Non-local interactions in quantum mechanics modeled by shifted Dirac delta functions

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Abstract. We introduce a new interaction formulated in terms of two shifted delta functions with non-local boundary conditions. For this purpose we use a method of self-adjoint extensions for a particular symmetric operator. We obtain bound states of the associated quantum mechanical problem and study its symmetry properties. We compare the results with the ground state of the quantum mechanical problem with a local interaction in terms of two delta functions.

1. Introduction

The Dirac delta distribution as a potential in quantum physics allows to construct approximate models with great historical and physical interest [1, 2, 3, 4, 5, 6, 7, 8, 9]. One of them is an approximate model of a hydrogen molecule ion, $H_2^+$. The ion is commonly formed in molecular clouds in space. The first successful quantum mechanical treatment of $H_2^+$ was published by O. Burrau in 1927 [1]. The electron of the molecule moves in an attractive potential generated by the two protons. The quantum analysis of the electron arises from the solution of the Schrödinger equation, where the potential can be approximated by two delta “functions” with negative coefficients describing the attractive interaction.

Another interesting application arises from an approximation to the Kronig-Penney [3] model, which describes some basic quantum effects in the conduction of electrical charges in metals. The potential is expressed as an infinite sequence of Dirac deltas with a finite separation between them, a combination of Dirac deltas. The delta interactions have been also used in toy models of quantum wires or nanowires.

In this work we introduce a new non-local interaction of the type of two delta functions. We will determine the bound states, in particular the ground state of the quantum mechanical hamiltonian and study its symmetry properties. A main motivation for this work is the formulation of non-local field theory, based on string field theory [10].

2. Preliminaries

Let us consider the differential operator $D = -d^2/dt^2$ acting on the real line $\mathbb{R}$ with the domain $D(D) = \{ y(t) \mid y(t), y'(t), y''(t) \in L^2(\mathbb{R}), y(0) = y'(0) = 0 \}.$
Then the adjoint operator $D^*$ has the domain
\[ \mathcal{D}(D^*) = \{ y(t) | y(t), y'(t), y''(t) |_{\mathbb{R}_+} \in L^2(\mathbb{R}_+), y(t), y'(t), y''(t) |_{\mathbb{R}_-} \in L^2(\mathbb{R}_-), \} , \]
where $\mathbb{R}_+ = \{ t | t \geq 0 \}$, $\mathbb{R}_- = \{ t | t \leq 0 \}$. Since $D \subset D^*$, for every self-adjoint extension $\tilde{D}$ of $D$ we have $D \subset \tilde{D} \subset D^*$ and an extension of $D$ can be reduced to a restriction of $D^*$. For $D^*$ we have $(D^* y, z) = -y'(-0)\bar{z}(-0) + y'(0)\bar{z}'(0) + y(-0)\bar{z}'(0) - y'(0)\bar{z}'(0) + (y, D^* z)$. So, we obtain a self-adjoint restriction if $y'(-0)\bar{z}(-0) + y'(0)\bar{z}'(0) + y(-0)\bar{z}'(0) - y'(0)\bar{z}'(0) = 0$. The space of boundary values in our case is four-dimensional, so for a self-adjoint restriction we need two linear homogeneous conditions. One of these conditions have the form (the same for $y$ and $z$) $y(-0) = y(+0)$ and $y'(0) - y'(0) = c = \text{const}$. In this case the first generalized derivative of $y(t)$ has a jump and, so, the second one has a generalized summand with the delta-function. Thus the corresponding extension $\tilde{D}$ can be naturally presented in the form $\tilde{D} y(t) = -d^2 y(t)/dt^2 + c \cdot y(t) \delta(t) = -d^2 y(t)/dt^2 + c \cdot y(t) \delta(t)$. If $c < 0$, then $\tilde{A}$ has the negative eigenvalue $\lambda = -c^2$ that corresponds to the eigenfunction $y(t) = e^{-c|t|}$. These facts are well known and can be find in the book of Albeverio et al [11].

In the present paper we study some generalization of the described above scheme for two boundary problems for points $-h, h$ and a behaviour of the corresponding extension if $h \to 0$. We shall show that this behaviour involves the derivatives of the delta-function.

Note that even for the one-point problem there are some self-adjoint extensions with one or two negative eigenvalues, naturally involve not only the delta-function but its first derivative. One of such extension is given by the boundary conditions ($\alpha > 0$, $\beta > 0$)
\[ y(+0) = \frac{1}{2} \left\{ -\left( \frac{1}{\alpha} + \frac{1}{\beta} \right) y'(+0) + \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) y'(-0) \right\}, \]
\[ y(-0) = \frac{1}{2} \left\{ -\left( \frac{1}{\alpha} + \frac{1}{\beta} \right) y'(+0) + \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) y'(-0) \right\}, \]
where $\int_{-\infty}^{0} (|y(t)|^2 + |y(t)'|^2 + |y(t)''|^2) dt + \int_{0}^{\infty} (|y(t)|^2 + |y(t)'|^2 + |y(t)''|^2) dt < \infty$. The eigenvalues for this extension are $-\alpha^2$ and $-\beta^2$, the corresponding eigenfunctions are $e^{-\alpha|t|}$ and $\text{Sgn}(t)e^{-\beta|t|}$ respectively. The extension $\tilde{D}$ has the representation
\[ \tilde{D} y(t) = -y''(t) - \frac{1}{\beta} \cdot \delta'(t) (y'(0) + y'(0)) - \alpha \cdot \delta(t) (y(0) + y(0)) . \]

3. Main constructions
3.1. Continuous function
Let the differential operator $D = -d^2 \cdot dt^2$ act on the real line $\mathbb{R}$ and have the domain $\mathcal{D}(D) =$ \[ \{ y(t) | y(t), y'(t), y''(t) \in L^2(\mathbb{R}), y(-h) = y'(-h), y(h) = y'(h) = 0 \}. \]

Then for the adjoint operator $D^*$ we have
\[ (D^* y, z) - (y, D^* z) = -y'(-h-0)\bar{z}(-h-0) + y'(-h+0)\bar{z}(-h+0) - y'(-h-0)\bar{z}(-h+0) + y'(h+0)\bar{z}(-h+0) + y(-h-0)\bar{z}'(-h-0) - y(-h+0)\bar{z}'(-h+0) + y(h+0)\bar{z}'(h+0). \]

Seeking self-adjoint extensions of $D$ let us suppose that
\[ y(-h-0) = y(-h+0), \quad y(h-0) = y(h+0). \]
and the same for \( z(t) \). Then the conditions of self-adjointness for extensions of \( D \) convert to
\[
(y'(-h+0) - y'(-h-0))z(-h) + y'(h+0) - y'(h-0))z(h) - y(h)(z'(-h+0) - z'(-h-0) -\]
\[
y(h)(z'(h+0) - z'(h-0)) = 0. \]
Let us seek self-adjoint extensions such that
\[
\left( \begin{array}{c}
(y'(-h+0) - y'(-h-0)) \\
y'(h+0) - y'(h-0)
\end{array} \right) = \left( \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array} \right) \left( \begin{array}{c}
y(-h) \\
y(h)
\end{array} \right).
\]
It easy to check that the corresponding extension will be self-adjoint if and only if the matrix
\[
A = \left( \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{array} \right)
\]
is symmetric. Note that the first derivative of \( y(t) \) can have jumps in the points \(-h\) and \( h \), so the second one (if \( y(t) \) is considered as a generalized function) two shifted \( \delta \)-summands:
\[
y''(t) = y''_d(t) + (a_{11}y(-h) + a_{12}y(h))\delta(t + h) + (a_{21}y(-h) + a_{22}y(h))\delta(t - h),
\]
where
\[
f''_d(t) = \begin{cases} f''(t) & \text{if } f''(t) \text{ exists in the classical sense}, \\
0 & \text{in the opposite case}.
\end{cases}
\]
Thus, the corresponding extension \( \tilde{D}_h \) of the operator \( D \) can be re-written as
\[
\tilde{D}_h y(t) = -y''(t) + (a_{11}y(-h) + a_{12}y(h))\delta(t + h) + (a_{21}y(-h) + a_{22}y(h))\delta(t - h).
\]
Note that due to presence of \( \alpha_{12} \) and \( \alpha_{21} \) in (7) the interaction in question is not local.

**Proposition 1** Let an extension \( \tilde{D} \) satisfies the conditions (4) and (5). Then \( \tilde{D} \) has no non-negative eigenvalues and the number of its negative eigenvalues doesn’t exceed the number of negative eigenvalues (taking in account the multiplicity) of matrix \( A \) from (6).

**Proof of Proposition 1** Let us suppose that \( \tilde{D} \) has a positive eigenvalue \( \omega^2 \), \( \omega > 0 \). Then a corresponding eigenfunction \( y(t) \) has a form
\[
y(t) = \begin{cases} a_1 \cos \omega t + b_1 \sin \omega t, & t \leq h, \\
a_2 \cos \omega t + b_2 \sin \omega t, & t \in (-h, h), \\
a_3 \cos \omega t + b_3 \sin \omega t, & t \geq h.
\end{cases}
\]
Since \( y(t) \in L^2(R) \), \( a_1 = b_1 = a_3 = b_3 = 0 \). Due to the condition (4) it means that \( y(-h) = y(h) = 0 \) and \( y'(-h-0) = y'(h-0) = 0 \), therefore (see (5)) \( y'(-h+0) = y'(h+0) = 0 \). From here it is easy to obtain \( a_2 = b_2 = 0 \), i.e. \( y(t) \equiv 0 \). Thus \( \omega^2 \) cannot be an eigenvalue for \( \tilde{D} \). In the same way one can show that 0 is not an eigenvalue for \( \tilde{D} \) too. The last assertion of Proposition 1 follows from the equality (\( \tilde{D} y(t), y(t) \)) = \( (y(-h), y(h)) \cdot A \cdot (y(-h), y(h))^T + \| y' \|_2^2 \).

Let us find the extension \( \tilde{D}_h \) of \( D \) such that it has two eigenvalues \( \lambda = -\alpha^2 \) and \( \mu = -\beta^2 \). In order to do it we construct a matrix \( A \) such that for every positive \( h \) the function
\[
f_h(t) = \begin{cases} e^{\alpha t}, & t \leq -h \\
e^{-\alpha h} + e^{\alpha t}, & t \in (-h, h) \\
e^{-\alpha t}, & t \geq h
\end{cases}
\]
would be an eigenfunction of the operator $\tilde{D}_h$. According to (5) we have

$$
\frac{2\alpha \cdot e^{-ah}}{1 + e^{-2ah}} = e^{-ah} \cdot (a_{11} + a_{12}),
\frac{2\alpha \cdot e^{-ah}}{1 + e^{-2ah}} = e^{-ah} \cdot (a_{21} + a_{22}).
$$

(9)

In the same way let us demand that for every positive $h$ the function

$$
g_h(t) = \begin{cases} 
    e^{\beta t}, & t \leq -h \\
    e^{-\beta h} \cdot (e^{\beta t} - e^{-\beta t}), & t \in (-h, h) \\
    -e^{-\beta t}, & t \geq h
\end{cases}
$$

(10)

would be an eigenfunction of the operator $\tilde{D}_h$. According to (5) we have

$$
\frac{2\beta \cdot e^{-\beta h}}{1 - e^{-2\beta h}} = e^{-\beta h} \cdot (a_{11} - a_{12}),
\frac{2\beta \cdot e^{-\beta h}}{1 - e^{-2\beta h}} = e^{-\beta h} \cdot (a_{21} - a_{22}).
$$

(11)

The system (9), (11) has a solution and we have

$$
a_{11} = a_{22} = -\left(\frac{\alpha}{1 + e^{-2ah}} + \frac{\beta}{1 - e^{-2\beta h}}\right),
a_{12} = a_{21} = \left(-\frac{\alpha}{1 + e^{-2ah}} + \frac{\beta}{1 - e^{-2\beta h}}\right).
$$

(12)

Then the expression (7) is converted to

$$
\tilde{D}_h y(t) = -y''(t) - \frac{\alpha(\delta(-h)+\delta(t+h))(y(-h)+y(h))}{1+e^{-2ah}} + \frac{\beta(\delta(t+h)-\delta(t-h))(y(h)-y(-h))}{1-e^{-2\beta h}}.
$$

(13)

Let us consider some particular cases. The interaction in (5) is not, generally speaking, local, because the jump of the derivative $y'$ in the point $-h$ depends of the values $y$ not only in the point $-h$ but also in the point $h$, etc. This interaction is local if and only if $a_{12} = a_{21} = 0$, that means

$$
\frac{\alpha}{1 + e^{-2ah}} = \frac{\beta}{1 - e^{-2\beta h}}.
$$

The latter brings $\beta < \beta \cdot (1 + e^{-2ah}) = \alpha \cdot (1 - e^{-2\beta h}) < \alpha$, so in the case of locality the bound state is given by (8). If $\alpha = \beta$ then $a_{12} > 0$ and the bound state is a linear combination of (10) (8). It can be asymmetric. If we consider (8)+(10) we obtain a lefthanded valued eigenfunction and if we take (8)-(10) we get a right handed valued eigenfunction.

Note that $a_{12} \to 0$ and $a_{11} \to \infty$ if $\alpha \to \infty$, so for relatively big $\alpha$ a violence of locality in (5) is weak.

Let us consider the behaviour of $\tilde{D}_h$ if $h \to 0$. For this aim we calculate the resolvent of $\tilde{D}_h$ for a fixed real point $-\gamma^2$, $\gamma > \alpha$, $\gamma > \beta$. Let

$$
G(t) = \frac{e^{-\gamma|t|}}{2\gamma}.
$$
Then for \( z(t) \in L^2(\mathbb{R}) \), such that

\[
\int_{-\infty}^{+\infty} z(\tau)G(-h - \tau)d\tau = 0, \quad \int_{-\infty}^{+\infty} z(\tau)G(h - \tau)d\tau = 0,
\]

and

\[
y(t) = \int_{-\infty}^{+\infty} z(\tau)G(t - \tau)d\tau
\]

we have (see (13))

\[
(\gamma^2 I + \tilde{D}_h)y(t) = z(t)
\]

thus,

\[
(\gamma^2 I + \tilde{D}_h)^{-1}z(t) = y(t) = \int_{-\infty}^{+\infty} z(\tau)G(t - \tau)d\tau.
\]

Let us introduce some notations. We put

\[
u_h(t) = G(-h - t) + G(h - t), \quad v_h(t) = G(-h - t) - G(h - t),
\]

\[M_h = \{ f(t) : f(t) \in L^2(\mathbb{R}), f(t) \perp u_h(t), f(t) \perp v_h(t) \}\]

and denote by \( N_h \) the linear span of \( f_h(t) \) and \( g_h(t) \). Then (note that this decomposition is not orthogonal)

\[L^2(\mathbb{R}) = M_h + N_h. \tag{15}\]

Let \( \mu_h = \int_{-\infty}^{+\infty} u_h(t)f_h(t)dt \) and \( \nu_h = \int_{-\infty}^{+\infty} v_h(t)g_h(t)dt \). Then for every \( f(t) \in L^2(\mathbb{R}) \) the decomposition (15) the form

\[
f(t) = (f(t) - \left( \frac{f_h(t)}{\mu_h} \int_{-\infty}^{+\infty} f(t)u_h(t)dt + \frac{g_h(t)}{\nu_h} \int_{-\infty}^{+\infty} f(t)v_h(t)dt \right)) + \left( \frac{f_h(t)}{\mu_h} \int_{-\infty}^{+\infty} f(t)u_h(t)dt + \frac{g_h(t)}{\nu_h} \int_{-\infty}^{+\infty} f(t)v_h(t)dt \right).
\]

Thus (see (14)),

\[
(\gamma^2 I + \tilde{D}_h)^{-1}f(t) = \left( \int_{-\infty}^{+\infty} f(\tau)G(t - \tau)d\tau - \left( \frac{f_h(t)}{\mu_h} \int_{-\infty}^{+\infty} f(t)u_h(t)dt + \frac{g_h(t)}{\nu_h} \int_{-\infty}^{+\infty} f(t)v_h(t)dt \right) \right) + \left( \frac{f_h(t)}{\mu_h} \int_{-\infty}^{+\infty} f(t)u_h(t)dt + \frac{g_h(t)}{\nu_h} \int_{-\infty}^{+\infty} f(t)v_h(t)dt \right). \tag{16}\]

Note that

\[
u_h(t) = \begin{cases} \frac{e^{-\gamma(t+h)} + e^{-\gamma(t-h)}}{2\gamma}, & t \leq -h \\ \frac{e^{-\gamma(t+h)} + e^{-\gamma(t-h)}}{2\gamma}, & t \in (-h, h) \\ \frac{e^{-\gamma(t+h)} + e^{-\gamma(t-h)}}{2\gamma}, & t \geq h \end{cases} \tag{17}\]
and
\[ v_h(t) = \begin{cases} 
\frac{e^{\gamma(t+h)} - e^{\gamma(t-h)}}{2\gamma}, & t \leq -h \\
\frac{e^{-\gamma(t+h)} - e^{\gamma(t-h)}}{2\gamma}, & t \in (-h, h) \\
\frac{e^{\gamma(t+h)} - e^{-\gamma(t-h)}}{2\gamma}, & t \geq h.
\end{cases} \]  
(18)

The direct calculation brings
\[
\mu_h = 2 \frac{\cosh(\gamma h)}{\gamma} \cdot e^{-(\alpha+\gamma)h} + \frac{\alpha + \gamma}{\gamma \cosh(\alpha h)} \cdot \left\{ \frac{\sinh(\alpha + \gamma)h}{(\alpha + \gamma)} + \frac{\sinh(\gamma - \alpha)h}{(\gamma - \alpha)} \right\},
\]
\[
\nu_h = 2 \frac{\sinh(\gamma h)}{\gamma} \cdot e^{-(\beta+\gamma)h} + \frac{\beta + \gamma}{\gamma \sinh(\beta h)} \cdot \left\{ \frac{\sinh(\beta + \gamma)h}{(\beta + \gamma)} - \frac{\sinh(\gamma - \beta)h}{(\gamma - \beta)} \right\},
\]
so for small \( h \) we have \( \nu_h = \frac{2h^{\alpha+\beta}}{\beta+\gamma} - \frac{2h^2}{3} + \ldots \).

The latter equalities bring
\[ \lim_{h \to 0} \frac{u_h(t)}{\mu_h} = \gamma(\alpha + \gamma)G(t), \quad \lim_{h \to 0} \frac{v_h(t)}{\nu_h} = -\gamma(\beta + \gamma)G(t) \text{ Sgn} (t) \]  
(19)

From (16) and (19) we have
\[
\lim_{h \to 0} (\gamma^2 I + \tilde{D}_h)^{-1} f(t) = \left( \int_{-\infty}^{+\infty} f(\tau)G(t-\tau)d\tau - \gamma(\alpha + \gamma) \int_{-\infty}^{+\infty} f(\tau)G(t-\tau)d\tau \int_{-\infty}^{+\infty} f(t)G(t)dt - (\beta + \gamma) \int_{-\infty}^{+\infty} g_0(\tau)G(t-\tau)d\tau \int_{-\infty}^{+\infty} f(t)G(t) \text{ Sgn} (t)dt \right) + \gamma \left( \frac{f_0(t)}{\gamma - \alpha} \int_{-\infty}^{+\infty} f(t)G(t)dt - \frac{g_0(t)}{\gamma - \beta} \int_{-\infty}^{+\infty} f(t)G(t) \text{ Sgn} (t)dt \right), \]
(20)

where \( f_0(t) = e^{-\alpha|t|} \) and \( g_0(t) = -e^{-\beta|t|} \text{ Sgn} (t) \). It is easy to check that the right part in (20) the resolvent of the operator (2) calculated in the point \(-\gamma^2\), thus this operator is a limit of \( \tilde{D}_h \) and, respectively, the boundary conditions (1) be treated as a limit of the boundary conditions (5), (12) for \( h \to 0 \). Let us note that this limit pass eliminates a possibility of a spontaneous symmetry breaking model because the condition \( \alpha = \beta \) transforms (1) to the conditions
\[ \alpha \cdot y(-0) = y'(-0) \text{ and } \alpha \cdot y(+0) = -y'(+0). \]

The latter means that waves on \( \mathbb{R}_- \) and \( \mathbb{R}_+ \) are independent, so it is a model of a non transitable barrier.

**Remark 1** There is an extension \( \tilde{D} \) that doesn’t satisfy the conditions (4), (5) and has a positive eigenvalue.

### 3.2. Continuous derivative

Seeking self-adjoint extensions of \( D \) let us suppose that
\[ y'(-h - 0) = y'(-h + 0), \quad y'(h - 0) = y'(h + 0) \]  
(21)
and the same for \( z(t) \). Then the conditions of self-adjointness for extensions of \( D \) convert to

\[
\begin{align*}
y'(-h) (-\bar{z}(-h - 0) + \bar{z}(-h + 0)) + y'(-h) (-\bar{z}(h - 0) + \bar{z}(h + 0)) + \\
(y(-h - 0) - y(-h + 0)) \bar{z}'(-h) + (y(h - 0) - y(h + 0)) \bar{z}'(h) &= 0.
\end{align*}
\]

Let us seek self-adjoint extensions such that

\[
\begin{pmatrix}
y(-h + 0) - y(-h - 0) \\
y(h + 0) - y(h - 0)
\end{pmatrix} =
\begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
\begin{pmatrix}
y(-h) \\
y'(h)
\end{pmatrix}.
\]

It easy to check that the corresponding extension will be self-adjoint if and only if the matrix

\[
B = \begin{pmatrix}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{pmatrix}
\]

is symmetric.

Let us consider a function \( y(t) \) under Conditions (22) as a generalized function (distribution). Then

\[
y''(t) = y''_d(t) + (b_{11} y'(-h) + b_{12} y'(h)) \delta'(t + h) + (b_{21} y'(-h) + b_{22} y'(h)) \delta'(t - h),
\]

where

\[
f''_d(t) = \begin{cases}
f''(t) & \text{if } f''(t) \text{ exists in the classical sense,} \\
0 & \text{in the opposite case,}
\end{cases}
\]

as before. Thus, the corresponding extension \( \tilde{D}_h \) of the operator \( D \) can be re-written as

\[
\tilde{D}_h y(t) = - y''(t) + (b_{11} y'(-h) + b_{12} y'(h)) \delta'(t + h) + (b_{21} y'(-h) + b_{22} y'(h)) \delta'(t - h).
\]

Let us construct a matrix \( B \) such that for every positive \( h \) the function

\[
f_h(t) = \begin{cases}
e^{-\alpha t}, & t \leq h \\
e^{-\alpha h} (e^{-\alpha t} + e^{\alpha t}) & t \in (-h, h) \\
e^{-\alpha t}, & t \geq h
\end{cases}
\]

would be an eigenfunction of the operator \( \tilde{D} \). It is clear that the corresponding eigenvalue is \( \lambda = -\alpha^2 \). Note that \( \int_{-h}^{h} f_h(t) dt = -\frac{2e^{-\alpha h}}{\alpha} \), so in the sense of distributions \( \lim_{h \to +0} f_h(t) = f_0(t) - \frac{2}{\alpha} \delta(t) \), where \( f_0(t) = e^{-\alpha^2 |t|} \), therefore in this case the limit generates a new boundary problem, that directly involves \( \delta(t) \). Let us show that there is an extension \( \tilde{D} \) that corresponds to \( f_h(t) \). According to (5) we have

\[
\frac{2e^{-\alpha h}}{1 - e^{-2\alpha h}} = \alpha \cdot e^{-\alpha h} \cdot (b_{12} - b_{11}),
\]

\[
\frac{2e^{-\alpha h}}{1 - e^{-2\alpha h}} = \alpha \cdot e^{-\alpha h} \cdot (b_{21} - b_{22}).
\]

The last system implies that the matrix elements \( b_{12} = b_{21} \) are real and \( b_{11} = b_{22} \). The second type of conditions we obtain by the hypothesis that \( \tilde{D} \) has another eigenvalue \( \mu = -\beta^2 \). If an
eigenfunction \( g_h(t) \) corresponds to \( \mu \), then it must be orthogonal to \( f_h(t) \). This condition, in particular, would be fulfilled if \( g_h(t) \) is odd. Put

\[
g_h(t) = \begin{cases} 
  e^{\beta t}, & t < h \\
  e^{-\beta h}(-e^{-\beta t} + e^{\beta t}) / (e^{\beta h} + e^{-\beta h}), & t \in (-h, h) \\
  -e^{-\beta t}, & t \geq h 
\end{cases}
\]  

(26)

According to (5) we have

\[
-\frac{2e^{-\beta h}}{1 + e^{-2\beta h}} = \beta \cdot e^{-\beta h} \cdot (b_{11} + b_{12}),
\]

\[
-\frac{2e^{-\beta h}}{1 + e^{-2\beta h}} = \beta \cdot e^{-\beta h} \cdot (b_{21} + b_{22}).
\]

So

\[
b_{11} = -\left( \frac{1}{\alpha(1 - e^{-2\alpha h})} + \frac{1}{\beta(1 + e^{-2\beta h})} \right),
\]

\[
b_{12} = \left( \frac{1}{\alpha(1 - e^{-2\alpha h})} - \frac{1}{\beta(1 + e^{-2\beta h})} \right).
\]

(27)

Then the expression (24) is converted to

\[
\tilde{D}_h y(t) = -y''(t) + \frac{(\delta'(t-h) - \delta'(t+h))(y'(-h) - y'(h))}{\alpha(1 - e^{-2\alpha h})} - \frac{\delta'(t-h) + \delta'(t+h)(y'(-h) + y'(h))}{\beta(1 + e^{-2\beta h})}.
\]

Strictly speaking in the latter expression we cannot pass to limit for \( h \to 0 \) because the domain of \( \tilde{D}_h \) depends on \( h \) and, moreover, the eigenfunction \( f_h(t) \) does not converge to any function of \( L^2(\mathbb{R}) \), but from some heuristic point of view we can say that \( \tilde{D}_h \to \tilde{D}_0 \), where

\[
\tilde{D}_0 y(t) = -y''(t) - \frac{\delta''(t)(y'(-0) - y'(+0))}{\alpha^2} - \frac{\delta'(t)(y'(-0) + y'(0))}{\beta}.
\]

The constructed operator \( \tilde{D}_0 \) is not well defined because its domain is not evident.

4. Conclusions
We introduced a new interaction of the type of two delta functions or their derivatives.

For this aim we used a method of self-adjoint extensions of a given symmetric operator. We determined the associated bound states. There exists a symmetric and an antisymmetric eigenfunction. We analyzed the dependence on the intensity of the interaction of the corresponding eigenvalues as well as the limit of degeneracy of the ground state. We compared the results with the corresponding ones for the interaction given by a potential with only two delta functions. We expect to extend this exotic non-local interaction to non-local field theory arising from string theory.

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References

[1] Ø. Burrau, Naturwissenschaften, Volume 15, Issue 1, 16-17 (1927).
[2] C. N. Yang, Phys. Rev. Lett. 19:13121315 (1967).
[3] R. de L. Kronig and W. G. Penney, Proc. Roy. Soc. (London) 130A, 499-513 (1931).
[4] A. A. Frost, J. Chem. Phys. 25, 1150-1154 (1956).
[5] A. A. Frost and F. E. Leland J. Chem. Phys. 25, 1154-1160 (1956).
[6] R. Ahlrichs and P. Claverie, J. Quant. Chem. 6, 1001-1009 (1972).
[7] P. R. Certain and W. Byers Brown, J. Quant. Chem. 6, 131-142 (1972).
[8] S. Albeverio, F. Gesztesy, R. Hoegh-Krohn and W. Kirsch, Operator Theory 12, 101-126. Erratum 13, 419 (1985).
[9] F. Gesztesy and H. Holden, J. Phys. A 20, 5157-5177 (1987).
[10] “Strings Theory and Fundamental Interactions”, Edited by M. Gasparini and J. Maharana, Lecture Notes in Physics 737, Springer, Berlin Heidelberg (2008).
[11] S. Albeverio, F. Gesztesy, R. Hoegh-Kron and H. Holden, “Solvable models in quantum mechanics”, AMS Chelsea Publishing, American Mathematical Society-Providence, Rhode Island (2004).