The applicability range of different forms of the radiation friction force in the ultrarelativistic electron interaction with electromagnetic wave (exact solutions)

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Abstract. When the effects of radiation reaction dominate the interaction of electrons with intense laser pulses, the electron dynamics changes qualitatively. The adequate theoretical description of this regime becomes crucially important with the use of the radiation friction force either in the Lorentz-Abraham-Dirac form, which possess unphysical runaway solutions, or in the Landau-Lifshitz form, which is a perturbation valid for relatively low electromagnetic wave amplitude. The goal of the present paper is to find the limits of the Landau-Lifshitz radiation force applicability in terms of the electromagnetic wave amplitude and frequency.

1. Introduction
The recent development of ultrahigh-intensity laser systems has generated a great amount of interest in a class of well-known theoretical problems involving the interaction of strong electromagnetic fields with with charged particles that have not been experimentally demonstrated. Among them the problem of the radiation friction effects on the charged particle dynamics, which has been attracting attention for more than a century [1, 2, 3, 4]. The radiation friction imposes constraints on the highest attainable energy of charged particles accelerated by standard accelerators [5] and in space [6], in particular, on the energy of the ultra high energy cosmic rays [7]. The effects of radiation reaction on electrons in a magnetically confined plasma lead to the phase space contraction [8]. The radiation generated by present day [9, 10] lasers approaches limits when the radiation friction force will change the scenario of the electromagnetic (EM) wave interaction with matter, i.e. at $I > I_{\text{rad}} = 10^{23}$ W/cm$^2$. The electron dynamics will become dissipative with fast conversion of the EM wave energy to hard EM radiation, which for typical lasers parameter is in the gamma-ray range [11, 12, 13, 14]. There are discussions of the modification of the electron acceleration in the laser wake field acceleration regime [15] and the ion acceleration in the radiation pressure dominated regime [16] due the radiation friction, which are mainly obtained with computer simulations [11, 17]. If the laser intensity substantially exceeds $I_{\text{rad}}$ novel physics of abundant electron-positron pair creation will come into play [18] (see also [13] and [19]) when the electron (positron) interaction with the EM field is principally determined by the radiation friction effects. The persistent interest towards the radiation friction effects stems from all these reasons [20, 21].

In order to self-consistently find the trajectory of the emitting electron, the so called
Minkowsky equations [3] should be solved with the radiation friction force taken into account

\[
m_\text{e}c \frac{du^\mu}{ds} = \frac{e}{c} F^{\mu\nu} u_\nu + g^\mu, \tag{1}
\]

\[
\frac{dx^\mu}{ds} = u^\mu. \tag{2}
\]

Here \(u_\mu = (\gamma, p/m_\text{e}c)\) is the four-velocity, \(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\) is the EM field tensor with \(A_\mu\) being the EM four-vector and \(\mu = 0, 1, 2, 3\), and \(s = c \int dt/\gamma\). The radiation friction force in the Lorentz-Abraham-Dirac (LAD) form [22, 23, 24] is given by

\[
g^\mu = \frac{2e^2}{3c} \left[ \frac{d^2 u^\mu}{ds^2} - u^\mu \left( \frac{du^\mu}{ds} \right)^2 \right]. \tag{3}
\]

As is well known, the equation (1) with the radiation friction force in the LAD form (3) possesses unphysical self-accelerating solutions (e.g. see Refs. [2, 3]). When the radiation friction force is taken to be in the Landau-Lifshitz (L-L) form,

\[
g^\mu = \frac{2e^3}{3m_\text{e}c^2} \left\{ \frac{\partial F^{\mu\nu}}{\partial x^\lambda} u_\nu u_\lambda - \frac{e}{m_\text{e}c^2} \left[ F^{\mu\lambda} F_{\nu\lambda} u^\nu - \left( F_{\nu\lambda} u^\lambda \right) (F^{\nu\rho} u_\rho) u^\mu \right] \right\}, \tag{4}
\]

the electron motion equations does not have pathological solutions, although they are not always consistent with energy-momentum conservation for an abruptly changing electromagnetic field [25]. Since the L-L friction force is derived as a perturbation to the equations of motion, this approximation is valid provided there exists a frame of reference, where the L-L radiation friction force is small compared to the Lorentz force, \(eF^{\mu\nu} u_\nu\), as noticed in Ref. [2]. Proving this frame of reference existence and finding the range of validity of the friction force in the L-L form is far from trivial. Below, using several exact analytical solutions to the electron motion equations in the EM field for the radiation friction force in the LAD and L-L forms, we discuss the validity range of the later approximation.

The electron motion equations with the LAD friction force admit exact solution for the stationary problem describing the electron motion in the rotating electric field (see Refs. [26, 12, 13]). This problem can also be solved for the case of the L-L force. Generalizing the electromagnetic field configuration, we consider the electric and magnetic field to be a superposition of the rotating with the frequency \(\omega\) homogeneous in space and time-independent components

\[
E = e_1 E_1 + e_2 [Dx_2 + E \cos(\omega t)] + e_3 [Dx_3 + E \sin(\omega t)], \tag{5}
\]

\[
B = e_1 B_1 + e_2 Jx_3 - e_3 Jx_2, \tag{6}
\]

where \(e_1, e_2\) and \(e_3\) are the unit vectors along the 1,2,3 axis. The EM field tensor is equal to

\[
F^{\mu\nu} = \begin{pmatrix}
0 & -E_1 & -Dx_2 - E \cos(\omega t) & -Dx_3 - E \sin(\omega t) \\
E_1 & 0 & -Jx_2 & Jx_3 \\
Dx_2 + E \cos(\omega t) & Jx_2 & 0 & -B_1 \\
Dx_3 + E \sin(\omega t) & -Jx_3 & B_1 & 0
\end{pmatrix}. \tag{7}
\]

In the case when \(E_1, B_1, J\) and \(D\) vanish the electric field can be realized in the antinodes, where the magnetic field vanishes, of a standing EM wave formed by two counter-propagating circularly polarized EM waves. Such an EM field configuration plays an important role in theoretical considerations of various nonlinear effects in quantum electrodynamics, e.g. see Refs. [13, 18, 27, 28]. This EM configuration corresponds also to the circularly polarized EM wave
propagating in the underdense plasma for the frame of reference moving with the wave group velocity \([29, 30]\). In this frame of reference the EM wave frequency is equal to the Langmuir frequency, \(\omega_{pe} = (4\pi n_0 e^2/m_e)^{1/2}\), where \(n_0\) is the plasma density and the wave has no magnetic field component. The static component of the magnetic field, \(B_1\), can be generated in the laser plasmas due to the inverse Faraday effect. Its effect on the charged particle motion has been studied in Ref. \([26]\). The radial component of the electric field, \(\mathbf{E}_2Dx_2 + \mathbf{E}_3Dx_3\), and azimuthal component of magnetic field, \(\mathbf{e}_2Jx_3 - \mathbf{e}_3Jx_2\), correspond to the plasma wave in the boosted frame of reference with \(E_1\) being the longitudinal component of the wake field.

It is convenient to write the electron momentum \(p = e_1p_1(t) + e_2p_2(t) + e_3p_3(t)\) and coordinates \(x = e_1x_1(t) + e_2x_2(t) + e_3x_3(t)\) as a combination of non-rotating and rotating with the angular frequency \(\omega\) vectors,

\[
\begin{pmatrix}
\tilde{u}_1 \\
\tilde{u}_2 \\
\tilde{u}_3 
\end{pmatrix} = \frac{1}{m_e c} \begin{pmatrix}
p_1 \\
p_\parallel \\
p_\perp
\end{pmatrix} = \frac{1}{m_e c} \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\omega t) & \sin(\omega t) \\
0 & -\sin(\omega t) & \cos(\omega t)
\end{pmatrix} \begin{pmatrix}
p_1 \\
p_2 \\
p_3
\end{pmatrix}
\]

(8)

and

\[
\begin{pmatrix}
\tilde{x}_1 \\
\tilde{x}_2 \\
\tilde{x}_3
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos(\omega t) & \sin(\omega t) \\
0 & -\sin(\omega t) & \cos(\omega t)
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}
\]

(9)

2. Stationary solution to the electron equations of motion with the radiation friction force in the Lorentz-Abraham-Dirac form

Stationary solution to equations (1) and (2), for which the vectors \(\mathbf{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\) and \(\mathbf{\tilde{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\) do not depend on time, with the radiation friction force in the LAD form (3) can be cast as

\[
0 = a_1 - \varepsilon_{rad}\tilde{u}_1 \gamma \left( \gamma^2 - 1 - \tilde{u}_1^2 \right),
\]

(10)

\[
\tilde{u}_2 = \left( d - \tilde{u}_1 \right) \frac{\tilde{u}_2}{\gamma} + \tilde{u}_3 \frac{\tilde{u}_3}{\gamma} + \varepsilon_{rad}\tilde{u}_3 \gamma \left( \gamma^2 - \tilde{u}_1^2 \right),
\]

(11)

\[
\tilde{u}_3 = \left( d - \tilde{u}_1 \right) \frac{\tilde{u}_3}{\gamma} - \tilde{u}_2 \frac{\tilde{u}_2}{\gamma} + a - \varepsilon_{rad}\tilde{u}_2 \gamma \left( \gamma^2 - \tilde{u}_1^2 \right),
\]

(12)

where we use the relationship between \(\tilde{x}_i\) and \(\tilde{u}_i\), with \(i = 1, 2, 3\), given by Eq. (2), which is

\[
\tilde{x}_1 = s\gamma \tilde{u}_1, \quad \tilde{x}_2 = \frac{\tilde{u}_3 c}{\gamma \omega_0}, \quad \text{and} \quad \tilde{x}_3 = -\frac{\tilde{u}_2 c}{\gamma \omega_0}.
\]

(13)

Here the dimensionless parameter,

\[
\varepsilon_{rad} = \frac{2e^2 \omega_0}{3m_e c^3}
\]

(14)

characterizes the radiation damping effect, \(a_1 = eE_1/m_e \omega_0 c\), \(a = eE/m_e \omega_0 c\), \(d = eD/m_e \omega_0 c\), \(j = eJ/m_e \omega_0 c\), and \(b = eB_1/m_e \omega_0 c\) are normalized longitudinal and transverse components of the electric and magnetic field, and \(\gamma\) is the electron relativistic Lorentz-factor equal to \((1 + \tilde{u}_1^2 + \tilde{u}_2^2 + \tilde{u}_3^2)^{1/2}\). The parameter \(\varepsilon_{rad}\) can also be written as \(\varepsilon_{rad} \approx 4\pi \epsilon_r e^2 / 3\lambda_0^3\) or \(\varepsilon_{rad} = 2\omega_0 \epsilon_r / 3\), where \(\epsilon_r = e^2 / m_e c^2\) is the classical electron radius, \(\epsilon_r = e^2 / c\), and \(\lambda_0 = 2\pi c / \omega_0\).

At first we analyze the most simple case with \(B_1 = J = D = 0\). The stationary solution to equations (1) and (2), for which the vectors \(\mathbf{\tilde{u}} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)\) and \(\mathbf{\tilde{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\) do not depend on time, with the radiation friction force in the LAD form (3) can be cast as

\[
0 = a_1 - \varepsilon_{rad}\tilde{u}_1 \gamma \left( \gamma^2 - 1 - \tilde{u}_1^2 \right),
\]

(15)
\[ \tilde{u}_2 = \varepsilon_{\text{rad}} \tilde{u}_3 \gamma (\gamma^2 - \tilde{u}_1^2), \]  
(16)

\[ \tilde{u}_3 = a - \varepsilon_{\text{rad}} \tilde{u}_2 \gamma (\gamma^2 - \tilde{u}_1^2). \]  
(17)

Multiplying Eq. (15) by \( \tilde{u}_1 \), Eq. (16) by \( \tilde{u}_2 \), and Eq. (17) by \( \tilde{u}_3 \), and adding them, we obtain

\[ a_1 \tilde{u}_1 + a \tilde{u}_2 = \varepsilon_{\text{rad}} \gamma^3 (\gamma^2 - 1 - \tilde{u}_1^2). \]  
(18)

The left hand side of this equation is proportional to the work produced by the electric field in the units of time and the right hand side is proportional to the energy dissipation rate due to the radiation losses.

Multiplying Eq. (16) by \( \tilde{u}_3 \) and Eq. (17) by \( \tilde{u}_2 \), and adding them, we obtain

\[ \tilde{u}_2^2 + \tilde{u}_3^2 = a \tilde{u}_3. \]  
(19)

3. Electron in the rotating electric field

3.1. Lorentz-Abraham-Dirac form of the radiation friction force

If, in addition, the longitudinal component of electric field vanishes, \( a_1 = 0 \) with \( \tilde{u}_1 = 0 \), we obtain from Eqs. (10 - 12)

\[ p_\parallel = \varepsilon_{\text{rad}} p_\perp \gamma^3 \quad \text{and} \quad p_\perp = m_e c a - \varepsilon_{\text{rad}} p_\parallel \gamma^3, \]  
(20)

where the parallel and perpendicular to the electric field components of the electron momentum defined by Eq. (8) are equal to

\[ p_\parallel = \frac{(\mathbf{p} \cdot \mathbf{E})}{|\mathbf{E}|} = m_e c \tilde{u}_2 \quad \text{and} \quad p_\perp = (p^2 - p_\parallel^2)^{1/2} = m_e c \tilde{u}_3, \]  
(21)

respectively. In this case equation (19) yields a relationship between \( p_\parallel \) and \( p_\perp \):

\[ p_\parallel^2 + p_\perp^2 = m_e c a p_\perp. \]  
(22)

The electron gamma-factor \( \gamma \) is equal to \((1 + \tilde{u}_2^2 + \tilde{u}_3^2)^{1/2} \equiv (1 + p_\parallel^2 + p_\perp^2)^{1/2}\).

As we see, from the relationship \( p_\parallel = [p_\perp (p_\perp - m_e c a)]^{1/2} \) it follows that the perpendicular to the electric field component of the electron momentum is always equal or less than \( a \). Multiplying the first equation in (20) by \( p_\perp \) and the second equation by \( p_\parallel \) and subtracting them, we find

\[ p_\parallel a = \varepsilon_{\text{rad}} p_\perp^3 \gamma^3, \]  
(23)

which corresponds to the energy balance equation (18) for \( a_1 = \tilde{u}_1 = 0 \).

If the EM field amplitude is relatively small, i.e. \( a \ll \varepsilon_{\text{rad}}^{-1/3} \) Eqs. (22) and (23) yield for the components of the electron momentum perpendicular and parallel to the electric field

\[ p_\perp \approx m_e c \left( a - \varepsilon_{\text{rad}}^2 a \right) \quad \text{and} \quad p_\parallel \approx m_e c \varepsilon_{\text{rad}} a^4. \]  
(24)

In the opposite limit, when \( a \gg \varepsilon_{\text{rad}}^{-1/3} \), we obtain

\[ p_\perp \approx \frac{m_e c}{(\varepsilon_{\text{rad}} a)^{1/2}} \quad \text{and} \quad p_\parallel \approx m_e c \left( a \frac{1}{\varepsilon_{\text{rad}}} \right)^{1/4}. \]  
(25)

In Fig. 1a we show a dependence of \( p_\perp \) and \( p_\parallel \) on the EM field amplitude, \( a \), for the dimensionless parameter \( \varepsilon_{\text{rad}} = 10^{-8} \), obtained by numerical solution of Eqs. (20). Here the horizontal axis is normalized by \( \varepsilon_{\text{rad}}^{-1/3} \) and the vertical axis is normalized by \((a_m/\varepsilon_{\text{rad}})^{1/4}\).
Figure 1. a) Dependence of perpendicular, $p_\perp$, and parallel, $p||$, to the electric field components of the electron momentum (normalized on $m_e c (a_m/\varepsilon_{rad})^{1/4}$) on the normalized EM field amplitude, $a \varepsilon_{rad}$, and b) dependence of $\varphi$ and normalized on $a \varepsilon_{rad}$ for $a_m = 2500$ and $\varepsilon_{rad} = 10^{-8}$.

As we see, the dependences of the components of the electron momentum perpendicular and parallel to the electric field correspond the asymptotics given by Eqs. (24) and (25). The perpendicular momentum reaches the maximum at $a \approx \varepsilon_{rad}^{-1/3}$ and then decreases. The parallel momentum component monotonously increases with the EM amplitude growth.

It is also convenient to represent the momentum components in the complex form

$$p|| + ip_\perp = p \exp(-i\varphi)$$

with $p = (p_\perp^2 + p||^2)^{1/2}$ and $\varphi$ being the momentum value and the phase between the rotating electric field and the momentum vector. Eqs. (20) can be rewritten as

$$m_e c a = p(1 + \varepsilon_{rad}^2 \gamma^6)^{1/2} \quad \text{and} \quad \tan \varphi = -\frac{1}{\varepsilon_{rad} \gamma^3},$$

where the electron gamma-factor $\gamma$ is equal to $(1 + p^2)^{1/2}$. These equations are the same as Eqs. (6) in Ref. [13].

In Fig. 1b we present the electron gamma factor $\gamma$ normalized by $(a_m/\varepsilon_{rad})^{1/4}$ and the angle $\varphi$ versus the EM field amplitude $a$ for $\varepsilon_{rad} = 10^{-8}$. The angle $\varphi$ changes from $\pi/2$ at $a = 0$, when the electron momentum is perpendicular to the electric field vector, to $0$ at $a \to \infty$, when the electron momentum becomes antiparallel to the electric field. The horizontal axis is normalized in the same way as in Fig. 1a.

3.2. Landau-Lifshitz form of the radiation friction force

We look for the solutions describing a stationary electron orbit in a rotating homogeneous electric field, i.e. $E_1, D, B_1, J$ vanish in Eqs. (5 - 7). From Eq. (4) we obtain for the $p||$ and $p_\perp$ momentum components the algebraic equations

$$p|| = \varepsilon_{rad} \frac{p_\perp}{\gamma} a^2 \left[ 1 + \left( \frac{p_\perp}{m_e c} \right)^2 \right] \quad \text{and} \quad p_\perp = m_e c a - \varepsilon_{rad} \frac{p||}{\gamma} a^2 \left[ 1 + \left( \frac{p_\perp}{m_e c} \right)^2 \right].$$

Using the variables $p$ and $\varphi$ defined by Eq. (26) we can present these equations in the form

$$m_e c a = \varepsilon_{rad} \frac{p}{\gamma} a^2 + \left( \frac{p}{m_e c} \right)^2 \quad \text{and} \quad \sin \varphi = -\frac{p}{m_e c a}.$$
In the range of the EM field amplitude, $0 < a \ll \varepsilon^{-1}_{rad}$ solution to these equations has the same asymptotic dependences as given by Eqs. (20) and (22). However, when the EM field amplitude approaches the value of $\varepsilon^{-1}_{rad}$, the solution qualitatively changes. According to Eq. (29), the electron momentum decreases as also shown in Fig. 2. In Fig. 2a we present the components of the electron momentum parallel and perpendicular to the instantaneous electric field as functions of the electric field amplitude. Fig. 2b shows the dependences of the angle, $\varphi$, and the electron gamma-factor, $\gamma$, on the electric field. The momentum and gamma-factor are normalized by $(a_m/\varepsilon_{rad})^{1/4}$, and the dimensionless electric field amplitude by $\varepsilon_{rad}$.

**Figure 2.** Solution of the electron motion equation with the radiation friction force in the LL form in the case of rotating homogeneous electric field: a) Dependence of the components of the electron momentum (normalized by $m_e c (a_m/\varepsilon_{rad})^{1/4}$) perpendicular, $p_\perp$, and parallel, $p_|$, to the electric field on the normalized EM field amplitude, $a \varepsilon_{rad}$, and b) dependence of $\varphi$ normalized by $(a_m/\varepsilon_{rad})^{1/4}$ and the electron gamma-factor, $\gamma$, on $a \varepsilon_{rad}$ for $a_m = 1500$ and $\varepsilon_{rad} = 7.5 \times 10^{-4}$.

In Fig. 3 we present the results of the solution of the electron motion equation in a rotating homogeneous electric field. Here the dependences of the electron gamma-factors on the electric field, $\gamma_LAD$ and $\gamma_LL$, correspond to the radiation friction force taken in the LAD and L-L form, respectively. The normalization is the same as in Fig. 2.

**Figure 3.** Solution of the electron motion equation in a rotating homogeneous electric field for $a_m = 1500$ and $\varepsilon_{rad} = 7.5 \times 10^{-4}$. Dependences of the electron gamma-factors on the electric field: $\gamma_LAD$ and $\gamma_LL$ correspond to the radiation friction force taken in the LAD and L-L form, respectively. The normalization is the same as in Fig. 2.
4. Electron in the superposition of rotating and radial electric fields

4.1. Lorentz-Abraham-Dirac form of the radiation friction force

Electron dynamics in the superposition of rotating and radial electric field corresponds to the case of the electron direct acceleration by the laser pulse propagating inside the self-focusing channel. In the frame of reference moving with the laser pulse group velocity equations of electron motion are Eqs. (10 - 12) with \( b = j = a_1 = 0 \):

\[
\tilde{u}_2 = d\tilde{u}_2 + \varepsilon_{rad}\tilde{u}_3\gamma^2, \tag{30}
\]

\[
\tilde{u}_3 = d\tilde{u}_3 + a - \varepsilon_{rad}\tilde{u}_2\gamma^3. \tag{31}
\]

For variables \( p \) and \( \varphi \) defined by Eq. (26) we can rewrite Eqs. (30, 31) as

\[
m_eca = p \left[ (1 - d/\gamma)^2 + \varepsilon_{rad}^2\gamma^6 \right]^{1/2} \quad \text{and} \quad \tan \varphi = -\frac{\gamma - d}{\varepsilon_{rad}\gamma^4}, \tag{32}
\]

with \( \gamma = (1 + p^2)^{1/2} \).

4.2. Landau-Lifshitz form of the radiation friction force

The equations of the electron motion in the superposition of rotating and radial electric fields \( (b = j = a_1 = 0) \) with the radiation friction force in the L-L form are

\[
(\gamma - d)\tilde{u}_2 = \varepsilon_{rad}\tilde{u}_3 \left[ (d\gamma + a)\tilde{u}_3 + a^2 - d^2 \right], \tag{33}
\]

\[
(\gamma - d)\tilde{u}_3 = a\gamma - \varepsilon_{rad}\tilde{u}_2 \left[ (d\gamma + a)\tilde{u}_3 + a^2 - d^2 \right]. \tag{34}
\]

We can rewrite Eqs. (33, 34) as

\[
a^2 = d^2 - \frac{(d - \gamma + \gamma^3)^2}{\gamma^2} + \frac{\gamma^2 - \sqrt{\gamma^4 \left[ 1 - 4\varepsilon_{rad}^2 (\gamma^2 - 1)^2 \right]}}{2\varepsilon_{rad}^2 (1 - \gamma^2)} \tag{35}
\]

\[
\tan \phi = \frac{2\varepsilon_{rad}(\gamma - d)(\gamma^2 - 1)}{\gamma^2 - \sqrt{\gamma^4 \left[ 1 - 4\varepsilon_{rad}^2 (\gamma^2 - 1)^2 \right]}}, \tag{36}
\]

As in the above considered case of rotating electric field, the dependence of the electron momentum on the electric field amplitude shows similar behavior in the cases of LAD and L-L forms of the radiation friction force, provided \( a \gg 1, \gamma \gg 1 \), and \( 2\varepsilon_{rad}\gamma^2 \ll 1 \).

5. Discussions and conclusions

As follows from consideration of the above presented exact solutions to the electron motion equations with the radiation friction force taken either in the LAD or in the L-L form, in the limit of relatively low electric field amplitude they show the same behavior, as seen in Fig. 3. When the electric field is strong, i.e. the normalized field amplitude \( a \) approaches the value of \( \varepsilon_{rad}^{-1} \), the solutions are drastically different.

The condition \( a = \varepsilon_{rad}^{-1} \) corresponds to the electric field equal to the critical electric field of classical electrodynamics, \( E_{cr} = m_e^2c^4/e^3 \). This electric field can produce a work equal to \( m_ec^2 \) over the distance of the classical electron radius \( r_e \).

\[ \]

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The radiation friction force in the Landau-Lifshitz form assumes the smallness of the EM field amplitude compared to the critical field of classical electrodynamics. Another parameter which should be small is the ratio of the EM field inhomogeneity scale-length to the classical electron radius, $r_e$. The time dependent EM fields should be slowly evolved on a timescale compared to $t_e = r_e/c$, as discussed in Ref. [25] devoted to the problem of classical electrodynamics applicability.

Obviously, the limit of the EM field amplitude of the order of $E_{cr}$ (and of the space- and time scales of the order of $r_e$ and $t_e = r_e/c$) is of pure academic interest, because the quantum mechanical effects become important at electric field amplitudes substantially lower (and at the spatial scale of the order of the electron Compton wavelength, $\hbar/m_ec$). Here $\alpha$ is the fine structure constant. For the electron motion in colliding EM waves, as shown in Ref. [12] the QED effects, due to the recoil from the photon emission, should be incorporated into the description of the electron interaction with the EM field for an even smaller EM wave amplitude. This field is of the order of $\alpha E_S$, which is equal to $\alpha^2 E_{cr}$, below which both the LAD and L-L forms for the radiation friction force give the same result.

Let us derive the extreme field limits, which mark the subsequent onset of the classical radiation reaction regime, the regime when quantum recoil becomes important, and the $e^+e^-$ pair production from vacuum, for two principle experimental schemes aimed at the study of particle physics effects at high laser intensity: (i) colliding laser pulses (all optical setup) and (ii) laser - e-beam interaction (See Figure 4).

In what follows we consider the behavior of an electron in the focus of two colliding circularly polarized laser pulses, i.e., in the antinode of a standing light wave with null magnetic field, and in the collision with the counterpropagating laser pulse and determine the thresholds for the classical radiation dominated regime of interaction and for the onset of the quantum regime. We also mention the threshold for the Schwinger process in the case of two colliding laser pulses.

The power emitted by the electron in the circularly polarized electric field in the ultra-relativistic limit is proportional to the forth power of its energy $P_{C,\gamma} = \varepsilon_{rad} \omega mc^2 r_e^2 (\gamma - 1) \sim \gamma^4 e^3$. In the non-radiative approximation, the electron can acquire the energy from the EM field with the rate $\approx \omega mc^2 a$. The condition of the balance between the acquired and emitted energy is $\varepsilon_{rad} \omega \sim 1$. Thus the radiation reaction (RR) effects become dominant at $a > a_{rad} = \varepsilon_{rad}^{-1/3}$. For a head-on collision of an energetic e-beam and an intense laser pulse with $\varepsilon_{rad} \gamma^2 \gg 1$ the longitudinal momentum is given by $p_x = -p_0 \left[ 1 + \varepsilon_{rad} \omega (p_0/m) \int_0^\gamma a^2 (-2\eta) d\eta \right]^{-1}$. In this case the interaction is purely dissipative and all the EM forces can be neglected except the radiation reaction. This means that the classical radiation effects become dominant at
a > a_{rad} = (\varepsilon_{rad}\omega_{\text{laser}}\gamma_{e,0})^{-1/2}, \text{ where } \tau_{\text{laser}} \text{ is the duration of the laser pulse and } \gamma_{e,0} \text{ is the initial energy of the e-beam.}

The quantum effects become important when the energy of a photon emitted by an electron is of order of the electron energy, i.e., \( \hbar \omega_m = \gamma_e m c^2 \). In the case of the electron circulating in the focus of two colliding laser pulses the electron emits photons with the energy \( \hbar \omega_m = \hbar \omega^3_e \). Thus the quantum recoil comes into play when \( a > a_Q = (2\alpha/3)^{-2}\varepsilon_{rad}^{-1} \). Here we took into account the fact that in the limit \( I > 10^{23} \text{ W/cm}^2 \) due to strong radiation damping effects the electron energy scales as \( mc^2(a/\varepsilon_{rad})^{1/4} \). For an electron colliding with a laser pulse a characteristic photon energy is \( \hbar \omega_m \approx \hbar \omega_{\alpha a}^3 \), which corresponds to the condition \( \chi_e = e\hbar \sqrt{(F_{\mu\nu}p_{\mu})^2/m^3c^4} \approx 1 \).

The parameter \( \chi_e \) has the meaning of the EM field strength in the rest frame of the particle. It is responsible for the magnitude of the quantum nonlinear effects. Thus the quantum recoil comes into play when \( a > a_Q = (2\alpha/3)^{-2}\varepsilon_{rad}^{-1} \). This condition is analogous to two conditions: \( \chi_e > 1 \) and \( \alpha a > 1 \) derived from the analysis of the \( e \rightarrow e\gamma \) process probability \([31]\). The probability of pair creation acquires its optimum value over the characteristic scale of the process when the electric field strength is of the order of the "critical" for quantum electrodynamics (QED) value \( a > a_S = (2\alpha/3)^{-2}\varepsilon_{rad}^{-1} \).

### Table 1. The extreme field limits for two colliding 0.8 \( \mu \) laser pulses and a 10 GeV electron beam colliding with a 0.8 \( \mu \) laser pulse.

|                    | Classical RR                              | Quantum regime                              | Schwinger limit                      |
|--------------------|--------------------------------------------|---------------------------------------------|-------------------------------------|
| two colliding      | \( a_{rad} = \varepsilon_{rad}^{1/3} \)    | \( a_Q = \left(\frac{2}{3}\alpha\right)^2\varepsilon_{rad}^{-1} \) | \( a_S = \left(\frac{2}{3}\alpha\right)^2\varepsilon_{rad}^{-1} \) |
| pulses             | \( a_{rad} = 400 \)                       | \( a_Q = 1.6 \times 10^{3} \)               | \( a_S = 3 \times 10^{5} \)         |
| \( I_{rad} \)      | \( 3.5 \times 10^{23} \text{ W/cm}^2 \)   | \( I_Q = 5.5 \times 10^{24} \text{ W/cm}^2 \) | \( I_S = 2.3 \times 10^{29} \text{ W/cm}^2 \) |
| e-beam and laser   | \( a_{rad} = (\varepsilon_{rad}\omega_{\text{laser}}\gamma_{e,0})^{-1/2} \) | \( a_Q = \left(\frac{2}{3}\alpha\right)\gamma_e^{-1}\varepsilon_{rad}^{-1} \) | \( a_Q = 20 \) |
| pulse             | \( a_{rad} = 10 \)                        | \( a_Q = 2 \)                               | \( a_S = 8.7 \times 10^{20} \text{ W/cm}^2 \) |
| \( I_{rad} \)      | \( 2.2 \times 10^{20} \text{ W/cm}^2 \)   |                                             |                                    |

Although conclusions following from the above presented consideration do not have the character of a rigorously proved mathematical theorem, they give an indication that for

\[
a < a_Q = \left(\frac{2}{3}\alpha\right)^2\varepsilon_{rad}^{-1}
\]  

(37)

the solutions of the electron equations of motion with the radiation friction force in the L-L form coincide with those obtained from solution of the equations of motion with the radiation friction force in the LAD form.

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