Universal relations for spin-orbit-coupled Fermi gases in two and three dimensions

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We present a comprehensive derivation of a set of universal relations for spin-orbit-coupled Fermi gases in three or two dimension, which follow from the short-range behavior of the two-body physics. Besides the adiabatic energy relations, the large-momentum distribution, the grand canonical potential and pressure relation derived in our previous work for three-dimensional systems [Phys. Rev. Lett. 120, 060408 (2018)], we further derive high-frequency tail of the radio-frequency spectroscopy and the short-range behavior of the pair correlation function. In addition, we also extend the derivation to two-dimensional systems with Rashba type of spin-orbit coupling. To simply demonstrate how the spin-orbit-coupling effect modifies the two-body short-range behavior, we solve the two-body problem in the sub-Hilbert space of zero center-of-mass momentum and zero total angular momentum, and perturbatively take the spin-orbit-coupling effect into account at short distance, since the strength of the spin-orbit coupling should be much smaller than the corresponding scale of the finite range of interatomic interactions. The universal asymptotic forms of the two-body wave function at short distance are then derived, which do not depend on the short-range details of interatomic potentials. We find that new scattering parameters need to be introduced because of spin-orbit coupling, besides the traditional s- and p-wave scattering length (volume) and effective ranges. This is a general and unique feature for spin-orbit-coupled systems. We show how these two-body parameters characterize the universal relations in the presence of spin-orbit coupling. This work probably shed light for understanding the profound properties of the many-body quantum systems in the presence of the spin-orbit coupling.

I. INTRODUCTION

Understanding strongly-interacting many-body systems is one of the most daunting challenges in modern physics. Owing to the development of the experimental technique, ultracold atomic gases acquire a high degree of control and tunability in interatomic interaction, geometry, purity, atomic species, and lattice constant (of optical lattices) [1–5]. To date, ultracold quantum gases have emerged as a versatile platform for exploring a broad variety of many-body phenomena as well as offering numerous examples of interesting many-body states [6–8]. Unlike conventional electric gases in condensed matters, atomic quantum gases are extremely dilute, and the mean distance between atoms is usually very large (on the order of $\mu m$), while the range of interatomic interactions is several orders smaller (on the order of several tens of nm). Therefore, the two-body correlations characterize the key properties of such many-body systems near scattering resonances, where the two-body interactions are simply described by the scattering length and become irrelevant to the specific form of interatomic potentials.

A set of universal relations, following from the short-range behavior of the two-body physics, govern some crucial features of ultracold atomic gases, and provide powerful constraints on the behavior of the system. Many of these relations were first derived by Shina Tan, such as the adiabatic energy relation, energy theorem, general virial theorem and pressure relation [9–11]. Afterwards, more universal behaviors were obtained by others, such as the radio-frequency (rf) spectroscopy, photoassociation, static structure factors and so on [12]. All these relations are characterized by the only universal quantity named contact, and therefore known as the contact theory. During past few years, the concept of contact theory was further generalized to higher-partial-wave interactions [13–20] as well as to low dimensions [21–20], and more contacts appear when additional two-body parameters are involved.

The reason why the contact theory is significantly important in ultracold atoms is attributed to its direct connection to the experimental measurements. Some of the universal relations were experimentally confirmed, involving various measurements of the contact itself. For two-component Fermi gases with s-wave interactions, D. S. Jin’s group measured the contact according to three different methods, i.e., the momentum distribution, photoemission spectroscopy, and rf spectroscopy, and tested the adiabatic energy relation when the interatomic in-
teraction was adiabatically swept \[30\]. The asymptotic behavior of the static structure factor at large momentum was confirmed by C. J. Vale’s group, by using Bragg spectroscopy technique \[31, 32\]. Recently, the temperature evolution of the contact was resolved independently by M. Zwierlein’s group and C. J. Vale’s group, especially the characteristic behavior of the contact across the superfluid transition \[33, 34\]. For single-component Fermi gases with \(p\)-wave interactions, the feasibility of generalizing the contact theory for higher-partial-wave scatterings was confirmed experimentally by Thywissen’s group \[35\], in which the anisotropic \(p\)-wave interaction was tuned according to the magnetic vector \[36\]. Nowadays, the contact gradually becomes one of fundamental concepts in ultracold atomic physics both theoretically and experimentally.

In the past decade, the realizations of the spin-orbit (SO) coupling in ultracold neutral atoms have sparked a great deal of interest \[37–44\]. It provides an ideal platform on which to study novel quantum phenomena resulted from SO coupling in a highly controllable and tunable way, such as topological insulators and superconductors \[6, 7\], and (spin) Hall effect \[45–47\]. Nevertheless, it is still challenging to theoretically deal with the many-body correlations for SO-coupled systems. Unlike the situation in condensed matters, the intrinsic short-range feature of interatomic potentials is unchanged for neutral atoms even in the presence of SO coupling. The natural question may be raised, from the point of view of the contact theory, as to whether the two-body physics could provide crucial constraints on many-body behaviors of SO-coupled atomic systems. In addition, it was pointed out that although the short-range feature remains, the SO-coupling effect does modify the short-range behavior of the two-body wave function \[48\]. Therefore, the existence and exact forms of universal relations for SO-coupled atomic systems attract a great deal of attention. In \[49\], we preliminarily discussed some of the universal relations for three-dimensional (3D) Fermi gases in the presence of 3D isotropic SO coupling. We proposed a simple way to construct the short-range wave function, in which the SO coupling effect could be taken into account perturbatively. Since SO-coupling in general couples different partial waves of the two-body scatterings, additional contact parameters appear in universal relations. Before long, our theory was verified by different groups near \(s\)-wave resonances \[50, 51\].

So far, the generalization of the contact theory in the presence of SO-coupling is mostly discussed in 3D, while the derivation of these universal relations is still elusive in two-dimensional (2D) systems. The short-range behavior of the two-body physics in 2D is different from that in 3D: the two-body wave function in 3D is power-law divergent, while one has to deal with the logarithmic divergence in 2D. From the point of view of the contact theory, different short-range correlations in two-body physics result in different forms of universal relations. Therefore, it requires a direct extension to 2D in the similar manner as in 3D in the presence of SO coupling.

The purpose of this article is to present a comprehensive derivation of universal relations for SO-coupled Fermi gases. Besides the adiabatic energy relations, the large-momentum distribution, the grand canonical potential and pressure relation derived in our previous work for 3D systems \[49\], we further derive high-frequency tail of the rf spectroscopy and the short-range behavior of the pair correlation function. Then we generalize the derivation of universal relations for 3D systems to 2D case with Rashba SO coupling in a similar way. For the convenience of the presentation, we still construct the short-range behavior of the two-body wave function in the sub-Hilbert space of zero center-of-mass (c.m.) momentum and zero total angular momentum as before, and then only \(s\) and \(p\)-wave scatterings are coupled \[49, 52, 53\]. Our results show that the SO coupling introduces a new contact and modifies the universal relations of many-body systems.

The remainder of this paper is organized as follows. In the next section, we present the derivations of the short-range behavior of two-body wave functions for SO-coupled Fermi gases in three and two dimensions, respectively. Subsequently, with the short-range behavior of the two-body wave functions in hands, we derive a set of universal relations for a 3D SO-coupled Fermi gases in Sec. III, and then generalize them to 2D SO-coupled Fermi gases in Sec. IV, including adiabatic energy relations, asymptotic behavior of the large-momentum distribution, the high-frequency behavior of the rf response, short-range behavior of the pair correlation function, grand canonical potential and pressure relation. Finally, the main results are summarized in Sec. V.

## II. Universal Short-Range Behavior of Two-Body Wave Functions

The ultracold atomic gases are dilute, while the range of interatomic potentials is extremely small. When two fermions get close enough to interact with each other, they usually far away from the others. If only these two-body correlations are taken into account, some key properties of many-body systems are characterized by the short-range two-body physics, which is the basic idea of the contact theory. In this section, we are going to discuss the short-range behavior of two-body wave functions for 3D Fermi gases in the presence of 3D SO coupling and 2D Fermi gases in the presence of 2D SO coupling, respectively. Let us consider spin-half SO-coupled Fermi gases, and the Hamiltonian of a single fermion is modeled as

\[
\hat{H}_1 = \frac{\hbar^2 \k_1^2}{2M} + \frac{\hbar^2 \lambda}{M} \chi + \frac{\hbar^2 \lambda^2}{2M},
\]

where \(\k_1 = -i \nabla\) is the single-particle momentum operator, \(M\) is the atomic mass, \(\hbar\) is the Planck’s constant divided by \(2\pi\). Here, the SO coupling is described by the term \(\hbar^2 \lambda \chi / M\) with the strength \(\lambda > 0\), and \(\chi\) takes
the isotropic form of $\hat{k}_1 \cdot \sigma$ in 3D or the Rashba form of $\sigma \times \hat{n}$ in 2D \cite{[54]}. where $\sigma$ is the Pauli operator, and $\hat{n}$ is the unit vector perpendicular to the $(x - y)$ plane.

Because of SO coupling, the orbital angular momentum of the relative motion of two fermions is no longer conserved, and then all the partial-wave scatterings are coupled \cite{[52]}. Fortunately, the c.m. momentum $\mathbf{K}$ of two fermions is still conserved as well as the total angular momentum $\mathbf{J}$. For simplicity, we may reasonably focus on the two-body problem in the subspace of $\mathbf{K} = 0$ and $\mathbf{J} = 0$, and then only $s$- and $p$-wave scatterings are involved \cite{[52],[53]}. Consequently, the Hamiltonian of two spin-half fermions can be written as

$$
\hat{H}_2 = \frac{\hbar^2 \hat{k}^2}{M} + \frac{\hbar^2 \lambda}{M} \hat{Q}(\mathbf{r}) + \frac{\hbar^2 \lambda^2}{M} + V(\mathbf{r}),
$$

where $\hat{k} = (\hat{k}_2 - \hat{k}_1)/2$ is the momentum operator for the relative motion $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$, $V(\mathbf{r})$ is the short-range interatomic interaction with a finite range $\epsilon$, $\hat{Q}(\mathbf{r}) = (\hat{\sigma}_2 - \hat{\sigma}_1) \cdot \hat{n}$ in 3D or $\hat{Q}(\mathbf{r}) = (\hat{\sigma}_2 - \hat{\sigma}_1) \times \hat{n}$ in 2D, and $\hat{\sigma}_i$ is the spin operator of the $i$th atom. In the followings, let us consider the two-body problems in the 3D systems with 3D SO coupling and 2D systems with 2D SO coupling, respectively.

### A. For 3D systems with 3D SO coupling

In the subspace of $\mathbf{K} = 0$ and $\mathbf{J} = 0$, we may choose the common eigenstates of the total Hamiltonian $\hat{H}_2$ and total angular momentum $\mathbf{J}(= 0)$ as the basis of Hilbert space, which take the forms of

$$
\Omega_0 (\hat{\mathbf{r}}) = Y_{00} (\hat{\mathbf{r}}) |S\rangle,
$$

$$
\Omega_1 (\hat{\mathbf{r}}) = -\frac{i}{\sqrt{3}} [Y_{11} (\hat{\mathbf{r}}) |\uparrow\uparrow\rangle
+ Y_{1-1} (\hat{\mathbf{r}}) |\downarrow\downarrow\rangle
- Y_1 (\hat{\mathbf{r}}) |T\rangle],
$$

where $Y_{lm} (\hat{\mathbf{r}})$ is the spherical harmonics, $\hat{\mathbf{r}} \equiv (\theta, \varphi)$ denotes the angular degree of freedom of the coordinate $\mathbf{r}$, and $|S\rangle = (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ and $\{ |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, |T\rangle = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \}$ are the spin-singlet and spin-triplet states with total spin $S = 0$ and 1, respectively. Then the two-body wave function can formally be written in the basis of $\{ \Omega_0 (\hat{\mathbf{r}}), \Omega_1 (\hat{\mathbf{r}}) \}$ as

$$
\Psi (\mathbf{r}) = \psi_0 (\mathbf{r}) \Omega_0 (\hat{\mathbf{r}}) + \psi_1 (\mathbf{r}) \Omega_1 (\hat{\mathbf{r}}),
$$

where $\psi_i (\mathbf{r})$ ($i = 0, 1$) is the radial part of the wave function. Note that here we consider an isotropic $p$-wave interaction and the radial wave function is identical for three scattering channels, i.e., $m = 0, \pm 1$.

Typically, the SO coupling strength (of the order $\mu \mathrm{m}^{-1}$) is pretty small compared to the inverse of the interaction range (of the order nm$^{-1}$) \cite{[38],[39]}, i.e., $\lambda \ll \epsilon^{-1}$. Moreover, in the low-energy scattering limit, the relative momentum $k = \sqrt{ME}/\hbar^2$ is also much smaller than $\epsilon^{-1}$. Thus, when two fermions get as close as the range of the interaction, i.e., $r \sim \epsilon$, the SO coupling can be treated as perturbation as well as the energy. We assume that the two-body wave function may take the form of the following ansatz \cite{[49]}

$$
\Psi (\mathbf{r}) \approx \phi (\mathbf{r}) + k^2 f (\mathbf{r}) - \lambda g (\mathbf{r}),
$$

as the distance of two fermions approaches $\epsilon$. Here, we keep up to the first-order terms of the energy ($k^2$) and SO coupling strength ($\lambda$). The advantage of this ansatz is that the functions $\phi (\mathbf{r}), f (\mathbf{r})$, and $g (\mathbf{r})$ are all independent on $k^2$ and $\lambda$. These functions are determined only by the short-range details of the interaction, and thus characterize the intrinsic properties of the interatomic potential. We expect that in the absence of SO coupling the conventional scattering length or volume is included in the zero-order term $\phi (\mathbf{r})$, while the effective range is included in $f (\mathbf{r})$, the coefficient of the first-order term of $k^2$. Interestingly, we may anticipate that new scattering parameters resulted from SO coupling appear in the first-order term of $\lambda \ln g (\mathbf{r})$. Conveniently, more scattering parameters may be introduced if higher-order terms of $k^2$ and $\lambda$ are perturbatively considered.

Inserting the ansatz (6) into the Schrödinger equation $\hat{H}_2 \Psi (\mathbf{r}) = E \Psi (\mathbf{r})$, and comparing the corresponding coefficients of $k^2$ and $\lambda$, we find

$$
\begin{cases}
- \nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2} \phi (\mathbf{r}) = 0, \\
- \nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2} \phi (\mathbf{r}) = \phi (\mathbf{r}), \\
- \nabla^2 + \frac{MV(\mathbf{r})}{\hbar^2} g (\mathbf{r}) = \hat{Q}(\mathbf{r}) \phi (\mathbf{r}).
\end{cases}
$$

These coupled equations can easily be solved for $r > \epsilon$, and we obtain

$$
\begin{align*}
\phi (\mathbf{r}) &= \alpha_0 \left( \frac{1}{r} - \frac{1}{a_0} \right) \Omega_0 (\hat{\mathbf{r}}) \\
+ \alpha_1 \left( \frac{1}{r^2} \right) - \frac{1}{3a_1} \hat{\sigma}_2 \Omega_1 (\hat{\mathbf{r}}) + O (r^2),
\end{align*}
$$

$$
\begin{align*}
f (\mathbf{r}) &= \alpha_0 \left( \frac{1}{2} b_0 - \frac{1}{2} \right) \Omega_0 (\hat{\mathbf{r}}) \\
+ \alpha_1 \left( \frac{1}{2} + \frac{b_1}{6} \right) \Omega_1 (\hat{\mathbf{r}}) + O (r^2),
\end{align*}
$$

$$
\begin{align*}
g (\mathbf{r}) &= - \alpha_0 \Omega_0 (\hat{\mathbf{r}}) - \alpha_0 (1 + vr) \Omega_1 (\hat{\mathbf{r}}) + O (r^2)
\end{align*}
$$

where $\alpha_0$ and $\alpha_1$ are two complex superposition coefficients, $a_i$ and $b_i$ are $s$-wave scattering length and effective range for $i = 0$, and $p$-wave scattering volume and effective range for $i = 1$, respectively. Interestingly, two
new scattering parameters $u$ and $v$ are involved as we anticipate. They are corrections from SO coupling to the short-range behavior of the two-body wave function in $s$- and $p$-wave channels, respectively.

In the absence of SO coupling, if atoms are initially prepared near an $s$-wave resonance, the contribution from the $p$-wave channel could be ignored, and we have $\alpha_1 \approx 0$. Naturally, the two-body wave function $\Psi(r)$ reduces to the known $s$-wave form of (up to a constant $\alpha_0$)

$$\Psi(r) = \frac{1}{r} \left( \frac{1}{a_0} + \frac{b_0 k^2}{2} - \frac{k^2}{2} r^2 \right) \Omega_0(\hat{r}) + O(r^2) \quad (13)$$

at short distance $r \geq \epsilon$. Subsequently, when SO coupling is switched on near the $s$-wave resonance, a considerable $p$-wave contribution is involved, and the two-body wave function becomes

$$\Psi_s(r) = \left( \frac{1}{r} - \frac{1}{a_0} + \frac{b_0 k^2}{2} - \frac{k^2}{2} r^2 \right) \Omega_0(\hat{r}) + (1 + vr) \Omega_1(\hat{r}) + O(r^2), \quad (14)$$

which recovers the modified Bethe-Peierls boundary condition of $[18]$ by noticing $\Omega_0(\hat{r}) = |S|/\sqrt{4\pi}$ and $\Omega_1(\hat{r}) = -i(\hat{\sigma}_2 - \hat{\sigma}_1) \cdot (r/r) |S|/\sqrt{16\pi}$. We can see that the parameter $v$ characterizes the hybridization of the $p$-wave component into the $s$-wave scattering due to SO coupling. If atoms are initially prepared near a $p$-wave resonance without SO coupling, the $s$-wave scattering could be ignored, then we have $\alpha_0 \approx 0$. The two-body wave function $\Psi(r)$ takes the known $p$-wave form at short distance, i.e.,

$$\Psi(r) = \left( \frac{1}{r^2} - \frac{1}{3a_1} + \frac{b_1 k^2}{6} + \frac{1}{6} \right) \Omega_1(\hat{r}) + O(r^2). \quad (15)$$

In the presence of SO coupling near the $p$-wave resonance, an $s$-wave component is introduced, and the two-body wave function becomes

$$\Psi_p(r) = \left[ \frac{1}{r^2} + \frac{k^2}{2} + \left( \frac{1}{3a_1} + \frac{b_1 k^2}{6} \right) r \right] \Omega_1(\hat{r}) + u \lambda \Omega_0(\hat{r}) + O(r^2) \quad (16)$$

at short distance. We can see that the parameter $u$ describes the hybridization of the $s$-wave component into the $p$-wave scattering due to SO coupling. In general, both $s$- and $p$-wave scatterings exist between atoms in the absence of SO coupling. Therefore, when SO coupling is introduced, the two-body wave function is generally the arbitrary superposition of Eqs. (14) and (16), and can be written as

$$\Psi_{3D}(r) = \alpha_0 \left( \frac{1}{r} - \frac{1}{a_0} + \frac{b_0 k^2}{2} - \frac{k^2}{2} r^2 \right) \Omega_0(\hat{r}) + \alpha_1 \left[ \frac{1}{r^2} + \frac{k^2}{2} + \frac{\alpha_0}{\alpha_1} \lambda + \left( -\frac{1}{3a_1} + \frac{b_1 k^2}{6} + \frac{\alpha_0}{\alpha_1} v \right) r \right] \Omega_1(\hat{r}) + O(r^2). \quad (17)$$

at short distance $r \geq \epsilon$. Eq. (17) can be treated as the short-range boundary condition for two-body wave functions in 3D in the presence of 3D SO coupling, when both $s$- and $p$-wave interactions are considered.

B. For 2D systems with 2D SO coupling

Let us consider two spin-half fermions scattering in the $x-y$ plane. We easily find that the total angular momentum $J$ perpendicular to the $x-y$ plane is conserved as well as the c.m. momentum $K$. Therefore, we may still focus on the two-body problem in the subspace of $K = 0$ and $J = 0$, which is spanned by the following three orthogonal basis

$$\Omega_0(\varphi) = \frac{1}{\sqrt{2\pi}} |S\rangle, \quad (18)$$

$$\Omega_{-1}(\varphi) = \frac{e^{-i\varphi}}{\sqrt{2\pi}} |\uparrow\uparrow\rangle, \quad (19)$$

$$\Omega_1(\varphi) = \frac{e^{i\varphi}}{\sqrt{2\pi}} |\downarrow\downarrow\rangle, \quad (20)$$

where $\varphi$ is the azimuthal angle of the relative coordinate $r$. Then the two-body wave function can formally be expanded as

$$\Psi(r) = \sum_{m=0,\pm 1} \psi_m(r) \Omega_m(\varphi), \quad (21)$$

and $\psi_m(r)$ is the radial wave function. Analogously, the strength of SO coupling as well as the energy can be taken into account perturbatively at short distance. We assume that the two-body wave function has the form
of the ansatz, and the corresponding functions to be determined can easily be solved out from the Schrödinger equation outside the range of the interatomic potential, i.e., \( r \geq \epsilon \). After straightforward algebra, we obtain

\[
\phi (r) = a_0 \left( \ln \frac{r}{2a_0} + \gamma \right) \Omega_0 (\varphi) + \left( \frac{1}{r} - \frac{\pi}{4a_1} \right) \sum_{m=\pm 1} \alpha_m \Omega_m (\varphi) + O (r^2),
\]

for \( r \geq \epsilon \). It is apparent that \( \psi_{2D} (r) \) naturally decouples to the s- and p-wave short-range boundary conditions in the absence of SO coupling. However, Rashba SO coupling mixes the s- and p-wave scatterings, and two new scattering parameters \( u \) and \( v \) are introduced. We should note that the short-range behaviors of the two-body wave function, i.e., Eqs. (17) and (25), are universal and does not depend on the specific form of interatomic potentials.

### III. Universal Relations in the Presence of Isotropic 3D SO Coupling

In the previous section, we have discussed the two-body problem in the presence of SO coupling, and obtained the short-range behaviors of the two-body wave functions. Then, we are ready to consider Tan’s universal relations of SO-coupled many-body systems, if only two-body correlations are taken into account. Owing to the short-range property of interactions between neutral atoms, when two fermions \( (i \text{ and } j) \) get as close as the range of interatomic potentials, all the other atoms are usually far away. In this case, the many-body wave functions approximately take the forms of Eq. (17) in 3D systems with 3D SO coupling, when the fermions \( i \) and \( j \) approach to each other. We need to pay attention that the arbitrary superposition coefficient \( \alpha_m (X) \) then becomes the functions of the c.m. coordinates of the pair \( (i, j) \) as well as those of the rest of the fermions, which we include into the variable \( X \). In the follows, we derive a set of universal relations for SO-coupled many-body systems by using Eqs. (17) for 3D SO-coupled Fermi gases. These relations include adiabatic energy relations, the large-momentum behavior of the momentum distribution, the high-frequency tail of the rf spectroscopy, the short-range behavior of the pair correlation function, the grand canonical potential and pressure relation. Let us consider a strongly interacting two-component Fermi gases with total atom number \( N \). For simplicity, we consider the case with \( b_0 \approx 0 \) for broad s-wave resonances in the follows.

#### A. Adiabatic energy relations

In order to investigate how the energy varies with the two-body interaction, let us consider two many-body wave functions \( \psi \) and \( \psi' \), corresponding to different interatomic interaction strengths. They satisfy the Schrödinger equation with different energies
\[
\sum_{i=1}^{N} \hat{H}^{(i)}_1 \Psi = E \Psi, \quad (26)
\]

\[
\sum_{i=1}^{N} \hat{H}^{(i)}_1 \Psi' = E' \Psi', \quad (27)
\]

if there is not any pair of atoms within the range of the interaction, where \( \hat{H}^{(i)}_1 \) denotes the single-atom Hamiltonian \( (1) \) for the \( i \)th fermion. By subtracting \( 27^* \times \Psi \) from \( \Psi'^* \times 26 \), and integrating over the domain \( D_\epsilon \), the set of all configurations \( (r_i, r_j) \) in which \( r = |r_i - r_j| > \epsilon \), we arrive at

\[
(E - E') \int \prod_{i=1}^{N} dr_i \Psi'^* \Psi = \\
- \frac{\hbar^2}{M} \int r > \epsilon dX dr \left[ \psi'^* \nabla^2 \psi - (\nabla^2 \psi'^*) \psi \right] \\
+ \frac{\hbar^2}{M} \int r > \epsilon \int dX dr \left[ \psi'^* \left( \hat{Q} \Psi \right) - \left( \hat{Q} \psi'^* \right)^* \Psi \right], \quad (28)
\]

where \( N = N(N-1)/2 \) is the number of all the possible ways to pair atom. Using the Gauss’ theorem, the first term on the right-hand side (RHS) can be written as

\[
- \frac{\hbar^2}{M} \int r > \epsilon dX dr \left[ \psi'^* \nabla^2 \psi - (\nabla^2 \psi'^*) \psi \right] \\
= - \frac{\hbar^2}{M} \int r > \epsilon dX dr \left[ \psi'^* \nabla \psi - (\nabla \psi'^*) \psi \right] \cdot \hat{n} dS \\
= \frac{\hbar^2}{M} \int r > \epsilon \int dX \left( \psi'^* \frac{\partial}{\partial r} \psi - \psi \frac{\partial}{\partial r} \psi'^* \right) r = \epsilon, \quad (29)
\]

where \( \hat{n} \) is the surface in which the distance between the two atoms in the pair \( (i,j) \) is \( \epsilon \) with, \( \hat{n} \) is the direction normal to \( \hat{S} \) but opposite to the radial direction, and \( \psi_0 \) (\( \psi \)) is the s-wave (p-wave) component of the radial two-body wave function. In addition, for the second term on the RHS of Eq. (28), we have

\[
\hat{Q} (r) \Psi = - \frac{2}{r^2} \frac{\partial}{\partial r} \left( r^2 \psi_1 \right) \Omega_0 (\hat{r}) + 2 \frac{\partial \psi_0}{\partial r} \Omega_1 (\hat{r}), \quad (30)
\]

then it becomes

\[
\frac{\hbar^2}{M} \int r > \epsilon dX dr \left[ \psi'^* \left( \hat{Q} (r) \Psi \right) - \left( \hat{Q} (r) \psi'^* \right)^* \Psi \right] = \frac{\hbar^2}{M} \int dX \left( \psi'^*_0 \psi - \psi^*_1 \psi \right)_{r = \epsilon}. \quad (31)
\]

Combining Eqs. (28), (29) and (31), we obtain

\[
(E - E') \prod_{i=1}^{N} dr_i \Psi'^* \Psi = \\
\frac{\hbar^2}{M} \int dX \left( \psi'^* \frac{\partial}{\partial r} \psi - \psi \frac{\partial}{\partial r} \psi'^* \right) r = \epsilon \\
+ \frac{2\lambda \hbar^2}{M} \int dX \left( \psi'^*_0 \psi - \psi^*_1 \psi \right) r = \epsilon \quad (32)
\]

Inserting the asymptotic form of the many-body wave function Eq. (17) into Eq. (32) and letting \( E' \to E \) and \( \Psi' \to \Psi \), we find

\[
\delta E \cdot \int \prod_{i=1}^{N} dr_i |\Psi|^2 = - \frac{\hbar^2}{M} \left( \mathcal{I}_a^{(1)} - \lambda \mathcal{I}_\lambda \right) \delta a_0^{(-1)} \\
- \frac{\hbar^2}{M} \mathcal{I}_a^{(1)} \delta a_1^{(-1)} + \frac{\epsilon_1}{2} \delta b_1 + \frac{3\lambda \hbar^2}{2M} \mathcal{I}_\lambda \delta v \\
- \frac{\hbar^2}{M} \left( 2 \lambda \mathcal{I}_a^{(1)} - \frac{1}{2} \mathcal{I}_\lambda \right) \delta u + \left( \frac{1}{\epsilon} + \frac{b_1}{2} \right) \mathcal{I}_a^{(1)} \delta E, \quad (33)
\]

where

\[
\mathcal{I}_a^{(m)} = N \int dX |a_{m} (X)|^2, \quad (34)
\]

\[
\mathcal{E}_m = N \int dX a_{m}^* (X) \left[ E - \hat{T} (X) \right] a_{m} (X), \quad (35)
\]

\[
\mathcal{I}_\lambda = N \int dX a_{0}^* (X) a_{1} (X) + c.c., \quad (36)
\]

\[
\mathcal{E}_\lambda = N \int dX a_{0}^* (X) \left[ E - \hat{T} (X) \right] a_{1} (X) + c.c.(37)
\]

for \( m = 0, 1 \), and \( \hat{T} (X) \) is the kinetic operator including the c.m. motion of the pair \( (i,j) \) and those of all the rest fermions. Using the normalization of the many-body wave function (see appendix A)

\[
\prod_{i=1}^{N} \int dr_i |\Psi|^2 = 1 + \left( \frac{1}{\epsilon} + \frac{b_1}{2} \right) \mathcal{I}_a^{(1)}, \quad (38)
\]

we can further simplify Eq. (33) as

\[
\delta E = - \frac{\hbar^2}{M} \left( \mathcal{I}_a^{(0)} - \lambda \mathcal{I}_\lambda \right) \delta a_0^{(-1)} \\
- \frac{\hbar^2}{M} \mathcal{I}_a^{(1)} \delta a_1^{(-1)} + \frac{\epsilon_1}{2} \delta b_1 + \frac{3\lambda \hbar^2}{2M} \mathcal{I}_\lambda \delta v \\
+ \frac{\hbar^2}{M} \left( 2 \lambda \mathcal{I}_a^{(1)} - \frac{1}{2} \mathcal{I}_\lambda \right) \delta u \quad (39)
\]

which yields the following set of adiabatic energy relations

\[
\delta E = - \frac{\hbar^2}{M} \left( \mathcal{I}_a^{(0)} - \lambda \mathcal{I}_\lambda \right) \delta a_0^{(-1)} \\
- \frac{\hbar^2}{M} \mathcal{I}_a^{(1)} \delta a_1^{(-1)} + \frac{\epsilon_1}{2} \delta b_1 + \frac{3\lambda \hbar^2}{2M} \mathcal{I}_\lambda \delta v \\
+ \frac{\hbar^2}{M} \left( 2 \lambda \mathcal{I}_a^{(1)} - \frac{1}{2} \mathcal{I}_\lambda \right) \delta u \quad (39)
\]
\[
\frac{\partial E}{\partial a_0} = -\frac{\hbar^2}{M} \left( T_a^{(0)} - \lambda I_\lambda \right),
\]
\[
\frac{\partial E}{\partial a_1} = -\frac{\hbar^2 T_a^{(1)}}{M},
\]
\[
\frac{\partial E}{\partial b_1} = \frac{\epsilon_1}{2},
\]
\[
\frac{\partial E}{\partial u} = \frac{\lambda \hbar^2}{2M} \left( I_\lambda - 4\lambda I_a^{(1)} \right),
\]
\[
\frac{\partial E}{\partial v} = \frac{3\lambda \hbar^2 I_\lambda}{2M}.
\]

Interestingly, two additional new adiabatic energy relations appear, i.e., Eqs. (13) and (14), which originate from new scattering parameters introduced by SO coupling. These relations demonstrate how the macroscopic internal energy of an SO-coupled many-body system varies with microscopic two-body scattering parameters.

B. Tail of the momentum distribution at large \( q \)

Let us then study the asymptotic behavior of the large momentum distribution for a many-body system with \( N \) fermions. The momentum distribution of the \( i \)th fermion is defined as

\[
n_i(q) = \int \prod_{\ell \neq i} dr_\ell \left| \tilde{\Psi}_i(q) \right|^2,
\]

where \( \tilde{\Psi}_i(q) \equiv \int dr \tilde{\Psi}_{3D}(r)e^{-iq\cdot r} \), and then the total momentum distribution can be written as \( n(q) = \sum_{i=1}^{N} n_i(q) \). When two fermions \((i,j)\) get close while all the other fermions are far away, we may write the many-body function \( \Psi_{3D} \) at \( r = |r_i - r_j| \approx 0 \) as the following ansatz

\[
\Psi_{3D}(X,r) = \left[ \frac{\alpha_0(X)}{r} + B_0(X) + C_0(X) r \right] \Omega_0(\bar{r}) + \left[ \frac{\alpha_1(X)}{r^2} + B_1(X) + C_1(X) r \right] \Omega_1(\bar{r}) + O(r^2),
\]

where \( \alpha_m, B_m \) and \( C_m \) \((m = 0,1)\) are all regular functions. Comparing Eq. (17) with (41) at small \( r \), we find

\[
B_0(X) = -\frac{\alpha_0}{a_0} + \alpha_1 u_\lambda,
\]
\[
B_1(X) = \frac{\alpha_1 k^2}{2} + \alpha_0 \lambda,
\]
\[
C_0(X) = -\frac{\epsilon_0 k^2}{2},
\]
\[
C_1(X) = -\frac{\epsilon_1}{3a_1} + \frac{\alpha_1 k^2}{6} + \epsilon_0 \lambda.
\]

The asymptotic form of the momentum distribution at large \( q \) but still smaller than \( \epsilon^{-1} \) is determined by the asymptotic behavior at short distance with respect to the two interacting fermions, then we have

\[
\tilde{\Psi}_i(q) \approx q^{-\alpha} \int dr \tilde{\Psi}_{3D}(X,r) e^{-iqr}. \quad (51)
\]

Using \( \nabla^2 (r^{-1}) = -4\pi \delta(r) \), we have the identity

\[
f(q) \equiv \int dr \frac{e^{-iqr}}{r} = \frac{4\pi}{q^2},
\]

so that

\[
\int dr \frac{\alpha_0(X)}{r} \Omega_0(\bar{r}) e^{-iqr} = \frac{4\pi}{q^2} \alpha_0(X) \Omega_0(q), \quad (53)
\]
\[
\int dr B_0(X) \Omega_0(\bar{r}) e^{-iqr} = 0, \quad (54)
\]
\[
\int dr C_0(X) r \Omega_0(\bar{r}) e^{-iqr} = -\frac{8\pi}{q^3} \alpha_0(X) \Omega_0(q), \quad (55)
\]
\[
\int dr \frac{\alpha_1(X)}{r^2} \Omega_1(\bar{r}) e^{-iqr} = -\frac{4\pi}{q^2} \alpha_1(X) \Omega_1(q), \quad (56)
\]
\[
\int dr B_1(X) \Omega_1(\bar{r}) e^{-iqr} = -\frac{8\pi}{q^3} B_1(X) \Omega_1(q), \quad (57)
\]
\[
\int dr C_1(X) r \Omega_1(\bar{r}) e^{-iqr} = 0. \quad (58)
\]

Inserting Eqs. (53)-(58) into (51), and then into Eq. (49), we find that the total momentum distribution \( n(q) \) at large \( q \) takes the form

\[
n_{3D}(q) \approx N \int dX \frac{32\pi^2 \alpha_1 \alpha_1^* \Omega_1(q) \Omega_1^*(q)}{q^2} + \frac{32\pi^2}{q^3} \left[ \alpha_0 \alpha_1^* \Omega_0(q) \Omega_0^*(q) - \alpha_0^* \alpha_1 \Omega_0(q) \Omega_1^*(q) \right]
\]
\[
+ \left[ 32\pi^2 \alpha_0 \alpha_0^* \Omega_0(q) \Omega_0^*(q) + 64\pi^2 k^2 \alpha_0 \alpha_1 \Omega_1(q) \Omega_1^*(q) \right]
\]
\[
+ 64\pi^2 \lambda \left( \alpha_0^* \alpha_1 + \alpha_0 \alpha_1^* \right) \Omega_1(q) \Omega_1^*(q) \frac{1}{q^2} + O(q^{-5}).
\]

If we are only interested in the dependence of the momentum distribution on the amplitude of \( q \), we may integrate over the direction of \( q \), and then we find all the odd-order terms of \( q^{-1} \) vanish. Finally, we obtain

\[
n_{3D}(q) = \frac{C_a^{(1)}}{q^2} + \left( C_a^{(0)} + C_b^{(1)} + \lambda \mathcal{P}_\lambda \right) \frac{1}{q^2} + O(q^{-6}),
\]

where the contacts are defined as

\[
C_a^{(m)} = 32\pi^2 T_a^{(m)}, \quad (m = 0,1),
\]
\[
C_b^{(1)} = \frac{64\pi^2 M}{\hbar^2} \mathcal{E}_1,
\]
\[
\mathcal{P}_\lambda = 64\pi^2 I_\lambda.
\]
With these definitions in hand, the adiabatic energy relations \( \frac{\partial E}{\partial \sigma_0} \) and \( \frac{\partial E}{\partial \sigma_1} \) can alternatively be written as

\[
\frac{\partial E}{\partial \sigma_0} = -\frac{\hbar^2 C_a^{(0)}}{32\pi^2 M} + \lambda \frac{\hbar^2 P_\lambda}{64\pi^2 M},
\]

(64)

\[
\frac{\partial E}{\partial \sigma_1} = -\frac{\hbar^2 C_a^{(1)}}{32\pi^2 M},
\]

(65)

\[
\frac{\partial E}{\partial b_i} = \frac{\hbar^2 C_b^{(1)}}{128\pi^2 M},
\]

(66)

\[
\frac{\partial E}{\partial v} = \frac{3\lambda \hbar^2 P_\lambda}{128\pi^2 M},
\]

(68)

In the absence of SO coupling, Eqs. (64), (65) and (66) simply reduce to the ordinary form of the adiabatic energy relations for \( s \)- and \( p \)-wave interactions \(^{10,17}\), with respect to the scattering length (or volume) as well as effective range. We should note that for the \( s \)-wave interaction, there is a difference of the factor \( 8\pi \) from the well-known form of adiabatic energy relations. This is because we include the spherical harmonics \( Y_{00}(\hat{r}) = 1 / \sqrt{4\pi} \) in the \( s \)-partial wave function. Besides, an additional factor \( 1/2 \) is introduced in order to keep the definition of contacts consistent with those in the tail of the momentum distribution at large \( q \). In the presence of SO coupling, two additional adiabatic energy relations appear, i.e., Eqs. (67) and (68), and a new contact \( P_\lambda \) is introduced.

### C. The high-frequency tail of the rf spectroscopy

Next, we discuss the asymptotic behavior of the rf spectroscopy at high frequency. The basic idea of the rf transition is as follows. For an atomic Fermi gas with two hyperfine states, denoted as \( |\uparrow\rangle \) and \( |\downarrow\rangle \), the rf field drives transitions between one of the hyperfine states (i.e. \( |\downarrow\rangle \)) and an empty hyperfine state \( |3\rangle \) with a bare atomic hyperfine energy difference \( \hbar \omega_{34} \) due to the magnetic field splitting \(^{56,57}\). The universal scaling behavior at high frequency of the rf response of the system is governed by contacts. In this subsection, we are going to show how the contacts defined by the adiabatic energy relations characterize such high-frequency scalings of the rf transition in 3D Fermi gases with 3D SO coupling. Here, we will present a two-body derivation first, which may avoid complicated notations as much as possible, and the results can easily be generalized to many-body systems later. The rf field driving the spin-down particle to the state \( |3\rangle \) is described by

\[
\hat{H}_{rf} = \gamma_{rf} \sum_k \left( e^{-i\omega t} c_{3k}^\dagger c_{4k} + e^{i\omega t} c_{4k}^\dagger c_{3k} \right),
\]

(69)

where \( \gamma_{rf} \) is the strength of the rf drive, \( \omega \) is the rf frequency, and \( c_{\sigma k}^\dagger \) and \( c_{\sigma k} \) are respectively the creation and annihilation operators for fermions with the momentum \( k \) in the spin states \(|\sigma\rangle\).

For any two-body state \(|\Psi_{2b}\rangle\), we may write it in the momentum space as

\[
|\Psi_{2b}\rangle = \sum_{\sigma_1,\sigma_2} \sum_{k_1,k_2} \tilde{\phi}_{\sigma_1,\sigma_2}(k_1,k_2) c_{\sigma_1,k_1}^\dagger c_{\sigma_2,k_2}^\dagger |0\rangle,
\]

(70)

where \( \tilde{\phi}_{\sigma_1,\sigma_2}(k_1,k_2) \) is the Fourier transform of \( \phi_{\sigma_1,\sigma_2}(r_1,r_2) \equiv \langle r_1,r_2; \sigma_1,\sigma_2 |\Psi_{2b}\rangle \), i.e.,

\[
\tilde{\phi}_{\sigma_1,\sigma_2}(k_1,k_2) = \int dr_1 dr_2 \phi_{\sigma_1,\sigma_2}(r_1,r_2) e^{-ik_1 \cdot r_1 - i k_2 \cdot r_2},
\]

(71)

and \( \sigma_i = \uparrow, \downarrow \) denotes the spin of the \( i \)-th particle. The specific form of \( \phi_{\sigma_1,\sigma_2}(k_1,k_2) \) can easily be obtained by using that of the two-body wave function \( \langle r_1,r_2; \sigma_1,\sigma_2 |\Psi_{2b}\rangle \) in the coordinate space, i.e., Eq. (59). Acting Eq. (69) onto (70), we obtain the two-body wave function after the rf transition,

\[
\hat{H}_{rf} |\Psi_{2b}\rangle = \gamma_{rf} e^{-i\omega t} \times \sum_{k_1,k_2} \left[ \tilde{\phi}_{\uparrow\uparrow}(k_1,k_2) c_{3k_1}^\dagger c_{4k_2} - \tilde{\phi}_{\uparrow\downarrow}(k_1,k_2) c_{3k_1}^\dagger c_{4k_2} - \tilde{\phi}_{\downarrow\uparrow}(k_1,k_2) c_{4k_1}^\dagger c_{3k_2} + \tilde{\phi}_{\downarrow\downarrow}(k_1,k_2) c_{4k_1}^\dagger c_{3k_2} \right] |0\rangle.
\]

(72)

The physical meaning of Eq. (72) is apparent: after the rf transition, the atom with initial spin state \(|\downarrow\rangle \) is driven to the empty spin state \(|3\rangle \), while the other one remains in the spin state \(|\uparrow\rangle \). Therefore, there are totally four possible final two-body states with, respectively, possibilities of \( |\uparrow\uparrow\rangle \), \( |\uparrow\downarrow\rangle \), \( |\downarrow\uparrow\rangle \), and \( |\downarrow\downarrow\rangle \). Taking all these final states into account, and according to the Fermi's golden rule \(^{24}\), the two-body rf transition rate is therefore given by the Franck-Condon factor,

\[
\Gamma_2(\omega) = \frac{2\pi \gamma_{rf}^2}{\hbar} \sum_{k_1,k_2} \left( |\tilde{\phi}_{\uparrow\uparrow}(k_1,k_2)|^2 + |\tilde{\phi}_{\uparrow\downarrow}(k_1,k_2)|^2 + |\tilde{\phi}_{\downarrow\uparrow}(k_1,k_2)|^2 + |\tilde{\phi}_{\downarrow\downarrow}(k_1,k_2)|^2 \right) \delta(\hbar \omega - \Delta E),
\]

(73)

where \( \Delta E \) is the energy difference between the final and initial states, and takes the form of

\[
\Delta E = \frac{\hbar k^2}{M} - \frac{\hbar^2 q^2}{M} + \hbar \omega_{34},
\]

(74)

where \( k = (k_1 - k_2) / 2 \), \( \hbar^2 q^2 / M \) is the relative energy of two fermions in the initial state, and \( \omega_{34} \equiv \omega_3 - \omega_4 \) is the bare hyperfine splitting between the spin states \(|3\rangle \) and \(|\downarrow\rangle \), and can be set to 0 without loss of generality. Now, we are interested in the asymptotic form of \( \Gamma_2(\omega) \) at large
ω but still small compared to \( h/Mc^2 \), which is determined by the short-range behavior when two fermions get as close as \( \epsilon \). Combining Eqs. (71) and (73), as well as the asymptotic form of the two-body wave function (17) at \( r = r_1 - r_2 \sim 0 \), we finally obtain the asymptotic behavior of the rf response of 3D SO-coupled Fermi gases at large \( \omega \).

\[
\Gamma_2(\omega) = \frac{M^2c^2}{16\pi^2\hbar^3} \left[ \frac{C_a^{(1)}}{(M\omega/\hbar)^{1/2}} + \frac{C_a^{(0)} + 3C_b^{(1)}/4 + \lambda p_\lambda}{(M\omega/\hbar)^{3/2}} \right],
\]

where \( C_a^{(0)} \), \( C_a^{(1)} \), \( C_b^{(1)} \) and \( p_\lambda \) are contacts for a two-body system with \( \mathcal{N} = 1 \) in the definitions (61)-(63).

For many-body systems, all possible \( \mathcal{N} = N(N-1)/2 \) pairs may contribute to the high-frequency tail of the rf spectroscopy, while high-order contributions from more than two fermions are ignored. Then we can generalize the above two-body picture to many-body systems by simply redefining the constant \( \mathcal{N} \) into the contacts, and then obtain

\[
\Gamma_N(\omega) = \frac{M^2c^2}{16\pi^2\hbar^3} \left[ \frac{C_a^{(1)}}{(M\omega/\hbar)^{1/2}} + \frac{C_a^{(0)} + 3C_b^{(1)}/4 + \lambda p_\lambda}{(M\omega/\hbar)^{3/2}} \right],
\]

where \( C_a^{(0)} \), \( C_a^{(1)} \), \( C_b^{(1)} \) and \( p_\lambda \) are corresponding contacts for many-body systems. In the absence of SO coupling, Eq. (70) simply reduces to the ordinary asymptotic behaviors of the rf response for s- and p-wave interactions, respectively [13, 58].

D. Pair correlation function at short distances

The pair correlation function \( g_2(s_1, s_2) \) gives the probability of finding two fermions with one at position \( s_1 \) and the other one at position \( s_2 \) simultaneously, i.e.,

\[
g_2(s_1, s_2) \equiv \langle \hat{\rho}(s_1) \hat{\rho}(s_2) \rangle, \quad \text{where } \hat{\rho}(s) = \sum_i \delta(s - r_i) \text{ is the density operator at the position } s. \]

For a pure many-body state |\( \Psi \rangle \) of \( N \) fermions, we have [20]

\[
g_2(s_1, s_2) = \int d\mathbf{r}_1d\mathbf{r}_2 \cdots d\mathbf{r}_N \langle \Psi | \hat{\rho}(s_1) \hat{\rho}(s_2) | \Psi \rangle = N(N-1) \int d\mathbf{X}' |\Psi(\mathbf{X}, \mathbf{r})|^2,
\]

where \( \mathbf{r} = s_1 - s_2 \) is relative coordinates of the pair fermions at positions \( s_1 \) and \( s_2 \), and \( \mathbf{X}' \) denotes the degrees of freedom of all the other fermions. If we further integrate over the c.m. coordinate of the pair, we can define the spatially integrated pair correlation function as

\[
G_2(\mathbf{r}) = \overline{N(N-1) \int d\mathbf{X} |\Psi(\mathbf{X}, \mathbf{r})|^2},
\]

and \( \mathbf{X} \) includes the c.m. coordinate \( \mathbf{R} = (s_1 + s_2)/2 \) of the pair besides \( \mathbf{X}' \). Inserting the short-range form of many-body wave functions for SO coupled Fermi gases, i.e. Eq. (17) into Eq. (78), we find

\[
G_2(\mathbf{r}) \approx N(N-1) \int d\mathbf{X} \left\{ \frac{\alpha_1^1\Omega_1^0\Omega_1^1}{r^4} \right. + \frac{\alpha^0_0\alpha^1_0\Omega^0_0\Omega_1^1 + \alpha_0\alpha^1_0\Omega^0_0\Omega_1^1}{r^3} + |\alpha^0_0\alpha^0_0\Omega^0_0\Omega_1^1\rangle \left( \frac{1}{r^4} + O(r^{-1}) \right) \}
\]

Further, if we are only care about the dependence of \( G_2(\mathbf{r}) \) on the amplitude of \( r = |\mathbf{r}| \), we can integrate over the direction of \( \mathbf{r} \), and use the definitions of contacts (61)-(63), then it yields

\[
G_2(\mathbf{r}) \approx \frac{1}{16\pi^2} \left[ \frac{C_a^{(1)}}{r^4} + \frac{C_a^{(0)} + C_b^{(1)}/2 + \lambda p_\lambda/2}{r^2} \right] \frac{1}{r} + \left( \frac{2C_a^{(0)}}{a_0} - \frac{2b_1C_b^{(1)}}{3a_1} + \lambda(u + v)p_\lambda/2 \right) \frac{1}{r}.
\]

which reduces to the results in the absence of the SO coupling for s- and p-wave interactions, respectively

E. Grand canonical potential and pressure relation

The adiabatic energy relations as well as the large-momentum distribution we obtained is valid for any pure energy eigenstate. Therefore, they should still hold for any incoherent mixed state statistically at finite temperature. Then the energy \( E \) and contacts then become their statistical average values. Now, let us look at the grand thermodynamic potential \( \mathcal{J} \) for a homogeneous system, which is defined as [61]

\[
\mathcal{J} \equiv -PV = E - TS - \mu N,
\]

where \( P, V, T, S, \mu, N \) are, respectively, the pressure, volume, temperature, entropy, chemical potential, and total particle number. The grand canonical potential \( \mathcal{J} \)
is the function of \( V, T, S \), and takes the following differential form

\[
dJ = -PdV - SdT - Ndmu. \tag{82}
\]

For the two-body microscopic parameters, we may evaluate their dimensions as \( a_0 \sim \text{Length}^1, a_1 \sim \text{Length}^3, b_1 \sim \text{Length}^{-1}, u \sim \text{Length}^{-1}, \) and \( v \sim \text{Length}^{-1} \). Therefore, there are basically following energy scales in the grand thermodynamic potential, i.e., \( k_BT, \mu, h^2/MV^2, h^2/Ma_0^2, h^2/Ma_1^{2/3}, h^2b_1^2/M, h^2u^2/M, h^2v^2/M \). Then we may express the thermodynamic potential \( J \) in the terms of a dimensionless function \( \bar{J} \) as \[21, 62\]

\[
\bar{J} (V, T, \mu, a_0, a_1, b_1, u, v) = k_BT \bar{J} \left( \frac{h^2/MV^{2/3}}{k_BT}, \frac{\mu}{k_BT}, \frac{h^2/Ma_0^2}{k_BT}, \frac{h^2/Ma_1^{2/3}}{k_BT}, \frac{h^2b_1^2/M}{k_BT}, \frac{h^2u^2/M}{k_BT}, \frac{h^2v^2/M}{k_BT} \right). \tag{83}
\]

Consequently, one can deduce the simple scaling law

\[
\bar{J} \left( \gamma^{-3/2}V, \gamma T, \gamma \mu, \gamma^{-1/2}a_0, \gamma^{-3/2}a_1, \gamma^{1/2}b_1, \gamma^{1/2}u, \gamma^{1/2}v \right) = \gamma \bar{J} (V, T, \mu, a_0, a_1, b_1, u, v). \tag{84}
\]

The derivative of Eq.\[84\] with respect to \( \gamma \) at \( \gamma = 1 \) simply yields

\[
\left( -\frac{3V}{2} \frac{\partial}{\partial V} + T \frac{\partial}{\partial T} + \mu \frac{\partial}{\partial \mu} - \frac{a_0}{2} \frac{\partial}{\partial a_0} - \frac{3a_1}{2} \frac{\partial}{\partial a_1} + \frac{b_1}{2} \frac{\partial}{\partial b_1} + \frac{u}{2} \frac{\partial}{\partial u} + \frac{v}{2} \frac{\partial}{\partial v} \right) \bar{J} = \bar{J}, \tag{85}
\]

where all the partial derivatives are to be understood as leaving all other system variables constant. Since

\[
\bar{J} - T \frac{\partial \bar{J}}{\partial T} - \mu \frac{\partial \bar{J}}{\partial \mu} = \bar{J} + TS + \mu N = E, \tag{86}
\]

and the variation of the grand thermodynamic potential \( \delta \bar{J} \) with respect to the two-body parameters at fixed volume \( V \), temperature \( T \) and chemical potential \( \mu \) is equal to that of the energy \( \delta E \) at fixed volume \( V \), entropy \( S \) and particle number \( N \), i.e., \( (\delta \bar{J})_{V, T, \mu} = (\delta E)_{V, S, N} \) and \( V dV = d \bar{J} \), we easily obtain from Eqs.\[83\] and \[86\]

\[
-\frac{3}{2} \bar{J} - a_0 \frac{\partial E}{\partial a_0} - 3a_1 \frac{\partial E}{\partial a_1} - b_1 \frac{\partial E}{\partial b_1} + u \frac{\partial E}{\partial u} + v \frac{\partial E}{\partial v} = E, \tag{87}
\]

Further by using adiabatic energy relations, Eq.\[87\] becomes

\[
\bar{J} = -\frac{2}{3}E - \frac{h^2}{96\pi^2Ma_0} \left( C_a^{(0)} - \frac{\lambda \rho \lambda}{2} \right) - \frac{h^2C_a^{(1)}}{32\pi^2Ma_1} + \frac{h^2b_1C_b^{(1)}}{384\pi^2M} - \frac{\lambda u h^2}{48\pi^2M} \left( \frac{\lambda C_a^{(1)}}{8} + \frac{\lambda v \rho \lambda h^2}{128\pi^2M} \right) \tag{88}
\]

or the pressure relation by dividing both sides of Eq.\[83\] by \(-V\), which respectively reduces to the well-known results in the absence of the spin-orbit coupling

\[
P = \frac{2E}{3V} + \frac{h^2C_a^{(0)}}{96\pi^2MVa_0} \tag{89}
\]

for \( s \)-wave interactions, which is consistent with the result of Ref.\[11, 61, 63\], and

\[
P = \frac{2E}{3V} + \frac{h^2C_a^{(1)}}{32\pi^2MVa_1} - \frac{b_1h^2C_b^{(1)}}{384\pi^2MV} \tag{90}
\]

for \( p \)-wave interactions, which is consistent with the result of Ref.\[13\].

IV. UNIVERSAL RELATIONS IN 2D SYSTEMS WITH RASHBA SO COUPLING

The derivation of the universal relations for 3D Fermi gases with 3D SO coupling can directly be generalized to those for 2D systems with 2D SO coupling. In this section, with the short-range form of the two-body wave function for 2D systems with 2D SO coupling in hands, i.e., Eq.\[26\], we are going to discuss Tan’s universal relations for 2D Fermi gases with 2D SO coupling, by taking into account only two-body correlations.

A. Adiabatic energy relations

Let us consider how the energy of the SO-coupled system varies with the two-body interaction in 2D systems with 2D SO coupling. The two wave functions of a many-body system \( \Psi (r) \) and \( \Psi^* (r) \), corresponding to different interatomic interaction strengths, satisfy the Schrödinger equation with different energies, i.e. formally as Eqs.\[26\] and \[27\]. Analogously, by subtracting \( \Psi^* \times \psi \) from \( \Psi^* \times \psi \), and integrating over the domain \( D_r \), the set of all configurations \( (r_i, r_j) \) in which \( r = |r_i - r_j| > \epsilon \), we obtain
\[(E - E') \int_{D_c} \prod_{i=1}^{N} dr_i \Psi^\ast \Psi =
- \frac{\hbar^2}{M} \int dX dr \left[ \Psi^\ast \nabla^2_r \Psi - (\nabla^2_r \Psi^\ast) \Psi \right]
+ \frac{\hbar^2 \lambda}{M} \int_{r > \epsilon} dX dr \left[ \Psi^\ast \left( \hat{Q} \Psi \right) - \left( \hat{Q} \Psi^\ast \right)^\ast \Psi \right],
\]
where \( N = N(N - 1)/2 \) is again the number of all the possible ways to pair atom. Using the Gauss’ theorem, the first term on the right-hand side (RHS) can be written as

\[- \frac{\hbar^2}{M} \int dX dr \left[ \Psi^\ast \nabla^2_r \Psi - (\nabla^2_r \Psi^\ast) \Psi \right]
= - \frac{\hbar^2}{M} \int dX \sum_{m=0, \pm 1} \left( \psi^\ast_m \frac{\partial}{\partial r} \psi_m - \psi^\ast_m \frac{\partial}{\partial r} \psi^\ast_m \right)_{r = \epsilon},
\]
where \( S \) is the boundary of \( D_c \) that the distance between the two fermions in the pair \((i, j)\) is \( \epsilon \), \( \hat{n} \) is the direction normal to \( S \), but is opposite to the radial direction, and \( \psi_0 (\psi_{\pm 1}) \) is the \( s \)-wave (\( p \)-wave) component of the two-body wave function as defined in Eq. (21). Since

\[\hat{Q} (\Psi) = \sum_{m=\pm 1} \left[ -\sqrt{2} \frac{\partial}{\partial r} (r \psi_m) \Omega_0 (\hat{r}) + \sqrt{2} \frac{\partial^2 \psi_m}{\partial r^2} \Omega_m (\hat{r}) \right],\]

we find that the second term on the right-hand side (RHS) of Eq. (21) can be written as

\[- \frac{\hbar^2 \lambda}{M} \int_{r > \epsilon} dX dr \left[ \Psi^\ast \left( \hat{Q} (\Psi) \right) - \left( \hat{Q} (\Psi^\ast) \right)^\ast \Psi \right]
= \frac{\sqrt{2} \lambda \hbar^2}{M} \int dX \sum_{m=0, \pm 1} (\psi^\ast_0 \psi_m - \psi^\ast_m \psi_0)_{r = \epsilon},\]
Combining Eqs. (21), (92) and (94), we have

\[(E - E') \int_{D_c} \prod_{i=1}^{N} dr_i \Psi^\ast \Psi =
\frac{\hbar^2}{M} \int dX dr \left[ \Psi^\ast \nabla^2_r \Psi - (\nabla^2_r \Psi^\ast) \Psi \right]
+ \frac{\hbar^2 \lambda}{M} \int_{r > \epsilon} dX dr \left[ \Psi^\ast \left( \hat{Q} \Psi \right) - \left( \hat{Q} \Psi^\ast \right)^\ast \Psi \right],\]
Inserting the asymptotic form of the many-body wave function Eq. (25) into Eq. (95), and letting \( E' \to E \) and \( \Psi' \to \Psi \), we arrive at

\[\delta E \cdot \int D_c \prod_{i=1}^{N} dr_i |\Psi|^2 = \frac{\hbar^2}{M} \left( T^{(0)}_a + \sum_{m=\pm 1} \frac{\sqrt{2}}{2} \lambda T^{(m)}_a \right) \delta \ln a_0
+ \sum_{m=\pm 1} \left\{ -\frac{\pi \hbar^2 T^{(m)}_a}{2M} \delta a_1^{-1} + \left( \delta \epsilon_m - \frac{\lambda \hbar^2 T^{(m)}_a}{\sqrt{2}M} \right) \delta \ln b_1
- \frac{\hbar^2}{M} \left[ \sqrt{2} \lambda T^{(m)}_a + \frac{\lambda T^{(m)}_a}{2} \right] \delta \ln \left( \frac{\epsilon}{2b_1} + \gamma \right) T^{(m)}_a \delta E \right\},\]
where

\[T^{(m)}_a = N \int dX |\alpha_m|^2,\]

\[\delta \epsilon_m = N \int dX \alpha_m \left( E - \hat{T} \right) \alpha_m\]
for \( m = 0, \pm 1, \)

\[T^{(\pm 1)}_a = N \int dX \alpha^\ast_0 \alpha_{\pm 1} + \text{c.c.,}\]

\[\delta \epsilon^{(\pm 1)}_a = N \int dX \alpha_0 \left( E - \hat{T} \right) \alpha_{\pm 1} + \text{c.c.,}\]

\[T_p = N \int dX \alpha^\ast_0 \alpha_1 + \text{c.c.,}\]
and \( \hat{T}(X) \) is the kinetic operator including the c.m. motion of the pair as well as those of all the rest fermions. Using the normalization of the wave function (see appendix B)

\[\int D_c \prod_{i=1}^{N} dr_i |\Psi|^2 = 1 - \sum_{m=\pm 1} \left( \ln \frac{\epsilon}{2b_1} + \gamma \right) T^{(m)}_a,\]
we can further simplify Eq. (96) as

\[\delta E = \frac{\hbar^2}{M} \left( T^{(0)}_a + \sum_{m=\pm 1} \frac{\lambda T^{(m)}_a}{\sqrt{2}} \right) \delta \ln a_0
+ \sum_{m=\pm 1} \left\{ -\frac{\pi \hbar^2 T^{(m)}_a}{2M} \delta a_1^{-1} + \left( \delta \epsilon_m - \frac{\lambda \hbar^2 T^{(m)}_a}{\sqrt{2}M} \right) \delta \ln b_1
- \frac{\hbar^2}{M} \left[ \sqrt{2} \lambda T^{(m)}_a + \frac{\lambda T^{(m)}_a}{2} \right] \delta \ln \left( \frac{\epsilon}{2b_1} + \gamma \right) T^{(m)}_a \delta E \right\},\]

(103)
which characterizes how the energy of a 2D system with 2D SO coupling varies as the scattering parameters adiabatically change, and yields the following set of adiabatic energy relations

\[
\frac{\partial E}{\partial \ln a_0} = \frac{\hbar^2}{M} \left( \mathcal{I}_a^{(0)} + \frac{\lambda}{\sqrt{2}} \sum_{m=\pm 1} \mathcal{I}_a^{(m)} \right),
\]

(104)

\[
\frac{\partial E}{\partial a_1} = -\frac{\pi \hbar^2}{2M} \sum_{m=\pm 1} \mathcal{I}_a^{(m)},
\]

(105)

\[
\frac{\partial E}{\partial \ln b_1} = \sum_{m=\pm 1} \left( \mathcal{E}_m - \frac{\lambda \hbar^2 \mathcal{I}_a^{(m)}}{\sqrt{2M}} \right),
\]

(106)

\[
\frac{\partial E}{\partial u} = \frac{\hbar^2 \lambda}{\sqrt{2M}} \sum_{m=\pm 1} \left[ \mathcal{I}_a^{(m)} \sqrt{2} + \lambda \left( 2 \mathcal{I}_a^{(m)} + \mathcal{I}_p \right) \right],
\]

(107)

\[
\frac{\partial E}{\partial v} = \frac{\hbar^2 \lambda}{M} \sum_{m=\pm 1} \mathcal{I}_a^{(m)}.
\]

(108)

Obviously, there are additional two new adiabatic energy relations appear, i.e. Eqs. (107) and (108), which originate from new scattering parameters introduced by SO coupling.

**B. Tail of the momentum distribution at large \(q\)**

In general, the momentum distribution at large \(q\) is determined by the short-range behavior of the many-body wave function when the fermions \(i\) and \(j\) are close. Similarly as in the 3D case, we can formally write the many-body wave function \(\Psi_{2D}\) at \(r = 0\) as the following ansatz

\[
\Psi_{2D}(X, r) = \left[ a_0 + C_0 \right] \Omega_0(\hat{r}) + \sum_m \left[ a_m \mathcal{F}_m \right] \Omega_m(\hat{r}) + O(r^2),
\]

(109)

where \(a_j, B_\lambda, C_\lambda,(j = 0, \pm 1)\) are all regular functions of \(X\). Comparing Eqs. (25) and (109) at small \(r\), we find that

\[
B_0(X) = a_0 (\gamma - \ln 2a_0) + \sum_{m=\pm 1} a_m \lambda u,
\]

(110)

\[
B_m(X) = -\frac{a_m k^2}{2} + \frac{\lambda a_0}{\sqrt{2}},
\]

(111)

\[
C_0(X) = -a_0 k^2 + \frac{\lambda a_0}{4},
\]

(112)

\[
C_m(X) = a_m \left( -\frac{\pi}{4a_1} + \frac{1 - 2\gamma}{4} k^2 \right)
+ \alpha_0 \lambda v + \left( \frac{\alpha_m k^2}{2} - \frac{\lambda a_0}{\sqrt{2}} \right) \ln 2 b_1.
\]

(113)

In the follows, we derive the momentum distribution at large \(q\) but still smaller than \(e^{-1}\). With the help of the plane-wave expansion

\[
e^{i \mathbf{q} \cdot \mathbf{r}} = \sqrt{2\pi} \sum_{m=0}^\infty \sum_{\sigma = 1}^2 \eta_m a^m_j \mathbf{J}_m(qr) e^{-i \sigma_m \varphi} \Omega_m(\gamma),
\]

(114)

where \(\eta_m = 1/2\) for \(m = 0\), and \(\eta_m = 1\) for \(m \geq 1\), and \(\varphi\) is the azimuthal angle of \(q\), we have

\[
\int d\mathbf{\alpha}_0 \ln r \Omega_0(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = -\frac{2\pi}{q^2} \alpha_0 \Omega_0(\hat{\mathbf{q}}),
\]

(115)

\[
\int d\mathbf{\alpha}_0 \Omega_0(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = 0,
\]

(116)

\[
\int d\mathbf{C}_0 \ln r \Omega_0(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = \frac{8\pi}{q^3} \mathbf{C}_0 \Omega_0(\hat{\mathbf{q}}),
\]

(117)

\[
\int d\mathbf{\alpha}_m \Omega_m(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = -\frac{2\pi}{q^2} \alpha_m \Omega_m(\hat{\mathbf{q}}),
\]

(118)

\[
\int d\mathbf{\alpha}_m \mathbf{B}_m \ln r \Omega_m(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = \frac{4\pi}{q^2} \mathbf{B}_m \Omega_m(\hat{\mathbf{q}}),
\]

(119)

\[
\int d\mathbf{\alpha}_m \mathbf{B}_m \ln r \Omega_m(\hat{\mathbf{r}}) e^{-i \mathbf{q} \cdot \mathbf{r}} = 0,
\]

(120)

where \(\hat{\mathbf{q}}\) is the angular part of \(\mathbf{q}\). Inserting Eqs. (115)-(120) into (109), we find that the total momentum distribution \(n_{2D}(\mathbf{q})\) at large \(\mathbf{q}\) takes the form of

\[
n_{2D}(\mathbf{q}) \approx \mathcal{N} \int d\mathbf{X} \sum_{m, m'} a_m a_{m'}^* \Omega_m(\hat{\mathbf{q}}) \Omega_{m'}^*(\hat{\mathbf{q}}) \frac{8\pi^2}{q^2}
+i \sum_m \left[ a_0^* a_m \Omega_0^*(\hat{\mathbf{q}}) \Omega_m(\hat{\mathbf{q}}) - a_0 a_m^* \Omega_m^*(\hat{\mathbf{q}}) \Omega_{m'}(\hat{\mathbf{q}}) \right] \frac{8\pi^2}{q^3}
+ \left\{ a_0 a_0^* \Omega_0(\hat{\mathbf{q}}) \Omega_0^*(\hat{\mathbf{q}}) + \sum_{m, m'} \left[ -\sqrt{2} \lambda (a_0 a_m \Omega_m^*(\hat{\mathbf{q}}) \Omega_{m'}(\hat{\mathbf{q}}) + a_0 a_m^* \Omega_m(\hat{\mathbf{q}}) \Omega_{m'}^*(\hat{\mathbf{q}})) \right] \times \frac{8\pi^2}{q^3} + O(q^{-5}) \right\}
\]

(121)

and the summations are over \(m, m' = \pm 1\). If we are only interested in the dependence of \(n_{2D}(\mathbf{q})\) on the amplitude of \(\mathbf{q}\), the expression can further be simplified by integrating \(n_{2D}(\mathbf{q})\) over the direction of \(\mathbf{q}\), and all the odd-order terms of \(q^{-1}\) vanish. Finally, we arrive at

\[
n_{2D}(q) = \frac{\sum_{m=\pm 1} C_a^{(m)}}{q^2}
+ \left( C_a^{(0)} + \sum_{m=\pm 1} \left( C_b^{(m)} - \lambda \mathcal{I}_a^{(m)} \right) \right) \frac{1}{q^2} + O(q^{-6}),
\]

(122)

where the contacts are defined as

\[
C_a^{(j)} = 8\pi^2 \mathcal{I}_a^{(j)}
\]

(123)
for $j = 0, \pm 1$, and

$$C_b^{(m)} = \frac{16\pi^2 M}{\hbar^2} \xi_m, \quad (124)$$

$$\mathcal{P}^{(m)}_\lambda = 8\sqrt{2\pi^2} \mathcal{P}_{\lambda}^{(m)} \quad (125)$$

for $m = \pm 1$. With these definitions in hands, the adiabatic energy relations \([104]-[108]\) can alternatively be written as

$$\frac{\partial E}{\partial \ln a_0} = \frac{\hbar^2}{8\pi^2 M} \left( C_a^{(0)} + \frac{\lambda}{2} \sum_{m=\pm 1} \mathcal{P}^{(m)}_\lambda \right), \quad (126)$$

$$\frac{\partial E}{\partial a_1} = -\frac{\hbar^2}{16\pi M} \sum_{m=\pm 1} C_a^{(m)}, \quad (127)$$

$$\frac{\partial E}{\partial \ln b_1} = \frac{\hbar^2}{16\pi^2 M} \sum_{m=\pm 1} \left( C_b^{(m)} - \lambda \mathcal{P}^{(m)}_\lambda \right), \quad (128)$$

$$\frac{\partial E}{\partial u} = -\frac{\hbar^2 \lambda}{16\sqrt{2\pi^2 M}} \sum_{m=\pm 1} \mathcal{P}^{(m)}_\lambda, \quad (129)$$

$$\frac{\partial E}{\partial v} = \frac{\hbar^2 \lambda}{8\sqrt{2\pi^2 M}} \sum_{m=\pm 1} \mathcal{P}^{(m)}_\lambda. \quad (130)$$

In the absence of SO coupling, Eqs. \((126), (127)\) and \((128)\) simply reduce to the ordinary form of the adiabatic energy relations for $s$- and $p$-wave interactions \([24, 29]\), with respect to the scattering length (or area) as well as effective range. And for the $s$-wave interaction, there is a difference of the factor $2\pi$ from the Ref.\([24]\), which is because we include the angular part $1/\sqrt{2\pi}$ in the $s$-partial wave function. In addition, two additional new adiabatic energy relations, i.e., Eqs. \((129)\) and \((130)\), and new contacts $\mathcal{P}^{(m)}_\lambda$ appear, due to SO coupling.

\section*{C. The high-frequency tail of the rf spectroscopy}

We may carry out the analogous procedure as that in 3D systems with 3D SO coupling, and the two-body rf transition rate takes the form

$$\Gamma_2(\omega) = \frac{2\pi \gamma \gamma^r f}{\hbar} \times \sum_{k_1,k_2} \left( |\tilde{\phi}_{\uparrow\downarrow}|^2 + |\tilde{\phi}_{\downarrow\uparrow}|^2 + 2 |\tilde{\phi}_{\uparrow\uparrow}|^2 \right) \delta (\hbar \omega - \Delta E), \quad (131)$$

where

$$\tilde{\phi}_{\sigma_1 \sigma_2}(k_1, k_2) = \int dr_1 dr_2 \hat{\phi}_{\sigma_1 \sigma_2}(r_1, r_2) e^{-i k_1 r_1} e^{-i k_2 r_2}. \quad (132)$$

If we are only interested in the high-frequency tail of the transition rate, we can use the asymptotic behavior of the two-body wave function for a 2D system with 2D SO coupling, i.e. Eq. \((25)\). Combining with Eqs.\((131)\) and \((132)\), we obtain the two-body rf transition rate $\Gamma_2(\omega)$ as

$$\Gamma_2(\omega) = \frac{M \gamma^2}{4\pi \hbar^3} \left[ \frac{c_a^{(1)}}{M \omega / \hbar} + \frac{c_a^{(0)} / 2 + c_b^{(0)} / 2 - \lambda \mathcal{P}^{(1)}_\lambda}{(M \omega / \hbar)^2} \right], \quad (133)$$

where $c_a^{(0)}, c_a^{(1)}, c_b^{(0)}, c_b^{(1)}$ and $\mathcal{P}^{(1)}_\lambda$ are contacts for a two-body system with $\mathcal{N} = 1$ in the definitions \((126)-(128)\).

For many-body systems, all $\mathcal{N} = N(N - 1)/2$ pairs contribute to the transition rate. Similarly, we can redefining the constant $\mathcal{N}$ into the contacts, and then obtain

$$\Gamma_N(\omega) = \frac{M^2 \gamma^2}{4\pi \hbar^3} \left[ \frac{c_a^{(1)}}{M \omega / \hbar} + \frac{c_a^{(0)} / 2 + c_b^{(1)} / 2 - \lambda \mathcal{P}^{(1)}_\lambda}{(M \omega / \hbar)^2} \right], \quad (134)$$

where $c_a^{(0)}, c_a^{(1)}, c_b^{(1)}$ and $\mathcal{P}^{(1)}_\lambda$ are corresponding contacts for many-body systems. In the absence of SO coupling, Eq. \((134)\) simply reduces to the ordinary results for $s$- and $p$-wave interactions, respectively \([24, 64]\).

\section*{D. Pair correlation function at short distances}

Let us then discuss the short-distance behavior of the pair correlation function for a 2D Fermi gas with 2D SO coupling. Inserting the asymptotic form of the many-body wave function at short distance, i.e. Eq. \((78)\) into the Eq. \((73)\), we easily obtain spatially integrated pair correlation function $G_2(r)$. If we are only interested in the dependence of $G_2(r)$ on the amplitude of $r = |r|$, we may integrate over the direction of $r$, and obtain

$$G_2(r) \approx \frac{1}{4\pi^2} \left[ \sum_{m=\pm 1} \frac{C_a^{(m)}}{r^2} + C_a^{(0)} \left( \ln \frac{r}{2a_0} \right)^2 \right]$$

$$+ \left( 2\gamma c_a^{(0)} + \frac{\lambda u}{\sqrt{2}} \sum_{m=\pm 1} \mathcal{P}^{(m)}_\lambda \right) \frac{r}{2a_0}$$

$$+ \sum_{m=\pm 1} \frac{1}{4\pi} \left( -C_b^{(m)} + \lambda \mathcal{P}^{(m)}_\lambda \right) \ln \frac{r}{2b_1}. \quad (135)$$

In the absence of SO coupling, Eq. \((135)\) simply reduces to the ordinary results for $s$- and $p$-wave interactions, respectively \([24, 29]\).
E. Grand canonical potential and pressure relation

Similarly, according to the dimension analysis, we easily obtain

$$-J - a_0 \frac{\partial E}{\partial a_0} - a_1 \frac{\partial E}{\partial a_1} - b_1 \frac{\partial E}{\partial b_1} = E. \quad (136)$$

Further by using adiabatic energy relations, Eq. (136) becomes

$$J = -E - \frac{\hbar^2}{16\pi^2 M} \left( C_a^{(0)} + \frac{\lambda}{2} \sum_{m=\pm1} P_{\lambda}^{(m)} \right)$$

$$- \frac{\hbar^2}{16\pi M} \sum_{m, p = \pm1} \left[ C_a^{(m)} \frac{1}{a_1} + \frac{1}{2\pi} \left( C_b^{(m)} - \lambda P_{\lambda}^{(m)} \right) \right]. \quad (137)$$

The pressure relation can be obtained by dividing both sides of Eq. (137) by $-V$, which respectively reduces to the results in the absence of SO coupling

$$P = \frac{E}{V} + \frac{\hbar^2 C_a^{(0)}}{16\pi^2 MV} \quad (138)$$

for $s$-wave interactions, which is consistent with the result of Ref. [22], and

$$P = \frac{E}{V} + \frac{\hbar^2}{16\pi MV} \left( \sum_{m=\pm1} \frac{C_a^{(m)}}{a_1} + \frac{C_b^{(m)}}{2\pi} \right) \quad (139)$$

for $p$-wave interactions, which is consistent with the result of Ref. [29],

V. CONCLUSIONS

In conclusion, we systematically study a set of universal relations for spin-orbit-coupled Fermi gases in three or two dimension, respectively. The universal short-range forms of two-body wave functions are analytically derived, by using a perturbation method, in the sub-Hilbert space of zero center-of-mass momentum and zero total angular momentum of pairs. The obtained short-range behaviors of two-body wave functions do not depend on the short-range details of interatomic potentials. We find that two new microscopic scattering parameters appear because of spin-orbit coupling, and then new contacts need to be introduced in both three- and two-dimensional systems. However, due to different short-range behaviors of two-body wave functions for three- and two-dimensional systems, the specific forms of universal relations are distinct in different dimensions. As we anticipate, the universal relations for spin-orbit-coupled systems, such as the adiabatic energy relations, the large-momentum distributions, the high-frequency behavior of the radio-frequency responses, short-range behaviors of the pair correlation functions, grand canonical potentials, and pressure relations, are fully captured by the contacts defined. In general, more partial-wave scatterings should be taken into account for nonzero center-of-mass momentum and nonzero total angular momentum of pairs. Consequently, we may expect more contacts to appear. Our results may shed some light for understanding the profound properties of the few- and many-body spin-orbit-coupled quantum gases.

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APPENDIX A: NORMALIZATION OF THE WAVE FUNCTION FOR 3D SYSTEMS WITH 3D SO COUPLING

In this section of Appendix A, we are going to derive

$$\int_{D_r} \prod_{i=1}^N d\mathbf{r}_i |\Psi|^2$$

for 3D many-body systems with 3D SO coupling. Let us consider two many-body wave functions $\Psi'$ and $\Psi$, corresponding to different energies $\hbar^2 k^2 / M$ and $\hbar^2 k'^2 / M$, respectively. They should be orthogonal, i.e., $\int_{D_r} \prod_{i=1}^N d\mathbf{r}_i \Psi'^{*}\Psi = 0$, and therefore we have

$$\int_{r<\epsilon} \prod_{i=1}^N d\mathbf{r}_i \Psi'^{*}\Psi = - \int_{r>\epsilon} \prod_{i=1}^N d\mathbf{r}_i \Psi'^{*}\Psi. \quad (140)$$

From the Schrödinger equation satisfied by $\Psi'$ and $\Psi$ outside the interaction potential, i.e., $r > \epsilon$, we easily obtain

$$\int_{r>\epsilon} \prod_{i=1}^N d\mathbf{r}_i \Psi'^{*}\Psi = \frac{\epsilon^2}{k^2 - k'^2} N \int d\mathbf{X} \int_{r=\epsilon} \prod_{i=1}^N d\mathbf{r}_i$$

$$\left( \Psi'^{*} \frac{\partial}{\partial r} \Psi - \Psi \frac{\partial}{\partial r} \Psi'^{*} \right) + \frac{\lambda}{2\pi} \left( \psi'^{*} \psi_1 - \psi_1^* \psi' \right). \quad (141)$$

In the presence of SO coupling, only $s$- and $p$-wave scatterings are involved in the subspace $K = 0$ and $J = 0$, and the wave function at short distance takes the form of Eq. (17). Using the asymptotic behavior of the wave function, we easily evaluate...
\[
\int_{r<\epsilon} \prod_{i=1}^{N} dr_i |\Psi|^2 = -\lim_{k' \to k} \frac{1}{2} \left( \int_{r<\epsilon} \prod_{i=1}^{N} dr_i \Psi^* \Psi + \int_{r>\epsilon} \prod_{i=1}^{N} dr_i \Psi \Psi^* \right)
= -N \int dX \left\{ |\alpha_1|^2 + \frac{|\alpha_1|^2 b_1}{c} \right\}
= - \left( \frac{1}{c} + \frac{b_1}{2} \right) T_a^{(1)}, \quad (142)
\]

which in turn yields
\[
\int_{D_r} \prod_{i=1}^{N} dr_i |\Psi|^2 = 1 + \left( \frac{1}{c} + \frac{b_1}{2} \right) T_a^{(1)}. \quad (143)
\]

**APPENDIX B: NORMALIZATION OF THE WAVE FUNCTION FOR 2D SYSTEM WITH 2D SO COUPLING**

In this section of Appendix B, we are going to derive \( \int_{D_r} \prod_{i=1}^{N} dr_i |\Psi|^2 \) for 2D many-body systems with 2D SO coupling. Let us consider two many-body wave functions \( \Psi' \) and \( \Psi \), corresponding to different energies \( \hbar^2 k'^2 / M \) and \( \hbar^2 k^2 / M \), respectively. They should be orthogonal, i.e., \( \int_{D_r} \prod_{i=1}^{N} dr_i \Psi'^* \Psi = 0 \), and therefore we have
\[
\int_{r<\epsilon} \prod_{i=1}^{N} dr_i \Psi'^* \Psi = -\int_{r>\epsilon} \prod_{i=1}^{N} dr_i \Psi'^* \Psi. \quad (144)
\]

From the Schrödinger equation satisfied by \( \Psi' \) and \( \Psi \) outside the interaction potential, i.e., \( r > \epsilon \), we easily obtain
\[
\int_{r<\epsilon} \prod_{i=1}^{N} dr_i |\Psi|^2 = \frac{\epsilon}{k^2 - k'^2} N \int dX \int_{r=\epsilon} d\vec{r} \left[ \left( \psi'^* \frac{\partial}{\partial r} \psi - \psi \frac{\partial}{\partial r} \psi'^* \right) + \sum_{m=\pm 1} \frac{\lambda}{\sqrt{2\pi}} (\psi'^* \psi_m - \psi'^I \psi) \right]. \quad (145)
\]

In the presence of SO coupling, only \( s- \) and \( p- \) wave scatterings are involved in the subspace \( K = 0 \) and \( J = 0 \), and the wave function at short distance takes the form of Eq. (25). Using the asymptotic behavior of the wave function, we easily evaluate
\[
\int_{r<\epsilon} \prod_{i=1}^{N} dr_i |\Psi|^2 = -\lim_{k' \to k} \frac{1}{2} \left( \int_{r<\epsilon} \prod_{i=1}^{N} dr_i \Psi'^* \Psi + \int_{r>\epsilon} \prod_{i=1}^{N} dr_i \Psi \Psi^* \right)
= N \int dX \left( \ln \frac{\epsilon}{2 b_1} + \gamma \right) |\alpha_m|^2
= \sum_{m=\pm 1} \left( \ln \frac{\epsilon}{2 b_1} + \gamma \right) T_a^{(m)}, \quad (146)
\]

which in turn yields
\[
\int_{D_r} \prod_{i=1}^{N} dr_i |\Psi|^2 = 1 - \sum_{m=\pm 1} \left( \ln \frac{\epsilon}{2 b_1} + \gamma \right) T_a^{(m)}. \quad (147)
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