THE BRAID GROUP ACTION ON EXCEPTIONAL SEQUENCES FOR WEIGHTED PROJECTIVE LINES

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We dedicate this work to the memory of Andrzeyj Skowroński

Abstract. We give a new and intrinsic proof of the transitivity of the braid group action on the set of full exceptional sequences of coherent sheaves on a weighted projective line. We do not use here the corresponding result of Crawley-Boevey for modules over hereditary algebras. As an application we prove that the strongest global dimension of the category of coherent sheaves on a weighted projective line $X$ does not depend on the parameters of $X$. Finally we prove that the determinant of the matrix obtained by taking the values of $n$ $\mathbb{Z}$-linear functions defined on the Grothendieck group $\text{K}_0(X) \cong \mathbb{Z}^n$ of the elements of a full exceptional sequence is an invariant, up to sign.

1. Introduction

Let $X$ be a weighted projective line in the sense of Geigle and Lenzing [GL1]. The braid group $B_n$ on $n$ strings acts on the set of full exceptional sequences in the category $\text{coh}(X)$ of coherent sheaves on $X$, where $n$ denotes the rank of the Grothendieck group $\text{K}_0(X)$ of $\text{coh}(X)$. This action is given by mutations in the sense of Gorodentsev and Rudakov [GR]. The following result was proved in [M1]

Theorem 1.1. The action of the braid group on the set of full exceptional sequences in the category of coherent sheaves on a weighted projective line $X$ is transitive.

The proof was based on induction on the rank of the Grothendieck group of $\text{coh}(X)$ and on the rather strong result of Crawley-Boevey [CB] which states that the braid group acts transitively on the set of full exceptional sequences in the category of finitely generated modules over a hereditary algebra over an algebraically closed field.

It is desirable to have in the geometric situation a purely sheaf-theoretical proof for the transitivity of the braid group operation. In this paper we show that this in fact can be done using perpendicular calculus of exceptional pairs. For this we calculate the left perpendicular category of the sum of two line bundles $L \oplus L(\vec{c})$ formed in the sheaf category, where $L$ is a line bundle and $\vec{c}$ the canonical element of the grading group of $X$. For the convenience of the reader we also state the unchanged parts of the original proof.

Furthermore, we give two applications of the transitivity of the braid group action. First we show that the strongest global dimension of a weighted projective line $X$, a notion which we defined in this paper, is independent of the parameters of $X$. This means that if $X = X(p, \lambda)$ and $X' = X(p, \lambda')$, are weighted projective lines with the same weight sequence $p$ and different parameter sequences $\lambda$ and $\lambda'$ then the strong global dimensions for $X$ and $X'$ are the same.

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Second we prove that the determinant of the matrix obtained by applying \( n \) additive functions defined on the Grothendieck group of \( \text{coh}(\mathcal{X}) \) to the sheaves of a full exceptional sequence on \( \mathcal{X} \) is independent of the exceptional sequence, up to sign. Finally, we calculate this invariant for taking the rank function, the degree function and \( n - 2 \) Euler forms with respect to simple exceptional sheaves.

2. Preliminaries

2.1. Weighted projective lines were introduced by Geigle and Lenzing in 1987 in order to give a geometric approach to Ringel’s canonical algebras \([R]\). We recall some of the basic facts and refer for details to \([GL1]\).

Let \( k \) be an algebraically closed field. A weight sequence \( \mathbf{p} = (p_1, \ldots, p_t) \) is a sequence of natural numbers \( p_i \) with \( p_i \geq 2 \). For a weight sequence \( \mathbf{p} \) denote by \( L(\mathbf{p}) \) the abelian group with generators \( \vec{x}_1, \ldots, \vec{x}_t \) and relations \( p_1 \vec{x}_1 = \cdots = p_t \vec{x}_1 := \vec{c} \). The element \( \vec{c} \) is called the canonical element. \( L(\mathbf{p}) \) is an ordered group with \( \sum_{i=1}^t \mathbb{N} \vec{x}_i \) as cone of non-negative elements. Furthermore, each element \( \vec{x} \) can be written, on a unique way, in normal form \( \vec{x} = l \vec{c} + \sum_{i=1}^t l_i \vec{x}_i \) with \( l \in \mathbb{Z} \) and \( 0 \leq l_i < p_i \). Consider further a sequence of parameters \( \lambda = (\lambda_1, \ldots, \lambda_t) \), that is the \( \lambda_i \) are non-zero and pairwise distinct elements from \( k \). We denote \( S = S(\mathbf{p}, \lambda) = k[X_1, \ldots, X_t]/(X_1^{p_1} + X_2^{p_2} + \lambda_3 X_3^{p_3} + \cdots) \). The algebra \( S(\mathbf{p}, \lambda) \) is \( L(\mathbf{p}) \)-graded by defining \( \text{deg} (X_i) = \vec{x}_i \). Then the weighted projective line \( \mathcal{X} = \mathcal{X}(\mathbf{p}, \lambda) \) is defined to be the \( L(\mathbf{p}) \)-graded projective scheme \( \text{Proj}^d(\mathcal{O}(S(\mathbf{p}, \lambda))) \) and the category \( \text{coh}(\mathcal{X}) \) of coherent sheaves on \( \mathcal{X} \) is the quotient of the category of finitely generated \( L(\mathbf{p}) \)-graded \( S \) modules modulo the \( L(\mathbf{p}) \)-graded \( S \) modules of finite length. The category \( \text{coh}(\mathcal{X}) \) is abelian, hereditary, that is \( \text{Ext}^1(A, B) = 0 \) for all \( A \) and \( B \) in \( \text{coh}(\mathcal{X}) \) and \( t \geq 2 \), and it has finite dimensional Hom and Ext\(^1\) spaces. Moreover, \( \text{coh}(\mathcal{X}) \) admits Serre duality in the form \( \text{Ext}^1(A, B) \cong \text{D Hom}(B, A(\vec{\omega})) \), where \( \vec{\omega} \) denotes the dualizing element \( (t - 2)\vec{c} - \sum_{i=1}^t \vec{x}_i \), and consequently \( \text{coh}(\mathcal{X}) \) has Auslander-Reiten sequences.

We denote the structure sheaf on \( \mathcal{X} \) by \( \mathcal{O} \). It is well known that the isomorphism class of line bundles on \( \mathcal{X} \) form a group, via the tensor product and this group is isomorphic to the group \( L(\mathbf{p}) \) via the map \( \vec{x} \mapsto \mathcal{O}(\vec{x}) \). Moreover, the isomorphism class of line bundles modulo the \( L(\mathbf{p}) \)-graded structure sheaf modulo the \( \mathcal{O}(\vec{x})=\sum_{i=1}^t l_i \vec{x}_i \) is in normal form, then \( \dim S_2 = l + 1 \) provided \( l \geq -1 \). For coherent sheaves on \( \mathcal{X} \) we have the rank and the degree function, which are defined on the Grothendieck group \( K_0(\mathcal{X}) \). The sheaves of rank 0 are those of finite length. One of the key results in \([GL1]\) is that the sheaf \( \bigoplus_{0 \leq i \leq d} \mathcal{O}(\vec{x}_i) \) is a tilting sheaf such that its endomorphism algebra is a canonical algebra.

2.2. Recall that an object in a hereditary \( k \)-category \( \mathcal{H} \) is called exceptional if \( \text{End}(E) = k \) and \( \text{Ext}^1(E, E) = 0 \). Moreover, a sequence of exceptional objects \( (E_1, \ldots, E_r) \) is called an exceptional sequence if \( \text{Hom}(E_j, E_i) = 0 = \text{Ext}^1(E_j, E_i) \) for all \( j > i \). If \( r = 2 \) then \( (E_1, E_2) \) is called an exceptional pair and if \( r \) equals the rank of the Grothendieck group \( K_0(\mathcal{H}) \) then \( (E_1, E_2) \) is called a full exceptional sequence. This nomenclature is justified since every exceptional sequence has at most \( K_0(\mathcal{H}) \) entries and any exceptional sequence can be extended to at least one full exceptional sequence.

Gorodentsiev and Rudakov defined mutations of exceptional sequences on \( \mathbb{P}^n \) which give rise to an operation of the braid group \( B_r = \langle \sigma_1, \ldots, \sigma_{r-1} | \sigma_i \sigma_j = \sigma_j \sigma_i \rangle \) for \( i - j \geq 2 \) and \( \sigma_i \sigma_{i+1} = \sigma_{i+1} \sigma_i \sigma_{i+1} \) on the set of (isomorphism classes) of exceptional sequences of length \( r \) \([GR]\). For a categorical treatment we refer to \([B]\).

We will study the action of the braid group \( B_n \), where \( n \) is the rank of \( K_0(\mathcal{X}) \), on the set of full exceptional sequences in \( \text{coh}(\mathcal{X}) \). In this case each line bundle is exceptional. Moreover, the simple exceptional sheaves of rank 0 fit in exact sequences

\[
0 \rightarrow \mathcal{O}(j \vec{x}_i) \rightarrow \mathcal{O}((j + 1) \vec{x}_i) \rightarrow S_{i,j} \rightarrow 0.
\]
We continue this section with the following lemma, which is probably well known, but for the sake of completeness we state and give a proof.

**Lemma 2.1.** Let $A$ be an abelian $k$-category and $M$ an object in it whose endomorphism ring is a finite dimension $k$-algebra, and $f$ an element in $\text{End}(M)$. Then the following are equivalent:

1. $f$ is a monomorphism,
2. $f$ is an epimorphism
3. $f$ is an isomorphism

**Proof.**

We show that (1) implies (3). Let $f$ be a monomorphism. We can assume that $f$ is non-zero. There is $n$ such that $\{f, f^2, \cdots, f^n\}$ is linearly dependent. So there is a non-trivial linear combination $\lambda_1 f^1 + \cdots + \lambda_n f^n = 0$ with $\lambda_i$ and $\lambda_n$ non-zero. So we have $f^n(\lambda_1 f^1 + \cdots + \lambda_n f^n) = 0$. Since $f^1$ is a monomorphism, we get that $(\lambda_i Id + \lambda_{i+1} f^i + \cdots + \lambda_n f^{n-i}) = 0$ which implies that $\lambda_i Id = -(\lambda_{i+1} f + \cdots + \lambda_n f^{n-i})$. Factoring out $f$ we get that $f$ is invertible, which shows that (1) implies (3) Analogously we have that (2) implies (3). Since clearly (3) implies (1) and (2), we have that the three assertions are equivalent.

**Corollary 2.2.** Let us assume the same hypotheses as in lemma 2.1. If $M$ and $N$ are objects in $\mathcal{A}$ and there are monomorphisms, $f : M \to N$ and $g : N \to M$ then $f$ and $g$ are isomorphisms. The analogous statement is valid for epimorphism. □

Given two sheaves $A$ and $B$ over a weighted projective line, we define the trace map $\text{can} : \text{Hom}(A, B) \otimes A \to B$ in the usual way, i.e. $\text{can}(f \otimes a) = f(a)$. In the literature, the image of can is also called the trace of $A$ in $B$.

Furthermore, if the space $\text{Hom}(A, B)$ is different from zero, then the canonical map $\text{can} : \text{Hom}(A, B) \otimes A \to B$ is surjective or injective but not bijective, the proof for this fact is similar to the proof of [HR, Lemma 4.1]. In order to make our text complete, we give it now.

**Lemma 2.3.** Let $A, B$ be an exceptional pair in $\text{coh}(\mathcal{X})$, then the trace map $\text{can} : \text{Hom}(A, B) \otimes A \to B$ is either a monomorphism or an epimorphism.

**Proof.** We let $U$ be the image of $\text{can}$, by $\mu$ the inclusion $\mu : U \to B$ and $\text{can} = \mu \delta$, where $\delta$ is induced by $\text{can}$.

So we get the following exact sequence:

\[(*) \quad 0 \to U \to B \to B/U \to 0\]

Using that $\text{Ext}^2 = 0$, we get an epimorphism $\text{Ext}^1(B/U, \text{Hom}(A, B) \otimes A) \to \text{Ext}^1(B/U, U)$. This shows that the short exact sequence $(*)$ comes from an extension in the group $\text{Ext}^1(B/U, \text{Hom}(A, B) \otimes A)$, i.e. we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \to & \text{Hom}(A, B) \otimes A & \overset{\mu'}\to & V & \overset{\delta'}\to & B/U & \to & 0 \\
& & \downarrow{\delta} & & \downarrow{\delta'} & & \downarrow{\text{Id}} & & \\
0 & \to & U & \overset{\mu}\to & B & \overset{\delta}\to & B/U & \to & 0
\end{array}
\]

where $\delta$ and $\delta'$ are epimorphisms. From this diagram we get the following exact sequence:

\[0 \to \text{Hom}(A, B) \otimes A \overset{(\delta, \mu')^r}\to U \oplus V \overset{\mu - \delta'}\to B \to 0.\]

Since $\text{Ext}^1(B, A) = 0$ the exact sequence above splits. We consider now two cases.
Case 2: $B$ is not a direct summand of $U$. Therefore, $U \cong A'$ for some $t$ and $U \oplus V \cong A' \oplus (A' \oplus B)$ where $t + s = \dim \text{Hom}(A, B)$. Since $U$ is the image of $\mathrm{can}$, we have $\text{Hom}(\text{Hom}(A, B) \otimes A, B) = \text{Hom}(\text{Hom}(A, B) \otimes A, U) \cong \text{Hom}(\text{Hom}(A, B) \otimes A, A')$ and $\dim \text{Hom}(A, B) = t \dim \text{Hom}(A, A)$. Therefore, $t = \dim \text{Hom}(A, B)$ and $s = 0$.

The morphism $\text{Hom}(A, B) \otimes A \xrightarrow{\delta} U$, the isomorphism $U \cong \text{Hom}(A, B) \otimes A$ and the Corollary 2.2 give us that $\delta$ is a monomorphism.

The left mutation of $(A, B)$ is the exceptional pair $(L_AB, A)$, where $L_AB$ is given by one of the following three exact sequences: if $\text{Hom}(A, B) \neq 0$ then

\[
0 \rightarrow L_AB \rightarrow \text{Hom}(A, B) \otimes_k A \xrightarrow{\text{can}} B \rightarrow 0,
\]

and if $\text{Ext}^1(A, B) \neq 0$ then

\[
0 \rightarrow B \rightarrow L_AB \rightarrow \text{Ext}^1(A, B) \otimes_k A \rightarrow 0,
\]

where the third sequence is the universal extension. If $\text{Hom}(A, B) = 0 = \text{Ext}^1(A, B)$ then $L_AB = B$ and the left mutation of the pair $(A, B)$ is called a transposition. Now, the generators of $B_n$ act on the set of full exceptional sequences in $\text{coh}(X)$ as follows:

\[
\sigma_i \cdot (E_1, \ldots, E_i-1, E_i, E_{i+1}, E_{i+2}, \ldots, E_n) = (E_1, \ldots, E_{i-1}, L_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n).
\]

Further the right mutation of an exceptional pair $(A, B)$ is the exceptional pair $(B, R_B A)$, where $R_B A$ is given by one of the following three exact sequences

\[
0 \rightarrow A \xrightarrow{\text{cocan}} \text{DHom}(A, B) \otimes_k B \rightarrow R_B A \rightarrow 0,
\]

\[
0 \rightarrow R_B A \rightarrow A \xrightarrow{\text{cocan}} \text{DHom}(A, B) \otimes_k B \rightarrow 0,
\]

\[
0 \rightarrow \text{DExt}^1(A, B) \otimes_k B \rightarrow R_B A \rightarrow A \rightarrow 0,
\]

where $D = \text{Hom}_{\mathcal{A}}(-, k)$, cocan denotes the co-canonical map and the third sequence is the universal extension. Then $\sigma_i^{-1}$ acts in the following way.

\[
\sigma_i^{-1} \cdot (E_1, \ldots, E_i-1, E_i, E_{i+1}, E_{i+2}, \ldots, E_n) = (E_1, \ldots, E_{i-1}, R_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots, E_n).
\]

The following lemma is a useful tool.

**Lemma 2.4.** We have

(i) $\sigma_1 \ldots \sigma_{n-1}(E_1, E_2, \ldots, E_n) = (E_n(\tilde{\omega}), E_1, E_2, \ldots, E_{n-1})$

(ii) $\sigma_{n-1} \ldots \sigma_1(E_1, E_2, \ldots, E_n) = (E_2, \ldots, E_{n-1}, E_1(-\tilde{\omega}))$

(iii) In the orbit of an exceptional sequence $(E_1, \ldots, E_n, E_{n+1}, \ldots)$ there is an exceptional sequence of the form $(E_a, E_{n+1}, \ldots)$

The proof for (i) and (ii) is given in [M1, Proposition 2.4] and (iii) is a consequence of (i) and (ii).

2.3. Recall that for an object $X$ in a hereditary category $\mathcal{H}$ the left perpendicular category with respect to $X$ is defined as the full subcategory of all objects $Y$ satisfying $\text{Hom}(Y, X) = 0$ and $\text{Ext}^1(Y, X) = 0$ (see [GL2]). The right perpendicular category is defined dually.
3. Proof of Theorem 1.1

3.1. In this section we will prove Theorem 1.1. Let $X$ be a weighted projective line of weight type $p = (p_1, \ldots, p_t)$ and rank of $K_0(X)$ equals $n$. We start with the following observation.

**Proposition 3.1.** (a) Let $(L, L')$ be an exceptional pair of line bundles in $\text{coh}(X)$ with $\dim \text{Hom}(L, L') \geq 2$. Then $L' \cong L(\tilde{c})$ and $\dim \text{Hom}(L, L') = 2$.

(b) The left perpendicular category with respect to $L \oplus L(\tilde{c})$ for a line bundle $L$, formed in $\text{coh}(X)$, consists only of finite length sheaves. Moreover, this perpendicular category is equivalent to the category of finite dimensional modules over the path algebra of the disjoint union of linearly oriented quivers of type $A_{p_i-1}$, $i = 1, \ldots, t$.

**Proof.** (a) We have $L' = L(\tilde{x})$ for some $\tilde{x}$. We write $\tilde{x}$ in normal form and, after renumbering the indices, if necessary, $\tilde{x} = l\tilde{c} + \sum_{j=1}^{r} l_j \tilde{x}_j$, where $l_1 \neq 0, \ldots, l_r \neq 0$ for some $r$. Since $\dim \text{Hom}_s(L, L') \geq 2$ we have $l \geq 1$. Using Serre duality and the fact that $(L, L(\tilde{x}))$ is an exceptional pair we have $0 = \text{Ext}^1(L(\tilde{x}), L) \cong \text{Hom}(L(\tilde{x}), L(\tilde{x} + \tilde{c})) \cong \text{Hom}(O, O(\tilde{x} + \tilde{c}))$. Now $\tilde{x} + \tilde{c} = l\tilde{c} + \sum_{j=1}^{r} l_j \tilde{x}_j + (t-2)\tilde{c} - \sum_{i=1}^{l} \tilde{x}_i = (l-2+r)\tilde{c} + \sum_{j=1}^{r} (l_j-1)\tilde{x}_j + \sum_{i=r+1}^{t} (p_i-1)\tilde{x}_i$. This element is in normal form and it follows that $l - 2 + r < 0$, hence $l = 1$ and $r = 0$. Consequently $\tilde{x} = \tilde{c}$.

(b) After renumbering the indices, if necessary, for the simple exceptional sheaves in the tubes we can assume that $\text{Ext}^1(S_{i_0}, L) \neq 0$ for $i = 1, \ldots, t$.

The Riemann-Roch formula [GL1] 2.9 applied to $S_{i_0}$ and $L$ yields

$$\sum_{j=0}^{p-1} \tau^j S_{i_0}, L = p(1-g) \text{rk}(S_{i_0}) \text{rk}(L) + \text{det} \begin{pmatrix} \text{rk}(S_{i_0}) & \text{rk}(L) \\ \text{deg}(S_{i_0}) & \text{deg}(L) \end{pmatrix}$$

where $p$ denotes the least common multiple of the weights $p_1, \ldots, p_t$, $g$ is the genus of the weighted projective line and $\langle A, B \rangle = \dim \text{Hom}(A, B) - \dim \text{Ext}^1(A, B)$ the Euler form. Since the $\tau$ period of $S_{i_0}$ is $p$, $\text{rk}(S_{i_0}) = 0$ and $\text{deg}(S_{i_0}) = \frac{1}{p_i}$. We conclude that

$$\frac{p}{p_i} \sum_{j=0}^{p_i-1} \tau^j S_{i_0}, L = 1.$$

Since there are no non-zero homomorphisms from finite length sheaves to vector bundles we obtain that $\sum_{j=0}^{p_i-1} \dim \text{Ext}^1(\tau^j S_{i_0}, L) = 1$ and therefore $\text{Ext}^1(\tau^j S_{i_0}, L) = 0$ for $j = 1, \ldots, p_i - 1$. Again using that there are no non-zero homomorphisms from finite length sheaves to vector bundles we see that the sheaves $S_{i,j}$ for $j = 1, \ldots, p_i - 1, i = 1, \ldots, t$ belong to the left perpendicular category, formed in $\text{coh}(X)$, $\text{coh}(L)$, formed in $\text{coh}(X)$. Since $S_{i,j} = S_{i,j}(\tilde{c})$, the same argument can be applied to the line bundle $L(\tilde{c})$. Therefore the finite length sheaves $S_{i,j}$ for $j = 1, \ldots, p_i - 1, i = 1, \ldots, t$ belong to the left perpendicular category $H^\perp(L(\oplus L(\tilde{c})))$.

The category $H$ can be obtained by forming first the left perpendicular category $H_1$ with respect to $L$ in $\text{coh}(X)$ and then the left perpendicular category $H_2$ with respect to $L(\tilde{c})$ in $H_1$. The category $H_1$ is known to be equivalent to the category of finitely generated modules over a hereditary algebra, in fact the path algebra of the quiver, obtained from the quiver of the canonical algebra $\text{End}(\bigoplus_{0 \leq \tilde{x} \leq \tilde{c}} L(\tilde{x}))$ by removing the vertex which corresponds to $L$, see [LP]. Then by a result of Happel [H], using the fact that $L(\tilde{c})$ considered in the module category $\text{mod}(H_1)$ is exceptional, the category $H_2$ is equivalent to the category of finitely generated modules over a hereditary algebra $H_2$. Moreover, both results together imply that the rank of the Grothendieck group $K_0(H_2)$ equals $n - 2$.

Denote by $[b]S_{i,1}$ the indecomposable sheaf with socle $S_{i,1}$ and quasi-length $j$. The sheaf $T = \bigoplus_{i=1}^{t} \bigoplus_{j=1}^{p_i-1} [b]S_{i,1}$ satisfies $\text{Ext}^1(T, T) = 0$ and consists of $n - 2$ indecomposable direct summands. Therefore $T$ is a tilting sheaf in $H$ and consequently $H$ consists of the objects of the wings for $[b]S_{p_i-1}, i = 1, \ldots, t$. This shows that the endomorphism algebra of $T$ is the disjoint union of linear quivers of type $A_{p_i-1}$, $i = 1, \ldots, t$. □
3.2. We will use the following three results of [M1]

**Lemma 3.2.** [M1 Lemma 2.7] Two distinct complete exceptional sequences, differs in at least two places.

**3.5** An exceptional sequence $\langle E_1, ..., E_n \rangle$ in $\text{coh}(\mathcal{X})$ is called orthogonal if $\text{Hom}(E_i, E_j) = 0$ for all $i \neq j$.

**Proposition 3.3.** [M1 Proposition 2.8] There are no orthogonal complete exceptional sequences in $\text{coh}(\mathcal{X})$.

**Lemma 3.4.** [M1 Lemma 3.1] Let $E_1, E_2, ..., E_n$ be an exceptional sequence in $\text{coh}(\mathcal{X})$ such that $\dim_k \text{Hom}(E_1, E_2) \geq 2$.

(i) Suppose that $LE_2 = LE_1, E_2$ is defined by an exact sequence

$$0 \to LE_2 \to \text{Hom}(E_1, E_2) \otimes E_1 \to E_2 \to 0.$$ 

Then morphisms $0 \neq h \in \text{Hom}(LE_2, E_1)$ and $0 \neq f \in \text{Hom}(E_1, E_2)$ are either both monomorphisms or both epimorphisms.

(ii) Suppose that $RE_1 = RE_1, E_1$ is defined by an exact sequence

$$0 \to E_1 \to D \text{Hom}_X(E_1, E_2) \otimes E_2 \to RE_1 \to 0.$$ 

Then morphisms $0 \neq h \in \text{Hom}(E_2, RE_1)$ and $0 \neq f \in \text{Hom}(E_1, E_2)$ are either both monomorphisms or both epimorphisms.

3.3. For an exceptional sequence $\epsilon = (E_1, ..., E_n)$ we define

$$||\epsilon|| = (\text{rk}(E_{\pi(1)}), ..., \text{rk}(E_{\pi(n)})],$$

where $\pi$ is a permutation of $1, ..., n$ such that $\text{rk}(E_{\pi(1)}) \geq ... \geq \text{rk}(E_{\pi(n)})$.

**Proposition 3.5.** Let $\mathcal{X}$ be a weighted projective line with at least one weight, i.e. $\mathcal{X} \neq \mathbb{P}^1$. Then in each orbit, under the braid group action, there is a complete exceptional sequence containing a simple sheaf of rank 0.

**Proof.** We show first the following claim: if $\epsilon = (E_1, ..., E_n)$ is a complete exceptional sequence in $\text{coh}(\mathcal{X})$ with $\text{rk}(E_i) \geq 1$ for all $i$ then there exists $\sigma \in S_n$ such that $||\sigma \cdot \epsilon|| < ||\epsilon||$.

Let $\epsilon = (E_1, ..., E_n)$ be a complete exceptional sequence in $\text{coh}(\mathcal{X})$ with $\text{rk}(E_i) \geq 1$ for all $i$. We know from (3.3) that $\epsilon$ is not orthogonal. Choose $a < b$ such that $\text{Hom}(E_a, E_b) \neq 0$, but $\text{Hom}(E_{\pi(i)}, E_j) = 0$ for the remaining $a \leq i < j \leq b$.

Let $f : E_a \to E_b$ a nonzero morphism. We know that $f$ is a monomorphism or an epimorphism, thus we distinguish two cases.

Case 1: $f$ is a monomorphism.

Then $f$ induces epimorphisms $\text{Ext}^1(E_a, E_i) \to \text{Ext}^1(E_a, E_i)$ for all $i$. Since the first Ext-group is zero for $i \leq b$ the second Ext-group also vanishes for these $i$. We see that both $\text{Hom}(E_a, E_i) = 0$, and $\text{Ext}^1(E_a, E_i) = 0$ for all $a < i < b$, therefore applying transpositions we obtain that

$$\sigma_a^{-1} \sigma_a^{-1} \sigma_a^{-1} \epsilon = (E_1, ..., E_{a-1}, E_{a+1}, ..., E_{b-1}, E_a, E_b, ..., E_n).$$

Moreover, using Lemma [2.4] we can assume that $a = 1$ and $b = 2$.

Now, the left mutation $LE_2 = LE_1, E_2$ is defined by an exact sequence being of the form

(i) $$0 \to \text{Hom}(E_1, E_2) \otimes E_1 \to E_2 \to LE_2 \to 0$$

or

(ii) $$0 \to LE_2 \to \text{Hom}(E_1, E_2) \otimes E_1 \to E_2 \to 0.$$ 

In the case (i) we have $\text{rk}(LE_2) < \text{rk}(E_2)$, hence $||\sigma_1 \epsilon|| < ||\epsilon||$ and we are done.
In the case (ii) there exists a nonzero morphism \( h : LE_2 \to E_1 \). Again, \( h \) is a monomorphism or an epimorphism. Because \( f \) is a monomorphism we infer from the sequence (ii) that \( \dim k \hom(E_1, E_2) \geq 2 \). But then, in view of Lemma \ref{lemma:condition}, \( h \) is a monomorphism. Thus 
\[
\rk(LE_2) \leq \rk(E_1) \leq \rk(E_2).
\]
If \( \rk(LE_2) < \rk(E_2) \) we apply \( \sigma_i^{-1} \) as before and obtain \( \|\sigma_i^{-1}\epsilon\| < \|\epsilon\| \).

Assume otherwise that \( \rk(LE_2) = \rk(E_2) \). Then also \( \rk(E_1) = \rk(E_2) \) and therefore \( \dim_M \hom(E_1, E_2) = 2 \).

Consider an exact sequence
\[
0 \to E_1 \overset{f}{\to} E_2 \to C \to 0
\]
where \( C = \coker(f) \). Clearly \( \rk(C) = 0 \). Furthermore, applying the functor \( \hom(E_i, -) \) we conclude that \( \dim M \hom(E_{i}, C) = 1 \), for \( i = 1, 2 \). Finally, applying the functor \( \hom(-, C) \) we obtain \( \hom(C, C) = k \) and \( \ext^1(C, C) = k \), in particular \( C \) is indecomposable.

We have to consider two cases. First assume that \( C \) is a finite length sheaf concentrated at an ordinary point. Now \( \end{L}(C) = k \) which implies that \( C \) is a simple sheaf. The Riemann-Roch formula yields \( \hom(L, C) = k \) for each line bundle \( L \). Thus using a line bundle filtration for \( E_1 \) we obtain \( \dim_M \hom(E_1, C) = \rk(E_1) \). We have shown before that \( \dim_M \hom(E_1, C) = 1 \). Thus we obtain \( \rk(E_1) = \rk(E_2) = 1 \) and we have also \( \dim \hom(E_1, E_2) = 2 \). But then we have \( \rk(E_i) = 0 \) for \( i > 2 \) by Proposition \ref{proposition:case2}.

Now, assume that \( C \) is a sheaf of finite length concentrated at an exceptional point, say \( \lambda_i \) of weight \( p_i \). It follows from \( \hom(C, C) = k \), \( \ext^1(C, C) = k \) and the tube structure of the Auslander-Reiten quiver that the length of \( C \) is \( p_i \), and therefore for the classes in the Grothendieck group \( K_0(X) \) group we have \( [C] = \sum_{j=0}^{p_i-1} [S_{i,j}] \) where \( S_{i,j} \) are the objects on the mouth of the tube. From the exact sequences stated in subsection \ref{subsection:exact sequences} we infer that \( [S_{i,j}] = [O(j + 1)\xi_i] - [O(j)\xi_i] \) for \( i = 1, \ldots, t \), \( j = 1, \ldots, p_i \). Hence \( \sum_{j=0}^{p_i-1} [S_{i,j}] = [O(\xi_i)] - [O] \). On the other hand there is an exact sequence \( 0 \to O \to O(\xi) \to S(\xi) \to 0 \) where \( S \) is a simple finite length sheaf concentrated in an ordinary point and consequently \( [C] = [S] \). We conclude that \( 1 = \dim_M \hom(E_1, C) = \chi([E_1], [C]) = \chi([E_1], [S]) = \dim_M \hom(E_1, S) = \rk(E_1) \), where \( \chi \) is the Euler form. Then we have \( \rk(E_1) = \rk(E_2) = 1 \) and \( \dim \hom(E_1, E_2) = 2 \), and consequently \( \rk(E_i) = 0 \) for \( i > 2 \) by Lemma \ref{lemma:case2}, contrary to our assumption.

Case 2: \( f \) is an epimorphism.

Then \( f \) induces epimorphisms \( \ext^1(E_i, E_a) \to \ext^1(E_i, E_b) \), for all \( i \). The first Ext-group is zero for \( i \geq a \), thus also the second Ext-group vanishes for these \( i \). We see that both \( \hom(E_i, E_b) = 0 \) and \( \ext^1(E_i, E_b) = 0 \) for all \( a < i < b \), and again applying transpositions we have
\[
\sigma_{a+1}^{-1} \cdots \sigma_{b-1}^{-1} \epsilon = (E_1, \ldots, E_{a-1}, E_a, E_b, E_{a+1}, \ldots, E_n).
\]
As before we can assume \( a = 1 \) and \( b = 2 \). Then \( RE_1 = R E_2 E_1 \) is defined by an exact sequence
\[
(i) \quad 0 \to RE_1 \to E_1 \to D \hom_X(E_1, E_2) \otimes E_2 \to 0
\]
or
\[
(ii) \quad 0 \to E_1 \to D \hom_X(E_1, E_2) \otimes E_2 \to RE_1 \to 0
\]
In the first case we have \( \rk(RE_1) < \rk(E_1) \), and consequently \( \|\sigma_i^{-1}\epsilon\| < \|\epsilon\| \). In the second case there is a nonzero map \( h : E_2 \to RE_1 \), which again is a monomorphism or an epimorphism. Since \( f \) is an epimorphism we conclude that \( \hom(E_1, E_2) \geq 2 \) and therefore \( h \) is an epimorphism by Lemma \ref{lemma:case2}.

Now, in this case,
\[
\rk(E_1) > \rk(E_2) > \rk(RE_1)
\]
and therefore again \( \|\sigma_i^{-1}\epsilon\| < \|\epsilon\| \).
So the claim is proved. We see that after applying successively the norm reduction above, if necessary, we can shift by a braid group element any full exceptional sequence to a sequence containing an exceptional sheaf of rank 0.

We will show now that in the same orbit there is an exceptional sequence containing a simple sheaf.

Now let $s$ be the minimal number with the property that the orbit of $\epsilon$ contains an exceptional sequence with a rank 0 sheaf $F$ of length $s$. By Lemma [2.4] we can assume that this exceptional sequence is of the form $(E_1, \ldots, E_{n-1}, F)$.

We have to show that $s = 1$. Assume contrary that $F$ is not simple and denote by $S$ the socle of $F$. We claim that $(E_1, \ldots, E_{n-1}, S)$ is an exceptional sequence, too. Indeed, we have $\text{Ext}^1(S, E_i) = 0$ for $1 \leq i \leq n - 1$, because the embedding $S \hookrightarrow F$ induces epimorphisms $\text{Ext}^1(F, E_i) \to \text{Ext}^1(S, E_i)$ and the first Ext-group vanishes by assumption. On the other hand, $\text{Hom}(S, E_i) = 0$ for $1 \leq i \leq n - 1$, because the existence of a nonzero morphism from $S$ to some $E_i$ implies that $E_i$ also has finite length, and equals therefore $[r]S$, for some $r$, the unique indecomposable finite length sheaf with socle $S$ and length $r$. Then $r \geq s$ by minimality of $s$. But this implies $\text{Hom}(F, E_i) \neq 0$, contrary to the fact that $(E_1, \ldots, E_{n-1}, F)$ is an exceptional sequence. Thus we have two exceptional sequences which coincide in the first $n - 1$ terms but are different in the last one. By Lemma [3.2] this is impossible.

3.4. Proof of Theorem We show by induction on the rank $n$ of $K_0(\mathcal{X})$ that the group $B_n$ acts transitively on the set of complete exceptional sequences in $\text{coh}(\mathcal{X})$.

If $n = 2$ then $\mathcal{X} = \mathbb{P}^1$. In this case an exceptional sequence is of the form $(\mathcal{O}(i), \mathcal{O}(i+1))$ for some $i \in \mathbb{Z}$ and the braid group $B_2 \cong \mathbb{Z}$ obviously acts transitively on the set of these exceptional sequences.

Now, suppose $n > 2$ and assume that $\epsilon = (E_1, \ldots, E_n)$ is a full exceptional sequence in $\text{coh}(\mathcal{X})$. By Proposition [3.5] and applying, if necessary, Lemma [2.4] we have $g.\epsilon = (E'_1, \ldots, E'_{n-1}, S)$ for some $g \in B_n$ and some simple exceptional finite length sheaf $S$. Denote by $\kappa = (\mathcal{O}, \mathcal{O}(\vec{x}_1), \mathcal{O}(2\vec{x}_1), \ldots, \mathcal{O}(p_1-1\vec{x}_1), \mathcal{O}(\vec{x}_2), \ldots, \mathcal{O}(p_n-1\vec{x}_1), \mathcal{O}(\vec{\tau}))$ the exceptional sequence corresponding to the canonical tilting sheaf. Since $S$ is exceptional simple, $S = S_{i,j}$ for some $i,j$. From the exact sequence

$$0 \to \mathcal{O}(j\vec{x}_1) \to \mathcal{O}(j+1\vec{x}_1) \to S_{i,j} \to 0$$

we see that the right mutation of the pair $(\mathcal{O}(j\vec{x}_1), \mathcal{O}(j+1\vec{x}_1))$ equals $(\mathcal{O}(j+1\vec{x}_1), S_{i,j})$. Thus, for some $g_1 \in B_n$ we get $g_1.\kappa = (\mathcal{O}, \ldots, \mathcal{O}(j+1\vec{x}_1), S_{i,j}, \ldots)$. Observe that in case $j = p_i-1$, $i \neq t$, we first can apply transpositions in order to arrange that $\mathcal{O}(j\vec{x}_1)$ and $\mathcal{O}(j+1\vec{x}_1)$ are neighbours. Applying if necessary, Lemma [2.2], we obtain $g_2.\kappa = (F_1, \ldots, F_{n-1}, S)$ for some $g_2 \in B_n$ and line bundles $F_1, \ldots, F_{n-1}$. Now, by [GL2] the right peripheral category $S^\perp$ is equivalent to a sheaf category $\text{coh}(\mathcal{X}')$ for a weighted projective line $\mathcal{X}' = \mathcal{X}(\vec{p}', \lambda)$ with weight sequence $\vec{p}' = (p_1, \ldots, p_{n-1}, p_{n-1}, p_{n+1}, \ldots, p_n)$. By induction $(E'_1, \ldots, E'_{n-1})$ and $(F_1, \ldots, F_{n-1})$, considered as complete exceptional sequences in $S^\perp$, are in the same orbit under the action of the braid group $B_{n-1}$ on the set of complete exceptional sequences in $\text{coh}(\mathcal{X}')$. We conclude that $\epsilon$ and $\kappa$ are in the same orbit, which finishes the proof.

4. The strongest global dimension of $\text{coh}(\mathcal{X})$

4.1. As an application of the transitivity of the braid group action in [M2] it was shown that several data are independent of the parameters of the weighted projective line. Here we are going to investigate the strongest global dimension of weighted projective lines by studying the spreading of tilting complexes in the derived category.

Definition 4.1. The strongest global dimension is the maximum of the strong global dimension of all algebras which are derived equivalent do $\text{coh}(\mathcal{X})$.

We have the following characterization of the strongest global dimension.
Proposition 4.2. The strongest global dimension of a weighted projective line $X$ is one if $X = \mathbb{P}^1$ or is the maximal number $m + 2$ such that there exists a tilting complex $T$ in the derived category of coh($X$) of the form $\bigoplus_{i=0}^{m} T_i[i]$ with $T_i \in$ coh($X$), $i \in \mathbb{Z}$ and $T_0 \neq 0 \neq T_m$.

Proof. See Theorem 1 in [ALM].

The strongest global dimension of $X$ will be denoted by $\text{st.gl.dim}X$. It follows from the definition that if the bounded derived category of an algebra $A$ is triangular equivalent to the bounded derived category of coh($X$), then the $\text{sgl.dim}A \leq \text{st.gl.dim}X$.

In [M2] it was shown that if $X$ has weight type $(2, 2, \ldots, 2)$, $(t$ entries) then $\text{st.gl.dim}X = 2$. For tubular weighted projective lines $X$ bounds of the strongest global dimension were given in [S].

Before the main theorem in this section we have the following remarks and facts. Recall that, if $(A, B)$ is an exceptional pair in coh($X$), then $\text{Hom}_X(A, B) = 0$ or $\text{Ext}^1(A, B) = 0$. First we assume that $\text{Hom}(A, B) \neq 0$. We have then two cases for the left mutation to consider:

$$(\alpha) : 0 \rightarrow L_AB \rightarrow \text{Hom}(A, B) \otimes_k A \rightarrow B \rightarrow 0,$$

$$(\beta) : 0 \rightarrow \text{Hom}(A, B) \otimes_k A \rightarrow B \otimes L_AB \rightarrow 0.$$

It is important to note that the surjectivity of the canonical map depends only on $\text{rk}(A)$, $\text{rk}(B)$ and on the dimension of the spaces $\text{Hom}_X(A, B)$. We have that

- if $\text{rk}(A) \neq 0$ then $\text{can}$ is surjective $\iff$ $\dim_k \text{Hom}(A, B) \cdot \text{rank} A > \text{rank} B$.
- if $\text{rk}(A) = 0$ $\text{can}$ is surjective $\iff$ $\dim_k \text{Hom}(A, B) \cdot \dim_k A > \dim_k B$.

Applying $\text{Hom}_{X}(A, A)$ in $(\alpha)$ we have $\text{Hom}_{X}(\alpha, A) :$

$$0 \rightarrow \text{Hom}(B, A) \rightarrow \text{Hom}(\text{Hom}(A, B) \otimes_k A, A) \rightarrow \text{Hom}(L_AB, A) \rightarrow 0$$

Applying $\text{Hom}_{X}(A, A)$ in $(\beta)$ we have $\text{Hom}_{X}(\beta, A) :$

$$0 \rightarrow \text{Hom}(L_AB, A) \rightarrow \text{Hom}(B, A) \rightarrow \text{Hom}(\text{Hom}(A, B) \otimes_k A, A) \rightarrow 0$$

Now, the following remarks follows from both long exact sequences:

Remark 4.3. The mutation of $(A, B)$ is the exceptional pair $(L_AB, A)$ and:

- In the case $(\alpha)$, the conditions $\text{Hom}_{X}(B, A) = 0 = \text{Ext}^1(B, A)$ imply that $\dim_k \text{Hom}(L_AB, A) = \dim_k \text{Hom}(A, B)$ and $\dim_k \text{Ext}^1(L_AB, A) = 0$.
- In the case $(\beta)$, the conditions $\text{Hom}_{X}(B, A) = 0 = \text{Ext}^1(B, A)$ imply that $\dim_k \text{Hom}(L_AB, A) = 0$ and $\dim_k \text{Ext}^1(L_AB, A) = \dim_k \text{Hom}(A, B)$.

Now assume that $\text{Ext}(A, B) \neq 0$. Then have the universal extension

$$(\gamma) : 0 \rightarrow B \rightarrow L_AB \rightarrow \text{Ext}(A, B) \otimes_k A \rightarrow 0.$$

Applying $\text{Hom}_{X}(\gamma, A)$ in $(\gamma)$ we have $\text{Hom}_{X}(\gamma, A) :$

$$0 \rightarrow \text{Hom}(\text{Ext}(A, B) \otimes_k A, A) \rightarrow \text{Hom}(L_AB, A) \rightarrow \text{Hom}(B, A) \rightarrow 0.$$
4.2. Summarizing, from [33] and [34] if \((A,B)\) is an exceptional pair, then on the mutation pair \((L_AB,A)\) we can compute the dimensions \(\dim_k \text{Hom}(L_{AB}, A), \dim_k \text{Ext}_1^1(L_{AB}, A)\), and \(\text{rk}(L_{AB})\) without using the parameters \(\lambda\).

**Lemma 4.4.** Let \(\epsilon = (E_1, \ldots, E_n)\) be a complete exceptional sequences in \(\text{coh}(\mathcal{X})\) and \(\sigma\) be the generator of the braid group \(B_n\) such that \(\sigma \cdot \epsilon = (E_1, \ldots, E_{k-1}, L_{E_k}, E_k, E_{k+2}, \ldots, E_n)\), where we write shortly \(L_{E_k}\) instead of \(L_{E_k}, E_k\). Then the respective dimensions \(\dim_k \text{Hom}(E_i, L_{E_k+1}), \dim_k \text{Ext}_1^1(E_i, L_{E_k+1})\) for \(1 \leq i \leq k-1\), \(\dim_k \text{Hom}_\mathcal{X}(L_{E_k+1}, E_i)\), \(\dim_k \text{Ext}_1^1(L_{E_k+1}, E_i)\) for \(i \in \{k, k+2, \ldots, n\}\) and \(\text{rk}(L_{E_k+1})\) depend only on the dimensions of the \(\text{Hom}, \text{Ext}^1\) and the ranks of the elements in \(\epsilon\).

**Proof.** In remarks [33] and [34] we have seen that the dimensions \(\dim_k \text{Hom}(L_{E_k+1}, E_k)\), \(\dim_k \text{Ext}_1^1(L_{E_k+1}, E_k)\) depend only of the dimension of \(\text{Hom}_\mathcal{X}(X_k, X_{k+1})\) or \(\text{Ext}_1^1(X_k, X_{k+1})\). Now we will prove the claim for \(\dim_k \text{Hom}(E_j, L_{E_k+1}), \dim_k \text{Ext}_1^1(E_j, L_{E_k+1})\) for \(1 \leq j \leq k-1\).

Suppose that the mutation is given by type \((\alpha)\), then we have the exact sequence

\[
0 \to L_{E_k+1} \to \text{Hom}(E_k, L_{E_k+1}) \otimes_k E_k \to \text{Ext}_1^1(E_k, L_{E_k+1}) \to 0,
\]

which induces a long exact sequence

\[
0 \to \text{Hom}(E_j, L_{E_k+1}) \to \text{Hom}(E_j, \text{Hom}(E_k, L_{E_k+1}) \otimes_k E_k) \to \text{Hom}(E_j, E_k) \to \text{Ext}_1^1(E_j, \text{Hom}(E_k, L_{E_k+1}) \otimes_k E_k) \to \text{Ext}_1^1(E_j, E_k) \to 0
\]

for \(1 \leq j \leq k-1\).

Since by Lemma 3.2.4 \(\text{Hom}_\mathcal{X}(E_j, E_{k+1}) = 0\) or \(\text{Ext}_1^1(E_j, E_{k+1}) = 0\) we have either

\[
\dim_k \text{Hom}(E_j, L_{E_k+1}) = \dim_k \text{Hom}(E_j, E_k) \cdot \dim_k \text{Hom}(E_k, E_{k+1})
\]

and

\[
\text{Ext}_1^1(E_j, L_{E_k+1}) = \dim_k \text{Ext}_1^1(E_j, E_k) \cdot \dim_k \text{Hom}(E_k, E_{k+1}) - \dim_k \text{Ext}_1^1(E_j, E_k)
\]

or

\[
\dim_k \text{Hom}(E_j, E_k) \cdot \dim_k \text{Hom}(E_k, E_{k+1}) + \dim_k \text{Ext}_1^1(E_j, L_{E_k+1}) = \dim_k \text{Hom}(E_j, E_k) + \dim_k \text{Hom}(E_j, E_{k+1}).
\]

Since \((E_j, L_{E_k+1})\) is an exceptional pair, we have \(\text{Hom}_\mathcal{X}(E_j, L_{E_k+1}) = 0\) or \(\text{Ext}_1^1(E_j, L_{E_k+1}) = 0\). Each one gives us that \(\dim_k \text{Hom}(E_j, L_{E_k+1})\) and \(\text{Ext}_1^1(E_j, L_{E_k+1})\) depend only of the dimensions of the \(\text{Hom}, \text{Ext}^1\) spaces of \(\epsilon\).

In the cases that the left mutation is given by type \((\beta)\) or type \((\gamma)\) the proof is similar. \(\Box\)

We have as a consequence of the previous discussion the following:

**Corollary 4.5.** Suppose that \(\epsilon = (E_1, \ldots, E_n)\) and \(\epsilon' = (E'_1, \ldots, E'_n)\) are complete exceptional sequences in \(\text{coh}(\mathcal{X})\) such that the following formulas are valid \(\dim_k \text{Hom}_\mathcal{X}(E_j, E_l) = \dim_k \text{Hom}_\mathcal{X}(E'_j, E'_l), \dim_k \text{Ext}_1^1(E_j, E_l) = \dim_k \text{Ext}_1^1(E'_j, E'_l)\) and \(\text{rk}(E_j) = \text{rk}(E'_j)\) for all \(1 \leq j, l \leq n\). Let \(\sigma \in B_n\) and \(\sigma \epsilon = (F_1, \ldots, F_n), \sigma \epsilon' = (F'_1, \ldots, F'_n)\). Then \(\dim_k \text{Hom}_\mathcal{X}(F_j, F_l) = \dim_k \text{Hom}_\mathcal{X}(F'_j, F'_l), \dim_k \text{Ext}_1^1(F_j, F_l) = \dim_k \text{Ext}_1^1(F'_j, F'_l)\) and \(\text{rk}(F_j) = \text{rk}(F'_j)\) for all \(1 \leq j, l \leq n\).

\(\Box\)

**Theorem 4.6.** Let \(\mathcal{X} = (p, \lambda)\) and \(\mathcal{X}' = (p, \lambda')\) be weighted projective lines with the same weight type. Then \(\text{st.gl.dim} \mathcal{X} = \text{st.gl.dim} \mathcal{X}'\).

**Proof.** Let \(m\) be maximal such that there exists a tilting complex \(T\) of the form \(\bigoplus_{i=0}^m T_i[i]\) with \(T_i \in \text{coh}(\mathcal{X})\) and \(T_0 \neq 0 \neq T_m\). Write \(T = \bigoplus_{l=0}^n T_l[n_l]\) with indecomposable sheaves \(E_j\) and \(n_j \in \mathbb{Z}\). The \(E_j\) can be ordered in such a way that they form a full exceptional sequence \(\epsilon\) in \(\text{coh}(\mathcal{X})\). By Theorem 1.1. there exists a braid group element \(\sigma \in B_n\) such that \(\epsilon = \sigma \cdot \kappa\) where \(\kappa = (O_{\mathcal{X}}, O_{\mathcal{X}}(\mathbf{x}_1), \ldots, O_{\mathcal{X}}((p_1-1)\mathbf{x}_1), \ldots, O_{\mathcal{X}}((p_n-1)\mathbf{x}_1), O_{\mathcal{X}}(\mathbf{x}_2))\) is the exceptional sequence obtained from the canonical tilting sheaf \(\bigoplus_{0 \leq \mathbf{x} \leq \mathbf{x}_2} O_{\mathcal{X}}(\mathbf{x})\) on \(\mathcal{X}\).
Now the application of the same braid group element $\sigma$ to the exceptional sequence $\kappa' = (\mathcal{O}_X, \mathcal{O}_X(\overline{x}_1), \ldots, \mathcal{O}_X((p_1-1)\overline{x}_1), \ldots, \mathcal{O}_X((p_t-1)\overline{x}_t), \mathcal{O}_X(\overline{c}))$ obtained from the canonical tilting sheaf $\bigoplus_{0 \leq i \leq t} \mathcal{O}_X(i\overline{x})$ on $\mathcal{X}'$ yields a full exceptional sequence $\epsilon'$ for the weighted projective line $\mathcal{X}'$.

The exceptional sheaves $\mathcal{O}_X(u\overline{x}_1)$ and $\mathcal{O}_X(s\overline{x}_i)$ of $\kappa$ and $\kappa'$, respectively, satisfy the same dimension for the Hom and Ext spaces that is,

$$\dim \text{Ext}^k_{\mathcal{X}}(\mathcal{O}_X(m\overline{x}_i), \mathcal{O}_X(n\overline{x}_j)) = \dim \text{Ext}^k_{\mathcal{X}'}(\mathcal{O}_{\mathcal{X}'}(m\overline{x}_i), \mathcal{O}_{\mathcal{X}'}(n\overline{x}_j))$$

for $i, j \in \{1, \ldots, t\}$, $m, n \in \{1, \ldots, \max\{p_1, \ldots, p_t\}\}$ and $k \in \{0, 1\}$. Moreover, the ranks of all sheaves of $\kappa$ and $\kappa'$ equal 1.

The sequence $\epsilon'$ is constructed from $\kappa'$ using successively the same kind of mutations as in the construction of $\epsilon$. Therefore, following Corollary 4.5, the exceptional sheaves $E_j'$ of $\epsilon'$ satisfy the same dimension formulas for the Hom, Ext and rank spaces as the exceptional sheaves $E_j$ of $\epsilon$.

Therefore the exceptional sheaves $E_j'$ of $\epsilon'$ satisfy the same dimension formulas for the Hom and Ext spaces as the exceptional sheaves $T_j$ of $\epsilon$. Hence the $E_j'$ can be shifted in the derived category of $\text{coh}(\mathcal{X}')$ as the $E_j$ which yields a tilting complex $\bigoplus_{i=0}^m T_i[i]$ with $T_j' \in \text{coh}(\mathcal{X}')$ and with $T_0' \neq 0 \neq T_m'$. Consequently $\text{st.gl.dim X} = m \leq \text{st.gl.dim X}'$. By symmetry, $\text{st.gl.dim X}' \leq \text{st.gl.dim X}$ and consequently $\text{st.gl.dim X} = \text{st.gl.dim X}'$.  

- Note that from Corollary 4.5 we obtain that the ordinary quivers of the algebras $\text{End T}$ and $\text{End T}'$ in the former theorem are the same which was already stated in [M2].

- The former proof also suggests the following:

**Conjecture:** Let $T$ be a tilting complex of the form $\bigoplus_{i=0}^m T_i[i]$ with $T_i \in \text{coh}(\mathcal{X})$ and $T_0 \neq 0 \neq T_m$ and $A = \text{End T}$. The strong global dimension of $A$ does not depend on the parameters.

The validity of this conjecture implies the statement of the Theorem 4.5.

### 5. Determinants for exceptional sequences

Let $f_1, \ldots, f_n$ be group homomorphisms defined on the Grothendieck group $K_0(\mathcal{X})$ of a weighted projective line with values in $\mathbb{Z}$. For a full exceptional sequence $\epsilon = (E_1, \ldots, E_n)$ on $\mathcal{X}$ we form the $n \times n$ matrix $M(\epsilon)$ whose coefficient at the place $(i, j)$ equals $f_i(E_j)$ and we consider the determinant of that matrix $\det(M(\epsilon))$.

**Theorem 5.1.** There exists a constant $c \in k$ such that $\det(M(\epsilon)) = c$ or $-c$ for all full exceptional sequences $\epsilon$ in $\text{coh}(\mathcal{X})$.

**Proof.** We are going to show that the determinant of the matrix does not change if we apply to the exceptional sequence in $\text{coh}(\mathcal{X})$ the left mutation $\sigma_i$. For right mutations the proof is analogous.

For a full exceptional sequence $\epsilon = (E_1, E_2, \ldots, E_n)$ we denote $\dim_k \text{Hom}(E_i, E_{i+1}) = h$ and $\dim_k \text{Ext}^1(E_i, E_{i+1}) = e$. Now, $\sigma_i \cdot \epsilon$ equals $(E_1, \ldots, E_{i-1}, LE_i, E_{i+1}, LE_{i+1}, E_{i+2}, \ldots, E_n)$ and we have $[L_E, E_{i+1}] = h[E_i] - [E_{i+1}]$, $[L_E, E_{i+1}] = [E_{i+1}] - h[E_i]$ or $[L_E, E_{i+1}] = e[E_i] + [E_{i+1}]$ depending on the type of the left mutation of the pair $(E_i, E_{i+1})$.

**The matrix for the exceptional sequence $\sigma_i \cdot \epsilon$ is obtained from that of $\epsilon$ by replacing the values in the $i$-th column by $f_j(E_{i+1}) - hf_j(E_i)$, $-f_j(E_{i+1}) + hf_j(E_i)$ or $f_j(E_{i+1}) + ef_j(E_i)$, $j = 1, \ldots, n$ and by replacing the values in the $i + 1$-th column by $f_j(E_i)$, $j = 1, \ldots, n$. Then the statement follows from the known rules for determinants.**

In particular we can apply the method above to the rank function, the degree function and the $n - 2$ Euler forms $(-, S_{ij})$, $j = 1, \ldots, p_i - 1$, $i = 1, \ldots, t$.  

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Corollary 5.2. For each full exceptional sequence $\epsilon = (E_1, E_2, \ldots, E_n)$ in col($X$) the determinant of the matrix

$$
M(\epsilon) = \begin{pmatrix}
\text{rk } E_1 & \text{rk } E_2 & \text{rk } E_3 & \cdots & \text{rk } E_{p_1+1} & \cdots & \text{rk } E_n \\
\deg E_1 & \deg E_2 & \deg E_3 & \cdots & \deg E_{p_1+1} & \cdots & \deg E_n \\
\langle E_1, S_1, 1 \rangle & \langle E_2, S_1, 1 \rangle & \langle E_3, S_1, 1 \rangle & \cdots & \langle E_{p_1+1}, S_1, 1 \rangle & \cdots & \langle E_n, S_1, 1 \rangle \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\langle E_1, S_{p_1-1}, 1 \rangle & \langle E_2, S_{p_1-1}, 1 \rangle & \langle E_3, S_{p_1-1}, 1 \rangle & \cdots & \langle E_{p_1}, S_{p_1-1}, 1 \rangle & \cdots & \langle E_n, S_{p_1-1}, 1 \rangle \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
\langle E_1, S_{p_1}, 1 \rangle & \langle E_2, S_{p_1}, 1 \rangle & \langle E_3, S_{p_1}, 1 \rangle & \cdots & \langle E_{p_1+1}, S_{p_1}, 1 \rangle & \cdots & \langle E_n, S_{p_1}, 1 \rangle \\
\end{pmatrix}
$$

equals p or $-p$. Recall that $p$ denotes the least common multiple of the weights $p_1, \ldots, p_t$.

Proof. The determinant is easily calculated to be $p$ or $-p$ for the exceptional sequence $(O, O(\overline{c}), S_1, 1, \ldots, S_{p_1-1}, 1, S_{p_1}, 1)$ using the block structure of the matrix and the fact that $\text{rk } O = \text{rk } O(\overline{c}) = 1$, $\deg O = 0$ and $\deg O(\overline{c}) = p$. Then the statement follows from Theorem 5.1.

Remark 5.3. The determinantal equation obtained in the way above can be interpreted as a diophantine equation for the weighted projective line $X$. Diophantine equations expressed for data in terms of exceptional sequences seems to be typical. So Rudakov showed that the ranks of the vector bundles of an exceptional triple on the projective plane satisfy the Markov equation $X^2 + Y^2 + Z^2 = 3XYZ$ [10]. Diophantine equations for partial tilting sequences on weighted projective lines were given in [2 Chapter 10.2].

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