The Greiner’s approach of heat kernel asymptotics and the variation formulas for the equivariant Ray-Singer metric

Yong Wang

Abstract

In this paper, using the Greiner’s approach of heat kernel asymptotics, we give proofs of the equivariant Gauss-Bonnet-Chern formula and the variation formulas for the equivariant Ray-Singer metric, which are originally due to J.M. Bismut and W. Zhang.

Keywords: Greiner’s heat kernel asymptotics; equivariant Gauss-Bonnet-Chern formula; equivariant Ray-Singer metric

1 Introduction

The first success of proving the Atiyah-Singer index theorem directly by heat kernel method was achieved by Patodi [Pa], who carried out the “fantastic cancellation” (cf. [MS]) for the Laplace operators and for the first time proved a local version of the Gauss-Bonnet-Chern theorem. Nextly several different direct heat kernel proofs of the Atiyah-Singer index theorem for Dirac operators have appeared independently: Bismut [Bi], Getzler [Ge1], [Ge2] and Yu [Yu], Ponge [Po]. All the proofs have their own advantages.

The Atiyah-Bott-Segal-Singer index formula is a generalization with group action of the Atiyah-Singer index theorem. In [BV], Berline and Vergne gave a heat kernel proof of the Atiyah-Bott-Segal-Singer index formula. In [LYZ], Lafferty, Yu and Zhang presented a very simple and direct geometric proof for equivariant index of the Dirac operator. In [PW], Ponge and Wang gave a different proof of the equivariant index formula by the Greiner’s approach of the heat kernel asymptotics. By the method in [LYZ], Zhou gave a direct geometric proof of the equivariant Gauss-Bonnet-Chern formula in [Zh]. The first purpose of this paper is to give another proof of the equivariant Gauss-Bonnet-Chern formula by the Greiner’s approach of the heat kernel asymptotics.

In [BW1], Bismut and Zhang extended the famous Cheeger-Müller theorem to the case where the metric on the auxiliary bundle is not flat and they proved anomaly formulas for Ray-Singer metrics. In [BW2], Bismut and Zhang extended their generalized Cheeger-Müller theorem to the equivariant case and gave anomaly formulas.
for equivariant Ray-Singer metrics. In [We], Weiss gave a new and detailed proof of the variation formulas for the equivariant Ray-Singer metric due to Bismut-Zhang by the method in [BV]. The proof of Weiss and Berline-Vergne lifted the operators to the principle bundle. The second purpose of this paper is to give another proof of anomaly formulas for the equivariant Ray-Singer metric due to Bismut-Zhang by the Greiner’s approach of the heat kernel asymptotics.

This paper is organized as follows: In Section 2, we give another proof of the equivariant Gauss-Bonnet-Chern formula. In Section 3, we give another proof of the variation formulas for the equivariant Ray-Singer metric.

2 The equivariant Gauss-Bonnet-Chern formula

Let \( M \) be a closed even dimensional \( n \) oriented Riemannian manifold and \( \phi \) is an isometry on \( M \) preserving the orientation. Then \( \phi \) induces a map \( \tilde{\phi} = \phi^{-1*} \): \( \wedge T^*M \to \wedge T^*_\phi M \) on the exterior algebra bundle \( \wedge T^*M \). Let \( d \) denote the exterior differential operator and \( \delta \) be its adjoint operator and \( D = d + \delta \) be the de-Rham Hodge operator. Let \( D^+ = D|_{\wedge \text{even} T^*M} \) and \( D^- = D|_{\wedge \text{odd} T^*M} \). Then \( \tilde{\phi}D = D\tilde{\phi} \) and we define the equivariant index

\[
\text{Ind}_\phi(D) = \text{Tr}(\tilde{\phi}|_{\ker D^+}) - \text{Tr}(\tilde{\phi}|_{\ker D^-}).
\]

(2.1)

We recall the Greiner’s approach of heat kernel asymptotics as in [Gr] and [BeGS], [Po]. Define the operator given by

\[
(Q_0u)(x,s) = \int_0^\infty e^{-sD^2}[u(x,t-s)]dt, \quad u \in \Gamma_c(M \times \mathbb{R}, \wedge T^*M),
\]

(2.2)

maps continuously \( u \) to \( D'(M \times \mathbb{R}, \wedge T^*M) \) which is the dual space of \( \Gamma_c(M \times \mathbb{R}, \wedge T^*M) \). We have

\[
(D^2 + \frac{\partial}{\partial t})Q_0u = Q_0(D^2 + \frac{\partial}{\partial t})u = u, \quad u \in \Gamma_c(M \times \mathbb{R}, \wedge T^*M)).
\]

(2.3)

Let \( (D^2 + \frac{\partial}{\partial t})^{-1} \) is the Volterra inverse of \( D^2 + \frac{\partial}{\partial t} \) as in [BeGS]. Then

\[
(D^2 + \frac{\partial}{\partial t})Q = I - R_1; \quad Q(D^2 + \frac{\partial}{\partial t}) = 1 - R_2,
\]

(2.4)

where \( R_1, R_2 \) are smooth operators. Let

\[
(Q_0u)(x,t) = \int_{M \times \mathbb{R}} K_{Q_0}(x,y,t-s)u(y,s)dyds,
\]

(2.5)

and \( k_t(x,y) \) is the heat kernel of \( e^{-tD^2} \). We get

\[
K_{Q_0}(x,y,t) = k_t(x,y) \text{ when } t > 0, \quad \text{ when } t < 0, \quad K_{Q_0}(x,y,t) = 0.
\]

(2.6)
**Definition 2.1** The operator $P$ is called the Volterra $\Psi DO$ if (i) $P$ has the Volterra property, i.e. it has a distribution kernel of the form $K_P(x, y, t - s)$ where $K_P(x, y, t)$ vanishes on the region $t < 0$.

(ii) The parabolic homogeneity of the heat operator $P + \frac{\partial}{\partial t}$, i.e. the homogeneity with respect to the dilations of $\mathbb{R}^n \times \mathbb{R}$ given by

$$
\lambda \cdot (\xi, \tau) = (\lambda \xi, \lambda^2 \tau), \quad (\xi, \tau) \in \mathbb{R}^n \times \mathbb{R}, \quad \lambda \neq 0.
$$

In the sequel for $g \in S(\mathbb{R}^{n+1})$ and $\lambda \neq 0$, we let $g_\lambda$ be the tempered distribution defined by

$$
g_\lambda(\xi, \tau), u(\xi, \tau)) = |\lambda|^{-(n+2)} \langle g_\lambda(\xi, \tau), u(\lambda^{-1} \xi, \lambda^{-2} \tau) \rangle, \quad u \in S(\mathbb{R}^{n+1}).
$$

**Definition 2.2** A distribution $g \in S(\mathbb{R}^{n+1})$ is parabolic homogeneous of degree $m$, $m \in \mathbb{Z}$, if for any $\lambda \neq 0$, we have $g_\lambda = \lambda^m g$.

Let $C_-$ denote the complex halfplane $\{\text{Im} \tau < 0\}$ with closure $\overline{C_-}$. Then:

**Lemma 2.3** ([BeGS, Prop. 1.9]). Let $q(\xi, \tau) \in C^\infty((\mathbb{R}^n \times \mathbb{R})/0)$ be a parabolic homogeneous symbol of degree $m$ such that:

(i) $q$ extends to a continuous function on $(\mathbb{R}^n \times C_-)/0$ in such way to be holomorphic in the last variable when the latter is restricted to $\overline{C_-}$.

Then there is a unique $g \in S(\mathbb{R}^{n+1})$ agreeing with $q$ on $\mathbb{R}^{n+1}/0$ so that:

(ii) $g$ is homogeneous of degree $m$;

(iii) The inverse Fourier transform $\hat{g}(x, t)$ vanishes for $t < 0$.

Let $U$ be an open subset of $\mathbb{R}^n$. We define Volterra symbols and Volterra $\Psi DO$s on $U \times \mathbb{R}^{n+1}/0$ as follows.

**Definition 2.4** $S^m_U(U \times \mathbb{R}^{n+1})$, $m \in \mathbb{Z}$, consists in smooth functions $q(x, \xi, \tau)$ on $U \times \mathbb{R}^n \times \mathbb{R}$ with an asymptotic expansion $q \sim \sum_{j \geq 0} q_{m-j}$, where:

- $q_l \in C^\infty(U \times [(\mathbb{R}^n \times \mathbb{R})/0]$ is a homogeneous Volterra symbol of degree $l$, i.e. $q_l$ is parabolic homogeneous of degree $l$ and satisfies the property (i) in Lemma 2.3 with respect to the last $n+1$ variables;

- The sign $\sim$ means that, for any integer $N$ and any compact $K$, $U$, there is a constant $C_{NK\alpha \beta k} > 0$ such that for $x \in K$ and for $|\xi| + |\tau|^{\frac{1}{2}} > 1$ we have

$$
|\partial_x^\alpha \partial_\xi^\beta \partial_\tau^k (q - \sum_{j \leq N} q_{m-j})(x, \xi, \tau)| \leq C_{NK\alpha \beta k}(|\xi| + |\tau|^{\frac{1}{2}})^{m-N-|\beta|-2k}.
$$
Definition 2.5 $\Psi^m_V(U \times \mathbb{R})$, $m \in \mathbb{Z}$, consists in continuous operators $Q$ from $C^\infty_c(U \times \mathbb{R})$ to $C^\infty(U \times \mathbb{R})$ such that:

(i) $Q$ has the Volterra property;

(ii) $Q = q(x, D_x, D_t) + R$ for some symbol $q$ in $S^m_V(U \times \mathbb{R})$ and some smooth operator $R$.

In the sequel if $Q$ is a Volterra $\Psi DO$, we let $K_Q(x, y, t - s)$ denote its distribution kernel, so that the distribution $K_Q(x, y, t)$ vanishes for $t < 0$.

Definition 2.6 Let $q_m(x, \xi, \tau) \in C^\infty(U \times (\mathbb{R}^{n+1}/0))$ be a homogeneous Volterra symbol of order $m$ and let $g_m \in C^\infty(U) \otimes S'(\mathbb{R}^{n+1})$ denote its unique homogeneous extension given by Lemma 2.3. Then:
- $\hat{q}_m(x, y, t)$ is the inverse Fourier transform of $g_m(x, \xi, \tau)$ in the last $n + 1$ variables;
- $\hat{q}_m(x, D_x, D_t)$ is the operator with kernel $\hat{q}_m(x, y - x, t)$.

Proposition 2.7 The following properties hold.

1) Composition. Let $Q_j \in \Psi^m_{V_j}(U \times \mathbb{R})$, $j = 1, 2$ have symbol $q_j$ and suppose that $Q_1$ or $Q_2$ is properly supported. Then $Q_1Q_2$ is a Volterra $\Psi DO$ of order $m_1 + m_2$ with symbol $q_1 \circ q_2 = \sum \frac{1}{\alpha!} \partial_{\xi}^\alpha q_1 \partial^{\alpha}_x q_2$.

2) Parametrices. An operator $Q$ is the order $m$ Volterra $\Psi DO$ with the paramatrix $P$ then

$$QP = 1 - R_1, \quad PQ = 1 - R_2$$

where $R_1$, $R_2$ are smooth operators.

Proposition 2.8 The differential operator $D^2 + \partial_t$ is invertible and its inverse $(D^2 + \partial_t)^{-1}$ is a Volterra $\Psi DO$ of order $-2$.

We denote by $M^\phi$ the fixed-point set of $\phi$, and for $a = 0, \ldots, n$, we let $M^\phi_a = \bigcup_{0 \leq a \leq n} M^\phi_a$, where $M^\phi_a$ is an $a$-dimensional submanifold. Given a fixed-point $x_0$ in a component $M^\phi_a$, consider some local coordinates $x = (x^1, \ldots, x^a)$ around $x_0$. Setting $b = n - a$, we may further assume that over the range of the domain of the local coordinates there is an orthonormal frame $e_1(x), \ldots, e_b(x)$ of $N^\phi_x$. This defines fiber coordinates $v = (v_1, \ldots, v_b)$. Composing with the map $(x, v) \in N^\phi(x_0) \to \exp_x(v)$ we then get local coordinates $x^1, \ldots, x^a, v^1, \ldots, v^b$ for $M_2$ near the fixed point $x_0$. We shall refer to this type of coordinates as tubular coordinates. Then $N^\phi(x_0)$ is homeomorphic with a tubular neighborhood of $M^\phi$.

By the McKean-Singer formula, we have,

$$\text{Ind}_\phi(D^+) = \text{Str}[\tilde{e} e^{-tD^2}] = \int_M \text{Str}[\tilde{\phi} k_t(x, \phi(x))] dx = \int_M \text{Str}[\tilde{\phi} K_{(D^2 + \partial_t)^{-1}}(x, \phi(x), t)] dx.$$  \hspace{1cm} (2.11)

Let $Q = (D^2 + \partial_t)^{-1}$. For $x \in M^\phi$ and $t > 0$ set

$$I_Q(x, t) := \tilde{\phi}(x)^{-1} \int_{N^\phi_{\epsilon}(x)} \phi(\exp_x v) K_Q(\exp_x v, \exp_x(\phi'(x), t) dv.$$  \hspace{1cm} (2.12)
Here we use the trivialization of $\Lambda(T^*M)$ about the tubular coordinates. Using the tubular coordinates, then

$$I_Q(x,t) = \int_{|v|<\varepsilon} \tilde{\phi}(x,0)^{-1}\tilde{\phi}(x,v)K_Q(x,v;x,\phi'(x)v;t)dv. \quad (2.13)$$

Let

$$q_m^{\wedge(T^*M)}(x,v;\xi,\nu;\tau) := \tilde{\phi}(x,0)^{-1}\tilde{\phi}(x,v)q_{m-j}(x,\xi,\nu;\tau). \quad (2.14)$$

Recall

**Proposition 2.9 ([PW])** Let $Q \in \Psi^m_V(M \times \mathbb{R}, \Lambda(T^*M))$, $m \in \mathbb{Z}$. Uniformly on each component $M_\phi$

$$I_Q(x,t) \sim \sum_{j \geq 0} t^{-\left(\frac{m}{2}+\frac{|\alpha|}{2}\right)} I_Q^j(x) \quad \text{as} \quad t \to 0^+, \quad (2.15)$$

where $I_Q^j(x)$ is defined by

$$I_Q^j(x) := \sum_{|\alpha| \leq m-\left[\frac{m}{2}\right]+2j} \int \frac{v^\alpha}{\alpha!} \left(\partial^\alpha_v q^{\wedge(T^*M)}_{2^{\left[\frac{m}{2}\right]}-2j+|\alpha|}\right)^\wedge (x,0;0,(1-\phi'(x)v);1)dv. \quad (2.16)$$

Let $e(TM^\phi, \nabla^{TM^\phi}) = Pf \left[ -\frac{1}{2\pi} R^{TM^\phi} \right]$ is the Euler form of $TM^\phi$ associated with $
abla^{TM^\phi}$, where $\nabla^{TM^\phi}$ is the Levi-Civita connection on $M^\phi$ and $R^{TM^\phi}$ its curvature. Then we have

**Theorem 2.10** (The equivariant Gauss-Bonnet-Chern theorem)

$$\text{Ind}_\phi(D^+) = \int_{M^\phi} e(TM^\phi, \nabla^{TM^\phi}). \quad (2.17)$$

Let $(V,q)$ be a finite dimensional real vector space equipped with a quadratic form. Let $C(V,q)$ be the associated Clifford algebra, i.e. the associative algebra generated by $V$ with the relations $v \cdot w + w \cdot v = -2q(v,w)$ for $v,w \in V$. Let $e_1 \cdots, e_n$ be the orthonormal basis of $(V,q)$. Let $C(V,q) \otimes C(V,-q)$ be the grading tensor product of $C(V,q)$ and $C(V,-q)$ and $\Lambda^*V \otimes \Lambda^*V$ be the grading tensor product of $\Lambda^*V$ and $\Lambda^*V$. Define the symbol map:

$$\sigma : C(V,q) \otimes C(V,-q) \to \Lambda^*V \otimes \Lambda^*V;$$

$$\sigma(e_{j_1} \cdots e_{j_l} \otimes 1) = e_{j_1} \wedge \cdots \wedge e_{j_l} \otimes 1; \quad \sigma(1 \otimes \tilde{c}(e_{j_1} \cdots \tilde{c}(e_{j_l})) = 1 \otimes \tilde{e}_{j_1} \wedge \cdots \wedge \tilde{e}_{j_l}. \quad (2.18)$$
Using the interior multiplication \( \iota(e_j) : \Lambda^* V \to \Lambda^{*-1} V \) and the exterior multiplication \( \varepsilon(e_j) : \Lambda^* V \to \Lambda^{*+1} V \), we define representations of \( C(V,q) \) and \( C(V,-q) \) on the exterior algebra:

\[
c : C(V,q) \to \text{End} \, \wedge V, \ e_j \mapsto c(e_j) \equiv \varepsilon(e_j) - \iota(e_j),
\]

\[
\tilde{c} : C(V,-q) \to \text{End} \, \wedge V, \ e_j \mapsto \tilde{c}(e_j) \equiv \varepsilon(e_j) + \iota(e_j),
\]

The tensor product of these representations yields an isomorphism of superalgebras

\[
c \otimes \tilde{c} : C(V,q) \otimes C(V,-q) \to \text{End} \, \wedge V.
\]

which we will also denote by \( c \). We obtain a supertrace (i.e. a linear functional vanishing on supercommutators) on \( C(V,q) \otimes C(V,-q) \) by setting \( \text{Str}(a) = \text{Str}_{\text{End} \, \wedge V}[c(a)] \) for \( a \in C(V,q) \otimes C(V,-q) \), where \( \text{Str}_{\text{End} \, \wedge V} \) is the canonical supertrace on \( \text{End} V \).

**Lemma 2.11** For \( 1 \leq i_1 < \cdots < i_p \leq n, \ 1 \leq j_1 < \cdots < j_q \leq n \), then

\[
\text{Str}[c(e_{i_1}) \cdots c(e_{i_p}) \tilde{c}(e_{j_1}) \cdots \tilde{c}(e_{j_q})] = (-1)^{\frac{p}{2} + 2n},
\]

when \( p = q = n \), otherwise equals zero.

We will also denote the volume element in \( \Lambda V \otimes \Lambda V \) by \( \omega = e^1 \wedge \cdots \wedge e^n \wedge \epsilon^1 \wedge \cdots \wedge \epsilon^n \) and for \( a \in \Lambda V \otimes \Lambda V \) let \( Ta \) be the coefficient of \( \omega \) in \( a \). The linear functional \( T : \Lambda V \otimes \Lambda V \to R \) is called the Berezin trace. Then for \( a \in C(V,q) \otimes C(V,-q) \), one has \( \text{Str}(a) = (-1)^{\frac{p}{2} + 2n}(T \sigma)(a) \). We define the Getzler order as follows:

\[
\deg \partial_j = \frac{1}{2} \deg \partial_t = 2 \deg e_j = 2 \deg \tilde{c}(e_j) = - \deg x^1 = 1. \tag{2.19}
\]

Let \( Q \in \Psi^*_V(\mathbb{R}^n \times \mathbb{R}, \Lambda^* T^* M) \) have symbol

\[
q(x, \xi, \tau) \sim \sum_{k \leq m^*} q_k(x, \xi, \tau), \tag{2.20}
\]

where \( q_k(x, \xi, \tau) \) is an order \( k \) symbol. Then taking components in each subspace \( \Lambda^* T^* M \) and using Taylor expansions at \( x = 0 \) give formal expansions

\[
\sigma[q(x, \xi, \tau)] \sim \sum_{j,k} \sigma[q_k(x, \xi, \tau)]^{(j)} \sim \sum_{j,k,\alpha} \frac{x^\alpha}{\alpha!} \sigma[\partial_\alpha^2 q_k(0, \xi, \tau)]^{(j)}. \tag{2.21}
\]

The symbol \( \frac{x^\alpha}{\alpha!} \sigma[\partial_\alpha^2 q_k(0, \xi, \tau)]^{(j)} \) is the Getzler homogeneous of \( k + \frac{j}{2} - |\alpha| \). So we can expand \( \sigma[q(x, \xi, \tau)] \) as

\[
\sigma[q(x, \xi, \tau)] \sim \sum_{j \geq 0} q_{(m-\frac{j}{2})}(x, \xi, \tau), \quad q_{(m)} \neq 0, \tag{2.22}
\]

where \( q_{(m-\frac{j}{2})} \) is a Getzler homogeneous symbol of degree \( m - \frac{j}{2} \).
Definition 2.12 The \( m \) is called as the Getzler order of \( Q \). The symbol \( q_{(m)} \) is the principle Getzler homogeneous symbol of \( Q \). The operator \( Q_{(m)} = q_{(m)}(x, D_x, D_t) \) is called as the model operator of \( Q \).

Let \( e_1, \ldots , e_n \) be an oriented orthonormal basis of \( T_{x_0}M \) such that \( e_1, \ldots , e_a \) span \( T_{x_0}M^\phi \) and \( e_{a+1}, \ldots , e_n \) span \( N_{x_0}^\phi \). This provides us with normal coordinates \((x_1, \ldots , x_n) \to \exp_{x_0}(x^1e_1 + \cdots + x^ne_n)\). Moreover using parallel translation enables us to construct a synchronous local oriented tangent frame \( e_1(x), \ldots , e_n(x) \) such that \( e_1(x), \ldots , e_a(x) \) form an oriented frame of \( TM^\phi_{x,a} \) and \( e_{a+1}(x), \ldots , e_n(x) \) form an (oriented) frame \( N^\phi \) (when both frames are restricted to \( M^\phi \)). This gives rise to trivializations of the tangent and exterior algebra bundles. Write

\[
\phi'(0) = \begin{pmatrix} 1 & 0 \\ 0 & \phi^N \end{pmatrix}.
\]

Let \( \wedge(n) = \wedge^* \mathbb{R}^n \) be the exterior algebra of \( \mathbb{R}^n \). We shall use the following gradings on \( \wedge(n) \otimes \wedge(n) \),

\[
\wedge(n) \otimes \wedge(n) = \bigoplus_{1 \leq k_1, k_2 \leq a} \wedge^{k_1, k_2}(n) \otimes \wedge^{k_1, k_2}(n),
\]

where \( \wedge^{k_1, k_2}(n) \) is the space of forms \( dx^{i_1} \wedge \cdots \wedge dx^{i_k} \) with \( 1 \leq i_1 < \cdots < i_k \leq a \) and \( a + 1 \leq i_{k+1} < \cdots < i_{k+l} \leq n \). Given a form \( \omega \in \wedge(n) \otimes \wedge(n) \) we shall denote by \( \omega^{(k_1, k_2)} \) its component in \( \wedge^{k_1, k_2}(n) \otimes \wedge^{k_1, k_2}(n) \). We denote by \( |\omega|((a,0),(a,0)) \) the Berezin integral \( |\omega|((*,0),(*,0))((a,0),(a,0)) \) of its component \( \omega((*,0),(*,0)) \) in \( \wedge((*,0),(*,0))(n) \).

By (2.19), similar to Lemma 3.6 in [Wa], we have

Lemma 2.13 \( Q \in \Psi^*_f(\mathbb{R}^n, x, \wedge(T^*M)) \) has the Getzler order \( m \) and model operator \( Q_{(m)} \). Let \( j \) be even, then as \( t \to 0^+ \)

(1) \( \sigma[Q(0,t)]^{(j)} = O(t^{\frac{\frac{m-a-1}{2}}{2}}) \) if \( m - \frac{j}{2} \) is odd.

(2) \( \sigma[Q(0,t)]^{(j)} = O(t^{\frac{\frac{m-a-2}{2}}{2}})I_{Q_{(m)}}(0,1)^{(j)} + O(t^{\frac{\frac{m-a}{2}}{2}}) \) if \( m - \frac{j}{2} \) is even.

In particular, for \( m = -2 \) and \( j = 2a \) and \( a \) is even we get

\[
\sigma[Q(0,t)]((a,0),(a,0)) = I_{Q(-2)}(0,1)((a,0),(a,0)) + O(t^2).
\]

(2.23)

By the Weitzenböck formula, we have

\[
D^2 = -\sum_{j=1}^{n} (\nabla^2 e_j - \nabla^2_{g_M} e_j) + \frac{r_M}{4} - \frac{1}{8} \sum_{1 \leq i,j,k,l \leq n} R_{ijkl} c(e_i)c(e_j)c(e_k)c(e_l).
\]

(2.24)

By (2.19) and (2.24), we get the model operator of \( \frac{\partial}{\partial t} + D^2 \) is \( \frac{\partial}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j} - \frac{1}{2} \hat{R} \), where \( \hat{R} = \frac{1}{4} \sum_{1 \leq i,j,k,l \leq n} R_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l \). By

\[
(\frac{\partial}{\partial t} - \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j} - \frac{1}{2} \hat{R}) K_{Q(-2)}(x,y,t) = 0,
\]

(2.25)
we get
\[ K_{Q(-2)}(x, y, t) = (4\pi t)^{\frac{n}{2}}\exp\left(-\frac{1}{4t}\|x - y\|^2\right) e^{\frac{\Delta t}{4}}. \] (2.26)

Similar to Lemma 9.13 in [PW], we get
\[ I_{Q(-2)}(0, t) = (4\pi t)^{-\frac{n}{2}}\det^{-1}(1 - \phi^N)e^{\frac{\Delta t}{4}}. \] (2.27)

Let the matrix \( \phi^N \) equal
\[ \phi^N = \text{diag}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}, \cdots, \begin{pmatrix} \cos\theta_n & \sin\theta_n \\ -\sin\theta_n & -\cos\theta_n \end{pmatrix}\right), \]

we have the lemma

**Lemma 2.14 ([Zh])**

\[ \tilde{\phi} = \left(\frac{1}{2}\right)^{n-a} \prod_{j = 1}^{\frac{n}{2}} [(1 + \cos\theta_j) - (1 - \cos\theta_j)c(e_{2j-1})c(e_{2j})c(e_{2j-1})c(e_{2j})] \]
\[ + \sin\theta_j(c(e_{2j-1})c(e_{2j}) - c(e_{2j-1})c(e_{2j}))]. \] (2.28)

By the lemma 2.14, we have
\[ \sigma(\tilde{\phi})^{((0,b),(0,b))} = \left(\frac{1}{2}\right)^{\frac{n}{2}a} \prod_{j = 1}^{\frac{n}{2}} (1 - \cos\theta_j)e^{a+1} \wedge \cdots \wedge e^n \wedge \tilde{e}^{a+1} \wedge \cdots \wedge \tilde{e}^n \]
\[ = (-\frac{1}{4})^{\frac{n}{2}} \det(1 - \phi^N)e^{a+1} \wedge \cdots \wedge e^n \wedge \tilde{e}^{a+1} \wedge \cdots \wedge \tilde{e}^n. \] (2.29)

So we get the following lemma

**Lemma 2.15** For \( A \in C(V, q) \tilde{\otimes} C(V, -q) \), we have
\[ \text{Str}[\tilde{\phi}A] = (-1)^{\frac{n}{2}} 2^n \left(\frac{1}{4}\right)^{\frac{n}{2}} \det(1 - \phi^N)|\sigma(A)|^{((0, 0), (a, 0))} \]
\[ + (-1)^{\frac{n}{2}} 2^n \sum_{l_1, l_2 < b} |\sigma(\tilde{\phi})^{((0,l_1), (0,l_2))}\sigma(A)^{(a, b-l_1, a, b-l_2))}|(n,n). \] (2.30)

By (2.27) and (2.30) and Lemma 2.13, we get
\[ \lim_{t \to 0} \text{Str}[\tilde{\phi}I_{D^2 + \partial_1^{-1}}(x_0, t)] = (-1)^{\frac{n}{2}} 2^n \left(\frac{1}{4}\right)^{\frac{n}{2}} (4\pi)^{-\frac{n}{2}} |\sigma(\tilde{\phi})^{((0,0), (a, 0))}| = e(TM\phi). \] (2.31)

So we get the theorem 2.10.
3 The variation formulas for the equivariant Ray-Singer metric

Let $M$ be an even dimensional $n$ oriented closed manifold and $(F, \nabla^F)$ be a flat complex vector bundle over $M$. Let further $g^{TM}$ be a Riemannian metric on $M$ and $h^F$ be a Hermitian metric on $F$. We will not assume $h^F$ to be parallel with respect to $\nabla^F$. Let $G$ be a compact Lie group acting smoothly on $M$ such that the metric $g^{TM}$, $h^F$ and the flat connection $\nabla^F$ are preserved. Let $A^*(M, F) = \Gamma(M, \wedge^* F \otimes F)$ denote the differential forms on $M$ with values in $F$. Let further $d^F : A^*(M, F) \to A^{*+1}(M, F)$ denote the exterior differential associated with the flat connection $\nabla^F$. The Hodge Laplacian is given by $\Delta^F = d^F d^F + d^{F*} d^F$, where $d^{F*}$ denotes the formal adjoint of $d^F$. For $t > 0$ let $\exp(-t\Delta^F)$ denote the heat operator. We consider 1-parameter families of $G$-invariant metrics: (1) $\varepsilon \mapsto g^{TM}(\varepsilon)$ with $g^{TM}(0) = g^{TM}$ (2) $\varepsilon \mapsto h^F(\varepsilon)$ with $h^F(0) = h^F$. Let $C = s^{-1} \ast$ and $V = (h^F)^{-1}h^F$. For a 1-parameter of Riemannian metrics $\varepsilon \mapsto g^{TM}(\varepsilon)$ we set

$$\dot{S} := \nabla^TM - \frac{1}{2}[\nabla^TM, (g^{TM})^{-1}\dot{g}^{TM}] \in A(M, so(TM)).$$

We define the transgression form

$$\varepsilon'(TM) := \frac{\partial}{\partial b} \bigg|_{b=0} \text{Pf} \left[ -\frac{1}{2\pi} (R^{TM} + b\dot{S}) \right].$$

Since $h^F$ is not necessarily parallel about $\nabla^F$, we may define a second flat connection $(\nabla^F)^T$ on $F$ by the formula $(\nabla^F)^T = (h^F)^{-1}\nabla^{F*}h^F$, where $\nabla^{F*}$ denotes the connection induced by $\nabla^F$ on $F^*$ and $h^F : F \to F^*$ is the isomorphism induced by $h^F$. Observe that $(\nabla^F)^T = \nabla^F$ if and only if $\nabla^F h^F = 0$. We set $\omega(F, h^F) := (\nabla^F)^T - \nabla^F \in A^1(M, \text{End}(F))$ and for $\phi \in G$, $\theta(\phi, F, h^F) := \text{Tr}[\phi \omega(F, h^F)] \in A^1(M^\phi)$. Then $\theta(\phi, F, h^F)$ is closed and that its cohomology class does not depend on $h^F$. In the following, we will prove

**Theorem 3.1 (Bismut-Zhang)** For $\phi \in G$, we have

$$\lim_{t \to 0} \text{Str}[\tilde{\phi} V \exp(-t\Delta^F)] = \int_{M^\phi} \text{Tr}[\phi F V] e(TM^\phi, \nabla^{TM^\phi}),$$

$$\lim_{t \to 0} \text{Str}[\tilde{\phi} C \exp(-t\Delta^F)] = -\int_{M^\phi} \theta(\phi, F, h^F) \varepsilon'(TM^\phi),$$

By the theorem 3.1, we can easily get the variation formulas for the equivariant Ray-Singer metric due to Bismut-Zhang (see [BZ2] or [We]). In general, neither of the connections $\nabla^F$ and $(\nabla^F)^T$ will preserve the metric $h^F$. As in [BZ1] we define a third connection $\nabla^{F,e} = \frac{1}{2}(\nabla^F + (\nabla^F)^T)$ on $F$. This connection will preserve $h^F$, but it will in general not be flat.

In the following we will write $\mathcal{E} = \wedge^* T M \otimes F$. We will also denote by $\nabla^{F,e}$ the tensor product connection $\nabla^{\wedge^* T M} \otimes 1 + 1 \otimes \nabla^{F,e}$ on $\mathcal{E}$, where $\nabla^{\wedge^* T M}$ is the connection...
on $\wedge T^* M$ induced by $\nabla^{TM}$. Let $\triangle^{E,e}$ denote the connection Laplacian on $E$ associated to the connection $\nabla^{F,e}$. Since $\nabla^{F,e}$ is a metric connection on $E$, the operator $\triangle^{E,e}$ will be formally selfadjoint.

**Proposition 3.2** ([BZ1]): (Lichnerowicz formula for $\triangle(F)$) One has

$$\triangle(F) = -\triangle^{E,e} + E$$

with $E \in \Gamma(M, \text{End} E)$ which w.r.t. a local ON-frame $e_j$ is given by

$$E = -\frac{1}{8} \sum_{i,j,k,l} (R^{TM}(e_i, e_j)e_k, e_l)c(e_i)c(e_j)\bar{c}(e_k)\bar{c}(e_l)$$

$$-\frac{1}{8} \sum_{i,j} c(e_i)c(e_j)\omega(F, h^F)^2(e_i, e_j) + \frac{1}{8} \sum_{i,j} \bar{c}(e_i)\bar{c}(e_j)\omega(F, h^F)^2(e_i, e_j)$$

$$-\frac{1}{2} \sum_{i,j} c(e_i)\bar{c}(e_j)[\nabla^{TM} \otimes \text{End} F \omega(F, h^F)(e_j) + \frac{1}{2} \omega(F, h^F)^2(e_i, e_j)]$$

$$+ \frac{1}{4} \sum_j (\omega(F, h^F)(e_j))^2 + \frac{1}{4} r^M,$$

where $r^M$ denotes the scalar curvature of $(M, g^{TM})$.

Let $D^0 = \sum_{j=1}^n c(e_j)\nabla^e_j$ be a selfadjoint twisted Dirac operator, then

$$(D^0)^2 = -\triangle^{E,e} - \frac{1}{8} \sum_{i,j,k,l} (R^{TM}(e_i, e_j)e_k, e_l)c(e_i)c(e_j)\bar{c}(e_k)\bar{c}(e_l)$$

$$-\frac{1}{8} \sum_{i,j} c(e_i)c(e_j)\omega(F, h^F)^2(e_i, e_j) + \frac{1}{4} r^M$$

(3.3)

Let

$$L(\omega) = \frac{1}{8} \sum_{i,j} \bar{c}(e_i)\bar{c}(e_j)\omega(F, h^F)^2(e_i, e_j) - \frac{1}{2} \sum_{i,j} c(e_i)\bar{c}(e_j)[\nabla^{TM} \otimes \text{End} F \omega(F, h^F)(e_j)]$$

$$+ \frac{1}{2} \omega(F, h^F)^2(e_i, e_j) + \frac{1}{4} \sum_j (\omega(F, h^F)(e_j))^2.$$ 

(3.4)

Then

$$\triangle(F) = (D^0)^2 + L(\omega).$$

(3.5)

**The Proof of (3.1):**

By (2.19), we get the model operator of $\frac{\partial}{\partial t} + \triangle(F)$ is still $\frac{\partial}{\partial t} - \sum_{j=1}^n \frac{\partial^2}{\partial y_j^2} - \frac{1}{2} \hat{R}$. Let $P$ be a differential operator, then we have

$$\text{Str}[\tilde{\phi} \text{Pexp}(-t \triangle(F))] = \int_{M^0} \text{Str}[\tilde{\phi} P(\frac{\partial}{\partial t} + \triangle(F))^{-1}(x, t)]dx + O(t^\infty).$$

(3.6)
When $P = V$ is a 0-order differential operator and $O_{G}(V) = 0$, we have $O_{G}(V(\frac{\partial}{\partial y}) + \Delta(F))^{-1} = -2$, and the model operator of $V(\frac{\partial}{\partial y} + \Delta(F))$ is $V \left[ \frac{\partial}{\partial y} - \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} - \frac{1}{2}R \right]$. Then similar to the proof of the theorem 2.10, we get (3.1). □

**Lemma 3.3** ([BZ1]) The endomorphism $C = *^{-1} \ast$ is given in terms of Clifford variable by

$$C = -\frac{1}{2} \sum_{i,j} ((g^{TM})^{-1}g^{TM} e_i, e_j) c(e_i) \tilde{c}(e_j), \quad (3.7)$$

$$\sigma(C) = -\frac{1}{2} \sum_{i,j} ((g^{TM})^{-1}g^{TM} e_i, e_j) e^i \wedge \tilde{e}^j. \quad (3.8)$$

As in [BZ1], we introduce an auxiliary even Clifford variable, such that $\sigma^2 = 1$ and $\sigma$ commutes with all the previously considered operators. Let $A, B \in \text{End}(\mathcal{E})$ be trace class. Then $A + \sigma B$ lies in $\text{End}(\mathcal{E}) \otimes \mathbb{R}(\sigma)$. Set

$$\text{Str}^\sigma[A + \sigma B] = \text{Str}[B]. \quad (3.9)$$

Let

$$(D^2)^{\text{odd}} := -\frac{1}{2} \sum_{i,j} c(e_i) \tilde{c}(e_j) [\nabla^{TM} \otimes \text{End} F \omega(F, h^F)(e_j) + \frac{1}{2} \omega(F, h^F)^2(e_i, e_j)], \quad (3.10)$$

$$(D^2)^{\text{even}} := \Delta(F) - (D^2)^{\text{odd}}. \quad (3.11)$$

By the theorem 4.18 in [BZ2], we have

$$\text{Str}[^\phi C \text{exp}(-t \Delta(F))] = \text{Str}^\sigma[^\phi C \text{exp}(-t((D^2)^{\text{even}} + \sigma(D^2)^{\text{odd}}))]. \quad (3.12)$$

Let

$$L(\omega, \sigma) = \frac{1}{8} \sum_{i,j} \tilde{c}(e_i) \tilde{c}(e_j) \omega(F, h^F)^2(e_i, e_j) - \frac{1}{2} \sigma \sum_{i,j} c(e_i) \tilde{c}(e_j) [\nabla^{TM} \otimes \text{End} F \omega(F, h^F)(e_j) + \frac{1}{2} \omega(F, h^F)^2(e_i, e_j)] + \frac{1}{4} \sum_{j} (\omega(F, h^F)(e_j))^2. \quad (3.13)$$

Then

$$(D^2)^{\text{even}} + \sigma(D^2)^{\text{odd}} = (D^0)^2 + L(\omega, \sigma). \quad (3.14)$$

By the Duhamel principle, we have when $t$ is small

$$\text{Str}^\sigma[^\phi C \text{exp}(-t((D^0)^2 + L(\omega, \sigma)))] = \sum_{k=1}^{\frac{d}{2}} \sum_{k > \frac{d}{2}} (-t)^k \int_{\Delta_k} \text{Str}^\sigma[^\phi C \cdot e^{-t(D^0)^2} L(\omega, \sigma) \cdots L(\omega, \sigma) e^{-t_k(D^0)^2}]d\text{vol}_{\Delta_k}. \quad (3.15)$$
By the Hölder inequality and the Weyl theorem, we can get

\[
\sum_{k \geq k_0} t^k \int_{\Delta_k} \left| \text{Str}^\sigma [\phi^e \cdot e^{-t_0 t(D^0)^2} L(\omega, \sigma) \cdots L(\omega, \sigma) e^{-t_k t(D^0)^2}] \right| \, \text{dvol}_k \\
\leq \sum_{k \geq k_0} t^k \frac{||L(\omega, \sigma)||_k}{k!} O(t^{-\frac{n}{2}}) = O(t^{\frac{3}{2}}).
\]

(3.16)

Let \( B \) be an operator and \( l \) be a positive integer. Write

\[
B^{[l]} = [(D^0)^2, B^{[l-1]}], \quad B^{[0]} = B.
\]

**Lemma 3.4** ([CH]) Let \( B \) a finite order differential operator, then for any \( s > 0 \), we have:

\[
e^{-s(D^0)^2} B = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l B^{[l]} e^{-s(D^0)^2} + (-1)^N s^N B^{[N]}(s),
\]

where \( B^{[N]}(s) \) is given by

\[
B^{[N]}(s) = \int_{\Delta_N} e^{-u_1 s(D^0)^2} B^{[N]} e^{-(1-u_1) s(D^0)^2} \, du_1 du_2 \cdots du_N.
\]

(3.17)

Similar to the theorem 1 in [CH], we have

\[
t^k \int_{\Delta_k} \text{Str}^\sigma [\phi^e \cdot e^{-t_0 t(D^0)^2} L(\omega, \sigma) \cdots L(\omega, \sigma) e^{-t_k t(D^0)^2}] \, \text{dvol}_k
\]

\[
= \sum_{0 \leq |\lambda| \leq n-k} c_{\lambda, k} t^{k+\lambda_1 + \cdots + \lambda_k} \int_{\Delta_k} \text{Str}^\sigma [\phi^e C[L(\omega, \sigma)]^{[\lambda_1]} \cdots L(\omega, \sigma)]^{[\lambda_k]} e^{-t(D^0)^2}] + O(t^{\frac{3}{2}}),
\]

where \( c_{\lambda, k} \) is a constant. Since \( O_G((D^0)^2) = 2 \) and \( O_G(L(\omega, \sigma)) = 1 \) and \( O_G(C) = 1 \), we have

\[
O_G(CL(\omega, \sigma)^{[\lambda_1]} \cdots L(\omega, \sigma)^{[\lambda_k]}) = k + 1 + 2(\lambda_1 + \cdots + \lambda_k).
\]

(3.20)

So when \( k > 1 \), by the lemma 2.13, we get

\[
t^k \int_{\Delta_k} \text{Str}^\sigma [\phi^e C \cdot e^{-t_0 t(D^0)^2} L(\omega, \sigma) \cdots L(\omega, \sigma) e^{-t_k t(D^0)^2}] \, \text{dvol}_k = O(t^{\frac{3}{2}}),
\]

(3.21)

So, we need only compute the term

\[
- \sum_{0 \leq \lambda_1 \leq n-k} c_{\lambda_1, k} t^{1+\lambda_1} \int_{\Delta_1} \text{Str}^\sigma [\phi^e C[L(\omega, \sigma)]^{[\lambda_1]} e^{-t(D^0)^2}].
\]

(3.22)
Since
\[(D^0)^2 = -\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2} - \frac{1}{2} \bar{R} + O_G(1),\] (3.23)
we can get \(O_G([(D^0)^2, L(\omega, \sigma)]) = 2\), then \(O_G(L(\omega, \sigma)^{[\lambda_1]} = 2\lambda_1\) for \(\lambda_1 > 0\). So by the lemma 2,13, we get
\[-\sum_{0 \leq \lambda_1 \leq n-k} c_{\lambda_1, k} t^{1+\lambda_1} \int_{\Delta_1} \text{Str}^\sigma [\tilde{\phi}C[L(\omega, \sigma)]^{[\lambda_1]} e^{-t(D^0)^2}] = -\text{Str}^\sigma [\tilde{\phi}CL(\omega, \sigma)e^{-t(D^0)^2}] + O(t^\frac{1}{2}) = -\text{Str}^\sigma [\tilde{\phi}C(D^2)^{\text{odd}} e^{-t(D^0)^2}] + O(t^\frac{1}{2}),\] (3.24)
Similar to the theorem 2.10, we can get
\[-\text{Str}^\sigma [\tilde{\phi}C(D^2)^{\text{odd}} e^{-t(D^0)^2}] = -\left(-\frac{1}{\pi}\right)^{\frac{a}{2}} \left\{ \sum_{i,j=1}^{a} \left[ \left( (g^{TM})^{-1} g^{TM} e_i, e_j \right) e^i \wedge \hat{e}^j \right] \right\} \wedge \exp(-\frac{\bar{R}^{TM\phi}}{2}) \wedge \frac{1}{2} \left\{ \sum_{i,j=1}^{a} e^i \wedge \hat{e}^j \text{Tr}[\phi F(\nabla e_i \omega(e_j))] \right\}^{(a,0),(a,0)},\] (3.25)
By (3.25), using the same calculations as in the non-equivariant case (see [BZ1]), we can prove (3.2).

**Acknowledgement.** This work was supported by NSFC. 11271062 and NCET-13-0721.

**References**

[BeGS] R. Beals, P. Greiner, N. Stanton, The heat equation on a CR manifold. J. Differential Geom. 20 (1984), 343-387.

[BV] N. Berline and M. Vergne: A computation of the equivariant index of the Dirac operators. Bull. Soc. Math. France 113(1985) 305-345.

[Bi1] J.-M. Bismut: The Atiyah-Singer theorems: A probabilistic approach. J. Func. Anal. 57(1984) 56-99.

[BZ1] J.-M. Bismut, W. Zhang, An extension of a theorem by Cheeger and Muller. Asterisque 205, 1992.

[BZ2] J.-M. Bismut, W. Zhang, Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle. GAFA 4, 1994, no. 2, 136-212.

[CH] S. Chern and X. Hu, Equivariant Chern character for the invariant Dirac operators, Michigan Math. J. 44(1997), 451-473.

[Ge1] E. Getzler: Pseudodifferential operators on supermanifolds and the Atiyah-Singer index theorem. Commun. Math. Phys. 92(1983) 163-178.

[Ge2] E. Getzler: A short proof of the local Atiyah-Singer index theorem. Topology 25(1986) 111-117.
[Gr] P. Greiner, An asymptotic expansion for the heat equation. Arch. Rational Mech. Anal. 41 (1971), 163-218.

[LYZ] J. D. Lafferty, Y. L. Yu and W. P. Zhang, A direct geometric proof of Lefschetz fixed point formulas, Trans. AMS. 329 (1992), 571-583.

[MS] H. P. Mckean; I. M. Singer: Curvature and the eigenvalue of the Laplacian, J. Diff. Geom., I (1967), 43-69.

[Pa] V. K. Patodi, Ourvatures and the eigenforms of the Laplace operator, J. Diff. Geom., 5 (1971), 238-249.

[Po] R. Ponge, A new short proof of the local index formula and some of its applications. Comm. Math. Phys. 241 (2003), 215-234.

[PW] R. Ponge and H. Wang, Noncommutative geometry, conformal geometry, and the local equivariant index theorem, arXiv:1210.2032.

[Wa] Y. Wang, The Greiners approach of heat kernel asymptotics, equivariant family JLO characters and equivariant eta forms, arXiv:1304.7354.

[We] H. Weiss, The variation formulas for the equivariant Ray-Singer metric. Internat. J. Math. 19 (2008), no. 9, 1021-1051.

[Yu] Y. Yu, Local index theorem for Dirac operators. Acta. Math. Sinica., vol.3 (1987), No.2, pp. 152-169.

[Zh] J. Zhou, A geometric proof of the Lefschetz fixed-point theorem for signature operators, (in Chinese) Acta Math. Sinica 35 (1992), no. 2, 230-239.

School of Mathematics and Statistics, Northeast Normal University, Changchun Jilin, 130024, China
E-mail: wangy581@nenu.edu.cn