Moduli of $G$-constellations and crepant resolutions II: the Craw-Ishii conjecture

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Abstract

For any given finite subgroup $G \subset SL_3(\mathbb{C})$, we show that every projective crepant resolution $X$ of the quotient variety $\mathbb{C}^3/G$ is isomorphic to the moduli space of $\theta$-stable $G$-constellations for a generic stability condition $\theta$, as conjectured by Craw and Ishii. We also show that generators of the Cox ring of $X$ can be obtained from semi-invariants for representations of the McKay quiver of $G$.

1 Introduction

Let $G$ be a finite subgroup of $SL_n(\mathbb{C})$. The philosophy of the McKay correspondence is that the geometric properties of a crepant resolution of $\mathbb{C}^n/G$ should reflect the representation-theoretic properties of $G$ (cf. [Re2]). A natural way to see the explicit correspondence is to realize a crepant resolution of $\mathbb{C}^n/G$ as the $G$-Hilbert scheme (see e.g. [G], [CCL]) or, more generally, a moduli space of $G$-constellations. In dimension two, it is well known that the unique crepant resolution is obtained as the $G$-Hilbert scheme. In dimension three, it was also proved by Craw and Ishii [CI] that every projective crepant resolution is obtained as the moduli space $M_\theta$ of $G$-constellations for a suitable stability condition $\theta$ when $G$ is abelian. In general $M_\theta$ for any finite subgroup $G \subset SL_3(\mathbb{C})$ is known to be a crepant resolution for any generic stability condition $\theta$ by the result of Bridgeland, King and Reid [BKR], and it is conjectured that the result by Craw and Ishii still holds for non-abelian $G$ [INdC, Conjecture 1.4]. This conjecture has been known to hold in special cases including resolutions whose fibers have dimension at most one [NdCS], [W], and iterated $G$-Hilbert schemes [INdC]. The main result of this article is that this conjecture is affirmative in general:

Theorem 1.1. (=Theorem 4.1) Let $G \subset SL_3(\mathbb{C})$ be a finite subgroup. Then every projective crepant resolution of $\mathbb{C}^3/G$ is isomorphic to the moduli space of $\theta$-stable $G$-constellations for a generic stability condition $\theta$.

In this paper we also give an explicit construction of a torus-equivariant morphism from the spectrum of the Cox ring of a crepant resolution of $\mathbb{C}^3/G$ to (a certain quotient of) a space of $G$-constellations, as a generalization of the construction in Part I [Y2] of the present paper. In [Y2], we gave a necessary and sufficient condition for a given (not-necessarily-projective) crepant resolution $X \to \mathbb{C}^n/G$ to admit a moduli description.

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in terms of a natural $G$-constellation family when $G$ is abelian. The key idea to prove this result was to construct an equivariant morphism between certain toric varieties so that it descends to a birational morphism from $X$ to a candidate of a fine moduli space. In this paper we generalize this construction to non-abelian $G$ so that we obtain an embedding $\mathcal{W} \hookrightarrow Y_\mathcal{W}$ of (a quotient of) a space of $G$-constellations into a toric variety such that it induces an embedding of $\mathcal{M}_\theta$ into a toric variety for any generic $\theta$ (Theorem 3.5). This embedding of $\mathcal{M}_\theta$ is so nice that it enables us to understand the behaviour of $\mathcal{M}_\theta$ under variation of $\theta$ through the change of the ambient toric variety. We will demonstrate this embedding and investigate the geometric structure of the moduli space explicitly via concrete examples in Section 3.5. The embedding will also play a key role to prove Theorem 1.1 so that we can overcome one of difficulties which arises when we extend the main result of [CI] to non-abelian cases.

The embedding $\mathcal{W} \hookrightarrow Y_\mathcal{W}$ will be constructed by using the Cox ring $\text{Cox}(X)$ of $X = \mathcal{M}_\theta$, whose generators play a similar role to homogeneous coordinates of a projective space. On the other hand $\mathcal{M}_\theta$ is regarded as a moduli space of certain representations of the McKay quiver of $G$, and semi- (or relative) invariants for these representations also play a role of homogeneous coordinates of $\mathcal{M}_\theta$. Then the equivariant morphism

$$X := \text{Spec} \text{Cox}(X) \to \mathcal{W}$$

is obtained by assigning to each semi-invariant a global section of a line bundle on $X$. Moreover, as an application of Theorem 1.1 we will show that generators of $\text{Cox}(X)$ can be computed (at least theoretically) once we know a generating system of semi-invariants for the McKay quiver. More precisely, there is a ring homomorphism $\varphi$ from the ring $\mathbb{C}[\mathcal{W}]$ of semi-invariants to the (semi-)invariant ring $\mathbb{C}[\mathbb{C}^3]^G$ such that the “associated elements” to the images of homogeneous generators of the integral closure of $\mathbb{C}[\mathcal{W}]$ under $\varphi$ give a generating system of $\text{Cox}(X)$ (see Proposition 4.14 for the precise statement).

The strategy for the proof of Theorem 1.1 is based on the one given by Craw and Ishii for abelian cases. To explain this, let us recall that each GIT-chamber $C \subset \Theta$ in the space of stability conditions gives a fine moduli space $X := \mathfrak{M}_C$ and its universal family $\mathcal{U}_C$. By the result of [BKR], the Fourier-Mukai transform by $\mathcal{U}_C$ gives an equivalence between the bounded derived category of coherent sheaves on $X$ and that of $G$-equivariant sheaves on $\mathbb{C}^3$. This induces an isomorphism

$$\phi^*_C : \Theta \to F^1$$

from the space of stability conditions to the subspace $F^1$ inside the Grothendieck group $K(X)_\mathbb{R}$ generated by sheaves whose supports have codimension at least one. The proof of the theorem will be done by studying the behaviour of $\phi^*_C(C)$ under crossing a wall of Type $\emptyset$ inside $\Theta$. Here a wall of Type $\emptyset$ means a codimension-one face $W$ of $\overline{C}$ whose associated morphism $X \to \mathfrak{M}_W$ induced by variation of GIT is an isomorphism. The significant fact to prove Theorem 1.1 is that, similarly to abelian cases, crossing such a wall gives a wall-crossing in $F^1$ as well, up to tensoring by a line bundle (Proposition 4.13). The corresponding result [CI, Proposition 7.3] for abelian $G$ was proven by using a certain rigidity result for which the abelian assumption was essential. In the present paper we bypass this issue by showing that the structure of the unstable locus $D_W$
associated to the wall $W$ is simple enough (Lemma 4.10) to prove Proposition 4.13 even if the rigidity does not hold.

Once Proposition 4.13 is established, we can find a wall $\tilde{B}$ in $\Theta$ which realizes the small contraction $X \rightarrow X_B$ for any given flop $X \rightarrow X_B \leftarrow X'$ after crossing walls of Type 0. Using technical Lemma 4.3 which is deduced from the embedding $\mathcal{W} \hookrightarrow \mathcal{Y}_W$, we can confirm that crossing the wall $\tilde{B}$ gives the flop $X \rightarrow X'$. Since any two projective crepant resolutions are connected by a sequence of flops (cf. [K]), this proves the theorem. In [CI] the fact that $\tilde{B}$ induces a flop was also proven, but again by relying essentially on the abelian assumption. Thus, we can avoid this issue by using the embedding $\mathcal{W} \hookrightarrow \mathcal{Y}_W$. We remark that, in the construction of the embedding, we essentially use the finite generation of $\text{Cox}(X)$, which is guaranteed by [BCHM] (see Subsection 3.1).

This paper is organized as follows. In Section 2 we review the construction of a moduli space of $G$-constellations and semi-invariants of quiver representations. In Section 3, we introduce the Cox ring of a crepant resolution and relate it with the semi-invariants for representations of the McKay quiver of $G$ so that we obtain a nice embedding of a moduli space into a toric variety (Theorem 3.5). We also give concrete examples in Subsection 3.5. In Section 4, we give a proof of the main result (Theorem 4.1). In the appendix, symbols used in §2-§4 are listed.

Conventions & Notations
In this paper all schemes are algebraic ones defined over $\mathbb{C}$. By a point of a scheme, we always mean a closed point. For an abelian group $A$, we denote by $A_\mathbb{R}$ its scalar extension $A \otimes_{\mathbb{Z}} \mathbb{R}$. Similarly, for an homomorphism $f$ of abelian groups, $f_\mathbb{R}$ denotes the scalar extension. For a finite dimensional $\mathbb{R}$-vector space $V$ and a subset $S \subset V$, we denote by $\text{Cone}(S)$ the cone generated by $S$. See the appendix for the list of symbols used in §2-§4.

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2 Moduli spaces of $G$-constellations

2.1 GIT-construction
In this section we review the construction of a moduli space of $G$-constellations for an arbitrary finite subgroup $G \subset SL(V)$ with $V = \mathbb{C}^n$ (cf. [CI] Section 2). See e.g. [CMT] and [Y2] for the special case where $G$ is abelian.

We first recall that a $G$-constellation $F$ is a $G$-equivariant $\mathbb{C}[V]$-module which is isomorphic to the regular representation $R = \bigoplus_{g \in G} \mathbb{C}g$ as a $\mathbb{C}G$-module where the $G$-action on $\mathbb{C}[V]$ comes from the inclusion $G \subset SL(V)$. Although $G$-constellations are
defined as $G$-equivariant coherent sheaves on $V$ in $[\mathcal{C}]$, we identify them with their global sections since $V$ is affine.

Let $\text{Irr}(G)$ be the set of isomorphism classes of irreducible representations of $G$. As is well known, $R$ is isomorphic to $\bigoplus_{\rho \in \text{Irr}(G)} \rho^\oplus \dim \rho$ as a $\mathbb{C}G$-module. We will construct a moduli space of $G$-constellations by taking a quotient of the affine scheme

$$\mathcal{N} = \{B \in \text{Hom}_{\mathbb{C}G}(V^* \otimes_{\mathbb{C}} R, R) \mid B \wedge B = 0\}$$

(2.1)

where $V^*$ is the dual representation of $V$ and

$$B \wedge B \in \text{Hom}_{\mathbb{C}[G]}(V^* \otimes_{\mathbb{C}} V^* \otimes_{\mathbb{C}} R, R)$$

(2.2)

is defined by

$$(B \wedge B)(x \otimes y \otimes a) = B(x \otimes B(y \otimes a)) - B(y \otimes B(x \otimes a)).$$

We may regard $\mathcal{N}$ as a space parametrizing all $G$-constellations, and two elements of $\mathcal{N}$ are isomorphic as $G$-equivariant $\mathbb{C}[V]$-modules if and only if they lie in the same $GL_R$-orbit where $GL_R := \text{Aut}_{\mathbb{C}G}(R)$ is the $G$-equivariant automorphism group of $R$, which acts on $\mathcal{N}$ by conjugation. By Schur’s lemma, $GL_R$ is written as a direct product $\prod_{\rho \in \text{Irr}(G)} GL_{\dim \rho}(\mathbb{C})$. Note that the diagonal subgroup $\mathbb{C}^*$ acts trivially on $\mathcal{N}$ and thus the action of $GL_R$ descends to that of $PGL_R := GL_R/\mathbb{C}^*$.

Since the orbit space $\mathcal{N}/PGL_R$ does not admit a reasonable structure as a scheme, we apply the construction of geometric invariant theory (GIT). Concretely, we consider a $PGL_R$-linearization of the trivial line bundle of $\mathcal{N}$ and take the GIT-quotient with respect to it. Such a linearization is given by a character of $PGL_R$, and the group $\chi(PGL_R)$ of characters can be identified with the space of stability conditions

$$\Theta := \{\theta \in \text{Hom}_{\mathbb{Z}}(R(G), \mathbb{Z}) \mid \theta(R) = 0\}$$

where $R(G) = \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}\rho$ is the representation ring of $G$. Note that $\Theta$ has rank $s - 1$ with $s$ being the number of conjugacy classes in $G$. The identification is given by

$$\chi(PGL_R) \mapsto \Theta$$

$$\chi = (\bar{g}_\rho \mapsto \text{det}(g_\rho)^{\theta_\rho}, \forall \rho \in \text{Irr}(G)) \mapsto \theta_\chi = (\rho \mapsto \theta_\rho, \forall \rho \in \text{Irr}(G)).$$

For a character $\chi \in \chi(PGL_R)$ we consider the $\chi$-semistable locus

$$\mathcal{N}^{\chi-ss} = \{p \in \mathcal{N} \mid f(p) \neq 0 \text{ for some } f \in A_{\chi} \text{ and } k \in \mathbb{N}\}$$

where $A_{\chi}$ is the $\mathbb{C}$-vector space consisting of regular functions $f$ on $\mathcal{N}$ such that $g \cdot f = \chi(g)f$ for any $g \in PGL_R$. As an open subset of $\mathcal{N}^{\chi-ss}$, the $\chi$-stable locus $\mathcal{N}^{\chi-s}$ is defined as the set of points $p \in \mathcal{N}^{\chi-ss}$ such that the orbit $PGL_R \cdot p$ is closed in $\mathcal{N}^{\chi-ss}$ and that the stabilizer subgroup of $p$ in $PGL_R$ is finite. We say that a character $\chi$ is generic if $\mathcal{N}^{\chi-ss} = \mathcal{N}^{\chi-s}$.

For any character $\chi$, we have a $PGL_R$-invariant morphism

$$\pi_\chi : \mathcal{N}^{\chi-ss} \to \mathcal{N}/\chi PGL_R := \text{Proj} \left( \bigoplus_{k=0}^{\infty} A_{\chi}^k \right)$$
which gives a one-to-one correspondence between the set of closed $PGL_R$-orbits in $\mathcal{N}^{x\text{-ss}}$ and the set of closed points in $\mathcal{N}/\!\!/\chi PGL_R$ (cf. [MF]). We call $\mathcal{N}/\!\!/\chi PGL_R$ the GIT-quotient of $\mathcal{N}$ by $PGL_R$ with respect to $\chi$. If $\chi$ is generic, then $\mathcal{N}/\!\!/\chi PGL_R$ is the orbit space of $\mathcal{N}^{x\text{-ss}}$, and in such a case $\mathcal{N}/\!\!/\chi PGL_R$ is called the geometric quotient. The stability of a point of $\mathcal{N}$ can be rephrased in representation-theoretic terms by the result of King:

**Proposition 2.1.** [Kin, Proposition 3.1] For a point $p \in \mathcal{N}$ and a character $\chi \in \chi(PGL_R)$, $p$ is in $\mathcal{N}^{x\text{-ss}}$ (resp. $\mathcal{N}^{x\text{-s}}$) if and only if the corresponding $G$-constellation $F_p$ is $\theta_{\chi}$-semistable (resp. $\theta_{\chi}$-stable), that is, $\theta_{\chi}(M) \geq 0$ (resp. $> 0$) for any nonzero $G$-equivariant $\mathbb{C}[V]$-submodule $M \subseteq F_p$.

We will use the scalar extension $\Theta := \bar{\Theta} \otimes_{\mathbb{Z}} \mathbb{R}$ as the space of stability conditions rather than $\Theta$ itself in order to consider polyhedral cones inside $\Theta$ (or $\chi(PGL_R)_{\mathbb{R}}$). For $\theta \in \Theta$ and the corresponding character $\chi_{\theta}$, we denote $\mathcal{N}/\!\!/_{\chi_{\theta}} PGL_R$ also by $\mathcal{M}_\theta$. From King’s result above, we see that $\Theta$ is divided into finitely many open chambers, on which the semistable locus is constant. We call such a chamber a GIT-chamber of $\Theta$. Since $G$-constellations are regular representations as $\mathbb{C}G$-modules, supporting hyperplanes of the GIT-chambers are of the form $\left\{ \sum_{\rho \in \text{Irr}(G)} a_\rho \theta_\rho = 0 \right\}$ with $a_\rho \in \{0, 1, \ldots, \dim \rho\}$.

Typical examples of $G$-constellations are given by the coordinate rings of free $G$-orbits, that is, orbits $G \cdot x \subset \mathbb{C}^n$ such that $\sharp(G \cdot x) = \sharp G$. Such $G$-constellations are simple $\mathbb{C}[V]$-modules and hence stable for any $\theta \in \Theta$. As a natural generalization of free $G$-orbits, $G$-constellations obtained as quotients of $\mathbb{C}[V]$ by its ideal are called $G$-clusters. Since $G$-clusters are characterized as the $G$-constellations generated by the trivial representation, the moduli space of $G$-clusters, called the $G$-Hilbert scheme, is realized as $\mathcal{M}_{\theta_+}$ for $\theta_+ \in \Theta$ satisfying $\theta_+(\rho) > 0$ for any nontrivial $\rho \in \text{Irr}(G)$.

By [CI, Proposition 2.2], there exists an irreducible component of $\mathcal{M}_0$ which is isomorphic to $\mathbb{C}^n/G$ such that its general points parametrize the coordinate rings of free $G$-orbits in $\mathbb{C}^n$. Accordingly, there is an irreducible component $\mathcal{V} \subset \mathcal{N}$ (with the reduced structure) such that the restriction of the quotient map $\mathcal{N} \to \mathcal{M}_0$ to $\mathcal{V}$ factors into

- a closed morphism $\mathcal{V} \to \mathbb{C}^n/G$ which sends a $G$-constellation to its support, and
- a closed immersion $\mathbb{C}^n/G \to \mathcal{M}_0$ onto an irreducible component of $\mathcal{M}_0$.

In Subsection 3.3 we will give a more explicit description of $\mathcal{V}$ (cf. Proposition [3.2]). Note that the morphism $\mathcal{M}_\theta := \mathcal{V}/\!\!/_{\chi_{\theta}} PGL_R \to \mathbb{C}^n/G$ induced by the inclusion $\mathcal{V} \cap \mathcal{N}^{x\text{-ss}} \hookrightarrow \mathcal{V}$ is a birational morphism for any $\theta \in \Theta$. The irreducible component $\mathcal{M}_\theta$ of $\mathcal{M}_0$ is sometimes called the coherent component. For a GIT-chamber $C \subset \Theta$, the moduli space $\mathcal{M}_\theta$ is constant for $\theta \in C$ and thus we also denote it by $\mathcal{M}_C$.

The main result of [BKR] shows that the $G$-Hilbert scheme for $G \subset SL_3(\mathbb{C})$ is a crepant resolution of $\mathbb{C}^3/G$. More generally, one can repeat the proof of this result to show that the moduli space $\mathcal{M}_\theta$ of $\theta$-semistable $G$-constellations for any generic $\theta \in \Theta$ is a crepant resolution of $\mathbb{C}^3/G$. In particular $\mathcal{M}_\theta$ is irreducible and hence $\mathcal{M}_\theta = \mathcal{M}_\theta$ in this case.
As explained in [CI, §2.1], the moduli space $\mathcal{M}_C$ for a GIT-chamber $C$ admits a universal family $\mathcal{U}_C$ on $\mathcal{M}_C \times \mathbb{C}^n$, and the associated tautological bundle $(p_{\mathcal{M}_C})_* \mathcal{U}_C$ on $\mathcal{M}_C$ admits a decomposition 

$$(p_{\mathcal{M}_C})_* \mathcal{U}_C = \bigoplus_{\rho \in \text{Irr}(G)} R_\rho \otimes \rho$$

into isotypic components where $p_{\mathcal{M}_C} : \mathcal{M}_C \times \mathbb{C}^n \to \mathcal{M}_C$ is the projection. Note that each $R_\rho$ is a vector bundle on $\mathcal{M}_C$ of rank $\text{dim} \rho$. Since tensoring by a line bundle to $(p_{\mathcal{M}_C})_* \mathcal{U}_C$ has no effect on the parametrized $G$-constellations, we assume that $R_{\rho_0} \cong \mathcal{O}_{\mathcal{M}_C}$ for the trivial representation $\rho_0$ whenever we talk about the universal family or the tautological bundle determined by $C$. Later we will identify $G$-constellations with representations of a quiver in Subsection 2.3. We will also use the associated tautological bundle $R_C$ on $\mathcal{M}_C$ as a moduli space of quiver representations rather than use the bundle $(p_{\mathcal{M}_C})_* \mathcal{U}_C$.

For later use, we introduce the notion of an orbit cone for an affine scheme acted on by an algebraic group, following [BH, Definition 2.1]:

**Definition 2.2.** Let $H$ be an algebraic group acting on an affine algebraic scheme $Z$. For a point $z \in Z$, the orbit cone of $z$ is the polyhedral cone $C_z \subset \chi(H)_\mathbb{R}$ generated by characters $\chi$ such that $z$ is $\chi$-semistable.

**Remark 2.1.** The orbit cone of $z \in Z$ depends only on the $H$-orbit of $z$. If $Z$ is the affine scheme $\mathcal{N}$ acted on by $H = PGL_R$ as above, the orbit cone of a point of $\mathcal{N}$ corresponding to a $G$-constellation $F$ is identified with a cone in $\Theta$ consisting of stability conditions $\theta$ such that $F$ is $\theta$-semistable. Moreover, for a GIT-chamber $C \subset \Theta$, its closure $\overline{C}$ is equal to the intersection of all the orbit cones for points of $\mathcal{N}^{G-ss}$.

### 2.2 Semi-invariants for quiver representations

Let us consider the subgroup $SL_R := \prod_{\rho \in \text{Irr}(G)} SL_{\text{dim} \rho}(\mathbb{C})$ of $GL_R$. Since $SL_R$ is reductive, there are finitely many generators of the invariant ring of the coordinate ring $\mathbb{C}[\mathcal{N}]$ of the affine scheme $\mathcal{N}$. We call elements of $\mathbb{C}[\mathcal{N}]^{SL_R}$ semi-invariants (with respect to the action of $GL_R$ on $\mathcal{N}$). Note that we have a decomposition of this ring into homogeneous components:

$$\mathbb{C}[\mathcal{N}]^{SL_R} = \bigoplus_{\chi \in \chi(PGL_R)} A_\chi.$$

As we will see in the next subsection, $\mathcal{N}$ is identified with a space of quiver representations (with relations). In general the ring of semi-invariants for representations of a quiver (in characteristic zero) is generated by determinantal semi-invariants (see Theorem 2.3 below). In this subsection we explain how to obtain such semi-invariants.

Firstly, we briefly recall the concept of representations of a quiver. See e.g. [DeW2], [Kir] for more details. Let $Q$ be a quiver, that is, an oriented graph consisting of a finite set $I$ of vertices and a finite set $A$ of arrows between two vertices. Here we allow $Q$ to have loops and oriented cycles. For any map $\alpha : I \to \mathbb{Z}_{\geq 0}$, we define a space $\text{Rep}(\alpha)$ of representations of $Q$ with a dimension vector $\alpha$ as

$$\text{Rep}(\alpha) = \bigoplus_{\alpha \in A} \text{Hom}_\mathbb{C} \left( \mathbb{C}^{\alpha(t_a)}, \mathbb{C}^{\alpha(h_a)} \right)$$

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where \( t_a \) and \( h_a \) denote the tail and the head of \( a \) respectively. For a representation \( H = (H_A)_{a \in A} \in \text{Rep}(\alpha) \) and a path

\[
p = a_1 \cdots a_k \quad (a_i \in A, h_{a_i} = t_{a_{i+1}}),
\]

we set \( H_p = H_{a_k} \circ \cdots \circ H_{a_1} \). For the trivial path \( e_v \) at a vertex \( v \), we set \( H_{e_v} = \text{id}_{C^\alpha(v)} \).

\( \text{Rep}(\alpha) \) is acted on by \( GL_\alpha := \prod_{v \in I} GL_{\alpha(v)} \) by conjugation.

Let \( v_1, \ldots, v_r, w_1, \ldots, w_s \) be vertices of \( Q \) with possible repetition such that

\[
\sum_{i=1}^r \alpha(v_i) = \sum_{j=1}^s \alpha(w_j),
\]

and let \( p_{i,j} \) be a \( C \)-linear combination of paths from \( v_i \) to \( w_j \). Then the function \( f : \text{Rep}(\alpha) \to C \) defined by

\[
H \mapsto \det \begin{pmatrix}
H_{p_{1,1}} & \cdots & H_{p_{1,s}} \\
\vdots & \ddots & \vdots \\
H_{p_{r,1}} & \cdots & H_{p_{r,s}}
\end{pmatrix}
\]

(2.3)
gives a semi-invariant satisfying

\[
(g_v)_{v \in I} \cdot f = \left( \prod_{v \in I} \det g_v^{c_v} \right) f
\]

for \( (g_v)_{v \in I} \in GL_\alpha \) where

\[
c_v = \# \{ j \mid w_j = v \} - \# \{ i \mid v_i = v \}.
\]

(2.4)

A semi-invariant obtained in this way is called a determinantal semi-invariant of weight \( (c_v)_{v \in I} \). Note that weights are identified with characters of \( GL_\alpha \).

**Theorem 2.3.** ([DeW1], [DZ], [SV]) The ring \( C[\text{Rep}(\alpha)]^{SL_\alpha} \) of semi-invariants on \( \text{Rep}(\alpha) \) with respect to the \( GL_\alpha \)-action is generated by determinantal semi-invariants.

This was first proved by Derksen and Weyman [DeW1] under the assumption that \( Q \) has no oriented cycles, and generalized by Domokos and Zubkov [DZ] and by Schofield and van den Bergh [SV]. We will apply this theorem to representations of the McKay quiver introduced in the next subsection.

### 2.3 \( G \)-constellations as representations of the McKay quiver

In this subsection we realize the affine scheme \( \mathcal{N} \) in (2.1) as a space of quiver representations.

The **McKay quiver** \( Q_G \) of \( G \) is the quiver having \( \text{Irr}(G) \) as the set of vertices and having \( a_{\rho, \rho'} := \dim_C(\text{Hom}_C(V^* \otimes_C \rho, \rho')) \) arrows from \( \rho \) to \( \rho' \) for any \( \rho, \rho' \in \text{Irr}(G) \). Decomposing \( R \) into irreducible summands, we see that \( \text{Hom}_C(V^* \otimes_C R, R) \) is isomorphic to the representation space \( \text{Rep}(\alpha_G) \) of \( Q_G \) for the dimension vector \( \alpha_G : \rho \mapsto \dim \rho \).
and two 2-dimensional representations}

We moreover prepare their copies

The moduli space of representations of \( Q_2 \) admits a decomposition \( R_2 = \bigoplus_{\rho \in \text{Irr}(G)} R_\rho \) into the isotypic components. Note that \( R_2 \) is equipped with a \( \mathbb{C}[V] \)-action whose restriction to \( \mathbb{C}[V]^G \) coincides with the \( H^0(O_{\mathcal{M}_2}) \)-action as a vector bundle on \( \mathcal{M}_2 \). Hereafter, by \textit{the tautological bundle on} \( \mathcal{M}_2 \), we mean the bundle \( R_2 \).

**Example 2.1.** \((D_5\text{-singularity})\) We describe the space \( \mathcal{N} \) explicitly for the case of the \( D_5 \)-singularity. This singularity is given as a quotient of \( \mathbb{C}^2 \) by a binary dihedral group \( G \) of order 12:

\[
G = \left\langle g_1 = \left( \zeta_6, 0 \right), g_2 = \left( 0, 1 \right) \right\rangle \subset SL_2(\mathbb{C})
\]

where \( \zeta_6 \) is the sixth root of unity. \( \text{Irr}(G) \) consists of four 1-dimensional representations \( \rho_k : G \to \mathbb{C}^* (k = 0, 1, 2, 3) \) defined by

\[
\rho_k(g_j) = \begin{cases} (-1)^k & \text{if } j = 1 \\ \zeta_k^j & \text{if } j = 2 \end{cases},
\]

and two 2-dimensional representations \( V_1, V_2 \); the first one is the inclusion \( G \subset SL_2(\mathbb{C}) \) presented above and the other is presented as

\[
g_1 \mapsto \left( \zeta_6^2, 0 \right), \quad g_2 \mapsto \left( 0, 1 \right).
\]

Let \( e_0, \ldots, e_3 \) be a basis of \( \rho_0, \ldots, \rho_3 \) respectively. Also, we let \( \{v_{11}, v_{12}\} \) (resp. \( \{v_{21}, v_{22}\} \)) be a basis of \( V_1 \) (resp. \( V_2 \)) such that it gives the matrix presentation above.

We moreover prepare their copies \( \{v'_{11}, v'_{12}\} \) and \( \{v'_{21}, v'_{22}\} \) for the second direct summands of \( V_1 \) and \( V_2 \) in \( R \) respectively. When we emphasize that \( v_{11} \) and \( v_{12} \) are elements of \( V^* \), we instead denote them by \( x \) and \( y \) respectively.

We fix isomorphisms of \( \mathbb{C}G \)-modules as follows:

\[
\begin{align*}
V^* \otimes \rho_0 & \cong V_1, & V^* \otimes \rho_1 & \cong V_2, & V^* \otimes \rho_2 & \cong V_1, & V^* \otimes \rho_3 & \cong V_2 \\
x \otimes e_0 & \mapsto v_{11} & x \otimes e_1 & \mapsto v_{22} & x \otimes e_2 & \mapsto v_{11} & x \otimes e_3 & \mapsto v_{22} \\
y \otimes e_0 & \mapsto v_{12} & y \otimes e_1 & \mapsto v_{21} & y \otimes e_2 & \mapsto v_{12} & y \otimes e_3 & \mapsto v_{21}
\end{align*}
\]

Thus the McKay quiver \( Q_2 \) is presented as...
Using the isomorphisms above, matrices associated to the arrows in $Q_G$ define an action of $\mathbb{C}[V]$ on the vector space $\mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_3 \oplus (\mathbb{C}v_{11} \oplus \mathbb{C}v_{12}) \oplus (\mathbb{C}v'_{11} \oplus \mathbb{C}v'_{12}) \oplus (\mathbb{C}v_{21} \oplus \mathbb{C}v_{22}) \oplus (\mathbb{C}v'_{21} \oplus \mathbb{C}v'_{22}) \cong R$.

For instance, a matrix $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ corresponding to the arrow $A_1$ in Figure 1 lets $V^*$ act on $\mathbb{C}e_1 = \rho_1$ as $x \cdot e_1 = a_1 v_{22} + a_2 v'_{22}$ and $y \cdot e_1 = a_1 iv_{21} + a_2 iv'_{21}$. Recall that the matrices must satisfy the condition $B \wedge B = 0$ (see (2.2)). By writing down the conditions $x \cdot (y \cdot v) - y \cdot (x \cdot v) = 0$ for each basis $v$ of $R$, one sees that this condition is expressed as the following matrix relations:

$$B_B A_B = B_B A_B = B_B A_B = B_B A_B = A_B B_B - A_B B_B - DC = A_B B_B - A_B B_B - iCD = 0. \quad (2.5)$$

**Remarks 2.1.** (1) Usually the quiver $Q_G$ itself can be obtained easily from the character table of $G$, without giving explicit $G$-equivariant homomorphisms $V^* \otimes \rho \to \rho'$ as above. However, this computation will be necessary in order to construct embeddings of moduli spaces into toric varieties, as we will see in the next section.

(2) The existence of an ideal $I_G$ of the path algebra $\mathbb{C}Q_G$ such that

$${\cal N} \cong \{ H \in \text{Rep}(\alpha_G) \mid H_p = 0, p \in I_G \}$$

as in the above example is regarded as a consequence of the Morita equivalence between the skew group algebra $\mathbb{C}[V]^* G$ and a quotient of $\mathbb{C}Q_G$. For another method to calculate $I_G$, see [BSW, Theorem 3.2 and §5].

### 3 Embedding into toric varieties

In this section we explain how we embed moduli spaces $M_\theta$ into toric varieties such that the birational geometry of the embedded varieties is inherited from that of the ambient toric varieties.

#### 3.1 The Cox ring of a resolution of a quotient singularity

In this subsection we treat the Cox ring, denoted by $\text{Cox}(X)$, of a crepant resolution $X$ of a quotient singularity $\mathbb{C}^n/G$ for $G \subset SL_n(\mathbb{C})$. $\text{Cox}(X)$ is the direct sum $\bigoplus_{L \in \text{Pic}(X)} H^0(X, L)$ as an $H^0(\mathcal{O}_{V/G})$-module and admits a multiplicative structure inherited from the tensor product of global sections of line bundles (see [ADHL] for generalities regarding Cox rings). Then $\text{Cox}(X)$ is a Pic($X$)-graded commutative ring and
is known to be finitely generated by the fundamental result of Birken-Cascini-Hacon-McKernan [BCHM]. See e.g. the proof of [BCRSW, Lemma 5.3] as to how [BCHM, Corollary 1.3.2] can be applied to our situation to deduce the finite generation of the Cox ring. The most important feature of the Cox ring is that $X$ can be recovered from \( \text{Cox}(X) \) as a GIT-quotient of Spec Cox(X) by an algebraic torus with respect to a generic stability parameter if $X \to \mathbb{C}^n/G$ is projective (see the next subsection). Note that the isomorphism class of Cox$(X)$ is independent of the choice of $X$ since crepant resolutions are mutually isomorphic in codimension one. The structure of Cox$(X)$ has been studied for special classes of $G$ in [D], [DW], [G] and so on. See [YT] for a general treatment. Moreover, algorithms finding generators of Cox$(X)$ are given in [DK] and [YT].

Now we review how to describe the structure of Cox$(X)$ explicitly. We refer the reader to [YT] for details. We first recall the notion of age. For an element $g \in G$, one can choose a basis $x_1, \ldots, x_n$ of the dual space $V^*$ of $V = \mathbb{C}^n$ such that $g \cdot x_j = e^{\frac{\pi i a_j}{r}} x_j$ with an integer $0 \leq a_j < r$ where $r$ is the order of $G$. The age of $g$ is defined as $\frac{1}{r} \sum_{j=1}^{n} a_j$, which is an integer since $g$ is in $\text{SL}_n(\mathbb{C})$. We call an element $g \in G$ junior if its age is one. Note that the age is invariant under conjugation. We also define a discrete valuation $\nu_g : \mathbb{C}(V)^* \to \mathbb{Z}$ by setting $\nu_g(x_j) = a_j$.

It is shown in [IR] that there is a one-to-one correspondence between the set of conjugacy classes of junior elements in $G$ and the set of irreducible exceptional divisors of the resolution $\pi : X \to V/G$, such that the divisorial valuation $\nu_E : \mathbb{C}(X)^* \to \mathbb{Z}$ along the $\pi$-exceptional divisor $E$ corresponding to $g$ coincides with the restriction of $\frac{1}{r} \nu_g$ to $\mathbb{C}(X)$ (via the birational map $\pi$). Let $E_1, \ldots, E_m$ be the set of $\pi$-exceptional irreducible divisors and let $g_k$ $(k = 1, \ldots, m)$ be a representative of the junior conjugacy class corresponding to $E_k$. We also put $\nu_k = \nu_{g_k}$ and $r_k = \nu(g_k)$.

The Cox ring has an embedding into the Laurent polynomial ring

$$\mathbb{C}[V]^{[G,G]}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$$

with $m$ variables over the invariant subring for the commutator group $[G,G]$ of $G$ [YT, Lemma 4.2]. Note that $\mathbb{C}[V]^{[G,G]}$ is naturally graded by the dual abelian group $\text{Ab}(G)^\vee = \text{Hom}_{\mathbb{Z}}(\text{Ab}(G), \mathbb{C}^*)$ of the abelianization $\text{Ab}(G) = G/[G,G]$ so that its degree-zero part is equal to $\mathbb{C}[V]^G$. For $\text{Ab}(G)^\vee$-homogeneous generators $f_1, \ldots, f_{\ell}$ of $\mathbb{C}[V]^{[G,G]}$, we associate to them the following set

$$\left\{ \widetilde{f}_i := f_i \prod_{k=1}^{m} t_k^{\nu_k(f_i)} \right\}_{i=1,\ldots,\ell} \cup \left\{ t_k^{-r_k} \right\}_{k=1,\ldots,m}$$

of $\ell + m$ elements of $\mathbb{C}[V]^{[G,G]}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, which lie in Cox$(X)$. This set is a natural candidate of a generating system of Cox$(X)$ but it is not a generating system in general. In fact it is a generating system if and only if $f_i$’s satisfy the valuation lifting condition cf. [DG, Theorem 2.2]. We do not explain this condition here as we will not use it later.

**Remark 3.1.** The ring $\mathbb{C}[V]^{[G,G]}$ treated here should be understood as the Cox ring Cox$(V/G)$ of $V/G$ ([AG, Theorem 3.1]), and the $\text{Ab}(G)^\vee$-grading also corresponds to the $\text{Cl}(V/G)$-grading on Cox$(V/G)$. Thus each $f_i$ above defines a Weil divisor $D_i$ on $V/G$. 


and the associated element $\tilde{f}_i$ to $f_i$ corresponds to a section defining the strict transform in $X$ of $D_i$ under the birational morphism $X \to V/G$.

The Pic($X$)-grading of $\text{Cox}(X) \subset \mathbb{C}[V][^G,G][t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$ can be read off from the $\mathbb{Z}^m$-grading given by the multi-degrees for the variables $t_1, \ldots, t_m$. More precisely, the Pic($X$)-grading is given so that the generator $t_i^{-r_i}$ lies in the homogeneous component corresponding to the divisor class of $E_i$. Note that the divisor classes of $E_1, \ldots, E_m$ are a $\mathbb{Q}$-basis (but usually not a $\mathbb{Z}$-basis) of Pic($X$).

### 3.2 Review of birational geometry of $X$

As mentioned in the previous subsection, any projective crepant resolution $X$ of $V/G$ has a finitely generated Cox ring. Such a variety is called a Mori dream space. This notion was originally introduced by Hu-Keel [HK] for a projective variety (over a field), but it makes sense in a relative setting (cf. [O]) so that the theory applies to the resolution $X \to V/G$. A remarkable property of a Mori dream space is that its birational contractions are completely controlled by polyhedral cones in the Picard group. In this section we review this briefly.

Hereafter we assume $X \to V/G$ is projective. We first consider the polyhedral cone generated by the classes of divisors in Pic($X$)$_{\mathbb{R}}$ whose base loci are of codimension greater than one. This cone is called the movable cone and denoted by Mov($X$). In our case, the movable cone is a convex cone which is the union of the nef cones of all relative minimal models $X' \to V/G$ (i.e. projective crepant birational morphisms with $X'$ having at worst $\mathbb{Q}$-factorial and terminal singularities). Here, the nef cone $\text{Nef}(X') \subset \text{Pic}(X')_{\mathbb{R}}$ is regarded as a cone inside Pic($X$)$_{\mathbb{R}}$ via the birational map $X \to X'$ which is an isomorphism in codimension one. Note that, in dimension three, relative minimal models are nothing but projective crepant resolutions.

The nef cones $\text{Nef}(X') \subset \text{Pic}(X')_{\mathbb{R}}$ are full-dimensional and have disjoint interiors. If we take any $L$ in the interior $\text{Amp}(X')$ of $\text{Nef}(X')$, the complete linear system of a sufficiently divisible power of $L$ gives the birational map $X \to X'$. Set

$$T_X = \text{Hom}(\text{Pic}(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^m,$$

which naturally acts on $\mathfrak{X} := \text{Spec} \text{Cox}(X)$. This birational map is the same as the one $X \dashrightarrow \mathfrak{X}/\chi T_X$ between the GIT-quotients with the character $\chi \in \chi(T_X)$ corresponding to $L$ via the identification $\chi(T_X)_{\mathbb{R}} = \text{Pic}(X)_{\mathbb{R}}$ (see the proof of [O, Theorem 6.7]). Moreover, the similar results hold for any rational contraction, that is, a rational map $f : X \dashrightarrow Z$ which is the composite of a sequence of flops $X \dashrightarrow X'$ and a proper birational morphism $f' : X' \to Z$ over $V/G$ such that

- $Z$ is a normal variety projective over $V/G$, and
- $f'$ has connected fibers, i.e. $f'_* \mathcal{O}_{X'} = \mathcal{O}_Z$.

More precisely, for such $f$, there exists $L \in \text{Mov}(X)$ whose linear system recovers $f$, and moreover $f$ equals the rational map $X \dashrightarrow \mathfrak{X}/\chi T_X$ induced by variation of GIT for the
character $\chi$ corresponding to $L$. Note that general points of a facet (i.e. codimension-one face) of $\text{Mov}(X)$ corresponds to a primitive divisorial rational contraction (i.e., a composite of a sequence of flops and a birational morphism contracting a single divisor which drops the Picard number by one). In contrast, small contractions are realized by taking divisors (or the corresponding characters) from the interior of $\text{Mov}(X)$. In particular every flop $X \dasharrow X'$ corresponds to the common facet $\text{Nef}(X) \cap \text{Nef}(X')$ of the nef cones of $X$ and $X'$.

**Remarks 3.1.** (1) We can also consider divisors (or corresponding characters) outside the movable cone, but they again give rational contractions (cf. [O, Corollary 5.8]) and thus considering the movable cone is essential.

(2) Classifying the GIT-quotients of $X$ (or equivalently, rational contractions of $X$) gives a fan structure on $\chi(T_X)_\mathbb{R}$ (or on $\text{Pic}(X)_\mathbb{R}$). This fan is called the GIT-fan for the $T_X$-action on $X$. When $X$ is toric, this is also called the secondary fan. See [CLS, Ch. 14] for a more detailed exposition of this fan.

**Example 3.1.** (Abelian group of type $\frac{1}{10}(1,3,6)$)

We illustrate how to compute the movable cone and nef cones for a crepant resolution of $\mathbb{C}^n/G$ for abelian $G$ through a particular example. Here we treat the case when $G \subset SL_3(\mathbb{C})$ is the cyclic group generated by

$$g = \begin{pmatrix} \zeta_{10} & 0 & 0 \\ 0 & \zeta_3 & 0 \\ 0 & 0 & \zeta_{10}^6 \end{pmatrix}$$

with $\zeta_{10}$ the primitive 10th root of unity. In this case $g, g^2, g^4, g^5, g^7$ are the junior elements, and the Cox ring of any crepant resolution $X$ is generated by the following 8 elements in $\mathbb{C}[x, y, z][t_1^{\pm 1}, \ldots, t_5^{\pm 1}]$:

$$xt_1t_2^2t_3^2t_4t_7, yt_1t_2^3t_3^3t_4t_5, zt_1^6t_2^2t_3^2t_5, t_1^{-10}, t_2^{-5}, t_3^{-5}, t_4^{-2}, t_5^{-10}$$

(3.2)

where $x, y$ and $z$ are the standard coordinates of $\mathbb{C}^3$. Moreover, $X$ is a toric variety corresponding to a fan $\Sigma$ obtained as a triangulation of the cone $\sigma_0 = \text{Cone}(e_1, e_2, e_3) \subset \mathbb{R}^3$ such that the set of rays (i.e. 1-dimensional cones) of $\Sigma$ is generated by $e_1, e_2, e_3, v_1, \ldots, v_5$ where $e_j$ is the standard basis of $\mathbb{R}^3$ and $v_k$'s are given as

$$v_1 = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_5 = \begin{pmatrix} 7 \\ 1 \\ 2 \end{pmatrix}$$

so that $v_k$ corresponds to $E_k$. See e.g. [Y2, §3] for details. One can check that there are six crepant resolutions, which are in fact all projective. These resolutions $X_1, \ldots, X_6$ fit into the following diagram of flops:

$$X_3 \xrightarrow{} X_4 \xrightarrow{} X_5 \xrightarrow{} X_6.$$
The cross sections by the hyperplane $\{(a, b, c) \in \mathbb{R}^3 \mid a + b + c = 1\}$ of the fans $\Sigma_1$ and $\Sigma_6$ of $X_1$ and $X_6$ respectively are given in Figure 2. Among $X_i$'s, the $G$-Hilbert scheme corresponds to $X_1$ (cf. [CR]).

As is well known, each $i$-dimensional cone $\tau \in \Sigma_1$ represents a torus orbit $O_\tau$ in $X_1$ of codimension $i$ (see e.g. [CLS, §3.2]). $\tau$ also defines a cone $C_\tau \subset \text{Pic}(X_1)_{\mathbb{R}}$ generated by line bundles having global sections which do not vanish on $O_\tau$. Under the identification of $\text{Pic}(X_1)_{\mathbb{R}}$ with $\chi(T_{X_1})_{\mathbb{R}}$, $C_\tau$ is the same as the orbit cone (see Definition 2.2) of some (and hence any) lift $\bar{x} \in \mathfrak{X}$ of some (and hence any) $x \in O_\tau$. $C_\tau$ is explicitly computed as the cone generated by the weights of the generators in (3.2) whose corresponding rays are not contained in $\tau$. For example, $C_\tau$ for $\tau = \text{Cone}(e_3, v_1) \in \Sigma_1$ is generated by the weights of the generators in (3.2) with the third and fourth ones removed.

The movable cone is obtained as the intersection $\bigcap_{i=1}^5 C_{\tau_i}$ where $\tau_i = \mathbb{R}_{\geq 0}v_i$. Similarly, the nef cone of $X_1$ is the intersection $\bigcap_{\tau \in \Sigma_1} C_\tau$. In our case, however, we do not have to compute $C_\tau$ for all $\tau \in \Sigma_1$. $\text{Nef}(X_1)$ is given as the intersection of $\text{Mov}(X)$ and just one cone $C_\tau$ for $\tau = \text{Cone}(v_2, v_4, v_5)$ since $\Sigma_1$ is clearly the only triangulation containing this three-dimensional cone.

The above computations of cones can be done, for instance by using the package “Polyhedra” of Macaulay2 [GS]. Then we see that $\text{Mov}(X_1)$ has 9 rays and 7 facets. The fact that $\text{Mov}(X_1)$ is non-simplicial comes from non-uniqueness of primitive rational contractions of exceptional divisors. For example, as we can see from Figure 2, the divisor $E_3 \subset X_1$ is mapped to a curve by a primitive contraction while it is mapped to a point by a primitive contraction inside $X_6$. We can also see that $\text{Nef}(X_1)$ is a simplicial cone having facets corresponding to 2 primitive divisorial contractions for $E_1, E_3$ and 3 flops along the curves $E_2 \cap E_4, E_2 \cap E_5$, and $E_4 \cap E_5$.

Later we will compare cones in $\text{Mov}(X_1)$ and orbit cones in $\Theta$ for moduli spaces of $G$-constellations (see Example 3.2).
3.3 Construction of embeddings

In this subsection we give a construction of nice embeddings of moduli spaces \( \mathcal{M}_\theta \) into toric varieties.

Let us consider the homomorphism \( \iota_X^* : \mathbb{C}[y_1, \ldots, y_{\ell+m}] \to \operatorname{Cox}(X) \) sending \( y_i \)'s to homogeneous generators of \( \operatorname{Cox}(X) \) of the form in (3.1) so that we obtain an embedding \( \iota_X \) of \( \mathcal{X} = \operatorname{Spec}(\operatorname{Cox}(X)) \) into the affine space \( \mathbb{A}^{\ell+m} \). We endow \( \mathbb{C}[y_1, \ldots, y_{\ell+m}] \) with a \( \mathbb{Z}^m \)-grading so that \( \iota_X \) preserves the grading. Recall that a character \( \chi \in \chi(T_X) = \operatorname{Pic}(X) \mathbb{R} \cong \mathbb{R}^m \) corresponding to an ample divisor on \( X \) satisfies \( \mathcal{X} / \chi T_X = X \) by the GIT construction (see the previous subsection). We can also consider the GIT-quotient \( X = \mathbb{A}^{\ell+m} / \chi T_X \) for the same character, which is a normal toric variety such that \( \operatorname{Pic}(Y) \cong \chi(T_X) \cong \operatorname{Pic}(X) \mathbb{C} \), and \( \iota_X \) induces an embedding \( \iota_X^* : X \hookrightarrow Y \) between the quotients.

**Remark 3.2.** If the generators of the Cox ring are minimal, then \( \iota_X \) is a neat embedding in the sense of [Ro]. See [Ro, §2.3] for details.

This embedding has a nice property from the viewpoint of birational geometry. As explained in the previous subsection, every rational contraction \( X \dashrightarrow X' \) is realized as the rational map \( X \dashrightarrow \mathcal{X} / \chi' T_X \) for some \( \chi' \). The same character \( \chi' \) gives a rational contraction \( Y \dashrightarrow Y' \) as well, and \( \iota_X \) again induces an embedding \( X' \hookrightarrow Y' \). Note that GIT-chambers for \( \mathbb{A}^{\ell+m} \) are finer than those for \( \mathcal{X} \) in general, and in particular \( Y \dashrightarrow Y' \) may not be an isomorphism even if \( X \dashrightarrow X' \) is an isomorphism. In other words, the inclusion \( \iota_X^*(\operatorname{Nef}(Y)) \subset \operatorname{Nef}(X) \) is strict in general. Note, however, that we have

\[
\iota_X^*(\operatorname{Mov}(Y)) = \operatorname{Mov}(X)
\]

since every exceptional divisor of \( Y \to \mathbb{A}^{\ell+m}/\chi T_X \) restricts to an exceptional divisor of \( X 
\to \mathbb{C}^n / G \) by the construction of \( Y \).

Next we construct an embedding of a moduli space \( \mathcal{M}_\theta \) into a toric variety. Recall that \( \mathcal{M}_\theta \) is obtained as the GIT-quotient of the variety \( \mathcal{V} \) by the action of \( \operatorname{PGL}_R \) with respect to the character \( \chi_\theta \in \chi(\operatorname{PGL}_R) \mathbb{R} \cong \Theta \). Let \( T_\Theta \subset \operatorname{PGL}_R \) be the algebraic torus consisting of (the images of) the component-wise scalar matrices. Since the restriction map \( \chi(\operatorname{PGL}_R \mathbb{R} \to \chi(T_\Theta) \mathbb{R} \) is an isomorphism, via this identification \( \mathcal{M}_\theta \) is isomorphic to \( \mathcal{W} / \chi_\theta T_\Theta \) with the affine categorical quotient

\[
\mathcal{W} = \mathcal{V} / \mathbb{C} \operatorname{SL}_R.
\]

Note that the points of \( \mathcal{W} \) are in bijection with the closed \( \mathbb{C} \operatorname{SL}_R \)-orbits in \( \mathcal{V} \).

In the proof of [Cl Proposition 2.2], it is shown that \( \mathcal{M}_0 = \mathcal{V} / \operatorname{PGL}_R \) admits an irreducible component which is isomorphic to \( \mathbb{C}^n / G \) (see also Subsection 2.1) by constructing a morphism \( V \to \mathcal{M}_0 \) induced from a \( G \)-constellation family \( \mathcal{F} \) over \( V = \mathbb{C}^n \). We will explicitly describe \( V \to \mathcal{M}_0 \) by lifting it to a morphism \( V \to \mathcal{V} \). To do this, let

\[
\mu : \mathbb{C}[V] \to \mathbb{C}[G] \otimes_{\mathbb{C}} \mathbb{C}[V]
\]

be the ring homomorphism corresponding to the action of \( G \subset \mathbb{C} \operatorname{SL}(V) \). We let \( G \) act on \( \mathbb{C}[G] \) as \( (g \cdot f)(h) = f(g^{-1} h) \) for any \( g, h \in G \) and \( f \in \mathbb{C}[G] \). In terms of algebras,
the \( \mathbb{C}[V] \)-action on each fiber \( \mathcal{F}_p := \mathbb{C}[G] (p \in V) \) of the family \( \mathcal{F} \) is given by the homomorphism

\[
\mathbb{C}[V] \otimes \mathbb{C}[G] \to \mathbb{C}[G]; \quad x \otimes f \mapsto (\mu(x)f \bmod m_p)
\]

where \( m_p \subset \mathbb{C}[V] \) is the maximal ideal for \( p \). This is indeed \( G \)-equivariant since we have

\[
(g \cdot (x \cdot f))(h) = (x \cdot f)(g^{-1}h) = (\mu(x)f \bmod m_p)(g^{-1}h) = x(g^{-1}h \cdot p)f(g^{-1}h)
\]

while

\[
((g \cdot x) \cdot (g \cdot f))(h) = (\mu(g \cdot x)(g \cdot f) \bmod m_p)(h) = ((g \cdot x)(h \cdot p))(g \cdot f)(h)) \nonumber
\]

\[
= x(g^{-1}h \cdot p)f(g^{-1}h)
\]

for any \( g, h \in G, x \in \mathbb{C}[V] \) and \( f \in \mathcal{F}_p = \mathbb{C}[G] \).

For each \( \rho \in \text{Irr}(G) \), we fix its \( \mathbb{C} \)-basis \( v_1, \ldots, v_{\dim \rho} \). The matrix presentation of the representation \( \rho \) with respect to this basis gives a matrix \( (f_{i,j})_{i,j} \) with entries in \( \mathbb{C}[G] \), that is, \( g \cdot v_j = \sum_{i=1}^{\dim \rho} f_{i,j}(g)v_i \) for all \( g \in G \). Let \( v_1^*, \ldots, v_{\dim \rho}^* \) be the dual basis to \( v_1, \ldots, v_{\dim \rho} \). Then the assignment \( f_{i,j} \mapsto v_i^* \) gives \( G \)-equivariant isomorphisms from the vector subspaces \( V_{\rho,j} := \bigoplus_{i=1}^{\dim \rho} \mathbb{C} f_{i,j} \subset \mathbb{C}[G] \) to the dual (or contragredient) representations \( \rho^* \) of \( \rho \). This can be checked by comparing the coefficients of \( v_i^* \)'s in

\[
g^{-1}h \cdot v_j = \sum_{i=1}^{\dim \rho} f_{i,j}(g^{-1}h)v_i = \sum_{i=1}^{\dim \rho} (g \cdot f_{i,j})(h)v_i \quad \text{and}
\]

\[
g^{-1} \cdot (h \cdot v_j) = g^{-1} \cdot \left( \sum_{i=1}^{\dim \rho} f_{i,j}(h)v_i \right) = \sum_{i=1}^{\dim \rho} f_{i,j}(h) \left( \sum_{k=1}^{\dim \rho} f_{k,i}(g^{-1}v_k) \right)
\]

with \( g, h \in G \). Since \( \mathbb{C}[G] \) is a direct sum of \( \{V_{\rho,j}\}_{\rho,j} \) by the character theory for finite groups, we obtain an explicit \( G \)-equivariant isomorphism \( \mathbb{C}[G] \to \bigoplus_{\rho \in \text{Irr}(G)} \rho^* \otimes \dim \rho \). We will describe the family \( \mathcal{F} \) as quiver representations with the matrix presentation with respect to the basis of \( \mathbb{C}[G] \) obtained in this way. To this end, we also fix a basis of the \( \alpha_{\rho,\rho'} \)-dimensional vector space \( \text{Hom}_{\mathbb{C}[G]}(V^* \otimes_{\mathbb{C}} \rho, \rho') \) for each pair \( \rho, \rho' \in \text{Irr}(G) \).

For each arrow \( a \in A \) from \( \rho \) to \( \rho' \) in \( Q_G \), we have an embedding \( \rho' \to V^* \otimes \rho \). Regarding the fixed basis for \( \rho \) and \( \rho' \), we obtain a \( (\dim \rho' \times \dim \rho) \)-matrix \( \tilde{H}_a \) with entries in \( V^* \). Then we obtain a set of matrices \( \{ H_a^* := (\tilde{H}_a \mod m_p) \}_{a \in A} \), which can be regarded as a set of representations of \( Q_G \) (see Subsection 2.3) parametrized by \( V \).

**Lemma 3.1.** The representation of \( Q_G \) corresponding to the \( G \)-constellation \( \mathcal{F}_p \) is isomorphic to \( \{ H_a^* \}_{a \in A} \) for any \( p \in V \).

**Proof.** We take any arrow \( a \in A \) from \( \rho \) to \( \rho' \) and set \( m = \dim \rho \) and \( n = \dim \rho' \).

Let \( (f_{i,j})_{1 \leq i,j \leq m} \) (resp. \( (g_{i,j})_{1 \leq i,j \leq n} \)) be the matrix presentation of \( \rho^* \) (resp. \( \rho'^\ast \)) with respect to the fixed basis. We write the \( n \times m \)-matrix \( \tilde{H}_a \) as \( (h_{i,j})_{i,j} \) with \( h_{i,j} \in V^* \).

Regarded as a representation of \( Q_G \), \( H_a^* \) defines an action of \( V^* \) on the vector space
\( V_{\rho', j} = \bigoplus_{i=1}^{m} \mathbb{C}f_{i,j} \subset \mathbb{C}[G], \) which is isomorphic to \( \rho \) as a \( \mathbb{C}G \)-module, so that

\[
\sum_{i=1}^{m} h_{1,i} \cdot f_{i,k} = \sum_{j=1}^{n} h_{j,k}(p)g_{1,j} \\
\sum_{i=1}^{m} h_{2,i} \cdot f_{i,k} = \sum_{j=1}^{n} h_{j,k}(p)g_{2,j} \\
\vdots \\
\sum_{i=1}^{m} h_{n,i} \cdot f_{i,k} = \sum_{j=1}^{n} h_{j,k}(p)g_{n,j}
\]

(3.3)

for each \( k = 1, \ldots, m \). We show that this action coincides with the one for \( F_p \). By the definition of the \( \mathbb{C}[V] \)-module structure on \( F_p = \mathbb{C}[G] \), the LHS of the \( j \)-th row of (3.3), as an element of \( F_p \), is the function

\[
g \mapsto h_{j,1}(g \cdot p)f_{1,k}(g) + h_{j,2}(g \cdot p)f_{2,k}(g) + \cdots + h_{j,m}(g \cdot p)f_{m,k}(g)
\]

on \( G \) for each \( j \). Note that the value of this function at the identity \( 1 \in G \) is \( h_{j,k}(p) \) and this is also equal to the value of the RHS of the \( j \)-th row of (3.3) (as a function on \( G \)) at \( 1 \in G \). By the choice of \( \bar{H}_a \), the LHSs and the RHSs of (3.3) give \( G \)-equivariantly isomorphic basis of \( \rho' \). Noticing that \( f(g) = g^{-1} \cdot f(1) \) for any \( f \in \mathbb{C}[G] \) and \( g \in G \), this implies that the two functions attain the same values for all \( g \in G \), and hence the claim.

We will give concrete examples of the representations \( \{ H_a \}_{a \in A} \) in Subsection 3.5.

The generic representation \( \{ \bar{H}_a \}_{a \in A} \) gives a morphism \( V \to \mathcal{N} \), which factors through \( V \to \mathcal{V} \) since \( V \) is mapped onto \( \mathcal{V}/G \subset \mathcal{M}_0 \). Let \( f \) be a semi-invariant function on \( \mathcal{N} \) and hence on \( \mathcal{V} \). Then the (scheme-theoretic) zero locus of \( f \) on \( \mathcal{V} \) descends to a Weil divisor \( D_f \) on \( \mathcal{V}/G \). Thus, sending \( f \) to the section of the divisorial sheaf \( \mathcal{O}_{\mathcal{V}/G}(D_f) \) defines a ring homomorphism

\[
\varphi : \mathbb{C}[\mathcal{N}]^{SL_R} \to \text{Cox}(\mathcal{V}/G) \cong \mathbb{C}[\mathcal{V}]^{[G,G]}
\]

(cf. Remark 3.1). More explicitly, for a determinantal semi-invariant \( f : H \mapsto \det(H_{p_{i,j}})_{i,j} \) defined by paths \( p_{i,j} \) as in (2.3), \( \varphi(f) \) is given as \( \mathcal{H} \in \mathbb{C}[\mathcal{V}] \) where \( \mathcal{H} = (\bar{H}_{p_{i,j}})_{i,j} \) is defined similarly to \( \bar{H}_a \). Note that if \( f \) has weight \( (\theta_{\rho})_{\rho \in \text{Irr}(G)} \), then \( \varphi(f) \) is acted on by \( G \) by the character

\[
\det \left( \bigoplus_{\rho \in \text{Irr}(G)} \rho^{\otimes \theta_{\rho}} \right) : G \to \mathbb{C}^*.
\]

In particular this naturally gives a homomorphism \( \bar{\Theta} \to \text{Ab}(G)^\vee \).

The homomorphism \( \varphi \) becomes more meaningful when we restrict our attention from \( \mathcal{N} \) to its irreducible component \( \mathcal{V} \). Recall that we have the following commutative
diagram of affine schemes and their GIT quotients

\[
\begin{array}{c}
\mathcal{N} \xrightarrow{\phi} \mathcal{N} \sslash_{SL_R} \to \mathcal{M}_0 \xrightarrow{\chi_{\theta}} \mathcal{M}_\theta \\
\mathcal{V} \xrightarrow{\phi} \mathcal{W} \sslash_{SL_R} \to \mathcal{V}/G \xrightarrow{\chi_{\theta}} \mathcal{M}_\theta
\end{array}
\]

for any \( \theta \in \Theta \) where the vertical arrows are all closed immersions of irreducible components. Using \( \phi \), we can give a description of the coordinate ring of \( \mathcal{W} = \mathcal{V} \sslash_{SL_R} \) similarly to the description of \( \text{Cox}(X) \) in (3.1). For this, let us consider the \( \bar{\Theta} \)-graded algebra \( \bigoplus_{\theta \in \bar{\Theta}} \mathbb{C}[V]_{[G,G]}^{t_\theta} \) where \( t_\theta \) is a formal variable indicating the graded component for \( \theta \in \bar{\Theta} \).

**Proposition 3.2.** The algebra \( \mathbb{C}[\mathcal{W}] = \mathbb{C}[V]_{SL_R} \) is isomorphic to the graded subring \( S_W \) of \( \bigoplus_{\theta \in \bar{\Theta}} \mathbb{C}[V]_{[G,G]}^{t_\theta} \) generated by \( \phi(h) t_\theta \) for semi-invariants \( h \in \mathbb{C}[\mathcal{N}]_{SL_R} \) of weight \( \theta \).

**Proof.** The \( T_\Theta \)-equivariant morphism

\( V/[G,G] \times T_\Theta \rightarrow \mathcal{N} \sslash_{SL_R} \)

corresponding to the homomorphism \( \mathbb{C}[\mathcal{N}]_{SL_R} \rightarrow \bigoplus_{\theta \in \bar{\Theta}} \mathbb{C}[V]_{[G,G]}^{t_\theta} \) is induced from the morphism \( V \rightarrow \mathcal{V} \subset \mathcal{N} \) (and thus descends to the closed immersion \( \mathcal{V}/G \rightarrow \mathcal{M}_0 \)). Then the claim follows since the spectrum of the integral domain \( S_W \) is isomorphic to the scheme-theoretic image of the equivariant morphism above, which is equal to \( \mathcal{V} \sslash_{SL_R} = \mathcal{W} \).

From now on we assume \( n = 3 \), i.e. \( V = \mathbb{C}^3 \). We fix any GIT-chamber \( C \subset \Theta \) and let \( X := \mathfrak{M}_C \) be a crepant resolution of \( V/G \). Similar to the construction of \( \phi \), we define a ring homomorphism

\[ \phi_C : \mathbb{C}[\mathcal{N}]_{SL_R} \rightarrow \text{Cox}(X) \]

so that, for a homogeneous \( h \in \mathbb{C}[\mathcal{N}]_{SL_R} \), the zero locus of \( h \) in \( \mathcal{N}_{C,ss} \) descends to a divisor \( D_h \) of \( X \) which is defined by \( \phi_C(h) \in H^0(\mathcal{O}_X(D_h)) \subset \text{Cox}(X) \). More explicitly, \( \phi_C(h) \) is given as

\[ \phi_C(h) = \varphi(h) \prod t_k^{-\text{ord}_{E_k}(h)} \]  

(3.4)

where \( \text{ord}_{E_k}(h) \) is the coefficient of the exceptional divisor \( E_k \) in \( D_h \). Note that \( \varphi \) fits into the following commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}[\mathcal{N}]_{SL_R} & \xrightarrow{\phi_C} & \mathbb{C}[V]_{SL_R} \\
\downarrow & & \downarrow \cong \\
\text{Cox}(X) & \xrightarrow{\varphi} & \mathbb{C}[V]_{[G,G]}^{t_1, \ldots, t_m} \xrightarrow{t_\theta+1} \mathbb{C}[V]_{[G,G]}
\end{array}
\]
where the upper right isomorphism comes from Proposition 3.2. Since $\mathcal{V}$ is the irreducible component of $\mathcal{N}$ which dominates $\mathcal{M}_C$, $\varphi_C$ factors through $\mathbb{C}[\mathcal{V}]^{SLR}$ and we thus obtain a morphism

$$\varphi_C^* : \mathcal{X} \to \mathcal{W}$$

which descends to the identification $X \to \mathcal{M}_C$.

The ring homomorphism $\varphi$ also induces a homomorphism

$$\psi_C : \Theta \to \text{Pic}(X)_R$$

such that the degree of $\varphi_C(f)$ for a homogeneous element $f$ of weight $\theta \in \Theta$ is equal to $\psi_C(\theta)$. This is indeed well-defined since two semi-invariants on $\mathcal{V}$ of the same degree differ by multiplication by a $G$-invariant rational function on $V$ and thus define the same class in $\text{Pic}(X)$. Later we will see that $\psi_C$ can be computed explicitly using an embedding of $\mathcal{W}$ into a toric variety. See Remarks 3.2(1) and examples in Subsection 3.3. We will also see in Lemma 4.8 that $\psi_C$ may be defined in terms of tautological bundles on the moduli space $X = \mathcal{M}_C$.

Now we are ready to define an embedding of $\mathcal{W}$ into a toric variety. We first choose $\text{Ab}(G)^\vee$-homogeneous generators $f_1, \ldots, f_\ell \in \mathbb{C}[\mathcal{V}]^{[G,G]}$ such that the associated elements in (3.1) generate $\text{Cox}(X)$. Let $\iota_X : \mathcal{X} \to \mathbb{A}^{\ell+m}$ be the associated embedding constructed in the beginning of this subsection. Set $N_X = \mathbb{Z}^\ell \oplus \text{Pic}(X)^\vee$ and let $\sigma_\Lambda \subset (N_X)_R \cong \mathbb{R}^{\ell+m}$ be the cone defining the affine toric variety $\mathbb{A}^{\ell+m}$ with the $T_X$-action. We also set $N_\mathcal{W} = \mathbb{Z}^\ell \oplus \bar{\Theta}^\vee$ and define $\sigma_\mathcal{W} \subset (N_\mathcal{W})_R \cong \mathbb{R}^{\ell+s-1}$ as the cone generated by the set of cones $\{(\text{id}_\mathcal{W} \oplus \psi^*_C)\sigma_\Lambda\}_C$ where $C$ runs through all GIT-chambers $C \subset \Theta$ and the choice of $f_i$’s is consistent regardless of the choice of $C$. Note that $\psi_C$ and its pullback $\psi_C^* : \text{Pic}(X)^\vee_R \to \Theta^\vee$ depend on $C$.

Let $\bar{Y}_\mathcal{W}$ be the affine normal toric variety defined by $\sigma_\mathcal{W}$. We show that the normalization $\bar{\mathcal{W}}$ of $\mathcal{W}$ has an embedding into $\bar{Y}_\mathcal{W}$. By definition, the coordinate ring of $\bar{Y}_\mathcal{W}$ is the semigroup algebra $\mathbb{C}[S_\mathcal{W}]$ for the semigroup $S_\mathcal{W} = M_\mathcal{W} \cap \sigma_\mathcal{W}^\vee$ where $M_\mathcal{W} = N_\mathcal{W}^\vee \cong \mathbb{Z}^{\ell+s-1}$ and $\sigma_\mathcal{W}^\vee$ is the dual cone

$$\{f \in (M_\mathcal{W})_R \mid f(v) \geq 0 \text{ for all } v \in \sigma_\mathcal{W}\}$$

to $\sigma_\mathcal{W}$. Since $f_i$’s generate $\mathbb{C}[\mathcal{V}]^{[G,G]}$, we can choose homogeneous generators $\{h_j\}_j$ of $\mathbb{C}[\mathcal{W}] = \mathbb{C}[\mathcal{V}]^{SLR}$ such that each $\varphi(h_j)$ is of the form $f_1^{a_1} \cdots f_\ell^{a_\ell}$. This presentation may not be unique in general, but we fix it for each $\varphi(h_j)$. Then $\mathcal{W}$ admits a closed immersion into a possibly non-normal affine toric variety

$$Y_\mathcal{W} = \text{Spec} \mathbb{C}[S'_\mathcal{W}]$$
defined by the semigroup $S'_\mathcal{W} \subset M_\mathcal{W}$ generated by $\{(a_1, \ldots, a_\ell, \theta_j)\}_j$ where $\theta_j \in \bar{\Theta}$ is the weight of $h_j$. This also induces a closed immersion of $\mathcal{W}$ into the normalization of $Y_\mathcal{W}$. Note that we have $S'_\mathcal{W} \subset S_\mathcal{W}$ and thus $\sigma_\mathcal{W}$ is contained in the cone dual to $S'_\mathcal{W}$.

**Lemma 3.3.** We have the following commutative diagram

$$
\begin{array}{ccc}
\mathbb{A}^{\ell+m} & \longrightarrow & \bar{Y}_\mathcal{W} \\
\iota_X \uparrow & & \uparrow \\
\mathcal{X} & \longrightarrow & \bar{\mathcal{W}} \\
\end{array}
$$

$$\begin{array}{ccc}
& & Y_\mathcal{W} \\
& & \uparrow \\
& & \mathcal{W}
\end{array}
$$

(3.6)
where the three vertical arrows are closed immersions and the two right-hand horizontal arrows are the normalization maps.

Proof. Given the above, it suffices to show that the semigroup algebra $\mathbb{C}[S_W]$ coincides with the integral closure of $\mathbb{C}[S'_W]$.

For each $f \in S_W$, it defines a rational function $\bar{f}$ on $\bar{W}$ via the above embedding of $\bar{W}$. It suffices to show that $\bar{f}$ is regular on $\bar{W}$. Since $\bar{W}$ is normal it suffices to show that $\bar{f}$ is regular outside a locus of codimension greater than one. By the construction of $\sigma_W$, $\bar{f}$ is regular at general points of any divisor $D \subset \bar{W}$ which descends to a divisor on a resolution $\bar{W}/T_\Theta$ of $V/G$ for some generic stability condition $\theta$. Then the claim follows by the lemma below.

Lemma 3.4. The complement of the open subset $\bigcup_{\theta: \text{generic}} W^{\theta-ss}$ in $W$ is of codimension at least two.

Proof. Assume that the complement contains an irreducible divisor $D \subset W$ in order to deduce a contradiction. Let $C \subset \Theta$ be the cone generated by the weights of semi-invariants of $V$ which do not vanish at general points of $D$. By the assumption, $C$ has positive codimension and, equivalently, general points of $D$ has a positive-dimensional stabilizer subgroup in $T_\Theta$. Then the image of $D$ under the quotient map $W^{\theta-ss} \to W/T_\Theta$ for $\theta$ in the relative interior of $C$ has dimension greater than or equal to $n = 3$. This is contrary to the irreducibility of $W$ since the $T_\Theta$-orbits of general points of $W$, which correspond to free $G$-orbits (see Subsection (2.1)), are closed and hence $W/T_\Theta$ has dimension three for any (non-generic) $\theta$.

For any $\theta \in C$, we obtain the following diagram of maps between quotients induced from (3.6):

$$
\begin{array}{ccc}
Y & \xrightarrow{\iota_X} & Y_{\bar{W}}/T_\Theta \\
\uparrow & & \uparrow \\
X & \xrightarrow{\iota_X} & W/T_\Theta
\end{array}
\quad (3.7)
$$

where vertical arrows are again closed immersions and horizontal arrows are all isomorphisms. The results obtained so far are summarized as follows:

Theorem 3.5. The irreducible component $W$ of $N/SL_R$ such that $W/T_\Theta \cong \mathbb{C}^3/G$ admits a $T_\Theta$-equivariant closed immersion $\iota_W : W \to Y_W$ into a possibly non-normal toric variety whose torus contains $T_\Theta$, satisfying the following property:

- for any generic $\theta \in \Theta$, the induced morphism $W/T_\Theta \to Y_{\bar{W}}/T_\Theta$ between the GIT-quotients is identified with a closed immersion $\iota_X : X \to Y$ of the crepant resolution $X := \mathcal{M}_\theta$ into a normal toric variety such that the restriction map $\iota_X^* : \text{Pic}(Y)_R \to \text{Pic}(X)_R$ is an isomorphism and $\text{Mov}(X) = \iota_X^*(\text{Mov}(Y))$.

In the next subsection we will explain how the (semi)stable locus of $Y_W$ for a given stability condition is determined in terms of the fan $\sigma_W$.
Remarks 3.2. (1) Let \( v_1, \ldots, v_m \in \sigma_\theta \) be the primitive generators of the rays corresponding to the exceptional divisors \( E_k \) of \( X = \mathfrak{M}_\theta \to V/G \). Recall that the order of zeros (or poles) of a rational function \( f \in M_\theta \) along the corresponding divisor to \( v_k \) is given by \( f(v_k) \), by toric geometry. Thus, the image of a homogeneous semi-invariant \( h := f_1^{a_1} \cdots f_m^{a_m} t_\theta(a_i \in \mathbb{Z}) \) under \( \varphi_C \) is given by

\[
\tilde{f}_1^{a_1} \cdots \tilde{f}_m^{a_m} \prod_{k=1}^m t_k^{-a(v_k)r_k} \quad (3.8)
\]

(see Remark \[3.1\] and \[3.4\]) where \( a = (a_1, \ldots, a_m, \theta) \in \mathbb{Z}^m \oplus \bar{\Theta} \) is identified with an element of \( M_\theta \). In particular, \( \psi_C(\theta) \in \text{Pic}(X) \) is given by the multi-degree of \( t_k \)'s in \( [3.8] \). Note that \( \psi_C \) depends only on \( v_k \)'s and therefore \( \psi_C' \) is equal to \( \psi_C \) as long as we take a GIT-chamber \( C' \) inside the intersection of the orbit cones of \( G \)-constellations corresponding to general points of the divisors \( E_1, \ldots, E_m \).

(2) The homomorphism \( \varphi_C : \mathbb{C}[N]^{SL_R} \to \text{Cox}(X) \) is never surjective unless \( G \) is trivial since each \( t_k^{-r_k} \in \text{Cox}(X) \) is clearly outside of the image. However, we will show that, for homogeneous generators \( \{h_i\} \) of \( \mathbb{C}[N]^{SL_R} \), the associated elements to the images \( \varphi(h_i) \) generate \( \text{Cox}(X) \) (Proposition \[4.14\]).

### 3.4 GIT-quotients of a toric variety by a subtorus-action

As we saw in the previous subsection, every crepant resolution of the form \( \mathfrak{M}_\theta \) is embedded in a quotient of the toric variety \( \bar{Y}_\theta \) (or \( Y_\theta \)) by the algebraic torus \( T_\theta \). In this subsection we show how to compute the \( \theta \)-semistable locus of \( \bar{Y}_\theta \) for \( \theta \in \Theta \) in terms of the fan of \( \bar{Y}_\theta \). This also determines the semistable locus of \( \bar{W} \) since we have

\[
\bar{Y}_\theta^{\theta-ss} \cap \bar{W} = \bar{W}^{\theta-ss}
\]

by the construction of \( \bar{Y}_\theta \). If \( \theta \) is generic, \( \bar{W}^{\theta-ss} \) is also identified with \( \mathcal{W}^{\theta-ss} \) since the geometric quotient of \( \mathcal{W}^{\theta-ss} \) is smooth.

To consider quotients of \( \bar{Y}_\theta \), first note that \( T_\Theta \) is regarded as a subtorus of the big torus \( T_\theta := \text{Hom}_\mathbb{Z}(N_\theta, \mathbb{C}^*) \subset Y_\theta \). Quotients of a toric variety by an action of a subtorus of the big torus have been studied in [KSZ], [AH], [Hu] and so on. When we consider GIT-quotients (or more generally good quotients), the quotient map naturally induces a one-to-one correspondence between the sets of the maximal cones for the toric variety and its quotient (see e.g. [AH] Proposition 3.2]). To state this more precisely for our situation, recall that the toric variety \( \bar{Y}_\theta \) is defined by the cone \( \sigma_\theta \) in \( (N_\theta)_\mathbb{R} \). For any \( \theta \in \Theta \), its semistable locus \( \bar{Y}_\theta^{\theta-ss} \) is \( T_\theta \)-invariant since functions on \( \bar{W} \) defining the semistable locus can be taken as homogeneous ones with respect to \( T_\theta \). Thus, \( \bar{Y}_\theta^{\theta-ss} \) is also a toric variety defined by a subfan \( \Sigma_\theta \) of \( \Sigma_\Theta \) where \( \Sigma_\Theta \) is the fan consisting of all faces of \( \sigma_\Theta \). Then the GIT-quotient \( Y_\theta := \bar{Y}_\theta / T_\theta \) is a toric variety defined by a fan \( \Sigma_\theta \) in \( \mathbb{R}^\ell \), and the quotient map \( \bar{Y}_\theta^{\theta-ss} \to Y_\theta \) is a toric morphism induced by the projection

\[
p : (N_\theta)_\mathbb{R} \to (N_\theta / \Theta^\vee)_\mathbb{R} = \mathbb{R}^\ell
\]

satisfying the following condition:
For any cone $\sigma \in \bar{\Sigma}_\theta$, the cone $p^{-1}(\sigma) \cap |\Sigma_\theta|$ belongs to $\Sigma_\theta$ where $|\Sigma_\theta| \subset (N_\mathcal{W})_\mathbb{R}$ is the support of the fan $\Sigma_\theta$. Moreover, if $\theta$ is generic, $p$ induces an isomorphism of $\Sigma_\theta$ and $\bar{\Sigma}_\theta$.

Now we give an explicit method to compute $\Sigma_\theta$ and hence $\bar{Y}_\mathcal{W}^{\theta,ss}$. To this end, it is sufficient to determine a general point of $p^{-1}(v) \cap |\Sigma_\theta|$ for each $v \in \mathbb{R}^\ell$ by the condition (3.4) above. Let $q : (N_\mathcal{W})_\mathbb{R} \cong \mathbb{R}^\ell \times \Theta^\vee \to \Theta^\vee$ be the second projection to the dual vector space of $\Theta \cong \mathbb{R}^{s-1}$ (while $p$ is regarded as the first projection). Then we set $P_v = q(p^{-1}(v) \cap |\Sigma_\mathcal{W}|) \subset \Theta^\vee$, which is a rational convex polyhedron since $\sigma_\mathcal{W}$ is rational and strongly convex. Note that $\theta$ is a linear function on $P_v$. The following result is regarded as a generalization of [CMT, Theorem 7.2] to the case where $G$ is non-abelian, see Example 3.5.1.

**Proposition 3.6.** For any $\theta \in \Theta$, a face $F$ of $\sigma_\mathcal{W}$ belongs to $\Sigma_\theta$ if and only if $q(F) \cap P_{p(\bar{v})}$ equals the set of points of $P_{p(\bar{v})}$ for which the function $\theta : P_{p(\bar{v})} \to \mathbb{R}$ is minimized for some (and hence any) point $\bar{v}$ of the relative interior $\text{relint}(F)$ of $F$.

This result can be proven by using [CM, Proposition 2.7], on which the proof of [CMT, Theorem 7.2] relies. Here we give another proof using the Hilbert-Mumford numerical criterion in order to understand the result in terms of one-parameter subgroups. The numerical criterion states that a point $x \in \bar{Y}_\mathcal{W}$ is $\theta$-semistable if and only if we have $\lambda \cdot \theta \geq 0$ for any 1-parameter subgroup (1-PS) $\lambda : \mathbb{C}^* \to T_\theta$ such that $\lim_{t \to 0}(\lambda(t) \cdot x)$ exists, where $\lambda \cdot \theta$ means the natural paring between a 1-PS and a character of $T_\theta$.

To apply this criterion to toric varieties, let us recall that faces $F$ of $\sigma_\mathcal{W}$ and $T_\mathcal{W}$-orbits of $W$ are in one-to-one correspondence which assigns to $F$ the orbit $O(F)$ of $x_F := \lim_{t \to 0} \lambda \bar{v}(t)$ for the 1-PS $\lambda : \mathbb{C}^* \to T_\mathcal{W}$ corresponding to $\bar{v} \in \text{relint}(F)$. Moreover, the closure of $O(F)$ is equal to $\bigcup_{F' \subset F' \in \Sigma_\mathcal{W}} O(F')$, and we have $\lim_{t \to 0}(\lambda u(t) \cdot x_F) \in O(F')$ for any $u \in \text{relint}(F')$ where $\lambda u$ is the 1-PS corresponding to $u$. See [CLS, Proposition 3.2.2] and its proof for these standard facts. We also observe that the limit $\lim_{t \to 0}(\lambda u(t) \cdot x_F)$ for $u \in N_\mathcal{W}$ exists if and only if there exist (sufficiently small) $h > 0$ and a face $F'$ of $\sigma_\mathcal{W}$ containing $F$ such that $u_1 + hu$ lies in $F'$ for some (and hence any) $u_1 \in \text{relint}(F)$.

**Proof of Proposition 3.6.**
According to the observation above, for a 1-PS $\lambda w$ of $T_\theta$ corresponding to a vector $w \in \tilde{\Theta}^\vee$ and any point $x$ of the orbit $O(F)$, the limit $\lim_{t \to 0}(\lambda w(t) \cdot x)$ exists if and only if there exist $h > 0$ and a face $F'$ containing $F$ such that $u_1 + hu \in F$ for some $u_1 \in \text{relint}(F)$.

Let us assume that $q(F) \cap P_{p(\bar{v})}$ does not minimize $\theta$. Then there is another face $F'$ of $\sigma_\mathcal{W}$ such that $q(F') \cap P_{p(\bar{v})}$ minimizes $\theta$, and $(w_1 - q(\bar{v})) \cdot \theta < 0$ for any $w_1 \in \text{relint}(q(F') \cap P_{p(\bar{v})})$. Since $q(\bar{v}) + 1 \cdot (w_1 - q(\bar{v})) = w_1$ is inside $P_{p(\bar{v})}$, the convexity of $P_{p(\bar{v})}$ implies that the limit for the 1-PS corresponding to $w_1$ exists and thus $F$ is not in $\Sigma_\theta$ by the numerical criterion.

Conversely, if $q(F) \cap P_{p(\bar{v})}$ minimizes $\theta$, then $q(F) \cap P_{p(\bar{v})}$ is a face of $P_{p(\bar{v})}$ having $\theta = (the \ minimal \ value) \subset \tilde{\Theta}^\vee$ as the supporting hyperplane. Therefore, $(w - q(\bar{v})) \cdot \theta \geq 0$ for any $w \in P_{p(\bar{v})}$ and $F$ is in $\Sigma_\theta$ by the numerical criterion again. 

\[\square\]
**Remark 3.3.** Whenever we are given an affine toric variety and a subtorus of the big torus, the similar statement to the above proposition applies to compute the semistable locus of the given toric variety with respect to a character of the subtorus. A description of the fan of the GIT quotient is also given in [CM §3].

### 3.5 Examples

In this subsection we demonstrate the construction of the toric embedding and the homomorphism $\varphi_C$ in the previous subsection via concrete examples.

#### 3.5.1 Abelian cases

We first consider the case where $G \subset SL_3(\mathbb{C})$ is a finite abelian subgroup. In this case all irreducible representations of $G$ are 1-dimensional, which particularly implies that $SL_R$ is trivial and hence $V = \mathcal{V}$. Then there is a natural choice of generators of the Cox ring of a crepant resolution $X$ and generators of the coordinate ring of $V$ as follows.

Once we fix a basis of $V = \mathbb{C}^3$ such that $G$ is a diagonal group, we obtain generators of $\mathbb{C}[V]_{G,G} = \mathbb{C}[V]$ as the standard basis $x_1, x_2, x_3$ of $V^*$ and also standard generators of $\text{Cox}(X)$ (cf. [Y2, Proposition 3.5]). As explained in [CMT] (see also [Y2]), the coordinate ring of $V$ is generated by $3r$ variables $\{ x_{i,j} \}_{0 \leq i \leq r-1, 1 \leq j \leq 3}$ with certain relations where $r$ is the order of $G$. In terms of the quiver representation, the variable $x_{i,j}$ is a homogeneous semi-invariant corresponding to an arrow $\rho_i$ to $\rho_i \otimes \chi_j$ of the McKay quiver of $G$ where $\text{Irr}(G) = \{ \rho_0, \ldots, \rho_{r-1} \}$ and we regard the character $\chi_j : G \to \mathbb{C}^*$ defined by $g \cdot x_j = \chi_j(g) x_j$ as a 1-dimensional representation of $G$. Then $\mathcal{V}$ is already a possibly non-normal toric variety [CMT, Theorem 3.10], and the ambient toric variety $\bar{Y}_V$ constructed from these generators is nothing but $V$ itself.

For each orbit $O \subset V$ by the big torus, it has the distinguished point $v \in O$ satisfying $x_{i,j}(v) = 1$ for all $i, j$ such that $x_{i,j}(O) \neq \{0\}$. For any $\theta$, the fan $\Sigma_\theta$ of the $\theta$-semistable locus of $\mathcal{V}$ is computed using Proposition 3.6, and one obtains the distinguished point corresponding to a cone $\sigma \in \Sigma_\theta$ by setting

$$x_{i,j}(v) = \begin{cases} 1 & \text{if } \sigma \subset H_{i,j} \\ 0 & \text{if } \sigma \not\subset H_{i,j} \end{cases}$$

where $H_{i,j} \subset (N_V)_\mathbb{R}$ is the hyperplane defined by $x_{i,j} \in M_V$. This is essentially the same procedure as the one given in [CMT Theorem 7.2], which enables us to obtain the distinguished point corresponding to the cone $\sigma_w \in \Sigma_\theta$ such that $p(\sigma_w)$ contains a given $w \in \mathbb{R}^3$ in its relative interior. Note that one can also consider distinguished points of $\bar{Y}_V$ for non-abelian $G$ but these points usually do not lie inside $\mathcal{W}$.

In [Y2], an explicit description of the homomorphism $\varphi_C : \mathbb{C}[V] \to \text{Cox}(X)$ is given in terms of the notion of a $G$-nat family introduced by Logvinenko [Lo]. Recall that a $G$-nat family on a crepant resolution $X \to \mathbb{C}^3/G$ is a family of $G$-constellations which extends to $X$ the family over the open subset of $\mathbb{C}^3/G$ parametrizing the free $G$-orbits. To give such a family is equivalent to choosing rays of $\sigma_V$ such that their images under $p : (N_V)_\mathbb{R} \to \mathbb{R}^3$ are the rays of the toric fan of $X$. The main result of [Y2] for $n = 3$
states that $X$ is realized as a fine moduli space of $G$-constellations (not-necessarily of the form $\mathfrak{M}_0$) if and only if $X$ admits a $G$-nat family $\mathcal{F}$ such that the maximal cones of the “associated” subfan $\Sigma_{X,\mathcal{F}}$ of $\Sigma_Y$ has the expected dimension 3 (see [Y2] §4 for details). Note that the results in [Y2] work for abelian $G \subset SL_n(\mathbb{C})$ with arbitrary $n$.

**Remark 3.4.** If we add redundant homogeneous generators (like $x_j^2 + x_{j'}$ if $\chi_j^2 = \chi_{j'}$) to the generators $x_1, \ldots, x_n$ of $\mathbb{C}[V]$, we obtain a strictly bigger toric variety $\bar{Y}_V$ than $V$.

**Example 3.2.** (Continuation of Example 3.1) We consider again the abelian $G$ treated in Example 3.1. In order to describe the normalization $\bar{V}$ of $V$ concretely, we fix coordinates $\theta_0, \ldots, \theta_9$ of $\theta \in \Theta$ so that $\theta_i = \theta(\rho^{\otimes i}) \in \mathbb{R}$ where $\rho: G \to \mathbb{C}^*$ is the one-dimensional representation defined by $\rho(g) = \zeta_{10}$. We identify $\Theta$ with $\mathbb{R}^9$ via $\theta \mapsto (\theta_1, \ldots, \theta_9)$ from now on. Note that $\sum_{i=0}^9 \theta_i = 0$. Then we see that the cone $\sigma_V$ admits 65 rays, whose images under $p: \mathbb{R}^3 \times \mathbb{R}^9 \to \mathbb{R}^3$ are equal to the rays $\mathbb{R}_{\geq 0} e_j (j = 1, 2, 3)$ and $\mathbb{R}_{\geq 0} v_k (k = 1, \ldots, 5)$, as expected.

In order to determine which rays of $\sigma_V$ correspond to exceptional divisors of $X_1$, recall that the $G$-Hilbert scheme is realized as $V//\theta_+ T\Theta$ for any $\theta_+ \in \Theta$ satisfying $(\theta_+)_i > 0$ for all $i > 0$. Note that GIT-chamber $C_1$ for the $G$-Hilbert scheme contains all such $\theta_+$. Using Proposition 3.6, we see that the rays generated by the following vectors in $\mathbb{R}^3 \times \mathbb{R}^9$ are the rays in $\Sigma_{\theta_+}$:

\[
\begin{align*}
    w_1 &= (1, 3, 6, -1, -2, -3, -4, -5, -6, -7, -8, -9) \\
    w_2 &= (1, 2, 3, -1, -2, -3, -4, -5, -1, -2, -3, -4) \\
    w_3 &= (2, 1, 2, -2, -4, -1, -3, -5, -2, -4, -6, -3) \\
    w_4 &= (1, 1, 0, -1, 0, -1, 0, -1, 0, -1) \\
    w_5 &= (7, 1, 2, -7, -4, -1, -8, -5, -2, -9, -6, -3).
\end{align*}
\]

Alternatively, one can determine the $w_i$’s by using the fact that the universal family for the $G$-Hilbert scheme is the *maximal shift family* in the sense of [La] §3.5. More explicitly, the $(3+j)$-th entry of $w_k$ is given as

\[-\min\{ad_{x,k} + bd_{y,k} + cd_{z,k} \mid a, b, c \in \mathbb{Z}_{\geq 0} \text{ s.t. } g \cdot x^a y^b z^c = \rho_j(g) x^a y^b z^c, \forall g \in G\} \quad (3.9)\]

where $d_{x,k}, d_{y,k}, d_{z,k} \in \mathbb{Z}_{\geq 0}$ are the exponents of $t_k$ for the generators in (3.2) associated to $x, y, z$ respectively.

Now we can explicitly describe $\varphi_{C_1}$ as explained in Remarks 3.2(1). If we embed $\mathrm{Pic}(X_1)$ into $\mathbb{Z}^5$ as we have been doing so far (cf. (3.2)), then the map $(\psi_{C_1})_R: \mathbb{R}^9 \to \mathbb{R}^5$ is given by the matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\
2 & 4 & 1 & 3 & 5 & 2 & 4 & 6 & 3 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
7 & 4 & 1 & 8 & 5 & 2 & 9 & 6 & 3
\end{pmatrix}.
\]

Note that the $(k, j)$-entry of this matrix equals the negative of the $(3+j)$-th entry of $w_k$. Indeed, a monomial $x^i y^b z^c$ which attains the minimum of (3.9) naturally defines an $j$-th semi-invariant which has weight $(0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^9$ and does not vanish along $E_k$. One can check that we have $(\psi_{C_1})_R(C_1) = \mathrm{Nef}(X_1)$.

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Now we can explicitly confirm that every crepant resolution of $\mathbb{C}^3/G$ is realized as $\mathcal{M}_\theta$ for some generic $\theta$, which is guaranteed by the main result of [CI]. As observed in Remarks 3.2(1), the map $\psi_C$ for a GIT-chamber $C$ coincides with $\psi_{C_1}$ as long as $C$ is contained in the intersection $C_+ := \bigcap_{k=1}^{5} C_{w_k}$ of the orbit cones

$$C_{w_k} = \text{Cone}\{\text{wt}(x_{i,j}) \in \Theta \mid w_k \subset \{x_{i,j} = 0\}\}.$$  

One sees that we have $(\psi_{C_1})_R(C_+) = \bigcup_{i=1}^{5} \text{Nef}(X_i)$, which implies that the interior of

$$C'_i := (\psi_{C_1})^{-1}_R(\text{Nef}(X_i)) \cap C_+ \quad (i = 1, \ldots, 5)$$

is a GIT-chamber $C'_i$ whose associated moduli space equals $X_i$. The resolution $X_6$ is also achieved as a moduli space by crossing a boundary of $C_+$. More precisely, one can check that the hyperplane $\{\theta_0 + \theta_1 + \theta_3 = 0\}$ forms a facet of both $C_+$ and $C_3$ and that the adjacent chamber $C'_3$ to $C_3$ regarding this facet corresponds to the ray of $\sigma V$ generated by

$$w'_3 = (2, 1, 2, -2, 1, -1, 2, 0, 3, 1, -1, 2).$$

Then the counterpart $C'_+ \subset C'_+$ obtained by replacing $C_3$ with $C'_3$ satisfies

$$(\psi_{C'_3})_R(C'_+) \supset \text{Nef}(X_6).$$

Therefore, we obtain a GIT-chamber $C_6 \subset C'_+$ such that $\mathcal{M}_{C_6} \cong X_6$.

3.5.2 $D_3$-singularity, continuation of Example 2.1

We next consider the case of $D_3$-singularity again (see Example 2.1). Note that the construction of embeddings in Subsection 3.3 is valid in dimension two since the moduli space for any generic $\theta$ is a crepant (or the minimal) resolution of $\mathbb{C}^2/G$ as well.

We use $x, y$ as the standard coordinates of $V = \mathbb{C}^2$ again. In terms of representations of the quiver $Q_G$ of Figure 1, the $G$-constellation family over $\mathbb{C}^2$ are given by the following generic matrices:

$$A_0 = \begin{pmatrix} x \\ y \end{pmatrix}, B_0 = \begin{pmatrix} -y & x \end{pmatrix}, A_1 = \begin{pmatrix} -iy \\ x \end{pmatrix}, B_1 = \begin{pmatrix} x & iy \end{pmatrix}, A_2 = \begin{pmatrix} x \\ -y \end{pmatrix}, B_2 = \begin{pmatrix} y & x \end{pmatrix},$$

$$A_3 = \begin{pmatrix} iy \\ x \end{pmatrix}, B_3 = \begin{pmatrix} x & -iy \end{pmatrix}, C = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, D = \begin{pmatrix} -y & 0 \\ 0 & x \end{pmatrix}$$

for the choice of basis of the vector spaces $\text{Hom}_{\mathbb{C}G}(V^* \otimes \rho, \rho')$ in Example 2.1. Here we regard the arrows of $Q_G$ as the corresponding matrices by abuse of notation. Note that these matrices satisfy the condition (2.5). We take $\text{Ab}(G)^\vee$-homogeneous elements

$$f_1 = x^3 + iy^3, \quad f_2 = xy, \quad f_3 = x^3 - iy^3$$

so that $\mathbb{C}[V]^{|G,G|} = \mathbb{C}[f_1, f_2, f_3]$. Note that $g \in G$ acts on $f_k$ via the character $\rho_k$. As is shown in [D], §6 and [Y1, Example 1], these $f_k$'s give rise to generators of the Cox ring of the minimal resolution $X$ of $V/G$. We will consider the embedding of $\mathcal{W}$ into a toric variety with respect to these $f_k$'s.
Let us consider the semi-invariants obtained as the determinants of the following 42 square (block) matrices:

\[
C, D, B_2A_0, B_0A_2, B_3A_1, B_1A_3, (A_0 \quad A_2), (A_1 \quad A_3), (B_1 \quad B_3), B_1CA_0, B_3CA_0,
B_1CA_2, B_3CA_2, B_0DA_1, B_2DA_1, B_0DA_3, B_0DA_3, (CA_0 \quad A_1), (CA_0 \quad A_3),
(A_2 \quad A_1), (CA_2 \quad A_3), (A_0 \quad DA_1), (A_0 \quad DA_3), (A_2 \quad DA_1), (A_2 \quad DA_3),
\]

These semi-invariants are chosen so that Condition (3.11) below is satisfied. (See Remark 3.6.) We will see that these semi-invariants give a generating system of \(\mathbb{C}[W] = \mathbb{C}[V]^{SL_2}\). Although it might be possible to check this directly (with help of a computer) by using algorithms such as the one in [DeK, 4.1], we adopt another approach.

In order to construct \(Y_W\) explicitly, we use coordinates \(\theta_0, \ldots, \theta_5\) of \(\theta \in \Theta\) such that \(\theta_i = \theta(\rho_i)\) for \(i = 0, 1, 2, 3\) and \(\theta_i = \theta(V_{-3})\) for \(i = 4, 5\). We then identify \(\Theta\) with \(\mathbb{R}^5\) via \(\theta \mapsto (\theta_1, \ldots, \theta_5)\). Note that we have \(\theta_0 + \theta_1 + \theta_2 + \theta_3 + 2\theta_4 + 2\theta_5 = 0\) in this case. Each of the 42 semi-invariants above is mapped to one of \(f_i\)'s under \(\varphi\) up to constant multiplication, and we construct an embedding \(W \hookrightarrow Y_W\) with respect to \(f_i\)'s. For example, the associated vector in \(M_W \subset \mathbb{Z}^{3+5}\) to the semi-invariant for the matrix \(C\) (or the corresponding arrow in \(Q_c\)), is given as \((0, 1, 0, 0, 0, 0, -1, 1)\) since \(\det C = f_2\) and the weight \(\theta \in \Theta\) of this semi-invariant satisfies \(\theta_1 = \theta_2 = \theta_3 = 0, \theta_4 = -1, \theta_5 = 1\) (see (2.4)). Similarly, the associated vector to \(\begin{pmatrix} B_2 & 0 \\ C & A_3 \end{pmatrix}\) is given as \((1, 0, 0, 0, 1, -1, -1, 1)\).

Let \(\sigma' \subset \mathbb{R}^{3+5}\) be the dual cone to the cone spanned by the vectors in \(M_W\) associated to the above semi-invariants. We will show that \(\sigma'\) equals the cone \(\sigma_W \subset \mathbb{R}^8\) defining \(Y_W\). Let \(W'\) be the spectrum of the subalgebra of \(\mathbb{C}[W]\) generated by the 42 semi-invariants above. Then we obtain an embedding of \(W'\) into a toric variety \(Y'_W\) whose normalization is defined by \(\sigma'\), similar to the construction of \(W \hookrightarrow Y_W\). In fact one can check that \(Y'_W\) is normal by comparing \(\sigma' \cap M_W\) and the semi-group generated by the elements of \(M_W\) associated to the 42 semi-invariants. This computation can be done e.g. by using the package “Normaliz” of Macaulay2 [GS].

We see, by calculation, that \(\sigma'\) admits 165 rays. The primitive generators in \(\mathbb{Z}^8\) of these rays include the first three coordinate vectors \(e_1, e_2, e_3\), and the other 162 vectors are divided into five types by looking at their first three coordinates:

\[
v_1 = (1, 2, 1), \ v_2 = (1, 1, 1), \ v_3 = (3, 2, 3), \ v_4 = (3, 2, 5), \text{ or } v_5 = (5, 2, 3). \quad (3.10)
\]

These five types correspond to the prime exceptional divisors \(E_1, \ldots, E_5\) of the minimal resolution \(X\) defined by the conjugacy classes of \(g_1, g_2^2, g_3^2, g_2, g_1g_2\) respectively (cf. [Y1 Example 1]). More precisely, the rays for \(E_1, \ldots, E_5\) in the fan of the ambient toric variety \(Y\) of \(X\) (with respect to \(f_i\)'s) are generated by the vectors in (3.10). Note that \(\sigma'\) particularly satisfies the following condition:

the polyhedral cone \(p^{-1}(\mathbb{R}_{\geq 0}v_k) \cap \sigma'\) is generated by rays of \(\sigma'\) for each \(k\). \quad (3.11)
This will be important when one computes a generating system of semi-invariants efficiently (see Remark 3.6).

The fact that the rays of $\sigma'$ are projected to those of the fan of $X$ implies that $Y_W//_\theta T_\Theta$ is isomorphic to the toric variety $Y$ for any generic $\theta \in \Theta$. Therefore, we have $\sigma_W = \sigma'$, and the natural toric morphism $Y_W \to Y'_W$ is an isomorphism. This also implies that the 42 semi-invariants generate $\mathbb{C}[W]$.

Since the $G$-Hilbert scheme is given by a stability condition $\theta_+$ satisfying $(\theta_+)_i > 0$ for all $i > 0$, we see that generators of the rays of $\sigma'_W$ corresponding to the exceptional divisors $E_1, \ldots, E_5$ of the $G$-Hilbert scheme are given by

$$
w_1 = (1, 2, 1, -1, -2, -1, -2, -2), \quad w_2 = (1, 1, 2, -1, -1, -1, -2, -2),$$
$$
w_3 = (3, 2, 3, -3, -2, -3, -4, -6), \quad w_4 = (3, 2, 5, -3, -2, -5, -4, -6),$$
$$
w_5 = (5, 2, 3, -5, -2, -3, -4, -6)
$$

respectively. From these data, one can also explicitly compute the map $(\psi_C)_R : \mathbb{R}^5 \to \mathbb{R}^5$ associated to the GIT-chamber $C$ containing $\theta_+$.

**Remark 3.5.** In [Le, §2], the $G$-Hilbert schemes $X$ for binary dihedral groups are studied, and in particular the author introduced distinguished $G$-clusters which give rise to open affine covering of $X$. For example, the distinguished $G$-cluster $F$ of type $B$ is defined by an homogeneous ideal of $\mathbb{C}[x, y]$ and is presented as the following $\mathbb{C}$-vector subspace:

$$
\mathbb{C} \oplus (C x \oplus C y) \oplus C x y \oplus (C x^2 \oplus C y^2) \oplus C x^3 \oplus C y^3 \oplus (C x^4 \oplus C y^4) \oplus (C x^5 \oplus C y^5).
$$

Its structure as a $\mathbb{C}[V]$-module is also presented as in the following diagram:

$$
\begin{array}{c}
\rho_0 \\
V_1 \downarrow \\
\rho_2 \\
\vdots \\
\rho_3 \downarrow \\
V_2 \\
\vdots \\
\rho_1 \downarrow \\
V_2 \rightarrow V_1
\end{array}
$$

where each arrow $\rho \to \rho'$ represents a nontrivial action of $V^*$ on $\rho$ into $\rho'$ similarly to the McKay quiver. One sees that, among the 42 semi-invariants, the ones $h_1, \ldots, h_9$ for the matrices

$$
B_1 C A_0, B_2 A_0, B_3 C A_0, (C A_0 \ A_1), (C A_0 \ A_3), (A_0 \ D A_1), (A_0 \ D A_3), \begin{pmatrix} A_0 & D \\ 0 & B_1 \end{pmatrix}, \begin{pmatrix} A_0 & D \\ 0 & B_3 \end{pmatrix}
$$

do not vanish at $F$, and thus the face of $\sigma_W$ corresponding to $F$ is given as the intersection $\sigma_W \cap \{w_{h_1} = \cdots = w_{h_9} = 0\}$ where $w_{h_i} \in M_W$ is the associated element to $h_i$. This is equal to $\text{Cone}(w_1, w_2)$, and thus $F$ corresponds to the the intersection point of the two curves $E_1$ and $E_2$. 

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3.5.3 Trihedral group of order 21

Here we deal with the following three-dimensional example:

\[ G = \left\{ g_1 = \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7^4 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} \subset SL_3(\mathbb{C}) \]

where \( \zeta_7 \) is the seventh root of unity. Then \( \text{Irr}(G) \) consists of three 1-dimensional representations \( \rho_k : G \rightarrow \mathbb{C}^* \) \((k = 0, 1, 2)\) defined by \( \rho_k(g_1) = 1 \), \( \rho_k(g_2) = \omega^k \) where \( \omega \) is the third root of unity, and two 3-dimensional representations \( V_1, V_2 \), the first one is the inclusion \( G \subset SL_3(\mathbb{C}) \) presented above and the other is presented as

\[ g_1 \mapsto \begin{pmatrix} \zeta_7^3 & 0 & 0 \\ 0 & \zeta_7^6 & 0 \\ 0 & 0 & \zeta_7^5 \end{pmatrix}, g_2 \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \]

We again fix the basis obtained in this way for the subsequent computations.

The McKay quiver \( Q_G \) for this case is as follows:

![McKay quiver diagram]

With the standard coordinates \( x, y, z \) of \( V = \mathbb{C}^3 \), the generic matrices are given as

\[ A_0 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B_0 = \begin{pmatrix} z & x & y \end{pmatrix}, A_1 = \begin{pmatrix} x \\ \omega y \\ \omega^2 z \end{pmatrix}, B_1 = \begin{pmatrix} \omega z & x & \omega^2 y \end{pmatrix}, \]

\[ A_2 = \begin{pmatrix} x \\ \omega^2 y \\ \omega z \end{pmatrix}, B_2 = \begin{pmatrix} \omega^2 z & x & \omega y \end{pmatrix}, C_1 = \begin{pmatrix} 0 & 0 & x \\ 0 & y & 0 \\ z & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} y & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & x \end{pmatrix}, \]

\[ D = \begin{pmatrix} 0 & y & 0 \\ 0 & 0 & z \\ x & 0 & 0 \end{pmatrix}, L_1 = \begin{pmatrix} 0 & z & 0 \\ x & 0 & 0 \\ 0 & y & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & x \\ 0 & y & 0 \end{pmatrix} \]

for some choice of basis \( \text{Hom}_{\mathbb{C}G}(\rho \otimes \mathbb{C} V^*, \rho') \) with \( \rho, \rho' \in \text{Irr}(G) \).

In this case, it seems hard to obtain generators of semi-invariants directly using the algorithm in [DeK, Ch. 4] since the input data are large. Instead we give a candidate of a system of generators and then confirm the correctness as we did in the previous example.

As computed in [DG, §5.1], there are \( Ab(G)^\vee \)-homogeneous polynomials \( f_1, \ldots, f_{13} \in \mathbb{C}[V]^{G,G} \) such that the associated elements

\[ \tilde{f}_1, \ldots, \tilde{f}_{13}, t_1^{-7}, t_2^{-3}, t_3^{-3} \in \mathbb{C}[V]^{G,G}[t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1}] \]

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generate the Cox ring of a crepant resolution $X$ of $V/G$, where we set the variables $t_1, t_2, t_3$ so that they correspond to junior elements $g_1, g_2, g_3^2$ respectively (see Subsection 3.1).

Similarly to the previous examples, we realize the cone $\sigma_W$ inside $\mathbb{R}^{13+5-1}$ by ordering the elements of $\text{Irr}(G)$ as $\rho_0, \rho_1, \rho_2, V_1, V_2$. Regarding the embedding $\iota_X : X \hookrightarrow Y$ with respect to $f_1, \ldots, f_{13}$, the generators of the rays in $\mathbb{R}^{13}$ corresponding to the exceptional divisors $E_1, E_2, E_3$ are given as

$$v_1 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1),$$
$$v_2 = (0, 0, 2, 1, 0, 2, 1, 3, 2, 1, 3, 2),$$
$$v_3 = (0, 0, 1, 2, 0, 1, 2, 3, 1, 2, 3, 4, 2)$$

respectively. Then we define $\sigma' \subset \mathbb{R}^{17}$ as the cone generated by the coordinate vectors $e_1, \ldots, e_{13}$ and the following 16 vectors:

$$v_{1,1} = (v_1, -1, -1, -1, -2), \quad v_{1,2} = (v_1, 0, -1, -1, -2), \quad v_{1,3} = (v_1, -1, 0, -1, -2),$$
$$v_{1,4} = (v_1, 0, 0, -1, -2), \quad v_{1,5} = (v_1, 0, 0, -1, 0), \quad v_{1,6} = (v_1, 0, 0, 0, 1),$$
$$v_{1,7} = (v_1, 0, 0, 2, 1), \quad v_{1,8} = (v_1, 0, 1, 2, 1), \quad v_{1,9} = (v_1, 1, 0, 2, 1), \quad v_{1,10} = (v_1, 1, 1, 2, 1),$$
$$v_{2,1} = (v_2, -1, -2, -3, -3), \quad v_{2,2} = (v_2, -1, 1, 0, 0), \quad v_{2,3} = (v_2, 2, 1, 3, 3),$$
$$v_{3,1} = (v_3, -2, -1, -3, -3), \quad v_{3,2} = (v_3, 1, -1, 0, 0), \quad v_{3,3} = (v_3, 1, 2, 3, 3).$$

By computing $\sigma'$ explicitly (with help of Macaulay2 [GS]), one sees that it has 400 facets and accordingly we obtain 400 primitive elements of $M_W$ defining these facets. In fact we can take determinantal semi-invariants corresponding to these primitive elements. One can also check that the toric variety $Y'_W$ constructed from these elements is already normal again. Since the rays of $\sigma'$ are projected to those of the fan of $X$, we have that $\sigma' = \sigma_W$ again and that the associated 400 semi-invariants generate $\mathbb{C}[W]$.

**Remark 3.6.** In general, a candidate $\sigma'$ of $\sigma_W$ can be given by computing determinantal semi-invariants $h$ (from ones with $\varphi(h)$ having smaller degrees) until the dual cone

$$\sigma'' := \bigcap_h \{v \in \mathbb{R}^\ell \oplus \Theta^\vee \mid w_h(v) \geq 0\}$$

satisfies the condition (3.11). This may be achieved by computing relatively small number of semi-invariants (compared to the number of generators of $\mathbb{C}[W]$). Then the candidate $\sigma'$ is defined as the cone generated by the rays of $\sigma''$ which are projected to the rays of the fan of $X$.

In the above case, such $\sigma''$ is obtained from determinantal semi-invariants coming from $3 \times 3$-matrices only, while the 400 generators of $\mathbb{C}[W]$ include ones coming from $6 \times 6$-matrices. One may also verify that $\sigma_W = \sigma'$ by checking the existence of $G$-constellations corresponding to rays of $\sigma'$, as will be done below.

Once we know the cone $\sigma_W$, one can investigate the structures of $G$-constellations presented by faces of $\sigma_W$. As a demonstration, we take the ray generated by $v_{1,1}$. If we
take the generators $v_{1,1}, \ldots, v_{1,10}$ of $\sigma_W$ whose initial coordinates vector is $v_1$, the orbit cone $C_{v_{1,j}} \subset \Theta$ of the corresponding divisor to $v_{1,j}$ is given as the cone

$$\{ \theta \in \Theta \mid q(v_{1,j}) \cdot \theta = \min_{j=1, \ldots, 10} q(v_{1,j}) \cdot \theta \}$$

by Proposition 3.6. Note that $\{C_{v_{1,j}}\}_{j=1, \ldots, 10}$ give a subdivision of $\Theta$ into polyhedral cones. The orbit cone $C_{v_{1,1}} \subset \Theta$ is defined by the following 5 inequalities:

$$\theta_1 \geq 0, \theta_2 \geq 0, \theta_1 + \theta_2 + 2\theta_4 \geq 0, \theta_1 + \theta_2 + 3\theta_3 + 3\theta_4 \geq 0, \theta_1 + \theta_2 + 3\theta_3 + 3\theta_4 \geq 0.$$

Notice that the coefficients of these inequalities are obtained as the direction vectors $-v_{1,1}v_{1,j}$ for $j$ such that the line segment $v_{1,1}v_{1,j}$ forms an edge of the convex hull $\text{Conv}(v_{1,1}, \ldots, v_{1,10})$. One can infer the structure of the $G$-constellation $F$ of a general point of $E_1$ from these inequalities. In this case the structure of $F$ is presented as the following diagram for some decomposition $F = \rho_0 \oplus \rho_1 \oplus \rho_2 \oplus V_1^{\oplus 3} \oplus V_2^{\oplus 3}$.

As another example, the orbit cone $C_{v_{1,6}}$ for $v_{1,6} = (v_1, 0, 0, 1)$ is defined by

$$\theta_3 \geq 0, \theta_1 + 2\theta_3 \geq 0, \theta_2 + 2\theta_3 \geq 0, \theta_1 + \theta_2 + 2\theta_3 \geq 0,$$

$$\theta_3 + \theta_4 \leq 0, \theta_1 + \theta_3 + 3\theta_4 \leq 0, \theta_2 + \theta_3 + 3\theta_4 \leq 0, \theta_1 + \theta_2 + \theta_3 + 3\theta_4 \leq 0$$

and the structure of the $G$-constellation $F$ for $v_{1,6}$ is presented as

As a more explicit description, $F$ can be realized as a quotient of the ideal of $\mathbb{C}[V]$ generated by $xy, yz$ and $zx$. Note that $\mathbb{C}xy \oplus \mathbb{C}yz \oplus \mathbb{C}zx$ is isomorphic to $V_2$ as a $\mathbb{C}G$-module.

Finally, we compute the homomorphism $\psi_C$ for certain chambers $C$ and see that every projective crepant resolution of $V/G$ is obtained as a moduli space of $G$-constellations. Let $C_{i,j} \subset \Theta$ be the orbit cones corresponding to the rays of $\sigma_W$ generated by $v_{i,j}$. There are 42 choices of a triple $(i, j, k)$ such that $C_{1,i} \cap C_{2,j} \cap C_{3,k}$ has the full dimension 4.
Among them, \(C_+ := \overline{C}_{1,1} \cap \overline{C}_{2,1} \cap \overline{C}_{3,1}\) is the one containing the GIT-chamber \(C_{++}\) for the \(G\)-Hilbert scheme. \(C_{++}\) is in fact defined by the inequalities \(\theta_i > 0\) for some generic \(\theta \in C_+\). In this case the movable cone \(\text{Mov}(X)\) is a three-dimensional simplicial cone divided into four nef cones (cf. [DG, Proposition 5.2]). The nef cone for the \(G\)-Hilbert scheme is given as \(\psi_{C_{++}}(\overline{C}_{++})\) and we see that it is the central one which is strictly smaller than \(\text{Mov}(X)\) for the cone \(C = \overline{C}_{1,2} \cap \overline{C}_{2,1} \cap \overline{C}_{3,1}\) adjacent to \(C_+\) and any GIT-chamber \(C' \subset C\).

### 4 Proof of the main result

In this section we give a proof of the following theorem:

**Theorem 4.1.** ([Theorem 1.1]) Let \(G \subset SL_3(\mathbb{C})\) be a finite subgroup and \(X \to \mathbb{C}^3/G\) a projective crepant resolution. Then there is a generic stability condition \(\theta \in \Theta\) such that \(X \cong \mathcal{M}_\theta\).

To prove Theorem 4.1, we will use a similar argument to the one in [CI, §8]. As explained in Introduction, the strategy is to show that, starting from any GIT-chamber \(C \subset \Theta\) and the associated moduli space \(\mathcal{M}_C\), one can reach a wall in \(\Theta\) which induces a given flop \(\mathcal{M}_C \to X'\) by crossing walls of a certain type. Here we say that the intersection \(W = \overline{C} \cap \overline{C}'\) for GIT-chambers \(C, C' \subset \Theta\) is a wall (of \(C\)) if \(W\) is a codimension-one face of \(C\) (and hence of \(C'\)). To carry out the strategy, we should know how the tautological bundle \(\mathcal{R}_C\) of \(\mathcal{M}_C\) changes when we cross a wall in \(\Theta\).

For any given GIT-chamber \(C \subset \Theta\), we choose any general point \(\theta_0\) of a wall \(W \subset C\). Then we have \(\mathcal{W}^{C-\text{ss}} \subset \mathcal{W}^{\text{ss}}\) and this inclusion induces a morphism \(\alpha : \mathcal{M}_C \to \mathcal{M}_{\theta_0}\) over \(\mathcal{M}_0 = \mathbb{C}^3/G\) where \(-\text{ss}\) denotes the normalization. Similarly to the case of \(\mathcal{M}_C\), we denote the moduli space \(\mathcal{M}_{\theta_0}\) by \(\mathcal{M}_W\). We also denote the crepant resolution \(\mathcal{M}_C\) by \(X\) as in previous sections.

Recall that each line bundle \(L \in \text{Pic}(X)\) defines a character \(\chi_L \in \chi(T_X)\) and its associated GIT-quotient \(X_L := \mathcal{X}/\chi_L T_X\) with a sufficiently divisible \(k > 0\). As observed in Subsection 3.2, \(X_L\) is the same as the image of the rational map obtained by the complete linear system of (some power of) \(L\). Note that \(X_0 = \mathbb{C}^3/G\) and that \(X_L\) makes sense for \(L \in \text{Pic}(X)\). The following lemma shows that one can determine \(\alpha : \mathcal{M}_C \to \mathcal{M}_W\) by looking at the homomorphism \(\psi_C : \Theta \to \text{Pic}(X)\) in (3.5).

**Lemma 4.2.** (cf. [CI, Lemma 3.3]) For any point \(\theta \in \overline{C}\), the normalized GIT-quotient \(\mathcal{M}_\theta\) is isomorphic to \(X_{\psi_C(\theta)}\).

**Proof.** Recall that \(X = \mathcal{M}_C\) is regarded as a GIT-quotient of \(\mathcal{X} = \text{Spec Cox}(X)\) via the restriction \(\mathcal{X}^{\psi_C(\text{ss})} \to \mathcal{W}^{\psi_C(\text{ss})}\) of the equivariant (affine) morphism \(\varphi_C : \mathcal{X} \to \mathcal{W}\) obtained from \(\varphi_C\) (see Subsection 3.3). Since \(\theta \in \overline{C}\), the line bundle \(L := \psi_C(\theta)\) induces a restriction \(\mathcal{X}^{L-\text{ss}} \to \mathcal{W}^{\text{ss}}\) of \(\varphi_C^*\) which extends \(\mathcal{X}^{\psi_C(\text{ss})} \to \mathcal{W}^{\psi_C(\text{ss})}\), by the construction of \(\psi_C\). Then the composition \(\mathcal{X}^{\psi_C(\text{ss})} \hookrightarrow \mathcal{X}^{L-\text{ss}} \to \mathcal{W}^{\text{ss}}\) induces a morphism \(X \to \mathcal{M}_\theta\) and
$L$ is just the pullback of the ample line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$ (induced by the GIT-construction) under this morphism. Thus, the claim follows since $X \to \mathcal{M}_0$ is identified with the morphism coming from the linear system of $L$. Note in particular that (a multiple of) $L$ is globally generated.

The following technical lemma will be an important step for the proof of the main theorem.

**Lemma 4.3.** Let $F$ be a face of an orbit cone $\tilde{C}_x \subset \Theta$ of the $G$-constellation corresponding to a point $x \in \mathcal{M}_C$. If the relative interior $\text{relint}(F)$ of $F$ intersects $\overline{C}$ and if $\psi_C(F) \subset \text{Pic}(X)_R$ is not of full dimension, then $\psi_C(F)$ is on the boundary of the orbit cone $C_x \subset \text{Pic}(X)_R$ of a point of $\mathcal{X}$ which is a lift of $x \in X$.

In particular, in this case, any point $\theta_0$ of $\text{relint}(F) \cap \overline{C}$ induces a nontrivial birational contraction $X \to \mathcal{M}_{\theta_0}$ such that every point $x' \in X$ whose orbit cone in $\Theta$ is equal to $\tilde{C}_x$ lies in the exceptional locus.

**Proof.** We first choose homogeneous generators $h_1, \ldots, h_{\ell}$ of $\mathbb{C}[\mathcal{W}] = \mathbb{C}[\mathcal{V}]^{SL_R}$ so that we have an associated embedding $\mathcal{W} \hookrightarrow Y_\mathcal{W}$ and its quotient $X \hookrightarrow Y$ as in Theorem 3.5. Let $\bar{x} \in \mathcal{W}$ be any lift of $x$. By reordering $h_i$'s, we may assume that

1. there exists $\ell_1 < \ell$ such that $h_1, \ldots, h_{\ell_1}$ do not vanish at $\bar{x}$ and their weights generate the cone $F$ and that
2. for any $h_k$ with $k > \ell_1$, either $h_k$ vanishes at $\bar{x}$ or $\text{wt}(h_k)$ is not in $F$.

The locally closed subset of $\mathcal{W}$ consisting of points having $F$ as their orbit cones is given by

$$W_F := \mathcal{W} \cap \{h_1, h_2, \ldots, h_{\ell_1} \neq 0\} \cap \{h_{\ell_1+1} = \cdots = h_{\ell} = 0\}.$$ 

Note that torus-orbits of the toric varieties $\mathcal{Y}$, $\mathcal{A}^{\ell+m}$, and $\mathcal{Y}_\mathcal{W}$ restrict to stratification of $X$, $\mathcal{X}$, and $\mathcal{W}$ respectively, and that $W_F$ is such a stratum of $\mathcal{W}$.

By construction, the equivariant map $\varphi^*_C : \mathcal{X} \to \mathcal{W}$ associated to $C$ is the restriction of the toric morphism $\mathcal{A}^{\ell+m} \to Y_\mathcal{W}$ which is obtained by sending the rays $\tilde{\tau}_k$ of the fan of $\mathcal{A}^{\ell+m}$ corresponding to the exceptional divisors $E_k$ to the rays $\tau_{C,k}$ of $\sigma_W$ corresponding to $E_k$ in $\mathcal{M}_C$. For $\theta_0 \in \text{relint}(F) \cap \overline{C}$, the restriction $X^{\psi_C(\theta_0)}_{\text{ss}} \to W^\theta_{\text{ss}}$ of $\varphi_C^*$ descends to an isomorphism $X_{\psi_C(\theta_0)} \to \mathcal{M}_{\theta_0}$ by Lemma 4.2. Note that $W_F$ is a closed subset of $W^\theta_{\text{ss}}$ which intersects the closure of the $T_\Theta$-orbit of $\bar{x}$. In particular, the categorical quotient of $W_F$ by $T_\Theta$ is again a closed subset in $\mathcal{M}_{\theta_0}$.

We “perturb” the map $\varphi^*_C$ to obtain another equivariant map $\varphi^{*\prime} : \mathcal{X} \to \mathcal{W}$ such that its restriction to the semistable loci still descends to an isomorphism $X_{\psi_C(\theta_0)} \to \mathcal{M}_{\theta_0}$ and that every point of $W_F \subset W^\theta_{\text{ss}}$ is inside a $T_\Theta$-orbit of a point in the image of $\varphi^{*\prime}$. Such $\varphi^{*\prime}$ is explicitly constructed as follows. For each $k = 1, \ldots, m$, let $\sigma_k$ be the cone (contained in $\sigma_W$) generated by $\tau_{C,k}$ and the rays $\tau$ of $\sigma_W$ (if any) such that $\tau$ is projected to $\tau_k$ under $p : (N_\mathcal{W})_R \to (N_\mathcal{W}/\Theta^\vee)_R$ (see Subsection 3.4) and that the orbit cone of $\tau$ contains $F$ as a face. Note that $\sigma_k$ is equal to $\overline{\tau_{C,k}}$ if $\tau_{C,k} \not\subset \tau$ and otherwise $\sigma_k$ is the intersection $\sigma_F \cap p^{-1}(\tau_k)$ where $\sigma_F$ is the face of $\sigma_W$ corresponding to the stratum $W_F$. Then $\varphi^{*\prime}$ is obtained as the restriction of the toric morphism $\mathcal{A}^{\ell+m} \to Y_\mathcal{W}$ obtained by sending each ray $\tilde{\tau}_k$ to a ray generated by a general point of $\sigma_k$. Note that $\varphi^{*\prime}$ indeed induces an isomorphism $X_{\psi_C(\theta_0)} \to \mathcal{M}_{\theta_0}$ since the projected cone $p(\sigma)$ of the image $\sigma$ of
each cone of the fan of $X^{\psi_C(\theta_0)-ss}$ under the induced homomorphism $\mathbb{R}^{\ell+m} \to \mathbb{R}^{\ell+s-1}$ by $\varphi'_C$ remains the same even if we replace $\varphi'_C$ by $\varphi'^*$. The map $\varphi'^*$ gives an associated homomorphism $\psi' : \Theta \to \text{Pic}(X)_\mathbb{R}$ similarly to $\psi_C$. More concretely, for a homogeneous semi-invariant $f \in \mathbb{C}[\mathcal{W}]$ of weight $c \in \Theta$, $\psi'(c)$ is the class of the sum of the strict transform of the image in $V/G$ of $\{f = 0\}$ and $\sum_k v_k(f) E_k$ where $v_k$ is the primitive element in $\sigma_k$ defining $\varphi'^*$ (cf. Remarks 3.1 and 3.2(1)).

One sees that $\psi'|_F = a\psi_C|_F$ for some constant $a \in \mathbb{Q}$ and in particular $\psi'(F)$ is not full-dimensional by the assumption. Let $\varphi' : \mathbb{C}[\mathcal{W}] \to \mathbb{C}[\mathcal{X}]$ be the ring homomorphism associated to $\varphi'^*$. The equivariant map $\varphi'^*$ restricts to a map from the nonempty (affine) locally closed subset

$$X_F := (\varphi'^*)^{-1}(\mathcal{W}_F) = \mathcal{X} \cap \{\varphi'(h_1 h_2 \cdots h_{\ell_1}) \neq 0\} \cap \{\varphi'(h_{\ell_1+1}) = \cdots = \varphi_C(h_{\ell}) = 0\}$$

of $\mathcal{X}$ to $\mathcal{W}_F$, which descends to an isomorphism between the (categorical) quotients by $T_X$ and $T_0$.

We assume that $\psi'(F)$ does not lie on the boundary of $C_x$ to deduce a contradiction. Then the quotients of $X_F$ and $\mathcal{W}_F$ are isomorphic to the stratum $S_x$ of $X$ containing $x$. If $X_F$ itself is a stratum of $X$, then it must be the set $X_x$ of points of $X$ whose orbit cones are equal to $C_x$. The fact that $\psi'(F)$ is not of full-dimension implies that every point of $X_x$ has a positive-dimensional stabilizer in $T_X$, which is a contradiction since $C_x$ is full-dimensional. If there exists a stratum $X_x' \subsetneq X_F$ not equal to $X_x$, then this is again a contradiction since the orbit cone of such $X_{x'}$ is a proper face of $C_x$ and in particular $X_{x'}$ cannot be mapped to $S_x$ under the quotient map. Note that proper faces of an orbit cone of $X$ always correspond to nontrivial contractions, in contrast to faces of orbit cones in $\Theta$ such as walls of Type 0 introduced below. The last claim also follows from this observation. \qed

If a wall $W = \overline{C} \cap \overline{C'}$ is not contained in the pullback under $\psi_C : \Theta \to \text{Pic}(X)_\mathbb{R}$ of a GIT wall for the action of $T_X$ on $X$, then the morphism $\mathcal{M}_C \to \mathcal{M}_W$ is an isomorphism by Lemma 4.2 and we call such a wall of Type 0 following [CI]. Conversely, for any wall $W$ whose image under $\psi_C$ is not of full dimension, the morphism $\mathcal{M}_C \to \mathcal{M}_W$ is a non-trivial contraction of either a divisor or a curve by Lemma 4.3. For a wall $W$ of Type 0, the two moduli spaces $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ are isomorphic but the tautological bundle changes. In fact $\mathcal{R}_{C'}$ is a modification of $\mathcal{R}_C$ by tensoring by a line bundle to a subbundle of $\mathcal{R}_C$ similar to the abelian cases [CI, Corollary 4.2], as we will see below.

In order to study what happens when we cross a wall $W$ of Type 0, we first see that the unstable locus

$$D_W = \{x \in X \mid \text{any lift } \bar{x} \in \mathcal{W} \text{ of } x \text{ is not } W\text{-stable}\} \subset X$$

with respect to $W$ is a compact divisor as a topological space. Later we will see that $D_W$ has a natural scheme structure as a certain moduli space and show that $D_W$ is reduced with respect to this scheme structure in Lemma 4.6. We will also show that $D_W$ is connected in Proposition 4.13.
Lemma 4.4. For any given GIT-chamber $C \subset \Theta$ and its wall $W$ of Type 0, the unstable locus $D_W \subset X$ associated to $W$ is a compact divisor.

Proof. We first show that $D_W$ is a divisor. We fix an embedding $W \hookrightarrow Y_W$ again, and let $\Sigma_\theta$ be the subfan of $\sigma_W$ defining $Y^\theta_{wss}$ for $\theta \in \Theta$ (see Subsection 3.4). We also consider the ambient toric variety $Y$ of $X$ and let $\sigma_x$ be the cone of the fan $\Sigma_X$ of $Y$ corresponding to a point $x \in X \subset Y$.

For any point $x \in D_W$, there exists a unique cone $\bar{\sigma}_x \in \Sigma_\theta$ for $\theta \in C$ such that $p(\bar{\sigma}_x) = \sigma_x$. Since $\mathcal{M}_C \rightarrow \mathcal{M}_W$ is an isomorphism, we have a cone $\bar{\sigma}_x \in \Sigma_{\theta_0}$ strictly containing $\sigma_x$ (as a facet) such that it satisfies $p(\bar{\sigma}_x) = \sigma_x$ where $\theta_0$ is a general point of $W$. Recall that the maximal cones of the fan of $\mathcal{W}^\theta_{wss}$ are projected under $p$ to the maximal cones of $\Sigma_X$ (see (3.4)). Since the image of any ray of $\sigma_W$ under $p$ is again a ray of $\Sigma_X$, there must be a ray $\tau$ of $\bar{\sigma}_x$ and a face $\bar{\tau} \subset \bar{\sigma}_x$ such that $\bar{\tau} \supseteq \tau$ and $p(\bar{\tau}) = p(\bar{\tau})$. This implies that $x$ is contained in the unstable divisor of $X$ corresponding to $p(\bar{\tau})$. Thus, $D_W$ is of pure codimension one.

For the compactness of $D_W$, one may apply exactly the same argument in [CI, Proposition 4.4].

Now we describe how $G$-constellations over unstable points change when we cross the wall $W$. We first observe that there is a unique maximal $\mathbb{C}G$-submodule $R_+$ of the regular representation $R$ of $G$ which satisfies $\theta_0(R^+) = 0$ and $\theta(R^+) > 0$ for a general $\theta_0 \in W$ and $\theta \in C$. For a $G$-constellation $F$ corresponding to any point $x \in D_W \subset X = \mathcal{M}_C$, we define $S \varsubsetneq F$ as the unique nonzero minimal $\mathbb{C}[V]$-submodule such that $\theta_0(S) = 0$ if $R_+$ is primitive in $R(G) = \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}\rho$ and otherwise we define $S \varsubsetneq F$ as the unique (nonzero) maximal $\mathbb{C}[V]$-submodule such that $\theta_0(S) = 0$. We also let $Q$ be the quotient of $F$ by $S$. Note that $R$ is primitive as an element of $R(G)$ and thus at least $R_+$ or its complement in $R$ is primitive in $R(G)$.

In abelian cases every irreducible representation of $G$ has multiplicity one in $R$ and thus the modules $S$ and $Q$ are clearly indecomposable by the $\theta_0$-semistability. When $G$ is non-abelian and either $S$ or $Q$, say $S$, is non-primitive in $R(G)$, the module $S$ may be decomposable in general. Even if $S$ is indecomposable, the $\theta_0$-polystable module $\bar{F}$ of $Q_G$ (i.e. the direct sum of $\theta_0$-stable modules) corresponding to the image of $x$ under $\mathcal{M}_C \rightarrow \mathcal{M}_W$ may not be isomorphic to the direct sum $S \oplus Q$. $S$ may degenerate in $\bar{F}$ to decompose nontrivially into a direct sum $\bigoplus_i \bar{S}_i$ satisfying $\theta_0(\bar{S}_i) = 0$.

We will identify $D_W$ with a certain moduli space $Z_Q$ which plays the same role as the moduli space $Z$ introduced in the proof of [CI, Lemma 3.10]. $Z_Q$ is constructed as follows. We first choose an irreducible component $D$ of $D_W$, and take its general point and the corresponding $\mathbb{C}[V]$-modules $S, Q$ as above. We then fix a direct summand $R_2$ of $R$ such that $R_2 \cong Q$ as a $\mathbb{C}G$-module. Similarly to the construction of $N$ in Subsection 2.1 we define $N_Q$ to be the space of $G$-equivariant $\mathbb{C}[V]$-module structures on $R$ such that $R_2$ is a quotient $\mathbb{C}[V]$-module. More precisely, if we regard $N$ as the space of quiver representations (see Subsection 2.3) and fix isotypic decompositions of $R_2$ and its complement $R_1 \cong S$ in $R$, then $N_Q$ is defined as the closed subscheme of $N$ given by vanishing of the coordinates (i.e. the entries of matrices) corresponding to arrows which send vectors in $R_1$ to those in $R_2$. Then $Z_Q$ is constructed as the (geometric) quotient
of the set \( \mathcal{N}^{C-ss}_Q = \mathcal{N}_Q \cap \mathcal{N}^{C-ss} \) of \( C \)-\( ( \text{semi}) \)stable points by the action of \( GL_{R_1} \times GL_{R_2} \), similar to the construction of \( \mathcal{M}_C \). Note that \( Z_Q \) admits a natural map \( Z_Q \to \mathcal{M}_C \), which will turn out to be a closed immersion.

**Lemma 4.5.** The scheme \( Z_Q \) is reduced.

**Proof.** It suffices to show that \( \mathcal{N}^{C-ss}_Q \) is reduced. Let

\[
I_N \subset A_G := \mathbb{C}[\text{Hom}_G(V^* \otimes \mathbb{C} \cdot R, R)]
\]

be the defining ideal of \( \mathcal{N} \) (see (2.2)). Note that \( A_G \) is identified with a polynomial ring whose variables are entries of matrices corresponding to arrows of \( Q_G \). As mentioned above, the variables of \( A_G \) corresponding to the arrows from \( R_1 \) to \( R_2 \) generate an ideal \( I_S \) such that the defining ideal of \( \mathcal{N}_Q \) is equal to \( I_N + I_S \). Then, by noticing that the defining equations of \( \mathcal{N} \) come from the commutativity of the actions of the variables \( x, y, z \in V^* \), one can check that \( I_N \) is generated by polynomials \( \{p_i\}_i \), such that every term of \( p_i \) is in \( I_S \) as long as \( p_i \) is in \( I_S \). (For example, if the action of \( x \otimes y \) sends a vector in \( R_1 \) to one in \( R_2 \), so does the action of \( y \otimes x \).) This implies that a power \( f^m (m > 0) \) of a polynomial \( f \in A_G \) lies in \( I_S + I_N \) only if the power \( f_0^m \) of the sum \( f_0 \) of terms of \( f \) not in \( I_S \) lies in \( I_N \). Since \( \mathcal{N}^{C-ss} \) is smooth and in particular reduced, it follows that \( \mathcal{N}^{C-ss}_Q \) is reduced as well. \( \square \)

By the choice of \( S \), there is no nonzero \( G \)-equivariant \( \mathbb{C}[V] \)-linear map \( S \to Q \) (i.e. \( G \text{-Hom}_{\mathbb{C}[V]}(S, Q) = 0 \)) and thus there exists a Zariski open subset of \( Z_Q \) which admits an immersion into \( D \) (cf. the proof of [CI, Lemma 3.10]). By performing the same operation on each irreducible component of \( D_W \), we get a (topologically) dense Zariski open subset \( U \) of \( D_W \) together with a universal quotient map \( \mathcal{R}_C|_U \to \tilde{Q} \) whose restriction to \( U \cap D \) equals the universal quotient map associated to \( U \subset Z_Q \). We choose \( U \) as the maximal one satisfying this condition, and denote by \( \mathfrak{Z} \) the complement of \( U \) in \( D_W \) as a topological space. Then \( \mathfrak{Z} \) parametrizes \( G \)-constellations whose associated quotient \( \mathbb{C}[V] \)-modules are strictly larger than \( R_2 \) as \( CG \)-modules. Note that this may a priori happen for non-abelian \( G \). Also, the rank of \( \tilde{Q} \) a priori depends on irreducible components of \( D_W \) but it will turn out that \( \mathfrak{Z} \) is empty and that \( R_2 \) is uniquely determined from \( W \) (see Remark 4.1).

We take a 1-PS \( \lambda : \mathbb{C}^* \to GL_R \) (for each irreducible component of \( D_W \)) defined by assigning weight one to the generators of the ideal \( I_S \subset A_G \) in the proof of Lemma 4.5 and weight zero to the other variables of \( A_G \). Then \( \lambda \) is positive on \( C \) and zero on \( W \) as a function on \( \Theta \cong \chi(PGL_R) \). \( \lambda \) induces a filtration of \( F \) by its submodules (cf. [Kin, §3]) such that it has two associated graded components one of which is isomorphic to \( Q \). This works for a family of \( G \)-constellations over the open subset \( U \) of \( D_W \) and thus we obtain an exact sequence

\[
0 \to S \to \mathcal{R}_C|_U \to \tilde{Q} \to 0 \quad (4.1)
\]

of flat families of \( \mathbb{C}[V] \)-modules. By construction, the quotient map \( \mathcal{R}_C|_U \to \tilde{Q} \) in (4.1) coincides with the one induced by the schemes \( Z_Q \). Note also that the sequence (4.1) is a nontrivial extension as \( \mathbb{C}[V] \)-modules even if it splits as vector bundles on \( U \). It will be shown that this sequence extends to whole \( D_W \) in Lemma 4.6.
Let $\mathcal{K}$ be the kernel of the natural surjection $\mathcal{R}_C|_{X \setminus 3} \to \mathcal{Q}$. Then $\mathcal{K}$ is a vector bundle on $X \setminus 3$ (with a $\mathbb{C}[V]$-action) called the elementary transformation of $\mathcal{R}_C|_{X \setminus 3}$ (associated to $\mathcal{R}_C|_{X \setminus 3} \to \mathcal{Q}$) in the sense of [M]. Applying the inverse of the elementary transformation to $\mathcal{K}$, we obtain an exact sequence

$$0 \to \mathcal{Q} \otimes \mathcal{O}_{X \setminus 3}(-U) \to \mathcal{K}|_U \to \mathcal{S} \to 0$$

(4.2)

of flat families of $\mathbb{C}[V]$-modules. Note that $\mathcal{K}$ parametrizes the same $G$-constellations as $\mathcal{R}_C|_{X \setminus 3}$ outside $U$ while sub and quotient modules are switched over $U$.

**Lemma 4.6.** The family $\mathcal{K}$ over $X \setminus 3$ extends to a family $\mathcal{K}$ over the whole $X$ which coincides with the tautological bundle $\mathcal{R}_{C'}$ for the GIT-chamber $C'$ up to tensoring by a line bundle on $X$.

**Proof.** First recall that the bundle $\mathcal{R}_{C'} = \bigoplus_{\rho \in \text{Irr}(G)} \mathcal{R}_{\rho}$ is normalized by tensoring by a line bundle so that the component $\mathcal{R}_{\rho}$ becomes the trivial line bundle. Since $3 \subset X$ is of codimension at least two and $X$ is smooth, it suffices to show that $\mathcal{K}$ parametrizes $\theta'$-(semi)stable $G$-constellations for $\theta' \in C'$. To this end, we only have to show that the fibers of $\mathcal{K}$ are indecomposable as $\mathbb{C}[V]$-modules.

Let $F$ be the $G$-constellations corresponding to a (general) point of $x \in U \subset \mathcal{M}_C$. If the fibers $S \subset F$ and $Q = F/S$ of $\mathcal{S}$ and $\mathcal{Q}$ respectively are primitive as elements in $R$, then they are indecomposable and the same argument as in the proof of [CI, Proposition 4.1] shows that the sequence (4.2) is a nontrivial extension of $\mathbb{C}[V]$-modules, which implies that the fibers of $\mathcal{K}$ are indecomposable as well.

If one of $S$ and $Q$ is not primitive in $R$, we may assume that $S$ is not primitive and $Q$ is primitive by switching the roles of $C$ and $C'$, if necessary. Then we have a filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \cdots \subset \mathcal{S}_{k-1} = \mathcal{S} \subset \mathcal{S}_k = \mathcal{R}_C|_U$$

consisting of $\mathbb{C}[V]$-equivariant subbundles of $\mathcal{R}_C|_U$ such that each fiber of the associated graded bundle $\bigoplus_{i=1}^k \mathcal{S}_i/\mathcal{S}_{i-1}$ represents the $W$-polystable module of the corresponding point in $\mathcal{M}_W$. This filtration can be obtained by an appropriate 1-PS, similarly to (4.1). Since fibers of each component $\mathcal{S}_i/\mathcal{S}_{i-1}$ are indecomposable, we can reduce the claim to the previous case by applying elementary transformations repeatedly to quotients by $\mathcal{S}_i/\mathcal{S}_{i-1}$ for $i = 1, \ldots, k-1$. 

**Lemma 4.7.** $D_W$ is isomorphic to a disjoint union of (the underlying topological spaces of) $Z_Q$'s for all possible $Q$.

**Proof.** By the previous lemma, the extended bundle $\mathcal{K}$ is isomorphic to a vector bundle of the form $\mathcal{R}_{C'} \otimes L$ for some $L \in \text{Pic}(X)$. Let $\overline{U}$ be the scheme-theoretic closure so that it a reduced Cartier divisor whose support is equal to $D_W$. Then there is a quotient morphism over $\overline{U}$ from $\mathcal{R}_C|_{\overline{U}}$ to the cokernel $\mathcal{Q}$ of $\mathcal{K} \to \mathcal{R}_C$, whose restriction to $U$ coincides with the universal quotient morphism on $U$. We show that each connected component of $\overline{U}$ is identified with some $Z_Q$ and that $\mathcal{R}_C|_{\overline{U}} \to \mathcal{Q}$ is identified with the universal quotient morphism for $Z_Q$. For this, it suffices to show that every fiber of $\mathcal{Q}$ on each connected component of $D_W$ has the same dimension. Indeed, this condition implies...
that every point of each connected component defines the same $Q$ as a $\mathbb{C}G$-module and hence the whole $Z_Q$ admits a closed immersion onto $\overline{U}$.

Let $S$ be the kernel of $R_C|_S \to Q$. Then every fiber of $S$ has the dimension equal to or less than $\dim \mathbb{C} S = \text{rank} R_C - \dim \mathbb{C} Q$. However, by considering the inverse of elementary transformation for the same wall $W$ with the roles of $C$ and $C'$ switched, we also see that every fiber of $S$ has the dimension at least $\dim \mathbb{C} S$. Therefore, every fiber of $S$ has the same dimension, and the same holds for $Q$ as well.

\begin{remark}
The previous lemma implies that the whole $Z_Q$ is identified with the scheme $Z$ introduced in \cite[Lemma 3.10]{CI}. As we will see in Proposition 4.13 that $D_W$ is connected and hence $D_W$ is in fact identified with a single reduced scheme $Z_Q$.
\end{remark}

Next we describe the behaviour of the Grothendieck group $K(X)$ of locally free sheaves on $X$ under the wall-crossing. As we will see below, it is basically the same as the abelian cases and this enables us to proceed the proof of the main result in a similar way to \cite{CI}. To this end we briefly review the necessary notions. See \cite[§2.4, §5, and §7.1]{CI} and also \cite[§9.2]{BKR} for details.

The result of \cite{BKR} shows that the Fourier-Mukai transform by the universal family $U_C$ gives an equivalence $\Phi_C : D(X) \cong D^G(\mathbb{C}^3)$ whose inverse $\Phi_C^{-1} : D^G(\mathbb{C}^3) \to D(X)$ is given by $\mathcal{O}_{\mathbb{C}^3} \otimes \rho \mapsto R_{\rho}^\vee$. $\Phi_C$ induces an isomorphism between $K(X)$ and the Grothendieck group $K^G(\mathbb{C}^3)$ of $G$-equivariant sheaves on $\mathbb{C}^3$ supported at the origin. With respect to this pairing, the classes of $\{ \mathcal{O}_{\mathbb{C}^3} \otimes \rho \}_{\rho \in \text{Irr}(G)}$ and $\{ \mathcal{O}_0 \otimes \rho \}_{\rho \in \text{Irr}(G)}$ give mutually dual basis of $K(\mathbb{C}^3)$ and $K_0(\mathbb{C}^3)$ respectively where $\mathcal{O}_0$ is the skyscraper sheaf at the origin of $\mathbb{C}^3$.

The equivalence $\Phi_C$ also induces an isomorphism $\phi_C : K_0(X) \to K^G_0(\mathbb{C}^3)$ and its pullback $\phi_C^* : K^G(\mathbb{C}^3) \to K(X)$. The vector space $\Theta$ is regarded as a subspace of $K^G(\mathbb{C}^3)_R$, and the restriction of $\phi_C^*$ to $\Theta$ gives an isomorphism $\Theta \to F^1$ where we have a filtration

$$K(X)_\mathbb{R} = F^0 \supset F^1 \supset F^2 \supset F^3 = 0$$

with $F^i$ being the subspace generated by sheaves whose supports are of codimension at least $i$. The homomorphism $\phi_C^*|_\Theta : \Theta \to F^1$ is explicitly given as

$$\theta = (\theta_\rho)_{\rho} \mapsto \sum_{\rho \in \text{Irr}(G)} \theta_\rho[R_{\rho}^\vee]$$

where $[R_{\rho}^\vee]$ denotes the class of the dual sheaf $R_{\rho}^\vee$ in $F^1$. We then obtain a homomorphism $L_C : \Theta \to \text{Pic}(X)_\mathbb{R}$ such that the following diagram of vector spaces is commuta-
\[ \Theta \xrightarrow{\phi_C} F^1 \]
\[ L_C \downarrow \quad \text{pr} \downarrow \]
\[ \text{Pic}(X) \xrightarrow{\text{det}^{-1}} F^1/F^2 \]

where pr is the projection and the lower horizontal arrow denotes the isomorphism \( F^1/F^2 \cong \text{Pic}(X) \) given by sending the class \( [F] \in F^1 \) to \( \text{det}(F)^{-1} \).

**Lemma 4.8.** \( L_C \) is equal to the homomorphism \( \psi_C : \Theta \to \text{Pic}(X) \) in (3.5).

**Proof.** By definition \( L_C \) is given as
\[ \theta = (\theta_\rho)_{\rho} \mapsto \det \left( \bigoplus_{\rho \in \text{Irr}(G)} R_\rho^{\perp \theta_\rho} \right) \]

For a determinantal semi-invariant \( f \) having \( \theta = (\theta_\rho)_\rho \) as its weight, \( \varphi_C(f) \) is regarded as a section of a line bundle \( L_C(\theta) \) by noticing that each arrow \( a_{\rho,\rho'} \in Q \) is regarded as a section of \( R_\rho \otimes R_{\rho'} \). Thus, \( L_C(\theta) \) is nothing but the line bundle \( \psi_C(\theta) \). Since \( \Theta \) is spanned by the weights of determinantal semi-invariants by Theorem 2.3, \( L_C \) and \( \psi_C \) are the same on the whole \( \Theta \). \( \square \)

When we cross a wall from a GIT-chamber \( C \) to an adjacent one \( C' \), the corresponding chambers \( \phi^*_C(C) \) and \( \phi^*_{C'}(C') \) in \( F^1 \) may not be adjacent in general. For example, in dimension two, the automorphism \( \phi^*_{C'} \circ (\phi_C)^{-1} \) of \( F^1 \cong \text{Pic}(X) \) is a reflection along the hyperplane containing \( \phi_C(W) \) for the wall \( W = \overline{C} \cap \overline{C}' \). In the proof of the main result of [CI], namely the abelian case of Theorem 4.1, the fact that, for a wall-crossing of Type 0, the chambers \( \phi^*_C(C) \) and \( \phi^*_{C'}(C') \) in \( F^1 \) are adjacent up to tensoring by a line bundle plays a crucial role. We will show that the same property holds for non-abelian groups as well (cf. Proposition 4.13).

Let us recall that a wall \( W = \overline{C} \cap \overline{C}' \) of Type 0 induces an exact sequence
\[ 0 \to S \to R_{C|D_W} \to Q \to 0 \] (4.4)
of flat family of \( G \)-equivariant \( \mathbb{C}[V] \)-modules on the unstable locus \( D_W \) which extends (4.1) (see the proof of Lemma 4.7).

For each connected component \( D \subseteq D_W \), we will show that either (1) one of \( S|_D \) and \( Q|_D \) parametrizes rigid \( \mathbb{C}[V] \)-modules (i.e. \( \mathbb{C}[V] \)-modules \( F \) with \( G \text{-Ext}^1_{\mathbb{C}[V]}(F, F) = 0 \)), or (2) \( D \) is a product of (possibly reducible) curves (Lemma 4.10). To do this, we first take the direct summands \( R_1, R_2 \) of \( R \) corresponding to \( x \in D \) as in the construction of \( Z_Q \). We then consider the closed subscheme \( \mathcal{V}_S \) (resp. \( \mathcal{V}_Q \)) of \( \mathcal{N} \) given by vanishing of coordinates corresponding to all arrows but ones from \( R_1 \) (resp. \( R_2 \)) to itself. Now we define coarse moduli space \( \mathcal{M}_S \) (resp. \( \mathcal{M}_Q \)) as the GIT-quotient
\[ \mathcal{V}_S//_{\theta_1} GL_{R_1} \] (resp. \( \mathcal{V}_Q//_{\theta_2} GL_{R_2} \))
where $\theta_i$ is the stability condition induced from $\theta_0 \in W$ via the natural inclusion $GL_{R_i} \to GL_{R}$. We denote by $\mathcal{M}_S$ (resp. $\mathcal{M}_Q$) the reduced induced subscheme of $\bar{\mathcal{M}}_S$ (resp. $\bar{\mathcal{M}}_Q$). Let $\bar{D}$ be the image of $D$ with the reduced structure under the morphism $\mathcal{M}_C \to \mathcal{M}_W$. Note that $\bar{D}$ is possibly non-normal.

**Lemma 4.9.** With the notation as above, there are natural projections from $D_W$ to $\bar{\mathcal{M}}_S$ and $\bar{\mathcal{M}}_Q$. Moreover, the bijective map $D \to \bar{D}$ factors through the induced map $D \to \mathcal{M}_S \times \mathcal{M}_Q$ which is also bijective.

**Proof.** First note that the projections are well-defined since fibers of $\mathcal{S}|_D$ and $\mathcal{Q}|_D$ are semistable with respect to $\theta_1$ and $\theta_2$ respectively. We also have a morphism $\mathcal{M}_S \times \mathcal{M}_Q \to \bar{D}$ which comes from the assignment $(F_1, F_2) \mapsto F_1 \oplus F_2$. This is bijective by the choice of $\theta_1, \theta_2$, and clearly the composition of this map with $D \to \mathcal{M}_S \times \mathcal{M}_Q$ is equal to the map $D \to \bar{D}$.

The following lemma indicates that there can be two essentially different situations for $D$.

**Lemma 4.10.** For each connected component $D \subset D_W$, exactly one of the following two situations occurs:

1. Either $\mathcal{S}|_D$ or $\mathcal{Q}|_D$ parametrizes rigid $C[V]$-modules.
2. $D$ is a product of two curves $C_1, C_2$ where each $C_i$ is a tree of rational curves of Dynkin type, that is, $C_i$ is isomorphic to the (reduced) exceptional locus of the minimal resolution of an ADE surface singularity.

**Proof.** Assume that there exists a general point $x \in D$ such that its corresponding $C[V]$-modules $S \subset F$ and $Q = F/S$ are not rigid. The 2-dimensional tangent space $T_xD$ of $D$ (as a relative Quot scheme) is isomorphic to the kernel of the natural map $G\text{-}\text{Ext}^1_{C[V]}(F, F) \to G\text{-}\text{Ext}^1_{C[V]}(S, Q)$ by the $G$-equivariant version of [S Proposition 4.4.4]. Since we have

$$G\text{-}\text{Hom}(S, Q) = 0 \quad \text{and} \quad G\text{-}\text{Ext}^1(S, Q) \cong G\text{-}\text{Ext}^1(Q, S) \cong \mathbb{C},$$

one sees that $T_xD$ is isomorphic to $G\text{-}\text{Ext}^1_{C[V]}(S, S) \oplus G\text{-}\text{Ext}^1_{C[V]}(Q, Q)$ and hence

$$\dim G\text{-}\text{Ext}^1_{C[V]}(S, S) = \dim G\text{-}\text{Ext}^1_{C[V]}(Q, Q) = 1.$$ 

Therefore, $D$ is a product of two curves. Note that even if $S$ and $Q$ are non-rigid, the moduli space $\mathcal{M}_S$ or $\mathcal{M}_Q$ might be one point with nonreduced structure.

By Lemma 4.9 we only have to show that $\mathcal{M}_S$ and $\mathcal{M}_Q$ are curves of Dynkin type when they are one-dimensional. Indeed, this particularly implies that each irreducible component of $\mathcal{M}_S \times \mathcal{M}_Q$ is smooth and thus $D \to \mathcal{M}_S \times \mathcal{M}_Q$ is an isomorphism.

By moving from the relative interior of $W$ to its smaller faces corresponding submodules of the fiber $\mathcal{S}_x$ for general $x \in D$, we obtain $\theta \in \Theta$ and its associated birational map $\alpha_\theta : \mathcal{M}_C \to \mathcal{M}_\Theta$ such that $\alpha_\theta$ is defined on $D$ and that $\alpha_\theta$ only collapses the first factor $\mathcal{M}_S$ of $D$ and in particular the image $\alpha_\theta(D)$ is the curve $\mathcal{M}_Q$. This is indeed
possible since we just have to avoid faces contained in facets of $W$ which induce divisorial contractions corresponding to curves in $\mathcal{M}_Q$. Note that such a facet is unique for each contraction and that moving from a face to its codimension-one face gives an isomorphism or a contraction of relative Picard number one between the corresponding (normalized) moduli spaces, and in particular each process does not contract the two factors at the same time.

Therefore, the curve $\mathcal{M}_S$ is identified with the fiber, under $\alpha_\theta$, over a general point of the one-dimensional singular locus of $\bar{\mathcal{M}}_\theta$. Then it must be a tree of rational curves of Dynkin type since $\bar{\mathcal{M}}_\theta$ has only canonical singularities and $\alpha_\theta$ is crepant (cf. [Re1, Corollary 1.14]). The same argument works for $\mathcal{M}_Q$ as well.

For given $E \in K_0(X)$, we consider the automorphism $\mathcal{T}_E$ of $K(X)$ defined by

$$\mathcal{T}_E(\xi) = \xi - \chi(\xi, E) \cdot E.$$

Note that $\mathcal{T}_E$ has positive determinant and fixes the hyperplane $E^\perp := \{\xi \in K(X) | \chi(\xi, E) = 0\}$ pointwise. We denote by $\mathcal{T}_E'$ the inverse of $\mathcal{T}_E$. We will show that the automorphism $\phi^*_\rho \circ (\phi^*_C)^{-1}$ of $F^1$ coincides with $\mathcal{T}_E$ or $\mathcal{T}_E'$ for some $E \in K_0(X)$ up to an automorphism obtained by tensoring by a line bundle of $X$ (Proposition [4.13]).

For the families $\mathcal{S}$ and $\mathcal{Q}$ in (4.4) associated to the wall $W$, let $\mathcal{S}_\rho$ and $\mathcal{Q}_\rho$ be their $\rho$-isotypic components respectively.

**Lemma 4.11.** For any $\rho, \sigma \in \text{Irr}(G)$, we have $H^i(D_W, \mathcal{S}^\vee \otimes \mathcal{Q}_\sigma) = 0$ for all $i$.

**Proof.** We first note that the $G$-equivariant bundle $\bigoplus_{\rho \in \text{Irr}(G)} \mathcal{R}_\rho^\vee \otimes \rho^*$ on $X \times \mathbb{C}^3$ can be regarded as the universal family of the moduli space $\mathcal{M}_\rho^*$ for $\rho^* \in \Theta$ defined by $\rho^*(\rho) = -\theta(\rho^*)$ with $\theta \in C$ by [CI, Lemma 2.6(ii)]. Note that $\mathcal{M}_\rho^* \cong \mathcal{M}_C$ and that the associated sub and quotient bundles $\mathcal{S}^*, \mathcal{Q}^*$ of $\mathcal{R}_{\rho^*}|_{D_W} \cong \mathcal{R}_C^\vee|_{D_W}$ are isomorphic to $\mathcal{Q}^\vee$ and $\mathcal{S}^\vee$ respectively.

By dualizing the inverse

$$0 \to \mathcal{R}_C(-D_W) \to \mathcal{K} \to \mathcal{S} \to 0$$

of the elementary transformation

$$0 \to \mathcal{K} \to \mathcal{R}_C \to \mathcal{Q} \to 0$$

and then tensoring with $\mathcal{O}_X(-D_W)$, we obtain a sequence

$$0 \to \mathcal{K}^\vee(-D_W) \to \mathcal{R}_C^\vee \to \mathcal{S}^\vee \to 0$$

which is identified with the tautological sequence for $\mathcal{M}_{\rho^*}$. Also, tensoring with $\mathcal{O}_X(D_W)$ to the inverse of the elementary transformation for this sequence, we obtain

$$0 \to \mathcal{R}_C^\vee \to \mathcal{K}^\vee \to \mathcal{Q}^\vee \otimes \mathcal{O}_X(D_W) \to 0.$$
Therefore, $\mathcal{R}_C^\vee|_{D_W}$ admits $Q^\vee$ (resp. $S^\vee$) as the associated sub (resp. quotient) bundle while $\mathcal{K}_D^\vee|_{D_W}$ admits $S^\vee$ (resp. $Q^\vee(D_W)$) as the associated sub (resp. quotient) bundle.

From (4.7), we obtain a long exact sequence

$$
\cdots \rightarrow \text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{R}_C^\vee) \rightarrow \text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee) \xrightarrow{f^i} \text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, Q^\vee(D_W)) \rightarrow \text{Ext}^{i+1}_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{R}_C^\vee) \rightarrow \cdots
$$

(4.8)

since $\mathcal{R}_C^\vee$ is locally free. For all $i$, the map $f^i$ in (4.8) is identified with the composition of the restriction map

$$
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee) \rightarrow \text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee|_{D_W})
$$

(4.9)

and the natural projection

$$
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee|_{D_W}) \cong H^i(D_W, \mathcal{R}_C \otimes \mathcal{K}_C^\vee|_{D_W}) \rightarrow H^i(D_W, \mathcal{R}_C \otimes Q^\vee(D_W)).
$$

By using the long exact sequence obtained from (4.6), we similarly have a map $g^i : \text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee) \rightarrow H^i(D_W, \mathcal{K}_C^\vee \otimes Q)$ which is the composition of (4.9) and the projection

$$
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{K}_C^\vee|_{D_W}) \rightarrow H^i(D_W, \mathcal{K}_C^\vee \otimes Q).
$$

The kernels of the two projections are the same since we have

$$
\text{Ext}^i_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{R}_C^\vee) = \text{Ext}^i_{\mathcal{O}_X}(\mathcal{K}_C^\vee, \mathcal{K}_C^\vee) = 0
$$

(4.10)

for $i > 0$ and

$$
\text{Hom}_{\mathcal{O}_X}(\mathcal{R}_C^\vee, \mathcal{R}_C^\vee) = \text{Hom}_{\mathcal{O}_X}(\mathcal{K}_C^\vee, \mathcal{K}_C^\vee)
$$

(4.11)

(cf. [C1, Lemma 5.4]). It follows that $H^i(D_W, S^\vee \otimes Q) = 0$ (and $H^i(D_W, S \otimes Q^\vee(D_W)) = 0$) for all $i$. 

Let $W = \mathcal{C} \cap \mathcal{C}'$ be a wall of Type 0 with the associated sequence (4.4) on the unstable locus $D_W$. Take the decomposition $D_W = \bigcup_i D_i$ into connected components and let $R_{1,i}$ (resp. $R_{2,i}$) be the $\mathbb{C}G$-modules for fibers of $S|_{D_i}$ (resp. $Q|_{D_i}$). Note that all $R_{1,i}$ (resp. $R_{2,i}$) are proportional in $R(G)$ since $D_i$’s are defined by the same wall $W$.

**Lemma 4.12.** For each $i$, there exists a class $E_i \in K_0(X)$ of a line bundle on $D_i$ such that the image $\phi_C(E_i) \in K_0(\mathbb{C}^3) \cong R(G)$ is proportional to either $E_{1,i}$ or $E_{2,i}$.

**Proof.** We consider the two cases (1), (2) in Lemma 4.10 separately. For the rigid case (1), we first assume that $Q|_{D_i}$ parametrizes rigid modules which are indecomposable so that there exists a line bundle $L$ on $D_i$ such that $Q|_{D_i} \cong L^\oplus_{\text{rank}(Q|_{D_i})}$ for all $\rho$. Note that $R_{2,i} = \bigoplus_{\rho \in \text{Irr}(G)} L^\oplus_{\text{rank}(Q|_{D_i})}$ in $R(G)$.  


We set \( \mathcal{E}_i := [L^{-1} \otimes \omega_{D_i}] \in K_0(X) \). The cohomology sheaves of \( \Phi_C(L^{-1} \otimes \omega_{D_i}) \), which are \( G \)-equivariant \( \mathbb{C}[V] \)-modules supported at the origin, are computed as

\[
\Phi^j_C(L^{-1} \otimes \omega_{D_i}) \cong \bigoplus_{\rho \in \text{Irr}(G)} H^j(X, \mathcal{R}_\rho \otimes L^{-1} \otimes \omega_{D_i}) \otimes_{\mathbb{C}} \rho
\]

\[
\cong \bigoplus_{\rho \in \text{Irr}(G)} H^{2-j}(D_i, \mathcal{R}_\rho^\vee \otimes L)^\vee \otimes_{\mathbb{C}} \rho
\]

\[
\cong \bigoplus_{\rho \in \text{Irr}(G)} H^{2-j}(D_i, ((L^{-1})^{\oplus \text{rank}(\mathcal{Q}_{\rho} \mid_{D_i})} \oplus \mathcal{S}_\rho^\vee) \otimes L)^\vee \otimes_{\mathbb{C}} \rho
\]

\[
\cong \bigoplus_{\rho \in \text{Irr}(G)} H^{2-j}(D_i, \mathcal{O}_{D_i}^{\oplus \text{rank}(\mathcal{Q}_{\rho} \mid_{D_i})})^\vee \otimes_{\mathbb{C}} \rho
\]

where the last isomorphism holds by Lemma 4.11. Therefore, the class \( \phi^*_C(\mathcal{E}_i) \) is equal to \( \chi(\mathcal{O}_{D_i}) \cdot R_{2,i} \in R(G) \).

If fibers of \( \mathcal{Q}_{\rho} \mid_{D_i} \) are rigid but not indecomposable, we can write \( \mathcal{Q}_{\rho} \mid_{D_i} \) as a direct sum \( \bigoplus_{j} L_i^j \otimes \mathbb{C} \) of line bundles. Then the similar computation as above shows that \( \phi^*_C(\mathcal{E}_i) \) is again proportional to \( R_{2,i} \in R(G) \) when we use any \( L_j \) instead of \( L \). Note that each direct summand of fibers of \( \mathcal{Q}_{\rho} \mid_{D_i} \) is proportional to \( R_{2,i} \in R(G) \). This implies that the line bundles \( L_j \) on \( D_i \) are all proportional in \( K(X) \). Then \( L_j \)'s are mutually isomorphic since they define the same class \( \mathcal{O}_X(-D_i) \in \text{Pic}(X) \). Therefore, we again have that \( \phi^*_C(\mathcal{E}_i) \) is equal to \( \chi(\mathcal{O}_{D_i}) \cdot R_{2,i} \in R(G) \) for \( \mathcal{E}_i := [L^{-1} \otimes \omega_{D_i}] \) with a line bundle \( L \) on \( D_i \).

When fibers of \( \mathcal{S}_{\rho} \mid_{D_i} \) are rigid, one can similarly show that there exists a line bundle \( L' \) on \( D_i \) such that \( \mathcal{S}_{\rho} \mid_{D_i} \cong L' \oplus \text{rank}(\mathcal{S}_{\rho} \mid_{D_i}) \) and that \( \phi^*_C([L'^{-1}]) \) is equal to \( \chi(\mathcal{O}_{D_i}) \cdot R_{1,i} \in R(G) \).

Next we consider the case (2) in Lemma 4.10. In this case each \( D_i \) is the product of curves \( C_1 \) and \( C_2 \) which are regarded as moduli spaces parametrizing fibers of \( \mathcal{S}_{\rho} \mid_{D_i} \) and \( \mathcal{Q}_{\rho} \mid_{D_i} \) respectively. Since each \( C_j \) is a union of \( \mathbb{P}^1 \)'s, the bundle \( \mathcal{S}_{\rho} \mid_{D_i} \) (resp. \( \mathcal{Q}_{\rho} \mid_{D_i} \)) on \( D_i \) is a direct sum of line bundles such that the restrictions of these line bundles to fibers of the projection \( D_i \to C_1 \) (resp. \( D_i \to C_2 \)) are mutually isomorphic. We take any line bundles \( L_1 \) and \( L_2 \) on \( D_i \) appearing as direct summands of \( \mathcal{S}_{\rho} \mid_{D_i} \) and \( \mathcal{Q}_{\rho'} \mid_{D_i} \) respectively for any \( \rho, \rho' \). By Lemma 4.11, we have \( H^j(D_i, L_1^{-1} \otimes L_2) = 0 \) for all \( j \). Then either

\[
H^j(C_1, (L_1^{-1} \otimes L_2)|_{C_1}) = 0 \quad \text{or} \quad H^j(C_2, (L_1^{-1} \otimes L_2)|_{C_2}) = 0 \quad \forall j
\]

holds where \( C_1 \) and \( C_2 \) are regarded as fibers of \( D_i \to C_2 \) and \( D_i \to C_1 \) respectively. From this we see that one has either \( H^j(C_2, (L_1^{-1} \otimes L_2)|_{C_2}) = 0 \) for all line bundles \( L_2 \) appearing as direct summands of \( \mathcal{Q}_{\rho} \mid_{D_i} \) or \( H^j(C_1, (L_1^{-1} \otimes L_2)|_{C_1}) = 0 \) for all line bundles \( L_1 \) appearing as direct summands of \( \mathcal{S}_{\rho} \mid_{D_i} \). In the former case one has \( \phi_C([L_2^{-1}]) = R_{2,i} \in R(G) \), and in the latter case \( \phi_C([L_1^{-1}] \otimes \omega_{D_i}) = R_{1,i} \). This can be checked by noticing that the Euler characteristics \( \chi(L_2^i|_{C_2}) \) (resp. \( \chi(L_1^i|_{C_1}) \)) are the same for all the direct summands \( L_2 \) of \( \mathcal{Q}_{\rho} \mid_{D_i} \) in the former case (resp. \( L_1 \) of \( \mathcal{S}_{\rho} \mid_{D_i} \) in the latter case). Note that the Euler characteristic of a line bundle \( L \) on \( C_i \) is equal to \( 1 + \sum_k \deg(L|_{C_{i,k}}) \) where \( C_{i,k} \) are the irreducible components of \( C_i \).
Proposition 4.13. The unstable locus $D_W$ is connected. Moreover, the automorphism of $K(X)$ induced by the autoequivalence $\Phi_{C'} \circ \Phi_C$ of $D(X)$ is equal to one of the four automorphisms: (a) $\mathcal{T}_C$, (b) $\mathcal{T}_{C'}$, (c) $\xi \mapsto \mathcal{T}_C(\xi) \otimes O(-D_W)$, and (d) $\xi \mapsto \mathcal{T}_{C'}(\xi \otimes O(D_W))$. In particular, $\phi_{C'}^*(C')$ is adjacent to $\phi_C^*(C) \otimes L$ for some $L$ in $\text{Pic}^c(X)$, the subgroup of $\text{Pic}(X)$ generated by the classes of compact divisors.

Proof. We have shown in the previous lemma that the image in $R(G)$ of a line bundle on $D_i$ is proportional to $R_{1,i}$ (and to $R_{2,i}$) modulo $\mathcal{Z}R$ in any case. Since $D_i$'s are linearly independent in $\text{Pic}(X) \cong F^1/F^2$, the classes of the line bundles on $D_i$'s in $K(X)$ are also linearly independent (cf. the proof of [CI, Proposition 5.5(ii)]). Therefore, $D_W$ must be connected. The similar argument in the proof of [CI, Proposition 5.5(iii)] also shows that we have $H^1(O_{D_W}) = H^2(O_{D_W}) = 0$ in the rigid case (1) in Lemma 4.10.

To summarize, the following two cases may occur:

(I) we have $\chi(\mathcal{R}_\rho^\vee, \mathcal{E}) = \text{rank}(\mathcal{Q}_\rho)$ for the class $\mathcal{E} = [L^{-1} \otimes \omega_{D_W}] \in K_0(X)$ with an arbitrary line bundle $L$ on $D_W$ which is a direct summand of $\mathcal{Q}$, and

(II) we have $\chi(\mathcal{R}_\rho^\vee, \mathcal{E}) = \text{rank}(\mathcal{S}_\rho)$ for the class $\mathcal{E} = [L^{-1}] \in K_0(X)$ with an arbitrary line bundle $L$ on $D_W$ which is a direct summand of $\mathcal{S}$.

In the former case (I), one has

$$\mathcal{T}_C([\mathcal{R}_\rho^\vee]) = [\mathcal{R}_\rho^\vee] - \text{rank}(\mathcal{Q}_\rho)[L^{-1} \otimes \omega_{D_W}] = [\mathcal{R}_\rho^\vee] - [\mathcal{Q}_\rho^\vee(D)]$$

in $K(X)$. By the sequence [L.7] and the fact that \{\mathcal{R}_\rho^\vee\}_\rho$ forms a basis of $K(X)$, we can conclude that $\mathcal{T}_C$ coincides with the automorphism induced by $\Phi_{C'} \circ \Phi_C$ or its composition with tensoring by $O_X(-D_W)$. Which of the cases (a) and (c) happens depends on whether $Q_{\rho_0} = 0$ or $\neq 0$. The latter case (II) can be treated similarly by switching the roles of $C$ and $C'$, so that we fall into the cases (b) and (d). This completes the proof.

Remark 4.2. The above proposition shows that the wall-crossing gives a combination of a spherical twist and tensoring by a line bundle at least at the level of the Grothendieck group. In the rigid case (1) in Lemma 4.10, one can use the same argument as in the proof of [CI, Proposition 7.3] to prove the similar statement at the level of the derived category. In the case (2), however, this argument does not work since the line bundles in $\mathcal{S}$ or $\mathcal{Q}$ are not mutually isomorphic (while they have the same Euler characteristics) in contrast to the rigid case. By analogy with the abelian case [CI, Corollary 4.6], it is expected that the case (2) does not occur.

Proof of Theorem 4.1

Since all projective crepant resolutions of $\mathbb{C}^3/G$ are connected with each other by a sequence of flops [K], we only have to show that every flop of $X = \mathcal{M}_C$ can be realized by (not necessarily single) wall-crossings. Let $B \subset \text{Amp}(X)$ be a facet corresponding to a flop $X \rightarrow X_B \leftarrow X'$. If we can reach a GIT-chamber $C''$ through walls of Type 0 such that $C''$ admits a wall $\tilde{B} \subset C''$ satisfying $\psi_{C''}(\tilde{B}) \subset B$, then crossing $\tilde{B}$ induces the desired flop. Indeed, otherwise $\tilde{B}$ would be contained in a facet of the orbit cone of a general point of a divisor since the homomorphism $\psi_{C''}$ must change when we cross $\tilde{B}$. In such a case, however, applying Lemma 4.3 to $\tilde{B}$ shows that $\tilde{B}$ would induce a divisorial contraction, which is a contradiction.
As shown in [CI] §8, each compact exceptional divisor \( E \subset X \) acts on \( F^1 \) by
\[
\xi \mapsto \xi \otimes O_X(E) = \xi + [\xi \otimes O_E]
\]
with \([\xi \otimes O_E] \in F^2\). Moreover, for the set of all irreducible compact exceptional divisors \( S_1, \ldots, S_d \) and for any \( \xi \in \text{pr}^{-1}(\text{Amp}(X) \cup \text{relint}(B)) \), the set \([\xi \otimes O_{S_1}], \ldots, [\xi \otimes O_{S_d}]\) forms a basis of \( F^2 \) where \( \text{Amp}(X) \) is identified with a cone in \( F^1/F^2 \) (see (4.3)).

The existence of the desired chamber \( C'' \) can be shown similarly to the proof of [CI] Proposition 8.2. More precisely, one can show from Proposition 4.13 that we have
\[
\text{pr}^{-1}(\text{Amp}(X) \cup \text{relint}(B)) \subset \bigcup_{L \in \text{Pic}^c(X)} \bigcup_{\mathfrak{M}_C \ni X} L \otimes \phi_C^*(\mathcal{C})
\]
(4.12)
noticing that GIT-chambers in \( \Theta \) are finite. Thus we obtain a chamber \( C'' \) and its wall \( \bar{B} \) such that \( \text{pr}(\phi_C^*(\bar{B})) \subset B \).

Theorem 4.1 together with (4.12) implies that \( \{\psi_C^*(\mathcal{C})\}_{C \in \Theta} \) chamber covers the movable cone of \( X \). Thus, every rational contraction \( X \rightarrow X' \) (see Subsection 3.2) is realized as the birational map \( \mathfrak{M}_C \rightarrow \mathfrak{M}_\theta \) induced by variation of GIT for some \( \theta \). This fact has the following application:

**Proposition 4.14.** Let \( h_1, \ldots, h_\ell \) be homogeneous generators of the normalization of \( \mathbb{C}[\mathcal{W}] \). Then the associated \( \ell + m \) elements
\[
\overline{\varphi(h_1)}, \ldots, \overline{\varphi(h_\ell)}, t_1^{-r_1}, \ldots, t_m^{-r_m}
\]
to \( \varphi(h_1), \ldots, \varphi(h_\ell) \in \mathbb{C}[V]^{[G,G]} \) (see (3.1)) generate the Cox ring \( \text{Cox}(X) \subset \mathbb{C}[V]^{[G,G]}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}] \) of any crepant resolution \( X \rightarrow \mathbb{C}^3/G \).

**Proof.** Let \( f_1, \ldots, f_\ell \in \mathbb{C}[V]^{[G,G]} \) be homogeneous elements such that the associated elements in (3.1) generate \( \text{Cox}(X) \). We consider the ambient toric varieties \( Y \) and \( \widetilde{Y}_W \) of \( \mathcal{W} \) associated to \( f_i \)'s (see Subsection 3.3). Note that each \( f_i \) corresponds to a non-exceptional torus-invariant divisor \( D_i \subset Y \) and that the image of the corresponding section \( f_{D_i} \in H^0(Y, O(-D_i)) \) (i.e. \( D_i = \{f_{D_i} = 0\}\) in \( \text{Cox}(X) \) is equal to \( f_i \) (cf. Remark 3.1). Regarding the weight of \( f_{D_i} \) as a character of \( T_X \), we consider its associated GIT-quotient \( Y_i \) of \( A^{\ell + m} \). In terms of a fan, \( Y_i \) is characterized as a birational model \( Y \rightarrow Y_i \) with \( \dim \text{Pic}(Y_i) = 1 \) such that the toric fan of \( Y_i \) contains the cone generated by the rays corresponding to the torus-invariant divisors of \( Y_i \) except \( D_i \).

For each \( i \), let \( X_i \subset Y_i \) be the induced GIT-quotient of Spec(\( \text{Cox}(X) \)). As observed above, there exists \( \theta \in \Theta \) such that \( X_i \cong \mathfrak{M}_\theta \). Since \( \widetilde{Y}_W \) is normal, there is a homogeneous semi-invariant \( f \) in \( \mathbb{C}[\mathcal{W}] \) such that its (scheme-theoretic) zero locus in \( \mathfrak{M}_\theta \cong X_i \) is equal to the strict transform of \( D_i \cap X \) under \( X \rightarrow X_i \). If we take a GIT-chamber \( C \) so that \( \mathcal{C} \) contains \( \theta \), this implies \( \varphi_C(f) \) is equal to \( \widetilde{f}_i \) (up to constant multiplication). This particularly implies that each \( \widetilde{f}_i \) can be obtained as the associated element of \( \varphi(h) \) for some homogeneous semi-invariant \( h \). Thus, the claim follows. \( \square \)
A List of Symbols

\( R \) the regular representation of \( G \) 
\( \text{Irr}(G) \) the set of irreducible representations of \( G \) 
\( N \) the affine scheme parametrizing all \( G \)-constellations 
\( GL_R \) := \( \text{Aut}_{CG}(R) = \prod_{\rho \in \text{Irr}(G)} GL_{\dim \rho}(\mathbb{C}) \) 
\( PGL_R \) := \( GL_R / \mathbb{C}^* \) 
\( \Theta \) := \( \{ \theta \in \text{Hom}_\mathbb{Z}(R(G), \mathbb{Z}) \mid \theta(R) = 0 \} \) 
\( R(G) \) := \( \bigoplus_{\rho \in \text{Irr}(G)} \mathbb{Z}\rho \) : the representation ring of \( G \) 
\( \Theta \) := \( \Theta \otimes_{\mathbb{Z}} \mathbb{R} \) : the space of stability conditions 
\( M_\Theta \) the moduli space of \( \theta \)-semistable \( G \)-constellations 
\( \mathcal{V} \subset N \) the main irreducible component 
\( \mathcal{M}_\Theta \) the coherent component of \( M_\Theta \) 
\( \mathcal{M}_C \) := \( \mathcal{M}_\theta \) for \( \theta \) in a GIT-chamber \( C \subset \Theta \) 
\( SL_R \) := \( \prod_{\rho \in \text{Irr}(G)} SL_{\dim \rho}(\mathbb{C}) \subset GL_R \) 
\( Q_G \) the McKay quiver of \( G \) 
\( \mathcal{R}_C \) the tautological bundle of \( \mathcal{M}_C \) 
\( \mathcal{R}_\rho \) the isotypic component of \( \mathcal{R}_C \) for \( \rho \in \text{Irr}(G) \) 
\( \text{Cox}(X) \) the Cox ring of a crepant resolution \( X \to \mathbb{C}^n / G \) 
\( Ab(G) \) := \( G/[G,G] \): the abelianization of \( G \) 
\( T_X \) the torus acting on \( X := \text{Spec} \text{Cox}(X) \) 
\( \iota_X : \mathfrak{X} \to \mathbb{A}^{\ell+M} \) the embedding given by \( f_1, \ldots, f_\ell \in \mathbb{C}[V]^G,G \) 
\( \iota_X : X \to Y \) the embedding into a toric variety associated to \( \iota_X \) 
\( \mathcal{W} \) := \( \mathcal{V} / SL_R \) 
\( T_\Theta \) the acting torus on \( \mathcal{W} \) 
\( \varphi \) the ring map \( \mathbb{C}[N]^{SL_R} \to \text{Cox}(V/G) \cong \mathbb{C}[V]^{[G,G]} \) 
\( \varphi_C \) the ring map \( \mathbb{C}[N]^{SL_R} \to \text{Cox}(X) \) 
\( \psi_C \) the \( \mathbb{R} \)-linear map \( \Theta \to \text{Pic}(X)_{\mathbb{R}} \) 
\( N_{\mathcal{W}} \) := \( \mathbb{Z}^\ell \oplus \Theta^V \) 
\( M_{\mathcal{W}} \) := \( \text{Hom}_\mathbb{Z}(N_{\mathcal{W}}, \mathbb{Z}) \) 
\( \mathcal{W} \hookrightarrow \mathcal{W}_W \) the embedding into a toric variety associated to \( \iota_X \) 
\( \mathcal{W} \hookrightarrow \mathcal{W}_W \) the normalization of \( \mathcal{W} \hookrightarrow \mathcal{W}_W \) 
\( W \subset C \) a wall (of Type 0, mostly) 
\( \mathcal{M}_W \) := \( \mathcal{M}_\theta \) for general \( \theta \in W \) 
\( \mathcal{M}_W \) the normalization of \( \mathcal{M}_W \) 
\( D_W \subset \mathcal{M}_C \) the unstable locus associated to \( W \) 
\( S, Q \) the associated sub and quotient bundles of \( \mathcal{R}_C|_{D_W} \) 
\( K(X) \) the Grothendieck group of \( X \) 
\( K_0(X) \) the Grothendieck group of \( X \) with compact support 
\( \mathcal{T}_E \) the automorphism of \( K(X) \) defined by \( E \in K_0(X) \) 
\( \mathcal{T}_E \) the inverse of \( \mathcal{T}_E \) 
\( \mathcal{S}_\rho \) the isotypic component of \( \mathcal{S} \) for \( \rho \in \text{Irr}(G) \) 
\( Q_\rho \) the isotypic component of \( Q \) for \( \rho \in \text{Irr}(G) \)
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