The curvature-induced gauge potential and the geometric momentum for a particle on
a hypersphere

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A particle that is constrained to freely move on a hyperspherical surface in an \( N (\geq 2) \) dimensional flat space experiences a curvature-induced gauge potential, whose form was given long ago (J. Math. Phys. 34(1993)2827). We demonstrate that the momentum for the particle on the hypersphere is the geometric one including the gauge potential and its components obey the commutation relations

\[
[p_i, p_j] = -i\hbar J_{ij}/r^2,
\]

in which \( \hbar \) is the Planck’s constant, and \( p_i \) (\( i, j = 1, 2, 3, \ldots N \)) denotes the \( i \)-th component of the geometric momentum, and \( J_{ij} \) specifies the \( ij \)-th component of the generalized angular momentum containing both the orbital part and the coupling of the generators of continuous rotational symmetry group \( SO(\ N-1) \) and curvature, and \( r \) denotes the radius of the \( N-1 \) dimensional hypersphere.

Keywords: curvature, gauge potential, hypersphere, geometric momentum

I. INTRODUCTION

In quantum mechanics, a constrained dynamical system is usually associated with a gauge structure, and it is quite well-understood in, for instance, gravitational field [1, 2], condensed matter physics [3, 4], quantum fields [5] and particle physics [6]. We are recently interested in the constrained motion, i.e., a particle remains and freely moves on a hypersurface [7–18]. There are also a lot of papers paying attention to the curvature-induced gauge structure for the system, and the form of the gauge potential is well-known [19–31]. The form of the gauge potential is clearly a coupling of the curvature and all generators of a rotational symmetry group, which in quantum mechanics is not necessarily understood to represent the spin though it is the important situation. Even so, the gauge potential is far beyond fully understood. For instance, once a classical bracket (c.f. Eq. (13)) involves both the momentum and the orbital angular momentum, the corresponding quantum commutator (c.f. Eq. (19)) based on the quantization of the classical bracket contains both the momentum operator and the angular momentum operator, and it appears transparent. However, the quantum commutator is highly non-trivial because the momentum explicitly contains the extrinsic curvature, and the previous studies unintentionally missed this geometric nature of the momentum till 2011 when the geometric momentum came into sight [8]. The main aim of the present study is to show that the relation (19) holds true with reasonable inclusion of the gauge potential into the geometric momentum.

This paper is organized in the following. In section II, how Ohnuki and Kitakado obtained the gauge potential is outlined and commented. In section III, the Dirac formalism of quantization for the particle on the sphere is invoked, where the Dirac brackets between the momentum components and the orbital angular momentum components play central role. To show that the fundamental quantum conditions in the Dirac scheme of quantization are completely compatible with the gauge potential, we must utilize the proper form of the momentum. In section IV and V, we explicitly prove two sets of fundamental quantum conditions in the Dirac scheme, respectively. The final section VI is a brief conclusion and discussion.

II. OHNUKI AND KITAKADO GAUGE POTENTIAL ON HYPERSPHERICAL SURFACE

By a hyperspherical surface \( S^{N-1} \) (\( N \geq 2 \)), we mean a spherical surface in \( N \) dimensional flat space \( E^N \). The surface equation is given by,

\[
x_i x_i = r^2, \quad (i = 1, 2, 3, \ldots, N), \quad n_i = \frac{x_i}{r},
\]

where \( r \) denotes the radius of the sphere, and \( x_i \) is the \( i \)-th coordinate and \( n_i \) is the \( i \)-th component of the normal vector. Throughout the paper, the Einstein’s summation convention is adopted, which implies the indices repeated...
twice in a term are summed over the range of the index unless specified, and the Roman indices \( i, j, k, l, m, n \) run from 1 to \( N \) to specify the coordinates of the surface in \( E^N \), while the Greek ones \( \alpha, \beta, \lambda, \mu \) run from 1 to \( N - 1 \) which can not be confused with the \( N - 1 \) local coordinates of the surface.

Ohnuki and Kitakado started from a fundamental algebra on the flat \( E^N \) space \([19]\), the corresponding Euclidean group \( E(N) \) whose generators \( \{ x_i, J_{jk} \} \) satisfy following relations,

\[
[x_i, x_j] = 0, [x_k, J_{ij}] = i\hbar (x_i \delta_{kj} - x_j \delta_{ki}),
\]

\[
[J_{jk}, J_{im}] = i\hbar (\delta_{jl}J_{km} - \delta_{jm}J_{kl} + \delta_{km}J_{jl} - \delta_{kl}J_{jm}).
\]

The \( SO(N) \) \([9]\) is a subgroup of \( E(N) \), whose generators are the generalized angular momentum,

\[
J_{ij} = L_{ij} + f_{ij},
\]

where \( L_{ij} = x_i (-i\hbar \partial_j) - x_j (-i\hbar \partial_i) \) is \( ij \)-th component of the usual orbital angular momentum, and \( f_{ij} \) is related to \( S_{\alpha \beta} \) that obey commutation relations \([3]\) as well and constitute the irreducible representation of \( SO(N - 1) \) group \([19]\),

\[
f_{\alpha \beta} = -f_{\beta \alpha} = S_{\alpha \beta},
\]

\[
f_{\alpha N} = -f_{N \alpha} = -\frac{r}{r + x_N} S_{\alpha \beta} n_\beta.
\]

Ohnuki and Kitakado obtained the form of \( f_{ij} \) by means of consideration of the transportation of a state from any point to another on \( S^{N-1} \) by successive infinitesimal unitary transformation \([19]\). The gauge structure is rooted in a coupling between the generators \( S_{\alpha \beta} \) of rotational symmetry group \( SO(N - 1) \) and curvature via relation \([8] \). To be explicit, the general form of the gauge potential \( A \) is in the following \([19]\),

\[
A_i = \frac{1}{r} f_{ij} n_j = \frac{1}{r + x_N} \begin{cases} \frac{1}{r} S_{\beta \alpha} n_\beta, & i \neq N \\ 0, & i = N \end{cases}.
\]

In the following, we point out three elementary properties of the gauge field \( A \). 1. Commutators \([A_\alpha, A_\beta]\) between the two different components of the gauge potential \( A \) indicate that the gauge field is non-Abelian,

\[
[A_\alpha, A_\beta] = \frac{n_\alpha n_\beta}{(r + x_N)^2} [S_{\alpha \mu}, S_{\beta \nu}] \]

\[
= \frac{i\hbar}{(r + x_N)^2} \left( \left( 1 - \frac{x_N^2}{r^2} \right) S_{\alpha \beta} + \left( n_\alpha S_{\beta \mu} - n_\beta S_{\alpha \mu} \right) n_\mu \right)
\]

\[
= \frac{i\hbar}{r^2 (r + x_N)} S_{\alpha \beta} + \frac{1}{r + x_N} \left( n_\alpha A_\beta - n_\beta A_\alpha \right)
\]

\[
= \frac{i\hbar}{r^2 (r + x_N)} S_{\alpha \beta} + \frac{1}{r + x_N} (n \wedge A)_{\alpha \beta} \neq 0.
\]

The field strength and the gauge transformation of the gauge field \( A \) are discussed in Ref. \([22]\).

2. It is easily to check that the \( N \)-th component of the gauge potential \( A \) is zero, i.e., \( A_N = n_\alpha S_{\alpha \beta} n_\beta/(r + x_N) = 0 \), which amounts to choose a convenient gauge condition. More importantly, the gauge potential \( A \) is defined on the tangential plane on \( S^{N-1} \) for we have,

\[
A \cdot n = 0, \text{ or, } A_\alpha n_\alpha = 0.
\]

3. A very important case is: Once \( S_{\alpha \beta} \) is understood as spin matrix, the gauge field \( A \) can be interpreted as spin-curvature coupling, where the curvature part \( n_\beta/(r + x_N) \) in \([2] \) is related to spin-connections, which is in detail discussed in Ref. \([24]\). Once the radius \( r \) is infinitely large, we have \( n_\beta/(r + x_N) \to 0 \). Thus, the spin is not directly the usual intrinsic angular momentum of the particle defined in flat space.

### III. Dirac Formalism of Quantization on the Sphere \( S^{N-1} \) and an \( SO(N,1) \) Algebra

Let us now consider following equation of the hyperspherical surface equation,

\[
f(x) = \frac{1}{2r} \left( x_i^2 - r^2 \right) = 0.
\]
From it we know \( n = \nabla f \). Dirac formalism for constrained motion gives following Dirac brackets

\[
[x_i, x_j]_D = 0, \tag{11}
\]

\[
[x_i, p_j]_D = (\delta_{ij} - n_in_j), \tag{12}
\]

\[
[p_i, p_j]_D = -\frac{L_{ij}}{r^2} = -\frac{x_ip_j - x_jp_i}{r^2}, \tag{13}
\]

where \( L_{ij} = x_ip_j - x_jp_i \) is the usual \( ij \)-component of the orbital angular momentum satisfying,

\[
[x_k, L_{ij}]_D = (x_i \delta_{kj} - x_j \delta_{ki}), \tag{14}
\]

\[
[p_k, L_{ij}]_D = (p_i \delta_{kj} - p_j \delta_{ki}), \tag{15}
\]

\[
[L_{jk}, L_{lm}]_D = (\delta_{jl}L_{km} - \delta_{jm}L_{kl} + \delta_{km}L_{jl} - \delta_{kl}L_{jm}). \tag{16}
\]

When transition to quantum mechanics, we utilize the full Dirac formalism of quantization procedure to define a quantum commutator for any pair of operators \( f \) and \( g \) is given by \( [f, g] = i\hbar \, O \{ [f, g]_D \} \). Thus, the proper form of quantum conditions from \( (11) \) to \( (16) \) must be, together with \( (2) \) and \( (3) \),

\[
[x_i, x_j] = 0, \tag{17}
\]

\[
[x_i, p_j] = i\hbar (\delta_{ij} - n_in_j), \tag{18}
\]

\[
[p_i, p_j] = -i\hbar \frac{J_{ij}}{r^2} \neq -i\hbar \frac{J_{ij}}{r^2} (x_ip_j - x_jp_i), \tag{19}
\]

\[
[p_k, J_{ij}] = i\hbar (p_i \delta_{kj} - p_j \delta_{ki}). \tag{20}
\]

Here, we replace the symbol \( L_{ij} \) by \( J_{ij} \), which denotes the generalized angular momentum, then the quantum momentum \( p_i \) in our problem is not the usual one. It is clear that operators \( (p_i, J_{ij}) \) form the \( SO(N, 1) \) algebra from relations \( (3) \), \( (19) \) and \( (20) \).

We will show that with \( p = \Pi - \mathbf{A} \), relations \( (19) \) and \( (20) \) are valid, in which the \( \mathbf{A} \)-independent part \( \Pi \) is a well-defined geometric momentum \( (7) \), \( (18) \). In general, the geometric momentum takes the form, \( \Pi = \Pi(x) = -i\hbar(\nabla_{\Sigma} + M\mathbf{n}/2) \) with \( \nabla_{\Sigma} \equiv \nabla - \mathbf{n} (\mathbf{n} \cdot \nabla) = \nabla - \mathbf{n} \partial_{\mathbf{n}} \) denoting the gradient operator on \( S^{N-1} \), and the mean curvature \( M = -\nabla_{\Sigma} \cdot \mathbf{n} \) is defined by the sum of the all principal curvatures. Notice that the mean curvature \( M \) is an extrinsic curvature, this form of momentum \( p = \Pi - \mathbf{A} \) is fundamentally different from the canonical ones in curvilinear coordinates for \( p \) depends on the geometric invariants whereas the the canonical ones do not. However, for a particle on the hyperspherical surface, the geometric momentum assumes a simpler form which can be obtained via a simple quantization of relation \( p_i = n_j L_{ji}/r \) which in quantum mechanics turns out to be,

\[
\Pi_i = \frac{1}{2r} \left( n_j L_{ji} + L_{ji}n_j \right), \quad \text{i.e.,} \quad \Pi = -i\hbar \left( \nabla_{\Sigma} - (N - 1)\frac{\mathbf{n}}{2r} \right), \tag{21}
\]

where the fact \( [n_j, L_{ji}] \neq 0 \) is noted. Now we have \( J_{ij} = L_{ij} + f_{ij} \). The similar quantization of the classical relation \( p_i = n_j L_{ji}/r \) gives explicitly how the gauge potential enters in the geometric momentum,

\[
p_i = \frac{1}{2r} \left( n_j J_{ji} + J_{ji}n_j \right) = \Pi_i - A_i, \tag{22}
\]

where \( A_i = f_{ij}n_j/r \) comes from its definition \( (7) \). What is more, the momentum still lies on the tangent plane on the surface for we have,

\[
\mathbf{p} \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{p} = 0, \tag{23}
\]

where \( A_i n_i = 0 \) directly arises from its definition \( (7) \) as well, and \( \Pi \cdot \mathbf{n} + \mathbf{n} \cdot \Pi = 0 \) is easily verifiable with definition of \( \Pi \) \( (21) \).

There is a curious fact. The form \( [p_i, p_j]_D = -L_{ij}/r^2 \) \( (13) \) is seldom used, and it is usually written in the form \( [p_i, p_j]_D = -(x_ip_j - x_jp_i)/r^2 \) whose quantization assumes the form \( [p_i, p_j] = -i\hbar (x_ip_j - x_jp_i)/r^2 \), see for instance Weinberg’s lectures on quantum mechanics \( [32] \) and also Ohnuki and Kitakado paper \( [24] \), in which only the orbital angular momentum presents. However, if starting from \( [p_i, p_j]_p = -L_{ij}/r^2 \) \( (13) \) which is explicitly expressed in terms of orbital angular momentum, the quantum angular momentum allows the spin-related angular momentum to enter because the angular momentum in quantum mechanics is determined by the algebra. In other words, the quantum conditions assume the form \( [p_i, p_j] = -i\hbar J_{ij}/r^2 \) \( (19) \) in which \( J_{ij} \) is known, then the proper form of the \( p_i \) is sought.

In section IV, we will show \( [p_i, p_j] = -i\hbar J_{ij}/r^2 \) \( (19) \) with \( p_i \) defined in \( (22) \); and in section V, we will prove \( [p_k, J_{ij}] = i\hbar (p_i \delta_{kj} - p_j \delta_{ki}) \) \( (20) \).
IV. A PROOF OF $[p_i, p_j] = -i\hbar J_{ij}/r^2$

We start from $p = \Pi - A$, from which the commutators $[p_i, p_j]$ are,

$$[p_i, p_j] = [\Pi_i - A_i, \Pi_j - A_j].$$

(24)

We split $N$ components $p_i$ of the momentum $p$ into two categories of $p_\alpha (\alpha = 1, 2, \cdots, N - 1)$ and $p_N$,

$$p_\alpha = \Pi_\alpha + \frac{1}{r^2} r_i f_i = \Pi_\alpha - A_\alpha,$$

$$p_N = \Pi_N - A_N = \Pi_N,$$

(25)

(26)

where we used $A_N = 0$. The calculations of $[p_i, p_j]$ will be done separately, and we first study $[p_\alpha, p_\beta]$ and secondly deal with $[p_\alpha, p_N]$.

In order to compute $[p_\alpha, p_\beta]$, we split the commutator $[p_\alpha, p_\beta]$ into four parts,

$$[p_\alpha, p_\beta] = [\Pi_\alpha - A_\alpha, \Pi_\beta - A_\beta] = [\Pi_\alpha, \Pi_\beta] - [\Pi_\alpha, A_\beta] - [A_\alpha, \Pi_\beta] + [A_\alpha, A_\beta].$$

(27)

In the right-hand-side of above expression, the first part is known and the result is given by,

$$[\Pi_i, \Pi_j] = -\frac{i\hbar}{r^2} L_{ij}. $$

(28)

The second and the third part is essentially the same for $[\Pi_\alpha, A_\beta] = -[A_\beta, \Pi_\alpha]$, and the last one is $[A_\alpha, A_\beta]$ which is also understood via Eq. (27). To carry out $[\Pi_\alpha, A_\beta]$, we start from $[\Pi_i, A_\beta]$ and then project onto the $\alpha$-direction. To note a relation $[\Pi_i, A_\beta] = -i\hbar [\nabla - n (n \cdot \nabla), A_\beta]$, we need thus to compute the commutator $[\nabla, A_\beta]$,

$$[\nabla, A_\beta] = \nabla A_\beta = \nabla \left( \frac{n_\mu}{r + x N} \right) S_{\beta \mu},$$

$$= \left( -\frac{n_\mu (n_\mu + 2)}{(r + x N)^2} \right) n - \left( \frac{n_\mu}{r (r + x N)} \right) e_N + \left( \frac{1}{r (r + x N)} \right) e_\mu \right) S_{\beta \mu}. $$

(29)

Then $n \cdot \nabla A_\beta$ is with use of $n \cdot e_N = n_N$ and $n \cdot e_\mu = n_\mu$,

$$n \cdot \nabla A_\beta = \left( \frac{2n_\mu (n_\mu + 1)}{(r + x N)^2} + \frac{n_\mu}{r (r + x N)} \right) S_{\beta \mu}. $$

(30)

Thus, we obtain,

$$[\nabla - n (n \cdot \nabla), A_\beta] = \nabla A_\beta - n (n \cdot \nabla) A_\beta$$

$$= \left( \frac{2n_\mu (n_\mu + 1)}{(r + x N)^2} + \frac{n_\mu}{r (r + x N)} \right) n - \left( \frac{n_\mu}{r (r + x N)} \right) e_N + \left( \frac{1}{r (r + x N)} \right) e_\mu \right) S_{\beta \mu}, $$

(31)

Projecting it onto the $\alpha$-direction, we get $[\Pi_\alpha, A_\beta]$, with $e_\alpha \cdot n = n_\alpha$, $e_\alpha \cdot e_N = 0$, and $e_\alpha \cdot e_\mu = \delta_{\alpha \mu}$,

$$[\Pi_\alpha, A_\beta] = -i\hbar e_\alpha \cdot [\nabla - n (n \cdot \nabla), A_\beta]$$

$$= -i\hbar \left( \frac{n_\mu (n_\mu + 1 - 2 n_\mu)}{(r + x N)^2} + \frac{1}{r (r + x N)} \delta_{\alpha \mu} \right) S_{\beta \mu}$$

$$= -i\hbar \left( \frac{n_\alpha}{r + x N} A_\beta + \frac{S_{\beta \alpha}}{r (r + x N)} \right). $$

(32)
So, two terms in right-hand-side of (27) \([\Pi_\alpha, A_\beta] + [A_\alpha, \Pi_\beta]\) is,

\[
[\Pi_\alpha, A_\beta] + [A_\alpha, \Pi_\beta] = [\Pi_\alpha, A_\beta] - [\Pi_\beta, A_\alpha]
\]

\[
= -i\hbar \left( -\frac{n_\alpha}{r + x_N} A_\beta + \frac{S_{\delta\alpha}}{r (r + x_N)} - \left( -\frac{n_\beta}{r + x_N} A_\alpha + \frac{S_{\alpha\delta}}{r (r + x_N)} \right) \right)
\]

\[
= -i\hbar \left( -\frac{1}{r + x_N} (n_\alpha A_\beta - n_\beta A_\alpha) + \frac{2S_{\delta\alpha}}{r (r + x_N)} \right).
\]  

Combining results (35) and (36) above gives the result (29).

Thus we have from Eqs. (28) and (34),

Similarly, we can prove,

\[
[p_\alpha, p_\beta] = -\frac{i\hbar}{r^2} (L_{\alpha\beta} + S_{\alpha\beta}) = -\frac{i\hbar}{r^2} J_{\alpha\beta}.
\]  

Combination of two equations (35) and (36) above gives the result (11). Q.E.D.

V. A PROOF OF \([p_\lambda, J_{ij}] = i\hbar (p_\lambda\delta_{kj} - p_j\delta_{ki})\)

The proof of fundamental quantum conditions \([p_\lambda, J_{ij}] = i\hbar (p_\lambda\delta_{kj} - p_j\delta_{ki})\) \([20]\) is also straightforward. To do it, we split \(N\) components \(p_i\) of the momentum \(p\) into two categories of \(p_\alpha (\alpha = 1, 2, \cdots, N-1)\) and \(p_N\). In consequence, the commutators \([p_\lambda, J_{ij}]\) are divided into two categories as \([p_\lambda, J_{ij}]\) and \([p_N, J_{ij}]\). To process \([p_\lambda, J_{ij}]\), we compute \([p_\lambda, J_{ij}]\) and \([p_\lambda, J_{iN}]\) one by one. The detailed steps of calculation of commutators \([p_\lambda, J_{i\beta}]\) is given in the following.

We split commutators \([p_\lambda, J_{i\beta}]\) into four parts,

\[
[p_\lambda, J_{i\beta}] = [\Pi_\lambda - A_\lambda, L_{\alpha\beta} + S_{\alpha\beta}] = [\Pi_\lambda, L_{\alpha\beta}] + [\Pi_\lambda, S_{\alpha\beta}] - [A_\lambda, L_{\alpha\beta}] - [A_\lambda, S_{\alpha\beta}].
\]  

The first part of the commutators in the right-hand-side was already treated \([9]\) and the results are given by,

\[
[\Pi_\lambda, L_{\alpha\beta}] = i\hbar (\Pi_\alpha\delta_{\beta\lambda} - \Pi_\beta\delta_{\alpha\lambda}).
\]  

The second part vanishes because of,

\[
[\Pi_\lambda, S_{\alpha\beta}] = 0.
\]  

The third part \([A_\lambda, L_{\alpha\beta}]\) is with \(L_{\alpha\beta} \equiv x_\alpha \Pi_\beta - x_\beta \Pi_\alpha\),

\[
[A_\lambda, L_{\alpha\beta}] = \frac{S_{\lambda\alpha} n_\tau}{r + x_N}, \frac{x_\alpha \Pi_\beta}{r + x_N}
\]

\[
= \frac{x_\alpha (S_{\lambda\alpha} n_\tau \Pi_\beta)}{r + x_N} - \frac{x_\beta (S_{\lambda\beta} n_\tau \Pi_\alpha)}{r + x_N}
\]

\[
= \frac{i\hbar (S_{\lambda\beta} n_\alpha - S_{\lambda\alpha} n_\beta)}{r + x_N}.
\]
Thus, the conjecture is true for the well-known system. With this understanding at hand, we can boldly guess that under investigations.

where we used results (32). The fourth part \([A_\lambda, S_{\alpha\beta}]\) in (37) can be easily carried out with use of the algebraic relations (3).

\[
[A_\lambda, S_{\alpha\beta}] = \frac{n_\lambda}{r + x_N}[S_{\lambda\tau}, S_{\alpha\beta}] = -i\hbar(\delta_{\alpha\lambda}A_\beta - \delta_{\beta\lambda}A_\alpha + \frac{n_\alpha S_{\lambda\beta}}{r + x_N} - \frac{n_\beta S_{\lambda\alpha}}{r + x_N}),
\]

(41)

Combination of results for the third and fourth part (40c) and (41) yields a simple result,

\[
-\left(\frac{r}{A_\lambda} - [A_\lambda, S_{\alpha\beta}]\right) = \frac{-i\hbar}{r + x_N}[S_{\lambda\alpha}, S_{\beta\gamma}] + \frac{\hbar}{r + x_N}[S_{\lambda\alpha}, S_{\beta\gamma}] + \frac{n_\alpha S_{\lambda\beta}}{r + x_N} - \frac{n_\beta S_{\lambda\alpha}}{r + x_N}.
\]

(42a)

Thus, the commutators \([p_\lambda, J_{\alpha\beta}]\) give,

\[
[p_\lambda, J_{\alpha\beta}] = i\hbar(\delta_{\alpha\lambda}A_\beta - \delta_{\beta\lambda}A_\alpha) + \frac{n_\alpha S_{\lambda\beta}}{r + x_N} - \frac{n_\beta S_{\lambda\alpha}}{r + x_N}.
\]

(42b)

Similarly, we can prove following two relations,

\[
[p_\lambda, J_{\alpha\beta}] = -i\hbar p_N d_\alpha d_\lambda,
\]

(43a)

and

\[
[p_\lambda, J_{\alpha\beta}] = i\hbar(p_\lambda d_{\alpha\beta} - p_\beta d_{\alpha\lambda}).
\]

(43b)

In final, from results (43a)-(45), we see that the proof of the fundamental quantum conditions (20) is complete. Q.E.D.

VI. CONCLUSIONS AND DISCUSSIONS

For a particle that is constrained on an \((N - 1)\)-dimensional hypersphere, the classical bracket is \([p_i, p_j]_D = -L_{ij}/r^2\).

In contrast to the usual understanding of the corresponding quantum condition remains \([p_i, p_j] = -i\hbar L_{ij}/r^2\) in which \(L_{ij}\) represents orbital angular momentum, we argue that it must be replaced by \([p_i, p_j] = -i\hbar J_{ij}/r^2\) in which \(J_{ij}\) includes two parts, and one of them is the usual orbital angular momentum \(L_{ij}\) and another comes from the gauge potential \(f_{ij}\). In other words, we adapt the definition of momentum such that \([p_i, p_j] = -i\hbar J_{ij}/r^2\) holds true. What is more, we see that another set of the fundamental quantum conditions \([p_k, J_{ij}] = i\hbar(p_k d_{ij} - p_j d_{ki})\) comes true as well. The momentum is the geometric one which came into our sight recently.

The present study gives us an insight into the noncommutativity between different components of momentum. Our general conjecture is: where there is such a noncommutativity, there is a generalized angular momentum which is possibly associated with both the gauge field angular momentum and the orbital one. The most familiar system is a charged particle in \(E^3\) under influence of uniform magnetic field \(B = B_0\delta z\), where we have \([p_x, p_y] = i\hbar qB_0\) in which \(p_x\) and \(p_y\) are \(x\)- and \(y\)-component of the kinetic momentum \(p = -i\hbar \nabla - qA\), and \(q\) is the charge. The magnetic field is a gauge field, but where is the angular momentum? A usually overlooked fact is that the \(z\)-component of the field angular momentum is \(L_z = qB_0\rho^2/2\) with \(\rho = \sqrt{x^2 + y^2}\), we immediately find that \([p_x, p_y] = 2i\hbar L_z/\rho^2\).

Thus, the conjecture is true for the well-known system. With this understanding at hand, we can boldly guess that the angular momentum part \(f_{ij}\) in (41) to be closely related to the angular momentum of the gauge field, which is still under investigations.

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