The geometry of electromagnetic curves on Riemannian manifolds

Sergio Islas-Ramírez,1, Cesar S. Lopez-Monsalvo,2 and José Antonio Eduardo Roa-Neri1

1 Universidad Autónoma Metropolitana Azcapotzalco, Avenida San Pablo Xalpa 180, Azcapotzalco, Reynosa Tamaulipas, C.P. 02200, Ciudad de México, Mexico
2 Conacyt-Universidad Autónoma Metropolitana Azcapotzalco, Avenida San Pablo Xalpa 180, Azcapotzalco, Reynosa Tamaulipas, C.P. 02200, Ciudad de México, Mexico

We present a concise definition of an electromagnetic curve on a Riemannian manifold and illustrate the explicit case of the motion of a charged particle on the unit sphere under the influence of a uniform magnetic field.

I. INTRODUCTION

Magnetic curves describe the motion of a charged particle under the influence of a magnetic field. That is, solutions to the equations of motion

\[
\frac{d\vec{p}}{dt} = q (\vec{v} \times \vec{B}) \quad \text{with} \quad \vec{p} = m\vec{v},
\]

(1)

Here, \(\vec{p}\) is the momentum of the particle, \(q\) and \(m\) represent its charge and mass, respectively, \(\vec{v}\) is its velocity and \(\vec{B}\) is a given magnetic field [1].

On the other hand, Riemannian geometry has proven to be extremely useful in describing the dynamics of a system subject to spatial constraints [2]. That is, situations in which the entire space is not available and the motion is confined to a given surface. In such case, a reformulation of the Principle of Inertia is in order, providing us with the conditions a curve must satisfy so that it describes free motion on the constraint surface.

In this work, we provide a concise definition of electromagnetic curves on general Riemannian manifolds aided with the geometric formulation of Maxwell’s theory [3], allowing us to present the dynamics for an arbitrary number of spatial dimensions, where the tools of vector calculus are no longer well defined.

II. INERTIAL MOTION ON A RIEMANNIAN MANIFOLD

Galilean inertia – rooted in Euclidean geometry – is a degenerate case of differential geometry in the sense that straight lines, auto-parallel curves and paths of minimal length, coincide. Soon after Lobachevsky and Bolyai ventured outside the realm of Euclid’s axioms, new possibilities for inertial motion emerged. In particular, one can formulate a Riemannian Principle of Inertia, namely

**Definition 1** (Riemannian Inertial Motion). Let \((M, g)\) be a Riemannian manifold where \(g\) denotes its metric. Consider a parametrized curve

\[
\gamma : [a, b] \subset \mathbb{R} \to M.
\]

(2)

We say \(\gamma\) is an inertial motion if:

1. It is an extremal of the arc-length functional

\[
\ell(\gamma) = \int_a^b \sqrt{g(\dot{\gamma}, \dot{\gamma})} \, dt \quad \text{where} \quad \dot{\gamma} \equiv \frac{d}{dt} \gamma
\]

(3)

2. Its velocity is uniform, i.e.

\[
\frac{d}{dt} |\dot{\gamma}|^2 = \mathcal{L}_\dot{\gamma} g(\dot{\gamma}, \dot{\gamma}) = 0 \quad \forall t \in [a, b],
\]

(4)

where \(\mathcal{L}\) denotes the Lie derivative [4].

It follows from (3) and (4) that acceleration measures the departure of a curve from being inertial. This is expressed through the notion of **covariant derivative** which, in the case of Riemannian geometry, is given by

\[
a \equiv \nabla_\dot{\gamma} \ddot{\gamma} = \frac{D}{dt} \ddot{\gamma} = \frac{D}{dt} \ddot{\gamma},
\]

(5)

where \(\nabla\) is the Levi-Civita connection compatible with the metric \(g\) and \(D/dt\) denotes its associated covariant derivative [5].
III. ELECTROMAGNETIC CURVES ON RIEMANNIAN MANIFOLDS

Maxwell’s equations on manifolds are expressed as two independent conservation laws [3], namely

\[ \oint_{\partial \Omega^3} F \overset{!}{=} 0 \quad \text{and} \quad \oint_{\partial \Omega^n} J \overset{!}{=} 0. \]  (6)

Here \( \Omega^3 \) and \( \Omega^n \) represent arbitrary three and \( n \)-dimensional regions with boundary, respectively. The boundary operator is expressed as \( \partial \) and the symbol \( \overset{!}{=} \) denotes a physical demand, in this case, that the electromagnetic flux \( F \) and current \( J \) are conserved. Stokes’ theorem together with the arbitrariness of the domains in (6) imply the local conservation laws

\[ dF = 0 \quad \text{and} \quad dJ = 0, \]  (7)

where \( d \) denotes the exterior derivative. Therefore, an electromagnetic field on a manifold is expressed by a closed 2-form \( F \) whilst an electromagnetic current is given by a closed \( (n-1) \)-form \( J \).

On a Riemannian manifold \((M, g)\), the metric tensor plays the rôle of a material medium, defining the constitutive relation

\[ H = \zeta \ast_g F \quad \text{with} \quad dH = J, \]  (8)

where \( \zeta \) denotes the medium impedance and \( \ast_g \) is the Hodge star operator associated with the metric \( g \) [4].

The motion of a charged particle under the influence of an electromagnetic field \( F \) satisfies the equation

\[ m \gamma'' = q \Phi(\dot{\gamma}), \]  (9)

where the left hand side (lhs) is the particle’s mass times its acceleration [cf. equation (5)] while the right hand side (rhs) is the Lorentz force defined by the compatibility condition

\[ g(\Phi(u), w) = F(u, w), \]  (10)

where \( u \) and \( w \) are two arbitrary vector fields defined on \( M \) [1].

Therefore, an electromagnetic curve on a Riemannian manifold can be defined as follows:

**Definition 2 (Electromagnetic curve).** Let \((M, g)\) be a Riemannian manifold, \( J \) a closed \((n-1)\)-form on \( M \). A parametrized curve

\[ \gamma : [a, b] \subset \mathbb{R} \to M \]  (11)

is called an electromagnetic curve if it satisfies [9] where the electromagnetic field \( F \) is a solution to Maxwell’s equations [8] and the Lorentz force satisfies the compatibility condition [10].

IV. MAGNETIC CURVES ON A SPHERE

Let us consider a vertically oriented uniform magnetic field defined in a region of \( \mathbb{R}^3 \) where the standard Euclidean inner product is assumed, together with a charged particle confined to move on a sphere. Let \((S^2, g)\) be the unit sphere canonically embedded in \( \mathbb{R}^3 \) where \( g \) is the induced metric given by

\[ g = d\theta \otimes d\theta + \sin^2(\theta) d\varphi \otimes d\varphi \]  (12)

and the flux 2-form on the sphere becomes

\[ F|_{S^2} = B_z \cos(\theta) \sin(\theta) \; d\theta \wedge d\varphi. \]  (13)

The particle’s acceleration [9] in the lhs of [9] is given by

\[ a = \left[ \frac{d^2\theta}{dt^2} + \sin(\theta) \cos(\theta) \left( \frac{d\varphi}{dt} \right)^2 \right] \frac{\partial}{\partial \theta} \]

\[ + \left[ \frac{d^2\varphi}{dt^2} + \cot(\theta) \frac{d\theta}{dt} \frac{d\varphi}{dt} \right] \frac{\partial}{\partial \varphi}, \]  (14)

while

\[ \Phi(\dot{\gamma}) = \left[ B_z \sin(\theta) \cos(\theta) \frac{d\Phi}{dt} \right] \frac{\partial}{\partial \theta} \]

\[ - \left[ B_z \cot(\theta) \frac{d\theta}{dt} \right] \frac{\partial}{\partial \varphi}. \]  (15)
Solving numerically equation (9) for $\gamma : t \mapsto [\theta(t), \varphi(t)]$ we obtain the curves shown in figure 2. Here, we present some representative cases of magnetic curves for a pair of initial conditions. The red and blue lines correspond to flipped initial velocities. Notice the directionality of the curves, they are not reversible. This fact motivates further exploration and provides us with a magnetic analogue to the Zermelo navigation problem [6].

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