form is quite complicated. However, it is not difficult to find the orthogonal polynomials. Thus, take \( T_k = t_k, \ k = 1, \ldots, N \), where \( t_k \) is a complex number in the open unit disk \( \mathbb{D} \) and define \( \phi : \mathcal{P}_{2N} \to \mathbb{C} \),

\[
(3.3) \quad \phi(X_{i_1} \ldots X_{i_n}) = t_{\epsilon(i_1) \ldots \epsilon(i_n)},
\]

where

\[
\epsilon(i) = \begin{cases} i & \text{if } 1 \leq i \leq N; \\
(i-N)^{-1} & \text{if } N < i \leq 2N
\end{cases}
\]

(if \( 1 \leq j \leq N \), then \( j \) is viewed as an element of \( \mathbb{F}_N \) with inverse \( j^{-1} \)). Then \( \phi(X_{\sigma}^+X_\tau) = K^+ (\sigma, \tau) \) for \( \sigma, \tau \in \mathbb{F}_N^+ \), so that \( \phi \) is a q-positive functional with moment kernel \( K^+ \). Since \( t_k \in \mathbb{D}, \ k = 1, \ldots, N \), it follows that \( \phi \) is a strictly q-positive functional on \( \mathcal{P}_{2N} \) and since \( G(\mathcal{P}_{2N}) = \mathbb{F}_N^+ \), we can consider \( \{\varphi_\sigma\}_{\sigma \in \mathbb{F}_N^+} \) the set of orthonormal polynomials associated with \( \phi \).

**Theorem 3.1.** The orthonormal polynomials of \( \phi \) are:

\[
\varphi_k = \frac{1}{d_{t_k}}(X_k - t_k), \quad k = 1, \ldots, N,
\]

and for \( \sigma \in \mathbb{F}_N^+ \),

\[
\varphi_{\sigma k} = X_{\sigma} \varphi_k, \quad k = 1, \ldots, N.
\]

**Proof.** Let \( \sigma, \tau \in \mathbb{F}_N^+, \ \tau \prec \sigma \). Then \( \sigma = \sigma'k \) for some \( \sigma' \in \mathbb{F}_N^+ \) and some \( k = 1, \ldots, N \). Consequently,

\[
\phi(X_\tau^+ \varphi_\sigma) = \phi(X_\tau^+ X_{\sigma'}^+ \varphi_k) = \frac{1}{d_{t_k}} (\phi(X_\tau^+ X_{\sigma'}^+ X_k) - t_k \phi(X_\tau^+ X_{\sigma'}^+)) = \frac{1}{d_{t_k}} (t_{\tau^{-1}\sigma'k} - t_k t_{\tau^{-1}\sigma'}) = 0.
\]
Also,

$$\phi(\varphi^+ \varphi) = \phi(\varphi^+ X^+ X' \varphi_k) = \frac{1}{d_{tk}}\phi \left( (X^+ - 7_k)X^+ X'(X_k - t_k) \right) = \frac{1}{d_{tk}}(1 - |t_k|^2 - |t_k|^2 + |t_k|^2) = 1.$$  

These relations show that \( \{ \varphi_\sigma \}_{\sigma \in \mathbb{F}^+_N} \) is indeed the family of orthonormal polynomials associated with \( \phi \).

Similar calculations will give an explicit formula of \( \varphi_\sigma(X_1, \ldots, X_N, l) \) for \( l \geq 1 \), at least for \( |\sigma| \) large enough. Thus, we introduce the following notation: \( r : \mathbb{F}^+_N \to \mathbb{N}_0 \) is the natural bijection between \( \mathbb{F}^+_N \) and \( \mathbb{N}_0 \), so that \( r(\emptyset) = 0 \), \( r(1) = 1 \), \( \ldots \), \( r(N) = N \), \( r(11) = N + 1 \), \ldots; then for \( l \geq 0 \) and \( \sigma \in \mathbb{F}^+ \), \( \sigma - l \) denotes the word in \( \mathbb{F}^+_N \) that is \( l \) steps ahead of \( \sigma \) (so \( \sigma - 1 \) is just the predecessor of \( \sigma \)). For \( r(\sigma) > l \), the word \( \sigma - l \) can be uniquely represented in the form \( \sigma - l = q(\sigma)p(\sigma) \) for some \( q(\sigma) \in \mathbb{F}^+_N \) and \( p(\sigma) \in \{ 1, \ldots, N \} \). With this notation, we can obtain as in the proof of Theorem \( \text{3.1} \) that for \( l \geq 1 \) and \( r(\sigma) > l \),

$$\varphi^l_\sigma = \frac{1}{d_{p(\sigma)}} \left( X_\sigma - t_{p(\sigma)}X_{q(\sigma)} \right). \tag{3.4}$$

As a consequence of \( \text{[3.4]} \) and Theorem 3.2 in \( \text{[3]} \) we obtain \( \gamma_{\sigma, \tau} = 0 \) for \( \sigma + r(\tau) + l \prec \tau \). This gives more information about the parameters of \( K^+ \) but still the remaining parameters look too complicated compared with the fact that \( K^+ \) is determined by just \( N \) complex numbers. A possibility to address this issue is to use parameters associated to \( K^+ \) along a fixed chordal sequence, as in Theorem 3.1 in \( \text{[1]} \). More precisely, we use the following construction. In general we use the notation \( G = (V, E) \) in order to denote an undirected graph with \( V \) the set of vertices and \( E \) the set of edges. For \( v, w \in V \), the notation \( (v, w) \) denotes the edge of \( G \) with endpoints \( v \) and \( w \). Let \( E^0_\emptyset = \emptyset \) and \( G^0_\emptyset = (\mathbb{F}^+_N, \emptyset) \). For \( \sigma \in \mathbb{F}^+_N \setminus \{ \emptyset \} \), \( k \in \{ 1, \ldots, N \} \), and \( 1 \leq l \leq r(\sigma k) \), we define

$$E^{l}_{\sigma k} = E^{r(\sigma k) - 1}_{\sigma k} \cup \{ (\sigma, \sigma k) \}$$

and for \( l > 1 \), the set \( E^{k}_{\sigma k} \) is obtained by adding one new edge \( (\tau, \sigma k) \) to \( E^{l-1}_{\sigma k} \), where \( \tau \neq \sigma \) and \( \tau \prec \sigma k \). Then define \( G^l_\sigma = (\mathbb{F}^+_N, E^l_{\sigma k}) \). It is easily seen that \( V_\sigma = \{ 0 \leq \tau \leq \sigma \} \) is a maximal clique in \( G^l_\sigma \), that is \( (V_\sigma, E^{r(\sigma)}_\sigma) \) is the complete graph and \( V_\sigma \) is maximal with this property. This implies that each \( G^l_\sigma \) is a chordal graph and if we order the family \( \{ G^l_\sigma \} \) by lexicographic order on the pairs \( (\sigma, l) \), \( \sigma \in \mathbb{F}^+_N \), \( 1 \leq l \leq r(\sigma) \), then \( \{ G^l_\sigma \} \) is a chordal sequence, according to the terminology in \( \text{[1]} \). By Theorem 3.1 in \( \text{[1]} \) (see also Theorem 7.2.7 in \( \text{[3]} \)), the kernel \( K^+ \) is uniquely determined by a family

$$\{ \gamma^l_\sigma : \sigma \in \mathbb{F}^+_N \setminus \{ \emptyset \}, 1 \leq l \leq r(\sigma) \} \cup \{ \gamma^l_\sigma \}$$

of complex numbers with \( |\gamma^l_\sigma| < 1 \). We call these numbers the parameters of \( K^+ \) along the chordal sequence \( \{ G^l_\sigma \} \).

**Theorem 3.2.** The parameters of \( K^+ \) along the chordal sequence \( \{ G^l_\sigma \} \) are given by:

$$\gamma^{l}_{\sigma k} = t_k \text{ for } k = 1, \ldots, N, \sigma \in \mathbb{F}^+_N \text{ and } \gamma^{l}_{\tau} = 0 \text{ for } l > 1 \text{ and } \tau \in \mathbb{F}^+_N \setminus \{ \emptyset \}.$$
Proof. For $V \subset \mathbb{F}_N^+$ we denote by $K_V^+$ the restriction of $K^+$ to $V$, that is, $K_V^+(\sigma, \tau) = K^+(\sigma, \tau)$ for $\sigma, \tau \in V$. We claim that $K^+_V\{0 \leq \tau \leq \sigma k\}$ is the Markov product of the kernels $K^+_\{0 \leq \tau \leq \sigma k-1\}$ and $K^+_\{\sigma, \sigma k\}$. Indeed, we have $\{0 \leq \tau \leq \sigma k\} \cap \{\sigma, \sigma k\} = \{\sigma\}$ and

$$K^+(\tau, \sigma k) = t_{\tau^{-1}\sigma k} = t_{\tau^{-1}\sigma} t_k = t_{\tau^{-1}\sigma} t_{\tau^{-1}\sigma k} = K^+(\tau, \sigma) K^+(\sigma, \sigma k).$$

Now an application of Theorem 2.3 concludes the proof. □

We could deal now with orthogonal polynomials along a chordal sequence such as the one above. However, we do not pursue this here, more details can be found in [2].

3.2. Moment kernels on $A_N^O$. We can see that the functional $\phi$ given by (3.3) induces a functional $\tilde{\phi}$ on $A_N^O$ such that $\tilde{\phi} \circ \pi_{A_N^O} = \phi$. This suggests that the invariant kernels are related to $A_N^O$ and the following result explains this connection. We use the notation introduced in Subsection 2.1.

**Theorem 3.3.** $K = K_\phi$ for some linear functional on $\mathcal{R}(A_N^O)$ if and only if $K$ is an invariant kernel.

**Proof.** Let $K = K_\phi$ for some linear functional $\phi$ on $\mathcal{R}(A_N^O)$ and let $\tau, \sigma, \sigma'$ be words in the index set of $A_N^O$, which is $\mathbb{F}_N^+$. Then

$$K(\tau \sigma, \tau \sigma') = \phi(X_{\tau \sigma}^+ X_{\tau \sigma'}) = \phi(X_{\sigma}^+ X_{\tau}^+ X_{\tau} X_{\sigma'}).$$

Since $X_{\tau}^+ X_{\tau} = 1$ in $\mathcal{R}(A_N^O)$, we deduce that

$$K(\tau \sigma, \tau \sigma') = K(\sigma, \sigma').$$

Conversely, let $K$ be an invariant kernel. Any element of $\mathcal{R}(A_N^O)$ is a linear combination of monomials $X_{i_1} \ldots X_{i_n}, i_1, \ldots, i_n \in \{1, \ldots, 2N\}$, with the property that there is no pair $(i_k, i_{k+1})$ with $i_k > N$ and $i_k - i_{k+1} = N$. We define $\phi$ on monomials as above which can be written in the form $X_{\sigma}^+ X_{\tau}$ by the formula

$$\phi(X_{\sigma}^+ X_{\tau}) = K(\sigma, \tau),$$

and arbitrarly on the other monomials in $\mathcal{R}(A_N^O)$. The invarince of $K$ insures that $\phi$ is well-defined. Then we extend $\phi$ by linearity to the whole $\mathcal{R}(A_N^O)$ and clearly $K = K_\phi$. □

This result explains that the study of orthogonal polynomials on $\mathcal{R}(A_N^O)$ reduces to the study of invariant kernels.
3.3. **Free products.** Since \( \mathcal{P}_{N} \) is a free product of \( N \) copies of \( \mathcal{P}_{1} \), it is quite natural to look at free products of q-positive functionals. Let \( \mathcal{R}(\mathcal{A}_{1}) \), \( \mathcal{R}(\mathcal{A}_{2}) \) be two algebras with sets of defining relations \( \mathcal{A}_{1} \), respectively, \( \mathcal{A}_{2} \). It is convenient to view \( \mathcal{R}(\mathcal{A}_{1}) \) as a quotient of \( \mathcal{P}_{2N} \) in the variables \( X_{1}, \ldots, X_{2N} \) and \( \mathcal{R}(\mathcal{A}_{2}) \) as a quotient of \( \mathcal{P}_{2M} \) in the variables \( Y_{1}, \ldots, Y_{2M} \). According to the notation in Subsection 2.1, let \( F_{\alpha} = \pi_{\mathcal{A}_{1}}(X_{\alpha}) \), \( \alpha \in G(\mathcal{A}_{1}) \), and \( G_{\beta} = \pi_{\mathcal{A}_{2}}(Y_{\beta}) \), \( \beta \in G(\mathcal{A}_{2}) \). Each of \( G(\mathcal{A}_{1}) \) and \( G(\mathcal{A}_{2}) \) contains words of length 1, otherwise the situation is degenerate, in the sense that \( G \) is scalar-valued, and also that \( K_{\mathcal{A}_{1}} \) and \( K_{\mathcal{A}_{2}} \) are isomorphic, in the sense that \( \mathcal{R}(\mathcal{A}_{1}) = \mathcal{R}(\mathcal{A}_{2}) = \mathbb{C} \).

In order to simplify the notation, but without loss of generality, we can assume that \( G(\mathcal{A}_{1}) \) contains all of 1, ..., \( N \) and \( G(\mathcal{A}_{2}) \) contains all of 1, ..., \( M \). In this way, \( \mathcal{R}(\mathcal{A}_{1}) \) is the set of polynomials in the variables \( F_{1}, \ldots, F_{N}, F_{1}^{+}, \ldots, F_{N}^{+} \) (satisfying the defining relations in \( \mathcal{A}_{1} \)), and similarly, \( \mathcal{R}(\mathcal{A}_{2}) \) is the set of polynomials in the variables \( G_{1}, \ldots, G_{M}, G_{1}^{+}, \ldots, G_{M}^{+} \) (satisfying the defining relations in \( \mathcal{A}_{2} \)). Let \( \mathcal{R}^{0}(\mathcal{A}_{i}), i = 1, 2 \), denote the set of polynomials in \( \mathcal{R}(\mathcal{A}_{i}) \) without constant term. Then

\[
\mathcal{R}_{1}(\mathcal{A}_{1}) \ast \mathcal{R}_{1}(\mathcal{A}_{2}) = \mathbb{C} \oplus \left( \oplus_{n \geq 1} \oplus_{i_{1} \neq i_{2}, \ldots, i_{n-1} \neq i_{n}} \mathcal{R}^{0}(\mathcal{A}_{i_{1}}) \otimes \ldots \otimes \mathcal{R}^{0}(\mathcal{A}_{i_{n}}) \right),
\]

and we notice that \( \mathcal{R}_{1}(\mathcal{A}_{1}) \ast \mathcal{R}_{1}(\mathcal{A}_{2}) \) is isomorphic to \( \mathcal{R}(\mathcal{A}_{1} + \mathcal{A}_{2}) \), where \( \mathcal{A}_{1} + \mathcal{A}_{2} \) is the disjoint union of \( \mathcal{A}_{1} \) and \( \mathcal{A}_{2} \) (due to our convention to view \( \mathcal{A}_{1} \) as a subset of \( \mathcal{P}_{2N} \) in the variables \( X_{1}, \ldots, X_{2N} \) and \( \mathcal{A}_{2} \) as a subset of \( \mathcal{P}_{2M} \) in the variables \( Y_{1}, \ldots, Y_{2M} \), the sets \( \mathcal{A}_{1} \) and \( \mathcal{A}_{2} \) are automatically disjoint).

Now let \( \phi_{1} \) be a q-positive functional on \( \mathcal{R}(\mathcal{A}_{1}) \) and \( \phi_{2} \) be a q-positive functional on \( \mathcal{R}(\mathcal{A}_{2}) \). Their free product \( \phi = \phi_{1} \ast \phi_{2} \) on \( \mathcal{R}(\mathcal{A}_{1}) \ast \mathcal{R}(\mathcal{A}_{2}) \) is defined by \( \phi(1) = 1 \) and \( \phi(P_{i_{1}} \ldots P_{i_{n}}) = \phi_{i_{1}}(P_{i_{1}}) \ldots \phi_{i_{n}}(P_{i_{n}}) \) for \( n \geq 1, i_{1} \neq i_{2}, \ldots, i_{n-1} \neq i_{n}, P_{i_{k}} \in \mathcal{R}^{0}(\mathcal{A}_{i_{k}}), \) and \( i_{k} \in \{1, 2\} \) for \( k = 1, \ldots, n \). The map \( \phi \) given by (3.3) is an example of a free product of \( N \) q-positive functionals.

Since in general a q-positive functional is not positive, the main result in [3] cannot be applied in order to conclude that the free product of two q-positive functionals is q-positive, however this follows from the more general result in [4]. Using Theorem 3.3 it follows that for strictly q-positive functionals \( \phi_{1} \) and \( \phi_{2} \) on \( \mathcal{R}(\mathcal{A}_{1}^{0}) \) and, respectively, \( \mathcal{R}(\mathcal{A}_{2}^{0}) \), the kernel \( K_{\phi_{1} \ast \phi_{2}} \) is a strictly positive definite invariant kernel. This construction produces a relatively large class of positive definite and strictly positive definite invariant kernels.

4. **THE DISPLACEMENT STRUCTURE OF IN Variant KERNELS**

From the examples in the previous section we see that it is difficult to explore the additional symmetry of a positive definite invariant kernel in terms of its parameters or of its orthogonal polynomials. In particular, the invariance is not encoded efficiently into the generators of the displacement equation (2.10). In this section we consider a different displacement structure of an invariant kernel. In order to avoid notational complications, we can assume that the positive definite invariant kernel \( K \) is scalar-valued, and also that \( K(\sigma, \sigma) = 1 \) for all \( \sigma \in \mathbb{P}^{1}_{N} \). For each \( n \geq 0 \) we introduce the following elements: the \( \sum_{k=0}^{n} N^{k} \times \sum_{k=0}^{n} N^{k} \) matrix \( F_{k,n} \), \( k = 1, \ldots, N \), whose action...
on the Hilbert space $\mathcal{F}_n$ of sequences $\{h_\sigma\}_{|\sigma| \leq n}$ (with Euclidean norm) is given by

$$F_{k,n}(\{h_\sigma\}_{|\sigma| \leq n}) = \{g_\sigma\}_{|\sigma| \leq n},$$

where

$$g_\tau = \begin{cases} 
    h_\sigma & \text{if } \tau = k\sigma \\
    0 & \text{otherwise}.
\end{cases}$$

Also, let $R_n = [K(\sigma, \tau)]_{|\sigma|, |\tau| \leq n}$ and define $Q_n = [Q_n(\sigma, \tau)]_{|\sigma|, |\tau| \leq n}$, where $Q_n(\sigma, \tau) = 0$ if $\sigma = \alpha\sigma'$, $\tau = \alpha\tau'$ for some $\alpha \in \mathbb{F}_N^+ - \{\emptyset\}$, $\sigma', \tau' \in \mathbb{F}_N^+$, and otherwise $Q_n(\sigma, \tau) = K(\sigma, \tau)$. The next result shows that the left hand side of the relation (4.1) sifts out all the redundancy in $K$ caused by its invariance.

**Lemma 4.1.** For each $n \geq 0$ the matrix $R_n$ satisfies the displacement equation (4.1)

$$R_n - \sum_{k=1}^N F_{k,n}R_nF_{k,n}^* = Q_n.$$

**Proof.** Let $\{e_\sigma\}_{|\sigma| \leq n}$ be the standard basis of the Hilbert space $\mathcal{F}_n$. Then

$$\langle (R_n - \sum_{k=1}^N F_{k,n}R_nF_{k,n}^*)e_\sigma, e_\tau \rangle$$

$$= \langle R_n e_\sigma, e_\tau \rangle - \sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_\sigma, e_\tau \rangle$$

$$= \langle R_n e_\sigma, e_\tau \rangle - \sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_\sigma, e_\tau \rangle.$$

If there is no $\alpha \in \mathbb{F}_N^+ - \{\emptyset\}$ such that $\sigma = \alpha\sigma'$ and $\tau = \alpha\tau'$, then the first letter of $\sigma$ is going to be different from the first letter of $\tau$, which implies that $\sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_\sigma, e_\tau \rangle = 0$ and

$$\langle (R_n - \sum_{k=1}^N F_{k,n}R_nF_{k,n}^*)e_\sigma, e_\tau \rangle$$

$$= \langle R_n e_\sigma, e_\tau \rangle - \sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_\sigma, e_\tau \rangle$$

$$= \langle R_n e_\sigma, e_\tau \rangle - \sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_\sigma, e_\tau \rangle.$$

If there is $\alpha \in \mathbb{F}_N^+ - \{\emptyset\}$ such that $\sigma = \alpha\sigma'$ or $\tau = \alpha\tau'$, this implies that there is $p \in \{1, \ldots, N\}$ such that $\sigma = p\sigma'$, $\tau = p\tau'$, and then

$$\langle (R_n - \sum_{k=1}^N F_{k,n}R_nF_{k,n}^*)e_\sigma, e_\tau \rangle$$

$$= K(p\tau', p\sigma') - \sum_{k=1}^N \langle F_{k,n}R_nF_{k,n}^* e_{p\sigma'}, e_{p\tau'} \rangle$$

$$= K(\tau', \sigma') - \langle R_n e_{\sigma'}, e_{\tau'} \rangle = 0 = Q_n(\tau, \sigma).$$

In conclusion, we obtained (4.1). $\square$
We now try to factorize $Q_n$ in the form $G_n J_n G_n^*$ for some symmetry $J_n$ ($J_n = J_n^* = J_n^{-1}$), but of course, $J_n$ is no longer $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. In order to obtain a result suitable for the displacement structure theory, $J_n$ should be the same for all $Q_n$ (that is, for all invariant kernels $K$).

Lemma 4.2. (a) Let $A = [A_{i,j}]_{i,j=1}^p$ be a selfadjoint block-matrix with $A_{k,k} = 0$ for all $k = 1, \ldots, p$. Then

\[ A = B J_{2p-2} B^*, \]

where

\[ B = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & I \\
A_{2,1} & 0 & \ldots & 0 & 0 & & & & I \\
A_{3,1} & A_{3,2} & \ldots & 0 & 0 & 0 & & & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \\
A_{p-1,1} & A_{p-1,2} & \ldots & 0 & I & 0 & \ldots & 0 & 0 \\
A_{p,1} & A_{p,2} & \ldots & A_{p,p-1} & 0 & 0 & \ldots & 0 & 0 \\
\end{bmatrix} \]

is a $p \times (2p - 2)$ block matrix and $J_k$ is a $k \times k$ block-matrix,

\[ J_k = \begin{bmatrix}
0 & 0 & \ldots & 0 & I \\
0 & 0 & \ldots & I & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & I & \ldots & 0 & 0 \\
I & 0 & \ldots & 0 & 0 \\
\end{bmatrix}. \]

(b) Assume all $A_{i,j}$ are complex numbers. Then $2p - 2$ is the minimal dimension of a symmetry $J$ with the property that for any selfadjoint matrix $A$ with zero diagonal, the relation (4.2) holds for some $B$.

Proof. (a) The formula (4.2) is easily verified by direct computations that can be omitted.

(b) Since $A_{k,k} = 0$ for $k = 1, \ldots, p$, $A$ cannot be positive or negative (excepting for the trivial case $A_{i,j} = 0$ for all $i, j$). Therefore $A$ can have (generically) at most $p - 1$ positive eigenvalues or at most $p - 1$ negative eigenvalues. This implies that the symmetry $J$ satisfying (4.2) for all selfadjoint $A$ with zero diagonal must have at least as many positive eigenvalue and, respectively, negative eigenvalues, which gives a total at least $2p - 2$ eigenvalues. The construction of (a) realizes this value, so $2p - 2$ is the minimal dimension of a symmetry $J$ satisfying (4.2) for any selfadjoint matrix with zero diagonal. \(\Box\)

Theorem 4.3. For each $n \geq 0$ the matrix $R_n$ satisfies the displacement equation

\[ R_n - \sum_{k=1}^{N} F_{k,n} R_n F_{k,n}^* = G_n J_n G_n^*, \]

where $J_n$ is a symmetry of dimension $2 + (2N - 2) \sum_{k=0}^{n-1} N^k$. 

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Proof. By Lemma 4.1, the matrix $R_n$ satisfies the displacement equation

$$R_n - \sum_{k=1}^{N} F_{k,n} R_n F_{k,n}^* = Q_n.$$ 

From the definition of $Q_n$, we deduce that

$$Q_n = \begin{bmatrix} 1 & S_n^* \\ S_n & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & L_n \end{bmatrix},$$

where $S_n = [K(0, \sigma)]_{|\sigma| \leq n}$, $L_n(\sigma, \tau) = 0$ if $\sigma = \alpha \sigma'$, $\tau = \alpha \tau'$ for some $\alpha \in \mathbb{F}_N^+ - \{0\}$, $\sigma', \tau' \in \mathbb{F}_N^+$, and otherwise, $L_n(\sigma, \tau) = K(\sigma, \tau)$ (note that $L_n(\sigma, \tau)$ is defined only for $\sigma, \tau \in \mathbb{F}_N^+ - \{0\}$). Since we have the factorization

$$\begin{bmatrix} 1 & S_n \\ S_n^* & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ S_n^* & S_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & S_n \\ 0 & S_n \end{bmatrix},$$

we need only to show that $L_n$ has a factorization of the form $L_n = G_n' J_n' G_n'^*$ with a symmetry $J_n'$ of dimension $(2N - 2) \sum_{k=0}^{n-1} N^k$. Then

$$Q_n = \begin{bmatrix} 1 & 0 \\ S_n^* & S_n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & S_n \\ 0 & S_n \end{bmatrix}$$

will be the required factorization of $Q_n$.

Now, for $k = 1, \ldots, N$ we define $A_k = \{k \tau \mid |\tau| \leq n - 1\}$. Then $\{A_k\}_{k=1}^{N}$ is a partition of the set $W_n = \{\sigma \in \mathbb{F}_N^+ \mid |\sigma| \leq n\}$. We reorder the elements of $W_n$ such that $\sigma < \tau$ if $\sigma \in A_k$, $\tau \in A_j$, $k < j$. Then $Q_n = [A_{i,j}]_{i,j=1}^{N}$ with $A_{k,k} = 0$ for all $k = 1, \ldots, N$. By Lemma 4.2,

$$Q_n = B_n J_{2N-2} B_n^*,$$

where $B_n$ is given by (4.3). We can define $J_n' = J_{2N-2}$, so that we obtain a factorization of $Q_n$ with the required dimension of the symmetry $J_n$. □

The symmetry $J_n$ in Theorem 1.3 is unitarily equivalent to the symmetry

$$\begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix},$$

where $p_n = 1 + (N - 1) \sum_{k=0}^{n-1} N^k$, so that we can rewrite equation (4.5) in the more familiar form

$$R_n - \sum_{k=1}^{N} F_{k,n} R_n F_{k,n}^* = G_n \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix} G_n^*,$$

for some new $G_n$, and the established results of the displacement structure theory can be used in order to explore the structure of $R_n$. In particular, we obtain a Schur type algorithm that better encodes the invariance of $K$. Still, we have to note the fact that due to the complexity of $K$, the number $p_n$ depends on $n$. Some more details in this direction can be found in [2].

We conclude by noticing that the moment kernel of a q-positive functional on $\mathcal{R}(A_{CT}^N)$ is characterized by the property that $L_n = 0$ for all $n \geq 0$. Thus, (4.5) appears as an extension of the displacement equation for $\mathcal{R}(A_{CT}^N)$ obtained in [7].
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Department of Mathematics, University of Texas at Dallas, Richardson, TX 75083
E-mail address: banks@utdallas.edu

Department of Mathematics, University of Texas at Dallas, Richardson, TX 75083
E-mail address: tiberiu@utdallas.edu

Department of Mathematics, University of Texas at Dallas, Richardson, TX 75083
E-mail address: