The algorithm of investigation of deformations of solids with contact interaction

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Abstract. The work is devoted to the construction of a computational algorithm for investigation of finite deformations of solids with contact interaction. Contact interaction of solids is an actual issue due to it is one of the most common methods of transferring external forces in practice and encounter in many engineering tasks. A master–slave algorithm is used to build a contact interaction model. To perform the contact conditions in finite element implementation of the penalty method is used. The contact problem is formulated in a variational formulation. A resolving equation in the actual state is obtained, which, with the help of linearization and the finite element method, reduces to a system of linear algebraic equations.

1. Introduction

The specialty of contact problems – the interaction of elastic solids is accompany by the appearance of contact zones between the bodies, through which forces are transferred from one body to another [1, 2]. The boundary conditions in the contact zone become specific, since the contact points in this region can either move identically or slip to each other. All of these problems complicate the boundary conditions for each of the contacting bodies, since both stresses and displacements of points in the contact zone become unknown. When solving the contact problem by the finite element method [3], the main difficulty is to fulfill the non-penetration conditions, as well as additional kinematic conditions in the case of a problem with friction at a common unknown boundary.

2. Research methods

2.1. Kinematics of a continuum

For a describing of finite deformations the deformation gradient \( \mathbf{F} \) is used [4], which written as

\[
\mathbf{F} = \frac{\partial \mathbf{y}_i}{\partial \mathbf{x}_j} \left( \mathbf{e}_i \mathbf{e}_j \right)
\]  

(1)

where \( x_i \) – components of position vector of a material point in current state, \( X_i \) – components of position vector of a material point in reference state, \( \mathbf{e}_i \) – unit vector of global Cartesian coordinate system.
Next tensors are used for describing of kinematics of a continuum, which depend of deformation gradient \( F \):

- the left Cauchy–Green tensor
  
  \[
  (B) = (F) \cdot (F)^T = \frac{\partial y_i}{\partial x_k} \frac{\partial y_j}{\partial x_k} (\dot{e}_i \dot{e}_j) = B_{ij} (\dot{e}_i \dot{e}_j)
  \]  

(2)

- the spatial gradient of rate tensor
  
  \[
  (h) = \frac{\partial y_i}{\partial x_j} (\dot{e}_i \dot{e}_j) = (\dot{F}) \cdot (F^{-1})
  \]  

(3)

- the deformation rate tensor
  
  \[
  (d) = \frac{1}{2} [(h) + (h)^T] = \frac{1}{2} [(\dot{F}) \cdot (F^{-1}) + (F^{-1})^T \cdot (\dot{F})^T] = d_{ij} (\dot{e}_i \dot{e}_j)
  \]  

(4)

2.2. Kinematics of contact interaction

The variational formulation of the contact problem is based on the satisfaction of relative non-penetration of bodies. To do this, consider the penetration function for the contacting bodies A and B. On the body B we select the point S and project it onto the surface of the body A [1, 5].

The distance CS, which is the shortest distance between the bodies, determines the penetration function. Define on the body A normal \( \vec{n} \)

![Figure 1. The definition of the function of penetration](image)

and then

\[
 p = (\vec{r}_i - \vec{\rho}) \cdot \vec{n}
\]  

(5)

The function defined in this way makes it possible to judge the contact of bodies: if \( p > 0 \), then there is no penetration, that is, bodies do not contact; if \( p \leq 0 \), then the body B penetrates into the body A; if \( p = 0 \), then the condition of non-penetration of one body into another is fulfilled.

The body B containing the point at which we will check the penetration is called the "slave" body, and the point S is a slave point. The body A, on whose surface we project a "slave" point, is called the "master" body, and the surface is the "master" surface. The method of solving contact problems based on specifying "slave" points and determining the amount of their penetration into the "master" surface was called the "master-slave" algorithm.

The role of the "slave" point will be played by the node of the finite element mesh, and the coordinates of the point C will be determined from the closest point projection algorithm, which reduces to the following extremal problem [1, 2]:

\[
 F(\xi^1, \xi^2) = \| \vec{r}_i - \vec{\rho}(\xi^1, \xi^2) \| = (\vec{r}_i - \vec{\rho}) \cdot (\vec{r}_i - \vec{\rho}) \rightarrow \min
\]  

(6)
where \( \tilde{r} \) - the radius vectors of the set of "slave" points, \( \rho(\xi^1, \xi^2) \) - the parameterization of the surface "master".

Assuming that the function \( F \) is twice differentiable, the Newton method, which reduces to the following procedure, is used to find the minimum of a function:

\[
\begin{align*}
\xi^{(n+1)} = & \xi^{(n)} + \Delta \xi + \xi^{(0)} - \text{defined} \\
\Delta \xi = & - (F^*)^{-1} \cdot F'
\end{align*}
\]  

(7)

2.3. Constitutive equations

The value of the potential deformation energy stored by the elementary volume of the body during its deformation will be determined by the function \( W \).

For an isotropic material, the function of the specific potential energy is written as:

\[
W = W(I_{1B}, I_{2B}, I_{3B})
\]  

(8)

where \( I_{1B} = tr(B^2) \cdot I, I_{2B} = tr(B) \cdot (B) = (B) \cdot (B), I_{3B} = \det(B) \) are the principal invariants of the tensor \( B \).

Then for the stress tensor of Cauchy-Euler the following relation holds

\[
(\sigma) = \frac{2}{J} (B) \left[ \frac{\partial W}{\partial B} \right]
\]  

(9)

and for the rate of change of Cauchy-Euler stresses:

\[
(\dot{\sigma}) = 2 \left[ \frac{1}{J} (\dot{B}) \left( \frac{\partial W}{\partial B} \right) + \frac{1}{J} (B) \left( \frac{\partial^2 W}{\partial B^2} \right) \cdot (\dot{B}) - \frac{1}{J} (B) \left( \frac{\partial W}{\partial B} \right) I_{1B} \right],
\]  

(10)

2.4. Variational formulation

To obtain the variational equation, let us consider two bodies contacting the sites \( S \) with the forces \( \tilde{T}_1 \) and \( \tilde{T}_2 \), respectively [8-12]:

\[ \text{Figure 2. Contact efforts} \]

where \( dS \) - an infinitesimal element of the contact line, \( \delta u \) - virtual movement of the element.

Then the equation of the principle of virtual displacements will have the form:

\[
\delta W = \int_{S_1} \tilde{T}_1 \delta u_1 dS_1 + \int_{S_2} \tilde{T}_2 \delta u_2 dS_2
\]

(11)

Taking into account the equilibrium condition in the contact zone, we obtain

\[
\delta W = \int_{S_1} \tilde{T}_1 (\delta u_1 - \delta u_2) dS_1,
\]

(12)

where \( \tilde{T} = N\delta + T\rho \). Then
\[ \delta W_c = \delta W_c^N + \delta W_c^T, \]  \hspace{1cm} (13)

where \( \delta W_c^N = \int_N (\delta r - \delta \rho) \rho dS = \int_N e_s H(-p) \delta p dS \), \( \delta W_c^T = \int_T a_{ij} (\delta r_i - \delta \rho) \cdot \dot{\rho} dS. \)

The functional \( \delta W_c^N \) contain penalty parameter \( e_s \), which determines the rigidity of the interaction, and the increase of which leads in the limit to the satisfaction of the conditions for the non-penetration of the bodies into each other and the Heaviside function \( H(-p) = \begin{cases} 1, & p \leq 0 \\ 0, & p > 0. \end{cases} \)

In the absence of tangential forces, we obtain the contact problem in the absence of friction: \( T_i = 0. \)

Contact functional \( \delta W_c^N \) is added to the functional of the specific potential energy for two bodies \( \delta W \). We obtain the complete variational equation:

\[ \delta W + \int_e e_s H(-p) \delta p dS = 0 \]  \hspace{1cm} (14)

We carry out the procedure for linearizing the functional \( \delta W_c^N \), we obtain

\[ D(\delta W_c^N) = \int e_s H(-p) (p \delta p + p \delta \dot{p}) dS = \]

\[ = \int e_s H(-p) (\delta r - \delta \rho) (\ddot{n} \times \ddot{n}) (\ddot{V} - \dot{V}) dS - \int e_s H(-p) p \delta \rho \ddot{a}^0 (\ddot{n} \times \ddot{\rho}) (\ddot{V} - \dot{V}) dS - \]

\[ - \int e_s H(-p) p (\delta \dot{r} - \delta \dot{\rho}) (\ddot{n} \times \ddot{n}) (\ddot{V} - \dot{V}) dS = \]  \hspace{1cm} (15)

Equation (15) is the initial equation for obtaining the contact stiffness matrix.

The linearization of the functional for the tangential force \([13, 17]\) is carried out taking into account the kinematic dependencies separately for the sticking and sliding forces.

2.5. Incremental method

To solve the problem, the incremental method is used \([18, 22, 33]\). According to this method, the deformation process is constructed from a sequence of equilibrium states, when the transition described by linear equations with respect to the increments of the displacement vector \( u \) or the configuration vector \( R \) from the current position to the subsequent is determined by the increment of the load.

The counting configuration will be current, and the variational equation of the virtual power principle will act as the resolving equation.

As a result, for the \( k \)-th step we obtain the following resolving equation:

\[ \int_{\Omega_k} \left( \frac{1}{2} \dot{h} \cdot \ddot{h} - \frac{1}{2} \dot{\sigma} \cdot \ddot{\sigma} + \dot{h} \delta h - \dot{\sigma} \delta \sigma \right) + \int_{\Sigma_k} \left( \dot{r}_n \cdot \ddot{V} - \dot{r}_n \delta \dot{V} \right) d\Omega_k + \int_{\delta \Omega_k} \left( \dot{\sigma} \cdot \ddot{d} - \dot{\sigma} \delta \dot{d} \right) = \]  \hspace{1cm} (16)

In the left side of equation (16), summands are assembled in which an unknown velocity vector \( \dot{\dot{V}} \) is present. The right side contains summands calculated in the current configuration.
The resulting equation is linear with respect to velocity $k^i v$. Therefore, after numerical sampling, it is possible to obtain a system of linear algebraic equations for the corresponding nodal values of the projections of velocities $k^iv$. Taking into account the fact that accelerations are not taken into account in the processes under study, and the time can be defined as a monotonically increasing parameter that determines the load change, as a result we obtain a resolving equation for the displacement increments $\Delta^i u$.

3. **Numerical results**

The numerical implementation is based on the finite element method. An 8-node brick element is used [4,18,23-33]. The contact finite element is approximated by 5-node finite element. On the "master" body 4 nodes are selected, the "slave" node of the body is the fifth node of the contact element.

As an example let us give the results of the calculation for the elastic potential given in the form

$$W = \frac{\lambda + 2\mu}{8}(I_{1B} - 3)^2 + \mu(I_{2B} - 3) - \frac{\mu}{2}(I_{3B} - 3),$$

(17)

where $\lambda, \mu$ – the Lamé parameters.

So the Cauchy-Euler stress tensor is as follows:

$$\sigma = \frac{2}{J}(B) \left( \frac{\partial W}{\partial B} \right) - \frac{1}{J} \left[ \frac{\lambda}{2}(I_{1B} - 3) - \mu \right] \cdot (B) + \mu(B)^2 \right]$$

(18)

As a test contact problem in the absence of tangential forces, a special case of the Hertz problem on the interaction of an infinite cylinder with a plane is considered.

An infinitely long cylinder is compressed by planes with a modulus of elasticity much greater than that of a cylinder, without friction with force $f = 7.5\,\text{kN}$. The radius of the cylinder is $R = 100\,\text{mm}$, the material is linearly elastic with the modulus of elasticity $E = 10^5\,\text{N/mm}^2$ and the Poisson's ratio $\nu = 0.3$. Since the problem is symmetric, it is sufficient to create a model of compression of the cylinder in the form of its quarter, with the assignment of the corresponding symmetry conditions.

As a finite element mesh, a mesh was chosen with a high concentration of elements in the contact zone, and slightly more rarefied in areas not directly exposed.

In figure 3 are the deformed state of the cylinder and the distribution of contact stresses throughout the body and in the contact area.

![Figure 3. The contact stresses in the contact area](image)
4. Conclusion

Thus, the paper suggests a coordinated technique for numerical investigation of the behaviour of three-dimensional bodies in their contact interaction. Linearized defining relations for the contact were obtained, an effective process of searching for the contact zone was realized, an effective numerical realization based on the finite element method was carried out. Solved test geometrically nonlinear problems which demonstrate the efficiency of the proposed technique. The stress-strain state and contact pressure are calculated, when compared with the analytical data of which we get a coincidence with acceptable errors. It should be noted that the calculations showed that, regardless of the choice of the initial approximation, the iterative process of finding the contact zone always converged.

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