A Quaternionic Bernstein Theorem

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Abstract. We prove a four-dimensional version of a Bernstein’s theorem, with complex polynomials being replaced by quaternionic polynomials. Moreover, using an Almansi-type decomposition of polynomials, we formulate the quaternionic Bernstein’s inequality in terms of four-dimensional zonal harmonics and Gegenbauer polynomials.

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1. Introduction

The famous Bernstein’s inequality for complex polynomials (first established in this form by M. Riesz in 1914) states that:

Theorem. (A) If \( p(z) \) is a complex polynomial of degree \( d \) and \( \max_{|z|=1} |p(z)| = M \), then \( |p'(z)| \leq d M \) for \( |z| = 1 \), with equality holding if and only if \( p(z) \) is a multiple of the power \( z^d \).

Recently [15], Bernstein’s inequality has been proved for quaternionic polynomials with coefficients on one side. The inequality in the complex case can be deduced from a more general theorem, proved by Bernstein [3] in 1930.

Theorem. (B) Let \( p(z) \) and \( q(z) \) be two complex polynomials with degree of \( p(z) \) not exceeding that of \( q(z) \). If \( q(z) \) has all its zeros in \( \{ |z| \leq 1 \} \) and \( |p(z)| \leq |q(z)| \) for \( |z| = 1 \), then \( |p'(z)| \leq |q'(z)| \) for \( |z| = 1 \).

It is then natural to pose the following question: Is it possible to extend Theorem (B) to quaternionic polynomials?

This short note gives an answer to this question. We show that a quaternionic version of Theorem (B) holds true only after imposing an assumption on the second polynomial (Theorem 2.1). We must require that the quaternionic polynomial \( Q \in \mathbb{H}[X] \) on the right-hand side of the inequality has every coefficients belonging to a fixed commutative subalgebra of \( \mathbb{H} \), i.e., to a
isomorphic copy of \( \mathbb{C} \). We also show in Proposition 2.7 that the assumption made on \( Q \) in Theorem 2.1 is necessary. This restricted version of the Bernstein Theorem is, however, sufficient to deduce, as in the complex case, the quaternionic Bernstein’s inequality: if \( P \in \mathbb{H}[X] \) is a quaternionic polynomial of degree \( d \), then the sup-norms satisfy \( \| P' \| \leq d \| P \| \) (Corollary 2.4).

In Sect. 3, we restate the inequality in terms of four-dimensional zonal harmonics and Gegenbauer polynomials. To obtain this form, we use results from [12] to obtain an Almansi-type decomposition of a quaternionic polynomial.

We refer the reader to [5,6,9] for definitions and properties concerning the algebra \( \mathbb{H} \) of quaternions and many aspects of the theory of quaternionic slice-regular functions, a class of functions which includes polynomials and convergent power series, and more generally for slice functions. The ring \( \mathbb{H}[X] \) of quaternionic polynomials is defined by fixing the position of the coefficients with respect to the indeterminate \( X \) (e.g., on the right) and by imposing commutativity of \( X \) with the coefficients when two polynomials are multiplied together (see, e.g., [11, Sect. 16.3]). Given two polynomials \( P,Q \in \mathbb{H}[X] \), let \( P \cdot Q \) denote the product obtained in this way. A direct computation (see [11, Sect. 16.3]) shows that if \( P(x) \neq 0 \), then

\[
(P \cdot Q)(x) = P(x)Q(P(x)^{-1}xp(x)),
\]

while \((P \cdot Q)(x) = 0\) if \( P(x) = 0 \). In particular, if \( P \) has real coefficients, then \((P \cdot Q)(x) = P(x)Q(x)\). In this setting, a (left) root or zero of a polynomial \( P(X) = \sum_{h=0}^{d} X^h a_h \) is an element \( x \in \mathbb{H} \), such that \( P(x) = \sum_{h=0}^{d} x^h a_h = 0 \).

A subset \( A \) of \( \mathbb{H} \) is called circular, or axially symmetric, if, for each \( x \in A \), \( A \) contains the whole set (a 2-sphere if \( x \notin \mathbb{R} \), a point if \( x \in \mathbb{R} \))

\[
S_x = \{ pxp^{-1} \in \mathbb{H} \mid p \in \mathbb{H}^* \},
\]

where \( \mathbb{H}^* := \mathbb{H} \setminus \{ 0 \} \). In particular, for any imaginary unit \( I \in \mathbb{H} \), \( S_I = S \) is the 2-sphere of all imaginary units in \( \mathbb{H} \). It is well known (see, e.g., [5, Sect. 3.3]) that if \( P \neq 0 \), the zero set \( V(P) \) consists of isolated points or isolated 2-spheres of the form (2).

2. A Bernstein-Type Theorem

Let \( I \in S \) and let \( \mathbb{C}_I \subset \mathbb{H} \) be the real subalgebra generated by \( I \), i.e., the complex plane generated by 1 and \( I \). If \( \mathbb{C}_I \) contains every coefficient of \( P \in \mathbb{H}[X] \), then we say that \( P \) is a \( \mathbb{C}_I \)-polynomial. Every \( \mathbb{C}_I \)-polynomial \( P \) is one-slice-preserving, i.e., \( P(\mathbb{C}_I) \subseteq \mathbb{C}_I \). If this property holds for two imaginary units \( I,J \), with \( I \neq \pm J \), then it holds for every unit and \( P \) is called slice-preserving. This happens exactly when all the coefficients of \( P \) are real.

Let \( P(X) = \sum_{k=0}^{d} X^k a_k \) be a polynomial of degree \( d \geq 1 \) with quaternionic coefficients. Let \( P'(X) = \sum_{k=1}^{d} X^{k-1} ka_k \) be the derivative of \( P \). For every \( I \in S \), let \( \pi_I : \mathbb{H} \to \mathbb{H} \) be the orthogonal projection onto \( \mathbb{C}_I \) and \( \pi_I^* = id - \pi_I \). Let \( P^I(X) := \sum_{k=1}^{d} X^k a_k, I \) be the \( \mathbb{C}_I \)-polynomial with coefficients \( a_k, I := \pi_I(a_k) \).
We denote by \( \mathbb{B} = \{ x \in \mathbb{H} \mid |x| < 1 \} \) the unit ball in \( \mathbb{H} \) and by \( \mathbb{S}^3 = \{ x \in \mathbb{H} \mid |x| = 1 \} \) the unit sphere.

We recall that a quaternionic polynomial, as any slice-regular function, satisfies the maximum modulus principle \([5, \text{Theorem 7.1}]\). Let
\[
\|P\| = \max_{|x|=1} |P(x)| = \max_{|x|\leq 1} |P(x)|
\]
denote the sup-norm of the polynomial \( P \in \mathbb{H}[X] \) on \( \mathbb{B} \). Given \( y \in \mathbb{S}^3 \), let us denote
\[
M_y(P) := \max_{z \in \mathbb{S}_y} |P(z)|, \quad m_y(P) := \min_{z \in \mathbb{S}_y} |P(z)|.
\]

**Theorem 2.1.** (Bernstein-type theorem) Let \( P, Q \in \mathbb{H}[X] \) be two quaternionic polynomials with degree of \( P \) not exceeding that of \( Q \). Assume that there exists \( I \in \mathbb{S} \), such that \( Q \) is a \( \mathbb{C}_I \)-polynomial. If \( V(Q) \subseteq \mathbb{B} \) and \( |P(x)| \leq |Q(x)| \) for \( x \in \mathbb{S}^3 \), then \( |P'(x)| \leq |Q'(x)| \) for \( x \in \mathbb{S}^3 \cap \mathbb{C}_I \). For every \( x = \alpha + J \beta \in \mathbb{S}^3 \), if \( P' \) is not identically zero on \( \mathbb{S}_x \), it holds
\[
|P'(x)| \leq \frac{\sqrt{2}M_x(P')}{\sqrt{M_x(P')^2 + m_x(P')^2}} \max \{|Q'(x_I)|, |Q'(x_I)|\}
\]
with \( x_I = \alpha + I \beta \). Moreover, it holds \( \|P'\| \leq \sqrt{2}\|Q'\| \).

Before proving the theorem, we state a technical lemma about a norm estimate that holds for quaternionic polynomials and more generally for any continuous slice function.

**Lemma 2.2.** Let \( P \in \mathbb{H}[X] \), \( y \in \mathbb{S}^3 \) with \( P \) not identically zero on \( \mathbb{S}_y \). Let \( I \in \mathbb{S} \) be fixed. Then, it holds
\[
|P(x)| \leq \frac{\sqrt{2}M_y(P)}{\sqrt{M_y(P)^2 + m_y(P)^2}} \max \{|P(x_I)|, |P(x_I)|\}
\]
for every \( x = \alpha + K \beta \in \mathbb{S}_y \), where \( x_I = \alpha + I \beta \in \mathbb{S}_y \cap \mathbb{C}_I \).

**Proof.** Let \( M := M_y(P) \), \( m := m_y(P) \). We may assume that \( y \) is not real. Since \( P \) is a slice function, it can be expressed as \( P(x) = P_\alpha^o(x) + \text{Im}(x)P_\alpha^i(x) \), where \( P_\alpha^o \) and \( P_\alpha^i \) are constant functions on \( \mathbb{S}_y \) (the spherical value and the spherical derivative of \( P \), respectively \([7, \text{Sect. 3.3}]\)). Let \( \langle u, v \rangle \) denote the Euclidean scalar product of \( u, v \in \mathbb{H} \). Then, for every \( x \in \mathbb{S}_y \), it holds
\[
|P(x)|^2 = C + 2\langle v, \text{Im}(x) \rangle,
\]
where \( C = |P_\alpha^o(y)|^2 + |\text{Im}(y)|^2|P_\alpha^i(y)|^2 \) and \( v := P_\alpha^o(y)\overline{P_\alpha^i(y)} \). If \( v \in \mathbb{C}_J \), then we get as in \([9, \text{Lemma 5.3}]\) that
\[
M = \max_{\mathbb{S}_y \cap \mathbb{C}_J} |P(x)| \quad \text{and} \quad m = \min_{\mathbb{S}_y \cap \mathbb{C}_J} |P(x)|.
\]
Therefore, it holds \( M^2 = C + 2|\text{Im}(v)||\text{Im}(y)| \), \( m^2 = C - 2|\text{Im}(v)||\text{Im}(y)| \), whence \( M^2 + m^2 = 2C \). Let \( \tilde{x} \in \mathbb{S}_y \cap \mathbb{C}_I = \{ x_I, \overline{x}_I \} \) be such that \( \langle v, \text{Im}(\tilde{x}) \rangle \geq 0 \). Then
\[
\max \{|P(x_I)|^2, |P(\overline{x}_I)|^2\} = |P(\tilde{x})|^2 \geq C = \frac{M^2 + m^2}{2},
\]
whence
\[ M^2 \leq \frac{2M^2}{M^2 + m^2} \max \{|P(x)|, |P(\bar{x})|\}, \]
which is equivalent to the thesis. \qed

**Example 2.3.** A simple example illustrating Lemma 2.2 is given by the linear polynomial \( P(X) = 2X - j - k \). Let \( y = i \). Then, \( S_i = S \) and \( M_1(P) = |P(-(j + k)/\sqrt{2})| = 2 + \sqrt{2}, \) \( m_2(P) = |P((j + k)/\sqrt{2})| = 2 - \sqrt{2} \). The inequality of the Lemma is then
\[ |P(x)| \leq \frac{1 + \sqrt{2}}{\sqrt{3}} \max \{|P(x)|, |P(\bar{x})|\} \quad \text{for every } x \in S. \]

Let \( I \) be orthogonal to \((j + k)/\sqrt{2}\); for example, \( I = i \). Then
\[ \max \{|P(x)|, |P(\bar{x})|\} = |P(i)| = \sqrt{6}, \]
showing that the constant \( \frac{1 + \sqrt{2}}{\sqrt{3}} \) is the best one in the estimate above.

**Proof of Theorem 2.1.** Let \( \lambda \in \mathbb{H} \) with \(|\lambda| > 1 \) and set \( R := Q - P\lambda^{-1} \in \mathbb{H}[X] \). The polynomials \( Q \) and \( R^I = Q - (P\lambda^{-1})^I \) are \( \mathcal{C}_I \)-polynomials, and then, they can be identified with elements of \( \mathcal{C}_I[X] \), with \( \deg(R^I) \leq \deg(Q) \). For every \( x \in \mathcal{C}_I \), it holds
\[ |R^I(x) - Q(x)| = |(P\lambda^{-1})^I(x)| = |\pi_I((P\lambda^{-1})(x))| \leq |(P\lambda^{-1})(x)| = \frac{|P(x)|}{|\lambda|}. \]

If \( x \in S^3 \cap \mathcal{C}_I = \{ x \in \mathcal{C}_I \ | \ x = 1 \} \), then
\[ |R^I(x) - Q(x)| \leq \frac{|P(x)|}{|\lambda|} \leq \frac{|Q(x)|}{|\lambda|} \leq |Q(x)|. \quad (4) \]

In view of Rouché’s Theorem for polynomials in \( \mathcal{C}_I[X] \), \( R^I \) and \( Q \) have the same zeros in the disc \( \{ x \in \mathcal{C}_I \ | \ x < 1 \} \). Moreover, if \( |x| = 1 \) and \( Q(x) = 0 \), the inequality (4) gives \( R^I(x) = 0 \). Since \( \deg(R^I) \leq \deg(Q) \) and \( V(Q) \subseteq \mathbb{H} \), we get that \( V(R^I) \cap \mathcal{C}_I \subseteq \mathbb{H} \cap \mathcal{C}_I \). From the complex Gauss–Lucas Theorem, we get \( V(R^I) \cap \mathcal{C}_I \subseteq V((R^I)^I) \cap \mathcal{C}_I \subseteq \mathbb{H} \cap \mathcal{C}_I \).

Now, let \( x \in \mathcal{C}_I \) with \(|x| > 1 \) be fixed and define \( \lambda := Q'(x)^{-1}P'(x) \in \mathbb{H} \). Observe that \( Q'(x) \neq 0 \) again from the complex Gauss–Lucas Theorem applied to the polynomial \( Q \) considered as element of \( \mathcal{C}_I[X] \). If \(|\lambda| > 1 \), the polynomial \( R = Q - P\lambda^{-1} \in \mathbb{H}[X] \) defined as above has zero derivative at \( x \): \( R'(x) = Q'(x) - P'(x)\lambda^{-1} = 0 \), contradicting what obtained before. Therefore, it must be \(|\lambda| \leq 1 \), i.e., \(|P'(x)|/|Q'(x)| \leq 1 \) for all \( x \in \mathcal{C}_I \) with \(|x| > 1 \). By continuity, \(|P'(x)| \leq |Q'(x)| \) for all \( x \in \mathcal{C}_I \) with \(|x| = 1 \).

To prove (3), we apply Lemma 2.2 to \( P' \) and use the inequalities \(|P'(x_1)| \leq |Q'(x_1)|, |P'(\bar{x}_1)| \leq |Q'(\bar{x}_1)| \). The last statement follows from a general property of slice functions (see again [9, Lemma 5.3]): since \( Q' \) is a \( \mathcal{C}_I \)-polynomial, its maximum modulus on the 2-sphere \( S_x \) is attained at one of the points \( x_1 = \alpha + I\beta, \bar{x}_1 = \alpha - I\beta \) of the intersection \( S_x \cap \mathcal{C}_I \). \qed

**Corollary 2.4.** (Bernstein’s inequality) If \( P \in \mathbb{H}[X] \) is a quaternionic polynomial of degree \( d \), then \( \|P'\| \leq d\|P\| \).
Proof. Let $M = \|P\|$ and apply the previous theorem to $P(X)$ and $Q(X) = MX^d$. Since $Q$ is slice-preserving, the first inequality in the thesis of Theorem 2.1 holds for every $I \in \mathbb{S}$.

Remark 2.5. The proof of Theorem 2.1 makes use of the complex Gauss–Lucas Theorem. One could hope to obtain a better estimate by means of a quaternionic Gauss–Lucas Theorem. Unfortunately, this last result is valid only for a small class of quaternionic polynomials, as it has been showed in [8].

The inequality of Corollary 2.4 is best possible with equality holding if and only if $P$ is a multiple of the power $X^d$. One implication is immediate. If $P(X) = X^d a$, with $a \in \mathbb{H}$ and $d \geq 1$, then $\|P\| = \|dX^d a\| = d|a| = d\|P\|$. We show the converse.

Proposition 2.6. If $P \in \mathbb{H}[X]$ is a quaternionic polynomial of degree $d$, and $|P'(y)| = d\|P\|$ at a point $y \in \mathbb{S}^3$, then $P(X) = X^d a$, for an $a \in \mathbb{H}$ with $|a| = \|P\|$.

Proof. We can assume that $P(X)$ is not constant. Let $b = P'(y)^{-1}$ and set $Q(X) := P(X)b = \sum_{k=1}^{d} X^k a_k$. Then, $Q'(y) = 1$, $\|Q\| = 1/d$ and $\|Q'\| \leq 1$. Let $I \in \mathbb{S}$, such that $C_I \ni y$. Then

$$1 = Q'(y) = \sum_k k y^{k-1} a_k = \pi_I(Q'(y)) = \sum_k k y^{k-1} \pi_I(a_k) = (Q^I)'(y).$$

If $x \in C_I \cap \mathbb{S}^3$, it holds

$$|(Q^I)'(x)| = \left| \sum_k k x^{k-1} \pi_I(a_k) \right| = \left| \pi_I \left( \sum_k k x^{k-1} a_k \right) \right| \leq \left| \sum_k k x^{k-1} a_k \right|$$

This means that the $C_I$-polynomial $Q^I$, considered as an element of $C_I[X]$, satisfies the equality in the classic Bernstein’s inequality. The same inequality implies that

$$1 = \max_{x \in C_I \cap \mathbb{S}^3} |(Q^I|_{C_I})'(x)| \leq d \max_{x \in C_I \cap \mathbb{S}^3} |Q^I|_{C_I}(x)| \leq d\|Q\| = 1,$$

i.e., $\max_{x \in C_I \cap \mathbb{S}^3} |Q^I|_{C_I}(x)| = 1/d$. Therefore, the restriction of $Q^I$ to $C_I$ coincides with the function $x^d c$, with $c \in C_I$, $|c| = 1/d$

$$Q^I(x) = \sum_{k=1}^{d} x^k \pi_I(a_k) = x^d c$$

for every $x \in C_I$.

This implies that $\pi_I(a_d) = c$, $\pi_I(a_k) = 0$ for each $k = 1, \ldots, d - 1$ and $Q$ can be written as $Q(X) = X^d c + \tilde{Q}(X)$, with the coefficients of $\tilde{Q}$ belonging to $C^\perp_I = \pi_I^\perp(\mathbb{H})$. When $x \in C_I \cap \mathbb{S}^3$, $\tilde{Q}(x) \in C^\perp_I$, and then

$$\frac{1}{d^2} \geq |Q(x)|^2 = |x^d c|^2 + |\tilde{Q}(x)|^2 = \frac{1}{d^2} + |\tilde{Q}(x)|^2.$$

This inequality forces $\tilde{Q}$ to be the zero polynomial, and then, $P(X) = Q(X)b^{-1} = X^d c b^{-1}$. □
We now show that in Theorem 2.1, the assumption on \( Q \) to be one-slice-preserving is necessary.

**Proposition 2.7.** Let
\[
P(X) = (X - i) \cdot (X - j) \cdot (X - k), \quad Q(X) = 2X \cdot (X - i) \cdot (X - j).
\]
Then, \( V(Q) = \{0, i\} \subseteq \mathbb{R} \) and \(|P(x)| \leq |Q(x)|\) for every \( x \in \mathbb{S}^3 \), but there exists \( y \in \mathbb{S}^3 \), such that \(|P'(y)| > |Q'(y)|\).

**Proof.** By a direct computation, we obtain
\[
P(X) = X^3 - X^2(i + j + k) + X(i - j + k) + 1,
\]
\[
Q(X) = 2X^3 - 2X^2(i + j) + 2Xk,
\]
\[
P'(X) = 3X^2 - 2X(i + j + k) + i - j + k,
\]
\[
Q'(X) = 6X^2 - 4X(i + j) + 2k.
\]
Let \( P_1(X) = X - k, \ Q_1(X) = 2X, \ P_2(X) = (X - j) \cdot P_1(X), \ Q_2(X) = (X - j) \cdot Q_1(X) \). Then, \( P(X) = (X - i) \cdot P_2(X) \) and \( Q(X) = (X - i) \cdot Q_2(X) \).

For every \( x \in \mathbb{S}^3 \setminus \{j\} \), using formula (1), we get
\[
|P_2(x)| = |x - j||(x - j)^{-1} x(x - j) - k| \leq 2|x - j| = |x - j|2x = |Q_2(x)|.
\]
Since \( P_2(j) = Q_2(j) = 0 \), the inequality holds also at \( j \). From this, we obtain, for each \( x \in \mathbb{S}^3 \setminus \{i\} \)
\[
|P(x)| = |x - i||P_2((x - i)^{-1} x(x - i))| \leq |x - i||Q_2((x - i)^{-1} x(x - i))| = |Q(x)|.
\]
Since \( P \) and \( Q \) vanish at \( i \), \(|P(x)| \leq |Q(x)|\) for every \( x \in \mathbb{S}^3 \).

Let \( y = \frac{1}{16}(1 + 9i + 4j - \sqrt{2}k) \in \mathbb{S}^3 \). An easy computation gives
\[
|P'(y)|^2 = \frac{7}{25}(5 + \sqrt{2}) \simeq 1.80, \quad |Q'(y)|^2 = \frac{4}{25}(10 - 3\sqrt{2}) \simeq 0.92.
\]

\( \square \)

3. Bernstein Inequality and Zonal Harmonics

Since the restriction of a complex variable power \( z^m \) to the unit circumference is equal to \( \cos(m\theta) + i\sin(m\theta) \), the classic Bernstein inequality for complex polynomials can be restated in terms of trigonometric polynomials. In this section, we show that a similar interpretation is possible in four dimensions, by means of an Almansi-type decomposition of quaternionic polynomials and its relation with zonal harmonics in \( \mathbb{R}^4 \).

Quaternionic polynomials, as any slice-regular function, are biharmonic with respect to the standard Laplacian of \( \mathbb{R}^4 \) [12, Theorem 6.3]. In view of Almansi’s Theorem (see e.g. [1, Proposition 1.3]), the four real components of such polynomials have a decomposition in terms of a pair of harmonic functions. The results of [12] can be applied to obtain a refined decomposition of the polynomial in terms of the quaternionic variable.

Let \( Z_k(x, a) \) denote the real four-dimensional (solid) zonal harmonic of degree \( k \) with pole \( a \in \mathbb{S}^3 \) (see, e.g., [2, Ch.5]). The symmetry properties of
zonal harmonics imply that \( Z_k(x, a) = Z_k(xa, 1) \) for every \( x \in \mathbb{H} \) and any \( a \in S^3 \). Moreover, it holds [12, Corollary 6.7(d)]

\[
x^k = \tilde{Z}_k(x) - \overline{x} \tilde{Z}_{k-1}(x) \quad \text{for every } x \in \mathbb{H} \text{ and } k \in \mathbb{N},
\]

where \( \tilde{Z}_k(x) \) is the real-valued zonal harmonic defined by \( \tilde{Z}_k(x) := \frac{1}{k+1} Z_k(x, 1) \) for any \( k \geq 0 \) and by \( \tilde{Z}_{-1} := 0 \).

In the following, we will consider polynomials in the four real variables \( x_0, x_1, x_2, x_3 \) of the form \( P(x) = \sum_{k=0}^d X_k c_k \), with quaternionic coefficients \( c_k \in \mathbb{H} \). They will be called zonal harmonic polynomials with pole 1. All these polynomials have an axial symmetry with respect to the real axis: for every orthogonal transformation \( T \) of \( \mathbb{H} \approx \mathbb{R}^4 \) fixing 1, it holds \( P \circ T = P \).

**Proposition 3.1.** (Almansi-type decomposition) Let \( P \in \mathbb{H}[X] \) be a quaternionic polynomial of degree \( d \geq 1 \). There exist two zonal harmonic polynomials \( A, B \) with pole 1, of degrees \( d \) and \( d - 1 \), respectively, such that

\[
P(x) = A(x) - \overline{x} B(x) \quad \text{for every } x \in \mathbb{H}.
\]

The restrictions of \( A \) and \( B \) to the unit sphere \( S^3 \) are spherical harmonics depending only on \( x_0 = \text{Re}(x) \).

**Proof.** Let \( P(X) = \sum_{k=0}^d X_k c_k \). Formula (6) follows immediately from (5) setting

\[
A(x) = \sum_{k=0}^d \tilde{Z}_k(x) c_k \quad \text{and} \quad B(x) = \sum_{k=0}^{d-1} \tilde{Z}_k(x) c_{k+1}.
\]

The restriction of \( \tilde{Z}_k(x) \) to the unit sphere \( S^3 \) is equal to the Gegenbauer (or Chebyshev of the second kind) polynomial \( C^{(1)}_k(x_0) \), where \( x_0 = \text{Re}(x) \) (see [12, Corollary 6.7(e)]). This property implies immediately the last statement. \( \square \)

**Remark 3.2.** See [13,14] for an extension of the Almansi decomposition to polynomials or more generally slice-regular functions on quaternions and Clifford algebras.

Thanks to the previous decomposition, the quaternionic Bernstein inequality of Corollary 2.4 can be restated in terms of Gegenbauer polynomials \( C^{(1)}_k(x_0) \). Let \( d \in \mathbb{N} \). For any \((d+1)\)-uple \( \alpha = (a_0, \ldots, a_d) \in \mathbb{H}^{d+1} \), let \( Q_\alpha : S^3 \rightarrow \mathbb{H} \) be defined by

\[
Q_\alpha(x) := \sum_{k=0}^d (C^{(1)}_k(x_0) - \overline{x} C^{(1)}_{k-1}(x_0)) a_k
\]

for any \( x = x_0 + ix_1 + jx_2 + kx_3 \in S^3 \) (where we set \( C^{(1)}_{-1} := 0 \)). Being the restriction to \( S^3 \) of the quaternionic polynomial \( P(X) = \sum_{k=0}^d X_k a_k \), which has biharmonic real components on \( \mathbb{H} \), \( Q_\alpha \) is a quaternionic valued spherical biharmonic of degree \( d \) (see, e.g., [10]).
Corollary 3.3. Let $\alpha = (a_0, \ldots, a_d)$ and $\alpha' = (a_1, 2a_2, \ldots, ka_k, \ldots, da_d, 0) \in \mathbb{H}^{d+1}$. Then, it holds
\[
\text{if } |Q_\alpha(x)| = \left| \sum_{k=0}^{d} \left( C_k^{(1)}(x_0) - \bar{\alpha} C_{k-1}^{(1)}(x_0) \right) a_k \right| \leq M \text{ for every } x \in S^3,
\]
then $|Q_{\alpha'}(x)| = \left| \sum_{k=0}^{d-1} \left( C_k^{(1)}(x_0) - \bar{\alpha} C_{k-1}^{(1)}(x_0) \right) (k+1)a_{k+1} \right| \leq dM
$
for every $x \in S^3$.

Proof. Let $P(X) = \sum_{k=0}^{d} X^k a_k$. From formula (5), it follows that the restriction of $P'$ to the unit sphere is the spherical biharmonic $Q_{\alpha'}$. Corollary 2.4 permits to conclude. □

Remark 3.4. Let $P \in \mathbb{H}[X]$ be a polynomial with Almansi-type decomposition $P(x) = A(x) - \bar{x}B(x)$ and let $y = \alpha + J\beta \in S^3$, $\alpha, \beta \in \mathbb{R}$, $\beta > 0$. Let $v = A(y)\bar{B}(y)$. It follows from general properties of slice functions [9, Lemma 5.3] that if $v \in \mathbb{R}$, then $|P|_{S_y}$ is constant, while if $v \not\in \mathbb{R}$, then the maximum modulus of $P$ on the 2-sphere $S_y \subset S^3$ is attained at the point $\alpha + I\beta$, with $I = \text{Im}(v)/|\text{Im}(v)|$, while the minimum modulus is attained at $\alpha - I\beta$. In principle, this reduces the problem of maximizing or minimizing the modulus of $P$ on the unit sphere (or ball) to a one-dimensional problem.

Example 3.5. Consider the polynomial $P(X) = (X - i) \cdot (X - j) \cdot (X - k)$ of Proposition 2.7. Since the first four zonal harmonics are
\[
\begin{align*}
\tilde{Z}_0(x) &= 1, \quad \tilde{Z}_1(x) = 2x_0, \quad \tilde{Z}_2(x) = 3x_0^2 - x_1^2 - x_2^2 - x_3^2, \\
\tilde{Z}_3(x) &= 4x_0(x_1^2 - x_2^2 - x_3^2),
\end{align*}
\]
the Almansi-type decomposition of $P$ is $P(x) = A(x) - \bar{x}B(x)$, with
\[
A(x) = \tilde{Z}_3(x) + \tilde{Z}_0(x) + (i+k) \left( \tilde{Z}_1(x) - \tilde{Z}_2(x) \right) - j \left( \tilde{Z}_1(x) + \tilde{Z}_2(x) \right)
\]
\[
= (1 + 4x_0^3 - 4x_0x_1^2 - 4x_0x_2^2 - 4x_0x_3^2) + (i+k)(2x_0 - 3x_0^2 + x_1^2 + x_2^2 + x_3^2)
\]
\[
- j(2x_0 + 3x_0^2 - x_1^2 - x_2^2 - x_3^2),
\]
\[
B(x) = (3x_0^2 - x_1^2 - x_2^2 - x_3^2) + i(1 - 2x_0) - j(1 + 2x_0) + k(1 - 2x_0)
\]
harmonic polynomials. Their restrictions to $S^3$ are the spherical harmonics
\[
A_{|S^3}(x) = (1 - 4x_0 + 8x_0^3) + i(1 + 2x_0 - 4x_0^2) + j(1 - 2x_0 - 4x_0^2)
\]
\[
+ k(1 + 2x_0 - 4x_0^2),
\]
\[
B_{|S^3}(x) = (-1 + 4x_0^2) + i(1 - 2x_0) - j(1 + 2x_0) + k(1 - 2x_0).
\]
Following the observation made in Remark 3.4, since $\text{Im}(A(y)\bar{B}(y)) = 4((\alpha - 1)i + \alpha k)$, where $\alpha = \text{Re}(y)$, $y \in S^3$, one can find the 2-sphere $S_y \subset S^3$ where the maximum modulus of $P$ is attained. A direct computation gives $\text{Re}(y) = (1 - \sqrt{19})/6 \sim -0.56$ and the corresponding maximum value $\|P\| \sim 4.70$ attained at the point $\bar{y} = (1 - \sqrt{19})/6 - i(5 + \sqrt{19})/12 + k(1 - \sqrt{19})/12$ of $S^3$. 

Some of the results presented in this note can be generalized to the general setting of real alternative *-algebras, where polynomials can be defined and share many of the properties valid on the quaternions (see [7]). The polynomials of Proposition 2.7 can be defined every time the algebra contains an Hamiltonian triple \(i, j, k\), i.e., when the algebra contains a subalgebra isomorphic to \(\mathbb{H}\) (see [4, Sect. 8.1]). This is true, e.g., for the algebra of octonions and for the Clifford algebras with signature \((0, n)\), with \(n \geq 2\). In all such algebras, one can repeat the previous proofs and get the analog of Theorem 2.1, as well as of the Bernstein inequality (see also [15] for this last result).

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**Declarations**

**Conflict of Interest** The author has no competing interests to declare that are relevant to the content of this article.

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