CABLE KNOTS ARE NOT THIN

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ABSTRACT. We prove that the \((p,q)\)-cable of a non-trivial knot is not Floer homologically thin. Using this and a theorem of Ian Zemke in [33], we find a larger class of satellite knots, containing non-cable knots as well, which are not Floer homologically thin.

1. Introduction

In his seminal work [31, 32], Thurston showed that a knot, based on the geometry on its complement, is either one of three types: torus, satellite, or hyperbolic. Apart from that classification, there is a family of knots that are easy to describe diagramatically, namely, alternating knots, which admit projections onto generic planes, that ‘alternate’ between under-passes and over-passes. It was proved by Menasco in [17] that an alternating knot is either a torus or a hyperbolic knot.

Theorem 1 ([17]). If \(L\) is a non-split prime alternating link, and if \(S \subset S^3 \setminus L\) is a closed incompressible surface, then \(S\) contains a circle which is isotopic in \(S^3 \setminus L\) to a meridian of \(L\).

The above theorem and the fact that the exterior of a satellite knot contains an incompressible torus implies that prime alternating knots are not satellite. Menasco’s proof of Theorem 1 makes direct use of alternating knot diagrams. More recently, Ozsváth and Szabó defined a larger class of knots, called quasi-alternating knots (see [23, Definition 3.1]).

Definition 2. Let \(Q\) denote the smallest set of links such that

- the unknot is a member of \(Q\).
- if \(L\) is a member of \(Q\), then there exists a projection of \(L\) and a crossing \(c\) in that projection such that
  1. both smoothings of \(L\) at \(c\) (see Figure 1), \(L_0\) and \(L_\infty\) are in \(Q\),
  2. \(\det(L) = \det(L_0) + \det(L_\infty)\).

The knot Floer homology of knots belonging to this set exhibit the same kind of characteristics as the knot Floer homology of alternating knots, hence the name quasi-alternating. Specifically, one of the characteristics is that quasi-alternating knots are Floer homologically thin, which is to say that the knot Floer homology of a quasi-alternating knot is supported in gradings where the difference between Alexander and Maslov gradings is fixed [16, Theorem 1.2] and they are completely determined by the signature of the knot and its Alexander polynomial. Furthermore, Ozsváth and Szabó proved that double branched
covers of quasi-alternating knots are $L$-spaces, i.e. $\widehat{HF}(\Sigma(K)) \cong \mathbb{Z}^{\det(K)}$ \cite[Proposition 3.3]{23}. More recently, Gordon and Lidman showed in \cite[Theorem 1.2; Theorem 1.3]{3} that for cable knots, double branched covers are not $L$-spaces, hence these knots are not quasi-alternating. This supports the following folklore conjecture:

**Conjecture 3.** Satellite knots are not quasi-alternating.

One could try to prove Conjecture 3 for quasi-alternating knots with fixed determinant. In this regard, combined results of Greene, Teragaito, Lidman and Sivek show that quasi-alternating knots with small determinant ($\leq 7$) are alternating (see \cite[30, 13]{4}). Therefore there are no satellite knots with small determinant which are also quasi-alternating by Theorem 1. However, their methods don’t generalize for knots with higher determinants. Ideally, one would like to prove an analog of Theorem 1 for quasi-alternating knots in the context of Heegaard Floer homology. But this remains a bit out of reach at the moment. Meanwhile, we can try to verify the above conjecture for certain classes of satellite knots.

In this regard, it is reasonable to try to show that a satellite knot fails at least one of the two aforementioned characteristics of quasi-alternating knots: that quasi-alternating knots are Floer homologically thin, and that their double branched covers are $L$-spaces.

Gordon and Lidman’s work shows that it’s not known how to characterize double branched cover of general satellite knots. Therefore in this article, we’ll focus on the property of Floer homology being thin. Showing that a non-trivial quasi-alternating knot with trivial Alexander polynomial cannot be satellite is quite straight-forward. Recall that for a knot $K$ with (symmetric) Alexander polynomial $\Delta_K(t) = \sum_{i=0}^{g} a_i (t^i + t^{-i})$, 

$$a_i = \chi(\widehat{HF}(S^3, K, i)) = \sum_{\alpha \in \widehat{HF}(S^3, K, i)} (-1)^{m_{\alpha}}$$

where $m_{\alpha}$ is the Maslov grading of $\alpha$. For Floer homologically thin knots, all elements in a fixed Alexander grading of its knot Floer homology have same Maslov grading (see \cite[Theorem 1.2]{16}), in other words, $\widehat{HF}(S^3, K, i) = \mathbb{Z}^{[a_i]}$. This and the fact that knot Floer homology detects the Seifert genus of a knot (i.e. $\widehat{HF}(S^3, K, g_3(K)) \neq 0$) imply that $K$ must be the unknot. Hence satellite knots with trivial Alexander polynomial, for example, Whitehead doubles, are not quasi-alternating. For a satellite knot with an arbitrary pattern
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and having non-trivial Alexander polynomial, this argument cannot be used. But for cable knots, we can prove the following.

**Theorem 4.** If the pattern is embedded inside the solid torus in such a way that it lies on a torus in standard way, in other words if it is $T_{p,q}$, then for any non-trivial companion $K$, the resulting cable knot $K_{p,q}$ is not Floer homologically thin.

This theorem verifies Conjecture 3 for cable knots, which can also be inferred from Gordon and Lidman’s work in [3].

**Corollary 5.** If $K$ is non-trivial, $K_{p,q}$ is not quasi-alternating.

In [33, Theorem 1.1], Zemke proves that if there is a ribbon concordance $C$ from $K_0$ to $K_1$, then there is an injection $F_C : \hat{HFK}(K_0) \to \hat{HFK}(K_1)$, which preserves both Alexander and Maslov gradings. Complied with Zemke’s result, Theorem 4 implies the following corollary.

**Corollary 6.** If there is a ribbon concordance between a cable knot $K_{p,q}$ and another knot $K'$, then $K'$ is not Floer homologically thin. In particular, $K'$ is not quasi-alternating.

Miyazaki in [18] proved that non-trivial band sums between knots are ribbon concordant to the trivial band sum between them i.e connected sum of those knots. One can start with a cable knot $K_{p,q}$ and place a number of unknots inside a regular neighborhood of $K$ such that they are unlinked to both the pattern and the companion, and then join them by some non-trivial bands them and the $T_{p,q}$ pattern sitting already inside that neighborhood, such that the bands stays inside the neighborhood. Now if one considers the resulting knot inside the solid torus as the pattern and take the companion as $K$, then the resulting satellite knot $P(K)$ is ribbon concordant to $K_{p,q}$ by Miyazaki’s result. Hence by Corollary 6, $P(K)$ is not thin and in particular, not quasi-alternating. This gives evidence to the affirmative answer to the Conjecture 3 for a large class of satellite knots containing the cable knots.

**Remark.** An alternative way to prove that a knot is not Floer homologically thin is by comparing $\tau$-invariant, a knot concordance invariant defined by Ozsváth and Szabo in [21] and Rasmussen in [27], and the signature of the knot, since for thin knots $\tau(K) = -$ negative of the half of its signature. In recent years, there has been much work studying effects of cabling on various knot invariants. To name a few, Shinohara in [28] studied signature and satellite operation, Hedden in [6], [7], Ina Petkova in [26], Jennifer Hom in [10] studied $\tau$ invariant and cabling operation, Wenzhao Chen in [2] studied $\Upsilon$ invariant and cabling, Apratim Chakraborty in [1] studied Legendrian knot invariant $\hat{\theta}$ and cabling. In particular for $\tau$ of a cable knot, Hedden provided an inequality in [7], later improved by Jennifer Hom in [10], in terms of the $\tau$ of the companion knots. Using that and Shinohara’s result [28, Theorem 9] about signature of cable knots, one can prove that a certain class of cable knots are not thin (eg. most of the iterated torus knots). This was pointed out to the author by Abhishek Mallick. We’ll take a different approach to prove Theorem 4 in full generality.
In order to prove Theorem 4, we will use bordered Floer homology package of Lipshitz, Ozsváth and Thurston in [14], which is tailor made to study satellite knots. We’ll discuss why and how a specific version of the ‘splicing theorem’ in [14] allows us to prove Theorem 4. In Section 2, we will briefly discuss the algebraic structure of bordered Floer homology. In Section 3, we will use the aforementioned splicing theorem to find two elements in the knot Floer homology of any cable knot such that the difference between their Alexander grading and that of their Maslov grading are not same, to deduce that they cannot be Floer homologically thin.

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2. Background on bordered Floer homology

We start by describing the features of bordered Floer setting we will be exploiting to proof the main theorem. We’ll only be interested in the case when the manifold has torus boundary. A bordered 3-manifold (with torus boundary) is a compact manifold with (torus) boundary, along with a diffeomorphism \( \phi : T^2 \to \partial Y \), up to isotopy fixing a neighborhood of a point, [8] and [15, Definition 1.4].

There are several versions of pairing theorem [14, Theorem 1.3] that come in handy to study Heegaard Floer holomogy a closed 3-manifold generated by splicing two manifolds with boundaries or a manifold cut along some closed surfaces (see [8],[5]). We’ll be interested in this specific splicing theorem:

**Theorem 7** (Theorem 11.19 in [14]). If \( K_1 \subset Y_1 \) and after gluing \( Y_1 \) and \( Y_2 \) along their boundaries \( \partial Y_1 = \partial Y_2 = F \), produces a null-homologous knot \( K \subset Y_1 \cup_F Y_2 \), then there is a homotopy equivalence of \( \mathbb{Z} \)-filtered chain complexes

\[
\hat{CFK}(Y,K) \simeq \hat{CFA}(Y_1,K_1) \boxtimes \hat{CFD}(Y_2)
\]

and the following equivalences of \( \mathbb{F}[U] \)-modules :

\[
gCFK^{-}(Y,K) \simeq CFA^{-}(Y_1,K_1) \boxtimes \hat{CFD}(Y_2)
\]

which respects the gradings, where \( gCFK^{-}(Y,K) \) denotes the associated graded object.

To make sense about the \( \hat{CFD} , \hat{CFA} , CFA^{-} \) mentioned above, we start by reviewing the bordered Floer Homology setting.

For a compact manifold \( Y \) with torus boundary, a bordered Heegaard diagram is a tuple \( \mathcal{H} : (\Sigma, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{g-1}, \beta_1, \ldots, \beta_g, z) \) such that

- \( \Sigma \) is a compact, oriented surface of genus \( g \) with one boundary component,
\begin{itemize}
  \item $\beta$ is a $g$-tuple of pairwise disjoint circles in the interior of $\Sigma_g$,
  \item $\alpha^c$ is a $(g - 1)$ tuple of pairwise disjoint circles in the interior of $\Sigma_g$,
  \item $\alpha^c$ is a 2-tuple of pairwise disjoint arcs in $\Sigma_g$ with boundary in $\partial \Sigma_g$,
  \item $z$ is a base point in $\partial \Sigma_g \setminus \alpha^a$,
  \item $\alpha^a \cap \alpha^c = \phi$,
  \item both $\Sigma_g \setminus (\alpha^a \cup \alpha^c)$ and $\Sigma_g \setminus \beta$ are connected.
\end{itemize}

To provide an example, given a knot $K \subset S^3$, we can find a bordered Heegaard diagram of $S^3 \setminus K$, by first starting with a specific Heegaard diagram of $S^3$, $(\Sigma, \alpha, \beta, w)$ and then adding an extra basepoint $z$ such that $(\Sigma, \alpha, \beta, z, w)$ is a doubly pointed Heegaard diagram of $K$ in $S^3$ i.e $w, z$ are two points on the Heegaard surface such that joining $w$ to $z$ in the complement of $\beta$ curves and joining $z$ to $w$ in the complement of $\alpha$ curves and subsequently pushing those arcs into the $\alpha$ and $\beta$ handle bodies, respectively, produces $K$.

Given a doubly pointed Heegaard diagram $(\Sigma, \alpha, \beta, z, w)$ of $K$ in $S^3$, first we stabilize the Heegaard surface by attaching a 2-handle with feet near the base points $w, z$. Now we can draw a longitude $\beta_g$ of the knot that goes over that newly attached handle (see [22, Fig.4]) and thus we get a meridian of the knot, living on the attached 2-handle, call it $\alpha_g$. Hence $(\Sigma', \alpha \cup \alpha_g, \beta \cup \beta_g, z, w)$ is a stabilized doubly pointed knot diagram. Take $\lambda$, a closed curve, parallel to $\beta_g$, intersecting $\alpha_g$ at one point, say $p$. Let $D$ be a small neighborhood disk around $p$. Then the complement of int$(D)$ specifies a bordered Heegaard diagram that represents a bordered Heegaard diagram of $S^3 \setminus K$, specifically $(\Sigma', \alpha \cup \alpha_g \cup \lambda', \beta \cup \beta_g, z')$, where $\alpha'_g = \alpha_g \setminus \{p\}$, $\lambda' = \lambda \setminus \{p\}$ and $z'$ lies on $\partial D$, away from the endpoints of the $\alpha$-arcs.

To get a bordered Heegaard diagram of an $n$-framed knot complement, instead of considering $\lambda$, we have to take $\lambda_n$, which we can get by winding $\lambda$ around $\alpha_g, n$ times. After that operation, one of the $\alpha$-arcs will be $\lambda'_n = \lambda_n \setminus \{p\}$, instead of $\lambda'$ mentioned earlier (see [12, Section 2.6]).

One can define a doubly pointed bordered Heegaard diagram of $K$ in a manifold $Y$, by finding a bordered Heegaard diagram of $Y$ and then adding an extra basepoint to represent $K \subset Y$, just as in a doubly pointed knot diagram.

Given a bordered Heegaard diagram of a manifold $\mathcal{H} : (\Sigma, \alpha, \beta, z)$, a generator $x = \{x_1, x_2, \cdots, x_g\} \in \alpha \cap \beta$ such that exactly one point can lie on each $\beta$ circle, exactly one point can lie on each $\alpha$-circle and at most one point can lie on each $\alpha$-arcs. Let $\mathcal{G}(\mathcal{H})$ be the set of all such generators.

The settings in place, we now discuss the algebraic preliminaries regarding bordered Floer homology that will eventually lead to explaining the components of the splicing Theorem 7.

Let $\mathcal{Z}$ denotes the boundary of $D$ in a bordered Heegaard diagram. Call $(\mathcal{Z}, a, M, z)$ a pointed matched circle, with $4k$ marked points $a = \{a_1, a'_1, a_2, a'_2, \cdots, a_{2k}, a'_{2k}\}$ and $M : a_i \to a'_i, i = 1, 2, \cdots, 2k$, a pairing of the points and $z$ being a base point on $\mathcal{Z}$. For $Y$, a compact 3-manifold with torus boundary, we only focus on the case where $k = 1$ and we call $a = \{a_0, a_1, a_2, a_3\}$ such that $M(a_0) = a_2$, $M(a_1) = a_3$ and the points $a_0, a_1, a_2, a_3$ are labeled on $\mathcal{Z}$ in a clock-wise direction.
Let $\alpha^a_1, \alpha^a_2$ denote the arcs from $a_0$ to $a_2$ and from $a_1$ to $a_3$, respectively (See Figure 2). A pointed matched circle represents a compact surface with one boundary component this way: consider $\mathbb{Z} \times [0, 1]$ and then add bands i.e one dimensional 2-handles with feet to pairs of matched points on the circle on $\mathbb{Z} \times \{0\}$ and then attach a disk to the new boundary component created after adding the bands. If we cap off the remaining boundary component with a disk, in general we call that surface $F(\mathbb{Z})$. In particular, when $k = 1$, $F(\mathbb{Z})$ is a torus.

For the case in hand i.e when $F(\mathbb{Z})$ represents a torus, $\mathcal{A}(\mathbb{Z})$ is a unital algebra over $\mathbb{F}_2$ with six ‘Reeb’ elements $\rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$ (in this case, one can think $\rho_1, \rho_2, \rho_3$ to be the arcs in $\mathbb{Z}$ between $a_0, a_1$, between $a_1, a_2$ and between $a_2, a_3$, respectively) and two idempotents $\iota_0, \iota_1$ such that $\iota_0 + \iota_1 = 1$ and these generators have these non-zero relations:

\[
\begin{align*}
\rho_1 &= \iota_0 \rho_1 = \rho_1 \iota_1 & \rho_2 &= \iota_1 \rho_2 = \rho_2 \iota_0 & \rho_3 &= \iota_0 \rho_3 = \rho_3 \iota_1 \\
\iota_0 \rho_{12} &= \rho_{12} \iota_0 &= \rho_{12} & \iota_1 \rho_{23} &= \rho_{23} \iota_1 &= \rho_{23} & \iota_0 \rho_{123} &= \rho_{123} \iota_1 &= \rho_{123} \\
\rho_1 \rho_2 &= \rho_{12} & \rho_2 \rho_3 &= \rho_{23} & \rho_{12} \rho_3 &= \rho_{12} \rho_{23} &= \rho_{123} \\
\end{align*}
\]

Through out the paper, if not mention otherwise, we’ll denote $\mathcal{A}(\mathbb{Z})$ by $\mathcal{A}$ only. For more in depth discussion on general $\mathcal{A}(\mathbb{Z})$, see Chapter 3 of [14] or Chapter 1.4 [15] for a short exposition.

In [14], Lipshitz, Ozsváth and Thurston associates to a bordered 3-manifold $(Y, \phi : \partial Y \to F(\mathbb{Z}))$, $\widehat{CFD}(Y)$ and $\widehat{CFA}(Y)$, which are right $\mathcal{A}_\infty \mathcal{A}(\mathbb{Z})$ and left $dg \mathcal{A}(\mathbb{Z})$ module, respectively. When $K \subset Y$, $CFA^{-}(Y, K)$ is also a $dg \mathcal{A}(\mathbb{Z})$ module. We’ll concern ourselves with $CFA^{-}(Y, K)$ and $\widehat{CFD}(Y)$ chain complexes for this paper. We discuss about these two kinds of modules next.

A vector space $M$ over $\mathbb{F}_2$ is said to have a (right) type A structure over $\mathcal{A}$ or is said to be a right $\mathcal{A}_\infty$-module over $\mathcal{A}$ (where $\mathcal{A}$ is a graded, unital algebra) if $M$ is equipped with

\[\text{Figure 2. pointed matched circle}\]
a right action of $\mathcal{I}$ (the set of all idempotents in $A$, in our case $\mathcal{I} = \{\iota_0, \iota_1\}$), such that $M = M\iota_0 \oplus M\iota_1$, as a vector space and, and multiplication maps
\[ m_{k+1} : M \otimes A^\otimes k \rightarrow M, \quad k \geq 0 \]
satisfying the $A_\infty$ relations
\[ 0 = \sum_{i=0}^{n} m_{k-i+1}(m_i(x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i) \otimes a_{i+1} \otimes \cdots \otimes a_k) + \]
\[ \sum_{i=1}^{n-1} m_{k-1}(x \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_k) \]
and the unital conditions
\[ m_2(x, 1) = x, \]
\[ m_k(x \otimes \cdots \cdot \cdot \cdot) = 0, \quad k > 2. \]

We say that $M$ is \textit{bounded} if $m_k = 0$ for sufficiently large $k$.

Now $CFA^-(Y, K)$ is a $\mathbb{F}[U]$-module generated by $\mathfrak{G}(\mathcal{H})$, where the right action by $\mathcal{I}$ is defined by
\[ x \cdot \iota_0 = \begin{cases} x & \text{if } x \text{ does occupy the arc } \alpha_1^a \\ 0 & \text{otherwise} \end{cases} \]
\[ x \cdot \iota_1 = \begin{cases} x & \text{if } x \text{ does occupy the arc } \alpha_2^a \\ 0 & \text{otherwise} \end{cases} \]
The right $A_\infty$-module structure on $CFA^-(Y, K)$ is determined by the multiplication maps
\[ m_{k+1} : CFA^-(Y, K) \otimes A^\otimes k \rightarrow CFA^-(Y, K) \]
defined by
\[ m_{k+1}(x, \rho_1, \cdots, \rho_i) = \sum_{y \in \mathfrak{G}(\mathcal{H})} \sum_{B \in \pi_2(x, y)} \#(M^B(x, y, (\rho_1, \cdots, \rho_i)) U^n w(B) y), \]
\[ m_2(x, 1) = x, \]
\[ m_{k+1}(x, \cdots, 1, \cdots) = 0 \quad \text{for } k > 0, \]
where $M(x, y, (\rho_1, \cdots, \rho_i))$ is described in [15, Chapter 2] as follows: let $S$ be a smooth surface with boundary and $2g$ punctures on its boundary and label $g$ of these punctures $-$, another $g$ punctures $+$, and the remaining punctures $e$. For $x, y \in \mathfrak{G}(\mathcal{H})$, $M(x, y, (\rho_1, \cdots, \rho_i))$ consists of maps
\[ u : (S, \partial S) \rightarrow (\Sigma \setminus \{z\}) \times [0, 1] \times \mathbb{R}, (\alpha \times \{1\} \times \mathbb{R}) \cup (\beta \times \{0\} \times \mathbb{R})) \]
such that
\begin{itemize}
  \item at the punctures $-$, $u$ is asymptotic to $x \times [0, 1] \times \{-\infty\}$,
  \item at the puncture $+$, $u$ is asymptotic to $y \times [0, 1] \times \{\infty\}$,
\end{itemize}
• at the punctures labeled $e$, $u$ is asymptotic to the chords $\rho_i \times (1, t_i) \in \partial \Sigma \times \{1\} \times \mathbb{R}$, and $t_1 < t_2 < \cdots < t_n$,
• $u$ is proper and extends to a proper map from $\bar{S} \to \bar{\Sigma}$, where both $\bar{S}$ and $\bar{\Sigma}$ are ‘filled in’ by disks,
• the composition of the extended map and projection to $[0, 1] \times \mathbb{R}$ is a $g$-fold branched cover and $u|_{\partial S}$ is injective.

If $\hat{\mathcal{M}}(x, y, (\rho_1, \rho_2, \cdots, \rho_n))$ denotes the moduli space of $J$-holomorphic maps satisfying the above properties. Then $\mathcal{M}(x, y, (\rho_1, \rho_2, \cdots, \rho_n)) = \hat{\mathcal{M}}(x, y, (\rho_1, \rho_2, \cdots, \rho_n))/\mathbb{R}$ (where $\mathbb{R}$ denotes the translation action in the image). Given $B \in \pi_2(x, y), \mathcal{M}^B(x, y, (\rho_1, \cdots, \rho_n))$ is the set of all such $J$-holomorphic maps which has the same homology as that of $B$. See [14, Chapter 5] for details. The above defined family of multiplication maps $\{m_i\}$ counts the number of holomorphic representative of such maps.

Now we describe the other module structure mentioned above. A vector space $N$ over $\mathbb{F}_2$ is said to have a (left) type D structure over $A$, equipped with a left action of $\mathcal{I}$ such that $N = \iota_0 N \oplus \iota_1 N$, as a vector space and a map
\[
\delta_1 : N \to A \otimes N
\]
satisfying the type D condition:
\[
(\mu \otimes id_N) \circ (id_A \otimes \delta_1) \circ \delta_1 = 0
\]
where $\mu : A \otimes A \to A$ is the multiplication map in $A$. Also, we inductively define maps
\[
\delta_k : N \to A^\otimes k \otimes N
\]
such that $\delta_0 = id_N, \delta_i = (id_A^\otimes (i - 1) \otimes \delta_1) \otimes \delta_{i - 1}$. We say $N$ is bounded if $\delta_k$ is zero for sufficiently large $k$.

This general definition in place, now $\widehat{CFD}(Y)$ is a $\mathbb{F}$-vector space generated by $\mathcal{G}(\mathcal{H})$, where the left action by $\mathcal{I}$ is defined by
\[
\iota_0 \cdot x = \begin{cases} x & \text{if } x \text{ does not occupy the arc } \alpha^a_1 \\ 0 & \text{otherwise} \end{cases}
\]
\[
\iota_1 \cdot x = \begin{cases} x & \text{if } x \text{ does not occupy the arc } \alpha^a_2 \\ 0 & \text{otherwise} \end{cases}
\]
and the left $A$-module structure on $\widehat{CFD}(Y)$ is defined by:
\[
\delta_1 : \widehat{CFD}(Y) \to A \otimes \widehat{CFD}(Y)
\]
defined by:
\[
\delta_1(x) = \sum_{y \in \mathcal{G}(\mathcal{H})} \sum_{B \in \pi_2(x, y)} \sum_{\text{ind}(B, (\rho_{i_1}, \cdots, \rho_{i_k})) = 1} #(\mathcal{M}^B(x, y, (\rho_{i_1}, \cdots, \rho_{i_k}))) \rho_{i_1} \cdots \rho_{i_k} y
\]
Recall that for each $x \in \mathcal{G}(\mathcal{H})$, $\pi_2(x, x)$ i.e collection of all $J$-holomorphic Whitney disks connecting $x$ to itself, forms a group where the multiplication is given by concatenation of disks. We can think of $B$ as living on the Heegaard surface and a linear combination of the regions in $\Sigma \setminus (\alpha \cup \beta)$. Elements of $\pi_2(x, x)$ are called periodic domains, which is naturally isomorphic to $H_2(Y, \partial Y)$. A non-trivial class $B \in \pi_2(x, y)$ is called positive if all its local multiplicities are non-negative. The Heegaard diagram $\mathcal{H}$ is called provincially admissible if it has no positive periodic domains with multiplicity 0 everywhere along $\partial B$. The Heegaard diagram $\mathcal{H}$ is called admissible if it has no positive periodic domains. Provincial admissibility of $\mathcal{H}$ ensures that the above mentioned maps $m_i$ and $\delta_1$ are well-defined. Admissibility of $\mathcal{H}$ ensures that $\hat{CFA}^{-}(Y, K)$ and $\hat{CFD}(Y, K)$ are bounded. Compare [25, Chapter 4,5] and [14, Chapter 4].

Now the definitions of $\hat{CFD}$ and $CFA^{-}$ in place, we describe the operation between these two modules, mentioned in Equation (1) and (2). If one of $CFA^{-}(Y, K)$, $\hat{CFD}$ is bounded, then the box tensor product $CFA(Y, K) \boxtimes \hat{CFD}$ is the $F[U]$-module $CFA^{-} \otimes \hat{CFD}$ equipped with the differential:

$$\partial^\mathcal{H}(x \otimes y) = \sum_{k=0}^{\infty} (m_{k+1} \otimes id|_{\hat{CFD}})(x \otimes \delta_k(y))$$

The finite-ness of the sum is ensured by bounded-ness of any one of $CFA^{-}(Y, K)$, $\hat{CFD}(Y, K)$.

For the case that we’re interested in, writing $\delta_1$ map in terms of the elements of $A(T^2)$ helps. Let $\rho_0 = \iota_0 + \iota_1 = 1$ and then rewrite $\delta_1$ as

$$\delta_1 = \sum_i \rho_i \otimes D_i$$

where $i$ runs over $\{0, 1, 2, 3, 12, 23, 123\}$ and $D_i : \hat{CFD} \to \hat{CFD}$ are called coefficient maps.

In this notation, the differential in $CFA^{-} \boxtimes \hat{CFD}$ can be written like this:

$$\partial^\mathcal{H}(x \otimes y) = \sum_{k} m_{k+1}(x, \rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}) \otimes D_{i_k} \circ \cdots \circ D_{i_2} \circ D_{i_1}(y)$$

where $k$ runs over all such sequence $i_1, i_2, \ldots, i_k$ of $k$ elements from $\{\varphi, 1, 2, 3, 12, 23, 123\}$ (including the empty sequence when $k = 0$).

3. Proof of Theorem 4

Through out the proof, whenever we talk about $T_{p,q} \in D^2 \times S^1$, we assume that $gcd(p,q) = 1, p > q, |q| \neq 1$. If we finish the proof of our main theorem for this case and for any framing of $K$, then we can see that if $q = mp + i, m > 0, p > i > 1$, then by choosing $m$ framing of $K$ as the companion and $T_{p,i}$ as the pattern, we’ll be done. For $q = \pm 1$ case, one can use the $A_\infty$ relations described in [10] and [26] and follow the same strategy that we follow to find the desired elements in the knot Floer homology of the cable.
We start our proof by discussing some materials that we need to borrow from the knot Floer homology setup. For an integer homology sphere $Y$, recall that for a knot $K \subset Y$, $\text{CFK}_\infty(Y)$ is a filtered chain complex, gotten from the Heegaard Floer chain complex of the ambient manifold $Y$ i.e $\text{CF}_\infty(Y)$ by introducing additional filtration, induced by Alexander grading, which is dictated by the knot and where $\text{CF}_\infty(Y)$ is a $\mathbb{F}[U]$-module over $\mathcal{G}(\mathcal{H})$ for a Heegaard diagram $\mathcal{H}$ of $Y$.

For a fixed Alexander grading $j$, $\hat{\text{HFK}}(Y,K,j)$ is the homology of the chain complex $\hat{\text{CFK}}(Y,K,j)$, where $\hat{\text{CFK}}(Y,K,j)$ is generated by $[x,0,j]$ where $x \in T_\alpha \cap T_\beta$, $((\Sigma, \alpha, \beta, w, z)$ is a doubly pointed knot diagram for $K \subset Y$) such that $A(x) = \frac{\langle c_1(s_{n_d}(x)), [\hat{F}] \rangle}{2}$ where $\hat{F}$ is a capped-off Seifert surface of $K$ in $Y$ (since $Y$ is a homology sphere, the definition is independent of choice of Seifert surface of $K$ in $Y$).

We will consider $\text{CFK}_-^-(K)$, a chain complex generated by $\mathcal{G}(\mathcal{H})$ and the differential in this bi-filtered chain complex is given by

$$\partial^{-}(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\{\phi \in \pi_2(x,y)|\mu(\phi)=1, n_z(\phi)=0\}} \#(\hat{\mathcal{M}}(\phi)) \cdot U^{n_w(\phi)} \cdot y$$

where $\hat{\mathcal{M}}(\phi)$ denotes the quotient of the moduli space of $J$-holomorphic disks representing the homotopy type of $\phi$, $\mathcal{M}(\phi)$, divided out by the natural action of $\mathbb{R}$ on this moduli space and $\mu(\phi)$ denotes the ‘expected dimension’ of $\mathcal{M}(\phi)$, see [22] for detailed discussions. Setting $U = 0$ in the above differential, defines the differential $\partial$ for $\hat{\text{CFK}}$, and the homology of that is denoted by $\hat{\text{HFK}}(S^3,K)$. By $g\text{CFK}^-$ we denote the associated graded object, filtered by the Alexander grading i.e $j$.

Given a null-homologous knot $K$ in $S^3$, it’s convenient to look at the knot Floer complex living inside $i,j$-plane, where $j$ is the additional grading i.e the Alexander grading induced by the knot and $i$ denotes the power of $U$. The general rule to draw the knot complex is this: on $i = 0$ axis, for a fixed $j$, we put $\text{dim}(\hat{\text{HFK}}(S^3,K,j))$-no.of points at $(0,j)$ co-ordinate and then extend that to the whole $(i,j)$ plane by translating through $U$ and $U^{-1}$ (eg. multiplying $U$ pushes some $[y,0,m]$ to $[y,-1,m-1]$ and $U^{-1}$ pushes that to $[y,1,m+1]$). That is we assume the complex is reduced which is same as saying $C(i,j) = \hat{\text{HFK}}(S^3,K,j-i)$. This is due to the fact that a filtered chain complex is always filtered chain homotopic to a reduced complex, see [9, Reduction Lemma]. We also draw the boundary maps $\partial^\infty$ by arrows emanating from some generator(s), pointing towards the generator(s) that live(s) in their boundary. A reduced chain complex will ensure that the arrows will be pointing downwards (i.e when the boundary map will strictly reduce the Alexander grading), pointing to the left (i.e when the boundary map will strictly reduce the $U$-power) or both (i.e when the boundary map points to south-west direction to itself).
One might see $\text{CFK}^{-}(S^3, K)$ as $C(i \leq 0)$. The Reduction Lemma implies

$$\dim_{\mathbb{F}[U]}(\text{CFK}^{-}(S^3, K)) = \dim_{\mathbb{F}}(\text{HFK}(S^3, K)) = 2n + 1, \text{for some } n \geq 0$$

For a $\mathbb{Z} \oplus \mathbb{Z}$-filtered complex $C$ and a given basis, we call an operation on the basis a filtered change of the basis if that operation replaces some basis element $x_j$ by $\sum_{i=1}^{m} a_i y_i$ such that both the filtrations of each $a_i y_i$ is less than equal to those of $x_j, i = 1, 2, \cdots, m$.

We call a $\mathbb{F}[U]$ basis $\{\xi_0, \xi_1, \xi_2, \cdots, \xi_{2n+1}\}$ of $\text{CFK}^{-}$ vertically simplified if

- $\partial_{\text{vert}}^{\eta}(\xi_{2i-1}) = \xi_{2i}$ (mod $U^-\text{C}$) for $i = 1, \cdots, n$.
- $A(\xi_{2i-1}) - A(\xi_{2i}) = k_i > 0$.
- $\xi_0$ is the generator of the vertical homology.

where $\xi_i \in C(i = 0), i = 0, 1, \cdots, 2n + 1$ and $\partial_{\text{vert}} = \partial_0 |_{C(i = 0)}$.

Similarly, we can also define a horizontally simplified basis of $\text{CFK}^{-} \{\eta_0, \eta_1, \eta_2, \cdots, \eta_{2n+1}\}$ where

- $\partial_{\text{hor}}^{\eta}(\eta_{2p-1}) = U^{l_p} \cdot \eta_{2p}$ (mod the associated graded object of $\text{CFK}^{-}$, where $j = A(\eta_{2p-1}) - 1$), for $p = 1, 2, \cdots, n$.
- $A(\eta_{2p}) - A(\eta_{2p-1}) = l_p > 0$.
- $\eta_0$ is the generator of the horizontal homology.

where $\eta_p \in C(j = 0), p = 0, 1, \cdots, 2n + 1$ and $\partial_{\text{hor}} = \partial_0 |_{C(j = 0)}$.

Lipshitz, Ozsváth, Thurston in [14, Theorem 11.57], Hom in [10, Lemma 2.1] proved that $\text{CFK}^{-}$ always admits vertically and horizontally simplified bases. Also the facts that $C(i = 0)$ and $C(j = 0)$ are isomorphic and $\{\ell_1, \ell_2, \cdots, \ell_n\} = \{k_1, k_2, \cdots, k_n\}$ are same sets, follow from the symmetry of knot Floer homology under reversing the roles of the marked points $w, z$, thus the orientation of the knot, see [22, Proposition 3.8].

Now we recall Lipshitz, Ozsváth, Thurston’s algorithm from [14, Theorem 11.26, A.11] to find the complete set of generators for $\widehat{\text{CFD}}(X_K, \phi_n)$ given $\text{CFK}^{-}(K)$, where $X_K = S^3 \setminus K$ and $\phi_n$ denotes a parametrization of the boundary of the knot exterior, where the knot is taken to be $m$-framed.

**Theorem 8** (Theorem 11.26, A.11 in [14]). With notation as above, if $X_K$ denotes the complement of the knot $K$ with an integer framing $r$, then $\widehat{\text{CFD}}(X_K, r)$ has the following description:

- $\iota_0(\widehat{\text{CFD}}(X_K))$ is of dimension $2n + 1$ and is generated by $\{\xi_0, \xi_1, \cdots, \xi_{2n+1}\}$ or $\{\eta_0, \eta_1, \cdots, \eta_{2n+1}\}$.
- $\iota_1(\widehat{\text{CFD}}(X_K))$ is generated by:
  $$\bigcup_{i \in \{1, 2, \cdots, n\}} \{\kappa^i_1, \kappa^i_2, \cdots, \kappa^i_{\ell_i}\} \bigcup_{j \in \{1, 2, \cdots, n\}} \{\lambda^j_1, \lambda^j_2, \cdots, \lambda^j_{k_j}\} \cup \{\mu_1, \cdots, \mu_1\},$$

- For each vertical arrow of length $\ell_i$, we have $\kappa^i_1, \cdots, \kappa^i_{\ell_i}$ (subspace generated by these is called vertical chain) with following coefficients:

$$\xi_{2i-1} \xrightarrow{D_{1i}} \kappa^i_1 \xleftarrow{D_{2i}} \cdots \xleftarrow{D_{2i}} \kappa^i_{\ell_i} \xrightarrow{D_{12i}} \xi_{2i}$$
For each horizontal arrow of length $k_j$, we have $\lambda^1_j, \ldots, \lambda^k_j$ (subspace generated by these is called horizontal chain) with following coefficients:

$$
\eta_{2j-1} \xrightarrow{D_3} \lambda^1_j \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \lambda^k_j \xrightarrow{D_2} \eta_{2j}
$$

If $t = 2\tau(K) - r$, then we have another additional set of generators $\{\mu_1, \ldots, \mu_t\}$ (subspace generated by these is called unstable chain) with following coefficients:

$$
\begin{align*}
\xi_0 \xrightarrow{D_1} \mu_1 & \xleftarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \mu_t \xleftarrow{D_3} \mu_0 & \text{if } t > 0 \\
\xi_0 \xrightarrow{D_{12}} \eta_0 & \xleftarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \mu_t \xleftarrow{D_2} \eta_0 & \text{if } t = 0 \\
\xi_0 \xrightarrow{D_{123}} \mu_1 & \xleftarrow{D_{23}} \mu_2 \xrightarrow{D_{23}} \ldots \xrightarrow{D_{23}} \mu_t \xleftarrow{D_2} \eta_0 & \text{if } t < 0
\end{align*}
$$

where $\tau(K) = \min\{p | i_p : C(i = 0, j \leq p) \to C(i = 0) \text{ induces surjection in homology}\}$, which is a concordance-invariant defined by Ozsváth and Szabó in [21] and Rasmussen in [27], independently.

The gradings are determined as follows:

- The grading set is $G/\lambda^{-1} \text{gr}(\rho_{12})^{-1} \text{gr}(\rho_{23})^{-r}$
- Grading of any element $x_0$ in $i_0(\hat{CFD}(X_K, r))$, represented by a generator of the knot Floer homology, is determined by Alexander grading $A$ and Maslov grading $M$ of $x_0$ in the knot Floer complex : $\text{gr}(x_0) = \lambda^{M-A} \text{gr}(\rho_{23})^{-A}$

We will discuss about the grading notations later.

Now the settings and the main ingredient in place, we discuss the main strategy to prove our main result. We’ll produce two non-zero elements in the knot Floer homology of a given cable knot, using the splicing theorem 7, such that the difference between their Alexander grading is not equal the difference between their Maslov grading. Thus by [16, Theorem 1.2], they are not Floer homologically thin. To do that we’ll look at $CFA^-\left(D^2 \times S^1, T_{p,q}\right)$ and $\widehat{CFK}(X_K)$ simultaneously to find two elements from both modules such that their box tensor is well defined. The splicing theorem allows us to find those two elements, living inside the knot Floer chain complex of the cable knot. Then we have to make sure that those elements are non-zero in the knot Floer homology of the cable. Then we calculate their grading to get the desired result.

Observe that to make sure that an element, say $\gamma$ in $gCFK^-$ is a non-zero element in knot Floer homology (i.e the homology of $(\widehat{CFK}, \partial)$), it is enough to check that all the elements in $\partial^- (\gamma)$ contains a non-zero $U$ power and there is no element in $gCFK^-$ whose boundary contains $U^0 \cdot \gamma$.

Petkova in [26] and Hom in [10] used a bordered Heegaard diagram of $(p, 1)$-pattern knot in the solid torus and looked at the lifts of the $\alpha$ arcs and the $\beta$ curve in the universal cover of the genus one Heegaard surface i.e Euclidean plane, to count the Whitney disks between the generators.

To find $CFA^-\left(D^2 \times S^1, T_{p,q}\right)$, we follow the same strategy. We will use Ording’s algorithm from [20, Theorem 3.5] to find a bordered Heegaard diagram of $T_{p,q}$ in the solid torus, and
then look at the lifts of $\alpha$ arcs and the $\beta$ curve in Euclidean plane to find our intended two elements.

Let $K$ be our companion knot in $S^3$. We are going to find the desired generators, separating our search in two cases: when $t = 2\tau(K) - r$ is zero and when $t$ is non-zero.

Case 1. Let $t = 2\tau(K) - r \neq 0$. For subcases of this case, we recall the definition of $\epsilon$, a knot concordance invariant, defined by Hom in [11]. To define that, we recall another knot concordance invariant $\nu$, defined by Ozsáth, Szabó, which is $\nu(K) = \min\{s|p_s : C\{\max(i = 0, j = s)\} \to C(i = 0) \text{ induces a surjection in homology}\}$, where $p_s$ is the projection map onto the $i$ co-ordinate. Recall that $\nu(K) = \tau(K)$ or $\tau(K) + 1$, see [21]. Then

$$\epsilon(K) = \begin{cases} -1 & \text{if } \nu(K) = \tau(K) + 1 \\ 0 & \text{if } \nu(K) = \tau(K) \text{ and } \nu(-K2) = \tau(-K) \\ 1 & \text{if } \nu(-K) = \tau(-K) + 1 \end{cases}$$

If $\epsilon(K) = 1$, we use [10, Lemma 3.2] to find $\{\xi_i\}$, a vertically simplified $\mathbb{F}[U]$ basis of $CFK^-$, with the following properties, after possible renaming,

- $U^k \cdot \xi_2$ is the generator of the homology of $C(j = 0)$, for some $k$.
- there exists $\xi_1$ such that $\partial^{vert} \xi_1 = \xi_2$.
- $\xi_0$ is the generator of the homology of $C(i = 0)$.

Here’s a sketch of the proof of [10, Lemma 3.2]: from the definition of $\epsilon$, $\epsilon(K) = 1 \Rightarrow \nu(K) = \tau(K)$ but $\nu(-K) = \tau(-K) + 1$ i.e $\xi_0$, the generator of the vertical homology, ‘gets killed’ in the horizontal complex. What we mean by that is: $U^{\tau(K)} \cdot \xi_0$ lives in the image of $\partial^{hor}$. Also recall the homology of $(C(i = 0), \partial^{vert})$ is isomorphic to the homology of $(C(j = 0), \partial^{hor})$. That fact and the symmetry of $\widehat{HFK}$ with respect to Alexander grading implies that some $\xi_2$, which lives in $\widehat{HFK}(S^3, K, -\tau(K))$, will be the generator of the horizontal complex i.e $U^{-\tau} \cdot \xi_2$ is the generator of the horizontal homology and also, there is some $\xi_1$ such that $\partial^{vert} \xi_1 = \xi_2$. For a detailed Proof, see [10, Section 3].

Now we turn our attention to $CFA^-(D^2 \times S^1, T_{pq})$. Petkova and Hom used the following doubly pointed bordered Heegaard diagram for $(p, 1)$ patterns in $D^2 \times S^1$, see Fig 3.

From the picture, one considers the indicated generator $a$. Now the observation that both $\xi_2$ and $a$ live in the $i_0$ part of the $\mathbb{F}$-vector spaces $\widehat{CFD}(X_K, r)$ and $CFA^-(D^2 \times S^1, T_{pq})$, respectively, allows to look at $a \otimes \xi_2$, which Hom does in [10] to find the $\tau$ of the cables $K_{p, q + 1}$. Hom shows that $a \otimes \xi_2$ survives in homology and thus an element of $\widehat{HFK}(K_{p, q + 1})$. To do this, both Petkova and Hom find all the generators and the $A_\infty$-relations in $CFA^-(D^2 \times S^1, T_{pq})$, from the diagram. See [26, Section 4], [10, Section 4.1].

We use Ordning’s ‘cat’s cradle’ algorithm from [20] to find a doubly pointed bordered diagram for $(D^2 \times S^1, T_{pq})$ and from that we’ll look for our desired generators.

In [20, Theorem 3.5], given a $(1, 1)$ knot, Ordning describes an algorithm to find a Heegaard normal form of a genus one knot diagram for the knot. First one starts with finding a standard form of the knot in torus and then draw the $\beta$ curve on the torus, isotopic to the standard longitude of the torus, such that it misses $p(t_\beta)$, where $t_\beta$ is the part of the...
Figure 3. doubly pointed bordered Heegaard diagram of \((p,1)\) torus knot in \(D^2 \times S^1\)

knot in the \(\beta\)-handle body (i.e. the part one can get by joining \(w\) to \(z\) without crossing \(\beta\)) and \(p\) is the projection map onto the torus. See [20, Figure 3.11] for a step-by-step pictures obtained after applying the algorithm for \(T_{5,3}\) and also for description on standard and normal forms of \((1,1)\) knots.

One way to view Ording’s algorithm is that one starts with a copy of the standard longitude and meridian of a torus and starts deforming the longitude in each step, keeping it isotopic to the longitude but missing the standard form of the \((1,1)\) knot. Now, one can observe that at each step when the new longitude hits the meridian, two new generators in the knot Floer homology of the knot, are born. The generator \(a\) is the one in knot Floer homology of \(T_{p,q}\) in \(S^3\), which is the generator of the vertical homology i.e the unique intersection point one starts with at the beginning of the algorithm.

After getting the \(\beta\) circle for the knot diagram of \(T_{p,q}\), we can cut out a neighborhood of the vertices of the fundamental domain of the torus and take the horizontal and vertical boundary components as the \(\alpha\)-arcs (where, one was a longitude and the other was a meridian of the torus, before cutting out). Now changing the name of \(w\) in Ording’s picture to \(z\) and placing another basepoint \(z\) at the bottom of the picture gives us a genus one doubly pointed bordered diagram for \(T_{p,q}\) in the solid torus. One might also place the \(w'\) around the middle of the initial longitude and start the algorithm of finding \(\beta\) with the initial points of the standard form of the knot at the middle of the meridian. Observe that to find the standard \((5,3)\) torus knot, by joining \(w\) to \(z\) in the complement of \(\beta\) curve and
then from \( z \) to \( w \) in the complement of the meridian, one has to ‘flip’ Ording’s picture, which would then look like Figure 6, on the torus and like Figure 7, on the lifted setting.

Below are two examples of doubly pointed bordered Heegaard diagrams of \((D^2 \times S^1, T_{p,q})\) on the fundamental domain and then in the lift, where \( p = 3, q = 2 \) in the first case (Figure 4,5) and \( p = 5, q = 3 \) (Figure 6,7) in the second case.

We describe one more intersection point of the doubly pointed bordered diagram of \( T_{p,q} \) in the solid torus. Observe that \( a \) lives in the \( \alpha_0 \) arc. For \( q > 0 \) now we describe the other intersection point we’ll consider from the bordered diagram. It’ll suffice since for \( q < 0 \), the doubly pointed bordered Heegaard diagram would be the reflection with respect to the meridian in the fundamental domain picture for \( q > 0 \) case. If we enumerate the intersection points of \( \beta \) curve with \( \alpha_1 \) arc starting from left, we call the first intersection point \( b_1 \). See Figure 4,6 for examples when \( q \neq 1 \) and also Figure 3, when \( q = 1 \).

To find the element which survives in the homology, Petkova and Hom found all the \( \mathcal{A}_\infty \) relations from a bordered diagram of \((p,1)\) pattern in \( D^2 \times S^1 \). For our case, we will only be interested in two generators and \( \mathcal{A}_\infty \) relations concerning them. For any \((p,q)\), it’s not always easy to find all the disks and thus all the \( \mathcal{A}_\infty \)-relations. Instead we will be looking into these two specific relations, coming from the bordered Heegaard diagram \( \mathcal{H}(p,q) \):

**Lemma 9.** In \( CFA^-(D^2 \times S^1, T_{p,q}) \), these are two specific \( \mathcal{A}_\infty \) relations:

\[
m_3(a, \rho_3, \rho_2) = U^{n_{zw}} \cdot a
\]

\[
m_4(a, \rho_3, \rho_2, \rho_1) = U \cdot b_1
\]

![Figure 4. A genus one bordered Heegaard diagram \( \mathcal{H}(3,2) \) of \( T_{3,2} \)](image-url)
Figure 5. A part of the lifted bordered Heegaard diagram $\mathcal{H}(3, 2)$ of $T_{3,2}$

Figure 6. A genus one bordered Heegaard diagram $\mathcal{H}(5, 3)$ of $T_{5,3}$
where \( n_w \) is the number of \( w \)'s in the primitive positive periodic domain of \( \mathcal{H}(p, q) \), where by primitive we mean the generator of \( \pi_2(a, a) \cong \mathbb{Z} \).

Proof. For a given genus one doubly pointed bordered Heegaard diagram, we can look at the fundamental domain of the torus and find a periodic domain joining \( a \), bounded by \( \alpha \)-arcs and \( \beta \) curve. We start with one of the representative of \( a \) on the domain and the boundary will have \( \rho_3 \) and \( \rho_2 \) crossed, to reach to another representative of \( a \) on the other vertical side. Then by following the whole \( \beta \) curve, one can reach the \( a \) representative one had started with, hence getting the first \( A_\infty \) relation.

Similarly, from the description of \( b_1 \), one can see a domain bounded by \( \alpha \)-arcs to the right and \( \beta \) curves to the left, joining \( a \) and \( b_1 \), crossing \( \rho_3, \rho_2, \rho_1 \) in the process. Also, the \( U \) power takes care of number of \( w \) encountered inside the domain. Combining these, we get the second \( A_\infty \) relation. \( \square \)

Lemma 10. Multiplicity of \( w \) in the primitive periodic domain of \( \mathcal{H}(p, q) \) is \( vx + 1 \), where \( x, y, u, v \) are unique positive integers such that \( p = x + y, q = u + v \) such that \( vx - uy = 1 \).

Proof. Instead of looking at the lifts of \( \alpha \) arcs, if we look at lifts of the \( \alpha \) circle such that after pulling tight \( \beta \), we get that \( \bar{\alpha} \) and \( \bar{\beta} \) intersecting at the lattice points (i.e \( \alpha \) and \( \beta \) intersect
at only one point), then \((\Sigma, \alpha, \beta, z, w)\) becomes a doubly pointed Heegaard diagram of the knot \(T_{p,q} \subset S^3\). Hence the intersections of \(\tilde{\beta}\) and \(\tilde{\alpha}\) equals to \(\text{dim}_F(\widehat{\text{HFK}}(T_{p,q}))\).

Corollary 2.6 in [29] states that the number of non-zero terms in the Alexander polynomial of \(T_{p,q}\) is \(2vx - 1\), where \(x, y, u, v\) are unique positive integers such that \(p = x + y, q = u + v\) such that \(vx - uy = 1\). Now since \(T_{p,q}\) are L-space knots (since positive surgery along torus knots with certain coefficient produces lens space, by [19]), Ozsváth and Szabó showed in [24] that the L-space knots forms a ‘staircase’ complex and hence each such \(j\) for which \(\widehat{\text{HFK}}(K, j) \neq 0\) is of dimension 1 i.e \(\text{dim}_F(\widehat{\text{HFK}}(T_{p,q}))\) is equal to the number of non-zero terms in \(\Delta_{p,q}\), which is \(2vx - 1\).

Now, in the lifted picture, whenever \(\tilde{\beta}\) crosses a \(\tilde{\alpha}\), since it creates two generators in \(\widehat{\text{HFK}}(T_{p,q})\) (and assuming that the complex is reduced i.e there is no disk formed which doesn’t contain \(z\) or \(w\)), there has to be a \(w\) that it’ll cross. Thus for \(2vx - 1\) number of intersections between \(\tilde{\beta}\) and \(\tilde{\alpha}\), barring the one that’ll generate the \(\widehat{\text{HF}}(S^3)\) (i.e the intersection point denoted \(a\)), for every two points, there will be a \(w\) crossed. Hence, the number of \(w\)’s in the primitive periodic domain is equal \(2vx - 1\), where one of the \(w\)’s comes from the \(w\) lying inside the fundamental domain of torus, bounded by the boundary of the periodic domain. The other \(w\) stays inside this periodic domain because of how the \(\beta\) occur according the algorithm of Ording in [20]. See Figure 8 where the initial and the end \(\beta\) strands of the boundary of this periodic domain is drawn. □

Next lemma will allow us to consider any element in the knot Floer complex of the cable, of the form \(b_1 \otimes \cdot\), irrespective of \(\epsilon(K)\). The reason is that all elements coming from the box tensor product of the form \(b_1 \otimes \cdot\) are non-zero in the knot Floer homology of cable.

**Lemma 11.** In \(\mathcal{H}(p,q)\), there is no Whitney disk connecting \(b_1\) that does not contain \(w\)’s. In other words, every \(A_\infty\) relation in \(\text{CF}\alpha^-((S^1 \times D^2), T_{p,q})\), concerning \(b_1\), has a non-zero \(U\) coefficients to it, when \(p > q\).

**Proof.** From the algorithm, observe that to prove the claim of the lemma one has to show that the second strand of \(\beta\) curve, i.e the strand that starts from \(b_1\), should lie to the left of \(w\). If that happens, then we can see that since no disks are ‘allowed’ to contain \(z\) in it, the only way any other disk connecting \(b_1\) can exist if and only if it contains a \(w\) inside it and hence the claim.

Now to prove the claim, we observe that since according to the algorithm, \(p(t_\beta)\) misses the standard form of the torus knot on the fundamental domain and also it is isotopic to the standard longitude of the torus, from the Figure 8 it’s easy to see that \(\beta\) will cross \(w\) from its left right after crossing \(b_1\), since each strand of the standard form of the knot has slope greater than 1 (as \(p > q\)) and thus the first strand of the knot will intersect the horizontal boundary of the fundamental domain to the left of \(w\). □

Hom showed in [10] that for a \((p, 1)\) pattern, when \(\epsilon(K) = 1\), the element \(a \otimes \xi_2\) survives in the homology and hence becomes a generator of \(\widehat{\text{HF}}(S^3)\) and calculated the Alexander grading of that generator to find the \(\tau\) of \((p, pn + 1)\) cables of \(K\).
Figure 8. The leftmost figure shows a normal form of $T_{p,q}$ on the fundamental domain where the $i$-th strand on the left vertical line, counting from top, gets identified with $p+i \mod (q-1)$-th strand on the right vertical line. Second picture shows a standard form of $T_{p,q}$ on the fundamental domain. The rightmost picture shows the eventual standard form on a ‘bordered’ fundamental domain, on which one can apply Ording’s algorithm to find $\mathcal{H}(p,q)$ i.e the doubly pointed bordered diagram of $T_{p,q} \subset D^2 \times S^1$. The dotted curve shows the start and the ending of the $\beta$ curve.

For general $(p,q)$ patterns, we observe that the same is true as well when $t \neq 0$ and $\epsilon(K) = 1$. This will makes sure that when $\epsilon(K) = 1$ and $t \neq 0$, then $a \otimes \xi_2$ will survive in the homology in the knot Floer complex of the $K_{p,q}$, when we take $U = 0$ in the differential $\partial^-$. In other words, $a \otimes \xi_2$ is a non-zero element of $\widehat{HF}(S^3, K_{p,q})$.

**Lemma 12.** When $\epsilon(K) = 1$ and $t \neq 0$, $a \otimes \xi_2$ is an element that survives in homology in box tensor.

**Proof.** First we look at the description of $\widehat{CFD}(X_K, r)$ generators and we observe the incoming and outgoing arrows and the corresponding coefficients to and from $\xi_2$ in it. The immediate incoming and outgoing arrows to and from $\xi_2$ consist of coefficients $D_{123}, D_3, D_2$.

To start with, there is no $A_{\infty}$ relation such as $m_k(a, \rho_{123}, \cdots) = c$ or $m_k(d, \cdots, \rho_{123}) = a$ for some $c, d \in CFD^-(D^2 \times S^1, T_{p,q})$, since around $a$, $\beta$ always has positive slope in $H(p,q)$ and since in the algorithm, $\beta$ always runs along the standard form of the knot and only changes its direction around $w$.

Also, there is no $A_{\infty}$-relation such as $m_k(c, \rho_1, \cdots) = a$ where $c \neq a$ i.e there is no Whitney disk which starts from $a$ and is bounded by $\alpha$’s to the left and $\tilde{\beta}$’s to the right and ends at $c$ (that doesn’t include $z$), where $c \neq a$.

Also, a $A_{\infty}$ relation $m_{k+1}(a, \rho_3, \rho_{23}, \rho_{23}, \cdots, \rho_{23}) = c$, for some $c$ is not possible, since a Whitney disk in $\mathcal{H}(p,q)$, starting from $a$ with $\alpha$’s to the right and $\beta$ to the left and having $\rho_{23}$ in it, should contain $\rho_2$ and $\rho_1$ as well.

Combining these observations completes the proof.  \qed
For this specific subcase, one of the desired generators is \( a \otimes \xi_2 \), while the other one will be \( b_1 \otimes \kappa_{\ell_i} \), for some \( \ell \). We shall figure out \( i \) during our eventual grading calculation.

At this moment, let us recall the grading scheme in bordered Floer homology from [14, Chapter 10]. The grading for elements of a bordered Floer complex, \( gr \), takes values in a non-commutative group \( G(Z) \), whose elements are triples of the form \((m; i, j)\) where \( m, i, j \in \frac{1}{2}\mathbb{Z}, \ i + j \in \mathbb{Z} \), where the half integer \( m \) is the Maslov component, the pair \((i, j)\) is the \( \text{spin}^c \)-component. We’ll also be interested in \( \tilde{G} = G(Z) \times \mathbb{Z} \), where the last component reflects the \( U \) grading. The group law is defined by :

\[
(m_1; i_1, j_1; n_1) \cdot (m_2; i_2, j_2; n_2) = (m_1 + m_2 + (i_1 j_2 - i_2 j_1); i_1 + i_2, j_1 + j_2; n_1 + n_2)
\]

\( G(Z) \) has these grading on Reeb elements :

\[
gr(\rho_1) = \left( -\frac{1}{2}; \frac{1}{2}; -\frac{1}{2} \right)
\]

\[
gr(\rho_2) = \left( -\frac{1}{2}; \frac{1}{2}; \frac{1}{2} \right)
\]

\[
gr(\rho_3) = \left( -\frac{1}{2}; -\frac{1}{2}; \frac{1}{2} \right)
\]

along with this rule that \( gr(\rho_1 \rho_2) = gr(\rho_1) gr(\rho_2) \) and \( gr(\rho_{IJ}) = \lambda gr(\rho_I) gr(\rho_J) \), (where \( IJ \in \{12, 23, 123\} \) ) where \( \lambda = (1; 0, 0) \in G(\mathbb{Z}) \).

Recall that the set of all periodic domain is isomorphic to \( H_2(Y, \partial Y) \cong \mathbb{Z} \).

If we call the image of the generator of this group in \( \tilde{G} \) by \( g \), Then for a multiplication map \( m_{k+1}(x, \rho_{i_1}, \ldots, \rho_{i_k}) = U^m y \) in \( CFA^-(Y, K) \), we have

\[
gr(y) = \lambda^{k-1} gr(x) gr(\rho_{i_1}) \cdots gr(\rho_{i_k}) \cdot u^{-m} \in (g) \bs \tilde{G}
\]

where \( gr(u) = (0; 0, 0; -1) \in \tilde{G} \) (Observe that both \( \lambda, u \) are in the centralizer of \( \tilde{G} \)).

If we call the image of the generator of periodic domains in \( G, h \), then if \( D_I \) is a coefficient map from \( x \) to \( y \) in \( CF\tilde{D}(Y, r) \), then we have

\[
gr(y) = \lambda^{-1} gr(\rho_I)^{-1} gr(x) \in \tilde{G}/\langle h \rangle
\]

The box tensor product between \( CF\tilde{D} \) and \( CFA^- \) of two manifolds with torus boundary is then graded by \( (g) / \tilde{G} \bs \langle h \rangle \). Every element in this double-coset space is uniquely equivalent to an element of the form \( \lambda^a u^b \), for some \( a, b \in \mathbb{Z} \) i.e the grading of that element takes form \((a; 0, 0; -b)\).

We recall that the \( z \)-normalized Maslov grading \( N \), defined by Lipshitz, Ozsváth, Thurston in [14, Section 11.3] can also be realized like this: \( N = M - 2A \) ([Equation 11.13][14] and \( N = 0 \) for the generator of \( H_*(gCFK^-(K)/U = 1) \cong \mathbb{Z} \) ([14, Equation 11.15]), where \( M \) denotes the Maslov grading and \( A \) denotes the Alexander grading of some element in \( gCFK^-(K) \). The first co-ordinate \( a \) from the discussion above, is the value of \( N \), upto an additive constant. The last co-ordinate \( b \) from above, is the Alexander grading, upto an additive constant. See [14, 11.9] for an example showing how one can find the exact
Maslov and Alexander grading, using Poincaré polynomial, using the fact that weighted Euler characteristics of knot Floer homology is the Poincaré polynomial of the knot.

Now, if we have two elements in knot Floer homology of a thin knot, from the splicing formula, whose grading reduce to $(a_1; 0; b_1)$ and $(a_2; 0; b_2)$, then $N_1 = a_1 + c_0 = M_1 - 2A_1, N_2 = a_2 + c_0 = M_2 - 2A_2$ and $A_1 = b_1 + d_0, A_2 = b_2 + d_0$ (where $c_0$ and $d_0$ are some additive constants for $N$ and $A$, respectively). Then since $M_1 - M_2$ should be equal $A_1 - A_2$, that implies that for thin knots,

$$a_1 - a_2 = b_2 - b_1$$

We’ll carry out the grading calculations and find two elements in the knot Floer homology, by the splicing formula, for which Equation (3) fails to hold.

Now we calculate the grading of the two elements $a \otimes \xi_2$ and $b_1 \otimes \kappa_i^{j_l}$. Since we’re just interested in showing that the difference in their Maslov grading is not equal the difference in their Alexander grading, the relative grading will do the job for us.

Let $\xi_{2i}$ be some element in the vertically simplified basis of $\text{CFK}^-(K)$ such that:

$$\xi_{2i} \xrightarrow{D_{123}} \kappa_i^{j_l} \xrightarrow{D_{23}} \cdots \xrightarrow{}$$

and let $A(\xi_{2i}) = d, M(\xi_{2i}) = m$ (by this we mean that A is the Alexander grading and M is the Maslov grading of the element of the knot Floer homology of $K$, that represents $\xi_{2i}$). Then, using [14, Theorem A.11], we get that

$$gr(\xi_{2i}) = \lambda^{m-2d} \cdot gr(\rho_{23})^{-d} \in G(\mathbb{Z}).$$

Let $\kappa_i^{j_l} = D_{123} \cdot \xi_{2i}$, then

$$gr(\kappa_i^{j_l}) = \lambda^{-1} \cdot gr(\rho_{123})^{-1} \cdot \lambda^{m-2d} \cdot gr(\rho_{23})^{-d}$$

Lemma 10 and Equation (3) imply that the grading set for $\text{CF}^-$ is isomorphic to

$$gr(a) \cdot \hat{G} = u^{-(v+1)} \cdot gr(\rho_{23}) \setminus \hat{G}.$$ Theorem 8 implies the grading set for $\text{CFD}(X_K, r)$ is $G/\lambda^{-1} \cdot gr(\rho_{12})^{-1} \cdot gr(\rho_{23})^{-r}$, if the framing of the companion knot is $r$.

Then Lemma 9 implies

$$gr(b_1) = \lambda \cdot u^{-1} \cdot gr(\rho_{23}) \cdot gr(\rho_1) \sim \lambda \cdot u^{-1} \cdot u^{v+1} \cdot gr(\rho_1) \sim \lambda \cdot u^{v} \cdot gr(\rho_1)$$

From the description of $\xi_2$ and using [14, Theorem 2], we get that

$$gr(\xi_2) = \lambda^{-2\tau_K + 2\tau_K} \cdot gr(\rho_{23})^{\tau_K} = gr(\rho_{23})^{\tau_K},$$

since the generator of homology of $C(j = 0)$ has Alexander grading $-\tau_K$ and Maslov grading $-2\tau_K$ and $\xi_2$ is a representative of that. Thus, since $gr(a) = (0; 0, 0; 0),

$$gr(a \otimes \xi_2) = gr(\rho_{23})^{\tau_K} \sim u^{\tau_K(v+1)} = (0; 0, 0; \tau_K(v+1))$$ (4)
and
\[ \text{gr}(b_1 \otimes \kappa_{\ell_i}^i) = \text{gr}(b) \cdot \text{gr}(\kappa_{\ell_i}^i) = \lambda \cdot u^{vx} \cdot \text{gr}(\rho_1) \cdot \lambda^{-1} \cdot \text{gr}(\rho_{123})^{-1} \cdot \lambda^{m-2d} \cdot \text{gr}(\rho_{23})^{-d} \]
\[ = \lambda^{m-2d} \cdot u^{vx} \cdot \text{gr}(\rho_{23})^{-1} \cdot \lambda^{-1} \cdot \text{gr}(\rho_{23})^{-d} \]
\[ = \lambda^{m-2d-1} \cdot u^{vx} \cdot \text{gr}(\rho_{23})^{-d-1} \sim \lambda^{m-2d-1} \cdot u^{-vdx} \]
\[ = (m - 2d - 1; 0; 0; -vdx) \]

Now, we can choose \( m = -2\tau_K, d = -\tau_K \), by choosing \( \ell = 1 \) i.e
\[ \text{gr}(b_1 \otimes \kappa_{\ell_i}^i) = (-1; 0, 0; vx\tau_K) \]
(5)

Getting back to proving that the difference between the relative Maslov grading of those two elements are not equal the difference between their relative Alexander grading, we see that if \( \tau_K \neq -1 \), by comparing with Equation (3) we’re done. Hence, we’re done with the case when \( t \neq 0, \epsilon(K) = 1 \) and \( \tau_K \neq -1 \).

For \( t \neq 0, \epsilon(K) = 1, \tau_K = -1 \), we look at the unstable chain of \( \iota_1 \) part of \( \widehat{CFD}(X_K, r) \), mentioned in Theorem 8. If \( t < 0 \), then \( \mu_1 = D_{123} \cdot \xi_0 \). If \( t > 0 \), then \( \mu_1 = D_1 \cdot \xi_0 \). We calculate the grading of \( \mu_1 \) for both cases.

By Theorem 8, \( \text{gr}(\xi_0) \) equals to \( \lambda^2 \cdot \rho_{23} \). Hence if \( t < 0 \),
\[ \mu_1 = D_{123} \cdot \xi_0 \Rightarrow \text{gr}(\mu_1) = \lambda^{-1} \cdot \text{gr}(\rho_{123})^{-1} \cdot \text{gr}(\xi_0) \]
\[ = \lambda^{-1} \cdot \text{gr}(\rho_{123})^{-1} \cdot \lambda^2 \cdot \text{gr}(\rho_{23}) \]
\[ = \lambda^{-1} \cdot \lambda^{-1} \cdot \text{gr}(\rho_1)^{-1} \cdot \text{gr}(\rho_{23})^{-1} \cdot \lambda^2 \cdot \text{gr}(\rho_{23}) \]
\[ \sim \text{gr}(\rho_1)^{-1} \]

Then
\[ \text{gr}(b_1 \otimes \mu_1) = \text{gr}(b_1) \cdot \text{gr}(\mu_1) \]
\[ = \lambda \cdot u^{vx} \cdot \text{gr}(\rho_1) \cdot \text{gr}(\rho_1)^{-1} \sim \lambda \cdot u^{vx} \]
\[ = (1; 0, 0; vx) \]

If \( t > 0 \),
\[ \mu_1 = D_1 \cdot \xi_0 \Rightarrow \text{gr}(\mu_1) = \lambda^{-1} \cdot \text{gr}(\rho_1)^{-1} \cdot \text{gr}(\xi_0) \]
\[ = \lambda^{-1} \cdot \text{gr}(\rho_1)^{-1} \cdot \lambda^2 \cdot \text{gr}(\rho_{23}) \]
\[ = \lambda \cdot \text{gr}(\rho_1)^{-1} \cdot \text{gr}(\rho_{23}) \]
Then
\[ gr(b_1 \otimes \mu_1) = gr(b_1) \cdot gr(\mu_1) = \lambda \cdot u^{vx} \cdot gr(\rho_1) \cdot \lambda \cdot gr(\rho_1)^{-1} \cdot gr(\rho_{23}) \sim \lambda^2 \cdot u^{2vx} = (2; 0, 0; 2vx). \]

Now comparing \( gr(a \otimes \xi_2) \) and \( gr(b \otimes \mu_1) \), we get that Equality (3) happens iff \( vx = -1 \).

Now \( \dim(\widehat{HFK}(S^3, T_{p,q})) = 2vx - 1 \geq 3 \) (see \([29, \text{Corollary 2.6}]\)). That implies that \( vx \) should be greater than or equal to 2. This is because the only torus knot(s) whose knot Floer homology has rank 3 are \( \pm T_{2,3} \). To see this, note that \( \widehat{HFK}(K, g(K)) \cong \widehat{HFK}(K, -g(K)) \neq 0 \) and \( \widehat{HFK}(K, 0) \neq 0 \) and the only genus 1 fibered knots are \( \pm T_{2,3} \) and figure eight knots, as \( \widehat{HFK}(K, g(K)) \cong \mathbb{Z} \) implies that \( K \subset S^3 \) is fibered but the figure eight knot is not a torus knot.

Hence we have a contradiction to the possible value of \( vx \). So we have \( a \otimes \xi_2 \) and \( b \otimes \mu_1 \) are non-zero elements (by Lemma 11 and 12) in the knot Floer homology of the cable such that difference between their Maslov grading cannot be equal the difference between their Alexander grading.

Now if \( \epsilon(K) = 0 \), then we consider an element \( \xi \), which lies in the lowest Alexander grading in \( \widehat{HFK}(K) \) i.e \( A(\xi) = -g \). Since \( \epsilon(K) = 0 \) implies \( \tau_K = 0 \), there is a \( \xi' \in \widehat{HFK}(K) \) such that \( \partial^\vert \xi' = \xi \). Then there is some \( \xi_{2s} \) in the vertically simplified basis of \( CFK^- \) such that \( A(\xi_{2s}) = A(\xi) = -g \) and let \( M(\xi_{2s}) = M(\xi) \), (see the Proof of \([10, \text{Lemma 2.1}]\)). We choose our \( \ell \) to be 2. Then
\[ gr(b_1 \otimes \kappa_{2s}^g) = (m + 2g - 1; 0, 0; vx_s) \]
\[ gr(a \otimes \xi_{2s}) = (m + 2g; 0, 0; vx_s). \]

**Lemma 13.** If \( \epsilon(K) = 0, t \neq 0 \), then \( a \otimes \xi_{2s} \) is non-trivial in the knot Floer homology of the cable \( K_{p,q} \).

**Proof.** Since \( \epsilon(K) = 0 \) implies \( \tau_K = 0 \) and \( \xi_{2s} \) is not a generator of either vertical or horizontal homology, we just have to check that there is no possibility of any \( A_\infty \) relations in \( CFK^-(D^2 \times S^1, T_{p,q}) \) involving \( a \), which has the same coefficients from the vertical and horizontal chain of \( CFD(X_K, r) \).

To do this, first we check the vertical chain in \( CFD(X_K, r) \) and the coefficient maps from Theorem (8). We said in the proof of Lemma 12, there is no disk connecting \( a \) and starting with coefficient \( \rho_{123} \), since in the algorithm, the \( \beta \) curve always lie along the standard form of the knot, and only changes its direction around \( w \). Hence there is no \( A_\infty \) relation such as \( m_k(a, \rho_{123}, \cdots) = c \), for some \( c \).

Now, we check the vertical chain in \( CFD(X_K, r) \) and the coefficient maps from Theorem 8. We can see that the only relation involving \( \rho_2, \rho_3 \) and \( \rho_{23} \) involving \( a \) is
Let $t = 2\tau(K) - r = 0$, then we follow the same strategy that we followed for the last subcase in $t \neq 0$ case. We consider an element $\xi'$, which is neither a generator of the vertical homology, nor that of the horizontal homology, and gets ‘killed’ in the vertical chain complex i.e there is a $\xi'' \in \overline{HF}_K(K)$ such that $\xi'' \in \partial^{vert}\xi''$. Then there is some $\xi_{2s}$, in the vertically simplified basis of $\overline{CFK}^-$ such that $A(\xi_{2s}) = A(\xi') = A$ and let $M(\xi_{2s}) = M(\xi') = M$, (see the proof of [10, Lemma 2.1]). We choose our $\ell = s$. Then

$$gr(b_1 \otimes \kappa_{\ell s}) = (M - 2A - 1; 0, 0; -vAx)$$
$$gr(a \otimes \xi_{2s}) = (m - 2A; 0, 0; -vAx)$$

For this case, we can see that Equation (3) will fail to happen and hence the difference between the Alexander grading and Maslov grading of $a \otimes \xi_{2s}$ and $b \otimes \kappa_{\ell s}$ are not equal. We claim this :

**Lemma 14.** If $t = 0$, then $a \otimes \xi_{2s}$ is non-trivial in the knot Floer homology of the cable.

**Proof.** Since $t \neq 0$ implies and $\xi_{2s}$ is not a generator of vertical homology, we just have to check that there is no possibility of any $A_{\infty}$ relations in $CFA^{-}(D^2 \times S^1, T_{p,q})$ involving $a$, which has exact same coefficients from the vertical and horizontal chain of $\overline{CFD}(X_K, r)$.

To do that, first we check the vertical chain in $\overline{CFD}(X_K, r)$ and the coefficient maps from Theorem 8. As we’ve said in the proof of Lemma 12, there is no disk connecting $a$ and starting with coefficient $\rho_{123}$, since in the algorithm, the $\beta$ curve always lie along the standard form of the knot, and only changes its direction around $w$. Hence there is no $A_{\infty}$ relation such as $m_k(a, \rho_{123}, \dots) = c$, for some $c$.

Now, we check the horizontal chain in $\overline{CFD}(X_K, r)$ and the coefficient maps from Theorem (8). We can see that the only relation involving $\rho_3, \rho_{32}$ and involving $a$ is

$$m_k(a, \rho_3, \rho_{23}, \ldots, \rho_2) = U^n \cdot a,$$

where $n$ is non-zero, as the said relation indicates a positive multiple of the primitive periodic domain. Thus the element $a \otimes \xi_{2s}$ is non-zero in the knot Floer homology of the cable.

Now, we check the unstable chain in $\overline{CFD}(X_K, r)$ and coefficient maps there when $t = 0$. There is an incoming $D_{12}$ map towards $\eta_0$, the generator of horizontal homology. Even if $\xi_{2s} = \eta_0$, one can check that $a \otimes \alpha_{2s}$ will still be non-trivial in the knot Floer homology of the cable, since there is $A_{\infty}$ relation such as $m_2(c, \rho_{12}) = a$, for any $c$, in other words, there is no Whitney disk, connecting $a$ and some other intersection point, which is bounded by $\beta$ curve to the right and $\alpha$-arcs to the left, starting with $a$. Hence our claim is proved.

**Also, by Lemma 11, $b \otimes \kappa_{\ell s}$ is non-zero in the knot Floer homology of the cable.**
Now, the remaining case: when $\epsilon(K) = -1$, then we can look at $-K$ instead. Since $\epsilon(-K) = -\epsilon(K) = 1$ and also $K_{p,q} = (-K)_{-p,q}$ i.e we’re already done with the case. This completes our proof. □

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