Five-point Correlation Numbers in One-Matrix Model

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Abstract

The five-point correlation numbers in the One-matrix model is calculated in the Liouville frame. Validity of the fusion rules for it is checked.

1 Introduction

There exist two approaches to the 2d Quantum gravity. One of them is the continuous approach. In this approach the theory is determined by the functional integral over all metrics [1]. Calculation of this integral in conformal gauge leads to the Liouville field theory. Therefore this approach is called the Liouville gravity.

The other way to describe sum over all surfaces is the discrete approach. It is based on the idea of approximation of two-dimensional geometry by the ensemble of planar graphs of big size. Technically the ensemble of graphs is usually defined by expansion into a series of perturbation theory of integral over matrixes. That is why this approach is called the Matrix models. References to the both approaches can be found in the review [2].

The main objects in these two approaches are the correlation numbers $\langle O_L^1...O_L^N \rangle$ of the observables $O_L^k$ in the Liouville gravity and the correlation numbers $\langle O_M^1...O_M^N \rangle$ of the observables $O_M^k$ in the Matrix models. It was a remarkable discovery that the spectra of the gravitational dimensions [2] in the both approaches coincide. It was reason to assume that both these theories describe the same variant of the 2d Quantum gravity and therefore the correlation numbers will coincide. However the attempt of naive identification of the correlation numbers doesn’t lead to the coincidence in general case.

It was remarked in [3], [4], that existence of the so-called resonances makes the identification of the correlation numbers ambiguous. In order to investigate this ambiguity it is convenient to pass from the correlation numbers to generating functions of correlators according to the formulas

$$F^L(\lambda_1, ..., \lambda_N) = \langle \exp(\sum \lambda_k O_L^k) \rangle, \quad Z(t_1, ..., t_N) = \langle \exp(\sum t_k O_M^k) \rangle.$$  

In [3], [4] a conjecture was proposed that there exists a “resonance” transformation $t_k = t_k(\{\lambda\})$ (from KdV frame to Liouville frame), such that the correlation functions in the both theories will coincide

$$F^L(\lambda_1, ..., \lambda_N) = Z(t_1(\{\lambda\}), ..., t_N(\{\lambda\})).$$

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The form of the transformation was conjectured in [4] for particular case, namely for the Minimal quantum gravity $\mathcal{MG}_{2/2p+1}$ and the One-matrix model with $p$ critical points. In loc. cit. the conjectured of the coincidence was checked up to four-point correlator.

In this paper we continue comparing of $\mathcal{MG}_{2/2p+1}$ and the One-matrix model. Using the “resonance” transformation [4] we find the five-point correlation numbers in One-matrix model in the Liouville frame. And we checked that the correlation numbers satisfy the fusion rules, which necessarily must be satisfied in the Minimal gravity $\mathcal{MG}_{2/2p+1}$.

The article is organized in the following way. In the first part of the article there is a brief summary of the results of the paper [4]. In the second part, an expression for the five-point correlation numbers in the Liouville frame is found, and validity of the fusion rules for it is proved.

2 One-matrix model

As it has been shown in the classical works on the Matrix models (see review [2]) in the scaling limit near the $p$-critical point the partition function of the One-matrix model can be described in terms of the solution of the “string equation”

$$\mathcal{P}(u) = 0,$$  

(2.1)

where $\mathcal{P}(u)$ is the polynomial of degree $p+1$ ($p$ is natural number)

$$\mathcal{P}(u) = u^{p+1} + t_0 u^{p-1} + \sum_{k=1}^{p-1} t_k u^{p-k-1},$$  

(2.2)

with the parameters $t_k$ describing deviation from the $p$-critical point. The singular part of the partition function in the Matrix models $Z(t_0,t_1,...t_{p-1})$ can be described according to (2.2), as

$$Z = \frac{1}{2} \int_{u_*}^{u_0} \mathcal{P}^2(u) du,$$  

(2.3)

where $u_* = u_*(t_0,t_1,...,t_{p-1})$ is the maximal real root [4] of the polynomial (2.2). Expression (2.3) gives only the singular part of the partition function in the Matrix models, but complete the matrix integral includes also the regular part, which is analytical with respect to all parameters $t_k$ at the point $\{t_1,...,t_{p-1}\} = 0$. It is not worth considering it. If we calculate the correlation numbers by taking derivatives in the parameters $t_k$ (KdV frame), according to the formula

$$\langle O_{k_1}^M O_{k_2}^M ... O_{k_N}^M \rangle = Z_{k_1...k_N} = \frac{\partial^N Z}{\partial t_{k_1}...\partial t_{k_N}} \bigg|_{t_1=...=t_{p-1}=0},$$  

(2.4)

that coincidence with results of the Minimal gravity $\mathcal{MG}_{2/2p+1}$ will not take place.

In the work [4] it was argued that among analytic and scale-invariant transformations $t_k = t_k(\lambda)$, exists the special one, after which the correlation numbers satisfy the fusion rules, which necessarily must be satisfied in the Minimal gravity $\mathcal{MG}_{2/2p+1}$. The fusion rules are expressed as follows

$$\langle O_{k_1} O_{k_2} ... O_{k_N} \rangle = 0, \quad \text{if} \quad \begin{cases} k_1 + ... + k_{N-1} < k_N, & \text{when } k_1 + ... + k_N \text{ is even,} \\ k_1 + ... + k_N < 2p - 1, & \text{when } k_1 + ... + k_N \text{ is odd,} \end{cases}$$  

(2.5)

Here it is assumed that $k_i$ run through the range $k_i = 0, 1, ..., p-1$ and that $k_N$ is the maximal index, i.e. $k_i \leq k_N$. Later we will say that we are in even(odd) sector, if $k_1 + ... + k_N$ is
even(odd). After this transformation the polynomial $\mathcal{P}(u)$, up to the factor $\frac{(p+1)!}{(2p)!} u_0^{p+1}$ takes the form [4]:

$$Q(x, \{s\}) = \sum_{n=0}^{\infty} \sum_{k_1...k_n=1}^{p-1} \frac{s_{k_1}...s_{k_n}}{n!} \frac{d^{n-1}}{dx^{n-1}} L_p - \sum_{k_i-n}(x),$$

(2.6)

here, we already pass from dimensional Liouville parameters $\{\lambda_k\}$ to dimensionless $\{s_k\}$, by the formula

$$s_k = \frac{g_k u_0^{k-2}}{g 2p + 1} \lambda_k,$$

where $g_k = \frac{(p - k - 1)!}{(2p - 2k - 3)!!}$, $g = \frac{(p + 1)!}{(2p + 1)!!}$

(2.7)

Also $x = u/u_0$, where $u_0$ is $u_*$ at $s_1, ..., s_{p-1} = 0$. The relation between $t_0$ and $u_0$ is

$$t_0 = -\frac{1}{2} \frac{p(p + 1)}{2p - 1} u_0^2.$$

(2.8)

And $L_n(x)$ are the Legendre polynomials (see Appendix A). We assume that $\frac{d}{dx}^{-1} L_p = \int L_p dx = \frac{L_{p+1} - L_{p-1}}{2p+1}$. Below we use the notations

$$Q_{k_1...k_n}(x) = \frac{d^{n-1}}{dx^{n-1}} L_p - \sum_{k_i-n}(x), \quad Q_0(x) = \frac{L_{p+1} - L_{p-1}}{2p + 1}.$$

(2.9)

This “resonance” transformation is expressed as

$$t_k = \frac{g_k u_0^{k-2}}{g 2p + 1} \cdot \sum_{n=1}^{\infty} \sum_{m_1,...,m_n=0}^\infty \frac{(2p - 2k - 5 + 2n)!!}{(2p - 2k - 3)!!} \cdot \frac{s_{m_1}...s_{m_n}}{n!},$$

(2.10)

where we introduce auxiliary parameter $s_0 = -\frac{1}{2}$.

Thus the singular part of the partition function in the Matrix models expressed in terms of the new parameters can be written as

$$Z = \frac{1}{2} \int_{0}^{x_*} Q^2(x) dx,$$

(2.11)

where $x_* = x_*(s_1, ..., s_{p-1})$ is the maximal real root of the polynomial $Q(x)$, and $x_*(0, 0, ..., 0) = 1$. The correlation numbers in the Liouville frame are defined as follows

$$\langle O_{k_1}^M O_{k_2}^M ... O_{k_N}^M \rangle = Z_{k_1...k_N} = \frac{\partial^N Z}{\partial s_{k_1}...\partial s_{k_N}} \bigg|_{s_1=...=s_{p-1}=0}.$$

(2.12)

These correlators will coincide with those of the Minimal gravity $\mathcal{MG}_{2/2p+1}$. In the next section we will give the answers derived from the formula (2.12) for the three- and four-point correlation numbers.

### 2.1 Three- and four-point correlation numbers

In the next sections we will use convenient notations

$$k = \sum_{i=1}^{N} k_i, \quad \text{and} \quad k_{j_i...j_m}^{i_1...i_n} = (k_{i_1} + ... + k_{i_n}) - (k_{j_1} + ... + k_{j_m}).$$

(2.13)
To simplify expressions we need a symbol of symmetrization, denoted by parentheses, for example
\[ Q_{(k_1 k_2)} = Q_{k_1 k_2} Q_{k_3} + Q_{k_1 k_3} Q_{k_2} + Q_{k_2 k_3} Q_{k_1}, \tag{2.14} \]
notice, that the order of indexes in each \( Q \)-term doesn’t matter.

The answer for the three-point correlation numbers can be written in the form
\[ Z_{k_1 k_2 k_3} = \frac{1}{2} \int_{-1}^{1} (Q_{(k_1 k_2)} + Q_0 Q_{k_1 k_2 k_3}) dx - \frac{Q_{k_1} (1) Q_{k_2} (1) Q_{k_3} (1)}{Q_0 (1)} - \frac{Q_0 (1) Q_{k_1 k_2} (1) Q_{k_3} (1)}{Q_0 (1)}. \tag{2.15} \]
Taking into account the values of the Legendre polynomials (see Appendix A) and their derivatives in the point \( x = 1 \), and also that \( Q_0 (x) \) is orthogonal to \( Q_{k_1 k_2 k_3} (x) \), we get
\[ Z_{k_1 k_2 k_3} = -1 + \frac{1}{2} \int_{-1}^{1} Q_{(k_1 k_2)} Q_{k_3} dx. \tag{2.16} \]
Now we give the general final answer for \( Z_{k_1 k_2 k_3} \), assuming that \( 0 \leq k_1 \leq k_2 \leq k_3 \leq p - 1 \)
\[ Z_{k_1 k_2 k_3} = \begin{cases} -1, & \text{if} \quad k_3 \leq k_{12}, \\ 0, & \text{if} \quad k_3 > k_{12}, \end{cases} \tag{2.17} \]
in the even sector, and
\[ Z_{k_1 k_2 k_3} = \begin{cases} -1, & \text{if} \quad k \geq 2p - 1, \\ \text{reg., if} \quad k < 2p - 1, \end{cases} \tag{2.18} \]
in the odd sector, where ”reg.” are the regular terms.
Here we give the answer for four-point correlation numbers
\[ Z_{k_1 k_2 k_3 k_4} = \frac{1}{2} \int_{-1}^{1} \left( Q_{(k_1 k_2 k_3 k_4)} + Q_{(k_1 k_2) Q_{k_3 k_4}} + Q_0 Q_{k_1 k_2 k_3 k_4} \right) dx - \frac{Q_{(k_1 k_2) Q_{k_3} Q_{k_4}}}{Q_0} + \frac{Q_0 (1) Q_{k_1 k_2} Q_{k_3} Q_{k_4}}{(Q_0)^2} - \frac{Q_0 Q_{k_3} Q_{k_2} Q_{k_3} Q_{k_4}}{(Q_0)^3}, \tag{2.19} \]
where all \( Q \)-terms after the integral are taken in the point \( x = 1 \). Now assuming, as usual, that \( 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq p - 1 \), in the even sector we have
\[ Z_{k_1 k_2 k_3 k_4} = \begin{cases} \frac{1}{2} (3p^2 - 5p - (2p - 1)k + \sum k_i^2), & p - 1 < k_{12} \\ (1 + k_1)(2p - 3 - k) + F(k_{14}) + F(k_{13}), & k_{12} \leq p - 1 < k_{13}, \\ (1 + k_1)(2p - 3 - k) + F(k_{14}), & k_{13} \leq p - 1 < k_{14}, k_{23}, \\ (1 + k_1)(2p - 3 - k), & k_{14} < k_{23}, k_{14} \leq p - 1, \\ \frac{1}{2} (2 + k_{123} - k_4)(2p - 3 - k), & k_4 - k_{123} > 0, \\ 0, & \end{cases} \tag{2.20} \]
where $F(k) = \frac{1}{2}(p-k-1)(p-k-2)$. And in the odd sector

$$Z_{k_1k_2k_3k_4} = \begin{cases} \frac{1}{2}(3p^2 - 5p - (2p - 1)k + \sum k_i^2), & p - 1 < k, \\ (1 + k_1)(2p - 3 - k) + F(k_{14}) + F(k_13), & k_12 \leq p - 1 < k, \\ (1 + k_1)(2p - 3 - k), & k_13 \leq p - 1 < k_{14}, \\ \frac{1}{2}(2 + k_{123} - k_1)(2p - 3 - k), & k_{14} \leq k_{23}, k_{23} \leq p - 1, k \geq 2p - 3, \\ \text{reg.}, & k \leq 2p - 5. \end{cases} \tag{2.21}$$

One can see, that near the ”critical” regions, i.e. the regions in which relations between $k_i$ are already similar to those for which the fusion rules must be valid, the correlation numbers are factorized and become the simple form. In other regions the correlators has very bulky form. This interesting feature of the ”simplification” of the correlator will be noted in the calculation of the five-point correlation numbers.

### 3 Five-point correlation numbers

We can at last begin to investigate the five-point correlation numbers in the Matrix models. The answer for it can be written as follows

$$Z_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} \left( Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5} + Q_{k_1 k_2} Q_{k_3} Q_{k_4} Q_{k_5} + Q_{0} Q_{k_1 k_2 k_3} Q_{k_4} Q_{k_5} \right) dx - \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5}}{Q_0} - \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5}}{Q_0}$$

\[ - \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5}}{(Q_0)^2} + \frac{(Q_0)^2}{(Q_0)^3} \frac{(Q_0)^2}{(Q_0)^3} - \frac{Q''_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5}}{(Q_0)^4} + \frac{3(Q''_0)^2}{Q_0} \frac{Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5}}{(Q_0)^4}, \tag{3.1} \]

where all $Q$-terms after the integral are taken in the point $x = 1$. After simplification (see Appendix C), one can get

$$Z_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} \left( Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5} + Q_{k_1 k_2} Q_{k_3} Q_{k_4} Q_{k_5} \right) dx + (Q''_0 - 3(Q''_0)^2) -$$

\[ + \sum_{i=1}^{5} (3Q''_0 Q_{k_i} - Q''_{k_i}) + \sum_{i < j} (Q''_{k_i} Q_{k_j}) - \frac{2Q''_{k_i} Q_{k_j} - Q''_{k_i} Q_{k_j}) +} \]

\[ + \sum_{i,j,l} Q_{k_i} Q_{k_j} - \sum_{i < j} Q_{k_i k_j} - \sum_{i,j,k,l} Q_{k_i k_j} Q_{k_i k_j}, \tag{3.2} \]

For convenience let us divide the five-point correlator in several parts

$$Z_{k_1k_2k_3k_4k_5} = Z^{(1)}_{k_1k_2k_3k_4k_5} + Z^{(J)}_{k_1k_2k_3k_4k_5} + Z^{(0)}_{k_1k_2k_3k_4k_5}, \tag{3.3}$$

where the first two integral terms are

$$Z^{(1)}_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5} dx, \quad Z^{(J)}_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} Q_{k_1} Q_{k_2} Q_{k_3} Q_{k_4} Q_{k_5} dx, \tag{3.4}$$
and the last term is given by the formula
\[ Z_{k_1k_2k_3k_4k_5}^{(0)} = (Q''_0 - 3(Q'_0)^2) + \sum_{i=1}^{5} (3Q''_0 Q'_k - Q''_k) - \sum_{i<j} 2Q'_k Q'_j + \]
\[ + \sum_{i<j} (Q'_{kji} - Q''_0 Q_{kij}) - \sum_{i<j<l} Q_{kij} Q_{klj} + \sum_{i,j,l} Q'_{kij} Q_{klj} - \sum_{i,j,k,l} Q_{kij} Q_{klm} \] (3.5)

After the further simplification (see Appendix C), one can find
\[ Z_{k_1k_2k_3k_4k_5}^{(0)} = \]
\[ = \frac{1}{8} (4 \sum_{i=1}^{5} k_i^2 - k^2 - 2k - 8 - \sum_{m<n} k_{ijl}(k_{ijl} + 2)\Theta(k_{mn} - p))(2p - 3 - k)(2p - 5 - k) + \]
\[ + \sum_{i<j<l} H(k_{ijl}) G_1 - \sum_{i,j,l,m} F(k_{ij}) F(k_{lm}) G_2, \] (3.6)

where \( H(k) = \frac{1}{8} \prod_{r=1}^{4} (p - r - k) \), and also two factors \( G_1 = \Theta(k_{ijl} - p) + \Theta(k_{mn} - p) - 1 \) and \( G_2 = \Theta(k_{ij} - p)\Theta(k_{lm} - p) \), and \( \Theta(a - b) \) is the step-function, which is defined as
\[ \Theta(a - b) = \begin{cases} 1, & \text{if } a > b, \\ 0, & \text{if } a \leq b. \end{cases} \] (3.7)

Now we get down to consideration of the correlation numbers in the particular sectors. Let us begin with the odd sector.

### 3.1 Odd Sector

In this sector \( k = k_1 + k_2 + k_3 + k_4 + k_5 \) is odd, and \( Z_{k_1k_2k_3k_4k_5}^{(1)} = Z_{k_1k_2k_3k_4k_5}^{(J)} = 0 \) due to oddness of the integrand. Thus one can get
\[ Z_{k_1k_2k_3k_4k_5}^{(odd)} = Z_{k_1k_2k_3k_4k_5}^{(0)} = \]
\[ = \frac{1}{8} (4 \sum_{i=1}^{5} k_i^2 - k^2 - 2k - 8 - \sum_{m<n} k_{ijl}(k_{ijl} + 2)\Theta(k_{mn} - p))(2p - 3 - k)(2p - 5 - k) + \]
\[ + \sum_{i<j<l} H(k_{ijl}) G_1 - \sum_{i,j,l,m} F(k_{ij}) F(k_{lm}) G_2. \] (3.8)

If \( k \leq 2p - 7 \), the terms of the partition function are regular [4]. Thus, due to the formula (3.5) we only need to check the fusion rules at the points \( k = 2p - 5 \) and \( k = 2p - 3 \). It is obvious, that if \( k = 2p - 3 \), and \( k = 2p - 5 \), the first term in the r.h.s. of the formula (3.8) equals zero. Also notice that \( G_2 = 0 \), because \( G_2 = 1 \) at least if \( k \geq 2p \). For \( k = 2p - 3, 2p - 5, G_1 = 0 \) or \( G_1 = -1 \). In case \( G_1 = -1 \) we have the inequalities \( k_{mn} < p \) and \( k_{ijl} < p \), therefore \( k_{ijl} \) can be equal only \( p - 4, p - 3, p - 2, p - 1 \), but for these values \( H(k_{ijl}) = 0 \). Thus we showed, that the second term in the formula (3.8) also equals zero for \( k = 2p - 3 \) and \( k = 2p - 5 \).

As the result we proved the validity of the fusion rules for \( Z_{k_1k_2k_3k_4k_5}^{(odd)} \), i.e.
\[ Z_{k_1k_2k_3k_4k_5}^{(odd)} = 0, \text{ when } k < 2p - 1. \] (3.9)

Now let us pass to consideration of the even sector.
3.2 Even Sector

In the even sector \( k = k_1 + k_2 + k_3 + k_4 + k_5 \) is even. After calculation \( Z^{(J)}_{k_1 k_2 k_3 k_4 k_5} \) (see formula (C.31)) and summation of it with \( Z^{(0)}_{k_1 k_2 k_3 k_4 k_5} \) one can get

\[
Z^{(even)}_{k_1 k_2 k_3 k_4 k_5} = Z^{(I)}_{k_1 k_2 k_3 k_4 k_5} + \frac{1}{8} \left( 4 \sum_{i=1}^{5} k_i^2 - k^2 - 2k - 8 - \sum_{m<n} G_3 k_{ijl}^m (k_{ijl}^m + 2) (2p - 3 - k)(2p - 5 - k) + \right.
\]
\[
+ \left. \sum_{i<j<l} H(k_{ijl}) - \sum_{i,j,l,m} F(k_{ij}) F(k_{lm}) \right) G_2, \quad (3.10)
\]

where \( G_3 = \Theta(k_{mn} - p) + \Theta(k_{mn} - k_{ijl} - 2)\Theta(p - 1 - k_{ijl})\Theta(p - 1 - k_{mn}) \) and \( G_2 = \Theta(k_{ij} - p)\Theta(k_{lm} - p) \).

We as usual assume the following ordering \( 0 \leq k_1 \leq k_2 \leq k_3 \leq k_4 \leq k_5 \leq p - 1 \). If \( k_5 > k_{1234} \) it is obvious, that \( G_2 = 0 \). Then by means of the simple reasoning one can prove, that \( G_3 = 0 \), when one of the indexes \( i, j, l \) equals five. In remaining cases \( G_3 = 1 \). Having calculated \( Z^{(I)}_{k_1 k_2 k_3 k_4 k_5} \) (see formula (C.30)), we derive the following formula

\[
Z^{(even)}_{k_1 k_2 k_3 k_4 k_5} = \frac{1}{8} \left( k_{1234}^{k_5} - 2\right) \left( k_{1234}^{k_5} - 4\right) (2p - 3 - k)(2p - 5 - k)(1 - \Theta(k_{5}^{1234} - 6)). \quad (3.11)
\]

From this formula we can see \( Z^{(even)}_{k_1 k_2 k_3 k_4 k_5} = 0 \) if \( k_5 > k_{1234} \). Thus the fusion rules are valid. Notice that when \( k_5 = k_{1234} = 2 \) and \( k_5 = k_{1234} + 4 \) nulling of the function take place automatically, without the integral term \( Z^{(I)}_{k_1 k_2 k_3 k_4 k_5} \).

4 Conclusion

In this paper the five-point correlation numbers have been calculated. These numbers are necessary for the several reasons. First it is one more test of the fusion rules.

Second, as we assume the correlation numbers in the two approaches are the same. Therefore the five-point correlation numbers which were found in this paper have to coincide with answers in continuous approach. But the five-point correlation numbers in continuous approach are still unknown.

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Appendix

A Legendre Polynomials

The Legendre polynomials \( L_n(x) \) are \( n \)-th order polynomials, which form an orthogonal system on the interval \([-1, 1]\) with the weight 1,

\[
\int_{-1}^{1} L_n(x)L_m(x)dx = \frac{2\delta_{m,n}}{2n+1}.
\] (A.1)

The standard normalization is such that

\[
L_n(1) = 1.
\] (A.2)

Explicit formula for \( L_n(x) \) is

\[
L_n(x) = \frac{2^{-n}}{n!} \frac{d^n}{dx^n}[x^2 - 1]^n = 2^{-n} \sum_{l=0}^{[n/2]} (-1)^l \frac{(2n - 2l)!}{l!(n - l)!(n - 2l)!}x^{n-2l}.
\] (A.3)

Expression of polynomial \( L_n(x) \) in terms of Hypergeometric is series

\[
L_n(x) = {2F1}\left(-n, n+1; 1; \frac{1-x}{2}\right),
\] (A.4)

from this we can obtain

\[
L'_n(1) = \frac{n(n+1)}{2}, \quad L''_n(1) = \frac{(n-1)n(n+1)(n+2)}{8}, \quad \text{etc}
\] (A.5)

Yet another closed form is in terms of the contour integral

\[
L_n(x) = \oint_0 \frac{(1-2xz+z^2)^{-1/2}}{z^{n+1}} \frac{dz}{2\pi i}.
\] (A.6)

The following relations are useful in our analysis:

\[
L'_{n+1}(x) - L'_{n-1}(x) = (2n+1)L_n(x),
\] (A.7)

they are valid for all \( n = 0, 1, 2, 3 \ldots \) if one assume, that \( L_{-1}(x) = 0 \). The following formulas are

\[
\frac{1}{2} \int_{-1}^{1} L'_n(x)L_m(x)dx = E_{n+m} \Theta_{n,m+1},
\] (A.8)

\[
\frac{1}{2} \int_{-1}^{1} L''_n(x)L_m(x)dx = E_{n+m} \Theta_{n,m+2} \frac{(n+m+1)(n-m)}{2},
\] (A.9)

and in general

\[
\frac{1}{2} \int_{-1}^{1} L'^{(l)}_n(x)L_m(x)dx = E_{n+m-l} \Theta_{n,m+l} \frac{2^{-l+1}}{(l-1)!} \prod_{s=0}^{l-2} (n+m+l-1-2s)(n-m+l-2-2s),
\] (A.10)

where \( L'^{(l)}_n(x) \) stands for the \( l \)-th derivative. Here \( \Theta_{n,m} = L_{n-m}(1) \) is the step function, and

\[
E_n = \begin{cases} 
1, & \text{if } n \text{ is even}, \\
0, & \text{if } n \text{ is odd}.
\end{cases}
\] (A.11)
Integrating (A.9) by parts, we have
\[ \frac{1}{2} \int_{-1}^{1} L_n'(x)L_m'(x)dx = E_{n+m} \left[ \Theta_{n,n} \frac{n(n+1)}{2} + \Theta_{n,m} \frac{m(m+1)}{2} \right]. \] (A.12)

The general formula which expresses \( m \)-th derivative of the Legendre polynomial of \( n \)-th order, by the sum of the Legendre polynomials is
\[ \frac{d^m}{dx^m} L_n(x) = \sum_{k=2}^{n-m} (2k+1)B_{n,k}^{(m)} L_k(x), \] (A.13)
where
\[ B_{n,k}^{(m)} = \frac{2^{m-1} \Gamma \left( \frac{n+k+m+1}{2} \right) \Gamma \left( \frac{n-k+m}{2} \right)}{(m-1)! \Gamma \left( \frac{n+k-m+3}{2} \right) \Gamma \left( \frac{n-k-m+2}{2} \right)}. \] (A.14)

notice, that (A.10) easily leads from this formula.

**B Evaluation of \( x_{k_i k_j} \) and \( x_{k_i k_j k_l} \)**

From ”string equation”
\[ Q(x, \{s\}) = 0, \] (B.1)
where
\[ Q(x, \{s\}) = Q_0(x) + \sum_{k=1}^{p-1} s_k Q_k(x) + ... + \sum_{k_i=1}^{p-1} \frac{s_{k_1}...s_{k_n}}{n!} Q_{k_1...k_n}(x) + ..., \] (B.2)
after differentiation, we have
\[ \frac{\partial Q}{\partial x} dx + \sum_{k=1}^{p-1} \frac{\partial Q}{\partial s_k} ds_k + ... + \sum_{k_i=1}^{p-1} \frac{\partial Q}{\partial s_{k_i}} ds_{k_i} = 0, \] (B.3)
and we can see that
\[ Q' \frac{\partial x}{\partial s_{k_i}} + \frac{\partial Q}{\partial s_{k_i}} = 0, \text{ therefore } x_{k_i} = -\frac{Q_{k_i}}{Q'}. \] (B.4)

In what follows we will use the formula
\[ \frac{\partial Q_{k_i}}{\partial s_{k_j}} = Q_{k_i}' x_{k_j} + Q_{k_i k_j}. \] (B.5)

Evaluation of \( x_{k_i k_j} \)
\[ x_{k_i k_j} = -\frac{\partial}{\partial s_{k_j}} \left( \frac{Q_{k_i}}{Q'} \right) = -\frac{Q_{k_i}' x_{k_j} + Q_{k_i k_j}}{Q'} + \frac{Q_{k_i} (Q'' x_{k_j} + Q_{k_j}')}{(Q')^2} = \]
\[ = -\frac{Q_{k_i k_j}}{Q'} + \frac{Q_{k_i}' Q_{k_j} + Q_{k_i} Q_{k_j}'}{(Q')^2} - \frac{Q'' Q_{k_i} Q_{k_j}}{(Q')^3}. \] (B.6)
Evaluation of $x_{k_i, k_j, k_l}$

$$
x_{k_i, k_j, k_l} = \frac{\partial}{\partial s_{k_l}} \left( \frac{Q'_{k_i} Q_{k_j} + Q'_{k_j} Q_{k_i}}{(Q')^2} - \frac{Q_{k_i, k_j}}{Q'} - \frac{Q''_{k_i} Q_{k_j}}{(Q')^3} \right) =
$$

$$
\left( \frac{Q''_{k_i} x_{k_i} + Q'_{k_i} Q_{k_j}}{Q'} \right) + \left( \frac{Q'_{k_i} x_{k_j} + Q_{k_i, k_j}}{Q'} \right) + \left( \frac{Q'_{k_j} x_{k_l} + Q_{k_j, k_l}}{Q'} \right) + \left( \frac{Q''_{k_i} x_{k_l} + Q'_{k_i} Q_{k_l}}{Q'} \right) -
$$

$$
\frac{2(Q'_{k_i} Q_{k_j} + Q_{k_i} Q'_{k_j}) (Q''_{x_{k_i}} + Q'_{k_i})}{(Q')^3} - \frac{Q_{k_i, k_j} x_{k_i} + Q_{k_i, k_j}}{Q'} + \frac{Q_{k_j, k_l} x_{k_l} + Q_{k_j, k_l}}{Q'} -
$$

$$
\frac{(Q_{k_j} Q_{k_l} (Q''_x_{k_i} + Q'_{k_l}) + (Q'_{k_l} x_{k_l} + Q_{k_l, k_l}) Q'' Q_{k_j} + Q_{k_l} Q'' (Q'_{k_i} x_{k_i} + Q_{k_i, k_i}))}{(Q')^4} +
$$

$$
+ \frac{3Q'' Q_{k_i} Q_{k_j}}{(Q')^4} (Q'' x_{k_i} + Q'_{k_l}). \quad (B.7)
$$

Replace $x_{k_i}, x_{k_j}$ and $x_{k_l}$ in this formula we obtain

$$
x_{k_i, k_j, k_l} = -\frac{Q_{k_i} Q_{k_j}}{Q'} + \frac{Q'_{k_i, k_j} Q_{k_j} + Q'_{k_j, k_i} Q_{k_i}}{(Q')^2} + \frac{Q'_{k_i, k_j} Q_{k_j} + Q'_{k_j, k_i} Q_{k_i}}{(Q')^2} -
$$

$$
- \frac{Q''_{k_i, k_j} Q_{k_j} + Q'_{k_i} Q_{k_j} + Q_{k_i, k_j}}{(Q')^3} + \frac{2(Q'_{k_i} Q'_{k_j} Q_{k_j} + Q'_{k_j} Q_{k_j} Q_{k_i} + Q_{k_i} Q'_{k_j} Q_{k_j})}{(Q')^3} -
$$

$$
- \frac{Q'' Q_{k_i, k_j} Q_{k_j} + Q_{k_i, k_j} Q_{k_j} + Q_{k_j, k_j} Q_{k_i} + Q'_{k_i} Q_{k_j} Q_{k_i}}{(Q')^4} +
$$

$$
+ \frac{3Q'' Q_{k_i} Q_{k_j}}{(Q')^4} - 3(Q'')^2 \frac{Q_{k_i} Q_{k_j} Q_{k_l}}{(Q')^5}. \quad (B.8)
$$

Using the notation of the symmetrization we have

$$
x_{k_i, k_j, k_l} = -\frac{Q_{k_i} Q_{k_j} k_l}{Q'} + \frac{Q'_{(k_i, k_j) Q_{k_l}}}{(Q')^2} + \frac{Q'_{(k_j, k_i) Q_{k_l}}}{(Q')^2} - \frac{Q''_{(k_i, k_j) Q_{k_l}}}{(Q')^3} - \frac{2Q'_{(k_i) Q'_{(k_j) Q_{k_l}}}}{(Q')^3} -
$$

$$
- Q''_{(k_i) Q_{(k_j) Q_{k_l}}} + \frac{3Q'' Q_{(k_i) Q_{(k_j) Q_{k_l}}}}{(Q')^4} + Q'' Q_{(k_i) Q_{(k_j) Q_{k_l}}}, \quad (Q')^4 - 3(Q'')^2 \frac{Q_{k_i} Q_{k_j} Q_{k_l}}{(Q')^5}. \quad (B.9)
$$
From equations (2.11) and (2.12) one can calculate that

\[
Z_{k_1 k_2 k_3 k_4 k_5} = \frac{1}{2} \int_{-1}^{1} \left( Q_{k_1 k_2 k_3 k_4 k_5} + Q_{k_1 k_2 k_3 k_4 k_5} + Q_{0 k_1 k_2 k_3 k_4 k_5} \right) dx +
+ \left( Q_{k_1 k_2 k_3 Q_k} + Q_{k_1 k_2 Q_k k_4} + Q_{0 Q_k k_2 k_3 k_4} \right) x_k +
+ \left( Q_{k_1 k_2 k_3 Q_k} + Q_{k_1 k_2 k_3 Q_k} + Q_{0 Q_k k_2 k_3 k_4} \right) x_k +
+ \left( Q'_{k_1 k_2 Q_k} + Q'_{k_1 k_2 Q_k} + Q'_{0 Q'_{k_1 k_2}} \right) x_k x_k +
+ \left( Q'_{k_1 k_2 Q_k} + Q'_{k_1 k_2 Q_k} + Q'_{0 Q'_{k_1 k_2}} \right) x_k x_k +
+ \left( Q'_{k_1 k_2 Q_k} + Q'_{k_1 k_2 Q_k} + Q'_{0 Q'_{k_1 k_2}} \right) x_k x_k +
+ \left( Q'_{k_1 k_2 Q_k} + Q'_{k_1 k_2 Q_k} + Q'_{0 Q'_{k_1 k_2}} \right) x_k x_k +
+ \left( Q''_{k_1 Q_k} + Q''_{k_1 Q_k} + 2 Q''_{Q_k k_2} + Q''_{0 Q'_{k_1 k_2}} \right) x_k x_k x_k +
+ \left( Q''_{k_1 Q_k} + Q''_{k_1 Q_k} + 2 Q''_{Q_k k_2} + Q''_{0 Q'_{k_1 k_2}} \right) x_k x_k x_k +
+ \left( Q''_{k_1 Q_k} + Q''_{k_1 Q_k} + 2 Q''_{Q_k k_2} + Q''_{0 Q'_{k_1 k_2}} \right) x_k x_k x_k.
\]  

(C.1)

All integrated terms are taken at the point \( x = 1 \). Substituting in this formula values for \( x_k \), \( x_{k_j k_l} \) and \( x_{k_j k_l k_m} \), calculated in Appendix B, after simplification we find

\[
Z_{k_1 k_2 k_3 k_4 k_5} = \frac{1}{2} \int_{-1}^{1} \left( Q_{k_1 k_2 k_3 k_4 k_5} + Q_{k_1 k_2 k_3 k_4 k_5} + Q_{0 Q_k k_2 k_3 k_4} \right) dx - \frac{Q_{k_1 k_2 k_3 Q_k k_4 k_5}}{Q_0^2} -
- \frac{Q'_{k_1 k_2 Q_k k_3 k_4}}{(Q_0')^2} + \frac{Q''_{k_1 k_2 Q_k}}{(Q_0'')^3} - \frac{Q''_{k_1 k_2 Q_k}}{(Q_0'')^3} - \frac{Q''_{k_1 k_2 Q_k}}{(Q_0'')^3} +
+ \frac{3Q''_{k_1 k_2 Q_k}}{(Q_0'')^4} + \left( Q''_0 - \frac{3(Q''_0)^2}{Q_0''} \right) \frac{Q_{k_1 k_2 Q_k k_3 Q_k k_4 Q_k}}{(Q_0'')^4}.
\]  

(C.2)

Because \( Q_{k_1 \ldots k_n}(x) = \frac{d^{n+1}}{dx^{n+1}} L_{p-k} \sum_{k-n}(x) \), using Appendix A, we find that

\[
Q_0(1) = 0, \quad (C.3)
\]

\[
Q_0'(1) = 1, \quad (C.4)
\]

\[
Q_0''(1) = 1, \quad (C.5)
\]

\[
Q''_0(1) = \frac{p(p + 1)}{2}, \quad (C.6)
\]

\[
Q''_0'(1) = \frac{1}{8}(p - 1)p(p + 1)(p + 2), \quad (C.7)
\]

\[
Q'_k(1) = F_\Theta(k_i - 1), \quad (C.8)
\]

\[
Q_{k_j k_l}(1) = F_\Theta(k_{ij}), \quad (C.9)
\]

\[
Q_{k_j k_l}(1) = H_\Theta(k_{ij}), \quad (C.10)
\]

\[
Q_{k_j k_l}(1) = H_\Theta(k_{ij} - 1), \quad (C.11)
\]

\[
Q''_k(1) = H_\Theta(k_i - 2), \quad (C.12)
\]
here new function were introduced
\[ F_\Theta(k) = \frac{1}{2}(p - 1 - k)(p - 2 - k)\Theta(p - 1 - k), \]  
\[ H_\Theta(k) = \frac{1}{2} F_\Theta(k) F_\Theta(k + 2) = \frac{1}{8} \Theta(p - 1 - k) \prod_{r=1}^{4} (p - r - k). \]  

After partial simplification we get
\[ Z_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} \left( Q_{k_1k_2k_3k_4k_5} + Q_{k_1k_2k_3k_4k_5} \right) dx + (Q''_0 - 3(Q''_0)^2) - \]
\[ + \sum_{i=1}^{5} (3Q''_0Q'_{k_i} - Q''_{k_i}) + \sum_{i<j} (Q'_{k_ik_j} - 2Q'_{k_ik_j} - Q''_{k_ik_j}) + \]
\[ + \sum_{i,j,l} Q'_{k_i} Q_{k_jk_i} - \sum_{i<j<l} Q_{k_i} Q_{k_j} - \sum_{i,j,k,l} Q_{k_ik_j} Q_{k_ik_l}. \]  

Also introduce the functions
\[ F(k) = \frac{1}{2}(p - 1 - k)(p - 2 - k), \]
\[ H(k) = \frac{1}{2} F(k) F(k + 2) = \frac{1}{8} \prod_{r=1}^{4} (p - r - k), \]

which already doesn’t depend from \( \Theta(p - 1 - k) \). Divide the five-point correlation numbers on several parts
\[ Z_{k_1k_2k_3k_4k_5} = Z^{(1)}_{k_1k_2k_3k_4k_5} + Z^{(2)}_{k_1k_2k_3k_4k_5} + Z^{(3)}_{k_1k_2k_3k_4k_5} + Z^{(4)}_{k_1k_2k_3k_4k_5}, \]  
where
\[ Z^{(1)}_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} Q_{k_1k_2k_3k_4k_5} dx, \quad Z^{(2)}_{k_1k_2k_3k_4k_5} = \frac{1}{2} \int_{-1}^{1} Q_{k_1k_2k_3k_4k_5} dx, \]  
\[ Z^{(1)}_{k_1k_2k_3k_4k_5} = \sum_{i=1}^{5} \left( \frac{3p(p + 1)}{2} F_\Theta(k_i - 1) - H_\Theta(k_i - 2) \right) - \]
\[ - 2 \sum_{i<j} F_\Theta(k_i - 1) F_\Theta(k_j - 1) - \frac{p(p + 1)(5p^2 + 5p + 2)}{8}, \]  
\[ Z^{(2)}_{k_1k_2k_3k_4k_5} = \sum_{i<j} \left( H_\Theta(k_{ij} - 1) - F_\Theta(k_{ij}) \frac{p(p + 1)}{2} + F_\Theta(k_{ij}) \sum_{l \neq i,j} F_\Theta(k_l - 1) \right) - \]
\[ - \sum_{i<j<l} H_\Theta(k_{ijl}) - \sum_{i,j,l,m} F_\Theta(k_{ij}) F_\Theta(k_{lm}). \]  

Part of partition function \( Z^{(1)}_{k_1k_2k_3k_4k_5} \) has such a difference, that one shouldn’t take into account influence on it of the factors \( \Theta(a - b) \), due to the inequalities \( k_i \leq p - 1 \). So we can rewrite it
by means of the functions $F(k)$ and $H(k)$, as follows

$$Z_{k_1k_2k_3k_4k_5}^{(1)} = \sum_{i=1}^{5} \left( \frac{3p(p+1)}{2} F(k_i - 1) - H(k_i - 2) \right) - 2 \sum_{i<j} F(k_i - 1)F(k_j - 1) - \frac{p(p+1)(5p^2 + 5p + 2)}{8}. \quad (C.22)$$

Then using that

$$H_\Theta(k_{ijl}) = H(k_{ijl})(1 - \Theta(k_{ijl} - p)),
H_\Theta(k_{ij}) = H(k_{ij}) (1 - \Theta(k_{ij} - p)),
F_\Theta(k_{ij}) = F(k_{ij}) (1 - \Theta(k_{ij} - p)),
F_\Theta(k_{ij})F_\Theta(k_{lm}) = F(k_{ij}) F(k_{lm}) (1 - (\Theta(k_{ij} - p) + \Theta(k_{lm} - p) - \Theta(k_{ij} - p) \Theta(k_{lm} - p))), \quad (C.23)$$

let us write $Z_{k_1k_2k_3k_4k_5}^{(2)}$, in the following way

$$Z_{k_1k_2k_3k_4k_5}^{(2)} = \tilde{Z}_{k_1k_2k_3k_4k_5}^{(2)} + Z_{k_1k_2k_3k_4k_5}^{(2\Theta)}, \quad (C.24)$$

where

$$\tilde{Z}_{k_1k_2k_3k_4k_5}^{(2)} = \sum_{i<j} \left( H(k_{ij} - 1) - F(k_{ij}) \frac{p(p+1)}{2} + F(k_{ij}) \sum_{l \neq i,j} F(k_l - 1) \right) - \sum_{i<j<l} H(k_{ijl}) - \sum_{i,j,l,m} F(k_{ij})F(k_{lm}), \quad (C.25)$$

which doesn’t already depend on the step-functions, and

$$Z_{k_1k_2k_3k_4k_5}^{(2\Theta)} = \sum_{m<n} \Theta(k_{mn} - p) \left[ F(k_{mn}) \left( \sum_{i<j} F(k_{ij}) - \sum_{l \neq m,n} F(k_l - 1) + \frac{p(p+1)}{2} \right) - H(k_{mn} - 1) \right] + \sum_{i<j<l} H(k_{ijl}) \Theta(k_{ijl} - p) - \sum_{i,j,l,m} F(k_{ij})F(k_{lm}) \Theta(k_{ij} - p) \Theta(k_{lm} - p), \quad (C.26)$$

in which there is all dependence on $\Theta(a - b)$. Having simplified, we have

$$Z_{k_1k_2k_3k_4k_5}^{(2\Theta)} = -\frac{1}{8} (2p - 3 - k) (2p - 5 - k) \sum_{m<n} k_{ijl}^{mn} (k_{ijl}^{mn} + 2) \Theta(k_{mn} - p) + \sum_{i<j<l} H(k_{ijl}) [\Theta(k_{ijl} - p) + \Theta(k_{mn} - p)] - \sum_{i,j,l,m} F(k_{ij})F(k_{lm}) \Theta(k_{ij} - p) \Theta(k_{lm} - p). \quad (C.27)$$

On the other hand one can calculate

$$Z_{k_1k_2k_3k_4k_5}^{(1)} + \tilde{Z}_{k_1k_2k_3k_4k_5}^{(2)} = \frac{1}{8} (4 \sum_{i=1}^{5} k_i^2 - k^2 - 2k - 8)(2p - 3 - k)(2p - 5 - k) - \sum_{i<j<l} H(k_{ijl}). \quad (C.28)$$
As the result we find that
\[ Z_{k_1 k_2 k_3 k_4 k_5}^{(1)} + Z_{k_1 k_2 k_3 k_4 k_5}^{(2)} = \]
\[ = \frac{1}{8} \left( 4 \sum_{i=1}^{5} k_i^2 - k^2 - 2k - 8 - \sum_{m<n} k_{ij}^{mn} (k_{ij}^{mn} + 2) \Theta(k_{mn} - p)(2p - 3 - k)(2p - 5 - k) + \right. \]
\[ + \left. \sum_{i<j<l} H(k_{ijl}) G_1 - \sum_{i,j,l,m} F(k_{ij}) F(k_{lm}) G_2 \right) \quad \text{(C.29)} \]

where \( G_1 = \Theta(k_{ijl} - p) + \Theta(k_{lm} - p) - 1 \) and \( G_2 = \Theta(k_{ij} - p) \Theta(k_{lm} - p) \).

In the even section we will need formulas
\[ \frac{1}{2} \int_{-1}^{1} Q_{k_{ijl}k_{mn}}(x) Q_{k_{km}}(x) dx = \]
\[ = \frac{1}{8} (k_{ijl}^{jm} - 2)(k_{n}^{lm} - 4)(2p - 3 - k)(2p - 5 - k) \Theta(k_{ijl}^{jm} - 6) \quad \text{(C.30)} \]
\[ \frac{1}{2} \int_{-1}^{1} Q_{k_{mn}k_{n}}(x) Q_{k_{ijl}k_{i}}(x) dx = \]
\[ = \left( H(k_{ijl}) - \frac{k_{ijl}^{mn} (k_{ijl}^{mn} + 2)(2p - 3 - k)(2p - 5 - k)}{8} \Theta(k_{mn}^{ijl} - 2) \right) \Theta(p-1-k_{ijl}) \Theta(p-1-k_{mn}). \quad \text{(C.31)} \]

References

[1] A. Polyakov, “Quantum Geometry of Bosonic Strings”, Phys. Lett. B103: 207-210, (1981).
[2] P.H. Ginsparg and G.W. Moore, “Lectures on 2-D gravity and 2-D string theory”, arXiv:9304011 [hep-th]; P. Di Francesco, P.H. Ginsparg, J. Zinn-Justin, “2-D Gravity and random matrices”, Phys.Rep.254:1-133,(1995); arXiv:9306153 [hep-th]
[3] G.W. Moore, N. Seiberg, M. Staudacher, “From loops to states in 2-D quantum gravity”, Nucl. Phys. B362, 665-709, (1991)
[4] A.A. Belavin and A.B. Zamolodchikov, “On correlation numbers in 2D minimal gravity and matrix models”, Jour. Phys. A42 (2009) 304004; arXiv:0811.0450 [hep-th]