Multiplicity of singularities is not a bi-Lipschitz invariant

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Abstract
It was conjectured that multiplicity of a singularity is bi-Lipschitz invariant. We disprove this conjecture constructing examples of bi-Lipschitz equivalent complex algebraic singularities with different values of multiplicity.

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1 Introduction

The famous multiplicity conjecture, stated by Zariski in 1971 (see [21]), is formulated as follows: if two germs of complex analytic hypersurfaces are ambient topological equivalent, then they have the same multiplicity. It was proved by Zariski [20] for germs of plane analytic curves. The results of Pham-Teisser in [18] show that this result can be extended in the following “metric” way: if the two germs of complex analytic curves

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in n-dimensional space are bi-Lipschitz equivalent with respect to the outer metric, then the germs of the space curves have the same multiplicity. Comte in [7] proves that the multiplicity of complex analytic germs (not necessarily codimension 1 sets) is invariant under bi-Lipschitz homeomorphism with Lipschitz constants close enough to 1 (this is a severe assumption). These results motivated the following question, closely related to the multiplicity conjecture: is the multiplicity of a germ of analytic singularity a bi-Lipschitz invariant? This question was stated as a conjecture in [4].

The Lipschitz Regularity Theorem in [1] shows that if the multiplicity of a complex analytic germ is equal to one, then it is a bi-Lipschitz invariant. Namely, if a germs of an analytic set is bi-Lipschitz equivalent to a smooth germ, then it is smooth itself. Later, Fernandes and Sampaio in [9] give a positive answer to this question for surfaces in 3-dimensional space with respect to the ambient bi-Lipschitz equivalence. More recently, Neumann and Pichon [15] showed that the multiplicity is an invariant under bi-Lipschitz equivalence, for germs of normal surface singularities. Another important result in [9] is the following: in order to prove (or disprove) the bi-Lipschitz invariance of the multiplicity, it is enough to prove it for the algebraic cones, i.e. for the algebraic sets, defined by homogeneous polynomials. In [4] the authors show that the conjecture has a positive answer for 1 or 2 dimensional complex analytic sets.

The present paper shows that the multiplicity of complex algebraic sets is not a bi-Lipschitz invariant for the sets of dimension bigger or equal to three. Moreover, we show that there exists an infinite family of 3-dimensional germs, such that all of them are bi-Lipschitz equivalent, but they have different multiplicities. The idea of the construction is to consider the complex cones over different embeddings of \( \mathbb{C}P^1 \times \mathbb{C}P^1 \) to complex projective spaces. Using the topology of Smale–Barden manifolds, we show that all the links of such singularities are diffeomorphic. That is why the germs of the corresponding cones are bi-Lipschitz equivalent. From the other hand, the multiplicities of the cones at the origin may be explicitly calculated in terms of the embeddings.

We finish this introduction pointing out that the examples mentioned above show also that the problem of the construction of Lipschitz Stratification and the problem of Lipschitz Classification of complex analytic germs are not equivalent. More precisely, the examples of complex analytic germs described here are bi-Lipschitz homeomorphic, but their multiplicities are different. From the other hand, the results of Comte [7] show that the multiplicity is constant along the strata of a Lipschitz Stratification. That is why the sets do not belong to the same stratum.

## 2 Smale–Barden manifolds

The classification of 5-manifolds is due to Smale [17] and Barden [3].

**Definition 2.1** A simply connected, compact, oriented 5-manifold is called Smale–Barden manifold.

The Smale–Barden manifolds are uniquely determined by their second Stiefel–Whitney class and the linking form.
Theorem 2.2 \[3\] Let \( X, X' \) be two Smale–Barden manifolds. Assume that \( H^2(X) = H^2(X') \) and this isomorphism is compatible with the linking form and preserves the second Stiefel–Whitney class. Then \( X \) is diffeomorphic to \( X' \).

Corollary 2.3 There exist only two Smale–Barden manifolds \( M \) with \( H^2(M) = \mathbb{Z} \), the product \( S^2 \times S^3 \) and the total space of a non-trivial \( S^3 \)-bundle over \( S^2 \) (see \[6\] for an introduction to Barden theory, where this manifold is formally introduced).

Proof Indeed, the linking form on \( \mathbb{Z} \) vanishes, therefore the manifold is uniquely determined by the Stiefel-Whitney class \( w_2(M) \). Hence we have only two possibilities: \( w_2(M) = 0 \) and \( w_2(M) \neq 0 \).

In early 2000-ies, the classification of 5-manifolds attracted interest coming from algebraic geometry, in the context of Sasakian geometry and geometry of generalized Seifert manifolds \[11,12\]. In the present paper we are interested in \( S^1 \)-bundles over \( \mathbb{C}P^1 \times \mathbb{C}P^1 \).

Proposition 2.4 Let \( M \) be a simply connected 5-manifold which is the total space of an \( S^1 \)-bundle over \( B = \mathbb{C}P^1 \times \mathbb{C}P^1 \). Then \( H^2(M) \) is torsion-free, and \( M \) is diffeomorphic to \( S^2 \times S^3 \).

Proof

Step 1: Universal coefficients formula gives an exact sequence

\[
0 \rightarrow \text{Ext}^1_{\mathbb{Z}}(H_1(M; \mathbb{Z}), \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z}) \rightarrow \text{Hom}_{\mathbb{Z}}(H_2(M; \mathbb{Z}), \mathbb{Z}) \rightarrow 0.
\]

This implies that \( H^2(M; \mathbb{Z}) \) is torsion-free.

Step 2: Consider the following exact sequence of homotopy groups

\[
0 \rightarrow \pi_2(M) \rightarrow \pi_2(B) \xrightarrow{\phi} \pi_1(S^1) \rightarrow \pi_1(M) \rightarrow 0
\]

Since \( \pi_1(M) = 0 \), the map \( \phi \), representing the first Chern class of this \( S^1 \)-bundle, is surjective. This exact sequence becomes

\[
0 \rightarrow \pi_2(M) \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z} \rightarrow 0
\]

giving \( \pi_2(M) = \mathbb{Z} \), and \( H^2(M) = \mathbb{Z} \) because \( H^2(M) \) is torsion-free.

Step 3: To deduce Proposition 2.4 from the Smale–Barden classification, it remains to show that \( w_2(M) = 0 \). In order to do this, let \( \pi : M \rightarrow B \) be the projection of an \( S^1 \)-bundle. By Lemma 36 in \[12\], we obtain that \( w_2(M) = \pi^*(w_2(B)) \). Since \( w_1(S^2) \) and \( w_2(S^2) \) vanish (see \[14\, \text{Example 1, p. 41}\]), by the Whitney Product Theorem (see \[14\, \text{Axiom 3, p. 37}\]), it follows that \( w_2(B) = w_2(S^2 \times S^2) = 0 \), which implies \( w_2(M) = 0 \).

\[ \Box \]
3 Multiplicity of homogeneous singularities

Let $A$ be a complex algebraic set of $\mathbb{C}^{n+1}$ and $x \in A$. The multiplicity of $A$ at $x$, denoted by $\text{mult}(A, x)$, is defined to be the multiplicity of the maximal ideal of the local ring $\mathcal{O}_{A,x}$. Given a projective variety $X \subset \mathbb{C}P^n$, we define the projective cone $C(X)$ to be the union of all the complex lines in $\mathbb{C}^{n+1}$ that represent the elements of $X$. We see that the multiplicity of $C(X)$ at the origin $0 \in \mathbb{C}^{n+1}$ coincides with the degree of $X$ (see [5], subsection 11.3).

Next, we shall be interested in the following geometric situation. Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Then the Picard group of $X = \mathbb{C}P^1 \times \mathbb{C}P^1$ is isomorphic to $\mathbb{Z}^2$, and a line bundle is determined by its bidegree. We shall denote a line bundle of bidegree $a, b$ by $\mathcal{O}(a, b)$. Let $S$ be the link of $C(X)$, i.e. $S = C(X) \cap S^{2n+1}$. Thus, we have a circle bundle $\pi : S \to X$ given by the Hopf mapping.

**Proposition 3.1** Let $X \subset \mathbb{C}P^n$ be a variety isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. Assume that $\mathcal{O}(1)|_X = \mathcal{O}(a, b)$. Then $X$ has degree $2ab$. If, in addition, $a$ and $b$ are relatively prime, the link of $C(X)$ is diffeomorphic to $S^2 \times S^3$.

**Proof**

**Step 1:** Since $c_1(\mathcal{O}(a, b))^2 = 2ab$, and degree of a subvariety $X \subset \mathbb{C}P^n$ is its intersection with the top power of $\mathcal{O}(1)|_X$, one has deg $X = 2ab$.

**Step 2:** Let $S = C(X) \cap S^{2n+1}$. Consider the homotopy exact sequence

$$0 \to \pi_2(S) \to \pi_2(X) \xrightarrow{\phi} \pi_1(S^1) \to \pi_1(S) \to \pi_1(X) \to 0$$

for the circle bundle $\pi : S \to X$ given by the Hopf mapping. Since the map $\phi$ represents the first Chern class of $\mathcal{O}(1)|_X$, it is obtained as a quotient of $\mathbb{Z}^2$ by a subgroup generated by $(a, b)$, and this map is surjective because $a$ and $b$ are relatively prime. Then $\pi_1(S) = \pi_1(X) = 0$ and, thus, $S$ is a simply connected 5-manifold which is the total space of an $S^1$-bundle over $\mathbb{C}P^1 \times \mathbb{C}P^1$. By Proposition 2.4, $S$ is diffeomorphic to $S^2 \times S^3$. \hfill \Box

4 Lipschitz invariance of singularities

Let $X \subset \mathbb{C}^n$ be a complex variety. The induced metric from the Euclidean distance on $\mathbb{C}^n$ gives a distance on $X$; it is called the outer metric on $X$.

**Definition 4.1** Let $X \subset \mathbb{C}^n$ and $X' \subset \mathbb{C}^n'$ be complex varieties equipped with the outer metrics, $x \in X, x' \in X'$ marked points. We say that $(X, x)$ is bi-Lipschitz equivalent to $(X', x')$ if there exist a neighborhoods $U$ of $x$ in $\mathbb{C}^n$ and $U'$ of $x$ in $\mathbb{C}^n'$, and a bi-Lipschitz homeomorphism of $X \cap U$ to $X' \cap U'$ mapping $x$ to $x'$.

**Definition 4.2** Let $X, X' \subset \mathbb{C}^n$ be complex varieties equipped with the outer metrics, $x \in X, x' \in X'$ marked points. We say that $(X, x)$ is ambient bi-Lipschitz equivalent
to \((X', x')\) if there exists a bi-Lipschitz equivalence of a neighbourhood \(U\) of \(x\) in \(\mathbb{C}^n\) and a neighbourhood \(U'\) of \(x\) in \(\mathbb{C}^n\) mapping \(X \cap U\) to \(X' \cap U'\) and \(x\) to \(x'\).

Actually, the two definitions above do not coincide. The ambient bi-Lipschitz equivalence implies bi-Lipschitz equivalence, but the examples presented in [2] show that the converse does not hold true in general.

As it was already mentioned in Introduction, it was conjectured in [4] that the multiplicity is a bi-Lipschitz invariant. We prove that this is false. Here is the main result of this paper.

**Theorem 4.3** For each \(n \geq 3\), there exists a family \(\{Y_i\}_{i \in \mathbb{Z}}\) of \(n\)-dimensional complex algebraic varieties \(Y_i \subset \mathbb{C}^{n_i + 1}\) such that:

(a) for each pair \(i \neq j\), the germs at the origin of \(Y_i \subset \mathbb{C}^{n_i + 1}\) and \(Y_j \subset \mathbb{C}^{n_j + 1}\) are bi-Lipschitz equivalent, but \((Y_i, 0)\) and \((Y_j, 0)\) have different multiplicity.

(b) for each pair \(i \neq j\), there are \(n\)-dimensional complex algebraic varieties \(Z_{ij}, \tilde{Z}_{ij} \subset \mathbb{C}^{n_i + n_j + 2}\) such that \((Z_{ij}, 0)\) and \((\tilde{Z}_{ij}, 0)\) are ambient bi-Lipschitz equivalent, but \(\text{mult}(Z_{ij}, 0) = \text{mult}(Y_i, 0)\) and \(\text{mult}(\tilde{Z}_{ij}, 0) = \text{mult}(Y_j, 0)\) and, in particular, they have different multiplicity.

**Proof** Let \(\{p_i\}_{i \in \mathbb{Z}}\) be the family of odd prime numbers. For each \(i \in \mathbb{Z}\), let \(L_i\) be a very ample bundle on \(X = \mathbb{C}P^1 \times \mathbb{C}P^1\) of bidegree \((2, p_i)\). Let \(X_i\) be the projective variety obtained by the embedding of the very ample bundle \(L_i\). So, \(X_i\) is embedded in \(\mathbb{C}P^{m_i}\), for some \(m_i\). Consider the link of the singularity \(S_i := C(X_i) \cap S^{2m_i + 1}\), where \(S^{2m_i + 1}\) is the unit sphere centered in \(0 \in \mathbb{C}^{m_i + 1}\). Then, for each pair \(i \neq j\) the links \(S_i, S_j\) are diffeomorphic to \(S^2 \times S^3\) (Proposition 3.1). In particular, \(S_i\) and \(S_j\) are bi-Lipschitz homeomorphic. Since a bi-Lipschitz map from \(S_i\) to \(S_j\) induces a bi-Lipschitz map of their cones, then the affine cones \((C(X_i), 0)\) and \((C(X_j), 0)\) are bi-Lipschitz equivalent, but \(\text{mult}(C(X_i), 0) = 4p_i\) and \(\text{mult}(C(X_j), 0) = 4p_j\) (Proposition 3.1). Thus, if for each \(i \in \mathbb{Z}\) we define \(Y_i := C(X_i) \times \mathbb{C}^{n-3}\), then we have that the family \(\{Y_i\}_{i \in \mathbb{Z}}\) satisfies the item (a), since \(\text{mult}(Y_i, 0) = \text{mult}(C(X_i), 0) = 4p_i\), for all \(i \in \mathbb{Z}\).

Concerning the item (b), let \(\phi_{ij} : Y_i \rightarrow Y_j\) be a bi-Lipschitz homeomorphism such that \(\phi_{ij}(0) = 0\). Let \(\tilde{\phi}_{ij} : \mathbb{C}^{n_i + 1} \rightarrow \mathbb{C}^{n_j + 1}\) (resp. \(\psi_{ij} : \mathbb{C}^{n_j + 1} \rightarrow \mathbb{C}^{n_i + 1}\)) be a Lipschitz extension of \(\phi_{ij}\) (resp. \(\psi_{ij} = \phi_{ij}^{-1}\)) (see [10,13] and [19]). Let us define \(\varphi, \psi : \mathbb{C}^{n_i + 1} \times \mathbb{C}^{n_j + 1} \rightarrow \mathbb{C}^{n_j + 1} \times \mathbb{C}^{n_i + 1}\) as follows:

\[
\varphi(x, y) = (x - \tilde{\psi}_{ij}(y + \phi_{ij}(x)), y + \tilde{\phi}_{ij}(x))
\]

and

\[
\psi(z, w) = (z + \tilde{\psi}_{ij}(w), w - \tilde{\phi}_{ij}(z + \tilde{\phi}_{ij}(w))).
\]

It easy to verify that \(\psi = \varphi^{-1}\) and since \(\varphi\) and \(\psi\) are composition of Lipschitz maps, they are also Lipschitz maps. Moreover, if \(Z_{ij} = Y_i \times \{0\}\) and \(\tilde{Z}_{ij} = \{0\} \times Y_j\), we obtain that \(\varphi(Z_{ij}) = \tilde{Z}_{ij}\) (see [16]). Therefore, \((Z_{ij}, 0)\) and \((\tilde{Z}_{ij}, 0)\) are bi-Lipschitz equivalent, but \(\text{mult}(Z_{ij}, 0) = \text{mult}(Y_i, 0)\) and \(\text{mult}(\tilde{Z}_{ij}, 0) = \text{mult}(Y_j, 0)\) and, in particular, they have different multiplicity. \(\square\)
Remark 4.4  Let us finish this paper by remarking that the question of the bi-Lipschitz invariance of the multiplicity of complex hypersurface germs of dimension $\geq 3$ remains open.

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