Supplemental document accompanying submission to Optica

Title: Conformal frequency conversion for arbitrary vectorial structured light

Authors: Zhihan Zhu, Hai-Jun Wu, Bing-Shi Yu, Wei Gao, Dong-Sheng Ding, Zhiyuan Zhou, Xiaopeng Hu, Carmelo Rosales, Yijie Shen, Bao-Sen Shi

Submitted: 9/28/2021 9:20:23 AM
Conformal frequency conversion for arbitrary vectorial structured light: supplemental document

1. Detailed Theoretical Framework

Spatial Modes. — In the simulation and data analysis, we used LG modes, denoted by $LG_{\ell}^p$, and their superpositions to represent general spatial mode. The spatial complex amplitude of the LG mode in the cylindrical coordinates $\{r, \varphi, z\}$, with the spatial indices of $\ell$ (azimuthal) and $p$ (radial), is given by [1]

$$LG_{\ell}^p(r, \varphi, z) = \sqrt{2p! \pi (p+|\ell|)! w(\zeta)} \frac{1}{w(z)} \left(\frac{2r^2}{w(z)}\right)^{|\ell|} \exp\left[\frac{-r^2}{w(z)^2}\right] L_{p}^{2\ell+1}\left(\frac{2r^2}{w(z)^2}\right) \exp\left[-i\left(kz + k\frac{r^2}{2R_c} + \ell \varphi - i\phi_0\right)\right],$$

where $w(z) = w_0\sqrt{1+(z/z_w)^2}$, $R_c = z^2 + z_w^2$, and $\phi_0 = (2p + |\ell| + 1)\arctan(z/z_w)$ denote the beam waist, radius of curvature, and Gouy phase upon propagation (here $z_w = k w_0^2/2$ is the Rayleigh length), respectively, and $L_{p}^{2\ell+1}(\cdot)$ is the Laguerre polynomial with mode orders $p$ and $\ell$, given by

$$L_{p}^{2\ell+1}(\gamma) = \sum_{k=0}^{\min(p,\ell)} \frac{(-\gamma)^k}{k!p!(\ell-k)!} \Gamma(p+\ell+1,k+1).$$

Ince-Gauss (IG) modes, used for representing the generalized SOC modes, are eigenfunctions of the paraxial wave equation in elliptical coordinates. A pair of IG modes with even and odd parity can be expressed as [2]

$$IG_{\eta,m}^\epsilon(r, z; \epsilon) = \gamma_\epsilon C_{\eta}^\epsilon(i\xi, \epsilon) C_{\eta}^\epsilon(\eta, \epsilon) \exp[G_{N}^\epsilon(r, z)],$$

$$IG_{\eta,m}^\epsilon(r, z; \epsilon) = \gamma_o S_{\eta}^\epsilon(i\xi, \epsilon) S_{\eta}^\epsilon(\eta, \epsilon) \exp[G_{N}^\epsilon(r, z)],$$

where $\xi \in [0, \infty)$ and $\eta \in [0, 2\pi)$ are the radial and the angular elliptic variables, respectively; $C_{\eta}^\epsilon(*)$ ($S_{\eta}^\epsilon(*)$) and $\gamma_\epsilon$ ($\gamma_o$) are even (odd) Ince polynomials and the associated normalization constants, respectively; and $G_{N}^\epsilon(r, z)$ is the amplitude envelop of Gaussian beams of order $N$. It should be noted that $\epsilon \in [0, \infty)$ defines the ellipticity of the coordinates, i.e., the elliptical coordinates are not unique and, particularly, the IG modes become LG and HG modes with same the orders for $\epsilon = 0$ and $\epsilon \to \infty$, respectively. For a given $\epsilon > 0$, $IG_{\eta,m}^\epsilon$ can be expressed as a superposition of $(N+1)$ LG modes with the same order $N = 2p + |\ell|$, given by

$$IG_{\eta,m}^\epsilon = \sum_p a_p LG_{\eta}^{p, \epsilon}, \quad p \in [0, 1, \ldots, (N-1)/2],$$

In addition, similar to the relation between the helical LG modes and their parity counterparts, IG mode can also exist in the helical manner, denoted as $IG_{\eta,m}^\epsilon$, that carry net OAM, and the corresponding relation is given by [3]

$$IG_{m}^{\epsilon} = \sqrt{2}(IG_{m}^{\epsilon} \pm iIG_{m}^{\epsilon}),$$

Thus, $IG_{\eta,m}^\epsilon$ with $\epsilon = 0, 2$, and $\epsilon$, used in the Sec. B of the main text, can be represented as LG superpositions modes by combined use of Eqs. (S1–S4), given by
\[
IG_{3,3}^{\alpha=0} = LG_{1}^{\pm 3} \\
IG_{3,3}^{\alpha=2} = 0.26LG_{1}^{0} + 0.798LG_{1}^{3} - 0.508 LG_{2}^{1} \\
-0.165LG_{3}^{1} - 0.103LG_{5}^{3} - 0.005 LG_{5}^{5}, \\
IG_{3,3}^{\alpha=-2} = 0.26LG_{3}^{0} + 0.798LG_{3}^{3} - 0.508 LG_{1}^{1}, \\
-0.165LG_{1}^{1} - 0.103LG_{3}^{3} - 0.005 LG_{5}^{5}. \\
IG_{5,3}^{\alpha=4} = 0.791LG_{5}^{0} - 0.5 LG_{2}^{5} - 0.354LG_{3}^{3} \\
IG_{5,3}^{\alpha=-4} = 0.791LG_{5}^{0} - 0.5 LG_{2}^{5} - 0.354LG_{3}^{3}
\]

from where we know that the average OAM carried by them are \( \pm 3\hbar, \pm 2.447\hbar, \) and \( \pm 3\hbar \) per photon, respectively.

**Modal Transformation.** — Here we derive the modal transformation of the general vector mode, shown in Eq. (1), in the Sagnac loop, i.e., \( E_{\alpha}^{\text{in}}(r,z) = E_{\alpha}^{\text{in}}(r,z)\hat{e}_{\theta} + e^{i\phi}E_{\alpha}^{\text{in}}(r,z)\hat{e}_{\varphi} \). By assuming the SoP relation \( \hat{e}_{\theta} = \sqrt{1 - \beta}\hat{e}_{\theta} + e^{i\alpha}\sqrt{1 - \beta}\hat{e}_{\varphi} \) and \( \hat{e}_{\varphi} = \sqrt{1 - \beta}\hat{e}_{\theta} - e^{i\alpha}\sqrt{1 - \beta}\hat{e}_{\varphi} \), we can obtain the SoP-dependent spatial modes with respective to \( \hat{e}_{\theta} \) and \( \hat{e}_{\varphi} \), given by

\[
E_{\alpha}^{\text{in}}(r,z) = [\sqrt{1 - \beta}]u_{\alpha}(r,z) + e^{i\alpha}(1 - \beta)u_{\alpha}(r,z)]e^{-i\phi(\alpha)z} \\
E_{\alpha}^{\text{in}}(r,z) = [\sqrt{1 - \beta}]u_{\alpha}(r,z) - e^{i\alpha}(1 - \beta)u_{\alpha}(r,z)]e^{-i\phi(\alpha)z}.
\]

Note that \( E_{\alpha}^{\text{in}}(r,z) \) and \( E_{\alpha}^{\text{in}}(r,z) \) are not usually orthogonal to each other, unless \( \alpha = 0.5 \) [4].

In addition, note that the premise to achieve the conformal upconversion, i.e., \( E_{\alpha}^{\text{in}}\hat{e}_{\theta} \rightarrow E_{\alpha}^{\text{in}}\hat{e}_{\theta} \) and \( E_{\alpha}^{\text{in}}\hat{e}_{\varphi} \rightarrow E_{\alpha}^{\text{in}}\hat{e}_{\varphi} \), is using flattop beam. Here we show briefly why the commonly used Gauss pump would lead to the angular spread of spatial spectrum, or rather, radial-modal degeneration for LG modes. The spatial amplitude of an SFG (with \( k_{1} = k_{1} + k_{2} \)) driven by two LG modes \( LG_{m}^{0}(r,\varphi) \) with \( k_{1} \) and \( LG_{n}^{0}(r,\varphi) \) with \( k_{2} \), can be expressed as

\[
E_{\alpha\beta\gamma}(r,\varphi) = LG_{m}^{0}(r,\varphi)LG_{n}^{0}(r,\varphi)
\]

where \( w_{1} \) and \( w_{2} \) denote the beam waists of two pumps. Note that Eq. (S7) usually are no long a propagation-invariant (eigen) mode and one can further represent it in the form of a superposition of LG modes, i.e., the modal selection rule, given by

\[
E_{\alpha\beta\gamma}(r,\varphi) = \sum_{n=m}^{m} a_{\alpha} LG_{n}^{0}(r,\varphi),
\]

where \( m = \ell_{1} + \ell_{2} \) and \( n = (|\ell_{1}| + |\ell_{2}| - |\ell_{1} + \ell_{2}|)/2 \). See Ref. 4 for more details. By assuming \( w_{1} = \ell_{1} \) and \( \ell_{2} = p_{2} = 0 \), we can obtain the modal transformation of LG modes in the upconversion pumped by a Gauss mode. For instance, the transformation of \( LG_{1}^{4}, LG_{0}^{4}, \) and \( LG_{2}^{4} \), i.e., a group data in Fig. 2(a), is shown below.
Super-Gauss Mode. — To realize the conformal frequency conversion, we need a perfect flattop beam — i.e., whose amplitude and phase are both spatially uniform — as the pump. Crucially, the intensity-flattop beam obtained via phase-only modulation that we used in Refs. 5 and 6, cannot provide a flattop wavefront. To address this, we design a perfect flattop beam based on super-Gauss distribution \([7]\), given by \( u_{SG} (r, \phi) = \exp \left(-\frac{r}{\omega} \right) \) , its intensity distribution is shown in Fig. S2 (a), where we see that the degree of ‘flattop’ increases with the order \(n\). Here we used the distribution with an order of \(n = 12\) to define the super-Gauss mode at the beam waist, given by

\[
E_{SG} (r, \phi, z_0) = u_{SG} (r, \phi) \exp \left(-i k z \right),
\]

and its spatial complex amplitude upon propagation including lens transformation can be calculated using Collins integral \([8]\). To generate the super-Gauss mode at the focal region in the crystal, we used computed holography based on complex-amplitude modulation and following with a lens Fourier transformation, as shown in Fig. S2(b). The complex-amplitude mask loaded on the surface of SLM was designed by a Fourier integral, given by

\[
E_{SG} (r, \phi, z_0) = \mathcal{F} \left\{ E_{SG} (r, \phi, z_0) \right\}.
\]

![FIG. S1. Modal transformation of high-order LG modes during the SFG pump by Gauss-mode pump.](image1)

![FIG. S2. (a) Intensity distribution of super-Gauss distribution as a function of order \(n\). (b) Super-Gauss mode generation using computed holography based on complex-amplitude modulation.](image2)
Beyond having a perfect flattop complex amplitude at the beam waist plane, the super-Gauss mode has also another important merit, i.e., maintaining a flattop wavefront over a short distance. For comparison, Fig. S3 shows beam propagations of the super-Gauss mode and the TEM$_{00}$ mode in a 20 mm PPKTP crystal, which were used in the experiment. We see that the wavefront of super-Gauss mode, especially for the center region, keep flat through the crystal. Yet, for a Gauss beam, sphere wavefronts appear at the two ends of the crystal.

![Image](a) (b)

FIG. S3. Beam propagation behavior of (a) the super-Gauss mode ($w_0=0.2$ mm) and (b) TEM$_{00}$ mode ($W_0=0.5$) within a 20 mm crystal, which are focused by a 100 mm lens.

2. Additional Results

For the setup demonstrated in the main text, we also measured the efficiency of conversion in the depleted region by using pulse light, where the pulses were obtained via intensity-modulated 1560 nm wave and its SHG (780nm) wave. The quasi-continuous pulses have a 4ns duration with 100:1 duty factor. Before examine the conformal upconversion, the conversion efficiency between Gauss modes was characterized, as shown in Fig. S4, where the average power of signal was fixed at 1 mW. It was shown that, as illustrated in the Sec. A of the main text, i) both $\eta_p$ and $\eta_q$ are linearly proportional to the pump power before the signal-depleted region; and ii) $\eta_p$ for 780 nm signal are double of that for 1560 nm, while $\eta_q$ is identical for the two. Besides, it is worthy to note that the upper limit of $\eta_q$ is below 100% in the Gauss-mode pumped SFG, even for a Gauss-mode signal.

![Image](a) (b)

FIG. S4. Measured power (a) and quantum (b) efficiencies in the (1 mW) small signal SFG, where pump and signal were both Gauss modes with the same beam waist at the center plane of the crystal. The curves and dots are theoretical prediction and experimental data, respectively.
Then, we first consider the small signal in the depleted region. Specifically, a CV mode with $t = 2$ at 780 nm was pumped by a flattop light at 1560 nm that can just well cover it, and the average power of the CV-mode signal was fixed at 1 mW. The results in Fig. S5 show that, unlike the max $\eta_q$ was limited ~80% shown in Fig. S2, the 100% $\eta_q$ can be obtained experimentally by using only a 1.175 W pump with a flattop mode. The vectorial transverse structure was also well maintained during the upconversion.

![FIG. S5. Conformal upconversion of CV modes in the (1 mW) small-signal-depleted region, where the complete MUBs are defined in the SOC space spanned by $|L\rangle=|\hat{e}_z,-2\rangle$ and $|R\rangle=|\hat{e}_z,2\rangle$. (a)–(c) show the theoretical vector profiles, the measured signals, and the corresponding SFG, respectively, at the 100% quantum efficiency, and corresponding positions of the MUBs on the HOPS are given in (d). (e) Measured quantum efficiency versus pump power.](image)

Finally, we consider cases in laser-based applications, that is, depleted SFG with a larger signal. Here, a group of vectorial IG modes based on $IG_{44}$ with $\nu = 1$ were played as signals, whose power was 200 mW at 1560 nm. According to the measured $\eta_q$ with respect to power of pump shown in Fig. S6(a), one can obtain a 200 mW structured laser at 520 nm with the same vector profiles by using a 513 mW pump.

![FIG. S6. Conformal upconversion of vectorial IG modes in the big-signal-depleted region, where the power of signal is 200 mw. (a) Measured power efficiency versus pump power. (b) and (c) show the SOC states on the sphere and associated vector profiles at the 100% power efficiency, respectively.](image)
References

1. Allen L, Beijersbergen M W, Spreeuw R J C, et al. Orbital angular momentum of light and the transformation of Laguerre-Gaussian laser modes. Phys. Rev. A 45(11): 8185-8189 (1992).
2. M. A. Bandres and J. C. Gutiérrez-Vega. Ince–Gaussian beams, Opt. Lett. 29, 144 (2004).
3. W. N. Plick, M. Krenn, R. Fickler, S. Ramelow, and A. Zeilinger. Quantum orbital angular momentum of elliptically symmetric light. Phys. Rev. A 87, 033806 (2013).
4. H.-J. Wu, L.-W. Mao, Y.-J. Yang, C. Rosales-Guzmán, W. Gao, B.-S. Shi, Z.-H. Zhu. Radial modal transitions of Laguerre-Gauss modes during parametric up-conversion: Towards the full-field selection rule of spatial modes, Phys. Rev. A 101, 063805 (2020).
5. H.-J. Wu, B. Zhao, C. R.-Guzmán, W. Gao, B.-S. Shi, and Z.-H. Zhu. Spatial-Polarization-Independent Parametric Up-Conversion of Vectorially Structured Light. Phys. Rev. Applied 13, 064041 (2020).
6. H.-R. Yang, H.-J. Wu, W. Gao, C. R.-Guzmán, and Z.-H. Zhu. Parametric upconversion of Ince-Gaussian modes. Opt. Lett. 45, 3034 (2020).
7. F. Gori. Flattened gaussian beams. Opt. Comm. 107(5-6):335-341 (1994).
8. S. A. Collins. Lens-system diffraction integral written in terms of matrix optics. J. Opt. Soc. Am. A, 60(9), 1168 (1970).