Symmetric Zinbiel superalgebras

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ABSTRACT
The notion of symmetric Zinbiel superalgebras is introduced. We prove that the nilpotency index of a symmetric Zinbiel superalgebra is not greater than 4 and describe two-generated symmetric Zinbiel algebras and odd generated superalgebras. We discuss identities of mono and binary symmetric Zinbiel and Leibniz algebras. It is proven that each quadratic Zinbiel algebra is 2-step nilpotent. Also, we study double extensions of symmetric Zinbiel algebras.

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1. Introduction
Loday introduced a class of symmetric operads generated by one bilinear operation subject to one relation making each left-normed product of three elements equal to a linear combination of right-normed products:

\[(a_1a_2)a_3 = \sum_{\sigma \in S_3} x_{\sigma}a_{\sigma(1)}(a_{\sigma(2)}a_{\sigma(3)})\]

such an operad is called a parametrized one-relation operad. For a particular choice of parameters \(\{x_{\sigma}\}\), this operad is said to be regular if each of its components is the regular representation of the symmetric group; equivalently, the corresponding free algebra on a vector space \(V\) is, as a graded vector space, isomorphic to the tensor algebra of \(V\). Bremner and Dotsenko classified, over an algebraically closed field of characteristic zero, all regular parametrized one-relation operads. In fact, they proved that each such operad is isomorphic to one of the following five operads: the left-nilpotent operad, the associative operad, the Leibniz operad, the Zinbiel operad, and the Poisson operad [7]. An algebra \(A\) is called a (left) Zinbiel algebra if it satisfies the identity

\((xy)z = x(yz + zy)\).

Zinbiel algebras were introduced by Loday in [29]. Under the Koszul duality, the operad of Zinbiel algebras is dual to the operad of Leibniz algebras. Zinbiel algebras are also known as pre-commutative algebras [28] and chronological algebras [27]. A Zinbiel algebra is equivalent to a commutative dendriform algebra [2]. It plays an important role in the definition of pre-Gerstenhaber algebras. The variety of Zinbiel algebras is a proper subvariety in the variety of right
commutative algebras. Each Zinbiel algebra with the commutator multiplication gives a Tortkara algebra [16], which has sprung up in unexpected areas of mathematics [13,14]. Recently, the notion of matching Zinbiel algebras was introduced in [21]. Zinbiel algebras also appeared in a study of rack cohomology [12], number theory [10] and in a construction of a Cartesian differential category [22]. In recent years, there has been a strong interest in the study of Zinbiel algebras in the algebraic and the operad context [4,9,15–18,21,24,26,30,31–33].

Free Zinbiel algebras were shown to be precisely the shuffle product algebra [30], which is under a certain interest until now [11]. Naurazbekova proved that, over a field of characteristic zero, free Zinbiel algebras are the free associative-commutative algebras (without unity) with respect to the symmetrization multiplication and their free generators are found; also she constructed examples of subalgebras of the two-generated free Zinbiel algebra that are free Zinbiel algebras of countable rank [32]. Nilpotent algebras play an important role in the class of Zinbiel algebras. So, Dzhumadildaev and Tulenbaev proved that each complex finite-dimensional Zinbiel algebra is nilpotent [17] and recently Towers proved that each finite-dimensional Zinbiel algebra is nilpotent [34]; Naurazbekova and Umirbaev proved that in characteristic zero any proper subvariety of the variety of Zinbiel algebras is nilpotent [33]. Finite-dimensional Zinbiel algebras with a “big” nilpotency index are classified in [1,9]. Central extensions of three-dimensional Zinbiel algebras were calculated in [4]. The full system of degenerations of complex four-dimensional Zinbiel algebras is given in [26] and the geometric classification of complex five-dimensional Zinbiel algebras is given in [5].

The present paper is about symmetric (left and right) Zinbiel (super)algebras. This variety of algebras is dual to symmetric Leibniz algebras (about symmetric Leibniz algebras see in [6] and references therein) and it is very thin. As we have proven each symmetric Zinbiel algebra is a 2-step nilpotent [33]. Finite-dimensional Zinbiel algebras with a symmetric Zinbiel superalgebras with two odd generators (Proposition 15) and symmetric Zinbiel algebras is dual to symmetric Leibniz algebras (about symmetric Leibniz algebras see in [6] and references therein) and it is very thin. As we have proven each symmetric Zinbiel algebra is a 2-step nilpotent or 3-step nilpotent algebra. We also provide a classification of two-generated symmetric Zinbiel algebras (Theorem 15) and symmetric Zinbiel superalgebras with two odd generators (Proposition 16). The next part of the paper is devoted to a study of mono and binary metric Zinbiel algebras (Theorem 15) and symmetric Zinbiel superalgebras with two odd generators (Proposition 16). The next part of the paper is devoted to a study of mono and binary metric Zinbiel algebras (Theorem 15) and symmetric Zinbiel superalgebras with two odd generators (Proposition 16). The next part of the paper is devoted to a study of mono and binary metric Zinbiel algebras (Theorem 15) and symmetric Zinbiel superalgebras with two odd generators (Proposition 16). The next part of the paper is devoted to a study of mono and binary metric Zinbiel (super)algebras. This variety of algebras is dual to symmetric Leibniz algebras (about symmetric Leibniz algebras see in [6] and references therein) and it is very thin. As we have proven each symmetric Zinbiel algebra is a 2-step nilpotent (Proposition 28). The notion of double extensions of quadratic Zinbiel algebras is introduced and described in the last section.

2. Definitions and preliminary results

Let \( V = V_0 \oplus V_1 \) be a \( \mathbb{Z}_2 \)-graded vector space over the field \( \mathbb{K} \). An element \( x \) in \( V \) is called homogeneous if \( x \in V_0 \) or \( x \in V_1 \). Throughout this paper, all elements are supposed to be homogeneous unless otherwise stated. For a homogeneous element \( x \) we shall use the standard notation \( [x] = x \in \mathbb{Z}_2 = \{0, 1\} \) to indicate its degree, i.e. whether it is contained in the even part \( ([x] = 0) \) or in the odd part \( ([x] = 1) \). All superalgebras considered in this paper are finite-dimensional.

Let \( V \) be a \( \mathbb{K} \)-vector space and \( \gamma := \wedge V \) be a Grassmann (or exterior) algebra of \( V \). We know that \( \Gamma := \oplus_{\ell \in \mathbb{Z}} \wedge^\ell V = \Gamma_0 \oplus \Gamma_1 \) is a \( \mathbb{Z}_2 \)-graded associative algebra, where \( \Gamma_0 := \oplus_{\ell \in \mathbb{Z}} \wedge^{2\ell} V \) and \( \Gamma_1 : = \oplus_{\ell \in \mathbb{Z}} \wedge^{2\ell+1} V \), such that \( X_2 X_\beta = (-1)^{2\ell} X_\beta X_2, \quad \forall (X_\alpha, X_\beta) \in \Gamma_\alpha \times \Gamma_\beta \), \( \forall \alpha, \beta \in \mathbb{Z}_2 \).

Let \( (\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1, \cdot) \) be a superalgebra and \( \Gamma(\mathcal{A}) \) its Grassmann enveloping algebra which is a subalgebra of \( \mathcal{A} \otimes \Gamma \) given by \( \Gamma(\mathcal{A}) = \mathcal{A}_0 \otimes \Gamma_0 \oplus \mathcal{A}_1 \otimes \Gamma_1 \). Let us assume now that \( \mathcal{B} \) is a homogeneous variety of algebras. Then, \( \mathcal{A} \) is said to be a \( \mathcal{B} \)-superalgebra if \( \Gamma(\mathcal{A}) \) belongs to \( \mathcal{B} \). So, \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) is a left (resp. right) Zinbiel superalgebra if \( \Gamma(\mathcal{A}) \) is a left (resp. right) Zinbiel algebra. Consequently, we get the following definition.

Definition 1. Let \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) be a \( \mathbb{Z}_2 \)-graded vector space and let “·” : \( \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) be a bilinear map on \( \mathcal{A} \) such that \( \mathcal{A}_i : \mathcal{A}_j \subset \mathcal{A}_{i+j}, \quad \forall i, j \in \mathbb{Z}_2 \).
1. \( A \) is called a left (resp. right) Zinbiel superalgebra if, for all \( x, y, z \in A_0 \cup A_1 \),

\[
(xy)z = x(yz) + (-1)^{|y||z|}x(zy),
\]

(1.1)

\[
(\text{resp. } x(yz) = (xy)z + (-1)^{|x||y|}(yx)z),
\]

(1.2)

or, equivalently, \( (x, y, z) = (-1)^{|y||z|}x(zy) \) (resp. \( (x, y, z) = (-1)^{|x||y|}(yx)z \)), where \( (x, y, z) = (xy)z - x(yz) \) is the associator associated to “\( \cdot \)”.  

2. \( A \) is called a symmetric Zinbiel superalgebra if it is both a left and a right Zinbiel superalgebra.

As usual, for \( x \in A_0 \cup A_1 \), we define the corresponding endomorphism of \( A \) by \( L_x(y) = xy \) (resp. \( R_x(y) = (-1)^{|x||y|}yx \)), \( \forall y \in A_0 \cup A_1 \), which is called the left (resp. right) multiplication by \( x \).

The following proposition can be verified by a direct calculation.

**Proposition 2.** Let \((A, \cdot)\) be a symmetric Zinbiel superalgebra. Then,

1. \((A, \cdot)\) is a LR-superalgebra (LR-superalgebras are also known as bicommutative superalgebras), i.e. it satisfies

\[
(xy)z = (-1)^{|y||z|}(xz)y \text{ and } x(yz) = (-1)^{|x||y|}y(xz), \text{ for all } x, y, z \in A_0 \cup A_1,
\]

2. \((A, \cdot)\) is an anti-flexible superalgebra, i.e. it satisfies

\[
(x, y, z) = (-1)^{|x||y|+|z||x|+|y||z|}(z, y, x), \text{ for all } x, y, z \in A_0 \cup A_1,
\]

3. \((A, \cdot)\) satisfies

\[
(xy)z = (-1)^{|x||y|+|z|}y(zx), \text{ for all } x, y, z \in A_0 \cup A_1,
\]

4. \((A, \cdot)\) satisfies

\[
(xy)z = (-1)^{|x||y|+|z|}z(yx), \text{ for all } x, y, z \in A_0 \cup A_1.
\]

In the following definition, we recall some results about representations of left and right Zinbiel superalgebras.

**Definition 3.** Let \((A, \cdot)\) be a non-associative superalgebra, \( V \) be a \( \mathbb{Z}_2 \)-graded vector space and \( r, l : A \to \text{End}(V) \) be two even linear maps. If \( A \) is a left (resp. right) Zinbiel superalgebra, then we say that \((r, l)\) is a left (resp. right) representation of \( A \) in \( V \) if for all \( x, y \in A_{[x]} \times A_{[y]} \):

\[
l(xy) = l(x)l(y) + l(x)r(y); \quad r(xy) = r(xy) + (-1)^{|x||y|}r(y)x; \quad l(x)l(y) = (-1)^{|x||y|}[r(y), l(x)],
\]

( resp. \( l(xy) = l(xy) + (-1)^{|x||y|}l(y)x; \quad l(x)l(y) = (-1)^{|x||y|}r(y)r(x) + r(y)l(x) \)).

**Example 4.** Let \((A, \cdot)\) be a left (resp. right) Zinbiel superalgebra. Consider the even maps \( L : A \to \text{End}(A) \) and \( R : A \to \text{End}(A) \) defined by: \( L(x) := L_x, R(x) := R_x, \forall x \in A \). Then, \((R, L)\) is a left (resp. right) representation of \( A \) in \( A \) called the left (resp. right) adjoint representation of \( A \).

**Proposition 5.** Let \((A, \cdot)\) be a left (resp. right) Zinbiel superalgebra and \( r, l : A \to \text{End}(V) \) be two even linear maps. Then, the \( \mathbb{Z}_2 \)-graded vector space \( \mathcal{A} = A \oplus V \) endowed with the product defined by: \( (x + u)(y + v) = xy + l(x)(v) + (-1)^{|x||y|}r(y)(u), \forall (x + u, y + v) \in \mathcal{A}_{[x]} \times \mathcal{A}_{[y]} \), is a left (resp. right) Zinbiel superalgebra if and only if \((r, l)\) is a left (resp. right) representation of \( A \) in \( V \).
**Definition 7.** Let \((\mathcal{A}, \cdot)\) be a symmetric Zinbiel superalgebra, \(\mathcal{V}\) be a \(\mathbb{Z}_2\)-graded vector space and \(r, l : \mathcal{A} \to \text{End}(\mathcal{V})\) be two even linear maps. Then, we say that \((r, l)\) is a representation of \(\mathcal{A}\) in \(\mathcal{V}\) if \((r, l)\) is a left and a right representation of \(\mathcal{A}\) in \(\mathcal{V}\). We denote by \(\text{Rep}(\mathcal{A}, \mathcal{V})\) the set of all representations of \(\mathcal{A}\) in \(\mathcal{V}\).

**Example 8.** If \((\mathcal{A}, \cdot)\) is a symmetric Zinbiel superalgebra, then:

1. \((R, L)\) (see Example 4) is a representation of \(\mathcal{A}\) in \(\mathcal{A}\) called the adjoint representation of \(\mathcal{A}\).
2. \((R^*, L^*)\) (see Proposition 6) is a representation of \(\mathcal{A}\) in \(\mathcal{A}^*\) called the co-adjoint representation of \(\mathcal{A}\).

**Remark 9.** Let \((\mathcal{A}, \cdot)\) be a symmetric Zinbiel superalgebra, \(\mathcal{V}\) be a \(\mathbb{Z}_2\)-graded vector space and \(r, l : \mathcal{A} \to \text{End}(\mathcal{V})\) be two even linear maps. Then, the \(\mathbb{Z}_2\)-graded vector space \(\mathcal{A} = \mathcal{A} \oplus \mathcal{V}\) endowed with the product defined by:

\[(x + u)(y + v) = xy + l(x)(v) + (-1)^{|x||y|}r(y)(u), \ \forall (x + u, y + v) \in \mathcal{A}_{[x]} \times \mathcal{A}_{[y]} ,\]

is a symmetric Zinbiel superalgebra if and only if \((r, l) \in \text{Rep}(\mathcal{A}, \mathcal{V})\).

### 2.1. Characterizations of symmetric Zinbiel superalgebras

This section will be devoted to the characterizations of symmetric Zinbiel superalgebras. In this section, we will consider algebras over the complex field.

**Definition 10.** Let \((\mathcal{A}, \cdot)\) be a superalgebra, then
2. \( (\mathcal{A}, \cdot) \) is a 2-step nilpotent superalgebra if for all elements \( x, y, z \in \mathcal{A}_0 \cup \mathcal{A}_1 \) we have \( (xy)z = x(yz) = 0 \), but the product of the algebra is nonzero.

3. \( (\mathcal{A}, \cdot) \) is a 3-step nilpotent superalgebra if for all elements \( x, y, z, t \in \mathcal{A}_0 \cup \mathcal{A}_1 \) we have
\[
(xy)(zt) = x((yz)t) = x(y(zt)) = ((xy)z)t = (x(yz))t = 0,
\]
but the algebra is non-2-step nilpotent.

**Example 11.** Each 2-step nilpotent superalgebra is a symmetric Zinbiel superalgebra.

**Theorem 12.** If \( (\mathcal{A}, \cdot) \) is a nonzero non-2-step nilpotent symmetric Zinbiel superalgebra, then

1. \( (\mathcal{A}, \cdot) \) is a 3-step nilpotent superalgebra, i.e. it is a central extension of a 2-step nilpotent superalgebra;
2. \( x^3 = 0 \) for each \( x \in \mathcal{A}_{[x]} \).

**Proof.** For the first part of the theorem we consider the following relations:
\[
(xy)(zt) = (-1)^{[(|x|+|y|)(|x|+|y|)+|x|]} z((xy)t) = -(1)^{[(|x|+|y|)(|x|+|y|)+|x|]} z(t(yx)),
\]
\[
(xy)(zt) = -(1)^{|x|+|y|(|x|+|y|)+|t|} (tz)(xy) = -(1)^{|x|+|y|(|x|+|y|)+|t|} (t(xy))z = -(1)^{|x|+|y|(|x|+|y|)+|t|} z(t(xy)).
\]
Hence,
\[
0 = z(t(xy) + (1)^{|x|+|y|} xy) = z((tx)y),
\]
which gives that all products of 4 elements are equal to zero and \( (\mathcal{A}, \cdot) \) is a 3-step nilpotent superalgebra.

Now, for the second part of the theorem, we are using (1.1) and (1.2) and substituting \( y = z = x \in \mathcal{A}_{[x]} \). We get:

a. If \( x \in \mathcal{A}_0 \), then \( x^2 x = 2x^2 \) and \( xx^2 = 2x^2 x \), which implies that \( x^2 x = 0 = xx^2 \).

b. If \( x \in \mathcal{A}_1 \), then \( x^2 x = 0 = xx^2 \). So, \( x^3 = 0 \).

**Corollary 13.** If \( (\mathcal{A}, \cdot) \) is a \( d \)-generated symmetric Zinbiel superalgebra, then \( \dim(\mathcal{A}, \cdot) \leq -d + d^2 + 2d^3 + d^4 \).

**Proof.** It is easy to see that the maximal dimension of a central extension \( \hat{\mathcal{A}} \) of the \( d \)-dimensional zero product algebra (with a basis \( \{e_1, \ldots, e_d\} \) ) is \( d + d^2 \). Let say that the last algebra has a basis \( \{e_1, \ldots, e_d, e_{1,1}, \ldots, e_{d,d}\} \), such that \( e_i e_j = e_j e_i \). \( \mathcal{A} \) is a central extension of \( \hat{\mathcal{A}} \). Note now, that \( x^3 = 0 \) for each \( x \in \mathcal{A} \), hence \( e_i e_i e_i = 0 \) and \( \dim \hat{\mathcal{A}}^2 = d^2 \). It is easy to see that the dimension of the cohomology space of \( \mathcal{A} \) is not greater than \( (d + d^2)^2 - d^2 - 2d \). Hence, the dimension of a central extension of a central extension of the \( d \)-generated zero product algebra is not greater than \( d + d^2 + (d + d^2)^2 - d^2 - 2d = -d + d^2 + 2d^3 + d^4 \).

**Corollary 14.** If \( (\mathcal{A}, \cdot) \) is a one-generated symmetric Zinbiel superalgebra, then there are only two following possibilities

1. \( \dim(\mathcal{A}, \cdot) = (2,0) \) and the multiplication table is given by \( e_1^2 = e_2 \).
2. \( \dim(\mathcal{A}, \cdot) = (1,1), \mathcal{A}_0 = \langle e_1 \rangle, \mathcal{A}_1 = \langle e_2 \rangle, \) and the multiplication table is given by \( e_2^2 = e_1 \).
Theorem 15. If $\mathcal{A}$ is a two-generated symmetric Zinbiel algebra, then there are only following opportunities:

1. if $\dim(\mathcal{A})$ is a 2-step nilpotent algebra, then $3 \leq \dim(\mathcal{A}) \leq 6$ and
   (a) if $\dim(\mathcal{A}) = 3$, then $(\mathcal{A}, \cdot)$ is isomorphic to one of the following algebras
      \[
      \mathcal{N}_1^3 : e_1 e_1 = e_2 \quad e_1 e_2 = 0 \quad e_2 e_1 = 0 \quad e_2 e_2 = 0 \\
      \mathcal{N}_2^3 : e_1 e_1 = e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = 0 \quad e_2 e_2 = e_3 \\
      \mathcal{N}_3^3 : e_1 e_1 = 0 \quad e_1 e_2 = e_3 \quad e_2 e_1 = -e_3 \quad e_2 e_2 = 0 \\
      \mathcal{N}_4^3 \downarrow \mathcal{N}_5 : e_1 e_1 = e_6 e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = e_3 \quad e_2 e_2 = e_3 \\
      (b) if $\dim(\mathcal{A}) = 4$, then $(\mathcal{A}, \cdot)$ is isomorphic to one of the following algebras
         \[
         \mathcal{N}_1^4 : e_1 e_1 = 0 \quad e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = 0 \\
         \mathcal{N}_2^4 : e_1 e_1 = e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = e_4 \quad e_2 e_2 = 0 \\
         \mathcal{N}_3^4 : e_1 e_1 = e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_4 \\
         \mathcal{N}_4^4 : e_1 e_1 = e_3 \quad e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_3 \\
         \mathcal{N}_5^4 \downarrow \mathcal{N}_6 : e_1 e_1 = e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_3 + \lambda e_4 \\
         \mathcal{N}_7^4 \downarrow \mathcal{N}_8 : e_1 e_1 = e_3 \quad e_1 e_2 = e_4 \quad e_2 e_1 = \lambda e_4 \quad e_2 e_2 = 0 \\
      (c) if $\dim(\mathcal{A}) = 5$, then $(\mathcal{A}, \cdot)$ is isomorphic to one of the following algebras
         \[
         \mathcal{N}_1^5 : e_1 e_1 = 0 \quad e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \\
         \mathcal{N}_2^5 : e_1 e_1 = e_3 \quad e_1 e_2 = 0 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \\
         \mathcal{N}_3^5 : e_1 e_1 = e_3 \quad e_1 e_2 = e_4 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \\
         \mathcal{N}_4^5 \downarrow \mathcal{N}_5 : e_1 e_1 = e_3 + \lambda e_5 \quad e_1 e_2 = e_4 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \\
      (d) if $\dim(\mathcal{A}) = 6$, then $(\mathcal{A}, \cdot)$ is isomorphic to the following algebra
         \[
         \mathcal{N}_6^6 : e_1 e_1 = e_3 \quad e_1 e_2 = e_4 \quad e_2 e_1 = e_5 \quad e_2 e_2 = e_6 \\
      2. if $(\mathcal{A}, \cdot)$ is a 3-step nilpotent algebra, then $6 \leq \dim(\mathcal{A}) \leq 8$ and
      (a) if $\dim(\mathcal{A}) = 6$, then $(\mathcal{A}, \cdot)$ is isomorphic to one of the following algebras:
         \[
         \mathcal{Z}_1^6 : e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \quad e_1 e_5 = e_6 \quad e_5 e_1 = -e_6 \\
         e_2 e_4 = -2e_6 \quad e_4 e_2 = -e_6 \quad e_2 e_3 = e_6 \quad e_3 e_2 = 2e_6 \\
         \mathcal{Z}_2^6 : e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 + e_6 \quad e_1 e_5 = e_6 \quad e_5 e_1 = -e_6 \\
         e_2 e_4 = -2e_6 \quad e_4 e_2 = -e_6 \quad e_2 e_3 = e_6 \quad e_3 e_2 = 2e_6 \\
      (b) if $\dim(\mathcal{A}) = 7$, then $(\mathcal{A}, \cdot)$ is isomorphic to the following algebra
         \[
         \mathcal{Z}_1^7 : e_1 e_1 = e_7 \quad e_1 e_2 = e_3 \quad e_2 e_1 = e_4 \quad e_2 e_2 = e_5 \quad e_1 e_5 = e_6 \\
         e_5 e_1 = -e_6 \quad e_2 e_4 = -2e_6 \quad e_4 e_2 = -e_6 \quad e_2 e_3 = e_6 \quad e_3 e_2 = 2e_6 \\
      (c) if $\dim(\mathcal{A}) = 8$, then $(\mathcal{A}, \cdot)$ is isomorphic to the following algebra
         \[
         \mathcal{Z}_1^8 : e_1 e_1 = e_3 \quad e_1 e_2 = e_4 \quad e_1 e_4 = 2e_7 \quad e_1 e_5 = -e_7 \quad e_1 e_6 = e_8 \quad e_2 e_1 = e_5 \\
         e_2 e_2 = e_6 \quad e_2 e_3 = -e_7 \quad e_2 e_4 = e_8 \quad e_2 e_5 = -2e_8 \quad e_3 e_2 = e_7 \quad e_4 e_1 = e_7 \\
         e_4 e_2 = 2e_8 \quad e_5 e_1 = -2e_7 \quad e_5 e_2 = -e_8 \quad e_6 e_1 = -e_8 \\

Proof. The first part of the theorem follows from [8]. The proof of the second part of the theorem will be done the usual way by calculations of symmetric Zinbiel central extensions (it is
known as the Skjelbred-Sund method (about the description of the method for an arbitrary variety see [25] and for Zinbiel case see [4]).

Thanks to [4], there are no non-trivial symmetric Zinbiel central extensions of algebras \( R^3_1 \) and \( R^4_1 \). Hence, we will consider only algebras \( R^5_1 \) and \( R^6_1 \).

- The algebra \( R^5_1 \) has only two 1-dimensional and one 2-dimensional central extensions:

\[
\begin{align*}
3^1_1 : & 
\begin{cases}
e_1 e_2 = e_3 \\
e_2 e_1 = e_4 \\
e_2 e_2 = e_5 \\
e_1 e_5 = e_6 \\
e_5 e_1 = -e_6 \\
e_2 e_4 = -2e_6 \\
e_4 e_2 = -e_6 \\
e_2 e_3 = e_6 \\
e_3 e_2 = 2e_6 \\
e_5 e_1 = -e_6 \\
e_2 e_4 = -2e_6 \\
e_4 e_2 = -e_6 \\
e_2 e_3 = e_6 \\
e_3 e_2 = 2e_6 \\
e_5 e_1 = -e_6 \\
e_2 e_4 = -2e_6 \\
e_4 e_2 = -e_6 \\
e_2 e_3 = e_6 \\
e_3 e_2 = 2e_6
\end{cases}
\]

- The algebra \( R^5_3 \) has no central extensions.
- The algebra \( R^5_3 \) and \( R^5_3 \) have no central extensions.

**Proposition 16.** If \((\mathcal{A}, \cdot)\) is a symmetric Zinbiel superalgebra with two odd generators, then \((\mathcal{A}, \cdot)\) is a 2-step nilpotent superalgebra given in the first part of the theorem 15 with the following grading \( \mathcal{A}_1 = \langle e_1, e_2 \rangle \) and \( \mathcal{A}_0 = \langle e_3, ..., e_{2+i} \rangle \), where \( i = \dim(\mathcal{A}_0) \).

**Proof.** The description of 2-step nilpotent symmetric Zinbiel superalgebras with two odd generators follows from the theorem 15.

If \((\mathcal{A}, \cdot)\) is a 3-step nilpotent symmetric Zinbiel superalgebra, then the product of three elements \( x, y, \) and \( z \) of \((\mathcal{A}, \cdot)\) is equal to zero if an element (or more elements) from it is not a linear combination of generators (i.e., one (or more) element is in \( \mathcal{A}^2 \)). Hence, superalgebra \((\mathcal{A}, \cdot)\) without grading is equivalent to an algebra with the following identities

\[
\Omega = \{(xy)z = x(yz - yx) \text{ and } x(yz) = (xy - yx)z\}.
\]

By the standard Skjelbred-Sund method, it is possible to verify, that there are no non-trivial extensions of algebras from the first part of the theorem 15 in the variety defined by identities from \( \Omega \). It follows that there are no 3-step nilpotent symmetric Zinbiel superalgebras with two odd generators.

### 3. Mono and binary symmetric Zinbiel and Leibniz algebras

In this section, we will consider algebras over the complex field.

**Definition 17.** Let \( \Omega \) be a variety of algebras defined by a family of polynomial identities, then we say that an algebra \( \mathcal{A} \in \Omega_i \) if and only if each \( i \)-generated subalgebra of \( \mathcal{A} \) gives an algebra from \( \Omega \). In particular, if \( \mathcal{A} \in \Omega_1 \), then \( \mathcal{A} \) is a mono-\( \Omega \) algebra, if \( \mathcal{A} \in \Omega_2 \), then \( \mathcal{A} \) is a binary-\( \Omega \) algebra.

For example, let \( \text{Ass} \) be the class of associative algebras. Then by Artin’s theorem, the class \( \text{Ass}_2 \) coincides with the class of alternative algebras. Albert’s theorem follows that the class \( \text{Ass}_1 \)
coincides with the class of power-associative algebras. It is easy to see that Lie$_1$ coincides with anticommutative algebras, i.e. they satisfy the identity $x^2 = 0$. The identities of Lie$_2$ are described by Gainov [19]. Below we will consider mono and binary symmetric Leibniz and Zinbiel algebras.

Thanks to [20,23], we have the description of mono and binary left Leibniz algebras.

**Theorem 18.** An algebra $A$ is mono left Leibniz if and only if it satisfies the following identities:

\[ x^2x = 0, \quad x^2x^2 = 0. \]

An algebra $A$ is binary left Leibniz if and only if it satisfies the following identities:

\[ x^2y = 0, \quad x(yx) = (xy)x + yx^2, \quad x(y(xy)) = (xy)(xy) + y(x(xy)). \]

Thanks to [24], we have the description of mono and binary left Zinbiel algebras.

**Theorem 19.** An algebra $A$ is mono left Zinbiel if and only if it satisfies the following identities:

\[ xx^2 = 2x^2x, \quad x^2x^2 = 3(x^2x)x. \]

An algebra $A$ is binary left Zinbiel if and only if it satisfies the following identities:

\[ x(yx) = (xy + yx)x, \quad x(xy) = 2x^2y. \]

As some trivial corollaries from the previous theorems, we have the following propositions.

**Proposition 20.** The variety of mono symmetric Zinbiel algebras coincides with the variety of mono symmetric Leibniz algebras and it is defined by the following identities

\[ x^2x = 0 \quad \text{and} \quad xx^2 = 0. \]

**Proof.** Theorems 18 and 19 give that variety of mono symmetric Zinbiel algebras coincides with the variety of mono symmetric Leibniz algebras and it is defined by the following identities

\[ x^2x = 0, \quad xx^2 = 0, \quad x^2x^2 = 0. \]

By the linearization process of the first identity, we have

\[ (x_1x_2)x_3 + (x_2x_1)x_3 + (x_1x_3)x_2 + (x_3x_1)x_2 + (x_2x_3)x_1 + (x_3x_2)x_1 = 0. \]

Hence, taking $x_1 = x^2$ and $x_2 = x_3 = x$, we have $x^2x^2 = 0$ and this identity follows from the other two identities. On the other side, identities $x^2x = 0$ and $xx^2 = 0$ are independent. \[\Box\]

**Proposition 21.** The variety of binary symmetric Zinbiel algebras is defined by the following identities

\[ x(yx) = (xy + yx)x, \quad x(xy) = 2x^2y \quad \text{and} \quad (yx)x = 2yx^2. \]

**Proposition 22.** The variety of binary symmetric Leibniz algebras is defined by the following identities

\[ yx^2 = 0, \quad x^2y = 0, \quad x(yx) = (xy)x \quad \text{and} \quad x(y(xy)) = (xy)(xy) + y(x(xy)). \]

**Proposition 23.** The intersection of varieties of binary symmetric Zinbiel and Leibniz algebras is defined by the following identities

\[ (x_1x_2)x_3 = (-1)^{\sigma}(x_{\sigma(1)}x_{\sigma(2)})x_{\sigma(3)} \quad \text{and} \quad x_1(x_2x_3) = (-1)^{\sigma}x_{\sigma(1)}(x_{\sigma(2)}x_{\sigma(3)}), \quad \sigma \in S_3. \]
Proof. Thanks to two previous propositions, we have that each algebra from the intersection of symmetric Zinbiel and Leibniz algebras satisfies the following identities

$$x^2y = 0, \ x(xy) = 0, \ yx^2 = 0, \ (yx)x = 0.$$ 

Hence, by linearization, it satisfies

$$(x_1x_2)x_3 = -(x_2x_1)x_3, \ x_1(x_2x_3) = -x_2(x_1x_3), \ x_1(x_2x_3) = -x_1(x_3x_2), \ (x_1x_2)x_3 = -(x_1x_3)x_2.$$ 

It is easy to see that each algebra that satisfies the previous identities also satisfies all identities from two previous propositions. Hence, we complete the proof of the statement. \qed

Theorem 24. Let $SZ$ and $SL$ be the varieties of symmetric Zinbiel and symmetric Leibniz algebras. Then, the following inclusions are strict.

$$\begin{array}{ccc}
SZ & \subset & SZ_2 \\
\subset & \cup & \subset \\
SL \cap SZ = SL \cap SZ_2 = SZ \cap SL_2 & \subset & \text{Ass} \cap \text{Lie}_1 \subset SL_2 \cap SZ_2 \subset SZ_1 = SL_1 \\
\subset & \cap & \subset \\
SL & \subset & SL_2
\end{array}$$

Proof. Let us prove that $SL \cap SZ_2$ is defined by $(xy)z = 0$ and $x(yz) = 0$. Obviously, that each algebra from $SL \cap SZ_2$ satisfies identities from the previous proposition and two Leibniz identities. Hence,

$$2(xy)z = (xy)z + (xz)y + x(yz) = x(yz) \text{ and } 2x(yz) = x(yz) + (xy)z + y(xz) = (xy)z,$$

which gives $(xy)z = 0$ and $x(yz) = 0$.

$SZ \cap SL_2$ is also defined by $(xy)z = 0$ and $x(yz) = 0$, because each algebra from $SZ \cap SL_2$ satisfies identities from the previous proposition. Hence, it satisfies

$$(xy)z = x(yz + zy) = 0 \text{ and } x(yz) = (xy + xy)z = 0.$$ 

Obviously, we have the equalities $SL \cap SZ = SL \cap SZ_2 = SZ \cap SL_2$.

All indicated inclusions are following from previous propositions and other known results. To prove the strict inclusion $\omega \subset \Omega$ of varieties of algebras, we will indicate an algebra $A \in \Omega \setminus \omega$. 

$$\begin{array}{cccc}
SL \cap SZ & \subset & SZ & e_1e_2 = e_3, \ e_1e_5 = e_6, \ e_2e_1 = e_4 \\
SL_2 \cap SZ_2 & \subset & SZ_2 & e_2e_2 = e_5, \ e_2e_3 = e_6, \ e_2e_4 = -2e_6 \\
SL_2 & \subset & SL_1 & e_3e_2 = 2e_6, \ e_4e_2 = -e_6, \ e_5e_1 = -e_6 \\
SL \cap SZ & \subset & SL & e_1e_2 = e_3, \ e_2e_1 = -e_3, \ e_2e_3 = e_4 \text{ or } e_3e_2 = -e_4 \\
SL \cap SZ & \subset & \text{Ass} \cap \text{Lie}_1 & e_1e_2 = e_4, \ e_1e_3 = e_5, \ e_1e_6 = e_7, \ e_2e_1 = -e_4 \\
& & & e_2e_3 = e_6, \ e_2e_5 = -e_7, \ e_3e_1 = -e_5, \ e_3e_2 = -e_6 \\
& & & e_3e_4 = e_7, \ e_4e_3 = e_7, \ e_5e_2 = -e_7, \ e_6e_1 = e_7 \\
\text{Ass} \cap \text{Lie}_1 & \subset & SL_2 \cap SZ_2 & e_1e_2 = e_4, \ e_1e_3 = e_5, \ e_1e_6 = e_7 \\
SL & \subset & SL_2 & e_2e_1 = -e_4, \ e_2e_3 = e_6, \ e_2e_5 = -e_7 \\
SZ & \subset & SZ_2 & e_3e_1 = -e_5, \ e_3e_2 = -e_6, \ e_3e_4 = e_7 \\
SZ_2 & \subset & SZ_1 & e_1e_2 = e_2 \\
SL_2 \cap SZ_2 & \subset & SL_2 & e_2e_1 = -e_2
\end{array}$$ 

\qed
From the proof of the present theorem it is easy to see that the variety \( SL_2 \cap SZ_2 \) does not contain \( SL \) and \( SZ \) as subvarieties. It gives the following interesting open question.

**Question 1.** Is \( SL_1 \) the minimal variety that contains the varieties \( SL \) and \( SZ \) (resp. \( SL_2 \) and \( SZ_2 \)) as subvarieties?

### 4. Quadratic Zinbiel superalgebras

**Definition 25.** A triple \((\mathcal{A}, \cdot, \mathcal{B})\) is said to be a quadratic Zinbiel superalgebra if \((\mathcal{A}, \cdot)\) is a left (or a right) Zinbiel superalgebra endowed with a non-degenerate bilinear form \( \mathcal{B} \) satisfying the following properties:

(i) \( \mathcal{B}(x, y) = (-1)^{|x||y|} \mathcal{B}(y, x) \), \( \forall x \in \mathcal{A}_{|x|}, y \in \mathcal{A}_{|y|} \), i.e. \( \mathcal{B} \) is supersymmetric;

(ii) \( \mathcal{B}(xy, z) = \mathcal{B}(x, yz) \), \( \forall x, y, z \in \mathcal{A} \), i.e. \( \mathcal{B} \) is invariant;

(iii) \( \mathcal{B}(\mathcal{A}_0, \mathcal{A}_1) = \mathcal{B}(\mathcal{A}_1, \mathcal{A}_0) = \{0\} \), i.e. \( \mathcal{B} \) is even.

In this case, \( \mathcal{B} \) is called an invariant scalar product on \( \mathcal{A} \).

**Proposition 26.** Let \((\mathcal{A}, \cdot)\) be a left (resp. right) Zinbiel superalgebra. If \( \mathcal{A} \) is quadratic, then \( \mathcal{A} \) is symmetric.

**Proof.** Suppose that \((\mathcal{A}, \cdot)\) is a left Zinbiel superalgebra. Then, for all \( x, y, z, t \in \mathcal{A}_0 \cup \mathcal{A}_1 \), we have

\[
0 = \mathcal{B}((xy)z - x(yz) - (-1)^{|y||z|} x(zy)t) = \mathcal{B}(x, yzt) - (yz)t - (-1)^{|y||z|}(zy)t.
\]

The fact that \( \mathcal{B} \) is non-degenerate implies that \( yzt - (yz)t - (-1)^{|y||z|}(zy)t = 0 \). Consequently, \( \mathcal{A} \) is a right Zinbiel superalgebra. By the same way, we prove the result for right Zinbiel superalgebra. \( \square \)

**Proposition 27.** Let \( \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \) be a symmetric Zinbiel superalgebra. Then, \( \mathcal{A} \) is quadratic if and only if the adjoint and the co-adjoint representations of \( \mathcal{A} \) are equivalent and \( \dim(\mathcal{A}_1) \) is even.

**Proof.** The proof is similar to that in [3]. \( \square \)

The following result comes from Proposition 6.

**Proposition 28.** Let \((\mathcal{A}, \cdot)\) be a left (resp. right) Zinbiel superalgebra. If \( \mathcal{A} \) is quadratic, then \( \mathcal{A} \) is 2-step nilpotent.

### 5. Extensions of quadratic Zinbiel superalgebras

In this section, we introduce the notion of a double extension of quadratic Zinbiel superalgebras by the one-dimensional \( \mathbb{Z}_2 \)-graded vector space, which is performed in two distinct steps, a central extension followed by a semi-direct product. In the first step, we are going to introduce the notion of central extension by the one-dimensional \( \mathbb{Z}_2 \)-graded vector space.

**Proposition 29.** Let \((\mathcal{A}, \cdot, \mathcal{B})\) be a quadratic Zinbiel superalgebra and \( \Omega : \mathcal{A} \times \mathcal{A} \to \mathbb{K} \) be a bilinear map of degree \( x \). Then, there exists a homogeneous endomorphism \( \delta \) of \( \mathcal{A} \) of degree \( x \) such that \( \Omega(x, y) = \mathcal{B}(\delta(x), y) \), \( \forall x, y \in \mathcal{A} \). The map \( \Omega \) is a homogeneous scalar Zinbiel 2-cocycle of \( \mathcal{A} \) of degree \( x \) if and only if, for all \( x, y \in \mathcal{A}_{|x|} \times \mathcal{A}_{|y|} \), the following assertions hold:
\[(i)\] \(\delta(xy) = \delta(x)y + (-1)^{|x||y|}y\delta(x)\) and \[(ii)\] \(\delta(xy) = -(1)^{|x|}x\delta(y)\).

**Proof.** (i) It is easy to see the following

\[\Omega(x, yz) + (-1)^{|y||z|}\Omega(x, zy) - \Omega(xy, z) = B(\delta(x)y + (-1)^{|x||y|}y\delta(x) - \delta(xy), z).\]

(ii) Note that \(\forall x, y, z \in A_{|x|} \times A_{|y|} \times A_{|z|}\) we have

\[\Omega(x, yz) + (-1)^{|x||y|}\Omega(2x, y) = (-1)^{|x||y|}z\delta(x) + \delta(xz).\]

\[\square\]

**Proposition 30.** Let \((A, \cdot, B)\) be a quadratic Zinbiel superalgebra, \(V = \mathbb{K}d\) be the one-dimensional \(Z_2\)-graded vector space and \(\delta\) be a homogeneous endomorphism of \(A\) of degree \(x\). Define the bilinear map \(\Omega: A \times A \rightarrow V^*\) by \(\Omega(x, y) = B(\delta(x), y)d^x\). Then, the \(Z_2\)-graded vector space \(\tilde{A} = A \oplus V^*\) endowed with the product defined by

\[(x + \lambda d^x) * (y + \lambda' d^y) = xy + \Omega(x, y), \text{ for all } (x + \lambda d^x), (y + \lambda' d^y) \in \tilde{A},\]

is a symmetric Zinbiel superalgebra if and only if \(\Omega\) is a homogeneous Zinbiel 2-cocycle of \(A\) on the trivial \(A\) – module \(V^*\) of degree \(x\). It is termed the central extension of \(A\) by \(V^*\) by means of \(\Omega\).

**Proof.** We only need calculation, where we use Proposition 29. \(\square\)

We will define, in the following, another type of extension of symmetric Zinbiel superalgebras. Let us consider \((A, \cdot)\) a symmetric Zinbiel superalgebra and \(V = \mathbb{K}d\) a one-dimensional \(Z_2\)-graded vector space. Let \(\delta, D\) be two homogeneous endomorphisms of \(A\) of degree \(|d|\) and \(a_0 \in \text{Ann}(A) \cap A_0\). Consider the \(Z_2\)-graded vector space \(\tilde{A} = A \oplus V\) on which we define the following product:

\[d \odot d = a_0; \quad x \odot y = xy; \quad d \odot x = \delta(x); \quad x \odot d = D(x), \quad \forall x, y \in A.\]  \(5.1\)

**Proposition 31.** The \(Z_2\)-graded vector space \(\tilde{A}\) endowed with the product (5.1) is a symmetric Zinbiel superalgebra if and only if, for all \(x, y \in A_{|x|} \times A_{|y|}\), the following conditions hold:

\[\delta(xy) = (-1)^{|d||x|}x\delta(y) = \delta(x)y + (-1)^{|x||y|}D(x)y = -(1)^{|y||x|}D(y)x;\]

\[\delta(x)y = (-1)^{|x||y|}\delta(y)x = -(1)^{|d||x|}D(x)y;\]

\[D(xy) = -(1)^{|d||y|}D(x)y;\]

\[\delta^2(x) = (-1)^{|d||y|}\delta^2(y) = -D^2(x) = (1 + (-1)^{|d|})a_0x;\]

\[\delta \circ D(x) = -(1)^{|d|}D \circ \delta(x) = -(1)^{|d||x|}x a_0 = -(1)^{|d||x|+|d|}a_0x; \quad \delta(a_0) = D(a_0) = 0.\]

In this case, \((\delta, D, a_0)\) is called “an admissible triple” and the symmetric Zinbiel superalgebra \(\tilde{A}\) is termed the semi-direct product of \(A\) by \(V\) by means of \((\delta, D, a_0)\).

**Proof.** The proof is a straightforward computation. \(\square\)

Now, we are in a position to introduce the double extensions of quadratic Zinbiel superalgebras. We start by presenting the concept of even double extension of these superalgebras by the one-dimensional Lie algebra.

**Theorem 32.** Let \((A, \cdot, B)\) be a quadratic Zinbiel superalgebra, \(V = \mathbb{K}d\) be the one-dimensional vector space and \((\delta, \delta^*, a_0)\) be an admissible triple of \(A\) such that \(\delta(A) \subseteq \text{Ann}(A), \delta(A^2) = \{0\}, \delta^2 = \delta \circ \delta^* = \delta^* \circ \delta = \{0\}, a_0 \in \text{Ann}(A) \cap A_0\) and \(B(a_0, a_0) = 0\), where \(\delta^*\) is the adjoint of \(\delta\) with respect to \(B\). Then, the \(Z_2\)-graded vector space \(\tilde{A} = V^* \oplus A \oplus V\) endowed with the following
product:
\[
d \circ d = a_0 + x d^*; \quad d \circ x = \delta(x) + B(x, a_0) d^*;
\]
\[
x \circ y = xy + B(\delta(x), y) d^*; \quad x \circ d = \delta^*(x) + B(x, a_0) d^*, \quad \forall x, y \in A, \; x \in K,
\]
is a symmetric Zinbiel superalgebra. Moreover, the even bilinear form \( B : A \times A \to K \), defined by \( B|_{A \times A} = B; \quad B(d, d^*) = B(d^*, d) = 1 \), is an associative scalar product on \( A \).

The Zinbiel superalgebra \((\tilde{A}, \tilde{B})\) is called the even double extension of \( A \) by \( V \) by means of the admissible triple \((\delta, \delta^*, a_0)\).

**Proof.** Since \( \delta(A^2) = \{0\} \) and \( \delta(A) \subseteq \text{Ann}(A) \), then Proposition 29 implies that the even map \( \Omega : A \times A \to V^* \), \( (x, y) \mapsto \Omega(x, y) = B(\delta(x), y) d^* \), is an even Zinbiel 2-cocycle of \( A \) on the trivial \( A - \) module \( V^* \). So, \( \tilde{A} = A \oplus V^* \) endowed with the product defined by:
\[
(x + f) \ast (y + g) = xy + B(\delta(x), y) d^*, \quad \forall x, y \in A, \; f, g \in V^*,
\]
is a symmetric Zinbiel superalgebra, central extension of \( A \) by means of \( \Omega \). Now, we define two even endomorphisms \( \delta_1 \) and \( \delta_2 \) of \( \tilde{A} \), respectively, by
\[
\delta_1(x + \lambda d^*) = \delta(x) + B(x, a_0) d^* \quad \text{and} \quad \delta_2(x + \lambda d^*) = \delta^*(x) + B(x, a_0) d^*, \quad \forall x \in A, \; \lambda \in K.
\]
Let us consider the element \( a_1 = a_0 + x d^* \in \tilde{A} \), where \( x \) is a fixed scalar in \( K \). So, \( a_1 \in \text{Ann}(\tilde{A}) \cap \tilde{A}_0 \).

Since
\[
\delta(\tilde{A}) \subseteq \text{Ann}(\tilde{A}), \quad \delta(\tilde{A}^2) = \{0\}, \quad \delta^2 = \delta \circ \delta^* = \delta^* \circ \delta = \{0\}, \quad a_0 \in \text{Ann}(\tilde{A}) \cap \tilde{A}_0, \quad B(a_0, a_0) = 0,
\]
then a simple computation proves that the triple \((\delta_1, \delta_2, a_1)\) is an admissible triple of \( \tilde{A} \). Consequently, we can consider the symmetric Zinbiel superalgebra \( \tilde{A} = \tilde{A} \oplus V = \tilde{V} \oplus A \oplus V \), semi-direct product of \( A \) by \( V \). In addition, it is easy to show that the bilinear form \( \tilde{B} : \tilde{A} \times \tilde{A} \to K \), defined by \( \tilde{B}|_{A \times A} = B; \quad \tilde{B}(d, d^*) = 1 \), is an associative scalar product on \( \tilde{A} \).

Our next goal is to establish the converse of Theorem 32.

**Theorem 33.** Let \((A, \circ, B)\) be a quadratic Zinbiel superalgebra of dimension \((n, m)\) such that \( n \geq 2 \). If \( \text{Ann}(A) \cap A_0 \neq \{0\} \), then \( A \) is isomorphic to an even double extension of a quadratic Zinbiel superalgebra \((H, \cdot, B_H)\) of dimension \((n - 2, m)\) by the one-dimensional vector space.

**Proof.** Suppose that \( \text{Ann}(A) \cap A_0 \neq \{0\} \), then there exists a non-zero element \( e \) of \( \text{Ann}(A) \cap A_0 \). Denote by \( J = Ke \) and by \( J^\perp \) the orthogonal of \( J \) with respect to \( B \). Then, \( J \subseteq J^\perp \). Since \( B \) is even and non-degenerate, then there exists \( d \in A_0 \setminus \{0\} \) such that \( B(e, d) = B(d, e) = 1 \). Let \( V = Kd \) and \( H = (J \oplus V)^\perp \), then \( A = J \oplus H \oplus V \) and \( J^\perp = J \oplus H \) is an ideal of \( \tilde{A} \). Let \( x, y \in H_{[x]} \times H_{[y]} \), then \( x \circ y = \Phi(x, y)e + xy \), where \( \Phi : H \times \tilde{H} \to K \) is an even bilinear form and “ \( \cdot \) : \( H \times \tilde{H} \to \tilde{H} \) is an even bilinear map. It is easy to show that \((H, \cdot)\) is a symmetric Zinbiel superalgebra and that the even bilinear form \( B_H = B|_{H \times H} \) is an associative scalar product on \((H, \cdot)\). Therefore, the even bilinear form \( \Phi \) is an element of \((Z^2_{\text{Zinbi}}(H, K))_0 \). The fact that \( J^\perp = J \oplus H \) is an ideal of \( \tilde{A} \) implies that
\[
d \circ d = x e + a_0 + \lambda d^*, \quad \text{where} \; \lambda, \lambda \in K, \quad a_0 \in H_0,
\]
\[
x \circ d = \Psi(x)e + D(x), \quad \text{where} \; D \in (\text{End}(H))_0, \; \Psi \in (H^*)_0,
\]
\[
d \circ x = \Theta(x)e + \delta(x), \quad \text{where} \; \delta \in (\text{End}(H))_0, \; \Theta \in (H^*)_0.
\]
Let \( x, y \in H_{[x]} \times H_{[y]} \). Since \( B \) is supersymmetric and associative, then \( B(x \circ y, d) = B(x, y \circ d) = B(d \circ x, y) \). It follows that \( \Phi(x, y) = B(x, D(y)) = B(\delta(x), y) \). So, \( D = \delta^* \). Moreover, \( \Psi(x) = B(x \circ d, d) = B(d, d \circ x) = \Theta(x) = B(x, a_0) \) and \( \lambda = B(d \circ d, e) = B(d, d \circ e) = 0.\)
Then, \(d \odot d = xe + a_0\). Since \(\mathcal{A}\) is a left Zinbiel superalgebra satisfying:

\[
\mathcal{X} \odot (Y \odot Z) + (-1)^{[x][y]}(Y \odot \mathcal{X}) \odot Y = 0, \quad \text{for all } \mathcal{X} \in \mathcal{A}_{[x]}, Y \in \mathcal{A}_{[y]}, \mathcal{X} \in \mathcal{A}_{[z]},
\]

then a simple computation proves that \((\delta, D, a_0)\) is an admissible triple of \(\mathcal{H}\), \(\delta(\mathcal{H}) \subseteq \text{Ann}(\mathcal{H}), \delta(\mathcal{H}^2) = \{0\}, \delta^2 = \delta \circ D = D \circ \delta = \{0\}, a_0 \in \text{Ann}(\mathcal{H}) \cap \mathcal{H}_0\) and \(\mathcal{B}(a_0, a_0) = 0\).

Therefore, we can consider the quadratic Zinbiel superalgebra \(\tilde{\mathcal{A}} = \mathcal{V}^* \oplus \mathcal{H} \oplus \mathcal{V}\), even double extension of \((\mathcal{H}, \cdot, \mathcal{B}_\mathcal{H})\) by \(\mathcal{V}\) by means of \((\delta, D, a_0)\).

It is clear that the map

\[
\Gamma : \mathcal{A} \rightarrow \tilde{\mathcal{A}}, \quad \lambda e + \lambda'd \mapsto \lambda d^* + x + \lambda'd,
\]

is an isomorphism of Zinbiel superalgebras. Then, \(\mathcal{A}\) is isomorphic to the even double extension of \(\mathcal{H}\) by \(\mathcal{V}\).

In the second part of this section, we will present the notion of an odd double extension of quadratic Zinbiel superalgebras by the one-dimensional \(\mathbb{Z}_2\)-graded vector space with even part zero.

**Theorem 34.** Let \((\mathcal{A}, \cdot, \mathcal{B})\) be a quadratic Zinbiel superalgebra, \(N = N_1 = \mathbb{K}d\) be the one-dimensional \(\mathbb{Z}_2\)-graded vector space with even part zero and \((\delta, D, a_0)\) be an admissible triple of \(\mathcal{A}\) such that

\[
\delta(\mathcal{A}) \subseteq \text{Ann}(\mathcal{A}), \quad \delta(\mathcal{A}^2) = \{0\}, \quad \delta \circ D = D \circ \delta = \{0\}, \quad a_0 \in \text{Ann}(\mathcal{A}) \cap \mathcal{A}_0 \quad \text{and} \quad \mathcal{B}(a_0, a_0) = 0,
\]

where \(D \in (\text{End}(\mathcal{A}))_1\) verifying \(\mathcal{B}(\delta(x), y) = (-1)^{|x||y|}\mathcal{B}(x, D(y))\) for all \(x, y \in \mathcal{A}_{[x]} \times \mathcal{A}_{[y]}\). Then,

the \(\mathbb{Z}_2\)-graded vector space \(\tilde{\mathcal{A}} = \mathcal{N}^* \oplus \mathcal{A} \oplus \mathcal{N}\) endowed with the following product:

\[
\begin{align*}
d \odot d &= a_0; \\
x \odot y &= xy - \mathcal{B}(\delta(x), y)d^*; \\
x \odot d &= D(x) + \mathcal{B}(x, a_0)d^*, \quad \forall x, y \in \mathcal{A},
\end{align*}
\]

is a symmetric Zinbiel superalgebra. Moreover, the even bilinear form \(\mathcal{B} : \tilde{\mathcal{A}} \times \tilde{\mathcal{A}} \rightarrow \mathbb{K}\), defined by \(\mathcal{B}|_{\mathcal{A} \times \mathcal{A}} = \mathcal{B}; \quad \mathcal{B}(d^*, d) = -\mathcal{B}(d, d^*) = 1\), is an associative scalar product on \(\tilde{\mathcal{A}}\).

The Zinbiel superalgebra \((\tilde{\mathcal{A}}, \mathcal{B})\) is termed the odd double extension of \(\mathcal{A}\) by \(\mathcal{N}\) by means of the admissible triple \((\delta, D, a_0)\).

**Proof.** The proof is similar to that of Theorem 32.

The following theorem is the converse of Theorem 34.

**Theorem 35.** Let \((\mathcal{A}, \odot, \mathcal{B})\) be a quadratic Zinbiel superalgebra of dimension \((n, m)\) such that \(m \geq 2\). If \(\text{Ann}(\mathcal{A}) \cap \mathcal{A}_1 \neq \{0\}\), then \(\mathcal{A}\) is isomorphic to an odd double extension of a quadratic Zinbiel superalgebra \((\mathcal{H}, \cdot, \mathcal{B}_\mathcal{H})\) of dimension \((n, m - 2)\) by the one-dimensional Lie superalgebra with even part zero.

**Proof.** Suppose that \(\text{Ann}(\mathcal{A}) \cap \mathcal{A}_1 \neq \{0\}\), then there exists a non-zero element \(e\) of \(\text{Ann}(\mathcal{A}) \cap \mathcal{A}_1\). Denote by \(\mathcal{J} = \mathbb{K}e\) and by \(\mathcal{J}^\perp\) the orthogonal of \(\mathcal{J}\) with respect to \(\mathcal{B}\). Then, \(\mathcal{J} \subseteq \mathcal{J}^\perp\). Since \(\mathcal{B}\) is even and non-degenerate, then there exists \(\mathcal{J} \subseteq \mathcal{J}^\perp\). Since \(\mathcal{B}\) is even and non-degenerate, then there exists \(d \in \mathcal{A}_1 \setminus \{0\}\) such that \(\mathcal{B}(e, d) = -\mathcal{B}(d, e) = 1\). Let \(\mathcal{N} = \mathbb{K}d\) and \(\mathcal{H} = (\mathcal{J} \oplus \mathcal{N})^\perp\), then \(\mathcal{A} = \mathcal{J} \oplus \mathcal{H} \oplus \mathcal{N}\) and \(\mathcal{J}^\perp = \mathcal{J} \oplus \mathcal{H}\) is an ideal of \(\mathcal{A}\). Let \(x, y \in \mathcal{H}_{[x]} \times \mathcal{H}_{[y]}\), then \(x \odot y = \Phi(x, y)e + xy\), where \(\Phi : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{K}\) is an odd bilinear form and " \cdot \cdot : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H}\) is an even bilinear map. It is easy to show that \((\mathcal{H}, \cdot)\) is a symmetric Zinbiel superalgebra and that the even bilinear form \(\mathcal{B}_\mathcal{H} = \mathcal{B}|_{\mathcal{H} \times \mathcal{H}}\) is an associative scalar product on \((\mathcal{H}, \cdot)\). Therefore, the odd bilinear form \(\Phi\) is an element of
Since $J^\perp = J \oplus H$ is an ideal of $A$, then
\[ d \circ d = a_0, \text{ where } a_0 \in H_0, \]
\[ x \circ d = \Psi(x)e + D(x), \text{ where } D \in (\text{End}(H))_1, \Psi \in (H^*)_1, \]
\[ d \circ x = \Theta(x)e + \delta(x), \text{ where } \delta \in (\text{End}(H^*)_1, \Theta \in (H^*)_0. \]

Let $x, y \in H_{|x|} \times H_{|y|}$. Since $\mathfrak{B}$ is supersymmetric and associative, then $\mathfrak{B}(d, x \circ y) = \mathfrak{B}(d \circ x, y) = (-1)^{|x|+|y|} \mathfrak{B}(x, y \circ d)$, which implies that $\Phi(x, y) = -\mathfrak{B}(\delta(x), y) = (-1)^{|x|+|y|} \mathfrak{B}(x, D(y))$. Furthermore, $\Psi(x) = \mathfrak{B}(x \circ d, d) = \mathfrak{B}(d, d \circ x) = -\Theta(x) = \mathfrak{B}(x, a_0)$. Since $A$ is a left Zinbiel superalgebra satisfying:
\[ \mathfrak{A} \circ (\mathfrak{B} \circ \mathfrak{A}) + (-1)^{|x|(|x|+|y|)} (\mathfrak{A} \circ \mathfrak{A}) \circ \mathfrak{B} = 0, \text{ for all } x \in A_{|x|}, y \in A_{|y|}, \mathfrak{A} \in A_{|\mathfrak{A}|}, \]
then an easy calculation proves that $(\delta, D, a_0)$ is an admissible triple of $H$,
\[ \delta(H) \subseteq \text{Ann}(H), \delta(H^2) = \{0\}, \delta \circ D = D \circ \delta = \{0\}, a_0 \in \text{Ann}(H) \cap H_0 \text{ and } \mathfrak{B}(a_0, a_0) = 0. \]

Consequently, we can consider the quadratic Zinbiel superalgebra $\mathfrak{A} = N^* \oplus H \oplus N$, odd double extension of $(\mathfrak{H}, \cdot, \mathfrak{B}_H)$ by $N$ by means of $(\delta, D, a_0)$.

Clearly, the map
\[ \Gamma : \mathfrak{A} \to \mathfrak{A}, \lambda e + x + \lambda d \mapsto \lambda d^* + x + \lambda d, \]
is an isomorphism of Zinbiel superalgebras. Then, $\mathfrak{A}$ is isomorphic to the odd double extension of $H$ by $N$.

\[ \square \]

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