On superposition of the autoBäcklund transformations for 
(2 + 1)-dimensional integrable systems

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The well known classical autoBäcklund transformation for the sine-Gordon equation

\[ u_{xt} = \sin u \]  

given by (cf. [13])

\[
\begin{align*}
\bar{u}_t &= u_t - 2\alpha \sin \frac{u + \bar{u}}{2} \\
\bar{u}_x &= -u_x + \frac{2}{\alpha} \sin \frac{u - \bar{u}}{2}
\end{align*}
\]  

(2)

(where \( \alpha = \text{const} \) is a parameter of the transformation), possesses the nonlinear superposition 
property connecting solutions \( u^{(0)}, u^{(1)}, u^{(2)} \) and \( u^{(12)} \) of (1) if the pairs \( (u^{(0)}, u^{(1)}), (u^{(2)}, u^{(12)}) \) 
are connected by (2) with \( \alpha = \alpha_1 \), and \( (u^{(0)}, u^{(2)}), (u^{(1)}, u^{(12)}) \) 
are connected by (2) with \( \alpha = \alpha_2 \):

\[ u^{(12)} = 4 \tan \left[ \frac{\alpha_1 + \alpha_2}{\alpha_1 - \alpha_2} \tan \left( \frac{u^{(1)} - u^{(2)}}{4} \right) \right] + u^{(0)} \]  

(3)

The formula (3) is schematically presented on Fig. 1.

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Fig. 1

Since (2) defines the result of the Bäcklund transformation up to 1 constant (the initial data for $\Pi$) we need the following precise formulation: among the Bäcklund transformations with the parameter $\alpha_2$, obtained from $u^{(1)}$ and among the Bäcklund transformations with the parameter $\alpha_1$, obtained from $u^{(2)}$ one can find exactly one common solution of (1) defined by the explicit formula (3). Hereafter we will use the word "transformation" instead of "auto transformation".

The aforementioned nonlinear superposition principle ("the permutability property") was discovered by L. Bianchi in the framework of the classical differential geometry when he studied the induced by the Bäcklund transformations deformations of constant curvature surfaces in $\mathbb{R}^3$. In that period L. Bianchi, G. Darboux and others (see [2, 3, 7]) found Bäcklund transformations also for the equation $u_{xt} = e^u - e^{-2u}$, the "3-wave resonant interaction" system

$$
\begin{align*}
\dot{u}_1 &= c_1 u_2^x + \kappa u_2^x u_3^x, \\
\dot{u}_2 &= c_2 u_3^x + \kappa u_1^x u_3^x, \\
\dot{u}_3 &= c_3 u_2^x + \kappa u_1^x u_2^x,
\end{align*}
$$

(the latter coincides with the theory of Egorov type curvilinear coordinate systems in $\mathbb{R}^3$) and some others. They also found the appropriate nonlinear superposition formulas for these systems. In the modern theory of (1 + 1)-dimensional nonlinear integrable systems of partial differential equations Bäcklund transformations (see [13, 14]) and the nonlinear superposition formulas were found for the majority of all known integrable systems.

For (2 + 1)-dimensional integrable systems (i.e. systems with 2 "spatial" and one "time"
independent variables) some examples of Bäcklund transformations were also found. In the classical differential geometry of XIX–beginning of the XX century the respective problems (described by partial differential systems with 3 independent variables) were also intensively studied: the theory of arbitrary curvilinear orthogonal coordinate systems in $\mathbb{R}^3$ (and in $\mathbb{R}^n$ which is again only a (2+1)-dimensional problem for any $n$), the theory of conjugate coordinate systems, numerous problems in the theory of congruences ([2, 3, 19, 5]). For these problems the proper theory of Bäcklund transformations was constructed alongside with the nonlinear superposition principle (“permutability property”). But their properties are more complicated and are given in Table 1 compared to the properties of Bäcklund transformations of (1 + 1)-dimensional systems.

|   | (1 + 1)-dim case | (2 + 1)-dim case |
|---|------------------|------------------|
| 1 | Solutions of the system | solutions are parameterized by (several) functions of 1 variable | solutions are parameterized by (several) functions of 2 variables |
| 2 | Bäcklund transformation | Gives a new solution parameterized by (several) constants (as a solution of ODE’s) | Gives a new solution parameterized by (several) functions of 1 variable |
| 3 | The solution $u^{(12)}$ on Fig. 1 | defined uniquely by an algebraic formula via $u^{(0)}$, $u^{(1)}$, $u^{(2)}$ | is parameterized by (several) constants (and usually found via a quadrature), i.e. we have no algebraic formula |

In the modern theory of (2 + 1)-dimensional integrable systems (see [12, 4, 15]) the appropriate Bäcklund transformations also possess the property 2 of Table 1, and in the cases where a superposition formula was given the property 3 holds. As noted in [4, 16] in the majority of these cases the Bäcklund transformations were induced by the so called Moutard transformation well known in the classical differential geometry (see below).

In the present paper we will show how in the case of (2 + 1)-dimensional integrable systems one can find an extended formula of nonlinear superposition such that the resulting solution will be found uniquely from the given previous solution with algebraic operations.

The exposition of the method will be given for the case of the so called Ribaucour transformations of triply orthogonal curvilinear coordinate systems in $\mathbb{R}^3$ — one of the (2+1)-dimensional Bäcklund transformations obtained in classical differential geometry as well as for the Moutard transformation.

Let us remind the respective definitions (cf. [1, 3]).

**Definition 1.** Two surfaces $S_1$ and $S_2$ are called related by a Ribaucour transformation if one can establish (locally) a one-to-one correspondence between their points $(x_1 \in S_1) \leftrightarrow (x_2 \in S_2)$, such that the normals to $S_i$ taken in the corresponding point intersect in a point $P = P(x_1)$ such that $|Px_1| = |Px_2|$ and the curvature lines of $S_1$ correspond to the curvature lines of $S_2$.

Hence $S_1$ and $S_2$ are two corresponding pieces of the enveloping surface of some 2-parametric family of spheres with the additional property of correspondence of curvature lines.
In order to generalize the Ribaucour transformation for the case of two triply orthogonal curvilinear coordinate systems in $\mathbb{R}^3$ we have to take three 3-parametric families of surfaces (or $n$-parametric for the $n$-dimensional case) tangent in the corresponding (i.e. having the same values of the curvilinear coordinates) points to the corresponding one-parametric families of coordinate surfaces of the both orthogonal coordinate systems. Since due to the Dupin theorem the coordinate lines (i.e. lines of intersection of the coordinate surfaces) of any triply orthogonal curvilinear coordinate system are the curvature lines on the coordinate surfaces the described correspondence will guarantee the correspondence of curvature lines. Let us give the formulas (see \[1, 3\]). Hereafter one shall NOT apply summation to repeated indices unless stated explicitly.

Let $x^i = x^i(u^1, u^2, u^3)$ be a curvilinear orthogonal coordinate system. The quantities $H_i = |\partial_i x|$, $\partial_i = \partial/\partial u^i$, are called its Lamé coefficients, and $\beta_{ik} = \partial_i H_k/H_i$, $i \neq k$, are called its rotation coefficients. $\Gamma_{ki}^k = \partial_i H_k/H_k$ are the Christoffel symbols of the corresponding diagonal metric $g_{ii}(u) = H_i^2(u)$ and $\vec{X}_i = \partial_i \vec{x}/H_i$ gives the orthonormal tangent frame at the point $\vec{x}$. We have the known Lamé system of equations (necessary and sufficient for $\beta_{ij}$ to be the rotation coefficients of some orthogonal curvilinear coordinate system):

$$
\begin{align*}
\partial_i \beta_{jk} &= \beta_{ji} \beta_{ik}, \quad i \neq j \neq k, \\
\partial_i \beta_{ik} + \partial_k \beta_{ji} + \beta_{ji} \beta_{jk} &= 0, \quad i \neq j \neq k.
\end{align*}
$$

The system (4) is an overdetermined system whose solutions are parameterized by 3 functions of 2 variables (see [1, 3]). In order to define a Ribaucour transformation in terms of the rotation coefficients we have to find a solution (“a potential”) of the following overdetermined compatible system

$$
\partial_i \partial_j \varphi = \Gamma_{ij}^s \partial_i \varphi + \Gamma_{ij}^j \partial_j \varphi, \quad i \neq j.
$$

Then the Ribaucour transformation will be given by the formulas [3]

$$
\begin{align*}
\vec{x} &= \vec{x} - \frac{2\varphi}{A} \sum_{i=1}^3 \gamma_i \vec{X}_i, \\
\vec{X}_i &= \vec{X}_i - \frac{2\gamma_i}{A} \sum_{s=1}^3 \gamma_s \vec{X}_s, \\
\vec{H}_i &= H_i - \frac{2\varphi}{A} (\partial_i \gamma_i + \sum_{s \neq i} \beta_{si} \gamma_s), \\
\vec{\beta}_{ik} &= \beta_{ik} - \frac{2\gamma_i}{A} (\partial_k \gamma_k + \sum_{s \neq k} \beta_{sk} \gamma_s),
\end{align*}
$$

where $\gamma_i = \partial_i \varphi/H_i$, $A = \sum (\gamma_i)^2$. The nonlinear superposition of two Ribaucour transformations $\vec{x}^{(0)} \xrightarrow{\varphi^{(1)}} \vec{x}^{(1)}$, $\vec{x}^{(0)} \xrightarrow{\varphi^{(2)}} \vec{x}^{(2)}$, i.e. transformations $\vec{x}^{(1)} \xrightarrow{\varphi^{(12)}} \vec{x}^{(12)}$, $\vec{x}^{(2)} \xrightarrow{\varphi^{(21)}} \vec{x}^{(12)}$, is given by (5, 7) where

$$
\begin{align*}
\varphi^{(12)} &= \frac{(a \tau^{(12)} + c) \varphi^{(1)}}{A^{(1)}} + a \varphi^{(2)}, \quad \varphi^{(21)} = -\frac{(a \tau^{(12)} + c + 2ab \tau^{(12)}) \varphi^{(2)}}{A^{(2)}} + a \varphi^{(1)}, \\
\gamma_i^{(12)} &= \frac{(a \tau^{(12)} + c) \gamma_i^{(1)}}{A^{(1)}} + a \gamma_i^{(2)}, \quad \gamma_i^{(21)} = -\frac{(a \tau^{(12)} + c + 2ab \tau^{(12)}) \gamma_i^{(2)}}{A^{(2)}} + a \gamma_i^{(1)},
\end{align*}
$$
\[ A(i) = \sum_s (\gamma_s(i))^2, \quad B^{(12)} = \sum_s (\gamma_s^{(1)} \gamma_s^{(2)}) \; ; \; a \text{ and } c \text{ are constants and } \tau^{(12)}(u) \text{ is found via a quadrature from} \]

\[ \partial_k \tau^{(12)} = -2\gamma_k^{(2)} \left( \partial_k \gamma_k^{(1)} + \sum_{s \neq k} \beta_{sk} \gamma_s^{(1)} \right), \quad (9) \]

\[ \tau^{(21)} = -(\tau^{(12)} + 2B^{(12)}) \]  

(line 3 in Table. 1).

The basis of our method is given by the following commutative diagram of Bäcklund transformations (Fig. 2) where we used the notations for the case of the Ribaucour transformations for the sake of simplicity.

As was shown in [18] the diagram on Fig. 2 is actually commutative for (1 + 1)-dimensional equations (sine-Gordon, KdV, the 3-wave system). In that case if \( u^{(0)}, \ u^{(1)}, \ u^{(2)} \) are given one can find the solutions \( u^{(12)}, \ u^{(13)}, \ u^{(23)} \) using the algebraic superposition formula and one had only to check that the last 8th solution \( u^{(123)} \) will coincide for all three subdiagrams \( u^{(12)} \rightarrow u^{(123)} \leftarrow u^{(13)}, \ u^{(12)} \rightarrow u^{(123)} \leftarrow u^{(23)} \), \( u^{(13)} \rightarrow u^{(123)} \leftarrow u^{(23)} \) obtainable again from the algebraic superposition formulas (Fig. 1). As we show in the following Theorem just the
diagram on Fig. 2 gives the possibility to avoid quadratures in the superposition formulas for 
$(2 + 1)$-dimensional systems and obtain an algebraic superposition principle ("the Bäcklund cube").

**Theorem 1** Let three Ribaucour transformations $\vec{x}^{(0)} \xrightarrow{\varphi^{(1)}} \vec{x}^{(1)}$, $\vec{x}^{(0)} \xrightarrow{\varphi^{(2)}} \vec{x}^{(2)}$, $\vec{x}^{(0)} \xrightarrow{\varphi^{(3)}} \vec{x}^{(3)}$ of solutions of (4) be given as well as obtained from them via \((8), \(9)\) solutions $\vec{x}^{(12)}$, $\vec{x}^{(13)}$, $\vec{x}^{(23)}$. Then there exists a unique triply orthogonal coordinate system $\vec{x}^{(123)}$ related by Ribaucour transformations to $\vec{x}^{(12)}$, $\vec{x}^{(13)}$, $\vec{x}^{(23)}$: $\vec{x}^{(12)} \xrightarrow{\varphi^{(123)}} \vec{x}^{(123)}$, $\vec{x}^{(13)} \xrightarrow{\varphi^{(132)}} \vec{x}^{(123)}$, $\vec{x}^{(23)} \xrightarrow{\varphi^{(231)}} \vec{x}^{(123)}$. This solution $\vec{x}^{(123)}$ as well as the corresponding quantities $H^{(123)}$, $\beta^{(123)}$ may be expressed with algebraic formulas comprising only the given $\vec{x}^{(12)}$, $\vec{x}^{(13)}$, $\vec{x}^{(23)}$, $\vec{x}^{(1)}$, $\vec{x}^{(2)}$, $\vec{x}^{(3)}$.

**Proof.** As shown in [3] for the solutions $\vec{x}^{(0)}(u)$, $\vec{x}^{(1)}(u)$, $\vec{x}^{(2)}(u)$, $\vec{x}^{(12)}(u)$ connected (see Fig. 1) by Ribaucour transformations $\vec{x}^{(0)}$, $\vec{x}^{(1)}$ the following holds: all four points lie on one circumference (depending on the parameters — curvilinear coordinates $u^1$, $u^2$, $u^3$). Thus if we construct (for some fixed values $u^1$, $u^2$, $u^3$) the sphere passing through the four points $x^{(0)}$, $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, then all the other given $x^{(12)}$, $x^{(13)}$, $x^{(23)}$ will also belong to this sphere. Draw on this sphere three circumferences passing through the triples $(x^{(1)}$, $x^{(12)}$, $x^{(13)})$, $(x^{(2)}$, $x^{(12)}$, $x^{(23)})$, $(x^{(3)}$, $x^{(13)}$, $x^{(23)})$. We will show that all three circumferences will have one common point — the point $x^{(123)}$ to be found. Indeed after a stereographic projection of the sphere onto a plane we get the configuration shown on Fig. 3. (We did not show on this figure only the circumference passing through $x^{(1)}$, $x^{(12)}$, $x^{(13)}$).
Now we have to apply the well known geometric theorem (see, e.g. [11, §10.9.7.2]): if for 4 circumferences $C_1$, $C_2$, $C_3$, $C_4$, $C_1 \cap C_2 = \{P, P'\}$, $C_2 \cap C_3 = \{Q, Q'\}$, $C_3 \cap C_4 = \{R, R'\}$, $C_4 \cap C_1 = \{S, S'\}$, the points $P$, $Q$, $R$, $S$ lie on one circumference (or a straight line) then also $P'$, $Q'$, $R'$, $S'$ lie on one circumference (or a straight line). So $x^{(1)}$, $x^{(12)}$, $x^{(13)}$, $x^{(123)}$ really lie on one circumference. In the case when the pole of the stereographic projection lie on one of the circumferences the corresponding projection becomes a straight line. Thus the statement about the uniqueness of $x^{(123)}$ is proved. Obviously the coordinates of the point $x^{(123)}$ are algebraically expressible in terms of the given $x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(12)}$, $x^{(13)}$, $x^{(23)}$.

We still have to show that the found $x^{(123)}(u)$ as a function of the parameters $u^i$ gives an orthogonal curvilinear coordinate system related to $x^{(12)}$, $x^{(13)}$, $x^{(23)}$ by Ribaucour transformations.

In order to prove this we will show that we can choose at least some set of potentials $\varphi^{(123)}$, $\varphi^{(132)}$, $\varphi^{(231)}$ — solutions of (5) with corresponding to $x^{(12)}$, $x^{(13)}$, $x^{(23)}$ Christoffel symbols, such
that substituting them into (1) we will get the common vertex \( x^{(123)} \): \( x^{(12)} \xrightarrow{\varphi^{(123)}} x^{(123)} \xrightarrow{\varphi^{(132)}} x^{(13)} \xrightarrow{\varphi^{(13)}} x^{(132)} \xrightarrow{\varphi^{(21)}} x^{(23)} \xrightarrow{\varphi^{(23)}} x^{(21)} \). A priori we can find using (8), (9) some potentials \( \varphi^{(123)} = (\tau^{(123)} \varphi^{(12)})/A^{(12)} + \varphi^{(13)} \), \( \varphi^{(213)} = (\tau^{(213)} \varphi^{(21)})/A^{(21)} + \varphi^{(23)} \), \( \varphi^{(312)} = (\tau^{(312)} \varphi^{(31)})/A^{(31)} + \varphi^{(32)} \), \( \varphi^{(321)} = (\tau^{(321)} \varphi^{(32)})/A^{(32)} + \varphi^{(31)} \), (where we set for simplicity and without loss of generality the constants \( a = 1 \) and \( c = 0 \) in (9),) which will give us three different final points \( x^{(12)} \xrightarrow{\varphi^{(123)}} x^{(123)} \xrightarrow{\varphi^{(132)}} x^{(13)} \xrightarrow{\varphi^{(213)}} x^{(213)} \xrightarrow{\varphi^{(312)}} x^{(312)} \xrightarrow{\varphi^{(321)}} x^{(23)} \). Here the functions \( \tau^{(ijk)} \) are (defined up to an additive constant) solutions of the corresponding equations (9):

\[
\partial_s \tau^{(ijk)} = -2\gamma_s^{(ik)} \left( \partial_s \gamma_s^{(ij)} + \sum_{q \neq s} \beta_s^{(iq)} \gamma_s^{(ij)} \right)
\]  

(10)

**Lemma 1** The quantities \( \tau^{(123)} = \tau^{(23)} + \tau^{(13)} \tau^{(21)} A^{(1)} = \tau^{(23)} - \tau^{(13)} \tau^{(21)} + 2B^{(12)}, \)

\[
\tau^{(13)} + \tau^{(12)} \tau^{(23)} A^{(2)}, \tau^{(312)} = \tau^{(12)} + \tau^{(13)} \tau^{(32)} A^{(3)}, \tau^{(132)} = \tau^{(32)} + \tau^{(312)} A^{(1)} = \tau^{(23)} + \tau^{(31)} = \tau^{(23)} \tau^{(31)} A^{(3)}
\]

\( \tau^{(321)} = \tau^{(31)} + \tau^{(32)} \tau^{(21)} A^{(2)} \) satisfy (10) and after substitution into the formulas for \( \varphi^{(ijk)} \) result in \( \varphi^{(123)} = \varphi^{(213)}, \varphi^{(132)} = \varphi^{(312)}, \varphi^{(321)} = \varphi^{(23)} \).

The proof is easily obtained with a direct computation.

Consequently choosing the given in the Lemma \( \tau^{(ijk)} \) we find ONE common resulting \( \bar{x}^{(123)} = \bar{x}^{(213)} = \bar{x}^{(312)} \). The uniqueness of such \( \bar{x}^{(123)} \) has been proved above. The proof is completed.

The exposed above method of construction of algebraic nonlinear superposition formulas is especially simple for the case of the Moutard transformation (see [2], [3]). This transformation played an important role in classical differential geometry in the theory of surface deformations, the theory of congruences and nets. We will formulate the classical definitions in the form convenient for us (see also the formulation of the Ribaucour transformation in similar form in [4]). Let a ”potential” \( M(x, y) \) (an arbitrary function of two variables) is given (for the case of Ribaucour transformations the ”potential” \( M(u^1, u^2, u^3) \) satisfies a 3rd order equation, see [1], [3]). Then a Moutard transformation is defined by a solution \( u = R(x, y) \) of the Moutard equation

\[
u_{xy} = M(x, y) u, \quad u = u(x, y),
\]

(11)

with the formulas

\[
M = \frac{2R_x R_y}{R^2} = R \left( \frac{1}{R} \right)_{xy}.
\]

(12)

If we have alongside with \( u(x, y) \) a second solution \( u = \varphi(x, y) \) of the same equation (11) with the potential \( M = M_0 \) then one can find using a quadrature a solution \( \vartheta \) of the equation (11) with a transformed potential \( \bar{M} = M_1 \):

\[
\left\{ \begin{array}{l}
(R \vartheta)_x = -R^2 \varphi_R, \\
(R \vartheta)_y = R^2 \varphi_R,
\end{array} \right.
\]

(13)
Let us interpret (13) as a diagram on Fig. 1 (cf. [3, v. II, p. II, §297]): if a potential $M_0$ is given as well as its two Moutard transformations $M_0 \xrightarrow{u=R} M_1$, $M_0 \xrightarrow{u=\varphi} M_2$ then using a quadrature we can find $M_{12} = M_1 - 2(\log \vartheta)_{xy}$, i.e. $M_1 \xrightarrow{\vartheta} M_{12}$, $M_2 \xrightarrow{\psi} M_{12}$, $\vartheta$ is found from (13), $\psi = -u\vartheta/\varphi$ (see Fig. 4). L. Bianchi in [3, v. II, p. II, §297] gives in another form the "Bäcklund cube" formula for the case of Moutard transformations as the following proposition: if $R_1$, $R_2$, $\vartheta$ are three solution of (11) with a given $M = M_0$, and $\vartheta_1$, $\vartheta_2$ (solutions of the transformed equations with the potentials $M_1 = M_0 - 2(\ln R_1)_{xy}$, $M_2 = M_0 - 2(\ln R_2)_{xy}$) are obtained from $\vartheta$ via (13) then there exists a unique solution $\vartheta'$ of the 4th Moutard equation $u_{xy} = M_{12}u$ connected to $\vartheta_1$, $\vartheta_2$ with Moutard transformations (13). This $\vartheta'$ is expressible with an algebraic formula

$$\vartheta' - \vartheta = \frac{R_1 R_2}{\lambda} (\vartheta_2 - \vartheta_1), \quad \lambda = R_1 R_1' = -R_2 R_2',$$

where $R_1'$, $R_2'$ are obtained from $R_2$, $R_1$ according to (13).
This is illustrated on Fig. 5 where one can easily tag the untagged connecting lines following the expression for $\psi$ on Fig. 4, commutativity of the diagrams on Figs. 4, 5 is easily verified. So the cubic diagram on Figs. 2, 5 provide us with the method of exclusion of quadratures in the superposition formulas for the cases of Ribaucour and Moutard transformations. One may suppose that this diagram gives some algebraic superposition formulas also for Bäcklund transformations of other $(2 + 1)$-dimensional integrable systems (provided they possess the "traditional" superposition, Fig. 1). Thus the Table 1 shall be completed with the following row (cf. [18] about the validity of the column corresponding to $(1 + 1)$-dimensional systems):

|   | Solution $x^{(123)}$ on Fig. 2 | exists and is algebraically expressible through the other given solutions | exists and is algebraically expressible through the other given solutions |
|---|-----------------------------------|-----------------------------------------------------------------------------|-----------------------------------------------------------------------------|

As one can check using (10), if we have a forth initial Ribaucour transformation $\vec{x}^{(0)} \xrightarrow{\phi^{(4)}} \vec{x}^{(4)}$ we obtain a commutative diagram of Ribaucour transformations (4-dimensional cube skeleton) comprising 16 vertices $\vec{x}^{(i)}, \vec{x}^{(ij)}, \vec{x}^{(ijk)}, \vec{x}^{(1234)}, 1 \leq i < j < k \leq 4$. One may conjecture validity of analogous commutative diagrams for the case of $n$ initial Ribaucour transformations $\vec{x}^{(0)} \xrightarrow{\phi^{(i)}} \vec{x}^{(i)}, 1 \leq i \leq n$. The last (but not the least) property of such "Bäcklund hypercube" formulas consists in the possibility to obtain wide classes of solutions of $(2 + 1)$-dimensional integrable systems in question using only algebraic formulas (and performing quadratures only on "the first level" which is usually trivial for a trivial initial seed solution $u^{(0)}$). As we have shown elsewhere (see [3, 4]), one can obtain in such a way "almost all" their solutions.

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