MISCELLANEOUS PROBLEMS ABOUT PACKING AND COVERING

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ABSTRACT. In this paper we discuss various special problems on packing and covering. Among others we survey the problems and results concerning finite arrangements, Minkowskian, saturated, compact, and totally separable packings. We discuss shortest path problems and questions about stability of packings.

0.1. Arranging houses. Suppose a large area is designated for a housing project in which the minimum distance between the congruent rectangular outlines of the houses is prescribed. Which arrangement of the rectangles allows for the greatest number of houses in the area? By (10,1), the problem is reduced to the determination of the densest lattice packing of the parallel domain of the rectangle. This problem was solved completely by L. FEJES TÓTH\footnote{The English translation of the book “Lagerungen in der Ebene, auf der Kugel und im Raum” by László Fejes Tóth will be published by Springer in the book series Grundlehren der mathematischen Wissenschaften under the title “Lagerungen—Arrangements in the Plane, on the Sphere and in Space”. Besides detailed notes to the original text the English edition contains eight self-contained new chapters surveying topics related to the subject of the book but not contained in it. This is a preprint of one of the new chapters.} and FLORIAN\footnote{The English translation of the book “Lagerungen in der Ebene, auf der Kugel und im Raum” by László Fejes Tóth will be published by Springer in the book series Grundlehren der mathematischen Wissenschaften under the title “Lagerungen—Arrangements in the Plane, on the Sphere and in Space”. Besides detailed notes to the original text the English edition contains eight self-contained new chapters surveying topics related to the subject of the book but not contained in it. This is a preprint of one of the new chapters.}\footnote{The English translation of the book “Lagerungen in der Ebene, auf der Kugel und im Raum” by László Fejes Tóth will be published by Springer in the book series Grundlehren der mathematischen Wissenschaften under the title “Lagerungen—Arrangements in the Plane, on the Sphere and in Space”. Besides detailed notes to the original text the English edition contains eight self-contained new chapters surveying topics related to the subject of the book but not contained in it. This is a preprint of one of the new chapters.}.

Let $a$ denote the length of the shorter side of the rectangle and $d$ the prescribed minimum distance between the houses. There are three types of optimal arrangement according as $a/d \leq 4 - \sqrt{12}$, $4 - \sqrt{12} < a/d < 2 - \sqrt{2}$, or $2 - \sqrt{2} \leq a/d$. Let $H$ denote the minimum area hexagon circumscribed about the parallel domain of the rectangle. If $a/d \leq 4 - \sqrt{12}$ then $H$ has bilateral symmetry about a line parallel to the longer side of the rectangle (see Figure 1). If $2 - \sqrt{2} \leq a/d$ then, besides a pair of sides parallel to the longer sides of the rectangle, $H$ also has a pair of sides parallel to the shorter sides of the rectangle (Figure 2). If $4 - \sqrt{12} < a/d < 2 - \sqrt{2}$ then $H$ has neither bilateral symmetry nor sides parallel to the shorter sides of the rectangle (Figure 3).
Figure 4 shows a portion of an imaginary town. The black circles represent cylindrical houses and the white ones landing platforms for helicopters. To every black circle a white one is assigned, tangent to it. Naturally, no black circle is allowed to overlap with another black circle or with a white one. But the white circles can overlap with each other in part or completely, so that several houses can share a common landing platform. All white circles are mutually congruent and so are the black ones, and their radii are prescribed. Under these conditions, the densest packing of the black circles is to be determined. Molnár [1964, 1966b] and Jucovič [1970] obtained a series of very nice arrangements of circles as the solution of this problem and of its variations. The same problem on the sphere was studied by Molnár [1975].

Figure 4

1. Packing barrels

Let us imagine that in a large area we want to place as many equal-sized barrels as possible, and so that each barrel can be moved away from the area without disturbing the other barrels. Thus the densest blocking-free packing of the plane with congruent circles is desired, where a packing is blocking-free if within the part of the plane not covered by the circles every circle can be moved arbitrarily far from its original position. G. Fejes Tóth, who raised this problem, stated the conjecture that in the best packing the circles are arranged in double rows, separated by aisles (Figure 5). The density of this packing equals \( \frac{\sqrt{5} - 1}{2} \cdot \frac{\pi}{\sqrt{12}} \). Heppes [1967a] further generalized the problem, asking only that the barrels be \( r \)-accessible, meaning that a cellarer whose vertical shadow on the floor is a circle of radius \( r \) can freely come into contact with every barrel. He conjectured that the densest \( r \)-accessible packing of unit circles consists of appropriately placed double-rows. He gave a density bound which brought him very close to the solution of the problem. The correctness of the conjecture was proved in a series of papers by G. Blind [1972, 1976, 1977]. A simpler proof was given by G. Blind and R. Blind [1979]. G. Blind [1981] extended the result to the sphere. The analogous problem in space was considered by G. Blind and R. Blind [1978].

Figure 5
2. Covering with a margin

A covering of the plane with unit circles has margin $\mu \in [0, 1]$ if the removal of any one of the circles creates empty space small enough to be covered by a circle of radius $1 - \mu$. Obviously, a covering with a large margin cannot be too thin. Hence, it is natural to ask: Determine the minimum density of a covering with unit circles with margin $\mu$ and the covering that attains this density. A covering with margin 0 is a double covering, while margin 1 does not mean any additional restriction for the covering. Thus, this problem connects the problems of thinnest covering and thinnest double covering by unit circles.

A. Bezdek and W. Kuperberg [1997], who raised the problem about thinnest covering with a margin, solved it restricted to the special case when the original arrangement is lattice-like. For $0 \leq \mu \leq \mu_1 = 0.56408 \ldots$ the optimal arrangement is a triangular lattice, for $\mu_1 \leq \mu \leq \mu_2 = 0.78608 \ldots$ the solution is a square lattice, and for $\mu_2 \leq \mu \leq 1$ the optimal lattice remains unchanged; it coincides with the thinnest double lattice of unit circles. Since the thinnest double packing of unit circles is not lattice-like, the solution of the problem is not lattice-like in general. However, it is conjectured that the solution is lattice-like for sufficiently small values of $\mu$.

We note that for lattice arrangements the hole resulting by removing a circle is symmetric about the center of the removed circle, thus the smallest circle that can cover the hole will be centered here, as well. In this case the problem can be interpreted as searching for the most economical distribution of transmitting towers over a large area, all towers having the same circular range, under the requirement that the region should be covered even if due to a partial power loss the range of radius of one of the towers is reduced by a factor of $1 - \mu$.

Heppes [2002] considered a dual problem. A packing of unit circles has expendability $\varepsilon > 0$ if for every circle $C$ of the packing there is a circle of radius $1 + \varepsilon$ intersecting $C$ but not overlapping with any of the other circles. Thus, removing $C$ and replacing it with the larger circle still creates a packing. Heppes determined the densest lattice packing of unit circles with expendability $\varepsilon$ for all $\varepsilon > 0$.

3. Finite packing and covering in 2 dimensions

Nice, particular problems arise when trying to pack a given number of congruent circles of maximum radius in a specific region or to cover the region with congruent circles of minimum radius. Most often, the chosen container is the square, the circle, or the equilateral triangle. The extensive literature on this subject consists mainly of articles treating a single, specific case of the general problem, too numerous to be listed here. Among them are several algorithmic results that present some good arrangements, however not confirmed to be optimal. A major goal in this field is to find algorithms that give or approximate the optimal solutions. Thus far, the only algorithm of this type was constructed by Peikert, Würtz, Monagan and de Groot [1992] (see also Peikert [1994]) for finding the densest packing of congruent circles in a square. Figure 6 illustrates the optimal packing of up to 20 congruent circles in a square container.
Consider a densest packing with a large number of translates of a convex disk in a large container, or under some other constraints that force finiteness. While it is expected that such packing should be close to a cluster taken from the densest packing of the whole plane or space, it seldom is identical with such a cluster. Schürmann [2002a] proved that the solutions to several finite packing problems are non-lattice if the number of the translates is sufficiently large. In particular, he proved this for the packings of circles that minimizes the diameter of their union, thereby confirming a conjecture of Erdős.

For further literature on packings in bounded containers, see Szabó et al. [2007] and Melissen [1997].

4. Finite arrangements in higher dimensions

There are several results about packing congruent balls in various containers. Schaer [1966a, 1966b, 1966c, 1994] considered the problem of densest packing of $k$ congruent balls in a cube, and solved it for $k \leq 10$. The notoriously difficult case of 14 balls in a cube was settled by Joós in [2009a] by proving that if 14 points are placed in the unit cube, then two of the points are no more than $\sqrt{2}/2$ away from each other. The results of Schaer and Joós confirm some of the conjectures stated by Goldberg [1971].

Golser [1977] studied the problem of packing $k$ congruent balls in the regular octahedron and solved it for $k \leq 7$. Böröczky Jr. and Wintsche [2000] generalized Golser’s result to higher dimensions. They proved that the maximum radius of $k \leq 2n + 1$ congruent balls packed in the regular $n$-dimensional cross-polytope does not depend on $n$, and for $4 \leq k \leq 2n$ the radius is constant. K. Bezdek [1987a] solved the problem of packing $k$ congruent balls in a regular tetrahedron, for $k = 5, 8, 9$ and 10. W. Kuperberg [2007] considered the problem of maximum radius of $k \leq 2n + 2$ $n$-dimensional congruent balls packed in a spherical container. While for $k \leq n + 1$ and $k = 2n + 2$, the optimal configurations of balls are unique
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(see Davenport and Hajós [1951] and Rankin [1955]), W. Kuperberg describes
the structure of the non-unique configurations for \( n + 2 \leq k \leq 2n + 1 \), in which the
radius remains constant. An alternate characterization of these configurations was
given by Musin [2019].

The problem of thinnest covering of the cube with \( k \) congruent balls was solved
by G. Kuperberg and W. Kuperberg for \( k = 2, 3, 4 \) and 8 and by Joós [2014a, 2014b] for \( k = 5 \) and 6. Joós [2008b, 2009b, 2018] proved that the minimum radius
of 8 congruent balls that can cover the unit cube is \( \sqrt{5/12} \) in 4 dimensions and
\( \sqrt{2/3} \) in 5 dimensions. The problem of covering the \( n \)-dimensional cross-polytope
with \( k \) congruent balls of minimum radius was studied by Böröczky, Jr., Fábián
and Wintsche [2006] who found the solution for \( k = 2, n, 2n \). Remarkably,
the solution of the cases \( k = 2 \) and \( k = n \) is substantially different for \( n = 3 \) and
\( n \neq 3 \).

Another finite packing problem asks to arrange \( k \) non-overlapping unit balls so
that the convex hull of their union is of minimum volume. L. Fejes Tóth [1975a] conjectured that in dimension \( n \geq 5 \) the balls’ centers should be collinear, so that
the convex hull of the union of the balls forms a “sausage-like” solid of length \( 2k \).

Figure 7

The conjecture, known as the sausage conjecture, attracted great interest and
generated intensive research on finite packing and covering (see Gritzmann and
Wills [1993] and Böröczky Jr. [2004]). The conjecture was verified for \( n \geq 13, 387 \) by Betke, Henk and Wills [1994]. Betke and Henk [1998] lowered the
bound for the dimension to \( n \geq 42 \).

5. SLAB, CYLINDER, TORUS

What is the densest packing of unit circles in a strip of width \( 2 \geq w \)? The answer
is trivial for \( 2 \leq w \leq 2 + \sqrt{3} \) but it becomes difficult for \( w \geq 2 + \sqrt{3} \). Extending a
result by Kertész [1982], who gave the solution for \( 2 + \sqrt{3} \leq w \leq 2 + 2\sqrt{2} \), Füredi [1991] solved the problem for \( 2 + \sqrt{3} \leq w \leq 2 + 2\sqrt{3} \). Molnár conjectured that the maximum density is

\[
\frac{(n + 1)(n + 2)\pi}{2w(n + 1) + 2\sqrt{1 - (w - \sqrt{3})^2}}, \quad n = \left\lfloor \frac{w - 2}{\sqrt{3}} \right\rfloor,
\]

and observed that for \( w = 2 + n\sqrt{3} \), \( n = 1, 2, \ldots \) his conjecture follows from the
theorem of Groemer [1960a] mentioned on page XX.

Horváth and Molnár [1967] studied the problem of densest packing of unit
balls in a slab of space bounded by a pair of parallel planes. They showed, among
other things, that such packings consisting of two hexagonal or square layers of
balls are extremal in a slab of the corresponding width. Molnár [1978] extended
this result by finding the densest packing in a slab of every width between 2 and
Horváth [1974] investigated the problem of densest packing of unit balls in a 4-dimensional slab of width $2 < w \leq 2 + \sqrt{2}$.

Packing and covering with congruent circles on the surface of the infinite circular cylinder was considered by L. Fejes Tóth [1962], Bleicher and L. Fejes Tóth [1964] and Mughal and Weaire [2014]. L. Fejes Tóth [1973e] gave an upper bound for the number of points with given minimal distance on the surface of polyhedra. The papers by Dickinson, Guillot, Keaton and Xhumari [2011a, 2011b], Connelly and Dickinson [2014], Connelly, Shen, and Smith [2014], Connelly, Funkhouser, V. Kuperberg and Solomonides [2017], Heppes [1999], Musin, Nikitenko [2016], Przeworski [2006] and Brandt, Dickinson, Ellsworth, Kenkel and Smith [2019] investigate packings of circles on the torus. The dual problem of the thinnest covering of the torus by congruent circles was treated by Joós [2019]. Joós and Nagy [2008a] determined the smallest upper bound for the radius of $k \leq 4$ congruent balls in the 3-dimensional cubical flat torus.

6. Close packings and loose coverings

L. Fejes Tóth [1976] defined another measure of efficiency, alternate to density. He considered the supremum of the radii of the circles that can be placed in the complement of the packing. The smaller that number is, the more close, or efficient, is the packing. The closeness of the packing is measured by the inverse of this supremum, and a close packing is one with largest possible closeness. Looseness of a covering is similarly determined by the inverse of the supremum of the radii of circles that can be placed in the intersection of two members of the covering, and a loose covering is one with largest possible looseness. L. Fejes Tóth [1978a] proved that the closeness of a packing by translates of a convex disk $K$ cannot exceed the closeness of the closest lattice packing of $K$. He also remarked that this remains true if positively-homothetic copies of another convex disk instead of a circle are used to measure closeness. This is a result analogous to the corresponding theorem about density. On the other hand, he produced a centrally symmetric convex disk and a packing consisting of translates of the disk and a rotated copy of it, with closeness greater than that of the closest lattice packing. An alternative example was constructed by A. Bezdek [1980].

Linhart [1978] observed that the natural way of measuring closeness of a packing and looseness of a covering with translates of a convex disk $K$ is by using the largest negatively-homothetic copy of $K$ instead of a circle. Then the problems of close packing and loose covering become equivalent. He proved that, measuring closeness and looseness this way, the triangle is the worst for both close packing and loose covering, with closeness 2 and looseness 3. This result corresponds to the theorems of Fáry concerning the “worst case” for densest packing and thinnest covering with translates of a convex disk, where again the triangle is the worst one in both cases. Linhart also conjectured that the worst case among centrally symmetric convex disks is the affine-regular octagon. Specifically, he conjectured that every centrally symmetric convex disk can pack the plane by translates with closeness at least $3 + 2\sqrt{2}$, and can cover the plane by translates with looseness at least $4 + 2\sqrt{2}$.

Zong [2008] considered the problem of closest packing, with closeness measured the Linhart way, though phrased in a slightly different manner, and he confirmed
Linhart’s conjecture about the affine regular octagon. The relation between Linhart’s approach to closeness and Zong’s so-called simultaneous packing and covering constant is as follows: For a given convex disk $K$, let $c(K)$ denote the minimum closeness (in Linhart’s sense) of a packing with translates of $K$, and let $\gamma(K)$ denote Zong’s minimum homothety coefficient for a transition from a packing to a covering with translates of $K$. Then $\gamma(K) = 1 + c(K)^{-1}$.

Similarly to his simultaneous packing and covering constant Zong [2003] introduced the simultaneous lattice packing and covering constant $\gamma^*(K)$ and he proved that $\gamma^*(K) \leq 7/4$ for every three-dimensional convex body $K$.

Confirming a conjecture of L. Fejes Tóth [1976], Böröczky [1986] proved that the closest packing with unit balls in space is unique and is obtained by placing the centers of the balls at the vertices and at the centers of all cubes of a cubic lattice of edge-length $4/\sqrt{3}$. Since for balls the problems of closest packing and loosest covering are equivalent, Böröczky’s solution settles both. Böröczky [2001] defined edge-closeness of a packing of congruent balls as the supremum of the distance of a line of an edge of some Dirichlet cell and the corresponding center of ball. He showed that the minimum edge-closeness of a packing of unit balls is $\sqrt{3}/2$. Again, the unique optimal arrangement is the body-centered cubic lattice.

Using techniques of Delone and Ryškov [1963] and Ryškov and Baranovskii [1975, 1976], Horváth [1980, 1986] solved the problem of the closest lattice packing with unit balls in dimension 4 and 5. Schürmann and Vallentin [2006] designed an algorithm for approximating the loosest lattice sphere covering with arbitrary accuracy. In 6-dimensional space, the algorithm produced the best known lattice for loose sphere covering.

7. ARRANGING REGULAR TETRAHEDRA

Since no integer multiple of the dihedral angle $\varphi = \arccos(1/3) = 1.23\ldots$ at the edges of the regular tetrahedron $T$ equals $2\pi$ ($5\varphi = 6.15\ldots$ is just slightly smaller than $2\pi$), we know that space cannot be tiled with congruent copies of $T$, hence $\delta(T) < 1$. Then, as Hilbert [1900] asked, how densely can space be packed with congruent regular tetrahedra? The question is also of interest in areas other than mathematics, e.g., physics (compacting loose particles), chemistry (material design), etc. The past few years brought an exciting development: A series of articles appeared, each providing a surprisingly dense—denser than any previously known—packing. Conway and Torquato [2006] presented a packing with density $0.717455\ldots$, which is almost twice the lattice packing density of the tetrahedron. After improvements by Chen [2008], Haji-Akbari, Engel, Keys, Zheng, Petscher, Palffy-Muhoray and Glotzer [2009], Kallus, Elser and Gravel [2010], and Torquato and Jiao [2009c], a packing of the currently highest known density, namely $4000/4671 = 0.856347\ldots$ was obtained by Chen, Engel and Glotzer [2010].

While it was known for a long time that the value of $\delta(T)$ must be smaller than 1, no explicit non-trivial (i.e., strictly below 1) upper bound for the packing density of the regular tetrahedron was presented until Gravel, Elser and Kallus [2011] gave such a bound, about $1 - 10^{-24}$. They also gave a similar upper bound for the packing density of the regular octahedron. The gap between the density of the best known packing and the upper bound remains quite wide, and it may be very hard to narrow it down substantially. On the other hand, there is hope for the
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determination of the translational packing density of tetrahedra for which ZONG
\cite{Zong2019} suggested a computer approach.

The papers by LAGARIAS and ZONG \cite{LagariasZong2012} and ZIEGLER \cite{Ziegler2010} survey the history
of packing regular tetrahedra. The thinnest known covering by regular tetrahedra constructed by
CONWAY and TORQUATO \cite{ConwayTorquato2006} has density $9/8$. FIDUCCIA, FORCADE and ZITO \cite{ForcadeZito1998}
and independently DOUGHERTY and FABER \cite{DoughertyFaber2004} found a body $T_{84}$ that admits a lattice
tiling of $E^3$ and is inscribed in a tetrahedron $T$ of volume $\text{vol}(T) = \frac{125}{63}\text{vol}(T_{84})$. It follows that $\vartheta_L(T) \leq \frac{125}{63}$.

Forcade and Lamoreaux \cite{ForcadeLamoreaux2000} conjectured that $\vartheta_L(T) = \frac{125}{63}$ and support the conjecture by proving that the lattice corresponding
to the tiling by copies of $T_{84}$ represents a local minimum of the density. There is
no non-trivial lower bound for the covering density of the regular tetrahedron, but
recently XUE and ZONG \cite{XueZong2018} established such a bound for lattice arrangements:
The lattice covering density of a simplex in $E^n$ satisfies
$$\vartheta_L(S) \geq 1 + \frac{1}{2^{3n+7}}.$$ 8. Packing cylinders

The first known non-tiling 3-dimensional convex solid to have its packing density
determined was the infinite circular cylinder $C$, that is, the Minkowski sum $B + L$
where $B$ is a circle and $L$ is a line, see A. BEZDEK and W. KUPERBERG \cite{BezdekKuperberg1990}. As
expected, $\delta(C) = \pi/\sqrt{12}$, the maximum density being attained when all cylinders
are parallel.

Concerning packings of unit cylinders in which no two are parallel, A. BEZDEK
and W. KUPERBERG \cite{BezdekKuperberg1991a} showed that for every such packing, the complement
of the packing contains a ball of radius $r > \rho = \frac{2}{\sqrt{3}} - 1$. In other words, the
closeness (see XX) of such a packing is greater than $1/\rho$. They also proved that
every point $p$ lying in the complement of the packing is within a distance of at
most $\sqrt{2}/3$ from the center of such a ball. Moreover, every ball of radius $\rho$ not
intersecting any of the cylinders can be moved continuously from its original position
to the position of any other such ball, while avoiding every cylinder during the
motion. It appeared that if no two of the cylinders in a packing are parallel, then
the density of the packing should be rather low, perhaps even zero. However,
K. KUPERBERG \cite{Kuperberg1990} constructed such a packing with positive density. GRAF
and PAUKOWITSCHE \cite{GrafPaukowitsch1997} improved the construction, reaching density 5/12. By
a further improvement ISMAILESCU and LASKAWIEC \cite{IsmailescuLaskawiec2019} reached density 1/2.
Moreover, they constructed packings of congruent cylinders with no two cylinders
parallel to each other whose local density in a ball of sufficiently large radius is
arbitrarily close to $\pi/\sqrt{12}$.

9. Obstructing light

H. Hornich posed the question of how many material unit balls (meaning closed
balls with mutually disjoint interiors) are needed to radially shield one such ball,
in the sense that every ray emanating from the center of the shielded ball must
meet a shielding one. The set of shielding balls is called a cloud. Let $H(r)$ denote
the smallest number of unit balls in a cloud for a ball of radius $r$. As a simple
consequence of (V.1.2) (page XX114), L. Fejes Tóth [1959c] showed the inequality
\[ H(r) > \frac{12\alpha}{6\alpha - \pi}, \quad \frac{\pi}{6} < \alpha = \arctan \frac{1 + r}{\sqrt{6r + 3r^2}} < \frac{\pi}{2}, \]
which yields the lower bound \( H(1) \geq 19 \). The bound was subsequently improved to \( H(1) \geq 24 \) by Heppes [1967b] and then raised again by Csóka [1977] to \( H(1) \geq 30 \). By a suitable construction, Danzer [1960] showed that \( H(1) \leq 42 \).

L. Fejes Tóth [1959c] showed that a point, and therefore also a sufficiently small ball, can be shielded by six unit balls. Grünbaum [1960] proved that five balls do not suffice. This can be interpreted as the equality \( H(0) = 6 \). Besides \( r = 0 \), the value of \( H(r) \) is not known for any \( r \). However, the above inequality yields \( H(r) > 8\pi r^2/\sqrt{27} \). For large values of \( r \) the estimate is asymptotically exact.

The notion of a cloud for a ball can be extended in various ways. We can consider clouds for balls of infinite radius, that is half-spaces. A cloud for a half-space is a packing in the complement of the half-space such that every line perpendicular to the bounding plane intersects a member of the packing. Of course, a cloud for a half-space contains infinitely many balls, so in this case, we are looking for the width of the cloud, that is for the minimal width of a slab containing the cloud. A packing disjoint from a set \( S \) that intersects all rays issuing from the boundary of \( S \) in the complement of \( S \) is a dark cloud for \( S \). If each such ray intersects the interior of a member of the cloud then the cloud is called deep. Finally, if the corresponding rays intersect at least \( k \) members of the cloud, then we have a \( k \)-fold cloud.

Consider \( k \) consecutive rows of the densest lattice packing of unit circles. These circles are contained in a strip of width \((k - 1)\sqrt{3} - 2\) and form a \( k \)-fold cloud for the half-plane parallel to the rows. Heppes [1961a] proved that this is the narrowest \( k \)-fold cloud of unit circles shielding a half-plane. An alternative proof of this statement is due to Hajós [1964]. Figure 8 shows the narrowest \( 4 \)-fold cloud of circles.

It is not difficult to show (L. Fejes Tóth [1959c]) that the width of any cloud of unit balls shielding a half-space in \( E^3 \) must be at least \( 2 + \sqrt{2} \). Equality holds only when the cloud consists of two square-pattern layers in contact with each other, so that each ball in one layer touches exactly four balls in the other layer. If we stack \( k \) horizontal clouds of this kind, we obtain a \( k \)-fold cloud of width \((2k - 1)\sqrt{2} + 2\). Heppes [1961a] improved this for every \( k > 1 \) by constructing a \( k \)-fold cloud of unit balls of width \((k + \left\lfloor \frac{k - 1}{3} \right\rfloor)\sqrt{3} + 2\). Jucović [1966] gave bounds for the number of circles in a \( k \)-fold cloud for a point.

For a convex body \( K \), let \( C(K) \) and \( C_T(K) \) denote the minimum cardinality of dark clouds for \( K \) consisting of congruent copies of \( K \) and of translates of \( K \), respectively.

The papers by Böröczky and Soltan [1998], Zong [1999b, 1999a, 1997b], Talata [2000a] contain bounds for \( C_T(K) \) for convex bodies \( K \) in \( E^n \). Zong [1997b] proved that \( C_T(K) \geq 2n \) with equality only for parallelepipeds. Finding upper bounds for
$C_T(K)$ is related to the following problem: Given $\varepsilon > 0$, find the minimum number $s(n, \varepsilon)$ such that for some packing of unit balls in $n$ dimensions, every line segment of length greater than $s(n, \varepsilon)$ must be closer than $\varepsilon$ to one of the centers of the balls. The bounds for $s(n, \varepsilon)$ given by HENK and ZONG [2000] were improved by Böröczky, Jr. and TÁRDS [2002], whose result implies that

$$C_T(K) \leq 3^{n^2+o(n^2)}$$

for every convex body $K \in E^n$ and

$$C_T(K) \leq 2^{n^2+o(n^2)}$$

for every centrally symmetric convex body $K \in E^n$.

Szabó and Ujváry-Menyhárt [2002] proved that the minimum cardinality of a deep cloud for a convex disk is at most 9 with equality only for the circle.

Heppes [1960a] observed that a lattice packing of balls is always penetrable by lines in three linearly independent directions, implying that no lattice packing of balls can form a dark cloud for a half-space. This shows that the existence of a dark cloud of congruent balls is not so trivial. Nevertheless, Böröczky [1967] succeeded in constructing such a cloud of a relatively small width. Take four consecutive hexagonal-pattern layers of balls, $s_0, s_1, s_2, s_3$ from the densest lattice packing of balls. Böröczky proved that these layers, adjoined by the mirror images $s_{-1}, s_{-2}, s_{-3}$ of $s_1, s_2, s_3$ reflected in the middle plane of $s_0$, form a dark cloud, and he asserted without proof that the same is true for $s_{-2}, s_{-1}, s_0, s_1, s_2$ already.

The aforementioned theorem of Heppes was strengthened by Hortobágyi [1971], who proved that every lattice packing of unit balls can be penetrated in three linearly independent directions by a cylindrical beam of light of radius $\frac{3\sqrt{2}}{4} - 1 = 0.06065\ldots$. As the example of the densest lattice packing of balls shows, this constant cannot be replaced with a larger one. This last statement is quite non-trivial, but it follows from Theorem II of Bambah and Woods [1994], see also Makai and Martini [2016], where a gap in the proof of this theorem was filled.

The observation of Heppes inspired further research. Hausel [1992], Henk and Zong [2000], Henk, Ziegler and Zong [2002] and Henk [2005] estimated the greatest number $k(n)$ with the property that for every lattice packing of the $n$-dimensional ball there exists a “free $k$-dimensional plane”, that is, a $k$-dimensional plane contained in the complement of the packing. Hausel [1992] proved that $k(n) \leq n - c\sqrt{n}$, for some constant $c > 0$. The best lower bound up to date, given by Henk [2005], is $n/\log_2 n \leq k(n)$, for $n$ sufficiently large. Horváth and Ryškov [1975a, 1975b] estimated the maximum radius of a cylinder around a line that can be inserted in the void of every lattice packing of the $n$-dimensional unit ball, and conjectured that their result is sharp for $n = 4$. Their conjecture was refuted by Makai and Martini [2016].

A related problem, posed by G. Fejes Tóth [1976b], concerns the thinnest lattice arrangement of balls that intersects every $k$-dimensional plane. For $k = 0$ the problem is about the thinnest lattice covering, hence it is solved for $n \leq 5$. For $n = 2$, $k = 1$ the problem was solved by L. Fejes Tóth and Makai [1974] by proving that the density of a lattice packing of circles intersecting every line is at least $\sqrt{3}\pi/8$. For $k = n - 1$ the problem turned out to be equivalent to finding the densest lattice packing of balls. This is a consequence of the following result of Makai [1978] also found independently Kannan and Lovász [1988]: Let $\rho(K)$
denote the infimum of the density of a lattice arrangement of a convex body $K$, such that every hyperplane intersects one of the members of the arrangement, and let $\tilde{K}$ denote the polar body of $\frac{1}{2}(K - K)$. Then

$$\rho(K)\delta_L(\tilde{K}) = \text{vol}(K)\text{vol}\left(\frac{1}{4}\tilde{K}\right).$$

This solves the problem for balls in dimensions $n \leq 8$ and $n = 24$. For $0 < k < n - 1$ only the case $k = 1$, $n = 3$ has been solved: Bambah and Woods [1994] showed that the thinnest lattice arrangement of balls intersecting every line arises from the densest lattice packing by enlarging the balls’ radius by a factor of $3\sqrt{2}/\pi$. Kannan and Lovász [1988] and González Merino and Schymura [2017] investigated the case $0 < k < n - 1$ further.

10. Avoiding obstacles

If convex disks are packed in a parallel strip of the plane, we call it a layer of disks. Let $w$ be the width of the strip and $l$ be the length of a path that connects the two edges of the strip without penetrating any of the disks. (L. Fejes Tóth [1966b]) defined the permeability $p$ of the layer as

$$p = w / \inf l.$$

He showed that for the permeability $p$ of a layer of congruent circles $p > \sqrt{27}/2\pi = 0.82699\ldots$ holds. For layers consisting of a large number of rows from the densest lattice packing of circles, $p$ comes arbitrarily close to this lower bound. For congruent circles the value of $\inf p$ is not known. However, by a rather complicated construction in the same article a layer of incongruent circles with $p = 0.82322\ldots < 2\pi/\sqrt{27}$ was found.

The situation is quite different if one considers layers of squares instead of circles. For layers consisting of congruent squares, $\inf p = 2/3$ holds. But the same holds even for layers of squares of arbitrary sizes and orientations. Moreover, it was shown by L. Fejes Tóth [1968a] that to any layer of similar copies of a parallelogram $P$ there is a layer consisting of translates of a replica of $P$ with the same permeability. It is an interesting question to consider which convex disks share this property with the parallelogram and which behave like the circle. Florian and Groemer [1985] showed that for every $m \geq 39$ regular $m$-gons belong to the latter group, that is there exists a layer of homothetic copies of them whose permeability is smaller than the infimum of the permeability of all layers of congruent regular $m$-gons.

The above mentioned results of L. Fejes Tóth were sharpened by Bollobás [1968a] and Florian [1978b]. Bollobás determining the infimum of the permeability of a layer of given width of similar copies of a given parallelogram and Florian determining the infimum of the permeability of a layer of given width of unit circles. Hortobágyi [1976a] proved that the permeability of a layer of translates of a disk of constant width is at least $\sqrt{27}/2\pi$. Florian [1979b] proved that the permeability of a layer of translates of a regular hexagon, and also of a layer of translates of a regular triangle is at least $3/4$. Subsequently, Florian [1980a] observed that these results are special cases of a general theorem. Namely, the infimum of the permeability of layers by translates of a convex disk $K$ equals the infimum of the permeability of layers by translates of the difference body $K - K$. A survey on permeability can be found in Florian [1980b].
L. Fejes Tóth [1978b] also raised the following, natural problem. Given a packing of the plane with convex “obstacles” and two points not in the interior of any of the obstacles, find or estimate the greatest possible size of a necessary detour caused by the obstacles in traveling from one point to the other. The first result in this direction was given by Pach [1977], who proved that for any packing of square obstacles of sides at most 1, any pair of points at distance $d$ outside the squares can be connected by an obstacle-avoiding path of length at most $3\frac{d}{2} + \sqrt{d} + 1$.

G. Fejes Tóth [1978] improved this bound to $(3\frac{d}{2} + 1)$ and proved the bound $(2\pi/\sqrt{27})(d - 2) + \pi$ for unit circular obstacles. Both of these bounds are sharp for infinitely many values of $d$.

The shortest path problem for balls was treated in G. Fejes Tóth [2013]. It turned out that in $E^n$, even for a packing of balls with arbitrary but bounded radii, where the obstacles’ density might be 1, we need not make a detour greater than $O(d/n)$ ($d \to \infty$) in order to connect two points lying at distance $d$ outside the balls by a path avoiding the balls. For a packing of congruent balls in $E^n$ the detour we have to make approaches zero exponentially with the dimension.

Algorithmic aspects of problems of this type have been studied by Papadimitriou and Yannakakis [1989, 1991], Chan and Lam [1993], and A. Bezdek [1999]. A. Bezdek also obtained a bound for the length of the shortest path in space with cubical obstacles.

An interesting variation of the obstacle-avoiding path problem was considered by L. Fejes Tóth [1993]: From a point outside the obstacles, one tries to escape in any direction to a distance $d$ away from the point. He proved that for any set of convex, open obstacles an escape path exists of length at most $d^2 + \frac{\pi}{2} \ln d + c$, and that some sets of obstacles require the length of $d^2 - (\frac{\pi}{2} + \frac{\pi}{3}) d + c$.

A dual problem arises when we consider a family of convex disks covering the plane and we wish to travel only within the part of the plane covered at least twice. G. Fejes Tóth [2013] stated the following conjecture. If the plane is covered by a family of unit circles, then for any two points, each covered at least twice, there is a path contained in the multiply covered region, connecting one point with the other, and of length at most $d\sqrt{2} + c$, where $d$ is the distance between the points and $c$ is a constant. Baggett and A. Bezdek [2003] confirmed the conjecture in the case when the circles form a lattice covering. Roldán-Pensado [2013] showed that two points at distance $d$ apart lying in the multiply covered part of the plane can be connected by a path that remains in the part of the plane covered at least twice and whose length is at most $(\pi/3 + \sqrt{3}) d + c$ for some constant $c < 17$. A. Bezdek and Yuan [2020] investigated a related problem, where they measured the length of the path by the number of passings from one circle into another.

### 11. Stability

We say that a packing of (not necessarily congruent) circles is stable if every circle is immobilized by the others, i.e., if in every circle the central angle $\lambda$ based on the largest “free” arc, that is containing no contact point, is smaller than $\pi$. The instability of the packing is defined as $\Lambda = \sup \lambda$, and its stability as $\pi - \Lambda$. We wish to find a thinnest packing among those with prescribed stability. The following theorem of L. Fejes Tóth [1960a] is related to this problem:

Given a stable packing of the plane with circles whose radii have a positive lower bound and a finite upper bound. If $d$ is the density and $\Lambda$ is the instability of the
packing, then
\[ d \geq \frac{\pi}{n \tan \frac{\Lambda}{2} + \tan \frac{2\pi - n\Lambda}{2}}, \quad n = \lfloor \frac{2\pi}{\Lambda} \rfloor. \]

This bound is reached in the case of congruent circles whose centers form the vertices of one of the tilings \( \{3, 6\} \), \( \{4, 4\} \), \( \{6, 3\} \), \( \{4, 8, 8\} \), or \( \{3, 12, 12\} \). The corresponding instabilities are \( \Lambda = \pi/3, 2\pi/3, 3\pi/4 \) and \( 5\pi/6 \). Surprisingly, Böröczky [1964] succeeded in constructing stable packings of density \( d = 0 \) consisting of congruent circles (Figure 9). His example shows that the above inequality is sharp even in the limiting case \( \Lambda = \pi \). Kahle [2012] modified Böröczky’s construction to obtain thin stable packings of congruent circles in rectangles and regular hexagons.

Dominyák [1964] gave bounds for the instability of stable circle packings in the spherical and hyperbolic plane. His bounds are sharp, in several cases characterizing regular and Archimedean tilings.

De Bruijn [1954, 1955] proved that the members of a packing of starlike sets in \( E^n \) can be moved apart arbitrarily far from each other, each translated continuously without overlapping with any of the others (Figure 10), unless each of the sets’ star-center is unique, and they all coincide in the packing (Figure 11). He also proved that in a finite packing of convex disks, for every direction, there is a disk that can be translated in that direction arbitrarily far from its original position without intersecting other members of the packing. L. Fejes Tóth and Heppes [1963] independently discovered the same results, and the result about starlike sets was also noticed by Dawson [1984]. In contrast to the case of the plane, in space there are finite packings of convex bodies such that no single member can move rigidly without disturbing the others. The example given by L. Fejes Tóth and Heppes consists of 12 tetrahedra packed around a rhombic dodecahedron. They conjectured that the tetrahedra alone, without the central dodecahedron, have the stated property. This was confirmed by Snoeyink and Stolfi [1994]. Shephard [1970] constructed a packing of twelve centrally symmetric polyhedra with the same property.
By the theorem of de Bruijn, the members of every finite packing of convex bodies can be moved arbitrarily far without disturbing the others by simultaneous translations. Natarajan [1988] conjectured that the members of such a packing can even be separated by translations with two hands that is, a proper subset of the packing exists that can be translated to infinity by applying a common translation to them without disturbing the members in the complement. This conjecture turned out to be false: Snoeyink and Stolfi [1994] gave a counterexample of 6 bodies and showed that a packing of at most 5 convex bodies can indeed be separated with two hands.

12. Minkowskian arrangements

A Minkowskian arrangement of similar copies of a centrally symmetric convex disk was defined by L. Fejes Tóth [1965b] as an arrangement in which no member contains the center of another one. He proved that the density of such an arrangement of circles cannot exceed $2\pi/\sqrt{3}$. A densest Minkowskian arrangement of circles consists of congruent circles, and is obtained by replacing every circle in a densest packing of congruent circles by a concentric one, with twice as large radius. This result is an extensive generalization of inequality (III.2,1). In fact, L. Fejes Tóth [1965b] proved that if finitely many circles form a Minkowskian arrangement then the density of the circles in their union cannot exceed $2\pi/\sqrt{3}$. In a subsequent paper [1967a] he gave an upper bound for the total area of such an arrangement, which is sharp in many cases. Molnár [1966a] extended that result to the sphere and to the hyperbolic plane.

The problem of the densest Minkowskian arrangement of circles was generalized by L. Fejes Tóth [1967a] as follows. Let $\mu$ be a positive number smaller than
1. We consider a set of circles $c_1, c_2, \ldots$ of radii $r_1, r_2, \ldots$. With each circle $c_i$ we associate a concentric circle with radius $\mu r_i$, and we call it the kernel of $c_i$. In a generalized Minkowskian arrangement of circles of order $\mu$ none of the circles is allowed to overlap the kernel of another. L. Fejes Tóth [1967a] conjectured that for $\mu \leq \bar{\mu} = \sqrt{3} - 1$ the densest arrangement consists of congruent circles, and each of them touches six kernels (Figure 12). In this conjecture $\bar{\mu}$ denotes the greatest value of $\mu$ under which this particular arrangement is a covering. Molnár [1967b] and Florian [1967] gave density bounds under some condition for the homogeneity of the arrangement. Böröczky and Szabó [2002] proved the conjecture in full generality. Kadlicskó and Lángi [2022] gave a sharp bound for the total area of the circles in a finite generalized Minkowskian arrangement of circles of order $\mu \leq \sqrt{3} - 1$.

Minkowskian arrangements of circles on the sphere were treated by L. Fejes Tóth [1999] who gave an upper bound for the total area of $n$ spherical caps in a Minkowskian arrangement, sharp for $n = 3, 4, 6$ and 12. The centers of the circles in the optimal arrangement form an equilateral triangle inscribed in a great circle, a regular tetrahedron, a regular octahedron and a regular icosahedron inscribed in the sphere, respectively. Also, the bound is asymptotically sharp for large $n$.

The density of a Minkowskian arrangement of homothetic copies of a centrally symmetric convex body in $\mathbb{E}^n$ is at most $2^n$. On the other hand, allowing similar copies, there is no universal upper bound for the density. This was noticed already by L. Fejes Tóth [1965b], who also observed that in order to achieve high density, the members of the arrangement must occur in many different orientations: A Minkowskian arrangement of similar bodies in $\mathbb{E}^n$ with at most $m$ distinct orientations can have density at most $m2^n$. Bleicher and Osborn [1967] showed that there are Minkowskian arrangements in $\mathbb{E}^n$ even of congruent copies in at most $m$ distinct orientations with densities arbitrarily close to $m2^n$.

What is the maximum number $M(n)$ of pairwise intersecting homothetic copies of a centrally symmetric convex body in $\mathbb{E}^n$? The example of the cube shows that $3^n \leq M(n)$. Füredi and Loeb [1994], who first considered this problem, proved that $M(n) \leq 5^n$. This upper bound was subsequently improved by Naszódi, Pach and Swanepoel [2017] to $O(3^n \ln n)$, by Polyanskii [2017] to $3^n + 1$, by Naszódi and Swanepoel [2018] to $2 \cdot 3^n$, and finally by Földvári [2020], who proved the sharp bound $3^n$ with equality only for parallelotopes.

13. Saturated arrangements

Minkowskian arrangements of circles are in a certain sense dual to saturated collections of circles. Let $S$ be a collection of closed circular disks and let $r > 0$ be the infimum of their radii. We say that $S$ is saturated if the part of the plane not covered by the circles contains no circle of radius $r$. It was conjectured by L. Fejes Tóth [1967a] that the density of a saturated collection of circles is always greater than or equal to $\pi / \sqrt{108}$. The conjecture was confirmed by Eggleston [1965] for families of mutually non-overlapping circles, and proved in general by Bambah and Woods [1968a]. Thus, a thinnest saturated arrangement of circles arises by replacing the circles in a thinnest covering of the plane by concentric circles of half size. This is a generalization of the inequalities (III,2,2) and (III,2,5). A corresponding theorem for a packing of homothetic copies of a centrally symmetric convex disk in
place of circles was proved by Bambah and Woods \[1968a\]. Dumir and Khassa \[1973a\] strengthened the above result for circles, and in \[1973b\] for arbitrary centrally symmetric convex disks as follows: No saturated arrangement of homothetic copies of a centrally symmetric convex disk \(K\) can cover a smaller portion of the plane than \(\vartheta(K)/4\). For the density of a saturated arrangement of homothetic copies of a (not necessary symmetric) convex disk \(K\) \[1975a\] proved the upper bound \(\text{area}(K)/t(K)\), where \(t(K)\) denotes the area of the largest triangle contained in \(K\). For the density of saturated packings of balls in 3 dimensions Khassa \[1975b\] established the upper bound \(3/32\).

Recall from Chapter 10 that a packing with congruent copies of a set \(K\) is \(k\)-saturated, if deleting \(k-1\) members of the packing never creates a void large enough to pack in \(k\) copies of \(K\). It is natural to ask for the infimum \(\Delta_k(K)\) of the densities of all \(k\)-saturated packings with replicas of \(K\). G. Fejes Tóth, G. Kuperberg and W. Kuperberg \[1998\] proved the asymptotic bound \(\Delta_k(K) \geq \delta(K) - O(k^{-1/n})\) for every body \(K\) in \(\mathbb{E}^n\). The determination of these quantities is difficult even for \(K = B^2\). The only result in this direction is due to Heppes \[2001b\], who determined the infimum of the densities of 2-saturated lattice packings of circular disks, supporting the conjecture that \(\Delta_2(B^2) = \pi(3 - \sqrt{5})/\sqrt{27} = 0.461873\ldots\)

Another notion of higher order saturation was studied by L. Fejes Tóth and Heppes \[1980\] and A. Bezdek \[1990\]. Here, we formulate the concept only for the special case of packings of congruent circular disks. A packing of disks of radius \(r\) is \(saturated\ of\ order\ \(k\) if every disk of radius \(r\) intersects at least \(k\) members of it. Saturation of order 1 means just saturation, and it is easily seen that the order of saturation of a packing of congruent circles is at most 3. L. Fejes Tóth and Heppes proved that the density of an order 3 saturated packing of congruent circles is at least \(\pi/(2 - \sqrt{3})\), and A. Bezdek proved that the density of an order 2 saturated packing of congruent circles is at least \(\pi/(\sqrt{27})\). The thinnest order 3 saturated packing arises by placing the centers in the vertices of a tiling \((3,3,4,3,4)\), and a thinnest order 2 saturated packing consists of the face-incircles of the tiling \(\{3,6\}\).

14. Compact packings

A packing of the plane is said to be \(compact\ if each member \(K\) of the packing satisfies the following three conditions:

(1) \(K\) has a finite number of neighbors,
(2) all neighbors of \(K\) can be ordered cyclically so that each of them touches its successor,
(3) the union of the neighbors of \(K\) contains a polygon enclosing \(K\).

L. Fejes Tóth \[1984b\] originated the study of compact packings by proving that if a compact packing of the plane with circular disks has positive homogeneity then its density is at least \(\pi/\sqrt{12}\). Further, if a compact packing of the plane with homothetic centrally symmetric convex disks has positive homogeneity, then its density is at least \(3/4\), where equality occurs only for packings with affine regular hexagons. A. Bezdek, K. Bezdek and Böröczky \[1986\] proved that if a compact packing of the plane with positively homothetic copies of a convex disk has positive homogeneity, then its density is at least \(1/2\), and equality occurs for various packings with homothetic triangles.

The only compact packing of congruent circular disks is the hexagonal lattice. Kennedy \[2006\] considered compact packings of circular disks of two different
radii, 1 and \( r < 1 \), and proved that there are only nine values of \( r \) for which such compact packings exist. He also described all packing configurations in the nine cases. \textsc{Messerschmidt [2020]} proved the upper bound 13617 for the number of pairs \( (r, s) \) that allow a compact packing by disks of radii 1, \( r \) and \( s \) \( (r < s < 1) \). In fact, there are much fewer such pairs: \textsc{Fernique, Hashemi and Sizova [2020]} enumerated all 164 compact packings consisting of three different sizes of circular disks. \textsc{Messerschmidt [2021]} proved that for every \( n \) there exist only finitely many tuples \( (r_1, \ldots, r_n) \) with \( 0 < r_1 < \ldots < r_n = 1 \) that can occur as the radii of the disks in any compact packing of the plane with \( n \) distinct sizes of disk.

It can be expected that compact packings of circles are the densest among all packings with the given radii. \textsc{Bédaride and Fernique [2020]} conjectured that if disks with \( n \) different radii allow a saturated compact packing in which at least one disk of each size appears, then the maximal density over all the packings by disks with these \( n \) radii is reached for a compact packing. The hypothesis of saturation is necessary for \( n \geq 3 \). The conjecture was proved for some cases, including all of the nine pairs of radii allowing compact packings of disks with two different sizes, by \textsc{Heppes [2000, 2003b], Kennedy [2004], Bédaride and Fernique [2020]} and \textsc{Fernique [2021a, 2021b]}, however \textsc{Fernique and Pchelina [2021]} showed that the density of one of the compact circle packings with three different radii is not maximal.

\textsc{Florian [1985]} considered compact packings with circular disks on the sphere and in the hyperbolic plane.

Compact packings in higher dimensions were investigated by \textsc{K. Bezdek [1987b]} and \textsc{K. Bezdek and Connelly [1991]}. A packing in \( E^n \) is \textit{compact} if each member \( A \) of the packing is enclosed by its neighbors in the sense that any curve, connecting a point of \( A \) with a point sufficiently far from \( A \), intersects the closure of a neighbor of \( A \). \textsc{K. Bezdek and Connelly [1991]} proved that the density of a compact packing in \( E^n \) consisting of homothetic centrally symmetric convex bodies with bounded homogeneity is at least \( (n+1)/2n \), and there is a compact lattice packing of centrally symmetric convex bodies where equality holds.

\textsc{Fernique [2021]} proposed a different generalization of the concept of compact packings. The contact graph of a compact packing of circles, i.e., the graph that connects the centers of adjacent circles, is a triangulation. By analogy, Fernique calls a packing of balls in \( E^n \) compact if its contact graph is the 1-skeleton of a face-to-face tiling by simplices. Since regular tetrahedra do not tile the space, there is no compact packing of congruent balls. \textsc{Fernique [2021]} determined the unique compact packing with two different sizes of balls and in \textsc{Fernique and Pchelina [2021]} he described all the four compact packings with balls of three different sizes.

15. \textbf{Totally separable packings}

A packing of convex bodies is \textit{totally separable} if each pair of the bodies can be separated by a hyperplane not intersecting the interior of any of the bodies. Given a convex disk, what is the maximum density of a totally separable packing with its congruent copies? \textsc{G. Fejes Tóth and L. Fejes Tóth [1973]} proved that the density of an arbitrary totally separable packing with congruent copies of a convex disk cannot exceed the ratio between the area of the disk and the minimum area of a quadrilateral containing the disk. If the disk is centrally symmetric, then that ratio is actually the maximum density of such a packing. Namely, the minimum area
of the quadrilateral containing a centrally symmetric disk can always be attained by a parallelogram, and congruent parallelograms admit a totally separable tiling. In particular, this yields that the density of a completely separable packing with congruent circles cannot exceed $\frac{\pi}{4}$. A. Bezdek [1983] proved this density bound under an assumption weaker than total separability, requiring only the packing to satisfy the following local separability condition: For every triple of circles there is a line separating one of them from the other two. He also showed that, in general, for non-circular disks local separability does not imply the similar parallelogram bound. K. Bezdek and Lángi [2020b] proved an analogue of Oler’s inequality for totally separable packings of translates of a convex disk. Vermes [1996] investigated totally separable tilings and totally separable packings of circles in the hyperbolic plane.

![Figure 13](image)

The interesting problem of maximum density of a totally separable packing of (not necessarily congruent) circular disks remains open. One can rephrase this problem in the following way: Start with a configuration of lines partitioning the plane into bounded regions and place a circle in each of the regions. What is the maximum density of a circle packing so obtained? In this phrasing, the analogous question about the minimum density of the covering by the circumcircles of the resulting cells can be asked. It is conjectured that the configuration of lines providing each of these extreme densities partitions the plane into the Archimedean tiling $(3, 6, 3, 6)$ (see Figure 13). G. Fejes Tóth [1987] proved that neither of the extreme densities in question are equal to 1.

K. Bezdek and Lángi [2022] and Vásárhelyi [2003] investigate totally separable packings and coverings of circles on the sphere.

The problem of the densest totally separable packing of $E^3$ with congruent balls was solved by Kertész [1988], who proved that if a cube of volume $V$ contains $N$ unit balls forming a totally separable packing, then $V \geq 8N$. Consequently, the cubic lattice packing is the densest one among all totally separable packings of $E^3$ with congruent balls, and the maximum density is $\pi/6$.

16. **Point-trapping lattices**

An arrangement of sets is point-trapping if every component of the complement of the union of the sets is bounded. It is natural to ask: What is the minimum density
of a point-trapping lattice arrangement of any \( n \)-dimensional convex body? Confirming the chessboard conjecture of L. Fejes Tóth [1975c], Böröczky, Bárány, Makai and Pach [1986] proved that the minimum is \( 1/2 \), attained in the “chessboard” lattice arrangement of cubes.

The problem about the minimum density of a point-trapping lattice of \( K \) can be posed for any specific convex body \( K \). Bleicher [1975] proved that the minimum density of a point-trapping lattice arrangement of three-dimensional unit balls is equal to \( \frac{128}{3} \pi \frac{1}{\sqrt{7142 + 180^2}} = 1.1104 \ldots \), and the extreme lattice is generated by three vectors, each of length \( \frac{1}{2} \sqrt{7 + \sqrt{17}} = 1.6676 \ldots \), and each two forming an angle of \( \arccos \frac{\sqrt{8}}{8} = 67.021^\circ \).

17. Connected arrangements

An arrangement of sets is said to be connected if the union of the sets is connected. The problem of the minimum density \( c(K) \) of a connected lattice arrangement of an \( n \)-dimensional convex body \( K \) has been explored by Groemer [1966b], who proved the inequalities

\[
\frac{1}{n!} \leq c(K) \leq \frac{\pi^{n/2}}{2^n \Gamma(\frac{n}{2} + 1)}.
\]

Both inequalities are sharp. The value \( c(K) = 1/n! \) is attained when \( K \) is a simplex or a cross-polytope, and the other extreme value of \( c(K) \) is attained when \( K \) is a ball. Groemer characterized those centrally symmetric bodies \( K \) for which \( c(K) = 1/n! \). Extending Groemer’s investigation, L. Fejes Tóth [1973d] characterized all \( n \)-dimensional convex bodies \( K \) for which the inequality \( c(K) \geq 1/n! \) turns into equality: They are the topological isomorphs of the regular cross-polytope and their limiting polytopes.

18. Points on the sphere

In Chapter XX we mentioned results about problems of finding the optima of sums of the form \( \sum_{i \neq j} f(|x_i - x_j|) \) for a given number \( N \) of points \( \{x_1, \ldots, x_N\} \) on the unit sphere. We continue to survey problems of this type. We start with two problems of this kind, of interest from the geometric point of view, and remarkably easy to solve for every value of \( N \geq 2 \).

Distribute points \( x_1, \ldots, x_N \) on \( S^2 \) so that:

1. the sum \( \sum |x_i - x_j|^2 \) is as large as possible;
2. the sum \( \sum \tilde{x}_i \tilde{x}_j \) is as large as possible, where \( \tilde{x}_i \tilde{x}_j = 2 \arcsin \left( \frac{1}{2} |x_i - x_j| \right) \) is the spherical distance between \( x_i \) and \( x_j \).

Concerning the first problem we have \( \sum |x_i - x_j|^2 \leq N^2 \) and equality occurs only if the vectors from the sphere’s center to the points are in equilibrium (see L. Fejes Tóth [1956b]).

For the solution of the second problem, the parity of \( N \) plays a role. If \( N = 2k \), then \( \sum \tilde{x}_i \tilde{x}_j \leq \pi k^2 \), and equality occurs precisely when the configuration of points is symmetric about the sphere’s center. For \( N = 2k + 1 \) we have \( \sum \tilde{x}_i \tilde{x}_j \leq \pi k(k + 1) \). Equality occurs here precisely when all points without an antipodal partner are distributed on a great circle in such a way that two open semicircles determined by any point contain the same number of points. The case \( N = 4 \) was solved by Frostman [1953], the cases \( N = 5 \) and \( 6 \) by L. Fejes Tóth [1959d], the cases
$N = 2k$ ($k = 1, 2, \ldots$) by Sperling [1960], and the general case by Nielsen [1965] (see also Larcher [1962]).

L. Fejes Tóth [1959d] also considered the corresponding problem in the elliptic plane, that is maximizing the sum of the non-obtuse angles formed by $N$ lines. He conjectured that the maximum of the angle sum is attained when the lines are evenly distributed between the three coordinate axis in which case the sum of the angles is asymptotically $\frac{N^2 \pi}{6} = N^2 \cdot 0.523 \ldots$. He solved the problem for $N \leq 6$ and proved the upper bound $\frac{k(N-1)}{5} \pi$ on the sum of the angles of $N$ lines.

Fodor, Vígh and Zarnócz [2016a] gave an improvement of this bound that is asymptotically equal to $\frac{3N^2}{16} \pi = N^2 \cdot 0.589 \ldots$ as $N \to \infty$. Bilyk and Matzke [2019] further improved the bound to $\left(\frac{\pi}{4} - \frac{69}{300}\right)N^2 = N^2 \cdot 0.555 \ldots$ and also gave a bound for the corresponding problem in higher dimensions.

Alexander [1972] considered another problem of this kind: Find the configurations of $N$ points on the $n$-dimensional unit sphere for which the sum of all distances between the points attains its maximum $S(N,n)$. Through an elegant integral-averaging technique he obtained the bounds $\frac{2}{3}N^2 - 10\sqrt{N} \leq S(N,2) \leq \frac{2}{3}N^2 - \frac{1}{2}$. Stolarsky [1972] extended Alexander’s result for powers of the distances between the points and generalized it to all dimensions. In 1973 Stolarsky proved a remarkable invariance theorem stating that the sum of all the distances between the points of a given set of $N$ points on the $n$-dimensional sphere plus the discrepancy of the set is a constant independent from the distribution of the points. Using this result, Stolarsky gave a sharper bound for $S(N,n)$. If $c_0(n)$ denotes the average distance from a variable point to a fixed one on the surface of the sphere, then there is a constant $c_1(n)$ such that $S(N,n) < c_0(n)N^2 - c_1(n)N^{1-1/n}$.

Beck [1984] showed exactness of the constant $c_0$ by proving that $c_0(n)N^2 - c_2(n)N^{1-1/n} < S(N,n)$ with a suitable constant $c_2(n)$. Alternative proofs of Stolarsky’s invariance theorem were given by Brauchart and Dick [2013] and Bilyk, Dai and Matzke [2018]. The latter authors proved various generalizations of the invariance principle, and applied them to problems of energy optimization.

In a similar vein, Witsenhausen [1974] stated the following problem: Under the constraint that the diameter of the set of points $x_1, x_2, \ldots, x_k$ in $n$-dimensional space must be smaller than 1, find their configuration that maximizes the sum $\sum_{i,j} |x_i - x_j|^2$. Witsenhausen conjectured that the maximum, denoted by $M(n,N)$, is attained when the points are distributed among the $n+1$ vertices of a regular simplex of edge-length 1 and supported the conjecture by proving the inequality $M(n,N) \leq N^2n/(n+1)$. The conjecture was confirmed for $n = 2$ by Pilllshammer [2000] and in arbitrary dimension by Benassi and Malagoli [2008].

19. ARRANGEMENTS OF GREAT CIRCLES

Maehara [1995] considered the following problem: Arrange $k$ great circles so that the maximum spherical distance between a point of the sphere and the nearest
crossing point of the great circles is as small as possible. He solved the problem for \( k = 3 \) and 4. The optimal arrangement in these cases occur when each circle is divided into \( 2(k - 1) \) equal arcs by the other \( k - 1 \) great circles.

L. Fejes Tóth \[1959b\] showed that the longest edge of a spherical tiling formed by four or five great circles takes its minimum for the Archimedean tiling \((3,4,3,4)\) and \((3,5,3,5)\), respectively. Heppes \[1958\] studied a related problem: He showed that the area of the face with the smallest area of a tiling formed by four great circles takes its maximum for the cuboctahedron \((3,4,3,4)\).

A conjecture of L. Fejes Tóth \[1987\] about arrangements of great circles on a sphere states: If \( k \) great circles are in general position (no three of them have a common point), then the ratio of the greatest among the areas of the regions into which the circles partition the sphere to the smallest one tends to infinity as \( k \to \infty \). In the same article a lower bound is given: For sufficiently large \( k \) the ratio is greater than 7.43.

Motivated by this conjecture, Ismailescu \[2003\] considered an analogous problem on the plane and proved the following: Consider an arrangement of \( k \) lines such that no three are concurrent and all intersection points lie inside a unit circular disk. Then among the \( 1 + k(k - 1) \) bounded cells of the subdivision of the plane by the lines, there is one whose area is at least \( \pi k \). As a corollary, it follows that the ratio \( q \) of the greatest area to the smallest area of cells is at least \( (k + 1)/8 \). The question whether the ratio \( q \) tends to infinity as \( k \to \infty \) if we do not restrict the points of intersections to lie in a circle, remains open.

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