Abstract. Let \( g \) be an affine Kac-Moody algebra with symmetric Cartan datum, \( n^+ \) be the maximal nilpotent subalgebra of \( g \). By the Hall algebra approach, we construct integral bases of the \( \mathbb{Z} \)-form of the enveloping algebra \( U(n^+) \). In particular, the representation theory of tame quivers is essentially used in this paper.

1. Introduction

1.1. There are remarkable connections between the representation theory of quivers and Lie theory. For a complex semisimple Lie algebra \( g \), let \( Q \) be the quiver given by orienting the Dynkin diagram of \( g \). In [Gab], P. Gabriel discovered that the set of isomorphic classes of indecomposable representations of \( Q \) is in one-to-one correspondence with the set of positive roots of \( g \). A direct construction of the Lie algebra \( g \) using the representations of quivers was given by C. M. Ringel. Roughly speaking, he defined the Hall algebra \( \mathcal{H}(Q) \) using representations of \( Q \) over a finite field and then proved that \( \mathcal{H}(Q) \) is isomorphic to the positive part of the quantum group \( U^+_v(g) \), see [R1] [R3]. This result was generalized to the case of Kac-Moody algebra in [Gr], where it was showed that the composition subalgebra \( \mathcal{C}(Q) \) provides a realization of \( U^+_v(g) \). Thus when \( v \) specializes to 1, the Hall algebra degenerates to the enveloping algebra \( U(n^+) \), where \( n^+ \) is the maximal nilpotent subalgebra of \( g \), see [R2].

In [L1], G. Lusztig gave a geometric definition of \( \mathcal{H}(Q) \), which is a modified version of A. Schofield [S]. Namely, he used the constructible functions on varieties of \( \mathbb{C}(Q) \)-modules. The Euler characteristics appeared in the definition of multiplication. Similar as in the quantum case, the composition subalgebra \( \mathcal{C}(Q) \) is isomorphic to the enveloping algebra \( U(n^+) \) (see Theorem 3.1). In the finite and affine case, a Chevalley basis of \( n^+ \) was reconstructed by the approach of Hall algebra in [FMV] (see Proposition 5.3). Thus the structure constants of this basis, which is the Euler characteristics of certain varieties, is given by the cocycles.

1.2. For any complex semisimple Lie algebra \( g \), Kostant defined a \( \mathbb{Z} \)-subalgebra \( U_\mathbb{Z} \) of the universal enveloping algebra \( U(g) \) using divided powers of the Chevalley basis, which is the well-known Kostant \( \mathbb{Z} \)-form [Ko]. Then he constructed a \( \mathbb{Z} \)-basis of \( U_\mathbb{Z} \). These results were generalized to the affine Kac-Moody algebra by H. Garland, in [Gal]. He defined the root vectors using the loop algebra structure of the affine Kac-Moody algebra. The \( \mathbb{Z} \)-form is defined as the \( \mathbb{Z} \)-subalgebra generated by the divided powers of all real root vectors. And he also construct a \( \mathbb{Z} \)-basis of \( U_\mathbb{Z} \). Given an order on the set of positive roots. The basis elements given in [Gal]

Key words and phrases. tame quiver, Hall algebra, affine enveloping algebra.
2000 Mathematics Subject Classification: 16G20, 17B35.
supported in part by NSF of China (No. 10631010) and by NKBRPC (No. 2006CB805905).
are ordered monomials of the following generators: the divided powers of real root vectors and certain functions of imaginary root vectors. The method in [Gal] is not representation-theoretic and the proof of the integrality of the basis is difficult, using some complicated combinatorial identities.

1.3. In this paper, we will construct \( \mathbb{Z} \)-bases of \( U_\mathbb{Z}(\mathfrak{g}) \) for affine Kac-Moody algebras by the Hall algebra approach. The representation theory of tame quivers, especially the structure of the Auslander-Reiten-quiver is essentially used in our method. The AR-quiver \( \Gamma_Q \), whose vertices are isomorphic classes of indecomposable \( \mathbb{C}Q \)-modules and arrows are irreducible morphisms, gives a nice description of the category of \( \mathbb{C}Q \)-modules. In [FMV], one can already see that the real root vectors not only come from the preprojective or preinjective components of \( \Gamma_Q \), but also come from the non-homogeneous tubes in the regular component. Furthermore, the behaviors of the imaginary root vectors arising from homogeneous tubes and non-homogeneous tubes are quite different.

Therefore, we construct basis elements from the components of the AR-quiver respectively. More precisely, the basis elements we construct arise from the preprojective component, the preinjective component, each non-homogeneous tube and an embedding of the module category of the Kronecker quiver respectively. Then the ordered monomials of those basis elements form the desired integral basis. In particular, the order is given by the structure of the AR-quiver. In this way, we generalize the main results of Frenkel, Malkin and Vybornov in [FMV] to the enveloping algebra level.

1.4. One key to prove the integrality of our bases is that the Euler characteristics are always integers. Thus when we evaluated the product of two characteristic functions of certain constructible sets at any point, we get an integer (see 3.4 for details). However, this is not enough since not every basis element can be made as a single characteristic function. Moreover, the supports of two basis elements may have common points. So we have to find good constructible functions to be the basis elements. The most difficult part is the choice in the homogeneous tubes, where our idea comes from both representation theory and the theory of symmetric functions (see 6.5).

1.5. Let’s say something about the quantum case. There are several results in constructing \( \mathbb{Z}[v, v^{-1}] \)-bases of the integral form of \( U_v^+ \) using the Hall algebra. Ringel has construct an integral PBW-basis of \( U_v^+ \) in the case of finite type [R4]. Later a \( \mathbb{Q}(v) \)-basis of type \( A_1^{(1)} \) was given in [Z] and it was improved to be a \( \mathbb{Z}[v, v^{-1}] \)-basis in [C]. For the affine case, in [LXZ], a PBW-basis of \( U_v^+ \) was given as a step to construct the canonical basis by an algebraic method. The method in the present paper obviously stems from that of [LXZ]. However, the basis given there is a \( \mathbb{Q}[v, v^{-1}] \)-basis. Although it has been proved in [LXZ] that this basis has a nice connection with the canonical basis, it seems difficult to prove that it is actually a \( \mathbb{Z}[v, v^{-1}] \)-basis only using algebraic methods.

1.6. The paper is organized as follows: In Section 2, we make necessary notations and recall some basic definitions. In Section 3, we recall the definition of Hall algebra. For convenience, we use the geometric version, following Lusztig [L1]. We also calculate the product of characteristic functions in two easy cases. A brief review of the representation theory of tame quivers is given in Section 4 for the
details one can see [DR]. In Section 5 we focus on the preprojective and preinjective
modules. We define two subalgebras \( \mathcal{C}(Q)^{prep} \), \( \mathcal{C}(Q)^{prei} \) and construct \( \mathbb{Z} \)-bases of
them respectively. The arguments in this section are similar to the quantum case,
see [R3] which considered the case of finite type. Moreover, the results in this section
are valid for arbitrary quiver without oriented cycles, not only tame. Sections 5
and 7 are devoted to the construction of basis elements arising from the regular
components. In Section 6 we consider the Kronecker quiver \( K \) and construct a \( \mathbb{Z} \-
basis of \( \mathbb{C}Z(K) \). The most important part of this section is the construction of basis
elements from the regular components. In Section 7 we consider the cyclic quiver
\( C_r \) and construct a \( \mathbb{Z} \)-basis of \( \mathbb{C}Z(C_r) \). This provides the basis elements arising
from the non-homogeneous tubes. The method we used in this section comes from
[DDX]. Finally, in Section 8 we combine the results from Section 5 to 7 to obtain
integral bases of \( \mathbb{C}Z(Q) \).

2. Notations and preliminaries

2.1. Cartan datum. Following Lusztig [L2], a Cartan datum is a pair \((I, (-,-))\)
consisting of a finite set \( I \) and a bilinear form \((-,-) : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z} \) which
satisfies the following conditions:

\[
(i, i) = 2, \text{ for all } i \in I; \\
(i, j) \leq 0, \text{ for all } i \neq j; \\
(i, j) = 0 \text{ if and only if } (j, i) = 0.
\]

Note that if we set \( a_{ij} = (i, j) \) then the matrix \( A = (a_{ij}) \) is a generalized Cartan
matrix.

A Cartan datum is said to be irreducible if the corresponding Cartan matrix
cannot be made block-diagonal by simultaneous permutations of rows and columns.
It is said to be symmetric if \((i, j) = (j, i)\) for all \( i, j \in I \). In this paper we always
assume that the Cartan datum is irreducible and symmetric.

A Cartan datum is said to be of finite type (resp. affine) if the corresponding
Cartan matrix is positive definite (resp. positive semi-definite). A Cartan datum
is said to be simply-laced if \((i, j) \in \{0, -1\} \) for all \( i, j \in I \).

2.2. Kac-Moody algebras and their enveloping algebras. For a given Cartan
datum there is the corresponding Kac-Moody algebra \( \mathfrak{g} \). \( \mathfrak{g} \) has the triangular
decomposition \( \mathfrak{g} \simeq \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \) with \( \mathfrak{n}^+ \), \( \mathfrak{n}^- \) the maximal nilpotent subalgebras
and \( \mathfrak{h} \) the Cartan subalgebra. The universal enveloping algebra \( U = U(\mathfrak{g}) \) is the
\( \mathbb{C} \)-algebra generated by \( \{e_i, f_i, h_i | i \in I\} \) with the following relations:

\[
[h_i, h_j] = 0, \text{ for all } i, j \in I; \\
[e_i, f_j] = \delta_{ij} h_i, \text{ for all } i, j \in I; \\
h_i, e_j = (i, j)e_j, \text{ for all } i, j \in I; \\
h_i, f_j = -(i, j)f_j, \text{ for all } i, j \in I;
\]

\[
\sum_{k=0}^{1-(i,j)} (-1)^k \binom{1-(i,j)}{k} e_i^k e_j e_i^{1-(i,j)-k} = 0, \text{ for } i \neq j;
\]

\[
\sum_{k=0}^{1-(i,j)} (-1)^k \binom{1-(i,j)}{k} f_i^k f_j f_i^{1-(i,j)-k} = 0, \text{ for } i \neq j.
\]
Let $U^+$ (resp. $U^-$, $U^0$) be the subalgebra of $U$ generated by $\{e_i\}_{i \in I}$ (resp. $\{f_i\}_{i \in I}$, $\{h_i\}_{i \in I}$). We know that $U \simeq U^+ \otimes U^0 \otimes U^-$. Actually $U^+$ (resp. $U^-$) is the universal enveloping algebra of $n^+$ (resp. $n^-$). The Kostant $\mathbb{Z}$-form $U_\mathbb{Z}$ is defined as the $\mathbb{Z}$-subalgebra of $U$ generated by $e_i^{(n)}$ and $f_i^{(n)}$, for all $i \in I$ and $n \in \mathbb{N}$, where $e_i^{(n)} = e_i^n/n!$, $f_i^{(n)} = f_i^n/n!$ are called divided powers. Let $U_\mathbb{Z}^+ = U^+ \cap U_\mathbb{Z}$ (resp. $U_\mathbb{Z}^- = U^- \cap U_\mathbb{Z}$) be the $\mathbb{Z}$-subalgebra of $U$ generated by $e_i^{(n)}$ (resp. $f_i^{(n)}$).

2.3. Quivers and their representations. A quiver is an oriented graph $Q = (I, \Omega, s, t)$ where $I$ is the set of vertices, $\Omega$ is the set of arrows, $s, t$ are two maps from $\Omega$ to $I$ denoting the starting and terminal vertex respectively. In this paper we consider quivers without loops (i.e. arrows from one vertex to itself).

A quiver is called of finite type (tame) if the corresponding Cartan datum is of finite type (resp. affine). Thus the underlying graph of a tame quiver is of type $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$ or $E_8^{(1)}$.

A representation of $Q$ over $\mathbb{C}$ is an $I$-graded $\mathbb{C}$-vector space $V = \oplus_{i \in I} V_i$ with a collection of linear maps $x = (x_h)_{h \in \Omega} \in \oplus_{h \in \Omega} \text{Hom}_{\mathbb{C}}(V_{s(h)}, V_{t(h)})$. A morphism from a representation $(V, x)$ to another representation $(V', x')$ is an $I$-graded $\mathbb{C}$-linear map $\phi : V \to V'$ such that $x_h' \phi_{s(h)} = \phi_{t(h)} x_h$ for any $h \in \Omega$.

The dimension vector of a representation $M = (V, x)$ is defined as a vector $\text{dim} M = \sum_{i \in I} (\text{dim}_{\mathbb{C}} V_i) i \in \mathbb{N}[I]$. A representation $(V, x)$ is called finite dimensional if $V_i$ is finite dimensional for all $i$.

A representation $(V, x)$ of $Q$ is called nilpotent if there exists $N \in \mathbb{N}$ such that $x_{h_N} \cdots x_{h_1} = 0$ for any sequence $h_1, \ldots, h_N \in \Omega$ with $t(h_i) = s(h_{i+1})$.

Denote by $\text{rep}(Q)$ the category of finite dimensional representations of $Q$. We know that $\text{rep}(Q)$ is equivalent to $\text{mod-}\mathbb{C}Q$, the category of finite dimensional left $\mathbb{C}Q$-modules, where $\mathbb{C}Q$ is the path algebra of $Q$. By this reason we will use $\mathbb{C}Q$-modules or representations of $Q$ freely in the sequel, and we will just write modules or representations for short when $Q$ is fixed. Denote by $\text{rep}_0(Q)$ the full subcategory of $\text{rep}(Q)$ consisting of all nilpotent representations. Note that if $Q$ has no oriented cycles, we have $\text{rep}(Q) = \text{rep}_0(Q)$.

The isomorphic classes of simple objects in $\text{rep}_0(Q)$ are in one-to-one correspondence with the vertices of $Q$. Namely, for each $i \in I$, set $V_i = \mathbb{C}$, $V_j = 0$ for $j \neq i$ and $x = 0$. Then the module $(V, x)$ is simple, denoted by $S_i$.

2.4. Euler forms. Let $(I, (-,-))$ be a Cartan datum, we have the corresponding Dynkin diagram, which is an unoriented graph. Giving any orientation of the graph we obtain a quiver $Q$. $Q$ is said to be a quiver corresponding to $(I, (-,-))$. Conversely, the Cartan datum $(I, (-,-))$ can be recovered from any quiver corresponding to it.

More precisely, we define a bilinear form $\langle -, - \rangle : \mathbb{Z}[I] \times \mathbb{Z}[I] \to \mathbb{Z}$ given by $\langle i, j \rangle = \delta_{ij} - \sharp\{h \in \Omega | s(h) = i, t(h) = j\}$, where $\delta_{ij}$ is the Kronecker symbol and $\sharp$ denote the number of elements in a set. This form is called the Euler form. We know that for any $M, N \in \text{rep}(Q)$,

$$\langle \text{dim} M, \text{dim} N \rangle = \dim \text{Hom}(M, N) - \dim \text{Ext}^1(M, N).$$

The symmetric Euler form is defined as $\langle \alpha, \beta \rangle = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$, for any $\alpha, \beta \in \mathbb{Z}[I]$. Then $(I, (-,-))$ is a Cartan datum, the corresponding Dynkin diagram of which is just the underlying graph of $Q$. 
In the sequel, we will write $g(Q)$ for the Kac-Moody algebra $g$ with Cartan datum corresponding to $Q$.

3. The Hall algebra

3.1. Constructible functions. Let $X$ be an algebraic variety over $\mathbb{C}$. A subset of $X$ is called locally closed if it is the intersection of an open and a closed subset. A constructible set in $X$ is a union of finite many locally closed subset of $X$. A function $f : X \to \mathbb{C}$ is called constructible if $f(X)$ is a finite set and $f^{-1}(m)$ is a constructible subset of $X$ for all $m \in \mathbb{C}$. We denote by $\mathcal{M}(X)$ the set of all constructible functions on $X$ with values in $\mathbb{C}$. $\mathcal{M}(X)$ is naturally a $\mathbb{C}$-vector space. Let $G$ be an algebraic group acting on $X$. We denote by $\mathcal{M}(X)^G$ the subspace of $\mathcal{M}(X)$ consisting of all $G$-invariant constructible functions.

Let $\phi : X \to Y$ be a morphism of algebraic varieties. We can define two linear maps $\phi^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ and $\phi_* : \mathcal{M}(X) \to \mathcal{M}(Y)$ as follows:

$$\phi^*(g)(x) = g(\phi(x))$$

for any $g \in \mathcal{M}(Y)$ and $x \in X$.

$$\phi_*(f)(y) = \sum_{a \in \mathbb{C}} a \chi(\phi^{-1}(y) \cap f^{-1}(a))$$

for any $f \in \mathcal{M}(X)$ and $y \in Y$, where $\chi$ denotes the Euler characteristic with compact support.

3.2. Varieties of representations. Given a quiver $Q$ and a fixed dimension vector $\alpha = \sum_{i \in \mathcal{I}} \alpha_i \in \mathbb{N}[\mathcal{I}]$, denote by $E_\alpha$ the set of all representations of $Q$ with dimension vector $\alpha$. i.e.

$$E_\alpha = \prod_{h \in \Omega} \text{Hom}_\mathbb{C}(C^\alpha_{(h)}, C^\alpha_{(h)})$$

Hence $E_\alpha$ is a affine space, in particular, an affine algebraic variety. Let $G_\alpha = \prod_{i \in \mathcal{I}} GL(\alpha_i, \mathbb{C})$. The group $G_\alpha$ acts on $E_\alpha$ by $g(x_h) = g_{(h)} x_h g_{(h)}^{-1}$, for any $g = (g_i)_{i \in \mathcal{I}}$, $x = (x_h)_{h \in \Omega}$. Let $E_n^G$ be the subset of all nilpotent representations in $E_\alpha$. It is easy to see that $E_n^G$ is a closed subvariety of $E_\alpha$. The group $G_\alpha$ also acts on $E_n^G$.

3.3. The Hall algebra. To simplify the notations, we write $\mathcal{M}(E_n^G)^{G_\alpha}$ as $\mathcal{H}_\alpha(Q)$. Let $\mathcal{H}(Q) = \bigoplus_{\alpha \in \mathbb{N}[\mathcal{I}]} \mathcal{H}_\alpha(Q)$.

Lusztig [L1] has defined a bilinear map

$$*: \mathcal{H}_\alpha(Q) \times \mathcal{H}_\beta(Q) \to \mathcal{H}_\gamma(Q)$$

for any $\alpha, \beta, \gamma \in \mathbb{N}[\mathcal{I}]$ such that $\alpha + \beta = \gamma$. Then an $\mathbb{N}[\mathcal{I}]$-graded multiplication can be endowed with $\mathcal{H}(Q)$.

The map $*$ is defined as follows: Consider the following diagram:

$$E_n^G \times E_n^G \xrightarrow{P_1} E' \xrightarrow{P_2} E'' \xrightarrow{P_3} E_n^G$$

where the notations are as follows:

$E'$ is the variety of all pairs $(W, x)$ consisting of $x \in E_n^G$ and an $x$-stable $I$-graded subspace of $\mathbb{C}^x$ such that $\text{dim} W = \beta$.
$\mathcal{E}\prime$ is the variety of all quadruples $(x,W,R^\alpha,R^\beta)$, where $(x,W) \in \mathcal{E}\prime$, $R^\beta$ is an isomorphism $C^\beta \simeq W$, $R^\alpha$ is an isomorphism $C^\alpha \simeq C^\gamma/W$.

$p_1(x,W,R^\alpha,R^\beta) = (x^\alpha,x^\beta)$, where $x^1R^\alpha_{s(h)} = R^\alpha_{t(h)}x_h^1$ and $x^1R^\beta_{s(h)} = R^\beta_{t(h)}x_h^1$,

$p_2(x,W,R^\alpha,R^\beta) = (x,W)$; $p_3(x,W) = x$.

Note that $p_1$ is smooth with connected fibres, $p_2$ is a principal $G_\alpha \times G_\beta$-bundle and $p_3$ is proper.

Now we can define a convolution product of constructible functions

For $f_\alpha \in \mathcal{H}_\alpha(Q)$, $f_\beta \in \mathcal{H}_\beta(Q)$, we let $f_1$ be a constructible function on $\mathcal{E}_{\alpha}^{nil} \times \mathcal{E}_{\beta}^{nil}$ given by $f_1(x_1,x_2) = f_\alpha(x_1)f_\beta(x_2)$ for any $x_1 \in \mathcal{E}_{\alpha}^{nil}$, $x_2 \in \mathcal{E}_{\beta}^{nil}$. Then there is a unique function $f_2 \in \mathcal{M}(\mathcal{E}\prime)$ such that $p_1^*f_1 = p_2^*f_2$. We define $f_\alpha * f_\beta$ as $(p_3)_!(f_2)$.

The $\mathbb{C}$-space $\mathcal{H}(Q)$ equipped with the multiplication $*$ is an $\mathbb{N}[I]$-graded associative $\mathbb{C}$-algebra, called the Hall algebra. In the sequel we will omit the operator $*$.

### 3.4. Characteristic functions.

For a fixed dimension vector $\alpha$ and a $G_\alpha$-invariant constructible subset $\mathcal{O}$ of $\mathcal{E}_{\alpha}^{nil}$, we have the characteristic function of $\mathcal{O}$, which is defined as the function taking the value 1 on $\mathcal{O}$ and 0 elsewhere. We denote the function by $1_\mathcal{O}$. It is obvious that $1_\mathcal{O} \in \mathcal{H}_\alpha(Q)$.

For any $M \in \mathcal{E}_{\alpha}^{nil}$, the $G_\alpha$-orbit of $M$ is denoted by $\mathcal{O}_M$. In particular, $\mathcal{O}_M$ is a constructible subset of $\mathcal{E}_{\alpha}^{nil}$. In this case we just write $1_M$ instead of $1_{\mathcal{O}_M}$.

Let $M \in \mathcal{E}_{\alpha}^{nil}$, $N \in \mathcal{E}_{\beta}^{nil}$. The definition of the multiplication yields

$$1_M1_N(L) = \chi(\mathcal{F}(M,N;L)),$$

where $\mathcal{F}(M,N;L)$ is the variety of all submodules $L'$ of $L$ such that $L' \simeq N$ and $L/L' \simeq M$.

In general, let $\mathcal{O}_1$ (resp. $\mathcal{O}_2$) be a $G_\alpha$ (resp. $G_\beta$)-invariant constructible subset of $\mathcal{E}_{\alpha}^{nil}$ (resp. $\mathcal{E}_{\beta}^{nil}$), we have

$$1_{\mathcal{O}_1}1_{\mathcal{O}_2}(M) = \chi(\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2;M)),$$

where $\mathcal{F}(\mathcal{O}_1,\mathcal{O}_2;M)$ is the variety of all submodules $M'$ of $M$ such that $M' \in \mathcal{O}_2$ and $M/M' \in \mathcal{O}_1$.

### 3.5. The composition algebra.

For any $i \in I$, the simple module $S_i$ is the unique module with dimension vector $i$. We write the characteristic function $1_{S_i}$ simply as $1_i$.

Let $\mathcal{C}(Q)$ be the $\mathbb{C}$-subalgebra of $\mathcal{H}(Q)$ generated by $1_i$, for all $i \in I$. $\mathcal{C}(Q)$ is called the composition algebra. The following theorem is well-known (for example, see [L1]):

**Theorem 3.1.** For any quiver $Q$ without loops, the composition algebra $\mathcal{C}(Q)$ is isomorphic to the positive part of the enveloping algebra $U^+=U^+(\mathfrak{g}(Q))$. This isomorphism is given by $1_i \mapsto e_i$ for any $i \in I$.

We can also define the $\mathbb{Z}$-form of the composition algebra (or the integral composition algebra) $\mathcal{C}_{\mathbb{Z}}(Q)$, which is the $\mathbb{Z}$-subalgebra of $\mathcal{H}(Q)$ generated by the divided powers $1_i^{(n)}$, for all $i \in I$ and $n \in \mathbb{N}$.

The following corollary can be seen immediately from the theorem.

**Corollary 3.2.** The integral composition algebra $\mathcal{C}_{\mathbb{Z}}(Q)$ is isomorphic to $U^+_{\mathbb{Z}}$. 
3.6. Some calculations. In general, the calculation of the Euler Characteristic of a variety is difficult. In this subsection, we give two formulas dealing with special cases which we will use later.

For any $M \in \text{rep}(Q)$, we denote by $tM$ the direct sum of $t$ copies of $M$. A module $M \in \text{rep}(Q)$ is called exceptional if $\text{Ext}^1(M, M) = 0$.

**Lemma 3.3.** For any exceptional module $M$ we have

$$1_{tM} = 1_M^{(t)}.$$

**Proof.** Since $M$ has no self-extensions, we have by definition

$$1_M = 1_{M_0} \supset 1_{M_1} \supset \cdots \supset 1_{M_t} = 0$$

where $F$ is the variety of all filtrations $tM = M_0 \supset M_1 \supset \cdots \supset M_t = 0$ with factors isomorphic to $M$.

It is easy to see that $\chi(F)$ is equal to the Euler characteristic of the variety of complete flags in $\mathbb{C}^t$. Hence $\chi(F) = t!$ and the lemma holds. □

**Lemma 3.4.** For any $M_1, \cdots, M_t \in \text{rep}(Q)$ such that $\text{Hom}(M_i, M_j) = 0$ and $\text{Ext}^1(M_j, M_i) = 0$ for all $i > j$. Then we have

$$1_M = 1_{M_1}1_{M_2} \cdots 1_{M_t}$$

where $M = \oplus_{i=1}^t M_i$.

**Proof.** It is sufficient to prove the case $t = 2$. The general case follows by induction.

So let $\text{Hom}(M_2, M_1) = 0$ and $\text{Ext}^1(M_1, M_2) = 0$, we need to prove $1_{M_1 \oplus M_2} = 1_{M_1}1_{M_2}$. Since $\text{Ext}^1(M_1, M_2) = 0$, we have

$$1_{M_1}1_{M_2} = \chi(G)1_{M_1 \oplus M_2}$$

where

$$G = \{ N \subset M_1 \oplus M_2 | N \simeq M_2 \text{ and } M_1 \oplus M_2 / N \simeq M_1 \}.$$

As $\text{Hom}(M_2, M_1) = 0$, we know that $G$ is a single point and hence $\chi(G) = 1$. □

4. Representation of tame quivers

In this section, we give a brief review of the representation theory of tame quivers. And in the first three subsections the results are valid for any quiver without oriented cycles, not only tame. For details one can see [DR], for example.

4.1. The classification of indecomposable modules. Let $Q$ be a quiver without oriented cycles (not necessarily be tame). Denote by $\text{ind}(Q)$ the set of isomorphic classes of indecomposable modules. For $M \in \text{rep}(Q)$, denote its isomorphic class by $[M]$.

The objects in $\text{ind}(Q)$ can be classified as follows: $M \in \text{ind}(Q)$ is called preprojective (resp. preinjective) if there exists a positive integer $n$ such that $\tau^n M = 0$ (resp. $\tau^{-n} M = 0$) where $\tau$ denotes the Auslander-Reiten translation. And $M$ is called regular if for any $n \in \mathbb{Z}$, $\tau^n M \neq 0$. We say a decomposable module is preprojective, preinjective or regular if each indecomposable summand is.

Let $\text{Prep}(Q), \text{Prei}(Q)$ and $\text{Reg}(Q)$ denote the full subcategory of $\text{rep}(Q)$ consisting of preprojective, preinjective and regular modules respectively. These three subcategories are extension-closed. Moreover, we have the following results:
Proposition 4.1. For any $P \in \text{Prep}(Q), R \in \text{Reg}(Q)$ and $I \in \text{Prei}(Q)$, we have
\[ \text{Hom}(I, P) = \text{Ext}^1(P, I) = 0; \]
\[ \text{Hom}(I, R) = \text{Ext}^1(R, I) = 0; \]
\[ \text{Hom}(R, P) = \text{Ext}^1(P, R) = 0. \]

Roughly speaking, the Auslander-Reiten-quiver (AR-quiver, for short) of $\text{rep}(Q)$ is the quiver whose vertices are objects in $\text{ind}(Q)$ and arrows are the irreducible morphisms between modules. We denote it by $\Gamma_Q$. Thus in the language of Auslander-Reiten theory, $\Gamma_Q$ can be divided into three components, called the preprojective, regular and preinjective components respectively.

We remark that if $Q$ is of finite type, there is no regular component in $\Gamma_Q$ and the preprojective component coincides with the preinjective component.

4.2. Root system and indecomposable modules. We denote by $\Delta$ the root system of $g(Q)$. For each $i \in I$, we identified it with the simple root $\alpha_i$. Thus the positive root lattice can be identified with $\mathbb{N}[I]$. And the set of positive roots is $\Delta^+ = \Delta \cap \mathbb{N}[I]$.

The set of real roots and imaginary roots are denoted by $\Delta^r$ and $\Delta^i$ respectively. Set $\Delta^+_r = \Delta^r \cap \Delta^+$, $\Delta^+_i = \Delta^i \cap \Delta^+$.

The following theorem is due to Kac, which is a generalization of Gabriel’s theorem.

Theorem 4.2. (1). For any $[M] \in \text{ind}(Q)$, $\dim M \in \Delta^+$.
(2). For any $\alpha \in \Delta^+_r$, there is a unique $[M] \in \text{ind}(Q)$ with $\dim M = \alpha$.
(3). For any $\alpha \in \Delta^+_i$, there are infinitely many $[M] \in \text{ind}(Q)$ with $\dim M = \alpha$.

For any $\alpha \in \Delta^+_r$, we denote the unique (up to isomorphism) indecomposable module with dimension vector $\alpha$ by $M(\alpha)$.

Denote by $\Delta^+_r$ (resp. $\Delta^+_i$, $\Delta^+_r$) the set of all positive roots which are dimension vectors of preprojective (resp. preinjective, regular) modules. It is known that $\Delta^+_r \cup \Delta^+_i \subset \Delta^+_r \cap \Delta^+_i \subset \Delta^+_i$ but in general they are not equal.

4.3. The preprojective and preinjective modules. To describe $\text{ind}(Q)$, we need to describe the preprojective, preinjective and the regular component respectively. The case of the preprojective or preinjective is easy. In fact it is similar to the case of finite type. We just recall the following representation-directed property:

Lemma 4.3. (1). $\Delta^+_r$ can be totally ordered as
\[ \Delta^+_r = \{\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_m \prec \cdots \} \]
such that
\[ \text{Hom}(M(\alpha_i), M(\alpha_j)) = 0, \text{ for all } i > j; \]
\[ \text{Ext}^1(M(\alpha_i), M(\alpha_j)) = 0, \text{ for all } i \leq j. \]

(2). $\Delta^+_i$ can be totally ordered as
\[ \Delta^+_i = \{\cdots \prec \beta_n \prec \cdots \prec \beta_2 \prec \beta_1 \} \]
such that
\[ \text{Hom}(M(\beta_i), M(\beta_j)) = 0, \text{ for all } i < j; \]
\[ \text{Ext}^1(M(\beta_i), M(\beta_j)) = 0, \text{ for all } i \geq j. \]
4.4. The Jordan quiver. The description of the regular component is more complicated. We need some preparations in this and the next subsection.

Let $C_1$ be the quiver with only one vertex and a loop arrow. This is the so-called Jordan quiver. Now a module in $\text{rep}_0(C_1)$ is just a pair $(V, x)$ where $V$ is a $\mathbb{C}$-space and $x$ is a nilpotent linear transformation on $V$.

The simple module is denoted by $S$. Any indecomposable module in $\text{rep}_0(C_1)$ with dimension $n$ is isomorphic to $S[n] = (\mathbb{C}^n, J_n)$, where $J_n$ is the $n \times n$ Jordan block with 0’s in the diagonal. And $\tau S[n] = S[n]$ for any $n$. The AR-quiver of $\text{rep}_0(C_1)$ is called a homogeneous tube.

4.5. The cyclic quiver. In this subsection we fix $r \in \mathbb{N}, r \geq 2$. Let $C_r = (I, \Omega, s, t)$ be a cyclic quiver with $r$ vertices, i.e. $I = \mathbb{Z}/r\mathbb{Z} = \{1, 2, \cdots, r\}$, $\Omega = \{\rho_i|1 \leq i \leq n\}$ where $s(\rho_i) = i, t(\rho_i) = i + 1$ for all $i$. Note that underline graph of this quiver is of type $A_{r-1}^{(1)}$, but it has an oriented cycle. So we consider the category $\text{rep}_0(C_r)$.

For any $i \in I$ and $l \geq 1$, there is a unique (up to isomorphism) indecomposable module in $\text{rep}_0(C_r)$ with top $S_i$ and length $l$, denoted by $S_i[l]$. And it is known that the set of isomorphic classes of indecomposable modules in $\text{rep}_0(C_r)$ is just \{ $S_i[l]|1 \leq i \leq r, l \geq 1$ \}. Moreover we have $\tau S_i[l] = S_{i+1}[l]$, hence $\tau^r S_i[l] = S_i[l]$ for all $i, l$. For this reason, the AR-quiver $\Gamma_{C_r}$ is called a non-homogeneous tube or more precisely, a tube of rank $r$ (So a homogeneous tube actually means a tube of rank 1).

Let $\delta = (1, 1, \cdots, 1)$ be the minimal imaginary root. We know that $\dim S_i[nr] = n\delta$ for any $i \in I$ and $n \geq 1$. And $\dim S_i[l] \in \Delta^*_+$ for any $i \in I$ and $r \not| l$.

Denote by $\mathcal{I}(C_r)$ the set of isomorphic classes of all modules in $\text{rep}_0(C_r)$. Let $\Pi$ be the set of $r$-tuples of partitions $\pi = (\pi^{(1)}, \pi^{(2)}, \cdots, \pi^{(r)})$ with each components $\pi^{(i)} = (\pi_1^{(i)} \geq \pi_2^{(i)} \geq \cdots)$ being a partition of a positive integer. For each $\pi \in \Pi$, we have a module

$$M(\pi) = \bigoplus_{i \in I, j \geq 1} S_i[\pi_j^{(i)}].$$

In this way we obtain a bijection between $\Pi$ and $\mathcal{I}(C_r)$.

4.6. Tame quivers and the regular modules. In this subsection let $Q = (I, \Omega, s, t)$ be a tame quiver without oriented cycles. Now we can describe the subcategory $\text{Reg}(Q)$. The following lemma shows that the regular component of $\Gamma_Q$ is a collection of tubes indexed by $\mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$. Moreover, there are only finite many non-homogeneous tubes and all the others are homogeneous.

Lemma 4.4.

$$\text{Reg}(Q) \cong (\prod_{j=1}^s T_j) \prod_{x \in \mathbb{P}^1 \setminus J} T_x$$

as coproduct of abelian categories, where $J$ is a subset of $\mathbb{P}^1$ containing $s$ elements, each $T_x$ is isomorphic to $\text{rep}_0(C_1)$ and each $T_i$ is isomorphic to $\text{rep}_0(C_{r_i})$ for some $r_i > 1$.

According to the lemma, For each $x \in \mathbb{P}^1$, let $F_x : \text{rep}_0(C_1) \to T_x, F_j : \text{rep}_0(C_{r_j}) \to T_j$ be the isomorphic functors respectively, for each $x \in \mathbb{P}^1 - J$ and $1 \leq j \leq s$. For any $x$ and $l \geq 1$, set $S_{x,l} = F_x(S[l])$ (see [4,3]). For any $1 \leq j \leq s$, $1 \leq i \leq r_i$ and $l \geq 1$, set $S_{j,i,l} = F_j(S_i[l])$ (see [4,3]).
The modules $S_{x,1}$ and $S_{j,i,1}$ are called quasi-simple for any $x$, $j$ and $i$. The module $S_{x,1}$ (resp. $S_{j,i,1}$) is the unique (up to isomorphism) module with quasi-top $S_{x,1}$ (resp. $S_{j,i,1}$) and quasi-length 1. Here the word quasi- means with respect to the subcategory $T_x$ or $T_i$. Note that for all $x$ and $i$, $T_x$ and $T_i$ are extension-closed abelian subcategories of $\text{rep}(Q)$.

We know that for an affine Cartan datum, $\Delta_{\text{im}} = (\mathbb{N} - \{0\})\delta$ where $\delta$ is the minimal imaginary root. The following lemma describes all indecomposable modules with dimension vector $n\delta$ and also tells us the difference between $\Delta_{\text{im}}$ and $\Delta_{\text{reg}}$.

**Lemma 4.5.** (1). For any $n \geq 1$, the set of isomorphic classes of indecomposable modules with dimension vector $n\delta$ is the following
\[
\{[S_{x,n}] | x \in P_1 \setminus J \} \cup \{[S_{j,i,nr_j}] | 1 \leq j \leq s, 1 \leq i \leq r_j \}
\]
(2). $\dim S_{j,i,l} \in \Delta_{\text{re}}^+$ for any $1 \leq j \leq s, 1 \leq i \leq r_j$ and $r_j \nmid l$.

In fact for each tame quiver $Q$, the number of non-homogeneous tubes and the rank of each tube can be determined precisely. Anyway we don’t need the details in this paper. But we need to mention one particular case: the Kronecker quiver $K$, which is the quiver with two vertices and two arrows pointing from one vertex to the other. In this case all tubes are homogeneous. Later (in section 6) we will discuss the composition algebra $C(K)$ in details.

At the end of this subsection, we recall a well-known result (see [CB], for example):

**Lemma 4.6.** We have:
\[
1 + \sum_{j=1}^{s} (r_j - 1) = |I| - 1
\]
and this number is equal to the multiplicity of any imaginary root.

5. Bases arising from preprojective and preinjective components

In this section we assume that $Q = (I, \Omega, s, t)$ is a quiver without oriented cycles (not necessarily be tame). We will discuss two subalgebras of the composition algebra $C(Q)$.

5.1. The subalgebra $C(Q)^{\text{prep}}$ and $C(Q)^{\text{prei}}$. Let $C(Q)^{\text{prep}}$ (resp. $C(Q)^{\text{prei}}$) be the subalgebra of $\mathcal{H}(Q)$ generated by $1_P$ for all $P \in \text{Prep}(Q)$ (resp. $1_I$ for all $I \in \text{Prei}(Q)$). In this section we will prove that $1_P$ and $1_I$ are actually in $C_{\mathbb{Z}}(Q)$. Thus $C(Q)^{\text{prep}}$ and $C(Q)^{\text{prei}}$ are subalgebras of the composition algebra $C(Q)$. Moreover, we can define $C_{\mathbb{Z}}(Q)^{\text{prep}}$ (resp. $C_{\mathbb{Z}}(Q)^{\text{prei}}$) to be the $\mathbb{Z}$-subalgebra of $C_{\mathbb{Z}}(Q)$ generated by $1_P$ for all $P \in \text{Prep}(Q)$ (resp. $1_I$ for all $I \in \text{Prei}(Q)$). At the end of this section we will construct $\mathbb{Z}$-bases of these two subalgebras.

5.2. Reflection functors. To prove $1_P$ and $1_I$ lie in the composition algebra we need to use some tools in representation theory.

Let $i$ be a sink of $Q$, i.e. there are no arrows starting from $i$. We define $\sigma_i Q$ to be the quiver obtained from $Q$ by reversing all the arrows connected to $i$. Following [BGP] we can define the reflection functors:

\[
\sigma_i^+ : \text{rep}(Q) \to \text{rep}(\sigma_i Q)
\]
The action of the functor $\sigma_i^+$ on objects is defined by $\sigma_i^+(V, x) = (V', x')$ where

\[
V'_k = V_k \text{ if } k \neq i,
\]

\[
V'_i = \ker(\sum_{t(h) = i} x_h : \sum_{t(h) = i} V_{s(h)} \to V_i),
\]

\[
x'_h = x_h \text{ if } t(h) \neq i,
\]

$x'_i$ is the composition $V'_i \to \sum_{t(h') = i} V_{s(h')} \to V_{s(h)}$ if $t(h) = i$.

The action of $\sigma_i^+$ on morphisms is the natural one.

Let $\text{rep}(Q)[i]$ be the subcategory of $\text{rep}(Q)$ consisting of all modules which do not have $S_i$ as direct summands. Note that since $i$ is a sink, $S_i$ is a simple projective module. Hence $\text{rep}(Q)[i]$ is closed under extensions. Then we can define $\mathcal{H}(Q)[i]$ to be the subalgebra of $\mathcal{H}(Q)$ generated by all constructible functions whose support are contained in $\text{rep}(Q)[i]$. The functor $\sigma_i^+$ induces an algebra homomorphism

\[\sigma_i : \mathcal{H}(Q)[i] \to \mathcal{H}(\sigma_i Q)[i]\]

defined by

\[\sigma_i(1_M) = 1_{\sigma_i^+ M}, \text{ for any } M \in \text{rep}(Q)[i]\]

Let $C(Q)[i] = \mathcal{H}(Q)[i] \cap C(Q)$, $C_2(Q)[i] = \mathcal{H}(Q)[i] \cap C_2(Q)$. Note that $C(Q)$ and $C(\sigma_i Q)$ (hence $C_2(Q)$ and $C_2(\sigma_i Q)$) are canonically isomorphic by fixing the Chevalley generators which correspond to the simple modules of $\text{rep}(Q)$ and $\text{rep}(\sigma_i Q)$ respectively.

We know that (see [BGP]) $\sigma_i^+$ restricts to an equivalence of categories:

\[\sigma_i^+ : \text{rep}(Q)[i] \xrightarrow{\sim} \text{rep}(\sigma_i Q)[i].\]

Hence it induces an isomorphism of algebras:

\[\sigma_i : C(Q)[i] \xrightarrow{\sim} C(Q)[i].\]

Dually for any source $i \in I$ we can define the reflection functor $\sigma_i^-$. We have similar results as above.

\[r_i = \exp(\text{ad} e_i) \exp(\text{ad} (-f_i)) \exp(\text{ad} e_i) : U \xrightarrow{\sim} U.\]

And we have

\[r_i(e_i) = -f_i\]

\[r_i(f_i) = -e_i\]

\[r_i(e_j) = (\text{ad} e_i)^{(-a_{ij})}(e_j), \text{ for } i \neq j\]

\[r_i(f_j) = -(\text{ad} f_i)^{(-a_{ij})}(f_j), \text{ for } i \neq j.\]

Thus $r_i$ is also an automorphism of $U_Z$.

**Lemma 5.1.** $\sigma_i$ restricting to $C_2(Q)[i]$ is a $\mathbb{Z}$-automorphism which equals to the restriction of $r_i$. 
Proof. We need a result proved in [XZZ] in which the reflection functors are generalized to the root category. It is proved that the reflection functor $\sigma_i$ induces an automorphism of the whole Kac-Moody algebra $\mathfrak{g}(Q)$ (and hence an automorphism of the enveloping algebra $U(\mathfrak{g})$) and this automorphism is just the same as $r_i$.

In our case we go back to the positive part $U^+$. When we restrict both automorphisms to $C_\mathbb{Z}(Q)[i] \subset U^+_{\mathbb{Z}}$, we get the result in the lemma. \hfill \Box

5.4. Admissible sequences. Let $i_1, \ldots, i_m$ be an admissible source sequence of $Q$, i.e. $i_1$ is a source of $Q$ and for any $1 < t \leq m$, the vertex $i_t$ is a source for $\sigma_{i_{t-1}} \cdots \sigma_{i_1} Q$.

Let $M \in \text{Pre}(Q)$ be indecomposable, then there exists an admissible source sequence $i_1, \ldots, i_m$ of $Q$ such that

$$M = \sigma_{i_1}^+ \cdots \sigma_{i_{m-1}}^+(S_{i_m})$$

where $S_{i_m}$ is the simple $C(\sigma_{i_{m-1}} \cdots \sigma_{i_1} Q)$-module corresponding to the vertex $i$. (See [BGP])

Lemma 5.2. Let $M \in \text{Pre}(Q)$ be indecomposable. Then there exists an admissible source sequence $i_1, \ldots, i_m$ of $Q$ such that

$$\mathbf{1}_M = r_{i_1} \cdots r_{i_{m-1}} \mathbf{1}_{i_m}$$

Proof. It is clear by the definition of $\sigma_i$ and lemma 5.1. \hfill \Box

The similar results can be proved for indecomposable preprojective modules using admissible sink sequences and $\sigma_i^-$ instead.

Thus by the above lemma we can see that for any indecomposable $M \in \text{Pre}(Q)$ or $\text{Prep}(Q)$, the corresponding $\mathbf{1}_M \in C_\mathbb{Z}(Q)$.

5.5. $\mathbb{Z}$-Bases of $C_\mathbb{Z}(Q)_{\text{prei}}$ and $C_\mathbb{Z}(Q)_{\text{prep}}$. We will use the notations in 4.3. Let $I$ be any preinjective $\mathbb{C}Q$-module, then it can be decomposed into a direct sum of indecomposable preinjective modules.

Lemma 5.3. For any $I \in \text{Pre}(Q)$, if it decomposes as

$$I = \bigoplus_{k=1}^m b_k M(\beta_k)$$

where $\beta_{i_m} < \cdots < \beta_{i_2} < \beta_{i_1} \in \Delta^\text{prei}_+$ and $b_k \neq 0$. Then we have

$$\mathbf{1}_I = \mathbf{1}_{M(\beta_{i_m})}^{(b_{i_m})} \cdots \mathbf{1}_{M(\beta_{i_2})}^{(b_{i_2})} \mathbf{1}_{M(\beta_{i_1})}^{(b_{i_1})}$$

Proof. The key is the representation-directed property (see 4.3). Then the result is clear by Lemma 5.3 and Lemma 5.4. \hfill \Box

From this lemma we can see that $\mathbf{1}_I \in C_\mathbb{Z}(Q)$ for all $I \in \text{Pre}(Q)$, which we have claimed in the beginning of this section. Now we can give a $\mathbb{Z}$-basis of $C_\mathbb{Z}(Q)_{\text{prei}}$.

Proposition 5.4. The set $\{\mathbf{1}_I | I \in \text{Pre}(Q)\}$ is a $\mathbb{Z}$-basis of $C_\mathbb{Z}(Q)_{\text{prei}}$ (hence also a $\mathbb{C}$-basis of $C(Q)_{\text{prei}}$).

Proof. Since the subcategory $\text{Pre}(Q)$ is extension-closed, we have for any $I_1, I_2 \in \text{Pre}(Q)$,

$$\mathbf{1}_{I_1} \mathbf{1}_{I_2} = \sum_{I \in \text{Pre}(Q); \dim I = \dim I_1 + \dim I_2} \chi(I; I_1, I_2) \mathbf{1}_I$$
Lemma 5.5. For any \( P \in \text{Prep}(Q) \), if it decomposes as
\[
P = \bigoplus_{k=1}^{m} a_{i_k} M(\alpha_{i_k})
\]
where \( \alpha_{i_1} \prec \alpha_{i_2} \prec \cdots \prec \alpha_{i_m} \in \Delta_+^{\text{prei}} \) and \( a_{i_k} \neq 0 \). Then we have
\[
1_P = 1^{(\alpha_{i_1})}_{M(\alpha_{i_1})} 1^{(\alpha_{i_2})}_{M(\alpha_{i_2})} \cdots 1^{(\alpha_{i_m})}_{M(\alpha_{i_m})}
\]

Proposition 5.6. The set \( \{1_P | P \in \text{Prep}(Q)\} \) is a \( \mathbb{Z} \)-basis of \( C_\mathbb{Z}(Q)^{\text{prep}} \) (hence a \( \mathbb{C} \)-basis of \( C(Q)^{\text{prep}} \)).

5.6. Remarks. (1). The arguments in this section are essentially the same as in the case of finite type. In fact when \( Q \) is of finite type, we have \( C(Q)^{\text{prep}} = C(Q)^{\text{rep}} = C(Q) \). Thus a \( \mathbb{Z} \)-basis of \( C_\mathbb{Z}(Q) \) has been given.

(2). The proofs of many results in this section is similar to the quantum case. For example, see [R3] (which discussed the case of finite type).

6. Integral Basis: The Case of the Kronecker Quiver

In this section we consider the simplest tame quiver, namely the Kronecker quiver \( K = (I, \Omega, s, t) \) where \( I = \{1, 2\} \), \( \Omega = \{\rho_1, \rho_2\} \), \( s(\rho_1) = s(\rho_2) = 1 \) and \( t(\rho_1) = t(\rho_2) = 2 \). Note that this quiver is the only non-simply-laced tame quiver.

6.1. Some notations. For convenience in this section we will identify \( \mathbb{N}^2 \) with \( \mathbb{N}[I] \) and write the dimension vectors as \((a, b) \in \mathbb{N}^2\). The set of positive roots are
\[
\Delta_+ = \{ (n, n+1), (m+1, m), (l+1, l+1) | n, m, l \in \mathbb{N} \}.
\]

And we have \( \Delta_+^{\text{prep}} = \{ (m+1, m) | m \in \mathbb{N} \} \), \( \Delta_+^{\text{prei}} = \{ (n, n+1) | n \in \mathbb{N} \} \), \( \Delta_+^{\text{reg}} = \{ (l+1, l+1) | l \in \mathbb{N} \} \) respectively. Note that the minimal imaginary root \( \delta = (1, 1) \).

Hence in this case \( \Delta_+^{\text{reg}} = \Delta_+^{\text{im}} \).

The order on \( \Delta_+^{\text{prei}} \) given by the representation-directed property (lemma 4.3) is
\[
\cdots \prec (n, n+1) \prec (1, 2) \prec (0, 1),
\]
and the order on \( \Delta_+^{\text{pre}} \) is
\[
(1, 0) \prec (2, 1) \prec \cdots \prec (m+1, m) \prec \cdots.
\]

Recall [4.6] that \( \text{Reg}(K) \simeq \bigsqcup_{x \in \mathcal{P}} T_x \), where \( T_x \simeq \text{rep}_0(C_1) \) for all \( x \).

6.2. A basis of \( n^+(K) \). In this section, for simplicity we will denote by \( 1_\alpha \) the characteristic function \( 1_{M(\alpha)} \) for any \( \alpha \in \Delta_+^{\text{reg}} \). Since \( \dim S_1 = (1, 0), \dim S_2 = (0, 1) \), we write \( 1_1 = 1_{(1,0)}, 1_2 = 1_{(0,1)}. \)

Following [FMV], for any \( n \geq 1 \), the set of all indecomposable regular modules with dimension vector \( n \delta \) is a constructible subset of \( E_{n\delta} \) (see 3.2). Let \( P_{n\delta} \) be the characteristic function of this set. Hence \( P_{n\delta} \in \mathcal{H}(Q) \).

The following results have been proved in [FMV]:
Proposition 6.1. The set
\[ \{1_{(m,m+1)}, 1_{(n+1,n)}, P_\delta | m, n \geq 0; k \geq 1 \} \]
is a basis of the maximal nilpotent subalgebra \( n^+(K) \) of the Lie algebra \( \mathfrak{g}(K) \).
Moreover, the structure constants with respect to the basis are clear:
\[
[P_{m\delta}, P_{n\delta}] = 0; \\
[1_{(n,n+1)}, 1_{(m,m+1)}] = 0; \\
[1_{(n+1,n)}, 1_{(m+1,m)}] = 0; \\
[P_{n\delta}, 1_{(m+1,m)}] = 21_{(m+n+1,m+n)}; \\
[1_{(m,m+1)}, P_{n\delta}] = 21_{(m+n,m+n+1)}; \\
[1_{(m,m+1)}, 1_{(n+1,n)}] = P_{(m+n+1)\delta};
\]
for any \( m, n \in \mathbb{N} \).

Since \( 1_{(m,m+1)}, 1_{(n+1,n)} \in C_\mathbb{Z}(K) \) (see section 6.3), by the last formula in the above proposition we can see that \( P_{n\delta} \in C_\mathbb{Z}(K) \) for any \( n \geq 1 \).

6.3. The function \( H_{n\delta} \). For \( n \geq 1 \), the set of all regular modules (may be decomposable) with dimension vector \( n\delta \) is also a constructible subset. Let \( H_{n\delta} \) be the characteristic function of this set. For convenience we also set \( H_{0\delta} = 1 \).

Lemma 6.2.
\[
1_2^{(n)} 1_1^{(n+1)} = 1_{(n+1,n)} + \sum_{l=1}^{n} 1_{(n+1-l,n-l)} H_{l\delta} + \sum_{P,I,l} 1_P H_{l\delta} 1_I; \\
1_2^{(n+1)} 1_1^{(n)} = 1_{(n,n+1)} + \sum_{l=1}^{n} H_{l\delta} 1_{(n-l,n+1-l)} + \sum_{P,I,l} 1_P H_{l\delta} 1_I; \\
1_2^{(n)} 1_1^{(n)} = H_{n\delta} + \sum_{P,I,l} 1_P H_{l\delta} 1_I,
\]
where in the formulas the last terms sum over all non-zero \( P \in \text{Prei}(Q) \), \( I \in \text{Prei}(Q) \) and \( 1 < l < n-1 \) such that \( \dim P + \dim I + (l,I) = (n+1,n), (n,n+1), (n,n) \) respectively.

Proof. We just prove (1), the proofs for (2) and (3) are similar.

By lemma 3.3 we know that
\[
1_2^{(n)} 1_1^{(n+1)} = 1_{nS_2} 1_{(n+1)S_1}.
\]

Note that Ext\(^1\)(\( S_2, S_1 \)) \( \neq 0 \) and Ext\(^1\)(\( S_1, S_2 \)) = 0. So each module with dimension vector \( (n+1,n) \) is in the support of \( 1_{nS_2} 1_{(n+1)S_1} \). Thus the support of \( 1_{nS_2} 1_{(n+1)S_1} \) contains infinitely many orbits of non-isomorphic modules. But for any such module \( M \) we have
\[
1_{nS_2} 1_{(n+1)S_1}(M) = \chi(\mathcal{F}(nS_2, (n+1)S_1; M)) = 1,
\]
since Hom\((S_1, S_2) = 0 \).

Note that each module can be decomposed into a direct sum of preprojective, regular and preinjective modules. Then using lemma 6.4 and the definition of \( H_{n\delta} \) we get the formula (1). \( \square \)
Corollary 6.3. $H_{n\delta} \in \mathcal{C}_K$, for any $n \geq 1$.

Proof. The left hand sides of the formulas in the above lemma are in $\mathcal{C}_K$. Also we know that for any $P \in \text{Pre}(K)$, $I \in \text{Prei}(K)$, $1_P, 1_I \in \mathcal{C}_K$. Then the corollary follows easily by induction on $n$. □

By concrete calculations we can find the relation between $H_{n\delta}$ and $P_{n\delta}$:

Lemma 6.4. For any $n \in \mathbb{N}, n \geq 1$,

$$H_{n\delta} = \frac{1}{n} \sum_{l=0}^{n-1} H_{l\delta} P_{(n-l)\delta}$$

Proof. By Lemma 6.2 we have

$$1_{(n-1,n)} 1_1 = 1_{2}^{(n)} 1_1^{(n-1)} 1_1 - \left( \sum_{l=1}^{n-1} H_{l\delta} 1_{(n-l-1,n-l)} 1_1 + \sum_{P,I} 1_P H_{l\delta} 1_I 1_1 \right)$$

$$= n 1_{2}^{(n)} 1_1^{(n)} - \sum_{l=1}^{n-1} H_{l\delta} 1_{(n-l-1,n-l)} 1_1 - \sum_{P,I} 1_P H_{l\delta} 1_I 1_1$$

$$= n H_{n\delta} + X,$$

where

$$X = n \sum_{P,I} 1_P H_{l\delta} 1_I - \sum_{l=1}^{n-1} H_{l\delta} 1_{(n-l-1,n-l)} 1_1 - \sum_{P,I} 1_P H_{l\delta} 1_I 1_1.$$  

and in the above formula the last term sums over all non-zero preprojective and preinjective modules $P, I$ and $1 < l < n - 1$ such that $\dim P + \dim I + (l, I) = (n-1, n)$.

Then by Proposition 6.1 we have

$$P_{n\delta} = 1_{(n-1,n)} 1_1 - 1_I 1_{(n-1,n)} = n H_{n\delta} + X - 1_I 1_{(n-1,n)}.$$  

Now we only need to prove

$$X - 1_I 1_{(n-1,n)} = - \sum_{l=1}^{n-1} H_{l\delta} P_{(n-l)\delta}.$$  

In fact, in the above formula, the left hand side is

$$n \sum_{P,I} 1_P H_{l\delta} 1_I - \sum_{l=1}^{n-1} H_{l\delta} 1_{(n-l-1,n-l)} 1_1 - \sum_{P,I} 1_P H_{l\delta} 1_I 1_1 - 1_I 1_{(n-1,n)}$$

$$= - \sum_{l=1}^{n-1} H_{l\delta} (1_{(n-l-1,n-l)} 1_1 - 1_I 1_{(n-l-1,n-l)}) + Y$$

$$= - \sum_{l=1}^{n-1} H_{l\delta} P_{(n-l)\delta} + Y,$$

where

$$Y = n \sum_{P,I} 1_P H_{l\delta} 1_I - \sum_{P,I} 1_P H_{l\delta} 1_I 1_1 - 1_I 1_{(n-1,n)} - \sum_{l=1}^{n-1} H_{l\delta} 1_I 1_{(n-l-1,n-l)}.$$
Thus it remains to prove \( Y = 0 \). If \( Y \neq 0 \), it is easy to see that \( Y \) is a non-zero preinjective summand. But on the other hand,

\[
Y = \sum_{i=0}^{n-1} H_{i \delta} P_{(n-i)\delta} - nH_{n\delta}
\]

whose support contains only regular modules, which is a contradiction. \( \square \)

From this lemma we also know that \( H_{n\delta} H_{m\delta} = H_{m\delta} H_{n\delta} \) for any \( n, m \in \mathbb{N} \).

6.4. **The subalgebra** \( C_\mathbb{Z}(K)^{\text{reg}} \). Let \( C_\mathbb{Z}(K)^{\text{reg}} \) (resp. \( C(K)^{\text{reg}} \)) be the \( \mathbb{Z} \)-subalgebra (resp. \( \mathbb{C} \)-subalgebra) of \( \mathcal{C}(Q) \) generated by \( \{ H_{n\delta} | n \in \mathbb{N} \} \).

For a positive integer \( n \), let \( \mathbf{P}(n) \) be the set of all partitions of \( n \). For any \( \lambda \in \mathbf{P}(n) \) we also denote by \( \lambda \vdash n \) and write \( |\lambda| = n \). For \( n = 0 \), we set \( \mathbf{P}(0) = \{ 0 \} \).

For any \( \omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n \), we define

\[
H_{\omega\delta} = H_{\omega_1\delta} H_{\omega_2\delta} \cdots H_{\omega_t\delta}.
\]

The following lemma is obvious.

**Lemma 6.5.**

\[
\begin{align*}
C(K)^{\text{reg}} &\simeq \mathbb{C}[H_{\delta}, H_{2\delta}, \cdots, H_{n\delta}, \cdots], \\
C_\mathbb{Z}(K)^{\text{reg}} &\simeq \mathbb{Z}[H_{\delta}, H_{2\delta}, \cdots, H_{n\delta}, \cdots].
\end{align*}
\]

And the set \( \{ H_{\omega\delta} | \omega \vdash n, n \in \mathbb{N} \} \) is a \( \mathbb{Z} \)-basis of \( C_\mathbb{Z}(K)^{\text{reg}} \) and a \( \mathbb{C} \)-basis of \( C(K)^{\text{reg}} \).

From this lemma we know that \( C(K)^{\text{reg}} \) (resp. \( C_\mathbb{Z}(K)^{\text{reg}} \)) is naturally \( \mathbb{N} \)-graded, namely

\[
C(K)^{\text{reg}} = \oplus_{n \in \mathbb{N}} C(K)^{\text{reg}}_n; \quad C_\mathbb{Z}(K)^{\text{reg}} = \oplus_{n \in \mathbb{N}} C_\mathbb{Z}(K)^{\text{reg}}_n,
\]

where \( C(K)^{\text{reg}}_n \) (resp. \( C_\mathbb{Z}(K)^{\text{reg}}_n \)) is the \( \mathbb{C} \)-subspace (resp. free \( \mathbb{Z} \)-submodule) generated by \( \{ H_{\omega\delta} | \omega \vdash n \} \). Equivalently, \( C(K)^{\text{reg}}_n \) (resp. \( C_\mathbb{Z}(K)^{\text{reg}}_n \)) is the \( \mathbb{C} \)-subspace (resp. free \( \mathbb{Z} \)-submodule) generated by constructible functions in \( C(K)^{\text{reg}} \) whose supports are contained in \( \mathbf{E}_{n\delta} \).

Then we know that the dimension of \( C(K)^{\text{reg}}_n \) (or the rank of \( C_\mathbb{Z}(K)^{\text{reg}}_n \)) is \( |\{ \omega | \omega \vdash n \}| \), which is a finite number.

6.5. **The functions** \( M_{\omega\delta} \) and \( E_{n\delta} \). For any \( n \geq 1 \) and \( \omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_t) \vdash n \), let \( \mathcal{S}_\omega \) be the constructible subset of \( \mathbf{E}_{n\delta} \) consisting of regular modules \( R \simeq R_1 \oplus R_2 \oplus \cdots \oplus R_s \) with \( \dim R_i = \omega_i \) and \( R_i \) indecomposable for all \( i \). We define \( M_{\omega\delta} \) to be the characteristic function of the set \( \mathcal{S}_\omega \). We also set \( M_{0\delta} = 1 \). By definition we have

**Lemma 6.6.** For any \( n \in \mathbb{N} \),

\[
H_{n\delta} = \sum_{\omega \vdash n} M_{\omega\delta}.
\]

We will prove that the set \( \{ M_{\omega\delta} | \omega \vdash n, n \in \mathbb{N} \} \) is also a \( \mathbb{Z} \)-basis of \( C_\mathbb{Z}(K)^{\text{reg}} \).

The idea comes from the theory of symmetric functions. Let’s recall some notations and results in [M]. Let \( \Lambda \) be the ring of symmetric functions in countably many independent variables with coefficients in \( \mathbb{Z} \). For \( n \in \mathbb{N}, n \geq 1 \), denote by \( h_n \) (resp. \( e_n \)) the \( n \)th complete symmetric function (resp. elementary symmetric function). We know that

\[
\Lambda \simeq \mathbb{Z}[h_1, h_2, \cdots, h_n, \cdots] \simeq \mathbb{Z}[e_1, e_2, \cdots, e_n, \cdots]
\]
Now we come back to $C_\omega(K)^{reg}$. We use the notations in [4.6]. For each $x \in \mathbb{P}^1$, denote by $h_{n,x}$ the characteristic function of all modules in $T_x$ with dimension vector $n\delta$, and let $e_{n,x}$ be the characteristic function of the module $nS_{x,1}$. Note that $S_{x,1}$ is the unique quasi-simple module in $T_x$. Let $H(T_x)$ be the subalgebra of $\mathcal{H}(Q)$ generated by all characteristic functions $1_M$ with $M \in T_x$. The following lemma also comes from [M]:

**Lemma 6.7.** For any $x \in \mathbb{P}^1$, there exists an isomorphism

$$\psi_x : H(T_x) \xrightarrow{\sim} A$$

where $\psi_x(h_{n,x}) = h_n$ and $\psi_x(e_{n,x}) = e_n$ for any $n \geq 1$.

For any $n \geq 1$, let $E_{n\delta}$ be the characteristic function of the set $S_{\{1^n\}}$ where $(1^n) = (1, 1, \ldots, 1) \vdash n$. So $E_{n\delta} = M_{\{1^n\}\delta}$. For convenience, set $E_{\emptyset \delta} = 1$. We also define $E_{\omega \delta} = E_{\omega_1 \delta}E_{\omega_2 \delta} \cdots E_{\omega_i \delta}$ for $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_i) \vdash n$.

**Lemma 6.8.** The set $\{E_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $C_\omega(K)^{reg}$ and a $\mathbb{C}$-basis of $C(K)^{reg}$.

**Proof.** First it is easy to see that the elements in the set $\{E_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ are $\mathbb{Z}$-linear independent.

Let $E(t) = 1 + \sum_{n \geq 1}^{} E_{n\delta} t^n$, $H(t) = 1 + \sum_{n \geq 1}^{} H_{n\delta} t^n$ be the generating functions. Also for each $x \in \mathbb{P}^1$ let $E_x(t) = 1 + \sum_{n \geq 1}^{} e_{n,x} t^n$ and $H_x(t) = 1 + \sum_{n \geq 1}^{} h_{n,x} t^n$.

By the definitions we can see that

$$E(t) = \prod_{x \in \mathbb{P}^1} E_x(t), \quad H(t) = \prod_{x \in \mathbb{P}^1} H_x(t).$$

By Lemma 6.7 and results in [M] (section I.2), we have $H_x(t)E_x(-t) = 1$ for any $x \in \mathbb{P}^1$. Thus

$$H(t)E(-t) = 1.$$

Equivalently, we have

$$\sum_{k=0}^{n} (-1)^k E_{k\delta} H_{(n-k)\delta} = 0$$

for all $n \geq 1$.

Now by induction we can see that for any $n$, $H_{n\delta}$ is in the $\mathbb{Z}$-span of $\{E_{\omega \delta} | \omega \vdash n\}$ and vice versa. Since the set $\{H_{n\delta} | \omega \vdash n\}$ is a $\mathbb{Z}$-basis of $C_\omega(K)^{reg}$, we see that $\{E_{\omega \delta} | \omega \vdash n\}$ is also a $\mathbb{Z}$-basis of $C_\omega(K)^{reg}$. Hence the lemma holds. \hfill \Box

For any partition $\lambda \vdash n$, let $\lambda'$ be the conjugate of $\lambda$. By definition $\lambda' \vdash n$ and the Young diagram of $\lambda'$ is the transpose of the one of $\lambda$. Recall that for any positive integer $n$, the dominance order on the set $\mathbb{P}(n)$ is defined as follows: $\lambda \leq \mu$ if and only if $\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$ for all $i \geq 1$.

**Lemma 6.9.** For any $\omega = (\omega_1 \geq \omega_2 \geq \cdots \geq \omega_i) \vdash n$, we have

$$E_{\omega \delta} = M_{\omega'} + \sum_{\mu \subset \lambda'} a_{\omega'\mu} M_{\mu \delta},$$

where $a_{\omega'\mu} \in \mathbb{Z}$. 
Proof. Note that $M_{\omega \delta} \in C(K)_n^{reg}$ for any fixed $\omega \vdash n$. Further, $\{M_{\omega \delta} | \omega \vdash n\}$ is a linearly independent set. Hence it is a $C$-basis of $C(K)_n^{reg}$. So $E_{\omega \delta}$ is a $C$-linear combination of $M_{\mu \delta}$, $\mu \vdash n$.

By definition

$$E_{\omega \delta} = E_{\omega_1 \delta} E_{\omega_2 \delta} \cdots E_{\omega_k \delta}. $$

For any $N \in \text{Reg}(K)$, let $F(\omega; N)$ be the set of all filtrations

$$0 = N_1 \subset N_2 \subset \cdots \subset N_0 = N$$
such that $N_{i-1}/N_i$ is isomorphic to a direct sum of $\omega_i$ quasi-simples. So we have

$$E_{\omega \delta}(N) = \chi(F(\omega; N)).$$

Suppose that $\omega' = (\omega'_1, \omega'_2, \cdots, \omega'_m)$. It is not difficult to see that $N$ is in the support of $E_{\omega}$ if and only if $N \in S_\mu$ for some $\mu \leq \omega'$. Thus

$$E_{\omega \delta} = \sum_{\mu \leq \omega'} a_{\omega' \mu} M_{\mu \delta}. $$

Choosing any $N_\mu \in S_\mu$, we have

$$E_{\omega \delta}(N_\mu) = \chi(F(\omega; N_\mu)) = a_{\omega' \mu} \in \mathbb{Z}. $$

Now it remains to prove $a_{\omega' \omega'} = 1$. This is equivalent to prove that for any $N_{\omega'} \in S_{\omega'}$, $\chi(F(\omega; N_{\omega'})) = 1$. But the only filtration of $N_{\omega'}$ in $F(\omega; N_{\omega'})$ is

$$0 = \text{rad}^1(N) \subset \cdots \subset \text{rad}^i(N) \subset \text{rad}^n(N) = N,$$

where rad denote the quasi-radical i.e. the radical in the subcategory $\text{Reg}(K)$. Hence $F(\omega; N_{\omega'})$ is a single point and we are done. \hfill $\square$

Finally we can prove the following:

**Lemma 6.10.** The set $\{M_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $C_\mathbb{Z}(K)^{reg}$ and a $C$-basis of $C(K)^{reg}$.

**Proof.** For any fixed $n \in \mathbb{N}$, $\{E_{\omega \delta} | \omega \vdash n\}$ is a $\mathbb{Z}$-basis of $C_\mathbb{Z}(K)_n^{reg}$. By the lemma above the transition matrix from $\{M_{\omega \delta} | \omega \vdash n\}$ to $\{E_{\omega \delta} | \omega \vdash n\}$ is upper triangular with 1’s in the diagonal. Thus for $\{M_{\omega \delta} | \omega \vdash n\}$ is also a $\mathbb{Z}$-basis of $C_\mathbb{Z}(K)_n^{reg}$. So $\{M_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of $C_\mathbb{Z}(K)^{reg}$. \hfill $\square$

### 6.6. Integral bases of $C_\mathbb{Z}(K)$

The main result of this section is the following:

**Proposition 6.11.** The set $\{1_P M_{\omega \delta} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$ is a $\mathbb{Z}$-basis of the algebra $C_\mathbb{Z}(K)$.

**Proof.** First we prove that the above set is a $\mathbb{C}$-basis of the algebra $C(K)$.

By Proposition 6.3 and the PBW-basis theorem, the set

$$\{1_P P_{\omega \delta} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{C}$-basis of $C(K)$, where $P_{\omega \delta} = P_{\omega_1 \delta} \cdots P_{\omega_k \delta}$. But from lemma 6.4 we can see that $\{P_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ and $\{H_{\omega \delta} | \omega \vdash n, n \in \mathbb{N}\}$ can be $\mathbb{C}$-expressed each other (actually the coefficients are in $\mathbb{Q}$). So the following set

$$\{1_P H_{\omega \delta} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}$$

is a $\mathbb{C}$-basis of $C(K)$. 


By Lemma 6.5 and Lemma 6.10 we can see that the set in the proposition is also a $\mathbb{C}$-basis of $\mathcal{C}(K)$.

Now consider the $\mathbb{Z}$-subalgebra of $\mathcal{C}(K)$ generated by
\[
\{1_P, M_{\omega \delta}, 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N}\}.
\]

We claim that this $\mathbb{Z}$-subalgebra is $\mathcal{C}_Z(K)$. First, by Lemma 5.5 and Lemma 6.10 the divided powers $1^{(l)}_1$ and $1^{(m)}_2$ for any $l, m \in \mathbb{N}$, are contained in the above generators. Second, from results in section 5.5 and 6.3 we know that the generators above are all in $\mathcal{C}_Z(K)$. Thus our claim holds.

It remains to prove that any product of the generators is in the $\mathbb{Z}$-span of the elements in the set. For the case $1_P 1_{P'}$ with $P, P' \in \text{Prep}(K)$ and $1_I 1_{I'}$ with $I, I' \in \text{Prei}(K)$, we have already done in 5.5. And Lemma 6.10 shows that for any $\lambda \vdash n, \mu \vdash m$, $M_{\lambda \delta} M_{\mu \delta}$ also has the desired property.

For any $P \in \text{Prep}(K), I \in \text{Prei}(K)$, since the set is a $\mathbb{C}$-basis of $\mathcal{C}(K)$, we have
\[
1_I 1_P = \sum_{P', \omega, I'} a_{P', \omega, I'} 1_{P'} M_{\omega \delta} 1_{I'},
\]
where $a_{P', \omega, I'} \in \mathbb{C}$ and
\[
\dim P' + |\omega| \delta + \dim I' = \dim P + \dim I.
\]

We need to prove all the coefficients $a_{P', \omega, I'}$ are integers. For any $P', \omega, I'$, the function $1_{P'} M_{\omega \delta} 1_{I'}$ is the characteristic function of the following set (recall the definition of $S_\omega$ in 6.3):
\[
\{M \simeq P' \oplus R \oplus I' | R \in S_\omega\}
\]

Denote the set by $S_{P', \omega, I'}$. It is easy to see that $S_{P', \omega, I'} \cap S_{P'', \mu, I''} \neq \emptyset$ if and only if $P' = P'', \omega = \mu$, and $I' = I''$.

Thus for any fixed $P', \omega, I'$, and any module $N_{P', \omega, I'} \in S_{P', \omega, I'}$ we have
\[
1_I 1_P (N_{P', \omega, I'}) = a_{P', \omega, I'}.
\]

But on the other hand by the definition of the multiplication in the Hall algebra,
\[
1_I 1_P (N_{P', \omega, I'}) = \chi(\mathcal{F}(I, P; N_{P', \omega, I'})).
\]

Hence $a_{P', \omega, I'} = \chi(\mathcal{F}(I, P; N_{P', \omega, I'})) \in \mathbb{Z}$.

Next we consider $M_{\omega \delta} 1_P$ for any $\lambda \vdash n$ and $P \in \text{Prep}(K)$. Since preinjective modules do not occur in the direct summands of the extension of a regular module by a preprojective module, so first we have
\[
M_{\omega \delta} 1_P = \sum_{P', \mu} b_{P', \mu} 1_{P'} M_{\mu \delta},
\]
where $b_{P', \mu} \in \mathbb{C}$ and $\dim P' + |\mu| \delta = \dim P + n \delta$.

For any module $N$, let $\mathcal{F}(\omega, P; N)$ be the set consisting of all submodules $L$ of $N$ such that $L \simeq P$ and $N/L \in S_\omega$. For any $P'$ and $\mu$, let $S_{P', \mu}$ be the set of all modules $N$ such that $N \simeq P' \oplus R, R \in S_\mu$.

By the same argument as in the case $1_I 1_P$ we can see that
\[
b_{P', \mu} = \chi(\mathcal{F}(\omega, P; N_{P', \mu})) \in \mathbb{Z},
\]
where $N_{P', \mu}$ is a module in $S_{P', \mu}$.

The case $1_I M_{\omega \delta}$ is completely similar.
Thus the set
\[ \{ 1_P M_{\alpha_0} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N} \} \]

is a \( \mathbb{Z} \)-basis of \( \mathcal{C}_Z(K) \).

**Corollary 6.12.** The following two sets
\[ \{ 1_P H_{\alpha_0} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N} \}, \]
\[ \{ 1_P E_{\alpha_0} 1_I | P \in \text{Prep}(K), I \in \text{Prei}(K), \omega \vdash n, n \in \mathbb{N} \} \]

are also \( \mathbb{Z} \)-bases of the algebra \( \mathcal{C}_Z(K) \).

**Proof.** Note that the elements in the above two sets are different from the one in Proposition 6.11 only in the regular part. However, they can be \( \mathbb{Z} \)-linear expressed by each other, see Lemma 6.5, 6.8 and 6.10. So the corollary holds. \( \square \)

The results above immediately implies that the algebra \( \mathcal{C}_Z(K) \) has an integral triangular decomposition:

**Corollary 6.13.**
\[ \mathcal{C}_Z(K) \simeq \mathcal{C}_Z(K)^{prep} \otimes \mathcal{C}_Z(K)^{reg} \otimes \mathcal{C}_Z(K)^{prei}. \]

**6.7. Remarks.** (1) The proofs of lemma 6.2, 6.4 are similar to the quantum case in [Z]. However, the bases we constructed in 6.6 are all different from \( \text{Gal} \).

(2) The relation between \( H_{k_0} \) and \( H_{k_0} \) given by lemma 6.4 is equivalent to the following:
\[ \sum_{i \geq 0} H_{i_0} t^i = \exp \left( \sum_{j \geq 1} \frac{P_{j_0}}{j} t^j \right). \]

This relation also appeared in the basis elements corresponding to imaginary roots in \( \text{Gal} \). However, the bases we constructed in 6.6 are all different from \( \text{Gal} \).

**7. Integral basis: The case of cyclic quivers**

In this section we consider the cyclic quiver \( C_r \). We will construct a \( \mathbb{Z} \)-basis of the integral composition algebra \( \mathcal{C}_Z(C_r) \). We use the notations in [4.5].

**7.1. Generic extensions.** Given any two modules \( M, N \) in \( \text{rep}_0(C_r) \), there exists a unique (up to isomorphism) extension \( L \) of \( M \) by \( N \) with maximal \( \dim \mathbb{Z}O_L \) (or equivalently, minimal \( \dim \mathbb{Z} \text{End}(L) \)), see [M]. This extension module \( L \) is called the generic extension of \( M \) by \( N \), denoted by \( L = M \bowtie N \). We can define \( [M] \bowtie [N] = [M \bowtie N] \) then it is known that the operator \( \bowtie \) is associative and \( (I(C_r), \bowtie) \) is a monoid with identity \( [0] \).

An \( n \)-tuple of partitions \( \pi = (\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(n)}) \) in \( \Pi \) is called aperiodic or separated if for each \( l \geq 1 \) there is some \( i = i(l) \in I \) such that \( \pi^{(i)}_j \neq l \) for all \( j \geq 1 \). We denote by \( \Pi^a \) the set of aperiodic \( n \)-tuples of partitions. A module \( M \) in \( \text{rep}_0(C_r) \) is called aperiodic if \( M \simeq M(\pi) \) for some \( \pi \in \Pi^a \). For any dimension vector \( [\alpha] \in \mathbb{N}[I] \), set \( \Pi^a_{\alpha} = \{ \lambda \in \Pi | \dim \mathbb{Z}M(\lambda) = \alpha \} \) and \( \Pi^a = \Pi^a \cap \Pi_{\alpha} \).

Let \( W \) be the set of all words on the alphabet \( I \). For each \( \omega = i_1 i_2 \cdots i_m \in W \), set
\[ M(\omega) = S_{i_1} \bowtie S_{i_2} \bowtie \cdots \bowtie S_{i_m}, \]

Then there is a unique \( \pi \in \Pi \) such that \( M(\pi) \simeq M(\omega) \) and we set \( \varphi(\omega) = \pi \). It has been proved that \( \pi = \varphi(\omega) \in \Pi^a \) and \( \varphi \) induces a surjection \( \varphi : W \rightarrow \Pi^a \).
7.2. Distinguished words. For \( \omega \in \mathcal{W} \), we write \( \omega \) in tight form: \( \omega = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \) with \( j_r \neq j_{r+1} \) for all \( r \). A word \( \omega \) is called distinguished if \( M(\varphi(\omega)) \) has a unique filtration

\[
M(\varphi(\omega)) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0
\]

with \( M_{t-1}/M_r \simeq e_r S_r \).

For \( \lambda \in \Pi \) and \( \omega = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \mathcal{W} \), let \( \chi^\lambda_\omega \) denote the Euler characteristic of the variety consisting of all filtrations of \( M(\lambda) \):

\[
M(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_{t-1} \supset M_t = 0
\]

with \( M_{t-1}/M_r \simeq e_r S_r \). Thus if \( \omega \) is distinguished then \( \chi^\omega_\omega(\omega) = 1 \).

The following proposition has been proved in [DDX]:

**Proposition 7.1.** For any \( \pi \in \Pi^\alpha \), there exists a distinguished word

\[
\omega_\pi = j_1^{e_1} j_2^{e_2} \cdots j_t^{e_t} \in \varphi^{-1}(\pi).
\]

For each \( \pi \in \Pi^\alpha \), we fix a distinguished word \( \omega_\pi \in \varphi^{-1}(\pi) \). The set \( \mathcal{D} = \{ \omega_\pi | \pi \in \Pi^\alpha \} \) is called a section of distinguished words of \( \varphi \) over \( \Pi^\alpha \).

7.3. Monomial bases. For \( \lambda \in \Pi \) and \( \omega = i_1 i_2 \cdots i_m \in \mathcal{W} \), we denote by \( \chi^\lambda_\omega \) the Euler characteristic of the variety consisting of all filtrations of \( M(\lambda) \)

\[
M(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_m = 0
\]

with \( M_{m-1}/M_r \simeq S_r \).

For each word \( \omega = i_1 i_2 \cdots i_m \in \mathcal{W} \) we define

\[
m_\omega = 1_{i_1} 1_{i_2} \cdots 1_{i_m}.
\]

The following results is proved in [DD]:

**Proposition 7.2.** Fix a distinguished section \( \mathcal{D} = \{ \omega_\pi | \pi \in \Pi^\alpha \} \) over \( \Pi^\alpha \). The set \( \{ m_\pi | \pi \in \Pi^\alpha \} \) is a \( \mathbb{C} \)-basis of \( \mathcal{C}(C_r) \).

Note that the proof in [DD] has used results in the quantum case. However, a self-contained proof in our case can be easily given by a similar method, which we omitted here.

7.4. A geometric order on \( \Pi \). We can define an order on the set \( \Pi \) as follows:

For \( \lambda, \mu \in \Pi \), set \( \lambda \leq \mu \) if and only if \( \mathcal{O}_M(\lambda) \subset \mathcal{O}_M(\mu) \). Of course this order can be endowed in \( I(C_r) \) by setting \( M(\lambda) \leq M(\mu) \) if and only if \( \lambda \leq \mu \).

The following lemma asserts that the order is compatible with the generic extension, see [DD]:

**Lemma 7.3.** \( M' \leq M, N' \leq N \) implies \( M' \circ N' \leq M \circ N \).

**Lemma 7.4.** For each \( \omega = i_1 i_2 \cdots i_m \in \mathcal{W} \), we have

\[
m_\omega = \sum_{\lambda \leq \varphi(\omega)} \chi^\lambda_\omega 1_{M(\lambda)}
\]

**Proof.** By the definition of \( m_\omega \), we just need to prove that \( \chi^\lambda_\omega \neq 0 \) implies \( \lambda \leq \varphi(\omega) \).

We prove by induction on \( m \).

If \( m = 1 \), there is nothing to prove. So let \( m > 1 \) and set \( \omega' = i_2 \cdots i_m \). Then

\[
M(\omega) = S_{i_1} \circ (S_{i_2} \circ \cdots \circ S_{i_m}) = S_{i_1} \circ M(\omega').
\]
Lemma 7.5. Let $\lambda \neq 0$, $M(\lambda)$ has a submodule $M'$ with $M(\lambda)/M' \simeq S_i$, and $M'$ has a composition series of type $\omega'$. By the inductive hypothesis, we have $M' \leq M(\omega')$. Hence $$M(\lambda) \leq S_i \circ M' \leq S_i \circ M(\omega') = M(\omega) = M(\varphi(\omega)).$$ That is, $\lambda \leq \varphi(\omega)$.

7.5. A $\mathbb{Z}$-basis of $C_2(C_r)$. For each $\omega = j_1^{(e_1)} j_2^{(e_2)} \cdots j_t^{(e_t)} \in \mathcal{W}$ in tight form, define $m(\omega) = \sum_{\lambda \leq \varphi(\omega)} \chi^\lambda_{\omega} 1_{M(\lambda)}$. Then we have $$m(\omega) = \sum_{\lambda \leq \varphi(\omega)} \chi^\lambda_{\omega} 1_{M(\lambda)}.$$ In particular, for a distinguished word $\omega_\pi \in \varphi^{-1}(\pi)$ with $\pi \in \Pi^a$, since $\tilde{\chi}_{\omega_\pi}^\pi = 1$, we have $$m(\omega_\pi) = 1_{M(\pi)} + \sum_{\lambda < \pi} \tilde{\chi}_{\omega_\pi}^\pi 1_{M(\lambda)}.$$ 

Lemma 7.5. Let $\mathcal{P}(C_r)$ be the $\mathbb{C}$-subspace of $\mathcal{H}(C_r)$ spanned by all $1_{M(\lambda)}$ with $\lambda \in \Pi \setminus \Pi^a$. Then as a vector space, $\mathcal{H}(C_r) = \mathcal{C}(C_r) \oplus \mathcal{P}(C_r)$.

Proof. Since $\mathcal{H}(C_r)$ and $\mathcal{C}(C_r)$ are $\mathbb{N}[I]$-graded, it suffices to prove that for each $\alpha \in \mathbb{N}[I]$, $\mathcal{H}(C_r)_\alpha = \mathcal{C}(C_r)_\alpha \oplus \mathcal{P}(C_r)_\alpha$, where $\mathcal{P}(C_r)_\alpha$ is the $\mathbb{C}$-subspace of $\mathcal{H}(C_r)_\alpha$ spanned by all $1_{M(\lambda)}$ with $\lambda \in \Pi_\alpha \setminus \Pi^a_\alpha$.

Now we show $\mathcal{C}(C_r)_\alpha \cap \mathcal{P}(C_r)_\alpha = \{0\}$. Once this is done, a dimension comparison forces $\mathcal{H}(C_r)_\alpha = \mathcal{C}(C_r)_\alpha \oplus \mathcal{P}(C_r)_\alpha$.

Take an $x \in \mathcal{C}(C_r)_\alpha \cap \mathcal{P}(C_r)_\alpha$ and suppose $x \neq 0$. Then we can write $$x = \sum_{\pi \in \Pi_\alpha^a} a_\pi m_{\omega_\pi}$$ for some $a_\pi \in \mathbb{C}$. Let $\mu \in \Pi_\alpha^a$ be maximal such that $a_\mu \neq 0$. We can rewrite $x = \sum_{\lambda \in \Pi_\alpha^a} b_\lambda 1_{M(\lambda)}$. By the maximality of $\mu$, we have $b_\mu = a_\mu \tilde{\chi}_{\omega_\mu}^\mu$, which contradicts the fact that $x \in \mathcal{P}(C_r)_\alpha$. \hfill \Box

Now we fix a section of distinguished words $D = \{\omega_\pi | \pi \in \Pi^a\}$, define inductively the elements $E_\pi$ as follows:

For any $\alpha \in \mathbb{N}[I]$ and $\pi \in \Pi_\alpha^a$, if $\pi$ is minimal, let $$E_\pi = m(\omega_\pi) \in \mathcal{C}_2(C_r)_\alpha.$$ In general, assume that $E_\lambda \in \mathcal{C}_2(C_r)_\alpha$ has been defined for all $\lambda \in \Pi_\alpha^a$ with $\lambda < \pi$, then we define $$E_\pi = m(\omega_\pi) - \sum_{\lambda < \pi, \lambda \in \Pi_\alpha^a} \tilde{\chi}_{\omega_\pi}^\lambda E_\lambda \in \mathcal{C}_2(C_r)_\alpha.$$

Lemma 7.6. Let $\{\omega_\pi | \pi \in \Pi^a\}$ be a given distinguished section. For each $\pi \in \Pi_\alpha^a$, we have $$E_\pi = 1_{M(\pi)} + \sum_{\lambda \in \Pi_\alpha \setminus \Pi_\alpha^a, \lambda < \pi} g_\lambda^\pi 1_{M(\lambda)}$$ for some $g_\lambda^\pi \in \mathbb{Z}$, and $$m(\omega_\pi) = E_\pi + \sum_{\lambda < \pi, \lambda \in \Pi_\alpha^a} \tilde{\chi}_{\omega_\pi}^\lambda E_\lambda.$$
Proof. The second formula follows immediately from the definition. The first assertion follows from induction and Lemma 7.5. □

Proposition 7.7. For each distinguished section $D = \{ \omega_\pi | \pi \in \Pi^a \}$ of $\varphi$ over $\Pi^a$, the set $\{ E_\pi | \pi \in \Pi^a \}$ is a $\mathbb{Z}$-basis of $C_\mathbb{Z}(C_r)$.

Proof. We have known that the elements in the set are $\mathbb{Z}$-linearly independent. So it suffices to prove that for any $\alpha \in \mathbb{N}[I]$, the $\mathbb{Z}$-module $C_\mathbb{Z}(C_r)_\alpha$ is spanned by $\{ E_\lambda | \lambda \in \Pi^a \}$.

Let $W_\alpha = \{ \omega \in W | \dim(M(\varphi(\omega))) = \alpha \}$. It is clear that $C_\mathbb{Z}(C_r)_\alpha$ is spanned by $m^{(\pi)}, \pi \in W_\alpha$. Thus it remains to prove that each $m^{(\pi)}$ is a $\mathbb{Z}$-linear combination of $E_\pi$, $\pi \in \Pi^a$.

Take arbitrary $\omega \in W_\alpha$, and set $\pi = \varphi(\omega) \in \Pi^a$. We have

$m^{(\pi)} = \sum_{\lambda \leq \pi} \bar{\lambda}_\omega 1_{M(\lambda)},$}

hence

$m^{(\pi)} - \sum_{\lambda \in \Pi^a, \lambda \leq \pi} \bar{\lambda}_\omega = \sum_{\lambda \in \Pi^a, \lambda \leq \pi} a^\pi_\lambda 1_{M(\lambda)},$

for some $a^\pi_\lambda \in \mathbb{Z}$.

The left hand side in the above formula is in $C_\mathbb{Z}(C_r)_\alpha$. Hence by Lemma 7.5 it must be zero. That yields

$m^{(\omega)} = \sum_{\lambda \in \Pi^a, \lambda \leq \pi} \bar{\lambda}_\omega E_\lambda.$

□

7.6. Connection with a basis of $n^+(C_r)$. We investigate the relation between the $\mathbb{Z}$-basis we constructed in Proposition 7.7 and the basis of $n^+(C_r)$ constructed in [FMV]:

Proposition 7.8. The union of the following two sets

$\{ 1_{S_i[l]} | 1 \leq i \leq r, r \nmid l \} \cup \{ 1_{S_i[nr]} - 1_{S_{i+1}[nr]} | n \geq 1, 1 \leq i \leq r - 1 \}$

is a basis of $n^+(C_r)$.

Note that in the proposition the elements in the first set are the real root vectors while those in the second are imaginary root vectors.

Lemma 7.9. (1). For fixed $i, l$ with $1 \leq i \leq r, r \nmid l$, we have

$1_{S_i[l]} = E_{\pi},$

where $\pi \in \Pi^a$ and $M(\pi) = S_i[l]$.

(2). For fixed $i, n$ with $n \geq 1, 1 \leq i \leq r - 1$, we have

$1_{S_i[nr]} - 1_{S_{i+1}[nr]} = E_{\pi_1} - E_{\pi_2},$

where $\pi_1, \pi_2 \in \Pi^a$ and $M(\pi_1) = S_i[nr], M(\pi_2) = S_{i+1}[nr].$
Lemma 8.2. The union of the following sets

\[ \{1_{s,j,i} \mid 1 \leq j \leq s, 1 \leq i \leq r_j, r_j \neq l \}, \]
\[ \{1_{s,j,i,k} - 1_{s,j,i+k,r_j} \mid 1 \leq j \leq s, 1 \leq i \leq r_j - 1, k \geq 1 \}, \]
\[ \{P_{n} \mid n \geq 1\}; \]

We have known that \( 1_{s,j,i} \in n^+(C_r) \subset C(C_r). \) But the second term in the formula above is in \( P(C_r). \) Thus by Lemma 7.3 it must be zero. \( \square \)

8. Integral Bases: The General Affine Case

Now we consider general tame quivers. In this section let \( Q = (I, \Omega, s, t) \) be a tame quiver without oriented cycles. We will use the notations in 4.6.

8.1. Embedding of the Module Category of Kronecker Quiver. Let \( K \) be the Kronecker quiver (see 4.6 and section 6). If \( Q \neq K \), the main difference between \( \text{rep}(K) \) and \( \text{rep}(Q) \) is that the regular component of \( \text{rep}(K) \) only consists of homogeneous tubes, while \( \text{rep}(Q) \) has \( n \) non-homogeneous tubes. A well-known result in representation theory of tame quivers is that \( \text{rep}(K) \) can be embedded into \( \text{rep}(Q) \).

To make it more precise, we need more notations. In the rest of this section \( \delta \) denotes the minimal imaginary root of \( Q \), and the minimal imaginary root of \( K \) is denoted by \( \delta_K \). For the modules in \( \text{rep}(K) \) and in \( \text{rep}(Q) \) we distinguish them by putting different superscripts \( K \) and \( Q \) respectively.

Lemma 8.1. There exists a fully faithful, exact functor \( F : \text{rep}(K) \hookrightarrow \text{rep}(Q) \) which satisfies

1. \( F(P^K) \in \text{Prep}(Q), F(I^K) \in \text{Prei}(Q) \) for all \( P^K \in \text{Prep}(K), I^K \in \text{Prei}(K) \).
2. \( F(S^K_{x,l}) = S^Q_{x,l} \) for all \( x \in \mathbb{P}^1 \setminus J \) and \( l \geq 1 \).
3. For each \( 1 \leq j \leq s \) there exists \( 1 \leq k_j \leq r_j \) such that \( F(S^K_{j,l}) = S^Q_{j,k_j,l,r_j} \) for all \( l \geq 1 \).

The embedding functor \( F : \text{rep}(K) \hookrightarrow \text{rep}(Q) \) gives rise to an injective morphism between the corresponding Hall algebras \( \text{H}(K) \hookrightarrow \text{H}(Q) \), which we still denote by \( F \). Namely \( F(1_{M^K}) = 1_{F(M^K)} \) for any \( M^K \in \text{rep}(K) \).

Note that by (1) in the above lemma, \( F(S^K_{1}) \in \text{Prep}(Q) \) or \( \text{Prei}(Q) \) for each simple module \( S^K \) in \( \text{rep}(K) \). Hence \( F(1_{S^K}) \in C_2(Q) \).

So we have proved the following:

Lemma 8.2. \( F : \text{H}(K) \hookrightarrow \text{H}(Q) \) restricts to an injective morphism \( F : \text{C}(K) \hookrightarrow \text{C}(Q) \) and also \( F : \text{C}_2(K) \hookrightarrow \text{C}_2(Q) \).

Recall that the sets \( \{M_{\omega \delta_K} \mid \omega \vdash n, n \in \mathbb{N} \}, \{H_{\omega \delta_K} \mid \omega \vdash n, n \in \mathbb{N} \}, \{E_{\omega \delta_K} \mid \omega \vdash n, n \in \mathbb{N} \} \) are \( \mathbb{Z} \)-bases of \( \text{C}_2(K) \) resp. (Proposition 6.11 Corollary 6.12). Set \( M_{\omega \delta} = F(M_{\omega \delta_K}), H_{\omega \delta} = F(H_{\omega \delta_K}) \) and \( E_{\omega \delta} = F(E_{\omega \delta_K}) \) for all \( \omega \vdash n, n \in \mathbb{N} \). We also define \( P_{n \delta} \) to be \( F(P_{n \delta_K}) \) for any \( n \in \mathbb{N} \). By the above lemma, \( M_{\omega \delta}, H_{\omega \delta}, E_{\omega \delta} \in \text{C}_2(Q) \).

8.2. A basis of \( n^+(Q) \). In [FMV], a basis of \( n^+(Q) \) has been given:

Proposition 8.3. The union of the following sets

\[ \{1_P, 1_I \mid P \in \text{Prep}(Q), I \in \text{Prei}(Q) \text{ and } P, I \text{ indecomposable} \}, \]
\[ \{1_{s,j,i} \mid 1 \leq j \leq s, 1 \leq i \leq r_j, r_j \neq l \}, \]
\[ \{1_{s,j,i,k} - 1_{s,j,i+k,r_j} \mid 1 \leq j \leq s, 1 \leq i \leq r_j - 1, k \geq 1 \}, \]
\[ \{P_{n} \mid n \geq 1\}; \]
forms a \( \mathbb{Z} \)-basis of \( n^+(Q) \).

Note that in this proposition, \( 1_P, 1_I \) and \( 1_{s,j,i} \) correspond to the real root vectors while \( 1_{s,j,kr_j} - 1_{s,j+1,kr_j} \) and \( P_n \delta \) correspond to the imaginary root vectors.

One can check the multiplicity, recall Lemma 4.6.

8.3. Basis elements arising from non-homogeneous tubes. For any non-homogeneous tube \( T_j \) \((1 \leq j \leq s)\), which is an extension-closed abelian subcategory of \( \text{rep}(Q) \) (see 4.6), we can define the following subalgebras of \( \mathcal{H}(Q) \): \( \mathcal{H}(T_j) \) is the \( \mathbb{C} \)-subalgebra generated by all constructible functions whose supports are contained in \( T_j \). \( \mathcal{C}(T_j) \) is the \( \mathbb{C} \)-subalgebra of generated by \( 1_{s,j,i} \) for all \( 1 \leq i \leq r_j \). And \( \mathcal{C}_2(T_j) \) is defined to be the \( \mathbb{Z} \)-subalgebra of \( \mathcal{H}(T_j) \) generated by \( 1_{s,j,i} \) for all \( 1 \leq i \leq r_j \) and \( t \geq 1 \).

We have the following result:

**Lemma 8.4.** For any \( 1 \leq j \leq s \), the subalgebra \( \mathcal{C}(T_j) \) is contained in \( \mathcal{C}(Q) \). And \( \mathcal{C}_2(T_j) \) is a \( \mathbb{Z} \)-subalgebra of \( \mathcal{C}_2(Q) \).

**Proof.** We claim that \( 1_{s,j,i} \in \mathcal{C}_2(Q) \) for all \( 1 \leq i \leq r_j \).

By Proposition 8.3 we see that \( 1_{s,j,1} \) is a real root vector of the Lie algebra \( \mathfrak{g}(Q) \), thus it can be obtained from \( 1_i \) for some \( i \) by a series of automorphisms \( r_k \) (see 5.3). This forces \( 1_{s,j,i} \in \mathcal{C}_2(Q) \).

The lemma follows immediately. \( \square \)

Note that for any homogeneous tube \( T_x, x \in \mathbb{P}^1 \setminus J \), we can define the subalgebras \( \mathcal{H}(T_x), \mathcal{C}(T_x) \) similarly. But \( \mathcal{C}(T_x) \) does not contained in \( \mathcal{C}(Q) \) any more.

Recall 4.6 that we have the isomorphic functor \( F_j : \text{rep}_0(C_{r_j}) \cong \mathcal{T}_j \). This induces an isomorphism of the corresponding Hall algebra \( \mathcal{H}(C_{r_j}) \cong \mathcal{H}(T_j) \), which we still denoted by \( F_j \). Obviously \( F_j \) restricts to an isomorphism \( \mathcal{C}_2(C_{r_j}) \cong \mathcal{C}_2(T_j) \).

We have to introduce more notations to distinguish objects in various \( \text{rep}_0(C_{r_j}) \) and \( \mathcal{H}(C_{r_j}) \). Let \( \Pi_j \) denote the set of aperiodic \( r_j \)-tuples of partitions. So for \( \pi \in \Pi_j, M(\pi) \in \text{rep}_0(C_{r_j}) \), let \( M(\pi)_j = F_j(M(\pi)) \in T_j \).

By Proposition 7.7 the set \( \{ E_{\pi_1}\pi_2 \in \Pi_j \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{C}_2(C_{r_j}) \). For \( 1 \leq j \leq s \) and \( \pi \in \Pi_j \), let \( E_{\pi,j} = F_j(E_{\pi}) \). Thus for any \( J \), \( \{ E_{\pi,j}\pi \in \Pi_j \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{C}_2(T_j) \).

8.4. Main result: \( \mathbb{Z} \)-bases of \( \mathcal{C}_2(Q) \). Now we can state the main result in this paper. Let \( \mathcal{J} \) be the set of quadruples \( c = (P_c, I_c, \pi_c, \omega_c) \) where \( P_c \in \text{Prep}(Q), I_c \in \text{Prei}(Q), \pi_c = (\pi_{c1}, \pi_{c2}, \cdots, \pi_{cs}), \) each \( \pi_{cj} \in \Pi_j \) and \( \omega_c \vdash n, n \in \mathbb{N} \).

For each \( c \in \mathcal{J} \) we define

\[
B_c = 1_{P_c} E_{\pi_{c1},1} E_{\pi_{c2},2} \cdots E_{\pi_{cs},s} M_{\omega_c} 1_{I_c}.
\]

**Theorem 8.5.** The set \( \{ B_c \mid c \in \mathcal{J} \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{C}_2(Q) \).

Note that for any \( c \in \mathcal{J} \), the modules in the support of \( B_c \) have the same dimension vectors. So we define

\[
\dim B_c = \dim P_c + \sum_{j=1}^s \dim M(\pi_{cj})_j + |\omega_c| \delta + \dim I_c.
\]

Once this theorem is proved, we have the following corollary. The proof is similar to Corollary 6.12.
Corollary 8.6. The following two sets
\[ \{ B_{j}^{c} = 1_{P_{c}} E_{\pi_{1},1} E_{\pi_{2},2} \cdots E_{\pi_{r},r} H_{\omega_{c} \delta} 1_{P_{c}} | c \in J \} \]
\[ \{ B_{j}^{c} = 1_{P_{c}} E_{\pi_{1},1} E_{\pi_{2},2} \cdots E_{\pi_{r},r} 1_{P_{c}} | c \in J \} \]
are also \( \mathbb{Z} \)-basis of \( \mathbb{C}_{\frac{2}{\pi}}(Q) \).

Define \( \mathbb{C}_{\mathbb{Z}}(Q)^{reg} \) (resp. \( \mathbb{C}_{\mathbb{Z}}(Q)^{reg} \)) to be the \( \mathbb{C} \)-subalgebra (resp. \( \mathbb{Z} \)-subalgebra) generated by \( \{ E_{\pi_{j}} | \pi \in \Pi_{j}^{+}, 1 \leq j \leq s \} \) and \( \{ M_{\omega_{c}} | \omega = n, n \in \mathbb{N} \} \). As in Corollary 8.7., we have a triangular decomposition of the integral composition algebra \( \mathbb{C}_{\mathbb{Z}}(Q) \):

Corollary 8.7.
\[ \mathbb{C}_{\mathbb{Z}}(Q) \simeq \mathbb{C}_{\mathbb{Z}}(Q)^{Prep} \otimes \mathbb{C}_{\mathbb{Z}}(Q)^{Reg} \otimes \mathbb{C}_{\mathbb{Z}}(Q)^{Prei}. \]

The rest of the paper is devoted to the proof of Theorem 8.3.

8.5. A \( \mathbb{C} \)-basis of \( \mathbb{C}(Q) \). In this subsection we prove the set \( \{ B_{c} | c \in J \} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}(Q) \), which is the first step to prove Theorem 8.3. We need the following lemma:

Lemma 8.8. (1) Fixed \( 1 \leq j \leq s \), for any \( 1 \leq i \leq r_{j} \) and \( r_{j} \not| \, i \) we have
\[ 1_{S_{j,i,1}} = E_{\pi_{j}} \]
where \( \pi \in \Pi_{j}^{+} \) such that \( M(\pi)_{j} = S_{j,i,1} \).

(2) Fixed \( 1 \leq j \leq s \), for any \( 1 \leq i \leq r_{j} - 1, \, n \geq 1 \) we have
\[ 1_{S_{j,i,nr_{j}}} - 1_{S_{j,i+1,nr_{j}}} = E_{\pi_{1},j} - E_{\pi_{2},j} \]
where \( \pi_{1}, \pi_{2} \in \Pi_{j}^{+} \) such that \( M(\pi_{1})_{j} = S_{j,i,nr_{j}}, \, M(\pi_{2})_{j} = S_{j,i+1,nr_{j}} \).

Proof. It follows immediately from Lemma 7.9. \( \square \)

By the PBW-theorem, the monomials in a fixed order on the basis elements of \( \mathbb{n}^{+}(Q) \) given in Proposition 5.3 form a \( \mathbb{C} \)-basis of \( \mathbb{C}(Q) \).

Note that \( \mathbb{C}(Q) \) is \( \mathbb{N}[I] \)-graded: \( \mathbb{C}(Q) = \oplus_{\alpha \in \mathbb{N}[I]} \mathbb{C}(Q)_{\alpha} \), where \( \mathbb{C}(Q)_{\alpha} \) is the subspace spanned by constructible functions in \( \mathbb{C}(Q) \) whose supports are in \( \mathbb{E}_{\alpha} \). The PBW-basis elements are of course homogeneous. By construction \( B_{c} \) is also homogeneous for any \( c \in J \).

By the results in sections 5.6 and 6.6, and lemma 8.8, we can see that for any \( \alpha \in \mathbb{N}[I] \), the basis of \( \mathbb{C}(Q)_{\alpha} \) can be expressed by \( \{ B_{c} | c \in J \} \). Moreover, by definition the elements in \( \{ B_{c} | c \in J \} \) is \( \mathbb{C} \)-linear independent. Hence \( \{ B_{c} | c \in J \} \) is a \( \mathbb{C} \)-basis of \( \mathbb{C}(Q) \).

8.6. Commutation relations. We have known that the elements in the set \( \{ B_{c} | c \in J \} \) are all in \( \mathbb{C}_{\mathbb{Z}}(Q) \). And the divided powers of \( 1_{S} \) for any simple module \( S \) are in the set \( \{ B_{c} | c \in J \} \). Thus the \( \mathbb{Z} \)-subalgebra generated by \( \{ B_{c} | c \in J \} \) is equal to \( \mathbb{C}_{\mathbb{Z}}(Q) \).

Therefore, to prove Theorem 8.3 we have to check the product of any two elements in \( \{ B_{c} | c \in J \} \) is still a \( \mathbb{Z} \)-combination of elements in \( \{ B_{c} | c \in J \} \). So the procedure is similar to the proof of Proposition 6.11. However, it is more complicated here since we have basis elements \( E_{\pi_{j}} \) arising from non-homogeneous tubes, moreover, the support of \( M_{\omega_{c}} \) contains modules not only in the homogeneous tubes but also non-homogeneous tubes.

For the case \( 1_{P} 1_{P'} \) and \( 1_{I} 1_{I'} \) with \( P, P' \in \text{Prep}(Q); \, I, I' \in \text{Prei}(Q) \), we have done in 6.3. And we have the case \( M_{\lambda_{1}} M_{\omega_{c}} \) for any \( \lambda \vdash n, \, \omega \vdash m \) done in 6.4.
Consider $E_{\pi_1,j}E_{\pi_2,k}$ for any $1 \leq j, k \leq s$, $\pi_1 \in \Pi^q_j, \pi_2 \in \Pi^q_s$. Since there are no non-trivial extensions between different tubes, $E_{\pi_1,j}E_{\pi_2,k} = E_{\pi_2,k}E_{\pi_1,j}$ for $j \neq k$. When $j = k$, we know that $E_{\pi_1,j}E_{\pi_2,j}$ must be a Z-combination of $\{E_{\pi,j}|j \in \Pi^q_j\}$, see [7.5]

**Lemma 8.9.** For any fixed $1 \leq j \leq s$, $\pi \in \Pi^q_j$ and $P \in \text{Prep}(Q)$,

$$E_{\pi,j}1_P = \sum_{P' \in \text{Prep}(Q), \pi' \in \Pi^q_j} a_{P', \pi', j}1_{P'}E_{\pi', j}$$

where $\dim M(\pi')_j + \dim P' = \dim P + \dim M(\pi)_j$ and $a_{P', \pi', j} \in \mathbb{Z}$.

*Proof.* For any module $M$ in the support of $E_{\pi,j}1_P$, the direct summand of $M$ only contains preprojective modules and regular modules in $\mathcal{T}_j$. So

$$E_{\pi,j}1_P = \sum_{P' \in \text{Prep}(Q), \pi' \in \Pi^q_j} a_{P', \pi', j}1_{P'}E_{\pi', j}$$

for some $a_{P', \pi', j} \in \mathbb{C}$. And by a comparison of the dimension vectors in both sides, we have $\dim M(\pi')_j + \dim P' = \dim P + \dim M(\pi)_j$.

By Lemma 7.6, we have

$$E_{\pi', j} = 1_{M(\pi')_j} + \sum_{\lambda \in \Pi^q_j \setminus \Pi^q_j, \lambda < \pi} g_{\lambda, j}1_{M(\lambda)_j}$$

with $g_{\lambda, j} \in \mathbb{Z}$.

For any fixed $P'$ and $\pi'$, let $M_{P', \pi', j} = P' \oplus M(\pi')_j$. We can see that $M_{P', \pi', j}$ is not contained in the support of any other $1_{P''}E_{\pi'', j}$. Thus

$$E_{\pi,j}1_P(M_{P', \pi', j}) = a_{P', \pi', j}1_{P'}E_{\pi', j}(M_{P', \pi', j}) = a_{P', \pi', j},$$

Again by Lemma 7.6

$$E_{\pi,j} = 1_{M(\pi)_j} + \sum_{\lambda \in \Pi^q_j \setminus \Pi^q_j, \lambda < \pi} g_{\lambda, j}1_{M(\lambda)_j}$$

with $g_{\lambda, j} \in \mathbb{Z}$.

This yields

$$a_{P', \pi', j} = \chi(\mathcal{F}(M(\pi)_j, P; M_{P', \pi', j})) + \sum_{\lambda \in \Pi^q_j \setminus \Pi^q_j, \lambda < \pi} g_{\lambda, j}\chi(\mathcal{F}(M(\lambda)_j, P; M_{P', \pi', j})).$$

Hence $a_{P', \pi', j} \in \mathbb{Z}$. \hfill \Box

Similarly we can prove

**Lemma 8.10.** For any fixed $1 \leq j \leq s$, $\pi \in \Pi^q_j$ and $I \in \text{Prei}(Q)$,

$$1_IE_{\pi,j} = \sum_{\pi', I'} b_{\pi', I', j}E_{\pi', j}1_{I'}$$

where $\dim M(\pi')_j + \dim I' = \dim I + \dim M(\pi)_j$ and $b_{\pi', I', j} \in \mathbb{Z}$.

For $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t) \vdash n$, denote by $S_\lambda$ the set of all regular modules $M \simeq \bigoplus_{i=1}^t R_i \in \text{rep}(Q)$ such that $R_i$ indecomposable homogeneous and $\dim R_i = \lambda_i\delta$. (note that this definition coincides with the one in [6.1] when $Q = K$).
Lemma 8.11. For any fixed \(1 \leq j \leq s\), \(\pi \in \Pi^a_j\) and \(\omega \vdash n, n \in \mathbb{N}\),

\[
M_{\omega \delta} E_{\pi,j} = \sum_{\pi' \in \Pi^a_j, \lambda' \vdash n} c_{\pi', \lambda,j} E_{\pi', j} M_{\lambda'}
\]

where \(\dim M(\pi') + k \delta = n \delta + \dim M(\pi)\) and \(c_{\pi', \lambda,j} \in \mathbb{Z}\).

Proof. Since there are no non-trivial extensions between different tubes, \(M_{\omega \delta} E_{\pi,j}\) has the desired expression with \(c_{\pi', \lambda,j} \in \mathbb{C}\).

We prove \(c_{\pi', \lambda,j} \in \mathbb{Z}\) by inverse induction.

We can find a positive integer \(m\) such that for any \(k > m\) and any \(\lambda' \vdash k\), \(\pi' \in \Pi^a_j\), the coefficient \(c_{\pi', \lambda,j} = 0\). Now fix \(\pi' \in \Pi^a_j\) and \(\lambda' \vdash m\), let \(N_{\pi', j, \lambda'}\) be a module isomorphic to the direct sum of \(M(\pi')\) and \(R\) where \(R \in S_{\lambda'}\). It is not difficult to see that \(N_{\pi', j, \lambda'}\) is not contained in the support of \(E_{\pi'', j} M_{\lambda'' \delta}\) unless \(\pi'' = \pi'\) and \(\lambda'' = \lambda\).

Hence we have

\[
M_{\omega \delta} E_{\pi,j} (N_{\pi', j, \lambda'}) = c_{\pi', \lambda,j} E_{\pi', j} M_{\lambda'} (N_{\pi', j, \lambda'}) = c_{\pi', \lambda,j}
\]

Since

\[
E_{\pi,j} = 1_{M(\pi)} + \sum_{\lambda \in \Pi^a_j \setminus \Pi^a_j, \lambda < \pi} g_{\lambda,j} 1_{M(\lambda)}
\]

with \(g_{\lambda,j} \in \mathbb{Z}\), we have

\[
c_{\pi', \lambda,j} = \chi(\mathcal{F}(\omega, M(\pi); N_{\pi', j, \lambda'})) + \sum_{\lambda \in \Pi^a_j \setminus \Pi^a_j, \lambda < \pi} g_{\lambda,j} \chi(\mathcal{F}(\omega, M(\lambda); N_{\pi', j, \lambda'})) \in \mathbb{Z}.
\]

Now we assume that \(c_{\pi', j} \in \mathbb{Z}\) for all \(\pi' \in \Pi^a_j\) and \(\lambda' \vdash k\), \(n + 1 \leq k \leq m\). Consider \(\pi' \vdash n\) and \(\pi' \in \Pi^a_j\). Again we choose a module \(N_{\pi', j, \lambda'} \cong M(\pi') \oplus R\) where \(R \in S_{\lambda'}\). We can see that \(N_{\pi', j, \lambda'}\) is in the support of \(E_{\pi'', j} M_{\lambda'' \delta}\) only if \(\pi'' = \pi'\) and \(\lambda'' = \lambda'\) for some \(k > n\).

Hence we have

\[
M_{\omega \delta} E_{\pi,j} (N_{\pi', j, \lambda'}) = c_{\pi', \lambda', j} E_{\pi', j} M_{\lambda'} (N_{\pi', j, \lambda'}) + \sum_{\pi'' \in \Pi^a_j, |\lambda''| > |\lambda'|} c_{\pi'', \lambda'', j} E_{\pi'', j} M_{\lambda'' \delta} (N_{\pi', j, \lambda'})
\]

\[
= c_{\pi', \lambda', j} + \sum_{\pi'' \in \Pi^a_j, |\lambda''| > |\lambda'|} c_{\pi'', \lambda'', j} E_{\pi'', j} M_{\lambda'' \delta} (N_{\pi', j, \lambda'})
\]

On the other hand, \(M_{\omega \delta} E_{\pi,j} (N_{\pi', j, \lambda'})\) and \(E_{\pi'', j} M_{\lambda'' \delta} (N_{\pi', j, \lambda'})\) are all in the \(\mathbb{Z}\)-form.

By the inductive hypothesis, \(c_{\pi'', \lambda'', j} \in \mathbb{Z}\) for all \(|\lambda''| > |\lambda'|\) and \(\pi'' \in \Pi^a_j\). Thus we know that \(c_{\pi', \lambda', j} \in \mathbb{Z}\).

Finally, by induction we can see all the coefficients \(c_{\pi', \lambda,j} \in \mathbb{Z}\). \(\square\)

Let \(\mathcal{J}(\hat{I})\) (resp. \(\mathcal{J}(\hat{P})\)) be the subset of \(\mathcal{J}\) consisting of \(c = (P_c, 0, \pi_c, \omega_c)\) (resp. \(c = (0, I_c, \pi_c, \omega_c)\)).

Let \(S_c\) be the set of all modules \(N \cong P_c \oplus M(\pi_{c1}) \oplus \cdots \oplus M(\pi_{cs}) \oplus R \oplus I_c\), where \(R \in S_{\pi_c}\) and all direct summands of \(R\) are in the homogeneous tubes.

Lemma 8.12. For any \(\omega \vdash n, n \in \mathbb{N}\) and \(P \in \text{Prep}(Q)\), we have

\[
M_{\omega \delta} 1_{P} = \sum_{e \in \mathcal{J}(\hat{I})} d_e B_e
\]
with \( \dim B_c = n\delta + \dim P \) and \( d_c \in \mathbb{Z} \).

**Proof.** The extension of a regular module by a preprojective contains no direct summands of preinjective modules. Note that the support of \( M_{\omega\delta} \) contains not only modules in the homogeneous tubes but also non-homogeneous tubes. So the terms \( E_{x_{c,j}}(1 \leq j \leq s) \) occur in the right hand side. Hence \( c \in J(\hat{I}) \).

We can prove \( d_c \in \mathbb{Z} \) by completely similar arguments as in the proof of Lemma 8.11.

By similar methods we can prove:

**Lemma 8.13.** For any \( \omega \vdash n, n \in \mathbb{N} \) and \( I \in \text{Pre}(Q) \), we have

\[
1_I M_{\omega\delta} = \sum_{c \in J(P)} e_c B_c
\]

with \( \dim B_c = \dim I + n\delta \) and \( e_c \in \mathbb{Z} \).

**Lemma 8.14.** For any \( P \in \text{Pre}(Q) \) and \( I \in \text{Pre}(Q) \), we have

\[
1_I 1_P = \sum_{c \in J} f_c B_c
\]

with \( \dim B_c = \dim I + \dim P \) and \( f_c \in \mathbb{Z} \).

Now by all the lemmas above, we see that any monomial of \( \{B_c | c \in J\} \) is still in the \( \mathbb{Z} \)-span of the set \( \{B_c | c \in J\} \). Therefore \( \{B_c | c \in J\} \) is a \( \mathbb{Z} \)-basis of \( C_Z(Q) \).

**Acknowledgments.** The authors would like to thank Professor J. Xiao for his ideas and encouraging supervision.

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