ON THE ERGODIC PROPERTIES OF CERTAIN ADDITIVE CELLULAR AUTOMATA OVER $\mathbb{Z}_m$

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Abstract. In this paper, we investigate some ergodic properties of $\mathbb{Z}^2$-actions $T_{p,n}$ generated by an additive cellular automata and shift acting on the space of all doubly-infinite sequences taking values in $\mathbb{Z}_m$.

1. Introduction

Mathematical study of cellular automata was initiated by Hedlund late 1960s. Hedlund determined the properties of endomorphisms and automorphisms of the shift dynamical system[2]. Sato studied linear cellular automata with-dimensional cell space as well as higher-dimensional cell space[3]. The properties of endomorphisms of subshifts of finite type were studied by Coven et al. [1]. Sinai gave a formula for directional entropy[5]. Ergodic properties of cellular automata have been investigated from various aspects by Shereshevsky and proved that if the automata map is bipermutative then associated CA-action is strongly-mixing[4].

In this paper, we shall restrict our attention to additive cellular automata over $\mathbb{Z}_m$. The organization of the paper is as follows: In section 2 we establish the basic formulation of problem necessary to state our main theorem. In section 3 we prove our main theorem and some results. Let us provide some notation and background.

2. Formulation of the problem

Let $\mathbb{Z}_m = \{0, 1, ..., m-1\}$ be a finite alphabet and $\Omega = \mathbb{Z}_m^\mathbb{Z}$ be the space of double-infinite sequences $x = (x_n)_{n=-\infty}^{\infty}$, $x_n \in \mathbb{Z}_m$, $\sigma$ is the shift in $\Omega$, i.e. $\sigma x = x' = (x'_n)$, $x'_n = x_{n+1}$, $x_n \in \mathbb{Z}_m$. A continuous map $f_\infty : \Omega \rightarrow \Omega$ commuting with the shift (i.e. such that $f_\infty \circ \sigma = \sigma \circ f_\infty$) is called a cellular automaton. It is well known (see[2], Theorem 3.4)) that $f_\infty : \Omega \rightarrow \Omega$ is a cellular automaton if and only if there exist $l, r \in \mathbb{Z}$ with $l \leq r$ and a mapping $f : [Z_r-l+1]_m \rightarrow [Z_m]$ such that $f_\infty(x) = (y_n)_{n=-\infty}^{\infty}$, $y_n = f(x_{n+l}, ..., x_{n+r})$ for all $x \in \Omega$. $n \in \mathbb{Z}$. It is called the mapping $f$ the rule of $f_\infty$ and the interval $[l, r]$ the range of $f_\infty$. In [5], it was assumed that $\sigma$ and $f_\infty$ generate an action of the group $\mathbb{Z}_2$ on $\Omega$: for $(m, n) \in \mathbb{Z}_2$ the corresponding transformation is $T_{p,n} = \sigma^p f_\infty^n$. Firstly, we consider additive cellular automata $f_\infty$ determined by an automation rule

$f(x_{n-k}, ..., x_{n+k}) = (\sum_{i=-k}^{k} \lambda_i x_{n+i})(modm)(\lambda_i \in \mathbb{Z}_m)$. 

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A cellular automaton (CA) defined on $\Omega$ is a map $F : \Omega \rightarrow \Omega$ such that for $x \in \Omega$ and $i \in \mathbb{Z}$, $(Fx)_i = f(x_{i-r}, \ldots, x_{i+r})$ where $r \in \mathbb{N}$ is radius and $f : \mathbb{Z}^{2r+1} \rightarrow \mathbb{Z}_m$ is a given local rule. Generally, we take as $(\lambda_i = 1)$. Let us consider a block $A = \{a_{-k}, \ldots, a_{a+k}\}$. The first preimage of the block $A$ under $f_{\infty}$ is 

$$y \in \Omega : y_{a-2k} = j_{a-2k}, \ldots, y_{a+2k} = j_{a+2k}, j_{a-2k}, \ldots, j_{a+2k} \in \mathbb{Z}_m$$

where $y_{a-2k} + \ldots + y_a = i_{a-k}(\text{modm})$,

$$\ldots$$

$$\ldots$$

$$y_{a-k} + \ldots + y_{a+k} = i_{a+k}(\text{modm})$$.

It is easy to see from this system of equations that $(f_{\infty})^{-1}(A)$ consists of $m^2k$ following blocks $(j_{a-2k}, \ldots, j_{a+2k})$. Now we calculate the measure

$$\mu((f_{\infty})^{-1}(A)) = m^{2k}\mu(y \in \Omega : y_{a-2k} = j_{a-2k}, \ldots, y_{a+2k} = j_{a+2k}, j_{a-2k}, \ldots, j_{a+2k} \in \mathbb{Z}_m)$$

$$= m^{2k}m^{-(4k+1)} = m^{-(2k+1)}$$,

**Example.** Let $A = \{0, 1\}$ and $f(x-2, x-1, x_0, x_1, x_2) = \left(\sum_{i=-2}^{2} x_i\right)(\text{mod } 2)$. Then

$$(f_{\infty})^{-1}(-2[10101]_2) = -3[111110000]_5 \cup -3[100000111]_5 \cup -3[010001011]_5 \cup -3[010110010]_5 \cup -3[001111100]_5 \cup -3[000011111]_5 \cup -3[001100100]_5 \cup -3[101100100]_5 \cup -3[100110110]_5 \cup -3[101010101]_5 \cup -3[010111100]_5 \cup -3[011011001]_5.$$ 

Thus we have 

$$\mu((f_{\infty})^{-1}(-2[10101]_2)) = 16\mu(-3[j_{-4}, \ldots, j_4]_5) = 2^42^{-9} = 2^{-5}.$$ 

If we continue this operation, by the same way, we can determine the measure of $(n-1)$st preimage of the block $A = \{a_{-k}, \ldots, a_{a+k}\}$ under $f_{\infty}$.

Evidently this $(n-1)$st preimage consist of such $(z_n)_{n=1}^{\infty}$, for which we have following system of equations:

$$z_{a-2k} + \ldots + z_{a-(n-1)k} + \ldots + z_{a-(n-2)k} = h_{a-(n-1)k}(\text{modm})$$

$$\ldots$$

$$\ldots$$

$$z_{a-k} + \ldots + z_a + \ldots + z_{a+k} = h_a(\text{modm})$$,

$$\ldots$$

$$\ldots$$
Bernoulli measure on $\Omega$, that is, $\mu_T$ by $(p, n + p > b)$ where $h_{a-(n-1)k}, \cdots, h_a, \cdots, h_{a+(n-1)k} \in Z_m$. So we can calculate the measure
$$\mu(f_{\infty(1)}(A)) = m^{2(n-1)k}m^{-(2nk+1)} = m^{-(2k+1)}.$$  

3. Results

Here we shall use the terminology of Sinai [5]. Let us consider as $Z^2$ – action $T_{p,n} = \sigma^p f_{\infty}^n$.

**Proposition:** Let $T_{p,n} = \sigma^p f_{\infty}^n$ be $Z^2$ – action as above and if $\mu$ is stationary Bernoulli measure on $\Omega$, that is, $\mu(i) = \frac{1}{r}$, $\forall i = 0, 1, \ldots, m - 1$, then both $f_{\infty}$ and $T_{p,n}$ are Bernoulli measure preserving transformations.

**Lemma:** The surjective CA-map $f_{\infty}$ generated by the rule
$$f(x_{n+1}, \ldots, x_{n+r}) = (\sum_{i=1}^r x_{n+i})(mod m)$$
is nonergodic with respect to the measure $\mu$, because the equality
$$\mu(b[\varepsilon_0, \ldots, \varepsilon_s]_{b+s} \cap f_{\infty}^{-n}(a[d_0, \ldots, d_k]_{a+k})) = \mu(b[\varepsilon_0, \ldots, \varepsilon_s]_{b+s})\mu(a[d_0, \ldots, d_k]_{a+k})$$
can’t be obtained sometimes. But we show that $Z^2$ – action $T_{p,n} = \sigma^p f_{\infty}^n$ defined by $(p, n) \mapsto T_{p,n} = \sigma^p f_{\infty}^n$ on $(\Omega, \mathcal{B}, \mu)$ is ergodic, weak-mixing and strong-mixing if $p > b + s + n\ell - a$.

**Theorem 1:** [6, Theorem 1.17] Let $(X, \mathcal{B}, \mu)$ be a measure space and let $\mathcal{A}$ be a semi-algebra that generates $\mathcal{B}$. Let $T: X \to X$ be a measure-preserving transformation. Then
(i) $T$ is ergodic iff $\forall A, B \in \mathcal{A}$
$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B),$$
(ii) $T$ is weak-mixing iff $\forall A, B \in \mathcal{A}$
$$\lim_{n \to \infty} \sum_{i=0}^{n-1} |\mu(T^{-i}A \cap B) - \mu(A)\mu(B)| = 0$$
and
(iii) $T$ is strongly-mixing iff $\forall A, B \in \mathcal{A}$
$$\lim_{n \to \infty} \mu(T^{-n}A \cap B) = \mu(A)\mu(B).$$

Now we can give main theorem.

**Theorem 2:** Let $Z_m = \{0, 1, \ldots, m - 1\}$ be a finite alphabet and $\Omega = Z_m^\infty$ be the space of double-infinite sequences $x = (x_n)_{n=-\infty}^\infty$, $x_n \in Z_m$. If additive cellular automata $f_{\infty}$ is given by the formula:
$$f_{\infty}(x) = (y_n)_{n=-\infty}^\infty, y_n = f(x_{n+\ell}, \ldots, x_{n+r}) = (\sum_{i=\ell}^r x_{n+i})(mod m)$$
for all $x \in \Omega$, $(p, n) \in Z^+ \times Z^+$, then $Z^2$ – action $T_{p,n} = \sigma^p f_{\infty}^n$ is ergodic, strongly-mixing and weak-mixing.
Proof. To prove that $T_{p,n}$ is ergodic it is sufficient to verify (Theorem 1,ii) for any two cylinder sets $A = a [d_0, ..., d_k]_{a+k}$ and $B = b [e_0, ..., e_s]_{b+s}$, we have

$$\lim_{p,n \to \infty} \frac{1}{pn} \sum_{(i,j) \in D} \mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k})) = \mu(b[e_0, ..., e_s]_{b+s}) \mu(a[d_0, ..., d_k]_{a+k}),$$

where $D = [0, p-1] \times [0, n-1] \cap Z^2$. For $i > b + s + j - a$ we have

$$\mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k})) = \mu(b[e_0, ..., e_s]_{b+s}) \mu(a[d_0, ..., d_k]_{a+k}).$$

On the other hand, we show that

$$\lim_{p,n \to \infty} \frac{1}{pn} \sum_{(i,j) \in D} \mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k}))$$

$$= \lim_{p,n \to \infty} \frac{1}{pn} \mu(b[e_0, ..., e_s]_{b+s}) \sum_{(i,j) \in D} f_\infty^{-j} \sigma^{-i} (a[d_0, ..., d_k]_{a+k}))$$

$$= \mu(b[e_0, ..., e_s]_{b+s}) \lim_{p,n \to \infty} \frac{1}{pn} \sum_{(i,j) \in D} f_\infty^{-j} (a[i,j,d_0, ..., d_k]_{a+k+i+1}))$$

$$= \mu(B) \lim_{p,n \to \infty} \frac{1}{pn} \sum_{j=0}^{n-1} (f_\infty^{-j} (a[d_0, ..., d_k]_{a+k}) + ... + f_\infty^{-j} (a[d_0, ..., d_k]_{a+k+p-1}))$$

$$= \mu(B) \lim_{p,n \to \infty} \frac{1}{pn} \sum_{i=0}^{n-1} [pm^{-k+1}]$$

$$= \mu(B) \mu(A).$$

So $Z^2 - \text{action } T_{p,n} = \sigma^p f_\infty^n$ is ergodic. Similarly for $i > b + s + j - a$ we have

$$\mu(b[e_0, ..., e_s]_{b+s} \cap T_{(-i,-j)}(a[d_0, ..., d_k]_{a+k})) = \mu(b[e_0, ..., e_s]_{b+s}) \mu(a[d_0, ..., d_k]_{a+k}).$$

Let $A = a [d_0, ..., d_k]_{a+k}$ and $B = b [e_0, ..., e_s]_{b+s}$ be any arbitrary cylinder sets. Then we have

$$\lim_{p,n \to \infty} \mu(T_{(-p,-n)}(A) \cap B) = \lim_{p,n \to \infty} \mu((f_\infty)^{-n}(a[p,d_0, ..., d_k]_{a+k+p} \cap B)$$

$$= \mu(B) \lim_{p,n \to \infty} (\mu f_\infty^{-n}(a[p,d_0, ..., d_k]_{a+k+p}))$$

$$= \mu(B) \mu(A).$$

Because every strongly-mixing transformation is weak-mixing, $T_{(p,n)}$ is weak-mixing.

One can prove that the natural extension of $T_{p,n} = \sigma^p f_\infty^n$ is ergodic and mixing.

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