A second order difference scheme for time fractional diffusion equation with generalized memory kernel

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Abstract

In the current work we build a difference analog of the Caputo fractional derivative with generalized memory kernel (\(\lambda L^2-1_\sigma\) formula). The fundamental features of this difference operator are studied and on its ground some difference schemes generating approximations of the second order in time for the generalized time-fractional diffusion equation with variable coefficients are worked out. We have proved stability and convergence of the given schemes in the grid \(L^2\) - norm with the rate equal to the order of the approximation error. The achieved results are supported by the numerical computations performed for some test problems.

Keywords: fractional derivative with generalized memory kernel, a priori estimates, fractional diffusion equation, finite difference scheme, stability, convergence

1 Introduction

Differential equations with fractional order derivatives represent a powerful mathematical tool for exact and realistic description of physical and chemical processes for which it is needed to take into consideration the background (memory) of the process [1, 2, 3, 4]. The patterns which take memory into consideration in such equations are the memory functions that are the kernels of integrals defining the operators of fractional integro-differentiation. For fractional integro-differentiation operators, the memory functions are namely power functions. The exponent of the power function of memory defines the order of the

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derivative and is connected with the fractal dimension of the environment in which the described process takes place. For more accurate description of the process in heterogeneous porous media, differential equations with fractional derivatives of distributed order are often used too. Processes of the memory can be described with the help of the memory function which has more complex structure than the power function.

In the rectangle $\bar{Q}_T = \{0 \leq x \leq 1, 0 \leq t \leq T\}$ we consider the Dirichlet boundary value problem for time fractional diffusion equation with generalized memory kernel and variable coefficients

$$\partial_{0t}^{\alpha,\lambda(t)} u = \mathcal{L} u + f(x, t), \quad 0 < x < 1, \quad 0 < t \leq T,$$

$$u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq T, \quad u(x, 0) = u_0(x), \quad 0 \leq x \leq 1,$$

where

$$\mathcal{L} u = \frac{\partial}{\partial x} \left( k(x, t) \frac{\partial u}{\partial x} \right) - q(x, t) u,$$

$$\partial_{0t}^{\alpha,\lambda(t)} u(x, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\lambda(t-\eta)}{(t-\eta)^\alpha} \frac{\partial u}{\partial \eta}(x, \eta) d\eta$$

is the generalized Caputo fractional derivative of order $\alpha$, $0 < \alpha < 1$ with weighting function $\lambda(t) \in \mathcal{C}^2[0, T]$, where $\lambda(t) > 0$, $\lambda'(t) \leq 0$ for all $t \in [0, T]$; $0 < c_1 \leq k(x, t) \leq c_2$, $q(x, t) \geq 0$ for all $(x, t) \in \bar{Q}_T$.

Diffusion and Fokker-Planck-Smoluchowski equations which have a generalized memory kernel were investigated in [5]. In this work it is demonstrated that the memory kernel appearing in the generalized diffusion equation has diverse potential forms which can describe a broad range of experimental phenomena.

With the help of the energy inequality method, a priori estimates for the solution of both differential and difference problems of the Dirichlet and Robin boundary value problems for the fractional, variable and distributed order diffusion equation with Caputo fractional derivative were derived in [6, 7, 8, 9, 10, 11, 12, 13]. A priori estimates for the difference problems analyzed in [14] by means of the maximum principle imply the stability and convergence of these difference schemes.

In this work, to construct difference schemes with the order of accuracy $O(\tau^2)$ in time we have to demand the existence of a sufficiently smooth solution of the original problem. It brings on a significant narrowing of the input data class of the problem for which we apply the proposed method. As it is well known (see for example [15, 16]), in the case of smooth input data for a time-fractional diffusion equation,
the solutions are not necessarily smooth in a closed domain, because
the derivatives of the function $u(x, t)$ with respect to $t$ might possess a
singularity at $t = 0$. In such cases, if possible, we present the solution
as the sum of two functions: one of which is known but not smooth,
whereas the other is smooth but not known, as it is illustrated in work [17].

In work [18], we consider a reaction-diffusion problem with a Caputo
time derivative of the order $\alpha \in (0,1)$. It is shown that the
solution of such a problem has in general a weak singularity near the
initial time $t = 0$, and we derive sharp pointwise bounds on certain
derivatives of this solution. We have given a new analysis of a stan-
dard finite difference method for the problem, taking into account this
initial singularity.

In [19], we study an analysis of the $L_1$ scheme for the subdiffusion
equation with nonsmooth data. In [20], error estimates for approxi-
mations of distributed order time fractional diffusion equation with
nonsmooth data were investigated.

In the current paper, a difference analog of the Caputo fractional
derivative with generalized memory kernel ($\lambda L_2-1$ formula) is built
up. The essential features of this difference operator are investigated
and on its ground some difference schemes generating approximations
of the second and fourth order in space and the second order in time
for the generalized time-fractional diffusion equation with variable co-
efficients are studied. Stability of the suggested schemes as well as
their convergence in the grid $L_2$ - norm with the rate equal to the
order of the approximation error are proven. The achieved results
are supported by the numerical computations performed for some test
problems.

2 Stability and convergence of the family of difference schemes

In this section, we consider some families of difference schemes in a
general form, set on a non-uniform time grid. A criterion of the sta-
bility of the difference schemes in the grid $L_2$ - norm is worked out.
The convergence of solutions of the difference schemes to the solution
of the corresponding differential problem with the rate equal to the
order of the approximation error is proven.

In the rectangle $Q_T = \{(x, t) : 0 \leq x \leq l, 0 \leq t \leq T\}$ we assign the
grid $\Omega_{\Gamma_T} = \Omega_h \times \Omega_{\tau}$, where $\Omega_h = \{x_i = ih, i = 0,1,\ldots,N; hN = l\}$,
$\Omega_{\tau} = \{t_j : 0 = t_0 < t_1 < t_2 < \ldots < t_{M-1} < t_M = T\}$.

The family of difference schemes, approximating problem (1) - (2)
on the grid $\mathcal{W}_{h\tau}$, mainly has the form

\[ g\Delta^\alpha_0 y_i = \Lambda y_i^{(\sigma_j+1)} + \nu_i^{j+1}, \quad i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, M-1, \]

\[ y(0, t) = 0, \quad y(l, t) = 0, \quad t \in \mathcal{W}_\tau, \quad y(x, 0) = u_0(x), \quad x \in \mathcal{W}_h, \]

where

\[ g\Delta^\alpha_0 y_i = \sum_{s=0}^j (y_i^{s+1} - y_i^{s}) g_s^{j+1}, \quad g_s^{j+1} > 0, \]

is a difference analog of the generalized Caputo fractional derivative of the order $\alpha$ with weighting function $\lambda(t) (0 < \alpha < 1, \lambda(t) > 0, \lambda'(t) \leq 0)$, $\Lambda$ is a difference operator which approximates the continuous operator $\mathcal{L}$, such that the operator $-\Lambda$ preserves its positive definiteness:

\[ (-\Lambda y, y) \geq \kappa \|y\|_0^2, \quad (y, v) = \sum_{i=1}^{N-1} y_i v_i h, \quad \|y\|_0^2 = (y, y), \quad \kappa > 0, \]

\[ y^{j+\sigma_j+1} = \sigma_j + 1 y^{j+1} + (1 - \sigma_j + 1) y^{j}, \quad 0 \leq \sigma_j + 1 \leq 1, \quad j = 0, 1, \ldots, M-1. \]

**Lemma 2.1** [S] If $g_j^{j+1} > g_{j-1}^{j+1} > \ldots > g_0^{j+1} > 0, \quad j = 0, 1, \ldots, M-1$ then for any function $v(t)$ defined on the grid $\mathcal{W}_\tau$ the following inequalities hold true

\[ v^{j+1} g\Delta^\alpha_0 y_i v \geq \frac{1}{2 g_0^{j+1}} \left( g\Delta^\alpha_0 y_i v \right)^2, \]

\[ v^j g\Delta^\alpha_0 y_i v \geq \frac{1}{2 g_j^{j+1}} \left( g\Delta^\alpha_0 y_i v \right)^2, \]

where $g_{-1}^{j+1} = 0$.

**Corollary 2.1** [S] If $g_j^{j+1} > g_{j-1}^{j+1} > \ldots > g_0^{j+1} > 0$ and $\frac{g_j^{j+1}}{2 g_j^{j+1} - g_{j-1}^{j+1}} \leq \sigma_{j+1} \leq 1$, where $j = 0, 1, \ldots, M-1, \quad g_{-1}^{j+1} = 0$, then for any function $v(t)$ defined on the grid $\mathcal{W}_\tau$ we have the inequality

\[ (\sigma_j + 1 v^{j+1} + (1 - \sigma_j v^{j+1}) g\Delta^\alpha_0 y_i v \geq \frac{1}{2 g_j^{j+1}} \left( g\Delta^\alpha_0 y_i v \right)^2. \]

**Theorem 2.1** [S] If

\[ g_j^{j+1} > g_{j-1}^{j+1} > \ldots > g_0^{j+1} \geq c_2 > 0, \quad \frac{g_j^{j+1}}{2 g_j^{j+1} - g_{j-1}^{j+1}} \leq \sigma_j + 1 \leq 1, \]

4
where \( j = 0, 1, \ldots, M - 1, g^1_{j-1} = 0 \), then the difference scheme (3)–(4) is unconditionally stable and its solution satisfies the following a priori estimate:

\[
\|y^j\|^2 \leq \|y^0\|^2 + \frac{1}{2\kappa c^2} \max_{0 \leq j \leq M} \|\varphi^j\|^2,
\]

(9)

A priori estimate (9) implies the stability of difference scheme (3)–(4).

Theorem 2.2 [8] If the conditions of Theorem (2.1) are fulfilled and difference scheme (3)–(4) has the approximation order \( O(N^{-r_1} + M^{-r_2}) \), where \( r_1 \) and \( r_2 \) are some known positive numbers, then the solution of difference scheme (3)–(4) converges to the solution of differential problem (1)–(2) in the grid \( L^2 \) - norm with the rate equal to the order of the approximation error \( O(N^{-r_1} + M^{-r_2}) \).

3 A second order numerical differentiation formula for the generalized Caputo fractional derivative

In this section, we construct a difference analog of the Caputo fractional derivative with the approximation order \( O(\tau^2) \) and investigate its essential properties.

Next we consider the uniform grid \( \bar{\omega}_\tau = \{t_j = j\tau, j = 0, 1, \ldots, M, \tau M = T\} \). Let us find the discrete analog of the \( \partial^{\alpha,\lambda}_{0t_{j+\sigma}} v(t) \) at the fixed point \( t_{j+\sigma}, j \in \{0, 1, \ldots, M - 1\} \), where \( v(t) \in C^3[0, T] \), \( \sigma = 1 - \alpha/2 \). For all \( \alpha \in (0, 1) \) and \( \lambda(t) > 0 (\lambda'(t) \leq 0, \lambda(t) \in C^2[0, T]) \) the following equalities hold true

\[
\partial^{\alpha,\lambda}_{0t_{j+\sigma}} v(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} v'(\eta)d\eta
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \left( \sum_{s=1}^{j} \int_{t_s}^{t_s} \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} v'(\eta)d\eta + \int_{t_j}^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} v'(\eta)d\eta \right)
\]

\[
= \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{j} \int_{t_s}^{t_s} \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} (\Pi_{2,s} v(\eta))' d\eta
\]
\[
\begin{align*}
&+ \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_s} \frac{\lambda(t_{j+\sigma}-\eta)}{(t_{j+\sigma}-\eta)^\alpha} (v(\eta) - \Pi_{2,s} v(\eta))' \, d\eta \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma}-\eta)}{(t_{j+\sigma}-\eta)^\alpha} (\Pi_{1,j} v(\eta))' \, d\eta \\
&+ \frac{1}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma}-\eta)}{(t_{j+\sigma}-\eta)^\alpha} (v(\eta) - \Pi_{1,j} v(\eta))' \, d\eta \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} v_{l,s-1} \int_{t_{s-1}}^{t_s} \frac{\lambda(t_{j+\sigma}-\eta)}{(t_{j+\sigma}-\eta)^\alpha} \, d\eta \\
&+ \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} v_{l,t} \int_{t_{s-1}}^{t_s} \frac{\lambda(t_{j+\sigma}-\eta)}{(t_{j+\sigma}-\eta)^\alpha} \, d\eta + R^{(1)}_{1j} + R^{(1)}_{2j} + R^{(2)}_{1j} + R^{(3)}_{1j} \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \left( v_{l,s-1} \int_{t_{s-1}}^{t_s} \frac{\lambda_{j-s+\sigma+1/2} - \lambda_{t_{j-s+\sigma}}(\eta - t_{s-1/2})}{(t_{j+\sigma}-\eta)^\alpha} \, d\eta ight. \\
&\left. + \lambda_{j-s+\sigma} v_{l,t} \int_{t_{s-1}}^{t_s} \frac{\eta - t_{s-1/2}}{(t_{j+\sigma}-\eta)^\alpha} \, d\eta \right) \\
&+ \frac{\lambda_{\sigma-1/2} v_{l,t}}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+\sigma}} \frac{\, d\eta}{(t_{j+\sigma}-\eta)^\alpha} + R^{(1)}_{1j} + R^{(1)}_{2j} + R^{(2)}_{1j} + R^{(3)}_{1j} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=1}^{j} \left( v_{l,s-1}(\lambda_{j-s+\sigma+1/2} - \lambda_{t_{j-s+\sigma+1}} + (\lambda_{j-s+\sigma} - \lambda_{j-s+\sigma+1})b_{j-s+1}^{(\alpha)} \\
+ \lambda_{j-s+\sigma} b_{j-s+1}(v_{l,t} - v_{l,s-1}) \right) + \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \lambda_{\sigma-1/2} a_{0}^{(\alpha)} v_{l,t} + R^{l+\sigma} \\
&= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \left( (\lambda_{j+\sigma-1/2} - \lambda_{j+\sigma} b_{j}^{(\alpha)}) v_{l,0} \right)
\end{align*}
\]
\begin{align*}
+ \sum_{s=1}^{j-1} \left( \lambda_{j-s+\sigma} a_{j-s}^{(a)} + \lambda_{j-s+\sigma} b_{j-s+1}^{(a)} - \lambda_{j-s+\sigma} b_{j-s}^{(a)} \right) v_{t,s} \\
+ (\lambda_{\sigma-1/2} a_0^{(a)} + \lambda_{\sigma} b_{1}^{(a)}) v_{t,j} \\
= \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} c_{j-s}^{(a)} v_{t,s} + R_{1j}^{1+\sigma}.
\end{align*}

where

\begin{align*}
& a_0^{(a)} = \sigma^{1-\alpha}, \quad a_l^{(a)} = (l + \sigma)^{1-\alpha} - (l - 1 + \sigma)^{1-\alpha}, \\
& b_l^{(a)} = \frac{1}{2 - \alpha} \left[ (l+\sigma)^{2-\alpha} - (l-1+\sigma)^{2-\alpha} \right] - \frac{1}{2} \left[ (l+\sigma)^{1-\alpha} + (l-1+\sigma)^{1-\alpha} \right], \quad l \geq 1,
\end{align*}

\begin{align*}
& \lambda_i = \lambda(t_s), \quad v_{t,s} = \frac{v(t_{s+1}) - v(t_s)}{\tau}, \quad v_{\tau t,s} = \frac{v(t_s) - v(t_{s-1})}{\tau}, \\
& \Pi_{1,sv}(t) = v(t_{s+1}) \frac{t - t_s}{\tau} + v(t_s) \frac{t_{s+1} - t}{\tau}, \\
& \Pi_{2,sv}(t) = v(t_{s+1}) \frac{(t - t_{s-1})(t - t_s)}{2\tau^2} + v(t_{s-1}) \frac{(t - t_s)(t - t_{s+1})}{2\tau^2}, \\
& R_{1j}^{1+\sigma} = R_{1j}^{(1)} + R_{jj+\sigma}^{(1)} + R_{1j}^{(2)} + R_{jj+\sigma}^{(2)} + R_{1j}^{(3)}, \\
& R_{1j}^{(1)} = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_s} \lambda(t_{j+\sigma} - \eta) \left( \frac{(t_{j+\sigma} - \eta)^{\alpha}}{(t_{j+\sigma} - \eta)^{\alpha}} \right) (v(\eta) - \Pi_{2,sv}(\eta))' d\eta, \\
& R_{jj+\sigma}^{(1)} = \frac{1}{\Gamma(1 - \alpha)} \int_{t_{j+\sigma}}^{t_{j+\sigma}+\eta} \lambda(t_{j+\sigma} - \eta) \left( \frac{(t_{j+\sigma} - \eta)^{\alpha}}{(t_{j+\sigma} - \eta)^{\alpha}} \right) (v(\eta) - \Pi_{1,sv}(\eta))' d\eta, \\
& R_{1j}^{(2)} = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{j} v_{t,s-1} \int_{t_{s-1}}^{t_s} \lambda(t_{j+\sigma} - \eta) - \lambda_{j-s+\sigma+1/2} + \lambda_{j-s+\sigma}(\eta - t_{s-1/2}) \left( \frac{(t_{j+\sigma} - \eta)^{\alpha}}{(t_{j+\sigma} - \eta)^{\alpha}} \right) d\eta, \\
& R_{jj+\sigma}^{(2)} = \frac{v_{t,j}}{\Gamma(1 - \alpha)} \int_{t_{j+\sigma}}^{t_{j+\sigma}+\eta} \lambda(t_{j+\sigma} - \eta) - \lambda_{\sigma-1/2} \left( \frac{(t_{j+\sigma} - \eta)^{\alpha}}{(t_{j+\sigma} - \eta)^{\alpha}} \right) d\eta, \\
& R_{1j}^{(3)} = \frac{1}{\Gamma(1 - \alpha)} \sum_{s=1}^{j} v_{\tau t,s} \int_{t_{s-1}}^{t_s} \lambda(t_{j+\sigma} - \eta) - \lambda_{j-s+\sigma}(\eta - t_{s-1/2}) \left( \frac{(t_{j+\sigma} - \eta)^{\alpha}}{(t_{j+\sigma} - \eta)^{\alpha}} \right) d\eta.
\end{align*}
Let us consider the below fractional numerical differentiation formula for the generalized Caputo fractional derivative of order \( \alpha \) with weighting function \( \lambda(t) \) \( (0 < \alpha < 1, \lambda(t) > 0, \lambda'(t) \leq 0) \)

\[
\Delta_{0^+}^{\alpha,\lambda(t)} v = \frac{\tau^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{s=0}^{j} c_{j-s}^{(a)} v_{t+s},
\]

where

\[
c_{0}^{(a)} = \lambda_{\sigma-1/2} a_{0}^{(a)}, \quad \text{for } \ j = 0; \quad \text{and for } \ j \geq 1,
\]

\[
c_{s}^{(a)} = \begin{cases} 
\lambda_{\sigma-1/2} a_{0}^{(a)} + \lambda_{\sigma} b_{1}^{(a)}, & s = 0, \\
\lambda_{s+\sigma-1/2} a_{s}^{(a)} + \lambda_{s+\sigma} b_{s+1}^{(a)} - \lambda_{s+\sigma} b_{s}^{(a)}, & 1 \leq s \leq j - 1, \\
\lambda_{j+\sigma-1/2} a_{j}^{(a)} - \lambda_{j+\sigma} b_{j}^{(a)}, & s = j.
\end{cases}
\]

We call (10) the \( \lambda L2-1_{\sigma} \) - formula for the generalized Caputo fractional derivative.

**Lemma 3.1** For any \( \alpha \in (0, 1) \) and \( v(t) \in C^{3}[0, t_{j+1}] \), it is true that

\[
\partial_{0^+}^{\alpha,\lambda(t)} v = \Delta_{0^+}^{\alpha,\lambda(t)} v + O(\tau^2),
\]

where \( \lambda(t) > 0, \lambda'(t) \leq 0 \) and \( \lambda(t) \in C^{2}[0, t_{j+1}] \).

**Proof.** We have \( \partial_{0^+}^{\alpha,\lambda(t)} v - \Delta_{0^+}^{\alpha,\lambda(t)} v = R_{1j}^{(1)} + R_{1j}^{(1)} + R_{1j}^{(2)} + R_{1j}^{(2)} + R_{1j}^{(3)} \).

Estimate the errors \( R_{1j}^{(1)}, R_{1j}^{(1)}, R_{1j}^{(2)}, R_{1j}^{(2)} \) and \( R_{1j}^{(3)} \):

\[
|R_{1j}^{(1)}| = \frac{1}{\Gamma(1-\alpha)} \left| \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} (v(\eta) - \Pi_{2,s} v(\eta))' d\eta \right|
\]

\[
\leq \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \left( -\frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} + \frac{\zeta(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^{\alpha+1}} \right) (v(\eta) - \Pi_{2,s} v(\eta)) d\eta
\]

\[
\leq \frac{M_{1j}^{1+3} \tau^3}{9 \sqrt{3} \Gamma(1-\alpha)} \sum_{s=1}^{j} \int_{t_{s-1}}^{t_{s}} \left( \frac{m_{1j}^{1+3}}{(t_{j+\sigma} - \eta)^\alpha} + \frac{\zeta(0)}{(t_{j+\sigma} - \eta)^{\alpha+1}} \right) d\eta
\]

\[
= \frac{M_{1j}^{1+3} \tau^3}{9 \sqrt{3} \Gamma(1-\alpha)} \int_{0}^{t_{j}} \left( \frac{m_{1j}^{1+3}}{(t_{j+\sigma} - \eta)^\alpha} + \frac{\zeta(0)}{(t_{j+\sigma} - \eta)^{\alpha+1}} \right) d\eta
\]
\[ R_{j,j+\sigma}^{(1)} = \frac{1}{\Gamma(1-\alpha)} \left| \int_{t_j}^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma} - \eta)\left(\Pi_{1,j} v(\eta) - v_{t,j}\right)}{(t_{j+\sigma} - \eta)^\alpha} d\eta \right| \]

\[ = \frac{\lambda(t_{j+\sigma} - \eta)(\eta - t_{j+1/2})}{(t_{j+\sigma} - \eta)^\alpha} d\eta + \mathcal{O}(\tau^{3-\alpha}) \]

\[ R_{i,j}^{(2)} = \frac{1}{\Gamma(1-\alpha)} \sum_{s=1}^{j} \left| \frac{\lambda(t_{j+\sigma} - \eta)}{(t_{j+\sigma} - \eta)^\alpha} \left(\Pi_{1,j} v(\eta) - v_{t,j}\right) d\eta \right| \]

\[ \leq \frac{M_{1}^{j+1} m_{2}^{j+1} \tau^2}{4\Gamma(1-\alpha)} \int_{t_{s-1}}^{t_{s}} \frac{d\eta}{(t_{j+\sigma} - \eta)^\alpha} = \frac{M_{1}^{j+1} m_{2}^{j+1} \tau^2}{4\Gamma(1-\alpha)} \int_{0}^{t_{j}} \frac{d\eta}{(t_{j+\sigma} - \eta)^\alpha} \]

\[ \leq \frac{M_{1}^{j+1} m_{2}^{j+1} \tau^2}{4\Gamma(1-\alpha)} = \mathcal{O}(\tau^2) \]

\[ R_{j,j+\sigma}^{(2)} = \left| \frac{v_{t,j}}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+\sigma}} \frac{\lambda(t_{j+\sigma} - \eta) - \lambda_{j-\sigma} + \lambda_{j-\sigma+1/2}}{(t_{j+\sigma} - \eta)^\alpha} d\eta \right| \]

\[ = \left| \frac{v_{t,j}}{\Gamma(1-\alpha)} \int_{t_j}^{t_{j+\sigma}} \frac{-\lambda(t_{j-\sigma} + (\eta - t_{j+1/2})^2)}{(t_{j+\sigma} - \eta)^\alpha} d\eta \right| \]

\[ \leq \frac{M_{1}^{j+1} m_{2}^{j+1} \tau^2}{4\Gamma(2-\alpha)} \tau^{3-\alpha} = \mathcal{O}(\tau^{3-\alpha}) \]
Proof. The validity of Lemma 3.2 results from the following equalities:

\[ |R_{1j}^{(3)}| = \frac{1}{\Gamma(1 - \alpha)} \left| \sum_{s=1}^{j} \nu_{ls} \int_{t_{s-1}}^{t_j} \frac{(\lambda(t_{j+\sigma} - \eta) - \lambda_{j-s+\sigma})(\eta - t_{s-1/2})}{(t_{j+\sigma} - \eta)^{\alpha}} \, d\eta \right| \]

\[ \leq \frac{M_{j+1}^{l+1} m_j^{l+1+2 + \sigma}}{2\Gamma(1 - \alpha)} \int_{0}^{t_j} \frac{d\eta}{(t_{j+\sigma} - \eta)^{\alpha}} = \frac{M_{j+1}^{l+1} m_j^{l+1+1-\alpha+2}}{2\Gamma(2 - \alpha)} = O(\tau^2) \]

where \( M_{j+1}^{l+1} = \max_{0 \leq t \leq t_{j+1}} |\xi^{(k)}(t)|, \) \( m_j^{l+1} = \max_{0 \leq t \leq t_{j+1}} |\lambda^{(k)}(t)|. \)

Lemma 3.2 For all \( \alpha \in (0, 1) \) and \( s = 1, 2, 3, \ldots \)

\[ \frac{1 - \alpha}{(s + \sigma)^{\alpha}} < a_s < \frac{1 - \alpha}{(s + \sigma - 1)^{\alpha}}, \quad (13) \]

\[ \frac{\alpha(1 - \alpha)}{(s + \sigma + 1)^{\alpha+1}} < a_s - a_{s+1} < \frac{\alpha(1 - \alpha)}{(s + \sigma - 1)^{\alpha+1}}, \quad (14) \]

\[ \frac{\alpha(1 - \alpha)}{12(s + \sigma)^{\alpha+1}} < b_s < \frac{\alpha(1 - \alpha)}{12(s + \sigma - 1)^{\alpha+1}}, \quad (15) \]

Proof. The validity of Lemma 3.2 results from the following equalities:

\[ a_s^{(\alpha)} = (1 - \alpha) \int_{0}^{1} \frac{d\xi}{(s + \sigma - 1 + \xi)^{\alpha}}, \]

\[ a_s^{(\alpha)} - a_{s+1}^{(\alpha)} = \alpha(1 - \alpha) \int_{0}^{1} \int_{0}^{1} \frac{d\xi}{(s + \sigma - 1 + \xi + \eta)^{\alpha+1}}, \]

\[ b_s^{(\alpha)} = \frac{\alpha(1 - \alpha)}{2^{2-\alpha}} \int_{0}^{1} \int_{0}^{2(s+\sigma)-1+\eta} \frac{d\xi}{\xi^{\alpha+1}}. \]

Lemma 3.3 For all \( \alpha \in (0, 1) \) and \( s = 1, 2, 3, \ldots \)

\[ a_s^{(\alpha)} - b_s^{(\alpha)} > \frac{1 - \alpha}{2} (s + \sigma)^{-\alpha}, \quad (16) \]

\[ (2\sigma - 1)(a_0^{(\alpha)} + b_1^{(\alpha)}) - \sigma(a_1^{(\alpha)} + b_2^{(\alpha)} - b_1^{(\alpha)}) > \frac{\alpha(1 - \alpha)}{4\sigma(1 + \sigma)^{\alpha}}. \quad (17) \]
Lemma 3.4 For any $\alpha \in (0, 1)$ and $c_s^{(a)}$ ($0 \leq s \leq j, j \geq 1$) defined in (11), the following is valid

$$c_j^{(a)} > \frac{1 - \alpha}{2} \frac{\lambda_{j+\sigma}}{(j + \sigma)^{\alpha}},$$  \hspace{1cm} (18)

$$(2\sigma - 1)c_0^{(a)} - \sigma c_1^{(a)} > 0,$$  \hspace{1cm} (19)

$$c_0^{(a)} > c_1^{(a)} > c_2^{(a)} > \ldots > c_{j-1}^{(a)} > c_j^{(a)},$$  \hspace{1cm} (20)

where $\sigma = 1 - \alpha/2 \in (1/2, 1)$.

Proof. The inequality (18) follows from the inequality (16) since

$$c_j^{(a)} = \lambda_{j+\sigma}a_j^{(a)} - \lambda_{j+\sigma}b_j^{(a)}$$

$$\geq \lambda_{j+\sigma}(a_j^{(a)} - b_j^{(a)}) > \frac{1 - \alpha}{2} \frac{\lambda_{j+\sigma}}{(j + \sigma)^{\alpha}}.$$  

The inequality (19) follows from the inequality (17) since

$$(2\sigma - 1)c_0^{(a)} - \sigma c_1^{(a)} = (2\sigma - 1)(\lambda_{\sigma-1/2}a_0^{(a)} + \lambda_{\sigma}b_1^{(a)})$$

$$-\sigma(\lambda_{\sigma+1/2}a_1^{(a)} + \lambda_{\sigma+1}b_2^{(a)} - \lambda_{\sigma+1}b_1^{(a)})$$

$$\geq \lambda_{\sigma}((2\sigma - 1)(a_0^{(a)} + b_1^{(a)}) - \sigma(a_1^{(a)} + b_2^{(a)} - b_1^{(a)}))$$

$$-\sigma(\lambda_{\sigma} - \lambda_{1+\sigma})b_1^{(a)} > \lambda_{\sigma} \frac{\alpha(1 - \alpha)}{4\sigma(1 + \sigma)^{\alpha}} - (\lambda_{\sigma} - \lambda_{1+\sigma}) \frac{\alpha(1 - \alpha)}{12\sigma^{\alpha}}$$

$$> (\lambda_{\sigma} - \lambda_{1+\sigma}) \frac{\alpha(1 - \alpha)}{12\sigma(1 + \sigma)^{\alpha}} (3 - \sigma^{-\alpha}(1 + \sigma)^{\alpha}) > 0.$$  

The inequality (20) for the case $c_0^{(a)} > c_1^{(a)}$, follows from the inequality (19). Let us prove the inequality $c_s^{(a)} > c_{s+1}^{(a)}$ for $s = 1, 2, \ldots, j$. The difference $c_s^{(a)} - c_{s+1}^{(a)}$ satisfies the following estimates

$$c_s^{(a)} - c_{s+1}^{(a)} = \lambda_{s+\sigma}a_s^{(a)} - \lambda_{s+\sigma}b_{s+1}^{(a)}$$

$$+ (\lambda_{s+\sigma} + \lambda_{s+\sigma+1})b_s^{(a)} - \lambda_{s+\sigma+1}b_{s+2}^{(a)}$$

$$> \lambda_{s+\sigma}(a_s^{(a)} - a_{s+1}^{(a)} - b_s^{(a)} + b_{s+1}^{(a)}) + \lambda_{s+\sigma+1}(b_s^{(a)} - b_{s+2}^{(a)}$$

$$> \lambda_{s+\sigma}(a_s^{(a)} - a_{s+1}^{(a)} - b_s^{(a)}$$

$$> \lambda_{s+\sigma} \left( \frac{\alpha(1 - \alpha)}{(s + \sigma + 1)^{\alpha+1}} - \frac{\alpha(1 - \alpha)}{12(s + \sigma - 1)^{\alpha+1}} \right)$$

$$= \frac{\alpha(1 - \alpha)\lambda_{s+\sigma}}{12(s + \sigma + 1)^{\alpha+1}} \left( 12 - \frac{(s + \sigma + 1)^{\alpha+1}}{(s + \sigma - 1)^{\alpha+1}} \right) > 0.$$
Corollary 3.1 For any function $v(t)$ defined on the grid $\mathcal{X}_\tau$ we have the inequality
\[
(\sigma v^{j+1} + (1-\sigma)v^j) \Delta^{\alpha,\lambda}_{0t_{j+\sigma}} v \geq \frac{1}{2} \Delta^{\alpha,\lambda}_{0t_{j+\sigma}} v^2.
\] (21)

4 A second order difference scheme for the generalized time-fractional diffusion equation

Suppose that a solution $u(x, t) \in C^{4,3}_{x,t}$ of problem (1)–(2) exists, and the coefficients of equation (1) and the functions $f(x, t)$ and $u_0(x)$ fulfill the conditions, necessary for the construction of difference schemes with the order of approximation $O(h^2 + \tau^2)$.

Consider the following difference scheme
\[
\Delta^{\alpha,\lambda}_{0t_{j+\sigma}} y_i = \Lambda y_i^{(\sigma)} + \varphi_i^{j+\sigma}, \quad i = 1, 2, \ldots, N-1, \quad j = 0, 1, \ldots, M-1,
\] (22)
\[
y(0, t) = 0, \quad y(l, t) = 0, \quad t \in \mathcal{X}_\tau, \quad y(x, 0) = u_0(x), \quad x \in \mathcal{X}_h, \quad (23)
\] where
\[
\Lambda y_i = \left((a y_x)_x - dy\right)_i = \frac{a_{i+1}y_{i+1} - (a_{i+1} + a_i)y_i + a_iy_{i-1}}{h} - d_iy_i, \quad i = 1, \ldots, N-1,
\]
y$^{j+\sigma} = \sigma y^{j+1} + (1-\sigma)y^j, \quad y_x,i = (y_i - y_{i-1})/h, \quad y_{x,i} = (y_{i+1} - y_i)/h, \quad a_i^{j+\sigma} = k(x_{i-1/2}, t_{j+\sigma}), \quad d_i^{j+\sigma} = q(x_i, t_{j+\sigma}), \quad \varphi_i^{j+\sigma} = f(x_i, t_{j+\sigma}).$

If the solution of problem (1)–(2) $u \in C^{4,3}_{x,t}$ then according to [22] and the formula (10), the order of approximation of difference scheme (22)–(23) is $O(h^2 + \tau^2)$.

Theorem 4.1 The difference scheme (22)–(23) is unconditionally stable and for its solution the following a priori estimate is valid:
\[
\|y^{j+1}\|_0^2 \leq \|y^0\|_0^2 + \frac{T^\alpha \Gamma(1-\alpha)}{2\lambda(T)c_1} \max_{0 \leq j \leq M} \|\varphi^j\|_0^2.
\] (24)

Proof. For the difference operator $\Lambda$ by means of Green’s first difference formula and the embedding theorem [22] for the functions vanishing at $x = 0$ and $x = 1$, we arrive at $(-\Lambda y, y) \geq 4c_1\|y\|_0^2$, that is for this operator we can take $\kappa = 4c_1$.  

12
Considering that difference scheme (22)–(23) has the form (3)–(4) where
\[ g_{j+1}^s = \frac{s^{\alpha}}{\Gamma(2-\alpha)} x_{j-s}^{(\alpha)} \]
then Lemma 5 implies validity of the following inequalities:
\[ g_{j+1}^0 > \frac{\lambda(t_{j+\sigma})}{2\Gamma(1-\alpha)t^{\alpha}_{j+\sigma}} > \frac{\lambda(T)}{2\Gamma(1-\alpha)T^{\alpha}}, \]
\[ g_{j+1}^j > g_{j+1}^{j-1} > \ldots > g_{0}^{j+1}, \quad \sigma = 1 - \alpha/2. \]
Therefore, validity of Theorem 4.1 follows from Theorem 2.1.

From Theorem 2.2 it results that if the solution of problem (1)–(2) is sufficiently smooth, the solution of difference scheme (22)–(23) converges to the solution of the differential problem with the rate equal to the order of the approximation error \( O(h^2 + \tau^2) \).

### 4.1 Numerical results

Numerical computations are carried out for a test problem when the function
\[ u(x, t) = \sin(\pi x) \left( 1 + \frac{6 - (6 + 6bt + 3b^2t^2 + b^3t^3)e^{-bt}}{b^4} \right) \]
is the exact solution of problem (1)–(2) with \( \lambda(t) = e^{-bt} \), \( b \geq 0 \) and the coefficients \( k(x, t) = 2 - \cos(xt) \), \( q(x, t) = 1 - \sin(xt) \), \( T = 1 \).

The errors \( z = y - u \) and convergence order (CO) in the norms \( \| \cdot \|_0 \) and \( \| \cdot \|_{C(\bar{\omega}h\tau)} \), where \( \| y \|_{C(\bar{\omega}h\tau)} = \max_{(x_i, t_j) \in \bar{\omega}h\tau} |y| \), are shown in Table 1.

Table 1 demonstrates that as the number of the spatial subintervals and time steps increases, keeping \( 3h = \tau \), the maximum error decreases, as it is expected and the convergence order of the approximate scheme is \( O(h^2) = O(\tau^2) \), where the convergence order is presented by the formula: \( \text{CO} = \log_{\tau_2} \frac{||z_1||}{||z_2||} \) (\( z_i \) is the error corresponding to \( \tau_i \)).

Table 2 demonstrates that if \( h = 1/10000 \), then while the number of time steps of our approximate scheme is increasing, the maximum error is decreasing, as one can expect and the convergence order of time is \( O(\tau^2) \), where the convergence order is presented by the following formula: \( \text{CO} = \log_{\tau_2} \frac{||z_1||}{||z_2||} \).
5 A compact difference scheme for the tempered time-fractional diffusion equation.

In the current section for problem (1)–(2) with a smooth solution, we build up a compact difference scheme with the approximation order $O(h^4 + \tau^2)$ for the case when $k = k(t)$ and $q = q(t)$. Next we prove the stability and convergence of the constructed difference scheme in the grid $L_2$ norm with the rate equal to the order of the approximation error. The achieved results are supported by the numerical computations performed for a test example.

To differential problem (1)–(2), we put into correspondence a difference scheme in the case when $k = k(t)$ and $q = q(t)$:

$$
\Delta_{0t_{j+\sigma}}^{\alpha,\lambda(t)} \mathcal{H}_h y_i = a^{j+\sigma} y_{j,i}^{(\sigma)} - d^{i+\sigma} \mathcal{H}_h y_i^{(\sigma)} + \mathcal{H}_h \varphi_{j+\sigma}^{i},
$$

(25)

where $\mathcal{H}_h v_i = v_i + h^2 v_{j,i} / 12$, $i = 1, \ldots, N - 1$, $a^{j+\sigma} = k(t_{j+\sigma})$, $d^{i+\sigma} = q(t_{j+\sigma})$, $\varphi_{j+\sigma}^{i} = f(x_i, t_{j+\sigma})$, $y_{j,i}^{(\sigma)} = \sigma y_{j,i}^{(1)} + (1 - \sigma)y_{j,i}^{(0)}$.

From [24] and Lemma 2 we deduce that if $u \in C^{h,3}_{x,t}$, then the difference scheme has the approximation order $O(\tau^2 + h^4)$.

**Theorem 5.1** The difference scheme (25)–(26) is unconditionally stable and for its solution the following a priori estimate is valid:

$$
\|\mathcal{H}_ha^{i+1}\|_0^2 \leq \|\mathcal{H}_ha^{i}\|_0^2 + \frac{T^2(1 - \alpha)}{\lambda(\tau)c_1} \max_{0 \leq t \leq M} \|\mathcal{H}_h \varphi^{i}\|_0^2,
$$

(27)

**Proof.** Taking the scalar product of the equation (25) with $\mathcal{H}_ha^{i} = (\mathcal{H}_hy)^{(\sigma)}$, we get

$$
(\mathcal{H}_hy^{(\sigma)}, \Delta_{0t_{j+\sigma}}^{\alpha,\lambda(t)} \mathcal{H}_h y) - a^{j+\sigma} (\mathcal{H}_h y^{(\sigma)}, y_{j,i}^{(\sigma)})
+ d^{i+\sigma} (\mathcal{H}_h y^{(\sigma)}, \mathcal{H}_h \varphi^{j+\sigma}) = (\mathcal{H}_h y^{(\sigma)}, \mathcal{H}_h \varphi^{j+\sigma}).
$$

(28)

Let us transform the terms in identity (28) as

$$
(\mathcal{H}_hy^{(\sigma)}, \Delta_{0t_{j+\sigma}}^{\alpha,\lambda(t)} \mathcal{H}_h y) \geq \frac{1}{2} \Delta_{0t_{j+\sigma}}^{\alpha,\lambda(t)} \|\mathcal{H}_hy^{(\sigma)}\|_0^2,
$$

$$
- (\mathcal{H}_h y^{(\sigma)}, y_{j,i}^{(\sigma)}) = - (y^{(\sigma)}, y_{j,i}^{(\sigma)}) - \frac{h^2}{12} \|y_{j,i}^{(\sigma)}\|_0^2
= \|y_{j,i}^{(\sigma)}\|_0^2 - \frac{1}{12} \sum_{i=1}^{N-1} (y_{j,i+1}^{(\sigma)} - y_{j,i}^{(\sigma)})^2 h
$$

14
\begin{align*}
\geq \|y^{(\sigma)}\|_0^2 - \frac{1}{3} \|y^{(\sigma)}\|_0^2 = \frac{2}{3} \|y^{(\sigma)}\|_0^2 \geq \frac{8}{3} \|y^{(\sigma)}\|_0^2, \quad \text{where} \quad \|y\|_0^2 = \sum_{i=1}^{N} y_i^2 h,
\end{align*}

\begin{align*}
(\mathcal{H}_h y^{(\sigma)}, \mathcal{H}_h \varphi^{j+\sigma}) \leq \varepsilon \|\mathcal{H}_h y^{(\sigma)}\|_0^2 + \frac{1}{4\varepsilon} \|\mathcal{H}_h \varphi^{j+\sigma}\|_0^2
\end{align*}

\begin{align*}
= \varepsilon \sum_{i=1}^{N-1} \left( \frac{y_i^{(\sigma)} + 10y_i + y_{i+1}^{(\sigma)}}{12} \right)^2 h + \frac{1}{4\varepsilon} \|\mathcal{H}_h \varphi^{j+\sigma}\|_0^2
\end{align*}

\begin{align*}
\leq \varepsilon \|y^{(\sigma)}\|_0^2 + \frac{1}{4\varepsilon} \|\mathcal{H}_h \varphi^{j+\sigma}\|_0^2.
\end{align*}

Taking into consideration the transformations above, from identity (28) with \( \varepsilon = \frac{8c_1}{3} \) we get the inequality

\begin{align*}
\Delta_{0l, j+1}^{\alpha, \lambda(t)} \|\mathcal{H}_h y\|_0^2 \leq \frac{1}{8c_1} \|\mathcal{H}_h \varphi^{j+1}\|_0^2.
\end{align*}

The following procedure is similar to the proof of Theorem 1 in [8], and it is left out. The norm \( \|\mathcal{H}_h y\|_0 \) is equivalent to the norm \( \|y\|_0 \), which results from the inequalities

\begin{align*}
\frac{5}{12} \|y\|_0^2 \leq \|\mathcal{H}_h y\|_0^2 \leq \|y\|_0^2.
\end{align*}

Likewise Theorem 2.2 we get the convergence result.

**Theorem 5.2** Suppose that \( u(x, t) \in C^{6,3}_{x,t} \) is the solution of problem (1)–(2) for the case when \( k = k(t) \), \( q = q(t) \), and \( \{y_{j}^i \mid 0 \leq i \leq N, 1 \leq j \leq M\} \) is the solution of difference scheme (25)–(26). Then the following holds true

\begin{align*}
\|u(\cdot, t_j) - y_j\|_0 \leq C_R (\tau^2 + h^4), \quad 1 \leq j \leq M,
\end{align*}

where \( C_R \) is a positive constant not depending on \( \tau \) and \( h \).

### 5.1 Numerical results

In this subsection we present a test example for a numerical research of difference scheme (25)–(26).

Examine the following problem:

\begin{align*}
\partial_{0t}^{\alpha, \lambda(t)} u = k(t) \partial_{x}^2 u - q(t) u + f(x, t), \quad 0 < x < 1, \quad 0 < t \leq 1, \quad (29)
\end{align*}

\begin{align*}
u(0, t) = 0, \quad u(1, t) = 0, \quad 0 \leq t \leq 1, \quad u(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1, \quad (30)
\end{align*}
where $\lambda(t) = e^{-bt}$, $b \geq 0$, $k(t) = 2 - \sin(3t)$, $q(t) = 1 - \cos(2t)$,

$$f(x,t) = \left[ \pi^2 g(t)k(t) + g(t)q(t) + \frac{2t^{3-\alpha}e^{-bt}}{\Gamma(4-\alpha)} \right] \sin(\pi x),$$

whose exact analytical solution is $u(x,t) = g(t) \sin(\pi x)$, where

$$g(t) = 1 + \frac{2 - (2 + 2bt + b^2t^2)e^{-bt}}{b^3}.$$

Table 3 presents the $L_2$-norm, the errors of the maximum norm and the time convergence order for $\alpha = 0.1, 0.5, 0.9$, where $h = 1/500$. By this we can see that the time convergence order is 2.

Table 4 shows the $L_2$-norm, the maximum norm errors and the time convergence order, where $\tau = 1/2000$. We can see that the order of convergence in space is 4.

Table 5 demonstrates that as the number of spatial subintervals and time steps increases keeping $\tau = 16h^2$, the maximum error is reduced, as it is expected, and the convergence order of the approximate of the scheme is $O(h^4 + \tau^2) = O(\tau^2)$.

6 Conclusion

In the current paper, we study the stability and convergence of a difference schemes which approximate the time fractional diffusion equation with generalized memory kernel. We have built a new difference approximation of the generalized Caputo fractional derivative with the approximation order $O(\tau^2)$. The essential features of this difference operator are investigated. We have also constructed some new difference schemes of the second and fourth approximation order in space and the second approximation order in time for the generalized time fractional diffusion equation with variable coefficients. The stability and convergence of these schemes in the grid $L_2$-norm with the rate equal to the order of the approximation error are proven as well. The method can be without difficulty expanded to other time fractional partial differential equations with any other boundary conditions.

Numerical tests thoroughly confirming the achieved theoretical results are implemented. In all the computations Julia v1.6.2 is used.

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Table 1: $L_2$ - norm and maximum norm error behavior versus grid size reduction when $\tau = 3h$. 

| $b$ | $\alpha$ | $h$ | $\max_{0 \leq n < M} \| z^n \|_0$ | CO in $\| \cdot \|_0$ | $\| z \|_{C(\bar{\omega}_h)}$ | CO in $\| \cdot \|_{C(\bar{\omega}_h)}$ |
|-----|----------|-----|----------------|----------------|----------------|----------------|
| 1.0 | 0.9      | 1/10| $4.853172e-4$ | $6.860735e-4$ | $1.195117e-4$ | $2.0218$ |
|     |          | 1/20| $1.956661e-5$ | $4.193765e-5$ | $1.047192e-5$ | $2.0103$ |
|     |          | 1/40| $1.853344e-6$ | $2.619972e-6$ | $1.196972e-6$ | $1.9989$ |
|     |          | 1/80| $4.639354e-7$ | $6.558408e-7$ | $1.9981$       | $1.9981$ |
|     |          | 1/160| $1.61322e-7$ | $1.641709e-7$ | $1.9981$       | $1.9981$ |
|     |          | 1/320| $2.904554e-8$ | $4.106038e-8$ | $1.9994$       | $1.9994$ |
| 2.0 | 0.5      | 1/10| $5.695428e-4$ | $8.053893e-4$ | $1.281254e-4$ | $2.1522$ |
|     |          | 1/20| $3.11526e-5$ | $4.387037e-5$ | $1.104282e-5$ | $1.9901$ |
|     |          | 1/40| $7.832071e-6$ | $2.778988e-6$ | $1.9921$       | $1.9921$ |
|     |          | 1/80| $1.970207e-6$ | $6.983969e-7$ | $1.9936$       | $1.9936$ |
|     |          | 1/160| $1.243664e-7$ | $1.753507e-7$ | $1.9925$       | $1.9925$ |
|     |          | 1/320| $3.125438e-8$ | $4.406668e-8$ | $1.9994$       | $1.9994$ |
| 3.0 | 0.1      | 1/10| $5.590468e-4$ | $7.905373e-4$ | $1.378485e-4$ | $2.0199$ |
|     |          | 1/20| $3.418923e-5$ | $4.820603e-5$ | $1.206419e-5$ | $1.9985$ |
|     |          | 1/40| $8.555678e-6$ | $3.018517e-6$ | $1.9988$       | $1.9988$ |
|     |          | 1/80| $2.140670e-6$ | $7.551986e-7$ | $1.9989$       | $1.9989$ |
|     |          | 1/160| $1.340154e-7$ | $1.889726e-7$ | $1.9987$       | $1.9987$ |
|     |          | 1/320| $3.349770e-8$ | $4.723392e-8$ | $2.0003$       | $2.0003$ |
Table 2: $L_2$ - norm and maximum norm error behavior versus $\tau$-grid size reduction when $h = 1/2000$.

| $b$ | $\alpha$ | $\frac{h}{2}$ | $\frac{h}{10}$ | $\frac{h}{50}$ | $\frac{h}{100}$ | $\frac{h}{200}$ |
|-----|-----------|----------------|----------------|----------------|----------------|----------------|
| 3.0 | 0.9       | 1/10           | 6.977406 $e^{-5}$ | 2.0363 $e^{-5}$ | 2.0491 $e^{-6}$ | 2.1639 $e^{-7}$ |
|     |           | 1/20           | 1.700981 $e^{-5}$ | 2.0363 $e^{-5}$ | 2.0491 $e^{-6}$ | 2.1639 $e^{-7}$ |
|     |           | 1/40           | 4.110301 $e^{-6}$ | 2.0491 $e^{-6}$ | 2.0491 $e^{-6}$ | 2.0491 $e^{-6}$ |
|     |           | 1/80           | 9.171973 $e^{-7}$ | 2.0491 $e^{-6}$ | 2.0491 $e^{-6}$ | 2.0491 $e^{-6}$ |
| 2.0 | 0.5       | 1/10           | 1.144134 $e^{-4}$ | 2.0177 $e^{-5}$ | 2.0318 $e^{-6}$ | 2.0912 $e^{-7}$ |
|     |           | 1/20           | 2.825404 $e^{-5}$ | 2.0177 $e^{-5}$ | 2.0318 $e^{-6}$ | 2.0912 $e^{-7}$ |
|     |           | 1/40           | 6.909733 $e^{-6}$ | 2.0318 $e^{-6}$ | 2.0318 $e^{-6}$ | 2.0318 $e^{-6}$ |
|     |           | 1/80           | 1.621574 $e^{-6}$ | 2.0318 $e^{-6}$ | 2.0318 $e^{-6}$ | 2.0318 $e^{-6}$ |
| 1.0 | 0.1       | 1/10           | 9.999960 $e^{-5}$ | 2.0026 $e^{-5}$ | 2.0124 $e^{-6}$ | 2.0174 $e^{-7}$ |
|     |           | 1/20           | 2.495408 $e^{-5}$ | 2.0026 $e^{-5}$ | 2.0124 $e^{-6}$ | 2.0174 $e^{-7}$ |
|     |           | 1/40           | 6.147966 $e^{-6}$ | 2.0026 $e^{-5}$ | 2.0124 $e^{-6}$ | 2.0174 $e^{-7}$ |
|     |           | 1/80           | 1.438581 $e^{-6}$ | 2.0026 $e^{-5}$ | 2.0124 $e^{-6}$ | 2.0174 $e^{-7}$ |

Table 3: $L_2$ - norm and maximum norm error behavior compared with $\tau$-grid size reduction when $h = 1/500$.

| $b$ | $\alpha$ | $\tau$ | $\frac{h}{2}$ | $\frac{h}{10}$ | $\frac{h}{50}$ | $\frac{h}{100}$ | $\frac{h}{200}$ |
|-----|-----------|--------|----------------|----------------|----------------|----------------|----------------|
| 1.0 | 0.9       | 1/10   | 3.870828 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
|     |           | 1/20   | 9.636762 $e^{-5}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
|     |           | 1/40   | 2.398099 $e^{-5}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
|     |           | 1/80   | 5.973624 $e^{-6}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
|     |           | 1/160  | 1.488446 $e^{-6}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
|     |           | 1/320  | 3.709923 $e^{-7}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ | 2.0060 $e^{-4}$ |
| 2.0 | 0.5       | 1/10   | 1.383725 $e^{-4}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/20   | 3.418301 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/40   | 8.442745 $e^{-6}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/80   | 2.092596 $e^{-6}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/160  | 5.200842 $e^{-7}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/320  | 1.295146 $e^{-7}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
| 3.0 | 0.1       | 1/10   | 2.622451 $e^{-5}$ | 3.708705 $e^{-5}$ | 2.1052 $e^{-6}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/20   | 6.094819 $e^{-6}$ | 2.1052 $e^{-6}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ | 2.0172 $e^{-5}$ |
|     |           | 1/40   | 1.451037 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ |
|     |           | 1/80   | 3.532982 $e^{-7}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ |
|     |           | 1/160  | 8.699997 $e^{-8}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ |
|     |           | 1/320  | 2.156752 $e^{-8}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ | 2.0704 $e^{-6}$ |
Table 4: $L_2$ - norm and maximum norm error behavior compared with grid size reduction when $\tau = 16h^2$.

| $b$ | $\alpha$ | $h$ | $\max_{0<\omega<h} ||z^n||_0$ | CO in $|| \cdot ||_0$ | $||z||_{C(\omega^h)}$ CO in $|| \cdot ||_{C(\omega^h)}$ |
|----|----|----|-----------------------------|----------------|----------------------------------|
| 1.0 | 0.9 | 1/4 | $1.216509e-3$ | $1.720403e-3$ | |
|     |     | 1/8 | $7.463500e-5$ | $4.0267$ | $1.055498e-4$ | $4.0267$ |
|     |     | 1/16| $4.635757e-6$ | $4.0089$ | $6.555951e-6$ | $4.0089$ |
|     |     | 1/32| $2.818584e-7$ | $4.0397$ | $3.986080e-7$ | $4.0397$ |
| 2.0 | 0.5 | 1/4 | $1.133742e-3$ | $1.603353e-3$ | |
|     |     | 1/8 | $6.956352e-5$ | $4.0266$ | $9.837767e-4$ | $4.0266$ |
|     |     | 1/16| $4.327824e-6$ | $4.0066$ | $6.120468e-6$ | $4.0066$ |
|     |     | 1/32| $2.702171e-7$ | $4.0014$ | $3.821448e-7$ | $4.0014$ |
| 3.0 | 0.1 | 1/4 | $1.086389e-3$ | $1.536387e-3$ | |
|     |     | 1/8 | $6.666005e-5$ | $4.0265$ | $9.427155e-4$ | $4.0265$ |
|     |     | 1/16| $4.147156e-6$ | $4.0066$ | $5.864965e-6$ | $4.0066$ |
|     |     | 1/32| $2.588975e-7$ | $4.0016$ | $3.661364e-7$ | $4.0016$ |
Table 5: $L_2$ - norm and maximum norm error behavior compared with the grid size reduction when $\tau = 16h^2$.

| $b$ | $\alpha$ | $\tau$ | $\max_{0 \leq n \leq M} \|z^n\|_0$ | CO in $\|\cdot\|_0$ | $\|z\|_{C(\omega_{h\tau})}$ | CO in $\|\cdot\|_{C(\omega_{h\tau})}$ |
|-----|-----|-----|-------|---------|---------|---------|
| 1.0 | 0.9 | 1/10 | $3.828076e-4$ | $5.413717e-4$ |
|     |     | 1/20 | $1.492480e-5$ | $1.362844e-4$ | $2.0163$ |
|     |     | 1/40 | $3.352703e-5$ | $3.327224e-5$ | $2.0079$ |
|     |     | 1/80 | $5.807158e-6$ | $8.212562e-6$ | $2.0184$ |
|     |     | 1/160| $1.450182e-6$ | $2.0015$ | $2.050867e-6$ | $2.0016$ |
|     |     | 1/320| $3.605780e-7$ | $2.0078$ | $5.099343e-7$ | $2.0078$ |
|     |     | 1/640| $9.010072e-8$ | $2.0007$ | $1.274216e-7$ | $2.0007$ |
|     |     | 1/1280| $2.241364e-8$ | $2.0071$ | $3.169767e-8$ | $2.0072$ |
|     |     | 1/2560| $5.591086e-9$ | $2.0031$ | $7.906995e-9$ | $2.0032$ |
| 2.0 | 0.5 | 1/10 | $1.342903e-4$ | $1.899152e-4$ |
|     |     | 1/20 | $3.253876e-5$ | $4.601676e-5$ | $2.0451$ |
|     |     | 1/40 | $8.015256e-6$ | $1.133528e-5$ | $2.0213$ |
|     |     | 1/80 | $1.935905e-6$ | $2.0497$ | $2.737783e-6$ | $2.0497$ |
|     |     | 1/160| $4.838928e-7$ | $2.0000$ | $6.844551e-7$ | $2.0000$ |
|     |     | 1/320| $1.196592e-7$ | $2.0160$ | $1.692237e-7$ | $2.0160$ |
|     |     | 1/640| $3.000207e-8$ | $1.9949$ | $4.245569e-8$ | $1.9949$ |
|     |     | 1/1280| $7.438279e-9$ | $2.0129$ | $1.051931e-8$ | $2.0129$ |
|     |     | 1/2560| $1.857629e-9$ | $2.0015$ | $2.627084e-9$ | $2.0015$ |
| 3.0 | 0.1 | 1/10 | $2.218725e-5$ | $3.137751e-5$ |
|     |     | 1/20 | $4.434359e-6$ | $6.271313e-6$ | $2.3229$ |
|     |     | 1/40 | $1.019302e-6$ | $1.441511e-6$ | $2.1211$ |
|     |     | 1/80 | $3.005858e-7$ | $1.7617$ | $4.250925e-7$ | $1.7617$ |
|     |     | 1/160| $7.429821e-8$ | $2.0163$ | $1.050735e-7$ | $2.0163$ |
|     |     | 1/320| $1.967234e-8$ | $1.9171$ | $2.782089e-8$ | $1.9171$ |
|     |     | 1/640| $4.756649e-9$ | $2.0481$ | $6.726918e-9$ | $2.0481$ |
|     |     | 1/1280| $1.243872e-9$ | $1.9351$ | $1.759102e-9$ | $1.9351$ |
|     |     | 1/2560| $3.110283e-10$ | $1.9997$ | $4.398604e-10$ | $1.9997$ |