SELF-AVERAGED SCALING LIMITS FOR RANDOM PARABOLIC WAVES

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Abstract. We consider 6 types of scaling limits for the Wigner-Moyal equation of the parabolic waves in random media, the limiting cases of which include the radiative transfer limit, the diffusion limit and the white-noise limit. We show under fairly general assumptions on the random refractive index field that sufficient amount of medium diversity (thus excluding the white-noise limit) leads to statistical stability or self-averaging in the sense that the limiting law is deterministic and is governed by various transport equations depending on the specific scaling involved. We obtain 6 different radiative transfer equations as limits.

1. Introduction

The celebrated Schrödinger equation
\[ i\hbar \frac{\partial \Psi}{\partial t} + \frac{\hbar^2}{2m} \Delta \Psi + \sigma V(t, x) \Psi = 0, \quad \Psi(0, x) = \Psi_0(x) \]
describes the evolution of the wave function \( \Psi \) of a quantum spin-less particle in a potential \( \sigma V \) where \( \sigma \) is the typical size of the variation.

A similar equation called the parabolic wave equation is also widely used to describe the propagation of the modulation of low intensity wave beam in turbulent or turbid media in the forward scattering approximation of the full wave equation \[19\]. In this connection the refractive index fluctuation plays the role of the potential in the equation. Nondimensionalized with respect to the propagation distances in the longitudinal and transverse directions, \( L_z \) and \( L_x \), respectively, the parabolic wave equation for the modulation function \( \Psi \) reads
\[ ik^{-1} L_z^{-1} \frac{\partial \Psi}{\partial z} + 2^{-1} k^{-2} L_x^{-2} \Delta \Psi + \sigma V(zL_z, xL_x) \Psi = 0, \quad \Psi(0, x) = \Psi_0(x) \]
where \( k \) is the carrier wavenumber, \( \Psi \) the amplitude modulation and \( \Delta \) the Laplacian operator in the transverse coordinates \( x \). Here we have assumed the random media has a constant background. In the sequel we will adopt the notation of \[11\].

In this paper we study the scaling regimes where the wave field experiences both longitudinal and transverse diversity of the random medium, represented by \( V \), whose fluctuation is assumed to be weak. This gives rise to random spread of wave energy in the transverse directions. To fix the idea, let us choose the units of the longitudinal and transverse coordinates such that the spectral support of the function \( V \) is \( O(1) \) while
\[ L_z \sim L_x \gg 1, \]
namely \( L_z \) and \( L_x \) are much larger than the correlation lengths of the medium in the longitudinal and transverse directions, respectively. There is no loss of generality in the choice of the hyperbolic scaling \[22\] (cf. Remark 2 below). Depending on the actual length scales and anisotropy of the medium, we adjust the intensity \( \sigma \) of the medium fluctuations to obtain a nontrivial limit.

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2. WIGNER DISTRIBUTION AND TIME REVERSAL

There has been a surge of interest in the radiative transfer limit in terms of the Wigner distribution (see below) because of its application to the spectacular phenomena related to time-reversal (or phase-conjugate) mirrors [1, 3, 5, 16].

The Wigner transform or distribution of the wave function $\Psi$ is defined as

$$W(z, x, p) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p} \cdot \mathbf{y}} \Psi(z, \mathbf{x} + \frac{\mathbf{y}}{2}) \Psi^{\ast}(z, \mathbf{x} - \frac{\mathbf{y}}{2}) d\mathbf{y}$$

$$= \frac{1}{(2\pi)^d} \int \int e^{-i\mathbf{p} \cdot (\mathbf{y}_1 - \mathbf{y}_2)} \delta(\mathbf{x} - \mathbf{y}_1 + \frac{\mathbf{y}_2}{2}) \rho(\mathbf{y}_1, \mathbf{y}_2) d\mathbf{y}_1 d\mathbf{y}_2$$

where $\rho(\mathbf{y}_1, \mathbf{y}_2) = \Psi(\mathbf{y}_1)\Psi^{\ast}(\mathbf{y}_2)$ is called the two-point function or the density matrix. As apparent from the definition the Wigner function contains all the information about $\rho$. The Wigner distribution have the simple properties

$$\int W(z, x, p) dx dp = \|\Psi\|_2^2, \quad \|W\|_2 = (2\pi)^{-d/2}\|\Psi\|_2^2.$$

This is the case of pure state Wigner distribution. What is the more pertinent for us is the so-called mixed state Wigner distribution.

Let us briefly review how a mixed state Wigner distribution arises in the time-reversal operation. Let $G_H(0, \mathbf{x}, z, \mathbf{y})$ be the Green’s function, with the point source located at $(z, \mathbf{y})$, for the reduced wave (Helmholtz) equation for which the Schrödinger equation is an approximation. By the self-adjointness of the Helmholtz equation, $G_H$ satisfies the symmetry property

$$G_H(0, \mathbf{x}, z, \mathbf{y}) = G_H(z, \mathbf{y}, 0, \mathbf{x}).$$

The wave field $\Psi_m$ received at the mirror is given by

$$\Psi_m(z, \mathbf{x}_m) = \chi_A(\mathbf{x}_m) \int G_H(0, \mathbf{x}_m, z, \mathbf{x}_s) \Psi_0(\mathbf{x}_s) d\mathbf{x}_s$$

$$= \chi_A(\mathbf{x}_m) \int G_H(z, \mathbf{x}_s, 0, \mathbf{x}_m) \Psi_0(\mathbf{x}_s) d\mathbf{x}_s$$

where $\chi_A$ is the aperture function of the phase-conjugating mirror $A$.

After phase conjugation and back-propagation we have at the source plane the wave field

$$\Psi^B(z, x; k) = \int G_H(z, x, 0, \mathbf{x}_m) \overline{G_H(z, \mathbf{x}_s, 0, \mathbf{x}_m) \chi_A(\mathbf{x}_m) \Psi_0(\mathbf{x}_s)} d\mathbf{x}_m d\mathbf{x}_s.$$

In the parabolic approximations the Green’s function $G_H(z, \mathbf{x}, 0, \mathbf{y})$ is approximated by $e^{ikz}G_S(z, \mathbf{x}, \mathbf{y})$ where $G_S(z, \mathbf{x}, \mathbf{y})$ is the propagator of the Schrödinger equation. Making the approximation in the above expression for the back-propagated field we obtain

$$\Psi^B(z, x; k) = \int G_S(z, x, \mathbf{x}_m) \overline{G_S(z, \mathbf{x}_s, \mathbf{x}_m) \Psi_0(\mathbf{x}_s) \chi_A(\mathbf{x}_m)} d\mathbf{x}_m d\mathbf{x}_s$$

$$= \int e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_s)} W(z, \frac{x + x_s}{2}, \mathbf{p}) \overline{\Psi_0(\mathbf{x}_s)} d\mathbf{p} d\mathbf{x}_s$$

(3)

where the Wigner distribution $W$ is given by

$$W(z, x, p)$$

(4)

$$= \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p} \cdot \mathbf{y}} G_S(z, x + \mathbf{y}/2, \mathbf{x}_m) \overline{G_S(z, x - \mathbf{y}/2, \mathbf{x}_m) \chi_A(\mathbf{x}_m)} d\mathbf{y} d\mathbf{x}_m.$$

This is a mixed-state Wigner distribution. In general, the integral in (4) should be interpreted in the distributional sense.
The Wigner distribution in (4) has the initial condition 
\[ W(0, x, p) = \frac{\chi_A(x)}{(2\pi)^d} \]
and can be treated as a generalized function on \( \mathbb{R}^{2d} \). Indeed, for any \( \Theta \in C^\infty_c(\mathbb{R}^d) \) we have
\[ \langle \Psi^B, \Theta \rangle = \int \int W(z, r, p) \Theta(r, p) dr dp \]
where the function \( \Theta \) is defined as
\[ \theta(r, p) = 2^d \int \Theta(y) e^{ip \cdot (y-r)/\gamma} \Psi_0(2r-y) dy. \]

If for instance \( \Psi_0 \in C^\infty_c(\mathbb{R}^d) \) then it is easy to see \( \Theta(y, p) \) is compactly supported in \( y \in \mathbb{R}^d \) and decays rapidly (faster than any power) in \( p \in \mathbb{R}^d \). As a result we can always approximate to arbitrary accuracy the distributional initial data such as (5) by square-integrable initial data.

The fluctuations of the back-propagated wave field is thus determined by the fluctuations of the Wigner distribution. The statistical stability or self-averaging of the Wigner distribution in turn explains, modulo the scaling limit, the persistence and stability of the super-focusing of the time-reversed, back-propagated wave field observed experimentally and numerically.

Our main results show that under various scaling limits, sufficient amount of spatial-transverse diversity experienced by the propagating wave pulse results in self-averaging and deterministic limiting laws.

From the perspective of the quantum stochastic dynamics in a random environment, our results say that due to the spatio- temporal diversity experienced by the wave function of the quantum particle the quantum dynamics has in the scaling limit a classical probabilistic description which is independent of the particular realization of the environment. The transition from a unitary evolution to an irreversible process is of course the outcome of the phase-space coarse-graining by the test functions. The results presented below are a rigorous demonstration of decoherence, a mechanism believed to be responsible for the emergence of the classical world from the quantum one [12], [20].

3. Assumptions

We assume \( V_z(x) = V(z, x) \) is \( x \)-homogeneous square-integrable process with the (partial) spectral measure \( \hat{V}(z, dq) \). which is an orthogonal random measure
\[ \mathbb{E}[\hat{V}(z, dp)\hat{V}(z, dq)] = \delta(p + q)\Phi_0(p) \ dp \ dq \]
and gives rise to the (partial) spectral representation of the refractive index field
\[ V_z(x) \equiv V(z, x) = \int \exp(ip \cdot x) \hat{V}(z, dp) . \]

In case that \( V(\vec{x}), \vec{x} \in \mathbb{R}^{d+1} \), is \( \vec{x} \)-homogeneous square-integrable random field with the full spectral density given by \( \Phi(\xi, k) \) we have the following relation
\[ \Phi_0(p) = \int \Phi(w, p) dw. \]

We also have the following relation between the partial and full spectral measures
\[ \hat{V}_z(dp) = \int e^{i\xi \cdot \vec{p}} \hat{V}(dw, dp) \]
such that
\[
E[\hat{V}_z(dp)\hat{V}_s(dq)] = \int e^{i(s-z)w} \Phi(w, p) \, dw \, \delta(p + q) \, dp \, dq
\]
\[= \Phi(s - z, p) \delta(p + q) \, dp \, dq\]
where
\[\Phi(s, p) = \int e^{isw} \Phi(w, p) \, dw.\]
Since \(\Phi(k) = \Phi(-k), \forall k \in \mathbb{R}^{d+1}\), we may assume in this case that
\[\Phi(w, q) = \Phi(-w, p) = \Phi(w, -p) = \Phi(-w, -p), \ \forall w, p\]
so that \(\Phi(s, p)\) is real-valued and \(\Phi(s, p) = \Phi(-s, p)\).

For simplicity of the analysis we assume that the spectrum \(\Phi\) is smooth and has a compact support. We also assume the sample field is almost surely smooth and its spatial derivatives all have finite moments. The conditions of smoothness and having a compact support are unnecessary; they are assumed here in order to make simple the key estimate (Proposition 5, 6, 7, 8).

Let \(F_z\) and \(F_z^+\) be the sigma-algebras generated by \(\{V_s : \forall s \leq z\}\) and \(\{V_s : \forall s \geq z\}\), respectively. Define the correlation coefficient
\[\rho(t) = \sup_{g \in F_z^+} \sup_{\Phi(\cdot, k) \in L^2(\mathbb{R})} \sup_{E[\cdot] = 1} E[hg].\]

**Assumption 1.** The correlation coefficient \(\rho(t)\) is integrable

When \(V_z\) is a Gaussian process, the correlation coefficient \(\rho(t)\) equals the linear correlation coefficient \(r(t)\) which has the following useful expression
\[r(t) = \sup_{g_1, g_2} \int \Phi(t - \tau_1 - \tau_2, k)g_1(\tau_1, k)g_2(\tau_2, k) \, dk \, d\tau_1 \, d\tau_2\]
where the supremum is taken over all \(g_1, g_2 \in L^2(\mathbb{R})\) which are supported on \((-\infty, 0] \times \mathbb{R}^d\) and satisfy the constraint
\[\int \Phi(t - t', k)g_1(t, k) \, dk \, dt'[k] \, dtk = 1.\]

There are various criteria for the decay rate of the linear correlation coefficients in the literature. For example, according to [11], Chapter V, Theorem 8 (after adaptation to the \(t\)-continuous version), even under the restrictive condition that \(\Phi \in C^\infty\) there is a large class of Gaussian processes whose correlation coefficient decays faster than any power law.

Secondly, we assume a 6-th order quasi-Gaussian property: Let
\[U_s^1(x) = V_s(x), \quad U_s^2(x) = E_s[V_s](x), \quad s \geq z.\]

**Assumption 2.** For any choices of \(\sigma_j \in \{1, 2\}, j = 1, 2, ..., N\) and a set of linear operators \(\{T_j\}\), there exists a finite constant \(C\)
\[E \left[ \prod_{j=1}^{N} T_j U_{s_j}^{\sigma_j}(x_j) \right] = 0, \quad N = 3, 5\]
\[E \left[ \prod_{j=1}^{N} T_j U_{s_j}^{\sigma_j}(x_j) \right] \leq C \sum_{\text{phasings}} \prod_{mn} E \left[ T_m U_{s_m}^{\sigma_m}(x_m) T_n U_{s_n}^{\sigma_n}(x_n) \right], \quad N = 4, 6\]
where the summation is over all possible pairings \(\{\text{phasings}\}\) among \(\{1, 2, ..., N\}\).
Finally we assume

**Assumption 3.** There exists a constant $C$ such that for Theorem 1, 2, 3 (i), (iii) and 4 (i), (iii)

\[ \lim_{\varepsilon \to 0} \mathbb{E} [\sup_{z < z_0} \| \hat{L}_z \xi \|_2^2] \leq \frac{C}{\varepsilon} \mathbb{E} [\| \hat{L}_z \xi \|_2^2], \quad \forall \theta \in C_c^\infty (\mathbb{R}^{2d}), \forall z_0 < \infty; \]

and for Theorem 3 (ii) and 4 (ii)

\[ \lim_{\varepsilon \to 0} \mathbb{E} [\sup_{z < z_0} \| \hat{L}_z \hat{L}_z \xi \|_2^2] \leq \frac{C}{\varepsilon^{2\alpha}} \mathbb{E} [\| \hat{L}_z \hat{L}_z \xi \|_2^2], \quad \forall \theta \in C_c^\infty (\mathbb{R}^{2d}), \forall z_0 < \infty; \]

where $\hat{L}_z$ is defined, respectively, by (66), (67), (100) and (121) and $\alpha \in (0, 1)$ as specified in the statements of the theorems.

Assumption 3 is readily satisfied for Gaussian random fields. This can be seen by first observing that $\hat{L}_z \xi$ is a Gaussian process and $\hat{L}_z \hat{L}_z \xi$ is a $\chi^2$-process and, secondly, by an application of Borell’s inequality [11] that the supremum over $z < z_0$ inside the expectation can be over-estimated by a log ($1/\varepsilon$) factor for excursion on the scale of any power of $1/\varepsilon$:

\[ \mathbb{E} [\sup_{z < z_0} \| \hat{L}_z \xi \|_2^2] \leq C \log \left( \frac{1}{\varepsilon} \right) \mathbb{E} [\| \hat{L}_z \xi \|_2^2]; \]

\[ \mathbb{E} [\sup_{z < z_0} \| \hat{L}_z \hat{L}_z \xi \|_2^2] \leq C \log^2 \left( \frac{1}{\varepsilon} \right) \mathbb{E} [\| \hat{L}_z \hat{L}_z \xi \|_2^2]. \]

4. Main results

In the standard scaling, we set

\[ L_z = L_x = \frac{1}{\varepsilon^2} \gg 1, \quad \sigma = \varepsilon. \]

To describe the small scale wave energy we consider the scaled version of the Wigner distribution

\[ W^\varepsilon (z, x, p) = \frac{1}{(2\pi)^d} \int e^{-ip \cdot y} \Psi (z, x + \frac{\varepsilon^2 y}{2}) \Psi^* (z, x - \frac{\varepsilon^2 y}{2}) dy \]

\[ = \frac{1}{(2\pi)^d} \int e^{-ip \cdot (y_1 - y_2)/\varepsilon^2} \delta (x - \frac{y_1 + y_2}{2}) \rho (y_1, y_2) dy_1 dy_2. \]

The Wigner distribution $W^\varepsilon$ has a limit as certain measure, the Wigner measure, introduced in [15]. But as remarked in the introduction, we always consider uniformly $L^2$ initial condition induced by a mixed-state density matrix $\rho$.

The Wigner distribution satisfies the Wigner-Moyal equation

\[ \frac{\partial W^\varepsilon}{\partial z} + \frac{p}{k} \cdot \nabla W^\varepsilon + \frac{k}{\varepsilon} \hat{L}_z W^\varepsilon = 0 \]

with $W^\varepsilon (x, p) = W^\varepsilon (z, x, p)$. Here the integral operator $\hat{L}_z$ is given by

\[ \hat{L}_z W^\varepsilon (x, \tilde{x}, p) = i \int e^{i q \cdot \tilde{x}} [W^\varepsilon (x, p + q/2) - W^\varepsilon (x, p - q/2)] \tilde{V} (\frac{z}{\varepsilon^2}, dq), \quad \tilde{x} = x \varepsilon^{2\alpha} \]

with $\alpha = 1$.

The more general case with $\alpha \in (0, 1)$ can be derived from a somewhat different scaling (cf. the scaling leading to Theorem 2): We probe a highly anisotropic medium $V (z, \varepsilon^{2-2\alpha} x)$ with the strength

\[ \sigma = \varepsilon^{2\alpha - 1} \]

with a wave beam composed of waves of lengths comparable to that of the medium, so we replace $k$ by $k\varepsilon^{2-2\alpha}$:

$$k \rightarrow k\varepsilon^{2-2\alpha}$$

in the parabolic wave equation. We then use the following definition of the Wigner distribution to resolve the wave energy:

$$W^\varepsilon(z, x, p) = \frac{1}{(2\pi)^d} \int e^{-ip \cdot y} \Psi(z, x + \varepsilon^{2\alpha}y)\Psi^*(z, x - \varepsilon^{2\alpha}y) dy.$$  

The difference in scaling between (19) and (18) is, of course, due to the rescaling of coordinates and its variants studied in the sequel are understood in the weak sense and we consider their norms of the initial data and produces a sequence of square-integrable as is assumed here. However, due to the weak limiting procedure, there is no guarantee that the $L^2$-norm of the initial data is preserved in the limit.

We will not address the uniqueness of solution for the Wigner-Moyal equation (20) but we will show that as $\varepsilon \rightarrow 0$ any sequence of weak solutions to eq. (20) converges in a suitable sense to the unique solution of a deterministic transport equation.

We state our first result in the following theorem.

**Remark 1.** Since Eq. (20) is linear, the existence of weak solutions can be established straightforwardly by the weak-star compactness argument. Let us briefly comment on this. First, we introduce truncation $N < \infty$

$$V_N(z, x) = V(z, x), \quad |V(z, x)| < N$$

and zero otherwise. Clearly, for such bounded $V_N$ the corresponding operator $\mathcal{L}_z^\varepsilon$ is a bounded self-adjoint operator on $L^2(\mathbb{R}^d)$. Hence the corresponding Wigner-Moyal equation preserves the $L^2$-norm of the initial data and produces a sequence of $L^2$-bounded weak solutions. Passing to the limit $N \rightarrow \infty$ we obtain a $L^2$-weak solution for the original Wigner-Moyal equation if $V$ is locally square-integrable as is assumed here. However, due to the weak limiting procedure, there is no guarantee that the $L^2$-norm of the initial data is preserved in the limit.
Theorem 1. Let $\Phi \in C_c^\infty$ and Assumption 1, 2, 3 be satisfied. Then the weak solution $W_z^\varepsilon$ of the Wigner-Moyal equation (20), (17) with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution $W_z$ of the radiative transfer equation

$$ \frac{\partial W_z(x,p)}{\partial z} + \frac{p}{k} \cdot \nabla W_z(x,p) = k^2 \mathcal{L} W_z(x,p) $$

with the initial condition $W_0$ and one of the following scattering operator $\mathcal{L}$:

Case (i): $0 < \alpha < 1$,

$$ \mathcal{L} W_z(x,p) = 2\pi \int \Phi(0, q - p) [W_z(x, q) - W_z(x, p)] dq; $$

Case (ii): $\alpha = 1$,

$$ \mathcal{L} W_z(x,p) = 2\pi \int \Phi \left( \frac{|q|^2 - |p|^2}{2k}, q - p \right) [W_z(x, q) - W_z(x, p)] dq; $$

Case (iii): $\alpha > 1$,

$$ \mathcal{L} W_z = 0. $$

The case of $\alpha = 0$ corresponds to the so-called white-noise scaling whose limit is a Markovian process [3]. Eq. (24) has recently been obtained in [3] for strongly mixing $z$-Markovian refractive index fields with a bounded generator.

In order to obtain a nontrivial scattering kernel for $\alpha > 1$ we need to boost up the intensity of $V$ (cf. Theorem 3).

Next we consider a second type of scaling limits which starts with the highly anisotropic medium $V(z, \varepsilon^{2-2\alpha}x)$. We then set

$$ L_z = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^{2\alpha - 1}, \quad 0 < \alpha < 1 $$

under which the parabolic wave equation becomes

$$ ik^{-1} \frac{\partial \Psi}{\partial z} + 2^{-1} k^2 \varepsilon^2 \Delta \Psi + \varepsilon^{2\alpha-3} V(z\varepsilon^{-2}, x\varepsilon^{-2\alpha}) \Psi = 0, \quad \Psi(0, x) = \Psi_0(x). $$

The radiative transfer scaling (24) is the limiting case $\alpha = 1$. The time-evolution of the Wigner function (15) is governed by the Wigner-Moyal equation (20) with the following operator $\mathcal{L}_z^\varepsilon$

$$ \mathcal{L}_z^\varepsilon W_z^\varepsilon(x, \tilde{x}, p) $$

$$ = \int e^{i q \cdot \tilde{x} - \varepsilon^{2\alpha-2} [W_z^\varepsilon(x, p + \varepsilon^{2-2\alpha} q/2) - W_z^\varepsilon(x, p - \varepsilon^{2-2\alpha} q/2)]} \tilde{\nu}_z^\varepsilon dq, \quad \tilde{x} = x\varepsilon^{-2\alpha}. $$

The partial Fourier transform of $\mathcal{L}_z^\varepsilon \theta$ is now given by (21) with the following $\delta_z W_z$

$$ \delta_z V_z^\varepsilon(\tilde{x}, y) = \varepsilon^{2\alpha-2} [V_z^\varepsilon(\tilde{x} + y\varepsilon^{2-2\alpha}/2) - V_z^\varepsilon(\tilde{x} - y\varepsilon^{2-2\alpha}/2)]. $$

We now state the result for the scaling limit (20), (27).

Theorem 2. Let $0 < \alpha < 1$. Let $\Phi \in C_c^\infty$ and Assumption 1, 2, 3 be satisfied. Then the weak solution $W_z^\varepsilon$ of the Wigner-Moyal equation (20), (27) with the initial condition $W_0 \in L^2(\mathbb{R}^{2d})$ converges in probability as the distribution-valued process to the deterministic limit given by the weak solution $W_z$ of the following advection-diffusion equations with the initial condition $W_0$:

$$ \frac{\partial W_z}{\partial z} + \frac{p}{k} \cdot \nabla W_z = k^2 \nabla_p \cdot D \nabla_p W_z, $$

with one of the following diffusion tensors $D$. 

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\[ \alpha \in (0, 1): \quad D = \pi \int \Phi(0, q)q \otimes q d q; \]

\[ \alpha > 1: \quad D = 0. \]

For \(\alpha = 1\) the limit is the same as that in Theorem 1 Case (ii). \(\alpha = 0\) gives rise to the white-noise limit for the Liouville equation. The advection-diffusion equation (30) can be obtained from (23) under the diffusion limit of the latter.

Let us consider yet another type of scaling limit parametrized by \(\beta\). We first assume a highly anisotropic medium \(V(\varepsilon^{2-2\beta}z, x)\) and set

\[ L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon \]

i.e. the radiative transfer scaling. The Schrödinger equation then becomes

\[ ik^{-1} \frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1} k^{-2} \varepsilon^2 \Delta \Psi^\varepsilon + \varepsilon^{-1} V(\varepsilon^{2-2\beta}, x\varepsilon^{-2}) \Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, x) = \Psi_0(x), \quad \beta < 1 \]

the corresponding Wigner-Moyal equation is

\[ \frac{\partial W^\varepsilon}{\partial z} + \frac{p}{k} \cdot \nabla W^\varepsilon + \frac{k}{\varepsilon^\beta} L^\varepsilon_z W^\varepsilon = 0 \]

with

\[ L^\varepsilon_z W^\varepsilon (x, \tilde{x}, p) = i \varepsilon^{\beta-1} \int e^{i q \cdot \tilde{x}} [W^\varepsilon(x, p + q/2) - W^\varepsilon(x, p - q/2)] \hat{V}(\frac{z}{\varepsilon^{2\beta}}, dq), \quad \tilde{x} = x \varepsilon^{-2}, \beta < 1. \]

Eq. (35) is a borderline case of the following family of scaling limits. Let us consider probing an anisotropic medium \(V(\varepsilon^{2-2\beta}z, \varepsilon^{2-2\alpha}x), \alpha > 0\), with a wave beam composed of waves of lengths comparable to that of the medium, so we switch to (18) and (19) for the formulation of scaling limits.

Two situations arise: Case (i) \(\alpha < \beta\) and Case (ii) \(\alpha > \beta\).

In the first case \(\alpha < \beta\) we set the strength of the medium fluctuation to be

\[ \sigma = \varepsilon^{2\alpha-\beta}. \]

The resulting equation is (34) with

\[ L^\varepsilon_z W^\varepsilon (x, \tilde{x}, p) = i \int e^{i q \cdot \tilde{x}} [W^\varepsilon(x, p + q/2) - W^\varepsilon(x, p - q/2)] \hat{V}(\frac{z}{\varepsilon^{2\beta}}, dq), \quad \tilde{x} = x \varepsilon^{-2\alpha} \]

In the second case \(\alpha > \beta\) we set the strength of the medium fluctuation to be

\[ \sigma = \varepsilon^\alpha. \]

The resulting equation is (34) with

\[ L^\varepsilon_z W^\varepsilon (x, \tilde{x}, p) = i \varepsilon^{\beta-\alpha} \int e^{i q \cdot \tilde{x}} [W^\varepsilon(x, p + q/2) - W^\varepsilon(x, p - q/2)] \hat{V}(\frac{z}{\varepsilon^{2\beta}}, dq), \quad \tilde{x} = x \varepsilon^{-2\alpha}. \]

We have the following theorem.

**Theorem 3.** Let \(\alpha, \beta > 0\). Let \(\Phi \in C^\infty\) and Assumption 1, 2, 3 be satisfied. Then the weak solution \(W^\varepsilon\) of the Wigner-Moyal equation (34) with Case (i): (37) or Case (ii): (36) and the initial condition \(W_0 \in L^2(\mathbb{R}^{2d})\) converges in probability as the distribution-valued process to the
The deterministic limit given by the weak solution $W_z$ of the transport equation with the initial condition $W_0$:

$$\frac{\partial W_z}{\partial z} + \frac{P_x}{k} \cdot \nabla W_z = k^2 \mathcal{L} W_z,$$

with one of the following the scattering operators $\mathcal{L}$.

**Case (i):** $\alpha < \beta$

$$\mathcal{L} W_z(x, p) = 2\pi \int \Phi(0, q - p) [W_z(x, q) - W_z(x, p)] \, dq.$$  \tag{39}

**Case (ii):** $1 < \alpha / \beta < 4/3, \, d \geq 3$

$$\mathcal{L} W_z(x, p) = 2\pi \int \delta(\frac{|q|^2 - |p|^2}{2k}) \left[ \int \Phi(w, q - p) \, dw \right] [W_z(x, q) - W_z(x, p)] \, dq.$$  \tag{40}

**Case (iii):** $\alpha = \beta$

$$\mathcal{L} W_z(x, p) = 2\pi \int \Phi(\frac{|q|^2 - |p|^2}{2k}, q - p) [W_z(x, q) - W_z(x, p)] \, dq.$$  \tag{41}

Theorem 3 (i) probably holds for $d = 2$ and $\alpha / \beta > 4/3$ but we do not pursue it here in order to keep the argument as simple as possible.

Earlier \cite{15}, \cite{5} have established the convergence of the mean field $\mathbb{E} W_z^\varepsilon$ for $\varepsilon$-independent Gaussian media and $d \geq 3$. Their transport equation can be viewed as a limiting case of (38) in which $\Phi(\xi, k)$ is a $\delta$-function concentrated at $\xi = 0$. See also \cite{17} for mean-field results for $\varepsilon$-finitely dependent potentials.

Unlike the transport equations (24), (23), the scattering kernel (40) is elastic in the sense that it preserves the kinetic energy of the scattered particle so that the incoming and outgoing momenta $q, p$ have the same magnitude.

Finally let us consider two other types of scaling limit starting with the slowly-varying refractive index field $V(\varepsilon^{2-2\beta} z, \varepsilon^{2-2\alpha} x)$, $\alpha, \beta \in (0, 1)$. In the first case

$$\beta > \alpha, \quad 0 < \alpha < 1$$

we set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^{2\alpha - \beta}$$

under which we have the following parabolic wave equation

$$i k^{-1} \frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1} k^{-2} \varepsilon^2 \Delta \Psi^\varepsilon + \varepsilon^{2\alpha - \beta - 2} V(\varepsilon^{2-2\beta} x, \varepsilon^{2-2\alpha} x) \Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, x) = \Psi_0(x),$$

and the corresponding Wigner-Moyal equation (41) with

$$i k^{-1} \frac{\partial W_z^\varepsilon}{\partial z} + 2^{-1} k^{-2} \varepsilon^2 \Delta W_z^\varepsilon + \varepsilon^{2\alpha - 2} V(\varepsilon^{2-2\beta} x, \varepsilon^{2-2\alpha} x) W_z^\varepsilon = 0, \quad W_z^\varepsilon(0, x, p) = \Psi_0(x, p).$$

In the second case

$$\alpha > \beta, \quad 0 < \alpha < 1,$$

we set

$$L_x = L_z = \varepsilon^{-2}, \quad \sigma = \varepsilon^\alpha.$$ 

After rescaling the parabolic wave equation reads

$$i k^{-1} \frac{\partial \Psi^\varepsilon}{\partial z} + 2^{-1} k^{-2} \varepsilon^2 \Delta \Psi^\varepsilon + \varepsilon^{2\alpha - 2} V(\varepsilon^{2-2\beta} x, \varepsilon^{2-2\alpha} x) \Psi^\varepsilon = 0, \quad \Psi^\varepsilon(0, x) = \Psi_0(x),$$
and the corresponding Wigner-Moyal equation takes the form of \( \mathcal{L}_\varepsilon W_\varepsilon(x, \tilde{x}, p) \) with
\[
(4 \mathcal{L}_\varepsilon W_\varepsilon(x, \tilde{x}, p) = i\varepsilon^{-\alpha} \int e^{i\xi \cdot \tilde{x}} e^{2\alpha - 2} \left[ W_\varepsilon(x, p + \varepsilon^{2-2\alpha} q/2) - W_\varepsilon(x, p - \varepsilon^{2-2\alpha} q/2) \right] \tilde{\chi}(\varepsilon^{-2\alpha}; dq), \tilde{x} = x \varepsilon^{-2\alpha}
\]

**Theorem 4.** Let \( \alpha, \beta \in (0, 1) \). Let \( \Phi \in C^\infty_\varepsilon \) and Assumption 1, 2, 3 be satisfied. Then the weak solution \( W_\varepsilon \) of the Wigner-Moyal equation \((34)\) with Case (i): \((45)\) or Case (ii): \((49)\) and the initial condition \( W_0 \in L^2(\mathbb{R}^2) \) converges in probability as the distribution-valued process to the deterministic limit given by the weak solution \( W \) of the advection-diffusion equation \((30)\) with the following diffusion tensors:

- **Case (i)-(42), (43):**
  \[
  D = \pi \int \Phi(0, q) q \otimes dq.
  \]
- **Case (ii)-(46), (47):**
  \[
  D(p) = \pi k |p|^{-1} \int \left[ \int \Phi(w, p) dw \right] p \otimes p dp,
  \]
  where \( p_\perp \in \mathbb{R}^{d-1}, p_\perp \cdot p = 0 \).
- **Case (iii):** \( \alpha = \beta \)
  \[
  D(p) = \pi \int \Phi(k^{-1} q) q \otimes dq
  \]

The advection-diffusion equation with \((50), (51)\) and \((52)\) are the diffusion limit of the transport equations \((40)\) and \((24)\), respectively. The limiting case of \( \alpha = 0 \) gives rise to the white-noise model of the Liouville equation. We believe that the result for Case (ii) can be extended to \( d = 2 \) and \( \beta/\alpha \in (0, 1) \).

**Remark 2.** Taken together, our results have roughly covered all the super-parabolic scaling
\[
L_x \gg \sqrt{L_z}.
\]
To see this, let us set
\[
L_z \sim L^2_\varepsilon = \varepsilon^{-2\gamma}, \quad 0 < \gamma < 2
\]
and define the following Wigner transform
\[
W_\varepsilon(z, x, p) = \frac{1}{(2\pi)^d} \int e^{-ip\cdot y} \Psi(z, x + \varepsilon^2 y/2) \Psi^*(z, x - \varepsilon^2 y/2) dy
\]
\[
= \frac{1}{(2\pi)^d} \int e^{-ip(\gamma - \zeta_2)/\varepsilon^2} \delta(x - \gamma_1 + \gamma_2) \rho(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2
\]
with the new parameter
\[
\bar{\varepsilon} = \varepsilon^{-2\gamma}
\]
and analyze analogously the preceding scaling limits as parametrized by \( \bar{\varepsilon} \). For a systematic treatment of scaling limits resulting in a transport equation, see [7].

Our approach is to use the conditional shift \([13]\) to formulate the corresponding martingale problem parametrized by \( \varepsilon \) and adapt the perturbed test function technique to the probabilistic setting to establish the convergence of the martingales. It then turns out that after subtracting the drift and the Stratonovitch correction term the limiting martingale has null quadratic variation (see Proposition \([6]\) implying that the limit is deterministic. The perturbed test functions constructed here (see e.g. \([7], [8] \) and \([82] \) are related to those in \([2], [3] \) but our analysis is carried out in a
more general framework as formulated in \[5\] and provides a unified treatment of a range of scaling limits from the radiative transfer to the diffusion limit and the white-noise limit.

5. Martingale formulation

We consider the weak formulation of the Wigner-Moyal equation:

\[
[W_{z}^\varepsilon, \theta] - [W_{0}, \theta] = k^{-1} \int_{0}^{z} [W_{s}^\varepsilon, p \cdot \nabla \theta] ds + \frac{k}{\varepsilon} \int_{0}^{z} [W_{s}^\varepsilon, \mathcal{L}_{z}^\varepsilon \theta] ds
\]

for any test function $\theta \in C_\infty(\mathbb{R}^{2d})$, which is a dense subspace in $L^2(\mathbb{R}^{2d})$. The tightness result (see below) implies that for $L^2$ initial data the limiting measure $\mathbb{P}$ is supported in $L^2([0, z_0]; L^2(\mathbb{R}^{2d}))$.

For tightness as well as identification of the limit, the following infinitesimal operator $\mathcal{A}^\varepsilon$ will play an important role. Let $V_{z}^\varepsilon \equiv V(z/\varepsilon^2, \cdot)$. Let $\mathcal{F}_{z}^\varepsilon$ be the $\sigma$-algebras generated by $\{V_{s}^\varepsilon, s \leq z\}$ and $\mathbb{E}_{z}^\varepsilon$ the corresponding conditional expectation w.r.t. $\mathcal{F}_{z}^\varepsilon$. Let $\mathcal{M}^\varepsilon$ be the space of measurable function adapted to $\{\mathcal{F}_{z}, \forall t\}$ such that $\sup_{t<z} \mathbb{E}|f(z)| < \infty$. We say $f(\cdot) \in D(\mathcal{A}^\varepsilon)$, the domain of $\mathcal{A}^\varepsilon$, and $\mathcal{A}^\varepsilon f = g$ if $f, g \in \mathcal{M}^\varepsilon$ and for $f^\delta(z) \equiv \delta^{-1}[\mathbb{E}_{z}^\varepsilon f(z + \delta) - f(z)]$ we have

\[
\sup_{z, \delta} \mathbb{E}|f^\delta(z)| < \infty \\
\lim_{\delta \to 0} \mathbb{E}|f^\delta(z) - g(z)| = 0, \quad \forall z.
\]

Consider a special class of admissible functions $f(z) = \phi([W_{z}^\varepsilon, \theta]), f'(z) = \phi'([W_{z}^\varepsilon, \theta]), \forall \phi \in C_\infty(\mathbb{R})$ we have the following expression from (53) and the chain rule

\[
\mathcal{A}^\varepsilon f(z) = f'(z) \left[ \frac{1}{k} [W_{z}^\varepsilon, p \cdot \nabla \theta] + \frac{k}{\varepsilon} [W_{z}^\varepsilon, \mathcal{L}_{z}^\varepsilon \theta] \right].
\]

In case of the test function $\theta$ that is also a functional of the media we have

\[
\mathcal{A}^\varepsilon f(z) = f'(z) \left[ \frac{1}{k} [W_{z}^\varepsilon, p \cdot \nabla \theta] + \frac{k}{\varepsilon} [W_{z}^\varepsilon, \mathcal{L}_{z}^\varepsilon \theta] + [W_{z}^\varepsilon, \mathcal{A}^\varepsilon \theta] \right]
\]

and when $\theta$ depends explicitly on the fast spatial variable

$\tilde{x} = x/\varepsilon^{2\alpha}$

the gradient $\nabla$ is conveniently decomposed into the gradient w.r.t. the slow variable $\nabla x$ and that w.r.t. the fast variable $\nabla \tilde{x}$

$\nabla = \nabla x + \varepsilon^{-2\alpha} \nabla \tilde{x}$.

A main property of $\mathcal{A}^\varepsilon$ is that

\[
f(z) - \int_{0}^{z} \mathcal{A}^\varepsilon f(s) ds \quad \text{is a } \mathcal{F}_{z}^\varepsilon\text{-martingale}, \quad \forall f \in D(\mathcal{A}^\varepsilon).
\]

Also,

\[
\mathbb{E}^\varepsilon_{s} f(z) - f(s) = \int_{s}^{z} \mathbb{E}^\varepsilon_{s} \mathcal{A}^\varepsilon f(\tau) d\tau \quad \forall s < z \quad \text{a.s.}
\]

(see \[5\]). We denote by $\mathcal{A}$ the infinitesimal operator corresponding to the unscaled process $V_{z}(\cdot) = V(z, \cdot)$. 

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6. Proof of Theorem 1

6.1. Tightness. In the sequel we will adopt the following notation

\[ f(z) \equiv \phi((W^z_\varepsilon, \theta)), \quad f'(z) \equiv \phi'(W^z_\varepsilon, \theta), \quad f''(z) \equiv \phi''((W^z_\varepsilon, \theta)), \quad \forall \phi \in C^\infty(\mathbb{R}). \]

Namely, the prime stands for the differentiation w.r.t. the original argument (not \( t \)) of \( f, f' \) etc.

Let \( D([0, \infty); L^2_{\text{w}}(\mathbb{R}^{2d})) \) be the \( L^2 \)-valued right continuous processes with left limits endowed with the Skorohod topology. A family of processes \{ \{W^\varepsilon, 0 < \varepsilon < 1\} \subset D([0, \infty); L^2_{\text{w}}(\mathbb{R}^{2d})) \) is tight if and only if the family of processes \{ \{W^\varepsilon, \theta, 0 < \varepsilon < 1\} \subset D([0, \infty); L^2_{\text{w}}(\mathbb{R}^{2d})) \) is tight for all \( \theta \in C^\infty_c[0, \infty] \) [10]. We use the tightness criterion of [14] (Chap. 3, Theorem 4), namely, we will prove: Firstly,

\[
\lim_{N \to \infty} \limsup_{\varepsilon \to 0} \mathbb{P}\{ \sup_{z < z_0} |(W^\varepsilon_{\gamma}, \theta)| \geq N \} = 0, \quad \forall z_0 < \infty.
\]

Secondly, for each \( \phi \in C^\infty(\mathbb{R}) \) there is a sequence \( f^\varepsilon(z) \in D(A^\varepsilon) \) such that for each \( z_0 < \infty \) \( \{A^\varepsilon f^\varepsilon(z), 0 < \varepsilon < 1, 0 < z < z_0\} \) is uniformly integrable and

\[
\lim_{\varepsilon \to 0} \mathbb{P}\{ \sup_{z < z_0} |f^\varepsilon(z) - \phi((W^\varepsilon_{\gamma}, \theta))| \geq \delta \} = 0, \quad \forall \delta > 0.
\]

Then it follows that the laws of \{ \{W^\varepsilon, \theta, 0 < \varepsilon < 1\} \} are tight in the space of \( D([0, \infty); \mathbb{R}) \).

First, condition (58) is satisfied because the \( L^2 \)-norm is uniformly bounded.

Let

\[
\hat{L}^\varepsilon_{\gamma} \theta(x, \bar{x}, p) \equiv i\varepsilon^{-2} \int_{-\infty}^{\infty} e^{i\frac{q\cdot x}{2}} [\theta(x, p + \frac{q}{2}) - \theta(x, p - \frac{q}{2})] e^{ik^{-1}(s-z)p\cdot q/\varepsilon^2 s^2} \hat{V}_z^\varepsilon(e^s dq) ds
\]

which has a compact support since both \( \theta \) and \( \Phi \) do. Note that the operator \( \hat{L}^\varepsilon_{\gamma} \) maps a real-valued function \( \theta \) to a real-valued function.

Lemma 1.

\[
\mathbb{E}[(\hat{L}^\varepsilon_{\gamma} \theta(x, \bar{x}, p))^2] \leq \left[ \int_0^{\infty} \rho(s) ds \right]^2 \int [\theta(x, p + \frac{q}{2}) - \theta(x, p - \frac{q}{2})]^2 \Phi(\xi, q) d\xi dq
\]

which has an \( \varepsilon \)-uniformly bounded support and is bounded uniformly in \( x, p, \varepsilon \).

Proof. Consider the following trial functions in the definition of the maximal correlation coefficient

\[
h = h_s(x, p) = i \int e^{iq\cdot x}/2a [\theta(x, p + \frac{q}{2}) - \theta(x, p - \frac{q}{2})] e^{ik^{-1}(s-z)p\cdot q/\varepsilon^2 s^2} \hat{V}_z^\varepsilon(e^s dq) \]

\[
g = g_t(x, p) = i \int e^{iq\cdot x}/2a [\theta(x, p + \frac{q}{2}) - \theta(x, p - \frac{q}{2})] e^{ik^{-1}(s-z)p\cdot q/\varepsilon^2 s^2} \hat{V}_z^\varepsilon(e^s dq)
\]

It is easy to see that

\[
h_s \in L^2(P, \Omega, \mathcal{F}_{\varepsilon^{-2} s})
\]

\[
g_t \in L^2(P, \Omega, \mathcal{F}_{\varepsilon^{-2} t})
\]

and their second moments are uniformly bounded in \( x, p, \varepsilon \) since

\[
\mathbb{E}[h_s^2](x, p) \leq \mathbb{E}[g_t^2](x, p)
\]

\[
\mathbb{E}[g_t^2](x, p) = \int [\theta(x, p + \frac{q}{2}) - \theta(x, p - \frac{q}{2})]^2 \Phi(\xi, q) d\xi dq
\]

which is uniformly bounded for any integrable spectral density \( \Phi \).
From the definition \[65\] we have
\[
|E[h_s(x, p)h_t(y, q)]| = |E[h_sg_t]| \leq \rho(\varepsilon^{-2}(t-z))E^{1/2}[h_s^2(x, p)]E^{1/2}[g_t^2(y, q)].
\]
Hence by setting \(s = t, x = y, p = q\) first and the Cauchy-Schwartz inequality we have
\[
E[h_s^2(x, p)] \leq \rho^2(\varepsilon^{-2}(s-z))E[g_t^2(x, p)]
\]
\[
|E[h_s(x, p)h_t(y, q)]| \leq \rho(\varepsilon^{-2}(t-z))\rho(\varepsilon^{-2}(s-z))E^{1/2}[g_t^2(x, p)]E^{1/2}[g_t^2(y, q)], \quad \forall s, t \geq z, \forall x, y.
\]
Hence
\[
\varepsilon^{-4}\int_0^\infty \int_0^\infty E[h_s(x, p)g_t(x, p)]ds dt \leq E[g_t^2](x, p)\left[\int_0^\infty \rho(s)ds\right]^2
\]
which together with \[68\] yields \[61\]. \hfill \Box

**Corollary 1.**

\[64\]
\[
E\left[\mathbf{p} \cdot \nabla_x \tilde{L}_z^\varepsilon \theta \right]^2(x, p)
\]
\[
\leq \left[\int_0^\infty \rho(s)ds\right]^2 \int [\mathbf{p} \cdot \nabla_x \theta(x, p + q/2) - \mathbf{p} \cdot \nabla_x \theta(x, p - q/2)]^2 \Phi(\xi, q)d\xi dq
\]

which has an \(\varepsilon\)-uniformly bounded support and is bounded uniformly in \(x, p, \varepsilon\).

Inequality \[64\] can be obtained from the expression
\[
\mathbf{p} \cdot \nabla_x \tilde{L}_z^\varepsilon \theta(x, \tilde{x}, p)
\]
\[
\equiv i\varepsilon^{-2}\int_\mathbb{R} [\tilde{\gamma} \cdot \nabla_{\tilde{x}} \theta(x, p + q/2) - \tilde{\gamma} \cdot \nabla_{\tilde{x}} \theta(x, p - q/2)]e^{ik^{-1}(s-z)p \cdot q/\varepsilon^{2\alpha}}E_{\varepsilon}^\gamma \tilde{\gamma} \varepsilon (d\gamma) ds
\]
as in Lemma \[1\].

The main property of \(\tilde{L}_z^\varepsilon \theta\) is that it solves the corrector equation
\[65\]
\[
\left[\varepsilon^{-2}\mathbf{p} / \kappa \cdot \nabla_{\tilde{x}} + A \right] \tilde{L}_z^\varepsilon \theta = \varepsilon^{-2} L_z^\varepsilon \theta.
\]

Eq. \[65\] can also be solved by using \[21\], yielding the solution
\[66\]
\[
\mathcal{F}_z^{-1} \tilde{L}_z^\varepsilon \theta(x, \tilde{x}, y) = \varepsilon^{-2} \int_\mathbb{R} e^{-i\varepsilon^{-2}\alpha k^{-1}(s-z)\nabla_{\tilde{x}} \nabla_{\tilde{x}}} [E_{\varepsilon}^\gamma \delta_{\varepsilon} V_x^\varepsilon] \mathcal{F}_z^{-1} \theta \] \(x, \tilde{x}, y)ds
\]

where
\[
\delta_{\varepsilon} V_x^\varepsilon(\tilde{x}, y) = V_x^\varepsilon(\tilde{x} + y/2) - V_x^\varepsilon(\tilde{x} - y/2).
\]

Recall that \(\nabla_{\tilde{x}}\) and \(\nabla_{\tilde{x}}\) are the gradients w.r.t. the fast variable \(\tilde{x}\) and the slow variable \(x\), respectively.

We will need to estimate the iteration of \(\tilde{L}_z^\varepsilon\) and \(\tilde{L}_z^\varepsilon\):

\[
\tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta(x, \tilde{x}, p) = -\varepsilon^{-2} \int_\mathbb{R} \int \tilde{\gamma} \varepsilon (d\gamma) E_{\varepsilon}^\gamma [\tilde{\gamma} \varepsilon (d\gamma')] e^{i\varepsilon^{-k^{-1}(s-z)p \cdot q/\varepsilon^{2\alpha}}}
\]

\[
\{[\theta(x, p + q'/2 + q/2) - \theta(x, p + q'/2 - q/2)] e^{ik^{-1}(s-z)q' \cdot q/(2\varepsilon^{2\alpha})}
\]

\[
- [\theta(x, p - q'/2 + q/2) - \theta(x, p - q'/2 - q/2)] e^{-ik^{-1}(s-z)q' \cdot q/(2\varepsilon^{2\alpha})}
\}
\]

\[
ds dt.
\]

\[
\tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta(x, \tilde{x}, p) = -\varepsilon^{-4} \int_\mathbb{R} \int \int E^\varepsilon_{\tilde{\gamma}} [\tilde{\gamma} \varepsilon (d\gamma)] E^\varepsilon_{\tilde{\gamma}} [\tilde{\gamma} \varepsilon (d\gamma')] e^{i\varepsilon^{-k^{-1}(s-z)p \cdot q/\varepsilon^{2\alpha}}} e^{ik^{-1}(t-z)p \cdot q/\varepsilon^{2\alpha}} e^{ik^{-1}(t-z)p \cdot q/\varepsilon^{2\alpha}}
\]

\[
\{[\theta(x, p + q'/2 + q/2) - \theta(x, p + q'/2 - q/2)] e^{ik^{-1}(s-z)q' \cdot q/(2\varepsilon^{2\alpha})}
\]

\[
- [\theta(x, p - q'/2 + q/2) - \theta(x, p - q'/2 - q/2)] e^{-ik^{-1}(s-z)q' \cdot q/(2\varepsilon^{2\alpha})}
\}
\]

\[
ds dt.
\]
Both have a compact support. Their second moments can be estimated as in Lemma I by using the 6-th order quasi-Gaussian property (Assumption 2). In order to carry out the same argument we need to approximate the terms of non-product form such as $\theta(x, p + q/2 + q/2)e^{ik^{-1}(s-z)q'q/(2\varepsilon^2\alpha^2)}$

by sum of product of functions of variables that are statistically coupled in the quasi-Gaussian pairing.

Since we do not need the pointwise estimate such as stated in Lemma II we shall demonstrate a simpler approach on the inverse Fourier transform:

$$\mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \left( x, \tilde{x}, y \right) = \varepsilon^{-2} \int_0^\infty \delta_z \delta_z e^{-i\varepsilon^{-2}\alpha^{-1}(s-z)\nabla_y \nabla_x} \left[ \mathbb{E}_x [\delta_x V_t] \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right] (x, \tilde{x}, y) ds$$

$$\mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \left( x, \tilde{x}, y \right) = -\varepsilon^{-4} \int_0^\infty e^{-i\varepsilon^{-2}\alpha^{-1}(t-z)\nabla_y \nabla_x} \left[ \mathbb{E}_x [\delta_x V_t] e^{-i\varepsilon^{-2}\alpha^{-1}(s-z)\nabla_y \nabla_x} \right] \left[ \mathbb{E}_x [\delta_x V_t] \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right] (x, \tilde{x}, y) ds dt.$$

**Lemma 2.**

$$\mathbb{E} \left\| \mathcal{L}_z^2 \mathcal{L}_z^2 \theta \right\|_2^2 \leq 8C \left( \int_0^\infty \rho(s) ds \right)^2 \mathbb{E} \left[ \mathcal{L}_z^2 \mathcal{L}_z^2 \theta \right]^2 \mathbb{E} \left[ \mathcal{L}_z^2 \mathcal{L}_z^2 \theta \right]^2 \int \left[ \theta(x, p + q/2) - \theta(x, p - q/2) \right]^2 \Phi(\xi, \eta) d\xi d\eta d\sigma d\tau$$

for some constant $C$ independent of $\varepsilon$.

**Proof.** The calculation for $\mathcal{L}_z^2 \mathcal{L}_z^2 \theta$ is simpler. Let us consider $\mathcal{L}_z^2 \mathcal{L}_z^2 \theta$.

By the Parseval theorem and the unitarity of $i\tau \nabla_y \nabla_x, \tau \in \mathbb{R}$,

$$\mathbb{E} \left\| \mathcal{L}_z^2 \mathcal{L}_z^2 \theta \right\|_2^2 = \mathbb{E} \left\| \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right\|_2^2$$

$$= \varepsilon^{-8} \int \mathbb{E} \left\{ \int_0^\infty e^{-i\varepsilon^{-2}\alpha^{-1}(t-z)\nabla_y \nabla_x} \left[ \mathbb{E}_x [\delta_x V_t] e^{-i\varepsilon^{-2}\alpha^{-1}(s-z)\nabla_y \nabla_x} \right] \left[ \mathbb{E}_x [\delta_x V_t] \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right] \right\} (x, \tilde{x}, y) ds dt$$

$$= \varepsilon^{-8} \int \mathbb{E} \left\{ \int_0^\infty e^{-i\varepsilon^{-2}\alpha^{-1}(t-z)\nabla_y \nabla_x} \left[ \mathbb{E}_x [\delta_x V_t] e^{-i\varepsilon^{-2}\alpha^{-1}(s-z)\nabla_y \nabla_x} \right] \left[ \mathbb{E}_x [\delta_x V_t] \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right] \right\} (x, \tilde{x}, y) ds dt$$

$$\leq C \varepsilon^{-8} \int \mathbb{E} \left\{ \left| \mathbb{E}_x [\delta_x V_t] e^{-i\varepsilon^{-2}\alpha^{-1}(s-z)\nabla_y \nabla_x} \left[ \mathbb{E}_x [\delta_x V_t] \mathcal{F}^{-1}_{\mathcal{L}_z^2 \mathcal{L}_z^2 \theta} \right] \right| (x, \tilde{x}, y) ds dt \right\} ds dt'$$

The last inequality follows from the quasi-Gaussian property. Note that in the $x$ integrals above the fast variable $\tilde{x}$ is integrated and not treated as independent of $x$. 

Let \( g(t) = \delta_z V_t^\varepsilon \) and
\[
h(s) = e^{-i\varepsilon 2\alpha k^{-1}(s-z)\nabla_y \cdot \nabla_x} \left[ \delta_z V_s^\varepsilon F_2^{-1} \theta \right].
\]
The same argument as that for Lemma 1 yields
\[
|\mathbb{E}[\mathbb{E}_z[g(t)]|\mathbb{E}_z[h(s)]]| \leq \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2]
\]
\[
\leq \rho(e^{-2}(t-s))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[h^2(s)], \quad t, s \geq z;
\]
\[
|\mathbb{E}[\mathbb{E}_z[g(t)]|\mathbb{E}_z[g(t')]| \leq \mathbb{E}^{1/2}[\mathbb{E}_z[g(t)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[g(t')]]^2]
\]
\[
\leq \rho(e^{-2}(t-s))\mathbb{E}^{1/2}[g^2(t)]\mathbb{E}^{1/2}[g^2(t')], \quad t, t' \geq z;
\]
\[
|\mathbb{E}[\mathbb{E}_z[h(s)]|\mathbb{E}_z[h(s')]] | \leq \mathbb{E}^{1/2}[\mathbb{E}_z[h(s)]^2]\mathbb{E}^{1/2}[\mathbb{E}_z[h(s')]]^2]
\]
\[
\leq \rho(e^{-2}(s-z))\mathbb{E}^{1/2}[h^2(s)]\mathbb{E}^{1/2}[h^2(s')], \quad s, s' \geq z.
\]
Combining the above estimates we get
\[
\mathbb{E}[\tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta]^2 \leq 2C \left( \int_0^\infty \rho(s)ds \right)^4 \int \mathbb{E}[g(z)]^2 \mathbb{E}[h(z)]^2 dx dy
\]
\[
\leq 2C \left( \int_0^\infty \rho(s)ds \right)^4 \int \mathbb{E}[\delta_z V_s^\varepsilon]^2 \mathbb{E} \left[ e^{-i\varepsilon 2\alpha k^{-1}(s-z)\nabla_y \cdot \nabla_x} \left[ \delta_z V_s^\varepsilon F_2^{-1} \theta \right] \right]^2 dx dy
\]
\[
\leq 8C \left( \int_0^\infty \rho(s)ds \right)^4 \int \mathbb{E}[V_s^\varepsilon]^2 \mathbb{E} \left[ e^{-i\varepsilon 2\alpha k^{-1}(s-z)\nabla_y \cdot \nabla_x} \left[ \delta_z V_s^\varepsilon F_2^{-1} \theta \right] \right]^2 dx dy
\]
\[
\leq 8C \left( \int_0^\infty \rho(s)ds \right)^4 \int \mathbb{E}[V_s^\varepsilon]^2 \mathbb{E} \left\{ e^{i\xi \tilde{X}}[\theta(x, p + q/2) - \theta(x, p - q/2)] \right\} dx dp
\]
\[
\times e^{i k^{-1}(s-z)p / \varepsilon} e^{i k^{-1}(s-z)q / \varepsilon} (dq) \right\}^2 dx dp
\]
\[
\leq 8C \left( \int_0^\infty \rho(s)ds \right)^4 \int \mathbb{E}[V_s^\varepsilon]^2 \mathbb{E} \left\{ \theta(x, p + q/2) - \theta(x, p - q/2) \right\}^2 \Phi(\xi, q) d\xi dx dq dp.
\]
\]
Eq. (70) is convenient for estimating the second moment of \( p \cdot \nabla_x \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta \) and \( \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta \) which by (70) and (21) have the following expressions
\[
\mathcal{F}^{-1}_2 \left\{ p \cdot \nabla_x \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta \right\}(x, \tilde{x}, y)
\]
\[
= i\varepsilon^{-2} \nabla_y \cdot \nabla_x \int_z^\infty e^{-i\varepsilon 2\alpha k^{-1} (t-z) \nabla_y \cdot \nabla_x} \left\{ \mathbb{E}_z[\delta_z V_t^\varepsilon e^{-i\varepsilon 2\alpha k^{-1} (s-z) \nabla_y \cdot \nabla_x} \left[ \mathbb{E}_z[\delta_z V_s^\varepsilon F_2^{-1} \theta] \right] \right\} (x, y) ds dt
\]
\[
= i\varepsilon^{-2} \int_z^\infty e^{-i\varepsilon 2\alpha k^{-1} (t-z) \nabla_y \cdot \nabla_x} \left\{ \mathbb{E}_z[\delta_z V_t^\varepsilon e^{-i\varepsilon 2\alpha k^{-1} (s-z) \nabla_y \cdot \nabla_x} \left[ \mathbb{E}_z[\nabla_y \delta_z V_s^\varepsilon F_2^{-1} \theta] \right] \right\} (x, y) ds dt
\]
\[
+ i\varepsilon^{-2} \int_z^\infty e^{-i\varepsilon 2\alpha k^{-1} (t-z) \nabla_y \cdot \nabla_x} \left\{ \mathbb{E}_z[\delta_z \nabla_y V_t^\varepsilon \cdot \nabla_y \delta_z V_s^\varepsilon F_2^{-1} \theta] \right\} (x, y) ds dt
\]
\[
\mathcal{F}^{-1}_2 \left\{ \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \tilde{L}_z^\varepsilon \theta \right\}(x, \tilde{x}, y)
\]
\[
= i\varepsilon^{-4} \delta_z V_s^\varepsilon (\tilde{x}, y) \int_z^\infty e^{-i\varepsilon 2\alpha k^{-1} (t-z) \nabla_y \cdot \nabla_x} \left\{ \mathbb{E}_z[\delta_z V_t^\varepsilon e^{-i\varepsilon 2\alpha k^{-1} (s-z) \nabla_y \cdot \nabla_x} \left[ \mathbb{E}_z[\delta_z V_s^\varepsilon F_2^{-1} \theta] \right] \right\} (x, y) ds dt.
\]
The same calculation as in Lemma 2 yields the following estimates:
Corollary 2.

\[ E\|p \cdot \nabla_x \hat{L}_x \hat{L}_x \theta \|^2 \leq 32C \left( \int_0^\infty \rho(s)ds \right)^4 \left\{ E[\nabla_x V_x^\varepsilon]^2 \int [\nabla_x \theta(x, p + q/2) - \nabla_x \theta(x, p - q/2)]^2 \Phi(\xi, q)d\xi dx dq dp \\
+ E[V_x^\varepsilon]^2 \int [\nabla_x \theta(x, p + q/2) - \nabla_x \theta(x, p - q/2)]^2 |p|^2 \Phi(\xi, q)d\xi dx dq dp \right\}; \]

\[ E\|\hat{L}_x \hat{L}_x \hat{L}_x \theta \|^2 \leq 32C \left( \int_0^\infty \rho(s)ds \right)^4 E[V_x^\varepsilon]^4 \int [\theta(x, p + q/2) - \theta(x, p - q/2)]^2 \Phi(\xi, q)d\xi dx dq dp \]

for some constant C independent of \( \varepsilon \).

Let

\[ f_1^\varepsilon(z) = k\varepsilon f'(z) \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \theta \right\rangle \]

be the 1-st perturbation of \( f(z) \).

**Proposition 1.**

\[ \lim_{\varepsilon \to 0} \sup_{z < z_0} E|f_1^\varepsilon(z)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{z < z_0} |f_1^\varepsilon(z)| = 0 \quad \text{in probability} \]

**Proof.** We have

\[ E|f_1^\varepsilon(z)| \leq \varepsilon \|f'\|_\infty \|W_0\|_2 E\|\hat{L}_x^\varepsilon \theta\|_2 \]

and

\[ \sup_{z < z_0} |f_1^\varepsilon(z)| \leq \varepsilon \|f'\|_\infty \|W_0\|_2 \sup_{z < z_0} \|\hat{L}_x^\varepsilon \theta\|_2. \]

The right side of (72) is \( O(\varepsilon) \) while the right side of (73) is \( o(1) \) in probability by Chebyshev’s inequality and Assumption 3.

Proposition 1 now follows from (72) and (73).

Set \( f^\varepsilon(z) = f(z) + f_1^\varepsilon(z) \). A straightforward calculation yields

\[ \mathcal{A}^\varepsilon f_1^\varepsilon = \varepsilon f'(z) \left\langle W_x^\varepsilon, p \cdot \nabla_x \hat{L}_x^\varepsilon \theta \right\rangle + \varepsilon f''(z) \left\langle W_x^\varepsilon, p \cdot \nabla_x \hat{L}_x^\varepsilon \theta \right\rangle + k^2 f'(z) \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \hat{L}_x^\varepsilon \theta \right\rangle + k^2 f''(z) \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \hat{L}_x^\varepsilon \theta \right\rangle \\
+ \frac{k}{\varepsilon} f'(z) \left\langle W_x^\varepsilon, \hat{L}_x \hat{L}_x^\varepsilon \theta \right\rangle \]

and, hence

\[ \mathcal{A}^\varepsilon f^\varepsilon(z) = \frac{1}{k} f'(z) \left\langle W_x^\varepsilon, p \cdot \nabla_x \theta \right\rangle + k^2 f'(z) \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \hat{L}_x^\varepsilon \theta \right\rangle + k^2 f''(z) \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \hat{L}_x^\varepsilon \theta \right\rangle \\
+ \varepsilon \left[ f'(z) \left\langle W_x^\varepsilon, p \cdot \nabla_x \hat{L}_x^\varepsilon \theta \right\rangle + f''(z) \left\langle W_x^\varepsilon, p \cdot \nabla_x \theta \right\rangle \right] \left\langle W_x^\varepsilon, \hat{L}_x^\varepsilon \theta \right\rangle \\
= A_0^\varepsilon(z) + A_1^\varepsilon(z) + A_2^\varepsilon(z) + R_1^\varepsilon(z) \]

where \( A_0^\varepsilon(z) \) and \( A_2^\varepsilon(z) \) are the \( O(1) \) statistical coupling terms.

**Proposition 2.**

\[ \lim_{\varepsilon \to 0} \sup_{z < z_0} E|R_1^\varepsilon(z)|^2 = 0 \]
Proof.

\[ |R_1^\varepsilon| \leq \varepsilon \left[ \|f''\|_\infty \|W_0\|_2 \|p \cdot \nabla_x \theta\|_2 + \|f'\|_\infty \|W_2^\varepsilon\|_2 \|p \cdot \nabla_x (\tilde{L}^\varepsilon \theta)\|_2 \right]. \]

Clearly

\[ \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}[|R_1^\varepsilon(z)|^2] = 0. \]

by Lemma 11 and Corollary 11.

For the tightness criterion stated in the beginnings of the section, it remains to show

**Proposition 3.** \( \{A^\varepsilon f^\varepsilon\} \) are uniformly integrable.

**Proof.** We show that \( \{A^\varepsilon_i\}, i = 0, 1, 2, 3 \) are uniformly integrable.

For this we have the following estimates:

\[
|A_0^\varepsilon(z)| \leq \frac{1}{k} \|f'\|_\infty \|W_0\|_2 \|p \cdot \nabla_x \theta\|_2 \\
|A_1^\varepsilon(z)| \leq k^2 \|f'\|_\infty \|W_0\|_2 \|\tilde{L}_\varepsilon \tilde{L}^\varepsilon \theta\|_2 \\
|A_2^\varepsilon(z)| \leq k^2 \|f''\|_\infty \|W_0\|_2 \|\tilde{L}_\varepsilon \tilde{L}^\varepsilon \theta\|_2. 
\]

The second moments of the right hand side of the above expressions are uniformly bounded as \( \varepsilon \to 0 \) by Lemmas 11 and 21 and hence \( A_0^\varepsilon(z), A_1^\varepsilon(z), A_2^\varepsilon(z) \) are uniformly integrable. By Proposition 12, \( R_1^\varepsilon \) is uniformly integrable.

**6.2. Identification of the limit.** Our strategy is to show directly that in passing to the weak limit the limiting process solves the martingale problem with zero quadratic variation. The uniqueness of the limiting deterministic problem then identifies the limit.

For this purpose, we introduce the next perturbations \( f_2^\varepsilon, f_3^\varepsilon \). Let

\[
A_2^{(1)}(\psi) \equiv \int \psi(x, p) Q_1(\theta \otimes \theta)(x, p, y, q) \psi(y, q) \, dx \, dp \, dy \, dq \\
A_1^{(1)}(\psi) \equiv \int Q'_1 \theta(x, p) \psi(x, p) \, dx \, dp, \quad \forall \psi \in L^2(\mathbb{R}^{2d})
\]

where

\[
Q_1(\theta \otimes \theta)(x, p, y, q) = \mathbb{E} \left[ \tilde{L}^\varepsilon \theta(x, p) \tilde{L}^\varepsilon \theta(y, q) \right]
\]

and

\[
Q'_1 \theta(x, p) = \mathbb{E} \left[ \tilde{L}^\varepsilon \tilde{L}^\varepsilon \theta(x, p) \right].
\]

Clearly,

\[
A_2^{(1)}(\psi) = \mathbb{E} \left[ \langle \psi, \tilde{L}^\varepsilon \theta \rangle \langle \psi, \tilde{L}^\varepsilon \theta \rangle \right].
\]

Let

\[
Q_2(\theta \otimes \theta)(x, p, y, q) \equiv \mathbb{E} \left[ \tilde{L}^\varepsilon \theta(x, p) \tilde{L}^\varepsilon \theta(y, q) \right]
\]

and

\[
Q'_2 \theta(x, p) = \mathbb{E} \left[ \tilde{L}^\varepsilon \tilde{L}^\varepsilon \theta(x, p) \right].
\]

Let

\[
A_2^{(2)}(\psi) \equiv \int \psi(x, p) Q_2(\theta \otimes \theta)(x, p, y, q) \psi(y, q) \, dx \, dp \, dy \, dq \\
A_1^{(2)}(\psi) \equiv \int Q'_2 \theta(x, p) \psi(x, p) \, dx \, dp
\]
Define

\[ f^e_2(z) = \frac{\varepsilon^2 k^2}{2} f''(z) \left( \left\langle W^e_z, \tilde{\mathcal{L}}^e_z \theta \right\rangle^2 - A^{(2)}_2(W^e_z) \right) \]

\[ f^e_3(z) = \frac{\varepsilon^2 k^2}{2} f'(z) \left( \left\langle W^e_z, \tilde{\mathcal{L}}^e_z \theta \right\rangle^2 - A^{(2)}_1(W^e_z) \right). \]

**Proposition 4.**

\[ \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}|f^e_2(z)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}|f^e_3(z)| = 0. \]

**Proof.** We have the bounds

\[ \sup_{z < z_0} \mathbb{E}|f^e_2(z)| \leq \sup_{z < z_0} \varepsilon^2 k^2 \|f''\|_\infty \left[ \|W_0\|^2 \mathbb{E} \|\tilde{\mathcal{L}}^e \theta\|_2^2 + \mathbb{E} [A^{(2)}_2(W^e)] \right] \]

\[ \sup_{z < z_0} \mathbb{E}|f^e_3(z)| \leq \sup_{z < z_0} \varepsilon^2 k^2 \|f'\|_\infty \left[ \|W_0\|_2 \mathbb{E} \|\tilde{\mathcal{L}}^e \tilde{\mathcal{L}}^e \theta\|_2 + \mathbb{E} [A^{(2)}_1(W^e)] \right]. \]

The right sides of both tend to zero as \( \varepsilon \to 0 \) by Lemma 1 and 2. \qed

We have

\[ A^e f^e_2(z) = k^2 f''(z) \left[ - \left\langle W^e_z, \mathcal{L}^e_z \theta \right\rangle \left\langle W^e_z, \tilde{\mathcal{L}}^e_z \theta \right\rangle + A^{(1)}_2(W^e_z) \right] + R^e_2(z) \]

\[ A^e f^e_3(z) = k^2 f'(z) \left[ - \left\langle W^e_z, \mathcal{L}^e_z (\tilde{\mathcal{L}}^e_z \theta) \right\rangle + A^{(1)}_1(W^e_z) \right] + R^e_3(z) \]

with

\[ R^e_2(z) = \varepsilon^2 k^2 \frac{f''(z)}{2} \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x \theta \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z \theta \right\rangle \right] \left[ \left\langle W^e_z, \tilde{\mathcal{L}}^e_z \theta \right\rangle^2 - A^{(2)}_2(W^e_z) \right] \]

\[ + \varepsilon^2 k^2 \frac{f''(z)}{2} \left\langle W^e_z, \mathcal{L}^e_z \theta \right\rangle \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x (\tilde{\mathcal{L}}^e_z \theta) \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z \mathcal{L}^e_z \theta \right\rangle \right] \]

\[ - \varepsilon^2 k^2 f'(z) \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x (\tilde{\mathcal{L}}^e_z \theta) \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z \tilde{\mathcal{L}}^e_z \theta \right\rangle \right]. \]

where \( G^{(2)}_\theta \) denotes the operator

\[ G^{(2)}_\theta \psi \equiv \int Q_2(\theta \otimes \theta)(x, p, y, q) \psi(y, q) dy dq. \]

Similarly

\[ R^e_3(z) = \varepsilon^2 k^2 \frac{f'(z)}{2} \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x (\tilde{\mathcal{L}}^e_z \tilde{\mathcal{L}}^e_z \theta) \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z \tilde{\mathcal{L}}^e_z \theta \right\rangle \right] \]

\[ + \varepsilon^2 k^2 \frac{f''(z)}{2} \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x \theta \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z \theta \right\rangle \right] \left[ \left\langle W^e_z, \tilde{\mathcal{L}}^e_z \tilde{\mathcal{L}}^e_z \theta \right\rangle - A^{(2)}_1(W^e_z) \right] \]

\[ - \varepsilon^2 k^2 f'(z) \left[ \frac{1}{k} \left\langle W^e_z, \mathbf{p} \cdot \nabla x (\tilde{\mathcal{L}}^e_z \theta) \right\rangle + \frac{k}{\varepsilon} \left\langle W^e_z, \mathcal{L}^e_z Q^e_\theta \right\rangle \right]. \]

**Proposition 5.**

\[ \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}|R^e_2(z)| = 0, \quad \lim_{\varepsilon \to 0} \sup_{z < z_0} \mathbb{E}|R^e_3(z)| = 0. \]

**Proof.** Part of the argument is analogous to that given for Proposition 4. The additional estimates that we need to consider are the following.
In $R_2^α$: First

$$
\sup_{z < z_0} \varepsilon^2 \mathbb{E} \left| \langle W^ε, \mathbf{p} \cdot \nabla_x (G^{(2)}_θ W^ε) \rangle \right|
= \varepsilon^2 \int \mathbb{E} [W^ε(x, \mathbf{p})W^ε(y, \mathbf{q})] \mathbb{E} \left[ \mathbf{p} \cdot \nabla_x \tilde{\zeta}_z θ(x, \mathbf{p})\tilde{\zeta}_z θ(y, \mathbf{q}) \right] dxdydpdq
\leq \varepsilon^2 \int \mathbb{E} [W^ε(x, \mathbf{p})W^ε(y, \mathbf{q})] \mathbb{E}^{1/2} \left[ |\mathbf{p} \cdot \nabla_x \tilde{\zeta}_z θ|^2 \right](x, \mathbf{p}) \mathbb{E}^{1/2} \left[ |\tilde{\zeta}_z θ|^2 \right](y, \mathbf{q}) dxdydpdq
$$

which is $O(\varepsilon^2)$ by using Lemma 2, Corollary 4 and the fact $\mathbb{E} [W^ε(x, \mathbf{p})W^ε(y, \mathbf{q})] \in L^2(\mathbb{R}^d)$ in conjunction with the same argument as in proof of Lemma 1; Secondly,

$$
\sup_{z < z_0} \varepsilon \mathbb{E} \left| \langle W^ε, \mathcal{L}^ε θ W^ε \rangle \right|
= \sup_{z < z_0} \varepsilon \|W_0\|_2 \mathbb{E} \|\mathcal{L}^ε θ W^ε\|_2
= \sup_{z < z_0} \varepsilon \|W_0\|_2 \mathbb{E} \left[ \mathcal{F}_2^{-1} \mathcal{L}^ε θ \mathcal{F}_2^{-1} \mathcal{L}^ε θ \right] \mathcal{F}_2^{-1} W^ε \|_2
$$

Let

$$
h_α = e^{-ik\varepsilon^{-2α}(s-z)\nabla_x \nabla_x [\delta V^ε x \mathcal{F}_2^{-1} θ]].
$$

We then have

$$
\mathbb{E} \mathcal{F}_2^{-1} \mathcal{L}^ε θ \left[ \mathcal{F}_2^{-1} \mathcal{L}^ε θ \mathcal{F}_2^{-1} \mathcal{L}^ε θ \right] \mathcal{F}_2^{-1} W^ε \|_2
= \mathbb{E} \left\{ \left\| \varepsilon^{-4} \int \int 2 ε V^ε(x, y) \mathbb{E} \left[ \mathbb{E} \left[ h_t(dx', dy') \right] \mathcal{F}_2^{-1} W^ε(x', y') dx' dy' dt \right]^2 dxdy \right\}^{1/2}
\leq \mathbb{E}^{1/2} \left\{ \int \int 2 ε V^ε(x, y) \rho(ε^{-2}(s-z))ρ(ε^{-2}(t-z)) \mathbb{E}^{1/2} [h_t(x, y)]^2 \int \int 2 dx' dy' dsdt \right\}^{1/2}
\leq \mathbb{E}^{1/2} \left\{ \int \int 2 ε V^ε(x, y) \rho(ε^{-2}(s-z))ρ(ε^{-2}(t-z)) \mathbb{E}^{1/2} [h_t(x, y)]^2 \right\}^{1/2}
\left( \int \mathbb{E} [h_t(dx', dy')] dx' dy' \right) \left( \int |W^ε(x', y')|^2 dx' dp' \right) dsdt \right\}^{1/2} dxdy.
$$

Recall that $\|W^ε\|_2 \leq \|W_0\|_2$ and

$$
\int \mathbb{E} [h_t(dx', dy')] dx' dy' = \int \left[ \theta(x, \mathbf{p} + \mathbf{q}/2) - \theta(x, \mathbf{p} - \mathbf{q}/2) \right]^2 \Phi(\xi, \mathbf{q}) dξ dq dx dp < \infty
$$
so that
\[
\mathbb{E}\|F^{-1}_2\mathcal{L}_2^\varepsilon\mathbb{E}\left[H^{-1}_2\tilde{\mathcal{L}}_2^\varepsilon\theta \otimes F^{-1}_2\tilde{\mathcal{L}}_2^\varepsilon\theta\right]F^{-1}_2W_2^\varepsilon\|_2 \\
\leq \|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^{3/2}\mathbb{E}^{1/2}\left\{\int_{\varepsilon^{-4}}^{\infty} \|\delta_2V_2^\varepsilon(x, y)\|_2\rho(\varepsilon^{-2}(s - z))\rho(\varepsilon^{-2}(t - z))\mathbb{E}^{1/2}\|h_s(x, y)\|_2^2dsdt\right\}^2 \|dxdy\}
\]
\[
\leq \|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^{2}\left(\sup_{x,y} \mathbb{E}\|\delta_2V_2^\varepsilon\|_2^2\right)\varepsilon^{-8}\int_{\varepsilon^{-4}}^{\infty} \|\delta_2V_2^\varepsilon\|_2^2\rho(\varepsilon^{-2}(s - z))\rho(\varepsilon^{-2}(t - z)) \times \rho(\varepsilon^{-2}(s' - z))\rho(\varepsilon^{-2}(t' - z))\mathbb{E}^{1/2}\|h_s\|_2^{2}\mathbb{E}^{1/2}\|h_s\|_2^{2}dsdt\|dtd'ds\}
\]
\[
\leq \|W_0\|_2\mathbb{E}^{1/2}\|h_s\|_2^{2}\left(\sup_{x,y} \mathbb{E}\|\delta_2V_2^\varepsilon\|_2^2\right)\left(\int_{0}^{\infty} \rho(s)ds\right)^2 < \infty.
\]
Recall from (63) that
\[
\mathbb{E}\|h_s\|_2^{2} = \int[\theta(x, p + q/2) - \theta(x, p - q/2)]^2 \Phi(\xi, q)d\xi dq dx dp < \infty.
\]
Hence
\[
\sup_{s < t_0} \mathbb{E}\left|\left(W_2^\varepsilon, \mathcal{L}_2^\varepsilon G_\theta^2 W_2^\varepsilon\right)\right| = O(\varepsilon).
\]
In $R_3^\varepsilon$,
\[
\sup_{s < t_0} \mathbb{E}\left|\left(W_2^\varepsilon, \mathcal{L}_2^\varepsilon \tilde{\mathcal{L}}_2^\varepsilon \tilde{\mathcal{L}}_2^\varepsilon\theta\right)\right| \leq \varepsilon\|W_0\|_2\sup_{s < t_0} \mathbb{E}\|\mathcal{L}_2^\varepsilon \tilde{\mathcal{L}}_2^\varepsilon \tilde{\mathcal{L}}_2^\varepsilon\|_2
\]
which is $O(\varepsilon)$ by Corollary 2.

The other two terms in $R_3^\varepsilon$
\[
\varepsilon^2\mathbb{E}\left|\left(W_2^\varepsilon, \mathcal{L}_2^\varepsilon \nabla_\theta(Q_2^\varepsilon\theta)\right)\right| \leq \varepsilon^2\|W_0\|_2\mathbb{E}^{1/2}\|\mathcal{L}_2^\varepsilon \mathcal{L}_2^\varepsilon\|_2
\]
\[
\leq \varepsilon^2\|W_0\|_2\mathbb{E}\|\nabla_\theta(Q_2^\varepsilon\theta)\|_2
\]
which is $O(\varepsilon^2)$ by Corollary 2.

\[
\varepsilon\mathbb{E}\left|\left(W_2^\varepsilon, \mathcal{L}_2^\varepsilon Q_2^\varepsilon\theta\right)\right| \leq \varepsilon\|W_0\|_2\mathbb{E}\|\mathcal{L}_2^\varepsilon \mathcal{L}_2^\varepsilon\|_2
\]
\[
\leq \varepsilon\|W_0\|_2\mathbb{E}\|\nabla_\theta(Q_2^\varepsilon\theta)\|_2
\]
which is $O(\varepsilon)$ by Lemma 2.

Consider the test function $f^\varepsilon(z) = f(z) + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z)$. We have
\[
\mathcal{A}^\varepsilon f^\varepsilon(z) = \frac{1}{k} f'(z) W_2^\varepsilon + k^2 f''(z) A_1^1(W_2^\varepsilon) + k^2 f' A_1^1(W_2^\varepsilon) + R_1^\varepsilon(z) + R_2^\varepsilon(z) + R_3^\varepsilon(z).
\]
Set
\[
R^\varepsilon(z) = R_1^\varepsilon(z) + R_2^\varepsilon(z) + R_3^\varepsilon(z).
\]
It follows from Propositions 3 and 5 that
\[
\lim_{\varepsilon \to 0} \sup_{s < t_0} \mathbb{E}|R^\varepsilon(z)| = 0.
\]
Proposition 6.

\[ \lim_{\varepsilon \to 0} \sup_{\varepsilon < 2\varepsilon \|\psi\|_1} A_2^{(1)}(\psi) = 0. \]

Proof. We have

\[
A_2^{(1)}(\psi) = \int \psi(x, p) Q_1(\theta \otimes \theta)(x, p, y, q) \psi(y, q) \, dx \, dp \, dq
\]

\[
= \frac{1}{2} \int \psi(x, p) [Q_1(\theta \otimes \theta)(y, q, x, p) + Q_1(\theta \otimes \theta)(x, p, y, q)] \psi(y, q) \, dx \, dp \, dq
\]

where the kernel \( Q_1(\theta \otimes \theta) \) can be written as

\[
Q_1(\theta \otimes \theta)(y, q, x, p) + Q_1(\theta \otimes \theta)(x, p, y, q)
\]

\[
= \int_{-\infty}^{\infty} ds \int dp \, \Phi(s, p') e^{ip' \cdot (x-y)/\varepsilon^{2\alpha}} e^{-ik^{-1}s \cdot p' \varepsilon^{2-2\alpha}} \left[ \theta(x, p + p'/2) - \theta(x, p - p'/2) \right]
\]

\[
\times \left[ \theta(y, q + p'/2) - \theta(y, q - p'/2) \right] \Phi(k^{-1}p \cdot p' \varepsilon^{2-2\alpha}, p') \, dp'
\]

which is uniformly compactly supported on \( \mathbb{R}^{4d} \). For smooth and compactly supported \( \Phi \), \( Q_1(\theta \otimes \theta) \) tends to zero fast than any power of \( \varepsilon \) uniformly outside any neighborhood of \( x = y \) while stays uniformly bounded everywhere. Therefore the \( L^2 \)-norm of \( Q_1(\theta \otimes \theta) \) tends to zero and the proposition follows. \( \square \)

Similar calculation leads to the following expression: For any real-valued, \( L^2 \)-weakly convergent sequence \( \psi^\varepsilon \to \psi \), we have

\[
\lim_{\varepsilon \to 0} A_2^{(1)}(\psi)
\]

\[
= \lim_{\varepsilon \to 0} \int_0^\infty ds \int dw dq dx dp \, \psi^\varepsilon(x, p) \Phi(w, q) e^{isw} e^{-ik^{-1}s \cdot p \cdot q \varepsilon^{2-2\alpha}} \left[ e^{-ik^{-1}|q|^2 \varepsilon^{2-2\alpha}/2} \left[ \theta(x, p + q) - \theta(x, p) \right] - e^{ik^{-1}|q|^2 \varepsilon^{2-2\alpha}/2} \left[ \theta(x, p) - \theta(x, p - q) \right] \right]
\]

\[
= \lim_{\varepsilon \to 0} \int_0^\infty ds \int dw dq dx dp \, \psi^\varepsilon(x, p) \Phi(s, q) e^{-ik^{-1}s \cdot p \cdot q \varepsilon^{2-2\alpha}} \left[ e^{-ik^{-1}|q|^2 \varepsilon^{2-2\alpha}/2} \left[ \theta(x, p + q) - \theta(x, p) \right] - e^{ik^{-1}|q|^2 \varepsilon^{2-2\alpha}/2} \left[ \theta(x, p) - \theta(x, p - q) \right] \right].
\]

Note that the integrand is invariant under the change of variables:

\[ s \to -s, \quad q \to -q. \]
Thus we can write

\[
\lim_{\varepsilon \to 0} A_{1}^{(1)}(\psi) = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{-\infty}^{\infty} ds \int dq dx dp \ \psi^\varepsilon(x, p) e^{-ik^{-1} sp} q e^{2\gamma} [\phi(x, p + q) - \phi(x, p)]
\]

with compact support, we have

\[
\Phi((2k)^{-1}(|q|^2 - |p|^2), q - p) \Rightarrow \Phi(0, q - p), \quad 0 < \alpha < 1
\]

\[
\Phi(0, q - p), \quad \alpha > 1
\]

in \( L^2(\mathbb{R}^d) \). Therefore,

\[
\lim_{\varepsilon \to 0} A_{1}^{(1)}(\psi^\varepsilon) = \int dq [\phi(x, q) - \phi(x, p)] \times \begin{cases}
\Phi((2k)^{-1}(|q|^2 - |p|^2), q - p), & \alpha = 1 \\
\Phi(0, q - p), & 0 < \alpha < 1 \\
0, & \alpha > 1
\end{cases}
\]

Recall that

\[
M_z^\varepsilon(\theta) = \int_0^z A^\varepsilon f^\varepsilon(s) ds
\]

\[
= f(z) + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z) - \int_0^z \frac{1}{k} f'(z) \langle W_z^\varepsilon, p \cdot \nabla x \theta \rangle ds
\]

\[
- \int_0^z k^2 \left[ f''(s) A_{1}^{(1)}(W_s^\varepsilon) + f'(s) A_{1}^{(1)}(W_s^\varepsilon) \right] ds - \int_0^z R^\varepsilon(s) ds
\]

is a martingale. The martingale property implies that for any finite sequence \( 0 < z_1 < z_2 < z_3 < \ldots < z_n \leq z \), \( C^2 \)-function \( f \) and bounded continuous function \( h \) with compact support, we have

\[
\mathbb{E} \{ h \left( \langle W_z^\varepsilon, \theta \rangle, \langle W_z^\varepsilon, \bar{\theta} \rangle, \ldots, \langle W_z^\varepsilon, \bar{\theta} \rangle \right) \} = 0,
\]

\[
\forall s > 0, \quad 2z \leq 2z \leq \ldots \leq z_n \leq z.
\]

Let

\[
\bar{A}^\varepsilon f(z) \equiv f'(s) \left[ \frac{1}{k} \langle W_z, p \cdot \nabla x \theta \rangle + k^2 A_{1}(W_z) \right].
\]

In view of the results of Propositions 1, 2, 3, 4 and 5 we see that \( f^\varepsilon(z) \) and \( A^\varepsilon f^\varepsilon(z) \) in (87) can be replaced by \( f(z) \) and \( \bar{A} f(z) \), respectively, modulo an error that vanishes as \( \varepsilon \to 0 \). With this and
the tightness of \( \{W_z^\varepsilon\} \) we can pass to the limit \( \varepsilon \to 0 \) in \( \mathbb{S} \). We see that the limiting process satisfies the martingale property that
\[
\mathbb{E} \{ h(\langle W_z^1, \theta \rangle, \langle W_z^2, \theta \rangle, \ldots, \langle W_z^n, \theta \rangle) [M_{z+s}(\theta) - M_z(\theta)] \} = 0, \quad \forall s > 0.
\]
where
\[
M_z(\theta) = f(z) - \int_0^z \bar{A} f(s) \, ds.
\]
Then it follows that
\[
\mathbb{E} [M_{z+s}(\theta) - M_z(\theta)|W_u, u \leq z] = 0, \quad \forall z, s > 0
\]
which proves that \( M_z(\theta) \) is a martingale given by
\[
M_z(\theta) = f(z) - \int_0^z \left\{ f'(s) \left[ \frac{1}{k} \langle W_s, \mathbf{p} \cdot \nabla \theta \rangle + k^2 \bar{A}_1(W_s) \right] \right\} ds.
\]
Choose \( f(r) = r \) and \( r^2 \) in \( (90) \) we see that
\[
M_z^{(1)}(\theta) = \langle W_z, \theta \rangle - \int_0^z \left[ \frac{1}{k} \langle W_s, \mathbf{p} \cdot \nabla \theta \rangle + k^2 \bar{A}_1(W_s) \right] ds
\]
is a martingale with the null quadratic variation
\[
\left[ M^{(1)}(\theta), M^{(1)}(\theta) \right]_z = 0.
\]
Thus
\[
f(z) - \int_0^z \left\{ f'(s) \left[ \frac{1}{k} \langle W_s, \mathbf{p} \cdot \nabla \theta \rangle + k^2 \bar{A}_1(W_s) \right] \right\} ds = f(0), \quad \forall z > 0.
\]
Since \( \langle W_z^\varepsilon, \theta \rangle \) is uniformly bounded
\[
|\langle W_z^\varepsilon, \theta \rangle| \leq \|W_0\|_2 \|\theta\|_2
\]
we have the convergence of the second moment
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left\{ \langle W_z^\varepsilon, \theta \rangle^2 \right\} = \langle W_z, \theta \rangle^2
\]
and hence the convergence in probability.

7. Proof of Theorem 2

7.1. Tightness. Instead of \( (60) \) we use the following the corrector
\[
(9) \hat{\theta}_z^\varepsilon(\mathbf{x}, \tilde{x}, \mathbf{p}) = \frac{i}{\varepsilon^2} \int_z^{\infty} \int e^{i\mathbf{q} \cdot \tilde{x}} e^{i\mathbf{k} \cdot (s-z) \mathbf{p}} \mathbf{q} / (2\varepsilon^2) \varepsilon^{2\alpha-2} [\theta(\mathbf{x}, \mathbf{p}) + \varepsilon^{2-2\alpha} \mathbf{q}/2] - \theta(\mathbf{x}, \mathbf{p} - \varepsilon^{2-2\alpha} \mathbf{q}/2)] \hat{V}_s^\varepsilon(\mathbf{q} d\mathbf{q})
\]
which satisfies the corrector equation \( (55) \). Its inverse Fourier transform is given by
\[
(92) \hat{F}^{-1} \hat{Z}_z^\varepsilon(\mathbf{x}, \tilde{x}, \mathbf{y}) = \varepsilon^{-2} \int_z^{\infty} e^{-is^{-2\alpha} k^{-1} (s-z) \nabla \mathbf{y} \cdot \nabla \tilde{x}} [\mathbb{E}_z [\delta_x V_s^\varepsilon] \hat{F}^{-1} \theta](\mathbf{x}, \mathbf{y}) ds.
\]
Instead of Lemma \( (1) \) Corollary \( (1) \) Lemma \( (2) \) and Corollary \( (2) \) we have

Lemma 3. Assumption 1 implies that
\[
(93) \lim_{\varepsilon \to 0} \sup \mathbb{E} \left[ \hat{Z}_z^\varepsilon(\mathbf{x}, \mathbf{p}) \right]^2 (\mathbf{x}, \mathbf{p}) \leq \left[ \int_0^{\infty} \rho(s) ds \right]^2 \int \left[ \mathbf{q} \cdot \nabla \theta(\mathbf{x}, \mathbf{p}) \right]^2 \Phi(\xi, \mathbf{q}) d\xi d\mathbf{q}
\]
which has a bounded support and is bounded uniformly in \( \mathbf{x}, \mathbf{p} \).
Corollary 3. 
\[
\lim_{\varepsilon \to 0} \sup_{\|\psi\|_2=1} A_2^{(1)}(\psi) = 0.
\]

Proof. The kernel \( Q_1(\theta \otimes \theta) \) has the following expressions
\[
Q_1(\theta \otimes \theta)(x, p, y, q) = \int_0^\infty \int_0^\infty \Phi(s, p') e^{ip'(x-y)}/\varepsilon^2 \alpha e^{-iks(p-p')^{2/2}} \varepsilon^{2\alpha - 2} [\theta(x, p + \varepsilon^{2-2\alpha} p'/2) - \theta(x, p - \varepsilon^{2-2\alpha} p'/2)] \\
\times \varepsilon^{2\alpha - 2} \theta(y, q + \varepsilon^{2-2\alpha} p'/2) - \theta(y, q - \varepsilon^{2-2\alpha} p'/2)] \quad dp' \quad ds
\]
\[
= \lim_{\varepsilon \to 0} \pi \int_0^\infty \Phi(k^{-1} \cdot p, p') e^{ip'(x-y)/\varepsilon^2} \varepsilon^{2\alpha - 2} \theta(x, p + \varepsilon^{2-2\alpha} p'/2) - \theta(x, p - \varepsilon^{2-2\alpha} p'/2)] \\
\times \varepsilon^{2\alpha - 2} [\theta(y, q + \varepsilon^{2-2\alpha} p'/2) - \theta(y, q - \varepsilon^{2-2\alpha} p'/2)] \quad dp'.
\]

Note that
\[
\varepsilon^{2\alpha - 2} \theta(x, p + \varepsilon^{2-2\alpha} p'/2) - \theta(x, p - \varepsilon^{2-2\alpha} p'/2)] \quad \rightarrow \quad p' \cdot \nabla \theta(x, p)
\]
in \( C^\infty_c(\mathbb{R}^d) \).

Thus the \( L^2 \)-norm of \( Q_1(\theta \otimes \theta) \) tends to zero for the same reason as given in the proof of Proposition 4.
To identify the limit we have the following straightforward calculation: For any real-valued, $L^2$-weakly convergent sequence $\psi^\varepsilon \to \psi$

$$\lim_{\varepsilon \to 0} A^{(1)}_1(\psi^\varepsilon) = \lim_{\varepsilon \to 0} \int_0^\infty ds \int dw dq dp \psi^\varepsilon(x, p) \Phi(w, q) e^{i sw} e^{-i k^{-1} sp q^2 - 2\alpha \varepsilon^4 - 4\alpha}$$

$$= \lim_{\varepsilon \to 0} \int_0^\infty ds \int dq dp \psi^\varepsilon(x, p) \Phi(s, q) e^{-i k^{-1} sp q^2 - 2\alpha \varepsilon^4 - 4\alpha}$$

$$= \pi \int dq dp \psi(x, p) \Phi(0, q)(q \cdot \nabla_p)^2 \theta(x, p)$$

$$= A_1(\psi)$$

for $\alpha \in (0, 1)$. For $\alpha = 1$ we have the same result as in Theorem 1 Case (ii); for $\alpha > 1$, the limit is identically zero.

8. Proof of Theorem 3

The proof of the result for Case (i) and (iii) is identical to that for Theorem 1, Case (i) and (iii), respectively. So in the sequel we focus on the second case $\alpha > \beta$.

Introducing a new parameter

$$\bar{\varepsilon} = \varepsilon^\beta$$

we can rewrite the equation as

$$\frac{\partial W^\varepsilon_z}{\partial z} + \frac{P}{k} \cdot \nabla W^\varepsilon_z + \frac{k}{\varepsilon} \mathcal{L}_z^\varepsilon W^\varepsilon_z = 0$$

with

$$\mathcal{L}_z^\varepsilon W^\varepsilon_z(x, \bar{x}, p) = i\bar{\varepsilon}^{1-\alpha/\beta} \int e^{i\bar{q} \cdot \bar{\bar{x}}} [W^\varepsilon_z(x, p + \bar{q}/2) - W^\varepsilon_z(x, p - \bar{q}/2)] \tilde{V}^\varepsilon_z(dq), \quad \bar{x} = x\bar{\varepsilon}^{-2\alpha/\beta}, \beta < 1$$

with

$$\tilde{V}^\varepsilon_z(dq) = \tilde{V}(\frac{z}{\bar{\varepsilon}^2}, dq), \quad V^\varepsilon_z(x) = V(\frac{z}{\bar{\varepsilon}^2}, x).$$

Note again that

$$\mathcal{F}_2^{-1} \mathcal{L}_z^\varepsilon \theta = -i\bar{\varepsilon}^{1-\alpha/\beta} \delta_x \mathcal{V}^\varepsilon_z(\bar{x}, y) \mathcal{F}_2^{-1} \theta$$

with

$$\delta_x \mathcal{V}^\varepsilon_z(\bar{x}, y) = V^\varepsilon_z(\bar{x} + y/2) - V^\varepsilon_z(\bar{x} - y/2).$$

We will work with (99) and (100) and construct the perturbed test function in the power of $\bar{\varepsilon}$.

First we note that

$$\mathbb{E} [\mathcal{L}_z^\varepsilon \theta]^2(x, p) = \bar{\varepsilon}^{2-2\alpha/\beta} \int [\theta(x, p + \bar{q}/2) - \theta(x, p - \bar{q}/2)]^2 \Phi(\bar{x}, q)d\bar{x} dq$$

which has an $\varepsilon$-uniformly bounded support.

Instead of (99) we define

$$\tilde{\mathcal{L}}^\varepsilon_x \theta(x, \bar{x}, p) = i\bar{\varepsilon}^{1+\alpha/\beta} \int e^{i\bar{q} \cdot \bar{x}} e^{i k^{-1}(s-z)p q \bar{\varepsilon}^{2\alpha/\beta}} [\theta(x, p + q/2) - \theta(x, p - q/2)] \mathbb{E}^\varepsilon \tilde{V}^\varepsilon_z(dq)$$
Lemma 5.

\[ \frac{\alpha}{\beta} \left( \int_0^\infty \rho(s) ds \right)^2 \int_\mathbb{R}^3 |\nabla \theta(x, p + q/2) - \nabla \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq \]

which has an \( \varepsilon \)-uniformly bounded support.

Corollary 5.

\[ \frac{\alpha}{\beta} \left( \int_0^\infty \rho(s) ds \right)^2 \int_\mathbb{R}^3 |p \cdot \nabla \theta(x, p + q/2) - p \cdot \nabla \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq \]

which has an \( \varepsilon \)-uniformly bounded support.

Lemma 6.

\[ \mathbb{E} \left\| L_2 \tilde{L}_2 \theta \right\|_2^2 \leq \varepsilon^{1-4\alpha/\beta} 8C \left( \int_0^\infty \rho(s) ds \right)^2 \mathbb{E} [V_2]^2 \int_\mathbb{R}^3 |\theta(x, p + q/2) - \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq d\rho_p d\rho_q \]

\[ \mathbb{E} \left\| L_2 \tilde{L}_2 \theta \right\|_2^2 \leq \varepsilon^{1-4\alpha/\beta} 8C \left( \int_0^\infty \rho(s) ds \right)^4 \mathbb{E} [V_2]^2 \int_\mathbb{R}^3 |\theta(x, p + q/2) - \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq d\rho_p d\rho_q \]

for some constant \( C \) independent of \( \varepsilon \).

Corollary 6.

\[ \mathbb{E} \left\| p \cdot \nabla \theta \tilde{L}_2 \tilde{L}_2 \theta \right\|_2^2 \leq \varepsilon^{1-4\alpha/\beta} 32C \left( \int_0^\infty \rho(s) ds \right)^4 \left\{ \mathbb{E} [\nabla_\theta V_2]^2 \int_\mathbb{R}^3 |\nabla \theta(x, p + q/2) - \nabla \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq d\rho_p d\rho_q + \mathbb{E} [V_2]^2 \int_\mathbb{R}^3 |\nabla \theta(x, p + q/2) - \nabla \theta(x, p - q/2)|^2 |p|^2 \Phi(\xi, q) d\xi dq d\rho_p d\rho_q \right\} ; \]

\[ \mathbb{E} \left\| L_2 \tilde{L}_2 \tilde{L}_2 \theta \right\|_2^2 \leq \varepsilon^{6-6\alpha/\beta} 32C \left( \int_0^\infty \rho(s) ds \right)^4 \mathbb{E} [V_2]^4 \int_\mathbb{R}^3 |\theta(x, p + q/2) - \theta(x, p - q/2)|^2 \Phi(\xi, q) d\xi dq d\rho_p d\rho_q \]

for some constant \( C \) independent of \( \varepsilon \).

As a consequence of the divergent factor \( \varepsilon^{1-1/\beta} \) in the above estimates the previous proof of uniform integrability of \( \mathcal{A}^\varepsilon [f(z) + \varepsilon f_1] \) (e.g. Proposition 3) breaks down. For both the tightness and the identification we shall use the test function

\[ f_\varepsilon(z) = f(z) + f_1^\varepsilon(z) + f_2^\varepsilon(z) + f_3^\varepsilon(z) \]

with

\[ \tilde{x} = x^{\varepsilon-2\alpha/\beta}. \]

After the partial inverse Fourier transform

\[ \mathcal{F}_2^{-1} \tilde{L}_2 \theta(x, \tilde{x}, y) = \frac{i}{\varepsilon^{1+\alpha/\beta}} \int_\mathbb{R} \varepsilon^{k-1}(s-z) \nabla_y \nabla_\tilde{x} \varepsilon^{-2\alpha/\beta} \left[ \mathbb{E}^\varepsilon [\delta_\tilde{x} V_2^\varepsilon] \mathcal{F}_2^{-1} \theta \right](x, \tilde{x}, y) ds \]

The corrector equation holds again:

\[ \left[ \varepsilon^{-2\alpha/\beta} \frac{\mathbf{P}}{k} \cdot \nabla \tilde{x} + \mathcal{A}^\varepsilon \right] \tilde{L}_2 \theta = \varepsilon^{-2} \tilde{L}_2 \theta. \]

Following the same argument as in the proof of Theorem 1 we have the following estimates:
where

\begin{align}
(104) \quad f_1'(z) &= k\bar{e}f'(z)\langle W^\varepsilon, \bar{L}^\varepsilon \theta \rangle \\
(105) \quad f_2'(z) &= \frac{\varepsilon^2 k^2}{2} f''(z) \left[ \left\langle W^\varepsilon, \bar{L}^\varepsilon \theta \right\rangle^2 - A_2^{(2)}(W^\varepsilon) \right] \\
(106) \quad f_3'(z) &= \frac{\varepsilon^2 k^2}{2} f'(z) \left[ \left\langle W^\varepsilon, \bar{L}^\varepsilon \bar{L}^\varepsilon \theta \right\rangle - A_1^{(2)}(W^\varepsilon) \right].
\end{align}

with $A_1^{(2)}, A_2^{(2)}$ given as before.

Following the same procedure as in the proof of Theorem 1 we obtain

\begin{align}
(107) \quad A^\varepsilon f^\varepsilon(z) &= \frac{1}{k} f'(z) \langle W^\varepsilon, p \cdot \nabla_x \theta \rangle + k^2 f''(z) A_2^{(1)}(W^\varepsilon) + k^2 f' A_1^{(1)}(W^\varepsilon) + R_2(z) + R_3(z) + A_3(z),
\end{align}

where

\begin{align}
(108) \quad R_2(z) &= \varepsilon^2 k^2 \frac{f'''(z)}{2} \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x \theta \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon \theta \rangle \right] \left[ \langle W^\varepsilon, \bar{L}^\varepsilon \theta \rangle^2 - A_2^{(2)}(W^\varepsilon) \right] \\
&\quad + \varepsilon^2 k^2 f''(z) \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x (L^\varepsilon \bar{L}^\varepsilon \theta) \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon \bar{L}^\varepsilon \theta \rangle \right] \\
&\quad - \varepsilon^2 k^2 f''(z) \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x (G^2(\bar{\theta}) W^\varepsilon) \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon G^2(\bar{\theta}) W^\varepsilon \rangle \right]
\end{align}

\begin{align}
(109) \quad R_3(z) &= \varepsilon^2 k^2 f'(z) \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x (\bar{L}^\varepsilon \bar{L}^\varepsilon \theta) \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon \bar{L}^\varepsilon \theta \rangle \right] \\
&\quad + \varepsilon^2 k^2 f''(z) \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x \theta \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon \theta \rangle \right] \left[ \langle W^\varepsilon, \bar{L}^\varepsilon \bar{L}^\varepsilon \theta \rangle - A_1^{(2)}(W^\varepsilon) \right] \\
&\quad - \varepsilon^2 k^2 f'(z) \left[ \frac{1}{k} \langle W^\varepsilon, p \cdot \nabla_x (Q_2^2 \theta) \rangle + \frac{k}{\bar{e}} \langle W^\varepsilon, L^\varepsilon Q_2^2 \theta \rangle \right]
\end{align}

and $A_1^\varepsilon, A_2^{(1)}, A_1^{(1)}, G^{(2)}_\theta, Q_2$ all have the same expressions as in the proof of Theorem 1.

With the assumption $\beta > 3/4$, Propositions 1, 2, 4, 5 and 6 hold true. Let us remark that the most severe terms due to the divergent factor $\bar{e}^{1-\alpha/\beta}$ are

\begin{align}
(110) \quad \sup_{z < z_0} \bar{e}E \left| \langle W^\varepsilon, L^\varepsilon G^{(2)}_\theta W^\varepsilon \rangle \right| &= O(\bar{e}^{4-3\alpha/\beta}), \\
(111) \quad \sup_{z < z_0} \bar{e}E \left| \langle W^\varepsilon, \bar{L}^\varepsilon \bar{L}^\varepsilon \theta \rangle \right| &= O(\bar{e}^{4-3\alpha/\beta})
\end{align}

(cf. Corollary 5).

To satisfy \textbullet we need to show

**Proposition 8.**

\[
\lim_{\bar{e} \to 0} \sup_{z < z_0} |f_j(z)| = 0, \quad j = 2, 3, \quad \text{in probability.}
\]

**Proof.** We have the estimates

\[
\sup_{z < z_0} |f_2(z)| \leq \sup_{z < z_0} \bar{e}^2 k^2 \|f''\|_\infty \left( \|W_0\|^2_2 \|\bar{L}^\varepsilon \theta\|^2_2 + A_2^{(2)}(W^\varepsilon) \right)
\]

\[
\sup_{z < z_0} |f_3(z)| \leq \sup_{z < z_0} \bar{e}^2 k^2 \|f'\|_\infty \left( \|W_0\|^2_2 \|\bar{L}^\varepsilon \bar{L}^\varepsilon \theta\|^2_2 + A_1^{(2)}(W^\varepsilon) \right)
\]

which vanish in probability by using Assumption 3, Lemma 5, 6 and Chebyshev’s inequality.
Proposition 9.

\[
\lim_{\varepsilon \to 0} \sup_{\xi < \varepsilon \|\psi\|_2 = 1} A^{(1)}_2(\psi) = 0.
\]

Proof. The kernel \( Q_1(\theta \otimes \theta) \) has the following expressions in terms of \( \varepsilon \) (not \( \bar{\varepsilon} \)):

\[
Q_1(\theta \otimes \theta)(x, p, y, q) = \varepsilon^{2\beta - 2\alpha} \pi \int_0^\infty \int \Phi(s, p') e^{i p' \cdot (\bar{x} - \bar{y})} e^{-is \cdot p' \varepsilon^{2\beta - 2\alpha}} [\theta(x, p + p'/2) - \theta(x, p - p'/2)] \\
\times [\theta(y, q + p'/2) - \theta(y, q - p'/2)] \, dp' \, ds
\]

Taking the \( L^2 \)-norm and passing to the limit we have

\[
\lim_{\varepsilon \to 0} \int dxdpdydq |Q_1(\theta \otimes \theta)(x, p, y, q)|^2
\]

\[
= \lim_{\varepsilon \to 0} \pi^2 \int dxdpdydq \left| \int \frac{P'}{k} \left[ \int \Phi(w, p') dw \right] e^{i p' \cdot (x - y)/\varepsilon^2} \right|^2 \\
\times \left[ \theta(x, p + p'/2) - \theta(x, p - p'/2) \right] \left[ \theta(y, q + p'/2) - \theta(y, q - p'/2) \right] \, dp' \right|^2
\]

\[
= \lim_{\varepsilon \to 0} \pi^2 \int dxdpdydq \left| \int \frac{k}{|P|} \left[ \int \Phi(w, p_\perp) dw \right] e^{i p_\perp \cdot (x - y)/\varepsilon^2} \right|^2 \\
\times \left[ \theta(x, p + p_\perp/2) - \theta(x, p - p_\perp/2) \right] \left[ \theta(y, q + p_\perp/2) - \theta(y, q - p_\perp/2) \right] \, dp_\perp \right|^2
\]

where \( p_\perp \cdot p = 0, p_\perp \in \mathbb{R}^{d-1} \). In passing to the limit the only problem is at the point \( p = 0 \). But the integrand in the above integral is bounded by

\[
(112) \quad C|p|^{-2}, \quad \text{some constant } C < \infty
\]

which is integrable in a neighborhood of zero if \( d \geq 3 \). Hence the \( L^2 \)-norm of \( Q_1(\theta \otimes \theta) \) tends to zero by the dominated convergence theorem. \( \square \)
We have the following straightforward calculation: For any real-valued, $L^2$-weakly convergent sequence $\psi^\varepsilon \to \psi$,

$$\lim_{\varepsilon \to 0} A_1^{(1)}(\psi^\varepsilon) = \lim_{\varepsilon \to 0} \int_0^\infty ds \int dq dx dp \ \psi^\varepsilon(x, p) \tilde{\Phi}(s, q) \varepsilon^{2\beta - 2\alpha}$$

$$= \lim_{\varepsilon \to 0} \pi \int dq dx dp \ \psi^\varepsilon(x, p) \left[ \Phi(k^{-1}(p + q/2) \cdot q \varepsilon^{2\beta - 2\alpha}, q) \varepsilon^{2\beta - 2\alpha} [\theta(x, p + q) - \theta(x, p)] - \Phi(k^{-1}(p - q/2) \cdot q \varepsilon^{2\beta - 2\alpha}, q) \varepsilon^{2\beta - 2\alpha} [\theta(x, p - q) - \theta(x, p)] \right]$$

$$= 2\pi \int dq dx dp \ \psi(x, p) \delta(\frac{\varepsilon - \varepsilon}{2k}) \left[ \int \Phi(w, q - p) dw \right] [\theta(x, p + q) - \theta(x, p)]$$

following from the strong convergence of

$$\int \left[ \Phi(\frac{\varepsilon - \varepsilon}{2k}, q - p) \varepsilon^{2\beta - 2\alpha} [\theta(x, p + q) - \theta(x, p)] - \Phi(\frac{\varepsilon - \varepsilon}{2k}, q - p) \varepsilon^{2\beta - 2\alpha} [\theta(x, p - q) - \theta(x, p)] \right] dq, \ \forall \Phi \in C_c^\infty(\mathbb{R}^{d+1}), \ \theta \in C_c^\infty(\mathbb{R}^d)$$

in $L^2(\mathbb{R}^d)$.

**Proposition 10.** $A^\varepsilon f^\varepsilon(z)$ is uniformly integrable.

This, of course, follows from the fact that each term in (107) has a uniformly bounded second moment. Therefore we have completed the tightness argument. Moreover, we have also identified the limiting equation.

9. **Proof of Theorem 4**

Introducing a new parameter

$$\tilde{\varepsilon} = \varepsilon^\beta,$$

the fast variable

$$\tilde{x} = x\varepsilon^{-2\alpha}$$

and the rescaled process

$$\tilde{V}_{\varepsilon}^\varepsilon(dq) = \tilde{V}(\tilde{z}, dq), \ \ V_{\varepsilon}^\varepsilon(x) = V(\tilde{\varepsilon}, x)$$

we rewrite the equation as

$$\frac{\partial W_{\varepsilon}^\varepsilon}{\partial \tilde{z}} + \frac{P}{k} \cdot \nabla W_{\varepsilon}^\varepsilon + \frac{k}{\varepsilon} C_{\varepsilon} W_{\varepsilon}^\varepsilon = 0$$
with, in the case (i),
\begin{equation}
\mathcal{L}_z^\varepsilon W_\varepsilon^\varepsilon(x, \tilde{x}, p) = i \int e^{i q \tilde{x}} \varepsilon^{2\alpha-2} \left[ W_\varepsilon^\varepsilon(x, p + \varepsilon^{2-2\alpha} q/2) - W_\varepsilon^\varepsilon(x, p - \varepsilon^{2-2\alpha} q/2) \right] \hat{V}_\varepsilon^\varepsilon(dq),
\end{equation}
and in the case (ii)
\begin{equation}
\mathcal{L}_z^\varepsilon W_\varepsilon^\varepsilon(x, \tilde{x}, p) = i \varepsilon^{1-\alpha/\beta} \int e^{i q \tilde{x}} \varepsilon^{2\alpha-2} \left[ W_\varepsilon^\varepsilon(x, p + \varepsilon^{2-2\alpha} q/2) - W_\varepsilon^\varepsilon(x, p - \varepsilon^{2-2\alpha} q/2) \right] \hat{V}_\varepsilon^\varepsilon(dq).
\end{equation}

Taking the partial inverse Fourier transform we get in the case (i)
\begin{equation}
\mathcal{F}^{-1}_2 \mathcal{L}_z^\varepsilon \theta(x, \tilde{x}, y) = -i \varepsilon \mathcal{F}^{-1}_2 V_\varepsilon^\varepsilon(x, y) \mathcal{F}^{-1}_2 \theta(x, y)
\end{equation}
and in the case (ii)
\begin{equation}
\mathcal{F}^{-1}_2 \mathcal{L}_z^\varepsilon \theta(x, \tilde{x}, y) = -i \varepsilon^{1-\alpha/\beta} \delta_\varepsilon V_\varepsilon^\varepsilon(x, y) \mathcal{F}^{-1}_2 \theta(x, y)
\end{equation}
with
\begin{equation}
\delta_\varepsilon V_\varepsilon^\varepsilon(x, y) = \varepsilon^{2\alpha-2} \left[ V_\varepsilon^\varepsilon(x + \varepsilon^{2-2\alpha} y/2, y) - V_\varepsilon^\varepsilon(x - \varepsilon^{2-2\alpha} y/2) \right].
\end{equation}

The proof for the case (i) is entirely analogous to that for Theorem 2 and we will focus on the case (ii) in the sequel. And we will work with (116) and (118) and construct the perturbed test with, in the case (i),
\begin{equation}
\text{Lemma 7.}
\end{equation}
\begin{equation}
\limsup_{\varepsilon \to 0} \varepsilon^{-2+2\alpha/\beta} \mathbb{E} \left[ \mathcal{L}_z^\varepsilon \theta \right]^2(x, p) = \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq
\end{equation}
which has an \(\varepsilon\)-uniformly bounded support.

As in the proof of Theorem 3, we carry the analysis in the power of \(\varepsilon = \varepsilon^\theta\). We consider the rescaled process (113) and its sigma algebras.

We introduce the corrector
\begin{equation}
\hat{\mathcal{L}}_z^\varepsilon \theta(x, \tilde{x}, p)
\end{equation}
which after the partial Fourier inversion becomes
\begin{equation}
\mathcal{F}^{-1}_2 \hat{\mathcal{L}}_z^\varepsilon \theta(x, \tilde{x}, y) = -i \varepsilon^{1+\alpha/\beta} \int e^{i(k+1)(s-z)} \nabla_y \cdot \nabla_x \varepsilon^{2\alpha/\beta} \mathbb{E}_z \left[ \mathcal{L}_z^\varepsilon \theta \right] \mathcal{F}^{-1}_2 \theta(x, \tilde{x}, y) ds.
\end{equation}
The corrector solves the corrector equation eq. (101).

Following the same argument as in the proof of Theorem 1 we have the following estimates:

\begin{equation}
\text{Lemma 7.}
\end{equation}
\begin{equation}
\limsup_{\varepsilon \to 0} \varepsilon^{-2+2\alpha/\beta} \mathbb{E} \left[ \hat{\mathcal{L}}_z^\varepsilon \theta \right]^2(x, p) \leq \int_0^\infty \rho(s) ds \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq
\end{equation}
which has a compact support.

\begin{equation}
\text{Corollary 7.}
\end{equation}
\begin{equation}
\limsup_{\varepsilon \to 0} \varepsilon^{-2+2\alpha/\beta} \mathbb{E} \left[ p \cdot \nabla_x \hat{\mathcal{L}}_z^\varepsilon \theta \right]^2(x, p) \leq \int_0^\infty \rho(s) ds \int [p \cdot \nabla_x q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq
\end{equation}
which has a compact support.
**Lemma 8.**

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-4+4\alpha/\beta} \mathbb{E} \| \mathcal{L}_x^\varepsilon \mathcal{L}_x^\varepsilon \theta \|_2^2 \leq 8C \left( \int_0^\infty \rho(s) ds \right)^2 \mathbb{E} \| V_2 \|_2^2 \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq dp
\]

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-4+4\alpha/\beta} \mathbb{E} \| \mathcal{L}_x^\varepsilon \mathcal{L}_x^\varepsilon \theta \|_2^2 \leq 8C \left( \int_0^\infty \rho(s) ds \right)^4 \mathbb{E} \| V_2 \|_2^4 \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq dp
\]

for some constant \(C\) independent of \(\varepsilon\).

**Corollary 8.**

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-4+4\alpha/\beta} \mathbb{E} \| \nabla_x \mathcal{L}_x^\varepsilon \mathcal{L}_x^\varepsilon \theta \|_2^2 \leq 32C \left( \int_0^\infty \rho(s) ds \right)^4 \mathbb{E} \| V_2 \|_2^4 \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq dp
\]

\[
\limsup_{\varepsilon \to 0} \varepsilon^{-6+6\alpha/\beta} \mathbb{E} \| \mathcal{L}_x^\varepsilon \mathcal{L}_x^\varepsilon \mathcal{L}_x^\varepsilon \theta \|_2^2 \leq 32C \left( \int_0^\infty \rho(s) ds \right)^4 \mathbb{E} \| V_2 \|_2^4 \int [q \cdot \nabla_p \theta(x, p)]^2 \Phi(\xi, q) d\xi dq dp
\]

for some constant \(C\) independent of \(\varepsilon\).

The rest of the argument follows the general outline of that of Theorem 3 Case (ii).

Let us know verify that the quadratic variation vanishes in the limit.

**Proposition 11.**

\[
\limsup_{\varepsilon \to 0} \sup_{\|\psi\|_2 = 1} A_2(\psi) = 0.
\]

**Proof.** The kernel \(Q_1(\theta \otimes \theta)\) can be calculated as follows.

\[
Q_1(\theta \otimes \theta)(x, p, y, q) = \varepsilon^{2\beta-2\alpha} \int_0^\infty d\xi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(s, p') e^{i p' \cdot (x-y)} e^{-i k^{-1} s p' \epsilon^{2\beta-2\alpha}} e^{-2\alpha s - 2} \left[ \theta(x, p + \epsilon^{2\alpha} p'/2) - \theta(x, p - \epsilon^{2\alpha} p'/2) \right] \times e^{2\alpha s - 2} \left[ \theta(y, q + \epsilon^{2\alpha} p'/2) - \theta(y, q - \epsilon^{2\alpha} p'/2) \right] dp' ds
\]

\[
= \pi \int \varepsilon^{2\beta-2\alpha} \Phi(k^{-1} p \cdot p' \epsilon^{2\beta-2\alpha}) e^{i p' \cdot (x-y)} e^{-2\alpha s - 2} \left[ \theta(x, p + \epsilon^{2\alpha} p'/2) - \theta(x, p - \epsilon^{2\alpha} p'/2) \right] \times e^{2\alpha s - 2} \left[ \theta(y, q + \epsilon^{2\alpha} p'/2) - \theta(y, q - \epsilon^{2\alpha} p'/2) \right] dp'
\]

whose \(L^2\)-norm has the following limit

\[
\lim_{\varepsilon \to 0} \pi^2 \int \left| \int_{p_\perp} k |p_\perp|^{-1} \left[ \int \Phi(w, p_\perp) dw \right] e^{i p_\perp \cdot (x-y) \varepsilon^{-2\alpha}} |q \cdot \nabla_p \theta(x, p)|^2 dp_\perp \right|^2 dx dp = 0
\]

if \(d \geq 3\) by the dominated convergence theorem because the integrand is bounded by the integrable function \(112\) in a neighborhood of zero.
To identify the limit, we have the following straightforward calculation: For any real-valued, $L^2$-weakly convergent sequence $\psi^\varepsilon \to \psi$,

$$\lim_{\varepsilon \to 0} A_1^{(1)}(\psi^\varepsilon) = \lim_{\varepsilon \to 0} \int_0^\infty ds \int dq dx dp \ \psi^\varepsilon(x, p) \Phi(s, q)e^{-ik^{-1}s\vec{q} \cdot \vec{q} e^{2\beta-2\alpha}}e^{2\beta-2\alpha} e^{4\alpha-4}$$

$$\left[ e^{-ik^{-1}s|q|^2} e^{2\beta-4\alpha} / 2 \left[ \theta(x, p + \varepsilon^{2-2\alpha} q) - \theta(x, p) \right] \right]$$

$$- e^{-ik^{-1}s|q|^2} e^{2\beta-4\alpha} / 2 \left[ \theta(x, p) - \theta(x, p - \varepsilon^{2-2\alpha} q) \right]$$

$$= \lim_{\varepsilon \to 0} \pi \int \varepsilon^{2\alpha-2} \varepsilon^{2\beta-2\alpha} \left[ \Phi(k^{-1}(\vec{p} + \varepsilon^{2-2\alpha} q/2) \cdot \vec{q} e^{2\beta-2\alpha}, \vec{q} e^{2\beta-2\alpha} \left[ \theta(x, p + \varepsilon^{2-2\alpha} q) - \theta(x, p) \right] \right]$$

$$- \Phi(k^{-1}(\vec{p} - \varepsilon^{2-2\alpha} q/2) \cdot \vec{q} e^{2\beta-2\alpha}, \vec{q} e^{2\beta-2\alpha} \left[ \theta(x, p) - \theta(x, p - \varepsilon^{2-2\alpha} q) \right] \right] \psi^\varepsilon(x, p) d\vec{q} dx dp$$

$$= \pi \int \vec{q} \cdot \nabla_\vec{p} \left[ \delta(k^{-1}\vec{p} \cdot \vec{q}) \left[ \int \Phi(w, q) dw \right] \right] \theta(x, p) \psi(x, p) d\vec{q} dx dp$$

$$= \pi \int k|\vec{p}|^{-1} \left[ \int \Phi(w, \vec{p} \cdot \nabla_\vec{p}) dw \right] (\vec{p} \cdot \nabla_\vec{p})^2 \theta(x, p) \psi(x, p) d\vec{p} d\vec{x} dp$$

$$\equiv A_1(\psi)$$

where $\vec{p} \perp \in \mathbb{R}^{d-1}, \vec{p} \perp \cdot \vec{p} = 0$.

**Case (iii):** $\alpha = \beta$.

$$\lim_{\varepsilon \to 0} A_1^{(1)}(\psi^\varepsilon) = \lim_{\varepsilon \to 0} \pi \int d\vec{q} dx dp \ \psi^\varepsilon(x, p) e^{4\alpha-4} \left[ \Phi(k^{-1}(\vec{p} + \varepsilon^{2-2\alpha} q, q) \cdot q, q) \right] \left[ \theta(x, p + \varepsilon^{2-2\alpha} q) - \theta(x, p) \right]$$

$$- \Phi(k^{-1}(\vec{p} - \varepsilon^{2-2\alpha} q/2) \cdot \vec{q} e^{2\beta-2\alpha}, \vec{q} e^{2\beta-2\alpha} \left[ \theta(x, p) - \theta(x, p - \varepsilon^{2-2\alpha} q) \right] \right]$$

$$= \pi \int d\vec{q} dx dp \ \psi(x, p) q \cdot \nabla_\vec{p} \left[ \Phi(k^{-1}\vec{p} \cdot q, q) \cdot \nabla_\vec{p} \right] \theta(x, p)$$

$$\equiv A_1(\psi)$$

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