QUANTUM CLIFFORD HOPF GEBRA
FOR QUANTUM FIELD THEORY

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We give arguments for the necessity to employ quantum Clifford Hopf gebras in quantum field theory. The role of the antipode is examined, Feynman diagrams are re-interpreted as tangles of graphical calculus. Regularization due to the design of convolution Hopf gebras is given as a program for future research.

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I. INTRODUCTION

Clifford algebras have been used to code QFT for several times, e.g. [1]. Our starting point is ‘functional QFT’ developed by Stumpf and coworkers [20]. In this approach quantum field theories are described by algebraic sources (Schwinger sources) spanning a functional (Fock) space. The coefficients of such sources are correlation functions w.r.t. the exact physical vacuum, as e.g. time-ordered $\tau$-functions or normal-ordered $\phi$-functions. The dynamics is given via the Schwinger–Dyson hierarchy which translates into a functional equation in the algebraic picture, see (5). Using methods closely related to $C^*$-algebras like GNS-states etc. the hierarchy translates into a functional Schrödinger equation which contains the whole information of the quantum dynamics. Unfortunately calculations in this formalism become cumbersome, if compared with e.g. path integral methods, which prevented their common usage, in spite of successful applications of the method in composite particle theory and the successful derivation of the $SU(1) \times SU(2)$ electroweak theory from sub-fermion models which are beyond the abilities of path integral methods [20].

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This computational success was the motivation to search for a compact formalism behind the method. In several works [4, 5, 6, 9, 11] a concise and powerful description of functional quantum field theory was developed employing Clifford algebras. However, it turned out that one had to use ‘Quantum Clifford Algebras’ (QCA) [10], i.e. Clifford algebras of an arbitrary bilinear form, which cover deformed structures as Hecke algebras, the Manin quantum plane, $q$-spinors etc. The first success of this treatment was the disappearance of certain singularities in vertex normal-ordering [4] just by a correct algebraic reformulation. A further point concerns interacting QFT which cannot be treated in Fock space, i.e. with path integral methods. Therefore such QFT has to select a unique vacuum for every dynamics [8]. It was shown, that this is the case if the antisymmetric part of the bilinear form of the QCA is identified with the (exact) propagator of the theory. Perturbation theory uses the free propagator and thus falsely Fock space.

In this paper, we give arguments, that a ‘Quantum Clifford Hopf Algebra’ (QCHG) should be used to describe QFT. The most important technique is Rota-Stein ‘Cliffordization’ [19], which allows to introduce Clifford products by product ‘mutation’ of the Grassmann product and yields closed and grade independent formulas. This process needs intrinsically the co-algebra structure.

The convolution product given canonically by a pair of product and co-product and the antipode constitute a Hopf algebra [3, 17]. Product, co-product and Convolution will be interpreted by quantum processes like absorption, emission and interaction [18]. The meaning of the antipode can be anticipated from [11], where it was identified with grade-involution in the Grassmann case. The antipode plays also an important role in renormalization theory of perturbative QFT [14]. However, as discussed above, we are searching for non-perturbative regular QFTs. In this spirit, we state a program how a Hopf-algebra could be designed for such a purpose.

**II. QUANTUM CLIFFORD ALGEBRAS FOR QFT**

The definition of QCAs can be found in [10, 13]. We give the basic definitions to show how this algebras fit for QFT. Using Chevalley deformation (see the criticism below) we
define \((x, y \in V, u, v, w \in V^\wedge \equiv \text{Ext}V)\)

\[
\begin{align*}
 i) \quad x \cdot_B y &= B(x, y) \\
 ii) \quad x \cdot_B (u \land v) &= (x \cdot_B u) \land v - S(u) \land (x \cdot_B y) \\
 iii) \quad u \cdot_B (v \cdot_B w) &= (u \land v) \cdot_B w \\
 vi) \quad S : V^\wedge \mapsto V^\wedge, \quad S(u) = (-1)^{\partial u}u,
\end{align*}
\]

where \(\partial u \in \mathbb{N}_0\) is the Grassmann grade of \(u\). It is obvious from this definitions that the bilinear form \(B\) needs to have no symmetry, while the usual commutator based definition \(\{x, y\}_+ = 2g(x, y)\) is by definition restricted to symmetric bilinear forms. We define \(g(x, y) := 1/2(B(x, y) + B(y, x))\) and \(F(x, y) := 1/2(B(x, y) - B(y, x))\). There is an Clifford algebra isomorphism \(\phi\), called Wick-isomorphism \([10]\), which connects \(\mathcal{C}(V, B)\) and \(\mathcal{C}(V, g)\). This results in a change of the Grassmann product and of the Grassmann grade. One defines

\[
x \cdot y := x \land y + F(x, y)
\]

on grade one elements which is obviously not grade preserving, i.e. \(\partial \neq \partial\), \([10], [11]\).

Applying this structure to QFT is done by the following identification. Let \(\psi_X\) be a field operator (or its dual) with all indices (discrete, continuous and conjugation) are put into a super-index \(X\). The field operators are identified with the Clifford map as \([7, 9]\):

\[
\begin{align*}
 \psi_X^L &:= X \cdot_B g + X \land + F(X, \ldots) \\
 \psi_X^R &:= X \cdot_B g - X \land + F(X, \ldots),
\end{align*}
\]

where \(L, R\) denotes right or left action. Given a possibly non-linear field equation \(\dot{\psi} = V(\psi)\) we obtain the normal-ordered generating functional \(|N(j, a)\rangle\) in Clifford terms via \([3]\)

\[
\begin{align*}
\dot{\psi} &= [H(\psi), \psi]_- \\
H[j, d] &= H(\psi^L)\psi - \psi H(\psi^R) \\
\partial_t |N(j, a)\rangle &= H[j, d] |N(j, a)\rangle,
\end{align*}
\]

where \(j\) and \(d\) algebraically span \(V^\wedge\) and its dual \(V^{\wedge*}\). One notes: Quantization is given by the symmetric part of the bilinear form, while the propagator is given by the antisymmetric part which fixes all freedom.
III. BIGEブラ STRUクTURE

Every space $V^\wedge$, dim $V = n$ underlying a Grassmann algebra has a $\mathbb{Z}_n$-grading, and there is a natural duality between grade $r$ and $n-r$ subspaces, e.g. $V$ and $\wedge^{n-1}V$. Beside the deformation into a Clifford algebra as in (1), one can think about introducing a Clifford product on $\wedge^{n-1}V$. Using product co-product duality (see Fig. 1), it is well known, that any product on the dual space induces a co-product on the former. A Grassmann algebra is in this way a convolution algebra obeying algebra and co-gebra structure. However, it can be proved that it is a biegebra possessing an antipode $S$ (given in (1-vi) !), rendering it a Hopfgebra. If an unital convolution algebra possess an antipode it can be proved to be a Hopfgebra [3, 18]. We re-interpret the product as an absorption (of a tensor factor) and co-product as an emission (of a tensor factor).

IV. CLIFFORD HOPF GEBRA

Since a Grassmann Hopfgebra is the classical case, one is interested to find a Clifford Hopfgebra as its quantization. Unfortunately, there are theorems [17, 19] that Clifford convolution algebras possess no antipode if product co-product duality, see Fig. 1, is employed. Moreover in [19] only the product was deformed, which is artificial.

A. Convolution Bigebra

A program was started [17] which breaks the product co-product duality and investigates the general case of a Clifford convolution algebra. However, since the antipode seems to play an important role in renormalization theory [14] and might be needed to model some notions of physics like expectation values, and the ‘split’ of a tangle, see below, we want to propose a different approach.

B. Antipode

Axioms for an antipode can be given for any unital convolution algebra, see Fig. 2. If an algebra is unital and a co-gebra is co-unital, the mutually defined convolution algebra is unital too. If an antipode can be found for a given pair of product and co-product, it is
unique. A quadruple $H = (V^\wedge, m, \Delta, S)$ of a space $V^\wedge$, a product $m$, a coproduct $\Delta$ and an antipode $S$ is a Hopf algebra \[3, 18\].

### C. Quantum Clifford Hopf Algebras

An idea developed in \[12\] to circumvent the non-existence theorem for the antipode is to employ QCA. Having an antisymmetric part in the bilinear form it is possible to arrange that $\det(g) \neq 0$ but $\det(B) = 0$. In this case, no product co-product duality is possible and we can freely adjust product and co-product. Indeed antipodes for 1+1 space-time have been found in such a setting.

### V. QFT By Clifford Hopf Algebras

Quantization in the sense of a transition from Grassmann to Clifford algebras has a major drawback if introduced by Chevalley deformation as done in (1). The first two formulas (1-i) and (1-ii) are applicable only for grade one elements. This means, that we have to fix a unique grade one space, which breaks some symmetries. Moreover, computations with higher grade elements have to be reduced recursively to this cases. But – this is also exactly the case in QFT if described in the ‘second quantization’ picture, or even in quantum mechanics if the Dirac creator, annihilator picture is used.

Creation is (wedge) concatenation by a grade one element, while annihilation is the antiderivation rule given in (1-ii). This is reflected by formulas (3) and (4) where we identified the field operators. But – as we have seen in (2), adding an antisymmetric part does not preserve the grading. This is the (Hopf algebraic) origin of the need of a Wick normal-ordering which literally does transform to that particular grading which is selected by the propagator (antisymmetric part) \[11\]. Using Rota–Stein’s ‘Cliffordization’ \[19\], one can introduce a Clifford product as a deformation of the Grassmann product for any bigegebra element. This deformation, shown in Fig. 3, replaces the product tangle by a more complicated tangle,
possessing an internal loop. Since it has two inputs and one output, it is still a product or an absorption. The same should be done with the co-product, by duality as shown in Fig. 1, or starting from a Clifford convolution algebra with two independent (might be singular) bilinear forms. Having such a general tool at our disposal one can break of the strong connection which ties grade one elements to absorption and emission or say annihilation and creation.

A. Feynman Diagrams as Tangles

We re-interpret, as also promoted by Oziewicz [18], a convolution bigebra tangle as physical process. This is quite different to the usual interpretation of such iconic notations. Perturbative QFT (pQFT) creates and annihilates single bare –or naked– particles or fields. Only after this unphysical processes has been defined the particles are dressed by an infinity set of possible internal vacuum polarization diagrams. However this method can be see to follow directly from the drawback of the definitions in Eqn. (1). Since only grade one elements can be used in the creation and annihilation process, any higher grade action has to be split into this basic building blocks. Such a behaviour can be found e.g. in Wick normal-ordering [4, 11]. If the base space $V$ is of infinite dimension, such an process is an infinite recursion. The solution may be taken from the Rota–Stein Cliffordization. Since this formula is grade independent, due to the usage of Hopf algebraic methods, such tangles are valid for any process, not just only creation and annihilation by grade one elements. This opens very general new insights into the structure of non-perturbative QFT. pQFT is moreover tied to the intrinsic grading obtained from the artificial distinction of grade one elements. A change to this grading e.g. by Wick normal-ordering, has to be performed, otherwise no numerical results can be obtained. The Hopf gebraic approach cannot suffer from this deficiency, since it is intrinsically invariant w.r.t. the grading.
The Rota-Stein Cliffordization depicted as ‘sausage’ in Fig. 3 shows a second feature which is of utmost importance for QFT. Adding a certain internal loop, mutates a product into a new one. Indeed, the tangles before and after mutation (i.e. quantization) work out completely different. This can be the starting point for an axiomatization of Hopf algebraic structures of convolution algebras of QFT. Of course this is a wide and open field of speculation, but looking carefully at the needs of QFT will surely generate such rules. The main problem will be to find some basic tangles, the processes of QFT, which may possess internal loops. Such basic tangles have to be selfconsistent. Afterwards they can be easily tied together to model interactions. In the case of the Clifford product this is already a truth. One should however be careful to think too simple minded. It will be most likely the case, that not a finite set of ‘basic’ tangles will do the job in any case. When proceeding from elementary objects to bound states, one should expect a new basic behaviour of such objects resulting in the need of new tangles with internal loops. ‘Elementary’ tangles might show an internal structure if ‘magnified’ like the Clifford product if seen as sausage tangle.

B. Renormalization

We discuss how such techniques are employed in the theory of renormalization. Connes, Kreimer and others have used the antipode of a rooted tree Hopf algebra \cite{13} to generate the counterterms of renormalization \cite{14}. This is analogous to the Wick normal-ordering using Hopf algebra techniques \cite{11}. However, both cases fail to hit the point, since we are interested in invariants. E.g. the artificial grade one dependence forcing normal-ordering is also artificial. The same argument should hold for renormalization, which is also an artificial rewriting of diagrams into a series of diagrams which have to be summed up. Such diagrams are not tangles in our sense!

C. Search for Finite QFT

We end this article proposing a program to search for a structure which may, this is our hope, lead ultimatively to finite QFT.

Suppose you have a loop as given in the LHS of Fig. 4. If such a tangle is seen as physical process it cannot be distinguished from a single line. Otherwise one would have to have
information about the internal structure of the tangle, but that would need a measurement and this would incorporate a different process as mere propagation.

If we calculate this tangle in a Clifford biconvolution bigebra \([3, 12, 18]\), i.e. with the requirement that the product and coproduct is Clifford and they are related by \(B = B^{-1}\), we get the RHS. We stay with a single line up to a scalar factor which equals the dimension of the space (infinite in QFT). In every tangle process which we want to calculate we are allowed to remove all such loops by multiplying with \(\dim V\). It can be asked, if we can find a Clifford biconvolution Hopf gebra which has a unit multiplicative factor. This reverses the arguments: \textit{physics models the Hopf gebraic axioms}, and not vice versa.

We insist in Hopf gebras, since the antipode seems to be a necessary part of such a framework. Looking at the antipode axiom, Fig. 2, (which might be modified also) the important feature is that the output of this tangle is a scalar times identity whatever one throws in. This antipode tangle could probably be used as an surrogate for an expectation value. Moreover, the antipode splits tangles into parts which is essential in some proofs.

\textbf{Problems:} We have lots of open problems to solve: Find all basic tangles of a realistic (even free) QFT. Give rules to extract all loops in such tangles, like Fig. 4, to basic loops or non-loops which can be boxed in basic tangles. Derive from such rules the product, coproduct and antipode of a Clifford Hopf gebra as a model. Find a translation between usual QFT and such a tangle based description, e.g. for vacuum expectation values, transition matrix elements, \ldots etc.

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