Generalized hole-particle transformations and spin reflection positivity in multi-orbital systems

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We propose a scheme combining spin reflection positivity and generalized hole-particle and orbital transformations to characterize the symmetry properties of the ground state for some correlated electron models on bipartite lattices. In particular, we rigorously determine at half-filling and for different regions of the parameter space the spin, orbital and $\eta$ pairing pseudospin of the ground state of generalized two-orbital Hubbard models which include the Hund’s rule coupling.

I. INTRODUCTION

Since the discovery of high temperature superconductors, a renewed interest has increased in models describing strongly correlated electrons. As the simplest one the Hubbard model has been intensively studied within the analysis of strongly correlated electron systems. Apart from it, various generalized Hubbard models have also attracted considerable interest. Indeed, by including some additional interaction terms into the Hubbard Hamiltonian, either the magnetic ordering phases or the superconducting phases may be stabilized in appropriate regions of the parameter space. The introduction in these models of orbital degrees of freedom enriches the phase diagram of the model, because the interplay of orbital and spin dynamics can yield a series of novel physical phenomena, such as unconventional metal-insulator transition, novel quantum spin and orbital ordered states, and transport property anomalies.

Generally, all of these models retain only part of the electron-electron interactions which arise as a result of the repulsion between two electrons having opposite spins and located at the same lattice site. In spite of this crude assumption, these models correspond to a many-body problem and in general cannot be exactly solved. Exact solutions are of great importance, since in some cases the errors introduced by the approximations may dominate the results to such an extent that one might end up with an incorrect description of the phenomenon under study. Considering the exact results with respect to the dimensionality of the model, we know that most of them have been derived in two limiting case: either in one dimension or in the infinity dimensions. For instance, the exact solution of the Hubbard model was given in the one dimensional case by means of the Bethe ansatz by Lieb and Wu. The other class of exact solutions belongs to the other limiting case, i.e. $d=\infty$, where the dynamical mean-field approximation becomes exact. The situation gets more complicated as physically interesting lower dimensional cases are considered. $d>1$ rules out the applicability of the well-established Bethe ansatz approach, while mean-field-like descriptions lead to qualitatively or quantitatively incorrect conclusions, because the effects of spatial and dynamical fluctuations are not properly taken into account. Nevertheless, exact results holding in any dimensions are available. We refer to, for instance, the Lieb theorem, the flat-band ferromagnetism, and the Nagaoka theorem. However, for extended versions of the Hubbard model exact results are still rare.

In this paper we rigorously prove some exact results for a generalized Hubbard model. Namely, we consider a two-orbital Hubbard Hamiltonian which incorporates also the Hund’s rule coupling. We notice that the study of relevant physical systems requires the use of such kind of model whose minimal constituents are the hopping term between different, or same, orbitals, and the on-site Coulomb and exchange interactions. This is for instance, to report few of them, the case of non-metallic Cu compounds (d^9 configuration, one hole in the two degenerate $e_g$-orbitals), low spin Ni^{3+}(d^7 configuration, one electron in $e_g$-orbitals), as well as Mn^{3+} and Cr^{2+} ions (high-spin d^4 configuration, one $e_g$ electron), and in the Ru^{14+} ions (d^4 configuration, two holes in $t_{2g}$ orbitals).

Here, our aim is to determine rigorously the symmetry properties of the above mentioned Hubbard Hamiltonian by investigating which values of the spin, orbital and $\eta$ pairing pseudospin quantum numbers occur in the ground state. A physical interpretation and discussion of the outcome is also presented. The following conditions are assumed for the validity of the results obtained below: i) the number of electrons is equal to twice number of sites in the lattice; ii) the lattice satisfies the connectivity condition; iii) the hopping amplitude is different from zero only for charge transfer between orbitals of the same type.

The scheme of the proof develops into three preliminary steps which are based on the use of the property of reflection positivity of the Hamiltonian in an opportune range of parameters and on the application of continuity arguments derived by the uniqueness of the ground state. Furthermore, the link introduced by some unitary transformations between the ground states in different regions of the space of parameters, allows to recover all the missing information about their symmetry properties.
II. THE MODEL

The explicit form of the Hamiltonian for a double orbitally degenerate Hubbard model on a connected bipartite lattice $\Gamma$ is written as:

$$H = H_t + H_{\text{int}},$$

(1)

where

$$H_t = -t \sum_{\langle i,j \rangle, \lambda, \sigma} d_{i\lambda\sigma}^\dagger d_{j\lambda\sigma}$$

(2)

$$H_{\text{int}} = U_0 \sum_{i,\lambda} n_{i\lambda\uparrow} n_{i\lambda\downarrow} + U \sum_{i,\sigma} n_{i\sigma} n_{i\bar{\sigma}} +$$

$$(U - J) \sum_{i,\sigma} n_{i\sigma} n_{i\bar{\sigma}} - J \sum_{i,\sigma} d_{i\sigma}^\dagger d_{i\bar{\sigma}} d_{i\bar{\sigma}}^\dagger d_{i\sigma}$$

(3)

Here $d_{i\lambda\sigma}^\dagger$ is the creation operator of correlated electrons with spin $\sigma$ at site $i$ on orbital $\lambda (=1,2)$, respectively and $n_{i\lambda\sigma}$ is the number operator for the electrons on the site $i$ and orbital $\lambda$. We have used the simplified notation $\bar{\sigma} = -\sigma$. The parameter $t$ is the hopping matrix between nearest-neighbor sites, and we assume that the $\lambda$ orbitals are not mixed by the hopping; $U_0$, $U$ and $J$ stand for the intra-orbital, inter-orbital Coulomb and Hund’s rule exchange interaction, respectively.

The results presented below are valid for any choice of the parameters in the Hamiltonian. Nevertheless, since the two orbitals are equivalent and can be interchanged by a properly chosen canonical transformation, we impose an additional condition on the set of the parameters, that is $U_0 = U + J$.\textsuperscript{13,14} Hereafter, we will use this relation together with the condition that $U$ and $J$ are positive. When will be needed, the sign of the Coulomb and Hund coupling will appear explicitly. Hence $H_{\text{int}}$ becomes,

$$H_{\text{int}} = (U + J) \sum_{i,\lambda} n_{i\lambda\uparrow} n_{i\lambda\downarrow} + U \sum_{i,\sigma} n_{i\sigma} n_{i\bar{\sigma}} +$$

$$(U - J) \sum_{i,\sigma} n_{i\sigma} n_{i\bar{\sigma}} - J \sum_{i,\sigma} d_{i\sigma}^\dagger d_{i\bar{\sigma}} d_{i\bar{\sigma}}^\dagger d_{i\sigma}$$

(4)

Let us introduce the following operators:

$$S = \frac{1}{2} \sum_{i,\sigma,\sigma',\lambda} d_{i\lambda\sigma}^\dagger (\sigma)_{\sigma\sigma'} d_{i\lambda\sigma'}$$

(5)

$$T = \frac{1}{2} \sum_{i,\sigma,\lambda,\lambda'} d_{i\lambda\sigma}^\dagger (\sigma)_{\lambda\lambda'} d_{i\lambda'\sigma}$$

(6)

$$\eta = \frac{1}{2} \sum_{i,\sigma,\sigma',\lambda} \epsilon(i) d_{i\lambda\sigma}^\dagger (\sigma)_{\sigma\sigma'} d_{i\lambda\sigma'}$$

(7)

where $\sigma$ are the Pauli matrices and $\epsilon(i) = \pm 1$ depending to which of the two subparts of the bipartite lattice the site $i$ belongs.

Here $S$ is the usual total spin operator, $T$ is the pseudospin operator and $\eta$ is the so called pairing operator introduced by Yang\textsuperscript{16}, extended to the case of two types of electrons. The $T$ operator has properties that are exactly analogous to the properties of the usual one-half spin operator. Indeed, its third component at each site assumes the values $\frac{1}{2}$ and $-\frac{1}{2}$ corresponding to the occupied orbitals $\lambda = 1$ and $2$, respectively. Besides, $T_i^+$ takes a fermion in the orbital $2$, at the lattice site $i$, and transports it to the orbital $1$ located at the same lattice point. Obviously, $T_i^-$ corresponds to the opposite process.

As for $S$ and $T$, the $\eta$ operators generate another $SU(2)$ algebra which has as base configurations those with one orbital doubly occupied or empty on each site.

In this model, the total spin operator $S$ commutes with the Hamiltonian $[S, H] = 0$ and is a good quantum number. Another symmetry is that one related to the orbital degree of freedom as defined by $T$. The square of the total orbital pseudospin operator $T$ and its third component $T^z$ commute with the Hamiltonian $H$, since there is no hopping between different orbitals in $H_t$. On the other hand, the square of the total $\eta$ pseudospin operator and its third component $\eta^z$ also commute with the Hamiltonian. Hence, these relations imply that these are conserved quantities and that the eigenstates of $H$ can be labelled as follows:

$$\bullet > = |E, S, S_z, T, T_z, \eta, \eta_z > .$$

We also notice that $T^-$ and $T^+$ commutes with $H$, namely the $T^-$ multiplets are degenerate in energy. This property is not valid for the $\eta^+$ and $\eta^-$ operators.

It is important pointing out that these algebras are not independent each-other and can be related by means of extended hole-particle and orbital type transformations.

Let us consider the following unitary transformation $G$:

$$G d_{i\uparrow}^\dagger G^{-1} = d_{i\uparrow}$$

(8)

$$G d_{i\downarrow}^\dagger G^{-1} = d_{i\downarrow}$$

(9)

$$G d_{i\downarrow} G^{-1} = d_{i\uparrow}$$

(10)

$$G d_{i\uparrow} G^{-1} = d_{i\downarrow}.$$  

(11)

Under this transformation one has

$$GSG^{-1} = T,$$

(12)

and the Hamiltonian $H$ is transformed as follows:

$$GHG^{-1} = (H_t + H_{\text{int}})G^{-1} = \tilde{H}_G.$$  

(13)
Here $\tilde{H}_G = H(t, J \rightarrow -J)$, i.e. $\tilde{H}_G$ can be obtained from $H$ replacing $J$ with $-J$, implying that $\tilde{H}_G$ and $H$ are related to each other.

There is also another transformation that will appear very useful later for the determination of the symmetry properties of the ground state. This is the hole-particle transformation of the base operators of creation and annihilation:

$$\mathcal{R}d_{i\uparrow}\mathcal{R}^{-1} = d_{i\uparrow}$$  \hspace{1cm} (14)

$$\mathcal{R}d_{i\downarrow}\mathcal{R}^{-1} = \varepsilon(i)d_{i\downarrow}$$  \hspace{1cm} (15)

$$\mathcal{R}d_{d\uparrow}\mathcal{R}^{-1} = d_{d\uparrow}$$  \hspace{1cm} (16)

$$\mathcal{R}d_{d\downarrow}\mathcal{R}^{-1} = \varepsilon(i)d_{d\downarrow}$$  \hspace{1cm} (17)

Under this transformation one has

$$\mathcal{R}\mathcal{S}\mathcal{R}^{-1} = \eta,$$  \hspace{1cm} (18)

and the Hamiltonian $H$ is transformed as follows:

$$\mathcal{R}HR\mathcal{R}^{-1} = \mathcal{R}(H_t + H_{\text{int}})\mathcal{R}^{-1} = \tilde{H}_R.$$  \hspace{1cm} (19)

Here $\tilde{H}_R = H(t, J \Rightarrow -J, U \Rightarrow -U)$, i.e. $\tilde{H}_R$ can be obtained from $H$ replacing $J$ with $-J$ and $U$ with $-U$, implying that $\tilde{H}_R$ and $H$ are related to each other.

Our purpose is to determine rigorously the symmetry properties of the two-band Hubbard model as defined above, by investigating which values of the spin, orbital and $\eta$ pairing pseudospin quantum numbers occur in the ground state.

As we have pointed out in the Introduction, the proof is based on three preliminary steps, followed by the application of the $\mathcal{R}$ and $\mathcal{G}$ transformations to connect the ground states in different regions of the space of parameters. We will describe how this scheme may allow to get rigorous information on the symmetry property of the ground state even where the of spin reflection positivity is not directly applicable, thus yielding a more general description of the ground state symmetry properties.

**Step I:**

Let consider firstly the case derived by applying the transformation $J \rightarrow -J$ and $U \rightarrow -U$ on $H(U, J)$ which changes $H(U, J) \rightarrow H(-U, -J)$. We remind that this transformation is realized by the application of the unitary transformation $\mathcal{R}$ on $H(U, J)$. Having this modified Hamiltonian the property of spin reflection positivity, it is possible to state that the ground state $\langle G \rangle$ of $H(-U, -J)$, for any positive value of $U$ and $J$, is unique and has total spin $S = 0$.

Moreover, for this case it is possible to determine the value of the orbital pseudospin quantum number, by looking at the relevant configurations which contribute to the ground state at values of $|U|/t \gg 1$. As $-U$ is large and negative, one ends up with states which contain only configurations with two electrons occupying the same orbital on one site. The total orbital pseudospin for this kind of situation is $T = 0$, since it is identically zero on any configuration with two electrons in the same orbital state, i.e. it is a singlet in the orbital space. It is worth pointing out that the previous statement can be generalized only at the condition that the ground state is non-degenerate. This property permits the use of continuity like arguments to extend the validity of the result from the extreme large negative limit to any finite value of $-U$.

**Step II:**

Let now analyze the parameter case relative to the Hamiltonian $H(U, J)$ which is the most relevant for systems of physical interest. It is worth pointing out that the ground state of $H(U, J)$ is unique as it is directly connected to that one of $H(-U, -J)$ by the application of the unitary transformation $\mathcal{R}$ defined in Eqs.14-19.

Moreover, by performing a perturbation up to second order in $U/t$, it is possible to show that the low energy processes of $H(U, J)$ are described by an effective Heisenberg model for spin one half on an array composed by two sublattices $A$ and $B$ defined depending on the sign of the magnetic exchange between the spin belonging to the two sublattices. The effective Hamiltonian reads as follows:

$$H_{11} = J_1 \sum_{i,j} (S_{i(A)}S_{j(A)} + S_{i(B)}S_{j(B)}) + J_2 \sum_{i,j} S_{i(A)}S_{j(B)}$$

where $J_1 = -J$, i.e. the direct Hund coupling and $J_2 = 4t^2/(U + J)$. For this Hamiltonian, it has been proved that the ground state is unique, apart from the usual $SU(2)$ degeneracy, and it has total spin $S = \frac{1}{2}|(A - B)|$, where $A(B)$ are the total number of lattice sites belonging to the sublattice $A(B)$, respectively. For building the mapping, one has to consider an array where each orbital $\lambda_i$ on the generic $j$ site is associated with an effective one, so that $\frac{1}{2}|(A - B)|$ is equal to $\frac{1}{2}(N_{\lambda_1}(A) - N_{\lambda_1}(B)) + (N_{\lambda_2}(A) - N_{\lambda_2}(B))$, $N_{\lambda_i}(K)$ being the number of $\lambda_i$ orbitals belonging to the sublattice $K$.

It is important to stress that it is again the uniqueness property of the ground state of $H_{11}$ and $H(U, J)$ that can permit the use of continuity arguments to state that the ground state $|\mathcal{G}\rangle_{11}$ of $H(U, J)$ is nondegenerate except for the $SU(2)$ symmetry, and it has total spin $S = \frac{1}{2}|(A - B)|$.

**Step III:**

Finally, let analyze the situation obtained by applying to the Hamiltonian $H(U, J)$ the transformation $\mathcal{R}$ which changes $J \rightarrow (-J)$. By means of this transformation, $H(U, J)$ is modified in $H(U, -J)$ where the Hund coupling assumes the form of antiferromagnetic instead than ferromagnetic type interaction.

As it has been performed in the step II, by means of an expansion up to the second order in $U/t$, the search of the symmetry properties of the ground state of $H(U, -J)$
may be carried into the analysis of the ground state for an effective Heisenberg model with two kinds of exchanges. The Hamiltonian for the lower energy processes is given by the following expression,

\[ H_{III} = J_1 \sum_{i,j} (S_{i}(A)S_{j}(A) + S_{i}(B)S_{j}(B)) + J_2 \sum_{i,j} S_{i}(A)S_{j}(B) \]

where \( J_1 = -J \), i.e. the direct Hund coupling with inverted sign and \( J_2 = 4t^2/(U + J) \). As for the previous case, by using the arguments of Lieb, one can show that the ground state is unique except for the \( SU(2) \) degeneracy and has total spin \( S = \frac{1}{2}(|A - B|) \). Following the same procedure of step I, one can state by means of continuity arguments that the ground state \( |G\rangle_{III} \) of \( H(U, -J) \) is unique and has total spin \( S = \frac{1}{2}(|A - B|) \). Let remind that the value of \( A(B) \) is related to the number of orbitals belonging to the sublattice \( A(B) \), respectively.

To figure out how to determine the missing information on the ground state, let remind that if we have a simultaneous eigenstate of the operator \( S \) and of \( H(U, J) \) with assigned spin eigenvalue given by \( s \), then the unitary operator \( G \) transforms this state into an eigenstate of \( H(U, -J) \) and of \( T \) whose pseudospin orbital eigenvalue is equal to \( s \) too. Then, one can proceed from case II to III and extract the information that the ground states for \( H(U, J) \) and \( H(U, -J) \) have the same value of the orbital pseudospin, i.e. \( \alpha \), so that \( |G\rangle_{II} = |S = 0, T = \alpha, \eta = ? > \) and \( |G\rangle_{III} = |S = 0, T = \alpha, \eta = ? > \).

In the same way, the unitary operator \( R \) is mapping an eigenstate of the total spin and of the Hamiltonian \( H(U, J) \) into an eigenstate with equal total \( \eta \) pairing pseudospin and of \( H(−U, −J) \).

Hence by linking via \( R \) the ground states of I and II whose one knows the value of the total spin, it is possible to deduce the pseudospin values which are now equal to 0 and \( \alpha \), respectively. After that, we have full knowledge of the symmetry character of the ground state in I and II as given by \( |G\rangle_{I} = |S = 0, T = 0, \eta = \alpha \rangle \) and \( |G\rangle_{II} = |S = \alpha, T = \alpha, \eta = 0 \rangle \), though it is still missing the information on the \( \eta \) pseudospin in the region III.

Fig. 1. Scheme of the symmetry properties of the ground state of the two-band Hubbard model for different regions of the parameter space as derived from the procedure of step II–III.

As a summary in Fig. 1 it has been reported the main findings out of the procedure performed above. Each box contains the information about the quantum numbers of the ground state and to which region of the parameter space it refers. The ket \( |S = \alpha, T = ?, \eta = ? > \) stands for the ground state, whose \( S, T, \) and \( \eta \) are the values of the total spin, orbital and \( \eta \) pairing pseudospin quantum numbers with \( \alpha \) given by \( \frac{1}{2}(|A - B|) \).

As one can see in Fig. 1, there are some quantum numbers which cannot be derived from the analysis presented in step I–III, and for this reason the missing values have been indicated with a question mark. At this point, it is crucial the use of the transformations \( G \) and \( R \) to link the different ground states and the correspondent quantum numbers. Indeed, taking advantage of these symmetry transformations, one can complete and extend the scenario as in Fig. 1 by building up the scheme as it is presented in Fig. 2.

Fig. 2. A sketch of the links introduced by \( R \) and \( G \) between the Hamiltonian and their respective ground states as discussed in step I–III. Each box contains now the complete information about the quantum numbers of the ground state in the different regions of parameter space.

It comes natural at this point to enlarge the range of investigation by moving to the case IV which corresponds to the region of parameter space where \( U \) is negative and \( J \) is positive as indicated by \( H(−U, J) \), where a priori we do not have any indication on the symmetry character of its ground state. However, the region IV can be reached either via I or III so that we can extract the complete information on all the quantum numbers and cover the missing points in II and III. The path from I to IV turns out to be useful for determining the spin and orbital pseudospin values while the way to IV via III can yield the information on the \( \eta \) pairing pseudospin so to complete the features for the ground state in these two regions, that is \( |G\rangle_{IV} = |S = 0, T = \alpha, \eta = 0 > \) and
\( (G)_{1IV} = |S = 0, T = 0, \eta = \alpha > \), respectively. We would like to point out that there is an alternative way to determine the total spin in the ground state which does not go through the perturbative arguments used in steps II-III and that it only requires the condition of semi-positive definiteness of the ground state(s). The theorem which refers to this method has been demonstrated and extensively discussed in Ref.\(^ {18} \) and it is based on the condition that a positive definite state and a positive semi-definite state built on the same basis vector space have a nonzero overlap. The key element of this proof is to show that the ground state of the assigned Hamiltonian is positive semi-definite and that the ground state of a related spin Hamiltonian built on the same lattice and on the same Hilbert space is positive definite. Hence the condition of zero overlap between those states ensures that they have the same value of the total spin. The relevance of this way of proceeding is that so doing one does not have to prove the non-degeneracy of the ground state which is fundamental in the proof that makes use of perturbative like arguments.

For the case of Hamiltonian (1), the proof of the semi-positive definiteness of the ground state is not direct because one has to use a set of complete and orthonormal basis built by means of the operators \( R \) and \( G \). Indeed, due to the property of spin reflection positivity and to the filling chosen, one can introduce a set of complete and orthonormal basis of the form \( \{|K|\phi^\alpha_\downarrow \rangle \otimes |\phi^\alpha_\uparrow \rangle\} \), where

\[
K = R(G) \text{ and } |\phi^\alpha_\downarrow \rangle = \prod_{i,\lambda,\alpha} d^\dagger_{i,\lambda,\alpha} |\phi^\alpha \rangle \text{ with } \alpha \text{ being one of the possible configurations. Then, it is possible to prove that states obtained by an expansion on the previous basis as } |W\rangle = \sum_{\alpha,\beta} W_{\alpha,\beta} |K|\phi^\alpha_\downarrow \rangle \otimes |\phi^\alpha_\uparrow \rangle \text{ have the properties of semi-positive definiteness, that is all the eigenvalues of the matrix } W_{\alpha,\beta} \text{ are not less than zero.}
\]

At this point it is possible to recover the results of steps II-III by choosing a spin Hamiltonian of the form of \( H_{II} \) \(-\( H_{III} \) whose the ground state is positive definite and use the condition of zero overlap to get the value of the total spin.

Let consider now few comments about the physical content of the symmetry character of the ground states in the regions I-IV. One important observation is that \( (G)_{II} \) is a novel quantum state which shows coexistence of unsaturated ferromagnetic spin and orbital order on a lattice assumed that \( A \neq B \), in the sense that the ground state exhibits a value of the total spin and orbital momentum which is different from zero and scales as the total number of lattice sites. Moreover, as discussed in Ref.\(^ {19} \), the state \( |G\rangle_I \) and \( |G\rangle_{IV} \) which have \( \eta = \alpha > \) can support ODLRO (off diagonal long range order) as it concerns the long distances pair correlations. It is striking that even in presence of positive Coulomb interaction (i.e region IV) one can have an \( \eta \) pairing in the ground state. This can be understood considering that the pairing for a multi-orbital system can come in a higher orbital momentum state induced by the magnetic exchange so that the effect of the local Coulomb repulsion is reduced.

Another interesting physical property comes from the \( \eta \) pairing symmetry of the ground state of the region II and III. Due to the value assumed by \( \eta \), it can be shown\(^ {20,21} \) that this state can support coexistence of charge density wave (LRD) and superconductivity (ODLRO) in a phase which is usually indicated as supersolid. The statement is true in the sense that if LRO appears then by symmetry the state manifests also ODLRO and viceversa.

Indeed, it comes that due to the rotational invariance of the ground state in the \( \eta \) space, the Fourier transform of the transverse correlation function at equal time \( \eta^+(\mathbf{q}) \) are proportional to the diagonal one \( \eta^2(\mathbf{q}) \), thus if one of the two has the property to stay finite for some value of the momentum \( \mathbf{q} \) in the thermodynamic limit, the same happens for the other one.

### III. CONCLUSIONS

In conclusion we have shown that the symmetry features of the ground state at half filling of a generalized Hubbard model which includes the Hund’s rule coupling, can be obtained in a large range of the parameter space, only by combining the property of spin reflection positivity and the use of special unitary transformations. In particular it has been possible to extract all the most relevant symmetry characteristics related to conservation laws of the Hamiltonian that are represented by the orbital, spin and \( \eta \) pairing pseudospin operators. We want to notice that the use of spin reflection positivity has been largely used in literature to rigorously determine the symmetry character and the correlation functions of the ground state for different model Hamiltonians. In our work this idea is generalized and enriched in a simple scheme which takes advantage of the positivity character of the ground state in one region of the parameter space and of the use of symmetry transformations to scan a larger part of the phase diagram.

Moreover, the procedure presented above gives also the opportunity to extract information on the phase diagram of the model in exam, such as the occurrence of a novel quantum state with coexistence of spin and orbital order, the manifestation of ODLRO, and where the ground state can be a supersolid. Finally, it is worth pointing out that a similar model has been exactly solved in the one-dimensional case in the case of strong coupling regime\(^ {22} \) i.e. when the Coulomb repulsion is such that the double occupancy of electrons on the same site of the same orbital is excluded.

\(^ {1} \) R. Micnas, J. Ranninger, and S. Robaszkiewicz, Rev. Mod. Phys. 62, 113 (1990) and references therein.

\(^ {2} \) J. de Boer and A. Schadschneider, Phys. Rev. Lett. 75,
4298 (1995); Z. Szabó, Phys. Rev. B 59, 10007 (1999); C. Dziurzik, A. Schadschneider, and J. Zittartz, EPJ 12, 209 (1999).

3 L. M. Roth, Phys. Rev. 149, 306 (1966); K. I. Kugel and D. I. Khomskii, Sov. Phys. JETF 37, 725 (1974); M. Cyrot and C. Lyon-Caen, J. Phys. (Paris) 36, 253 (1975).

4 M. Imada, A. Fujimori, and Y. Tokura, Rev. Mod. Phys. 70, 1039 (1998).

5 E. H. Lieb and F. Y. Wu, Phys. Rev. Lett. 20, 1445 (1968).

6 W. Metzner and D. Vollhardt, Phys. Rev. Lett. 62, 324 (1989); E. Müller-Hartmann, Z. Phys. B 74, 507 (1989); 76, 211 (1989).

7 E. H. Lieb, Phys. Rev. Lett. 62, 1201 (1989).

8 A. Mielke, J. Phys. A 24, L73 (1991); 24, 3311 (1991); 25, 4335 (1991); H. Tasaki, Phys. Rev. Lett. 69, 1608 (1992); A. Mielke, Phys. Rev. Lett. 82, 4312 (1999).

9 Y. Nagaoka, Phys. Rev. 147, 392 (1966); H. Tasaki, Phys. Rev. B 40, 9192 (1989).

10 G. Su, Solid State Comm. 104, 541 (1997).

11 For a review, see: Exactly Solvable Models of Strongly Correlated Electrons, edited by V. E. Korepin and F. H. L. Essler (World Scientific Publishing, Singapore, 1994).

12 S. Feldkemper, W. Weber, J. Schulenburg, and J. Richter, Phys. Rev. B 52, 313 (1995).

13 A. M. Oles, L. F. Feiner and J. Zaanen, Phys. Rev. B 61, 6257 (2000) and references therein.

14 J. S. Griffith, The Theory of Transition Metal Ions, Cambridge University Press, Cambridge, 1971.

15 Y. Maeno, H. Hashimoto, K. Yoshida, S. Nishizaki, T. Fujita, J. G. Bednorz, and F. Lichtenberg, Nature 372, 532 (1994).

16 C. N. Yang and S. C. Zhang, Mod. Phys. Lett. 34, 759 (1990); C. N. Yang, Phys. Lett. A 161, 292 (1991); M. Pernici, Europhys. Lett. 12, 75 (1990).

17 E. Lieb and D. Mattis, J. Math. Phys. 3, 749 (1962).

18 S. Q. Shen, Int. J. Mod. Phys. B 12, 709 (1998).

19 S. Q. Shen, Z. M. Qiu, Phys. Rev. Lett. 71, 4238 (1993).

20 F. C. Pu, and S. Q. Shen, Phys. Rev. B 50, 16086 (1994).

21 M. Cuoco and C. Noce, Phys. Rev. B 59, 14831 (1999).

22 S. Q. Shen, Phys. Rev. B 57, 6474 (1998).