On modified tachyon DBI action

Mohammad R. Garousi
Department of Physics, Ferdowsi University of Mashhad,
P.O. Box 1436, Mashhad, Iran
School of Physics, Institute for research in fundamental sciences (IPM),
P.O. Box 19395-5531, Tehran, Iran.

ABSTRACT

Recently a modification of the tachyon DBI action has been proposed in which the tachyon carries the internal CP matrix $\sigma_1$ and $\sigma_2$. In this paper, we find the momentum expansion of the disk level S-matrix element of four tachyons and one gauge field in superstring theory and show that the first and second order terms of the expansion are in perfect agreement with the above tachyon DBI action.
1 Introduction

A natural mechanism for inflation in string theory is to have D-brane-anti-D-brane separated along an extra dimension [1, 2, 3, 4]. In this model, the brane separation plays the role of inflaton. When the separation is smaller than the string scale, the open string stretching between the branes become tachyonic at its ground state [5, 6, 7]. At this time the inflation stops and universe undergoes the reheating period when the energy of inflaton decays to the particles of the Standard Model [8, 9]. To understand this period, one needs to know the effective action of brane-antibrane. Even though the mass scale of the tachyon and the mass scale of the massive modes of the open string are the same order, there are various arguments that indicate there must be an effective action for brane-antibrane in which there are only tachyon and massless fields [10]. The effective action of brane-antibrane may be found from the effective action of two non-BPS branes by \((-1)^{F_L}\) orbifolding [11].

The effective theory of non-BPS branes has two parts, i.e.,

\[ S_{\text{non-BPS}} = S_{\text{DBI}} + S_{\text{WZ}} \]

where \(S_{\text{DBI}}/S_{\text{WZ}}\) must be an extension of the DBI/WZ action of BPS branes in which the tachyon modes of the non-BPS branes are included appropriately. There are various methods to study these actions. It has been shown in \([12, 13]\) that the tachyon DBI action \([14, 15, 16, 17]\) can capture many properties of decay of non-BPS branes around the stable point of the tachyon potential. In \([18]\), using the S-matrix method, an extension for the WZ action of non-BPS branes/brane-antibrane has been proposed in which the gauge field has been replaced by the superconnection. This action is consistent with the leading order terms of the momentum expansion of the S-matrix element of one RR and two tachyons \([18]\). This form of action has been then confirmed by the BSFT method in \([19, 20]\). See \([21, 22]\) for consistency of the WZ action with other S-matrix elements.

However, the BSFT and the S-matrix methods produce different actions for the DBI part which may be related to each other by some field redefinition. The S-matrix method indicates that the leading order terms of the momentum expansion of any S-matrix element which involves two tachyon vertex operators are consistent with the usual tachyon DBI action \([15]\). However, the momentum expansion of any S-matrix element that involves four or more tachyon vertex operators must be consistent with the modification of the tachyon DBI action in which the tachyon carries the Pauli matrices \(\sigma_1\) and/or \(\sigma_2\) \([22, 23]\). This action in flat background is

\[
S_{\text{DBI}} \sim \int d^{p+1}\sigma \text{Str} \left( V(T_i T_i) \sqrt{1 + \frac{1}{2}[T_i, T_j][T_j, T_i]} \right. \\
\left. \times \sqrt{-\det(\eta_{ab} + 2\pi \alpha' F_{ab} + 2\pi \alpha' D_a T_i (Q^{-1})_{ij} D_b T_j)} \right),
\]

(1)
where \( V(T_i T_i) = e^{\pi \alpha' m^2 T_i T_i / 2} \) and \( m^2 = -1/(2\alpha') \). In above action

\[
Q_{ij} = I \delta_{ij} - i[T_i, T_j]
\]

(2)

The subscripts \( i, j = 1, 2 \), i.e., \( T_1 = T\sigma_1, T_2 = T\sigma_2 \). In above action there is no sum over \( i, j \). To fix these indices, one must first expand the square roots and then choose two of indices to be 2 and all others to be 1. The trace in above equation should be completely symmetric between all matrices of the form \( F_{ab}, D_a T_i, [T_i, T_j] \) and individual \( T_i \) of the potential \( V(T_i T_i) \). This symmetric trace makes the above rule for fixing the indices \( i, j \) to be unambiguous. Around the stable point of the tachyon potential and for abelian gauge group, the above action reduces to the usual tachyon DBI action with the potential \( T^4 V \) (see the Discussion section).

The above modification to the tachyon DBI action has been found in [22] by studying the S-matrix element of one RR and three tachyons. The tachyon pole in this amplitude must be reproduced in field theory by considering appropriate four-tachyon couplings. The four-tachyon couplings in the usual tachyon DBI action can not reproduce the tachyon pole of the S-matrix element. However, by assuming the tachyon in the action carries the Pauli matrices as above, one can produce the tachyon pole exactly [22]. This rule is consistent with the observation that the open string states of non-BPS branes carry internal matrices [24]. To see this, we note that the tachyon of non-BPS branes in picture (0) carries internal matrix \( \sigma_1 \) [24]. The picture changing operator, on the other hand, carries the internal matrix \( \sigma_3 \) [25], hence, the tachyon in picture \((-1)\) must carry internal matrix \( \sigma_2 \). Now, using the fact that the superghost charge of the disk level amplitude is \(-2\), i.e., two of the vertex operators must be in \((-1)\) picture and all other must be in \((0)\) picture, and using the observation that the momentum expansion of the string theory S-matrix element of tachyons in which two of the tachyon vertex operators to be in \((-1)\) picture and all other to be in \((0)\) picture is very similar to the momentum expansion of the massless transverse scalars [22, 23], one finds the above rule for fixing the Pauli matrices in the modified tachyon DBI action (1).

In this paper we would like to compare the action (1) with the S-matrix element of four tachyons and one gauge field. This study can confirm the presence of covariant derivative in the action and more importantly this can confirm the presence of \( Q_{ij} \) which does not appear in the four tachyon couplings studied in [22]. So in the next section we write the S-matrix element by including the internal matrices in the vertex operators, and find the momentum expansion of the amplitude. The amplitude has massless/tachyon poles and contact terms at each order. In section 3, using the action (1), we reproduce exactly the massless/tachyon poles and the contact terms of the S-matrix element at the first and second orders. Section 4 devoted to a brief discussion.
2 Tachyon amplitude in superstring theory

The S-matrix element of four tachyons and one gauge field of N non-BPS D$p$-branes in superstring theory may be given by the following correlation function:

$$\sum_{\text{non-cyclic}} \int dx_1 dx_2 dx_3 dx_4 dx_5 \left( (V_{-1}^T(x_1)V_{-1}^T(x_2)V_0^T(x_3)V_0^T(x_4)V_0^A(x_5)) \right)$$

(3)

where the vertex operators for tachyons and gauge field are given by

$$V_0^T = (2ik \cdot \psi) e^{2ikX} \lambda \otimes \sigma_1$$

$$V_{-1}^T = e^{2ikX} e^{-\Phi} \lambda \otimes \sigma_2$$

$$V_0^A = \xi_a (\partial X^a + 2ik \cdot \psi^a) e^{2ikX} \lambda \otimes I$$

where $\xi_a$ is the polarization of gauge field. The matrix $\lambda$ is the external Chan-Paton matrix and $\sigma_1, \sigma_2, I$ are the internal CP matrices. In the above vertexes, $k$ is the world volume momentum of the open string states. The on-shell condition for tachyon is $k^2 = 1/(2\alpha')$, and for gauge field is $\xi_a k^a = 0$ and $k^2 = 0$.

Momentum expansion of the above S-matrix element should produce an effective field theory in which the massless fields carry identity internal matrix, because when tachyon is set to zero the effective action of non-BPS branes must be reduced to the effective action of BPS branes that has no internal CP matrix. Hence, one finds that the momentum expansion of above amplitude should have massless pole in $s_{12}, s_{34}$, i.e., $s_{12}, s_{34} \rightarrow 0$. In particular, it does not have massless pole in $s_{23}$, i.e., to have a massless pole in this channel, the gauge field should carry $\sigma_3$ which is forbidden. To have a S-matrix element whose momentum expansion is consistent with the effective field theory, one has to add two other correlators in (3), i.e.,

$$\mathcal{A} \sim \sum_{\text{non-cyclic}} \int dx_1 dx_2 dx_3 dx_4 dx_5 \left( (V_{-1}^T(x_1)V_{-1}^T(x_2)V_0^T(x_3)V_0^T(x_4)V_0^A(x_5)) \right.\left. + \langle V_{-1}^T(x_1)V_0^T(x_2)V_0^T(x_3)V_0^T(x_4)V_0^A(x_5) \rangle \right)$$

(4)

The momentum expansion of the second correlator should have massless pole in $s_{13}, s_{24}$, i.e., $s_{13}, s_{24} \rightarrow 0$, and the last correlator should have massless pole in $s_{14}, s_{23}$, i.e., $s_{14}, s_{23} \rightarrow 0$. All above correlators should have tachyonic pole in $s_{15}, s_{25}, s_{35}$ and $s_{45}$, and massive poles in all other channels. Note that the above correlators for different ways of distributing the superconformal ghost charge are equivalent when the internal CP factors are included [23], however, their momentum expansion are different.

\footnote{Our conventions in string theory side set $\alpha' = 2$.}
Alternatively, the S-matrix element can be given by the following correlator:

$$A \sim \sum_{\text{non-cyclic}} \int dx_1 dx_2 dx_3 dx_4 dx_5 \left( \langle V_0^T(x_1)V_0^T(x_2)V_0^T(x_3)V_0^T(x_4)V_A^A(x_5) \rangle \right)$$  \hfill (5)

In this case, the gauge field carries the identity internal matrix and all tachyon vertex operators carry $\sigma_1$. The momentum expansion of this amplitude has the same massless/tachyon poles as the amplitude (4). The amplitude is symmetric under interchanging the tachyons, so the expansion here should be around

$$(s_{12}, s_{34}) \to 0 \quad + \quad (s_{13}, s_{24}) \to 0 \quad + \quad (s_{14}, s_{23}) \to 0$$  \hfill (6)

which is symmetric under interchanging the tachyons. The first, second and the third term above are corresponding to the momentum expansion of the first, second and the third correlator in (4), respectively.

The definition of Mandelstam variables is

$$s_{ij} = -\alpha'(k_i + k_j)^2$$  \hfill (7)

The number of independent kinematic factors in the scattering amplitude of $n$ states is $\frac{1}{2}(n - 3)$ \[26\]. In the present case, there are 5 independent kinematic factors. One may choose them to be $s_{12}, s_{23}, s_{34}, s_{45}, s_{15}$. Using conservation of momentums and the on-shell conditions $k_1^2 = k_2^2 = k_3^2 = k_4^2 = 1/(2\alpha'), k_5^2 = 0$, one finds that the other kinematic factors $s_{13}, s_{14}, s_{24}, s_{25}, s_{35}$ can be written in terms of the independent ones as

$$s_{13} = s_{45} - s_{12} - s_{23} - \frac{3}{2}$$
$$s_{14} = s_{23} - s_{15} - s_{45} - 1$$
$$s_{24} = s_{15} - s_{23} - s_{34} - \frac{3}{2}$$
$$s_{25} = s_{34} - s_{12} - s_{15} - 1$$
$$s_{35} = s_{12} - s_{45} - s_{34} - 1$$  \hfill (8)

One can show that the Mandelstam variables satisfy the following relation:

$$\sum_{i<j} s_{ij} = -6$$  \hfill (9)

One should write the momentum expansion of the above amplitude in terms of independent variables. In \[27\], it has been argued that the momentum expansion of a S-matrix element should be, in general, around $(k_i + k_j)^2 \to 0$ and/or $k_i \cdot k_j \to 0$ where $k_i$ is the momentum of $i$-th particle. The expansion $(k_i + k_j)^2 \to 0$ should be corresponding to the massless poles. So, in the first correlator in (4) the expansion should be around

$$(k_1 + k_2)^2, (k_3 + k_4)^2 \to 0, \quad \text{and} \quad (k_2 \cdot k_3, k_1 \cdot k_5, k_4 \cdot k_5) \to 0$$  \hfill (10)
In the second correlator in (4), since there is no massless pole for the independent variables the expansion should be around

\[(k_1 \cdot k_2, k_3 \cdot k_4, k_2 \cdot k_3, k_1 \cdot k_5, k_4 \cdot k_5) \to 0\] (11)

In the last correlator in (4), there is massless pole only in the \(s_{23}\)-channel so the expansion should be around

\[(k_2 + k_3)^2, \to 0, \text{ and } (k_1 \cdot k_2, k_3 \cdot k_4, k_1 \cdot k_5, k_4 \cdot k_5) \to 0\] (12)

We now try to calculate the correlators in (4). It has been shown in [23], that when the internal CP factors are included, the S-matrix elements in different arrangement of the picture of the vertex operators are identical. So each of the terms in (4) should be the same as

\[
\mathcal{A}' \sim \sum_{\text{non-cyclic}} \int dx_1dx_2dx_3dx_4dx_5 \langle V_0^T(x_1)V_0^T(x_2)V_0^T(x_3)V_1^T(x_4)V_{-1}^A(x_5) \rangle (13)
\]

The only difference is in their internal CP factors. The CP factor for 12345 ordering has been calculated in [29]. The result has three terms which are proportional to \(k_1 \cdot \xi, k_2 \cdot \xi\) and \(k_3 \cdot \xi\). The one which is proportional to \(k_1 \cdot \xi\) is

\[
\mathcal{A}' = -2\alpha' T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \text{Tr}(\sigma_2 \sigma_2 \sigma_1 \sigma_2 \sigma_3) k_1 \cdot \xi \\
\times (-s_{23} - 1) \beta(-s_{12}, -s_{23} - 1)\beta(-s_{45} - \frac{1}{2}, -s_{34}) \\
\times F_2(-s_{12}, -s_{25} - \frac{1}{2}, s_{15} - s_{23} - s_{34} - \frac{1}{2}; -s_{12} - s_{23} - 1, -s_{34} - s_{45} - \frac{1}{2}; 1)
\]

So the \(\text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)k_1 \cdot \xi\) part of the amplitude (4) should be equal to

\[
\mathcal{A} = i2\alpha' T_p \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)(\text{Tr}(\sigma_2 \sigma_2 \sigma_1 \sigma_1 I) - \text{Tr}(\sigma_2 \sigma_1 \sigma_2 \sigma_1 I) + \text{Tr}(\sigma_2 \sigma_1 \sigma_1 \sigma_2 I)) k_1 \cdot \xi \\
\times (-s_{23} - 1) \beta(-s_{12}, -s_{23} - 1)\beta(-s_{45} - \frac{1}{2}, -s_{34}) \\
\times F_2(-s_{12}, -s_{25} - \frac{1}{2}, s_{15} - s_{23} - s_{34} - \frac{1}{2}; -s_{12} - s_{23} - 1, -s_{34} - s_{45} - \frac{1}{2}; 1) (14)
\]

The first, second and the third term above correspond to the first, second and the third correlator in (4), respectively. Note that the above amplitude is similar to the amplitude of four massless transverse scalars and one gauge field, in particular, the appearance of minus sign in the second term above [29]. There is similar symmetry in the S-matrix element of four tachyons [22, 23].
From the poles of the Beta and Hypergeometric functions, one realizes that the amplitude has tachyon, massless and infinite tower of massive poles. The tachyon and massless poles should be reproduced by effective field theory. The momentum expansion for Beta and the Hypergeometric functions must keep only the tachyon and the massless poles and expands all other poles to produce infinite number of contact terms which are ordered in terms of the momenta of the external states.

The momentum expansion in equations (10), (11) and (12) in terms of the independent Mandelstam variables are \((s_{12}, s_{34} \rightarrow 0, s_{23} \rightarrow -1, s_{45}, s_{15} \rightarrow -1/2), (s_{23}, s_{12}, s_{34} \rightarrow -1, s_{45}, s_{15} \rightarrow -1/2)\) and \((s_{23} \rightarrow 0, s_{12}, s_{34} \rightarrow -1, s_{45}, s_{15} \rightarrow -1/2)\), respectively. One can easily check that these limits are consistent with the constraint (9). Using the package HypExp \(^2\) for expanding the Hypergeometric functions, one can expand the amplitude (14) around the above points. The result is:

\[
A^{TTTA} = 4\alpha' T_T T_T (\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \xi \cdot k_1
\]

\[
\left[ \frac{s_{23} + 1}{(s_{45} + \frac{1}{2}) s_{12}} + \frac{s_{23} + 1}{(s_{15} + \frac{1}{2}) s_{34}} + \frac{s_{12} + 1}{s_{23} (s_{45} + \frac{1}{2}) s_{12} s_{34}} + \frac{s_{34} + 1}{s_{23} (s_{15} + \frac{1}{2})} \right]
\]

\[
-\zeta(2) \left( \frac{s_{12}}{(s_{45} + \frac{1}{2})} + \frac{s_{23}}{(s_{45} + \frac{1}{2})} + \frac{3s_{12}s_{23}}{(s_{45} + \frac{1}{2})} + \frac{4s_{12}}{(s_{45} + \frac{1}{2})} + \frac{4s_{23}}{(s_{45} + \frac{1}{2})} + \frac{3}{(s_{45} + \frac{1}{2})} \right)
\]

\[
+ \frac{s_{23}s_{12}}{(s_{15} + \frac{1}{2}) s_{34}} + \frac{s_{23}s_{15}}{(s_{15} + \frac{1}{2}) s_{34}} + \frac{3s_{23}s_{34}}{(s_{15} + \frac{1}{2}) s_{34}} + \frac{4s_{23}}{(s_{15} + \frac{1}{2}) s_{34}} + \frac{4s_{34}}{(s_{15} + \frac{1}{2}) (s_{15} + \frac{1}{2})} + \frac{3}{(s_{15} + \frac{1}{2})}
\]

\[
+ \frac{s_{34}}{s_{12}} + \frac{s_{23}s_{34}}{s_{34}} + \frac{3s_{23}s_{34}}{s_{34}} + \frac{4s_{23}}{s_{12}} + \frac{4s_{34}}{s_{23}} + \frac{3}{s_{12}}
\]

\[
+ \frac{(s_{45} + \frac{1}{2})}{s_{23}} + \frac{s_{12}(s_{15} + \frac{1}{2})}{s_{23}} + \frac{s_{23}(s_{45} + \frac{1}{2})}{s_{23}} + \frac{(s_{15} + \frac{1}{2})(s_{45} + \frac{1}{2})}{s_{23}} - \frac{(s_{15} + \frac{1}{2})}{s_{23}} \right]
\]

As we have anticipated, the above expansion keeps the tachyon and the massless poles of the amplitude (14) and expands all other poles. Obviously, one can rewrite the expansion in terms of the momenta of the external states, because the expansion (10), (11) and (12) are in terms of the momenta of the external states. The terms in the second and third lines above are \(1/\alpha'\), the terms proportional to \(\zeta(2)\) are \(\alpha'\) order, and dots refers to the higher order of \(\alpha'\). In terms of \(s_{ij}\), the above terms have different \(\alpha'\) order. This indicates that the effective field theory that should reproduce them must have couplings with different \(\alpha'\) order. In the next section, we shall show that the terms in the second and the third lines above are reproduced by the terms in the first line of (16), and the terms proportional to \(\zeta(2)\) are reproduced by all other terms in (16) which have obviously different \(\alpha'\) order.
3 Amplitude in effective field theory

Now we would like to compare the above terms of the expansion of the S-matrix element with the modified tachyon DBI action (11). One can expand the square roots in the action (11) to find various couplings. The terms of the expansion which has contribution to the S-matrix element of one gauge field and four tachyons are the following:

\[
L = -\frac{T_p}{2} \text{Tr} \left( (\pi\alpha') m^2 T_2 T_2 + (\pi\alpha') D_a T_2 D^a T_2 - (\pi\alpha')^2 F_{ab} F^{ba} + \frac{1}{4} T_1 T_2 [T_2, T_1]/4 \right) - \frac{T_p}{2} (2\pi\alpha')^2 \text{Str} \left( \frac{m^4}{8} T_1 T_1 T_2 T_2 + \frac{m^2}{4} T_1 T_1 D_a T_2 D^a T_2 \right. \\
\left. - \frac{1}{4} (D_a T_1 D_b T_1 D^b T_2 D^a T_2) + \frac{1}{8} (D_a T_1 D^a T_1) (D_b T_2 D^b T_2) \right) - \frac{T_p}{2} (2\pi\alpha')^2 \text{Str} \left( \frac{i}{2} D_a T_1 [T_1, T_2] D_b T_2 F^{ba} \right) \\
- \frac{T_p}{2} (2\pi\alpha')^3 \frac{3}{2} - \text{Str} \left( D^a T_2 D_b T_2 F^{bc} F_{ca} - \frac{1}{4} D^a T_2 D_a T_2 F^{bc} F_{cb} - \frac{m^2}{4} T_2 T_2 F^{ab} F_{ba} \right)
\]

Note that the above terms are not ordered in terms of power of \(\alpha'\). This is consistent with the momentum expansion of the S-matrix element (15). We shall show that the terms in the first line above which we call them kinetic order terms, reproduce the first leading order terms in (15), and the other terms reproduce the terms in (15) which are proportional to \(\zeta(2)\).

3.1 Kinetic order terms

Performing the trace over the internal CP matrices in the first line of (16), one finds

\[
- T_p \text{Tr} \left( (\pi\alpha') m^2 T T + (\pi\alpha') D_a T D^a T - (\pi\alpha')^2 F_{ab} F^{ba} + T^4 \right)
\]

The non-abelian field strength and covariant derivative of tachyon are, respectively,

\[
F^{ab} = \partial^a A^b - \partial^b A^a - i[A^a, A^b], \quad D_a T_j = \partial_a T_j - i[A_a, T_j]
\]

Using the couplings (17), one can calculate the Feynman amplitude corresponding to the massless and tachyon poles of the string theory S-matrix element (15). Similar calculation has been done in [29]. However, the last term in the above equation does not appear in the couplings considered in [29]. The Feynman amplitude at the kinetic order term which results from this term is given by

\[
V^\alpha(T_k A_5 T) G_{\alpha\beta}(T) V^\beta((T T_1 T_2 T_3) + V^\alpha(T_1 A_5 T) G_{\alpha\beta}(T) V^\beta(T T_2 T_3 T_4) \\
= 4i\alpha' T_p \xi_5 \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5) \left( \frac{2}{s_{45} + \frac{1}{2}} + \frac{2}{s_{15} + \frac{1}{2}} \right) + \cdots
\]
where we have kept only the terms that are proportional to \( k_1 \cdot \xi_5 \text{Tr}(\lambda_1 \lambda_2 \lambda_3 \lambda_4 \lambda_5) \). Adding them to the result in [29] (eq.(23) of [29]), one finds

\[
A^{TTTT}_k = (i\alpha' T_p) k_1 \cdot \xi_5 \left( \frac{12}{s_{45} + \frac{1}{2}} + \frac{12}{s_{15} + \frac{1}{2}} - \frac{4}{s_{23}} + \frac{4s_{23}}{(s_{45} + \frac{1}{2})s_{12}} + \frac{4s_{23}}{(s_{15} + \frac{1}{2})s_{34}} + \frac{4s_{34}}{(s_{15} + \frac{1}{2})s_{23}} + \frac{4s_{12}}{(s_{45} + \frac{1}{2})s_{23}} + \frac{4s_{23}}{(s_{34}s_{12})} + \frac{4}{(s_{15} + \frac{1}{2})s_{23}} + \frac{4}{(s_{45} + \frac{1}{2})s_{23}} + \frac{4}{(s_{34}s_{12})} \right) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) + \cdots
\]  

(19)

where dots refers to the terms which have coefficients other than \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5)k_1 \cdot \xi_5 \). This is exactly the leading order terms of the momentum expansion of the S-matrix element in the string theory side (15). The next order terms in (15) are proportional to \( \xi^2 \). We now turn to the Feynman amplitudes in field theory that are proportional to \( \xi^2 \).

### 3.2 \( \xi^2 \) order terms

There are three Feynman amplitudes in the field theory at this order. One amplitude is produced by one vertex from the kinetic term of tachyon, one tachyon propagator and one vertex from four-tachyon couplings in (16). The second amplitude is produced by one vertex from the kinetic term of tachyon, one gauge field propagator and one vertex from the two-gauge-two-tachyon couplings in (16). And the last amplitude is the contact term of one-gauge-four-tachyon couplings in (16). We write them, respectively, as

\[
A^{TTTT}_\xi^{(2)} = A^{(2)} + A'_{\xi^{(2)}} + A''_{\xi^{(2)}}
\]

The Feynman amplitude \( A_{\xi^{(2)}}(T_1T_2T_3T_4A_5) \) is given by

\[
A_{\xi^{(2)}}(T_1T_2T_3T_4A_5) = V_\alpha(T_1T_2T_3T)G_{\alpha\beta}(T)V_\beta(TT_4A_5)
\]

(20)

The vertex of four-tachyons should be read from the different terms in the second line of (16). To do this, one should first perform the symmetric trace and perform the trace over the internal CP matrices, i.e.,

\[
\mathcal{L}^{TTTT} = -\frac{T_p}{2} (2\pi \alpha')^2 \text{Tr} \left( \frac{m^4}{24} T^4 + \frac{m^2}{4} \left( \frac{2}{3} TTD_a T D_a T - \frac{1}{3} T D_a T T D^a T \right) + \frac{1}{24} \left( 2 D_a T D^a T D_b T D^b T - 3 D_a T D^a T D^b T \right) \right)
\]

(21)

Now we can read the vertex of three on-shell and one off-shell tachyons from the above couplings. The vertex of four-tachyon contains terms that have group factors \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha), \text{Tr}(\lambda^2 \lambda^1 \lambda^3 \Lambda^\alpha), \text{Tr}(\lambda^3 \lambda^2 \lambda^1 \Lambda^\alpha), \text{Tr}(\lambda^1 \lambda^3 \lambda^2 \Lambda^\alpha), \text{Tr}(\lambda^3 \lambda^1 \lambda^2 \Lambda^\alpha), \text{Tr}(\lambda^3 \lambda^1 \lambda^2 \Lambda^\alpha). \) After replacing them in (20), only the first factor will produce the desired ordering \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \).
So, we consider only the terms in the vertex that have factor \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha) \). Hence, the vertex is

\[
V_\alpha(T_1 T_2 T_3 T) = -T_p i (2\pi \alpha')^2 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \Lambda^\alpha) \\
\times \left( \frac{m^4}{6} - \frac{m^2}{6}(k_3 \cdot p + k_1 \cdot p + k_1 \cdot k_2 + k_2 \cdot k_3 - k_2 \cdot p - k_1 \cdot k_3) \\
+ \frac{1}{6}(k_1 \cdot k_2 k_3 \cdot p + k_2 \cdot k_3 k_1 \cdot p - 3k_1 \cdot k_3 k_2 \cdot p) \right) + \cdots
\]

(22)

where \( p \) is momentum of off-shell tachyon. Replacing this and the other vertex and propagator in (20), one finds

\[
A_{\zeta(2)}(T_1 T_2 T_3 T_4 A_5) = 2i\alpha' T_p \zeta(2) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^5) k_1 \cdot \xi_5 \\
\times \left( -\frac{2s_{12}^2}{s_{45} + \frac{1}{2}} - \frac{6s_{12}s_{23}}{s_{45} + \frac{1}{2}} - \frac{2s_{23}^2}{s_{45} + \frac{1}{2}} - \frac{7s_{12}}{s_{45} + \frac{1}{2}} - \frac{7s_{23}}{s_{45} + \frac{1}{2}} - \frac{4}{s_{45} + \frac{1}{2}} \\
+ \frac{2s_{45}s_{12}}{s_{45} + \frac{1}{2}} + \frac{2s_{45}s_{23}}{s_{45} + \frac{1}{2}} + \frac{4s_{45}}{s_{45} + \frac{1}{2}} \right) + \cdots
\]

(23)

where we have used relation (8) and \( \pi^2/6 = \zeta(2) \).

The other distinct diagram that produces the ordering \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \) is

\[
A_{\zeta(2)}(T_2 T_3 T_4 T_1 A_5) = 2i\alpha' T_p \zeta(2) \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k_1 \cdot \xi_5 \\
\times \left( -\frac{2s_{23}^2}{s_{15} + \frac{1}{2}} - \frac{6s_{34}s_{23}}{s_{15} + \frac{1}{2}} - \frac{2s_{34}^2}{s_{15} + \frac{1}{2}} - \frac{7s_{23}}{s_{15} + \frac{1}{2}} - \frac{7s_{34}}{s_{15} + \frac{1}{2}} - \frac{4}{s_{15} + \frac{1}{2}} \\
+ \frac{2s_{15}s_{23}}{s_{15} + \frac{1}{2}} + \frac{2s_{15}s_{34}}{s_{15} + \frac{1}{2}} + \frac{4s_{15}}{s_{15} + \frac{1}{2}} \right) + \cdots
\]

(24)

where again dots refer to the terms that have coefficient \( k_2 \cdot \xi_5, k_3 \cdot \xi_5, \) and have group factor other than \( \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \).

We now consider the amplitude \( A_{\zeta(2)}' \). The amplitude \( A_{\zeta(2)}'(A_5 T_1 T_2 T_3 T_4) \) is given by

\[
A_{\zeta(2)}'(A_5 T_1 T_2 T_3 T_4) = V_\alpha^a (A_5 T_1 T_2 A) G_{\alpha \beta}^a (A) V_\beta^b (AT_3 T_4)
\]

(25)

where the vertex of two-gauge-two-tachyon should be read from the terms in the last line of (16). There are three different amplitudes here that have been calculated in [29], i.e.,

\[
A_{\zeta(2)}'(A_5 T_1 T_2 T_3 T_4) = i\alpha' T_p \zeta(2) k_1 \cdot \xi_5 \text{Tr}(\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \\
\times \left( -\frac{4s_{23}(s_{15} + \frac{1}{2})}{s_{34}} - \frac{4s_{12}s_{23}}{s_{34}} - \frac{4(s_{15} + \frac{1}{2})}{s_{34}} - \frac{4s_{12}}{s_{34}} \right)
\]
\[ A_{(2)}' \left(T_3 T_4 A_5 T_1 T_2\right) = V_{\alpha}^a (T_3 T_4 A_5 A) G_{\alpha \beta}^{ab} (A) V_{\beta}^b (A T_1 T_2) \]
\[ = i \alpha' T_9 (2) k_1 \cdot \xi_5 \text{Tr} (\lambda^1 \lambda^2 \lambda^3 \lambda^5) \]
\[ \times \left( - \frac{4 s_{23} (s_{45} + \frac{1}{2})}{s_{12}} - \frac{4 s_{23} s_{34}}{s_{12}} - \frac{4 (s_{45} + \frac{1}{2})}{s_{12}} - \frac{4 s_{34}}{s_{12}} \right) \]
\[ - 2 (s_{45} + \frac{1}{2}) + 2 s_{12} + 4 s_{23} - 2 s_{34} + 4 \right) + \cdots \]
(26)

\[ A_{(2)}' \left(T_4 A_5 T_1 T_2 T_3\right) = V_{\alpha}^a (T_4 A_5 T_1 A) G_{\alpha \beta}^{ab} (A) V_{\beta}^b (A T_2 T_3) \]
\[ = i \alpha' T_9 (2) k_1 \cdot \xi_5 \text{Tr} (\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) \]
\[ \times \left( \frac{4 (s_{15} + \frac{1}{2}) (s_{45} + \frac{1}{2})}{s_{23}} - \frac{4 (s_{15} + \frac{1}{2}) s_{12}}{s_{23}} - \frac{4 (s_{45} + \frac{1}{2}) s_{34}}{s_{23}} \right) \]
\[ - \frac{4 (s_{15} + \frac{1}{2})}{s_{23}} - \frac{4 (s_{45} + \frac{1}{2})}{s_{23}} - 2 (s_{15} + \frac{1}{2}) - 2 (s_{45} + \frac{1}{2}) \right) + \cdots \]
(27)

Finally, we consider the amplitude \( A_{(2)}'' \). The couplings in (21) produce the following contact terms of four tachyons and one gauge field:
\[ A_{(2)}'' (DTDTDTDT) = 4 i \alpha' T_9 (2) k_1 \cdot \xi_5 \text{Tr} (\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) s_{23} \]
(29)

Other contact terms are coming from coupling in the fourth line of (16). In this term, the commutator \([T_1, T_2]\) is coming from \((Q^{-1})_{ij}\) in (1). Writing the symmetric trace in terms of ordinary trace and performing the trace over the internal CP matrices, one finds
\[ \frac{i}{3} T_p (2 \pi \alpha')^2 \text{Tr} (D_a T D_b T T T F^{ba} + D_a T D_b T F^{ba} T T - D_a T T T D_b T F^{ba}) \]
The above couplings produce the following contact terms:
\[ A_{(2)}'' (DTDTTTT) = -8 i \alpha' T_9 (2) k_1 \cdot \xi_5 \text{Tr} (\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) (1 + s_{15} + s_{45}) \]
(30)

Now comparing the field theory amplitudes with the string theory amplitude (15), one finds exact agreement, i.e.,
\[ A_{(2)} TT TT AA - A_k TT TT AA - A_{(2)} TT TT AA = 0 + \cdots \]
(31)
where dots refer to the terms with coefficients \(\zeta(3)\) which are \((\alpha')^2\) order, \(\zeta(4)\) which are \((\alpha')^3\) order, and so on. The above consistency, in particular, confirms the presence of \((Q^{-1})_{ij}\) in the tachyon action (1). This ends our illustration of consistency between the momentum expansion of the S-matrix element (15) and the non-abelian tachyon DBI action (1).
4 Discussion

In this paper we have shown that the leading order terms, and next to the leading order terms of the momentum expansion of the S-matrix element of four tachyons and one gauge field are reproduced exactly by the modified tachyon DBI action that recently has been proposed in [22]. The next order terms in the expansion (15) are \((\alpha')^2\) order. They contains massless/tachyon poles and contact terms. The higher derivative corrections in general have field redefinition freedom [30, 31], so one may choose this freedom to relate them to the couplings in field theory which have second derivative of \(T\), i.e., couplings that include \(\partial \partial T\). Similarly, the \((\alpha')^3\) terms of (15) may be related to the couplings that include \(\partial \partial \partial T\), and so on. It would be interesting to finds these higher derivative terms explicitly.

The couplings in (16) have on-shell ambiguity/freedom, i.e., \(m^2 T \sim D_a D^a T\). This ambiguity/freedom can not be fixed even by studying the S-matrix element in which the tachyon appears as off-shell in the tachyon pole. If one replaces \(T\) with \(D_a D^a T\) it does not change the tachyon poles but produces an extra contact terms. These contact terms however cancels the contact terms that are resulted from replacing \(T\) with \(DD T\). For example, one may rewrite the first term of (21) as

\[-T_p (2\pi \alpha')^2 Tr \left( \frac{m^4}{24} T^4 \right) \rightarrow -T_p (2\pi \alpha')^2 Tr \left( \frac{m^4}{24} T^4 + \frac{\beta m^2}{24} TTT D_a D^a T \right.
\left. + \frac{\lambda}{24} TTT D_a D_b D^a D^b T + \frac{\sigma}{24} TTT D_a D_b D^a D^b T + \frac{\gamma}{24} TTT D_a D_b D^a D^b T \right) \]

(32)

where \(\alpha + \beta + \gamma + \lambda + \sigma = 1\). The couplings on the right hand side reproduce exactly the same S-matrix element as the coupling on the left hand side. To see this, we note that the right hand side gives the following one-gauge-four-tachyon contact terms:

\[i T_p (2\pi \alpha')^2 k \cdot \xi_5 Tr (\lambda^1 \lambda^2 \lambda^3 \lambda^5) \left( \frac{(\gamma + \lambda + \sigma)}{24} (1 - s_{15} - s_{45}) - \beta m^2 \right) \]

(33)

However, these couplings change also the vertex (22) and so modifies the amplitudes (23) and (24). They gives the following extra contact terms:

\[2i \alpha' T_p \zeta_5 (2) Tr (\lambda^1 \lambda^2 \lambda^3 \lambda^4 \lambda^5) k \cdot \xi_5 (-\beta + (\gamma + \lambda + \sigma) (s_{45} + s_{15} - 1)) \]

(34)

They cancel exactly the contact terms in (33). Hence, one has freedom to choose the constants \(\alpha, \beta, \gamma, \lambda, \sigma\). The S-matrix method can only fix \(\alpha + \beta + \gamma + \lambda + \sigma = 1\). Assuming the effective field theory at the DBI order has no couplings which has \(D_a D^a T\) terms, fixes the constants to \(\beta = \gamma = \lambda = \sigma = 0\) which is consistent with the tachyon DBI action.

The modified tachyon DBI action (11) has couplings which are different from the usual tachyon DBI action. This action has been found from studying the S-matrix element around the unstable point of the tachyon potential, i.e., \(T = 0\). One may extrapolate the action
to the stable point of the tachyon potential, i.e., $T \to \infty$. Around this point the action reduces to the usual tachyon DBI action. To see this, we note the following:

\[
\sqrt{1 + \frac{1}{2}[T_i, T_j][T_j, T_i]} = 1 + \frac{1}{4}[T_1, T_2][T_2, T_1] - \frac{1}{32}[T_1, T_2][T_2, T_1][T_1, T_2][T_2, T_1] + \cdots
\]

\[
= 1 + \frac{1}{4}[T_1, T_2][T_2, T_1]
\]

(35)

where in the second line we have used the prescription given for the modified tachyon DBI action (11) that only two of the tachyons must be $T_2$. All terms of the square root in the second line of (11) and the terms of tachyon potential $V(T)$ that multiply the second term above must have $T_1$. So the square root in the second line of (11) that multiplied the second term above is the usual tachyon DBI action. On the other hand, around $T \to \infty$ only the second term in the above expansion is important. Therefore, the action (11) reduces to the usual tachyon DBI action with potential $T^4V(T^2)$. This potential goes to zero at $T \to \infty$ as expected from the tachyon condensation [14].

We have found the momentum expansion of the amplitude (11) by finding the massless poles of each correlator, i.e., in the first correlator $(k_1 + k_2)^2, (k_3 + k_4)^2 \to 0$, in the second correlator $(k_1 + k_3)^2, (k_2 + k_4)^2 \to 0$ and in the last correlator $(k_1 + k_4)^2, (k_2 + k_3)^2 \to 0$. All other Mandelstam variables go as $k_i k_j \to 0$. The momentum expansion of the amplitude (11) is similar. The correlator here can have massless pole in all $s_{12}, s_{34}, s_{24}, s_{14}, s_{23}$ channels. However, the on-shell constraint (8) does not allow the correlator to have massless pole in these channels at the same time. In this case, one has to send once $(k_1 + k_2)^2, (k_3 + k_4)^2 \to 0$, once $(k_1 + k_3)^2, (k_2 + k_4)^2 \to 0$ and once $(k_1 + k_4)^2, (k_2 + k_3)^2 \to 0$ as in (13). This can easily be extended to the n-point function. For example, the S-matrix element of six tachyons is given by

\[
\sum_{\text{non-cyclic}} \int dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \left( \langle V^T_0 (x_1) V^T_0 (x_2) V^T_0 (x_3) V^T_0 (x_4) V^T_0 (x_5) V^T_0 (x_6) \rangle \right)
\]

(36)

The result must be symmetric under interchanging the tachyons. The momentum expansion of this amplitude is the following:

\[
(s_{12}, s_{34}, s_{56}) \to 0 + (s_{12}, s_{35}, s_{46}) \to 0 + (s_{12}, s_{36}, s_{45}) \to 0 + \\
(s_{13}, s_{24}, s_{56}) \to 0 + (s_{13}, s_{25}, s_{46}) \to 0 + (s_{13}, s_{26}, s_{45}) \to 0 + \\
(s_{14}, s_{23}, s_{56}) \to 0 + (s_{14}, s_{25}, s_{36}) \to 0 + (s_{14}, s_{26}, s_{35}) \to 0 + \\
(s_{15}, s_{23}, s_{46}) \to 0 + (s_{15}, s_{24}, s_{36}) \to 0 + (s_{15}, s_{26}, s_{34}) \to 0 + \\
(s_{16}, s_{23}, s_{45}) \to 0 + (s_{16}, s_{24}, s_{35}) \to 0 + (s_{16}, s_{25}, s_{34}) \to 0
\]

(37)

which is symmetric under interchanging the tachyons. In each case all other Mandelstam variables go as $k_i k_j \to 0$. Of course, not all the Mandelstam variables are independent. In each case it is easy to find the expansion in terms of the independent variables, e.g., see the expansions in (10), (11) and (12).
The internal CP matrix for all the vertex operator in the above amplitude is $\sigma_1$. Alternatively, the amplitude can be written as

$$\sum_{\text{non-cyclic}} \int dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \left( (V^-_1(x_1)V^T_1(x_2)V^T_0(x_3)V^T_0(x_4)V^T_0(x_5)V^T_0(x_6)) + (V^T_1(x_1)V^T_0(x_2)V^T_0(x_3)V^T_0(x_4)V^T_0(x_5)V^T_0(x_6)) + (V^-_1(x_1)V^T_0(x_2)V^T(x_3)V^T_0(x_4)V^T_0(x_5)V^T_0(x_6)) + (V^-_1(x_1)V^T_0(x_2)V^T(x_3)V^T_0(x_4)V^T_0(x_5)V^T_0(x_6)) \right)$$

For the first correlator, the Mandelstam variables go as in the first line of (37), for the second correlator, the Mandelstam variables go as in the second line of (37), and so on. If one performs the correlators in the above amplitude and then use the above expansion, one should find consistency between the leading order terms of the expansion and the six-tachyon couplings in (1). It would be interesting to perform this calculation.

**Acknowledgements:** This work was supported by a grant from Ferdowsi University of Mashhad.
References

[1] G. R. Dvali and S. H. H. Tye, Phys. Lett. B 450, 72 (1999) arXiv:hep-ph/9812483.
[2] G. R. Dvali, Q. Shafi and S. Solganik, arXiv:hep-th/0105203.
[3] C. P. Burgess, M. Majumdar, D. Nolte, F. Quevedo, G. Rajesh and R. J. Zhang, JHEP 0107, 047 (2001) arXiv:hep-th/0105204.
[4] S. Kachru, R. Kallosh, A. Linde, J. M. Maldacena, L. P. McAllister and S. P. Trivedi, JCAP 0310, 013 (2003) arXiv:hep-th/0308055.
[5] T. Banks and L. Susskind, arXiv:hep-th/9511194.
[6] A. Sen, JHEP 9808, 010 (1998) arXiv:hep-th/9805019.
[7] S. P. de Alwis, Phys. Lett. B 461, 329 (1999) arXiv:hep-th/9905080.
[8] L. Kofman, A. D. Linde and A. A. Starobinsky, Phys. Rev. D 56, 3258 (1997) arXiv:hep-ph/9704452.
[9] R. H. Brandenberger, A. R. Frey and L. C. Lorenz, arXiv:0712.2178 [hep-th].
[10] A. Sen, Int. J. Mod. Phys. A 20, 5513 (2005) arXiv:hep-th/0410103.
[11] M. R. Garousi, JHEP 0501, 029 (2005) arXiv:hep-th/0411222.
[12] A. Sen, JHEP 0207, 065 (2002) arXiv:hep-th/0203265.
[13] A. Sen, Mod. Phys. Lett. A 17, 1797 (2002) arXiv:hep-th/0204143.
[14] A. Sen, JHEP 9910, 008 (1999) arXiv:hep-th/9909062.
[15] M. R. Garousi, Nucl. Phys. B 584, 284 (2000) arXiv:hep-th/0003122.
[16] E. A. Bergshoeff, M. de Roo, T. C. de Wit, E. Eyras and S. Panda, JHEP 0005, 009 (2000) arXiv:hep-th/0003221.
[17] J. Kluson, Phys. Rev. D 62, 126003 (2000) arXiv:hep-th/0004106.
[18] C. Kennedy and A. Wilkins, Phys. Lett. B 464, 206 (1999) arXiv:hep-th/9905195.
[19] P. Kraus and F. Larsen, Phys. Rev. D 63, 106004 (2001) arXiv:hep-th/0012198.
[20] T. Takayanagi, S. Terashima and T. Uesugi, JHEP 0103, 019 (2001) arXiv:hep-th/0012210.
[21] M. R. Garousi and E. Hatefi, arXiv:0710.5875 [hep-th].
[22] M. R. Garousi, arXiv:0802.2784 [hep-th].

[23] M. R. Garousi, arXiv:0810.2256 [hep-th].

[24] A. Sen, arXiv:hep-th/9904207.

[25] P. J. De Smet and J. Raeymaekers, JHEP 0008, 020 (2000) arXiv:hep-th/0004112.

[26] Z. Koba and H.B. Nielsen, Nucl. Phys. B 10, 633 (1969); Nucl. Phys. B 12, 517 (1969).

[27] M. R. Garousi, JHEP 0802, 109 (2008) arXiv:0712.1954 [hep-th].

[28] T. Huber and D. Maitre, Comput. Phys. Commun. 175, 122 (2006) arXiv:hep-ph/0507094.

[29] K. Bitaghsir-Fadafan and M. R. Garousi, Nucl. Phys. B 760, 197 (2007) arXiv:hep-th/0607249.

[30] A.A. Tseytlin, Nucl. Phys. B 276 (1987) 391.

[31] A.A. Tseytlin, Phys. Lett. B 176 (1986) 92; R.R. Metsaev, A.A. Tseytlin, Phys. Lett. B 185 (1987) 52.