A modelling approach for virtual development of wave based SHM systems

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Abstract. Within the last decade many SHM approaches utilizing elastic waves have been developed. In case of thin-walled structures especially guided waves - excited and sensed by surface-bonded or integrated piezoelectric elements - have shown a high potential. Despite of considerable efforts, some fundamental questions are still to be tackled for industrial application. One way of improving this situation and gaining a deeper insight into wave propagation and wave scattering phenomena is a detailed modelling of the systems. Very promising for this purpose seems to be the spectral element method.

This contribution presents a modelling approach for flat shell CFRP structures based on this methodology. Special attention is paid to the incorporation of delamination type damage. The delaminated area is modelled using separated upper and lower elements with conforming meshgrid. After introducing a contact formulation that preserves the high computational efficiency of the approach, the influence of contact on the local displacement field within the delaminated area is analysed. The simulation is validated by comparison to experimental data.

1. Introduction

The purpose of Structural Health Monitoring (SHM) is to evaluate the “state” of load-bearing structures. In the beginning mostly vibration-based techniques have been developed for global monitoring. On a more local level conventional non-destructive testing (NDT) methods based on ultrasonic waves have been used. A bit later wave-based techniques have been combined with permanently installed sensors and thereby adopted to the concept of SHM. Pioneering work in this area was performed in the group of Chang [1]. In case of thin-walled structures especially guided waves that are excited and sensed by surface-bonded or integrated piezoelectric elements (PZTs) have shown a high potential: they can travel over noteworthy distances and even small discontinuities lead to measurable wave scattering. The permanently installed PZTs enable a continuous monitoring.

Despite these advantages and considerable efforts, some fundamental questions are still to be tackled and the technology has to be improved for industrial application [2]. One major point of interest seems to be a more profound understanding of the complexity of the technology itself: The setup of such an SHM system requires a deep knowledge of wave propagation and scattering phenomena. In addition the development is usually very laborious and costly, because many parameters, e.g. excitation signals, damage evaluation algorithms and actuator/sensor distribution have to be optimized.

Hence there is a growing interest in efficient and accurate simulation tools to be able to perform virtual SHM system design instead of costly pretests. For the modelling of wave propagation phenomena a variety of methods is utilized: Among others the finite difference method (FDM) [3], the
pseudospectral method (PSM) [4], the finite element method (FEM) [5], the boundary element method (BEM) [6] and the local interaction simulation approach (LISA) [7] are used.

As most of the systems utilize a frequency range in between 30kHz and 250kHz conventional finite element modelling becomes computationally very expensive because the associated short wavelength demands for a dense meshgrid. A more promising method is the spectral element method (SEM) [8]. While modelling of guided waves requires a full 3D-model in theory, a 2D-approach can be used for thin shells up to a certain frequency-thickness product, see [9]. Typical applications for thin shells are composite materials. Different types of damages can occur in CFRP-plates and it is important to detect these failures. [10] and [11] select a Lamb wave approach to detect delaminations in composite materials. They show the influence of the delamination on wave propagation. A contact formulation between the delaminated layers was neglected.

Within this contribution the formulation of a flat shell spectral element is presented. For a material layup that is symmetrical with regard to the density, these elements exhibit a diagonal mass matrix leading to significant savings of memory and to a reduction of complexity of the time integration algorithm. Both in-plane and out-of-plane waves can be handled so that the fundamental $A_0$ and $S_0$ Lamb-modes and the $SH_0$ shear horizontal waves can be investigated. Regarding reinforced plastics the incorporation of material attenuation is quite relevant to be able to compare modelled and measured datasets. For that reason the modelling of material damping is discussed in detail.

The most important failure mechanisms in laminated structures are delamination and debonding. For that reason these damage cases are analyzed in detail. Special attention is paid to the effect of contact within the delaminated area. Moreover, some effects of the delamination depths within the cross-section and the shape of the delaminated area are analyzed. Finally, numerical results are compared to experimental data for the propagation of waves in a CFRP-plate including a delamination.

2. General theory of flat shell spectral elements

The Gauss-Lobatto-Legendre (GLL) spectral element discretization based upon quadrangular elements is essentially a higher order finite element technique. It combines the geometric flexibility of FEM with the accuracy and exponential convergence of spectral methods. The procedure is as follows: a mesh of $n_{el}$ non-overlapping elements $\Omega$ is defined on the domain $\Omega$. These elements are subsequently mapped individually on a reference element $\Omega^\text{ref} : \xi \in [-1,1] \times \eta \in [-1,1]$ using an invertible local mapping. On each element a set of GLL nodes is defined. Within the reference element these nodes are the $(N+1)$ roots of the polynomials $(1-\xi^2)L_{N-1}(\xi) = 0$ and $(1-\eta^2)L_{N-1}(\eta) = 0$, where $L_{N-1}$ denotes the Lobatto polynomial of order $N-1$. In contrast to classical lower order finite elements the distribution of nodes is not evenly spaced as it is shown in fig. 1 for elements with five and nine nodes per edge respectively.

Figure 1. Examples of nodal distribution within the reference element for 25 and 81 nodes.
The spectral shell element is based on the first order shear deformation theory (FSDT) with out-of-plane displacement $w$, independent rotations $\theta_x$ and $\theta_y$, and in-plane displacements $u$ and $v$, see fig. 2.

**Figure 2.** Illustration of the kinematics of a flat shell element.

The three independent displacements $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$ can be expressed as

$$\tilde{u}(x,y,z,t) = u(x,y,t) + \frac{\partial \zeta_0}{\partial x} \cdot w(x,y,z,t) + z \cdot \theta_y(x,y,t),$$  \hspace{1cm} (1)

$$\tilde{v}(x,y,z,t) = v(x,y,t) + \frac{\partial \zeta_0}{\partial y} \cdot w(x,y,z,t) - z \cdot \theta_x(x,y,t),$$ \hspace{1cm} (2)

$$\tilde{w}(x,y,z,t) = w(x,y,z,t).$$ \hspace{1cm} (3)

The second term of eq. (1) and (2) incorporates the curvature in form of a pre-deformation, see [12]. It has to be mentioned that this expression is valid for slightly curved shells that can be described in Cartesian coordinates only. The basic equations of motions resulting from this theory can be found in many textbooks, e.g. [13]. On the nodal base defined above, Lagrange interpolation polynomials can be used as shape functions, leading to an expression of the displacement field in the following form

$$\begin{bmatrix}
    w(\xi,\eta) \\
    \theta_x(\xi,\eta) \\
    \theta_y(\xi,\eta) \\
    u(\xi,\eta) \\
    v(\xi,\eta)
\end{bmatrix} = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} \Psi_{ij}^{(e)} \begin{bmatrix}
    \psi_i(\xi) \\
    \psi_j(\eta)
\end{bmatrix} \begin{bmatrix}
    \tilde{u}(\xi,\eta) \\
    \tilde{\theta}_x(\xi,\eta) \\
    \tilde{\theta}_y(\xi,\eta) \\
    \tilde{w}(\xi,\eta)
\end{bmatrix}.$$ \hspace{1cm} (4)

$\psi_1(\xi)$ denotes the $i$-th 1D Lagrange interpolation function, the hat indicates nodal degrees of freedom. An important property of these interpolation polynomials is the discrete orthogonality $\psi_i(\xi) = \delta_{ij}$, where $\delta_i$ denotes the Kronecker delta. Examples of selected shape functions can be seen in fig. 3. Utilizing this kind of shape functions based on the GLL-nodes leads to the highest interpolation accuracy [14]. For that reason only five to six nodes (depending on the degree of the interpolation polynomial) per shortest wavelength of the excited frequency range are necessary to capture the structural behaviour with the same accuracy as 15-30 nodes, which are needed using lower order FE [15].

**Figure 3.** Example of three typical shape functions of a spectral element with 25 nodes

3
Derivation of weak form and assembly of mass-, damping- and stiffness matrix follows the standard FE procedures. The stiffness matrix $\mathbf{K}$ can be formulated as

$$
\mathbf{K}^{(e)} = \int_{\Omega^e} [\mathbf{B}(x, y)]^T \mathbf{D} \mathbf{B}(x, y) \det(\mathbf{J}) \, d\Omega = \sum_{i=1}^{N_X} \sum_{j=1}^{N_Y} \overline{w}_{ij} \sum_{i=1}^{N_x} \overline{w}_x \sum_{j=1}^{N_y} \overline{w}_y \mathbf{B}(x_i, y_j)^T \mathbf{D} \mathbf{B}(x_i, y_j) \det(\mathbf{J}).
$$

(5)

where $\overline{w}$ denotes the integration weights. The strain-displacement matrix $\mathbf{B}$ and the material stiffness matrix $\mathbf{D}$ can be formulated according to

$$
\mathbf{B}_{ij} = \begin{pmatrix}
0 & 0 & -\Psi_{ij,x} & 0 & 0 \\
0 & \Psi_{ij,y} & 0 & 0 & 0 \\
0 & \Psi_{ij,x} & -\Psi_{ij,y} & 0 & 0 \\
\Psi_{ij,x} & 0 & 1 & 0 & 0 \\
\Psi_{ij,y} & -1 & 0 & 0 & 0 \\
\Psi_{ij,x} \cdot z_{0,x} & 0 & 0 & \Psi_{ij,x} & 0 \\
\Psi_{ij,y} \cdot z_{0,y} & 0 & 0 & 0 & \Psi_{ij,y} \\
\Psi_{ij,x} \cdot z_{0,y} + \Psi_{ij,y} \cdot z_{0,x} & 0 & 0 & \Psi_{ij,x} & \Psi_{ij,y}
\end{pmatrix}
$$

(6)

and

$$
\mathbf{D} = \begin{pmatrix}
D_{11} & D_{12} & D_{16} & 0 & 0 & B_{11} & B_{12} & B_{16} \\
D_{12} & D_{22} & D_{26} & 0 & 0 & B_{12} & B_{22} & B_{26} \\
D_{16} & D_{26} & D_{66} & 0 & 0 & B_{16} & B_{26} & B_{66} \\
0 & 0 & 0 & \kappa A_{55} & \kappa A_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & \kappa A_{45} & \kappa A_{44} & 0 & 0 & 0 \\
B_{11} & B_{12} & B_{16} & 0 & 0 & A_{11} & A_{12} & A_{16} \\
B_{12} & B_{22} & B_{26} & 0 & 0 & A_{12} & A_{22} & A_{26} \\
B_{16} & B_{26} & B_{66} & 0 & 0 & A_{16} & A_{26} & A_{66}
\end{pmatrix}.
$$

(7)

The notation $\cdot$ denotes the derivative with respect to $x$. $A_{ij}, B_{ij}$ and $D_{ij}$ are extensional-, coupling- and bending coefficients respectively and $\kappa$ is a shear correction factor. The mass matrix can be stated as

$$
\mathbf{M}^{(e)} = \int_{\Omega^e} [\Psi(x, y)]^T \mathbf{H} \Psi(x, y) \det(\mathbf{J}) \, d\Omega = \sum_{i=1}^{N_X} \sum_{j=1}^{N_Y} \overline{w}_{ij} \sum_{i=1}^{N_x} \overline{w}_x \sum_{j=1}^{N_y} \overline{w}_y [\Psi(x_i, y_j)]^T \mathbf{H} \Psi(x_i, y_j) \det(\mathbf{J}).
$$

(8)

where $\Psi$ is the matrix of shape functions and the matrix $\mathbf{H}$ contains the inertia terms:

$$
\mathbf{H} = \begin{pmatrix}
I_0 & 0 & 0 & 0 & 0 \\
0 & I_2 & 0 & 0 & -I_1 \\
0 & 0 & I_2 & I_1 & 0 \\
0 & 0 & I_1 & I_0 & 0 \\
0 & -I_1 & 0 & 0 & I_0
\end{pmatrix}.
$$

(9)

With the element thickness $h^e$ these inertia terms are defined as

$$
I_0 = \int_{h^e} \rho(z) \, dz, \quad I_1 = \int_{h^e} z \rho(z) \, dz \quad \text{and} \quad I_2 = \int_{h^e} z^2 \rho(z) \, dz.
$$

(10)

For the general case of anisotropic laminates, this formulation leads to an optimally concentrated, but non-diagonal mass matrix because of coupling terms between in-plane and rotational degrees of freedom. Fortunately, laminates with symmetrical layup are used in most applications and in this case
I1 vanishes. In that case the discrete orthogonality of the shape functions in conjunction with the application of the Gauss-Lobatto integration rule leads to a completely diagonal mass matrix.

In contrast to metallic materials, most CFRP or GFRP laminates exhibit a comparatively large amount of attenuation. To be able to incorporate this behaviour into the simulation approach, a material damping matrix is defined on element level. While proportional damping is assumed for reasons of simplicity in many papers, within this contribution another approach is used: Different damping coefficients are defined for in-plane and out of plane behaviour and for the fibre- and the transverse direction of each layer. For the k-th layer this leads to the following matrix structure:

\[
C_m^{(k)} = \begin{bmatrix}
\frac{C_{if}^{(k)} + C_{f}^{(k)}}{2} & 0 & 0 & 0 \\
0 & C_{if}^{(k)} & 0 & 0 \\
0 & 0 & C_{f}^{(k)} & 0 \\
0 & 0 & 0 & C_{im}^{(k)}
\end{bmatrix}.
\]

(11)

After transforming this matrix to the laminate coordinate system the damping matrix \(C_m^{(e)}\) is summed up from the contributions of each layer weighted by its thickness, leading to a material damping matrix \(C_m\). The element damping matrix can finally be constructed similar to the mass matrix:

\[
C^{(e)} = \int_{\Omega} \left[ \Psi(x, y) \right]^T C_m \Psi(x, y) \det(J) d\Omega = \sum_{i=1}^{N_y} \sum_{j=1}^{N_x} \bar{w}_i \bar{w}_j \left[ \Psi(x_i, y_j) \right]^T C_m \Psi(x_i, y_j) \det(J).
\]

(12)

After the assembly of all elements, this results in a linear system of 2nd-order differential equations that is very similar to a dynamic system resulting from conventional FE but with the advantageous property of diagonal mass- and damping matrices for laminates with symmetrical layup:

\[
M \ddot{q} + C \dot{q} + K q = F.
\]

(13)

A rotational degree of freedom (dof) about local z-axis \(\theta_z\) is not used to formulate the membrane behaviour of the element, see eq. (4). Though, to be able to transform the element matrices in space between local and global coordinates, it is meaningful to introduce this additional dof. It is possible to use elements with explicit \(\theta_z\)-dof but this would complicate the matters unnecessarily. Instead, small artificial values for the stiffness, mass and damping of this dof are defined following [16].

By using the central difference scheme, the resulting system of equations can be solved very rapidly. Moreover, by incorporating the electromechanical coupling of piezoelectric elements as actuators and sensors into the model, it is possible to simulate an SHM-system from a given input voltage up to the resulting sensor output voltage. The coupling, that is based on the fundamental piezoelectric equations, is not discussed here. The detailed equations can be found in [9].

3. Modelling of delamination-type damage

As mentioned above, the most frequent damage of layered fibre reinforced materials are delamination of plies and debonding. For that reason it is of great importance to incorporate an accurate and efficient delamination-model into the proposed spectral element framework.

Within the spectral element model the delaminated area is simulated by using separated upper and lower elements as sketched in fig. 4. The element nodes remain in the midplane of the plate, but by introducing offsets the effective midplane is shifted towards each of the two separated layers, see fig. 4 on the left. It is worth to note that these elements have no longer a symmetrical layup, so additional coupling between in-plane and out-of-plane modes may occur. This allows to capture the well-known effects of mode-conversion at delamination edges. In case of a modelled gap between the delaminated layers, the element nodes are also physically separated, see fig. 4 on the right.

If no gap between the layers is used and no contact conditions are formulated, the model can lead to interpenetration of upper and lower elements in the delaminated region.
Hence this is not the case for a real structure, a contact formulation is incorporated: within each
timestep during the time integration it is checked, if the condition $p = w_{up} - w_{lo} < 0$ is fulfilled. Index $up$
indicates a node in the upper element layer, $lo$ the corresponding node in the lower layer. If this
condition is satisfied, penetration occurs. To calculate contact forces and displacements, the so called
"forward increment Lagrange multiplier method" as introduced by [17] is applied and proceeds as

Influence of contact. If the contact condition is fulfilled for at least one nodal pair, the resulting
contact forces $F_{cont}$ are determined in the second step:

$$F_{cont} = \left\{ \Delta t^2 \cdot G^{t+\Delta t} M^{-1} \left[ G^{t+\Delta t} \right]^T \right\}^{-1} \cdot G^{t+\Delta t} \cdot q^{t+\Delta t}_{woc}.$$

Here $M$ is the mass matrix and $G^{t+\Delta t}$ is the contact displacement constraint matrix that can be
-calculated by expressing the contact condition in the form $G^{t+\Delta t} \cdot q^{t+\Delta t}_{woc} = 0$. Although the inverse of
-the mass matrix is used in equation (14), the calculation of contact forces can be performed rapidly,
because matrix $G$ consists of components of local out-of-plane displacement only. The corresponding
-entries of $M$ are guaranteed to build a diagonal matrix. Using $F_{cont}$ corrections of the displacement
-vector can be calculated in a third step:

$$q^{t+\Delta t}_{cor} = -\Delta t^2 \cdot M^{-1} \left[ G^{t+\Delta t} \right]^T F_{cont}.$$

Finally, the correct displacements at timestep $t+\Delta t$ are found as $q^{t+\Delta t} = q^{t+\Delta t}_{woc} + q^{t+\Delta t}_{cor}$. The advantage of
-this relatively straightforward kind of contact formulation is the possibility to incorporate the
-formulation efficiently into the central difference scheme as an explicit time integration algorithm.

4. Applications
Within this contribution, four different application examples are covered: First the model is validated
-by comparison to experimental data. Afterwards three numerical studies are presented: a comparison
-of scattered waves for two differently shaped delaminated areas, an analysis of the influence of contact
-on the local wavefield an on sensor output voltage and the effects of a debonded stiffener of a panel.

4.1. Validation - Comparison to experimental data
To validate the simulation approach, a CFRP plate with delamination is analysed. It is made of 16
equal layers resulting in a total thickness of 4.2mm with a stacking of $[0 \ 90 \ -45 \ 45 \ 0 \ 90 \ -45 \ 45]$. Nominal material parameters of the UD layers are the following: $E_1=155$GPa, $E_2=8.5$GPa, $G_{12}=G_{13}=G_{23}=4$GPa. The density is about 1600kg/m$^3$. 

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Figure 4. Modelling of delamination using separated upper and lower element layers

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Fig. 5a shows the plate of approximately 500mm x 500mm instrumented with nine PZTs. After several measurements in undamaged state, a low velocity impact with an energy of 15J is introduced between PZTs 5 and 2 using the impactor shown in fig. 5b. The resulting delaminated area is measured using an ultrasonic transducer and is additionally drawn into fig. 5a. This measurement also indicates a gap between the delaminated layers, so that contact is not considered within this study.

**Figure 5.** CFRP-plate with delamination (a) and impactor (b).

A comparison of modelled and measured sensor data for an excitation of P5 with centre frequency 60kHz and 120kHz is shown in fig. 6. A satisfactory agreement for the undamaged case and the effects of delamination are demonstrated. The simulation shows the same behaviour as the measured data, which is a very small change in the $S_0$-mode, a larger time-shift in the $A_0$-mode for excitation with 60kHz and a partly annihilation of the $A_0$-mode for excitation with 120kHz.

4.2. **Comparison of scattered waves from differently shaped delaminations**

One possible application of the proposed SEM modelling strategy is the detailed analysis of delamination with various shapes. In order to suppress effects from anisotropy, a simulation model of a plate of approximately 400mm x 400mm made from eight isotropic layers of 0.5mm thickness is

**Figure 6.** Sensor output voltage for P2 for an excitation of P5 with a five cycle tone-burst of centre frequency 60kHz (a) and 120kHz (b).
considered as an example. A delamination is introduced at pos. C (see fig. 7b) on the one hand within the dark-grey square in the middle of the plate (fig. 7a) and on the other hand within the smaller circle in the middle of this area (see fig. 7c).

Figure 7. Simulation model to analyse delaminations of different shape

In fig. 8 snapshots of the propagating transverse waves are shown for the same time instance 95µs. It can be clearly seen, that the shape of the delamination has a significant influence on the scattered waves: while the rectangular delamination leads to comparatively strong scattering, the amplitude of the scattered waves at the circular delamination is lower. Moreover the “dead zones” behind the delamination are qualitatively different: parallel to the delamination edges (see fig. 8a) versus radial towards the delamination centre (fig. 8b).

Figure 8. Snapshots of propagating waves after 95µs, square (a) and circular delamination (b).
4.3. Influence of contact within the delaminated area
The same spectral element model is used to analyze the influence of the contact formulation on propagating waves. In fig. 9 snapshots of the out-of-plane displacement of the central part of this plate including the rectangular delaminated area are shown. In fig. 9a the contact formulation is applied, in fig. 9b the same time instance is shown without taking into account the contact conditions, which leads to interpenetration of the upper and the lower layer within the delaminated area. Comparison of both figures leads to the fact, that the contact condition has a large influence on the displacement field within the delaminated area, but only a small effect on the waves within the surrounding area of the healthy structure at least at this displacement level.

![Figure 9](image)

**Figure 9.** Delaminated plate with contact (a) and without contact (b). Upper layer of delaminated area is displayed in red, lower layer in blue and healthy structure in green.

4.4. Debonding of a stringer of a stiffened panel
As a more advanced use case of the presented approach, the numerical model of a stiffened panel is considered. The model consists of a base plate of thickness 2mm with three stiffeners. Isotropic material properties of aluminium are used. The stiffener in the middle is assumed to be T-shaped and perfectly bonded to the structure in the undamaged case, see fig. 10a. This results in an increased thickness of 3mm for some elements on both sides of the stiffener, see the additional lines in fig. 10. The red coloured PZT is actuated using a three cycle tone burst of 75kHz.

![Figure 10](image)

**Figure 10.** z-displacement of waves in a stiffened panel, (a) undamaged and (b) with debonded stiffener. Additional wave packages passing the stiffener at the debonded area are noticed.
A stiffener debonding - a serious damage case in real aircraft structures - is simulated in a small area close to the middle of the stiffener. In fig. 10 snapshots of the $z$-displacement of the induced waves at $t=0.17$ ms for the undamaged (fig 10a) and debonded cases (fig. 10b) are shown.

While the main energy of the out-of-plane wave remains inside the area between two stiffeners, comparison of both snapshots clearly indicates several additional wave packages that can pass the stiffener in the debonded area. The amplitude of these wave packages is relatively large in comparison to the waves passing the stiffener in the undamaged case. The additional energy can be sensed in the adjacent area indicating this type of damage.

5. Conclusions

Within this contribution, a modelling approach for virtual development of wave based SHM systems has been presented. The developed flat shell spectral element that is based on FSDT offers several advantageous features compared to conventional finite elements regarding accuracy and numerical efficiency. An approach to implement material damping is presented to be able to cover the strong attenuation of composite materials. Special attention is paid to an efficient contact algorithm to be able to simulate the effect of contact within the delaminated area.

Several application examples are given including an experimental validation and three numerical studies including the effect of contact and the debonding of stringer of a stiffened panel to demonstrate possible applications.

The proposed spectral element modelling strategy seems very promising for the early development stage of SHM-systems: the simulation allows to change system parameters as excitation signals, and the location of PZT actuators and sensors easily. By running a large number of simulations a major part of the system development process can be performed virtually, for example by generating a database for the development of advanced damage diagnosis algorithms. This might improve the technology readiness of wave-based SHM-systems in the future.

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