On the fermion spectrum of spontaneously generated fuzzy extra dimensions with fluxes

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Abstract

We consider certain vacua of four-dimensional $SU(N)$ gauge theory with the same field content as the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, resulting from potentials which break the $\mathcal{N} = 4$ supersymmetry as well as its global $SO(6)$ symmetry down to $SO(3) \times SO(3)$. We show that the theory behaves at intermediate scales as Yang-Mills theory on $M^4 \times S^2_L \times S^2_R$, where the extra dimensions are fuzzy spheres with magnetic fluxes. We determine in particular the structure of the zero modes due to the fluxes, which leads to low-energy mirror models.
1 Introduction

The unification of the fundamental interactions has always been one of the main goals of theoretical physics. Several approaches have been employed in order to achieve this goal, one of the most exciting ones being the proposal that extra dimensions may exist in nature. The realization that superstring theories can be consistently defined only in ten dimensions has led to an intensive study of possible compactifications of these theories with the hope that phenomenologically viable four-dimensional vacua can be revealed upon dimensional reduction. Recently a surprising new approach was proposed in [1], where the above procedure was inversed. In particular it was found that extra dimensions can arise dynamically within a four-dimensional renormalizable and asymptotically-free gauge theory, as an effective description valid up to some energy scale. This has become known under the name of deconstruction.

A simple realization of the idea of a spontaneous generation of extra dimensions was given in [2] and [3], inspired by an earlier work [4] where the general ideas of [5, 6] were followed. In particular, it was shown in [2] that starting with the $SU(N)$ Yang-Mills theory on the Minkowski spacetime $M^4$ for some generic (large) $N \in \mathbb{N}$ coupled with three scalars and adding the most general renormalizable $SO(3)$-invariant potential, an extra-dimensional fuzzy sphere is formed via the Higgs effect. The unbroken gauge group is generically $SU(n_1) \times SU(n_2) \times U(1)$, or possibly $SU(n)$. In [3] fermions were added in the previous model and their effective description from both the 6D and 4D point of view was worked out. The most interesting feature is that the extra-dimensional sphere then automatically carries a magnetic flux, which couples to the fermions transforming in the bifundamental of $SU(n_1) \times SU(n_2)$. In view of this feature, the possibility to obtain chiral
4D models was studied. The outcome of this analysis was that only a picture of mirror fermions can be achieved in four dimensions.\footnote{For a discussion on phenomenological aspects of such models see e.g. [7].}

In the present work we explore the dynamical generation of a product of two fuzzy spheres. In particular, we start with the $SU(N)$ Yang-Mills theory in four dimensions, coupled to six scalars and four Majorana spinors, i.e. with the particle spectrum of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM). Adding an explicit $R$-symmetry-breaking potential, thus breaking the $\mathcal{N} = 4$ supersymmetry partially or completely, we reveal stable $M^4 \times S^2_L \times S^2_R$ vacua. In the most interesting case we include magnetic fluxes on the extra-dimensional fuzzy spheres and study the fermion spectrum, in particular the zero modes of the Dirac operator. The outcome of our analysis is that we obtain again a mirror model in low energies. However, in the present case we are able to single out an action which leads to exact separation of the two chiral mirror sectors which arise in the model.

One of the reasons for considering the $\mathcal{N} = 4$ SYM is that it is closely related to the IIB matrix model [8], which is a candidate for a quantum theory of fundamental interactions including gravity. The mechanism for gravity in that model was clarified recently in [9], where space-time is described by a noncommutative space. On $\mathbb{R}^4_\theta$, the matrix model is nothing but noncommutative $\mathcal{N} = 4$ SYM. Since extra dimensions can be realized in terms of deconstruction starting from a four-dimensional gauge theory, it is natural to look for stringy constructions such as branes in extra dimensions (see e.g. [10] and references therein), with the aim to recover the standard model or some extension of it. In the present work we take some new steps in that direction, starting from commutative $\mathcal{N} = 4$ SYM. We recover products of branes (in the form of fuzzy spheres) with fluxes, and matter fields realized as bi-modules connecting the branes. While some aspects of the brane constructions are still missing, the present approach also offers advantages. In particular, our results are obtained within a renormalizable four-dimensional gauge theory, and the vacua are at least local minima without flat directions and unstable moduli.

There has been considerable work on fuzzy geometries arising in matrix models, see e.g. [11–13] and references therein. In particular, the case of fuzzy $S^2 \times S^2$ has been studied in [14], and its gauge theory in [15]. The novel aspect of the present paper is to take into account fluxes on these fuzzy spaces, and to study the fermionic zero modes due to these fluxes in extra dimensions. For further literature on fuzzy spaces with fluxes see e.g. [11,16–22], and related work on the reduction of Yang-Mills models on $M^4 \times S^2$ in the presence of fluxes was given in [23, 24].

The outline of this article is as follows. In section 2 the action for the four-dimensional model that we consider is presented and the dynamical generation of the product of two fuzzy spheres is discussed. Moreover, the operators on $S^2_L \times S^2_R$ which are relevant in our analysis are defined, including the Dirac operator. In section 3 we add magnetic fluxes on the extra-dimensional fuzzy spheres and treat in detail the zero modes of the Dirac operator in order to discuss the chirality issue. It is shown that a mirror model is obtained and the separation of the two exactly chiral mirror sectors is discussed. In section 4 we present our conclusions and discuss the prospects of further work on the subject. Finally, in appendix A our Clifford algebra conventions are collected, while in appendix B the Dirac operator and its eigenmodes on the fuzzy sphere are presented.
2 Yang-Mills gauge theory and spontaneously generated fuzzy $S^2 \times S^2$

2.1 The action

We consider the $SU(N)$ Yang-Mills gauge theory in four-dimensional Minkowski spacetime, coupled to six scalars $\Phi_a = \Phi_a^\dagger$ ($a = 1 \ldots 6$) and four Majorana spinors $\chi_p$ ($p = 1 \ldots 4$) in the adjoint representation of the $SU(N)$. Moreover, we assume that the $\Phi_a$ transform in the vector representation of a global $SU(4) \cong SO(6)$ group and the $\chi_p$ in the fundamental of the $SU(4)$. The above particle spectrum coincides with the spectrum of the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) [25], where the global $SU(4)$ is the $R$-symmetry of the theory. The corresponding action, which is a modification of the $\mathcal{N} = 4$ SYM theory, is given by

$$S_{YM} = \int d^4x \left[ \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \sum_{a=1}^{6} D^\mu \Phi_a D_\mu \Phi_a - V(\Phi) \right) + \frac{1}{2} \text{Tr} \left( i \bar{\chi}_p D \chi_p + g_4 (\Delta^a_R)_{pq} \bar{\chi}_p R[\Phi_a, \chi_q] - g_4 (\Delta^a_L)_{pq} \bar{\chi}_p L[\Phi_a, \chi_q] \right) \right], \quad (1)$$

where the potential has the form

$$V(\Phi) = V_{\mathcal{N}=4}(\Phi) + V_{\text{break}}(\Phi). \quad (2)$$

In (2) the first term corresponds to the potential of the $\mathcal{N} = 4$ SYM theory, which is explicitly given by

$$V_{\mathcal{N}=4}(\Phi) = -\frac{1}{4} g_4^2 \sum_{a,b} [\Phi_a, \Phi_b]^2, \quad (3)$$

while the second term corresponds to an explicit $R$-symmetry-breaking potential, which breaks the $\mathcal{N} = 4$ supersymmetry as well as the global $SU(4)$ symmetry. Let us mention that here and in the following we use the conventions of [26].

In the above expressions $\mu, \nu = 0, 1, 2, 3$ are four-dimensional spacetime indices and $D_\mu = \partial_\mu - ig[A_\mu, \cdot]$ is the four-dimensional covariant derivative in the adjoint representation. The projection operators $L$ and $R$ are, as usual, defined as $L = \frac{1}{2}(1 - \gamma_5)$ and $R = \frac{1}{2}(1 + \gamma_5)$. The $(\Delta^a_R)_{pq}$ and $(\Delta^a_L)_{pq}$ are the intertwiners of the $4 \times 4 \rightarrow 6$ and $\bar{4} \times \bar{4} \rightarrow 6$ respectively, namely they are Clebsch-Gordan coefficients that couple two $4$s to a $6$. The Yukawa interactions in (1) are separately invariant under the $SU(4)$, since the $R\chi_p$ transforms in the $4$ and the $L\chi_p$ in the $\bar{4}$ of the $SU(4)$.

The action (1) and the considerations below are best understood through dimensional reduction, starting from the $\mathcal{N} = 1$ SYM theory in ten dimensions with a ten-dimensional Majorana-Weyl spinor $\Psi$ and action

$$S_{D=10} = -\frac{1}{4 g_4^2} \int d^{10}x \text{Tr} F_{MN} F^{MN} + \frac{1}{2} \int d^{10}x \text{Tr} \bar{\Psi} i \Gamma^M D_M \Psi, \quad (4)$$

where

$$D_M = \partial_M - ig[A_M, \cdot] \quad (5)$$
and capital Latin letters denote ten-dimensional indices, i.e. $M = 0, \ldots, 9$. Considering a compactification of the form $M^4 \times Y$, the scalars are obtained from the internal components of the higher-dimensional gauge field according to the splitting

$$A_M = (A_\mu, \Phi_{3+a}), \quad a = 1, \ldots, 6.$$  \hspace{1cm} (6)

The ten-dimensional Clifford algebra, generated by $\Gamma_M$, naturally separates into a four-dimensional and a six-dimensional one as follows,

$$\Gamma_M = (\Gamma_\mu, \Gamma_{3+a}),$$  
$$\Gamma_\mu = \gamma_\mu \otimes \mathds{1}_8,$$  
$$\Gamma_{3+a} = \gamma_5 \otimes \Delta_a. \hspace{1cm} (7)$$

Here the $\gamma_\mu$ define the four-dimensional Clifford algebra and they are chosen to be purely imaginary, corresponding to the Majorana representation in four dimensions (see Appendix A), while the $\Delta_a$ define the six-dimensional Euclidean Clifford algebra and they are chosen to be real and antisymmetric. Then it is straightforward to see that $\gamma_0 = \gamma_0^\dagger = -\gamma_0^T$ and $\gamma_i = -\gamma_i^\dagger = \gamma_i^T$. The ten-dimensional chirality operator is

$$\Gamma^{(11)} = \gamma_5 \otimes \Gamma^{(Y)}, \hspace{1cm} (8)$$

where the four- and six-dimensional chirality operators are defined as

$$\gamma_5 = -i\gamma_0 \ldots \gamma_3 = \gamma_5^\dagger = -\gamma_5^T,$$  
$$\Gamma^{(Y)} = -i\Delta_1 \ldots \Delta_6 = (\Gamma^{(Y)})^\dagger = -(\Gamma^{(Y)})^T. \hspace{1cm} (9)$$

Let us denote the ten-dimensional charge conjugation operator as

$$\mathcal{C} = C^{(4)} \otimes C^{(6)}, \hspace{1cm} (10)$$

where $C^{(4)}$ is the four-dimensional charge conjugation operator and $C^{(6)} = \mathds{1}_8$ in our conventions. This operator satisfies, as usual, the relation

$$\mathcal{C} \Gamma^M \mathcal{C}^{-1} = -(\Gamma^M)^T. \hspace{1cm} (11)$$

Then the Majorana-Weyl condition in ten dimensions is

$$\Psi^C = \mathcal{C} \bar{\Psi}^T \Psi = \Psi, \hspace{1cm} (12)$$

where

$$\bar{\Psi} = \Psi^T \mathcal{C}^T,$$  
$$\Psi^\dagger = \Psi^T \mathcal{C}^T \gamma_0 = \Psi^T. \hspace{1cm} (13)$$

Let us note that in the Majorana representation, where the $\gamma_\mu$ are imaginary, the four-dimensional charge conjugation operator is $C^{(4)} = -\gamma_0$.

\footnote{Note that $T$ transposes only the spinor.}
Performing a trivial dimensional reduction from ten to four dimensions, i.e. assuming that all fields do not depend on the internal coordinates, it is well-known that the Yang-Mills part of the ten-dimensional action leads to the bosonic part of the $\mathcal{N} = 4$ SYM in four dimensions, as in (11), with the potential term having the form (3). The couplings $g_4$ and $g_{10}$ are related through the volume $V$ of the internal six-dimensional torus as $g_4 = g_{10} \sqrt{V}$.

The reduction of the Dirac term is performed similarly. The Majorana-Weyl spinor $\Psi$ has the form

$$\Psi = \sum_{p=1}^{4} \left( R \chi_p \otimes \eta_p + L \chi_p \otimes \eta_p^* \right),$$

$$\bar{\Psi} = \sum_{p=1}^{4} \left( \bar{\chi}_p L \otimes \eta_p^* + \bar{\chi}_p R \otimes \eta_p^T \right),$$

(14)

where the $\chi_p$ are four-dimensional Majorana spinors and the $\eta_p$ are the four complex eigenvectors of the $\Gamma^{(Y)}$ with eigenvalue $+1$. Since the $\Gamma^{(Y)}$ is purely imaginary the $\eta_p^*$ have eigenvalue $-1$. Assuming that the spinor is independent of the extra-dimensional coordinates, the dimensional reduction of the Dirac term of the ten-dimensional action leads in four dimensions to the kinetic term for the spinor $\chi_p$ and the Yukawa couplings as they appear in (1). In particular, the Yukawa couplings arise from the term

$$\text{Tr} \, \bar{\Psi} i \mathcal{D}_{(6)} \Psi = \text{Tr} \, \bar{\Psi} \Delta^a [\Phi_a, \Psi],$$

(15)

where $\mathcal{D}_{(6)}$ denotes the Dirac operator on the internal space, which satisfies

$$\{ \mathcal{D}_{(6)}, \Gamma^{(Y)} \} = 0$$

(16)

and it will be related in the ensuing to the effective Dirac operator on the fuzzy extra dimensions.

The above procedure, based on a trivial dimensional reduction, leads to the $\mathcal{N} = 4$ SYM theory in four dimensions. However, in general, non-trivial dimensional reduction schemes [5, 27, 28] are known to lead in four dimensions to actions preserving less supersymmetry or no supersymmetry at all. The choice of the reduction scheme is reflected in the four-dimensional vacuum. Correspondingly, in our case we add to the potential of the four-dimensional theory the term $V_{\text{break}}$, with the obvious requirement that it is renormalizable and moreover that it leads to vacua which can be interpreted as switching on fluxes in the internal extra-dimensional manifold. This is expected to be appropriate in order to obtain chiral zero modes.

### 2.2 Type I vacuum and Dirac operator on fuzzy $S^2 \times S^2$

#### 2.2.1 The type I vacuum

We now assume that the renormalizable potential in the four-dimensional action admits vacua corresponding to the product of two fuzzy spheres [29], i.e.

$$\Phi^L_i \equiv \Phi_i = \alpha_L \lambda_i^{(NL)} \otimes 1_N \otimes 1_n, \quad i = 1, 2, 3,$$

$$\Phi^R_i \equiv \Phi_{3+i} = \alpha_R 1_N \otimes \lambda_i^{(NR)} \otimes 1_n, \quad i = 1, 2, 3,$$

(17)
where \( \lambda_i^{(N)} \) denotes the generator of the \( N \)-dimensional irreducible representation of \( SU(2) \) and therefore
\[
[\Phi^L_i, \Phi^L_j] = i\alpha_L \varepsilon_{ijk} \Phi^L_k,
\]
\[
\Phi^L_i \Phi^L_i = \frac{\alpha^2 L^2}{4} \left( N_L^2 - 1 \right),
\]
and similarly for the \( \Phi^R_i \). Moreover the two algebras commute with each other, i.e.
\[
[\Phi^L_i, \Phi^R_j] = 0.
\]
This vacuum preserves the global \( SO(3)_L \times SO(3)_R \) symmetry up to gauge transformations:
\[
g \triangleright \Phi_i = U \Phi_i U^{-1},
\]
where \( g \triangleright \) denotes the action of some \( g \in SO(3)_L \times SO(3)_R \) in the vector indices of the \( \Phi_i \) and \( U \in U(N) \) is the corresponding intertwining gauge group action.

The vacuum (17) can be obtained by choosing the potential \( V(\Phi) \) to have the following form:
\[
V[\Phi] = a^2_L (\Phi^L_i \Phi^L_i + b_L \mathbf{1})^2 + a^2_R (\Phi^R_i \Phi^R_i + b_R \mathbf{1})^2 + \frac{1}{g^2_L} F^L_{ij} F^L_{ij} + \frac{1}{g^2_R} F^R_{ij} F^R_{ij},
\]
where
\[
F^L_{ij} = [\Phi^L_i, \Phi^L_j] - i\varepsilon_{ijk} \Phi^L_k,
\]
\[
F^R_{ij} = [\Phi^R_i, \Phi^R_j] - i\varepsilon_{ijk} \Phi^R_k,
\]
which will be interpreted as field strengths on the spontaneously generated fuzzy spheres. The potential (21) breaks the global \( SO(6) \) symmetry down to \( SO(3)_L \times SO(3)_R \) and for suitable parameters \( a_{L/R}, b_{L/R}, g_{L/R} \), its stable global minimum is indeed given by (17) up to \( U(N) \) gauge transformations, provided that
\[
N = N_L N_R \cdot n.
\]
Such a vacuum should be interpreted as a stack of \( n \) fuzzy branes with geometry \( S^2_L \times S^2_R \).

In the present construction it breaks the gauge group \( SU(N) \) down to \( SU(n) \). The \( \Phi_i^{L/R} \) are interpreted as quantization of the coordinate functions \( x^i \) on \( S^2_L/R \subset \mathbb{R}^3 \) with radius \( R_{L/R} \). More generally, there is a de-quantization map
\[
\text{Mat}(N, \mathbb{C}) \twoheadrightarrow \mathcal{C}(S^2)
\]
\[
\Phi_i \mapsto x_i
\]
which extends to the spherical harmonics (defined as symmetric traceless polynomials in \( \Phi_i \) or, respectively, in \( x_i \)) up to a cutoff. In this way, the matrices \( \text{Mat}(N, \mathbb{C}) \) can be

\[ ^3 \text{This includes purely soft deformations from the } N = 4 \text{ potential, which are recovered via } a_L \to 0, a_L b_L = \text{const}. \text{ The present form emphasizes the case of true minima without flat directions. In general, the potential breaks supersymmetry completely, while some SUSY may be preserved for special choices of the parameters [30].} \]
identified with functions on $S^2$, which is the basis of the mechanism under consideration. This semi-classical limit, i.e. the limit $N \to \infty$, will be denoted as $\sim$ in the following. In complete analogy to previous work [2, 3], it is not hard to see that the model (1) can be interpreted in such a vacuum as $SU(n)$ gauge theory on $S^2_L \times S^2_R$, via the identification

$$
\text{Mat}(N_L N_R, \mathbb{C}) \leftrightarrow \mathcal{C}(S^2_L \times S^2_R) \\
f(\Phi_a) \mapsto f(x^L_i, x^R_i)
$$

(25)

so that $\text{Mat}(N, \mathbb{C})$ can be interpreted as $U(n)$-valued functions on $S^2_L \times S^2_R$. This is not surprising in view of the origin of (1) from dimensional reduction of $D = 10$ Yang-Mills theory.

### 2.2.2 Operators on $S^2_L \times S^2_R$

Having in mind the compactification on $S^2_L \times S^2_R \subset \mathbb{R}^6$, we organize the internal $SO(6)$ structure according to its subgroup $SO(3)_L \times SO(3)_R$ and we adopt the notation

$$
\Delta^L_i = \Delta_i, \quad \Delta^R_i = \Delta_{3+i}, \quad i = 1, 2, 3.
$$

(26)

Let us note that we have to work with the six-dimensional Clifford algebra acting on $\mathbb{C}^8$, which does not separate into a tangential and transversal part. This is typical for fuzzy geometries. In the following we shall rewrite the Dirac operator $\slashed{D}_{(6)}$ in terms of fuzzy Dirac operators on $S^2_L \times S^2_R$, which allows to organize the fermionic Hilbert space in terms of Kaluza-Klein modes. Moreover, this allows to make explicit the role of the would-be zero modes in the presence of fluxes which are crucial in our context. However, the separation of tangential and perpendicular quantities with respect to $S^2_L \times S^2_R \subset \mathbb{R}^6$ is somewhat intricate and non-standard.

Let us consider the following $SO(3)_L \times SO(3)_R$ invariant operators on each sphere [15],

$$
\chi_L = \frac{i}{2R_L} \Delta^L_i \{\Phi^L_i, \cdot\} \sim \frac{i}{R_L} \Delta^L_i x^L_i, \\
\chi_{L,\text{tang}} = \Gamma^{(Y)}_L \chi_L, \\
\Gamma^{(Y)}_L = \Delta_1 \Delta_2 \Delta_3,
$$

(27)

where

$$
R_L = \alpha_L N_L
$$

(28)

denotes the radius of $S^2_L$ and $\chi_R$, $\chi_{R,\text{tang}}$ and $\Gamma^{(Y)}_R$ are defined similarly. These operators are hermitian, i.e.

$$
(\chi_{L/R})^\dagger = \chi_{L/R}, \\
(\chi_{L/R,\text{tang}})^\dagger = \chi_{L/R,\text{tang}}, \\
(\Gamma^{(Y)}_{L/R})^\dagger = \Gamma^{(Y)}_{L/R},
$$

(29)

and they satisfy the relations

$$
\{\chi_{L/R}, \Gamma^{(Y)}\} = [\chi_{L/R}, \Gamma^{(Y)}] = 0
$$

(30)
and
\begin{align}
\{\chi_L, \chi_R\} &= 0, \\
[\chi_{L,tang}, \chi_{R,tang}] &= 0, \\
\chi_{L/R}^2 &\sim 1 \sim \chi_{L/R,tang}^2
\end{align}
(31)
in a $S^2_L \times S^2_R$ vacuum (17). In order to understand their meaning, let us consider e.g. $S^2_L$ as discussed in Appendix B. On the north pole with $x_1 \sim 0, x_2 \sim 0, x_3 \sim R_L$, the tangential chirality operator is given by $\chi_{L,tang} \sim i\Delta_1 \Delta_2$, while the operator $\chi_L \sim i\Delta_3$ is perpendicular. Therefore the $SU(2)_L \times SU(2)_R$-invariant operator
\begin{equation}
\chi_{\perp} := i\chi_L \chi_R,
\end{equation}
which squares to one, $(\chi_{\perp})^2 \sim 1$, corresponds to the chirality operator on the two-dimensional space which is perpendicular to $S^2_L \times S^2_R \subset \mathbb{R}^6$. In addition, the operator
\begin{equation}
\chi_{tang} := \Gamma(Y) \chi_{\perp} = -\chi_{L,tang} \chi_{R,tang} \sim \Delta_1 \Delta_2 \Delta_4 \Delta_5,
\end{equation}
is the tangential chirality operator on $S^2_L \times S^2_R$.

2.2.3 The Dirac operator

In order to understand the fuzzy Kaluza-Klein modes and especially the would-be zero modes, it is important to understand the relation of $\slashed{D}_{(6)}$ with the fuzzy Dirac operators on $S^2_L$ and $S^2_R$. We note that in the Majorana representation of the six-dimensional Clifford algebra, we have
\begin{equation}
-i\Gamma(Y)_L \Delta_i^L = \mathbb{1}_2 \otimes \gamma_i^L,
\end{equation}
where $\gamma_i^L = U^{-1}(\sigma_i \otimes \mathbb{1}_2)U$ is essentially a double-degenerate representation of the three-dimensional Clifford algebra. This allows to write
\begin{equation}
\Delta_i^L[\Psi_i, \Psi] + i\alpha_L \Gamma(Y)_L = i\Gamma(Y)_L \slashed{D}_{S^2_L},
\end{equation}
where $\slashed{D}_{S^2_L}$ is the standard Dirac operator on the fuzzy $S^2_L$ (see Appendix B for a short review). Note that one usually works with two-component spinors on the fuzzy sphere, where the tangential chirality operator is given by $\sigma_1 \sigma_2 = i\sigma_3$. Here $\Delta_3$ is independent of $\Delta_1 \Delta_2$ and therefore $\chi_{L,tang}$ is the proper tangential chirality operator on the $S^2_L$, rather than $\chi_L$. In particular, the would-be zero modes $\Psi_{(m)}^{1,2}$ on the $S^2_L$ with magnetic flux, discussed in Appendix B, are eigenvectors of $\chi_{L,tang}$ and not eigenvectors of $\chi_L$. We thus obtain the relation of $\slashed{D}_{(6)}$ with a “tangential” Dirac operator on $S^2 \times S^2 \subset \mathbb{R}^6$:
\begin{equation}
\slashed{D}_{(6)} = \slashed{D}_{S^2 \times S^2} - \alpha_L \Gamma(Y)_L - \alpha_R \Gamma(Y)_R,
\end{equation}
(36)
where
\begin{equation}
\slashed{D}_{S^2 \times S^2} = \Gamma(Y)_L \slashed{D}_{S^2_L} + \Gamma(Y)_R \slashed{D}_{S^2_R}.
\end{equation}
(37)

Note that the generator of $SU(2)_L$ on the spinors is given by $[\Delta_i^L, \Delta_j^L] = -2i\epsilon^{ijk} \mathbb{1}_2 \otimes \gamma_k^L$. 

4
Then the Yukawa term becomes

\[ S_{yuk} = \int \bar{\Psi} i D S^2 \times S^2 \Psi + S_{shift}, \]  

(38)

where the shift action

\[ S_{shift} = \int i Tr \bar{\Psi} \gamma_5 (\alpha_L \Gamma_L^{(Y)} + \alpha_R \Gamma_R^{(Y)}) \Psi \]  

(39)

is recognized as curvature effect. One can show that \( D S^2 \times S^2 \) reduces in the semi-classical limit to the Dirac operator on \( S_L^2 \times S_R^2 \) in the above background geometry (17) [15]. Note also that

\[ S_{shift,L}^* = \int (i Tr \bar{\Psi}^{\dagger} \gamma_0 \gamma_5 \alpha_L \Gamma_L^{(Y)} \Psi) \]  

(40)

and similarly for the \( S_{shift,R} \), as expected since the original action is hermitian.

3 Magnetic fluxes and chirality

3.1 The type II vacuum and the zero-modes

In order to obtain massless fermions, it is necessary to add magnetic fluxes \( m_L, m_R \) on the two spheres. As explained in [3], this is realized in a slightly modified class of vacua, called “type II vacua”. In the present case such a vacuum has the form

\[
\Phi_i = \begin{pmatrix}
\alpha_1 \lambda_i^{(N_L)} \otimes 1_{N_R^1} \otimes 1_{N_2^1} \\
0 \\
\alpha_2 \lambda_i^{(N_L)} \otimes 1_{N_R^2} \otimes 1_{N_2^2}
\end{pmatrix},
\]

\[
\Phi_{3+i} = \begin{pmatrix}
\alpha_3 1_{N_L^1} \otimes \lambda_i^{(N_R^1)} \otimes 1_{N_2^1} \\
0 \\
\alpha_4 1_{N_L^2} \otimes \lambda_i^{(N_R^2)} \otimes 1_{N_2^2}
\end{pmatrix}, \quad i = 1, 2, 3.
\]

(41)

The commutant of these generators, i.e. the unbroken gauge group, is \( SU(n_1) \times SU(n_2) \times U(1)_Q \), where the \( U(1)_Q \) has generator

\[ Q = \begin{pmatrix}
\frac{1}{N_R^1 N_L^{1 n_1}} 1 & 0 \\
0 & -\frac{1}{N_R^2 N_L^{2 n_2}} 1
\end{pmatrix}. \]

(42)

This vacuum corresponds to a splitting

\[ N = n_1 N_L^{1} N_R^1 + n_2 N_L^{2} N_R^2 \]  

(43)

and, respectively,

\[ \mathcal{H} = \mathbb{C}^{n_1 N_L^{1} N_R^1} \oplus \mathbb{C}^{n_2 N_L^{2} N_R^2} \]  

(44)

for the Hilbert space, which is more generic than (23). It determines a splitting of the fermionic wavefunction

\[ \Psi = \begin{pmatrix}
\Psi_{1,1} \\
\Psi_{1,2} \\
\Psi_{2,1} \\
\Psi_{2,2}
\end{pmatrix}, \]  

(45)
where \( \Psi^{1,2} \) transforms in the bifundamental representation \((n_1) \otimes (\bar{n}_2)\) of the \(SU(n_1) \times SU(n_2)\) and \(\Psi^{2,1}\) in the \((\bar{n}_1) \otimes (n_2)\). The Majorana condition \(\Psi^+ \equiv \Psi^T = \Psi\) implies

\[
(\Psi^{1,1})^+ = \Psi^{1,1}, \quad (\Psi^{2,2})^+ = \Psi^{2,2}, \quad (\Psi^{1,2})^+ = \Psi^{2,1}.
\]

(46)

The interpretation of this vacuum is as a stack of \(n_1\) fuzzy branes and a stack of \(n_2\) fuzzy branes\(^5\) with geometry \(S^2_L \times S^2_R\). However, these fuzzy spheres carry magnetic flux under the unbroken \(U(1)_Q\) given by [18]

\[
m_L = N^1_L - N^2_L, \quad m_R = N^1_R - N^2_R,
\]

(47)
on \(S^2_L\) and \(S^2_R\) respectively. Since the fermions \(\Psi\) transform in the adjoint representation, the diagonal components \(\Psi^{1,1}\) and \(\Psi^{2,2}\) are unaffected, but the off-diagonal components \(\Psi^{1,2}\) and \(\Psi^{2,1}\) feel this magnetic flux and develop chiral zero modes according to the index theorem. This can also be seen very explicitly in the fuzzy case [3], see Appendix B. For example, a flux \(m_L > 0\) on \(S^2_L\) implies that there are (would-be) zero modes \(\Psi^{1,2}_{(m_L)}\) for \(\mathcal{D}_S^L\) with \(\chi_L,\text{tang} = +1\), and \(\Psi^{2,1}_{(m_R)}\) with \(\chi_L,\text{tang} = -1\).

To be specific, assume that \(m_L > 0\) and \(m_R > 0\). As explained in Appendix B, then there exist ("would-be", approximate) zero modes \(\Psi^{1,2}_{(m_L,m_R)}\) of both \(\mathcal{D}_S^L\) and \(\mathcal{D}_S^R\) and therefore of \(\mathcal{D}_{S^2 \times S^2}\), with definite chirality\(^6\)

\[
\chi_L,\text{tang} \Psi^{1,2}_{(m_L,m_R)} = \Psi^{1,2}_{(m_L,m_R)} = \chi_R,\text{tang} \Psi^{1,2}_{(m_L,m_R)},
\]

\[
\chi_{\text{tang}} \Psi^{2,1}_{(m_L,m_R)} = \Psi^{2,1}_{(m_L,m_R)},
\]

(48)

There are also the "conjugate" zero modes \(\Psi^{2,1}_{(m_L,m_R)}\), which satisfy

\[
\chi_{\text{tang}} \Psi^{2,1}_{(m_L,m_R)} = -\Psi^{2,1}_{(m_L,m_R)} = \chi_R,\text{tang} \Psi^{2,1}_{(m_L,m_R)},
\]

\[
\chi_{\text{tang}} \Psi^{1,2}_{(m_L,m_R)} = \Psi^{1,2}_{(m_L,m_R)}.
\]

(49)

Thus, in general, we have

\[
\chi_{\text{tang}} \Psi_{(m_L,m_R)} = (-1)^{m_L+m_R} \Psi_{(m_L,m_R)}.
\]

(50)

All the other Kaluza-Klein modes have both chiralities and acquire a mass due to \(\mathcal{D}_{S^2 \times S^2}\).

Motivated by the properties of the zero modes which are encoded in (48) and (49) let us now define the following operators,

\[
\Pi_L \Psi := \gamma_5 \chi_{L,\text{tang}} \Psi,
\]

\[
\Pi_R \Psi := \gamma_5 \chi_{R,\text{tang}} \Psi.
\]

(51)

---

\(^5\)This is quite reminiscent of standard constructions in the context of string theory, see e.g. [10]. However, there are several differences to the situation in string theory. Notably, the branes do not live in a ten-dimensional space; even though \(M^4 \times S^2_L \times S^2_R \subset M^4 \times \mathbb{R}^6\), the two "missing dimensions" in \(\mathbb{R}^6\) have no physical meaning here and carry no physical degrees of freedom.

\(^6\)To simplify the notation we assume that the operators \(\chi, \mathcal{D}_{S^2}\) are defined appropriately such that these relations hold exactly, see Appendix B. Otherwise the stated eigenvalues of \(\chi\) and \(\mathcal{D}_{S^2}\) are approximate up to \(O(\frac{1}{N})\) corrections. Since we are mainly interested in the structure of the would-be zero modes, we do not keep track of these \(O(\frac{1}{N})\) corrections here.
which satisfy
\[ \Pi_L^2 \sim 1 \sim \Pi_R^2. \] (52)

They are clearly compatible with the ten-dimensional Weyl condition and also with the ten-dimensional Majorana condition \( \Psi^\dagger = \Psi^T \), since
\[
(\Pi_L \Psi)^\dagger = (\Pi_L \Psi)^T, \\
(\Pi_R \Psi)^\dagger = (\Pi_R \Psi)^T.
\] (53)

Consequently they are well-defined and as we shall exhibit in the following they will select the chiral sectors of our model. In order to understand the qualitative structure of the zero modes, in particular their chirality from the four-dimensional point of view, it is enough to consider the semi-classical limit. On the north pole we have, for the adapted representation given in the Appendix A,
\[
\chi_{L,tang} \sim i \Delta_1 \Delta_2 = 1 \otimes \sigma^3 \otimes \sigma^2, \\
\chi_{R,tang} \sim i \Delta_4 \Delta_5 = 1 \otimes \sigma^2 \otimes \sigma^3, \\
\chi_{tang} \sim 1 \otimes \sigma^1 \otimes \sigma^1, \\
\chi_{\perp} \sim \sigma^2 \otimes \sigma^1 \otimes \sigma^1.
\] (54-57)

Then, the unique solution of (48) has the form
\[
\Psi_{(m_L,m_R)}^{1,2} \sim \left( \rho_{1,2}^{1,2} \right) \otimes \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) - i \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right),
\] (58)

where \( \rho_{1,2} \), \( \eta_{1,2} \) are four-dimensional Dirac spinors. Similarly, the unique solution of (49) has the form
\[
\Psi_{(m_L,m_R)}^{2,1} \sim \left( \rho_{2,1}^{2,1} \right) \otimes \left( \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \otimes \left( \begin{array}{c} 0 \\ 1 \end{array} \right) + i \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \otimes \left( \begin{array}{c} 1 \\ -1 \end{array} \right) \right).
\] (59)

The Weyl condition \( \Gamma^{(11)} \Psi = \Psi \) implies
\[
\gamma_5 \Psi = \Gamma^{(1)} \Psi = -\left( \sigma_2 \otimes 1 \otimes 1 \right) \Psi, \\
i \eta_{1,2} = \gamma_5 \rho_{1,2}, \\
i \eta_{2,1} = \gamma_5 \rho_{2,1},
\] (60)
so that the would-be zero mode reduces essentially to
\[
\Psi_{(m)}^{1,2} \sim \left( \rho_{1,2}^{1,2} - i \gamma_5 \rho_{1,2} \right), \\
\Psi_{(m)}^{1,2} \sim \left( \rho_{1,2}^{1,2}, -i \rho_{1,2} \gamma_5 \right),
\] (61)

dropping the remaining tensor factors in (58). The Majorana condition, \( \Psi^+ := \Psi^{\dagger T} = \Psi \), in the present representation implies
\[
\rho_{1,2}^{1,2} = (\rho_{2,1}^{2,1})^+, 
\] (62)
and it relates the upper-diagonal and lower-diagonal components. This amounts to a single four-dimensional Dirac spinor \( \rho_{1,2} \) and the model is non-chiral.
3.2 Chiral low-energy theory.

In general, a model is chiral if all left-handed Weyl spinors $\psi_\alpha$ (including those obtained by conjugation of right-handed ones) live in the same complex representation of the gauge group and therefore a mass term $\sim \psi_\alpha \chi \varepsilon^{\alpha\beta}$ cannot be written down. However, here we work with Majorana spinors, which necessarily involve both chiralities, and one has to be somewhat careful.

Let us consider again the would-be zero modes. The cleanest way to understand their structure is to look at the upper (respectively lower) triangular matrices $\Psi^{1,2}$ (respectively $\Psi^{2,1}$), which are in inequivalent (conjugate) complex representations of the unbroken gauge group $SU(n_1) \times SU(n_2)$. They are related by the Majorana condition $\Psi^{1,2} = (\Psi^{2,1})^\dagger$. Therefore, we certainly cannot expect to have $\Psi^{2,1} = 0$. What is actually needed is that $\gamma_5|\Psi^{1,2} = +1$ and $\gamma_5|\Psi^{2,1} = -1$, so that the upper-triangular matrices have chirality +1, and the lower-triangular matrices are their charge conjugate modes; this is then a chiral theory.

Now let us show how to realize this. We recall that we have two fuzzy spheres with fluxes and we have assumed already that $m_L > 0$ and $m_R > 0$. Then, the relations (48) and (49) can be written as

\[
\chi_{L,tang}|\Psi^{1,2} = \chi_{R,tang}|\Psi^{1,2} = +1, \\
\chi_{L,tang}|\Psi^{2,1} = \chi_{R,tang}|\Psi^{2,1} = -1.
\]  

(63)

It follows from (48) and (49) that the operators $\Pi_L$ and $\Pi_R$, defined in (51), actually coincide on the space of zero modes. Hence the full fermionic Hilbert space can be separated in two sectors as follows,

\[
\mathcal{H}_+ = \{ \Psi; \, \Pi_L \Psi = \Psi \} \quad \text{and} \quad \mathcal{H}_- = \{ \Psi; \, \Pi_L \Psi = -\Psi \}.
\]  

(64)

Then it is clear that $\Psi^{1,2}$ and $\Psi^{2,1}$ have opposite four-dimensional chirality in each sector, which is the desired result as it was explained above. Therefore we end up with two exactly chiral mirror sectors, which are separated according to (64). One can show that in a type II vacuum the fermion spectrum of the standard model can be accommodated in these sectors [31]. Then, if the electroweak symmetry breaking occurs in one sector at a scale which is much higher than in the other sector, such a model might turn out to be realistic, which remains to be studied.

However, in order to be able to proceed further with the electroweak symmetry breaking, the two chiral sectors that we have presented above have to decouple. Concerning the kinetic part of the Yukawa couplings it is straightforward to show that

\[
Tr\Pi_L \overline{\Psi} \gamma_5 iD_{S^2 \times S^2} \Psi' = Tr\overline{\Psi} \gamma_5 iD_{S^2 \times S^2} \Pi_L \Psi'.
\]  

(65)

Therefore the Yukawa term due to $D_{S^2 \times S^2}$ couples only modes with the same eigenvalue of $\Pi_L$. This is as expected for a kinetic term for spinors on $S^2 \times S^2$.

Finally, one might worry that the would-be zero modes are not massless due to the presence of the shift action (39) in the model and therefore that the chiral sectors are not exactly separate. Indeed, the shift action couples opposite eigenvalues of $\Pi_L$, since

\[
Tr\Pi_L \overline{\Psi} \gamma_5 i\Gamma_L^{(Y)} \Psi' = -Tr\overline{\Psi} \gamma_5 i\Gamma_L^{(Y)} \Pi_L \Psi'.
\]  

(66)
Therefore, in order to make the would-be zero modes massless, we have to modify the model by adding a mass term which breaks $SO(6) \cong SU(4) \rightarrow SU(2)_L \times SU(2)_R$, in which case the correct action for a chiral model is given by

$$
S_{\text{chiral}}[\Psi] = S - S_{\text{shift}} = \int \bar{\Psi} i\not{D}_0 \Psi - \bar{\Psi} i\gamma_5 (\alpha_L \Gamma^{(\gamma)}_L + \alpha_R \Gamma^{(\gamma)}_R) \Psi
$$

$$
= \int \bar{\Psi} i\not{D}_{S^2 \times S^2} \Psi.
$$

This action is singled out by the fact that the would-be zero modes are exactly massless at tree level and therefore the model enjoys unbroken $U(1)_{\text{axial}}$ symmetry generated by $\Pi_L$. This means that quantum corrections will not induce any mass terms since they are protected by chiral symmetry. The two chiral sectors $\mathcal{H}_+$ and $\mathcal{H}_-$ are now exactly separated and one can proceed to study the electroweak symmetry breaking in each one. This will be pursued elsewhere.

## 4 Discussion and conclusions

Generalizing previous work [2–4, 30], we have shown how an effective extra-dimensional space with the geometry of fuzzy $S^2 \times S^2$ can arise within a renormalizable four-dimensional $SU(N)$ gauge theory with the matter content of $\mathcal{N} = 4$ SYM theory. The underlying mechanism is simply the standard Higgs effect and spontaneous symmetry breaking of the theory. The model behaves as an eight-dimensional Yang-Mills theory on $M^4 \times S^2_N \times S^2_N$, for energies below some cutoff given by the fuzzy nature of the extra-dimensional spheres. It represents a particularly simple yet rich realization of the idea of deconstructing dimensions [1], taking advantage of results from noncommutative field theory. This allows to consider ideas of compactification and dimensional reduction within a renormalizable framework.

We focus on the effects of fluxes on these fuzzy spheres and on the corresponding fermionic zero modes, i.e. the low-energy sector of the model.

Even though the fluxes on $S^2 \times S^2$ lead indeed to the expected zero modes, the model nevertheless turns out to be non-chiral a priori. More precisely, we find essentially mirror models, where two chiral sectors arise with opposite chirality. This means that each would-be zero mode from $\Psi_1, 2$ has a mirror partner from $\Psi_2, 1$, with opposite chirality and gauge quantum numbers. The reason for this is that the fuzzy geometry is four-dimensional but in some sense embedded in six extra dimensions. The missing two (“shadow”) dimensions are reflected in extra components of the spinors, which do not see the flux respectively the chirality on $S^2 \times S^2$. This is a crucial difference of our model comparing with models based on commutative extra dimensions, where chiral Lagrangians are easier to obtain [6, 32].

Thus we arrive essentially at a picture of mirror fermions discussed e.g. in [7] from a phenomenological point of view. While this may still be physically interesting since the “mirror fermions” may have larger mass than the ones we see at low energies, it would be desirable to find a chiral version with similar features. Thus the present work can be seen as a step in the direction of realistic models in this framework, suggesting the need of additional geometrical structures in the extra dimensions. There are indeed many possible directions for generalizations, and we hope to report on progress in this direction soon.

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7 see however [33].
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Appendices

5 Appendix A: Clifford algebra and $SO(3)_L \times SO(3)_R$

In this appendix we collect our conventions on the representations of the Clifford algebras we have used.

The gamma matrices $\gamma_\mu$ define the four-dimensional Clifford algebra and they are chosen to be purely imaginary, corresponding to the Majorana representation in four dimensions. In our conventions this representation is explicitly given by the matrices

\begin{align*}
\gamma_0 &= \sigma_0 \otimes \sigma_2, \\
\gamma_1 &= i\sigma_0 \otimes \sigma_3, \\
\gamma_2 &= i\sigma_1 \otimes \sigma_1, \\
\gamma_3 &= i\sigma_3 \otimes \sigma_1, 
\end{align*}

where $\sigma_0 := 1_2$ is the identity matrix.

Moreover, we give here the explicit Majorana representation of the six-dimensional Clifford algebra, which is known to exist in six Euclidean dimensions. This is naturally adapted to $SO(3)_L \times SO(3)_R \subset SO(6)$, and closely related to other constructions, see for example the ref. [15]. First we consider the matrices

\begin{align*}
\gamma_1^L &= \sigma^2 \otimes \sigma^1, \\
\gamma_2^L &= \sigma^2 \otimes 1, \\
\gamma_3^L &= i\sigma_2 \otimes \sigma^1, \\
\gamma_4^R &= \sigma^2 \otimes \sigma^2, \\
\gamma_5^R &= i\sigma_2 \otimes \sigma^3, \\
\gamma_6^R &= 1 \otimes \sigma^2, 
\end{align*}

which are antisymmetric and purely imaginary, hence hermitian, and they satisfy

\begin{align*}
[\gamma_i^L, \gamma_j^L] &= \delta^{ij} + i\epsilon^{ijk} \gamma_k^L, \\
[\gamma_i^R, \gamma_j^R] &= \delta^{ij} + i\epsilon^{ijk} \gamma_k^R, \\
\{\gamma_i^L, \gamma_j^R\} &= 0.
\end{align*}

Then the following matrices define a representation of the $SO(6)$ Clifford algebra

\begin{align*}
\Delta_i &= i\sigma_1 \otimes \gamma_i^L, \\
\Delta_{3+i} &= i\sigma_3 \otimes \gamma_i^L, 
\end{align*}

satisfying the desired relation

\begin{align*}
\{\Delta^\mu, \Delta^\nu\} &= -2\delta^{\mu\nu}.
\end{align*}

They are manifestly anti-symmetric and real, hence they furnish a Majorana representation. The left and right chiral projections are given by

\begin{align*}
\Gamma_L(Y) &= \Delta_1 \Delta_2 \Delta_3 = \sigma_1 \otimes 1, \\
\Gamma_R(Y) &= \Delta_4 \Delta_5 \Delta_6 = \sigma_3 \otimes 1
\end{align*}

and the six-dimensional chirality operator is

\begin{align*}
\Gamma(Y) &= -i\Gamma_L(Y) \Gamma_R(Y) = -\sigma_2 \otimes 1.
\end{align*}
6 Appendix B: Dirac operator and eigenmodes on fuzzy $S^2$

Assume that $\phi_i$ satisfies the relations of the fuzzy sphere

$$[\phi_i, \phi_j] = i\alpha \varepsilon_{ijk} \phi_k, \quad \phi_i^2 = \alpha^2 \frac{N^2 - 1}{4}, \quad (B-1)$$

and consider the Dirac operator on $S^2_N$ defined through (cf. [3, 34, 35])

$$\bar{D}_{S^2} \psi = \sigma^i [\phi_i, \psi] + \alpha \psi. \quad (B-2)$$

Since $\bar{D}_{S^2}$ commutes with the $SU(2)$ group of rotations, the eigenmodes of $\bar{D}_{S^2}$ are obtained by decomposing the spinors into irreducible representations of $SU(2)$,

$$\psi \in (2) \otimes (N) \otimes (N) = (2) \otimes ((1) \oplus (3) \oplus \ldots \oplus (2N - 1)) \oplus (2) \otimes (2N - 2) \oplus \ldots \oplus (2N - 2)$$

This decomposition defines the spinor harmonics $\psi_{\pm,(n)}$ which live in the $n$-dimensional representation of $SU(2)$ denoted by $(n)$ for $n = 2, 4, \ldots, 2N$, excluding $\psi_{-(2N)}$. The eigenvalue of $\bar{D}_{S^2}$ acting on these states can be determined easily using some $SU(2)$ algebra,

$$\bar{D}_{S^2} \psi_{\pm,(n)} = E_{\delta=\pm,(n)} \psi_{\pm,(n)}, \quad (B-3)$$

where

$$E_{\delta=\pm,(n)} = \frac{\alpha}{2} \begin{cases} n, & \delta = 1, \quad n = 2, 4, \ldots, 2N \\ -n, & \delta = -1, \quad n = 2, 4, \ldots, 2N - 2 \end{cases}. \quad (B-4)$$

We note that with the exception of $\psi_{+(2N)}$, all eigenstates come in pairs $(\psi_{+(n)}, \psi_{-(n)})$ for $n = 2, 4, \ldots, 2N - 2$, which have opposite eigenvalues $\pm \frac{\alpha}{2} n$ of $\bar{D}_{S^2}$. They are interchanged through the fuzzy chirality operator $\chi = \frac{1}{2\alpha^2} \sigma^i \{ \phi_i, \ldots \}$,

$$\chi \left( \begin{array}{c} \psi_{+(n)} \\ \psi_{-(n)} \end{array} \right) = c \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} \psi_{+(n)} \\ \psi_{-(n)} \end{array} \right), \quad (B-5)$$

for some $c \approx 1$, by virtue of the anticommutativity relation

$$\bar{D}_{S^2} \chi + \chi \bar{D}_{S^2} = 0. \quad (B-6)$$

Moreover, the chirality operator for the top mode vanishes, $\chi \psi_{+(2N)} = 0$.

Let us now consider a type II vacuum analogous to (41) in section 3, but for a simple fuzzy sphere [3]:

$$\phi_i = \left( \begin{array}{cc} \alpha_1 \lambda_1^{(N_1)} & 0 \\ 0 & \alpha_2 \lambda_1^{(N_2)} \end{array} \right), \quad (B-7)$$

with $m = N_1 - N_2$. This can be interpreted as a two $U(1)$ gauge fields on $S^2$ which differ by a magnetic flux $m$ [18]. Then there are $m$ “would-be” zero-modes as expected by the
index theorem, which have a simple group-theoretical realization. The point is that the off-
diagonal blocks are rectangular matrices, e.g. \( \psi^{1,2} \in Mat(N_1 \times N_2) \). To be specific assume
that \( m > 0 \). We decompose this module, tensored with the spinorial \((2)\), into irreducible
representations of the rotation group \( SU(2) \),

\[
(2) \otimes Mat(N_1 \times N_2) \cong (2) \otimes (N_1) \otimes (N_2) \\
\cong (|N_1 - N_2|) \oplus 2 \times (|N_1 - N_2| + 3) \\
\oplus \ldots \oplus 2 \times (N_1 + N_2 - 2) \oplus (N_1 + N_2). \tag{B-9}
\]

Note that there is indeed a single irreducible representation \((m)\), which corresponds to a
(would-be) zero mode that we denote as \( \Psi^{1,2}_{(m)} \). All the other irreducible representations
come in pairs with opposite chirality\(^8\). Together with \( SU(2) \) invariance, it follows that \( \Psi^{1,2}_{(m)} \)
is an eigenstate of both the fuzzy Dirac operator \( \mathcal{D}_{S^2} \) and the fuzzy chirality operator \( \chi \).

In general, these would-be zero modes will be only approximate zero modes,

\[
\mathcal{D}_{S^2} \Psi^{1,2}_{(m)} = E_0 \Psi^{1,2}_{(m)}, \quad E_0 = O\left( \frac{1}{N} \right), \\
\chi \Psi^{1,2}_{(m)} = c \Psi^{1,2}_{(m)}, \quad c \approx \pm 1 \tag{B-10}
\]

assuming \( \alpha_i = 1 + O\left( \frac{1}{N} \right) \). One can either leave things in this approximate form, or cast it
in a precise form assuming that \( \alpha_1 N_1 = \alpha_2 N_2 =: R \). In that case, it is useful to consider
\( \phi_0 = \frac{1}{2} \begin{pmatrix} \alpha_1 \mathbb{1} & 0 \\ 0 & \alpha_2 \mathbb{1} \end{pmatrix} \) in the vacuum \((B-8)\), which satisfies

\[
\Phi = \phi_i \sigma^i + \phi_0 \sigma_0, \quad \Phi^2 = \frac{R^2}{4}. \tag{B-11}
\]

Then the following refined Dirac resp. chirality operators

\[
\mathcal{D}_{S^2} \psi = \sigma^i [\phi_i, \psi] + \{\phi_0, \psi\}, \\
\chi \psi = \frac{1}{R} (\sigma^i [\phi_i, \psi] + [\phi_0, \psi]), \tag{B-12}
\]

satisfy

\[
\{ \mathcal{D}_{S^2}, \chi \} = 0 \tag{B-13}
\]

exactly. This implies that \( \Psi^{1,2}_{(m)} \) is an exact zero mode either of \( \mathcal{D}_{S^2} \) or of \( \chi \). It is easy to
see (cf. below) that the latter does not hold, therefore

\[
\mathcal{D}_{S^2} \Psi^{1,2}_{(m)} = 0. \tag{B-14}
\]

We note that such zero modes on a sphere only occur in the presence of magnetic flux. The
eigenvalue of \( \chi^2 \) can be computed as follows: we have \( 2 \Phi \Psi^{1,2}_{-(m)} = (R \chi + \mathcal{D}_{S^2}) \Psi^{1,2}_{-(m)} = R \chi \Psi^{1,2}_{-(m)} \) on the zero modes. Using \( 4 \Phi^2 = R^2 \) it follows that

\[
\chi^2 \Psi^{1,2}_{-(m)} = \Psi^{1,2}_{-(m)} \tag{B-15}
\]

\(^8\)Except for the top modes, where \( \chi = 0 \).
The sign of χ on Ψ₀,₁,₂ (m) is easy to obtain. Note that (B-9) involves the “anti-parallel” (respectively “parallel”) tensor product of (2) ⊗ (N₁), and the “parallel” (respectively “anti-parallel”) tensor product of (2) ⊗ (N₂). Therefore χ ∼ σᵢ xᵢ ≈ sign(m) when it acts on Ψ₀,₁,₂ (m) and similarly χ ≈ −sign(m) when it acts on Ψ₀,₂,₁ (m). It follows that

χΨ₀,₁,₂ (m) = +Ψ₀,₁,₂ (m),
χΨ₀,₂,₁ (m) = −Ψ₀,₂,₁ (m).

(B-16)

The inclusion of the second sphere is straightforward. There are m_L m_R (would-be) zero modes of D₅S² × S² in the presence of a flux m_L ≠ 0 on S²_L and a flux m_R ≠ 0 on S²_R, given simply by the product of the above zero modes.

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