Extending silted algebras to cluster-tilted algebras

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Abstract
It is well known that the relation-extensions of tilted algebras are cluster-tilted algebras. In this paper, we extend the result to silted algebras and prove some extension of silted algebras are cluster-tilted algebras.

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1 Introduction
Cluster-tilted algebras were introduced by Buan, Marsh, Reiten and et al. [3], and also in [7] for type A. Let A be a triangular algebra whose global dimension is at most two over an algebraically closed field k. The trivial extension of A by the A-A-bimodule Ext^2_A(DA, A) is called the relation-extension [1] of A, where D=Hom_k(−, k) is the standard duality. It is proved that the relation-extension of every tilted algebra is cluster-tilted, and every cluster-tilted algebra is of this form in [1].

The concept of silting complexes originated from [13] and 2-term silting complexes are of particular interest and important for the representation of algebra. In [6], the endomorphism algebras of 2-term silting complexes were introduced by Buan and Zhou. They also defined the concept of the silted algebra [5] which is the endomorphism algebras of 2-term silting complex over the derived category of hereditary algebras and proved that an algebra is silted if and only if it is shod [8] (projective dimension or injective dimension of every indecomposable module is at most one). In particular, tilted algebras are silted, indeed, the minimal projective presentation of a tilting module T over the hereditary algebra H gives rise to a 2-term silting complex P in K^b(proj H), and that there is an isomorphism of algebras End_H(T) ≅ End_{D^b(H)}(P).

As a generalization of tilting modules, support τ-tilting modules were introduced by Adachi, Iyama and Reiten [2]. They also shown that there is a bijection between support τ-tilting modules and 2-term silting complexes(see, [2, Theorem 3.2]). This result provided that every silted algebra can be described as the triangular matrix algebra \left( \begin{array}{cc} B & 0 \\ M & H_1 \end{array} \right) \right) where B is a tilted algebra, H_1 is a hereditary algebra and M is a H_1-B-bimodule (see, Proposition 3.2). It is a natural question whether silted algebras can be extended to cluster-tilted algebras.

In this paper, we give a positive answer and construct cluster-tilted algebras from silted algebras. We call a silted algebra A with respect to (T, P) for some hereditary algebra H if

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there exists a 2-term silting complex $P$ in $\mathcal{D}^b(H)$ which corresponding to the support $\tau$-tilting pair $(T, P)$ in $\text{mod} \, H$ such that $A \cong \text{End}_{\mathcal{D}^b(H)}(P)$. Our main results as follows.

**Theorem 1.1.** Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. Then the matrix algebra $\begin{pmatrix} B \otimes \text{Ext}^1_H(T, \tau^{-1}T) & \text{Hom}_H(P, \tau^{-1}T) \\ M & H_1 \end{pmatrix}$ is a cluster-tilted algebra.

As a consequence, we have the following result.

**Theorem 1.2.** Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. If $\text{Hom}_H(P, \tau^{-1}T) = 0$, then the triangular matrix algebra $\begin{pmatrix} B \otimes \text{Ext}^2_B(DB, B) & 0 \\ M & H_1 \end{pmatrix}$ is a cluster-tilted algebra.

Note that a tilted algebra is exactly silted algebra with respect to $(T, 0)$ for some hereditary algebra $H$, we can easy get the relation-extension of every tilted algebra is cluster-tilted.

Throughout this paper, all algebras are finite dimensional algebras over an algebraically closed field $k$. For an algebra $A$, we denoted by $\text{mod} \, A$ the category of finitely generated right $A$-modules and $\text{proj} \, A$ the category of finitely generated projective right $A$-modules. $K^b(\text{proj} \, A)$ will stand for the bounded homotopy category of finitely generated projective right $A$-modules and $\mathcal{D}^b(A)$ is the bounded derived category of finitely generated right $A$-modules. For a $A$-module $M$, $|M|$ is the number of pairwise non-isomorphic direct summands of $M$. All modules considered basic.

## 2 Preliminaries

### 2.1 Tilted algebras

Let $A$ be an algebra. An $A$-module $T$ is called tilting if (1) the projective dimension of $T$ is at most one, (2) $\text{Ext}^1_A(T, T) = 0$ and (3) $|T| = |A|$. The endomorphism algebra of a tilting module over a hereditary algebra is called a tilted algebra [10]. The following result is very useful.

**Theorem 2.1.** [9] Let $H$ be a hereditary algebra, $T$ a tilting $H$-module and $B = \text{End}_H(T)$ the corresponding tilted algebra. Then we have

1. The derived functor $\text{RHom}_H(T, -) : \mathcal{D}^b(H) \to \mathcal{D}^b(B)$ is an equivalence which maps $T$ to $B$.

2. $\text{RHom}_H(T, -)$ commutes with the Auslander-Reiten translations and the shifts in the respective categories.

### 2.2 Silted algebras

**Definition 2.2.** ([2, Definition 0.1]) Let $T \in \text{mod} \, A$.

1. $T$ is called $\tau$-rigid if $\text{Hom}_A(T, \tau T) = 0$.

2. $T$ is called $\tau$-tilting if it is $\tau$-rigid and $|T| = |A|$.
(3) $T$ is called support $\tau$-tilting if it is a $\tau$-tilting $A/eA$-module for some idempotent $e$ of $A$.

Sometimes, it is convenient to view support $\tau$-tilting modules and $\tau$-rigid modules as certain pairs of modules in $\mod A$.

**Definition 2.3.** ([2] Definition 0.3) Let $(T, P)$ be a pair in $\mod A$ with $P \in \proj A$.

1. $(T, P)$ is called a $\tau$-rigid pair if $M$ is $\tau$-rigid and $\Hom_A(T, M) = 0$.

2. $(T, P)$ is called a support $\tau$-tilting pair if $T$ is $\tau$-rigid and $|T| + |P| = |A|$.

It was showed in [2] Proposition 2.3] that $(T, P)$ is a support $\tau$-tilting pair in $\mod A$ if and only if $T$ is a $\tau$-tilting $A/eA$-module with $eA \cong P$.

Let $P$ be a complex in $K^b(\proj A)$. Recall that $P$ is silting if $\Hom_{K^b(\proj A)}(P, P[i]) = 0$ for $i > 0$, and if $P$ generates $K^b(\proj A)$ as a triangulated category. Moreover, $P$ is called 2-term if it only has non-zero terms in degree 0 and $-1$.

The next result show that the relationship between support $\tau$-tilting modules and 2-term silting complexes. For convenience, we denote by $st\tau$-$tilt$ $A$ all support $\tau$-tilting modules over the algebra $A$ and 2-silt $A$ all 2-term silting complexes over $K^b(\proj A)$.

**Theorem 2.4.** ([2] theorem 3.2] There exists a bijection between $st\tau$-$tilt$ $A$ and 2-silt $A$ given by $(T, P) \in st\tau$-$tilt A \rightarrow P_1 \oplus P \rightarrow P_0 \in 2$-$silt$ $A$ and $P \in 2$-$silt$ $A \rightarrow H^b(P) \in st\tau$-$tilt$ $A$, where $P_1 \rightarrow P_0$ is a minimal projective presentation of $T$.

We call an algebra $A$ is silted if there is a hereditary algebra $H$ and $P \in 2$-$silt$ $H$ such that $A \cong \End_{D^b(H)}(P)$.

### 2.3 Cluster-tilted algebras

The cluster category $\mathcal{C}_H$ of a hereditary algebra $H$ is the quotient category $\mathcal{D}^b(H)/F$ where $F = \tau^{-1}_{\mathcal{D}}[1]$ and $\tau^{-1}_{\mathcal{D}}$ is the inverse of the AuslanderReiten translation in $\mathcal{D}^b(H)$. The space of morphisms from $\tilde{X}$ to $\tilde{Y}$ in $\mathcal{C}_H$ is given by $\Hom_{\mathcal{C}_H}(\tilde{X}, \tilde{Y}) = \oplus_{i \in \mathbb{Z}} \Hom_{\mathcal{D}^b(H)}(X, F^iY)$. It is shown that $\mathcal{C}_H$ is a triangulated category. An object $\tilde{T} \in \mathcal{C}_H$ is called tilting if $\Ext_{\mathcal{C}_H}^1(\tilde{T}, \tilde{T}) = 0$ and the number of isomorphism classes of indecomposable summands of $\tilde{T}$ equals $|H|$. The algebra of endomorphisms $C = \End_{\mathcal{C}_H}(T)$ is called cluster-tilted. It is proved that the relation-extension of every tilted algebra is cluster-tiled, and every cluster-tilted algebra is of this form in $[1]$.

### 3 Main results

In this section, we prove our main results and give an example to illustrate our results.

**Definition 3.1.** We call a silted algebra $A$ with respect to $(T, P)$ for some hereditary algebra $H$ if there exists $P \in 2$-$silt$ $H$ which corresponding to $(T, P) \in st\tau$-$tilt$ $H$ such that $A \cong \End_{D^b(H)}(P)$.

**Proposition 3.2.** Let $A$ be a silted algebra with respect to $(T, P)$. Then $A$ is a triangular matrix algebra $\begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ where $B$ is a tilted algebra, $H_1$ is a hereditary algebra and $M$ is a $H_1$-$B$-bimodule.
Proof. Suppose that there is a hereditary algebra $H$ and $P \in 2$-silt $H$ which corresponding to $(T, P) \in s\tau$-tilt $H$ such that $A \cong \text{End}_{\mathcal{D}^b(H)}(P)$, then we have

$$A \cong \text{End}_{\mathcal{D}^b(H)}(P) \cong \text{End}_{\mathcal{D}^b(H)}(T \oplus P[1]) \text{(by Theorem 2.4)}$$

Take $H' = H/HeH$, we have $H'$ is a hereditary algebra, where $eH \cong P$. Therefore, $T$ is a tilting $H'$-module and $B = \text{End}_{\mathcal{D}^b(H)}(T) \cong \text{End}_H(T) \cong \text{End}_{H'}(T)$ is a tilted algebra. Moreover, $H_1 = \text{End}_{\mathcal{D}^b(H)}(P[1]) \cong \text{End}_H(P) \cong eHe$ is a hereditary algebra. Note that $\text{Hom}_{\mathcal{D}^b(H)}(P[1], T) = 0$ since $P$ is projective and $M = \text{Hom}_{\mathcal{D}^b(H)}(T, P[1]) \cong \text{Ext}_H(T, P)$ is a $H_1$-$B$-bimodule, we have $A$ is a triangular matrix algebra.

**Lemma 3.3.** Let $\mathcal{C}_H$ be a cluster category of a hereditary algebra $H$ and $T \in \text{mod} H$. Then we have

$$\text{End}_{\mathcal{C}_H}(\tilde{T}, \tilde{T}) \cong \text{End}_{\mathcal{D}^b(H)}(T) \ltimes \text{Hom}_{\mathcal{D}^b(H)}(T, FT),$$

where $\ltimes$ stand for the trivial extension.

**Proof.** It follows from [1, Lemma 3.3].

**Theorem 3.4.** Let $A = \left( \begin{array}{cc} B & 0 \\ M & H_1 \end{array} \right)$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. Then the matrix algebra $\left( \begin{array}{cc} B \ltimes \text{Ext}_H^1(T, \tau_H^{-1}T) & \text{Hom}_H(P, \tau_H^{-1}T) \\ M & H_1 \end{array} \right)$ is a cluster-tilted algebra.

**Proof.** Let $A = \left( \begin{array}{cc} B & 0 \\ M & H_1 \end{array} \right)$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. Then $\tilde{T} \oplus \tilde{P}[1]$ is a cluster-tilting object in $\mathcal{C}_H$. For any two $H$-modules $X$ and $Y$, we have $\text{Hom}_{\mathcal{D}^b(H)}(X, Y[i]) = 0$ for all $i \geq 2$ since $H$ is hereditary. Hence, we have

$$\text{End}_{\mathcal{C}_H}(\tilde{T} \oplus \tilde{P}[1]) \cong \left( \begin{array}{cc} \text{End}_{\mathcal{C}_H}(\tilde{T}) & \text{Hom}_{\mathcal{C}_H}(\tilde{P}[1], \tilde{T}) \\ \text{Hom}_{\mathcal{C}_H}(\tilde{T}, \tilde{P}[1]) & \text{End}_{\mathcal{C}_H}(\tilde{P}[1]) \end{array} \right)$$

which is a cluster-tilted algebra.

As a consequence, we have the following result.

**Corollary 3.5.** Let $A$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. If $T$ is injective, then $A$ is hereditary. In particular, a tilted algebra which induced by a injective tilting module is hereditary.
Proof. Since $T$ is injective, we have $\tau^{-1}_H T = 0$. By Theorem 3.4, $A$ is a cluster-tilted algebra whose global dimension is at most three. Note that every cluster-tilted algebra is 1-Gorenstein \cite{12}. Since the projective dimension of every module over a 1-Gorenstein algebra is at most one or infinite, we get the global dimension of $A$ is at most one, and so $A$ is hereditary.

\[ \text{Theorem 3.6.} \] Let $A = \begin{pmatrix} B & 0 \\ M & H_1 \end{pmatrix}$ be a silted algebra with respect to $(T, P)$ for some hereditary algebra $H$. If $\text{Hom}_H(P, \tau^{-1}_H T) = 0$, then the triangular matrix algebra $\begin{pmatrix} B \times \text{Ext}^2_B(DB, B) & 0 \\ M & H_1 \end{pmatrix}$ is a cluster-tilted algebra.

Proof. Take $H' = H/eH$, we have $\tau^{-1}_H T$ is a $H'$-module since $\text{Hom}_H(P, \tau^{-1}_H T) = 0$ where $eH \cong P$. Therefore, we have

\[
\text{Ext}^1_H(T, \tau^{-1}_H T) \cong \text{Ext}^1_H(T, \tau^{-1}_H T) \\
\cong \text{Hom}_{\mathcal{D}^b(H')}(T, F'T) \\
\cong \text{Hom}_{\mathcal{D}^b(B)}(B, F''B)\text{(by Lemma 2.1)} \\
\cong \text{Hom}_{\mathcal{D}^b(B)}(\tau_{\mathcal{D}^b(B)}B[1], B[2]) \\
\cong \text{Hom}_{\mathcal{D}^b(B)}(DB, B[2]) \\
\cong \text{Ext}^2_B(DB, B)
\]

where $F' = \tau^{-1}_{\mathcal{D}^b(H')}[1]$ and $F'' = \tau^{-1}_{\mathcal{D}^b(B)}[1]$ is the functor corresponding to $F'$ in the derived category $\mathcal{D}^b(B)$.

Note that a tilted algebra is exactly silted algebra with respect to $(T, 0)$ for some hereditary algebra $H$, we can easy get the following result.

\[ \text{Corollary 3.7.} \] The relation-extension of every tilted algebra is cluster-tilted.

\[ \text{Example 3.8.} \] Let $H$ be a hereditary algebra given by the following quiver,

\[ \begin{array}{c}
1 \\
\downarrow \\
3 \quad 4 \\
\downarrow \\
2
\end{array} \]

The support $\tau$-tilting pair $(T, P) = (P_4 \oplus P_1 \oplus S_1, P_2)$ corresponding to the 2-term silting complex $0 \to P_4 \oplus 0 \to P_1 \oplus P_3 \to P_1 \oplus P_2 \to 0$ which induced a silted algebra given as follows,

\[ 1 \xleftarrow{\gamma} 2 \xrightarrow{\beta} 3 \xrightarrow{\alpha} 4 \]

with the relations $\alpha \beta = 0$ and $\beta \gamma = 0$. Note that

\[ \dim_k \text{Ext}^1_H(T, \tau^{-1}_H T) = 2, \quad \dim_k \text{Hom}_H(P, \tau^{-1}_H T) = 1. \]
(in fact, $\dim_k \text{Ext}^1_H(S_1, \tau_H^{-1}P_1) = 1$, $\dim_k \text{Ext}^1_H(S_1, \tau_H^{-1}P_1) = 1$, $\dim_k \text{Hom}_H(P_2, \tau_H^{-1}P_1) = 1$.)

By Theorem 3.4, we can construct a cluster-tilted algebra given by the following quiver,

\[ \begin{array}{c}
1 \\
\delta \\
\beta \\
\gamma \\
\alpha \\
\beta \\
\epsilon \\
\delta \\
2 \\
3 \\
4 \\
\end{array} \]

with relations $\gamma \delta = \epsilon \alpha, \alpha \beta = 0, \beta \gamma = 0, \beta \epsilon = 0, \delta \beta = 0$.

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