BACH-FLAT CRITICAL METRICS OF THE VOLUME FUNCTIONAL ON COMPACT MANIFOLDS WITH BOUNDARY

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Abstract. The purpose of this article is to investigate Bach-flat critical metrics of the volume functional on a compact manifold $M$ with boundary $\partial M$. Here, we prove that a Bach-flat critical metric of the volume functional on a simply connected 4-dimensional manifold with boundary isometric to a standard sphere must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{R}^4$ or $S^4$. Moreover, in dimension three the result still is true replacing the Bach-Flat assumption by weaker condition that $M$ has divergence-free Bach tensor.

1. Introduction

A successful problem in Riemannian geometry is to study the critical points of the volume functional associated to space of smooth Riemannian structures. In the last decades very much attention has been given to study the critical points of the volume functional. Here, we shall study the space of smooth Riemannian structures on compact manifolds with boundary that satisfies a critical point equation associated to a boundary value problem.

Recently, inspired in a result obtained in [13] as well as in the characterization of the critical points of the scalar curvature functional, Miao and Tam studied variational properties of the volume functional constraint to the space of metrics of constant scalar curvature on a given compact manifold with boundary. For more details, we refer the reader to [18] and [19]. Afterward, in a celebrated article [12] Corvino-Eichmair-Miao studied this problem in a general context. In fact, they studied the modified problem of finding stationary points for the volume functional on the space of metrics whose scalar curvature is equal to a given constant. To do this, they localized a condition satisfied by such stationary points to smooth bounded domains.

We now recall the definition of critical metrics studied by Miao and Tam. Here, for simplicity, these metrics will be called Miao-Tam critical metrics.

Definition 1. A Miao-Tam critical metric is a 3-tuple $(M^n, g, f)$, where $(M^n, g)$, is a compact Riemannian manifold of dimension at least three with a smooth boundary $\partial M$ and $f : M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

\begin{equation}
\mathcal{L}_g^* (f) = g.
\end{equation}

Here, $\mathcal{L}_g^*$ is the formal $L^2$-adjoint of the linearized of the scalar curvature operator $\mathcal{L}_g$. Such a $f$ is called potential function.

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We recall that $L^*_g(f) = -(\Delta_g f) g + \text{Hess}_g f - f \text{Ric}_g$. So, the definition of Miao-Tam critical metric can be seen as the following boundary value problem

\begin{equation}
\begin{cases}
-(\Delta_g f) g + \text{Hess}_g f - f \text{Ric}_g = g & \text{in } M, \\
f = 0 & \text{on } \partial M.
\end{cases}
\end{equation}

We highlight that some explicit examples of Miao-Tam critical metrics are in the form of warped products. Those examples include the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres (cf. Corollaries 3.1 and 3.2 in [19]).

In 2009, Miao and Tam [18] were able to prove that these metrics arise as critical points of the volume functional on $M^n$ when restricted to the class of metrics $g$ on $M^n$ such that $g|_{\partial M} = h$, for a prescribed Riemannian metric $h$ on the boundary and $g$ has constant scalar curvature. In particular, they proved that critical metrics always have constant scalar curvature.

Here, we call attention to the paragraph where Miao and Tam [19] wrote:

"we want to know if there exist non-constant sectional curvature critical metrics on a compact manifold whose boundary is isometric to standard sphere. If yes, what can we say about the structure of such metrics?"

Indeed, they studied these critical metrics under Einstein and conformally flat assumptions. In special, they proved that a connected, compact, Einstein manifold $(M^n, g)$ with smooth boundary that satisfies (1.2) must be isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$, $\mathbb{H}^n$, or $\mathbb{S}^n$. Moreover, based in the techniques developed in a work of Kobayashi and Obata [10], Miao and Tam showed that the result even is true replacing the Einstein assumption by $(M^n, g)$ to be locally conformally flat with boundary isometric to a standard sphere. For more details see [19].

It should be emphasized that the hypothesis that the boundary of $M^n$ is isometric to a standard sphere $\mathbb{S}^{n-1}$ considered by Miao-Tam is not artificial. To clarify this, we consider that the boundary of $M^n$ is totally geodesic and is isometric to a standard sphere $\mathbb{S}^{n-1}$. Under these conditions, motivated by the positive mass theorem, Min-Oo conjectured that if $M^n$ has scalar curvature at least $n(n-1)$, then $M^n$ must be isometric to the hemisphere $\mathbb{S}^n_+$ with standard metric (cf. [20]). However, an elegant article due to Brendle-Marques-Neves shows counterexamples to Min-Oo’s Conjecture in dimensions $n \geq 3$. For more details see [6]. We also highlight that $\mathbb{S}^n_+$ satisfies (1.2) for a suitable potential function (cf. [18] p. 153).

We now recall that the Bach tensor on a Riemannian manifold $(M^n, g)$, $n \geq 4$, which was introduced in the early 1920s to study conformal relativity, see [3], is defined in term of the components of the Weyl tensor $W_{ikjl}$ as follows

\begin{equation}
B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{i\;k\;j\;l},
\end{equation}

while for $n = 3$ it is given by

\begin{equation}
B_{ij} = \nabla^k C_{kij}.
\end{equation}

We say that $(M^n, g)$ is Bach-flat when $B_{ij} = 0$. On 4-dimensional compact manifolds, Bach-flat metrics are precisely critical points of the conformally invariant functional on the space of smooth Riemannian structures

$$W(g) = \int_M |W_g|^2 dM_g,$$

where $W_g$ denotes the Weyl tensor of $g$. It is not difficult to check that locally conformally flat metrics as well as Einstein metrics are Bach-flat. Recently, Cao-Chen have studied Bach-flat gradient Ricci solitons, more precisely, they showed a stronger classification for
gradient Ricci solitons under Bach-flat assumption. For more details, we refer the reader to [7] and [8].

It is well-known that 4-dimensional compact Riemannian manifolds have special behavior; for more details see for instance [1], [5] and [22]. Here, we shall investigate Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary. More exactly, we replace the assumption of locally conformally flat in the Miao-Tam result by Bach-flat which is more weaker than the former. So, we announce our first result.

Theorem 1. Let \((M^4, g, f)\) be a simply connected, compact Miao-Tam critical metric with boundary isometric to a standard sphere \(S^3\). Then \((M^4, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^4, \mathbb{H}^4\) or \(S^4\) provided

\[
\int_M f^2 B(\nabla f, \nabla f) dM_g \geq 0,
\]

where \(B\) is the Bach tensor.

The proof of Theorem 1 was inspired in the trend developed by Cao-Chen in [7]. In the sequel, as an immediate consequence of Theorem 1 we deduce the following corollary.

Corollary 1. Let \((M^4, g, f)\) be a Bach-flat simply connected, compact Miao-Tam critical metric with boundary isometric to a standard sphere \(S^3\). Then \((M^4, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^4, \mathbb{H}^4\) or \(S^4\).

Based in the previous result, it is natural to ask what occurs in lower dimension. To do so, inspired in the ideas developed in [9] (see also [8] and [7]) we shall prove a rigidity result for a 3-dimensional Miao-Tam critical metric with divergence-free Bach tensor, i.e. \(\text{div}B = 0\), and boundary isometric to a standard sphere \(S^2\). Clearly, the assumption of divergence-free Bach tensor is weaker than Bach-flat assumption considered in Theorem 1. More exactly, we have the following theorem.

Theorem 2. Let \((M^3, g, f)\) be a simply connected, compact Miao-Tam critical metric with boundary isometric to a standard sphere \(S^2\). If \(\text{div}B(\nabla f) = 0\) in \(M\), where \(B\) is the Bach tensor, then \((M^3, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^3, \mathbb{H}^3\) or \(S^3\).

Finally, we get the following rigidity result.

Corollary 2. Let \((M^3, g, f)\) be a simply connected, compact Miao-Tam critical metric with divergence-free Bach tensor and boundary isometric to a standard sphere \(S^2\). Then \((M^3, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{R}^3, \mathbb{H}^3\) or \(S^3\).

2. Preliminaries and Key Lemmas

In this section we shall present a couple of lemmas that will be useful in the proof of our main results. We begin recalling that

\[
\Sigma^*_g(f) = - \langle \Delta_g f \rangle g + \text{Hess}_g f - f \text{Ric}_g.
\]

So, the fundamental equation of Miao-Tam critical metric (1.1) becomes

\[
-(\Delta f) g + \text{Hess} f - f \text{Ric} = g.
\]

Tracing (2.1) we arrive at

\[
(n-1) \Delta f + Rf = -n.
\]

Moreover, by using (2.2) it is not difficult to check that

\[
f \text{Ric} = \hat{\text{Hess}} f,
\]

where \(\hat{T}\) stands for the traceless of \(T\).
For simplicity, we now rewrite equation (2.1) in the tensorial language as follows

\[ (\Delta f)g_{ij} + \nabla_i \nabla_j f - f R_{ij} = g_{ij}. \]

Next, since a Miao-Tam critical metric has constant scalar curvature (cf. [18]), we use the last identity in order to obtain the following lemma.

**Lemma 1.** Let \((M^n, g, f)\) be a Miao-Tam critical metric. Then

\[ f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijks} \nabla_s f + R_{nk} - \frac{1}{n-1} g_{ij} R_{gjk}. \]

**Proof.** Computing \(\nabla_i (f R_{jk})\) with the aid of (2.4) we infer

\[ (\nabla_i f) R_{jk} + f \nabla_i R_{jk} = \nabla_i \nabla_j \nabla_k f - (\nabla_i \Delta f) g_{jk}. \]

Since \(R\) is constant (2.2) yields \(\nabla_i \Delta f = -\frac{R}{n-1} \nabla_i f\). Hence we use this data in (2.5) to deduce

\[ f \nabla_i R_{jk} = -(\nabla_i f) R_{jk} + \nabla_i \nabla_j \nabla_k f + \frac{R}{n-1} \nabla_i f g_{jk}. \]

Therefore, it suffices to apply the Ricci identity to finish the proof of the lemma. \(\square\)

To fix notations we recall three special tensors in the study of curvature for a Riemannian manifold \((M^n, g)\), \(n \geq 3\). The first one is the Weyl tensor \(W\) which is defined by the following decomposition formula

\[ R_{ijkl} = W_{ijkl} + \frac{1}{n-2} (R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il}) \]

\[ - \frac{R}{(n-1)(n-2)} (g_{jl} g_{ik} - g_{il} g_{jk}), \]

where \(R_{ijkl}\) stands for the Riemannian curvature operator, whereas the second one is the Cotton tensor \(C\) given by

\[ C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (\nabla_i R_{gjk} - \nabla_j R_{gik}). \]

These two tensors are related as follows

\[ C_{ijk} = -\frac{(n-2)}{(n-3)} \nabla_i W_{ijkl}, \]

provided \(n \geq 4\). Finally, the Schouten tensor \(A\) is defined by

\[ A_{ij} = \frac{1}{n-2} \left( R_{ij} - \frac{R}{2(n-1)} g_{ij} \right). \]

Combining equations (2.7) and (2.10) we have the next split

\[ R_{ijkl} = (A \odot g)_{ijkl} + W_{ijkl}, \]

where \(\odot\) is the Kulkarni-Nomizu product. For more details about these tensors we address to [5].

From now on we introduce the covariant 3-tensor \(T_{ijk}\) by

\[ T_{ijk} = \frac{n}{n-2} (R_{ik} \nabla_j f - R_{jk} \nabla_i f) - \frac{R}{n-2} (g_{ik} \nabla_j f - g_{jk} \nabla_i f) \]

\[ + \frac{1}{n-2} (g_{ik} R_{jx} \nabla^x f - g_{jk} R_{ix} \nabla^x f). \]

It is important to highlight that \(T_{ijk}\) was defined similarly to \(D_{ijk}\) in [7]. Now, we may announce our second lemma.
Lemma 2. Let \((M^n, g, f)\) be a Miao-Tam critical metric. Then the following identity holds:

\[ fC_{ijk} = T_{ijk} + W_{ijk} \nabla^s f. \]

Proof. First of all, we compare (2.8) with Lemma 1 to arrive at

\[ fC_{ijk} = R_{ijk} \nabla^s f + \frac{R}{n-1} (\nabla_i fg_{jk} - \nabla_j fg_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}). \]

On the other hand, from (2.7) we obtain

\[ R_{ijk} \nabla^s f = W_{ijk} \nabla^s f + \frac{1}{n-2} \left( R_{ik} g_{js} + R_{js} g_{ik} - R_{is} g_{jk} - R_{jk} g_{is} \right) \nabla^s f \]

\[ - \frac{R}{(n-1)(n-2)} (g_{js} g_{ik} - g_{is} g_{jk}) \nabla^s f. \]

From what it follows that

\[ fC_{ijk} = W_{ijk} \nabla^s f + \frac{1}{n-2} \left( R_{ik} \nabla_j f - R_{jk} \nabla_i f \right) \]

\[ + \frac{1}{n-2} \left( g_{ik} R_{js} \nabla^s f - g_{jk} R_{is} \nabla^s f \right) \]

\[ = T_{ijk} + W_{ijk} \nabla^s f, \]

which concludes the proof of the lemma. □

To simplify some computations we shall define a function \(\rho\) on \(M^n\) by

\[ \rho = |\nabla f|^2 + \frac{2}{n-1} f + \frac{R}{n-1} f^2. \]

We claim that

\[ \frac{1}{2} \nabla \rho = f \text{Ric}(\nabla f). \]

Indeed, since \(R\) is constant we have \(\frac{1}{2} \nabla \rho = Hess f(\nabla f) + \frac{1}{n-1} \nabla f + \frac{R}{n-1} f \nabla f.\) Next, we use (2.1) to obtain

\[ \frac{1}{2} \nabla \rho = Hess f(\nabla f) - (\Delta f + 1) \nabla f = f \text{Ric}(\nabla f), \]

which settles our claim.

We now follow the trend of Cao and Chen (cf. [7] and [8]) to study the level sets of the potential function of Miao-Tam critical metrics. To this end, first, we deduce a similar result concerning to the tensor \(T\) defined by (2.12) on the next lemma.

Lemma 3. Let \((M^n, g, f)\) be a Miao-Tam critical metric. Let \(\Sigma = \{f = f(p)\}\) be a level set of \(f\). If \(g_{ab}\) denotes the induced metric on \(\Sigma\), then, at any point where \(\nabla f \neq 0\), we have

\[ |fT|^2 = \frac{2(n-1)^2}{(n-2)^2} |\nabla f|^4 \sum_{a,b=2}^n |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{n-1}{2(n-2)} |\nabla^\Sigma \rho|^2, \]

where \(\rho\) is given by (2.14), \(h_{ab}\) and \(H\) are the second fundamental form and the mean curvature, respectively, while \(\nabla^\Sigma\) is the Riemannian connection of \(\Sigma\).
Proof. We consider \( \{e_1, e_2, \ldots, e_n\} \) be an orthonormal frame with \( e_1 = \nabla f \) and \( e_2, \ldots, e_n \) tangent to \( \Sigma \). A straightforward computation allows us to deduce

\[
|T|^2 = \frac{2(n-1)^2}{(n-2)^2}(|Ric|^2 - |\nabla f|^2) + \frac{2(n-1)R^2}{(n-2)^2} |\nabla f|^2 \\
+ \frac{2(n-1)}{(n-2)^2} |\nabla f|^2 - \frac{4(n-1)R}{(n-2)^2} (R |\nabla f|^2 - Ric(\nabla f, \nabla f)) \\
+ \frac{4(n-1)}{(n-2)^2} (RRic(\nabla f, \nabla f) - |Ric(\nabla f)|^2) - \frac{4(n-1)R}{(n-2)^2} Ric(\nabla f, \nabla f).
\]

Proceeding we can use (2.15) to obtain

\[
|fT|^2 = \frac{2(n-1)^2}{(n-2)^2} f^2 |\nabla f|^2 |Ric|^2 - \frac{n(n-1)}{2(n-2)^2} |\nabla \rho|^2 \\
- \frac{2(n-1)R^2}{(n-2)^2} f^2 |\nabla f|^2 + \frac{2(n-1)R}{(n-2)^2} f^2 (\nabla \rho, \nabla f).
\tag{2.16}
\]

On the other hand, the second fundamental form \( h_{ab} \) of the level set \( \Sigma \) as well as its mean curvature \( H \), are given respectively by

\[
h_{ab} = \left\langle \nabla_{e_a} \left( \frac{\nabla f}{|\nabla f|} \right), e_b \right\rangle = \frac{1}{|\nabla f|} \nabla_a \nabla_b f = \frac{1}{|\nabla f|} \left[ fR_{ab} - \left( \frac{1}{n-1} + \frac{fR}{n-1} \right) g_{ab} \right]
\]

and

\[
H = \frac{1}{|\nabla f|} (fR - fR_{11} - fR - 1) = -\frac{1}{|\nabla f|} (fR_{11} + 1).
\]

Whence, we deduce

\[
|h|^2 = \frac{1}{|\nabla f|^2} \left[ f^2 |Ric|^2 - 2f^2 \sum_{a=2}^{n} R_{1a}^2 - f^2 R_{11}^2 - 2f(R - R_{11}) \left( \frac{1}{n-1} + \frac{fR}{n-1} \right) \right] \\
+ \frac{1}{|\nabla f|^2} \left( fR + 1 \right)^2 \frac{n}{n-1}
\]

and

\[
H^2 = \frac{1}{|\nabla f|^2} \left( f^2 R_{11}^2 + 2fR_{11} + 1 \right).
\]

After some computations we obtain

\[
\sum_{a,b=2}^{n} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 = \frac{1}{|\nabla f|^2} \left[ f^2 |Ric|^2 - \left( \frac{nR_{11}^2 f^2 + 2f^2 R_{11}^2 - 2f^2 R_{11}}{n-1} \right) \right] \\
- \frac{2f^2}{|\nabla f|^2} \sum_{a=2}^{n} R_{1a}^2.
\tag{2.17}
\]

On the other hand, by using once more (2.15) we get

\[
fR_{11} = \frac{1}{|\nabla f|^2} fRic(\nabla f, \nabla f) = \frac{1}{2|\nabla f|^2} \langle \nabla \rho, \nabla f \rangle
\]

and

\[
fR_{1a} = \frac{1}{|\nabla f|^2} fRic(\nabla f, e_a) = \frac{1}{2|\nabla f|} \langle \nabla \rho, e_a \rangle = \frac{1}{2|\nabla f|} \nabla_a \rho.
\]
Substituting the last two identities in (2.17) we obtain

$$\sum_{a,b=2}^{n} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 = \frac{1}{|\nabla f|^2} \left( f^2 |\text{Ric}|^2 - \frac{1}{2|\nabla f|^2} |\nabla^2 \rho|^2 - \frac{R^2 f^2}{n-1} \right)$$

$$- \frac{n}{4(n-1)|\nabla f|^4} (\nabla \rho, \nabla f)^2 + \frac{R f}{(n-1)|\nabla f|^2} (\nabla \rho, \nabla f),$$

which can be rewritten as

$$f^2 |\text{Ric}|^2 = |\nabla f|^2 \sum_{a,b=2}^{n} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{n}{4(n-1)|\nabla f|^4} (\nabla \rho, \nabla f)^2$$

(2.18)

$$+ \frac{1}{2|\nabla f|^2} |\nabla^2 \rho|^2 + \frac{R f^2}{n-1} - \frac{R f}{(n-1)|\nabla f|^2} (\nabla \rho, \nabla f).$$

Finally, comparing (2.16) with (2.18), we deduce

$$|f T|^2 = \frac{2(n-1)^2}{(n-2)^2} |\nabla f|^4 \sum_{a,b=2}^{n} |h_{ab} - \frac{H}{n-1} g_{ab}|^2 + \frac{n-1}{2(n-2)} |\nabla^2 \rho|^2,$$

which completes the proof of the lemma. □

We point out that some of these calculations above was also done in [41] and [23] in a different context to study the CPE Conjecture posed in “Besse book” (cf. [5] page 128).

Next, as a consequence of Lemma 3 we derive the following properties concerning to a level set of the quoted metrics.

**Proposition 1.** Let \((M^n, g, f)\) be a Miao-Tam critical metric with \(T \equiv 0\). Let \(c\) be a regular value of \(f\) and \(\Sigma = \{p \in M; f(p) = c\}\) be a level set of \(f\). We consider \(e_1 = \frac{\nabla f}{|\nabla f|}\) and choose an orthonormal frame \(\{e_2, \ldots, e_n\}\) tangent to \(\Sigma\). Under these conditions the following assertions occur:

1. The second fundamental form \(h_{ab}\) of \(\Sigma\) is \(h_{ab} = \frac{H}{n-1} g_{ab}\).
2. \(\nabla f\) is constant on \(\Sigma\).
3. \(R_{1a} = 0\) for any \(a \geq 2\) and \(e_1\) is an eigenvector of \(\text{Ric}\).
4. The mean curvature of \(\Sigma\) is constant.
5. On \(\Sigma\), the Ricci tensor either has a unique eigenvalue or two distinct eigenvalues with multiplicity 1 and \(n-1\), respectively. Moreover, the eigenvalue with multiplicity 1 is in the direction of \(\nabla f\).
6. \(R_{1abc} = 0\), for \(a, b, c \in \{2, \ldots, n\}\).

**Proof.** The first two items follow directly from Lemma 3 jointly with (2.14). Since \(T \equiv 0\) we may use (2.12) to deduce

$$0 = T(e_i, \nabla f, \nabla f)$$

$$= \text{Ric}(e_i, \nabla f) |\nabla f|^2 - \text{Ric}(\nabla f, \nabla f) (\nabla f, e_i),$$

in other words

$$\text{Ric}(e_i, \nabla f) |\nabla f|^2 = \text{Ric}(\nabla f, \nabla f) (\nabla f, e_i).$$

So, for \(i = a \geq 2\), we obtain \(R_{1a} = 0\). Furthermore \(\text{Ric}(e_1) = g^{ij} R_{ij} e_i = R_{11} e_1\). Therefore, \(e_1 = \frac{\nabla f}{|\nabla f|}\) is an eigenvector of \(\text{Ric}\), which gives the establishment of the third assertion.

Proceeding we consider the Codazzi equation

(2.19)

$$R_{1abc} = \nabla^2 \Sigma h_{ca} - \nabla^2 \Sigma h_{ba}, \quad a, b, c = 2, \ldots, n.$$

By contracting (2.19) with respect to indices \(a\) and \(c\), we get

$$R_{1b} = \nabla^2 \Sigma H - g^{ac} \nabla^2 \Sigma h_{ba} = \frac{n-2}{n-1} \nabla^2 \Sigma H.$$
Now, we use that $R_{1b} = 0$ to conclude that $H$ is constant on $\Sigma$, which gives the fourth item. Next, since $e_1 = \frac{\nabla f}{|\nabla f|}$ is an eigenvector of $\text{Ric}$, we may choose the frame $\{e_1 = \frac{\nabla f}{|\nabla f|}, e_2, \ldots, e_n\}$ diagonalizing $\text{Ric}$ such that $\text{Ric}(e_k) = \lambda_k e_k$ for $k = 1, 2, \ldots, n$. Using once more that $T \equiv 0$, we have for all $a, b \geq 2$

$$0 = T_{a1b} = \frac{n-1}{n-2}(R_{ab}\nabla_1 f - R_{b1}\nabla_a f) - \frac{R}{n-2}(g_{ab}\nabla_1 f - g_{1b}\nabla_a f)$$

$$+ \frac{1}{n-2}(g_{ab}R_{1a}\nabla^s f - g_{1b}R_{a}\nabla^s f)$$

$$= \frac{n-1}{n-2}R_{ab}|\nabla f| - \frac{R}{n-2}g_{ab}|\nabla f| + \frac{1}{n-2}g_{ab}\lambda_1|\nabla f|.$$  

From what it follows that $R_{ab} = \frac{R-\lambda_1}{n-1}g_{ab}$ and then $\lambda_2 = \ldots = \lambda_n = \frac{R-\lambda_1}{n-1}$, which gives the fifth assertion. Finally, we use (2.19) and the fourth assertion to obtain the last one. So, we complete the proof of the proposition. \qed

Proceeding with such a metric with $T \equiv 0$ we obtain the following lemma.

**Lemma 4.** Let $(M^n, g, f)$ be a Miao-Tam critical metric with $T \equiv 0$. Then $C \equiv 0$, namely, $(M^n, g)$ has harmonic Weyl tensor.

**Proof.** The first part of the proof is standard and it will follows Lemma 4.2 of [7]. Here we present its proof for the sake of completeness.

First, since $T \equiv 0$ we invoke Lemma 2 to deduce $fC_{ijk} = W_{ijkl}\nabla^l f$, which implies

$$fC_{ijk}\nabla^k f = 0.$$  

We now consider a regular point $p \in M^n$, with associated level set $\Sigma$. We choose any local coordinates $(\theta^2, \ldots, \theta^n)$ on $\Sigma$ and split the metric in the local coordinates $(f, \theta^2, \ldots, \theta^n)$ as

$$g = \frac{1}{|\nabla f|^2}df^2 + g_{ab}(f, \theta)d\theta^a d\theta^b.$$  

Denoting $\partial_f = \partial_1 = \frac{\nabla f}{|\nabla f|}$ we get

$$\nabla_1 f = 1 \quad \text{and} \quad \nabla_a f = 0, \quad \text{for} \quad a \geq 2.$$  

From (2.20) we have $fC_{ij1} = 0$ for all $i, j = 1, \ldots, n$. Moreover, for $a, b, c \geq 2$, by using Codazzi equation jointly with first and fourth items of Proposition 1 we have

$$R_{1abc} = \nabla_1^c h_{ac} - \nabla^c_c h_{ab} = 0.$$  

In particular, using $R_{1a} = 0$ we get

$$W_{1abc} = R_{1abc} = 0.$$  

Whence, we obtain for $a, b, c \geq 2$

$$fC_{abc} = W_{abc}\nabla^a f = W_{abc}\nabla^1 f = 0.$$  

We now claim that $fC_{1ab} = 0$ for all $a, b \geq 2$. To do this, first, we notice that

$$fC_{1ab} = W_{1ab}\nabla^a f = W_{1ab}\nabla^s f = W_{1ab}|\nabla f|^2 = -\frac{1}{|\nabla f|^2}W(\nabla f, \partial_a, \nabla f, \partial_b).$$  

On the other hand, from (2.19) we have

$$\frac{1}{|\nabla f|^2}W(\nabla f, \partial_a, \nabla f, \partial_b) = \frac{1}{|\nabla f|^2}R(\nabla f, \partial_a, \nabla f, \partial_b) - \frac{1}{(n-2)}\left(\frac{1}{|\nabla f|^2}Ric(\nabla f, \nabla f)g_{ab} + R_{ab}\right)$$

$$+ \frac{R}{(n-1)(n-2)}g_{ab}.$$  

(2.21)
We now analyze the second fundamental form in the local coordinate \((f, \theta^2, \ldots, \theta^n)\). It is easy to see that

\[
h_{ab} = \frac{1}{|\nabla f|} \langle \nabla f, \nabla_a \partial_b \rangle = \frac{1}{|\nabla f|} \langle \nabla f, \Gamma^1_{ab} \partial_f \rangle = \frac{\Gamma^1_{ab}}{|\nabla f|},
\]

Moreover, a standard computation allows us to obtain

\[
\Gamma^1_{ab} = \frac{1}{2} g^{ij} \left( \partial_a g_{bj} + \partial_b g_{ja} - \partial_j g_{ab} \right) = \frac{1}{2} g^{11} \left( \partial_a g_{1b} + \partial_b g_{1a} - \partial_f g_{ab} \right) = \frac{1}{2} \nabla f (g_{ab}).
\]

From what it follows that

\[
(2.22) \quad h_{ab} = -\frac{\nabla f}{2|\nabla f|} (g_{ab}).
\]

Next, we use once more Proposition 1 to conclude that \(|\nabla f|\) is constant on \(\Sigma\), which implies that

\[
(2.23) \quad [\partial_a, \nabla f] = 0.
\]

Since \(\langle \nabla f, \partial_a \rangle = 0\) we have \(\nabla \frac{\partial_a}{\nabla f} \nabla f = 0\). Hence, we can use (2.22) and (2.23) to arrive at

\[
\frac{1}{|\nabla f|^2} R(\nabla f, \partial_a, \nabla f, \partial_b) = \frac{1}{|\nabla f|} \langle \nabla \frac{\partial_a}{\nabla f} \nabla_a \partial_b - \nabla_a \nabla \frac{\partial_a}{\nabla f} \partial_b, \nabla f \rangle = \frac{1}{|\nabla f|^2} \langle \nabla \nabla_f (\nabla_a \partial_b + \nabla_b \partial_a), \nabla f \rangle - \frac{1}{|\nabla f|} \langle \nabla_a \nabla \frac{\partial_a}{\nabla f} \partial_b, \nabla f \rangle = \frac{\nabla f}{|\nabla f|} (h_{ab}) + h_{ac} h_c^b.
\]

From this, we deduce

\[
(2.24) \quad \frac{1}{|\nabla f|^2} R(\nabla f, \partial_a, \nabla f, \partial_b) = \frac{\nabla f}{(n-1)|\nabla f|} H g_{ab} + \frac{H^2}{(n-1)^2} g_{ab}.
\]

In particular, taking the trace in (2.24) with respect to \(a\) and \(b\) we have

\[
\frac{1}{|\nabla f|^2} \text{Ric}(\nabla f, \nabla f) = \frac{\nabla f}{|\nabla f|} H + \frac{H^2}{(n-1)}
\]

and then (2.24) can be written as

\[
(2.25) \quad R(\nabla f, \partial_a, \nabla f, \partial_b) = \frac{\text{Ric}(\nabla f, \nabla f)}{(n-1)} g_{ab}.
\]

By using Proposition 1 we have

\[
\frac{1}{|\nabla f|^2} \text{Ric}(\nabla f, \nabla f) = \lambda
\]

and

\[
\text{Ric}(\partial_a, \partial_b) = \mu g_{ab},
\]

for \(a, b \geq 2\).
Therefore, substituting (2.25) in (2.21) we get
\[ fC_{1a}b = -\frac{1}{|\nabla f|^2} W(\nabla f, \partial_a, \nabla f, \partial_b) \]
\[ = -\frac{1}{|\nabla f|^2} \frac{\text{Ric}(\nabla f, \nabla f)}{(n-1)} g_{ab} + \frac{1}{|\nabla f|^2} \frac{\text{Ric}(\nabla f, \nabla f)}{(n-2)} g_{ab} + \frac{R_{ab}}{(n-1)(n-2)} g_{ab} \]
\[ = -\frac{\lambda}{(n-1)} g_{ab} + \frac{\mu}{(n-2)} g_{ab} - \frac{\lambda + (n-1)\mu}{(n-1)(n-2)} g_{ab} \]
\[ = 0, \]
which completes our claim.

Finally, we have \( fC_{ij} = 0 \) at a point \( p \) where \( \nabla f(p) \neq 0 \). Therefore, we use Lemma 2 to conclude that \( fC_{ij} = 0 \) in \( M^n \). From what it follows that \( C_{ij} = 0 \) in \( M \setminus \partial M \) and then the proof of the lemma follows from the continuity of the Cotton tensor.

To finish this section, we shall present a fundamental integral formula.

**Lemma 5.** Let \( (M^n, g, f) \) be a Miao-Tam critical metric. Then
\[ \int_M f^2 B(\nabla f, \nabla f) dM = -\frac{1}{2(n-1)} \int_M f^2 |T|^2 dM_g. \]

**Proof.** From (2.9) we can write the Bach tensor as
\[ B_{ij} = \frac{1}{n-2} (\nabla_k C_{kij} + R_{kl} W_{ikjl}). \]

Under this notation we get
\[ f^2 B_{ij} = \frac{1}{n-2} (f^2 \nabla_k C_{kij} + f^2 R_{kl} W_{ikjl}) \]
\[ = \frac{1}{n-2} (\nabla_k (f^2 C_{kij}) - 2fC_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl}). \]

We now use Lemma 2 jointly with (2.9) to obtain
\[ f^2 B_{ij} = \frac{1}{n-2} \left( \nabla_k \left[ f(W_{ikjl} \nabla_i f + T_{kij}) \right] - 2fC_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl} \right) \]
\[ = \frac{1}{n-2} \left( \nabla_k (fT_{kij}) + f\nabla_k W_{kij} \nabla_i f + fW_{kij} \nabla_k \nabla_i f + W_{kij} \nabla_k f \nabla_i f - 2fC_{kij} \nabla_k f + f^2 R_{kl} W_{ikjl} \right) \]
\[ = \frac{1}{n-2} \left( \nabla_k (fT_{kij}) + \frac{n-3}{n-2} fC_{jki} \nabla_k f + f(\nabla_k \nabla_i f - fR_{kl}) W_{ikjl} + W_{kij} \nabla_k f \nabla_i f + 2fC_{ikj} \nabla_k f \right). \]

In particular, by using (2.3) we deduce
\[ f^2 B_{ij} = \frac{1}{n-2} \left[ \nabla_k (fT_{kij}) + \frac{n-3}{n-2} fC_{jki} \nabla_k f + 2fC_{ikj} \nabla_k f + W_{kij} \nabla_k f \nabla_i f \right]. \]

Whence,
\[ f^2 B(\nabla f, \nabla f) = \frac{1}{n-2} \nabla_k (fT_{kij}) \nabla_i f \nabla_j f. \]

On the other hand, we notice that
\[ \nabla_k (fT_{kij}) \nabla_i f \nabla_j f = \nabla_k (fT_{kij} \nabla_i f \nabla_j f) - fT_{kij} \nabla_k \nabla_i f \nabla_j f - fT_{kij} \nabla_i f \nabla_k \nabla_j f. \]
Now, on integrating (2.26) over \( M \) and using Stokes formula we arrive at
\[
\int_M f^2 B(\nabla f, \nabla f) dM_g = -\frac{1}{n-2} \left( \int_M fT_{kij} \nabla_k \nabla_i f \nabla_j f dM_g + \int_M fT_{kij} \nabla_i f \nabla_j f dM_g \right) 
= -\frac{1}{n-2} \left( \int_M f^2 T_{kij} R_{ki} \nabla_j f dM_g + \int_M f^2 T_{kij} R_{kj} \nabla_i f dM_g \right),
\]

in the last equality we have used once more (2.4). Finally, we change \( k \) by \( i \) above and then using the properties of \( T \) we achieve

\[
\int_M f^2 B(\nabla f, \nabla f) dM_g = -\frac{1}{2(n-2)} \left( \int_M f^2 T_{kij} R_{ki} \nabla_j f dM_g + \int_M f^2 T_{kij} R_{kj} \nabla_i f dM_g \right) 
- \frac{1}{2(n-2)} \left( \int_M f^2 T_{ikj} R_{ik} \nabla_j f dM_g + \int_M f^2 T_{ikj} R_{ij} \nabla_k f dM_g \right) 
= \frac{1}{2(n-2)} \int_M f^2 T_{kij} (R_{ki} \nabla_i f - R_{ij} \nabla_k f) dM_g 
= -\frac{1}{2(n-1)} \int_M f^2 |T|^2 dM_g,
\]

that was to be proved. \( \square \)

3. Proof of the Results

3.1. The Proof of Theorem 1

Proof. First, since \( M^4 \) satisfies \( \int_M f^2 B(\nabla f, \nabla f) dM \geq 0 \) we invoke Lemma [5] to conclude that \( T \equiv 0 \). Therefore, from Lemma [4] we have \( C \equiv 0 \). Hence, we use Lemma [2] to obtain

\[
W_{ijk} \nabla^i f = 0.
\]

We now consider a point \( p \in M^4 \) such that \( \nabla f(p) \neq 0 \). Choosing an orthonormal frame \( \{e_1, e_2, \ldots, e_4\} \) with \( e_1 = |\nabla f| \) we arrive at

\[
W_{ijk} = 0.
\]

From a standard computation we conclude that \( W_{ijkl} = 0 \) whenever \( \nabla f(p) \neq 0 \). Now, choosing appropriate coordinates (e.g. harmonic one) we conclude that \( f \) and \( g \) are analytics, see for instance Theorem 2.8 in [11] (or Proposition 2.1 in [12]). Whence, we conclude that \( f \) can not vanish identically in a non-empty open set. So, the set of regular points is dense in \( M^4 \). This allows us to conclude that \( M^4 \) is locally conformally flat and we are in position to use Theorem 1.2 due to Miao and Tam [19] to conclude that \( (M^4, g) \) is isometric to a geodesic ball in a simply connected space form \( \mathbb{R}^4, \mathbb{H}^4 \) or \( S^4 \). This is what we wanted to prove. \( \square \)

3.2. Proof of Theorem 2

Proof. The first part of the proof will follow [9]. In fact, we recall that the Cotton tensor can be written as \( C_{ijk} = \nabla_i A_{jk} - \nabla_j A_{ik} \). From what it follows that

\[
\nabla_i B_{ij} = \nabla_i \nabla_k C_{kij} 
= \nabla_i \nabla_k (\nabla_k A_{ij} - \nabla_i A_{kj}) 
= \nabla_i \nabla_k \nabla_k A_{ij} - \nabla_i \nabla_k \nabla_i A_{kj} 
= \nabla_i \nabla_k \nabla_k A_{ij} - \nabla_i \nabla_i \nabla_k A_{kj}. 
\]

From the Ricci identity we have

\[
\nabla_i B_{ij} \nabla_j f = R_{ikjl} \nabla_k A_{il} \nabla_j f.
\]
We now remember that $W \equiv 0$ in dimension three. So, we may use (2.7) to obtain
\[
\nabla_i B_{ij} \nabla_j f = A_{ik} C_{kji} \nabla_j f + A_{ik} A_{kl} A_{lj} \nabla_j f + A_{jl} A_{il} A_{kl} \nabla_j f + A_{lk} A_{ij} C_{kli} \nabla_j f
\]
Proceeding, we use this last information to infer
\[
(div B)(\nabla f) = \nabla_i B_{ij} \nabla_j f = -C_{jki} R_{ik} \nabla_j f
\]
Since $C$ has trace-free in any two indices we may use (2.12) to obtain
\[
(div B)(\nabla f) = \frac{1}{4} C_{kji} T_{kji}
\]
Therefore, our assumption implies that $C \equiv 0$ in $M \setminus \partial M$ and from the continuity of the Cotton tensor we infer $C \equiv 0$ in $M$. From what it follows that $(M^3, g)$ is locally conformally flat. Now, it is sufficient to use Theorem 1.2 of [19] to get the promised result. □

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