The structure of 3-manifolds with 2-generated fundamental group

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The main purpose of this article is to describe compact orientable irreducible 3-manifolds that have a non-trivial JSJ-decomposition and 2-generated fundamental group. The rank of a group is the minimal number of elements needed to generate it. A natural question is whether the Heegaard genus of such a manifold is equal to 2.

When the JSJ-decomposition is empty, there are examples of closed 3-manifolds of Heegaard genus 3 that have 2-generated fundamental group [BZg]. These are Seifert fibered manifolds. Furthermore the second named author [W2] has recently found graph manifolds with the same property. At this point these are the only known examples of 3-manifolds that have 2-generated fundamental group but are not of Heegaard genus 2. In particular, it is still an open problem to find such examples that admit a complete hyperbolic structure. There are also examples of Seifert manifolds with Heegaard genus \( g + 1 \geq 3 \) and fundamental group of rank \( g \) [MSc].

When the JSJ-decomposition is non-trivial, our main result with respect to this question is the following; we will denote the base spaces by their topological type, followed by a list with the orders of their cone points. We denote the Möbius band by \( \mathbb{M} \), the disk by \( D \), the annulus by \( A \) and the 2-punctured disk by \( \Sigma \). We furthermore denote by \( Q^3 \) the orientable circle bundle over the Möbius band.

**Theorem 1** Let \( M \) be a compact, orientable, irreducible 3-manifold with rank \( \pi_1(M) = 2 \). If \( M \) has a non-trivial JSJ-decomposition then one of the following holds:

1. \( M \) is of Heegaard genus 2.

2. \( M = S \cup_T H \) where \( S \) is a Seifert manifold with basis \( D(p,q) \) or \( A(p) \), \( H \) is a hyperbolic manifold and \( \pi_1(H) \) is generated by a pair of elements with a single parabolic element. The gluing map identifies the fibre of \( S \) with the curve corresponding to the parabolic generator of \( \pi_1(H) \).

3. \( M = S_1 \cup_T S_2 \) where \( S_1 \) is a Seifert manifold over \( \mathbb{M} \) or \( \mathbb{M}(p) \) and \( S_2 \) is a Seifert manifold over \( D(2,2l+1) \). The gluing map identifies the fibre of \( S_1 \) with a curve on the boundary of \( S_2 \) that has intersection number one with the fibre of \( S_2 \).

4. \( M = Q^3 \cup H \) where \( H \) is a hyperbolic manifold that admits a finite-sheeted irregular covering by the exterior of a hyperbolic 2-bridge link.
Remark 1 It follows from T. Kobayashi’s work [Ko1] that the three manifolds of type 3) do not have Heegaard genus two unless either $S_1$ is the exterior of a 1-bridge knot in a lens space and the meridian is glued with the fibre of $S_2$, or $S_2$ is the exterior of a 2-bridge knot in $\mathbb{S}^3$ and the meridian is glued with the fibre of $S_1$ (see [W2]). We do not yet know an example of manifolds of type 2) that does not have Heegaard genus two. Moreover we conjecture that there is no example of type 4).

It is known that a Heegaard genus 2 closed 3-manifold is a 2-fold branched covering of the 3-sphere $\mathbb{S}^3$ [BH]. For a closed 3-manifolds with a 2-generated fundamental group, that is geometric or has a non-trivial JSJ-decomposition, we obtain the following related result:

Corollary 1 Let $M^3$ be a closed, orientable, irreducible 3-manifold with rank $\pi_1(M^3) = 2$. If $M$ is geometric or has a non-trivial JSJ-decomposition, then $M$ is a 2-fold branched covering of a homotopy sphere.

Remark 2 In general, we do not know how to show that these homotopy spheres are the true sphere $\mathbb{S}^3$.

With respect to Thurston’s geometrization conjecture, we have the following corollary for closed 3-manifolds containing a two-generator knot:

Corollary 2 Let $M^3$ be a closed, orientable, irreducible 3-manifold. If there is a two-generator knot $k \subset M$ that is not in a ball, then either $M$ is geometric or has a non-trivial JSJ-decomposition.

It follows from Thurston’s orbifold theorem ([BP],[CHK]) that a Heegaard genus two closed 3-manifolds has a geometric decomposition in the sense of Thurston. Since by [JS1, Lemma 5.4], a compact, orientable, irreducible 3-manifold with rank two fundamental group is either a handlebody of genus two or has Euler characteristic zero, Corollary 2 gives the following algebraic version of this result:

Corollary 3 Let $M^3$ be a closed, orientable, irreducible 3-manifold which is the union of two non-empty, compact, connected, irreducible, orientable 3-manifolds $U$ and $V$ along their boundaries. If both $1 \leq \text{rank} \pi_1(U) \leq 2$ and $1 \leq \text{rank} \pi_1(V) \leq 2$, then either $M^3$ is geometric or has a non-trivial JSJ-decomposition.

F.H. Norwood [N] has shown that a two-generator knot in $\mathbb{S}^3$ is prime. Later on M. Scharlemann [Schar] has shown that a tunnel number one (i.e. Heegaard genus two) knot in $\mathbb{S}^3$ is Conway irreducible (i.e doubly prime). The following corollary extends this last result to the case of a two-generator knot in $\mathbb{S}^3$.

Corollary 4 A two-generator knot $k \subset \mathbb{S}^3$ is prime and Conway irreducible.

Theorem 1 follows from the classification of 3-manifolds with non-trivial JSJ-decomposition and Heegaard genus 2 as given by T. Kobayashi’s work [Ko1], [Ko2], [Ko3] and the following theorem that describes (almost) precisely the JSJ-decomposition of a compact orientable 3-manifold with 2-generated fundamental group:
Theorem 2 Let $M^3$ be a compact, orientable, irreducible 3-manifold with non-trivial JSJ-decomposition. If rank $(\pi_1(M^3)) = 2$ then the JSJ-decomposition is of one of the following types. If $M^3$ is of one of the types (1)-(9) the converse holds:

1. $M_1$ is a Seifert manifold with base $A(p)$ or $D(p,q)$ and $M_2$ is a hyperbolic manifold whose fundamental group has a generating pair $\{g,h\}$ where $g$ is parabolic and $M^3$ is obtained from $M_1$ and $M_2$ by gluing boundary components such that the fibre of $M_1$ gets identified with the curve corresponding to the parabolic generator.

2. $M_1$ is a Seifert manifold with base $A(p)$ or $D(p,q)$ and $M_2$ is the exterior of a 1-bridge knot in a lens space which is Seifert and where the meridian is not the fibre. $M^3$ is obtained from $M_1$ and $M_2$ by gluing boundary components such that the fibre of $M_1$ gets identified with the meridian curve of $M_2$.

3. $M_1$ is a Seifert manifold with base $A(p)$ or $D(p,q)$ and $M_2$ is a Seifert space with base space $A(p)$, $D(p,q)$, $D(p,q,r)$, $A(p)$, $A(p,q)$, $\Sigma$, $\Sigma(p)$ or the once punctured Möbius band with at most one cone point. and $M_2$ is the exterior of a 2-bridge knot. Boundary components are glued such that the fibre of $M_1$ is identified with the meridian curve of $M_2$.

4. $M_1$ is a Seifert space with base space $M\tilde{o}$, $M\tilde{o}(p)$, $M\tilde{o}(p,q)$, $D(p,q)$, $D(p,q,r)$, $A(p)$, $A(p,q)$, $\Sigma$, $\Sigma(p)$ or the once punctured Möbius band with at most one cone point. and $M_2$ is the exterior of a 2-bridge knot. Boundary components are glued such that the fibre of $M_1$ is identified with the meridian curve of $M_2$.

5. $M_1$ is a Seifert space with base space $M\tilde{o}$ or $M\tilde{o}(p)$ and $M_2$ is a Seifert space over $D(2,2l+1)$. Boundary components are glued such that the fibre of $M_1$ has intersection number 1 with the fibre of $M_2$.

6. $M_1$ and $M_2$ are Seifert space with base of type $A(p)$ or $D(p,q)$ and $M_3$ is the exterior of a 2-bridge link. Boundary components are glued such that the fibres of $M_1$ and $M_2$ get identified with the meridian curves of $M_3$.

7. $M_1$ is a Seifert space with base space of type $A(p)$, $\Sigma$, $A(p,q)$ or $\Sigma(p)$ and $M_2$ is the exterior of a 2-bridge link. Two boundary components of $M_1$ are glued to the boundary components of $M_2$ such that fibre of $M_1$ gets identified with the meridian curves of $M_2$.

8. $M$ is the exterior of a 2-bridge link and $M^3$ is obtained from $M$ by identifying the boundaries such that the meridians get identified.

9. $M$ is a Seifert space with base $A(p)$, $\Sigma$, $A(p,q)$ or $\Sigma(p)$ and two boundary components are identified such that the fibre in one component has intersection number 1 with the fibre in the other.

10. $M$ is obtained from $M_1$ and $M_2$ by gluing along their boundary where $M_1 = Q^3$ and $M_2$ is a hyperbolic manifold that admits a finite-sheeted irregular covering by the exterior of a hyperbolic 2-bridge link.
For the last case see the discussion in later chapters on the restrictions of the gluing map. We conjecture that this case does not occur.

We give some results which are not direct consequences of Theorem 1 or Theorem 2 but which are immediate consequences of their proof.

**Corollary 5** Let $M^3$ be a compact orientable 3-manifold. Any 2-generated subgroup $U$ of $\pi_1(M^3)$ is of one of the following types:

1. $U$ is of finite index in $\pi_1(M^3)$
2. $U$ is free or free Abelian
3. $U$ is the fundamental group of a Seifert fibered manifold
4. $U$ is a lattice in $\text{PSL}(2,\mathbb{C})$
5. $U$ is the fundamental group of one of the 3-manifolds described in Theorem 2

If $M^3$ is not closed then (1) implies one of the cases (2)-(5)

**Corollary 6** Let $M^3$ be a compact, orientable, irreducible 3-manifold with incompressible boundary. If $\pi_1(M^3)$ is generated by two peripheral elements then $M^3$ is homeomorphic to the exterior of a 2-bridge knot or link in $S^3$.

**Corollary 7** Let $M^3$ be a compact orientable, irreducible 3-manifold with incompressible boundary. If $\pi_1(M^3)$ is generated by two elements one of which is peripheral then either $M^3$ has Heegaard genus 2 or $M^3$ is hyperbolic.

**Remark 3** A stronger version of Corollary 4 has been obtained by S. Bleiler and A. Jones [BJ2] in the case of a knot exterior in $S^3$ (cf. also [B3]).

Tunnel number one satellite knots has been classified by Morimoto and Sakuma [MSa]: their exteriors are obtained by gluing a torus knot exterior to a 2-bridge link exterior along a boundary component, such that the gluing map identifies a regular fibre of the torus knot exterior with a meridian of the link exterior.

The following corollary shows that the existence of a two generator satellite knot which has tunnel number at least two reduces to the existence of a two generator, but not 2-bridge, hyperbolic link, with a single meridional generator and with both components unknotted.

If such a link has tunnel number one, it has been shown that it is a 2-bridge link. (cf. [Ku]).

**Corollary 8** Let $k \subset S^3$ be a satellite knot with rank $\pi_1(E(k)) = 2$. Then one of the following holds:

1. Either $k$ has tunnel number one, or
2. \( E(k) = M_1 \cup M_2 \) where \( M_1 \) is the exterior of a \((p,q)\) torus knot and \( M_2 \) is the exterior of a hyperbolic link \( L = k_1 \cup k_2 \subset S^3 \) with two unknotted components and tunnel number \( \geq 2 \). Furthermore \( \pi_1(M_2) \) is generated by two elements one of which corresponds to a meridian \( m \) and the gluing map identifies the fibre of \( M_1 \) with \( m \).

Corollary 8 is already a consequence of the work of S. Bleiler and A. Jones [BJ2] (cf. also [B3]). Some of the results of this paper have also been obtained independently by D. Bachman, S. Bleiler and A. Jones [BBJ], using a different combinatorial group theoretical approach.

1 Some combinatorial tools

In this section we recall the results of [KW] on generating pairs of fundamental groups of graphs of groups and draw some conclusions. Some other lemmas that are needed later are also shown.

In [KW] the action of 2-generated groups on simplicial trees is investigated. Suppose that a group \( G \) acts simplicially on a simplicial tree \( T \) and that \( g \in G \). We define \( T_g = \{ x \in T | g^z x = x \text{ for some } z \in Z \text{ with } g^z \neq 1 \} \). It is clear that \( T_g \neq \emptyset \) if and only if \( g \) acts with a fixed point. With this notation the main result of [KW] immediately implies the following:

**Theorem 3** Let \( G \) be a torsion-free non-free 2-generated group acting simplicially and without inversion on the simplicial tree \( T \). Then any generating pair is Nielsen-equivalent to a pair \( \{g, h\} \) such that either

1. \( T_g \cap T_h \neq \emptyset \) or
2. \( T_g \cap hT_g \neq \emptyset \).

**Remark** It should be noted that the proof actually shows that once there is generating pair \( \{g, h'\} \) such that \( g \) acts with a fixed point then one of the two statements hold for a pair \( \{g, h\} \) where \( h = g^{z_1}h'g^{z_2} \) for some \( z_1, z_2 \in Z \).

Following [Se] we say that a splitting of a group \( G \) as a fundamental group of a graph of groups is 2-acylindrical if no non-trivial element \( g \in G \) fixes a segment of length greater than 2 in the corresponding Bass-Serre tree. We will sometimes use the following simple fact:

**Lemma 1** Suppose that \( G \) is the fundamental group of a 2-acylindrical graph of groups and that \( T \) is the corresponding Bass-Serre tree.

Then every vertex \( v \) that is fixed under the action of a non-trivial power of an element \( g \) is in distance at most 1 of a vertex fixed under the action of \( g \).

**Proof** Choose \( n \) such that \( g^n \neq 1 \) and that \( g^n v = v \). Let \( w \) be the vertex of \( T \) that is closest to \( v \) such that \( gw = w \). Clearly \( g^n \) fixes the segment \([v, w] \). The minimality of the distance between \( v \) and \( w \) guarantees that \( g \) fixes no point of \([v, w] \) except \( w \). It follows that the \([v, w] \cap g[v, w] = [v, w] \cap [gw, gw] = [v, w] \cap [gv, w] = \{w\} \). In particular \((v, gv) = 2d(v, w) \). Now \( g^n g[v, w] = g^{n+1}g[v, w] \) i.e. \( g \) fixes \([v, w] \) and \([g, v, w] \) and therefore \([v, gw] \). As the action of \( G \)
is 2-acylindrical this implies that $d(v, gv) \leq 2$ and therefore $d(v, w) \leq 1$ which proves the assertion. 

The following Lemma is a generalization of Corollary 3.2 of [KW] which is in turn a generalization of the main result of [BJ1].

**Lemma 2** Suppose that $G = A \ast_C B$ with $C \neq 1$ is torsion-free and that this splitting is 2-acylindrical. If $G$ is 2-generated then there exists a generating pair \{g, h\} such that (possibly after exchanging $A$ and $B$) $g \in A$ with $g^n \in C - 1$ and that one of the following holds:

1. $h \in B - C$.
2. $h = ab$ with $a \in A - C$, $b \in B - C$ and $a^{-1}g^ma \in C$ for some $m \in \mathbb{N}$.
3. $h = bab^{-1}$ with $a^m \in C - 1$ for some $m \in \mathbb{N}$.

**Proof** Applying Theorem 3 to an arbitrary generating pair guarantees the existence of a generating pair \{g, h\} such that g acts with a fixed point and that either $T_g \cap T_h \neq \emptyset$ or $T_g \cap hT_g \neq \emptyset$.

Since the splitting is 2-acylindrical we know that there exists a vertex $v \in T_g$ such that $gv = v$ and that for every vertex $w \in T_g$ we have $d(v, w) \leq 1$. In the case that $h$ also acts with a fixed point, $T_h$ clearly has the same structure. After conjugation we have that $v$ is the vertex fixed under the action of $A$ or $B$, w.l.o.g. we assume that $Av = v$, i.e. $g \in A$.

1. **case:** $T_g \cap T_h \neq \emptyset$. Choose $z \in T_g \cap T_h$ and $x \in T_h$ with $d(z, x) \leq 1$ such that $hx = x$. Note that $d(v, x) \leq 2$. If $v = x$ then $(g, h) \in A$ which contradicts our assumption that \{g, h\} is a generating set of $G$. If $d(v, x) = 1$ then we can assume after conjugation with an element of $A$ that $[x, v]$ is the edge fixed under the action of $C$ and $v$ is the vertex fixed by $B$, in particular $h \in B$. It now follows that either a power of $g$ or a power of $h$ must fix $[x, v]$ since otherwise $T_g \cap T_h = \emptyset$, i.e. that either $g^n \in C - 1$ or $h^n \in C - 1$ for some power of $g$ or $h$. If $g^n \in C - 1$ then we are in situation 1 of the lemma, otherwise we are in situation 1 after exchanging $A$ and $B$ and $g$ and $h$. If $d(x, v) = 2$ then $[x, v] = [x, z] \cup [z, v]$ and after conjugation with an element of $A$ we can assume that $[v, z]$ is fixed under the action of $C$. This implies that $g^n \in C - 1$ for some $n \in \mathbb{N}$. It is clear that $x = bv$ for some $b \in B$, it follows that $h = bab^{-1}$, i.e. that $h = bab^{-1}$ for some $a \in A$. Since a non-trivial power of $h$ fixes $z$ it follows that $h^m = ba^m b^{-1} \in B - 1$ for some $m \in \mathbb{N}$, i.e. $a^m \in C - 1$ for some $m \in \mathbb{N}$. This gives situation 3.

2. **case:** $T_g \cap hT_g \neq \emptyset$. Chose $x, y \in T_g$ such that $hx = y$. Since in the Bass-Serre tree of an amalgamated product $G$-equivalent vertices are always is even distance we can assume that either $d(x, y) = 0$ or $d(x, y) = 2$. If $d(x, y) = 0$ we are in the first case, i.e. we can assume that $d(x, y) = 2$. It is clear that $d(v, x) = d(v, y) = 1$ and after conjugation with an element of $A$ we can assume that $x$ is the vertex fixed under the action of $B$, in particular $[v, x]$ is fixed under the action of $C$ which implies that $g^n \in C - 1$ for some $n \in \mathbb{N}$. Now it is clear that $ax = y$ for some $a \in A$. This implies that $h = ba$ for some $b \in \text{Stab} y = B$ and $b \in B$. Since some power $g^k$ of $g$ fixes $[v, y]$ it follows that $g^{nk}$ fixes $[v, x]$ and $[v, y] = a[v, x]$. This implies in particular that $a^{-1}g^{nk}a \in C$. This gives situation 2. 

\[\square\]
The next lemma gives a bound on the number of vertex groups if the group is 2-generated and the graph underlying the splitting is homeomorphic to a circle.

**Lemma 3** Let $G$ be a torsion-free non-free 2-generated group. Then $G$ does not admit a 2-acylindrical splitting whose underlying graph is homeomorphic to a circle and has more than 2 vertices.

**Proof** Suppose that the underlying graph has at least three vertices. We apply Proposition 3, i.e. we can assume that there exists a generating pair $\{g, h\}$ such that either $T_g \cap T_h \neq \emptyset$ or that $T_g \cap hT_g \neq \emptyset$.

The case $T_g \cap T_h \neq \emptyset$ cannot occur since all elements that act with a fixed point lie in the kernel of the quotient map that quotients out all vertex groups. In our case however this quotient is an infinite cyclic group since there lies one edge outside a maximal tree of the graph underlying the splitting, a contradiction.

It remains to rule out the case $T_g \cap hT_g \neq \emptyset$, i.e. that there exists $x, y \in T_g$ such that $hx = y$. The diameter of $T_g$ is at most 2 since the splitting is 2-acylindrical, in particular $d(x, y) \leq 2$. This however implies that $\langle g, h \rangle$ lies in a vertex group of the graph of group obtained from the original graph of groups after collapsing the at most two edges corresponding to the edges of $[x, y]$. This however implies that $\langle g, h \rangle \neq G$ since the resulting graph of groups still contains at least one edge.

The next two lemmas give some information on the generators if there are only two vertices and if there is only one vertex, respectively.

**Lemma 4** Let $A$ and $B$ be two torsion-free groups and $C_1, C_2 \subset A$ and $C_3, C_4 \subset B$ subgroups such that there exist isomorphisms $\phi_1 : C_1 \rightarrow C_3$ and $\phi_2 : C_2 \rightarrow C_4$. Let $G = (A * C_1 C_2 B) * C_3 C_4$, i.e. $G = (A, B, t | \phi_1(c_1) = c_1, \phi_2(c_2) = tc_2t^{-1})$ and suppose that this splitting is 2-acylindrical. Then there exists a generating pair $\{g, h\}$ such that (possibly after exchanging $A$ and $B$)

1. $g \in B$, $g^n \in C_3 - 1 = C_1 - 1$ and $g^m \in bC_4b^{-1} - 1 = btC_2t^{-1}b^{-1}$ for some $b \in B$ and $n, m \in \mathbb{N}$ and

2. $h = bta$ for some $a \in A$.

**Proof** We study the action on the associated Bass-Serre tree. As in the proof of Lemma 3 we apply Theorem 3 and exclude the case that $g$ and $h$ act with a fixed point, i.e. we can assume that $T_g \cap hT_g \neq \emptyset$ and that $h$ acts without fixed point. Choose a vertex $v \in T_g$ such that $gv = v$; this implies that $d(v, x) \leq 1$ for all $x \in T_g$ since the action is 2-acylindrical. After conjugation and possibly interchanging $A$ and $B$ we can assume that this vertex is fixed by $B$, i.e. that $g \in B$. Choose further $x, y \in T_g$ such that $hx = y$. Note that $d(x, y)$ is even since the segment $[x, y]$ must map onto a closed path on the quotient graph. Since we assume that $h$ acts without fixed point this implies that $d(x, y) = 2$, i.e. that $[x, y] = [v, x] \cup [v, y]$ where $[v, x]$ and $[v, y]$ are edges. We can assume that there exists no $b \in B$ such that $b[v, x] = [v, y]$ since $h$ would then be a product of elliptic elements and therefore together with $g$ lie in the kernel of the map that quotients out the vertex groups. It follows that we can assume that $[v, x]$ is $B$-equivalent to the edge associated to $C_3$ and $[v, y]$ is $B$-equivalent to the edge associated to $C_4$. After conjugation in $B$ we can assume that $[v, x]$ is the edge associated to $C_3$, i.e. that $x$ is fixed under $A$. It follows that $[v, x]$ is
fixed by $C_3$ and $[v, y]$ is fixed under $bC_4b^{-1}$ where $b \in B$ is such that $btx = y$. This guarantees the first conclusion of the lemma. Since $y = btx$ it follows that $h = sbt$ for some $s \in \text{Stab } y = bt(\text{Stab } x)t^{-1}h^{-1} = btat^{-1}b^{-1}$. This implies that $sbt = (bt^{-1}b^{-1})bt = bta$ for some $a \in A$ which proves the Lemma.

Lemma 5 Let $A$ be a group, $C_1, C_2$ be two isomorphic subgroups with an isomorphism $\phi : C_1 \rightarrow C_2$. Let $G = A \ast_{C_1, C_2} = \langle A, t \mid tc_1t^{-1} = \phi(c_1) \rangle$ and suppose this splitting is 2-acylindrical. Then there exists a generating pair $\{g, h\}$ such that (possibly after exchanging $C_1$ and $C_2$) $g \in A$ and $g^\phi \in C_1$ and $h = at$ for some $a \in A$.

Proof As in the proof of Lemma 4 we can assume that there is a generating pair $\{g, h\}$ such that $T_g \cap hT_g \neq \emptyset$ and that $h$ acts without fixed point. Chose $x, y \in T_g$ such that $hx = y$. It is clear that $d(x, y)$ is odd since otherwise the exponent sum of all occurrences of $t$ in $h$ would be even and therefore not lie in $\{-1, 1\}$, which implies that the images of $g$ and $h$ do not generate the cyclic quotient of $G$ by $N_G(A)$ which is generated by the image of $t$. Since the diameter of $T_g$ is at most 2 this implies that $d(x, y) = 1$. Possibly after exchanging $x$ and $y$ we can assume that $gx = x$ since the action is 2-acylindrical. After conjugation we can assume that $x$ is fixed under the action of $A$ and that $[x, y]$ is fixed under the action of either $C_1$ or $C_2$. Possibly after exchanging $C_1$ and $C_2$ we can assume that $[x, y]$ is fixed by $C_1$, i.e. that $y = t^{-1}x$. It follows that $h = st^{-1}$ where $s \in \text{Stab } y = t^{-1}At$, i.e. $h = (t^{-1}at)t^{-1} = t^{-1}a$ for some $a \in A$. After replacing $h$ with its inverse this proves the lemma.

In some instances it is important to see that a set $S$ does not generate a given group $G$. If $G$ is given as a fundamental group of a graph of groups this can be seen if the induced splitting of the subgroup generated by $S$ is distinct from the original splitting of $G$. In $\text{FRW}$ situations where investigated when the induced splitting can be read of a particular generating set. The following Proposition describes the situations needed in the course of this paper. It is a direct consequence of the discussion in $\text{FRW}$.

Proposition 1 Let $G$ be a group that acts simplicially without inversion on a simplicial tree $T$. Let $ET$ be the set of edges of $T$.

1. Let $e = [v, w] \in ET$. Suppose that $G_v, G_w \subset G$ such that $G_vv = v$, $G_ww = w$ and $G_v \neq G_v \cap \text{Stab } e = G_w \cap \text{Stab } e \neq G_w$. Then the induced splitting of $U = \langle G_v, G_w \rangle$ is $G_v \ast_{G_v \cap \text{Stab } e} G_w$.

2. Let $e_1 = [v, w], e_2 = [v, z] \in ET$. Suppose that $G_v, G_w, G_z \subset G$ such that $G_vv = v$, $G_ww = w$ and $G_zz = z$. Suppose further that $G_v \cap \text{Stab } e_1 = G_w \cap \text{Stab } e_1 \neq G_v$, that $G_v \cap \text{Stab } e_2 = G_z \cap \text{Stab } e_2 \neq G_z$ and that $e_1$ and $e_2$ are not $G_v$ equivalent. Then the induced splitting of $U = \langle G_v, G_w, G_z \rangle$ is $G_w \ast_{G_v \cap \text{Stab } e_1} G_v \ast_{G_v \cap \text{Stab } e_2} G_z$.

3. Let $e = [v, w] \in ET$. Suppose that $G_v, G_w \subset G$ and $h \in G$ such that $G_vv = v$, $G_ww = w$, $G_v \cap \text{Stab } e = G_w \cap \text{Stab } e$, $hv = w$ and $h^{-1}G_vh = G_v$. Suppose further that $U = \langle G_v, h \rangle$. Then the induced splitting of $U$ is the HNN-extension of $G_v$ along $G_v \cap \text{Stab } e$. 

8
4. Let \(e_1 = [v, w], e_2 = [v, z] \in ET\). Suppose that \(G_v, G_w, G_z \subset G\) and \(h \in G\) such that \(G_v, G_w = v, G_vw, G_z = z, G_v \cap \text{Stab} e_1 = G_w \cap \text{Stab} e_1,\) \(G_v \cap \text{Stab} e_2 = G_z \cap \text{Stab} e_2\), \(w = hz, G_z = hG_wh^{-1}\), that \(e_1\) and \(e_2\) are not \(G_v\)-equivalent and that \(e_1\) and \(he_2\) are not \(G_w\)-equivalent. The induced splitting of \(U = (G_v, G_w, h)\) is \((G_v *_{C_1} G_w) *_{C_2} G_z\) where \(C_1 = G_v \cap \text{Stab} e_1 = G_w \cap \text{Stab} e_1\) and \(C_2 = G_v \cap \text{Stab} e_2 = G_z \cap \text{Stab} e_2\). □

2 Some topological lemmas

If \(M^3\) is a Seifert manifold with base orbifold \(O\) we denote \(\pi_1(M^3)\) by \(G\), the element of \(G\) corresponding to the fibre by \(f\) and \(\pi_1(O)\) by \(F\). We then have the following exact sequence

\[
1 \to \langle f \rangle \to G \xrightarrow{\pi} F \to 1.
\]

The subgroups of \(F\) corresponding to boundary curves of the base orbifold are the images of the subgroups of \(G\) corresponding to the boundary components of \(M^3\).

The only Seifert manifolds with boundary whose fibration is not unique up to isotopy are \(\mathbb{R} \times S^1 \times S^1\) and the orientable circle bundle over the Möbius band, denoted as \(Q\) by Waldhausen [Wald], which can also be fibered over \(D(2, 2)\). If \(Q\) is fibered over the Möbius band we denote the fibre by \(fQ\) and if \(Q\) is fibered over \(D(2, 2)\), we denote the fibre by \(fQ_1\).

Considering \(Q^3\) as the orientable circle bundle over the Möbius band yields the presentation \(\pi_1(Q^3) = \langle fQ, s | sfQs^{-1} = fQ^{-1} \rangle\), the standard presentation of the Klein bottle group. Considering it as the Seifert manifold over \(D(2, 2)\) we get the presentation \(\pi_1(Q^3) = \langle x, y, fQ, [x, fQ], [y, fQ], x^2 = fQ, y^2 = fQ \rangle\). The isomorphism is given by \(fQ = xyfQ^{-1}\) and \(s = x\).

Let \(C\) be the normal subgroup of \(\pi_1(Q^3)\) corresponding to the boundary which is generated by \(\{s^3, fQ\}\) and \(\{xy, fQ\}\) respectively. We see that for any \(g \in \pi_1(Q^3) - C\) we have that \(g^2 \in \langle fQ \rangle\) and \(g(xy)^nfQ^{-m}g^{-1} = x^{-1}(xy)^nfQ^{-m}x = (xy)^{-n}fQ^{-m+2n}\).

Lemma 6 Let \(M^3\) be a \(\partial\)-incompressible Seifert-manifold, \(T\) a boundary component and \(P \subset G\) be a subgroup corresponding to \(T\). Suppose further that \(g \in G\) generates a maximal cyclic subgroup such that \(\langle g \rangle \cap P = \langle g^n \rangle\) for some \(n \in \mathbb{N}\) and that \(P_1 \subset P\) with \(g^n \in P_1\). Then \(\langle P_1, g \rangle \cap P = P_1\).

Proof If \(g \in P\) the assertion is trivial, i.e. we can assume that \(g \in G - P\). If \(M^3\) is \(Q^3\) we write it as the Seifert space over \(D(2, 2)\). Let now \(w\) be the element of \(F\) that corresponds to the boundary curve of \(O\) that comes from \(T\). Since \(O\) is not the Möbius band we know that all roots of \(w^n\) are of type \(w^k\). This implies that for any element \(g \in G\) such that \(\pi(g^n) \in \langle w \rangle - 1\) for some \(n \geq 1\) we have \(g \in P\). It follows that \(\pi(g^n) = 1\), i.e. that \(g^n \in \langle f \rangle\). Since we assume that \(g\) generates a maximal cyclic subgroup we have \(g^n = f^{\pm 1}\), possibly after replacing \(g\) with its inverse we can assume that \(g^n = f\).

The fundamental group \(F\) of the base space is a free product of cyclic groups, i.e. \(F = \langle s_1, \ldots, s_n | s_1^{n_1}, \ldots, s_n^{n_k} \rangle\) with \(n_k \in \mathbb{N} \cup \{\infty\}\). It is clear that \(P_1 = \langle f, m^k \rangle\) for some \(k \in \mathbb{N}\) where \(m \in P\) is chosen such that \(P = \langle f, m \rangle\).
Choose now \( l \) such that \( \langle P_1, g \rangle \cap P = \langle f, m^l \rangle \); we have to show that \( k = l \). The projection \( \pi \) is an injection when restricted to \( \langle m \rangle \), has kernel \( \langle f \rangle \), maps \( m \) onto \( w \) (or \( w^{-1} \)) and \( g \) onto a conjugate of \( s_i \) for some \( i \in \{ 1, \ldots, r \} \). This means that \( \pi(P_1) = \langle \pi(m^k) \rangle = \langle w^\pm k \rangle \) and \( \pi(\langle P_1, g \rangle) \cap \langle \pi(m) \rangle = \langle \pi(m^l) \rangle = \langle w^\pm l \rangle \). The assertion now follows from the fact that \( \pi(\langle P_1, g \rangle) \cap \langle \pi(m) \rangle = \langle w^k, s \rangle \cap \langle w \rangle \) for some \( s \in F \) that is conjugate to some \( s_i \) and that in \( F \) the statement \( \langle w^k, s \rangle \cap \langle w \rangle = \langle w^k \rangle \) holds for any element \( s \) that is conjugate to one of the \( s_i \). This can be seen by looking at the orbifold group obtained by adding the relation \( w^k \) to \( F \): The resulting orbifold is not bad since \( O \) was not a disk with less than two cone points, and the image of \( s \) in the quotient is not conjugate to an elliptic element corresponding to the new cone point.

**Lemma 7** Let \( M^3 \) be a \( \partial \)-incompressible Seifert-manifold with boundary component \( T \) and \( P \subset G \) be a corresponding subgroup. Suppose further that \( g \in G \) such that \( g^n \in P \) for some \( n \in \mathbb{N} \) and that \( G = \langle P, g \rangle \). Then \( M^3 \) is a Seifert manifold with basis \( D(p, q) \) or \( A(p) \) and \( g \) is a root of the fibre.

**Proof** The fact that \( g \) is a root of the fibre follows as in the proof of Lemma 3, possibly after rewriting \( Q^3 \) as the Seifert space with base space \( D(2, 2) \). Now the base space must have at least one boundary component and be generated by the element corresponding to the boundary and a torsion element. The only base spaces with this property are \( D(p, q) \) and \( A(p) \).

**Lemma 8** Let \( M^3 \) be an orientable Seifert-manifold which is not \( T^2 \times I \). Suppose that \( T_1 \) and \( T_2 \) are two boundary components with corresponding subgroups \( P_1 \) and \( P_2 \). Suppose that \( g \in P_1 \) and \( h \in P_2 \), that neither \( g \) nor \( h \) correspond to the fibre and that \( \langle g, h \rangle \) is not free. Then the following hold:

1. \( O \) is of type \( A(p) \), i.e. \( G = \langle s, x, f | [s, f], [x, f], x^p = f^h \rangle \) with \( 1 \leq b < |p|/2 \) and \( (p, b) = 1 \) and after conjugation we have \( g = s f^m \) and \( h = x s f^n \) for some \( m, n \in \mathbb{Z} \). In particular \( \langle g, h \rangle \) maps surjectively onto \( F \).

2. If \( \langle g, h \rangle = G \) then we have additionally that \( b = 1 \) and after conjugation we have \( g = s f^m \) and \( h = x s f^n \) for some \( m \in \mathbb{Z} \). In particular \( M^3 \) is the exterior of a 2-bridge link and \( g \) and \( h \) correspond to the meridians.

**Proof** We first show that \( H := \pi(\langle g, h \rangle) = \langle \pi(g), \pi(h) \rangle \subset F \) is free in \( \pi(g) \) and \( \pi(h) \) unless \( O \) is of type \( A(p) \) and \( \pi(g) \) and \( \pi(h) \) correspond to the boundary curves of \( O \). This clearly implies the first assertion of the lemma.

Note that \( \pi(g) \) and \( \pi(h) \) are powers of elements that correspond to the boundary curves of \( O \). Since \( H \) is a subgroup of a free product of cyclic group we know by Kurosh's subgroup theorem that \( H \) itself is a free product of cyclic groups. It follows that \( H \) is free if and only if \( H \) is torsion-free.

Suppose now that either \( O \) is not of type \( A(p) \) or that \( O \) is of type \( A(p) \) and \( \pi(g) \) (the case of \( \pi(h) \) is analogous) is a proper power of a boundary curve. We look at the quotient map \( \phi : F \to F/\langle \pi(g) \rangle \). It is clear that no torsion element lies in the kernel of \( \phi \) since \( O \) has at least two boundary components and the resulting orbifold is therefore good. In particular \( \phi(H) \) has torsion if \( H \) has torsion. It is clear that \( \phi(H) = \langle \phi(\pi(h)) \rangle \). The group \( \langle \phi(\pi(h)) \rangle \) however is infinite cyclic since it is a subgroup of the cyclic subgroup of an orbifold group corresponding to a boundary curve which is infinite because the orbifold
is different from $D$ (since $O$ was assumed to not be of type $A$) and $D(p)$ (note, that if $O$ was of type $A(p)$ the new orbifold is of type $D(p, q)$ since we assumed that $g$ was a proper power of the boundary curve). It follows that $H$ is torsion free and therefore free.

It follows that either $⟨g, h⟩$ is free or that $G = ⟨s, x, f | [s, f], [x, f], x^p = f^b⟩$ for some $p > b > 0$ with $b < |p/2|$ and $(b, p) = 1$ and that $g$ is conjugate to an element of type $sf^k$ and $h$ is conjugate to an element of type $(xs)f^l$ for some $k, l \in \mathbb{Z}$.

**Claim:** Either $⟨\pi(g), \pi(h)⟩$ is free in $\pi(g)$ and $\pi(h)$ or after conjugation $\pi(g) = s$ and $\pi(h) = xs$.

After a suitable conjugation $\pi(g)$ is in the desired form, i.e. $\pi(g) = s$ and $\pi(h) = w(xs)w^{-1}$ for some $w \in F = ⟨s, x | x^p⟩ = ⟨s | −⟩ * ⟨x | x^p⟩$. It is easy to see that $\pi(h)$ has normal form of one of the types $y_1 \cdots (y_2)_{y_1} \cdots y_1$, $y_1 \cdots (y_2)_{x(y_2)} \cdots y_1$, $y_1 \cdots y_2 x(y_2) \cdots y_1$ or $y_1 \cdots y_2 s(x^{-1}) \cdots y_1$. If $y_1 \in ⟨s⟩$ we can conjugate $\pi(g)$ and $\pi(h)$ by $y_1$. This conjugation does not change $\pi(g)$ but reduces the length of $\pi(h)$; i.e. we can assume that $y_1 \in ⟨x⟩$. If $l \geq 2$ we see that no cancellation occurs in products of powers of $\pi(g)$ and $\pi(h)$, since both $\pi(g)$ and $\pi(h)$ are of infinite order this implies that $⟨\pi(g), \pi(h)⟩$ is free in $\pi(g)$ and $\pi(h)$. If $l = 1$ cancellation occurs if and only if $y_1 = y_1 = 1$. It follows that either $\pi(h) = xs$ or $\pi(h) = sx$. After conjugation of the pair $⟨g, h⟩$ we have $\pi(h) = xs$ which proves the claim.

If $⟨g, h⟩$ is not free in $g$ and $h$ it follows that $⟨\pi(g), \pi(h)⟩$ is not free in $\pi(g)$ and $\pi(h)$, i.e. after conjugation $g = sf^k$ and $h = (xs)f^l$. It is clear that $⟨g, h⟩ = (sf^k, (xs)f^l) = (sf^k, x^{f^l-k})$ maps surjectively to $F$, it remains to verify the second assertion, i.e. we have to determine the situations where in addition $⟨g, h⟩ \cap ⟨f⟩ = ⟨f⟩$. Now a freely reduces products in $x$ and $s$ is trivial in $F$ if and only if it is a product of conjugates of $p_{th}$ powers of $x$. It follows that a freely reduced product in $sf^k$ and $x^{f^l-k}$ lies in the kernel of $\pi$ if and only if it is a product of conjugates of $p_{th}$ powers of $xf^{l-k}$. Since $w(xf^{l-k})w^{-1} = wxf^{p(l-k)}w^{-1} = wfp^{l-k}w^{-1} = w^{f^{b+a(l-k)}}w^{-1} = f^{b+a(l-k)}$ for any $w \in F$ it follows that $⟨g, h⟩ \cap ⟨f⟩ = ⟨f^{b+a(l-k)}⟩$. This implies that $b + a(l - k) = ±1$ and therefore $b = 1$ and $l = k$. 

**Lemma 9** Let $M^3$ be an orientable $\partial$-incompressible Seifert-manifold which is not $T^2 \times I$. Suppose that $T$ is a boundary component. Let $P \subset \pi_1(M^3)$ be the corresponding subgroup and $g \in P$ where $g$ is primitive in $P$ and does not correspond to the fibre. Then there exists an element $h \in \pi_1(M^3)$ such that $⟨g, h⟩ = \pi_1(M^3)$ if and only if one of the following holds (after conjugation):

1. $O$ is of type $A(p)$, i.e. $G = ⟨s, x, f | [s, f], [x, s], x^p = f^b⟩$ and $g = sf^k$ or $g = xsf^k$ for some $k$.

2. $O$ is of type $D(p, q)$, i.e. $G = ⟨x, y, f | [x, f], [y, f], x^p = f^{b_1}, y^q = f^{b_2}⟩$ and $g = xyf^n$ for some $n \in \mathbb{N}$.

3. $O$ is of type $M\partial(p)$, i.e. $G = ⟨x, s, f | [x, f], x^p = f^{b_1}, sfs^{-1} = f^{-1}⟩$ and either $b = 1$ and $g = s^ax$ or $a = 1$, $b = 1$ and $g = s^axf^{-1}$.

4. $O$ is of type $M\partial$, i.e. $G = ⟨s, f | sfs^{-1} = f^{-1}⟩$ and $g = s^{2l}f^\pm 1$ for some $l \in \mathbb{Z}$. 

11
Except in the first case this implies that \( M^3 \) is the exterior of a 1-bridge knot in a lens space and \( g \) corresponds to the meridian.

**Proof** Let \( M(g) \) be the manifold obtained by a Dehn filling of \( M \) along the (simple) curve on \( T \) corresponding to \( g \). Since \( g \) is not a power of the fibre we can extend the Seifert fibration of \( M \) to \( M(g) \). The classification of small Seifert manifolds shows that \( M(g) \) has cyclic fundamental group, i.e. is a lens space, if and only if \( M \) and \( g \) are as in one of the cases (1)-(4) of Lemma 9. Except in the first cases this implies that \( M^3 \) is the exterior of a 1-bridge knot in a lens space and \( g \) corresponds to the meridian, see Lemma 1 of [W3]. This guarantees in particular the existence of the appropriate \( h \). In the first case \( h \) can be chosen as a generator of the cyclic subgroup \( \langle x, f \rangle \). \( \Box \)

**Lemma 10** Let \( M^3 \) be an orientable \( \partial \)-incompressible Seifert-manifold that is not \( T^2 \times I \) with a boundary component \( T \) and \( P \subset G = \pi_1(M^3) \) be a corresponding subgroup.

Suppose further that \( g \in P \) and \( h \in wPw^{-1} - P \) for some \( w \in G \) and that \( g \) does not correspond to a power of the fibre. Denote the intersection number of \( g \) and \( h \) with the fibre on \( T \) by \( n_g \) and \( n_h \).

Then \( \langle g, h \rangle \) is free in \( g \) and \( h \) or \( M^3 \) is a Seifert space with base space of type \( D(2,q) \), \( \min(|n_g|,|n_h|) = 1 \) and \( \max(|n_g|,|n_h|) \leq 3 \). If \( \max(|n_g|,|n_h|) > 1 \) or \( q \) is even we further have that \( h \) is not conjugate to an element of \( P \) in \( \langle g, h \rangle \).

**Proof** Possibly after conjugation and exchanging \( g \) and \( h \) and replacing \( g \) or \( h \) by their inverses we can assume that \( 1 \leq n_g \leq n_h \) since neither \( g \) nor \( h \) corresponds to a power of the fibre. We can assume that \( M^3 \) is not \( Q^3 \) since in this case \( wPw^{-1} = P \) for all \( w \in G \). We first show that \( \langle g, h \rangle \) is free in \( g \) and \( h \) unless the base \( \mathcal{O} \) is of type \( D(p,q) \). We actually show that the projections \( \pi(g) \) and \( \pi(h) \) generate a free subgroup in the base group. Note first that \( \langle \pi(g), \pi(h) \rangle \) is not cyclic since the cyclic subgroup corresponding to the boundary curve is malnormal and therefore the unique maximal cyclic subgroup that contains \( \pi(g) \). However \( \pi(h) \) does not lie in this subgroup. Since the base group is a free product of cyclic it follows from Kurosh’s theorem that \( \langle \pi(g), \pi(h) \rangle \) is the free product of cyclics. To conclude we need to show that \( \langle \pi(g), \pi(h) \rangle \) is torsion-free unless \( \mathcal{O} \) is of type \( D(p,q) \). It is clear that \( \langle \pi(g), \pi(h) \rangle \) lies in the kernel of the map that quotients out the element corresponding to the boundary curve of \( \mathcal{O} \) that corresponds to \( T \). This kernel however only contains torsion elements if the resulting orbifold is bad, i.e. if \( \mathcal{O} \) was of type \( D, D(p) \) or \( D(p,q) \). The first two cases cannot occur since we assume \( M^3 \) to be \( \partial \)-incompressible, i.e. the claim holds.

Suppose now that \( \mathcal{O} \) is of type \( D(p,q) \). The base group \( F \) has the presentation \( \langle a, b | a^p, b^q \rangle = \langle a | a^p \rangle * \langle b | b^q \rangle \) and \( \pi(g) = \langle ab \rangle^n, \pi(h) = v(ab)^{n_h}v^{-1} \) for some \( v \in F \). We write \( v = x_1 \cdots x_k \) as a normal form with respect to the free product \( \langle a | a^p \rangle * \langle b | b^q \rangle \). It follows that \( \pi(h) = x_1 \cdots x_k(ab)^{n_h}x_k^{-1} \cdots x_1^{-1} \).

W.l.o.g. we can assume that the free product length of \( \pi(h) \) is minimal with respect to conjugation with powers of \( ab \) since this corresponds to conjugation of the pair \( \langle \pi(g), \pi(h) \rangle \). The normal form of \( \pi(h) \) is clearly of one of the types \( x_1 \cdots x_k(ab)^{n_h}x_k^{-1} \cdots x_1^{-1} \) with \( x_1a \in \langle a \rangle - 1 \), \( x_1 \cdots x_k(ab)^{n_h}a(bx_1^{-1}) \cdots x_1^{-1} \) with \( bx_1^{-1} \in \langle b \rangle - 1 \), \( x_1 \cdots x_k(ba)^{n_h}a(ba)^{-1}x_1^{-1} \cdots x_1^{-1} \) with \( xa_1 \in \langle a \rangle - 1 \) or \( x_1 \cdots x_k(ba)^{n_h}b(ax_1^{-1}) \cdots x_1^{-1} \) with \( ax_1^{-1} \in \langle a \rangle - 1 \).
If \( l \geq 2 \) then \( x_1 \neq b^{-1} \) and \( x_1 \neq a \) since otherwise we could reduce the length of \( \pi(h) \) by conjugation with \( ab \) or \( (ab)^{-1} \). It follows that no cancellation occurs in products in \( \pi(g) \) and \( \pi(h) \) with implies that \( \langle \pi(g), \pi(h) \rangle \) is free in \( \pi(g) \) and \( \pi(h) \).

Suppose now that \( l = 1 \). We carry out the case \( \pi(h) = (x_1a)b(ab)^{n_h-1}x_1^{-1}, \) the other cases are analogous. If no cancellation occurs we argue as before, i.e. we only have to study the cases (i) \( x_1a = a \), i.e. \( x_1 = 1 \) and (ii) \( x_1 = a \). In the first case this gives a contradiction to the assumption that \( \pi(h) = (x_1a)b(ab)^{n_h-1}x_1^{-1} \) is a normal form. In the second case we have \( \pi(h) = a^2(ba)^{n_h-1}ba^{-1} \) and after conjugation of \( \{\pi(g), \pi(h)\} \) with \( a \) we have \( \pi(g) = (ba)^{n_g} \) and \( \pi(h) = (ab)^{n_h} \). Since we assume that \( (p, q) \neq (2, 2) \) we easily see that in a product of length two in \( \pi(g) \) and \( \pi(h) \) at most one letter cancels.

This shows that \( \langle \pi(g), \pi(h) \rangle \) is free in \( \pi(g) \) and \( \pi(h) \) if \( \min(n_g, n_h) \geq 2 \), i.e. if the length of a reduced form is at least 4.

Suppose that \( n_g = 1 \), i.e. \( \pi(g) = ab \) and \( \pi(h) = (ba)^{n_h} \). If \( p \neq 2 \) and \( q \neq 2 \) we see as before that \( \langle \pi(g), \pi(h) \rangle \) is free in \( \pi(g) \) and \( \pi(h) \) since no cancellation occurs.

Suppose that \( p = 2 \), the case \( q = 2 \) is analogous. It follows that \( \langle \pi(g), \pi(h) \rangle = \langle ab, (ba)^{n_h} \rangle = \langle ab, (ba)^{n_h} \cdot ab \rangle = \langle ab, (ba)^{n_h-1}b^2 \rangle \). If \( n_h \geq 4 \) or \( q \geq 4 \) and \( |n_h| \geq 2 \) then \( (ba)^{n_h-1}b^2 \) is of infinite order and any power has normal form that starts with \( b \) and ends with \( b^2 \). Again no cancellation occurs and we see that \( \langle \pi(g), \pi(h) \rangle \) is free.

If \( q = 3 \) and \( n_h \in \{2, 3\} \) or \( q \geq 4 \) even and \( n_h = 1 \) then cancellation arguments show that \( \langle \pi(g), \pi(h) \rangle = \langle ab, (ba)^{n_h-1}b^2 \rangle = \langle ab \rangle \ast \langle (ba)^{n_h-1}b^2 \rangle \cong \mathbb{Z} \ast \mathbb{Z}_k \) for some \( k \in \mathbb{Z} \) (which depends on the situation we are in). The normal form of \( \pi(h) \) with respect to this free product has length \( 2 \) and is clearly not conjugate to an element of \( \langle ab \rangle \), which have normal form of length \( 1 \). This implies that \( h \) is not conjugate to an element of \( P \) in \( \langle g, h \rangle \).

\[ \square \]

**Lemma 11**: Let \( M^3 \) be an orientable \( \partial \)-incompressible Seifert-manifold which is not \( T^2 \times I \) and not \( Q^3 \) with a boundary component \( T \) and \( P \subset G = \pi_1(M^3) \) be a corresponding subgroup. Let \( g \in P \) such that \( g \) is not the fibre. Suppose that \( h \in \pi_1(M^3) \setminus P \) such that \( \langle g, hgh^{-1} \rangle \) is not free. Then one of the following holds:

1. If \( \langle g, hgh^{-1} \rangle = G \) for some \( h \in G \) then \( M^3 \) is the exterior of the \( (2,p) \)-torus knot and \( g \) corresponds to a meridian. In particular \( M^3 \) is the exterior of a 2-bridge knot.

2. If \( \langle g, hgh^{-1} \rangle \) maps surjectively onto the base group for some \( h \in G \) then the base orbifold is of type \( D(2,p) \) with odd \( p \) and \( g \) maps onto the element of the base group corresponding to a boundary curve.

3. The base manifold is of type \( D(2,2l) \) and \( g \) maps onto the element of the base group corresponding to a boundary curve. In this case \( hgh^{-1} \) is not conjugate to an element of \( P \) in \( \langle g, hgh^{-1} \rangle \).

**Proof**: It is clear that \( g \) and \( hgh^{-1} \) have the same non-trivial intersection number \( n \) with the fibre. It follows therefore immediately from Lemma 10 that the base is of type \( D(2,n) \). It further follows from the proof of Lemma 10 that we can assume that \( \pi(g) = ab \) and \( \pi(h) = ba \). If \( n \) is odd this implies that \( \pi(g) \) and
\( \pi(h) \) generate \( F \) which puts us into situation 2. If \( n \) is even Lemma 10 implies that we are in situation 3.

To show that the first statement holds we look at the manifold \( M^3(g) \) obtained from \( M^3 \) by a Dehn filling killing the curve corresponding to \( g \). Since \( g \) has intersection number one with the fibre, we can extend the Seifert fibration of \( M^3 \) to a Seifert fibration of \( M^3(g) \). Since \( M^3(g) \) has trivial fundamental group it must be \( S^3 \). It follows that \( M^3 \) is a Seifert fibered knot exterior in \( S^3 \) that has base space \( D(2,q) \). This implies that \( M^3 \) is the exterior of the \((2,p)\)-torus knot. Now \( g \) must correspond to a meridian since torus knots have property P.  

**Proposition 2** Let \( M^3 \) be a compact orientable 3-manifold with a complete hyperbolic structure of finite volume on its interior. Suppose that \( U \) is a subgroup of \( \pi_1(M^3) \) which is generated by two parabolic primitive elements. Suppose furthermore that these two parabolic elements are conjugated in \( \pi_1(M^3) \) if \( \partial M^3 \) is connected.

Then either \( U \) is free or \( U \) is Abelian or one of the following holds:

1. \( U = \pi_1(M^3) \) and \( M^3 \) is the exterior of a 2-bridge knot or link in \( S^3 \).

2. \( |\pi_1(M^3) : U| = 2 \) and the covering space \( \hat{M}^3 \) of \( M^3 \) corresponding to \( U \) is the exterior of a 2-bridge link in \( S^3 \) (with 2 components).

Moreover the two parabolic generators of \( U \) correspond to meridian curves of the 2-bridge knot or link.

**Proof** Assume that \( U \) is neither Abelian, nor free. Since \( M^3 \) is irreducible and atoroidal, by [JS2, Thm.VI.4.1] \( U \) must be of finite index in \( \pi_1(H^3) \). Hence \( U \cong \pi_1(M^3) \), where \( M^3 \) is a finite covering of \( M^3 \). In particular the interior of \( \hat{M}^3 \) admits a complete hyperbolic structure with finite volume. The proof of Proposition follows now from Lemma 12, Lemma 13 and the work of M. Sakuma on symmetries of spherical Montesinos links [Sa2].

**Lemma 12** \( \hat{M}^3 \) is homeomorphic to the exterior of a 2-bridge knot or link \( L \subset S^3 \) and the two parabolic generators correspond to meridians of the knot or the link \( L \).

**Proof** Since \( \pi_1(\hat{M}^3) \) is generated by two parabolic elements, the proof follows essentially from [Ad, Thm.3.3], together with [BZm, Prop.3.2]. We show here how to use Thurston’s orbifold theorem (cf.[BP], [CHK]) to avoid to assume Poincaré conjecture in the proof.

A homological argument shows that \( \partial \hat{M}^3 \) has at most two components. These must be tori, since \( M^3 \) admits a complete hyperbolic metric of finite volume on its interior. According to [Ad, Corollary 3.2], the two generators are conjugate if and only if \( \hat{M}^3 \) has only one torus boundary component. Moreover by [Ad, Thm.2.2] each generator corresponds (up to conjugation) to a simple loop on \( \partial M^3 \). By gluing a solid torus to each boundary component so that a meridian of the solid torus goes to the boundary curve corresponding to a parabolic generator, one obtains a homotopy sphere \( \Sigma^3 \), that may or may not be irreducible.

Hence \( \hat{M}^3 \) is the exterior of a hyperbolic knot or link \( L \) in the homotopy sphere \( \Sigma^3 \). Moreover, \( \pi_1(\Sigma^3 - L) \) is generated by two meridians. To show that
\(\Sigma^3\) is in fact the true sphere \(S^3\) and \(L\) is a 2-bridge knot or link, we follows essentially the arguments in [BZ, Prop.3.2].

Let \(V^3\) be the 2-fold covering of \(\Sigma^3\) branched along \(L\). Then one has the exact sequence: \(\{1\} \to \pi_1(V) \to \pi_1(\Sigma^3 - L)/N \to \mathbb{Z}_2 \to \{1\}\), where \(N\) is the subgroup of \(\pi_1(\Sigma^3 - L)\) normally generated by the squares of all meridians of \(L\). Then the group \(\pi_1(\Sigma^3 - L)/N\) is a dihedral group or a cyclic group of order 2. In the last case, by the proof of the Smith conjecture, \(L\) would be a trivial knot contradicting that it is a hyperbolic knot or link. Therefore \(\pi_1(V)\) must be cyclic.

Since \(L\) is a hyperbolic knot or link, \(\Sigma^3 - L\) is irreducible and does not contain any essential properly embedded annulus. Hence by the equivariant sphere theorem [DD], \(V\) is irreducible and \(\pi_1(V)\) is finite cyclic. By Thurston’s orbifold theorem (cf.[BP],[CHK]), \(V\) is geometric, hence it is a lens space. Moreover the covering involution is conjugated to an isometry of the spherical structure on \(V\). Hence the quotient \(\Sigma^3\) is the true sphere \(S^3\) and the branching set \(L\) is a 2-bridge knot or link.

We use now that the two parabolic generators of \(U = \pi(\hat{M}^3)\) are primitive in \(\pi_1(M^3)\), and conjugated in \(\pi_1(M^3)\) if \(\partial M^3\) is connected, to show: \(\Sigma^3 - L\) is regular.

**Lemma 13** The finite covering \(p : \hat{M}^3 \to M^3\) is regular.

**Proof** Denote the two parabolic generators by \(g\) and \(h\). By Lemma [12] \(\hat{M}^3\) is the exterior of a 2-bridge knot or link \(L \subset S^3\) such that the two parabolic generators \(g\) and \(h\) correspond to simple closed meridian curves of \(L\) on \(\partial M^3\).

If \(\hat{M}^3\) is the exterior of a 2-bridge knot we choose a meridian curve \(\mu \subset \partial \hat{M}^3\) that corresponds to the free homotopy class of \(g\) and \(h\). If \(\hat{M}^3\) is the exterior of a 2-bridge link then we choose two meridian curves \(\hat{\mu}_1\) and \(\hat{\mu}_2\) on the different components of \(\partial \hat{M}^3\) that represent the free homotopy classes of \(g\) and \(h\).

Since \(g\) and \(h\) are primitive in \(\pi_1(M^3)\), analogously we can choose either a closed simple curve \(\mu\) or closed simple curves \(\mu_1\) and \(\mu_2\) on \(\partial M^3\), depending on whether \(\partial M^3\) is connected, which represent the free homotopy classes of \(g\) and \(h\) in \(\pi_1(M^3)\).

The choice of the curves clearly guarantees that either \(p(\hat{\mu}_i)\) if parallel to \(\mu\) on \(\partial M^3\), or that \(p(\hat{\mu}_1)\) and \(p(\hat{\mu}_2)\) are parallel to \(\mu\) or that \(p(\hat{\mu}_1)\) is parallel to \(\mu_1\) and that \(p(\hat{\mu}_2)\) is parallel to \(\mu_2\).

Moreover the fact that \(g\) and \(h\) are primitive in \(\pi_1(M^3)\) implies that in any case each component of \(p^{-1}(\hat{\mu}_i))\) or \(p^{-1}(\hat{\mu}_i))\), \(i \in \{1, 2\}\) is mapped homeomorphically under \(p\) on to \(p(\hat{\mu}_i)\) or \(p(\hat{\mu}_i)\), \(i \in \{1, 2\}\).

Hence, the covering map \(p : M^3 \to M^3\) extends to a true covering map \(\bar{p} : \mathbb{S}^3 \to M^3(\mu)\) (or \(\bar{p} : \mathbb{S}^3 \to M^3(\mu_1, \mu_2)\)), where \(M^3(\mu)\) (\(M^3(\mu_1, \mu_2)\)) is the closed orientable 3-manifold obtained by gluing a solid torus (two solid tori) to \(\partial M^3\), so that a meridian of the solid torus goes to the boundary curve \(\mu\) (the meridians of the solid tori go to \(\mu_1\) and \(\mu_2\)). Such a covering \(\bar{p}\) is regular, hence the covering \(p\) is regular.

We need now the following lemma:

**Lemma 14** Any orientation preserving finite order symmetry of \(\hat{H}^3\), without fixed point on \(\partial \hat{M}^3\) extends to a finite order symmetry of \(S^3\) preserving a 2-bridge knot or link.
Since any two bridge link admits an order two symmetry that exchanges its components, it suffices to prove the lemma when the symmetry preserves each component of \( \partial \hat{M}^3 \). Let \( L \subset S^3 \) be a 2-bridge knot or link whose exterior is \( \hat{M}^3 \). Let \( f : H^3 \to M^3 \) be a free finite order orientation preserving symmetry, that preserves each component of \( \partial \hat{M}^3 \).

A free orientation preserving diffeomorphism of order \( n \) of a torus \( T^2 \equiv S^1 \times S^1 \) is conjugated by a diffeomorphism isotopic to the identity to a map of the form:

\[
g(\theta, \phi) = (\theta + 2\pi r/p, \phi + 2\pi s/q),
\]

where \((r, p) = (s, q) = 1\) and \( \text{lcm}(p, q) = n \). Hence it is isotopic to the identity on \( T^2 \).

It follows that the restriction of \( f \) to each component of \( \partial \hat{M}^3 \) is isotopic to the identity. Hence it preserves the isotopy class of the meridian curves of \( L \) on each component of \( \partial M^3 \). Thus, \( f \) extends to a symmetry of \( S^3 \) preserving \( L \).

\[\square\]

**Remark 4** Lemma 13 and Lemma 14 show that, when \( \partial M^3 \) is connected, but \( \partial \hat{M}^3 \) has two components, \( p : \hat{M}^3 \to M^3 \) is a regular covering if and only if the two primitive parabolic generators of \( \pi(\hat{M}^3) \) are conjugate in \( \pi_1(M^3) \).

The following lemma finishes the proof of Proposition 2.

**Lemma 15** The finite covering \( p : \hat{M}^3 \to M^3 \) is either 1-sheeted or 2-sheeted in the case that \( M^3 \) is the exterior of a 2-bridge link \( L \subset S^3 \).

**Proof** Since the finite covering \( p : \hat{M}^3 \to M^3 \) is regular, the finite group of covering transformations extends to a finite group of free symmetries of \( S^3 \) preserving the 2-bridge knot or link with exterior \( M^3 \).

The symmetry group of a hyperbolic 2-bridge knot or link is known by M. Sakuma’s work (\cite{Sa2}), using Thurston’s orbifold Theorem (cf.\cite{BP},\cite{CHK}). In particular, the orientation preserving symmetry subgroup, acting freely on the exterior of 2-bridge knot or link, has order at most two. Moreover, it may have order two only in the case of a 2-bridge link, since a hyperbolic 2-bridge knot does not admit a free symmetry (\cite{GLM},\cite{Ha},\cite{Sa2}).

\[\square\]

### 3 The proof of Theorem 2

A first observation is that the splitting of \( G = \pi_1(M^3) \) that corresponds to the JSJ-decomposition of the 3-manifold \( M^3 \) is 2-acylindrical unless one of the Seifert pieces is homeomorphic to \( Q^3 \). This follows easily form the following three facts:

1. All properly embedded annuli in a Seifert manifold different from \( Q^3 \) are either parallel to the boundary or are vertical meaning that the intersection with the boundary components are the fibre, this implies that only the element corresponding to the fibre can fix two edges emanating at a vertex in the Bass-Serre tree corresponding to a Seifert piece.
2. For the hyperbolic (acylindrical) pieces there is no such element at all.

3. The fibre is never glued to the fibre at an essential JSJ-torus separating two Seifert pieces.

We now proceed with the proof of Theorem 2 for the case where the JSJ-decomposition does not contain any pieces that are homeomorphic to $Q^3$. We first consider the case where the JSJ-decomposition of the given 3-manifold has a separating torus and then where it does not. We conclude by dealing with the case that there are pieces of type $Q^3$.

### 3.1 The JSJ-decomposition has a separating torus $T$ and no piece of type $Q^3$

The torus $T$ splits the 3-manifold in two pieces $M_A$ and $M_B$. The fundamental group $G = \pi_1(M^3)$ then splits as an amalgamated product $A *_C B$ with amalgam $C \cong \mathbb{Z} \oplus \mathbb{Z}$ and $A$ and $B$ the fundamental groups of $M_A$ and $M_B$. This amalgamated product satisfies the condition of Lemma 3. We therefore have to investigate the following different cases, where $\{g, h\}$ is a generating set of $G$. It is clear that we can always assume $g$ to be primitive.

1. **Case**: $g \in A$ and $g^n \in C$ and $h \in B - 1$.

   It is clear that $g \in A - C$ since otherwise $(g, h) \in B$ which implies that $g$ and $h$ do not generate $G$. In particular it follows that $g$ lies in the fundamental group of the piece $M_A$ of the JSJ-decomposition of $M_A$ that contains $T$ since the splitting is 2-acylindrical. Since $g \notin C$ this implies that $M_A$, is Seifert and that $g$ is a root of the fibre. This also implies that $M_A = M_{A_1}$ since $(g, h) \subset \pi_1(M_{A_1} \cup_T M_B)$ and therefore $M_A$ cannot contain a piece besides $M_{A_1}$. We define $C' = (g^n, h) \cap C \subset B$. Now by Lemma 5 we get that $(g^n, h) \cap C = C'$. Proposition 1 (1) therefore implies that the induced splitting of $(g, h)$ is $(g, C') *_{C'} (g^n, h)$. Since $g$ and $h$ generate $G$ this implies that $C' = C$, that $A = (g, C)$ and that $B = (g^n, h)$. By Lemma 5 $A = (g, C)$ implies that $M_A$ is a Seifert manifold with base $D(p, q)$ or $A(p)$.

   It remains to analyze what the manifold $M_B$ can be. If $M_B$ has a trivial JSJ-decomposition and the only piece is hyperbolic we cannot say anything more, this puts us into situation 1 of Theorem 2.

   If $M_B$ consists of a single Seifert piece then by Lemma 5 $M_B$ must either be the exterior of a 1-bridge knot in a lens space and $g^n$ corresponds to a meridian of the knot or a Seifert space with base space $A(p)$ and $g^n$ corresponds to a curve that has intersection number one with the fibre. This puts us in situation 2 and 3, respectively, of Theorem 2.

   If $M_B$ has a non-trivial JSJ-decomposition then we study the action of $(g^n, h)$ on the Bass-Serre tree of the corresponding splitting of $B$. It is clear that $g^n$ acts with a fixed point since it lies in the vertex group that corresponds to the piece $M_{B_1}$ of the JSJ-decomposition of $M_B$ that has $T$ as a boundary component. It is also easy to see that $T_{g^n}$ consists of a single point: This is clear if $M_{B_1}$ is acylindrical (hyperbolic). If $M_{B_1}$ is Seifert it follows from the fact that no power of $g^n$ corresponds to a power of the fibre of $M_{B_1}$, otherwise $T$ wouldn’t be a torus of the JSJ-decomposition, but merely an essential torus in a Seifert piece. By the remark after Theorem 3 we can assume that either $T_{g^n} \cap T_h \neq \emptyset$ or that $T_{g^n} \cap hT_{g^n} \neq \emptyset$. In the second case $h$ must fix the same vertex as
$g^n$ and therefore lie in the same vertex group as $g^n$. This implies that $\langle g^n, h \rangle$ does only intersect the vertex group that corresponds to $M_{B_1}$ which implies that $g^n$ and $h$ do not generate $B$ since $M_B$ was assumed to have non-trivial JSJ-decomposition, a contradiction.

It remains to verify the case $T_{g^n} \cap T_h \neq \emptyset$: Since $g^n$ and $h$ act with a fixed point it follows that the JSJ-decomposition of $M_B$ does not contain a non-separating torus since otherwise $g^n$ and $h$ cannot generate $B$ by the argument given in the proof of Lemma 2. $T_{g^n} \cap T_h \neq \emptyset$ implies that a power $h^m$ of $h$ fixes the single vertex $v$ of $T_{g^n}$. By the above argument we can assume that $h$ does not fix $v$. Since the action is 2-acylindrical this implies that $h$ fixes a vertex $w$ that is in distance one from $v$. As before we argue that the piece $M_{B_2}$ of the JSJ-decomposition of $M_B$ corresponding to $w$ is a Seifert piece and $h$ is a root of the fibre. Let $T'$ be the torus of the JSJ-decomposition of $M_B$ that separates $M_{B_1}$ and $M_{B_2}$ and let $C'$ the corresponding edge group. As before we now see that the induced splitting of $(g^n, h)$ is $(g^n, h^m) *_{C'} (C', h)$ where $C' = C \cap (g^n, h^m)$ and that we must have $C' = C$. Now this implies that $M_{B_1}$ is generated by two element that come from subgroups that correspond to two different torus boundary components and is therefore a 2-bridge link by either Lemma 3 or Lemma 4 depending whether the piece is Seifert or hyperbolic. Furthermore the two generators must correspond to meridians. As before we further see that $M_{B_2}$ must be of type $D(p, q)$ or $A(p)$ by Lemma 5. This puts us in situation 6 of Theorem 2.

2. Case: $g \in A$ and $g^n \in C$ (we assume that $n$ is chosen minimal with this property) and $h = ab$ with $a \in A - C$, $b \in B - C$ and $a^{-1}g^ma \in C$ for some $m \in \mathbb{N}$.

Let $M_{A_1}$ be the piece of the JSJ-decomposition of $M_A$ that contains $T$. It is clear that $g^n \in C$ and that $a^{-1}g^ma \in C$. The properly immersed annulus corresponding to $a^{-1}g^ma = c \in C$ can be homotoped into the piece $M_{A_1}$. This implies that $(g, h) \subset \pi_1(M_A \cup_T M_B)$ and therefore $M_A = M_{A_1}$. Now $M_A$ cannot be acylindrical since otherwise the annulus must be boundary parallel which implies that $a \in C$ which contradicts our assumptions. Thus $M_A$ is Seifert. Since we assume that $M_A$ is not of type $Q^3$ this implies that $g^mn$ is a power of the fibre of $M_A$. After replacing $g$ by a generator of the cyclic group $\langle g, f_A \rangle$ we can assume that $g$ is a root of the fibre $f_A$ of $M_A$. In particular we have $f_A = g^n$ and $a^{-1}g^ma = g^\pm n$ and therefore $h^{-1}g^n h = b^{-1}g^{\pm n}b$.

We first show that we can restrict ourselves to the case that $(g^n, b^{-1}g^nb)$ is not free. Suppose that $(g^n, b^{-1}g^nb)$ is free. We look at the Bass-Serre tree with respect to the decomposition $G = A *_C B$. Let $w$ be the vertex fixed under the action of $A$ and $v$ be the vertex fixed under the action of $B$. It is clear that the vertex $z = h^{-1}w = b^{-1}a^{-1}w = b^{-1}w$ is different from $w$ and in distance 1 from $v$. We denote the edge $\langle v, w \rangle$ by $e_1$ and the edge $\langle v, z \rangle$ by $e_2$. It is clear that that $b^{-1}e_1 = e_2$. We define $G_w = \langle g \rangle$, $G_z = \langle h^{-1}gh \rangle$ and $G_b = \langle g^n, b^{-1}g^nb \rangle$. The freeness of $G_w$ and the minimality of $n$ guarantees that $G_w \cap \text{Stab}_{e_1} = \langle g^n \rangle = G_w \cap \text{Stab}_{e_2}$, that $G_z \cap \text{Stab}_{e_2} = \langle b^{-1}g^nb \rangle = G_z \cap \text{Stab}_{e_2}$ and that $e_1$ and $e_2$ are not $G_w$-equivalent. Since $hv \neq v$ we have that $he_2 = ab^{-1}e_1 = ae_1 \neq e_1$. Now $ae_1$ and $e_1$ are not $(g)$-equivalent unless $ac \in \langle g \rangle$ for some $c \in C$ in which case we can replace $h$ by an element of $B$ after left multiplication with a power of $g$. This however puts us into the first case, we can therefore assume that $ae_1$ and $e_1$ are not $(g)$-equivalent. It follows that all the conditions of Proposition 4

18
(2) are fulfilled, i.e. the induced splitting of \( \langle g, h \rangle \) has two edges and vertices with cyclic edge groups. This however means that \( \langle g, h \rangle \neq G \).

We next show that the JSJ-decomposition of \( M_B \) must be trivial, i.e. that \( M_B = M_{B_1} \) is geometric, where \( M_{B_1} \) is the piece of the JSJ of \( B \) containing \( T \). It clearly suffices to show that the subgroup \((C, b)\) lies in the subgroup \( B_1 \) of \( B \) corresponding to \( M_{B_1} \). We study the action of \( B \) on the Bass-Serre tree associated to the splitting of \( B \) that corresponds to the JSJ-decomposition of \( M_B \). Clearly \( g^n \) acts with a fixed point and since \( g^n \) is not the fibre of a Seifert piece of \( M_B \) it follows that it is not conjugate to an element of one of the edge groups of the decomposition of \( B \), nor is one of its powers. It follows that \( T_{g^n} \) consists of a single vertex and so does \( T_{h^{-1}g^n h} = h^{-1}T_{g^n} \). Now by Lemma 2.1 of [KW] either \( \langle g^n, h^{-1}g^n h \rangle \) is free or \( T_{g^n} \cap T_{h^{-1}g^n h} \neq \emptyset \). Since we already dealt with the first case we can assume that the second case holds. It follows that \( h \) fixes the vertex which is fixed under the action of \( g^n \), hence \( b \in B_1 \) which proves our assertion.

We distinguish the cases that \( M_B \) is a Seifert manifold and that \( M_B \) is hyperbolic.

Suppose that \( M_B \) is a Seifert manifold. We have to investigate the following cases because of Lemma 1.

1. \( M_B \) is the complement of a 2-bridge knot, \( g^n \) corresponds to a meridian, \( a^{-1}g^n a = g^\pm n \), i.e. \( g^n = f_A^{\pm 1} \) and \( \langle g^n, h^{-1}g^n h \rangle = B \).

We get that \( (g, h = ab) = (g, a, b) \) since \( b \in B = \langle g^n, h^{-1}g^n h \rangle \). It follows that \( (g, h) = G \) if and only if \( A = \langle g, a, C \rangle \) where \( g \) is a root of the fibre (or the fibre itself). This clearly implies that \( M_A \) is a Seifert space with base space \( M\tilde{\circ}(p), M\tilde{\circ}(p, q), D(p, q), D(p, q, r), A(p), A(p, q), \Sigma, \Sigma(p) \) or the once punctured Möbius band with at most one cone point which puts us into situation 4 of Theorem 3. This follows since the group obtained by killing the boundary curve must be two generated with one elliptic generator. The only two-generated closed 2-orbifold group not in this list is the group of the base \( D(2, 2, 2, 2l+1) \) which however has no generating pair that contains an elliptic element [PRZ].

2. \( M_B \) is a Seifert manifold with base \( D(2, 2l + 1) \), \( g^n \) corresponds to a curve that has intersection number one with the fibre of \( M_B \), \( a^{-1}g^n a = g^\pm n \) and \( \langle g^n, h^{-1}g^n h \rangle \) maps surjectively onto the base group and \( \langle g^n, h^{-1}g^n h \rangle \neq B \).

Note that \( |B : \langle g^n, h^{-1}g^n h \rangle| = k < \infty \) where \( k > 1 \) is such that \( (f_B) \cap \langle g^n, h^{-1}g^n h \rangle = \langle f^n_B \rangle \). Let \( m \geq 2 \) be a prime such that \( m|k \). We add the generator \( f^n_B \) to the generating pair \( \{g, h\} \). We clearly have \( \langle f_B \rangle \cap \langle g^n, h^{-1}g^n h, f^n_B \rangle = \langle f^n_B \rangle \). Since \( \langle g^n, h^{-1}g^n h \rangle \) maps surjectively onto the base group we know that \( f^n_B b \in \langle g^n, h^{-1}g^n h \rangle \) for some \( t \in \mathbb{Z} \). We then rewrite the reduced form of \( h = ab \) in the form \( h = a(f_B^{-1}f_B^t)b = (af_B^{-1})(f_B^tb) = a'b' \) and denote \( a' \) and \( b' \) again by \( a \) and \( b \). We then have that \( (g, h = ab, f_B^t) = (g, a, b, f_B^t) \). Let now \( B' = \langle a, g, f_B^t \rangle \). Now either \( B' \cap C = \langle g^n, f_B^t \rangle \) or \( B' \cap C = C \) since there is no intermediate group between \( \langle g^n, f_B^t \rangle \) and \( C \) since \( m \) is prime. We now apply Proposition 1 (1). In the first case we get that the induced splitting of \( \langle g, h, f_B^t \rangle \) is \( \langle g, a, f_B^t \rangle \) and in particular that \( \langle g, h, f_B^t \rangle \cap C = \langle g^n, f_B^t \rangle \neq C \) which contradicts the fact that \( G = \langle g, h, f_B^t \rangle \). In the second case we get that the induced splitting is of type \( \langle g, a, f_B^t \rangle \ast_C B \).

If \( g \) and \( h \) generate \( G \) this means that \( A = \langle g, a, f_B^t \rangle \). This implies that the quotient of the base group by the \( nth \) power of the boundary curve is two generated where one generator is elliptic. This implies that the original base
was of one of the types $M_0$, $M_0(p)$, $D(p, q)$ or $A(p)$. The first two cases yield situation 5 of Theorem \[\text{[Ko2]}\] the last two cases yield situation 2. Note that these groups are in fact 2-generated by $[\text{Ko2}]$ and $[\text{V2}]$.

(3) $M_B$ is a Seifert manifold with base $D(2, 2l)$, $g^n$ corresponds to a curve that has intersection number one with the fibre of $M_B$. It follows from Lemma 11 that $g^n$ and $h^{-1}g^n h = b^{-1}g^n b$ are not conjugate in $(g^n, h^{-1}g^n h)$, nor are their images under $\pi$ conjugate in the image of $(g^n, h^{-1}g^n h)$. Hence they are also not conjugate in $(g^n, h^{-1}g^n h, f_B)$ when we add the generator $f_B$. In particular $b \notin (g^n, h^{-1}g^n h, f_B)$. We distinguish the cases where $a \in (g, f_B, af_Ba^{-1})$ and $a \notin (g, f_B, af_Ba^{-1})$.

In the first case we have that $G = \langle g, h = ab, f_B \rangle = \langle g, a, f_B, B \rangle$ and Proposition 3 (1) yields that $G = \langle g, h, f_B \rangle = \langle g, a, f_B \rangle *_{(g\pi, f_B)} \langle g^n, f_B, b \rangle = \langle g, a, C \rangle *_{C} \langle C, b \rangle$. In particular $\langle g, a, C \rangle = \langle g, f_B, af_Ba^{-1} \rangle = A$ and $\langle C, b \rangle = B$. Let $N_A(C)$ be the normal closure of $C$ in $A$. Now $A/N_A(C)$ must be generated by one element ($f_B$ and $af_Ba^{-1}$ lie in the kernel), namely the image of $g$. This however implies that the base is of type $D(p, q)$ or $A(p)$ since the only other cyclic 2-orbifold group is the projective plane with at most one singularity, but here the generator is not elliptic. This puts us into situation 2 of Theorem \[\text{[Ko2]}\]. Again we know that $G$ is in fact 2-generated since the manifold is of genus 2 by $[\text{Ko2}]$.

In the second case we choose $v$, $w$, $z$, $e_1$ and $e_2$ as in the beginning of case (2) above. We define $G_v = \langle g, f_B, af_Ba^{-1} \rangle$, $G_w = \langle f_B, g^n, b^{-1}g^n b \rangle$ and $G_z = \langle b^{-1}g^n b, b^{-1}a^{-1}f_Bab, b^{-1}a^{-1}gab \rangle$. It is clear that $e_2 = be_1$ and that $he_2 = ae_1$. The facts that $b \notin (f_B, g^n, b^{-1}g^n b)$ and that $a \notin (g, f_B, af_Ba^{-1})$ imply that $e_1$ and $e_2$ are $G_v$-inequivalent and that $e_1$ and $he_2$ are $G_v$-inequivalent. Proposition 3 (4) then implies that the induced splitting of $(g, h, f_B)$ has two edge groups and two vertex groups, which implies that it is not the induced splitting of $G$, i.e. $(g, h, f_B) \neq G$ and therefore $(g, h) \neq G$.

Suppose now that $M_B$ is hyperbolic.

Since $U := \langle g^n, h^{-1}g^n h \rangle = \langle g^n, b^{-1}g^n b \rangle$ is neither free nor Abelian it follows from Proposition 3 that either $M_B$ is the complement of a 2-bride knot or link, that $U = B$ and that $g^n$ and $b^{-1}g^n b$ correspond to meridians or that $U$ is a subgroup of $B$ of index 2 and that the covering space $M_B$ corresponding to $U$ is homeomorphic to the exterior of a 2-bride link in $S^3$ (with two components).

In the first case we argue precisely as in case (1) above. In the second case we can assume that the covering $p : M_B \to M_B$ is a homeomorphism when restricted to a boundary component since the degree of the covering is 2 and since the two boundary components of $M_B$ get mapped onto the same boundary component of $M^3$. It follows that $C \cap U = C$, $h^{-1}Ch \cap U = h^{-1}Ch$ and $b \notin U$ since the two meridians are not conjugate in $U$. It follows that $B = \langle g^n, b \rangle$ where $g^n$ is parabolic. We can now argue precisely as in case (3) above and either obtain that $(g, h) \neq G$ or that $M_A$ is a Seifert space over $D(p, q)$ or $A(p)$.

Since the fibre $f_{fa} = g^n$ of $M_A$ gets identified with the parabolic generator of $B$ this puts us into situation 1 of Theorem \[\text{[Ko2]}\].

3. Case: $g \in A$ and $g^n \in C$ and $h = bab^{-1}$ with $a^n \in C - 1$ for some $m \in \mathbb{N}$. We can assume that $g \notin C$ since we are otherwise in the first case after conjugation with $b$. As in the case before it follows that $M_A$ is Seifert and that we can assume that $a, g \in A - C$ are roots of the fibre $f_a$. 

20
We distinguish the cases that $bc \in \langle g^n, ba^m b^{-1} \rangle \subset B$ for some $c \in C$ and that $bc \notin \langle g^n, ba^m b^{-1} \rangle$ for all $c \in C$.

(a) Suppose that $bc \in \langle g^n, ba^m b^{-1} \rangle \subset B$ for some $c \in C$. After rewriting the reduced form of $h = bab^{-1}$ as $(bc)(c^{-1}ac)(c^{-1}b^{-1})$ we have that $b \in \langle g^n, ba^m b^{-1} \rangle$. It follows that $\langle g, h \rangle = \langle b, a, c \rangle$. Since $g$ and $a$ are both roots of the fibre it follows that $g^n = a^m = f_A$. This implies that $ba^m b^{-1} = (ba)g^n(a^{-1}b^{-1})$. We show that $\langle g, h \rangle = \langle g, ab \rangle$ which puts us into the second case. Since $ba^m b^{-1} = b f_A b^{-1} = (ba) f_A (ba)^{-1} = (ba) g^n (ba)^{-1} \in \langle g, ab \rangle$ it follows that $b \in \langle g, ab \rangle$ and therefore also $a \in \langle g, ab \rangle$. We have shown that $\langle g, ab \rangle = \langle a, b, g \rangle = \langle g, h \rangle$.

(b) Suppose that $bc \notin \langle g^n, ba^m b^{-1} \rangle$ for all $c \in C$. We study the action on the Bass-Serre tree associated to the splitting $G = A *_C B$. Let $x$ be the vertex fixed under the action of $A$, $y$ be the vertex fixed by $B$ and $z = bx$ be the vertex fixed under the action of $bab^{-1}$. We further define $e_1 = [x, y]$ and $e_2 = [y, z]$. Note that $\text{Stab}(e_1) = C$ and that $\text{Stab}(e_2) = bCb^{-1}$. We define $G_y = \langle g^n, ba^m b^{-1} \rangle$, $G_x = \langle g, G \cap C \rangle$ and $G_z = \langle h, bC^{-1} \cap G \rangle$. Lemma 6 guarantees that $G_y \cap G_z = G_y \cap C$ and that $G_y \cap bC^{-1} = G_z \cap bC^{-1}$. We further have that $e_1$ and $e_2$ are $G_g$-inequivalent since $bc \notin G_y$ for all $c \in C$. This implies that all hypothesis of Proposition 8(2) are fulfilled, i.e. that the induced splitting of $\langle g, h \rangle$ has two edge groups. This however implies that $\langle g, h \rangle \neq G$.

3.2 The JSJ-decomposition has a non-separating torus

We can assume that there is no separating torus because there is no 3-manifold that has 2-generated fundamental group such that the JSJ contains a separating and a non-separating torus. In the case when there is no piece of type $Q^2$ this follows from section 2.3 in the case with a piece of type $Q^3$ we will see this in section 3.3. This implies that the JSJ-graph is homeomorphic to a circle and we only have to look at the cases where we have one or two pieces because of Lemma 8.

1. case: The JSJ-decomposition has one piece $N$ with one non-separating torus.

Let $A$ be the fundamental group of $N$. We can then write $G$ as the HNN-extension $\langle A, t | c_1 t^{-1} = \phi(c_1) \rangle$ for $c_1 \in C_1$ where $\phi : C_1 \to C_2$ is the isomorphism between the two torus subgroups induced by the JSJ-decomposition of the manifold. Because of Lemma 8 we can assume that there exists a generating pair $\langle g, h \rangle$ such that $g \in A$ and that $g^n \in C_1$ and that $h = at$ for some $a \in A$. Note that $hg^n h^{-1} \in A$. We first look at the case that $N$ is Seifert and then at the case that $N$ is a hyperbolic piece. In the first case we distinguish the cases that neither $g^n$ nor $hg^n h^{-1}$ corresponds to the fibre of $N$ and that either $g^n$ or $hg^n h^{-1}$ corresponds to the fibre of $N$. We can clearly choose $g$ such that $g$ generates a maximal cyclic subgroup and $n$ minimal such that $g^n \in C_1$.

(a) Suppose that $N$ is Seifert and neither $g^n$ nor $hg^n h^{-1}$ correspond to the fibre. This implies that $g \in C_1$, otherwise $g$ would be a root of the fibre and $g^n = f_N$. If $\langle g, hgh^{-1} \rangle$ is free then $\langle g, hgh^{-1} \rangle \cap C_1 = \langle g \rangle$ and $h^{-1} \langle g, hgh^{-1} \rangle h \cap C_1 = \langle h^{-1} gh, g \rangle \cap C_1 = \langle g \rangle$. Proposition 8(3) therefore implies that the induced splitting of $\langle g, hgh^{-1} \rangle$ is an HNN-extension with cyclic edge group which implies that $\langle g, hgh^{-1} \rangle \neq G$. It therefore follows from Lemma 8 that the base $O$ of $N$ is of type $A(p)$. If $\langle g, hgh^{-1} \rangle$ is not free we can by Lemma 8 assume that
On the boundary that have intersection number 1 with the fibre. If \( \langle g, hgh^{-1} \rangle = A \) then \( N \) is a 2-bridge link by Lemma 3 and \( g \) and \( hgh^{-1} \) correspond to the meridians, in particular \( M^3 \) is obtained from \( N \) by identifying the boundary components such that the meridians get identified, i.e. we are in situation 8 of Theorem 2.

If \( \langle g, hgh^{-1} \rangle \neq A \) then \(|A : \langle g, hgh^{-1} \rangle| = k \leq \infty \) where \( k \) is such that \((f) \cap \langle g, hgh^{-1} \rangle = \langle f^k \rangle \). This is clear since \( \langle g, hgh^{-1} \rangle \) maps surjectively onto the base group. It follows that \( C_1 \cap \langle g, hgh^{-1} \rangle = \langle f^k, g \rangle \) and that \( C_1 \cap h^{-1} \langle g, hgh^{-1} \rangle h = C_1 \cap (h^{-1} gh, g) = \langle f^k, g \rangle \). It then follows from Proposition 4(3) that the induced splitting of \( \langle g, h \rangle \) has one edge group which is a proper subgroup of the edge group of the splitting of \( G \) which implies that \( G \neq \langle g, h \rangle \).

(b) \( N \) is Seifert and either \( g^n \) or \( hgh^{-1} \) correspond to the fibre. Without loss of generality we can assume that \( g^n \) corresponds to a fibre, i.e. \( g \) maps onto an elliptic element of the base group and \( g^n = f \). Note that \( h^{-1} fh = t^{-1} a^{-1} fah = t^{-1} f^\pm t \in t^{-1} C_2 t = C_1 \). In particular we have \( \langle f, h^{-1} fh \rangle \subset C_1 \). We distinguish the cases where \( C_1 = \langle f, h^{-1} fh \rangle \) and \( C_1 \neq \langle f, h^{-1} fh \rangle \).

Suppose that \( C_1 = \langle f, h^{-1} fh \rangle \). Note that \( C_1 \subset \langle g, hfh^{-1}, h^{-1} fh \rangle \subset A \) and \( C_1 \subset h^{-1} \langle g, hfh^{-1}, h^{-1} fh \rangle h = \langle h^{-1} gh, f, h^{-2} fh^2 \rangle \subset h^{-1} Ah \). Proposition 4(3) therefore implies that the induced splitting of \( \langle g, h \rangle \) is a HNN-extension with base \( \langle g, hfh^{-1}, h^{-1} fh \rangle \) and edge group \( C_1 \). Note that \( C_1 = \langle f, h^{-1} fh \rangle \) implies that \( hfh^{-1} \) and \( h^{-1} fh \) correspond to boundary curves that have intersection number one with the fibre. It follows that \( G = \langle g, h \rangle \) if and only if \( F \) is generated by one elliptic element and two elements that correspond to boundary curves. The only such orbifolds are \( A(p), A(p, q), \Sigma \) and \( \Sigma(p) \), i.e. \( M^3 \) is obtained from a Seifert manifold with base \( A(p), A(p, q), \Sigma \) or \( \Sigma(p) \) where the boundary components are glued such that the fibre on one component is glued with a curve on the other component that has intersection number one with the fibre \( f \). This puts us into situation 9 of Theorem 2.

Suppose that \( C_1 \neq \langle f, h^{-1} fh \rangle \). This means that \( C_1 \cap \langle f, h^{-1} fh \rangle = \langle f, m^k \rangle \) where \( m \) corresponds to a curve with intersection number one with the fibre and \( k \geq 2 \). In particular we have \( \langle f, m^k \rangle \subset \langle g, hfh^{-1}, h^{-1} fh \rangle \) and \( \langle f, m^k \rangle \subset h^{-1} \langle g, hfh^{-1}, h^{-1} fh \rangle h = \langle h^{-1} gh, f, h^{-2} fh^2 \rangle \). It suffices to show that \( C_1 \cap \langle g, hfh^{-1}, h^{-1} fh \rangle = C_1 \cap h^{-1} \langle g, hfh^{-1}, h^{-1} fh \rangle h = \langle f, m^k \rangle \) since we then see as in the case before that the induced splitting of \( \langle g, h \rangle \) is a HNN-extension with edge group \( \langle f, m^k \rangle \) which implies that \( \langle g, h \rangle \neq G \). We show that \( C_1 \cap \langle g, hfh^{-1}, h^{-1} fh \rangle = \langle f, m^k \rangle \), the argument for the other statement is analogous. This can be seen by quotienting the normal closure \( \overline{N} \) of \( \langle g, hfh^{-1}, h^{-1} fh \rangle \) out of \( A \). The quotient map maps \( A \) onto the fundamental group of the orbifold that is obtained from the base space by replacing the two boundary components by cone points of order \( k \). It is clear that \( m \) gets mapped onto an elliptic element of order \( k \). This implies that \( \langle m \rangle \cap \overline{N} = \langle m^k \rangle \) and therefore \( \langle m \rangle \cap \langle g, hfh^{-1}, h^{-1} fh \rangle = \langle m^k \rangle \) which proves the assertion.

(c) Suppose that \( N \) is hyperbolic. Since \( C_1 \) is malnormal in \( A \) this implies that \( g \in C_1 \) since \( g^n \in C_1 \). As in case (a) we see that \( \langle g, hgh^{-1} \rangle \) cannot be free. Since \( \langle g, hgh^{-1} \rangle \) is also not Abelian it follows from Proposition 2 that either \( N \) is the complement of a 2-bridge link and \( \langle g, hgh^{-1} \rangle = A \) or that \( \langle ghgh^{-1} \rangle \) is a subgroup of \( A \) of index two and the corresponding covering is the exterior of a 2-bridge link. In the first case \( g \) and \( hgh^{-1} \) correspond to meridians by
Proposition 2 which puts us into situation 8 of Theorem 2. In the second case we can argue as in the second part of (a) to show that \( \langle g, h \rangle \neq G \). This is possible since the degree of the covering is 2 when restricted to either boundary components.

2. case: The JSJ-decomposition has two pieces and two non-separating tori. Let \( M_A \) and \( M_B \) be the two pieces of the decomposition and denote \( \pi_1(M_A) \) by \( A \) and \( \pi_1(M_B) \) by \( B \). Let \( C_1 \) and \( C_2 \) be the subgroups of \( A \) corresponding to the boundary components of \( M_A \) glued with boundary components of \( M_B \). Let \( C_3 \) and \( C_4 \) be the corresponding subgroups of \( B \). Suppose further that \( \phi_1 : C_1 \to C_3 \) and \( \phi_2 : C_2 \to C_4 \) are the isomorphisms that are induced by the gluing. Then \( G = \langle A, B, t | \phi_1(c_1) = c_1, \phi_2(c_2) = t c_2 t^{-1} \rangle \). Since this splitting is 2-acylindrical Lemma 3 guarantees (possibly after exchanging \( A \) and \( B \)) that \( g \in B, \ g^n \in C_3 - 1 \) and \( g^m \in b C_3 b^{-1} - 1 \) for some \( b \in B \) and \( n, m \in \mathbb{N} \) and \( h = b t a \) for some \( a \in A \).

It is clear that \( g^{nm} \in C_3 - 1 \) and \( g^{mn} \in b C_3 b^{-1} \). It follows from the annulus Theorem that there exists an essential annulus in \( M_B \). This implies that \( M_B \) is Seifert and that \( g^{nm} \) corresponds to a power of the fibre. After replacing \( g \) with a generator of a maximal cyclic subgroup containing the original \( g \) we can therefore assume that \( n = m \) and \( g^m = f_B \).

We now look at the action of \( G \) on the Bass-Serre tree corresponding to the above splitting. Let \( w \) be the vertex fixed by \( B \), \( v \) be the vertex fixed by \( A \) and \( z = h^{-1} w = a^{-1} t^{-1} h^{-1} w = a^{-1} t^{-1} w \) be the vertex fixed by \( (b t a)^{-1} B b t a \) = \( (t a)^{-1} B \). Note that \( e_1 = [v, w] \) and \( e_2 = [v, z] \) are \( A \)-inequivalent and that \( e_1 \) and \( (b t a) e_2 \) are \( B \)-inequivalent since they project onto distinct edges of the quotient graph. This implies that for three groups \( G_w \), \( G_v \) and \( G_z \) fixing \( w \), \( v \) and \( z \), respectively, we only need to show that \( G_v \cap \text{Stab} e_1 = G_w \cap \text{Stab} e_1, G_v \cap \text{Stab} e_2 = G_z \cap \text{Stab} e_2 \) and that \( h G_w h^{-1} = G_z \) in order to verify that the hypothesis of Proposition 2 (4) are fulfilled.

Note that \( \langle f_B, h^{-1} f_B h \rangle = \langle f_B, (b t a)^{-1} f_B b t a \rangle = \langle f_B, a^{-1} t^{-1} f_B^-1 t a \rangle \subset A \) and that \( f_B \) and \( h^{-1} f_B h \) correspond to elements of different boundary components of \( M_A \). Using the analysis of (a) and (c) above we have to distinguish three cases (i) where \( \langle f_B, h^{-1} f_B h \rangle \) is free, (ii) where \( \langle f_B, h^{-1} f_B h \rangle = A \) and \( M_A \) is a 2-bridge link and \( f_B \) and \( h^{-1} f_B h \) correspond to meridians and (iii) where \( \langle f_B, h^{-1} f_B h \rangle \) is of finite index in \( A \) and \( |C_1 : C'_1| = |C_4 : C'_4| = |A : \langle f_B, h f_B h^{-1} \rangle | \) where \( C'_1 = C_1 \cap \langle f_B, h f_B h^{-1} \rangle \) and \( C'_4 = C_4 \cap a \langle f_B, h f_B h^{-1} \rangle a^{-1} \).

(i) If \( A' = \langle f_B, h^{-1} f_B h \rangle \subset A \) is free then clearly \( A' \cap C_1 = \langle f_B \rangle \) and \( A' \cap h^{-1} C_2 h = \langle h^{-1} f_B h \rangle \). This implies that if we define \( G_w = \langle g \rangle, G_v = \langle f_B, h^{-1} f_B h \rangle \) and \( G_z = \langle h^{-1} g h \rangle \) the hypothesis of Proposition 2 (4) are fulfilled and therefore the induced splitting of \( \langle g, h \rangle \) has cyclic edge groups which implies that \( \langle g, h \rangle \neq G \).

(ii) If \( \langle f_B, h^{-1} f_B h \rangle = A \) then in particular \( C_1 \subset \langle f_B, h^{-1} f_B h \rangle \) and \( a^{-1} C_4 a \subset \langle f_B, h^{-1} f_B h \rangle \). We define \( G_w = \langle g, C_1, b C_2 b^{-1} \rangle, G_v = \langle f_B, h^{-1} f_B h \rangle = A \) and \( G_z = h^{-1} G_w h \). This implies that the induced splitting of \( \langle g, h \rangle \) consist of two vertex groups where one equals \( A \) and the other to \( G_w \) and the edge groups are the edge groups of the original group. This means that \( \langle g, h \rangle = G \) if and only if \( G_w = B \). This however implies that \( B \) is generated by the subgroups corresponding to the boundary components and a root of the fibre, i.e. the base is generated by two boundary curves and one elliptic element, as before we see that this implies that the base space is of type \( A(p), A(p, q), \Sigma \) or \( \Sigma(p) \). This means
that $M^3$ is obtained from a Seifert manifold with base space $A(p)$, $A(p,q)$, $\Sigma$ or $\Sigma(p)$ and a 2-bridge link complement where the boundary components are glued such that the fibre is glued to meridians of the 2-bridge link complement. This puts us into situation 7 of Theorem \[ii\].

(iii) Let $C'_2 = tC'b^{-1}$. We define $G_w = (g, C'_1, bC'_2b^{-1})$, $G_v = (f_B, h, bh^{-1})$ and $G_z = h^{-1}(g, C'_1, bC'_2b^{-1})$. The same argument as in case 1 (c) shows that $G_w \cap \text{Stab}(e_1) = C'_1$ and that $G_z \cap \text{Stab}(e_2) = a^{-1}C'_2a$, i.e. that the hypothesis of Proposition \[i\] (4) is fulfilled. This implies that one of the vertex groups of the induced splitting of $(g, h)$ is a proper subgroup of $A$ and the other a proper subgroup of $B$. It follows that $(g, h) \neq G.$

3.3 Dealing with the existence of pieces of type $Q^3$

We start by the following lemma that will be useful in the course of our investigation.

Lemma 16 Let $M^3$ be a 3-manifold whose JSJ-decomposition consists of a piece homeomorphic to $Q^3$ and a Seifert piece $M_B$ with base $D(p,q)$ or $A(p)$. Suppose further that $\pi_1(M^3) = (g, h)$ where $g$ is a root of the fibre $f_B$ of $M_B$. Then the intersection number of $f'_Q$ and $f_B$ is one.

Proof We use the notation as in the beginning of section \[ii\]. Let $G = A \ast_C B$ be the splitting of $G$ that corresponds to the JSJ-decomposition of $M^3$. Suppose that the intersection number of $f'_Q$ and $f_B$ is greater than one, i.e. that $f_B = (xy)^nf'_Q$ for some $n \geq 2$. It follows that $xf_Bx^{-1} = (xy)^n f'_Q^{m+2n}$. We clearly have $\langle f_B, x f_B x^{-1} \rangle \subset \langle (xy)^n, f'_Q \rangle = U \neq C$ and $U$ is normal in $A$ with $A/U \cong D_{2n}$ and $C/U \cong \mathbb{Z}_n$. In particular $g^2 = 1$ for all $g \in A/U - C/U$.

Since $(xy)^n$ is a $n$th power and since $f'_Q$ has intersection number $n$ with $f_B$ it follows that the intersection number of any element of $U$ with $f_B$ is a multiple of $n$. It follows that $B/N_B(U)$ is the fundamental group of the orbifold obtained by replacing a boundary curve of the base space of $M_B$ with a cone point of order $n$. If the base space was $D(p,q)$ the resulting orbifold is $S^2(p,q,n)$ if it was $A(p)$ it becomes $D(p,n)$.

It follows that $G' = G/N_G(U)$ can again be written as an amalgamated product $G' = A' \ast_{C'} B'$ where $A' = A/U$, $C' = C/U$ and $B' = B/N_B(U)$. Denote the quotient map $G \rightarrow G'$ by $\phi$. Since $G'$ is a proper amalgamated product and therefore not cyclic it follows that $\phi(g)$ cannot be trivial. It is clear that no non-trivial power of $\phi(g)$ is conjugate to an element of $C'$ since in a triangle groups elliptic elements that correspond to different cone points in the orbifold are not conjugate and the amalgam corresponds to the new cone points. This implies that $T_{\phi(g)}$ consists of a single vertex if we look at the action of $G'$ with respect to the splitting $G' = A' \ast_{C'} B'$.

Note that $G'$ is not the free product of two cyclic groups. By the main result of \[iii\] we can now choose $h$ such that either $T_{\phi(g)} \cap T_{\phi(h)} \neq \emptyset$ or that $T_{\phi(g)} \cap hT_{\phi(g)}h^{-1} \neq \emptyset$. The case $T_{\phi(g)} \cap \phi(h)T_{\phi(g)} \neq \emptyset$ cannot occur since then $\phi(h)$ would also fix the single vertex of $T_{\phi(g)}$ which implies that $(\phi(g), \phi(h)) \subset B$.

It follows that $T_{\phi(g)} \cap T_{\phi(h)} \neq \emptyset$. In particular $\phi(h)$ acts with a fixed point. If $\phi(h)$ is conjugate to an element of $B'$ then $(\phi(g), \phi(h))$ lies in the kernel of $G' \rightarrow G'/B' \cong \mathbb{Z}_2$, i.e. $G' \neq (\phi(g), \phi(h))$. We can therefore assume that $\phi(h)$ is conjugate to an element of $A - C$. Clearly these elements are of order two.
and fix no edge. This implies that \( T_{\phi(h)} \) consists of a single vertex. It then follows from \( T_{\phi(g)} \cap T_{\phi(h)} \neq \emptyset \) that \( T_{\phi(g)} = T_{\phi(h)} \) since \( T_{\phi(g)} \) consists of a single vertex also. This however gives a contradiction since the vertices correspond to different factors of the amalgamated product. 

The preceding lemma allows us to deal with the case that the JSJ-decomposition has two pieces that are both homeomorphic to \( \mathbb{Q}^3 \); the following Lemma implies that manifolds of this type that have 2-generated fundamental group fall into situation 2 of Theorem 2.

**Lemma 17** Let \( M^3 \) be a 3-manifold whose JSJ-decomposition consists of two pieces that are homeomorphic to \( \mathbb{Q}^3 \). Then rank \( \pi_1(M^3) = 2 \) if and only if the fibre of one piece has intersection number one with the fibre of the other piece (when we look at the pieces as the Seifert space over \( D(2, 2) \)).

**Proof** Let \( G = A \ast_C B \) be the splitting of \( G \) that corresponds to the JSJ-decomposition of \( M^3 \). Suppose that \( G \) is 2-generated. It follows from Theorem 6 that \( G \) is generated by a pair of elements \( \{g, h\} \) such that \( g \) acts with a fixed point. It is clear that \( G \) is not conjugate to an element of \( C \), since it would otherwise lie in the kernel of the quotient map \( G \to G/C \cong \mathbb{Z}_2 \ast \mathbb{Z}_2 \) which clearly implies that \( \{g, h\} \) do not generate \( G \). It follows that after conjugation either \( g \in A - C \) or \( g \in B - C \), in particular after replacing \( g \) with a primitive element we have that \( g^2 \) is the fibre of one of the pieces. The assertion now follows from Lemma 16. □

Next we show that basically the same result as in Lemma 6 holds if one of the pieces is homeomorphic to \( \mathbb{Q}^3 \).

**Lemma 18** Suppose that \( M^3 \) is a 3-manifold, \( T \) a separating torus of the JSJ-decomposition such that \( M^3 = Q^3 \cup_T M_B \) where \( M_B \) is not homeomorphic to \( Q^3 \). Let \( G = A \ast_C B \) be the corresponding decomposition of \( G = \pi_1(M^3) \). Suppose that rank \( G = 2 \). Then either \( M^3 \) falls into case 2 of Theorem 6 or there exists a generating pair \( \{g, h\} \) such that either 1 or 2 hold:

1. \( g \in A - C, g^2 \in C \) and one of the following holds:
   (i) \( h \in B - C \).
   (ii) \( h = ab \) with \( a \in A - C, b \in B - C \).
   (iii) \( h = bab^{-1} \) with \( a \in A - C \) and \( a^2 \in C - 1 \).

2. \( M_B \) is Seifert, \( g \in B \) and \( h = ab \) with \( a \in A - C, b \in B - C \). Furthermore \( g^n \in C \) for some \( n \in \mathbb{N} \) and \( g^n \) corresponds to a power of the fiber.

3. \( M_B \) has trivial JSJ-decomposition, \( g \in C \) and \( g \) corresponds neither to the fiber of \( H^3_Q \) of \( Q^3 \) nor to the fiber of \( M_B \) (if \( M_B \) is Seifert).

**Proof** We study the action of \( G \) on the Bass-Serre tree corresponding to the splitting \( G = A \ast_C B \). We first investigate the structure of the trees \( T_g \) of elliptic elements of \( G \). After conjugation we can assume that \( g \in A \) or \( g \in B \).

If \( g \in A - C \) then the tree consists of the vertex \( x \) fixed under the action of \( A \) and all edges emanating from \( x \). This is clear since in this case \( g^2 \) corresponds to a power of the fibre of \( Q^3 \) and does therefore not correspond to the fibre of the piece \( M_{B_1} \) of \( M_B \) that contains \( T \) (if \( M_{B_1} \) is Seifert).
If \( g \in B \) then either \( T_g = y \) where \( y \) is the vertex fixed under \( B \) or after conjugation \( g^n \in C \) for some \( n \in \mathbb{N} \). If \( g \in C \), \( M_{B_1} \) is Seifert and \( g \) is in \( G \) conjugate to a power of \( f_{B_1} \), we conjugate \( g \) such that \( g = f_{B_1}^k \) for some \( k \in \mathbb{Z} \). Note that for \( a \in A - C \) we get that \( af_{B_1}a^{-1} \notin \langle f_{B_1} \rangle \) since we assume that \( f_{B_1} \neq f_A \). Note further that \( bcb^{-1} \notin C \) for all \( b \in B - C \) and \( c \in C - \langle f_{B_1} \rangle \) since we assume that \( M_B \) is not homeomorphic to \( Q^3 \). This follows from the annulus theorem, the fact that all properly immersed annuli can be homotoped into a Seifert piece and since properly immersed annuli in Seifert pieces are homotopic to vertical annuli. Therefore all essential properly immersed annuli in \( M_B \) with boundary in \( T \) can be homotoped to be vertical in \( M_{B_1} \). Together this implies that \( T_g \) must be contained in the 2-neighborhood of \( y \).

As in the proof of Lemma 2 we have to distinguish the cases that \( T_g \cap T_h \neq \emptyset \) and that \( T_g \cap hT_g \neq \emptyset \).

If \( T_g \cap T_h \neq \emptyset \) we can assume that either \( g \) or \( h \) is conjugate to an element of \( A - C \), otherwise \( \langle g, h \rangle \) lies in the kernel of the quotient map \( G \to G/N_G(B) \cong \mathbb{Z}_2 \) which implies that \( \langle g, h \rangle \neq G \). If \( g \) and \( h \) are conjugate to elements of \( A - C \) then we can argue as in the proof of Lemma 2 to obtain situation (iii) of 1. We are left with the case that \( g \in A - C \) and that \( h \) is conjugate to an element of \( B \). The structure of the trees \( T_g \) and \( T_h \) guarantee that there exists a vertex \( z \) with \( hz = z \) such that either \( d(x, z) = 1 \) or \( d(x, z) = 3 \) where \( x \) is chosen such that \( Ax = x \).

If \( d(x, z) = 1 \) we have after conjugation with an element of \( A \) that \( h \in B \), since further \( g^2 \in C \) we are in situation (i) of case 1.

If \( d(x, z) = 3 \) then a power \( h^k \) of \( h \) must be conjugate to an element of \( C \) since it fixes an edge of the Bass-Serre tree. Note that this implies that \( h \) is conjugate to an element of \( B_1 \), the subgroup of \( B \) corresponding to the piece \( M_{B_1} \) of the JSJ-decomposition of \( M_B \) containing the torus \( T \). This is clear since otherwise this \( h^k \) lies in two different edge groups that are incident with \( M_{B_1} \), which implies that \( h^k \) is conjugate to a power of the fibre of \( M_{B_1} \), in particular \( M_{B_1} \) is Seifert. Such element however have no roots outside \( B_1 \).

We now study the action of \( G \) on the Bass-Serre tree corresponding to the splitting of \( G \) that corresponds to the full JSJ-decomposition of \( M^3 \). The trees \( T_g \) and \( T_h \) are now defined with respect to the new action of \( G \). Since \( g \in A \) and a power of \( h \) is conjugate to an element of \( C \) it follows that \( g \) and \( h \) again act with fixed points. Since the subgroup generated by \( g \) and \( h \) is not free it follows that \( T_g \cap T_h \neq \emptyset \). As in the proof of Lemma 2 we conclude that the underlying graph of the JSJ-decomposition must be a tree since the elliptic elements \( g \) and \( h \) generate \( G \). It is also clear that there is no other piece in the JSJ-decomposition of \( M^3 \) that is homeomorphic \( Q^3 \), otherwise we could define a surjective homomorphism \( \phi : G \to \mathbb{Z}_2 \) whose kernel contains \( g \) and \( h \).

It is easy to see that the trees \( T_g \) and \( T_h \) have the same structure as before since we assume that \( M_B \) is not homeomorphic to \( Q^3 \). Thus \( T_g \) lies in the 1-neighborhood of \( \tilde{x} \) where \( \tilde{x} \) is chosen such that \( \tilde{x} = Ax = g\tilde{x} \) and \( T_h \) lies in the 2-neighborhood of \( \tilde{z} \) where \( \tilde{z} \) is such that \( h\tilde{z} = \tilde{z} \) and \( \tilde{z} \) is fixed by a conjugate of \( B_1 \).

It follows that there exists a vertex \( \tilde{y} \in T_g \cap T_h \) with \( d(\tilde{x}, \tilde{y}) \leq 1 \) and \( d(\tilde{z}, \tilde{y}) \leq 2 \). In particular we have \( d(\tilde{x}, \tilde{z}) \leq 3 \). Since \( \tilde{x} \) and \( \tilde{z} \) map onto vertices of distance 1 in the quotient tree (the tree underlying the JSJ-decomposition and the corresponding graph of groups) we must have that \( d(\tilde{x}, \tilde{z}) = 1 \) or \( d(\tilde{x}, \tilde{z}) = 3 \). In the first case we argue as before. If \( d(\tilde{x}, \tilde{z}) = 3 \) then \( d(\tilde{z}, \tilde{y}) = 2 \) and we
denote the midpoint of \([\bar{y}, \bar{z}]\) by \(p\). Since \(h\) fixes \(\bar{z}\) there must be a power of \(h\) that fixes \([\bar{y}, \bar{z}]\). We first show that \([\bar{y}, \bar{z}]\) maps onto the edge of the JSJ-graph that corresponds to \(T\), i.e. that \(p\) is fixed by a conjugate of \(A\). Suppose that \(p\) corresponds to a piece \(M'\) of the JSJ-decomposition of \(M^3\) that is different from \(M_A\). It is clear that \(M'\) is Seifert since \(h^n\) lies in distinct conjugates of a peripheral subgroup of \(\pi_1(M')\). It further follows that \(h^n\) corresponds to a power of \(f_M'\) since \(M'\) is not homeomorphic to \(Q^3\). This however yields a contradiction since then \(h\) cannot be conjugate to an element of \(C\). This implies that \(\langle g, h \rangle \subset \langle A, B_1 \rangle\), in particular the JSJ-decomposition of \(M_B\) must consist of \(M_{B_1}\) alone. Thus we have \(M_B = M_{B_1}, B_1 = B\) and also \(\bar{x} = x, \bar{y} = y\) and \(\bar{z} = z\).

We now look at the graph \([x, z] = [x, y] \cup [y, z]\). After conjugation by an element of \(A\) we can assume that \(By = y\). Note that we can consider the segment \([y, z] = [y, p] \cup [p, z]\) as a single edge since \(\text{Stab} [y, z] \cap \langle h \rangle = \text{Stab} [y, p] \cap \langle h \rangle = \text{Stab} [p, z] \cap \langle h \rangle\). We define \(G_x = \langle g \rangle, G_y = \langle g^2, h^n \rangle\) where \(n\) is chosen minimal such that \(h^n\) fixes \([y, z]\) and \(G_z = \langle h \rangle\). We can assume that \(n \geq 2\), otherwise \(h\) fixes \(y\) and we are in situation (i) of 1. In particular we can assume that \(h\) is a root of the fibre of some conjugate of \(B\) and that \(h^n\) is the fibre of \(Stab z \cong B\). This implies that \(h^n\) (considered as an element of \(\text{Stab} y\)) has non-trivial and even intersection number with the fibre of \(B = Stab y\) since for any \(c \in C\) and \(a \in A - C\) the elements \(c\) and \(aca^{-1}\) differ by an even multiple of the fibre \(f_A\) (considered as the Seifert space over \(D(2, 2)\)). Since \(f_A\) has by assumption non-trivial intersection number (on \(T\)) with \(f_B\) we have that \(af_Ba^{-1}\) has even intersection number with \(f_B\) for all \(a \in A - C\). We can exclude the situation that \(M_B\) is a Seifert space over \(D(p, q)\) and that \(g^2\) (a power of the fibre of \(A\)) has intersection number 1 with the fibre of \(M_B\) since this puts us into case 2 of Theorem 3. It then follows from Lemma 10 that \(G_y = \langle g^2, h^n \rangle\) is free in \(g^2\) and \(h^n\). This implies that \([x, y]\) and \([y, p]\) are \(G_y\)-inequivalent since otherwise \(a^2\) and a conjugate of \(h^n\) would commute which cannot happen if \(\langle g^2, h^n \rangle\) is free in \(g^2\) and \(h^n\). It is further clear that \(G_y \cap \text{Stab} [x, y] = \langle g^2 \rangle\) and that \(G_y \cap \text{Stab} [y, p] = \langle h^n \rangle\) since primitive elements generated maximal Abelian subgroups in free groups. By Proposition 1(2) we therefore know that the induced splitting of \(\langle g, h \rangle\) has cyclic edge groups which implies that \(\langle g, h \rangle \neq G\).

If \(T_g \cap hT_g \neq \emptyset\) we distinguish the cases that \(g\) lies (after conjugation) in \(A - C\) or in \(B\). If \(g \in A - C\) then the structure of \(T_g\) allows us argue as in the proof of Lemma 9 to get in situation (ii) of case 1.

Suppose that \(g \in B\). If \(g \in C\) and \(g\) has a root that is conjugate to an element of \(A - C\) we can clearly replace \(g\) by this element and argue as before. We can therefore assume that \(g\) is not conjugate to a power of \(f'_G\). If \(T_g\) consists of a single vertex then we argue as before that \(\langle g, h \rangle\) fixes this vertex which implies that \(\langle g, h \rangle \neq G\). It follows that (after conjugation) we can assume that \(g^k \in C\) for some \(k \in \mathbb{Z}\) and argue as before to see that \(g \in B_1\) where \(B_1\) is as above. Again we assume that \(g\) is a root of the fibre of \(B_1\) if \(B_1\) is Seifert and \(g\) is in \(G\) conjugate to a root of the fibre of \(B_1\). We can assume that the JSJ-decomposition of \(M^3\) has no piece that is homeomorphic to \(Q^3\) besides \(M_A\), otherwise we could define a surjective homomorphism from \(G\) to \(\mathbb{Z}_2 * \mathbb{Z}_2\) such that \(g\) lies in the kernel which clearly implies that \(\langle g, h \rangle \neq G\). A similar argument shows that the JSJ-graph must be a tree since we could otherwise define a surjective homomorphism from \(G\) onto \(\mathbb{Z}_2 * \mathbb{Z}\) with \(g\) in the kernel.
We now study the action on the Bass-Serre tree associated to the splitting that corresponds to the full JSJ-decomposition of $M^3$. It is clear that $g$ again acts with a fixed point. By the remark after Proposition 1 we can (after replacing $h$ by $g^kh$ for some $k$) that either $T_g \cap T_h \neq \emptyset$ or $T_g \cap hT_g \neq \emptyset$. If $T_g \cap T_h \neq \emptyset$ we are back in the first case.

Suppose that $T_g \cap hT_g \neq \emptyset$. Let $y$ be the vertex fixed under the action of $B_1$. As before we see that $T_g$ is contained in the 2-neighbourhood of $y$. Since $M_B$ contains no piece that is homeomorphic $Q^3$ it is further clear that for any vertex $v \in T_g$ with $d(y, v) = 2$ we see as before that $[y, v]$ maps onto the edge of the JSJ-decomposition that corresponds to $T$. We also have that $y$ is the only vertex of $T_g$ that can be of valence greater than 2 since $C$ is of index 2 in $A$.

Choose two vertices $p, q \in T_g$ such that $hp = q$. Since $p$ and $q$ project onto the same vertex of the JSJ-tree it follows that $d(p, q)$ is even. If $d(p, q) = 0$ then $h$ acts with a fixed point and we are in the case before. We have to deal with the cases that $d(p, q) = 2$ and that $d(p, q) = 4$.

If $d(p, q) = 2$ then either $y \in \{p, q\}$ or $d(y, p) = d(y, q) = 1$. In both cases we have that $h$ lies in the subgroup of $G$ that is generated by $B_1$ and the vertex group that corresponds to the vertex that is the midpoint of $[p, q]$ in the first case or that corresponds to $p$ and $q$ in the second case. If $g$ and $h$ generate $G$ this implies that this vertex corresponds to $A$ and that the JSJ-decomposition of $M_B$ is trivial. In both cases we see that after conjugation with an element of $B_1 = B$ we have that the edge fixed by $C$ lies in $[p, q]$, it follows that $g^n \in C$ for some $n \in \mathbb{N}$. It is further clear that $h = ab$ for some $a \in A$ and $b \in B$ since $ap = q$ for some $a \in A$. If $g \notin C$, i.e. $n \geq 2$ we have that $M_B$ is Seifert fibered and that $g$ is a root of the fibre. This puts us into situation 2 or 3.

Suppose now that $d(p, q) = 4$. It is clear that $M_{B_1}$ is a Seifert piece and that $g$ is a root of the fibre $f_{B_1}$ since otherwise no power of $g$ could fix two edges emanating at $y$ (we assume that $M_B$ is not homeomorphic to $Q^3$). As before we argue that $a f_B a^{-1}$ has non-trivial even intersection number with $f_B$ for all $a \in A - C$. Now $g^n$ and $h g^n h^{-1}$ fix $q$ and in $Stab y \cong B$ they correspond to boundary curves with even intersection number with the fibre that lie in different conjugacy classes of the peripheral subgroup of $B$ corresponding to $T$. By Lemma 18 this implies that $(g^n, h g^n h^{-1})$ is free in $g^n$ and $h g^n h^{-1}$. We define $G_q = \langle g^n, h g^n h^{-1} \rangle$, $G_p = \langle g^n, h^{-1} g^n h \rangle$ and $G_q = \langle g \rangle$. The freeness of $G_q$ and $G_q$ and the minimality of $n$ guarantee as before that $Stab [y, q] \cap G_q = Stab [y, q] \cap G_q$, that $Stab [y, p] \cap G_y = Stab [y, p] \cap G_p$ and that $[y, q]$ and $h[y, p] = [h, q]$ are $G_y$-inequivalent. We can further assume that $[y, p]$ and $[y, p]$ are $G_y$-inequivalent since otherwise we had that $g^kh$ acts with a fixed point for some $k \in \mathbb{N}$ which puts us into the first situation. Now by Proposition 3 we have that the induced splitting of $\langle g, h \rangle$ has cyclic edge groups, i.e. $\langle g, h \rangle \neq G$. □

We conclude the proof of Theorem 2. We have the different cases of Lemma 18. In case (i)-(iii) of situation 1 the argument is the same as in the case without pieces of type $Q^3$. We therefore only have to investigate the cases that $h = ab$ and that either

(a) $g \in C$ and that $g$ corresponds to neither the fibre $f_Q$ of $Q^3$ nor to the fibre of $M_B$ (if $M_B$ is Seifert) or

(b) $g \in B$ where $M_B$ is Seifert and $g$ is a root of the fibre $f_B$.

We can assume that $g$ does not correspond to $f_Q$, the fibre in the fibration of $A$ as the orientable surface bundle over the Möbius band, since we can then
argue as in the case without $Q^3$. We can further assume that $a^{-1}ga$ does not correspond to the fibre of $M_B$, where $M_B$ is the piece of $M_B$ containing $T$ since otherwise the proof of Lemma 2 produces situation (b). The same arguments as in the case without pieces of type $Q^3$ show that $(g, h) \notin G$ if $(g, h^{-1}gh) \subset B$ is free. Also as in the case without $Q^3$ we see that $M_B$ must have a trivial JSJ-decomposition. Note, that $g \in C$ and therefore $a^{-1}ga = \bar{g} \in C$ since $C$ is normal in $A$, therefore $h^{-1}gh = b^{-1}gb \in B$.

Suppose that $M_B$ is a hyperbolic piece. Now $(g, b^{-1}gb)$ cannot be Abelian since it does not lie in a parabolic subgroup. Since we assume that $g$ is neither a power of $f_Q$ nor of $f'_Q$ it follows that $g \neq \bar{g}^\pm 1$ which implies that $g$ and $b^{-1}gb$ are not conjugate in $B$ since $C$ is malnormal in $B$. By Lemma 12 we know that the covering of $M_B$ corresponding to $(g, b^{-1}gb)$ must be homeomorphic to the exterior of a two-bridge link. This covering can however not be regular since $g$ and $\bar{g}$ are not conjugate in $B$. This puts us into case 10 of Theorem 2.

Suppose that $M_B$ is Seifert. If $g$ and $\bar{g}$ have intersection number greater than 1 with $f_B$ then $(g, b^{-1}gb)$ is free by Lemma 14 and it follows that $(g, h) \notin G$. The same holds if $M_B$ is not a Seifert piece over a base space of type $D(2, n)$. It remains to check the case that the base space of $M_B$ is of type $D(2, n)$ and that either $g$ or $\bar{g}$ has intersection number one with $f_B$. We can assume that the intersection number of $f_Q$ with $f_B$ is not $\pm 1$ since we are otherwise in case 2 of Theorem 2. W.l.o.g. we can assume that $g$ has intersection number 1 with $f_B$. We use the notation as in the beginning of section 2. We write $g = (xy)^n f'_Q^{-m}$, it follows that $\bar{g}^{-1} = (xy)^n f'_Q^{-m-2n}$, i.e. that $g$ and $\bar{g}$ differ by $2(n + m)f'_Q$. Since we assume that $f'_Q$ has at least intersection number 2 with $f_B$ this implies that $\bar{g}$ has intersection number at least $4(n + m) - 1$ with $f_B$. It follows that the intersection number of $\bar{g}$ with $f_B$ is greater than 3 unless $|n + m| \leq 1$ in which case we argue as before since $(g, b^{-1}gb)$ is free by Lemma 14. If $n + m = 0$ then $g$ corresponds to the fibre $f_Q$ which we have already excluded. It remains to check the case that $|n + m| = 1$. W.l.o.g. we can assume that $n + m = 1$, i.e. $m = -n + 1$. The only case we have to deal with is that $g = (xy)^n f'_Q^{-n+1}$ has intersection number 1 with $f_B$ and that $\bar{g}^{-1} = (xy)^n f'_Q^{-n-1}$ has (algebraic) intersection number $-3$ with $f_B$. This however implies that $(xy)^n f'_Q^{-n}$ has intersection number $1 (-1)$ with $f_B$. This clearly implies that $n = \pm 1$, i.e. that $g$ corresponds to $f_B$ which we have already excluded.

(b) If the base space of $M_B$ is of type $D(p, q)$ or $A(p)$ and the intersection number of the fibres are one then we are in case 2 of Theorem 2, otherwise the fundamental group cannot be generated by $g$ and $h$ by Lemma 14.

If the base space is not of one of the above types then we consider the quotient of $G$ by $N_G(C)$. The quotient map $\phi : G \to G/N_G(C)$ maps $G$ onto the free product $A/C \ast B/N_B(C)$. It is clear that $A/N_A(C) = A/C \cong \mathbb{Z}_2$ and that $B/N_B(C)$ is a orbifold group that is not generated by an elliptic element. Furthermore $\phi(g)$ is elliptic in the quotient orbifold group. The proof of Grusko’s theorem implies that $\{\phi(g), \phi(h)\}$ is Nielsen-equivalent to a pair $\{\phi(g), \hat{h}\}$ such that $\hat{h} \in A/C \cup B/N_B(C)$. Such $\phi(g)$ and $\hat{h}$ however cannot generate $A/C \ast B/N_B(C)$ since $\phi(g)$ does not generate $B/N_B(C)$. It follows that $\phi(g)$ and $\phi(h)$ cannot generate $A/C \ast B/N_B(C)$. Thus $g$ and $h$ do not generate $G$. \qed
4 Heegaard genus

In this section we deduce Theorem 1 from Theorem 2. We will use the work of T. Kobayashi ([Ko1,2,3]).

In [Ko2], there is a complete list of closed orientable irreducible 3-manifolds with a genus 2 Heegaard splitting and with a non-trivial JSJ-decomposition. Later on, T. Kobayashi showed in [Ko3,§3] how to extend the results of [Ko1,2] in the context of Heegaard splittings of irreducible 3-manifolds with incompressible toral boundary. These results show that, except the 3-manifolds given in Theorem 1, all the other ones described in Theorem 2 have Heegaard genus 2.

The fact that an irreducible Heegaard genus two splitting is strongly irreducible, implies the following key lemma (cf. [Mor, Lemma 1.1], [MSa, Lemma 2.2] or [RuS2,§6]):

**Lemma 19** Let $M^3$ be a compact orientable irreducible 3-manifold with incompressible toral boundary and which admits a Heegaard decomposition $(V_1, V_2)$ of genus two. If $M$ has a non-trivial JSJ decomposition, then the JSJ family of tori $\Sigma$ can be isotoped so that it intersects $V_1$ and $V_2$ in a collection of essential annuli whose boundaries are also essential on the tori $\Sigma$.

A proof of this lemma follows from the sweep-out of $M^3$ by the Heegaard surface $F = \partial V_1 = \partial V_2$ as described in [RuS1].

Now the description of Heegaard genus two, compact, orientable, irreducible 3-manifolds with incompressible toral boundary follows, like in the closed case, from a careful analysis of the possible intersections of the JSJ family $\Sigma$ and the genus two compression-bodies $V_1$ and $V_2$ of the Heegaard decomposition (cf. [Ko2,§3], [Mor, Lemma 1.5], [MSa, Theorem 2.1 and Lemma 2.2], [RuS2,§5 and §6]).

For example if $M^3$ has one boundary component, case 1) of Theorem 2 corresponds to case 3-b) in the proof of Theorem 2.1 in [MSa]. In the same way, case 2) (respectively cases 3) and 4) ) of Theorem 2 corresponds to case 2-b) (respectively 3-a) and 1) ) of the proof of Theorem 2.1 in [MSa]. The analysis when $M^3$ has two boundary components is similar for these cases.

Case 6) of Theorem 2 is explained in [Ko3, Lemma 6.1].

The cases 7), 8) and 9) of of Theorem 2 corresponds to the case where the JSJ family $\Sigma$ contains only non-separating tori. The analysis is the same as the one carried for the closed case in [Ko1, Thm 2] (cf. also [RuS2,§6, cases 2) and 3)).

One can also deduce the classification of Heegaard genus 2 compact irreducible and $\partial$-irreducible 3-manifolds, with non-empty toral boundary and non-trivial JSJ-decomposition, from the closed case by using the following lemma. This lemma is a direct consequence of the work of Y. Rieck and E. Sedgwick [RiS1], [RiS2].

**Lemma 20** Let $M^3$ be a compact orientable irreducible 3-manifold with incompressible toral boundary. Let $T^2 \subset \partial M^3$ be a boundary component. If infinitely many Dehn fillings on $T^2$ yield a manifold with a Heegaard splitting of genus 2, then $M^3$ has a Heegaard splitting of genus 2.
Proof By the work of Y. Rieck and E. Sedgwick [RiS2, Cor. 6.6] there are infinitely many Dehn fillings on \(T^2\) such that the core \(\gamma\) of the attached solid torus can be isotoped into the genus 2 Heegaard surface \(\Sigma^2\) of the resulting manifold. Since a Heegaard genus two splitting is strongly irreducible, by [RiS1, Thm.4.6] either:

1. \(\Sigma^2\) can be isotoped to a Heegaard surface of \(M^3\); or
2. the surface \(\Sigma^2 \cap M^3\) is incompressible and \(\partial\)-incompressible in \(M^3\) or compresses to such an essential surface.

In the latter case, the filling slope is the boundary slope of an essential surface, with all its boundary in \(T^2\). By Hatcher’s finiteness result (cf. [Hat]), there are only finitely many such slopes on \(T^2\).

\[\Box\]

5 Two-generated subgroups of 3-manifold fundamental groups

In this section we prove Corollaries 4 to 7 which are direct consequence of the proof of Theorem 2.

The proof of Corollary follows immediately from the following Lemma:

Lemma 21 Let \(M^3\) be a compact orientable 3-manifold and let \(U\) be a 2-generated subgroup of \(\pi_1(M^3)\) that is neither free or free abelian. If \(U\) is of infinite index in \(\pi_1(M^3)\), then \(U\) is either the fundamental group of a Seifert fibered manifold, or of a complete hyperbolic manifold with finite volume, or of one of the manifolds described in Theorem 4.

Proof Let \(p : \hat{M}^3 \to M^3\) be the covering of \(M^3\) with fundamental group \(\pi_1(\hat{M}^3) = U\). By Scott’s compact core theorem, there is a compact submanifold \(\hat{N}^3 \subset \hat{M}^3\) such that the inclusion map \(\hat{N}^3 \hookrightarrow \hat{M}^3\) induces an isomorphism between \(\pi_1(\hat{N}^3)\) and \(U\).

If \(U = U_1 * U_2\) is a non-trivial free product, then one factor, say \(U_1\), is a finite cyclic group since \(U\) is not a free group. In this case, since \(M^3\) is orientable, by Epstein’s theorem [Ep] (cf. [He, Thm.9.8]) \(M^3 = M^3_1 \# R^3\) where \(R^3\) is closed and orientable, \(\pi_1(R^3)\) is finite. Moreover, after conjugation, one may assume that \(U_1\) is a subgroup of \(\pi_1(R^3)\). Since the covering \(p : M^3 \to M^3\) is of infinite index, \(p^{-1}(R^3)\) would lift to infinitely many connected sum factors of \(M^3\) with non-trivial fundamental group \(U_1\), contradicting the fact that \(\pi_1(M^3) = U\) is of rank two.

Therefore, we can always assume that \(U\) is freely indecomposable. Since \(U = \pi_1(\hat{N}^3)\) is not infinite cyclic, according to [He, Lemma 10.1] there is a compact orientable and irreducible 3-manifold \(N^3_0\) with fundamental group \(\pi_1(N^3_0) = U\). It is the Poincaré associate of \(\hat{N}^3\), i.e. the only non-simply connected prime factor of the 3-manifold obtained by capping off the 2-spheres in \(\partial \hat{N}^3\) by 3-balls.

Since \(U\) is of infinite index in \(\pi_1(M^3)\), its cohomological dimension is smaller or equal to two. Hence it cannot be the fundamental group of a closed irreducible
3-manifold. Therefore the compact, orientable, irreducible 3-manifold $N_3^0$ has a non-empty incompressible boundary, since $U$ is not a free product. Hence $N_3^0$ is a Haken 3-manifold with incompressible boundary. According to [JS2, Lemma 5.4], all components of $\partial N_3^0$ are tori.

If $N_3^0$ is atoroidal (i.e. has a trivial JSJ decomposition), it follows from Thurston’s hyperbolization theorem that it is either a Seifert fibered 3-manifold or a complete hyperbolic 3-manifold with finite volume.

If $N_3^0$ has a non-trivial JSJ-decomposition then $N_3^0$ is one of the manifold described in Theorem 2.

\[ \square \]

Proof of Corollary 6

The following can be extracted from the proof of Theorem 2 but a direct argument is just as easy. We first show that $M^3$ must have a trivial JSJ-decomposition if $\pi_1(M^3)$ is generated by two peripheral elements. Suppose that $\pi_1(M^3) = \langle g, h \rangle$ where $g$ and $h$ are peripheral. It is clear that $g$ and $h$ act as elliptic elements on the Bass-Serre tree associated to the splitting of $\pi_1(M^3)$ corresponding to the JSJ-decomposition. It then follows from [KW] that powers of $g$ and $h$ must have a common fixed point. Since $g$ and $h$ are peripheral they are not proper roots of the fiber of some Seifert piece in the JSJ-decomposition. Thus $g(h)$ fixes every point in the Bass-Serre tree that is fixed by a power of $g(h)$. It follows that $g$ and $h$ have a common fixed point, i.e. $\pi_1(M^3) = \langle g, h \rangle$ is conjugate to a vertex stabilizer. It follows that the splitting and therefore the JSJ-splitting of $M^3$ is trivial.

Since it has an incompressible boundary, according to [JS2, Lemma 5.4] all components of $\partial M^3$ are tori. Thus by Thurston’s hyperbolization theorem $M^3$ is either a Seifert fibered 3-manifold or a complete hyperbolic 3-manifold with finite volume. In the Seifert fibered case, the proof follows from Lemmas 8 and 11. In the hyperbolic case it follows from Lemma 12. \[ \square \]

Proof of Corollary 7

Let $M^3$ be a compact orientable, irreducible 3-manifold with incompressible boundary. If $M^3$ is not hyperbolic and $\pi_1(M^3)$ is generated by two elements one of which is peripheral then either $M^3$ has Heegaard genus 2 or $M^3$ belongs to case 2) of Theorem 1.

So we can assume that $M^3 = S \cup_T H$. Here $H$ is a hyperbolic 3-manifold with $\pi_1(H)$ generated by a pair of elements with a parabolic generator belonging to the parabolic subgroup associated to the boundary component $T$. The Seifert manifold $S$ has basis $D(p, q)$ or $A(p)$. The gluing map identifies the fibre of $S$ on $T$ with the simple closed curve corresponding to the parabolic generator of $\pi_1(H)$.

We will show that $H$ is homeomorphic to the exterior of a two bridge link and that the parabolic generator corresponds to a meridian. It then follows from [Ko2] that $M^3$ has Heegaard genus two.

Suppose now that $\pi_1(M^3) = \langle g, h \rangle$ where $g$ is peripheral. As before we see that $g$ acts with fixed point on the associated Bass-Serre tree and it follows from [KW] that we can choose $h$ such that either $T_g \cap T_h \neq \emptyset$ or $T_g \cap hT_g \neq \emptyset$. In particular the conclusion of Lemma 2 and Lemma 18 hold with this particular $g$ except possibly in the case that $h$ also acts with a fixed point; here we might have to exchange $g$ and $h$. It now suffices to observe that in the proof of Theorem 2 either $H$ was the exterior of a 2-bridge link or $g$ was a proper root of the fibre. The last case however is impossible if $g$ and $h$ were not interchanged since a peripheral element cannot be a proper root of the fibre. If $g$ and $h$ were
exchanged then the proof of Theorem 2 implies that $g$ is a root of the fibre and
$\pi_1(H)$ is conjugate to $\langle h, g^n \rangle$ where $g^n$ is a power of $g$ that lies in the conjugate
of some edge group. In particular $g^n$ is peripheral in $\pi_1(H)$. Since $h$ is also
peripheral it follows from Corollary 6 that $H$ is the exterior of a 2-bridge link
and that $h$ corresponds to a meridian.

\[ \square \]

Proof of Corollary 8 Let $k \subset S^3$ be a 2-generator satellite knot. It follows from
Theorem 1 that $k$ is tunnel number one or its exterior $E(k)$ belongs to case 2)
of Theorem 1.

It means that $E(k) = M_1 \cup T M_2$, where $M_1$ is a Seifert manifold and $M_2$ is
a hyperbolic 3-manifold.

The Seifert manifold $M_1$ has basis $D(p, q)$ or $A(p)$, hence it is the exterior
of a $(p, q)$-torus knot in $S^3$, or $k$ is a cable knot [Ja, Lemma IX.22]. But, by
using the cyclic surgery Theorem [CGLS], Bleiler [Ble2] has shown that a 2-
generator cable knot is an iterated torus knot, contradicting the fact that $M_2$
is hyperbolic.

Therefore $M_1$ is the exterior of a torus knot and $\partial M_2 = T \cup \partial E(k)$ has
two components. The splice decomposition of the satellite knot $k$ shows that
$M_2$ is homeomorphic to the exterior of a link $L' = k_0' \cup k_1'$ obtained in the
way described in [EN, Prop. 2.1]. The torus $T$ bounds in $S^3$ a solid torus $V_0$
containing $k$ and $M_2$ is homeomorphic to the exterior of $k$ in $V_0$. By gluing a
solid torus $V_1$ to $T$ in such a way that the preferred longitude of the torus
knot exterior on $T$ is identified with the boundary of a meridian of $V_1$ one gets
$S^3$. Then the image of $k$ in $S^3$ by this desplcing operation is $k_0'$ and $L'$ is the
link formed by $k_0'$ and the core $k_1'$ of the unknotted solid torus $V_1$. It follows
that $k_1'$ is unknotted in $S^3$

The hyperbolic 3-manifold $M_2$ has a fundamental group $\pi_1(M_2)$ generated by
a pair of elements with a single parabolic generator $a$ belonging to the parabolic
subgroup associated to the boundary component $T$. The gluing map identifies
the fibre of $S$ on $T$ with the simple closed curve $\alpha \subset T$ corresponding to the
parabolic generator $a$.

Because of the gluing instruction, this curve $\alpha$ intersects on $T$ the meridian
of the torus knot and hence the meridian of $k_1'$ only once. Therefore a Dehn
surgery along $k_1'$ with slope $\alpha$ yields back $S^3$. Let $k_1$ be the core of this Dehn
surgery and $k_0 \subset S^3$ be the image of $k_0'$ after this Dehn surgery. Then the
exterior of $L = k_0 \cup k_1$ is homeomorphic to the exterior of $L'$, hence to $M_2$.

Moreover, $\pi_1(M_2)$ is generated by two elements where one is a meridian of
the component $k_1$ of $L$. It follows that the fundamental group of the exterior
of the other component $k_0$ is cyclic, hence $k_0$ is also unknotted by the loop
theorem.

Then it follows from [Ku] that $L$ has tunnel number one if and only if it is
a 2-bridge link. But in this case $K$ has tunnel number one (cf. [MSa]).

\[ \square \]

6 Involutions on 3-manifolds with a 2-generated fundamental group

In this section we prove Corollaries and 3.
6.1 2-fold branched coverings of homotopy spheres

For the proof of Corollary 1 we distinguish the two cases that \( M^3 \) is geometric and that \( M^3 \) has a non-trivial JSJ-decomposition.

1. case: \( M \) is geometric.

Then it is either a Seifert fibred, a Sol or a hyperbolic 3-manifold with rank \( \pi_1(M^3) = 2 \).

In the Seifert fibred case, the proof follows from the determination of rank 2 closed Seifert 3-manifolds [BZg] and the construction of the fibre preserving Montesinos involution on these manifolds which shows that they are 2-fold branched coverings of \( S^3 \) ([Mon1],[Mon2, §4.7]).

For Sol 3-manifolds, the proof follows from the determination of rank 2 orientable torus bundles [TO, Lemma 1] (cf. also [Sa2]).

In the hyperbolic case, the proof follows from the following proposition which includes the case with boundary. We will need it when the JSJ-decomposition is not trivial. We recall that a \( n \)-times punctured homotopy 3-sphere is a 3-dimensional homotopy sphere minus the interior of \( n \) disjoint embedded 3-balls.

**Proposition 3** Let \( M^3 \) be a compact orientable complete hyperbolic 3-manifold with finite volume. If rank \( \pi_1(M^3) = 2 \) and \( \pi_1(M^3) \) is not abelian, then \( M^3 \) is a 2-fold branched covering of a \( n \)-times punctured homotopy sphere with \( n \leq 2 \).

**Proof** Let \( \alpha \) and \( \beta \) two elements in \( \pi_1(M) \) that generate the group. The proof follows from the following lemma due to Jorgensen (cf. [Th, chap.5, Prop.5.4.1 and 5.4.2]):

**Lemma 22** Any complete hyperbolic 3-manifolds \( M^3 \) whose fundamental group is not abelian and generated by two elements \( \alpha \) and \( \beta \) admits a non-free, orientation preserving isometry \( \tau \) of order 2 which takes \( \alpha \) to \( \alpha^{-1} \) and \( \beta \) to \( \beta^{-1} \). Moreover if the two generators are conjugate in \( \pi_1(M^3) \), there is an orientation preserving, isometric \( \mathbb{Z}^2 \oplus \mathbb{Z}^2 \) action on \( M^3 \) generated by \( \tau \), together with an involution \( \rho \) which interchanges \( \alpha \) and \( \beta \).

**Proof of Lemma 22** The involution \( \tau \) is induced by conjugation by the rotation \( t \) of angle \( \pi \) about the the following line \( \Delta_0 \) in the hyperbolic space \( \mathbb{H}^3 \):

(i) \( \Delta_0 \) is the unique common perpendicular to the axis of \( \alpha \) and \( \beta \) if they are both loxodromic elements in \( PSL_2(\mathbb{C}) \);

(ii) \( \Delta_0 \) is the unique line through the fixed points at infinity of \( \alpha \) and \( \beta \), if they are both parabolic elements in \( PSL_2(\mathbb{C}) \), (the two fixed points are distinct, because \( \pi_1(M) \) is not abelian);

(iii) \( \Delta_0 \) is the unique perpendicular to the axis of \( \alpha \) through the fixed point at infinity of \( \beta \), if \( \alpha \) is loxodromic and \( \beta \) is parabolic.

In particular, \( t\alpha^{-1} = \alpha^{-1} \) and \( t\beta^{-1} = \beta^{-1} \). Therefore, if \( w(\alpha, \beta) = 1 \) is any relation in the group \( \pi_1(M) \), then \( w(\alpha^{-1}, \beta^{-1}) = 1 \). It follows that the rotation \( t \) conjugates the group \( \pi_1(M) \) to itself, and hence induces a non-free, orientation preserving, isometric involution \( \tau \) on \( M^3 \) which takes \( \alpha \) to \( \alpha^{-1} \) and \( \beta \) to \( \beta^{-1} \).

If \( \alpha \) and \( \beta \) are conjugated in \( \pi_1(M^3) \), then they are either both loxodromic elements or both parabolic elements (with distinct fixed point) in \( PSL_2(\mathbb{C}) \). Then the two involutions \( \rho \) and \( \tau \circ \rho \) are induced by conjugation by rotations
r_1 and r_2 of angle π about two lines \( \Delta_1 \) and \( \Delta_2 \) that are perpendicular to each other and perpendicular to the line \( \Delta_0 \) in the hyperbolic space \( \mathbb{H}^3 \):

(iv) If \( \alpha \) and \( \beta \) are both loxodromic elements in \( \text{PSL}_2(\mathbb{C}) \), \( \Delta_1 \) and \( \Delta_2 \) intersects \( \Delta_0 \) at the mid point between the axis \( \Delta_\alpha \) and \( \Delta_\beta \) of \( \alpha \) and \( \beta \). Moreover the planes \( (\Delta_0, \Delta_1) \) and \( (\Delta_0, \Delta_2) \) are the two bisector planes between the two planes \( (\Delta_0, \Delta_\alpha) \) and \( (\Delta_0, \Delta_\beta) \).

(v) If \( \alpha \) and \( \beta \) are both parabolic elements in \( \text{PSL}_2(\mathbb{C}) \), \( \Delta_1 \) and \( \Delta_2 \) intersect \( \Delta_0 \) at the unique point \( p \subset \Delta_0 \) such that \( d(p, \alpha(p)) = d(p, \beta(p)) \).

In particular the two rotations \( r_1 \) and \( r_2 \) around \( \Delta_1 \) and \( \Delta_2 \) commute with the rotation \( t \) around \( \Delta_0 \).

One of these rotations, say \( r_1 \) conjugates \( \alpha \) to \( \beta \), while the second one \( r_2 = t \circ r_1 \) conjugates \( \alpha \) to \( \beta^{-1} \). Therefore, if \( w(\alpha, \beta) = 1 \) is any relation in the group \( \pi_1(M) \), then \( w(\beta, \alpha) = 1 \). So the rotation \( r_1 \) conjugates the group \( \pi_1(M) \) to itself, and hence induces a non-free, orientation preserving, isometric involution \( \rho \) on \( M^3 \) which takes \( \alpha \) to \( \beta^{-1} \) and commutes with the involution \( \tau \).

We finish now the proof of Proposition \[\text{Proposition 3}\]. Let \( \Gamma \) be the subgroup of \( \text{PSL}_2(\mathbb{C}) \) generated by \( \alpha, \beta \) and \( t \). Then \( \pi_1(M) \) is a subgroup of index at most 2 in \( \Gamma \). Hence \( \Gamma \) is a discrete cocompact subgroup of \( \text{PSL}_2(\mathbb{C}) \) and \( \mathcal{O} = \mathbb{H}^3/\Gamma \) is the compact orientable hyperbolic 3-orbifold \( \mathcal{O} = M/\tau \) obtained as the quotient of \( M^3 \) by the orientation preserving isometric involution \( \tau \). The orbifold fundamental group of \( \mathcal{O} \) is \( \Gamma \).

Since \( \Gamma \) is generated by the three orientation preserving isometric involutions \( t\alpha \), \( t\beta \) and \( t \) of the hyperbolic space \( \mathbb{H}^3 \), the fundamental group of the underlying space \( |\mathcal{O}| \) of \( \mathcal{O} \) is trivial (cf. [Th, Chap.13]). In particular \( |\mathcal{O}| \) is a \( n \)-times punctured homotopy sphere, and \( M^3 \) is a 2-fold branched covering of it.

Since rank \( \pi_1(M^3) = 2 \), \( M^3 \) has at most two boundary components and they are tori. Therefore \( \partial \mathcal{O} \) has at most two boundary components and they are pillows (i.e. 2-spheres with 4 branching points of order 2). Thus \( |\mathcal{O}| \) is a \( n \)-times punctured homotopy sphere with \( n \leq 2 \). In particular the restriction of the involution \( \tau \) on each torus boundary component is conjugated by an isotopy to the Weierstrass involution.

\[\text{Remark 5}\] Except in the case where \( M^3 \) has Heegaard genus 2, we cannot prove that \( |\mathcal{O}| \) is the true sphere \( S^3 \), the true ball \( B^3 \) or the product \( S^2 \times [0, 1] \).

2. case: \( M^3 \) has a non-trivial JSJ-decomposition.

Then either \( M^3 \) has Heegaard genus two or belongs to one of the cases 2) to 4) described in Theorem \[\text{Theorem 4}\].

If \( M^3 \) has Heegaard genus two, then it is a 2-fold branched covering of \( S^3 \) by Birman and Hilden [BH].

If \( M^3 \) belongs to case 2), \( M^3 = S \cup_H H \), where \( H \) is a hyperbolic 3-manifold with a 2-generated fundamental group and \( S \) is a Seifert 3-manifold over \( D(p, q) \). By Proposition \[\text{Proposition 5}\] \( H^3 \) is a 2-fold branched covering of a homotopy ball and the restriction of the covering involution to \( T = \partial H \) is conjugated by an isotopy to the Weierstrass involution.

A Seifert manifold \( S \) with basis \( D(p, q) \) admits an orientation preserving Montesinos involution with quotient a 3-ball, which is fiber-preserving and re-
verses the orientation of the fiber. The restriction of this involution to \( T = \partial S \) is also conjugated by an isotopy to the Weierstrass involution.

Since any gluing homeomorphism \( f : T \to T \) is isotopic to one which commutes with the Weierstrass involutions on \( T \), one can glue the two involutions to get an involution on \( M^3 \) with quotient a homotopy sphere.

If \( M^3 \) belongs to case 3), \( M^3 = S_1 \cup_T S_2 \), where \( S_1 \) is a Seifert 3-manifold over \( M\tilde{o} \) or \( M\tilde{o}(p) \) and \( S \) is a Seifert 3-manifold over \( D(2, 2l + 1) \). Both sides admit an orientation preserving Montesinos involution whose restriction to the torus boundary is conjugated by an isotopy to the Weierstrass involution. The same argument as in case 2) shows that \( M^3 \) is a 2-fold branched covering of \( S^3 \).

If \( M^3 \) belongs to case 4), \( M^3 = Q \cup_T H \), where \( H \) is a hyperbolic 3-manifold and \( Q \) is the orientable circle bundle over \( M\tilde{o} \). The difference here is that we do not know that the fundamental group of the hyperbolic piece is generated by two elements. However, the restrictions on \( \pi_1(H) \) given at the end of §3.3 allow to show:

**Lemma 23** In case 4) of Theorem 1, the fundamental group of the hyperbolic piece \( H \) is generated by two conjugated peripheral subgroups \( P = \pi_1(\partial H) \) and \( hP_1h^{-1} \), with \( h \in \pi_1(H) \).

By an hyperbolic Dehn filling argument we deduce from Lemma 23 the following corollary:

**Corollary 9** In case 4) of Theorem 1, the hyperbolic piece \( H \) is a 2-fold branched covering of a homotopy ball.

**Proof of Corollary 9** Let \( \alpha \subset \partial H \) be a simple closed curve and let \( H(\alpha) \) be the closed 3-manifold obtained by gluing a solid torus along \( \partial H \) in such way that \( \alpha \) is identified with the boundary of a meridian disk. It follows from Lemma 23 \( \pi_1(H(\alpha)) \) is generated by two conjugated elements.

By Thurston’s hyperbolic Dehn filling theorem [Th] (cf.[BP],[CHK]), for almost all simple curves \( \alpha \subset \partial H \), \( H(\alpha) \) is hyperbolic and the core \( k_\alpha \) of the Dehn filling is the shortest geodesic in \( H(\alpha) \).

By Lemma 23 there is an orientation preserving, isometric \( \mathbb{Z}^2 \oplus \mathbb{Z}^2 \) action on \( H(\alpha) \) generated by an involution \( \tau \) which take each generator of \( \pi_1(H(\alpha)) \) to its inverse, together with an involution \( \rho \) which interchanges the two generators. Moreover, this \( \mathbb{Z}^2 \oplus \mathbb{Z}^2 \) action preserves the shortest geodesic \( k_\alpha \). Hence, it induces a \( \mathbb{Z}^2 \oplus \mathbb{Z}^2 \) action on the exterior \( H \) of \( k_\alpha \). Moreover, the involutions \( \tau \) reverses the orientation of \( k_\alpha \), since it takes each generator to its inverse. In particular, the quotient of an invariant tubular neighborhood of \( k_\alpha \) by \( \tau \) is a 3-ball.

By Proposition 3 the quotient of \( H(\alpha) \) by the involution \( \tau \) is a homotopy sphere, hence the quotient of \( H \) by the restriction of \( \tau \) is a homotopy ball.

Using Corollary 23 and the fiber preserving Montesinos involution on \( Q \), viewed as a Seifert manifold over \( D(2, 2) \), one can conclude in case 4) like in case 2). This finishes the proof of Corollary 9.
Examples of closed hyperbolic 3-manifolds with a 2-generated fundamental group can be obtained by hyperbolic Dehn filling along a 2-generator knot in \( S^3 \), for examples a 2-bridge knot.

Using hyperbolic Dehn fillings of once punctured torus bundles with pseudo-Anosov monodromy, A. Reid [Rei] has been able to construct infinitely many closed 2- or 3-generator hyperbolic 3-manifolds which have a proper finite sheeted cover with a 2-generated fundamental group. From his construction, he gave also examples of closed Haken hyperbolic 3-manifolds which are neither a surface bundle nor double covered by a surface bundle, but which have a finite cover which has 2-generated fundamental group.

However, a 2-generator once punctured torus bundle has Heegaard genus 2 by [TO, Lemma 1] (cf. also [Sa2]).

We give now an example of hyperbolic genus 2 surface bundle with a 2-generated fundamental group, due to Nielsen.

Let \( F \) be a compact orientable surface with negative Euler characteristic. We say that a diffeomorphism \( \phi \in \text{Diff}^+(F) \) fills up \( \pi_1(F) \) if there is an element \( \gamma \in \pi_1(F) \) such that its orbit \( \{ \phi^n(\gamma) \}_{n \in \mathbb{Z}} \) generates \( \pi_1(F) \).

It is an easy observation that the mapping torus of a diffeomorphism \( \phi \in \text{Diff}^+(F) \) that fills up \( \pi_1(F) \) has a 2-generated fundamental group.

The following example (Nielsen’s example \( \# 13 \) in [Nie]; cf. [Gil]) is a pseudo-Anosov diffeomorphism of a one-punctured surface \( F \) of genus 2 which fills up \( \pi_1(F) \). Hence its mapping torus \( M_\phi \) is a complete hyperbolic 3-manifold with finite volume and one cusp.

The closed genus 2 surface bundle, obtained by Dehn filing \( \partial M_\phi \) along the the closed curve \( \partial F \), has still a pseudo-Anosov monodromy and hence has a 2-generated fundamental group.

A diffeomorphism \( \phi \in \text{Diff}^+(F) \) is determined, up to isotopy, by its induced action on \( \pi_1(F) = \langle a, b, c, d \rangle \), where the conjugacy class of \( [a, b][d, c] \) represents the simple loop \( \partial F \).

Let \( \phi_* : \pi_1(F) \to \pi_1(F) \) be the automorphism given by:

\[
\phi_*(a) = c^{-1}a^{-1}; \quad \phi_*(b) = b^{-1}a^{-1}
\]

\[
\phi_*(c) = b^{-1}a^{-1}d; \quad \phi_*(d) = c^{-1}.
\]

One easily verifies that:

\[
\phi_*([a, b][c, d]) = (abc)^{-1}[a, b][c, d](abc).
\]

That is the necessary and sufficient condition for \( \phi_* \) to be induced by a diffeomorphism \( \phi \in \text{Diff}^+(F) \).

Moreover, it is easy to check that \( \pi_1(F) \) is generated by \( \langle b, \phi_*(b), \phi_2^*(b), \phi_3^*(b) \rangle \). Hence \( \phi \) fills up \( \pi_1(F) \).

The Nielsen’s classification of diffeomorphisms of surfaces [Nie], based on the Nielsen types of the lifts of the diffeomorphism to the unit disk allows to show that \( \phi \) is pseudo-Anosov (cf. [Gil]).

### 6.2 2-generator knots in 3-manifolds

The proof of corollary follows from Thurston’s orbifold theorem (cf. [BP], [CHK]) and the following lemma:
Lemma 24 Let $k \subset M^3$ be a 2-generator knot in a closed, orientable, irreducible 3-manifold $M^3$. If $k$ is not contained in a ball, then $k$ is strongly invertible: i.e. there is an orientation preserving involution on $M^3$ that preserves $k$ while reversing its orientation.

Proof Since $M^3$ is irreducible and $k \subset M$ is not contained in a ball, the exterior $E(k)$ of $k$ is irreducible. It follows from Theorem 1 that $E(k)$ has Heegaard genus two or belongs to the case 2) described in Theorem 1.

If $E(k)$ has Heegaard genus 2, then the hyperelliptic involution on the genus 2 Heegaard surface extends to both sides since the attaching curve for the 2-handle can be choosen, up to isotopy, to be invariant under the hyperelliptic involution. This gives a non-free, orientation preserving involution $\tau$ on $E(k)$ whose restriction to $\partial E(k)$ is conjugated to the Weierstrass involution with quotient a 2-sphere with 4 branching points. In particular it extends to an orientation preserving, non-free involution $\tilde{\tau}$ on $M$, preserving $k$ and with two fixed points on $k$.

If $E(k)$ belongs to the case 2) described in Theorem 1, $E(k) = S \cup_T H$, where $H$ is a hyperbolic 3-manifold with a 2-generated fundamental group and $S$ is a Seifert 3-manifold over $D(p,q)$ or $A(p)$.

By Proposition 4 $H$ is a 2-fold branched covering of a homotopy ball or twice punctured homotopy sphere, according whether $\partial H$ has one or two components. Moreover the restriction of the covering involution to each component $T \subset \partial H$ is conjugated by an isotopy to the Weierstrass involution.

A Seifert manifold $S$ with basis $D(p,q)$ (respectively $A(p)$) admits an orientation preserving Montesinos involution with quotient a 3-ball (respectively a product $S^2 \times [0,1]$), which is fiber-preserving and reverses the orientation of the fiber. The restriction of this involution to each component $T \subset \partial S$ is also conjugated by an isotopy to the Weierstrass involution.

Then a gluing argument, analogous to the one used in the proof of Corollary 1 shows that $E(k)$ admits an orientation preserving, non-free involution $\tau$ whose restriction to $\partial E(k)$ is conjugated to the Weierstrass involution. As above it extends to an orientation preserving, non-free involution $\tilde{\tau}$ on $M$, preserving $k$ and with two fixed points on $k$. 

Proof of corollary 3 Since $M$ is irreducible, by the proof of the Smith conjecture ([MB],[Wa3]) the orbifold $M/\tilde{\tau}$ is irreducible. Then, by Thurston’s orbifold theorem ([BP], CHK) either $M/\tilde{\tau}$ is geometric, or $M$ has a non-trivial JSJ-decomposition.

7 2-generator knots in $\mathbb{S}^3$

In this section we prove corollary 3, which extends to 2-generator knots a result shown for tunnel number one knots by M.Scharlemann ([Scha]).

We recall that a knot $k \subset \mathbb{S}^3$ is prime if there is no sphere $S^2$ that meets $k$ transversally in two points and intersects the exterior $E(k) = \mathbb{S}^3 - \text{int}(N(k))$ in an essential annulus.

A Conway sphere for a knot $k$ is sphere $S^2$ that meets $k$ transversally in four points and gives in the exterior $E(k)$ of $k$ an essential planar surface. The knot $k$ is Conway irreducible (or doubly prime) if there is no Conway sphere for $k$. 

38
The fact that 2-generator knots are prime follows already from [N] (see also [W1]). So in the remaining we assume hat the knot \( k \) is prime. Then by [BS] for a prime knot \( k \subset S^3 \) there is a finite characteristic collection of disjoint, non-parallel tori and Conway spheres such that: (i) the collection of tori corresponds to the JSJ-family of tori of the exterior \( E(k) \) of \( k \); (ii) the collection of tori and Conway spheres lifts to the JSJ-family of tori of the 2-fold branched covering of \( k \).

In particular a knot \( k \subset S^3 \) is prime and Conway irreducible if and only if its 2-fold branched covering is irreducible and topologically atoroidal (cf. [Ble1]).

Starting with a 2-generator knot \( k \), we consider its Bonahon-Siebenmann characteristic collection \( C \) of tori and Conway spheres, and we assume that \( C \) contains at least one Conway sphere. This Conway sphere will avoid the JSJ family of tori \( T \) in \( C \).

Let \( X^3 \subset E(k) \) be the closure of the connected component of \( E(k) - T \) that contains \( \partial E(k) \). Then the track of the Conway sphere in \( E(k) \) is an essential, properly embedded four punctured sphere \((\Sigma^2, \partial \Sigma^2) \hookrightarrow (X^3, \partial E(k)) \).

If \( \text{rank} (\pi_1(E(k))) = 2 \), it follows from Theorem [2] that either \( X^3 \) is Seifert fibered, or it is hyperbolic with \( \text{rank} (\pi_1(X^3)) = 2 \).

If \( X^3 \) is Seifert fibered, the only essential surfaces in \( X^3 \), with non-empty boundary, are either saturated annuli or horizontal surfaces (i.e branched covering of the basis of the Seifert fibration). Hence it is an annulus or it meets each boundary component of \( \partial X^3 \). In both cases, it cannot be a four punctured sphere with only meridional boundary components (cf. also [GL, Lemma 3.1], since \( X^3 \) must be a cable space).

If \( X^3 \) is hyperbolic and \( \text{rank} (\pi_1(X^3)) = 2 \), one uses the following well-known lemma:

**Lemma 25** Let \( M^3 \) be a compact, orientable, irreducible and atoroidal 3-manifold with empty or incompressible boundary. If \( \text{rank} (\pi_1(M^3)) = 2 \), then \( M^3 \) does not contain any properly embedded, essential, acylindrical, compact orientable separating surface.

A properly embedded, compact orientable surface \((F^2, \partial F^2) \hookrightarrow (M^3, \partial M^3)\) is acylindrical if any embedded incompressible annulus \((A^2, \partial A^2) \hookrightarrow (M^3, F^2)\), such that \( A^2 \cap F^2 = \partial A^2 \), can be homotoped into \( F^2 \), by a homotopy supported in the side of the surface containing \( A^2 \).

By the annulus Theorem (cf. [Ja, Chap.VIII], [Sco]), a properly embedded, essential and acylindrical, compact, orientable and separating surface in \( M^3 \) induces a malnormal (i.e. 1-acylindrical) amalgamated splitting of \( \pi_1(M^3) \). By a result of A. Karrass and D. Solitar [KS], a non-free, two generated group cannot admit such a malnormal splitting.

To finish the proof of Corollary 3 when \( X^3 \) is hyperbolic, it remains to show that the properly embedded, essential, four punctured sphere \((\Sigma^2, \partial \Sigma^2) \hookrightarrow (X^3, \partial E(k)) \) is acylindrical in \( X^3 \).

If \((A^2, \partial A^2) \hookrightarrow (X^3, \Sigma^2)\), such that \( A^2 \cap \Sigma^2 = \partial A^2 \), is an incompressible annulus, then the two components of \( \partial A^2 \) are parallel essential simple curves on the four punctured sphere \( \Sigma^2 \). Otherwise, one component of \( \partial A^2 \) must be boundary parallel on \( \Sigma^2 \). Since \( k \) is a prime knot, the other component of \( \partial A^2 \) cannot be parallel to the boundary on \( \Sigma^2 \). Hence, this curve must be bound a disk with two holes, that does not contain the other component of \( \partial A^2 \). This
disk with two holes, together with the annulus $A^2$ gives an embedded three-punctured sphere, with meridional boundary components, in $E(k)$, which is impossible.

Now, the annulus $A^2$, together with the annulus on $\Sigma^2$ bounded by $\partial A^2$, give a torus $T^2$. This torus must be compressible in the side of $\Sigma^2$ that contains $A^2$, otherwise it would be incompressible in $X^3$, since $\Sigma^2$ is essential in $X^3$. Since $X^3$ is irreducible and the components of $\partial A^2$ are not contained in a ball in $X^3$, $T^2$ bounds a solid torus in one side of $\Sigma^2$, which shows that the annulus $A^2$ can be pushed on $\Sigma^2$.  

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