Spatiotemporal spread of perturbations in power-law models at low temperatures: Exact results for classical out-of-time-order correlators

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Phys. Rev. E 104, 044117 — Published 18 October 2021

DOI: 10.1103/PhysRevE.104.044117
Spatio-temporal spread of perturbations in power-law models at low temperatures:

Exact results for classical OTOC

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(Dated: September 27, 2021)

We present exact results for the classical version of the Out-of-Time-Order Commutator (OTOC) for a family of power-law models consisting of $N$ particles in one dimension and confined by an external harmonic potential. These particles are interacting via power-law interaction of the form $\propto \sum_{i,j=1}^{N} |x_i - x_j|^{-k} \forall k > 1$ where $x_i$ is the position of the $i^{th}$ particle. We present numerical results for the OTOC for finite $N$ at low temperatures and short enough times so that the system is well approximated by the linearized dynamics around the many body ground state. In the large-$N$ limit, we compute the ground-state dispersion relation in the absence of external harmonic potential exactly and use it to arrive at analytical results for OTOC. We find excellent agreement between our analytical results and the numerics. We further obtain analytical results in the limit where only linear and leading nonlinear (in momentum) terms in the dispersion relation are included. The resulting OTOC is in agreement with numerics in the vicinity of the edge of the “light cone”. We find remarkably distinct features in OTOC below and above $k = 3$ in terms of going from non-Airy behaviour ($1 < k < 3$) to an Airy universality class ($k > 3$). We present certain additional rich features for the case $k = 2$ that stem from the underlying integrability of the Calogero-Moser model. We present a field theory approach that also assists in understanding certain aspects of OTOC such as the sound speed. Our findings are a step forward towards a more general understanding of the spatio-temporal spread of perturbations in long-range interacting systems.

I. INTRODUCTION

Collective behaviour of many particle systems far-from-equilibrium has been a central issue of interest [1–5]. In particular, the role of integrability and its breaking in the dynamical behaviour of a system is of great interest both from a theoretical [6–9] and an experimental perspective [10–12]. More generally, chaos which characterises extreme sensitivity to arbitrarily small perturbations in initial conditions has been extensively studied both in classical [13, 14] and quantum systems [15, 16]. Recently, long-ranged systems have taken a special place as a platform for studying collective behaviour and have become a promising avenue for experimental research. Notable examples of long-ranged systems include, one dimensional one component plasma [17–20], Dyson’s Log gas [21], Calogero-Moser Systems [22–24], dipolar Bose gas [25, 26], ionic systems [27–29], 3D Coulomb gas confined in one dimension [30], Yukawa gas [31] to name a few. Two main ingredients for understanding sensitivity to initial conditions in long-ranged systems are (i) the availability of a family of long-ranged models which contain in them both generic and integrable cases with preferably having reasonably well understood classical and quantum limits and (ii) the availability of diagnostics which can characterize dynamical phenomena and has both classical and quantum counterparts.

The Riesz gas [32, 33] is one such platform which encompasses a family of long ranged models and we consider the case when it is trapped in an external trapping potential. This family contains in it several models which have themselves been a subject of great interest both from a physics and mathematics perspective. Familiar examples for specific values of $k$ include Dyson’s log gas ($k \to 0$), Integrable Calogero-Moser system ($k = 2$), 1D One Component Plasma ($k=-1$), Coulomb gas confined to 1D ($k = 1$), dipolar gas ($k = 3$) and hard rods ($k \to \infty$). The parameter $k$ which characterises the power-law interaction spans from “relatively long-ranged” to “relatively short-ranged” as we increase $k$. Recently, collective field theory [33] has been provided for the Riesz gas and its finite ranged generalization [34]. Such a collective description is an important step forward to study nonlinear hydrodynamics [35, 36].

The second ingredient, i.e., a suitable diagnostic which can characterize dynamical phenomena is the classical version of the quantum Out-of-Time Ordered Correlator (OTOC) [37–43] which quantifies growth/decay of perturbations in time and their spread in space. This quantity is precisely suited to explore questions on chaos, aspects of integrability, entropy and nonlinearity to name a few. In recent years, classical OTOC has been employed as an insightful diagnostic tool to study various extended classical systems such as classical one dimensional spin chains [44], thermalised fluid obeying truncated Burgers equation [45], classical interacting spins on Kagome lattice [46, 47], disordered systems [48], two-dimensional anisotropic XXZ model [49], spherical p-spin glass model [50], discrete non-linear Schrodinger equation [51] and open systems such as the driven-dissipative duffing chain[52]. The classical OTOC shines light on how perturbations spread in space and grow/decay in time. The more conventional ways of probing classical perturbations such as Lyapunov exponents can be deduced from OTOC although the OTOC is well suited for probing extended many particle systems.
It is worth noting that, to the best of our knowledge, all works on classical OTOC so far have been restricted to the case of short ranged interactions (essentially nearest neighbour) and away from any integrable points [44–49, 51, 52]. In fact, even for short ranged models, although there have been studies of OTOC in quantum integrable systems [43, 53–57], there has been no work reported on OTOC in classical integrable models to the best of our understanding. In this work, we aim to fill this important gap in our understanding by studying low temperature OTOC of a family of power-law models. Our key results can be summarized as follows: (i) We found exact analytical computation of the dispersion relation in absence of external harmonic potential and utilised it to compute analytical results for OTOC at low temperatures and short enough times (ii) We performed direct numerics and demonstrated excellent agreement with the results obtained after using dispersion relation. (iii) We obtained exact results for OTOC at the integrable point \( k = 2 \). (iv) We introduced a field theory approach that paved an alternate path to the investigative aspects of OTOC. A summary of key results is presented in Table I.

| Asymptotic Scaling of \( D(x,t) \), for large \( \eta_{\pm} \) or \( \Delta_{\pm} \) | \( 1/x^{k+2} \) |
|---|---|
| Spread of perturbations | Ballistic |
| Dispersion relation, \( \omega_k(q) \) | \( \alpha_k q - \beta_k q^k \), for \( 1 < k < 3 \) |
| | \( \alpha_k q + \gamma_3 q^3 \log(qa) \), for \( k = 3 \) |
| | \( \alpha_k q - \delta_k q^3 - \beta_k q^k \), for \( 3 < k < 5 \) |
| Profile of the envelope of perturbations | Convex for \( k < 2 \) |
| | Flat for \( k = 2 \) |
| | Concave for \( k > 2 \) |

TABLE I: Summary of key results \((k > 1)\). Note that \( \alpha_k, \beta_k, \delta_k, \gamma_3 \) are all given in Eq. 5.

II. MODEL AND DEFINITIONS

We consider \( N \) classical particles in 1D with pairwise interaction confined by an external harmonic trap. This

FIG. 1: (a) Heat map of the OTOC (for clear visualization, we choose \( N = 65 \)) from direct numerical simulation of Eq. 7 for the Calogero-Moser case \((k = 2)\). The solid black line separating the white and blue (gray) regions depicts the light cone. Time axis is in units of \( a/v_B \). (b) 2D heatmap showing a ballistic light-cone \((N = 65)\). Note that each point in the plot corresponds to the amplitude of the OTOC. (c) Comparing OTOC from direct numerical simulation of Eq. 7 with the analytical expression in Eq. 9 at a time snapshot, \( t \approx 164 \) (in units of \( a/v_B \)). Note that, for visualisation purpose, we have plotted only the positive x-axis and the results are mirror symmetric on the other side. The time \( t \) is chosen such that the front moves about 10% from the center to make sure that we are far enough from the edge of the cloud. Here, \( a = 0.0347, v_B = 90.5207 \).
is the so called Riesz gas [32] given by,

\[ H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V_k(\{x_j\}) \]  

(1a)

\[ V_k(\{x_j\}) = \sum_{i=1}^{N} \left[ \frac{m \omega^2}{2} x_i^2 + \frac{J}{2} \sum_{j \neq i} \frac{1}{|x_i - x_j|^k} \right] \]  

(1b)

Here \( x_i \) is the position of the \( i \)th particle (such that \( i < j \iff x_i < x_j \), i.e., ordering is maintained), \( p_i \) is the corresponding conjugate momentum, \( m \) is the mass of each particle, \( J \) is the interaction strength and \( \omega \) is the frequency of the external trap. Therefore, equations of motion become \( \dot{x}_i = \frac{p_i}{m} \) and

\[ \dot{p}_i = -m \omega^2 x_i + \frac{Jk}{2} \sum_{j \neq i} \text{sgn}(x_i - x_j) \frac{1}{|x_i - x_j|^{k+1}} \]  

(2)

For a set of initial conditions \( \{x_i(0), p_i(0)\} \), one can in principle solve the above \( N \) ordinary differential equations each consisting of \( (N-1) \) pairing terms thereby rendering it highly nonlinear and nonlocal.

## III. Dispersion Relation

In the ground state in the absence of external trap, we find the dispersion relation (using small oscillation analysis) to be (Appendix B),

\[ \omega_k(q) = \sqrt{\frac{Jk(k+1)}{m a^k+2} \left[ 2 \zeta(k+2) - P(k, q) \right]} \]  

(3)

where \( P(k, q) = L_{k+2}(e^{-iqa}) + L_{k+2}(e^{iqa}) \) with \( L_n(z) = \sum_{p=1}^{\infty} z^p/p^n \) being the Polylogarithm function. Here \( a \) is the lattice spacing/inverse density (i.e., equilibrium is achieved when \( x_i(t) = ai \) and \( p_i(0) = 0 \)) and \( \zeta(z) = \sum_{n=1}^{\infty} 1/n^z \) is the Riemann zeta function. The lattice spacing ‘\( a \)’ can be introduced in the following manner. Let us say that we have \( N \) particles confined in a Harmonic trap of frequency \( \omega \). The minimum energy configuration is such that the density takes a dome shape \[58\] and the inter-particle distance at the centre is given by \[58\]

\[ a = 2^{1/k+2} \left( J(k+1)/\zeta(k)/m \right)^{1/2} \left( \frac{\omega}{B[1+1/k, 1+1/k]} \right)^{1/2} \]

where \( B[a, b] = \int_0^1 dw w^{-1}(1-w)^{k-1} \) is the standard Beta function. The homogeneous limit can thus be realised by simultaneously taking a careful limit \( N \to \infty \) and \( \omega \to 0 \) keeping \( \omega N \) to be a constant. For a finite \( \omega \), the dome shape survives in the large-\( N \) limit.

It turns out that the above dispersion relation (Eq. 3) is periodic with period \( 2\pi/a \) and it has a maxima at \( q = \pi/a \). Using the remarkable property of Polylogarithm function \[59\], \( L_n(z) = \Gamma(1-n) \log((1/z)^{n-1} + \sum_{i=0}^{\infty} \zeta(n-i) \log(z)^i) \) for \( n \notin \mathbb{Z} \) and \( |\log z| < 2\pi \), we find that the above exact dispersion relation (Eq. 3) for \( k > 1 \) has the following small-\( q \) expansion up to the next leading relevant order (Appendix B),

\[ \omega_k(q) \approx \begin{cases} \alpha_k q - \beta_k q^k, & 1 < k < 3 \\ \alpha_k q + \gamma_k q^3 \log(qa), & k = 3 \\ \alpha_k q - \delta_k q^3 - \beta_k q^k, & 3 < k < 5 \end{cases} \]  

(4)

with

\[ \alpha_k = \sqrt{\frac{Jk(k+1)}{m a^k}} \zeta(k), \quad \gamma_3 = 1 - \frac{1}{2} \left[ \frac{a}{3m \zeta(3)} \right]^{1/2} \]

\[ \delta_k = \frac{1}{24} \left( \frac{Jk(k+1)}{m \zeta(k)} \right) \frac{\zeta(2k-4)}{a^{(k-4)/2}} \]

\[ \beta_k = \sqrt{\left( \frac{Jk(k+1)}{m \zeta(k)} \right) \cos \left( \frac{\pi}{2} (k+1) \right) a^{(k-1)/2-1} \Gamma(1-k) \]}

\[ \text{where } \Gamma(z) \text{ is the gamma function. If } k \in \mathbb{Z} \text{ one can resort to the conventional definition of Polylogarithm function to get the above small-} q \text{ expansion (Appendix B). Note also that in the regime } 3 < k < 5, \text{ we wrote down the next-to-next leading order term (Eq. 4) since this is what remarkably results in power-law asymptotic behaviour of the OTOC to be discussed later and hence, it is a relevant term.}

## IV. OUT-OF-TIME-ORDER COMMUTATOR (OTOC)

The key diagnostic for us is the classical version of the well known quantum OTOC. In the Heisenberg picture, the quantum OTOC can be defined \[44, 60\] as \( D(x, t) = \langle [\hat{A}_x(t), \hat{B}_0(0)]^2 \rangle \) where \( \hat{A}_x(t) \) and \( \hat{B}_0(0) \) are local operators at position \( x \) and the origin, respectively. The average \( \langle \ldots \rangle \), is over a given quantum state. This quantity captures the effect of an operator \( \hat{B}_0(0) \) on another operator \( \hat{A}_x(t) \) at a different position and time. We now replace the commutator by a Poisson bracket, \( \{\hat{A}_x(t), \hat{B}_0(0)\} \). For our purposes, if one makes the identification, \( \hat{A}_x(t) \equiv x_i(t) \) and \( \hat{B}_0(0) \equiv p_i(0) \), then the Poisson bracket is \( \{x_i(t), p_i(0)\} \approx \delta_{x_i}(t) \delta_{p_i}(0) \). Therefore, the classical OTOC in our variables becomes \( D(i, t) = \langle \{x_i(t), p_i(0)\}^2 \rangle \approx \langle \delta_{x_i}(t) \delta_{p_i}(0) \rangle^2 \). Here, \( \langle \ldots \rangle \) is the average over a thermal ensemble of initial conditions at a given temperature \( T \). However, for low enough temperatures \( (k_B T \ll J/a^k) \) where \( k_B \) is the Boltzmann constant), the initial conditions are very close to the true ground state (global minima) which is characterised by the set \( \{x_i(0) = y, p_i(0) = 0\} \) that minimises the energy in Eq. 1. Therefore, for low enough temperature we do not need to make an ensemble average. \( D(i, t) \) can be interpreted as follows: Take two copies \( (I, II) \) of a system with identical initial conditions. Now, we infinitesimally
perturb the position (by $\epsilon$) of one particle (say the middle one) in only one of the copies. In such a case, the OTOC is,

$$D(i, t) = \left| \frac{x_i^1(t) - x_i^1(t)}{x_i^1(N+1)(0) - x_i^1(N+1)(0)} \right|^2 = \frac{\delta x_i(t)}{\epsilon}^2$$  \hspace{1cm} (6)

where we assume $N$ is an odd integer just for convenience. Since, here we are in a regime of sufficiently low temperature, one can invoke a Hessian description, which yields, $\delta \dot{x}(t) = -M \delta x(t)$ where $\delta x(t)$ is a $N \times 1$ column vector consisting of elements $\delta x_i(t)$ and $M$ is a $N \times N$ Hessian matrix given by $M_{ij} = \left[ \frac{\partial^2 V(y)}{\partial x_i \partial x_j} \right]_{x=y}$ where $y$ is the equilibrium solution that minimizes Eq. 1. In this Hessian limit, Eq. 6 becomes (Appendix A)

$$D(i, t) = \left| \sum_{\alpha=1}^{N} \langle \lambda_{\alpha} | \frac{N+1}{2} \rangle (e_i | \lambda_{\alpha}) \cos (\omega_{\alpha} t) \right|^2$$  \hspace{1cm} (7)

where $|\lambda_{\alpha}\rangle$ is the $\alpha^{th}$ eigenvector of $M$ and $\omega_{\alpha}^2$ is the corresponding eigenvalue. The set $\{e_i\}$ is the standard basis for $\mathbb{R}^N$. We have chosen the initial conditions, $\langle e_i | \delta x(t = 0) \rangle = \delta_0 \omega_{\alpha}$ and $\langle e_i | \delta \dot{x}(t = 0) \rangle = 0$ such that $D(i, 0) = \delta_0 \omega_{\alpha}^2$. In general, neither the equilibrium positions $\{y_i\}$ nor the eigenvectors/eigenvalues are easy to find. Barring the exceptional integrable case [61] of $k = 2$, we resort to direct numerics.

V. DIRECT NUMERICS

In order to compute the OTOC (Eq. 7) numerically, we first need to find the set $\{y_i\}$. This is done via the Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm [62, 63] which is an efficient way for energy minimization of Eq. 1 when $N$ is large (Appendix E). All numerical results presented here are for $N = 4097$. After getting the positions of the minima ($\{y_i\}$) one can compute the Hessian matrix $M$, its eigenvectors ($|\lambda_{\alpha}\rangle$) and eigenvalues ($\omega_{\alpha}^2$) thereby aiding the computation of OTOC (Eq. 7). It is also important to mention that for sufficiently large-$N$ the resulting density profile is sufficiently flat near the centre. Hence, for comparing these direct numerical results with analytics (discussed later), we can ignore the harmonic trap as long as we are studying features relatively far from the edges.

VI. ANALYTICAL APPROACH

The dispersion relation (Eq. 3), along with a plane wave ansatz (Appendix C), gives us [52],

$$D(x, t) = \left| \frac{a}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dq \cos \left( qx - \omega_k(q) t \right) \right|^2$$  \hspace{1cm} (8)

FIG. 2: (a) Comparing OTOC ($k = 1.5$) from direct numerical simulation of Eq. 7 with the analytical expression in Eq. 8 at a time snapshot, $t \approx 151$ (in units of $a/v_B$). Note again that, for visualisation purpose, we have plotted only the positive x-axis and the results are mirror symmetric on the other side. The time $t$ is again chosen such that the front moves about 10% from the center to make sure that we are far enough from the edge. Note the difference in the profiles of the OTOC. While $k = 2$ has a relatively flat envelope (Fig. 1(c)), $k = 1.5$ shows a downward trending envelope and $k = 4.5$ has an upward trending envelope (Appendix C). (b) 2D heatmap showing a ballistic lightcone. (c) Log-Log plot showing the asymptotic power-law behaviour for $k = 1.5$. This plot is the zoomed version of the plot in the top panel near the right front, with $x \in [5.5, 8.5]$. Here, $a \sim 0.02237$ and $v_B \sim 53.9296$. 

where $\omega_k(q)$ is the full dispersion relation given in Eq. 3. It is to be noted that the limits of the integral are chosen to be where the dispersion relation reaches a maximum (zero group velocity). Eq. 8 encodes both the left and the right movers. In other words, Eq. 8 can be split into two pieces of integral comprising of negative $(-\pi/a,0)$ and positive $(0, \pi/a)$ momentum. Next, we will present the results for various values of $k$. Owing to a rich mathematical structure rooted in integrability, we will present the $k=2$ case first.

$k = 2$ (Integrable Calogero-Moser Model): It remarkably turns out that, when $k = 2$, the expansion of Eq. 3 in $q$ terminates to exactly yield, $\omega_2(q) = \sqrt{\frac{J}{m}} \left( \frac{\pi a}{a} - \frac{q^2}{2} \right)$. Using this, the OTOC Eq. 8 gives (Appendix D),

$$D(x,t) = \left| \frac{a}{2\pi} [D_R(x,t) + D_L(x,t)] \right|^2 \tag{9}$$

where, $D_R/D_L$ are the left and right moving perturbations respectively and are given by,

$$D_{R,L} = \sqrt{\frac{\pi}{t \nu}} \cos \left( \frac{\nu t^2}{2u_t^2} \right) \left[ \pm C \left( \frac{v}{u_t \sqrt{\nu}} \right) \mp C \left( \frac{\eta_\pi}{u_t \sqrt{\nu}} \right) \right] + \sqrt{\frac{\pi}{t \nu}} \sin \left( \frac{\nu t^2}{2u_t^2} \right) \left[ \pm S \left( \frac{v}{u_t \sqrt{\nu}} \right) \mp S \left( \frac{\eta_\pi}{u_t \sqrt{\nu}} \right) \right] \tag{10}$$

where $u_t = (\sqrt{J/\sqrt{mt}})^{1/2}$ and $C(y)$ and $S(y)$ are the Fresnel Cosine and Fresnel Sine integrals respectively (Appendix D). Here $v = x/t$ and $\eta_\pm = v \pm \sqrt{J/m(\pi/a)}$. The velocities $\eta_+$ and $\eta_-$ are indeed the velocities of the fronts at right and left respectively. The notation $u_t$ is introduced for convenience and one can infer that $u_t$ dictates the length scale of the oscillations as it appears inside the Fresnel functions as well as in the overall amplitude. In Fig. 1, we show the OTOC for the case $k=2$. Exact agreement between direct numerics (Eq. 7) and analytical expression (Eq. 9) is established in Fig. 1(c). The slope of the heat map (Fig. 1(b)) is precisely the butterfly velocity (which in this ground state case is the zero group velocity). The slope of the heat map (Fig. 1(b)) is precisely the butterfly velocity (which in this ground state case is the zero group velocity). The slope of the heat map (Fig. 1(b)) is precisely the butterfly velocity (which in this ground state case is the zero group velocity). The slope of the heat map (Fig. 1(b)) is precisely the butterfly velocity (which in this ground state case is the zero group velocity). The slope of the heat map (Fig. 1(b)) is precisely the butterfly velocity (which in this ground state case is the zero group velocity).

Eq. 16 express $\Delta_{k,\pm}$ for various values of $k$. Note that when the lattice spacing $a \rightarrow 0$ (which is essentially the large-$N$ limit), then the upper limit of the integral in Eq. 12 becomes $+\infty$. This can be thought of as a $k \neq 3$ generalization of the Airy Integral. In stark contrast to the Airy Integral, in the regime $1 < k < 3$, we find, $B_k(y) \sim 1/y^{k+1}$ for large $y$. This implies that, $D_{L,R}(x,t) \propto 1/\Delta_k^{k+1}$ where $\pm$ indicates whether we are probing the left or the right front respectively. Note that, throughout the paper, for the sake of brevity by $\Delta_k$ we mean $\Delta_{k,\pm}$. In Fig. 2(c), we demonstrate that this powerlaw prediction (for $k = 1.5$ which will give $D \sim 1/\Delta_3$) is consistent with direct numerical results.

3 $< k < 5$: In this case, if we consider the lowest and the next order term in the dispersion relation (Eq. 4) we
in the dispersion relation of the form

\[ \Delta_k = \frac{x - \alpha t}{3t_k^{1/k}} \]

(13)

where \( \text{Ai}(z) \) is the Airy function. Refer to Eq. 16 to see the definition of \( \Delta_{k,\pm} \) in various regimes of \( k \). This would imply that the asymptotic behaviour of the OTOC (Eq. 8) would be characterised by exponential since large argument behaviour of Airy function is \( \text{Ai}(z) \sim e^{-\frac{2}{3}z^{3/2}}/2\sqrt{\pi}z^{1/4} \).

However, this is because we stopped at \( O(q^3) \) in Eq. 4. Knowing that we have a power-law model, we expect that the asymptotic behaviour of OTOC should be characterised by power-laws. It turns out that this is captured by considering higher orders in the dispersion relation (Eq. 8). For \( 3 < k < 5 \), the next order term after \( q^3 \) in Eq. 4 would be \( q^5 \) and this will yield a power law tail \( D_{\text{L},R}(x,t) \propto 1/\Delta_{k,5}^{1/5} \). For e.g., for the case \( k = 4.5 \) (see Fig. 4), we demonstrate (Appendix D) that \( D_{\text{L},R}(x,t) \propto 1/\Delta_{k,5}^{1/5} \) (see Fig. 5).

We briefly comment on the case \( 5 < k < \infty \) and \( k \notin \text{odd integer} \). The term in the dispersion expansion that results in power law is \( \delta_k q^k \) which will again yield \( D_{\text{L},R}(x,t) \propto 1/\Delta_{k,5}^{k+1} \). However, to see this power law, one needs to go to very large asymptotic values since this will happen only after all the exponential behaviours (arising due to \( O(q^{2k+1}) \)) where \( Z \) are odd integers) are suppressed. Next, we will discuss the case when \( k \) is an odd integer. In particular, we will discuss the case of \( k = 3 \) but our method can be adapted for all odd integers.

\( k = 3 \): In this case, we see a logarithmic term in Eq. 4, i.e., \( q^3 \log(q) \). Note that without the logarithmic piece, we would have ended up with Airy function for the OTOC which would have resulted in exponential asymptotics. However, we now get,

\[ D_{\text{R}}(x,t) = \frac{a \text{Ai}(\Delta_{k,-})}{2\pi (3t\gamma_m)^{1/3}} \]

(14)

where,

\[ B_3(y) := \int_0^{(3t\gamma_m)^{1/3} \frac{3}{2}} ds \cos \left( ys + \frac{s^3}{3} \log \left( \frac{s}{(3t\gamma_m)^{1/3}} \right) \right) \]

(15)

We find that the large \( y \) behaviour is \( B_3(y) \sim 1/y^4 \) which implies \( D_{\text{L},R}(x,t) \propto 1/\Delta_{4,5}^{1/5} \). It is remarkable to note that the expected power-law behaviour is recovered as a result of the intricate role played by the Logarithmic term in the dispersion relation for \( k = 3 \). We also find that for odd integers, i.e., \( k \in 2Z + 1 \) where \( Z \) are positive integers, the power-law is recovered by a term in the dispersion relation of the form \( \beta_k q^k \log(q) \).

It is worth recollecting that the details of \( \Delta_{\pm} = \Delta_{k,\pm} \) depends on the regime of \( k \) we are investigating. This is summarised as follows.

\[ \Delta_{\pm} = \begin{cases} \frac{x - \alpha t}{(3t\gamma_m)^{1/k}}, & 1 < k < 3 \\ \frac{x - \alpha t}{(3t\gamma_m)^{1/k}}, & 3 < k < 5 \end{cases} \]

(16)

VII. FIELD THEORY

An alternative approach to studying the large \( N \) behaviour of this system is to investigate the collective field theory. Recently, a systematic derivation of large-\( N \) field theory \([58]\) was achieved. Here, we will show that certain aspects of spatio-temporal spread of correlations such as the butterfly velocity can be obtained by a field theory. Let us define a density field, \( \rho(x) = \sum_{i=1}^{N}(\delta(x-x_i)) \) where \( \langle \cdots \rangle \) denotes an average with respect to a Boltzmann measure. We also define a momentum field, \( j(x) = \sum_{i=1}^{N}(p_i \delta(x-x_i)) \). We will introduce a velocity field \( v(x) \) such that \( j(x) = \rho(x)v(x) \). In large-\( N \) at sufficiently low temperatures, the field theory is given by, \( H[\rho(x)] = \frac{m}{2} \int \rho(x)v(x)^2 dx + \frac{j}{m} \int \rho(x)\zeta(k)dx \). This in conjugation with Poisson Brackets, \( \{\rho(x_1),v(x_2)\} = \frac{1}{m} \delta'(x_1-x_2) \) gives us

\[ \dot{\rho} = -\partial_x(\rho v), \quad \dot{v} = -\partial_x \left( \frac{v^2}{2} + \frac{j}{m} \zeta(k)(k+1)\rho^k + \ldots \right) \]

(17)

One can linearise the above Continuity and Euler equations by using \( \rho(x,t) = \rho_0 + \delta \rho(y,t), v(x,t) = 0 + \delta v(x,t) \) to get a wave equation with sound speed given by \( c = \sqrt{J/mK(k+1)} \). This is precisely the butterfly velocity in agreement with \( \alpha_k \) given in Eq. 5. The background density \( \langle \rho_0 \rangle \) or the inverse lattice spacing \( (a^{-1}) \) has already been discussed before.

VIII. CONCLUSIONS

In summary, we studied a family of power-law models at low temperature. In particular, we probed in detail the spatio-temporal spread of perturbations. We could analytically compute the dispersion relation in absence of external harmonic potential and utilise it to get analytical results for OTOC at low temperatures. We then performed direct numerics with the aid of BFGS algorithm and demonstrated excellent agreement with the results obtained after using dispersion relation. Exact results for OTOC at the integrable point \( (k = 2) \) were obtained. We also presented a collective field theory approach to understand certain features of the OTOC such as the butterfly speed. The main observations are summarised in Table. I.

Our work was restricted to low enough temperatures and short enough times so that the system is still in the linear regime. Therefore, although there was spread of perturbations, we restricted ourselves to temperatures
and time scales where there was no growth in magnitude of perturbations. Precisely quantifying the limits of this regime as well as exploring beyond it is part of our planned future work. It is worth noting that for the temperature range considered in our manuscript, the quantum analog would be the probe of the ground state \cite{64–66} rather than chaos. Being a long-ranged system of particles, high temperature studies are considerably numerically intense and it will be interesting to study classical OTOC and the largest Lyapunov exponent using methods of Ref. \citenum{67}. In such high temperature cases one expects exponential (non-integrable, \(k \neq 2\)) or power-law (integrable, \(k = 2\)) growth and this will be addressed in a future work. Understanding aspects of integrability (\(k = 2\)) and its breaking through the lens of OTOC still remains largely unexplored and is an interesting future direction. The analogous quantum case is a fascinating and challenging problem especially given the fact that quantum long ranged systems exhibit rich OTOC features depending on the exponent \((k)\) of the power law interactions \cite{68}. Needless to mention, the energy, Government of India, under project no. RTI4001.

IX. ACKNOWLEDGEMENTS

We would like to thank A. Dhar, A. Kundu and A. K. Chatterjee for useful discussions. MK would like to acknowledge support from the project 6004-1 of the Indo-French Centre for the Promotion of Advanced Research (IFCPAR), Ramanujan Fellowship (SB/S2/RJN-114/2016), SERB Early Career Research Award (ECR/2018/002085) and SERB Matrics Grant (MTR/2019/001101) from the Science and Engineering Research Board (SERB), Department of Science and Technology, Government of India. DH is supported in part by (USA) DOE grant DE-SC0016244. BKS and MK acknowledge support of the Department of Atomic Energy, Government of India, under project no. RTI4001.

Appendix A: OTOC in Hessian approximation valid in the low \(T\) regime

Here, we present a derivation of the OTOC in the Hessian approximation. Let us recap that the Hamiltonian is given by

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + V_k(\{x_j\}) \tag{A1a}
\]

\[
V_k(\{x_j\}) = \sum_{i=1}^{N} \left[ \frac{m_0^2}{2} x_i^2 + \frac{J}{2} \sum_{j \neq i} \frac{1}{|x_i - x_j|^k} \right] \tag{A1b}
\]

which therefore yields the \(N \times N\) Hessian matrix,

\[
M_{ij} = \left[ \frac{\partial^2 V_k}{\partial x_i \partial x_j} \right]_{x=y} \tag{A2}
\]

where \(y\) is the equilibrium solution that minimizes Eq. A1. Let us define a column vector \((N \times 1)\) representing the relative displacement (again a \(N \times 1\) column vector) of corresponding particles of copy \(I\) and copy \(II\)

\[
|\delta x(t)\rangle = |x'(t)\rangle - |x''(t)\rangle \tag{A3}
\]

Let \(M\) be the Hessian matrix. We therefore have

\[
|\delta \hat{x}(t)\rangle = -M |\delta x(t)\rangle \tag{A4}
\]

We assume initial condition for velocities \(\langle e_i | \delta \hat{x}(t=0) \rangle = 0\) where \(\{e_i\}\) is the standard basis for \(\mathbb{R}^N\). Let \(|\lambda_\alpha\rangle\) be the eigenvectors and \(\omega_\alpha^2\) be the corresponding eigenvalues of \(M\). In the eigenbasis, the solution is

\[
|\delta x(t)\rangle = \sum_{\alpha=1}^{N} \langle \lambda_\alpha | \delta x(0)\rangle |\lambda_\alpha\rangle \cos(\omega_\alpha t) \tag{A5}
\]

With our chosen initial condition \(\langle e_i | \delta x(0)\rangle = \epsilon \delta_i, \frac{\alpha+1}{2}\) the OTOC finally becomes

\[
D(i,t) = \left| \frac{\langle e_i | \delta \hat{x}(t) \rangle}{\epsilon} \right|^2 = \sum_{\alpha=1}^{N} \langle \lambda_\alpha | \frac{N+1}{2} \rangle \langle e_i | \lambda_\alpha \rangle \cos(\omega_\alpha t) \tag{A6}
\]

Appendix B: Derivation of the dispersion relation

Here, we present a detailed derivation of the dispersion relation. Let us consider the Riesz gas (Eq. A1) without the Harmonic trap. We will also consider the case of \(N \to \infty\) and label the particles from \(-\infty\) to \(+\infty\) without loss of generality. The corresponding equation of motion are

\[
\hat{x}_i = p_i/m, \quad \hat{p}_i = \frac{Jk}{2} \sum_{j=-\infty}^{\infty} \frac{\text{sgn}(x_i - x_j)}{|x_i - x_j|^k} \tag{B1}
\]

The equilibrium solution takes the form

\[
x_i(t = 0) = ia, \quad p_i(t = 0) = 0 \tag{B2}
\]
where $a$ is a chosen lattice spacing. The above solution (Eq. B2) does not evolve and therefore it is an equilibrium solution for Eq. B1. This is so because Eq. B1 is an odd sum. In other words

$$\sum_{j=-\infty}^{\infty} \frac{\text{sgn}(x_i - x_j)}{|x_i - x_j|^{k+1}} = \frac{1}{a^{k+1}} \sum_{j=-\infty}^{\infty} \frac{\text{sgn}(i - j)}{|i - j|^{k+1}} = 0 \quad (\text{B3})$$

Having found an equilibrium background, we do a small oscillation analysis of Eq. B1. Using the ansatz,

$$x_i(t) = ai + \epsilon \cos(qai - \omega t) \quad (\text{B4})$$

we get from Eq. B1

$$- m\omega^2 \epsilon \cos(qai - \omega t) = \frac{Jk}{2} \sum_{j \neq i} \frac{\text{sgn}(x_i - x_j)}{|a(i - j)|} \left[ 1 + \frac{\epsilon}{\sqrt{|a(i - j)|}} \frac{\cos(qai - \omega t) - \cos(qaj - \omega t)}{\cos(qai - \omega t) - \cos(qaj - \omega t)} \right]^{k+1}$$

(B5)

Note that, from Eq. B5 onwards, for convenience of notation we suppress the subscript $k$ in $\omega_k$ and it is restored in the end. Also, $\epsilon$ introduced for the small oscillation analysis in Eq. B4 should not be confused with the $\epsilon$ used in the definition of OTOC (Eq. 6).

Splitting the summation above (Eq. B5) gives

$$- m\omega^2 \epsilon \cos(qai - \omega t) = \sum_{j < i} \frac{Jk}{2} \frac{1}{|a(i - j)|^{k+1}} \left[ 1 + \frac{\epsilon}{\sqrt{|a(i - j)|}} \frac{\cos(qai - \omega t) - \cos(qaj - \omega t)}{\cos(qai - \omega t) - \cos(qaj - \omega t)} \right]^{k+1}$$

$$- \sum_{j > i} \frac{Jk}{2} \frac{1}{|a(i - j)|^{k+1}} \left[ 1 + \frac{\epsilon}{\sqrt{|a(i - j)|}} \frac{\cos(qai - \omega t) - \cos(qaj - \omega t)}{\cos(qai - \omega t) - \cos(qaj - \omega t)} \right]^{k+1}$$

(B6)

Since $\frac{\epsilon}{a} << 1$ (small oscillation theory), we use the binomial expansion. Remember that our choice of labelling, $x_i < x_j \iff i < j$, means that the absolute value can be removed once the sum is broken into parts as done in Eq. B6. We keep terms only up to leading order in $\epsilon$. Simple algebra and trigonometric identities simplify the expression to give

$$m\omega^2 \cos(qai - \omega t) = \sum_{j=-\infty}^{i-1} \frac{-2Jk(k + 1)}{|a(i - j)|^{k+2}} \left[ \sin(qa \frac{i + j}{2} - \omega t) \sin(qa \frac{i - j}{2}) \right]$$

$$+ \sum_{j=i+1}^{\infty} \frac{2Jk(k + 1)}{|a(j - i)|^{k+2}} \left[ \sin(qa \frac{j + i}{2} - \omega t) \sin(qa \frac{j - i}{2}) \right]$$

(B7)

Let us define integers $i - j \equiv x_1$ and $j - i \equiv x_2$ in the first and the second sum in Eq. B7 respectively. Further simplification yields...
\( m\omega^2 \cos(q\omega - \omega t) = \sum_{x_1=1}^{\infty} \frac{-2Jk(k+1)}{(a x_1)^{k+2}} \left[ \sin(q\omega - \omega t) \cos \left( \frac{qax_1}{2} \right) \sin \left( \frac{qa}{2} x_1 \right) - \cos(q\omega - \omega t) \sin^2 \left( \frac{qa}{2} x_1 \right) \right] \)
\[ \text{+ } \sum_{x_2=1}^{\infty} \frac{2Jk(k+1)}{(a x_2)^{k+2}} \left[ \sin(q\omega - \omega t) \cos \left( \frac{qa}{2} x_2 \right) \sin \left( \frac{qa}{2} x_2 \right) + \cos(q\omega - \omega t) \sin^2 \left( \frac{qa}{2} x_2 \right) \right] \]  

(B8)

In above Eq. B8, \( x_1 \) and \( x_2 \) are dummy variables and it is easy to see that the first terms in each of the sums cancels each other. The expression (Eq. B8) easily simplifies further to (restoring the subscript notation from Eq. B4)

\[ \omega_k^2(q) = \frac{4Jk(k+1)}{ma^{k+2}} \sum_{x_1=1}^{\infty} \frac{\sin^2 \left( \frac{qa}{2} x_1 \right)}{x_1^{k+2}} \]  

(B9)

where \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) is the Riemann zeta function and \( \text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \) is the Polylogarithm function. Note that Eq. B10 is the exact dispersion relation. Fig. 1 (Left) shows a plot of \( \omega_k(q) \) for \( k = 2 \). Further, using the following remarkable identity of Polylogarithm function [59]

\[ L_n(z) = \Gamma(1-n) \log \left( \frac{1}{z} \right)^{n-1} + \sum_{l=0}^{\infty} \zeta(n-l) \frac{\log(z)^l}{l!} \]  

(B11)

for \( n \in \mathbb{Z} \) and \( \left| \log z \right| < 2\pi \), one can expand Eq. B10 (to the next-to-next leading order) as a power series in \( q \) to obtain

\[ \omega_k(q) \approx \begin{cases} \alpha_k q - \beta_k q^3 - \delta_k q^5, & 1 < k < 3 \\ \alpha_k q + \gamma_3 q^3 \log(qa) - \tilde{\gamma}_3 q^3, & k = 3 \\ \alpha_k q - \delta_k q^3 - \beta_k q^5, & 3 < k < 5 \end{cases} \]  

(B12)

with

\[ \alpha_k = \sqrt{\frac{Jk(k+1)}{ma^k} \zeta(k)}, \quad \gamma_3 = \frac{1}{4} \sqrt{ \frac{Ja}{3m\zeta(3)} } \]
\[ \delta_k = \frac{1}{24} \sqrt{ \frac{Jk(k+1)}{m\zeta(k)} \frac{\zeta(k-2)}{a^{(k-4)/2}} } \]
\[ \beta_k = \sqrt{ \frac{Jk(k+1)}{m\zeta(k)} } \cos \left( \frac{\pi}{2} [k+1] \right) a^{k/2-1} \Gamma(-1-k) \]
\[ \tilde{\gamma}_3 = \frac{25\gamma_3}{12} \]  

(B13)

Using the relation \( \sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \), we see that Eq. B9 can be simplified to

\[ \omega_k(q) = \sqrt{\frac{Jk(k+1)}{ma^{k+2}} \left[ 2\zeta(k+2) - L_{k+2}(e^{-iqa}) - L_{k+2}(e^{iqa}) \right]} \]  

(B10)

It is to be noted that, if \( k \in \mathbb{Z} \) one can resort to the conventional definition of Polylogarithm function to get the above small-\( q \) expansion. In Eq. B13, we see that both \( \beta_k \) and \( \delta_k \) terms diverge in the limit \( k \to 3 \). But upon a careful computation of this limit, it is seen that these divergences in fact cancel each other and give rise to the logarithm term in the dispersion relation for \( k = 3 \). More precisely,

\[ \lim_{k \to 3} (\beta_k q^k + \delta_k q^3) = \lim_{\epsilon \to 0} \left( \beta_{3\pm\eta} q^{3\pm\eta} + \delta_{3\pm\eta} q^3 \right) \]
\[ = \sqrt{ \frac{Ja3}{3m\zeta(3)} } q^3 \lim_{\epsilon \to 0} \left( q^\epsilon a^{\epsilon/2} \Gamma(-4 + \epsilon) \right) + \frac{1}{24} a^{-\epsilon/2} \zeta(1 \pm \epsilon) \]
\[ = 6 \sqrt{ \frac{Ja}{3m\zeta(3)} } \left( \frac{25 - 12 \log(qa)}{288} \right) q^3 \]
\[ = -\gamma_3 q^3 \log(qa) + \tilde{\gamma}_3 q^3 \]  

(B14)

Moreover, this kind of cancellation is seen for all odd integer values of \( k \), leading to a logarithm term \( \sim q^k \log(q) \). This is a crucial term in the dispersion relation because this is precisely the term which leads to a power-law decay of the OTOC beyond the cone, for odd integer values of \( k \).

**Appendix C: OTOC in planewave basis for large-\( N \)**

Here, we briefly derive the large-\( N \) expression of the OTOC in the plane wave basis. Let, \( \{\delta x(t)\} \) be a vector
FIG. 3: (a) Dispersion relation over one time period in $q$, for $k = 2$, $q \in [0, 2\pi/a]$. (b) Comparing OTOC from direct numerical simulation with the analytical expression at a time snapshot, $t \approx 164$ (in units of $a/v_B$). Note that, for visualisation purpose, we have plotted only the positive x-axis and the results are mirror symmetric on the other side. (c) Power law decay of OTOC beyond the cone for $k = 2$ and $N = 4097$. The slope depicts the exponent of the power law. Note that the asymptotics agrees with Eq. D5 including coefficients. Here, $a \sim 0.0347$, $v_B \sim 90.5207$.

containing the displacements $\delta x_j(t)$. We can expand this vector in terms of the plane-wave phonons of our system as

$$|\delta x(t)\rangle = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dq \langle q|\delta x(t)\rangle |q\rangle \quad (C1)$$

which implies,

$$\langle e_j|\delta x(t)\rangle = \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dq \langle q|\delta x(t)\rangle \langle e_j|q\rangle \quad (C2)$$

where $\langle e_j|\delta x(t)\rangle = \delta x_j(t)$. Here we remind that $\{|e_i\rangle\}$ is the standard basis for $\mathbb{R}^N$. At $t = 0$, since $\langle e_j|\delta x(t = 0)\rangle = e \delta_{j,0}$ and $\langle e_j|q\rangle = \text{Real} [e^{(qja-\omega_k(q)t)}]$ at time $t$, we have $\langle q|\delta x(t = 0)\rangle = \epsilon$ (constant) which ensures that Eq. C1 is satisfied at $t = 0$. The noninteracting phonons simply evolve freely, so, using $ja = x$, we can write

$$\langle e_j|\delta x(t)\rangle = \frac{\epsilon a}{2\pi} \text{Real} \left[ \int_{-\pi/a}^{\pi/a} dq e^{i(qx - \omega_k(q)t)} \right] \quad (C3)$$

We therefore finally get,

$$D(x,t) = \left| \frac{\langle e_j|\delta x(t)\rangle^2}{\epsilon} \right| = \left| \frac{a}{2\pi} \int_{-\pi/a}^{\pi/a} dq \cos \left( qx - \omega_k(q)t \right) \right|^2 \quad (C4)$$

where $\omega_k(q)$ is given in Eq. B10. Figs. 1 (c) and 4 (c) show a plot of Eq. C4 for cases $k = 2$ and $k = 4.5$ respectively. The figures only show the right sector ($x > 0$), the left sector ($x < 0$) being just the mirror image.

Appendix D: Asymptotic analysis of OTOC

In this section we provide the asymptotic analysis of the OTOC at either the left or the right front. We divide this section into $k = 2$ (integrable case) and $k \neq 2$ (non-integrable case).

1. Integrable case, $k = 2$

In this case, the dispersion (Eq.B10) remarkably has a finite expansion,

$$\omega_2(q) = \sqrt{\frac{J}{m}} \left( \frac{\pi q}{a} - \frac{q^2}{2} \right), \quad \text{for} \quad k = 2 \quad (D1)$$

This allows us to cast the OTOC in terms of the Fresnel functions without resorting to any approximations. We would like to understand the behaviour of the OTOC in the large $\eta_k$ limit where $\eta_k = v \pm \sqrt{\frac{J}{m}} \frac{\pi}{2}$. For e.g., large $\eta_-$ means that we are probing the asymptotics of the right front. For this, we look at the asymptotics of the...
The following equations.

\[
D_{R,L} = \sqrt{\pi} \frac{u_t}{u_1 t} \cos \left( \frac{\eta^2}{2u_1^2} \right) \left[ \pm C \left( \frac{v}{\sqrt{\pi} u_1} \right) \right.
+ \sqrt{\pi} \frac{u_1}{u_t} \sin \left( \frac{\eta^2}{2u_1^2} \right) \left[ \right. \pm S \left( \frac{v}{\sqrt{\pi} u_1} \right) \right.
\]

where \( C(r) = \int_0^r dt \cos(\pi t^2/2) \) and \( S(r) = \int_0^r dt \sin(\pi t^2/2) \) are the Fresnel Cosine and Sine integrals, respectively. Recall that \( u_t = \sqrt{J/\sqrt{m}} \).

For our purpose we use the large argument asymptotic form of Fresnel integrals. In the large \( r \) limit, we have

\[
S(r) = \frac{1}{2} \cos \left( \frac{\pi r^2}{2} + O(r^{-4}) \right) \left[ -\frac{1}{\pi r^3} + O(r^{-4}) \right]
+ \sin \left( \frac{\pi r^2}{2} + O(r^{-4}) \right) \left[ -\frac{1}{\pi r^3} + O(r^{-4}) \right]
\]

\[
C(r) = \frac{1}{2} + \sin \left( \frac{\pi r^2}{2} + O(r^{-4}) \right) \left[ \frac{1}{\pi r^3} + O(r^{-4}) \right]
\]

We re-emphasize here that to get the correct asymptotic behaviour in the right sector, we go to the large \( \eta_- \) limit. Note that \( v \gg \eta_- \). This is because \( v = v_B + \eta_- \) and \( v_B \) is very large in the limit that lattice spacing is very small. Therefore, we can take the limit \( v \to \infty \) while assuming \( \eta_- \) to be large enough to be able to do asymptotics. We use Eqs. D3 to get the large \( \eta_- \) expansion for \( D_R \) and set \( v \sqrt{\eta_-} \to \infty \) in the argument of the Fresnel functions. This finally yields

\[
D_R \approx \frac{\sqrt{\pi}}{u_T} \cos \left( \frac{\eta^2}{2u_T^2} \right) \sqrt{\frac{\pi}{8}} \sqrt{\frac{\pi}{8}} + \frac{\sqrt{\pi}}{u_T} \sin \left( \frac{\eta^2}{2u_T^2} \right) \left[ \frac{\pi}{8} + \frac{\pi}{8} \right]
- \frac{\sqrt{\pi}}{u_T} \cos \left( \frac{\eta^2}{2u_T^2} \right) \left[ \frac{\pi}{8} \right] - \frac{\sqrt{\pi}}{u_T} \sin \left( \frac{\eta^2}{2u_T^2} \right) \left[ \frac{\pi}{8} \right]
\]

\[
D(x, t) \approx \frac{a}{2 \beta} \left( \frac{u_T^2}{\eta_-^2} \right)^k, \text{ for } \eta_- > 0 \text{ and large}
\]

We find that beyond the front, there is a power-law decay \( (\propto \eta_-^0) \). We can see that this result (Eq. D5) agrees with the numerical simulations [Fig. 3 (c)] not only in terms of the exponent of the power law but also in terms of the coefficients given in Eq. D5.

2. Non-integrable case, \( k \neq 2 \)

a. \( 1 < k < 3 \)

Now turning our attention to the next case, we try to analyse the asymptotic behaviour of

\[
D_{R,L}(x, t) = \frac{a}{2 \beta} B_k \left( \Delta \right), \text{ with } \Delta_{k,\pm} = \frac{x + \alpha t}{(3t \beta k)^{1/k}}
\]

where we define a special function,

\[
B_k(y) := \int_0^{(3t \beta k)^{1/k} \pi} ds \cos \left( y s + \frac{s^k}{3} \right)
\]

The reader is referred to Eq. 16 to find the definition of \( \Delta_{k,\pm} \) for different values of \( k \). Again, we are interested in this analysis for large \( N \) (equivalently, \( a \to 0^+ \)) and large \( \Delta_{k,\pm} \) limits of the system. The strategy would be to again look at the asymptotic form of Eq. D7. But in the absence of an analytical expression we resort to numerical methods. We find a power-law decay of the form

\[
D(x, t) \propto |\Delta_{\pm}^{-(k+1)}|^2
\]

where \(-/+\) corresponds to the right/left sectors, respectively. Therefore, the OTOC outside the light cone decays as a power law with exponent \( 2k + 2 \) (see Fig. 2(c) for \( k = 1.5 \)).

b. \( 3 < k < 5 \)

As we have discussed, just considering the first two terms in the expansion Eq. B12 does not capture the power-law decay of the OTOC. This is because we effectively end up with Airy function which has an exponentially decaying tail. So in order to observe the power-law nature of the OTOC, one has to consider the next order term in Eq. B12 as well, in addition to the first two terms.

The expectation is that once the Airy-like behaviour of the dominant term washes away, the power-law nature of the sub-dominant term will take over. This amounts to using Eq. B12 in Eq. C4. We find that including the next order term \( \sim q^{k} \) recovers the power-law from an Airy-like behaviour. For e.g., this claim about a power-law behaviour is verified, for \( k = 4.5 \), by the direct numerical simulations and asymptotic analysis of the OTOC using the full dispersion relation (Eq. B10), as is seen in Fig. 5. We find a decay \( \propto |\Delta_{\pm}|^{-11} \) in this case.

Notice the drastic difference in the nature of the envelope of the OTOC profile as one goes from \( k < 2 \) (peaking at the centre and then having a downward envelope) to \( k = 2 \) (essentially flat) and \( k > 2 \) (peaking
at the edges and having an upward envelope). The integrable model \((k = 2)\) serves as a transition point for a change in the nature of propagation of the perturbations through the system.

c. \(k = 3\)

The case of \(k = 3\), and more generally, odd integer values of \(k\) is a little subtle from an analytical analysis stand point. As elaborated in Eq. B14, the subtle cancellation of divergences arising due to the Riemann zeta function and the Gamma function gives rise to a logarithm term. This is a crucial term as it gives rise to the power-law decay of the OTOC. As is clear, in the absence of this term, we obtain Airy-like behaviour for \(k = 3\), due to the \(q^3\) term in Eq. B12. For \(k = 3\), using Eq. B12 in Eq. C4 produces a power-law decay of the OTOC \(\propto |\Delta|^{-8}\).

These observations strongly suggest that the long-range nature of Riesz gas family of models is what gives rise to the power-law decay of the OTOC (for \(k > 1\)). A particular term in the expansion of \(\omega_k(q)\) at \(O(q^k)\), \(\forall k > 1\) explains the power-law.

**Appendix E: Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm**

We have implemented the BFGS algorithm [62, 63] to find the global minimum energy configuration of our system (Eq. A1) for system sizes upto \(N = 4097\). This is an iterative algorithm which can be used for non-linear, unconstrained optimisation problems. Below we provide the algorithm. Let \(V\{x_i\}\) (for e.g., Eq. A1) be a scalar function on \(\mathbb{R}^N\) which is to be minimised. \(\nabla V\{x_i\}\) is the corresponding gradient. \(\mathbf{M}\) is the Hessian matrix (for e.g., Eq. A2).

1. Make an initial guess for the minimising configuration \(x_0 = \{x_0^i\}\). Using this compute \(\mathbf{M}_0 = \mathbf{M}(x_0)\) and \(\nabla V_0 = \nabla V(x_0)\).

2. Find the minimising direction by solving: \(\mathbf{M}_n\mathbf{p}_n = \nabla V_n\). Here \(\mathbf{p}_n\) is the minimisation direction at the \(n^{th}\) iterative step. Therefore, we solve the inverse problem \(\mathbf{p}_n = \mathbf{M}_n^{-1}\nabla V_n\).

3. Now, we try to find the optimum step size \(\alpha_n\) to take in the minimisation direction. One could use the back-tracking line-search method for this purpose. We start with a sufficiently large initial step size \(\alpha_0\) and iteratively reduce it, i.e., \(\alpha_{j+1} = \eta \alpha_j\) where \(\eta \in (0, 1)\) is a control parameter. This is done until the Armijo-Goldstein condition is satisfied, namely, \(V(x_n) - V(x_n + \alpha_j \mathbf{p}_n) > \alpha_j \gamma\), where \(\gamma = -c\delta\) and \(\delta = \nabla V(x_n)\). Here, similar to \(\eta\), \(c \sim 0.1346\), \(v_B \sim 466.514\).
$c \in (0,1)$ is another control parameter. So, we finally find $\alpha_n$ such that it is the largest of the set $\alpha'_s$ until the Armijo-Goldstein condition is still satisfied.

4. Updating procedure:

i) $x_{n+1} = x_n + \alpha_n p_n$.

ii) $s_n = \alpha_n p_n$.

iii) $y_n = \nabla V_{n+1} - \nabla V_n$.

iv) $M_{n+1} = M_n + \frac{y_n y_n^T}{y_n s_n} - \frac{M_n s_n s_n^T M_n}{s_n^T M_n s_n}$

5. Repeat steps 2 – 4.

FIG. 5: Power law decay of OTOC beyond the cone for $k = 4.5$ and $N = 4097$. The slope depicting the exponent of the power law. Here, $a \sim 0.1346$, $v_B \sim 466.514$

Appendix F: A comparison with the 2-point correlator

The two point correlator is given by the expression $\langle x_i(t)x_0(0) \rangle$, where $x_i(t)$ is the position of the $i^{th}$ particle at time $t$. On the other hand the classical OTOC is given by $\langle |\frac{x_i(t) - x_0(t)}{\epsilon}|^2 \rangle$. Here, $\langle ... \rangle$ is the average over a thermal ensemble of initial conditions at a given temperature $T$. As mentioned before, for low enough temperatures, we do not need to make an ensemble average.

In our setup, $x_i^I(0) = y_i$ (except the middle particle position which is perturbed by $\epsilon$). On the other hand, we take $x_i^{II}(0) = y_i$. Here, $y_i$ is the equilibrium position of the $i^{th}$ particle. This immediately implies that $x_i^{II}(t) = y_i$ at all times. We can now relabel $x_i^I(t)$ as $x_i(t)$ and rewrite the OTOC as $|\frac{x_i(t) - y_i}{\epsilon}|^2$. In this form, it is apparent that the 2-point correlator and the classical OTOC probe similar physics. However, it should be emphasised that this similarity is because of the low temperature effects. In a more general setup at higher temperatures, the classical OTOC is better suited to probe the perturbations in extended classical systems.

[1] G. L. Sewell, Quantum theory of collective phenomena (Courier Corporation, 2014).
[2] G. Schütz, 1 - Exactly Solvable Models for Many-Body Systems Far from Equilibrium, edited by C. Domb and J. Lebowitz, Phase Transitions and Critical Phenomena, Vol. 19 (Academic Press, 2001) pp. 1–251.
[3] L. F. Cugliandolo, Comptes Rendus Physique 14, 685 (2013).
[4] K. Kaneko and I. Tsuda, Complex systems: chaos and beyond: a constructive approach with applications in life sciences (Springer Science & Business Media, 2011).
[5] H. Stanley, Introduction to Phase Transitions and Critical Phenomena (Oxford University Press, Oxford and New York, 1987).
[6] O. Babelon, D. Bernard, and M. Talon, Introduction to classical integrable systems (Cambridge University Press, 2003).
[7] A. Das, Integrable models, Vol. 30 (World scientific, 1989).
[8] M. A. Olshanetsky and A. M. Perelomov, Physics Reports 71, 313 (1981).
[9] M. Olshanetsky and A. Perelomov, Physics Reports 94, 313 (1983).
[10] T. Kinoshita, T. Wenger, and D. S. Weiss, Nature 440,
L. Edward, R. D. H. Rojas, C. S. H. Calva, and I. P. Castillo, Phys. M. C. Gutzwiller, Chaos in classical and quantum mechanics. 1994
S. H. Strogatz, Nonlinear dynamics and chaos. 1994
F. Haake, in Quantum Coherence in Mesoscopic Systems
L. Yan, W. Wan, L. Chen, F. Zhou, S. Gong, X. Tong, and M. Feng, Scientific reports 6, 21547 (2016).
D. H. Dubin, Physical Review E 55, 4017 (1997).
F. D. Cunden, P. Facchi, M. Ligabò, and P. Vivo, Journal of Physics A: Mathematical and Theoretical 51, 35LT01 (2018).
M. Riesz, Acta Sci. Math. Szeged 9, 1 (1938).
S. Agarwal, A. Dhar, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, and G. Schehr, Physical Review Letters 123, 100603 (2019).
A. Kumar, M. Kulkarni, and A. Kundu, Physical Review E 102, 032128 (2020).
M. Kulkarni and A. G. Abanov, Physical Review A 86, 033614 (2012).
J. Joseph, J. E. Thomas, M. Kulkarni, and A. G. Abanov, Physical Review Letters 106, 150401 (2011).
Y. Sekino and L. Susskind, Journal of High Energy Physics 2008, 065 (2008).
S. H. Shenker and D. Stanford, Journal of High Energy Physics 2014, 67 (2014).
E. B. Rozenbaum, S. Ganeshan, and V. Galitski, Physical review letters 118, 086801 (2017).
I. Kukuljan, S. Grozdanov, and T. Prosen, Physical Review B 96, 060301 (2017).
A. Bohrdt, C. B. Mendel, M. Endres, and M. Knap, New Journal of Physics 19, 063001 (2017).
A. Lakshminarayan, Physical Review E 99, 012201 (2019).
V. Khemani, D. A. Huse, and A. Nahum, Physical Review B 98, 144304 (2018).
A. Das, S. Chakrabarty, A. Dhar, A. Kundu, D. A. Huse, R. Moessner, S. S. Ray, and S. Bhattacharjee, Phys. Rev. Lett. 121, 024101 (2018).
D. Kumar, S. Bhattacharjee, and S. Sankar Ray, arXiv:1906.00016 (2019).
T. Blitewski, S. Bhattacharjee, and R. Moessner, Physical review letters 121, 250602 (2018).
T. Blitewski, S. Bhattacharjee, and R. Moessner, arXiv preprint arXiv:2011.04700 (2020).
M. Kumar, A. Kundu, M. Kulkarni, D. A. Huse, and A. Dhar, Physical Review E 102, 022130 (2020).
S. Ruidas and S. Banerjee, arXiv preprint arXiv:2007.12708 (2020).
S. Bera, K. Lokesh, and S. Banerjee, arXiv preprint arXiv:2105.13376 (2021).
A. K. Chatterjee, M. Kulkarni, and A. Kundu, arXiv preprint arXiv:2106.01267 (2021).
A. K. Chatterjee, A. Kundu, and M. Kulkarni, Physical Review E 102, 052103 (2020).
S. Gopalakrishnan, D. A. Huse, V. Khemani, and R. Vasseur, Physical Review B 98, 220303 (2018).
C.-J. Lin and O. I. Motrunich, Physical Review B 97, 144304 (2018).
C. B. Dağ, L.-M. Duan, and K. Sun, Physical Review B 101, 104415 (2020).
S. Xu and B. Swingle, Nature Physics 16, 199 (2020).
M. Gäßterner, J. G. Bohnet, A. Safavi-Naini, M. L. Wall, J. J. Bollinger, and A. M. Rey, Nature Physics 13, 781 (2017).
S. Agarwal, A. Dhar, M. Kulkarni, A. Kundu, S. N. Majumdar, D. Mukamel, and G. Schehr, Phys. Rev. Lett. 123, 100603 (2019).
G. Szegö et al., Bulletin of the American Mathematical Society 60, 405 (1954).
J. Maldacena, S. H. Shenker, and D. Stanford, Journal of High Energy Physics 2016, 1 (2016).
S. Agarwal, M. Kulkarni, and A. Dhar, Journal of Statistical Physics 176, 1463 (2019).
R. Fletcher, The computer journal 13, 317 (1970).
C. G. Broyden, IMA Journal of Applied Mathematics 6, 76 (1970).
M. Heyl, F. Pollmann, and B. Dóra, Physical review letters 121, 016801 (2018).
C. B. Dağ, K. Sun, and L.-M. Duan, Physical review letters 123, 140602 (2019).
R. J. Lewis-Swan, S. R. Muleady, and A. M. Rey, Physical Review Letters 125, 240605 (2020).
G. Benettin, L. Galgani, and J.-M. Strelcyn, Phys. Rev. A 14, 2338 (1976).
L. Colmenero and D. J. Luitz, Physical Review Research 2, 043047 (2020).