HYPERBOLIC VOLUME OF LINK FAMILIES

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Abstract

For families of knots and links given in Conway notation we compute lower
maximal and upper minimal bound of hyperbolic volume by using source links
and augmented links.

In this paper we consider families of knots and links (KLs) given in Conway
notation and their hyperbolic volume. Hyperbolic volume of the complement of a
link \( L \) will be shortly denoted as \( \text{Vol}(L) \). First we define a KL family [1]:

**Definition 1.** For a link or knot \( L \) given in an unreduced* Conway notation
\( C(L) \) denote by \( S \) a set of numbers in the Conway symbol excluding numbers
denoting basic polyhedra and zeros (denoting the position of tangles in the vertices of poly-
hedra). For \( C(L) \) and an arbitrary (non-empty) subset \( \tilde{S} \) of \( S \) the family \( F_{\tilde{S}}(L) \) of
knots or links derived from \( L \) is constructed by substituting each \( a \in S \), \( a \neq 1 \), by
\( sgn(a)(|a| + k_a) \) for \( k_a \in \mathbb{N} \).

**Definition 2.** A KL with single bigons, or equivalently, a KL given by Conway
symbol containing only tangles \( 1, -1, 2, \) or \( -2 \) is called a source link.

This means that all KLs generated from a source link \( \tilde{S} \) by substituting single
bigons by chains of bigons make a family generated from \( \tilde{S} \).

**Definition 3.** Let \( D \) be a reduced link diagram. Two crossings of \( D \) are twist
equivalent if there exist a flype connecting these two crossings into a bigon. The
**twist number of diagram** \( D \) is the number of its twist equivalence classes. The **twist
number** \( t(L) = t_{\text{min}}(L) \) of a link \( L \) is the minimal twist number over all diagrams
of \( L \) [2,3].

Since Conway symbols of links are twist-reduced, for a link diagram given in
Conway notation twist number \( t_D \) is the number of parameters plus the number
of single (isolated) crossings in the Conway symbol. All diagrams in a family of
alternating link diagrams have the same twist number \( t_D \). For example, twist

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*The Conway notation is called unreduced if in symbols of polyhedral KLs elementary tangles 1 in single vertices are not omitted.
number \( t_D \) of all diagrams of the family \( pq \) \((p \geq 2, q \geq 2)\) is \( t_D(pq) = 2\), for the family of the diagrams \( p\, 1\, q \) \((p \geq 2, q \geq 2)\) it is \( t_D(p\, 1\, q) = 3\), etc. However, there are alternating links with non-alternating diagrams of smaller twist number than the twist number of their alternating diagrams \([2]\).

**Conjecture 1.** For every link \( L \) given by a minimal twist-reduced diagram \( D \), the twist number \( t(L) \geq \lfloor \frac{t_D}{2} \rfloor + 1 \).

In many papers the twist number \( t(L) \) is used as the main tool to determine the upper and lower bound of a hyperbolic volume.

In the book *LinKnot* \([3]\) it is pointed out that different knot and link \((KL)\) invariants (unlinking number, signature, coefficients of Alexander and Jones polynomial, etc.) can be expressed as the functions of parameters from Conway symbols of \( KL \) families. Experimental results suggest that the same holds for hyperbolic volume.

Hyperbolic volumes of families of \( KLs \) from the program *LinKnot*, given in Conway notation for \( KLs \) up to 49 crossings, are computed by *Knotscape* and Windows version of *Snap Pea*, so all computations are done with a limited precision.

Simplest one-parameter family of knots is the family \( 2p+1 \) \((p \geq 1)\) which consists of non-hyperbolic knots \( 3_1, 5_1, \ldots \).

\[\text{Figure 1: (a) Point plot of the hyperbolic volumes for the family } (p+1)(p+1) \text{ (} p \geq 1 \text{); (b) interpolation by a function given by equation 1.1 for } n = 4.\]

For \( KLs \) from one-parameter subfamily \((p+1)(p+1) \geq 1\) of two-parameter link family \( pq \) \((p \geq 2, q \geq 2)\) with \( p \leq 23\), hyperbolic volumes are given in the following table:
The list plot (Fig. 1a) was obtained by Mathematica 6.0. All discrete list plots for one-parameter families are interpolated by functions of the form
\[
\sum_{i=0}^{n} a_i x^{2i} + c
\]
with \( n = 3 \) or \( n = 4 \). In this example, the best interpolation is obtained by Mathematica function FindFit for \( n = 4 \), with the maximal error \( 2.17782 \times 10^{-9} \) (Fig. 1b). The other possible interpolation by simpler functions of the form
\[
a_0 + \sum_{i=1}^{n} \frac{a_i}{(x+c)^{2i}}
\]
in some cases is less precise.

From the functions interpolating hyperbolic volumes of families we can make conclusion about their asymptotic behavior. For example, in the interpolating function for the family \( pp \) (\( p \geq 2 \)) obtained for \( n = 4 \) the coefficients are \( a_8 = 2.349132472871824 \), \( b_8 = 0.5358878957172603 \), and \( c = 2.944097878883564 \), so the interpolating function converges to \( \frac{a_8}{b_8} + c = Vol(6^\ast) = 7.32772\ldots \), which is the hyperbolic volume of Borromean rings complement. This result is in the complete agreement with the results of C. Petronio and A. Vesnin [4, Proposition 1], which showed that hyperbolic volume of the family \( pq \) (\( p \geq 2, q \geq 2 \)) converges to \( V_2 = Vol(6^\ast) = 7.32772\ldots \) when \( p \to \infty \) and \( q \to \infty \). The same result we obtained for the family \( p1p \) (\( p \geq 2 \)), i.e., for the family \( p1q \) when \( p \to \infty \) and \( q \to \infty \).

For the families of rational KLs of the form \( p \ldots p \) (\( p \geq 2 \)) where \( p \) occurs \( n \) times, shortly denoted as \( p^n \), \( \lim_{p \to \infty} Vol(p^n) = (n-1)V_2 \). The same holds for the rational link families of the form \( p1p, p1p1p, p1p1p1p, \ldots \) where \( p \) occurs \( n \) times.

Augmented links are used for obtaining bounds of hyperbolic volumes [5,6,7]. In the language of Conway symbols and chains of bigons, augmentation of a bigon chain (tangle) \( p \) (\( p \geq 2 \)) is its replacement by tangle (2, –2) 0 (Fig. 2).
Figure 2: Augmentation of a tangle $p$ ($p \geq 2$).

**Definition 4.** For a family of alternating links $L$ given by the Conway symbol

$C(p_1, ..., p_k, ..., p_n)$

containing chain of bigons $p_k$ ($p_k \geq 2$), subfamily with changing parameter $p_k$ ($p_k = 2, 3, 4, ...$) and all other $p$-s fixed is called $p_k$-*subfamily*, link obtained by replacing $p_k$ ($k \in \{1, 2, ..., n\}$, $p_k \geq 2$) by a tangle $(2, -2)0$ is called $p_k$-*augmented link* and denoted by $C(p_1, ..., \overline{p_k}, ..., p_n)$, and link obtained by replacing $p_k$ ($k \in \{1, 2, ..., n\}$, $p_k \geq 2$) by a single bigon 2 is called $p_k$-*source link* and denoted by $C(p_1, ..., p_k, ..., p_n)$. In the replacements single crossings remain unchanged.

**Theorem 1.** For every alternating algebraic link $L$ with at least two chains of bigons, or polyhedral link with at least one chain of bigons,

$Vol(C(p_1, ..., p_k, ..., p_n)) \leq Vol(L) < Vol(C(p_1, ..., p_k + 1, ..., p_n)) \leq Vol(C(p_1, ..., p, ..., \overline{p_k}, ..., p_n)).$

In general,

$Vol(L) \leq Vol(\overline{L}) \leq Vol(L),$

where $L$ is the source link, $\overline{L}$ is completely augmented link with all chains of bigons replaced by tangles $(2, -2)0$ and

$\lim Vol(C(p_1, ..., p_n)) = \overline{L},$

where the limes is taken when all chains of bigons tend to infinity [5, Corollary 2].

For example, let’s consider two-parameter family $L = 8^*p.0.q.0$ ($p \geq 2$, $q \geq 2$). For fixed $q = 2, 3, 4, 5, ...$ we obtain sequence of one-parameter subfamilies: $f_2 = 8^*p.0.20$, $f_3 = 8^*p.0.30$, $f_4 = 8^*p.0.40$, $f_5 = 8^*p.0.50$, ... satisfying the relations:

$16.6380380564 \ldots \leq Vol(f_2) \leq Vol(8^*(2, -2)0) = 19.29865114 \ldots$

$17.7392681473 \ldots \leq Vol(8^*2.0.30) \leq Vol(f_3) \leq Vol(8^*(2, -2).30) = 20.559914 \ldots$
From the interpolating functions for $f_2$, $f_3$, $f_4$, $f_5$ we obtain asymptotic values: 19.2972, ..., 20.5586, ..., 21.2112, ..., 21.574, ..., respectively.

For the whole family $8^p 0 q 0$ from the subfamily $8^p 0 p 0$ and interpolating function 1.1 with $n = 4$, $a_8 = 17.378499561645388$, $b_8 = 10.894820228608358$, $c = 20.771542391115194$, we obtain the asymptotic value $\frac{a_8}{b_8} + c = 22.3667\ldots$, and $Vol(L) = Vol(8^*(2, -2), (2, -2)) = 22.36710548\ldots$. Hence, for the family $L = 8^p 0 q 0$ ($p \geq 2$, $q \geq 2$) holds the following relation:

$$16.6380380564\ldots \leq Vol(L) \leq Vol(L) \leq Vol(\overline{L}) = 22.36710548\ldots$$

Conjecture 2 can be applied also to non-alternating KLs, but only positive chains of bigons can be varied and augmented. For example, let’s consider two-parameter family of non-alternating knots given by minimal diagrams $L = 8^p 0 q 0$ ($p \geq 2$, $q \geq 2$). For fixed $q = 2, 3, 4, 5\ldots$ we obtain sequence of one-parameter subfamilies: $f_2^q = 8^p 0. - 20$, $f_3^q = 8^p 0. - 30$, $f_4^q = 8^p 0. - 40$, $f_5^q = 8^p 0. - 50$, ... satisfying the relations:

$$13.2900030686\ldots \leq Vol(8^p 0 q 0) \leq Vol(f_2^q) \leq Vol(8^*(2, -2). -20) = 16.69568447\ldots$$

$$17.7392681473\ldots \leq Vol(8^p 0 q 0) \leq Vol(f_3^q) \leq Vol(8^*(2, -2). -30) = 19.29865114\ldots$$

$$18.3010568281\ldots \leq Vol(8^p 0 q 0) \leq Vol(f_4^q) \leq Vol(8^*(2, -2). -40) = 20.559914\ldots$$

$$18.6120521177\ldots \leq Vol(8^p 0 q 0) \leq Vol(f_5^q) \leq Vol(8^*(2, -2). -50) = 21.21212466\ldots$$

From the interpolating functions for $f_2$, $f_3$, $f_4$, $f_5$ we obtain asymptotic values: $16.6957\ldots$, $19.2961\ldots$, $20.56\ldots$, $21.2093\ldots$, respectively.

However, we cannot obtain correct results by fixing $p$, and varying $q$. This means that we need to fix negative chains and vary only positive ones. For negative chains, this problem can be solved by taking mirror image of a link $L$ or another minimal diagram of the same link with positive chains of bigons.

For completely augmented links signs of bigon chains are not relevant. For example, the family $L = 8^p 0 q 0$ ($p \geq 2$, $q \geq 2$) also can be given by the minimal
diagram $L' = 9^*(p-1)0.−1.−1.q0.−1.−1 : −1.−1$. From completely augmented link $\overline{L}$ we obtain minimal upper bound $\overline{\text{Vol}} = \text{Vol}(\overline{L}) = \text{Vol}(9^*(2, −2).−1.−1.(2, −2).−1.−1 : −1.−1) = 22.36710548\ldots$, so we conclude that the family $L = 8^*p0.−q0 (p \geq 2, q \geq 2)$ holds the following relation:

$$13.2900030686\ldots = \text{Vol}(L) \leq \text{Vol}L \leq \text{Vol}(\overline{L}) = 22.36710548\ldots$$

Hence, the families $8^*p0.q0$ and $8^*p0.−q0 (p \geq 2, q \geq 2)$ have the same minimal upper bound.

In the same way, we can consider the minimal diagrams $8^*p.−1.q.−1.r.−1 : −1$ and $q.−p.−(r+1).20 : −1 (p, q, r \geq 2)$ of the same non-alternating link. Their corresponding completely augmented links $8^*(2, −2)0.−1.(2, −2)0.−1.(2, −2)0.−1 : −1$ and $(2, −2)0.(2, −2)0.(2, −2)0.20 : −1$ have the same hyperbolic volume $18.83168337\ldots$ (Fig. 3).

Figure 3: (a) Point plot and (b) list plot 3D of the hyperbolic volumes for the family $p q (p \geq 2), q \geq 2$. 
Figure 4: (a) Point plot and (b) list plot 3D of the hyperbolic volumes for the family $8^*p.0.q.0$ ($p \geq 2, q \geq 2$).

As the last example of this kind we can consider three families of knots given by minimal diagrams $10^*p.0::q.0$, $10^*p.0::q.0$, $10^*p.0::q.0$ ($p \geq 2, q \geq 2$). Despite of the fact that the first is the family of alternating links, and other two are non-alternating, their hyperbolic volumes converge to the hyperbolic volume of the same completely augmented knot $10^*(2, -2) :: (2, -2)$ with the hyperbolic volume $26.3062315\ldots$ (Fig. 5).

Figure 5: Completely augmented links $8^*(2, -2)0. -1.(2, -2)0. -1.(2, -2)0. -1 : -1$ and $.(2, -2)0.(2, -2)0.(2, -2)0.2:0 : -1$ with the same hyperbolic volume.
Figure 6: (a) Completely augmented knot $10^*(2, -2) :: (2, -2)$; (b) list plot 3D of the hyperbolic volumes for the families $10^*p0 :: q0, 10^*p0 :: -q0, 10^*p0 :: -q0 (p \geq 2, q \geq 2)$ converging to the same minimal upper bound.

Two-parameter families and their hyperbolic volumes are visualized using Mathematica functions \texttt{ListPointPlot3D} and \texttt{ListPlot3D}. For the $KL$ family $pq (p \geq 2, q \geq 2)$, beginning with knot 22 ($4_1$) the point plot (\texttt{ListPointPlot3D}) and the plot (\texttt{ListPlot3D}) of the list $(p, q, Vol(p,q))$ are shown in Fig. 3. The same data for the family $8^*p0q0 (p \geq 2, q \geq 2)$ are shown on Fig. 4. For all alternating two-parameter $KL$ families we obtained smooth surfaces with one-parameter subfamilies which can be interpolated by the preceding class of functions.

Similar results are obtained for two-parameter families of non-alternating knots, up to 49 crossings. The most interesting of all are point and list plots of hyperbolic volumes of knots belonging to the family $6^* - (2p + 1).(2q). - 2.2. - 2$ where both positive and negative values for $p$ and $q$ are allowed $^\dagger$.

In order to visually represent hyperbolic volumes of knots belonging to three-parameter families, we fix one parameter and vary remaining two. Fig. 8 shows three-dimensional list plots of hyperbolic volumes for pretzel knots $p, q, r$ for $p = 2, p = 3, p = 4, p = 5$ and $p = 6$ simultaneously, where values of $q$ and $r$ are varied. Computations include alternating and non-alternating pretzel knots $p, q, r$ up to 49 crossings.

Similar results are obtained for the hyperbolic volume of the families of basic polyhedra computed for the family of antiprismatic basic polyhedra $6^*, 8^*, \ldots, 48^*$.

$^\dagger$This family contains Lorenz knots for $p \geq 2, q \geq 1$, with additional condition $p \leq q + 1$. 

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Figure 6: (a) Completely augmented knot $10^*(2, -2) :: (2, -2)$; (b) list plot 3D of the hyperbolic volumes for the families $10^*p0 :: q0, 10^*p0 :: -q0, 10^*p0 :: -q0 (p \geq 2, q \geq 2)$ converging to the same minimal upper bound.
Figure 7: (a) Point plot and (b) list plot 3D of the hyperbolic volumes for the family $\mathcal{K}^* - (2p + 1)(2q)$. –2.2; −2.

given by braid words of the form $(aB)^n$ ($n \geq 2$)† (Fig. 9) and for the family of basic polyhedra $9^*$ $\mathcal{B} = (AbACbACbC)$, $10^{**}$ $\mathcal{B} = (AbAbCbACbC)$, $11^{**}$ $\mathcal{B} = (AbAbACbACbC)$, $12F\mathcal{B} = (AbAbCbACbC)$, etc. (computed up to 48 crossings).

The obtained results suggest that hyperbolic volume of $\mathcal{K}$s given in Conway notation, with arbitrarily large number of crossings can be computed (or at least approximated) directly from the hyperbolic volume of the $\mathcal{K}$ family it belongs to provided we have the interpolating function, and that maximal lower and minimal upper bound of the hyperbolic volume for a given family can be simply computed from source link and augmented link.

The lower and upper bounds of hyperbolic volumes for all alternating $\mathcal{K}$ families derived from source links with at most $n = 9$ crossings are given in the following table. As the main referential volumes are used volume of ideal hyperbolic tetrahedron $V_0 = 1.0149416064...$, hyperbolic volume of Whitehead link $V_1 = \text{Vol}(212) = 3.663862377...$ and the hyperbolic volume of Borromean rings $V_2 = 2V_1 = \text{Vol}(6^*) = 7.327724753...$ For every link family is given its source link in classical and Conway notation [8], Conway symbol of the family, and lower and upper bound of hyperbolic volume. Families which have the same lower and upper bound are given in pairs or triples. All parameters $p, q, \ldots$ are greater or equal 2.

†For $n = 2$ we obtain, as a limiting case, figure-eight knot.
Figure 8: List plot 3D of the hyperbolic volumes for the three-parameter family of pretzel knots $p, q, r$ for fixed $p \in \{2, 3, 4, 5, 6\}$.

|   | $p$  | $q$  | $r$  | $V_0$     | $V_1$     |
|---|------|------|------|-----------|-----------|
| 1 | $4_1$ | 2    | 2    | $pq$      | $V_0$     |
| 2 | $5_1^2$ | 2    | 1    | 2      | $1q$      |
| 3 | $6_3^2$ | 2    | 2    | 2      | $pqr$     |
| 4 | $6_1^2$ | 2    | 2    | 2      | $p,q,r$   |
| 5 | $6_3$  | 2    | 1    | 2    | $p11q$    |
| 6 | $7_6$  | 2    | 2    | 1    | $pq1r$    |
| 7 | $7_7$  | 2    | 1    | 1    | $p111q$   |
| 8 | $7_5^2$ | 2    | 2    | 2    | $p1,q,r$  |
| 9 | $7_3^2$ | 2    | 2    | 2+   | $p,q,r+$  |
| 10| $7_6^2$ | 2    | 2    | 2    | $p,q,s$   |
| 11| $8_{12}$ | 2    | 2    | 2    | $pqr,s$   |
| 12| $8_2$  | 2    | 2    | 2    | $p1q1r$   |
| 13| $8_3^2$ | 2    | 1    | 1    | $p1111q$  |
| 14| $8_3^2$ | 2    | 2    | 2    | $p,q,r,s$ |
| 15| $8_3^2$ | 2    | 2    | 2    | $(p,q)(r,s)$ |
| 16| $8_3^2$ | 2    | 2    | 2    | $pq,r,s$  |
| 17| $8_3^3$ | 2    | 2    | 2    | $p,q,r++$ |
| 18| $8_{10}$ | 2    | 1    | 2    | $p11q,r$  |
| 19| $8_{12}$ | 2    | 1    | 2    | $p1,q,r+$ |
| 20| $8_{15}$ | 2    | 1    | 2    | $p1,q1,r$ |
| 21| $8_{13}$ | 2    | 1    | 2    | $.p1$     |
| 22| $8_{14}$ | 2    | 2    | 2    | $.p:q$    |
| 23| $8_{16}$ | 2    | 2    | 2    | $.p:q0$   |
| 24| $8_{16}$ | 2    | 2    | 2    | $.p.q$    |
| 25| $8_{17}$ | 2    | 2    | 2    | $.p.q$    |
Figure 9: (a) Point plot of the hyperbolic volumes for the family of the antiprismatic basic polyhedra \((2n)^*\) \((n = 2, \ldots 24)\); (b) its interpolation function given by equation (1.1) for \(n = 4\).

|   |   |   |   |   |
|---|---|---|---|---|
| 26 | 923 | 22122 | \(pqrs\) | 10.6113483 \ldots | \(6V_1\) |
| 27 | 911 | 22212 | \(pqr\) | 10.75904664 \ldots | \(6V_1\) |
| 28 | 927 | 221112 | \(pqr\) | 10.99998096 \ldots | 17.47714082 \ldots |
| 29 | 923 | 221112 | \(pqr\) | 11.1884778 \ldots | 19.0793609 \ldots |
| 30 | 923 | 21, 2, 21 | \(p1qr\) | 12.046092 \ldots | 18.8316834 \ldots |
| 31 | 925 | 221, 2 | \(pqr\) | 11.3817861 \ldots | 19.5826692 \ldots |
| 32 | 925 | 221, 2 | \(pqr\) | 11.39030515 \ldots | 23.3281935 \ldots |
| 33 | 925 | 212, 2, 2 | \(pqr\) | 10.74025676 \ldots | \(6V_1\) |
| 34 | 928 | 21, 2, 2++ | \(pqr\) | 10.74025676 \ldots | \(6V_1\) |
| 35 | 926 | 211, 2, 2 | \(p1q\) | 11.76223429 \ldots | 19.5826692 \ldots |
| 36 | 926 | 211, 2, 2+ | \(pq\) | 11.76223429 \ldots | 19.5826692 \ldots |
| 37 | 924 | 21, 2, 2, 2 | \(pqr\) | 12.2765628 \ldots | 24.55255516 \ldots |
| 38 | 929 | 21, 2, 2 | \(p1qr\) | 12.2765628 \ldots | 24.55255516 \ldots |
| 39 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 40 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 41 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 42 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 43 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 44 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 45 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 46 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 47 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 48 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 49 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 50 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 51 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 52 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 53 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 54 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 55 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 56 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 57 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 58 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
| 59 | 930 | 211, 21, 2 | \(p1q1r\) | 11.95452697 \ldots | 19.58266925 \ldots |
Some of non-alternating source links from the preceding table are non-hyperbolic, so their hyperbolic volume is 0. In the family 60), link $p, q, -r$ is non-hyperbolic for $r = 2$, and hyperbolic otherwise; in the family 66) all links except source link $2, 2, -2, -2$ are hyperbolic. The same holds for the family 67) and its source link $(2, 2) - (2, 2)$, and for family 80) and its source link $2 : -20 : -20$.

It is interesting to notice that all families with the same lower and upper bound of hyperbolic volume have subfamilies of distinct links with the same hyperbolic volume, which can be distinguished by Alexander and Jones polynomial. Alternating link families 3) and 4) have the subfamilies $p q 2$ and $2, 2, 2$ of links which cannot be distinguished by hyperbolic volume, 8) and 9) have the subfamilies $p, 1, 2$ and $p, 2, 2 +, 14$) and 15) the subfamilies $q, p, 2$ and $(p, q)(2, 2), 18$ and 19) the subfamilies $p 1 1, 2, 2$ and $p 1 1, 2, 2 +, 22$) and 23) the subfamilies $p : 2$ and $p : 20, 31$ and 32) the subfamilies $p 1 q, 2$ and $p 1 q, 2 +, 35$) and 36) the subfamilies $p 1 1, 2, 2$ and $p 1 1, 2, 2 +, 35$) and 36) the subfamilies $q, p, 2$ and $(p, q)(2, 2), 40, 41$ and 42) the subfamilies $(2, 2) p(2, 2), 2, 2, 2, 2 + p$ and $(2, 2 + p)(2, 2)$, 46 and 47) have the subfamilies $p 1 : 2$ and $p 1 : 20$ with the same property. The same holds for non-alternating link families 65) and 66) and their subfamilies $q, p, 2, -2$ and $(p, q)(2, -2), 71$ and 72) and their subfamilies $p 2 1, 2, -2$

\(^3\)Sequence of $p$ pluses is denoted by $+p$. 

| 60 | $6^3_3$ | $2, 2, -2$ | $p, q, -r$ | 0 | $4V_1$ |
| 61 | $7^3_8$ | $2 1, 2, -2$ | $p 1, q, -r$ | $V_1$ | 16.0004687... |
| 62 | $8_{21}$ | $2 1, 2, -2$ | $p 1, q 1, -r$ | 6.78371352... | 17.6277542... |
| 63 | $8^7_{15}$ | $2, 2, -2$ | $p q, r, -s$ | $V_1$ | 0 |
| 64 | $8^8_{16}$ | $2 1, 2, -2$ | $p 1 1, q, -r$ | 5.3334895696... | 17.6277542... |
| 65 | $8^4_3$ | $2, 2, -2$ | $p q, r, -s$ | $2V_1$ | 24.09218408... |
| 66 | $8^3_9$ | $(2, 2) - (2, 2)$ | $(p, q)(r, -s)$ | $2V_1$ | 24.09218408... |
| 67 | $8^3_7$ | $2, 2, -2, -2$ | $p, q, r, -s$ | 0 | 24.09218408... |
| 68 | $8^3_{10}$ | $2, 2, -2, -2$ | $(p, q) - (r, s)$ | 0 | 24.09218408... |
| 69 | $9_{44}$ | $2 2, 2 1, -2$ | $p q, r 1, -s$ | 7.40676752724... | 23.32819345... |
| 70 | $9_{45}$ | $2 1, 2 1, -2$ | $p 1 1, q, -r$ | 8.6020031166... | 19.58266925... |
| 71 | $9^5_{48}$ | $2 2 1, 2, -2$ | $p q, r, -s$ | 7.706911803... | 19.58266922... |
| 72 | $9^3_{14}$ | $2 1 1, 2, -2$ | $p 1 1, q, -r$ | 7.706911803... | 19.58266922... |
| 73 | $9^3_{13}$ | $2 1, 2, -2$ | $p 1 q, r, -s$ | 5.3334895696... | 6V_1 |
| 74 | $9^3_{16}$ | $2 1, 2, -2$ | $p 1, q, -s$ | 9.96651188... | 24.55255516... |
| 75 | $9^5_{58}$ | $(2, 1, -2)(2, 2)$ | $(p, q)(r, s)$ | 9.96651188... | 24.55255516... |
| 76 | $9^5_{60}$ | $(2, 1, 2)(2, -2)$ | $(p, q)(r, -s)$ | 8.997359144... | 24.55255516... |
| 77 | $9^5_{60}$ | $(2, 1, 2)(2, -2)$ | $(p, q) - (r, s)$ | 5.333489567... | 24.55255516... |
| 78 | $9^3_{18}$ | $(2, 2)(2, -2)$ | $(p, q)(r, -s)$ | 2V_1 | 24.55255516... |
| 79 | $9^3_{19}$ | $(2, 2)(2, -2)$ | $(p, q) - (r, s)$ | 2V_1 | 24.55255516... |
| 80 | $9^3_{61}$ | $2 : -20 : -20$ | $p : -q 0 : -r 0$ | 0 | 22.07660239... |
| 81 | $9^3_{49}$ | $-20 : -20 : -20$ | $-p 0 : -q 0 : -r 0$ | 9.427073628... | 21.1717152... |
| 82 | $9^3_{20}$ | $(2, -2)$ | $- (p, q)$ | 3V_1 | 19.66433108... |
| 83 | $9^3_{31}$ | $- (2, 2)$ | $- (p, q)$ | 0 | 21.1717152... |
| 84 | $9^3_{47}$ | $8^* - 20$ | $8^* - p 0$ | 10.0499579... | 16.69568447... |

\(^3\)Sequence of $p$ pluses is denoted by $+p$. 


Conjecture 2. Every two families with the same lower and upper bound of hyperbolic volume have subfamilies of links with the same hyperbolic volumes.

Theorem 2. Hyperbolic volume completely distinguishes alternating links belonging to the same family.

This theorem does not hold for non-alternating links: for example, links of the subfamilies \( p, 3, -2 \) and \( p - 6, 2, -3 \) \((p \geq 8)\) belonging to the family 60) \( p, q, -r \) have the same hyperbolic volume. These links can be distinguished by Alexander and Jones polynomial.

For \( n \leq 12 \) there are no source knots with the same hyperbolic volume, so we propose the following conjecture:

Conjecture 3. Hyperbolic volume completely distinguishes source knots.

Definition 5. The replacement of a tangle \((2, 2)\) by \((2, 2)\) 0, or \((2, -2)\) by \((2, -2)\) 0, or \(- (2, 2)\) by \(- (2, 2)\) 0 and vice versa will be called \((2, 2)\)-reversal. Two links are called \((2, 2)\)-equivalent if one can be obtained from the other by \((2, 2)\)-reversals.

Theorem 3. \((2, 2)\)-equivalent links have the same hyperbolic volume. The same holds for their corresponding augmented links.

From the Theorem 3 we can make many conclusions about links and their hyperbolic volume. For example, pretzel links \( 2, 2, -p \) \((p \geq 3)\) and rational links \( 2, (p-2) 2 \) have the same hyperbolic volume and its upper bound \( 4V_1 \). We can also conclude that all links belonging to the family \( 2, 2, -p \) are hyperbolic, except the link \( 2, 2, -2 \) which is \((2, 2)\)-equivalent with non-hyperbolic rational link \( 2 - 2 2 = 4 \).

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*Except mutant source knots.*
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