Research Article

The S-Transform of Distributions

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Parseval’s formula and inversion formula for the S-transform are given. A relation between the S-transform and pseudodifferential operators is obtained. The S-transform is studied on the spaces $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$.

1. Introduction

The S-transform was first used by Stockwell et al. [1] in 1996. If $\omega(t, \xi)$ is a window function, then the continuous S-transform of $\phi(t)$ with respect to $\omega$ is defined as [2]

$$
(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} \phi(t) \omega(\tau - t, \xi) e^{-i2\pi(\tau, \xi)} dt,
$$

for $x, \xi \in \mathbb{R}^n$. (1)

In signal analysis, at least in dimension $n = 1$, $\mathbb{R}^{2n}$ is called the time-frequency plane, and in physics $\mathbb{R}^{2n}$ is called the phase space.

Equation (1) can be rewritten as a convolution as

$$
(S_\omega \phi)(\tau, \xi) = \left( \phi(\cdot) e^{-i2\pi(\cdot, \xi)} * \omega(\cdot, \xi) \right)(\tau).
$$

Now, we define the translation, modulation, and involution operators, respectively, by

$$
T_\tau \phi(t) = \phi(t - \tau) \quad \text{(translation)}
$$

$$
M_\xi \phi(t) = e^{i2\pi(\xi, t)} \phi(t) \quad \text{(modulation)}
$$

$$
I \phi(t) = \phi(-t) \quad \text{(involution)},
$$

where $t, \tau, \xi \in \mathbb{R}^n$.

Definition 1. If $\phi(t)$ is defined on $\mathbb{R}^n$, then the Fourier transform of $\phi$ is given by

$$
\mathcal{F}[\phi(t)](\xi) = \int_{\mathbb{R}^n} \phi(t) e^{-i2\pi(\xi, t)} dt,
$$

where $\langle t, \xi \rangle = \sum_{j=1}^n t_j \xi_j$ is the usual inner product on $\mathbb{R}^n$.

Definition 2. If $\phi(t, x)$ is defined on $\mathbb{R}^{2n}$, then the partial Fourier transform of $\phi(t, x)$ with respect to the first coordinate is given by

$$
\mathcal{F}_1[\phi(t, x)](\xi, x) = \int_{\mathbb{R}^n} \phi(t, x) e^{-i2\pi(\xi, t)} dt,
$$

and the partial Fourier transform of $\phi(t, x)$ with respect to the second coordinate is given by

$$
\mathcal{F}_2[\phi(t, x)](t, \eta) = \int_{\mathbb{R}^n} \phi(t, x) e^{-i2\pi(x, \eta)} dx.
$$

Applying the convolution property for the Fourier transform in (2), we obtain

$$
(S_\omega \phi)(\tau, \xi) = \mathcal{F}_1^{-1} \left[ \mathcal{F}_1(\phi)(\alpha + \xi) \mathcal{F}_1(\omega)(\alpha, \xi) \right](\tau, \xi),
$$

where $\mathcal{F}_1^{-1}$ is the inverse Fourier transform.

Now, we define the translation, modulation, and involution operators, respectively, by

$$
T_\tau \phi(t) = \phi(t - \tau) \quad \text{(translation)}
$$

$$
M_\xi \phi(t) = e^{i2\pi(\xi, t)} \phi(t) \quad \text{(modulation)}
$$

$$
I \phi(t) = \phi(-t) \quad \text{(involution)},
$$

where $t, \tau, \xi \in \mathbb{R}^n$.

Definition 3 (the Dirac delta). The Dirac delta function is defined by

$$
\int_{\mathbb{R}^n} \phi(t, x) \delta(t, x) dt dx = \phi(0, 0).
$$

Definition 4 (tempered distribution). A function $\phi \in C_c^\infty(\mathbb{R}^n)$ is said to be rapidly decreasing if

$$
\gamma_{a, b}(\phi) = \sup_{x \in \mathbb{R}^n} |x^a D^b \phi(x)| < \infty,
$$

where $a, b \in \mathbb{Z}$.
for all pairs of multi-indices $\alpha, \beta \in \mathbb{N}_0^n$. The space of all rapidly decreasing functions on $\mathbb{R}^n$ is denoted by $\mathcal{S}'(\mathbb{R}^n)$ or simply $\mathcal{S}'$. Elements in the dual space $\mathcal{S}'$ of $\mathcal{S}$ are called tempered distribution.

2. Some Important Properties of S-Transform

Some properties of S-transform can be found in [3–8] and certain properties of S-transform are obtained in this section. By definition, we have

\[
(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} \omega(t - \tau, \xi) \phi(t) e^{-i2\pi(t, \xi)} dt
\]

\[
= \int_{\mathbb{R}^n} \omega(-x, \xi) \phi(t + x) e^{-i2\pi(t + x, \xi)} dx
\]

\[
= \int_{\mathbb{R}^n} \omega(-x, \xi) M_{-x} \phi(x) dx
\]

\[
= \int_{\mathbb{R}^n} \omega(-x, \xi) T_{-x} M_{-x} \phi(x) dx.
\]

Thus, the S-transform $S_\omega$ appears as a superposition of time-frequency shifts as follows:

\[
S_\omega := \int_{\mathbb{R}^n} \omega(-x, \xi) T_{-x} M_{-x} \phi(x) dx.
\]

Example 5. If $\omega(t, \xi) = m(\xi)$, that is, independent of $t$, then

\[
(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} m(\xi) \phi(t) e^{-i2\pi(t, \xi)} dt
\]

\[
= m(\xi) (\mathcal{F}\phi)(\xi).
\]

So $S_\omega$ is a multiplication operator. In particular, if $\omega(t, \xi) = 1$, then $(S_\omega \phi)(\tau, \xi) = (\mathcal{F}\phi)(\xi)$.

Example 6. If $\omega(t, \xi) = m(t)$, then

\[
(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} m(t - x, \xi) \phi(t) e^{-i2\pi(t, \xi)} dt
\]

\[
= \int_{\mathbb{R}^n} T_{-x} m(-t) \phi(t) e^{-i2\pi(t, \xi)} dt
\]

\[
= \int_{\mathbb{R}^n} T_{-x} \mathcal{F} m(t) \phi(t) e^{-i2\pi(t, \xi)} dt
\]

\[
= \mathcal{F} (\phi T_{-x} \mathcal{F} m)(\xi).
\]

Theorem 7 (Parseval’s formula). Let $\omega_1$ and $\omega_2$ be the window functions such that

\[
\int_{\mathbb{R}^n} \omega_1(t, \xi) \omega_2(t, \xi) dt d\xi = \delta(x - t, \xi - \eta).
\]

Let $\phi_1, \phi_2 \in L^2(\mathbb{R}^n)$ and let $(S_{\omega_1} \phi_1)$ and $(S_{\omega_2} \phi_2)$ be the S-transforms of $\phi_1$ and $\phi_2$, respectively. Then

\[
\int \int (S_{\omega_1} \phi_1)(\tau, \xi) (S_{\omega_2} \phi_2)(\tau, \xi) d\tau d\xi
\]

\[
= \int \int \phi_1(t) \phi_2(t) dt.
\]

This immediately implies the Plancherel formula

\[
\|S_\omega \phi\|_{L^2(\mathbb{R}^n)} = \|\phi\|_{L^2(\mathbb{R}^n)}.
\]

Proof. Consider

\[
\int \int (S_{\omega_1} \phi_1)(\tau, \xi) (S_{\omega_2} \phi_2)(\tau, \xi) d\tau d\xi
\]

\[
= \int \int \phi_1(t) (\omega_1(t, \xi) e^{-i2\pi(t, \xi)}) dt
\]

\[
\times \int \left( \int \phi_2(x) \omega_2(t - x, \xi) e^{-i2\pi(t - x, \xi)} dx \right) d\tau d\xi
\]

\[
= \int \int \phi_1(t) \phi_2(x) e^{i2\pi((t - x), \xi)} \delta(x - t, \xi) d\tau d\xi
\]

\[
= \int \phi_1(t) \phi_2(t) dt.
\]

Theorem 8 (inversion formula). If $\phi \in L^2(\mathbb{R}^n)$ and window function $\omega$ satisfy the condition (14) of the previous theorem, then

\[
\phi(t) = \int \int (S_{\omega} \phi)(\tau, \xi) \omega(t - \tau, \xi) e^{i2\pi(t, \xi)} d\tau d\xi.
\]

Proof. By the previous theorem we can write

\[
\langle \phi_1, \phi_2 \rangle = \int \int (S_{\omega_1} \phi_1)(\tau, \xi) (S_{\omega_2} \phi_2)(\tau, \xi) d\tau d\xi
\]

\[
= \int \int (S_{\omega_1} \phi_1)(\tau, \xi)
\]

\[
\times \left( \int \mathcal{F} m(t) \phi(t) e^{i2\pi(t, \xi)} dt \right) d\tau d\xi
\]

\[
= \int \mathcal{F} (\phi T_{-x} \mathcal{F} m)(\xi) \delta(x - t, \xi) d\tau d\xi.
\]

Hence

\[
\phi(t) = \int \int (S_{\omega} \phi_1)(\tau, \xi) \omega(t - \tau, \xi) \phi_2(t) d\tau d\xi.
\]

\[
= \int (S_{\omega} \phi_1)(\tau, \xi) \omega(t - \tau, \xi) \phi_2(t) d\tau d\xi.
\]

Definition 9. Let $\omega$ be a window function and $S_\omega$ is the S-transform. Then the transform $S_\omega^*$ defined by

\[
\langle S_\omega \phi, \psi \rangle = \langle \phi, S_\omega^* \psi \rangle
\]
is called the adjoint of $S_\omega$. If $\phi \in L^2(\mathbb{R}^n)$ and $\psi \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then (21) implies that
\[
(S_\omega^* \psi)(t) = \int_{\mathbb{R}^n} \psi(\tau, \xi) \omega(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi,
\] (22)
where $t, \tau, \xi \in \mathbb{R}^n$.

**Theorem 10** (Parseval's formula for $S_\omega^*$). Let $\omega_1$ and $\omega_2$ be the window functions that satisfy the condition (14). If $\psi_1, \psi_2 \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, then
\[
\int_{\mathbb{R}^n} (S_\omega^* \psi_1)(t)(S_\omega^* \psi_2)(t) dt = \int_{\mathbb{R}^n} \psi_1(\tau, \xi) \psi_2(\tau, \xi) d\tau d\xi,
\] (23)
and the Plancherel formula is
\[
\|S_\omega^* \psi\|_{L^2(\mathbb{R}^n)} = \|\psi\|_{L^2(\mathbb{R}^n \times \mathbb{R}^n)}.
\] (24)

**Proof.** Consider
\[
\int_{\mathbb{R}^n} (S_\omega^* \psi_1)(t)(S_\omega^* \psi_2)(t) dt = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \psi_1(\tau, \xi) \omega_1(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi \right) \times \left( \int_{\mathbb{R}^n} \psi_2(\tau, \xi) \omega_2(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi \right) dt
\]
\[
= \int_{\mathbb{R}^n} \psi_1(\tau, \xi) \omega_1(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi \times \delta(\tau-t) dt
\]
\[
= \int_{\mathbb{R}^n} \psi_1(\tau, \xi) \omega_1(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi.
\]
(25)
This proves the theorem.

**Theorem 11.** If the window function $\omega$ satisfies the condition (14), then
\[
S_\omega S_\omega^* = I = S_\omega^* S_\omega,
\] (26)
where $I$ is the identity operator.

**Proof.** By definition
\[
(S_\omega^* \psi)(t) = \int_{\mathbb{R}^n} \psi(\tau, \xi) \omega(\tau-t, \xi)e^{i2\pi \langle \tau, \xi \rangle} d\tau d\xi
\]
\[
(S_\omega^* \psi)(t) = S_\omega^* \left[ S_{\omega^*} \psi \right](t).
\] (27)
Thus
\[
S_\omega^* = S_\omega^{-1}.
\] (28)
This proves the theorem.

**Definition 12** (pseudodifferential operator). Let $\sigma$ be a (measurable) function or a tempered distribution on $\mathbb{R}^n$. Then the operator
\[
K_\sigma \phi(t) = \int_{\mathbb{R}^n} \sigma(t, \xi) \hat{\phi}(\xi) e^{i2\pi \langle t, \xi \rangle} d\xi
\] (29)
is called the pseudodifferential operator.

The pseudodifferential operator plays an important role in the theory of partial differential equations. The pseudodifferential operator has been studied on function and distribution spaces by many authors. Details of the concept can be found in [9, 10].

**2.1. Relation between the S-Transform and Pseudodifferential Operator.** Here we give a direct relation between S-transform and pseudodifferential operator which may be very useful in the study of S-transform of distribution spaces. The continuous S-transform of a function $\phi$ with respect to a window function $\omega$ is given by
\[
(S_\omega \phi)(\tau, \xi) = \int_{\mathbb{R}^n} \phi(t) \omega(\tau-t, \xi)e^{i2\pi \langle t, \xi \rangle} dt
\]
\[
= \int_{\mathbb{R}^n} \phi(t-x) \omega(x, \xi)e^{-i2\pi \langle t-x, \xi \rangle} dx
\]
\[
e^{-i2\pi(\tau, \xi)} \int_{\mathbb{R}^n} T_{-\tau} \phi(-x) \omega(x, \xi)e^{i2\pi(\xi, \xi)} dx
\]
\[
e^{-i2\pi(\tau, \xi)} \int_{\mathbb{R}^n} \mathcal{F} \left[ \mathcal{F}(T_{-\tau} \phi) \right](x)
\]
\[
\times \omega(x, \xi)e^{i2\pi(\xi, \xi)} dx
\]
\[
e^{-i2\pi(\tau, \xi)} K_\sigma \left[ \mathcal{F}(T_{-\tau} \phi) \right](\xi),
\] (30)
where $\sigma(\xi, x) = \omega(x, \xi)$.

**3. The S-Transform of Distributions**

In this section we will investigate the S-transform of tempered distribution by means of the Fourier transform.

**Theorem 13.** If $\omega \in S(\mathbb{R}^{2n})$, then $S_\omega$ maps $S(\mathbb{R}^n)$ into $S(\mathbb{R}^{2n})$.

**Proof.** By (6) we have
\[
(S_\omega \phi)(\tau, \xi) = \mathcal{F}_1^{-1} \left[ \mathcal{F}_1(\phi)(\alpha + \xi) \right] \mathcal{F}_1(\omega)(\alpha, \xi)(\tau, \xi).
\] (31)
Thus, $(S_\omega \phi) \in S(\mathbb{R}^{2n})$, since the Fourier transform is continuous isomorphism from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$, and its inverse is also a continuous isomorphism from $S(\mathbb{R}^n)$ to $S(\mathbb{R}^n)$ (see [11], page 66–67).
Theorem 14. If \( \omega \in \mathcal{S}(\mathbb{R}^{2n}) \), then \( S_\omega \) maps \( \mathcal{S}'(\mathbb{R}^n) \) into \( \mathcal{S}'(\mathbb{R}^{2n}) \).

Proof. For any \( f \in \mathcal{S}'(\mathbb{R}^n) \) and \( \psi \in \mathcal{S}(\mathbb{R}^{2n}) \), we have

\[
\langle (S_\omega f)(\tau, \xi), \psi(\tau, \xi) \rangle = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} f(t) \omega(t - \tau, \xi) e^{-i2\pi \langle t, \xi \rangle} dt \right) \overline{\psi(\tau, \xi)} d\tau d\xi
\]

(in fact \( \overline{\psi} = \psi \), since \( \psi \in \mathcal{S} \))

\[
= \int_{\mathbb{R}^n} f(t) \left( \int_{\mathbb{R}^n} \overline{\psi(\tau, \xi)} \omega(t - \tau, \xi) e^{-i2\pi \langle t, \xi \rangle} d\tau d\xi \right) dt
\]

\[
= \langle f, \phi \rangle ,
\]

where

\[
\phi(t) = \int_{\mathbb{R}^n} \overline{\psi(\tau, \xi)} \omega(t - \tau, \xi) e^{i2\pi \langle t, \xi \rangle} d\tau d\xi
\]

\[
= \mathcal{F}_2^{-1} (\psi * (\mathcal{F} \omega))(t) \in \mathcal{S}(\mathbb{R}^n) .
\]

Thus, \( (S_\omega f)(\tau, \xi) \in \mathcal{S}'(\mathbb{R}^{2n}) \).

Theorem 15. If \( \omega \in \mathcal{S}'(\mathbb{R}^{2n}) \), then \( S_\omega \) maps \( \mathcal{S}(\mathbb{R}^n) \) into \( \mathcal{S}'(\mathbb{R}^{2n}) \).

Proof. If \( \phi \in \mathcal{S}(\mathbb{R}^n) \) and \( \psi \in \mathcal{S}(\mathbb{R}^{2n}) \), then

\[
U_{\phi, \psi}(\tau, \xi) := \mathcal{F}_2(\phi)(\alpha + \xi) \psi(\alpha, \xi) \in \mathcal{S}(\mathbb{R}^{2n}) .
\]

Thus for any \( \alpha \in \mathcal{S}(\mathbb{R}^{2n}) \), we have

\[
\langle (\mathcal{F}_1 \omega)(\alpha, \xi), U_{\phi, \psi}(\alpha, \xi) \rangle
\]

\[
= \int_{\mathbb{R}^n} (\mathcal{F}_1 \omega)(\alpha, \xi) (\mathcal{F}_1 \phi)(\alpha + \xi) \overline{\psi(\alpha, \xi)} d\alpha d\xi
\]

\[
= \int_{\mathbb{R}^n} (\mathcal{F}_1 (S_\omega \phi))(\alpha, \xi) \overline{\psi(\alpha, \xi)} d\alpha d\xi
\]

\[
= \langle (\mathcal{F}_1 (S_\omega \phi))(\alpha, \xi), (\mathcal{F}_1 \phi)(\alpha, \xi) \rangle .
\]

Thus \( \mathcal{F}_1 (S_\omega \phi)(\alpha, \xi) \in \mathcal{S}'(\mathbb{R}^{2n}) \) and hence \( S_\omega \phi \in \mathcal{S}'(\mathbb{R}^{2n}) \).

Conflict of Interests

The authors declare that there is no conflict of interests.

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References

[1] R. G. Stockwell, L. Mansinha, and R. P. Lowe, “Localization of the complex spectrum: the S transform,” IEEE Transactions on Signal Processing, vol. 44, no. 4, pp. 998–1001, 1996.

[2] S. Ventosa, C. Simon, M. Schimmel, J. J. Danobeitia, and A. Manuel, “The S-transform from a wavelet point of view,” IEEE Transactions on Signal Processing, vol. 56, no. 7, pp. 2771–2780, 2008.

[3] S. K. Singh, “The S-Transform on spaces of type S,” Integral Transforms and Special Functions, vol. 23, no. 7, pp. 481–494, 2012.

[4] S. K. Singh, “The S-Transform on spaces of type W,” Integral Transforms and Special Functions, vol. 23, no. 12, pp. 891–899, 2012.

[5] S. K. Singh, “The fractional S-Transform of tempered ultradistributions,” Investigations in Mathematical Sciences, vol. 2, no. 2, pp. 315–325, 2012.

[6] S. K. Singh, “The fractional S-Transform on spaces of type S,” Journal of Mathematics, vol. 2013, Article ID 105848, 9 pages, 2013.

[7] S. K. Singh, “The fractional S-Transform on spaces of type W,” Journal of Pseudo-Differential Operators and Applications, vol. 4, no. 2, pp. 251–265, 2013.

[8] S. K. Singh, “A new integral transform: theory part,” Investigations in Mathematical Sciences, vol. 3, no. 1, pp. 135–139, 2013.

[9] K. Grochenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, Mass, USA, 2001.

[10] S. Zaidman, Distributions and Pseudo-Differential Operators, Logman, Essex, UK, 1991.

[11] R. S. Pathak, A Course in Distribution Theory and Applications, Narosa Publishing House, New Delhi, India, 2009.