Exact low-energy effective actions for hypermultiplets in four dimensions

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Abstract

We consider the general hypermultiplet Low-Energy Effective Action (LEEA) that may appear in quantized, four-dimensional, N=2 supersymmetric, gauge theories, e.g. in the Coulomb and Higgs branches. Our main purpose is a description of the exact LEEA of \( n \) magnetically charged hypermultiplets. The hypermultiplet LEEA is given by the N=2 supersymmetric Non-Linear Sigma-Model (NLSM) with a 4\( n \)-dimensional hyper-Kähler metric, subject to non-anomalous symmetries. Harmonic Superspace (HSS) and the NLSM isometries are very useful to constrain the hyper-Kähler geometry of the LEEA. We use N=2 supersymmetric projections of HSS superfields to N=2 linear (tensor) \( O(2) \) and \( O(4) \) multiplets in N=2 Projective Superspace (PSS) to deduce the explicit form of the LEEA in some particular cases. As the by-product, a simple new classification of all multi-monopole moduli space metrics having \( su(2)_R \) symmetry is proposed in terms of real quartic polynomials of 2\( n \) variables, modulo \( Sp(n) \) transformations. The 4d hypermultiplet LEEA for \( n = 2 \) can be encoded in terms of an elliptic curve.

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1 Introduction

The seminal work of Seiberg and Witten [1] gave many important insights into the non-perturbative dynamics of the four-dimensional N=2 supersymmetric SU(2) Yang-Mills theory whose gauge symmetry is spontaneously broken to its abelian subgroup (in the so-called Coulomb branch). It was subsequently generalized to other simply-laced gauge groups and N=2 super-QCD as well [2] (see also ref. [3] for a review or an introduction). Seiberg-Witten theory deals with the exact Low-Energy Effective Action (LEEA) in terms of abelian N=2 vector multiplets, which includes both perturbative (one-loop) and non-perturbative (instanton) corrections. To fix those corrections, N=2 extended supersymmetry plays the important rôle that amounts to restricting the N=2 vector gauge LEEA to the general ‘Ansatz’ governed by a single holomorphic function $F(W^A)$ of the N=2 abelian vector superfield strengths $W^A$, where $A = 1, 2, \ldots, N_c - 1$ [4, 5]. This Ansatz is manifestly N=2 supersymmetric, gauge-invariant and model-independent due to the off-shell nature of the N=2 restricted chiral superfields $W^A$. According to the Riemann-Hilbert theorem, a (multi-valued) holomorphic function is fully determined by its singularity structure and monodromy (or asymptotics). The number of singularities is dictated by Witten index [6]. However, a calculation of the Witten index in massless gauge field theories is plagued with ambiguities, so that its value is usually postulated from physical considerations and consistency. This is closely related to the existence of BPS monopoles representing the non-perturbative degrees of freedom in the non-abelian N=2 gauge theory under consideration [3]. Indeed, any non-abelian N=2 supersymmetric gauge field theory can be considered as a particular Yang-Mills-Higgs system (with adjoint fermions) whose field equations admit the solitonic solutions labelled by magnetic charge, and whose scalar potential is fixed by N=2 supersymmetry. The monodromy behaviour is obtained from perturbative calculations and electric-magnetic (S) duality. For example, the chiral anomaly of the underlying N=2 gauge theory fully determines the perturbative (logarithmic) contribution to the second derivative of $F(W)$, which is equivalent to knowing the monodromy around the infinity in the quantum moduli space parametrized by vacuum expectation values of the Higgs fields. The S duality and global consistency conditions fix the remaining monodromies and, hence, the whole function $F(W)$.

Due to the holomorphicity of the Seiberg-Witten LEEA, the non-perturbative contributions to $F(W)$ take the form of an infinite sum over all instanton numbers. In particular, no instanton/anti-instanton (i.e. non-holomorphic) contributions can appear in the Seiberg-Witten LEEA, in contrast to the standard (N=0) QCD. Yet
another special and very remarkable feature of the Seiberg-Witten-type solutions is the fact that any of them (e.g., in the $SU(N_c)$-based theory) can be nicely encoded in terms of the auxiliary (hyperelliptic) curve (or Riemann surface) $\Sigma_{SW}$ of genus $(N_c - 1)$ \(^1\)\(^2\). The function $\mathcal{F}(W)$ is most naturally represented in the parametric form, in terms of certain abelian differential $\lambda_{SW}$ (of the 3rd kind) integrated over the periods of $\Sigma_{SW}$ \(^3\).

N=2 supersymmetric gauge field theories can have only two types of rigid N=2 supermultiplets, an N=2 \textit{vector} multiplet comprising the helicities $(\pm 1, \pm \frac{1}{2}, 0)$, and a \textit{hypermultiplet} comprising the helicities $(\pm \frac{1}{2}, \pm 0)$. The two types of N=2 multiplets are truly different in \textit{four} spacetime dimensions (4d), but they become dual to each other in \textit{three} dimensions (3d). The latter observation can be elevated to the co-called ‘c-map’ relating the special Kähler geometry of the vector multiplet moduli space to the hyper-Kähler geometry of the hypermultiplet moduli space \(^7\) or, even further, to the mirror symmetry between certain type-II superstring compactifications on Calabi-Yau manifolds \(^8\). The LEHA of the 3d, N=4 supersymmetric non-abelian gauge field theories can, therefore, be deduced along the lines of the 4d Seiberg-Witten theory \(^9\). One of the most remarkable developments in the 3d, N=4 Seiberg-Witten theory was the proposed equivalence between the \textit{quantum} moduli space of the 3d, N=4 supersymmetric pure $SU(n)$ gauge theory (in the Coulomb branch) and the \textit{classical} moduli space of $n$ BPS monopoles in the 4d, $SU(2)$-based Yang-Mills-Higgs system \(^10\)\(^11\). In particular, as was argued by Hanany and Witten \(^11\), this equivalence between the moduli spaces of \textit{different} field theories in \textit{different} dimensions, for all $n$, can be understood as a consequence of S-duality applied to certain configurations of intersecting (Dirichlet) 3-branes and 5-branes in type-IIB superstring theory. Though the brane technology is very efficient in explaining the equivalence between the apparently different moduli spaces, it is not powerful enough in deriving explicit metrics on them. In this paper we consider those metrics possessing an $su(2)_R$ isometry. Another class of the $so(2)$-invariant hyper-Kähler metrics arises in the quantum moduli spaces of the 3d, N=4 supersymmetric $SU(2)$ gauge theories with $k$ fundamental matter hypermultiplets in the Coulomb branch. The latter can be identified with $D_k$ gravitational instantons in four Euclidean dimensions. The $D_k$ metrics in (almost) explicit form were recently calculated in ref. \(^12\) from the standard Nahm construction \(^13\), see also refs. \(^14\)\(^15\) for the related work. In the purely gauge 3d, N=4 theory with $N_c = 2$ the quantum moduli space metric is known to be given by the $su(2)_R$ invariant Atiyah-Hitchin metric \(^9\)\(^16\). The \textit{asymptotical} metrics in the classical multi-monopole moduli spaces for well-separated monopoles in 4d are available in their explicit form \(^17\)\(^18\), whereas the \textit{exact} metrics are only
known up to certain algebraic (Ercolani-Sinha) constraints \[19, 20\].

A derivation of the exact hypermultiplet LEEA in four spacetime dimensions (4d) requires the techniques that are very different from the ones used in the Seiberg-Witten theory. The main reason is the different status of an N=2 scalar multiplet (called a hypermultiplet) versus an N=2 vector multiplet. To appreciate this fact, it is worth mentioning the well-known fact that there exist no off-shell, manifestly N=2 supersymmetric (i.e. model-independent) formulation of the most fundamental Fayet-Sohnius (FS) hypermultiplet in the conventional N=2 extended superspace (see, e.g., ref. [21] for a review of N=2 superfields, and the references therein). Some restricted (non-universal) off-shell versions of a hypermultiplet, nevertheless, exist in N=2 Projective Superspace (PSS) invented by Karlhede, Lindström and Roček [22], where they are known as projective [22, 23] or (generalised) tensor N=2 multiplets [24]. The PSS construction gives up the manifest $su(2)_R$ internal symmetry rotating N=2 supersymmetry charges, while it also implies vector fields amongst the N=2 projective superfield components. The most symmetric approach to N=2 supersymmetry is provided by Harmonic Superspace (HSS) invented by Galperin, Ivanov, Kalitzin, Ogievetsky and Sokatchev [25], by using the infinite number of auxiliary fields. Unlike the PSS approach, both N=2 vector multiplets and hypermultiplets can be introduced in HSS on equal footing. Moreover, HSS allows one to keep both N=2 supersymmetry and its $su(2)_R$ automorphisms manifest. In the PSS approach, one adds a holomorphic (projective) coordinate to the conventional N=2 superspace, and then one uses hidden holomorphicity of the N=2 superspace constraints defining the N=2 projective multiplets. In the HSS description, one adds group-valued twistors and uses hidden Grassmann analyticity of the off-shell HSS constraints defining N=2 vector multiplets and hypermultiplets. In our view, as far as N=2 supersymmetry in 4d is concerned, the Grassmann analyticity is more fundamental than holomorphicity. The PSS and HSS descriptions of the hypermultiplet LEEA are closely related, most notably, by the N=2 supersymmetric projections of HSS superfields onto the PSS superfields.

The group of analyticity-preserving field reparametrizations (in HSS) is much larger than the group of holomorphicity-preserving reparametrizations (in PSS). Accordingly, a derivation of the exact hypermultiplet LEEA is more complicated or, at least, different, from solving a Seiberg-Witten (or Riemann-Hilbert) problem, just because more data is needed to fix a (Grassmann) analytic function versus a holomorphic one. A formal solution to the Riemann-Hilbert problem is given by the linear system of Picard-Fuchs differential equations [26]. As is demonstrated in this paper, a formal solution to the hypermultiplet LEEA can be given in HSS, in the form of an analytic hyper-Kähler potential of the metric. A derivation of the metric from
the hyper-Kähler potential requires an elimination of all the auxiliary fields hidden in analytic hypermultiplet superfields, which amounts to solving the (infinite) linear system of differential equations on a two-sphere.

From the physical point of view, it is important to understand the origin of a non-trivial hypermultiplet self-interaction in the LEEA. For example, as was argued by Seiberg and Witten [1], the exact effective NLSM metric for ‘fundamental’ hypermultiplets with vanishing magnetic charges in the Higgs branch of a 4d, N=2 gauge field theory is flat, i.e. there is no self-interaction at all. However, magnetically charged (massive) hypermultiplets can have a non-trivial self-interaction [27, 28]. This observation is consistent with the brane technology [29]. The corresponding LEEA just describes the low-energy dynamics of the BPS monopoles representing nonperturbative degrees of freedom, in the Lorentz-invariant way (cf. ref. [30]). For instance, the non-trivial NLSM corrections to the perturbative hypermultiplet LEEA found in refs. [27, 28] (see sect. 3 too) are all proportional to the squared absolute value of a central charge in N=2 supersymmetry algebra. The central charge itself is given by a linear combination of abelian charges of the underlying (spontaneously broken) non-abelian N=2 gauge theory. The non-vanishing central charge is also responsible for a dynamical generation of the non-trivial scalar potential associated with the hypermultiplet LEEA [28, 31]. Those important features were not fully appreciated in the earlier investigations of quantized 4d, N=2 supersymmetric field theories, which were either limited to renormalizable N=2 field theories or didn’t include N=2 central charges into the propagators (see, however, ref. [5] where the scalar potentials based on ‘active’ central charges were investigated in 2d, N=4 NLSM).

The paper is organized as follows. In sect. 2 we review a superspace derivation of the known relation between N=2 supersymmetry and hyper-Kähler geometry in 4d NLSM, which plays the important rôle in our investigation. We remind the reader about the formulation of N=2 NLSM in N=1 superspace and then motivate the use of PSS and HSS for a resolution of the hyper-Kähler constraints and isometries. In sect. 3 we discuss in detail the most general 4d LEEA of a single hypermultiplet, and demonstrate that it is given by the N=2 NLSM with the Taub-NUT or Atiyah-Hitchin metric. Sect. 4 is devoted to a discussion of the LEEA for many hypermultiplets and related multi-monopole moduli space metrics. Sect. 5 comprises our conclusion. Our presentation is self-contained. Basic facts about hyper-Kähler geometry are collected in Appendix A. Projective superspace (PSS) is introduced in Appendix B. Harmonic superspace (HSS) is defined in Appendix C. The classical moduli spaces of solitonic solutions in the 4d, SU(2)-based Yang-Mills-Higgs system are briefly reviewed in Appendix D.
2 Supersymmetry and hyper-Kähler geometry

The most natural description of 4d supersymmetry is provided by superspace. The natural framework for N=1 supersymmetry is given by N=1 superspace [32]. N=2 extended supersymmetry can be manifestly realized in N=2 superspace that has three different versions (standard, projective and harmonic). As regards general N=2 NLSM, in subsects. 2.1 and 2.2 we first recall their description in N=1 superspace [33] and then in N=2 HSS [34, 35], in order to remind the reader about the equivalence between rigid N=2 supersymmetry and hyper-Kähler geometry. Isometries in general N=2 NLSM are discussed in subsect. 2.3. The whole section establishes our setup and provides a technical introduction to the rest of the paper.

2.1 N=2 NLSM in N=1 superspace

N=1 scalar (chiral) multiplets are described by the N=1 complex chiral superfields \( \Phi_i \) and their conjugates \( \bar{\Phi}_i \), \( i = 1, 2, \ldots, k \), satisfying the off-shell constraints

\[
D_\alpha \Phi_i = 0 \ , \quad D_\alpha \bar{\Phi}_i = 0 \ , \quad (2.1)
\]

where we have introduced the covariant spinor derivatives \( D_\alpha \) and \( \bar{D}_\alpha \) in flat N=1 superspace \( \mathbb{Z} = (x^\mu, \theta^\alpha, \bar{\theta}^\dagger \alpha) \). They obey the basic anticommutation relations of N=1 supersymmetry,

\[
\{ D_\alpha, \bar{D}_\alpha \} = i\partial_\alpha \ , \quad \{ D_\alpha, D_\beta \} = \{ \bar{D}_\alpha, \bar{D}_\beta \} = 0 \ . \quad (2.2)
\]

We use the two-component spinor notation that is standard in 4d supersymmetry [32]. The field components of the chiral superfield \( \Phi^i \) are

\[
A^i = |\Phi^i| \ , \quad \psi^i = \partial_\alpha |\Phi^i| \ , \quad F^i = \frac{1}{2} D_\alpha D_\beta |\Phi^i| \ , \quad (2.3)
\]

where \(|\) means taking the first (leading, or \( \theta \)-independent) component of a superfield or an operator. The scalars \( A \) and the spinors \( \psi \) are the propagating fields, whereas the scalars \( F \) are the auxiliary fields.

The general 4d NLSM is described by an action

\[
S_{NLSM}[A] = \frac{1}{2\kappa^2} \int d^4 x \ g_{ij}(A) \partial_\mu A^i \partial^\mu A^j \ . \quad (2.4)
\]

It has an N=1 supersymmetric extension if and only if the NLSM metric \( g_{ij}(A) \) is Kähler [36]. Indeed, the most general N=1 supersymmetric action, in terms of N=1
chiral superfields and of the second order in spacetime derivatives of the physical scalars, reads

\[ S = \int d^4x d^4\theta K(\Phi^i, \bar{\Phi}_j) = -\frac{1}{2} \int d^4x \, K_{ij}^{\alpha} \partial^{\alpha} \partial_{\alpha} \bar{A}_j + \ldots, \quad (2.5) \]

where we have explicitly written down the leading bosonic NLSM term. We use the notation \[33\]

\[ K_{i_1 \cdots i_n j_1 \cdots j_m} \equiv \frac{\partial}{\partial A^{i_1}} \cdots \frac{\partial}{\partial A^{i_n}} \frac{\partial}{\partial \bar{A}_{j_1}} \cdots \frac{\partial}{\partial \bar{A}_{j_m}} K(A, \bar{A}), \quad (2.6) \]

and similarly for \[ K(\Phi^i, \bar{\Phi}_j) \]. The right-hand-side of eq. (2.5) has the standard NLSM form (2.4) with the restricted (=Kähler) metric

\[ ds^2 = K_{ij} (A, \bar{A}) dA^i d\bar{A}_j. \quad (2.7) \]

A complex manifold whose metric can be written down in the form (2.7) with a locally defined potential \( K \) is called the Kähler manifold \[34\]. The form of the superfield NLSM action (2.5) is preserved under arbitrary reparametrizations of \( \Phi^i \) and \( \bar{\Phi}_j \).

The line element (2.7) is only preserved under holomorphic reparametrizations of \( A^i \) and \( \bar{A}_j \), \( A^i \to f^i(A^j) \), while this can be extended to holomorphic transformations of the chiral superfields, \( \Phi^i \to f^i(\bar{\Phi}_j) \). The Ricci tensor of a Kähler metric reads

\[ R_{ij} = \{ \ln \det(K_{km}) \}_{,i,j}. \quad (2.8) \]

Having constructed the most general N=1 supersymmetric NLSM action (2.5) in terms of a Kähler potential \( K(\Phi, \bar{\Phi}) \), one can further impose extra non-manifest (non-linear) supersymmetry on the action (2.5), in order to get N=2 NLSM. In the absence of N=2 auxiliary fields, the extended N=2 supersymmetry algebra can be closed only on-shell, i.e. on the equations of motion for the NLSM fields.

The most general ‘Ansatz’ for the transformation law of extra supersymmetry is given by

\[ \delta \Phi^i = \bar{D}^2 (\varepsilon \bar{\Omega}^i), \quad \delta \bar{\Phi}_i = D^2 (\varepsilon \Omega^i), \quad (2.9) \]

where \( \varepsilon \) is a constant chiral superfield parameter, \( \bar{D}_\alpha \varepsilon = D^2 \varepsilon = \partial_i \varepsilon = 0 \), and \( \bar{\Omega} \) is a function of \( \Phi \) and \( \bar{\Phi} \) (modulo an additive chiral term). The on-shell closure of the supersymmetry transformations (2.9) implies the relations \[33\]

\[ \Omega_{i,j} \bar{\Omega}^{j,k} = \Omega_{j,i} \bar{\Omega}^{k,i} = -\delta^k_i, \]

\[ \bar{\Omega}^{[i \mid [j \mid k]} \bar{\Omega}^{j,k]} = 0, \quad (2.10) \]

\[ \bar{D}^2 \bar{\Omega}^i = \bar{\Omega}^{i,j} \bar{D}^2 \bar{\Phi}_j - \frac{i}{2} \bar{\Omega}^{i,jk} \bar{D}_\alpha \bar{\Phi}_j \bar{D}^\alpha \bar{\Phi}_k = 0, \]
and their complex conjugates. The N=1 NLSM action (2.5) is invariant under the transformations (2.9) provided that

\[
\bar{\omega}^{jm} \equiv K_i^j \Omega^{i,m} = - \bar{\omega}^{mj},
\]

\[
K_i^j \bar{\Omega}^{i,mk} + K_i^{mk} \bar{\Omega}^{j,m} = 0,
\]

\[
K_i^j \bar{\Omega}^{i,m}_k + K_i^{km} \bar{\Omega}^{j,m} = 0.
\]

This is to be compared to the field equations following from the action (2.5),

\[
D^2 K_i = K_i^j D^2 \Phi_j + \frac{1}{2} K_i^{jk} D \Phi_j D^2 \Phi_k = 0.
\]

The first two lines of eq. (2.11) imply that the third line of eq. (2.10) is equivalent to the equation of motion (2.12), which confirms the on-shell closure of the N=2 supersymmetry algebra [33].

It is now straightforward to check that eqs. (2.10) and (2.11) together amount to hyper-Kähler geometry (Appendix A). In particular, the quaternionic structure (A.12) comprises the canonical complex structure \( J^{(3)} \) of eq. (A.6) and two non-canonical complex structures,

\[
J^{(1)} = \begin{pmatrix} 0 & \Omega_{ji} & \\ -\bar{\Omega}_{ji} & 0 & \end{pmatrix}
\]

and

\[
J^{(2)} = \begin{pmatrix} 0 & i\Omega_{ji} \\ -i\bar{\Omega}_{ji} & 0 & \end{pmatrix},
\]

with mixed (one covariant and one contravariant) indices — see the first line of eq. (2.10). Both \( J^{(1)} \) and \( J^{(2)} \) are integrable due to the second line of eq. (2.10), while they are covariantly constant due to the second and third lines of eq. (2.11). Finally, the NLSM metric is hermitian with respect to all three complex structures due to the first line of eq. (2.11). According to Appendix A, this precisely amounts to the hyper-Kähler structure.

The canonical complex structure, \( J^{(3)} \), is obviously related to the given Kähler structure of the N=1 NLSM that we started with. The coordinate system, where the metric takes the Kähler form with respect to a non-canonical complex structure, is, therefore, related to the preferred one by a nonholomorphic coordinate transformation.

Eq. (2.13) gives the complex structures in terms of the derivatives of the nonholomorphic functions \( \Omega_i \) and \( \bar{\Omega}^i \) introduced in eq. (2.9). These functions can be reconstructed from a given Kähler potential and one of the noncanonical complex structures \( J \). Since \( J \) anticommutes with the canonical complex structure, the former can be written down in the form

\[
J_{i}^j = \begin{pmatrix} 0 & \Omega_{ji} \\ -\bar{\Omega}_{ji} & 0 & \end{pmatrix}
\]
with some matrix $\Omega$ and its complex conjugate $\bar{\Omega}$. It implies that $J^{ij}$ is block-diagonal,

$$J^{ij} = \begin{pmatrix} \bar{\gamma}^{ij} & 0 \\ 0 & \gamma^{ij} \end{pmatrix}, \quad \text{where} \quad \bar{\gamma}^{ij} \equiv (K^{-1})^i_k \bar{\Omega}^{kj}. \quad (2.15)$$

The covariant constancy of $J$ yields that $\bar{\gamma}^{jk}$ is holomorphic whereas $\gamma^{jk}$ is antiholomorphic, $\bar{\partial}^i \bar{\gamma}^{jk} = \partial_i \gamma^{jk} = 0$. Similarly one finds that $\bar{\omega}^{ij}$, introduced in the first line of eq. (2.11), is antiholomorphic, whereas $\omega^{ij} \equiv K^k_i \Omega_{kj}$ is holomorphic.

It is straightforward to verify that the functions

$$\bar{\Omega}^i \equiv \bar{\gamma}^{ij} K_j \quad (2.16)$$

obey the desired relation

$$\bar{\Omega}^{i,j} = \bar{\Omega}^{ij}, \quad (2.17)$$

while they satisfy all eqs. (2.10) valid for any hyper-Kähler manifold. Eqs. (2.15) and (2.16) further imply that

$$K_i \bar{\Omega}^i = 0 \quad \text{and} \quad \nabla_i \bar{\Omega}^i = 0. \quad (2.18)$$

One concludes that the hyper-Kähler structure plays the most fundamental role in the hyper-Kähler geometry, since all other geometrical quantities can be constructed in terms of it.

The N=1 superspace approach remains the most popular method in 4d supersymmetry, mainly because it has a very simple and clear connection to the component approach. However, N=1 superspace is clearly inadequate for N=2 supersymmetry since only one of the supersymmetries can be manifestly realized there, whereas another supersymmetry is necessarily hidden, being non-linearly realized. The differential constraints implied by the second supersymmetry (see, e.g., eq. (A.13) in Appendix A) are to be solved, since their presence does not allow one to formulate the most general ‘Ansatz’ for the N=2 hypermultiplet LEEA or, equivalently, the most general hyper-Kähler NLSM. Perhaps, most importantly, any treatment of isometries of N=2 NLSM is very complicated in N=1 superspace. The N=2 NLSM isometries are, however, going to be crucial for our purposes in the next sections.

### 2.2 N=2 superspace and NLSM

The 4d, N=2 superalgebra $SUSY_4^2$ is a graded extension of the 4d Poincaré algebra. In addition to the generators of the Poincaré algebra $(P_\mu, M_{\lambda\rho})$, $\mu = 0, 1, 2, 3$, the
superalgebra $SUSY^2_4$ contains two Majorana spinor generators $(Q^i_\alpha, \bar{Q}^i_\dot{\alpha})$, $i = 1, 2$, and the generators $A^i_{aj}$ of the $U(2)_R = SU(2)_R \times U(1)_R$ automorphisms. Together they satisfy the (anti)commutation relations

$$\frac{1}{4} [M_{\mu\nu}, M_{\rho\lambda}] = \eta_{\mu\rho} M_{\nu\lambda} - \eta_{\mu\lambda} M_{\nu\rho} + \eta_{\nu\rho} M_{\mu\lambda} - \eta_{\nu\lambda} M_{\mu\rho} ,$$
$$\frac{1}{4} [P_\mu, M_{\nu\lambda}] = \eta_{\mu\nu} P_{\lambda} - \eta_{\mu\lambda} P_{\nu} , \quad [P_\mu, P_\nu] = 0 ,$$
$$[Q^i_\alpha, M_{\mu\nu}] = \frac{1}{4} (\sigma_{\mu\nu} Q^i)_\alpha , \quad [\bar{Q}^i_\dot{\alpha}, M_{\mu\nu}] = \frac{1}{4} (\bar{\sigma}_{\mu\nu} \bar{Q}^i)_\dot{\alpha} ,$$
$$[P_\mu, Q^i_\alpha] = [P_\mu, \bar{Q}^i_\dot{\alpha}] = \{ Q^i_\alpha, Q^j_\beta \} = \{ \bar{Q}^i_\dot{\alpha}, \bar{Q}^j_\dot{\beta} \} = 0 ,$$
$$\{ Q^i_\alpha, \bar{Q}^j_\dot{\alpha} \} = \delta^i_j \sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu ,$$
$$[A^i_{aj}, A^j_{bm}] = \delta^i_j A^i_{bm} - \delta^i_m A^j_{aj} ,$$
$$[A^i_{aj}, Q^l_\alpha] = \delta^i_j Q^l_\alpha , \quad [A^i_{aj}, \bar{Q}^l_\dot{\alpha}] = - \delta^i_j \bar{Q}^l_\dot{\alpha} ,$$
$$[A^i_{aj}, M_{\mu\nu}] = [A^i_{aj}, P_\mu] = 0 . \quad (2.19)$$

The $U(1)_R$ generator $B \equiv A^i_i$ has the commutation relations

$$[B, Q^i_\alpha] = Q^i_\alpha , \quad [B, \bar{Q}^i_\dot{\alpha}] = - \bar{Q}^i_\dot{\alpha} . \quad (2.20)$$

In flat $N=2$ superspace with the coordinates $Z = (x^\mu, \theta^i_\alpha, \bar{\theta}^i_\dot{\alpha})$ one can also introduce the $N=2$ covariant spinor derivatives $(\tilde{D}^i_\alpha, \tilde{D}^i_\dot{\alpha})$ anticommuting with the $N=2$ supersymmetry charges and satisfying the same algebra,

$$\{ \tilde{D}^i_\alpha, \tilde{D}^j_{\dot{\alpha}} \} = \delta^i_j \sigma^{\mu}_{\alpha\dot{\alpha}} P_\mu , \quad \{ \tilde{D}^i_\alpha, \tilde{D}^j_{\dot{\alpha}} \} = 0 . \quad (2.21)$$

Their explicit realization reads

$$\tilde{D}^i_\alpha = \frac{\partial}{\partial \theta^i_\alpha} - \frac{i}{2} \sigma^{\mu}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}i} , \quad \tilde{D}^i_\dot{\alpha} = \frac{\partial}{\partial \bar{\theta}^i_\dot{\alpha}} + \frac{i}{2} \sigma^{\mu}_{\alpha\dot{\alpha}} \theta^{\alpha i} . \quad (2.22)$$

A general $N=2$ superfield is the irreducible representation of the enlarged superalgebra $SUSY^{2D}_4$ defined by adding the covariant derivatives $(\tilde{D}^i_\alpha, \tilde{D}^i_\dot{\alpha})$ to the generators of $SUSY^2_4$. The same superfield is, however, reducible with respect to the $N=2$ supersymmetry algebra $SUSY^2_4$, while its irreducible constituents can be defined either by imposing certain $N=2$ superspace constraints or by using superprojectors [38, 21]. For example, the abelian $N=2$ superfield strength of an $N=2$ vector multiplet is described by the restricted chiral $N=2$ superfield $W$ subject to the $N=2$ superspace (off-shell) constraints [38, 21]

$$\tilde{D}^i_\alpha W = 0 , \quad D^4 W = \square \bar{W} , \quad \text{where} \quad D^4 \equiv \prod_{i,\alpha}^4 D^i_\alpha . \quad (2.23)$$
Solving the constraints (2.23) is fully straightforward, and it results in
\[
W = \exp \left\{ -\frac{i}{2} \theta_i \theta^i \right\} \left[ A + \theta^\alpha \psi^i_\alpha - \frac{1}{2} \theta^\alpha_i (\tau_m)^i_j \theta^j_\alpha C_m + \frac{1}{8} \theta^\alpha_i (\sigma_{\mu\nu})^\beta_\alpha \theta^\beta_i F^{\mu\nu} \right.
\]
\[
- i(\theta^3)^{i\alpha} \partial_{\alpha\beta} \bar{\psi}^\beta_i + \theta^4 \Box \bar{A} \right]\ ,
\]
(2.24)
in terms of the complex scalar $A$, the Majorana douplet $\psi^i_\alpha$, the real auxiliary $SU(2)$ triplet $C_m$, and the real antisymmetric tensor $F^{\mu\nu}$ subject to the ‘Bianchi identity’,
\[
\varepsilon^{\mu\nu\lambda\rho} \partial_\nu F_{\lambda\rho} = 0 ,
\]
(2.25)
whose solution is $F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The free equations of motion for the $N=2$ vector multiplet in the standard $N=2$ superspace are given by
\[
D_{ij} W = 0 \ , \ \text{where} \ D_{ij} \equiv D^\alpha_i D_\alpha j .
\]
(2.26)

Let’s now introduce the $N=2$ source superfield $L_{ij} = L_{ji}$ into eq. (2.26),
\[
D_{ij} W = L_{ij} .
\]
(2.27)
The constraints (2.23) imply the constraints on the $N=2$ superfield $L_{ij}$,
\[
D^j_\alpha L^{ij}_k = \overline{D}\alpha^i L^{jk}_\alpha = 0 \quad \text{and} \quad L^{ij}_\alpha = \varepsilon^{ik} \varepsilon^{jl} T_{kl} .
\]
(2.28)
The constraints (2.28) define an off-shell $N=2$ tensor multiplet in the standard $N=2$ superspace [22].

Eq. (2.28) can be further generalized to $N=2$ superfields $L^{i_1\cdots i_n}$ that are totally symmetric with respect to their $SU(2)_R$ indices and satisfy the constraints [24]
\[
D^k_\alpha L^{i_1\cdots i_n}_\alpha = \overline{D}\alpha^k L^{i_1\cdots i_n}_\alpha = 0 .
\]
(2.29)
In the case of an even number of indices, $n = 2p$, the superfields $L^{i_1\cdots i_{2p}}$ can satisfy the reality condition
\[
\overline{T}_{i_1\cdots i_{2p}} \equiv (L^{i_1\cdots i_{2p}})^* = \varepsilon_{i_1 j_1} \cdots \varepsilon_{i_{2p} j_{2p}} L^{j_1\cdots j_{2p}} .
\]
(2.30)

Eqs. (2.29) and (2.30) define the projective (generalized tensor) $N=2$ multiplets for all $n \geq 2$ [22, 24]. They are irreducible off-shell representations of $N=2$ extended 4d supersymmetry of superspin $Y = 0$ and superisospin $I = (n-2)/2$, having $8(n-1)$ bosonic and the same number of fermionic (off-shell) field components [24],
\[
\left\{ L^{i_1\cdots i_n}_\alpha , \ \psi^{i_2\cdots i_n}_\alpha , \ N^{i_3\cdots i_n}_\alpha , \ V^{i_3\cdots i_n}_{\alpha\beta} , \ \bar{\psi}^{i_4\cdots i_n}_\beta , \ C^{i_5\cdots i_n}_\alpha \right\} .
\]
(2.31)
It follows from matching the bosonic and fermionic degrees of freedom that the real vector $V_{\alpha}{}^\alpha$ in an $N=2$ tensor multiplet has to be conserved, $\partial^\alpha V_{\alpha}{}^\alpha = 0$, whereas the vector fields $V_{i_1\cdots i_n}{}^{\alpha}{}_{\alpha}$ of the projective $N=2$ multiplets with $n > 2$ are all unconstrained.

Choosing $n = 1$ in eq. (2.29) results in the on-shell constraints defining the Fayet-Sohnius (FS) hypermultiplet (of vanishing central charge) whose independent components have helicities $(\pm \frac{1}{2}, \pm 0)$, as is required for the ‘true’ hypermultiplet [10]. The failure to incorporate the off-shell (i.e. model-independent) FS hypermultiplet within the framework of the standard $N=2$ superspace has far reaching consequences in $N=2$ supersymmetry. In particular, as regards $N=2$ NLSM, it does not allow one to formulate the most general, manifestly $N=2$ supersymmetric NLSM in the standard $N=2$ superspace. Since $N=2$ supersymmetry amounts to the hyper-Kähler NLSM geometry, the roots of the problem can be traced back to the basic properties of the hyper-Kähler structure (Appendix A). To the end of this subsection, we argue that the twistor approach [34, 35] is indeed the natural way to solve this problem.

Given a hyper-Kähler manifold $\mathcal{M}$, a linear combination, $aJ^{(1)} + bJ^{(2)} + cJ^{(3)}$, of its three, linearly independent and covariantly constant, complex structures satisfying the quaternionic algebra (A.12), with arbitrary real parameters $(a, b, c)$, is also the covariantly constant complex structure provided that $a^2 + b^2 + c^2 = 1$. Hence, a hyper-Kähler manifold $\mathcal{M}$ possess the variety of non-canonical complex structures (on the top of the canonical one), worthy of a two-sphere $S^2$. This feature is crucial for the efficiency of the twistor space [41] in monopole physics, whereas its PSS and HSS extensions provide the natural framework for an explicit construction of hyper-Kähler metrics from superspace. In the HSS approach one extends the ordinary $N=2$ superspace by the two-sphere $S^2$. Because of the isomorphism $S^2 \sim SU(2)/U(1)$, one can actually add the group $SU(2)$ instead of the coset, by restricting the HSS superfields to the ones that are equivariant with respect to the $U(1)$ symmetry — this mathematical construction is the very particular realization of a flag manifold [42]. The $SU(2)$ symmetry can be identified with the $N=2$ supersymmetry automorphisms $SU(2)_R$ that can be made manifest in HSS. In the PSS construction [22, 23] one uses another isomorphism $S^2 \sim CP(1)$ by adding a (complex) projective line $CP(1)$ to the standard $N=2$ superspace. The $SU(2)_R$ automorphisms are realized in PSS by the (non-linear) projective transformations. Though the HSS approach is the most symmetric and universal one, it also implies the infinite number of auxiliary fields, e.g. in the off-shell formulation of the FS hypermultiplet. This makes the relation between the HSS superfields and the component approach to be highly non-trivial. The PSS approach can be formulated with a finite number of auxiliary fields for a restricted class of $N=2$ NLSM, by assuming a holomorphic (polynomial) dependence.
upon the $CP(1)$ coordinate. Allowing a more general (e.g., meromorphic) dependence of projective N=2 multiplets upon the $CP(1)$ coordinate makes the PSS method to be essentially equivalent to the HSS one \[13\] (see Appendix B for a technical introduction into PSS, and Appendix C for a technical introduction into HSS).

The PSS construction of the N=2 NLSM metrics can be summarized into a short prescription known as the generalized Legendre transform \[44\]. One considers sections of $O(2p)$ line bundles over $CP(1)$, defined by all holomorphic polynomials of the $CP(1)$ projective coordinate $\xi$, of order $2p$ (cf. eq. (B.2))

$$Q_{(2p)}(\xi) = z + \nu \xi + w_2 \xi^2 + \ldots + w_{2p-2} \xi^{2p-2} + (-1)^{p-1} \bar{v} \xi^{2p-1} + (-1)^p \bar{z} \xi^{2p}, \quad (2.32)$$

subject to the reality condition (cf. eq. (2.30))

$$\overline{Q_{(2p)}(\xi)} = (-1)^p \bar{\xi}^{2p} Q_{(2p)}(-1/\bar{\xi}). \quad (2.33)$$

One introduces the contour integral (cf. eqs. (B.10) and (B.25))

$$F = \frac{1}{2\pi i} \oint_C d\xi \xi^{-2} G(Q(\xi), \xi) \quad (2.34)$$

in terms of a holomorphic function $G(Q(\xi), \xi)$. A Kähler potential $K$ of the hyper-Kähler metric associated with the holomorphic input $(G, C)$ is found by performing the complex Legendre transform with respect to $\nu$ and $\bar{v}$ \[44\] (cf. eq. (B.26)),

$$K(z, \bar{z}, u, \bar{u}) = F(z, \bar{z}, v, \bar{v}, w_a) - uv - \bar{u}\bar{v}, \quad (2.35)$$

subject to

$$u \equiv \frac{\partial F}{\partial \nu}, \quad \bar{u} \equiv \frac{\partial F}{\partial \bar{v}}, \quad (2.36)$$

when simultaneously extremizing $F$ with respect to all $w_a$, where $a = 2, \ldots, 2p - 2$ (cf. eq. (B.23),

$$\frac{\partial F}{\partial w_a} = 0. \quad (2.37)$$

The generalized Legendre transform provides the very powerful method for an explicit construction of hyper-Kähler metrics, especially after taking into account all $O(2p)$ sections over $CP(1)$ with $p = 1, 2, \ldots, \infty$. In fact, one has to take all of them in the most general case. However, because of the complicated (highly non-linear) algebraic relations associated with the generalized Legendre transform, it seems to be very difficult to classify all hyper-Kähler metrics (e.g., according to their isometries) by using this method.
Harmonic superspace (HSS) can be independently justified by ‘relaxing’ (i.e. lifting off-shell) the Fayet-Sohnius hypermultiplet constraints (2.29) in the on-shell case of \( n = 1 \). It can be achieved with the infinite chain,

\[
\begin{align*}
D^{(i} L^{j)} &= D_{\alpha k} L^{ijk} , \\
\bar{D}^{(i} L^{j)} &= D_{\alpha k} L^{ijk} , \\
D^{(ijkl)} &= D_{\alpha m} L^{ijklm} , \\
\bar{D}^{(ijkl)} &= D_{\alpha m} L^{ijklm} , \\
\end{align*}
\]

which involves all complex projective multiplets \( L^{(i_1 \cdots i_{2p+1})} \) up to \( p = \infty \). The hypermultiplet HSS superfield \( q^+ \) (see Appendix C) is equivalent to eq. (2.38). Though it is definitely possible to incorporate all relaxed projective supermultiplets, including the one of eq. (2.38), into the framework of the PSS construction (Appendix B), it is hardly convenient in practice.

2.3 Isometries of \( N=2 \) NLSM

As regards the general bosonic NLSM (2.4), its isometry (symmetry) Lie group \( G \) may have an isotropy subgroup \( H \) consisting of those transformations of \( G \) that leave a point \( \{ A^i \} \) in the NLSM target space fixed. The remaining symmetries of \( G \) move the point \( \{ A^i \} \), being non-linearly realised. The infinitesimal action of \( G \) reads

\[
\delta A^i = \lambda^M k^i_M (A) ,
\]

where \( k^i_M (A) \), \( M = 1, 2, \ldots, \dim G \), are the Killing vectors generating the group \( G \) and satisfying the Killing equation \( k^{(ij)}_M = 0 \), while \( \lambda^M \) are (Lie algebra) parameters. The isotropy subgroup \( H(A) \) depends upon the point \( \{ A^i \} \) chosen. In adapted local coordinates, associated with a given point \( A^i \), the group \( H(A) \) acts linearly, i.e.

\[
\delta A^i = i \lambda^X (T_X)^j_i A^j ,
\]

where \( T_X \) are the hermitian generators of \( H(A) \). We follow ref. [33] here.

A description of isometries of a Kähler manifold has some special features related to the invariance of its complex structure under holomorphic transformations that do not mix \( A^i \) and \( \bar{A}^j \). It is, therefore, natural to distinguish between the holomorphic isometries possessing the same property, and the isometries that do not. The Lie derivative associated with a holomorphic Killing vector leaves both the Kähler metric
and the complex structure invariant [45]. The action of the holomorphic isotropy subgroup in adapted coordinates reads

\[ \delta A^i = i \lambda^i_j A^j , \quad \delta \bar{A}_i = -i \bar{A}_j \lambda^j_i , \]  
where \[ \lambda^i_j = \lambda^X (T^X)_i^j . \] (2.41)

The Kähler potential is generically invariant under the isometry modulo a Kähler gauge transformation,

\[ \delta K = \eta(A) + \bar{\eta}(\bar{A}) , \] (2.42)

with a holomorphic function \( \eta(A) \). In the isotropic case one can always choose the Kähler gauge where \( \eta \) vanishes, so that

\[ \delta K = i \lambda^i_j (K^i_j A^j - K^j_i \bar{A}_i) = 0 . \] (2.43)

To describe general holomorphic isometries, one introduces Killing vectors \( k_a \) with holomorphic components \( k^i_a (A) \) and their complex conjugates \( \bar{k}_{ai} (\bar{A}) \) [33],

\[ \delta A^i = \lambda^a_k k^i_a \equiv L_{\lambda^a_k} A^i , \quad \delta \bar{A}_i = \lambda^a_{\bar{\lambda}} \bar{k}_{ai} \equiv L_{\lambda^a_{\bar{\lambda}}} \bar{A}_i . \] (2.44)

In adapted coordinates, eq. (2.44) takes the form of eq. (2.41). The holomorphic and antiholomorphic components of the Killing vectors generate two separate isometry algebras, and, in general coordinates, obey the Killing equations

\[ K^i_{j;k} k^{j;k}_{a} + K^j_{k;k} \bar{k}_{ai} = 0 . \] (2.45)

Eq. (2.45) implies the existence of the Killing potential associated with a holomorphic Killing vector [45, 33].

Since a hyper-Kähler metric is fully characterized by its quaternionic or triholomorphic structure, it is also quite natural to distinguish between the triholomorphic isometries preserving all three complex structures, and the non-triholomorphic isometries that do not share this property. The triholomorphic isometries are also known in the literature as translational, while they commute with N=2 supersymmetry in the N=2 NLSM. The non-triholomorphic isometries are sometimes called rotational since the action of the corresponding Killing vector on the complex structures amounts to their rotation, while they do not commute with N=2 supersymmetry.

In real coordinates (Appendix A), a triholomorphic Killing vector \( k^m \) satisfies the equation

\[ P_{\pm i}^j \left( P^n_{\mp m} k^m \right)_{ij} = 0 , \] (2.46)
where we have used the projection operators of eq. (A.2). Eq. (2.46) is equivalent to the vanishing Lie derivative of each complex structure $J$,

$$L_k J^i_j \equiv k^m J^i_j ,m - k^j_m J^i_m + k^m ,i J^j_m = 0 .$$

(2.47)

In the special coordinates, where the complex structures have the form (A.6) and (2.13), and the Killing vector $k^m$ are manifestly holomorphic with respect to the canonical complex structure, the triholomorphicity condition reads

$$\bar{\Omega}^{ij} \bar{k}_j;m - \bar{\Omega}^{jm} \bar{k}_i;j = 0 ,$$

(2.48)

or, equivalently,

$$\bar{\omega}^{[j[i} \bar{k}^{;m]} = 0 .$$

(2.49)

The triholomorphic condition (2.46) or (2.47) can be considered as the integrability condition for the existence of a real Killing potential $X^{(J)}$ for each complex structure $J$, which satisfies the differential equation

$$k^i J_{ij} = -X^{(J)} ,j .$$

(2.50)

In the special coordinates, the existence of a real Killing potential $X^{(J)}$ amounts to the existence of a holomorphic Killing potential $P$ and an antiholomorphic Killing potential $\bar{P}$, which are defined with respect to $J^{(1)} \mp iJ^{(2)}$, respectively.

Triholomorphic isometries of a hyper-Kähler metric significantly simplify an explicit construction of the metric. Given an N=2 tensor multiplet amongst the arguments of the PSS construction (Appendix B) or, equivalently, a section of the line bundle $O(2)$ in the generalized Legendre transform, it always implies a translational (or triholomorphic) isometry. This can be understood as the result of dualization of the conserved vector amongst the components of the N=2 tensor multiplet. The translational isometry is manifest in the corresponding Kähler potential of eq. (B.13).

In the HSS approach, an analytic hypermultiplet superfield $q^+$ and its analytic conjugate $\bar{q}^+$ naturally form an $Sp(1)$ doublet, $q^{a+}$, where $a = 1, 2$. The free hypermultiplet action (C.7) has the manifest $Sp(1)$ isometry, in addition to the manifest $SU(2)_R$ isometry. The $U(1)$ subgroup of the $Sp(1)$ rotations is given by

$$q^+ \to e^{i\alpha} q^+ \quad \text{and} \quad \bar{q}^+ \to e^{-i\alpha} \bar{q}^+ ,$$

(2.51)

and it can be identified with the $U(1)_R$ symmetry.
In order to get the most general N=2 NLSM in HSS, let’s make the same trick as in general relativity, where one goes from a flat space metric in general coordinates to the truly curved space metric. In HSS we can assign extra \( Sp(n) \) indices to the HSS superfields,

\[
q^{a+} \to q^{A+}, \quad A = 1, 2, \ldots, 2n ,
\]

and apply a reparametrization,

\[
q^{A+} \to q^{A+}' = f^{A+}(q, u), \quad \text{with} \quad u^{\pm i} \text{ inert},
\]

to the free HSS action (Appendix C)

\[
S_{\text{free}}[q^A] = \frac{1}{2} \int_{\text{analytic}} q^{A+} D^{++} q^{A+}.
\]

One easily finds that the free action (2.54) gets transformed into

\[
S[q^A] = \frac{1}{2} \int_{\text{analytic}} \left\{ F_A^{+}(q, u) D^{++} q^{A+} + G^{(+4)}(q, u) \right\} \equiv \frac{1}{2} \int_{\text{analytic}} L^{(+4)}(q, u),
\]

with the particular functions \( F_A^{+} \) and \( G^{(+4)} \) given by

\[
F_A^{+} = f_B^{+} \frac{\partial f^{B+}}{\partial q^{A+}} \quad \text{and} \quad G^{(+4)} = f_B^{+} \partial^{++} f^{B+}.
\]

It is therefore eq. (2.56), with arbitrary complex functions \( F_A^{+}(q, u) \) and a generic real function \( G^{(+4)}(q, u) \), that represents the most general N=2 NLSM in HSS (cf. ref. [46]). The action (2.56) is invariant under infinitesimal field reparametrizations,

\[
\delta q^{A+} = \rho^{A+}(q, u), \quad \delta u^{\pm i} = 0,
\]

provided that

\[
\delta F_A^{+} = F_B^{+} \frac{\partial f^{B+}}{\partial q^{A+}} \quad \text{and} \quad \delta G^{(+4)} = F_A^{+} \partial^{++} \rho^{A+}.
\]

The HSS ‘vielbein’ \( F_A \) is the pure gauge field with respect to the HSS reparametrizations, and it can be gauge-fixed to the ‘canonical’ form, \( F_A^{+} = q_A^{+} \), in adapted coordinates on the NLSM target space. It results in the standard form of the most general N=2 NLSM action in HSS [34, 35],

\[
S_{\text{NLSM}}[q] = \frac{1}{2} \int_{\text{analytic}} \left\{ q^{A+} D^{++} q^{A+} + K^{(+4)}(q, u) \right\}.
\]

The function \( K^{(+4)}(q, u) \) is called a hyper-Kähler potential [34, 35].

Since isometries are the symmetries of the NLSM action, not of the NLSM Lagrangian, in HSS the latter may vary into a total harmonic derivative, \( \delta L^{(+4)} = D^{++} \Lambda^{++}(q, u) \), because of the identity

\[
\int_{\text{analytic}} D^{++} \Lambda^{++} \equiv \int_{\text{analytic}} \left[ \partial^{++} \Lambda^{++} + \frac{\partial \Lambda^{++}}{\partial q^{A+}} D^{++} q^{A+} \right] = 0.
\]
The action (2.55) is invariant under an infinitesimal isometry transformation

$$\delta q^A = \varepsilon^X \rho^{XA+}$$  \hspace{1cm} (2.61)

with some constant parameters $\varepsilon_X$, $X = 1, 2, \ldots, \dim H$, and the triholomorphic Killing vectors $\rho^{XA+}(q, u)$ provided that [16]

$$\left( \frac{\partial F_A^+}{\partial q_{B+}} + \frac{\partial F_B^+}{\partial q_{A+}} \right) \rho^{X_{B+}} = \frac{\partial \Lambda^{X_{++}}}{\partial q^{A+}},$$

$$\left( \frac{\partial \mathcal{V}^{(+4)}}{\partial q^{A+}} - \partial^{++} F_A^+ \right) \rho^{X_{A+}} = \partial^{++} \Lambda^{X_{++}}. \hspace{1cm} (2.62)$$

In adapted coordinates eq. (2.62) simplifies to

$$\frac{\partial \Lambda^{X_{++}}}{\partial q^{A+}} = -2\rho^X_A, \quad -2\partial^{++} \Lambda^{X_{++}} = \frac{\partial \mathcal{K}^{(+4)}}{\partial q^+_A} \frac{\partial \Lambda^{X_{++}}}{\partial q^{A+}}. \hspace{1cm} (2.63)$$

By analogy with the N=1 superspace description of N=2 NLSM isometries, the analytic HSS superfield $\Lambda^{X_{++}}$ is called the (triholomorphic) Killing potential of the (triholomorphic) Killing vector $\rho^{XA+}$ in HSS [16].

If the N=2 NLSM action (2.59) in adapted coordinates has a linearly realised isometry, eq. (2.61) takes the form (cf. eqs. (2.40) and (2.41))

$$\delta q^A = i\varepsilon^X (T_X)^A B q_{B+}, \hspace{1cm} (2.64)$$

where $T_X$ are the generators of $H$. The corresponding Killing vectors are linear in $q$, whereas their Killing potentials are quadratic [16],

$$\Lambda^{X_{++}} = -i q^+_A (T_X)^A B q_{B+}. \hspace{1cm} (2.65)$$

3 Exact dynamics of a single hypermultiplet

The general 4d, N=2 NLSM Lagrangian for a single hypermultiplet in HSS reads (subsect. 2.3)

$$-L^{(+4)} = \mathcal{K}^{(+4)}(\mathbf{q}^+, q^+; u^+) \hspace{1cm}, \hspace{1cm} (3.1)$$

where the hyper-Kähler potential $\mathcal{K}^{(+4)}$ is an arbitrary function of an unconstrained analytic HSS superfield $q^+$, its analytic conjugate $\mathbf{q}^+$, and harmonics $u^\pm$. A function $\mathcal{K}^{(+4)}$ should be of $U(1)$ charge (+4) in order to cancel the opposite $U(1)$ charge of the analytic measure in HSS (Appendix C). As regards the hypermultiplet LEA also having the form of eq. (3.1), its hyper-Kähler potential plays the role similar to
that of the holomorphic Seiberg-Witten potential $F$ in the abelian N=2 vector LEEA \cite{1}. Because of manifest N=2 supersymmetry of eq. (3.1) describing the propagating Fayet-Sohnius hypermultiplet degrees of freedom only, the equations of motion for the HSS action (3.1) determine (at least, in principle) the component hyper-Kähler NLSM metric in terms of a single HSS potential $K^{(+4)}$. It is not known how to deduce an explicit NLSM metric from eq. (3.1) in the case of a generic hyper–Kähler potential, though some explicit examples are available (Appendix C).

A crucial simplification arises when the $SU(2)_R$ symmetry is not broken, which is expected to be the case for the hypermultiplet LEEA (sect. 1). Since the $SU(2)_R$ transformations are linearly realised in HSS, the $SU(2)_R$ isometry of the hypermultiplet LEEA just means that the corresponding hyper-Kähler potential $K^{(+4)}$ should be independent upon harmonics. This observation gives rise to the most general invariant ‘Ansatz’ for the LEEA of a single hypermultiplet, in the form of the most general, of $U(1)$ charge (+4), harmonic-independent hyper-Kähler potential (cf. ref. \cite{34}),

$$K^{(+4)} = \frac{\lambda}{2} (q^+)^2 (q^+)^2 + \left[ \gamma (q^+)^4 + \beta (q^+)^3 q^+ + h.c. \right], \quad (3.2)$$

with one real ($\lambda$) and two complex ($\beta, \gamma$) parameters. The $Sp(1) = SU(2)_{PG}$ transformations of $q^+_a$ leave the form of eq. (3.2) invariant but not the coefficients. Since $SU(2)_{PG}$ is the symmetry of a free hypermultiplet action (C.7), it can be used to reduce the number of coupling constants in the family of hyper-Kähler metrics associated with the hyper-Kähler potential (3.2) from five to two. In addition, eq. (3.2) implies the conservation law \cite{34}

$$D^{++} K^{(+4)} = 0 \quad (3.3)$$

that is valid on the equations of motion of the hypermultiplet HSS superfield,

$$D^{++} q^+_a = \partial K^{(+4)}/\partial q^+_a \quad \text{and} \quad D^{++} q^+_a = -\partial K^{(+4)}/\partial q^+_a. \quad (3.4)$$

### 3.1 Hypermultiplet LEEA in the Coulomb branch

The manifestly N=2 supersymmetric HSS description of the hypermultiplet LEEA allows us to exploit the constraints imposed by unbroken N=2 supersymmetry and its automorphism symmetry in the very efficient and transparent way. For example, as regards a perturbation theory in 4d, N=2 supersymmetric QED (or in the Coulomb branch of N=2 supersymmetric $SU(2)$ Yang-Mills theory \cite{1}), the unbroken symmetry is given by

$$SU(2)_R, \text{global} \times U(1)_{\text{local}}. \quad (3.5)$$
Fig. 1. The one-loop harmonic supergraph contributing to the induced hypermultiplet self-interaction.

The unique hypermultiplet self-interaction consistent with N=2 supersymmetry and the symmetry (3.5) in HSS is described by the hyper-Kähler potential

$$K^{(+4)}_{\text{Taub-NUT}} = \frac{\lambda}{2} \left( \frac{q^+ q^+}{q^+ q^+} \right)^2,$$

just because this is the only function of U(1) charge (+4) that is independent upon harmonics, being invariant under the U(1) phase transformations (2.51) too.

The induced coupling constant $\lambda$ of eq. (3.6) in the one-loop approximation can be determined from a calculation of the HSS graph shown in Fig. 1, after taking into account central charges [28]. The analytic propagator (the wave lines in Fig. 1) of the N=2 vector HSS superfield $V^{++}$ in N=2 supersymmetric Feynman gauge is given by [47]

$$i \langle V^{++}(1)V^{++}(2) \rangle = \frac{1}{\Box_1} (D_1^{+})^4 \delta^{12}(Z_1 - Z_2) \delta^{(-2,2)}(u_1, u_2),$$

where the harmonic delta-function $\delta^{(-2,2)}(u_1, u_2)$ has been introduced [47]. The hypermultiplet analytic propagator (the solid lines in Fig. 1) with non-vanishing central
Fig. 2. Multi-loop corrections to the hypermultiplet self-interaction.

The propagator is more complicated \[28\],

\[
i \left\langle q^+(1)q^+(2) \right\rangle = \frac{-1}{\Box_1 + m^2} \frac{(D_1^+)^4(D_2^+)^4}{(u_1^2 u_2^2)^3} e^{-\epsilon_3[v(2) - v(1)]} \delta^{12}(Z_1 - Z_2), \tag{3.8}
\]

where we have used the ‘pseudo-real’ \( Sp(1) \) notation, see eq. (C.7). The ‘bridge’ \( v \) satisfies an equation \( D^{++}e^v = 0 \), whereas \( m^2 = |Z|^2 \) is the hypermultiplet BPS mass. One finds

\[
i v = -Z(\bar{\theta}^+ \bar{\theta}^-) - \bar{Z}(\theta^+ \theta^-). \tag{3.9}
\]

A calculation of the LEEA from the one-loop HSS graph in Fig. 1 is straightforward, while \( Z \neq 0 \) is essential \[28\]. We find the predicted form (3.6) of the induced hyper-Kähler potential with the one-loop induced NLSM coupling constant

\[
\lambda = \frac{g^4}{\pi^2} \left[ \frac{1}{m^2} \ln \left( 1 + \frac{m^2}{\Lambda^2} \right) - \frac{1}{\Lambda^2 + m^2} \right] \tag{3.10}
\]
in terms of the abelian coupling constant \( g \), the bare BPS mass \( m^2 \) and the IR-cutoff \( \Lambda \).

Note that \( \lambda \neq 0 \) only when \( Z \neq 0 \). The naive ‘non-renormalization theorem’ forbids the appearance of the quantum corrections given by integrals over a subspace of the full \( N=2 \) superspace, like the one in eq. (3.6). However, this ‘non-renormalization theorem’ does not apply in the case under consideration, because of the non-vanishing central charge \( Z \) (cf. ref. \[17\]).

The infra-red divergence of the one-loop induced effective coupling \( \lambda \) in eq. (3.10) may cause concern about the consistency of our approach. It is worth mentioning, however, that eq. (3.6) is manifestly \( N=2 \) supersymmetric at any \( \lambda \). In other words, the infra-red cutoff is consistent with \( N=2 \) supersymmetry of the hypermultiplet LEEA. We expect the IR divergences to disappear after summing up (IR-divergent) higher-loop Feynman supergraphs with four external FS hypermultiplets (Fig. 2).

It seems to be rather difficult to calculate the exact dependence of \( \lambda \) upon the fundamental parameters of the underlying gauge theory, such as the Yang-Mills coupling

\[2\text{Eq. (3.8) reduces to the HSS hypermultiplet propagator found in ref. [17] when } Z = 0.\]
constant, an N=2 central charge and a Higgs expectation value, by using quantum perturbation theory. The perturbative expansion in terms of Feynman graphs assumes the validity of the weak-coupling description associated with a particular choice of fields and coupling constants in (a portion of) the quantum moduli space, whereas the parameter $\lambda$ is essentially non-perturbative (cf. the Fermi constant $F_\pi$ in QCD). Nevertheless, the exact geometrical (Taub-NUT) nature of the result (3.6), associated with a fundamental monopole belonging to the hypermultiplet under consideration, is very clear in the HSS approach. The coefficient $\lambda$ is simply related to the Taub-NUT mass, $\lambda = \frac{1}{4}M^{-2}$ (see Appendix C).

To understand the hyper-Kähler geometry associated with the hyper-Kähler potential (3.6), one may perform an N=2 supersymmetric reduction of the FS hypermultiplet to an N=2 tensor multiplet, and then rewrite the corresponding dual HSS action into PSS. Unlike the off-shell FS hypermultiplet, the off-shell N=2 tensor multiplet has the finite number of the auxiliary fields — see eq. (2.31). The N=2 tensor multiplet constraints (2.28) can be rewritten to HSS as

\[ D^{++}L^{++} = 0 \quad \text{and} \quad \bar{T}^{++} = L^{++}, \quad (3.11) \]

where $L^{++} = u_i^+ u_j^+ L^{ij}(Z)$. Let’s substitute (we temporarily set $\lambda = 1$)

\[ \mathcal{K}^{(++)}_{TN} = \frac{1}{2}(\bar{q}^+ q^+)^2 = -2(L^{++})^2, \quad \text{or, equivalently,} \quad \bar{q}^+ q^+ = 2iL^{++}, \quad (3.12) \]

which is certainly allowed because of eq. (3.3). The constraints (2.28) can be taken into account off-shell, by using an extra real analytic HSS superfield $\omega$ as the Lagrange multiplier. Changing the variables from $(\bar{q}^+, q^+)$ to $(L^{++}, \omega)$ amounts to the N=2 duality transformation in HSS. An explicit solution to eq. (3.12) is known [48],

\[ q^+ = -i \left(2u_1^+ + if^{++}u_1^-\right) e^{-i\omega/2}, \quad \bar{q}^+ = i \left(2u_2^+ - if^{++}u_2^-\right) e^{i\omega/2}, \quad (3.13) \]

where the function $f^{++}$ is given by

\[ f^{++}(L, u) = \frac{2(L^{++} - 2iu_1^+ u_2^+)}{1 + \sqrt{1 - 4u_1^+ u_2^+ u_1^- u_2^- - 2iL^{++}u_1^- u_2^-}}. \quad (3.14) \]

It is straightforward to rewrite the free (massless) HSS action (C.7) in terms of the new variables. This results in the improved (i.e. N=2 superconformally invariant) N=2 tensor multiplet action [48],

\[ S_{\text{improved}} = \frac{1}{2} \int d\zeta^{(4)} du(f^{++})^2. \quad (3.15) \]
The action dual to the NLSM action defined by eq. (C.10) is, therefore, given by a sum of the non-improved (quadratic) and improved (non-polynomial) HSS actions for the N=2 tensor multiplet \[ [23, 48] \],

\[
S_{\text{Taub-NUT}}[L; \omega] = S_{\text{improved}} + \frac{1}{2} \int d\zeta^{(-4)} du \left[ (L^{++})^2 + \omega D^{++} L^{++} \right]. \tag{3.16}
\]

To understand the peculiar structure of the improved action defined by eqs. (3.14) and (3.15), let’s extract a constant ‘vacuum expectation value’ \( c^{ij} \) out of \( L_{ij} \) by rewriting it to the form

\[
L^{++}(\zeta, u) = c^{++} + l^{++}(\zeta, u). \tag{3.17}
\]

We use the notation

\[
c^{\pm \pm} = c^{ij} u^\pm_i u^\pm_j, \quad (c^{ij}) = \varepsilon_{ik} \varepsilon_{jl} c^{kl}, \quad c^2 = \frac{1}{2} c^{ij} c^{ij} \neq 0,
\]

\[
f^{++}(L, u) \equiv l^{++} f(y), \quad y = l^{++} c^{--}. \tag{3.18}
\]

The function \( f(y) \) then appears to be a solution to the quadratic equation,

\[
\frac{1}{f(y)} = 1 + \frac{y f(y)}{4 c^2}. \tag{3.19}
\]

It can be shown that this equation follows from the rigid N=2 superconformal invariance on the improved action [18]. The improved action, defined by eqs. (3.15) and (3.19), does not really depend upon \( c^{ij} \) because of its \( SU(2)_{\text{conf}} \) invariance.

It is also straightforward to rewrite eq. (3.16) to N=2 PSS and even to N=1 superspace, where it takes the form of eqs. (B.3) and (B.10), respectively. After restoring the dependence upon \( \lambda \equiv \frac{1}{4} M^{-2} \) in eq. (3.16), we thus reproduce the PSS action of ref. [22], with

\[
\oint G = M \oint_{C_0} \frac{Q_{(2)}^2}{2 \xi} + \oint_{C_r} Q_{(2)} (\ln Q_{(2)} - 1), \tag{3.20}
\]

where the contour \( C_0 \) goes around the origin, whereas the contour \( C_r \) encircles the roots of a quadratic equation,

\[
Q_{(2)}(\xi) = 0, \tag{3.21}
\]

in complex \( \xi \)-plane. The hyper-Kähler metric of the N=2 NLSM defined by eqs. (3.16) and (3.20) is equivalent to the Taub-NUT metric after the N=1 superspace Legendre transform (see Appendix B) [23], with the mass parameter \( M = \frac{1}{2} \lambda^{-1/2} \). Stated differently, eq. (3.6) describes the hyper-Kähler potential of the Taub-NUT metric in HSS (Appendix C).
The known $U(2) = SU(2) \times U(1)$ isometry of the Taub-NUT metric is clearly consistent with eq. (3.5). It is instructive to investigate the realization of this internal symmetry in the various formulations of the $N=2$ Taub-NUT NLSM mentioned in this paper. The hyper-Kähler potential (3.6) in terms of the FS hypermultiplet $q^+$ provides the manifestly invariant formulation. In the dual HSS form, in terms of $(L^{++}, \omega)$, the non-abelian factor $SU(2)$ is represented by the $SU(2)_{\text{conf}}$, whereas the abelian factor $U(1)$ is realized by constant shifts of $\omega$. The $SU(2)^R$ transformations act in PSS in the form of projective (fractional) transformations (B.7).

Eqs. (B.8) and (B.9) imply that the $SU(2)^R$ invariant PSS potential $G(Q_{(2)})$ should be ‘almost’ linear in $Q_{(2)}$, like in the second term of eq. (3.20) where the extra logarithmic factor is merely responsible for defining the (closed) integration contour. The transition $u_i \to \xi_i = (1, \xi)$ describes a holomorphic projection of HSS to PSS where the analytic superfield $L^{++}(\zeta, u)$ is replaced by a holomorphic (with respect to $\xi$) section $Q_{(2)}(L, \xi)$ of the line bundle $O(2)$ whose fiber is parametrized by the constrained N=1 superfields, $\chi$ and $g$. The equation $y^2 = \chi - i\xi g + \xi^2 \bar{\chi}$ defines the Riemann sphere in $\mathbb{C}^2$ parametrized by $(y, \xi)$, where $y^2 \equiv |Q_{(2)}|^2$.

### 3.2 Hypermultiplet LEEA in the Higgs branch

There are no instantons in an abelian $N=2$ supersymmetric quantum field theory. This means that the perturbative result of subsect. 3.1 about the hypermultiplet LEEA described by the Taub-NUT metric is, in fact, exact in the abelian case. If, however, the underlying $N=2$ gauge field theory has a non-abelian gauge group whose rank is larger than one (say, $SU(3)$), one may expect nonperturbative contributions to the hypermultiplet LEEA (in the Higgs branch) from instantons and anti-instantons, which break the $U(1)$ symmetry in eq. (3.5) (cf. ref. [11]).

Given the most general $SU(2)^R$-invariant hyper-Kähler potential (3.2), let’s make a substitution [49]

\[
\mathcal{K}^{(+4)}(q, \bar{q}) \equiv \lambda \frac{1}{2} (q^+ \bar{q})^2 (q^+)^2 + \left[ \gamma (q^+)^4 + \beta (q^+)^3 q^+ + \text{h.c.} \right] = L^{++++}(\zeta, u) , \tag{3.22}
\]

where the real analytic superfield $L^{++++}$ satisfies the conservation law (3.3), i.e.

\[
D^{++}L^{++++} = 0 . \tag{3.23}
\]

Eq. (3.23) can be recognized as the off-shell $N=2$ superspace constraints

\[
D^{(iL^{jklm})} = \tilde{D}^{(iL^{jklm})} = 0 , \tag{3.24}
\]

24
where \( L^{++++} = u_i^+ u_j^+ u_k^+ u_l^+ L^{ijkl}(Z) \), while eq. (3.22) implies the reality condition

\[
L^{ijkl} = \varepsilon_{im} \varepsilon_{jn} \varepsilon_{kp} \varepsilon_{lq} L^{mnpq},
\]

(3.25)
defining together the \( O(4) \) projective \( N=2 \) supermultiplet (see subsect. 2.2. and Appendix B). Unlike the \( O(2) \) tensor multiplet, the \( O(4) \) multiplet does not have a conserved vector (or a gauge antisymmetric tensor) amongst its field components, which implies the absence of the \( U(1) \) triholomorphic isometry in the \( N=2 \) NLSM to be constructed in terms of \( L^{++++} \).

The \( N=2 \) supersymmetric PSS construction of the invariant actions (Appendix B), in terms of a PSS potential \( G(Q(\xi), \xi) \), equally applies to the projective \( O(4) \) supermultiplets, while \( L^{++++} \) should enter the universal PSS action (B.3) via the argument

\[
Q_{(4)}(Z, \xi) = \xi_i \xi_j Q^{ijkl}(Z), \quad \xi_i = (1, \xi).
\]

(3.26)
The \( N=1 \) superspace projections of eqs. (3.26) and (B.3) are given by eqs. (B.24) and (B.25), respectively, in terms of the \( N=1 \) chiral superfield \( \chi \), the \( N=1 \) complex linear superfield \( W \), and the \( N=1 \) general (unconstrained) real superfield \( V \).

The auxiliary field component \( C \) of the \( O(4) \) projective multiplet, defined by eq. (2.31) (or, equivalently, the \( N=1 \) superfield \( V \) in the \( N=1 \) superspace reformulation) enters the PSS action (B.3) as the Lagrange multiplier, whose elimination via its ‘equation of motion’ gives rise to an algebraic constraint,

\[
Re \int \frac{\partial G}{\partial Q_{(4)}} = 0.
\]

(3.27)
Eq. (3.27) reduces the number of independent physical real scalars from five to four, which is consistent with the fact that the real dimension of any hyper-Kähler manifold is a multiple of four (Appendix A). After solving the constraint (3.27), the complex linear \( N=1 \) superfield \( W \) can be traded for yet another \( N=1 \) chiral superfield \( \psi \), by the use of the \( N=1 \) superfield Legendre transform. It results in the \( N=1 \) superspace Kähler potential \( K(\chi, \bar{\chi}, \psi, \bar{\psi}) \) of the \( N=2 \) supersymmetric NLSM (Appendix B).

The most straightforward procedure of calculating the dependence \( q(L) \) out of the definition (3.22), as well as performing an explicit \( N=2 \) supersymmetric duality transformation of the free FS action (C.7) into the dual (improved) action of the constrained \( N=2 \) superfield \( L^{ijkl} \) defined by eqs. (3.24) and (3.25), needs the explicit roots of the quartic polynomial. Though it is possible to calculate the roots, the results are not very illuminating. In fact, the explicit roots are not even necessary to determine the explicit form of the dual \( N=2 \) PSS action. Within the manifestly \( N=2 \)
supersymmetric approach, it is the $SU(2)_R$ symmetry and regularity requirements that are sufficient to fix the action in question, either in HSS or PSS (cf. ref. [15]). The one real and two complex constants, $(\lambda, \beta, \gamma)$, respectively, parametrizing the hyper-Kähler potential (3.22) can be naturally united into an $SU(2)$ 5-plet $c^{ijkl}$ subject to the reality condition (3.25). After extracting a constant piece out of $q^+$, say, $q^+_{a} = u^+_a + \tilde{q}^+_a$ and $u_a = (1, \xi)$, and collecting all constant pieces on the left-hand-side of eq. (3.22), we can identify their sum with a constant piece $c^{++++} = c^{ijkl}u^+_iu^+_j\tilde{u}^+_k\tilde{u}^+_l$ of $L^{++++}$ on the right-hand-side of eq. (3.22), representing the constant vacuum expectation values of the N=1 superfield components of $L^{++++}$,

$$\lambda = \langle V \rangle , \quad \beta = \langle W \rangle , \quad \gamma = \langle \chi \rangle . \quad (3.28)$$

The $SU(2)_R$ transformations in PSS are the projective transformations (B.7), so that the PSS potential $G$ of the ‘improved’ $O(4)$ projective multiplet action having the form (B.3) must be proportional to $\sqrt{Q(4)}$ just because of the transformation rules

$$G(Q'_4(\xi'),\xi') = \frac{1}{(a+b\xi)^2}G(Q_4(\xi),\xi) \quad \text{and} \quad Q'_4(\xi') = \frac{1}{(a+b\xi)^4}Q(4) \xi . \quad (3.29)$$

The most general non-trivial contour $C_r$ in complex $\xi$-plane, whose definition is compatible with the projective $SU(2)$ symmetry, is the one encircling the roots of the quartic (cf. eq. (3.21)),

$$Q_4(\xi) = p + \xi q + \xi^2 r - \xi^3 \tilde{q} + \xi^4 \tilde{p} , \quad (3.30)$$

with one real ($r$) and two complex ($p,q$) additional parameters belonging to yet another 5-plet of $SU(2)_{PG} = Sp(1)$. The projective $SU(2)$ invariance can be used to reduce the number of independent parameters in the corresponding family of the hyper-Kähler metrics from five to two, which is consistent with the HSS predictions. We didn’t attempt to establish an explicit relation between the HSS coefficients $(\lambda, \gamma, \beta)$ and the PSS coefficients $(r,q,p)$. The most natural contour $C_r$, surrounding roots of the equation

$$Q_4(\xi) = 0 , \quad (3.31)$$

just leads to the only non-singular hyper-Kähler NLSM metric (see subsect. 3.3).

Taking together the above considerations implies that the $SU(2)$-invariant PSS action, dual to the HSS action defined by eqs. (3.1) and (3.2), is given by

$$\frac{1}{2\pi i} \oint G = -\frac{1}{2\pi i} \oint_{C_0} \frac{Q_4}{\xi} + \oint_{C_r} \sqrt{\frac{Q_4}{\xi}} . \quad (3.32)$$
The constraint (3.27) reads in the case as follows:

\[ \oint_{C_r} d\xi \sqrt{Q(4)} = 1 . \quad (3.33) \]

The generalized Legendre transform of the function (3.32) is known [50], so we can simply ‘borrow’ some of the results of ref. [50] here.

Due to the reality condition (3.25), the quartic equation (3.31) has two pairs of roots \( (\rho, -1/\bar{\rho}) \) related by an \( SL(2, \mathbb{Z}) \) transformation and satisfying the defining relation

\[ Q(4)(\xi) = c(\xi - \rho_1)(\bar{\rho}_1 \xi + 1)(\xi - \rho_2)(\bar{\rho}_2 \xi + 1) . \quad (3.34) \]

The branch cuts in complex \( \xi \)-plane can be chosen to run from \( \rho_1 \) to \(-1/\bar{\rho}_2\) and from \( \rho_2 \) to \(-1/\bar{\rho}_1\). The contour integration in eq. (3.33) can then be reduced to the complete elliptic integral (in the Legendre normal form) over the branch cut [50],

\[ \frac{4}{\sqrt{c(1 + |\rho_1|^2)(1 + |\rho_2|^2)}} \int_0^1 \frac{d\xi}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)}} = 1 , \quad (3.35) \]

with the modulus

\[ k^2 = \frac{(1 + \rho_1\bar{\rho}_2)(1 + \rho_2\bar{\rho}_1)}{(1 + |\rho_1|^2)(1 + |\rho_2|^2)} . \quad (3.36) \]

The constraint (3.35) can be explicitly solved in terms of the complete elliptic integrals,

\[ K(k) = \int_0^{\pi/2} \frac{d\gamma}{\sqrt{1 - k^2 \sin^2 \gamma}} , \quad E(k) = \int_0^{\pi/2} d\gamma \sqrt{1 - k^2 \sin^2 \gamma} , \quad (3.37) \]

of the first and second kind, respectively, by using the following parametrization [50]:

\[ \Phi = 2e^{2i\varphi} \left[ \cos(2\psi)(1 + \cos^2 \vartheta) 
+ 2i \sin(2\psi) \cos \vartheta + (2k^2 - 1) \sin^2 \vartheta \right] K^2(k) , \]

\[ H = 8e^{i\varphi} \sin \vartheta \left[ \sin(2\psi) 
- i \cos(2\psi) \cos \vartheta + i(2k^2 - 1) \cos \vartheta \right] K^2(k) , \]

\[ V = 4 \left[ -3 \cos(2\psi) \sin^2 \vartheta + (2k^2 - 1)(1 - 3 \cos^2 \vartheta) \right] K^2(k) , \quad (3.38) \]

where the Euler ‘angles’ \( (\vartheta, \psi, \varphi) \) have been introduced. Together with the modulus \( k \) they represent the independent (superfield) coordinates in the N=2 NLSM under consideration.

Being applied to the function (3.32), the Legendre transform (2.35) with respect to \( W \), on the constraint (2.37) having of the form (3.35), gives rise to the (double cover
of) Atiyah-Hitchin (AH) space $M_2$ as the NLSM target space. It was demonstrated in ref. [50] by comparing the hyper-Kähler structures, which is enough to claim the equivalence between the N=2 NLSM metrics in accordance with the general discussion in subsect. 2.1.

The AH space $M_2$ was originally introduced as the (centered) moduli space of two (fundamental) BPS $SU(2)$ monopoles [11]. The metric of the AH space is known to be the only regular hyper-Kähler metric with the entirely non-triholomorphic $SO(3)$ symmetry rotating hyper-Kähler complex structures [11]. In the Donaldson description [51] of the AH space (see subsection 4.2 for basic definitions), the AH space $M_2$ is described by the quotient of an algebraic curve in $C^3$,

$$x^2 - zy^2 = 1 , \quad \text{where} \quad x, y, z \in C ,$$

(3.39)

under $Z_2 : (x, y, z) \equiv (-x, -y, z)$. Eq. (3.39) thus describes the $SU(2)$-symmetric universal (2-fold) covering $\tilde{M}_2$ of the AH space. Since our N=2 superspace techniques are purely local, we are unable to distinguish between $M_2$ and $\tilde{M}_2$. Accordingly, we make no distinction between $SO(3)$ and $SU(2)$ in our considerations.  

The line element of any four-dimensional (Euclidean) metric having $SO(3)$ isometry can be written down in the Bianchi IX formalism as follows:

$$ds^2 = f^2(t)dt^2 + A^2(t)\sigma_1^2 + B^2(t)\sigma_2^2 + C^2(t)\sigma_3^2 ,$$

(3.40)

where $f(t) = \frac{1}{2}ABC$, while $\sigma_i$ stand for the $SO(3)$-invariant 1-forms

$$\sigma_1 = + \frac{1}{2} (\sin \psi d\vartheta - \sin \vartheta \cos \psi d\varphi) ,$$

$$\sigma_2 = - \frac{1}{2} (\cos \psi d\vartheta + \sin \vartheta \sin \psi d\varphi) ,$$

$$\sigma_3 = + \frac{1}{2} (d\psi + \cos \vartheta d\varphi) ,$$

(3.41)

in terms of the Euler angles ($\vartheta, \psi, \varphi$). The one-forms (3.41) obey the relations

$$\sigma_i \wedge \sigma_j = \frac{1}{2} \varepsilon_{ijk} d\sigma_k , \quad i, j, k = 1, 2, 3 .$$

(3.42)

The standard parametrization of the AH metric uses the complete elliptic integrals (3.37) and the modulus $k$ related to the variable $t$ of eq. (3.40) via the relation

$$t = -\frac{2K(k')}{\pi K(k)} ,$$

(3.43)

\footnote{See, however, ref. [3] for a discussion of the global issues associated with $M_2$ and $\tilde{M}_2$.}
where \( k' \) is known as the complementary modulus, \( k'^2 = 1 - k^2 \). The AH metric in terms of the independent coordinates \((k; \vartheta, \psi, \varphi)\) reads \[41\]

\[
ds_{AH}^2 = \frac{1}{4} A^2 B^2 C^2 \left(\frac{dk}{kk'k''K} \right)^2 + A^2(k)\sigma_1^2 + B^2(k)\sigma_2^2 + C^2(k)\sigma_3^2 ,
\]

(3.44)

where the coefficient functions satisfy the relations \[41\]

\[
AB = -K(k) \left[ E(k) - K(k) \right], \\
BC = -K(k) \left[ E(k) - k'^2K(k) \right], \\
AC = -K(k)E(k).
\]

(3.45)

The parametrization (3.38) leads to the AH metric in the form (3.44) whose coefficient functions \((A, B, C)\) are given by a cyclic permutation of those in eq. (3.45) \[50\]. The Kähler potential of the AH metric was calculated in ref. \[52\].

In the limit \( k \to 1 \) (or, equivalently, \( k' \to 0 \)), one has an asymptotic expansion

\[
K(k) \approx -\log k' \left[1 + \frac{(k')^2}{4}\right] + \ldots ,
\]

(3.46)

which suggests us to make a redefinition

\[
k' = \sqrt{1 - k^2} \approx 4 \exp \left(\frac{1}{\gamma}\right),
\]

(3.47)

and describe the same limit at \( \gamma \to 0^- \). After substituting eq. (3.46) into eq. (3.45) one finds that the AH metric becomes exponentially close to the Taub-NUT metric in the form (3.40) subject to the additional relations

\[
A^2 \approx B^2 \approx \frac{1 + \gamma}{\gamma^2} , \quad C^2 \approx \frac{1}{1 + \gamma},
\]

(3.48)

and with the negative mass parameter \( M = -\frac{1}{2} \) \[11\]. In terms of the general hyper-Kähler potential (3.2) whose ‘Taub-NUT’ parameter \( \lambda \) is fixed, we are left with the one-parameter family of the hyper-Kähler metrics, all having the same perturbative behaviour. In this context, the unique solution given by the AH metric follows from a calculation of the one-instanton contribution to the LEEM, which unambiguously determines the last parameter \[16\] (see subsect. 3.3. for another argument). It also implies that \( |\gamma|^{-1} \) is proportional to the one-instanton action.

The extra \( U(1) \) symmetry of the Taub-NUT metric (when compared to the AH metric) is the direct consequence of the relation \( A^2 = B^2 \) arising in the asymptotic limit described by eq. (3.48). The vicinity of \( k' \approx 0^+ \) describes the region of
hypermultiplet moduli space where quantum perturbation theory applies, with the exponentially small (nonholomorphic) AH corrections to the Taub-NUT metric being interpreted as the instanton/anti-instanton contributions. Those nonperturbative corrections are supposed to be related to the existence of BPS monopoles in the underlying non-abelian N=2 field theory. The AH metric, as the metric in the hypermultiplet quantum moduli space, was proposed by Seiberg and Witten [9] in the context of 3d, N=4 supersymmetric gauge field theories, where it can be related to the (Seiberg-Witten) gauge LEEA [1] via the c-map in three spacetime dimensions (sect. 1).

From the N=2 PSS viewpoint, the transition from the perturbative hypermultiplet LEEA to the nonperturbative one thus corresponds to the transition from the $O(2)$ holomorphic line bundle associated with the standard N=2 tensor multiplet to the $O(4)$ holomorphic line bundle associated with the N=2 projective $O(4)$ multiplet. The two holomorphic bundles are topologically different: with respect to the standard covering of $CP(1)$ by two open affine sets, the $O(2)$ bundle has the transition functions $\xi^{-2}$, whereas the $O(4)$ bundle has the transition functions $\xi^{-4}$. In general, the variable $Q_{(2j)}$ is the coordinate in the fiber of the $O(2j)$ line bundle.

### 3.3 Atiyah-Hitchin metric and elliptic curve

The quadratic dependence of $Q_{(2)}$ upon $\xi$ in eqs. (B.2) and (B.10) allows us to interpret $Q_{(2)}(\xi)$ as a holomorphic (of degree 2) section of PSS, fibered by the superfields $(\chi, g)$ and topologically equivalent to the Riemann sphere described by an algebraic equation $y^2 = \chi - i\xi g + \xi^2 \chi$ with $y^2 \equiv Q_{(2)}$. Similarly, the quartic dependence of $Q_{(4)}$ upon $\xi$ in eqs. (B.2) and (3.26) allows us to interpret $Q_{(4)}(\xi)$ as a holomorphic (of degree 4) section of PSS, fibered by the superfields $(\chi, W, V)$ and topologically equivalent to the elliptic curve $\Sigma_{\text{Hyper}}$ defined by an algebraic equation

$$y^2(\xi) = \chi + \xi W + \xi^2 V - \xi^3 \chi + \xi^4 \chi,$$

where $y^2 \equiv Q_{(4)}$. The non-perturbative hypermultiplet LEEA can, therefore, be encoded in terms of the genus-one Riemann surface $\Sigma_{\text{Hyper}}$. This result is quite similar to the famous Seiberg-Witten description [1] of the exact LEEA in the $SU(2)$ N=2 super-Yang-Mills theory, in terms of the elliptic curve $\Sigma_{\text{SW}}$.

The twistor construction of the AH metric [11] is known to be closely related to the spectral curve $\Sigma_{\text{H}}$. The elliptic curve $\Sigma_{\text{H}}$ naturally arises in the uniformization process of the algebraic curve (3.39) in the Donaldson description of the AH space [52]. This actually provides enough evidence to identify $\Sigma_{\text{H}}$ with $\Sigma_{\text{Hyper}}$. [19].
The defining equation (3.49) can be put into the normal (Hurtubise) form \[41\]

\[
\tilde{y}^2(\tilde{\xi}) = K^2(k)\tilde{\xi} \left[ kk'(\tilde{\xi}^2 - 1) + (k^2 - k'^2)\tilde{\xi} \right].
\] (3.50)

Eq. (3.50) is simply related to another standard (Weierstrass) form, \( y^2 = 4x^3 - g_2x - g_3 \) \[55\]. Therefore, in accordance with refs. \[41, 55\], the real period \( \omega \) of \( \Sigma_H \) is

\[
\omega \equiv 4k_1, \quad \text{where} \quad 4k_1^2 = kk'K^2(k),
\] (3.51)

whereas the complex period 'matrix' of \( \Sigma_H \) is given by

\[
\tau = \frac{iK(k')}{K(k)}.
\] (3.52)

The normal form (3.50) is related to that of eq. (3.49) by the projective \( SU(2) \) transformation (cf. eqs. (B.7) and (B.9))

\[
\xi = \frac{\tilde{\alpha} + \tilde{\beta}}{\beta \xi + a}, \quad y = \frac{\tilde{y}}{(\beta \xi + a)^2}, \quad |a|^2 + |b|^2 = 1,
\] (3.53)

whose complex parameters \((a, b)\) are functions of the Euler 'angles' \((\vartheta, \psi, \phi)\) (see eq. (3.38) in subsect. 3.2) \[54\],

\[
a = e^{i\varphi} \left[ \sqrt{\frac{1 - k}{2}} \sin \frac{\vartheta}{2} e^{i\frac{\psi}{2}} - i \sqrt{\frac{1 - k}{2}} \cos \frac{\vartheta}{2} e^{i\frac{\psi}{2}} \right],
\]

\[
b = e^{i\varphi} \left[ -\sqrt{\frac{1 + k}{2}} \sin \frac{\vartheta}{2} e^{-i\frac{\psi}{2}} - i \sqrt{\frac{1 - k}{2}} \cos \frac{\vartheta}{2} e^{i\frac{\psi}{2}} \right].
\] (3.54)

Eq. (3.54) implies another parametrization of the quartic (49) \[54\],

\[
\Phi = -\frac{1}{4} K^2(k)e^{-2i\varphi} \sin^2 \vartheta \left( 1 + k^2 \sinh^2 \nu \right),
\]

\[
H = -\frac{1}{2} K^2(k) \sin(2\varphi)e^{-i\varphi} \left( 1 + k^2 \cos \psi \tan \vartheta \sinh \nu \right),
\]

\[
V = +\frac{1}{2} K^2(k) \left[ 2 - k'^2 + 3 \sin^2 \vartheta \left( k'^2 \cos^2 \psi - 1 \right) \right],
\] (3.55)

where \( \nu \equiv \log \left( \tan \frac{\vartheta}{2} \right) + i\psi \). The parametrization (3.55) is closely related to that of eq. (3.38), by a cyclic permutation of the coefficient functions \((A, B, C)\) in eq. (3.44).

At generic values of the AH modulus \(k\), \(0 < k < 1\), the roots of the Weierstrass form are all different from each other, while they all lie on the real axis, say, at \( e_3 < e_2 < e_1 < \infty = (e_4) \). Accordingly, the branch cuts are running from \( e_3 \) to \( e_2 \) and from \( e_1 \) to \( \infty \). The \( C_r \) integration contour in the PSS formulation of the exact
hypermultiplet \text{LEEA} in eq. (3.32) can now be interpreted as the contour integral over the non-contractible \(\alpha\)-cycle of the elliptic curve \(\Sigma_H\) \cite{54}, again in very close analogy to the standard writing of the Seiberg-Witten solution \cite{1} in terms of the abelian differential \(\lambda_{SW}\) integrated over the periods of \(\Sigma_{SW}\).

The perturbative (Taub-NUT) limit \(k \to 1\) corresponds to the situation when \(e_2 \to e_1\), so that the \(\beta\)-cycle of \(\Sigma_H\) degenerates. The curve (3.50) then asymptotically approaches a complex line, \(\tilde{y} \sim \pm K \tilde{\xi}\). Another limit, \(k \to 0\), leads to a coordinate \textit{bolt-type singularity} of the AH metric in the standard parametrization (3.44) \cite{41}. In the context of monopole physics, \(k \to 0\) corresponds to the coincidence limit of two centered monopoles. In the context of the hypermultiplet \text{LEEA}, \(k \to 0\) implies \(e_2 \to e_3\), so that the \(\alpha\)-cycle of \(\Sigma_H\) degenerates, as well as the whole hypermultiplet action associated with eq. (3.32). The two limits, \(k \to 1\) and \(k \to 0\), are related by the modular transformation exchanging \(k\) with \(k'\), and \(\alpha\)-cycle with \(\beta\)-cycle \cite{54}.

The AH metric is known to be \textit{only} regular and complete four-dimensional hyper-Kähler metric with the purely rotational \(SO(3)\) isometry \cite{11, 53, 57}. Being regular means the absence of singularities, while completeness means that every curve of finite length in the hyper-Kähler manifold under consideration has a limiting point. It is worth mentioning that only regular (i.e. globally well-defined) hyper-Kähler metrics can be interpreted as the metrics governing the hypermultiplet \text{LEEA}. Since the most general four-dimensional \(su(2)_{K}\)-invariant hyper-Kähler metrics in HSS are given by the two-parametric family described by the hyper-Kähler potential (3.2), it is important to find a simple way by which one can distinguish the regular AH metric amongst them. This problem is also present in the generalized Legendre transform construction of hyper-Kähler metrics. The underlying elliptic curve provides a nice interpretation to this problem \cite{49}. The most general integration contour \(C_r\) in the PSS construction (3.32) is described by eq. (3.30). As is clear from the results of this subsection, any non-trivial contour \(C_r\) can be equally interpreted as a linear combination, \(C_r = n_1 \alpha + n_2 \beta\), of the fundamental cycles, \(\alpha\) and \(\beta\), of the underlying elliptic curve \(S_H\), with some integral coefficients, \(n_1\) and \(n_2\). An integration over the \(\beta\)-cycle is known to yield a bolt-type singularity \cite{41}, and it thus has be excluded. The contour integration over the \(\alpha\)-cycle yields the regular AH solution. The value \(n_1\) of the ‘winding’ number is obviously not relevant for regularity of the metric. \footnote{The \(D_k\) gravitational instanton metrics are not regular \cite{12, 14, 15}, and they only possess \(U(1)\) rotational isometry.}

A generalization of the ‘regular’ 1-cycle to the higher-genus spectral curve associated with the charge-\(n\) (centered) monopole space (subsect. 4.2) just leads to the Ercolani-Sinha constraints \cite{19} mentioned in the Introduction.
4 More hypermultiplets and larger gauge groups

If the underlying N=2 supersymmetric gauge field theory has a larger (of rank \( r > 1 \)) simple gauge group, such as \( SU(n + 1) \), \( n > 1 \), there may be more (magnetically charged) massive hypermultiplets \( q^{A+} \), \( A = 1, 2, \ldots, 2n \), in the LEEA, either in the Coulomb branch or in the Higgs branch. Their exact self-interaction is described by the hyper-Kähler NLSM governed by a hyper-Kähler potential \( \mathcal{K}^{(+4)}(q^{A+}; u_i^\pm) \) in HSS,

\[
-\mathcal{L}_{\text{LEE}A} = q_A^+ D^{++} q^{A+} + \mathcal{K}^{(+4)}(q^{A+}; u_i^\pm) .
\]  

(4.1)

The non-anomalous \( SU(2)_R \) symmetry further implies that the hyper-Kähler potential \( \mathcal{K}^{(+4)} \) should be independent upon harmonics. Therefore, the analytic function \( \mathcal{K}^{(+4)} \) is given by a real quartic polynomial of the analytic HSS superfields \( q^{A+} \),

\[
\mathcal{K}^{(+4)}(q) = P^{(+4)}(q) \equiv \lambda_{ABCD} q^{A+} q^{B+} q^{C+} q^{D+} ,
\]  

(4.2)

whose coefficients \( \lambda_{(ABCD)} \) are totally symmetric and are subject to the reality condition, \( \overline{P}^{(+4)} = P^{(+4)} \). In a bit more explicit notation, \( q^{A+} = (q^{+}_{M}, \overline{q}^{+}_{\bar{M}}) \), we have

\[
P^{(+4)} \equiv \lambda_{MN,PQ} q^{+}_{M} q^{+}_{N} \overline{q}^{+}_{P} \overline{q}^{+}_{Q} + \tilde{\lambda}_{NMPQ} q^{+}_{M} q^{+}_{N} q^{+}_{P} \overline{q}^{+}_{Q} + h.c. ,
\]  

(4.3)

with some constants \( \lambda_{(MN),(PQ)} \), \( \tilde{\lambda}_{(NMPQ)} \) and \( \tilde{\lambda}_{(NMP)} \).

Not all of the constants are really significant since the kinetic terms in eq. (4.1) have the manifest global \( Sp(n) \) symmetry under the transformations of \( q^{A+} \). Hence, the space of all \( SU(2)_R \)-invariant hyper-Kähler metrics in \( 4n \) real dimensions is parametrized by the quotient

\[
\mathcal{T}_n = \{ P^{(+4)}(\lambda) \}/Sp(n) .
\]  

(4.4)

It may not be accidental that the \( Sp(n) \) factor in eq. (4.4) coincides with the maximal \( Sp(n) \) holonomy group of hyper-Kähler manifolds in \( 4n \) real dimensions.

4.1 Exact hypermultiplet LEEA in the Coulomb branch

In the Coulomb branch of the N=2 super-Yang-Mills theory with the \( G = SU(n + 1) \) gauge group, the gauge symmetry \( G \) is (generically) spontaneously broken to its maximal Cartan subgroup \( H = U(1)^n \), due to a non-vanishing vacuum expectation value, \( \langle \phi \rangle \neq 0 \), of the adjoint Higgs field. Since \( \pi_2[SU(n + 1)/U(1)^n] = \pi_1[U(1)^n] = 33 \)
\textbf{Z}^n$, one expects rank($G$) = $n$ different types of magnetic monopoles associated to each of the Cartan generators and belonging to hypermultiplets. The corresponding classical solitonic solutions can be obtained via embedding the known $SU(2)$ solutions (see Appendix D) along the simple root directions in $G$ (see ref. [17], or a recent review [58]). As regards the LEEA of the hypermultiplets corresponding to different simple roots, it should possess the tri-holomorphic (or translational) $U(1)^n$ isometry, in addition to the rotational $SU(2)_R$ isometry discussed above. In this subsection, we present a very simple derivation of the corresponding hyper-Kähler potential in HSS, and give the associated hyper-Kähler metric in components.

First, let’s recall some basic facts about Lie algebras and monopoles [59]. The generators of a rank-$r$ Lie algebra $\mathfrak{g}$ can be naturally divided into $r$ commuting Cartan generators $\{H_i\}$, and the raising and lowering operators, $E_{\vec{\alpha}}$ and $E_{-\vec{\alpha}}$, for each of the $r$-component root vectors $\{\vec{\alpha}\}$,

$$ [H_i, H_j] = 0 \ , \quad [E_{\vec{\alpha}}, \vec{H}] = \vec{\alpha} E_{\vec{\alpha}}, \quad [E_{\vec{\alpha}}, E_{-\vec{\alpha}}] = \vec{\alpha} \cdot \vec{H} \ .$$

(4.5)

We assume the standard normalization, $\text{tr}(H_i H_j) = \delta_{ij}$. The root vectors define a root lattice, while any root vector can be decomposed with respect to a basis of simple (fundamental) roots $\{\vec{\alpha}_i\}$ with non-negative integral coefficients,

$$ \vec{\alpha} = \sum_{i=1}^{r} n_i \vec{\alpha}_i \ .$$

(4.6)

The simple root basis is not unique, with all choices being related by the Weyl group transformations. The dual root lattice is defined by

$$ \vec{\alpha}^* = \sum_{j=1}^{r} n_j^* \vec{\alpha}_j^* \ , \quad \text{where} \quad \vec{\alpha}_j^* = \vec{\alpha}_j / \vec{\alpha}_j^2 \ .$$

(4.7)

Given $G = SU(n+1)$, one has $\vec{\alpha}_i \cdot \vec{\alpha}_i > 0$ and $\vec{\alpha}_i \cdot \vec{\alpha}_j \leq 0$ for $i \neq j$; while $r = n$.

The vacuum expectation value of the Higgs field can always be assigned to the Cartan subalgebra,

$$ \langle \phi \rangle = \vec{h} \cdot \vec{H} \ .$$

(4.8)

If $\vec{h} \in H$ has non-vanishing products with all roots of $G$, as we assume here, the group $G$ is maximally broken to its Cartan subgroup. The unique set of simple roots is naturally distinguished by the condition

$$ \vec{h} \cdot \vec{\alpha}_i > 0 \ , \quad \text{where} \quad 1 \leq i \leq r \ .$$

(4.9)

\footnote{Most of our considerations apply to any simple Lie group $G$. We use $SU(n+1)$ for definiteness.}
A magnetic charge is characterized by another vector $\vec{g} \in H$. The topological (Dirac) quantization condition on the magnetic charge implies that $\vec{g}$ belongs to the dual lattice \( (\text{cf. Appendix D}) \),

$$\vec{g} = \frac{4\pi}{e} \sum_{j=1}^{r} k_j \vec{\alpha}_j^*$$

where \( \{k_j\} \) are topological charges belonging to the integral homotopy \( \mathbb{Z}^r \), and \( g = \frac{4\pi}{e} \) is the unit of magnetic charge. The general BPS mass formula reads

$$M = \left| \vec{\alpha} \cdot \vec{g} \right|,$$

so that the mass spectrum of the fundamental monopoles (of unit charge) is given by

$$m_j = \left| \vec{\alpha} \cdot \vec{g}_j \right|, \quad j = 1, 2, \ldots, r.$$ (4.12)

The long-range force between two different monopoles \( i \) and \( j \) obeys the standard (inverse radius squared) law, whose strength is given by the inner product of magnetic charges, \( \vec{g}_i \cdot \vec{g}_j / 4\pi \). The classical solitonic solutions describing \( SU(n+1) \) monopoles can be constructed, in principle, by the ADHMN method \( [13] \), whose (Nahm) data is, however, highly constrained. It makes a direct construction of the multi-monopole moduli space metrics via the so-called ‘moduli space approximation’ \( [30] \) to be highly non-trivial.

Each of \( n \) (translational) commuting \( U(1) \) isometries can be realized by rigid phase rotations of a single \( q \)-hypermultiplet in eq. (4.3) \( (\text{cf. eq. (2.51)}) \),

$$q_m^+ \rightarrow e^{i\alpha_m} q_m^+ \quad \text{and} \quad \vec{q}_m^+ \rightarrow e^{-i\alpha_m} \vec{q}_m^+, \quad \text{where} \quad m = 1, 2, \ldots, n.$$ (4.13)

This implies that the most general quartic polynomial of eq. (4.3) reduces in this case to a merely quadratic real function of the invariants

$$iL_{m}^{++} \equiv \vec{q}_m^+ q_m^+,$$ (4.14)

which reads

$$P^{(4)} = \frac{1}{2} \sum_{p,q=1}^{n} \lambda_{pq} L_p^{++} L_q^{++} \equiv \frac{1}{2} \sum_{j=1}^{n} \lambda_j (\vec{q}_j^+ q_j^+)^2 \quad \text{and} \quad \frac{1}{2} \sum_{i<j}^{n} \lambda_{ij} (\vec{q}_i^+ q_i^+)(\vec{q}_j^+ q_j^+).$$ (4.15)

A perturbative calculation of the HSS multi-hypermultiplet one-loop diagram (Fig. 1) in the \( N=2 \) dual \( U(1)^n \) gauge theory also yields \( [28] \)

$$P^{(4)} \propto \sum_{i,j=1}^{n} \left( \vec{q}_i^+ \vec{g}_i q_i^+ \right) \cdot \left( \vec{q}_j^+ \vec{g}_j q_j^+ \right).$$ (4.16)
We thus conclude, from the symmetry considerations alone, that the hypermultiplet LEEA decomposes into the sum of the single hypermultiplet self-interactions and their pairwise interactions.

Moreover, the equations of motion for the \( q \)-superfields, in the presence of any \( SU(2)_R \)-invariant self-interaction \( \mathcal{K}^{(+)}(q) \), imply the conservation laws \( D^{++} L_m^{++} = 0 \) for the composite HSS superfields (4.14) subject to the reality condition \( \overline{L_m^{++}} = L_m^{++} \). In other words, \( L_m^{++} \) can be recognized as the \( N=2 \) tensor multiplets (cf. sect. 3).

This confirms the claim \[50\] that the exact metric on the moduli space of \( n \) distinct monopoles in a spontaneously broken \( SU(n+1) \) pure gauge theory can be obtained by the generalized Legendre transform in terms of sections of the \( O(2) \) holomorphic bundle only. In our terms, this result comes after projecting the HSS action (4.15) into PSS by using the \( N=2 \) duality transformation (4.14). The equivalent PSS action is given by a sum of the non-improved and improved terms \[50\],

\[
\oint G = \sum_{j=1}^{n} m_j \oint_{C_0} \frac{Q_j^{(2)}}{2 \xi} + \sum_{i<j} \frac{1}{2\pi} (\overline{g}_i \cdot \overline{g}_j) \oint_{C_r} (Q_i^{(2)} - Q_j^{(2)}) \left\{ \ln(Q_i^{(2)} - Q_j^{(2)}) - 1 \right\},
\]

in agreement with eqs. (4.15) and (4.16).

The generalized Legendre transform (Appendix B) of the \( N=2 \) PSS action defined by eqs. (B.3) and (4.17) yields the \( N=2 \) NLSM whose hyper-Kähler metric is given by \[50, 50\]

\[
ds_{m^{TN}}^2 = M_{ij} d\vec{r}_i \cdot d\vec{r}_j + \frac{g^4}{(4\pi)^2} (M^{-1})_{ij} (d\xi_i + \vec{W}_{ik} \cdot d\vec{r}_k)(d\xi_j + \vec{W}_{jl} \cdot d\vec{r}_l),
\]

where

\begin{equation}
M_{ij} = \begin{cases} 
m_i - \sum_{k \neq i} \frac{\overline{g}_i \cdot \overline{g}_k}{4\pi r_{ik}}, & i = j, \\
\frac{\overline{g}_i \cdot \overline{g}_j}{4\pi r_{ij}}, & i \neq j,
\end{cases}
\end{equation}

and

\begin{equation}
\vec{W}_{ij} = \begin{cases} 
-\sum_{k \neq i} \overline{\alpha}_i^* \cdot \overline{\alpha}_k^* \overline{w}_{ik}, & i = j, \\
\overline{\alpha}_i^* \cdot \overline{\alpha}_j^* \overline{w}_{ij}, & i \neq j.
\end{cases}
\end{equation}

The quantity \( \overline{w}_{ij} \) is the Dirac potential from the \( j \)-th monopole evaluated at the position \( \vec{r}_i \), where the \( i \)-th Dirac monopole is located. The function \( \overline{w}_{ij}(\vec{r}_i - \vec{r}_j) \) satisfies an equation

\[
\vec{\nabla}_i \times \overline{w}_{ij}(\vec{r}_i - \vec{r}_j) = \frac{(\vec{r}_i - \vec{r}_j)}{r_{ij}^3}.
\]

The metric (4.18)–(4.21) was first identified by Lee, Weinberg and Yi (LWY) \[17\] as the asymptotic metric in the multi-monopole moduli space for the \( n \) distinct fundamental monopoles corresponding to all different simple (or fundamental) roots.

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Because of its origin, the LWY metric may also be called the multi-dimensional Taub-NUT metric. Since the inner products of the dual fundamental roots in the case of $SU(n+1)$ can only be either zero or negative, the hyper-Kähler metric (4.18)–(4.21) appears to be regular and geodesically complete [17]. Subsequently, it was further argued by the same authors [17] that the metric found may actually be exact in this case (see ref. [58] too). As was later demonstrated by Chalmers [60], by the use of the generalized Legendre transform based on eq. (4.17), the LWY metric is unique indeed. Our HSS approach greatly simplifies the proof of this statement.

If one deals with $n > 1$ similar monopoles (or hypermultiplets) labelled by the same root (say, $g_1$), the multi-dimensional Taub-NUT metric develops a singularity in the ‘core’ region at $r = g_1^2/(4\pi m)$, where the similar monopoles approach each other [17]. It means that the asymptotic LWY metric cannot be exact in this case. This situation is apparently similar to the single charge-2 case described by the AH metric whose asymptotic behaviour is given by the Taub-NUT metric with the negative mass parameter (sect. 3).

The $SU(2)_R$ invariant family of hyper-Kähler potentials in eq. (4.2) describes the exact metrics on the multi-monopole moduli space if there are no more than two similar monopoles belonging to the same root, $k_i \leq 2$. In other words, this situation corresponds to ‘gluing’ together $n$ Atiyah-Hitchin or Taub-NUT metrics, each being associated with a simple root.

A generic real quartic polynomial $P^{(+4)}(q)$ may have no translational isometries at all. The problem of extracting the explicit hyper-Kähler metric associated with a hyper-Kähler potential (4.2) essentially amounts to determination of the set of composite analytic HSS superfields of the second- and forth-order in $q^+$, which would allow one to make an N=2 supersymmetric reduction from HSS to PSS, and thus to get rid of the auxiliary fields. One such variable is obviously given by the hyper-Kähler potential $P^{(+4)}(q)$ itself, because the conservation law (3.3) is valid for any number of the FS hypermultiplets subject to their equations of motion (3.4). This gives rise to the $O(4)$ projective multiplet $L^{++++} = P^{(+4)}(q)$ as one of the ‘good’ variables. If $P^{(+4)}(q)$ can be represented as a quadratic polynomial squared, the role of $L^{++++}$ is replaced by an N=2 tensor multiplet. One can easily imagine mixed situations, where dualizing the HSS potential $P^{(+4)}(q)$ can be performed by using both $O(2)$ and $O(4)$ projective N=2 multiplets. Some of those ‘mixed’ PSS actions were considered in ref. [60].

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7 The case of $k_i > 2$ is discussed in the next subsect. 4.2.
4.2 Non-abelian LEA, rational maps, and spectral curve

It should be emphasized that in the HSS-based approach we may easily treat all types of the gauge symmetry breaking. We only discussed above the maximal symmetry breaking of the gauge group $G$ down to its Cartan subgroup. More general patterns of the gauge symmetry breaking are possible, with the unbroken subgroup being of the form $K \times U(1)^{r-k}$, where $K$ is a non-abelian (of rank $k < r$) subgroup of $G$. It happens when the vector $\vec{h} \in H$, defining the vacuum expectation value of the Higgs field in eq. (4.8), is orthogonal to the simple roots of $K$. The unbroken symmetry should be simply imposed on the hypermultiplet LEA as its isometry. It just forces the hyper-Kähler potential (4.3) to be a function of the $K \times U(1)^{r-k}$ invariants only, which considerably restricts the form of the quartic (4.3). In addition, the magnetic charge $\vec{g}$ is supposed to commute with the unbroken symmetry, which implies \[ \vec{g} \cdot \vec{\gamma}_i = 0 \] for all simple roots $\vec{\gamma}_i$ of $K$, $i = 1, 2, \ldots, k$.

In the case of massless monopoles and hypermultiplets, some of the diagonal coefficients of the quartic (4.3) become infinite. It does not, however, imply that the HSS description becomes invalid. Although it means that some rescalings are necessary. This is particularly clear in the PSS description, in terms of the N=2 tensor multiplets, where eq. (4.17) apparently continues to be valid, even if some of the masses $m_i$ vanish.

Given the existence of the elliptic curve behind the AH metric associated with magnetic charge 2, it is not very surprising that there also exist the Riemann surface $\Sigma_g$ of genus $g = (n - 1)$ in the case of magnetic charge $n$ (or, equivalently, $n$ similar fundamental monopoles). The $\Sigma_g$ is known as the spectral curve in the twistor theory of monopoles [61]. We now briefly describe a relation between the PSS and HSS constructions of the multi-monopole moduli space metrics and the spectral curve. The good starting point is the Donaldson classification [51] of the charge-$n$ (BPS) monopoles in the $SU(2)$ Yang-Mills-Higgs system (Appendix D). It is just opposite to the situation discussed in the preceding subsect. 4.1, since now all monopoles are supposed to be similar, being assigned to a single root of $G$.

The Donaldson classification [51] makes use of the rational maps defined by

\[ S(z) = \frac{p(z)}{q(z)} \equiv \frac{\sum_{j=0}^{n-1} a_j z^j}{z^n + \sum_{j=0}^{n-1} b_{-j} z^j} \] (4.23)
in \( R_n(z) \). It is assumed that the roots \( \beta_j \) of \( q(z) \) are all different, while the residues \( p(\beta_j) \neq 0 \) for all \( j \). According to ref. [41], there exist a closed 2-form,

\[
\Omega = \sum_{j=1}^{n} d\beta_j \wedge d\ln p(\beta_j) ,
\]

that is symmetric under any exchange of \( \beta_i \), symplectic in \( R_n(z) \), and holomorphic, being an \( O(2) \)-section over \( CP(1) \). The Donaldson theorem [51] claims the existence of a one-to-one correspondence between the (universal cover of) charge-\( n \) monopole configurations and the rational maps \( S(z) \). For example, the \( O(2) \)-section \( \Omega \) can be (locally) written down as

\[
\Omega = (J^{(2)} + iJ^{(3)}) + 2\xi J^{(1)} - \xi^2 (J^{(2)} - iJ^{(3)}) ,
\]

where \( (J^{(1)}, J^{(2)}, J^{(3)}) \) is the hyper-Kähler structure [41].

It is worth mentioning that the input provided by the rational map (4.23) naively has \( 4n \) real parameters given by the real and imaginary parts of the complex coefficients \( (a_i, b_j) \). Since the function \( S(z) \) transforms as the (chiral) 2d conformal field [62] under the transformations

\[
a_i \rightarrow \lambda^i a_i , \quad b_i \rightarrow \lambda^{-i} b_i ,
\]

with the complex parameter \( \lambda \), the map \( S(z) \) is actually dependent upon \( (4n - 2) \) real parameters. The gauge symmetry (4.26) can be fixed, for example, by choosing a gauge \( b_{n-1} = 0 \). The equation

\[
\Delta_n(p, q) \equiv \prod_{j=1}^{n} p(\beta_j) = 1 ,
\]

where \( \Delta_n(p, q) \) is known as the resultant of the map \( S(z) \), defines the algebraic manifold \( \widetilde{M}_n \) of real dimension \( 4(n-1) \). The resultant also has a cyclic symmetry under the transformations

\[
\Gamma_n : \quad a_i \rightarrow qa_i , \quad \text{where} \quad q^n = 1 .
\]

As is explained in ref. [51], it is the quotient space \( \widetilde{M}_n/\Gamma_n \) that is supposed to be identified with the charge-\( n \) (centered) monopole moduli space (cf. eq. (3.39) for the AH space). Setting \( \Delta_n = 1 \) and \( b_{n-1} = 0 \) means choosing the overall phase and location of the \( n \)-monopole to be zero [41]. It is now easy to verify that the Donaldson description of the AH space in eq. (3.39) follows from eq. (4.27) in the case of \( n = 2 \).

In the general case, all \( \{ \beta_i, p(\beta_j) \} \) are the good complex (Donaldson) coordinates on the multi-monopole moduli space, while \( b_j \) can be recognized as the \( O(2j) \) section.
over $CP(1)$, $b_j(\xi) = Q(2j)(\xi)$, that is real in the sense of eq. (2.33). The (Hitchin) spectral curve $\Sigma_g$ of genus $g = n - 1$ can now be written down in the form

$$y^n + \sum_{j=1}^n \xi^{n-j} Q(2j)(\xi) = 0 . \quad (4.29)$$

The Riemann surface (4.29) has the full information about a charge-$n$ monopole. The equivalent PSS data $(G, C)$ in the form of eq. (B.3) and the related HSS potential $\mathcal{K}^{(+4)}$ in eq. (2.59) can, therefore, be completely encoded in terms of the spectral curve (4.29). In particular, the corresponding PSS potential was recently found by Houghton [63], and it leads to the N=2 PSS NLSM action dictated by

$$\frac{1}{2\pi i} \oint G = \oint_C y - \frac{1}{2\pi i} \oint_O \frac{y^2}{\xi} , \quad (4.30)$$

where $C$ is a non-trivial 1-cycle on $\Sigma_g$ (see, e.g., ref. [19] for details), and $O = \sum_{j=1}^n 0_j$ where $0_j$ encircles the origin of the $j$-th sheet of $\Sigma_g$. The corresponding HSS potential $\mathcal{K}^{(+4)}(q, u)$ is a finite polynomial in $q$ of order $2n$, while it must be explicitly dependent upon harmonics in the case of $n > 2$. This means that the $SU(2)_R$ symmetry is necessarily broken for $n > 2$. The asymptotical behavior of the multi-monopole moduli space metric is described by the limit where the spectral curve $\Sigma_g$ degenerates (with exponential accuracy) into the product of $g$ spheres [63]. It also implies that $Q(2j) \to Q_j^{(2)}$ that brings us back to the preceding subsect. 4.1.

Since the HSS method can be considered as the very particular application of flag manifolds in the twistor approach to a construction of the hyper-Kähler metrics on the multi-monopole moduli spaces, the HSS method may also be useful for a construction of the classical monopole solutions themselves. For instance, a correspondence between the $SU(n+1)$ monopoles and the rational maps of the Riemann sphere into flag manifolds was recently noticed in ref. [64], whereas the relevance of harmonic maps in this connection was recently emphasized in ref. [65]. It would be interesting to explore their relations to HSS.

## 5 Conclusion

In this paper we reviewed many aspects of the hypermultiplet low-energy effective action, including the PSS and HSS technology. Our results are summarized in the Abstract. We conclude with a few remarks of general nature.

The 4d twistor approach has many similarities with the inverse scattering method in the theory of (lower dimensional) integrable systems. It its turn, the integrable systems are known to be closely connected to the hyper-Kähler geometry and the theory
of Riemann surfaces [66]. In our approach to a construction of the 4d hypermultiplet LEEA, it was the passage from the N=2 harmonic superspace (HSS) to the N=2 projective superspace (PSS) that provided the link of harmonic analysis to complex analysis and thus allowed us to introduce the holomorphic quantities. The latter were then interpreted in terms of the elliptic curve $\Sigma_H$. In addition, the uniformization of the AH algebraic curve (3.39) describing the AH space is known to be closely related to the continual 3d Toda equation [54] that, in its turn, naturally arises in the large $N$ limit of 2d conformal field theories [62].

It is also remarkable that the very simple (quartic polynomial) ‘Ansatz’ (4.2) for the hyper-Kähler potential in HSS provides the exhaustive description of the highly non-trivial class of the $SU(2)_R$-invariant hyper-Kähler metrics that naturally generalize the standard (four-dimensional) Taub-NUT and Atiyah-Hitchin metrics to higher dimensions. The related hypermultiplet LEEA has manifest N=2 supersymmetry and the $SU(2)_R$ internal symmetry. In particular, the N=2 NLSM in HSS with the Atiyah-Hitchin metric has the natural holomorphic projection to N=2 PSS, where it is associated with the holomorphic $O(4)$ line bundle. Similar projections exist in the more general case of $n$ fundamental monopoles.

The non-perturbative (instanton) corrections to the hypermultiplet LEEA are dictated by the hidden (in 4d) spectral curve parametrizing the exact solution. Since the Seiberg-Witten curve $\Sigma_{SW}$ is known to have the simple geometrical interpretation in M-theory, where it can be considered as the part of a magnetically charged five-brane worldvolume wrapped about $\Sigma_{SW}$ [67], it is conceivable that the hypermultiplet spectral curve $\Sigma_H$ may have, perhaps, a similar geometrical interpretation which is presumably related to (Dirichlet) 6-branes in ten-dimensional spacetime [29].

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Appendix A: ABC of hyper-Kähler geometry

A full account of complex geometry, including the hyper-Kähler one, is available in the mathematical literature, see e.g., the books [37, 68]. We follow here the presentation of ref. [33] adapted for physicists.

Let’s consider a manifold $\mathcal{M}$ of real dimension $2k$, covered by a system of coordinate neighbourhoods (charts) $\{x^i\}$. The coordinates in an intersection of two charts are supposed to be related by smooth and locally invertible transition functions.

A mixed second-rank tensor $J_{ij}$ with real components is called an almost complex structure if it satisfies the relation

$$J^i_j J^j_k = -\delta^i_k. \quad (A.1)$$

The manifold $\mathcal{M}$ equipped with an almost complex structure is called an almost complex manifold of complex dimension $k$. The almost complex structure defines a multiplication by ‘$i$’ on vectors in every coordinate chart. The projectors

$$P_\pm = \frac{1}{2}(1 \pm iJ) \quad (A.2)$$

can be used to split any vector $V^i$ into two projections $V^i_\pm$. In particular, the basis 1-forms $dx^j$ can be split into

$$\omega^i_\pm = P^i_\pm dx^j. \quad (A.3)$$

The almost complex structure is said to be integrable if

$$(d\omega^i_\pm)_j = P^i_\pm^k dP^j_\pm^l dx^l = 0. \quad (A.4)$$

In this case $J$ is called a complex structure, while $\mathcal{M}$ is called a complex manifold. The complex manifold is characterized by holomorphic transition functions. Eq. (A.4) is equivalent to the vanishing torsion on $\mathcal{M}$,

$$N_{ij}^k \equiv J^\ell_{[i}[n} \partial_n J^\ell_{j]}^k + J^n_{[i}^k \partial_{j]} J^n_{]i} = 0, \quad (A.5)$$

where $N_{ij}^k$ is known as Nijenhuis tensor.

After introducing holomorphic and antiholomorphic coordinates $z^i$ and $\bar{z}^i$ in $\mathcal{M}$, eq. (A.4) or (A.5) allows one to put the complex structure $J$ into the canonical form

$$J^i_j = \begin{pmatrix} i\delta^i_j & 0 \\ 0 & -i\delta^i_j \end{pmatrix} \quad (A.6).$$
The existence of a system of complex coordinate charts in which the almost complex structure has the form (A.6) is equivalent to the integrability condition (A.4) \[37\].

Given a Riemannian manifold $\mathcal{M}$ with a metric $g_{ij}$ and an (almost) complex structure $J_{ij}$, the invariance of the metric with respect to the complex structure means

$$J^k_J^m g_{km} = g_{ij}, \quad \text{or, equivalently,} \quad J_{ij} \equiv g_{jk}J^k_i = -J_{ji}. \quad (A.7)$$

The metric satisfying eq. (A.7) is called hermitian.

An (almost) complex manifold equipped with a hermitian metric is called an (almost) hermitian manifold. The hermitian manifold thus possesses the fundamental 2-form

$$\omega \equiv J_{ij} dx^i \wedge dx^j. \quad (A.8)$$

In the special coordinates (where the complex structure $J$ is of the canonical form), the fundamental 2-form reads

$$\omega = 2i g_{i\bar{j}} dz^i \wedge d\bar{z}^\bar{j}. \quad (A.9)$$

An (almost) Kähler manifold is the (almost) hermitian manifold with the closed fundamental form,

$$d\omega = 0 \quad \text{or, equivalently,} \quad J_{[ij,k]} = 0. \quad (A.10)$$

As is clear from eqs. (A.9) and (A.10), the metric of a Kähler manifold can be (locally) expressed in terms of a Kähler potential. The conditions (A.5) and (A.10) together are equivalent to the covariant constancy of the complex structure,

$$\nabla_i J^k_j = 0. \quad (A.11)$$

An (almost) quaternionic structure is the set of three linearly independent (almost) complex structures $J_{i(A)}^j, A = 1, 2, 3$, satisfying the quaternionic algebra,

$$J_{i(A)}^k J_{k(B)}^j = -\delta_{ik}^j + \epsilon^{ABC} J_{i(C)}^j \quad (A.12)$$

A manifold with the quaternionic structure is called a quaternionic manifold. If a quaternionic manifold possesses a metric that is hermitian with respect to all three covariantly constant complex structures, the manifold is called hyper-Kähler. In other words, a hyper-Kähler manifold is characterized by three linearly independent complex structures satisfying the quaternionic algebra, while there exists the coordinate system for any given complex structure where it takes the canonical form (A.6).
Any hyper-Kähler manifold is Ricci-flat. In four dimensions the hyper-Kähler condition is, in fact, equivalent to the Ricci-flatness and Kähler conditions together. Imposing the Ricci-flatness condition on a Kähler potential $K$ — see eq. (2.8) — results in the non-linear partial differential equation

$$\det(K_{ij}) = 1$$

known as the Monge-Ampère equation [23].

Perhaps, the most elegant description of hyper-Kähler geometry is possible in terms of holonomy, whose generators are the components of the Riemannian curvature. Given a Riemannian manifold of real dimension $4n$, it is hyper-Kähler if and only if its holonomy is a subgroup of $Sp(n)$. The case of $n = 1$ is special since the most general 4d holonomy group factorizes, $O(4) \cong SU(2) \times SU(2)$. Self-duality of the Riemann curvature means that the holonomy is a subgroup of $SU(2)$. Since $SU(2) \cong Sp(1)$, it is clear that a four-dimensional hyper-Kähler manifold is always (anti)self-dual and vice versa [11].

Appendix B: N=2 projective superspace (PSS)

The idea of N=2 projective superspace (PSS) naturally comes out of the efforts to construct an N=2 supersymmetric self-interaction of the projective $O(n)$ multiplets satisfying the constraints (2.29) and (2.30) [22, 24]. Let’s introduce a function $G(L_A; \xi, \eta)$, whose arguments are given by some number ($A = 1, 2, \ldots$) of the $O(n)$ projective multiplets (of any type with $n \geq 2$) and two projective (complex) $CP(1)$ coordinates, $\xi$ and $\eta$, and constrain it by four linear differential equations in N=2 superspace,

$$\nabla_\alpha G \equiv (\overline{D}_\alpha + \xi D_\alpha^2)G = 0, \quad \Delta_{\bullet\bullet} G \equiv (\overline{D}_{\bullet\bullet} + \eta \overline{D}_{\bullet\bullet}^2)G = 0.$$

(B.1)

It is straightforward to verify that a general solution to eq. (B.1) reads

$$G = G(Q_A(\xi); \xi), \quad \eta = \xi, \quad Q_{(n)}(\xi) \equiv \xi_{i_1} \cdots \xi_{i_n} L^{i_1 \cdots i_n}, \quad \xi_i \equiv (1, \xi),$$

in terms of an arbitrary function $G(Q_A; \xi)$. Of course, in deriving eq. (B.2) we have used the defining constraints (2.29).

Since the function $G$ does not depend upon a half of the Grassmann coordinates of N=2 superspace by its definition (B.1), its integration over the rest of the N=2 superspace coordinates is invariant under N=2 supersymmetry. This leads to the following N=2 invariant action [22, 24]:

$$S[L_A] = \int d^4 x \frac{1}{2\pi i} \oint_C d\xi \ (1 + \xi^2)^{-\frac{n}{4}} \nabla^2 \Delta G(Q_A, \xi) + \text{h.c.},$$

(B.3)
where we have introduced the new superspace derivatives,
\[
\tilde{\nabla}_\alpha \equiv \xi D^1_{\alpha} - D^2_{\alpha} , \quad \tilde{\Delta}^{\star}_{\alpha} \equiv \xi D^1_{\alpha} - D^2_{\alpha},
\] (B.4)
in the directions orthogonal to the ‘vanishing’ directions defined by eq. (B.1). The integration contour \(C\) in the complex \(\xi\)-plane is supposed to make the action (B.3) non-trivial. The points \(\xi_{\pm} = \pm i\), where the linear independence of the derivatives (B.1) and (B.4) breaks down, should be outside of the contour \(C\).

The form of the PSS action (B.3) is universal in the sense that it applies to any set of the projective multiplets \(L_{(n)}\) entering the action via the corresponding function \(Q_{(n)}(\xi)\) defined by eq. (B.2), while the whole action is governed by a single holomorphic potential \(G\). This construction is easily generalizable to the case of the so-called relaxed hypermultiplets. For example, in the case of the relaxed Howe-Stelle-Townsend hypermultiplet defined by the N=2 superspace constraints [69]
\[
D^{(i} L^{j)k} = D_{aL} L^{ijkl} , \quad D^{(i} L^{jklm)} = 0 ,
\] (B.5)
and their conjugates, one should merely substitute \(Q_{(2)}\) by \(Q_{(2)}^{\text{rel}}\).

\[
Q_{(2),\text{rel}} = Q_{(2)} - \frac{5}{4} \frac{\partial Q_{(4)}}{\partial \xi}.
\] (B.6)

Some comments are in order. In the odd case of \(n = 2p+1\), the conjugated superfields \(\overline{T}^{1\cdots i_n}\) may enter the action (B.3) via the corresponding polynomial \(\overline{Q}_{(2p+1)}(\xi)\). However, this makes the potential \(G\) to be nonholomorphic, so that we exclude those superfiels from our consideration. The factor \((1 + \xi^2)^{-4}\) in the action (B.3) was introduced to simplify the transformation properties of the integrand under the \(SU(2)_R\) automorphisms of the N=2 supersymmetry algebra, as well as the corresponding expressions in N=1 superspace (see below).

It is worth mentioning that the PSS construction above is not invariant under the \(SU(2)_R\) rotations. The \(CP(1)\) variable \(\xi\) has the rational transformation law,
\[
\xi' = \frac{\bar{a} \xi - \bar{b}}{a + b \xi},
\] (B.7)
whose complex \(SU(2)_R\) transformation parameters \((a, b)\) are constrained by the condition \(|a|^2 + |b|^2 = 1\). Nevertheless, the N=2 supersymmetric action (B.3) is going to be \(SU(2)_R\) invariant too, provided that the function \(G\) transforms as
\[
G(Q', \xi') = \frac{1}{(a + b \xi)^2} G(Q, \xi)
\] (B.8)
under the projective transformations (B.7), modulo an additive total derivative. Since we have

\[ Q'_{(n)}(\xi') = \frac{1}{(a + b\xi)^n} Q_{(n)}(\xi) \, , \]  

(B.9)

(B.8) puts the severe restriction on a choice of \( G \). For example, the \( SU(2)_R \) invariant PSS Lagrangian of an \( O(2) \) multiplet should be linear in \( Q_2 \) outside of the origin of the complex \( \xi \)-plane (in fact, up a logarithmical factor), whereas in the case of an \( O(4) \) multiplet, its \( SU(2)_R \) invariant PSS Lagrangian should be proportional to \( \nu \).

The invariant action (B.3) of \( N=2 \) tensor multiplets \( (n = 2) \) can be easily rewritten to \( N=1 \) superspace \([22]\). One finds \( Q_{(2)} = \chi - i\xi g + \xi^2 \bar{\chi} \) and

\[ S = \int d^4 x d^4 \theta \frac{1}{2\pi i} \oint_C d\xi \xi^{-2} G(\chi - i\xi g + \xi^2 \bar{\chi}, \xi) + \text{h.c.} \, , \]  

(B.10)
in terms of the \( N=1 \) complex chiral superfield \( \chi = L^{11} \) and the \( N=1 \) real linear superfield \( g = L^{12} \), where \( | \) denotes the \((\theta_2, \theta_1)\)-independent part of a superfield or an operator. The \( N=1 \) multiplets \( \chi \) and \( g \) together constitute an \( N=2 \) tensor multiplet in 4d. The \( N=1 \) superspace covariant derivatives are given by \( D = D^\alpha | \) and \( \bar{D} = \bar{D}^\alpha | \), whereas the \( N=1 \) superfields \( \chi \) and \( g \) satisfy the constraints

\[ \bar{D}^\alpha \chi = D_\alpha \bar{\chi} = 0 \, , \]  

(B.11)

and

\[ \bar{D}_\alpha D^\alpha g = D_\alpha D_\alpha g = 0 \, , \]  

(B.12)

respectively. The Legendre (duality) transform in \( N=1 \) superspace allows one \([22]\) to trade the \( N=1 \) linear superfield \( g \) for yet another \( N=1 \) chiral superfield \( \psi \) (cf. eq. (B.26) and the discussion below). When being applied to the action (B.10), it yields the dual NLSM action

\[ S = \int d^4 x d^4 \theta K(\psi + \bar{\psi}, \chi, \bar{\chi}) \, , \]  

(B.13)

whose hyper-Kähler NLSM metric has a Kähler potential

\[ K = \left[ \frac{1}{2\pi i} \oint_C d\xi \xi^{-2} G(\chi - i\xi H + \xi^2 \bar{\chi}, \xi) + \text{h.c.} \right] + (\psi + \bar{\psi})H \, , \]  

(B.14)

where the function \( H(\chi, \bar{\chi}, \psi + \bar{\psi}) \) is a solution to the algebraic equation

\[ \psi + \bar{\psi} = \frac{1}{2\pi i} \oint_C d\xi \xi^{-1} \frac{\partial G}{\partial Q}(Q, \xi) + \text{h.c.} \, , \quad Q = \chi - i\xi H + \xi^2 \bar{\chi} \, . \]  

(B.15)

As a simple illustration, let’s take \( G(Q, \xi) = F(Q)/\xi \), in terms of a holomorphic function \( F(Q^A) \) of some number \( (A) \) of \( N=2 \) tensor multiplets, with the contour \( C \)
encircling the origin \[70\]. Eq. (B.10) yields in this case the N=1 superspace action

\[
S[\chi, \bar{\chi}; H] = \int d^4x d^4\theta \left\{ F_A(\chi)\bar{\chi}^A - \frac{1}{2}F_{AB}(\chi)H^A H^B + \text{h.c.} \right\}
\]

\[
= \int d^4x d^4\theta \left\{ K(\chi, \bar{\chi}) - \frac{1}{2}g_{AB}(\chi, \bar{\chi})H^A H^B \right\},
\]

where we have used the notation

\[
F_A = \frac{\partial F}{\partial Q^A}, \quad F_{AB} = \frac{\partial^2 F}{\partial Q^A \partial Q^B},
\]

and

\[
K(\chi, \bar{\chi}) = F_A \bar{\chi}^A + \bar{F}_A \chi^A, \quad g_{AB}(\chi, \bar{\chi}) = F_A (\chi) + \bar{F}_A (\bar{\chi}).
\]

The N=1 superspace Legendre transform eliminates all the N=1 linear superfields \(H^A\) in favor of the N=1 chiral superfields \(\psi_A\) according to the algebraic equation (B.15),

\[
\psi_A + \bar{\psi}_A = g_{AB}(\chi, \bar{\chi})H^B,
\]

whose solution is

\[
H^A = g^{AB}(\chi, \bar{\chi})(\psi_A + \bar{\psi}_A),
\]

in terms of the inverse matrix \(g^{AB}\). Substituting the solution (B.20) back into eq. (B.16) yields a Kähler potential in N=1 superspace,

\[
K(\chi, \bar{\chi}, \psi, \bar{\psi}) = K(\chi, \bar{\chi}) + \frac{1}{2}g^{AB}(\chi, \bar{\chi})(\psi_A + \bar{\psi}_A)(\psi_B + \bar{\psi}_B).
\]

By construction, the Kähler potential (B.21) is parametrized by the holomorphic potential \(F(\chi)\), while it yields a hyper-Kähler NLSM metric because of (on-shell) N=2 supersymmetry. In the context of the ‘c-map’ (sect. 1), the N=1 chiral superfields \(\psi\) can be interpreted as the co-vectors associated with the dual special Kähler manifold \[70\].

It is straightforward to reduce the N=2 PSS self-interaction of other projective multiplets with \(n > 2\) in eq. (B.3) to N=1 superspace. For example, the components of the projective \(O(4)\) superfield \(L^{ijkl}\) defined by the constraints (2.29) and (2.30) are given by

\[
L^{ijkl}; \quad \lambda_{\alpha}^{ijk} = D_{\alpha l} L^{ijkl}, \quad \bar{\lambda}_{\dot{\alpha}}^{ijk} = \bar{D}_{\dot{\alpha} l} L^{ijkl};
\]

\[
M^{ij} = -2D_{kl} L^{ijkl}, \quad \bar{M}^{ij} = -2\bar{D}_{kl} L^{ijkl}; \quad V_{\alpha \dot{\alpha}}^{ij} = i[D_{\alpha k}, D_{\dot{\alpha} l} L^{ijkl}];
\]

\[
\chi_{\alpha k} = D_{\alpha}^{kl} M_{kl}, \quad \bar{\chi}_{\dot{\alpha} k} = \bar{D}_{\dot{\alpha} l} \bar{M}_{kl}; \quad C = -2\bar{D}_{ij} D^{kl} L^{ijkl}.
\]

(B.22)
The fields \( (L^{ijkl}, \lambda^{ijk}_\alpha) \) are physical, the fields \( (M^{ij}, V^{ij}_\alpha \alpha) \) are auxiliary, while the fields \( (C, \chi^\alpha_i) \) play the role of Lagrange multipliers. Varying the action (B.3) with respect to \( C \) yields an algebraic constraint
\[
\text{Re} \int_C d\xi \frac{\partial G}{\partial Q} = 0 \quad (B.23)
\]
that reduces the number of the independent bosonic degrees of freedom on-shell. Indeed, the scalars \( L^{ijkl} \) comprise five real bosonic components, while eq. (B.23) reduces their number by one, in agreement with the known fact that the dimension of a hyper-Kähler manifold is always multiple to four.

The N=1 superspace reformulation of the interacting theory (B.3) in terms of the projective \( O(4) \) multiplet is obtained after a decomposition of the N=2 extended superfield \( L^{ijkl} \) in terms of its N=1 superfield constituents,
\[
L^{1111} = \chi, \quad L^{2222} = \bar{\chi}, \quad 4L^{1112} = W, \quad 6L^{1122} = V, \quad (B.24)
\]
where \( \chi \) is the N=1 complex chiral superfield satisfying the constraints (2.1), \( W \) is the N=1 complex linear superfield, \( \bar{D}_\alpha^* W = 0 \), and \( V \) is the general N=1 real scalar superfield. It gives rise to the N=1 superspace action \cite{2}
\[
S_1 = \int d^4x d^4\theta \frac{1}{2\pi i} \oint_C \frac{d\xi}{\xi^2} G(\chi + \xi W + \xi^2 V - \xi^3 \bar{W} + \xi^4 \bar{\chi}, \xi ) + \text{h.c.} \quad (B.25)
\]
The N=1 complex linear multiplet \( W \) is dual to an N=1 chiral multiplet \( \psi \), while it can be made manifest after introducing the N=1 chiral Lagrange multiplier \( \psi \) into the action (B.25). This yields the master action
\[
S = S_1 + \int d^4x d^4\theta \left( \bar{\psi} \bar{W} + \psi W \right) , \quad (B.26)
\]
where \( W \) is now the general complex scalar N=1 superfield. Varying eq. (B.26) with respect to \( \psi \) yields back the constraint on \( W \) and the action \( S_1 \). Varying eq. (B.26) with respect to \( W \) instead (i.e. performing the Legendre transform) yields an algebraic equation on \( W \), which can (at least, in principle) be solved in terms of the other superfields. After being substituted back into the action (B.26), it results in the dual N=1 supersymmetric action
\[
S_{\text{dual}} = \int d^4x d^4\theta K(\chi, \bar{\chi}, \psi, \bar{\psi}, V) , \quad (B.27)
\]
with certain function \( K \). The general superfield \( V \) can now be determined from eq. (B.27), by the use of its algebraic equation of motion. Substituting the result back into eq. (B.27) yields the N=2 NLSM hyper-Kähler potential \( K_{h-K}(\chi, \psi; \bar{\chi}, \bar{\psi}) \) that is only dependent upon the N=1 chiral superfields and their conjugates.
As the simplest example, let’s consider a free theory defined by

\[ G(Q_4, \xi) \propto \frac{Q_4^2}{\xi^3}, \quad \text{where} \quad Q_4 = \chi + \xi W + \xi^2 V - \xi^3 W + \xi^4 \chi, \quad (B.28) \]

with the contour \( C \) encircling the origin in complex \( \xi \)-plane. The corresponding \( \mathcal{N}=1 \) superspace action reads

\[ S_1 = \int d^4x d^4\theta \left( \chi \overline{\chi} - W \overline{W} + V^2 \right), \quad (B.29) \]

while its dual chiral (on-shell equivalent) action is given by

\[ S_{\text{dual}} = \int d^4x d^4\theta \left( \chi \overline{\chi} + \psi \overline{\psi} \right). \quad (B.30) \]

A non-trivial example is given by the \( O(4) \) tensor multiplet PSS selfinteraction parametrized by a holomorphic potential \[70,\]

\[ G(Q^A_4; \xi) = \frac{F(Q^A_4)}{\xi^3}, \quad (B.31) \]

where \( F(Q) \) is a holomorphic function of \( Q^A_4 \). Eqs. (B.3) and (B.31) lead to the \( \mathcal{N}=1 \) action

\[ S_1 = \int d^4x d^4\theta \left\{ K(\chi, \bar{\chi}) + g_{AB}(\chi, \bar{\chi}) \left[ \frac{1}{2} V^A V^B - W^A \overline{W}^B \right] \\
+ \frac{1}{2} \left[ F_{ABC}(\chi) W^B W^C + \text{h.c.} \right] V^A \\
+ \frac{1}{4} \left[ F_{ABCD}(\chi) W^A W^B W^C W^D + \text{h.c.} \right] \right\}, \quad (B.32) \]

where we have used the notation of eqs. (B.17) and (B.18). The action (B.32) is quadratic in the general \( V \) superfields that can be eliminated according to their algebraic equations of motion,

\[ V^D = -\frac{1}{2g^{AD}(\chi, \bar{\chi})} \left[ F_{ABC}(\chi) W^B W^C + \text{h.c.} \right]. \quad (B.33) \]

After being substituted back into the action (B.32), it yields the \( \mathcal{N}=1 \) action \[70\]

\[ S = \int d^4x d^4\theta \left\{ K(\chi, \bar{\chi}) - g_{AB}(\chi, \bar{\chi}) W^A \overline{W}^B \\
- \frac{1}{2} g_{EF}(\chi, \bar{\chi}) F_{ABE}(\chi) \overline{F}_{CDF}(\chi) W^A W^B W^C W^D \\
+ \frac{1}{4} \left[ F_{ABCD}(\chi, \bar{\chi}) W^A W^B W^C W^D + \text{h.c.} \right] \right\}, \quad (B.34) \]

where we have used the notation

\[ F_{ABCD}(\chi, \bar{\chi}) = F_{ABCD}(\chi) - 3F_{ABE}(\chi) g_{EF}(\chi, \bar{\chi}) F_{CDF}(\chi). \quad (B.35) \]

Unfortunately, dualizing the complex linear superfields \( W^A \) in favor of \( \mathcal{N}=1 \) chiral superfields by the use of the \( \mathcal{N}=1 \) superfield Legendre transform in the general case of eq. (B.34) does not seem to allow any simple solution.
Appendix C: N=2 harmonic superspace (HSS)

In the HSS approach \cite{24} one adds harmonics (or twistors) $u^{\pm i}$ parametrizing the Riemann sphere $S^2 \sim SU(2)/U(1)$ and satisfying the relations

$$
\begin{pmatrix}
  u_{+i} \\
  u_{-i}
\end{pmatrix} \in SU(2),
$$

$$
  u^{+i} u_{-i} = 1, \quad u^{+i} u^{+j} = u^{-i} u^{-j} = 0, \quad \bar{u}^{\pm} = u_{\mp}.
$$

Then one can make manifest the hidden analyticity structure of the standard N=2 superspace constraints (subsect. 2.2) defining both N=2 vector multiplets and Fayet-Sohnius hypermultiplets, and find their manifestly N=2 supersymmetric solutions in terms of \textit{unconstrained} (analytic) N=2 superfields, while preserving the $SU(2)_R$ (linearly realized) symmetry. We follow ref. \cite{24} here.

Instead of using an explicit parametrization of the sphere $S^2$, it is more convenient to deal with the (equivariant) functions of the harmonics, which carry a definite $U(1)$ charge defined by

$$
U(u^{\pm i}) = \pm 1.
$$

The simple harmonic integration rules,

$$
\int du = 1 \quad \text{and} \quad \int du u^{+(i_1} \cdots u^{+(i_m} u^{-j_1} \cdots u^{-j_n)} = 0 \quad \text{otherwise}, \quad \tag{C.2}
$$

are similar to the (Berezin) integration rules in superspace. It is obvious that any harmonic integral over a $U(1)$-charged quantity vanishes.

The usual complex conjugation does not preserve analyticity (see below). However, it does, after being combined with another (star) conjugation that only acts on $U(1)$ indices as $(u_{+i}^*) = u_{-i}$ and $(u_{-i}^*) = -u_{+i}$. The harmonic covariant derivatives preserving the defining equations (C.1) are given by

$$
D^{++} = u^{+i} \frac{\partial}{\partial u_{-i}} \equiv \partial^{++}, \quad D^{-+} = u^{-i} \frac{\partial}{\partial u_{+i}}, \quad D^0 = u^{+i} \frac{\partial}{\partial u_{+i}} - u^{-i} \frac{\partial}{\partial u_{-i}}. \quad \tag{C.3}
$$

It is easy to check that they satisfy an $su(2)$ algebra and commute with the standard N=2 superspace covariant derivatives $D^i_{\alpha}$ and $\bar{D}_{\dot{\alpha}}^i$.

The key feature of HSS is the existence of an \textit{analytic} subspace parametrized by

$$
(\zeta, u) = \left\{ x^{\mu}_{\text{analytic}} = x^{\mu} - 2i \theta^{(i} \sigma^{\mu} \bar{\theta}^{j)} u_{+i} u_{-j}, \quad \theta^+_\alpha = \theta^i_\alpha u^+_{i}, \quad \bar{\theta}^+_\dot{\alpha} = \bar{\theta}^i_{\dot{\alpha}} u^+_{i}, \quad u^\pm \right\}, \quad \tag{C.4}
$$

which is invariant under N=2 supersymmetry and is closed under the analyticity-preserving conjugation \cite{25}. This allows one to define \textit{analytic} N=2 superfields of any non-negative and integral $U(1)$ charge by the analyticity conditions (\textit{cf.} the defining conditions (2.1) of N=1 chiral superfields)

$$
D^+_\alpha \phi^{(q)} = \bar{D}^+_{\dot{\alpha}} \phi^{(q)} = 0, \quad \text{where} \quad D^+_\alpha = D^i_{\alpha} u^+_{i} \quad \text{and} \quad \bar{D}^+_{\dot{\alpha}} = \bar{D}_{\dot{\alpha}}^i u^+_{i}. \quad \tag{C.5}
$$

50
The analytic measure reads \( d\zeta(-4)du \equiv d^4x_\text{analytic}^\mu d^2\theta^+ d^2\bar{\theta}^+ du \) and carries the \( U(1) \) charge \(-4\). The harmonic derivative \( D^{++} \) in the analytic basis (C.4) takes the form

\[
D^{++}_\text{analytic} = \partial^{++} - 2i\theta^+ \sigma^\mu \bar{\theta}^+ \partial_\mu , \tag{C.6}
\]

it preserves analyticity, and it allows one to integrate by parts. In what follows we omit the explicit reference to the analytic basis, in order to simplify our notation.

It is the advantage of HSS that both an off-shell \( N=2 \) vector multiplet and an off-shell hypermultiplet can be introduced there \emph{on equal footing}. For example, the Fayet-Sohnius hypermultiplet is naturally described in HSS by an unconstrained complex analytic superfield \( q^+ \) of \( U(1) \) charge \((+1)\). An \( N=2 \) vector multiplet in HSS is described by an unconstrained analytic gauge superfield \( V^{++} \) of \( U(1) \) charge \((+2)\). The HSS superfield \( V^{++} \) is real with respect to the analyticity preserving conjugation, \( \overline{V^{++}} = V^{++} \), while it can be naturally introduced as a connection to the harmonic derivative \( D^{++} \) \cite{25}.

An \( N=2 \) manifestly supersymmetric free hypermultiplet HSS action reads \cite{25} \footnote{Our superfields are all dimensionless, whereas the constant \( \kappa \) in front of our actions has dimension of length. We set \( \kappa = 1 \) for notational simplicity.}

\[
S[q] = -\frac{1}{\kappa^2} \int d\zeta(-4)du \, \bar{q} D^{++} q^+ = -\frac{1}{2\kappa^2} \int d\zeta(-4)du \, q^a D^{++} q^+_a , \tag{C.7}
\]

where we have introduced the notation \( q^+_a = (\bar{q}^+, q^+) \), \( q^+_a = \varepsilon^{ab} q^+_b \) and \( a = 1, 2 \). Its minimal coupling to an abelian \( N=2 \) gauge superfield \( V^{++} \) reads \cite{25}

\[
S[q,V] = -\int d\zeta(-4)du \, \bar{q}^+ (D^{++} + iV^{++}) q^+ . \tag{C.8}
\]

Both actions are also manifestly invariant under the \( SU(2)_R \) automorphisms, just because of the absence of explicit dependence of the HSS Lagrangian upon harmonics. The action (C.7), in fact, possesses the extra \( SU(2)_{\text{PG}} \) internal symmetry rotating the doublet \( q^+_a \), whereas this symmetry is apparently broken in eq. (C.8) to its \( U(1) \) subgroup. The massless HSS action (C.7) is actually invariant under the full \( N=2 \) superconformal symmetry isomorphic to \( SU(2,2|2) \) that also leaves the analytic subspace invariant. The supergroup \( SU(2,2|2) \) contains yet another \( SU(2)_{\text{conf}} \) that can be identified on-shell with \( SU(2)_R \).

A hypermultiplet (BPS) mass can only come from central charges in \( N=2 \) supersymmetry algebra. The most natural way to introduce central charges \( (Z, \bar{Z}) \) is to identify them with spontaneously broken \( U(1)_R \) generators of dimensional reduction from six dimensions via the Scherk-Schwarz mechanism \cite{28}. After being rewritten
to six dimensions and then ‘compactified’ down to four dimensions, the harmonic derivative (C.6) receives an additional ‘connection’ term in 4d,
\[ \mathcal{D}^{++} = D^{++} + v^{++}, \quad \text{where} \quad v^{++} = i(\theta^+ \theta^+) \bar{Z} + i(\bar{\theta}^+ \bar{\theta}^+) Z. \quad (C.9) \]

The N=2 central charges can, therefore, be equivalently treated as the abelian N=2 vector superfield background with the covariantly constant N=2 gauge superfield strength \( \langle W \rangle = Z \). The non-vanishing N=2 central charges break the rigid \( U(1)_R \) symmetry, \( \theta^i_\alpha \rightarrow e^{-i \gamma} \theta^i_\alpha, \ \bar{\theta}^{i\dot{\alpha}} \rightarrow e^{+i \gamma} \bar{\theta}^{i\dot{\alpha}} \), of a massless N=2 supersymmetric field theory. This symmetry breaking results in the appearance of anomalous terms in the N=2 gauge LEFA \([1]\), and in the hypermultiplet LEFA as well \([28]\).

The general procedure of getting the component NLSM metric from a self-interacting hypermultiplet action in HSS has the following steps \([34]\):

- expand the equations of motion in Grassmann variables, and ignore all the fermionic field components,
- solve the kinematical differential equations (on the sphere \( S^2 \sim SU(2)/U(1) \)) for the auxiliary field components, thus eliminating the infinite tower of them in the harmonic expansion of the hypermultiplet HSS superfields,
- substitute the solution back into the original hypermultiplet action in HSS, and integrate over all the anticommuting and harmonic coordinates.

This most straightforward calculation is very difficult in practice. Nevertheless, in was actually done in some special cases of the N=2 NLSM in HSS, including the ones with the four-dimensional Taub-NUT \([34]\), Eguchi-Hanson \([71]\) and Gibbons-Hawking (multi-centre) \([72]\) metrics.

For example, in the Taub-NUT case, the corresponding HSS hypermultiplet action reads \([34, 28]\)
\[ S_{\text{Taub-NUT}}[q] = - \int d\zeta d^{(-4)}du \left[ \bar{q}^+ \mathcal{D}^{++} q^+ + \frac{\lambda}{2}(\bar{q}^+)^2(q^+)^2 \right]. \quad (C.10) \]

The HSS equations of motion for the analytic superfield \( q^+(\zeta, u) \) are given by
\[ \mathcal{D}^{++} q^+ + \lambda(\bar{q}^+ q^+)q^+ = 0, \quad (C.11) \]
where the analytic harmonic derivative \( \mathcal{D}^{++} \) with central charges is
\[ \mathcal{D}^{++} = \partial^{++} - 2i \theta^+ \sigma^m \bar{\theta}^+ \partial_m + i(\theta^+)^2 Z + i(\bar{\theta}^+)^2 Z. \quad (C.12) \]
The bosonic terms in the $\theta$-expansion of $q^+$ read [34]

$$
q^+(\zeta, u) = F^+(x_A, u) + i\theta^+ \sigma^m \bar{\theta}^- A^-_m(x_A, u) + \theta^+ \theta^+ M^-(x_A, u) + \bar{\theta}^+ N^-(x_A, u) + \theta^+ \bar{\theta}^+ \bar{\theta}^+ P(-3)(x_A, u).
$$

(C.13)

The kinematical equations of motion in the $(x^-_{\text{analytic}}, u)$ space, in the presence of central charges, are given by [28]

$$
\begin{align*}
\partial^{++} F^+ &= -\lambda(\bar{F}^+ F^+) F^+, \\
\partial^{++} A^-_m &= 2\partial_m F^+ - \lambda(\bar{F}^+ F^+) A^-_m - \lambda(F^+)^2 \bar{A}^-_m, \\
\partial^{++} M^- &= -\lambda(F^+)^2 N^- - 2\lambda(\bar{F}^+ F^+) M^- - i\bar{Z} F^+, \\
\partial^{++} N^- &= -\lambda(F^+)^2 M^- - 2\lambda(\bar{F}^+ F^+) N^- - iZ F^+.
\end{align*}
$$

(C.14)

After integrating over the Grassmann variables in the action (C.10) and using the kinematical equations of motion, one finds that the bosonic action reduces to

$$
S_B = \frac{1}{2} \int d^3x du \left[ A^-_m \partial^m F^+ - \bar{A}_m \partial^m F^+ - i(\bar{N}^- Z + \bar{M}^- \bar{Z}) F^+ - i\bar{F}^+ (Z M^- - \bar{Z} N^-) \right]
$$

(C.15)

The kinematical equations for $F^+$ and $A^-_m$ can be easily solved. Using the convenient parametrization [34]

$$
F^+(x, u) = f^i(x) u^+_i \exp \left[ \lambda f^{ij}(x) \bar{f}^k(x) u^+_j u^+_k \right],
$$

(C.16)

one finds that

$$
S_B = \int d^4x \left\{ g_{ij} \partial_m f^i \partial^m f^j + g^{ij} \partial_m \bar{f}^i \partial^m \bar{f}^j + h^i_j \partial_m f^j \partial^m \bar{f}^i - V(f) \right\},
$$

(C.17)

where the metric is given by [34]

$$
\begin{align*}
g_{ij} &= \frac{\lambda(2 + \lambda f \bar{f})}{4(1 + \lambda f \bar{f})} f^i f^j, & g^{ij} &= \frac{\lambda(2 + \lambda f \bar{f})}{4(1 + \lambda f \bar{f})} f^i f^j, \\
h^i_j &= \delta^i_j (1 + \lambda f \bar{f}) - \frac{\lambda(2 + \lambda f \bar{f})}{2(1 + \lambda f \bar{f})} f^i \bar{f}^j, & f \bar{f} &\equiv f^i \bar{f}^i.
\end{align*}

(C.18)

The metric (C.18) takes the standard Taub-NUT form [73]

$$
ds^2 = \frac{r + M}{2(r - M)} dr^2 + \frac{1}{2}(r^2 - M^2)(d\psi^2 + \sin^2 \psi d\varphi^2) + 2M^2 \left( \frac{r - M}{r + M} \right) (d\psi + \cos \varphi d\varphi)^2,
$$

(C.19)

after the change of variables [34]

$$
\begin{align*}
f^1 &= \sqrt{2M(r - M)} \cos \frac{\psi}{2} \exp \frac{i}{2}(\psi + \varphi), \\
f^2 &= \sqrt{2M(r - M)} \sin \frac{\psi}{2} \exp \frac{i}{2}(\psi - \varphi).
\end{align*}
$$

(C.20)
with
\[ f \bar{f} = 2M(r - M) , \quad r \geq M \equiv \frac{1}{2\sqrt{\lambda}} , \]  
(C.21)
and \( M \) being the Taub-NUT mass parameter. The non-vanishing auxiliary fields \( M^- \) and \( N^- \) lead, in addition, to a non-trivial scalar potential
\[ V(f) = |Z|^2 \frac{f \bar{f}}{1 + \lambda f \bar{f}} . \]  
(C.22)
Further examples can be found in refs. [71, 72]. The non-vanishing central charges in the hypermultiplet LEEA may lead to spontaneous supersymmetry breaking via the dynamical generation of a scalar potential, e.g. in the Eguchi-Hanson case [31].

**Appendix D: BPS monopoles in the \( SU(2) \) Yang-Mills-Higgs system, and classical moduli spaces**

The Lagrangian of the \( SU(2) \) Yang-Mills-Higgs (YMH) system in \( 1 + 3 \) spacetime dimensions reads
\[ \mathcal{L}_{YMH} = -\frac{1}{4} \text{tr}(F_{\mu \nu}^2) + \frac{1}{2} \text{tr}(D_\mu \Phi)^2 - V(\Phi) , \]  
(D.1)
where both the \( SU(2) \) Yang-Mills field \( A_\mu \) and the Higgs field \( \Phi \) are valued in the Lie algebra of \( SU(2) \),
\[ F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu] , \quad D_\mu \Phi = \partial_\mu \Phi + ie[A_\mu, \Phi] , \]  
(D.2)
with the coupling constant \( e \), while the Higgs potential is of the form
\[ V(\Phi) = \lambda \text{tr}(\Phi^2 - v^2)^2 . \]  
(D.3)
A non-vanishing expectation value, \( \langle \Phi \rangle = v \), of the Higgs field breaks the gauge symmetry \( SU(2) \) down to \( U(1) \). The YMH equations of motion are known to admit the solitonic solutions given by field configurations of finite energy, with the boundary condition \( \Phi^2 \to v^2 \) at the special infinity \( S^2 \). The (gauge-inequivalent) solitonic solutions form the finite-dimensional space labelled by integral topological charge \( n \) belonging to the homotopy group \( \pi_2[SU(2)/U(1)] = \pi_2[S^2] = \pi_1[U(1)] = \mathbb{Z} \), which is simply related to the magnetic charge \( g_{\text{magnetic}} = 4\pi n/e \).

9 The notation in this Appendix may differ from the one used in the rest of the paper.
The most fundamental classical solution, corresponding to the static (t’Hooft-Polyakov) monopole \[74\] of unit magnetic charge \((n = 1)\), is usually written down in the spherically-symmetric form \((a = 1, 2, 3)\)

\[
A^a_i = \varepsilon_{iak}\hat{r}_k \left[1 - \frac{u(r)}{er}\right], \quad \Phi^a = \hat{r}_a h(r),
\]

where \(\hat{r}\) is the unit radial vector, while the functions \(u(r)\) and \(h(r)\) are supposed to satisfy the boundary conditions, \(u(0) = 1, u(\infty) = 0, h(0) = 0\) and \(h(\infty) = v\), in order to avoid singularities and have a finite energy. In the so-called Bogomol’nyi-Prasad-Sommerfeld (BPS) limit \[75\], defined by sending \(\lambda \to 0\) while maintaining all the boundary conditions, one finds

\[
u(r) = \frac{v}{\sinh(erv)}, \quad h(r) = v \coth(erv) - \frac{1}{er},
\]

which implies for the magnetic field

\[
B^a_i = (D_i\Phi)^a = \frac{\hat{r}_i \hat{r}_a}{er^2} + O(1/r^3).
\]

A fundamental (t’Hooft-Polyakov) monopole has four collective coordinates, called moduli. They comprise three translational components, defining a spacial position of the monopole, and one angular component, describing the monopole orientation with respect to the unbroken \(U(1)\) gauge group and associated with electric charge of the monopole. The fundamental monopole moduli space is thus given by a non-trivial bundle \(M_1 = R^3 \times S^1\), while it must also be hyper-Kähler \[41\]. The only candidate hyper-Kähler metric on \(M_1\) is given by the Taub-NUT metric, just because of its isometries and regularity.

The BPS solitons of higher magnetic charge, \(n > 1\), can be understood (in asymptotic regions) as being composed of \(n\) fundamental monopoles. In other words, the multi-monopole states should not be considered as the new states in quantum theory, but they should rather be interpreted as multi-particle states \[58\]. The multi-monopole moduli space is also hyper-Kähler of real dimension \(4n\) \[41\]. The spacial coordinates, describing the ‘center-of-mass’ of a charge-\(n\) monopole configuration, can always be introduced to factorize the associated \(R^3\) factor in the corresponding moduli space \(M_n\). The total charge conservation, associated with the (unbroken) rigid \(U(1)\) rotations, further implies the presence of an \(S^1\) factor in \(M_n\). \[10\] The \(SU(2)\)-based charge-\(n\) monopole moduli space \(M_n\) is, therefore, of the form \[41\]

\[
M_n = R^3 \times S^1 \times \tilde{M}_n / \Gamma_n,
\]

\[10\] In the case of higher gauge groups of rank \(r > 1\), the total charge may no longer be periodic, which implies the \(R^1\) factor instead of \(S^1\) in the decomposition \((D.7)\) \[17\].
where \( \tilde{M}_n / \Gamma_n \) is called the centered (or reduced) multi-monopole moduli space of real dimension \( 4(n - 1) \), while \( \Gamma_n \) stands for a discrete subgroup of its isometry group. The \( n \)-cover \( \tilde{M}_n \) of the centered moduli space is a hyper-Kähler manifold too. Its modding by \( \Gamma_n \) in eq. (D.7) is necessary to get the right spectrum of quantized charges [41].

A hyper-Kähler metric on the manifold \( \tilde{M}_n \) is also supposed to be regular and complete. In the case of a charge-2 monopole (or, equivalently, two identical fundamental monopoles), the centered moduli space is of real dimension four, while it has a rotational isometry \( SO(3) \). The \( \tilde{M}_2 \) can thus be identified with the AH manifold.

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