THE ASYMPTOTIC DIMENSION OF BOX SPACES OF VIRTUALLY NILPOTENT GROUPS

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ABSTRACT. We show that every box space of a virtually nilpotent group has asymptotic dimension equal to the Hirsch length of that group.

1. INTRODUCTION

The principal objects of study of this note are so-called box spaces. These are metric spaces formed by stitching together certain finite quotients of residually finite groups, and have been the subject of much recent research, not least thanks to their utility in constructing metric spaces with unusual properties. For example, box spaces can be used to construct expanders \cite{Mar73}, as well as to construct groups without property A \cite{Gro03,AD08,Osa14}.

Of particular interest to us will be the so-called asymptotic dimension of a box space. Defined by Gromov \cite{Gromov}, the asymptotic dimension is a coarse version of the topological dimension. There has been a fair amount of recent work on computing the asymptotic dimension of various box spaces, and this note represents a contribution to that body of work.

Given a finitely generated residually finite infinite group \(G\), a filtration of \(G\) is a sequence \((N_n)_{n=1}^\infty\) of nested, normal, finite-index subgroups \(N_n\) of \(G\) such that \(\bigcap_{n} N_n = \{e\}\).

**Definition 1.1 (box space).** The box space \(\Box_{(N_i)} G\) of a finitely generated residually finite infinite group \(G\) with respect to a filtration \((N_n)_{n=1}^\infty\) of \(G\) and a finite generating set \(S\) of \(G\) is the metric space on the disjoint union of the quotients \((G/N_n)_n\) in which the metric on each component \(G/N_n\) is the Cayley-graph metric induced by \(S\), and the distance between \(x \in G/N_m\) and \(y \in G/N_n\) with \(m \neq n\) is defined to be the sum of the diameters of \(G/N_m\) and \(G/N_n\).

Now let \(X\) be a metric space. Given \(R > 0\), a family \(U\) of subsets of \(X\) is said to be \(R\)-disjoint if the distance between every pair of distinct sets in \(U\) is at least \(R\). Given \(R,S > 0\), the \((R,S)\)-dimension \((R,S)\)-dim\((X)\) of \(X\) is defined to be the least \(n \in \mathbb{Z}\) such that there exist families \(U_0,U_1,\ldots,U_n\) of subsets of \(X\) such that
- \(\bigcup_{j=0}^{n} U_j\) covers \(X\),
- \(U_j\) is \(R\)-disjoint for every \(j \in \{0,1,\ldots,n\}\), and
- \(\text{diam}(U) \leq S\) for every \(U \in U_j\) and \(j \in \{0,1,\ldots,n\}\).

**Definition 1.2 (asymptotic dimension).** The asymptotic dimension \(\text{asdim} X\) of a metric space \(X\) is defined to be the least \(n \in \mathbb{Z}\) such that for every \(R > 0\) there exists \(S > 0\) such that \((R,S)\)-dim\((X)\) \(\leq n\).

It is known that a virtually polycyclic group with a Cayley-graph metric has finite asymptotic dimension. Indeed, given a virtually polycyclic group \(G\) we write \(h(G)\) for the Hirsch length of \(G\),

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which is to say the number of infinite factors in a normal polycyclic series of a finite-index polycyclic subgroup of \( G \). Dranishnikov and Smith [DS06, Theorem 3.5] show that if \( G \) is residually finite and virtually polycyclic then

\[
\text{asdim} \ G = h(G).
\]

It is not unreasonable to expect that box spaces of residually finite virtually polycyclic groups should also have finite asymptotic dimension, and indeed there have been some results in this direction. For example, Szabó, Wu and Zacharias [SWZ14] show that every finitely generated virtually nilpotent group has some box space with finite asymptotic dimension. Finn-Sell and Wu [FSW15] show moreover that for certain box spaces of virtually polycyclic groups the asymptotic dimension of the box space is, like that of the group itself, equal to the Hirsch length of the group.

The main purpose of the present note is to strengthen these results in the case of virtually nilpotent groups. Indeed, we show that if \( G \) is a finitely generated virtually nilpotent group then in fact every box space of \( G \) has asymptotic dimension equal to the Hirsch length of \( G \), as follows.

**Theorem 1.3.** Let \( G \) be a finitely generated residually finite virtually nilpotent group and let \( (N_n)_n \) be a filtration of \( G \). Then \( \text{asdim} \; \sqcup(N_n)G = h(G) \).

The note is organised as follows. In Section 2 we present some basic facts about box spaces and about asymptotic dimension; in Section 3 we compute the asymptotic dimension of certain box spaces in terms of the asymptotic dimensions of the groups they are constructed from; and then finally, in Section 4 we bound the asymptotic dimension of box spaces of groups of polynomial growth in terms of the growth rate and deduce Theorem 1.3.

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## 2. Background

In this section we collect together various results about box spaces and asymptotic dimension.

**Coarse disjoint unions and box spaces.** Given a sequence \( (X_n)_{n=1}^{\infty} \) of sets we write \( \bigsqcup_n X_n \) for their disjoint union. If \( (X_n, d_n) \) and \( (\bigsqcup_n X_n, d) \) are all metric spaces then \( \bigsqcup_n X_n \) is said to be a coarse disjoint union of the \( X_n \) if

1. for each \( n \) the metric \( d \) restricts to \( d_n \) on \( X_n \),
2. whenever \( m \neq n \) the distance between \( X_m \) and \( X_n \) is at least \( \text{diam} \ X_m + \text{diam} \ X_n \), and
3. for every \( r \) the number of pairs \( m, n \) with \( d(X_m, X_n) < r \) is finite.

Note that if \( \text{diam} \ X_i \to \infty \) then property (2) of this definition implies property (3).

**Remark.** The exact distance between the components \( X_n \) of a coarse disjoint union does not change the coarse equivalence class. Since asymptotic dimension is a coarse invariant [BD08 Proposition 22], neither does it change the asymptotic dimension.

In the introduction we defined a box space \( \sqcup(N_i)G \) of a finitely generated residually finite group \( G \) with respect to a filtration \( (N_n) \) of \( G \) and a finite generating set \( S \) of \( G \) to be the disjoint union of the quotients \( (G/N_n) \) with the metric that restricts to the Cayley-graph metric induced by \( S \) on each \( G/N_n \), and such that for \( x \in G/N_m \) and \( y \in G/N_n \) with \( m \neq n \) the distance between \( x \)
and $y$ is precisely $\text{diam}(G/N_m) + \text{diam}(G/N_n)$. Note in particular that $\Box_{(N_i)}G$ is a coarse disjoint union of the $G/N_i$.

We also define the full box space $\Box fG$ of $G$, setting it to be the disjoint union of all finite quotients of $G$ with the metric that restricts to the Cayley-graph metric induced by $S$ on each quotient, and such that for $x \in G/N$ and $y \in G/N'$ with $N \neq N'$ the distance between $x$ and $y$ is precisely $\text{diam}(G/N) + \text{diam}(G/N')$.

The following standard result says that the components of a box space locally ‘look like’ the group.

**Proposition 2.1.** Let $G$ be a residually finite, finitely generated group and let $(N_n)_n$ be a filtration of $G$. Then there exists an increasing sequence $(i_n)_n$ such that for every $k \in \mathbb{N}$ the balls of radius $k$ of $G$ are isometric to the balls of radius $k$ of $G/N_i$, where $i \geq i_n$.

**Proof.** For a given $k$ and large enough $i$ we have $B_G(e, 2k) \cap N_i = \{1\}$, which in turn implies that $B_G(e, k)$ is isometric to $B_{G/N}(e, k)$. \qed

### Asymptotic Dimension

There exist various equivalent alternative definitions of asymptotic dimension (see [BD08, Theorem 19], for example), and it will be useful for us to record one of these alternatives. First, given $r > 0$ and a family $\mathcal{U}$ of subsets of a metric space $X$, we define the $r$-multiplicity of $\mathcal{U}$ to be equal to

$$\max_{x \in X} |\{U \in \mathcal{U} : U \cap B(x, r) \neq \emptyset\}|.$$  

Then we have the following result.

**Proposition 2.2** ([BD08, Theorem 19]). Let $X$ be a metric space. Then $\text{asdim}(X) \leq n$ if and only if for every $R > 0$ there exists $S > 0$ and a covering $\mathcal{U}$ of $X$ such that $\mathcal{U}$ has $R$-multiplicity at most $n + 1$ and $\text{diam}(U) \leq S$ for every $U \in \mathcal{U}$.

**Lemma 2.3** ([BD01, Finite Union Theorem]). Let $X$ be a metric space and let $X_1, \ldots, X_n$ be a finite partition of $X$. Then $\text{asdim} X = \max \{\text{asdim} X_i\}$.

Let $\mathcal{U}$ be a family of metric spaces. We say that $\mathcal{U}$ has asymptotic dimension at most $n$ uniformly, and write $\text{asdim} \mathcal{U} \leq_{\text{unif}} n$, if for every $R > 0$ there exists $S > 0$ such that $(R, S)$-dim $X \leq n$ for every $X \in \mathcal{U}$. This definition is particularly useful to us in light of the following result.

**Lemma 2.4** ([BD01, Theorem 1]). Let $X$ be a metric space, and let $\mathcal{U}$ be a family of subspaces that covers $X$. Suppose that $\text{asdim} \mathcal{U} \leq_{\text{unif}} n$, and that for every $k \in \mathbb{N}$ there exists $F_k \subset X$ with $\text{asdim} F_k \leq n$ such that the family $\{Y \setminus F_k : Y \in \mathcal{U}\}$ is $k$-disjoint. Then $\text{asdim} X \leq n$.

**Corollary 2.5.** If $X$ is a coarse disjoint union of metric spaces $(X_n)_{n=1}^\infty$ then $\text{asdim}(X_n) \leq_{\text{unif}} m$ if and only if $\text{asdim} X \leq m$.

We also record the following trivial fact as a lemma for ease of later reference.

**Lemma 2.6.** Let $X$ be a metric space, and let $\mathcal{U}$ be a family of metric spaces each of which is isometric to a subspace of $X$. Then $\text{asdim} \mathcal{U} \leq_{\text{unif}} \text{asdim} X$.

The following result is presumably well known, although we could not find a reference.

**Lemma 2.7.** Let $G$ be a finitely generated infinite group, and let $B$ be a coarse disjoint union of the balls $B_G(e, r)$ as $r$ ranges over the natural numbers. Then $\text{asdim} B = \text{asdim} G$. 

Proof. The fact that \( \text{asdim} B \leq \text{asdim} G \) follows from Corollary 2.5 and Lemma 2.6.

To prove that \( \text{asdim} G \leq \text{asdim} B \) it suffices to show that \((R, S) - \dim G \leq (R, S) - \dim B\) for every \( R, S > 0 \). To that end, fix \( R, S > 0 \) and suppose that \((R, S) - \dim B = n \in \mathbb{Z}\), so that there exist \( R\)-disjoint families \( U_0, U_1, \ldots, U_n \) of subsets of \( B \) that cover \( B \) such that \( \text{diam}(U) \leq S \) for every \( U \in U_j \) and \( j \in \{0, 1, \ldots, n\} \).

We partition \( G \) into sets \( U_0, \ldots, U_n \) as follows. First, enumerate the elements of \( G \) as \( x_1, x_2, x_3, \ldots \) in such a way that \( |x_m| \) is non-decreasing. We will specify for the \( x_m \) in turn which set \( U_j \) will contain \( x_m \). Note that for each \( m \) and each \( r \geq |x_m| \) there is a copy of \( x_m \) lying in the component \( B_G(e, r) \) of \( B \).

There exists \( i_1 \) and an infinite sequence \( r_{1,1} < r_{1,2} < \ldots \) such that for each \( r_{1,j} \) the copy of \( x_1 \) in the component \( B_G(e, r_{1,j}) \) lies in a set belonging to \( U_{i_1} \). We declare that \( x_1 \in U_{i_1} \). Similarly, there exists \( i_2 \) and an infinite subsequence \( r_{2,1} < r_{2,2} < \ldots \) of \( r_{1,1} < r_{1,2} < \ldots \) such that for each \( r_{2,j} \) the copy of \( x_2 \) in the component \( B_G(e, r_{2,j}) \) lies in a set belonging to \( U_{i_2} \). We declare that \( x_2 \in U_{i_2} \). Continuing in this way, for each \( m \) in turn there exists \( i_m \) and an infinite subsequence \( r_{m,1} < r_{m,2} < \ldots \) of \( r_{m-1,1} < r_{m-1,2} < \ldots \) such that for each \( r_{m,j} \) the copy of \( x_m \) in the component \( B_G(e, r_{m,j}) \) lies in a set belonging to \( U_{i_m} \). We declare that \( x_m \in U_{i_m} \).

It follows from the definition of the \( U_i \) that each \( U_i \) can be partitioned into subsets of diameter at most \( S \) that are \( R\)-disjoint, and this completes the proof. \( \square \)

3. THE ASYMPTOTIC DIMENSION OF ARBITRARY BOX SPACES

Yamauchi [Yam17, Theorem 1.3] shows that a coarse disjoint union of a sequence of graphs with girth tending to infinity has asymptotic dimension either infinite or at most 1. The following is an adaptation of his argument. We are grateful to Rufus Willett (private communication) for pointing it out to us.

**Proposition 3.1.** Let \( G \) be an infinite, residually finite, finitely generated group and let \( N_n \) be a filtration. Then \( \text{asdim} (\Box (N_n) G) \) is either infinite or equal to \( \text{asdim} G \).

**Proof.** We may assume that \( \text{asdim} (\Box (N_n) G) < \infty \), and so \( \text{asdim} (\Box (N_n) G) = m \) for some \( m \in \mathbb{N} \). We first prove that \( \text{asdim} (\Box (N_n) G) \leq \text{asdim} G \).

By definition, for every \( k \in \mathbb{N} \) there exists \( S_k \in \mathbb{N} \) such that \( (k, S_k) - \dim (\Box (N_n) G) \leq m \). For every such \( k \) there therefore exist \( k\)-disjoint families \( U_{i_1}^k, U_{i_2}^k, \ldots, U_{i_m}^k \) of subsets of \( (N_n) G \), such that \( \text{diam}(U) \leq S_k \) for every \( U \in U_j^k \) and such that \( \bigcup_{j=1}^m U_j^k \) covers \( \Box (N_n) G \). By Proposition 2.1 we can take a sequence \( i_k \to \infty \) such that for every \( i \geq i_k \) the balls of radius \( \max \{k, S_k\} \) in \( G/N_i \) are isometric to balls of radius \( \max \{k, S_k\} \) in \( G \). Without loss of generality we may assume that \( (i_k)_k \) is a non-decreasing sequence.

If \( U \in U_j^k \) satisfies \( U \subset G/N_i \) for some \( i \) then \( U \) is contained in a ball of radius at most \( S_k \) inside \( G/N_i \), and if this \( i \) is at least \( i_k \) then \( U \) is isometric to a subspace of \( G \). Lemma 2.6 therefore implies that

\[
\text{asdim} \left( \bigcup_{j,k} \{U \in U_j^k : U \subset G/N_i \text{ for some } i \geq i_k\} \right) \leq \text{unit} \ \text{asdim} G.
\]

Now if \( i \geq i_k \) then \( G/N_i \) is at a distance greater than \( S_k \) from its complement in \( \Box (N_n) G \), and so if \( U \in U_j^k \) intersects \( G/N_i \) non-trivially then in fact \( U \subset G/N_i \). This implies that for every \( i \geq i_k \)
the set $G/N_i$ is covered by the sets $U \in \mathcal{U}^k_j$ with $U \subset G/N_i$ and $j \in \{0, \ldots, m\}$. This means that, defining families $\mathcal{Y}^k_j$ of subsets of $\mathcal{B}(N_i)G$ for $j \in \{0, \ldots, m\}$ and $k \in \mathbb{N}$ via
\[
\mathcal{Y}^k_j = \{U \in \mathcal{U}^k_j : U \subset G/N_i \text{ for some } i \in [i_k, i_{k+1})\},
\]
and then defining families $\mathcal{Y}^{\geq k}_j$ via
\[
\mathcal{Y}^{\geq k}_j = \bigcup_{k' \geq k} \mathcal{Y}^{k'}_j,
\]
for each $k$ the set $\bigcup_{i=i_k}^{\infty} G/N_i$ is covered by the families $\mathcal{Y}^{\geq k}_j$ with $j \in \{0, \ldots, m\}$. In particular, setting $F = \bigcup_{i=i_1}^{i_{n-1}} G/N_i$ and $X_j = \bigcup_{Y \in \mathcal{Y}^{\geq 1}_j} Y$ we have
\[
\mathcal{B}(N_n)G = F \cup \bigcup_{j=0}^{m} X_j.
\]

Note that for each $j$ we have $\text{asdim } \mathcal{Y}^{\geq 1}_j \leq \text{unif asdim } G$ by (3.1). Note also that each family $\mathcal{Y}^{\geq k}_j$ is $k$-disjoint by the definitions of $\mathcal{U}^k_j$ and $i_k$. Finally, note that if we write $F_k = \bigcup_{i=i_1}^{i_{n-1}} G/N_i$ then we have $\mathcal{Y}^{\geq k}_j = \{Y \setminus F_k : Y \in \mathcal{Y}^{\geq 1}_j\}$. Since $F_k$ is finite, and hence of asymptotic dimension 0, it therefore follows from Lemma 2.4 that $\text{asdim } X_j \leq \text{asdim } G$, and then from (3.2) and Lemma 2.3 that
\[
\text{asdim } (\mathcal{B}(N_n)G) \leq \text{asdim } G,
\]
as desired.

Conversely, since $N_n$ is a filtration there is a subspace $B$ of $\mathcal{B}(N_n)G$ that is isometric to a coarse disjoint union of the balls $B_G(e, R)$ as $R$ ranges over the natural numbers. It follows from Lemma 2.7 that $\text{asdim } B = \text{asdim } G$, and since $B$ is a subspace of $\mathcal{B}(N_n)G$ this implies that
\[
\text{asdim } (\mathcal{B}(N_n)G) \geq \text{asdim } G,
\]
which completes the proof. \hfill \square

\section{4. Coarse Disjoint Unions of Groups of Polynomial Growth}

Given a group $G$ with a fixed finite generating set $S$ we write $B_G(x, R)$ for the ball of radius $R$ about the element $x \in G$ in the Cayley graph $\text{Cay}(G, S)$. The group $G$ is said to have \textit{polynomial growth of degree $d$} if there exists $C > 0$ such that $|B_G(e, r)| \leq Cr^d$ for every $r \geq 1$. It is well known and easy to check that this notion does not depend on the choice of finite generating set.

We say that a family $(G_\alpha)_{\alpha \in A}$ of groups with fixed generating sets $S_\alpha$ has \textit{uniform polynomial growth of degree at most $d$} if there exists $C > 0$ such that $|B_{G_\alpha}(e, r)| \leq Cr^d$ for every $r \geq 1$ for every $\alpha \in A$.

\textbf{Proposition 4.1.} Let $G_n$ be a sequence of finite groups with generating sets $S_n$, and suppose that $(\text{Cay}(G_n, S_n))_{n=1}^{\infty}$ has uniform polynomial growth of degree at most $d$. Let $X$ be a coarse disjoint union of the Cayley graphs $\text{Cay}(G_n, S_n)$. Then $\text{asdim } X \leq 4^d$.

\textbf{Proof.} We start with the standard observation that a polynomial growth bound implies a so-called \textit{doubling condition}, as used by Gromov \cite{Gro81} in his proof of his polynomial-growth theorem, for example. Specifically, let $K = 4^d + 1$, let $R > 0$, and take $S_0 = 4^{m+1}R$ with $m$ such that $(K/4^d)^m \geq CR^d$. Then we claim that for every $n$ there exists an $R_n$ such that $R \leq R_n \leq \frac{8}{K}$ and $|B_{G_n}(4R_n)| \leq$
Indeed, if this were not the case then $K_i |B_n(R)| < |B_n(4^i R)| \leq C 4^i d R^d$ for every $i$ with $4^i R \leq S_0$, and so setting $i = m$ would imply that $C 4^m d R^d > K_m |B_n(R)| \geq C 4^m d R^d |B_n(R)|$, contradicting the growth assumption.

Following Ruzsa [Ru29], for every $n \in \mathbb{N}$ with diam($G_n$) $> R$ we take $X_n$ maximal in $G_n$ such that $B_n(x, R_n)$ and $B_n(y, R_n)$ are disjoint for every $x$ and $y$ in $X_n$. We then define $F_R = \bigcup_{n: \text{diam}(G_n) \leq R} G_n$, take

$$U = \{F_R \} \cup \bigcup_{n: \text{diam}(G_n) > R} \{B_n(x, 2R_n) : x \in X_n\},$$

and set $S = \max \{S_0, \text{diam} F_R \}$.

First we show that $U$ is a covering of $\bigcup_{n} G_n$. Let $z \in G_m$ for some $m$. If diam($G_m$) $\leq R$, then $z \in F_R$. If diam($G_m$) $> R$, then as $X_m$ is maximal, there exists an $x \in X_m$ such that $B_n(z, R_m) \cap B_m(x, R_m)$ is non-empty, so $z \in B_m(x, 2R_m)$.

Next we note that diam($U$) $\leq S$ for every $U \in U$. For $U = F_R$ this is true by definition of $S$. On the other hand, for $U \in U$ with $U \neq F_R$ then $U \subset G_m$ for some $m$ and diam($U$) $= 4R_m \leq S_0 \leq S$.

Finally, we show that $U$ has $R$-multiplicity at most $K$. The $R$-multiplicity of $U$ in $G_n$ with diam($G_n$) $\leq R$ is $1 \leq K$, so take $z \in G/N_m$ with $m$ such that diam($G_m$) $\geq R$. Now for every $B_n(z, 2R_m) \in U$ which has an element at a distance at most $R$ to $z$, we have that $x \in B_n(z, 2R_m + R) \subset B_n(z, 3R_m)$. Now consider $B_n(z, 3R_m) \cap X_m$. As $B_n(z, R_m)$ and $B_n(y, R_m)$ are disjoint for any $x$ and $y$ in $X_m$, we have that $|B_n(z, 3R_m) \cap X_m| \leq |B_n(z, R_m)| / |B_n(z, R_m)| \leq K$. Therefore the $R$-multiplicity of $U$ is at most $K = 4^d + 1$, and so asdim $\bigcup_{n} G_n \leq 4^d$ by Proposition 2.2.

**Corollary 4.2.** Let $G$ be a virtually finitely generated virtually nilpotent group. Then $\square_f G$ has finite asymptotic dimension.

**Proof.** As $G$ is virtually nilpotent there exist constants $k$ and $C$ such that $|B_G(e, r)| \leq C r^k$ for every $r \geq 1$. This also means that $|B_G/N(e, r)| \leq C r^k$ for every $r \geq 1$ for every $N < G$, and so the result follows from Proposition 4.1. $\square$

**Proof of Theorem 4.3.** Since $\square_{(N_n)} G$ is a subspace of $\square_f G$ we have asdim $\square_{(N_n)} G \leq \text{asdim} \square_f G < \infty$ by Corollary 4.2. Proposition 3.1 therefore implies that asdim $\square_{(N_n)} G = \text{asdim}(G)$. Since $G$ is virtually polycyclic, the result therefore follows from (1.1). $\square$

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