THIRD ORDER OPERATORS WITH THREE-POINT CONDITIONS ASSOCIATED WITH BOUSSINESQ’S EQUATION

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Abstract. We consider a non-self-adjoint third order operator on the interval [0, 2] with real 1-periodic coefficients and three-point Dirichlet conditions at the points 0, 1 and 2. The eigenvalues of this operator consist an auxiliary spectrum for the inverse spectral problem associated with the good Boussinesq equation. We determine eigenvalue asymptotics at high energy and the trace formula for the operator.

1. Introduction and main results

We consider a non-self-adjoint operator $H$ acting on $L^2(0, 2)$ and given by

$$Hy = (y'' + py')' + py' + qy, \quad y(0) = y(1) = y(2) = 0,$$

(1.1)

where $p, q$ are real 1-periodic coefficients $p, q \in L^1(T), T = \mathbb{R}/\mathbb{Z}$. The operator is defined on the domain

$$\text{Dom}(H) = \{ y \in L^2(0, 2) : (y'' + py')' + py' + qy \in L^2(0, 2), y''(0), (y'' + py')' \in L^1(0, 2), y(0) = y(1) = y(2) = 0 \}.$$  

(1.2)

The multi-point problems for linear ordinary differential operators are well known, see, e.g., the papers [EHH92], [L68], [Po08] and references therein. Papanicolaou [Pa03], [Pa05] considered the non-linear equation associated with the linear fourth order Euler-Bernoulli operators on the circle and studied the multi-point problem for this operator. Trace formulas for multipoint problems for two-term 2n-order differential operators were obtained by Belabbasi [Be83].

The operator $H$ is used in the integration of the so-called good Boussinesq equation on the circle,

$$p_t = -\frac{1}{3}(p_{xxx} + 4(p^2)_{xx}), \quad p_t = q_x,$$

see [McK81] and references therein. It is equivalent to the Lax equation $L_t = LA - AL$, where the operators $L, A$ act on $L^2(T)$ and have the form $L = \partial^3 + p\partial + \partial p + q, A = -\partial^2 - \frac{4}{3}p$. Kalantarov and Ladyzhenskaja [KL77] showed that the good Boussinesq equation has blow-up solutions. The spectrum of the operator $H$ is an auxiliary spectrum for the Boussinesq equation similar to the Dirichlet spectrum on the unit interval for the Korteweg-de Vries equation on the circle. McKean [McK81] considered the operator $H$ with the coefficients $p, q \in C^\infty(T)$. The auxiliary spectrum in the finite-gap case was discussed by Dickson, Gesztesy and Unterkofler [DGU99], [DGU99x]. In addition, there are given some interesting exactly
calculating examples. Self-adjoint third order operators $i\partial^3 + ip\partial + i\partial p + q$ associated with the bad Boussinesq equations on the circle was studied by Badanin and Korotyaev [BK12, BK14]. The inverse scattering theory for the self-adjoint third order operator with decreasing coefficients was developed in [DT82]. Korotyaev [K16] considered resonances for third-order operator.

Consider the differential equation

$$y''' + (py)' + qy = \lambda y, \quad \lambda \in \mathbb{C}. \quad (1.3)$$

Rewrite this equation in the vector form

$$y' = \mathcal{P}y, \quad \text{where} \quad y = \begin{pmatrix} y \\ y' \\ y'' + py \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 & 1 & 0 \\ -p & 0 & 1 \\ \lambda - q & -p & 0 \end{pmatrix}. \quad (1.4)$$

The matrix-valued solution $M(x, \lambda), (x, \lambda) \in \mathbb{C} \times \mathbb{R}$, of the initial problem

$$M' = \mathcal{P}M, \quad M(0, \lambda) = \mathbb{I}_3, \quad (1.5)$$

is called the fundamental matrix, here and below $\mathbb{I}_3$ is the $3 \times 3$ identity matrix. It has the form

$$M = \begin{pmatrix} \varphi_1 & \varphi_2 & \varphi_3 \\ \varphi'_1 & \varphi'_2 & \varphi'_3 \\ \varphi''_1 + p\varphi_1 & \varphi''_2 + p\varphi_2 & \varphi''_3 + p\varphi_3 \end{pmatrix}, \quad (1.6)$$

where $\varphi_1, \varphi_2, \varphi_3$ are the fundamental solutions of equation $(1.3)$ satisfying the initial conditions $(1.4)$. Each matrix-valued function $M(x, \cdot), x \in \mathbb{R}$, is entire, real for $\lambda \in \mathbb{R}$ and satisfies the Liouville identity $\det M(x, \lambda) = 1$ for all $\lambda \in \mathbb{C}$.

It is well known that the spectrum $\sigma(H)$ of $H$ is pure discrete and satisfies

$$\sigma(H) = \{ \lambda \in \mathbb{C} : D(\lambda) = 0 \}. \quad (1.7)$$

Here $D$ is the entire function given by

$$D(\lambda) = \det \begin{pmatrix} \varphi_2(1, \lambda) & \varphi_3(1, \lambda) \\ \varphi_2(2, \lambda) & \varphi_3(2, \lambda) \end{pmatrix}, \quad (1.8)$$

see, e.g., [W18], where Green’s function is studied. The spectrum consists of eigenvalues $\mu_n, n \in \mathbb{Z} \setminus \{0\}$, labeled by

$$... \leq \text{Re} \mu_{-2} \leq \text{Re} \mu_{-1} \leq \text{Re} \mu_1 \leq \text{Re} \mu_2 \leq ...$$

counting with algebraic multiplicities. Note that some eigenvalues may be non-real, see Fig. 1 and we have no information on how large the algebraic multiplicity of the eigenvalue can be. In the unperturbed case $p = q = 0$ all eigenvalues $\mu_n$ are simple, real and have the form

$$\mu_n = (\nu n)^3, \quad n \in \mathbb{Z} \setminus \{0\}, \quad \nu = \frac{2\pi}{\sqrt{3}}. \quad (1.9)$$

For the function $f \in L^1(\mathbb{T})$ we introduce the Fourier coefficients

$$\widetilde{f}_n = \frac{2}{\sqrt{3}} \int_{0}^{1} f(x) \cos \left(2\pi nx + \frac{\pi}{6} \right) dx, \quad n \in \mathbb{N}. \quad (1.10)$$

We formulate our first main results about asymptotics of the eigenvalues.
Figure 1. The spectrum of the operator \( H \). The spectrum out of the large disc is real.

**Theorem 1.1.** Let \( p, q \in L^1(\mathbb{T}) \). Then each eigenvalue \( \mu_n \) with \( |n| \) large enough is real and has algebraic multiplicity one. Moreover,

\[
\mu_n = \mu_n^0 - 2\nu np_0 + \nu n\tilde{p}_n + O(n^{\frac{1}{2}}),
\]

as \( n \to \pm \infty \). If, in addition, \( p' \in L^1(\mathbb{T}) \), then the eigenvalues \( \mu_n \) satisfy

\[
\mu_n = \mu_n^0 - 2\nu np_0 + \nu n\tilde{p}_n + q_0 - \tilde{q}_n + O(n^{-\frac{1}{2}}).
\]

Moreover, if \( p'', q' \in L^1(\mathbb{T}) \), then

\[
\mu_n = \mu_n^0 - 2\nu np_0 + \nu n\tilde{p}_n + q_0 - \tilde{q}_n + \frac{4p_0^2}{3\nu n} + O(n^{-\frac{3}{2}}).
\]

In the nice paper [McK81] McKean studied inverse problems for third order operators on the circle. In particular, he determined the trace formula for the operator \( H \) in the case \( p, q \in C^\infty(\mathbb{T}) \). We extend this formula to a larger class of coefficients. Consider the shifted operator \( H_t = H(p_t, q_t) \), given in (1.1), where \( t \in \mathbb{T} \),

\[
p_t(x) = p(x + t), \quad q_t(x) = q(x + t) \quad \forall \ x \in \mathbb{R}.
\]

The spectrum consists of eigenvalues \( \mu_n(t), n \in \mathbb{Z} \setminus \{0\}, \ldots \leq \text{Re} \mu_{-2}(t) \leq \text{Re} \mu_{-1}(t) \leq \text{Re} \mu_1(t) \leq \text{Re} \mu_2(t) \leq \ldots \) counting with multiplicities.

**Theorem 1.2.** Let \( p'', q'' \in L^1(\mathbb{T}) \). Then there exists \( N = N(p, q) \in \mathbb{N} \) such that the functions \( \sum_{n=1}^{N} \mu_n(t) \) and each \( \mu_n(t), n > N \), belong to the space \( C^1(\mathbb{T}) \). Moreover, the following trace formula holds true:

\[
\sum_{n=-\infty}^{\infty} (\mu_n(t) - \mu_n(0)) = V(0) - V(t),
\]

the series converges absolutely and uniformly in \( t \in \mathbb{T} \), where

\[
V = q - \frac{p'}{3}.
\]

In particular, assume that we know \( \mu_n(t) \) for all \( (n, t) \in \mathbb{Z} \times \mathbb{T} \). Then

a) If, in addition, we know \( p \) and \( q(0) \), then we can recover \( q \).

b) If, in addition, we know \( q, p'(0) \) and \( p(0) \), then we can recover \( p \).
Remark. 1) McKean [McK81] obtained the trace formula in the case \( p, q \in C^\infty(\mathbb{T}) \), however, he does not discuss convergence of the series.

2) The proof of Theorem uses the methods from our paper [BK15].

3) Due to Lemma 5.2 ii), each eigenvalue \( \mu_n(t) \) with \(|n| \) large enough is simple and it is a smooth function of \( t \in \mathbb{T} \). Moreover, it is shown in [McK81] that in the case small \( p, q \in C^\infty(\mathbb{T}) \) the motion of the large eigenvalues is quite similar to the motion of the Dirichlet eigenvalues for the Schrödinger operators \(-y'' + qy\), see Trubowitz [T77] for the potential \( q \in C^3(\mathbb{T}) \) and Korotyaev [K99] for the potential \( q \in L^2(\mathbb{T}) \). The situation for the eigenvalues inside the bounded disc is more complicated, see the example with the \( \delta \)-coefficients in Section 3.3.

However, due to Rouché’s theorem, we can control their sum and it is smooth.

3) Let \( D(\lambda, t) = D(\lambda, pt, qt) \) and let \( n \in \mathbb{Z} \setminus \{0\} \). The identity \( D(\mu_n(t), t) = 0 \) gives that each function \( \mu_n(t) \) satisfies the so-called Dubrovin equation for the operator \( H \)

\[
\dot{\mu}_n(t) = -\frac{\dot{D}(\lambda, t)}{D'(\lambda, t)} \Big|_{\lambda=\mu_n(t)}, \quad t \in \mathbb{T},
\]

where \( \dot{D} = \frac{\partial D}{\partial t}, \quad D' = \frac{\partial D}{\partial \lambda} \). Dynamics of the eigenvalues plays an important role in solving the Boussinesq equation. It will be discussed in a separate paper.

The plan of the paper is as follows. The proof of Theorem 1.1 is rather complicated and in Section 2 we give a sketch of proof of this Theorem. Section 3 contains some preliminary simplest relations for the characteristic function \( D \) and for the fundamental matrix \( M \). Moreover, the unperturbed case \( p = q = 0 \) and the example with \( p = 0 \) and \( q \) is the periodic \( \delta \)-function are considered there. Section 4 is devoted to the Birkhoff method, which is a main tool of our proofs of the eigenvalue asymptotics. We present this method in a general formulation, since in the next parts of the paper it will be applied in three different cases, depending on the smoothness of the coefficients. In Section 5 we consider the eigenvalues for the case \( p, q \in L^1(\mathbb{T}) \). There we prove the Counting result and determine the asymptotics for this case. The eigenvalue asymptotics for the case \( p', q \in L^1(\mathbb{T}) \) is determined in Section 6 and for the case \( p'', q' \in L^1(\mathbb{T}) \) in Section 7. We prove the trace formula in Section 8.

2. Sketch of proofs

2.1. Factorization formula. In this Section we describe briefly our proof of Theorem 1.1. Our main tool is an asymptotic analysis of the fundamental matrix \( M \) at large \( |\lambda| \). Such analysis is standard for Schrödinger operators. For this case (even with the matrix coefficients) all entries of the fundamental matrix are bounded for \( \lambda \to +\infty \) (in the unperturbed case they have the form \( \cos \sqrt{\lambda}x, \sin \sqrt{\lambda}x \)). But in our case we meet additional difficulties. For our third order operator all entries of the fundamental matrix are unbounded as \( \lambda \to +\infty \) (in the unperturbed case they have the form \( (3.4) \)). In order to obtain the asymptotics of the fundamental matrix we use the method developed by Birkhoff [B08]. Now we give a brief description of this method, see the details in Section 4.

Let \( z = \lambda^\frac{1}{3} \in \mathcal{Z}_+ \), where

\[
\mathcal{Z}_+ = \left\{ z \in \mathbb{C} : \arg z \in \left[0, \frac{\pi}{3}\right) \right\}.
\]

Introduce the diagonal matrix \( \mathcal{T} \) and the matrix \( \Omega \) by

\[
\mathcal{T} = \text{diag}(\tau_1, \tau_2, \tau_3) = \text{diag}(\tau, \tau^2, 1), \quad \tau = e^{\frac{2\pi}{3}},
\]

(2.2)
\[
\Omega = \begin{pmatrix}
1 & 1 & 1 \\
\tau z & \tau z & 1 \\
\tau^2 z & \tau z & z^2 \\
\end{pmatrix}.
\] (2.3)

The key point in our proof of the asymptotics is the following factorization formula (2.4) for the matrix-valued solution of equation (1.4).

**Theorem 2.1.** Let \( p, q \in L^1(\mathbb{T}) \). Then there exists a matrix-valued solution \( A(x, z) \) of equation (1.4) such that each function \( A(x, \cdot), x \in [0, 2] \) is analytic in \( Z_+ \) for \(|z| \) large enough and satisfies

\[
A(x, \lambda) = \Omega(z) \left(1 + O(z^{-1})\right)e^{zxT},
\] (2.4)

as \(|z| \to \infty, z \in Z_+\), uniformly in \((\arg z, x) \in [0, \pi/4] \times [0, 2] \).

The factorization formula (2.4) represents the matrix-valued solution of equation (1.4) in the form of product of the bounded matrices and the diagonal matrix, which contains all exponentially increasing and exponentially decreasing factors.

### 2.2. Sketch of proof of asymptotics (2.4)

Let \( z \in Z_+ \) and let \( A(x, z), x \in \mathbb{R} \), be a matrix-valued solution of equation (1.4). We show how to determine the solution \( A \) with the needed asymptotics (2.4) using three steps:

**Step 1.** We introduce the matrix-valued function \( Y(x, z) \) by

\[
A(x, z) = \Omega(z)Y(x, z).
\] (2.5)

Substituting the definition (2.5) into equation (1.4) and using the identity

\[
\Omega^{-1}P\Omega = zT + \frac{Q}{z}, \quad Q = -\frac{p}{3} \begin{pmatrix}
\tau^2 & -\tau & -1 \\
-\tau^2 & \tau & -1 \\
-\tau^2 & -\tau & 1 \\
\end{pmatrix} - \frac{q}{3z} \begin{pmatrix}
\tau & \tau^2 & \tau^2 \\
\tau^2 & \tau & 1 \\
\tau^2 & 1 & 1 \\
\end{pmatrix},
\]

we obtain that \( Y \) satisfies the equation

\[
Y' - zTY = \frac{Q}{z}Y.
\] (2.6)

It is easy to find the inverse operator for the operator on the left side of equation (2.6). This would be sufficient in the case of a second order operator. However, for higher order operators, additional steps are required.

**Step 2.** We introduce a matrix-valued function \( X(x, z) \) by

\[
Y(x, z) = X(x, z)e^{zxT}.
\] (2.7)

Then \( X \) is a solution of the differential equation

\[
X' + z(XT - TX) = \frac{Q}{z}X.
\] (2.8)

Equation (2.8), as well as the equivalent equations (1.4) and (2.6), has many solutions. In order to find the unique one we choose the solution under the two-side initial conditions

\[
X_{jk}(0, z) = 0, \quad j < k, \quad X_{jk}(2, z) = \delta_{jk}, \quad j \geq k.
\]

This choice of solution \( X \) will lead us to the desired solution \( A \) of equation (1.4) satisfying the asymptotics (2.4) and this is a crucial point of the method.
Step 3. Let \( z \in \mathbb{Z}_+ \) and let \(|z|\) be large enough. It is shown in Theorem 4.5 that \( X \) is the unique solution of the integral equation

\[
X = 1 + \frac{1}{z} K X,
\]

where

\[
(KX)_{\ell j}(x, z) = \int_0^2 K_{\ell j}(x, s, z) (QX)_{\ell j}(s, z) ds \quad \forall \ \ell, j = 1, 2, 3,
\]

\[
K_{\ell j}(x, s, z) = \begin{cases} 
  e^{z(x-s)(\tau_\ell - \tau_j)} \chi(x - s), & \ell < j \\
  -e^{z(x-s)(\tau_\ell - \tau_j)} \chi(s - x), & \ell \geq j,
\end{cases}
\]

\( \chi(s) = \begin{cases} 
  1, & s \geq 0 \\
  0, & s < 0
\end{cases} \).

Note that

\[
\text{Re}(\tau_1 z) \leq \text{Re}(\tau_2 z) \leq \text{Re}(\tau_3 z) \quad \forall \ z \in \mathbb{Z}_+.
\]  

(2.10)

Then the kernel of the integral operator \( K \) satisfies

\[
|K_{\ell j}(x, s, z)| \leq 1 \text{ for all } \ell, j = 1, 2, 3 \text{ and } (x, s, z) \in [0, 2] \times \mathbb{Z}_+.
\]

Iterations of the integral equation (2.9) give the asymptotics

\[
X(x, z) = 1 + O(z^{-1})
\]

as \(|z| \to \infty, z \in \mathbb{Z}_+\) uniformly in \( x \in [0, 2] \). Then the definition (2.7) gives

\[
Y(x, z) = (1 + O(z^{-1})) e^{zxT}.
\]  

(2.12)

Substituting this asymptotics into (2.5) we obtain the asymptotics (2.4).

2.3. Asymptotics of the characteristic function. Let \((x, z) \in [0, 2] \times \mathbb{Z}_+\) and let \(|z|\) be large enough. Then the fundamental matrix \( M(x, \lambda) \) satisfies the identity

\[
M(x, \lambda) = A(x, z) A^{-1}(0, z), \quad \lambda = z^3.
\]  

(2.13)

The functions

\[
\phi_j(x, z) = A_{1j}(x, z), \quad j = 1, 2, 3,
\]

are solutions of equation (1.3). In Lemma 3.2 we will prove that

\[
D(\lambda) = \frac{\det \phi(z)}{\det A(0, z)},
\]  

(2.15)

where

\[
\phi(z) = \begin{pmatrix} 
  \phi_1(0, z) & \phi_2(0, z) & \phi_3(0, z) \\
  \phi_1(1, z) & \phi_2(1, z) & \phi_3(1, z) \\
  \phi_1(2, z) & \phi_2(2, z) & \phi_3(2, z)
\end{pmatrix}.
\]  

(2.16)

The asymptotics (2.4) (see Lemma 5.2 i)) gives

\[
\text{det} A(0, z) = -i 3 \sqrt{3} z^3 \left( 1 + O(z^{-1}) \right),
\]

as \(|z| \to \infty, z \in \mathbb{Z}_+\). This asymptotics and the identity (2.15) show that the large positive zeros of the function \( D \) coincide with the zeros of the third order determinant \( \det \phi(z) \).

Moreover, substituting the asymptotics (2.4) into the definition (2.14) and using (2.8) we obtain

\[
\phi_j(x, z) = e^{zx\tau_j} \left( 1 + O(z^{-1}) \right),
\]

as \(|z| \to \infty, z \in \mathbb{Z}_+\) uniformly in \( x \in [0, 2] \). These asymptotics show that the solutions \( \phi_j \) are Jost type solutions of equation (1.3).
For simplicity, consider the real $z \to +\infty$, see Lemma 5.3 for the situation in the whole sector $Z_+$. The definition (2.12) gives that $\phi_1, \phi_2, \phi_3$ satisfy

$$|\phi_1(x, z)| = |e^{zx}| (1 + O(z^{-1})) = e^{-\frac{1}{2}zx} (1 + O(z^{-1})), $$

$$|\phi_2(x, z)| = |e^{zx^2}| (1 + O(z^{-1})) = e^{-\frac{1}{2}zx} (1 + O(z^{-1})), $$

and

$$|\phi_3(x, z)| = e^{x^2} (1 + O(z^{-1})). $$

Substituting these asymptotics into the definition (2.16) we obtain

$$\det \phi(z) = \phi_3(2, z) \left( \det \begin{pmatrix} \phi_1(0, z) & \phi_2(0, z) \\ \phi_1(1, z) & \phi_2(1, z) \end{pmatrix} + O(e^{-z}) \right).$$

Substituting the last asymptotics and the asymptotics (2.17) into the identity (2.15) we get the following asymptotic representation for the function $D$:

$$D(\lambda) = \frac{i \phi_3(2, z)}{3\sqrt{3z^3}} \left( \det \begin{pmatrix} \phi_1(0, z) & \phi_2(0, z) \\ \phi_1(1, z) & \phi_2(1, z) \end{pmatrix} + O(e^{-z}) \right), \quad (2.19)$$

as $z \to +\infty$. Thus, we reduce the asymptotic analysis of the zeros of the third-order determinant $\det \phi(z)$ in the identity (2.15) to the analysis of the zeros of the simpler second-order determinant in (2.19).

Of course, the rough asymptotics (2.18) does not give us asymptotics of the zeros of the function $D$. However, we can take the next terms in the iteration series for the solution of the integral equation (2.9). It improves the asymptotics of the matrix-valued function $A$ and then the asymptotics of $\phi_1$ and $\phi_2$, due to the definition (2.14). In this way, using the identity (2.19) we determine the asymptotics (1.11)–(1.13) of the zeros of the function $D$.

3. Characteristic function $D$

3.1. **Properties of the characteristic function.** In the following Lemma we establish some simple properties of the function $D$. Moreover, there we prove for completeness that the spectrum of the operator $H$ is the set of zeros of the entire function $D(\lambda)$ given by the definition (1.7).

Denote by $D(\lambda, p, q)$ the function $D(\lambda)$ for the operator $H = H(p, q)$ given by (1.1)–(1.2) and let $\mu_n(p, q)$ be zeros of the function $D(\lambda, p, q)$.

**Lemma 3.1.** Let $p, q \in L^1(\mathbb{T})$. Then

i) The spectrum $\sigma(H)$ of the operator $H$ satisfies (1.4).

ii) The function $D(\lambda)$ is real for $\lambda \in \mathbb{R}$ and satisfies

$$D(\lambda, p, q) = D(-\lambda, p^-, -q^-) \quad \forall \quad \lambda \in \mathbb{C}, \quad (3.1)$$

where $p^-(t) = p(-t), q^-(t) = q(-t), t \in \mathbb{R}$. The zeros $\mu_n(p, q)$ of the function $D(\lambda, p, q)$ satisfy

$$\mu_{-n}(p, q) = -\mu_n(p^-, -q^-) \quad \forall \quad n \in \mathbb{Z} \setminus \{0\}. \quad (3.2)$$

**Proof.** i) Assume that $y(x, \lambda), (x, \lambda) \in [0, 2] \times \mathbb{C}$ is the solution of equation (1.3) satisfying the three-point conditions

$$y(0, \lambda) = y(1, \lambda) = y(2, \lambda) = 0. \quad (3.3)$$

Then

$$y(x, \lambda) = C_1 \varphi_1(x, \lambda) + C_2 \varphi_2(x, \lambda) + C_3 \varphi_3(x, \lambda),$$
where \( \varphi_j(x, \lambda) \) are the fundamental solutions and \( C_1, C_2, C_3 \) are complex constant. The conditions (3.3) yield \( C_1 = 0 \) and
\[
C_2 \varphi_2(1, \lambda) + C_3 \varphi_3(1, \lambda) = 0, \quad C_2 \varphi_2(2, \lambda) + C_3 \varphi_3(2, \lambda) = 0.
\]
The solution \( y(x, \lambda) \) is non-trivial iff \( D(\lambda) = 0 \). The function \( D \) is entire, then the spectrum is pure discrete and satisfies (1.6).

ii) The definition (1.7) shows that \( D(\lambda) \) is real for \( \lambda \in \mathbb{R} \). Substituting \(-t\) instead of \( t\) in equation (1.3) we obtain that the operator \( H(p, q) \) is unitarily equivalent to the operator \(-H(p^-, q^-)\). This yields (3.1). The identity (3.2) follows.

**Remark.** Using these results we reduce the analysis of the function \( D \) in \( \mathbb{C} \) to the analysis in the domain \( \text{Re} \lambda \geq 0, \text{Im} \lambda \geq 0 \) and the analysis of the eigenvalues \( \mu_n \) at \( n \to -\infty \) to the analysis at \( n \to +\infty \).

### 3.2. The unperturbed case.
If \( p = q = 0 \), then the fundamental solutions have the forms
\[
\begin{align*}
\varphi_1^0(x, \lambda) &= \frac{1}{3}(e^{zx} + e^{\tau zx} + e^{\tau^2 zx}), \\
\varphi_2^0(x, \lambda) &= \frac{1}{3z}(e^{zx} + \tau^2 e^{\tau zx} + \tau e^{\tau^2 zx}), \\
\varphi_3^0(x, \lambda) &= \frac{1}{3z^2}(e^{zx} + \tau e^{\tau zx} + \tau^2 e^{\tau^2 zx})
\end{align*}
\]
(3.4)

here and below
\( \tau = e^{i2\pi/3}, \quad z = \lambda^{1/3}, \quad z \in \overline{\mathbb{Z}} \),
\[
\overline{\mathbb{Z}} = \left\{ z \in \mathbb{C} : \arg z \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \right\},
\]
\[
\mathbb{Z} = \left\{ z \in \mathbb{C} : \arg z \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \right\}.
\]

Then for the function \( D \) at \( p = q = 0 \) we have
\[
D^0 = \frac{1}{9\lambda} \det \begin{pmatrix}
e^z + \tau e^{\tau z} + \tau e^{\tau^2 z} & e^z + \tau e^{\tau z} + \tau e^{\tau^2 z} \\
e^{2z} + \tau^2 e^{2\tau z} + \tau e^{2\tau^2 z} & e^{2z} + \tau^2 e^{2\tau z} + \tau e^{2\tau^2 z}
\end{pmatrix}
= \frac{\tau - \tau^2}{9\lambda} \det \begin{pmatrix}
e^z & 1 \\
e^{2z} & e^{2\tau z}
\end{pmatrix} = \frac{i}{3\sqrt{3}\lambda} \det \begin{pmatrix}
e^z & 1 \\
e^{2z} & e^{2\tau z}
\end{pmatrix}.
\]
(3.5)

The standard formula for the Vandermonde determinant gives
\[
D^0 = \frac{i}{3\sqrt{3}\lambda} (e^z - e^{\tau z})(e^z - e^{\tau^2 z})(e^z - e^{\tau^2 z}) = \frac{8}{3\sqrt{3}\lambda} \sin \frac{\sqrt{3}z}{2} \sin \frac{\sqrt{3}\tau z}{2} \sin \frac{\sqrt{3}\tau^2 z}{2}.
\]

Here only the first factor \( \sin \frac{\sqrt{3}z}{2} \) has zeros in \( \overline{\mathbb{Z}} \) and then the zeros \( \mu_n^0, n \in \mathbb{Z} \setminus \{0\} \), of \( D^0 \) are simple and have the form (1.9).
3.3. Example: \(\delta\)-coefficient. In order to illustrate the movement of eigenvalues we consider an example. Consider the operator \(H_t\) with a periodic \(\delta\)-potential, where \(p = 0, q = \gamma \sum_{n=-\infty}^{\infty} \delta(x + t - n)\). The standard calculations show that the fundamental solutions \(\phi_2, \phi_3\) satisfy

\[
\phi_j(1, \lambda, t) = \phi_j(1, \lambda) - \gamma \phi_j(t, \lambda) \phi_3'(1 - t, \lambda),
\]

\[
\phi_j(2, \lambda, t) = \phi_j(2, \lambda) - \gamma \phi_j(t, \lambda) \phi_3'(2 - t, \lambda) - \gamma \phi_3(1 - t, \lambda)(\phi_j(t + 1, \lambda) - \gamma \phi_j(t, \lambda) \phi_3(1, \lambda)).
\]

The results of numeric analysis of the zeros \(\mu_{\pm 1}(t)\) of the function \(D(\lambda, t)\), defined by (1.7), are shown in Fig. 2. Here we take \(\gamma = 40\) and have

\[
\mu_{-1}(0) = \mu_{-1}(1) = -\left(\frac{2\pi}{\sqrt{3}}\right)^3,
\]

\[
\mu_{1}(0) = \mu_{1}(1) = \left(\frac{2\pi}{\sqrt{3}}\right)^3,
\]

\[
\mu_{-1}(t_1) = \mu_{1}(t_1), \quad \mu_{-1}(t_2) = \mu_{1}(t_2),
\]

for some \(0 < t_1 < t_2 < 1\) and

\[
\mu_{-1}(t) = \overline{\mu}_1(t) \quad \forall \ t \in (t_1, t_2).
\]

Moreover, the extreme positions of \(\mu_{\pm 1}(t)\) coincide with the branch points \(r_{\pm 2}\) of the so called multiplier curve for the operator with the periodic coefficients on the axis, see [McK81]. Thus, the eigenvalue \(\mu_1(t)\):

1) starts as \(t = 0\) from the point \(\left(\frac{2\pi}{\sqrt{3}}\right)^3\) moving to the left,
2) collides with the eigenvalue \(\mu_{-1}(t)\) at the point \(a_1\) as \(t = t_1\),
3) leaves the real axis and moves along the oval curve in the complex plane,
4) returning to the real axis it collides with the eigenvalue \(\mu_{-1}(t)\) again at the point \(a_2\) as \(t = t_2\),
5) moves along the real axis to the right until the branch point \(r_2\),
6) turns back and returns to the point \(\left(\frac{2\pi}{\sqrt{3}}\right)^3\) as \(t = 1\).
The motion of $\mu_{-1}(t)$ is similar, see Fig. 2. The other eigenvalues move similarly to the eigenvalues of the Dirichlet problem for the Schrödinger operator associated with the Korteweg-de Vries equation. Branch points of the multiplier curve satisfy $... < r_{-2} < r_{-1}^+ < r_{-1}^- < r_{-2}^+ < r_{-3}^- < r_{-3}^+ ...$. If $t$ changes from 0 to 1, then the eigenvalue $\mu_n(t), n = \pm 2, \pm 3, ...$ runs all interval $[r_n^-, r_n^+]$ making $n$ complete revolutions.

Thus, the movement of the first eigenvalues is complicated. It can start moving along the real axis, then go out into the complex plane, and then return to the real axis again. When the parameter $\gamma$ becomes larger, the movement of the eigenvalues becomes even more complicated. Therefore, the analysis of such eigenvalues is difficult.

3.4. Fundamental solutions. The matrix-valued function $M(\lambda)$ is entire, however it is difficult to obtain its asymptotics at large $|\lambda|$. We introduce other matrix-valued solutions of equation (1.4) that differ from the solution $M$. We take a $3 \times 3$ matrix-valued function $A(x, z), z \in \mathbb{C}$, such that

1) $A(\cdot, z)$ satisfies equation (1.4) on $[0, 2]$,
2) each $A(x, \cdot), x \in [0, 2]$ is analytic on a domain $\mathcal{D} \subset \mathbb{C}$ and
3) $\det A(0, z) \neq 0$ for all $z \in \mathcal{D}$.

We can always achieve the fulfillment of the condition 3) by choosing the domain $\mathcal{D}$ that is sufficiently small.

The fundamental matrix $M$ satisfies the identity

$$M(x, \lambda) = A(x, z)A^{-1}(0, z), \quad \forall \quad (x, z) \in [0, 2] \times \mathcal{D}. \quad (3.6)$$

More often we need the entries from the first line of the matrix-valued function $A$. The functions

$$\phi_j(x, z) = A_{1j}(x, z), \quad j = 1, 2, 3, \quad (3.7)$$

are fundamental solutions of equation (3.4). Introduce the matrix-valued function $\phi$ by

$$\phi(z) = \begin{pmatrix} \phi_1(0, z) & \phi_2(0, z) & \phi_3(0, z) \\ \phi(1, z) & \phi_2(1, z) & \phi_3(1, z) \\ \phi_3(2, z) & \phi_3(2, z) \end{pmatrix}. \quad (3.8)$$

The functions $\det \phi(z)$ and $\det A(0, z)$ are analytic in $\mathcal{D}$, but, in general, they are not entire.

Lemma 3.2. Let $p, q \in L^1(\mathbb{T})$. Then the function $D$ has the form

$$D(\lambda) = \frac{\det \phi(z)}{\det A(0, z)}, \quad z = \lambda \frac{1}{\epsilon} \in \mathcal{D}, \quad (3.9)$$

and the function $\frac{\det \phi(z)}{\det A(0, z)}$ has an analytic extension from $\mathcal{D}$ onto the whole sector $\mathcal{Z}$. Thus, the identity (3.7) holds for all $z \in \mathcal{Z}, \lambda = z^3 \in \mathbb{C}$.

Proof. The function $A$ is a solution of equation (1.4), then

$$A = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 \\ \phi_1' & \phi_2' & \phi_3' \\ \eta_1 & \eta_2 & \eta_3 \end{pmatrix}, \quad \eta_j = \phi_j'' + p\phi_j. \quad (3.10)$$

For each $j = 1, 2, 3$ we have

$$\phi_j(x, z) = \phi_j(0, z)\varphi_1(x, z) + \phi_j'(0, z)\varphi_2(x, z) + \eta_j(0, z)\varphi_3(x, z).$$
Then
\[
\begin{pmatrix}
\phi_1(0, z) & \phi_2(0, z) & \phi_3(0, z) \\
\phi_1(1, z) & \phi_2(1, z) & \phi_3(1, z) \\
\phi_1(2, z) & \phi_2(2, z) & \phi_3(2, z)
\end{pmatrix}
\begin{pmatrix}
\varphi_1(0, \lambda) & \varphi_2(0, \lambda) & \varphi_3(0, \lambda) \\
\varphi_1(1, \lambda) & \varphi_2(1, \lambda) & \varphi_3(1, \lambda) \\
\varphi_1(2, \lambda) & \varphi_2(2, \lambda) & \varphi_3(2, \lambda)
\end{pmatrix}
\begin{pmatrix}
\phi_1 & \phi_2 & \phi_3 \\
\phi_1' & \phi_2' & \phi_3' \\
\eta_1 & \eta_2 & \eta_3
\end{pmatrix}
= (0, z).
\]

(3.11)

The definition (1.7) implies the identity
\[
D(\lambda) = \det \begin{pmatrix}
1 & 0 & 0 \\
\varphi_1(1, \lambda) & \varphi_2(1, \lambda) & \varphi_3(1, \lambda) \\
\varphi_1(2, \lambda) & \varphi_2(2, \lambda) & \varphi_3(2, \lambda)
\end{pmatrix}
= \det \begin{pmatrix}
\varphi_1(0, \lambda) & \varphi_2(0, \lambda) & \varphi_3(0, \lambda) \\
\varphi_1(1, \lambda) & \varphi_2(1, \lambda) & \varphi_3(1, \lambda) \\
\varphi_1(2, \lambda) & \varphi_2(2, \lambda) & \varphi_3(2, \lambda)
\end{pmatrix}.
\]

(3.12)

Substituting the identities (3.10) and (3.12) into (3.11) we obtain (3.9).

Remark. The asymptotics (2.17) shows that \( \det A(0, z) \) does not vanish at large \( |z| \) and then the zeros of the function \( D(z^3) \) coincide with the zeros of \( \det \phi(z) \) at high energy.

4. Birkhoff’s method

4.1. Birkhoff’s differential equation. In Section 2 we rewrote the problem (1.4) in the form (2.1), i.e.,
\[
Y' - z\mathcal{T}Y = -\frac{1}{3z} \left( pP + \frac{q}{z}Q \right) Y, \quad Y(0, z) = I_3,
\]

(4.1)

where
\[
P = \begin{pmatrix}
\tau^2 & -\tau & -1 \\
-\tau^2 & \tau & -1 \\
-\tau^2 & -\tau & 1
\end{pmatrix}, \quad Q = \begin{pmatrix}
\tau & \tau & \tau \\
\tau^2 & \tau^2 & \tau^2 \\
1 & 1 & 1
\end{pmatrix}.
\]

(4.2)

We analyze equation (4.1) by using our version of the Birkhoff method of asymptotic analysis of higher-order equations, see [BK11], [BK14a]. Now we present this method for third order case.

Introduce the domains
\[
\mathcal{Z}_+(r) = \{ z \in \mathcal{Z}_+ : |z| > r \} \subset \mathcal{Z}_+, \quad r > 0,
\]

(4.3)

where the sector \( \mathcal{Z}_+ \) has the form (2.1).

Condition 4.1. Let a \( 3 \times 3 \) - matrix-valued functions \( \Phi(x, z) \) and \( \Theta(x, z) \), \( (x, z) \in T \times \mathcal{Z}_+ \), satisfy:
1) \( \Theta = \text{diag}(\Theta_1, \Theta_2, \Theta_3) \) is diagonal,
2) \( \Phi(\cdot, z) \in L^1(T) \) and \( \Theta(\cdot, z) \in L^1(T) \) for all \( z \in \mathcal{Z}_+(r) \), where \( r > 0 \) is large enough,
3) \( \Phi(x, z) \) and \( \Theta(x, z) \) are analytic in \( \mathcal{Z}_+(r) \) and
\[
\Phi(x, z) = \mathcal{F}(x) + O(z^{-1}), \quad \Theta(x, z) = \mathcal{T} + O(z^{-1})
\]
\]

as \( |z| \to \infty, z \in \mathcal{Z}_+ \), uniformly in \( x \), where \( \mathcal{T} \) is given by (2.2) and \( \mathcal{F} \in L^1(T) \),
4) \( \mathcal{F} \) is off-diagonal, that is \( \mathcal{F}_{jj} = 0, j = 1, 2, 3 \).

We consider the following differential equation on \( \mathbb{R} \):
\[
A' - z\Theta A = \frac{1}{z^m}\Phi A,
\]

(4.5)

where \( z \in \mathcal{Z}_+(r), r > 0 \) is large enough, \( m \in \mathbb{N} \), and the \( 3 \times 3 \)-matrix-valued functions \( \Theta(x, z) \) and \( \Phi(x, z) \) satisfy Conditions 1.1. Below \( \Theta(x, z), \Phi(x, z) \) and \( m \) have different forms.
depending on smoothness of the coefficients \( p, q \). For example, if \( p, q \in L^1(\mathbb{T}) \), then equation (4.5) has the form (4.1).

**Remark.** 1) Here and always below our asymptotics at large \(|z|\) are uniform in \( \arg z \).

2) The hypothesis 4 in Condition 4.1 is assumed without loss of generality, since we can replace the diagonal part of \( F \) into the left hand side of equation (4.5).

Birkhoff \[B08\] developed a method for obtaining asymptotics of fundamental solutions of higher-order linear differential equations. He found a way to rewrite the differential equation (1.4) in the form of a Fredholm integral equation with a small kernel at high energy. Birkhoff used a scalar form of the differential equation. However, the matrix form (4.5) is more convenient for analysis. Below we give our version of the Birkhoff method for third order differential equations, the case of fourth order equations see in \[BK14x\].

First of all, we rewrite equation (4.5) in the form (4.7) convenient for the application of the Birkhoff method.

**Lemma 4.2.** Let the \( 3 \times 3 \) - matrix-valued functions \( \Phi \) and \( \Theta \) satisfy Condition 4.1 and let \( z \in \mathbb{Z}_+(r) \), where \( r > 0 \) is large enough. Let a matrix-valued function \( A \) and a matrix-valued function \( X \) satisfy

\[
A(x, z) = X(x, z)e^{\int_0^x \Theta(s, z) ds}, \quad x \in \mathbb{R}.
\]

Then \( A \) is a solution of the differential equation (4.5) if and only if \( X \) is a solution of the differential equation

\[
X' + z(X\Theta - \Theta X) = \frac{1}{z^m} \Phi X
\]

**Proof.** Let the matrix-valued function \( A \) satisfy the differential equation (4.5) and let (4.6) be fulfilled. Then

\[
(Xe^{\int_0^x \Theta ds})' - z\Theta Xe^{\int_0^x \Theta ds} = \frac{1}{z^m} \Phi Xe^{\int_0^x \Theta ds},
\]

which gives

\[
X' e^{\int_0^x \Theta ds} + zX\Theta e^{\int_0^x \Theta ds} - z\Theta Xe^{\int_0^x \Theta ds} = \frac{1}{z^m} \Phi Xe^{\int_0^x \Theta ds}.
\]

Thus, \( X \) satisfies the differential equation (4.7).

Conversely, let the matrix-valued function \( A \) have the form (4.6) and let \( X \) satisfy the differential equation (4.7). The identities

\[
A' = zX\Theta e^{\int_0^x \Theta ds} + X'e^{\int_0^x \Theta ds} = zX\Theta e^{\int_0^x \Theta ds} - \left( z(X\Theta - \Theta X) - \frac{1}{z^m} \Phi X \right) e^{\int_0^x \Theta ds}
\]

\[
= \left( z\Theta X + \frac{1}{z^m} \Phi X \right) e^{\int_0^x \Theta ds} = z\Theta A + \frac{1}{z^m} \Phi A
\]

show that \( A \) satisfies the differential equation (4.5). \( \square \)

4.2. **Birkhoff’s integral equation.** Below in Theorem 4.5 i) we will prove that the differential equation (4.7) is equivalent in some sense to an integral equation. But first we will study this integral equation. We introduce the following class of the Birkhoff operators.

**Definition 4.3.** Let \( K \) be an integral operator in the space \( C[0, 2] \) of \( 3 \times 3 \) continuous matrix-valued functions on \([0, 2]\) given by

\[
(KX)_{\ell j}(x, z) = \int_0^2 K_{\ell j}(x, s, z)(\Phi X)_{\ell j}(s, z) ds \quad \forall \ \ell, j = 1, 2, 3,
\]
where Φ and Θ satisfy Condition 4.1 and

\[ K_{ij}(x, s, z) = \begin{cases} \int e^{z(x^2 - s^2)} \chi(x - s), & \ell < j \\ -e^{-z(x^2 - s^2)} \chi(s - x), & \ell \geq j \end{cases}, \quad \chi(s) = \begin{cases} 1, & s \geq 0 \\ 0, & s < 0 \end{cases}. \] (4.9)

If \(|z|\) is large, then, due to (2.10), the kernel of the integral operator \(K\) is bounded. Therefore, the matrix-valued integral equation

\[ \mathcal{X} = \mathbf{I}_3 + \frac{1}{z^m} K \mathcal{X}, \quad m \in \mathbb{N}, \] (4.10)

that we call the Birkhoff equation, has a unique solution for large \(|z|\).

**Lemma 4.4.** Let the \(3 \times 3\) - matrix-valued functions \(\Phi\) and \(\Theta\) satisfy the Condition 4.1 and let \(K\) be the integral operator given by Definition 4.3.

i) Let \(z \in \mathcal{Z}_+(r)\), where \(r > 0\) is large enough, and let \(m \in \mathbb{N}\). Then the integral equation (4.10) has the unique solution \(\mathcal{X}(\cdot, z) \in C[0, 2]\). Moreover, each matrix-valued function \(\mathcal{X}(x, \cdot), x \in [0, 2]\), is analytic in \(\mathcal{Z}_+(r)\) and satisfies

\[ \mathcal{X}(x, z) = \mathbf{I}_3 + \frac{B(x, z)}{z^m} + O(1), \quad \text{as } |z| \to \infty, z \in \mathcal{Z}_+, \text{ uniformly on } x \in [0, 2], \] (4.11)

as \(|z| \to \infty, z \in \mathcal{Z}_+\), uniformly on \(x \in [0, 2]\), where

\[ B = K \mathbf{I}_3. \] (4.12)

ii) The function \(B = K \mathbf{I}_3\) satisfies

\[ B_{ij}(x, z) = O(z^{-1}), \quad j = 1, 2, 3, \] (4.13)

\[ B_{ij}(x, z) = -\int_x^2 e^{-z(s-x)(\tau-x)} F_{ij}(s) ds + O(z^{-1}), \quad 1 \leq j < \ell \leq 3, \] (4.14)

as \(|z| \to \infty, z \in \mathcal{Z}_+\) uniformly in \(x \in [0, 2]\). Moreover, the matrix-valued functions

\[ \eta_j(x, z) = \sum_{k=1}^3 B_{kj}(x, z), \quad j = 1, 2, \] (4.15)

satisfy

\[ \eta_1(x, z) = \int_0^x e^{2\pi i n(x-s)} F_{12}(s) ds + O(n^{-1}), \] (4.16a)

\[ \eta_2(x, z) = -\int_x^2 e^{2\pi i n(s-x)} F_{21}(s) ds + O(n^{-1}), \] (4.16b)

as \(z = 2\pi n + O(n^{-1}), n \to +\infty\) uniformly in \(x \in [0, 2]\).

**Proof.** i) Let \(z \in \mathcal{Z}_+(r)\), where \(r > 0\) is large enough. Due to the asymptotics (4.4) and the estimates (2.10), the function \(K(x, s, z)\) satisfies \(|K(x, s, z)| \leq C\) for all \((x, s, z) \in [0, 2]^2 \times \mathcal{Z}_+(r)\) and for some \(C > 0\). Then the operator \(K\) is bounded in \(C[0, 2]\). Then the operator \(\frac{1}{z^m} K\) in equation (4.10) is a contraction. Consider the iteration series

\[ \mathcal{X}(x, z) = \sum_{n=0}^{\infty} \frac{(K^n \mathbf{I}_3)(x, z)}{z^{nm}} \quad \forall \quad (x, z) \in [0, 2] \times \mathcal{Z}_+(r). \] (4.17)
The matrix-valued functions $K^n\mathbb{1}_3, n \geq 0$, satisfy $\|K^n\mathbb{1}_3\|_{C[0,2]} \leq C$ for some $C > 0$, where $\|\cdot\|_{C[0,2]}$ is any matrix norm. Then the series (4.17) converges absolutely and uniformly on any compact set in $[0, 2] \times \mathbb{Z}_+(r)$. Each matrix valued function $K^n\mathbb{1}_3(x, \cdot), n \geq 0, t \in [0, 2]$, is analytic in $\mathbb{Z}_+(r)$. Then each matrix valued function $\mathcal{X}(x, \cdot), x \in [0, 2]$, is analytic in $\mathbb{Z}_+(r)$ and satisfies the asymptotics (4.11).

ii) The definitions (4.8) and (4.12) and the asymptotics (4.4) give

$$B_{\ell j}(x, z) = \int_0^2 K_{\ell j}(x, s, z)F_{\ell j}(s)ds + O(z^{-1}), \quad \ell, j = 1, 2, 3.$$  (4.18)

Moreover,

$$\left| \int_x^2 e^{-(\sqrt{3}-i)\pi n(s-x)}F_{31}(s)ds \right| \leq \int_x^2 e^{-\sqrt{3}0n(s-x)}|F_{31}(s)|ds$$

$$\leq \left( \int_x^2 e^{-2\sqrt{3}0n(s-x)}ds \right)^{\frac{1}{2}} \left( \int_x^2 |F_{31}(s)|^2ds \right)^{\frac{1}{2}} = O(n^{-\frac{1}{2}})$$

uniformly in $x$. Then the asymptotics (4.19) implies

$$B_{31}(x, z) = O(n^{-\frac{1}{2}}), \quad \text{uniformly in } x \in [0, 2].$$  (4.20)

Similarly,

$$B_{32}(x, z) = O(n^{-\frac{1}{2}}). \quad \text{ (4.21)}$$

The asymptotics (4.13), (4.18), (4.20) and (4.21) yield the asymptotics (4.16).

4.3. **Factorization theorem.** Let $z \in \mathbb{Z}_+(r)$, where $r > 0$ is large enough. Introduce the fundamental matrix-valued solution $M$ of equation (4.5) by the condition

$$M(0, z) = \mathbb{1}_3.$$  (4.22)

Then

$$M(x, z) = \mathcal{A}(x, z)\mathcal{A}^{-1}(0, z).$$  (4.23)

Now we prove the basic result of this Section, the factorization formula (4.24) for the solution $M$. In fact, this formula together with the asymptotics (4.11) gives the high energy asymptotics of the fundamental matrix, see Remark below.

**Theorem 4.5.** Let the $3 \times 3$ - matrix-valued functions $\Phi$ and $\Theta$ satisfy Condition (4.1) and let $z \in \mathbb{Z}_+(r)$, where $r > 0$ is large enough.
i) The matrix-valued function \( \mathcal{X}(x,z), x \in [0,2], \) satisfies the integral equation (4.10) if and only if it satisfies the differential equation (4.7) and the initial conditions (4.23).

\[ \mathcal{X}_{jk}(0,z) = 0, \quad j < k, \quad \mathcal{X}_{jk}(2,z) = \delta_{jk}, \quad j \geq k. \]  

(4.23)

ii) Let a matrix-valued function \( \mathcal{X}(x,z) \) and a matrix-valued function \( \mathcal{M}(x,z) \) satisfy

\[ \mathcal{M}(x,z) = \mathcal{X}(x,z)e^{\int_0^z \Theta(s,z)ds} \mathcal{X}^{-1}(0,z), \quad \forall x \in [0,2]. \]  

(4.24)

Then \( \mathcal{M}(x,z) \) is the solution of the initial problem \( \mathcal{M}(0,z) = \mathbb{I}_3 \) for equation (4.3) if and only if \( \mathcal{X}(x,z) \) is the solution of the integral equation (4.10).

**Proof.** i) Let \( \mathcal{X} \) satisfy equation (4.7). In terms of matrix entries we have

\[ \mathcal{X}'_{jk} + z(\mathcal{X}_j \Theta_k - \Theta_j \mathcal{X}_k) = \frac{1}{z^m} \sum_{\ell=1}^{3} \Phi_{j\ell} \mathcal{X}_{\ell k}, \quad j, k = 1, 2, 3. \]  

(4.25)

Assume, in addition, that \( \mathcal{X} \) satisfies the initial conditions (4.23). Integrating equation (4.25) and using the conditions (4.23) we obtain

\[ \mathcal{X}_{jk}(x) = \frac{1}{z^m} \int_0^x e^{\int_0^s (\Theta_h(u)-\Theta_j(u))du} \sum_{\ell=1}^{3} \Phi_{j\ell} \mathcal{X}_{\ell k}(s)ds, \quad j < k \]  

\[ \mathcal{X}_{jk}(x) = -\frac{1}{z^m} \int_x^2 e^{\int_0^s (\Theta_h(u)-\Theta_j(u))du} \sum_{\ell=1}^{3} \Phi_{j\ell} \mathcal{X}_{\ell k}(s)ds, \quad j > k \]  

(4.26)

for all \( j, k = 1, 2, 3 \), and

\[ \mathcal{X}_{jj}(x) = 1 - \frac{1}{z^m} \int_x^2 \sum_{\ell=1}^{3} \Phi_{j\ell} \mathcal{X}_{\ell j}(s)ds, \quad j = 1, 2, 3. \]  

(4.27)

Using the definitions (4.9) we rewrite these equations in the form

\[ \mathcal{X}_{jk} = \delta_{jk} + \frac{1}{z^m} \int_0^2 K_{jk}(x,s,z)(\Phi \mathcal{X})_{jk}(s,z)ds, \quad j, k = 1, 2, 3. \]  

(4.28)

The definition (4.8) yields that the function \( \mathcal{X} \) satisfies equation (4.10). Conversely, let \( \mathcal{X} \) satisfy equation (4.10). In terms of matrix entries equation (4.10) has the form (4.28). Substituting the definitions (4.9) into (4.28) we obtain that the functions \( \mathcal{X}_{jk} \) satisfy (4.20), (4.27), which yields (4.23). Differentiating the identities (4.20), (4.27), we get that \( \mathcal{X}_{jk} \) satisfy equations (4.25). Then \( \mathcal{X} \) satisfies equation (4.7).

ii) Let the matrix-valued function \( \mathcal{X} \) be the solution of the integral equation (4.10). Then \( \mathcal{X} \) satisfies the differential equation (4.7) and the initial conditions (4.23). Then the matrix \( \mathcal{M} \), given by (4.22), satisfies the differential equation (4.5) and the initial condition \( \mathcal{M}(0,z) = \mathbb{I}_3 \).

Conversely, let \( \mathcal{M} \) satisfy equation (4.5) and let \( \mathcal{M} \) has the the form (4.24). Substituting (4.24) into (4.5) we obtain that \( \mathcal{X} \) satisfies the differential equation (4.7), and then \( \mathcal{X}(x,z) \) is the solution of the integral equation (4.10).

**Remark.** Recall that the matrix-valued functions \( \mathcal{X}(x,z) \) and \( \mathcal{X}^{-1}(x,z) \) are uniformly bounded on \( [0,2] \times \mathcal{Z}_+(r) \), see Lemma (4.4) i). Thus the formula (4.24) represents the fundamental matrix in the form of product of the bounded matrices \( \mathcal{X}(x,z) \) and \( \mathcal{X}^{-1}(0,z) \) and the diagonal matrix \( e^{\int_0^z \Theta(s,z)ds} \) which contains all exponentially increasing and exponentially
4.4. Asymptotics of the solutions $\phi_j$. Always in this paper, see (5.2), (6.9) and (7.8), the matrix-valued function $A$ in (3.6) has the form

$$A(x, z) = \Omega(z)(\mathbb{I}_3 + \frac{1}{z^2} \mathcal{W}(x, z)) \mathcal{X}(x, z)e^{\int_0^x \Theta(s, z)ds}, \quad (x, z) \in [0, 2] \times \mathbb{Z}_+,$$  

(4.29)

where $\mathcal{W} = 0$ in (5.2), $\mathcal{W} = \frac{m}{3} W_1$ in (6.9) and $\mathcal{W} = \frac{m}{3} W_1 + \frac{1}{3z} W_2$ in (7.8). In the following Lemma we determine asymptotics of the Jost type solutions $\phi_j$, given by (3.7), when $A$ satisfies (4.29).

**Lemma 4.6.** Let $p, q \in L^1(\mathbb{T})$. Let the $3 \times 3$ matrix-valued function $\mathcal{W}(x, z)$ satisfies: $\mathcal{W}(\cdot, z) \in L^1(\mathbb{T})$ for all $z \in \mathbb{Z}_+$, $\mathcal{W}(x, \cdot)$ is analytic in $\mathbb{Z}_+$ and $\mathcal{W}(x, z) = O(1)$ as $|z| \to \infty$, $z \in \mathbb{Z}_+$ uniformly in $x \in \mathbb{T}$. Let the matrix-valued function $\mathcal{X}$ be the solution of equation (4.10) for some $m = 1, 2, 3$ and $\Theta$ and $\Phi$ satisfying Condition 4.1. Let the matrix-valued function $A$ satisfies (4.29), and let $\phi_j = A_{1j}, j = 1, 2, 3$. Then

$$\phi_j(x, z) = e^{\int_0^x \Theta_j(s, z)ds} \left(1 + \zeta_j(x, z) + O(z^{-m-2})\right),$$  

(4.30)

as $|z| \to \infty, z \in \mathbb{Z}_+$, uniformly in $x \in [0, 2]$, where

$$\zeta_j(x, z) = \frac{\eta_j(x, z)}{z^m} + \frac{1}{z^2} \sum_{k=1}^3 \mathcal{W}_{kj}(x, z),$$  

(4.31)

and $\eta_j$ are given by (4.15).

**Proof.** Let $|z| \to \infty, z \in \mathbb{Z}_+$, and let $m = 1, 2, 3$. Substituting the definition (2.3) and the asymptotics (4.11) into the definition (4.29) we obtain

$$\phi_j(x, z) = A_{1j}(x, z) = e^{\int_0^x \Theta_j(s, z)ds} \sum_{k, \ell=1}^3 \left(\delta_{k\ell} + \frac{1}{z^2} \mathcal{W}_{k\ell}(x, z)\right) \left(\delta_{\ell j} + \frac{1}{z^m} \mathcal{B}_{\ell j}(x, z) + O(z^{-2m})\right)$$

$$= e^{\int_0^x \Theta_j(s, z)ds} \sum_{k, \ell=1}^3 \left(\delta_{k\ell}\delta_{\ell j} + \frac{1}{z^2} \mathcal{W}_{k\ell}(x, z)\delta_{\ell j} + \frac{1}{z^m} \delta_{k\ell}\mathcal{B}_{\ell j}(x, z) + O(z^{-2m})\right),$$

uniformly in $x \in [0, 2]$, which yields (4.30). $\blacksquare$

5. Eigenvalues for $p, q \in L^1(\mathbb{T})$

5.1. Factorization of the fundamental matrix. In this Section we consider the case $p, q \in L^1(\mathbb{T})$. We apply Theorem 4.3 to equation (4.1) in order to obtain the factorization of the fundamental matrix. Preliminary, we have to replace the diagonal part of the matrix $P$ from the right side of (4.1) onto the left one in order to get the off-diagonal matrix $\mathcal{F}$ in the asymptotics of $\Phi$ in (4.4) (see (5.4), where the matrix $P - 2\mathcal{T}^2$ is off-diagonal). Then we obtain the following corollary of Theorem 4.5 on the factorization of the fundamental matrix.

**Corollary 5.1.** Let $p, q \in L^1(\mathbb{T})$. Let $r > 0$ be large enough and let $z \in \mathbb{Z}_+(r)$. Then the following representation of the fundamental solution holds true:

$$M(x, \lambda) = A(x, z)A^{-1}(0, z),$$  

(5.1)
for all \( x \in [0, 2] \), where \( \lambda = z^3 \),

\[
A(x, z) = \Omega(z) \mathcal{X}(x, z) e^{z \int_0^x T_1(s, z) \, ds},
\]

where \( \Omega \) is given by the definition (2.3), the diagonal \( 3 \times 3 \) matrix-valued function \( T_1 \) has the form

\[
T_1 = T - \frac{2p}{3z^2} T^2,
\]

and \( \mathcal{X} \) is the solution of the integral equation (4.10) with \( m, \Theta \) and \( \Phi \) given by

\[
m = 1, \quad \Theta = T_1, \quad \Phi = -\frac{1}{3} \left( p(P - 2T^2) + \frac{q}{z} Q \right).
\]

Each function \( A(x, \cdot) \), \( x \in [0, 2] \), is analytic in \( \mathbb{Z}_+(r) \).

**Proof.** Rewrite equation (4.1) in the form

\[
Y' - zT_1 Y = -\frac{1}{3z} \left( p(P - 2T^2) + \frac{q}{z} Q \right) Y.
\]

Equation (5.5) has the form (4.3) with \( m, \Theta, \Phi \) satisfying (5.4). Let \( Y \) be the solution of equation (5.5) satisfying the condition \( Y(0) = I_3 \). Theorem 1.5 shows that \( Y \) satisfies the identity (4.24). Substituting (4.24) into the definition (2.5) we obtain the representation (5.1).

Theorem 2.1 follows immediately from the previous result.

**Proof of Theorem 2.1.** The asymptotics (4.11) gives (2.11). Substituting the asymptotics (2.11) into the identity (5.2) we obtain (2.4).
5.2. **Counting Lemma.** Let $p, q \in L^1(\mathbb{T})$. Let the matrix-valued function $A(x, z)$ be given by (5.2). Now we prove the Counting Lemma for the eigenvalues. Introduce the domains $D_n, n \in \mathbb{Z},$ by

$$D_n = \left\{ \lambda \in \mathbb{C} : |z - \nu n| < \frac{\nu}{4} \right\},$$

$$D_{-n} = \left\{ \lambda \in \mathbb{C} : |z - e^{i\pi} \nu n| < \frac{\nu}{4} \right\} \cup \left\{ \lambda \in \mathbb{C} : |z - e^{-i\pi} \nu n| < \frac{\nu}{4} \right\},$$

where $n \geq 0, \nu = \frac{2\pi}{\sqrt{3}},$ see (1.9), and the contours

$$C_a = \{ \lambda \in \mathbb{C} : |z - a| = r \}, \quad a \in \mathbb{C}, \quad r > 0.$$ Below we need the identity

$$\det \Omega = 3(\tau^2 - \tau)z^3 = -i3\sqrt{3}z^3,$$

which follows from the definition (2.3).

**Lemma 5.2.** Let $p, q \in L^1(\mathbb{T}).$ Then

i) The solutions $\phi_j$ of equation (1.3) and the function $\det A(0, z)$ satisfy

$$\det A(0, z) = -i3\sqrt{3}z^3(1 + O(z^{-1})), \quad \phi_j(x, z) = e^{z \tau_j} (1 + O(z^{-1})),$$

as $j = 1, 2, 3, |z| \to \infty, z \in \mathbb{Z},$ uniformly in $x \in [0, 2]$. Moreover,

$$D(\lambda) = D^0(\lambda)(1 + O(z^{-1})).$$

as $|\lambda| \to \infty$ and $\lambda \in \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} D_n$.

ii) For each integer $N > 0$ large enough the function $D$ has (counting with multiplicities) $2N + 1$ zeros on the domain $\{|\lambda| < \nu^2(N + \frac{1}{2})^3\}$ and for each $|n| > N$ exactly one simple zero in the domain $D_n$. There are no other zeros.

iii) There exists $N \in \mathbb{N}$ such that each $\mu_n, |n| > N$, is real.

**Proof.** i) We have $m = 1, \Theta = T_1 = \mathcal{T} + O(z^{-2})$ and $\mathcal{W} = 0$ in Lemma 4.6. Then the asymptotics (4.30) and (4.31) give (5.9). Then

$$\phi(z) = \begin{pmatrix} 1 + O(z^{-1}) & 1 + O(z^{-1}) & 1 + O(z^{-1}) \\ e^{\tau_1 z}(1 + O(z^{-1})) & e^{\tau_2 z}(1 + O(z^{-1})) & e^{\tau_3 z}(1 + O(z^{-1})) \\ e^{2\tau_1 z}(1 + O(z^{-1})) & e^{2\tau_2 z}(1 + O(z^{-1})) & e^{2\tau_3 z}(1 + O(z^{-1})) \end{pmatrix}.$$ 

Let $|z| \to \infty, z \in \mathbb{Z}_+, \lambda \in \mathbb{C} \setminus \cup_{n \in \mathbb{Z}} D_n$. Then the identity (5.5) gives

$$\det \phi(z) = -i3\sqrt{3}z^3D^0(\lambda)(1 + O(z^{-1})).$$

Moreover,

$$\det A(0, z) = \det \Omega(z) \det X(0, z) = -i3\sqrt{3}z^3(1 + O(z^{-1})), $$

which yields (5.8), here we used $\det \Omega = -i3\sqrt{3}z^3$, see (5.7). Substituting these asymptotics into the identity (5.9) we obtain (5.10) for $\lambda \in \mathbb{C}_+$. The identity $D(\lambda) = D^0(\lambda)$ gives (5.10) for $\lambda \in \mathbb{C}_-.$
ii) Let \( N \geq 0 \) be integer and large enough and let \( N' > N \) be another integer. Let \( \lambda \) belong to the contours \( C_{0}(\nu^{3}(N + \frac{1}{2})^{3}), C_{0}(\nu^{3}(N' + \frac{1}{2})^{3}) \) and \( \partial D_{n} \) for all \(|n| > N\). Asymptotics (5.10) yields

\[
|D(\lambda) - D^{0}(\lambda)| = |D^{0}(\lambda)| \left| \frac{D(\lambda)}{D^{0}(\lambda)} - 1 \right| = |D^{0}(\lambda)|O(|z|^{-1}) < |D^{0}(\lambda)|
\]
on all contours. Hence, by Rouché’s theorem, \( D \) has as many zeros, as \( D^{0} \) in each of the bounded domains and the remaining unbounded domain. Since \( D^{0} \) has exactly one simple zero in each \( D_{n}, n \in \mathbb{Z} \), and since \( N' > N \) can be chosen arbitrarily large, the statement follows.

iii) Let \(|n| > N\). The definition (1.8) and the statement i) show that the zero \( \mu_{n} \) of the function \( D \) satisfies: \( \mu_{n} \in D_{n} \). If \( \mu_{n} \notin \mathbb{R} \), then \( \overline{\mu}_{n} \) is also a zero of \( D \) and \( \overline{\mu}_{n} \in D_{n} \). Then there are two zeros of \( D \) in \( D_{n} \), which contradicts the statement i).

5.3. Rough eigenvalue asymptotics. Recall that the eigenvalues of the operator \( H \) are zeros of the entire function \( D \) given by the definition (1.7). The identity (3.9) and the asymptotics (5.8) show that the large eigenvalues are zeros of the function \( \det \phi(z) \). The function \( \det \phi(z) \) is analytic in \( Z_{+}(r) \), where \( r > 0 \) is large enough. Using the asymptotic behavior of the function \( \phi_{3}(x, z) \) at high energy we reduce in Lemma 5.3 the determinant of the \( 3 \times 3 \)-matrix \( \phi(z) \) to the determinant of a \( 2 \times 2 \)-matrix.

The asymptotics (5.9) and the definitions (2.2) give

\[
|\phi_{3}(x, z)| = e^{\Re z}(1 + O(z^{-1})), \quad (5.12)
\]
as \( |z| \to \infty, z \in Z_{+} \). The asymptotics (5.12) show that \(|\phi_{3}(x, z)| > 0 \) for all \((x, z) \in [0, 2] \times Z_{+}(r)\). Then the function \( \xi(z) \), given by

\[
\xi(z) = \frac{\det \phi(z)}{\phi_{3}(2, z)}, \quad (5.13)
\]
is analytic in \( Z_{+}(r) \), where \( r > 0 \) is large enough. Let \( \mu_{n} \) be the eigenvalue of the operator \( H \). Then the identity (3.9) and the asymptotics (5.8) give

\[
\xi(\mu_{n}^{1/3}) = 0 \quad (5.14)
\]
for all \( n \in \mathbb{N} \) large enough. The symmetry (3.2) shows that it is sufficiently to determine the asymptotics for the large positive eigenvalues. Introduce the sector

\[
Z_{+}^{+} = \{ z \in \mathbb{C} : \arg z \in [0, \frac{\pi}{6}] \}.
\]

**Lemma 5.3.** Let \( p, q \in L^{1}(\mathbb{T}) \).

i) Let, in addition, \(|z| \to \infty \) and \( z \in Z_{+}^{+} \). Then

\[
\xi(z) = \det \begin{pmatrix} \phi_{1}(0, z) & \phi_{2}(0, z) \\ \phi_{1}(1, z) & \phi_{2}(1, z) \end{pmatrix} + O(e^{-\Re z}). \quad (5.15)
\]

ii) The eigenvalues \( \mu_{n} \) satisfy

\[
\mu_{n} = (\nu n + O(n^{-1}))^{3}, \quad (5.16)
\]
as \( n \to +\infty, \nu = \frac{2\pi}{\sqrt{3}} \).
Proof. i) Let $|z| \to \infty, z \in \mathbb{Z}$. The asymptotics (5.9) and the definitions (2.2) give
\[
|\phi_1(x, z)| = e^{-Re z^{-3/2}(1 + O(z^{-1}))}, \quad |\phi_2(x, z)| = e^{-Re z^{-3/2}(1 + O(z^{-1}))}.
\]
Let $|z| \to \infty, z \in \mathbb{Z}^+$. Then $\text{Im} z \leq \frac{Re z}{\sqrt{3}}$ and substituting these asymptotics into the definition (5.8) we obtain
\[
\det \phi(z) = \begin{pmatrix}
\phi_1(0, z) & \phi_2(0, z) & O(1) \\
\phi_1(1, z) & \phi_2(1, z) & e^{Re z}O(1) \\
e^{-Re z^{-3/2}Im z}O(1) & e^{-Re z^{-3/2}Im z}O(1) & 1
\end{pmatrix},
\]
The asymptotics (5.12) implies
\[
\xi(z) = \begin{pmatrix}
\phi_1(0, z) & \phi_2(0, z) & e^{-2Re z}O(1) \\
\phi_1(1, z) & \phi_2(1, z) & e^{-Re z}O(1) \\
e^{-Re z^{-3/2}Im z}O(1) & e^{-Re z^{-3/2}Im z}O(1) & 1
\end{pmatrix},
\]
which yields (5.15).

ii) Let $\lambda = z^n = \mu_n, n \to +\infty$. Lemma 5.2 ii) yields $z = \nu n + \delta_n, \delta_n = O(1)$ and $\delta_n$ is real. The asymptotics (5.9) implies
\[
\phi_1(0, z) = 1 + O(n^{-1}), \quad \phi_2(0, z) = 1 + O(n^{-1}),
\]
\[
\phi_1(1, z) = e^{-\frac{1 + i\sqrt{3}}{2}(\nu n + \delta_n)}(1 + O(n^{-1})) = (-1)^n e^{-\frac{\pi}{2}n} \left(1 - \frac{1 - i\sqrt{3}}{2} \delta_n + O(\delta_n^2) + O(n^{-1})\right),
\]
\[
\phi_2(1, z) = e^{-\frac{1 + i\sqrt{3}}{2}(\nu n + \delta_n)}(1 + O(n^{-1})) = (-1)^n e^{-\frac{\pi}{2}n} \left(1 + \frac{1 + i\sqrt{3}}{2} \delta_n + O(\delta_n^2) + O(n^{-1})\right).
\]
Substituting these asymptotics into (5.15) we obtain
\[
\xi(z) = (-1)^n e^{-\frac{\pi}{2}n} \left(-i\sqrt{3} \delta_n + O(\delta_n^2) + O(n^{-1})\right).
\]
The identity $\xi(z) = 0$, see (5.14), gives $\delta_n = O(n^{-1})$, which yields (5.16).

5.4. Eigenvalue asymptotics. Now we determine eigenvalue asymptotics for the case $p, q \in L^1(\mathbb{T})$. Introduce the Fourier coefficients
\[
f_0 = \int_0^1 f(x)dx, \quad \hat{f}_n = \int_0^1 e^{-i2\pi nx}f(x)dx, \quad n \in \mathbb{Z}.
\]
Note that the coefficients, given by (1.10), satisfy
\[
\hat{f}_n = \text{Re} \hat{f}_n - \frac{\text{Im} \hat{f}_n}{\sqrt{3}}, \quad n \in \mathbb{N}.
\]

Lemma 5.4. Let $p, q \in L^1(\mathbb{T})$. Then the eigenvalues $\mu_n$ satisfy the asymptotics (1.11).

Proof. Let $p, q \in L^1(\mathbb{T})$ and let the matrix-valued function $A$ be given by (5.2). Then $\mathcal{W} = 0$ in the asymptotics (4.20) and in the considered case we have
\[
m = 1, \quad \mathcal{F} = -\frac{p}{3}(p - 2\mathcal{T}), \quad \Theta = \mathcal{T} - \frac{2p}{3z^2}\mathcal{T},
\]
see (5.4) and (5.3). The identity (4.31) gives
\[
\zeta_j(x, z) = \frac{\eta_j(x, z)}{z}, \quad j = 1, 2, 3, \quad (5.17)
\]
where \( \eta_j \) are given by (4.15). Let

\[
\lambda = \frac{x^3}{\sqrt{\pi n}}, \quad n \to +\infty.
\]

Lemma (5.2) v) shows \( z = \frac{2\pi n}{\sqrt{\Delta}} + \delta_n, \delta_n = O(n^{-1}) \). Substituting the asymptotics (1.16) into the identity (5.14) and using the definitions (4.2) we obtain

\[
\zeta_1(x, z) = \frac{1}{3z} \int_x^2 e^{i2\pi n(s-x)}p(s)P_{21}ds + O(n^{-\frac{3}{2}}),
\]

\[
\zeta_2(x, z) = \frac{1}{3z} \int_0^x e^{i2\pi n(x-s)}p(s)P_{12}ds + O(n^{-\frac{3}{2}}),
\]

Then

\[
\zeta_1(0, z) = -\frac{2\tau^2}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}), \quad \zeta_2(0, z) = O(n^{-\frac{3}{2}}),
\]

\[
\zeta_1(1, z) = -\frac{\tau^2}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}), \quad \zeta_2(1, z) = \frac{\tau}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}),
\]

and the asymptotics (4.30) gives

\[
\phi_1(0, z) = 1 - \frac{2\tau^2}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}), \quad \phi_2(0, z) = 1 + O(n^{-\frac{3}{2}}),
\]

\[
\phi_1(1, z) = e^{\tau z - \frac{2\pi n}{3x^2} \tau^2} \left( 1 - \frac{\tau^2}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}) \right), \quad \phi_2(1, z) = e^{\tau z - \frac{2\pi n}{3x^2} \tau^2} \left( 1 + \frac{\tau}{3z} \hat{p}_n + O(n^{-\frac{3}{2}}) \right).
\]

Substituting these asymptotics into (5.15) and using the identities

\[
e^{\tau z} = e^{-\frac{\tau^2}{2} + i\frac{\tau^2}{2}z}, \quad e^{\tau z} = (-1)^n e^{-\frac{\tau^2}{2} + i\frac{\tau^2}{2}z},
\]

we obtain

\[
\xi(z) = (-1)^n e^{-\frac{\tau^2}{2} + i\frac{\tau^2}{2}z} \left( \xi_n + O(e^{-\frac{3\tau}{3x^2}}) \right), \quad \xi_n = \det \left( \begin{array}{cc}
1 - \frac{2\pi n}{3x^2} \hat{p}_n + O(n^{-\frac{3}{2}}) & 1 + O(n^{-\frac{3}{2}}) \\
e^{i\frac{\tau^2}{2}z} \hat{p}_n + O(n^{-\frac{3}{2}}) & e^{-i\frac{\tau^2}{2}z} \hat{p}_n + O(n^{-\frac{3}{2}}) \end{array} \right)
\]

Using the asymptotics \( e^{i\frac{\tau^2}{2}z} \hat{p}_n = 1 \pm i \sqrt{3} \delta_n + O(n^{-2}) \) we obtain

\[
\xi_n = \det \left( \begin{array}{cc}
1 - \frac{2\pi n}{3x^2} \hat{p}_n + O(n^{-\frac{3}{2}}) & 1 + O(n^{-\frac{3}{2}}) \\
1 + i\frac{\sqrt{3}}{2} \delta_n + i\frac{\pi n}{\sqrt{3}} & \frac{\sqrt{3}}{2} \hat{p}_n + O(n^{-\frac{3}{2}}) \end{array} \right)
\]

\[
= -i \sqrt{3} \delta_n - \frac{i2\pi n}{3\sqrt{3}z} + \tau \hat{p}_n + O(n^{-\frac{3}{2}}) = -i \sqrt{3} \left( \delta_n + \frac{2\pi n \tau}{3\sqrt{3}z} + \frac{2 \Im(\tau \hat{p}_n)}{3\sqrt{3}z} \right) + O(n^{-\frac{3}{2}}).
\]

The identities (5.14) and (5.18) imply \( \xi_n = O(e^{-\frac{3\tau}{3x^2}}) \), which gives

\[
\delta_n = -\frac{2\pi n \tau}{3z} + \frac{2 \Im(\tau \hat{p}_n)}{3\sqrt{3}z} + O(n^{-\frac{3}{2}}) = -\frac{p_0}{3\sqrt{3}z} + \Im(\hat{p}_n) + O(n^{-\frac{3}{2}}).
\]

Using the identity \( \Im(\tau \hat{p}_n) = \frac{1}{2} (\sqrt{3} \Re \hat{p}_n - \Im \hat{p}_n) \) we obtain

\[
z = \nu n - \frac{p_0}{3\sqrt{3}z} + \frac{\sqrt{3} \Re \hat{p}_n - \Im \hat{p}_n}{6\pi n} + O(n^{-\frac{3}{2}}),
\]

which yields the asymptotics (1.11) as \( n \to +\infty \).
This identity (3.2) and \( \hat{\beta}_n = \hat{\beta}_{-n} \) give the asymptotics (1.11) for \( n \to -\infty \).

6. THE CASE \( p', q \in L^1(\mathbb{T}) \)

6.1. Transformation of the differential equation. Integrating by part in the integral operator \( K \) of the integral equation (4.10), given by the definition (4.8), and repeating the previous analysis we obtain the asymptotics for the case of smooth coefficients. However, in order to simplify the calculations we use the other method based on an additional transformation of the differential equation, see Fedoryuk [F12, Ch V.1.3].

Let \( p', q \in L^1(\mathbb{T}) \), let \( r > 0 \) be large enough and let \( z \in \mathbb{Z}_+(r) \). Introduce the matrix-valued function \( U_1(x, z), x \in \mathbb{R} \) by

\[
U_1(x, z) = \frac{i}{\sqrt{3}} \begin{pmatrix} 0 & \tau & -\tau \\ -\tau^2 & 0 & \tau^2 \\ 1 & -1 & 0 \end{pmatrix},
\]

(6.1)

The matrix \( U_1(x, z) \) is invertible at large \(|z|\) and we introduce the matrix \( Y_1(x, z) \) by the identity

\[
Y(x, z) = U_1(x, z)Y_1(x, z)U_1^{-1}(0, z), \quad (x, z) \in \mathbb{R} \times \mathbb{Z}_+(r),
\]

(6.3)

where \( Y \) the solution of the problem (4.1).

**Lemma 6.1.** Let \( p', q \in L^1(\mathbb{T}) \) and let \( z \in \mathbb{Z}_+(r) \), where \( r > 0 \) is large enough. Then the matrix-valued function \( Y_1(x, z) \), given by (6.3), satisfies the equation

\[
Y_1' - zT_1Y_1 = \frac{1}{z^2}\Phi_1 Y_1, \quad Y_1(0, z) = \mathbb{I}_3,
\]

(6.4)

where

\[
\Phi_1(x, z) = P_1(x) - \frac{p^2(x)}{9z}(PW_1 - 2W_1T^2) + O(z^{-2})
\]

the asymptotics is uniform on \( x \in \mathbb{R}, T_1 \) is given by the definition (5.3) and

\[
P_1(x) = -\frac{1}{3}(q(x)Q + p'(x)W_1).
\]

(6.5)

**Proof.** Let \( z \in \mathbb{Z}_+(r) \). Substituting (6.3) into equation (4.1) we obtain

\[
Y_1' = U_1^{-1}A_1 Y_1, \quad A_1(x, z) = \left( zT - \frac{p(x)}{3z}P - \frac{q(x)}{3z^2}Q \right) U_1(x, z) - \frac{p'(x)}{3z^2}W_1.
\]

(6.6)

Let \(|z| \to \infty\). Then

\[
A_1(x, z) = zT + \frac{p(x)}{3z}(TW_1 - P) + \frac{P_1(x)}{z^2} - \frac{p^2(x)}{9z^3}PW_1 + O(z^{-4}),
\]

and

\[
U_1^{-1}(x, z) = \mathbb{I}_3 - \frac{p(x)}{3z^2}W_1 + \frac{p^2(x)}{9z^4}W_1^2 + O(z^{-6}).
\]
uniformly in \( x \). We have

\[
U^{-1}_1(x, z)A_1(x, z) = z \mathcal{T} - \frac{p(x)}{3z}(W_1, \mathcal{T}) + P + \frac{P_1(x)}{z^2} + \frac{p^2(x)}{9z^3}(W_1[W_1, \mathcal{T}] + [W_1, P]) + O(z^{-4}),
\]

(6.7)

here and below \([A, B] = AB - BA\). The matrix \( W_1 \) satisfies

\[
[W_1, \mathcal{T}] + P = \text{diag}(P_{jj})_{j=1}^3 = 2\mathcal{T}^2.
\]

(6.8)

Substituting (6.8) into (6.7) we obtain

\[
U^{-1}_1(x, z)A_1(x, z) = z \mathcal{T} - 2\frac{p(x)}{3z}\mathcal{T}^2 + \frac{P_1(x)}{z^2} - \frac{p^2(x)}{9z^3}(PW_1 - 2W_1\mathcal{T}^2) + O(z^{-4}).
\]

Substituting this identity into equation (6.6) we obtain (6.4).

6.2. Factorization of the fundamental matrix. Now we apply Theorem 4.5 to equation (6.4) in order to obtain the factorization of the fundamental matrix. Preliminary, as well as in the case of equation (4.1) (see Sect 5.1) we have to replace the diagonal part \(-\frac{q}{3} \mathcal{T}\) of the matrix \( P_1 \) in the right side of (6.4) into the left one in order to get the off-diagonal matrix \( F \) in the asymptotics of \( \Phi \) in (4.4). Then we obtain the following corollary of Theorem 4.5 on the factorization of the fundamental matrix.

**Corollary 6.2.** Let \( p', q \in L^1(\mathbb{T}) \) and let \( z \in \mathbb{Z}_+(r) \), where \( r > 0 \) is large enough. Then the fundamental matrix \( M \) has the representation \( M(x, \lambda) = A(x, z)A^{-1}(0, z) \), where \( \lambda = z^3 \) and the matrix-valued function \( A \) has the form

\[
A(x, z) = \Omega(z)U_1(x, z)\mathcal{X}(x, z)e^{\int_0^x T_2(s, z)ds}, \quad (x, z) \in [0, 2] \times \mathbb{Z}_+(r).
\]

(6.9)

\[
T_2 = \mathcal{T} - \frac{2p(x)}{3z^2}\mathcal{T}^2 - \frac{q}{3z^3}\mathcal{T},
\]

(6.10)

and \( \mathcal{X} \) is the solution of the integral equation (4.10) with

\[
m = 2, \quad \Theta = T_2, \quad \Phi = \Phi_1 + \frac{q}{3}\mathcal{T}.
\]

(6.11)

Each function \( A(x, \cdot), x \in [0, 2], \) is analytic in \( \mathbb{Z}_+(r) \), where \( r > 0 \) is large enough.

**Proof.** Rewrite equation (6.4) in the form

\[
Y'_1 - zT_2Y_1 = \frac{1}{z^2}\left(\Phi_1 + \frac{q}{3}\mathcal{T}\right)Y_1,
\]

(6.12)

Equation (6.12) has the form (4.15) with \( m, \Theta, \Phi \) satisfying (6.11). Let \( Y_1 \) be the solution of equation (6.12) satisfying the condition \( Y_1(0) = \mathbb{I}_r \). Theorem 4.5 shows that \( Y_1 \) satisfies the identity (4.24). The identities (2.30) and (6.3) imply the identity (6.6), where \( A \) satisfies (6.9).
Lemma 6.3. Let \( p', q \in L^1(\mathbb{T}) \) and let \( n \to \pm \infty \). Then the eigenvalues \( \mu_n \) satisfy
\[
\mu_n = \mu_n^0 - 2\nu np_0 + q_0 + \text{Re} \left( \frac{\hat{P}_n}{3} - \hat{q}_n \right) + \frac{\text{Im} ( \hat{q}_n + \hat{p}_n )}{\sqrt{3}} + O(n^{-\frac{3}{2}}). \tag{6.13}
\]

Proof. Let \( p', q \in L^1(\mathbb{T}) \). Without loss of generality we assume that \( \int_0^1 q(t) dt = 0 \). Let the matrix-valued function \( \Phi \) be given by (6.9). Then \( \mathcal{W} = \frac{i}{8} \mathcal{W}_1 \) in the asymptotics (4.29) and, due to (6.11), in the considered case we have
\[
m = 2, \quad \mathcal{F} = P_1 + \frac{q}{3} \mathcal{T}, \quad \Theta = \mathcal{T} - \frac{2p}{3z^2} \mathcal{T}_2 - \frac{q}{3z^3} \mathcal{T},
\]
\( P_1 \) is given by (6.5). The identity (4.31) gives
\[
\zeta_j(x, z) = \frac{\eta_j(x, z)}{z^2} + \frac{p(x) \eta_{1,j}}{3z^2}, \quad \eta_{1,j} = \sum_{m=1}^3 W_{1,mj},
\]
and \( \eta_j \) are given by the definition (4.15). The definition (6.2) gives
\[
\eta_{1,1} = \frac{\varphi}{\sqrt{3}}, \quad \eta_{1,2} = \frac{\varphi}{\sqrt{3}}. \tag{6.16}
\]
Let \( \lambda = z^3 = \mu_n, n \to +\infty \). The asymptotics (1.11) show that \( z = \nu n - \frac{\nu n}{\sqrt{3} \nu n} + \delta_n, \delta_n = o(n^{-1}) \). The identity (4.16) and the definitions (4.2) and (6.2) imply
\[
\eta_1(x, z) = -\int_x^2 e^{i2\pi n(s-x)} P_{1,21}(s) ds + O(n^{-\frac{3}{2}}) = -\frac{i}{3\sqrt{3}} \int_x^2 e^{i2\pi n(s-x)} h(s) ds + O(n^{-\frac{3}{2}}),
\]
\[
\eta_2(x, z) = \int_0^x e^{i2\pi n(x-s)} P_{1,12}(s) ds + O(n^{-\frac{3}{2}}) = -\frac{i}{3\sqrt{3}} \int_0^x e^{i2\pi n(x-s)} h(s) ds + O(n^{-\frac{3}{2}}),
\]
where we used the identities
\[
P_{1,21} = \frac{i \varphi}{3\sqrt{3}}, \quad P_{1,12} = -\frac{i \varphi}{3\sqrt{3}}.
\]
Substituting these asymptotics into the identity (6.15) and using the identities (6.16) we obtain
\[
\zeta_1(x, z) = \frac{\varphi p(x)}{3\sqrt{3}z^2} - \frac{i}{3\sqrt{3}z^2} \int_x^2 e^{i2\pi n(s-x)} h(s) ds + O(n^{-\frac{3}{2}}),
\]
\[
\zeta_2(x, z) = \frac{\varphi p(x)}{3\sqrt{3}z^2} - \frac{i}{3\sqrt{3}z^2} \int_0^x e^{i2\pi n(x-s)} h(s) ds + O(n^{-\frac{3}{2}}).
\]
Then the asymptotics (1.30) gives
\[
\phi_1(0, z) = 1 + \frac{\varphi p(0) - 2i \varphi n}{3\sqrt{3}z^2} + O(n^{-\frac{3}{2}}), \quad \phi_1(1, z) = e^{\tau z - 2\pi n z^2} \left( 1 + \frac{\varphi p(0) - i \varphi n}{3\sqrt{3}z^2} + O(n^{-\frac{3}{2}}) \right),
\]
\[
\phi_2(0, z) = 1 + \frac{\varphi p(0)}{3\sqrt{3}z^2} + O(n^{-\frac{3}{2}}), \quad \phi_2(1, z) = e^{\tau z - 2\pi n z^2} \left( 1 + \frac{\varphi p(0) - i \varphi n}{3\sqrt{3}z^2} + O(n^{-\frac{3}{2}}) \right)
\]
where we used (5.3). Substituting these asymptotics into (5.15) and using the identities
\[ e^{-\frac{2\pi}{3}z^2} = (-1)^ne^{-\frac{2\pi}{4} + \frac{2\pi}{3}e^{i\frac{\sqrt{3}}{2}\delta_n}}, \quad e^{-\frac{2\pi}{3}z^2} = (-1)^ne^{-\frac{2\pi}{4} + \frac{2\pi}{3}e^{-i\frac{\sqrt{3}}{2}\delta_n}} \]
and \( \int_0^1 q(s)ds = 0 \) we obtain
\[ \xi(z) = (-1)^n e^{-\frac{2\pi}{4} + \frac{2\pi}{3}(\xi_n + e^{-\frac{2\pi}{3}O(1)}), \]
where
\[ \xi_n = \text{det} \left( 1 + \frac{\mp[(0)-ih_n]}{3\sqrt{3}z^2} + O(n^{-\frac{5}{2}}) \right) \]
\[ = -i\sqrt{3}\delta_n - \frac{i(h_n + \hat{h}_n)}{3\sqrt{3}z^2} + O(n^{-\frac{5}{2}}) = -i\sqrt{3}\delta_n - \frac{i2\text{Re}\hat{h}_n}{\sqrt{3}(2\pi n)^2} + O(n^{-\frac{5}{2}}). \]
The identity (5.14) implies \( \xi_n + e^{-\frac{2\pi}{3}O(1)} = 0 \), which gives
\[ \delta_n = -\frac{2\text{Re}\hat{h}_n}{3(2\pi n)^2} + O(n^{-\frac{5}{2}}). \] (6.18)
The definition (6.13) gives
\[ \text{Re}\hat{h}_n = \sqrt{3}\text{Im}(\tau\hat{q}_n) + \text{Re}(\tau\hat{p}_n). \]
Using the identities
\[ \text{Im}(\tau\hat{q}_n) = \frac{1}{2}(\sqrt{3}\text{Re}\hat{q}_n - \text{Im}\hat{q}_n), \quad \text{Re}(\tau\hat{p}_n) = -\frac{1}{2}(\sqrt{3}\text{Im}\hat{p}_n + \text{Re}\hat{p}_n) \]
we obtain
\[ \text{Re}\hat{h}_n = \frac{\sqrt{3}}{2}\left( \sqrt{3}\text{Re}\left(\hat{q}_n - \frac{\hat{p}_n}{3}\right) - \text{Im}(\hat{p}_n + \hat{q}_n) \right). \]
Substituting this identity into (6.18) we obtain
\[ \delta_n = -\frac{1}{(2\pi n)^2}\left( \text{Re}\left(\hat{q}_n - \frac{\hat{p}_n}{3}\right) - \frac{\text{Im}(\hat{p}_n + \hat{q}_n)}{\sqrt{3}} \right) + O(n^{-\frac{5}{2}}), \]
which gives
\[ z = \nu n - \frac{p_0}{\sqrt{3}\pi} - \frac{1}{(2\pi n)^2}\left( \text{Re}\left(\hat{q}_n - \frac{\hat{p}_n}{3}\right) - \frac{\text{Im}(\hat{p}_n + \hat{q}_n)}{\sqrt{3}} \right) + O(n^{-\frac{5}{2}}). \]
This yields the asymptotics (6.14) as \( n \to +\infty \). The identities (312) and \( \hat{q}_n = \hat{q}_{-n}, (p^-)_n = -p'_{-n} \) give the asymptotics (6.14) for \( n \to -\infty \).

7. The case \( p'', q' \in L^1(\mathbb{T}) \)

7.1. Transformation of the differential equation. In the previous Section we determined eigenvalue asymptotics for the case \( p', q \in L^1(\mathbb{T}) \). Using the similar arguments in this Section we determine eigenvalue asymptotics for the case \( p'', q' \in L^1(\mathbb{T}) \). First of all, in order to avoid the difficulties arising from the application of integration by parts, we transform the differential equation.
Let \( p'', q' \in L^1(\mathbb{T}) \). Let \( r > 0 \) be large enough and let \( z \in \mathcal{Z}_+(r) \). Introduce the matrix-valued function
\[
U_2(x, z) = U_1(x, z) + \frac{W_2(x)}{3z^3} = \mathbb{I}_3 + \frac{p(x)W_1}{3z^2} + \frac{W_2(x)}{3z^3}, \quad x \in \mathbb{R},
\]
where
\[
W_2 = \frac{1}{3} \begin{pmatrix}
0 & h & \bar{h} \\
\bar{h} & 0 & h \\
h & \bar{h} & 0
\end{pmatrix},
\]
(7.2)
h is given by (6.13). The matrix \( U_2(x, z) \) is invertible at large \( |z| \) and we introduce the matrix \( Y_2(x, z) \) by the identity
\[
Y(x, z) = U_2(x, z)Y_2(x, z)U^{-1}_2(0, z), \quad (x, z) \in \mathbb{R} \times \mathcal{Z}_+(r),
\]
where \( Y \) is the solution of the problem (4.1). Let \((x, z) \in \mathbb{R} \times \mathcal{Z}_+(r)\). Then the matrix-valued function \( Y_2(x, z) \), given by (7.3), satisfies the equation
\[
Y_2' - zT_2Y_2 = \frac{1}{z^3} \Phi_2 Y_2, \quad Y_2(0, z) = \mathbb{I}_3,
\]
(7.4)
where \( T_2 \) is given by the definition (6.10),
\[
\Phi_2(x, z) = -\frac{1}{3} W_2''(x) - \frac{p^2(x)}{9} (PW_1 - 2W_1 T^2) + O(z^{-1}),
\]
(7.5)
uniformly on \( x \in \mathbb{R} \).

**Proof.** Let \((x, z) \in \mathbb{R} \times \mathcal{Z}_+(r)\). The definitions (6.3) and (7.3) give
\[
Y_1(x, z) = U_3(x, z)Y_2(x, z)U^{-1}_3(0, z), \quad U_3 = U^{-1}_1 U_2.
\]
Substituting this identity into equation (6.1) we obtain
\[
Y_2' = U^{-1}_3 \left( zT_1 + \frac{P_1}{z^2} - \frac{p^2}{9z^3} Q_1 \right) U_3 - U''_3 + O(z^{-4}) \right) Y_2,
\]
(7.6)
where
\[
P_1 = -\frac{1}{3} (qQ + p'W_1), \quad Q_1 = PW_1 - 2W_1 T^2.
\]
The definitions (6.1) and (7.1) imply
\[
U_3(x, z) = \mathbb{I}_3 + \frac{W_2(x)}{3z^3} + O(z^{-5}), \quad U''_3(x, z) = \frac{W_2''(x)}{3z^3} + O(z^{-5}).
\]
Substituting this asymptotics into equation (7.6) we obtain
\[
Y_2' = \left( zT_1 + \frac{1}{3z^2} (3P_1 + [\mathcal{T}, W_2]) - \frac{1}{9z^3} (p^2 Q_1 + 3W_2'') + O(z^{-4}) \right) Y_2,
\]
(7.7)
where we used the definition (5.3). The matrix \( W_2 \) satisfies
\[
[\mathcal{T}, W_2] + 3P_1 = -q \mathcal{T}.
\]
Substituting this identity into equation (7.7) we obtain
\[
Y_2' = \left( zT_1 - \frac{qT}{3z^2} - \frac{1}{9z^3} (p^2 Q_1 + 3W_2'') + O(z^{-4}) \right) Y_2.
\]
The identity \( T_2 = T_1 - \frac{q}{3z^2} \mathcal{T} \) gives (7.4).
7.2. Factorization of the fundamental matrix. We apply Theorem 4.5 to equation (7.4) in order to obtain the factorization of the fundamental matrix. In this case the matrix $-\frac{1}{3}W'_2(x) - \frac{1}{9}p^2(x)(PW_1 - 2W_1T^2)$ in the right side is off-diagonal and we can apply Theorem 4.5 immediately to equation (7.4). Then we obtain the following corollary of Theorem 4.5 on the factorization of the fundamental matrix.

Corollary 7.2. Let $p''$, $q' \in L^1(\mathbb{T})$ and let $r \in \mathbb{Z}_+(r)$, where $r > 0$ is large enough. Then the fundamental matrix $M$ has the representation $M(x, \lambda) = A(x, z)A^{-1}(0, z)$, where $\lambda = z^3$ and the matrix-valued function $A$ has the form

$$A(x, z) = \Omega(z)U_2(x, z)\mathcal{X}(x, z)e^{\int_0^1 T_2(s, z)ds}, \quad (x, z) \in [0, 2] \times \mathbb{Z}_+(r), \quad (7.8)$$

and $\mathcal{X}$ is the solution of the integral equation (4.10) with

$$m = 3, \quad \Theta = T_2, \quad \Phi = \Phi_2. \quad (7.9)$$

Each function $A(x, \cdot), x \in [0, 2]$, is analytic in $\mathbb{Z}_+(r)$, where $r > 0$ is large enough.

\textbf{Proof.} Equation (7.4) has the form (4.5) with $m, \Theta, \Phi$ satisfying (7.9). Let $Y_2$ be the solution of equation (7.4) satisfying the condition $Y_2(0) = \mathbb{I}_3$. Theorem 4.5 shows that $Y_2$ satisfies the identity (4.24). The identities (2.5), (7.3) imply the identity (3.6), where $A$ satisfies (7.8). \hfill \blacksquare

7.3. Eigenvalue asymptotics. Now we determine the eigenvalue asymptotics for the case $p''$, $q' \in L^1(\mathbb{T})$.

Lemma 7.3. Let $p''$, $q' \in L^1(\mathbb{T})$ and let $n \to \pm \infty$. Then the eigenvalues $\mu_n$ satisfy

$$\mu_n = \mu_n'' - 2npn_0 + q_0 + \frac{2p_n^2}{\sqrt{3\pi n}} + \frac{1}{2\pi n} \left( \text{Im} \left( \frac{p_n''}{3} - \hat{q}_n \right) - \text{Re} \left( \frac{p_n''}{3} + \hat{q}_n \right) \right) + O(n^{-\frac{3}{2}}). \quad (7.10)$$

\textbf{Proof.} Let $p''$, $q' \in L^1(\mathbb{T})$. Without loss of generality we assume that $\int_0^1 q(t)dt = 0$. Let the matrix-valued function $A$ be given by (7.8). Then $\mathcal{W}(x) = \frac{2}{3}W_1 + \frac{1}{3z}W_2$ in the asymptotics (4.29) and, due to Lemma 7.1 ii), in the considered case we have

$$m = 3, \quad \mathcal{F} = -\frac{1}{3}(W' + \frac{2p}{3}Q_1), \quad \Theta = T - \frac{2p}{3z^2}T^2 - \frac{q}{3z^3}T,$$

where $Q_1 = PW_1 - 2W_1T^2$. The identity (4.31) gives

$$\zeta_j(x, z) = \frac{p(x)\eta_{1,j}}{3z^2} + \frac{\eta_{2,j}(x)}{3z^3} + \frac{\eta_j(x, z)}{z^3}, \quad \eta_{2,j}(x) = \sum_{m=1}^3 W_{2,mj}(x), \quad (7.11)$$

where $\eta_{1,j}$ are given by (6.15). The definition (7.2) implies

$$\eta_{2,1}(x) = \frac{1}{3}(h + \overline{h}) = q - \frac{p'}{3} = V, \quad (7.12)$$

where $h$ is given by (6.13) and we used (1.15). Let $\lambda = z^3 = \mu_n, n \to +\infty$. Integrating by parts in the asymptotics (6.14) we obtain

$$z = \nu n - \frac{p_0}{\sqrt{3\pi n}} + \delta_n, \quad \delta_n = O(n^{-\frac{3}{2}}).$$

The definition (7.2) provides

$$\mathcal{F}_{12} = -\frac{h'}{9} - \frac{p''}{9}Q_{1,12}, \quad \mathcal{F}_{21} = -\frac{p'}{9} - \frac{p''}{9}Q_{1,21} \quad (7.13)$$
Substituting this identity into (4.16) and using the asymptotics

\[ \int_0^x e^{-i2\pi n_s}p^2(s) = O(n^{-1}), \quad \int_x^{2} e^{-i2\pi n_s}p^2(s) = O(n^{-1}), \]

we obtain

\[ \eta_1(x, z) = \frac{1}{9} \int_x^{2} e^{i2\pi n(s-x)}h'(s)ds + O(n^{-\frac{1}{2}}), \]

\[ \eta_2(x, z) = -\frac{1}{9} \int_0^x e^{i2\pi n(s-x)}h'(s)ds + O(n^{-\frac{1}{2}}). \]

Substituting this asymptotics into the definition (7.11) and using the identities (6.10) and (7.12) we obtain

\[ \zeta_1(x, z) = \frac{\varphi p(x)}{3\sqrt{3}z^2} + \frac{V(x)}{3z^3} + \frac{1}{9z^2} \int_x^{2} e^{i2\pi n(s-x)}h'(s)ds + O(n^{-\frac{1}{2}}), \]

\[ \zeta_2(x, z) = \frac{\varphi p(x)}{3\sqrt{3}z^2} + \frac{V(x)}{3z^3} - \frac{1}{9z^3} \int_0^x e^{i2\pi n(x-s)}h'(s)ds + O(n^{-\frac{1}{2}}), \]

where \( \varphi \) is given by (6.13). Then the asymptotics (4.30) gives

\[ \phi_1(0, z) = 1 + \beta(z) + \frac{2h'}{9z^3} + O(n^{-\frac{5}{2}}), \quad \phi_1(1, z) = e^{z(\tau - \frac{2n\pi z^2}{3s^2})} \left(1 + \beta(z) + \frac{h'}{9z^3} + O(n^{-\frac{5}{2}})\right), \]

\[ \phi_2(0, z) = 1 + \alpha(z) + O(n^{-\frac{5}{2}}), \quad \phi_2(1, z) = e^{z(\tau - \frac{2n\pi z^2}{3s^2})} \left(1 + \alpha(z) - \frac{h'}{9z^3} + O(n^{-\frac{5}{2}})\right), \]

where

\[ \alpha(z) = \frac{\varphi p(0)}{3\sqrt{3}z^2} + \frac{V(0)}{3z^3}, \quad \beta(z) = \frac{\varphi p(0)}{3\sqrt{3}z^2} + \frac{V(0)}{3z^3} \]

and we used (6.10) and the identity \( \int_0^1 q(s)ds = 0 \). Substituting the asymptotics (7.14) and (7.15) into (5.15) and using (6.17) and the asymptotics \( \delta_n = O(n^{-\frac{5}{2}}) \) we obtain

\[ \xi(z) = (-1)^n e^{-\frac{5}{2} + \frac{\pi i}{3}} (\xi_n + e^{-\frac{3\pi i}{6}} O(1)), \]

where

\[ \xi_n = \det \begin{pmatrix} 1 + \beta(z) + \frac{2h'}{9z^3} + O(n^{-\frac{5}{2}}) & 1 + \alpha(z) + O(n^{-\frac{5}{2}}) \\ 1 + i\frac{\sqrt{3}}{2} \delta_n + \beta(z) + \frac{h'}{9z^3} + O(n^{-\frac{5}{2}}) & 1 - i\frac{\sqrt{3}}{2} \delta_n + \alpha(z) - \frac{h'}{9z^3} + O(n^{-\frac{5}{2}}) \end{pmatrix} = -i\sqrt{3} \delta_n - \frac{2i \Im \tilde{h}_n'}{\sqrt{3}(2\pi n)^{3}} + O(n^{-\frac{5}{2}}). \]

The identity (5.13) implies \( \xi_n + e^{-\frac{3\pi i}{6}} O(1) = 0 \), which gives

\[ \delta_n = -\frac{2 \Im \tilde{h}_n'}{3(2\pi n)^{3}} + O(n^{-\frac{5}{2}}). \]

The definition (6.13) gives

\[ \Im \tilde{h}_n' = \Im (\tau \tilde{p}_n') - \sqrt{3} \Re (\tau q_n'). \]
Substituting this identity into (7.17) and using
\[ \text{Im} (\tau \hat{p}''_n) = \frac{1}{2} (\sqrt{3} \text{Re} \hat{p}''_n - \text{Im} \hat{\varphi}_n) \quad \text{Re} (\tau \hat{q}''_n) = \frac{1}{2} (-\sqrt{3} \text{Im} \hat{q}''_n - \text{Re} \hat{\varphi}_n), \]
we obtain
\[ \delta_n = \frac{1}{3(2\pi n)^3} \left( \sqrt{3}(-\sqrt{3} \text{Im} \hat{q}''_n - \text{Re} \hat{\varphi}_n) - (\sqrt{3} \text{Re} \hat{p}''_n - \text{Im} \hat{\varphi}_n) \right) + O(n^{-\frac{7}{2}}), \]
which gives
\[ z = \nu n - \frac{p_0}{\sqrt{3} \pi n} + \frac{1}{3(2\pi n)^3} \left( \sqrt{3}(-\sqrt{3} \text{Im} \hat{q}''_n - \text{Re} \hat{\varphi}_n) - (\sqrt{3} \text{Re} \hat{p}''_n - \text{Im} \hat{\varphi}_n) \right) + O(n^{-\frac{7}{2}}). \]
This yields the asymptotics (7.10) as \( n \to +\infty \). The identities (5.2) and \((q^-)'_n = -q'_n, (p^-)'_n = \hat{p}'_n\) give the asymptotics (7.10) for \( n \to -\infty \). ■

**Proof of Theorem 1.1.** Lemma 5.2 iii) shows that there exists \( N \in \mathbb{N} \) such that each \( \mu_n, |n| > N \), is real. The asymptotics (1.11) is proved in Lemma 5.4. Substituting the identities
\[ \hat{p}'_n = \int_0^1 e^{-i2\pi nx} p'(x) dx = i2\pi n \hat{\varphi}_n, \quad \hat{\varphi}_n = i2\pi n \hat{\varphi}_n, \quad \hat{p}'_n = -(2\pi n)^2 \hat{\varphi}_n \]
into the asymptotics (6.14) and (7.10) we obtain (1.12) and (1.13). ■

8. Trace formula

8.1. Asymptotics of the characteristic function. Below we need the following result about the characteristic function.

**Lemma 8.1.** Let \( p''', q'' \in L^1(\mathbb{T}) \). Then
\[ D(\lambda) = \frac{2e^{z(\frac{1}{3} - \frac{1}{3\lambda})}}{3\sqrt{3}\lambda} \left( 1 + \frac{V(0)}{\lambda} \right) \left( \sin \left( \frac{\sqrt{3}}{2} z + \frac{p_0}{\sqrt{3}z} \right) + e^{\frac{2\pi}{3} \text{Im} z O(z^{-4})} \right) \quad (8.1) \]
as \( |\lambda| \to \infty, \lambda \in \mathbb{C}_+, \text{Re} \lambda \geq 0 \), where \( V = q - \frac{p'}{3} \), see (1.13).

**Proof.** Let \( p''', q'' \in L^1(\mathbb{T}) \). Let \( |z| \to \infty, z \in \mathbb{Z}_+ \). Integrating by parts in the asymptotics (4.14) we obtain
\[ B_{ij}(x, z) = O(z^{-1}), \quad 1 \leq j, \ell \leq 1, 2, 3, 4, \quad j \neq \ell. \]
Substituting these asymptotics into the definition (7.11) and using the definitions (4.2), (6.2) and (7.2) we obtain
\[ \zeta_1(x, z) = \frac{p(x)}{3\sqrt{3}z^2} + \frac{V(x)}{3z^3} + O(z^{-4}), \]
\[ \zeta_2(x, z) = \frac{p(x)}{3\sqrt{3}z^2} + \frac{V(x)}{3z^3} + O(z^{-4}), \]
\[ \zeta_3(x, z) = \frac{p(x)}{3\sqrt{3}z^2} + \frac{V(x)}{3z^3} + O(z^{-4}), \]
where \( V(x) = \frac{1}{8}(h(x) + \overline{h(x)}) = q(x) - \frac{p'(x)}{3} \) and \( h, \overline{h} \) are given by (6.13). Then the asymptotics (4.31) gives
\[ \phi_1(0, z) = \alpha(z) + O(z^{-4}), \quad \phi_1(1, z) = e^{z(\frac{2p_0}{3\sqrt{3}z^2} + \frac{1}{3}) \left( \alpha(z) + O(z^{-4}) \right)}, \]
\[ \phi_2(0, z) = \beta(z) + O(z^{-4}), \quad \phi_2(1, z) = e^{z^2 - \frac{2n_i}{z^2}}(\beta(z) + O(z^{-4})) , \]
\[ \phi_3(2, z) = e^{2z(1 - \frac{2\alpha}{z^3})} + \frac{p(0)}{3z^2} + \frac{V(0)}{3z^3} + O(z^{-4}) , \tag{8.2} \]
where
\[ \alpha(z) = 1 + \frac{\zeta p(0)}{3\sqrt[3]{2z^2}} + \frac{V(0)}{3z^3} , \quad \beta(z) = 1 + \frac{\zeta p(0)}{3\sqrt[3]{2z^2}} + \frac{V(0)}{3z^3} \tag{8.3} \]

Let \(|z| \to \infty, z \in \mathbb{Z}^+\). Substituting these asymptotics into the asymptotics (5.15) and using the definition (5.13) we obtain
\[ \det \phi(z) = \phi_3(2, z)e^{z(-\frac{1}{2} + \frac{2\pi i}{3\pi})}(\xi(z) + e^{\frac{\pi}{2}\text{Im}zO(z^{-4}))} . \tag{8.4} \]
where
\[ \xi(z) = \det \left( \begin{array}{cc} \frac{\alpha(z)}{\alpha(z)e^{iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}} & \frac{\beta(z)}{\beta(z)e^{-iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}} \\ \frac{\alpha(z)e^{-iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}}{\beta(z)e^{iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}} & \frac{\beta(z)e^{-iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}}{\beta(z)e^{iz(\frac{\pi}{2} + \frac{\pi}{3\pi})}} \end{array} \right) = -2i\alpha(z)\beta(z)\sin\left(\frac{\sqrt{3}}{2}z + \frac{p_0}{\sqrt{3}z}\right) . \]
The definitions (8.3) imply
\[ \xi(z) = -2i\left(1 - \frac{p(0)}{3z^2} + \frac{V(0)}{3z^3}\right)\left(\sin\left(\frac{\sqrt{3}}{2}z + \frac{p_0}{\sqrt{3}z}\right) + e^{\frac{\pi}{2}\text{Im}zO(z^{-4}))}\right) . \]
Substituting this asymptotics and (8.2) into (8.4) we obtain
\[ \det \phi(z) = -2ie^{z(\frac{\pi}{2} + \frac{\pi}{3\pi})}\left(1 + \frac{V(0)}{3z^3}\right)\left(\sin\left(\frac{\sqrt{3}}{2}z + \frac{p_0}{\sqrt{3}z}\right) + e^{\frac{\pi}{2}\text{Im}zO(z^{-4}))}\right) . \tag{8.5} \]

The asymptotics (4.11) and (4.13) imply
\[ \det \mathcal{X}(x, z) = 1 + \text{Tr} \mathcal{B}(x, z) + O(z^{-6}) = 1 + O(z^{-4}) \quad \forall \quad x \in \mathbb{R} . \]
Substituting these asymptotics and (5.7) into (7.8) we obtain
\[ \det A(0, z) = -i3\sqrt{3}z^3(1 + O(z^{-4})) . \]
Substituting the last asymptotics and the asymptotics (8.5) into the identity (3.9) we obtain the asymptotics (8.1). □

8.2. Trace formula. We prove the trace formula for the operators \(H_1, H\).

Proof of Theorem 1.2 Let \(p''' \in L^1(\mathbb{T})\). Introduce the function \(F(t) = \sum_{n=-N}^N \mu_n(t), t \in \mathbb{T}\), where \(N\) is given in Lemma (5.2 ii), and the resolvents \(R_t(\lambda) = (H_t - \lambda)^{-1}\). Then we have
\[ F(t_1) - F(t_2) = \frac{1}{2\pi i} \int_{\Gamma_N} \lambda \text{Tr} \left(R_{t_1}(\lambda) - R_{t_2}(\lambda)\right) d\lambda , \]
\[ \mu_n(t_1) - \mu_n(t_2) = \frac{1}{2\pi i} \int_{\ell_n} \lambda \text{Tr} \left(R_{t_1}(\lambda) - R_{t_2}(\lambda)\right) d\lambda \quad \forall \quad n > N , \]
where the contours \(\Gamma_N\) and \(\ell_n\) are given by
\[ \Gamma_N = \left\{ \lambda \in \mathbb{C} : |\lambda| = \left(\frac{2\pi(N + \frac{1}{2})}{\sqrt{3}2}\right)^{\frac{1}{3}} \right\} , \quad \ell_n = \left\{ \lambda \in \mathbb{C} : \left|z - \frac{2\pi n}{\sqrt{3}2}\right| = \frac{\pi}{4} \right\} . \]
Using the identities

\[ R_{t_2}(\lambda) - R_{t_1}(\lambda) = R_{t_1}(\lambda)\left(H_{t_2} - H_{t_1}\right)R_{t_2}(\lambda) = R_{t_1}(\lambda)\left(\partial(p_{t_2} - p_{t_1}) + (p_{t_2} - p_{t_1})\partial + q_{t_2} - q_{t_1}\right)R_{t_2}(\lambda), \]

where \( f_t = f(\cdot + t) \), we obtain

\[ |F^{(k)}(t_1) - F^{(k)}(t_2)| \leq C \left( \|p_{t_2}^{(k+1)} - p_{t_1}^{(k+1)}\|_{\infty} + \|p_{t_2}^{(k)} - p_{t_1}^{(k)}\|_{\infty} + \|q_{t_2}^{(k)} - q_{t_1}^{(k)}\|_{\infty} \right) \]

for some constant \( C > 0 \) and \( k = 0, 1 \), where \( f(0) = f, f^{(k)} = d^k f/dt^k \). These estimates imply \( F \in C^1(\mathbb{T}) \). The similar arguments show that \( \mu_n \in C^1(\mathbb{T}) \) for all \( n > N \).

The asymptotics (7.10) shows that the series (1.14) converges absolutely and uniformly in \( t \in \mathbb{T} \).

Let \( D(\lambda, t) = D(\lambda, p_t, q_t) \) and \( D(\lambda) = D(\lambda, 0) \). The asymptotics (5.10) shows that \( \log D(\lambda, t)/D(\lambda) \) is well defined on the contours \( \Gamma_N \) for large \( N \in \mathbb{N} \) by the condition \( \log 1 = 0 \) and for \( N \in \mathbb{N} \) large enough we have

\[ \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \, d\log \frac{D(\lambda, t)}{D(\lambda)} = \frac{1}{2\pi i} \oint_{\Gamma_N} \sum_{n \in \mathbb{Z}} \left( \frac{\lambda}{\lambda - \mu_n(t)} - \frac{\lambda}{\lambda - \mu_n(0)} \right) d\lambda = \sum_{n=\pm N}^{N} \left( \mu_n(t) - \mu_n(0) \right). \]

Let \( \lambda \in \Gamma_N, N \to \infty \). The asymptotics (8.1) and the identity (3.1) give

\[ \frac{D(\lambda, t)}{D(\lambda)} = 1 + \frac{\sqrt{3}}{\lambda} \left( V(t) - V(0) \right) + O(z^{-4}), \]

which yields

\[ \log \frac{D(\lambda, t)}{D(\lambda)} = \frac{\sqrt{3}}{\lambda} \left( V(t) - V(0) \right) + O(z^{-4}). \]

Integrating by parts we obtain

\[ \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \, d\log \frac{D(\lambda, t)}{D(\lambda)} = \frac{1}{2\pi i} \oint_{\Gamma_N} \log \frac{D(\lambda, t)}{D(\lambda)} d\lambda. \]

Using the estimate

\[ \lim_{N \to \infty} \oint_{\Gamma_N} O(z^{-4}) d\lambda = 0 \]

we obtain

\[ \lim_{N \to \infty} \frac{1}{2\pi i} \oint_{\Gamma_N} \lambda \, d\log \frac{D(\lambda, t)}{D(\lambda)} = V(0) - V(t). \]

The identity (8.6) gives the identity (1.14).

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