Dynamics of flat actions on totally disconnected, locally compact groups

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Abstract

Let $G$ be a totally disconnected, locally compact group and let $H$ be a virtually flat (for example, polycyclic) group of automorphisms of $G$. We study the structure of, and relationships between, various subgroups of $G$ defined by the dynamics of $H$. In particular, we consider the following four subgroups: the intersection of all tidy subgroups for $H$ on $G$ (in the case that $H$ is flat); the intersection of all $H$-invariant open subgroups of $G$; the smallest closed $H$-invariant subgroup $D$ such that $H$ acts distally on $G/D$; and the group generated by the closures of contraction groups of elements of $H$ on $G$.

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1 Introduction

1.1 Background

Since the groundbreaking article [20] of G. Willis in 1994, a suite of tools for studying totally disconnected, locally compact (t.d.l.c.) groups $G$ has been developed using the dynamics of the action of automorphisms of $G$ on the space of compact open subgroups of $G$. The key concepts are the scale, which is a measure of how far an automorphism $\alpha$ fails to normalize a compact open subgroup, and tidy subgroups, which are the compact open subgroups that have the least displacement under $\alpha$. The scale is a numerical invariant that can be thought of as analogous to the spectral radius in operator theory, and moreover it turns out that the tidy subgroups form a class of subgroups on which the action of $\alpha$ is especially well-behaved, with important structural characterizations. This area of research may thus be termed scale theory or tidy theory. The trivial case of tidy theory is when there exist arbitrarily small compact open subgroups that are $\alpha$-invariant; in this case, we say $\alpha$ is anisotropic. More generally, a group of automorphisms is defined to be anisotropic if every element is anisotropic, and $G$ is anisotropic if $\text{Inn}(G)$ is anisotropic.

Tidy theory has since been generalized from actions of cyclic groups to endomorphisms ([24]) and also to flat group actions, which are defined to be actions of a group $H$ on the
t.d.l.c. group $G$, such that there exists a compact open subgroup $U$ that is tidy for every element of $H$. The theory of flat groups was introduced in [22], although the term ‘flat’ itself appeared slightly later (see [2], which also gives a more geometric presentation of the results in [22]). The class of flat groups is surprisingly large: for instance all finitely generated nilpotent groups of automorphisms are flat, and all polycyclic groups of automorphisms are virtually flat. Nevertheless, flat groups possess a special structure: given a flat group $H$, the set of uniscalar elements $H_u$ (that is, the normalizer of any compact open subgroup that is tidy for $H$) forms a normal subgroup of $H$, and the quotient $H/H_u$ is a torsion-free abelian group. If $H/H_u$ is finitely generated, then the tidy subgroups for $H$ admit something akin to an eigenspace decomposition.

Tidy theory has also been deepened, especially in the case of actions of $\mathbb{Z}$, by the investigation of the role played by certain subgroups in controlling the dynamics. In particular, the contraction group $\text{con}(\alpha)$ of an automorphism $\alpha$, that is, the set of elements $x \in G$ such that $\alpha^n(x)$ converges to the identity, plays a critical role in tidy theory. One can show that $\alpha$ is anisotropic if and only if both $\alpha$ and $\alpha^{-1}$ have trivial contraction group.

An important fact for the theory of t.d.l.c. groups (which does not hold for connected locally compact groups) is the result of Baumgartner–Willis and Jaworski ([1],[10]) that the contraction group also controls contraction relative to a closed subgroup: specifically, if $K$ is an $\alpha$-invariant closed subgroup of $G$, then the set of elements $x \in G$ such that $\alpha^n(x)K$ converges to $K$ in the coset space $G/K$ is precisely $\text{con}(\alpha)K$. This suggests the idea of decomposing the action of $\alpha$ into an ‘anisotropic’ action on the coset space $G/K$, where $K$ is the smallest closed subgroup containing $\text{con}(\alpha)$ and $\text{con}(\alpha^{-1})$, and a residual action on the subgroup $K$ itself. (As we shall see, this idea can be usefully generalized to flat group actions.)

1.2 The relative Tits core

Contraction groups were used in [6] to define the Tits core $G^\dagger$ of $G$:

$$G^\dagger := \langle \text{con}(\alpha) \mid \alpha \in \text{Inn}(G) \rangle.$$ 

In this paper, we consider the notion of the relative Tits core of the set $A$ of automorphisms of $G$ (or a subset of $G$):

$$G_A^\dagger := \langle \text{con}(\alpha) \mid \alpha \in A \cup A^{-1} \rangle.$$ 

We focus on the case that $G$ is totally disconnected and locally compact. Of particular interest is the case when $A$ is a singleton (in which case we define $G^\dagger_\alpha = G^\dagger_{\langle \alpha \rangle}$, and it will transpire that $G^\dagger_\alpha = G^\dagger_{\langle \alpha \rangle}$, or when $A$ is a flat group of automorphisms. In fact, the invariance properties of the relative Tits core will allow us to work in many cases with subgroups $A$ of $G$ that are almost flat, that is, such that some closed cocompact subgroup of $\overline{A}$ is flat on $G$. (In particular, virtually flat groups of automorphisms can be interpreted as almost flat in this sense.)
The relative Tits core is defined in terms of the contraction groups of individual elements of \( A \). However, in the case that \( A \) is a flat subgroup such that \( A/A_u \) is finitely generated (or \( A \) contains a cocompact subgroup of this form), we shall see that \( G^\dagger_A \) plays an important role in the action of \( A \) as a whole.

Using the results of [1] and [6], we will obtain some invariance properties of the relative Tits core. Like the scale function, the relative Tits core \( G_x \) of \( x \in G \) remains constant under sufficiently small perturbations of \( x \).

**Theorem 1.1** (See Theorem 3.7 and Lemma 3.8). Let \( G \) be a t.d.l.c. group.

(i) Let \( x \in G \) and let \( U \) be a compact open subgroup of \( G \) that is tidy for \( x \). Let \( u, v \in U \) and let \( n \in \mathbb{Z} \setminus \{0\} \). Then

\[
G_x^\dagger = G_{ux^n v}^\dagger.
\]

Consequently, \( G_x^\dagger = G_X^\dagger \), where \( X = \bigcup_{n \in \mathbb{Z}} Ux^n U \).

(ii) Let \( X \) be a subset of \( G \) and let \( Y \) be the set of all elements \( y \in G \) such that \( \text{con}(y) \leq G_X^\dagger \). Then \( Y \) is a clopen subset of \( G \). In particular, \( G_X^\dagger = G_X^\dagger \).

(iii) Let \( H \) be a subgroup of \( G \), and let \( K \) be a cocompact subgroup of \( \overline{H} \) such that \( K \) is flat. Then \( G_H^\dagger = G_H^\dagger = G_K^\dagger \).

**Corollary 1.2.** Let \( G \) be a t.d.l.c. group and let \( H \) be an almost flat subgroup of \( G \). Then \( N_G(G_H^\dagger) \) is open in \( G \).

In particular, \( G_g^\dagger \) has open normalizer for all \( g \in G \). This contrasts with the normalizers of \( \text{con}(g) \) and \( \text{nub}(g) \): see §3.3.

In [6], it was shown that if \( D \) is a dense subgroup of the t.d.l.c. group \( G \) that is normalized by \( G^\dagger \), then \( G^\dagger \leq D \). Here is a relative version of this result.

**Theorem 1.3** (See §3.3). Let \( G \) be a t.d.l.c. group, let \( D \) be a subgroup of \( G \) (not necessarily closed), and let \( X \subseteq \overline{D} \). Suppose that there is an open subgroup \( U \) of \( G \) such that

\[
U \cap G_X^\dagger \leq N_G(D).
\]

Then \( G_X^\dagger \leq D \).

### 1.3 The nub of a flat group

Let \( H \) be a flat group of automorphisms of the t.d.l.c. group \( G \). The nub \( \text{nub}(H) \) of \( H \) is the intersection of all tidy subgroups for \( H \). This generalizes the notion of the nub of an automorphism introduced in [23]; in particular, \( \text{nub}(\alpha) = \text{nub}(\langle \alpha \rangle) \).
For a general flat group, the nub is less well-behaved than in the cyclic case. For instance, it is possible that \( \text{nub}(H) \) has proper \( H \)-invariant open subgroups (see Example 4.1). However, we are able to obtain some structural results.

The nubs of the subgroups of \( H \) are all normal in \( \text{nub}(H) \), and nubs of uniscalar flat groups have open normalizer (see Corollary 4.6).

For flat groups \( H \) of finite rank (that is, \( H/H_u \) is finitely generated), the nub can be analysed as the product of \( \text{nub}(H_u) \) with the nubs of individual elements of \( H \).

**Theorem 1.4** (See §4.4). Let \( G \) be a t.d.l.c. group and let \( H \) be a flat group of automorphisms of \( G \). Let \( L \) be a uniscalar normal subgroup of \( H \) such that \( H/L \) is polycyclic. Then there is a finite subset \( X \) of \( H \) such that the following holds:

Let \( U \) be a compact open subgroup of \( G \) such that \( \text{nub}(L) \leq U \) and \( \text{nub}(\alpha) \leq H \) for all \( \alpha \in X \). Then there is a finite subset \( Y \) of \( H \) such that \( V = \bigcap_{\alpha \in Y} \alpha(H) \) is tidy for \( H \).

In particular, writing \( X = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \), we have

\[ \text{nub}(H) = \text{nub}(L)\text{nub}(\alpha_1)\text{nub}(\alpha_2)\ldots\text{nub}(\alpha_n). \]

### 1.4 Residuals

Let \( G \) be a topological group. The **discrete residual** \( \text{Res}(G) \) of \( G \) is the intersection of all open normal subgroups of \( G \). More generally, given a group \( H \) of automorphisms of \( G \), one can define \( \text{Res}_G(H) \), the **discrete residual of** \( H \) **on** \( G \), to be the intersection of all open \( H \)-invariant subgroups of \( G \).

One can iterate the process of taking the discrete residual of an action, to produce a (possibly transfinite) descending chain of closed subgroups of \( G \) such that \( H \) has residually discrete action on each factor, terminating in a group \( \text{Res}_\infty^G(H) \), which is the largest \( H \)-invariant subgroup of \( G \) that has no proper open \( H \)-invariant subgroup. It is straightforward to show (see Lemma 5.2) that the action of \( H \) on the coset space \( G/\text{Res}_\infty^G(H) \) is **distal**, in other words, no non-trivial \( H \)-orbit accumulates at the trivial coset. One can also define the **distal residual** \( \text{Dist}_G(H) \), which is the smallest closed \( H \)-invariant subgroup of \( G \) such that \( H \) acts distally on \( G/\text{Dist}_G(H) \).

Let \( K = \text{Dist}_G(H) \). Then \( K \) is \( H \)-invariant and \( \text{Dist}_K(H) = K \), in other words, \( H \) does not act distally on any coset space of \( K \). Consider the semidirect product \( K \rtimes H \), and observe that \( H \) exerts considerable control over certain actions of \( K \rtimes H \). Specifically, a version of the Mautner phenomenon holds, as follows:

**Theorem 1.5** (See §5.2). Let \( G \) be a topological group and let \( H \) be a subgroup of \( G \) such that \( G = \text{Dist}_G(H)H \).

Let \( X \) be a topological space admitting an action of \( G \) by homeomorphisms, such that the
map $G \to X; g \mapsto gx$ is continuous for all $x \in X$. Let $x \in X$; suppose that $x$ is fixed by $H$, and that no $H$-orbit on $X \setminus \{x\}$ accumulates at $x$. Then $x$ is fixed by $G$.

Evidently $\text{Dist}_G(H)$ contains the contraction group of every element of $H$. In general, one thus has the following sequence of inclusions:

$$G^1_H \subseteq \text{Dist}_G(H) \subseteq \text{Res}_G^\infty(H) \subseteq \text{Res}_G(H).$$

(1)

If $G$ is a t.d.l.c. group and $H$ is a compactly generated flat subgroup of $G$ (or more generally, $H$ has a cocompact subgroup of this kind), we can say more about the relationships between these subgroups using tidy theory. In particular, if $H$ has a polycyclic subgroup with cocompact closure in $H$, we see that all the groups in (1) are actually equal.

**Theorem 1.6** (See §5.4). Let $G$ be a t.d.l.c. group, let $H \leq G$, and suppose there is a cocompact subgroup $K$ of $\overline{H}$ such that $K$ is flat on $G$ and $K/K_u$ is finitely generated.

(i) We have the following expressions for $\text{Res}_G(H)$:

$$\text{Res}_G(H) = \text{Res}_G(K) = G^1_K \text{nub}_G(K) = G^1_K \text{nub}_G(K_u).$$

(ii) The normalizer of $\text{Res}_G(H)$ in $G$ is open. Indeed, $\text{Res}_G(H)$ is normalized by every tidy subgroup for the action of $K$ on $G$.

(iii) $H$ is anisotropic and flat on $N_G(G^1_H)/G^1_H$.

(iv) $G^1_H$ is a cocompact normal subgroup of $\text{Res}_G(H)$. Indeed, $\text{Res}_G(H)/G^1_H$ is the nub of the action of $H$ on $N_G(G^1_H)/G^1_H$.

**Corollary 1.7.** Let $G$ be a t.d.l.c. group, let $H \leq G$, and suppose there is a polycyclic subgroup $K$ of $\overline{H}$ such that $K$ is cocompact in $\overline{H}$. Then $\text{Res}_G(H) = G^1_H$.

We also give a sufficient condition for a compactly generated subgroup to act non-distally, without any flatness assumptions (see Theorem §5.21).

### 1.5 Reduced envelopes of flat subgroups

Let $G$ be a t.d.l.c. group and let $X \subseteq G$. An **envelope** of $X$ in $G$ is an open subgroup of $G$ that contains $X$. Say an envelope $E$ of $X$ is **reduced** if, whenever $E_2$ is an envelope of $X$, then $|E : E \cap E_2|$ is finite.

Observe that reduced envelopes, if they exist, are unique up to commensurability. If $H$ is a flat subgroup of $G$, a natural candidate for a reduced envelope for $H$ is the group
\( \langle U, H \rangle \), where \( U \) is tidy for \( H \). We confirm that \( \langle U, H \rangle \) is indeed reduced provided that \( H/H_u \) is finitely generated. In fact, we obtain a reduced envelope for \( H \leq G \) whenever \( \overline{H} \) has a closed cocompact subgroup \( K \) such that \( K \) is flat and \( K/K_u \) is finitely generated.

**Theorem 1.8** (See § 6.1). Let \( G \) be a t.d.l.c. group and let \( K \) be a closed flat subgroup of \( G \) such that \( K/K_u \) is finitely generated. Let \( U \) be a compact open subgroup that is tidy for \( K \) and let \( U_0 = \bigcap_{k \in K} kUk^{-1} \).

(i) The product \( G^\dagger_K U_0 \) is the group generated by all \( K \)-conjugates of \( U \). Hence \( \langle K, U \rangle \) is a reduced envelope for \( K \) in \( G \), and moreover

\[
\langle K, U \rangle = G^\dagger_K U_0 K.
\]

(ii) Let \( H \leq G \) such that \( K \) is cocompact in \( \overline{H} \). Then \( H \) has a reduced envelope in \( G \), and every reduced envelope for \( H \) in \( G \) is also a reduced envelope for \( K \) in \( G \).

(iii) Suppose that \( K \) has finite covolume in \( G \). Then

\[
G^\dagger_K \subseteq KVK
\]

for any open subgroup \( V \) of \( U \), and there is a finite product \( P = \prod_{i=1}^n \text{con}(k_i) \) of contraction groups of elements \( k_1, k_2, \ldots, k_n \in K \) such that

\[
\langle K, U \rangle = PU_0 K.
\]

A special case of the theory of flat groups is when a t.d.l.c. group \( G \) is flat on itself. Here we obtain a structural result under the assumption that \( G/G_u \) is finitely generated.

**Theorem 1.9** (See § 6.3). Let \( G \) be a t.d.l.c. group such that \( G \) is flat on itself and \( G/G_u \) is finitely generated. Let \( U \) be a compact open subgroup of \( G \) that is tidy for \( G \) and let \( U_0 \) be the core of \( U \) in \( G \). Then \( G \) has a series of closed normal subgroups

\[
\text{Res}_{U_0}(G) \leq \text{nub}(G) \leq U_0 \leq T \leq G_u \leq G
\]

such that:

(i) \( T = G^\dagger U_0 \) and \( T/U_0 = (G/U_0)^\dagger \);

(ii) \( T \) is the smallest normal subgroup of \( G \) that contains \( U \), and the smallest open normal subgroup of \( G \) that contains \( U_0 \);

(iii) \( U \) is normal in \( G_u \), and hence \( [U, gUg^{-1}] \leq U \) for all \( g \in G \);

(iv) Given a compact normal subgroup \( K \) of \( G \), then \( |KU_0 : U_0| \) is finite and \( [G^\dagger, K] \leq \text{nub}(G) \).
1.6 Non-closed contraction groups

We can use the structure of reduced envelopes and results from [8] to show that for a certain class of t.d.l.c. groups $G$, either all contraction groups in $G$ are trivial or there exists a non-closed contraction group in $G$.

**Theorem 1.10** (See §6.4). Let $G$ be a t.d.l.c. group. Suppose that $G$ has a non-degenerate faithful weakly decomposable action on a Boolean algebra. Then exactly one of the following holds:

(i) $G$ is anisotropic, and given any compactly generated subgroup $K$ of $G$ such that $N_G(K)$ is open, then $K$ has arbitrarily small compact normal subgroups.

(ii) There exists $g \in G$ such that $nub(g)$ is non-trivial, in other words, $con(g)$ is not closed.

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2 Preliminaries

Given a topological group $G$, $Aut(G)$ denotes the group of topological automorphisms of $G$, that is, permutations of $G$ that are both group automorphisms and homeomorphisms. Given $X \subseteq G$ and $Y \subseteq Aut(G)$, we say $X$ is $Y$-invariant if $\alpha(X) = X$ for all $\alpha \in Y$.

Throughout, we adopt the convention that any definition given for automorphisms of a group $G$ also applies to an element $g$ of the group, acting via the automorphism $x \mapsto gxg^{-1}$. Similarly, definitions given for sets of automorphisms also apply to subsets of the group itself. In fact, the distinction between subgroups and automorphisms will turn out to be largely inconsequential, since a t.d.l.c. group $G$ with a group of automorphisms $H$ can be extended to a t.d.l.c. group $G \rtimes H$ in which $G$ is open, and we are concerned with properties of the action of $H$ that can be recovered from the action on any open $H$-invariant subgroup of $G$.

The following classical result is a defining feature of the theory of t.d.l.c. groups, and will be frequently used without comment.

**Theorem 2.1** (Van Dantzig, [9]). Let $G$ be a t.d.l.c. group. Then the compact open subgroups of $G$ form a base of neighbourhoods of the identity.
2.1 Tidy theory for a single automorphism

Let $G$ be a t.d.l.c. group, let $\alpha \in \text{Aut}(G)$, and let $U$ be a compact open subgroup of $G$. Define the subgroups

$$U_+ = \bigcap_{n \geq 0} \alpha^n(U); \quad U_- = \bigcap_{n \leq 0} \alpha^n(U).$$

Then $U$ is tidy above for $\alpha$ if $U = U_+ U_-$. Equivalently, there exist subgroups $V$ and $W$ of $U$ such that $U = VW$, $\alpha(V) \geq V$ and $\alpha(W) \leq W$. It is tidy below for $\alpha$ if the group $U_{++} := \{ g \in G \mid \forall n \ll 0 : \alpha^n(g) \}$ is closed.

A tidy subgroup for $\alpha$ is a compact open subgroup that is both tidy above and tidy below. More generally, a compact open subgroup $U$ is said to be tidy (above, below) for a set of automorphisms $A$ if it is tidy (above, below) for each element $\alpha \in A$. Some caution is required here, as a compact open subgroup $U$ may be tidy for $A$ without being tidy for the group generated by $A$ (see [22, Example 3.5]).

The scale $s(\alpha)$ is the minimum value of the (necessarily finite) index $|\alpha(U) : \alpha(U) \cap U|$ as $U$ ranges over the compact open subgroups of $G$. We say $\alpha$ is uniscalar if $s(\alpha) = s(\alpha^{-1}) = 1$; equivalently, $\alpha$ is uniscalar if it leaves invariant a compact open subgroup of $G$.

These concepts originate in [20], where it was shown that a tidy subgroup exists for every topological automorphism of a t.d.l.c. group.

Theorem 2.2 ([20] Theorem 1 and [21] Theorem 3.1). Let $G$ be a t.d.l.c. group and let $\alpha \in \text{Aut}(G)$. Then there exists a tidy subgroup for $\alpha$. Indeed, given a compact open subgroup $U$ of $G$, then $U$ is tidy for $\alpha$ if and only if $|\alpha(U) : \alpha(U) \cap U| = s(\alpha)$.

Some equivalent formulations of the tidy below property are effectively given in [20].

Lemma 2.3 (See [20] Lemma 3 and its corollary). Let $G$ be a t.d.l.c. group and let $\alpha \in \text{Aut}(G)$. Define

$$U_{++} := \{ g \in G \mid \forall n \ll 0 \alpha^n(g) \}; \quad U_{--} := \{ g \in G \mid \forall n \gg 0 \alpha^n(g) \}; \quad \mathcal{L}_U := U_{++} \cap U_{--}.$$

Then the following are equivalent.

(i) $U_{++}$ is closed;
(ii) $U_{--}$ is closed;
(iii) $\mathcal{L}_U \leq U$;
(iv) $U_{++} \cap U = U_+$;
(v) $U_{--} \cap U = U_-$.
Proof. In [20], conditions (i)–(iv) are shown to be equivalent to the condition that $U$ is tidy, under the assumption that $U$ is tidy above. However, we can bypass the assumption that $U$ is tidy above by noting (as in [20]) that any compact open subgroup $V$ can be replaced with the tidy above subgroup $U = \bigcap_{i=0}^{n} \alpha^i(V)$ for $n$ large enough. We see that $U_+ = V_+, U_{++} = V_{++}, U_- = V_-$, $\mathcal{L}_U = \mathcal{L}_V$ and $\mathcal{L}_V \cap V \leq U$. So $V$ is tidy below if and only if it is tidy below if and only if any one of the equivalent statements (i)–(iv) is satisfied, which can all be translated to corresponding statements for $V$. One can see the equivalence of (iv) and (v) by noting that replacing $\alpha$ with $\alpha^{-1}$ reverses the roles of (iv) and (v), but has no effect on (iii).

We see that if $U$ is tidy (above, below) for $\alpha$, then it is also tidy (above,below) for $\alpha^n$, for any $n \in \mathbb{Z} \setminus \{0\}$. (The converse is false: for example, if $\alpha$ acts on the group $\mathbb{Z}_p \times \mathbb{Z}_p$ by swapping the two copies of $\mathbb{Z}_p$, then $\mathbb{Z}_p \times p\mathbb{Z}_p$ is tidy for $\alpha^2$, but not for $\alpha$.)

Lemma 2.4. Let $G$ be a t.d.l.c. group, let $\alpha \in \text{Aut}(G)$, let $n \in \mathbb{Z} \setminus \{0\}$ and let $U$ be a compact open subgroup of $G$.

(i) If $U$ is tidy above for $\alpha$, then it is tidy above for $\alpha^n$.

(ii) If $U$ is tidy below for $\alpha$, then it is tidy below for $\alpha^n$.

Proof. Observe that $\alpha$ and $\alpha^{-1}$ play symmetrical roles in the definitions of tidy above and tidy below. Thus we may assume $n > 0$.

If $U$ is tidy above for $\alpha$, then $U = VW$ with $\alpha(V) \geq V$, so $\alpha^n(V) \geq V$, and $\alpha(W) \leq W$, so $\alpha^n(W) \leq W$. Thus $U$ is tidy above for $\alpha^n$, proving (i).

If instead $U$ is tidy below for $\alpha$, define $\mathcal{L}_U$ as above and set

$$\mathcal{L}'_U := \{g \in G \mid \forall |k| \gg 0 : \alpha^{nk}(g) \in U\}.$$ 

Then $\mathcal{L}'_U \subseteq \mathcal{L}_U \subseteq U$, so $U$ is tidy below for $\alpha^n$.

There are strong restrictions on the dynamics of $\alpha$ on orbits that intersect a tidy subgroup. In particular, an $\alpha$-orbit cannot leave the tidy subgroup $U$ and then return to it, and any forward or backward $\alpha$-orbit that escapes from $U$ is necessarily unbounded.

Lemma 2.5 ([22] Lemma 2.6). Let $G$ be a t.d.l.c. group, let $\alpha \in \text{Aut}(G)$, let $U$ be a compact open subgroup of $G$ that is tidy for $\alpha$ and let $u \in U$.

(i) The set $\{\alpha^n(u) \mid n \geq 0\}$ is bounded (that is, relatively compact in $G$) if and only if $u \in U_-$.

(ii) The set $\{n \in \mathbb{Z} \mid \alpha^n(u) \in U\}$ is an interval in $\mathbb{Z}$. 

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For a fixed automorphism \( \alpha \), the behaviours of the classes of tidy above and tidy below subgroups are somewhat divergent. Tidy above subgroups can be thought of as ‘small enough’; in particular, they form a base of identity neighbourhoods, by the following result:

**Proposition 2.6** ([20] Lemma 1). *Let \( G \) be a t.d.l.c. group, let \( \alpha \in \text{Aut}(G) \) and let \( U \) be a compact open subgroup of \( G \). Then there exists \( n \) (depending on \( U \) and \( \alpha \)) such that for all intervals \( I \in \mathbb{Z} \) of length at least \( n \), the intersection \( \bigcap_{i \in I} \alpha^i(U) \) is tidy above for \( \alpha \).*

Tidy below subgroups are instead ‘large enough’, in a way that is characterized by the nub \( \text{nub}(\alpha) \) of \( \alpha \). The nub is the intersection of all tidy subgroups for \( \alpha \); it also admits several other equivalent definitions, as described in [23].

**Proposition 2.7** ([23] Corollary 4.2). *Let \( G \) be a t.d.l.c. group, let \( \alpha \in \text{Aut}(G) \) and let \( U \) be a compact open subgroup of \( G \). Then \( U \) is tidy below for \( \alpha \) if and only if \( \text{nub}(\alpha) \leq U \). In particular, if \( U \) is tidy below for \( \alpha \) and \( V \) is a compact subgroup of \( G \) such that \( V \geq U \), then \( V \) is tidy below for \( \alpha \).*

**Theorem 2.8** ([23] Theorem 4.1). *Let \( G \) be a t.d.l.c. group and let \( \alpha \in \text{Aut}(G) \). Then \( \text{nub}(\alpha) \) is the largest closed subgroup of \( G \) on which \( \alpha \) acts ergodically and the largest compact subgroup of \( G \) that has no open \( \alpha \)-invariant subgroups. In addition, \( \text{nub}(\alpha) = \text{con}(\alpha) \cap \text{con}(\alpha^{-1}) \).*

A characterization for the situation when \( \text{nub}(\alpha) \) is trivial was given in [1].

**Theorem 2.9** (See [1] Corollary 3.30 and Theorem 3.32). *Let \( G \) be a t.d.l.c. group and let \( \alpha \in \text{Aut}(G) \). Then \( \text{con}(\alpha) = \text{con}(\alpha)\text{nub}(\alpha) \), and \( \text{nub}(\alpha) = 1 \) if and only if \( \text{con}(\alpha) \) is a closed subgroup of \( G \).*

Applying the scale function to inner automorphisms defines a function from \( G \) to the positive integers. This function is continuous (with respect to the discrete topology on \( \mathbb{N} \)) and satisfies \( s(g^n) = s(g^n) \) for all \( g \in G \) and \( n \geq 0 \). This is due to the stability properties of the tidy subgroups.

**Theorem 2.10** (Willis [20]). *Let \( G \) be a t.d.l.c. group.

(i) Let \( U \) be a compact open subgroup of \( G \). Let \( X \) be the set of elements \( x \in X \) such that \( U \) is tidy for \( x \). Then \( X \) is invariant under left and right translations by \( U \), in other words, \( X \) is a union of \( (U, U) \)-double cosets. In particular, \( X \) is a clopen subset of \( G \). In addition, for all \( n \in \mathbb{Z} \), if \( x \in X \) then \( x^n \in X \).

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In [1], the authors often assume that the t.d.l.c. group \( G \) is metrizable, but only do so in order to appeal to [1, Theorem 3.8], where metrizability is used in the proof. Since [1, Theorem 3.8] was subsequently shown by Jaworski ([10, Theorem 1]) to be true without the assumption of metrizability, the remaining results of [1] are also valid for t.d.l.c. groups in general.
(ii) The function $s : G \to \mathbb{N}$ is continuous when $\mathbb{N}$ is equipped with the discrete topology. Indeed, if $U$ is tidy for $x \in G$, then $s(x) = s(y)$ for all $y \in UxU$.

(iii) Let $\alpha$ be an automorphism of $G$. Then the collection of tidy subgroups for $\alpha$ is invariant under the action of $\alpha$ and closed under finite intersections.

Proof. (i) $X$ is a union of $(U, U)$-double cosets by \cite{20} Theorem 3, and any union of left cosets of a fixed open subgroup is clopen. Given $x \in X$, then $x^n \in X$ for all $n \in \mathbb{Z}$ by Lemma \cite{23}.

(ii) is \cite{20} Theorem 3 and Corollary 4.

(iii) It is clear that the collection of tidy subgroups for $\alpha$ is invariant under the action of $\alpha$. The fact that this collection is closed under finite intersections is \cite{20} Lemma 10. □

The scale can be determined asymptotically, starting with any compact open subgroup.

**Theorem 2.11** \cite{13} Theorem 7.7. Let $G$ be a t.d.l.c. group, let $\alpha$ be an automorphism of $G$, and let $U$ be a compact open subgroup of $G$. Then

$$|\alpha^n(U) : \alpha^n(U) \cap U|^{1/n} \to s(\alpha)$$

as $n \to +\infty$.

We derive the following result from Theorem 2.11; it can also be derived easily from \cite{21}, Proposition 4.3.

**Corollary 2.12.** Let $G$ be a t.d.l.c. group, let $\alpha$ be an automorphism of $G$ and let $K$ be an open subgroup of $G$ such that $\alpha(K) = K$. Then $s_G(\alpha) = s_K(\alpha)$, and every compact open subgroup of $K$ that is tidy for $\alpha$ on $K$ is also tidy for $\alpha$ on $G$.

Proof. Let $V$ be a compact open subgroup of $H$. Then $V$ is open in $G$, so by Theorem 2.11 we have

$$s_G(\alpha) = \lim_{n \to \infty} |\alpha^n(V) : \alpha^n(V) \cap U|^{1/n} = s_K(\alpha).$$

The assertion about tidy subgroups follows from Theorem 2.11 □

An automorphism $\alpha$ is **anisotropic** if the set of compact open $\alpha$-invariant subgroups of $G$ forms a base of identity neighbourhoods. Given a t.d.l.c. group $G$ and a group $H$ acting on $G$ (or a subgroup $H$ of $G$), we say $H$ is uniscalar or anisotropic respectively on $G$ if all the automorphisms of $G$ induced by $H$ are so. ‘Uniscalar/anisotropic subgroup’ should be understood in this relative sense.

Anisotropic automorphisms are necessarily uniscalar. In general, a uniscalar automorphism need not be anisotropic, however certain local structures of the group $G$ can force all uniscalar automorphisms to be anisotropic: for example, if every (equivalently, some)
compact open subgroup $U$ of $G$ is topologically finitely generated and virtually pro-$p$, then $U$ admits a base of identity neighbourhoods consisting of characteristic subgroups, so any automorphism leaving $U$ invariant must be anisotropic.

Contraction groups and the nub can be used to characterize when an automorphism is uniscalar or anisotropic.

**Proposition 2.13.** Let $G$ be a t.d.l.c. group and let $\alpha \in \text{Aut}(G)$.

(i) We have $s(\alpha) = 1$ if and only if $\text{con}(\alpha^{-1})$ is relatively compact.

(ii) Suppose that $\alpha$ is uniscalar. Then $\alpha$ is anisotropic if and only if $\text{nub}(\alpha)$ is trivial.

(iii) If $\text{con}(\alpha) = \text{con}(\alpha^{-1}) = \{1\}$, then $\alpha$ is anisotropic (and conversely).

**Proof.** For part (i), see [1, Proposition 3.24].

Suppose that $\alpha$ is uniscalar. Then a compact open subgroup of $G$ is tidy for $\alpha$ if and only if it is $\alpha$-invariant. If $\alpha$ is anisotropic, then evidently the intersection of all $\alpha$-invariant subgroups for $\alpha$ is trivial, so $\text{nub}(\alpha) = \{1\}$. Conversely if $\text{nub}(\alpha) = \{1\}$, consider a compact open subgroup $U$ of $G$ and an $\alpha$-invariant compact open subgroup $V$ of $G$. Then by the compactness of $V \setminus U$, there exists a finite set $\{V_1, V_2, \ldots, V_n\}$ of $\alpha$-invariant compact open subgroups of $G$ such that $W = V \cap \bigcap_{i=1}^n V_i \leq U$. Now $W$ is an $\alpha$-invariant compact open subgroup; since $U$ was an arbitrary compact open subgroup of $G$, we conclude by Van Dantzig’s theorem that there exist arbitrarily small compact open $\alpha$-invariant subgroups of $G$, that is, $\alpha$ is anisotropic, proving (ii).

If $\alpha$ is anisotropic, then clearly $\text{con}(\alpha) = \text{con}(\alpha^{-1}) = \{1\}$. Conversely, if $\text{con}(\alpha) = \text{con}(\alpha^{-1}) = \{1\}$, then $\alpha$ is uniscalar by part (i) and $\text{nub}(\alpha) = \{1\}$ by Theorem 2.8, so $\alpha$ is anisotropic, completing the proof of (iii).

2.2 Flat groups

A group of automorphisms $H$ of $G$ is **flat** if there exists a compact open subgroup $U$ of $G$ such that $U$ is tidy for $H$, that is, for all $\alpha \in H$, $U$ is tidy for $\alpha$. More generally, any group acting on $G$ (such as a subgroup of $G$) is said to be flat on $G$ if it induces a flat group of automorphisms, and ‘flat subgroup’ should be understood in this relative sense. (Note that if $H$ is a closed subgroup of $G$, then $H$ may be flat on itself without being flat on $G$.)

A class of groups that are evidently flat are groups $H \leq \text{Aut}(G)$ such that $H$ leaves invariant a compact open subgroup of $G$. More generally, it is easily seen that in any flat group $H$, and given any tidy subgroup $U$ for $H$, the set of elements of $H$ that leave $U$ invariant form a normal subgroup, the **uniscal part** $H_u$ of $H$, which does not depend on the choice of $U$.  

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The uniscalar part itself could potentially be any group that acts by automorphisms on a compact open subgroup of $G$. However, the quotient $H/H_u$ has a special structure, as first described by Willis in [22]. In particular, the following holds:

**Theorem 2.14** ([22] Theorem 4.15). Let $G$ be a t.d.l.c. group and let $H$ be a flat group of automorphisms of $G$. Then $H/H_u$ is a torsion-free abelian group, and every non-identity element of $H/H_u$ is a finite power of an indivisible element.

The **(flat) rank** of a flat group is the minimum number of generators of $H/H_u$.

Some sufficient conditions for a group to be a finite-rank flat group were given in [22], with further generalizations in [18].

**Theorem 2.15** ([18] Theorems 4.9 and 4.13). Let $G$ be a t.d.l.c. group, let $H$ be a group of automorphisms of $G$, and let $K$ be a normal subgroup of $H$ such that $K$ leaves invariant a compact open subgroup of $G$.

(i) If $H/K$ is finitely generated and nilpotent, then $H$ is flat.

(ii) If $H/K$ is polycyclic, then $H$ has a flat subgroup of finite index.

In particular, Theorem 2.15 implies that, given an abstract finitely generated nilpotent group $N$, then given any homomorphism $N \to \text{Aut}(G)$, where $G$ is a t.d.l.c. group, then the derived group $[N, N]$ will leave invariant a compact open subgroup. Example 2.19 below shows that finitely generated polycyclic groups need not be flat, and an example given after [18, Theorem 4.13] shows that finitely generated soluble groups need not be virtually flat.

We also see from Theorem 2.15 that flatness of finite rank persists on restricting the action to a closed invariant subgroup.

**Corollary 2.16.** Let $G$ be a t.d.l.c. group and let $H$ be a flat group of automorphisms of $G$ of finite rank. Let $K$ be a closed $H$-invariant subgroup of $G$. Then $H$ is flat of finite rank on $K$.

**Proof.** Let $U$ be a compact open subgroup of $G$ that is tidy for $H$ and let $L$ be the uniscalar part of $H$ acting on $G$. Then $U$ is $L$-invariant, so $U \cap K$ is also $L$-invariant, and $H/L$ is finitely generated and abelian. Hence $H$ is flat of finite rank on $K$ by Theorem 2.15.

In discussions of flat subgroups of t.d.l.c. groups, it is convenient to work with closed subgroups. We note that the flat property is well-behaved under closure.

**Lemma 2.17.** Let $G$ be a t.d.l.c. group and let $H$ be a flat subgroup of $G$. Then $H$ is a flat subgroup of $G$ and $H_u$ is an open subgroup of $H$. 14
Proof. We see that $H_u$ is open by Theorem 2.10 since it consists of the uniscalar elements of $H$. Also by Theorem 2.10 any compact open subgroup that is tidy for $H$ is also tidy for $\overline{H}$, so $\overline{H}$ is flat.

Definition 2.18. A subgroup $H$ of a t.d.l.c. group $G$ is almost flat (on $G$) if $\overline{H}$ has a closed cocompact subgroup $K$ such that $K$ is flat on $G$. Say $H$ is almost finite-rank flat if in addition $K$ can be chosen so that $K/K_u$ is finitely generated.

It is not clear at present whether an almost flat subgroup is necessarily virtually flat, that is, has a subgroup of finite index that is flat on $G$. Virtually flat subgroups however need not be flat, as the next example shows. In any case, almost (finite-rank) flat subgroups will be sufficiently well-behaved for most purposes in the present paper.

Example 2.19. Let $K = \mathbb{Q}_p \rtimes \langle t \rangle$, where $\mathbb{Q}_p$ is open in $K$ and $t$ acts on $\mathbb{Q}_p$ as multiplication by $p$, let $G = K \wr C$ where $C$ is a finite non-trivial cyclic group acting regularly, and let $H$ be the polycyclic subgroup $\langle t \rangle \wr C = B \rtimes C$, where $B \cong \mathbb{Z}^n$. Observe that no non-trivial element of $B$ is uniscalar, so in particular the derived group of $H$ is not uniscalar. Hence $H$ is not flat. However, the finite index subgroup $B$ of $H$ is flat: indeed, there are arbitrarily small tidy subgroups for $B$ of the form $\mathbb{Z}_p^n$.

Note that if $H$ is a closed compactly generated subgroup of $G$ that is almost flat, then it is almost finite-rank flat: any cocompact flat subgroup $K$ is compactly generated, so that $K/K_u$ is finitely generated.

We also introduce a notion that is stronger than being flat (and is not in general satisfied even by cyclic groups.

Definition 2.20. A group of automorphisms $H$ of a t.d.l.c. group $G$ is smooth (on $G$) if the tidy subgroups for $H$ on $G$ form a base of neighbourhoods of the identity.

In one situation, the (relative) flat and smooth properties is clearly inherited from cocompact subgroups.

Lemma 2.21. Let $G$ be a t.d.l.c. group and let $H$ be a closed subgroup of $G$. Suppose there is a closed subgroup $K$ of $H$ such that $K$ is cocompact in $H$ and such that $K$ is flat and uniscalar on $G$. Then $H$ is flat and uniscalar on $G$. If in addition $K$ is smooth on $G$, then so is $H$.

Proof. Suppose that $K$ is flat on $G$. Then there is a compact open subgroup $U$ of $G$ that is tidy for $K$, and if $K$ is smooth, then $U$ can be made arbitrarily small. Since $K$ is uniscalar, in fact $K$ normalizes $U$. Now $H = XK$, where $X$ is a compact set, so

$$V = \bigcap_{h \in H} hUh^{-1} = \bigcap_{x \in X} xUx^{-1}$$

I thank George Willis for pointing out this example.
is a compact open subgroup normalized by $H$ such that $V \leq U$. In particular $H$ is unisclar on $G$, and also $V$ is tidy for $H$, so $H$ is flat on $G$. If $K$ is smooth, then $V$ can be made arbitrarily small, so $H$ is smooth.

2.3 Metrizability

A topological space (or group) is **metrizable** if it is homeomorphic to a metric space. Not all t.d.l.c. groups are metrizable, and for the most part we do not need to restrict to the metrizable case, but occasionally it will be necessary to do so. Here are some equivalent conditions.

**Lemma 2.22.** Let $G$ be a t.d.l.c. group. Then the following are equivalent.

(i) $G$ is metrizable;

(ii) $G$ is first countable, that is, there is a countable base of neighbourhoods of the identity;

(iii) $G$ contains a Polish (that is, separable and completely metrizable) open subgroup;

(iv) Every compact subgroup of $G$ has only countably many open subgroups;

(v) Every non-discrete compact subgroup of $G$ is homeomorphic to the Cantor set;

(vi) $G$ is either discrete or homeomorphic to a disjoint union of copies of the Cantor set.

**Proof.** It is clear that if $U$ is a compact open subgroup of $G$, then $G$ is homeomorphic to a disjoint union of copies of $U$ (namely the left cosets of $U$), so $G$ is metrizable if and only if $U$ is metrizable. The other properties are also stable on passing between $G$ and $U$. Hence we may assume $G$ is profinite. It is also clear that each of the conditions (iii), (iv), (v) and (vi) implies metrizability.

By [25, Proposition 4.1.3], $G$ is metrizable if and only if it is an inverse limit of a countable sequence of finite groups. An inverse limit of countably many finite groups is evidently first countable. Conversely, by Van Dantzig’s theorem any local base at the identity in a t.d.l.c. group can be replaced by one of the same size consisting of compact open subgroups, so a first countable profinite group $G$ has a local base at the identity consisting of countably many open subgroups, from which we conclude that $G$ is an inverse limit of a countable sequence of finite groups, and that $G$ has only countably many open subgroups in total (since only finitely many open subgroups of a compact group can contain a given open subgroup). Hence (i) and (ii) are equivalent and (i) implies (iv).
Under the assumption that \( G \) is an inverse limit of a countable sequence of finite groups, it is easily verified that \( G \) is either finite or homeomorphic to the Cantor set, and thus \( G \) is Polish; moreover, every closed subgroup of a Polish group is Polish. So (i) implies (iii), (v) and (vi), completing the proof that all six conditions are equivalent.

3 The relative Tits core

Contraction groups in t.d.l.c. groups have useful stability properties, which translate well to the context of relative Tits cores. In particular, the group \( G^1_X \) is less sensitive to the choice of \( X \) than one might expect, as will be shown in Theorem 3.7 below. First, we recall some prior work on stability properties of contraction groups.

3.1 Prior results on stability of the contraction group

The following result on contraction groups was proved by Baumgartner–Willis for metrizable t.d.l.c. groups, then extended to the general t.d.l.c. case by Jaworski. (The analogous assertion does not hold in general for connected locally compact groups: see [11, Example 4.1].)

**Theorem 3.1** ([1] Theorem 3.8, [10] Theorem 1). Let \( G \) be a t.d.l.c. group, let \( \alpha \in \text{Aut}(G) \) and let \( H \) be a closed subgroup of \( G \) such that \( \alpha(H) = H \). Let \( \mathcal{O}(G) \) be the set of all identity neighbourhoods in \( G \). Define

\[
\text{con}_{G/H}(\alpha) := \{ x \in G \mid \forall U \in \mathcal{O}(G) \exists n \forall n' \geq n : \alpha^{n'}(x) \in UH \}. 
\]

Then \( \text{con}_{G/H}(\alpha) = \text{con}(\alpha)H \).

In particular, combining Theorem 3.1 with Proposition 2.13, we have a criterion for an automorphism to have anisotropic action on a subquotient of \( G \).

**Corollary 3.2.** Let \( G \) be a t.d.l.c. group, let \( \alpha \in \text{Aut}(G) \), and let \( H \) and \( K \) be closed \( \alpha \)-invariant subgroups of \( G \) such that \( K \) is normal in \( H \). A sufficient condition for \( \alpha \) to have anisotropic action on \( H/K \) is that \( G^1_\alpha \leq K \). If \( H \) is open in \( G \), this condition is also necessary.

The contraction group of an automorphism is invariant under raising the automorphism to a positive power.

**Lemma 3.3.** Let \( G \) be a topological group, let \( \alpha \in \text{Aut}(G) \) and let \( n \) be a positive integer. Then

\[
\text{con}(\alpha) = \text{con}(\alpha^n). 
\]
Proof. We have \(\text{con}(\alpha^n) \geq \text{con}(\alpha)\), since \(\alpha^{ni}\) is a subsequence of \((\alpha^i)_{i \in \mathbb{N}}\). Let \(x \in \text{con}(\alpha^n)\) and set \(x_i = \alpha^i(x)\). Then the sequence \((x_{ni})_{i \in \mathbb{N}}\) converges to the identity. Since \(\alpha^j\) is a continuous automorphism and \(\alpha^j(x_k) = x_{j+k}\) for all \(j, k \in \mathbb{Z}\), it follows that the sequence \((x_{j+ni})_{i \in \mathbb{N}}\) converges to \(\alpha^j(1) = 1\). Hence \((x_i)_{i \in \mathbb{N}}\) converges to the identity, since it can be partitioned into finitely many subsequences \((x_{j+ni})_{i \in \mathbb{N}}\) for \(0 \leq j < n\), each of which converges to the identity. In other words, \(x \in \text{con}(\alpha)\). \(\square\)

The stability of contraction groups was also investigated in \([6]\).

**Proposition 3.4** (\([6]\), Lemma 4.1 and Corollary 4.2). Let \(G\) be a t.d.l.c. group. Let \(g \in G\) and let \(U\) be a compact open subgroup of \(G\) that is tidy above for \(g\). Then for every \(u \in U\), there exists \(t \in U_+ \cap \text{con}(g^{-1})\) such that

\[
\text{con}(gu) = t\text{con}(g)t^{-1}.
\]

**Proposition 3.5** (\([6]\), Proposition 5.1). Let \(G\) be a t.d.l.c. group and let \(A\) be an abstract subgroup of \(G\). Given any \(g \in A\), if \(\text{con}(g)\) normalizes \(A\), then \(\text{con}(g) \leq A\). In particular, any (abstract) normal subgroup of \(G\) containing \(g\) also contains \(\text{con}(g)\).

There is a straightforward condition for when the contraction group of an element is the same as its contraction group acting on a closed subgroup.

**Lemma 3.6.** Let \(G\) be a t.d.l.c. group, let \(g \in G\) and let \(K\) be a closed \((g)\)-invariant subgroup of \(G\). Then \(\text{con}_K(g) = \text{con}(g) \cap K\). In particular, we have \(K^+ = G^+_K\) if and only if \(G^+_K \leq K\).

Proof. Let \(g \in G\). Given \(u \in \text{con}_K(g)\), then for all open subgroups \(U\) of \(G\), we have \(g^nug^{-n} \in K \cap U \leq U\) for \(n\) sufficiently large, since \(K \cap U\) is an open subgroup of \(K\). Thus \(u \in \text{con}(g) \cap K\). Conversely, given \(u \in \text{con}(g) \cap K\), then \(g^nug^{-n} \in K\) for all \(n \geq 0\) by hypothesis, so given an open subgroup \(U\) of \(G\), we have \(g^nug^{-n} \in K \cap U\) for \(n\) sufficiently large. Since the subgroups \(K \cap U\) form a base of identity neighbourhoods in \(K\) as \(U\) ranges over the open subgroups of \(G\), it follows that \(u \in \text{con}_K(g)\). Thus \(\text{con}_K(g) = \text{con}(g) \cap K\). The last conclusion is clear. \(\square\)

### 3.2 Invariance of the relative Tits core

We are now able to give several invariance properties of relative Tits cores.

**Theorem 3.7.** Let \(G\) be a t.d.l.c. group and let \(X\) be a subset of \(G\). Suppose that for all \(g \in G\), we have \(g \in X\) whenever \(\text{con}(g) \leq G^+_X\). Then the following properties hold.

(i) \(X\) is a clopen subset of \(G\) that contains all anisotropic elements of \(G\).

(ii) Let \(g \in G\) and let \(n\) be a non-zero integer. Then \(g \in X\) if and only if \(g^n \in X\).
(iii) The normalizer of $X$ is closed in $G$ and is equal to the normalizer of $G^\dagger_X$ in $G$.

(iv) Let $H$ be a closed subgroup of $G$ and let $K$ be a cocompact subgroup of $H$. Then $H \subseteq X$ if and only if $X$ contains every $H$-conjugate of $K$.

(v) Let $Y \subseteq G$ be such that there exists a compact open subgroup $U$ of $G$ such that for all $g \in Y$, there exists $V \supseteq U$ such that $V$ is tidy for $g$. (For example, $Y$ could be a union of finitely many flat subgroups of $G$.) Then $N_G(G^\dagger_Y)$ is open in $G$.

(vi) Let $R = G^\dagger_X$. Then

$$R^\dagger = G^\dagger_R \leq G^\dagger_X,$$

so $R \subseteq X$. In particular, if $G^\dagger_X$ is dense in $G$, then $G^\dagger_X = G^\dagger$.

We begin the proof of the theorem with a lemma.

**Lemma 3.8.** Let $G$ be a t.d.l.c. group. Given $g \in G$, let $L_g = \langle \text{con}(g), \text{con}(g^{-1}) \rangle$. Let $g \in G$, let $U$ be an open subgroup of $G$ that is tidy for $g$, let $u, u' \in U$ and let $n \in \mathbb{Z} \setminus \{0\}$. Then $L_{ug^n u'} = L_g$ and $G^\dagger_{ug^n u'} = G^\dagger_g$. In particular, the normalizers of $L_g$ and $G^\dagger_g$ both contain $U$, and are thus open subgroups of $G$.

**Proof.** By Lemma 3.3 we have $\{\text{con}(g), \text{con}(g^{-1})\} = \{\text{con}(g^n), \text{con}(g^{-n})\}$ for all non-zero integers $n$, and by Lemma 2.4 $U$ is tidy for $g^n$; thus we may assume $n = 1$. By Proposition 3.4, $\text{con}(gu')$ is conjugate to $\text{con}(g)$ under the action of $\text{con}(g^{-1})$, so $\text{con}(gu') \leq L_g$. Similarly, $\text{con}(g^{-1} u^{-1})$ is conjugate to $\text{con}(g^{-1})$ under the action of $\text{con}(g)$, so $\text{con}(g^{-1} u^{-1}) \leq L_g$. Now $U$ is tidy for $ug$ by Theorem 2.10, so $\text{con}(ugu')$ is conjugate to $\text{con}(ug)$ under the action of $\text{con}(g^{-1} u^{-1})$, so $\text{con}(ugu') \leq L_g$. Similarly, $\text{con}(u'g^{-1} u^{-1}) \leq L_g$. Thus $L_{ugu'} \leq L$. By the same argument $L_g \leq L_{ugu'}$, since $U$ is tidy for $ugu'$, so $L_{ugu'} = L$.

The proof that $G^\dagger_{ug^n u'} = G^\dagger_g$ is similar. \qed

We introduce the following concept for the purposes of the proof.

Given a subset $X$ of $G$, define the **Tits-core closure** $TC_G(X)$ to be the set

$$TC_G(X) := \{ x \in G | \text{con}(x) \leq G^\dagger_X \}.$$

Say $X$ is **Tits-core closed** in $G$ if $X = TC_G(X)$. Note that $TC_G(TC_G(X)) = TC_G(X)$ for any subset $X$ of $G$, since $G^\dagger_X = G^\dagger_{TC_G(X)}$.

**Proof of Theorem 3.7.** (i) We see from Lemma 3.8 that $X$ is open. Evidently $X$ contains all anistropic elements of $G$.

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Let $k \in X$. Then there is a compact open subgroup $U$ of $G$ that is tidy for $k$, and moreover we have $h \in kU$ for some $h \in X$. Hence $G^U_k = G^U_h$ by Lemma 5.3, so $G^U_k \leq G^U_X$, in other words $k \in X$. Hence $X$ is closed.

(ii) follows immediately from Lemma 5.3.

(iii) We have $N_G(X) \leq N_G(G^U_X)$, since $G^U_X$ is determined by $G$ and $X$. Given $g \in N_G(G^U_X)$ and $x \in X$, then

$$G^U_{gxg^{-1}} = gG^U_xg^{-1} \leq gG^U_Xg^{-1} = G^U_X,$$

so $gxg^{-1} \in X$. Since $x \in X$ was arbitrary we have $gXg^{-1} \subseteq X$, and by symmetry in fact $gXg^{-1} = X$. So $g \in N_G(X)$ and hence $N_G(X) = N_G(G^U_X)$.

Let $N = N_G(X)$, let $r \in \mathcal{N}$ and let $x \in X$. Then $r$ can be approximated in $G$ by elements of $N$, so given a compact open subgroup $U$ of $G$ that is tidy for $r^{-1}x$, there exists $s \in N$ such that $s \in Ur$. By Lemma 5.3 we have $G^U_{r^{-1}x} = G^U_{sx^{-1}r}$, and since $s \in N_G(X)$ we have $G^U_{sx^{-1}r} \leq G^U_X$. Since $x \in X$ was arbitrary (in particular, independent of the choice of $r$), we have $rG^U_Xr^{-1} \leq G^U_X$, and by symmetry in fact $rG^U_Xr^{-1} = G^U_X$, so $r \in N$. Hence $N$ is closed.

(iv) If $H \subseteq X$, then clearly $X$ contains every $H$-conjugate of $K \leq H$. Conversely, suppose that $X$ contains every $H$-conjugate of $K$. Let $g \in H$ and let $U$ be a compact open subgroup of $G$ that is tidy for $g$. Then the sequence $(g^iK)_{i \in \mathbb{N}}$ lies in the compact coset space $H/K$, so has an accumulation point $hK$ say. In particular, there exist $i, j \in \mathbb{Z}$ distinct and $h \in H$ such that $\{g^i, g^j\} \subseteq UhK$, and hence $g^{-i-j} \in UhKh^{-1}U$, that is, $g^{-i-j} = uyv$ for $u, v \in U$ and $y \in hk^{-1} \subseteq X$. Then $G^U_{g^{-i-j}} = G^U_y$ by Lemma 5.3 so $g^{-i-j} \in X$, and hence $g \in X$ by part (ii). Since $g \in H$ was arbitrary, $H \subseteq X$.

(v) Let $U$ and $Y$ be as in the statement. Then by Lemma 5.3 we have $G^U_{uyv} = G^U_y$ for all $u, v \in U$ and $y \in Y$. Hence $G^U_Y = G^U_{UYU}$, so $G^U_Y$ is normalized by $U$. Hence $N_G(G^U_Y)$ is open.

(vi) Let $g \in G^U_X$. Then $g = u_1u_2 \ldots u_n$, where $u_i \in G^U_{x_i}$ for some $x_i \in X$. Thus $g \in G^U_Y$, where $Y$ is a finite subset of $X$. By part (v), $H = N_G(G^U_Y)$ is open. We see that $\text{con}(g) \leq H$, since $H$ is a $g$-invariant neighbourhood of the identity, and hence $\text{con}(g) \leq N_G(G^U_Y)$. By Proposition 5.10 it follows that $\text{con}(g) \leq G^U_Y$, and hence $\text{con}(g) \leq G^U_X$. Since $g \in G^U_X$ was arbitrary, we conclude that $G^U_X \subseteq X$, and since $X$ is closed, we in fact have $R \subseteq X$, that is, $G^U_R \leq G^U_X$.

Given $r \in R$, we have seen that $\text{con}_G(r) \leq G^U_X$, so $\text{con}_G(r) \leq R$. It follows from Lemma 5.3 that in fact $\text{con}_G(r) = \text{con}_R(r)$. Hence $R^U = R^U_R = G^U_R$. \[\square\]

We observe that the relative Tits core is invariant under passing to a cocompact subgroup that is subnormal, flat or anisotropic.
Corollary 3.9. Let $G$ be a t.d.l.c. group, let $H$ be a closed subgroup of $G$, and let $K$ be a cocompact subgroup of $H$. Suppose that $K$ is at least one of: subnormal in $H$, flat on $G$, or anisotropic on $G$. Then $G^\dagger_H = G^\dagger_K$.

Proof. Suppose $K$ is subnormal in $H$. By induction on the subnormal depth, we may assume that $K$ is normal in $H$. The conclusion then follows by Theorem 3.7(iv).

Suppose instead that $K$ is either flat on $G$ or anisotropic on $G$. If $K$ is flat, then $G^\dagger_K$ has open normalizer in $G$, by Theorem 3.7(v); if $K$ is anisotropic, then $G^\dagger_K$ is trivial, so $G^\dagger_K$ is normal in $G$. In either case, let $L$ be the normalizer of $G^\dagger_K$ in $H$. Then $L$ is open in $H$ and $K \leq L$, so $L$ has finite index in $H$. By Theorem 3.7(ii), $G^\dagger_H = G^\dagger_L$. In turn, $L$ normalizes $X = \text{TC}_G(K)$ and $K$ is cocompact in $L$, so $G^\dagger_L = G^\dagger_K$ by Theorem 3.7(iv).

The following corollary is now clear.

Corollary 3.10. Let $G$ be a t.d.l.c. group and let $H$ be an almost flat subgroup of $G$. Then $N_G(G^\dagger_H)$ is open in $G$.

Question 1. Let $G$ be a t.d.l.c. group and let $H$ be a cocompact subgroup in $G$. Is it necessarily the case that $G^\dagger = G^\dagger_H$? (By the results we have so far, it is equivalent to showing that $G^\dagger_H$ is normal in $G$.)

3.3 Subgroups containing relative Tits cores

There is no reason for an arbitrary closed subgroup $K$ of $G$ to contain $G^\dagger_K$. For example, if $G$ is the automorphism group of a locally finite tree, then $G^\dagger_g$ is open in $G$ for every hyperbolic element $g \in G$ (see Example 3.14), so certainly $G^\dagger_g \not\leq \langle g \rangle$. However, we can ensure $G^\dagger_K \leq K$ under certain circumstances, as stated in Theorem 1.3.

Proof of Theorem 1.3. We may assume that $X = X^{-1}$. Let $U$ be an open subgroup of $G$ such that $U \cap G^\dagger_X \leq N_G(D)$.

Let $x \in X$. By Lemma 3.8 we have $G^\dagger_x = G^\dagger_d$ for all $d \in VxV$, where $V$ is a compact open subgroup of $G$ that is tidy for $x$. Since $X \subseteq D$, there exists $d \in VxV \cap D$: for this $d$, we see that $U \cap G^\dagger_d = U \cap G^\dagger_x \leq N_G(D)$.

Let $u \in \text{con}(d)$. Then for $n \geq 0$ sufficiently large, we have $d^nu^{-n}u^{-n} \in U$, and thus $d^nu^{-n}u^{-n} \in N_G(D)$. But $N_G(D)$ is $D$-invariant, so $u \in N_G(D)$, and hence $\text{con}(d) \leq N_G(D)$. In addition, $N_G(D)$ contains the open subgroup $U \cap \text{nub}(d)$ of $\text{nub}(d)$. By Theorem 2.8 there are no proper $d$-invariant open subgroups of $\text{nub}(d)$, so $\text{nub}(d) \leq N_G(D)$. Thus $\text{con}(d) \leq N_G(D)$ by Theorem 2.9. The same argument shows that $\text{con}(d^{-1}) \leq N_G(D)$, so in fact $G^\dagger_d \leq N_G(D)$. Hence by Proposition 3.5 we have $G^\dagger_d \leq D$, so $G^\dagger_x \leq D$. 21
As $x \in X$ was arbitrary, we conclude that $G^+_X \subseteq D$. Finally we have $G^+_X = G^+_X$ by Theorem 3.7.

**Corollary 3.11.** Let $G$ be a t.d.l.c. group, let $A$ be a subgroup of $G$, and let $B \subseteq A$. Then the following are equivalent:

(i) $G^+_B \leq A$;

(ii) There exists a subgroup $H$ of $G$ such that $A$ is subnormal in $H$ and $G^+_B \leq H$.

**Proof.** Clearly (i) implies (ii), as we can take $A = H$ in this case. Suppose (ii) holds, and let

$$A = A_0 \lhd A_1 \lhd \ldots \lhd A_n = H$$

be a subnormal series from $A$ to $H$. Suppose $n > 0$. Then by Theorem 1.3, we have $G^+_B \leq A_{n-1}$, since $G^+_B$ normalizes $A_{n-1}$ and $B \subseteq A_{n-1}$. The conclusion now follows by induction on the subnormal degree of $B$ in $H$.

The following is now clear from Corollaries 3.11 and 3.9.

**Corollary 3.12.** Let $G$ be a t.d.l.c. group and let $H$ be a subnormal subgroup of $G$. Then $G^+_H = (H)^\dagger$. If in addition $H$ is closed and cocompact in $G$, then $G^+_H = H^\dagger$.

In particular, we have the following strengthening of [3, Corollary 1.2].

**Corollary 3.13.** Let $G$ be a t.d.l.c. group and let $S$ be the set of subnormal subgroups $S$ of $G$ such that $\overline{S}$ is cocompact in $G$. Then

$$G^\dagger \leq \bigcap_{S \in S} S.$$

3.4 Examples

We give two basic examples that illustrate how the relative Tits core $G^+_g$ can depend very little on the choice of element $g$, even though the groups $\text{con}(g)$ and $\text{con}(g^{-1})$ are sensitive to the choice of $g$.

**Example 3.14.** Let $T$ be a locally finite regular tree of degree at least 3, let $G = \text{Aut}(T)$, and let $g \in G$. If $g$ is elliptic, that is, $g$ fixes a vertex or inverts an edge, then $\text{con}(g) = \text{con}(g^{-1}) = \{1\}$. Otherwise $g$ is hyperbolic, and the set of vertices $v$ such that $d(v, gv)$ is minimised forms a bi-infinite path $L$ in $T$, the axis of $g$. Identify $L$ with $\mathbb{Z}$, so that $g(0) > 0$ and $d_T(i, j) = |j - i|$, and let $P$ be the nearest point projection from...
$T$ to $L = \mathbb{Z}$. Let $K^-_n$ be the fixator of the set $P^{-1}(\infty, n)$ and let $K^+_n$ be the fixator of the set $P^{-1}((n, \infty))$. Then we see that 

$$\text{con}(g) = \bigcup_{n \in \mathbb{Z}} K^-_n \text{ and } \text{con}(g^{-1}) = \bigcup_{n \in \mathbb{Z}} K^+_n.$$ 

It is easily seen that $\text{con}(g)$ is not closed in $G$: in fact, the closure $\overline{\text{con}(g)}$ consists of all elliptic elements fixing a certain end $\omega$ of the axis of $g$. The normalizer of $\text{con}(g)$ (and also of $\overline{\text{con}(g)}$) is the stabilizer of $\omega$, so $N_G(\text{con}(g))$ is a closed but not open subgroup of $G$. Similarly, $\text{nub}(g)$ is the pointwise stabilizer of the axis of $g$, so $N_G(\text{nub}(g))$ is not open. However the subgroups $K^-_n$ and $K^+_n$ are each compact, and the product $K^-_nK^+_{n+1}$ is a compact open subgroup of $G$, being the stabilizer in $G$ of the directed edge $(n, n+1)$. In turn, it is easily seen that the group generated by the stabilizers of the directed edges $(n, n+1)$ as $n$ ranges over $L$ is in fact the subgroup $G^+$ of $G$ generated by all directed edge stabilizers, which is an open subgroup of $G$ of index 2. So in this case, every element $g \in G$ satisfies either $G^g_1 = G^+ = G^{\dagger}$ (if $g$ is hyperbolic) or $G^g_1 = \{1\}$ (if $G$ is not hyperbolic).

**Example 3.15.** (See [16] for a more detailed treatment of a class of examples including this one.)

Let $G = \text{GL}_n(\mathbb{Q}_p)$ and let $g$ be the diagonal matrix $\text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Suppose that 

$$|\lambda_1|_p \geq |\lambda_2|_p \geq \cdots \geq |\lambda_n|_p.$$ 

Then $\text{con}(g)$ is closed in this case: it is the group of matrices of the form $1 + u$, where

$$u_{ij} = 0 \text{ whenever } |\lambda_i|_p \leq |\lambda_j|_p.$$ 

In other words, $\text{con}(g)$ is a group of block upper unitriangular matrices, with the blocks corresponding to intervals of $(\lambda_1, \ldots, \lambda_n)$ on which $|\lambda_i|_p$ is constant. Thus $\text{con}(g)$ is the unipotent radical of a parabolic subgroup $P$, where $P$ consists of all elements $a$ of $G$ such that $a_{ij} = 0$ whenever $|\lambda_i|_p > |\lambda_j|_p$. In fact $P$ can be characterized directly in terms of the dynamics of $g$: it consists of those elements $a \in G$ such that $\{g^nag^{-n} \mid n \geq 0\}$ is relatively compact. We see also that $\text{con}(g^{-1})$ is simply the image of $\text{con}(g)$ under matrix transposition.

A similar description of contraction groups can be given for all elements of $G$ that have non-trivial contraction group. So a typical non-trivial relative Tits core $G^g_1$ in $G$ is of the form $(U, U')$, where $U$ is the unipotent radical of a parabolic subgroup $P$ and $U'$ is the unipotent radical of a parabolic subgroup opposite to $P$. By [4, Proposition 6.2(v)], the group $(U, U')$ does not depend on which parabolic $P$ we have, as long as it is proper (in other words, as long as $U$ is non-trivial). So in fact $G^g_1 = G^{\dagger}$ whenever $\text{con}(g) \neq \{1\}$; in the present example, $G^{\dagger} = \text{SL}_n(\mathbb{Q}_p)$.

## 4 The nub of a flat group

Let $G$ be a t.d.l.c. group and let $H$ be a group of automorphisms of $G$. We define the **lower nub** $\text{lnub}(H)$ to be the closure of the group generated by $\text{nub}(\alpha)$ as $\alpha$ ranges
over $H$. If $H$ is flat, the nub $\text{nub}(H)$ is the intersection of all compact open subgroups of $U$ that are tidy for $H$. Recall that the action of $H$ is said to be smooth if the tidy subgroups for the action form a base of neighbourhoods of the identity; in other words, a flat group $H$ is smooth if and only if $\text{nub}(H)$ is trivial.

It is clear from Theorem 2.7 that $\text{lnub}(H) \leq \text{nub}(H)$ whenever $H$ is flat. The following example illustrates that $\text{lnub}(H)$ need not be the same as $\text{nub}(H)$, even for uniscalar flat groups, and that the action of $H$ on $\text{nub}(H)$ does not in general have the dynamical properties observed in [23] in the cyclic case.

**Example 4.1.** Let $V = \mathbb{F}_p[[t]]$, regarded as a profinite vector space over $\mathbb{F}_p$, and let $W$ be a closed subspace of $V$. Let $H$ be the group of continuous $\mathbb{F}_p$-linear maps from $V$ to $W$ under pointwise addition, and define an action $\rho$ of $H$ on $G = V \oplus W$ by setting $\rho(h)(v, w) = (v, w + h(v))$. Then $\rho(H)$ is a subgroup of $\text{Aut}(G)$ (necessarily flat, since $G$ is compact), and $\text{nub}(\rho(H)) = W$, since for every subspace $V'$ of $V$ of finite codimension, there exists $h \in H$ such that $h(V') = W$. However, $\rho(H)$ acts trivially on $W$ and the group $G \rtimes \rho H$ is nilpotent, so there is no non-trivial subgroup $K$ of $G$ such that $N_{\rho(H)}(K)$ acts ergodically on $K$, and in particular $\text{nub}(\rho(h))$ is trivial for every $h \in H$.

Both the nub and the lower nub are well-behaved under closures.

**Lemma 4.2.** Let $G$ be a t.d.l.c. group and let $H$ be a subgroup of $G$. Suppose $\text{lnub}(H)$ is compact. Then $\text{lnub}(H) = \text{lnub}(\overline{H})$. If $H$ is flat, then $\overline{H}$ is flat and $\text{nub}(H) = \text{nub}(\overline{H})$.

**Proof.** Suppose $\text{lnub}(H)$ is compact and let $U$ be a compact open subgroup such that $\text{lnub}(H) \leq U$, so that $\text{nub}(h) \leq U$ for all $h \in H$. Let $a \in \overline{H}$. Then by [6, Theorem 1.5], for some open subgroup $V$ of $U$, then $\text{nub}(a)$ is $V$-conjugate to $\text{nub}(h)$ for all $h \in aV$. Since $aV \cap H$ is non-empty, we conclude that $\text{nub}(a) \leq U$. So in fact

$$\text{lnub}(H) \leq U \Rightarrow \text{lnub}(\overline{H}) \leq U$$

for all compact open subgroups $U$, and the converse implication also clearly holds. Hence $\text{lnub}(H) = \text{lnub}(\overline{H})$.

Now suppose $H$ is flat on $G$. Then any tidy subgroup $U$ for $H$ is also tidy for $\overline{H}$ by Theorem 2.10 and conversely. Hence $\text{nub}(H) = \text{nub}(\overline{H})$. \hfill \Box

The nub provides a simple criterion for when a flat group of automorphisms of $G$ has flat action on an open subgroup of $G$.

**Lemma 4.3.** Let $G$ be a t.d.l.c. group, let $H$ be a flat group of automorphisms of $G$, and let $K$ be an open $H$-invariant subgroup of $G$. Then $H$ is flat on $K$ if and only if $\text{nub}(H) \leq K$. If $\text{nub}(H) \leq K$, then $\text{nub}(H) = \text{nub}(H)$.
Proof. By Corollary 2.12, we have \( s_G(\alpha) = s_K(\alpha) \) for all \( \alpha \in H \), so the tidy subgroups for \( H \) on \( K \) are precisely the tidy subgroups \( U \) for \( H \) on \( G \) such that \( U \leq K \). If \( \text{nub}_G(H) \nsubseteq K \), then no such tidy subgroup can exist, so \( H \) is not flat on \( K \). So suppose that \( \text{nub}_G(H) \leq K \). Let \( V \) be a compact open subgroup of \( K \) such that \( \text{nub}_G(H) \leq V \). Let \( U \) be a tidy subgroup for \( H \) on \( G \). Then \( U \cap V \) has finite index in the compact group \( U \), and the collection of tidy subgroups for \( H \) on \( G \) is closed under finite intersections, so there exists \( W \leq V \) such that \( W \) is tidy for \( H \) on \( G \), and hence also for \( H \) on \( K \). Thus \( H \) is flat on \( K \); we see that given tidy subgroups \( A \) and \( B \) for \( H \) on \( G \) and on \( K \) respectively, then \( A \cap B \) is tidy for \( H \) both on \( G \) and on \( K \), so \( \text{nub}_G(H) = \text{nub}_K(H) \). \( \square \)

The following is an immediate consequence of Corollary 2.16 and Lemma 4.3.

Corollary 4.4. Let \( G \) be a t.d.l.c. group and let \( H \) be a flat group of automorphisms of \( G \) of finite rank. Let \( K \) be an open \( H \)-invariant subgroup of \( G \). Then \( \text{nub}_G(H) = \text{nub}_K(H) \).

4.1 Invariant uniscalar subgroups

We now prove a result on the effect of \( H \)-invariant subgroups on tidy subgroups for \( H \), which will allow us to establish some properties of \( N_G(\text{nub}(H)) \). This result is a variation on [22, Theorem 3.3].

Theorem 4.5. Let \( G \) be a t.d.l.c. group and let \( H \) be a flat group of automorphisms of \( G \). Let \( K \) be an \( H \)-invariant subgroup of \( G \) such that \( |K : N_K(U)| \) is finite, and let \( V = \bigcap_{k \in K} kUk^{-1} \).

(i) If \( U \) is tidy below for \( H \), then \( V \) is tidy below for \( H \).

(ii) If \( U \) is tidy for \( H \), then \( V \) is tidy for \( H \). If in addition \( K \) is compact, then \( VK \) is also tidy for \( H \).

Proof. It suffices to consider elements \( \alpha \in H \) individually, so fix \( \alpha \in H \). Suppose for the time being that \( U \) is tidy for \( H \).

Since \( |K : N_K(U)| \) is finite, \( V \) is the intersection of finitely many conjugates of \( U \), and hence \( V \) is a compact open subgroup of \( G \). Note also that the set \( \bigcup_{k \in K} kUk^{-1} \) is compact. Since \( K \) is \( \alpha \)-invariant, for all \( n \in \mathbb{Z} \) we have

\[
\alpha^n(V) = \bigcap_{k \in K} \alpha^n(kUk^{-1}) = \bigcap_{k \in K} k\alpha^n(U)k^{-1}.
\]

Define

\[
U_+ := \bigcap_{n \geq 0} \alpha^n(U), \quad U_- := \bigcap_{n \leq 0} \alpha^n(U) \quad \text{and} \quad U_{++} = \{ g \in G \mid \forall n \gg 0 : \alpha^{-n}(g) \in U \}.
\]
The subgroups $V_+$, $V_-$ and $V_{++}$ are defined similarly, with $V$ in place of $U$. We see that $V_+ = \bigcap_{k \in K} kU_+ k^{-1}$ and $V_- = \bigcap_{k \in K} kU_- k^{-1}$. Let $v \in V \cap U_+$ and let $k \in K$. Then for all $n \leq 0$, we have
\[ \alpha^n(kv k^{-1}) = \alpha^n(k) \alpha^n(v) \alpha^n(k^{-1}) \in \bigcup_{k \in K} kU_+ k^{-1}. \]

So the backward $\alpha$-orbit $\{ \alpha^n(kv k^{-1}) \mid n \leq 0 \}$ is confined to the relatively compact set $\bigcup_{k \in K} kU_+ k^{-1}$. Moreover, we have $kv k^{-1} \in U$ since $v \in V$. Thus by Lemma 2.3, $kv k^{-1} \in U_+$, so in fact $kv k^{-1} \in V \cap U_+$. Since $k \in K$ was arbitrary, we conclude that $V \cap U_+$ is normalized by $K$. In particular, it follows that
\[ V \cap U_+ = V \cap \bigcap_{k \in K} kU_+ k^{-1} = V_+. \]

Similarly, $V \cap U_- = V_-$. Suppose $U$ is tidy below for $\alpha$. Note that $V_{++} \leq U_{++}$, so
\[ V \cap V_{++} \leq V \cap U_{++} \leq V \cap U \cap U_{++} = V \cap U_+ = V_+, \]
and hence $V$ is tidy below for $\alpha$ by Lemma 2.4.

Given $v \in V$, there exist $u_+ \in U_+$ and $u_- \in U_-$ such that $v = u_+ u_-$. Then the sequence $(\alpha^{-n}(u_+))_{n \geq 0}$ is confined to the compact set $U_+$, so has an accumulation point $x$ say. For all $r \in \mathbb{Z}$, we see that $\alpha^{-r}(x)$ is an accumulation point of the sequence $(\alpha^{-n}(u_+))_{n \geq r}$, and hence also of $(\alpha^{-n}(u_+))_{n \geq 0}$ (which accounts for all but finitely many of the terms of $(\alpha^{-n}(u_+))_{n \geq r}$). Thus $\alpha^r(x) \in U$ for all $r \in \mathbb{Z}$ and in fact $\alpha^r(x) \in U_+ \cap U_-$ for all $r \in \mathbb{Z}$. Let $p \geq 0$ be such that $\alpha^{-p}(u_+) \in V x$, that is, $\alpha^{-p}(u_+) = v' x$ for some $v' \in V$. Then $v'$ is also an element of $U_+$, so in fact $v' \in V \cap U_+ = V_+$. Now $v = \alpha^p(v') \alpha^p(x) u_-;$ we see that $v \in \alpha^p(v') U_-$ (since $\alpha^p(x) \in U_-$) and also that $\alpha^p(v') = u_+ (\alpha^p(x))^{-1} \in U_+$ (since $\alpha^p(x) \in U_+$), and hence we could have chosen $u_+$ and $u_-$ so that $u_+ = \alpha^p(v')$ for some $p \geq 0$ and $v' \in V_+$. Let us assume that we have done so, and let $k \in K$. Then
\[ \alpha^{-p}(ku_+ k^{-1}) = \alpha^{-p}(k) v' \alpha^{-p}(k^{-1}); \]
since $\alpha^{-p}(k) \in K$ and $V_+ = V \cap U_+$ is normalized by $K$, we see that $\alpha^{-p}(ku_+ k^{-1}) \in V_+$, in other words $ku_+ k^{-1} = u'$ for some $u' \in \alpha^p(V_+)$, so $kv k^{-1} = u' ku_+ k^{-1}$. At the same time, $kv k^{-1} \in U$, so $kv k^{-1} = w_+ w_-$ for $w_+ \in U_+$ and $w_- \in U_-$. Consider the element $w_+^{-1} u'$. Since $u' ku_+ k^{-1} = w_+ w_-$, we have
\[ w_+^{-1} u' = w_- (ku_- k^{-1})^{-1} \in \bigcup_{k \in K} (U_- kU_- k^{-1}); \]
since $K$ is $\alpha$-invariant and $\alpha(U_-) \leq U_-$, it follows that the sequence $(\alpha^n(w_+^{-1} u'))_{n \geq 0}$ is confined to a compact set. In addition
\[ \alpha^{-p}(w_+^{-1} u') \in \alpha^{-p}(U_+ V_+) \subseteq U_+, \]
so $\alpha^{-p}(w_-^{+1}u')$ is an element of $U_+$ whose forward $\alpha$-orbit is bounded; it follows from Lemma 2.4 that $\alpha^{-p}(w_-^{+1}u') \in U_+ \cap U_-$, so $w_-^{+1}u' \in U_+ \cap U_-$. In particular, $u' = w_+(w_-^{+1}u') \in U_+$. Since $u' = ku_+k^{-1}$ and the choice of $k \in K$ was arbitrary, we conclude that $u_+ \in V \cap U_+$, so $u_+ \in V_+$. Hence $u_- \in V$ as well (since $u_- = u_+^{-1}v$), so $u_- \in V \cap U_-$, and hence $u_- \in V_-$. Thus we have expressed an arbitrary $v \in V$ as a product of an element of $V_+$ and an element of $V_-$, so $V$ is tidy above for $\alpha$, completing the proof that $V$ is tidy for $\alpha$.

Now suppose $K$ is compact. Since $K$ is $\alpha$-invariant, we see that

$$|\alpha(VK) : \alpha(VK) \cap VK| \leq |\alpha(V) : \alpha(V) \cap V|.$$ 

Since $V$ is tidy for $H$, the minimum value for $|\alpha(W) : \alpha(W) \cap W|$ (for $W$ a compact open subgroup of $G$) is already attained by $V$, so $VK$ is tidy for $H$ by Theorem 2.2. This completes the proof of (ii).

Finally, let us relax the assumption that $U$ is tidy, and instead assume that $U$ is tidy below for $\alpha$. Then there exists $U' = \bigcap_{i=0}^{n} \alpha^i(U)$ that is tidy above for $\alpha$, by Proposition 2.6 in fact $U'$ is also tidy below for $\alpha$ by Proposition 2.7. Each of the groups $\alpha^i(U)$ has only finitely many $K$-conjugates, because $K$ is $\alpha$-invariant, so $|K : \text{N}_K(U')|$ is finite. We now apply part (ii) to conclude that $V' = \bigcap_{k \in K} kU'k^{-1}$ is tidy for $\alpha$. Now $V \geq V'$ since $U \geq U'$, so $V$ is tidy below for $\alpha$ by Proposition 2.7, proving (i).

**Corollary 4.6.** Let $G$ be a t.d.l.c. group and let $H$ be a flat group of automorphisms of $G$.

(i) Let $L$ be the closure of the group generated by all $H$-invariant compact subgroups of $G$. Then $\text{nub}(H)$ is a normal subgroup of $L$.

(ii) Let $H'$ be a subgroup of $H$. Then $\text{nub}(H')$ is a normal subgroup of $\text{nub}(H)$.

(iii) Suppose $H$ is uniscalar. Then $\text{nub}(H)$ is normalized by every compact open subgroup of $G$ that is tidy for $H$. In particular, $\text{N}_G(\text{nub}(H))$ is open in $G$.

**Proof.** Theorem 4.5 implies that whenever $K$ is an $H$-invariant compact subgroup of $G$ and $U$ is a tidy subgroup for $H$, then $U$ contains a $K$-invariant tidy subgroup for $H$. Thus for any $H$-invariant compact subgroup $K$ of $G$, then $\text{nub}(H)$ can be expressed as an intersection of $K$-invariant compact open subgroups, so $K$ normalizes $\text{nub}(H)$. Hence the normalizer of $\text{nub}(H)$ contains a dense subgroup of $L$. Since $\text{nub}(H)$ is compact and $H$-invariant, in fact $\text{nub}(H) \leq L$ and $\text{N}_G(\text{nub}(H))$ is closed, so $L \leq \text{N}_G(\text{nub}(H))$, proving (i).

Given a subgroup $H'$ of $H$, then every compact open subgroup of $G$ that is tidy for $H$ is also tidy for $H'$; hence $\text{nub}(H') \leq \text{nub}(H)$. Since $\text{nub}(H)$ is an $H'$-invariant compact subgroup of $G$, it follows from part (i) that $\text{nub}(H')$ is normalized by $\text{nub}(H)$, proving (ii).
Suppose $H$ is uniscalar. Then a compact open subgroup $V$ of $G$ is tidy for $H$ if and only if $V$ is $H$-invariant; in this case we have $V \leq N_G(\text{nub}(H))$ by part (i), proving (iii).

The following lemma and corollary can be used to enlarge the uniscalar part of a flat subgroup.

**Lemma 4.7.** Let $G$ be a t.d.l.c. group and let $H$ and $K$ be flat subgroups of $G$ such that $H \leq N_G(K)$ and $K$ is uniscalar. Let $U$ be a compact open subgroup of $G$. Then $U$ is tidy for $HK$ if and only if $U$ is tidy for both $H$ and $K$. Moreover, if $HK$ is flat, then $s_G(hk) = s_G(h)$ for all $h \in H$ and $k \in K$.

**Proof.** If $U$ is tidy for $HK$, then clearly it is tidy for both $H$ and $K$. Conversely, suppose that $U$ is tidy for $H$ and for $K$, and let $h \in H$ and $k \in K$. Then $U$ is normalized by $K$ since $K$ is uniscalar, and for all $n \in \mathbb{Z}$ we have $(hk)^n \in h^nK = Kh^n$, since $K$ is normalized by $H$. Thus $h^nUh^{-n} = (hk)^nU(hk)^{-n}$ for all $n \in \mathbb{Z}$. By Theorem 2.11 it follows that $s_G(h) = s_G(hk)$, and hence by Theorem 2.2 $U$ is tidy for $hk$.

**Corollary 4.8.** Let $G$ be a t.d.l.c. group and let $H$ be a flat subgroup of $G$. Let $K$ be a compact $H$-invariant subgroup of $G$. Then $HK$ is flat, and for all $h \in H$ and $k \in K$ we have $s_G(hk) = s_G(h)$.

**Proof.** By Theorem 4.5, there is a compact open subgroup $V$ of $G$ that is normalized by $K$ and is tidy for $H$. The conclusion now follows from Lemma 4.7.

### 4.2 Tidy subgroups in quotients

The (lower) nub is not in general preserved under passing to quotients, because a compact open subgroup that is tidy (below) does not necessarily remain tidy below on passing to a quotient. Indeed, [21, Example 6.5] gives an example of the following situation: there is a t.d.l.c. group $G$, an automorphism $\alpha$ and a closed $\alpha$-invariant subgroup $K$ of $G$, such that $\alpha$ has arbitrarily small tidy subgroups (so $\text{nub}_G(\alpha)$ is trivial), and yet for every tidy subgroup $U$ for $\alpha$ on $G$, the group $UK/K$ is not tidy for $\alpha$ on $G/K$, because $UK/K$ fails to be tidy below (in other words, $UK/K$ does not contain $\text{nub}_{G/K}(\alpha)$).

However, under certain conditions there is good control over the tidy subgroups, and hence the nub, when passing to a quotient. In particular, it suffices for the scale to be preserved, as the following lemma shows.

**Lemma 4.9.** Let $G$ be a t.d.l.c. group, let $\alpha$ be an automorphism of $G$ and let $U$ be a compact open subgroup of $G$. Let $K$ be a closed subgroup of $G$, such that $U \leq N_G(K)$ and $\alpha(K) = K$.

(i) We have $s_{N_G(K)/K}(\alpha) \leq s_{N_G(K)}(\alpha)$. Indeed, $s_{N_G(K)/K}(\alpha)$ divides $s_{N_G(K)}(\alpha)$.
(ii) If $U$ is tidy above for $\alpha$, then $UK/K$ is tidy above for the action of $\alpha$ on $NG(K)/K$.

(iii) Suppose $s_{NG(K)/K}(\alpha) = s_{NG(K)}(\alpha)$ and that $U$ is tidy for $\alpha$ on $NG(K)$. Then $UK/K$ is tidy for $\alpha$ on $NG(K)/K$.

(iv) Suppose $K$ is compact. Then $s_G(\alpha) = s_{NG(K)}(\alpha) = s_{NG(K)/K}(\alpha)$;

moreover, given any $V/K \in NG(K)/K$ such that $V/K$ is tidy for $\alpha$ on $NG(K)/K$, then $V$ is tidy for the action of $\alpha$ on $G$.

**Proof.** (i) See [21, Proposition 4.7].

(ii) Let $U_+ = \bigcap_{n \geq 0} \alpha^n(U)$ and $U_- = \bigcap_{n \leq 0} \alpha^n(U)$. Suppose $U$ is tidy above for $\alpha$. Then $UK/K = (U_+K/K)(U_-K/K)$, and we have $\alpha(U_+K/K) \geq U_+K/K$ and $\alpha(U_-K/K) \leq U_-K/K$. So $UK/K$ is tidy above for $\alpha$.

(iii) It is clear that $|\alpha(UK/K) : \alpha(UK/K) \cap UK/K| \leq |\alpha(U) : \alpha(U) \cap U| = s_{NG(K)}(\alpha)$.

Since $s_{NG(K)}(\alpha)$ is already the minimum possible value for the index of $\alpha(UK/K) \cap UK/K$ in $\alpha(UK/K)$, it follows that $UK/K$ is tidy for $\alpha$ by Theorem 2.2.

(iv) Let $K \leq V \leq NG(K)$ such that $V/K$ is tidy for $\alpha$ on $NG(K)/K$. Then $V$ is a compact open subgroup of $G$. Since $K$ is $\alpha$-invariant, we have $|\alpha^n(V/K) : \alpha^n(V/K) \cap V/K| = |\alpha^n(V) : \alpha^n(V) \cap V|$, for all $n \in \mathbb{Z}$. Hence $s_G(\alpha) = s_{NG(K)}(\alpha) = s_{NG(K)/K}(\alpha)$, by Theorem 2.11, so $V$ is tidy for $\alpha$ on $G$. $\square$

We also note as a general point that intersections of compact subgroups of t.d.l.c. groups are well-behaved under continuous homomorphisms.

**Lemma 4.10.** Let $G$ be a t.d.l.c. group, let $C$ be a collection of compact subgroups of $G$ that is closed under finite intersections and let $\phi : G \to H$ be a continuous homomorphism to some t.d.l.c. group $H$. Then

$$\bigcap_{C \in C}(\phi(C)) = \phi(\bigcap_{C \in C} C).$$

**Proof.** For each $C \in C$, we see that $\phi(C)$ is profinite, so $\bigcap_{C \in C}(\phi(C))$ is profinite. Thus $\bigcap_{C \in C}(\phi(C))$ is expressible as an intersection of compact open subgroups of $H$. Writing $D = \bigcap_{C \in C} C$, it is clear that $\phi(D)$ is compact, hence closed, and that

$$\bigcap_{C \in C}(\phi(C)) \geq \phi(D).$$
If $K = L = N$, then $s$ is also tidy for $G$, and let $\alpha$ be a closed $G$-invariant subgroup of $H$. Consider the set $C = \{ C \setminus \phi^{-1}(U) \mid C \in C \}$. Then $C'$ consists of compact sets; it is closed under finite intersections; and the intersection $\bigcap_{E \in C'} E$ is empty, since $D \leq \phi^{-1}(U)$. Thus $\emptyset \in C'$, in other words, $C \leq \phi^{-1}(U)$ for some $C \in C$. But then $\phi(C) \leq U$, and hence $\bigcap_{C \in C'} (\phi(C)) \leq U$ as desired.

Combining the previous two lemmas, we obtain the following stability properties of tidy subgroups and the nub under passing to a quotient.

**Corollary 4.11.** Let $G$ be a t.d.l.c. group, let $H$ be a group of automorphisms of $G$ and let $K$ be a closed $H$-invariant subgroup of $G$. Suppose that $H$ is flat on $N_G(K)$, and let $L = N_G(K)/K$.

If $K$ is compact, then we have $s_L(\alpha) = s_G(\alpha)$ for all $\alpha \in H$, and also $\nub_L(H) = \nub_G(H)K/K$.

If $s_L(\alpha) = s_G(\alpha)$ for all $\alpha \in H$, then the following holds:

(i) The action of $H$ on $L$ is flat. Indeed, whenever $U$ is tidy for $H$ on $N_G(K)$, then $UK/K$ is tidy for $H$ on $L$.

(ii) We have $\nub_L(H) \leq \nub_G(H)K/K$.

In particular, if $\nub_G(H) \leq K$, then the action of $H$ on $L$ is smooth.

**Proof.** It is clear that $\nub_N(G)(H) = \nub_G(H)$, and by Corollary 2.12, we have $s_G(\alpha) = s_N(\alpha)$ for all $\alpha \in H$. (Corollary 2.12 also ensures that $H$ is flat on $G$.) So we may assume $G = N_G(K)$.

If $K$ is compact, then for all $\alpha \in H$ we have $s_L(\alpha) = s_G(\alpha)$ by Lemma 4.9(iv).

Now assume that $s_L(\alpha) = s_G(\alpha)$ for all $\alpha \in H$. Then by Lemma 4.9(iii), if $U$ is a compact open subgroup of $G$ that is tidy for every $\alpha \in H$, then $UK/K$ is also tidy for every $\alpha \in H$, proving (i). Thus $\nub_L(H) \leq UK/K$ for every tidy subgroup $U$ for $H$ on $G$, so by Lemma 1.10, we conclude that $\nub_L(H) \leq \nub_G(H)K/K$, proving (ii). If $K$ is compact, then by Lemma 4.9(iv), every tidy subgroup $V/K$ of $H$ on $G/K$ is the image of a tidy subgroup $V$ of $H$ on $G$, so $\nub_G(H) \leq V$ for all such $V$, and hence $\nub_G(H)K/K = \nub_L(H)$.

The hypotheses of Corollary 4.11 hold in particular in the case when $H$ is a flat group of automorphisms and $K = \nub(H_u)$. So we obtain an action of $H$ on a quotient $L = N_G(K)/K$ of an open subgroup of $G$, such that the action of $H$ on $L$ retains the important properties of the action of $H$ on $G$, such as flatness and the scale function,
but now the uniscal part of $H$ (which is the same subgroup, whether we define it with respect to $G$ or with respect to $L$) acts smoothly.

4.3 Flatness below

Say $H$ is flat below if there exists a compact open subgroup $U$ of $G$ such that $U$ is tidy below for all $\alpha \in H$; equivalently (given Propositions 2.6 and 2.7), for all $\alpha \in H$ there exists $V \leq U$ depending on $\alpha$ such that $V$ is tidy for $\alpha$. By Proposition 2.7, $H$ is flat below if and only if $\text{lnub}(H)$ is compact, and in this case $\text{lnub}(H)$ is the intersection of all compact open subgroups of $G$ that are tidy below for $H$.

We note that unlike the flat property, flatness below is inherited from cocompact normal subgroups, so in particular any virtually flat below group is flat below. Flatness below is also stable on restricting the action to a closed subgroup.

Lemma 4.12. Let $G$ be a t.d.l.c. group, let $H$ be a closed subgroup of $G$ and let $K$ be a closed cocompact normal subgroup of $H$. Then $H$ is flat below if and only if $K$ is flat below.

Proof. If $H$ is flat below, then clearly $K \leq H$ is as well, so we may assume $K$ is flat below. Let $U$ be a compact open subgroup of $G$ such that $\text{lnub}(K) \leq U$, and let $h \in H$. Then the sequence $(h^nK)_{n \geq 0}$ accumulates at the identity in the compact group $H/K$. Let $V$ be a compact open subgroup of $G$ that is tidy above for $h$ and let $W$ be a compact open subgroup of $G$ that is tidy above for $k$, such that $W \leq V \leq U$. Then there exist distinct integers $i, j \in \mathbb{Z}$ such that $\{h^i, h^j\} \subseteq VK$, so $h^{i-j} = rks$ for some $r, s \in W$ and $k \in K$. By [6] Lemma 4.3 and Corollary 4.4, there exists $v \in V$ such that $\text{nub}(r^{-1}h^{i-j}) = v\text{ub}(h^{i-j})v^{-1}$, and there exists $w \in W$ such that $\text{nub}(ks) = w\text{ub}(k)w^{-1}$. Thus $\text{nub}(h^{i-j})$ is $V$-conjugate to $\text{nub}(k) \leq U$, so $\text{nub}(h^{i-j}) \leq U$. Moreover $\text{nub}(h^{i-j}) = \text{nub}(h)$ by Lemma 2.4, so $\text{nub}(h) \leq U$. Since $h \in H$ was arbitrary, we conclude that $\text{lnub}(H) \leq U$, so $H$ is flat below.

Lemma 4.13. Let $G$ be a t.d.l.c. group, let $H$ be a subgroup of $G$ and let $K$ be a closed $H$-invariant subgroup of $G$. Then $\text{lnub}_K(H) \leq \text{lnub}_G(H)$, and if $K$ is open then $\text{lnub}_K(H) = \text{lnub}_G(H)$. In particular, if $H$ is flat below on $G$, then it is flat below on $K$.

Proof. Let $h \in H$. By [21] Lemma 4.1, any tidy subgroup for $h$ on $G$ contains a tidy subgroup for $h$ on $K$. Thus $\text{nub}_K(h) \leq \text{nub}_G(h)$. If $K$ is open, then $\text{nub}_G(h) \cap K$ is an open $h$-invariant subgroup of $\text{nub}_G(h)$, so by Theorem 2.3 we have $\text{nub}_K(h) \leq K$. Since $\text{nub}_K(h)$ is the largest $h$-invariant subgroup of $K$ on which $h$ acts ergodically, we must have $\text{nub}_K(h) = \text{nub}_G(h)$. The remaining conclusions are clear.
We saw in Example 2.19 that a finitely generated non-flat group $H$ can potentially be virtually flat (and hence also flat below); indeed, in this example, $H$ even has non-uniscalar derived group. As we shall see, the latter property is no coincidence: when considering a finitely generated group of automorphisms $H$, the difference between ‘flat below’ and ‘flat’ comes down to the existence or otherwise of an $[H, H]$-invariant compact open subgroup.

Here are some basic observations about the role of $[H, H]$-invariant compact open subgroups in the tidy theory of $H$.

**Lemma 4.14.** Let $G$ be a t.d.l.c. group, let $H$ be a group of automorphisms of $G$ and let $L$ be a subgroup of $H$ such that $[H, H] \leq L$. Let $U$ be a compact open subgroup of $G$, and suppose that $U$ is $L$-invariant.

(i) Let $X$ be a subset of $H$. Then $\bigcap_{\alpha \in X} \alpha(U)$ is $L$-invariant.

(ii) Let $\alpha \in H$ and let $\beta \in L\alpha$. Then $U$ is tidy above for $\beta$ if and only if it is tidy above for $\alpha$, and $U$ is tidy below for $\beta$ if and only if it is tidy below for $\alpha$.

(iii) Let $\alpha, \beta \in H$ and suppose $U$ is tidy (above, below) for $\alpha$. Then $\beta(U)$ is tidy (above, below) for $\alpha$.

**Proof.** (i) We see that $L$ is normal in $H$, since $[H, H] \leq L$. Hence the set of $L$-invariant compact open subgroups of $G$ is preserved by $H$. Consequently, if $U$ is $L$-invariant, then so is $\bigcap_{\alpha \in X} \alpha(U)$.

(ii) Observe that the sets $U_+ := \bigcap_{n \geq 0} \beta^n(U)$ and $U_- := \bigcap_{n \leq 0} \beta^n(U)$ do not depend on the choice of $\beta$ inside $L\alpha$, because $L$ leaves $\alpha^n(U)$ invariant for all $n \in \mathbb{Z}$. Thus the validity of the equation $U = U_+ U_-$ does not depend on the choice of $\beta$ inside $L\alpha$. Similarly, the set

$$U_{++} := \bigcup_{i \geq 0} \bigcap_{n \geq i} \beta^n(U),$$

does not depend on the choice of $\beta$ inside $L\alpha$. Consequently, $U$ is tidy (above, below) for $\beta$ if and only if it is tidy (above, below) for $\alpha$.

(iii) We see that $\beta(U)$ is tidy (above, below) for $\beta \alpha \beta^{-1} \in L\alpha$. Hence $\beta(U)$ is tidy (above, below) for $\alpha$ by part (ii).

**Definition 4.15.** Let $G$ be a t.d.l.c. group and let $H$ be a group of automorphisms of $G$. A *tidying set* for $H$ is a subset $X$ of $H$ with the following property:

(*) Let $U$ be a compact open subgroup of $G$, and suppose $\alpha(U)$ is tidy for $\beta$, for all $\alpha \in H$ and $\beta \in X$. Then $U$ is tidy for $H$.

Given a finitely generated group $H$ of automorphisms, not all finite generating sets for $H$ are tidying sets (see [22, Example 3.5]). However, the following is effectively established in the proof of [22, Theorem 5.5].

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Theorem 4.16 (See [22], §5). Let $G$ be a t.d.l.c. group and let $H$ be a finitely generated group of automorphisms of $G$. Then there exists a finite subset $X$ of $H$ that is a tidying set for $H$ on $G$.

We thus obtain a ‘tidying above procedure’ for actions of finitely generated groups under certain circumstances.

Lemma 4.17. Let $G$ be a t.d.l.c. group, let $H$ be a finitely generated group of automorphisms of $G$, and let $X$ be a finite tidying set for $H$ on $G$. Let $U$ be a compact open subgroup of $G$ such that $U$ is tidy below for $\alpha$, for all $\alpha \in X$, and such that $U$ has only finitely many conjugates under the action of $[H,H]$. Then there is a finite intersection of $H$-conjugates of $U$ that is tidy for $H$ on $G$.

Proof. By Theorem 4.5, the intersection of all $[H,H]$-conjugates of $U$ is tidy below for all $\alpha \in X$ (since $[H,H]$ is invariant under the action of each $\alpha \in X$). So we may assume that $U$ is $[H,H]$-invariant. By Lemma 4.14 if $U$ is $[H,H]$-invariant and tidy below for some $\alpha \in H$, then $U$ is also tidy below for every $H$-conjugate of $\alpha$, and so any $H$-conjugate of $U$ is tidy below for $\alpha$. It is clear from Proposition 2.7 that the property of being tidy below for $\alpha$ is closed under finite intersections; hence any finite intersection of $H$-conjugates of $U$ is tidy below for $\alpha$.

Fix a compact open subgroup $U$ of $G$ such that $U$ is tidy below for $\alpha$, for all $\alpha \in X$, and such that $U$ is $[H,H]$-invariant. Let $X = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. We define a sequence of subgroups $U(i)$ as follows: $U(0) = U$, and thereafter $U(i) = \bigcap_{|\alpha| \leq k_i} \alpha^\alpha(U(i-1))$, where $k_i$ is large enough so that $U(i)$ is tidy above for $\alpha_i$ (such a $k_i$ exists by Lemma 2.6). Then $U(i)$ is tidy below for $\alpha_i$ by Lemma 4.14(iii), since it is a finite intersection of $H$-conjugates of $U$, so in fact $U(i)$ is tidy for $\alpha_i$. Given $j > i$, then $U(j)$ is a finite intersection of $H$-conjugates of $U(i)$; each $H$-conjugate of $U(i)$ is tidy for $\alpha_i$ by Lemma 4.14(iii). Hence $U(j)$ is tidy for $\alpha_i$. In particular, $U(m)$ is tidy for every element of $X$. By Lemma 4.14(iii), for all $\gamma \in M$, the conjugate $\gamma(U(m))$ is tidy for every element of $X$. Hence $V = U(m)$ is tidy for $H$.

4.4 A decomposition theorem for the nub

We can now prove Theorem 1.4.

Proof of Theorem 1.4. By induction on the derived length of $H/L$, we have

$$\nub(H) = \nub(H^*) \nub(H) = \nub(H^*) \prod_{\alpha \in X'} \nub(\alpha),$$

where $X'$ is some finite subset of $H$ and $H^*/L$ is the last non-trivial term in the derived series of $H/L$; note that $H^*/L$ is finitely generated. Hence we may assume that $H/L$ is abelian.
In light of Lemma 4.9 and Corollary 4.11 we may assume that \( nub(L) \) is trivial, in other words, \( L \) is smooth.

Let \( M \) be a finitely generated subgroup of \( H \) such that \( H = LM \) and let \( X = \{ \alpha_1, \ldots, \alpha_n \} \) be a finite tidying set for \( H \), as given by Theorem 4.16. We wish to show that any compact open subgroup \( U \) containing \( \nub(\alpha) \) for all \( \alpha \in X \) is tidy below for \( H \).

Let \( U \) be a compact open subgroup such that \( \nub(\alpha) \leq U \) for all \( \alpha \in X \). Let \( U'' = U''nub(H) \), where \( U'' \) is the intersection of all \( L \)-conjugates of \( U \). Then \( U'' \) is a compact open \( L \)-invariant subgroup of \( G \) and we have \( \nub(\alpha) \leq U \cap U'' \) for all \( \alpha \in X \). So we may assume \( U \leq U'' \). It then follows that \( U \) has only finitely many \( L \)-conjugates (since there are only finitely many subgroups between \( U' \) and \( U'' \)). Using Theorem 4.5 we can therefore replace \( U \) with an \( L \)-invariant subgroup \( U'' \) of \( U \) that still contains \( \nub(\alpha) \) for all \( \alpha \in X \). (The topological structure of \( L \) is irrelevant to this application of Theorem 4.5, so there is no harm in treating \( L \) as a subgroup of \( G \) by replacing \( G \) with \( G \rtimes L \).)

Given Lemma 4.17 there is a finite intersection \( V \) of \( M \)-conjugates of \( U'' \) that is tidy for \( M \) on \( G \). Notice that \( V \) inherits \( L \)-invariance from \( U \), since \( M \) normalizes \( L \). We conclude by Lemma 4.7 that \( V \) is tidy for \( H \). Hence \( U \) is tidy below as desired.

By Corollary 4.6 the groups \( \nub(L) \) and \( \nub(\alpha) \) for \( \alpha \in H \) are normal subgroups of \( \nub(H) \). Thus the product

\[
K = \nub(L)\nub(\alpha_1)\nub(\alpha_2) \ldots \nub(\alpha_n)
\]

is compact subgroup of \( G \) that does not depend on the ordering of the factors. We have seen that every compact open subgroup of \( G \) that contains \( K \) also contains \( \nub(H) \), and clearly

\[
K \subseteq \nub(L)\nub(H) \subseteq \nub(H).
\]

Since in a t.d.l.c. group, every compact subgroup is expressible as the intersection of the compact open subgroups that contain it, we conclude that \( K = \nub(H) \), and in fact

\[
K = \nub(L)\nub(H) = \nub(H).
\]

We highlight the following special cases of Theorem 1.4. (Here the product should be understood as a permutably product of subsets of \( G \), which is not necessarily a direct product.)

**Corollary 4.18.** Let \( G \) be a t.d.l.c. group and let \( H \) be a flat group of automorphisms of \( G \).

(i) Suppose that \( H/H_\alpha \) is finitely generated. Then

\[
\nub(H) = \nub(H_\alpha) \prod_{\alpha \in X} \nub(\alpha)
\]

for some finite subset \( X \) of \( H \).
(ii) Suppose that $H/[H,H]$ is finitely generated. Then
\[ \text{nub}(H) = \text{nub}([H,H]) \prod_{\alpha \in X} \text{nub}(\alpha) \]
for some finite subset $X$ of $H$.

(iii) Suppose that $H$ has a smooth normal subgroup $K$, such that $H/K$ is polycyclic. Then
\[ \text{nub}(H) = \prod_{\alpha \in X} \text{nub}(\alpha) \]
for some finite subset $X$ of $H$.

5 Residuals

5.1 Generalities

**Definition 5.1.** Let $G$ be a topological group. Define the set $\text{prox}_G(H)$ of elements of $G$ that are **proximal** to the identity under the action of $H$, that is, $x \in \text{prox}_G(H)$ if every neighbourhood of the identity in $G$ contains $\alpha(x)$ for some $\alpha \in H$. In other words, $\text{prox}_G(H)$ is the intersection of all $H$-invariant identity neighbourhoods in $G$. More generally, given an $H$-invariant closed subgroup $K$ of $G$ (so that $H$ has an induced action on the left coset space $G/K$), we define $\text{prox}_{G/K}(H)$ to be the set of elements of the left coset space $G/K$ that are proximal to $K$ under the action of $H$. The action of $H$ is **distal** if $\text{prox}_G(H) = \{1\}$ (or $\text{prox}_{G/K}(H) = \{K\}$). We also define $\text{prox}(G) := \text{prox}_G(\text{Inn}(G))$ and say $G$ is distal if the action of $\text{Inn}(G)$ is distal.

We observe that distal action is a residual property, and also a property closed under extensions.

**Lemma 5.2.** Let $G$ be a topological group and let $H$ be a group of automorphisms of $G$.

(i) Let $K$ be a collection of closed $H$-invariant subgroups of $G$ such that $H$ is distal on $G/K$ for all $K \in K$. Then $H$ is distal on $G/L$, where $L = \bigcap_{K \in K} K$.

(ii) Let $(K_\alpha)_{\alpha<\lambda}$ be a well-ordered descending chain of closed $H$-invariant subgroups of $G$, such that $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$ whenever $\alpha$ is a non-zero limit ordinal. Suppose that $H$ is distal on the coset space $K_\alpha/K_{\alpha+1}$ for all $\alpha$ such that $\alpha + 1 < \lambda$. Then $H$ is distal on $K_\alpha/K_{\alpha}$ for all $\alpha < \lambda$.

**Proof.** (i) Let $x \in G$ and suppose $xL \in \text{prox}_{G/L}(H)$. Then there is a convergent net $(\alpha_i(x)L)_{i \in I}$ in $G/L$ with limit $L$. Given $K \in K$, then $(\alpha_i(x)K)_{i \in I}$ converges to $K$, owing
to the natural continuous quotient map $G/L \to G/K$. Since the action of $H$ on $G/K$ is distal, we must have $x \in K$. Since $K \in \mathcal{K}$ was arbitrary, we in fact have $x \in L$. Thus the action of $H$ on $G/L$ is distal.

(ii) Suppose there is some $\alpha < \lambda$ such that $H$ is not distal on $K_0/K_\alpha$; let $\alpha$ be the least ordinal for which this occurs. Clearly $\alpha > 0$.

If $\alpha$ is a successor ordinal, say $\alpha = \beta + 1$, then $H$ is distal on $K_0/K_\beta$, ensuring that $\text{prox}_{K_0/K_\alpha}(H) \subseteq K_\beta/K_\alpha$, and $H$ is distal on $K_\beta/K_\alpha$, so $\text{prox}_{K_0/K_\alpha}(H)$ is trivial. Thus $H$ is distal on $K_0/K_\alpha$.

If $\alpha$ is a limit ordinal, then $H$ is distal on $K_0/K_\beta$ for all $\beta < \alpha$, so $H$ is distal on $K_0/K_\alpha$ by part (i).

In either case, we obtain a contradiction to the assumption that $H$ was not distal on $K_0/K_\alpha$ for some $\alpha < \lambda$.  

We define the **distal residual** $\text{Dist}_G(H)$ to be the intersection of all $H$-invariant closed subgroups $K$ of $G$ such that $\text{prox}_{K_0/K}(H)$ is trivial, and $\text{Dist}(G) := \text{Dist}_G(\text{Inn}(G))$. Equivalently in light of Lemma 5.2(i), the subgroup $D = \text{Dist}_G(H)$ is the smallest $H$-invariant closed subgroup of $G$ such that the conjugation action of $H$ on the coset space $G/D$ is distal. By Lemma 5.2(ii), in fact $D = \text{Dist}_D(H)$. We can also define the distal residual of a coset space: given an $H$-invariant subgroup $K$ of $G$, define $\text{Dist}_{G/K}(H) := D/K$, where $D$ is the smallest closed $H$-invariant subgroup of $G$ such that $K \leq D$ and the conjugation action of $H$ on $G/D$ is distal.

**Definition 5.3.** Let $G$ be a topological group. The **discrete residual** $\text{Res}(G)$ of $G$ is the intersection of all open normal subgroups of $G$, and $G$ is **residually discrete** if $\text{Res}(G) = \{1\}$. More generally, given a subgroup $H$ of $G$, one can define $\text{Res}_G(H)$, the **discrete residual** of $H$ on $G$, to be the intersection of all open $H$-invariant subgroups of $G$, and the action of $H$ on $G$ is **residually discrete** if $\text{Res}_G(H) = \{1\}$.

Clearly $\text{Dist}_G(H) \subseteq \text{Res}_G(H)$, and Example 4.1 shows that this inequality can be strict.

Define $\text{Res}^0_G(H)$ as $\alpha$ ranges over the ordinals as follows: $\text{Res}^0_G(H) := G$; $\text{Res}^{\alpha+1}_G(H) := \text{Res}_G(\text{Res}_G^\alpha(H))$; and if $\alpha$ is a non-zero limit ordinal, $\text{Res}_G^\alpha(H) := \bigcap_{\beta < \alpha} \text{Res}_G^\beta(H)$. Then the groups $\text{Res}_G^\alpha(H)$ form a descending chain of closed $H$-invariant subgroups of $G$, eventually terminating at some group $\text{Res}_G^\infty(H)$ that has no proper $H$-invariant open subgroups. Indeed, $\text{Res}_G^\infty(H)$ can be characterized as the unique largest closed $H$-invariant subgroup $K$ of $G$ such that $K$ has no proper open $H$-invariant subgroups.

**Lemma 5.4.** Let $G$ be a topological group and let $H$ be a group of automorphisms of $G$. Let $K$ be a closed $H$-invariant subgroup of $G$, and suppose that $K$ has no proper $H$-invariant subgroups. Then $K \leq \text{Res}_G^\infty(H)$.  

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Proof. It is enough to show that $K \leq \text{Res}_G^\alpha(H)$ for every ordinal $\alpha$. We proceed by induction on $\alpha$.

The case $\alpha = 0$ is immediate.

If $\alpha$ is a non-zero limit ordinal, then $K \leq \text{Res}_G^{\beta}(H)$ for all $\beta < \alpha$ by the inductive hypothesis, so $K \leq \text{Res}_G^\alpha(H)$.

If $\alpha = \beta + 1$ for some ordinal $\beta$, then $K \leq L = \text{Res}_G^{\beta}(H)$ by the inductive hypothesis. Given an open $H$-invariant subgroup $U$ of $L$, then $K \cap U$ is an open $H$-invariant subgroup of $K$, so $K \cap U = K$, that is, $K \leq U$. Thus $K \leq \text{Res}_L(H) = \text{Res}_G^\alpha(H)$. \qed

By Lemma 5.2(ii), the action of $H$ on $G/\text{Res}_G^\infty(H)$ is distal, so we have Dist$_G(H) \leq \text{Res}_G^\infty(H)$. It is also the case that Dist$_G(H)$ has no proper open $H$-invariant subgroups; indeed, Dist$_G(H)$ does not have any non-trivial coset spaces on which $H$ acts distally.

Let us now assume that $G$ is a t.d.l.c. group. Since contraction groups are well-behaved with respect to coset spaces by Theorem 3.1 we have $G^\dagger_H \leq \text{Dist}_G(H)$. So we have an ascending chain
\[
G^\dagger_H \subseteq \text{Dist}_G(H) \subseteq \text{Res}_G^\infty(H) \subseteq \text{Res}_G(H)
\]
of closed $H$-invariant subgroups of $G$.

In the case of a group of automorphisms of a profinite group, the (non-)existence of invariant open subgroups is closely related to whether or not the action is ergodic. The next two results are based on results by Jaworski (III) for actions on compact groups in general.

**Proposition 5.5** (See [11] Proposition 2.1). Let $G$ be a profinite group and let $H$ be a group of automorphisms of $G$. Then the following are equivalent:

(i) $H$ does not act ergodically.

(ii) There exists a proper open normal $H$-invariant subgroup $N$ of $G$.

(iii) There exists a compact $H$-invariant identity neighbourhood $U$ in $G$ such that $U^2 \neq G$.

**Proposition 5.6.** Let $G$ be a t.d.l.c. group and let $H$ be a group of automorphisms of $G$. Suppose that $\text{Res}_G^\infty(H)$ is compact. Then the following holds.

(i) $\text{Res}_G^\infty(H)$ is the largest closed subgroup of $G$ on which $H$ acts ergodically.

(ii) $\text{Res}_G^\infty(H)$ is normalized by every compact $H$-invariant subgroup of $G$.

(iii) If $\text{Res}_G^\infty(H)$ is metrizable, then $\text{Res}_G^\infty(H) = \text{Dist}_G(H)$. 37
Proof. Suppose that \( K \) is a closed \( H \)-invariant subgroup on which \( H \) acts ergodically. Then \( K \) cannot have a proper open \( H \)-invariant subgroup, so \( \text{Res}_K(H) = K \), and hence \( K \leq \text{Res}_G^\infty(H) \); in particular, \( K \) is compact. Observe that if \( L \) is an \( H \)-invariant subgroup of \( G \) and \( \text{Res}_G^\infty(H) \leq L \), then \( \text{Res}_G^\infty(H) = \text{Res}_L^\infty(H) \). Part (i) follows from Proposition 5.5 and part (ii) follows from [11, Theorem 2.6].

By Lemma 5.2(ii), we have \( \text{Dist}_G(H) \leq \text{Res}_G^\infty(H) \). Since \( H \) acts ergodically on \( \text{Res}_G^\infty(H) \), the action of \( H \) on the coset space \( \text{Res}_G^\infty(H)/\text{Dist}_G(H) \) is also ergodic. However, the action of \( H \) on \( \text{Res}_G^\infty(H)/\text{Dist}_G(H) \) is also distal.

Suppose \( R = \text{Res}_G^\infty(H) \) is metrizable, in other words, \( R \) has only countably many open subgroups. Then \( \text{prox}_{R}(H) \) can be expressed as the intersection of countably many open \( H \)-invariant subsets of \( R \). Since \( H \) acts ergodically on \( R \), every open \( H \)-invariant subset of \( R \) is dense, so by the Baire category theorem, \( \text{prox}_{R}(H) \) is dense in \( R \). This implies that \( \text{prox}_{R/D}(H) \) is dense for any closed \( H \)-invariant subgroup \( D \) of \( R \), and hence \( \text{Dist}_G(H) = \text{Res}_G^\infty(H) \).

5.2 The Mautner phenomenon

The Mautner phenomenon is a collection of related results of the following form: given a suitable action of a group \( G \) on a set \( X \), and a point \( x \in X \) that is fixed by some subgroup \( H \leq G \), then the stabilizer of \( x \) in \( G \) necessarily contains not just \( H \), but a much larger subgroup (often \( G \) itself) that depends on the dynamics of the conjugation action of \( H \) on \( G \). The concept originates in the ergodic theory of flows on manifolds, and also plays an important role in the representation theory of locally compact groups; see for instance [13], [15] and [19]. We can define the phenomenon for topological groups in general terms as follows.

Definition 5.7. Let \( G \) be a group acting on a Hausdorff topological space \( X \), and let \( x \in X \) be a fixed point of the action. Then \( x \in X \) is an **isolated point** of the action of \( G \) if for all \( y \in X \setminus \{x\} \), the closure of the \( G \)-orbit of \( y \) does not contain \( x \); in other words, no orbit of the action of \( G \) on \( X \) accumulates at \( x \).

Let \( G \) be a topological group and let \( H \leq G \). We say that \( H \) exhibits the Mautner phenomenon in \( G \) if the following condition holds:

(*) Let \( X \) be a Hausdorff topological space admitting a \( G \)-action by homeomorphisms such that the map \( G \to X; g \mapsto gx \) is continuous for all \( x \in X \). Suppose \( x \in X \) is an isolated point of the action of \( H \). Then \( x \) is a fixed point of the action of \( G \).

We can extract some more familiar versions of the Mautner phenomenon from this definition.

Proposition 5.8. Let \( G \) be a topological group and let \( H \) be a subgroup of \( G \) that exhibits the Mautner phenomenon. Then the following holds.
(i) Let $X$ be a metrizable space on which $G$ acts continuously, and suppose $H$ acts distally with respect to some metric $d$ for $X$ (that is, $\inf \{d(hx, hy) \mid h \in H\} > 0$ for any pair $(x, y)$ of distinct points). Then every point fixed by $H$ is fixed by $G$.

(ii) Let $X$ be a topological space admitting a Borel probability measure, such that $G$ acts continuously and ergodically by measure-preserving maps. Then the action of $H$ on $X$ is ergodic.

Proof. (i) Let $x \in X$ be a fixed point of $H$ and let $y \in X \setminus \{x\}$. Then since $(x, y)$ is not a proximal pair for $H$, the $H$-orbit of $y$ does not accumulate at $x$. Thus $x$ is an isolated fixed point of $H$, so $x$ is fixed by $G$.

(ii) Assume for a contradiction that there exists a measurable subset $Y$ of $X$ such that $0 < \mu(Y) < 1$ and $\mu(hY \setminus Y) = 0$ for all $h \in H$, and consider the space $L^\infty(X)$ of essentially bounded functions from $X$ to $\mathbb{C}$ modulo essentially zero functions. Then the indicator function $\phi_Y$ of $Y$ is (a representative of) a non-zero element of $L^\infty(X)$ that is fixed by $H$. Now $L^\infty(X)$ is a normed vector space, so in particular a metric space, on which $G$ acts continuously by isometries, so $\phi_Y$ is fixed by $G$ by part (i). But then $\mu(gY \setminus Y) = 0$ for all $g \in G$, so the action of $G$ on $X$ is not ergodic, a contradiction.

A natural criterion for the Mautner phenomenon can be expressed in terms of a distal residual. (Note that if $H \leq D \leq G$, then the translation action of $H$ on $G/D$ is the same as the conjugation action of $H$ on $G/D$.)

**Proposition 5.9.** Let $G$ be a topological group and let $H$ be a subgroup of $G$. Let $D/H = \text{Dist}_{G/H}(H)$. Then $H$ exhibits the Mautner phenomenon in $D$. Moreover, $H$ exhibits the Mautner phenomenon in $G$ if and only if $D = G$.

Proof. Suppose $H$ exhibits the Mautner phenomenon in $G$, and let $R$ be a closed subgroup of $G$ such that $H \leq R \leq G$ and $H$ acts distally on $G/R$ by translation. Then the map $G \to G/R; g \mapsto gxR$ is continuous for all $x \in G$, and $R$ is an isolated point of the action of $H$ on $G/R$. Hence $R$ is a fixed point of the action of $G$ on $G/R$ by translation, in other words $R = G$, and hence $\text{Dist}_{G/H}(H) = G/H$.

Let $D/H = \text{Dist}_{G/H}(H)$. It remains to show that $H$ exhibits the Mautner phenomenon in $D$.

Let $X$ be a Hausdorff topological space admitting a $D$-action by homeomorphisms such that the map $D \to X; g \mapsto gx$ is continuous for all $x \in X$. Suppose $x \in X$ is an isolated point of the action of $H$. Then the stabilizer $D_x$ is a closed subgroup of $D$ such that $H \leq D_x$.

Suppose $D_x < D$. Then by hypothesis, $H$ does not act distally on $G/D_x$. Since $H$ acts distally on $G/D$, it follows that $H$ does not act distally on $D/D_x$, that is, there exists $g \in D \setminus D_x$ such that the set $\{hgD_x \mid h \in H\}$ accumulates at $D_x$. Then there
are nets \((h_i)_{i \in I}\) and \((k_i)_{i \in I}\) in \(D_x\) such that \((h_igk_i)_{i \in I}\) converges to the identity, and thus \((h_igk_i)_{i \in I}\) converges to \(x\). Since \(x\) is fixed by \(D_x\), in fact \((h_ig)_{i \in I}\) converges to \(x\), where \(y = gx\). But \(x\) is an isolated point of \(H\), so we must have \(y = x\). Thus \(g \in D_x\), a contradiction. Hence our assumption that \(D_x < D\) was false, in other words, \(x\) is fixed by \(D\), proving that \(H\) exhibits the Mautner phenomenon in \(D\).

**Corollary 5.10.** Let \(G\) be a t.d.l.c. group and let \(H\) be a subgroup of \(G\). Then \(H\) exhibits the Mautner phenomenon in \(\text{Dist}(\overline{H})\) and in \(G_H\).

Here is a characterization of the Mautner phenomenon for metrizable t.d.l.c. groups in the case when \(H\) has finite covolume in \(G\).

**Proposition 5.11.** Let \(G\) be a t.d.l.c. group and let \(H\) be a subgroup of \(G\) such that \(\overline{H}\) has finite covolume in \(G\). A necessary condition for \(H\) to exhibit the Mautner phenomenon in \(G\) is that

\[
G = \overline{H^UH}
\]

for any open subgroup \(U\) of \(G\). In other words, for all \(g \in G\) there exist nets \((h_i)_{i \in I}\), \((h'_i)_{i \in I}\) and \((u_i)_{i \in I}\) such that: \(h_i, h'_i \in H\), \(u_i \to 1\) and \(h_iu_ih'_i \to g\).

If \(G\) is metrizable, this condition is also sufficient.

**Proof.** Without loss of generality, we may assume \(H\) is closed in \(G\).

Suppose that \(H\) exhibits the Mautner phenomenon in \(G\). We observe that the coset space \(G/H\) is a Borel probability space, on which \(G\) acts continuously and ergodically (indeed, transitively) by measure-preserving maps. Thus by Proposition 5.9, \(H\) acts ergodically on \(G/H\). In particular, let \(U\) be an open subgroup of \(G\). Then \(UH/H\) is a subspace of non-zero measure, so the \(H\)-invariant subspace \(HUH/H\) has a complement of zero measure. Let \(V\) be a compact open subgroup of \(G\), and suppose there is \(g \in G\) such that \(gV \cap HUH = \emptyset\). Then \(gVH \cap HUH = \emptyset\), since \(HUH\) is invariant under right translation by \(H\). In other words, \(gVH/H\) is disjoint from \(HUH/H\) in the coset space \(G/H\). But then \(gVH/H\) has zero measure, which is absurd. This contradiction implies that \(HUH\) is dense in \(G\). The conclusion about convergent nets is clear.

Conversely, suppose that \(H\) does not exhibit the Mautner phenomenon in \(G\). Then by Proposition 5.9 there exists a proper closed subgroup \(D\) of \(G\) such that \(H \leq D\) and \(H\) acts distally on \(G/D\). If \(G\) is metrizable, then \(G/D\) is a locally compact Hausdorff metrizable space, so the fact that no \(H\)-orbit accumulates at \(D\) ensures that there is a proper \(H\)-invariant neighbourhood \(K/D\) of \(D\) in \(G/D\) that is not dense in \(G/D\): for instance, if we specify a metric on \(G/D\), and \(B_n\) is the open ball of radius \(1/n\) around \(D\) with respect to this metric, then by the Baire category theorem, there exists \(n \in \mathbb{N}\) such that the set \(\bigcup_{h \in H} hB_n\) is not dense. Now \(K\) is a neighbourhood of the identity in \(G\), so there is a compact open subgroup \(U\) of \(G\) such that \(U \subseteq K\). By construction, \(K = KD\), so certainly \(K = KH\), and since \(K/D\) is \(H\)-invariant, in fact \(K = HKH\). Hence \(HUH \subseteq K\); in particular, \(HUH\) is not dense in \(G\). \(\square\)
We record the following consequence, which will be used later.

**Corollary 5.12.** Let $G$ be a t.d.l.c. group, let $H$ be a subgroup of $G$ such that $H$ has finite covolume and let $U$ be an open subgroup of $G^1_H$. Then

$$G^1_H = HUH.$$  

### 5.3 Eigenfactors

We recall some of the theory of eigenfactors as set out in [22].

Let $H$ be a flat subgroup of the t.d.l.c. group $G$, and let $U$ be a compact open subgroup of $G$ that is tidy for $U$. A $(U)$-eigenfactor for $H$ is a closed subgroup $K$ of $U$ with the following properties:

(a) $K$ is commensurated by $H$.

(b) The set $\{hKh^{-1} \mid h \in H\}$ is totally ordered by inclusion.

(c) $K$ is the intersection of the set $\{hUh^{-1} \mid h \in H, \alpha(K) \geq K\}$.

**Theorem 5.13** ([22] Lemma 6.2 and Theorem 6.8). Let $G$ be a t.d.l.c. group, let $H$ be a flat subgroup of $G$ such that $H/H_\alpha$ is finitely generated and let $U$ be a compact open subgroup of $G$ that is tidy for $H$. Then there are only finitely many $U$-eigenfactors for $H$, and $U$ can be expressed as a product of the distinct $U$-eigenfactors (in some order).

Let $U_0 = \bigcap_{h \in H} hUh^{-1}$. Then $U_0$ is a $U$-eigenfactor of $H$; moreover it is the only $U$-eigenfactor that is normalized by $H$. For any other $U$-eigenfactor $K$, we see that $\{hKh^{-1} \mid h \in H\}$ under inclusion is order-isomorphic to $\mathbb{Z}$, and given $g \in H$ such that $gKg^{-1} < K$, then

$$U_0 = \bigcap_{h \in H} hKh^{-1} = \bigcap_{n \geq 0} g^nKg^{-n}.$$  

In other words, $g$ induces a contracting self-map on the coset space $K/U_0$. By Theorem 3.11 it follows that $K \subseteq \text{con}(g)U_0$, so $K = (\text{con}(g) \cap K)U_0$. In particular, the following holds.

**Corollary 5.14.** Let $G$ be a t.d.l.c. group, let $H$ be a flat subgroup of $G$ such that $H/H_\alpha$ is finitely generated and let $U$ be a compact open subgroup of $G$ that is tidy for $H$. Let $U_0 = \bigcap_{h \in H} hUh^{-1}$. Then $G^1_HU_0$ is the group generated by all $H$-conjugates of $U$. In particular, $G^1_HU_0$ is an open subgroup of $G$, and $UG^1_H/G^1_H$ is normalized by $H$.  

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Proof. By Lemma 3.8, $G_H^\dagger$ is normalized by $U_0$, so $G_H^\dagger U_0$ is a group. We see that $G_H^\dagger U_0$ contains every $U$-eigenfactor, so by Theorem 5.13, $U \leq G_H^\dagger U_0$. Thus $G_H^\dagger U_0 = G_H^\dagger U$. Clearly, $G_H^\dagger U_0$ is $H$-invariant, so the quotient $UG_H^\dagger / G_H^\dagger$ is normalized by $H$.

Let $K$ be the group generated by all $H$-conjugates of $U$. Then $K \leq G_H^\dagger U$ since $G_H^\dagger U$ is $H$-invariant. On the other hand $G_H^\dagger$ is clearly a subgroup of $K$, since $K$ is open and $H$-invariant, and also $U \leq K$, so in fact we must have $K = G_H^\dagger U$.

5.4 Almost flat actions

Let $G$ be a t.d.l.c. group and let $H$ be an almost finite-rank flat subgroup of $G$. Then $\text{Res}_G(H)$ is expressible in terms of nubs and contraction groups, as stated in Theorem 1.6.

We begin the proof with the case where $H$ is flat and uniscalar, a situation which has several equivalent characterizations.

Lemma 5.15. Let $G$ be a t.d.l.c. group and let $H$ be a flat subgroup of $G$. Then the following are equivalent:

(i) $H$ is uniscalar;
(ii) $\text{Res}_G(H)$ is compact;
(iii) $\text{Res}_G(H) = \text{nub}_G(H)$;
(iv) $G_H^\dagger = \text{lnub}_G(H)$.

Proof. Suppose that $H$ is uniscalar. Then there exists an $H$-invariant compact open subgroup $U$. Thus (i) implies (ii). Moreover, every $H$-invariant open subgroup $O$ contains a compact open $H$-invariant subgroup $O \cap U$, and since $H$ is uniscalar, the nub of $H$ is precisely the intersection of all $H$-invariant compact open subgroups. Thus (i) implies (iii). For each $h \in H$, we have $\text{nub}(h) = \text{con}(h)$ by [23, Proposition 5.4], so (i) implies (iv).

Conversely, suppose that at least one of (ii), (iii) and (iv) holds. Then $\text{con}(\alpha)$ is relatively compact for all $\alpha \in H$, since we have $\text{con}(\alpha) \leq G_H^\dagger$ and $\text{con}(\alpha) \leq \text{Res}_G(H)$, and both $\text{lnub}_G(H)$ and $\text{nub}_G(H)$ are compact. By Proposition 2.13 it follows that $H$ is uniscalar, so each of (ii), (iii) and (iv) implies (i). Hence (i), (ii), (iii) and (iv) are all equivalent as required. □

The following is immediately apparent from the proof of Lemma 2.24.

Lemma 5.16. Let $G$ be a t.d.l.c. group and let $H$ be a subgroup of $G$. Suppose that $H$ has a cocompact subgroup $K$ that is flat and uniscalar on $G$. Then $\text{Res}_G(H) = \text{Res}_G(K)$.
Proof of Theorem 1.6. Since the relative Tits cores, discrete residual and nub defined with respect to the action of a subgroup $H$ on a t.d.l.c. group $G$ are all unaffected by replacing $H$ with $\overline{H}$, we may assume that $H$ is closed.

Let $K$ be a cocompact subgroup of $H$ such that $K$ is flat on $G$ and $K/K_u$ is finitely generated. Consider the set

$$\mathcal{N} = \{ U \leq G \mid U \text{ is tidy for } K \text{ on } G \}.$$

By Corollary 5.14 the group $G^t_K U$ is a $K$-invariant open subgroup of $G$, for all $U \in \mathcal{N}$. By definition, $\text{nub}_G(K) = \bigcap_{U \in \mathcal{N}} U$, so by Lemma 4.10 we have

$$\bigcap_{U \in \mathcal{N}} (G^t_K U) = \overline{G^t_K \text{nub}_G(K)}.$$

In particular, we see that

$$\text{Res}_G(K) \leq \overline{G^t_K \text{nub}_G(K)}.$$

By Theorem 1.4 we have $\text{nub}_G(K) = \text{nub}_G(K) \text{nub}_G(K_u)$, and since $\text{nub}(k) \leq \text{con}(k)$ for each $k \in K$, we see that $\text{nub}_G(K) \leq G^t_K$. Thus

$$\overline{G^t_K \text{nub}_G(K)} = \overline{G^t_K \text{nub}_G(K_u)}.$$

Let $Y$ be a $K$-invariant open subgroup of $G$, and let $U$ be a tidy subgroup for $K$ on $G$. Then $U$ is $K_u$-invariant, so $Y \cap U$ is also $K_u$-invariant, and hence $Y \cap U$ is tidy for $K_u$. Thus $\text{nub}_G(K_u) \leq Y$. In addition, $\text{con}(k) \leq Y$ for all $k \in K$, since $Y$ is an open $K$-invariant identity neighbourhood. Hence $Y \geq \overline{G^t_K \text{nub}_G(K_u)}$. We conclude that

$$\text{Res}_G(K) = \overline{G^t_K \text{nub}_G(K)} = \overline{G^t_K \text{nub}_G(K_u)}.$$

Write $T = \overline{G^t_H}$ and $O = N_G(G^t_H)$; note that $T$ and $O$ are both $H$-invariant. By Corollary 3.9 $G^t_H = G^t_K$. By Lemma 5.8 $O$ contains a tidy subgroup $U$ for the action of $K$ on $G$. In particular, $O$ is open in $G$, so $\text{Res}_G(H) \leq O$ and $\text{Res}_G(K) \leq O$. The image $UT/T$ is normalized by $K$ by Corollary 5.14 so $K$ is uniscalar and flat on $O/T$. By Lemma 2.21 $H$ is also flat on $O/T$, and $H$ is anisotropic on $O/T$ by Corollary 5.2 proving (iii).

By Lemma 5.16 we see that $\text{Res}_{O/T}(H) = \text{Res}_{O/T}(K)$. Moreover, every $K$-invariant open subgroup of $G$ contains $T$, so in fact

$$\text{Res}_{O/T}(K) = \text{Res}_G(K)/T$$

and

$$\text{Res}_{O/T}(H) = \text{Res}_G(H)/T.$$ 

Thus $\text{Res}_G(H) = \text{Res}_G(K)$, completing the proof of (i).

We have seen that $\text{Res}_G(H) = T \text{nub}_G(K_u)$. Given a tidy subgroup $U$ for $K$ on $G$, then $U$ normalizes $T$ by Lemma 5.8 and $U$ normalizes $\text{nub}_G(K_u)$ by Corollary 4.6. Thus $U$ normalizes $\text{Res}_G(H)$, proving (ii).
Since \( H \) is flat and uniscalar on \( O/T \), by Lemma 5.15 it follows that \( \text{Res}_G(H)/T = \text{nub}_{O/T}(H) \), proving (iv).

Proof of Corollary 1.7. By Theorem 2.15 there is a polycyclic subgroup \( L \) of \( K \) such that \( L \) is flat on \( G \) and \( L \) has finite index in \( K \). We therefore have \( \text{Res}_G(H) = G/L \text{nub}_G(L) \) by Theorem 1.6(i). Since \( L \) is polycyclic, we have \( G/L \geq \text{nub}_G(L) \) by Theorem 1.4, and \( G/L = G/H \) by Corollary 3.9. Hence \( \text{Res}_G(H) = G/H \) as required.

From Theorem 1.6, we see that if \( H \) is almost finite-rank flat, then the action of \( H \) on \( N_G(\text{Res}_G(H))/\text{Res}_G(H) \) is uniscalar and smooth, in other words, the \( H \)-invariant open subgroups of \( N_G(\text{Res}_G(H))/\text{Res}_G(H) \) form a base of identity neighbourhoods. In fact, this provides another characterization of the discrete residual.

Corollary 5.17. Let \( G \) be a t.d.l.c. group and let \( H \) be an almost finite-rank flat subgroup of \( G \). Then \( \text{Res}_G(H) = R \) is the smallest closed subgroup of \( G \) with both of the following properties:

(a) \( N_G(R) \) is open in \( G \) and contains \( H \);

(b) The action of \( H \) on \( N_G(R)/R \) is uniscalar and smooth.

Proof. Let \( R \) be a closed subgroup of \( G \) satisfying (a) and (b). Then \( H \) normalizes arbitrarily small (compact) open subgroups \( U/R \) of \( N_G(R)/R \), and if \( U/R \) is such a subgroup, then \( U \) is \( H \)-invariant and open in \( G \), so \( \text{Res}_G(H) \leq U \). Hence \( \text{Res}_G(H) \leq R \).

Now consider \( \text{Res}_G(H) = R \) itself. By Theorem 1.6(ii), \( R \) satisfies (a). By Theorem 1.6(iii), the action of \( H \) on \( O/T \) is anisotropic and flat, where \( T = G/H \) and \( O = N_G(G/H) \). We have \( R/T = \text{nub}_{O/T}(H) \) by Theorem 1.6(iv), so by Corollary 4.11 the action of \( H \) on \( O/R \) is smooth; this action is also uniscalar by Lemma 4.9. Thus the action on \( N_G(R)/R \) is also uniscalar and smooth.

The fact that an almost finite-rank flat group has a smooth uniscalar action on \( O/\text{Res}_G(H) \), where \( O \) is a suitable open subgroup of \( G \), can be compared to [5, Corollary 4.1], which shows that given a compactly generated t.d.l.c. group \( G \), then \( G/\text{Res}(G) \) is a SIN group, in other words, the action of \( G \) on \( G/\text{Res}_G(G) \) is smooth and uniscalar. We will prove another result related to [5, Corollary 4.1] later; see §5.5.

We also note the following dichotomy for the action of polycyclic groups \( H \): either the action of \( H \) is smooth uniscalar (so in particular distal), or some element of \( H \) has non-trivial contraction group. Compare [17, Theorem 4.1].
The fact that $\overline{G^t_H}$ is cocompact and normal in $\text{Res}_G(H)$ reduces some natural questions about the structure of $\text{Res}_G(H)$ to the compact case. In particular, we obtain the following from Theorem 1.6 and Proposition 5.6.

**Corollary 5.18.** Let $G$ be a t.d.l.c. group and let $H$ be an almost finite-rank flat subgroup of $G$.

(i) $N_G(\text{Res}_G^\infty(H))$ is open in $G$ and contains both $H$ and $\text{Res}_G(H)$.

(ii) The action of $H$ on $\overline{\text{Res}_G^\infty(H)/G^t_H}$ is ergodic. Indeed, $\overline{\text{Res}_G^\infty(H)/G^t_H}$ is the largest closed subgroup of $N_G(\overline{G^t_H})/\overline{G^t_H}$ on which $H$ acts ergodically.

(iii) If $G$ is metrizable, then $\overline{\text{Res}_G^\infty(H)} = \text{Dist}_G(H)$.

**Proof.** Without loss of generality, we can replace $G$ with $N_G(\overline{G^t_H})$, and then assume $\overline{G^t_H} = \{1\}$. Hence by Theorem 1.6, $H$ is anisotropic and flat, and $\text{Res}_G(H) = \text{nub}_G(H)$ is compact. In particular, $\overline{\text{Res}_G^\infty(H)/G^t_H}$ is compact, so by Proposition 5.6(ii), $\overline{\text{Res}_G^\infty(H)/G^t_H}$ is normalized by every compact $H$-invariant subgroup of $G$. In particular $\overline{\text{Res}_G^\infty(H)/G^t_H}$ is normalized by $\text{Res}_G(H)$, and also by any compact open subgroup of $G$ that is tidy for $H$ (since $H$ is anisotropic), so $N_G(\overline{\text{Res}_G^\infty(H)})$ is open in $G$. Clearly also $H \leq N_G(\overline{\text{Res}_G^\infty(H)})$.

Thus (i) holds.

Parts (ii) and (iii) are now immediate consequences of Proposition 5.6, noting that in the class of t.d.l.c. groups, metrizability is stable under closed subgroups and quotients by closed normal subgroups.

As noted in the introduction, if $H$ is polycyclic, then Theorem 1.6 implies that

$$\overline{G^t_H} = \text{Dist}_G(H) = \overline{\text{Res}_G^\infty(H)} = \text{Res}_G(H).$$

The equality of these four groups can be rephrased in terms of their definitions, and whether or not a given closed subgroup contains them.

**Corollary 5.19.** Let $G$ be a t.d.l.c. group, let $K$ be a closed subgroup of $G$ and let $H \leq G$ be polycyclic. Then the following are equivalent:

(i) $\text{con}(h) \leq K$ for all $h \in H$;

(ii) There is a closed subgroup $L$ of $G$ such that $L \leq K$, $H \leq N_G(L)$ and $H$ acts distally on $G/L$ by conjugation;

(iii) There is a set of open subgroups $U$ of $G$ such that $\bigcap_{U \in U} U \leq K$ and such that $H \leq N_G(U)$ for all $U \in U$;

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Whenever $M$ is a closed $H$-invariant subgroup of $G$ such that $M$ has no proper $H$-invariant open subgroups, then $M \leq K$.

More generally, the fact that $G_H^\dagger$ is cocompact in $\text{Res}_G(H)$ allows us to prove a stability result for discrete residuals on quotients.

**Proposition 5.20.** Let $G$ be a t.d.l.c. group, let $H$ be an almost finite-rank flat subgroup of $G$ and let $K$ be a closed normal $H$-invariant subgroup of $G$. Then

$$\text{Res}_{G/K}(H) = \overline{\text{Res}_G(H)K/K}.$$  

**Proof.** Let $N = N_G(G_H^\dagger)$, and note that $N$ is open in $G$ by Corollary 3.10. We have $\text{Res}_{G/K}(H) \leq UK/K$ for any open $H$-invariant subgroup $U$ of $G$, since $UK/K$ is also open and $H$-invariant. In particular, $\text{Res}_{G/K}(H)$ is contained in $NK/K$; similarly, $\text{Res}_G(H) \leq N$. Moreover, $H$ is an almost finite-rank flat subgroup of $NK$. Thus we may assume $G = NK$.

Then $R = G_H^\dagger K$ is normal in $G$, and since every open $H$-invariant subgroup of $G$ contains $G_H^\dagger$, we see that

$$\text{Res}_{G/K}(H)/R = \text{Res}_{G/R}(H)$$

and

$$\overline{\text{Res}_G(H)K} \geq R.$$  

We claim that $\text{Res}_{G/R}(H) \leq \text{Res}_G(H)R/R$, which suffices to finish the proof. (The reverse inequality $\text{Res}_{G/R}(H) \geq \text{Res}_G(H)R/R$ is clear, since any $H$-invariant open subgroup of $G/R$ is the image of an $H$-invariant open subgroup of $G$.)

Let $T = G_H^\dagger$. By Theorem 1.6(iii), $H$ is uniscalar and flat on $N/T$, in other words there is a compact open subgroup $U/T$ of $N/T$ such that $U/T$ is normalized by $H$. Thus $\text{Res}_G(H)/T = \text{mb}_{N/T}(H)$, in other words, $\text{Res}_G(H)/T$ is an intersection of compact open subgroups of $N/T$.

Considering the natural homomorphism from $N/T$ to $NR/R$, we see by Lemma 4.10 that

$$\text{Res}_G(H)R/R = \bigcap_{U \in \mathcal{U}} (UR/R),$$

where $\mathcal{U}$ is the set of $H$-invariant open subgroups of $N$ such that $T \leq U \leq N$ and $U/T$ is compact. Given $U \in \mathcal{U}$, then $UR/R$ is an $H$-invariant open subgroup of $G/R$, so $UR/R \geq \text{Res}_{G/R}(H)$. Hence

$$\text{Res}_G(H)R/R \geq \text{Res}_{G/R}(H),$$

proving the claim.  

\[\square\]
5.5 A criterion for a non-distal action

We give here a sufficient condition for a compactly generated subgroup (not necessarily flat) to act non-distally.

Theorem 5.21. Let $G$ be a t.d.l.c. group and let $H$ be a compactly generated subgroup of $G$. Suppose that there is an infinite compact subgroup $K$ of $G$ such that the following conditions hold:

(i) $K$ is commensurated by $H$.

(ii) $N_H(K)$ is open in $H$.

(iii) The intersection of all $H$-conjugates of $K$ is trivial.

Then $K \cap \text{prox}_G(H)$ contains a non-trivial element. In particular, $H$ does not act distally on $G$.

Proof. Let $L$ be a compact open subgroup of $N_H(K)$. Then $KL$ is a profinite group in which $K$ is normal, which implies that every open subgroup of $K$ has open normalizer in $L$.

Let $X$ be a compact generating set for $H$ such that $1 \in X$ and $x \in X \iff x^{-1} \in X$. Write $X^n$ for the set of elements of $H$ of the form $x_1x_2 \ldots x_n$ for $x_i \in X$; note that $X^n$ is compact for all $n$. Set $K(n) = \bigcap_{y \in X^n} y^{-1}Ky$; since $X^n$ is compact, we see that $X^n$ is contained in the union of finitely many (right) cosets of the open subgroup $L$ of $H$, and hence $K(n)$ is the intersection of finitely many conjugates of $K$. Since $K$ is commensurated by $H$, in fact $K(n)$ is an open subgroup of $K$. The same argument shows that there are only finitely many distinct subgroups of the form $y^{-1}K(n)y$ for $m, n \in \mathbb{N}$ and $y \in X^m$.

Thus we have a descending chain $K = K(0) \supseteq K(1) \supseteq \ldots$ of open subgroups of $K$ with trivial intersection, which thus forms a base of identity neighbourhoods in $K$. Note that given $k, l > 0$ we have $K(k + l) = \bigcap_{y \in X^k} y^{-1}K(l)y$.

For $n > 0$, let $P(n) = \bigcup_{y \in X^n} yK(n)y^{-1}$. Then by construction, $P(1) \supseteq P(2) \supseteq \ldots$ is a descending chain of subsets of $K$. Each $P(n)$ is closed, since it is the union of finitely many closed subgroups. Suppose that $P(m) \subseteq K(1)$ for some $m$, in other words, for all $y \in X^m$ we have $yK(m)y^{-1} \subseteq K(1)$. Then $K(m) \subseteq y^{-1}K(1)y$ for all $y \in X^m$, so $K(m) \subseteq \bigcap_{y \in X^m} y^{-1}K(1)y = K(m + 1)$. We also have $K(m') \subseteq K(1)$ for all $m' > m$, so by the same argument $K(m') = K(m' + 1)$ for all $m' \geq m$. But this is impossible, as $K$ is an infinite profinite group and the set $\{ K(m) \mid m \in \mathbb{N} \}$ is a base of neighbourhoods of the identity in $K$. Hence $P(m)$ is not a subset of $K(1)$, for all $m \in \mathbb{N}$. Since $K \setminus K(1)$ is compact, it follows that $P = \bigcap_{n=1}^{\infty} P(n)$ is not a subset of $K(1)$. In particular, there is some non-identity element $x$ of $K$ such that $x \in P(n)$ for all $n$, and so for all $n$ there

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exists $x_n \in K(n)$ such that $x_n$ is $H$-conjugate to $x$. Since the subgroups $K(n)$ form a base of identity neighbourhoods in $K$, we have $x_n \to 1$ with respect to the topology of $K$, and hence $x_n \to 1$ in the topology of $G$. We conclude that $x \in \text{prox}_G(H)$, so the action of $H$ on $G$ is not distal.

The hypotheses of Theorem 5.21 are general enough that the (relative) Tits core $G^\dagger_H$ can be trivial, even if $H = G$: see Example 5.24 below. This is in contrast to the special case when $K$ is open and $H$ is flat, in which case we could use Corollary 5.14 to conclude that $G^\dagger_H$ is open.

Just the fact that certain actions of compactly generated t.d.l.c. groups are non-distal does however give some general results about their structure.

**Corollary 5.22.** Let $G$ be a compactly generated t.d.l.c. group. Suppose there is a compact open subgroup of $G$ with trivial core. Then $G$ is distal if and only if $G$ is discrete.

*Proof.* Let $K$ be a compact open subgroup of $G$ with trivial core and let $H = G$. If $K$ is finite, then $G$ is discrete (and hence distal). If $K$ is infinite, then all the hypotheses of Theorem 5.21 are satisfied, so $G$ is not distal. 

The following is a variant of [5, Corollary 4.1].

**Corollary 5.23.** Let $G$ be a compactly generated t.d.l.c. group. The following are equivalent:

(i) $G$ is residually discrete;

(ii) $G$ is a SIN group, in other words $\text{Inn}(G)$ is smooth and uniscalar;

(iii) For every compact normal subgroup $K$ of $G$, the quotient $G/K$ is distal.

*Proof.* Given Van Dantzig’s theorem, it is clear that (ii) implies (i).

Suppose (i) holds, let $K$ be a compact normal subgroup of $G$ and let $x \in G$. Suppose that $xK \in \text{prox}_{G/K}(G)$. By replacing $x$ with a conjugate of $x$ in $G$, we may assume $x \in U$, where $U$ is a compact open subgroup of $G$ containing $K$. We also have $x \in NK$, where $N$ is any open normal subgroup of $G$, so $x \in NK \cap U = (N \cap U)K$. By assumption, $\text{Res}(G)$ is trivial, so for every open subgroup $V$ of $U$, the intersection $\text{Res}(G) \cap (U \setminus V)$ is empty. By compactness, it follows that there is an open normal subgroup $N$ of $G$ such that $N \cap (U \setminus V)$ is empty, that is, $N \cap U \leq V$, and hence $x \in VK$. Since $V$ could be any open subgroup of $U$, we conclude that $x \in K$, so $G/K$ is distal, that is, (i) implies (iii).
If (iii) holds, then in particular if $K$ is the core of a compact open subgroup of $G$, then $G/K$ is discrete by Corollary 5.22, in other words $K$ is open. Thus the compact open normal subgroups of $G$ form a base of identity neighbourhoods, that is, $G$ is a SIN group. So (iii) implies (ii) and the cycle of implications is complete.

**Example 5.24.** This example is due to Kepert–Willis and Bhattacharjee–MacPherson ([12], [3]).

Let $F$ be a non-abelian finite simple group. Let $R = \prod \mathbb{Z} F$. Then we can form a semidirect product $R \rtimes \text{Sym} (\mathbb{Z})$, where $\text{Sym} (\mathbb{Z})$ acts by permuting the copies of $F$. Let $A$ be a subgroup of $\text{Sym} (\mathbb{Z})$ with the following properties:

(a) $A$ is transitive on $\mathbb{Z}$;
(b) $A$ is finitely generated as an abstract group;
(c) For all $a \in A$, every orbit of $\langle a \rangle$ on $\mathbb{Z}$ is finite;
(d) For all $a \in A$, the symmetric difference of $N$ and $aN$ is finite.

Such a permutation group was obtained by Bhattacharjee and MacPherson: they show ([3, Theorem 1.2]) that the free group on 2 generators has a faithful transitive action on $\mathbb{Z}$ with the required properties.

For each $i \in \mathbb{Z}$ let $S_i$ be the subgroup $\prod_{j \geq i} F$ of $R$; and let $S$ be the ascending union $\bigcup_{i \leq 0} S_i$, equipped with the topology extending the natural topology of $S_0$ (so $S_0$ is embedded in $S$ as a compact open subgroup). Condition (d) ensures that $A$ normalizes $S$, preserving the topology of $S$, so that there is a subgroup $G = S \rtimes A$ of $R \rtimes \text{Sym} (\mathbb{Z})$, and moreover the subgroups $S_i$ generate a group topology on $G$, under which $G$ is a t.d.l.c. group. Indeed, given conditions (c) and (d), we see that each element $a \in A$ preserves intervals $[j, +\infty)$ in $\mathbb{Z}$ where $j$ can be made arbitrarily large, and consequently $a$ normalizes subgroups $S_j$ such that $j \rightarrow +\infty$. Such a collection of subgroups forms a base of neighbourhoods of the identity in $G$, so each $a \in A$ is anisotropic, and indeed $G$ as a whole is anisotropic, that is, $G^\dagger = \{1\}$. Conditions (a) and (b) ensure that $G$ is compactly generated (it is generated by $S_0$ and $A$) and also that $G$ does not have any non-trivial compact normal subgroups: indeed, using the transitivity of $A$, it can be seen that every non-trivial normal subgroup of $G$ contains $T = \bigoplus \mathbb{Z} [F, F] = \bigoplus \mathbb{Z} F$, which already fails to be relatively compact in $G$.

Although $G^\dagger$ is trivial, we can easily see that this example does not contradict Theorem 5.21. $A$ does not act distally on $G$, because $T$ is a subset of $\text{prox}_G (H)$.

The following related questions are still open:

**Question 2.** Let $G$ be a compactly generated t.d.l.c. group such that $\text{Res} (G) = G$. Is $G^\dagger$ necessarily dense in $G$? (By Theorem 5.1 and [3, Theorem A], it suffices to answer
this question in the case that \( G \) is topologically simple, so it also suffices to determine whether or not \( G^\dagger \) can be trivial.)

**Question 3.** Let \( G \) be a compactly generated distal t.d.l.c. group. Is \( G \) necessarily a SIN group?

### 6 Open envelopes

#### 6.1 Reduced envelope of an almost flat subgroup

Within the class of subgroups normalized by the almost finite-rank flat subgroup \( H \), we can consider the open subgroups of \( G \) that actually contain \( H \).

**Definition 6.1.** Let \( G \) be a t.d.l.c. group and let \( X \subseteq G \). An envelope of \( X \) in \( G \) is an open subgroup of \( G \) that contains \( X \). Say an envelope \( E \) of \( X \) is reduced if, whenever \( E_2 \) is an envelope of \( X \) in \( G \), then \( |E : E \cap E_2| \) is finite.

The following observations are immediate from the definitions, together with Van Dantzig’s theorem.

**Lemma 6.2.** Let \( G \) be a t.d.l.c. group and let \( X \subseteq G \).

(i) Let \( E \) be an envelope for \( X \). Then \( \text{Res}_G(X) \leq E \). If \( X = H \) is a subgroup of \( G \), then \( \text{Res}_G(H)H \) is a subgroup of \( E \).

(ii) Suppose that \( X \) has a reduced envelope \( E \) in \( G \). Then all reduced envelopes of \( X \) in \( G \) are commensurate to \( E \), and there is a reduced envelope \( E_2 \leq E \) of the form \( E_2 = \langle U, X \rangle \), where \( U \) is a compact open subgroup of \( G \).

We now prove the theorem on reduced envelopes from the introduction.

**Proof of Theorem \[1.8\]** By Corollary \[5.14\] the group \( U_0G^\dagger_K \) is the smallest \( K \)-invariant subgroup of \( G \) that contains \( U \). Hence

\[
\langle K, U \rangle = G^\dagger_KU_0K.
\]

Clearly \( \langle K, U \rangle \) is an envelope for \( K \) in \( G \). Since every envelope for \( K \) contains \( G^\dagger_KK \), which is a cocompact subgroup of \( \langle K, U \rangle \), we see that \( \langle K, U \rangle \) is reduced, proving (i).

Now consider \( H \leq G \) such that \( K \) is cocompact in \( H \). Then \( G^\dagger_H = G^\dagger_K \) by Corollary \[3.9\]. Let \( O = N_G(G^\dagger_H) \). Since the action of \( H \) on \( O/G^\dagger_H \) is uniscalar and flat, there exists an \( H \)-invariant subgroup \( A \) of \( G \) such that \( G^\dagger_H \leq A \) and \( A/G^\dagger_H \) is a compact open subgroup.
of \(OG_H\). Thus \(E = AH\) is an envelope for \(H\) in \(G\). Since \(G^1_H\) is cocompact in \(E\), in fact \(E\) is a reduced envelope for \(H\). Since \(K\) is cocompact in \(H\), the open subgroup \(AK\) has finite index in \(E\); since every envelope of \(K\) contains \(G^1_K = G^1_H K\), we see that \(AK\) is a reduced envelope for \(K\), so \(E\) is a reduced envelope for \(K\), completing the proof of (ii).

Now suppose that \(K\) has finite covolume in \(G\), and let \(E = \langle K, U \rangle\). By Corollary 5.12, we know that \(\overline{G^1_K K} = \overline{KY} K\), where \(Y\) is any open subgroup of \(\overline{G^1_K} K\). In particular, \(\overline{G^1_K} K \subseteq \overline{KV} K\) for any open subgroup \(V\) of \(U\). By part (i), we have \(E = \overline{G^1_K U_0 K} = \overline{G^1_K K U_0}\). Hence \(E = \overline{KV K U_0} = \overline{KV U_0} K\). It is now clear that \(E = \overline{V K U_0 K}\).

In particular, we have \(E = UKUK\). Let \(R\) be the union of all \(K\)-conjugates of \(U\). Then by Theorem 5.13, we can write \(R = QU_0\), where \(Q\) is a finite product of contraction groups of elements of \(K\). In turn, it is easily seen that \(\text{con}(k)\) is invariant under conjugation by \(U_0\), for all \(k \in K\). Thus we have

\[
E = UKUK = QU_0 QU_0 K = Q^2 U_0 K
\]

where \(Q^2\) is a finite product of contraction groups of elements of \(K\), completing the proof of (iii). \(\square\)

We observe that up to taking closures, the relative Tits core of an almost finite-rank flat subgroup is realized as the Tits core of any reduced envelope.

**Proposition 6.3.** Let \(G\) be a t.d.l.c. group and let \(H\) be an almost finite-rank flat subgroup of \(G\). Let \(E\) be a reduced envelope for \(H\) in \(G\). Then

\[
\overline{G^1_H} = \overline{E^1_H} = \overline{E}.
\]

**Proof.** Let \(K\) be a cocompact subgroup of \(\overline{H}\) such that \(K\) is flat on \(G\) and \(K/K_u\) is finitely generated. By Theorem 5.7 we have \(G^1_H = G^1_K\), so by Theorem 1.8 we may assume that \(H = K\). Clearly \(G^1_H = E^1_H\), since \(E\) is an open \(H\)-invariant subgroup of \(G\). Since the Tits core is invariant on passing to an open subgroup of finite index, the choice of \(E\) is inconsequential, and we can arrange for \(E\) to normalize \(G^1_H\) and contain a tidy subgroup for \(H\), so that \(H\) is flat on \(E\). Thus we may assume \(G = E\) and that \(G^1_H\) is normal in \(G\).

Let \(S = G^1_H H\). Then \(G^1_S \leq \overline{G^1_H}\), since the action of \(S\) on \(G^1_H/G^1\) is anisotropic, and also \(G^1_H \leq G^1_S\) since \(H \leq S\). So \(G^1_S = G^1_H\), and in fact \(G^1_S = G^1_H\).

By Theorem 1.8 \(S\) is cocompact in \(G\), and \(\overline{S/G^1_H}\) is anisotropic. Hence \(G^1_H/G^1_H\) is anisotropic by Corollary 3.9 so \(G^1 \leq G^1_H\), and hence \(\overline{G^1} = \overline{G^1_H}\). \(\square\)
6.2 Compact normal subgroups of reduced envelopes

We can use discrete residuals of the $H$-action to restrict the action of the relative Tits core of $H$ on compact subgroups.

**Proposition 6.4.** Let $G$ be a t.d.l.c. group, let $H \leq G$, and let $S = HG_H^\dagger$. Let $K$ be a compact subgroup of $G$ that is normalized by $S$.

(i) $N_{SK}(L)$ is open in $SK$, for every open subgroup $L$ of $K$. In particular, every open subgroup of $K$ that is normalized by $H$ is also normalized by $G_H^\dagger$, so $\text{Res}_K(H) = \text{Res}_K(HG_H^\dagger)$.

(ii) The commutator group $[G_H^\dagger, K]$ is contained in $\text{Res}_K(H)$.

**Proof.** We may assume $G = SK$, since $(SK)_H^\dagger = G_H^\dagger$. Thus $K$ is normal in $G$.

Let $U$ be a compact open subgroup of $G$ such that $K \leq U$, and let $L$ be an open subgroup of $K$.

Since $U$ is profinite, there exists an open normal subgroup $V$ of $U$ such that $K \cap V \leq L$. Note that $K \cap V$ and $K$ are both normal in $U$, and there are only finitely many subgroups between $K \cap V$ and $K$, and in particular $L$ has finitely many conjugates in $U$, and hence $L$ is normalized by an open subgroup of $U$, proving that $N_G(L)$ is open. If $H \leq N_G(L)$, then $N_G(L)$ is an $H$-invariant open subgroup of $G$, so $\text{Res}_G(H) \leq N_G(L)$ and in particular $G_H^\dagger \leq N_G(L)$; in other words, every open subgroup of $K$ that is normalized by $H$ is also normalized by $G_H^\dagger$. Hence $\text{Res}_G(H) = \text{Res}_K(HG_H^\dagger)$, completing the proof of (i).

Let $L$ be an open subgroup of $K$ that is normalized by $H$. Since $N_G(L)$ is open, there is a compact open subgroup $V$ of $G$ such that $[V, K] \leq L$.

Let $h \in H$, let $u \in \text{con}(h)$ and let $k \in K$. Then for $n$ sufficiently large we have $g^nug^{-n} \in V$, so

$h^n[k, u]h^{-n} = h^nkh^{-n}, h^nuh^{-n}] \in [K, V] \leq L$.

Since $L$ is normalized by $h$, in fact $[k, u] \leq L$, so $[K, \text{con}(h)] \leq L$. In other words, $\text{con}(h) \leq C_G(K/L)$. Since $h \in H$ was arbitrary and $C_G(K/L)$ is closed, it follows that $G_H^\dagger \leq C_G(K/L)$, in other words $[G_H^\dagger, K] \leq L$. Applying this argument to all open $H$-invariant subgroups $L$ of $K$, we conclude that $[G_H^\dagger, K] \leq \text{Res}_K(H)$.

From now on, assume that $H$ is flat and $H/H_u$ is finitely generated. Define $\text{nub}_G^\star(H)$ to be the intersection of all $G_H^\dagger$-conjugates of $\text{nub}_G(H)$. (If $H = \langle h \rangle$ is cyclic, define $\text{nub}_G^\star(h) = \text{nub}_G^\star(H)$..) The group $\text{nub}_G^\star(H)$ plays a critical role in the structure of compact $(HG_H^\dagger)$-invariant subgroups of $G$.

We observe the following consequence of Theorem 1.8.
Lemma 6.5. Let $G$ be a t.d.l.c. group, let $H$ be a flat subgroup of $G$ and let $U$ be a compact open subgroup of $G$ that is tidy for $H$. Suppose $H/H_u$ is finitely generated. Then $\nub_G^*(H)$ is a normal subgroup of $\langle H, U \rangle$.

Proof. We have $\nub_G^*(H) \leq U$, so certainly $\nub_G^*(H)$ is a subgroup of $\langle H, U \rangle$. Clearly $H$ normalizes $\nub_G^*(H)$, and by Corollary 4.6 $\nub_G^*(H)$ is also normalized by $U_0 = \bigcap_{h \in H} hUh^{-1}$. Thus by Theorem 1.8 the set of $\langle H, U \rangle$-conjugates of $\nub_G^*(H)$ is the same as the set of $G^1_H$-conjugates of $\nub_G^*(H)$. Hence $\nub_G^*(H)$ is normal in $\langle H, U \rangle$.

By Theorem 4.5, every $H$-invariant compact subgroup $K$ of $G$ is contained in a tidy subgroup $U$ for $H$, so in fact $K \leq U_0$, where $U_0$ is the intersection of all $H$-conjugates of $U$. We have good control over $\Res_{U_0}(H)$ thanks to the following:

Lemma 6.6 (See [22] Lemma 4.11 and Lemma 6.2). Let $G$ be a t.d.l.c. group, let $H$ be a flat subgroup of $G$ such that $H/H_u$ is finitely generated, and let $U$ and $V$ be compact open subgroups of $G$ that are tidy for $H$. Let $U_0 = \bigcap_{h \in H} hUh^{-1}$ and let $V_0 = \bigcap_{h \in H} hVh^{-1}$. Then $U \cap V_0 = V \cap U_0$.

In particular, $U_0 \cap V_0$ is an open $H$-invariant subgroup of $U_0$.

Corollary 6.7. Let $G$ be a t.d.l.c. group, let $H$ be a flat subgroup of $G$ such that $H/H_u$ is finitely generated and let $K$ be a compact $H$-invariant subgroup of $G$. Then $\Res_K(H) \leq \nub_G(H)$. If $K$ is $HG^1_H$-invariant, then $\Res_K(H) \leq \nub_G^*(H)$.

Proof. By Theorem 4.5 there is a tidy subgroup $U$ for $H$ such that $K \leq U$, and hence $K \leq U_0$ where $U_0 = \bigcap_{h \in H} hUh^{-1}$. Certainly $\Res_K(H) \leq \Res_{U_0}(H)$ in this case, so we may assume that $K = U_0$. Given any tidy subgroup $V$ for $H$, then $U_0 \cap V_0$ is an open $H$-invariant subgroup of $U_0$ by Lemma 6.6 so $\Res_K(H) \leq V$. So $\Res_K(H)$ is contained in every tidy subgroup for $H$, hence $\Res_K(H) \leq \nub_G(H)$.

Now suppose $K$ is $HG^1_H$-invariant. Then $\Res_K(H) = \Res_K(HG^1_H)$ by Proposition 6.4, so $\Res_K(H)$ is also $HG^1_H$-invariant. Hence $\Res_K(H) \leq \nub_G^*(H)$.

Combining Proposition 6.4 with Corollary 6.7 we obtain the following restriction on compact $(HG^1_H)$-invariant subgroups of $G$.

Proposition 6.8. Let $G$ be a t.d.l.c. group and let $H$ be a flat subgroup of $G$ such that $H/H_u$ is finitely generated.

(i) Let $K$ be a compact $HG^1_H$-invariant subgroup of $G$. Then

$$[G^1_H, K] \leq \nub_G^*(H).$$
(ii) Let \( R = G_H^{G G_H(H)}, \) and let \( K/nub^*_G(H) \) be a compact \( H \)-invariant normal subgroup of \( R/nub^*_G(H) \). Then \( K/nub^*_G(H) \) is central in \( R/nub^*_G(H) \).

Proof. By Proposition 6.4, we have \( [G_H^{G G_H}, K] \leq \text{Res}_K(H) \), and by Corollary 6.7 we have \( \text{Res}_K(H) \leq nub^*_G(H) \). Hence \( [G_H^{G G_H}, K] \leq nub^*_G(H) \), proving (i).

(ii) follows immediately from (i), noting that if \( K/nub^*_G(H) \) is compact, then \( K \) is compact.

6.3 Cocompact envelopes and flat t.d.l.c. groups

One special case of envelopes is a cocompact envelope, where the group \( H \) has co-compact closure in the envelope \( E \). If a cocompact envelope exists, it is clearly reduced. For example, we see from Theorem 1.8 that if \( H \) is flat and uniscalar, then any reduced envelope \( E \) for \( H \) is cocompact.

Consider the situation that the almost finite-rank flat subgroup \( H \) of \( G \) is subnormal in some (reduced) envelope. This can only occur under special circumstances, and in particular we find that any reduced envelope is a cocompact envelope.

Proposition 6.9. Let \( G \) be a t.d.l.c. group and let \( H \) be an almost finite-rank flat subgroup of \( G \). Suppose that there is an envelope \( L \) of \( H \) in \( G \) such that \( H \) is subnormal in \( L \). Let \( E \) be a reduced envelope of \( H \). Then \( E \) is cocompact in \( E \), and \( E/R \) is anisotropic, where \( R \) is the core of \( H \) in \( E \).

Proof. Let \( E \) be a reduced envelope of \( H \). Observe that \( E_H \) does not depend on the choice of \( E \); in particular \( E_H = (E \cap L)_H \). Then \( H \) is subnormal in \( E \cap L \), so by Corollary 3.11 we have \( E_H \leq H \), and in fact \( E_H = \overline{(H)} \) by Corollary 3.12.

By Theorem 1.8(i), the group \( E_H \) is cocompact in some, hence every, reduced envelope for \( H \), so \( E_H \) is cocompact in \( E \). Since \( E_H \leq \overline{H} \), it follows that \( H \) is cocompact in \( E \). Hence \( \overline{H} = E \) by Corollary 3.9.

Clearly \( E \) is normal in \( E \), so \( E \) is contained in the core \( R \) of \( H \) in \( E \). Thus \( E/R \) is anisotropic by Corollary 3.2.

Another special case is that the flat subgroup \( H \) of \( G \) is actually already open in \( G \); this reduces to considering the situation of a t.d.l.c. group \( G \) that has flat action on itself, as in Theorem 1.9.

Proof of Theorem 1.9. Note that all the hypotheses for \( G \) are also satisfied by the quotient \( G/U_0 \).
Let $T = G^\dagger U_0$. Then $U \leq T$ by Theorem 1.8; in particular $T$ is open and hence closed. It follows from Theorem 3.1 that $G/T$ is the largest anisotropic quotient of $G/U_0$, so $T/U_0 = (G/U_0)^\dagger$; the same argument as for $G$ shows that $(G/U_0)^\dagger$ is closed in $G/U_0$, so in fact $T/U_0 = (G/U_0)^\dagger$, completing the proof of (i).

Since $G^\dagger \leq \text{Res}(G)$ for any t.d.l.c. group $G$, we see that in fact $T$ is the smallest open normal subgroup of $G$ that contains $U_0$, so in particular $T$ is the normal closure of $U$, proving (ii). Since $G_u$ is open in $G$ and contains $U_0$, it follows that $T$ is uniscalar. Since $U$ is tidy for $G$, it is also tidy for $G_u$ on $G$, and hence normalized by $G_u$; since $G_u$ is normal in $G$, the $G$-conjugates of $U$ normalize each other, proving (iii).

Let $K$ be a compact normal subgroup of $G$. Then by Theorem 4.5, there is a tidy subgroup $V$ for $G$ such that $K \leq V$, and hence $K \leq V_0$ where $V_0 = \bigcap_{g \in G} gVg^{-1}$. By Lemma 6.6 $V_0$ is commensurate to $U_0$, so $|KU_0 : U_0|$ is finite. We have $[G^\dagger, K] \leq \text{nub}(G)$ by Proposition 6.8.

We have $\text{Res}_{U_0}(G) \leq \text{nub}(G)$ by Corollary 6.7. All the other inequalities are now clear.

### 6.4 Faithful weakly decomposable groups

We apply the results of §6.2 to a class of t.d.l.c. groups considered in [7] and [8].

**Definition 6.10.** An action of a t.d.l.c. group $G$ on a Boolean algebra $\mathcal{A}$ is (non-degenerate) **faithful weakly decomposable** if it is faithful, such that for all $\alpha \in \mathcal{A} \setminus \{0\}$, the stabilizer in $G$ of $\alpha$ is open, and the pointwise stabilizer of the set $\{\beta \in \mathcal{A} \mid \alpha \wedge \beta = 0\}$ is non-discrete. We say $G$ is **faithful weakly decomposable** if it has a non-degenerate faithful weakly decomposable action on some Boolean algebra.

The faithful weakly decomposable property implies several structural properties of $G$, as described in [7] §5. In particular, if $G$ is faithful weakly decomposable, it follows from results in [4] that $G$ has trivial quasi-centre, and given $\{1\} \neq K \leq G$ such that $N_G(K)$ is open, then $K$ is not abelian. The Boolean algebra $\mathcal{A}$ can always be taken to be the (global) centralizer lattice of $G$, that is the set

$$\{C_G(K) \mid K \leq G, \ N_G(K) \text{ is open}\};$$

given $K, L \leq G$ such that $N_G(K)$ and $N_G(L)$ are open, if $C_G(K)$ and $C_G(L)$ have an open subgroup in common, then $C_G(K) = C_G(L)$.

Moreover, if $H$ is a closed subgroup of $G$ such that $N_G(H)$ is open, then $H$ is also faithful weakly decomposable (by [7] Proposition 5.22)).

The faithful weakly decomposable property also has implications for the local dynamics of $G$, as investigated in [8] §6. In particular, we recall the following.
**Proposition 6.11.** Let $G$ be a t.d.l.c. group acting on a Boolean algebra $A$. Suppose the action is faithful weakly decomposable.

(i) (See [3, Theorem 6.11]) Suppose that $G$ is compactly generated and that there is an identity neighbourhood in $G$ that contains no non-trivial compact normal subgroups of $G$. Then there exists $g \in G$ and $\alpha \in A$ such that $g\alpha < \alpha$.

(ii) (See [3, Proposition 6.7]) Let $g \in G$, and suppose there exists $\alpha \in A$ such that $g\alpha < \alpha$. Then $\nub(g)$ is non-trivial; in other words, $\con(g)$ is not closed.

We can use this result to establish a dichotomy for faithful weakly decomposable t.d.l.c. groups, as stated in the introduction. We begin the proof of Theorem 1.10 with a lemma.

**Lemma 6.12.** Let $G$ be a faithful weakly decomposable t.d.l.c. group and let $\alpha$ be an automorphism of $G$ that is not anisotropic on $G$. Let $S = G^1_{\alpha} \rtimes \langle \alpha \rangle$, equipped with a topology such that $G^1_{\alpha}$ is embedded as an open subgroup of $S$. Suppose $\nub_{G^1_{\alpha}}(\alpha) = \{1\}$. Then $S$ is compactly generated and has no non-trivial compact normal subgroup, and there exists $g \in S$ such that $\nub_{G^1_{\alpha}}(g)$ is non-trivial.

**Proof.** By Theorem 3.7, the group $G^1_{\alpha}$ has open normalizer in $G$. By Proposition 6.8, every compact $\langle \alpha, G^1_{\alpha} \rangle$-invariant subgroup of $G$ commutes with $G^1_{\alpha}$. Let $A$ be the global centralizer lattice of $G$, on which the action of $G$ is faithful weakly decomposable by [7, Theorem 5.18]. As explained in [7, Proposition 5.22], whenever $L$ is a closed subgroup of $G$ such that $N_G(L)$ is open, there is a principal ideal $I$ of $A$, such that the action of $L$ on $I$ is faithful weakly decomposable. In particular, this argument applies to $L = G^1_{\alpha}$. Moreover $I$ is obtained from $A$ in a canonical way, so the action extends to an action of $S$ on $I$. The latter action is not necessarily faithful, but clearly the kernel $K$ of the action is discrete so $K \leq QZ(S)$, and in fact $QZ(S)$ acts trivially on $I$, so $K = QZ(S)$ and $G^1_{\alpha} \cap QZ(S) = \{1\}$. Thus $QZ(S)$ is a discrete subgroup of $S$; also $QZ(S)$ is isomorphic to a subgroup of $\langle \alpha \rangle$, so $QZ(S)$ is torsion-free, and hence $QZ(S)$ has trivial intersection with every compact subgroup of $S$. Since $G^1_{\alpha}$ and $QZ(S)$ normalize each other, in fact $QZ(S) = \overline{CS(G^1_{\alpha})}$. Observe that $\alpha$ does not leave invariant any proper open subgroup of $G^1_{\alpha}$, so $S = \langle \alpha, U \rangle$, where $U$ is any compact open subgroup of $G^1_{\alpha}$; in particular $S$ is compactly generated. Given a compact normal subgroup $N$ of $S$, then $N$ commutes with $G^1_{\alpha}$, so $N \leq QZ(S)$ and hence $N$ is trivial.

Let $T = G^1_{\alpha}$. If $QZ(S) = \{1\}$, then $S$ is faithful weakly decomposable and there is an identity neighbourhood (namely $T$) that contains no non-trivial compact normal subgroup of $S$, so by Proposition 6.11 there exists $g \in S$ such that $\nub_{S}(g)$ is non-trivial; clearly $\nub_{S}(g) = \nub_{T}(g)$. If instead $QZ(S) > \{1\}$, then there exists $\beta \in QZ(S)$.
and \( n > 0 \) such that \( \beta \alpha^n \in T \). Since \( \beta \) centralizes \( T \), we see that
\[
T_{(\beta \alpha^n)}^\dagger = G_{\alpha^n}^\dagger = G_{\alpha}^\dagger \quad \text{and} \quad \text{nub}_{T}(\beta \alpha^n) = \text{nub}_{G}^\dagger(\alpha^n) = \text{nub}_{G}(\alpha) = \{1\}.
\]
The same argument as used for \( S \) now shows that \( T \) is compactly generated and has no non-trivial compact normal subgroups. Hence by Proposition 6.11 there exists \( g \in T \) such that \( \text{nub}_{T}(g) \) is non-trivial.

Proof of Theorem 1.10. It is clear that (i) and (ii) are mutually exclusive. We may suppose that (ii) fails, that is, every contraction group in \( G \) is closed.

Let \( K \) be a compactly generated subgroup of \( G \) such that \( N_{G}(K) \) is open. Then \( K \) is faithful weakly decomposable by [7 Proposition 5.22]. Note that by Lemma 3.6 we have \( \text{con}_{G}(g) = \text{con}_{K}(g) \) for all \( g \in K \), so in particular every contraction group in \( K \) is closed. Thus by Proposition 6.11, \( K \) has arbitrarily small compact normal subgroups.

If \( G \) is anisotropic, we are done, so let us suppose \( h \in G \) is such that \( \text{con}(h) \neq 1 \), so \( h \) is not anisotropic on \( G \). Since \( \text{con}(h) \) is closed, \( \text{nub}(h) = \{1\} \), so certainly \( \text{nub}_{G}^\dagger(h) \) is trivial. Thus by Lemma 6.12 there exists \( g \in S \) such that \( \text{nub}_{G}^\dagger_{h}(g) \) is non-trivial, where \( S = G_{h}^\dagger \rtimes \langle h \rangle \). Let \( g' \in G \), with the same action on \( G_{h}^\dagger \) as \( g \) has. Then \( \text{nub}_{G}^\dagger_{h}(g') \) is a non-trivial compact subgroup of \( G \) on which \( g' \) acts ergodically, so \( g' \) has non-trivial \( \text{nub} \) on \( G \). But then \( \text{con}(g') \) is not closed, a contradiction.

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