CR Singularities: Another Normal Form

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Abstract. Let \((z,w)\) be the coordinates in \(\mathbb{C}^2\). We construct a normal form for a class of real-formal surfaces \(M \subset \mathbb{C}^2\) defined near a degenerate CR singularity \(p = 0\) as follows
\[ w = z^2 + \overline{z}^2 + O(3). \]

1. Introduction and Main Result

The study of Real Submanifolds in Complex Space, near a CR singularity goes back to Bishop\[1\]. A point \(p \in M\) is called a CR singularity if it is a jumping discontinuity point for the map \(M \ni q \rightarrow \text{dim} T_q M\) defined near \(p\). Bishop\[1\] considered the case when there exist coordinates \((z,w)\) in \(\mathbb{C}^2\) such that near a CR singularity \(p = 0\), the surface \(M \subset \mathbb{C}^2\) is defined locally by
\[ w = z^2 + \lambda (z^2 + \overline{w}^2) + O(3), \]
where \(\lambda \in [0, \infty)\) is a holomorphic invariant called the Bishop invariant. When \(\lambda = \infty\), \(M\) is understood to be defined by the equation \(w = z^2 + \overline{z}^2 + O(3)\). If \(\lambda\) is non-exceptional, Moser-Webster\[9\] proved that there exists a formal transformation that sends \(M\) into the following normal form
\[ w = z^2 + (\lambda + \epsilon u^2) (z^2 + \overline{z}^2), \quad \epsilon \in \{0, -1, +1\}, \quad \epsilon \in \mathbb{N}, \]
where \(w = u + iv\). Moser\[10\] constructed when \(\lambda = 0\) the following partial normal form:
\[ w = z^2 + 2\Re \left\{ \sum_{j \geq 2} a_j z^j \right\}. \]

Here \(s := \min \{j \in \mathbb{N}; \ a_j \neq 0\}\) is the simplest higher order invariant, known as the Moser invariant. When \(s < \infty\), Huang-Yin\[5\] proved that \[13\] can be formally transformed into the following normal form
\[ w = z^2 + 2\Re \left\{ \sum_{j \geq 2} a_j z^j \right\}, \quad a_s = 1, \quad a_j = 0, \quad \text{if} \quad j = 0, 1 \mod s, \quad j > s. \]

In this paper, we continue the study of the C.-R. Singular Real Submanifolds in Complex Spaces considering certain Classes of Real-Submanifolds using motivation from Moser-Webster\[9\]. They\[9\] considered the following class of real-analytic surfaces
\[ w = z^2 + \overline{z}^2 + \sum_{m+n \geq 3} a_{m,n} z^m \overline{z}^n, \]
where \((z,w)\) are the coordinates in \(\mathbb{C}^2\).

Regardless of its apparent simplicity, \[13\] defines also a very interesting class of C.-R. Singular Submanifolds in Complex Spaces. In particular, it requires a similar approach depending on the Fischer decomposition\[8\] that has been applied by Zaitsev\[11\], Huang-Yin\[5\],\[12\],\[13\] in other situations, and also by the author recently\[8\]. In order to develop a partial normal form, we define the following Fischer-normalization space:

Before beginning, we introduce by \[8\] the following notation
\[ P^r = \sum_{m+n=k_0} p_{m,n} \frac{\partial^{m+n}}{\partial z^m \partial \overline{z}^n}, \quad \text{if} \ P(z, \overline{z}) = \sum_{m+n=k_0} p_{m,n} z^m \overline{w}^n. \]

In particular, we use the following polynomial
\[ Q(z, \overline{w}) = z^2 + \overline{w}^2, \]
and in consequence the following differential operator
\[ \text{tr} = \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \overline{z}^2}. \]

Recalling the Fischer Decomposition from Shapiro\[8\], we consider the following Fischer Decompositions
\[ z^k = A(z, \overline{w}) Q(z, \overline{w}) + C(z, \overline{w}), \quad \text{where} \ \text{tr} (C(z, \overline{w})) = 0, \forall k > 2 \ \text{natural number.} \]

We define
\[ S_p, \quad \text{for all} \ p \geq 3, \]

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which consists in real-valued polynomials $P(z, \overline{z})$ of degree $p \geq 1$ in $(z, \overline{z})$ satisfying the normalizations:

$$P_k^{(p)}(z, \overline{z}) = R_{k+1}^{(p)}(z, \overline{z})Q(z, \overline{z}) + R_{k+2}^{(p)}(z, \overline{z}),$$

for all $k = 0, \ldots, \left\lfloor \frac{p-1}{2} \right\rfloor$ and given $P_0^{(p)}(z, \overline{z}) = P(z, \overline{z}),$

such that

\begin{equation}
R_{k+2}^{(p)}(z, \overline{z}) \in \left( \ker C_k \cap \ker \overline{C}_k \cap \ker \text{tr} \right).
\end{equation}

Furthermore, we assume that

\begin{equation}
W(z, \overline{z}) \neq 0,
\end{equation}

where we have used the following notation

\begin{equation}
\sum_{m+n=3} a_{m,n} z^m \overline{z}^n \mod (C_3, \overline{C}_3) W(z, \overline{z}).
\end{equation}

The main result, of this note, is the following

**Theorem 1.1.** Let $M \subset \mathbb{C}^2$ be a formal surface defined near $p = 0$ by \eqref{2.1} satisfying the nondegeneracy condition \eqref{1.10}. Then, there exists a unique formal transformation of the following type

\begin{equation}
(z', w') = \left( z + \sum_{k+l \geq 2} f_{k,l} z^k w^l, \quad w + \sum_{k+l \geq 2} g_{k,l} z^k w^l \right),
\end{equation}

that transforms $M$ into the following formal normal form:

\begin{equation}
w' = P(z', \overline{z}') + \sum_{m+n \geq 3} a'_{m,n} z^m \overline{z}^n,
\end{equation}

where the following Fischer normalization conditions are satisfied

\begin{equation}
\text{Im} \left( \sum_{m+n=p} a'_{m,n} z^m \overline{z}^n \right) \in S_{p-1}, \quad \text{Re} \left( \sum_{m+n=p} a'_{m,n} z^m \overline{z}^n \right) \in S_p, \quad \text{for all } p \geq 3,
\end{equation}

where $S_p$ is defined in \eqref{1.12}, and as well the following normalization conditions holds

\begin{equation}
W^* \left( R_{3k}^{(3)}(z, \overline{z}) \right) = 0, \quad \text{for all } k > 2.
\end{equation}

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2. Proof of Theorem \eqref{1.12}

2.1. Notations. Let $(z, w)$ be the holomorphic coordinates in $\mathbb{C}^2$. Throughout this note, we use the following notations

$$a_{l \geq 1}(z, \overline{z}) = \sum_{m+n=l} a_{m,n} z^m \overline{z}^n, \quad a_1(z, \overline{z}) = \sum_{m+n=1} a_{m,n} z^m \overline{z}^n, \quad \text{for all } l \geq 3.$$

2.2. Transformation Equations. Let $M \subset \mathbb{C}^2$ be the real-formal surface defined near $p = 0$ by

\begin{equation}
w = Q(z, \overline{z}) + \sum_{m+n \geq 3} a_{m,n} z^m \overline{z}^n.
\end{equation}

Let $M' \subset \mathbb{C}^2$ be another real-formal surface defined near $p' = 0$ by

\begin{equation}
w' = Q(z', \overline{z}') + \sum_{m+n \geq 3} a'_{m,n} z^m \overline{z}^n.
\end{equation}

We consider

$$z' = f(z, w), \quad w' = g(z, w),$$

a formal transformation which sends $M$ into $M'$ and that fixes the point $0 \in \mathbb{C}^2$. It follows by \eqref{2.1} that

\begin{equation}
g(z, w) = Q(f(z, w), \overline{f(z, w)}) + \sum_{m+n \geq 3} a'_{m,n} (f(z, w))^m (\overline{f(z, w)})^n,
\end{equation}

where $w$ is defined by \eqref{2.1}. Writing as follows

\begin{equation}
f(z, w) = \sum_{m+n \geq 0} f_{m,n} z^m w^n, \quad g(z, w) = \sum_{m+n \geq 0} g_{m,n} z^m w^n,
\end{equation}

it follows by \eqref{2.1} that
\[
\sum_{m+n \geq 0} g_{m,n} z^m (Q(z, \overline{z}) + a_{\geq 3}(z, \overline{z}))^n = Q \left( \sum_{m+n \geq 0} f_{m,n} z^m (Q(z, \overline{z}) + a_{\geq 3}(z, \overline{z}))^n \right) + a_{\geq 3} \left( \sum_{m+n \geq 0} f_{m,n} z^m (Q(z, \overline{z}) + a_{\geq 3}(z, \overline{z}))^n \right).
\]

(2.4)

Since our map fixes the point \( 0 \in \mathbb{C}^2 \), it follows that \( g_{0,0} = 0 \) and \( f_{0,0} = 0 \). Collecting the terms of bidegree \((m, 0)\) in \((z, \overline{z})\) in (2.4), for all \( m < 2 \), it follows that \( g_{m,0} = 0 \), for all \( m < 2 \). Collecting the sums of bidegree \((m, n)\) in \((z, \overline{z})\) with \( m + n = 2 \) in (2.4), it follows that

\[
g_{0,1} Q(z, \overline{z}) = Q \left( f_1, a z, \overline{f_1}, \overline{a z} \right).
\]

Then, (2.5) describes all the possible values of \( g_{0,1} \) and \( f_{1,0} \) and in particular we obtain that \( \text{Im} g_{0,1} = 0 \). By composing with \( n \) linear automorphism of the model manifold \( \text{Re} w = Q(z, \overline{z}) \), we can assume that \( g_{0,1} = 1 \), \( f_{1,0} = 1 \). By a careful analysis of the terms interactions in (2.4), we conclude that in order to put suitable normalization conditions, we have to consider the following terms

\[
g_{m,n} z^m (P(z, \overline{z}))^n, \quad f_{m,n} z^m Q(z, \overline{z}) (Q(z, \overline{z}))^n, \quad f_{m,n} z^m Q(z, \overline{z}) (Q(z, \overline{z}))^n.
\]

Collecting the sum of terms of bidegree \((m, n)\) in \((z, \overline{z})\) with \( T = m + n \) in (2.4), it follows that \( g_T (z, Q(z, \overline{z})) = \text{Re} \left( Q(z, \overline{z}) f_T \left( z, Q(z, \overline{z}) \right) \right) + \ldots \),

where we have used the following notations

\[
g_T(z, w) = \sum_{m+n = T} g_{m,n} z^m w^n, \quad f_T(z, w) = \sum_{m+n = T} f_{m,n} z^m w^n,
\]

and where the terms defined by \( \ldots \), depend on \( f_{k,l} \) with \( k + 2l - 1 < T - 1 \), and as well on \( g_{k,l} \) with \( k + 2l < T \).

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