Abstract

We propose a new approach, two-dimensional binary fused compressive sensing (2DBFCS) to recover 2D sparse piece-wise signals from 1-bit measurements, exploiting group sparsity in 2D 1-bit compressive sensing. The proposed method is a modified 2D version of the previous binary iterative hard thresholding (2DBIHT) algorithm, where, the objective function consists of a 2D one-sided $\ell_1$ (or $\ell_2$) function and an indicator function of $K$-sparsity and an indicator function of total variation (TV) or modified TV (MTV) constraint (the MTV favors both sparsity and piece/wise smoothness while the TV promotes the whole smoothness). The subgradient of 2D one-sided $\ell_1$ (or $\ell_2$) barrier and the projection onto the $K$-sparsity and TV or MTV constraint set are easy to compute, such that the forward-backward splitting can be applied in 2DBFCS efficiently. Experiments on the recovery of 2D sparse piece-wise smooth signals show that the proposed 2DBFCS with the TV or MTV is able to take advantage of the piece-wise smoothness of the original signal, achieving more accurate recovery than 2DBIHT. Especially, the 2DBFCS with the MTV and the $\ell_2$ barrier performs best amongst the algorithms.

1 Introduction

We focus on recovering a sparse piece-wise smooth two-dimensional (2D) signal (an image) $X$ from 1-bit measurements,

$$Y = \text{sign}(AX + W),$$  

(1)

where $Y \in \{-1, 1\}^{M \times L}$ is the measurement matrix, sign is the element-wise sign function that returns $+1$ for positive arguments and $-1$ otherwise, $A \in \mathbb{R}^{M \times N}$ is the known sensing matrix, $X \in \mathbb{R}^{N \times L}$ is the original 2D signal, and $W$ is additive noise. Unlike in conventional compressive sensing (CS), 1-bit measurements lose any information about the magnitude of the original signal $X$. The goal is then to recover $X$, but only up to an unknown and unrecoverable magnitude \[ \|X\|_0 = k \].

Our innovations, with respect to 1-bit CS as proposed in [2], are twofold: a) we address the 2D case; b) more importantly, we introduce a new regularizer favoring both sparsity and piece-wise smoothness, which can be seen as a modified 2D version of fused lasso [3]. This new regularizer is the indicator of a union of convex subspaces (total-variation balls) of the canonical subspaces, simultaneously enforcing sparsity and smoothness within each connected subset of non-zero elements. The rationale is that, when imposing smoothness and sparseness, smoothness should not interfere with sparsity, i.e., it should not be imposed across the transitions from zero to non-zero elements. The proposed regularizer promotes sparseness and smoothness and (although it is non-convex) has a computationally feasible projection, based on which we propose a modified version of the binary iterative hard thresholding (BIHT) algorithm [4].

2 2D Binary Iterative Hard Thresholding (2DBIHT)

To recover $X$ from $Y$, we first consider a 2D version of the criterion proposed by Jacques et al. [4]

$$\min_{X} f(Y \odot AX) + \tau_{\xi_{C}}(X)$$  

subject to $\|X\|_2 <= 1$,  

(2)

where: the operation $\odot$ denotes element-wise (Hadamard) product; $\tau_{C}(X)$ denotes the indicator function of set $C$.

$$\tau_{C}(X) = \begin{cases} 0, & X \in C; \\ \infty, & X \notin C; \end{cases}$$  

(3)

$K = \{X \in \mathbb{R}^{N \times L} : \|X\|_0 <= K\}$ (with $\|V\|_0$ denoting the number of non-zero components in $V$) is the set of $K$-sparse $N \times L$ images; $\|X\|_2 = (\sum_{i,j}X_{i,j}^2)^{1/2}$ is the Euclidean norm, and $f$ is one of the penalty functions defined next. To penalize linearly the violations of the sign consistency between the observations and the estimate $\hat{X}$, the barrier function is chosen as $f(Z) = 2\|Z\|_1$, where $Z = \min\{Z, 0\}$ (with the minimum applied entry-wise and the factor 2 included for later convenience) and $\|V\|_1 = \sum_{i,j}|V_{i,j}|$ is the $\ell_1$ norm of $V$. A quadratic barrier for sign violations (see [5]) is achieved by using $f(Z) = \|Z\|_2^2$, where the factor $1/2$ is also included for convenience. The iterative hard thresholding (IHT) [1] algorithm applied to $\hat{X}$ (ignoring the norm constraint during the iterations) leads to the 2DBIHT algorithm, which is a 2D version of the binary iterative hard thresholding (BIHT) [4].

Algorithm 2DBIHT

1. Set $k = 0, \tau > 0, X_0$ and $K$  
2. repeat  
3. $V_{k+1} = X_k - \tau f(Y \odot (AX_k))$  
4. $X_{k+1} = P_{\tau\xi}(V_{k+1})$  
5. $k \leftarrow k + 1$  

In this algorithm, $\partial f$ denotes the subgradient of the objective (see [4], for details), which is given by

$$\partial f(Y \odot (AX)) = \begin{cases} A^T (\text{sign}(AX) - Y), & \ell_1 \text{ barrier}, \\ A^T (Y \odot (Y \odot (AX))^{-1}), & \ell_2 \text{ barrier}, \end{cases}$$  

(4)

Step 3 corresponds to a sub-gradient descent step (with step-size $\tau$), while Step 4 performs the projection onto the non-convex set $\xi_k$, which corresponds to computing the best $K$-term approximation of $V$, that is, keeping $K$ largest components in magnitude and setting the others to zero. Finally, the returned solution is projected onto the unit sphere to satisfy the constraint $\|X\|_2 = 1$. The versions of BIHT with $\ell_1$ and $\ell_2$ objectives are referred to as 2DBIHT-$\ell_1$ and 2DBIHT-$\ell_2$, respectively.

3 2D Fused Binary Compressive Sensing (2DBFCS)

The proposed formulation essentially adds a new constraint of low (modified) total variation to the criterion of 2DBIHT [4], which encourages 4-neighbor elements to be similar, justifying the term “fused”.

3.1 2DBFCS with Total Variation

We first propose the following model:

$$\min_{X} \frac{f(Y \odot AX) + \tau_{\xi_{C}}(X) + \tau_T(X)}{X}$$  

subject to $\|X\|_2 = 1$,  

(5)

where $T_e = \{X \in \mathbb{R}^{N \times L} : \text{TV}(X) \leq \varepsilon\}$, with $\text{TV}(X)$ denoting the total variation (TV), which in the two-dimensional (2D) case is defined as

$$\text{TV}(X) = \sum_{i=1}^{N-1} \sum_{j=1}^{L-1} \left|X_{i+1,j} - X_{i,j}\right| + \left|X_{i,j+1} - X_{i,j}\right|,$$  

(6)

and $\varepsilon$ is a positive parameter. In the same vein as 2DBIHT, the proposed algorithm is as follows:

Algorithm 2DBFCS-TV

1. Set $\tau > 0, \varepsilon > 0, K$, and $X_0$  
2. repeat  
3. $V_{k+1} = X_k - \tau f(Y \odot (AX_k))$  
4. $X_{k+1} = P_{\tau\xi}(P_{\varepsilon}(V_{k+1}))$  
5. $k \leftarrow k + 1$  

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Exploiting Group Sparsity in 2D 1-Bit Compressive Sensing

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6. until some stopping criterion is satisfied.
7. return \( X / \| X \|_2 \)

where line 4 is to compute the projection onto \( T_\varepsilon \), which can be obtained by using the algorithm proposed by Fadili and Peyré [3]. The versions of the 2DBFCS-TV algorithm with \( \ell_1 \) and \( \ell_2 \) objectives are referred to as 2DBFCS-TV-\( \ell_1 \) and 2DBFCS-TV-\( \ell_2 \), respectively.

### 3.2 2DFBCS with Modified Total Variation

We propose the following problem formulation of 2DFBCS:

\[
\begin{align*}
\min_{X \in \mathbb{R}^{N \times L}} & \ f(Y \odot (AX)) + t_{F_k} (X) \\
\text{subject to} & \ \|X\|_2 = 1,
\end{align*}
\]

where the set \( F_k \) requires a more careful explanation. As usual, define \( \Sigma_K = \{ X \in \mathbb{R}^{N \times L} : \|X\|_0 \leq K \} \) (with \( \|X\|_0 \) denoting the number of non-zeros in \( X \)) as the set of \( K \)-sparse \( N \times L \) images. Consider the undirected 4-nearest-neighbors graph on the sites of \( N \times L \) images, i.e., \( G = (\mathcal{N}, \mathcal{E}) \), where \( \mathcal{N} = \{(i,j), i = 1, \ldots, N, j = 1, \ldots, L\} \) and \( (i,j), (k,l) \in \mathcal{E} \Leftrightarrow \{(i = k) \wedge ((j = l) \vee ((i = k) \wedge ((j = l) \wedge (i = j))\} \). Given some \( V \in \mathbb{R}^{N \times L} \), let \( \tilde{G}(V) = (\tilde{\mathcal{N}}, \tilde{\mathcal{E}}(V)) \) be the subgraph of \( G \) obtained by removing all the nodes corresponding to zero elements of \( V \) (that is, \( (i,j) \in \tilde{\mathcal{N}}(V) \Leftrightarrow V_{i,j} \neq 0 \)), as well as the corresponding edges. Naturally, \( \tilde{G}(V) \) may not be a connected graph; define \( \tilde{G}_i(V) \) as the set of connected subgraphs of \( \tilde{G}(V) \), where \( \tilde{G}_i(V) = \{ \tilde{G}_1(V), \ldots, \tilde{G}_i(V) \} \). Define the normalized TV of the sub-image of \( V \) corresponding to each of these connected subgraphs as

\[
\tilde{\text{TV}}(\tilde{G}_i(V)) = \frac{\|E_i(V)\|}{\|\tilde{X}\|_1} = \frac{\sum_{(i,j),(k,l) \in \tilde{E}(i)} |V_{i,j} - V_{k,l}|}{\|\tilde{X}\|_1}
\]

(assuming \( |\tilde{E}(i)| > 0 \)) where \( V_{ij} \) is the subgraph indexed by \( \tilde{G}_i(V) \). Finally, the set \( \tilde{F}_k \subseteq \Sigma_K \) is defined as

\[
\tilde{F}_k = \{ X \in \Sigma_K : \tilde{\text{TV}}(X_{\tilde{G}_i(X)}) \leq \varepsilon, k = 1, \ldots, K(X) \}
\]

In short, \( \tilde{F}_k \) is the set of \( K \)-sparse images such that the normalized TV of each of its connected blocks of non-zeros doesn’t exceed \( \varepsilon \). Notice that this is different from the intersection of a TV ball with \( \Sigma_K \), as considered in [3].

In the same vein as the 2DBIHT, we propose the following BIHT-type algorithm to solve \( \tilde{F}_k \):

**Algorithm 2DFBCS-MTV**

1. Set \( \tau > 0, \theta > 0, K, \) and \( X_0 \)
2. repeat
3. \( V_{k+1} = X_k - \tau \partial f (X \odot (AX_k)) \)
4. \( X_{k+1} = P_{\tilde{F}_k} (V_{k+1}) \)
5. \( k \leftarrow k + 1 \)
6. until some stopping criterion is satisfied.
7. return \( X / \| X \|_2 \)

In this algorithm, line 3 is also a sub-gradient descent step, where \( \partial f \) is defined as \( \tilde{F}_k \) while line 4 performs the projection onto \( \tilde{F}_k \). Although \( \tilde{F}_k \) is non-convex, here we can briefly show that \( \tilde{F}_k \) can be computed as the follows (the details of computing \( \tilde{F}_k \) are shown in Appendix): first, project onto \( \Sigma_K \), i.e., \( U = P_{\Sigma_K} (V_k) \); then, \( X \in \tilde{F}_k \) is obtained by projecting every connected group of non-zeros in \( U \) onto the \( \varepsilon \)-radius normalized TV ball \( B_\varepsilon \):

\[
B_\varepsilon = \{ X_{\tilde{G}_i(X)} \in \mathbb{R}^{N \times L} : \tilde{\text{TV}}(X_{\tilde{G}_i(X)}) \leq \varepsilon, k = 1, \ldots, K(X) \}
\]

### 4 Experiments

In this section, we report results of experiments aimed at comparing the performance of 2DFBCS with that of 2DBHIT. Without loss of generality, we assume the original group-sparse image \( X \in \mathbb{R}^{400 \times 100} \) in which, 10 line-groups are randomly generated, and each line-group has 9 elements valued by 10 or -10, and then it is followed by a normalized operation \( X = X / \| X \|_2 \). The sensing matrix \( A \) is a \( 200 \times 400 \) matrix whose components are sampled from the standard normal distribution. And the variance of white Gaussian noise \( W \in \mathbb{R}^{200 \times 100} \) is 0.01. Then the observations \( Y \) are obtained by \( \mathbf{Y} = AX + W \).

We run the aforementioned six algorithms, the stepsizes of 2DBHIT-\( \ell_1 \) and 2DBHIT-\( \ell_2 \) are set as \( \tau = 1 \) and \( 1/m \), respectively, and the parameters of 2DFBCS-TV-\( \ell_1 \), 2DFBCS-TV-\( \ell_2 \), 2DBFCS-MTV-\( \ell_1 \) and 2DBFCS-MTV-\( \ell_2 \) are hand tuned for the best improvement in signal-to-noise. The recovered signals are shown in Figure 1 from which, we can clearly see that the proposed 2DBFCS basically performs better than 2DBHIT. In general, the algorithms with the \( \ell_2 \) barrier outperforms that with the \( \ell_1 \) barrier. Especially, the 2DFBCS-MTV-\( \ell_2 \) shows its superiority over other algorithms, and nevertheless, the 2DFBCS-TV-\( \ell_2 \) is also good at recovering sparse piece-wise images.

### 5 Conclusions

We have proposed the 2D binary fused compressive sensing (2DFBCS) to recover 2D sparse piece-wise smooth signals from 2D 1-bit compressive measurements. We have shown that if the original signals are in fact sparse and piece-wise smooth, the proposed method, is able to take advantage of the piece-wise smoothness of the original signal, outperforms (under several accuracy measures) the 2D version of the previous method binary iterative hard thresholding (termed 2DBHIT), which relies only on sparsity of the original signal. Future work will involve using the technique of detecting sign flips to obtain a robust version of 2DBFCS.

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