On Synchronous, Asynchronous, and Randomized Best-Response schemes for computing equilibria in Stochastic Nash games

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Abstract

In this paper, we consider a stochastic Nash game in which each player minimizes a parameterized expectation-valued convex objective function. In deterministic regimes, proximal best-response (BR) schemes have been shown to be convergent under a suitable spectral property associated with the proximal BR map. However, a direct application of this scheme to stochastic settings requires obtaining exact solutions to stochastic optimization problems at each iteration. Instead, we propose an inexact generalization of this scheme in which an inexact solution to the BR problem is computed in an expected-value sense via a stochastic approximation (SA) scheme. On the basis of this framework, we present three inexact BR schemes: (i) First, we propose a synchronous inexact BR scheme where all players simultaneously update their strategies; (ii) Second, we extend this to a randomized setting where a subset of players is randomly chosen to update their strategies while the other players keep their strategies invariant; (iii) Third, we propose an asynchronous scheme, where each player chooses its update frequency while using outdated rival-specific data in updating its strategy. Under a suitable contractive property on the proximal BR map, we proceed to derive a.s. convergence of the iterates to the Nash equilibrium (NE) for (i) and (ii) and mean-convergence for (i)–(iii). In addition, we show that for (i)–(iii), the generated iterates converge to the unique equilibrium in mean at a linear rate with a prescribed constant rather than a sub-linear rate. Finally, we establish the overall iteration complexity of the scheme in terms of projected stochastic gradient (SG) steps for computing an ε−Nash equilibrium and note that in all settings, the iteration complexity is $O(1/\epsilon^{2(1+c)+\delta})$ where $c = 0$ in the context of (i) and represents the positive cost of randomization in (ii) and asynchronicity and delay in (iii). Notably, in the synchronous regime, we achieve a near-optimal rate from the standpoint of solving stochastic convex optimization problems by SA schemes. The schemes are further extended to settings where players solve two-stage stochastic Nash games with linear and quadratic recourse. Finally, preliminary numerics developed on a multi-portfolio investment problem and a two-stage capacity expansion game support the rate and complexity statements.

1 Introduction

Nash games represent an important subclass of noncooperative games \cite{6,23} and are rooted in the seminal work by Nash in 1950 \cite{33}. In the Nash equilibrium problem (NEP), there is a finite set of players, where each player aims at minimizing its own payoff function over a player-specific strategy set, given the rivals’ strategies. Nash’s eponymous solution concept requires that at an equilibrium, no player can improve its payoff by unilaterally deviating from its equilibrium strategy. Over the last several decades, there has been a surge of interest in utilizing Nash games to model a range of problems in control theory and decision-making with applications in communication networks, signal processing, electricity markets, (cf. \cite{5,55}). In recent years, stochastic Nash equilibrium models have found particular relevance in power markets \cite{17,24,26,27,39,49}. Motivated by those applications, we consider an $N$-player stochastic Nash game, where each player involves solving a stochastic constrained optimization problem parameterized by the rivals’ strategies. We aim to analyze a breadth of distributed BR schemes

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for computing a Nash equilibrium in regimes complicated by uncertainty, delay, asynchronicity, and randomized update rules.

Computation of Nash equilibria has been a compelling concern for the last several decades. As such game-theoretic models have gained wider usage in networked regimes, the need for distributed algorithms has become paramount. In particular, such algorithms require that the schemes respect the privacy concerns of users and are implementable over networked regimes. For instance, in flow control and routing problems in communication networks, gradient-based schemes have proved particularly useful [5, 25, 36, 55]. Such schemes generally impose a suitable monotonicity property on the map to ensure global convergence of iterates. Gradient-based schemes are characterized by ease of implementation and lower complexity in terms of each player step. Yet, such schemes are not “fully rational” since selfish players may deviate the update schemes unless they are forced by some authority to follow such gradient-based schemes. In BR schemes, players select their BR, given the current strategies of its rivals’ (cf. [6, 22]). While there has been some effort to extend such schemes to engineered settings (cf. [47]), in which the BR of each player can be expressed in a closed form. Proximal BR schemes appear to have been first discussed by Facchinei and Pang in 2009 [19]. In subsequent work, asynchronous extensions to these proximal schemes were applied to a class of optimization problems [46] while in [45], a framework that relied on solving a sequence of NEPs was proposed to solve a class of monotone Nash games. In regimes with expectation-valued payoffs, there has been far less computational research. Sampled and smoothed counterparts of the iterative regularization schemes presented in [25] were considered in [28] and [56]. However, much of this work considered settings where the integrands of the expectation were differentiable, ruling out the incorporation of two-stage recourse. Further, much of the past work in computing Nash equilibria has focused on risk-neutral problems with the sole exception being [27], where gradient methods were combined with cutting-plane schemes. In recent work, the last three authors have considered Nash games characterized by risk-aversion as well as two-stage recourse and have examined a smoothing-based sampling scheme that is more aligned with sample average approximation (SAA) techniques [37]. Yet, the question of computing a Nash equilibrium via natively distributed BR schemes in stochastic regimes in settings complicated by delay, asynchronicity, and two-stage recourse remains a compelling open question, motivating the current research.

**Challenge and motivation:** A natural question is why does it remain challenging to extend BR schemes to stochastic regimes? When contending with Nash games where each player’s payoff function is expectation-valued, computing a BR requires solving a stochastic optimization exactly or accurately. Unfortunately, unless this BR problem is convex and the expectation as well as its gradients are available as closed-form expressions, extensions of BR schemes to the stochastic regime remain impractical since the BR problem is essentially a stochastic optimization problem and requires Monte-Carlo sampling schemes [51]; in effect, this leads to a two-loop scheme in which the upper loop represents the BR iterations while the lower loop captures the solution of the BR problem. Practical implementations of such schemes that provide asymptotically accurate solutions at each step remain unavailable and at best, this avenue provides a rather coarse approximation of the equilibrium.

**A single-loop approach:** Instead, we consider developing single-loop *inexact* BR schemes that are practically implementable; here the BR is computed *inexactly* in an expected-value sense but require that the inexactness sequence be driven to zero. This is achieved by utilizing a SA scheme that utilizes an increasing number of projected SG steps. Note that in comparison with a two-loop scheme, in this setting, the “inside loop” requires a known number of iterations, thereby making this scheme a single-loop scheme. While this is a relatively simple scheme, the resulting convergence of iterates is by no means guaranteed and claiming optimality of the rate is far from immediate. In fact, convergence of BR schemes holds under diverse settings requiring either that the proximal-response map (defined later) admits a suitable contractive property or that player objectives admit a potential function (cf. [19, 22]). Our work draws inspiration from the work by Facchinei and Pang [19] in which the contractive property of the proximal BR map is employed as a tool to ensure convergence of the iterates produced by the BR scheme.

**Contributions:** Given the inherent challenges in solving this problem in finite time, we consider several *inexact* variants of this scheme and make the following contributions that are summarized in Table 1.
(i) **Synchronous schemes:** We propose a synchronous inexact proximal BR scheme to find the Nash equilibrium, and prove that the generated iterates converge almost surely and in mean to the unique Nash equilibrium when the proximal BR map is contractive in 2-norm. Furthermore, when the inexactness is dropped at a suitable rate, the rate of convergence of the iterates is provably linear or geometric rather than sub-linear. Based on this linear rate of convergence for the BR iterates and by assuming that an inexact solution is computed via a SA, we derive the overall iteration complexity in projected SG steps for computing an $\epsilon$-NE$_2$ (see definition (26)) and show that this overall complexity is $O(N^{1+\delta/2} / \epsilon^{2+\delta})$ where $\delta > 0$. In addition, the lower bound of the derived iteration complexity satisfies $\Omega(N / \epsilon^2)$. Furthermore, for a specific selection of the algorithmic parameters, the overall complexity is shown to be exactly of $O(N / \epsilon^2)$, which is optimal for the resolution of stochastic convex optimization by standard SA schemes.

(ii) **Randomized extension:** Subsequently, a randomized inexact proximal BR (BR) scheme motivated by [14,34] is proposed, in which a subset of players is randomly chosen to update at each iteration. To be specific, player $i$ is chosen with some positive probability $p_i$ at each major iteration. This randomized protocol allows players to initiate an update according to a local clock, e.g., the Poisson clock, and locally choose the inexactness sequence. With a suitably selected inexactness sequence, the estimates are shown to converge to the Nash equilibrium almost surely and in mean at a prescribed linear rate when the proximal BR map is contractive in 2-norm. The expectation of the total number of projected SG steps to compute an $\epsilon$-NE$_2$ is shown to be $O\left((\sqrt{N} / \epsilon)^2 \ln(\tilde{\eta}_0 \tilde{\eta}) / \ln(\tilde{\eta}^{-1}) + \delta\right)$ for some $\delta > 0$, where $\tilde{\eta}$ and $\tilde{\eta}_0$ are defined in (33) and (36), respectively. Further, as noted in Remark 7, this bound is worse than that shown for the synchronous algorithm, a consequence of the randomized index selection scheme that accommodates much flexibility into the update scheme.

(iii) **Asynchronous and delay-tolerant schemes:** Synchronous algorithms require players to update their strategies simultaneously, but this is often difficult to mandate in networked settings with a large collection of noncooperative players. Additionally, players may often not have access to their rivals’ latest strategies. Motivated by the asynchronous algorithm specified in [9], we propose an asynchronous inexact proximal BR scheme, where each player determines its update time while using possibly outdated rival information. Yet, we assume that each player updates at least once in any time interval of length $B_1$, and that the communication delays are uniformly bounded by $B_2$. When the proximal BR map is contractive in $\infty$-norm, the iterates are proved to converge in mean to the unique equilibrium at a linear rate for an appropriately chosen inexactness sequence. Furthermore, we derive the overall iteration complexity of the projected SG steps for computing an $\epsilon$-NE$_\infty$ (see definition (44)) and show that the complexity bound is of $O\left((1/\epsilon)^{2B_1} \left(1+\left[\frac{B_2}{p_I}\right]\right)^{+\delta}\right)$ for some $\delta > 0$. Specially, if the players update in a cyclic fashion, then the complexity bound improves to $O\left((1/\epsilon)^{2\left(1+\left[\frac{B_2}{p_I}\right]\right)}\right)$.

(iv) **Incorporating private two-stage recourse:** To show that all the aforementioned avenues can be extended to accommodate recourse-based objectives arising from two-stage stochastic programming [15], we consider an extended stochastic NEP where each player solves a two-stage stochastic program with recourse. We separately investigate the case of linear and quadratic recourse with a particular accent on computing subgradients of the random recourse function. On the basis of the existence and boundedness of the stochastic subgradient, the proposed synchronous, randomized, and asynchronous inexact proximal BR algorithms are still applicable and the formulated convergence results hold as well.

(v) **Preliminary numerics:** Finally, our numerical studies on competitive portfolio selection problems and two-stage competitive capacity expansion problems suggest that the empirical behavior corresponds well with the rate statements and the complexity bounds.

**Organization:** The remainder of this paper is organized as follows: In Section 2 we formulate the stochastic Nash game, provide some basic assumptions as well as some background on the proximal BR map. In Section 3
we introduce a synchronous inexact proximal BR scheme, derive rate statements, and analyze the overall iteration complexity when an inexact solution to the BR problem is solved via a SA scheme. A randomized and an asynchronous inexact proximal BR scheme are proposed in Sections 4 and 5 respectively, where the rate and complexity statements established. Finally, all the proposed avenues are extended to accommodate recourse-based objectives in Section 6. We present some numerical results in Section 7 and conclude in Section 8 with a brief summary of our main findings.

Notations: When referring to a vector $x$, it is assumed to be a column vector while $x^T$ denotes its transpose. Generally, $\|x\|$ denotes the Euclidean vector norm, i.e., $\|x\| = \sqrt{x^T x}$, while other norms will be specified appropriately (such as the 1-norm or the $\infty$-norm). We use $\Pi_X[x]$ to denote the Euclidean projection of a vector $x$ on a set $X$, i.e., $\Pi_X [x] = \min_{y \in X} \|x - y\|$. We abbreviate “almost surely” by $a.s.$ and use $\mathbb{E}[z]$ to denote the expectation of a random variable $z$. For a square matrix $A$, we denote by $\rho(A)$ the spectral radius and by $\lambda_{\min}(A)$ the smallest eigenvalue of $\frac{A + A^T}{2}$. We denote by $\otimes$ and $\mathbf{I}_m$ the Kronecker product and $m \times m$ identity matrix, respectively.

## 2 Problem Formulation and Preliminaries

### 2.1 Problem Statement

There exists a set of $N$ players indexed by $i$ where $i \in \mathcal{N} \triangleq \{1, \ldots, N\}$. For any $i \in \mathcal{N}$, the $i$th player has a strategy set $X_i \subseteq \mathbb{R}^{n_i}$ and a payoff function $f_i(x_i, x_{-i})$ depending on its own strategy $x_i$ and on the vector of rivals’ strategies $x_{-i} \triangleq \{x_j \mid j \neq i\}$. Suppose $n \triangleq \sum_{i=1}^N n_i$, $X \triangleq \prod_i X_i$ and $X_{-i} \triangleq \prod_{j \neq i} X_j$. Let us consider a stochastic setting of the NEP denoted by $(\text{SNash})$, in which the objective of player $i$, given rivals’ strategies $x_{-i}$, is to solve the following constrained stochastic program

$$\min_{x_i \in X_i} f_i(x_i, x_{-i}) \triangleq \mathbb{E} [\psi_i(x_i, x_{-i}; \xi(\omega))],$$

$$(\text{SNash}_i(x_{-i}))$$

where $\psi_i(\cdot): X \times \mathbb{R}^d \to \mathbb{R}$ is a scalar-valued function, and the expectation is taken with respect to the random vector $\xi: \Omega \to \mathbb{R}^d$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. An NE of the stochastic Nash game (SNash) $x^* = \{x^*_i\}_{i=1}^N$ satisfies the following:

$$x^*_i \text{ solves } (\text{SNash}_i(x_{-i}^*)), \ \forall i \in \mathcal{N}.$$  

In other words, $x^*$ is an NE if no player can improve the payoff by unilaterally deviating from the equilibrium strategy $x^*_i$. For notational simplicity, $\xi$ is used to denote $\xi(\omega)$ throughout the paper. In this work, we assume that for any $i \in \mathcal{N}$, $f_i(x_i, x_{-i})$ and $\psi_i(x_i, x_{-i}; \xi)$ is smooth in $x_i$ for every given $x_{-i}$ and $\xi$. Further, there exists a stochastic oracle such that for any $i \in \mathcal{N}$ and every given $x, \xi$ returns a sample $\nabla x_i \psi_i(x_i, x_{-i}; \xi)$, which is an unbiased estimator of $\nabla x_i f_i(x_i, x_{-i})$. We impose the following conditions on the stochastic Nash game.

| Update scheme         | Asymptotic convergence                  | Rate of convergence     | Iteration complexity                                      |
|-----------------------|-----------------------------------------|-------------------------|----------------------------------------------------------|
| Synchronous (Algorithm 1) (using $\|\cdot\|_2$ norm) | a.s. convergence (Proposition 2)        | geometric (Proposition 4) | $\epsilon$-NE$_2$: \(O \left( (\sqrt{N}/\epsilon)^{2+\delta} \right) \) Corollary 4: \(O(N/\epsilon^2)\) |
| Randomized (Algorithm 2) (using $\|\cdot\|_2$ norm) | a.s. convergence (Lemma 4)              | geometric (Lemma 5)     | $\epsilon$-NE$_2$: \(O \left( (\sqrt{N}/\epsilon)^{2\ln(\eta^{-1})/\ln(\eta^{-1})+\delta} \right) \) |
| Asynchronous (Algorithm 3) (using $\|\cdot\|_\infty$ norm) | convergence in mean (Lemma 7)           | geometric (Lemma 7)     | $\epsilon$-NE$_\infty$: \(O \left( (1/\epsilon)^{2B_i(1+\frac{B_i}{m})+\delta} \right) \) Theorem 4(a): \(O \left( (1/\epsilon)^{2(1+\frac{B_i}{m})+\delta} \right) \) Theorem 4(b): \(O \left( (1/\epsilon)^{2(1+\frac{B_i}{m})+\delta} \right) \) |

Table 1: Summary of Contributions
Remark 1  Let the following hold.
(a) $X_i$ is a closed, compact, and convex set;
(b) $f_i(x_i, x_{-i})$ is convex and twice continuously differentiable in $x_i$ over an open set containing $X_i$ for every $x_{-i} \in X_{-i}$;
(c) For all $x_{-i} \in X_{-i}$ and any $\omega \in \Omega$, $\psi_i(x_i, x_{-i}; \xi(\omega))$ is differentiable in $x_i$ over an open set containing $X_i$ such that $\nabla_{x_i}f_i(x_i, x_{-i}) = \mathbb{E}[\nabla_{x_i}\psi_i(x_i, x_{-i}; \xi);$
(d) For any $i \in N$ and all $x \in X$, there exists a constant $M_i > 0$ such that $\mathbb{E}[\|\nabla_{x_i}\psi_i(x_i, x_{-i}; \xi)\|^2] \leq M_i^2$.

### 2.2 Background on proximal BR maps

In this paper, we consider the class of stochastic Nash games in which the proximal BR map (which is defined subsequently) admits a contractive property \cite{19}. Suppose $\hat{x}(y)$ is defined as follows:

$$\hat{x}(y) \triangleq \arg \min_{x \in X} \left[ \sum_{i=1}^{N} \mathbb{E} \left[ \psi_i(x_i, y_{-i}; \xi) \right] + \frac{\mu}{2} \|x - y\|^2 \right]. \quad (1)$$

It is easily seen that the objective function is now separable in $x_i$ and \eqref{1} reduces to a set of player-specific proximal BR problems, in which the $i$th player’s problem is given by the following:

$$\hat{x}_i(y) \triangleq \arg \min_{x_i \in X_i} \left[ \mathbb{E} \left[ \psi_i(x_i, y_{-i}; \xi) \right] + \frac{\mu}{2} \|x_i - y_i\|^2 \right]. \quad (2)$$

Analogous to the avenue adopted in \cite{19}, we may define the $N \times N$ real matrix $\Gamma = [\gamma_{ij}]_{i,j=1}^{N}:

$$\Gamma \triangleq \begin{pmatrix}
\frac{\mu}{\bar{\zeta}_{11}} & \frac{\bar{\zeta}_{12}}{\bar{\zeta}_{11}} & \ldots & \frac{\bar{\zeta}_{1N}}{\bar{\zeta}_{11}} \\
\frac{\bar{\zeta}_{21}}{\bar{\zeta}_{22}} & \frac{\mu}{\bar{\zeta}_{22}} & \ldots & \frac{\bar{\zeta}_{2N}}{\bar{\zeta}_{22}} \\
\vdots & \ddots & \ddots & \vdots \\
\frac{\bar{\zeta}_{N1}}{\bar{\zeta}_{N1}} & \frac{\bar{\zeta}_{N2}}{\bar{\zeta}_{N2}} & \ldots & \frac{\mu}{\bar{\zeta}_{NN}}
\end{pmatrix} \quad (3)
$$

with

$$\bar{\zeta}_{ij,\min} \triangleq \inf_{x \in X} \lambda_{\min} (\nabla^2_{x_i} f_i(x)) \quad \text{and} \quad \bar{\zeta}_{ij,\max} \triangleq \sup_{x \in X} \|\nabla^2_{x_i,x_j} f_i(x)\| \quad \forall j \neq i. \quad (4)$$

Then we obtain the following inequality:

$$\begin{pmatrix}
\|\hat{x}_1(y') - \hat{x}_1(y)\| \\
\vdots \\
\|\hat{x}_N(y') - \hat{x}_N(y)\|
\end{pmatrix} \leq \Gamma \begin{pmatrix}
\|y'_1 - y_1\| \\
\vdots \\
\|y'_N - y_N\|
\end{pmatrix}. \quad (5)$$

If the spectral radius $\rho(\Gamma) < 1$, then there exist a scalar $a \in (0, 1)$ and a monotonic norm $\| \| \cdot \| \|$ such that

$$\begin{pmatrix}
\|\hat{x}_1(y') - \hat{x}_1(y)\| \\
\vdots \\
\|\hat{x}_N(y') - \hat{x}_N(y)\|
\end{pmatrix} \leq a \begin{pmatrix}
\|y'_1 - y_1\| \\
\vdots \\
\|y'_N - y_N\|
\end{pmatrix}. \quad (6)$$

Note that sufficient conditions for the contractive property of the BR map $\hat{x}(\bullet)$ are provided in \cite{19} Proposition 12.17.

Remark 1  Let $D_i \in \mathbb{R}^{n_i \times n_i} \forall i \in N$ be any arbitrary chosen nonsingular matrices. Define

$$\bar{\zeta}_{i,\min} \triangleq \inf_{x \in X} \lambda_{\min} \left( D_i^T \nabla^2_{x_i} f_i(x) D_i \right), \quad \bar{\zeta}_{ij,\max} \triangleq \sup_{x \in X} \| D_i^T \nabla^2_{x_i,x_j} f_i(x) D_j \| \quad \forall j \neq i.$$


It is easily seen that definition (4) is only a special case with \( D_i \in \mathbb{R}^{n_i \times n_i} \) being an identity matrix. By making simple modifications to the proof given in [19, Section 12.6.1] we are able to obtain the inequality (5) with \( \Gamma \) defined by (3). As discussed in [45], matrices \( D_i \forall i \in \mathcal{N} \) provide an additional degree of freedom in deriving the contraction property of the proximal BR map \( \tilde{x}(\bullet) \).

Recall that the proximal BR \( \tilde{x}_i(y) \) requires solving a stochastic optimization problem defined in (2). Stochastic optimization problems have been studied extensively over the last several decades through Monte-Carlo sampling schemes such as SAA and SA. SAA provides a foundation for relating the estimators of an expectation-valued problem obtained by solving the deterministic sample-average approximation. Asymptotic convergence and error bounds for the SAA estimators have been extensively investigated in [51]. SA was first considered by Robbins and Monro [41] for seeking roots of a regression function with noisy observations. Such schemes have found wide applications in stochastic problems such as convex optimization, variational inequality problems, and systems and control problems. Here, we apply the SA scheme to estimate the optimal solution of the stochastic BR problem (2).

3 A Synchronous Inexact Proximal BR Scheme

In this section, we present a synchronous inexact proximal BR scheme for which both asymptotic convergence analysis, rate statements, and overall iteration complexity results are provided.

3.1 Description of algorithm

Recall that the exact proximal BR \( \tilde{x}_i(y_k) \) is defined as the following:

\[
\tilde{x}_i(y_k) \triangleq \arg\min_{x_i \in X_i} [f_i(x_i, y_{-i,k}) + \frac{\mu}{2} \|x_i - y_{i,k}\|^2].
\]  

(7)

Since \( f_i(x_i, y_{-i,k}) \) is the expectation of \( \psi_i(x_i, y_{-i,k}; \xi) \) with respect to \( \xi \), a closed-form expression of \( \tilde{x}_i(y_k) \) is unavailable. Instead we propose an inexact proximal BR scheme (Algorithm 1) computing an inexact BR that satisfies (8) rather than computing the exact BR, which is available only in an asymptotic sense. It is worth pointing out that \( \mathcal{F}_k \) in equation (8) denotes the \( \sigma \)-field of the entire information used by the algorithm up to (and including) the update of \( x_k \). We will further define \( \mathcal{F}_k \) in Section 3.4 and specify the deterministic sequence \( \{\alpha_{i,k}\}_{k \geq 0} \) in Algorithm 1 when we proceed to investigate the convergence properties.

**Algorithm 1 Inexact proximal BR scheme for stochastic Nash games**

Set \( k := 0 \). Let \( y_{i,0} = x_{i,0} \in X_i \), and \( \{\alpha_{i,k}\}_{k \geq 0} \) be a given deterministic sequence for \( i = 1, \ldots, N \).

1. For \( i = 1, \ldots, N \), let \( x_{i,k+1} \) satisfy the following

\[
x_{i,k+1} \in \{ z \in X_i : \mathbb{E} \left[ \|z - \tilde{x}_i(y_k)\|^2 \left| \mathcal{F}_k \right. \right] \leq \alpha_{i,k}^2 \text{ a.s.} \}.
\]  

(8)

2. For \( i = 1, \ldots, N \), \( y_{i,k+1} := x_{i,k+1} \);

3. \( k := k + 1 \); If \( k < K \), return to (1); else STOP.

3.2 Convergence analysis

We now proceed to analyze this scheme in greater detail and initiate our discussion by noting the convergence of the exact BR scheme.

**Proposition 1** ([19]) Consider the stochastic Nash game in which the \( i \)th player solves \( (SNash_i(x_{-i})) \), given \( x_{-i} \in X_{-i} \). Let the sequence of exact responses be denoted by \( \{y_k^{\infty}\}_{k=1}^{\infty} \). Assume that the matrix \( \Gamma \) defined by (5) satisfies \( \rho(\Gamma) < 1 \). Then the following hold:
(i) The contraction given by (10) holds for the proximal BR map.

(ii) For \( i = 1, \cdots, N \), the BR iterates \( y_{i,k}^{\infty} \rightarrow x_i^* \) as \( k \rightarrow \infty \).

Naturally, one may derive the uniqueness of the Nash equilibrium from the existence of this contractive property (cf. [19]). We prove that the sequence \( \{x_k\}_{k \geq 0} \) generated by Algorithm (1) converges almost surely and in expectation to the unique Nash equilibrium of the stochastic Nash game in Propositions (2) and (3), respectively. The results rely on comparing the inexact responses with the exact BR map, which is known to be contractive by the following sufficient condition:

**Assumption 2** The \( \Gamma \) matrix defined in (5) satisfies \( \|\Gamma\| < 1 \).

Define \( a \triangleq \|\Gamma\| \). Then \( a \in (0, 1) \) by Assumption 2 and hence by (5) we obtain the following:

\[
\begin{bmatrix}
\|\tilde{x}_1(y') - \tilde{x}_1(y)\| \\
\vdots \\
\|\tilde{x}_N(y') - \tilde{x}_N(y)\|
\end{bmatrix} \leq a \begin{bmatrix}
\|y_1' - y_1\| \\
\vdots \\
\|y_N' - y_N\|
\end{bmatrix} \quad \forall y, y' \in X. \tag{9}
\]

We utilize the following lemma for random and deterministic recursions to prove the almost sure convergence and convergence in mean, respectively.

**Lemma 1** (a) [38 Lemma 2.2.10] Let \( \{v_k\}_{k \geq 0} \) be a sequence of random variables, \( v_k \geq 0, \mathbb{E}[v_0] < \infty \), such that \( \mathbb{E}[v_{k+1}|v_0, \cdots, v_k] \leq q_k v_k + \xi_k \), a.s. where \( 0 \leq q_k < 1, \xi_k \geq 0, \sum_{k=0}^{\infty} (1 - q_k) = \infty, \sum_{k=0}^{\infty} \xi_k < \infty \), and \( \lim_{k \rightarrow \infty} \frac{\xi_k}{1 - q_k} = 0 \). Then \( v_k \rightarrow 0 \) a.s. (b) [38 Lemma 2.2.3] Let \( u_{k+1} \leq q_k u_k + \xi_k, 0 \leq q_k < 1, \xi_k \geq 0 \), where \( 0 \leq q_k < 1, \xi_k \geq 0 \), \( \sum_{k=0}^{\infty} (1 - q_k) = \infty \), and \( \lim_{k \rightarrow \infty} \frac{\xi_k}{1 - q_k} = 0 \). Then \( \limsup_{k \rightarrow \infty} u_k \leq 0 \). In particular, if \( u_k \geq 0 \), then \( \lim_{k \rightarrow \infty} u_k = 0 \).

**Proposition 2** (Almost Sure Convergence) Let the sequence \( \{x_k\}_{k \geq 0} \) be generated by Algorithm (1) Suppose Assumptions 1 and 2 hold, and that \( \alpha_{i,k} \geq 0 \) with \( \sum_{k=0}^{\infty} \alpha_{i,k} < \infty \) for any \( i \in \mathcal{N} \). Then for any \( i \in \mathcal{N} \),

\[
\lim_{k \rightarrow \infty} x_{i,k} = x_i^* \quad \text{a.s.}
\]

**Proof.** We begin by noting that \( x_i^* = \tilde{x}_i(x^*) \) and examine \( \|x_{i,k+1} - x_i^*\|\):

\[
\|x_{i,k+1} - x_i^*\| \leq \|x_{i,k+1} - \tilde{x}_i(y_k)\| + \|\tilde{x}_i(y_k) - \tilde{x}_i(x^*)\|, \tag{10}
\]

where the inequality follows from the triangle inequality. Furthermore, by \( y_k = x_k \) and the contractive property (9) of \( \tilde{x}(\bullet) \), we have the following bound:

\[
\begin{bmatrix}
\|\tilde{x}_1(y_k) - \tilde{x}_1(x^*)\| \\
\vdots \\
\|\tilde{x}_N(y_k) - \tilde{x}_N(x^*)\|
\end{bmatrix} = \begin{bmatrix}
\|\tilde{x}_1(x_k) - \tilde{x}_1(x^*)\| \\
\vdots \\
\|\tilde{x}_N(x_k) - \tilde{x}_N(x^*)\|
\end{bmatrix} \leq a \begin{bmatrix}
\|x_{1,k} - x_1^*\| \\
\vdots \\
\|x_{N,k} - x_N^*\|
\end{bmatrix}. \tag{11}
\]

Consequently, from (10) by the triangle inequality, we obtain the following inequality:

\[
v_{k+1} \triangleq \begin{bmatrix}
\|x_{1,k+1} - x_1^*\| \\
\vdots \\
\|x_{N,k+1} - x_N^*\|
\end{bmatrix} \leq a \begin{bmatrix}
\|x_{1,k} - x_1^*\| \\
\vdots \\
\|x_{N,k} - x_N^*\|
\end{bmatrix} + \begin{bmatrix}
\|x_{1,k+1} - \tilde{x}_1(y_k)\| \\
\vdots \\
\|x_{N,k+1} - \tilde{x}_N(y_k)\|
\end{bmatrix}. \tag{12}
\]
Denote by $\alpha_{\text{max},k} \triangleq \max_i \alpha_{i,k}$. Then by (8) and conditional Jensen’s inequality, we have that for all $k \geq 0$:

$$
\mathbb{E} \left[ \left( \|x_{1,k+1} - \hat{x}_1(y_k)\| \right) \right| \mathcal{F}_k] \leq \left( \mathbb{E} \left[ \sum_{i=1}^N \|x_{i,k+1} - \hat{x}_i(y_k)\|^2 \right] \right| \mathcal{F}_k] \leq \sum_{i=1}^N \alpha_{i,k}^2 \leq \sqrt{N} \alpha_{\text{max},k} \ a.s.
$$

(13)

By taking expectations conditioned on $\mathcal{F}_k$ with respect to both sides of (12), from (13) we have that

$$
\mathbb{E}[v_{k+1}|\mathcal{F}_k] \leq au_k + \sqrt{N} \alpha_{\text{max},k} \ a.s.
$$

(14)

Since $\sum_k (1 - a) = \infty$ and $\lim_{k \to \infty} \frac{\alpha_{\text{max},k}}{(1-a)} = 0$, by Lemma 1(a) and the summability of $\alpha_{\text{max},k}$, we conclude that $v_k \to 0$ a.s. Consequently, the result follows by $v_k$ defined in (12).

\begin{proposition}
Let the sequence $\{x_k\}_{k \geq 0}$ be generated by Algorithm 1. Suppose Assumptions 1 and 2 hold, and that $\alpha_{i,k} \geq 0$ with $\lim_{k \to \infty} \alpha_{i,k} = 0$ for any $i \in \mathcal{N}$. Then we have the following assertions:

(a) (convergence of the variance of $x_k$) $\lim_{k \to \infty} \text{Var}(x_k) = 0$.

(b) (convergence in mean) $\lim_{k \to \infty} \mathbb{E}[\|x_{i,k} - x_i^*\|] = 0 \ \forall i \in \mathcal{N}$.

\end{proposition}

Proof. (a) Note that the variance of $x_k$ may be bounded as follows:

$$
\text{Var}(x_k) = \mathbb{E} \left[ \|\mathbb{E}[x_k] - x_k\|^2 \right] = \mathbb{E} \left[ \|\mathbb{E}[x_k - x^*] - (x_k - x^*)\|^2 \right] \leq \mathbb{E} \left[ \|x_k - x^*\|^2 \right].
$$

(15)

By Assumption 1(a), we define the diameter of the $X_i$ as follows:

$$
D_X_i \triangleq \sup \{d(x_i, x'_i) : x_i, x'_i \in X_i \} < \infty.
$$

(16)

By (12), we have the following:

$$
\|x_{k+1} - x^*\|^2 \leq \sum_{i=1}^N \|x_{i,k+1} - \hat{x}_i(y_k)\|^2 + a^2 \|x_k - x^*\|^2 + 2a \left( \left\|\left( \|x_{1,k} - x_1^*\| \right) \right\| \right) + \left( \left\|\left( \|x_{N,k} - x_N^*\| \right) \right\| \right).
$$

Thus, by taking expectations conditioned on $\mathcal{F}_k$, from (8), (13), and (16) we have that

$$
\mathbb{E} \left[ \|x_{k+1} - x^*\|^2 \right| \mathcal{F}_k] \leq a^2 \|x_k - x^*\|^2 + \sum_{i=1}^N \alpha_{i,k}^2 + 2\sqrt{\sum_{i=1}^N D_X_i^2} \left( \sum_{i=1}^N \alpha_{i,k}^2 \right) \ a.s.
$$

(17)

Since $\alpha_{i,k}$ is deterministic and $\lim_{k \to \infty} \alpha_{i,k} = 0$, by taking expectations on both sides of (17) and by applying Lemma 1(b) we have that $\lim_{k \to \infty} \mathbb{E}[\|x_k - x^*\|^2] = 0$. Thus, the result (a) follows by the expression (15).

(b) By Jensen’s inequality, we have that $\mathbb{E}[\|x_{i,k} - x_i^*\|] \leq \sqrt{\mathbb{E}[\|x_{i,k} - x_i^*\|^2]}$, which implies result (b) by invoking the fact that $\lim_{k \to \infty} \mathbb{E}[\|x_{i,k} - x_i^*\|^2] = 0 \ \forall i \in \mathcal{N}$. \qed
3.3 Rate of convergence
We begin with a supporting lemma that bounds a sub-linear rate by a prescribed linear rate. This has been proven in [3] but a simpler proof is provided here.

Lemma 2 Given a function $zc^2$ with $0 < c < 1$. Then for all $z \geq 0$, we have that

$$zc^2 \leq Dq^2,$$

where $c < q < 1$ and $D \geq \frac{1}{\ln(q/c)}$.

Proof. Since $zc^2 \leq Dq^2$ for all $z \geq 0$. If $p \triangleq c/q < 1$, it follows that $\max_{z \geq 0} z p^2 \leq D$. Then a maximizer of $zp^2$ satisfies the following: $p^2 + zp^2 \ln(p) = 0$ implying that $z = -1/\ln(p) > 0$. ($z$ can be seen to be a maximizer by taking second derivatives). It follows that

$$\max_{z \geq 0} zp^2 = \frac{1}{\ln(p)} \frac{1}{p} = \frac{1}{\ln(p)} e^{-1} = \frac{1}{\ln(q/c)}.$$  

Proposition 4 (Geometric rate of convergence) Consider the synchronous inexact proximal BR scheme (Algorithm 1) where $\mathbb{E}[\|x_i,0 - x^*_i\|] \leq C$ and $\alpha_{i,k} = \eta^{k+1}$ for some $\eta \in (0, 1)$. Suppose Assumptions 1 and 2 hold. Define $c \triangleq \max\{a, \eta\}$, and

$$u_k \triangleq \mathbb{E} \left[ \left\| \begin{array}{c} \|x_{1,k} - x^*_1\| \\
\vdots \\
\|x_{N,k} - x^*_N\| \end{array} \right\| \right].$$ (18)

Then the following holds for any $q \in (c, 1)$ and $D \triangleq 1/\ln((q/c)^e)$

$$u_k \leq \sqrt{N}(C + D)q^k \quad \forall k \geq 0.$$ (19)

Proof. By taking expectations on both sides of (12), from (13) we obtain the following recursion:

$$u_{k+1} \leq au_k + \sqrt{N} \alpha_{\text{max},k} \quad \forall k \geq 0.$$ (20)

Since $\alpha_{\text{max},k} = \eta^{k+1}$, by (20) we obtain the following sequence of inequalities:

$$u_k \leq au_{k-1} + \sqrt{N} \eta^k \leq a \eta^0 u_0 + \sqrt{N} \sum_{j=1}^{k} a^{k-j} \eta^j \leq c^k u_0 + \sqrt{N} \left( \sum_{j=1}^{k} c^j \right) = (u_0 + \sqrt{N}k) c^k.$$ (21)

By Lemma 2 we have that $ke^k \leq Dq^k$ for any $q \in (c, 1)$ and $D \geq 1/\ln((q/c)^e)$. Then by (21), we get

$$(u_0 + \sqrt{N}k) c^k \leq (u_0 + \sqrt{N}D)q^k.$$ 

Since $u_0 \leq \sqrt{NC} \text{ by } \mathbb{E}[\|x_{i,0} - x^*_i\|] \leq C \text{ for all } i \in \mathcal{N}$, it follows that $u_k \leq \sqrt{N}(C + D)q^k$.  

Remark 2 This result shows that if the inexactness sequence is driven to zero at a suitable rate, there is no degradation in the overall rate of convergence; in effect, the rate stays linear and can be precisely specified. Notably, the impact of the number of players on the rate appears only in terms of the constant. While there exists prior research where a geometric rate of convergence is retained by solving subproblems accurately enough (cf. [27][47]), this result appears to be the first such result in the context of inexact proximal best-response schemes for stochastic Nash games.
Lemma 3

Let Assumption 1 hold. Consider the synchronous inexact proximal BR scheme given by Algorithm (7). We derive an error bound for point \((z_{i,t}, x_{i,k})\) by the nonexpansive property of the projection operator we have

\[ z_{i,t+1} := \Pi_{X_i} \left[ z_{i,t} - \gamma_t \left( \nabla_x \psi_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) + \mu(z_{i,t} - y_{i,k}) \right) \right], \quad (SA_{i,k}) \]

where \(z_{i,1} = x_{i,k}\), and \(\gamma_t = 1/\mu(t + 1)\). Note that \(\nabla_x \psi_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t)\) denotes the sampled gradient of \(f_i(\cdot)\) at point \((z_{i,t}, x_{i,k})\) while \(\gamma_t\) is a non-summable but square summable sequence.

Let the SA scheme \((SA_{i,k})\) from \(t = 1, \ldots, j_{i,k}\) be employed for obtaining \(x_{i,k+1}\). Define \(\xi_{i,k} \triangleq (\xi_{i,k}^1, \ldots, \xi_{i,k}^{j_{i,k}})\), \(\xi_{i,k}^{[l]} = (\xi_{i,k}^1, \ldots, \xi_{i,k}^{l})\), and \(F_k \triangleq \sigma(x_0, \xi_{i,k}, i \in N, 0 \leq l \leq k-1)\). Then by Algorithm 1 we see that \(x_k\) is adapted to \(F_k\), and \(\tilde{x}_i(y_k)\) is also adapted to \(F_k\) since \(\tilde{x}_i(y_k) = \tilde{x}_i(x_k)\) is an optimal solution to the parameterized problem (7). We derive an error bound for \((SA_{i,k})\) as specified by the following lemma.

**Lemma 3** Let Assumption 1 hold. Consider the synchronous inexact proximal BR scheme given by Algorithm 1. Assume that for any \(t = 1, \ldots, j_{i,k}\), \(E \left[ \psi_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \right] = \nabla_x \tilde{f}_i(z_{i,t}, y_{i,k}) \) a.s., and \(E \left[ \| \psi_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \|^2 \right] \leq Q_i/2\) a.s. Define \(Q_i \triangleq \frac{2M^2}{\mu^2} + 2D_X^2\). Then the following holds for any \(t = 1, \ldots, j_{i,k}\):

\[ E \left[ \| z_{i,t} - \tilde{x}_i(y_k) \|^2 \left| F_k \right. \right] \leq Q_i/(t + 1) \quad a.s. \]

**Proof.** Set \(A_t = \| z_{i,t} - \tilde{x}_i(y_k) \|^2 \) and \(a_t = E \left[ A_{t+1} \right] \). Since \(X_i\) is a convex set and \(\tilde{x}_i(y_k) \in X_i\), from algorithm \((SA_{i,k})\) by the nonexpansive property of the projection operator we have

\[ A_{t+1} \leq A_t + \gamma_t^2 \| G_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \|^2 - 2\gamma_t (z_{i,t} - \tilde{x}_i(y_k))^T G_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \]

(22)

Since \(x_k\) is adapted to \(F_k\), by \((SA_{i,k})\) it is seen that \(z_{i,t}\) is adapted to \(\sigma(F_k, \xi_{i,k}^{[t-1]})\). Then by invoking \(E \left[ \psi_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \right] \) a.s., and the tower property of conditional expectation we have that for any \(t = 1, \ldots, j_{i,k}\):

\[ E \left[ (z_{i,t} - \tilde{x}_i(y_k))^T G_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \right] = E \left[ (z_{i,t} - \tilde{x}_i(y_k))^T G_i(z_{i,t}, y_{i,k}; \xi_{i,k}^t) \right] \]

(23)

Note that \(\tilde{f}_i(x_i) = f_i(x_i, y_{i,k}) + \frac{\mu}{2} \| x_i - y_{i,k} \|^2\) is strongly convex with parameter \(\mu\) and \(\tilde{x}_i(y_k)\) denotes the optimal solution of \(\min_{x_i \in X_i} \tilde{f}_i(x_i)\). Then by \(z_{i,t} \in X_i\), we may conclude that for any \(t = 1, \ldots, j_{i,k}\):

\[ (z_{i,t} - \tilde{x}_i(y_k))^T \nabla \tilde{f}_i(\tilde{x}_i(y_k)) \geq 0, \]

and \( (z_{i,t} - \tilde{x}_i(y_k))^T (\nabla \tilde{f}_i(z_{i,t}) - \nabla \tilde{f}_i(\tilde{x}_i(y_k))) \geq \mu \| z_{i,t} - \tilde{x}_i(y_k) \|^2 = \mu A_t. \]

Consequently, by adding the two inequalities we obtain the following:

\[ (z_{i,t} - \tilde{x}_i(y_k))^T (\nabla_x f_i(z_{i,t}, y_{i,k}) + \mu(z_{i,t} - y_{i,k})) = (z_{i,t} - \tilde{x}_i(y_k))^T \nabla \tilde{f}_i(z_{i,t}) \geq \mu A_t. \]
After taking conditional expectations and invoking (23), we obtain
\[ \mathbb{E} \left[ (z_{i,t} - \tilde{x}_i(y_k))^T G_i(z_{i,t}, y_k, \xi^i_{t,k}) | \mathcal{F}_k \right] \geq \mu \mathbb{E} [ A_i | \mathcal{F}_k ] = \mu a_t \text{ a.s.} \quad (24) \]

Since \( \|G_i(z_{i,t}, y_k, \xi^i_{t,k})\|^2 \leq 2\|\nabla_x \psi_i(z_{i,t}, x_{-i,k}, \xi^i_{t,k})\|^2 + 2\|\mu(z_{i,t} - y_k)\|^2 \), by Assumptions I(d) and (16), and by invoking \( \mathbb{E} \left[ \|\psi_i(z_{i,t}, y_{-i,k}, \xi^i_{t,k})\|^2 | \mathcal{F}_k, \xi^i_{t-1} \right] = \|\nabla_x f_i(z_{i,t}, y_{-i,k})\|^2 \text{ a.s.} \), we have that
\[ \mathbb{E} \left[ \|G_i(z_{i,t}, y_k, \xi^i_{t,k})\|^2 | \mathcal{F}_k \right] \leq 2M_i^2 + 2\|\mu(z_{i,t} - y_k)\|^2 \leq 2(M_i^2 + \mu^2 D_X^2) \text{ a.s.} \]
Then by the tower property of conditional expectations we obtain the following:
\[ \mathbb{E} \left[ \|G_i(z_{i,t}, y_k, \xi^i_{t,k})\|^2 | \mathcal{F}_k \right] \leq 2(M_i^2 + \mu^2 D_X^2) \text{ a.s.} \quad (25) \]

By taking expectations conditioned on \( \mathcal{F}_k \) on both sides of (22), from (24) and (25) we have
\[ a_{t+1} \leq (1 - 2\mu \gamma_t) a_t + 2(M_i^2 + \mu^2 D_X^2) \gamma_t^2 \text{ a.s.} \]

By substituting \( \gamma_t = \frac{1}{\mu(t+1)} \) we obtain \( a_{t+1} \leq (1 - \frac{2}{t+1}) a_t + \frac{Q_i}{(t+1)^2} \text{ a.s.} \), where \( Q_i \triangleq \frac{2M_i^2}{\mu^2} + 2D_X^2 \).

We inductively prove that \( a_t \leq \frac{Q_i}{(t+1)^2} \text{ a.s.} \). By (16) we have that \( A_1 \leq D_X^2 \), and hence \( a_1 \leq \frac{Q_i}{2} \). Suppose that \( a_t \leq \frac{Q_i}{(t+1)^2} \text{ a.s.} \) holds for some \( t \geq 1 \). It remains to show that \( a_{t+1} \leq \frac{Q_i}{(t+2)^2} \text{ a.s.} \). Note that
\[ a_{t+1} \leq \left( 1 - \frac{2}{t+1} \right) + \frac{Q_i}{(t+1)^2} \leq \frac{t}{(t+1)^2} \leq \frac{1}{t+2} \text{ a.s.,} \]
where the last inequality holds by the fact that \( t(t+2) \leq (t+1)^2 \). Consequently, we conclude that \( a_t \leq \frac{Q_i}{(t+1)^2} \text{ a.s.} \), and hence the lemma follows.

To address strategies that are an approximate Nash equilibrium, we use the concept of \( \epsilon \)-Nash equilibrium. A random strategy profile \( x : \Omega \rightarrow \mathbb{R}^n \) is an \( \epsilon \)-NE2 if
\[ \mathbb{E} \left( \left\| x_i - x_i^* \right\| \right) \leq \epsilon. \quad (26) \]

Next, we derive a bound on the overall iteration complexity for the synchronous inexact BR scheme.

**Theorem 1** Let Algorithm I be applied to the \( N \) -player stochastic Nash game (SNash), where \( \mathbb{E}[\|x_i,0 - x_i^*\|] \leq C \) and \( \alpha_{i,k} = \eta^{k+1} \) for some \( \eta \in (0,1) \). Assume that an inexact solution to the proximal BR problem given by (3) is computed via (SA_{e,k}). Suppose Assumptions I and II hold. Then the following hold.

(a) **(Overall iteration complexity)** Define \( c \triangleq \max \{ a, \eta \} \), let \( q \in (c,1) \), and \( D = 1/\ln((q/c)^c) \). Then the number of projected stochastic gradient steps for player \( i \) to compute an \( \epsilon \)-NE2 is no greater than
\[ \ell_i(\eta) \triangleq \frac{Q_i}{\eta^4 \ln(1/\eta^2)} \left( \frac{\sqrt{N}(C + D)}{\epsilon} \right)^{\ln(1/\eta^2) / \ln(1/q)} + \left[ \ln \left( \frac{\sqrt{N}(C + D)/\epsilon}{\ln(1/q)} \right) \right] \cdot (27) \]

(b) **(Bounds on complexity)** The expression (27) satisfies \( \ell_i(\eta) = \Omega(1/\epsilon^2) \). By setting \( \eta = a \) and \( D = \frac{1+2\delta^{-1}}{\epsilon \ln(a^{-1})} \), \( \ell_i(a) \) satisfies the following for any given \( \delta > 0 \):
\[ \ell_i(a) \leq \frac{Q_i}{a^4 \ln(1/a^2)} \left( \frac{\sqrt{N}(C + D)}{\epsilon} \right)^{2+\delta} + \ln \left( \frac{\sqrt{N}(C + D)/\epsilon}{2 \ln(a^{-1})} \right) \cdot (28) \]
Proof. (a) Suppose $j_{i,k} = \left\lceil \frac{Q_i}{\eta^2(k+1)} \right\rceil$ steps of the SA scheme ($SA_{i,k}$) are taken at major iteration $k$ to obtain $x_{i,k+1}$, i.e., $x_{i,k+1} = z_{i,j_{i,k}}$. Then by Lemma 3 and $j_{i,k} \geq \frac{Q_i}{\eta^2(k+1)}$, we have that

$$\mathbb{E} \left[ \|x_{i,k+1} - \bar{x}_i(y_k)\|^2 \right] = \mathbb{E} \left[ \|z_{i,j_{i,k}} - \bar{x}_i(y_k)\|^2 \right] \leq \frac{Q_i}{j_{i,k}} \leq \eta^{2(k+1)} = o_i^2, \quad \forall i \in \mathcal{N}.$$ 

Based on (19), we have that

$$u_{k+1} \leq \sqrt{N(C + D)} q^{k+1} \leq \epsilon \Rightarrow q^{k+1} \leq \frac{\epsilon}{\sqrt{N(C + D)}} \triangleq \bar{\epsilon} \Rightarrow k \geq \frac{\ln(1/\bar{\epsilon})}{\ln(1/q)} - 1. \quad (29)$$

By recalling that for $\beta > 1$, the following holds:

$$\sum_{k=1}^{K+1} \beta^k \leq \int_1^{K+1} \beta^x dx \leq \frac{\beta^{K+1}}{\ln(\beta)}, \quad (30)$$

which allows us to bound the overall iteration complexity of player $i$ as follows:

$$\sum_{k=0}^{\ln(1/\bar{\epsilon})/\ln(1/q)} (1 + \frac{Q_i}{\eta^2(k+1)}) = \sum_{k=1}^{\frac{\ln(1/\bar{\epsilon})}{\ln(1/q)}} \frac{Q_i}{\eta^2k} + \left\lceil \frac{\ln(1/\bar{\epsilon})}{\ln(1/q)} \right\rceil \leq \frac{Q_i}{\ln(1/\bar{\epsilon})} \eta^{-2\left(\frac{\ln(1/\bar{\epsilon})/\ln(1/q)}{\ln(1/\bar{\epsilon})}\right)^2} + \left\lceil \frac{\ln(1/\bar{\epsilon})}{\ln(1/q)} \right\rceil.$$ 

Note that

$$\eta^{-2\ln(1/\bar{\epsilon})/\ln(1/q)} = (e^{\ln(\eta^{-2})})^{\frac{\ln(1/\bar{\epsilon})}{\ln(1/q)}} = e^{\ln(1/\bar{\epsilon})} \frac{\ln(\eta^{-2})}{\ln(1/q)} = (1/\bar{\epsilon})^{\frac{\ln(1/\bar{\epsilon})}{\ln(1/q)}}.$$ 

Therefore, the overall iteration complexity of player $i$ is bounded by the following:

$$\frac{Q_i}{\eta^4 \ln(1/\bar{\epsilon}^2)} \left( 1 + \frac{\ln(1/\bar{\epsilon})}{\ln(1/q)} \right) + \left\lceil \frac{\ln(1/\bar{\epsilon})}{\ln(1/q)} \right\rceil.$$ 

Then the result (a) follows by the definition of $\bar{\epsilon}$ in (29).

(b). The proof of the first assertion $\ell_i(\eta) = O(1/\epsilon^2)$ can be found in [50, Proposition 4(b)].

Define $q = ce^{\delta_0/2}$, where $\delta_0 = \frac{\delta \ln(a^{-1})}{1 + \delta/2}$. Since $c = a$ by $\eta = a$, we obtain the following:

$$\ln(1/q) = \ln(a^{-1}) - \delta_0/2 = \frac{2 \ln(a^{-1})}{2 + \delta} \leq \frac{2 \ln(a^{-1})}{2} = 2 + \frac{\delta a}{\ln(a^{-1})} = 2 + \delta,$$

and

$$D = 1/\ln((q/c)^{\delta_0}) = 1/\ln(c^{\delta_0}) = 2 + \frac{2 \delta^{-1}}{e \ln(a^{-1})}.$$ 

Consequently, by the expression $\ell_i(\eta)$ defined in (27), we obtain (28) when $\eta$ is set as $a$. \hfill \Box

Remark 3 Several points need reinforcement in light of the obtained complexity statements:

(i) First, the iteration complexity bound grows slowly in the number of players, a desirable feature of any distributed algorithm employed for games with a large collection of agents.

(ii) Second, when $N = 1$, the complexity bound reduces to $\mathcal{O}(1/\epsilon^2 + \delta)$ which can be made arbitrarily close to the bound for standard stochastic convex programs suggesting that the bounds may well be optimal.

(iii) Third, in an effort to examine the tightness of the bound, we show that our bound on the derived iteration complexity is bounded from below by a term that is of the order $\mathcal{O}(N/\epsilon^2)$. Note this is not a lower bound on the complexity itself which we leave as future research.
Corollary 1  Let Algorithm 1 be applied to the $N$–player stochastic Nash game (SNash), where $\alpha_{i,k} = \eta^{k+1}$ for some $\eta \in (a, 1)$, $\mathbb{E}[\|x_{i,0} - x_i\|] \leq C$, and an inexact proximal BR solution satisfying (8) is computed via (SA$_i$, $k$). Suppose Assumptions 1 and 2 hold. Then the number of projected gradient steps for player $i$ to compute an $\epsilon$–NE is no greater than

$$Q_i \eta^k \ln(1/\eta^2) \left( \frac{\sqrt{N} \left( C + \frac{\eta}{\eta-a} \right)}{\epsilon} \right)^2 + \left[ \ln \frac{\sqrt{N} \left( C + \frac{\eta}{\eta-a} \right)}{\ln(1/\eta)} \right].$$

(32)

Proof. Since $\frac{a}{\eta} \in (0, 1)$ by $\eta \in (a, 1)$, we have that

$$\sum_{j=1}^{k} a^{k-j} \eta^j = \eta^k \sum_{j=0}^{k-1} (a/\eta)^j \leq \frac{1 - (a/\eta)^k}{1 - a/\eta} \eta^k = 1 - a/\eta.$$

Then by $u_0 \leq \sqrt{NC}, a < \eta$, and by invoking the second inequality in (21), we obtain the following bound:

$$u_k \leq \sqrt{NCa^k} + \sqrt{N}\eta \left( \eta^k - a^k \right) \leq \left( \sqrt{NC} + \sqrt{N}\eta \right) \eta^k.$$

Thus, by setting $\tilde{k} = \left[ \frac{\ln(1/\epsilon)}{\ln(1/\eta)} \right]$ with $\bar{c} = \frac{\epsilon}{\sqrt{N}(C + \eta/(\eta - a))}$, we get $u_{k+1} \leq \epsilon \forall k \geq \tilde{k} - 1$. Suppose $j_{i,k} = \left[ \frac{Q_i}{\eta^k (1 + \beta)} \right]$ steps of (SA$_i$, $k$) are taken at major iteration $k$ to obtain $x_{i,k+1}$. Then by Lemma 3, we have that $\mathbb{E} \left[ \|x_{i,k+1} - \bar{x}_i(y_k)\|^2 | F_k \right] \leq \alpha_{i,k}^2$ for all $i \in \mathcal{N}$. By invoking (30) for $\beta > 1$, we may bound the iteration complexity of player $i$ as follows:

$$\sum_{k=0}^{\ln(1/\epsilon) / \ln(1/\eta)} j_{i,k} = \sum_{k=1}^{\ln(1/\epsilon) / \ln(1/\eta)} \frac{Q_i}{\eta^k} + \left[ \ln(1/\epsilon) \right] \leq \frac{Q_i}{\ln(1/\epsilon)} \eta^{-2 \ln(1/\epsilon) \ln(1/\eta)^2} \left[ \ln(1/\epsilon) \right]$$

which results in (32) by utilizing (31) and by recalling $\bar{c} = \frac{\epsilon}{\sqrt{N}(C + \eta/(\eta - a))}$. \hfill $\square$

Remark 4  It seems that Corollary 1 gives better results than those of Theorem 1 since it shows that the iteration complexity is exactly of $\mathcal{O}(1/\epsilon^2)$ instead of $\mathcal{O}(1/\epsilon^{2+\delta})$. However, Corollary 1 restricts the parameter $\eta$ to be $\eta \in (a, 1)$ while Theorem 1 provides more flexibility on the selection of $\eta$ only by requiring $\eta \in (0, 1)$. As a matter of fact, the numerical results demonstrated in Table 3 indicate that smaller $\eta \in (0, 1)$ may give better empirical iteration complexity. In the following two sections, we establish the iteration complexity for the randomized and the asynchronous algorithms through the same path of Theorem 1.

We now provide a related result for a probabilistically $\epsilon$-Nash equilibrium. Specifically, a random strategy profile $x : \Omega \rightarrow \mathbb{R}^n$ is an $\epsilon_P\delta$–NE if

$$\mathbb{P} \left( \omega : \left( \begin{array}{c} \|x_1(\omega) - x_1^*\| \\
\vdots \\
\|x_N(\omega) - x_N^*\| \end{array} \right) \leq \epsilon \right) \geq (1 - \delta).$$

Corollary 2  Let Algorithm 1 be applied to the $N$–player stochastic Nash game (SNash), where $\alpha_{i,k} = \eta^{k+1}$ for some $\eta \in (a, 1)$, $\mathbb{E}[\|x_{i,0} - x_i^*\|] \leq C$, and an inexact proximal BR solution satisfying (8) is computed via (SA$_i$, $k$). Suppose Assumptions 1 and 2 hold. Then the iteration complexity for player $i$ to compute an $\epsilon_P\delta$–NE is bounded by (28) with $\epsilon$ replaced by $\epsilon_P\delta$.

Note. This follows because $u_k \leq \epsilon\delta$ implies that $\mathbb{P}(v_k \geq \epsilon) \leq \mathbb{E}[v_k]/\epsilon = u_k/\epsilon \leq \delta$ by the Markov inequality, where $v_k$ and $u_k$ are defined by (12) and (18), respectively. Then the $\epsilon_P\delta$–NE $x$ is also an $\epsilon_P\delta$–NE solution.
4 A Randomized Inexact Proximal BR Algorithm

In this section, we propose a randomized inexact proximal BR algorithm, where a subset of players is randomly chosen to update their strategies in each major iteration, and the inexact proximal BR solutions are achieved via the SA scheme.

4.1 Algorithm description

The randomized block coordinate descent method was proposed in [34] to solve large-scale optimization problems, where the coordinates are partitioned into several blocks and at each iteration only one block of variables is randomly chosen to update while the other blocks are kept invariant. Subsequently, this convergence was analyzed to nonconvex and fixed-point regimes [14, 31, 34, 40]. Motivated by these research, we design a randomized inexact proximal BR scheme as follows: For any \( i \in N \), let \( \{\chi_{i,k}\}_{k \geq 0} \) be a sequence of i.i.d Bernoulli random variables taking values in \( \{0, 1\} \). The variable \( \chi_{i,k} \) signals whether the \( i \)th player updates at major iteration \( k \): if \( \chi_{i,k} = 1 \), then player \( i \) initiates an update at major iteration \( k \) and computes an inexact proximal BR solution satisfying (33), where \( \mathcal{F}_k \) is the \( \sigma \)-field of the entire information used by the algorithm up to (and including) the update of \( x_k \), and \( \alpha_{i,k} \) is a nonnegative random variable adapted to \( \mathcal{F}_k \). It is worth noticing that \( \mathcal{F}_k \) contains the sample \( \chi_{i,l} \) for any \( i \in N \) and \( l : 0 \leq l \leq k - 1 \), and \( \alpha_{i,k} \) may also depend on the samples \( \{\chi_{i,l}\}_{0 \leq l \leq k-1} \). We define \( \mathcal{F}_k \) in Section [1.3] and specify the selection of the sequence \( \{\alpha_{i,k}\}_{k \geq 0} \) when we proceed to investigate the convergence properties for which we need the following condition in the convergence analysis.

Algorithm 2 Randomized inexact proximal BR scheme

Let \( k := 0 \), \( y_{i,0} = x_{i,0} \in X_i \) for \( i = 1, \ldots, N \).

1. If \( \chi_{i,k} = 1 \), then player \( i \) updates \( x_{i,k+1} \) satisfying the following:

\[
x_{i,k+1} \in \{ z \in X_i : \mathbb{E} \left[ ||z - \hat{x}_i(y_k)||^2 | \mathcal{F}_k \right] \leq \alpha_{i,k}^2 \text{ a.s.} \},
\]

where \( \hat{x}_i(y_k) \) is defined in (7). Otherwise, \( x_{i,k+1} = x_{i,k} \).

2. For \( i = 1, \ldots, N \), \( y_{i,k+1} := x_{i,k+1} \).

3. \( k := k + 1 \); If \( k < K \), return to (1); else STOP.

Assumption 3 For any \( i \in N \), \( \mathbb{P}(\chi_{i,k} = 1) = p_i > 0 \) and \( \chi_{i,k} \) is independent of \( \mathcal{F}_k \).

Remark 5 We make the following clarifications on Assumption 3:

(i) The condition \( \mathbb{P}(\chi_{i,k} = 1) = p_i > 0 \) guarantees that each player initiates an update with positive probability at major iteration \( k \) of Algorithm 2. It accommodates the special case where only one player is randomly chosen with positive probability to update in each iteration.

(ii) The Poisson model [2, 11] is a special case: Each player \( i \in N \) is activated according to a local Poisson clock, which ticks according to a Poisson process with rate \( \varrho_i > 0 \). Suppose that there is a virtual global clock which ticks whenever any of the local Poisson clocks tick. Suppose that the local Poisson clocks are independent, then the global clock ticks according to a Poisson process with rate \( \sum_{i=1}^{N} \varrho_i \). Let \( Z_k \) denote the time of the \( k \)-th tick of the global clock, and \( I_k \in \{1, \ldots, N\} \) denote the set of players whose clocks tick at time \( Z_k \). Since the local Poisson clocks are independent, with probability one, \( I_k \) contains a single element and \( \mathbb{P}(I_k = i) = \frac{\varrho_i}{\sum_{i=1}^{N} \varrho_i} \triangleq p_i \). Besides, the memoryless property of the Poisson process indicates that the process \( \{I_k\}_{k \geq 0} \) is i.i.d. As a result, the processes \( \{\chi_{i,k}\}_{i \in N} \) are mutually independent, and for each \( i \in N \), \( \{\chi_{i,k}\} \) is an i.i.d sequence with \( \mathbb{P}(\chi_{i,k} = 1) = p_i > 0 \) and \( \sum_{i=1}^{N} p_i = 1 \).
4.2 Convergence analysis

We now establish the almost sure convergence and the geometric rate of convergence of the \( \{x_k\} \) to \( x^\ast \) under suitable conditions on the inexactness sequences \( \{\alpha_{i,k}\}, i \in \mathcal{N} \).

**Lemma 4** (a.s. convergence) Let \( \{x_k\}_{k \geq 0} \) be generated by Algorithm 2. Suppose Assumptions 1-3 hold. For any \( i \in \mathcal{N}, 0 \leq \alpha_{i,k} < 1 \) and \( \sum_{k=0}^{\infty} \alpha_{i,k} < \infty \) a.s. Then for any \( i \in \mathcal{N} \), \( \lim_{k \to \infty} x_{i,k} = x_i^\ast \) a.s.

**Proof.** See Appendix A.

Define \( \beta_{i,0} = 0 \) and \( \beta_{i,k} = \sum_{p=0}^{k-1} \chi_{i,p} \) for all \( k \geq 1 \). Thus, \( \beta_{i,k} \) is adapted to \( \mathcal{F}_k \). Note that for the Poisson model described in Remark 5, players may not able to know the exact number of ticks of the global clock. Then it is impractical to set \( \alpha_{i,k} \) be a function of \( k \), instead, a function of \( \beta_{i,k} \) is appropriate.

**Lemma 5** (Geometric Convergence) Let \( \{x_k\}_{k \geq 0} \) be generated by Algorithm 2 where \( \mathbb{E}[\|x_{i,0} - x_i^\ast\|] \leq C \) and \( \alpha_{i,k} = \eta^{\beta_{i,k}+1} \) for some \( \eta \in (0, 1) \). Suppose Assumptions 1-3 hold. Define \( \bar{c} \equiv \max \{ \bar{a}, \bar{\eta} \} \) with \( \bar{a} \) and \( \bar{\eta} \) defined by (B.1) and (B.3), respectively. Then the following holds for any \( q \in (\bar{c}, 1) \):

\[
\mathbb{E}[\|x_k - x^\ast\|_P] \leq \sqrt{N(\bar{C} + \bar{D})q^k} \quad \forall k \geq 0,
\]

where \( \| \cdot \|_P \) is defined in (A.2), \( \bar{D} \equiv 1/\ln((q/\bar{c})^q) \), \( \bar{C} = C \left( \sum_{i=1}^{N} N^{-1} p_i^{-1} \right)^{1/2} \), and \( \bar{D} = D\bar{\eta}^{-1} \).

**Proof.** See Appendix B.

**Remark 6** By \( \| \cdot \|_P \) defined in (A.2), we get \( \|x_k\|_P \geq \frac{1}{p_{\text{max}}} \sum_{i=1}^{N} \|x_i\|^2 \), where \( p_{\text{max}} = \max_{i \in \mathcal{N}} p_i \). Thus,

\[
\|x_k - x^\ast\|_P \geq \frac{1}{p_{\text{max}}} \sum_{i=1}^{N} \|x_{i,k} - x_i^\ast\|^2 = \frac{1}{p_{\text{max}}} v_k,
\]

where \( v_k \) is defined in (12). Then by Lemma 3 and \( u_k \) is defined in (18), we have that \( u_k \leq (p_{\text{max}})^{1/2} \mathbb{E}[\|x_k - x^\ast\|_P] \leq (Np_{\text{max}})^{1/2}(\bar{C} + \bar{D})q^k \).

4.3 Overall iteration complexity

We proceed to estimate the iteration complexity, where the inexact proximal BR solution is computed via a SA scheme. At the major iteration \( k \): if \( \chi_{i,k} = 1 \), then player \( i \) takes SG steps (SA_{i,k}) from \( t = 1, \ldots, j_i,k \). We define \( \xi_{i,k}, I_k, \mathcal{F}_k \) as \( \xi_{i,k} \equiv (\xi_i^{(1)}, \cdots, \xi_i^{(j_i,k)}) \), \( I_k \equiv \{ i \in \mathcal{N} : \chi_{i,k} = 1 \} \), and \( \mathcal{F}_k \equiv \sigma \{ x_0, \xi_i^{(i)}, \chi_{i,k} \}_{i \in I_k}, \chi_{i,k} \}_{i \in \mathcal{N}, 0 \leq l \leq k - 1} \), respectively. Then, by Algorithm 2, \( x_k \) is adapted to \( \mathcal{F}_k \). Similar to the proof of Lemma 3 we obtain the following result for the SA scheme (SA_{i,k}).

**Lemma 6** Let Assumption 1 hold. Assume that for any \( i \in I_k \) the random variables \( \{\xi_i^{(t)}\}_{1 \leq t \leq j_i,k} \) are i.i.d. and, in addition, that the random vector \( \xi_{i,k} \) is independent of \( \mathcal{F}_k \). Then for any \( i \in I_k \) and any \( t \geq 1 \) we have that \( \mathbb{E}[\|x_{i,t} - \bar{x}i(y_k)\|^2 | \mathcal{F}_k] \leq Q_i/(t + 1) \) a.s.

**Theorem 2** Let the sequence \( \{x_k\}_{k \geq 0} \) be generated by Algorithm 2 where \( \alpha_{i,k} = \eta^{\beta_{i,k}+1} \) for some \( \eta \in (0, 1) \) and \( \mathbb{E}[\|x_{i,0} - x_i^\ast\|] \leq C \). Suppose that an inexact solution characterized by (33) is computed via (SA_{i,k}). Let Assumptions 1-3 hold. Suppose \( \bar{a}, \bar{\eta} \) and \( \bar{\eta}_0 \) are defined by (B.1), (B.3) and (36). Define \( \bar{c} \equiv \max \{ \bar{a}, \bar{\eta} \} \) and let \( \bar{q} \in (\bar{c}, 1) \). Then expectation of the number of projected gradient steps for player \( i \) to compute an \( \epsilon - \text{NE} \) is no greater than

\[
\bar{\ell}_i(\eta) = \frac{p_i Q_i}{\eta^2 \bar{\eta}_0 ^2 \ln(1/\bar{\eta}_0^2)} \left( \frac{\ln(1/\bar{\eta}_0^2)}{\ln(1/\bar{\eta}_0)} \right) + \left( \frac{\ln(1/\bar{c})}{\ln(1/\bar{q})} \right),
\]

(34)
where \( \tilde{\epsilon} \triangleq \frac{\epsilon}{(Np_{\text{max}})^{1/2}(C + D)} \) with \( \tilde{C} = C \left( \sum_{i=1}^{N} N^{-1} p_i^{-1} \right)^{1/2} \) and \( \tilde{D} = D \eta^{-1} \) with \( D \triangleq \frac{1}{\ln((\tilde{q}/\tilde{c})^e)} \). If \( \eta = a \), then given any \( \delta > 0 \), \( \tilde{\epsilon}_i(\eta) \) satisfies the following upper bound:

\[
\tilde{\epsilon}_i(\eta) \leq \frac{p_i Q_i}{\eta^2 \eta^2 \ln(1/\eta^2)} \left( \frac{1}{\tilde{\epsilon}} \right)^{2 \ln(\tilde{\eta}_0^{-1})/\ln(\tilde{\eta}) + \delta} + \left[ \ln(1/\tilde{\epsilon}) \left( \frac{1}{\ln(\tilde{\eta})} + \frac{\delta}{2 \ln(\tilde{\eta}_0^{-1})} \right) \right].
\]

(35)

**Proof.** Suppose \( j_{i,k} = \left[ \frac{Q_i}{\eta^{\epsilon(i_{i,k} + 1)}(\tilde{\eta}_0^{-1})} \right] \) steps of the SA scheme (SA\(_{i,k}\)) are taken at major iteration \( k \) to obtain \( x_{i,k+1} \), i.e., \( x_{i,k+1} = z_{i,j_{i,k},k} \). Then by Lemma 6, it follows that

\[
\mathbb{E} \left[ \|x_{i,k+1} - \bar{x}_i(y_k)\|^2 | F_k \right] = \mathbb{E} \left[ \|z_{i,j_{i,k},k} - \bar{x}_i(y_k)\|^2 | F_k \right] \leq \frac{Q_i}{\eta_{j_{i,k}}^2} \leq \alpha_{i,k}^2 \forall i \in \mathcal{N}.
\]

By Remark 6 we have that

\[
u_{k+1} \leq (Np_{\text{max}})^{1/2}(C + D) \eta^k \leq \epsilon \Rightarrow k \geq \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} - 1,
\]

where \( \tilde{\epsilon} \triangleq \frac{\epsilon}{(Np_{\text{max}})^{1/2}(C + D)} \). Similar to (B.3) we have that

\[
\mathbb{E}[\eta^{-2\beta_{i,k}}] = (p_i \eta^{-2} + 1 - p_i)^k = (p_i (\eta^{-2} - 1) + 1)^k \leq (p_{\text{max}}(\eta^{-2} - 1) + 1)^k \triangleq \tilde{\eta}_0^{-2k} \forall i \in \mathcal{N}.
\]

(36)

Then by invoking (B.5) for \( \beta > 1 \), the expectation of the overall iteration complexity of player \( i \) is bounded by the following:

\[
\mathbb{E} \left[ \sum_{k=0}^{\left[ \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} \right] - 1} j_{i,k} \chi_{i,k} \right] = \sum_{k=0}^{\left[ \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} \right] - 1} \mathbb{E} \left[ j_{i,k} \right] \mathbb{E} \left[ \chi_{i,k} \right] \quad \text{(since } \chi_{i,k} \text{ is independent of } j_{i,k})
\]

\[
\leq \sum_{k=0}^{\left[ \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} \right] - 1} \left( \frac{p_i Q_i}{\tilde{\eta}_0^{2\eta_0^2} \eta^2} + 1 \right) \frac{\eta_0^2}{\eta^2} \sum_{k=1}^{\left[ \frac{\ln(1/\tilde{\epsilon})}{\ln(1/\tilde{q})} \right]} \frac{p_i Q_i}{\tilde{\eta}_0^{2\eta_0^2} \eta^2} \ln(1/\tilde{\eta}) \leq \frac{2}{\eta_0^2} \frac{p_i Q_i}{\tilde{\eta}_0^{2\eta_0^2}} \left( \frac{1}{\ln(1/\tilde{\epsilon})} \right) + \left[ \ln(1/\tilde{\epsilon}) \left( \frac{1}{\ln(1/\tilde{\eta})} + \frac{\delta}{2 \ln(\tilde{\eta}_0^{-1})} \right) \right],
\]

which results in (34) by (31). If \( \eta = a \), then by \( \tilde{a} \) and \( \tilde{\eta} \) defined in (B.1) and (B.3), we have that \( \tilde{a} = \tilde{\eta} \) and \( \tilde{\epsilon} = \tilde{a} \). Define \( \tilde{q} = \tilde{\eta} e^{\delta_0/2} \), where \( \delta_0 = \frac{\delta}{\ln(\tilde{\eta}_0^{-1})} \). Then we obtain the following:

\[
\ln(1/\tilde{q}) = \ln(\tilde{\eta}) - \delta_0/2 = \frac{\ln(\tilde{\eta}_0^{-1})}{\ln(\tilde{\eta}_0^{-1})/\ln(\tilde{\eta}) + \delta/2} = \frac{1}{1/\ln(\tilde{\eta}) + \delta/2 \ln(\tilde{\eta}_0^{-1})},
\]

\[
\ln(1/\tilde{q}^2) = \frac{2 \ln(\tilde{\eta}_0^{-1})}{\ln(1/\tilde{q})} = \frac{2 \ln(\tilde{\eta}_0^{-1})}{\ln(\tilde{\eta})} + \delta,
\]

and \( D = 1/\ln((\tilde{q}/\tilde{c})^e) = 1/\ln(e^{\delta_0/2}) = \frac{2}{\delta_0 e} = \frac{\ln(\tilde{\eta}_0^{-1})/\ln(\tilde{\eta}) + 2\delta^{-1}}{e \ln(\tilde{\eta})}. \)

(37)

Consequently, the result (35) follows by the expression \( \tilde{\epsilon}_i(\tilde{\eta}) \) defined in (34).

\( \Box \)

**Remark 7** We make the following illustrations on Theorem 2:

(i) If \( p_i = 1 \forall i \in \mathcal{N} \), then \( Q = Q \) and \( p_{\text{min}} = 1 \). Thus, from (B.1), (B.3), (36), we see that \( \tilde{a} = a, \tilde{\eta} = \eta, \tilde{\eta}_0 = \eta, \) and hence \( \tilde{C} = C, \tilde{D} = D \). As a result, the results of Lemma 5 and Theorem 2 for the randomized scheme reduce to those given in Proposition 4 and Theorem 7 for the synchronous scheme, respectively.
Note that the iteration complexity in Theorem 2 of the randomized algorithm is described via the expected number of projected gradient steps, since the number of gradient steps to get an inexact solution is a random variable dependent on the realization of updates in the associated trajectory.

By definitions (B.3) and (36) we have that

\[
(\hat{\eta}/\eta_0)^2 = 1 + p_{\text{max}}(\eta^{-2} - 1)p_{\text{min}} \left( \frac{1}{p_{\text{min}}} - \frac{\eta^2}{p_{\text{max}}} - (1 - \eta^2) \right) \geq 1 + p_{\text{max}}(1 - p_{\text{min}})\eta^{-2}(1 - \eta^2)^2 \geq 1,
\]

where the equality holds only if \( p_{\text{min}} = 1 \). Thus, for the case \( p_{\text{min}} < 1 \), the complexity bound is greater than the bounds derived in Proposition 4 and Theorem 7, a consequence of the cost of randomization.

5 An Asynchronous Inexact Proximal BR Algorithm

Asynchronous methods date back to [12] when they were employed for the solution to systems of linear equations. Subsequently, they were used in optimization problems, in which a partially asynchronous gradient projection algorithm is proposed in [9], while the convergence rate is analyzed in [52]. Here, we adapt the scheme developed in [9] to stochastic Nash games and propose an asynchronous inexact proximal BR algorithm. Recall that in [9], the \( \infty \) norm is utilized in the rate analysis of the asynchronous schemes for problems with maximum norm contraction mappings. By assuming that the proximal BR map is contractive in the \( \infty \) norm, we obtain a geometric rate of convergence for an appropriately chosen inexactness sequence, establish the overall iteration complexity in terms of the number of projected gradient steps, and analyze the associated complexity bound.

5.1 Algorithm design

The synchronous algorithm designed in the previous section requires that all players update their strategies simultaneously. In a network with a large collection of noncooperative players, players might not be able to make simultaneous updates nor may they have access to their rivals' latest information. In this context, we propose the following asynchronous scheme. Let \( T = \{0, 1, 2, \cdots \} \) be a set of epochs at which one or more players update their strategies. Denote by \( I_k \subset N \) the set of players which update their strategies at time \( k \). For any \( i \in I_k \), player \( i \) may not obtain its rivals’ latest information, instead, the outdated data \( y_k^i = (x_{1,k-\tau_{i1}(k)}, \cdots, x_{N,k-\tau_{iN}(k)}) \) is available to player \( i \), where \( \tau_{ij}(k) \leq k, j = 1, \cdots, N \) are random nonnegative integers representing communication delays from player \( j \) to player \( i \) at time \( k \). Set \( \tau_{ii}(k) = 0 \) without loss of generality.

The following Algorithm 3 presents the asynchronous inexact proximal BR scheme, where \( F_k = \sigma \{ F'_k, \tau_{ij}(k) \}_{i \in I_k, j \in N} \) with \( F'_k \) being the \( \sigma \)-field of the entire information employed by the algorithm up to (and including) the update of \( x_k \). It is worth noticing that \( y_k^i \) is adapted to \( F_k \) while is not adapted to \( F'_k \) since \( y_k^i \) depends on \( \tau_{ij}(k) \). We define \( F_k \) in Section 5.3 while the sequence \( \{\alpha_{i,k}\} \) will be specified when analyzing the convergence properties of Algorithm 3.

**Algorithm 3** Asynchronous inexact proximal BR scheme

Let \( k := 0, x_{i,0} \in X_i \) for \( i \in N \).

1. For any \( i \in I_k \), set \( y_k^i = (x_{1,k-\tau_{i1}(k)}, \cdots, x_{N,k-\tau_{iN}(k)}) \).

2. If \( i \in I_k \), then player \( i \) updates \( x_{i,k+1} \) that satisfies the following:

\[
x_{i,k+1} \in \{ z \in X_i : \mathbb{E} \left[ \| z - \hat{x}_i(y_k^i) \|^2 \mid F_k \right] \leq \alpha_{i,k}^2 \text{ a.s.} \},
\]

where \( \hat{x}_i(y_k^i) \) is defined in (7) with \( y_k \) replaced by \( y_k^i \). Otherwise, \( x_{i,k+1} = x_{i,k} \).

3. \( k := k + 1 \); If \( k < K \), return to (1); else STOP.
We impose the following assumptions on the asynchronous protocol, and on the parameters $\zeta_{i,\min}, \zeta_{ij,\max}, i, j = 1, \cdots, N$ that are defined in (4).

**Assumption 4**  (a) The sequence of sets $\{I_k\}_{k \geq 0}$ is deterministic.
(b) Each player $i \in N$ updates its strategy at least once during any time interval of length $b_i$. In addition, there exists a positive integer $B_1$ such that $b_i \leq B_1 \forall i \in N$;
(c) There exists a random variable $\tau_{ij}$ such that $\tau_{ij}(k) \leq \tau_{ij} \forall k \geq 0, i \in I_k, j \in N$ a.s. Furthermore, there exists a nonnegative integer $B_2$ such that $\tau_{ij} \leq B_2 \forall i, j \in N$ a.s.

By Assumption (b) we see that for any $k \geq 0$ and any $i \in I_k, j \in N$

\[
\max\{0, k - B_2\} \leq \max\{0, k - \tau_{ij}\} \leq k - \tau_{ij}(k) \leq k \quad \text{a.s.}
\]  \hspace{1cm} (39)

**Assumption 5**  (Strict Diagonal Dominance) For any $i = 1, \cdots, N$, $\zeta_{i,\min} > \sum_{j \neq i} \zeta_{ij,\max}$.

Then from this assumption by $\Gamma$ defined in (3), we have that $a_{\infty} \triangleq \|\Gamma\|_{\infty} < 1$. Let $y : \Omega \rightarrow X$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. Then by taking expectations on both sides of (5), and by taking the infinity norm, we have that

\[
\left\| \left( \begin{array}{c}
\mathbb{E}[\|\hat{x}_1(y) - \hat{x}_1(x^*)\|]\n\vdots
\mathbb{E}[\|\hat{x}_N(y) - \hat{x}_N(x^*)\|]
\end{array} \right) \right\|_{\infty} \leq a_{\infty} \left\| \left( \begin{array}{c}
\mathbb{E}[\|y_1 - x_1^*\|]\n\vdots
\mathbb{E}[\|y_N - x_N^*\|]
\end{array} \right) \right\|_{\infty}.
\]

As a result, by $x^*_i = \hat{x}_i(x^*)$ we see that for any $i = 1, \cdots, N$:

\[
\mathbb{E}[\|\hat{x}_i(y) - x_i^*\|] = \mathbb{E}[\|\hat{x}_i(y) - \hat{x}_i(x^*)\|] \leq a_{\infty} \max_{i \in N} \mathbb{E}[\|y_i - x_i^*\|].
\]  \hspace{1cm} (40)

### 5.2 Rate of convergence

Denote by $\beta_{i,k}$ the number of updates player $i$ has carried out up to (and including) major iteration $k$.

**Lemma 7**  Let the asynchronous inexact proximal BR scheme (Algorithm 3) be applied to the $N$-player stochastic Nash game (SNash), where $\alpha_{i,k} = \eta^{\beta_{i,k}}$ for some scalar $\eta \in (0, 1)$ and $\mathbb{E}[\|x_{i,0} - x_i^*\|] \leq C$. Suppose Assumptions 4 and 5 hold. Then we have the following for any $k \geq 0$:

\[
\max_{i \in N} \mathbb{E}[\|x_{i,k} - x_i^*\|] \leq (C + k)\rho^{\frac{\beta_{i,k}}{B_i}},
\]  \hspace{1cm} (41)

where $\rho = (\max\{a_{\infty}, \eta\})^{1/(n_0 + 1)}$ with $n_0 \triangleq \left\lceil \frac{B_2}{B_1} \right\rceil$. Furthermore, if $q > c \triangleq \rho^{\frac{1}{B_1}}$, and $D \geq 1/\ln((q/c)^c)$,

\[
\max_{i \in N} \mathbb{E}[\|x_{i,k} - x_i^*\|] \leq \rho^{-\frac{n_0 - 1}{B_1} + (C + D)q^k}, \quad \forall k \geq 0.
\]  \hspace{1cm} (42)

**Proof.** See Appendix C

### 5.3 Overall iteration complexity analysis

We proceed to derive a bound on the overall iteration complexity, when the inexact proximal BR solution is computed via SA. At major iteration $k$, if $i \in I_k$, then given $y^t_k$, the $i$th player takes the following sequence of stochastic gradient steps from $t = 1, \cdots, I_k$:

\[
z_{i,t+1} := X_i \left[ z_{i,t} - \gamma_t \left( \nabla x_i \psi_i(z_{i,t}, y^t_{i,k}; \xi_{i,k}) + \mu(z_{i,t} - x_{i,k}) \right) \right],
\]  \hspace{1cm} (43)

where $z_{i,1} = x_{i,k}$, and $\gamma_t = 1/\mu(t + 1)$. Define $\mathcal{F}^l_k \triangleq \sigma\{x_0, \xi_{i,l}, \{\tau_{i}(l)\}_{l \in I_k}, 0 \leq l \leq k - 1\}$ and $\mathcal{F}_k \triangleq \sigma\{\mathcal{F}^l_k, \{\tau_{i}(k)\}_{i \in I_k}\}$, where $\tau_{i}(k) \triangleq \tau_{i,1}(k) \cdots \tau_{i,N}(k)$. Then by Algorithm 3 we see that $x_k$ is adapted to $\mathcal{F}_k$ while $\{y^t_k\}_{t \in I_k}$ is adapted to $\mathcal{F}_k$. Since $\mathcal{F}^l_k \subset \mathcal{F}_k$, similar to Lemma 3, we also have the following result for the scheme (43).
Lemma 8  Let Assumption 1 hold. Consider the asynchronous inexact proximal BR scheme given by Algorithm 3. Assume that for any $i \in I_k$ and any $t = 1, \ldots, j_{i,k}$, $\mathbb{E}\left[\psi_i(z_{i,t}, y_{i,t}^j; \xi_t^{i,k})\sigma\{\mathcal{F}_k, \xi_t^{i,k}\} \right] = \nabla x_i f_i(z_{i,t}, y_{i,t}^j)$ a.s., and $\mathbb{E}\left[\|\psi_i(z_{i,t}, y_{i,t}^j; \xi_t^{i,k})\|^2 \sigma\{\mathcal{F}_k, \xi_t^{i,k}\} \right] = \|\nabla x_i f_i(z_{i,t}, y_{i,t}^j)\|^2$ a.s. Then for any $i \in I_k$ and any $t = 1, \ldots, j_{i,k}$, we obtain that $\mathbb{E}[\|z_{i,t} - \hat{x}_i(y_t^j)\|^2 | \mathcal{F}_k] \leq Q_i/(t + 1)$ a.s.

Distinct from the definition of $\epsilon-$Nash equilibrium given in (26), for the asynchronous algorithm a random strategy profile $x : \Omega \rightarrow \mathbb{R}^n$ is called an $\epsilon-$NE when

$$\max_{i \in N} \mathbb{E}[\|x_i - x_i^\ast\|] \leq \epsilon.$$  (44)

Theorem 3  Let Algorithm 3 be applied to the stochastic Nash game (SNash), where $\alpha_{i,k} = \eta^{\beta_{i,k}}$ for some scalar $\eta \in (0, 1)$ and $\mathbb{E}[\|x_{0i} - x_i^\ast\|^2] \leq C^2$. Suppose that an inexact solution characterized by (38) is computed via (SA$_{i,k}$). Let Assumptions 1, 4 and 5 hold. Suppose $\rho = (\max\{a_\infty, \eta\})^{1/(n_0 + 1)}$ with $n_0 = \left[\frac{B_2}{B_1}\right]$. Define $c \triangleq \rho^{\frac{1}{B_1}}$ and let $q \in (c, 1)$. Then the number of projected gradient steps for player $i$ to compute an $\epsilon-$NE is no greater than

$$\hat{\ell}^{(1)}(\eta) \triangleq \frac{Q_i}{\eta^3 \ln(1/\eta^2)} \left(\frac{\ln(1/\eta^2)}{\ln(1/q)}\right) + \left[\ln(1/\epsilon)\right],$$  (45)

where $\hat{\ell}$ is defined in (46) with $D \geq \ln((q/c)^{\epsilon})$. In particular, if $B_1 = 1, B_2 = 0$, then (45) reduces to

$$\frac{Q_i}{\eta^3 \ln(1/\eta^2)} \left(\frac{C + D}{\epsilon}\right)^{\frac{1}{\ln(1/q)}}$$

with $q > c \triangleq \max\{a_\infty, \eta\}$ and $D \geq \ln((q/c)^{\epsilon})$.

Proof. For any $i \in I_k$, let $j_{i,k} = \left[\frac{Q_i}{\eta^3 (t + 1)}\right]$ be the number of steps of the scheme (43) taken by player $i$ in major iteration $k$ to obtain $x_{i,k+1}$, i.e., $x_{i,k+1} : = z_{i,j_{i,k}}$. Since $\beta_{i,k} \leq k + 1$ and $\eta \in (0, 1)$, by Lemma 8, we have the following for any $i \in I_k$:

$$\mathbb{E}\left[\|x_{i,k+1} - \hat{x}_i(y_{i,k}^j)\|^2 | \mathcal{F}_k\right] = \mathbb{E}\left[\|z_{i,j_{i,k}} - \hat{x}_i(y_{i,k}^j)\|^2 | \mathcal{F}_k\right] \leq \frac{Q_i}{j_{i,k}} \leq \eta^{2(k+1)} \leq \eta^{2\beta_{i,k}} = \alpha_{i,k}^2$$ a.s.

By (42), we may obtain a lower bound on $k$ as follows:

$$\max_{i \in N} \mathbb{E}[\|x_{i,k+1} - x_i^\ast\|] \leq \rho^{-\frac{B_2}{B_1} - 1} (C + D)q^{k+1} \leq \epsilon \Rightarrow q^{k+1} \leq \frac{\epsilon}{C + D} \rho^{-\frac{B_2}{B_1} - 1} \triangleq \hat{\ell} \Rightarrow k \geq \frac{\ln(1/\epsilon)}{\ln(1/q)} - 1.$$  (46)

Then by invoking (30) for $\beta > 1$, we derive the following complexity bound for player $i$:

$$\sum_{k=0}^{\left[\ln(1/\epsilon)\right] - 1} j_{i,k} \leq \sum_{k=0}^{\left[\ln(1/\epsilon)\right] - 1} \left(\frac{Q_i}{\eta^{2(k+1)}} + 1\right) = \sum_{k=1}^{\left[\ln(1/\epsilon)\right]} \frac{Q_i}{\eta^{2k}} + \left[\ln(1/\epsilon)\right]$$

$$\leq \frac{Q_i}{\ln(1/\eta^2)} \eta^{-2\left[\ln(1/\epsilon)\right]} + \left[\ln(1/\epsilon)\right],$$

which results in (45) by invoking (31). \hfill \Box

Coordinate descent methods advocate that only a small block of variables are updated in each iteration while others are kept fixed [53]. The blocks of variables can be updated in a cyclic, random, or greedy fashion. In Section 4 we examined randomized update schemes, whereas here we consider a cyclic update scheme to broader problem settings, and analyze the iteration complexity.
Lemma 3, we obtain that Then by similar procedure for deriving (45), we may obtain the following iteration complexity for player $i$.

Proof. Let Algorithm 3 be applied to the stochastic Nash game (SNash), where Corollary 3. Theorem 4. Set $\ell$ iteration $k$ once among $n$ steps. As a result, $\beta, \alpha$ hold. Suppose that the players are chosen to update cyclicly, and $\rho = (\max\{a_{\infty}, \eta\})^{1/(m+1)}$ with $n_0 = \lceil B_k \rceil$.

Let Assumptions 4 and 5 hold. Suppose that an inexact solution characterized by $c$. Define $\epsilon = \frac{\epsilon}{C+D}$. Suppose that $\gamma = \eta^{1/N}$, then the number of projected gradient steps for player $i$ to compute an $\epsilon$-$NE_\infty$ is no greater than

$$
\ell_i^{(2)}(\eta) \triangleq \frac{Q_i}{\eta^2 \eta^2 \ln(1/\eta^2)} \left( \frac{1}{\epsilon} \right) + \left\lceil \frac{\ln(1/\epsilon)}{\ln(1/\eta^2)} \right\rceil.
$$

Thus, the result follows by $\tilde{\eta} = \eta^{1/N}$ and by $\bar{\eta}$ defined in (46) with $B_1 = N$.

5.4 Complexity bound

In the following theorem, we drive the upper bound for $\ell_i^{(1)}(\eta)$ and $\ell_i^{(2)}(\eta)$ defined in Theorem 3 and Corollary 3 respectively.

Theorem 4. Set $\eta \triangleq a_{\infty}$. Then we have the following bounds on the expression $\ell_i^{(1)}(\eta)$ and $\ell_i^{(2)}(\eta)$.

a) Define $n_0 = \lceil B_2 \rceil$ and $n' = B_1(1 + n_0)$. Then the following holds for any given $\delta > 0$: 

$$
\ell_i^{(1)}(\eta) \leq \frac{Q_i}{\eta^2 \ln(1/\eta^2)} \left( \frac{1}{\epsilon} \right)^{2n' + \delta} + \left\lceil \frac{\ln(1/\epsilon)}{2 \ln(\eta^{-1})} \right\rceil,
$$

where $\bar{\eta} = \frac{\epsilon}{C+D} \eta^{-\beta_i/k+1} \eta^{1/n_0+1}$ with $D$ defined in (48). In particular, if $B_1 = 1$, then 

$$
\ell_i^{(1)}(\eta) \leq \frac{Q_i}{\eta^2 \ln(1/\eta^2)} \left( \frac{C + D}{\epsilon} \right)^{2(1+B_2)+\delta} + \left\lceil \frac{\ln(1+D)}{2 \ln(\eta^{-1})} \right\rceil,
$$

where $D = \frac{(1+B_2)^2 + (1+B_2)^2 \delta^{-1}}{2 \ln(\eta^{-1})}$. If $D = \frac{1+2\delta^{-1}}{e \ln(\eta^{-1})}$ and $B_2 = 0$, then given any $\delta > 0$, we have that 

$$
\ell_i^{(1)}(\eta) \leq \frac{Q_i}{\eta^2 \ln(1/\eta^2)} \left( \frac{C + D}{\epsilon} \right)^{2+\delta} + \left\lceil \frac{\ln(1+D)}{2 \ln(\eta^{-1})} \right\rceil,
$$

(47)
b) Given any $\delta > 0$, $\hat{\epsilon} \triangleq \epsilon \frac{\epsilon}{\epsilon + D} \rho^{N_{1} + n_{0}}$, $n_{0} = \lceil \frac{B_{2}}{N} \rceil$, $D$ given by (49), we have that

$$
\ell^{(2)}_{i} (\eta) \leq \frac{Q_{i}}{\eta^{\frac{2 + \delta}{\pi}}} \left( \frac{1}{\epsilon} \right)^{2(n_{0} + 1) + \delta} \left[ \ln \left( \frac{1}{\epsilon} \right) \frac{2(n_{0} + 1) + \delta}{2 n_{0} \ln(\eta^{-1})} \right].
$$

In particular, if $B_{2} = 0$, then given any $\delta > 0$, $\hat{\epsilon} \triangleq \frac{\epsilon}{\epsilon + D} \rho^{N_{1}}$ and $D = \frac{N_{1}}{\epsilon \ln(\eta^{-1})}$, we have that

$$
\ell^{(2)}_{i} (\eta) \leq \frac{Q_{i}}{\eta^{\frac{2 + \delta}{\pi}}} \left( \frac{1}{\epsilon} \right)^{2 + \delta} \left[ \ln \left( \frac{1}{\epsilon} \right) \frac{(2 + \delta) \eta}{2 n_{0} \ln(\eta^{-1})} \right].
$$

Proof. a) Define $q = c e^{\delta_{0}/2 n'}$ with $\delta_{0} = \frac{\delta \ln(\eta^{-1})}{2 n' + \delta / 2}$. By $\eta = a_{\infty}$, we get $\rho^{n_{0} + 1} = \eta$ and $c = \rho^{\frac{1}{2n'}} = \eta^{\frac{1}{2}}$. Then

$$
\ln(1/q) = \ln \left( \eta^{-1} e^{-\delta_{0}/2 n'} \right) = \frac{\ln(\eta^{-1})}{n'} - \frac{\delta_{0}}{2 n'} = \frac{2 \ln(\eta^{-1})}{2 n' + \delta},
$$

and $D = 1/\ln((q/c)^{e}) = 1/\ln(e^{\delta_{0}/2 n'}) = \frac{2 n'}{\delta_{0} e} = \frac{n' + 2(n')^{2} \delta^{-1}}{e \ln(\eta^{-1})}$. (48)

Consequently, by the expression $\ell^{(1)}_{i} (\eta)$ defined in (45), we show the first assertion of part (a). The remaining two assertions can be easily followed by the definitions of $n_{0}$, $n'$, and $D$. 

b) Define $q = c e^{\delta_{0}/2 N(1+n_{0})}$ with $\delta_{0} = \frac{\delta \ln(\eta^{-1})}{1+n_{0} + \delta / 2}$. By $\eta = a_{\infty}$, we get $\rho^{n_{0} + 1} = \eta$ and $c = \rho^{\frac{1}{2N}} = \eta^{\frac{1}{2N}}$. Then by $\tilde{\eta} = \eta^{1/N}$ we have the following:

$$
\ln(1/q) = \ln \left( \eta^{-1} e^{-\delta_{0}/2 N(1+n_{0})} \right) = \frac{\ln(\eta^{-1}) - \delta_{0} / 2}{N (1 + n_{0})} = \frac{2 \ln(\eta^{-1})}{2 (n_{0} + 1) + \delta},
$$

$$
\frac{\ln(1/\eta^{2})}{\ln(1/q)} = \frac{\ln(1/\eta^{2/N})}{\ln(1/q)} = 2 \frac{\ln(\eta^{-1})}{N} = 2 (n_{0} + 1) + \delta,
$$

and $D = 1/\ln((q/c)^{e}) = 1/\ln(e^{\delta_{0}/2 N(1+n_{0})}) = \frac{2 N (1 + n_{0})}{\delta_{0} e} = \frac{N (1 + n_{0}) + 2(1 + n_{0}) \delta^{-1}}{e \ln(\eta^{-1})}$. (49)

Then by $\ell^{(2)}_{i} (\eta)$ defined in Corollary 3, we obtain the first assertion of part (b). If $B_{2} = 0$, then by definitions of $n_{0}$ and $D$, we get $n_{0} = 0$ and $D = \frac{N_{1} + \delta^{-1}}{e \ln(\eta^{-1})}$. Thus, the remaining assertion of part (b) holds.

The complexity bounds given in the part (a) and part (b) of Theorem 4 are summarized as follows:

| (a)-almost cyclic rule |  | (b)-cyclic rule |
|------------------------|------------------|
| update frequency       | delay            | complexity bound |
| $B_{1}$                | $B_{2}$          | $O \left( \frac{1}{\epsilon}^{2B_{1} \left( 1 + \frac{B_{2}}{2N} \right)} \right)$ |
| 1                      | $B_{2}$          | $O \left( \frac{1}{\epsilon}^{2 \left( 1 + \frac{B_{2}}{2N} \right)} \right)$ |
| $B_{2}$                | 1                | $O \left( \frac{1}{\epsilon}^{2B_{2}} \right)$ |
| 0                      | $B_{2}$          | $O \left( \frac{1}{\epsilon}^{2 \left( 1 + \frac{B_{2}}{2N} \right)} \right)$ |

Table 2: Summary of Complexity Bounds

Remark 8 Based on established results, we make the following clarifications:

(i) If $B_{1} = 1$, $B_{2} = 0$, then the asynchronous algorithm degenerates to the synchronous algorithm. Notably, we achieve the complexity bound $O \left( \frac{1}{\epsilon}^{2 + \delta} \right)$ given by (47) in Theorem 4(a) compared with $O \left( \sqrt{N/\epsilon} \right)^{2 + \delta}$ given in Theorem 4(b). This is because the definition of $\epsilon N_{2}$ by (26) (resp. $\epsilon N_{\infty}$ by (44)) as well as the analysis of Theorem 4(b) and Theorem 4(a) are based on two-norm and infinity norm, respectively;

(ii) Assume that the players update in a cyclic manner and $B_{2} = 0$. Then the upper bound of iteration complexity is shown to be $O \left( \frac{1}{\epsilon}^{2 + \delta} \right)$ in Theorem 4(b), which is of the same order as that of the synchronous case given by (47) in Theorem 4(a).
(iii) From Corollary 3 and Theorem 2(b) we infer that for some specific selection of update indices in each iteration, we can get a much better upper bound than that given in Theorem 2(a). So, the update schemes of players are critical in improving the complexity bound. Nevertheless, these findings imply that the complexity bounds established in Theorem 2 for Algorithm 3 are optimal to some extent.

(iv) From the analysis of Algorithm 3, the geometric rate of convergence and the iteration complexity can also be established for Algorithm 2 and Algorithm 3 by replacing the assumption $||\Gamma|| < 1$ with $||\Gamma||_{\infty} < 1$.

Remark 9 The assumption $||\Gamma||_{\infty} < 1$ can be further weakened to $\rho(\Gamma) < 1$ based on the following: By [9, Corollary 2.6.1], there exists a nonnegative vector $w \in \mathbb{R}^N$ such that $a_\infty \triangleq \max_{i \in N} \frac{1}{w_i} \sum_{j=1}^{N} \gamma_{ij} w_j < 1$. Then by [5], we have that

$$\text{diag}\{w\}^{-1} \begin{pmatrix} \|\hat{x}_1(y') - \hat{x}_1(y)\| \\ \vdots \\ \|\hat{x}_N(y') - \hat{x}_N(y)\| \end{pmatrix} \leq \text{diag}\{w\}^{-1} \text{diag}\{w\} \text{diag}\{w\}^{-1} \begin{pmatrix} \|y_1' - y_1\| \\ \vdots \\ \|y_N' - y_N\| \end{pmatrix},$$

where $\text{diag}\{w\}$ is a diagonal matrix with the diagonal entries being $w_i$ $i \in N$. Similar to that given by the proof [45, Proposition 5] we have the following inequality:

$$\left\| \text{diag}\{w\}^{-1} \begin{pmatrix} \|\hat{x}_1(y') - \hat{x}_1(y)\| \\ \vdots \\ \|\hat{x}_N(y') - \hat{x}_N(y)\| \end{pmatrix} \right\|_{\infty} \leq a_\infty \left\| \text{diag}\{w\}^{-1} \begin{pmatrix} \|y_1' - y_1\| \\ \vdots \\ \|y_N' - y_N\| \end{pmatrix} \right\|_{\infty},$$

by which we are able to establish the geometric rate of convergence and the same order complexity bound for Algorithm 3 but with different constants.

6 Two-stage Recourse

In the previous sections, we assumed that the functions $f_i(x)$ are twice differentiable, a requirement that is essential in deriving the contractive properties of the proximal BR map. We weaken this assumption by considering a stochastic Nash game in the following form:

$$\min_{x_i \in X_i} f_i(x_i, x_{-i}) + g_i(x_i),$$

where $f_i(x_i, x_{-i}) \triangleq \mathbb{E} \left[ \psi_i(x_i, x_{-i}; \xi(x)) \right]$ and $g_i(x_i)$ can be merely convex, possibly nonsmooth, and expectation-valued. It is relatively simple to show that if the proximal BR map, constructed using the functions $f_1, \ldots, f_N$, admits a contractive property, then this modified problem also admits such a property. Consequently, the only challenge that emerges is the need to get inexact solutions to the proximal BR problem with an objective $f_i(x) + g_i(x_i)$ instead of merely $f_i(x)$. In this section, we consider a two-stage recourse-based model to introduce much needed flexibility into this framework.

Two-stage recourse-based stochastic programs originate from the work by Dantzig [15] and Beale [7] in the 50s. In the first stage, the decision-maker makes a decision prior to the revelation of the uncertainty, while in the second-stage, a scenario-specific decision is made contingent on the first-stage decision and on the realization of uncertainty. This second-stage decision is referred to as a recourse decision, which originates from the notion that in the second-stage, the decision-maker takes recourse based on the realization of uncertainty and on the first-stage decision. The expected cost of second-stage decisions is incorporated into the first-stage problem through the recourse function. This model has tremendous utility and finds applicability in financial planning, inventory control, power systems operation, etc (cf. [10, 51]). Nevertheless, such models and algorithms are not designed to accommodate multiple decision makers who do not cooperate in a multi-agent system. Recently, in [37], the authors consider two-stage non-cooperative games, where each agent is risk-averse and solves a rival-parameterized
stochastic program with quadratic recourse. To handle the nonsmooth recourse functions, the authors propose smoothing schemes leading to differentiable approximations, then design an iterative BR scheme for the smoothed problem and show its convergence. We propose a different avenue that leverages stochastic approximation and allows for deriving rate and complexity statements, that are unavailable in [37]. Next, we incorporate two-stage linear and quadratic recourse into this framework and derive iteration complexity statements for the proposed BR schemes.

6.1 Linear recourse

Consider the following two-stage stochastic Nash game:

$$\min_{x_i \in X_i} \mathbb{E} [\psi_i(x_i, x_{-i}; \xi(\omega)) + c_i(x_i) + \mathbb{E}[Q_i(x_i, \omega)],$$

(SNash$_{\text{rec}}(x_{-i}))$

where $c_i(x_i)$ is the continuously differentiable convex cost of the first-stage decision $x_i$ and

$$Q_i(x_i, \omega) \triangleq \min_{q_{i,\omega}} \{ q_i^T x_i + W_i q_i, \omega = h_i, q_i, \omega \geq 0 \}.$$ 

(Rec$_{\text{LP}}(x_i))$

The function $Q_i(x_i, \omega)$, being the cost of taking recourse given decision $x_i$ and scenario $\omega$, is shown to be convex in $x_i \in X_i$ for any given $\omega \in \Omega$ (See [51] Proposition 2.1).

Given a strategy $y \in X$, player $i$ can compute a proximal BR $\hat{x}_i(y)$ by solving the following stochastic optimization problem

$$\hat{x}_i(y) \triangleq \arg\min_{x_i \in X_i} \left[ \mathbb{E} [\psi_i(x_i, y_{-i}; \omega)] + c_i(x_i) + \mathbb{E}[Q_i(x_i, \omega)] + \frac{\mu}{2} \| x_i - y_i \|^2 \right].$$

(50)

It is easily seen that $\hat{x}(\cdot)$ defined by (50) is a contractive map since $g_i(x_i) := c_i(x_i) + \mathbb{E}[Q_i(x_i, \omega)]$ is convex. We can also use Algorithms [1 2] and 3 to find the Nash equilibrium of the problem (SNash$_{\text{rec}}(x_{-i}))$ but with $\hat{x}_i(\bullet)$ defined by (50) instead of $\hat{x}(\bullet)$. We intend to use the SG method to obtain an inexact proximal BR solution. So, we impose the following condition on the problem (Rec$_{\text{LP}}(x_i)$) to guarantee the existence and boundedness of the stochastic subgradient for function $\mathbb{E}[Q_i(x_i, \omega)]$.

**Assumption 6** For almost all $\omega \in \Omega$ and all $x_i \in X_i,$

a) there exists a strictly positive vector $q_{i,\omega}$ such that $T_{i,\omega} x_i + W_i q_{i,\omega} = h_{i,\omega},$

b) for any $v_{i,\omega} \geq 0$ with $W_i v_{i,\omega} = 0$, it holds that $d_{i,\omega}^T v_{i,\omega} \geq 0,$

c) $\mathbb{E}[\| T_{i,\omega} \|] < \infty$ for all $i \in \mathcal{N}.$

We adapt the following result from [51] to analyze the subgradient of $Q(x_i, \omega)$.

**Lemma 9** Let Assumption 6 hold. Then for almost all $\omega \in \Omega$ and all $x_i \in X_i,$

(i) $Q_i(x_i, \omega)$ is finite and its subdifferential at $x_i$ is given by

$$\partial x_i Q_i(x_i, \omega) = -T_{i,\omega}^T D_i(x_i, \omega),$$

where $D_i(x_i, \omega) \triangleq \arg\max_{\pi_{i,\omega} : W_i^T \pi_{i,\omega} \leq d_{i,\omega}} (h_{i,\omega} - T_{i,\omega} x_i)^T \pi_{i,\omega};$

(ii) $D_i(x_i, \omega)$ defined by (51) is bounded;

(iii) there exists a positive constant $M_s > 0$ such that

$$E[\| s_i \|^2] \leq M_s^2, \quad \forall s_i \in \partial Q_i(x_i, \omega).$$

(52)

**Proof.**

(i) Assumption 6(a) implies the existence of a feasible solution to problem (Rec$_{\text{LP}}(x_i)$). Then by Assumption 6(b), we see that for almost all $\omega \in \Omega$ and all $x_i \in X_i$, (Rec$_{\text{LP}}(x_i)$) has an optimal solution and $Q_i(x_i, \omega)$ is finite. Thus, by [51] Proposition 2.2] we obtain the result.
(ii) Note that \( h_{i,\omega} - T_{i,\omega} x_i \) is an interior point of the positive hull of matrix \( W_i \) by Assumption \( 6(a) \). It is shown in [51, p. 29] that the optimal solution set \( D_i(x_i, \omega) \) of (51) is bounded.

(iii) By Assumption \( 6(c) \) and the boundedness of \( D_i(x_i, \omega) \) we have the result.

Since \( Q_i(x_i, \omega) \) is subdifferentiable, the following SA scheme is utilized to approximate the proximal BR \( \tilde{x}_i(y_k) \):

\[
\tilde{x}_{i,t+1} := \Pi_{X_i} \left[ z_{i,t} - \tau \left( \nabla \psi_i(z_{i,t}) + \nabla_x \psi_i(z_{i,t}, y_{i,\omega}; \lambda_{i,k}^t) + \mu (z_{i,t} - y_{i,\omega}) + s_{i,t} \right) \right],
\]

where \( z_{i,1} = x_{i,k}, \tau_t = \gamma \mu (t + 1) \) and \( s_{i,t} \in \partial Q_i(z_{i,t}, \omega_{i,k}) \). In fact, the SA scheme \( (SA^\text{rec}_i) \) requires a solution of the second-stage dual problem \( (\text{Rec}^\text{rec}_i) \) to obtain a stochastic subgradient. Since \( \nabla c_i(\cdot) \) is continuous and \( X_i \) is compact, there exists \( M_c > 0 \) such that \( \|\nabla c_i(x_i)\| \leq M_c \) \( \forall x_i \in X_i \). Combined with Assumption \( 1(d) \), (16) and (52) yields that for any \( x_i \in X_i, y \in X \)

\[
\mathbb{E} \left[ \|\nabla c_i(x_i) + \nabla_x \psi_i(x_i, y_{i,\omega}; \xi) + \mu (x_i - y_i) + s_{i,t} \| \right] \leq 4(M_t^2 + M_s^2 + M_t^2 + \mu^2 D_{X_i}^2).
\]

By [51, Theorem 7.47], under suitable regularity conditions, it is seen that \( \mathbb{E} \left[ \partial Q_i(x_i, \omega) \right] = \partial \mathbb{E} \left[ Q_i(x_i, \omega) \right] \), and hence \( s_{i,t} \in \partial Q_i(z_{i,t}, \omega_{i,k}) \) is an unbiased estimate for some subgradient of \( \mathbb{E} \left[ Q_i(x_i, \omega) \right] \) at point \( z_{i,t} \).

Consequently, Algorithms 1, 2, and 3 are applicable to the stochastic Nash game with linear-recourse. Further, the convergence results all hold for the problem \( \text{SNash}_{\text{rec}}(x_{-i}) \) with \( Q_i(x_i, \omega) \) defined by (\text{Rec}_{\text{LP}}(x_i)) except that the constant \( Q_i \triangleq \frac{2M_t^2}{\mu^2} + 2D_{X_i}^2 \) is replaced by \( Q_i \triangleq \frac{4(M_t^2 + M_s^2 + M_t^2)}{\mu^2} + 4D_{X_i}^2 \).

### 6.2 Quadratic recourse

In the past, two-stage stochastic quadratic programs with fixed recourse have been considered in [13, 43], where the first-stage term is a quadratic function and the second-stage term is the expectation of the minimum value of a quadratic program. In [43] and [13], the authors work with the recourse subproblems’ dual that is approximated by a sequence of quadratic program subproblems, and propose a Lagrangian finite generation technique and a Newton’s method to solve the problem, respectively. In [29, 48], Dorn duality is employed in developing a Benders scheme. Motivated by this, we incorporate quadratic recourse in the stochastic Nash game and consider the two-stage problem \( \text{SNash}_{\text{rec}}(x_{-i}) \), for which \( c_i(x_i) \) is a continuously differentiable convex cost of \( x_i \), and the recourse function \( Q_i(x_i, \omega) \) is the optimal value of a convex quadratic program parameterized by the decision \( x_i \) and scenario \( \omega \):

\[
\min_{q_i, \omega} \left\{ d_{i,\omega}^T q_i, \omega + \frac{1}{2} q_i, \omega^T H_{i,\omega} q_i, \omega \mid q_i, \omega \in T_{i,\omega} \right\}, \quad (\text{Rec}_{\text{QP}}(x_i))
\]

where \( H_{i,\omega} \) is a symmetric positive semidefinite matrix. By Dorn duality [16], a dual problem to \( (\text{Rec}_{\text{QP}}(x_i)) \) has the following form:

\[
\max_{u_i, \omega, \pi_i, \omega} \phi_i(x_i, u_i, \omega, \pi_i, \omega) \triangleq (h_{i,\omega} - T_{i,\omega} x_i)^T \pi_i, \omega - \frac{1}{2} u_i^T H_{i,\omega} u_i, \omega
\]

subject to \( (u_i, \omega, \pi_i, \omega) \in S_{i,\omega} \triangleq \{ u_i, \omega, \pi_i, \omega : W_i^T \pi_i, \omega - H_{i,\omega} u_i, \omega \leq d_i, \omega \} \).

**Assumption 7** For any \( x_i \in X_i \) and almost all \( \omega \in \Omega \),

a) there exists a strictly positive vector \( q_i, \omega \) such that \( T_{i,\omega} x_i + W_i q_i, \omega = h_{i,\omega} \);

b) for any \( q_i, \omega \in T_{i,\omega}, v_i, \omega \geq 0 \) with \( W_i v_i, \omega = 0, v_i^T H_{i,\omega} v_i, \omega = 0 \), it holds that \( (H_{i,\omega} q_i, \omega + d_{i,\omega})^T v_i, \omega \geq 0 \);

c) \( \mathbb{E}[\|T_{i,\omega}\|] < \infty \) for all \( i \in N \).

Before proceeding to show the finiteness of \( Q_i(x_i, \omega) \), we first provide a lemma concerning the existence of optimal solutions for quadratic programs. Consider a quadratic program in the following form:

\[
\min_{x \in \mathbb{R}^n} \left\{ \frac{1}{2} x^T D x + c^T x \mid x \in \Delta(A, b) \right\},
\]

where \( D \in \mathbb{R}^{n \times n} \) is positive semidefinite. Then by [30, Corollary 2.6], we have the following lemma.
Lemma 10  The problem (54) has a solution if and only if $\Delta(A, b)$ is nonempty and the following condition is satisfied:

If $x, v \in \mathbb{R}^n$ are such that $x \in \Delta(A, b), v \geq 0, Av = 0,$ and $v^T Dv = 0$, then $(Dx + c)^T v \geq 0$.

Before providing the result, we state a simple result about the quadratic programming which may be proved directly or by invoking [20, Cor. 2.3.7].

Lemma 11  Assume that problem (54) has a solution set $S^*$. Then the set $DS^*$ contains a singleton.

We now analyze the subdifferential of the recourse function.

Lemma 12  Let Assumption [7] hold. Then we may claim the following:

(a) The optimal solution set $S_{i,\omega}^{\text{opt}}(x_i)$ of the dual problem (53) is nonempty, and $Q_i(x_i, \omega) = \max_{(u_i, \pi_i, \omega) \in S_{i,\omega}} \phi_i(x_i, u_i, \pi_i, \omega)$.

Additionally, $D_i(x_i, \omega) = \left\{ \rho_{i,\omega}(x_i) : \left( u_{i,\omega}^{\text{opt}}(x_i), \pi_{i,\omega}^{\text{opt}}(x_i) \right) \in S_{i,\omega}^{\text{opt}}(x_i) \right\}$ is a bounded set;

(b) $Q_i(x_i, \omega)$ is convex in $x_i \in X_i$ for any given $\omega \in \Omega$;

(c) $Q_i(x_i, \omega)$ is subdifferentiable at $x_i$ with $\partial_{x_i} Q_i(x_i, \omega) = -T_i^{\top} D_i(x_i, \omega)$. In particular, if $D_i(x_i, \omega)$ consists of a unique point $\pi_{i,\omega}^{\text{opt}}(x_i)$, then $Q_i(x_i, \omega)$ is differentiable at $x_i$ with $\nabla_{x_i} Q_i(x_i, \omega) = -T_i^{\top} \pi_{i,\omega}^{\text{opt}}(x_i)$;

(d) there exists a positive constant $M_0 > 0$ such that $E[\|s_i\|^2] \leq M_0^2 \forall s_i \in \partial Q_i(x_i, \omega)$.

Proof. (a) By invoking Assumption [7] together with Lemma [10] we have that $Q_i(x_i, \omega)$ defined by problem (RecQ$_i^{\omega}(x_i)$) has at least one optimal solution $q_{i,\omega}^\star$. Note that

$$T_i x_i + W_i \pi_i = h_i \iff \left( \begin{array}{c} W_i \\ -W_i \end{array} \right) \pi_i \geq \left( \begin{array}{c} h_i \omega - T_i x_i \\ - (h_i \omega - T_i x_i) \end{array} \right).$$

Then by [16, Theorem (Dual)-i] it is easily seen that a solution $\left( u_{i,\omega}^{\text{opt}}(x_i), \pi_{i,\omega}^{\text{opt}}(x_i) \right) \in S_{i,\omega}^{\text{opt}}(x_i)$ exists to the dual problem (53) and $Q_i(x_i, \omega) = \max_{(u_i, \pi_i, \omega) \in S_{i,\omega}} \phi_i(x_i, u_i, \pi_i, \omega)$.

By [16, Theorem (Dual)-ii] we see that for any $\left( u_{i,\omega}^{\text{opt}}(x_i), \pi_{i,\omega}^{\text{opt}}(x_i) \right) \in S_{i,\omega}^{\text{opt}}(x_i)$, a solution $q_{i,\omega}^\star$ satisfying $H_{i,\omega, q_{i,\omega}^\star} = H_{i,\omega} u_{i,\omega}^{\text{opt}}(x_i)$ to problem (RecQ$_i^{\omega}(x_i)$) also exists. We denote by $O_{i,\omega}^{\text{opt}}$ the optimal solution set of problem (RecQ$_i^{\omega}(x_i)$). Then by Lemma [11], the set $H_{i,\omega} O_{i,\omega}^{\text{opt}}$ contains a singleton denoted by $H_{i,\omega} q_{i,\omega}^\star$. Thus, for any $\left( u_{i,\omega}^{\text{opt}}(x_i), \pi_{i,\omega}^{\text{opt}}(x_i) \right) \in S_{i,\omega}^{\text{opt}}(x_i)$, $H_{i,\omega} u_{i,\omega}^{\text{opt}}(x_i)$ equals $H_{i,\omega} q_{i,\omega}^\star$. As a result, by the optimality condition for problem (53) we see that for any $\left( u_{i,\omega}^{\text{opt}}(x_i), \pi_{i,\omega}^{\text{opt}}(x_i) \right) \in S_{i,\omega}^{\text{opt}}(x_i)$, $\pi_{i,\omega}^{\text{opt}}(x_i)$ is a solution of the following linear program

$$\max_{\pi_{i,\omega}} \left\{ (h_i \omega - T_i x_i)^T \pi_{i,\omega} \mid W_i^{\top} \pi_{i,\omega} \leq H_{i,\omega} q_{i,\omega}^\star + d_{i,\omega} \right\}. \quad (55)$$

Note that $h_i \omega - T_i x_i$ is an interior point of the positive hull of matrix $W_i$ by Assumption [7], and from [51, p. 29], the optimal solution set of problem (55) must be bounded. Hence $D_i(x_i, \omega)$ is bounded.

(b) Since $\phi_i(x_i, u_i, \pi_i, \omega)$ is convex in $x_i$ for all $(u_i, \pi_i, \omega) \in S_{i,\omega}$, for any $\alpha \in [0, 1]$ and any $x_i, x'_i \in X_i$

$$\phi_i(\alpha x_i + (1 - \alpha) x'_i, u_i, \pi_i, \omega) \leq \alpha \phi_i(x_i, u_i, \pi_i, \omega) + (1 - \alpha) \phi_i(x'_i, u_i, \pi_i, \omega) \leq \alpha Q_i(x_i, \omega) + (1 - \alpha) Q_i(x'_i, \omega),$$

where the last inequality follows from part (a). Taking the maximum with respect to $(u_i, \pi_i, \omega) \in S_{i,\omega}$ we obtain $Q_i(\alpha x_i + (1 - \alpha) x'_i, \omega) \leq \alpha Q_i(x_i, \omega) + (1 - \alpha) Q_i(x'_i, \omega)$, and hence the recourse function $Q_i(x_i, \omega)$ is convex in $x_i \in X_i$ for every $\omega \in \Omega$. 

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7.1 Competitive portfolio selection

descriptions of our simulations are provided in Section 7.2. In Section 7.3, we formulate a two-stage competitive
In this section, we demonstrate the performance of the proposed algorithms through an asset management problem

7 Numerics

admit a contractive property, \( \rho \). Assume that the parameters \( \rho \) may invest an amount \( x_{ij} \) in asset \( j \). Suppose that asset \( j = 1, \ldots, n \) has a return rate \( r_j \), a random variable with expectation \( \nu_j = \mathbb{E}[r_j] \). Denote by \( \hat{R} = \mathbb{E}[(r - \nu)(r - \nu)^T] \) \in \( \mathbb{R}^{n \times n} \) the covariance matrix, where \( r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n \) and \( \nu = (\nu_1, \ldots, \nu_n)^T \in \mathbb{R}^n \). Then for investor \( i \in \mathcal{N} \), the expected return is \( \sum_{j=1}^{n} \nu_j x_{ij} = \nu^T x_i \), while the variance (risk) of the return is \( \mathbb{E}[(r^T x_i - \nu^T x_i)^2] = x_i^T \hat{R} x_i \), where \( x_i = (x_{i1}, \ldots, x_{in})^T \in \mathbb{R}^n \). The expected return and variance of each asset \( i \) can be estimated from the history. We consider a setting where each investor trades off between return and risk through the following utility:

\[
\sum_{j=1}^{n} x_{ij}^2 \nu_j = \nu^T x_i \]

\[
\mathbb{E}[(r^T x_i - \nu^T x_i)^2] = x_i^T \hat{R} x_i \]

We consider a setting where each investor trades off between return and risk through the following utility: \( U_i(x_i) = \rho_i x_i^T \hat{R} x_i - \nu^T x_i \), where \( \rho_i > 0 \) is the risk-aversion parameter of investor \( i \), and larger \( \rho_i \) results in a larger emphasis on risk.

In a practical market incorporating multiple investors, trades of diverse investors are usually pooled and executed together \[35,54\]. The transaction cost for a single investor may depend on the overall trading levels in the market and not just its own trading. Each investor \( i \) has an initial holding \( x_i^0 \), the transaction price is a function of trade size \( x_j - x_j^0 \) \forall j \in \mathcal{N} \) from all investors, e.g., the price takes the form \( \phi(\omega) \sum_{j=1}^{n} (x_j - x_j^0) \), where \( \phi : \Omega \rightarrow \mathbb{R}^{n \times n} \) is a diagonal random matrix with diagonal entries being positive. Besides, the transaction cost of each investor is proportional to its trade size. Based on the above considerations, the portfolio selection problem of investor \( i \) can be modeled as the following stochastic program:

\[
\min_{x_i \in X_i} f_i(x_i, x_{-i}) = U_i(x_i) + \mathbb{E} \left[ (x_i - x_i^0)^T \phi(\omega) \sum_{j=1}^{n} (x_j - x_j^0) \right],
\]

where \( X_i \) denotes the set of feasible portfolios of player \( i \). Denote by \( \Phi \triangleq \mathbb{E}[\phi(\omega)] \). Then for any \( i \in \mathcal{N} \),

\[
\nabla^2_{x_i} f_i(x) = 2 \rho_i R + 2 \Phi \quad \nabla^2_{x_i} f_i(x) = \Phi \quad \forall j \neq i.
\]

Assume that the parameters \( \rho_i, R, \Phi \) satisfy the following condition

\[
\lambda_{\min}(2 \rho_i R + 2 \Phi > (N - 1) \|\Phi\| \quad \forall i \in \mathcal{N}.
\]

7.1 Competitive portfolio selection

The portfolio selection problem has a long history in the field of modern financial theory, dating back to the seminal work of Markowitz \[32\]. We consider the case with \( N \) investors where investor \( i \in \mathcal{N} = \{1, \ldots, N\} \) may invest an amount \( x_{ij} \) in asset \( j \in \{1, \ldots, n\} \). Suppose that asset \( j = 1, \ldots, n \) has a return rate \( r_j \), a random variable with expectation \( \nu_j = \mathbb{E}[r_j] \). Denote by \( \hat{R} = \mathbb{E}[(r - \nu)(r - \nu)^T] \) \in \( \mathbb{R}^{n \times n} \) the covariance matrix, where \( r = (r_1, \ldots, r_n)^T \in \mathbb{R}^n \) and \( \nu = (\nu_1, \ldots, \nu_n)^T \in \mathbb{R}^n \). Then for investor \( i \in \mathcal{N} \), the expected return is \( \sum_{j=1}^{n} \nu_j x_{ij} = \nu^T x_i \), while the variance (risk) of the return is \( \mathbb{E}[(r^T x_i - \nu^T x_i)^2] = x_i^T \hat{R} x_i \), where \( x_i = (x_{i1}, \ldots, x_{in})^T \in \mathbb{R}^n \). The expected return and variance of each asset \( i \) can be estimated from the history. We consider a setting where each investor trades off between return and risk through the following utility:

\[
U_i(x_i) = \rho_i x_i^T \hat{R} x_i - \nu^T x_i.
\]

Denote by \( \Phi \triangleq \mathbb{E}[\phi(\omega)] \). Then for any \( i \in \mathcal{N} \),

\[
\nabla^2_{x_i} f_i(x) = 2 \rho_i R + 2 \Phi \quad \nabla^2_{x_i} f_i(x) = \Phi \quad \forall j \neq i.
\]

Assume that the parameters \( \rho_i, R, \Phi \) satisfy the following condition

\[
\lambda_{\min}(2 \rho_i R + 2 \Phi > (N - 1) \|\Phi\| \quad \forall i \in \mathcal{N}.
\]
If condition (57) holds, then \( \|\Gamma\|_\infty < 1 \). Since both \( R \) and \( \Phi \) are symmetric and positive definite matrices, all of their eigenvalues are strictly positive. Then for sufficiently large \( \rho_i \), condition (57) certainly holds. Notably, large \( \rho_i \) means that investor \( i \) is more risk-averse.

### 7.2 Numerical results

We consider a market of 6 investors and 4 assets, i.e., \( N = 6, n = 4 \). Set \( \nu = (0.5, 0.35, 0.4, 0.3) \) and let \( R \) be a diagonal matrix with \( R_{11} = 0.16, R_{22} = 0.1, R_{33} = 0.12, R_{44} = 0.09 \). Let each diagonal element of \( \phi(\omega) \) be a random variable satisfying a uniform distribution \([0.12, 0.18] \), and hence \( \Phi = 0.15I_4 \). Set \( \rho_i = 3 + i/N \). Let \( X_i = \{x_i \in \mathbb{R}^n : x_{ij} \geq 0, x_{ij} \leq \text{cap}_{ij}, j = 1, \cdots, n \} \). We set \( \text{cap}_{ij} = 0.5 \) and the initial holdings \( x^0_{ij} = 0 \) for all \( i \in \mathcal{N} \) and \( j = 1, \cdots, n \). Then Assumptions 1, 2 and 5 hold. Throughout this section, we assume that the empirical mean of the error is calculated by averaging across 50 trajectories.

#### 7.2.1 Convergence of the Synchronous Algorithm

We now present simulations for Algorithm 1 and examine its empirical convergence rate and iteration complexity. Suppose \( j_{i,k} = \left\lfloor \frac{1}{\eta_k^i} \right\rfloor \) steps of (SA\(_{i,k}\)) are taken at major iteration \( k \) to get an inexact solution to (7), where \( \eta = \alpha^{\kappa/2} \) for some \( \kappa > 0 \). Then \( \alpha_{i,k} \leq \eta^k \) by Lemma 3, and hence the sequence \( \{\alpha_{i,k}\} \) is summable by noting that \( \eta \in (0, 1) \). We carry out simulations for different selections of \( \mu, \kappa \), for each case the smallest number of projected SG steps each player has carried out to make \( x^*_i = (\text{SA}_{i,k}) \) is shown in Figure 1, which demonstrates that the iterates converge in mean to the unique equilibrium at a linear rate. We further examine the convergence behavior of Algorithm 1 with \( \mu = 2, \eta = \alpha \) by plotting the error bar of \( x_{11,k} \), where the blue line denotes the trajectory of the mean sequence computed via the sample average, the black dashed line denotes the equilibrium strategy \( x^*_1 \), while the red bars capture the variance of \( x_{11,k} \). Figure 2 shows that the equilibrium estimate approaches zero both with probability one and in mean with the variance across the samples decaying to zero. This corresponds well with the theoretical findings that the iterates converge a.s. to the NE and that the variance converges to zero by Proposition 2 and Proposition 3(b), respectively. The empirical relation between \( \epsilon \) and \( K(\epsilon) \) is shown in Figures 4(a) and 4(b) where \( K(\epsilon) \) denotes the smallest number of projected SG steps the player has carried out to make \( u_k < \epsilon \). The red solid curve represents the empirical data, while the blue dashed curve demonstrates the corresponding quadratic fit, which aligns well with the empirical data. It indicates that the empirical iteration complexity is of order \((1/\epsilon^2)\) in this simulation setting.

We plot the error bar of \( x_{11,k} \) in Figure 3 where the blue line denotes the trajectory of the mean sequence, the black dashed line denotes the equilibrium strategy \( x^*_1 \), and the red bars denotes the variance of \( x_{11,k} \).
It shows that the variance across the samples decays to zero, which is consistent with Proposition 3(b) by noting that $\alpha_{i,k} \to 0$. Figure 3 also displays that the estimates appear to converge a.s. to the NE though we cannot theoretically claim this from Proposition 2 since $\sum_{k=1}^{\infty} \alpha_{i,k} = \infty$. Further, the empirical relation between $\epsilon$ and $K(\epsilon)$ is shown in Figure 4(c) with the red solid curve representing the empirical data and the blue dashed curve demonstrating its quadratic fit. It is easily seen that the complexity bound of non-summable $\{\alpha_{i,k}\}$ is worse than that of summable $\{\alpha_{i,k}\}$ as shown in Figure 4.

Comparisons with stochastic gradient method: Set $\mu = 2.5$ and suppose $j_{i,k} = \left\lceil \frac{1}{\mu} \right\rceil$ steps of (SA$_{i,k}$) are taken at major iteration $k$ to get an inexact solution to (7). We compare both Algorithm 1 and the standard SG method for computing an NE in terms of the iteration complexity and communication overhead for achieving the same accuracy. The empirically observed relationship between $\epsilon$ and $K(\epsilon)$ for both methods are shown in Figure 5. From the figure, it can be observed that the iteration complexity are of the same orders while the constant of the SG method is superior to that of Algorithm 1. Note that in the delay-free SG method, each player performs a single projected gradient step by invoking a communication with its rivals, consequently, the resulting communication overhead is proportional to the total number of projected gradient steps. In contrast, the synchronous inexact proximal algorithm carries out an increasing number of player-specific projected gradient steps after a single round of communication with its rivals. Note that this communication overhead is expected to be less in stochastic gradient schemes with delay [2]. The communication overhead of the SG method and Algorithm 1 is compared in Figure 6. From the results demonstrated in Figure 5 and Figure 6, we conclude that Algorithm 1 compares well with the SG method in terms of overall projected gradient steps while showing markedly less communication overhead. In fact, in certain applications, high communication overhead tends to render a scheme impractical.
7.2.2 Convergence of the Randomized and Asynchronous Schemes

We now run simulations for Algorithm 2, where \( p_i = 1/N \forall i \in \mathcal{N} \). Suppose \( j_{i,k} = \left\lceil \frac{1}{\eta^2(\beta_{i,k} + 1)} \right\rceil \) steps of \((\text{SA}_{i,k})\) are taken at major iteration \( k \) to get an inexact solution satisfying (33), where \( \eta = \frac{a}{\kappa^2} \) for some \( \kappa > 0 \). The trajectories of \( u_k \) are shown in Figure 7, while the empirical mean of the number of projected SG steps is displayed in Figure 8. It is seen that randomized algorithm still displays linear convergence but its empirical iteration complexity of randomized algorithm is larger than that of the synchronous algorithm, a less surprising observation.

Next, we simulate the performance of Algorithm 3, where for any \( k \geq 0, i \in I_k, j \in \mathcal{N}, \) the communication delays \( \tau_{ij}(k) \) are independently generated from a uniform distribution on the set \( \{0, 1, \cdots, B_2\} \). Set \( \mu = 2, \eta = a_{\infty}, B_1 = 1 \). Suppose that \( j_{i,k} = \left\lceil \frac{1}{\eta^2(B_{i,k} + 1)} \right\rceil \) steps of (43) are taken at major iteration \( k \) to get an inexact solution to problem (38). We carry out simulations for different communication delays \( B_2 = 0, 4, 8, 12 \). The trajectories of \( u_k \) are shown in Figure 9, from which it is observed that the iterates produced by Algorithm 3 converge in mean to the unique equilibrium at a linear rate; however, the trajectories tend to display less of the monotonically decreasing behavior that characterizes the synchronous schemes. The empirically observed iteration complexity is plotted in Figure 10 and we note that this worsens as \( B_2 \) grows.

7.2.3 Comparisons of Empirical and Theoretical Results

Set \( \rho_i = 4 \forall i \in \mathcal{N} \). Let each player take 40 BR steps and the inexact proximal BR solution be computed via a SA scheme. The theoretical and empirical rate of convergence are shown in Table 4 from which it is seen that the theoretical error bound is relatively conservative while the empirical error is seen to be far smaller in practice.
Consider a set of $N$ players denoted by $\mathcal{N} \triangleq \{1, \cdots, N\}$, where the $i$-th player solves the following two-stage problem

$$\min_{x_i:0 \leq x_i \leq \text{cap}_i} \left( C_i(x_i) - P(x)x_i + \mathbb{E}[Q_i(x_i, \omega)] \right), \quad \text{where } Q_i(x_i, \omega) \triangleq \max_{q_i, \omega:0 \leq q_i, \omega \leq x_i} d_i q_i - \frac{h_i}{2} q_i^2.$$

(58)

where $P(x) = a - b \sum_{i=1}^N x_i$, $C_i(\cdot)$ is a twice continuously differentiable and $\eta_i$-strongly convex function of $x_i$. This can be viewed as a game where $N$ firms compete in Cournot in a capacity market defined by an inverse demand function $P(x)$ and subsequently make production decisions subject to a demand’s capacity constraints while faced by random prices and costs, where $P(x)$, the market price, is a decreasing function of total production, and $C_i(x_i)$ is the cost function of firm $i$. Capacity markets are utilized to price generation capacity in power markets [1][18].

Note that

$$\nabla^2 f = \begin{pmatrix} \nabla^2 f_1 & \cdots & \nabla^2 f_N \\ \vdots & \ddots & \vdots \\ \nabla^2 f_1 & \cdots & \nabla^2 f_N \end{pmatrix} = \begin{pmatrix} \nabla^2 C_1 \\ \vdots \\ \nabla^2 C_N \end{pmatrix} + b \left( I_N + N_1\mathbf{1}_N^T \right).$$

Since $C_i(\cdot)$ is a twice continuously differentiable and $\eta_i$-strongly convex function of $x_i$, by definition [4] we have that for any $i \in \mathcal{N}$, $\zeta_i_{\min} = \eta_i + 2b$ and $\zeta_i_{\max} = b \forall j \neq i$. Then by (3) the following holds:

$$\Gamma = \begin{pmatrix} \mu & \mu+\eta_1+2b & \cdots & \mu+\eta_N+2b \\ b & \mu+\eta_1+2b & \cdots & \mu+\eta_N+2b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & \mu+\eta_N+2b \end{pmatrix}.$$

Since for any $i \in \mathcal{N}$ and for any given $\omega$, the function $Q_i(\cdot, \omega)$ is convex. Then $Q_i(\cdot)$ is convex in $x_i$, and hence $\hat{x}(\cdot)$ is contractive when the spectral radius $\rho(\Gamma) < 1$, for which one of the sufficient conditions is $\|\Gamma\|_\infty < 1$. As such, the assumption $\min_{i \in } \eta_i > (N-3)b$ ensure that the proximal BR map is contractive.

By Dorn duality [16], a dual problem to the second stage problem (58) is as follows:

$$\min_{u_i, \omega} \left\{ \phi_i(x_i, u_i, \omega, \omega) = \frac{h_i}{2} u_i^2 + x_i v_i, \quad h_i u_i + v_i \geq d_i, \quad v_i \geq 0 \right\}. \quad (59)$$

Similar to Lemma 12 we can also show that the optimal solution set $S^{opt}_{i,\omega}(x_i)$ of (59) is bounded. Note that

$$\frac{\partial \phi_i(x_i, u_i, \omega, \omega)}{\partial x_i} = v_i, \quad \text{by invoking Daskin’s theorem, we obtain that } \frac{\partial Q_i(x_i, \omega)}{\partial x_i} = \text{conv}\{v_i, u_i, v_i, \omega) \in S^{opt}_{i,\omega}(x_i)\}.$$

In the simulations, we set $N = 5$, $\mu = 1$, $a = 2$, $b = 0.5$, $\text{cap}_i = 0.3 + 0.1 \sqrt{i}$, and $\eta_i = (N - 2.5)b$. 

| parameters \( \mu \) | synchronous empirical | theoretical empirical | randomized empirical | theoretical empirical | asynchronous empirical | theoretical empirical |
|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| \( \eta = a^{*} \) | 2.03 | 1.89 | 2.64e-03 | 1.98e+01 | 1.43e-03 | 1.36e+01 |
| \( \eta = a^{*+} \) | 4.76e-04 | 7.18e-01 | 7.42e-04 | 1.73e+01 | 3.24e-04 | 1.36e+01 |
| \( \eta = a^{*--} \) | 1.33e-04 | 3.18e-01 | 2.27e-04 | 1.53e+01 | 7.94e-05 | 1.37e+01 |
| \( \eta = a^{*--} \) | 2.26e-03 | 3.89e+00 | 3.3e-03 | 2.2e+00 | 4.33e-03 | 3.69e+01 |
| \( \eta = a^{*--} \) | 9.39e-04 | 2.33e+00 | 3.3e-03 | 2.09e+00 | 1.72e-03 | 3.69e+01 |
| \( \eta = a^{*--} \) | 1.33e-04 | 2.01e+01 | 2.06e-02 | 3.16e+01 | 1.11e-02 | 9.62e+01 |
| \( \eta = a^{*--} \) | 7.53e-03 | 2.01e+01 | 2.06e-02 | 3.16e+01 | 6.55e-03 | 9.62e+01 |

Table 4: Comparison of theoretical and empirical error
Suppose \( C_i(x_i) = \frac{\eta_i x_i^2}{2} \). Let \( h_{i,\omega} \) and \( d_{i,\omega} \) be random variables satisfying the uniform distributions \([0.45, 0.55]\) and \([0.3, 0.4]\), respectively. We now present simulations for the synchronous algorithm and examine its empirical convergence rate and iteration complexity, where the empirical results are obtained by averaging across 50 trajectories. Suppose \( j_{i,k} = \left\lceil \frac{1}{a_k} \right\rceil \) steps of \((\text{SA}_{i,k})\) are taken at major iteration \( k \) to get an inexact solution to (7). The rate of convergence of \( u_k \) is shown in Figure 11, which demonstrates that the iterates converge in mean to the unique equilibrium at a linear rate. The empirical relation between \( \epsilon \) and \( K(\epsilon) \) is shown in Figure 12, from which it is seen that the empirical data aligns well with the corresponding quadratic fit. It demonstrates that the empirical iteration complexity is still of order \( O(1/\epsilon^2) \) in the settings of a two-stage stochastic Nash game.

\[ \text{Figure 11: Linear Convergence} \quad \text{Figure 12: Iteration Complexity} \]

8 Concluding Remarks

This paper considers a class of Nash games where each player’s payoff function is expectation-valued. We propose a synchronous inexact proximal BR scheme to solve the problem as well as a randomized and an asynchronous variant. Under suitable contractive properties on the proximal BR map, we separately prove that all proposed schemes produce iterates that converge in mean to the unique equilibrium at a linear rate. In addition, we derive the overall iteration complexity for computing an \( \epsilon \)-Nash equilibrium in terms of projected gradient steps. Furthermore, we consider generalizations that allow for private recourse by allowing for each player to solve a two-stage stochastic Nash game with either linear or quadratic second-stage problems. Finally, preliminary numerical studies support the theoretical findings in terms of asymptotic behavior and rate statements.

A Proof of Lemma 4

Define \( t_{i,k} \triangleq x_i + \chi \left( \hat{x}_i(x_k) - x_i \right) \), and \( e_{i,k} \triangleq \chi \left( x_{i,k+1} - \hat{x}_i(x_k) \right) \). Then \( x_{i,k+1} = t_{i,k} + e_{i,k} \). By Assumption 3, we have that

\[
\mathbb{E} \left[ \| t_{i,k} - x_i \|^2 | \mathcal{F}_k \right] = (1 - p_i) \| x_{i,k} - x_i \|^2 + p_i \| \hat{x}_i(x_k) - x_i \|^2. \tag{A.1}
\]

For \( x = \{x_i\}_{i=1}^N \in X \) and \( P = \text{diag} \left\{ \frac{1}{p_1} \otimes I_{n_1}, \ldots, \frac{1}{p_N} \otimes I_{n_N} \right\} \), we define the weighted norm

\[
\| x \|^2_P = x^T P x = \sum_{i=1}^N \| x_i \|^2 / p_i. \tag{A.2}
\]

Then by denoting \( t_k = \{t_{i,k}\}_{i=1}^N \), from (A.1) and (A.2) it follows that

\[
\mathbb{E} \left[ \| t_k - x^* \|^2_P | \mathcal{F}_k \right] = \sum_{i=1}^N \frac{1}{p_i} \mathbb{E} \left[ \| t_{i,k} - x_i^* \|^2 | \mathcal{F}_k \right] = \| x_k - x^* \|^2_P + \sum_{i=1}^N \| \hat{x}_i(x_k) - x_i^* \|^2 - \sum_{i=1}^N \| x_{i,k} - x_i^* \|^2 \leq \| x_k - x^* \|^2_P - (1 - a^2) \| x_k - x^* \|^2, \tag{A.3}
\]

\[ 31 \]
where the last inequality follows by \( \hat{x}(x^*) = x^* \) and the inequality (9). By (33) and Assumption \( 5 \) we see that for any \( i \in \mathcal{N} \)

\[
E \left[ \left\| e_{i,k} \right\|_F^2 | \mathcal{F}_k \right] = p_i E \left[ \left\| x_{i,k+1} - \hat{x}_i(x_k) \right\|_F^2 | \mathcal{F}_k \right] \leq p_i \alpha_{i,k}^2 \ \text{a.s.} \quad (A.4)
\]

Then by the definition (A.2), we obtain the following:

\[
E \left[ \left\| e_k \right\|_F^2 | \mathcal{F}_k \right] \leq \sum_{i=1}^N \alpha_{i,k}^2 \ \text{a.s.} \quad (A.5)
\]

Note that by the Cauchy-Schwarz inequality, \( \left\| x_{i,k+1} - x_i^* \right\|^2 \leq \left\| t_{i,k} - x_i^* \right\|^2 + \left\| e_{i,k} \right\|^2 + 2 \| t_{i,k} - x_i^* \| \| e_{i,k} \| \). Then by taking expectations conditioned on \( \mathcal{F}_k \), and by the condition Jensen’s inequality, we have that

\[
E \left[ \left\| x_{i,k+1} - x_i^* \right\|^2 | \mathcal{F}_k \right] \leq E \left[ \left\| t_{i,k} - x_i^* \right\|^2 | \mathcal{F}_k \right] + E \left[ \left\| e_{i,k} \right\|^2 | \mathcal{F}_k \right] + 2 \sqrt{E \left[ \left\| t_{i,k} - x_i^* \right\|^2 | \mathcal{F}_k \right] E \left[ \left\| e_{i,k} \right\|^2 | \mathcal{F}_k \right]}
\]

Since \( t_{i,k} \in X_i \), by (16) and invoking the definition of weighted norm (A.2) we have that

\[
E \left[ \left\| x_{k+1} - x^* \right\|^2 | \mathcal{F}_k \right] \leq E \left[ \left\| t_k - x^* \right\|^2 | \mathcal{F}_k \right] + E \left[ \left\| e_k \right\|^2 | \mathcal{F}_k \right] + 2 \sum_{i=1}^N \frac{D_{i,k}}{p_i} \sqrt{E \left[ \left\| e_{i,k} \right\|^2 | \mathcal{F}_k \right]}
\]

\[
\leq \left\| x_k - x^* \right\|^2_2 - (1 - a^2) \left\| x_k - x^* \right\|^2_2 + \sum_{i=1}^N \alpha_{i,k}^2 + 2 \sum_{i=1}^N \frac{D_{i,k}}{p_i} \alpha_{i,k},
\]

where the last inequality is derived by (A.3), (A.4), and (A.5). Since \( 0 \leq \alpha_{i,k} < 1 \) and \( \sum_{k=0}^{\infty} \alpha_{i,k} < \infty \ a.s \), we have that \( \sum_{k=0}^{\infty} \alpha_{i,k}^2 < \infty \ a.s \). Then by the Robbins-Siegmund theorem (42, Theorem 1), \( \| x_k - x^* \|^2_2 \) converges almost surely and \( \sum_{k=0}^{\infty} (1 - a^2) \left\| x_k - x^* \right\|^2 < \infty \ a.s \). Consequently, \( \| x_k - x^* \|^2 \) converges to zero almost surely, and hence we obtain the result.

\[\Box\]

### B Proof of Lemma \([5]\)

Note that \( \| x_k - x^* \|^2 \geq p_{\text{min}} \sum_{i=1}^N \| x_{i,k} - x_i^* \|^2 / p_i = p_{\text{min}} \| x_k - x^* \|^2_2 \), where \( p_{\text{min}} = \min_{i \in \mathcal{N}} p_i \). Then by (A.3), we get

\[
E \left[ \| t_k - x^* \|^2_2 \big| \mathcal{F}_k \right] \leq \left( 1 - p_{\text{min}}(1 - a^2) \right) \| x_k - x^* \|^2_2 \ \text{a.s.,}
\]

and hence by the conditional Jensen’s inequality, we obtain the following bound: \( E \left[ \| t_k - x^* \|_P \big| \mathcal{F}_k \right] \leq \hat{a} \| x_k - x^* \|_P \ \text{a.s.} \) Then by invoking that \( x_{k+1} = t_k + e_k \), the triangle inequality and Jensen’s inequality, we have that

\[
E \left[ \| x_{k+1} - x^* \|_P \right] \leq E \left[ \| t_k - x^* \|_P \right] + E \left[ \| e_k \|_P \right] \leq \hat{a} E \left[ \| x_k - x^* \|_P \right] + \sqrt{E \left[ \| e_k \|^2_2 \right]}.
\]

Since \( \mathbb{P}(\beta_{i,k} = m) = \binom{k}{m} p_i^m (1 - p_i)^{k-m} \) for all \( k \geq 1 \). Then for any \( k \geq 1 \),

\[
E[\eta^{2\beta_{i,k}}] = \sum_{m=0}^k \eta^{2m} \mathbb{P}(\beta_{i,k} = m) = \sum_{m=0}^k \binom{k}{m} p_i^m (1 - p_i)^{k-m} \eta^{2m}
\]

\[
= \sum_{m=0}^k \binom{k}{m} (p_i \eta^2)^m (1 - p_i)^{k-m} = (p_i \eta^2 + 1 - p_i)^k
\]

\[
= (1 - p_i(1 - \eta^2))^k \leq (1 - p_{\text{min}}(1 - \eta^2))^k \triangleq \eta^{2k} \forall i \in \mathcal{N}.
\]
Thus, by (B.2), we have the following inequality for any $k \geq 0$, we obtain $E(\alpha_{i,k}^2) \leq \eta^2 \eta^{2k}$, and hence by (A.5) we have that

$$E[\|x_k\|^2] \leq \sum_{i=1}^N E[\alpha_{i,k}^2] \leq N \eta^2 \eta^{2k} \forall k \geq 0.$$  

Thus, by (B.2), we have the following inequality for any $k \geq 1$:

$$E[\|x_k - x^*\|_P] \leq \tilde{a}E[\|x_{k-1} - x^*\|_P + \sqrt{N} \eta \tilde{\eta}^{-1/2} \tilde{\eta}].$$

$$\leq \tilde{a}^k \|x_0 - x^*\| + \sqrt{N} \eta \tilde{\eta}^{-1} \sum_{j=1}^k \tilde{a}^{k-j} \tilde{\eta}^j \leq C \left( \sum_{i=1}^N \tilde{p}_i^{-1} \right)^{1/2} \tilde{\eta}^k + \sqrt{N} \eta \tilde{\eta}^{-1} k \tilde{\eta},$$

where $\tilde{\eta} \triangleq \max\{\tilde{\eta}, \tilde{\eta}\}$. Then by Lemma 2 we have that

$$E[\|x_k - x^*\|_P] \leq C \left( \sum_{i=1}^N \tilde{p}_i^{-1} \right)^{1/2} \tilde{q}^k + \sqrt{N} \eta \tilde{\eta}^{-1} D \tilde{q}^k \leq \sqrt{N}(\tilde{C} + \tilde{D}) \tilde{q}^k,$$

where $\tilde{q} > \tilde{\eta}$, $D \triangleq 1/\ln((\tilde{q}/\tilde{\eta})^e)$, $\tilde{C} \triangleq C \left( \sum_{i=1}^N N^{-1} \tilde{p}_i^{-1} \right)^{1/2}$, and $\tilde{D} \triangleq D\eta \tilde{\eta}^{-1}$.

\section{Proof of Lemma 7}

For any $i \in I_k$, by the triangle inequality we have that

$$\|x_{i,k+1} - x^*_i\| \leq \|x_{i,k+1} - \tilde{x}_i(y^*_k)\| + \|\tilde{x}_i(y^*_k) - \tilde{x}_i(x^*)\|.$$  

Then by taking expectations conditioned on $F_k$, by (38) and the conditional Jensen’s inequality, we obtain:

$$\mathbb{E}[\|x_{i,k+1} - x^*_i\|_F] \leq \alpha_{i,k} + \mathbb{E}[\|\tilde{x}_i(y^*_k) - \tilde{x}_i(x^*)\|_F] \quad \text{a.s.} \quad (C.1)$$

Since $\alpha_{i,k}$ is deterministic by Assumption 4 (a), by taking unconditional expectations on both sides of (C.1), and by invoking $y^*_k = (x_{1,k-\tau_1(k)}, \ldots, x_{N,k-\tau_N(k)})$ and (40), we have that for any $i \in I_k$:

$$\mathbb{E}[\|x_{i,k+1} - x^*_i\|] \leq \alpha_{i,k} + \max_{j \in N} \mathbb{E}[\|x_{i,k-\tau_j(k)} - x^*_j\|]. \quad (C.2)$$

We now prove inequality (41) by induction. It is obvious that (41) holds for $k = 0$ by $\mathbb{E}[\|x_{i,0} - x^*_i\|] \leq C$ for all $i \in N$. Inductively, we assume that (41) holds for all $k$ up to some nonnegative integer $k$.

We first prove the following inequality:

$$\max_{j \in N} \mathbb{E}[\|x_{j,k-\tau_j(k)} - x^*_j\|] \leq (C + k) \rho^{\max\{0,p-n_0\}} \forall k \in [pB_1, \tilde{k}]. \quad (C.3)$$

Notice that $\lfloor \tilde{k}/B_1 \rfloor = p$. Then by the induction that (41) holds for all $k \leq \tilde{k}$, we have

$$\max_{j \in N} \mathbb{E}[\|x_{j,k} - x^*_j\|] \leq (C + k) \rho^p \forall k \in [pB_1, \tilde{k}] \quad (C.4)$$

If $B_2 = 0$, then $k - \tau_j(k) = k$ and $n_0 = 0$ by its definition. Then by (C.4), we obtain the following:

$$\max_{j \in N} \mathbb{E}[\|x_{j,k-\tau_j(k)} - x^*_j\|] = \max_{j \in N} \mathbb{E}[\|x_{j,k} - x^*_j\|] \leq (C + k) \rho^p = (C + k) \rho^{\max\{0,p-n_0\}} \forall k \in [pB_1, \tilde{k}].$$
Thus, (C.3) holds for $B_2 = 0$. If $B_2$ satisfies $(n_0 - 1)B_1 + 1 \leq B_2 \leq n_0B_1$ for some positive integer $n_0 \geq 1$ and $k \in [pB_1, k]$, then by (10), we obtain

$$\max \{0, (p - n_0)B_1\} \leq \max \{0, k - n_0B_1\} \leq \max \{0, k - B_2\} \leq k - \tau_2(k) \leq k \text{ a.s.}$$

Then we have the following for all $k \in [pB_1, \bar{k}]$:

$$\max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,k-\tau_2(k)} - x^*_{j}\|] \leq \max_{0 \leq t \leq k} \max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,t} - x^*_{j}\|]. \quad (C.5)$$

We consider the following two possible cases:

i) If $\max \{0, (p - n_0)B_1\} = 0$, then by the inductive assumption, we have that for any $k \in [pB_1, \bar{k}]$:

$$\max_{0 \leq t \leq k} \max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,t} - x^*_{j}\|] \leq \max_{0 \leq t \leq k} (C + t)^{\frac{1}{\beta B_1}} \leq C + k \quad \text{(since } \rho \in (0,1)).$$

ii) If $\max \{0, (p - n_0)B_1\} = (p - n_0)B_1$, then by the inductive assumption, we have that for any $k \in [pB_1, \bar{k}]$:

$$\max_{0 \leq t \leq k} \max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,t} - x^*_{j}\|] \leq \max_{0 \leq t \leq k} (C + t)^{\frac{1}{\beta B_1}} \leq (C + k)^{p - n_0} \quad \text{(since } \rho \in (0,1)).$$

Combining cases i) and ii), by (C.5), we have the following bound: $\max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,k-\tau_2(k)} - x^*_{j}\|] \leq (C + k)^{p - n_0}$, and hence (C.3) holds for $B_2 \geq 1$. Consequently, we have shown (C.3).

We will validate that (41) holds for $k = \bar{k} + 1$ by considering two cases: $k = (p + 1)B_1 - 1$ or $\bar{k} \in [pB_1, (p + 1)B_1 - 1)$ for some nonnegative integer $p$.

**Case 1:** $\bar{k} = (p + 1)B_1 - 1$. By the inductive assumption, it is seen that (41) holds for any $k \in [pB_1, (p + 1)B_1)$. Note that each player updates at least once during any time interval of length $B_1$ by Assumption 4. Then player $i$ updates its strategy at least $p$ times in the time interval $[0, pB_1)$, and there exists at least one integer $k_i \in [pB_1, (p + 1)B_1)$ such that $i \in I_{k_i}$. Set $k_i$ to be the largest integer in the set $[pB_1, (p + 1)B_1)$ such that $i \in I_{k_i}$. Then $\beta_i,k_i \geq p + 1$, hence by (C.2) and (C.3), we derive the following:

$$\mathbb{E}[\|x_{i,k_i+1} - x^*_i\|] \leq \alpha_{i,k_i} + a_{\infty} \max_{j \in \mathcal{N}} \mathbb{E}[\|x_{j,k_i-\tau_i(k_i)} - x^*_j\|]$$

$$\leq \eta^{p+1} + a_{\infty}(C + k_i)^{\rho \max\{p-n_0,0\}} \quad \text{(since } \alpha_{i,k_i} = \eta_a^i,k_i \text{ and } \eta \in (0,1))$$

$$\leq \rho^{p+1} + (C + k_i)^{p+1} \quad \text{(since } 1 \geq \rho \geq \max\{a_{\infty},\rho^{-n_0}\eta\})$$

$$\leq (C + k_i + 1)^{p+1}.$$

Therefore, for any $i \in \mathcal{N}$, by the selection of $k_i$ and the definition of Algorithm 3, we know that

$$\mathbb{E}[\|x_{i,k+1} - x^*_i\|] = \mathbb{E}[\|x_{i,k_i+1} - x^*_i\|] \leq (C + k_i + 1)^{p+1} \leq (C + k + 1)^{p+1} \forall k : k_i \leq k < (p + 1)B_1.$$
For any \( i \notin I_k \), by (C.4), we have
\[
E[\|x_{i,k+1} - x_i^*\|] = E[\|x_{i,k} - x_i^*\|] \leq (C + \bar{k})\rho^k < (C + \bar{k} + 1)\rho^k, \text{ and hence (41) holds for } k = k + 1.
\]

By combing Cases 1 and 2, (41) holds for \( k = \bar{k} + 1 \). Thus, by induction, we obtain (41) for any \( k \geq 0 \).

Since
\[
\left\lfloor \frac{k}{B_1} \right\rfloor \geq \frac{k}{B_1} - \frac{B_1 - 1}{B_1} \text{ and } 0 < \rho < 1, \text{ by (41) we derive}
\]
\[
\max_{i \in N} E[\|x_{i,k} - x_i^*\|] \leq (C + k)\rho^{B_1 - \frac{B_1 - 1}{B_1}} = \rho^{\frac{k}{B_1}} (C + k)\rho^{\frac{k}{B_1}} = \rho^{\frac{B_1 - 1}{B_1}} (C + k)c^k, \quad \forall k \geq 0. \quad \text{(C.7)}
\]

From Lemma 2 there exist scalars \( q \) and \( D \) satisfying \( q \in (c, 1) \) and \( D \geq 1/\ln((q/c)^e) \) such that \( kc^k \leq Dq^k \) \( \forall k \geq 0 \), which incorporating with (C.7) and \( c^k < q^k \) yields (42).

\[\square\]

References

[1] Abada I, de Maere d’Aertrycke G, Smeers Y (2017) On the multiplicity of solutions in generation capacity investment models with incomplete markets: a risk-averse stochastic equilibrium approach. Math. Program. 165(1):5–69.

[2] Agarwal A, Duchi JC (2011) Distributed delayed stochastic optimization. Advances in Neural Information Processing Systems, 873–881.

[3] Ahmadi H (2016) On the Analysis of Data-driven and Distributed Algorithms for Convex Optimization Problems. Ph.D. thesis, Department of Industrial and Manufacturing Engineering, Pennsylvania State University.

[4] Aysal TC, Yildiz ME, Sarwate AD, Scaglione A (2009) Broadcast gossip algorithms for consensus. IEEE Transactions on Signal processing 57(7):2748–2761.

[5] Başar T (2007) Control and game-theoretic tools for communication networks. Appl. Comput. Math. 6(2):104–125, ISSN 1683-3511.

[6] Basar T, Olsder GJ (1999) Dynamic noncooperative game theory, volume 23 (SIAM).

[7] Beale EML (1955) On minimizing a convex function subject to linear inequalities. J. Roy. Statist. Soc. Ser. B. 17:173–184; discussion, 194–203, (Symposium on linear programming.).

[8] Bertsekas DP (1971) Control of uncertain systems with a set-membership description of the uncertainty. Technical report, DTIC Document.

[9] Bertsekas DP, Tsitsiklis JN (1989) Parallel and distributed computation: numerical methods, volume 23 (Prentice hall Englewood Cliffs, NJ).

[10] Birge JR, Louveaux F (1997) Introduction to Stochastic Programming: Springer Series in Operations Research (Springer).

[11] Boyd S, Ghosh A, Prabhakar B, Shah D (2006) Randomized gossip algorithms. IEEE/ACM Transactions on Networking (TON) 14(S1):2508–2530.

[12] Chazan D, Miranker W (1969) Chaotic relaxation. Linear algebra and its applications 2(2):199–222.

[13] Chen X, Qi L, Womersley RS (1995) Newton’s method for quadratic stochastic programs with recourse. Journal of Computational and Applied Mathematics 60(1-2):29–46.

[14] Combettes PL, Pesquet JC (2015) Stochastic quasi-fejér block-coordinate fixed point iterations with random sweeping. SIAM Journal on Optimization 25(2):1221–1248.
[15] Dantzig GB (1955) Linear programming under uncertainty. Management science 1(3-4):197–206.

[16] Dorn WS (1960) Duality in quadratic programming. Quarterly of Applied Mathematics 18(2):155–162.

[17] Ehrenmann A, Smeers Y (2011a) Generation capacity expansion in a risky environment: a stochastic equilibrium analysis. Operations research 59(6):1332–1346.

[18] Ehrenmann A, Smeers Y (2011b) Generation capacity expansion in a risky environment: A stochastic equilibrium analysis. Operations Research 59(6):1332–1346.

[19] Facchinei F, Pang J (2009) Nash equilibria: The variational approach. Convex Optimization in Signal Processing and Communications (Cambridge University Press (Cambridge, England)).

[20] Facchinei F, Pang JS (2003) Finite-dimensional variational inequalities and complementarity problems. Vol. I. Springer Series in Operations Research (New York: Springer-Verlag). ISBN 0-387-95580-1.

[21] Friedlander MP, Schmidt M (2012) Hybrid deterministic-stochastic methods for data fitting. SIAM Journal on Scientific Computing 34(3):A1380–A1405.

[22] Fudenberg D, Levine DK (1998) The theory of learning in games, volume 2 of MIT Press Series on Economic Learning and Social Evolution (Cambridge, MA: MIT Press), ISBN 0-262-06194-5.

[23] Fudenberg D, Tirole J (1991) Game Theory (MIT Press).

[24] Gürkan G, Pang JS (2009) Approximations of Nash equilibria. Mathematical Programming 117(1-2):223–253.

[25] Kannan A, Shanbhag UV (2012) Distributed computation of equilibria in monotone Nash games via iterative regularization techniques. SIAM Journal on Optimization 22(4):1177–1205.

[26] Kannan A, Shanbhag UV, Kim HM (2011) Strategic behavior in power markets under uncertainty. Energy Systems 2(2):115.

[27] Kannan A, Shanbhag UV, Kim HM (2013) Addressing supply-side risk in uncertain power markets: stochastic Nash models, scalable algorithms and error analysis. Optimization Methods and Software 28(5):1095–1138.

[28] Koshal J, Nedic A, Shanbhag UV (2013) Regularized iterative stochastic approximation methods for stochastic variational inequality problems. IEEE Transactions on Automatic Control 58(3):594–609.

[29] Kulkarni AA, Shanbhag UV (2012) Recourse-based stochastic nonlinear programming: properties and Benders-SQP algorithms. Computational Optimization and Applications 51(1):77–123.

[30] Lee GM, Tam NN, Yen ND (2006) Quadratic programming and affine variational inequalities: a qualitative study, volume 78 (Springer Science & Business Media).

[31] Lu Z, Xiao L (2013) Randomized block coordinate non-monotone gradient method for a class of nonlinear programming. arXiv preprint [arXiv:1306.5918].

[32] Markowitz H (1952) Portfolio selection. The journal of finance 7(1):77–91.

[33] Nash JF Jr (1950) Equilibrium points in n-person games. Proc. Nat. Acad. Sci. U. S. A. 36:48–49.

[34] Nesterov Y (2012) Efficiency of coordinate descent methods on huge-scale optimization problems. SIAM Journal on Optimization 22(2):341–362.
[35] O’Cinneide C, Scherer B, Xu X (2006) Pooling trades in a quantitative investment process. *The Journal of Portfolio Management* 32(4):33–43.

[36] Pang JS, Scutari G, Palomar DP, Facchinei F (2010) Design of cognitive radio systems under temperature-interference constraints: A variational inequality approach. *IEEE Transactions on Signal Processing* 58(6):3251–3271.

[37] Pang JS, Sen S, Shanbhag UV (2017) Two-stage non-cooperative games with risk-averse players. *Mathematical Programming* 165(1):235–290.

[38] Polyak B (1987) *Introduction to optimization* (New York: Optimization Software, Inc.).

[39] Ravat U, Shanbhag UV (2011) On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games. *SIAM Journal on Optimization* 21(3):1168–1199.

[40] Richtárik P, Takáč M (2014) Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function. *Mathematical Programming* 144(1-2):1–38.

[41] Robbins H, Monro S (1951) A stochastic approximation method. *The annals of mathematical statistics* 400–407.

[42] Robbins H, Siegmund D (1985) A convergence theorem for non negative almost supermartingales and some applications. *Herbert Robbins Selected Papers*, 111–135 (Springer).

[43] Rockafellar RT, Wets RB (1986) A lagrangian finite generation technique for solving linear-quadratic problems in stochastic programming. *Stochastic Programming 84 Part II*, 63–93 (Springer).

[44] Schmidt M, Roux NL, Bach FR (2011) Convergence rates of inexact proximal-gradient methods for convex optimization. *Advances in neural information processing systems*, 1458–1466.

[45] Scutari G, Facchinei F, Pang JS, Palomar DP (2014) Real and complex monotone communication games. *IEEE Transactions on Information Theory* 60(7):4197–4231.

[46] Scutari G, Facchinei F, Song P, Palomar DP, Pang JS (2013) Decomposition by partial linearization in multiuser systems. *Acoustics, Speech and Signal Processing (ICASSP), 2013 IEEE International Conference on*, 4424–4428 (IEEE).

[47] Scutari G, Palomar DP, Barbarossa S (2009) The mimo iterative waterfilling algorithm. *IEEE Transactions on Signal Processing* 57(5):1917–1935.

[48] Shanbhag UV (2006) Decomposition and sampling methods for stochastic equilibrium problems. *Department of Management Science and Engineering (Operations Research), Stanford University. Ph. D. thesis*.

[49] Shanbhag UV, Infanger G, Glynn PW (2011) A complementarity framework for forward contracting under uncertainty. *Operations Research* 59(4):810–834.

[50] Shanbhag UV, Pang JS, Sen S (2016) Inexact best-response schemes for stochastic Nash games: Linear convergence and iteration complexity analysis. *Decision and Control (CDC), 2016 IEEE 55th Conference on*, 3591–3596 (IEEE).

[51] Shapiro A, Dentcheva D, Ruszczynski A (2009) Lectures on stochastic programming. *MPS-SIAM series on optimization* 9(1).

[52] Tseng P (1991) On the rate of convergence of a partially asynchronous gradient projection algorithm. *SIAM Journal on Optimization* 1(4):603–619.
[53] Xu Y, Yin W (2015) Block stochastic gradient iteration for convex and nonconvex optimization. *SIAM Journal on Optimization* 25(3):1686–1716.

[54] Yang Y, Rubio F, Scutari G, Palomar DP (2013) Multi-portfolio optimization: A potential game approach. *IEEE Transactions on Signal Processing* 61(22):5590–5602.

[55] Yin H, Shanbhag UV, Mehta PG (2011) Nash equilibrium problems with scaled congestion costs and shared constraints. *IEEE Transactions on Automatic Control* 56(7):1702–1708.

[56] Yousefian F, Nedić A, Shanbhag UV (2016) Self-tuned stochastic approximation schemes for non-lipschitzian stochastic multi-user optimization and Nash games. *IEEE Transactions on Automatic Control* 61(7):1753–1766.