The Maximum Number of Subset Divisors of a Given Size

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1 Abstract

If $s$ is a positive integer and $A$ is a set of positive integers, we say that $B$ is an $s$-divisor of $A$ if $\sum_{b \in B} b \mid s \sum_{a \in A} a$. We study the maximal number of $k$-subsets of an $n$-element set that can be $s$-divisors. We provide a counterexample to a conjecture of Huynh that for $s = 1$, the answer is $\binom{n-1}{k}$ with only finitely many exceptions, but prove that adding a necessary condition makes this true. Moreover, we show that under a similar condition, the answer is $\binom{n-1}{k}$ with only finitely many exceptions for each $s$.

2 Introduction

If $X$ is a set of positive integers, let $\sum X$ denote $\sum_{x \in X} x$. Let $A$ be a finite subset of the positive integers. The elements of $A$ are $a_1 < a_2 < \cdots < a_n$ and let $B$ be a subset of $A$. We say that $B$ is a divisor of $A$ if $\sum B \mid \sum A$. We define $d_k(A)$ to be the number of $k$-subset divisors of $A$ and let $d(k,n)$ be the maximum value of $d_k(A)$ over all sets $A$ of $n$ positive integers.

Similarly, for $s \geq 1$ a positive integer, we say that $B$ is an $s$-divisor of $A$ if $\sum B \mid s \sum A$. We define $d_k^s(A)$ to be the number of $k$-subset $s$-divisors of $A$ and let $d^s(k,n)$ be the maximum value of $d_k^s(A)$ over all sets $A$ of $n$ positive integers.

Note that the concepts of divisor and 1-divisor coincide. Also, if $B$ is a divisor of $A$, then $B$ is an $s$-divisor of $A$ for all $s$, so $d_k^s(A) \geq d_k(A)$ and $d^s(k,n) \geq d(k,n)$.

Huynh [6] notes that for all values of $a_1, \ldots, a_{n-1}$, we can pick an $a_n$ and set $A = \{a_1, \ldots, a_{n-1}, a_n\}$ so that every $k$-subset of $\{a_1, \ldots, a_{n-1}\}$ is an $A$-divisor. Therefore $d(k,n) \geq \binom{n-1}{k}$ for all $1 \leq k \leq n$. This motivates the definition that $A$ is a k-anti-pencil if the set of $k$-subset divisors of $A$ is $\binom{\Delta \{a_n\}}{k}$. We similarly define $A$ to be a $(k,s)$-anti-pencil if the set of $k$-subset $s$-divisors of $A$ is $\binom{\Delta \{a_n\}}{k}$.

Huynh [6] also formulates the following conjecture (Conjecture 22).
Conjecture 1. For all but finitely many values of $k$ and $n$, $d(k, n) = \binom{n-1}{k}$.

In this paper, we provide infinite families of counterexamples, but prove that, with the exception of these families, the conjecture is true. This gives us the following modified form.

Conjecture 2. For all but finitely many integer pairs $(k, n)$ with $1 < k < n$, $d(k, n) = \binom{n-1}{k}$.

For convenience, we now rescale, dividing every element of $A$ by $\sum A$, so that now the elements of $A$ are positive rational numbers and $\sum A = 1$. Under this rescaling, $B \subseteq A$ is a divisor of $A$ if and only if $\sum B = 1$ for some positive integer $m$ and $B$ is an $s$-divisor of $A$ if and only if $\sum B = \frac{1}{s}$ for some positive integer $m$. Clearly, the values of $d(k, n)$ and $d_s(k, n)$ do not change.

The $k < n$ condition in Conjecture 2 is necessary since it is easy to see that $d(n, n) > \binom{n-1}{1}$. Also, if

$$A = \left\{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^{n-2}}, \frac{1}{3(2^{n-2})}, \frac{1}{3(2^{n-3})}\right\}$$

then $\sum A = 1$, so $d_1(A) = n$ and $d(1, n) > \binom{n-1}{1}$. Therefore the $1 < k$ condition is necessary.

However, we prove that these families of values $(k, n)$ cover all but finitely many exceptions.

Theorem 3. For all but finitely many pairs $(k, n)$, if $1 < k < n$, $|A| = n$, and $d(k, A) \geq \binom{n-1}{k}$, then $A$ is a $k$-anti-pencil.

Note that this immediately implies Conjecture 2.

If we are interested in $s$-divisors, we get another family of exceptions. If $s \geq 2$, $a_{n-1} = \frac{1}{s+1}$ and $a_n = \frac{2}{s+2}$, then $d_s(n-1, A) \geq 2$, so $d_s(n-1, n) \geq 2 > \binom{n-1}{1}$. However, we prove that these cover all but finitely many exceptions.

Theorem 4. Fix $s \geq 1$. For all but finitely many pairs $(k, n)$ (with the number of these pairs depending on $s$), if $1 < k < n - 1$, $|A| = n$, and $d_s(k, A) \geq \binom{n-1}{k}$, then $A$ is a $(k, s)$-anti-pencil.

Note that this immediately implies the following corollary.

Corollary 5. Fix $s \geq 1$. Then $d_s(k, n) = \binom{n-1}{k}$ for all but finitely many pairs $(k, n)$ with $1 < k < n - 1$ (with the number of these pairs depending on $s$).

We will prove Theorem 4. In the $s = 1$ case, where $k = n - 1$, if $i \leq n - 1$, then $\sum (A \setminus \{a_i\}) > \frac{1}{2}$, so $A \setminus \{a_i\}$ is not a divisor of $A$. This, together with the $s = 1$ case of Theorem 3, gives us Theorem 3.
3 Lemmas

We will need a lemma about a certain poset. First, we present some general definitions and theorems (all the definitions and results up to the lemma statement can be found in [4]).

The width of a poset is the size of its largest antichain. If \( P \) is a finite poset, we say that \( P \) is ranked if there exists a function \( \rho : P \to \mathbb{Z} \) satisfying \( \rho(y) = \rho(x) + 1 \) if \( y \) covers \( x \) in \( P \) (i.e. \( y > x \), and there is no \( z \in P \) with \( y > z > x \)). If \( \rho(x) = i \), then \( x \) is said to have rank \( i \). Let \( P_i \) denote the set of elements of \( P \) of rank \( i \). We say \( P \) is rank-symmetric rank-unimodal if there exists some \( c \in \mathbb{Z} \) with \( |P_i| \leq |P_{i+1}| \) when \( i < c \) and \( |P_{2c-i}| = |P_i| \) for all \( i \in \mathbb{Z} \). A ranked poset \( P \) is called strongly Sperner if for every positive integer \( s \), the largest subset of \( P \) that has no \((s+1)\)-chain is the union of the \( s \) largest \( P_i \).

Proctor, Saks, and Sturtevant [9] prove that the class of rank-symmetric rank-unimodal strongly Sperner posets is closed under products. A finite product of finite linear orders is called a chain product. Since a linear ordering of length \( n \) is rank-symmetric rank-unimodal strongly Sperner, so is any chain product.

We now take a \( d \)-dimensional lattice cube with \( n \) lattice points per edge. Define a poset on the lattice points by \((x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)\) if \( x_i \leq y_i \) for all \( i \). This is a chain product, so it is rank-symmetric rank-unimodal strongly Sperner.

**Lemma 6.** The largest antichain in this poset has at most \((n + d - 2)^{d-1}\sqrt{\frac{2}{d}}\) elements.

**Proof.** Center the cube on the origin by translation in \( \mathbb{R}^d \). Let \( U \) be the set of elements whose coordinates sum to 0. Since the poset is rank-symmetric rank-unimodal strongly Sperner, its width is at most the size of \( P_c \), which is \(|U|\).

For each \( y = (y_1, \ldots, y_d) \in U \), let \( S_y \) be the set of points \((x_1, \ldots, x_d)\) with \(|x_i - y_i| < \frac{1}{2}\) for \( 1 \leq i \leq d-1 \) (note that this does not include the last index) which lie on the hyperplane given by \( x_1 + \cdots + x_d = 0 \). If \( y, z \) are distinct elements of \( U \), then \( S_y \) and \( S_z \) are clearly disjoint. Also, the projection of \( S_y \) onto the hyperplane given by \( x_d = 0 \) is a unit \((d-1)\)-dimensional hypercube, which has volume 1. Thus the volume of \( S_y \) is \( \sqrt{d} \) and the volume of \( \bigcup_{y \in U} S_y \) is \(|U| \sqrt{d} \).

On the other hand, if \((x_1, \ldots, x_d) \in S_y\), then \(|x_i - y_i| < \frac{1}{2}\) for \( 1 \leq i \leq d-1 \) and \(|x_d - y_d| \leq \sum_{i=1}^{d-1} |x_i - y_i| < \frac{1}{2}(d-1) \). Thus \((x_1, \ldots, x_d)\) lies in the cube of edge length \((n-1) + (d-1) = n + d - 2\) centered at the origin. Therefore \( \bigcup_{y \in U} S_y \) lies in the intersection of a cube of edge length \( n + d - 2 \) with a hyperplane through its center (the origin).

Ball [1] shows that the volume of the intersection of a unit hypercube of arbitrary dimension with a hyperplane through its center is at most \( \sqrt{2} \). Therefore the volume of \( \bigcup_{y \in U} S_y \) is at most \((n + d - 2)^{d-1}\sqrt{2} \), so

\[
|U| \leq (n + d - 2)^{d-1}\sqrt{\frac{2}{d}}.
\]

\(\square\)
Let $X = \{x_1 < \cdots < x_n\}$ be a set of positive integers. If $B, C \subseteq \binom{X}{d}$, then we say that $B \leq C$ if we can write $B = \{b_1, \ldots, b_d\}$ and $C = \{c_1, \ldots, c_d\}$ with $b_i \leq c_i$ for all $1 \leq i \leq d$. Whenever we compare subsets of $A$, we will be using this partial order.

**Lemma 7.** Fix $d > 1$. For $n$ sufficiently large, the width of the partial order defined above is less than $\frac{1}{d!} \left(\frac{n!}{n^d}\right)$.

**Proof.** Let $U$ be a maximum antichain of the partial order. Take the partial order of $X^d$, which coincides with the cube partial order. Let $U' = \{(y_1, \ldots, y_d) \in X^d \mid \{y_1, \ldots, y_d\} \subseteq U\}$. Note that this means, in particular, that every $d$-tuple in $U'$ has its elements distinct. If $(y_1, \ldots, y_d), (z_1, \ldots, z_d) \in U'$ with $(y_1, \ldots, y_d) < (z_1, \ldots, z_d)$, then we get that $\{y_i\} \subseteq \{z_i\}$ and $\sum_{i=1}^d y_i < \sum_{i=1}^d z_i$, so $\{y_i\} \neq \{z_i\}$, so $\{y_i\} < \{z_i\}$, which is impossible. Thus $U'$ is an antichain of $X^d$ of size $d!|U|$ and

$$|U| \leq \frac{1}{d!} (n + 2d - 2)^{d-1} \sqrt{\frac{2}{d}}.$$  

Then $\left|\binom{X}{d}\right| = \binom{n}{d}$ gives us

$$\frac{|U|}{\left|\binom{X}{d}\right|} \leq \frac{(n + 2d - 2)^{d-1} \sqrt{\frac{2}{d}}}{n(n-1) \cdots (n-d+1)}.$$  

For sufficiently large $n$, $(\frac{n+2d-2}{n-d+1})^{d-1} < \sqrt{2}$, so $\frac{|U|}{\left|\binom{X}{d}\right|} < \frac{2}{\sqrt{2n}}$. \hfill $\square$

Let $d(n)$ denote the number of divisors of $n$. Then we have the following lemma, which is proven as Theorem 315 in [5].

**Lemma 8.** For each positive integer $k$, $d(n) = O(n^{\frac{1}{k}})$.

**Lemma 9.** Fix positive integers $k, m, a, b$. Then for positive integers $n$, the number of pairs of positive integers $(x, y)$ such that $\frac{m}{n} = \frac{a}{x} + \frac{b}{y}$ and all three fractions are in lowest terms is at most $O(n^{\frac{1}{k}})$.

**Proof.** Assume $\frac{m}{n} = \frac{a}{x} + \frac{b}{y}$. Let $p = \gcd(n, x)$, with $n = tp$ and $x = wp$. Then

$$\frac{b}{y} = \frac{m}{n} - \frac{a}{x} = \frac{mw - at}{twp}.$$  

Letting $q = \gcd(mw - at, twp)$, we get

$$mw - at = qb.$$  

(1)

For each choice of $n, p, q$, (1) gives at most one possible value of $w$, thus at most one value of $x$, and thus at most one value of $(x, y)$.

The definition of $q$ gives us $q \mid p$. Then for a given $n$, both $p$ and $q$ are divisors of $n$, so by Lemma 8 there are $O(n^{\frac{1}{k'}})$ possible values for $p$ and $O(n^{\frac{1}{k'}})$ values for $q$, so there are $O(n^{\frac{1}{k'}})$ values for $(p, q)$ and $O(n^{\frac{1}{k'}})$ pairs of numbers $(x, y)$. \hfill $\square$
4 Proof of Theorem 4

We split the proof up into several cases, each of which gives us one of the following propositions:

**Proposition 10.** For all \( k \geq 2 \), there exists \( n_0 = n_0(k, s) \) such that for all \( n \geq n_0 \), if a set \( A \) of positive rational numbers with \( |A| = n \) and \( \sum A = 1 \) satisfies \( d_k^n(A) \geq (\frac{n-1}{k}) \), then \( A \) is a \((k,s)\)-anti-pencil.

**Proposition 11.** There exists \( k_1 = k_1(s) \) such that for every pair \((k,n)\) with \( k \geq k_1 \) and \( n \geq 3k/2 \), if a set \( A \) of positive rational numbers with \( |A| = n \) and \( \sum A = 1 \) satisfies \( d_k^n(A) \geq (\frac{n-1}{k}) \), then \( A \) is a \((k,s)\)-anti-pencil.

**Proposition 12.** There exists \( k_2 = k_2(s) \) such that for every pair \((k,n)\) with \( k \geq k_2 \) and \( 2n/3 < k < n - (6s^2+3s)^2 \), if a set \( A \) of positive rational numbers with \( |A| = n \) and \( \sum A = 1 \) satisfies \( d_k^n(A) \geq (\frac{n-1}{k}) \), then \( A \) is a \((k,s)\)-anti-pencil.

**Proposition 13.** There exists \( k_3 = k_3(s) \) such that for every pair \((k,n)\) with \( k \geq k_3 \) and \( n - (6s^2+3s)^2 \leq k < n-1 \), if a set \( A \) of positive rational numbers with \( |A| = n \) and \( \sum A = 1 \) satisfies \( d_k^n(A) \geq (\frac{n-1}{k}) \), then \( A \) is a \((k,s)\)-anti-pencil.

Assume the above four propositions. Let \( K = \max(k_1, k_2, k_3) \) and let \( N = \max_{2 \leq k \leq K} n_0(k, s) \). For any pair of positive integers \((k,n)\) with \( 1 < k < n \), if \( k \geq K \), then Theorem 4 holds for the pair \((k,n)\) by Proposition 11, 12, or 13. If \( k < K \) and \( n \geq N \), then Theorem 4 holds for the pair \((k,n)\) by Proposition 10. Since there are only finitely many pairs \((k,n)\) with \( k < K \) and \( n < N \), this proves Theorem 4.

For all 4 cases, we will take \( A \) with \( |A| = n \) and \( d_k^n(A) \geq (\frac{n-1}{k}) \) and assume that \( A \) is not a \((k,s)\)-anti-pencil. Note that then some \( B \ni a_n \) has \( \sum B \leq \frac{s}{s+1} \), so since \( 1 < k \), we have \( a_n < \frac{s}{s+1} \). We will use this in all the cases below. Also, the number of \( k \)-subsets of \( A \) that are not \( s \)-divisors is at most \( \left( \frac{n}{k}\right) - \left( \frac{n-k}{k-1}\right) = \left( \frac{n-1}{k-1}\right) \).

**Remark 14.** If \( B \) and \( C \) are \( k \)-subsets of \( A \) with \( B < C \), then \( \sum B < \sum C \). Note that if \( B_0 < B_1 < \cdots < B_m \) are all divisors of \( A \) and \( \sum B_m < s/q \), then \( \sum B_0 < s/(q+m) \). Therefore if \( a \in B_0 \), then \( a < s/(q+m) \). Since \( k < n \), \( \sum B_m < s/s \), so we automatically get that \( a < s/(s+m) \).

4.1 Proof of Proposition 10 \((k \text{ small})\)

Fix \( k \geq 2 \) and let \( n \gg k \), and assume that \( A \) is not a \((k,s)\)-anti-pencil.

For \( i_1, \ldots, i_k \leq n \), call the ordered \( k \)-tuple \((i_1, \ldots, i_k)\) **repetitive** if not all entries are distinct. Call it **good** if all entries are distinct and \( \{a_{i_j}\} \) is an \( s \)-divisor. Otherwise, call the ordered \( k \)-tuple **bad**.

In order to bound \( a_{n-1} \), we will first restrict our attention to \( k \)-tuples where \( i_k \geq n-1 \). Among these, \( O(n^{k-2}) \) are repetitive. Also, \( O(n^{k-2}) \) include both \( n \) and \( n-1 \) among their components. Of the remainder, at most \( (k-1)\frac{n-1}{k-1} \leq n^{k-1} \) are bad. Thus at least \( 1/3 \) of the \( k \)-tuples \((i_1, \ldots, i_k)\) satisfying \( i_k \geq n-1 \) are good.
By the Pigeonhole Principle, there are some values \( j_2, \ldots, j_k \) with \( j_k \geq n - 1 \) such that the chain \( \{(1, j_2, \ldots, j_k), \ldots, (n, j_2, \ldots, j_k)\} \) has at least \( n/3 \) good \( k \)-tuples. This gives us a chain of \( k \)-subset \( s \)-divisors of length at least \( n/3 \). Thus \( a_{n-1} \leq \frac{3s}{n} \).

We now consider all \( k \)-tuples again. Let \( B = \{|a_i| > (1 - \frac{1}{9s^2}) n\} \). If \( a_i \in B \), then

\[
1 = \sum A = \sum_{j=1}^{n} a_j \leq \sum_{j=1}^{i} a_i + \sum_{j=i}^{n-1} a_{n-1} + a_n < na_i + \frac{n}{9s^2} a_{n-1} + a_n < na_i + \frac{1}{3s} + \frac{s}{s+1}
\]

so \( na_i > \frac{1}{6s} \) and \( a_i > \frac{1}{6sn} \).

Thus an \( s \)-divisor that is a subset of \( B \) must sum to some \( \frac{a}{m} > \frac{1}{6sn} \), so there are at most \( 6s^2n \) distinct values that \( m \) can take. Thus there are at most \( 6s^2n \) distinct values that an \( s \)-divisor that is a subset of \( B \) can sum to.

If \( D \in \binom{B}{k} \) and \( r = \frac{s}{m} \) for some positive integer \( m \), call \( D \) an \( r \)-stem if there are at least \( \frac{1}{10000s^6n} \) pairs \( \{x, y\} \subset B \setminus D \) with \( \sum (D \cup \{x, y\}) = r \). Call such pairs tails of \( D \). If two tails of \( D \) are \( \{x, y\} \) and \( \{x, z\} \), then the sum condition gives us \( y = z \), so tails of \( D \) are pairwise disjoint.

We want to find a large number of disjoint stems in \( B \). Specifically, for \( 1 \leq i \leq \frac{1}{20ks^2}n \), we will attempt to choose \( D_i \subset B \setminus \bigcup_{j=1}^{i-1} D_j \) to be an \( r_i \)-stem for some \( r_i = \frac{s}{m_i} \).

If we let \( B_i = B \setminus \bigcup_{j=1}^{i-1} D_j \), then \( |B_i| \leq |B|/2 \). Also, \( B_i \) has at most \( \binom{n-1}{k-1} \) \( k \)-subsets which are not \( s \)-divisors of \( A \), so it has at least \( \frac{1}{2} \binom{|B_i|}{k} \) \( k \)-subsets that are \( s \)-divisors. Since these take on at most \( 6s^2n \) values, there must be some positive integer \( m_i \) such that at least \( \frac{1}{12s^2n} \binom{|B_i|}{k} \) \( k \)-subsets of \( B_i \) sum to \( r_i = \frac{s}{m_i} \). If we randomly choose \( D_i \in \binom{B_i}{k-2} \), the expected value for the number of pairs \( \{x, y\} \subset B_i \setminus D_i \) with \( \sum (D_i \cup \{x, y\}) = r_i \) is at least \( \frac{1}{12s^2n} \binom{|B_i|}{(k-2)} \).

Since \( i \leq \frac{1}{20ks^2}n \), we have

\[
\frac{1}{12s^2n} \binom{|B_i|}{(k-2)} \geq \frac{1}{25s^2n} \binom{|B_i|}{k} > \frac{1}{25s^2n(20s)^2} n^2 \geq \frac{1}{10000s^6n}.
\]

By choosing some \( D_i \) that has at least the expected number of tails, we get that \( D_i \) satisfies the definition of an \( r_i \)-stem. Thus we can inductively construct a large family of disjoint \( r_i \)-stems.

Since the number of \( k \)-subsets of \( A \) which are not \( s \)-divisors is less than \( \binom{1}{20ks^2}n \), we know that there must exist disjoint \( D_{i_1}, \ldots, D_{i_k} \) such that every set consisting of one element of each \( D_{i_j} \) is an \( s \)-divisor. Note that in the \( k = 2 \) case, \( D_{i_1} = D_{i_2} = \emptyset \). Partition \( \bigcup_{j=1}^{k} D_{i_j} \) into \( k - 2 \) such sets \( C_1, \ldots, C_{k-2} \).

Let \( p = \lceil \frac{1}{10000s^6n/(2k)} \rceil = \lceil \frac{1}{20000s^6k}n \rceil \). For \( 1 \leq j \leq k \), we want to choose \( T_{i_1}^j, \ldots, T_{i_j}^j \) to be tails of \( D_{i_j} \). We will choose them for \( j = 1 \), then for \( j = 2 \), and so on. When we choose \( \{T_{i_j}^j\} \), we will make each of these tails disjoint from each of the \( k \) stems, as well as from the already chosen tails. This is possible since

\[
\Big| \bigcup_{h=1}^{k} D_{i_h} \cup \bigcup_{h=1}^{p} T_{i_1}^h \Big| = \Big| \bigcup_{h=1}^{k} D_{i_h} \Big| + \sum_{h=1}^{p} \Big| \bigcup_{\ell=1}^{k} T_{i_1}^\ell \Big| = k(k-2) + 2(j-1)p \leq k(k-2) + 2(k-1)p.
\]
Since every element of a stem or of a previously chosen tail can be in at most one tail of $D_{ij}$, at most $k(k - 2) + 2(k - 1)p$ tails are eliminated, so there must be at least $p$ tails still available to choose from.

We say that a choice of $k$ tails $\{T_{ij}^k\}_{j=1}^k$ for each stem is fortuitous if $\{x_{ij}^k\}_{j=1}^k$ and $\{y_{ij}^k\}_{j=1}^k$ are both $s$-divisors. There are $p^k > n^k/(20000s^6k)^k$ choices of tails, and at most $(\binom{n-1}{k-1})$ of them are not fortuitous. Thus at least $\frac{1}{2}$ of possible choices are fortuitous.

By the Pigeonhole Principle, we can choose $i_1, \ldots, i_{k-1}$ so that there are at least $p/2$ choices for $i$ which make $\{T_{1i}^k, \ldots, T_{k-1i}^k, T_{ki}^k\}$ fortuitous.

Note that different choices of $i$ give us different values of $x_i^k$ and therefore different values of $\sum_{j=1}^k x_{ij}^j$, so $\sum_{j=1}^k x_{ij}^j$ takes on at least $p/2 = \Omega(n)$ different values.

On the other hand, if we are given a fortuitous choice of tails $\{T_{ij}^k\}$, then

$$\sum_{j=1}^{k-2} C_j + \sum_{j=1}^k x_{ij}^j + \sum_{j=1}^k y_{ij}^j = \sum_{j=1}^k \sum_{j=1}^k \left(D_{ij} \cup \{x_{ij}^j, y_{ij}^j\}\right) - \sum_{j=1}^k \sum_{j=1}^{k-2} C_j.$$

The right hand side does not depend on our choice of tails. Also, since each $r_{ij}$ and each $\sum C_j$ has denominator at most $6s^2n$, the right hand side has denominator at most $(6s^2n)^{2k}$. Since both $\sum_{j=1}^k x_{ij}^j$ and $\sum_{j=1}^k y_{ij}^j$ are $s$-divisors, there are at most $s^2$ possibilities for their numerators. For each such possibility, by Lemma 9, $\sum_{j=1}^k x_{ij}^j$ can take on at most

$$O \left(\left(6s^2n\right)^{2k} \right)^\frac{1}{14} = O \left(sn^{\frac{1}{2}}\right)$$

different values. Thus $\sum_{j=1}^k x_{ij}^j$ can take on at most

$$O \left(s^3n^{\frac{1}{2}}\right)$$

different values, contradicting the lower bound above. Thus Proposition 10 holds.

4.2 Proof of Proposition 11 ($n \geq \frac{3}{2}k$, $k$ sufficiently large)

Let $d = [(s(s+1)/0.03)^2]$. Assume that $k$ is sufficiently large relative to $d$ and that $n \geq \frac{3}{2}k$, and assume that $A$ is not a $(k,s)$-antipencil.

Let $T_2$ be the set of $k$-subsets of $A$ that include both $a_{n-1}$ and $a_n$. Let $T_1$ be the set of $k$-subsets of $A$ that include one of $a_{n-1}$ or $a_n$, but not both. Define $U_1$ and $U_2$ similarly, but with $(k-d)$-subsets.
For $S \in U_t$, let $P_S = \{B \in T_t \mid S \subset B\}$ (the set of $k$-subsets obtainable by adding $d$ elements of $A$ less than $a_{n-1}$ to $S$). Note that an element of $T_t$ is contained in $P_S$ for exactly $(k-t)$ values of $S$. Thus if $\alpha|T_t|$ elements of $T_t$ are $s$-divisors, then there is some $S \in U_t$ so that at least $\alpha|P_S|$ elements of $P_S$ are $s$-divisors.

Now note that the disjoint union $T_1 \cup T_2$ is the set of all $k$-subsets whose greatest element is at least $a_{n-1}$, so

$$|T_1 \cup T_2| = \binom{n}{k} - \binom{n-2}{k}$$

and the fraction of the elements of $T_1 \cup T_2$ which are not $s$-divisors is at most

$$\frac{(n) - \binom{n-1}{k}}{(n) - \binom{n-2}{k}} = \frac{n-1}{2n-k-1} \leq 0.76$$

for sufficiently large $k$. Therefore, if we set $\alpha = 0.24$, then for $t = 1$ or $t = 2$, the fraction of elements of $T_t$ that are $s$-divisors is at least $\alpha$, so for some $S$, the fraction of elements of $P_S$ which are $s$-divisors is at least $\alpha = 0.24$.

Note that the partial order of $P_S$ is the same as the partial order of $(A\setminus S \setminus \{a_{n-1}, a_n\})$, so by Lemma 7, its width is at most $\frac{2}{\sqrt{d \frac{n-k-2}{n-k-2}}} |P_S|$. Then, by Dilworth’s theorem (Theorem 4.0.1 in [4]), there is a chain of $k$-subset $s$-divisors in $P_S$ of length at least

$$\frac{\alpha |P_S|}{\frac{2}{\sqrt{d \frac{n-k-2}{n-k-2}} |P_S|}} = 0.12\sqrt{d}(n-k-2) \geq (0.03\sqrt{d})n.$$ 

But the first element of the chain includes $a_{n-1}$ or $a_n$, so by Remark 14, $a_{n-1} \leq \frac{s}{(0.03\sqrt{d})n}$. Then

$$\sum_{i=1}^{n-1} a_i \leq \frac{s}{0.03\sqrt{d}}$$

and, since $a_n < \frac{s}{s+1}$,

$$\sum_{i=1}^{n} a_i < 1$$

yielding a contradiction. Thus Proposition 11 holds.

### 4.3 Proof of Proposition 12 ($\frac{2}{3}n < k < n - (6s^2 + 3s)^2$, $k$ sufficiently large)

Let $d = (6s^2 + 3s)^2$. Assume that $k$ is sufficiently large and that $\frac{2}{3}n < k < n - d$, and assume that $A$ is not a $(k, s)$-antipencil.

Randomly arrange the elements of $A$ around a circle. Let $M$ be the set of $k$-subsets of $A$ consisting of $k$ consecutive elements around the circle, and let $N = \{B \in M \mid \sum B \leq$
If $B, C \in N$ and they are shifted relative to each other by at least $n - k - 1$, then $|A \setminus (B \cup C)| \leq 1$, so

$$\sum A \leq \sum (A \setminus (B \cup C)) + \sum B + \sum C < \frac{s}{s+1} + \frac{1}{2(s+1)} + \frac{1}{2(s+1)} = 1,$$

which is impossible.

Thus every pair of elements of $N$ are shifted by at most $n - k - 2$ around the circle. This gives us $|N| \leq n - k - 1$. Since each $k$-subset of $A$ summing to at most $\frac{1}{2(s+1)}$ has equal probability of being in $N$, this tells us that the number of $k$-subsets with sum at most $\frac{1}{2(s+1)}$ is at most $\frac{n-k-1}{n} \binom{n}{k}$. Thus there are at least

$$\left( \frac{n-1}{k} \right) - \frac{n-k-1}{n} \binom{n}{k} = \frac{n-k}{n} \binom{n}{k} - \frac{n-k-1}{n} \binom{n}{k} = \frac{1}{n} \binom{n}{k}$$

$k$-subsets which are $s$-divisors of $A$ and have a sum of elements greater than $\frac{1}{2(s+1)}$. The sum of elements of such a set is $\frac{s}{m} > \frac{1}{2(s+1)}$, so it can take on one of $2s(s+1) - s = 2s^2 + s$ values, so there must be some integer $m$ so that at least $\frac{1}{(2s^2+s)n} \binom{n}{k}$ of the $k$-subsets of $A$ sum to $\frac{s}{m}$. Thus at least $\frac{1}{(2s^2+s)n} \binom{n}{n-k}$ of the $(n-k)$-subsets of $A$ sum to $1 - \frac{s}{m}$.

If $S \in \binom{A}{n-k}$, let $P_S$ be the set of $(n-k)$-subsets obtainable by adding $d$ elements of $A$ to $S$. Note that every $(n-k)$-subset of $A$ is contained in $P_S$ for exactly $\binom{n-k}{d}$ values of $S$, so there is some $S$ so that at least

$$\frac{1}{(2s^2+s)n} |P_S|$$

elements of $P_S$ sum to $1 - \frac{s}{m}$. They must then form an antichain.

However, the partial order of $P_S$ is the same as the partial order of $\binom{A \setminus S}{d}$, so by Lemma 7, its largest antichain has size less than

$$\frac{2}{\sqrt{d}} \frac{1}{k+d} |P_S| < \frac{3}{n\sqrt{d}} |P_S| = \frac{1}{(2s^2+s)n} |P_S|,$$

yielding a contradiction. Thus Proposition 12 holds.

### 4.4 Proof of Proposition 13 ($n - (6s^2 + 3s)^2 \leq k < n - 1$, $k$ sufficiently large)

Assume that $k$ is sufficiently large and $n - (6s^2 + 3s)^2 \leq k < n - 1$, and assume that $A$ is not a $(k,s)$-antipencil. Let $u = n - k$. Thus $1 < u \leq (6s^2 + 3s)^2$, so $u$ can take on only finitely many values. Assume that $k$ is sufficiently large relative to those values. Let

$$Y = \left\{ B \in \binom{A}{u} \mid A \setminus B \text{ is an } s\text{-divisor of } A \right\}.$$
By assumption, $|\mathcal{Y}| \geq \binom{n-1}{k} = \binom{n-1}{u-1}$.

Let $q$ be as small as possible so that $a_{n-q} < \frac{1}{u(s+1)}$. Note that $q < u(s+1)$. If $B \in \binom{A}{u}$ and $b \leq a_{n-q}$ for all $b \in B$, then $\sum B < \frac{1}{s+1}$, so $\sum(A \setminus B) > \frac{s}{s+1}$ and $B \notin \mathcal{Y}$. Thus every $B \in \mathcal{Y}$ contains at least one of the $q$ greatest elements of $A$.

The number of $u$-subsets of $A$ containing at least 2 of the $q$ greatest elements of $A$ is bounded by

$$2^q \binom{n-q}{u-2} < 2^u(s+1) \binom{n-1}{u-2} < \frac{1}{2} \cdot |\mathcal{Y}|,$$

so at least half of the elements of $\mathcal{Y}$ contain exactly one of the $q$ greatest elements of $A$.

Thus there must be some $a_i$ which is one of the $q$ greatest elements of $A$ such that at least $\frac{1}{2u(s+1)} \binom{n-1}{u-1}$ elements of $\mathcal{Y}$ include $a_i$ and no other of the $q$ largest elements.

If $B$ is such an element of $\mathcal{Y}$, then

$$\sum B < a_i + (u-1) \frac{1}{u(s+1)} < \frac{s}{s+1} + \frac{u-1}{u(s+1)} = 1 - \frac{1}{u(s+1)}.$$

Since $\sum B$ must be of the form $1 - \frac{s}{m}$ for some positive integer $m$, we get fewer than $s(s+1)u$ possible values of $m$. Thus there must be some value of $m$ so that there are at least

$$\frac{1}{2u^2 s(s+1)^2} \binom{n-1}{u-1}$$

different $u$-subsets of $A$ which include $a_i$ and sum to $1 - \frac{s}{m}$. However, if we have a collection of that many $u$-subsets of $A$ that contain $a_i$, then some 2 of them will have $u-1$ elements in common and thus will have different sum. This gives us a contradiction. Thus Proposition 13 holds.

5 Conclusion

In the statement of Theorem 3 and Theorem 4, “all but finitely many” cannot be omitted. For example, Huynh [6] notes that $n = 4, k = 2, A = \{1/24, 5/24, 7/24, 11/24\}$ gives $d_k(A) = 4 > \binom{n-1}{k}$. As $s$ increases, the number of such exceptions grows; in fact, it is easy to see that any $n, k, A$, will be an exception for sufficiently large $s$.

We could follow the proof and trace out the upper bounds on $n$ such that $(k, n)$ is an exception; however these will probably be far from optimal (for instance, for $s = 1$, $(2, 4)$ is likely the only exception). It would be interesting to get a good bound on the number of such exceptions, or on how large $n$ can be in terms of $s$ for $(k, n)$ to be an exception.

In this paper, we are counting $B \in \binom{A}{k}$ such that $\sum B = \frac{s}{m}$. If we instead counted $B$ such that $\sum B < \frac{k}{n}$, this problem becomes equivalent to the Manickam-Miklós-Singhi conjecture:

**Conjecture 15.** For positive integers $n, k$ with $n \geq 4k$, every set of $n$ real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ $k$-element subsets whose sum is also nonnegative.
The equivalence is given by taking the complement of $B$ and applying a linear transformation.

The MMS conjecture has been proven for $k \mid n$ [7], $n \geq 10^{46}k$ [8], and $n \geq 8k^2$ [3], and in general by Blinovskyy [2]. However, there are pairs $(n,k)$ with $n < 4k$ such that the bound of $\binom{n-1}{k-1}$ in the conjecture does not hold. This suggests a more general problem.

Problem 16. Fix $S \subseteq [0,1]$ and positive integers $n$ and $k$. If $A$ is a set of positive reals, let $d_k(S,A)$ be the number of subsets $B \in \binom{A}{k}$ such that $\sum B \in S$. Let $d(S,k,n)$ be the maximal value of $d(S,k,n)$ over all $A$ with $|A| = n$ and $\sum A = 1$. For what $S,k,n$ do we get $d(S,k,n) = \binom{n-1}{k}$? Furthermore, when does $d_k(S,A) \geq \binom{n-1}{k}$ imply that $A$ is an $k$-anti-pencil?

This paper addresses this problem for $S = \{\frac{m}{n} \mid m \in \mathbb{Z}^+\}$, while the MMS conjecture deals with this problem for $S = (0,k/n)$. Another example of a set for which this problem might be interesting is a set of the form $S = (0,\alpha k/n) \cup \{\frac{m}{n} \mid m \in \mathbb{Z}^+\}$, which combines the theorem of this paper with the MMS conjecture.

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References

[1] K. Ball. Cube slicing in $\mathbb{R}^n$. Proc. Amer. Math. Soc., 97(3):465–472, 1986.

[2] V. Blinovskyy. Minimal number of edges in hypergraph guaranteeing perfect fractional matching and MMS conjecture. Prob. Inf. Trans., 50(4):340–349, 2015.

[3] A. Chowdhury, G. Sarkis, and S. Shahriari. A new quadratic bound for the Manickam-Miklós-Singhi conjecture. 2014. preprint, http://arxiv.org/abs/1403.1844.

[4] K. Engel. Sperner Theory, volume 65 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1997.

[5] G.H. Hardy and E.M. Wright. An Introduction to the Theory of Numbers. Clarendon Press, Oxford, 5th edition, 1979.

[6] T. Huynh. Extremal problems for subset divisors. Elect. J. Combin., 21:P1.42, 2014.
[7] N. Manickam and D. Miklós. On the number of nonnegative partial sums of a nonnegative sum. In Combinatorics (Eger, 1987), Colloq. Math. Soc. János Bolyai 52 (North-Holland, 1988), 385-392.

[8] A. Pokrovskiy. A linear bound on the Manickam-Miklos-Singhi conjecture. *J. Combin. Theory Ser. A*, 133:280–306, 2015.

[9] R. A. Proctor, M. Saks, and D. Sturtevant. Product partial orders with the Sperner property. *Discrete Math.*, 30:173–180, 1980.