The Univalent Function Created by the Meromorphic Functions Where Defined on the Period Lattice

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Abstract
The function \( \xi(z) \) is obtained from the logarithmic derivative function \( \sigma(z) \). The elliptic function \( \wp(z) \) is also derived from the \( \xi(z) \) function. The function \( \wp(z) \) is a function of double periodic and meromorphic function on lattices region. The function \( \wp(z) \) is also double function. The function \( \wp(z) \) meromorphic and univalent function was obtained by the serial expansion of the function \( \wp(z) \). The function \( \wp(z) \) obtained here is shown to be a convex function.

Keywords: Convex function, Elliptic function, Lattices, Meromorphic function
2010 AMS: 30C45

1. Introduction

We begin this important paper by introducing some important functions and some important classes.

Definition 1.1. A get the subset of complex numbers \( \mathbb{C} \). If \( A \) is a group according to the collection process, then \( A \) in called a module defined on the ring of integers \( \mathbb{Z} \).

Definition 1.2. If the module \( A \) does not have a stack point in the finite plane, then this module \( A \) is called a lattice. Lattices can be divided into three groups as follows.

i. Zero dimensional lattices;
\[
W_m = \{m \omega : m = 0 \in \mathbb{Z}, \omega \neq 0 \in \mathbb{C}\}
\]

ii. One dimensional lattices;
\[
W_m = \{m \omega_1 : m \neq 0 \in \mathbb{Z}, \omega_1 \neq 0 \in \mathbb{C}\}
\]

iii. Two dimensional lattices;
\[
W_{m,n} = \{m \omega_1 + n \omega_2 : m \neq 0, n \neq 0 \in \mathbb{Z}, \omega_1 \neq 0, \omega_2 \neq 0 \in \mathbb{C}\}
\]

Lemma 1.3. The function \( \xi(z) \) is absolute and uniform convergence [1].
Proof.

\[ \xi(z) = \frac{1}{z} + \sum_{m,n \neq (0,0)} \left( \frac{1}{z-W} + \frac{1}{W} + \frac{z}{W^2} \right) \]

where

\[
\sum_{m,n \neq (0,0)} = \sum_{m,n} \left| \frac{1}{z-W} + \frac{1}{W} - \frac{z}{(z-W)(W^2)} \right| = \left| \frac{(Wmn)^2 + (z-Wmn)Wmn + (1-Wmn)z}{(z-Wmn)(Wmn)^2} \right| = \left| \frac{z}{(z-Wmn)(Wmn)^2} \right|
\]

For all m,n such that \(|W| > 2|z|\) the series under consideration in therefore absolutely and convergent. Thus, function \(\xi(z)\) has a simple pole at point \(z = W\). In that case, \(\xi(z)\) is meromorphic. On the other hand it is clear that \(\xi(z)\) in the odd function so \(\xi(z) = -\xi(-z)\).

\[\square\]

**Theorem 1.4.** The function \(\xi(z)\) has following the power series for point \(z = 0\).

\[ \xi(z) = \frac{1}{z} - \frac{A_2}{3} - \frac{A_4}{5} - \ldots = \frac{1}{z} - \sum_{k \geq 2} \frac{A_{2k-2}}{2k-1}z^{2k-1} \]

**Proof.** Let

\[ \xi(z) = \frac{1}{z} + \sum_{m,n \neq (0,0)} \left( \frac{1}{z-W} + \frac{1}{W} + \frac{z}{W^2} \right) \]

then

\[ \xi(z) = \frac{1}{z} + \sum_{m,n \neq (0,0)} \left[ -\frac{1}{W} \left( 1 + \frac{z}{W} + \frac{z^2}{W^2} + \ldots \right) \right] \]

\[ = \frac{1}{z} + \sum_{m,n \neq (0,0)} \frac{1}{\Delta_{mn}} \left[ 1 + \frac{z}{\Delta_{mn}} + \left( \frac{z}{\Delta_{mn}} \right)^2 + \ldots + \frac{1}{\Delta_{mn}} + \left( \frac{z}{\Delta_{mn}} \right)^2 \right] \]

\[ = \frac{1}{z} - \sum_{m,n \neq (0,0)} \frac{1}{\Delta_{mn}} \left[ \frac{z^2}{\Delta_{mn}^3} + \frac{z^3}{\Delta_{mn}^4} + \frac{z^4}{\Delta_{mn}^5} + \ldots \right] \]

\[ = \frac{1}{z} + \sum_{m,n \neq (0,0)} \frac{1}{W} \left[ \frac{z^2}{W^3} + \frac{z^3}{W^4} + \frac{z^4}{W^5} + \ldots \right] \]

\[ = \frac{1}{z} - \sum_{k \geq 2} \frac{1}{W^{k+1}} z^k = \frac{1}{z} - \sum_{k \geq 2} A_{k+1-1}z^k \]

\[ = \frac{1}{z} - \sum_{k \geq 2} (z^2 + z^3 + z^4 + \ldots) A_{k+1} \]

where \(A_{k+1} = \sum_{m,n \neq (0,0)} \).

Coefficients of \(m,n\) \(z^{2k}\) in evidently zero for \(k=1,2,3\), since the functions \(\xi(z)\) is an odd function, ie equality is as follows

\[ \xi(z) = \frac{1}{z} - \frac{A_2}{3} - \frac{A_4}{5} - \ldots = \frac{1}{z} - \sum_{k \geq 2} \frac{A_{2k-2}}{2k-1}z^{2k-1}. \]

\[\square\]
Definition 1.5. Weierstrass’s function $\wp(z)$ is defined by the double series as

$$\wp(z) = \frac{1}{z^2} + \sum_{m,n \neq (0,0)} \left[ \frac{1}{(z - w)^2} + \frac{1}{W^2} \right]$$

$$- \frac{d}{dz} \xi(z) = \wp(z)$$

where $\wp(z)$ equality can be seen here. That is to say $\wp(z)$ is double function [1].

The function $\wp(z)$ is meromorphic function in the complex plan ($|z| < 1$) with second order poles at the lattices points $z = W$. It is in double periodic with periods $\omega_1$ and $\omega_2$. This mean that $\wp(z)$ satisfies. Considering the following equality $\wp(z) = \frac{1}{z^2} + \sum_{k \geq 2} A_{2k-2}z^{2k-2}$ for $\frac{1}{z} - \sum_{k \geq 2} \frac{A_{2k-2}z^{2k-1}}{2k - 1}$ where $- \frac{d}{dz} \xi(z) = \wp(z)$. The functions $\wp(z)$ is a meromorphic and elliptic funtion which has $z = W$ second order pole points.

Theorem 1.6. The series $\wp(z)$ is absolutely and uniformly convergent for every $z = W$.

Proof.

$$\left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \left| \frac{W^2 - (z-W)^2}{(z-W)^2W^2} \right| = \left| \frac{2W - z}{(z-W)^2W^2} \right| \leq \frac{|z| \left( 2|W| + \frac{|W|}{2} \right)}{1 - W^2} = \frac{10|z|}{|W|^3}$$

where $|z| < \frac{1}{2}|W|$. Thus,

$$\sum_{m,n \neq (0,0)} \left| \frac{1}{(z-W)^2} - \frac{1}{W^2} \right| = \sum_{m,n \neq (0,0)} \frac{10|z|}{|W|^2}$$

The function $\wp(z)$ is meromorphic region $|z| < 1$ whether the function $\wp(z)$ is not analytical region $|z| < 1$. If we get consecutive derivatives from the equation as

$$\wp(z) = \frac{1}{z^2} + \sum_{k \geq 2} A_{2k-2}z^{2k-2}$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{k \geq 2} (2k - 2)A_{2k-2}z^{2k-3}$$

$$\wp''(z) = -\frac{6}{z^4} + \sum_{k \geq 2} (2k - 2)(2k - 3)A_{2k-2}z^{2k-4}$$

$$\wp'''(z) = -\frac{12}{z^5} + \sum_{k \geq 2} (2k - 2)(2k - 3)(2k - 4)A_{2k-2}z^{2k-5}$$

$$\wp^{(4)}(z) = (-1)^n \frac{(n+1)!}{z^{n+2}} + \sum_{k \geq 2} (2k - 2)(2k - 3)\cdots(2k - (n+1))A_{2k-2}z^{2k-(n+1)}.$$

In that case

$$\wp^{(2n-1)}(z) = \frac{(2n)!}{z^{2n+1}} + \sum_{k \geq 2} (2k - 2)(2k - 3)\cdots(2k - 2n)A_{2k-2}z^{2k-(2n)}$$

$$\wp^{(2n)}(z) = \frac{(2n)!}{z^{2n+1}} + \sum_{k \geq 2} (2k - 2)(2k - 3)\cdots(2k - 2n)A_{2k-2}z^{2k-(2n)}$$

$$\wp^{(2n-2)}(z) = \frac{(n-1)!}{z^{2n+1}} + \sum_{k \geq 2} (2k - 2)(2k - 3)\cdots(2k - (2n-1))A_{2k-2}z^{2k-(2n-1)}$$
Theorem 1.7. If \( \alpha_i \) and \( \beta_i \) \( (i = 1, 2, \ldots, r) \) be the zeros and poles respectively of an elliptic function \( f(z) \) in a cell, then

\[
\sum_{i=1}^{r} \alpha_i \equiv \sum_{i=1}^{r} \beta_i \pmod{2\omega_1, 2\omega_2}
\]

where every zero or pole is counted as many times as the multiplicity indicates.

Proof. We have

\[
\sum_{i=1}^{r} \alpha_i - \sum_{i=1}^{r} \beta_i = \frac{1}{2\pi i} \int f'(z) \frac{f(z)}{f(z)} dz \quad (P \text{ is any suitably chosen contour})
\]

\[
= \frac{1}{2\pi i} \left[ \int_{z_0}^{z_0+2\omega_1} \frac{f'(z)}{f(z)} dz + \int_{z_0+2\omega_1}^{z_0+2\omega_1+2\omega_2} \frac{f'(z)}{f(z)} dz + \int_{z_0+2\omega_1+2\omega_2}^{z_0} \frac{f'(z)}{f(z)} dz + \int_{z_0}^{z_0+2\omega_2} \frac{f'(z)}{f(z)} dz \right]
\]

\[
= \frac{1}{2\pi i} \left[ \int_{z_0}^{z_0+2\omega_1} (z - (z + 2\omega_1)) f'(z) \frac{f(z)}{f(z)} dz + \int_{z_0+2\omega_1}^{z_0+2\omega_1+2\omega_2} (z + 2\omega_1 - z) f'(z) \frac{f(z)}{f(z)} dz \right]
\]

\[
= \frac{1}{2\pi i} \left[ 2\omega_1 \int_{z_0}^{z_0+2\omega_1} f'(z) \frac{f(z)}{f(z)} dz - 2\omega_2 \int_{z_0}^{z_0+2\omega_1} f'(z) \frac{f(z)}{f(z)} dz \right]
\]

\[
= \frac{1}{2\pi i} \left\{ 2\omega_1 \left[ \log f(z) \right]_{z_0}^{z_0+2\omega_1} - 2\omega_2 \left[ \log f(z) \right]_{z_0}^{z_0+2\omega_1} \right\} = \frac{1}{2\pi i} (4\pi im\omega_1 - 4\pi in\omega_2) = (m2\omega_1 + 2n\omega_2) \quad (n = -n).
\]

Hence we conclude

\[
\sum_{i=1}^{r} \alpha_i \equiv \sum_{i=1}^{r} \beta_i \pmod{2\omega_1, 2\omega_2}[1].
\]

\( \square \)

Theorem 1.8. The sum, difference, product and the quotient of any two co-periodic elliptic functions are also elliptic function of the same period.

Proof. Since \( f_i(z + 2\omega) = f_i(z) \), where \( 2\omega = 2\omega_1 \) and \( 2\omega_2 \) \( (i = 1, 2) \) therefore

\[
f_1(z + 2\omega) \pm f_2(z + 2\omega) = f_1(z) \pm f_2(z)
\]

\[
f_1(z + 2\omega).f_2(z + 2\omega) = f_1(z).f_2(z)
\]

\[
f_1(z + 2\omega)/f_2(z + 2\omega) = f_1(z)/f_2(z).
\]

Again since the set of all meromorphic functions forms a field and \( f_1(z) \pm f_2(z) \), \( f_1(z).f_2(z) \) and \( f_1(z)/f_2(z) \) are meromorphic and periodic with periods \( 2\omega_1 \) and \( 2\omega_2 \). So they are elliptic functions with the same periods \([1]\). \( \square \)

Theorem 1.9. Let \( f(z) \) be regular and univalent in the closed disk \( D : |z| \leq R \). Then \( f(z) \) maps \( D \) onto a convex domain if and only if

\[
Re \left[ 1 + \frac{zf'(z)}{f(z)} \right] \geq 0, \quad \text{for } z \text{ on } D : |z| \leq R.
\]

Suppose further that \( f(0) = 0 \). Then \( f(z) \) maps \( D \) onto a region that is starlike with respect to \( w = 0 \) if and only if

\[
Re \left[ \frac{zf'(z)}{f(z)} \right] \geq 0, \quad \text{for } z \text{ on } D : |z| \leq R.
\]
Theorem 1.10. The function \( \varphi(z) \) and the function \( \xi(z) \) have the following equality

\[
\frac{\varphi^{2n-1}(z_1)}{\varphi^{2n-2}(z_1) - \varphi^{2n-2}(z_2)} = 2\xi(z_2 - z_1) - 2n(\xi(z_1) - \xi(z_2)).
\]

Lemma 1.11. The sum, difference, product and quotient of any co-periodic elliptic functions are also elliptic function of the same period.

Lemma 1.12. If the elliptic function \( f(z) \) has simple pole at and only at the points \( \beta_1, \beta_2, \beta_3, \ldots, \beta_n \) in cell with residues \( A_1, A_2, A_3, \ldots, A_n \), then

\[
\varphi(z) = A_0 + \sum_{r=1}^{s} (z - r)A_r,
\]

where \( A_0 \) is a constant. It is in the fact that a constant \( A_0 \) in zero. In that case, the function

\[
\frac{\varphi^{2n-1}(z)}{\varphi^{2n-2}(z) - \varphi^{2n-2}(z_2)}
\]

is an elliptical function with poles at \( z_2, z_2, 0 \) with residues \( 1, 1, -2n \) respectively. If the last equation is written in place of \( z \), then the following equation is found

\[
\frac{\varphi^{2n-1}(z)}{\varphi^{2n-2}(z) - \varphi^{2n-2}(z_2)} = A_0 + \xi(z - z_2) + \xi(z - z_2) - 2n\xi(z).
\]

If in the above equation \( z \) is written instead of \( -z \) then \( \varphi \) is an even function and \( \xi(z) \) is an odd function

\[
-\frac{\varphi^{2n-1}(z)}{\varphi^{2n-2}(z) - \varphi^{2n-2}(z_2)} = A_0 - \xi(z + z_2) - \xi(z - z_2) + 2n\xi(z).
\]

\[
-\frac{\varphi^{2n-1}(z)}{\varphi^{2n-2}(z) - \varphi^{2n-2}(z_2)} = -A_0 + \xi(z + z_2) + \xi(z - z_2) - 2n\xi(z)
\]

equations are obtained. If \( A_0 = 0 \) and \( z_1 \) are written instead of \( z \) then the following equation is continue

\[
\frac{\varphi^{2n-1}(z)}{\varphi^{2n-2}(z) - \varphi^{2n-2}(z_2)} = \xi(z_1 + z_2) + \xi(z_1 - z_2) - 2n\xi(z_1).
\]

The function \( \varphi(z) \) defined as follows

\[
\varphi(z) = \varphi(z) + \frac{z^3 - 1}{z^4} = z + \sum_{k \geq 2} A_{2k-2}z^{2k-2} = z + A_2z^2 + A_4z^4 + \ldots
\]

The function \( \varphi(z) \) is an analytical function for every \( z \in |z| < 1 \). Also because of its \( \varphi(0) = 0 \) and \( \varphi'(0) = 1 \), this function is class \( A \).
2. Main Theorem

Theorem 2.1. The function $\varphi(z)$ is an univalent function.

Proof. If $\varphi(z_1) - \varphi(z_2) = 0$, then

$$
\varphi(z) - \varphi(z) = z_1 + \sum_{k \geq 2} A_{2k-2} z_1^{2k-2} - z_2 - \sum_{k \geq 2} A_{2k-2} z_2^{2k-2} = 0
$$

Thus, the function $\varphi(z)$ is in class $S$. The subclass of $S$ consisting of the convex functions is defined by $K$, and $S^*$ denotes the subclass of starlike functions. Thus $K \subset S^* \subset S$ [3].

We can do this proof in another way as follows: $|z| < 1$ is clear that there is convex region.

Note that $\varphi(z_1) - \varphi(z_2) = \frac{z_2}{z_1}$

If

$$
\eta = tz_2 + (1-t)z_1, 0 \leq t \leq 1, \text{ then } z_1 - \varphi(z_2) = \int_0^1 \varphi'(tz_2 + (1-t)z_1)d\eta.
$$

Because,

$$
\eta = (tz_2 + (1-t)z_1) \in |z| < 1 \text{ and } Re\varphi'(z) = Re\varphi'(tz_2 + (1-t)z_1) > 0.
$$

Thus

$$
\varphi'(\eta) = \varphi'(tz_2 + (1-t)z_1) \neq 0. \text{ Therefore, if } z_1 - z_2 \neq 0, \text{ then } \varphi(z_1) - \varphi(z_2) \neq 0. \text{ This means that } \varphi(z) \text{ is univalent in } |z| < 1. \text{ On the other hand,}
$$

$$
Re\left(1 + \frac{z\varphi'(z)}{\varphi'(z)}\right) = Re\left(\frac{1 + 4A_2 z + 14A_4 z^3 + 36A_6 z^5 + ...}{1 + 2A_2 z + 4A_4 z^3 + 6A_6 z^5 + 8A_8 z^7 + ...}\right) = Re\left(1 + 2A_2 z - 4A_2 A_2 z^2 + (10A_4 + 8A_2 A_2 A_2) z^3 + ...\right) > 0
$$

since for every $z \in |z| < 1$.

\[\square\]

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