A PROJECTIVE-TO-CONFORMAL FEFFERMAN-TYPE CONSTRUCTION

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ABSTRACT. We study a Fefferman-type construction based on the inclusion of Lie groups SL($n+1$) into Spin($n+1$, $n+1$). The construction associates a split-signature ($n$, $n$)-conformal spin structure to a projective structure of dimension $n$. We prove the existence of a canonical pure twistor spinor and a light-like conformal Killing field on the constructed conformal space. We obtain a complete characterisation of the constructed conformal spaces in terms of these solutions to overdetermined equations and an integrability condition on the Weyl curvature. The Fefferman-type construction presented here can be understood as an alternative approach to study a conformal version of classical Patterson–Walker metrics as discussed in recent works by Dunajski–Tod and by the authors. The present work therefore gives a complete exposition of conformal Patterson–Walker metrics from the viewpoint of parabolic geometry.

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1. INTRODUCTION

In conformal geometry the geometric structure is given by an equivalence class of pseudo-Riemannian metrics: two metrics $g$ and $\hat{g}$ are considered to be equivalent if they differ by a positive smooth rescaling, $\hat{g} = e^{2f} g$. In projective geometry the geometric structure is given by an equivalence class of torsion-free affine connections: two connections $D$ and $\hat{D}$ are considered...
as equivalent if they share the same geodesics (as unparametrised curves). While conformal and projective structures both determine a corresponding class of affine connections, neither of them induces a single distinguished connection on the tangent bundle. Instead, both structures have canonically associated Cartan connections that govern the respective geometries and encode prolonged geometric data of the respective structures. It is therefore often useful when studying projective and conformal structures to work in the framework of Cartan geometries.

The present paper investigates a geometric construction that relates projective and conformal structures which is based on an inclusion of the respective Cartan structure groups $\text{SL}(n+1) \hookrightarrow \text{Spin}(n+1, n+1)$. The construction can be understood as an instance of a Fefferman-type construction as formalised in [ˇCap06, ˇCS09]; it produces an interesting conformal class of split-signature metrics coming from a projective class of connections. After reviewing the basic facts on projective and conformal structures as parabolic geometries in Section 2, the construction of a conformal from a projective structure and its algebraic and geometric features are described in detail in Section 3. In Section 4 we discuss in addition a closely related Fefferman-type construction, which starts from a Lagrangean contact structure, and which can be regarded as an intermediate step in the projective-to-conformal construction in the lowest dimensional case $n = 2$.

In the general situation $n \geq 3$ the Fefferman-type construction turns out to be non-normal, and the present work is, to our knowledge, the first comprehensive treatment of such a non-normal construction. In particular, we are able to obtain, also in this case, a local characterisation of the resulting conformal structures: In Section 5 we identify the special properties of split-signature conformal structures that guarantee local equivalence to a conformal structure which is obtained from a projective structure via the geometric construction. Our main characterisation results are then Theorem 5.17 and Theorem 5.18. We remark that techniques provided and developed in this section should have considerable scope for applications for other non-normal Fefferman-type constructions. A particularly interesting case of this sort consists of (not necessarily integrable) Cauchy-Riemann (CR)-structures - cf. [ˇCG10]) for a discussion of the integrable case.

Another version of these characterisation theorems is obtained in Theorem 6.7 in Section 6. This alternative, equivalent characterisation was also obtained by the authors in [HSSTZ16] by different means: The approach of [HSSTZ16] was based on spin calculus (see [PR86, Tag16]) and direct computations. The main advantage of the present paper and its Cartan geometric approach is that the properties of the induced conformal spaces can be obtained in an algebraic manner. Moreover, this description explains relations between solutions of BGG-equations and special properties of the induced conformal structures: Several such relationships were obtained in [HSSTZ16], and a comprehensive treatment of further relations is subject to forthcoming work. For this, the explicit formula for the difference between the induced and the normal Cartan connections obtained in Theorem 6.5 will be fundamental.
The projective-to-conformal construction studied in this paper should be understood as a generalization of the classical Patterson–Walker–Riemann extensions of affine spaces by E.M. Patterson and A. G. Walker (see [PW52, Wal54]). One of the authors first motivations to study this construction was [DT10], where the Patterson–Walker construction was generalized to a projectively invariant setting in dimension $n = 2$. Also, in [NS03] a construction of conformal structures of signature $(2, 2)$ using Cartan connections was presented. A generalisation of this approach to higher dimensions was obtained in [Nur12]. In Section 7 we show that the different approaches are in fact all equivalent to the Cartan-geometric construction presented here and we make the relationships precise. Moreover, we explicitly identify special metrics in the conformal class as Patterson–Walker metrics.

Acknowledgements. The authors express special thanks to Maciej Dunajski for motivating the study of this construction and for a number of enlightening discussions on this and adjacent topics. KS would also like to thank Paweł Nurowski for drawing her interest to the subject and for many useful conversations. MH gratefully acknowledges support by project P23244-N13 of the Austrian Science Fund (FWF) and by Forschungsnetzwerk Ost’ of the University of Greifswald. KS gratefully acknowledges support from grant J3071-N13 of the Austrian Science Fund (FWF). JS was supported by the Czech science foundation (GAČR) under grant P201/12/G028. AT-C was funded by GAČR post-doctoral grant GP14-27885P. VŽ was supported by GAČR grant GA201/08/0397.

2. Projective and conformal parabolic geometries

The standard reference for the background material on Cartan and parabolic geometries presented here is [CS09].

2.1. Cartan and parabolic geometries. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $P \subseteq G$ a closed subgroup with Lie algebra $\mathfrak{p}$. A Cartan geometry $(G, \omega)$ of type $(G, P)$ over a smooth manifold $M$ consists of a $P$-principal bundle $G \to M$ together with a Cartan connection $\omega \in \Omega^1(G, \mathfrak{g})$, i.e., a $\mathfrak{g}$-valued 1-form on $G$ that (i) is $P$-equivariant, (ii) maps each fundamental vector field $\zeta_X$ to its generator $X \in \mathfrak{p}$, and (iii) defines a linear isomorphism $\omega(u) : T_uG \to \mathfrak{g}$ for each $u \in G$. The canonical principal bundle $G \to G/P$ endowed with the Maurer–Cartan form constitutes the homogeneous model for Cartan geometries of type $(G, P)$.

The curvature of a Cartan connection $\omega$ is the 2-form $K \in \Omega^2(G, \mathfrak{g})$ defined as

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)],$$

for $\xi, \eta \in \mathfrak{x}(G)$. Since the curvature is strictly horizontal, it is equivalently encoded in the curvature function, the $P$-equivariant mapping $\kappa : G \to \Lambda^2(\mathfrak{g/}\mathfrak{p})^* \otimes \mathfrak{g}$ given by

$$\kappa(u)(X + \mathfrak{p}, Y + \mathfrak{p}) := K(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)).$$

(1)

The curvature is a complete obstruction to a local equivalence with the homogeneous model. If the image of $\kappa$ is contained in $\Lambda^2(\mathfrak{g/}\mathfrak{p})^* \otimes \mathfrak{p}$ the Cartan geometry is called torsion-free.
A parabolic geometry is a Cartan geometry of type \((G,P)\), where \(G\) is a semi-simple Lie group and \(P \subseteq G\) is a parabolic subgroup; a subalgebra \(p \subseteq g\) is parabolic if and only if its maximal nilpotent ideal, called nilradical \(p_+\), coincides with the orthogonal complement \(p^\perp\) of \(p \subseteq g\) with respect to the Killing form. In particular, this yields an isomorphism \((g/p)^* \cong p_+\) of \(P\)-modules. The quotient \(g_0 = p/p_+\) is called the Levi factor; it is reductive and decomposes as \(g_0 = g_0^s \oplus z(g_0)\) into a semi-simple part \(g_0^s = \{g_0, g_0\}\) and the center \(z(g_0)\). An identification of \(g_0\) with a subalgebra in \(p\) yields a grading \(g = g_- \oplus g_0 \oplus p_+\). The respective Lie groups are \(G_0^{ss} \subseteq G_0 \subseteq P\) and \(P_+ \subseteq P\) so that \(P = G_0 \ltimes P_+\) and \(P_+ = \exp(p_+)\).

In the case that the nilradical \(p_+\) is abelian, the representation of \(g_0\) on \(p_+\) is irreducible. However if \(p_+\) is not abelian, the previous grading of \(g\) may be refined; if \(k\) is the depth of the finest possible grading the parabolic geometry is called \([k]\)-graded. The grading of \(g\) induces a grading on \(\Lambda^2 p_+ \otimes g \cong \Lambda^2 (g/p)^* \otimes g\). A parabolic geometry is called regular, if the curvature function \(\kappa\) takes values only in the components of positive homogeneity. In particular, any torsion-free or \([1]\)-graded parabolic geometry is regular.

Given a \(g\)-module \(V\), there is a natural \(p\)-equivariant map, the \textit{Kostant co-differential} \([\text{Kos61}]\),
\[
\partial^* : \Lambda^k(g/p)^* \otimes V \to \Lambda^{k-1}(g/p)^* \otimes V,
\]
defining the Lie algebra homology of \(p_+\) with values in \(V\); see e.g. \([\text{CS09}]\) Section 3.3.1 for the explicit form. For \(V = g\), this gives rise to a natural normalisation condition: parabolic geometries satisfying \(\partial^*(\kappa) = 0\) are called normal. The harmonic curvature \(\kappa_H\) of a normal parabolic geometry is the image of \(\kappa\) under the projection \(\ker \partial^* \to \ker \partial^*/\im \partial^*\). For regular and normal parabolic geometries, the entire curvature \(\kappa\) is completely determined just by \(\kappa_H\).

2.2. Tractor bundles and connections. Every Cartan connection \(\omega\) on \(\mathcal{G} \to M\) naturally extends to a principal connection \(\hat{\omega}\) on the \(G\)-principal bundle \(\hat{\mathcal{G}} := \mathcal{G} \times_P G \to M\), which further induces a linear connection \(\nabla\mathcal{V}\) on any associated vector bundle \(\mathcal{V} := \mathcal{G} \times_P V = \hat{\mathcal{G}} \times_G V\) for a \(G\)-representation \(V\). Bundles and connections arising in this way are called tractor bundles and tractor connections. The tractor connections induced by normal Cartan connections are called normal tractor connections.

In particular, for \(V = g\) and the adjoint representation we obtain the \textit{adjoint tractor bundle} \(\mathcal{AM} := \mathcal{G} \times_P g\). The canonical projection \(g \to g/p\) and the identification \(TM \cong \mathcal{G} \times_P (g/p)\) yield a bundle projection \(\Pi : \mathcal{AM} \to TM\); the inclusion \(p_+ \subseteq g\) and the identification \(p_+ \cong (g/p)^*\) yield a bundle inclusion \(T^* \mathcal{M} \to \mathcal{AM}\). This allows us to interpret the Cartan curvature \(\kappa\) from \((\Pi)\) as a 2-form \(\Omega\) on \(M\) with values in \(\mathcal{AM}\). For a general \(G\)-representation \(V\), its derivative \(g \times V \to V\) induces a natural bundle map \(\bullet : \mathcal{AM} \times \mathcal{V} \to \mathcal{V}\). The relation between the curvature \(\Omega \in \Omega^2(M, \mathcal{AM})\) of the Cartan connection \(\omega\) and the curvature \(\Omega^\mathcal{V} \in \Omega^2(M, \End(\mathcal{V}))\) of the induced tractor connection \(\nabla\mathcal{V}\) is given by
\[
\Omega^\mathcal{V}(\xi, \eta)(s) = \Omega(\xi, \eta) \bullet s,
\]
for any \(\xi, \eta \in \mathfrak{X}(M)\) and \(s \in \Gamma(\mathcal{V})\).
The holonomy group of the principal connection \( \hat{\omega} \) is by definition the holonomy of the Cartan connection \( \omega \), i.e.
\[
\text{Hol}(\omega) := \text{Hol}(\hat{\omega}) \subseteq G.
\]
The holonomy group of a tractor connection \( \nabla^V \) coincides with the image of \( \text{Hol}(\hat{\omega}) \) under the defining representation \( G \to GL(V) \). Parallel sections of tractor bundles always lead to reductions of holonomies.

2.3. BGG-operators and first BGG-equations. In [CSS01], and later in a simplified manner in [CD01], it was shown that for a tractor bundle \( \mathcal{V} = \mathcal{G} \times_P V \) one can associate a sequence of differential operators, which are intrinsic to the given parabolic geometry \((\mathcal{G}, \omega)\).

The operators \( \Theta^V_k \) are the BGG-operators and they operate between the sub-quotients \( \mathcal{H}_k = \ker \partial^*/\text{im} \partial^* \) of the bundles of \( V \)-valued \( k \)-forms, where \( \partial^*: \Lambda^k T^*M \otimes V \to \Lambda^{k-1} T^*M \otimes V \) denotes the bundle map induced by the Kostant co-differential [2]. A key ingredient in their construction is the covariant exterior derivative \( d^V: \Omega^k(M, \mathcal{V}) \to \Omega^{k+1}(M, \mathcal{V}) \) given by the tractor connection \( \nabla^V \). We remark that \( \Theta^V_k \) form a complex if and only if the parabolic geometry \((\mathcal{G}, \omega)\) is locally flat.

The first BGG-operator \( \Theta^V_0 : \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{H}_1) \) is constructed as follows. The bundle \( \mathcal{H}_0 \) is simply the quotient \( \mathcal{V}/\mathcal{V}' \), where \( \mathcal{V}' \subseteq \mathcal{V} \) is the sub-bundle corresponding to the largest \( P \)-invariant filtration component in the \( G \)-representation \( V \). It turns out, there is a distinguished differential operator that splits the projection \( \Pi_0: \mathcal{V} \to \mathcal{H}_0 \), namely, the splitting operator, which is the unique map \( L_0^V: \Gamma(\mathcal{H}_0) \to \Gamma(\mathcal{V}) \) satisfying
\[
\Pi_0(L_0^V(\sigma)) = \sigma \quad \text{and} \quad \partial^*(d^V L_0^V(\sigma)) = 0, \quad \text{for any } \sigma \in \Gamma(\mathcal{H}_0). \quad (4)
\]
The latter condition allows to define the first BGG-operator by
\[
\Theta^V_0 := \Pi_1 \circ d^V \circ L_0^V,
\]
where \( \Pi_1 : \ker \partial^* \to \Gamma(\mathcal{H}_1) \). The first BGG-operator defines an overdetermined system of differential equations on \( \sigma \in \Gamma(\mathcal{H}_0) \), \( \Theta^V_0(\sigma) = 0 \), which is termed the first BGG-equation. For the projective and conformal structures we discuss below, a number of interesting geometric equations are encoded as first BGG-equations.

2.4. Weyl structures and connections. A Weyl structure of a parabolic geometry \((\mathcal{G}, \omega)\) over \( M \) is a reduction of the \( P \)-principal bundle \( \mathcal{G} \to M \) to the Levi subgroup \( G_0 \subseteq P \); the corresponding \( G_0 \)-bundle is denoted by \( \mathcal{G}_0 \to M \) and the embedding by \( j : \mathcal{G}_0 \to \mathcal{G} \). The class of all Weyl structures, which are parametrised by one-forms on \( M \), includes a particularly important subclass of exact Weyl structures, which are parametrised by functions on \( M \): For \([1]\)-graded parabolic geometries, these correspond to further reductions of \( \mathcal{G}_0 \to M \) just to the semi-simple part \( G_0^{ss} \) of \( G_0 \) or, equivalently, to sections of the principal \( \mathbb{R}_+ \)-bundle \( \mathcal{G}_0/G_0^{ss} \to M \). The latter bundle is called the bundle of scales and its sections are the scales.
For a Weyl structure $j : \mathcal{G}_0 \to \mathcal{G}$, the pullback of the Cartan connection $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ may be decomposed according to the grading $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+$:

$$j^* \omega = j^* \omega_- + j^* \omega_0 + j^* \omega_+.$$  

The $\mathfrak{g}_0$-part of $j^* \omega$ is a principal connection on the $G_0$-bundle $\mathcal{G}_0 \to M$; it induces connections on all associated bundles, which are called (exact) Weyl connections. In particular, any exact Weyl connection preserves the Weyl connections $\mathfrak{g}$.

The $\mathfrak{p}_+$-part of $j^* \omega$ is the so called Schouten tensor denoted by $P$; it is interpreted as a one-form on $M$ with values in $T^*M$, i.e. as a section of $T^*M \otimes T^*M$.

Likewise, we may decompose the pullback of the curvature. For normal $[1]$-graded parabolic geometries we employ the following notation: The $\mathfrak{g}_-$-part of $j^* K$ is the torsion, $T$, of the Weyl connection $D$, the $\mathfrak{g}_0$- and $\mathfrak{p}_+$-parts of $j^* K$ are the so called Weyl curvature, $W$, and Cotton tensor, $Y$, respectively. In the previous terms, they may be expressed as

$$W = R + \partial P, \quad Y = d^D P,$$

where $\partial$ is the bundle map induced by the standard differential that is adjoint to $[\xi]$, and $d^D$ is the covariant exterior derivative determined by $D$. Note that first non-zero tensor in the sequence $(T, W, Y)$ is independent of the choice of Weyl structure as it corresponds to the lowest non-zero homogeneous component of the harmonic curvature $\kappa_H$. In the case of projective or conformal structures, the torsion $T$ of any Weyl connection vanishes identically.

2.5. Notations and conventions. In order to distinguish various objects related to projective and conformal structures, the symbols referring to conformal data will always be endowed with tildes. To write down explicit formulae, we employ abstract index notation, cf. e.g. [PR84]. Furthermore, we will use different types of indices for projective and conformal manifolds. E.g., on a projective manifold $M$ we write $E_A := T^*M$, $E^A := TM$, and multiple indices denote tensor products, as in $E_A^B := T^*M \otimes TM$. Indices between squared brackets are skew, as in $E_{[AB]} := \Lambda^2 T^*M$, and indices between round brackets are symmetric, as in $E^{(AB)} := S^2 TM$. Analogously, on a conformal manifold $\tilde{M}$ we write $\tilde{E}_a := T^*\tilde{M}$, $\tilde{E}^a := T\tilde{M}$ etc.

By $E(w)$ and $\tilde{E}[w]$ we denote the density bundle over $M$ and $\tilde{M}$, respectively, with specific standard and structure-adapted normalisation which will be introduced later. Tensor products with another natural bundles are denoted as $E_A(w) := E_A \otimes E(w)$, $E_{[ab]}[w] := E_{[ab]} \otimes E[w]$, and the like.

2.6. Projective structures. Let $M$ be a smooth manifold of dimension $n \geq 2$. A projective structure on $M$ is given by a class, $\mathfrak{p}$, of torsion-free projectively equivalent affine connections: two connections $D$ and $\tilde{D}$ are projectively equivalent if they have the same geodesics as unparametrised curves. This is the case if and only if there is a one-form $Y_A \in \Gamma(E_A)$ such that, for all $\xi^A \in \Gamma(E^A)$,

$$\hat{D}_A \xi^B = D_A \xi^B + T_A \xi^B + \gamma_{B} \delta \xi^B,$$  

(6)
where \( \delta_{AB} \) is the Kronecker symbol for the identity map on the tangent bundle.

An oriented projective structure \((M, p)\), which is a projective structure \( p \) on an oriented manifold \( M \), is equivalently encoded as a normal parabolic geometry of type \((G, P)\), where \( G = \text{SL}(n+1) \) and \( P = \text{GL}(n) \rtimes \mathbb{R}^{n*} \) is the stabiliser of a ray in the standard representation \( \mathbb{R}^{n+1} \). The homogeneous model \( G/P \) is the standard projective sphere \( S^n \).

Affine connections from the projective class \( p \) are precisely the Weyl connections of the corresponding parabolic geometry. Exact Weyl connections are those \( D \in p \) which preserve a volume form — these are also known as special affine connections. In particular, a choice of \( D \in p \) reduces the structure group to \( G_0 = \text{GL}(n) \), if \( D \) is special, the structure group is further reduced to \( G_{ss}^0 = \text{SL}(n) \).

For later purposes we now give explicit expressions of the main curvature quantities, cf. e.g. \cite{Eas08, BEG94}. For \( D \in p \), the Schouten tensor is determined by the Ricci curvature of \( D \); if \( D \) is special, then the Schouten tensor is

\[
P_{AB} = \frac{1}{n-1} R_{PA}^B,
\]

in particular, it is symmetric. The projective Weyl curvature and the Cotton tensor are

\[
W_{AB}^C \equiv R_{AB}^C + P_{AB} \delta^C_B - P_{BC} \delta^A_B,
\]

\[
Y_{CAB} = 2 D_{[A} P_{B]C}.
\]

Note that for \( n = 2 \), the Weyl curvature vanishes identically, hence the only obstruction to the local flatness is the Cotton tensor (in particular, it does not depend on the choice of Weyl structure).

Henceforth, we use a suitable normalisation of densities so that the line bundle associated to the canonical one-dimensional representation of \( P \) has projective weight \(-1\). Hence, comparing with the usual notation, the density bundle of projective weight \( w \), denoted by \( \mathcal{E}(w) \), is just the bundle of ordinary \((\frac{n-1}{n})\)-densities. As an associated bundle to \( G \to M \), \( \mathcal{E}(w) \) corresponds to the 1-dimensional representation of \( P \) given by

\[
\text{GL}(n) \times \mathbb{R}^{n*} \to \mathbb{R}_+, \quad (A, X) \mapsto |\det(A)|^w.
\]

The projective standard tractor bundle is the tractor bundle associated to the standard representation of \( G = \text{SL}(n+1) \). The projective dual standard tractor bundle is denoted by \( T^* \), i.e. \( T^* := G \times_p \mathbb{R}^{n+1*} \). With respect to a choice of \( D \in p \)

\[
T^* = \begin{pmatrix} \mathcal{E}_A(1) \\ \mathcal{E}(1) \end{pmatrix}.
\]

The projective standard tractor connection is given by

\[
\nabla^T_{\varphi} \left( \frac{\varphi_A}{\sigma} \right) = \begin{pmatrix} D_C \varphi_A + P_C A \sigma \\ D_C \sigma - \varphi_C \end{pmatrix}.
\]
2.7. Conformal spin structures and tractor formulas. Let $\tilde{M}$ be a smooth manifold of dimension $2n \geq 4$. A conformal structure of signature $(n, n)$ on $\tilde{M}$ is given by a class, $\mathcal{C}$, of conformally equivalent pseudo-Riemannian metrics of signature $(n, n)$: two metrics $g$ and $\tilde{g}$ are conformally equivalent if $\tilde{g} = f^2 g$ for a nowhere-vanishing smooth function $f$ on $\tilde{M}$. It may be equivalently described as a reduction of the principal fibre bundle over $\tilde{M}$ to the structure group $\text{CO}(n, n) = \mathbb{R}_+ \times \text{SO}(n, n)$. An oriented conformal structure of signature $(n, n)$ is a conformal structure of signature $(n, n)$ together with fixed orientations both in time-like and space-like directions, equivalently, a reduction of the principal fibre bundle to the group $\text{CO}_o(n, n) = \mathbb{R}_+ \times \text{SO}_o(n, n)$, the connected component of the identity. A conformal spin structure $(\tilde{M}, \mathcal{C})$ of signature $(n, n)$ is a reduction of the principal fibre bundle over $\tilde{M}$ to the structure group $\text{CSpin}(n, n) = \mathbb{R}_+ \times \text{Spin}(n, n)$, the 2-fold covering of CO$_o(n, n)$.

A conformal spin structure of signature $(n, n)$ is equivalently encoded as a normal parabolic geometry of type $(\tilde{G}, \tilde{\mathcal{P}})$, where $\tilde{G} = \text{Spin}(n+1, n+1)$ and $\tilde{\mathcal{P}} = \text{CSpin}(n, n) \ltimes \mathbb{R}^{n,n,*}$ is the stabiliser of an isotropic ray in the standard representation $\mathbb{R}^{n+1,n+1}$. The homogeneous model $\tilde{G}/\tilde{\mathcal{P}}$ is the product $\mathbb{S}^n \times \mathbb{S}^n$ of two standard spheres (with round metrics of opposite signs).

A general Weyl connection is a torsion-free affine connection $\tilde{\nabla}$ such that $\tilde{\nabla} \tilde{g} \in \mathcal{C}$ for any $g \in \mathcal{C}$. Any choice of $g \in \mathcal{C}$ yields the canonical torsion-free Levi-Civita connection $\tilde{\nabla}^g = 0$, which is an exact Weyl connection. A choice of Weyl connection reduces the structure group to $\tilde{G}_0 = \text{CSpin}(n, n)$.

If the Weyl connection is exact, i.e., if it is a Levi-Civita connection of a metric from the conformal class, the structure group is further reduced to $\tilde{G}_{ss}^ss = \text{Spin}(n, n)$.

Now we briefly introduce the main curvature quantities of conformal structures, cf. e.g. [Eas96]. For $g \in \mathcal{C}$, the Schouten tensor,

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(g) = \frac{1}{2n-2}(\tilde{\text{Ric}}(g) - \frac{\tilde{\text{Sc}}(g)}{2(2n-1)} g)$$

is a trace modification of the Ricci curvature $\tilde{\text{Ric}}(g)$ by a multiple of the scalar curvature $\tilde{\text{Sc}}(g)$. The full trace of the Schouten tensor is denoted $\tilde{J} = g^{pq} \tilde{\mathcal{P}}_{pq}$. The conformal Weyl curvature and the Cotton tensors are

$$\tilde{W}^{ab} = \frac{\tilde{R}^{ab} - 2\delta^{[a}_{[c} \tilde{\mathcal{P}}^{b]}_{d]} + 2g_{[a]d} \tilde{\mathcal{P}}^{c}_{d]} c}{g^{ab}},$$

$$\tilde{Y}^{abc} = 2\tilde{D}_{[a} \tilde{\mathcal{P}}_{b]c}.$$
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representation of $\tilde{P}$ given by

$$(\mathbb{R}_+ \times \text{Spin}(n, n)) \times \mathbb{R}^{2n*} \to \mathbb{R}_+, \quad (a, A, Z) \mapsto a^{-w}.$$  (11)

In particular, the conformal structure may be seen as a section of $\tilde{E}_{ab}[2]$, which is called the conformal metric and denoted by $g_{ab}$. Contracting with $g_{ab}$ we have an identification $\tilde{E}_a \sim = \tilde{E}_a[2]$.

The spin bundles corresponding to the irreducible spin representations of $\text{Spin}(n, n)$ are denoted by $\tilde{S}_+$ and $\tilde{S}_-$, and $\tilde{S} = \tilde{S}_+ \oplus \tilde{S}_-$. We employ the weighted conformal gamma matrix $\gamma \in \Gamma(\tilde{E}_a \otimes (\text{End}\tilde{S})[1])$ such that $\gamma_p \gamma_q + \gamma_q \gamma_p = -2g_{pq}$. For $\xi \in \mathfrak{X}(\tilde{M})$ and $\chi \in \Gamma(\tilde{S})$, the Clifford multiplication of $\xi$ on $\chi$ is then written as $\xi \cdot \chi = \xi^p \gamma_p \chi$.

At this point we introduce the basic notation for the tractor bundles for conformal geometries.

The conformal standard tractor bundle is the associated bundle $\tilde{T} := \tilde{G} \times \tilde{P} \mathbb{R}^{n+1,n+1}$ with respect to the standard representation. With respect to a choice of metric $g \in \mathfrak{c}$, $\tilde{T}$ carries the conformal standard tractor connection $\tilde{\nabla}^{\tilde{T}}$, which preserves $h$ and is given by

$$\tilde{\nabla}^{\tilde{T}} \left( \begin{array}{c} \rho \\ \varphi_a \\ \sigma \end{array} \right) = \left( \begin{array}{c} \tilde{D}_c \rho - \tilde{P}_c^b \varphi_b \\ \tilde{D}_c \varphi_a + \sigma \tilde{P}_{ca} + \rho g_{ca} \\ \tilde{D}_c \sigma - \varphi_c \end{array} \right).$$  (13)

The curvature $\tilde{\Omega}_{ab} \in \Omega^2(\tilde{M}, \text{End}(\tilde{T}))$ of $\tilde{\nabla}^{\tilde{T}}$ is

$$\tilde{\Omega}_{ab} = \left( \begin{array}{ccc} 0 & -\tilde{Y}_{ab} & 0 \\ 0 & \tilde{W}^{c}_{ab} & \tilde{Y}_{ab}^c \\ 0 & 0 & 0 \end{array} \right),$$  (14)

and the BGG-splitting operator $L^{\tilde{T}}_0 : \Gamma(\tilde{E}_a[1]) \to \Gamma(\tilde{T})$ is given by

$$\sigma \mapsto \left( \begin{array}{c} \frac{1}{2n}(-\tilde{D}^p \tilde{D}_p - \tilde{J})\sigma \\ \tilde{D}_a \sigma \\ \sigma \end{array} \right).$$  (15)

The spin tractor bundle is the associated bundle $\tilde{S} := \tilde{G} \times \tilde{P} \Delta^{n+1,n+1}$, where $\Delta^{n+1,n+1}$ is the spin representation of $\tilde{G} = \text{Spin}(n+1, n+1)$. Since we work in even signature, this decomposes into irreducibles $\Delta^{n+1,n+1} =$
\[ \Delta_{n+1,n+1}^+ \oplus \Delta_{n+1,n+1}^-; \] the corresponding bundles are denoted by \( \tilde{S}_\pm = \tilde{G} \times \tilde{P} \). Under a choice of \( g \in c \) the spin tractor bundles decomposes as

\[ \tilde{S}_\pm = \tilde{G} \times \tilde{P} \]

where \( \tilde{S}_\pm \) are the natural spin bundles as before. The Clifford action of the conformal standard tractor bundle \( \tilde{T} \) on \( \tilde{S}_\pm \) is

\[ \left( \begin{array}{c} \rho \\ \varphi_a \\ \sigma \\ \tau \\ \chi \end{array} \right) \cdot \left( \begin{array}{c} \tau \\ \chi \end{array} \right) = \left( \begin{array}{c} -\varphi_a \gamma^a \tau + \sqrt{2} \rho \chi \\ \varphi_a \gamma^a \chi - \sqrt{2} \sigma \tau \end{array} \right), \quad (16) \]

ca. [Ham12]. \( \tilde{S} = \tilde{S}_+ \oplus \tilde{S}_- \) carries the spin tractor connections that are induced from the standard tractor connection on \( \tilde{T} \),

\[ \tilde{\nabla}_{\tilde{c}} \left( \begin{array}{c} \tau \\ \chi \end{array} \right) = \left( \begin{array}{c} \tilde{D}_c \tau + \frac{1}{\sqrt{2}} \tilde{P}_{cp} \gamma^p \chi \\ \tilde{D}_c \chi + \frac{1}{\sqrt{2}} \gamma^c \tau \end{array} \right). \]

The BGG-splitting operator of \( \tilde{S}_\pm \) is

\[ L_0^{\tilde{S}_\pm} : \Gamma(\tilde{S}_\pm[\frac{1}{2}]) \to \Gamma(\tilde{S}_\pm), \quad \chi \mapsto \left( \begin{array}{c} \frac{1}{\sqrt{2n}} \tilde{D} \chi \\ \chi \end{array} \right), \quad (17) \]

where

\[ \tilde{D} : \Gamma(\tilde{S}_\pm) \to \Gamma(\tilde{S}_\pm), \quad \tilde{D} := \gamma^p \tilde{D}_p, \]

is the Dirac operator. The first BGG-operator associated to \( \tilde{S}_\pm \) is

\[ \Theta_0^{\tilde{S}_\pm} : \Gamma(\tilde{S}_\pm[\frac{1}{2}]) \to \Gamma(\tilde{E}_a \otimes \tilde{S}_\pm[\frac{1}{2}]), \quad \chi \mapsto \tilde{D}_a \chi + \frac{1}{\sqrt{2n}} \gamma_a \tilde{D} \chi. \]

This is the twistor operator (cf. e.g. [BFGK90]), which is alternatively described as the projection of the Levi-Civita derivative of a spinor to the kernel of Clifford multiplication. Solutions of the twistor spinor equation, which are elements in the kernel of \( \Theta_0^{\tilde{S}} \), are called twistor spinors, and it is well known that \( \Pi^{\tilde{S}} \) induces an isomorphism between \( \tilde{\nabla}_{\tilde{S}} \)-parallel sections of \( \tilde{S} \) with \( \ker \Theta_0^{\tilde{S}} \).

The adjoint tractor bundle is the associated bundle \( \tilde{A}\tilde{M} := \tilde{G} \times \tilde{P} \tilde{g} \) with respect to the adjoint representation of \( \tilde{G} \) on \( \tilde{g} = \mathfrak{so}(n+1,n+1) \cong \Lambda^2 \mathbb{R}^{n+1,n+1} \).

Henceforth we identify \( \tilde{A}\tilde{M} \) with \( \Lambda^2 \tilde{T} \). With respect to a \( g \in c \),

\[ \tilde{A}\tilde{M} = \begin{pmatrix} \tilde{E}_a[0] \\ \tilde{E}_a[1] \end{pmatrix}, \quad (18) \]

The standard pairing on \( \tilde{A}\tilde{M} \) induced by the Killing form on \( \tilde{g} \) is denoted as \( \langle \cdot, \cdot \rangle : \tilde{A}\tilde{M} \times \tilde{A}\tilde{M} \to \mathbb{R} \). The standard representation of \( \tilde{g} \) on \( \mathbb{R}^{n+1,n+1} \) gives rise to an action \( \bullet \) of \( \tilde{g} \) on any associated tractor bundle \( \tilde{G} \times \tilde{P} V \). E.g., we
have for the conformal standard tractor bundle the map \( \bullet : A\tilde{M} \otimes \tilde{T} \to \tilde{T} \) given by
\[
\begin{pmatrix}
\rho_a \\
\mu_{aa1} \\
\beta_a
\end{pmatrix} \cdot \begin{pmatrix}
\nu \\
\omega_b \\
\sigma
\end{pmatrix} = \begin{pmatrix}
\rho^\sigma \omega_\sigma - \varphi \nu \\
\mu_b^a \omega_\sigma - \sigma \rho_b - \nu \beta_b \\
\beta^\sigma \omega_\sigma + \varphi \sigma
\end{pmatrix}.
\]
(19)
The normal tractor connection is given by
\[
\nabla^A\tilde{M} \left( \begin{pmatrix}
\rho_a \\
\mu_{aa1} \\
\kappa_a
\end{pmatrix} \cdot \begin{pmatrix}
\varphi
\end{pmatrix} \right) = \begin{pmatrix}
\tilde{D}_c \rho_a - \tilde{T}_c \mu_{pa} - \tilde{T}_{ca} \varphi \\
\left( \tilde{D}_c \mu_{aa1} + 2 \tilde{T}_c g_{[a]} \rho_{a1} \right) + 2 \tilde{T}_c g_{[a]} k_{a1} \\
\tilde{D}_c k_a - \mu_{ca} + g_{ca} \varphi
\end{pmatrix}.
\]
(20)
Written as a two-form \( \tilde{\Omega} \) with values in \( \Lambda^2 \tilde{T} \), the curvature of the standard tractor connection \( \nabla^\tilde{T} \) is
\[
\tilde{\Omega}_{a[0c1]} = \left( \begin{pmatrix}
-\tilde{\nabla}_{a0c1}
\end{pmatrix} \right) \in \Gamma(\tilde{E}_{[a0c1]} \otimes A\tilde{M}).
\]
(21)
The BGG-splitting operator
\[
L_0^A\tilde{M} : \Gamma(\tilde{E}^a) = \Gamma(\tilde{E}_a[2]) \to \Gamma(A\tilde{M}), \ k_a \mapsto \begin{pmatrix}
\rho_a \\
\mu_{aa1} \\
\kappa_a
\end{pmatrix}
\]
is determined by
\[
\mu_{aa1} = \tilde{D}_{[a} k_{a1]}, \ \varphi = -\frac{1}{2n} g^p q \tilde{D}_p k_q, \ \rho_a = -\frac{1}{4n} \tilde{D}^p \tilde{D}_p k_a + \frac{1}{4n} \tilde{D}^p \tilde{D}_p k_p + \frac{1}{4n} \tilde{D}_a \tilde{D}^p k_p + \frac{1}{n} \tilde{T}_a \tilde{D}^p k_p - \frac{1}{2n} \tilde{J} k_a.
\]
(22)
The first BGG-operator of \( A\tilde{M} \) is computed as
\[
\Theta_0^A\tilde{M} : \Gamma(\tilde{E}_a[2]) \to \Gamma(\tilde{E}_{(ab)}[2]), \ \xi_a \mapsto \tilde{D}_c(\xi_a)_{[c}.
\]
(23)
Thus \( \Theta_0^A\tilde{M} \) is the conformal Killing operator and the solutions to the first BGG-equation are the conformal Killing fields. In a prolonged form, the conformal Killing field equations are equivalent to
\[
\nabla^A\tilde{M}s = \xi^a \tilde{\Omega}_{ab},
\]
(24)
where \( s = L_0^A\tilde{M}(\xi) \), see [Gov06, Cap08].

3. The Fefferman-type construction

The construction of split-signature conformal structures from projective structures discussed in this section fits into a general scheme relating parabolic geometries of different types. Namely, it is an instance of the so-called Fefferman-type construction, whose name and general procedure is motivated by Fefferman’s construction of a canonical conformal structure induced by a CR structure, see [Cap06 and CS09] for a detailed discussion.
3.1. General procedure. Suppose we have two pairs of semi-simple Lie groups and parabolic subgroups, \((G, P)\) and \((\tilde{G}, \tilde{P})\), and a Lie group homomorphism \(i : G \to \tilde{G}\) such that the derivative \(i' : \mathfrak{g} \to \mathfrak{g}\) is injective. Assume further that the \(G\)-orbit of the origin in \(\tilde{G}/\tilde{P}\) is open and that the parabolic \(P \subseteq G\) contains \(Q := i^{-1}(\tilde{P})\), the preimage of \(\tilde{P} \subseteq \tilde{G}\).

Given a parabolic geometry \((G \to M, \omega)\) of type \((G, P)\), one first forms the correspondence space \(\tilde{M} := G/Q = G \times_P P/Q\), (25) which is also called the Fefferman space. Then \((G \to \tilde{M}, \omega)\) is automatically a Cartan geometry of type \((G, Q)\). As a next step, one considers the extended bundle \(\tilde{G} := G \times_Q \tilde{P}\) (26) with respect to the homomorphism \(Q \to \tilde{P}\). This is a principal bundle over \(\tilde{M}\) with structure group \(\tilde{P}\) and \(j : G \hookrightarrow \tilde{G}\) denotes the natural inclusion.

The equivariant extension of \(\omega \in \Omega^1(G, \mathfrak{g})\) yields a unique Cartan connection \(\tilde{\omega} \in \Omega^1(\tilde{G}, \tilde{\mathfrak{g}})\) of type \((\tilde{G}, \tilde{P})\) such that \(j^*\tilde{\omega} = i' \circ \omega\). (27)

Altogether, one obtains a functor from parabolic geometries \((G \to M, \omega)\) of type \((G, P)\) to parabolic geometries \((\tilde{G} \to \tilde{M}, \tilde{\omega})\) of type \((\tilde{G}, \tilde{P})\).

The relation between the corresponding curvatures is as follows: The previous assumptions yield a linear isomorphism \(\mathfrak{g}/\mathfrak{q} \cong \mathfrak{g}/\mathfrak{p}\) and an obvious projection \(\mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}\), where \(\mathfrak{q} \subseteq \mathfrak{p}\) is the Lie algebra of \(Q \subseteq P\). Composing these two maps one obtains a linear projection \(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \to \mathfrak{g}/\mathfrak{p}\), whose dual map is denoted as \(\varphi : (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*\). Since \(i' : \mathfrak{g} \to \tilde{\mathfrak{g}}\) is a homomorphism of Lie algebras, the curvature function \(\tilde{\kappa} : \tilde{G} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}\) is related to \(\kappa : G \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}\) by

\[\tilde{\kappa} \circ j = (\Lambda^2(\varphi \otimes i') \circ \kappa).\] (28)

We note that \(\tilde{\kappa}\) is fully determined by this formula.

Since \(i'\) is an embedding, the notation is in most cases simplified such that we write \(\mathfrak{g} \subseteq \tilde{\mathfrak{g}}, \mathfrak{q} = \mathfrak{g} \cap \mathfrak{p}\), etc. We proceed similarly on the level of Lie groups, provided that the map \(i\) is an embedding.

3.2. Algebraic setup. Here we specify the general setup for Fefferman-type constructions from Subsection 2.7 according to the description of oriented projective and conformal spin structures given in Subsection 2.6 and 2.7 respectively. Let \(\mathbb{R}^{n+1,n+1}\) be the real vector space \(\mathbb{R}^{2n+2}\) with an inner product, \(h\), of split-signature. Let \(\Delta^+_{n+1,n+1}\) and \(\Delta^-_{n+1,n+1}\) be the irreducible spin representations of \(\tilde{G} := \text{Spin}(n+1, n+1)\) as in Subsection 2.7. We fix two pure spinors \(s_F \in \Delta^-_{n+1,n+1}\) and \(s_E \in \Delta^+_{n+1,n+1}\) with non-trivial pairing, which is assigned for later use to be

\[\langle s_E, s_F \rangle = -\frac{1}{2}.\]
Note that $s_E$ lies in $\Delta_{n+1,n+1}$ if $n$ is even or in $\Delta_{n+1,n+1}^-$ if $n$ is odd.

Let us denote by $E, F \subseteq \mathbb{R}^{n+1,n+1}$ the kernels of $s_E, s_F$ with respect to the Clifford multiplication, i.e.

$$E := \{ X \in \mathbb{R}^{n+1,n+1} : X \cdot s_E = 0 \},$$

$$F := \{ X \in \mathbb{R}^{n+1,n+1} : X \cdot s_F = 0 \}.$$

The purity of $s_E$ and $s_F$ means that $E$ and $F$ are maximally isotropic subspaces in $\mathbb{R}^{n+1,n+1}$. The other assumptions guarantee that $E$ and $F$ are complementary and dual each other via the inner product $h$. Hence we use the decomposition

$$\mathbb{R}^{n+1,n+1} = E \oplus F \cong \mathbb{R}^{n+1} \oplus \mathbb{R}^{n+1*}$$

(29)

to identify the spinor representation $\Delta_{n+1,n+1} = \Delta_{n+1,n+1}^+ \oplus \Delta_{n+1,n+1}^-$ with the exterior power algebra $\Lambda^* E \cong \Lambda^* \mathbb{R}^{n+1}$, whose irreducible subrepresentations are $\Delta_{n+1,n+1}^+ \cong \Lambda^{even} \mathbb{R}^{n+1}$ and $\Delta_{n+1,n+1}^- \cong \Lambda^{odd} \mathbb{R}^{n+1}$. When $n$ is even, respectively, odd, we can identify $(\Delta_{n+1,n+1}^*)^* \cong \Delta_{n+1,n+1}^+$, respectively $(\Delta_{n+1,n+1}^*)^* \cong \Delta_{n+1,n+1}^-.$

Now, let us consider the subgroup in $\tilde{G}$ defined by

$$G := \{ g \in \text{Spin}(n+1,n+1) : g \cdot s_E = s_E, \ g \cdot s_F = s_F \}.$$

$G$ preserves the decomposition (29), so that the restriction of the action to $F$ is dual to the restriction to $E$, and the volume form is determined by $s_E$ and $s_F$. Hence $G$ is isomorphic to $\text{SL}(n+1)$ and this defines an embedding $i : G \hookrightarrow \tilde{G}$.

The $G$-invariant decomposition (29) determines a $G$-invariant skew-symmetric involution $K \in \mathfrak{so}(n+1,n+1)$ acting by the identity on $E$ and minus the identity on $F$. The relationship among $K, s_E$ and $s_F$ may be expressed as

$$h(X, K(Y)) = -h(K(X), Y) = 2 \langle s_E, (X \wedge Y) \cdot s_F \rangle,$$

(30)

where

$$(X \wedge Y) \cdot s_F = \frac{1}{2} (X \cdot Y \cdot s_F - Y \cdot X \cdot s_F) = X \cdot Y \cdot s_F + h(X,Y)s_F.$$

The spin action of $\tilde{\mathfrak{g}}$ is denoted by $\bullet$, and thus $A \bullet s = -\frac{1}{4} A \cdot s$, for any $A \in \tilde{\mathfrak{g}}$ and $s \in \Delta$. In particular,

$$K \bullet s_F = -\frac{1}{2}(n+1)s_F, \quad K \bullet s_E = \frac{1}{2}(n+1)s_E.$$

Here we identify $\tilde{\mathfrak{g}} = \mathfrak{so}(n+1,n+1)$, the Lie algebra of $\tilde{G}$, with $\Lambda^2 \mathbb{R}^{n+1,n+1}$. It is convenient to split $\tilde{\mathfrak{g}}$ in terms of irreducible $\mathfrak{g}$-modules as

$$\tilde{\mathfrak{g}} = \Lambda^2 (E \oplus F) = (E \otimes F)_{0,Tr} \oplus (E \otimes F)_{T_\mathfrak{g}} \oplus \Lambda^2E \oplus \Lambda^2F,$$

(31)

where $(E \otimes F)_{T_\mathfrak{g}} = \mathbb{R}K$, and $K$ acts as

$$[K, \phi] = 2\phi, \quad [K, \psi] = -2\psi, \quad [K, \lambda] = 0,$$

(32)
for any $\phi \in \Lambda^2 E$, $\psi \in \Lambda^2 F$ and $\lambda \in E \otimes F$. Further, the annihilators of $s_E$ and $s_F$ in $\tilde{g}$ are the subalgebras

$$\ker s_E := \{ \phi \in \tilde{g} : \phi \cdot s_E = 0 \} = \mathfrak{sl}(n+1) \oplus \Lambda^2 E,$$

$$\ker s_F := \{ \phi \in \tilde{g} : \phi \cdot s_F = 0 \} = \mathfrak{sl}(n+1) \oplus \Lambda^2 F. \quad (33)$$

3.3. Homogeneous model. The homogeneous model for conformal spin structures of signature $(n, n)$, $\tilde{G}/\tilde{P} \cong S^n \times S^n$, is the space of isotropic rays in $\mathbb{R}^{n+1, n+1}$. The subgroup $G \subseteq \tilde{G}$ does not act transitively on that space — according to the decomposition (29), there are three orbits: the set of rays contained in $E$, the set of rays contained in $F$, and the set of isotropic rays that are neither contained in $E$ nor in $F$. Note that only the last orbit is open in $\tilde{G}/\tilde{P}$, which is one of the requirements from 3.1.

Therefore, we define $\tilde{P} \subseteq \tilde{G}$ to be the stabiliser of a ray through a light-like vector $\tilde{v} \in \mathbb{R}^{n+1, n+1} \setminus (E \cup F)$. Denoting by $Q = i^{-1}(\tilde{P})$ the stabiliser of the ray $\mathbb{R}_+ \tilde{v}$ in $G$, we have the identification of $G/Q$ with the open orbit of the origin in $\tilde{G}/\tilde{P}$. The subgroup $Q$, which is not parabolic, is contained in the parabolic subgroup $P \subseteq G$ defined as the stabiliser in $G$ of the ray through the projection of $\tilde{v}$ to $E$,

$$v = (\tilde{v})_E.$$

In particular, $G/P$ is the standard projective sphere $S^n$, the homogeneous model of oriented projective structures of dimension $n$, and $G/Q \to G/P$ is the canonical fibration with the standard fibre $P/Q$, whose total space is the model Fefferman space.

Let us denote by $L = \mathbb{R}\tilde{v}$ the line spanned by the light-like vector $\tilde{v}$ and let $L^\perp$ be the orthogonal complement in $\mathbb{R}^{n+1, n+1}$ with respect to $h$. The tangent space of $G/Q$ at the origin can be seen in three different ways, namely,

$$(L^\perp/L)[1] \cong g/q \cong \tilde{g}/\tilde{p}.$$

The latter isomorphism is induced by the embedding $g \subseteq \tilde{g}$, the former one by the standard action of $g \subseteq \tilde{g}$ on the vector $\tilde{v} \in \mathbb{R}^{n+1, n+1}$. The particular weight in the first term is chosen so that both these identifications are $Q$-equivariant.

There are several natural $Q$-invariant objects that in turn yield distinguished geometric objects on the general Fefferman space. The $n$-dimensional $Q$-invariant subspace

$$f := ((\tilde{F} + L)/L)[1] \subseteq (L^\perp/L)[1] \quad \text{where} \quad \tilde{F} := F \cap L^\perp,$$

which is, under one of the previous identifications,

$$f \cong p/q \subseteq g/q,$$

the kernel of the projection $g/q \to g/p$. Another $n$-dimensional $Q$-invariant subspace is

$$e := ((\tilde{E} + L)/L)[1] \subseteq (L^\perp/L)[1] \quad \text{where} \quad \tilde{E} := E \cap L^\perp.$$
The intersection $e \cap f$ is 1-dimensional with a distinguished $Q$-invariant generator that corresponds to the $G$-invariant involution $K \in \tilde{\mathfrak{g}}$,

$$k := K + \tilde{\mathfrak{p}} \in \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}.$$  

Note that all these objects are isotropic with respect to the natural conformal class induced by the restriction of $h$ to $L^+ \subseteq \mathbb{R}^{n+1,n+1}$. In particular, both $e$ and $f$ are maximally isotropic subspaces such that

$$k \in e \cap f \subseteq k^\perp = e + f. \quad (34)$$

In Subsection 3.3, we introduced a map $\varphi : (\mathfrak{g}/\mathfrak{p})^* \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$, the dual map to the projection $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q} \to \mathfrak{g}/\mathfrak{p}$. The kernel of this projection is just $f$ and the image of $\varphi$ is identified with its annihilator, which will be denoted by $f^\circ$. Since $f$ is a maximally isotropic subspace in $\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}} \cong \mathfrak{g}/\mathfrak{q}$,

$$f^\circ \cong f[-2]. \quad (35)$$

Since $(\mathfrak{g}/\mathfrak{p})^* \cong \tilde{\mathfrak{p}}_+$, we may conclude with the help of explicit matrix realizations from Appendix A that

$$f^\circ = \tilde{\mathfrak{p}}_+ \cap \ker s_F. \quad (36)$$

Moreover, we note that

$$\left(\tilde{\mathfrak{p}}_+ \cap \ker s_F\right)_{E^\otimes F} = \mathfrak{p}_+, \quad \left(\tilde{\mathfrak{p}} \cap \ker s_F\right)_{E^\otimes F} = \mathfrak{p}, \quad (37)$$

$$\Lambda^2 F \cap \tilde{\mathfrak{p}} = \Lambda^2 \tilde{F} \subseteq \tilde{\mathfrak{g}}_0, \quad [\tilde{\mathfrak{p}}_+, \Lambda^2 \tilde{F}] = f^\circ, \quad [f^\circ, \Lambda^2 \tilde{F}] = 0. \quad (38)$$

3.4. **The Fefferman space and the induced structure.** The pairs of Lie groups $(G, P)$ and $(\tilde{G}, \tilde{P})$ from the previous subsection satisfy all the properties to launch the Fefferman-type construction.

**Proposition 3.1.** *The Fefferman-type construction for the pairs of Lie groups $(G, P)$ and $(\tilde{G}, \tilde{P})$ yields a natural construction of conformal spin structures $(\tilde{M}, \tilde{c})$ of signature $(n, n)$ from $n$-dimensional oriented projective structures $(M, \mathfrak{p})$. The Fefferman space $\tilde{M}$ is identified with the total space of the weighted cotangent bundle without the zero section $T^*M(2) \setminus \{0\}$.***

**Proof.** The first part of the statement is obvious from the general setting for Fefferman-type constructions and the Cartan-geometric description of oriented projective and conformal spin structures.

The second part is shown due to two natural identifications: On the one hand, the Fefferman space is by (25) equal to the total space of the associated bundle $\tilde{M} \cong \mathcal{G} \times_\mathfrak{p} P/Q$ over $M$. On the other hand, the weighted cotangent bundle to $M$ is identified with the associated bundle $T^*M(2) \cong \mathcal{G} \times_\mathfrak{p} (\mathfrak{g}/\mathfrak{p})^*(2)$ with respect to action of $P$ induced by the adjoint action and the representation (11) for $w = 2$. Hence it remains to verify that $(\mathfrak{g}/\mathfrak{p})^*(2) \setminus \{0\} \cong \mathfrak{p}_+(2)$ is the homogeneous space of $P/Q$, i.e. that the action of $P$ is transitive and $Q$ is a stabiliser of a non-zero element. But this is a purely algebraic task, which may be easily checked in a concrete matrix realisation. According to the choices made in Appendix A, one observes that $Q$ is the stabiliser of an element from (92) corresponding to $U = 0$ and $w = 1$. \hfill \Box

The induced conformal structure on $\tilde{M}$ has a number of specific features:
Proposition 3.2. Let \((\tilde{M}, c)\) be the conformal spin structure induced from an oriented projective structure \((M, p)\) by the Fefferman-type construction. Then \(\tilde{M}\) carries the following tractor fields, which are all parallel with respect to the induced tractor connections \(\nabla^{\text{ind}}\) on the respective bundles:

(a) pure tractor spinors \(s_E \in \Gamma(\tilde{S}_\pm)\) and \(s_F \in \Gamma(\tilde{S}_-)\) with non-trivial pairing,

(b) a tractor endomorphism \(K = L_0^M(k)\) which is an involution, i.e. \(K^2 = \text{id}_\tilde{T}\),

(c) \(K\) acts by the identity on \(\tilde{E} := \ker s_E\) and by minus the identity on \(\tilde{F} := \ker s_F\).

In particular, \(\tilde{E}, \tilde{F}\) are \(\nabla^{\text{ind}}\)-parallel, maximally isotropic subbundles and \(\tilde{T} = \tilde{E} \oplus \tilde{F}\).

The tractor fields \(s_E, s_F\), respectively \(K\), descend to pure spinors \(\eta \in \Gamma(\tilde{S}_\pm)\), \(\chi \in \Gamma(\tilde{S}_-)\), respectively a light-like vector field \(k \in \Gamma(TM)\), satisfying:

(d) the kernel \(\tilde{f} := \ker \chi\) coincides with the vertical subbundle of the projection \(\tilde{M} \to M\) and the kernel \(\tilde{c} := \ker \eta\) intersects \(\ker \chi\) in a 1-dimensional subspace,

(e) the vector field \(k\) is nowhere-vanishing and is a section to the intersection \(\ker \chi \cap \ker \eta\).

Proof. The \(G\)-invariant objects \(s_E, s_F\) and \(K\) introduced above induce the tractor fields \(s_E, s_F\) and \(K\) such that \(s_E \in \Gamma(\tilde{S}_\pm = \tilde{G} \times_P \Delta_\pm)\) corresponds to the constant \((P\text{-equivariant})\) map \(\tilde{G} \to \Delta_\pm\) and similarly for the other objects. The purity and non-trivial pairing of \(s_E\) and \(s_F\) follows from the purity and non-trivial pairing of \(s_E\) and \(s_F\). The kernels of \(s_E\) and \(s_F\) correspond to the kernels of \(s_E\) and \(s_F\) so that \(\tilde{E} = \tilde{G} \times_P E\) and \(\tilde{F} = \tilde{G} \times_P F\), respectively, and the decomposition \(\tilde{T} = \tilde{E} \oplus \tilde{F}\) corresponds to the \(G\)-invariant decomposition (29). Hence all these tractor objects are automatically parallel with respect to the induced tractor connections. Let \(\eta = \Pi_0^E(s_E), \chi = \Pi_0^\tilde{E}(s_F)\) and \(k = \Pi_0^{AM}(K)\) be the corresponding underlying objects on \(\tilde{M}\).

The filtration \(L \subseteq L^\perp \subseteq \mathbb{R}^{n+1,n+1}\) from (33) gives rise to the filtration of the standard tractor bundle, which can be written as

\[
\begin{pmatrix}
\mathbb{E}^{[-1]} \\
0 \\
0
\end{pmatrix} \subseteq
\begin{pmatrix}
\mathbb{E}_{\upsilon}^{[-1]} \\
\mathbb{E}_{\upsilon}[1] \\
\mathbb{E}[1]
\end{pmatrix} \subseteq
\begin{pmatrix}
\mathbb{E}_{\upsilon}^{[-1]} \\
\mathbb{E}_{\upsilon}[1] \\
\mathbb{E}[1]
\end{pmatrix} = \tilde{T}.
\]

In particular, the subspaces \(\tilde{E}, \tilde{F} \subseteq L^\perp\) are distinguished by the middle slot.

The \(Q\)-invariant maximally isotropic subspaces \(e, f \subseteq g/q\) determine the distributions in \(T\tilde{M}\), namely, \(\tilde{G} \times_Q e\) and \(\tilde{G} \times_Q f\). Now, according to the tractor Clifford action (18), it follows that these are precisely the kernels of the spinors \(\eta\) and \(\chi\). Since these subspaces are maximally isotropic, the corresponding spinors are pure. Since \(f \cong p/q\) is the kernel of the projection \(g/q \to g/p\), the corresponding subbundle \(\tilde{f}\) is identified with the vertical
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subbundle of the projection $\tilde{M} \to M$. Finally, the intersection $e \cap f$ is 1-
dimensional and it is generated by the projection of $K \in \mathfrak{g}$ to $\mathfrak{g}/\mathfrak{p}$. Hence
the corresponding vector field $k$ on $\tilde{M}$ belongs to the intersection $e \cap \tilde{f}$.

It remains to show that the spinors $\eta, \chi$ and the vector field $k$ are nowhere-vanishing. Since the skew-symmetric involution $K$ acts by the identity on $E$ and minus the identity on $F$, it cannot be contained in $\tilde{p}$ (this would
imply that it preserves the line $L = \mathbb{R}\tilde{v}$, which is impossible as $\tilde{v}$ is neither contained in $E$ nor in $F$). Hence $k$ is nowhere-vanishing. But then, also the underlying spinors are nowhere-vanishing.

\[\square\]

3.5. Normality. Now we show that, in the non-flat case, our Fefferman-
type construction is normal if and only if $\dim M = 2$. In the proof of this
result we will employ the following proposition, which is stated in a slightly
more general setting than needed there.

The Kostant co-differential $\partial^\ast$, see (2), of a parabolic geometry $(G \to M, \omega)$ of type $(G, P)$ can be explicitly expressed as follows. Let $X_1, \ldots, X_n \in \mathfrak{g}$ be elements projecting to a basis of $\mathfrak{g}/\mathfrak{p}$, and let $Z_1, \ldots, Z_n \in \mathfrak{p}_+ \cong (\mathfrak{g}/\mathfrak{p})^\ast$ form the dual basis. Then, for any $\phi \in \Lambda^2(\mathfrak{g}/\mathfrak{p})^\ast \otimes \mathfrak{g}$, $\partial^\ast \phi = \partial^1 \phi + \partial^2 \phi$, where

\[
\partial^1_\phi (X) := 2 \sum_{i=1}^n [\phi(X_i, X), Z_i], \quad \partial^2_\phi (X) := \sum_{i=1}^n \phi(X_i, [Z_i, X]),
\]

(40)

see \cite{CS09} Lemma 3.1.11. Note that $\partial^2$ vanishes identically for $[1]$-graded Lie algebras, since in case of a $[1]$-grading $[\mathfrak{p}_+, \mathfrak{g}] \subseteq \mathfrak{p}$.

Proposition 3.3. Consider a Fefferman-type construction from parabolic geometries of type $(G, P)$ to $[1]$-graded parabolic geometries of type $(\tilde{G}, \tilde{P})$ corresponding to a Lie group homomorphism $i : G \to \tilde{G}$ between simple groups. Suppose that the curvature function $\kappa$ of the parabolic geometry $(G, \omega)$ takes values in $\Lambda^2(\mathfrak{g}/\mathfrak{p})^\ast \otimes (\mathfrak{g} \cap \mathfrak{p})$. Suppose further that the two sum-
mands in the normality condition vanish separately, i.e. $\partial_\phi^1 \kappa = \partial_\phi^2 \kappa = 0$.

Then the induced geometry $(\tilde{G}, \tilde{\omega})$ is normal.

Proof. We use the Killing form $\tilde{B}$ of $\tilde{g}$ (which restricts to the Killing form $B$ of $g$ up to a constant multiple) to identify $(\mathfrak{g}/\mathfrak{p})^\ast \cong \mathfrak{p}_+$ and $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^\ast \cong \tilde{\mathfrak{p}}_+$. Let $X_1, \ldots, X_n \in \mathfrak{g}$ be elements projecting to a basis of $\mathfrak{g}/\mathfrak{p}$ and extend these by $X_{n+1}, \ldots, X_m \in \mathfrak{p}$ such that $X_1, \ldots, X_m$ project to a basis of $\mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Let $\{Z_1, \ldots, Z_n\}$, $Z_i \in \mathfrak{p}_+$, be the dual basis of $\{X_1 + \mathfrak{p}, \ldots, X_n + \mathfrak{p}\}$ and let $\{\tilde{Z}_1, \ldots, \tilde{Z}_m\}$, $\tilde{Z}_i \in \tilde{\mathfrak{p}}_+$ be the dual basis of $\{X_1 + \tilde{\mathfrak{p}}, \ldots, X_m + \tilde{\mathfrak{p}}\}$. Then $\tilde{Z}_j - Z_j$, for $j = 1, \ldots, n$, are contained in the orthogonal complement $\mathfrak{g}^\perp \subseteq \tilde{\mathfrak{g}}$ to $\mathfrak{g}$ with respect to the Killing form: for $i = 1, \ldots, n$, we have

\[
\tilde{B}(X_i, \tilde{Z}_j - Z_j) = \tilde{B}(X_i, \tilde{Z}_j) - \tilde{B}(X_i, Z_j) = \delta_{i,j} = \delta_{i,j} = 0.
\]

For $i = n+1, \ldots, m$, we have $\tilde{B}(X_i, \tilde{Z}_j) = 0$ since $i \neq j$ and $\tilde{B}(X_i, Z_j) = 0$ since $X_i \in \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$. Finally, for $Y \in \mathfrak{q}$ we have $\tilde{B}(Y, \tilde{Z}_j) = 0$ since $\mathfrak{q} \subseteq \tilde{\mathfrak{p}}$ and $\tilde{Z}_j \in \tilde{\mathfrak{p}}_+$ and $\tilde{B}(Y, Z_j) = 0$ since $\mathfrak{q} \subseteq \mathfrak{p}$ and $Z_j \in \mathfrak{p}_+$.

Now suppose $\kappa : G \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^\ast \otimes \mathfrak{g}$ is the curvature function of $(G, \omega)$, and let $\tilde{\kappa} : \tilde{G} \to \Lambda^2(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^\ast \otimes \tilde{\mathfrak{g}}$ be the curvature function of the geometry $(\tilde{G}, \tilde{\omega})$. 
The latter geometry is normal if and only if
\[ \partial^* \tilde{\kappa}(\tilde{u})(X) = 2 \sum_{i=1}^{m} [\tilde{\kappa}(\tilde{u})(X_i, X), \tilde{Z}_i] = 0 \] (41)
for all \( \tilde{u} \in \tilde{G} \) and \( X \in \tilde{g} \).

By construction, we know that \( \tilde{\kappa} \) is a \( \tilde{P} \)-equivariant extension of \( \kappa \) and so elements of \( \mathfrak{p} \) insert trivially into \( \tilde{\kappa} \). Since also \( \partial^* \) is \( \tilde{P} \)-equivariant, to show normality of \( \tilde{\kappa} \) it suffices to verify that
\[ \partial^* \tilde{\kappa}(u)(X) = 2 \sum_{i=1}^{n} [\kappa(u)(X_i, X), \tilde{Z}_i] = 2 \sum_{i=1}^{n} [\kappa(u)(X_i, X), \tilde{Z}_i] \] (42)
vanishes for all \( u \in G \) and \( X \in g \). Since by assumption
\[ \partial^*_i \kappa(u)(X) = 2 \sum_{i=1}^{n} [\kappa(u)(X_i, X), Z_i] = 0, \]
we can rewrite \( \partial^* \tilde{\kappa}(u)(X) \) as
\[ 2 \sum_{i=1}^{n} [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i]. \] (43)

We have observed that \( \tilde{Z}_i - Z_i \in g^\perp \) and by construction \( \kappa(u)(X_i, X) \in g \).
Since the decomposition \( \tilde{g} = g \oplus g^\perp \) is invariant under the action of \( g \), this implies that \( \partial^* \tilde{\kappa}(u)(X) = \sum_{i=1}^{n} [\kappa(u)(X_i, X), \tilde{Z}_i - Z_i] \in g^\perp \). On the other hand, since by assumption \( \kappa(u)(X_i, X) \in \mathfrak{p} \) and \( \tilde{Z}_i \in \tilde{\mathfrak{p}}_+ \), we have \( \partial^* \tilde{\kappa}(u)(X) \in \tilde{\mathfrak{p}}_+ \). But the intersection \( g^\perp \cap \tilde{\mathfrak{p}}_+ \) is zero: Note that \( \tilde{\mathfrak{p}}_+ = \tilde{\mathfrak{p}}^\perp \), so any element in \( g^\perp \cap \tilde{\mathfrak{p}}_+ \) is orthogonal to \( g^\perp \mathfrak{p} = \tilde{g} \). Since the Killing form is non-degenerate this implies \( g^\perp \cap \tilde{\mathfrak{p}}_+ = 0 \) and we conclude that \( \partial^* \circ \tilde{\kappa} = 0 \).

**Proposition 3.4.** Let \( (G \to M, \omega) \) be a normal projective parabolic geometry and let \( (\tilde{G} \to \tilde{M}, \tilde{\omega}^{ind}) \) be the conformal parabolic geometry obtained by the Fefferman-type construction.

(a) If \( \dim M = 2 \) then \( \tilde{\omega}^{ind} \) is normal.

(b) If \( \dim M > 2 \) then \( \tilde{\omega}^{ind} \) is normal if and only if \( \omega \) is flat.

Moreover, independently of the dimension of \( M \), \( \tilde{\omega}^{ind} \) is flat if and only if \( \omega \) is flat.

**Proof.** It is a general feature of Fefferman-type constructions that \( \tilde{\omega}^{ind} \) is flat if and only if \( \omega \) is flat. This immediately follows from the relation between the respective curvatures that is expressed in [28].

(a) In the special case of a projective structure in dimension \( n = 2 \) the curvature function of a normal projective Cartan connection takes values in \( \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}_+ \), see e.g. [CS09]. It is easily seen from the explicit matrices that \( \mathfrak{p}_+ \subseteq \mathfrak{p} \cap \mathfrak{g} \). We can thus apply Proposition 3.3 in this case, which proves the statement.

(b) If \( \omega \) is flat, then \( \tilde{\omega}^{ind} \) is also flat, in particular, it is normal. If \( \tilde{\omega}^{ind} \) is normal, then it is torsion-free, i.e. the curvature function \( \tilde{\kappa}^{ind} \) takes values in \( \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes (\mathfrak{p} \cap \mathfrak{g}) \). But this is only possible if the harmonic curvature of the original projective geometry takes values in a \( P \)-submodule of \( \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{p}_+ \).
The construction for the lowest admissible dimension $n = 2$ is indeed specific. Hence the geometric conclusions and characterisations are relatively easy to gain, and we lay this out in Subsection 4.2. The construction of the normal conformal Cartan connection is highly technical and is laid out in detail in Section 5. In the rest of this section we describe two important aspects of our construction: We discuss how tractorial objects on the projective structure relate to corresponding objects on the conformal structure, and we show how Weyl structures for the projective structure induce special Weyl structures for the conformal structure.

3.6. Relation between sections of tractor bundles. The discussion of tractor relations for the Fefferman construction in [ˇCG08] admits a straightforward generalisation to general Fefferman-type constructions for which $P/Q$ is connected, and thus in particular to the construction presented here.

Suppose $V$ is a $\tilde{G}$-representation, which is then also a $G$-representation, since $G \subseteq \tilde{G}$. Let $\nabla^V$ and $\nabla^{\tilde{V}}$ be the associated tractor bundles. Let $\nabla^{\tilde{V}}$ and $\nabla^V$ be the tractor connections induced by $\omega$ and $\tilde{\omega}$, where $\tilde{\omega}$ is induced by $\omega$ via the Fefferman-type construction. Sections of $V$ bijectively correspond to $P$-equivariant functions $f : G \rightarrow V$, while sections of $\tilde{V}$ correspond to $Q$-equivariant functions $f : \tilde{G} \rightarrow V$. In particular, since $Q \subseteq P$, every section of $V$ gives rise to a section of $\tilde{V}$, and we can view $\Gamma(V) \subseteq \Gamma(\tilde{V})$.

Conversely, analogously to Proposition 3.3 in [ˇCG08] we have:

**Proposition 3.5.** A section $s \in \Gamma(\tilde{V})$ is contained in $\Gamma(V)$ (i.e. the corresponding $Q$-equivariant function $f$ is indeed $P$-equivariant) if and only if $\nabla^{\tilde{V}}s = 0$ for $v \in \Gamma(\ker \chi)$.

**Proof.** Since $P/Q$ is connected, equivariance of $f$ can be checked infinitesimally, which means that $\zeta_X(\chi) \cdot f = -X(f(u))$ for any $u \in \tilde{G}$ and representative $X$ of $p/q$. This is equivalent to $\nabla^{\tilde{V}}s = 0$ for all sections $\xi$ of the vertical bundle $\tilde{f} = \tilde{G} \times_Q p/q$.

The tractor connection $\nabla^{\tilde{V}}$ restricts to a connection on $\Gamma(V) \subseteq \Gamma(\tilde{V})$, which coincides with $\nabla^V$. This implies a bijective correspondence between $\nabla^{\tilde{V}}$-parallel tractors in $\Gamma(\tilde{V})$ and $\nabla^V$-parallel tractors in $\Gamma(V)$. If $V$ is irreducible as a $\tilde{G}$-representation but has a $G$-invariant subspace $W \subseteq V$, then this correspondence restricts to a bijective correspondence between parallel sections of $\tilde{W} = \tilde{G} \times_Q W \rightarrow \tilde{M}$ and parallel sections of $W = G \times_P W \rightarrow M$.

3.7. Reduced Weyl structures and reduced scales. As a technical preliminary for further study we now relate the Weyl structures of the original projective Cartan geometry $(G, \omega)$ on $M$ and the induced conformal geometry $(\tilde{G}, \tilde{\omega})$ on the Fefferman space $\tilde{M}$.
Proposition 3.6. Any projective (exact) Weyl structure on $M$ induces a conformal (exact) Weyl structure on the Fefferman space $\tilde{M}$. In particular, any projective scale induces a conformal scale.

Proof. Fix a Levi subgroup $G_0 \subseteq P$, such that $G_0 \cong P/P_+$. Let $Q_0$ be the intersection of $G_0$ with $Q = G \cap P$. Since $P_+ \subseteq Q$, $Q_0 \cong Q/P_+$, and thus $G_0/Q_0 \cong P/Q$. Therefore a reduction $G_0 \rightarrowtail \mathcal{G}$ from $P$ to $G_0$ over the manifold $M$ induces a reduction from $Q_0$ over $\tilde{M}$. Composing the embedding $G_0 \rightarrowtail \mathcal{G}$ with the natural embedding $G \rightarrowtail \tilde{G} = G \times_Q \tilde{P}$ over $\tilde{M}$, one obtains a reduction $G_0 \rightarrowtail \tilde{G}$ from $\tilde{P}$ to $Q_0$ over $\tilde{M}$.

We can choose a Levi subgroup $\tilde{G}_0 \subseteq \tilde{P}$, $\tilde{G}_0 \cong \tilde{P}/\tilde{P}_+$, that contains $Q_0$, see Appendix A for an explicit realisation. Having done that, we define $\tilde{G}_0 := G_0 \times_{Q_0} \tilde{G}_0$, a $\tilde{G}_0$-principal bundle over $\tilde{M}$. Then $G_0$ is naturally embedded into $\tilde{G}_0$ and the above embedding $\tilde{j}$ canonically extends to an embedding of the $\tilde{G}_0$-bundle $\tilde{G}_0$ into the $\tilde{P}$-bundle $\tilde{G}$ over $\tilde{M}$, which we simply denote by $\tilde{j}$ again. Altogether, for a projective Weyl structure $G_0 \rightarrowtail \mathcal{G}$ we have constructed a conformal Weyl structure $\tilde{G}_0 \rightarrowtail \tilde{G}$.

Similar arguments hold true also for exact Weyl structures, i.e. the reductions of the respective bundles to the subgroups $G_0^{ss} \subseteq P$ and $\tilde{G}_0^{ss} \subseteq \tilde{P}$, the semi-simple parts of $G_0$ and $\tilde{G}_0$. (According to the previous choices, the subgroup $\tilde{G}_0^{ss}$ already contains $G_0^{ss} \cap Q$.)

□

A version of the above result in a more general context was proved in [Alt10].

Definition 3.7. Conformal Weyl structures induced by projective Weyl structures as above will be called reduced Weyl structures. Similar terminology applies also to exact Weyl structures; the corresponding conformal scales will be called reduced scales.

In Section 2.6 and 2.7 we have fixed suitable parametrisations of density bundles on projective and conformal manifolds, which are denoted by $E(w) \rightarrow M$ and $\tilde{E}[w] \rightarrow \tilde{M}$, respectively. Each of these bundles may as well be described as an associated bundle to the bundle of scales (only the
central part of the reductive subgroup $G_0$, respectively $\tilde{G}_0$, acts nontrivially). Hence everywhere positive sections of each of these bundles may as well be considered as scales; in the following we do not really distinguish between these two concepts. On that account, we need to understand the relation between the weights of projective and conformal densities for our Fefferman-type construction:

**Lemma 3.8.** Any projective $w$-density on $M$ induces a conformal $w$-density on the Fefferman space $\tilde{M}$.

**Proof.** The projective and conformal density bundles are defined via the representation of $P$ and $\tilde{P}$ as in (10) and (11), respectively. In order to compare the corresponding weights in the case of the Fefferman construction, it suffices to restrict the representations to $Q$, which is a subgroup both in $P$ and $\tilde{P}$ (in accord with the embedding $i : G \hookrightarrow \tilde{G}$). A direct check reveals the statement. □

An intrinsic characterisation of reduced scales among all conformal ones is formulated in Proposition 6.2.

### 4. Lagrangean contact structures and special dimension $n = 2$

There is a natural intermediate step to the construction of Section 3.4, namely, the canonical Lagrangean contact structure on the projectivised cotangent bundle $\mathcal{P}(T^*M)$ of a projective manifold $M$. In this section we shall briefly discuss the Fefferman-type construction for torsion-free Lagrangean contact structures, which is closely related to the projective to conformal construction but generally complementary to it except in dimension $n = 2$.

#### 4.1. The Fefferman-type construction for Lagrangean contact structures

A Lagrangean contact structure on $M'$ consists of a contact distribution $\mathcal{H} \subseteq TM'$ together with a decomposition $\mathcal{H} = e' \oplus f'$ into two subbundles that are maximally isotropic with respect to the Levi form $\mathcal{H} \times \mathcal{H} \rightarrow TM'/\mathcal{H}$. Such a structure is equivalently encoded as a normal parabolic geometry of type $(G, P')$, where $G = SL(n+1)$, $\dim M' = 2n - 1$, and $P' \subseteq G$ is the stabiliser of a flag of type line–hyperplane in the standard representation $\mathbb{R}^{n+1}$. For suitable choices as in Appendix A, the Lie algebra to $P'$ consists of matrices of the form

$$p' = \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & 0 & c \end{pmatrix} \subseteq p,$$

(44)

where $p \subseteq g$, the Lie algebra of $P \subseteq G$, appertains to projective structures as before. Given a projective Cartan geometry $(\mathcal{G} \rightarrow M, \omega)$, it turns out the correspondence space $M' := \mathcal{G}/P'$ can be identified with $\mathcal{P}(T^*M)$ and the Cartan geometry $(\mathcal{G} \rightarrow M', \omega)$ of type $(G, P')$ covers the natural Lagrangean contact structure. See e.g. [Cap05, CS09] for details.

The previous construction was based on the inclusion $P' \subseteq P$. On the other hand, there is an inclusion $Q \subseteq P'$, where $Q = i^{-1}(\tilde{P})$, $\tilde{P} \subseteq \tilde{G} = \text{Spin}(n+1, n+1)$, and $i : G \rightarrow \tilde{G}$ as before. This allows us to consider a
Fefferman-type construction for the pairs $(G, P)$ and $(\tilde{G}, \tilde{P})$, which induces a conformal spin structure on $\tilde{M} = G/Q$ to a Lagrangean contact structure on $M'$. This construction is indeed very similar to the original Fefferman construction; one deals with different real forms of the same complex Lie groups in the two cases. That is why the following statements and their proofs are analogous to those in the literature for the CR case, cf. e.g. [Gra87, CG10, Lei08].

The projective–to–conformal construction can thus be regarded as the composition of the correspondence space construction from a projective structure on $M$ to a Lagrangean contact structure on $M' = P(T^*M)$ with the Fefferman-type construction to a conformal spin structure on $\tilde{M} = T^*M(2) \setminus \{0\}$. The induced conformal objects from Proposition 3.2 correspond to the induced objects on $M'$, so that the vertical subbundle of the projection $\tilde{M} \to M'$ is spanned by $k$ and the decomposition $k^{\perp} = \tilde{e} \oplus \tilde{f} \subseteq T\tilde{M}$ descends to the decomposition $H = e' \oplus f' \subseteq TM'$ (cf. (34) and the respective matrix realisations).

**Proposition 4.1.** Let $(G \to M', \omega)$ be the normal parabolic geometry encoding a Lagrangean contact structure and let $(\tilde{G} \to \tilde{M}, \tilde{\omega})$ be the conformal parabolic geometry obtained by the Fefferman-type construction described in the previous paragraph. Then $\tilde{\omega}$ is normal if and only if $\omega$ is torsion-free.

**Proof.** Let $\omega$ be torsion-free and let $\kappa$ be the corresponding curvature function. It is easy to show that the two summands in the normality condition (40) vanish separately on $\kappa$ and that $\kappa$ takes values in the $P'$-submodule $\Lambda^2(g/p')^* \otimes (g_0'^{ss} \oplus p'_+^\circ) \subseteq \Lambda^2(g/p')^* \otimes p'$, see e.g. Subsection 3.8 in [CZ09]. Since $g_0'^{ss} \oplus p'_+ \subseteq q$ and $q = g \cap \tilde{p}$, all assumptions of Proposition 3.3 are satisfied, hence the normality of $\tilde{\omega}$ follows.

The converse direction is obvious since normal $\tilde{\omega}$ is torsion-free and concurrently $q \cap \tilde{p} = q$ is contained in $p' \subseteq g$. Hence $\kappa$ takes values in $\Lambda^2(g/p')^* \otimes p'$, i.e. $\omega$ is torsion-free. □

**Remark 4.2.**
(a) To relate Proposition 4.1 to Proposition 3.4, we remark that any three dimensional Lagrangean contact structure is automatically torsion-free, while in higher dimensions the Lagrangean contact structure induced by a projective structure as above is torsion-free if and only if it is flat, or equivalently, if and only if the initial projective structure is flat (see e.g. Proposition 4.4.2. in [CS09]). Note that the torsion-freeness of general Lagrange contact structure is equivalent to the integrability of both distributions $e', f' \subseteq TM'$.

(b) Since the Fefferman-type construction turns out to be normal in this case, we denote induced the Cartan connection form simply $\tilde{\omega}$.

The aim of the rest of the section is to derive a local characterisation of split-signature conformal structures arising from torsion-free Lagrangean contact structures.

**Proposition 4.3.** Let $(M, c)$ be a conformal spin manifold of split-signature $(n,n)$. Then the following conditions are locally equivalent:
(a) The spin tractor bundle admits two pure parallel tractor spinors \( s_E \in \Gamma(\tilde{S}_+) \) and \( s_F \in \Gamma(\tilde{S}_-) \) with non-trivial pairing.

(b) The conformal holonomy \( \text{Hol}(c) \) reduces to \( \text{SL}(n+1) \subseteq \text{Spin}(n+1,n+1) \) preserving a decomposition into maximally isotropic subspaces \( E \oplus F = \mathbb{R}^{n+1,n+1} \).

(c) The adjoint tractor bundle admits a parallel involution, i.e. \( K \in \Gamma(A\tilde{M}) \) such that \( K^2 = 1 \) and \( \tilde{\nabla}K = 0 \).

Proof. Given two parallel pure tractor spinors \( s_E, s_F \) with non-trivial pairing, the conformal holonomy \( \text{Hol}(c) \) is contained in \( \text{SL}(n+1) \subseteq \text{Spin}(n+1,n+1) \), which preserves the decomposition into the kernels \( E = \text{ker} s_E \) and \( F = \text{ker} s_F \), respectively. Thus (a) implies (b). Given such a holonomy invariant decomposition, we obtain a skew-symmetric involution \( K \) by defining it as the identity on \( E \) and minus the identity on \( F \), and this determines a parallel adjoint tractor \( K \). Thus (b) implies (c).

To verify that, locally, (c) implies (a), first notice that the skew symmetry of \( K \) implies that the eigenspaces to +1 and -1 have the same dimension and are dual each other with respect to the tractor metric; thus the conformal holonomy \( \text{Hol}(c) \) is reduced to \( \text{GL}(n+1) \). Next, we can use a result of Proposition 2.1 in [ˇCG10] which shows that if \( s \in \Gamma(\tilde{M}) \) is a parallel adjoint tractor, then the pairing with the normal conformal curvature vanishes. In particular for \( s = K \), \( \langle \tilde{\Omega}, K \rangle = 0 \), which implies that the curvature has values in \( \mathfrak{s}(n+1) \). Since \( \mathfrak{s}(n+1) \subseteq \mathfrak{g}(n+1) \) is an ideal, this implies that the conformal holonomy Lie algebra is contained in \( \mathfrak{s}(n+1) \) - see e.g. [Lei08] for an analogous argument in the CR case. Thus there are, locally, parallel pure tractor spinors \( s_E \) and \( s_F \).

Corollary 4.4. Let \((\tilde{M}, c)\) be the induced conformal spin structure on the Fefferman space over a torsion-free Lagrangean contact manifold. Then \( \tilde{M} \) carries tractor fields \( s_E, s_F, K \) satisfying the properties from Proposition 4.3. The underlying objects are pure twistor spinors \( \eta, \chi \), and a light-like conformal Killing field \( k \), which are all nowhere-vanishing.

Proof. Completely analogously to the proof of Proposition 3.2 one shows that the Fefferman space over a Lagrangean contact manifold carries induced tractor objects \( s_E, s_F, K \) having the required algebraic properties and non-vanishing projecting slots. By construction, they are parallel with respect to the induced tractor connection \( \tilde{\nabla} \) and, by Proposition 4.1, we know that this is already the normal conformal tractor connection. Thus they are parallel for the normal tractor connection and the corresponding underlying objects are pure twistor spinors and a light-like conformal Killing field, respectively.

Applying results from [ˇCG14] and [Cap05] easily yields a converse to Corollary 4.3. Suppose we are given a 2n-dimensional split-signature conformal structure \((M, c)\) and a holonomy reduction to \( \text{SL}(n+1) \subseteq \text{Spin}(n+1,n+1) \) determined by two parallel pure tractor spinors \( s_E \in \Gamma(\tilde{S}_+) \) and \( s_F \in \Gamma(\tilde{S}_-) \). Let \( \chi, \eta \) be the underlying twistor spinors and \( k \) be the underlying conformal Killing field. Recall from Section 3.3 that \( \tilde{G}/\tilde{P} \) decomposes under \( G \) into three orbits: an open one and two closed \( n \)-dimensional ones.
Theorem 2.6 in [CGH14] shows that, correspondingly, we have a decomposition of $\tilde{M}$ into so-called curved orbits and, provided they are non-empty, each of them carries an induced Cartan geometry of the same type as the corresponding orbit in the homogeneous model. Here this means that the closed $n$-dimensional curved orbits carry induced Cartan geometries of type $(\text{SL}(n+1), P)$ and thus inherit projective structures. The open curved orbit carries an induced Cartan geometry of type $(\text{SL}(n+1), Q)$. In terms of underlying data the decomposition of $\tilde{M}$ can be described as the decomposition into the zero sets of $\chi$ and $\eta$, respectively, and the open subset where both spinors, and thus $k$, are non-vanishing.

Proposition 4.5. Let $(\tilde{M}, c)$ be a conformal spin manifold of dimension $2n$ satisfying the conditions from Proposition 4.3 and assume that the underlying twistor spinors $\eta$, $\chi$ and the conformal Killing field $k$ are nowhere-vanishing. Then, around every point, $(\tilde{M}, c)$ is locally equivalent to the induced conformal structure on the Fefferman space over a torsion-free Lagrangean contact manifold of dimension $2n-1$.

Proof. Since $k$ is nowhere-vanishing, around each point one can form a local leaf space, $M'$, for the one-dimensional distribution spanned by $k$. Further, since the conformal Killing field $k$ corresponds to a parallel adjoint tractor, it inserts trivially into the curvature of the normal conformal Cartan connection, see Corollary 3.5 in [Cap08]. By the discussion above, the conformal Cartan geometry $(\tilde{G} \to \tilde{M}, \tilde{\omega})$ reduces to a Cartan geometry $(G \to \tilde{M}, \omega)$ of type $(G, Q)$. Hence $k$ inserts trivially into the curvature of $\omega$, i.e. $i_k \Omega = 0$.

Since, under the identification $T\tilde{M} = G \times_Q (g/q)$, the distribution spanned by $k$ corresponds to $p'/q \subseteq g/q$, this is precisely the curvature restriction from Corollary 2.7 in [Cap05]. By connectedness of $P'/Q$, we can thus conclude that $M'$ inherits a Cartan geometry of type $(G, P')$, and the correspondence space of that geometry is locally isomorphic to $\tilde{M}$. Since $\tilde{\omega}$ is torsion-free and $\tilde{g} \cap \tilde{p} \subseteq p'$, the parabolic geometry $(G \to M', \omega)$ is torsion-free and regular. Consequently, it determines a torsion-free Lagrangean contact structure on $M'$. The Fefferman-type construction discussed in this section then recovers the original conformal structure. □

Remark 4.6. By Theorem 3.1 in [CG10], if $K \in \Gamma(A\tilde{M})$ is a parallel section and the underlying conformal Killing field $k$ is light-like, then $K^2 = \lambda \text{id}$ where $\lambda$ is a constant explicitly given as

$$\lambda = \frac{1}{(2n)^2} (\tilde{D}_ak^a)^2 - k^a \tilde{P}_{ab}k^b - \frac{1}{2n} k^a \tilde{D}_a \tilde{D}_bk^b.$$ 

Up to a positive multiple, this is $(K, K)$. This allows us to rephrase the characterisation result Proposition 4.5 in underlying terms: A split-signature conformal structure $(\tilde{M}, c)$ is locally induced by a torsion-free Lagrangean contact structure if and only if it admits a light-like conformal Killing field $k$ that inserts trivially into Weyl and Cotton tensor such that $\lambda$ as defined above is positive.

4.2. Exceptional case: Dimension $n = 2$. Conformal structures induced from 2-dimensional projective structures are special and well-studied see e.g. [NS03], [DT10], [CS07]. The fact that for $n = 2$ normality is preserved (see
Proposition 3.4 implies that the induced 4-dimensional conformal structures satisfy the condition from Proposition 4.3 with nowhere-vanishing underlying twistor spinors and conformal Killing field. Moreover, the fact that they come from projective structures immediately implies that the induced curvature must be horizontal, i.e. \( i_X \tilde{\kappa} = 0 \) for all \( X \in f \).

**Proposition 4.7.** A conformal structure of signature \((2, 2)\) is locally equivalent to the induced conformal structure on the Fefferman space over a 2-dimensional projective structure if and only if the properties from Corollary 4.4 hold and the curvature of the normal conformal Cartan connection satisfies

\[
i_X \tilde{\kappa} = 0 \quad \text{for all} \quad X \in f.
\]

**Proof.** The properties from Corollary 4.4 imply that the normal conformal Cartan geometry \((\tilde{\mathcal{G}}, \tilde{\omega})\) can be reduced to a Cartan geometry \((G, \omega)\) of type \((G, Q)\). This Cartan geometry is locally the correspondence space over a Cartan geometry of type \((G, P)\), i.e. a projective geometry, if and only if \( i_X \tilde{\kappa} = 0 \) for all \( X \in f \sim p/q \), see [Cap05]. \(\square\)

**Remark 4.8.** For \( n = 2 \) the intermediate 3-dimensional Lagrangean contact structure can be equivalently viewed as a path geometry (or geometry of second order ODEs modulo point transformations). This geometry comes from a projective structure, i.e. the paths are the unparametrised geodesics of a projective class of connections, if and only if one of the two harmonic curvatures vanishes. It is shown in [NS03, Nur05] that this is equivalent to vanishing of a part (self dual or anti-selfdual) of the Weyl curvature of the induced conformal structure. In particular, the condition that \( i_X \tilde{\kappa} = 0 \) for all \( X \in f \) can be replaced by the condition that the conformal structure be anti-selfdual.

5. Normalisation and characterisation

By Proposition 3.4 for \( n \geq 3 \), the induced conformal Cartan connection associated to a non-flat \( n \)-dimensional projective structure differs from the normal conformal Cartan connection for the induced conformal structure. In this section we will analyse the form of the difference and thus derive properties of the induced conformal structures. Furthermore, we will show that any split-signature conformal manifold having these properties is locally equivalent to the conformal structure on the Fefferman space over a projective manifold.

5.1. The normalisation process. We are going to normalise the conformal Cartan connection \( \tilde{\omega}^{ind} \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}}) \) that is induced by a normal projective Cartan connection \( \omega \in \Omega^1(\mathcal{G}, \mathfrak{g}) \). Any other conformal Cartan connection \( \tilde{\omega}' \) differs from \( \tilde{\omega}^{ind} \) by some \( \Psi \in \Omega^1(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}}) \) so that \( \tilde{\omega}' = \tilde{\omega}^{ind} + \Psi \).

This \( \Psi \) must vanish on vertical fields and be \( \bar{P} \)-equivariant. The condition on \( \tilde{\omega}' \) to induce the same conformal structure on \( \tilde{M} \) as \( \tilde{\omega}^{ind} \) is that \( \Psi \) has values in \( \bar{\mathfrak{p}} \subseteq \tilde{\mathfrak{g}} \). One can therefore regard \( \Psi \) as a \( \bar{P} \)-equivariant function \( \Psi : \tilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\bar{\mathfrak{p}})^* \otimes \bar{\mathfrak{p}} \). According to the general theory of the normalisation process for parabolic geometries as outlined in [CS09, Section 3.1.13] there is
a unique such $\Psi$ such that the curvature function $\tilde{\kappa}'$ satisfies $\tilde{\partial}^*\tilde{\kappa}' = 0$, and then $\tilde{\omega}'$ is the normal conformal Cartan connection $\tilde{\omega}_{\text{nor}}$.

The failure of $\tilde{\omega}^{\text{ind}}$ to be normal is given by $\tilde{\partial}^*\tilde{\kappa}^{\text{ind}} : \tilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$. The normalisation of $\tilde{\omega}^{\text{ind}}$ proceeds by homogeneity of $(\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{p}$, which decomposes into two homogeneous components according to the decomposition $\tilde{\mathfrak{p}} = \tilde{\mathfrak{g}}_0 \oplus \mathfrak{p}_+$. In the first step of normalisation one looks for a $\Psi^1$ such that $\tilde{\omega}^1 = \tilde{\omega} + \Psi^1$ has $\tilde{\partial}^*\tilde{\kappa}^1$ taking values in the highest homogeneity, i.e. $\tilde{\partial}^*\tilde{\kappa}^1 : \tilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \mathfrak{p}_+$.

To write down this first normalisation we employ a Weyl structure $\tilde{\mathcal{G}}_0 \xrightarrow{\sim} \tilde{\mathcal{G}}$, and by Proposition 3.5 we can take a reduced Weyl structure, i.e. one that is induced by a $Q_0$-reduction

$$\mathcal{G}_0 \xrightarrow{\sim} \mathcal{G} \xrightarrow{\sim} \tilde{\mathcal{G}}.$$ 

This allows us to project $\tilde{\partial}^*\tilde{\kappa}^{\text{ind}}$ to $(\tilde{\partial}^*\tilde{\kappa}^{\text{ind}})_0 : \mathcal{G}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$ and to employ the $\tilde{\mathcal{G}}_0$-equivariant Kostant Laplacian $\tilde{\Box} : (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0 \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0$, $\Box := \tilde{\partial} \circ \tilde{\partial}^* + \tilde{\partial}^* \circ \tilde{\partial}$. For the first normalisation step we need to form a map $\Psi^1 : \tilde{\mathcal{G}} \to (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}$ that agrees with $-\tilde{\Box}^{-1}(\tilde{\partial}^*\tilde{\kappa}^{\text{ind}})_0$ in the $\tilde{\mathfrak{g}}_0$-component. If we have formed any such $\Psi^1$ along $\mathcal{G}_0 \xrightarrow{\sim} \tilde{\mathcal{G}}$ we can just equivariantly extend this to all of $\tilde{\mathcal{G}}$.

To proceed with the analysis of the normalisation we need to establish a couple of technical Lemmas. As before, we denote by $f^0 \subset \mathfrak{p}_+ \cong (\tilde{\mathfrak{g}}/\tilde{\mathfrak{p}})^*$ the annihilator of $f = p/q \subset \mathfrak{g}/\mathfrak{q} \cong \tilde{\mathfrak{g}}/\tilde{\mathfrak{p}}$. Recall that $f^0 = \varphi(\mathfrak{p}_+) \cong f[-2]$.

**Lemma 5.1.** Let $V$ be a $\mathfrak{g}$-representation contained in a $\tilde{\mathfrak{g}}$-representation $\tilde{V}$ and denote by $\phi \mapsto \tilde{\phi}$ the inclusion $\Lambda^k\mathfrak{p}_+ \otimes V \hookrightarrow \Lambda^k\tilde{\mathfrak{p}}_+ \otimes \tilde{V}$ induced by $\varphi : \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$ and $V \hookrightarrow \tilde{V}$. Then, for any $\phi \in \Lambda^k\mathfrak{p}_+ \otimes V$,

$$\tilde{\partial}^*\phi - \tilde{\partial}^*\tilde{\phi} \in \Lambda^{k-1}f^0 \otimes (\Lambda^2\tilde{F} \bullet V) \subseteq \Lambda^{k-1}\tilde{\mathfrak{p}}_+ \otimes \tilde{V}.$$ 

In particular, for the adjoint representations, $\partial^*\phi = 0$ if and only if $\tilde{\partial}^*\tilde{\phi} \in \Lambda^{k-1}f^0 \otimes \Lambda^2\tilde{F}$.

**Proof.** For the sake of presentation, assume that $\tilde{\phi}$ is decomposable, i.e. of the form $\phi = Z_1 \wedge \cdots \wedge Z_k \otimes v$, where $Z_i \in \mathfrak{p}_+$ and $v \in \tilde{V}$. Let us denote by the same symbols also the images of these elements under the inclusion $\mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}}$ and $V \hookrightarrow \tilde{V}$, i.e. $Z_i \in \mathfrak{p}$ and $v \in \tilde{V}$, respectively. Let $\tilde{Z}_i \in f^0$ be the images of $Z_i$ under the inclusion $\varphi : \mathfrak{p}_+ \to \tilde{\mathfrak{p}}_+$. Now, by definition of the Kostant co-differential (40), the difference $\tilde{\partial}^*\tilde{\phi} - \tilde{\partial}^*\tilde{\phi}$ evaluated on any $k-1$ elements from $\mathfrak{g}/\tilde{\mathfrak{p}}$ is a linear combination of terms of the form

$$(Z_i - \tilde{Z}_i) \bullet v. \quad (46)$$

However, the differences $Z_i - \tilde{Z}_i \in \tilde{\mathfrak{p}}$ are represented by the matrices as in (SS) where only the $Z$-entries are non-vanishing and hence contained in $\Lambda^2\tilde{F} \cap \tilde{\mathfrak{p}} = \Lambda^2\tilde{F}$. Thus (46) belong to the image of $\bullet : \Lambda^2\tilde{F} \times V \to \tilde{V}$ and the first claim follows.

For the second claim we use that $\Lambda^2\tilde{F} \bullet \mathfrak{g} = [\Lambda^2\tilde{F}, \mathfrak{g}] \subseteq \Lambda^2\tilde{F}$ and $\Lambda^2\tilde{F} \cap \mathfrak{g} = 0$: since $\tilde{\partial}^*\tilde{\phi}$ (evaluated on any $k-1$ elements from $\mathfrak{g}/\tilde{\mathfrak{p}}$) has values in $\mathfrak{g} \subset \mathfrak{g}$,
Lemma 5.3. For any \( A \) and \( \Lambda \) subspace in \( \tilde{\psi} \) is a sum of terms of the form \( Z_1 \wedge Z_2 \otimes A \), where \( Z_1 \in \tilde{p}_+, Z_2 \in f^\circ \) and \( A \in \Lambda^2 \tilde{F} \). Applying the Kostant co-differential gives

\[
\tilde{\partial}^* (Z_1 \wedge Z_2 \otimes A) = Z_1 \otimes [Z_2, A] - Z_2 \otimes [Z_1, A].
\]

Now \([Z_2, A]\) belongs to \([f^\circ, \Lambda^2 \tilde{F}] = 0\) and \([Z_1, A]\) belongs to \([\tilde{p}_+, \Lambda^2 \tilde{F}] = f^\circ\), hence the claim follows. \(\square\)

The following Lemma contains the crucial information which is necessary to perform our normalization. We are going to specify the curvature function \( \tilde{\kappa}^{ind} \) (later also \( \tilde{\kappa}^{nor} \)) by describing its values along the natural \( Q \)-reduction \( G \hookrightarrow \tilde{G} \) over \( \tilde{M} \). Recall from subsection 3.3 that \( \Lambda^2 \tilde{F} \) is a \( Q \)-invariant subspace in \( \tilde{g}_0 \), which can be identified with \((\Lambda^2 f)[-2]\).

Lemma 5.3. For any \( u \in \tilde{G} \), we have

\[
\tilde{\partial}^* \tilde{\kappa}^{ind}(u) \in f^\circ \otimes \Lambda^2 \tilde{F} \subseteq \tilde{p}_+ \otimes \tilde{g}_0.
\]

Identifying \( \Lambda^2 \tilde{F} \cong (\Lambda^2 f)[-2] \) and \( f^\circ \cong f[-2] \), we have in fact

\[
\tilde{\partial}^* \tilde{\kappa}^{ind}(u) \in (f \otimes \Lambda^2 f)[-4],
\]

i.e. \( \tilde{\partial}^* \tilde{\kappa}^{ind}(u) \) is contained in the kernel of the alteration map

\[
\text{alt} : (f \otimes \Lambda^2 f)[-4] \to (\Lambda^3 f)[-4].
\]

Proof. It is a general assumption that \( \tilde{\kappa}^{ind} \) is induced by a normal projective Cartan connection on \( G \), i.e. \( \partial^* \kappa(u) = 0 \), for any \( u \in \tilde{G} \). Hence it follows from Lemma 5.1 that \( \tilde{\partial}^* \tilde{\kappa}^{ind}(u) \) belongs to \( f^\circ \otimes \Lambda^2 \tilde{F} \cong (f \otimes \Lambda^2 f)[-4] \).

Further we need a finer discussion involving the properties of \( \kappa : G \to \Lambda^2 \tilde{p}_+ \otimes \tilde{g} \) to show that \( \kappa(u) \) belongs to the kernel of the \( Q \)-equivariant map \( \Lambda^2 \tilde{p}_+ \otimes \tilde{g} \to (\Lambda^3 f)[-4] \) given by

\[
\phi \mapsto \text{alt}(\tilde{\partial}^* \phi).
\]

Note that any element \( \phi \in \Lambda^2 \tilde{p}_+ \otimes \tilde{g}_0 \) for which \( \tilde{\partial}^* \phi = 0 \) is mapped to zero: since \( \phi \in \Lambda^2 \tilde{p}_+ \otimes \tilde{p} \) and \( [\tilde{p}_+ \otimes \tilde{p}] = \tilde{p}_+ \), the co-differential \( \tilde{\partial}^* \phi \) has values in \( f^\circ \otimes \tilde{p}_+ \). But, by Lemma 5.1, it also has values in \( f^\circ \otimes \Lambda^2 \tilde{F} \) and \( \tilde{p}_+ \cap \Lambda^2 \tilde{F} = 0 \).

Thus it suffices to consider the harmonic elements from \( \Lambda^2 \tilde{p}_+ \otimes \tilde{g}_0 \), i.e. the ones corresponding to the projective Weyl tensor. For that purpose we consider the simple part of \( Q_0 = Q \cap G_0 \) which is isomorphic to \( \text{SL}(n - 1) \), cf. the matrix realisation (93) where it corresponds to the \( A \)-block. Considering both \( \Lambda^2 \tilde{p}_+ \otimes \tilde{g}_0 \cong \Lambda^2 \mathbb{R}^n^* \otimes \mathbb{R}^n \otimes \mathbb{R}^n \) and \( (\Lambda^3 f)[-4] \cong \Lambda^3 \mathbb{R}^n^* \) as representations of \( \text{SL}(n - 1) \), the map (99) is either trivial or an isomorphism on each \( \text{SL}(n - 1) \)-irreducible component.

One can check that there is only one \( \text{SL}(n - 1) \)-irreducible component that occurs in both spaces, and it is isomorphic to \( \Lambda^2 \mathbb{R}^{n-1}^* \). Hence it suffices to compute (99) on one element contained in such component. Let \( X_n \in \tilde{g}_- \).
and $Z_n \in \mathfrak{p}_+$ be the two dual basis vectors stabilised by $\text{SL}(n - 1)$ and consider an element

$$\phi = Z_1 \wedge Z_2 \otimes X_n \otimes Z_n - Z_1 \wedge Z_2 \otimes X_1 \otimes Z_1 + Z_n \wedge Z_2 \otimes X_n \otimes Z_1.$$  (50)

Indeed $\phi$ is completely trace-free, satisfies the algebraic Bianchi identity and the $\text{SL}(n - 1)$-orbit of $\phi$ is isomorphic to $\Lambda^2 \mathbb{R}^{n-1}$. Now,

$$\widetilde{\partial}^* \widetilde{\phi} = -\widetilde{Z}_1 \otimes \widetilde{Z}_n \wedge \widetilde{Z}_2 - \widetilde{Z}_n \otimes \widetilde{Z}_1 \wedge \widetilde{Z}_2,$$

which indeed lies in the kernel of the alternation map. Hence the statement follows. \hfill \Box

We can now determine the form of the normal conformal Cartan connection:

**Proposition 5.4.** The normal conformal Cartan connection is of the form

$$\tilde{\omega}^\text{nor} = \tilde{\omega}^\text{ind} + \Psi^1 + \Psi^2$$  (51)

where $\Psi^1 = -\frac{1}{2} \widetilde{\partial}^* \kappa^\text{ind} \in \Omega^1_{\text{hor}}(\tilde{G}, \tilde{\mathfrak{p}})$ and $\Psi^2 \in \Omega^1_{\text{hor}}(\tilde{G}, \tilde{\mathfrak{p}}_+)$. Furthermore, along the reduction $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ we have

$$\Psi^1 \in \Omega^1_{\text{hor}}(\mathcal{G}, \Lambda^2 \bar{\mathcal{F}}), \quad \Psi^2 \in \Omega^1_{\text{hor}}(\mathcal{G}, f^o).$$

**Remark 5.5.** Since $\Psi^1$ and $\Psi^2$ are horizontal, they may equivalently be regarded as bundle-valued 1-forms on $\tilde{M}$. Denoting by $\Lambda^2 \bar{\mathcal{F}}$ the associated bundle $\mathcal{G} \times_Q \Lambda^2 \bar{\mathcal{F}}$ over $\tilde{M}$ and by $f^o \subseteq T^* \tilde{M}$ the annihilator of $f = \ker \chi \subseteq T\tilde{M}$, Proposition 5.4 says

$$\Psi^1 \in \Omega^1(\tilde{M}, \Lambda^2 \bar{\mathcal{F}}), \quad \Psi^2 \in \Omega^1(\tilde{M}, f^o) \quad \text{and} \quad (52)$$

$$\Psi^1(v) = 0, \quad \Psi^2(v) = 0 \quad \text{for all } v \in \Gamma(\ker \chi).$$  (53)

Below we also use the corresponding frame forms, i.e. the $\tilde{\mathfrak{p}}$-equivariant functions

$$\phi^1 : \tilde{G} \to (\tilde{\mathfrak{g}} / \tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}, \quad \phi^2 : \tilde{G} \to (\tilde{\mathfrak{g}} / \tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{p}}_+$$

such that, for any $u \in \tilde{G}$,

$$\Psi^1 = \phi^1(u) \circ \tilde{\omega}^\text{ind}, \quad \Psi^2 = \phi^2(u) \circ \tilde{\omega}^\text{ind}.$$  

In these terms, the Proposition means that along the reduction $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ these maps restrict to $Q$-equivariant functions

$$\phi^1 : \mathcal{G} \to f^o \otimes \Lambda^2 \bar{\mathcal{F}}, \quad \phi^2 : \mathcal{G} \to f^o \otimes f^o.$$  (54)

Further we put $\Psi = \Psi^1 + \Psi^2$ and $\phi = \phi^1 + \phi^2$.

**Proof.** The Kostant Laplacian $\tilde{\Box}$ restricts to an invertible endomorphism of $((\tilde{\mathfrak{g}} / \tilde{\mathfrak{p}})^* \otimes \tilde{\mathfrak{g}}_0) \cap \text{im} \tilde{\partial}^*$ that acts by scalar multiplication on each of the $\tilde{\mathfrak{g}}_0$-irreducible components. Now, restricting to $\mathcal{G} \hookrightarrow \tilde{\mathcal{G}}$ and suppressing all arguments $u \in \mathcal{G}$, it was shown in Lemma 5.3 that $\tilde{\partial}^* \kappa^\text{ind}$ is contained in one of the irreducible components, namely in $(f \cap \Lambda^2 f)[-4]$. On this component
acts by multiplication by 2. Thus, the modification map accomplishing the first normalisation step is
\[ \phi^1 : \mathcal{G} \to f^o \otimes \Lambda^2 \tilde{F}, \quad \phi^1 := -\frac{1}{2} \tilde{\Phi} \kappa^{\text{ind}} = -\tilde{\Phi}^{-1} \partial \kappa^{\text{ind}}. \]

Now, let \( \tilde{\omega}^1 := \tilde{\omega}^{\text{ind}} + \phi^1 \circ \tilde{\omega}^{\text{ind}} \) be the modified Cartan connection. The corresponding curvature function \( \tilde{\kappa}^1 \) can be expressed in terms of \( \tilde{\kappa}^{\text{ind}}, \phi^1 \) and its differential \( d\phi^1 \) so that
\[ \tilde{\kappa}^1(X,Y) = \tilde{\kappa}^{\text{ind}}(X,Y) + [X, \phi^1(Y)] - [Y, \phi^1(X)] \]
\[ + d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]) + [\phi^1(X), \phi^1(Y)], \]
where \( X, Y \in \mathfrak{g} \) and \( \xi = (\tilde{\omega}^{\text{ind}})^{-1}(X), \eta = (\tilde{\omega}^{\text{ind}})^{-1}(Y) \), cf. formula (3.1) in [CS09]. For the last term we have \([\phi^1(X), \phi^1(Y)] = 0\) since \( \phi^1(X) \) has values in \( \Lambda^2 \tilde{F} \). The first three terms are
\[ \tilde{\kappa}^{\text{ind}}(X,Y) + [X, \phi^1(Y)] - [Y, \phi^1(X)] = \tilde{\kappa}^{\text{ind}}(X,Y) + \partial \phi^1(X,Y), \]
which by construction vanishes upon application of the Kostant co-differential, i.e. \( \partial \tilde{\kappa}^{\text{ind}} + \partial \phi^1 \). The remaining terms in (55) can be combined into a map \( \Lambda^2 \tilde{g} \to \Lambda^2 \tilde{F} \),
\[ (X,Y) \mapsto d\phi^1(\xi)(Y) - d\phi^1(\eta)(X) - \phi^1([X,Y]), \]
which vanishes upon insertion of two elements \( X, Y \in \mathfrak{p} \). Therefore, applying Lemma 5.2 we conclude that \( \partial \tilde{\kappa}^1 \) has values in \( f^o \otimes f^o \). Thus, the second modification map is
\[ \phi^2 : \mathcal{G} \to f^o \otimes f^o, \quad \phi^2 := -\tilde{\Phi}^{-1} \partial \kappa^1. \]

The information provided in the previous Proposition allows us to determine the properties satisfied by the normal conformal Cartan curvature:

**Proposition 5.6.** The normal conformal Cartan curvature \( \tilde{\kappa}^{\text{nor}} \) restricts to a map
\[ \tilde{\kappa}^{\text{nor}} : \mathcal{G} \to \Lambda^2(\tilde{g}/\tilde{\mathfrak{p}})^* \otimes (\mathfrak{sl}(n+1) \oplus \Lambda^2 \tilde{F}). \]
Moreover, the following integrability condition holds:
\[ i_X \tilde{\kappa}^{\text{nor}}(u) \in f^o \otimes (\Lambda^2 \tilde{F} \oplus f^o) \quad \text{for all } X \in f, u \in \mathcal{G}. \]

**Proof.** Let \( \tilde{\kappa}^{\text{nor}} \) be the curvature function of the normal Cartan connection \( \tilde{\omega}^{\text{nor}} = \tilde{\omega}^{\text{ind}} + \phi \circ \tilde{\omega}^{\text{ind}} \), where \( \phi = \phi^1 + \phi^2 \). With the same conventions as in the proof of Proposition 5.1 formula (3.1) in [CS09] yields
\[ \tilde{\kappa}^{\text{nor}}(X,Y) = \tilde{\kappa}^{\text{ind}}(X,Y) + [X, \phi(Y)] - [Y, \phi(X)] \]
\[ + d\phi(\xi)(Y) - d\phi(\eta)(X) - \phi([X,Y]) + [\phi(X), \phi(Y)]. \]
Clearly, \( \tilde{\kappa}^{\text{ind}}(X,Y) \) has values in \( \mathfrak{sl}(n+1) \) and vanishes upon insertion of \( X \in \mathfrak{p} \). A term of the form \([X, \phi(Y)]\) vanishes if \( Y \in \mathfrak{p} \) and has values in \([\mathfrak{p}, \Lambda^2 \tilde{F} \oplus f^o] \subseteq \Lambda^2 \tilde{F} \oplus f^o \) for \( X \in \mathfrak{p} \). A term of the form \( d\phi(\xi)(Y) \) has values in \( \Lambda^2 \tilde{F} \oplus f^o \) and vanishes for \( Y \in \mathfrak{p} \). The term \( \phi([X,Y]) \) has values in \( \Lambda^2 \tilde{F} \oplus f^o \) and vanishes for \( X, Y \in \mathfrak{p} \). The last term \([\phi(X), \phi(Y)]\) vanishes for all \( X, Y \in \mathfrak{g} \) since \( \phi(X) \) has values in \( \Lambda^2 \tilde{F} \oplus f^o \). Altogether, we obtain (56) and (57). \( \square \)
We observe here that it follows directly from (56) that the pairing of $\kappa^{\text{nor}}$ with the involution $K$ vanishes,

$$\langle \kappa^{\text{nor}}, K \rangle = 0.$$  (59)

To derive properties of induced tractorial and underlying objects on the conformal structure we will need the following preparatory Lemma:

**Lemma 5.7.** Let $V$ be a $\tilde{G}$-representation and $v \in V$ an element which is stabilized under $G \subseteq \tilde{G}$. Let $v \in \Gamma(\tilde{\mathcal{V}})$ be the section of the associated tractor bundle $\tilde{\mathcal{G}} \times \tilde{p} \tilde{\mathcal{V}}$ corresponding to the constant function $G \to V, u \mapsto v$

along $G$. Then the covariant derivative $\tilde{\nabla}^{\text{nor}} v$ corresponds to the $Q$-equivariant function

$$G \to f^o \otimes V, \ u \mapsto \phi^1(u) \cdot v + \phi^2(u) \cdot v.$$  (60)

**Proof.** The covariant derivative $\tilde{\nabla}^{\text{nor}} v$ corresponds to the map

$$X \in \mathfrak{g} \mapsto (\tilde{\omega}^{\text{nor}})^{-1}(X) \cdot v + X \cdot v.$$  (60)

The first term in (60) vanishes since it is the directional derivative of the constant function $v$. Now $\tilde{\omega}^{\text{nor}} = \tilde{\omega}^{\text{ind}} + \phi^1 + \phi^2$, and since $X \cdot v = 0$ the claim follows. $\square$

We now show that the tractors $s_E, s_F$ and $K$ from Proposition 3.2 are all given as BGG-splittings from their underlying objects $\eta = \Pi_0(s_E), \chi = \Pi_0(s_F)$ and $k = \Pi_0(K)$: According to (4) we have to verify

$$\tilde{\partial}^o \tilde{\nabla}^{\text{nor}} s_F = 0, \quad \tilde{\partial}^o \tilde{\nabla}^{\text{nor}} s_E = 0 \quad \text{and} \quad \tilde{\partial}^o \tilde{\nabla}^{\text{nor}} K = 0.$$  (60)

The following propositions show these (and further) properties.

**Proposition 5.8.** The tractor spinor $s_F \in \Gamma(\tilde{S}_-) \text{ is parallel},$

$$\tilde{\nabla}^{\text{nor}} s_F = 0.$$  (60)

**Proof.** Since $\phi^1, \phi^2$ have values in ker $s_F$ we have $\phi^1 \cdot s_F + \phi^2 \cdot s_F = 0$, and thus $\tilde{\nabla}^{\text{nor}} s_F = 0$. $\square$

**Corollary 5.9.** The conformal holonomy of $c$ is contained in the isotropy subgroup of $s_F \in \Delta^{n+1,n+1} \subseteq \text{Spin}(n+1,n+1)$,

$$\text{Hol}(c) = \text{SL}(n+1) \ltimes \Lambda^2(\mathbb{R}^{n+1})^* \subseteq \text{Spin}(n+1,n+1).$$

**Proposition 5.10.** The tractor spinor $s_E \in \Gamma(\tilde{S}_\pm)$ is a BGG-splitting,

$$\tilde{\partial}^o (\tilde{\nabla}^{\text{nor}} s_E) = 0.$$  (60)

**Proof.** Since $\phi^1, \phi^2$ have values in ker $s_F$ we have $\phi^1 \cdot s_F + \phi^2 \cdot s_F = 0$, and thus $\tilde{\nabla}^{\text{nor}} s_F = 0$. $\square$
Proof. The spinor $s_E$ is of the form $s_E = \begin{pmatrix} * \\ \eta \end{pmatrix} \in \Delta_\pm$. According to Lemma 5.3 and Lemma 5.7 $\phi^1$ has values in $(f \odot \Lambda^2 f)[-4]$ and $\tilde{\partial}^* (\tilde{\nabla}^\text{nor} s_E)$ corresponds to

\[ \tilde{\partial}^* (\phi^1 \cdot s_E) = \begin{pmatrix} (\phi^1 \cdot \eta) \tilde{s}_\pm[-\frac{1}{2}] \\ 0 \end{pmatrix}. \]

The projection $(\phi^1 \cdot \eta) \tilde{s}_\pm[-\frac{1}{2}]$ can be realised as the full (triple) Clifford action on $\phi^1(u) \in (\bigotimes^3 f)[-4]$, where $u \in G$. Now it is easy to see that this action must vanish for a $\phi^1(u) \in (f \odot \Lambda^2 f)[-4]$: We realise $\phi^1(u)$ equivalently in $(S^2 f \otimes f)[-4]$ by symmetrisation in the first two slots, then the complete Clifford action on $\eta$ vanishes because the action of the first two slots is just a (trivial) trace multiplication. \[\square\]

Proposition 5.11. The tractor spinor $K \in \Gamma(\tilde{S}_\pm)$ projects to a conformal Killing field $k = \Pi_0(K)$ and is a BGG-splitting, $K = L^\Lambda^2 F(\tilde{\tau})$. In particular, $K$ satisfies

\[ \tilde{\nabla}^\text{nor} K = i_k \tilde{\Omega}^\text{nor} \]

Moreover, we have

\[ i_k \tilde{\kappa}_\text{nor} = 2\phi^2 \Lambda_2 F. \]  

Proof. According to Lemma 5.7, $\tilde{\nabla}^\text{nor} K$ corresponds to

\[ \phi^1 \cdot K + \phi^2 \cdot K. \]  

Now $K/\hat{\mathfrak{p}} = k \in f$, and therefore (63) lies in $\hat{\mathfrak{p}}$. In particular, $\tilde{\nabla}^\text{nor} K$ has trivial projecting slot, and thus $k = \Pi_0(K)$ is a conformal Killing field. Since $\phi^2 \cdot K \in \hat{\mathfrak{p}}_+$, we have that $\tilde{\partial}^* (\tilde{\nabla}^\text{nor} K)$ corresponds to

\[ \tilde{\partial}^* (\phi^1 \cdot K). \]  

Now $\phi^1 \cdot K = -K \cdot \phi^1 = 2\phi^1$, since $K$ acts by multiplication with $-2$ on $\Lambda^2 F$. But $\phi^1 \in \im \tilde{\partial}^* \subseteq \ker \tilde{\partial}^*$, and the expression (64) therefore vanishes. Display (61) is (24) for the conformal Killing field $k$ with its BGG-splitting $K$. In terms of the $Q$-equivariant functions $\phi = \phi^1 + \phi^2$ and $\kappa_\text{nor}$ along $G \hookrightarrow \tilde{G}$, (61) can be expressed as

\[ \phi \cdot K = i_k \kappa_\text{nor}, \]

which yields (62). \[\square\]

We now collect the essential information about the induced conformal structure $(\tilde{M}, c)$ which we derived:

Proposition 5.12. Let $(\tilde{M}, c)$ be the conformal spin structure induced from an oriented projective structure $(M, p)$ via the Fefferman-type construction. Then the following properties are satisfied:

(a) $(\tilde{M}, c)$ admits a nowhere-vanishing light-like conformal Killing field $k$ such that the corresponding tractor endomorphism $K = L_0^\Lambda \tilde{M}(k)$ is an involution, i.e. $K^2 = \im \tilde{\tau}$. 

(b) \((\widetilde{M}, c)\) admits a pure twistor spinor \(\chi \in \Gamma (\widetilde{S} - \frac{1}{2})\) with \(k \in \Gamma (\ker \chi)\) such that the corresponding parallel tractor spinor \(s_F = L_0^\chi (\chi)\) is pure.

(c) \(K\) acts by minus the identity on \(\ker s_F\).

(d) The following integrability condition holds:

\[ v^a w^c \widetilde{W}_{abcd} = 0, \quad \text{for all } v, w \in \Gamma (\ker \chi). \quad \tag{W} \]

The only thing left to show for Proposition 5.12 is that the integrability condition \((57)\) is equivalent to the condition \((W)\) on the Weyl tensor:

**Lemma 5.13.** Let \((\widetilde{M}, c)\) be a split-signature conformal spin structure endowed with tractors \(s_F, s_{\mathcal{L}}\) and \(K\) satisfying conditions (a) and (b) from Proposition 5.12. Then condition \((57)\) is equivalent to \((W)\).

**Proof.** The implication \((57) \implies (W)\) is obvious. It remains to prove the converse implication \((W) \implies (57)\).

By \((W)\), one has that \((i_{X_{\tilde{K}}\tilde{\chi}})_{\tilde{p}_0} (u) \in (f \otimes \Lambda^2 f)[-4] \subseteq f^0 \otimes \Lambda^2 \tilde{F}\) for \(X \in \mathfrak{f}, u \in \mathfrak{G}\). Since \(s_F\) is parallel with respect to \(\widetilde{V}^{\operatorname{nor}}\), we have \(\tilde{K}^{\operatorname{nor}} (u) \in \Lambda^2 (\tilde{g} / \tilde{p})^* \otimes (\tilde{p} \cap \ker s_F)\). The projection of \(\tilde{p} \cap \ker s_F\) to \(\tilde{p}_+\) is precisely \(f^0\), hence it follows that \((i_{X_{\tilde{K}}\tilde{\chi}})_{\tilde{p}_0} (u) \in (\tilde{g} / \tilde{p})^* \otimes f^0\), and we obtain \(i_{X_{\tilde{K}}\tilde{\chi}} (u) \in (\tilde{g} / \tilde{p})^* \otimes (\Lambda^2 \tilde{F} \oplus f^0)\).

We now prove that \(i_{X_1} i_{X_2} \tilde{K}^{\operatorname{nor}} = 0\) for all \(X_1, X_2 \in \mathfrak{f}\). For this purpose it will be useful to work with the curvature form \(\tilde{\Omega}^{\operatorname{nor}}\), which we can represent as

\[ \tilde{\Omega}_{ab}^{\operatorname{nor}} = \begin{pmatrix} 0 & -\tilde{W}_{ab}^c & 0 \\ 0 & \tilde{W}_{ab}^c & \tilde{\chi}_{ab} \\ 0 & 0 & 0 \end{pmatrix}, \]

according to \((14)\). By \((W)\) and the algebraic Bianchi identity, \(\tilde{W}_{abcd}\) vanishes upon insertion of \(v, w \in \Gamma (\ker \chi)\) into any two slots, and in particular \(v^a w^b \tilde{W}_{abcd} = 0\). Thus, it remains to check that \(v^a w^b \tilde{\chi}_{ab} = 0\). As in the proof of Proposition 3.2, a vector field \(w \in \Gamma (\ker \chi)\) corresponds to a section \(\begin{pmatrix} w^d \\ \frac{w}{0} \end{pmatrix} \in \Gamma (\tilde{F})\). According to \((19)\),

\[ v^a \tilde{\Omega}_{ab}^{\operatorname{nor}} \circ \begin{pmatrix} * \\ w^d \end{pmatrix} = v^a \tilde{W}_{ab}^c d^c w^d \in \Gamma (\tilde{\mathcal{E}}_b \otimes \tilde{T}). \]

Since \(i_w \tilde{\Omega}_{ab}^{\operatorname{nor}}\) annihilates \(\tilde{T}\), it follows that \(v^a w^c \tilde{\chi}_{ab} = 0\). Using \(\tilde{\chi}_{ab} = -\tilde{\chi}_{bra} - \tilde{\chi}_{abr}\), we obtain also \(v^a w^b \tilde{\chi}_{ab} = 0\).

**\(\Box\)**

5.2. **Characterisation.** We are now going to characterise the induced conformal structures. For this purpose it would be useful to have a formula that describes the induced Cartan connection \(\tilde{\omega}^{\operatorname{ind}}\) in terms of purely conformal data, i.e. \(\tilde{\omega}^{\operatorname{nor}}\) and \(\tilde{K}^{\operatorname{nor}}\). We will achieve such an explicit formula in the next section after a more refined analysis, see Theorem 6.15. However, to obtain a characterisation, it will in fact be sufficient for now to work with the following (‘intermediate’) Cartan connection form:

\[ \tilde{\omega}' := \tilde{\omega}^{\operatorname{nor}} - \frac{1}{2} L_0^\chi \tilde{K}^{\operatorname{nor}}. \quad \tag{65} \]
The following observation then follows immediately from Proposition 5.12 and (65):

**Lemma 5.14.** The pullbacks of the Cartan connection forms \( \tilde{\omega}' \) \( \in \Omega^1_{hor}(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}}) \), \( \tilde{\omega}^{nor} \in \Omega^1_{hor}(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}}) \) and \( \tilde{\omega}^{ind} \in \Omega^1_{hor}(\tilde{\mathcal{G}}, \tilde{\mathfrak{g}}) \) to \( \mathcal{G} \hookrightarrow \tilde{\mathcal{G}} \) agree modulo forms with values in \( \mathfrak{p}_+ \subseteq \mathfrak{sl}(n+1) \subseteq \tilde{\mathfrak{g}} \).

For the rest of this section, we will start with a given split-signature conformal spin structure \((\tilde{M}, \mathbf{c})\) satisfying all the properties of Proposition 5.12. In particular, \(\tilde{M}\) is endowed with a conformal Killing field \( k \in \Gamma(T\tilde{M}) \), and we can still use formula (65) to define a Cartan connection \( \tilde{\nabla}' \).

The corresponding tractor connection will be denoted by \( \nabla' \) and the curvature by \( \Omega' \) or \( \tilde{\kappa}' \). The following proposition now shows that the so constructed Cartan connection \( \tilde{\omega}' \) is in fact an \( SL(n+1) \)-connection.

**Proposition 5.15.** Let \((\tilde{M}, \mathbf{c})\) be a split-signature conformal (spin) structure satisfying all the properties of Proposition 5.12. Then the sections \( s_F \) and \( K \) are parallel with respect to the tractor connection \( \nabla' \), i.e.

\[
\nabla's_F = 0, \quad \nabla'K = 0. \tag{66}
\]

In particular, \( \text{Hol}(\tilde{\omega}') \subseteq SL(n+1) \) and \( \tilde{\omega}' \) pulls back to a Cartan connection of type \( (SL(n+1), Q) \) with respect to the \( Q \)-reduction \( \mathcal{G} \hookrightarrow \tilde{\mathcal{G}} \).

Along the reduction \( \mathcal{G} \hookrightarrow \tilde{\mathcal{G}} \), the curvature functions \( \tilde{\kappa}' \) and \( \tilde{\kappa}^{nor} \) are related according to

\[
\tilde{\kappa}' = (\tilde{\kappa}^{nor})_{\mathfrak{sl}(n+1)}. \tag{67}
\]

Moreover, \( \tilde{\kappa}' \) satisfies the following integrability condition:

\[
i_X \tilde{\kappa}'(u) \in f^o \otimes \mathfrak{p}_+ \quad \text{for } X \in f, \ u \in \mathcal{G}. \tag{68}
\]

**Proof.** A tractor connection induced by \( \tilde{\omega}' \) can be written as \( \nabla' = \tilde{\nabla'} + \Psi \) with \( \Psi = -\frac{1}{2}i_k \tilde{\Omega}' \). That \( \nabla's_F = 0 \) follows immediately from the fact that \( \tilde{\nabla}' - \tilde{\nabla}^{nor} = -\frac{1}{2}i_k \tilde{\Omega}' \) has values in \( \Lambda^2 \tilde{\mathcal{F}} \). Since \( K \) is a conformal Killing field we have \( \nabla^{nor}K = i_k \tilde{\Omega}^{nor} \). By definition

\[
\nabla'K = \nabla^{nor}K - \frac{1}{2}i_k \tilde{\Omega}^{nor} \cdot K,
\]

which vanishes, since \( i_k \tilde{\Omega}^{nor} \) has values in \( \Lambda^2 \tilde{\mathcal{F}} \) and therefore \( \frac{1}{2}i_k \tilde{\Omega}^{nor} \cdot K = i_k \tilde{\Omega}^{nor} \). As in Proposition 5.12, we write the decomposition of \( \tilde{\mathcal{F}} \) into maximally isotropic eigenspaces of \( K \) with eigenvalues \( \pm 1 \) as \( \tilde{\mathcal{F}} \oplus \tilde{\mathcal{F}} \).

Since \( K \) is \( \tilde{\nabla}' \)-parallel, it follows that this decomposition is preserved by \( \nabla' \).

Moreover, since \( \tilde{\mathcal{F}} \) is the kernel of the pure tractor spinor \( s_F \) it follows that \( \text{Hol}(\tilde{\omega}') \subseteq SL(n+1) \). In particular, \( \tilde{\omega}' \) reduces to a Cartan connection of type \( (SL(n+1), Q) \) on a \( Q \)-principal bundle \( \mathcal{G} \subseteq \tilde{\mathcal{G}} \).

We further compute that

\[
\tilde{\Omega}' = \tilde{\Omega}^{nor} - \frac{1}{2} d \tilde{\Xi}^{nor} i_k \tilde{\Omega}^{nor} = \tilde{\Omega}^{nor} - \frac{1}{2} d \tilde{\Xi}^{nor} \tilde{\nabla}^{nor} K
\]

\[
= \tilde{\Omega}^{nor} - \frac{1}{2} \Omega^{nor} \cdot K = \tilde{\Omega}^{nor} + \frac{1}{2} K \cdot \tilde{\Omega}^{nor} = (\tilde{\Omega}^{nor})(\tilde{\mathcal{F}} \oplus \tilde{\mathcal{F}}), \tag{69}
\]
where we are again using $\tilde{\nabla}^{nor} K = i_k \tilde{\Omega}^{nor}$ for the conformal Killing field $k$ and that $\Omega^{nor}$ has values in $\mathcal{E} \otimes F \oplus \Lambda^2 \tilde{F}$. Stated for the corresponding curvature functions, this yields $\kappa' = (\tilde{\kappa}^{nor})_{sl(n+1)}$. Moreover, since $\tilde{\kappa}^{nor}$ has values in $\ker s_F \cap \mathfrak{p}$, it follows from $(\mathfrak{p} \cap \ker s_F)_{sl(n+1)} = \mathfrak{p}$ (see (57)) that $\tilde{\kappa}'$ has values in $\mathfrak{p}$.

We know from (57) that $i_X \tilde{\kappa}^{nor}$ has values in $\Lambda^2 \tilde{F} \oplus \mathfrak{f}^0$ for $X \in f$. But since $(\Lambda^2 \tilde{F})_{sl(n+1)} = \{0\}$ and $(\mathfrak{p}_+ \cap \ker s_F)_{sl(n+1)} = \mathfrak{p}_+$ (see again (57)), we obtain that $(i_X i_X \tilde{\kappa}^{nor})_{sl(n+1)}$ has values in $\mathfrak{p}_+$. Finally, $(i_X i_X i_X \tilde{\kappa}^{nor})_{sl(n+1)} = 0$ for $X_1, X_2 \in f$ follows immediately from (57), and altogether we obtain (58).

Next, before stating the main characterisation Theorems 5.17 and 5.18, we will show the following proposition on factorisations of particular Cartan geometries. This proposition can be understood as an adapted variant of Theorem 2.7 of [Cap05].

**Proposition 5.16.** Let $(G \to \tilde{M}, \omega)$ be a Cartan geometry of type $(SL(n+1), Q)$ with curvature $\kappa : G \to \Lambda^2 (g/q)^* \otimes g$. We assume the following:

$$i_{X_1} i_{X_2} \kappa(u) \in \mathfrak{p}$$

for $u \in G$, $X_1, X_2 \in g/q$,

$$i_{X_1} i_{X_2} \kappa(u) \in \mathfrak{p}_+$$

for $X_1 \in g/q, X_2 \in g/q$,

$$i_{X_1} i_{X_2} \kappa(u) = 0$$

for $X_1, X_2 \in g/q$.

Then $G$ is locally a $P$-bundle over $M = G/P$ and $\omega$ defines a canonical projective structure on $M$.

**Proof.** The fact that $i_{X_1} i_{X_2} \kappa = 0$ for all $X_1, X_2 \in g/q$ implies that $G$ is locally a $P$-bundle $G \to M$ by [Cap05]. We will restrict $G$ to assume this globally. We define $M = G/P$ and $G_0 = G/P_+$.

Let $\sigma : G_0 \to G$ be a $G_0$-equivariant splitting. It follows from $\kappa(X, \cdot) \in \mathfrak{p}_+$, for all $X \in g/q$, that

$$L_{\zeta_X} \omega = -\text{ad}(X) \circ \omega \mod \mathfrak{p}_+,$$

for all $X \in \mathfrak{p}$. Now define $\theta \in \Omega^1(G_0, g_-), \gamma \in \Omega^1(G_0, g_0)$ and $\rho \in \Omega^1(G_0, \mathfrak{p}_+)$ via the decomposition $\sigma^*(\omega) = \theta \oplus \gamma \oplus \rho$. Since $\sigma$ is $G_0$-equivariant and the Lie derivative is compatible with pullbacks it follows that

$$L_{\zeta_X} (\theta \oplus \gamma) = -\text{ad}(X) \circ (\theta \oplus \gamma)$$

for all $X \in \mathfrak{g}_0$. In particular $\theta$ and $\gamma$ are $G_0$-equivariant and define a (reductive) Cartan geometry $(G_0 \to M, \theta \oplus \gamma)$ of type $(\mathbb{R}^n \times SL(n), SL(n))$, i.e. an affine connection on $M$. Since by assumption $\Omega$ has values in $\mathfrak{p}$, $\theta \oplus \gamma$ is torsion-free and so is the affine connection.

Now take another splitting $\sigma' = \sigma \cdot \exp(\Upsilon)$ for some $\Upsilon : G \to \mathfrak{p}_+$. Then since $\text{Ad}(\exp(\Upsilon))$ acts by the identity on $g_- = g/\mathfrak{p}$ one has $(r^{\exp(\Upsilon)})^* \omega = \omega$ modulo $\mathfrak{p}$, and thus $\theta$ is independent of the choice of splitting. Let $\sigma^*(\omega) = \theta \oplus \gamma \oplus \rho$. Then $\sigma^*(\omega) = \theta \oplus \gamma \oplus \rho'$ and $\theta \oplus \gamma' = \text{Ad}(\exp(\Upsilon)) \circ (\theta \oplus \gamma)$ (projected to $g_- \oplus \mathfrak{g}_0$). But since $\exp(\Upsilon) \in P_+$, this shows that $\gamma'$ is projectively equivalent to $\gamma$. We thus obtain a well-defined projective structure on $M$.

Since $\omega$ is $P$-torsionfree and $P$-equivariant modulo $\mathfrak{p}_+$, it can be (uniquely) modified to a normal Cartan connection $\omega^{nor} \in \Omega^1(G, \mathfrak{g})$ with $\omega^{nor} - \omega \in$
Theorem 5.17. A split-signature \((n,n)\) conformal spin structure \(c\) on a manifold \(\tilde{M}\) is (locally) induced by an \(n\)-dimensional projective structure via the Fefferman-type construction if and only if the properties stated in Proposition 5.12 are satisfied.

Proof. Starting with a projective structure \((M,p)\), it follows from Proposition 5.12 that the induced conformal structure \((\tilde{M},c)\) has all the stated properties. On the other hand, let \((\tilde{M},c)\) be a conformal structure with the stated properties. Then, by Proposition 5.15, \(\tilde{\omega}'\) restricts to a \(Q\)-equivariant Cartan connection form with values in \(\mathfrak{sl}(n+1)\) on the reduction \(\tilde{G} \hookrightarrow \tilde{\mathcal{G}}\).

The corresponding curvature \(\tilde{\kappa}'\) takes values in \(p\) and for \(v \in f\) we have that \(i_v\tilde{\omega}'\) takes values in \(p_+\). It follows from Proposition 5.16 that \(\tilde{\omega}'\) factorises to a projective structure \(p\) on the leaf space \(M\).

Let us now show that the two constructions are inverse to each other. Assume first that a conformal structure \((\tilde{M},c)\) is induced by a projective structure \((M,p)\). Then according to Lemma 5.14 \(\tilde{\omega}'\) and \(\tilde{\omega}'_{\text{nor}}\) agree modulo \(p_+\). This implies that the projective structure defined by \(\tilde{\omega}'\) is equal to the original projective structure. Conversely, assume now that \((M,p)\) is a projective structure with associated Cartan geometry \((\tilde{G},\omega')\) that is induced from a conformal structure \((\tilde{M},c)\) with associated Cartan geometry \((\tilde{G},\tilde{\omega}'_{\text{nor}})\). Since \(\tilde{\omega}'\) is not normal, but torsion-free, there is \(\varphi \in \Omega^1_{\text{hor}}(\tilde{G},p_+)\) such that \(\omega' + \varphi\) is the normal projective Cartan connection. Since \(p_+ \subseteq \tilde{p}\) the induced conformal structure on \(\tilde{M}\) agrees with the original conformal structure. Thus, the Fefferman-type construction (with normalisation) and the described factorisation are (locally) inverse to each other. □

We will now rephrase the assumption of the characterisation theorem in terms of underlying objects.

Theorem 5.18. A split-signature \((n,n)\) conformal spin structure \(c\) on a manifold \(\tilde{M}\) is (locally) induced by an \(n\)-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:

(a) \((\tilde{M},c)\) admits a nowhere-vanishing light-like conformal Killing field \(k^a\) such that relations

\[
\begin{align*}
 k^a k_a &= 0, & \rho^a \rho_a &= 0, \\
 \mu^a_b k^b &= \varphi k^a, & \mu^a_b \rho^b &= -\varphi \rho^a, \\
 k^a \rho_a &= \varphi^2 - 1, & \mu^a_c \mu_{cb} &= g_{ab} + 2 k^a (\rho_b)
\end{align*}
\]

hold, with \(\mu, \varphi\) and \(\rho\) determined by \(k\) as in (21).

(b) \((M,c)\) admits a pure twistor spinor \(\chi \in \Gamma(\tilde{S}_{_{\frac{1}{2}}})\) with \(k \in \Gamma(\ker \chi)\) such that the tractor spinor \(\left(\frac{1}{\sqrt{2\pi}} D_\chi\right) \in \Gamma(\tilde{S}_{_{\frac{1}{2}}})\) is pure.

(c) The Lie derivative of \(\chi\) with respect to the conformal Killing field \(k\) is

\[
\mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi.
\]
(d) The following integrability condition holds:
\[ v^r w^s \tilde{W}_{arbs} = 0 \quad \text{for all } v^r, w^s \in \Gamma(\ker \chi). \] (W)

In this case, i.e., when \((\tilde{M}, c)\) is induced by a projective structure, \(\tilde{f} = \ker \chi\) is an integrable distribution and the original projective structure is defined on the leaf space of \(\tilde{f}\).

**Proof.** We first show that the stated properties are equivalent to those of Proposition 5.12 and thus obtain the result by an application of Theorem 5.17.

(a) According to calculations (95) in Appendix B.1, equations (70) in terms of \(\mu, \varphi\) and \(\rho\) are equivalent to \(K := L_0 \Lambda^2 \tilde{T}(k)\) being an involution.

(b+c) Since \(k\) is a conformal Killing field which splits to \(K\) and \(\chi\) is a twistor spinor which splits to \(s_F\), \(L_k \chi = -\frac{1}{2} (n+1) \chi\) is equivalent to \(K \cdot s_F = -\frac{1}{2} (n+1) s_F\). Since the tractor spinor \(s_F\) is pure it has an \((n+1)\)-dimensional, maximally isotropic kernel \(\ker s_F\). Now \(K \cdot s_F = -\frac{1}{2} (n+1) s_F\) is equivalent to \(K\) acting by \(-1\) on \(s_F\), which therefore coincides with the \((-1)\)-eigenspace of \(K\).

(d) According to Lemma 5.13, if (a) and (b) hold, the stated integrability condition (W) on the Weyl tensor is equivalent to the integrability condition (77) of Proposition 5.12.

Now, if \((\tilde{M}, c)\) is (locally) the Fefferman-type space of a projective structure, it follows immediately that \(\tilde{f} = \ker \chi\) is integrable and the original projective structure is defined on the leaf space of \(\tilde{f}\), cf. Proposition 3.2, (d). □

**Remark 5.19.**

(a) Since \(\chi\) is by assumption a twistor spinor, the tractor spinor \(s_F = (\tilde{\chi}) \in \Gamma(\tilde{S}_-)\) is parallel with respect to \(\tilde{\nabla}^{\text{nor}}\). In particular, purity of \(s_F\) can be checked at one point. If \(\tilde{\chi} = 0\), this tractor spinor will be pure whenever \(\chi\) is pure. Otherwise, in case \(\tilde{\chi} \neq 0\), purity of \((\tilde{\chi})\) is equivalent to \(\chi\) and \(\tilde{\chi}\) being pure and their kernels having \((n-1)\)-dimensional intersection, cf. Proposition III-1.12 in [Che54] or [HM88, Tag17].

(b) To compute the Lie derivative of \(\chi\) with respect to the conformal Killing field one may use the formula
\[ \mathcal{L}_k \chi = \tilde{D}_k \chi - \frac{1}{4} (\tilde{D}_{(a} k_{b)}) \gamma^{a} \gamma^{b} \chi - \frac{1}{4n} (\tilde{D}_p k^p) \chi. \] (71)

This is the Lie derivative of the (weighted) spinor \(\chi\) with respect to \(k\), see e.g. [FFFG96, Ham12].

6. **Reduced scales, explicit normalization and an alternative characterisation**

Although we obtained the desired characterisation in Theorem 5.18 we do not yet know the explicit relationship between the induced Cartan connection form \(\tilde{\omega}^{\text{ind}}\) and the normal conformal Cartan connection form \(\tilde{\omega}^{\text{nor}}\). One of aims of the present section is to obtain a formula for this difference, which is achieved in Theorem 6.5. As a consequence, we also obtain an explicit
formula for the curvature $\hat{\Omega}^{\text{ind}}$ in terms of the normal conformal Cartan curvature $\hat{\Omega}^{\text{nor}}$ in Corollary 6.6. In this more refined analysis, reduced scales will play an important role. We start with their characterisation and some computational consequences.

6.1. Characterisation of reduced scales. The notion of reduced Weyl structures and reduced scales is introduced in Subsection 3.7. Here we shall find an intrinsic characterisation (i.e. using the conformal structure only) of reduced scales and discuss their properties.

As the scale bundle on the projective manifold $M$ we may consider the positive elements in the density bundle $\tilde{E}(1)$, which is the projecting part of the conformal standard tractor bundle $\tilde{T}^*$, see Subsection 2.6. Similarly, on the Fefferman space $\tilde{M}$ we take the positive elements in the density bundle $\tilde{E}(1)$, the projecting part of the conformal standard tractor bundle $\tilde{T}$. Hence for a projective scale $\rho \in \Gamma(\tilde{E}_+(1))$ we have the tractor $L_0^\rho (\rho) \in \Gamma(\tilde{T}^*)$; similarly, for a conformal scale $\sigma \in \Gamma(\tilde{E}_+(1))$ we have the tractor $L_0^\rho (\sigma) \in \Gamma(\tilde{T})$. These will be termed scale tractors.

On the one hand, sections of $\tilde{E}(1) \to M$ form a subset of all sections of $\tilde{E}(1) \to \tilde{M}$, see Lemma 3.8. On the other hand, sections of $\tilde{T}^* \to M$ are understood as specific sections of the bundle $\tilde{F} \to \tilde{M}$, which is a subbundle in $\tilde{T} \to \tilde{M}$, see the generalities in Subsection 3.6 and the setup of our construction in Subsection 3.2. It follows that these two natural inclusions commute with the BGG-splitting operators.

**Lemma 6.1.** Full arrows in the following diagram commute:

$$
\begin{array}{c}
\Gamma(T^*) \longrightarrow \Gamma(\tilde{T}) \\
L_0^T \left( \begin{array}{c} \Pi_0 \\
\tilde{\Pi}_0 \\
\end{array} \right) \longrightarrow L_0^\tilde{T} \\
\Gamma(E(1)) \longrightarrow \Gamma(\tilde{E}[1])
\end{array}
$$

**Proof.** Consider a projective density $\rho \in \Gamma(\tilde{E}(1))$ on $M$, the corresponding tractor $L_0^\rho (\rho) \in \Gamma(T^*)$, and its extension to $\tilde{F} \subseteq \tilde{T}$, which is denoted by $s'$. The extension of $\rho \in \Gamma(\tilde{E}(1))$ to $\tilde{E}[1]$ obviously coincides with the projection $\tilde{\Pi}_0(s')$, and it is denoted by $\sigma$. We need to show that $s' = L_0^\rho (\sigma)$, i.e. that $\bar{\partial}^* \nabla^{\text{nor}} s' = 0$. Since, according to Proposition 5.4 $\tilde{\omega}^{\text{nor}} = \tilde{\omega}^{\text{ind}} + \Psi^1 + \Psi^2$ with $\Psi^1 \in \Omega^1(\tilde{M}, \Lambda^2 \tilde{T}^*)$, $\Psi^2 \in \Omega^1(\tilde{M}, \tilde{F}^*)$, we have

$$
\nabla^{\text{nor}} s' = \nabla^{\text{ind}} s' + \Psi^1 \bullet s' + \Psi^2 \bullet s'.
$$

Since $\Lambda^2 \tilde{T}^*$ acts trivially on $\tilde{F} \subseteq \tilde{T}$, we have $\Psi^1 \bullet s' = 0$. Since $\tilde{F}^0 \subseteq T^\pi \tilde{M}$, it follows that $\bar{\partial}^*(\Psi^2 \bullet s') = 0$. It thus follows that $\bar{\partial}^*(\nabla^{\text{nor}} s') = \bar{\partial}^*(\nabla^{\text{ind}} s')$. Let $\phi$ be the frame form of $\nabla^{\text{nor}} L_0^\rho (\rho)$. Then, according to Lemma 5.1 we have that $\bar{\partial}^* \phi = 0$ since $\Lambda^2 F \bullet F = 0$, and in particular $\bar{\partial}^*(\nabla^{\text{nor}} s') = 0$. \qed

We can now characterise reduced scales in terms of the corresponding scale tractors:

**Proposition 6.2.** Suppose $(\tilde{M}, c)$ is a conformal spin structure of signature $(n, n)$ associated to an $n$-dimensional oriented projective structure $(\tilde{M}, p)$ via
the Fefferman-type construction. Let \( \sigma \in \Gamma(\tilde{E}_+[1]) \) be a conformal scale and let

\[ s := L_0^T(\sigma) \in \Gamma(\tilde{T}) \]

be the corresponding scale tractor. The following statements are equivalent:

(a) The scale \( \sigma \) is reduced.

(b) The tractor \( s \) is a section of \( \tilde{F} \subseteq \tilde{T} \) and both \( \tilde{\nabla}^{nor} s \) and \( \tilde{\nabla}^{ind} s \) is strictly horizontal, i.e. \( v^a \tilde{\nabla}_a^{nor} s = v^a \tilde{\nabla}_a^{ind} s = 0 \) for every \( v \in \Gamma(\ker \chi) \).

(c) The tractor \( s \) is a section of \( \tilde{F} \subseteq \tilde{T} \).

(d) The twistor spinor \( \chi \) is parallel with respect to the Levi-Civita connection \( \tilde{D} \) of the metric corresponding to the scale \( \sigma \).

Furthermore, in reduced scales, the Schouten tensor is strictly horizontal, i.e. it satisfies \( v^a \tilde{\nabla}^{ind} s = 0 \) for every \( v \in \Gamma(\ker \chi) \). In particular the scalar curvature \( \tilde{J} \) vanishes.

Proof. The equivalence (a) \( \iff \) (b) follows immediately from Lemma 6.1 and Proposition 3.5.

The implication (b) \( \implies \) (c) is obvious.

The implication (c) \( \implies \) (d) follows from computation in slots which we shall do in the Levi-Civita connection \( \tilde{D} \) corresponding to the scale \( \sigma \), i.e. \( \tilde{D}\sigma = 0 \). Since \( \chi \) is a twistor spinor, we only need to show that \( \tilde{D}\chi = 0 \) in this scale. Condition (c) means that \( s \cdot s_F = 0 \).

We next need the explicit formulas for the BGG-splitting operators \( L_0^T \) and \( L_0^\tilde{S} \), as given in (15) and (17), respectively. Then, using the formula for the tractor Clifford action (16), we have

\[
s \cdot s_F = L_0^T(\sigma) \cdot L_0^\tilde{S}(\chi) = \begin{pmatrix} 1 \nabla & \frac{1}{n \sqrt{2}} \tilde{D} \chi \\ 0 & \chi \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{n} \tilde{J} \chi \\ -\frac{\tilde{J} \chi}{\sqrt{2} n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\] (72)

Finally, we prove the implication (d) \( \implies \) (b): Since \( \tilde{D}\chi = 0 \), the form of \( s_F \) simplifies to \( s_F = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \). Inspecting the top slot of \( \tilde{\nabla}_a s_F = 0 \), we obtain \( \tilde{P}_{ac} \chi = 0 \) hence \( \tilde{P}_{ac} \in \Gamma(S^2 \tilde{f}) \) which in particular means \( \tilde{J} = 0 \).

Summarising, we have

\[
s = L_0^T(\sigma) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad s_F = L_0^\tilde{S}(\chi) = \begin{pmatrix} 0 \\ \chi \end{pmatrix},
\] (73)

respectively. Hence \( s \cdot s_F = 0 \), i.e. \( s \) is a section of \( \tilde{F} \subseteq \tilde{T} \).

According to (13) and the previous reasoning, we further have

\[
\tilde{\nabla}^{nor}_a s = \begin{pmatrix} 0 \\ \tilde{P}_{ab} \end{pmatrix}
\]

and hence \( v^a \tilde{\nabla}_a^{nor} s = 0 \) for any \( v \in \Gamma(\ker \chi) \). Since \( \tilde{\nabla}^{nor} = \tilde{\nabla}^{ind} + \Psi \) and the horizontality of \( \Psi \), it follows that \( v^a \tilde{\nabla}_a^{ind} s = 0 \). \( \square \)
6.2. Explicit normalization formula. So far we discussed three Cartan connections on the Fefferman space \( \tilde{M} \): \( \tilde{\omega}^{\text{ind}} \) — the one induced by the projective normal Cartan connection \( \omega \) on \( M \), see Subsection 5.3; \( \tilde{\omega}^{\text{nor}} \) — the corresponding conformal normal Cartan connection, see Subsection 5.1; and \( \tilde{\omega}' \) — the modified auxiliary Cartan connection, see Subsection 5.2. Various properties of these and derived objects are enumerated in Propositions 5.12 and 5.15; the following proposition refines the integrability conditions included there.

We will need some properties of the conformal curvature quantities in reduced scales. The computations of these and other properties are straightforward but rather long and deferred to Appendix B.2 for reasons of readability. Concretely, we will use:

\[
\begin{align*}
\tilde{W}_{abcd}^{\mu \nu} &= 0, \quad \text{where} \; \mu_{ab} = \tilde{D}_{(a} k_{b)} , \\
v^f \tilde{Y}_{abc} &= 0, \quad \text{for all} \; v^a \in \Gamma(\ker \chi).
\end{align*}
\]

Alternatively one can deduce (74) from (75) and (59).

**Proposition 6.3.** Let \( (\tilde{M}, \mathfrak{c}) \) be the conformal spin structure induced from an oriented projective structure \( (M, \mathfrak{p}) \) via the Fefferman-type construction. Then, along the reduction \( \mathcal{G} \hookrightarrow \tilde{\mathcal{G}} \),

\[
\begin{align*}
\iota_X \tilde{\kappa}^{\text{nor}}(u) &\in f^o \otimes \Lambda^2 \tilde{F} \quad \text{for} \; X \in f , \; u \in \mathcal{G} \quad (76) \\
\iota_X \tilde{\kappa}'(u) &\in f^o \otimes \Lambda^2 \tilde{F} \quad \text{for} \; X \in f , \; u \in \mathcal{G}. \quad (77)
\end{align*}
\]

**Proof.** From (57) we already know that \( \iota_X \tilde{\kappa}^{\text{nor}} \) has values in \( f^o \otimes (\Lambda^2 \tilde{F} \oplus f^o) \). We note that the top slot of sections of \( \Lambda^2 \tilde{F} \) vanishes in reduced scales, cf. (58). Thus the part in \( f^o \) corresponds to \( v^f \tilde{Y}_{abc} \) for a \( v \in \Gamma(f) \), which however has to vanish by (75). Hence (76) follows. The last condition (77) follows from \( \tilde{\kappa}' = (\tilde{\kappa}^{\text{nor}})_{\mathfrak{sl}(n+1)} \), cf. Proposition 5.15. \( \blacksquare \)

Since \( \tilde{\omega}' \) is an \( \mathfrak{sl}(n+1) \)-connection on \( \tilde{\mathcal{G}} \to \tilde{M} \), it is the extension of a Cartan connection \( \omega' \), on \( \mathcal{G} \to M \). Now, due to (77), any section \( v \in \Gamma(\ker \chi) \) inserts trivially into its curvature. But this is the standard condition on the connection \( \omega' \) to be a Cartan connection also on the bundle \( \mathcal{G} \to M \), i.e. to be a projective Cartan connection, cf. [Cap05].

Furthermore, we will show that the descended Cartan connection is normal, i.e. \( \omega' = \omega \). To do this, we first compute \( \tilde{\partial}^* \tilde{\kappa}' \) and then use the relation between the co-differentials \( \tilde{\partial}^* \) on \( M \) and \( \tilde{\partial}^* \) on \( \tilde{M} \) discussed in Lemma 5.1

**Proposition 6.4.** The curvature \( \tilde{\kappa}' \) satisfies

\[
\tilde{\partial}^* \tilde{\kappa}'(u) = \iota_u \tilde{\kappa}^{\text{nor}}(u) \in f^o \otimes \Lambda^2 \tilde{F} \quad \text{for all} \; u \in \mathcal{G}. \quad (78)
\]

**Proof.** We shall compute \( \tilde{\partial}^* \tilde{\mathcal{Y}}' \) directly. First observe that using Proposition 5.15 we have \( \tilde{\mathcal{Y}}' = (\tilde{\mathcal{Y}}^{\text{nor}})_{\mathfrak{sl}(n+1)} = \tilde{\mathcal{Y}}^{\text{nor}} + \frac{1}{2} \mathbf{K} \cdot \tilde{\mathcal{Y}}^{\text{nor}} \), hence \( \tilde{\partial}^* \tilde{\mathcal{Y}}' = \frac{1}{2} \tilde{\partial}^* (\mathbf{K} \cdot \tilde{\mathcal{Y}}^{\text{nor}}) \), since \( \tilde{\partial}^* \tilde{\mathcal{Y}}^{\text{nor}} = 0 \). The tractor \( \mathbf{K} \) has the form

\[
\mathbf{K} = \begin{pmatrix} \rho_c \\ \mu_{a\alpha c} \\ k_c \end{pmatrix} | \varphi , \quad \text{where} \; \mu_{ab} = \tilde{D}_{(a} k_{b)} \; \text{and} \; k = \frac{1}{2\rho} \tilde{D}^r k_r. \quad (79)
\]
according to \((22)\), which also provides an explicit form of \(\rho_a\). We compute
\[K \cdot \tilde{\Omega}_{ab}^\text{nor}\]
as
\[
\begin{pmatrix}
\rho_c \\
\mu_{c1} | \varphi \\
k_c
\end{pmatrix} \begin{pmatrix}
\tilde{Y}_{dab} \\
\tilde{W}_{abcd1} | 0
\end{pmatrix} = \begin{pmatrix}
\partial^r \tilde{W}_{abc} - \mu_c \tilde{Y}_{rab} + \varphi \tilde{Y}_{cab} \\
-2W_{ab}^r [\rho_{c1} | r] + 2k_{[\rho_{c1}]} ab | k^r \tilde{Y}_{rab}
\end{pmatrix}.
\]
Using this together with \((74)\) and \((75)\) we compute
\[
\tilde{\partial}^r (K \cdot \tilde{\Omega}_{ab}^\text{nor}) = \begin{pmatrix}
0 \\
2k^r \tilde{W}_{rac0c1} | 0
\end{pmatrix} = 2k^r \tilde{\Omega}_{ra}^\text{nor},
\]
which yields \((80)\).

\[\square\]

**Theorem 6.5.** Let \((\mathcal{G}, \omega)\) be a projective normal Cartan geometry over \(M\) and let \((\mathcal{G}, \tilde{\omega}^\text{ind})\) be the conformal Cartan geometry over \(\tilde{M}\) induced via the Fefferman-type construction. Then:

(a) \(\tilde{\omega}^\text{ind} = \omega' = \tilde{\omega}^\text{nor} - \frac{1}{2} i_k \tilde{\omega}^\text{nor}\)

(b) \(\tilde{\omega}^\text{nor} = \tilde{\omega}^\text{ind} + \Psi^1\), where \(\Psi^1 = -\frac{1}{2} \tilde{\partial}^r \tilde{\kappa}^\text{ind} = i_k \tilde{\omega}^\text{nor}\).

**Proof.**

(a) We use that \(i_X \tilde{\kappa}' = 0\) for all \(X \in \mathfrak{f}\) according to \((77)\). Then Proposition 6.4 together with Lemma 5.1 imply that \(\tilde{\partial}^r \tilde{\kappa}' = 0\). Thus \(\omega'\) is projectively normal, and therefore we obtain \(\tilde{\omega}' = \tilde{\omega}^\text{ind}\).

(b) The normalisation process of Proposition 5.4 provides \(\Psi = \Psi^1 + \Psi^2\) such that \(\tilde{\omega}^\text{nor} = \tilde{\omega}^\text{ind} + \Psi\), where \(\Psi^1\), \(\Psi^2\) are the first and second normalisation steps. However since \(\tilde{\omega}' = \tilde{\omega}^\text{ind}\), it follows from Proposition 6.4 and \((65)\) that \(\tilde{\partial}^r \tilde{\kappa}' = \tilde{\partial}^r \tilde{\kappa}^\text{ind}\) is, up to a constant multiple, the difference between \(\tilde{\omega}^\text{nor}\) and \(\tilde{\omega}^\text{ind}\). Therefore already the first normalisation step completes the normalisation, i.e. \(\Psi^2 = 0\).

\[\square\]

Using the explicit relationship provided in Theorem 6.5 we can also obtain a detailed description of the difference between the induced and the normal Cartan curvatures:

**Corollary 6.6.** In a reduced scale, we have the following relation between the curvatures of the induced and the normal conformal Cartan connection:
\[
\tilde{\Omega}_{ab}^\text{ind} = \tilde{\Omega}_{ab}^\text{nor} + \frac{1}{2} K \cdot \tilde{\Omega}_{ab}^\text{nor} = \begin{pmatrix}
\tilde{W}_{abc0c1} - \tilde{W}_{ab}^r [\rho_{c1} | r] + k_{[\rho_{c1}]} Y_{cab} | 0 \\
\frac{1}{2} k^r \tilde{W}_{abc}
\end{pmatrix}.
\]

In particular, \(\frac{1}{2} i_k \tilde{W}\) is the torsion of the Cartan connection \(\tilde{\omega}^\text{ind}\).

**Proof.** Choosing a reduced scale, \(\tilde{D} \chi = 0\), and in particular \(\tilde{\chi} = 0\), and thus the equivalent properties of Lemma 3.1 of Appendix 3.2 hold. Thus, the slots of \(K\) satisfy \(\rho_a = 0\), \(\varphi = -1\) and \(\mu_a \varphi v_r = -v_r\) for any \(v \in \Gamma(\ker \chi)\) using the notation as in \((79)\). Using also \((75)\) and the form of \(K \cdot \tilde{\Omega}_{ab}^\text{nor}\) obtained in the proof of Proposition 6.4 a short computation yields \((80)\).
6.3. **An alternative characterisation.** We have characterised split-signature \((n,n)\) conformal structures \(c\) on \(\tilde{M}\) induced by an \(n\)-dimensional projective structure via the Fefferman-type construction in Theorem 5.1. Our aim here is to show that this can be equivalently formulated by the following characterisation which was obtained by direct computations and spin calculus in [HSSTZ16]:

**Theorem 6.7.** A split-signature \((n,n)\) conformal spin structure \(c\) on a manifold \(\tilde{M}\) is (locally) induced by an \(n\)-dimensional projective structure via the Fefferman-type construction if and only if the following properties are satisfied:

(a) \((\tilde{M},c)\) admits a nowhere-vanishing light-like conformal Killing field \(k\).
(b) \((\tilde{M},c)\) admits a pure twistor spinor \(\chi\) such that \(\widetilde{f} = \ker \chi\) is integrable and \(k \in \Gamma(\widetilde{f})\).
(c) The Lie derivative of \(\chi\) with respect to the conformal Killing field \(k\) is

\[ \mathcal{L}_k \chi = -\frac{1}{2}(n+1)\chi. \]

(d) The following integrability condition holds:

\[ v^r w^s \widetilde{W}_{arb} s = 0 \quad \text{for } v^r, w^s \in \Gamma(\ker \chi). \]

To prove Theorem 6.7 we employ the following result from [HSSTZ16]:

**Proposition 6.8.** Let \(\chi\) be a pure real twistor-spinor on a conformal pseudo-Riemannian manifold \((\tilde{M},c)\) of signature \((n,n)\), with associated totally isotropic \(n\)-plane distribution \(\widetilde{f} = \ker \chi\). Suppose \(\widetilde{f}\) is integrable. Then there exists a conformal rescaling such that locally, \(\chi\) is parallel, i.e. \(\tilde{D}\chi = 0\).

In particular, the assumption on the pure twistor spinor \(\chi\) of Theorem 6.7 guarantees the existence of a suitable compatible metric as in the above proposition, and we shall assume \(\tilde{D}\chi = 0\) in the following.

**Lemma 6.9.** Assuming properties (a)–(d) of Theorem 6.7 the tractor endomorphism \(K = L^{\Lambda^2 T}_0 (k) \subseteq \text{End}(\tilde{T})\) satisfies

\[ K^2 = \lambda \text{id}_{\tilde{T}}. \]

for some \(\lambda \in C^\infty(\tilde{M})\).

We remark that after having shown Theorem 6.7 it follows in particular that \(\lambda\) is actually a constant and hence \(K^2\) is parallel.

**Proof.** We have \(\tilde{\Omega}_{cab}^{\text{nor}} \bullet L_0^{\tilde{S}}(\chi) = 0\) since \(\chi\) is a twistor spinor. Thus working in a scale such that \(\tilde{D}\chi = 0\) we have \(\tilde{\Omega}_{cab}^{\text{nor}} \bullet \left( 0_{\chi} \right) = 0\). Here the top slot means \(\tilde{Y}_{cab}^{\gamma} \chi = 0\). Hence it follows from properties of pure spinors (cf. the discussion around (97)) that \(v^c \tilde{Y}_{cab} = 0\) for \(v \in \Gamma(\tilde{f})\) which in particular means \(k^c \tilde{Y}_{cab} = 0\). Further, using the assumption (d) of Theorem 6.7 we have \(\tilde{D}^b v^c \tilde{W}_{arb}s k^s = 0\). Since \(\tilde{f} = \ker \chi\) is preserved by \(\tilde{D}\), this means

\[ v^r \tilde{W}_{rast} \mu^s t = 0 \quad \text{for } v^a \in \Gamma(\tilde{f}) \text{ and } \mu_{ab} = \tilde{D}_{[a} k_{b]}. \]
We shall use the notation $K^2 = K \otimes K$ where $\otimes$ is the projection $\otimes : \Lambda^2 T \otimes \Lambda^2 T \to S^2 T$. The main step of the proof is to show that $K \otimes K$ is a BGG-splitting, i.e. condition $\nabla^{nor}(K \otimes K) = 0$ is satisfied. Using (61), we have

$$\nabla^{nor}_a(K \otimes K) = 2k^r \nabla^{nor}_{ra} \otimes K = 2 \left( \begin{array}{c} \rho_c \\ \mu_{ca} k \end{array} \right) \otimes \left( \begin{array}{c} -k^r \bar{Y}_{bra} \\ k^r \bar{W}_{rab,b1} \end{array} \right)$$

where we use notation for slots of $K$ as in (19). Computing the slots of this projection we obtain

$$\nabla^{nor}_a(K \otimes K) = 2 \left( \begin{array}{c} \alpha_{ab} \\ \beta_{ab,b1} \end{array} \right) \in \Gamma(\bar{E}_a \otimes \bar{E}_{(ab)2} \mid \bar{E})$$

where $\alpha_{ab} = k^r \bar{W}_{rab} \bar{Y}_{sra} - \frac{1}{2}k^r \bar{Y}_{sra}\bar{W}_{ab} + \frac{1}{2}k^r \bar{Y}_{sra}\varphi$, $\beta_{ab,b1} = -k^r \bar{W}_{ra}(b_0) \bar{Y}_{b1} + \frac{1}{2}k^r k_{(b_0)ra}$, and $*$ denoted a term which we do not need to specify. For all non-zero slots, application of $\bar{\nabla}$ requires taking a trace. Using $k^r \bar{Y}_{rab}$, (31) and trace freeness of the Weyl and Cotton tensors we conclude that all traces of $\alpha_{ab}$ and $\beta_{ab,b1}$ vanish. Thus $\bar{\nabla}^{nor}(K \otimes K) = 0$, i.e. $K$ is a BGG-splitting.

We can decompose $K^2 \in \Gamma(S^2 T)$ into the trace free part $(K^2)_0 \in S^2 T$ and a multiple of the identity by a function $\lambda$:

$$K^2 = (K^2)_0 + \lambda \operatorname{id}_T.$$  

Since $K^2$ is a BGG-splitting, both summands on the right hand side are BGG-splittings. Since the projecting slot of $(K^2)_0$ is $-\frac{1}{4}k^r k_r$, cf. (32), and $k^a$ is light-like, we have shown

$$(K^2)_0 = L^2_k T(-k^r k_r) = 0.$$

We can now prove Theorem 6.7.

Proof. In the case where $\bar{M}, c$ is induced by a projective structure the stated properties hold according to Theorem 5.18.

In the converse direction, we employ the Levi-Civita connection $\bar{D}$ from Proposition 5.8. Then in particular $\bar{D}\chi = 0$ hence $s_F = L^0_k T(\chi) = \left( \begin{array}{c} 0 \\ \chi \end{array} \right)$. Thus it follows easily from (16) that the tractor spinor $s_F$ is pure since $\chi$ is pure. Using assumptions (a) and (c) of Theorem 6.7 we have $K = L^0_k T(k)$, which satisfies $K \bullet s_F = -\frac{1}{2}(n+1)s_F$. Thus the endomorphism $K$ preserves the (totally isotropic, $(n+1)$-dimensional) annihilator $\mathcal{F}$ of $s_F$ and moreover the trace of this restriction is $\operatorname{tr}(K|\mathcal{F}) = -(n+1)$. The next ingredient is Lemma 6.9 which in particular says that $K^2 = \lambda \operatorname{id}_T$ for some function
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\[ \lambda \in C^{\infty}(\tilde{M}). \] Now fix a point \( x \in \tilde{M} \) and consider the value \( \lambda(x) \in \mathbb{R} \). If \( \lambda(x) < 0 \) then endomorphism \( K|_{\tilde{F}} \) gives rise to a (multiple of a) complex structure, i.e. \( \text{tr}(K|_{\tilde{F}}) = 0 \) which is a contradiction. A similar reasoning excludes the case \( \lambda(x) = 0 \). Thus \( \lambda(x) > 0 \) and \( \tilde{T} \) decomposes to \( \tilde{T}^{+} \oplus \tilde{T}^{-} \) at the point \( x \in \tilde{M} \) according to eigenvalues \( \pm \sqrt{\lambda(x)} \). Since \( K \) is skew-endomorphism and we are in the split-signature, \( \tilde{T}^{+} \) and \( \tilde{T}^{-} \) are both totally isotropic and both of the dimension \( n+1 \). Using once more the condition \( \text{tr}(K|_{\tilde{F}}) = -(n+1) \), we conclude \( \tilde{F} = \tilde{T}^{-} \) at \( x \) and \( \lambda(x) = 1 \). Since this is true for any point \( x \in \tilde{M} \), we have \( K^{2} = \text{id}_{\tilde{T}} \), which implies Theorem 5.18(a) via equations (95) of Appendix B.1. Thus, all the properties (a)–(d) of Theorem 5.18 are satisfied. \( \square \)

7. COMPARISON WITH PATTERSON–WALKER METRICS

In this section we will show that the Fefferman-type construction based on the group inclusion \( \text{SL}(n+1) \hookrightarrow \text{Spin}(n+1, n+1) \) is closely related to the construction of so called Patterson–Walker metrics. These are Riemann extensions of affine connected spaces, first described in [PW52]. A conformal version of this construction was obtained by [DT10] for dimension \( n = 2 \), and was treated by the authors of the present article in general dimension in [HSSTZ16].

Let \( M \) be a smooth manifold and \( p : T^{\ast}M \to M \) its cotangent bundle. The vertical subbundle \( V \subseteq T(T^{\ast}M) \) of this projection is canonically isomorphic to \( T^{\ast}M \). An affine connection \( D \) on \( M \) determines a complementary horizontal distribution \( H \subseteq T(T^{\ast}M) \) that is isomorphic to \( TM \) via the tangent map of \( p \).

**Definition 7.1.** The Riemann extension or the Patterson–Walker metric associated to a torsion-free affine connection \( D \) on \( M \) is the pseudo-Riemannian metric \( g \) on \( T^{\ast}M \) fully determined by the following conditions:

(a) both \( V \) and \( H \) are isotropic with respect to \( g \),
(b) the value of \( g \) with one entry from \( V \) and another entry from \( H \) is given by the natural pairing between \( V \cong T^{\ast}M \) and \( H \cong TM \).

It follows that \( V \) is parallel with respect to the Levi-Civita connection of the just constructed metric. Hence Patterson–Walker metrics are special cases of Walker metrics, i.e. metrics admitting a parallel isotropic distribution.

The previous definition can be adapted to weighted cotangent bundles \( T^{\ast}M(w) = T^{\ast}M \otimes \mathcal{E}(w) \), provided that \( M \) is oriented and \( D \) is special, i.e. preserving a volume form on \( M \), which allows a trivialisation of \( \mathcal{E}(w) \). It turns out that Patterson–Walker metrics induced by projectively equivalent connections are conformally equivalent if and only if \( w = 2 \) (interpreted as a projective weight according to the conventions from Subsection 2.6). Altogether, we have a natural split-signature conformal structure on \( T^{\ast}M(2) \) induced by an oriented projective structure \( (M, p) \).

From Section 3.5 we know, that \( \tilde{M} = T^{\ast}M(2) \setminus \{0\} \) is the Fefferman space of the construction occupying the main part of this article. Special affine connections from \( p \) are just the exact Weyl connections of the corresponding
parabolic geometry. The corresponding objects on \( \tilde{M} \) are the reduced Weyl connections, respectively reduced scales, which correspond to distinguished metrics in the conformal class, see Subsection 3.7. We are going to show that these metrics are just the Patterson–Walker metrics.

By definition, the Fefferman space is \( \tilde{M} = G/Q \), where \((G \to M, \omega)\) is the Cartan geometry of type \((G,P)\) associated to the projective structure on \( M \) and \( Q \subseteq P \) is as in Appendix A. Under the identification \( T\tilde{M} \cong G \times Q \frac{g}{q} \), conformally invariant objects on \( \tilde{M} \) corresponds to \( Q \)-invariant data on \( \frac{g}{q} \). Objects related to the choice of a reduced Weyl structure, i.e. affine connection from \( p \), corresponds to data on \( \frac{g}{q} \) invariant under \( Q_0 = G_0 \cap Q \). Similarly, objects related to the choice of a reduced scale, i.e. special affine connection from \( p \), correspond to data invariant under \( G^{ss}_0 \cap Q \).

**Proposition 7.2.** Let \((\tilde{M}, c)\) be the conformal structure of signature \((n,n)\) associated to an \( n \)-dimensional projective structure \((M, p)\) via the Fefferman-type construction. Then any metric in \( c \) corresponding to a reduced scale is a Patterson–Walker metric.

**Proof.** Within the proof we refer to explicit matrix realisations from Appendix A.

The conformal structure on the Fefferman space \( \tilde{M} \) corresponds to the canonical \( \tilde{P} \)-invariant conformal class of inner products on \( \tilde{g}/\tilde{p} \). The embedding \( g = \mathfrak{sl}(n+1, \mathbb{R}) \hookrightarrow \mathfrak{so}(n+1, n+1) = \tilde{g} \) in (91) induces an isomorphism \( g/q \cong \tilde{g}/\tilde{p} \) of \( Q \)-modules. Under this identification, the conformal class on \( \tilde{g}/\tilde{p} \) determines a \( Q \)-invariant conformal class on \( g/q \) that we need to express explicitly. On that account, elements in \( g/q \) will be represented by matrices of the form

\[
\begin{pmatrix}
-\tilde{z} & * & * \\
X & * & * \\
w & Y^t & -\tilde{z}
\end{pmatrix},
\]

where \( z, w \in \mathbb{R} \) and \( X, Y \in \mathbb{R}^{n-1} \). Now it turns out that the conformal class on \( g/q \) is represented by the quadratic form, whose value on the entry from (83) is

\[
Y^t X - zw.
\]

This can be either checked by a straightforward application of the above mentioned isomorphism, or alternatively, one can show that the only quadratic form on \( g/q \), which changes conformally under the \( Q \)-action, is the one given by (84) up to a constant multiple. By the same reasoning it further follows that this quadratic form is invariant under the action of \( G^{ss}_0 \cap Q \). Altogether, any metric in the conformal class on \( \tilde{M} \) determined be a reduced scale corresponds to the quadratic form (84) in suitable frame.

In order to show that any such metric is Patterson–Walker, we have to reinterpret the characterisation from definition 7.1 in current terms. Firstly, the vertical subbundle \( V \subseteq T\tilde{M} \) corresponds to the \( Q \)-invariant subspace \( f = p/q \subseteq g/q \) so that \( V \cong \mathcal{G} \times_Q f \). According to the description in (83), \( f \) is given by \( X = 0 \) and \( w = 0 \). The canonical identification \( V \cong T^*\tilde{M}(2) \) corresponds to an isomorphism \( f \cong (g/p)^*(2) \) of \( Q \)-modules. As before, we identify \((g/p)^*(2)\) with \( p_+(2) \) via the Killing form on \( g \). It is an easy exercise
to show that the desired isomorphism is provided by the mapping \( f \to p_+(2) \) given by
\[
\begin{pmatrix}
-\frac{z}{2} & * & * \\
0 & * & * \\
0 & Y^t & -\frac{z}{2}
\end{pmatrix} \mapsto \begin{pmatrix} 0 & Y^t & -z \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]
(85)

Secondly, it turns out there is a unique \( Q_0 \)-invariant subspace, \( h \), which is complementary to \( f \) in \( g/q \). According to the previous description, it is given by \( Y = 0 \) and \( z = 0 \). This subspace corresponds to the horizontal distributions induced by the linear connections from \( p \) so that \( H \cong G_0 \times Q_0 h \).

The identification \( H \cong TM \) corresponds to the obvious isomorphism \( h \cong g/p \) of \( Q_0 \)-modules.

Now, both \( f \) and \( h \) are isotropic subspaces with respect to the inner product determined by (84), therefore the condition (a) from 7.1 is satisfied. Furthermore, for any \( v \in f \) and \( u \in h \), the inner product of \( v \) and \( u \) coincides with the pairing of the corresponding element \( v \in p_+(2) \) with \( u \in g/p \) via the Killing form. Hence also the condition (b) from 7.1 is satisfied. This completes the proof. \( \square \)

**Appendix A. Explicit matrix realisations**

Here we provide explicit realisations of the Lie algebras introduced in Subsection 3.2 in terms of matrices. We will consider the inner product \( h \) and the involution \( K \) on \( \mathbb{R}^{n+1,n+1} \) given by the block matrices
\[
h := \begin{pmatrix} 0 & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix} \quad \text{and} \quad K := \begin{pmatrix} I_{n+1} & 0 \\ 0 & -I_{n+1} \end{pmatrix} \quad (86)
\]
with respect to the standard basis \( (e_1, \ldots, e_{2n+2}) \). Then \( E = \langle e_1, \ldots, e_{n+1} \rangle \) and \( F = \langle e_{n+2}, \ldots, e_{2n+2} \rangle \) and the decomposition (31) can be written as
\[
\tilde{g} = \Lambda^2 (E \oplus F) = \begin{pmatrix} E \otimes F & \Lambda^2 E \\ \Lambda^2 F & E \otimes F \end{pmatrix} .
\]
(87)

For \( \tilde{v} := e_1 + e_{2n+2} \), the Lie algebra \( \tilde{p} \) of the parabolic subgroup \( \tilde{P} \subseteq \tilde{G} \) is of the following form
\[
\tilde{p} = \begin{pmatrix} a & U^t & w \\ X & B & V \\ 0 & Y^t & c \end{pmatrix} \begin{pmatrix} 0 & -W^t & -b \\ W & C & -X \\ b & X^t & 0 \end{pmatrix} ,
\]
(88)

where \( a, b, c, d, w \in \mathbb{R} \) with \( a - b = d - c \), \( U, V, W, X, Y, Z \in \mathbb{R}^{n-1} \), \( B \in gl(n-1) \) and \( C, D \in so(n-1) \). The nilradical \( \tilde{p}_+ = \tilde{p}^+ \) is then of the form
\[
\tilde{p}_+ = \begin{pmatrix} a & U^t & w \\ 0 & 0 & V \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} 0 & -V^t & -a \\ V & 0 & 0 \\ a & 0 & 0 \end{pmatrix}.
\]
(89)
A choice of Levi subalgebra \( \tilde{\mathfrak{g}}_0 \subseteq \tilde{\mathfrak{p}} \) determines a grading \( \mathfrak{g} = \mathfrak{g}_- \oplus \tilde{\mathfrak{g}}_0 \oplus \tilde{\mathfrak{p}}_+ \). We shall choose \( \tilde{\mathfrak{g}}_0 = \tilde{\mathfrak{p}} \cap \tilde{\mathfrak{p}}_{op} \), where \( \tilde{\mathfrak{p}}_{op} \subseteq \tilde{\mathfrak{g}} \) is the stabiliser of the light-like vector \( e_{n+2} \). Explicitly,

\[
\tilde{\mathfrak{g}}_0 = \begin{pmatrix}
a & 0 & 0 & 0 \\
X & B & V & 0 \\
0 & Y^t & c & 0 \\
a + c & Z^t & 0 & 0 \\
\end{pmatrix}.
\] (90)

The embedding \( \iota' : \mathfrak{g} \hookrightarrow \tilde{\mathfrak{g}} \) of Lie algebras has the form

\[
\mathfrak{sl}(n+1) \hookrightarrow \mathfrak{so}(n+1, n+1), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & -A^t \end{pmatrix}.
\] (91)

The subgroup \( Q = i^{-1}(\tilde{\mathfrak{P}}) \) is contained in \( P \), the stabiliser in \( G \) of \( v = (\tilde{v})_E = e_1 \); the inclusion of corresponding Lie algebras is

\[
\mathfrak{q} = \mathfrak{g} \cap \tilde{\mathfrak{p}} = \begin{pmatrix} a & U^t & w \\ 0 & A & V \\ 0 & 0 & -a \end{pmatrix} \subseteq \begin{pmatrix} a & U^t & w \\ 0 & B & V \\ 0 & 0 & c \end{pmatrix} = \mathfrak{p},
\]

where \( \text{tr}(A) = 0 \) and \( a + \text{tr}(B) + c = 0 \). The standard projective grading \( \mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{p}_+ \),

\[
\mathfrak{g}_- = \begin{pmatrix} 0 & 0 & 0 \\
X & 0 & 0 \\
y & 0 & 0 \\
\end{pmatrix}, \quad \mathfrak{g}_0 = \begin{pmatrix} a & 0 & 0 \\
0 & B & V \\
0 & 0 & c \end{pmatrix}, \quad \mathfrak{p}_+ = \begin{pmatrix} 0 & U^t & w \\
0 & 0 & 0 \\
0 & 0 & 0 \end{pmatrix},
\] (92)

is compatible with the previous conformal grading so that the reduced Lie subalgebra \( \mathfrak{q}_0 := \mathfrak{q} \cap \mathfrak{g}_0 \) coincides with the intersection of \( \mathfrak{g}_0 \cap \tilde{\mathfrak{g}}_0 \). Explicitly,

\[
\mathfrak{q}_0 = \begin{pmatrix} a & 0 & 0 \\
0 & A & V \\
0 & 0 & -a \end{pmatrix}
\] (93)

where \( \text{tr}(A) = 0 \).

**Appendix B. Explicit Formulas for Underlying Objects**

In this Appendix we focus on irreducible components (or “slots”) of tractor fields \( \mathbf{K} \in \Gamma(\Lambda^2 \tilde{T}) \subseteq \text{End} \tilde{T} \), \( \mathbf{s}_E \in \Gamma(\tilde{S}_+^\ast) \) and \( \mathbf{s}_F \in \Gamma(\tilde{S}_-) \) referred to in the main text. Recall we work on the split-signature conformal structure \((\tilde{M}, \mathbf{c})\). Given \( g \in \mathbf{c} \), we have

\[
\mathbf{K} = \begin{pmatrix} \rho_a \\ \mu_{ab} | \varphi \\
\kappa_a \end{pmatrix} \in \begin{pmatrix} \tilde{\mathbb{E}}_a \\ \tilde{\mathbb{E}}_{ab}[2] \oplus \mathbb{E}_a[2] \end{pmatrix},
\] (94)

\[
\mathbf{s}_E = \begin{pmatrix} \eta \\
\eta \end{pmatrix} \in \left( \tilde{S}_+^\ast \left[ -\frac{1}{2} \right] \right) \quad \text{and} \quad \mathbf{s}_F = \begin{pmatrix} \chi \\ \chi \end{pmatrix} \in \left( \tilde{S}_- \left[ -\frac{1}{2} \right] \right)
\]

where components of \( \mathbf{K} \), \( \mathbf{s}_E \) and \( \mathbf{s}_F \) are given by BGG-splitting operators, cf. \((22)\) and \((27)\).

In Section \( B.1 \) we study properties from Proposition \( 3.2 \) and derive a number of algebraic consequences for their constituents. Next, in Section
we will analyze the consequences of the additional conditions imposed on the tractors in Proposition 5.12.

B.1. Algebraic properties. First observe that $K^2 = \text{id}_\mathcal{F}$ if and only if

$$
\begin{align*}
k^a k_a &= 0, & \rho^a \rho_a &= 0, \\
\mu^a b k_b &= \varphi k^a, & \mu^a b \rho_b &= -\varphi \rho^a, \\
k^a \rho_a &= \varphi^2 - 1, & \mu_a^c \mu_{cb} &= g_{ab} + 2 k_i(a \rho_b).
\end{align*}
$$

In particular, $k^a$ and $\rho^a$ are light-like vector fields. This property further means that $\tilde{T}$ splits into the two $\pm 1$-eigentractor bundles $\tilde{E}$ and $\tilde{F}$ of the same rank $n+1$ (due to skew-symmetry of $K$). That is, for $V \in \Gamma(\tilde{F})$ and $U \in \Gamma(\tilde{E}),$

$$KV = -V, \quad KU = U. \quad (96)$$

Let $s_E \in \Gamma(\tilde{S}_+)$ and $s_F \in \Gamma(\tilde{S}_-)$ be the pure tractor-spinors annihilating $\tilde{E}$ and $\tilde{F}$ respectively. As in the setup of Section 5.2 we shall choose the normalisation $\langle s_E, s_F \rangle = -\frac{1}{2}$ so that (30) holds. These properties are reflected in slots of these tractor fields denoted as in (94). Then, the purity of $s_F$ entails that the weighted spinors $\chi, \bar{\chi}$ are pure with maximally intersecting totally isotropic $n$-plane distributions provided $\bar{\chi}$ is non-zero, and similarly for $s_F.$

Before we proceed, it will be convenient to introduce abstract index notation on spinor fields, cf. [PR86, Tag16]. Sections of the irreducible spinor bundles $\tilde{S}_+$ and $\tilde{S}_-$ will be adorned with primed and unprimed upper-case Roman indices, so that the above spinors will be denoted $\chi^A$ and $\bar{\chi}^A$, and similarly for dual bundles, i.e. $\bar{\eta}_A$ and $\eta_A.$ In abstract indices, the generators of the Clifford algebra of $(TM, g)$ will be denoted $\gamma_a B^A$ and $\gamma_a B^A.$ To streamline notation further, we shall write

$$\chi^A := \gamma_a^A B^B \chi^B : \Gamma(TM) \to \Gamma(\tilde{S}_-), \quad \bar{\chi}^A := \gamma_a^A B^B \bar{\chi}^B : \Gamma(TM) \to \Gamma(\tilde{S}_+),$$

$$\eta_{aA} := \eta_b \gamma_a^B \gamma^B A : \Gamma(TM) \to \Gamma(\tilde{S}_+), \quad \bar{\eta}_{aA} := \bar{\eta}_b \gamma_a^B \gamma^B \bar{A} : \Gamma(TM) \to \Gamma(\tilde{S}_+).$$

In particular, the totally isotropic $n$-plane distributions $\tilde{e} := \ker \eta$ and $\tilde{f} := \ker \chi$ correspond to the kernels of the maps $\eta_{aA}$ and $\chi^A$ respectively. As noted in [HM88, Tag16] and proved in Tag17 (see Remark 5.19), purity of $s_F$ is equivalent to $\chi$ and $\bar{\chi}$ being pure with their associated distributions intersecting maximally. These properties can be expressed algebraically as

$$\chi^{aA} \chi^B_a = 0, \quad \bar{\chi}^{aA} \bar{\chi}^B_a = 0, \quad \chi^{aA} \bar{\chi}^B_a = -2 \chi^B \bar{\chi}^{A}, \quad (97)$$

respectively, and similarly for $\eta$ and $\bar{\eta}.$

Now, any vector $v^a$ tangent to $\tilde{f}$ can be expressed as $v^a = \alpha_A \chi^a A$ for some spinor $\alpha_A.$ Similarly, any section $V \in \Gamma(\tilde{F})$ and $U \in \Gamma(\tilde{E})$ can be expressed as $h(V, X) = \langle v, X \cdot s_F \rangle$ and $h(U, X) = \langle u, X \cdot s_F \rangle.$ Using these properties and form of the Clifford action (10) in terms of the splitting, we find that
the eigentractor equations (96) are equivalent to
\[
\begin{align*}
  k^a \chi^B_a &= 0, & k^a \eta_a B &= 0, \\
  k^a \chi^B_a - \sqrt{2}(\varphi + 1) \chi^B_a &= 0, & k^a \eta_a B + \sqrt{2}(\varphi - 1) \eta_a B &= 0, \\
  (\mu^a_b + \delta^a_b) \chi^B_a + \sqrt{2} \chi^B_a k^a &= 0, & (\mu^a_b - \delta^a_b) \eta^b_A - \sqrt{2} \eta_a A k^a &= 0, \\
  (\mu^a_b + \delta^a_b) \chi^B_a + \sqrt{2} \chi^B_a \rho^a &= 0, & (\mu^a_b - \delta^a_b) \eta^b_A - \sqrt{2} \eta_a A \rho^a &= 0, \\
  \rho^a \chi^B_a + \sqrt{2}(\varphi - 1) \chi^B_a &= 0, & \rho^a \eta_a C = -\sqrt{2}(\varphi + 1) \eta_a C &= 0, \\
  \rho^a \chi^B_a &= 0, & \rho^a \eta_a B &= 0.
\end{align*}
\]

By inspection, we immediately obtain

**Lemma B.1.** For a given metric \( g \in c \), the following statements are equivalent:

(a) \( \bar{\chi} = 0 \);
(b) \( \mu^a_b \chi^B = -v^a \) for all \( v^a \in \Gamma(\ker \chi) \);
(c) \( \rho_a = 0 \) and \( \varphi = -1 \).

If any of these statements holds, then \( \bar{\eta} \) is a pure spinor dual to \( \chi \), and \( \mu^a_b \) acts as the identify on the annihilator of \( \eta \).

In particular, \( \mu^a_b \) is a skew-symmetric endomorphism squaring to the identity, i.e. \( \mu^a_c \mu^c_b = \delta^a_b \). Further, our choice of normalisation \( (s_E, s_F) = -\frac{1}{2} \) implies that \( \chi^A \eta^A = -\frac{1}{2} \), so that (98) leads to
\[
\begin{align*}
  \gamma^a_{ab} A^A &= 2 \sqrt{2} \eta^a A \chi^A, \\
  \mu_{ab} &= 2 \eta^A \gamma^a_{ab} A^A \chi^B.
\end{align*}
\]

where \( \gamma^a_{ab} A^A := \gamma^a_{[a} A^{b]} \chi^B \).

**B.2. Differential properties.** We shall now examine the consequences of further (differential) assumptions on the tractor objects \( s_F \) and \( K \) above.

We shall first assume that \( s_F \) is parallel with respect to the normal tractor connection. We shall then add the assumption that the projecting part \( k^a \) of \( K \) is a conformal Killing field. Finally, in addition to these two assumptions, we shall impose an additional curvature condition on the Weyl tensor.

Assume that \( s_F \) is parallel with respect to the normal Cartan connection, i.e.
\[
\bar{\nabla}^{nor} s_F = 0,
\]

so that \( \chi \) is a twistor spinor. By definition, (100) tells us that the curvature of the normal connection takes values in \((\bar{E} \otimes \bar{F})_0 \oplus \Lambda^2 \bar{F} \). Using purity of \( s_F \), \cite{Tag16} implies that the distribution \( \bar{f} = \ker \chi \) is integrable. This is used in \cite{HSSTZ16} to show, that there exist conformal scales for which \( \chi \) is parallel, see Proposition 6.8. Note that this is just the condition (a) of Lemma B.1.

This is an alternative way to show integrability of \( \bar{f} \). See also \cite{Lis15} for a similar result.

Let us now assume that in addition to (100), \( K \) is parallel with respect to the prolongation connection, i.e.
\[
\bar{\nabla}^{nor} b = k^a \bar{\Omega}^{nor}_{ab} k^a,
\]
so that $k^a$ is a conformal Killing vector field. By differentiating the eigentractor equation (96) for elements of $\tilde{\mathcal{F}}$, we immediately deduce from (100) and (101) that $i_k \tilde{\Omega}^{nor}$ takes values in $\Lambda^2 \tilde{\mathcal{F}}$. We shall not be too concerned with the tractor-spinor $s_E$. It suffices to say that taking the tractor covariant derivative of (30) and using (100) and (101), we obtain

$$\tilde{\nabla}_b s_E - \frac{1}{8} k^a \tilde{\Omega}^{nor}_{ab} \cdot s_E = 0.$$  \hspace{2cm} (102)

Re-expressing (102) in terms of the slots of $s_E$, we obtain

$$\tilde{\nabla}_a \kappa_{bc} - (1 - 2\sqrt{2}) \tilde{\Omega}^{nor}_{ab} \cdot s_E = 0,$$

(103)

$$\tilde{\nabla}_a \kappa_{bc} - (1 - 2\sqrt{2}) \tilde{\Omega}^{nor}_{ab} \cdot s_E = 0.$$  \hspace{2cm} (104)

**Proposition B.2.** Let $s_F$ be a pure tractor spinor satisfying (100) and $K$ be an adjoint tractor satisfying (101) and $K^2 = \text{id}_{\tilde{\mathcal{F}}}$. Then, in a scale such that $\chi$ is parallel,

$$\tilde{\nabla}_a k_b - \frac{1}{2} \tilde{\nabla}_a k_b = 0,$$

(105)

$$\tilde{\nabla}_a \mu_{bc} + 2 \tilde{\nabla}_a [k_{bc}] = 0,$$

(106)

$$\tilde{\nabla}_a \kappa_{bc} = 0,$$

(107)

$$\tilde{\nabla}_a \kappa_{bc} = 0.$$  \hspace{2cm} (108)

Further, the following conditions hold

$$\tilde{P}_{ab} \kappa_b = 0, \text{ for all } \kappa^a \in \Gamma(\ker(\chi)),$$

(109)

$$\tilde{Y}_{abc} k^c = 0,$$

(110)

$$k^a \tilde{W}_{abcd} \kappa^c = 0, \text{ for all } \kappa^a \in \Gamma(\ker(\chi)),$$

(111)

$$\tilde{W}_{abcd} \mu^{cd} = 0,$$

(112)

$$\tilde{W}_{abcd} \kappa^{cd} = 0, \text{ for all } \kappa^a, \kappa^b \in \Gamma(\ker(\chi)).$$

(113)

$$\kappa^a \tilde{Y}_{abc} = 0,$$

(114)

**Proof.** We assume the conditions given Lemma B.1. Then first four relations follows from (20). Similarly, (100) yields $\tilde{P}_{ab} k^b = 0$, and thus (109). From (108) and Lemma B.1 (b), we now immediately get (110). Contracting equation (106) with any $v^a \in \Gamma(\ker(\chi))$ and making use of the fact that $\tilde{D}_a$ preserves $\ker(\chi)$ (since $\chi$ is parallel with respect to $\tilde{D}_a$) lead to (111).

In slots, the integrability condition $\tilde{\Omega}^{nor}_{ab} \cdot s_F = 0$ reads as

$$\tilde{W}_{abcd} \gamma^{cd} \chi = 0,$$

(115)

$$\tilde{W}_{abcd} \gamma^{cd} B \chi = 2 \sqrt{2} \chi \gamma^A \tilde{Y}_{ab} = 0.$$  \hspace{2cm} (116)

Now, (112) follows from (115) and (99). Further, (113) is shown in [Tag16]. Finally, the condition (116) reduces to (114) under our assumption $\tilde{\chi} = 0$. 

□
At this stage, we impose the additional curvature condition
\[ \widetilde{W}_{abcd} v^a w^d = 0, \quad \text{for all } v^a, w^a \in \Gamma(\ker \chi). \] (117)

If we let \( M \) denote the (local) leaf space of the foliation tangent to the distribution \( \widetilde{f} = \ker \chi \), this condition allows the conformal structure \( c \) on \( \widetilde{M} \) to descend to a projective structure \( p \) on \( M \). In particular, by Proposition 6.2, scales in which \( \chi \) is parallel are none other than the reduced scales involved in the Fefferman construction of the main text, and which provide a tighter connection between the geometry of \((\widetilde{M}, c)\) and that of \((M, p)\).

As a consequence of the twistor spinor equation on \( \chi \), we obtain

**Lemma B.3.** Let \( \chi \in \Gamma(\widetilde{S}_-[\frac{1}{2}]) \) be a pure twistor spinor with associated integrable \( n \)-plane distribution \( \widetilde{f} \). Suppose that (117) holds. Then
\[ v^a w^b \widetilde{Y}_{abc} = 0, \quad \text{for all } v^a, w^a \in \Gamma(\ker \chi). \] (118)

Further, in a scale such that \( \chi \) is parallel, we have
\[ v^a \widetilde{Y}_{abc} = 0, \quad \text{for all } v^a \in \Gamma(\ker \chi). \] (119)

**Proof.** From the integrability condition (116) of \( \chi \), we have
\[ v^a \widetilde{W}_{abcd} \gamma^c A_{B \chi} = 2 \sqrt{2} \chi \gamma^A \widetilde{Y}_{cab} v^a, \quad \text{for all } v^a \in \Gamma(\ker \chi). \]

We must show that the left hand side vanishes under the assumption (117), which is itself equivalent to \( v^a \widetilde{W}_{abcd} \in \Gamma(\widetilde{f} \otimes \Lambda^2 \widetilde{f}) \) for all \( v^a \in \Gamma(\widetilde{f}) \). It is therefore enough to show that \( \phi_{ab} \gamma^a \gamma^b \widetilde{X} = 0 \) for any 2-form \( \phi_{ab} \in \Gamma(\Lambda^2 \widetilde{f}) \).

Writing \( \phi_{ab} = \phi_{[AB]} \chi^A \chi^B \) for some \( \phi_{[AB]} \), and using (97), we can immediately compute \( \phi_{ab} \gamma^{abc} D \chi^D = 0 \), thus establishing (118).

To show (119), recall that in a reduced scale, for which \( \chi \) is parallel, the distribution \( \widetilde{f} \) is preserved by \( \widetilde{D}_a \). Henceforth, \( v^a \) and \( w^b \) will denote two arbitrary sections of \( \widetilde{f} \). Conditions (109) and (117) can be expressed as \( \widetilde{R}_{abcd} v^a w^d = 0 \). Now, contract \( v^a \) and \( w^e \) into the Bianchi identity
\[ \widetilde{D}_a \widetilde{R}_{bcde} = 0 \]
where we have used the fact that \( \widetilde{f} \) is parallel. In particular, this implies that \( 0 = v^a \widetilde{D}_a \widetilde{R}_{bcde} \), where \( \widetilde{R}_{bcde} := \widetilde{R}_{a[b} \gamma^{ade} \) is the Ricci curvature. To see this, we note that we can always regard \( g^{ab} \) as a section of \( \widetilde{f} \otimes \widetilde{f}^* \). Since \( \widetilde{J} = 0 \), we have that \( \widetilde{P}_{ab} \) is proportional to \( \widetilde{Ric}_{ab} \) by a constant factor, and thus
\[ v^a \widetilde{D}_a \widetilde{P}_{bc} = 0, \]
The claim (119) now follows from the definition of the Cotton tensor \( \widetilde{Y}_{cab} = 2 \widetilde{D}_{[d} \widetilde{P}_{b]c} \) and another application of (109). \( \square \)

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