The Cheshire Cat Principle from Holography

Holger Bech Nielsen\textsuperscript{a} and Ismail Zahed\textsuperscript{b}

\textsuperscript{a} Niels Bohr Institute, 17 Blegdamsvej, Copenhagen, Denmark
\textsuperscript{b} Department of Physics and Astronomy, SUNY Stony-Brook, NY 11794

April 2, 2009

Abstract

The Cheshire cat principle states that hadronic observables at low energy do not distinguish between hard (quark) or soft (meson) constituents. As a result, the delineation between hard/soft (bag radius) is like the Cheshire cat smile in Alice in wonderland. This principle reemerges from current holographic descriptions of chiral baryons whereby the smile appears in the holographic direction. We illustrate this point for the baryonic form factor.

\textsuperscript{1}To appear in \textit{Multifaceted Skyrmion}, Eds. G. E. Brown and M. Rho, World Scientific
1 Introduction

Back in the eighties, quark bag models were proposed as models for hadrons that capture the essentials of asymptotic freedom through weakly interacting quarks and gluons within a bag, and the tenets of nuclear physics through strongly interacting mesons at the boundary. The delineation or bag radius was considered as a fundamental and physically measurable scale that separates ultraviolet from infrared QCD [1].

The Cheshire cat principle [2] suggested that this delineation was unphysical in low energy physics, whereby fermion and color degrees of freedom could readily leak through the bag radius, making the latter immaterial. In a way, the bag radius was like the smile of the Cheshire cat in Alice in wonderland. The leakage of the fundamental charges was the result of quantum effects or anomalies [3].

In 1+1 dimensions exact bosonization shows that a fermion can translate to a boson and vice-versa making the separation between a fermionic or quark and a bosonic or meson degree of freedom arbitrary. In 3+1 dimensions there is no known exact bozonization transcription, but in large $N_c$ the Skyrme model has shown that baryons can be decently described by topological mesons. The Skyrmion is the ultimate topological bag model with zero size bag radius [4], lending further credence to the Cheshire cat principle.

The Skyrme model was recently seen to emerge from holographic QCD once chiral symmetry is enforced in bulk [5]. In holography, the Skyrmion is dual to a flavor instanton in bulk at large $N_c$ and strong t’Hooft coupling $\lambda = g^2 N_c$ [5, 6]. The chiral Skyrme field is just the holonomy of the instanton in the conformal direction. This construction shows how a flavor instanton with instanton number one in bulk, transmutes to a baryon with fermion number one at the boundary.

Of course, QCD is not yet in a true correspondence with a known string theory, as $N = 4$ SYM happens to be according to Maldacena’s conjecture [7]. Perhaps, one way to achieve this is through the down-up string approach advocated in [8]. Throughout, we will assume that the correspondence when established will result in a model perhaps like the one suggested in [5] for the light mesons and to which we refer to as holographic QCD.

With this in mind, holographic QCD provides a simple realization of the Cheshire cat principle at strong coupling. In section 2, we review briefly the holonomy construction for the Skyrmion in holography and illustrate the Cheshire cat principle. In section 3 we outline the holographic model. In section 4 we construct the baryonic current. In section 5 we derive the baryonic form factor. Many of the points presented in this review are borrowed from recent arguments in [9].
2 The Principle and Holography

In holographic QCD, a baryon is initially described as a flavor instanton in the holographic \( Z \)-direction. The latter is warped by gravity. For large \( Z \), the warped instanton configuration is not known. However, at large \( \lambda = g^2 N_c \) the warped instanton configuration is forced to \( Z \sim 1/\sqrt{\lambda} \) due to the high cost in gravitational energy. As a result, the instanton in leading order is just the ADHM configuration with an additional \( U(1) \) baryonic field, with gauge components \[5\]

\[
\begin{align*}
\widetilde{A}_0 &= -\frac{1}{8\pi^2 a \lambda} \frac{2\rho^2 + \xi^2}{(\rho^2 + \xi^2)^2}, \\
\widetilde{A}_M &= \eta_{MN} \frac{\sigma_i}{2} \frac{2x_N}{\xi^2 + \rho^2},
\end{align*}
\]

with all other gauge components zero. The size is \( \rho \sim 1/\sqrt{\lambda} \). We refer to \[5\] (last reference) for more details on the relevance of this configuration for baryons. The ADHM configuration has maximal spherical symmetry and satisfies

\[
(\mathbb{R}A)_Z = A_Z(\mathbb{R}\vec{x}), \quad (\mathbb{R}^{ab}A^b)_i = \mathbb{R}^{T}_{ij}A^a_j(\mathbb{R}\vec{x}),
\]

with \( \mathbb{R}^{ab}\tau^b = \Lambda^+ \tau^a \Lambda \) a rigid \( \text{SO}(3) \) rotation, and \( \Lambda \) is \( \text{SU}(2) \) analogue.

The holographic baryon is just the holonomy of (1) along the gravity bearing and conformal direction \( Z \),

\[
U^{\mathbb{R}}(x) = \Lambda P \exp \left( -i \int_{-\infty}^{+\infty} dZ A_Z \right) \Lambda^+.
\]

The corresponding Skyrmion in large \( N_c \) and leading order in the strong coupling \( \lambda \) is \( U(\vec{x}) = e^{i\vec{r}.\vec{F}(\vec{x})} \) with the profile

\[
F(\vec{x}) = \frac{\pi |\vec{x}|}{\sqrt{\vec{x}^2 + \rho^2}}.
\]

In a way, the holonomy (3) is just the fermion propagator for an infinitly heavy flavored quark with the conformal direction playing the role of time. (3) is the bosonization of this conformal quark in 3+1 dimensions.

The ADHM configuration in bulk acts as a point-like Skyrmion on the boundary. The
baryon emerges from a semiclassical organization of the quantum fluctuations around the point-like source (3). To achieve this, we define

\[ A_M(t, x, Z) = R(t)(A_M(x - X_0(t), Z - Z_0(t)) + C_M(t, x - X_0(t), Z - Z_0(t))) , \]  

(5)

The collective coordinates \( R, X_0, Z_0, \rho \) and the fluctuations \( C \) in (5) form a redundant set. The redundancy is lifted by constraining the fluctuations to be orthogonal to the zero modes. This can be achieved either rigidly [10] or non-rigidly [11]. We choose the latter as it is causality friendly. For the collective iso-rotations the non-rigid constraint reads

\[ \int_{x=Z=0} d\hat{\xi} C G^B A_M , \]  

(6)

with \( (G^B)^{ab} = \epsilon^{aBb} \) the real generators of \( R \).

For \( Z \) and \( \rho \) the non-rigid constraints are more natural to implement since these modes are only soft near the origin at large \( \lambda \). The vector fluctuations at the origin linearize through the modes

\[ d^2\psi_n/dZ^2 = -\lambda_n \psi_n , \]  

(7)

with \( \psi_n(Z) \sim e^{-i\sqrt{\lambda_n}Z} \). In the spin-isospin 1 channel they are easily confused with \( \partial_Z A_i \) near the origin as we show in Fig. 1. Using the non-rigid constraint, the double counting is
removed by removing the origin from the vector mode functions

$$
\psi'_n(Z) = \theta(|Z| - Z_C)\psi_n(Z) ,
$$

with $Z_C \sim \rho \sim 1/\sqrt{\lambda}$ which becomes the origin for large $\lambda$. In the non-rigid semiclassical framework, the baryon at small $\xi < |Z_C|$ is described by a flat or uncurved instanton located at the origin of $R^4$ and rattling in the vicinity of $Z_C$. At large $\xi > |Z_C|$, the rattling instanton sources the vector meson fields described by a semi-classical expansion with non-rigid Dirac constraints. Changes in $Z_C$ (the core boundary) are reabsorbed by a residual gauge transformation on the core instanton. This is a holographic realization of the Cheshire cat principle [2] where $Z_C$ plays the role of the Cheshire cat smile.

### 3 The Holographic Model

To illustrate the Cheshire cat mechanism more quantitatively, we now summarize the holographic Yang-Mills-Chern-Simons action in 5D curved background. This is the leading term in a $1/\lambda$ expansion of the D-brane Born-Infeld (DBI) action on D8 [5],

$$
S = S_{YM} + S_{CS} ,
$$

$$
S_{YM} = -\kappa \int d^4x dZ \text{ tr } \left[ \frac{1}{2} K^{-1/3} F_{\mu\nu} + M_{KK}^2 K F_{\mu Z}^2 \right] ,
$$

$$
S_{CS} = \frac{N_c}{24\pi^2} \int_{M^4 \times R} \omega_{U(N_f)}^5(\mathcal{A}) ,
$$

where $\mu, \nu = 0, 1, 2, 3$ are 4D indices and the fifth(internal) coordinate $Z$ is dimensionless. There are three things which are inherited by the holographic dual gravity theory: $M_{KK}, \kappa,$ and $K$. $M_{KK}$ is the Kaluza-Klein scale and we will set $M_{KK} = 1$ as our unit. $\kappa$ and $K$ are defined as

$$
\kappa = \lambda N_c \frac{1}{216\pi^3} \equiv \lambda N_c a , \quad K = 1 + Z^2 .
$$

$\mathcal{A}$ is the 5D $U(N_f)$ 1-form gauge field and $F_{\mu\nu}$ and $F_{\mu Z}$ are the components of the 2-form field strength $\mathcal{F} = d\mathcal{A} - i\mathcal{A} \wedge \mathcal{A}$. $\omega_{U(N_f)}^5(\mathcal{A})$ is the Chern-Simons 5-form for the $U(N_f)$ gauge
field

$$\omega_s^{U(N_f)}(A) = \text{tr} \left( A F^2 + \frac{i}{2} A^3 F - \frac{1}{10} A^5 \right),$$

(13)

We note that $S_{YM}$ is of order $\lambda$, while $S_{CS}$ is of order $\lambda^0$. These terms are sufficient to carry a semiclassical expansion around the holonomy (3) with $\hbar = 1/\kappa$ as we now illustrate it for the baryon current.

### 4 The Baryon Current

To extract the baryon current, we source the reduced action with $\hat{V}_\mu(x)$ a $U(1)_V$ flavor field on the boundary in the presence of the vector fluctuations ($C = \hat{v}$). The effective action for the $U(1)_V$ source to order $\hbar^0$ reads

$$S_{\text{eff}}[\hat{V}_\mu] = \sum_{n=1}^\infty \int d^4x \left[ -\frac{1}{4} \left( \partial_\mu \hat{v}_\nu^n - \partial_\nu \hat{v}_\mu^n \right)^2 - \frac{1}{2} m_{\hat{v}^n}^2 (\hat{v}_\mu^n)^2 - \kappa K Z^\mu \hat{v}_\mu^n \psi_{2n-1} \right]_{Z=B},$$

(14)

The first line is the free action of the massive vector meson which is

$$\Delta_{\mu\nu}^{mn}(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \left[ \frac{-g_{\mu\nu} - p_\mu p_\nu/m_{\hat{v}^n}^2}{p^2 + m_{\hat{v}^n}^2} \delta_{mn} \right],$$

(15)

in Lorentz gauge. The second line is the direct coupling between the core instanton and the $U(1)_V$ source as displayed in Fig.2a while the last line corresponds to the vector omega, omega’, … mediated couplings (VMD) as displayed in Fig.2b. These couplings are

$$\kappa K Z^\mu \hat{v}_\mu^n \psi_{2n-1},$$

(16)

which are large and of order $1/\sqrt{\hbar}$ since $\psi_{2n-1} \sim \sqrt{\hbar}$. When $\rho$ is set to $1/\sqrt{\lambda}$ after the book-keeping noted above, the coupling scales like $\lambda \sqrt{N_c}$, or $\sqrt{N_c}$ in the large $N_c$ limit taken first.
The direct coupling drops by the sum rule

$$\sum_{n=1}^{\infty} \alpha v_n \psi_{2n-1} = 1 ,$$ (17)

following from closure in curved space

$$\delta(Z - Z') = \sum_{n=1}^{\infty} \kappa \psi_{2n-1}(Z) \psi_{2n-1}(Z') K^{-1/3}(Z') .$$ (18)

in complete analogy with VMD for the pion in holography [5].

Baryonic VMD is exact in holography provided that an infinite tower of radial omega’s are included in the mediation of the $U(1)_V$ current. To order $\hbar^0$ the baryon current is

$$J^\mu_B(x) = - \sum_{n,m} m_{vn}^2 a_{vn} \psi_{2n-1} \int d^4 y \, \kappa \hat{K}_{Z\nu}(y, Z) \Delta \frac{^n_{\mu} m}{mn}(y - x) \bigg|_{Z=B} .$$ (19)

This point is in agreement with the effective holographic approach described in [13]. The static baryon charge distribution is

$$J^0_B(x) = - \sum_n \int d\bar{y} \, \frac{2}{N_c} \kappa \hat{K}_{Z0}^{\bar{y}}(\bar{y}, Z) \Delta_n(\bar{y} - x) \, a_{vn} m_{vn}^2 \psi_{2n-1} \bigg|_{Z=B} ,$$ (20)

with

$$\Delta_n(\bar{y} - x) \equiv \int \frac{d\vec{p}}{(2\pi)^3} \frac{e^{-i\vec{p} \cdot (\bar{y} - x)}}{\vec{p}^2 + m_{vn}^2} .$$ (21)

The extra $2/N_c$ follows the normalization $\hat{V}_\mu = \delta_{\mu0} \frac{\sqrt{2N_f}}{N_c} \hat{B}_0(x)$ for the baryon number source.
5 Baryonic Form Factor

The static baryon form factor is a purely surface contribution from

\[ J_B^0(\vec{q}) = \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} J_B^0(x) \]

\[ = -\sum_{n} \int dZ \partial Z \left[ \left( \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} \Omega_0(x, Z) \right) \psi_{2n-1} \right] \frac{a_n m^2_{v_n}}{q^2 + m^2_{v_n}} \]  \hspace{1cm} (22)

\[ = \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} \sum_{n} \frac{a_n m^2_{v_n}}{q^2 + m^2_{v_n}} \psi_{2n-1}(Z_C) 2\Omega_0(\vec{x}, Z_C) , \] \hspace{1cm} (23)

with

\[ \Omega_0(x, Z) \equiv \frac{1}{N_c} \kappa K \hat{F}_Z(x, Z) . \] \hspace{1cm} (24)

The boundary contribution at \( Z = \infty \) vanishes since \( \psi_{2n-1} \sim 1/Z \) for large \( Z \). In the limit \( q \to 0 \) we pick the baryon charge

\[ \int d\vec{x} e^{i\vec{q} \cdot \vec{x}} 2\Omega_0(\vec{x}, Z_C) , \] \hspace{1cm} (25)

due to the sum rule (17), with the limits \( \lim_{q \to 0} \lim_{Z \to 0} \) understood sequentially.

The surface density follows from the U(1) bulk equation

\[ \frac{4}{N_c} \kappa K \hat{F}_Z(Z_c) = \int_{-Z_c}^{Z_c} dZ \frac{1}{32\pi^2} \epsilon_{MNPQ} \left( \text{tr}(\hat{F}_{MN}\hat{F}_{PQ}) + \frac{1}{2} \hat{F}_{MN}\hat{F}_{PQ} \right) \]
\[ + \frac{2}{N_c} \int_{-Z_c}^{Z_c} dZ \kappa K^{-1/3} \partial^i \hat{F}_{0i} , \] \hspace{1cm} (26)

The baryon number density lodged in \( |Z| < Z_c \) integrates to 1 since

\[ B = \int d\vec{x} J_B^0(\vec{x}) = \int d\vec{x} 2\Omega_0(\vec{x}, Z_c) = \int d\vec{x} \int_{-Z_c}^{Z_c} dZ \frac{1}{32\pi^2} \epsilon_{MNPQ} \text{tr}(\hat{F}_{MN}\hat{F}_{PQ}) = 1 , \] \hspace{1cm} (27)

as the spatial flux vanishes on \( R^3_x \) is zero for a sufficiently localized SU(2) instanton in \( R^3_x \times R_Z \).

The isoscalar charge radius, can be read from the \( q^2 \) terms of the form factor
\begin{equation}
\langle r^2 \rangle_0 = \frac{3}{2} \frac{Z_c \rho^2}{\sqrt{Z_c^2 + \rho^2}} + \int dZ \Delta_C(Z, Z_c)
\end{equation}

with \( r \equiv \sqrt{\langle \vec{x}^2 \rangle} \). The first contribution is from the core and of order \( 1/\lambda \),

\begin{equation}
\int d\vec{x} r^2 2Q_0(\vec{x}, Z_c) = \frac{3}{2} \frac{Z_c \rho^2}{\sqrt{Z_c^2 + \rho^2}} \to \frac{3}{2} \rho^2.
\end{equation}

The second contribution is from the cloud and of order \( \lambda^0 \),

\begin{equation}
\sum_{n=1}^{\infty} \frac{\alpha_{n,0} \psi_{2n-1}(Z_c)}{m_n^2} = \int dZ \Delta_C(Z, Z_c)
\end{equation}

with \( \Delta_C = \Box^{-1} \equiv -\partial_Z^{-1} K^{-1} \partial_Z^{-1} K^{-1/3} \) the inverse vector meson propagator in bulk.

The results presented in this section were derived in [9] using the cheshire cat descriptive. They were independently arrived at in [12] using the strong coupling source quantization. They also support, the effective 5-dimensional nucleon approach described in [13] using the heavy nucleon expansion.

6 Conclusions

The holography model presented here provides a simple realization of the Cheshire principle, whereby a zero size Skyrmion emerges to order \( 1/\hbar = \kappa \) through a holonomy in 5 dimensions. The latter is a bosonized form of a heavy quark sitting still in the conformal direction viewed as time. The baryon has zero size.

To order \( \hbar^0 \), the core Skyrmion is dressed by an infinite tower of vector mesons which couple in the holographic direction a distance \( Z_C \) away from the core. The emergence of \( Z_C \) follows from a non-rigid semiclassical quantization constraint to prevent double counting. \( Z_C \) divides the holographic direction into a core dominated by an instanton and a cloud described by vector mesons.

Observables are \( Z_C \) independent provided that the curvature in both the core and the cloud is correctly accounted for. This is the Cheshire cat mechanism in holography with \( Z_C \) playing the role of the Cheshire cat smile. We have illustrated this point using the
baryon form factor, where $Z_C$ was taken to zero using the uncurved or flat ADHM instanton. The curved instanton is not known. Most of these observations carry to other baryonic observables [9, 12] and baryonic matter [14] (and references therein).

7 Acknowledgments

IZ thanks Keun-Young Kim for his collaboration on numerous aspects of holographic QCD. This work was supported in part by US-DOE grants DE-FG02-88ER40388 and DE-FG03-97ER4014.

References

[1] A. Chodos, R. Jaffe, K. Johnson and C. Thorn, “Baryon Structure in the Bag Theory”, Phys. Rev. D 10, 2599 (1974);
G.E. Brown and M. Rho, “The Little Bag,” Phys. Lett. B 82, 177 (1979).

[2] S. Nadkarni, H. B. Nielsen and I. Zahed, “Bosonization Relations As Bag Boundary Conditions,” Nucl. Phys. B 253, 308 (1985).

[3] H.B. Nielsen, M. Rho, A. Wirzba and I. Zahed, “Color Anomaly in Hybrid Bag Model,” Phys. Lett. B 269, 389 (1991);
M. Rho, “The Cheshire Cat Hadrons Revisited,” Phys. Rep. 240, 1 (1994).

[4] I. Zahed and G.E. Brown, “The Skyrme Model”, Phys. Rep. 142, 1 (19986).

[5] T. Sakai and S. Sugimoto, “Low energy hadron physics in holographic QCD,” Prog. Theor. Phys. 113, 843 (2005);
T. Sakai and S. Sugimoto, “More on a holographic dual of QCD,” Prog. Theor. Phys. 114, 1083 (2006);
H. Hata, T. Sakai, S. Sugimoto and S. Yamato, “Baryons from instantons in holographic QCD,” arXiv:hep-th/0701280.

[6] D. Hong, M. Rho, H. Yee and P. Yi, “Chiral Dynamics of Baryons from String Theory” Phys. Rev. D 76, 061901 (2007).

[7] J. Maldacena, “The Large N limit of Superconformal Field Theories and Supergravity” Adv. Theor. Math. Phys. 2, 231 (1998).
[8] C. Csaki and M. Reece, “Toward a Systematic Holographic QCD: A Braneless Approach”, arXiv:hep-th/0608266.

[9] K. Y. Kim and I. Zahed, “Electromagnetic Baryon Form Factors from Holographic QCD.” JHEP 0809, 007 (1998).

[10] C. Adami and I. Zahed, “Soliton quantization in chiral models with vector mesons,” Phys. Lett. B 215 (1988) 387.

[11] H. Verschelde and H. Verbeke, “Nonrigid quantization of the skyrmion,” Nucl. Phys. A 495 (1989) 523.

[12] K. Hashimoto, T. Sakai and S. Sugimoto, “Holographic Baryons: Static Properties and Form Factors from Gauge/String Duality,” arXiv:0806.3122 [hep-th].

[13] D. K. Hong, M. Rho, H. U. Yee and P. Yi, “Nucleon Form Factors and Hidden Symmetry in Holographic QCD,” arXiv:0710.4615 [hep-ph]; M. Rho, “Baryons and Vector Dominance in Holographic Dual QCD,” arXiv:0805.3342 [hep-ph].

[14] K.Y. Kim, S.J. Sin and I. Zahed, “Dense Holographic QCD in the Wigner Seitz Approximation,” JHEP 0809, 001 (2008).