Wigner’s theorem says that symmetry transformations in quantum mechanics must be unitary or antiunitary [1]. The identity is unitary and symmetry, hence any symmetry that can become the identity by continuously changing a parameter has a unitary representation. On the contrary antiunitary maps are not connected to the identity, so it is not suitable to describe quantum physics. Therefore nature chooses unitary dynamics as its reasonable description. Nevertheless antiunitary transformations still have many interesting properties. The well-known Kramer’s degeneracy comes from the time-reverse symmetry of quantum systems which contain an odd number of fermions [2]. Such time-reversal transformation is antiunitary. Usually an antiunitary operator can be decomposed into an antilinear transformation multiplied by a unitary operator. Many strange properties come from the antilinear part of such an operator. Recent progress in quantum information reveals that antilinear operator may play an important role during the study of quantum entanglement. In fact the famous Positive Partial Transpose (PPT) criterion [3], where antilinear map acts on the second particle of a bipartite density operator, provides a useful condition for testing quantum separability. Also antilinear map can be used as a useful technique to construct superoperators [4]. In the case of 2-dimensional Hilbert space, antilinear map is directly related to the universal-flip of a quantum state [5]. For high dimensional case, universal-flip operator doesn’t exist. However, the map $|\phi\rangle = \sum_i \alpha_i |i\rangle \rightarrow |\phi^*\rangle = \sum_i \alpha_i^* |i\rangle$ with respect to a specific basis $\{|i\rangle\}$ still has many interesting properties. Here the state vector $|\phi^*\rangle$ is often called as the phase-conjugate state of $|\phi\rangle$.

In the case of continuous quantum variables, Cerf et al. [6] have pointed out that phase conjugate of an unknown Gaussian state can be realized by measurement procedure. We think that a similar result exists for any finite dimension case. In this paper, we simply prove that an antilinear map is essentially a classical operation. That is, if the input states are composed of $N$ copies of $|\phi\rangle$, a quantum operation can be implemented by classical ways. We next consider the information of a quantum state $|\phi\rangle$ contained in phase-conjugate pair $|\phi\rangle|\phi^*\rangle$. We find that there is more information about a quantum state encoded in phase-conjugate pair than in parallel pair. In the two-level case, when the number of the output copies is sufficiently large, quantum cloning with two antiparallel spins $\vec{m}_- - \vec{m}_+$ can get higher fidelity than with parallel spins [7]. Our proof reveals that such result still holds in the high-dimension case.

Consider a $d$-level system with $N$ copies. The whole state of such system can be expressed as $|\phi\rangle \otimes |\phi^*\rangle$ and $|\phi\rangle \in \mathcal{H}$ (Here and the following, without loss of generality, we assume that the Hilbert space $\mathcal{H}$ is $C^d$). The space spanned by all these states, which is often called as the “Bose subspace” of $\mathcal{H}^\otimes N$ and denoted by $\mathcal{H}^\otimes N_+$ [8], is invariant under permutation $S_N$. Our aim is just to find a trace-preserving completely positive (CP) map $\xi : \mathcal{H}^\otimes N \rightarrow \mathcal{H}$, which can maximize the mean fidelity

$$F = \int d\phi F(\phi) = \int d\phi \text{Tr}(|\phi^*\rangle\langle\phi^*|\xi(|\phi\rangle\langle\phi^*|\otimes N)).$$

(1)

For a quantum operation $\xi$ it is always possible to find a set of operators which satisfy $\xi(\rho) = \sum_\mu A_\mu \rho A_\mu^\dagger$ with the normalization condition $\sum_\mu A_\mu^\dagger A_\mu = I$. This is also known as Kraus representation [9] of quantum operation. By substituting this into Eq. (1) we can obtain

$$F = \int d\phi \sum_\mu \text{Tr}[A_\mu|\phi\rangle\langle\phi^*|A_\mu^\dagger|\phi^*\rangle].$$

(2)

Before proceeding let us introduce the natural isomorphism between operators $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ and vectors $|A\rangle$ in $\mathcal{H}_2 \otimes \mathcal{H}_1$ which is defined by $|A\rangle = \sum_{i,j} A_{ij}|i\rangle_2 |j\rangle_1$ (Here $\mathcal{H}_1$ and $\mathcal{H}_2$ are not required to have the same dimension). This method has been used in many related works [10, 11] and can greatly simplify the question we consider here. It is not difficult to testify that the following identities are satisfied

$$M \otimes N |A\rangle\langle B| = |A\rangle\langle B|M_N^\dagger,$$

(3)

$$\text{Tr}_1[|A\rangle\langle B|] = AB^\dagger,$$

(4)

$$\text{Tr}_2[|A\rangle\langle B|] = A^\dagger B^\dagger,$$

(5)

$$\text{Tr}[AM_1 A_2^\dagger M_2] = \text{Tr}[|A\rangle\langle A|M_2 \otimes M_1^\dagger].$$

(6)
Here $\tau$ and $*$ represent the transposition and complex conjugation with respect to the fixed basis, while $\text{Tr}_i$ denotes the partial trace over the Hilbert space $\mathcal{H}_i$. Hence by introducing this new notation Eq. (2) now can be rewritten as

$$F = \sum_{\mu} \text{Tr}[|A^{\dagger}_{\mu}|] \langle A^{\dagger}_{\mu}| \int d\phi \langle \phi| \langle \phi|^\otimes N+1 \rangle$$

$$= \frac{1}{d[N+1]} \sum_{\mu} \text{Tr}[|A^{\dagger}_{\mu}|] \langle A^{\dagger}_{\mu}| I_{\mathcal{H}^\otimes N+1} \rangle,$$  

(7)

where $d[N+1] = \frac{(N+d)(N+1)}{(d-1)(N+1)}$ represents the dimension of the “Bose subspace” $\mathcal{H}^\otimes N+1$. Since $\{A_{\mu}\}$ composes a complete quantum operation, under the natural isomorphism the normalization condition becomes

$$\sum_{\mu} \text{Tr}[|A^{\dagger}_{\mu}|] \langle A^{\dagger}_{\mu}| = I_{\mathcal{H}^\otimes N}$$  

(8)

Substituting Eq. (8) into Eq. (7) one can easily find that

$$F \leq \frac{1}{d[N+1]} \text{Tr}[|A^{\dagger}_{\mu}|] \langle A^{\dagger}_{\mu}| = \frac{d[N]}{d[N+1]} = \frac{N+1}{N+d},$$

(9)

Interestingly the right-hand-side of Eq. (9) is just the optimal fidelity for state estimation from $N$ parallel input copies [12]. Usually quantum physics is governed by unitary operations. Any physical accessible operations can be understood from the unitary evolution plus projective measurements process. In the most cases, quantum operations have been demonstrated superior to their classical correspondence. However Eq. (9) reveals that the fidelity of antilinear operation is bounded by the amount of classical information distillable from the input states. This means one can construct the phase-conjugate states of the inputs through a classical measurement-based scenario. It should be addressed that the irreducibility of the input state space plays an important role in the derivation (see Eq. (7)). Recently it has been pointed out by Buscemi et al [13] that for equatorial states the optimal phase covariant time-reversal states cannot be achieved via a measurement-preparation procedure.

In the case of 2-level system, Positive Operator Value Measure (POVM) acting on $|\phi>|\phi^*\rangle$ can get more information than two parallel states $|\phi>|\phi\rangle$ [14, 15]. This result is often considered as an evidence of “nonlocality without entanglement”. In fact, as we will mention below, in high dimensional system $d \geq 3$, such result will still hold. The method we use here can be regarded as a generalized version of Ref. [14, 15] in high-dimension case.

For phase-conjugate pair $|\phi>|\phi^*\rangle$, the density matrix is connected with $\rho(\phi, \phi^*) = |\phi\rangle<\phi| \otimes |\phi^*\rangle<\phi^*|$ by

$$\rho(\phi, \phi^*) = |\phi\rangle<\phi| \otimes |\phi^*\rangle<\phi^*| = \rho(\phi, \phi)^T,$$  

(10)

where $^T$ denotes the partial transpose of the second particles. Suppose there exists a set of Hermitian operators $\hat{a}_i$ which satisfies the following identity $\sum_i \hat{a}_i = I$. Since $\hat{a}_i$ is Hermitian, one can always express it as

$$\hat{a}_i = u_i^{(i)} I \otimes I + \sum_{m,n} t_{mn}^{(i)} \hat{\lambda}_m \otimes \hat{\lambda}_n$$

$$+ \sum_m (r_m^{(i)} \cdot \hat{\lambda}_m \otimes I + s_m^{(i)} \cdot I \otimes \hat{\lambda}_m),$$  

(11)

where $\hat{\lambda}_m$ represent the generators of the unitary group $\text{SU}(d)$. The explicit form of $\hat{\lambda}_m$ can be found elsewhere [16]. When $\hat{a}_i \geq 0$, which indicates that $\hat{a}_i$ are physical accessible operations, the set $\{\hat{a}_i\}$ constitutes a complete POVM and $\text{Tr}(\rho(\phi, \phi^*) \hat{a}_i)$ corresponds to the probability of getting the measurement outcome $i$. Now consider the passive transformation of $\hat{a}_i^T$ on $\{\hat{a}_i\}$, which is defined by

$$\text{Tr}(\hat{a}_i^T \rho) = \text{Tr}(\hat{a}_i^T \rho).$$

(12)

If $\hat{a}_i^T \geq 0$ for all $i$, $\{\hat{a}_i^T\}$ constitute a complete POVM for the input state $\rho(\phi, \phi^*)$. The probability of getting the outcome $i$ for input state $|\phi\rangle<\phi|$ now becomes $\text{Tr}(\hat{a}_i^T \rho(\phi, \phi^*)) = \text{Tr}(\hat{a}_i \rho(\phi, \phi))$. Therefore by introducing the $\hat{a}_i^T$ operator all quantities we are concerned about can be uniformly expressed in the same form except for different positive conditions.

Consider the input state $\rho_0 = |0\rangle<0| \otimes |\phi\rangle<\phi|$. We assume the POVM is covariant and symmetric. Mathematically this is equivalent to say that

$$\text{Tr}(\hat{a}_0 \rho_0 |0\rangle<0| \otimes |\phi\rangle<\phi|) = \text{Tr}(\hat{a}_0 \rho(\phi, \phi))$$  

(13)

and $\hat{a}_0$ is invariant under the permutation group $S_2$. The covariance of the POVM can greatly simplify the explicit form of $\hat{a}_0$. Since the state $\rho_0$ is invariant under $u_g \otimes u_g$ with [17]

$$u_g = \begin{pmatrix} e^{i\theta} & 0 & \ldots & 0 \\ 0 & u_g^{d-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & u_g \end{pmatrix},$$

(14)

where $u_g^{d-1}$ is a $(d-1)$-dimensional unitary matrix with $\det(u_g^{d-1}) = e^{-i\theta}u_g$, the operator $\hat{a}_0$ must be commutable with $u_g \otimes u_g$. A detailed analysis can give the explicit form of $\hat{a}_0$. Here we choose $\hat{a}_0$ with the following form which is enough for our consideration

$$\hat{a}_0 = I + \alpha (T^{(3)} \otimes I + I \otimes T^{(3)}) + \beta T^{(3)} \otimes T^{(3)}$$

$$+ \gamma \sum_{m=1}^{d-1} (T^{(1)} \otimes T^{(1)} \otimes T^{(2)} \otimes T^{(2)})$$

$$+ \delta \sum_{m,n=1}^{d-1} (T^{(1)} \otimes T^{(2)} \otimes T^{(2)} \otimes T^{(2)})$$

$$+ \frac{2}{d-2} T^{(3)} \otimes T^{(3)}),$$

(15)

where $T^{(3)} = \text{diag}(1-d, 1, 1, \ldots, 1)$, $T^{(1)}_{mn} = |m\rangle<|n| + |n\rangle<|m|$, $T^{(2)}_{mn} = -i|m\rangle<|n| + i|n\rangle<|m|$, and $T^{(3)}_{mn} = |m\rangle<|m| -$
expression as Substituting Eq. (15) into Eq. (18) we can simplify the
obtain
\[ \beta d(d - 1) + 4\gamma(d - 1) + 2\delta d(d - 2) = 0. \]  
(17)
When we get a measurement result \( r \) corresponding to
We can guess the input state to be \( |\phi_r \rangle \). The whole information distilled from the measurement
results can be described by the following mean fidelity
\[ F = \int \phi \frac{\text{Tr} [\hat{a}_r \rho(\phi, \phi)] |\langle \phi_r | \phi \rangle|^2}{\langle \phi_r | \phi \rangle}, \]
(18)
Substituting Eq. (15) into Eq. (18) we can simplify the
expression as
\[ F = \frac{1}{d} - \frac{\alpha(d + 2)(d - 1) - 2\beta(d - 1)(d - 2) + 2\delta d(d - 2)}{2d(d + 1)(d + 2)}. \]  
(19)
Now our aim is just to maximize the fidelity \( F \) under the
constraints of Eq. (15, 17) with \( \hat{a}_0 \) satisfying different
positive conditions.
\textbf{Case one:} When the input state is \( |\phi_i \rangle = |\phi \rangle |\phi \rangle \),
positivity of \( \hat{a}_0 \) gives
\[ \begin{cases} 
1 - 2\alpha(d - 1) + \beta(d - 1)^2 \geq 0, \\
1 + 2\alpha + \beta + 2\delta \frac{d-2}{d-1} \geq 0, \\
1 + 2\alpha + \beta - d(\alpha + \beta) \geq 2|\gamma|, \\
1 + 2\alpha + \beta - 2\delta \frac{d-2}{d-1} \geq |2\delta|. 
\end{cases} \]  
(20)
Maximizing \( F \) now becomes a usual linear programming
problem. A simple algebra reveals the maximum of the fidelity
\( F_{||} = \frac{3}{d^2} \) is obtained at \( \alpha = -\frac{1}{d}, \beta = \frac{1}{2}, \delta = 0, \) and \( \gamma = -\frac{1}{d} \), which is consistent with the result of Ref. [12].
The corresponding \( \hat{a}_0 \) now can be written as
\[ \hat{a}_0 = \frac{d(d + 1)}{2} |00\rangle \langle 00| + \frac{d - 1}{2} \sum_{i=1}^{d - 1} |\psi_i \rangle \langle \psi_i |, \]  
(21)
with \( |\psi_i \rangle = \frac{1}{\sqrt{2}} (|0i\rangle - |i0\rangle) \).
\textbf{Case two:} For input state \( |\phi_i \rangle = |\phi \rangle |\phi \rangle \), the operator
\( \hat{a}_0^* \) should be positive. This is equivalent to
\[ \begin{cases} 
1 - \alpha(d - 2) - \beta(d - 1) \geq 0, \\
1 + 2\alpha + \beta - 2\delta \frac{d-2}{d-1} \geq 0, \\
1 + 2\alpha + \beta + 2\delta \frac{d-2}{d-1} \geq |2\delta|, \\
1 + 2\alpha + \beta + 2\delta \frac{d-2}{d-1} \left[1 - 2\alpha(d - 1) + \beta(d - 1)^2 \right] \geq |2\gamma|^2. 
\end{cases} \]  
(22)
Since Eq. (22) contains a nonlinear term, maximizing
\( F \) corresponds to a nonlinear programming problem,
which make the question a little complicated. Before
giving an analytical expression, here we concentrate on a
very special case. By setting Eq. (22) to be equality
constraints, one can find a local extremal point of
Eq. (22) is
\[ \alpha = \frac{1}{(1 + \sqrt{1 + d})^2} - 1, \beta = \frac{4 + (d - 2)(d^2 + 2 \sqrt{1 + d})}{d^2(d - 1)}, \]
\[ \delta = \frac{(\sqrt{1 + d} - 1)^2}{d}, \gamma = \frac{2 - d^2 - (d - 2)\sqrt{1 + d}}{2d}. \]  
This indicates that \( \hat{a}_0 = |\psi_{local}\rangle \langle \psi_{local}| \) is a rank-1 operator with
\[ |\psi_{local}\rangle = \frac{1}{\sqrt{d}} \{ [(d - 1) \sqrt{1 + d} + 1]|00\rangle - (\sqrt{1 + d} - 1) \sum_{i=1}^{d - 1} |ii\rangle \}. \]  
(23)
The corresponding fidelity becomes
\[ F_{local} = \frac{2(1 + 2d)}{(1 + d)(2 + d)} - \frac{(d - 1)(\sqrt{1 + d} + 1)^2}{d^2(1 + d)}, \]  
(24)
which is larger than \( \frac{3}{d^2} \) for arbitrary integer \( d \geq 2 \).
Interestingly, in the case of \( d = 2 \), such rank-1 operator is
also global optimal [15]. Hence by considering the local
extreme value of \( F \), it is enough to show that phase-
conjugate pair can encode more information than parallel
pair.
In the general case, the global maximum of the fidelity
for this phase-conjugate input pair can be obtained when
\( \alpha = \frac{A_\perp - 1}{4(d - 1)}, \beta = \frac{(d - 1)A_\perp + 4}{4(d - 1)^2}, \delta = \frac{A_\perp}{8(d - 1)}, \) and \( \gamma = -\frac{1}{2} \),
with \( A_\perp = 2d \pm \sqrt{2d(d + 1)} \). The maximum fidelity can be formulated as
\[ F_\perp = \frac{1}{d + 2} \left( 2 + \sqrt{\frac{2d}{d + 1}} \right). \]  
(25)
And the measurement operator now becomes
\[ \hat{a}_0 = \frac{1}{2d} \sum_{\sum_{i=1}^{d - 1} |ii\rangle \langle ii| + |\psi_\perp\rangle \langle \psi_\perp|, \]  
(26)
with \( |\psi_\perp\rangle = \sqrt{\frac{dA_\perp}{d + 1}} |00\rangle - \sqrt{\frac{dA_\perp}{2d}} \sum_{i=1}^{d - 1} |ii\rangle \). In table. 1
we give some explicit numerical results on these different
fidelities. One can easily get that \( F \) \( \geq \) \( F_{local} \) and \( F \) in the most case the local extrema are very close to the
corresponding global maxima.
TABLE I: Some numerical results of the information distilled from two different input states (from Eq. (22)). Here $d$ is the dimension of $\mathcal{H}$. Compared with parallel pair ($F_{||}$), phase-conjugate input states ($F_{-}$ and $F_{\text{local}}$) can encode more information for our figure of merits. One can also find that the local extreme values are very close to the global maxima.

| $d$ | 2   | 3   | 4   | 5   | 6   | 11  | 17  |
|-----|-----|-----|-----|-----|-----|-----|-----|
| $F_{||}$ | 0.75 | 0.6 | 0.5 | 0.4286 | 0.375 | 0.2308 | 0.1579 |
| $F_{\text{local}}$ | 0.7887 | 0.6444 | 0.5427 | 0.4678 | 0.4195 | 0.2531 | 0.1723 |
| $F_{-}$ | 0.7887 | 0.6449 | 0.5442 | 0.4701 | 0.4137 | 0.2580 | 0.1776 |

It has been found that the optimal procedure to encode a quantum state depends only on the dimension of the encoding space [18]. Phase-conjugate pair span the whole Hilbert space of two particles while the space spanned by parallel pair is only $\frac{d(d+1)}{2}$-dimension. So it might not be a surprise that phase-conjugate pair encode more information. However, since the optimal encoding state is often an entangled state, while in our consideration, the two kinds of inputs are both direct-product states, which make the whole question not so obvious.

Many interesting problems arise from these results. One may consider, for example, the cloning machine with phase-conjugate pair. It has been demonstrated that for a 2-level system, quantum cloning with anti-parallel spins can get better results than parallel input spins when the number of the output copies is sufficiently large [7]. Generally state estimation can be considered as the limit case of quantum cloning [19], so it would be expected that in high dimension case, quantum cloning with phase-conjugate pair can also get higher fidelity. One can also consider generalizing this to the case of having $N$ parallel input states and $M$ phase-conjugate states [20]. Another point is that we have only considered the case that the input states are $n$-fold, actually to find the condition under which quantum operation is bounded by classical information is still an interesting question.

In conclusion, we have shown that an antilinear map is essentially a classical operation for $N$ parallel input states. The fidelity of such operation is bounded by the classical information distillable from the input state. We have also considered the classical information contained in two different input states ($|\phi\rangle|\phi\rangle$ and $|\phi\rangle|\phi^*\rangle$). Compared with the parallel pair, more information can be encoded in phase-conjugate pair for our figure of merit. We expect our work will be helpful to explore the role played by antilinear map within quantum information.

This work was funded by the National Fundamental Research Program (2001CB309300), the National Natural Science Foundation of China (10304017, 60121503), the Innovation Funds from the Chinese Academy of Sciences, and Program for New Century Excellent Talents in University.

Note added: Very recently J. Fiurášek [21] has got similar results about the state estimation from phase-conjugate pair $|\phi\rangle|\phi^*\rangle$, and he has also given a physical explanation about the results that we have got in this work. Interestingly the global optimal fidelity we got corresponds to the result of the optimal probabilistic estimation strategy, while the local extreme value agrees with the result of the optimal deterministic estimation strategy.

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