On Computing Min-Degree Elimination Orderings

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Abstract

We study faster algorithms for producing the minimum degree ordering used to speed up Gaussian elimination. This ordering is based on viewing the non-zero elements of a symmetric positive definite matrix as edges of an undirected graph, and aims at reducing the additional non-zeros (fill) in the matrix by repeatedly removing the vertex of minimum degree. It is one of the most widely used primitives for pre-processing sparse matrices in scientific computing.

Our result is in part motivated by the observation that sub-quadratic time algorithms for finding min-degree orderings are unlikely, assuming the strong exponential time hypothesis (SETH). This provides justification for the lack of provably efficient algorithms for generating such orderings, and leads us to study speedups via degree-restricted algorithms as well as approximations. Our two main results are: (1) an algorithm that produces a min-degree ordering whose maximum degree is bounded by $\Delta$ in $O(m\Delta \log^3 n)$ time, and (2) an algorithm that finds an $(1 + \epsilon)$-approximate marginal min-degree ordering in $O(m \log^5 n \epsilon^{-2})$ time.

Both of our algorithms rely on a host of randomization tools related to the $\ell_0$-estimator by [Cohen ‘97]. A key technical issue for the final nearly-linear time algorithm are the dependencies of the vertex removed on the randomness in the data structures. To address this, we provide a method for generating a pseudo-deterministic access sequence, which then allows the incorporation of data structures that only work under the oblivious adversary model.
1 Introduction

Many algorithms in numerical analysis and scientific computing benefit from speedups using combinatorial graph theory [NS12, HP07]. Such connections are due to the correspondence between non-zero entries of matrices and edges of graphs. The minimum degree algorithm is a classic heuristic for minimizing the space and time cost of Gaussian elimination, which solves a system of linear equations by adding and subtracting rows to eliminate variables. As its name suggests, it repeatedly pivots on the variable involved in the fewest number of equations [GL89]. There are many situations where this is suboptimal. Nonetheless, it is still a widely used and effective heuristic in practice [ADD04, DGLN04]. It is integral to the direct methods for solving linear systems exactly in LaPack [ABD90], which is in turn called by the “\” command for solving linear systems in MATLAB [Mat17]. It is also a critical part of the linear algebra suite in Julia [BKSE12].

While the best theoretical running times for solving such systems either rely on fast matrix multiplication [LG14] or iterative methods [ST14, KMP12], direct methods and their speedups are preferred in many cases. For such elimination-based methods, performances better than the general $O(n^3)$ bound for naive Gaussian elimination are known only when the non-zero graph has additional separators [LT79, LRT79, GT87] or hierarchical structure [PCD17]. Nonetheless, these methods are still preferable for a variety of reasons. They only depend on the non-zero structure, and have fewer numerical issues. More importantly, direct methods also benefit more from the inherent sparsity in many real-world input instances. For an input matrix and a given elimination order of the variables, the non-zero structure that arises over the course of the elimination steps has a simple characterization graph theoretically [Ros73, RTL76, LRT79, GT87].

This characterization of additional non-zero entries, known as fill, is at the core of elimination trees, which allow one to precisely allocate memory for the duration of the algorithm in $n\alpha(n)$ time [GNP94]. The reliable performance of elimination-based methods has led to the study of elimination-based methods for solving more structured linear systems [KS16]. However, recent hardness results seem to indicate that speedups via additional numerical structure may be limited to families of specific problems instead of all sparse matrices arising in scientific computing and numerical analysis [KZ17].

Although computing an elimination ordering that minimizes the total cost is NP-hard in general [BS90, Yan81], the minimum degree heuristic is exceptionally useful in practice. When the non-zeros of the matrix are viewed as edges of a graph, eliminating a vertex is equivalent to creating a clique on its neighborhood and then deleting this vertex. With this view in mind, the traditional min-degree algorithm can be viewed as: (1) find the vertex $v$ with minimum degree (which we term the fill-degree to avoid confusion with the original graph) in $O(n)$ time; (2) add a clique among all its neighbors in $O(n^2)$ time; (3) remove it together with all its edges from the graph in $O(n)$ time.

This leads to a running time that is $O(n^3)$—as high as the cost of Gaussian elimination itself. Somewhat surprisingly, despite the wide use of the min-degree heuristic in practice, there have been very few works on provably faster algorithms for producing this ordering. Instead, heuristics such as AMD (approximate-minimum degree ordering) [ADD96] aim to produce orderings similar to minimum-degree orderings in provably faster times such as $O(nm)$ without degree pivot size bounds.

Our investigation in this paper revolves around the question of finding provably more efficient

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1We will assume the system is symmetric positive definite (SPD) and thus the diagonal will remain strictly positive, allowing for any pivot order.
algorithms for producing exact and approximate min-degree orderings. We combine sketching with implicit representations of the fill structure to obtain provably $O(nm \log n)$ time algorithms. These algorithms utilize representations of intermediate non-zero structures related to elimination trees in order to implicitly examine the fill, which may be much larger. We also uncover a direct but nonetheless surprising connection between finding min-degree vertices and popular hardness assumptions. In particular, we show that computing the vertex of minimum degree after several specified pivot steps cannot be done faster than $O(n^2)$ time, assuming the widely-believed strong exponential time hypothesis [Wil05].

Nevertheless, we are able to extend various tools from sketching and sampling to give several improved bounds for computing and approximating minimum degree orderings. We show that our use of sketching can be much more efficient when the maximum degree is not too large. This in turn enables us to use sampling to construct data structures that accurately approximate the fill-degrees of vertices in graphs in polylog($n$) time, even under pivoting of additional vertices. Leveraging such approximate data structures, we obtain an algorithm for producing an approximate marginal minimum degree ordering, which at each step pivots a vertex whose degree is close to minimum, in nearly-linear time. Our main result is:

**Theorem 1.1.** Given an $n \times n$ matrix $A$ with non-zero graph structure $G$ containing $m$ non-zeros, we can produce an $\epsilon$-approximate greedy min-degree ordering in $O(m \log^{5} \frac{n}{\epsilon})$ time.

Our algorithms combine classical ideas in streaming algorithms and data structures, such as $\ell_0$-samplers [Coh97], wedge sampling [KPT17, ELRS17], and exponential start-time clustering [MPX13, MPVX15]. Until now these tools have not been rigorously studied in the context of scientific computing due to their dependency on randomization. However, we believe there are many other algorithms and heuristics in scientific computing that can benefit from the use of these techniques.

Furthermore, our overall algorithm critically relies on dissociating the randomness from the pivot steps, as the update is dependent on the randomness in the data structures. In Section 3.4 we give an example of how such correlations can “amplify” errors in the data structures. To address this issue, we define a pseudo-deterministic sequence of pivots based on a second degree-estimation scheme, which we discuss in Section 3.5.

Our paper is organized as follows. We will formalize the implicit representation of fill and definitions of exact, capped, and approximate min-degree orderings in Section 2. Then in Section 3 we give an overview of our results and discuss our main decorrelation technique in Subsection 3.5. Our main hardness results are in Section 4 while the use of sketching and sampling to obtain exact and approximate algorithms are in Sections 5 and 6 respectively. Further details on the graph theoretic building blocks are in Sections 7 and 8. They respectively cover the estimation of fill-degree of a single vertex and the maintenance of sketches as vertices are pivoted.

## 2 Preliminaries

We work in the pointer model, where function arguments are pointers to objects instead of the objects themselves. Therefore, we do not assume that passing an object of size $O(n)$ costs $O(n)$ time and space. This is essentially the “pass by reference” construct in high-level programming languages.
2.1 Gaussian Elimination and Fill

Gaussian elimination is the process of repeatedly eliminating variables from a system of linear equations, while maintaining an equivalent system on the remaining variables. Algebraically, this involves taking one equation involving some target variable and subtracting (a scaled version of) this equation from all others involving the target variable. Since our systems are SPD, we can also apply these operations to the columns and drop the variable, which gives the Schur complement.

A particularly interesting fact about Gaussian elimination is that the numerical Schur complement is unique irrespective of the ordering of pivoting. Under the now standard assumption that non-zero elements do not cancel each other out \cite{GL89}, this commutative property also holds for the combinatorial non-zero structure. Since the non-zero structure of a matrix corresponds to a graph, we can define the combinatorial change to the non-zero structure of the matrix as a graph theoretic operation. We start with the notation from Gilbert, Ng, and Peyton \cite{GNP94}. For a symmetric matrix, they use 

$$G(A)$$

to denote the undirected graph formed by its non-zero structure.

Gilbert, Ng, and Peyton \cite{GNP94} worked with a known elimination ordering and treated the entire fill pattern statically. Because we work with partially eliminated states, we will need to distinguish between the eliminated and remaining vertices in $G$ by implicitly associating vertices with two states:

- Eliminated vertices will be denoted using $x$ and $y$.
- Remaining vertices will be denoted using $u$, $v$, and $w$.

Then we use the fill graph

$$G^+$$

to denote the graph on the remaining vertices, where we add an edge $\{u, v\}$ between any pair of remaining vertices $u$ and $v$ connected via a path of eliminated vertices. We can also iteratively form the fill graph $G^+$ from $G$ by repeatedly removing an eliminated vertex $w$ and its incident edges, and then adding edges between all of the neighbors of $w$ to form a clique. This characterization of fill means that we can readily compute the fill-degree of a single vertex in a partially eliminated state without explicitly constructing the matrix.

**Lemma 2.1.** For any graph $G$ and vertex $v \in V$, given an elimination ordering $S$ we can compute in $O(m)$ time the value $\text{deg}(v)$ in $G^+$ when $v$ is eliminated.

**Proof.** Color the vertices in the sequence before $v$ red, and color all remaining vertices green. Run a depth-first search from $v$ that terminates at green vertices $u \neq v$. Let $D$ be the set of green vertices at which the search terminated. It follows from the definition of $G^+$ that $\text{deg}(v) = |D|$. \qed

This kind of path finding among eliminated vertices adds an additional layer of complexity to our structures. To overcome this, we contract eliminated vertices into their connected components, leading to the notion of the component graph. We use

$$G^\circ$$

to denote such a graph where we contract all edges $\{x, y\}$ between eliminated vertices $x$ and $y$. We will denote the vertices corresponding to such components by $c$. Note that $G^\circ$ is a quasi-bipartite
graph, because the contraction rule implies there are no edges between the component vertices. It is also useful to denote the neighborhood of different kinds of vertices in a component graph:

- \( N_{\text{remaining}}(c) \) or \( N_{\text{remaining}}(u) \): For a component \( c \) or a remaining vertex \( u \) in the component graph \( G^0 \), we use \( N_{\text{remaining}}(\cdot) \) to denote the neighbors that are remaining vertices.
- \( N_{\text{component}}(u) \): For a remaining vertex \( u \), this is the set of component vertices adjacent to \( u \).
- \( N_{\text{fill}}(u) \): For a remaining vertex \( u \), this denotes the neighbors of \( u \) in \( G^+ \), which is
  \[
  \left( \bigcup_{c \in N_{\text{component}}(u)} N_{\text{remaining}}(c) \right) \cup N_{\text{remaining}}(u) \cup \{u\}.
  \]

Note that the fill-degree of a remaining vertex \( u \) (its degree in \( G^+ \)) is precisely \(|N_{\text{fill}}(u)|\). Additionally, we use the restricted degrees:

- \( d_{\text{remain}}(c) \) or \( d_{\text{remain}}(u) \) to denote the size of \( N_{\text{remaining}}(c) \) or \( N_{\text{remaining}}(u) \), respectively.
- \( d_{\text{component}}(u) \) to denote the size of \( N_{\text{component}}(u) \) for some remaining vertex \( u \).

### 2.2 Min-Degree Orderings: Greedy, Capped, and Approximate

For an elimination ordering \( u_1, u_2, \ldots, u_n \), we define \( G_i \) as the graph with vertices \( u_1, u_2, \ldots, u_i \) marked as eliminated and \( u_{i+1}, u_{i+2}, \ldots, u_n \) marked as remaining. Furthermore, we say such a permutation is a minimum degree permutation if at each step \( i \), the vertex \( u_i \) has the minimum fill-degree in the non-zero structure graph \( G_{i-1} \). Concretely,

\[
\deg^{G^+_{i-1}}(u_i) = \min_{v \in V(G^+_{i-1})} \{ \deg^{G^+_{i-1}}(v) \}.
\]

Because the performance of our algorithm degrades over time as the minimum degree increases, we define the notion of a \( \Delta \)-capped minimum degree ordering, where degrees are truncated to \( \Delta \) before making a comparison. We first define \( \Delta \)-capped equality where \( \Delta \) is an integer.

**Definition 2.2.** We use the notation \( p = \_\Delta q \) to denote \( \min\{p, \Delta\} = \min\{q, \Delta\} \).

Now we can modify the definition of minimum degree in Equation 1 to specify that the elimination sequence \( u_1, u_2, \ldots, u_n \) satisfies the \( \Delta \)-capped minimum degree property at each time step:

\[
\deg^{G^+_{i-1}}(u_i) = \_\Delta \min_{v \in V(G^+_{i-1})} \{ \deg^{G^+_{i-1}}(v) \}.
\]

Our algorithm for finding the minimum (\( \Delta \)-capped) degrees is randomized, so we need to be careful to not introduce dependencies between different steps when several remaining vertices are of minimum degree. To bypass this problem, we require that the lexicographically least vertex be eliminated at each step in the event of a tie. This simple condition is critical for arguing that our randomized routines do not introduce dependencies as the algorithm progresses.

Lastly, our notion of approximating the min-degree ordering is based on finding the vertex whose fill-degree is approximately minimum in the current graph \( G^+ \). This decision process has no look-ahead, and therefore does not in any way approximate the minimum possible total fill.
**Definition 2.3.** An ordering of vertices \( u_1, u_2, \ldots, u_n \) is a \((1 + \epsilon)\)-approximate greedy min-degree ordering if for all steps \( 1 \leq i \leq n \) we have

\[
\deg_{G_{i-1}}^+(u_i) \leq (1 + \epsilon) \min_{v \in V(G_{i-1})} \left\{ \deg_{G_{i-1}}^+(v) \right\}.
\]

\[(3)\]

### 2.3 Randomized Tools

All of our algorithms are randomized, and their analyses involve tools such as the union bound, concentration bounds, and explicit calculations and approximations of expected values. We say an event happens with high probability (w.h.p.) if for any constant \( c > 0 \) there is a setting of constants (hidden by big-\( O \) notation) so that this event occurs with probability at least \( 1 - 1/n^c \).

We also make extensive applications of backward analysis \[Sei93\], which calculates the probabilities of events locally using the current state of the data structures.

Our final algorithm for producing \( \epsilon \)-approximate marginal min-degree orderings relies heavily on properties of the exponential distribution in order to decorrelate updates to the data structures and the results that it produces. Properties of the exponential random variable are formalized in Section 6 and we discuss its role in our algorithm in the overview in Section 3.5.

The analysis of our algorithms critically hinges on viewing all randomness as being generated before-hand, based on the (potential) index in which the procedure gets called. This is opposed to having a single source of randomness that we query sequentially as the procedures are invoked. For procedures such as the fill-degree estimator in Section 7.1 this method leads to a simplified analysis by viewing the output of a randomized sub-routine as a fixed distribution. Such a view of randomization is also a core idea in our decorrelation routine, which defines a random distribution on \( n \) elements, but only queries \( O(1) \) of them in expectation. This view is helpful for arguing that the randomness we query is independent of the indices that we ignored.

### 2.4 Related Works

**Fill from Gaussian Elimination and Pivot Orderings**

The study of better pivoting orderings is one of the foundational questions in combinatorial scientific computing. Work by George \[Geo73\] led to the study of nested dissection algorithms, which utilize separators to give provably smaller fill bounds for planar \[RTL76\], \[LRT79\] and separable graphs \[GT87\], \[AY10\]. One side effect of such a study is the far better (implicit) characterization of fill entries discussed in Section 2.4. This representation was used to compute the total amount of fill of a specific elimination ordering \[GNP94\]. It is also used to construct elimination trees, which are widely used in combinatorial scientific computing to both pre-allocate memory and optimize cache behaviors \[Liu90\].

**Finding Low Fill-in Orderings**

The ability to compute total fill for a given ordering raises the natural question of whether orderings with near-optimal fills can be computed. NP-hardness results for finding the minimum fill-in ordering \[Yan81\], \[BS90\] were followed by works for approximating the minimum total fill \[NSS00\], as well as algorithms \[KST99\], \[FV13\] and hardness results for parameterized variants \[WAPL14\], \[BCK+16\], \[CSI17\].
Partially due to the higher overhead of these methods, the minimum degree method remains one of the most widely used methods for producing orderings with small fill \cite{GL89}. Somewhat surprisingly, we were not able to find prior works that compute the exact minimum degree ordering in times faster than $O(n^3)$, or ones that utilize the implicit representation of fill provided by elimination trees\footnote{We use speculative language here due to the vastness of the literature on variants of minimum degree algorithms.}. On the other hand, there are various approximate schemes for producing min-degree like orderings. These include multiple minimum degree (MMD) \cite{Liu85} and an approximate minimum degree algorithm (AMD), the latter of which is used in MATLAB \cite{ADD96}. While both of these methods run extremely well in practice, theoretically they have tight performances of $O(n^2m)$ for MMD and $O(nm)$ for AMD \cite{HEKP01}. Furthermore, AMD can be viewed as a different version of the min-degree heuristic, as it is not always guaranteed to produce a vertex of approximate minimum degree.

### Estimating and Sketching Sizes of Sets

The core difficulty of our algorithms is in estimating the cardinality of sets (neighborhoods of eliminated components or component vertices in component graphs $G^o$) under union and deletion of elements. Many cardinality estimation algorithms have been proposed in the streaming algorithm literature using similar ideas \cite{FM85,CM05}. These algorithms often trade off accuracy for space, whereas we trade space for accuracy and efficiency in updates and queries.

Also closely related is another size-estimation framework for reachability problems by Cohen \cite{Coh97}. This work utilized $\ell_0$-estimators, which propagate random sketch values along neighborhoods to estimate the size of reachable sets. Our sketching method in Section 5 propagates the exact same set of values. However, we need to maintain this propagation under vertex pivots, which is akin to contracting edges in the component graph. This leads to a layer of intricacies that we resolve using amortized analysis in Section 8.

### Removing Dependencies in Randomized Algorithms

Lastly, our use of size estimators is dynamic—the choice of pivots, which in turn affects the subsequent graph eliminate states, is a result of the randomness used to generate the results of previous steps. The independence between the access sequence and randomness is a common requirement in recent works on data structures that maintain spanning trees and matchings \cite{BGS15,KKM13,Sol16}. There this assumption is known as the oblivious adversarial model, which states that the adversary can choose the graph and the sequence of updates, but it cannot choose updates adaptively in response to the randomly guided choices of the algorithm.

There have been recent works that re-inject randomness to preserve “independence” of randomized dimensionality-reduction procedures \cite{LS15}. The amount of “loss” in randomness has been characterized via mutual information in a recent work \cite{KNP+17}. Their bounds require an additional factor of $k$ of randomness in order to handle $k$ adversarially injected information, which as stated is too much for handling $n$ pivots adversarially. Our work also has some tenuous connections to recent works that utilize matrix martingales to analyze repeated introductions of randomness in graph algorithms \cite{KSI16,KPPS17}. However, our work utilizes more algorithmic tools than the martingale-based ones.
3 Overview

The starting point of our investigation uses sketching to design an efficient data structure for maintaining fill-degrees under pivot operations. This corresponds to edge contractions in the component graph and is based on the observation that $\ell_0$-estimators propagate well along edges of graphs. For any $n \times n$ matrix with $m$ non-zero entries, this algorithm takes $O(nm)$ time.

In our attempts to improve the running time of an exact algorithm, we came to the somewhat surprising realization that it is hard to compute the minimum degree in certain partially eliminated graphs in time $O(n^{2-\theta})$, for any $\theta > 0$, assuming the strong exponential time hypothesis. We extend this observation to give super-linear hardness for computing minimum degree orderings.

This hardness result for exact minimum degree sequences then motivated us to parameterize the performance of min-degree algorithms in a new way. Inspired by the behavior of AMD, we parameterize the performance of our algorithm in terms of intermediate degrees. Letting the minimum degree of the $i$-th pivot be $\Delta_i$ and the number of edges at that time be $m_i$, we improve the performance of our algorithm to $O(\max_i m_i \Delta_i)$. For many important real-world graphs such as grids and cube meshes, this bound is sub-quadratic. We then proceed to give a nearly-linear time algorithm for computing an $\epsilon$-approximate marginal min-degree ordering, where at each step the eliminated vertex has fill degree close to the current minimum.

3.1 Sketching the Fill Graph

We first explain the connection between computing fill-degrees and estimating the size of reachable sets. Assume for simplicity that no edges exist between the remaining vertices. Consider duplicating the remaining vertices so that each remaining vertex $u$ splits into $u_1, u_2$, and any edge $\{u, x\}$ in the component graph becomes two directed edges $(u_1 \rightarrow x)$ and $(x \rightarrow u_2)$. Then the fill-degree of $u$ is the number of remaining vertices $v_2$ reachable from $u_1$. Estimating the size of reachable sets is a well-studied problem for which Cohen [Coh97] gave a nearly-linear time algorithm using $\ell_0$-estimators. Adapting this framework to our setting for fill graphs (without duplication of vertices) leads to the following $\ell_0$-sketch structure.

**Definition 3.1.** An $\ell_0$-sketch structure consists of:

1. Each remaining vertex $u$ generating a random number $x_u$.

2. Each remaining vertex $u$ then computing the minimum $x_v$ among its neighbors in $G^+$ (including itself), which is equivalent to

$$\min_{v \in N_{\text{reachable}}(u)} x_v.$$  

In Section 8 we demonstrate that a copy of this structure can be maintained efficiently through any sequence of pivots in nearly-linear time. As the priorities $x_u$ are chosen independently and uniformly at random, we effectively assign each vertex $u$ a random vertex from its reachable set $N_{\text{reachable}}(u)$. Therefore, if we maintain $O(n \log n)$ independent copies of this $\ell_0$-sketch data structure, by a coupon-collector argument each vertex has a list of all its distinct neighbors. Adding together the cost of these $O(n \log n)$ copies leads to an $O(mn \log^2 n)$ time algorithm for computing a minimum degree sequence, which to the best of our knowledge is the fastest such algorithm.
3.2 SETH-Hardness of Computing Min-Degree Elimination Orderings

Our hardness results for computing the minimum fill degree and the min-degree ordering are based on the strong exponential time hypothesis (SETH), which states that for all $\theta > 0$ there exists a $k$ such that solving $k$-SAT requires $\Omega(2^{(1-\theta)n})$ time. Many hardness results based on SETH, including ours, go through the ORTHOGONAL VECTORS problem and make use of the following result.

**Theorem 3.2** ([Wil15]). Assuming SETH, for any $\theta > 0$, there does not exist an $O(n^{2-\theta})$ time algorithm that takes $n$ binary vectors with $\Theta(\log^2 n)$ bits and decides if there is an orthogonal pair.

We remark that ORTHOGONAL VECTORS is often stated as deciding if there exists a pair of orthogonal vectors from two different sets ([Wil15]), but we can reduce the problem to a single set by appending $[1;0]$ to all vectors in the first set and $[0;1]$ to all vectors in the second set.

Our hardness observation for computing the minimum degree of a vertex in the fill graph of some partially eliminated state is a direct reduction to ORTHOGONAL VECTORS. We give a bipartite graph construction that demonstrates how ORTHOGONAL VECTORS can be interpreted as deciding if a union of cliques covers a clique on the remaining vertices of a partially eliminated graph.

**Lemma 3.3.** Assuming SETH, for any $\theta > 0$, there does not exist an $O(m^{2-\theta})$ time algorithm that takes any partially eliminated graph $G$ and computes the minimum fill degree in $G^+$.

**Proof.** Consider an ORTHOGONAL VECTORS instance with $n$ vectors $a(1), a(2), \ldots, a(n) \in \{0,1\}^d$. Construct a bipartite graph $G = (V_{vec}, V_{dim}, E)$ such that each vertex in $V_{vec}$ corresponds to a vector $a(i)$ and each vertex in $V_{dim}$ uniquely corresponds to a dimension $1 \leq j \leq d$. For the edges, we connect vertices $i \in V_{vec}$ with $j \in V_{dim}$ if and only if $a(i)_j = 1$.

Consider the graph state with all of $V_{dim}$ eliminated and all of $V_{vec}$ remaining. We claim that there exists a pair of orthogonal vectors among $a(1), a(2), \ldots, a(n)$ if and only if there exists a remaining vertex $v \in V(G^+)$ with $\deg(v) < n - 1$. Let $u, v \in V_{vec}$ be any two different vertices, and let $a(u)$ and $a(v)$ be their corresponding vectors. The vertices $u$ and $v$ are adjacent in $G^+$ if and only if there exists a dimension $1 \leq j \leq d$ such that $a(u)_j = a(v)_j = 1$.

Suppose there exists an $O(m^{2-\theta})$ time algorithm for finding the minimum degree in a partially eliminated graph for some $\theta > 0$. Then for $d = \Theta(\log^2 n)$, we can use this algorithm to compute the vertex with minimum fill degree in the graph described above in time

$$O \left( m^{2-\theta} \right) = O \left( (n \log^2 n)^{2-\theta} \right) = O \left( n^{2-\theta/2} \right),$$

which contradicts SETH by Theorem 3.2.

In Section 4, we extend this observation to show that an $O(m^{4/3-\theta})$ algorithm for computing the min-degree elimination ordering does not exist, assuming SETH. This is based on constructing a graph where the bipartite graph in the proof of Lemma 3.3 appears in an intermediate step. The main overhead is adding more vertices and edges to force the vertices in $V_{dim}$ to be eliminated first. To do this, we first split such vertices into $\Theta(n)$ stars of degree $O(\sqrt{n})$. Then we fully connect $V_{vec}$ to an additional clique of size $\Theta(\sqrt{n})$ to ensure that the (split) vertices in $V_{dim}$ are the first to be pivoted. There are $O(n^{3/2}d)$ edges in this construction, which leads to the $m^{4/3-\theta}$-hardness. However, we believe this is suboptimal and that $m^{2-\theta}$-hardness is more likely.
### 3.3 Δ-capped and Approximately Marginal Min-Degree Ordering

This lower bound assuming SETH suggests that it is unlikely to obtain a nearly-linear, or even sub-quadratic, time algorithms for the min-degree ordering of a graph. As a result, we turn our attention towards approximations and output-sensitive algorithms.

Our first observation is that the size of $N_{\text{reachable}}(u)$ can be bounded by $\Delta$, so $O(\Delta \log n)$ copies of the sketches as discussed in Section 3.1 suffice for “coupon collecting” all $\Delta$ distinct values instead of $O(n \log n)$ copies. This leads to bounds that depend on the maximum intermediate fill-degrees, which on large sparse graphs are often significantly less than $\Theta(n)$. We also show how to maintain $O(\log n)$ copies of the data structure and use the $(1/e)$-th order statistic to approximate the number of entries in the set. This leads to procedures that maintain approximate minimum degree vertices for fixed sequences of updates. This type of estimation is the same as using $\ell_0$-estimators to approximate the size of reachable sets [Coh97].

This procedure of repeatedly pivoting out the approximate minimum degree vertices given by sketching yields a nearly-linear time algorithm for producing an $\epsilon$-approximate greedy min-degree ordering. Initially, however, we were unable to analyze it because the input sequence is not oblivious to the randomness of the data structure. In particular, the choice of pivots is dependent on the randomness of the sketches. Compared to the other recent works that analyze sequential randomness in graph sparsification [KS16, KPPS17], our accumulation of dependencies differs in that it affects the order in which vertices are removed, instead of just the approximations in matrices.

### 3.4 Correlation Under Non-Oblivious Adversaries

The general issue of correlations (or dependencies) between the randomness of a data structure and access patterns to it can be remarkably problematic. We consider a simple example where deciding future updates based on the output of previous queries results in a continual amplification of errors. This can be understood as adversarially correlating the update sequence with results of the randomness. Consider the data structure in Figure 1 for maintaining a sequence of sets $S_1, S_2, \ldots, S_m \subseteq \{1, 2, \ldots, n\}$ under insertion/deletions and returns the one with minimum size up to an additive error of $\epsilon n$.

#### Global Variables:

1. $K$, a subset of $O(\log n \epsilon^{-2})$ elements picked uniformly at random upon initialization.
2. $K_1, K_2, \ldots, K_m$, which are $S_i \cap K$ respectively.
3. A global priority queue that tracks the cardinalities of the sets $K_i$.

#### INSERT/DELETE $(i, x)$

1. If $x \in K$, update $K_i$ and its size in the global heap.

Figure 1: Instance of a randomized data structure that can be adversarially correlated.
For a non-adaptive sequence fixed ahead of time and a single set $S$, Chernoff bounds give a result that is an $\epsilon n$ approximation with high probability. Therefore, we can utilize this to build a data structure that maintains a series of sets under insertion/deletion and returns a set of approximate minimum cardinality (up to an additive $\epsilon n$). Furthermore, to remove ambiguity, we assume this data structure breaks ties lexicographically when the intersection of two sets with $K$ have equal cardinality. With a similar invocation of Chernoff bounds, we can show that this augmented data structure is correct under the oblivious adversary model. As we maintain $k = O(\log n \epsilon^{-2})$ elements from each set $K_i$, the total space usage of this data structure is $O(m \log n \epsilon^{-2})$.

On the other hand, an adaptive adversary can use the results of previous queries to infer the set of secret keys $K$ in $O(n)$ queries. Consider the following sequence of updates:

1. Start with two sets, $S_1$ and $S_2$, both initially equal to $\{1, 2, \ldots, n\}$.
2. For $x = 1, 2, \ldots, n$:
   (a) Delete $x$ from $S_2$.
   (b) If $S_2$ is the set of approximate minimum size (the one with the smallest cardinality $|K_i|$), insert $x$ back into $S_2$.

At the end of this sequence of updates, the only elements in $S_2$ are those in $K$, which is a substantially worse result than what we can guarantee under the oblivious adversary model.

Our use of sketching to find a minimum degree vertex clearly does not perform updates that are this adversarial, but it does act on the minimum value generated by the randomized routine so the final result can be reasonably inaccurate. Moreover, any accounting of correlation (in the standard sense) allows for the worst-case type of adaptive behavior described above. In the next subsection, we describe an algorithmic approach to fix this issue.

3.5 Decorrelating Sketches and Updates

Our correlation removal method is motivated by a third routine that estimates the fill-degree of a remaining vertex in time that is close to the degree of the vertex. We then define an approximate, greedy min-degree sequence using this routine. At each step we choose the pivot vertex to be the minimizer of

$$\left(1 - \frac{\epsilon \text{Exp} (1)}{\mathcal{O}(\log n)}\right) \cdot \text{ESTIMATEDEGREE} \left( u, \frac{\epsilon}{\mathcal{O}(\log n)} \right),$$

which is the $\epsilon$-decayed minimum over all the estimates returned by the degree estimation routine.

We then utilize an $\ell_0$-estimation structure to maintain approximate degrees throughout this update procedure. By doing this, the randomness in the $\ell_0$-estimation data structure is no longer correlated with the updates. This sequence is defined with the randomness that is independent of the $\ell_0$-estimators, and (after removing the probability of incorrectness) may as well be considered deterministic. On the other hand, evaluating such a sequence using only calls to ESTIMATEDEGREE is expensive: it requires one call per vertex, leading to a total of at least $\Omega(n^2)$. Here we reincorporate the $\ell_0$-estimation data structure via the following observations about the initial perturbation term involving the random variable $\text{Exp}(1)$.

1. For a set of vertices whose degrees are within $1 \pm \epsilon / \mathcal{O}(\log n)$ of each other, it suffices to randomly select and consider $\mathcal{O}(1)$ of them (by generating the highest order statistics for exponential random variables in decreasing order).
2. By the memoryless property of the exponential distribution, if we call \text{EstimateDegree}, with constant probability it will be for the pivoted vertex. Therefore, we can “charge” the cost of these evaluations to the overall edge count and retain the nearly-linear time bounds.

At a high level, we improve a data structure that only works under the oblivious adversary model by providing it with a fixed input using a second, more local, size-estimation routine. Our generation of this “fixed” update sequence can still benefit from the approximate bucketing created in the data structure. The key idea is that any dependencies on the \ell_0-sketch structure stop after these candidates are generated—their answers only depend on the randomness of the separate size-estimation procedures.

This approach has close connections to pseudo-deterministic algorithms [GG11, Gol12, GGR13], which formalize randomized algorithms whose output sequences are fixed. Such pseudo-deterministic update sequences seem particularly useful for expanding the settings in which data structures designed for the oblivious adversary model can be used. We hope to formalize such connections in the near future. However, the lack of a counterexample for directly using \ell_0-sketching structures, or a proof of its correctness, suggests that some ideas are still missing for the min-degree problem.

4 SETH-Hardness of Computing Min-Degree Orderings

We showed in Section 3.2 that computing the minimum fill degree of a partially eliminated graph cannot be done in \(O(m^{2-\theta})\) time, for any \(\theta > 0\), assuming the strong exponential time hypothesis (SETH). In this section, we augment this result to show that an exact linear-time algorithm for computing min-degree elimination orderings is unlikely. In particular, our main hardness result is:

**Theorem 4.1.** Assuming SETH, for any \(\theta > 0\), there does not exist an \(O(m^{4/3-\theta})\) time algorithm for producing a min-degree elimination ordering.

The main idea of our construction is to modify the bipartite graph in Subsection 3.2 so that a minimum degree ordering has the effect of necessarily eliminating the \(d\) vertices in \(V_{dim}\) before any vector vertex in \(V_{vec}\). This allows us to use a minimum degree ordering on the graph to efficiently solve an \text{OrthogonalVectors} instance. The main bottleneck in our initial approach is that vertices in \(V_{dim}\) can have degree as large as \(n\), so requiring that they are removed first is difficult. We address this by breaking these vertices apart into \(\Theta(n)\) vertices, each with degree \(O(\sqrt{n})\), using the following construction which we call a covering set system.

**Lemma 4.2.** Given any positive integer \(n\), we can construct in \(O(n^{3/2})\) time a covering set system of the integers \([n] = \{1, 2, \ldots, n\}\). This system is collection of subsets \(I_1, I_2, \ldots, I_k \subseteq [n]\) such that:

1. The number of subsets \(k = O(n)\).
2. The cardinality \(|I_j| \leq 10\sqrt{n}\), for all \(1 \leq j \leq k\).
3. For each \((i_1, i_2) \in [n]^2\) there exists a subset \(I_j\) such that \(i_1, i_2 \in I_j\).

Next we pad each of the vertices in \(G_{vec}\) with \(\Omega(\sqrt{n})\) edges to ensure that they are eliminated after the vertices introduced by the covering set systems. We outline this construction in Figure 2.
1. Create one vertex per input vector \( \mathbf{a}(1), \mathbf{a}(2), \ldots, \mathbf{a}(n) \), and let these vertices be \( \mathbf{V}_{\text{vec}} \).

2. For each dimension \( 1 \leq j \leq d \):
   (a) Construct a covering set system for \( [n] \).
   (b) Create a vertex in \( \mathbf{V}_{\text{dim}} \) for each subset in this covering set system.
   (c) For each vector \( \mathbf{a}(i) \) such that \( \mathbf{a}(i)_j = 1 \), add an edge between its vertex in \( \mathbf{V}_{\text{vec}} \) and every vertex corresponding to a subset in this covering system that contains \( i \).

3. Introduce \( 20\sqrt{n} \) extra vertices called \( \mathbf{V}_{\text{pad}} \):
   (a) Connect all pairs of vertices in \( \mathbf{V}_{\text{pad}} \).
   (b) Connect every vertex in \( \mathbf{V}_{\text{pad}} \) with every vertex in \( \mathbf{V}_{\text{vec}} \).

Figure 2: Construction for reducing OrthogonalVectors to MinDegreeOrdering.

Lemma 4.3. Let \( G \) be the graph produced by the construction in Figure 2 for an instance of OrthogonalVectors with \( n \) vectors of dimension \( d \). We have \( |V| = O(nd) \) and \( |E| = O(n^{3/2}d) \).

Proof. The number of vertices in \( G \) is
\[
|V| = 20\sqrt{n} + n + d \cdot O(n) = O(nd).
\]

Similarly, an upper bound on the number of edges in \( G \) is
\[
|E| = \binom{20\sqrt{n}}{2} + 20\sqrt{n} \cdot n + d \cdot 10\sqrt{n} \cdot O(n) = O\left(n^{3/2}d\right),
\]
where the terms on the left-hand side of the final equality correspond to edges contained in \( \mathbf{V}_{\text{pad}} \), the edges between \( \mathbf{V}_{\text{pad}} \) and \( \mathbf{V}_{\text{vec}} \), and edges between \( \mathbf{V}_{\text{vec}} \) and \( \mathbf{V}_{\text{dim}} \), respectively. \( \square \)

Lemma 4.4. Consider a graph \( G \) constructed from an OrthogonalVectors instance as described in Figure 2. For any min-degree ordering of \( G \), the first vertices to be eliminated are those in \( \mathbf{V}_{\text{dim}} \). The fill degree of the next eliminated vertex is \( \min_{v \in \mathbf{V}_{\text{vec}}} \deg(v) \).

Proof. Let the graph be \( G = (V, E) \), where \( V \) is partitioned into
\[
\mathbf{V}_{\text{vec}} \cup \mathbf{V}_{\text{dim}} \cup \mathbf{V}_{\text{pad}}
\]
as described in Figure 2. Initially, for every vertex \( v_{\text{pad}} \in \mathbf{V}_{\text{pad}} \) we have
\[
\deg(v_{\text{pad}}) = \left(20\sqrt{n} - 1\right) + n.
\]
For every vertex \( v_{\text{vec}} \in \mathbf{V}_{\text{vec}} \) we have
\[
\deg(v_{\text{vec}}) = 20\sqrt{n} + |E(v_{\text{vec}}, \mathbf{V}_{\text{dim}})| \geq 20\sqrt{n},
\]

and for every vertex \( v_{\text{dim}} \in V_{\text{dim}} \) we have
\[
\deg(v_{\text{dim}}) \leq 10\sqrt{n}.
\]

Pivoting out a vertex in \( V_{\text{dim}} \) does not increase the degree of any other vertex in \( V_{\text{dim}} \), because no two vertices in \( V_{\text{dim}} \) are adjacent. As these vertices are pivoted, we still maintain
\[
\deg(v) \geq 20\sqrt{n},
\]
for all \( v \in V_{\text{vec}} \). Therefore, the first vertices to be pivoted must be all \( v \in V_{\text{dim}} \). After all vertices in \( V_{\text{dim}} \) have been pivoted, the next vertex must have fill degree \( \min_{v \in V_{\text{vec}}} \deg(v) \), because either a vertex in \( V_{\text{vec}} \) will be eliminated or all remaining vertices have fill degree \( 20\sqrt{n} + n - 1 \).

Proof of Theorem 4.1. Suppose for some \( \theta > 0 \) there exists an \( O(m^{4/3-\theta}) \) time algorithm for \textsc{Min-DegreeOrdering}. Construct the graph \( G = (V, E) \) with covering sets as described in Figure 2. For \( d = \Theta(\log^2 n) \), it follows from Lemma 4.3 that \( |V| = O(n \log^2 n) \) and \( |E| = O(n^{3/2} \log^2 n) \). Therefore by the assumption, we can obtain a min-degree ordering of \( G \) in time
\[
O \left( m^{4/3-\theta} \right) = O \left( \left( n^{3/2} \log^2 n \right)^{4/3-\theta} \right) = O \left( n^{2-\theta} \right).
\]

By Lemma 4.4, the state of the elimination steps after the first \( |V_{\text{dim}}| \) vertices have been pivoted is essentially identical to the partially eliminated state from Lemma 3.3. Then by Lemma 2.1, we can compute the degree of the next vertex to be eliminated in \( O(m) = O(n^{2-\delta}) \) time. Checking whether the degree of that vertex is \( 20\sqrt{n} + n - 1 \) allows us to solve \textsc{OrthogonalVectors} in time \( O(n^{2-\theta}) \), which contradicts SETH.

It remains to efficiently construct the covering set systems as defined in Lemma 4.2, which we can interpret as a strategy for covering all the edges of \( K_n \) with \( O(n) K_{10\sqrt{n}} \) subgraphs. We also note that our construction of covering set systems is related to existence results for the covering problem with fixed-size subgraphs [CCLW13, CY98].

Proof of Lemma 4.2. Let \( p = \text{NextPrime}(\sqrt{n}) \). Bertrand’s postulate asserts that \( p < 4\sqrt{n} \), so we can compute \( p \) in \( O(n) \) time. Clearly we have \( [n] \subseteq [p^2] \), so it suffices to find a covering for \( [p^2] \). Map the elements of \( [p^2] \) to the coordinates of a \( p \times p \) array in the canonical way so that
\[
1 \mapsto (0,0) \\
2 \mapsto (0,1) \\
\vdots \\
p^2 \mapsto (p - 1, p - 1).
\]

For all \( (a, b) \in \{0, 1, \ldots, p - 1\}^2 \), define
\[
D(a, b) \overset{\text{def}}{=} \left\{ (x, y) \in \{0, 1, \ldots, p - 1\}^2 : y \equiv ax + b \pmod{p} \right\}
\]
to be the diagonal subsets of the array, and define
\[
R(a) \overset{\text{def}}{=} \left\{ (x, y) \in \{0, 1, \ldots, p - 1\}^2 : x \equiv a \pmod{p} \right\}
\]
to be the row subsets of the array. Let the collection of these subsets be
\[ S = \{ D(\alpha, \beta) : \alpha, \beta \in \{0, 1, \ldots, p - 1\}\} \cup \{ R(\alpha) : \alpha \in \{0, 1, \ldots, p - 1\}\}. \]

The construction clearly satisfies the first two conditions. Consider any \((\alpha, \beta) \in [p^2]^2\) and their coordinates in the array \((x_1, y_1)\) and \((x_2, y_2)\). If \(x_1 = x_2\), then \((x_1, y_1), (x_2, y_2) \in R(x_1)\). Otherwise, it follows that \((x_1, y_1)\) and \((x_2, y_2)\) are solutions to the line
\[ y \equiv \frac{y_1 - y_2}{x_1 - x_2} \cdot (x - x_1) + y_1 \pmod{p}, \]
so the third condition is satisfied.

\[\]

5 Sketching Based Algorithms for Computing Degrees

Let us recall a few relevant definitions from Section 2 for convenience. For a given vertex elimination sequence
\[ u^{(1)}, u^{(2)}, \ldots, u^{(n)}, \]
let \(G^{+}(t)\) denote the fill graph obtained by pivoting vertices \(u^{(1)}, u^{(2)}, \ldots, u^{(t)}\). Let \(\delta(t)\) denote the minimum degree of a vertex in \(G^{+}(t)\). An \(\ell_0\)-sketch data structure consists of the following:

- Each remaining vertex \(u\) generates a random number \(x_u\).
- Each remaining vertex \(u\) computes the vertex with the minimum \(x_v\) value among its neighbors in \(G^{+}(t)\) and itself (which we call the minimizer of \(u\)).

In this section we show that if an \(\ell_0\)-sketch data structure can be maintained efficiently for a dynamic graph, then we can use a set of copies of this data structure to find the vertex with minimum fill degree at each step and pivot out this vertex. Combining this with data structures for efficiently propagating sketch values from Section 8 gives a faster algorithm for computing minimum degree orderings on graphs. We use this technique in three different cases.

First, we consider the case where the minimum degree at each step is bounded. In this case, we choose a fixed number of copies of the \(\ell_0\)-sketch data structure and look at the minimizers over all the copies.

**Theorem 5.1.** There is an algorithm \(\text{DeltaCappedMinDegree}\) that, when given a graph with a lexicographically-first min-degree ordering such that the minimum degree is always bounded by \(\Delta\), outputs the ordering with high probability in expected time \(O(m\Delta \log^3 n)\) and uses space \(O(m\Delta \log n)\).

Next, we eliminate the condition on the minimum degrees and allow the time and space bounds of the algorithm to be output sensitive. In this case, we adaptively increase the number of copies of the \(\ell_0\)-data structure.

**Theorem 5.2.** There is an algorithm \(\text{OutputSensitiveMinDegree}\) that, when given a graph with a lexicographically-first min-degree ordering \(\delta^{(1)}, \delta^{(2)}, \ldots, \delta^{(n)}\), outputs this ordering with high probability in expected time \(O(m \cdot \max_{1 \leq t \leq n} \delta^{(t)} \cdot \log^3 n)\) and uses space \(O(m \cdot \max_{1 \leq t \leq n} \delta^{(t)} \cdot \log n)\).

Lastly, we modify the algorithm to compute the approximate minimum degree at each step. In this case, we use \(O(\log ne^{-2})\) copies of the data structure and use the reciprocal of the \((1 - 1/e)\)-th percentile among the \(x_v\) values of its minimizers as an effective approximate of the vertex degree.
**Theorem 5.3.** There is a data structure $\text{ApproxDegreeDS}$ that supports the following two operations:

1. $\text{ApproxDegreeDS} \_\text{Pivot}(u)$, which pivots a remaining vertex $u$.
2. $\text{ApproxDegreeDS} \_\text{Report}()$, which provides balanced binary search tree (BST) containers $V_1, V_2, \ldots, V_B$

such that all vertices in the bucket $V_i$ have degrees in the range $\left[(1 + \epsilon)^i - 2, (1 + \epsilon)^i + 2\right]$.

The memory usage of this data structure is $O(m \log n \epsilon^{-2})$. Moreover, if the pivots are picked independently from the randomness used in this data structure (i.e., we work under the oblivious adversary model) then:

1. The total cost of all the calls to $\text{ApproxDegreeDS} \_\text{Pivot}$ is bounded by $O(m \log^3 n \epsilon^{-2})$.
2. The cost of each call to $\text{ApproxDegreeDS} \_\text{Report}$ is bounded by $O(\log^2 n \epsilon^{-1})$.

### 5.1 Computing Exact Min-Degree

We consider the case where the minimum degree in each of the fill graphs $G^+(t)$ is at most $\Delta$. In this case, we maintain $k = O(\Delta \log n)$ copies of the $\ell_0$-sketch data structure. By the coupon-collector argument, any vertex with degree at most $\Delta$ has a list of all its distinct neighbors with high probability. This implies that for each $1 \leq t \leq n$, we can obtain the exact min-degree in $G^+(t)$ with high probability. Figure 3 gives a brief description of the data structures we will maintain for this version of the algorithm.

---

**Global Variables:**
- Graph $G$ that undergoes pivots, degree cap $\Delta$.
- $k$, the number of copies set to $O(\Delta \log n)$.
- $k$ copies of the $\ell_0$-sketch data structure $\text{DynamicL0Sketch}^{(1)}, \text{DynamicL0Sketch}^{(2)}, \ldots, \text{DynamicL0Sketch}^{(k)}$.
- For each vertex $u$, a balanced binary search tree $\text{minimizers}(u)$ that stores the minimizers of $u$ across all $k$ copies of the data structure.
- A balanced binary tree $\text{bst\_size\_of\_minimizers}$ on all vertices $u$ with the key of $u$ set to the number of different elements in $\text{minimizers}(u)$.

---

Figure 3: Global variables for the $\Delta$-capped min-degree algorithm $\text{DeltaCappedMinDegree}$.  

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Note that if we can efficiently maintain the data structures in Figure 3, simply finding the minimum element in $bst_{size\_of\_minimizers}$ gives us the vertex with minimum degree. Theorem 5.4 shows that this data structure can indeed be maintained efficiently.

**Theorem 5.4.** Given i.i.d. random variables $x_v$ associated with each vertex $v \in V(G^+(t))$, there is a data structure DynamicL0Sketch that, for each vertex $u$, maintains the vertex with minimum $x_v$ among itself and its neighbors in $G^+(t)$. This data structure supports the following methods:

- **QUERYMIN**$(u)$, which returns $x_{\min}(N_{fill}^G(t)(u))$ for a remaining vertex $u$ in $O(1)$ time.
- **PivotVertex**$(u)$, which pivots a remaining vertex $u$ and returns the list of all remaining vertices $v$ whose values of $x_{\min}(N_{fill}^G(t)(v))$ have changed just after this pivot.

The memory usage of this data structure is $O(m)$. Moreover, for any choice of $x$ values for vertices:

1. The total cost of all the pivots is $O(m \log^2 n)$.
2. For $1 \leq t \leq n$, the total size of all lists returned by **PivotVertex**$(u(t))$ is $O(m \log n)$.

This theorem relies on data structures described in Section 8, so we defer the proof to the end of that section.

Now consider a vertex $w$ with fill degree $d \leq \Delta$. By symmetry of the $x_u$ values, each vertex in $|N_{fill}(w)|$ is the minimizer of $w$ with probability $1/d$. As a result, maintaining $O(\Delta \log n)$ copies of the $\ell_0$-sketch data structure would ensure that we have an accurate estimation of the minimum fill degree. The pseudocode for this routine is given in Figure 4. The probability guarantees are formalized in Lemma 5.5, which is essentially a restatement of [Coh97, Theorem 2.1].

**Lemma 5.5.** For a remaining vertex $w$ with fill degree $d \leq \Delta$, with high probability we have

$$bst_{size\_of\_minimizers}[w] = d.$$ 

**Proof.** The only case where $bst_{size\_of\_minimizers}[w] \neq d$ is when at least one neighbor of $w$ is not chosen in $minimizers(w)$. Let $w'$ be an arbitrary neighbor of $w$ in the fill graph $G^+$. The probability of $w'$ not being chosen in any of the $k$ copies is

$$(1 - \frac{1}{d})^k.$$ 

Now, using the assumption that $d \leq \Delta$ and $k = O(\Delta \log n)$, we have

$$\Pr_{x_1, x_2, \ldots, x_n \sim [0, 1)} [w' \text{ not selected in any copy}] \leq \left(1 - \frac{1}{\Delta}\right)^{O(\Delta \log n)} \leq e^{-O(\log n)} \leq n^{-O(1)}.$$ 

Using a union bound over all neighbors, we can upper bound the probability that at least one of them is left out by

$$|N_{fill}(w)| \cdot n^{-O(1)} \leq n^{-O(1)},$$

which completes the proof. 

**DeltaCappedMinDegree**\((G, \Delta)\)

**Input:** graph \(G = (V, E)\), threshold \(\Delta\).

**Output:** exact lexicographically-first min-degree ordering \(u^{(1)}, u^{(2)}, \ldots, u^{(n)}\).

1. For \(t = 1\) to \(|V|\):
   (a) Set \(u^{(t)} \leftarrow \min(bst\_size\_of\_minimizers)\).
   (b) *DeltaCappedMinDegree_Pivot*(\(u^{(t)}\)).

**DeltaCappedMinDegree_Pivot**\((u)\)

**Input:** vertex to be pivoted \(u\).

**Output:** updated global state.

1. For copies \(1 \leq i \leq k\):
   (a) \((v_1, v_2, \ldots, v_l) \leftarrow \text{Dynamic}\ell_0\text{Sketch}^{(i)}\).\text{PivotVertex}(u)\), the set of vertices in copy \(i\) whose minimizers changed after we pivot out \(u\).
   (b) For each \(1 \leq j \leq l\):
      i. Update the corresponding values to copy \(i\) in \(minimizers(v_j)\).
      ii. Update the entry corresponding to \(v_j\) in \(bst\_size\_of\_minimizers\) with the new size of \(minimizers(v_j)\).

Figure 4: Pseudocode for \(\Delta\)-capped exact min-degree algorithm, which utilizes the global data structures for \(\text{DeltaCappedMinDegree}\) defined in Figure 3.

**Proof of Theorem 5.1** We prove the space bound first. By Theorem 5.4, each of the \(k\) copies of the data structure use \(O(m)\) memory. Each copy of \(minimizers\) can take space up to \(O(k \log k)\), and \(bst\_size\_of\_minimizers\) can use up to \(O(n \log n)\) space. Therefore, total space used is
\[
O(mk + nk \log k + n \log n) = O(mk) = O(m \Delta \log n).
\]

We now analyze the running time. Theorem 5.4 gives a direct cost of \(O(m \log^2 n)\) across all pivots, and in turn a total cost of \(O(m \Delta \log^3 n)\) across all copies. Furthermore, this implies that the sum of \(l\) (the length of the update lists in \(v\)) across all steps is at most \(O(m \log n)\). Each of these updates may lead to one BST update, so the total overhead is \(O(m \log^2 n)\), which is a lower order term.

**5.2 Output-Sensitive Running Time**

If we do away with the condition that minimum fill degrees are bounded above by \(\Delta\), the number of copies of the \(\ell_0\)-sketch data structure needed depends on the actual values of the minimum fill degree at each step. Therefore, to be more efficient, we can adaptively maintain the required number of copies of the \(\ell_0\)-sketch data structure.
For the graph $G^{+}(t)$, we need to have at least $\Omega(\delta(t) \log n)$ copies of the $\ell_0$-sketch data structure. However, we do not know the values of $\delta(t)$ a priori. Therefore, consider the following scheme that adaptively keeps a sufficient number of copies of the sketch structures:

1. Let $C = \delta(t-1)$. We will ensure that we have $O(C \log n)$ copies at all times. (Note that this is initially true.)

2. Let $\delta_C$ be the “computed” minimum degree in $G^{+}(t)$ using $O(C \log n)$ copies of the data structure.

3. If $\delta_C > C/2$, set $C \leftarrow 2C$ and repeat.

The core idea of the above routine is that if the “computed” min-degree is at most $C/2$, then with high probability the actual min-degree is at most $C$. Then, because we have $O(C \log n)$ copies of the data structure, the correctness of the algorithm follows.

Proof of Theorem 5.2. The proof follows analogously to that of Theorem 5.1, except that our upper bound for the minimum degrees can be simply given by $\Delta = 2 \cdot \max_{1 \leq t \leq n} \delta(t)$. With this, the claimed space and time bounds follow.

5.3 Computing Approximate Min-Degree

To avoid artificial conditions such as bounds on minimum fill degree and to make running times independent of the output, we modify the algorithm to obtain an approximate min-degree vertex at each step. To do this, we reduce the number of copies of DynamicL0Sketch and use the reciprocal of the $(1 - 1/e)$-th percentile of a set to approximate its size.\(^3\) However, there is a subtle issue with the randomness that this algorithm uses. A necessary condition for the algorithm to succeed as intended is that each step must be independent of its past decisions. Therefore, we must remove any dependencies between previous and current queries. Section 3.4 gives an example of such a correlation between steps of an algorithm. To circumvent this problem, we need to decorrelate the sketches we construct and the updates to the data structure from pivoting vertices. Section 6 tackles this issue. Rather than simply selecting a vertex with approximate min-degree, this algorithm requires access to all vertices whose estimated degree is within a certain range of values. It follows that this version of the algorithm utilizes such a data structure, as opposed to the previous two versions which just output the vertex to be pivoted.

Figure 5 gives a description of the data structures for this version of the algorithm.

\(^3\)Note that we use $e$ to refer to the base of the natural logarithm; it should not be confused with edges in a graph.
Global Variables: graph $G$, error tolerance $\epsilon > 0$.

1. Number of copies, $k = \mathcal{O}(\log n\epsilon^{-2})$.
2. $k$ copies of the $\ell_0$-sketch data structure $\text{DynamicL0Sketch}^{(1)}, \text{DynamicL0Sketch}^{(2)}, \ldots, \text{DynamicL0Sketch}^{(k)}$.
3. For each vertex $u$, a balanced binary search tree $\text{minimizers}(u)$ that stores the minimizers of the $\ell_0$-sketch at $u$ across all $k$ copies, and maintains the element in $\text{minimizers}(u)$ with rank $k \left(1 - \frac{1}{e}\right)$.
4. A balanced binary tree $\text{bst\_quantile}$ over all vertices $u$ whose key is the $\lfloor k (1 - 1/e) \rfloor$-ranked element in $\text{minimizers}(u)$.

Figure 5: Global variables and data structures for APPROXDEGREE_DS, which returns (implicit) partitions of vertices into buckets with $\epsilon$-approximate degrees.

To achieve our goal of using fewer copies of the data structure, we use a sampling-based algorithm. In particular, we make use of the following lemma.

**Lemma 5.6.** Suppose that we have $k$ copies of the $\ell_0$-sketch data structure for some $k \geq \mathcal{O}(\log n\epsilon^{-2})$. Let $w$ be a vertex with degree $d > k$, and let $q(w)$ denote the $\lfloor k (1 - 1/e) \rfloor$-ranked element in $\text{minimizers}(w)$. Then, with high probability, we have

$$\frac{1 - \epsilon}{d} \leq q(w) \leq \frac{1 + \epsilon}{d}.$$ 

Lemma 5.6 is simply a restatement of [Coh97, Propositions 7.1 and 7.2]. However, [Coh97] assumes that the random variables $x_u$ are drawn from the exponential distribution (and hence also their minimum), whereas we assume that $x_u$ is independently drawn from the uniform distribution. When $d$ is large though, the minimums of $d$ elements from both distributions are almost identically distributed. For the sake of completeness, we provide the proof for the $x_u$ variables being uniformly distributed in Appendix A.

This leads to the following result for providing implicit access to all vertices with approximately the same degree, which is crucial for our overall nearly-linear time algorithm in Section 6. We give its pseudocode in Figure 6.
ApproxDegreeDS_Pivot(u)

Input: vertex to be pivoted, \( u \).
Output: updated global state.

1. For each copy \( 1 \leq i \leq k \):
   (a) \( (v_1, v_2, \ldots, v_l) \leftarrow \text{DynamicL0Sketch}^{(i)} \).PivotVertex\( (u) \), the set of vertices in copy \( i \) whose minimizers changed after we pivot out \( u \).
   (b) For each \( 1 \leq j \leq l \):
      i. Update the value of corresponding to copy \( i \) in \( \text{minimizers}(v_j) \), which in turn updates its \( \lfloor k(1 - 1/e) \rfloor \)-ranked quantile.
      ii. Update the entry corresponding to \( v_j \) in \( \text{bst} \) quantile with the new value of the \( \lfloor k(1 - 1/e) \rfloor \)-ranked quantile of \( \text{minimizers}(v_j) \).

ApproxDegreeDS_Report()

Output: approximate bucketing of the vertices by their fill-degrees.

1. For each \( i \) from 0 to \( B = O(\log n \epsilon^{-1}) \):
   (a) Set \( S_i \) to be the split binary tree from \( \text{bst} \) quantile that contains all nodes with \( \lfloor k(1 - 1/e) \rfloor \)-ranked quantiles in the range

\[
(1 + \epsilon)^{-i-1}, (1 + \epsilon)^{-i} \]

2. Return the tuple \( (S_1, S_2, \ldots, S_B) \).

Figure 6: Pseudocode for data structure that returns pointers to binary trees containing partitions of remaining vertices into sets with \( \epsilon \)-approximate degrees. Its corresponding global variables are defined in Figure 5.

When interacting with ApproxDegreeDS_Report(), note that the maximum degree is \( O(n) \), so we have \( B = O(\log n \epsilon^{-1}) \). Therefore, the data structure can simply return pointers to “samplers” for the partition \( V_1, V_2, \ldots, V_B \).

Proof of Theorem 5.3. By construction, all vertices in \( V_i \) have their \( \lfloor k(1 - 1/e) \rfloor \)-ranked quantile in the range

\[
(1 + \epsilon)^{-i-1}, (1 + \epsilon)^{-i} \]

Subsequently from Lemma 5.6, the fill-degree of a vertex \( w \in V_i \) is in the range

\[
[(1 - \epsilon)(1 + \epsilon)^i, (1 + \epsilon)(1 + \epsilon)^{i+1}] ,
\]

which is within the claimed range for \( \epsilon \leq 1/2 \).
The proof of time and space bounds is again analogous to that of Theorem 5.1. Substituting in the new number of copies \( k = O(\log n \epsilon^{-2}) \) instead of \( \Delta \) proves the space complexity.

The main difference in these data structures is that we now need to store information about the \( \lfloor k(1 - 1/e) \rfloor \)-ranked quantile. These can be supported in \( O(\log n) \) time by augmenting the balanced binary search trees with information about sizes of the subtrees in standard ways (e.g. [CLRS09, Chapter 14]). A \( O(\log n) \) time splitting operation is also standard to most binary search tree data structures (e.g. treaps [SA96]).

Note that there may be some overlaps between the allowed ranges of the buckets; vertices on the boundary of the buckets may be a bit ambiguous.

An immediate corollary of Theorem 5.3 is that we can provide access to approximate minimum-degree vertices for a fixed sequence of updates by always returning some entry from the first non-empty bucket.

**Corollary 5.7.** For a fixed sequence of pivots \( u^{(1)}, u^{(2)}, \ldots, u^{(n)} \), we can find \((1 + \epsilon)\)-approx min-degree vertices in each of the intermediate states in \( O(m \log^3 n \epsilon^{-2}) \) time.

### 6 Generating Decorrelated Sequences

In this section we show our nearly-linear \((1 + \epsilon)\)-approximate min degree algorithm. The algorithm crucially uses the APPROXDEGREE DS data structure constructed in Section 5.3.

**Theorem 6.1.** There is an algorithm APPROXMINDEGREE SEQUENCE that produces an \( \epsilon \)-approximate greedy min-degree sequence in expected \( O(m \log^2 n \epsilon^{-2}) \) time with high probability.

The algorithm is based on the degree approximation routines using sketching, as described in Theorem 5.3. Theorem 5.3 provides us access to vertex buckets, where the \( i \)-th bucket contains vertices with fill degrees in the range \( [(1 + \epsilon)^i, (1 + \epsilon)^{i+2}] \). At any point, reporting any member of the first non-empty bucket gives an approximate minimum degree choice. However, such a choice must not have dependencies on the randomness used to generate this step, or more importantly, subsequent steps.

To address this issue, we use an additional layer of randomization, which decorrelates the \( \ell_0 \)-sketch data structures and the choice of vertex pivots. Figure 7 contains the pseudocode for the top-level algorithm to compute a nearly-linear \((1 + \epsilon)\)-approximate minimum degree sequence. The algorithm makes calls the following routines and data structures:

- **APPROXDEGREE DS**: Access to buckets of vertices with approximately equal degrees (Section 5.3).
- **EXPDECAYEDCANDIDATES**: Takes a set whose values are within \( 1 \pm \epsilon \) of each other, randomly perturbs its elements, and returns this \((\epsilon\text{-decayed})\) set.
- **ESTIMATEDEGREE**: Gives an \( \epsilon \)-approximation to the fill-degree of any given vertex (Section 7). The formal statement is given in Theorem 6.2.

**Theorem 6.2.** There is a data structure that maintains a component graph \( G^0 \) under (adversarial) vertex pivots in a total of \( O(m \log^2 n) \) time, and supports the operation \( \text{ESTIMATEDEGREE}(G^0, u, \epsilon) \), which given a vertex \( u \) and error threshold \( \epsilon > 0 \), returns with high probability an \( \epsilon \)-approximation to the fill-degree of \( u \) by making \( O(d_{\text{component}}^0(u) \log^2 n \epsilon^{-2}) \) oracle queries to \( G^0 \).
ApproxMinDegreeSequence$(G, \epsilon)$

**Input**: graph $G$ with $n$ vertices, error $\epsilon$.

**Output**: with high probability an $\epsilon$-min-degree sequence $u_1, u_2, \ldots, u_n$.

1. Set smaller error
   \[
   \hat{\epsilon} \leftarrow \frac{\epsilon}{O(\log n)}.
   \]

2. Initiate the approximate degree reporting data structure with $G$ and error $\hat{\epsilon}$.

3. For each $t = 1, 2, \ldots, n$:
   
   (a) Compute approximate “buckets” of degrees (implicitly),
   \[
   \left( S_1^{(t)}, S_2^{(t)}, \ldots, S_k^{(t)} \right) \leftarrow \text{APPROXDEGREEDS REPORT}() .
   \]

   (b) Let $i_{\min}$ be the index of the minimum non-empty bucket.

   (c) Initialize $\text{global candidates}^{(t)} \leftarrow \emptyset$.

   (d) For each $i = i_{\min}, i_{\min} + 1, \ldots, i_{\min} + O(\log n/\hat{\epsilon})$, generate the exponential distributions to form the candidates
   \[
   \text{global candidates}^{(t)} \leftarrow \text{global candidates}^{(t)} \cup \text{EXPDECAYEDCANDIDATES} (S_i, \hat{\epsilon}) .
   \]

   (e) Trim $\text{global candidates}^{(t)}$ to only contain entries $(\delta_v^{(t)}, v, i)$ with
   \[
   \left( 1 - \delta_v^{(t)} \right) (1 + \hat{\epsilon})^i \leq \min_{(\delta_w^{(t)}, w, j) \in \text{global candidates}^{(t)}} \left( 1 - \delta_w^{(t)} \right) (1 + \hat{\epsilon})^j .
   \]

   (f) Set $u^{(t)}$ to be the vertex that is the minimizer of
   \[
   \left( 1 - \delta_i^{(t)} \right) \text{ESTIMATEDEGREE} (v_i, \epsilon)
   \]
   over all $(\delta_i^{(t)}, v_i, i) \in \text{global candidates}^{(t)}$.

   (g) $\text{APPROXDEGREEDS PIVOT} (u^{(t)})$.

Figure 7: Pseudocode for approximate min-degree algorithm.

The most important part of the algorithm is arguably the use of exponential distributions to form candidates in a way that it is completely decorrelated with the randomness used to generate the $\ell_0$-sketch data structure and in the choice of previous vertex pivots. The following subsection summarizes some of the desirable properties of exponential random variables that we exploit in our algorithm.
6.1 Exponential Random Variables

The exponential distribution has been well-studied. In particular, we use properties of its order statistics, which arise in the study of fault tolerance and distributed graph decomposition [MPX13]. For a parameter $\lambda$, this distribution is defined by the probability density function (PDF)

$$f_{\text{Exp}}(x; \lambda) = \begin{cases} \lambda \exp(-\lambda x) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We denote this distribution by $\text{Exp}(\lambda)$, and also make use of its cumulative density function (CDF)

$$F_{\text{Exp}}(x; \lambda) = \begin{cases} 1 - \exp(-\lambda x) & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

A crucial fact about the exponential distribution is that it is memoryless. That is, if we condition on $\text{Exp}(\lambda) \geq t$, then $\text{Exp}(\lambda) - t$ follows the same distribution. A substantial part of our analysis relies on the order statistics of exponential random variables. Given $n$ random variables $X_1, X_2, \ldots, X_n$, the $i$-th order statistic is the value of the $i$-th minimum random variable. A useful property of exponential distributions is that the difference between its order statistics also follows an exponential distribution, which we exploit when sampling exponential random variables in decreasing order.

Lemma 6.3 ([Fel71]). Let $X^n_{(i)}$ denote the $i$-th order statistic of $n$ i.i.d. random variables drawn from the distribution $\text{Exp}(\lambda)$. Then, the $n$ variables $X_{(1)}, X_{(2)} - X_{(1)}, \ldots, X_{(n)} - X_{(n-1)}$ are independent, and the density of $X_{(k+1)} - X_{(k)}$ is given by the distribution $\text{Exp}((n-k)\lambda)$.

One approach to prove Lemma 6.3 uses the i.i.d. assumption to show that the CDF of $X^n_{(1)}$ is

$$F_{X^n_{(1)}}(x) = 1 - (1 - F(x))^n = 1 - \exp(-n\lambda x),$$

where $F(x) = 1 - \exp(-\lambda x)$ is the CDF of $\text{Exp}(\lambda)$. This proves that $X^n_{(1)}$ follows an exponential distribution with mean $1/(n\lambda)$. Then conditioning on $X^n_{(1)}$, we see that $X_{(2)} - X_{(1)}$ again follows an exponential distribution equal to $X^{n-1}_{(1)}$ because of the memoryless property. We can repeat this argument to get the density of $X^n_{(k+1)} - X^n_{(k)}$ for all $k$ up to $n - 1$.

The key definition in this section is a sequence defined by exponential perturbations. It is motivated by Theorem 5.3 which states that all the vertices are grouped approximately by degrees. In the following definition, $n$ is global and equal to the original number of vertices in the graph. Also, we let $c_1 > 1$ be some fixed constant.

Definition 6.4. Given a set of values $\{x_1, x_2, \ldots, x_k\}$, an $\epsilon$-decayed minimum of this set is generated by independently drawing the exponential random variables

$$\delta_i \sim \hat{\epsilon} \cdot \text{Exp}(1),$$

where $\hat{\epsilon} = \epsilon/(c_1 \cdot \log n)$ (line 1 in APPROXMINDEGREESEQUENCE), and returning

$$\min_i (1 - \delta_i) x_i.$$
Definition 6.5. Given a parameter $\epsilon$ and an $\epsilon$-degree estimation routine $\text{EstimateDegree}(G, \cdot)$, we define an $\epsilon$-decayed min-degree sequence as a sequence such that:

1. The next vertex to be pivoted, $u(t)$, is the one corresponding to the $\epsilon$-decayed minimum of the values $\text{EstimateDegree}(G(t-1), v)$, over all remaining vertices $v$ in $G(t-1)$.

2. $G(t)$ is the graph obtained after pivoting $u(t)$ from $G(t-1)$.

Importantly, note that the randomness of this degree estimator is regenerated at each step, thus removing all dependencies. Section 6.2 describes how to generate this distribution implicitly. We first show that this approximation is well-behaved.

Lemma 6.6. Let $Y$ be an $\epsilon$-decayed minimum of $\{x_1, x_2, \ldots, x_k\}$. With high probability, we have

$$Y \geq (1 - \epsilon) \min\{x_1, x_2, \ldots, x_k\}.$$ 

Proof. Let us bound the probability of the complementary event $Y < (1 - \epsilon) \min\{x_1, x_2, \ldots, x_k\}$.

Observe that we can upper bound this probability by the probability that some $x_i$ decreases to less than $1 - \epsilon$ times its original value. Recall that we set $\hat{\epsilon} = \epsilon/(c_1 \cdot \epsilon)$ for some constant $c_1 > 1$ (Line 1 in $\text{ApproxMinDegreeSequence}$). Consider $k$ i.i.d. exponential random variables $X_1, X_2, \ldots, X_k \sim \text{Exp}(1)$, and let

$$\delta_i = \hat{\epsilon} \cdot X_i,$$

as in the definition of an $\epsilon$-decayed minimum (Definition 6.4). By the CDF of the exponential function, for each $1 \leq i \leq k$, we have

$$\Pr_{\delta_i} [\delta_i > \epsilon] = \Pr_{X_i} \left[ \frac{\epsilon}{c_1 \log n} \cdot X_i > \epsilon \right] = \Pr_{X_i} [X_i > c_1 \log n] = \exp (-c_1 \log n).$$

By a union bound, it follows that

$$\Pr_{\delta_1, \ldots, \delta_k} \left[ \max_{1 \leq i \leq k} \delta_i > \epsilon \right] = \Pr_{\delta_1, \ldots, \delta_k} [\text{there exists } 1 \leq i \leq k \text{ such that } \delta_i > \epsilon] \leq \sum_{1 \leq i \leq n} \Pr_{\delta_1} [\delta_1 > \epsilon].$$

Substituting in the bound from the CDF for each $\delta_i$ gives

$$\Pr_{\delta_1, \ldots, \delta_k} \left[ \max_{1 \leq i \leq k} \delta_i > \epsilon \right] \leq n \cdot \exp (-c_1 \log n) = n^{1-c_1}.$$

Considering the complementary event gives the result for $c_1 > 1$. □
The above lemma implies that, to produce an $\epsilon$-approximate greedy minimum degree sequence (as defined in Definition 2.3), it suffices to compute an $\epsilon$-decayed minimum-degree sequence. Specifically, at each iteration, we only need to find the $\epsilon$-decayed minimum among the (approximate) degrees of the remaining vertices.

It turns out, however, that computing the approximate degrees for each remaining vertex during every iteration is rather expensive. Section 6.2 shows how we can tackle this problem by carefully choosing a candidate subset of vertices at each iteration.

### 6.2 Implicitly Generating the $\epsilon$-Decayed Minimum

We now consider the problem of finding the $\epsilon$-decayed minimum of a set of vertex degrees. Since the number of these vertices can be huge, our first step is to find a small candidate subset. This is done via the routine $\text{ExpDecayedCandidates}$, and its pseudocode is given in Figure 8. Once again, let $\hat{\epsilon}$ denote $\epsilon/(c_1 \log n)$, for some constant $c_1 > 1$.

---

**ExpDecayedCandidates** ($S = \{s_1, s_2, \ldots, s_k\}, \epsilon$)

Input: set $S$ of $k$ elements $s_1, s_2, \ldots, s_k$ whose values are within a factor of $(1 + c_2 \hat{\epsilon})$ of each other for some constant $c_2$.

Output: candidates for the $\epsilon$-decayed minimum of $S$.

1. Sample order statistics from $\text{Exp}(1)$ in decreasing order that are within an additive factor of $c_2$ from $X^{k}_{(k)}$:

   \[
   \left( X^{k}_{(k)}, X^{k}_{(k-1)}, \ldots, X^{k}_{(k-(m-1))} \right) = \text{SampleDecreasingExponentials}(k, c_2).
   \]

2. For each $i = 1, 2, \ldots, m$, let

   \[\delta_i = \hat{\epsilon} \cdot X^{k}_{(k-(i-1))}.\]

3. Assign each $\delta_i$ to a random element of $S$, $s_{\pi(i)}$, without replacement.

4. Return the set \{($\delta_i, s_{\pi(i)}$)\}.

---

Figure 8: Pseudocode for returning an $O(1)$-sized candidate subset for the $\epsilon$-decayed minimum of a given set of values that are within $(1 + c_2 \hat{\epsilon})$ of each other.

Notice that the input requires all the elements to be within a factor of $(1 + c_2 \hat{\epsilon})$ of each other. The way we achieve this is simply using the vertex buckets produced by our algorithm $\text{ApproxDegrees}$ in Section 5.3, using $\hat{\epsilon}$ as the tolerance value. The following lemma shows that approximate vertex degrees in one such bucket satisfies the required condition on the input.

**Lemma 6.7.** For an arbitrary bucket $B_i^{(t)}$, there exists a constant $c_2$ such that all its approximate degree values are within a factor of $(1 + c_2 \hat{\epsilon})$ (or alternatively, within a factor of $(1 + \hat{\epsilon})^7$) of each other.
Proof. From Theorem 5.3, the $i$-th bucket has vertices with degrees in the range

$$[(1 + \hat{\epsilon})^{i-2}, (1 + \hat{\epsilon})^{i+2}]$$.

From Theorem 6.2, we have oracle access to the graph $G^{o(t)}$ and can therefore invoke EstimateDegree on it. Instead of treating each call to EstimateDegree and the values of $\delta_u$ used to generate the $\epsilon$-decayed minimum as random variables, we consider them as fixed values after removing the bad cases using w.h.p. That is, we define

$$\tilde{d}^{(t)}_{fill}(u) \overset{\text{def}}{=} \text{EstimateDegree}\left(G^{o(t-1)}, u, \hat{\epsilon}\right).$$

By Theorem 6.2 with high probability, every call to EstimateDegree is correct, so we have

$$(1 - \epsilon) d^{G^{o(t-1)}}_{fill}(u) \leq \tilde{d}^{(t)}_{fill}(u) \leq (1 + \epsilon) d^{G^{o(t-1)}}_{fill}(u).$$

This implies that the values are in the range

$$[(1 + \hat{\epsilon})^{i-4}, (1 + \hat{\epsilon})^{i+3}]$$.

Hence, all values in a bucket are within a factor of

$$\frac{(1 + \hat{\epsilon})^{i+3}}{(1 + \hat{\epsilon})^{i-4}} = (1 + \hat{\epsilon})^7 \leq (1 + c_2 \hat{\epsilon})$$

of each other. \qed

Recall that $X^k_{(i)}$ denotes the distribution of the $i$-th smallest value among $k$ identically sampled variables. The most important part of the above algorithm is sampling the required order statistics efficiently. Figure 9 shows how to sample the order statistics

$$X^k_{(k)}, X^k_{(k-1)}, \ldots, X^k_{(1)}$$

from $\text{Exp}(1)$ in decreasing order iteratively using Lemma 6.3.
**SampleDecreasingExponentials**($k, c_2$)

**Input:** an integer $k \geq 0$, a real-values threshold $c_2 > 0$.

**Output:** realized order statistics $X_{(k)}^k, X_{(k-1)}^k, \ldots, X_{(m)}^k$ from Exp(1) such that $X_{(m)}^k \geq X_{(k)}^k - c_2$ and $X_{(m-1)}^k < X_{(k)}^k - c_2$.

1. Sample $X_{(k)}^k$ using binary search and the CDF
   \[ \Pr \left[ X_{(k)}^k \leq x \right] = \left(1 - e^{-x}\right)^k. \]

2. For each $i = 1, 2, \ldots, k - 1$:
   - Sample the difference $Y \sim \text{Exp}(i)$ and let
     \[ X_{(k-i)}^k = X_{(k-(i-1))}^k - Y. \]
   - If $X_{(k-i)}^k < X_{(k)}^k - c_2$, exit the loop (and let this last value of $i$ be denoted by $m$).

3. Return the tuple
   \[ \left( X_{(k)}^k, X_{(k-1)}^k, \ldots, X_{(k-(m-1))}^k \right). \]

---

**Figure 9:** Pseudocode for iteratively generating order statistics of exponential random variables in decreasing order within a threshold $c_2$ of the maximum value $X_{(k)}^k$.

To show that our algorithm is correct, we must prove two parts: that the algorithm picks $O(1)$ candidates in expectation, and with high probability the actual $\varepsilon$-decayed minimum belongs to this candidate set.

**Lemma 6.8.** Suppose $x_1, x_2, \ldots, x_k$ are within a factor of $(1+c_2\varepsilon)$ of each other. Then the $\varepsilon$-decayed minimum of this set is among the candidates returned by \text{ExpDecayedCandidates}(\{x_1, x_2, \ldots, x_k\}, \varepsilon). Furthermore, the expected number of candidates returned is $O(1)$.

**Proof.** Let $X_1, X_2, \ldots, X_k \sim \text{Exp}(1)$ be i.i.d., and the order statistic $X_{(k)}^k = \max\{X_1, X_2, \ldots, X_k\}$. First let us verify the correctness of \text{SampleDecreasingExponentials}. Observe that the CDF of $X_{(k)}^k$ is

\[
F_{X_{(k)}^k}(x) = \Pr \left[ \max\{X_1, X_2, \ldots, X_n\} \leq x \right] = \prod_{i=1}^{k} \Pr [X_i \leq x] = \left(1 - e^{-x}\right)^k.
\]

Therefore, $X_{(k)}^k$ is sampled correctly in the algorithm. Using the memoryless property of the exponential distribution in Lemma 6.3 we can iteratively generate $X_{(k-1)}^k, X_{(k-2)}^k, \ldots, X_{(1)}^k$ by
sampling their differences from varying exponential distributions. Let \( i^* = \arg \min_i 1 - \delta_i \). Now, to ensure that the \( \epsilon \)-decayed minimum is not left out from our candidate set, we need to sample every \( \delta_j \) such that
\[
\delta_j \geq \delta_{i^*} - c_2 \epsilon.
\]
To verify that this suffices, suppose that the \( \epsilon \)-decayed minimum (say \( x_{\pi(\ell)} \)) is not included in our candidate set. Then,
\[
(1 - \delta_{i^*}) x_{\pi(i^*)} \leq (1 - \delta_{i^*} + c_2 \epsilon)(x_{i^*} - c_2 \epsilon) \leq (1 - \delta_{\ell}) x_{\pi(\ell)},
\]
which is a contradiction.

Lastly, to count the number of elements included in the candidate set, we count the number of such \( \delta_j \) samples. Equivalently, we bound the number of random variables \( X_j \sim \text{Exp}(1) \) such that
\[
X_j \geq X_{(k)}^k - c_2,
\]
using Definition 6.4. Let \( Z_i \) be the indicator variable which equals 1 when \( X_j \geq X_{(k)}^k - c_2 \), and let \( Z = \sum_{1 \leq i \leq k} Z_i \) indicate the size of our candidate subset. Using the memoryless property of exponentials,
\[
\mathbb{E}[Z] = \sum_{i=1}^{k} \mathbb{E}[Z_i] = \sum_{i=1}^{k} \mathbb{P}[X_{(i)}^k \geq X_{(k)}^k - c_2] = 1 + \sum_{i=1}^{k-1} \mathbb{P}[X_{(i)}^k \geq X_{(k)}^k - c_2] = 1 + \sum_{i=1}^{k-1} \mathbb{P}[X_{(i)}^k \leq c_2] = 1 + \sum_{i=1}^{k-1} (1 - e^{-c_2})^i \leq e^{c_2},
\]
where the final equality sums the geometric series. Therefore, \( O(1) \) exponential random variables are generated as we sample backwards from the maximum.

Note that we cannot simply work with the smallest bucket, because the randomness introduces a \( 1 \pm \epsilon \) perturbation. Even the bucket location of the vertex of minimum degree is dependent on the randomness of the sketches used to generate them (discussed in Theorem 5.3). So, the algorithm finds this \( O(1) \)-sized candidate set for \( O(\log n \epsilon^{-1}) \) buckets, which suffices since the \( \epsilon \)-decayed minimum cannot be in the latter buckets. However, this increases our global candidate set to size \( O(\log n \epsilon^{-1}) \). As a final step before computing degrees of vertices, we show that we can trim this set carefully, while still keeping the \( \epsilon \)-decayed minimum in it with high probability.
Lemma 6.9. Let \((\delta_v^{(j)}, v, i)\) be the entry that corresponds to the \(\epsilon\)-decayed minimum in the set \(\text{global\_candidates}^{(t)}\). Then with high probability, we have

\[
\left(1 - \delta_v^{(t)}\right) (1 + \hat{\epsilon})^4 \leq (1 + \hat{\epsilon})^7 \min_{(\delta_w^{(t)}, w, j) \in \text{global\_candidates}^{(t)}} \left(1 - \delta_w^{(t)}\right) (1 + \hat{\epsilon})^j.
\]

Proof. Let \((\delta_u^{(j)}, u, j)\) be an arbitrary entry in the set \(\text{global\_candidates}^{(t)}\). We know that

\[
\left(1 - \delta_v^{(t)}\right) d_{\text{fill}}(v) \leq \left(1 - \delta_u^{(t)}\right) d_{\text{fill}}(u).
\]

From Lemma 6.7,

\[
d_{\text{fill}}(v) \leq (1 + \hat{\epsilon})^{i+3}
\]

and

\[
d_{\text{fill}}(u) \geq (1 + \hat{\epsilon})^{j-4}.
\]

Substituting these into the previous inequality gives us the result.

Now we use all our building blocks from this section to prove the correctness of the algorithm.

Lemma 6.10. For any graph \(G\) and any error \(\epsilon\), the output of \(\text{APPROXMINDEGREESEQUENCE}(G, \epsilon)\) is with high probability an \(\epsilon\)-approximate greedy min-degree sequence.

Proof. We prove by induction that for some constant \(c\), we can show that after \(t\) steps, our sequence is an \(\epsilon\)-approximate greedy min-degree sequence with probability at least \(1 - t \cdot n^{-c}\). The base case of \(t = 0\) follows because nothing has happened so far. As the inductive hypothesis, suppose we have an \(\epsilon\)-approximate greedy min-degree sequence \(u^{(1)}, u^{(2)}, \ldots, u^{(t)}\). Then consider the graph \(G^{(t)}\) where these vertices are marked as eliminated and the rest of the vertices are marked as remaining.

From Lemma 6.7, all values in a bucket are within a factor of \(1 + c_2 \hat{\epsilon}\) of each other. Thus we can use the guarantees of Lemma 6.8 to compute the \(\epsilon\)-decayed candidates of each bucket. That is, with high probability we did indeed return the \(\epsilon\)-decayed minimum of each bucket \(S_1^{(t)}\). The \(\epsilon\)-decayed minimum of \(k\) sets is the minimum of their respective \(\epsilon\)-decayed minimums. Additionally, Lemma 6.9 shows that trimming our set does not remove the \(\epsilon\)-decayed minimum from the set. So, we have that \(u^{(t+1)}\) is the \(\epsilon\)-decayed minimum over all the values of \(d_{\text{fill}}(u)\) with high probability.

Lastly, invoking the bound on distortions incurred by \(\epsilon\)-decay from Lemma 6.6 as well as the approximation error of \(\text{ESTIMATEDEGREE}\), gives that w.h.p. the fill degree of \(u^{(t+1)}\) is within \(1 + \epsilon\) of the minimum fill degree in \(G^{(t)}\). From all the above high probability claims, we get a failure probability of at most \(n^{-c}\). So the inductive hypothesis holds for \(t + 1\) as well.

We now consider the cost of the algorithms. For this, we show that if a vertex is close to the \(\epsilon\)-decayed minimum, then there is a high chance that it is the \(\epsilon\)-decayed minimum. That is to say, if the algorithm queries the approximate degree of a vertex, there is a good chance that this vertex belongs to the \(\epsilon\)-decayed approximate degree sequence.

Lemma 6.11. For any constant \(c_3\), a choice of \(\hat{\epsilon}\) (as in line \(1\) of \(\text{APPROXMINDEGREESEQUENCE}\)), a set of values \(d_1, d_2, \ldots, d_n\), and any index \(i\), we have

\[
\Pr_{\delta_1 \ldots \delta_n \sim \hat{\epsilon}\cdot\text{Exp}(1)} \left[ i \text{ is the } \epsilon\text{-decayed minimum of } d_1, d_2, \ldots, d_n \right] \geq \exp\left(-2c_3\right) \Pr_{\delta_1 \ldots \delta_n \sim \hat{\epsilon}\cdot\text{Exp}(1)} \left[ (1 - \delta_i) d_i \leq \left(1 + \frac{\epsilon}{c_1 \log n}\right) \min_j (1 - \delta_j) d_j \right].
\]

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Proof. Consider generating $\delta_i$ last. Then consider the value

$$m_{\setminus i} \overset{\text{def}}{=} \min_{j \neq i} (1 - \delta_j) d_j.$$  

If $m_{\setminus i} \geq d_i$, then both sides are 1 and the result holds trivially. Otherwise, we condition on

$$(1 - \delta_i) d_i \leq (1 + \hat{\epsilon})^{c_3} m_{\setminus i},$$

or equivalently

$$\delta_i \geq \hat{\gamma},$$

for some $\hat{\gamma}$ such that

$$(1 - \hat{\gamma}) d_i = (1 + \hat{\epsilon})^{c_3} m_{\setminus i}.$$  

Then by the memoryless property of exponentials from Lemma 6.3 with probability at least $\exp(-2c_3)$, we have

$$\delta_i \geq \hat{\gamma} + 2c_3,$$

which when substituted back in gives

$$\begin{align*}
(1 - \delta_i) d_i &\leq (1 - \hat{\gamma} - 2c_3) d_i \\
&\leq (1 - 2c_3) (1 - \hat{\gamma}) d_i \\
&= (1 - 2c_3) (1 + \hat{\epsilon})^{c_3} m \\
&\leq m.
\end{align*}$$

So conditioned on the decayed value of $i$ being within the given threshold of the minimum, it would decay below the minimum with probability at least $\exp(-2c_3)$.

Substituting the value of $c_3 = 7$ as in algorithm APPROXMINDEGREESEQUENCE in Figure 7, we get the following corollary.

**Corollary 6.12.** If a vertex $v$ is in global candidates after line 3e of APPROXMINDEGREESEQUENCE, then with constant ($\exp(-14)$) probability, $v$ is the $\epsilon$-decayed minimum.

We can now prove our main result.

**Proof of Theorem 6.1.** The correctness follows from Lemma 6.10. Theorem 5.3 allows us to maintain access to all the buckets in a total time of

$$O(m \log^3 n \hat{\epsilon}^{-2}) = O(m \log^5 n \epsilon^{-2})$$

across the sequence of pivots.

It remains to bound the costs of the calls to ESTIMATEDEGREE. By Theorem 6.2, the total costs of maintaining the graphs under pivots is $O(m \log^2 n)$, and comes out to be a lower order term. For the cost of the calls to ESTIMATEDEGREE, we utilize Corollary 6.12 which states that if a vertex is in global candidates, then it is the one pivoted with constant probability. Specifically, we prove inductively based on the number of vertices that remain that the expected cost of calling ESTIMATEDEGREE is bounded by

$$c_4 \cdot \left( \sum_{u \in V_{\text{remain}}^{(t)}} d_{V_{\text{remain}}^{(t)}}^r(u) \right) \log^2 n \cdot \hat{\epsilon}^{-2},$$

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for some constant $c_4$.

The base case of $t = n$ follows from the lack of vertices remaining. Suppose the result is true for $t + 1$ vertices. Then the cost of the next step is bounded by

$$
\sum_u \Pr_{\delta_v : v \in V} [u \text{ is the } \epsilon\text{-decayed minimum}] \cdot c_4 \cdot \left( \sum_{w \in V_{\text{remain}}^{G(t)}} d_{\text{remain}}^{V_G(t)} (w) - d_{\text{remain}}^{V_G(t)} (u) \right) \log^2 n \cdot \hat{\epsilon}^{-2}
$$

$$
= c_4 \left( \sum_{u \in V_{\text{remain}}^{G(t)}} d_{\text{remain}}^{V_G(t)} (u) \right) - c_4 \cdot \sum_u \Pr_{\delta_v : v \in V} [u \text{ is the } \epsilon\text{-decayed minimum}] \cdot \left( d_{\text{remain}}^{V_G(t)} (u) \right) \log^2 n \cdot \hat{\epsilon}^{-2}.
$$

On the other hand, we evaluate $\text{EstimateDegree}(u, \hat{\epsilon})$ on $G^{(t)}$ if $u \in \text{global}\_\text{candidates}^{(t)}$. By Corollary 6.12, we have

$$
\Pr_{\delta_v : v \in V} \left[ u \in \text{global}\_\text{candidates}^{(t)} \right] \leq \exp(14) \cdot \Pr_{\delta_v : v \in V} [u \text{ is the } \epsilon\text{-decayed minimum}] .
$$

Therefore, the expected cost of these calls is

$$
\sum_u \Pr_{\delta_v : v \in V} [u \in \text{global}\_\text{candidates}^{(t)}] \cdot c_3 \left( d_{\text{remain}}^{V_G(t)} (u) \right) \log^2 n \cdot \hat{\epsilon}^{-2}
$$

$$
\leq c_3 \cdot \exp(14) \cdot \sum_u \Pr_{\delta_v : v \in V} [u \text{ is the } \epsilon\text{-decayed minimum}] \cdot \left( d_{\text{remain}}^{V_G(t)} (u) \right) \log^2 n \cdot \hat{\epsilon}^{-2},
$$

so the inductive hypothesis holds for $t$ as well by letting

$$
c_4 = c_3 \cdot \exp(14).
$$

As the initial total of remaining degrees is $O(m)$, the total cost of these steps is

$$
O \left( m \log^2 n \hat{\epsilon}^{-2} \right) = O \left( m \log^4 n \epsilon^{-2} \right),
$$

which completes the proof.

\section{Estimating the Fill Degree of a Single Vertex}

This section discusses routines for approximating the fill-degree of a single vertex in a partially eliminated graph. Additionally, we also need to maintain this partially eliminated graph throughout the course of the algorithm. Specifically, we prove Theorem 5.2.

Note that in this partially eliminated graph (which we call the ‘component graph’), connected components of the eliminated vertices are contracted into single vertices, which we now call ‘component’ vertices, while the rest of the vertices are termed ‘remaining’ vertices. Hence, we can think of the state of the graph as one where the component vertices form an independent set. Also, we are only trying to approximate the fill degree of a single remaining vertex $u$. The fill degree of $u$ is simply the number of remaining neighbors of $u$ in addition to the number of remaining neighbors of any component neighbor of $u$. Since, the former is easy to compute, the object in question is the
cardinality of the unions of the remaining neighbors of the neighbors of $u$. This set-of-sets structure also has a natural interpretation as a matrix.

In particular, if we write out the neighbors of $u$ as rows of some matrix $A$, and view all remaining vertices as columns of this matrix, the problem can be viewed as querying for the number of non-zero columns in a 0/1 matrix.

Given a 0/1 matrix $A$ with $r$ rows, our goal is to estimate the number of non-zero columns, or columns with at least one entry, by making the following two types of queries:

1. **RowSize**($i$): return the number of non-zero elements in $i$-th row of $A$;
2. **SampleFromRow**($i$): return a column index $j$ uniformly sampled among all non-zero entries in row $i$ of $A$.
3. **QueryValue**($i, j$): returns the value of $A(i, j)$.

The main result as a matrix sampler is:

**Lemma 7.1.** There is a routine **EstimateNonZeroColumns** that takes (implicit) access to a matrix $A$, along with access to the three operations above, **RowSize**, **SampleColumn**, **QueryValue**, along with an error threshold $\epsilon$, returns a value that’s an $1 \pm \epsilon$ approximation to the number of non-zero columns in $A$ with high probability. Furthermore, the expected total number of operations called is $O(r \log^2 n \epsilon^{-2})$ where $r$ is the number of rows of $A$.

First, we will prove a weaker version of this result in Section 7.1. This algorithm relies on a routine to estimate the mean of a distribution, which is detailed in Section 7.2. Finally, by a more careful analysis of both these algorithms, we prove the exact claim in Lemma 7.1 in Section 7.3.

But, before proving this matrix based result, we first verify that this matrix game can be ported back to the graph theoretic setting as stated in Theorem 6.2. To do so, we need the following tools for querying degrees and sampling neighbors in a component graph as it undergoes pivots.

**Lemma 7.2.** We can maintain a component graph under pivoting of vertices in a total time of $O(m \log^2 n)$ so that the operations described in Theorem 6.2 can be performed in $O(\log n)$ time each. This component graph grants oracle access that allows for:

- querying the state of a vertex,
- querying the component or remaining degree of a vertex.
- sampling a random remaining neighbor of either a component or remaining vertex.
- sampling a random component vertex.

We defer the proof of this lemma to Section 8 along with the corresponding running time guarantees. Assuming the correctness of Lemmas 7.1 and 7.2 matching the above operations with the required matrix operations described in Theorem 6.2 then gives its proof.

**Proof.** (Of Theorem 6.2)

The provided graph theoretic operations can simulate the matrix operations by.

1. Generating a list of all the component neighbors of $u$. 

2. For each component neighbor $x$, finding the remaining degree of $x$.

3. Finding a random non-zero in some row corresponding is the same as sampling a remaining neighbor of the component vertex corresponding to it.

4. To query whether some row/column pair $x$ and $u$ are connected, we search for $u$ in the list of neighbors for $x$. Maintaining all neighbors in a searchable data structure such as a binary search tree resolves this.

Substituting in the runtime bounds gives the desired result.

7.1 Column Count Approximator Using Distribution Mean Estimators

We start by defining an overall estimator which is what we eventually sample. Consider weighting each entry $(i, j)$ by

$$\frac{1}{\text{ColumnSum}_A(j)}$$

where

$$\text{ColumnSum}_A(j) \overset{\text{def}}{=} \sum_i A(i, j).$$

This can be checked to be an unbiased estimator of the number of non-zero columns.

**Lemma 7.3.** The number of non-zero columns of a 0/1-matrix $A$ equals

$$\sum_{(i, j) : A(i, j) = 1} \frac{1}{\text{ColumnSum}_A(j)}.$$ 

**Proof.**

$$\sum_{(i, j) : A(i, j) = 1} \frac{1}{\text{ColumnSum}_A(j)} = \sum_{j : \text{column } j \text{ of } A \text{ is non-zero}} \sum_{1 \leq i \leq r} \frac{A(i, j)}{\text{ColumnSum}_A(j)}.$$

By the definition of column sum,

$$\sum_{1 \leq i \leq r} A(i, j) = \text{ColumnSum}_A(j).$$

Substituting this back gives us

$$\sum_{(i, j) : A(i, j) \neq 0} \frac{1}{\text{ColumnSum}_A(j)} = |\{j : \text{column } j \text{ of } A \text{ is non-zero}\}|.$$ 

This implies that we only need to estimate column sums. The way we will actually compute this approximation is to estimate the mean of some appropriately chosen distribution. The pseudocode for such a routine (ESTIMATEMEAN) is given in Figure 10.
EstimateMean($D, \sigma$)
Input: access to a distribution $D$, cutoff point $\sigma$.
Output: estimate of mean.

1. Initialize counter ← 0 and sum ← 0;
2. While sum < $\sigma$
   (a) Generate $x \sim D$,
   (b) sum ← sum + $x$,
   (c) Increment counter, counter ← counter + 1.
3. Return $\sigma$/counter;

Figure 10: Pseudocode for Mean Estimation

The following lemma bounds the accuracy and the running time of EstimateMean. Its proof is detailed in Section 7.2.

**Lemma 7.4.** Let $D$ be any arbitrary distribution on $[0, 1]$ with (unknown) mean $\mu$, and a cutoff parameter $\sigma > 0$. Then for any $\epsilon$, running EstimateMean($D, \sigma$), with probability at least $1 - 2 \exp(-\frac{\epsilon^2 \sigma}{5})$:

1. queries $D$ at most
   \[ O\left(\frac{\sigma}{\mu}\right) \]
   times;
2. Produces an output $\tilde{\mu}$ such that for any $\epsilon$, we have
   \[ (1 - \epsilon)\mu \leq \tilde{\mu} \leq (1 + \epsilon)\mu. \]

An immediate corollary of this is a routine for estimating the sum of all elements in a column (ApproxColumnSum), where the runtime depends on the column sum itself. Its pseudocode is given in Figure 11. Lemma 7.5 gives the correctness and running time of ApproxColumnSum.
ApproxColumnSum($A, r, j, \epsilon, \delta$)

Input: matrix $A$ with $r$ rows, column id $j$,
error $\epsilon$ and failure probability $\delta$.
Implicit access to the overall number of vertices $\pi$.
Output: approximation to $\text{ColumnSum}_A(j)$

1. Let $D_{\text{Col}(j)}$ denote the random variable formed by:
   (a) picking a random row $1 \leq i \leq r$, and
   (b) returning $A(i, j)$.
2. Set $\sigma \leftarrow O\left(\epsilon^{-2} \log\left(\frac{1}{\delta}\right)\right)$.
3. Return $r \cdot \text{EstimateMean}(D_{\text{Col}(j)}, \sigma)$.

Figure 11: Pseudocode for Estimating the Column Sum of a Matrix

Lemma 7.5. For any a matrix $A$, any column ID $j$, any error $\epsilon > 0$, and any failure probability $\delta$, a call to ApproxColumnSum returns with probability at least $1 - \delta$ an $(1 + \epsilon)$ approximation to $\text{ColumnSum}_A(j)$ while making

$$O\left(\frac{r \log(1/\delta)}{\text{ColumnSum}_A(j)\epsilon^2}\right).$$

oracle accesses to the matrix $A$ in expectation.

Proof. As in the pseudocode of ApproxColumnSum in Figure 11, we define the random variable $D_{\text{Col}(j)}$ as:

1. picking a random row $1 \leq i \leq r$, and
2. returning $A(i, j)$.

This gives a Bernoulli distribution that is:

- 1 with probability $\frac{\text{ColumnSum}_A(j)}{r}$, and
- 0 otherwise.

The mean of $D_{\text{Col}(j)}$ is its probability of being 1:

$$\frac{\text{ColumnSum}_A(j)}{r}.$$

The cost of the call to MeanEstimation is then given by Lemma 7.4. It gives that the number of accesses to the matrix $A$ via QueryValue in Algorithm 11 is upper-bounded by

$$O\left(\frac{\sigma}{(\text{ColumnSum}_A(j)/r)}\right) = O\left(\frac{r \log(1/\delta)}{\text{ColumnSum}_A(j)\epsilon^2}\right).$$

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This performance means that we can just treat \( (1 / \text{ColumnSum}_A(j)) \) as a random variable, and sample enough entries so that the sum is approximately \( O(\log n \epsilon^{-2}) \). The running time as given in Lemma 7.5 adds an extra factor of \( n \) to this, giving the claimed running time in Lemma 7.1.

Pseudocode of the overall algorithm is given in Figure 12.

```
SLOWERESTIMATENONZERO_COLUMNS(A, \epsilon)
Input: oracle access to the matrix A with \( r \) rows and \( n \) columns, error threshold \( \epsilon > 0 \).
Output: Estimate for the number of non-zero columns of \( A \).

1. Compute the total number of non-zeros in \( A \), \( nnz \).
2. Let \( D_{\text{global}} \) be the random distribution that:
   (a) Picks a random non-zero entry of \( A \), \((i, j)\) (by first picking a row with probability proportional to its number of non-zeros, and then picking a random non-zero entry from that row).
   (b) Return \( \frac{1}{\text{APPROXCOLUMNSUM}(A, j, \epsilon, n^{-O(1)})} \).

   Where the value of \( \text{APPROXCOLUMNSUM}(A, j, \epsilon, n^{-O(1)}) \) is generated once per each column \( j \), and reused on subsequent repeated calls (via e.g. storage in binary search trees).
3. Return \( nnz \cdot \text{ESTIMATEMEAN}(D_{\text{global}}, \epsilon) \).
```

Figure 12: Pseudocode for Estimating the Number of Non-Zero Columns of a Matrix

We first bound the correctness of the result returned by \( \text{SLOWERESTIMATENONZERO_COLUMNS}(A, \epsilon) \), and the expected number times it samples \( D_{\text{global}}(j) \).

**Lemma 7.6.** With high probability, the estimate returned by \( \text{SLOWERESTIMATENONZERO_COLUMNS}(A, \epsilon) \) is within \( 1 \pm \epsilon \) of the number of non-zero columns.

**Proof.** To begin with, we explicitly extract out all the randomness in Algorithm 12 considering running all calls to \( \text{APPROXCOLUMNSUM}(A, r, j, \epsilon, n^{-O(1)}) \) beforehand.

By Lemma 7.5, we have that with high probability we have for each \( j \),

\[
(1 - \epsilon) \text{COLUMNSUM}_A(j) \leq \text{APPROXCOLUMNSUM}(A, r, j, \epsilon, n^{-O(1)}) \leq (1 + \epsilon) \text{COLUMNSUM}_A(j).
\]

So by Lemma 7.3, we have that the expectation of \( D_{\text{global}}; \mu(D_{\text{global}}) \), is within \( 1 \pm \epsilon \) of the true value with high probability. Formally,

\[
(1 - \epsilon) \frac{|\{c : A_{j,c} \neq 0\}|}{nnz} \leq \mu(D_{\text{global}}) \leq (1 + \epsilon) \frac{|\{c : A_{j,c} \neq 0\}|}{nnz}.
\]

Incorporating the guarantee of Lemma 7.3 part 2 then gives:

\[
(1 - 3\epsilon) \frac{|\{c : A_{j,c} \neq 0\}|}{nnz} \leq \text{ESTIMATEMEAN}(\mu(D_{\text{global}}), \epsilon) \leq (1 + 3\epsilon) \frac{|\{c : A_{j,c} \neq 0\}|}{nnz}.
\]

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The desired bound then follows from halving \( \epsilon \), and the final multiplication by \( r \) on the last line. \[\square\]

**Proof.** (Of Lemma 7.1 with a worse factor of \( O(\log^2 n \epsilon^{-4}) \))

The correctness follows from Lemma 7.6 so it suffices to bound the total number of queries made to entries of \( A \). Furthermore, Part 1 of Lemma 7.4 gives that the expected number of queries made to \( D_{\text{global}} \) is:

\[
O \left( \frac{nnz \cdot \log n}{|\{c : A_{c} \neq 0\}| \epsilon^2} \right).
\]

So it suffices to bound the expected cost of each evaluation of \( D_{\text{global}} \).

Applying Lemma 7.5 to every column \( j \) gives that w.h.p. the number of queries made by ColumnSum\(_A\)(\(j\)) is at most

\[
O \left( \frac{r \log n}{\text{ColumnSum}_A(j) \epsilon^2} \right).
\]

Summing this over all ColumnSum\(_A\)(\(j\)) entries in that column, as well as all the non-zero columns gives that the expected number of queries when querying for a single entry of \( D_{\text{global}} \) is:

\[
\frac{1}{nnz} \sum_{j: \text{ColumnSum}_A(j) \neq 0} \sum_{i:A_{i,j} \neq 0} O \left( \frac{r \log n}{\text{ColumnSum}_A(j) \epsilon^2} \right)
= \frac{1}{nnz} \sum_{j: \text{ColumnSum}_A(j) \neq 0} O \left( \frac{r \log n}{\epsilon^2} \right)
= O \left( \frac{|\{c : A_{c} \neq 0\}| \cdot r \log n}{nnz \cdot \epsilon^2} \right).
\]

Multiplying this with the expected number of queries to \( D_{\text{global}} \) then gives the overall result. \[\square\]

We remark that the runtime bound also holds with high probability instead of in expectation if we invoke Chernoff bounds. This is because the cost of each query to \( D_{\text{global}} \) is bounded by \( O(r \log n \epsilon^{-2}) \), and the total cost bound is larger by a factor of at least \( \log n \).

### 7.2 Estimating Mean of a Distribution

We now provide the details of the mean estimation algorithm, which also gives the correctness of the column sum estimation scheme.

We analyze an equivalent scheme which generates the same output:

1. Generate a stream of infinite i.i.d. samples from \( D \), denoted as \( X_1, X_2, \cdots \);

2. Let \( \text{counter} \) be \( \arg \min_{t \geq 0} \left\{ \sum_{i \leq t} X_i \geq \sigma \right\} \);

3. Output \( \sigma / \text{counter} \).

This process evaluates more samples than \text{EstimateMean} (from Figure 10). However, the extra evaluations happen after the termination of that process. So it does not affect the outcome.
We will bound the success probability by bounding the partial sum of \( \{X_i\} \) at two points. These two points are defined based on the (hidden) value of \( \mu \), the expectation of the distribution \( D \). For a distribution \( D \), and an error \( \epsilon > 0 \), we make two marks at:

\[
L_{D,\epsilon} \overset{\text{def}}{=} \frac{\sigma}{(1 + \epsilon)\mu},
\]

and

\[
R_{D,\epsilon} \overset{\text{def}}{=} \frac{\sigma}{(1 - \epsilon)\mu}.
\]

By some algebra, we can check that if we terminate with \( L_{D,\epsilon} \leq \text{counter} \leq R_{D,\epsilon} \), then the final outcome is good. So it suffices to bound the probability of \( \text{counter} < L \) and \( \text{counter} > R \) separately.

**Lemma 7.7.** For any sequence \( X_1, X_2 \ldots \) generated by taking i.i.d. copies of a random variable \( D \), and with we have

\[
\Pr_{X_1, X_2, \ldots} \left[ \sum_{1 \leq i \leq L_{D,\epsilon}} X_i \geq \sigma \right] \leq \exp \left( -\frac{\epsilon^2 \sigma^2}{4} \right).
\]

**Proof.** Linearity of expectation gives:

\[
\mathbb{E}_{X_1, X_2, \ldots} \left[ \sum_{1 \leq i \leq L_{D,\epsilon}} X_i \right] = \mu L_{D,\epsilon}.
\]

So as \( X_1 \ldots X_{L_{D,\epsilon}} \) are i.i.d., we get

\[
\Pr_{X_1, X_2, \ldots} \left[ \sum_{1 \leq i \leq L_{D,\epsilon}} X_i \geq (1 + \epsilon) \mu L_{D,\epsilon} \right] \leq \exp \left( -\frac{\epsilon^2 \mu L_{D,\epsilon}}{3} \right)
\]

which directly implies the lemma by taking \((1+\epsilon)\mu L_{D,\epsilon} = \sigma\) into the left-hand side and \(\frac{\mu L_{D,\epsilon}}{3} > \frac{\sigma}{4}\) for small enough \(\epsilon\) into the right-hand side. \(\square\)

**Lemma 7.8.** For any sequence \( X_1, X_2 \ldots \) generated by taking i.i.d. copies of a random variable \( D \), and with we have

\[
\Pr_{X_1, X_2, \ldots} \left[ \sum_{1 \leq i \leq R_{D,\epsilon}} X_i \leq \sigma \right] \leq \exp \left( -\frac{\epsilon^2 \sigma^2}{4} \right).
\]

**Proof.** Similiar to proof of Lemma 7.7, but with lower end of Chernoff bounds. \(\square\)

**Proof.** (Of Lemma 7.4 with an additional overhead of \( \epsilon^{-2} \))
Consider the routine \texttt{SlowerEstimateNonZeroColumns} whose pseudocode is in Figure 12. The running time (Part 1) is an immediate consequence of the bound on \( \text{counter} \leq R_{D, \epsilon} \) from Lemma 7.8. So it remains to bound the (Part 2). Recall that the estimator is

\[
\text{counter} \leq \frac{\sigma}{\epsilon}
\]

Apply union bound over Lemma 7.6 and Lemma 7.7, we get

\[
\Pr_{X_1, X_2, \ldots} [L_{D, \epsilon} \leq \text{counter} \leq R_{D, \epsilon}] \geq 1 - 2 \exp \left( -\frac{\epsilon^2 \sigma^4}{4} \right) \geq 1 - \exp \left( -\frac{\epsilon^2 \sigma^5}{5} \right).
\]

Putting in the definition of \( L_{D, \epsilon} \) and \( R_{\epsilon} \) gives this is the same as

\[
\frac{\sigma}{(1 + \epsilon) \mu} \leq \text{counter} \leq \frac{\sigma}{(1 - \epsilon) \mu},
\]

which is in turn equivalent to

\[
(1 - \epsilon) \mu \leq \frac{\sigma}{\text{counter}} \leq (1 + \epsilon) \mu,
\]

or the estimator \( \frac{\sigma}{\text{counter}} \) is a good approximation to \( \mu \).

\[\square\]

### 7.3 More holistic analysis with better bounds

We now give a better running time bound by combining the analyses of the two estimators in a more global, holistic analysis. Pseudocode of the final routine is in Figure 13.
EstimateNonZeroColumns($A, \epsilon$)

Input: oracle access to the matrix $A$ with $r$ rows and $n$ columns, error threshold $\epsilon > 0$.

Output: Estimate for the number of non-zeros of $A$.

1. Compute the total number of non-zeros in $A$, $nnz$.
2. Let $lim \leftarrow O(r \log n)$;
3. Define the distribution $D_{combined}$ as the followings:
   (a) Picks a random non-zero entry of $A$, $(i, j)$ (by first picking a row with probability proportional to its number of non-zeros, and then picking a random non-zero entry from that row).
   (b) Initialize $counter \leftarrow 0$
   (c) While $counter < lim$,
      i. $counter \leftarrow counter + 1$,
      ii. Sample $i'$ uniformly from 1 to $r$
      iii. If $A_{i', j} \neq 0$ break.
   (d) Return $\frac{counter}{lim}$.
4. $\sigma \leftarrow O(\epsilon^{-2} \log^2 n)$.
5. Return $\text{ESTIMATEMEAN}(D_{combined}, \sigma) \cdot \frac{nnz \cdot r}{lim}$.

Figure 13: Pseudocode for Fast Estimating the Number of Non-Zero Columns of a Matrix

At the core of this algorithm is the following simpler, combined distribution.

Definition 7.9. We define the combined distribution, $D_{combined}$ as the distribution given by:

1. Sampling a non-zero entry $(i, j)$ from $A$ uniformly at random.
2. Return $1/r$ times the minimum of $O(r \log n)$ or the number of random $i$'s picked until a $A_{i', j}$ is non-zero.

This combined distribution is artificially capped at 1. More importantly, we can precisely calculate its expected value, up to a $1/poly(n)$ perturbation due to the truncation at $lim$.

Lemma 7.10. The distribution $D_{combined}$ as defined in Definition 7.9 has expectation

$$\left(1 - \frac{1}{n}\right) r \cdot \frac{|\{c : A_{c} \neq 0\}|}{nnz \cdot lim} \leq \mathbb{E}[D_{simple}] \leq \frac{r \cdot |\{c : A_{c} \neq 0\}|}{nnz \cdot lim},$$

40
and the expected cost of sampling $D_{\text{simple}}$ is

$$O\left(\frac{r \cdot |\{ c : A_{\cdot c} \neq 0\}|}{\text{nnz}}\right).$$

**Proof.** For each column $j$, denote $n_j$ to be the number of non-zeros in column $j$ of $A$. For the ease of representation, we use $p$ to denote the probability of picking a non-zero from this column,

$$p \overset{\text{def}}{=} \frac{n_j}{r}.$$

As there is at least one non-zero entry on the column $j$ in step 3, we should assume $n_j \geq 1$ for all time. Hence, $p$ is always positive.

Next, define the event $\text{Hit}_j$ as getting a non-zero $A_{i',j}$ by uniformly sampling $i'$ at 3(a). Let $h_j$ be the number of independent repeats of $\text{Hit}_j$ to make one happening without restriction of the iterations. Then we have, for any integer $k$,

$$\Pr[h_j = k] = (1 - p)^{k-1} \cdot p$$

So its expectation is given by

$$\mathbb{E}[h_j] = \sum_{1 \leq k} \Pr[h_j = k] \cdot k = \sum_{1 \leq k} (1 - p)^{k-1} \cdot p \cdot k$$

To compute this value, consider the generating function

$$G(x) \overset{\text{def}}{=} \sum_{1 \leq k} x^{k-1} \cdot k.$$  

Its integral is:

$$\int_0^x G(y) \, dy = \int_0^x \left( \sum_{1 \leq k} y^{k-1} \cdot k \right) \, dy = \sum_{1 \leq k} \int_0^x y^{k-1} \, dy = \sum_{1 \leq k} \frac{x^k}{1-x}.$$  

Then we have

$$G(x) = \frac{d}{dx} \left( \int_0^x G(y) \, dy \right) = \frac{d}{dx} \left( \frac{x}{1-x} \right) = \frac{1}{(1-x)^2}.$$  

Taking it back to the expectation, we get

$$\mathbb{E}[h_j] = p \cdot G(1-p) = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$  

This means if our initial entry is in column $j$, the expected value of

To account for the truncation, note that

$$\Pr[h_j \geq m] = (1 - p)^m = O\left(n^{-O(1)}\right).$$

So this changes the expectation by at most $n^{-O(1)}$. So we have

$$\frac{r}{n_j \cdot \lim} - n^{-O(1)} \leq \mathbb{E}[D_{\text{combined}} | \text{column } j \text{ is picked } D_{\text{simple}}] \leq \frac{r}{n_j \cdot \lim}.$$
and substituting back the probability that we pick column \( j \) with probability

\[
\frac{n_j}{\text{nnz}}
\]
gives

\[
\frac{r \cdot |\{ c : A, c \neq 0 \}|}{\text{nnz} \cdot \text{lim}} - n^{-O(1)} \leq \mathbb{E}[D_{\text{combined}}] \leq \frac{r \cdot |\{ c : A, c \neq 0 \}|}{\text{nnz} \cdot \text{lim}},
\]

and accuracy bound follows from the observation that the numerator is at least 1 and the denominator is bounded by \( \text{nnz} \cdot \text{lim} \leq n^4 \).

The expected running time also follows similarly, except we do not divide the number of terms sampled by \( \text{lim} \).

This means we can then invoke \textsc{EstimateMean} to approximate the mean of this distribution, and thus gives our overall guarantees.

Proof. (Of Lemma 7.1)

Notice that the value of \( D_{\text{simple}} \) is always between 0 and 1 due to the truncation by \( \text{lim} \), and then dividing by it. By Lemma 7.4, Part 2 we have that with high probability we obtain a \( 1 + \epsilon \) approximation of its expectation, which after the multiplication by \( \text{nnz} \cdot \text{lim} \) gives an \( 1 + \epsilon \) approximation to the number of non-zero columns.

It remains to bound the running time costs. Lemma 7.4, Part 1 gives that the expected number of times we sample \( D_{\text{combined}} \) is at most

\[
\frac{\text{nnz} \cdot \text{lim} \cdot \log n \epsilon^{-2}}{r \cdot |\{ c : A, c \neq 0 \}|},
\]

while the expected cost per call is

\[
O\left( \frac{r \cdot |\{ c : A, c \neq 0 \}|}{\text{nnz}} \right).
\]

Multiplying these gives that the expected total cost is

\[
O \left( \text{lim} \log n \epsilon^{-2} \right) = O \left( r \log^2 n \epsilon^{-2} \right).
\]

Furthermore, since the cost per call is capped at \( \text{lim} = O(r \log n) \), we also get that the runtime cost is concentrated around this value with high probability. \( \square \)

8 Maintaining Graphs Under Pivots

In this section we show that both the random graph access operations described in Theorem 6.2 and the \( \ell_0 \)-estimator as described in Definition 3.1 can be maintained efficiently under pivoting operations.

We start by checking that the component graph can be maintained under pivoting of new vertices while providing random-access to remaining and component neighbors of any vertex.
Proof. (of Lemma 7.2) We will maintain the adjacency list of \( G^0 \) explicitly, with each node storing its state as well as its neighbors in balanced binary search trees.

When we pivot a vertex, we examine all its neighbors that are component vertices, and merge the neighborhood lists of these vertices. By always inserting elements from the smaller list into the larger, we can guarantee that each element is inserted at most \( O(\log n) \) times. So across all \( m \) edges the total cost is \( O(m \log^2 n) \).

When a vertex is pivoted, we also need to move it from the remaining vertex list to the component vertex list in all of its neighbors. This can be done by going through all the edges of the vertex once. Its cost is \( O(m) \) because each vertex is pivoted at most once, and prior to pivoting no edges are added to it.

These binary search trees with all the neighbors allow for the sampling of a random component/remaining neighbors in \( O(\log n) \) time. A global list maintaining all the component/remaining vertices also allow them to be sampled in \( O(\log n) \) time.

The \( \ell_0 \)-sketches are maintained similarly as the graph changes. However, the minimum sketch value in each neighborhood keeps changing, and we deal with this by propagating new values proactively across to neighbors.

The algorithm is based on the notion of an eager-propagation routine: every time the \( x_{\min} \) at some vertex changes, it informs its neighbors of this change, and then in turn propagates this value.

This routine in the static case is the same as the \( \ell_0 \) estimators. Our main modification is to make it dynamic: after each pivot, the minimum per vertex can increase. Figure 14 contains a brief description of data structures we use to achieve this.

**Figure 14: Global Variables for maintaining data structures under vertex pivots**

**Maintain graphs under pivots**

Additional Variables: graph \( G \) that’s undergoing pivots, .

1. Set \( V_{\text{remaining}} \) containing the remaining vertices.
2. Set \( V_{\text{component}} \) containing the component vertices.
3. For each component vertex \( z \), an associated min-heap

   \[ z.\text{remaining} \]

   that contains the keys of its remaining neighbors.
4. For each remaining vertex \( u \), a min-heap

   \[ u.fill \]

   that contains the union of \( z.\text{remaining}.\text{min} \) for each component vertex \( z \) adjacent to \( u \) as well as the keys of \( u \)’s remaining neighbors.

In the case where nothing gets deleted, the following observation is sufficient for bounding the
cost of the propagations.

**Lemma 8.1.** For any sequence of increasing sets $S^{(0)}, S^{(1)}, \ldots, S^{(t)}$, the expected number of different minimums over a random labeling of the elements to $[0,1]$ is $O(\log n)$.

The major difficulty dealing with this is that deletions reduce degrees. In particular, it is possible for the min at some vertex to change $\Omega(n)$ times due to repeated deletions.

As a result, we can only bound the total, or average number of propagations. This leads to a much more involved amortized analysis, where we also use backwards analysis to explicitly bound the probability of each informing operation.

Given a component graph $G^{(t)}$ and a (remaining) vertex $v$ to be pivoted, we use the routine PivotVertex to produce a new graph $G^{(t+1)}$. In terms of the structure of the graph, our routine does the same thing as the traditional quotient graph model for symmetric factorization [GL81].

Therefore we turn our attention to the problem of maintaining the minimum sketch values of the neighborhoods, specifically the values $x_{\text{min}}(N_{\text{remaining}}^{G^{(t)}}(v))$ and $x_{\text{min}}(N_{\text{fill}}^{G^{(t)}}(v))$. This update procedure is basically a notification mechanism. When the status of a vertex changes, we update the data structures of its neighbors correspondingly. The Fill heap will then give $x_{\text{min}}(N_{\text{fill}}(u))$ and be used to estimate the fill-degree of each remaining vertex as described in Section 5.

These two sets of heaps are then maintained via a notification mechanism. Suppose a remaining vertex $v$ is pivoted. Then, for a component vertex $w$, the content of $w.\text{remaining}$ changes only if $v$ is its neighbor. Pseudocode of this update is given in Figure 15. In particular, since $v$ is no longer a remaining vertex, its entry needs to be removed from $z.\text{remaining}$. Furthermore, if $x_v$ was the minimum element in $w.\text{remaining}$, this is no longer the case and the other remaining neighbors of $w$ need to be notified of this (so they can update their fill heaps). This is done via the call to Informremaining in Line 2(b)i of the algorithm. The last step consists of melding the (now component) vertex $v$ with its existing component neighbors via calls to Meld.

The routine Informremaining (Algorithm 16) is responsible for updating the contents in the Fill heaps of remaining vertices. We break down its cost into two parts: when it is invoked by PivotVertex, and when it is invoked by Meld. The first type of calls happens only when a remaining vertex $v$ is pivoted, and $v$ is the minimum entry of the remaining heap of a component vertex. The following lemma gives an upper bound to the expected cost of such calls to Informremaining by arguing that this event happens with low probability.

**Lemma 8.2.** The expected total number of updates to remaining vertices made by Informremaining when invoked from PivotVertex (Line 2(b)i) over any sequence of $n$ pivots that are independent of the values of the $x_v$s is $O(m)$.

**Proof.** Let $G$ be any state during the sequence, and let $v$ be the remaining vertex to be pivoted with $w$ as a neighboring component vertex. We only invoke Informremaining if $x_v$ is the minimum value in $w.\text{remaining}$, which happens with probability $1/|N_{\text{remaining}}^{G}(w)|$ and would cost $O(|N_{\text{remaining}}^{G}(w)| \log n)$. Therefore the expected cost is only $O(\log n)$ for a pair of remaining vertex $v$ and neighboring component vertex $w$. When a remaining vertex $v$ is pivoted, its degree is the same as in the original graph. Therefore the number of such $v,w$ pairs is bounded by the degree of $v$ and hence the total expected cost is $O\left(\sum_{v \in V} \deg(v) \log n\right) = O(m \log n)$.

\qed
PivotVertex($u$)

Input: (implicitly as a global variable) a graph state $G = \langle V_{\text{Remaining}}, V_{\text{Component}}, E \rangle$ along with associated data structures.

A vertex $v \in V_{\text{Remaining}}$ to be pivoted,

Output: A list of vertices whose values of $x_{\text{min}}(N_{\text{fill}}(u))$ have changed.

1. Initialize $\text{changed}_\text{list} \leftarrow \emptyset$.
2. For each vertex $w \in N_{\text{Component}}^G(v)$ in lexicographical order
   (a) $w.\text{remaining}.\text{Remove}(x_v)$.
   (b) If $x_v$ was the old minimum in $w.\text{remaining}$
      i. $\text{changed}_\text{list} \leftarrow \text{changed}_\text{list} \cup \text{INFORMREMAINING}(w, x_v, w.\text{remaining}.\text{top})$.
   (c) $v.\text{remaining} \leftarrow \text{MELD}(v, w)$;
3. Update edges and $V_{\text{Component}}$ and $V_{\text{Remaining}}$ to form $G'$;
4. Return $\text{changed}_\text{list}$.

Figure 15: Pseudocode for pivoting a vertex

INFORMREMAINING($w, x_{\text{old}}, x_{\text{new}}$)

Input: (implicitly as a global variable) a graph state $G = \langle V_{\text{Remaining}}, V_{\text{Component}}, E \rangle$ along with associated data structures.

A ‘source’ component vertex $w \in V_{\text{Component}}$ that’s causing updates, old and new values $x_{\text{old}}$ and $x_{\text{new}}$.

Output: A list of vertices whose $v.\text{fill}.\text{min}$ changed.

1. Initialize $\text{changed}_\text{list} \leftarrow \emptyset$.
2. For each $v \in N_{\text{remaining}}^G(w)$
   (a) Remove the entry with key $x_{\text{old}}$ associated to $w$ from $v.\text{fill}$;
   (b) Add an entry with key $x_{\text{new}}$ associated to $w$ to $v.\text{fill}$;
   (c) If $v.\text{fill}.\text{min}$ changed, $\text{changed}_\text{list} \leftarrow \text{changed}_\text{list} \cup \{v\}$.
3. Return $\text{changed}_\text{list}$.

Figure 16: Pseudocode for propagating to remaining vertex neighbors
**Meld**

**Input:** (implicitly as a global variable) A graph state $G = \langle V_{\text{Remaining}}, V_{\text{Component}}, E \rangle$ along with associated data structures.

Two component vertices $u$ and $v$ to be melded.

**Output:** None. The algorithm simply updates the global state.

1. If $u.\text{remaining.min} < v.\text{remaining.min}$
   
   (a) **Informremaining**($v, v.\text{remaining.min}, u.\text{remaining.min}$);

2. Else If $v.\text{remaining.min} < u.\text{remaining.min}$

   (a) **Informremaining**($u, u.\text{remaining.min}, v.\text{remaining.min}$);

3. **HeapMeld**(v.\text{remaining}, u.\text{remaining});

Figure 17: Pseudocode for melding two component vertices, and informing their neighbors of any changes in the minimizers of $N_{\text{remaining}}$.

The calls to **Meld** are the primary bottlenecks in the running time, but will be handled similarly. Its pseudocode is given in Figure 17.

We will show that the expected number of vertices updated by **Informremaining** that result from any fixed sequence of calls to **Meld** is bounded by $O(m \log n)$. We first analyze the number of updates during a single meld in the following lemma.

**Lemma 8.3.** Let $u$ and $v$ be two component vertices in a graph stage $G^{(t)}$. Then the expected number of updates to vertices by **Informremaining** when melding $u$ and $v$, assuming that all the sketch values are generated independently, is at most:

$$2 \frac{|N_{\text{remaining}}^{G^{(t)}}(u) \cdot N_{\text{remaining}}^{G^{(t)}}(v)|}{|N_{\text{remaining}}^{G^{(t)}}(u) + N_{\text{remaining}}^{G^{(t)}}(v)|},$$

Proof. Let’s define:

- $n_{\text{common}} = \frac{N_{\text{remaining}}^{G^{(t)}}(u) \cap N_{\text{remaining}}^{G^{(t)}}(v)}{2}$,
- $n_u = \frac{N_{\text{remaining}}^{G^{(t)}}(u) \setminus N_{\text{remaining}}^{G^{(t)}}(v)}{2}$,
- $n_v = \frac{N_{\text{remaining}}^{G^{(t)}}(u) \setminus N_{\text{remaining}}^{G^{(t)}}(v)}{2}$.

If the minimum sketch value is generated by a vertex from $N_{\text{remaining}}^{G^{(t)}}(u) \cap N_{\text{remaining}}^{G^{(t)}}(v)$, then no cost is incurred. If it is generated by a vertex from $N_{\text{remaining}}^{G^{(t)}}(u) \setminus N_{\text{remaining}}^{G^{(t)}}(v)$, we need to update
the every vertex in $N_{\text{remaining}}^{G_t}(v)$ (line 1a). This happens with probability

$$\frac{n_u}{n_{\text{common}} + n_u + n_v} \leq \frac{1}{2n_{\text{common}} + n_u + n_v} \leq \frac{\left|N_{\text{remaining}}^{G_t}(u)\right|}{\left|N_{\text{remaining}}^{G_t}(u)\right| + \left|N_{\text{remaining}}^{G_t}(v)\right|}.$$ 

Therefore the expected number of updates is bounded by:

$$\frac{\left|N_{\text{remaining}}^{G_t}(u)\right|}{\left|N_{\text{remaining}}^{G_t}(u)\right| + \left|N_{\text{remaining}}^{G_t}(v)\right|},$$

and we get the other term (for updating $u$’s neighborhood) similarly. 

This allows us to carry out an amortized analysis for the number of updates to remaining vertices. We will define the potential function of an intermediate state during elimination in terms of the degrees of component vertices in the original graph $G^{(0)}$, in which adjacent component vertices are not contracted.

$$\Phi(G^{(t)}) \overset{\text{def}}{=} \sum_{u \in V_{\text{component}}(G^{(t)})} D^{G^{(0)}}(u) \log(D^{G^{(0)}}(u)),$$

where $D^{G^{(0)}}(u)$ for a component vertex $u \in V_{\text{component}}(G^{(t)})$ is defined to be

$$D^{G^{(0)}}(u) = \sum_{v \in V(G), \text{Comp}(v) = u} \deg_{G^{(0)}}(u).$$

This function starts out at 0, and can be at most $m \log n$.

**Lemma 8.4.** The total potential decrease caused by turning remaining vertices into component vertices is at most $O(m \log n)$.

**Proof.** When we turn a remaining vertex $v$ into a component vertex, we decrease the value of $D(u)$ for every $u$ in $N_{\text{remaining}}^{G}(v)$. This causes a total potential decrease of at most

$$\left|N_{\text{remaining}}^{G}(v)\right| \log n.$$

Since $\left|N_{\text{remaining}}^{G}(v)\right|$ is at most its degree, and each vertex can only be turned into a component vertex once, the total decrease in the potential is at most $O(m \log n)$. 

**Lemma 8.5.** When melding two neighboring component vertices in a graph $G^{(t)}$ to create $G^{(t+1)}$, we have

$$\mathbb{E}_{x_t \sim u \in V} \left[\text{number of remaining vertices updated}\right] \leq 2 \left( \Phi(G^{(t+1)}) - \Phi(G^{(t)}) \right).$$
Proof. When melding two component vertices \( u \) and \( v \) in \( G \) to form \( G' \), the change in potential \( \Phi(G') - \Phi(G) \) is given by
\[
(D(u) + D(v)) \log(D(u) + D(v)) - D(u) \log D(u) - D(v) \log D(v).
\]
On the other hand, by Lemma 8.3, the expected number of remaining vertices updated is
\[
2 \left| \left| N_{\text{remaining}}^{G(t)}(u) \right| \cdot \left| N_{\text{remaining}}^{G(t)}(v) \right| \right| \leq \frac{2D(u)D(v)}{D(u) + D(v)}.
\]
Now it suffices to show the following algebraic identity:
\[
2x \log x + 2y \log y + \frac{2xy}{x + y} \leq 2(x + y) \log (x + y),
\]
and let \( x = D(u) \) and \( y = D(v) \). By symmetry, we can assume \( x \leq y \) without loss of generality. Then we get
\[
\frac{xy}{x + y} \leq \frac{xy}{y} = y \cdot \frac{x}{y} \leq y \cdot \log \left(1 + \frac{x}{y}\right),
\]
where the last inequality follows from \( \log(1 + z) \geq z \) when \( z \leq 1 \). Plugging this in then gives:
\[
2x \log x + 2y \log y + \frac{2xy}{x + y} \leq 2x \log x + 2y \left( \log y + \log \left(1 + \frac{x}{y}\right)\right)
= 2x \log x + 2y \log (x + y)
\leq 2(x + y) \log (x + y).
\]

Lemma 8.6. Over any fixed sequence of calls to MELD, the expected number of updates to the Fill heaps in remaining vertices (lines 14 and 24) is bounded by \( O(m \log n) \).

Proof. By Lemma 8.5, the number of updates is within a constant of the potential increase. Since our potential function \( \Phi \) is bounded between 0 and \( O(m \log n) \), and by Lemma 8.4 does not decrease by more than \( O(m \log n) \), the total number of updates is also bounded by \( O(m \log n) \).

Combining the above lemmas gives our main theorem from Section 5 on maintaining one copy of the sketch.

Proof. (of Theorem 5.4) Given any graph \( G \) and a fixed sequence of vertices for pivoting, we use the \textsc{PivotVertex} routine to produce the sequence of graph states
\[
G^{(0)} = G, G^{(1)}, G^{(2)}, \ldots, G^{(n)}.
\]
Recall that the goal is to maintain \( x_{\min}(N_{\text{remaining}}^{G(t)}(v)) \) for all \( v \in V_{\text{component}}(G^{(t)}) \) and \( x_{\min}(N_{\text{fill}}^{G(t)}(v)) \) for all \( v \in V_{\text{remaining}}(G^{(t)}) \). This is achieved by maintaining the two min-heaps, \texttt{remaining} and \texttt{Fill}.
When pivoting a remaining vertex $v$, PivotVertex first removes it from the remaining heaps among $v$’s component vertex neighbors (line 2a), which are at most as many as the original degree of $v$. Therefore the total cost of this part of the algorithm is $O(m \log n)$. The rest of the running time cost is incurred by updates to the Fill heaps in InformRemaining. By Lemma 8.2 and Lemma 8.6, the number of updates is bounded by $O(m \log n)$. As each update is a $O(\log n)$ operation on a heap, the total running time is $O(m \log^2 n)$. The final step of a meld consists of merging the remaining heaps, and the cost of this step can be similarly bounded by $O(m \log^2 n)$.

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References

[ABD+90] E. Anderson, Z. Bai, J. Dongarra, A. Greenbaum, A. McKenney, J. Du Croz, S. Hammarling, J. Demmel, C. Bischof, and D. Sorensen. Lapack: A portable linear algebra library for high-performance computers. In Proceedings of the 1990 ACM/IEEE Conference on Supercomputing, Supercomputing ’90, pages 2–11, Los Alamitos, CA, USA, 1990. IEEE Computer Society Press.

[ADD96] Patrick R. Amestoy, Timothy A. Davis, and Iain S. Duff. An approximate minimum degree ordering algorithm. SIAM J. Matrix Anal. Appl., 17(4):886–905, October 1996.

[ADD04] Patrick R. Amestoy, Timothy A. Davis, and Iain S. Duff. Algorithm 837: Amd, an approximate minimum degree ordering algorithm. ACM Trans. Math. Softw., 30(3):381–388, September 2004.

[AY10] Noga Alon and Raphael Yuster. Solving linear systems through nested dissection. In Foundations of Computer Science (FOCS), 2010 51st Annual IEEE Symposium on, pages 225–234. IEEE, 2010.

[BCK+16] Ivan Bliznets, Marek Cygan, Paweł Komosa, Lukáš Mach, and Michal Pilipczuk. Lower bounds for the parameterized complexity of minimum fill-in and other completion problems. In Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1132–1151. Society for Industrial and Applied Mathematics, 2016.

[BGS15] Surender Baswana, Manoj Gupta, and Sandeep Sen. Fully dynamic maximal matching in $O(\log n)$ update time. SIAM Journal on Computing, 44(1):88–113, 2015. Announced at FOCS’11.

[BKSE12] Jeff Bezanson, Stefan Karpinski, Viral B Shah, and Alan Edelman. Julia: A fast dynamic language for technical computing. arXiv preprint arXiv:1209.5145, 2012. Available at: https://arxiv.org/abs/1209.5145.

[BS90] Piotr Berman and Georg Schnitger. On the performance of the minimum degree ordering for gaussian elimination. SIAM J. Matrix Anal. Appl., 11(1):83–88, 1990.
[CCLW13] Yeow Meng Chee, Charles J Colbourn, Alan CH Ling, and Richard M Wilson. Covering and packing for pairs. *Journal of Combinatorial Theory, Series A*, 120(7):1440–1449, 2013.

[CLRS09] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms, Third Edition*. The MIT Press, 3rd edition, 2009.

[CM05] Graham Cormode and S. Muthukrishnan. An improved data stream summary: The count-min sketch and its applications. *J. Algorithms*, 55(1):58–75, April 2005.

[Coh97] Edith Cohen. Size-estimation framework with applications to transitive closure and reachability. *J. Comput. Syst. Sci.*, 55(3):441–453, December 1997.

[CS17] Yixin Cao and R. B. Sandeep. Minimum fill-in: Inapproximability and almost tight lower bounds. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, SODA ’17, pages 875–880, 2017.

[CY98] Yair Caro and Raphael Yuster. Covering graphs: The covering problem solved. *Journal of Combinatorial Theory, Series A*, 83(2):273–282, 1998.

[DGLN04] Timothy A. Davis, John R. Gilbert, Stefan I. Larimore, and Esmond G. Ng. A column approximate minimum degree ordering algorithm. *ACM Trans. Math. Softw.*, 30(3):353–376, 2004.

[ELRS17] Talya Eden, Amit Levi, Dana Ron, and C Seshadhri. Approximately counting triangles in sublinear time. *SIAM Journal on Computing*, 46(5):1603–1646, 2017.

[Fel71] William Feller. *An introduction to probability theory and its applications. Vol. II*. Second edition. John Wiley & Sons Inc., New York, 1971.

[FM85] Philippe Flajolet and G. Nigel Martin. Probabilistic counting algorithms for data base applications. *J. Comput. Syst. Sci.*, 31(2):182–209, September 1985.

[FV13] Fedor V Fomin and Yngve Villanger. Subexponential parameterized algorithm for minimum fill-in. *SIAM Journal on Computing*, 42(6):2197–2216, 2013.

[Geo73] Alan George. Nested dissection of a regular finite element mesh. *SIAM Journal on Numerical Analysis*, 10(2):345–363, 1973.

[GG11] Eran Gat and Shafi Goldwasser. Probabilistic search algorithms with unique answers and their cryptographic applications. *Electronic Colloquium on Computational Complexity (ECCC)*, 18:136, 2011. Available at: http://eccc.hpi-web.de/report/2011/136.

[GGR13] Oded Goldreich, Shafi Goldwasser, and Dana Ron. On the possibilities and limitations of pseudodeterministic algorithms. In *Proceedings of the 4th Conference on Innovations in Theoretical Computer Science*, ITCS ’13, pages 127–138, New York, NY, USA, 2013. ACM.

[GL81] Alan George and Joseph W. Liu. *Computer Solution of Large Sparse Positive Definite*. Prentice Hall Professional Technical Reference, 1981.
[GL89] A. George and W. H. Liu. The evolution of the minimum degree ordering algorithm. *SIAM Rev.*, 31(1):1–19, March 1989.

[GNP94] John R Gilbert, Esmond G Ng, and Barry W Peyton. An efficient algorithm to compute row and column counts for sparse Cholesky factorization. *SIAM Journal on Matrix Analysis and Applications*, 15(4):1075–1091, 1994.

[Gol12] Shafi Goldwasser. Pseudo-deterministic algorithms. In *STACS’12 (29th Symposium on Theoretical Aspects of Computer Science)*, volume 14, pages 29–29. LIPIcs, 2012.

[GT87] J. R. Gilbert and R. E. Tarjan. The analysis of a nested dissection algorithm. *Numerische Mathematik*, 50(4):377–404, 1987.

[HEKP01] Pinar Heggernes, SC Eisestat, Gary Kumfert, and Alex Pothen. The computational complexity of the minimum degree algorithm. Technical report, INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING HAMPTON VA, 2001.

[HP07] Bruce Hendrickson and Alex Pothen. Combinatorial scientific computing: The enabling power of discrete algorithms in computational science. In *Proceedings of the 7th International Conference on High Performance Computing for Computational Science, VECPAR’06*, pages 260–280, 2007.

[KKM13] Bruce M Kapron, Valerie King, and Ben Mountjoy. Dynamic graph connectivity in polylogarithmic worst case time. In *Proceedings of the twenty-fourth annual ACM-SIAM symposium on Discrete algorithms*, pages 1131–1142. Society for Industrial and Applied Mathematics, 2013.

[KMP12] Ioannis Koutis, Gary L. Miller, and Richard Peng. A fast solver for a class of linear systems. *Communications of the ACM*, 55(10):99–107, October 2012. Available at http://ecom.uprprp.edu/~ikoutis/papers/CACM-KMP.pdf.

[KNP+17] Michael Kapralov, Jelani Nelson, Jakub Pachocki, Zhengyu Wang, David P Woodruff, and Mobin Yahyazadeh. Optimal lower bounds for universal relation, and for samplers and finding duplicates in streams. *arXiv preprint arXiv:1704.00633*, 2017.

[KP17] John Kallaugher and Erie Price. A hybrid sampling scheme for triangle counting. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1778–1797. SIAM, 2017. Available at: https://arxiv.org/abs/1610.02066.

[KPPS17] Rasmus Kyng, Jakub Pachocki, Richard Peng, and Sushant Sachdeva. A framework for analyzing resparsification algorithms. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2032–2043. SIAM, 2017. Available at: https://arxiv.org/abs/1611.06940.

[KS16] Rasmus Kyng and Sushant Sachdeva. Approximate gaussian elimination for laplacians-fast, sparse, and simple. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, pages 573–582. IEEE, 2016. Available at: https://arxiv.org/abs/1605.02353.
[KST99] Haim Kaplan, Ron Shamir, and Robert E. Tarjan. Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. *SIAM Journal on Computing*, 28(5):1906–1922, 1999.

[KZ17] Rasmus Kyng and Peng Zhang. Hardness results for structured linear systems. *CoRR*, abs/1705.02944, 2017.

[LG14] François Le Gall. Powers of tensors and fast matrix multiplication. In *Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*, ISSAC ’14, pages 296–303, 2014. Available at: https://arxiv.org/abs/1401.7714.

[Liu85] Joseph WH Liu. Modification of the minimum-degree algorithm by multiple elimination. *ACM Transactions on Mathematical Software (TOMS)*, 11(2):141–153, 1985.

[Liu90] Joseph WH Liu. The role of elimination trees in sparse factorization. *SIAM Journal on Matrix Analysis and Applications*, 11(1):134–172, 1990.

[LRT79] R. J. Lipton, D. J. Rose, and R. E. Tarjan. Generalized nested dissection. *SIAM J. on Numerical Analysis*, 16:346–358, 1979.

[LS15] Yin Tat Lee and Aaron Sidford. Efficient inverse maintenance and faster algorithms for linear programming. In *Proceedings of the 2015 IEEE 56th Annual Symposium on Foundations of Computer Science (FOCS)*, FOCS ’15, pages 230–249, 2015. Available at: https://arxiv.org/abs/1503.01752.

[LT79] R. J. Lipton and R. E. Tarjan. A planar separator theorem. *SIAM J. Appl. Math.*, 36:177–189, 1979.

[Mat17] MATLAB optimization toolbox, 2017. The MathWorks, Natick, MA, USA.

[MPVX15] Gary L Miller, Richard Peng, Adrian Vladu, and Shen Chen Xu. Improved parallel algorithms for spanners and hopsets. In *Proceedings of the 27th ACM Symposium on Parallelism in Algorithms and Architectures*, pages 192–201. ACM, 2015. Available at: https://arxiv.org/abs/1309.3545.

[MPX13] Gary L. Miller, Richard Peng, and Shen Chen Xu. Parallel graph decompositions using random shifts. In *25th ACM Symposium on Parallelism in Algorithms and Architectures, SPAA ’13, Montreal, QC, Canada - July 23 - 25, 2013*, pages 196–203, 2013. Available at: https://arxiv.org/abs/1307.3692.

[NS12] Uwe Naumann and Olaf Schenk. *Combinatorial Scientific Computing*. Chapman & Hall/CRC, 1st edition, 2012.

[NSS00] Assaf Natanzon, Ron Shamir, and Roded Sharan. A polynomial approximation algorithm for the minimum fill-in problem. *SIAM Journal on Computing*, 30(4):1067–1079, 2000.

[PCD17] Hadi Pouransari, Pieter Coulier, and Eric Darve. Fast hierarchical solvers for sparse matrices using extended sparsification and low-rank approximation. *SIAM Journal on Scientific Computing*, 39(3):A797–A830, 2017.
This section contains the proofs to Lemma 5.6, which claims that the reciprocal of the $\lfloor k(1 - 1/e) \rfloor$-ranked element in $\text{minimizers}(w)$ is a good approximation for the degree of a vertex $w$. The proofs follow similarly as in [Coh97] - with the difference being the distribution of the underlying $x$ variables.

We start by stating Hoeffding’s tail bounds.

**Lemma A.1** (Hoeffding’s inequality). Let $b_1, b_2, \ldots, b_n$ be i.i.d. Bernoulli random variables such
that $Pr[b_i = 1] = p$ and $Pr[b_i = 0] = 1 - p$. Then,

$$Pr \left[ \sum_{1 \leq i \leq n} b_i \leq (p - \delta)n \right] \leq \exp(-2\delta^2 n),$$

$$Pr \left[ \sum_{1 \leq i \leq n} b_i \geq (p + \delta)n \right] \leq \exp(-2\delta^2 n).$$

To apply Hoeffding’s bounds, we will also need the following numerical result.

**Lemma A.2.** Let $-0.1 < \epsilon < 0.1$ and $d \geq 1$ be some parameter. Then we have:

$$\exp \left( -1 + \epsilon - \frac{2}{d} \right) \leq \left( 1 - \frac{1 - \epsilon}{d} \right)^d \leq \exp (-1 + \epsilon)$$

**Proof.** The Maclaurin series for $\log(1 - x)$ is

$$\log (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$$

When $|x| \leq 0.1$, we have

$$\left| \frac{x}{3} + \frac{x^2}{4} + \ldots \right| \leq \frac{0.1}{3} + \frac{0.01}{4} + \ldots \leq \frac{1}{2},$$

so we have

$$-x - x^2 \leq \log (1 - x) \leq -x$$

Apply this with

$$x \leftarrow \frac{1 - \epsilon}{d}$$

gives

$$-\frac{1 - \epsilon}{d} - \frac{2}{d^2} \leq \log \left( 1 - \frac{1 - \epsilon}{d} \right) \leq -\frac{1 - \epsilon}{d},$$

which when exponentiated and taken to the $d^{th}$ power gives the result. \(\square\)

We are now well equipped to prove Lemma 5.6 which, for the sake of convenience, we split into the following two lemmas.

Let $k \geq \Omega(\log ne^{-2})$ denote the number of copies of the $\ell_0$-sketch data structure. Let $w$ be a vertex with degree $d > k$, and let $q(w)$ denote the $\lfloor k(1 - 1/e) \rfloor$-ranked element in $\text{Minimizers}(w)$.

**Lemma A.3.** With high probability,

$$\frac{1 - \epsilon}{d} \leq q(w).$$

**Proof.** For any $1 \leq i \leq k$,

$$Pr \left[ \text{Minimizer}(w)[i] \geq \frac{1 - \epsilon}{d} \right] = \prod_{v \in \text{nbr}(w)} Pr \left[ x_v^{[i]} \geq \frac{1 - \epsilon}{d} \right] = \left( 1 - \frac{1 - \epsilon}{d} \right)^d.$$
Let $J_i$ be the indicator variable which equals 1 when $\text{Minimizer}(w)[i] \geq \frac{1 - \epsilon}{d}$ and equals 0 otherwise. So, $E[J_i] = \left(1 - \frac{1 - \epsilon}{d}\right)^d$ and,

$$\Pr\left[q(w) \leq \frac{1 - \epsilon}{d}\right] = \Pr\left[\sum_{1 \leq i \leq k} I_i \leq k/e\right].$$

Using Lemma A.1,

$$\Pr\left[\sum_{1 \leq i \leq k} J_i \leq k/e\right] \leq \exp(-2k\delta^2) = \exp(-2 \log n(\delta/e)^2)$$

where $\delta = E[J_i] - 1/e$. So,

$$\frac{\delta}{e} = \left(1 - \frac{1 - \epsilon}{d}\right)^d - \frac{1}{e} \geq \frac{1/e^{(1 - \epsilon + 2/d)}}{e} - \frac{1}{e} \geq \frac{e^{\epsilon - 2/d} - 1}{e \cdot e},$$

where the first inequality follows from Lemma A.2. Since $d > k \geq \Omega(\log n \epsilon^{-2})$, we have $d \geq 4/e$, and can substitute to get:

$$\frac{\delta}{e} \geq \frac{e^{\epsilon/2} - 1}{\epsilon \cdot e} \geq \frac{1}{2e}.$$

This gives us that

$$\Pr\left[\sum_{1 \leq i \leq k} J_i \leq k/e\right] \leq (1/n)^c$$

for some constant $c > 0$. \qed

**Lemma A.4.** With high probability,

$$\frac{1 + \epsilon}{d} \geq q(w).$$

**Proof.** For any $1 \leq i \leq k$,

$$\Pr\left[\text{Minimizer}(w)[i] \geq \frac{1 + \epsilon}{d}\right] = \prod_{v \in \text{nbr}(w)} \Pr\left[x^v[i] \geq \frac{1 + \epsilon}{d}\right] = \left(1 - \frac{1 + \epsilon}{d}\right)^d.$$

Let $I_i$ be the indicator variable which equals 1 when $\text{Minimizer}(w)[i] \geq \frac{1 + \epsilon}{d}$ and equals 0 otherwise. So, $E[I_i] = \left(1 - \frac{1 + \epsilon}{d}\right)^d$ and,

$$\Pr\left[q(w) \geq \frac{1 + \epsilon}{d}\right] = \Pr\left[\sum_{1 \leq i \leq k} I_i \geq k/e\right]$$
Using Lemma A.1

\[
\Pr \left[ \sum_{1 \leq i \leq k} I_i \geq k/e \right] \leq \exp(-2k\delta^2) = \exp(-2 \log n(\delta/\epsilon)^2),
\]

where \( \delta = 1/e - E[I_i] \). So,

\[
\delta/\epsilon = \frac{1/e - \left(1 - \frac{1+\epsilon}{d}\right)^d}{\epsilon} \geq \frac{1/e - 1/e^{1+\epsilon}}{\epsilon} \geq \frac{e^\epsilon - 1}{\epsilon \cdot e^{1+\epsilon}} \geq \frac{1}{e^{1+\epsilon}},
\]

where the first inequality follows from Lemma A.2. So,

\[
\Pr \left[ \sum_{1 \leq i \leq k} I_i \geq k/e \right] \leq (1/n)^c
\]

for some constant \( c > 0 \). \(\square\)