On Approximating the $l_p$ Distances for $p > 2$
(When $p$ Is Even)

Ping Li
Department of Statistical Science
Faculty of Computing and Information Science
Cornell University, Ithaca, NY 14850

June 27, 2008

Abstract

Many applications in machine learning and data mining require computing pairwise $l_p$ distances in a data matrix $A \in \mathbb{R}^{n \times D}$. For massive high-dimensional data, computing all pairwise distances of $A$ can be infeasible. In fact, even storing $A$ or all pairwise distances of $A$ in the memory may be also infeasible.

For $0 < p \leq 2$, efficient small space algorithms exist, for example, based on the method of stable random projections, which unfortunately is not directly applicable to $p = 3, 4, 5, 6, ...$. This paper proposes a simple method for $p = 2, 4, 6, ...$ We first decompose the $l_p$ (where $p$ is even) distances into a sum of 2 marginal norms and $p - 1$ “inner products” at different orders. Then we apply normal or sub-Gaussian random projections to approximate the resultant “inner products,” assuming that the marginal norms can be computed exactly by a linear scan.

We propose two strategies for applying random projections. The basic projection strategy requires only one projection matrix but it is more difficult to analyze, while the alternative projection strategy requires $p - 1$ projection matrices but its theoretical analysis is much easier. In terms of the accuracy, at least for $p = 4$, the basic strategy is always more accurate than the alternative strategy if the data are non-negative, which is common in reality.

1 Introduction

This study proposes a simple method for efficiently computing the $l_p$ distances in a massive data matrix $A \in \mathbb{R}^{n \times D}$ for $p > 2$ (where $p$ is even), using random projections[22].

While many previous work on random projections focused on approximating the $l_2$ distances (and inner products), the method of symmetric stable random projections[8, 12, 18, 15] is applicable to approximating the $l_p$ distances for all $0 < p \leq 2$. This work proposes using random projections for $p > 2$, at least for some special cases.

1First draft Dec. 2007. Slightly revised June 2008.
Machine learning algorithms often operate on the $l_p$ distances of $A$ instead of the original data. A straightforward application would be searching for the nearest neighbors using $l_p$ distance. The $l_p$ distance is also a basic loss functions for quality measure. The widely used “kernel trick,” (e.g., for support vector machines (SVM)), is often constructed on top of the $l_p$ distances[21].

Here we can treat $p$ as a tuning parameter. It is common to take $p = 2$ (Euclidian distance), or $p = \infty$ (infinity distance), $p = 1$ (Manhattan distance), or $p = 0$ (Hamming distance); but in principle any $p$ values are possible. In fact, if there is an efficient mechanism to compute the $l_p$ distances, then it becomes affordable to tune learning algorithms for many values of $p$ for the best performance.

In modern data mining and learning applications, the ubiquitous phenomenon of “massive data” imposes challenges. For example, pre-computing and storing all pairwise $l_p$ distances in memory at the cost $O(n^2)$ can be infeasible when $n > 10^5$ (or even just $10^3$)[5]. For ultra high-dimensional data, even just storing the whole data matrix can be infeasible. In the meanwhile, modern applications can routinely involve millions of observations; and developing scalable learning and data mining algorithms has been an active research direction. One commonly used strategy in current practice is to compute the distances on the fly[5], in stead of storing all pairwise $l_p$ distances.

Data reduction algorithms such as sampling or sketching methods are also popular. While there have been extensive studies on approximating the $l_p$ distances for $0 < p \leq 2$, $p > 2$ can be useful too. For example, because the normal distribution is completely determined by its first two moments (mean and variance), we can identify the non-normal components of the data by analyzing higher moments, in particular, the fourth moments (i.e., kurtosis). Thus, the fourth moments are critical, for example, in the field of Independent Component Analysis (ICA)[11]. Therefore, it is viable to use the $l_p$ distance for $p > 2$ when lower order distances can not efficiently differentiate data.

It is unfortunate that the family of stable distributions[24] is limited to $0 < p \leq 2$ and hence we can not directly using stable distributions for approximating the $l_p$ distances. In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances in the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.

1.1 The Methodology

Given a giant data matrix $A \in \mathbb{R}^{n \times D}$, we assume that a linear scan of the data is feasible, but computing all pairwise interactions is not, either due to computational budget constraints or memory limits. Also, we only consider even $p = 4, 6, ..., $ among which $p = 4$ is probably the most important.

Interestingly, our method is based only on normal (or normal-like) projections. The observation is that, when $p$ is even, the $l_p$ distance can be decomposed into marginal

\footnote{It is well-known that the radial basis kernel using the $l_p$ distance with $0 < p \leq 2$ satisfies the Mercer’s condition. However, we can still use the $l_p$ distance with $p > 2$ as kernels, although in this case it is not guaranteed to find the “most optimal” solution. For very large-scale learning, we usually will not find the “most optimal” solution any way.}
\( l_p \) norms and “inner products” of various orders. For example, for two \( D \)-dimensional vectors \( x \) and \( y \), when \( p = 4 \), then

\[
d(p) = \sum_{i=1}^{D} |x_i - y_i|^p = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3.
\]

Since we assume that a linear scan of the data is feasible, we can compute \( \sum_{i=1}^{D} x_i^4 \) and \( \sum_{i=1}^{D} y_i^4 \) exactly. We can approximate the interaction terms \( \sum_{i=1}^{D} x_i^2 y_i^2 \), \( \sum_{i=1}^{D} x_i^3 y_i \), and \( \sum_{i=1}^{D} x_i y_i^3 \) using normal (or normal-like) random projections. Therefore, for \( p \) being even, we are able to efficiently approximate the \( l_p \) distances.

### 1.2 Paper Organization

Section 2 concerns using normal random projections for approximating \( l_4 \) distances. We introduce two projection strategies and the concept of utilizing the marginal norms to improve the estimates. Section 3 extends this approach to approximating \( l_6 \) distances. Section 4 analyzes the effect of replacing normal projections by sub-Gaussian projections.

### 2 Normal Random Projections for \( p = 4 \)

The goal is to efficiently compute all pairwise \( l_p \) (\( p = 4 \)) distances in \( A \in \mathbb{R}^{n \times D} \). It suffices to consider any two rows of \( A \), say \( x \) and \( y \), where \( x, y \in \mathbb{R}^D \). We need to estimate the \( l_p \) distance between \( x \) and \( y \)

\[
d(p) = \sum_{i=1}^{D} |x_i - y_i|^p.
\]

which, when \( p = 4 \), becomes

\[
d(4) = \sum_{i=1}^{D} |x_i - y_i|^4 = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3.
\]

In one pass, we can compute \( \sum_{i=1}^{D} x_i^4 \) and \( \sum_{i=1}^{D} y_i^4 \) easily, but computing the interactions is more difficult. We resort to random projections for approximating \( \sum_{i=1}^{D} x_i^2 y_i^2 \), \( \sum_{i=1}^{D} x_i^3 y_i \), and \( \sum_{i=1}^{D} x_i y_i^3 \). Since there are three “inner products” of different orders, we can choose either only one projection matrix for all three terms (the basic projection strategy), or three independent projection matrices (the alternative projection strategy).
2.1 The Basic Projection Strategy

First, generate a random matrix \( R \in \mathbb{R}^{D \times k} \) (\( k \ll D \)), with i.i.d. entries from a standard normal, i.e.,

\[
    r_{ij} \sim N(0, 1), \quad \mathbb{E}(r_{ij}) = 0, \quad \mathbb{E}(r_{ij}^2) = 1, \quad \mathbb{E}(r_{ij}^4) = 3.
\]

Using random projections, we generate six vectors in \( k \) dimensions, \( u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^k \):

\[
    u_{1,j} = \sum_{i=1}^{D} x_i r_{ij}, \quad u_{2,j} = \sum_{i=1}^{D} x_i^2 r_{ij}, \quad u_{3,j} = \sum_{i=1}^{D} x_i^3 r_{ij},
\]

\[
    v_{1,j} = \sum_{i=1}^{D} y_i r_{ij}, \quad v_{2,j} = \sum_{i=1}^{D} y_i^2 r_{ij}, \quad v_{3,j} = \sum_{i=1}^{D} y_i^3 r_{ij}.
\]

We have a simple unbiased estimator of \( d^{(4)} \)

\[
    \hat{d}^{(4)} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_2^T v_2 - 4u_3^T v_1 - 4u_1^T v_3 \right).
\]

Lemma 1

\[
    \mathbb{E}(\hat{d}^{(4)}) = d^{(4)}.
\]

\[
    \text{Var}(\hat{d}^{(4)}) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right) + \Delta_4
\]

\[
    \Delta_4 = -\frac{48}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i y_i^2 \right) - \frac{48}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^2 y_i^3 \right) + \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^3 \right).
\]

\(^3\)It is possible to relax the requirement of i.i.d samples. In fact, to prove unbiasedness of the estimates only needs pairwise independence, and to derive the variance formula requires four-wise independence.
Proof 1 See Appendix A □

The basic projection strategy is simple but its analysis is quite involved, especially when $p > 4$. Also, if we are interested in higher order moments (other than variance) of the estimator, the analysis becomes very tedious.

2.2 The Alternative Projection Strategy

Instead of one projection matrix $R$, we generate three, $R^{(a)}$, $R^{(b)}$, $R^{(c)}$, independently. By random projections, we generate six vectors in $k$ dimensions, $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^k$, such that

\[
\begin{align*}
    u_{1,j} &= \sum_{i=1}^{D} x_i r_{ij}^{(c)}, \\
    u_{2,j} &= \sum_{i=1}^{D} x_i^2 r_{ij}^{(a)}, \\
    u_{3,j} &= \sum_{i=1}^{D} x_i^3 r_{ij}^{(b)}, \\
    v_{1,j} &= \sum_{i=1}^{D} y_i r_{ij}^{(b)}, \\
    v_{2,j} &= \sum_{i=1}^{D} y_i^2 r_{ij}^{(a)}, \\
    v_{3,j} &= \sum_{i=1}^{D} y_i^3 r_{ij}^{(c)}. 
\end{align*}
\]

Here we abuse the notation slightly by using the same $u$ and $v$ for both projection strategies.

Again, we have an unbiased estimator, denoted by $\hat{d}_{(4),a}$

\[
\hat{d}_{(4),a} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_2^Tv_2 - 4u_4^Tv_1 - 4u_4^Tv_3 \right)
\]

Lemma 2

\[
E \left( \hat{d}_{(4),a} \right) = d_{(4)},
\]

\[
\text{Var} \left( \hat{d}_{(4),a} \right) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i \right)^2 \right).
\]

Proof 2 The proof basically follows from that of Lemma 7.
Compared with \( \text{Var}(\hat{d}_{(4)}) \) in Lemma 1, the difference would be 
\[
\text{Var}(\hat{d}_{(4)}) - \text{Var}(\hat{d}_{(4),a}) = \Delta_4
\]
which can be either negative or positive. For example, when all \( x_i \)'s are negative and all \( y_i \)'s are positive, then \( \Delta_4 \geq 0 \), i.e., the alternative projections strategy results in smaller variance and hence it should be adopted.

We can show in Lemma 3 that when the data are non-negative (which is more likely the reality), the difference in (1) will never exceed zero, suggesting that the basic strategy would be preferable, which is also operationally simpler (although more sophisticated in the analysis).

\textbf{Lemma 3} \textit{If all entries of} \( x \) \textit{and} \( y \) \textit{are non-negative, then}
\[
\text{Var}(\hat{d}_{(4)}) - \text{Var}(\hat{d}_{(4),a}) = \Delta_4 \leq 0. 
\]
\textit{Proof 3} \textit{See Appendix B.} \hfill \square

Thus, the main advantage of the alternative projection strategy is that it simplifies the analysis, especially true when \( p > 4 \). Also, analyzing the alternative projection strategy may provide an estimate for the basic projection strategy. For example, the variance of \( \hat{d}_{(4),a} \) is an upper bound of the variance of \( \hat{d}_{(4)} \) in non-negative data.

In the next subsection, we show that the alternative strategy make the analysis feasible when we take advantage of the marginal information.

### 2.3 Improving the Estimates Using Margins

Since we assume that a linear scan of the data is feasible and in fact the estimators in both strategies already take advantage of the marginal \( l_4 \) norms, \( \sum_{i=1}^{D} x_i^4 \) and \( \sum_{i=1}^{D} y_i^4 \), we might as well compute other marginal norms and try to take advantage of them in a systematic manner.

\textbf{Lemma 4} demonstrates such a method for improving estimates using margins. For simplicity, we assume in Lemma 4 that we adopt the alternative projection strategy, in order to carry out the (asymptotic) analysis of the variance.
Lemma 4 Suppose we use the alternative projection strategy described in Section 2.2 to generate samples \( u_{1,j}, u_{2,j}, u_{3,j}, v_{1,j}, v_{2,j}, \text{ and } v_{3,j} \). We estimate \( d_{(4)} \) by

\[
\hat{d}_{(4),a,mle} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6\hat{a}_{2,2} - 4\hat{a}_{3,1} - 4\hat{a}_{1,3},
\]

where \( \hat{a}_{2,2}, \hat{a}_{3,1}, \hat{a}_{1,3} \), are respectively, the solutions to the following three cubic equations:

\[
a_{2,2}^3 - \frac{a_{2,2}^2}{k} u_2 v_2 - \frac{1}{k} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 u_2 v_2
\]

\[+ a_{2,2} \left( -\sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 \right) + \frac{a_{2,2}}{k} \left( \sum_{i=1}^{D} x_i^4 ||v_2||^2 + \sum_{i=1}^{D} y_i^4 ||u_2||^2 \right) = 0.
\]

\[
a_{3,1}^3 - \frac{a_{3,1}^2}{k} u_1 v_1 - \frac{1}{k} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 u_1 v_1
\]

\[+ a_{3,1} \left( -\sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 \right) + \frac{a_{3,1}}{k} \left( \sum_{i=1}^{D} x_i^4 ||v_1||^2 + \sum_{i=1}^{D} y_i^4 ||u_1||^2 \right) = 0.
\]

\[
a_{1,3}^3 - \frac{a_{1,3}^2}{k} u_3 v_3 - \frac{1}{k} \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^4 u_3 v_3
\]

\[+ a_{1,3} \left( -\sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^4 \right) + \frac{a_{1,3}}{k} \left( \sum_{i=1}^{D} x_i^6 ||v_3||^2 + \sum_{i=1}^{D} y_i^4 ||u_3||^2 \right) = 0.
\]

Asymptotically (as \( k \to \infty \)), the variance would be

\[
\text{Var} \left( \hat{d}_{(4),a,mle} \right) = 36 \text{Var} (\hat{a}_{2,2}) + 16 \text{Var} (\hat{a}_{2,2}) + 16 \text{Var} (\hat{a}_{2,2})
\]

\[
= 36 \left( \frac{\sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 - \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2}{k} \right)^2
\]

\[+ 16 \left( \frac{\sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 - \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2}{k} \right)^2
\]

\[+ 16 \left( \frac{\sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 - \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2}{k} \right)^2 + O \left( \frac{1}{k^2} \right)
\]
Proof 4 [16, 17] proposed taking advantage of the marginal \( l_2 \) norms to improve the estimates of \( l_2 \) distances and inner products. Because we assume the alternative projection strategy, we can analyze \( \hat{a}_{2,2}, \hat{a}_{3,1}, \) and \( \hat{a}_{1,3} \), independently and then combine the results; and hence we skip the detailed proof.

Of course, in practice, we probably still prefer the basic projection strategy, i.e., only one projection matrix instead of three. In this case, we still solve three cubic equations, but the precise analysis of the variance becomes much more difficult. When the data are non-negative, we believe that \( \text{Var} \left( \hat{d}_{(4),a,mle} \right) \) will also be the upper bound of the estimation variance using the basic projection strategy, which can be easily verified by empirical results (not included in the current report).

Solving cubic equations is easy, as there are closed-form solutions. We can also solve the equations by iterative methods. In fact, it is common practice to do only a one-step iteration (starting with the solution without using margins), called "one-step Newton-Rhapson" in statistics.

3 Normal Random Projections for \( P=6 \)

For higher \( p \) (where \( p \) is even), we can follow basically the same procedure as for \( p = 4 \). To illustrate this, we work out an example for \( p = 6 \). We only demonstrate the basic projection strategy.

The \( l_6 \) distance can be decomposed into 2 marginal norms and 5 inner products at various orders:

\[
d_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 - 20 \sum_{i=1}^{D} x_i^3 y_i^3 \\
+ 15 \sum_{i=1}^{D} x_i^2 y_i^4 + 15 \sum_{i=1}^{D} x_i^4 y_i^2 - 6 \sum_{i=1}^{D} x_i^5 y_i - 6 \sum_{i=1}^{D} x_i y_i^5
\]

Generate one random projection matrix \( R \in \mathbb{R}^{D \times k} \), and

\[
u_{1,j} = \sum_{i=1}^{D} x_i r_{ij}, \quad u_{2,j} = \sum_{i=1}^{D} x_i^2 r_{ij}, \quad u_{3,j} = \sum_{i=1}^{D} x_i^3 r_{ij}, \\
u_{4,j} = \sum_{i=1}^{D} x_i^4 r_{ij}, \quad u_{5,j} = \sum_{i=1}^{D} x_i^5 r_{ij}, \\
u_{1,j} = \sum_{i=1}^{D} y_i r_{ij}, \quad v_{2,j} = \sum_{i=1}^{D} y_i^2 r_{ij}, \quad v_{3,j} = \sum_{i=1}^{D} y_i^3 r_{ij}, \\
u_{4,j} = \sum_{i=1}^{D} y_i^4 r_{ij}, \quad v_{5,j} = \sum_{i=1}^{D} y_i^5 r_{ij}.
\]
Lemma 5 provide the variance of the following unbiased estimator of $\hat{d}(\theta)$:

$$\hat{d}(\theta) = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \left( -20u_3 v_3 + 15u_1 v_2 + 15u_2 v_4 - 6u_5 v_3 - 6u_1^T v_5 \right)$$

$$= \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \sum_{j=1}^{k} -20u_{3,j} v_{3,j} + 15u_{1,j} v_{4,j} + 15u_{4,j} v_{2,j} - 6u_{1,j} v_{3,j} - 6u_{5,j} v_{1,j}.$$  

Lemma 5

$$\text{Var} \left( \hat{d}(\theta) \right) = \frac{400}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^3 y_i^3 \right)^2 \right) + \frac{225}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^8 + \left( \sum_{i=1}^{D} x_i^2 y_i^4 \right)^2 \right)$$

$$+ \frac{225}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^3 y_i^3 \right)^2 \right) + \frac{36}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^{10} + \left( \sum_{i=1}^{D} x_i y_i^4 \right)^2 \right) + \Delta_5$$

where

$$\Delta_5 = - \frac{600}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2 y_i^5 + \sum_{i=1}^{D} x_i y_i^3 \right) - \frac{600}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^4 \sum_{i=1}^{D} x_i y_i^3 \right)$$

$$+ \frac{240}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^8 + \sum_{i=1}^{D} x_i^3 y_i^5 + \sum_{i=1}^{D} x_i y_i^4 \right) + \frac{240}{k} \left( \sum_{i=1}^{D} x_i^8 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i^2 y_i^5 \sum_{i=1}^{D} x_i y_i^3 \right)$$

$$+ \frac{450}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^2 y_i^4 \sum_{i=1}^{D} x_i y_i^3 \right) - \frac{180}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^4 \sum_{i=1}^{D} x_i y_i^3 \right)$$

$$- \frac{180}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2 y_i^5 \sum_{i=1}^{D} x_i y_i^3 \right) - \frac{180}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^4 \sum_{i=1}^{D} x_i y_i^3 \right).$$

Proof 5 See Appendix C □.

When all entries of $x$ and $y$ are non-negative, we believe it is true that $\Delta_5 \leq 0$, but we did not proceed with the proof.

Of course, it is again a good idea to take advantage of the marginal norms, but we skip the analysis.

4 Sub-Gaussian Random Projections

It is well-known that it is not necessary to sample $r_{ij} \sim N(0, 1)$. In fact, to have an unbiased estimator, it suffices to sample $r_{ij}$ from any distribution with zero mean (and
unit variance). For good higher-order behaviors, it is often a good idea to sample from a sub-Gaussian distribution, of which a zero-mean normal distribution is a special case.

The theory of sub-Gaussian distributions was developed in the 1950’s. See [6] and references therein. A random variable $x$ is sub-Gaussian if there exists a constant $g > 0$ such that for all $t \in \mathbb{R}$:

$$
E(\exp(\sigma^2t^2)) \leq \exp\left(\frac{g^2t^2}{2}\right).
$$

In this section, we sample $r_{ij}$ from a sub-Gaussian distribution with the following restrictions:

$$
E(r_{ij}) = 0, \quad E(r_{ij}) = 1, \quad E(r_{ij}^4) = s,
$$

and we denote $r_{ij} \sim SubG(s)$. It can be shown that we must restrict $s \geq 1$.

One example would be the $r_{ij} \sim \text{Uniform}(-\sqrt{3}, \sqrt{3})$, for which $s = \frac{9}{5}$. Although the uniform distribution is simpler than normal, it is now well-known that we should sample from the following three-point sub-Gaussian distributions[1].

$$
r_{ij} = \sqrt{s} \times \begin{cases} 1 \text{ with prob. } \frac{1}{2} \\ 0 \text{ with prob. } 1 - \frac{1}{s} \\ -1 \text{ with prob. } \frac{1}{2s} \end{cases}, \quad s \geq 1
$$

In our analysis, we do not have to specify the exact distribution of $r_{ij}$ and we can simply express the estimation variance as a function of $s$.

Here, we consider the basic projections strategy, by generating one random projection matrix $R \in \mathbb{R}^{n \times D}$ with i.i.d. entries $r_{ij} \sim SubG(s)$, and

$$
\begin{align*}
    u_{1,j} &= \sum_{i=1}^{D} x_{i} r_{ij}, \quad u_{2,j} = \sum_{i=1}^{D} x_{i}^2 r_{ij}, \quad u_{3,j} = \sum_{i=1}^{D} x_{i}^3 r_{ij}, \\
    v_{1,j} &= \sum_{i=1}^{D} y_{i} r_{ij}, \quad v_{2,j} = \sum_{i=1}^{D} y_{i}^2 r_{ij}, \quad v_{3,j} = \sum_{i=1}^{D} y_{i}^3 r_{ij}.
\end{align*}
$$

We again have a simple unbiased estimator of $d_{(4)}$

$$
\hat{d}_{(4),s} = \sum_{i=1}^{D} x_{i}^4 + \sum_{i=1}^{D} y_{i}^4 + \frac{1}{10} (6u_{2}v_{2} - 4u_{3}v_{1} - 4u_{1}v_{3}).
$$
Lemma 6

\[
\text{Var}\left(\hat{d}(4,s)\right) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right) - \frac{48}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^2 y_i + \left( s - 3 \right) \sum_{i=1}^{D} x_i^5 y_i \right) - \frac{48}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + \left( s - 3 \right) \sum_{i=1}^{D} x_i^3 y_i \right) + \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + \left( s - 3 \right) \sum_{i=1}^{D} x_i^4 y_i \right)
\]

\textbf{Proof 6} See Appendix \[D\]. □

5 Conclusions

It has been an active research topic on approximating \(l_p\) distances in massive high-dimensional data, for example, a giant “data matrix” \(A \in \mathbb{R}^{n \times D}\). While a linear scan on \(A\) may be feasible, it can be prohibitive (or even infeasible) to compute and store all pairwise \(l_p\) distances. Using random projections can reduce the cost of computing all pairwise distances from \(O(n^2 D)\) to \(O(n^2 k)\) where \(k \ll D\). The data size is reduced from \(O(nD)\) to \(O(nk)\) and hence it may be possible to store the reduced data in memory.

While the well-known method of \textit{stable random projections} is applicable to \(0 < p \leq 2\), not directly to \(p > 2\), we propose a practical approach for approximating the \(l_p\) distances in massive data for \(p = 2, 4, 6, \ldots\), based on the simple fact that, when \(p\) is even, the \(l_p\) distances can be decomposed into 2 marginal norms and \(p - 1\) “inner products” of various orders. Two projection strategies are proposed to approximate these “inner products” as well as the \(l_p\) distances; and we show the basic projection strategy (which is simpler) is always preferable over the alternative strategy in terms of the accuracy, at least for \(p = 4\) in non-negative data. We also propose utilizing the marginal norms (which can be easily computed exactly) to further improve the estimates. Finally, we analyze the performance using sub-Gaussian random projections.
A Proof of Lemma 1

\[ \hat{d}_{(4)} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} \left( 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \right) \]

\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} \left( 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \right) \]

\[ u_{2,j}v_{2,j} = \left( \sum_{i=1}^{D} x_i^2 r_{ij} \right) \left( \sum_{i=1}^{D} y_i^2 r_{ij} \right) = \sum_{i=1}^{D} x_i^2 y_i^2 r_{ij}^2 + \sum_{i \neq i'} x_i^2 y_i^2 r_{i'j}^2 \]

Thus

\[ E (u_{2,j}v_{2,j}) = \sum_{i=1}^{D} x_i^2 y_i. \]

Similarly, we can show

\[ E (u_{3,j}v_{1,j}) = \sum_{i=1}^{D} x_i^3 y_i, \quad E (u_{1,j}v_{3,j}) = \sum_{i=1}^{D} x_i y_i^3. \]

Therefore,

\[ E (\hat{d}_{(4)}) = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} \left( 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \right) \]

\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} \left( 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3 \right) \right) = \hat{d}_{(4)}. \]

To derive the variance, we need to analyze the expectation

\[ (6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j})^2 \]

\[ = 36u_{2,j}^2 v_{2,j}^2 + 16u_{3,j}^2 v_{1,j}^2 + 16u_{1,j}^2 v_{3,j}^2 - 48u_{2,j}u_{3,j}v_{2,j}v_{1,j} \]

\[ - 48u_{2,j}u_{1,j}v_{2,j}v_{3,j} + 32u_{3,j}u_{1,j}v_{2,j}v_{3,j}. \]

To simplify the expression, we will skip the terms that will be zeros when taking expectations.

\[ E (u_{2,j}^2 v_{2,j}^2) = E \left( \left( \sum_{i=1}^{D} x_i^2 y_i^2 r_{ij}^2 + \sum_{i \neq i'} x_i^2 y_i^2 r_{i'j}^2 \right)^2 \right) \]

\[ = E \left( \sum_{i=1}^{D} x_i^4 y_i^4 + 2 \sum_{i \neq i'} x_i^2 y_i^2 x_i'^2 y_i'^2 + \sum_{i \neq i'} x_i^2 y_i^2 y_i'^2 r_{ij}^2 \right) \]

\[ = \sum_{i=1}^{D} 3x_i^4 y_i^4 + 2 \sum_{i \neq i'} x_i^2 y_i^2 x_i'^2 y_i'^2 + \sum_{i \neq i'} x_i^2 y_i^2 y_i'^2 \]

\[ = \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + 2 \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2. \]
Similarly

\[ E \left( u_{3,j}^2\nu_{1,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + 2 \left( \sum_{i=1}^{D} x_i y_i \right)^2, \]

\[ E \left( u_{3,j}^2\nu_{1,j}^2 \right) = \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i y_i \right)^2. \]

\[
E \left( u_{2,j} u_{3,j} \nu_{2,j} \nu_{1,j} \right) \\
= E \left( \sum_{i=1}^{D} D_{ij} D_{i'j'} \sum_{i=1}^{D} D_{ij} D_{i'j'} \sum_{i=1}^{D} D_{ij} D_{i'j'} \sum_{i=1}^{D} D_{ij} D_{i'j'} \right) \\
= E \left( \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \right) \\
= E \left( \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \right) \\
= \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij}. \\
\]

Similarly

\[ E \left( u_{2,j} u_{1,j} \nu_{2,j} \nu_{3,j} \right) \\
= \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij}, \]

\[ E \left( u_{3,j} u_{1,j} \nu_{3,j} \right) \\
= \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij} \sum_{i=1}^{D} D_{ij} D_{ij}. \]
Therefore,

\[ \text{Var}(6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j}) = 36 \sum_{i=1}^{D} x_i^4 y_i^4 + 72 \left( \frac{1}{D} \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 + 16 \sum_{i=1}^{D} x_i^6 y_i^6 + 32 \left( \frac{1}{D} \sum_{i=1}^{D} x_i^3 y_i \right)^2 + 16 \sum_{i=1}^{D} x_i^2 y_i^6 + 32 \left( \frac{1}{D} \sum_{i=1}^{D} x_i y_i^3 \right)^2 - 48 \left( \sum_{i=1}^{D} x_i^5 y_i^3 + \sum_{i=1}^{D} x_i^3 y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^4 + \sum_{i=1}^{D} x_i y_i^2 \right)^2 \]

from which it follows that

\[ \text{Var}(\hat{d}_{(4)}) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 y_i^4 + \left( \sum_{i=1}^{D} \frac{1}{x_i} y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 y_i^6 + \left( \sum_{i=1}^{D} \frac{1}{x_i} y_i^3 \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right) - \frac{48}{k} \left( \sum_{i=1}^{D} x_i^5 y_i^3 + \sum_{i=1}^{D} x_i^3 y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^4 + \sum_{i=1}^{D} x_i y_i^2 \right)^2 \]
B  Proof of Lemma 3

It suffices to show that

\[
\left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i^2 \right) \\
+ \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^5 y_i^3 \right) \\
- \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^1 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^2 \right) \geq 0.
\]

We need to use the arithmetic-geometric mean inequality:

\[
\sum_{i=1}^{n} w_i \geq \left( \prod_{i=1}^{n} w_i \right)^{1/n}, \quad \text{provided } w_i \geq 0.
\]

Because

\[
x_i^5 y_i^3 + x_i^3 y_i^5 \geq 2 \sqrt{x_i^3 y_i^5} = 2 x_i^2 y_i,
\]

\[
\sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^2 \geq 0.
\]

Thus it only remains to show that

\[
\sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^4 y_i^3 - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^3 \geq 0,
\]

for which it suffices to show that

\[
2 \sum_{i=1}^{D} x_i^{3/2} y_i^{3/2} \sum_{i=1}^{D} x_i^{5/2} y_i^{7/2} - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^3 \geq 0,
\]

or equivalently, to show that, if \( z_i \geq 0 \) \( \forall i \in [1, D] \), then

\[
f(z_i, i = 1, 2, \ldots, D) = 2 \sum_{i=1}^{D} z_i^3 \sum_{i=1}^{D} z_i^5 - \sum_{i=1}^{D} z_i^3 \sum_{i=1}^{D} z_i^6 \geq 0. \tag{3}
\]

Obviously, (3) holds for \( D = 1 \) and \( D = 2 \). To see that it is true for \( D > 2 \), we notice that only at \((z_1 = 0, z_2 = 0, \ldots, z_D = 0)\), the first derivative of \( f(z_i) \) is zero. We can also check that \( f(z_i = 1, i = 1, 2, \ldots, D) > 0 \). Since \( f(z_i) \) is a continuous function, we know \( f(z_i) \geq 0 \) must hold if \( z_i > 0 \) for all \( i \). There is no need to worry about the boundary case that \( z_j = 0 \) and \( z_i \geq 0 \) because it is reduced to a small problem with \( D' = D - 1 \) and we have already shown the base case when \( D = 1 \) and \( D = 2 \). Thus, we complete the proof.
Proof of Lemma 5

\[ d_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 - 20 \sum_{i=1}^{D} x_i^3 y_i^3 + 15 \sum_{i=1}^{D} x_i^2 y_i^4 + 15 \sum_{i=1}^{D} x_i^4 y_i^2 - 6 \sum_{i=1}^{D} x_i^7 y_i^3 - 6 \sum_{i=1}^{D} x_i y_i^7 \]

\[ d_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \left( -20 u_3^T v_3 + 15 u_4^T v_2 + 15 u_2^T v_4 - 6 u_5^T v_3 - 6 u_1^T v_5 \right) \]

To derive the variance, we need to analyze the expectation of

\[ (-20 u_3, v_3, j + 15 u_2, v_2, j + 15 u_4, v_2, j - 6 u_1, v_5, j - 6 u_5, v_1, j)^2 \]

\[ = 400 u_3^2, v_3^2, j + 225 u_2^2, v_2^2, j + 225 u_4^2, v_2^2, j + 36 u_1^2, v_5^2, j + 36 u_5^2, v_2^2, j \]

\[ - 600 u_3, v_3, u_2, v_4, j - 600 u_3, v_3, u_4, v_2, j + 240 u_3, v_3, u_1, v_5, j \]

\[ + 240 u_3, v_3, u_5, v_1, j + 450 u_2, v_4, u_4, v_2, j - 180 u_2, v_4, u_1, v_5, j \]

\[ - 180 u_2, v_4, u_5, v_1, j - 180 u_4, v_2, u_1, v_5, j - 180 u_4, v_2, u_5, v_1, j \]

\[ + 72 u_1, v_5, v_5, v_1, j \]

Skipping the detail, we can show that

\[ \mathbb{E} \left( u_{3,j}^2 v_{3,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i^3 y_i^3 \right)^2 \]

\[ \mathbb{E} \left( u_{2,j}^2 v_{4,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i^4 y_i^2 \right)^2 \]

\[ \mathbb{E} \left( u_{4,j}^2 v_{2,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i^7 y_i^3 \right)^2 \]

\[ \mathbb{E} \left( u_{1,j}^2 v_{5,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i y_i^7 \right)^2 \]

\[ \mathbb{E} \left( u_{5,j}^2 v_{1,j}^2 \right) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i y_i^7 \right)^2 \]
\[ E(u_{3,j}u_{2,j}v_{3,j}v_{4,j}) \]
\[ = \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^2 y_i^4 + \sum_{i=1}^{D} x_i^4 y_i^4 \sum_{i=1}^{D} x_i^2 y_i^3, \]
\[ E(u_{3,j}u_{4,j}v_{3,j}v_{2,j}) \]
\[ = \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i^2 + \sum_{i=1}^{D} x_i^4 y_i^4 \sum_{i=1}^{D} x_i^4 y_i^3, \]
\[ E(u_{3,j}v_{1,j}v_{3,j}v_{5,j}) \]
\[ = \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i y_i^5 + \sum_{i=1}^{D} x_i y_i^5 \sum_{i=1}^{D} x_i y_i^3, \]
\[ E(u_{2,j}u_{4,j}v_{4,j}v_{2,j}) \]
\[ = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i^4 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i^4 y_i^2, \]
\[ E(u_{2,j}u_{5,j}v_{4,j}v_{1,j}) \]
\[ = \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^5 + \sum_{i=1}^{D} x_i y_i^5 \sum_{i=1}^{D} x_i y_i^2, \]
\[ E(u_{4,j}u_{1,j}v_{2,j}v_{5,j}) \]
\[ = \sum_{i=1}^{D} x_i^9 \sum_{i=1}^{D} y_i^9 + \sum_{i=1}^{D} x_i^4 y_i^2 \sum_{i=1}^{D} x_i^7 y_i + \sum_{i=1}^{D} x_i^7 y_i \sum_{i=1}^{D} x_i^4 y_i^2, \]
\[ E(u_{4,j}v_{5,j}v_{2,j}v_{1,j}) \]
\[ = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i, \]
\[ E(u_{1,j}u_{5,j}v_{5,j}v_{1,j}) \]
\[ = \sum_{i=1}^{D} x_i^9 \sum_{i=1}^{D} y_i^9 + \sum_{i=1}^{D} x_i^4 y_i^2 \sum_{i=1}^{D} x_i^7 y_i + \sum_{i=1}^{D} x_i^7 y_i \sum_{i=1}^{D} x_i^4 y_i^2, \]
\[ E(u_{1,j}v_{5,j}v_{5,j}v_{1,j}) \]
\[ = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i y_i, \]
Combining the results, we obtain

\[
\text{Var} \left( \hat{d}(\delta) \right) = \frac{400}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right)
\]
\[+ \frac{225}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^8 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right)
\]
\[+ \frac{225}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^4 y_i \right)^2 \right)
\]
\[+ \frac{36}{k} \left( \sum_{i=1}^{D} x_i^8 \sum_{i=1}^{D} y_i^{10} + \left( \sum_{i=1}^{D} x_i^5 y_i \right)^2 \right)
\]
\[+ \frac{36}{k} \left( \sum_{i=1}^{D} x_i^{10} \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^7 y_i \right)^2 \right) + \Delta_{\delta}
\]

where

\[
k\Delta_{\delta}/6 = -100 \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^7 y_i^3 \right)
\]
\[-100 \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^7 y_i^3 \right)
\]
\[+ 40 \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^4 y_i^2 \sum_{i=1}^{D} x_i y_i \right)
\]
\[+ 40 \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^4 y_i^2 \sum_{i=1}^{D} x_i y_i \right)
\]
\[+ 75 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[-30 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[-30 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[-30 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[-30 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[-30 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right)
\]
\[+ 12 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i^4 y_i \right).
\]
D Proof of Lemma 6

\[ \hat{d}_{(4,s)} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_{i,2}^2v_2 - 4u_{i,1}^2v_1 - 4u_{i,3}^2v_3 \right) \]

\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} 6u_{i,2,j}v_{2,j} - 4u_{i,3,j}v_{1,j} - 4u_{i,1,j}v_{3,j} \right) \]

\[ E\left( u_{2,j}^2v_{2,j}^2 \right) = E\left( \left( \sum_{i=1}^{D} x_i^2y_i^2r_{ij} + \sum_{i \neq j'} x_i^2r_{ij}y_{i'}^2r_{ij'} \right)^2 \right) \]

\[ = E\left( \sum_{i=1}^{D} x_i^4y_i^4 + 2 \sum_{i \neq i'} x_i^2y_i^2x_{i'}^2y_{i'}^2 + \sum_{i \neq j'} x_i^4r_{ij}y_{i'}^2r_{ij'} \right) \]

\[ = \sum_{i=1}^{D} \left( x_i^4 + 2 \sum_{i \neq i'} x_i^2y_i^2 + \sum_{i \neq j'} x_i^4r_{ij} \right) + (s-3) \sum_{i=1}^{D} x_i^4y_i^4. \]

Similarly,

\[ E\left( u_{3,j}^2v_{1,j}^2 \right) = \sum_{i=1}^{D} x_i^6 + 2 \left( \sum_{i=1}^{D} x_i^3y_i \right)^2 + (s-3) \sum_{i=1}^{D} x_i^6y_i, \]

\[ E\left( u_{3,j}^2v_{1,j}^2 \right) = \sum_{i=1}^{D} x_i^6 + 2 \left( \sum_{i=1}^{D} x_i^3y_i \right)^2 + (s-3) \sum_{i=1}^{D} x_i^6y_i. \]

\[ E\left( u_{2,j}u_{3,j}v_{2,j}v_{1,j} \right) = \sum_{i=1}^{D} x_i^7 + \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2y_i^2 \sum_{i=1}^{D} x_i^3y_i^3 \]

\[ + \sum_{i=1}^{D} x_i^2y_i^2 \sum_{i=1}^{D} x_i^3y_i^3 + (s-3) \sum_{i=1}^{D} x_i^5y_i^5, \]

\[ E\left( u_{2,j}u_{1,j}v_{2,j}v_{3,j} \right) = \sum_{i=1}^{D} x_i^5 + \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_iy_i^3 \sum_{i=1}^{D} x_iy_i^3 \]

\[ + \sum_{i=1}^{D} x_iy_i^3 \sum_{i=1}^{D} x_iy_i^3 + (s-3) \sum_{i=1}^{D} x_iy_i^5, \]

\[ E\left( u_{3,j}u_{1,j}v_{3,j}v_{3,j} \right) = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_iy_i^3 \sum_{i=1}^{D} x_iy_i^3 \]

\[ + \sum_{i=1}^{D} x_iy_i^3 \sum_{i=1}^{D} x_iy_i^3 + (s-3) \sum_{i=1}^{D} x_iy_i^3. \]
Therefore,

\[
\text{Var}(6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j})
= 36 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + 72 \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 + 36(s-3) \sum_{i=1}^{D} x_i^4 y_i^4
+ 16 \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + 32 \left( \sum_{i=1}^{D} x_i^4 y_i \right)^2 + 36(s-3) \sum_{i=1}^{D} x_i^6 y_i^2
+ 16 \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^4 + 32 \left( \sum_{i=1}^{D} x_i^2 y_i^3 \right)^2 + 36(s-3) \sum_{i=1}^{D} x_i^2 y_i^6
- 48 \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
- 48 \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
+ 32 \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
- \left( 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^2 y_i^3 \right)^2
\]

from which it follows that

\[
\text{Var}(\hat{d}_{4},w) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 + (s-3) \sum_{i=1}^{D} x_i^4 y_i^4 \right)
+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^4 y_i \right)^2 + (s-3) \sum_{i=1}^{D} x_i^6 y_i^2 \right)
+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^3 \right)^2 + (s-3) \sum_{i=1}^{D} x_i^6 y_i^6 \right)
- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
+ \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^6 y_i^3 \right)
\]

References

[1] D. Achlioptas. Database-friendly random projections. In PODS, pages 274–281, Santa Barbara, CA, 2001.

[2] N. Alon, Y. Matias, and M. Szegedy. The space complexity of approximating the frequency moments. In STOC, pages 20–29, Philadelphia, PA, 1996.
[3] B. Babcock, S. Babu, M. Datar, R. Motwani, and J. Widom. Models and issues in data stream systems. In *PODS*, pages 1–16, Madison, WI, 2002.

[4] Z. Bar-Yossef, T. S. Jayram, R. Kumar, and D. Sivakumar. An information statistics approach to data stream and communication complexity. In *FOCS*, pages 209–218, Vancouver, BC, Canada, 2002.

[5] L. Bottou, O. Chapelle, D. DeCoste, and J. Weston, editors. *Large-Scale Kernel Machines*. The MIT Press, Cambridge, MA, 2007.

[6] V. V. Buldygin and Y. V. Kozachenko. *Metric Characterization of Random Variables and Random Processes*. American Mathematical Society, Providence, RI, 2000.

[7] G. Cormode, M. Datar, P. Indyk, and S. Muthukrishnan. Comparing data streams using hamming norms (how to zero in). In *VLDB*, pages 335–345, Hong Kong, China, 2002.

[8] G. Cormode, M. Datar, P. Indyk, and S. Muthukrishnan. Comparing data streams using hamming norms (how to zero in). *IEEE Transactions on Knowledge and Data Engineering*, 15(3):529–540, 2003.

[9] J. Feigenbaum, S. Kannan, M. Strauss, and M. Viswanathan. An approximate $l_1$-difference algorithm for massive data streams. In *FOCS*, pages 501–511, New York, 1999.

[10] M. R. Henzinger, P. Raghavan, and S. Rajagopalan. *Computing on Data Streams*. American Mathematical Society, Boston, MA, USA, 1999.

[11] A. Hyvärinen, J. Karhunen, and E. Oja. *Independent Component Analysis*. John Wiley & Sons, New York, 2001.

[12] P. Indyk. Stable distributions, pseudorandom generators, embeddings and data stream computation. In *FOCS*, pages 189–197, Redondo Beach, CA, 2000.

[13] P. Indyk. Stable distributions, pseudorandom generators, embeddings, and data stream computation. *Journal of ACM*, 53(3):307–323, 2006.

[14] P. Indyk and D. P. Woodruff. Optimal approximations of the frequency moments of data streams. In *STOC*, pages 202–208, Baltimore, MD, 2005.

[15] P. Li. Estimators and tail bounds for dimension reduction in $l_\alpha$ ($0 < \alpha \leq 2$) using stable random projections. In *SODA*, pages 10–19, 2008.

[16] P. Li, T. J. Hastie, and K. W. Church. Improving random projections using marginal information. In *COLT*, pages 635–649, Pittsburgh, PA, 2006.

[17] P. Li, T. J. Hastie, and K. W. Church. Very sparse random projections. In *KDD*, pages 287–296, Philadelphia, PA, 2006.

[18] P. Li, T. J. Hastie, and K. W. Church. Nonlinear estimators and tail bounds for dimensional reduction in $l_1$ using cauchy random projections. *Journal of Machine Learning Research*, 8:2497–2532, 2007.

[19] S. Muthukrishnan. Data streams: Algorithms and applications. *Foundations and Trends in Theoretical Computer Science*, 1:117–236, 2 2005.

[20] M. E. Saks and X. Sun. Space lower bounds for distance approximation in the data stream model. In *STOC*, pages 360–369, Montreal, Quebec, Canada, 2002.

[21] B. Schölkopf and A. J. Smola. *Learning with Kernels*. The MIT Press, Cambridge, MA, 2002.

[22] S. Vempala. *The Random Projection Method*. American Mathematical Society, Providence, RI, 2004.
[23] D. P. Woodruff. Optimal space lower bounds for all frequency moments. In SODA, pages 167–175, New Orleans, LA, 2004.

[24] V. M. Zolotarev. One-dimensional Stable Distributions. American Mathematical Society, Providence, RI, 1986.
On Approximating the $l_p$ Distances for $p > 2$
(When $p$ Is Even)

Ping Li
Department of Statistical Science
Faculty of Computing and Information Science
Cornell University, Ithaca, NY 14850

June 27, 2008

Abstract

Many applications in machine learning and data mining require computing pairwise $l_p$ distances in a data matrix $A \in \mathbb{R}^{n \times D}$. For massive high-dimensional data, computing all pairwise distances of $A$ can be infeasible. In fact, even storing $A$ or all pairwise distances of $A$ in the memory may be also infeasible.

For $0 < p \leq 2$, efficient small space algorithms exist, for example, based on the method of stable random projections, which unfortunately is not directly applicable to $p = 3, 4, 5, 6, ...$. This paper proposes a simple method for $p = 2, 4, 6, ...$ We first decompose the $l_p$ (where $p$ is even) distances into a sum of $2$ marginal norms and $p - 1$ “inner products” at different orders. Then we apply normal or sub-Gaussian random projections to approximate the resultant “inner products,” assuming that the marginal norms can be computed exactly by a linear scan.

We propose two strategies for applying random projections. The basic projection strategy requires only one projection matrix but it is more difficult to analyze, while the alternative projection strategy requires $p - 1$ projection matrices but its theoretical analysis is much easier. In terms of the accuracy, at least for $p = 4$, the basic strategy is always more accurate than the alternative strategy if the data are non-negative, which is common in reality.

1 Introduction

This study proposes a simple method for efficiently computing the $l_p$ distances in a massive data matrix $A \in \mathbb{R}^{n \times D}$ for $p > 2$ (where $p$ is even), using random projections[1].

While many previous work on random projections focused on approximating the $l_2$ distances (and inner products), the method of symmetric stable random projections[2, 3, 4] is applicable to approximating the $l_p$ distances for all $0 < p \leq 2$. This work proposes using random projections for $p > 2$, at least for some special cases.

1First draft Dec. 2007. Slightly revised June 2008.
Machine learning algorithms often operate on the $l_p$ distances of $A$ instead of the original data. A straightforward application would be searching for the nearest neighbors using $l_p$ distance. The $l_p$ distance is also a basic loss function for quality measure. The widely used “kernel trick,” (e.g., for support vector machines (SVM)), is often constructed on top of the $l_p$ distancesootnote{It is well-known that the radial basis kernel using the $l_p$ distance with $0 < p \leq 2$ satisfies the Mercer’s condition. However, we can still use the $l_p$ distance with $p > 2$ as kernels, although in this case it is not guaranteed to find the “most optimal” solution. For very large-scale learning, we usually will not find the “most optimal” solution any way.}.

Here we can treat $p$ as a tuning parameter. It is common to take $p = 2$ (Euclidean distance), or $p = \infty$ (infinity distance), $p = 1$ (Manhattan distance), or $p = 0$ (Hamming distance); but in principle any $p$ values are possible. In fact, if there is an efficient mechanism to compute the $l_p$ distances, then it becomes affordable to tune learning algorithms for many values of $p$ for the best performance.

In modern data mining and learning applications, the ubiquitous phenomenon of “massive data” imposes challenges. For example, pre-computing and storing all pairwise $l_p$ distances in memory at the cost $O(n^2)$ can be infeasible when $n > 10^6$ (or even just $10^5$)ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}. For ultra high-dimensional data, even just storing the whole data matrix can be infeasible. In the meanwhile, modern applications can routinely involve millions of observations; and developing scalable learning and data mining algorithms has been an active research direction. One commonly used strategy in current practice is to compute the distances on the fly\footnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, in stead of storing all pairwise $l_p$ distances.

Data reduction algorithms such as sampling or sketching methods are also popular. While there have been extensive studies on approximating the $l_p$ distances for $0 < p \leq 2$, $p > 2$ can be useful too. For example, because the normal distribution is completely determined by its first two moments (mean and variance), we can identify the non-normal components of the data by analyzing higher moments, in particular, the fourth moments (i.e., kurtosis). Thus, the fourth moments are critical, for example, in the field of Independent Component Analysis (ICA)\footnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}. Therefore, it is viable to use the $l_p$ distance for $p > 2$ when lower order distances can not efficiently differentiate data.

It is unfortunate that the family of stable distributions\footnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.} is limited to $0 < p \leq 2$ and hence we can not directly using stable distributions for approximating the $l_p$ distances. In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}ootnote{In the theoretical CS community, there have been many studies on approximating the $l_p$ norms and distances\footnote{Some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}, some of which also applicable to the $l_p$ distances (e.g., comparing two long vectors). Those papers proved that small space ($\tilde{O}(1)$) algorithms exist only for $0 < p \leq 2$.}.

1.1 The Methodology

Given a giant data matrix $A \in \mathbb{R}^{n \times D}$, we assume that a linear scan of the data is feasible, but computing all pairwise interactions is not, either due to computational budget constraints or memory limits. Also, we only consider even $p = 4, 6, ..., $ among which $p = 4$ is probably the most important.

Interestingly, our method is based only on normal (or normal-like) projections. The observation is that, when $p$ is even, the $l_p$ distance can be decomposed into marginal
\(l_p\) norms and “inner products” of various orders. For example, for two \(D\)-dimensional vectors \(x\) and \(y\), when \(p = 4\), then

\[
d(p) = \sum_{i=1}^{D} |x_i - y_i|^p = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3.
\]

Since we assume that a linear scan of the data is feasible, we can compute \(\sum_{i=1}^{D} x_i^4\) and \(\sum_{i=1}^{D} y_i^4\) exactly. We can approximate the interaction terms \(\sum_{i=1}^{D} x_i^2 y_i^2\), \(\sum_{i=1}^{D} x_i^3 y_i\), and \(\sum_{i=1}^{D} x_i y_i^3\) using normal (or normal-like) random projections. Therefore, for \(p\) being even, we are able to efficiently approximate the \(l_p\) distances.

1.2 Paper Organization

Section 2 concerns using normal random projections for approximating \(l_4\) distances. We introduce two projection strategies and the concept of utilizing the marginal norms to improve the estimates. Section 3 extends this approach to approximating \(l_6\) distances. Section 4 analyzes the effect of replacing normal projections by sub-Gaussian projections.

2 Normal Random Projections for \(p = 4\)

The goal is to efficiently compute all pairwise \(l_p\) (\(p = 4\)) distances in \(A \in \mathbb{R}^{n \times D}\). It suffices to consider any two rows of \(A\), say \(x\) and \(y\), where \(x, y \in \mathbb{R}^D\). We need to estimate the \(l_p\) distance between \(x\) and \(y\)

\[
d(p) = \sum_{i=1}^{D} |x_i - y_i|^p.
\]

which, when \(p = 4\), becomes

\[
d(4) = \sum_{i=1}^{D} |x_i - y_i|^4 = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3.
\]

In one pass, we can compute \(\sum_{i=1}^{D} x_i^4\) and \(\sum_{i=1}^{D} y_i^4\) easily, but computing the interactions is more difficult. We resort to random projections for approximating \(\sum_{i=1}^{D} x_i^2 y_i^2\), \(\sum_{i=1}^{D} x_i y_i\), and \(\sum_{i=1}^{D} x_i y_i^3\). Since there are three “inner products” of different orders, we can choose either only one projection matrix for all three terms (the basic projection strategy), or three independent projection matrices (the alternative projection strategy).
2.1 The Basic Projection Strategy

First, generate a random matrix $\mathbf{R} \in \mathbb{R}^{D \times k} (k \ll D)$, with i.i.d. entries from a standard normal, i.e.,

$$r_{ij} \sim N(0, 1), \ E(r_{ij}) = 0, \ E(r_{ij}^2) = 1, \ E(r_{ij}^4) = 3.$$  \hspace{1cm} \E \left( r_{ij}^s r_{ij}^{s'} \right) = 0, \text{ if } t \text{ or } s \text{ is odd, and } i \neq i' \text{ or } j \neq j'.

Using random projections, we generate six vectors in $k$ dimensions, $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^k$:

$$u_{1,j} = \sum_{i=1}^{D} x_i r_{ij}, \quad u_{2,j} = \sum_{i=1}^{D} x_i^2 r_{ij}, \quad u_{3,j} = \sum_{i=1}^{D} x_i^3 r_{ij},$$

$$v_{1,j} = \sum_{i=1}^{D} y_i r_{ij}, \quad v_{2,j} = \sum_{i=1}^{D} y_i^2 r_{ij}, \quad v_{3,j} = \sum_{i=1}^{D} y_i^3 r_{ij}.$$

We have a simple unbiased estimator of $d(4)$

$$\hat{d}(4) = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_2^T v_2 - 4u_3^T v_1 - 4u_1^T v_3 \right).$$

Lemma 1

$$E \left( \hat{d}(4) \right) = d(4),$$

$$\text{Var} \left( \hat{d}(4) \right) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right)$$

$$+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right)$$

$$+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right) + \Delta_4$$

$$\Delta_4 = -\frac{48}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i \right)$$

$$- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i \right)$$

$$+ \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i \right).$$

It is possible to relax the requirement of i.i.d samples. In fact, to prove unbiasedness of the estimates only needs pairwise independence, and to derive the variance formula requires four-wise independence.
Proof 1  See Appendix A □

The basic projection strategy is simple but its analysis is quite involved, especially when \( p > 4 \). Also, if we are interested in higher order moments (other than variance) of the estimator, the analysis becomes very tedious.

2.2 The Alternative Projection Strategy

Instead of one projection matrix \( R \), we generate three, \( R^{(a)}, R^{(b)}, R^{(c)} \), independently. By random projections, we generate six vectors in \( k \) dimensions, \( u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^k \), such that

\[
\begin{align*}
    u_{1,j} &= \sum_{i=1}^{D} x_i r_{ij}^{(c)}, \quad u_{2,j} = \sum_{i=1}^{D} x_i^2 r_{ij}^{(a)}, \quad u_{3,j} = \sum_{i=1}^{D} x_i^3 r_{ij}^{(b)}, \\
    v_{1,j} &= \sum_{i=1}^{D} y_i r_{ij}^{(b)}, \quad v_{2,j} = \sum_{i=1}^{D} y_i^2 r_{ij}^{(a)}, \quad v_{3,j} = \sum_{i=1}^{D} y_i^3 r_{ij}^{(c)}.
\end{align*}
\]

Here we abuse the notation slightly by using the same \( u \) and \( v \) for both projection strategies.

Again, we have an unbiased estimator, denoted by \( \hat{d}_{(4),a} \)

\[
\hat{d}_{(4),a} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_2^Tv_2 - 4u_4^Tv_1 - 4u_4^Tv_3 \right)
\]

Lemma 2

\[
E \left( \hat{d}_{(4),a} \right) = d_{(4)},
\]

\[
\text{Var} \left( \hat{d}_{(4),a} \right) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right) + \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right).
\]

Proof 2  The proof basically follows from that of Lemma 7
Compared with $\text{Var}\left(\hat{d}_{(4)}\right)$ in Lemma 1, the difference would be 

$$\Delta_4 = \text{Var}\left(\hat{d}_{(4)}\right) - \text{Var}\left(\hat{d}_{(4),a}\right) = \Delta_4$$

which can be either negative or positive. For example, when all $x_i$’s are negative and all $y_i$’s are positive, then $\Delta_4 \geq 0$, i.e., the alternative projection strategy results in smaller variance and hence it should be adopted.

We can show in Lemma 3 that when the data are non-negative (which is more likely the reality), the difference in (1) will never exceed zero, suggesting that the basic strategy would be preferable, which is also operationally simpler (although more sophisticated in the analysis).

**Lemma 3** If all entries of $x$ and $y$ are non-negative, then

$$\text{Var}\left(\hat{d}_{(4)}\right) - \text{Var}\left(\hat{d}_{(4),a}\right) = \Delta_4 \leq 0. \quad (2)$$

**Proof 3** See Appendix B.

Thus, the main advantage of the alternative projection strategy is that it simplifies the analysis, especially true when $p > 4$. Also, analyzing the alternative projection strategy may provide an estimate for the basic projection strategy. For example, the variance of $\hat{d}_{(4),a}$ is an upper bound of the variance of $\hat{d}_{(4)}$ in non-negative data.

In the next subsection, we show that the alternative strategy make the analysis feasible when we take advantage of the marginal information.

### 2.3 Improving the Estimates Using Margins

Since we assume that a linear scan of the data is feasible and in fact the estimators in both strategies already take advantage of the marginal $l_4$ norms, $\sum_{i=1}^{D} x_i^4$ and $\sum_{i=1}^{D} y_i^4$, we might as well compute other marginal norms and try to take advantage of them in a systematic manner.

Lemma 4 demonstrates such a method for improving estimates using margins. For simplicity, we assume in Lemma 4 that we adopt the alternative projection strategy, in order to carry out the (asymptotic) analysis of the variance.
Lemma 4 Suppose we use the alternative projection strategy described in Section 2.2 to generate samples \( u_{1,j}, u_{2,j}, u_{3,j}, v_{1,j}, v_{2,j}, \) and \( v_{3,j} \). We estimate \( d(4) \) by

\[
\hat{d}(4)_{a,mle} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + 6\hat{a}_{2,2} - 4\hat{a}_{3,1} - 4\hat{a}_{1,3},
\]

where \( \hat{a}_{2,2}, \hat{a}_{3,1}, \hat{a}_{1,3} \), are respectively, the solutions to the following three cubic equations:

\[
a_{2,2}^3 - \frac{a_{2,2}^2}{k} u_2 v_2 - \frac{1}{k} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 = 0,
\]

\[
a_{3,1}^3 - \frac{a_{3,1}^2}{k} u_1 v_1 - \frac{1}{k} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 = 0,
\]

\[
a_{1,3}^3 - \frac{a_{1,3}^2}{k} u_3 v_3 - \frac{1}{k} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 = 0.
\]

Asymptotically (as \( k \to \infty \)), the variance would be

\[
\text{Var} \left( \hat{d}(4)_{a,mle} \right) = 36 \text{Var} \left( \hat{a}_{2,2} \right) + 16 \text{Var} \left( \hat{a}_{2,2} \right) + 16 \text{Var} \left( \hat{a}_{2,2} \right)
\]

\[
= \frac{36}{k} \left( \frac{\sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 - \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2}{\sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2} \right)
\]

\[
+ \frac{16}{k} \left( \frac{\sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 - \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2}{\sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2} \right)
\]

\[
+ \frac{16}{k} \left( \frac{\sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 - \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2}{\sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2} \right) + O \left( \frac{1}{k^2} \right)
\]
Proof 4 [?,?] proposed taking advantage of the marginal \( l_2 \) norms to improve the estimates of \( l_2 \) distances and inner products. Because we assume the alternative projection strategy, we can analyze \( \hat{a}_{2,2}, \hat{a}_{3,1}, \) and \( \hat{a}_{1,3} \), independently and then combine the results; and hence we skip the detailed proof.

Of course, in practice, we probably still prefer the basic projection strategy, i.e., only one projection matrix instead of three. In this case, we still solve three cubic equations, but the precise analysis of the variance becomes much more difficult. When the data are non-negative, we believe that \( \text{Var} \left( \hat{d}_{(4),a,mle} \right) \) will also be the upper bound of the estimation variance using the basic projection strategy, which can be easily verified by empirical results (not included in the current report).

Solving cubic equations is easy, as there are closed-form solutions. We can also solve the equations by iterative methods. In fact, it is common practice to do only a one-step iteration (starting with the solution without using margins), called “one-step Newton-Rhapson” in statistics.

3 Normal Random Projections for \( P=6 \)

For higher \( p \) (where \( p \) is even), we can follow basically the same procedure as for \( p = 4 \). To illustrate this, we work out an example for \( p = 6 \). We only demonstrate the basic projection strategy.

The \( l_6 \) distance can be decomposed into 2 marginal norms and 5 inner products at various orders:

\[
d_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 - 20 \sum_{i=1}^{D} x_i^3 y_i^3 \\
+ 15 \sum_{i=1}^{D} x_i^2 y_i^4 + 15 \sum_{i=1}^{D} x_i^4 y_i^2 - 6 \sum_{i=1}^{D} x_i^5 y_i - 6 \sum_{i=1}^{D} x_i y_i^5
\]

Generate one random projection matrix \( R \in \mathbb{R}^{D \times k} \), and

\[
u_{1,j} = \sum_{i=1}^{D} x_i r_{ij}, \quad u_{2,j} = \sum_{i=1}^{D} x_i^2 r_{ij}, \quad u_{3,j} = \sum_{i=1}^{D} x_i^3 r_{ij}, \\
u_{4,j} = \sum_{i=1}^{D} x_i^4 r_{ij}, \quad u_{5,j} = \sum_{i=1}^{D} x_i^5 r_{ij}, \\
u_{1,j} = \sum_{i=1}^{D} y_i r_{ij}, \quad v_{2,j} = \sum_{i=1}^{D} y_i^2 r_{ij}, \quad v_{3,j} = \sum_{i=1}^{D} y_i^3 r_{ij}, \\
u_{4,j} = \sum_{i=1}^{D} y_i^4 r_{ij}, \quad v_{5,j} = \sum_{i=1}^{D} y_i^5 r_{ij},
\]
Lemma 5 provide the variance of the following unbiased estimator of $\hat{d}(6)$:

$$\hat{d}(6) = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \left( -20u_3^Tv_3 + 15u_4^Tv_2 + 15u_2^Tv_4 - 6u_5^Tv_3 - 6u_1^Tv_5 \right)$$

$$= \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \sum_{j=1}^{k} \left( -20u_{3,j}^Tv_3 + 15u_{2,j}^Tv_2 + 15u_{4,j}^Tv_4 - 6u_{5,j}^Tv_3 - 6u_{1,j}^Tv_5 \right).$$

**Lemma 5**

$$\text{Var} \left( \hat{d}(6) \right) = \frac{400}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right) + \frac{225}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right) + \frac{225}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right) + \frac{36}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \right) + \Delta_6$$

where

$$\Delta_6 = - \frac{600}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^2 y_i \right) - \frac{600}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^3 y_i \right)$$

$$+ \frac{240}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i y_i \right) + \frac{240}{k} \left( \sum_{i=1}^{D} x_i^8 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^3 y_i \sum_{i=1}^{D} x_i^5 y_i \right)$$

$$+ \frac{450}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i y_i \right) - \frac{180}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i^3 y_i \right)$$

$$- \frac{180}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i^3 y_i \right) - \frac{180}{k} \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i^3 y_i \right) - \frac{72}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i^5 y_i \sum_{i=1}^{D} x_i^2 y_i \right).$$

**Proof 5** See Appendix C. □.

When all entries of $x$ and $y$ are non-negative, we believe it is true that $\Delta_6 \leq 0$, but we did not proceed with the proof.

Of course, it is again a good idea to take advantage of the marginal norms, but we skip the analysis.

### 4 Sub-Gaussian Random Projections

It is well-known that it is not necessary to sample $r_{ij} \sim N(0, 1)$. In fact, to have an unbiased estimator, it suffices to sample $r_{ij}$ from any distribution with zero mean (and
unit variance). For good higher-order behaviors, it is often a good idea to sample from a sub-Gaussian distribution, of which a zero-mean normal distribution is a special case.

The theory of sub-Gaussian distributions was developed in the 1950’s. See [?] and references therein. A random variable \( x \) is sub-Gaussian if there exists a constant \( g > 0 \) such that for all \( t \in \mathbb{R} \):

\[
E(\exp(xt)) \leq \exp\left(\frac{g^2t^2}{2}\right).
\]

In this section, we sample \( r_{ij} \) from a sub-Gaussian distribution with the following restrictions:

\[
E(r_{ij}) = 0, \quad E(r_{ij}) = 1, \quad E(r_{ij}^4) = s,
\]

and we denote \( r_{ij} \sim \text{SubG}(s) \). It can be shown that we must restrict \( s \geq 1 \).

One example would be the \( r_{ij} \sim \text{Uniform}(-\sqrt{3}, \sqrt{3}) \), for which \( s = \frac{9}{5} \). Although the uniform distribution is simpler than normal, it is now well-known that we should sample from the following three-point sub-Gaussian distributions[?].

\[
r_{ij} = \sqrt{s} \times \begin{cases} 
1 & \text{with prob. } \frac{1}{2s} \\
0 & \text{with prob. } 1 - \frac{1}{s} \\
-1 & \text{with prob. } \frac{1}{2s}
\end{cases}, \quad s \geq 1
\]

In our analysis, we do not have to specify the exact distribution of \( r_{ij} \) and we can simply express the estimation variance as a function of \( s \).

Here, we consider the basic projections strategy, by generating one random projection matrix \( R \in \mathbb{R}^{n \times D} \) with i.i.d. entries \( r_{ij} \sim \text{SubG}(s) \), and

\[
\begin{align*}
u_{1,j} &= \sum_{i=1}^{D} x_i r_{ij}, \\
u_{2,j} &= \sum_{i=1}^{D} x_i^2 r_{ij}, \\
u_{3,j} &= \sum_{i=1}^{D} x_i^3 r_{ij}, \\
v_{1,j} &= \sum_{i=1}^{D} y_i r_{ij}, \\
v_{2,j} &= \sum_{i=1}^{D} y_i^2 r_{ij}, \\
v_{3,j} &= \sum_{i=1}^{D} y_i^3 r_{ij}.
\end{align*}
\]

We again have a simple unbiased estimator of \( d(4) \)

\[
\hat{d}_{(4),s} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} (6u_2^Tv_2 - 4u_3^Tv_1 - 4u_1^Tv_3)
\]
Lemma 6

$$\text{Var} \left( \hat{d}_{(4),s} \right) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 + (s-3) \sum_{i=1}^{D} x_i y_i \right)$$

$$+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 + (s-3) \sum_{i=1}^{D} x_i^2 y_i \right)$$

$$+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i \right)^3 + (s-3) \sum_{i=1}^{D} x_i y_i \right)$$

$$- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^5 y_i \right)$$

$$- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^2 y_i + (s-3) \sum_{i=1}^{D} x_i^3 y_i \right)$$

$$+ \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + (s-3) \sum_{i=1}^{D} x_i^4 y_i \right) .$$

**Proof 6** See Appendix D $\blacksquare$.

5 Conclusions

It has been an active research topic on approximating $l_p$ distances in massive high-dimensional data, for example, a giant “data matrix” $A \in \mathbb{R}^{n \times D}$. While a linear scan on $A$ may be feasible, it can be prohibitive (or even infeasible) to compute and store all pairwise $l_p$ distances. Using random projections can reduce the cost of computing all pairwise distances from $O(n^2 D)$ to $(n^2 k)$ where $k \ll D$. The data size is reduced from $O(n D)$ to $O(n k)$ and hence it may be possible to store the reduced data in memory.

While the well-known method of stable random projections is applicable to $0 < p \leq 2$, not directly to $p > 2$, we propose a practical approach for approximating the $l_p$ distances in massive data for $p = 2, 4, 6, \ldots$, based on the simple fact that, when $p$ is even, the $l_p$ distances can be decomposed into 2 marginal norms and $p - 1$ “inner products” of various orders. Two projection strategies are proposed to approximate these “inner products” as well as the $l_p$ distances; and we show the basic projection strategy (which is simpler) is always preferable over the alternative strategy in terms of the accuracy, at least for $p = 4$ in non-negative data. We also propose utilizing the marginal norms (which can be easily computed exactly) to further improve the estimates. Finally, we analyze the performance using sub-Gaussian random projections.
A Proof of Lemma 1

\[ \hat{d}_{(4)} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \]

\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \]

\[ u_{2,j}v_{2,j} = \left( \sum_{i=1}^{D} x_i^2 r_{ij} \right) \left( \sum_{i=1}^{D} y_i^2 r_{ij} \right) = \sum_{i=1}^{D} x_i^2 y_i^2 r_{ij}^2 + \sum_{i \neq i'} x_i^2 y_{i'}^2 r_{ij} r_{i'j} \]

Thus

\[ E \left( u_{2,j}v_{2,j} \right) = \sum_{i=1}^{D} x_i^2 y_i. \]

Similarly, we can show

\[ E \left( u_{3,j}v_{1,j} \right) = \sum_{i=1}^{D} x_i^3 y_i, \quad E \left( u_{1,j}v_{3,j} \right) = \sum_{i=1}^{D} x_i y_i^3. \]

Therefore,

\[ E \left( \hat{d}_{(4)} \right) = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} E \left( 6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j} \right) \right) \]

\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} \left( 6 \sum_{i=1}^{D} x_i^2 y_i^2 - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i y_i^3 \right) \right) = \hat{d}_{(4)}. \]

To derive the variance, we need to analyze the expectation

\[ (6u_{2,j}v_{2,j} - 4u_{3,j}v_{1,j} - 4u_{1,j}v_{3,j})^2 \]

\[ = 36u_{2,j}^2 v_{2,j}^2 + 16u_{3,j}^2 v_{1,j}^2 + 16u_{1,j}^2 v_{3,j}^2 - 48u_{2,j}u_{3,j}v_{1,j}v_{3,j} \]

\[ - 48u_{2,j}u_{1,j}v_{2,j}v_{3,j} + 32u_{3,j}u_{1,j}v_{1,j}v_{3,j}. \]

To simplify the expression, we will skip the terms that will be zeros when taking expectations.

\[ E \left( u_{2,j}^2 v_{2,j}^2 \right) = E \left( \left( \sum_{i=1}^{D} x_i^2 y_i^2 r_{ij}^2 + \sum_{i \neq i'} x_i^2 y_{i'}^2 r_{ij} r_{i'j} \right)^2 \right) \]

\[ = E \left( \sum_{i=1}^{D} x_i^4 y_i^4 r_{ij}^4 + 2 \sum_{i \neq i'} x_i^2 y_i^2 r_{ij}^2 x_i^2 y_{i'}^2 r_{i'j}^2 + \sum_{i \neq i'} x_i^2 y_{i'}^2 r_{ij} r_{i'j} \right) \]

\[ = \sum_{i=1}^{D} 3x_i^4 y_i^4 + 2 \sum_{i \neq i'} x_i^2 y_i^2 y_{i'}^2 + \sum_{i \neq i'} x_i^4 y_{i'}^2 \]

\[ = \sum_{i=1}^{D} 3x_i^4 y_i^4 + 2 \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2. \]
Similarly

\[ E(u_{3,j}^2 v_{1,j}^2) = \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + 2 \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2, \]

\[ E(u_{3,j}^2 v_{1,j}^3) = \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2. \]
Therefore,

\[
\text{Var} (6u_{2,j} v_{2,j} - 4u_{3,j} v_{1,j} - 4u_{1,j} v_{3,j}) \\
= 36 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + 72 \left( \sum_{i=1}^{D} x_i^3 y_i^2 \right)^2 \\
+ 16 \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + 32 \left( \sum_{i=1}^{D} x_i y_i \right)^2 \\
+ 16 \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + 32 \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \\
- 48 \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i^3 y_i \right) \\
- 48 \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i y_i \right) \\
+ 32 \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i^2 y_i^6 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i^3 y_i \right)
\]

from which it follows that

\[
\text{Var} \ (\hat{d}_{(4)}) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \right) \\
+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i y_i \right)^2 \right) \\
+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 \right) \\
- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i^3 y_i \right) \\
- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^3 \sum_{i=1}^{D} x_i y_i \right) \\
+ \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \sum_{i=1}^{D} x_i^2 y_i^6 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i^3 y_i \right)
\]
B  Proof of Lemma 3

It suffices to show that

\[
\begin{align*}
&\left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^2 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^2 \right) \\
&+ \left( \sum_{i=1}^{D} x_i^{3/2} y_i^{1/2} \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^{3/2} y_i^{1/2} \right) \\
&- \left( \sum_{i=1}^{D} x_i^{1/2} y_i^{1/2} \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^{1/2} y_i^{1/2} \right) \geq 0.
\end{align*}
\]

We need to use the arithmetic-geometric mean inequality:

\[
\sum_{i=1}^{n} w_i \geq n \left( \prod_{i=1}^{n} w_i \right)^{1/n}, \quad \text{provided } w_i \geq 0.
\]

Because

\[
x_i^5 y_j^3 + x_i^3 y_j^5 \geq 2\sqrt{x_i^5 y_j^3} = 2x_i y_j,
\]

\[
\sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^2 \geq 0.
\]

Thus it only remains to show that

\[
\sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i^2 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^2 y_i^3 - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i^3 \geq 0,
\]

for which it suffices to show that

\[
2 \sum_{i=1}^{D} x_i^{3/2} y_i^{1/2} \sum_{i=1}^{D} x_i^{5/2} y_i^{1/2} - \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^{3/2} y_i^{1/2} \geq 0,
\]

or equivalently, to show that, if \( z_i \geq 0 \forall i \in [1, D] \), then

\[
f(z_i, i = 1, 2, ..., D) = 2 \sum_{i=1}^{D} z_i^3 \sum_{i=1}^{D} z_i^5 - \sum_{i=1}^{D} z_i^2 \sum_{i=1}^{D} z_i^6 \geq 0. \quad (3)
\]

Obviously, (3) holds for \( D = 1 \) and \( D = 2 \). To see that it is true for \( D > 2 \), we notice that only at \((z_1 = 0, z_2 = 0, ..., z_D = 0)\), the first derivative of \( f(z_i) \) is zero. We can also check that \( f(z_i = 1, i = 1, 2, ..., D) > 0 \). Since \( f(z_i) \) is a continuous function, we know \( f(z_i) \geq 0 \) must hold if \( z_i > 0 \) for all \( i \). There is no need to worry about the boundary case that \( z_j = 0 \) and \( z_i \geq 0 \) because it is reduced to a small problem with \( D' = D - 1 \) and we have already shown the base case when \( D = 1 \) and \( D = 2 \). Thus, we complete the proof.
Proof of Lemma 5

\[ d_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 - 20 \sum_{i=1}^{D} x_i^3 y_i^3 \]
\[ + 15 \sum_{i=1}^{D} x_i^2 y_i^4 + 15 \sum_{i=1}^{D} x_i y_i^2 - 6 \sum_{i=1}^{D} x_i^2 y_i - 6 \sum_{i=1}^{D} x_i y_i^2 \]

\[ \tilde{d}_{(6)} = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \left( -20u_3^Tv_3 + 15u_4^Tv_2 + 15u_2^Tv_4 - 6u_5^Tv_3 - 6u_1^Tv_5 \right) \]
\[ = \sum_{i=1}^{D} x_i^6 + \sum_{i=1}^{D} y_i^6 + \frac{1}{k} \left( -20u_{3,j}v_{3,j} + 15u_{4,j}v_{2,j} + 15u_{2,j}v_{4,j} - 6u_{5,j}v_{5,j} - 6u_{1,j}v_{1,j} \right) \]

To derive the variance, we need to analyze the expectation of

\[ (-20u_{3,j}v_{3,j} + 15u_{2,j}v_{4,j} + 15u_{4,j}v_{2,j} - 6u_{1,j}v_{5,j} - 6u_{5,j}v_{1,j})^2 \]
\[ = 400u_{3,j}^2v_{3,j}^2 + 225u_{2,j}^2v_{4,j}^2 + 225u_{4,j}^2v_{2,j}^2 + 36u_{1,j}^2v_{5,j}^2 + 36u_{5,j}^2v_{1,j}^2 \]
\[ - 600u_{3,j}u_{2,j}v_{4,j}v_{3,j} - 600u_{3,j}u_{5,j}v_{3,j}v_{5,j} + 240u_{3,j}u_{4,j}v_{2,j}v_{3,j} + 240u_{3,j}u_{1,j}v_{5,j}v_{3,j} \]
\[ - 180u_{2,j}u_{4,j}v_{5,j}v_{3,j} - 180u_{4,j}u_{2,j}v_{5,j}v_{3,j} + 180u_{2,j}u_{4,j}v_{5,j}v_{1,j} \]

Skipping the detail, we can show that

\[ \mathbb{E}(u_{3,j}^2v_{3,j}^2) = \frac{1}{D} \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + 2 \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 \]
\[ \mathbb{E}(u_{2,j}^2v_{4,j}^2) = \frac{1}{D} \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^8 + 2 \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 \]
\[ \mathbb{E}(u_{4,j}^2v_{2,j}^2) = \frac{1}{D} \sum_{i=1}^{D} x_i^8 \sum_{i=1}^{D} y_i^4 + 2 \left( \sum_{i=1}^{D} x_i y_i \right)^2 \]
\[ \mathbb{E}(u_{1,j}^2v_{5,j}^2) = \frac{1}{D} \sum_{i=1}^{D} x_i^{10} \sum_{i=1}^{D} y_i^2 + 2 \left( \sum_{i=1}^{D} x_i y_i \right)^2 \]
\[ \mathbb{E}(u_{5,j}^2v_{1,j}^2) = \frac{1}{D} \sum_{i=1}^{D} x_i^{10} \sum_{i=1}^{D} y_i^2 + 2 \left( \sum_{i=1}^{D} x_i y_i \right)^2 \]
And

\[
E(u_{3,j}u_{2,j}v_{3,j}v_{4,j}) = \sum_{i=1}^{D} x_{i}^{5} \sum_{i=1}^{D} y_{i}^{7} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{2} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{2} y_{i}^{3},
\]

\[
E(u_{3,j}u_{4,j}v_{3,j}v_{2,j}) = \sum_{i=1}^{D} x_{i}^{7} \sum_{i=1}^{D} y_{i}^{9} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{2} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{3} y_{i}^{3},
\]

\[
E(u_{3,j}u_{1,j}v_{3,j}v_{2,j}) = \sum_{i=1}^{D} x_{i}^{8} \sum_{i=1}^{D} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{1} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{3} y_{i}^{4} \sum_{i=1}^{D} x_{i}^{3} y_{i}^{3},
\]

\[
E(u_{4,j}u_{3,j}v_{2,j}v_{1,j}) = \sum_{i=1}^{D} x_{i}^{9} \sum_{i=1}^{D} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{1} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2},
\]

\[
E(u_{4,j}u_{5,j}v_{2,j}v_{1,j}) = \sum_{i=1}^{D} x_{i}^{9} \sum_{i=1}^{D} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{1} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2},
\]

\[
E(u_{1,j}u_{5,j}v_{2,j}v_{1,j}) = \sum_{i=1}^{D} x_{i}^{9} \sum_{i=1}^{D} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{1} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2},
\]

\[
E(u_{1,j}u_{3,j}v_{2,j}v_{1,j}) = \sum_{i=1}^{D} x_{i}^{9} \sum_{i=1}^{D} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{1} y_{i}^{4} + \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2} \sum_{i=1}^{D} x_{i}^{4} y_{i}^{2},
\]

17
Combining the results, we obtain

\[
\text{Var} \left( \hat{d}_{(6)} \right) = \frac{400}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^3 y_i^3 \right)^2 \right) \\
+ \frac{225}{k} \left( \sum_{i=1}^{D} x_i^8 \sum_{i=1}^{D} y_i^8 + \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right) \\
+ \frac{225}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^4 y_i^2 \right)^2 \right) \\
+ \frac{36}{k} \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^{10} + \left( \sum_{i=1}^{D} x_i y_i^5 \right)^2 \right) \\
+ \frac{36}{k} \left( \sum_{i=1}^{D} x_i^{10} \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^5 y_i \right)^2 \right) + \Delta_6
\]

where

\[
k \Delta_6 / 6 = -100 \left( \sum_{i=1}^{D} x_i^5 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2 y_i^5 \sum_{i=1}^{D} x_i^7 y_i^2 \right) \\
- 100 \left( \sum_{i=1}^{D} x_i^7 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^4 y_i^3 \right) \\
+ 40 \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^8 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i y_i^3 \right) \\
+ 40 \left( \sum_{i=1}^{D} x_i^2 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^3 y_i^2 \sum_{i=1}^{D} x_i^4 y_i^2 \right) \\
+ 75 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^6 + \sum_{i=1}^{D} x_i^2 y_i^2 \sum_{i=1}^{D} x_i^4 y_i^4 \right) \\
- 30 \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i^5 y_i^3 \right) \\
- 30 \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i^5 y_i^2 \right) \\
- 30 \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i^5 y_i \right) \\
- 30 \left( \sum_{i=1}^{D} x_i^3 \sum_{i=1}^{D} y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^5 \right) \\
+ 12 \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^5 \right)
\]
D Proof of Lemma 6

\[ \hat{d}_{(4),s} = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( 6u_{2,j}^2 v_2 - 4u_1^2 v_1 - 4u_1^2 v_3 \right) \]
\[ = \sum_{i=1}^{D} x_i^4 + \sum_{i=1}^{D} y_i^4 + \frac{1}{k} \left( \sum_{j=1}^{k} 6u_{2,j}^2 v_2 - 4u_{3,j} v_1 - 4u_{1,j} v_3 \right) \]

\[ \mathbb{E} \left( u_{2,j}^2 v_{2,j}^2 \right) = \mathbb{E} \left( \left( \sum_{i=1}^{D} x_i^2 y_i^2 r_{ij} + \sum_{i \neq j'} x_i^2 r_{ij} y_{j'}^2 r_{ij'} \right)^2 \right) \]
\[ = \mathbb{E} \left( \sum_{i=1}^{D} x_i^4 y_i^4 + 2 \sum_{i \neq j'} x_i^2 y_i^2 x_{j'}^2 y_{j'}^2 + \sum_{i \neq j'} x_i^2 r_{ij} y_{j'}^2 r_{ij'} \right) \]
\[ = \sum_{i=1}^{D} s x_i^4 y_i^4 + 2 \sum_{i \neq j'} x_i^2 y_i^2 x_{j'}^2 y_{j'}^2 + \sum_{i \neq j'} x_i^4 y_i^4 \]
\[ = \sum_{i=1}^{D} x_i^4 \left( \sum_{i=1}^{D} y_i^4 + 2 \left( \sum_{i=1}^{D} x_i^2 y_i^2 \right)^2 \right) + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^4. \]

Similarly,

\[ \mathbb{E} \left( u_{3,j}^2 v_{1,j}^2 \right) = \mathbb{E} \left( \sum_{i=1}^{D} x_i^6 y_i^2 + 2 \left( \sum_{i=1}^{D} x_i^3 y_i \right)^2 + (s - 3) \sum_{i=1}^{D} x_i^6 y_i^2 \right) \]
\[ \mathbb{E} \left( u_{3,j}^2 v_{1,j}^2 \right) = \mathbb{E} \left( \sum_{i=1}^{D} x_i^2 y_i^6 + 2 \left( \sum_{i=1}^{D} x_i y_i^3 \right)^2 + (s - 3) \sum_{i=1}^{D} x_i^2 y_i^6 \right) \]

\[ \mathbb{E} \left( u_{2,j} u_{3,j} v_{2,j} v_{1,j} \right) = \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i^3 \sum_{i=1}^{D} x_i y_i \]
\[ + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^2 y_i + (s - 3) \sum_{i=1}^{D} x_i^5 y_i^3 \]
\[ \mathbb{E} \left( u_{2,j} u_{1,j} v_{2,j} v_{3,j} \right) = \sum_{i=1}^{D} x_i^2 y_i^5 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i y_i^3 \sum_{i=1}^{D} x_i^2 y_i^2 \]
\[ + \sum_{i=1}^{D} x_i y_i^2 \sum_{i=1}^{D} x_i^2 y_i^3 + (s - 3) \sum_{i=1}^{D} x_i^3 y_i^5 \]
\[ \mathbb{E} \left( u_{3,j} u_{1,j} v_{1,j} v_{3,j} \right) = \sum_{i=1}^{D} x_i^2 y_i^7 \sum_{i=1}^{D} y_i^7 + \sum_{i=1}^{D} x_i y_i^5 \sum_{i=1}^{D} x_i^2 y_i \]
\[ + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^2 y_i^3 + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^4. \]
Therefore,

\[
\text{Var}(6u_{2,j}v_{2,j} - 4u_{3,j}v_{3,j}) = 36 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + 72 \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 + 36(s - 3) \sum_{i=1}^{D} x_i^4 y_i^4
\]

\[+ 16 \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + 32 \left( \sum_{i=1}^{D} x_i^4 y_i \right)^2 + 36(s - 3) \sum_{i=1}^{D} x_i^6 y_i^2
\]

\[+ 16 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^6 + 32 \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 + 36(s - 3) \sum_{i=1}^{D} x_i^4 y_i^6
\]

\[- 48 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^5 y_i^3
\]

\[- 48 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^5 y_i^5
\]

\[+ 32 \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^3
\]

\[- \left( 6 \sum_{i=1}^{D} x_i^2 y_i - 4 \sum_{i=1}^{D} x_i^3 y_i - 4 \sum_{i=1}^{D} x_i^4 y_i \right)^2
\]

from which it follows that

\[
\text{Var}(d_{4,s}) = \frac{36}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^4 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^4 \right)
\]

\[+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^6 \sum_{i=1}^{D} y_i^2 + \left( \sum_{i=1}^{D} x_i^4 y_i \right)^2 + (s - 3) \sum_{i=1}^{D} x_i^6 y_i^2 \right)
\]

\[+ \frac{16}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^6 + \left( \sum_{i=1}^{D} x_i^2 y_i \right)^2 + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^6 \right)
\]

\[- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^5 y_i^3 \right)
\]

\[- \frac{48}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^5 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^5 y_i^5 \right)
\]

\[+ \frac{32}{k} \left( \sum_{i=1}^{D} x_i^4 \sum_{i=1}^{D} y_i^3 + \sum_{i=1}^{D} x_i y_i \sum_{i=1}^{D} x_i^3 y_i + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^2 y_i^2 + \sum_{i=1}^{D} x_i^2 y_i \sum_{i=1}^{D} x_i^3 y_i + (s - 3) \sum_{i=1}^{D} x_i^4 y_i^3 \right)
\]