Gauge theories hold a most prominent place in physics. They appear as effective low energy descriptions at different instances in condensed matter physics and nuclear physics. But far and foremost they lie at the root of our understanding of the four fundamental interactions that are each mediated by the gauge fields corresponding to a particular gauge symmetry. At the perturbative quantum level, this picture translates to the Feynman diagrammatic approach that has produced physical predictions with unlevelled precision, most famously in quantum electrodynamics (QED). However the perturbative approach miserably fails once the interactions become strong. This problem is most pressing for quantum chromodynamics (QCD), where all low energy features like quark confinement, chiral symmetry breaking and mass generation are essentially non-perturbative.

Lattice QCD, which is based on Monte Carlo sampling of Wilson’s Euclidean lattice version of gauge theories, has historically been by far the most successfull method in tackling this strongly coupled regime. Using up a sizable fraction of the global supercomputer time, state of the art calculations have now reached impressive accuracy, for instance in the ab initio determination of the light hadron masses [1]. But in spite of its clear superiority, the lattice Monte Carlo sampling also suffers from a few drawbacks. There is the infamous sign problem that prevents application to systems with large fermionic densities. In addition, the use of Euclidean time as opposed to real time, presents a serious barrier for the understanding of dynamical non-equilibrium phenomena. Over the last few years there has been a growing experimental and theoretical interest in precisely these elusive regimes, e.g. in the study of heavy ion collisions or early time cosmology.

In this letter we study the application of tensor network states (TNS) as a possible complementary approach to the numerical simulation of gauge theories focusing on the Schwinger model. This is highly relevant as this Hamiltonian method is free from the sign problem and allows for real-time dynamics. For the Schwinger model the TNS approach has been studied before by Byrnes et al [2] and Bañuls et al [3]. By integrating out the gauge field (which one can only do for $d=1+1$), the model was reduced to an ordinary spin model, yet with a non-local Hamiltonian. Our approach is conceptually different, as we keep the gauge field degrees of freedom, which enables us to take the thermodynamic limit, with the relevant global symmetries exact. TNS have been considered also for the discrete $Z_2$ gauge theory, for $d=1+1$ by Sugihara [4], and for $d=2+1$ by Tagliacozzo and Vidal [5].

Over the last decade the TNS framework has emerged as a powerful tool for the study of local quantum many body systems, exploring the fact that physical states (i.e. ground states and their low energy excitations) only occupy a tiny corner of the full Hilbert space [6]. This is exemplified by the relatively small amount of quantum entanglement that these states possess. TNS are then trial quantum states that precisely capture this feature, allowing for relatively low cost numerical variational calculations. In one spatial dimension, they also go by the name of matrix product states (MPS), underlying the well known density matrix renormalization group algorithm (DMRG) [7]. At present MPS/DMRG is the state of the art method in the numerical study of both static and dynamical properties of $d=1+1$ strongly correlated condensed matter systems. And also in higher dimensions the TNS framework [8], although less developed, is considered to be a promising candidate for the numerical simulation of strongly interacting quantum many body systems.

The essential new ingredient with respect to the usual MPS applications on quantum many body systems is that for the Hamiltonian formulation of gauge theories, of the full Hilbert space only the subspace of gauge invariant states is actually physical. Although, due to Elitzur’s theorem [9] gauge invariance will not be broken on the...
full Hilbert space, the low-energy excitations on the full space will typically be completely different than those on the constrained physical subspace. It is therefore crucial to restrict the variational MPS manifold to this physical subspace. Notice that the very same issue poses itself in the context of the simulation of gauge theories with ultracold atoms. See \[10\] for a recent proposal to implement gauge invariance in that case.

The massive Schwinger model is QED in 1+1 dimensions, with one flavor of fermionic particles with mass $m$, interacting through a $U(1)$ gauge field with coupling $g$ (which has mass dimension one for $d=1+1$). This model shares some interesting features with QCD, most notably the fermions are confined into zero charge bound state. Furthermore in the continuum it can be studied by a strong coupling expansion \[11\][12], which makes it a perfect benchmark model. We will apply our gauge invariant MPS construction on the Hamiltonian lattice formulation of the model, focusing on the strongly coupled regime $g/m \gtrsim 1$, and extrapolating our results to the continuum. We determine the ground state and stable bound states. In addition, we use our formalism to simulate the full real time quantum dynamics induced by a background electric field. This set up was recently also considered by Hebenstreit et al \[13\], with classical-statistical simulations for the gauge fields.

The Schwinger Hamiltonian To write down a lattice Hamiltonian for the Schwinger model, one starts from the Lagrangian density in the continuum:

$$\mathcal{L} = \bar{\psi} (\gamma^\mu (i\partial_\mu + g A_\mu) - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (1)$$

One can then perform a Hamiltonization quantization in the time-like axial gauge ($A_0 = 0$), which can be turned in a lattice system by the Kogut-Susskind spatial discretization \[14\] with the two-component fermions sited on a staggered lattice. These fermionic degrees of freedom can then finally be converted to spin 1/2 degrees of freedom by a Jordan-Wigner transformation, leading to the gauged spin Hamiltonian (see \[2\] for more details):

$$H = \frac{g}{2\sqrt{2}} \left( \sum_{n \in \mathbb{Z}} L(n)^2 + \frac{\mu}{2} \sum_{n \in \mathbb{Z}} (-1)^n (\sigma_z(n) + (-1)^n) \right)$$
$$+ x \sum_{n \in \mathbb{Z}} (\sigma^+ (n) e^{i\theta(n)} \sigma^- (n + 1) + h.c.). \quad (2)$$

Here we have introduced the parameters $x \equiv 1/(g^2 a^2)$ and $\mu \equiv 2\sqrt{m}/g$, with $a$ the lattice spacing.

The spins live on the sites of the lattice, with $\sigma_z(n) | s_n \rangle = s_n | s_n \rangle (s_n = \pm 1)$, and $\sigma^\pm \equiv 1/2(\sigma_x \pm i\sigma_y)$ the spin ladder operators. Notice the different second (mass) term in the Hamiltonian for even and odd sites. This can be traced back to the staggered formulation, with the even sites being reserved for the 'positrons' and the odd sites for the 'electrons'. For the even positron sites $s_{2n} = +1$ can be viewed as an occupied state, while $s_{2n} = -1$ corresponds to an empty state, and vice versa for the odd electron sites. The gauge fields $\theta(n) = agA_1(na/2)$, live on the links between the sites. Their conjugate momenta $L(n)$, with $[\theta(n), L(n')] = i\delta_{n,n'}$, correspond to the electric field, $gL(n) = E(na/2)$. Since $\theta(n)$ is an angular variable, $L(n)$ will have integer charge eigenvalues $p_n \in \mathbb{Z}$. The local Hilbert space, spanned by the corresponding eigenkets $| p_n \rangle$ is therefore infinite, but in practice we will do a truncation and consider $| p_n \rangle \leq p_{\text{max}}$ in a numerical scheme. For our calculations we take $p_{\text{max}} = 3$.

The Hamiltonian \[2\] is invariant under $T^2$, a translation over two sites, and the corresponding eigenvalues read $T^2 = e^{2ika}$, where $k$ is the physical momentum of the state. Another symmetry that will be useful is $CT$, obtained by a translation over one site, followed by a charge conjugation, $C | s_n, p_n \rangle = (-s_n, -p_n)$. Since $C^2 = 1$, we will have $CT = \pm e^{ika}$. The states with positive sign then correspond to the scalar sector, while the negative sign corresponds to the vector sector.

In addition, the Hamiltonian is invariant under the residual time-independent local gauge transformations, generated by

$$G_n = L(n) - L(n - 1) - \frac{1}{2}(\sigma_z(n) + (-1)^n) \quad (3)$$

It is this gauge invariance that sets the Hamiltonian quantization of gauge theories apart from the Hamiltonian quantization of ordinary systems. For gauge theories only the subspace of gauge-invariant states will be physical: $G_n | \Psi \rangle_{\text{phys}} = 0$ for every $n$. This is called the Gauss’ law constraint, as $G_n = 0$ is indeed the discretized version of $\partial \cdot E = \rho$. And we will now show how one can tailor the MPS formalism towards a constrained variational method on this physical gauge-invariant subspace.

Gauge invariant MPS. A general, not necessarily gauge-invariant MPS for the lattice spin-gauge system \[2\] has the form:

$$\sum_{s_n, \mathbf{p_n}} Tr[B_1^\alpha \cdots B_{2N}^\alpha C_1^\beta \cdots C_{2N}^\beta \cdots C_{2N}^{\alpha_2} | s_1, p_1, s_2, p_2, \ldots, p_{2N} \rangle \rangle,$$

where for now we consider a finite lattice of $2N$ sites. Here, each $B_n^\alpha$ (and $C_n^\beta$) is a complex $D \times D$ matrix with components $[B_n^\alpha]_{\alpha\beta}$, that constitute the variational parameters of the trial state. The indices $\alpha, \beta = 1, \ldots, D$ are referred to as virtual indices, and $D$ is called the bond dimension.

Gauss’ law (see \[3\]) prescribes how to update the electric field $L(n)$ at the right link of a site $n$: either staying with the value $L(n - 1)$ at the left in case there is no charge at the site, or adding/subtracting one unit in case there is a positive/negative charge at the site. This can be conveyed by the matrix multiplications in an MPS by giving the virtual indices a multiple index structure.
\[ \alpha \rightarrow (q, \alpha_q), \] where \( q \) labels the charge, and taking the matrices of the form:

\[
[B^n_m]_{(q, \alpha_q), (r, \beta_r)} = [b^n_m]_{n, q, \alpha_q, \beta_r} \delta_{q+(s_a+(-1)^s)/2, r} \\
[C^n_m]_{(q, \alpha_q), (r, \beta_r)} = [c^n_m]_{n, q, \alpha_q, \beta_r} \delta_{q, p_n} \delta_{r, p_n}. \tag{5}
\]

One can readily verify that an MPS \([4]\) with matrices of this form, indeed obeys the Gauss’ law constraint at every site. Conversely, we show in the appendix that every gauge-invariant state \( |\Psi\rangle \), obeying \( G_n |\Psi\rangle = 0 \) for every \( n \), has an MPS representation of the form \([5]\). We now show how to apply the time-dependent variational principle (TDVP) of \([20]\) to obtain a ground-state approximation in the thermodynamic limit \( (N \rightarrow \infty) \), taking into account the gauge invariance and anticipating \( CT = 0 \) (see the appendix for the details). It will be useful to block a site and link into one site with local Hilbert spaced spanned by the states \( |q_{2n-1}\rangle = \{|s_{2n-1}, p_{2n-1}\}\) and \( |q_{2n}\rangle = \{|s_{2n}, p_{2n}\}\). The ground-state ansatz then takes a form similar to a uniform MPS (uMPS) \([20]\):

\[
|\Psi(A)\rangle = \sum_{q_n} v_L^q \left( \prod_{n \in \mathbb{Z}} A^n_{q_n} \right) v_R |q^n\rangle, \tag{6}
\]

where \( |q^n\rangle = \{|(-1)^n q_n\}_{n \in \mathbb{Z}}\), \( v_L, v_R \in \mathbb{C}^D\), and \( A^n \in \mathbb{C}^{D \times D} \) as follows from \([3]\) of the general form (see appendix):

\[
[A^{(s,p)}]_{(q, \alpha_q); (r, \beta_r)} = [a^{s,p}]_{\alpha_q, \beta_r, \delta_{p,q+(s+1)/2}, \delta_{r,-p}}. \tag{7}
\]

The variational freedom of the gauge invariant state \( |\Psi(A)\rangle \) then lies within the matrices \([a^{s,p}] \in \mathbb{C}^{D_a \times D_r}\) and the total bond dimension of the uMPS equals \( D = \sum_{q \in \mathbb{Z}} D_q \).

The TDVP method evolves the Schrödinger equation (SE), \( i\hbar \partial_t |\Psi(A)\rangle = H |\Psi(A)\rangle \), within the manifold of uMPS. To this end the right-hand side of the SE is replaced by \( |\Phi(B_H(A), A)\rangle \), where \( |\Phi(B, A)\rangle \) is given by

\[
\sum_{m \in \mathbb{Z}} \sum_{q_n} v_L^q \left( \prod_{n \in m} A^n_{q_n} \right) B^{q_m} \left( \prod_{n > m} A^n_{q_n} \right) v_R |q^n\rangle, \tag{8}
\]

with \( B^q \) also of the block structure form \([4]\), \( B_H(A) = \arg\min_B \| |\Phi(B, A)\rangle - H |\Phi(A)\rangle \| \) and \( |\Phi(B_H(A), A)\rangle \perp |\Psi(A)\rangle \). The SE then boils down to an ordinary differential equation for the variational parameters, \( i\alpha = b_H(a) \), where \( b_H(a) \) can be calculated in \( \mathcal{O}\left((2p_{\text{max}}+1) \max_p \delta_D^3\right) \) time. Starting from a random state \( |\Psi(A)\rangle \) we will then evolve towards the ground state by an imaginary time evolution \( \tau = it \), that we stop once the state has converged.

Once we have a good approximation for the ground state, we can use the method of \([21]\) to obtain the one-particle excited states. The excitations are labelled by their (physical) momentum \( k \in [-\pi/2a, \pi/2a] \) and their CT quantum number \( \gamma = \pm 1 \). For a given ground-state approximation we then take the following ansatz state \( |\Phi_k, \gamma(B, A)\rangle \) for the one-particle excitations:

\[
\sum_{m \in \mathbb{Z}} \sum_{q_n} v_L^q \left( \prod_{n \in m} A^n_{q_n} \right) B^{q_m} \left( \prod_{n > m} A^n_{q_n} \right) v_R |q^n\rangle, \tag{9}
\]

with \( B^q \) again of the gauge-invariant form \([7]\) with general matrices \( b^{s,p} \). These are determined variationally by minimizing their energy in the ansatz subspace which leads to a generalized eigenvalue problem. For a given momentum and CT quantum number we typically find different local minima of which only one or two are stable under variation of the bond dimension \( D \). It are these stable states that we can interpret as approximations to actual physical one-particle excitations.

**Results.** The continuum limit \( a \rightarrow 0 \) of the Schwinzer model corresponds to the limit \( x \rightarrow \infty \) in \([2]\). To obtain the energies of the ground state and of the one-particle excitations in this limit, we have calculated these quantities for values of \( x = 100, 200, 300, 400, 600, 800 \). At every \( x \) we considered different values of \( D \) till convergence was reached at some \( D_{\text{max}} \). We estimated the truncation error on \( D \) from comparison of the result for \( D = D_{\text{max}} \) with the result for the next to largest value of \( D \). Larger values of \( x \) typically required larger values of \( D \) for the same order of the error. For instance for \( m/g = 0.5 \) our maximal \( D \) varied from 185 for \( x = 100 \) to 358 for \( x = 800 \). This scaling of \( D \) is not surprising, as it is well known that MPS representations require larger \( D \) for systems with larger correlation lengths \( \xi \) (in units of the lattice spacing) \([6]\). For the Schwinger model \( \xi \) indeed diverges in the \( x \rightarrow \infty \) limit.

It was also important to choose the distribution of \( D_q \) wisely, according to the relative weight of the different charge sectors. As illustrated in fig.1a, this is done by looking at the Schmidt coefficients for an arbitrary cut, and demanding that the smallest coefficients of each sector coincide more or less. The resulting distribution of

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*FIG. 1: Results for \( m/g = 0.75 \). Left (a): distribution of the logarithm of the Schmidt-coefficients in every charge sector. Right (b): Fit of the Einstein-dispersion relation \( E^2 = k^2 + M^2_{\perp,1}(x) \) for \( x = 100, 300, 800 \) (dashed lines) to the data (small circles). The stars represent the estimated continuum values, the full line (lowest lying curve) is the curve \( E^2 = k^2 + M^2_{\perp,1} \).*
TABLE I: Energy density and masses of the one-particle excitations (in units $g = 1$) for different $m/g$. The last column displays the result for the heavy vector boson, compatible with the prediction of Coleman [11] [12].

| $\omega_0$ | $M_{\nu,1}$ | $M_{\nu,2}$ |
|------------|-------------|-------------|
| 0.0000     | 0.56418(2)  |             |
| 0.125      | 0.78991(8)  | 1.472(4)    |
| 0.25       | 1.01917(2)  | 2.10(2)     |
| 0.5        | 1.48747(3)  | 2.778(2)    |
| 0.75       | 1.96347(3)  | 3.2043(2)   |
| 1.00       | 2.44441(1)  | 3.640(4)    |

$D_q$ is peaked around $q = 0$, and justifies our $p_{\text{max}} = 3$ truncation that corresponds to $D_q = 0$ for $|q| > 3$.

To extrapolate towards $x \to \infty$ we used a third order polynomial fit in $1/\sqrt{x}$ through the largest five $x$-values. Similar to [2] our extrapolation error is then estimated by considering a third and fourth order polynomial through all six points, taking the error to be the maximal difference with the original inferred value.

In table 1 we display our resulting values for the ground state energy density and the mass of the different one-particle excitations. For $m/g = 0$ this can be compared with the exact result that follows from bosonization [15]. In this limit the model reduces to a free theory, of one bosonic vector ($\gamma = -1$) particle with mass $M_{\nu,1} = 1/\sqrt{\pi} = 0.56419$ and with a ground-state energy density $\omega_0 = -1/\pi = -0.318310$ (both in units $g = 1$).

Furthermore, in the strong coupling expansion $g/m \gg 1$ on this exact result, it is found that the vector boson becomes an interacting particle, leading to two more stable bound states. There appears one scalar boson that is a stable bound state of two vectors and one more vector boson, that is best interpreted as a bound state of the scalar and the original lowest mass vector [11] [12]. For $g/m \neq 0$ we also find three excited states, one scalar and two vectors, with the hierarchy of masses $M_{\nu,1} < M_{\nu,1} < M_{\nu,2}$ matching that of the strong coupling result. But notice that for our values of $g/m$, the strong coupling expansion result is not reliable anymore, making a quantitative comparison useless. One can also show that in the continuum limit the ground-state energy is independent of $g/m$ which is compatible with our findings.

This is the first time that the second vector excitation has been found numerically. For the energy density and the two lowest mass excitations our results are consistent with the previous most precise simulations [2] [3], with a similar or sometimes better accuracy.

A nice cross-check of our method also follows from calculating the excitation energies for non-zero momenta $k$. The Schwinger model is Lorentz invariant in the continuum limit, so we should have an approximate Einstein dispersion relation at finite lattice spacing $a$, for small momenta $ka \ll 1$. As shown in fig.1b, this is precisely what we find.

Finally, we have investigated the non-equilibrium dynamics induced by applying a uniform electric field $E$ on the ground-state $|\Psi_0\rangle$ at time $t = 0$. Here we show some first results, a more detailed analysis will be presented elsewhere [15]. Physically, the situation corresponds to the so called Schwinger particle creation mechanism [17], but now for a confining theory. This subject was recently also explored in the AdS/CFT set-up [15]. In our set-up this can be simulated by applying a uniform quench, replacing $L(n) \rightarrow L(n) + \alpha$ in the Hamiltonian [9], where $E = g \alpha$. Again we used TDVP, but now for real-time evolution with the quench Hamiltonian. As the background field breaks $CT$ invariance our ansatz is now defined simply by blocking two sites and two links into one site and taking a gauge-invariant form for $A^\mu$ that follows from [5]. In fig. 2 one can see our results for the evolution of the electric field, $\langle \Psi_0(t)|L(2n-1) + L(2n)|\Psi_0(t)\rangle$ and for the current density, $\sqrt{\gamma} \langle \Psi_0(t)|\sigma^+(2n-1)e^{i\theta(2n-1)}\sigma^-(2n) + h.c.|\Psi_0(t)\rangle$, and this for $\alpha = 0.75$. We observe the typical plasma oscillations that are damped over time, which is a typical feature of thermalization. This is corroborated by the linear growth of the half-system entanglement entropy that we find in this case. It is precisely this growth of the entropy that will at some point invalidate the MPS approximation for a given $D$. We can determine this point self-consistently by looking at variations of the result for different $D$. From fig.2 we can infer then that at $t \approx 15$, our $D = 239$ MPS-result starts to become less reliable.

Conclusions. In this letter we have demonstrated the potential of MPS as numerical method for gauge theories. It is clear that we have only scratched the surface of this approach and that even within the Schwinger model there are many other types of calculations one could do, like for instance the construction of two-particle scattering states. Looking further afield, one can easily generalize our gauge invariant MPS ansatz to other
gauge groups like SU($N$) and also to higher dimensions. Explicitly for $d=2+1$, the gauge-invariant 2d PEPS construction now involves five-leg tensors with four virtual indices and one physical index ($c =$ charge) on the sites, of the form $[B^c]_{(q_l, \alpha_l, q_r, \alpha_r, q_d, \alpha_d)} = [B^c_{q_l,q_r,q_d}]_{\alpha_l,\alpha_r,\alpha_d} \delta_{q_l+q_r+q_d, q_u}$, while on the links we get a three-leg tensor with two virtual indices and one physical index ($p =$ electric field unit) $[C^p]_{(q, \alpha_q, q_r, \alpha_r)} = [C^p_{q,\alpha_q}]_{\alpha_q,\alpha_r} \delta_{q, p} \delta_{q_r, p}$.

While preparing our manuscript the paper [19] appeared with an approach that is conceptually close to ours. There the authors use a quantum link model to write down gauge invariant MPS for the Schwinger model.

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A1: Gauge invariant states

Consider a lattice with $2N$ sites. The basis of the total Hilbert space $\mathcal{H}$ is \( \{|q\rangle \equiv \{|s_n,p_n\rangle \}_{1\leq n \leq 2N}: s_n = \pm 1, p_n \in \mathbb{Z} \} \).

A general state $|\Psi\rangle \in \mathcal{H}$ can be written as a MPS in the canonical form (see [22, theorem 1]):

\[
|\Psi\rangle = \sum_{\{s\}=\pm 1} \sum_{\{p\} \in \mathbb{Z}} \prod_{n=1}^{N} B_{2n-1} C_{2n-1}^{p_{2n-1}} B_{2n} C_{2n}^{p_{2n}} |q\rangle,
\]

where $B_{2n}^{s} \in \mathbb{C}^{D^s \times D^l}$, $C_{2n}^{p} \in \mathbb{C}^{D^k \times D^k+1}$ and $D^1 = D^{2N+1} = 1$. By ‘being in its canonical form’ we mean that

\[
\sum_{s=\pm 1} B_{n}^{s} B_{n}^{s\dagger} = \mathds{1}, \sum_{p \in \mathbb{Z}} C_{n}^{p} C_{n}^{p\dagger} = \mathds{1}
\]

and that there exist positive definite diagonal matrices $l_{n}^{C}$ and $l_{n}^{B}$ such that

\[
\sum_{s=\pm 1} B_{n}^{s\dagger} B_{n}^{s} = l_{n}^{C}, \sum_{p \in \mathbb{Z}} C_{n}^{p\dagger} C_{n}^{p} = l_{n}^{B}.
\]

Because QED is a gauge theory we have to restrict to $\mathcal{H}_{\text{phys}}$, the set of all gauge invariant states. This means that every $|\Psi\rangle \in \mathcal{H}_{\text{phys}}$ has to satisfy

\[
|\Psi\rangle = \exp(-i\varphi_{l} G_{l}) |\Psi\rangle, l = 1, \ldots, 2N.
\]

The right-hand side can also be written as a MPS with the same bond dimension:

\[
\exp(-i\varphi_{l} G_{l}) |\Psi\rangle = \sum_{\{s\}=\pm 1} \sum_{\{p\} \in \mathbb{Z}} \text{tr}\left( \prod_{n=1}^{N} B_{2n-1}^{s_n-1} C_{2n-1}^{p_{2n-1}} B_{2n} C_{2n}^{p_{2n}} \right) |q\rangle
\]

where if $l > 1$: $\tilde{B}_{k}^{s_k} = B_{k}^{s_k}$ for $k \neq l$, $\tilde{C}_{k}^{p_k} = C_{k}^{p_k}$ for $k \neq l - 1, l$ and $\hat{C}_{l-1}^{p} = e^{-i\varphi_{l} p} C_{l-1}^{p}$, $\hat{B}_{l}^{s} = e^{-i\varphi_{l} (s |l - 1)} B_{l}^{s}$, $\hat{C}_{l}^{p} = e^{i\varphi_{l} p} C_{l}^{p}$ and if $l = 1$: $\tilde{B}_{k}^{s_k} = B_{k}^{s_k}$, $\tilde{C}_{k}^{p_k} = C_{k}^{p_k}$ for $k \neq 1$ and $\hat{B}_{1}^{s} = e^{-i\varphi_{1} (s |1)} B_{1}^{s}$, $\hat{C}_{1}^{p} = e^{i\varphi_{1} p} C_{1}^{p}$. Because the two MPS with identical bond dimension have to represent the same state $|\Psi\rangle$ and (10) is assumed to be in the canonical form, it follows by [22, theorem 2], that there exists invertible square matrices $U_{n}$ and $V_{n}$ such that $\tilde{B}_{n}^{s_n} = U_{n}^{-1} B_{n}^{s_n} V_{n}$, $\tilde{C}_{n}^{p_n} = V_{n}^{-1} C_{n}^{p_n} U_{n+1}$ where $U_{1} = 1$ and $U_{2N+1} = 1$. Note that $U_{n}$ and $V_{n}$ depend on $\varphi_{l}$ and $l$.

The matrices $U_{n}$ and $V_{n}$ are unitary matrices. Indeed, it’s not hard to check that $\tilde{B}_{n}^{s_n}$ and $\tilde{C}_{n}^{p_n}$ also obey (11) and (12). For $n = 2N$ we have that $\tilde{C}_{2N}^{p} = V_{2N} C_{2N}^{p}$ which implies that $V_{2N} V_{2N}^{\dagger} = \mathds{1}$. Using this, $B_{2N}^{p} = U_{2N}^{-1} B_{2N} C_{2N}^{p}$ and the fact that $\tilde{B}_{2N}^{s_n}$ and $\tilde{C}_{2N}^{p_n}$ obey (11), it follows that $U_{2N} U_{2N}^{\dagger} = \mathds{1}$. Proceeding in the same way from $n = 2N$ till $n = 1$ one sees that all the matrices $U_{n}$ and $V_{n}$ are unitary.

Now we will prove that $U_{n} V_{n} = \mathds{1}$ for $n \neq l$. If $n < l$ we may assume that $l > 1$. Note that $U_{1} = 1$ and that $B_{1}^{s} = B_{1}^{s} V_{1}$. Using (12) it follows that $V_{1} = (l_{1}^{-1})^{-1} \sum_{n=1}^{N} B_{n}^{s\dagger} B_{n}^{s} = \mathds{1}$. Assume now $V_{n-1} = \mathds{1}$ ($n < l - 1$) then $C_{n-1}^{p} = \tilde{C}_{n-1}^{p} = C_{n-1}^{p} U_{n}$, which implies $U_{n} = (l_{n}^{-1})^{-1} \sum_{n=1}^{N} C_{n-1}^{p\dagger} C_{n-1}^{p} = \mathds{1}$, i.e., $V_{n-1} = \mathds{1}$ implies that $U_{n} = \mathds{1}$ for $n < l - 1$. Using the same ideas one proves that $U_{n} = \mathds{1}$ implies $V_{n} = \mathds{1}$ ($n < l$). This concludes the case $n < l$. For $n > l$ one starts from $\tilde{C}_{n}^{p} = C_{n}^{p} U_{2N} C_{2N}^{p}$. From (11), we obtain that $V_{2N} = \mathds{1}$. As a consequence $\tilde{B}_{2N}^{s} = B_{2N}^{s} = U_{2N} B_{2N}^{s}$ holds. By (11) it follows that $U_{2N} = \mathds{1}$. One can now repeat this reasoning and see that $U_{n} V_{n} = \mathds{1}$ for all $n > l$.

So the MPS (10) is gauge invariant iff for every $l = 1, \ldots, 2N$ there exist unitary matrices $U_{l}$ and $V_{l}$ (depending on $\varphi_{l}$) such that

\[
U_{l}^{\dagger} B_{l}^{s} V_{l} = e^{-i\varphi_{l} (s |l - 1)} B_{l}^{s}, C_{l-1}^{p} U_{l} = e^{-i\varphi_{l} p} C_{l-1}^{p} (l > 1), V_{l}^{\dagger} C_{l}^{p} = e^{i\varphi_{l} p} C_{l}^{p}.
\]

Consider now the case $\varphi_{l} = 1$, then the matrices $U_{l}$ and $V_{l}$ do not depend on $\varphi_{l}$ anymore. The unitary matrices can be diagonalized (as exponential of a Hermitian matrix): $U_{l} = W_{l}^{\dagger} \Delta_{U_{l}} W_{l}$, $V_{l} = X_{l}^{\dagger} \Delta_{V_{l}} X_{l}$, where $W_{l}$, $X_{l}$ are unitary matrices and $\Delta_{U_{l}}$ and $\Delta_{V_{l}}$ are diagonal matrices where all the diagonal-elements have modulus one. If we perform the following MPS-gauge transformation:

\[
B_{l}^{s} \rightarrow \tilde{B}_{l}^{s} = W_{l} B_{l}^{s} X_{l}^{\dagger}, C_{l}^{p} \rightarrow \tilde{C}_{l}^{p} = X_{l} C_{l}^{p} W_{l+1}^{\dagger}, W_{l} = W_{2N+1} = 1
\]
the MPS (10) is unaffected and the conditions (15) now read
\[ \Delta^1_{U_2} B^2_{\gamma} \Delta_{V_1} = e^{-i(\pi + (-1)^l)/2} B^2_{\gamma}, \quad C^p_{l-1} \Delta_{U_1} = e^{-i p C^p_{l-1}} (l > 1), \quad \Delta^1_{V_1} C^p_l = e^{i p C^p_l}. \] (17)

The property (11) will also hold for \( \tilde{B} \) and \( \tilde{C} \), however the property (12) is modified in the sense that \( t^B_l \) and \( t^C_l \) are not diagonal anymore (but they remain positive definite). We will denote this matrices with \( \tilde{t}^B_l \) and \( \tilde{t}^C_l \). As already mentioned, the entries of the diagonal matrices \( \Delta_{U_1} \) and \( \Delta_{V_1} \) are complex phase factors. Let \( e^{-i \lambda_{l,j}}, \ j = 1, \ldots, n_{u_l}, \) be the eigenvalues of \( \Delta_{U_1} \) with multiplicity \( m(\lambda_{l,j}) \) respectively of \( \Delta_{V_1} \) with multiplicity \( m(\mu_{l,j}) \),
\[ \Delta_{U_1} = \sum_{j=1}^{n_{u_l}} \sum_{\alpha_j=1}^{m(\lambda_{l,j})} e^{-i \lambda_{l,j}} |\lambda_{l,j}, \alpha_j \rangle \langle \lambda_{l,j}, \alpha_j |, \quad \Delta_{V_1} = \sum_{j=1}^{n_{u_l}} \sum_{\alpha_j=1}^{m(\mu_{l,j})} e^{-i \mu_{l,j}} |\mu_{l,j}, \alpha_j \rangle \langle \mu_{l,j}, \alpha_j |, \] (18)
then we can write \( \tilde{B} \) and \( \tilde{C} \) as
\[ \tilde{B}^p_l = \sum_{j=1}^{n_{u_l}} \sum_{k=1}^{n_{u_{l+1}}} \sum_{\alpha_j=1}^{m(\lambda_{l,j})} \sum_{\alpha_k=1}^{m(\lambda_{l+1,k})} [\tilde{B}^p_l](\lambda_{l,j}, \alpha_j);(\lambda_{l+1,k}, \alpha_k) |\lambda_{l,j}, \alpha_j \rangle \langle \lambda_{l,j}, \alpha_j | \lambda_{l+1,k}, \alpha_k \rangle \langle \lambda_{l+1,k}, \alpha_k |, \] (19a)
\[ \tilde{C}^p_l = \sum_{j=1}^{n_{u_l}} \sum_{k=1}^{n_{u_{l+1}}} \sum_{\alpha_j=1}^{m(\mu_{l,j})} \sum_{\alpha_k=1}^{m(\mu_{l+1,k})} [\tilde{C}^p_l](\mu_{l,j}, \alpha_j);(\mu_{l+1,k}, \alpha_k) |\mu_{l,j}, \alpha_j \rangle \langle \mu_{l,j}, \alpha_j | \mu_{l+1,k}, \alpha_k \rangle \langle \mu_{l+1,k}, \alpha_k |, \] (19b)
\[ \tilde{B}^1_l = \sum_{k=1}^{n_{u_{l+1}}} \sum_{\beta_k=1}^{m(\lambda_{l,k})} \tilde{B}^1_l 1_{\lambda_{l,k}, \beta_k} |\mu_{l,k}, \beta_k \rangle \langle \mu_{l,k}, \beta_k |, \quad \tilde{C}^p_{2N} = \sum_{j=1}^{n_{u_{2N}}} \sum_{\alpha_j=1}^{m(\mu_{2N,j})} \tilde{C}^p_{2N} 1_{\mu_{2N,j}, \alpha_j} |\mu_{2N,j}, \alpha_j \rangle \langle \mu_{2N,j}, \alpha_j |. \] (19c)
Using (17) it follows that
\[ (e^{-i(p-\lambda_{l+1,k})} - 1)[\tilde{C}^p_l](\mu_{l,j}, \alpha_j);(\lambda_{l+1,k}, \beta_k) = 0, \quad (e^{-i(p-\mu_{l,j})} - 1)[\tilde{C}^p_l](\mu_{l,j}, \alpha_j);(\lambda_{l+1,k}, \beta_k) = 0. \] (20)
\[ (e^{-i(p-\mu_{2N,j})} - 1)[\tilde{C}^p_{2N}](\mu_{2N,j}, \alpha_j);1 = 0, \] (21)
so
\[ [\tilde{C}^p_l](\mu_{l,j}, \alpha_j);(\lambda_{l+1,k}, \beta_k) = \delta_{\mu_{l,j}, \mu_{l+1,k}} \delta_{\alpha_j, \beta_k} \tilde{C}^p_l 1_{\lambda_{l+1,k}, \beta_k} 1_{\mu_{l,j}, \alpha_j};1 = \delta_{\mu_{l,j}, \mu_{2N,j}} \tilde{C}^p_{2N} 1_{\mu_{2N,j}, \alpha_j} 1_{1} = 0. \] (22)
Note that \( \lambda_{l,j} \) and \( \mu_{l,j} \) are only unique up to a multiple of \( 2\pi \). By writing \( \delta_{\mu_{l,j}} \) we mean that we must take for \( \mu_{l,j} \) up to a multiple of \( 2\pi \) the value \( p \). Of course this will not influence the eigenvalue \( e^{-i \lambda_{l,j}} \).

Assume now that there would exist a \( \lambda_{l+1,k_0} \ (l < 2N) \) with \( \lambda_{l+1,k_0} \neq p, \forall p \in \mathbb{Z} \). Then it follows by (22) that
\[ [\tilde{C}^p_l](\mu_{l,j}, \alpha_j);(\lambda_{l+1,k_0}, \beta_k) = 0, \] (23)
\[ \forall p \in \mathbb{Z}, \forall j = 1, \ldots, n_{u_l}, \forall \alpha_j = 1, \ldots, m(\mu_{l,j}). \] If we now consider the non-singular matrix \( t^C_l \), see (12), then
\[ \left( \sum_{p \in \mathbb{Z}} (\tilde{C}^p_l)^\dagger t^C_l \tilde{C}^p_l \right)_{(l \lambda_{l+1,k_0}, \alpha_{k_0});(l \lambda_{l+1,k}, \beta_k)} = 0, \] (24)
\[ \forall \alpha_{k_0} = 1, \ldots, m(\lambda_{l,k_0}), \forall k = 1, \ldots, n_{u_{l+1}}, \forall \beta_k = 1, \ldots, m(\lambda_{l+1,k}). \] By (12) this would mean that \( t^B_{l+1} \) has a zero-row and would be singular which is a contradiction because \( t^B_{l+1} \) is positive definite. As a consequence all the \( \lambda_{l,k} \) are integers. In the same way, but now by using the condition (11) one proves that all the \( \mu_{l,j} \) are integers.

We can write (18) as
\[ \Delta_{u_l} = \sum_{q \in \mathbb{Z}} \sum_{\alpha_q=1}^{D^l_l} e^{-i q |q, \alpha_q \rangle \langle q, \alpha_q |}, \quad \Delta_{v_l} = \sum_{q \in \mathbb{Z}} \sum_{\alpha_q=1}^{D^l_l} e^{-i q |q, \alpha_q \rangle \langle q, \alpha_q |}, \] (25)
and expand $\tilde{B}, \tilde{C}$:

$$
\tilde{B}_1^q = \sum_{q, r \in \mathbb{Z}} \sum_{\beta, r = 1}^{D_q^l} \sum_{\alpha, q = 1}^{D_q^l} [\tilde{B}_1^q(q, \alpha, q); (r, \beta, r)] q, \alpha_q \{r, \beta_r, \tilde{C}_1^p = \sum_{q, r \in \mathbb{Z}} \sum_{\beta, r = 1}^{D_q^l} \sum_{\alpha, q = 1}^{D_q^l} [\tilde{C}_1^p(q, \alpha, q); (r, \beta, r)] q, \alpha_q \{r, \beta_r, (26)
$$

$$
\bar{B}_1^q = \sum_{q, r \in \mathbb{Z}} \sum_{\beta, r = 1}^{D_q^l} [\bar{B}_1^q(q, \alpha, q); (r, \beta, r)] q, \alpha_q \{r, \beta_r, \bar{C}_2^p = \sum_{q, r \in \mathbb{Z}} \sum_{\beta, r = 1}^{D_q^l} \sum_{\alpha, q = 1}^{D_q^l} [\bar{C}_2^p(q, \alpha, q); (r, \beta, r)] q, \alpha_q \{r, \beta_r, (27)
$$

where $D_q^l$ respectively $D_q^l$ denotes the multiplicity of the eigenvalue $q$ in the matrix $u_l$ respectively $v_l$. Note that $D_q^l = \sum_q D_q^l$ and $D_q^l = \sum_q D_q^l$. We have already proven, see (22), that

$$
[C_1^p(q, \alpha, q); (r, \beta, r)] = \delta_{q, p} \delta_{q, \beta r} c_p^q \delta_{q, \beta r} \{c_p^q \alpha_q^r, 1 = \delta_{q, p} \alpha_q^r, (28)
$$

where $c_p^q \in \mathbb{C} \times D_q^l$. Finally, if we substitute (25) in (19a), we obtain

$$
(e^{-i[(s-1)/2 + q - r]} - 1)[\tilde{B}_1^q(q, \alpha, q); (r, \beta, r)] = 0, (l > 1), (e^{-i[(s-1)/2 - r]} - 1)[\tilde{B}_1^q(q, \alpha, q); (r, \beta, r)] = 0
$$

meaning that

$$
[\tilde{B}_1^q(q, \alpha, q); (r, \beta, r)] = \delta_{r, q} \delta_{s, (s-1)/2} b_q^s \delta_{q, \beta r} \{b_q^s \alpha_q^r, \bar{B}_1^q(q, \alpha, q); (r, \beta, r)] = \delta_{r, (s-1)/2} b_q^s \delta_{q, \beta r} \{b_q^s \alpha_q^r, (29)
$$

$$
\bar{B}_1^q = U_l^q \bar{V}_1^q, \bar{C}_1^p = U_l^q \bar{C}_1^p = V_l^q \bar{C}_1^p, e^{-i\varphi} \bar{B}_1^q = U_l^q \bar{V}_1^q, \bar{C}_1^p = U_l^q \bar{C}_1^p, (31)
$$

where $b_q^s \in \mathbb{C} \times D_q^l$. Note that

$$
[U_l^q(q, \alpha, q); (r, \beta, r)] = \delta_{q, \beta r} e^{-i\varphi} \alpha_q^r \delta_{q, \beta r} \{e^{-i\varphi} \alpha_q^r, (32)
$$

solves this problem. This proves that every gauge-invariant state can be brought in the form (28) and (30) by a MPS-gauge transformation and, conversely, that every MPS in the form (28) and (30) is gauge invariant.

**A2: A MPS ansatz for CT-invariant systems in the thermodynamic limit**

Consider a one-dimensional lattice of size $4N$ where every site $n, n \in \{−2N + 1, \ldots, 2N\}$, contains a $d$-dimensional Hilbert space $\mathcal{H}_n$ spanned by the basis $\{|q_n\rangle \rangle_n : q_n = 1, \ldots, d\}$. The total Hilbert space is spanned by $\{|q \rangle \equiv \{|q_{-2N+1} \ldots q_{2N}\} : q_n = 1, \ldots, d\}}$. We will take the thermodynamic limit ($N \rightarrow +\infty$). Let $H$ be a Hamiltonian which can be written as

$$
H = \sum_{n \in \mathbb{Z}} (CT)^n \sum_{k=1}^{m} \left( h_k^{(k)} \otimes \left( C h_2^{(k)} C \right) \right) (CT)^{-n}, \quad \text{(33)}
$$

where $h_k^{(k)}$ has only support on one site $(i = 1, 2; k = 1, \ldots, m)$, $C = \otimes_{n \in \mathbb{Z}} C$, $C$ is an idempotent Hermitian operator which induces a permutation $c$ on the basis vectors $\langle C |q_n\rangle \rangle_n = |c(q_n)\rangle \rangle_n$, $c^2 = 1$ and $T$ is the translation operator. One can think of $C$ being the charge conjugation. It is clear that the Hamiltonian is invariant under the transformation $\text{CT} \equiv CT : H = (CT) H(CT)^T$. Further one notes that the Hamiltonian is invariant under translations over an even number of sites. As a consequence it is possible to label the eigenstates of the Hamiltonian by the quantum numbers $k \in [-\pi, \pi]$ and $\gamma \in \{-1, +1\}$: $H |k, \gamma\rangle = E_{k, \gamma} |k, \gamma\rangle$ where $CT |k, \gamma\rangle = \gamma e^{-ik/2} |k, \gamma\rangle, (34)$ and $T^2 |k, \gamma\rangle = e^{-ik} |k, \gamma\rangle$. The number $k$ corresponds to the momentum of the excitation (for translations over two sites). The excitations with
quantum number \( \gamma = +1 \) will be referred to as \textit{scalar} particles and the excitations with quantum number \( \gamma = -1 \) will be referred to as \textit{vector} particles.

When the ground state of the Hamiltonian \( H \) does not suffer from spontaneous symmetry breaking of the \( CT \)-symmetry, we can write down an ansatz which resembles a uniform MPS but is \( CT \)-invariant instead of translation invariant. Thereto we define \( |q\rangle \equiv |(q_{-2N+1}),q_{-2N+2},\ldots,c(q_{2N-1}),q_{2N},\ldots,c(q_{2N-1}),q_{2N}\rangle (N \to +\infty) \) which can be obtained by letting \( C \) act on the odd components of \( |q\rangle \). The MPS \( |\Psi(A)\rangle \) is now defined as

\[
|\Psi(A)\rangle \equiv \sum_{\{q_n\}_{n \in \mathbb{Z}}} v^\dagger_R \left( \prod_{n \in \mathbb{Z}} A^{q_n} \right) v_L |q^c\rangle, \quad A \in \mathbb{C}^{D \otimes D}, \quad v_R, v_L \in \mathbb{C}^{D \times 1}.
\]  

One easily verifies that this state is \( CT \)-invariant and that it can be obtained from the uniform MPS

\[
|\Psi^u(A)\rangle \equiv \sum_{\{q_n\}_{n \in \mathbb{Z}}} v^\dagger_L \left( \prod_{n \in \mathbb{Z}} A^{q_n} \right) v_R |q\rangle,
\]

by letting \( C \) act on the odd sites.

Once we have obtained (to sufficient accuracy) the ground state \( |\Psi(A)\rangle \) corresponding to the ground state energy density \( E_0 \) one can look for the excited states. For one-particle excited states an ansatz with quantum numbers \( k \) and \( \gamma \) \((k \in [-\pi, \pi], \gamma \in \{-1, +1\})\) is

\[
|\Phi_{k,\gamma}(B, A)\rangle = \sum_{n \in \mathbb{Z}} \gamma_n e^{i(k/2)n} \sum_{\{q\}} v^\dagger_L \left( \prod_{m<n} A^{q_m} \right) B^{q_n} \left( \prod_{m>n} A^{q_m} \right) v_R |q\rangle .
\]

It is not hard to see that \( CT |\Phi_{k,\gamma}(B, A)\rangle = \gamma e^{-ik/2} |\Phi_{k,\gamma}(B, A)\rangle \) and \( T^2 |\Phi_{k,\gamma}(B, A)\rangle = e^{-ik} |\Phi_{k,\gamma}(B, A)\rangle \). These states can be obtained by letting \( C \) act on the odd sites of the states \( |\Phi_{(k+(1-\gamma)\pi)/2}(B, A)\rangle \) where

\[
|\Phi_{k}^u(B, A)\rangle = \sum_{n \in \mathbb{Z}} e^{i\theta_n} \sum_{\{q\}} v^\dagger_L \left( \prod_{m<n} A^{q_m} \right) B^{q_n} \left( \prod_{m>n} A^{q_m} \right) v_R |q\rangle , \forall \theta \in [-\pi, \pi].
\]

The states \( |\Phi_{k}^u(B, A)\rangle \) were introduced in [23] as ansatz for momentum-\( l \) particles for translational invariant systems.

To obtain the ground state of \( H \) with ansatz \( |\Psi(A)\rangle \), we will use the TDVP method [20]. It can be verified that this is equivalent with applying the TDVP method to the translational invariant Hamiltonian

\[
H^u = \sum_{n \in \mathbb{Z}} T^n \sum_{\{q\}} \left( h^{1(k)} \otimes h^{2(k)} \right) T^{-n}
\]

where we take as ansatz \( |\Psi^u(A)\rangle \). The procedure is explained in [21], [24]. The \( A \) that we will obtain as the tensor corresponding to the ground state \( |\Psi^u(A)\rangle \) of \( H^u \) will also correspond to the ground state \( |\Psi(A)\rangle \) of \( H \).

To find the excited states we will apply the Rayleigh-Ritz method and find \( B \) in such a way that it minimizes

\[
\langle \Phi_{k,\gamma}(B^*, A^*) | H | \Phi_{k,\gamma}(B, A) \rangle / \langle \Phi_{k,\gamma}(B^*, A^*) | \Phi_{k,\gamma}(B, A) \rangle .
\]

By noting that

\[
\langle \Phi_{k,\gamma}(B^*, A^*) | H | \Phi_{k,\gamma}(B, A) \rangle / \langle \Phi_{k,\gamma}(B^*, A^*) | \Phi_{k,\gamma}(B, A) \rangle = \langle \Phi_{k+(1-\gamma)\pi/2}(B^*, A^*) | H | \Phi_{k+(1-\gamma)\pi/2}(B, A) \rangle / \langle \Phi_{k+(1-\gamma)\pi/2}(B^*, A^*) | \Phi_{k+(1-\gamma)\pi/2}(B, A) \rangle
\]

this problem is mapped to an analogue problem for uniform MPS. It is extensively discussed in [20], [21] how to deal with this and how to obtain an approximation for the excited states and their energy by minimizing the right-hand side.

For the Schwinger model, the Hamiltonian can be written as

\[
H = \frac{g}{2\sqrt{2}} \sum_{n \in \mathbb{Z}} (CT)^n \left( L(1)^2 + \frac{\mu}{2} (\sigma_z(1)+1) + x(\sigma^- (1)e^{-i\theta(1)}[C\sigma^- (2)C] + h.c.) \right) (CT)^{-n},
\]

where \( C \) is the charge conjugation: \( C |s, q\rangle = |-s, -q\rangle \). This implies that we have to apply the TDVP method for uniform MPS to the translational invariant Hamiltonian

\[
H^u = \frac{g}{2\sqrt{2}} \sum_{n \in \mathbb{Z}} T^n \left( L(1)^2 + \frac{\mu}{2} (\sigma_z(1)+1) + x(\sigma^- (1)e^{-i\theta(1)}\sigma^- (2) + h.c.) \right) T^{-n}.
\]
We will now construct an ansatz of the form \( (34) \) which is gauge invariant. We start from a MPS invariant under translations over an even number of sites and perform a charge conjugation on the odd sites:

\[
\sum_{\{s_n\} = \pm 1} \sum_{\{p_n\} \in \mathbb{Z}} v_L^i \left\{ \prod_{n=-N}^{N} B_1^{-s_2n-1} C_1^{-p_2n-1} B_2^{2s_2n} C_2^{p_2n} \right\} v_R \left\{ \{s_2n-1, p_2n-1, s_2n, p_2n\} \right\},
\]

where \( N \to +\infty \). To make the state gauge invariant, we will require that they have the form \( (28) \) and \( (30) \):

\[
[B_l^{(-1)^l}]_{(q,\alpha_q);(r,\beta_r)} = \delta_{r,q+(s+(1)^l)/2} [\varepsilon_l]_{\alpha_q,\beta_r}, [C_l^{(-1)^l}]_{(q,\alpha_q);(r,\beta_r)} = \delta_{q,p} \delta_{q,r} c^l_{\alpha_q,\beta_r},
\]

where \( l = 1, 2 \). We will now perform the following MPS-gauge transformation: \( B_l^s \rightarrow B_l^s, C_l^p \rightarrow C_l^p, B_2^s \rightarrow B_2^s, C_2^p \rightarrow C_2^p U^l \) where \( [U]_{(p,\alpha):(q,\beta)} = \delta_{p,-q} \delta_{\alpha,\beta} \). It follows that

\[
[B_1^s C_1^p]_{(q,\alpha_q);(r,\beta_r)} = b_{1,-q}^{-s} b_{2,q}^{-s} c_1^{-q-(s+1)/2} \delta_{p,q+(s+1)/2} \delta_{r,-q-(s+1)/2},
\]

\[
[B_2^s C_2^p]_{(q,\alpha_q);(r,\beta_r)} = b_{2,q}^{-q+(s+1)/2} \delta_{p,q+(s+1)/2} \delta_{r,-q-(s+1)/2}.
\]

By taking \( b_{1,-q}^{-s} = b_{2,q}^{-q} = b_{q}^s, c_1^{-q} = c_2^{-q} = c^p \) and defining \( [A^{s,p}]_{(q,\alpha_q);(r,\beta_r)} = [a^{(p,s)}]_{\alpha_q,\beta_r} \delta_{p,q+(s+1)/2} \delta_{r,-q-(s+1)/2} \) where \( a^{(p,s)} = b_{p,-(s+1)/2}^s c^p \), the state \( (41) \) can be written as

\[
|\Psi(A)\rangle = \sum_{\{s_n\} = \pm 1} \sum_{\{p_n\} \in \mathbb{Z}} v_L^i \left\{ \prod_{n=-N}^{N} A_{s_n,p_n}^{s,n} \right\} v_R \left\{ \{-s_{2n-1}, -p_{2n-1}, s_{2n}, p_{2n}\} \right\},
\]

which is a gauge- and CT-invariant ansatz. Once we have obtained the ground state, we can use the ansätze \( (36) \) to approximate the excited states. If we put \( [B^{s,p}]_{(q,\alpha_q);(r,\beta_r)} = [b^{(p,s)}]_{\alpha_q,\beta_r} \delta_{p,q+(s+1)/2} \delta_{r,-q-(s+1)/2} \) they will also be automatically gauge invariant. The variational freedom then lies within the matrices \( b^{(p,s)} \). Note that in \( (36) \) the energy corresponding to the physical momentum \( k \) is obtained by substituting \( k \to 2ka \).