THE MICROSCOPIC DERIVATION OF THE STOCHASTIC KELLER-SEGEL EQUATION

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Abstract. In this paper, we are the first to propose a stochastic aggregation-diffusion equation of the Keller-Segel (KS) type for modeling the chemotaxis in dimensions $d = 2, 3$. As opposed to the classical deterministic KS equation only allowing for the idiosyncratic noises, the stochastic KS equation is derived from an interacting particle system subject to both idiosyncratic and common noises. The unique existence of solutions to the stochastic KS equation and the propagation of chaos result are addressed.

Keywords: Mean-field limit, conditional expectation, coupling method, stochastic partial differential equation

1. Introduction

Many bacteria, such as Escherichia coli, Rhodobacter sphaeroides and Bacillus subtilus are able to direct their movements according to the surrounding environment by a biased random walk. For example, bacteria try to swim toward the highest concentration of nutrition or to flee from poisons. In biology, this phenomenon is called chemotaxis, which describes the directed movement of cells and organisms in response to chemical gradients. Chemotaxis is also observed in other biological fields, for instance movement of sperm towards the egg during fertilization, migration of neurons or lymphocytes and inflammatory processes.

Mathematically, one of the most classical models for studying chemotaxis is the Keller-Segel (KS) equation that is originally proposed in [18] to characterize the aggregation of the slime mold amoebae. The classical parabolic-elliptic type KS equation is of the following form:

$$
\begin{align*}
\partial_t \rho_t &= \Delta \rho_t - \nabla \cdot (\rho_t \nabla c_t), \quad x \in \mathbb{R}^d, t > 0, \\
-\Delta c_t + c_t &= \rho_t, \\
\rho_0 &\text{ is given,}
\end{align*}
$$

(1.1)

where $\rho_t(x)$ denotes the bacteria density, and $c_t(x)$ represents the chemical substance concentration. A feature of this equation is the competition between the diffusion and the nonlocal aggregation. Depending on the choice of the initial data, the solutions to the Keller-Segel equation may exist globally or blow up at finite time; see for example [7, 16]. There is an extensive literature on KS systems and their variations, and the readers are referred to [21] Chapter 5] for a comprehensive review. It is also well known that the KS equation (1.1) may be derived from a system of interacting particles $\{(X^i_t)_{t \geq 0}\}_{i=1}^N$ satisfying the following stochastic differential equations (SDEs):

$$
dX^i_t = \frac{1}{N-1} \sum_{j \neq i}^N F(X^i_t - X^j_t) \, dt + dB^i_t, \quad i = 1, \cdots, N, \quad t > 0,
$$

(1.2)

where the function $F$ models the pairwise interaction between particles and $\{(B^i_t)_{t \geq 0}\}_{i=1}^N$ are $N$ independent Wiener processes. The rigorous derivation of the KS equation such as (1.1) from the microscopic particle
system like \([1.2]\) through the propagation of chaos as \(N \to \infty\) may be found in \([9,14]\). For a review of the topic of the propagation of chaos, we refer the readers to \([4,15]\) and the references therein.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space where the \(d'\)-dimensional Wiener processes \(\{B_i(t)\}_{i=1}^{N}\) are independent of each other as well as of a \(d'\)-dimensional Wiener process \((W_t)_{t \geq 0}\). The initial data \(\zeta^i, i = 1, 2, \ldots, N\) are independently and identically distributed (i.i.d.) with a common density \(\rho_0\) and is independent of \(\{B_i(t)\}_{i=1}^{N}\) and \((W_t)_{t \geq 0}\). Denote by \((\mathcal{F}_t^W)_{t \geq 0}\) the augmented filtration generated by \((W_t)_{t \geq 0}\).

In this paper, we study a stochastic aggregation-diffusion equation of Keller-Segel (KS) type that may be derived as the mean-field limit from the interacting particle systems allowing for both idiosyncratic and common noises. Precisely, the stochastic KS equation is of the following form:

\[
\begin{aligned}
&d\rho_t = \frac{1}{2} \sum_{i,j} D_{ij} \left( \rho_t \sum_k (\nu_{ik}^{ij} \nu_{jk}^{ij} + \sigma_{ij}^{ik} \sigma_{ij}^{jk}) \right) dt - \nabla \cdot (\nabla c_t \rho_t) dt - \sum_i D_i \left( \rho_t \sum_k \sigma_{ik}^j dW_t^k \right), \\
&-\Delta c_t + c_t = \rho_t, \\
&\rho_0 \text{ is given},
\end{aligned}
\]

(1.3)

where \(D_{ij} := \frac{\partial}{\partial x_i \partial x_j}, D_i := \frac{\partial}{\partial x_i}, \) and the leading coefficients \(\nu\) and \(\sigma\) are deterministic functions from \([0,T] \times \mathbb{R}^d\) to \(\mathbb{R}^{d \times d'}\).

We may solve the second equation for the chemical concentration:

\[
c_t = (I - \Delta)^{-1} \rho_t = G * \rho_t(x),
\]

(1.4)

with \(G\) being the Bessel potential, and it follows that \(\nabla c_t = \nabla G * \rho_t\) where \(\nabla G\) is called the interaction force.

The associated interacting particle system has the form:

\[
\begin{aligned}
&dX_{t}^{i,\varepsilon} = \frac{1}{N-1} \sum_{j \neq i} \nabla G_{\varepsilon}(X_{t}^{i,\varepsilon} - X_{t}^{j,\varepsilon}) dt + \nu_t(X_{t}^{i,\varepsilon}) dB_t^i + \sigma_t(X_{t}^{i,\varepsilon}) dW_t, \quad i = 1 \ldots, N, \quad t > 0, \\
&X_{0}^{i,\varepsilon} = \zeta^i,
\end{aligned}
\]

(1.5)

where

\[
G_{\varepsilon}(x) = \psi_{\varepsilon} * G(x) = \int_{\mathbb{R}^d} G(y) \psi_{\varepsilon}(x-y) dy, \quad x \in \mathbb{R}^d, \quad \varepsilon > 0,
\]

is the regularized Bessel potential with the blob function \(\psi_{\varepsilon}(x) := \frac{1}{\varepsilon^d} \psi(\frac{x}{\varepsilon})\) satisfying

\[
0 \leq \psi \in C^\infty_c(\mathbb{R}^d), \quad \text{supp } \psi \subseteq B(0,1), \quad \int_{B(0,1)} \psi(x) dx = 1.
\]

(1.6)

In contrast with the classical KS models \((1.1)\) and \((1.2)\) only allowing for the idiosyncratic noise \((B_i(t))_{t \geq 0}\) that is independent from one particle to another, the stochastic systems \((1.3)\) and \((1.4)\) are additionally subject to a common noise \((W_t)_{t \geq 0}\), accounting for the common environment where the particles evolve. Moreover, the diffusion coefficients \(\sigma\) and \(\nu\) are time-state dependent.

In this paper, we first prove the existence and uniqueness of the solution to SPDE \((1.3)\) over a given finite time interval \([0,T]\) when the \(L^4\)-norm of \(\rho_0\) is sufficiently small (see Theorem \(3.3\)) and then it is verified that the following stochastic differential equations (SDEs) of McKean-Vlasov type:

\[
\begin{aligned}
&dY_t^i = \nabla G * \rho_t^i(Y_t^i) dt + \nu(Y_t^i) dB_t^i + \sigma(Y_t^i) dW_t, \quad i = 1, \ldots, N, \quad t > 0, \\
&\rho_t^i \text{ is the conditional density of } Y_t^i \text{ given } \mathcal{F}_t^W, \\
&Y_0^i = \zeta^i,
\end{aligned}
\]

(1.7)

has a unique solution with the conditional density \(\rho_t^i\) of \(Y_t^i\) given the common noise \(W_t\) existing and satisfying SPDE \((1.3)\); see Theorem \(4.4\). Here by the conditional density \(\rho_t^i\) of \(Y_t^i\) given \(\mathcal{F}_t^W\), we mean that

\[
\mathbb{E}[Y_t^i \in dx | \mathcal{F}_t^W] = \rho_t^i(x) dx,
\]

1The dimension of Wiener process \(W\) may be different from \(d'\); we assume \(d'\) for notational simplicity.
i.e., for any \( \varphi \in C_b(\mathbb{R}^d) \), it holds that
\[
\mathbb{E}[\varphi(Y_t^i)|\mathbb{F}_t^W] = \int_{\mathbb{R}^d} \varphi(x) \rho_t^i(x) \, dx.
\]

Finally, we prove that the solution \( \{(X_t^i)_{i \geq 0}\}_{i=1}^N \) of the particle system (1.5) well approximates that of (1.7), which indicates a result of propagation of chaos, i.e., the empirical measure
\[
\rho_t^{\varepsilon,N} := \frac{1}{N} \sum_{i=1}^N \delta_{X_t^i},
\]
associated to the particle system (1.5) converges weakly to the unique solution \( \rho_t \) to SPDE (1.3) as \( N \to \infty \) and \( \varepsilon \to 0^+ \); see Theorem 5.1 and Corollary 5.2.

In view of SPDE (1.3) and the approximating system (1.5), one may see that when the particle number \( N \) tends to infinity, the effect of the idiosyncratic noises averages out while the effect of common noises does not, leading to a stochastic nature of the limit distributions described by SPDE (1.3). Such properties have been investigated in the literature; refer to [1, 3, 6] for instance. In particular, in a quite related work [6], the authors study the propagation of chaos for an interacting particle system subject to a common environmental noise but with a uniformly Lipschitz continuous potential, and in [5], the stochastic mean-field limit of the Cucker-Smale flocking particles with multiplicative noises is obtained. Different from the existing literature with common noise, the main difficulties in dealing with our stochastic KS models are from the Bessel potential \( G \) which incurs the singularity of the drift of SDE (1.7) and the nonlinearity of SPDE (1.3), and for which the regularization with a blob function is introduced in the particle system (1.5) and a divergence-free assumption is imposed (see (iii) of assumption 1). The approaches adopted in this work mixes stochastic analysis and PDE theory.

The rest of the paper is organized as follows. In Section 2, we set some notations, present some auxiliary results and give the standing assumptions on the diffusion coefficients. Section 3 is then devoted to the proof of the existence and uniqueness of the solution to stochastic KS equation (1.3) in certain regular spaces. On the basis of the well-posedness of SPDE (1.3), we prove the existence and uniqueness of the strong solution to SDEs (1.7) in Section 4. Finally, in Section 5, the propagation of chaos result is addressed.

2. Preliminaries

2.1. Notations. The set of all the integers is denoted by \( \mathbb{Z} \), with \( \mathbb{Z}^+ \) the subset of the strictly positive elements. Denote by \( | \cdot | \) (respectively, \( \langle \cdot, \cdot \rangle \) or \( \cdot \) the usual norm (respectively, scalar product) in finite-dimensional Hilbert space such as \( \mathbb{R}, \mathbb{R}^k, \mathbb{R}^{k \times l}, k, l \in \mathbb{Z}^+ \).

Define the set of multi-indices
\[
\mathcal{A} := \{ \alpha = (\alpha_1, \ldots, \alpha_d) : \alpha_1, \ldots, \alpha_d \text{ are nonnegative integers} \}.
\]
For any \( \alpha \in \mathcal{A} \) and \( x = (x_1, \ldots, x_d) \in \mathbb{R}^d \), denote
\[
|\alpha| = \sum_{i=1}^d \alpha_i, \quad x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \quad D^\alpha := \frac{\partial|\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}.
\]

For each Banach space \( (\mathcal{X}, \| \cdot \|_{\mathcal{X}}) \), real \( q \in [1, \infty] \), and \( 0 \leq t < \tau \leq T \), we denote by \( S^q_T([t, \tau]; \mathcal{X}) \) the set of \( \mathcal{X} \)-valued, \( \mathcal{F}_t \)-adapted and continuous processes \( \{X_s\}_{s \in [t, \tau]} \) such that
\[
\left\| X \right\|_{S^q_T([t, \tau]; \mathcal{X})} := \left\{ \left( E \left[ \sup_{s \in [t, \tau]} \|X_s\|^q_{\mathcal{X}} \right] \right)^{1/q}, \quad q \in [1, \infty); \right. \\
\left. \left( \text{ess sup}_{\omega \in \Omega} \sup_{s \in [t, \tau]} \|X_s\|_{\mathcal{X}} \right)^q, \quad q = \infty. \right.
\]
\( L^q_f(t, \tau; \mathcal{X}) \) denotes the set of (equivalent classes of) \( \mathcal{X} \)-valued predictable processes \( \{X_s\}_{s \in [t, \tau]} \) such that

\[
\|X\|_{L^q_f(t, \tau; \mathcal{X})} := \left( \mathbb{E} \left[ \int_t^\tau \|X_s\|_{\mathcal{X}}^q \, ds \right] \right)^{1/q}, \quad q \in [1, \infty);
\]

\[
\sup_{(\omega, t) \in \Omega \times [t, \tau]} \|X_s\|_{\mathcal{X}}, \quad q = \infty.
\]

Both \( (S^q_f([t, \tau]; \mathcal{X}), \| \cdot \|_{S^q_f([t, \tau]; \mathcal{X})}) \) and \( (L^q_f(t, \tau; \mathcal{X}), \| \cdot \|_{L^q_f(t, \tau; \mathcal{X})}) \) are Banach spaces, and they may be defined well with the filtration \( (\mathcal{F}_t)_{t \geq 0} \) replaced by \( (\mathcal{F}_t^W)_{t \geq 0} \).

### 2.2. Auxiliaries and assumptions.

We first recall some properties of the Bessel potential introduced in [1, 4]. For \( p \in [1, \infty] \), denote by \( L^p = L^p(\mathbb{R}^d) \) the usual Lebesgue integrable spaces with norm \( \| \cdot \|_p \). Then for \( p \in (1, \infty) \) and \( m \in \mathbb{R} \), we may define the Bessel potential spaces as \( B^{m,p} = (I - \Delta)^{-m/2}L^p \). In [1, 4], if \( \rho_t \in L^p \) with \( 1 < p < \infty \), then \( c_t \in B^{2,p} \). In addition, it holds that

\[
\|c_t\|_{B^{2,p}} = \|\mathcal{F}^{-1}[(1 + |\omega|^2)\mathcal{F}[c_t]]\|_{L^p} = \|\rho_t\|_{L^p},
\]

where \( \mathcal{F} \) is the Fourier transformation. Due to the equivalence between the Bessel potential space \( B^{2,p} \) and the Sobolev space \( W^{2,p} \), we know that

\[
\|G * \rho_t\|_{W^{2,p}} = \|c_t\|_{W^{2,p}} \leq C\|\rho_t\|_{L^p}. \tag{2.1}
\]

On the other side, notice that

\[
(I - \Delta)^{-1} = (-\Delta)^{-1} - (I - \Delta)^{-1} = (-\Delta)^{-1}I - \Delta^{-1}.
\]

Thus, we may split the Bessel potential into the Newtonian potential \( \Phi \) and a function \( \Psi \) such that

\[
\mathcal{F}(\Psi)(\omega) = -\frac{1}{\omega^2(1 + \omega^2)},
\]

which implies that \( \Psi \in L^\infty(\mathbb{R}^d) \) \((d = 3)\) or \( \nabla \Psi \in L^\infty(\mathbb{R}^d) \) \((d = 2)\). Namely, one has

\[
G(x) = \Phi(x) + \Psi(x), \tag{2.2}
\]

where

\[
\Phi(x) = \begin{cases} \frac{C_{\alpha}}{|x|^{d-\alpha}}, & \text{if } d \geq 3 \\ \frac{C_{\alpha}}{|2x|^{d-\alpha}} \ln |x|, & \text{if } d = 2 \end{cases}
\]

is the Newtonian potential and \( \Psi \in L^\infty(\mathbb{R}^d) \) \((d = 3)\) or \( \nabla \Psi \in L^\infty(\mathbb{R}^d) \) \((d = 2)\). It then follows that for any \( \alpha \in \mathcal{A} \) with \( |\alpha| \geq 1 \), there holds

\[
\|D^\alpha(\nabla G_x)\|_\infty \leq C_{\alpha} \varepsilon^{1-d-|\alpha|} + \begin{cases} C_{\alpha,\|\Psi\|_\infty} \varepsilon^{-1-|\alpha|}, & \text{when } d = 3 \\ C_{\alpha,\|\nabla \Psi\|_\infty} \varepsilon^{-|\alpha|}, & \text{when } d = 2 \end{cases} \leq C_{\alpha} \varepsilon^{1-d-|\alpha|}. \tag{2.3}
\]

Here, we have used the estimate \( \|D^\alpha(\nabla G_x)\|_\infty \leq C_{\alpha} \varepsilon^{1-d-|\alpha|} \) from [13, Lemma 2.1].

Then, following are the standing assumptions on the coefficients \( \nu \) and \( \sigma \).

**Assumption 1.** The measurable diffusion coefficients \( \sigma, \mu : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d'} \) satisfy

(i) There exists a positive constant \( \lambda \) such that

\[
\sum_{i,j=1}^{d} \sum_{k=1}^{d'} \nu^{i,k}(x)\nu^{j,k}(x)\xi^i\xi^j \geq \lambda |\xi|^2
\]

holds for all \( x, \xi \in \mathbb{R}^d \) and all \( t \geq 0 \);
(ii) There exist $m \in \mathbb{Z}^+$ and real $\Lambda > 0$ such that for all $t \geq 0$ there holds
\[
\nu^{ik}_t(\cdot), \sigma^{ik}_t(\cdot) \in C^m, \quad \text{and} \quad \|\sigma^{ik}_t(\cdot)\|_{C^m} + \|\nu^{ik}_t(\cdot)\|_{C^m} \leq \Lambda,
\]
for $i = 1, \ldots, d$, and $k = 1, \ldots, d'$.

(iii) For all $(t, x) \in [0, \infty) \times \mathbb{R}^d$ and $k = 1, 2, \ldots, d'$,
\[
\sum_{i=1}^{d} D_i \sigma^{ik}_t(x) = 0.
\]

Remark 2.1. The assumption (i) ensures the superparabolicity of the concerned SPDEs, and the boundedness and regularity requirements in (ii) are placed for unique existence of certain regular solutions of SPDEs. The readers are referred to [19] for more discussions. The divergence-free condition (iii) is a technical one for the wellposedness of SPDE (1.3) (see Remark 3.1), and in fact, such kind of divergence-free conditions have been existing in the literature for more clear and elegant arguments (see [2, 6] for instance).

In the remaining part of the work, we shall use $C$ to denote a generic constant whose value may vary from line to line, and when needed, a bracket will follow immediately after $C$ to indicate what parameters $C$ depend on. By $A \hookrightarrow B$ we mean that normed space $(A, \| \cdot \|_A)$ is embedded into $(B, \| \cdot \|_B)$ with a constant $C$ such that
\[
\|f\|_B \leq C\|f\|_A, \quad \forall f \in A.
\]
For readers’ convenience, we list Sobolev’s embedding theorem in the following lemma.

**Lemma 2.1.** There holds the following assertions:

(i) For integer $n > d/q + k$ with $k \in \mathbb{N}$ and $q \in (1, \infty)$, we have $W^{n,q} \hookrightarrow C^{k,\delta}$, for any $\delta \in (0, (n - d/q) - k) \wedge 1$.

(ii) If $1 < r < s < \infty$ and $m, n \in \mathbb{N}$ such that $\frac{d}{s} - m = \frac{d}{r} - n$, then $W^{n,r} \hookrightarrow W^{m,s}$.

3. **Existence and uniqueness of the solution to SPDE (1.3)**

This section is devoted to the global existence and uniqueness of the solution to nonlinear SPDE (1.7). As already noted in (2.1), if $\rho_t \in L^4$, then it holds that
\[
\|c_t\|_{W^{2,4}} = \|G \ast \rho_s\|_{W^{2,4}} \leq C\|\rho_t\|_{L^4}.
\]
A direct result of Sobolev’s imbedding theorem implies
\[
\|c_t\|_{W^{1,\infty}} = \|G \ast \rho_s\|_{W^{1,\infty}} \leq C\|G \ast \rho_s\|_{W^{2,4}} \leq C\|\rho_t\|_{L^4},
\]
where $C$ depends only on $d$.

Before stating the theorem about the wellposedness, we introduce the definition of solutions to SPDE (1.3). Denote by $C^2_c(\mathbb{R}^d)$ the space of compactly supported functions having up to second-order continuous derivatives.

**Definition 3.1.** A family of random functions $\{\rho_t(\omega) : t \geq 0, \omega \in \Omega\}$ lying in $S^\infty_T((0,T]; L^1 \cap L^4)$ is a solution to equation (1.3) if $\rho_t$ satisfies the following stochastic integral equation for all $\varphi \in C^2_c(\mathbb{R}^d)$,
\[
\langle \rho_t, \varphi \rangle = \langle \rho_0, \varphi \rangle + \int_0^t \langle \rho_s, \nabla \varphi \cdot \nabla c_s \rangle \, ds + \int_0^t \langle \rho_s, \sum_{i=1}^{d} D_i \varphi \sum_{k=1}^{d'} \sigma^{ik}_s \, dW^k_s \rangle + \int_0^t \frac{1}{2} \sum_{i,j=1}^{d} \int_{ij} \langle \rho_s, (\nu^{ik}_s \nu^{jk}_s + \sigma^{ik}_s \sigma^{jk}_s) \rangle \, ds.
\]
\textbf{Theorem 3.2.} Let Assumption 1 hold with $m = 2$. Assume $0 \leq \rho_0 \in L^1 \cap B^{\frac{d}{2}, 4}$ ($d = 2, 3$) with $\|\rho_0\|_1 = 1$. For any $T > 0$, there exists a $\kappa > 0$ depending only on $T, \lambda, \Lambda$ and $d$ such that if $\|\rho_0\|_4 \leq \kappa$, SPDE (1.3) admits a unique nonnegative solution in

$$M := L^2_{T,W}(0, T; W^{1,2}) \cap L^4_{T,W}(0, T; W^{1,4}) \cap S_{T,W}^\infty([0, T]; L^1 \cap L^4).$$

\textit{Proof.} We use the standard Banach fixed-point theorem to prove the well-posedness of the SPDE (1.3). Let

$$B := \{u \in S_{T,W}^\infty([0, T]; L^1 \cap L^4) : \|u\|_{S_{T,W}^\infty([0, T]; L^1 \cap L^4)} \leq 2\kappa, \|u(\cdot, t)\|_1 = \|u_0\|_1 = 1\},$$

with metric $d(u, v) := \|u - v\|_{S_{T,W}^\infty([0, T]; L^1 \cap L^4)}$ and $\kappa$ to be determined later.

Now we define a map $T : B \to B$ as follows: For each $\xi \in B$, set $T(\xi) := \rho^\xi$ the solution to the following linear SPDE:

$$\begin{cases} d\rho_t = \frac{1}{2} \sum_{ij} D_{ij}(\rho_t \sum_k (\nu_{ik}^j \nu_{jk}^i + \sigma_{ik}^j \sigma_{jk}^i)) dt - \nabla \cdot ((\nabla \rho \ast \xi_t) \rho_t) dt - \sum_i D_i(\rho_t \sum_k \sigma_{ik}^j dW_t^j), \\
\rho_0 \text{ is given}.
\end{cases}$$

Suppose $\|\rho_0\|_4 \leq \kappa$, $\kappa$ to be determined later.

For each $\xi \in B$ and $\rho_t \in L^p(\mathbb{R}^d)$ with $p \in \{2, 4\}$, relation (3.2) indicates that

$$\|((\nabla \rho \ast \xi_t) \rho_t)\|_p \leq \|\nabla \rho \ast \xi_t\|_\infty \|\rho_t\|_p \leq C \|\xi_t\|_4 \|\rho_t\|_p \leq C(d)\kappa \|\rho_t\|_p, \text{ a.s., for all } t \in [0, T].$$

Thus, by the $L^p$-theory of SPDEs (see [19] Theorem 5.1 and [20] Theorem 2.1) and the maximum principle (19.1 Theorem 5.12), linear SPDE (3.4) admits a unique solution $\rho^\xi$ which is nonnegative and lying in $L^p_{T,W}(0, T; W^{1,p}) \cap S_{T,W}^\infty([0, T]; L^p)$, $p \in \{2, 4\}$.

Next we check that $\rho^\xi \in S_{T,W}^\infty([0, T]; L^1 \cap L^4)$ and without causing confusion we drop the superscript $\xi$. It is easy to see that the solution of (3.4) has the property of conservation of mass, i.e.,

$$\|\rho_t(\cdot)\|_1 = \|\rho_0\|_1 = 1.$$

Applying the Itô formula for $L^p$-norms in [20] Theorem 2.1 we have

$$\begin{align*}
\|\rho_t\|_4^4 - \|\rho_0\|_4^4 &= \int_0^t \left( \sum_{ijk} \langle \rho_s^6, D_i^2 \rho_s, D_j ((\nu_{ik}^j \nu_{jk}^i + \sigma_{ik}^j \sigma_{jk}^i) \rho_s) \rangle + 6 \sum_k \left| \int_0^t \langle \rho_s, D_j (\rho_s \sigma_{ik}^j)^2 \rangle \right| ds + 12 \int_0^t \langle \rho_s (\nabla \rho_s), (\nabla \xi_t \ast \rho_s)^2 \rangle ds \right) \nu_{ik}^j ds \ \\
&= \int_0^t 12 \left( \sum_{ijk} \langle |\rho_s|^2 D_i \rho_s, \sigma_{ik}^j \rho_s \rangle = 3 \sum_i \langle D_i (|\rho_s|^4), \sigma_{ik}^j \rangle = -3 \left| \sum_i \langle D_i \sigma_{ik}^j \rangle \right| \right) \nu_{ik}^j ds.
\end{align*}$$

Due to (iii) in Assumption 1 we know that for $k = 1, 2, \ldots, d$,

$$12 \sum_i \langle |\rho_s|^2 D_i \rho_s, \sigma_{ik}^j \rho_s \rangle = 3 \sum_i \langle D_i (|\rho_s|^4), \sigma_{ik}^j \rangle = -3 \left| \sum_i \langle D_i \sigma_{ik}^j \rangle \right| = 0.$$
Using (iii) in Assumption \(1\) again leads to
\[
\sum_j D_j(\sigma^{jk} \rho) = \sum_j \rho D_j \sigma^{jk} + \sum_j D_j \rho \sigma^{jk} = \sum_j D_j \rho \sigma^{jk}.
\]
Therefore it holds that
\[
\| \rho_t \|_4^4 - \| \rho_0 \|_4^4 = - \int_0^t \sum_{ijk} \langle |\rho_s|^2 D_i \rho_s, (\nu_s^{ik} \nu_s^{jk}) D_j \rho_s \rangle \, ds - \int_0^t \sum_{ijk} \langle |\rho_s|^2 D_i \rho_s, D_j (\nu_s^{ik} \nu_s^{jk} + \sigma_s^{ik} \sigma_s^{jk}) \rho_s \rangle \, ds
\]
\[
+ 12 \int_0^t \langle \rho_s (\nabla \rho_s), (\nabla G \ast \xi_s) \rho_s^2 \rangle \, ds, \quad \text{a.s.}
\]
Notice that for \(0 \leq s \leq t\), one has
\[
12 \langle \rho_s (\nabla \rho_s), (\nabla G \ast \xi_s) \rho_s^2 \rangle \leq 12 \| \rho_s \nabla \rho_s \|_2^2 \| \rho_s \|_4^2 \| \nabla G \ast \xi_s \|_\infty
\]
(by relation (3.6)) \leq 12 \cdot (d) \| \rho_s \nabla \rho_s \|_2^2 \| \rho_s \|_4^2 \| \xi_s \|_4
\]
\[
\leq C(d) \cdot \kappa \| \rho_s \nabla \rho_s \|_2^2 \| \rho_s \|_4^2
\]
(by Young’s inequality) \leq 2 \lambda \| \rho_s \nabla \rho_s \|_2^2 + C(d, \lambda) \kappa^2 \| \rho_s \|_4^2
\]
and by (ii) in Assumption \(1\) one has
\[
- \sum_{ijk} \langle |\rho_s|^2 D_i \rho_s, D_j (\nu_s^{ik} \nu_s^{jk} + \sigma_s^{ik} \sigma_s^{jk}) \rho_s \rangle \leq 4 \Lambda^2 \langle \rho_s | \nabla \rho_s, \rho_s \rangle.
\]
We further have by (3.6)
\[
\| \rho_t \|_4^4 - \| \rho_0 \|_4^4 \leq - 6 \lambda \int_0^t \| \rho_s \nabla \rho_s \|_2^2 \, ds + 24 \Lambda^2 \int_0^t \langle |\rho_s|^2 | \nabla \rho_s, \rho_s \rangle \, ds + \int_0^t 2 \lambda \| \rho_s \nabla \rho_s \|_2^2 + C(d, \lambda) \kappa^2 \| \rho_s \|_4^2 \, ds
\]
\[
\leq - 4 \lambda \int_0^t \| \rho_s \nabla \rho_s \|_2^2 \, ds + 2 \lambda \int_0^t \| \rho_s \nabla \rho_s \|_2^2 \, ds + C(d, \lambda, \Lambda) \kappa^2 \int_0^t \| \rho_s \|_4^2 \, ds
\]
\[
\leq C(d, \lambda, \Lambda) \kappa^2 \int_0^t \| \rho_s \|_4^2 \, ds. \quad \text{a.s.}
\]
Then by Gronwall’s inequality it yields that
\[
\sup_{t \in [0, T]} \| \rho_t \|_4 \leq \| \rho_0 \|_4 \left( 1 + C \kappa^2 T e^{C \kappa^2 T} \right)^{\frac{1}{4}}, \quad \text{a.s.}
\]
where \(C\) depends only on \(d, \lambda\) and \(\Lambda\). Then there exists some \(\kappa_0\) depending only on \(d, T, \lambda\) and \(\Lambda\) such that for all \(\kappa \leq \kappa_0\)
\[
\left( 1 + C \kappa^2 T e^{C \kappa^2 T} \right)^{\frac{1}{4}} \leq 2.
\]
Hence it concludes that
\[
\sup_{t \in [0, T]} \| \rho_t \|_4 \leq 2 \| \rho_0 \|_4 \leq 2 \kappa, \quad \text{a.s.}
\]
which means that \(\rho \in S^\infty_{\text{loc}} (0, T; L^4(\mathbb{R}^d))\).

For all \(\xi \in \mathbb{B}\), let \(\rho^\xi\) be the unique solution of the linear SPDE (3.4). From the discussion above, we get the solution map
\[
\mathcal{T} : \mathbb{B} \rightarrow \mathbb{B}, \quad \xi \mapsto \rho^\xi.
\]
Next we show that the map \(\mathcal{T}\) is a contraction.
For any $\xi, \hat{\xi} \in \mathbb{B}$, set $\delta \rho = \rho - \hat{\rho}$ and $\delta \xi = \xi - \hat{\xi}$. As before, we apply Itô formula for the $L^4$-norm of $\delta \rho$:

$$
\|\delta \rho_t\|_4^4 = \int_0^t \left( - \sum_{ijk} \langle 6|\delta \rho_s|^2 D_i \delta \rho_s, D_j \left( \left( \nu_s^{ik} \nu_s^{jk} + \sigma_s^{ik} \sigma_s^{jk} \right) \delta \rho_s \right) \rangle + 6 \sum_k \left( |\delta \rho_s|^2, | \sum_j D_j (\delta \rho_s \sigma_s^{jk}) |^2 \right) \right) ds
+ \int_0^t \left( \langle |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \xi_s \rho^*_s - \nabla G * \hat{\xi}_s \hat{\rho}^*_s \rangle ds + 12 \sum_{ik} \langle |\delta \rho_s|^2 D_i \delta \rho_s, \sigma_s^{ik} \delta \rho_s \rangle dW^k_s \right) \right) ds

= \int_0^t \left( - \sum_{ijk} \langle 6|\delta \rho_s|^2 D_i \delta \rho_s, D_j \left( \left( \nu_s^{ik} \nu_s^{jk} + \sigma_s^{ik} \sigma_s^{jk} \right) \delta \rho_s \right) \rangle + 6 \sum_k \left( |\delta \rho_s|^2, | \sum_j D_j (\delta \rho_s \sigma_s^{jk}) |^2 \right) \right) ds

+ \int_0^t \left( \langle |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \xi_s \rho^*_s - \nabla G * \hat{\xi}_s \hat{\rho}^*_s \rangle ds \right)

\leq -3\lambda \int_0^t \|\delta \rho_s \nabla \delta \rho_s\|_2^2 ds + C(d, \lambda, \Lambda) \int_0^t \|\delta \rho_s\|_4^4 ds

+ \int_0^t \left( \langle |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \xi_s \rho^*_s - \nabla G * \hat{\xi}_s \hat{\rho}^*_s \rangle ds \right), \text{ a.s.} \quad (3.10)

In a similar way to (3.7), we have

$$
12 \left( |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \xi_s \rho^*_s - \nabla G * \hat{\xi}_s \hat{\rho}^*_s \right) \leq 12 \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\delta \rho_s\|_4^2 \|\nabla G * \xi_s \rho^*_s\|_\infty

\text{ (by relation } 3.2) \leq C(d) \kappa \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\delta \rho_s\|_4^2

\text{ (by Young's inequality)} \leq 2\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^4.

Thus, applying Hölder’s inequalities, Young’s inequalities and relation 3.2, we have

$$
12 \left( |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \xi_s \rho^*_s - \nabla G * \hat{\xi}_s \hat{\rho}^*_s \right)

= 12 \left( |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \delta \xi_s \rho^*_s + \nabla G * \delta \hat{\xi}_s \delta \rho_s \right)

\leq 12 \left( |\delta \rho_s|^2 \nabla \delta \rho_s, \nabla G * \delta \xi_s \rho^*_s \right) + 2\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\nabla G * \delta \xi_s \rho^*_s\|_\infty + 2\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^4

\leq C(d) \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\delta \rho_s\|_4^2 \|\delta \xi_s \rho^*_s\|_4 \|\nabla \delta \xi_s \rho^*_s\|_4 + 2\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2 \|\nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^4

\leq 3\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \xi_s \rho^*_s\|_4 \|\delta \rho_s\|_4^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^2

\leq 3\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \xi_s \rho^*_s\|_4 \|\delta \rho_s\|_4^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^2

= 3\lambda \|\delta \rho_s \nabla \delta \rho_s\|_2^2 + C(d, \lambda) \kappa^2 \|\delta \xi_s \rho^*_s\|_4 \|\delta \rho_s\|_4^2 + C(d, \lambda) \kappa^2 \|\delta \rho_s\|_4^2,

$$

which together with (3.10) implies

$$
\|\delta \rho_t\|_4^4 \leq C(d, \lambda, \Lambda) \kappa^2 \int_0^t \|\delta \rho_s\|_4^2 ds + \int_0^t C(d, \lambda) \kappa^2 \|\delta \xi_s\|_4^2 ds \text{ a.s.} \quad (3.11)
$$

By Gronwall’s inequality, we get

$$
\|\delta \rho\|_{S_{W}^{\infty}([0,T]; L^4)} \leq C(\Lambda, d, T, \kappa) \|\delta \xi\|_{S_{W}^{\infty}([0,T]; L^1)}, \quad (3.12)
$$

where $C(\Lambda, d, T, \kappa) = |T^{1/4} C(d, \lambda)|^{1/4} e^{C(d, \lambda, \Lambda) \kappa^2 T}$ is continuous and increasing with respect to $\kappa$. Hence, whenever $\kappa > 0$ is so small that $C(\Lambda, d, T, \kappa) < 1$, the solution map $T$ is a contraction mapping on the complete metric space $\mathbb{B}$, and it admits a unique fixed point $\rho = \rho^*$ which is the unique solution to SPDE (1.3). Hence, we complete the proof. \(\square\)
Remark 3.1. In view of equation \[[3.5]\] and the computation that follows, one may see that the stochastic integral there equals zero because of the divergence-free condition (iii) of assumption \[\Pi\]. This further allows us to obtain \(\rho \in S^\infty_{p,w}(0,T;L^4(\mathbb{R}^d))\) which finally yields the conclusions in Theorem \[3.2\] with a deterministic \(\kappa\). Without (iii) of assumption \[\Pi\] one may try the standard localization method which, however, may incur cumbersome arguments not just for the wellposedness of SPDE \[1.3\] in this section, but also for the subsequent sections.

**Theorem 3.3.** Let Assumption \[\Pi\] hold with \(m = 3\). Suppose further \(\rho_0 \in L^1 \cap W^{2,2}\). Then for any \(T > 0\), there exists \(\kappa > 0\) depending on \(T, \Lambda\) and \(d\) such that if \(\|\rho_0\|_4 \leq \kappa\), SPDE \[1.3\] admits a unique nonnegative solution in

\[
M_2 := L^2_{p,w}(0,T;W^{3,2}) \cap S^2_{p,w}(0,T;W^{2,2}) \cap L^4_{p,w}(0,T;W^{1,4}) \cap S^\infty_{p,w}(0,T;L^1 \cap L^4).
\]

**Proof.** Notice that \(W^{2,2} \hookrightarrow B^{\frac{1}{2},4} \hookrightarrow L^4\) for \(d = 2\) or 3. Comparing Theorem \[3.3\] and Theorem \[3.2\] we only need to prove that the obtained unique solution \(\rho\) in Theorem \[3.2\] is also lying in \(L^2_{p,w}(0,T;W^{3,2}) \cap S^2_{p,w}(0,T;W^{2,2})\). In fact, \(\rho \in \mathbb{M}\) is the solution of the following linear SPDE:

\[
\begin{aligned}
\rho_t &= \left(\frac{1}{4} \sum_{ij} D_{ij}(\rho \sum_k (\nu^{ik} \psi^{jk} + \sigma^{ik} \sigma^{jk})) + f_t \right) dt - \sum_i D_i (\rho \sum_k \sigma^{ik}) dW^i_t \\
\rho_0 &\text{ is given,}
\end{aligned}
\]

with

\[
f_t = -\nabla \cdot (\rho_t \nabla c_t) = -\nabla \rho t \cdot \nabla c_t + \rho_t^2 - \mu c_t.
\]

As \(\rho \in \mathbb{M}\), it follows that

\[
\|f\|_{L^2_{p,w}(0,T;L^2)} \leq C(d) \|\rho\|_{S^\infty_{p,w}(0,T;L^1)} \|\rho\|_{L^2_{p,w}(0,T;W^{1,2})} + \|\rho\|_{L^4_{p,w}(0,T;L^4)} < \infty.
\]

The \(L^p\)-theory of SPDE (see [19, Theorem 5.1]) and Theorem \[3.2\] imply that

\[
\rho \in L^2_{p,w}(0,T;W^{2,2}) \cap S^2_{p,w}(0,T;W^{1,2}) \cap \mathbb{M}.
\]

Similarly, for \(j = 1, \ldots, d\) one has

\[
\|D_j f_t\|_2 \leq C \|\rho\|_{W^{2,2}} \|\rho\|_4 + C \|\rho\|_{W^{1,4}} \|\rho\|_4,
\]

which together with \[8.16\] and \[8.41\] implies that

\[
\|f\|_{L^2_{p,w}(0,T;W^{1,2})} < \infty.
\]

Hence, applying the \(L^p\)-theory of SPDE (see [19, Theorem 5.1]) and Theorem \[3.2\] again, we conclude

\[
\rho \in L^2_{p,w}(0,T;W^{3,2}) \cap S^2_{p,w}(0,T;W^{2,2}) \cap L^4_{p,w}(0,T;W^{1,4}) \cap S^\infty_{p,w}(0,T;L^1 \cap L^4).
\]

The proof is completed. \(\square\)
4. WELL-POSEDNESS OF THE NONLINEAR SDE

Let us consider the following SDE:

\[
\begin{align*}
    dY_t &= \nabla G \ast \rho_t(Y_t) \, dt + \nu_t(Y_t) \, dB_t + \sigma_t(Y_t) \, dW_t, \quad t > 0, \\
    \rho_0 &= \text{the conditional density of } Y_t \text{ given } \mathcal{F}_t^W, \\
    Y_0 &= \zeta^1,
\end{align*}
\]

where we take \( B = B^1 \) in this section as a \( d' \)-dimensional Wiener process independent of \( W \) and \( \zeta^1 \). In the following, we prove the well-posedness of the nonlinear SDE (4.1) which actually shares the same solvability as SDE (1.3) for each \( i \in \mathbb{Z}^+ \).

**Theorem 4.1.** (Well-posedness of the SDE) Let Assumption \( \Box \) hold with \( m = 3 \) and \( \rho \) be the regular solution to the SPDE (1.3) obtained in Theorem 3.3. Then the nonlinear SDE (4.1) has a unique strong solution \((Y_t)_{t \geq 0}\) for \( \rho \in S^2_{\mathcal{F}^W}([0, T]; W^{2,2}) \cap S^\infty_{\mathcal{F}^W}([0, T]; L^4) \) being its conditional density under filtration \((\mathcal{F}_t^W)_{t \geq 0}\).

**Proof.** For the solution \( \rho_t \in S^2_{\mathcal{F}^W}([0, T]; W^{2,2}) \cap S^\infty_{\mathcal{F}^W}([0, T]; L^4) \) of the SPDE (1.3) given in Theorem 3.3 by embedding theorems, we have

\[
\nabla G \ast \rho \in S^2_{\mathcal{F}^W}([0, T]; W^{2,2}) \cap S^\infty_{\mathcal{F}^W}([0, T]; L^4) \quad \text{and} \quad S^2_{\mathcal{F}^W}([0, T]; W^{1,4}) \hookrightarrow S^2_{\mathcal{F}^W}([0, T]; W^{1,\infty}) \cap S^\infty_{\mathcal{F}^W}([0, T]; L^\infty),
\]

which ensures the existence and uniqueness of strong solution \((\mathcal{Y}_t)_{t \geq 0}\) to the following linear SDE:

\[
\begin{align*}
    d\mathcal{Y}_t &= \nabla G \ast \rho_t(\mathcal{Y}_t) \, dt + \nu_t(\mathcal{Y}_t) \, dB_t + \sigma_t(\mathcal{Y}_t) \, dW_t, \quad t > 0, \\
    \mathcal{Y}_0 &= \zeta^1.
\end{align*}
\]

To prove that the conditional density given \( \mathcal{F}_t^W \) of \((\mathcal{Y}_t)_{t \geq 0}\) exists and is the solution to SPDE (1.3), we need the following result on backward SPDEs and associated probabilistic representation.

**Lemma 4.1.** Let Assumption \( \Box \) hold with \( m = 3 \), \( \rho \in S^2_{\mathcal{F}^W}([0, T]; W^{2,2}) \cap S^\infty_{\mathcal{F}^W}([0, T]; L^4) \) and \( T_1 \in (0, T] \). Then for each \( G \in L^2(\Omega, \mathcal{F}_{T_1}; W^{2,2}) \), the following backward SPDE:

\[
\begin{align*}
    -du &= \frac{1}{2} \sum_{i,j,k} (\nu^{ik} \nu^{jk} + \sigma^{ik} \sigma^{jk}) D_{ij}u + \sum_i D_i G \ast \rho D_i u + \sum_{i,k} \sigma^{ik} D_i \psi_k \, dt - \sum_k \psi_k \, dW_t^k, \\
    u_{T_1} &= G,
\end{align*}
\]

admits a unique solution

\[
(u, \psi) \in \left( L^2_{\mathcal{F}^W}(0, T; W^{3,2}) \cap S^2_{\mathcal{F}^W}([0, T]; W^{2,2}) \right) \times L^2_{\mathcal{F}^W}(0, T; W^{2,2}),
\]

i.e., for any \( \varphi \in C^2_c(\mathbb{R}^d) \), there holds for each \( t \in [0, T_1] \),

\[
\langle u_t, \varphi \rangle = \langle \varphi, G \rangle + \int_t^{T_1} \left\langle \varphi, \frac{1}{2} \sum_{i,j,k} (\nu^{ik} \nu^{jk} + \sigma^{ik} \sigma^{jk}) D_{ij}u_s + \sum_i D_i G \ast \rho_s D_i u_s + \sum_{i,k} \sigma^{ik} D_i \psi_k^s \right\rangle ds \\
- \int_t^{T_1} \sum_k \langle \varphi, \psi_k^s \rangle \, dW_s^k, \quad a.s.
\]

Moreover, for this solution, we have

\[
(u_t(y) = E \left[ G(\mathcal{Y}_{T_1}) | \mathcal{Y}_t = y, \mathcal{F}_t^W \right], \quad a.s. \text{ for any } t \in [0, T_1].
\]

For each \( T_1 \in (0, T] \), take an arbitrary \( \xi \in L^\infty(\Omega, \mathcal{F}_{T_1}) \) and \( \phi \in C^\infty(\mathbb{R}^d) \). In view of the SPDE (1.3), applying the Itô formula to \( \langle u_t, \rho_t \rangle \) (the duality analysis on the (1.3) and (4.1) as in [8,23]) gives

\[
\langle u_0, \rho_0 \rangle = \langle \xi \phi, \rho_{T_1} \rangle - \int_0^{T_1} \sum_{i,k} \langle u_t, D_i (\sigma_t^{ik} \rho_t) \rangle \, dW_t^k - \int_0^{T_1} \sum_k \langle \rho_t, \psi_k^t \rangle \, dW_t^k, \quad a.s.,
\]
where \((u, \psi)\) is the solution in Lemma \([4.1]\) with \(G = \xi \phi\). Then we have by taking expectations on both sides,

\[
\langle u_0, \rho_0 \rangle = E[\langle \xi \phi, \rho_{T_1} \rangle] = E[\xi \langle \phi, \rho_{T_1} \rangle].
\]

On the other hand, in view of the probabilistic representation \([4.5]\), we have

\[
(u_0, \rho_0) = \int_{\mathbb{R}^d} E \left[ G(\bar{Y}_{T_1}) \right] \Phi^- 0 \rho_0(y) \, dy = E[\xi \phi(\bar{Y}_{T_1})] = E[\xi E[\phi(\bar{Y}_{T_1}) | F^- _{T_1}]].
\]

Therefore,

\[
E[\xi \langle \phi, \rho_{T_1} \rangle] = E \left[ E[\xi E[\phi(\bar{Y}_{T_1}) | F^- _{T_1}]] \right],
\]

which by the arbitrariness of \((T_1, \xi, \phi)\) implies that \(\rho_i\) is the conditional density of \(\bar{Y}_i\) given \(F^- _{T_1}\) for each \(t \in [0, T]\), and shows the existence of strong solution to SDE \((4.1)\). In fact, this also means that each strong solution of SDE \((4.4)\) with \(\rho \in S^2_{W^2}(0, T; W^{1,2}) \cap S^\infty_{W^2}(0, T; L^4)\) must have the conditional density \(\rho\) being the solution to SPDE \((4.3)\), and thus, the strong solution is unique. We complete the proof.

**Proof of Lemma \([4.1]\)** Embedding theorem gives \((4.2)\) which by the \(L^2\)-theory of backward SPDEs (see \([8, 23]\)) implies that backward SPDE \((4.1)\) has a unique solution \((u, \psi) \in \{L^2_{W^2}(0, T; W^{1,2}) \cap S^2_{W^2}([0, T]; L^2)\} \times L^2_{W^2}(0, T; W^{2,2})\). Then we need to show that the solution \((u, \psi)\) has higher regularity as it is done in the proof of Theorem \([5.3]\). In fact, we have for each \(i = 1, \ldots, d\),

\[
\|D_i G \ast \rho_s D_i u_s\|_2 \leq \|D_i G \ast \rho_s\|_{\infty} \cdot \|D_i u_s\|_2 \leq \|\rho_s\|_4 \cdot \|D_i u_s\|_2,
\]

and thus, \(D_i G \ast \rho_s D_i u \in L^2_{W^2}(0, T; L^2)\), which by \(L^2\)-theory of backward SPDEs indicated further

\[
(u, \psi) \in \{L^2_{W^2}(0, T; W^{2,2}) \cap S^2_{W^2}([0, T]; W^{1,2})\} \times L^2_{W^2}(0, T; W^{2,2}).
\]

Taking derivatives gives further

\[
\|D_i (D_i G \ast \rho_s D_i u_s)\|_2 \leq \|D_i J G \ast \rho_s D_i u_s\|_2 + \|D_i G \ast \rho_s D_i u_s\|_2
\]

\[
\leq \|D_i G \ast \rho_s\|_4 \cdot \|D_i u_s|_4 + \|D_i G \ast \rho_s\|_{\infty} \|D_i u_s\|_2
\]

\[
\leq \|\rho_s\|_4 \|D_i u_s\|^{1/2}_2 + \|\rho_s\|_4 \|D_i u_s\|^{3/4}_6 + \|\rho_s\|_4 \|D_i u_s\|_2
\]

\[
\leq \|\rho_s\|_4 \|u_s\|_{W^{2,2}} + \|\rho_s\|_4 \|u_s\|_{W^{2,2}},
\]

and thus, \(D_i G \ast \rho_s D_i u_s \in L^2_{W^2}(0, T; W^{2,2})\), \(i = 1, \ldots, d\). Applying the \(L^2\)-theory again, we arrive at

\[
(u, \psi) \in \{L^2_{W^2}(0, T; W^{3,2}) \cap S^2_{W^2}([0, T]; W^{2,2})\} \times L^2_{W^2}(0, T; W^{2,2}).
\]

W.l.o.g., we prove the probabilistic representation \([4.5]\) for the case when \(t = 0\). In fact, a straightforward application of \([22\) Theorem 3.1] yields that

\[
u_0(y) = G(\bar{Y}_{T_1}) - \int_0^{T_1} \left( \sum_k \psi^k_s(\Y_s) + \sum_i \sigma^i_s(\Y_s) D_i u_s(\Y_s) \right) dW^k_s, \quad \text{a.s.}
\]

Noticing that by embedding theorem it holds that \(L^2_{W^2}(0, T; W^{2,2}) \hookrightarrow L^2_{W^2}(0, T; C^{1/4}(\mathbb{R}^d))\), we may easily check that the stochastic integral in the above equality is mean-zero. Therefore, we have \(u_0(y) = E \left[ G(\bar{Y}_{T_i}) | \bar{Y}_0 = y, F^- _0 \right] \) by taking conditional expectation on both sides. For general \(t \in (0, T_1]\), the proof of \([4.3]\) follows similarly.

---

\(^3\)The fact \(u \in S^2_{W^2}([0, T]; L^2)\) is not claimed in \([8, 23]\) but it follows straightforwardly from \([17\) Theorem A.2] for Itô’s formula of square norms. It is similar in the relation \([4.5]\).
5. Propagation of chaos

To prove the propagation of chaos, we recall the following auxiliary stochastic dynamics \( \{Y^i_t\}_{t \geq 0} \) as defined in (1.10):

\[
\begin{align*}
\text{d}Y^i_t &= \nabla G \ast \rho_t(Y^i_t) \, \text{d}t + \nu_t(Y^i_t) \, \text{d}B^i_t + \sigma_t(Y^i_t) \, \text{d}W_t, \quad t > 0, \quad i = 1, \ldots, N, \\
\rho_t &\text{ is the conditional density of } Y^i_t \text{ given } \mathcal{F}_t^W \text{ for all } i = 1, \ldots, N. \\
Y^i_0 &= \zeta^i.
\end{align*}
\]

This means that \( \{Y^i_t\}_{t \geq 0} \) are \( N \) copies of solutions to the nonlinear SDE (1.10), and they are conditional i.i.d. given \( W_t \).

Our main theorem of propagation of chaos states that the mean-field dynamics \( \{Y^i_t\}_{t \geq 0} \) well approximate the regularized interacting particle system \( \{(X^i_t)_{t \geq 0}\}_{i = 1}^N \) in (1.5).

**Theorem 5.1.** Let Assumption 1 hold with \( m = 3 \), and \( \{X^i_t\}_{t \geq 0} \) and \( \{Y^i_t\}_{t \geq 0} \) satisfy the interacting particle system (1.5) and the mean-field dynamics (5.1) respectively. Then for any fixed \( 0 < \delta \ll 1 \), such that \( \varepsilon^{-\delta} \leq \delta \ln(N) \) it holds that

\[
\sup_{t \in [0, T]} \sup_{i = 1, \ldots, N} \mathbb{E} \left[ |X^i_t - Y^i_t|^2 \right] \leq C \left( \frac{\delta \ln(N)}{N^{1-\delta}} \right),
\]

where \( C \) is a constant depending only on \( T, \lambda, \Lambda \) and \( \|\rho_0\|_{W^{2,2}([0,\varepsilon])} \).

**Proof.** Applying Itô’s formula yields that

\[
\begin{align*}
|X^i_t - Y^i_t|^2 &= \int_0^t 2(X^i_s - Y^i_s) \cdot \left( \frac{1}{N-1} \sum_{j \neq i}^N \nabla G_s(X^{i\varepsilon}_s - X^{j\varepsilon}_s) - \nabla G \ast \rho_s(Y^i_s) \right) \, \text{d}s \\
&\quad + \int_0^t 2(X^i_s - Y^i_s) \cdot (\nu_s(X^{i\varepsilon}_s) - \nu_s(Y^i_s)) \, \text{d}B^i_s + \int_0^t 2(X^i_s - Y^i_s) \cdot (\sigma_s(X^{i\varepsilon}_s) - \sigma_s(Y^i_s)) \, \text{d}W_s \\
&\quad + \int_0^t \sum_j \sum_k^d (\nu_s^{jk}(X^{i\varepsilon}_s) - \nu_s^{jk}(Y^i_s))^2 \, \text{d}s + \int_0^t \sum_j \sum_k^d (\sigma_s^{jk}(X^{i\varepsilon}_s) - \sigma_s^{jk}(Y^i_s))^2 \, \text{d}s.
\end{align*}
\]

Taking expectations on both sides one has

\[
\begin{align*}
\mathbb{E} \left[ |X^i_t - Y^i_t|^2 \right] &\leq \mathbb{E} \left[ \int_0^t 2(X^i_s - Y^i_s) \cdot \left( \frac{1}{N-1} \sum_{j \neq i}^N \nabla G_s(X^{i\varepsilon}_s - X^{j\varepsilon}_s) - \nabla G \ast \rho_s(Y^i_s) \right) \, \text{d}s \right] \\
&\quad + C(d, d', \Lambda) \int_0^t \mathbb{E} \left[ |X^i_s - Y^i_s|^2 \right] \, \text{d}s,
\end{align*}
\]

where we have used the fact that

\[
\mathbb{E} \left[ \int_0^t 2(X^i_s - Y^i_s) \cdot (\nu_s(X^{i\varepsilon}_s) - \nu_s(Y^i_s)) \, \text{d}B^i_s \right] = \mathbb{E} \left[ \int_0^t 2(X^i_s - Y^i_s) \cdot (\sigma_s(X^{i\varepsilon}_s) - \sigma_s(Y^i_s)) \, \text{d}W_s \right] = 0,
\]

and (ii) in Assumption 1.

To continue, we split the error

\[
\begin{align*}
\mathbb{E} \left[ \int_0^t 2(X^i_s - Y^i_s) \cdot \left( \frac{1}{N-1} \sum_{j \neq i}^N \nabla G_s(X^{i\varepsilon}_s - X^{j\varepsilon}_s) - \nabla G \ast \rho_s(Y^i_s) \right) \, \text{d}s \right]
\end{align*}
\]
into three parts. Notice that
\[
\frac{1}{N-1} \sum_{j \neq i}^{N} \nabla G_e(X_{s}^{i,e} - X_{s}^{j,e}) - \nabla G * \rho_s(Y_{s}^{i}) = \frac{1}{N-1} \left( \sum_{j \neq i}^{N} \nabla G_e(X_{s}^{i,e} - X_{s}^{j,e}) - \sum_{j \neq i}^{N} \nabla G_e(Y_{s}^{i} - Y_{s}^{j}) \right)
+ \frac{1}{N-1} \sum_{j \neq i}^{N} \nabla G_e(Y_{s}^{i} - Y_{s}^{j}) - \nabla G * \rho_s(Y_{s}^{i})
+ \nabla G_e * \rho_s(Y_{s}^{i}) - \nabla G * \rho_s(Y_{s}^{i})
=: I_{11}^{s} + I_{12}^{s} + I_{13}^{s}.
\]
First we compute
\[
\int_{0}^{t} 2(X_{s}^{i,e} - Y_{s}^{i}) \cdot I_{11}^{s} \, ds \leq 2 \int_{0}^{t} |X_{s}^{i,e} - Y_{s}^{i}| \frac{1}{N-1} \| \nabla G_e \|_{W_{1,\infty}} \sum_{j=1}^{N} |X_{s}^{j,e} - Y_{s}^{j}| \, ds,
\]
which leads to
\[
E \left[ \int_{0}^{t} 2(X_{s}^{i,e} - Y_{s}^{i}) \cdot I_{11}^{s} \, ds \right] \leq \frac{C_{e}^{-d}}{N-1} \int_{0}^{t} \sum_{j=1}^{N} |X_{s}^{j,e} - Y_{s}^{j}| \, ds
\leq C_{e}^{-d} \int_{0}^{t} \sup_{i=1, \ldots, N} E \left[ |X_{s}^{i,e} - Y_{s}^{i}|^{2} \right] \, ds. \tag{5.4}
\]
To estimate the second term, we rewrite
\[
I_{12}^{s} = \frac{1}{N-1} \sum_{j \neq i}^{N} (\nabla G_e(Y_{s}^{i} - Y_{s}^{j}) - \nabla G_e * \rho_s(Y_{s}^{i})) =: \frac{1}{N-1} \sum_{j \neq i}^{N} Z_{j},
\]
where
\[
Z_{j} = \nabla G_e(Y_{s}^{i} - Y_{s}^{j}) - \nabla G_e * \rho_s(Y_{s}^{i}), \quad j \neq i.
\]
It is easy to check that
\[
E[Z_{j}^{i}|F_{t}^{W}, Y_{s}^{i}] = \nabla G_e * \rho_s(Y_{s}^{i}) - \nabla G_e * \rho_s(Y_{s}^{i}) = 0,
\]
since \( \{Y_{j}^{i}\}_{j=1}^{N} \) are conditional i.i.d. with common conditional density \( \rho_s \) given \( F_{t}^{W} \). Thus one concludes that
\[
E[|I_{12}^{s}|^2] = \frac{1}{(N-1)^2} E \left[ \left( \sum_{j \neq i}^{N} Z_{j} \right) \left( \sum_{k \neq i}^{N} Z_{k} \right) \right]
= \frac{1}{(N-1)^2} E \left[ \left( \sum_{j \neq i}^{N} Z_{j} \right) \left( \sum_{k \neq i}^{N} Z_{k} \right) |F_{t}^{W}, Y_{s}^{i} \right]
= \frac{1}{(N-1)^2} E \left[ \sum_{j \neq i}^{N} |Z_{j}^{i}|^2 |F_{t}^{W}, Y_{s}^{i} \right] = \frac{1}{N-1} E[|Z_{1}^{2}|].
\]
Due to the fact that
\[
E[|Z_{2}^{i}|^2] = E \left[ \left( \nabla G_e(Y_{s}^{i} - Y_{s}^{j}) - \nabla G_e * \rho_s(Y_{s}^{i}) \right)^2 \right] \leq C_{e}^{-2(d-1)},
\]
one has
\[ \mathbb{E}[|I_{12}^s|^2] \leq \frac{C\varepsilon^{2(d-1)}}{N-1}. \]
Thus we concludes
\[
\mathbb{E} \left[ \int_0^t 2(X_{s}^{i,\varepsilon} - Y_{s}^j) \cdot I_{12}^s \, ds \right] \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + \int_0^t \mathbb{E} \left[ |I_{12}^s|^2 \right] \, ds \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + C\varepsilon \int_0^t \mathbb{E} \left[ \|\nabla c_s\|_{W^{1,\infty}}^2 \right] \, ds \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + C\varepsilon.
\]
Lastly, we compute
\[
|I_{13}^s| = |\psi_\varepsilon \ast \nabla c_s(Y_{s}^j) - \nabla c_s(Y_{s}^i)| = \left| \int_{\mathbb{R}^d} \psi_\varepsilon(y) |\nabla c_s(Y_{s}^j - y) - \nabla c_s(Y_{s}^i)| \, dy \right| \\
\leq \|\nabla c_s\|_{W^{1,\infty}} \int_{\mathbb{R}^d} |y| \psi_\varepsilon(y) \, dy \leq C\varepsilon \|\nabla c_s\|_{W^{1,\infty}}.
\]
Then it yields that
\[
\mathbb{E} \left[ \int_0^t 2(X_{s}^{i,\varepsilon} - Y_{s}^j) \cdot I_{13}^s \, ds \right] \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + \int_0^t \mathbb{E} \left[ |I_{13}^s|^2 \right] \, ds \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + C\varepsilon \int_0^t \mathbb{E} \left[ \|\nabla c_s\|_{W^{1,\infty}}^2 \right] \, ds \\
\leq \int_0^t \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + C\varepsilon.
\]
Now collecting estimates (5.4), (5.5) and (5.6) implies
\[
\mathbb{E} \left[ \int_0^t 2(X_{s}^{i,\varepsilon} - Y_{s}^j) \cdot \left( \frac{1}{N-1} \sum_{j \neq i}^N \nabla G_{\varepsilon}(X_{s}^{i,\varepsilon} - X_{s}^{j,\varepsilon}) - \nabla G \ast \rho_s(Y_{s}^i) \right) \right] \\
\leq C\varepsilon \int_0^t \sup_{i=1,\ldots,N} \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + \frac{C\varepsilon^{2(d-1)}}{N-1} + C\varepsilon
\]
which together with (5.3) leads to
\[
\sup_{i=1,\ldots,N} \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \leq C_1\varepsilon \int_0^t \sup_{i=1,\ldots,N} \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \, ds + \frac{C_2\varepsilon^{2(d-1)}}{N-1} + C_3\varepsilon.
\]
Applying Gronwall’s inequality further yields that
\[
\sup_{t \in [0,T]} \sup_{i=1,\ldots,N} \mathbb{E} \left[ |X_{s}^{i,\varepsilon} - Y_{s}^j|^2 \right] \leq \left( \frac{C_2\varepsilon^{2(d-1)}}{N-1} + C_3\varepsilon \right) \left( 1 + C_1\varepsilon^{-d} T e^{C_1\varepsilon^{-d} T} \right) \\
\leq C \varepsilon^{3d+2} \frac{e^{C\varepsilon^{-d}}}{N-1} \\
\leq C \left( \frac{\delta \ln(N)}{N^{1+\delta}} \right)^{3d-2},
\]
where we let \( e^{\varepsilon^{-d}} \leq N^\delta \) for any fixed \( 0 < \delta \ll 1 \). The proof is completed.

Theorem 5.1 implies the propagation of chaos result in the following sense:

**Corollary 5.2.** Under the same assumption as in Theorem 5.1, the empirical measure
\[
\bar{\rho}^\varepsilon_{t,N} := \frac{1}{N} \sum_{i=1}^N \delta_{X_{s}^{i,\varepsilon}}
\]
associated to the stochastic particle system \((\ref{1.5})\) converges weakly to unique solution \(\rho_t\) to the nonlinear SPDE \((\ref{1.3})\). More precisely, for any fixed \(0 < \delta \ll 1\), such that \(\varepsilon^{-d} \leq \delta \ln(N)\), it holds that
\[
\lim_{N \to \infty} \mathbb{E} \left[ \left| \langle \rho_{t}^{\varepsilon,N}, \phi \rangle - \langle \rho_{t}, \phi \rangle \right|^2 \right] = 0, \tag{5.9}
\]
for all \(\phi \in C_{b}^{1}(\mathbb{R}^{d})\).

**Proof.** Let us compute
\[
\mathbb{E} \left[ \left| \langle \rho_{t}^{\varepsilon,N}, \phi \rangle - \langle \rho_{t}, \phi \rangle \right|^2 \right] = \mathbb{E} \left[ \left| \frac{1}{N} \sum_{i=1}^{N} \phi(X_{t}^{i,\varepsilon}) - \int_{\mathbb{R}^{d}} \phi(x) \rho_{t}(x) \, dx \right|^2 \right]
\leq 2\mathbb{E} \left[ \left| \phi(X_{t}^{1,\varepsilon}) - \phi(Y_{t}^{1}) \right|^2 \right] + 2\mathbb{E} \left[ \left| \frac{1}{N} \sum_{i=1}^{N} \phi(Y_{t}^{i}) - \int_{\mathbb{R}^{d}} \phi(x) \rho_{t}(x) \, dx \right|^2 \right]
=: I_{1} + I_{2}. \tag{5.10}
\]

According to \((\ref{5.2})\), one has
\[
I_{1} \leq 2 \left\| \nabla \phi \right\|_{\infty}^{2} \mathbb{E} \left[ \left| X_{t}^{1,\varepsilon} - Y_{t}^{1} \right|^2 \right] \leq C \left( \frac{\delta \ln(N)}{N^{1-\varepsilon}} \right)^{\frac{2d-2}{1-\varepsilon}}, \tag{5.11}
\]
where \(C\) depends only on \(\|\nabla \phi\|_{\infty}, T, \Lambda\) and \(\|\rho_{0}\|_{L^2 \cap W^{2,2}(\mathbb{R}^{d})}\). To estimate \(I_{2}\), we compute that
\[
\mathbb{E} \left[ \left| \frac{1}{N} \sum_{i=1}^{N} \phi(Y_{t}^{i}) - \int_{\mathbb{R}^{d}} \phi(x) \rho_{t}(x) \, dx \right|^2 \right] \leq \frac{1}{N^2} \sum_{i=1}^{N} \mathbb{E} \left[ \left| \phi(Y_{t}^{i}) - \int_{\mathbb{R}^{d}} \phi(x) \rho_{t}(x) \, dx \right|^2 \right] \tag{5.12}
\leq C \frac{1}{N}, \tag{5.13}
\]
where \(C\) depends only on \(\|\phi\|_{\infty}\). This combining with \((\ref{5.11})\) implies \((\ref{5.9})\). \(\Box\)

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