New physics in the charged relativistic Bose gas using zeta-function regularization?\textsuperscript{1}

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Abstract

The multiplicative anomaly, recently introduced in QFT, plays a fundamental role in solving some mathematical inconsistencies of the widely used zeta-function regularization method. Its physical relevance is still an open question and is here analyzed in the light of a non-perturbative method. Even in this approach the “different physics” seems to hold and not to be easily removable by renormalization.

1 Introduction

The evaluation of functional determinants of pseudo-differential operators is often a fundamental point in quantum field theory calculations. As these operators are unbounded the corresponding determinant is undefined, unless a rigorous regularization procedure is applied. One of the most successful is the zeta-function regularization method \cite{1}-\cite{5}. This uses the so called “generalized Riemann zeta function” \(\zeta(s|A) = \text{Tr} A^{-s}\), (\(A\) a pseudo-differential operator) which is well defined for a sufficiently large real part of \(s\) and can be analytically continued to a function meromorphic in all the plane and analytic at \(s = 0\). As such its derivative with respect to \(s\) at zero is well defined and the logarithm of the zeta-function regularized functional determinant will then be defined by

\[
\ln \det \frac{A}{\sigma^2} = -\zeta'(0|A) - \zeta(0|A) \ln \sigma^2,
\]

where \(\sigma^2\) is a renormalization scale mass.

The recent introduction \cite{6}-\cite{9} of the long-overlooked multiplicative anomaly \cite{10}-\cite{12} in this framework has provoked a lively debate \cite{13}-\cite{20}. Its mathematical rigor and importance for quantum field theory has been widely proved by this and other authors \cite{21} but the question of its physical relevance is still open.

The multiplicative anomaly can appear when we evaluate a quantity of the form \(\ln \det(AB)\), with \(A\) and \(B\) two commuting pseudo-differential operators. The fact is that it is not always true that the equality \(\ln \det(AB) = \ln \det(A) + \ln \det(B)\) holds, as normally assumed in quantum field theory until recently. On the contrary, using

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zeta-function regularization, an additional term \( a(A, B) \), called the ‘multiplicative anomaly, may be present on the right hand-side. The anomaly can rarely be computed directly as the difference of the two sides. Fortunately Wodzicki, produced a very useful indirect formula for its evaluation (see [10]).

The fact that the anomaly can add an “anomalous” term to the “standard” result, the controversial physical relevance of which is still under scrutiny, will be the subject of this work.

For some of the very simple models studied up to now the multiplicative anomaly is not always present, while in others its “anomalous term” can be easily renormalized away at one-loop by a finite shift in the counterterms [6]. Renormalizability is therefore a fundamental issue to analyze to understand the physical content of the multiplicative anomaly.

To this end I will need to go beyond the one-loop approximation used until now. Since we finally want to analyze the symmetry breaking transition, the appropriate candidate seems a self-consistent approach similar to the large \( N \) expansion [22].

The relativistic charged Bose gas at finite temperature has long been of interest on its own [23]-[32]. The large \( N \) limit of the \( 0(N) \) interacting field can be found in Haber and Weldon. None of these remarkable papers adopts a fully regularized approach as there are formal manipulations of infinite sums involved. We showed how the anomaly is crucial for the consistency of the zeta-function method and how it creates an unexpected “anomalous term”. I refer the reader to the relevant papers [7]-[9] for details and results.

## 2 The non-perturbative approach

The inclusion of the chemical potential \( \mu \) in a Hamiltonian representation of the grand-canonical partition function is usually (see [23, 27, 29, 32]) realized by defining an effective Hamiltonian \( H = H_o + \mu Q \) where \( Q \) is the charge density operator. Through integration on the momenta the functional integral can be cast in Lagrangian form. Although I am only interested in \( N = 2 \) I will leave the \( N \) in the interacting term for clarity, its sum over repeated indices \( a = 1, 2 \) understood. The starting action is therefore

\[
S[\phi] = \int_0^{\beta} d\tau \int d^3x \left( \partial_\tau \phi_a \partial_\tau \phi_a + m_o^2 \phi_a^2 + \frac{\lambda_o}{4N} \phi^4 - \mu^2 \phi^2 + 2i\mu (\phi_2 \partial_\tau \phi_1 - \phi_1 \partial_\tau \phi_2) \right)
\]  

I can then insert in the functional integral a Gaussian representation of unity, through an auxiliary field \( B(x) \) [24]. Considering constant sources \( J_a \) for the \( \phi_a \) fields, the partition function then becomes

\[
Z[J] = \int \mathcal{D}B \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-\frac{1}{2} \int dx \left[ A_{ab}(x) \phi_a(x) \phi_b(x) - \frac{N}{2} B^2(x) \right]}
\]

where I defined the matrix valued differential operator

\[
A(x) = \begin{pmatrix}
-\partial_\tau^2 - \nabla^2 + m_o^2 + B(x) - \mu^2 & -2i\mu \partial_\tau \\
2i\mu \partial_\tau & -\partial_\tau^2 - \nabla^2 + m_o^2 + B(x) - \mu^2
\end{pmatrix}
\]
The functional integral in $\phi_a$ can be recast in exponential form to contribute in an effective action for the $B(x)$ field. It is also clear that this effective action is of order $N$. We can therefore apply a saddle point approximation in the field $B(x)$, such that

$$Z[J] \simeq \int D\phi D\phi_2 e^{-\frac{1}{2}S[\phi,B] + \frac{1}{\beta} J_a \int dx \phi_a(x)}$$  \hspace{1cm} (5)

where constant $\bar{B}$ is the large-N saddle-point.

At this point it proves useful to turn to the momentum representation,

$$Z[J] = e^{\frac{\beta V N}{2\lambda_o} \bar{B}^2 + \frac{\lambda_o}{2\beta^2} J_a A^{-1}_{ab} J_b e^{-\frac{1}{2} \log \det(A\sigma^2)}}$$  \hspace{1cm} (6)

where I denote the eigenvalues matrix as $A(n,k)$ and $A(0,0)$ as $A_0$. Then

$$W = -\frac{1}{\beta V} \log Z = -\frac{N}{2\lambda_o} \bar{B}^2 - \frac{1}{2\beta^2} \bar{J}_a A^{-1}_{ab} \bar{J}_b + \frac{1}{2\beta V} \log \det(A\sigma^2)$$  \hspace{1cm} (7)

where $\bar{B}$ is

$$\frac{N}{\lambda_o} \bar{B} + \frac{1}{\beta V} \frac{\partial}{\partial \bar{B}} \left( \frac{1}{2} \log \det(A\sigma^2) \right) - \frac{1}{2\beta^2} \left( J_1 (A^{-1}_{011}) J_1 + J_2 (A^{-1}_{022}) J_2 \right) = 0$$  \hspace{1cm} (8)

Going back to (4) we can see that $\bar{B}$ gives a contribution to the mass of the field $\phi$, $M^2 = m_o^2 + \bar{B}$. With this substitution the operator $A$ has the same form as the one for the non-interacting field. Calculations are then highly simplified and we can resort to previous results in [7]-[9], for which zeta-function regularization was used and in which the multiplicative anomaly contribution is properly accounted for:

$$-\frac{1}{2} \log \det(A\sigma^2) = -\frac{\beta V}{32\pi^2} M^4 \left( \log(M^2\sigma^2) - \frac{3}{2} \right) + S - \beta V \frac{16\pi^2}{3} \bar{\phi}_a \bar{\phi}_a$$  \hspace{1cm} (9)

where $S$ is the standard particles-antiparticles thermal term, and the underlined part is the one resulting from the multiplicative anomaly.

The constant semi-classical fields are

$$\bar{\phi}_1 = \frac{\beta}{\beta V} \frac{\partial W}{\partial J_1} = -\frac{N}{2\lambda_o} \bar{B} - \frac{1}{2\beta^2} J_a A^{-1}_{011} J_b + \frac{1}{2\beta V} \log \det(A\sigma^2)$$ \hspace{1cm} (10)

so that, using (9), we can express $M^2$ in terms of $\bar{\phi}^2$ as

$$M^2 = m_o^2 + \frac{1}{16\pi^2} \frac{\lambda_o}{N} M^2 \log(M^2\sigma^2) - \frac{\lambda_o}{N\beta V} \frac{\partial S}{\partial M^2} + \frac{1}{2N} \bar{\phi}_a \bar{\phi}_a + \frac{1}{16\pi^2 \lambda_o} \left( M^2 - \frac{\mu^2}{3} \right)$$  \hspace{1cm} (11)

and

$$\Gamma[\phi] = M^2 \left( \frac{N}{\lambda_o} m_o^2 + \frac{1}{2} \bar{\phi}^2 \right) + M^4 \left[ \frac{1}{32\pi^2} \log(M^2\sigma^2) - \frac{1}{64\pi^2} - \frac{N}{2\lambda_o} \right]$$  \hspace{1cm} (12)
where we have redefined the mass scale $\sigma^2 \to \sigma^2e$ so as to bring our notation into correspondence with that of Haber and Weldon. It is now time to put $N = 2$ for good. Performing standard renormalization techniques it seems that the anomaly cannot be renormalised away. This aspect needs to be further analyzed, and details will be given elsewhere.

Explicit calculation shows that the unbroken phase charge density is

$$\rho = -\frac{\mu M^2}{8\pi^2} + \frac{\mu^3}{12\pi^2} + \frac{1}{\beta V} \left. \frac{\partial S}{\partial \mu} \right|_{M^2 = \mu^2}$$

(13)

For the broken phase where, after the phase transition, $\mu = M$, we can find the expression for $\phi^2$ and the charge density takes the form

$$\rho = \mu^4 \frac{4}{\lambda_o} \left[ \mu^2 - m_o^2 \frac{\lambda_o}{32\pi^2} \mu^2 \log(\mu^2 \sigma^2) + \frac{\lambda_o}{2\beta V} \left. \frac{\partial S}{\partial M^2} \right|_{M^2 = \mu^2} - \frac{\lambda_o}{32\pi^2} \right]$$

(14)

It seems like the anomaly could change the “standard” transition temperature and, for fixed charge $\rho$, we can see a difference in physical observables (e.g. pressure) in the two phases.

3 Comments

The relevance of zeta-function regularization in QFT can not be easily dismissed. It is as reliable regularization technique as others. The multiplicative anomaly is indisputably necessary for mathematical consistency. These results seem to show that the “anomalous term” it creates in certain conditions is not trivially removable and could play a role in the physical measurables of the system. To my knowledge this is the first extension of the zeta-function regularization method in a non-perturbative calculation, within a neat procedure that could be useful on other models as well. It is clearly a non-trivial problem, and also very interesting as it goes to the roots of one of the most used regularization methods and of the functional integral approach itself.

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