Particle production from oscillating scalar backgrounds in an FLRW universe

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Abstract

We study particle production from oscillating scalar backgrounds in a spatially flat Friedmann–Lemaître–Robertson–Walker universe using non-equilibrium quantum field theory which fully captures the thermal effects and backreaction effects. To be concrete, we consider a $Z_2$-symmetric two-scalar model with quartic interactions. For quasi-harmonic oscillations, we adopt the multi-scale analysis to obtain analytical approximate expressions for the self-consistent evolution of the scalar background and the energy density of the produced particles in terms of the retarded self-energy and retarded proper four-vertex function, whose imaginary parts characterize different condensate decay channels and lead to dissipation. We find that reheating in this model can be complete if the imaginary part of the retarded self-energy is not negligible, which causes dissipation only at finite temperature through Landau damping. Our results can be generalized to more complicated models and thus could provide a general framework for studying the perturbative reheating process in the early Universe or perturbative production of Dark Matter from an oscillating inflaton.

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1 Introduction

Scalar fields play prominent roles in both particle physics and cosmology. In the standard model (SM), there is a known scalar field, the Higgs field, which is crucial for mass generation. Scalar fields are also assumed to be crucial in explaining a variety of phenomena beyond the SM. For example, in the Peccei-Quinn mechanism [1], which can elegantly solve the strong CP problem [2, 3], a (pseudo)scalar field called axion [5, 6] is generally predicted. In the inflationary scenario for the very early Universe [7–10], most of the inflation models assume that the inflaton is a scalar field [11]. Further, gauge singlet scalars [12] could also account for part or all of the Dark Matter [13].

In the standard single-field slow-roll inflation, the inflaton field initially has a non-vanishing field displacement from its equilibrium value and slowly rolls down to the latter. Near the min-

\footnote{It is argued recently in Ref. [4] that the strong CP problem may not exist at all.}
imum, it oscillates and transfers its kinetic and potential energy to perturbative fluctuations of fields coupled to it, leading to a dramatic production of particles. The associated thermalization process reheats the Universe and the standard big-bang picture of the Universe then follows. For large elongations, oscillations break adiabaticity, leading to the phenomenon of parametric resonance [14–18]. Particle production during this period is non-perturbative and the corresponding reheating process is usually called preheating. After preheating, the oscillation amplitude becomes small enough such that the process of particle production would finally become perturbative. In this work, we study the dissipation of oscillating scalar backgrounds in the latter regime of small elongations such that one can apply the small-field expansion in which the condensate-dependent masses of fluctuation fields are treated as perturbative terms compared to the condensate-independent masses.

Parametric resonance due to oscillating backgrounds has been studied thoroughly in the aforementioned classic papers on preheating. Generically, studies on non-perturbative particle production are usually based on the Bogoliubov method [31], the functional Schrödinger approach [32], in particular the Floquet theory on analyzing equations of motion for mode functions. Perturbative particle production is usually studied by the time-dependent perturbation theory, see, e.g., Ref. [33, 34]. These methods typically assume a fixed classical background and the computation is usually done at zero temperature, thus missing the backreaction effects and thermal effects. Reformulating the approaches in terms of particle distributions offers a way to incorporate thermal effects [35–37]. A natural framework with the above two effects being simultaneously taken into account does exist and is provided by the Closed-Time-Path (CTP) formalism [38, 39]. Some earlier studies on dissipation due to particle production using the CTP formalism can be found in Refs. [16, 40–42], while Refs. [43–45] focus on particular oscillating backgrounds. In principle, the non-equilibrium dynamics can be understood by solving the coupled equations of motion for the one-point function (of the scalar field that forms condensate) and for the two-point functions (the Kadanoff-Baym equations). However, due to the limited ability to solve these equations analytically, clear analytical relations between the particle production rates or the condensate evolution and the various microscopic quantities like self-energies and proper four-vertex functions, have never been rigorously derived.

Important progress has been made recently in Ref. [46] where the authors were able to solve the equation of motion for the scalar condensate analytically in the small-field regime and when the oscillation is quasi-harmonic. A crucial development made in Ref. [46] is introducing the multi-scale analysis [48, 49] to solving the non-local equation of motion for the condensate. This method assumes that physical processes happen on different time scales. Specifically, it assumes

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2 A different mechanism for non-perturbative particle production is the tachyonic preheating due to the spinodal instability [19–22]. We do not consider it in this work. Other works on backreaction effects in non-oscillating scalar backgrounds in the Yukawa theory are Refs. [23–25].

3 For the dissipative behavior of a scalar condensate in the slow-roll phase, see Refs. [26–30].

4 For an excellent complementary understanding to Ref. [46], see the recent work [47] where the authors study the condensate evolution beyond the small-field regime and solve the coupled equations of motion for the condensate and two-point functions numerically, capturing effects from both parametric resonance and spinodal instability.
that the oscillating amplitude and frequency change slightly during one single oscillation and the
window provided by the kernels of the non-local terms in the condensate equation of motion.
This is usually the case for the dissipating system under study when the coupling constants are
perturbatively small. The obtained condensate evolution is expressed in terms of the retarded
self-energy and retarded proper four-vertex function for the scalar that forms condensate.

In this paper we generalize the work [46] to a flat Friedmann–Lemaître–Robertson–Walker
(FLRW) universe. This generalization is of closer relevance to the perturbative reheating process
in the early Universe. Although particle production via parametric resonance is typically much
more efficient than the perturbative particle production, the latter still determines whether the
dissipation of inflaton is complete or not as well as some initial conditions right after reheat-
ing. This work also serves as a first-principle, and perhaps also more rigorous than alternative
methods, foundation for studying perturbative production of Dark Matter from an oscillating in-
falon [50–54].

The outline of the paper is as follows. In the next section, we introduce our model
and review the derivation of the condensate equation of motion using the two-particle-irreducible
(2PI) effective action [46, 57]. In Sec. 3, we solve the condensate equation of motion using the
multi-scale analysis and discuss its cosmological meanings in different situations. Finally, we
present our conclusions in Sec. 4.

2 The model and the condensate equation of motion

We consider the following action

\[ S[\Phi, \chi] = \int d^4x \sqrt{-g} \left[ \frac{1}{2} (\partial_\mu \Phi)(\partial^\mu \Phi) + \frac{1}{2} (\partial_\mu \chi)(\partial^\mu \chi) - V(\Phi, \chi) \right], \]  

where

\[ V(\Phi, \chi) = \frac{m^2}{2} \Phi^2 + \frac{m^2}{2} \chi^2 + \frac{\lambda}{4!} \Phi^4 + \frac{\lambda}{4!} \chi^4 + \frac{g}{4} \Phi^2 \chi^2. \]  

Here \( \Phi \) and \( \chi \) are two real scalars. This model is frequently considered in phenomenological
studies as one of the Higgs portal models [58]. For simplicity, we do not include the non-minimal
coupling with gravity. For a spatially flat FLRW universe

\[ ds^2 = dt^2 - a^2(t)dx^2, \]

and \( \sqrt{-g} = a^3(t) \). In this paper, we will take the spacetime background as fixed.

The scalar field \( \Phi \) is assumed to possess a non-vanishing expectation value \( \langle \Phi \rangle \equiv \varphi \) and will
be called the inflaton. Expanding \( \Phi = \varphi + \phi \), we can view \( \varphi \) as a background and \( \phi, \chi \) as
fluctuations about this background. Particles are then defined as excitations of the fluctuation

\[ 5Examinations on this scenario focusing on non-perturbative production of Dark Matter are given in Refs. [55,
56]. \]
fields $\phi$ and $\chi$. The total action can be written as $S[\Phi, \chi] = S_\varphi[\varphi] + S_{\phi\chi}[\phi, \chi; \varphi]$ where
\[
S_\varphi[\varphi] = \int d^4x a^3(t) \left[ \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m_\varphi^2 \varphi^2 - \frac{\lambda_\varphi}{4!} \varphi^4 \right],
\]
(4a)

$S_{\phi\chi}[\phi, \chi; \varphi] = \int d^4x a^3(t) \left[ \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} (\partial_\mu \chi)(\partial^\mu \chi) - V_{\phi\chi}(\phi, \chi) \right],
\]
(4b)

with
\[
V_{\phi\chi}(\phi, \chi) = \frac{1}{2} \left( m_\phi^2 + \frac{\lambda_\phi}{2} \varphi^2 \right) \phi^2 + \frac{1}{2} \left( m_\chi^2 + \frac{g}{2} \varphi^2 \right) \chi^2 + \frac{\lambda_\phi}{3!} \varphi^3 + \frac{g}{2} \varphi^2 \chi^2 + \frac{\lambda_\phi}{4!} \phi^4 + \frac{\lambda_\chi}{4!} \chi^4 + \frac{g}{4} \phi^2 \chi^2 + \text{(linear terms in fluctuations)}.
\]
(5)

The linear terms in fluctuations would not contribute to the perturbative diagrammatic expansion of the effective action.\(^6\) The oscillation of $\varphi$ could induce particle production for both $\phi$ and $\chi$, either through the time-dependent mass terms or the interacting terms $\varphi\phi^3$, $\varphi\phi\chi^2$. The small-field regime we consider in this work is defined by the requirement that the $\varphi$-dependent mass term is much smaller than the $\varphi$-independent mass term corresponding to the same particle type. (At finite temperature, one may consider thermal corrections to the $\varphi$-independent mass terms.) Therefore, we can take the $\varphi$-dependent mass terms as perturbations such that in the perturbative expansion of the effective action they are on an equal footing with the other interacting terms. In particular, this means that one can expand the two-point functions in the background field $\varphi$ and at the leading order they are independent of $\varphi$. Below we briefly review the derivation of the equation of motion for $\varphi$ using the CTP formalism [46]. A reader not concerned with this derivation may take Eq. (25) as a starting point.

The CTP formalism has been widely applied to the studies of baryogenesis, especially leptogenesis (see, e.g., Refs. [60–69]). For reviews, see Refs. [70–72]. In the conventional zero-temperature quantum field theory, there are known asymptotic states at far past and far future. Most physical problems are about the transition amplitudes between the asymptotic states. They belong to the boundary-value problem. In non-equilibrium quantum field theory, the initial states are prepared and the final states are unknown \textit{a priori} and can only be determined by the evolution itself. Thus the non-equilibrium dynamics is an initial-value problem. This is the reason why a closed-time path needs be introduced in non-equilibrium quantum field theory.

The initial value is usually given by a density matrix at a given time $\rho_D(t_i)$ in a mixed ($\text{Tr}\{\rho_D^2(t_i)\} < 1$) or pure ($\text{Tr}\{\rho_D^2(t_i)\} = 1$) state. In the Heisenberg picture, operators evolve with time while the states do not. The expectation value of an observable $\mathcal{O}$ at time $t$ is given by
\[
\langle \mathcal{O}(t) \rangle = \text{Tr} \{ \rho_D(t_i) \mathcal{O}(t) \},
\]
(6)

where $\mathcal{O}(t) = \exp(iH(t - t_i))\mathcal{O}(t_i)\exp(-iH(t - t_i))$. Viewed from the Schrödinger picture, the state evolves first forward in time from $t_i$ to $t$ and then backward in time to $t_i$. Expectation

\(^6\)For a clear explanation on this point in the case of 1PI effective action, see, e.g., Ref. [59].
values can thus be obtained by a generating functional formulated on a closed time contour \( C \) (named as the Keldysh contour), as illustrated in Fig. 1. The full information in a quantum field system is encoded in all its correlation functions. For most purposes, it is sufficient to study the one-point function, \( \varphi \), and the connected two-point functions, \( \Delta_\phi(x,y) \equiv \langle \Phi(x)\Phi(y) \rangle_c \) and \( \Delta_\chi(x,y) \equiv \langle \chi(x)\chi(y) \rangle_c \). Therefore the non-equilibrium dynamics is usually given by the coupled equations of motion for the one- and two-point functions. The equations of motion for these quantities are determined by the 2PI effective action, \( \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi] \) [57, 72],

\[
\frac{\delta \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi]}{\delta \varphi(x)} = 0, \quad (7a) \\
\frac{\delta \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi]}{\delta \Delta_\phi(x,y)} = 0, \quad (7b) \\
\frac{\delta \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi]}{\delta \Delta_\chi(x,y)} = 0. \quad (7c)
\]

Solving on-shell for the two-point functions as functionals of the condensate

\[
\left. \frac{\delta \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi]}{\delta \Delta_\phi} \right|_{\Delta_\phi=\Delta_\phi[\varphi]} = 0, \quad 0 = \left. \frac{\delta \Gamma_{2\text{PI}}[\varphi, \Delta_\phi, \Delta_\chi]}{\delta \Delta_\chi} \right|_{\Delta_\chi=\Delta_\chi[\varphi]} = 0, \quad (8)
\]

and plugging them back into the 2PI effective action produces the 2PI-resummed effective action for \( \varphi \)

\[
\Gamma_{2\text{PI}}^{\text{res}}[\varphi] \equiv \Gamma_{2\text{PI}}[\varphi, \Delta_\phi[\varphi], \Delta_\chi[\varphi]]. \quad (9)
\]

Then the equation of motion of the condensate is given by

\[
\frac{\delta \Gamma_{2\text{PI}}^{\text{res}}[\varphi]}{\delta \varphi(x)} = 0. \quad (10)
\]

Note that we have assumed that all external sources are vanishing. Otherwise the above equations of motion would have nonvanishing terms on the RHS related to the sources [73, 74]. In some discussions for dissipation, one would have to introduce nonvanishing sources to switch on perturbations at the initial time, see e.g., Ref. [75].

\[ t_i \quad \longrightarrow \quad \longrightarrow \quad \longrightarrow \quad t_f \rightarrow \infty \]

Figure 1: The Keldysh contour \( C \) for the generating functional in the CTP formalism.

When the oscillation amplitudes are small, one can perform a perturbative expansion in \( \varphi \) in Eqs. (8) and at the leading-order, the connected two-point functions are independent of \( \varphi \).

\footnote{Remember in the equation of motion for \( \varphi \), one has to take the limit \( \varphi^+(x) = \varphi^-(x) \) where \( \pm \) indicate the forward and backward branches of the Keldysh contour, respectively [71].}
Substituting the leading-order connected two-point functions into Eqs. (9) and (10), one obtains an equation of motion for the condensate in which propagators are the free thermal ones (see Eqs. (13)). Properly truncating the 2PI-resummed effective action for $\varphi$, the condensate equation of motion in flat spacetime is given as [46]

$$\ddot{\varphi}(t) + M_\phi^2 \dot{\varphi}(t) + \frac{\lambda_\phi \varphi^3(t)}{6} + \int_{t_i}^t dt' \pi_R(t - t') \varphi(t') + \frac{\varphi(t)}{6} \int_{t_i}^t dt' v_R(t - t') \varphi^2(t') = 0, \quad (11)$$

where $t_i$ is the initial time at which initial conditions $\varphi(t_i)$ and $\dot{\varphi}(t_i)$ are specified, $M_\phi^2$ is the thermal-corrected mass for the $\Phi$ scalar, $\pi_R$ is the retarded self-energy and, $\nu_R$ is the retarded proper four-vertex function. These quantities will be defined specifically below. As we shall see in the next section, the last two quantities play distinguished roles in the inflaton decay. Note that since we have performed the small-field expansion for the two-point functions and taken the leading order results. The propagators in the 2PI-resummed effective action $\Gamma_{2PI}^{\text{res}}$ are independent of $\varphi$ and therefore the background field only appears in the vertices but not in the propagators. As a result, in Eq. (11) we do not have the usual effective potential [76, 77]. For example, the familiar one-loop effective potential would be obtained when the propagators have a dependence on the background field that is approximated as constant.

We shall assume that the thermal bath is large such that it is in equilibrium all the time. Diagrammatically, the two-point functions at the leading level of the small-field expansion and the two-loop level of the 2PI effective action are simply given by the free thermal equilibrium propagators (the Schwinger-Keldysh polarity indices $A, B$ take values of $+, -$)

$$\begin{align*}
\overrightarrow{(x,A)} \overleftarrow{(x',B)} & \equiv D_{\phi}^{AB}(x - x'), \\
\overleftarrow{(x,A)} \overrightarrow{(x',B)} & \equiv D_{\chi}^{AB}(x - x'),
\end{align*} \quad (12)$$

where

$$\begin{align*}
D_{\phi,\chi}^{++}(x - x') &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \left[ \frac{i}{k^2 - M_{\phi,\chi}^2 + i\varepsilon} + 2\pi f_B(|k_0|) \delta \left( k^2 - M_{\phi,\chi}^2 \right) \right], \quad (13a) \\
D_{\phi,\chi}^{+-}(x - x') &= \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} \left[ 2\pi f_B(k_0) \text{sign}(k_0) \delta \left( k^2 - M_{\phi,\chi}^2 \right) \right], \quad (13b)
\end{align*}$$

with the remaining components being complex conjugates, $D_{\phi,\chi}^{--}(x - x') = [D_{\phi,\chi}^{++}(x - x')]^\ast$, $D_{\phi,\chi}^{+-}(x - x') = [D_{\phi,\chi}^{--}(x - x')]^\ast$. Here $f_B(\omega) = 1/(e^{\omega/T} - 1)$ is the Bose-Einstein distribution and the thermal masses are

$$\begin{align*}
M_{\phi}^2 &= m_{\phi}^2 - \frac{i\lambda_{\phi}}{2}, \\
M_{\chi}^2 &= m_{\chi}^2 - \frac{i\lambda_{\chi}}{2},
\end{align*} \quad (14a) \quad \begin{align*}
\frac{M_{\phi}^2}{24} &= m_{\phi}^2 + \frac{(\lambda_{\phi} + g)T^2}{24}, \\
\frac{M_{\chi}^2}{24} &= m_{\chi}^2 + \frac{(\lambda_{\chi} + g)T^2}{24}. \quad (14b)
\end{align*}$$
where the last step in each equation is only satisfied in the high-temperature limit. The various self-energies and proper four-vertex functions read

\[
\Pi_{AB}(x - x') = -i(AB)\lambda^2_\phi \frac{1}{6} \left( (x,A) \bigodot (x',B) - (x,A) \bigodot (x',B) \right),
\]

\[
V_{AB}(x - x') = -3i(AB)\lambda^2_\phi \frac{1}{2} \left( (x,A) \bigodot (x',B) - (x,A) \bigodot (x',B) \right).
\]

The retarded self-energy and retarded proper four-vertex function are then defined as

\[
\Pi_R(x - x') = \Pi^{++}(x - x') + \Pi^{+-}(x - x') = \theta(t-t') \left[ -\Pi^{++}(x - x') + \Pi^{+-}(x - x') \right],
\]

\[
V_R(x - x') = V^{++}(x - x') + V^{+-}(x - x') = \theta(t-t') \left[ -V^{++}(x - x') + V^{+-}(x - x') \right].
\]

Since the condensate is homogeneous, we have the spatial translation symmetry. We therefore define

\[
\pi_R(t - t') = \int d^3x \, \Pi_R(x - x') , \quad v_R(t - t') = \int d^3x \, V_R(x - x').
\]

As we shall see, it is the Fourier transforms of the retarded self-energy and four-vertex function evaluated at specific frequencies that will appear in the condensate evolution, see Eqs. (42) and (44) below.

It is usually difficult to obtain closed-form expressions for these Fourier transforms. For the model we consider, some estimates and discussions on them can be found in, e.g., Refs. [78–81]. The imaginary part of the proper four-vertex is known exactly [78],

\[
\text{Im} \left[ \tilde{v}_R(\omega) \right] = -\theta \left( \omega^2 - 4M^2_\phi \right) \frac{3\lambda^2_\phi}{32\pi} \sqrt{1 - \frac{4M^2_\phi}{\omega^2}} \left[ 1 + 2f_B \left( \frac{\omega}{2} \right) \right] - \theta \left( \omega^2 - 4M^2_\chi \right) \frac{3g^2}{32\pi} \sqrt{1 - \frac{4M^2_\chi}{\omega^2}} \left[ 1 + 2f_B \left( \frac{\omega}{2} \right) \right].
\]

In the limit of \( T = 0 \),

\[
\text{Im} \left[ \tilde{v}_R(\omega) \right] = -\theta \left( \omega^2 - 4m^2_\phi \right) \frac{3\lambda^2_\phi}{32\pi} \sqrt{1 - \frac{4m^2_\phi}{\omega^2}} - \theta \left( \omega^2 - 4m^2_\chi \right) \frac{3g^2}{32\pi} \sqrt{1 - \frac{4m^2_\chi}{\omega^2}}.
\]

The real part of the proper four-vertex has been studied in, e.g., Ref. [79] and can be calculated from the imaginary part via the Kramers–Kronig relation [82]

\[
\text{Re} \left[ \tilde{v}_R(\omega) \right] = \mathcal{P} \int \frac{d\omega'}{\pi} \frac{\text{Im} \left[ \tilde{v}_R(\omega') \right]}{\omega' - \omega},
\]

where \( \mathcal{P} \) denotes the Cauchy principal value. The self-energy corresponding to the “setting sun” diagrams is much more complicated. Closed-form expressions for the self-energy \( \tilde{\pi}_R(\omega) \), to our best of knowledge, are not known in the literature. In a work that is still in progress, we are
trying to obtain closed-form expressions for the imaginary and real parts of the self-energy. We have obtained a closed-form expression for \( \text{Im}[\pi_R(\omega)] \) with only the contribution from the first diagram in Eq. (15a) when \( \omega = M_\phi \) (this is the only relevant case, see Eq. (44)) \cite{83},

\[
\text{Im}[\pi_{R,\phi}(M_\phi)] = -\frac{\lambda^2_\phi T^2}{128\pi^3} \text{Li}_2 \left( e^{-M_\phi/T} \right),
\]

where \( \text{Li}_2(z) \) is the dilogarithm function defined as

\[
\text{Li}_2(z) = -\int_0^1 \frac{dx}{x} \ln(1 - z x),
\]

for \( z \in (-\infty, 1) \). The dilogarithm function has the following asymptotic behaviors

\[
\begin{align*}
\text{Li}_2(e^{-M_\phi/T}) &= \frac{\pi^2}{6} + O \left[ \frac{M_\phi}{T} \ln(M_\phi/T) \right] \quad \text{for } T \gg M_\phi, \\
\text{Li}_2(e^{-M_\phi/T}) &= e^{-M_\phi/T} + O \left( e^{-2M_\phi/T} \right) \quad \text{for } T \ll M_\phi.
\end{align*}
\]

The high-temperature limit of Eq. (21) is consistent with the result given in Refs. [80, 81]. For \( T \rightarrow 0 \) the imaginary part of the self-energy evaluated at \( \omega = M_\phi \) vanishes. This is natural because the imaginary part has the optical-theorem interpretation \cite{59} and at zero temperature, the particles in the loop in the diagrams of Eq. (15a) (more correctly, of Eq. (60) below) cannot be on-shell when the external particle has four-momentum \((M_\phi, 0)\). For more discussions on the microscopic interpretation of the condensate dissipation, see Sec. 3.1.2.

Apparently, the quantities in Eq. (44) depend on the plasma temperature. In this work, to avoid applying uncertain values of the quantities in Eq. (44), we will simply take them as input parameters when looking into the behaviors of the condensate evolution. This also assumes that the temperature is fixed and thus effects from an evolving temperature (see, e.g., Refs. [84–87]) are neglected. Although these effects are important in studying perturbative reheating, it is only possible to self-consistently track the evolution of the temperature until the exact dependence on \( T \) in all the Fourier transforms in Eq. (44) are known. We thus leave a more dedicate study of perturbative reheating applying our theoretical methods developed here for future work \cite{83}.

In an expanding universe, one in principle needs to rederive the condensate equation of motion following Ref. [46]. However, this would make the problem too complicated. For example, because of the time-dependent metric, the free thermal equilibrium propagators would take a time-translation-variant form, depending on the specific form of \( a(t) \). Below, we shall assume that the microscopic time scales are much smaller than the Hubble time so that, one can assume a flat spacetime background in the non-local terms in Eq. (11) but only take into account the effects from the expanding universe in the classical part. Therefore, we will use the following equation of motion for the condensate \( \varphi \),

\[
\ddot{\varphi}(t) + M_\phi^2 \varphi(t) + 3H \dot{\varphi}(t) + \frac{\lambda_\phi \varphi^3(t)}{6} + \int_{t_i}^t dt' \pi_R(t - t') \varphi(t') + \frac{\varphi(t)}{6} \int_{t_i}^t dt' v_R(t - t') \varphi^2(t') = 0. 
\]

We shall note that all the masses and couplings in the above equation are assumed to be the renormalized ones. In the next section, we solve Eq. (25) using the multi-scale analysis \cite{46, 48}. 

9
3  Solving the condensate equation of motion

Following [46], we assume that the last three terms in Eq. (25) are small compared with the mass term and the $\ddot{\varphi}(t)$ term. We will also assume that the Hubble friction term proportional to the Hubble constant is small compared with the first two terms. In a theoretical viewpoint, this assumption sets the range of validity of our results. However, for realistic applications in the perturbative reheating process after preheating, the Hubble friction term can be shown to be small as follows. Typically, after preheating the Universe is filled with a high-temperature plasma and can be well assumed to be radiation dominated. Therefore, the Hubble constant is

$$H^2 \approx \frac{8\pi}{3M_{\text{pl}}^2} \frac{g_* \pi^2}{30} T^4, \quad (26)$$

where $M_{\text{pl}}$ is the Planck mass and $g_*$ is the number of relativistic degrees of freedom. $g_*$ is model-dependent but to have an estimate we can take it to be $O(100)$. On the other hand, at high temperature $M_\phi$ can be approximated as

$$M_\phi^2 \approx \frac{(\lambda_\phi + g)T^2}{24}. \quad (27)$$

The first and second terms in the equation of motion are $\sim M_\phi^2 ||\varphi||$ where $||\varphi||$ denotes the oscillation amplitude, while the Hubble friction term is

$$H\dot{\varphi} \sim \frac{M_\phi}{M_{\text{pl}}(\lambda_\phi + g)} M_\phi^2 ||\varphi||. \quad (28)$$

Therefore for mildly small couplings, the Hubble friction term is suppressed by a factor of $M_\phi/M_{\text{pl}}$ compared with the first and second terms. Now we solve Eq. (25) step by step.

First, for bookkeeping purposes, all small terms will be multiplied by a parameter $\varepsilon$. The equation of motion for the condensate $\varphi$ then reads

$$\ddot{\varphi}(t) + M_\phi^2 \varphi(t) + 3\varepsilon H\dot{\varphi}(t) + \varepsilon \frac{\lambda_\phi \varphi^3(t)}{6} + \varepsilon \int_{t_i}^{t} dt' \pi_R(t-t')\varphi(t') + \varepsilon \frac{\varphi(t)}{6} \int_{t_i}^{t} dt' v_R(t-t')\varphi^2(t') = 0. \quad (29)$$

Further, we expect that there is a hierarchy for the time scales in the evolution of the condensate. The shorter time scale corresponds to the oscillation frequency $1/M_\phi$, and the longer time scale corresponds to the damping time scale. Therefore, we introduce two time variables, $t$ and $\tau = \varepsilon t$ [46]. Introducing a slow time variable $\tau$ allows us to correctly arrange terms in a perturbative expansion when there are terms with time derivatives. Once the perturbative calculation is done at a particular order, we will take $\varepsilon = 1$ finally and then $\tau$ becomes the common physical time. The solution for the condensate $\varphi$ then takes the following form

$$\varphi(t) = \varphi(t, \tau; \varepsilon). \quad (30)$$
With this assumption, we are able to organize (29) in powers of $\varepsilon$ as
\[
\frac{\partial^2 \varphi(t, \tau; \varepsilon)}{\partial t^2} + 2\varepsilon \frac{\partial^2 \varphi(t, \tau; \varepsilon)}{\partial t \partial \tau} + \varepsilon^2 \frac{\partial^2 \varphi(t, \tau; \varepsilon)}{\partial \tau^2} + M_φ^2 \varphi(t, \tau; \varepsilon)
\]
\[+ 3\varepsilon H(\tau) \frac{\partial \varphi(t, \tau; \varepsilon)}{\partial t} + 3\varepsilon^2 H(\tau) \frac{\partial \varphi(t, \tau; \varepsilon)}{\partial \tau} + \varepsilon \frac{\lambda_φ}{6} \varphi^3(t, \tau; \varepsilon)
\]
\[+ \varepsilon \int_{t_i}^{t} dt' \pi_R(t - t') \varphi(t', \tau'; \varepsilon) + \varepsilon \frac{\varphi(t, \tau; \varepsilon)}{6} \int_{t_i}^{t} dt' v_R(t - t') \varphi^2(t', \tau'; \varepsilon) = 0,
\]
where we have assumed that the Hubble parameter depends only on the slow time, i.e.,
\[H = H(\tau).\]
(32)

This does not necessarily mean that the Hubble time scale is the same as the damping time scale. It can be the case where the former is longer than latter.

Second, one can assume a power series solution of the form
\[\varphi(t, \tau; \varepsilon) = \varphi_0(t, \tau) + \varepsilon \varphi_1(t, \tau) + \varepsilon^2 \varphi_2(t, \tau) + \ldots,
\]
(33)
where
\[\varphi_n(t, \tau) = \left. \frac{1}{n!} \frac{\partial^n \varphi(t, \tau; \varepsilon)}{\partial \varepsilon^n} \right|_{\varepsilon=0}
\]
are the coefficients. We aim to obtain the lead-order solution $\varphi_0$.

Third, to simplify the last two terms on the LHS of Eq. (31), we assume that the solution for the condensate $\varphi$ varies only slightly during the window provided by the kernel of the non-local terms. With this assumption, in the integrand of the non-local terms, we can Taylor-expand $\varphi(t', \tau'; \varepsilon)$ at $\tau$ as [46]
\[\varphi(t', \tau'; \varepsilon) = \varphi(t', \tau; \varepsilon) + \varepsilon(t' - \tau) \frac{\partial \varphi(t', \tau'; \varepsilon)}{\partial \tau} + \varepsilon^2 \frac{(t' - \tau)^2}{2} \frac{\partial^2 \varphi(t', \tau'; \varepsilon)}{\partial \tau^2} + \ldots \]
(35)
where we have used the equality $\tau^{(i)} = \varepsilon t^{(i)}$. Then, up to the second order in $\varepsilon$, the non-local terms in the LHS of Eq. (31) could be written as
\[\varepsilon \int_{t_i}^{t} dt' \pi_R(t - t') \varphi(t', \tau'; \varepsilon)
\]
\[= \varepsilon \int_{t_i}^{t} dt' \pi_R(t - t') \varphi(t', \tau; \varepsilon) - \varepsilon^2 \int_{t_i}^{t} dt' (t - t') \pi_R(t - t') \frac{\partial \varphi(t', \tau; \varepsilon)}{\partial \tau} + O(\varepsilon^3),
\]
(36a)
and
\[\varepsilon \frac{\varphi(t, \tau; \varepsilon)}{6} \int_{t_i}^{t} dt' v_R(t - t') \varphi^2(t', \tau'; \varepsilon)
\]
\[= \varepsilon \frac{\varphi(t, \tau; \varepsilon)}{6} \int_{t_i}^{t} dt' v_R(t - t') \varphi^2(t', \tau; \varepsilon)
\]
\[+ \varepsilon^2 \frac{\varphi(t, \tau; \varepsilon)}{3} \int_{t_i}^{t} dt' (t - t') v_R(t - t') \varphi(t', \tau; \varepsilon) \frac{\partial \varphi(t', \tau; \varepsilon)}{\partial \tau} + O(\varepsilon^3).
\]
(36b)
Fourth, with Eq. (33) and Eq. (36), one can organize Eq. (31) in powers of \( \varepsilon \). The leading order is given by

\[
\frac{\partial^2 \varphi_0(t, \tau)}{\partial t^2} + M^2 \varphi_0(t, \tau) = 0,
\]

which is a harmonic oscillator equation. The subleading equation is

\[
\frac{\partial^2 \varphi_1(t, \tau)}{\partial t^2} + M^2 \varphi_1(t, \tau) = -2 \frac{\partial^2 \varphi_0(t, \tau)}{\partial t \partial \tau} - 3H(\tau) \frac{\partial \varphi_0(t, \tau)}{\partial t} - \frac{\lambda_\phi}{6} \varphi_0^3(t, \tau) - \int_{t_i}^t dt' \pi_R(t - t') \varphi_0(t', \tau) - \frac{\varphi_0(t, \tau)}{6} \int_{t_i}^t dt' v_R(t - t') \varphi_0^2(t', \tau).
\]

The solution for the harmonic oscillator equation (37) has the form

\[
\varphi_0(t, \tau) = \text{Re} \left[ R(\tau)e^{-iM_\phi t} \right].
\]

The integrals on the RHS of Eqs. (40) and (41) have the same form as Fourier transforms except for the upper and lower limits. Following Ref. [46], we make the approximation of neglecting any early-time transient effects due to initial conditions in the non-local terms. This allows us to take \( t_i \to -\infty \), eliminating the explicit finite \( t_i \) dependence appearing in the lower limit of the integrals. On the other hand, the appearance of the Heaviside step function in the definitions of the retarded self-energy and the retarded four-vertex function (cf., Eq. (16)) allows us to replace \( t \) by \( +\infty \) in the upper limit of the integrals.

So, we could actually recognize the integrals on the RHS of Eqs. (40) and (41) as Fourier
transforms. More explicitly, we have

$$\int_{t_i}^{t} dt' \tau_R(t - t') e^{iM_\phi(t-t')} \approx \int_{-\infty}^{+\infty} dt' \tau_R(t - t') e^{iM_\phi(t-t')} = \tilde{\tau}_R(M_\phi),$$

(42a)

$$\int_{t_i}^{t} dt' v_R(t - t') \approx \int_{-\infty}^{+\infty} dt' v_R(t - t') e^{i0(t-t')} = \tilde{v}_R(0),$$

(42b)

$$\int_{t_i}^{t} dt' v_R(t - t') e^{2iM_\phi(t-t')} \approx \int_{-\infty}^{+\infty} dt' v_R(t - t') e^{2iM_\phi(t-t')} = \tilde{v}_R(2M_\phi).$$

(42c)

Therefore, the subleading equation (38) now reads

$$\frac{\partial^2 \phi_1(t, \tau)}{\partial t^2} + M_\phi^2 \phi_1(t, \tau) = 2 \Re \left\{ iM_\phi e^{-iM_\phi t} \left[ \frac{d\tau_R(\tau)}{d\tau} + \frac{3}{2} H(\tau) R(\tau) + \frac{i\lambda_\phi}{16M_\phi} [R(\tau)]^2 R^*(\tau) \right. \right.$$

$$\left. + \frac{iR(\tau)}{2M_\phi} \tilde{\tau}_R(M_\phi) + \frac{i}{24M_\phi} \tilde{v}_R(0) [R(\tau)]^2 R^*(\tau) + \frac{i}{48M_\phi} \tilde{v}_R(2M_\phi) [R(\tau)]^2 R^*(\tau) \right\}$$

$$+ 2 \Re \left\{ iM_\phi e^{-i3M_\phi t} \left[ \frac{i\lambda_\phi}{48M_\phi} + \frac{i}{48M_\phi} \tilde{v}_R(2M_\phi) \right] [R(\tau)]^3 \right\}. \tag{43}$$

Defining

$$\mu \equiv \frac{\Re[\tilde{\tau}_R(M_\phi)]}{2M_\phi}, \quad \gamma \equiv -\frac{\Im[\tilde{\tau}_R(M_\phi)]}{2M_\phi}, \quad \sigma \equiv -\frac{\Im[\tilde{v}_R(2M_\phi)]}{24M_\phi},$$

$$\alpha_1 \equiv \frac{\Re[\tilde{v}_R(0)]}{24M_\phi}, \quad \alpha_2 \equiv \frac{\Re[\tilde{v}_R(2M_\phi)]}{48M_\phi}, \quad \alpha \equiv \alpha_1 + \alpha_2, \tag{44}$$

Eq. (43) can be written as

$$\frac{\partial^2 \phi_1(t, \tau)}{\partial t^2} + M_\phi^2 \phi_1(t, \tau) = 2 \Re \left\{ iM_\phi e^{-iM_\phi t} \left[ \frac{d\tau_R(\tau)}{d\tau} + \left( \frac{3}{2} H(\tau) + i\mu + \gamma \right) R(\tau) \right. \right.$$

$$\left. + \left( \frac{i\lambda_\phi}{16M_\phi} + i\alpha + \frac{\sigma}{2} \right) [R(\tau)]^2 R^*(\tau) \right\}$$

$$+ 2 \Re \left\{ iM_\phi e^{-i3M_\phi t} \left( \frac{i\lambda_\phi}{48M_\phi} + i\alpha_2 + \frac{\sigma}{2} \right) [R(\tau)]^3 \right\}, \tag{45}$$

where we used

$$\Im[\tilde{v}_R(0)] = 0, \tag{46}$$

which is due to that the imaginary part of the Fourier-transformed proper four-vertex is anti-symmetric.

Finally, we determine $R(\tau)$ from Eq. (45). We have organized the equation of motion for $\phi_1(t, \tau)$ in such a way that it describes the evolution of a harmonic oscillator $\phi_1(t, \tau)$ driven by
two oscillating forces. One force described by the first two lines on the RHS of Eq. (45) has a
frequency $M_\phi$, which is the same as the natural frequency of $\varphi_1(t, \tau)$. The other force given by
the last line has a frequency $3M_\phi$. The second force just modifies the original oscillating behavior
by adding different oscillations with constant amplitudes. However, it is well known that the first
force can lead to a resonant behavior where the final amplitude of $\varphi_1(t, \tau)$ would be proportional
to the time $t$, i.e., the amplitude of $\varphi_1(t, \tau)$ will increase without bound. As a standard procedure
of the multi-scale analysis [48], to avoid the non-physical spurious resonances we require that

$$\frac{dR(\tau)}{d\tau} + \left(\frac{3}{2} H(\tau) + i\mu + \gamma\right) R(\tau) + \left(\frac{i\lambda_\phi}{16M_\phi} + i\alpha + \frac{\sigma}{2}\right) [R(\tau)]^2 R^*(\tau) = 0 \quad (47)$$

such that the first force vanishes. Making the following Ansatz

$$R(\tau) = A(\tau) e^{-i f(\tau)}, \quad (48)$$

where $A(\tau)$ and $f(\tau)$ are real functions of $\tau$, we can split the complex equation (47) into the
following two real equations,

$$\frac{dA(\tau)}{d\tau} + \left(\gamma + \frac{3}{2} H(\tau)\right) A(\tau) + \frac{\sigma}{2} [A(\tau)]^3 = 0 \quad (49a)$$

$$\frac{df(\tau)}{d\tau} - \mu - \left(\frac{\lambda_\phi}{16M_\phi} + \alpha\right) [A(\tau)]^2 = 0 \quad (49b)$$

The solution for $A(\tau)$ depends on $\gamma$, $H(\tau)$ and $\sigma$. Other than these three parameters, the solution
for $f(\tau)$ also depends on $\mu$, $\lambda_\phi$, and $\alpha$.

To be general, we parameterize the Hubble constant as

$$H(\tau) = \frac{\zeta}{\tau}, \quad (50)$$

where $\zeta$ is a non-negative real number. Typical universes with different $\zeta$ will be discussed case
by case below. The solution of $A(\tau)$ can be written as

$$A(\tau) = \frac{\sqrt{2}}{e^{\pi(2\tau)} \zeta \sqrt{-\gamma^{3\zeta - 1}\sigma \Gamma(1 - 3\zeta, 2\gamma\tau) + c_0}}, \quad (51)$$

where $c_0$ is a constant of integration and $\Gamma(a, x)$ is the incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^{+\infty} dy e^{-y} y^{a-1}. \quad (52)$$

The leading approximation for the solution of Eq. (25) then reads

$$\varphi(t) \approx \varphi_0(t, t) = \frac{\sqrt{2}}{e^{\pi t} (2t)^{\frac{3\zeta}{2}} \sqrt{-\gamma^{3\zeta - 1}\sigma \Gamma(1 - 3\zeta, 2\gamma t) + c_0}} \times \cos [f(t) + M_\phi t]. \quad (53)$$
The solution of \( f(t) \) cannot be expressed by fundamental functions for unspecified \( \zeta \). It is already clear from Eq. (53) that the expansion of the universe \((\zeta > 0)\) induces a power-law damping behavior for the condensate. The leading approximation of the energy density for the condensate \( \varphi \) is given by

\[
\rho_\varphi = \frac{1}{2} \dot{\varphi}_0^2 + \frac{1}{2} M^2_\phi \varphi_0^2. \tag{54}
\]

This quantity will be used for the discussion of particle production. In what follows, we shall study the solution Eq. (53) in different situations.

## 3.1 Static universe

Taking \( \zeta = 0 \) gives a static universe. For future use, we discuss separately the cases when \( \gamma \) (the imaginary part of the self-energy) is negligible and when not. The reason for doing this is that \( \gamma \) and \( \sigma \) have very different effects on the evolution of the condensate in an expanding universe. In flat spacetime or a static universe, regardless of whether \( \gamma \) is vanishing or not, the decay of the condensate is always complete. We will see that the conclusion is completely different in an expanding universe. As \( \gamma \) is suppressed in the low temperature (see Eqs. (21) and (24)) while \( \sigma \) is nonvanishing even at zero temperature, there could be some range for the temperature in which \( \gamma \) is negligible. In such a case, we simply take \( \gamma = 0 \).

### 3.1.1 Non-negligible \( \gamma \)

We first consider the case of non-negligible \( \gamma \). In this case, we have the solution of \( A(t) \),

\[
A(t) = \frac{A_0 e^{-\gamma t}}{\sqrt{1 + \frac{\sigma A_0^2}{2\gamma}(1 - e^{-2\gamma t})}}, \tag{55}
\]

where we have used the identity \( \Gamma(1, x) = e^{-x} \) and \( A_0 \) is related to \( c_0 \) via \( 1/A_0^2 + \sigma/(2\gamma) = c_0/2 \).

The solution of \( f(t) \) reads

\[
f(t) = f_0 + \mu t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln \left[ 1 + \frac{\sigma A_0^2}{2\gamma}(1 - e^{-2\gamma t}) \right]}{16M_\phi \sigma}. \tag{56}
\]

The constants of integration are determined by the initial conditions for the condensate. Note that our solution given by Eqs. (55) and (56) agrees with the result in Ref. [46].

The energy density of the oscillating scalar field can be approximated as

\[
\rho_\varphi \approx \frac{1}{2} \left[ \dot{A}(t) \cos(f + M_\phi t) - A(t) \sin(f + M_\phi t) \right] + \frac{1}{2} M^2_\phi \dot{A}(t)[\cos(f + M_\phi t)]^2 \\
\approx \frac{1}{2} M^2_\phi [A(t)]^2 \approx \frac{1}{2} \frac{A_0^2 M^2_\phi}{2} \cdot \frac{e^{-2\gamma t}}{1 + \frac{\sigma A_0^2}{2\gamma}(1 - e^{-2\gamma t})}. \tag{57}
\]

See Eq.(3.92) of that reference. Note that the \( \gamma \) defined in our paper is one half of that in Ref. [46].
To get the second line in Eq. (57), we have used the assumption that

\[ \frac{\dot{A}}{A} \ll M_\phi, \quad \frac{df}{dt} \ll M_\phi, \quad (58) \]

which implies that the amplitude and the frequency of oscillations do not change much during one oscillation. From Eq. (57), we see that the energy density of the condensate experiences first power-law damping \([1/(1 + \sigma A_0^2 t)]\) at early times and then dominantly exponential damping at late times. More explicitly, we have

\[ \frac{1}{\rho_\varphi} \frac{d\rho_\varphi}{dt} \approx -2\gamma - \sigma [A(t)]^2, \quad (59) \]

from which we observe that the decay rate of the energy density \(\rho_\varphi\) can be separated into two parts; one is a constant proportional to \(\gamma\) and the other one is proportional to \(\sigma\) which falls off as \([A(t)]^2\). At late times, the decay rate is dominated by the constant term.

### 3.1.2 Microscopic interpretation

The damping of the condensate oscillations and the decrease of its energy density can be interpreted as particle production. Actually, the self-energy corresponds to the following diagrams in the effective action,

\[ \text{\begin{tikzpicture}
        \draw (-0.5,0) -- (0.5,0);
        \draw (0,0) circle (0.1);
        \filldraw[black] (0,0) circle (0.05);
    \end{tikzpicture}} \quad \text{\begin{tikzpicture}
        \draw (-0.5,0) -- (0.5,0);
        \draw (0,0) circle (0.1);
        \filldraw[black] (0,0) circle (0.05);
        \draw (0,0) circle (0.05);
        \end{tikzpicture}} \quad (60) \]

where a line ended with a wheel cross denotes the scalar background \(\varphi\). If \(\varphi\) is a perturbative particle, the cutting rules [88–94] suggest that processes contributing to a non-vanishing imaginary part of the retarded self-energy consist of \(\varphi \leftrightarrow \phi\phi\), \(\varphi \leftrightarrow \phi\chi\) and rearrangements thereof (e.g., \(\varphi\chi \leftrightarrow \phi\chi\)). However, for the condensate there are some subtleties.\(^9\) First, the condensate quanta have exactly zero momentum and their energy are equal to the mass of \(\phi\)-particles, i.e., every \(\varphi\) quantum has four-momentum \((M_\phi, 0)\). Thus some processes cannot satisfy the on-shell conditions. Apparently, \(\varphi \leftrightarrow \phi\phi\), \(\varphi \leftrightarrow \phi\chi\) are not kinematically possible. However, the processes \(\varphi\chi \leftrightarrow \phi\), \(\phi\chi \leftrightarrow \chi\), \(\varphi\phi \leftrightarrow \phi\) also cannot be on-shell.\(^10\) Second, the condensate has zero entropy and it is unlikely to spontaneously produce quanta of the condensate in the interactions in the plasma. Therefore we suspect that the condensate quanta can only appear as parent “particles”. Put in another way, producing an exactly zero-momentum \(\Phi\)-particle is seldom and can be neglected in the whole phase space. The processes contributing to the \(\gamma\) dissipation are therefore

\[ \text{\textbf{\gamma channels : \quad \varphi\phi} \rightarrow \phi\phi, \quad \varphi\chi \rightarrow \phi\chi, \quad \varphi\phi \rightarrow \chi\chi. \quad (61) \]

\(^9\)These subtleties have been overlooked in Ref. [46] and the microscopic interpretation given there should be taken with care.

\(^{10}\)A detailed derivation for the self-energy (e.g., Eq. (21)) that makes this statement more apparent will be given in Ref. [83].
These are the condensate decay channels with one condensate mode. We call them the $\gamma$ channels. These processes are possible at finite temperature because the plasma contains many $\chi$- and $\phi$-particles. The damping caused by them are called Landau damping [82]. The $\gamma$ channels are absent at zero temperature.

The diagrams in the effective action corresponding to the proper four-vertex function are

$$\begin{array}{c}
\includegraphics[width=0.3\textwidth]{diagram1.png} \\
\includegraphics[width=0.3\textwidth]{diagram2.png}
\end{array}$$

Similarly, the cutting rules suggest that processes contributing to the $\sigma$ dissipation coefficient (due to a non-vanishing imaginary part of the retarded proper four-vertex function) consist of

$$\sigma \text{ channels : } (\varphi\varphi) \rightarrow \phi\phi, \quad (\varphi\varphi) \rightarrow \chi\chi.$$  

Note that the processes $(\varphi\varphi)\phi \rightarrow \phi$, $(\varphi\varphi)\chi \rightarrow \chi$ cannot satisfy the on-shell conditions because the $\phi$ particles (or $\chi$ particles) on the LHS and RHS would have the same momentum but different energy and thus cannot be on-shell simultaneously. The processes in Eq. (63) are the decay channels with two condensate modes and we call them the $\sigma$ channels. They survive at zero temperature.

### 3.1.3 Negligible $\gamma$

Now we consider the case where the imaginary part of the self-energy is negligible. This would mean that we take $\gamma = 0$ in Eq. (49) ($\mu$ in general is not vanishing). Then we obtain the following solutions for $A(t)$ and $f(t),$

$$A(t) = \frac{1}{\sqrt{\sigma t + c_0}},$$

$$f(t) = f_0 + \mu t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln(c_0 + \sigma t)}{16M_\phi \sigma},$$

where $c_0$ and $f_0$ are constants of integration. Here and in what follows, $c_0$ and $f_0$ in the solutions for different situations are not necessarily identical. An example of this solution is shown by the brown line in Fig. 3.

The energy density of the oscillating scalar field can be approximated as

$$\rho_\varphi \approx \frac{1}{2} M_\phi^2 |A(t)|^2 \approx \frac{M_\phi^2}{2(\sigma t + c_0)}.$$

where again we have assumed that the amplitude and the frequency of oscillations do not change much during one oscillation. The energy density of the condensate goes to 0 when $t \to \infty$, see Fig. 2. The energy transfer from the condensate to the produced particles is efficient even if there are only the $\sigma$ channels and no the $\gamma$ channels. However, as we shall see shortly, the picture will be totally different if an expanding universe is considered.
3.2 Radiation-dominated universe

Let us now discuss a radiation-dominated universe for which $\zeta = 1/2$. This is the most relevant situation as we are interested in the radiation-dominated period after preheating. To have a comprehensive understanding of particle production in an expanding universe, below we discuss three different cases.

3.2.1 No interactions

The evolution of the condensate without interactions in a radiation-dominated universe can be obtained by setting $\gamma = \sigma = \lambda_\phi = \mu = \alpha = 0$ and $\zeta = 1/2$. Then, the leading approximation for the solution of $\varphi(t)$ is given by

$$\varphi(t) \approx \frac{1}{2^{1/4} \sqrt{c_0}} t^{-3/4} \cos (f_0 + M_\phi t) , \quad (66)$$

where $c_0$ and $f_0$ are constants of integration. In this situation, the oscillation amplitude decreases as $[a(t)]^{-3/2}$ with $a(t) \propto \sqrt{t}$. This damping is simply due to the expansion of the universe. The total energy of the condensate in a given physical volume $V$ is given by

$$\rho_\varphi V \approx \frac{V}{2} M_\phi^2 A^2 . \quad (67)$$
Note that in a spatially flat FLRW universe, the physical volume is given by \( V = a^3 V_c \) where \( V_c \) is the constant comoving volume. Hence, the total energy of the condensate is reflected by the quantity \( a^3 \rho_\phi \equiv \tilde{\rho}_\phi \) which will be called the comoving energy density in this paper. Since \( A(t) \propto [a(t)]^{-3/2} \), we have \( \rho_\phi \propto a^{-3} \) and

\[
\tilde{\rho}_\phi \equiv a^3 \rho_\phi = \text{constant},
\]

which implies that the total energy of the condensate is conserved.

Below, we shall discuss situations with interactions in which particle production would occur and the condensate comoving energy density is not conserved anymore. We will assume that the produced particles are relativistic (this could be the case if the plasma temperature is high compared to all the thermal masses). Let \( \rho_{pp} \) denote the energy density of the produced particles. Then the total energy for the produced particles is characterized by \( \tilde{\rho}_{pp} \equiv a^4 \rho_{pp} \). Without interactions, \( \tilde{\rho}_{pp} \) would be a conserved quantity. With interactions, we have the following relation between \( \tilde{\rho}_{pp} \) and \( \tilde{\rho}_\phi \),

\[
\dot{\rho}_{pp} + 4H \rho_{pp} = - (\dot{\rho}_\phi + 3H \rho_\phi) \quad \Rightarrow \quad \frac{d\tilde{\rho}_{pp}}{dt} = -a \frac{d\tilde{\rho}_\phi}{dt}. \tag{69}
\]

Thus knowing the evolution of the condensate comoving energy density \( \tilde{\rho}_\phi \), one can integrate the above equation to obtain \( \tilde{\rho}_{pp} \). To avoid assuming some constants in \( a(t) \) for different universes, below when discussing the energy transfer from the condensate to the produced particles, we simply compare the condensate comoving energy density with the energy loss defined as

\[
\tilde{\rho}_{\text{loss}}(t) \equiv \tilde{\rho}_\phi(t_i) - \tilde{\rho}_\phi(t). \tag{70}
\]

Note that the amplitude \( A(t) \) in Eq. (66) is divergent at \( t \to 0^+ \), which comes from the divergent behaviour of the Hubble parameter at \( t \to 0^+ \). This is not surprising since the time \( t \) in Eq. (66) is the cosmic time and \( t \to 0^+ \) corresponds to the so-called big-bang singularity. Thus, the solution for the condensate in an expanding universe should be applied for times larger than zero.

### 3.2.2 Negligible \( \gamma \)

If we take \( \gamma = 0 \), the solutions of \( A(t) \) and \( f(t) \) are given by\(^{11}\)

\[
A(t) = \frac{1}{\sqrt{c_0 t^{3/2} - 2\sigma t}}, \tag{71a}
\]

\[
f(t) = f_0 + \mu t - \frac{(\lambda_\phi + 16\alpha M_\phi)}{32 M_\phi \sigma} \left[ \ln(t) - 2 \ln \left( c_0 \sqrt{t} - 2\sigma \right) \right], \tag{71b}
\]

where \( c_0 \) and \( f_0 \) are constants of integration.

\(^{11}\)The relation \( \Gamma(-1/2, x) \to 2/\sqrt{x} \) as \( x \to 0 \) is needed if one derives the solution for \( A(t) \) by taking the limit \( \gamma \to 0 \) in Eq. (53).
Figure 3: Solutions of the condensate equation (25) with the imaginary part of the self-energy neglected ($\gamma = 0$). Analytic approximation in a radiation-dominated universe, i.e., Eqs. (71), is drawn in the red line with $\alpha = -0.01/M_\phi$, $\sigma = 0.005/M_\phi$, $\mu = 0.001M_\phi$, and $\lambda = 0.05$. Analytic approximation in a static universe with the same parameters, given by Eqs. (64), is drawn in the brown line. Analytic approximation without interactions in a radiation-dominated universe, i.e., Eq. (66), is drawn in the purple line.

The leading approximation for the condensate evolution is plotted in Fig. 3, which shows that the damping of the oscillations is dominated by the expansion of the universe. This observation suggests that reheating in this situation may not be complete. Indeed, the comoving energy density of the condensate decreases as

$$a^3 \rho_\phi \propto \frac{1}{c_0 - \frac{2\sigma}{\sqrt{t}}},$$

which approaches $1/c_0$ as $t \to \infty$. Different from the case of a complete decay in a static universe (cf. Eq. (57)), Eq. (72) implies that, due to the expansion of the universe, the condensate will never decay completely if the $\gamma$ channels are absent (see Fig. 4 and compare it with Fig. 2). The remnant energy stored in the condensate depends highly on the initial conditions. For $t \gg t_i$, one has

$$\rho_\phi(t) \approx \frac{t_i^{3/2} \rho_\phi(t_i)}{t^{3/2}} \left(1 - \frac{4\sigma t_i \rho_\phi(t_i)}{M_\phi^2 + 4\sigma t_i \rho_\phi(t_i)}\right).$$
Figure 4: Energy transfer from the condensate to the produced particles in a radiation-dominated universe without the self-energy correction. The evolution of the condensate comoving energy density and of the energy loss are drawn in the red and blue lines, respectively. Compared with Fig. 2, the energy transfer from the condensate to the produced particles in a radiation-dominated universe is inefficient in absence of the $\gamma$ channels.

3.2.3 Non-negligible $\gamma$

Now let us consider the full condensate equation of motion (25). For $\zeta = 1/2$, the solution for the amplitude $A(t)$ in a radiation-dominated universe is given by

$$A(t) = \frac{e^{-\gamma t}}{2^{1/4} t^{3/4} \sqrt{-\sigma \sqrt{\gamma} \Gamma \left(-\frac{1}{2}, 2\gamma t\right)} + c_0},$$

(74)

from which we get the solution for $f(t)$,

$$f(t) = f_0 + \mu t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln \left[-\sigma \sqrt{\gamma} \Gamma \left(-\frac{1}{2}, 2\gamma t\right) + c_0\right]}{16M_\phi \sigma}.$$  

(75)

At the leading approximation we thus have

$$\varphi(t) \approx \frac{e^{-\gamma t}}{2^{1/4} t^{3/4} \sqrt{-\sigma \sqrt{\gamma} \Gamma \left(-\frac{1}{2}, 2\gamma t\right)} + c_0} \times \cos \left\{ f_0 + (M_\phi + \mu)t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln \left[-\sigma \sqrt{\gamma} \Gamma \left(-\frac{1}{2}, 2\gamma t\right) + c_0\right]}{16M_\phi \sigma} \right\}.$$  

(76)

The solution (76) with different initial times are drawn in the red lines in Figs. 5 and 6. The earlier the condensate starts oscillating, the more important role the universe expansion plays in the damping.
Figure 5: Solutions of the condensate Eq. (25). Analytic approximation in a radiation-dominated universe, i.e., Eq. (76), is drawn in the red line with $\sigma = -\alpha = 0.01/M_\phi$, $\mu = 0.001M_\phi$, $\gamma = 0.005M_\phi$, and $\lambda_\phi = 0.05$. Analytic approximation in a static universe, given by Eqs. (55) and (56), is drawn in the brown line with the same parameters. Analytic approximation without interactions in a radiation-dominated universe, i.e., Eq. (66), is drawn in the purple line.

With the assumption Eq. (58), the energy density of the condensate is given by

$$\rho_\phi(t) \approx \frac{M_\phi^2}{2\sqrt{2}t^{3/2}} \cdot \frac{e^{-2\gamma t}}{c_0 - \sigma \sqrt{\Gamma(-\frac{1}{2}, 2\gamma t)}}. \quad (77)$$

We could also obtain the following equation for the comoving energy density of the condensate

$$\frac{1}{a^3 \rho_\phi} \frac{d(a^3 \rho_\phi)}{dt} \approx -2\gamma - \sigma [A(t)]^2. \quad (78)$$

The RHS of Eq. (78) has the same form as the RHS of Eq. (59), but with a different expression of $A(t)$.

To understand the late-time evolution of the condensate, we consider the limit $t \gg 1/\gamma$. Then

$$A(t) \approx \frac{t^{-3/4}}{21/4\sqrt{c_0}} e^{-\gamma t}, \quad (79)$$

and

$$f(t) \approx f_0 + \mu t + \frac{\lambda_\phi + 16\alpha M_\phi}{16M_\phi \sigma} \ln c_0, \quad (80)$$

where we have used the approximation that

$$\Gamma\left(-\frac{1}{2}, 2\gamma t\right) \approx e^{-2\gamma t}(2\gamma t)^{-3/2}. \quad (81)$$
The solution of the condensate $\varphi$ in the late-time limit is then given by
\[
\varphi(t) \approx \frac{t^{-3/4}}{2^{1/4} \sqrt{c_0}} e^{-\gamma t} \times \cos \left[ f_0 + (\mu + M_\phi) t + \frac{(\lambda_\phi + 16\alpha M_\phi)}{16M_\phi \sigma} \ln c_0 \right].
\] (82)

The comoving energy density of the condensate satisfies the following equation
\[
\frac{1}{a^3 \rho_\varphi} \frac{d(a^3 \rho_\varphi)}{dt} \approx -2\gamma \text{ for } t \gg 1/\gamma,
\] (83)
which indicates that, in the late-time limit, the comoving energy density $a^3 \rho_\varphi$ exponentially decreases with the decay rate $2\gamma$.

Comparison between the evolution of the (comoving) energy density in a static universe and in a radiation-dominated universe is given in Figs. 7 and 8. From Fig. 8, we can see that the difference, denoted as $\Delta \tilde{\rho}_\varphi(t)$, grows at early times and decreases at late times. The explanation is as follows. The cubic non-local correction plays an important role in the early decay of the (comoving) energy density, cf. the $\sigma A^2$ term in Eq. (59) and Eq. (78). Since the amplitude $A(t)$ decreases differently in a static and expanding universe, the (comoving) energy density also decay differently. At late times, their decay are both dominated by the constant $\gamma$ term, so the difference between them vanishes gradually.
3.3 Matter-dominated universe

Now we discuss the solution of the condensate in a matter-dominated universe \((\zeta = 2/3)\). In a matter-dominated universe and for processes whenever the Hubble expansion is relevant, the
temperature is typically negligibly small. In this case, $\gamma$ is vanishing and Landau damping through the processes in Eq. (63) is absent. However, for theoretical interest, we will still discuss the situation with non-vanishing $\gamma$ below.

### 3.3.1 No interactions

We first consider the evolution of a free massive scalar field in the background of a matter-dominated universe. This could be realized by setting $\gamma = \sigma = \lambda_\phi = \mu = \alpha = 0$ in Eq. (53) and taking $\zeta = 2/3$. Then we have

\begin{equation}
A(t) = \frac{1}{\sqrt{c_0} t}, \tag{84a}
\end{equation}

\begin{equation}
f(t) = f_0, \tag{84b}
\end{equation}

where $f_0$ is a constant of integration. In this case, the leading approximation for the solution of $\varphi(t)$ is given by

\begin{equation}
\varphi(t) \approx \frac{1}{\sqrt{c_0} t} \times \cos (f_0 + M_\rho t), \tag{85}
\end{equation}

which agrees with the result given in Ref. [95].

The amplitude of the oscillations, given in Eq. (84a), falls off as $[a(t)]^{-3/2}$. As in the case of a radiation-dominated universe the total energy of the condensate, proportional to $a^3 \rho_\varphi$, is conserved.

\footnote{In our calculation, we have treated the spacetime as a fixed background. However, a universe with only an oscillating scalar background field with a quadratic potential does behave like a matter-dominated universe.}
3.3.2 Negligible $\gamma$

Now we consider the case where the imaginary part of the self-energy is negligible. Then, we find

$$A(t) = \frac{1}{\sqrt{-\sigma t + c_0 t^2}},$$

(86a)

$$f(t) = f_0 + \mu t - \frac{(\lambda_\phi + 16\alpha M_\phi) [\ln(t) - \ln(c_0 t - \sigma)]}{16 M_\phi \sigma},$$

(86b)

where $c_0$ and $f_0$ are constants of integration. The leading approximation for the solution of condensate is plotted in the red curve in Fig. 10. The damping of the oscillation is dominated by the expansion of the universe.

![Figure 10: Solutions of the condensate equation (25) with the imaginary part of the self-energy neglected $\gamma = 0$. Analytic approximation in a matter-dominated universe, i.e., Eq. (86a), is drawn in the red line with $\sigma = -\alpha = 0.01/M_\phi$, $\lambda_\phi = 0.05$, $\gamma = 0$, and $\mu = 0.001M_\phi$. Analytic approximation in a static universe with the same parameters, given by Eqs. (64), is drawn in the brown line. Analytic approximation without interactions in a matter-dominated universe, i.e., Eq. (85), is drawn in the purple line.](image)

The comoving energy density of the condensate decreases as

$$a^3 \rho_\phi \propto \frac{1}{c_0 - \sigma t},$$

(87)

which approaches $1/c_0$ as $t \to \infty$. The comoving energy density of the condensate will never decay completely, see Fig. 11. For $t \gg t_i$,

$$\rho_\phi(t) \approx \frac{t_i^2 \rho_\phi(t_i)}{t^2} \left(1 - \frac{2\sigma t_i \rho_\phi(t_i)}{M_\phi^2 + 2\sigma t_i \rho_\phi(t_i)}\right).$$

(88)
Figure 11: Energy transfer from the condensate to the produced particles in a matter-dominated universe without the self-energy correction. The evolution of the condensate comoving energy density and of the energy loss are drawn in the red and blue lines, respectively. Comparing with Fig. 2, the energy transfer from the condensate to the produced particles in a matter-dominated universe is inefficient if the $\gamma$ dissipation is absent.

3.3.3 Non-negligible $\gamma$

Finally, let us consider the full condensate equation of motion (25) in a matter-dominated universe. We find

$$A(t) = \frac{e^{-\gamma t}}{t \sqrt{c_0 - 2\sigma \gamma \Gamma(-1, 2\gamma t)}},$$

$$f(t) = f_0 + \mu t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln [c_0 - 2\sigma \gamma \Gamma(-1, 2\gamma t)]}{16M_\phi \sigma},$$

where $c_0$ and $f_0$ are constants of integration. Then, the leading approximation for the solution of Eq. (25) in a matter-dominated universe reads

$$\varphi \approx \frac{e^{-\gamma t}}{t \sqrt{c_0 - 2\sigma \gamma \Gamma(-1, 2\gamma t)}} \cos \left\{ f_0 + (M_\phi + \mu)t + \frac{(\lambda_\phi + 16\alpha M_\phi) \ln [c_0 - 2\sigma \gamma \Gamma(-1, 2\gamma t)]}{16M_\phi \sigma} \right\}.$$

The solution Eq. (90) with different initial times are drawn in the red curves in Fig. 12 and Fig. 13. The damping due to the expansion of the universe can be neglected at late times.
Figure 12: Solutions of the condensate equation (25). Analytic approximation in a matter-dominated universe, i.e., Eq. (90), is drawn in the red line with $\sigma = -\alpha = 0.01/M_\phi$, $\mu = 0.01M_\phi$, $\gamma = 0.005M_\phi$, $\lambda_\phi = 0.05$. Analytic approximation in a static universe with the same parameters, given by Eqs. (55) and (56), is drawn in the brown line. Analytic approximate solution without interactions in a matter-dominated universe, i.e., Eq. (85), is drawn in the purple line.

Figure 13: Same as Fig. 12 but now with initial time $t_iM_\phi = 500$.

The energy density of the condensate in a matter-dominated universe can be approximated as

$$\rho_\phi(t) \approx \frac{M_\phi^2}{2t^2}, \quad e^{-2\gamma t},$$

which still satisfies Eq. (78), but with $A(t)$ given by Eq. (89a). In the late-time limit $\gamma t \gg 1$, the comoving energy density of the condensate exponentially decreases with the decay rate $2\gamma$. The energy transfer from the condensate to the produced particles in a matter-dominated universe is
drawn in Fig. 14. In presence of the $\gamma$ dissipation, the decay of the condensate comoving energy density is complete.

![Figure 14: Energy transfer from the condensate to the produced particles in a matter-dominated universe. The evolution of the condensate comoving energy density and of the energy loss from the condensate are shown in the red and blue lines, respectively. The energy transfer from the condensate to the produced particles in a matter-dominated universe is efficient in presence of a non-vanishing self-energy correction.](image)

4 Conclusions

In this paper we studied the damping of scalar condensate oscillations in a spatially flat FLRW universe in the mildly non-linear regime, generalizing the work [46]. We use non-equilibrium quantum field theory which naturally accommodates quantum statistical effects of the plasma as well as the backreaction effects from particle production. Even though the framework is quite general, we used a $Z_2$-symmetric two-scalar model with quartic interactions as a concrete example. The starting point of our analysis is the non-local equation of motion (25). The information about particular interactions is encoded in the integral kernels $\pi_R(t-t')$ and $v_R(t-t')$, which are closely related to the retarded self-energies and proper four-vertex functions in a given particle physics model. We applied the multi-scale analysis to solve the non-local equation (25), and were able to find leading-order analytic approximations for the solution in terms of the Fourier-transformed retarded self-energy and proper four-vertex function evaluated at particular energies that are related to the oscillation frequency. We carry out the analysis in static, radiation-dominated, and matter-dominated universes. The solutions take the general form $\varphi(t) \approx A(t) \times \cos[f(t) + M_\varphi t]$ and are concluded in Table 1. Assuming that the scalar condensate is the inflaton, the theoretical study given here may be used to study the perturbative reheating in the early Universe. However, for that purpose, one needs to obtain the closed-form expressions for all the Fourier transforms in Eq. (44) as a function of the temperature and the couplings and also track the evolution of the temperature self-consistently, as done in, e.g., Ref. [96].
As in flat spacetime, the dissipation is due to particle production from the decaying oscillating condensate. The decay channels can be classified into two classes. The first class is characterized by a non-vanishing imaginary part of the retarded self-energy, the $\gamma$ defined in Eq. (44), and contains the processes given in Eq. (61). We call them the $\gamma$ channels. These channels give rise to Landau damping with one condensate mode and manifest them in the equation of motion through the linear non-local term associated with $\pi_R(t-t')$. The second class is characterized by a non-vanishing imaginary part of the retarded four-vertex function, the $\sigma$ defined in Eq. (44), and contains the processes given in Eq. (63). We call them the $\sigma$ channels. These channels can also happen at zero temperature. They are encoded in the cubic non-local term in the equation of motion.

$$A(t) = \frac{A_0 e^{-\gamma t}}{\sqrt{1 + \frac{\sigma A_0^2}{2\gamma^2} (1 - e^{-2\gamma t})}}$$

$$f(t) = f_0 + \mu t + \frac{(\lambda_0 + 16\alpha M_\phi) \ln \left[ 1 + \frac{\sigma A_0^2}{2\gamma^2} (1 - e^{-2\gamma t}) \right]}{16M_\phi \sigma}$$

$$\text{radiation-dominated}$$

$$A(t) = \frac{e^{-\gamma t}}{2^{1/4} \sqrt{\frac{\gamma}{\gamma}} \Gamma\left(-\frac{1}{2},2\gamma t\right) + c_0}$$

$$f(t) = f_0 + \mu t + \frac{(\lambda_0 + 16\alpha M_\phi) \ln \left[ -\sigma \sqrt{\gamma} \Gamma\left(-\frac{1}{2},2\gamma t\right) + c_0 \right]}{16M_\phi \sigma}$$

$$\text{matter-dominated}$$

$$A(t) = \frac{e^{-\gamma t}}{t \sqrt{c_0 - 2\sigma \gamma \Gamma\left(-1,2\gamma t\right)}}$$

$$f(t) = f_0 + \mu t + \frac{(\lambda_0 + 16\alpha M_\phi) \ln \left[ c_0 - 2\sigma \gamma \Gamma\left(-1,2\gamma t\right) \right]}{16M_\phi \sigma}$$

Table 1: Solutions for the condensate evolution in a static, radiation-dominated, and matter-dominated universe. The constants $A_0$, $c_0$, and $f_0$ can be determined by the initial conditions for $\varphi$ and $\dot{\varphi}$.

An important observation made in this work is that the $\gamma$ channels are necessary to ensure a complete reheating. The evolution of the condensate energy can be generally described by the following equation

$$\frac{1}{a^3 \rho_\varphi} \frac{d(a^3 \rho_\varphi)}{dt} \approx -2\gamma - \sigma[A(t)]^2.$$  \hspace{1cm} (92)

The $\gamma$ term is constant and thus induces exponential damping for the comoving energy density $a^3 \rho_\varphi$, while the $\sigma$ term falls off with the amplitude squared and induces power-law damping. For early times when the oscillation amplitude is still large, the comoving energy density first experiences a power-law damping behavior and at later times an exponential damping behavior. In flat spacetime, the decay of the condensate is complete even if $\gamma$ is negligible, i.e., $\gamma = 0$. The conclusion is completely different in an expanding universe. When the $\gamma$ dissipation is absent, the energy density of the condensate in an expanding universe satisfies

$$\frac{1}{\rho_\varphi(t)} \frac{d\rho_\varphi(t)}{dt} \approx -3H(t) - \sigma[A(t)]^2.$$  \hspace{1cm} (93)

In an expanding universe, $[A(t)]^2$ decreases faster than the Hubble constant (for a radiation-dominated universe, $[A(t)]^2$ decreases as $t^{-3/2}$ and for a matter-dominated universe it decreases as $t^{-2}$). The decrease of the energy density at late times is thus dominantly caused by the
expansion of the universe and the energy transfer from the condensate to the produced particles never completes. This can also be very clearly seen from our explicit solutions whose behaviors have been extensively shown in various plots throughout the paper.

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