GENERALIZED FRACTIONAL ISOPERIMETRIC PROBLEM
OF SEVERAL VARIABLES

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Abstract. This work deals with the generalized fractional calculus of variations of several variables. Precisely, we prove a sufficient optimality condition for the fundamental problem and a necessary optimality condition for the isoperimetric problem. Our results cover important particular cases of problems with constant and variable order fractional operators.

1. Introduction. Within the years, several methods were proposed to solve mechanical problems with nonconservative forces, e.g., Rayleigh dissipation function method, technique introducing an auxiliary coordinate or approach including the microscopic details of the dissipation directly in the Lagrangian. Although, all mentioned methods are correct, they are not as direct and simple as it is in the case of conservative systems. In the notes from 1996-1997, Riewe presented a new approach to nonconservative forces [22, 23], where he claimed that friction forces follow from Lagrangians containing terms proportional to fractional derivatives. Riewe considered energy functionals containing fractional derivatives and with his works he initiated the mathematical field that is now called Fractional Calculus of Variations (FCV).

FCV unifies the calculus of variations and the fractional calculus by inserting fractional derivatives (and/or integrals) into the variational functionals. Fractional integrals and derivatives can be defined in different ways, and in each of the cases one needs to study different variational problems. By now several approaches were developed and the results include problems depending on e.g., Caputo fractional derivatives, Riemann–Liouville fractional derivatives, Riesz fractional derivatives, Hadamard fractional derivatives, variable order fractional derivatives [2–10, 14, 16, 17]. Therefore, in order to unify available results, in this work we study more general variational problems, where integral functionals depend on an unknown function (or functions) of several variables and its generalized partial fractional derivatives and/or generalized partial fractional integrals. Precisely, we are interested in the generalized fractional isoperimetric problem of several variables, where admissible functions need to satisfy certain boundary conditions and an isoperimetric constraint.

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The work is organized as follows. In Section 2 we give basic definitions of ordinary and partial generalized fractional operators as well as generalized fractional integration by parts formulas. Section 3 is devoted to the generalized fractional variational problems of several variables. Precisely, we study fundamental and isoperimetric problems: proving a necessary (Theorem 3.4 and Theorem 3.6) and sufficient optimality conditions (Theorem 3.3 and Theorem 3.5). Except main theorems, several corollaries concerning standard fractional operators are given.

2. Preliminaries. This section describes definitions of generalized fractional operators. In special cases, these operators simplify to the classical fractional integrals and derivatives. Interested reader can find more information in the following works [1, 15, 19–21].

2.1. Generalized fractional operators. In order to define generalized fractional operators let us introduce the following triangle

$$\Delta := \{ (t, \tau) \in \mathbb{R}^2 : a \leq t < \tau \leq b \}.$$ 

Definition 2.1. Let us consider a function $k : \Delta \to \mathbb{R}$. For any function $f : (a, b) \to \mathbb{R}$, the generalized fractional integral operator $K_P$ is defined for all $t \in (a, b)$ by:

$$K_P[f](t) = \lambda \int_a^t k(t, \tau) f(\tau) d\tau + \mu \int_t^b k(\tau, t) f(\tau) d\tau,$$  \hspace{1cm} (1)

with $P = (a, t, b, \lambda, \mu)$, $\lambda, \mu \in \mathbb{R}$.

The following example shows that in particular cases, for suitably chosen kernels $k(t, \tau)$ and sets $P$, kernel operators $K_P$, reduce to the classical Riemann–Liouville, Hadamard, Kilbas, Riesz, Katugampola or variable order fractional integrals.

Example 1. (a) Let $k^\alpha(t-\tau) = \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ and $0 < \alpha < 1$. If $P = (a, t, b, 1, 0)$, then

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau =: a^\alpha I_a^\alpha[f](t)$$

is the left Riemann–Liouville fractional integral of order $\alpha$; if $P = (a, t, b, 0, 1)$, then

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau) d\tau =: b^\alpha I_b^\alpha[f](t)$$

is the right Riemann–Liouville fractional integral of order $\alpha$.

(b) For $k^\alpha(t, \tau) = \frac{1}{\Gamma(\alpha(t, \tau))}(t-\tau)^{\alpha(t, \tau)-1}$, $\alpha \in C^1(\Delta; \mathbb{R})$ and $P = (a, t, b, 1, 0)$,

$$K_P[f](t) = \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))} (t-\tau)^{\alpha(t, \tau)-1} f(\tau) d\tau =: a^\alpha I_a^\alpha[\cdot](f)(t)$$

is the left Riemann–Liouville fractional integral of order $\alpha(\cdot, \cdot)$ and for $P = (a, t, b, 0, 1)$

$$K_P[f](t) = \int_t^b \frac{1}{\Gamma(\alpha(t, \tau))} (\tau-t)^{\alpha(t, \tau)-1} f(\tau) d\tau =: b^\alpha I_b^\alpha[\cdot](f)(t)$$

is the right Riemann–Liouville fractional integral of order $\alpha(\cdot, \cdot)$.
(c) For any $0 < \alpha < 1$, kernel $k^{\alpha}(t, \tau) = \frac{1}{\Gamma(1-\alpha)} \left( \log \frac{t}{\tau} \right)^{-\alpha} \frac{1}{\tau}$ and $P = \langle a, t, b, 1, 0 \rangle$, the general operator $K_P$ reduces to the left Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \log \frac{t}{\tau} \right)^{-\alpha} \frac{f(\tau)}{\tau} d\tau =: aJ^\alpha_t[f](t);$$

and for $P = \langle a, t, b, 0, 1 \rangle$ operator $K_P$ reduces to the right Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_t^b \left( \log \frac{\tau}{t} \right)^{-\alpha} \frac{f(\tau)}{\tau} d\tau =: bJ^\alpha_b[f](t).$$

(d) Generalized fractional integrals can be also reduced to, e.g., Riesz, Katugampola or Kilbas fractional operators. Their definitions can be found in [11–13].

The generalized differential operators $A_P$ and $B_P$ are defined with the help of the operator $K_P$.

**Definition 2.2.** The generalized fractional derivative of Riemann–Liouville type, denoted by $A_P$, is defined by

$$A_P = \frac{d}{dt} \circ K_P.$$  

The next differential operator is obtained by interchanging the order of the operators in the composition that defines $A_P$.

**Definition 2.3.** The general kernel differential operator of Caputo type, denoted by $B_P$, is given by

$$B_P = K_P \circ \frac{d}{dt}.$$  

**Example 2.** The standard Riemann–Liouville and Caputo fractional derivatives (see, e.g., [13,14]) are easily obtained from the general kernel operators $A_P$ and $B_P$, respectively. Let $k^{\alpha}(t, \tau) = \frac{1}{\Gamma(1-\alpha)} (t-\tau)^{-\alpha}$, $\alpha \in (0, 1)$. If $P = \langle a, t, b, 1, 0 \rangle$, then

$$A_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau) d\tau =: A_D^\alpha_t[f](t)$$

is the standard left Riemann–Liouville fractional derivative of order $\alpha$, while

$$B_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau) d\tau =: B_D^\alpha_t[f](t)$$

is the standard left Caputo fractional derivative of order $\alpha$; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$-A_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha} f(\tau) d\tau =: _tD^\alpha_b[f](t)$$

is the standard right Riemann–Liouville fractional derivative of order $\alpha$, while

$$-B_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} f'(\tau) d\tau =: _tD^\alpha_b[f](t)$$

is the standard right Caputo fractional derivative of order $\alpha$. 
2.2. **Generalized partial fractional operators.** In this section, we introduce notions of generalized partial fractional integrals and derivatives, in a multidimensional finite domain. They are natural generalizations of the corresponding fractional operators in the single variable case. Furthermore, similarly as in the integer order case, computation of partial fractional derivatives and integrals is reduced to the computation of one-variable fractional operators. Along the work, for $i = 1, \ldots, n$, let $a_i, b_i$ and $\alpha_i$ be numbers in $\mathbb{R}$ and $t = (t_1, \ldots, t_n)$ be such that $t \in \Omega_n$, where $\Omega_n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ is a subset of $\mathbb{R}^n$. Moreover, let us define the following sets:

$$\Delta_i := \{(t_i, \tau) \in \mathbb{R}^2 : a_i \leq \tau < t_i \leq b_i\}, \quad i = 1, \ldots, n.$$ Let us assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ are in $\mathbb{R}^n$. We shall present definitions of generalized partial fractional integrals and derivatives. Let $k_i : \Delta_i \rightarrow \mathbb{R}$, $i = 1, \ldots, n$ and $t \in \Omega_n$.

**Definition 2.4.** Let $f : \Omega_n \rightarrow \mathbb{R}$. The generalized partial integral $K_{P_i}$ is defined for almost all $t_i \in (a_i, b_i)$ by:

$$K_{P_i}[f](t) := \lambda_i \int_{a_i}^{t_i} k_i(t_i, \tau)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau + \mu_i \int_{t_i}^{b_i} k_i(\tau, t_i)f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau,$$

where $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$.

**Definition 2.5.** The generalized partial fractional derivative of Riemann–Liouville type with respect to the $i$th variable $t_i$ is given by

$$A_{P_i} := \frac{\partial}{\partial t_i} \circ K_{P_i}.$$

**Definition 2.6.** The generalized partial fractional derivative of Caputo type with respect to the $i$th variable $t_i$ is given by

$$B_{P_i} := K_{P_i} \circ \frac{\partial}{\partial t_i}.$$

**Example 3.** Similarly, as in the one-dimensional case, partial operators $K$, $A$ and $B$ reduce to the standard partial fractional integrals and derivatives. The left- or right-sided Riemann–Liouville partial fractional integral with respect to the $i$th variable $t_i$ is obtained by choosing the kernel $k^a_{\alpha_i}(t_i, \tau) = \frac{1}{\Gamma(\alpha_i)}(t_i - \tau)^{\alpha_i-1}$. That is,

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{\alpha_i-1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau = : a_i F_{t_i}^a[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, and

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{\alpha_i-1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)d\tau = : t_i F_{t_i}^b[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 0, 1 \rangle$. The standard left- and right-sided Riemann–Liouville and Caputo partial fractional derivatives with respect to $i$th variable $t_i$ are received by choosing the kernel $k^\alpha_{\alpha_i}(t_i, \tau) = \frac{1}{\Gamma(1-\alpha_i)}(t_i - \tau)^{-\alpha_i}$. If $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, then
\[ A_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \frac{\partial}{\partial t_i} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau \]

\[ =: a_i D_{t_i}^{\alpha_i}[f](t), \]

\[ B_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau \]

\[ =: c_i D_{t_i}^{\alpha_i}[f](t). \]

If \( P_i = \langle a_i, t_i, b_i, 0, 1 \rangle \), then

\[ -A_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau \]

\[ =: b_i D_{t_i}^{\alpha_i}[f](t), \]

\[ -B_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n) d\tau \]

\[ =: c_i D_{t_i}^{\alpha_i}[f](t). \]

Moreover, one can easily check, that also variable order partial fractional integrals and derivatives are particular cases of operators \( K_{P_i}, A_{P_i} \) and \( B_{P_i} \). Definitions of variable order partial fractional operators can be found in [18].

In classical calculus, integration by parts formula relates the integral of a product of functions to the integral of their derivative and antiderivative. As we can see below, this formula works also for generalized fractional operators, however it changes the type of differentiation: generalized partial fractional integrals \( K_{P_i} \) are transformed into \( K_{P_i}^* \) and generalized partial fractional Caputo derivatives \( B_{P_i} \) are transformed into \( A_{P_i}^* \).

In our setting, integration by parts changes a given parameter set \( P \) into its dual \( P^* \). The term \textit{duality} comes from the fact that \( P^{**} = P \).

\textbf{Definition 2.7} (Dual parameter set). Given a parameter set \( P = \langle a, t, b, \lambda, \mu \rangle \) we denote by \( P^* \) the parameter set \( P^* = \langle a, t, b, \lambda, \mu \rangle \). We say that \( P^* \) is the dual of \( P \).

In the following, we shall write by \( dt = dt_1 \ldots dt_n \) and take arbitrary but fixed \( i \in \{1, \ldots, n\} \).

\textbf{Theorem 2.8} (cf. [19]). Let \( P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle \) be the parameter set and let \( K_{P_i} \) be the generalized partial fractional integral with \( k_i \) being a difference kernel such that \( k_i \in L^1(0, b_i - a_i; \mathbb{R}) \). If \( f : \mathbb{R}^n \to \mathbb{R} \) and \( \eta : \mathbb{R}^n \to \mathbb{R} \), \( f, \eta \in C(\bar{\Omega}_n; \mathbb{R}) \), then the generalized partial fractional integrals satisfy the following identity:

\[ \int_{\Omega_n} f(t) \cdot K_{P_i}[\eta](t) \, dt = \int_{\Omega_n} \eta(t) \cdot K_{P_i}^* [f](t) \, dt, \]

where \( P_i^* \) is the dual of \( P_i \).
Theorem 2.9 (cf. [19]). Let \( P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle \) be the parameter set and \( f, \eta \in C^1(\bar{\Omega}_n; \mathbb{R}) \). Moreover, let \( B_{P_i} = \frac{d}{dt} \circ K_{P_i} \), where \( K_{P_i} \) is the generalized partial fractional integral with difference kernel i.e., \( k_i = k_i(t_i - \tau) \) such that \( k_i \in L^1(0, b_i - a_i; \mathbb{R}) \), and \( K_{P_i}^\star [f] \in C^1(\bar{\Omega}_n; \mathbb{R}) \). Then

\[
\int_{\bar{\Omega}_n} f(t) \cdot B_{P_i}^\star [\eta](t) \, dt = \int_{\partial \Omega_n} \eta(t) \cdot K_{P_i}^\star [/f](t) \cdot \nu^i \, d(\partial \Omega_n) - \int_{\Omega_n} \eta(t) \cdot A_{P_i}^\star [f](t) \, dt,
\]

where \( \nu^i \) is the outward pointing unit normal to \( \partial \Omega_n \).

3. Main results. In this section we study multidimensional generalized fractional variational problems. First we discuss the fundamental problem, for which we recall the necessary optimality condition and prove a sufficient condition for minimizer.

Next, we consider the generalized fractional isoperimetric problem i.e., we want to minimize the generalized fractional variational functional subject to the boundary conditions and the generalized fractional isoperimetric constraint. We show the necessary optimality condition of the Euler–Lagrange type and, in particular cases, obtain results concerning constant and variable order fractional variational problems.

3.1. Fundamental problem. In order to state the generalized fundamental problem let us introduce the notion of a generalized fractional gradient.

Definition 3.1. Let \( n \in \mathbb{N} \) and \( P = (P_1, \ldots, P_n) \), \( P_1 = \langle a_i, t, b_i, \lambda_i, \mu_i \rangle \). We define a generalized fractional gradient of a function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) with respect to the generalized fractional operator \( T \) by

\[
\nabla_{T_{P_i}} [f] := \sum_{i=1}^n e_i \cdot T_{P_i} [f],
\]

where \( \{ e_i : i = 1, \ldots, n \} \) denotes the standard basis in \( \mathbb{R}^n \).

For \( n \in \mathbb{N} \) let us assume that \( P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle \) and \( P = (P_1, \ldots, P_n) \), \( y : \mathbb{R}^n \rightarrow \mathbb{R} \), and \( \zeta : \partial \Omega_n \rightarrow \mathbb{R} \) is a given function. Consider the following functional:

\[
\mathcal{I} : \mathcal{A}(\zeta) \rightarrow \mathbb{R}
\]

\[
y \mapsto \int_{\Omega_n} F(y(t), \nabla_{K_{P_i}} [y](t), \nabla [y](t), \nabla_{B_{P_i}} [y](t), t) \, dt
\]

where

\[
\mathcal{A}(\zeta) := \{ y \in C^1(\bar{\Omega}_n; \mathbb{R}) : y|_{\partial \Omega_n} = \zeta, \ K_{P_i} [y], B_{P_i} [y] \in C(\bar{\Omega}_n; \mathbb{R}), i = 1, \ldots, n \},
\]

\( \nabla \) denotes the classical gradient operator, \( \nabla_{K_{P_i}} \) and \( \nabla_{B_{P_i}} \) are generalized fractional gradient operators such that \( K_{P_i} \) is the generalized partial fractional integral with the kernel \( k_i = k_i(t_i - \tau) \), \( k_i \in L^1(0, b_i - a_i; \mathbb{R}) \) and \( B_{P_i} \) is the generalized partial fractional derivative of Caputo type satisfying \( B_{P_i} = K_{P_i} \circ \frac{\partial}{\partial t} \), for \( i = 1, \ldots, n \). Moreover, we assume that \( F \) is a Lagrangian of class \( C^1 \):

\[
F : \mathbb{R}^{4n} \times \bar{\Omega}_n \rightarrow \mathbb{R}
\]

\[
(x_1, x_2, x_3, x_4, t) \mapsto F(x_1, x_2, x_3, x_4, t),
\]

and

- \( K_{P_i}^\star [\partial_{1+i} F(y(\tau), \nabla_{K_{P_i}} [y](\tau), \nabla [y](\tau), \nabla_{B_{P_i}} [y](\tau), \tau)] \in C(\bar{\Omega}_n; \mathbb{R}) \),
- \( t \mapsto \partial_{1+n+i} F(y(t), \nabla_{K_{P_i}} [y](t), \nabla [y](t), \nabla_{B_{P_i}} [y](t), t) \in C^1(\bar{\Omega}_n; \mathbb{R}) \),
\[ K_{P_i}^n \left[ \partial_{1+n+i} F(y(t), \nabla K_P[y](\tau), \nabla B_P[y](\tau), \tau) \right] \in C^1(\bar{\Omega}_n; \mathbb{R}), \]
where \( i = 1, \ldots, n. \)

The next theorem states that if a function minimizes functional (2), then it necessarily must satisfy (3). This means that equation (3) determines candidates to solve problem of minimizing functional (2).

**Theorem 3.2 (cf. [19]).** Suppose that \( \bar{y} \in \mathcal{A}(\zeta) \) is a minimizer of (2). Then, \( \bar{y} \) satisfies the following generalized Euler–Lagrange equation:

\[
\partial_t F(y(t)) + \sum_{i=1}^{n} \left( K_{P_i}^n \left[ \partial_{1+n+i} F(y(t)) \right] \right)(t)
- \frac{\partial}{\partial t_i} \left( \partial_{1+n+i} F(y(t)) - A_{P_i} \left[ \partial_{1+2n+i} F(y(t)) \right] \right) = 0, \quad t \in \Omega_n, \tag{3}
\]

where \( F(y(t)) = (y(t), \nabla K_P[y](t), \nabla B_P[y](t), t). \)

The next theorem gives a sufficient condition for a minimizer of functional (2).

**Theorem 3.3.** Suppose that \( \bar{y} \in \mathcal{A}(\zeta) \) satisfies (3) and function \( (x_1, x_2, x_3, x_4) \rightarrow F(x_1, x_2, x_3, x_4, t) \) is convex for every \( t \in \Omega_n. \) Then, \( \bar{y} \) is a minimizer of functional (2).

**Proof.** Let \( \bar{y} \in \mathcal{A}(\zeta) \) be a function satisfying equation (3) and such that \( (x_1, x_2, x_3, x_4) \rightarrow F(x_1, x_2, x_3, x_4, t) \) is convex for every \( t \in \Omega_n. \) Then, the following inequality holds:

\[
\mathcal{I}(y) \geq \mathcal{I}(\bar{y}) + \int_{\Omega_n} \left( \partial_t F(y) - \bar{y} \right)
+ \sum_{i=1}^{n} \left[ \partial_{1+i} F \cdot \nabla K_P[y] \bar{y} + \partial_{1+n+i} F \frac{\partial}{\partial t_i} \bar{y} + \partial_{1+2n+i} F \cdot B_P[y] \bar{y} \right] \right) dt,
\]

where functions \( \partial_t F \) are evaluated at \( (\bar{y}, \nabla K_P[\bar{y}], \nabla B_P[\bar{y}], t), \) for \( i = 1, \ldots, 3n + 1. \) Moreover, using the classical integration by parts formula, as well as Theorem 2.8 and the fact that \( y - \bar{y} \big|_{\partial \Omega_n} = 0, \) we obtain

\[
\mathcal{I}(y) \geq \mathcal{I}(\bar{y})
+ \int_{\Omega_n} \left( \partial_t F + \sum_{i=1}^{n} \left[ K_{P_i}^n \left[ \partial_{1+i} F \right] + \frac{\partial}{\partial t_i} \left( \partial_{1+n+i} F \right) + A_{P_i} \left[ \partial_{1+2n+i} F \right] \right] \right)(y - \bar{y}) dt.
\]

Finally, applying equation (3), we have \( \mathcal{I}(y) \geq \mathcal{I}(\bar{y}) \) for any \( y \in \mathcal{A}(\zeta) \) and the proof is complete. \( \square \)

3.2. Isoperimetric problem. Suppose that \( y : \mathbb{R}^n \rightarrow \mathbb{R}, \ P = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle, \ P = (P_1, \ldots, P_n) \) and \( \zeta : \partial \Omega_n \rightarrow \mathbb{R} \) is a given curve. Let us define the following functional

\[
\mathcal{J} : \mathcal{A}(\zeta) \rightarrow \mathbb{R}
\]
\[
y \mapsto \int_{\Omega_n} G(y(t), \nabla K_P[y](t), \nabla B_P[y](t), t) dt,
\]

where operators \( \nabla K_P, \nabla, \nabla B_P \) and function \( G \) are the same as in the case of functional (2).
Theorem 3.4. Let us assume that $\bar{y}$ minimizes functional (2) on the following set:

$$\mathcal{A}_{\epsilon}(\zeta) := \{ y \in \mathcal{A}(\zeta) : \mathcal{J}(y) = \xi \}.$$ 

Then, one can find a real constant $\lambda_0$ such that, for $H = F - \lambda_0 G$, equation

$$\partial_t H(*y)(t) + \sum_{i=1}^{n} \left( K_{P_i^*} [\partial_{1+i} H(*y)(\tau)](t) - \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(*y)(t)) - A_{P_i^*}[\partial_{1+2n+i} H(*y)(\tau)](t) \right) = 0 \quad (5)$$

holds, provided that

$$\partial_t G(*y)(t) + \sum_{i=1}^{n} \left( K_{P_i^*} [\partial_{1+i} G(*y)(\tau)](t) - \frac{\partial}{\partial t_i} (\partial_{1+n+i} G(*y)(t)) - A_{P_i^*}[\partial_{1+2n+i} G(*y)(\tau)](t) \right) \neq 0, \quad (6)$$

where $(*)y(t) = (\bar{y}(t), \nabla_{K_P} [\bar{y}](t), \nabla [\bar{y}](t), \nabla_{B_P} [\bar{y}](t), t)$.

Proof. The fundamental lemma of the calculus of variations and hypothesis (6) imply, that there exists $\eta_2 \in \mathcal{A}(0)$ such that

$$\int_{\Omega_n} \left( \partial_t G(*y)(t) + \sum_{i=1}^{n} K_{P_i^*} [\partial_{1+i} G(*y)(\tau)](t) \right) \cdot \eta_2(t) + \sum_{i=1}^{n} \left( \partial_{1+n+i} G(*y)(t) + K_{P_i^*} [\partial_{1+2n+i} G(*y)(\tau)](t) \right) \cdot \frac{\partial \eta_2(t)}{\partial t_i} \, dt = 1.$$ 

Now, with function $\eta_2$ and an arbitrary $\eta_1 \in \mathcal{A}(0)$, let us define

$$\phi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] \rightarrow \mathbb{R}$$

$$\quad (h_1, h_2) \mapsto \mathcal{J}(\bar{y} + h_1 \eta_1 + h_2 \eta_2)$$

and

$$\psi : [-\varepsilon_1, \varepsilon_1] \times [-\varepsilon_2, \varepsilon_2] \rightarrow \mathbb{R}$$

$$\quad (h_1, h_2) \mapsto \mathcal{J}(\bar{y} + h_1 \eta_1 + h_2 \eta_2) - \xi$$

Notice that, $\psi(0,0) = 0$ and that

$$\frac{\partial \psi}{\partial h_2} \big|_{(0,0)} = \int_{\Omega_n} \left( \partial_t G(*y)(t) + \sum_{i=1}^{n} K_{P_i^*} [\partial_{1+i} G(*y)(\tau)](t) \right) \cdot \eta_2(t) + \sum_{i=1}^{n} \left( \partial_{1+n+i} G(*y)(t) + K_{P_i^*} [\partial_{1+2n+i} G(*y)(\tau)](t) \right) \cdot \frac{\partial \eta_2(t)}{\partial t_i} \, dt = 1.$$ 

The implicit function theorem implies, that there is $\varepsilon_0 > 0$ and a function $s \in C^1([-\varepsilon_0, \varepsilon_0]\mathbb{R})$ with $s(0) = 0$ such that

$$\psi(h_1, s(h_1)) = 0, \ |h_1| \leq \varepsilon_0,$$
and then $\bar{y} + h_1 \eta_1 + s(h_1) \eta_2 \in A_{\xi}(\zeta)$. Moreover,
\[
\frac{\partial \psi}{\partial h_1} + \frac{\partial \psi}{\partial h_2} \cdot s'(h_1) = 0, \quad |h_1| \leq \varepsilon_0,
\]
and then
\[
s'(0) = -\frac{\partial \psi}{\partial h_1}(0,0) \cdot \eta_1.
\]
Because $\bar{y} \in A(\zeta)$ is a minimizer of $\mathcal{I}$ we have
\[
\phi(0,0) \leq \phi(h_1, s(h_1)), \quad |h_1| \leq \varepsilon_0,
\]
and hence
\[
\left.\frac{\partial \phi}{\partial h_1}\right|_{(0,0)} + \left.\frac{\partial \phi}{\partial h_2}\right|_{(0,0)} \cdot s'(0) = 0.
\]
Letting $\lambda_0 = \left.\frac{\partial \phi}{\partial h_2}\right|_{(0,0)}$ be the Lagrange multiplier we find
\[
\left.\frac{\partial \phi}{\partial h_1}\right|_{(0,0)} - \lambda_0 \left.\frac{\partial \psi}{\partial h_1}\right|_{(0,0)} = 0
\]
or, in other words,
\[
\int_{\Omega_n} \left[ \left. \left( \partial_1 F(\bar{y}) + \sum_{i=1}^n K_{P_1^*} \left[ \partial_{1+i} F(\bar{y}) \right] \right) \right|_{(0,0)} \cdot \eta_1(t) \right. 
+ \sum_{i=1}^n \left( \partial_1 G(\bar{y})(t) + K_{P_1^*} \left[ \partial_{1+i} G(\bar{y}) \right] \right) \cdot \eta_1(t) 
- \lambda_0 \left. \left( \partial_1 F(\bar{y}) + \sum_{i=1}^n K_{P_1^*} \left[ \partial_{1+i} F(\bar{y}) \right] \right) \right|_{(0,0)} \cdot \eta_1(t) 
+ \sum_{i=1}^n \left. \left( \partial_{1+i} H(\bar{y})(t) + K_{P_1^*} \left[ \partial_{1+i} H(\bar{y}) \right] \right) \right|_{(0,0)} \cdot \eta_1(t) \right] dt = 0.
\]
Finally, applying the integration by parts formula, and by the fundamental lemma of the calculus of variations we obtain (5).

**Remark 1.** Notice that, if the main boundary condition $y|_{\partial\Omega_n} = \zeta$ is not prescribed, then an extra boundary condition
\[
\sum_{i=1}^n \partial_{1+i} H(\bar{y})(t) + K_{P_1^*} \left[ \partial_{1+2n+i} H(\bar{y}) \right] (t) = 0, \quad \text{on } \partial\Omega
\]
holds.

**Corollary 1.** Let us assume that $\alpha = (\alpha_1, \ldots, \alpha_n) \in (0,1)^n$ and $\bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R})$ is a minimizer of the functional
\[
\mathcal{I}(y) = \int_a^b F(\bar{y})(t) \, dt,
\]
subject to an isoperimetric constraint
\[
\mathcal{J}(y) = \int_{\Omega_n} G(\bar{y})(t) \, dt = \xi
\]
and the boundary condition
\[ y(t)|_{\partial \Omega_n} = \zeta(t), \quad (9) \]
where \( \zeta : \partial \Omega_n \to \mathbb{R} \) is a given function and we use the notation \((\phi_y)(t) = (y(t), \nabla_{1-\alpha} y(t), \nabla y(t), \nabla_{D^\alpha} y(t), t)\) with
\[
\nabla_{1-\alpha} = \sum_{i=1}^n e_i \cdot a_i t_i^{-\alpha}, \quad \nabla_{D^\alpha} = \sum_{i=1}^n e_i \cdot C_i a_i t_i^\alpha.
\]
Moreover
- \( F, G \) are of class \( C^1 \),
- \( i_i b_i^{-\alpha_i} [\partial_{1+i} F(\phi_y)(\tau)], t_i b_i^{-\alpha_i} [\partial_{1+i} G(\phi_y)(\tau)] \) are continuous on \( \bar{\Omega}_n \),
- \( t \mapsto \partial_{1+n+i} F(\phi_y)(t), t \mapsto \partial_{1+n+i} G(\phi_y)(t) \) are continuously differentiable on \( \bar{\Omega}_n \),
- \( i_i b_i^{-\alpha_i} [\partial_{1+2n+i} F(\phi_y)(\tau)], t_i b_i^{-\alpha_i} [\partial_{1+2n+i} G(\phi_y)(\tau)] \) are continuously differentiable on \( \bar{\Omega}_n \).

Then, if \( \bar{y} \) is not an extremal for functional \((8)\), we can find \( \lambda_0 \in \mathbb{R} \) such that the following equation:
\[
\partial_1 H(\phi_y)(t) + \sum_{i=1}^n \left( t_i b_i^{-\alpha_i} [\partial_{1+i} H(\phi_y)(\tau)] (t) \right.
\]
\[
\left. - \frac{\partial}{\partial \tau} (\partial_{1+n+i} H(\phi_y)(t)) + t_i D_i^{\alpha_i} [\partial_{1+2n+i} H(\phi_y)(\tau)] (t) \right) = 0, \quad t \in \Omega_n,
\]
is satisfied, where \( H = F - \lambda_0 G \).

**Corollary 2.** Suppose that \( \alpha_i : [0, b_i - a_i] \to [0, 1] \) and let
\[
\nabla_I = \sum_{i=1}^n e_i \cdot a_i t_i^{1-\alpha_i}, \quad \nabla_D = \sum_{i=1}^n e_i \cdot C_i a_i t_i^{\alpha_i}.
\]
If \( \bar{y} \in C^1(\bar{\Omega}_n; \mathbb{R}) \) minimizes
\[
\mathcal{I}(y) = \int_{\Omega_n} F(\phi_y)(t) \, dt,
\]
subject to \((9)\) and
\[
\mathcal{J}(y) = \int_{\Omega_n} G(\phi_y)(t) \, dt = \xi,
\]
where \( (\phi_y)(t) = (y(t), \nabla_I y(t), \nabla y(t), \nabla_{D^\alpha} y(t), t) \), integral \( a_i t_i^{1-\alpha_i} [y] \) and derivative \( a_i t_i^{\alpha_i} \) are continuous on \( \bar{\Omega}_n \), and
- \( F, G \) are of class \( C^1 \),
- \( i_i b_i^{-\alpha_i} (t_i^{1-\alpha_i}) [\partial_{1+i} F(\phi_y)(\tau)], t_i b_i^{-\alpha_i} (t_i^{1-\alpha_i}) [\partial_{1+i} G(\phi_y)(\tau)] \) are continuous on \( \bar{\Omega}_n \),
- \( t \mapsto \partial_{1+n+i} F(\phi_y)(t), t \mapsto \partial_{1+n+i} G(\phi_y)(t) \) are continuously differentiable on \( \bar{\Omega}_n \),
- \( i_i b_i^{-\alpha_i} (t_i^{1-\alpha_i}) [\partial_{1+2n+i} F(\phi_y)(\tau)], t_i b_i^{-\alpha_i} (t_i^{1-\alpha_i}) [\partial_{1+2n+i} G(\phi_y)(\tau)] \) are continuously differentiable on \( \bar{\Omega}_n \),
for $i = 1, \ldots, n$. Then, there is $\lambda_0 \in \mathbb{R}$ such that, for $H = F - \lambda_0 G$, $\bar{y}$ is a solution to the following equation:

$$
\partial_t H(\bar{y})(t) + \sum_{i=1}^{n} \left( i, b_i^{\alpha_i} [\partial_{1+i} H(\bar{y})](\tau) \right)(t)
- \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(\bar{y})(t) + i, D_{b_i}^{\alpha_i} [\partial_{1+2n+i} H(\bar{y})](\tau))(t) = 0, \ t \in \Omega_n,
$$

provided that $\bar{y}$ is not a solution to Euler–Lagrange equation associated to $\mathcal{J}$.

**Example 4.** Consider the problem of minimizing the following functional

$$
\mathcal{I} : \mathcal{A}(\zeta) \rightarrow \mathbb{R},
\quad y \mapsto \int_{\Omega_n} \frac{1}{2} |\nabla B_{\rho} y(t)|^2 dt,
$$

subject to the boundary condition $y|_{\partial \Omega_n} = \zeta$ and an isoperimetric constraint $\mathcal{J}(y) = 0$, where

$$
\mathcal{J} : \mathcal{A}(\zeta) \rightarrow \mathbb{R},
\quad y \mapsto \int_{\Omega_n} \rho y dt,
$$

where $\rho = \rho(t) > 0$ is a weight function. Since $\rho > 0$ there is no solution to the Euler–Lagrange equation for functional $\mathcal{J}$. The augmented Lagrangian is

$$
H(x_1, x_2, x_3, x_4, t) = \frac{1}{2} |x_4|^2 - \lambda_0 \rho x_1,
$$

where $x_i = (x_i^1, \ldots, x_i^n), \ i = 2, 3, 4$, and by Theorem 3.4, the Euler–Lagrange equation for problem (10)–(11) is

$$
\sum_{i=1}^{n} A_{P_i} [B_{P_i}[y]] = \lambda_0 \rho.
$$

The next theorem gives sufficient condition for minimizer of functional (2) on the set $\mathcal{A}_\zeta(\zeta)$.

**Theorem 3.5.** Suppose that $\bar{y} \in \mathcal{A}_\zeta(\zeta)$ satisfies (5) and function $(x_1, x_2, x_3, x_4) \rightarrow H(x_1, x_2, x_3, x_4, t)$ is convex for every $t \in \Omega_n$. Then, $\bar{y}$ is a minimizer of functional (2) on the set $\mathcal{A}_\zeta(\zeta)$.

**Proof.** Let $\bar{y} \in \mathcal{A}_\zeta(\zeta)$ be a function satisfying equation (5) and such that $(x_1, x_2, x_3, x_4) \rightarrow H(x_1, x_2, x_3, x_4, t)$ is convex for every $t \in \Omega_n$. Then, the following inequality holds:

$$
\mathcal{I}(y) - \lambda_0 \mathcal{J}(y) \geq \mathcal{I}(\bar{y}) - \lambda_0 \mathcal{J}(\bar{y}) + \int_{\Omega_n} \left( \partial_t H(y - \bar{y})
+ \sum_{i=1}^{n} \left[ \partial_{1+i} H \cdot K_{P_i}[y - \bar{y}] + \partial_{1+n+i} \frac{\partial}{\partial t_i} [y - \bar{y}] + \partial_{1+2n+i} H \cdot B_{P_i}[y - \bar{y}] \right] \right) dt,
$$

where functions $\partial_t H$ are evaluated at $(\bar{y}, \nabla K_{P_i}[\bar{y}], \nabla[\bar{y}], \nabla B_{P_i}[\bar{y}], t)$, for $i = 1, \ldots, 3n + 1$. Moreover, using the classical integration by parts formula, as well as Theorem
2.8 and the fact that \( y - \bar{y}|_{\partial A_n} = 0 \), we obtain

\[
\mathcal{I}(y) - \lambda_0 \mathcal{J}(y) \geq \mathcal{I}(\bar{y}) - \lambda_0 \mathcal{J}(\bar{y}) + \int_{\Omega_n} \left( \partial_1 H + \sum_{i=1}^n \left[ K_{P_i} [\partial_{1+i} H] + \frac{\partial}{\partial t_i} (\partial_{1+n+i} H) + A_{P_i} [\partial_{1+2n+i} H] \right] \right) (y - \bar{y}) \, dt.
\]

Finally, applying equation (3) and having in mind that \( \mathcal{J}(y) = \mathcal{J}(\bar{y}) = \xi \), we have \( \mathcal{I}(y) \geq \mathcal{I}(\bar{y}) \) for any \( y \in \mathcal{A}_\xi(\zeta) \) and the proof is complete. \( \square \)

Note that one can readily extend Theorem 3.4 to cope with abnormal problems: when the optimal solution is an extremal for the isoperimetric constraint. Because generalized isoperimetric problem can be related to a finite–dimensional optimization problem we can simply apply the abnormal version of the Lagrange multipliers introducing an additional multiplier \( \lambda_1 \). Consequently the following result is true.

**Theorem 3.6.** Let us assume that \( \bar{y} \) minimizes functional (2) on the following set:

\[
\mathcal{A}_\xi(\zeta) := \{ y \in \mathcal{A}(\zeta) : \mathcal{J}(y) = \xi \}.
\]

Then, one can find real constants \( \lambda_0 \) and \( \lambda_1 \), \( \lambda_0, \lambda_1 \neq 0 \), such that, for \( H = \lambda_1 F - \lambda_0 G \), equation

\[
\partial_t H(s)(t) + \sum_{i=1}^n \left( K_{P_i} [\partial_{1+i} H(s)(\tau)](t) \right)
- \frac{\partial}{\partial t_i} (\partial_{1+n+i} H(s)(t)) - A_{P_i} [\partial_{1+2n+i} H(s)(\tau)](t) = 0
\]

holds, where \( (s)(t) = (\bar{y}(t), \nabla_{K_P}[\bar{y}](t), \nabla[\bar{y}](t), \nabla_{B_P}[\bar{y}](t), t) \). If \( \bar{y} \) is not an extremal for \( \mathcal{J} \) then we may take \( \lambda_1 = 1 \). If \( \bar{y} \) is an extremal for \( \mathcal{J} \) then we take \( \lambda_1 = 0 \), unless \( \bar{y} \) is also an extremal for \( \mathcal{I} \). In the latter case neither \( \lambda_0 \) nor \( \lambda_1 \) is determined.

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