Sleeping Combinatorial Bandits

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Abstract

In this paper, we study an interesting combination of sleeping and combinatorial stochastic bandits. In the mixed model studied here, at each discrete time instant, an arbitrary availability set is generated from a fixed set of base arms. An algorithm can select a subset of arms from the availability set (sleeping bandits) and receive the corresponding reward along with semi-bandit feedback (combinatorial bandits). We adapt the well-known CUCB algorithm in the sleeping combinatorial bandits setting and refer to it as CS-UCB. We prove — under mild smoothness conditions — that the CS-UCB algorithm achieves an $O(\log(T))$ instance-dependent regret guarantee. We further prove that (i) when the range of the rewards is bounded, the regret guarantee of CS-UCB algorithm is $O(\sqrt{T \log(T)})$ and (ii) the instance-independent regret is $O(\sqrt{T^2 \log(T)})$ in a general setting. Our results are quite general and hold under general environments — such as non-additive reward functions, volatile arm availability, a variable number of base-arms to be pulled — arising in practical applications. We validate the proven theoretical guarantees through experiments.

1 Introduction

The stochastic multi-armed bandit (MAB) problem is one of the fundamental online learning problems that captures the classic exploration vs. exploitation dilemma. A MAB algorithm, operating in an uncertain environment, is expected to optimally trade-off acquisition of new information with optimal use of information-at-hand to choose an action that maximizes the expected reward or, equivalently, minimizes the expected regret. In a classical stochastic MAB setup, an algorithm has to pull (aka select) a single arm (aka choice) at each time instant and receive a reward corresponding to a pulled arm. The reward from each arm is an independent sample from a fixed but unknown stochastic distribution. The goal is to minimize the expected regret; the difference between the expected cumulative reward of the best offline algorithm with known distributions and the expected cumulative reward of the algorithm.

In this paper, we study a combination of two well studied extensions of classical stochastic MABs, namely sleeping bandits [KNMS10] and combinatorial bandits [GKJ12]. In the sleeping bandits setting,
only a subset of base arms is available at each time instant. This variant, sometimes also known as volatile bandits [BPSF13] or mortal bandits [CKRU09], models many real-world scenarios such as crowdsourcing [LLJ19], online advertising [CKRU09], and network routing [KNMS10, BPSF13] where an algorithm is restricted to select from only the available set of choices.

Another well studied generalization of the classical MAB setting is the combinatorial MAB (CMAB) problem [AB09, GJK10, CBL12, CTMSP115, WC18]. Similar to sleeping bandits, this variant too provides an abstraction to many real-world decision problems. For instance, in an online advertising setup, the platform selects multiple ads to display at any point in time [GJK10]; in crowdsourcing, the requester chooses multiple crowd workers at the same time [uHC16] and in network routing algorithm has to choose a path instead of a single edge [TZC17, KWAS15a]. Studying the two settings together presents interesting and non-trivial technical challenges.

We consider the semi-bandit feedback model and a general reward function (under mild smoothness constraints). In the semi-bandit feedback model, an algorithm observes the reward realizations corresponding to each of the selected arms along with the overall reward for pulling the subset of arms. The smoothness properties on the reward functions studied in this paper are similar to those in [CWY13]. It is worth mentioning here that in the sleeping MAB setting, the conventional definition of regret is not appropriate as the best arm (or the best subset of arms in the combinatorial sleeping MAB case) may not be available at all time instants. Hence, we evaluate the performance of an algorithm in terms of its sleeping regret [KNMS10], defined as the difference between the expected reward obtained from best available arm and the arm pulled by the algorithm.

The paper is organized as follows: In Section 2, we formally introduce the sleeping combinatorial bandits problem and define Lipschitz smoothness and Bounded smoothness assumptions. The required notational setup is introduced and CS-UCB algorithm is given in Section 3. In Section 4, we provide regret analysis of CS-UCB under Lipschitz smoothness and Bounded smoothness assumptions. In Section 5, we provide an in-depth verify of the theoretical results on simulated data with few reward functions. The related literature is discussed in Section 6 and in Section 7, we conclude our paper with a brief discussion on the results and future directions.

2 Model and Assumptions

In a classical stochastic multi-armed bands (MAB) problem, at each discrete time step $t$, an algorithm pulls a single arm $i_t \in [k]$ and observes a random reward $X_{i_t,t}$. The random variables $(X_{i,t})_t$ are identical and independently distributed according to a distribution $D_i(\mu_i)$. Here, $\mu_i$ is the mean of distribution $D_i$. Note that the reward corresponding to arms $j \neq i_t$ is not observed. The reward distributions $(D_i)_{i \in [k]}$ are unknown to the algorithm. Throughout this paper we consider that the reward distributions have a bounded support. The algorithm’s objective is to minimize expected regret defined as, $R_{A}(T) = \mathbb{E}[(\sum_{t=1}^T X_{j^*,t} - X_{i_t,t})]$. Here, $j^* = \arg \max_i \mu_i$ denotes the best arm.

In this paper we consider a sleeping combinatorial bandits problem with $[k] := \{1, 2, \cdots, k\}$ denoting the set of base arms and $\mu \in [0, 1]^k$, the vector of unknown mean qualities of the base arms. Similar to the classical stochastic MAB problem, each base arm $i$ corresponds to an unknown distribution $D_i$ with mean $\mu_i \in [0, 1]$ over its quality. At each time instant $t$, a subset $A_t \subseteq [k]$ of the base arms become available. Throughout the paper, we consider that $A_t$ is an arbitrary non-empty
subset. A decision maker (i.e. an algorithm) can pull any non-empty subset \( S_t \subseteq A_t \) of arms and receive a reward \( R_t := R(S_t, \mu) \). The reward depends upon the selected subset \( S_t \) and the mean qualities of the arms, \( \mu \). We define, \( R_S := R(S, \mu) \) whenever the quality vector is clear from the context. Furthermore, the reward depends only on the qualities of pulled arms \( S_t \)\(^1\). We remark here that the classical stochastic bandits setting is a special case of our setting with \( A_t = [k], |S_t| = 1 \) and \( R_t = X_{S_t,t} \) for all \( t \).

For a given reward function \( R \) the problem reduces to finding a reward maximizing subset of arms. This problem, even when the qualities of the base arms are known, is known to be NP-hard in general [WN99]. However, many important settings, such as submodular reward functions, admit a polynomial time approximation schemes that provides a decent approximation guarantee. To demarcate the computational problem of finding an optimal set of arms from effectively learning the quality distributions (and hence the learning an optimal set of arms to be pulled) we assume the existence of an \((\gamma, \beta)\)-approximation oracle (denoted by \((\gamma, \beta)\)-ORACLE), which, given an availability set \( A \) and a quality vector \( \mu \), outputs a set \( S \) such that \( R_S(\mu) \geq \gamma \cdot R_{S'}(\mu) \), for all \( S' \subseteq \mathbb{2}^A \) with the probability of at least \( \beta \), with \( \gamma, \beta \in (0, 1] \). The computation oracle separates the learning task from the offline computation task and is extensively used in the literature [GKJ12, CWY13, CHL+16].

For the semi-bandit feedback to work effectively, we assume some smoothness properties on the reward function. These smoothness properties ensure that when the learning parameters are estimated with a certain precision, one can approximate the true reward with high accuracy. Formally, the reward function \( R_S(\mu) \), as a function of stochastic parameter \( \mu \), satisfies the following properties.

**Property 1. Monotonicity:** Let \( \mu, \mu' \in [0, 1]^k \) be two vectors such that \( \mu'_i \geq \mu_i \) for all \( i \in [k] \) then, for any \( S \subseteq [k], R_S(\mu') \geq R_S(\mu) \).

The monotonicity property implies that the reward from any subset increase if the mean qualities of an base arms increase.

**Property 2. Lipschitz Continuity:** There exists real valued constant \( C \geq 1 \) such that for all \( S \subseteq [k], \) we have \( |R_S(\mu) - R_S(\mu')| \leq C \max_{i \in S} |\mu_i - \mu'_i|\).

**Property 3. Bounded Smoothness:** There exists a strictly increasing function \( f \) such that for any \( S \subseteq [k], \) \( |R_S(\mu) - R_S(\mu')| \leq f(\Lambda) \text{ whenever } \max_{i \in S} |\mu_i - \mu'_i| \leq \Lambda \).

In our first setting we study the reward function \( R(\cdot) \) satisfying monotonicity (Property 1) and the Lipschitz continuity property (Property 2) whereas in the setting setting we consider Property 1 and Property 3. With a slight abuse of terminology, we call the first setup as Lipschitz smoothness and the second setup (i.e. monotonicity and bounded smoothness) as the Bounded smoothness.

The reward assumptions and regret notion considered in the paper encompass many specialized settings studied in literature as a special case. For instance, additive rewards with a fixed number of arms to pull [KWAS15b], submodular rewards with volatile bandits [CXL18], average reward, and so on. However, we remark here that the technical treatment of this problem requires newer proof techniques as the existing proof techniques from combinatorial bandits setup do not generalize trivially to the sleeping combinatorial bandits setting.

\(^1\)That is for any set \( S \subseteq [k], R_S(\mu) = R_S(\mu') \) if \( \mu_i = \mu'_i \) for all \( i \in S \).
Main Results of the Paper

• In the Lipschitz smoothness setting, we show that CS-UCB achieves $O(\log(T)/\Delta_{\min})$ instance-dependent regret guarantee (See Theorem 1). Here, $\Delta_{\min}$ is the difference between the reward from an optimal super-arm and a sub-optimal super-arm with maximum reward.

• Note that for smaller values of $\Delta_{\min}$, the regret guarantee of Theorem 1 regret guarantee is vacuous. In Theorem 2, we show that CS-UCB attains an instance-dependent regret guarantee of $O(\sqrt{\sigma kT \log(T)})$. Here, $\sigma = \Delta_{\max}/\Delta_{\min}$. Note that, in contrast with Theorem 1 this result depends only on the range of rewards of super-arms. In particular, if the best and worst super-arms do not have a large reward ratio, the result in Theorem 2 is tight. We refer to this setting as weak instance-dependent.

• Next, in Theorem 3, we obtain a $O(\sqrt{T^2 \log(T)})$ instance-independent regret guarantee without any dependence on $\sigma$ in a Lipschitz smoothness setting.

• Finally, in a Bounded smoothness setting, in Theorem 4, we show that CS-UCB attains $O(\log(T))$ regret guarantee. Though a similar result exists for the non-sleeping case [CWY13]; the regret analysis does not trivially generalize to the combinatorial sleeping MAB setting.

3 The Setting

In this paper, we consider that only a subset $A_t \subseteq [k]$ of arms is available at time $t$. Note that, $A_t$ is revealed only at time $t$. Further, let $S_t \subseteq A_t$ be the set of arms pulled by the algorithm at time $t$. The set $S_t$ is also called as a super-arm. To evaluate the performance of an algorithm with limited availability of arms, we extend the notion of regret considered for classical CMAB problem appropriately and call it a sleeping regret given by $R_{\text{Alg}}(T) := \max_{(A_t)_{t=1}^T} \mathbb{E}_{\text{Alg}}[\sum_{t=1}^T (R_{S_t^*} - R_{S_t})]$. Here, $S_t^* \in \arg\max_{S\subseteq A_t} R_S$. Note that when $A_t = [k]$ for all $t$, we recover the setting of [CWY13].

Next, we define the regret in the presence of $(\gamma, \beta)$-oracle. Let, $B_t$ be the event that an oracle returns an $\gamma$-approximate solution at time $t$ i.e. $B_t = \{R_{S_t^*} \geq \gamma \cdot R_{S_t^*}\}$. Note that $P(B_t) \geq \beta$. The expected sleeping regret of Alg with oracle access is given by,

$$R_{\text{Alg}}(T) = \max_{(A_t)_{t=1}^T} \mathbb{E}_{\text{Alg}}[\sum_{t=1}^T (\gamma \cdot \beta \cdot R_{S_t^*} - R_{S_t})]. \tag{1}$$

Notational Setup

We begin with the additional notation required to prove the results. For each base arm $i \in [k]$, let $N_{i,t}$ denotes the number of times arm $i$ is pulled till time $t$ and $\mu_{i,t}$ be the average reward obtained from arm $i$ till (and excluding) time $t$. Let

$$\bar{\mu}_{i,t} := \mu_{i,t} + \sqrt{3 \log(t)/2N_{i,t}}. \tag{2}$$

Following a standard terminology, we call $\tilde{\mu}_{i,t}$ as the UCB estimate of arm $i$ at time $t$. Furthermore, let $\Delta_S := \gamma \cdot \text{OPT}_A - R_S$ be the regret incurred by pulling super-arm $S$. Here, $\text{OPT}_A := R_{S^*} = \max_{S \subseteq A} R_S$ denotes the optimal reward when the set of available arms is $A$. A super-arm $S \subseteq A$ is bad (sub-optimal), if $\Delta_S > 0$. For a given $A \subseteq [k]$, we define the set of bad super-arms as $S_B(A) = \{S \subseteq A | \Delta_S > 0\}$. Further, for a given $A \subseteq [k]$, define

$$\Delta_{\min}(A) = \gamma \cdot \text{OPT}_A - \max_{S \in S_B(A)} R_S \quad \text{and},$$
\[ \Delta_{\text{max}}(A) = \gamma \cdot \text{OPT}_A - \min_{S \in \mathcal{S}_B(A)} R_S. \]

Note that, for any availability set \( A \), we have \( \Delta_{\text{max}}(A) \geq \Delta_{\text{min}}(A) > 0 \). The strict inequality follows from the definition of \( \mathcal{S}_B(A) \). Next, define \( \Delta_{\text{max}} = \max_{A \subseteq [k]} \Delta_{\text{max}}(A) \) and \( \Delta_{\text{min}} = \min_{A \subseteq [k]} \Delta_{\text{min}}(A) \). Arm \( i \) is called saturated if it is pulled for sufficiently many number of time steps, i.e., \( N_{i,t} \geq \ell_t \), where \( \ell_t \) works as threshold exploration. Note that a saturated arm at any time instant may become unsaturated in future. Further, we call a set \( A_t \) explored if all the arms in \( A_t \) are saturated, i.e., \( N_{i,t} \geq \ell_t \) for all \( i \in A_t \). First observe that if \( A_t \) is either empty or a singleton set then CS-UCB incurs a zero regret. Hence, without loss of generality we assume that \( |A_t| \geq 2 \) for all \( t \leq T \). We first make following useful observation.

**Observation 1.** Let \( \ell_t \in \mathbb{R}_+ \) be a positive number, \( S \subseteq [k] \) be any non-empty set of arms and such that \( N_{i,t} \geq \ell_t \) for all \( i \in S \) and \( \varepsilon_t := \sqrt{\frac{3 \log(t)}{2t}} \), then \( \mathbb{P}\left\{ \max_{i \in S} |\mu_{i,t} - \mu_i| < 2\varepsilon_t \right\} \geq 1 - 2|S|/t^3 \).

The proof of Observation 1 follows from Hoeffding’s inequality and is presented in the supplementary material for completeness.

**CS-UCB**

Note that the proposed CS-UCB algorithm is the same as CUCB [CWY13] except that at each time, only a subset of the arms is available, and the regret notion considered is sleeping regret instead of conventional regret. Similar to CUCB, we assume that the algorithm has access to a \((\gamma, \beta)\)-approximation oracle.

At each time \( t \), CS-UCB receives the set of available arms \( A_t \). If there is a base arm in \( A_t \) which is not pulled previously, an algorithm pulls all the available arms. For each time instances where all available arms are pulled atleast once, CS-UCB obtains \( S_t = \text{ORACLE}(\overline{\mu}_t, A_t) \). Here, \( \overline{\mu}_t \) represent the vector of UCB estimates given by Equation 2. The algorithm then pulls a super-arm \( S_t \) and obtain rewards \( R_{S_t}(\mu) \) and an individual base arm rewards (semi-bandit feedback) \( X_{i,t} \) for each \( i \in S_t \). Finally, CS-UCB update parameters

- \( N_{i,t+1} = N_{i,t} + 1 \) (\( i \in S_t \))
- \( \overline{\mu}_{i,t+1} = \frac{N_{i,t}\overline{\mu}_{i,t} + 1(i \in S_t) - X_{i,t}}{N_{i,t} + 1(i \in S_t)} + \sqrt{\frac{3 \log(t)}{N_{i,t} + 1(i \in S_t)}}. \)

Note that the regret (Equation 1) depends on the rewards from the base arm \( X_{i,t} \) only through \( R(\cdot) \). Further, observe that when \( A_t = [k] \) for all \( t \), the sleeping regret is same as conventional regret guarantee and CS-UCB is same as CUCB; hence the regret guarantees of [CWY13] will apply.

### 4 Regret Analysis of CS-UCB

In our first result, we show that under the Lipschitz smoothness setting, CS-UCB incurs a logarithmic instance-dependent regret. However, note that the regret depends inversely on the \( \Delta_{\text{min}} \) value. That is, for arbitrarily smaller values of \( \Delta_{\text{min}} \), the regret bound is vacuous. In Theorem 2, we prove that the weak instance-dependent regret of the proposed algorithm is \( O(\sqrt{\sigma kT \log(T)}) \). Here,
\[ \sigma = \Delta_{\text{max}}/\Delta_{\text{min}}; \] i.e., this result depends only on the ratio of the maximum and minimum achievable rewards. Finally, in Theorem 3, we show that the instance-independent regret of the proposed algorithm is \( O\left(\sqrt{\frac{kT^2 \log(T)}{2}}\right) \) in general. We begin with the following observation.

**Observation 2.** For all time instances \( t \) such that \( N_{i,t} > 0 \) and the reward function satisfies monotonicity and Lipschitz continuity (Properties 1 and 2), \( \Delta_{S_t} \leq C \left(1 + \sqrt{3 \log(T)/2}\right) \).

We are ready to present our first result.

**Theorem 1.** The expected sleeping regret incurred by CS-UCB when the reward function satisfies Lipschitz condition (Properties 1 and 2) is given by

\[
R_{\text{CS-UCB}}(T) \leq 2\beta k C \left[ \zeta(3)(1 + \sqrt{\frac{3 \log(T)}{2}}) + 3 \frac{\sigma C \log(T)}{\Delta_{\text{min}}} \right]
\]

Here, \( \zeta \) is the Riemann zeta function and \( \sigma = \Delta_{\text{max}}/\Delta_{\text{min}} \).

**Proof Outline:** Set \( \ell_t := 6C^2 \log(t)/\Delta_{\text{min}}^2 \) and \( \epsilon_t := \sqrt{3 \log(t)/2\ell_t} \) and divide the time instants into sets \( T_e \) and \( T_u \) as described as follows. Let \( T_e \) be the set of time instances \( t \) such that \( A_t \) is explored, i.e., \( T_e = \{t \leq T : N_{i,t} \geq \ell_t, \forall i \in A_t\} \) and \( T_u = [T] \setminus T_e \). Further let, for \( t \in T_u \), \( A_{e,t} \) be the set of saturated arms that are available at time \( t \), i.e., \( A_{e,t} := \{i | N_{i,t} \geq \ell_t\} \) and \( A_{u,t} := A_t \setminus A_{e,t} \). We have

\[
T_u = \left\{ t : \exists j \in A_{u,t} \right\} = \left\{ t : \exists j \in A_{u,t} \cap S_t \right\} \cup \left\{ t : \forall j \in A_{u,t}, j \notin S_t \right\}.
\]

We bound the sleeping regret incurred in disjoint sets \( T_e, D \) and \( E \) separately. Recall that \( B_t \) is an event that the oracle returns \( \gamma \)-approximate solution i.e. \( R_{S_t}(\mu_t) \geq \gamma \cdot R_{S^*}(\mu_t) \). We begin with following supporting lemmas.

**Lemma 1.** For all \( t \in T_e \) we have \( \mathbb{P}\{S_t \in S_B(A_t)|B_t\} \leq 2|A_t|T^{-3} \).

**Lemma 2.** \( |D| \leq k\ell_T \).

**Lemma 3.** For all \( t \in E \) we have, \( \mathbb{P}\{S_t \in S_B(A_t)|B_t\} \leq 2|S_t|/t^3 \).

Lemma 1 establishes that when all the base arms in the availability set are sufficiently explored then the set \( S_t \) returned by the oracle is an optimal set with high probability. This result follows from the fact that, as all the available arms are sufficiently pulled in the past, \( \mu_t \) is sufficiently close to \( \mu \). Lemma 2 follows directly from the fact that each base arm remains unsaturated till at most \( \ell_T \) pulls.

Finally, in Lemma 3 we handle the case that the availability set contains both saturated and unsaturated base arms. Note that the previous two lemmas also hold for CMAB settings. However, in contrast with CMAB, in our setting, the availability sequence may be such that at each time instant only a few explored arms are available and this may lead to high regret. Lemma 3 dismisses this hypothesis. First, note that only those base arms that are available but not-pulled are responsible for the regret. Furthermore, if an arm is available and it is not pulled for many time instances, its UCB
estimate increases and hence increasing its chances of getting pulled in the future due to the monotonicity assumption. This means that an optimal subset of the availability set will be pulled after some time with high probability. The detailed proof of Lemmas 1, 2 and 3 are given in supplementary material.

**Putting everything together:** For a given arbitrary availability sequence \( (A_t)_{t=1}^T \), the regret of CS-UCB is given as

\[
\mathcal{R}_{\text{CS-UCB}}(T) = \mathbb{E} \left[ \sum_{t \in [T]} \gamma \cdot \beta \cdot R_{S_t^*} - R_{S_t} \right]
\]

\[
\leq \sum_{t \in [T]} \gamma \cdot R_{S_t^*} - R_{S_t} \mid B_t \cdot \beta
\]

\[
\leq \left[ \sum_{t \in T_e \cup E} \mathbb{P} \{ S_t \in S_B(A_t) \mid B_t \} \cdot \Delta_s_t + \sum_{t \in D} \mathbb{P} \{ S_t \in S_B(A_t) \mid B_t \} \cdot \Delta_{\text{max}} \right] \cdot \beta
\]

\[
\leq \left[ \sum_{t \in T_e \cup E} \frac{2|A_t|}{\ell^3} \Delta_s_t + k\ell_T \Delta_{\text{max}} \right] \cdot \beta \quad \text{(From Lemmas 1, 2 and 3)}
\]

\[
\leq 2C \left( 1 + \sqrt{\frac{3 \log(T)}{2}} \right) \sum_{t=1}^{\infty} \frac{1}{t^3} + \frac{6kC^2 \log(T)}{\Delta_{\text{min}}} \cdot \Delta_{\text{max}} \right] \cdot \beta \quad \text{(from Observation 2)}
\]

\[
\leq \left[ 2kC \zeta(3) \left( 1 + \sqrt{\frac{3 \log(T)}{2}} \right) + \frac{6C^2 k \sigma \log(T)}{\Delta_{\text{min}}} \right] \cdot \beta
\]

Since the last equation holds for any arbitrary sequence \( (A_t)_{t=1}^T \), it also holds for an adversarially chosen availability sequence. This completes the proof of the theorem. \( \Box \)

Notice that the regret guarantee in Theorem 1 depends on the value of \( \Delta_{\text{min}} \). If this value is sufficiently low the regret guarantee is vacuous. In the next result, we show a weak instance-dependent regret guarantee where the regret is given in terms of the ratio \( \Delta_{\text{max}} / \Delta_{\text{min}} \).

**Theorem 2.** The weak instance-dependent sleeping regret of CS-UCB when the reward function satisfies Lipschitz condition (properties 1 and 2) is given by

\[
\mathcal{R}_{\text{CS-UCB}}(T) \leq 4C \sqrt{6k\sigma T \log(T)} + 2kC \zeta(3).
\]

Here, \( \zeta(.) \) is a Reimann zeta function and \( \sigma = \Delta_{\text{max}} / \Delta_{\text{min}} \).

It is easy to see that for large values of \( \Delta_{\text{min}} \) one can use the result of Theorem 1 to obtain the desired bound of Theorem 2. However, when \( \Delta_{\text{min}} \) is small i.e. \( \Delta_{\text{min}} < C \sqrt{6k\sigma \log(T)/T} \), the upper bound on regret is obtained by parametrized analysis with selecting parameter \( \eta \in (\Delta_{\text{min}}, \Delta_{\text{max}}] \) appropriately to minimize the regret. The detailed proof of Theorem 2 is given in supplementary material. Observe that the regret dependence of Theorem 2 on time horizon increase from \( O(\log(T)) \) to \( O(\sqrt{T \log(T)}) \) when we consider the weak instance-dependent regret guarantee. In the next result, we further relax the dependence on instance parameters \( \sigma \) to obtain a strong instance-independent regret guarantee of \( O(\sqrt[3]{T^2 \log(T)}) \).
Theorem 3. The instance-independent sleeping regret of CS-UCB when the reward function satisfies Lipschitz condition (Properties 1 and 2) is given by

\[ R_{\text{CS-UCB}}(T) \leq C(1 + \lambda) \cdot \sqrt{6kT^2 \log(T)} + 2k\lambda C\zeta(3) \]

where \( \lambda = (1 + \sqrt{3 \log(T)}/2) \).

First, using Theorem 1 we establish that the said instance-independent upper bound holds in this setting if \( \Delta_{\text{min}} \geq \left( \frac{T}{\log(T)} \right)^{-1/3} \). Then, similar to Theorem 2, we split the regret at any time \( t \) into two parts, where the per time regret is at most \( \eta \) and larger than \( \eta \) with \( \eta \geq \left( \frac{T}{\log(T)} \right)^{-1/3} \). As stated previously, this result provides the instance-independent regret guarantee without any additional restrictions on minimum and maximum rewards.

Theorem 4. The expected sleeping regret incurred by CS-UCB when the reward function satisfies bounded smoothness condition (Properties 1 and 3), is upper bounded by

\[ R_{\text{CS-UCB}}(T) \leq \left[ \frac{6 \log(T)}{(f^{-1}(\Delta_{\text{min}}))^2} + 2\zeta(3) \right] k \cdot \Delta_{\text{max}}. \]

A detailed proof is provided in supplementary material. Note that the proof technique closely follow Theorem 1. We remark here that we recover the regret bound of [CWY13] for non-sleeping combinatorial bandits case i.e., when \( A_t = [k] \) for all \( t \). Further, observe that when rewards are additive, i.e. \( R_{S_t} = \sum_{i \in S_t} X_{i,t} \) one can achieve \( \tilde{O}(\sqrt{T}) \) regret bound [KWAS15b]; however it is not clear if the \( \tilde{O}(\sqrt{T}) \) regret upper bound holds under bounded smoothness assumption. Finally, the instance-independent regret (Theorem 2 and Theorem 3) guarantee under bounded smoothness condition follows trivially by choosing \( C = \sup_{x \in [0,1]} f(x) \).

5 Simulation Results

In this section we validate the theoretical results of the paper using different reward functions on simulated data. In particular, we perform experiments on two different combinatorial bandits settings studied in the literature [JGB+18, KWAS15a]. In the first setting, the average quality of base arm \( i \) takes the form \( a_i \cdot \mu_i - b_i \); here \( \mu_i \) is a mean of the Bernoulli random variable and \( a_i \) and \( b_i \) are unknown constants. In this setting, the quality is also referred to as the utility from arm \( i \); where \( a_i \cdot X_i \) being random reward with mean \( \mu_i \) and \( b_i \) being the fixed cost corresponding to arm \( i \). We call this reward setting \(^2\) as UtilReward. The goal is to select all the available base arms with positive quality. In the second setting (which we call TopKReward), we consider the problem of pulling top \( K \) (in terms of quality) available arms and the reward function is additive \(^3\). In this setting, note that, if at some time \( t, |A_t| \leq K \), all the available arms are pulled and the regret at time instant \( t \) is zero. Further observe that, both the settings admit polynomial time exact oracles; i.e. \( (1, 1)\) ORACLE.

Simulation Setup and Observations

We run two experiments for each of the reward settings mentioned above. In the first experiment which we call ExpOne, the quality parameter \( \mu_i \) of each of the base arms \( i \) is chosen independently

\(^2\)See [JGB+18] for detailed motivation and applications of this setting.

\(^3\)More details and the regret analysis in non-sleeping case is given in [KWAS15a].
Figure 1
Regret Vs Time Plots For UtilReward: From L to R, (a) ExpOne: Instance-dependent regret with randomly generated qualities (Theorem 1) (b) ExpOne: Instance-dependent guarantee for $\Delta_{\text{min}} = 0.001$ (c) ExpTwo: Weak instance-dependent guarantee (Theorem 2) (d) ExpTwo: Instance-independent regret guarantee (Theorem 3).

from uniform distribution over interval $[0.3, 0.8]$. A quality feedback from the base arm $i \in S_t$ is an independent sample from a Bernoulli distribution with mean $\mu_i$. The availability parameter corresponding to arm $i$ is uniformly sampled from $[0.4, 0.9]$. Similar to the quality feedback, availability of $i$ is decided by a random draw from a Bernoulli distribution with a given availability parameter.

The second experiment, ExpTwo, is designed to validate the results of Theorem 2 and 3. The availability of base arms is generated using same approach as in the first experiment. However, the qualities of base arms is fixed to be close to each other. We validate the result of Theorem 2, by fixing the value of $\sigma := \Delta_{\text{max}}/\Delta_{\text{min}}$ and varying the values of $\Delta_{\text{min}}$ and Theorem 3 by varying the values of $\Delta_{\text{min}}$. Each of the experiments is executed over time horizon $T = 10^6$ and the average rewards from 50 independent runs.

We present the plots associated to UtilReward reward function in Fig. 1. The first two plots in Fig. 1 show that as $\Delta_{\text{min}}$ value decreases, the expected regret guarantee of Theorem 1 becomes vacuous. The next two plots show that the the regret dependence on time horizon increases from $\sqrt{T}$ to $\frac{3}{2} \sqrt{T^{2.5}}$ for similar values of $\Delta_{\text{min}}$ when we fix $\sigma$ and change $\Delta_{\text{min}}$ to arbitrary values of $\Delta_{\text{min}}$ and $\Delta_{\text{max}}$. Similar results were observed for different values of $\sigma, k, \Delta_{\text{min}}$ and reward function TopKReward (Fig. 2).

6 Related Work

The stochastic bandits problem has been extensively studied in the literature [LR85, ACBF02, Tho33, AG12]. We refer the reader to [LS18, Sli19] for a book exposition on multi-armed bandits and their applications. Most previous work in literature— with few exceptions such as [CGJ+17, KNMS10, LLJ19] — assume that all the arms are available at all time instants. It is shown that the classical
Figure 2

Regret Vs Time Plots For TopKReward: From L to R, (a) ExpOne: Instance-dependent regret with randomly generated qualities (Theorem 1) (b) ExpOne: Instance-dependent guarantee for $\Delta_{\text{min}} = 0.001$ (c) ExpTwo: Weak instance-dependent guarantee (Theorem 2) (d) ExpTwo: Instance-independent regret guarantee (Theorem 3).

algorithms, adapted appropriately, are also optimal in a sleeping bandits setting [CGJ+17, KNMS10].

Combinatorial multi-armed bandits (CMAB) is another well studied variant of stochastic MAB problem which considers multi-pull setup [CBL12, CLK+14, CWY13, CTMSPl15, GJK10, GJK12, KWAS15b, LLJ19, Ont13, WC18, WKA15]. [CWY13] consider a general reward function with some smoothness condition and proposed CUCB, a UCB-style algorithm. In contrast, we consider arbitrary arm availability and general rewards and show that CUCB when extended to sleeping bandits setting achieves optimal regret guarantee. We also remark here that their analysis does not generalize to combinatorial sleeping bandits, and hence we need novel proof techniques to bound the sleeping regret in a CMAB setting. To the best of our knowledge, we are the first to address combinatorial sleeping MAB with a general reward structure and provide instance-dependent as well as instance-independent regret upper bound.

The closest work to this work is [CXL18]. Similar to their work we consider semi-bandit feedback and combinatorial sleeping bandits framework. However, [CXL18] considers contextual bandits setting, whereas we study a sleeping combinatorial MAB setting. The proposed algorithms (CC-MAB and CS-MAB, respectively) differ crucially in how they carry out exploration. CC-MAB explores the
subset of available arms if it contains at least a single unsaturated base arm ([CXL18], Algorithm 2, Line 7). Hence, the exploitation is carried only if all the available base arms are saturated. In contrast, CS-UCB does not demarcate the exploration and exploitation in this manner. So, even if some “obviously” bad super-arms are not explored, CS-UCB does not pull them. Also, there are following two important differences in the setting considered. Firstly, [CXL18] consider that the reward function is submodular, whereas we consider general reward functions. Indeed, if the reward function satisfies submodularity, our results can be extended easily by considering \((1 - \frac{1}{e})\)-approximation oracle. Secondly, they consider that the time horizon is a-priori known to the algorithm, which may be an unrealistic assumption in many practical cases. Note that we provide an any-time regret guarantee, i.e., \(T\) is not given as an input to the algorithm. Also, they proved that CC-MAB achieves the regret of \(O(\sqrt{T^2 \log(T)})\) for the specific case of the submodular reward function, whereas we provide similar regret bound with more general reward functions. The recent work of [NET20] also studies contextual combinatorial bandits set up with sleeping arms and semi-bandit feedback. The authors consider a setting where the arms are differentiated based on the context.

7 Conclusion and Future Work

In this paper, we considered combinatorial sleeping multi-armed bandits setting where a subset of arms is available at a given time instant. We analyzed the CS-UCB algorithm and analyzed its regret guarantee under two setups; Lipschitz smoothness and Bounded smoothness. We showed that under Lipschitz smoothness setting, CS-UCB achieves \(O(\log(T)/\Delta_{\min})\) instance-dependent sleeping regret guarantee. Additionally, we prove that CS-UCB achieves \(O(\sqrt{T \log(T)})\) weak instance-dependent regret under the assumption that the ratio of maximum and minimum achievable rewards is bounded. Also, we provide \(O(\sqrt{T^2 \log(T)})\) instance-independent regret in the most general case. We also show that CS-UCB in Bounded smoothness setting matches the conventional regret guarantee for the combinatorial MAB setting under the same set of assumptions (i.e., \(O(\log(T))\)). Finally, we validate the proven theoretical guarantees through experiments.

The instance-independent regret guarantee under Bounded smoothness setting remains an interesting open problem. Also, a finely tuned analysis with availability specific regret guarantees is an interesting future direction. This setup could be used together with other MAB settings, for instance, rotting bandits [LCM17], where the arm pulling strategy may lead to the dropping of the arms.

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A Preliminaries

Definition 1. Let $X_1, X_2, \ldots, X_n$ be $n$ independent random variables, and $S_n = X_1 + X_2 + \ldots + X_n$, where $\forall i, X_i \in [a_i, b_i]$, then according to Hoeffding’s inequality,

$$\Pr \{ S_n - \mathbb{E}[S_n] \geq t \} \leq e^{-2t^2 \sum (b_i - a_i)^2}$$

A.1 Notation

| Symbol | Definition |
|--------|------------|
| $[k]$  | Set of arms. |
| $T$    | Time horizon (or the number time steps). |
| $A_t$  | Set of arms available at time $t$. |
| $\mu_i$ | Bernoulli parameter (or mean) for arm $i$. |
| $\mu_t$ | The vector of mean rewards of each arm in set $A_t \subset [k]$ at time $t$. |
| $\overline{\mu}_t$ | The vector of UCB estimates of means (unknown) for each arm in set $A_t \subset [k]$ at time $t$. |
| $S_t$  | The subset of arms (super-arm) pulled at time $t$. |
| $R_{S_t}$ | Reward obtained when super-arm $S_t$ is pulled at time $t$. |
| $S^*_t$ | $\arg\max_{S \subseteq A_t} R_S$. |
| $S_B(A)$ | The set of all bad super-arms i.e. $S_B(A) = \{ S \subseteq A | \Delta_S > 0 \}$. |
| $X_{i,t}$ | A random reward obtained at time $t$ from arm $i$. |
| $N_{i,t}$ | Number of times arm $i$ is pulled till $t$ time steps. |
| $\hat{\mu}_{i,t}$ | $X_{i,1:t}/N_{i,t}$; Empirical estimate of arm $i$ till time step $t$. |
| $\varepsilon_{i,t}$ | $\sqrt{3 \log(t)/2N_{i,t}}$; UCB confidence interval of arm $i$ till step $t$. |
| $\varepsilon_t$ | $\sqrt{3 \log(t)/2T}$, |
| $T_e$  | The set of time instances $t$ such that $A_t$ is explored, i.e., all arms in $A_t$ are saturated. |
| $T_u$  | $[T] \setminus T_e$. |
| $A_{e,t}$ | The set of saturated arms that are available at time $t$. |
| $A_{u,t}$ | $A_t \setminus A_{e,t}$ |
| $\Delta_{S_t}$ | $\gamma \cdot R_{S^*_t}(\mu) - R_{S_t}(\mu)$; Quantitative measure for sub-optimality of super-arm $S_t$. |

Table 1

Notation Table

A.2 Algorithm: CS-UCB
Algorithm 1 CS-UCB

1: Initialization:
2: for $i \in [k]$ do
3:     $N_{i,0} = 0, \overline{\mu}_{i,0} = 1, X_{i,0} = 0$
4: end for
5: for $t = 1, 2, 3, \ldots$ do
6:     Observe set of available arms as $A_t$
7:     if $\exists j \in A_t$, such that, $N_{j,t} = 0$ then
8:         Select $S_t = A_t$
9:     else
10:         $S_t = \text{ORACLE}(A_t, \overline{\mu}_t)$
11: end if
12: Observe: Semi-bandit feedback as $X_{j,t} \in \{0, 1\}, \forall j \in S_t$ and $R_{S_t}(\mu)$
13: Update:
14:     $N_{i,t} = \begin{cases} N_{i,t-1} - 1 & \text{if } \forall i \notin S_t \\ N_{i,t-1} + 1 & \text{if } \forall i \in S_t \end{cases}$
15:     $\overline{\mu}_{i,t} = \frac{X_{i,t}}{N_{i,t}} + \sqrt{\frac{3 \log(t)}{2N_{i,t}}}$
16:     $X_{i,t} = \begin{cases} X_{i,t-1} - 1 & \text{if } \forall i \notin S_t \\ X_{i,t-1} + X_{i,t} & \text{if } \forall i \in S_t \end{cases}$
17: end for

B Omitted Proofs

We begin with introducing additional notation used in rest of the paper. Let $R_S(\overline{\mu}; \mu)$ denote the reward obtained according to the quality vector $\mu$ from the set $S$ which also satisfies the condition that $R_S(\overline{\mu}) \geq \gamma \cdot R_{S^*}(\overline{\mu})$. Here, $R_{S^*} = \max_{S \in A} R_S(\overline{\mu})$.

Observation 1. Let $\ell_t \in \mathbb{R}_+$ be a positive number, $S \subseteq [k]$ be any non-empty set of arms and such that $N_{i,t} \geq \ell_t$ for all $i \in S$ and $\varepsilon_t := \sqrt{\frac{3 \log(t)}{2\ell_t}}$, then $\mathbb{P}\{\max_{i \in S} |\overline{\mu}_{i,t} - \mu_i| < 2\varepsilon_t \} \geq 1 - 2|S|/t^3$.

Proof. Using Hoeffding’s lemma [Hoe63] we have,

$$\mathbb{P}\{|\hat{\mu}_{i,t} - \mu_i| \geq \varepsilon_t \} \leq 2e^{-2N_{i,t}\varepsilon_t^2} = 2e^{-3N_{i,t}\log(t)/\ell_t} \leq 2/t^3.$$  \hfill (3)

Here, $\hat{\mu}_{i,t}$ is the empirical mean reward of arm $i$ till time $t$. The last inequality follows from the fact that $N_{i,t} \geq \ell_t$. Further, from the definition of $\overline{\mu}$, with probability at least $1 - \frac{2}{t^3}$,

$$\varepsilon_t > |\hat{\mu}_{i,t} - \mu_i| = |\overline{\mu}_{i,t} - \mu_i - \varepsilon_t| \geq |\overline{\mu}_{i,t} - \mu_i| - \varepsilon_t.$$  \hfill (4)

Here, (i) follows from Equation 3, (ii) is immediate from the definition of $\mu$ and finally (iii) follows from the triangle inequality. Thus, we have $|\overline{\mu}_{i,t} - \mu_i| < 2\varepsilon_t$ with probability at least $1 - 2|S|/t^3$. \hfill \(\square\)
\textbf{Observation 2.} For all time instances \( t \) such that \( N_{i,t} > 0 \) and the reward function satisfies monotonicity and Lipschitz continuity (Properties 1 and 2), \( \Delta_{S_t} \leq C(1 + \sqrt{3 \log(T)/2}) \).

\textit{Proof.} The monotonicity property and Lipschitz smoothness implies that,
\[
|R_{S_t}(\overline{\mu}_t) - R_{S_t}(\mu)| = R_{S_t}(\overline{\mu}) - R_{S_t}(\mu) \leq C \max_{i \in S_t} |\overline{\mu}_{i,t} - \mu_i|
\]
(Monotonicity property and Lipschitz property)

However,
\[
R_{S_t}(\overline{\mu}_t) - R_{S_t}(\mu) \geq \gamma \cdot R_{S_t}'(\overline{\mu}_t) - R_{S_t}(\mu) \geq \gamma \cdot R_{S_t}'(\mu) - R_{S_t}(\mu) = \Delta_{S_t}.
\]
(5)

Further, from the definition of \( \overline{\mu}_{i,t} = \hat{\mu}_{i,t} + \varepsilon_{i,t} \), where \( \varepsilon_{i,t} = \sqrt{3 \log(T)/2N_{i,t}} \) for arm \( i \) till time \( t \), we have
\[
|\overline{\mu}_{i,t} - \mu_i| \leq |\hat{\mu}_{i,t} - \mu_i| + \varepsilon_{i,t} \leq 1 + \sqrt{3 \log(T)/2}.
\]
(6)

From Eq. 3 and Eq. 4, observe that \( \Delta_{S_t} \leq C \max_{i \in S_t} |\overline{\mu}_{i,t} - \mu_i| \leq C(1 + \sqrt{3 \log(T)/2}) \).

\textbf{Lemma 1.} For all \( t \in T_c \) we have \( \mathbb{P}\{S_t \in S_B(A_t)|B_t\} \leq 2|A_t|t^{-3} \).

\textit{Proof.} Let \( S_t' = \arg\max_{S \subseteq A_t} R_S \) and the event \( B_t \) has occurred. We prove the lemma using the following supporting claim.

\textbf{Claim 1.} Let \( t \in T_c \) and \( S_t = \text{ORACLE}(A_t, \overline{\mu}) \) and \( S'_t = \text{ORACLE}(A_t, \mu) \). Then \( \mathbb{P}\{R_{S'_t}(\mu) = R_{S_t}(\mu)\} \geq 1 - 2|A_t|/t^3 \).

To see the proof of the lemma observe that
\[
R_{S_t}(\mu) = R_{S_t}'(\mu) \geq \gamma \cdot R_{S_t}'(\mu) = \gamma \cdot \text{OPT}_\mu(A_t).
\]

The first equality in the above equation is true with probability atleast \( 1 - 2|A_t|/t^3 \) from Claim 1. The first inequality holds from the fact that the event \( B_t \) has occurred. Hence, we have \( \mathbb{P}\{S_t \notin S_B(A_t)|B_t\} \geq 1 - 2|A_t|/t^3 \). This completes the proof of the lemma.

\textit{Proof of Claim 1.} First note that, it is enough to show that \( S_t' = S_t \). However, these sets might not be unique and hence we assume \( S_t' \neq S_t \). Let \( Q_t' \in \arg\max_{S \subseteq A_t} R_S(\mu) \) and \( S_t^* \in \arg\max_{S \subseteq A_t} R_S(\overline{\mu}) \).

From the monotonicity property of \( R \) and the definition of \( \overline{\mu} \) it holds that
\[
R_{S_t}(\overline{\mu}_t) \geq R_{S_t'}(\overline{\mu}_t) \geq R_{S_t}'(\mu)
\]
(7)

Here, the first inequality follows from the optimality of \( S_t \) with respect to \( \overline{\mu}_t \) and the second inequality follows from the monotonicity property. For contradiction, let us assume that \( S_t \neq S_t' \) and \( R_{S_t'}(\mu) > R_{S_t}(\mu) \). Using this inequality with Equation 7 we get \( R_{S_t}(\overline{\mu}_t) > R_{S_t}(\mu) \). From Lipschitz property we have,
\[
R_{S_t}(\overline{\mu}_t) - R_{S_t}(\mu) = |R_{S_t}(\overline{\mu}_t) - R_{S_t}(\mu)| \leq C \max_{i \in S_t} |\overline{\mu}_{i,t} - \mu_i|.
\]
(8)
Let \( \ell_t = 6C^2 \log(t)/\Delta_{\min}^2 \). As \( t \in T_e \), we have \( N_{i,t} \geq \ell_t \) for all \( i \in A_t \). Hence, from Observation 1, with probability at least \( 1 - \frac{2|A_t|}{\ell_t^2} \), we have, \( \max_{i \in S_t} |\pi_{i,t} - \mu_i| \leq \max_{i \in A_t} |\pi_{i,t} - \mu_i| < 2\varepsilon_t \). This gives, \( R_{S_t}(\pi_t) - R_{S_t}(\mu) < 2C \cdot \varepsilon_t = \Delta_{\min} \). To see the last inequality recall from Observation 1 that \( \varepsilon_t = \sqrt{\frac{3 \log(t)}{2\ell_t}} \). Hence, we have \( \Delta_{\min} > \gamma \cdot R_{S_t}(\pi) - R_{S_t}(\mu) \geq \gamma \cdot R_{S_t}(\mu) - R_{S_t}(\mu) \). This contradicts the definition of \( \Delta_{\min} \). Thus, with probability at least \( 1 - \frac{2|A_t|}{\ell_t^2} \) we have that \( R_{S_t}^{\star}(\pi_t) = R_{S_t}(\mu_t) \). This completes the proof of the claim.

**Lemma 2.** \( |D| \leq k\ell_T \).

**Proof.** Recall that by definition, we have \( |D| := |\{ t : \exists j \text{ such that } j \in A_{u,t}, j \in S_t \}| \). Hence we have,
\[
|D| = |\{ t : \exists j \text{ such that } N_{j,T} < \ell_T \}| \leq \sum_{j=1}^k |\{ t : N_{j,T} \leq \ell_T \}| \leq k\ell_T.
\]

**Lemma 3.** For all \( t \in E \) we have, \( \mathbb{P}\{ S_t \in S_B(A_t) | B_t \} \leq 2|S_t|/t^3 \).

**Proof.** Consider \( t \in E \) and recall from Lemma 1 that \( Q_t^{\star} = \arg \max_{S \in A_t} R_S(\mu) \). We have,
\[
R_{S_{t}}(\pi_{t}) \geq \gamma \cdot R_{S}(\pi_{t}^{\star}), \quad \forall S \subseteq A_t.
\]
For all \( j \in S_t \) we have \( N_{j,t} \geq \ell_t \). Hence from Observation 1, with probability at least \( 1 - \frac{2|S_t|}{\ell_t^2} \) we get,
\[
\max_{j \in S_t} |\pi_{j,t} - \mu_j| < 2\varepsilon_t.
\]
This implies,
\[
|R_{S_{t}}(\pi_{t}) - R_{S_{t}}(\mu)| < \Delta_{\min} \quad \text{(Lipschitz property (Property 2))}
\]
\[
\implies R_{S_{t}}(\pi_{t}) - R_{S_{t}}(\mu) < \Delta_{\min} \leq \Delta_{\min}(A_t)
\]
\[
\gamma \cdot R_{S_{t}}(\pi_{t}) - R_{S_{t}}(\mu) < \Delta_{\min}(A_t).
\]
(As, \( R_{S_{t}}(\pi_{t}) \geq \gamma \cdot R_{S_{t}}(\pi_{t}) \))
From the definition of \( \Delta_{\min}(A_t) \) and monotonicity property (Property 1) we have, a contradiction. Hence, \( S_t \notin S_B(A_t) \), which implies that \( \mathbb{P}\{ S_t \in S_B(A_t) \} \leq \frac{2|S_t|}{t^3} \) for all \( t \in E \). Hence, \( \mathbb{P}\{ S_t \in S_B(A_t) | B_t \} \leq 2|S_t|/t^3 \).

**Theorem 2.** The weak instance-dependent sleeping regret of CS-UCB when the reward function satisfies Lipschitz condition (properties 1 and 2) is given by
\[
\mathcal{R}_{\text{CS-UCB}}(T) \leq 4C \sqrt{6kT \log(T)} + 2kC\zeta(3).
\]
Here, \( \zeta(.) \) is a Reimann zeta function and \( \sigma = \Delta_{\max}/\Delta_{\min} \).

**Proof.** First, consider the case \( \Delta_{\min} \geq C \sqrt{\frac{6kT \log(T)}{T}} \). From Theorem 1 we have,
\[
\mathcal{R}_{\text{CS-UCB}}(T) \leq \left[ \frac{6C^2kT \log(T)}{\Delta_{\min}} + 2kC\zeta(3) \left(1 + \sqrt{3 \log(T)/2}\right) \right]
\]
\[ \leq C \sqrt{6k\sigma T \log(T)} + 2k\zeta(3)C(1 + \sqrt{3\log(T)/2}) \quad \text{(as, } \Delta_{\min} \geq C \sqrt{6k\sigma \log(T)/T}) \]

\[ \leq 3C \sqrt{6k\sigma T \log(T)} + 2kC\zeta(3). \]

The last inequality follows for all \( T \geq k \), from the fact that \( \sqrt{6\log(T)kC\zeta(3)} \leq 2 \sqrt{6\log(T)kC} \leq 2C \sqrt{6k\sigma T \log(T)}. \)

Next, let \( \Delta_{\min} < C \sqrt{6k\sigma \log(T)/T} \). Further, let \( \eta \geq C \sqrt{6k\sigma \log(T)/T} \) be a constant. We decompose the regret \( \Delta_{S_t} \) at any time \( t \) into two parts; i.e. \( \Delta_{S_t} \geq \eta \) and \( \Delta_{S_t} < \eta \), respectively. Thus, instance-independent sleeping regret of CS-UCB,

\[ \mathcal{R}_{CS-UCB}(T) = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}(S_t \in S_B(A_t))\Delta_{S_t} \right] = \mathbb{E} \left[ \sum_{t=1}^{T} \mathbb{1}(S_t \in S_B(A_t), \Delta_{S_t} < \eta) + \mathbb{1}(S_t \in S_B(A_t), \Delta_{S_t} \geq \eta) \right] \Delta_{S_t} \].

The first term is upper bounded by \( \eta T \). To bound the second term, consider a CSMAB instance such that \( S_B(A) = S_B(A) \cap \{ S \subseteq A | \Delta_{S_t} \geq \eta \} \). In this instance we have \( \Delta_{\max}' = \Delta_{\max} \) and \( \Delta_{\min}' = \eta \). Hence,

\[ \mathcal{R}_{CS-UCB}(T) \leq \eta T + \sum_{t=1}^{T} \mathbb{P} \{ S_t \in S_B'(A_t) \} \Delta_{S_t} \]

\[ \leq \eta T + \frac{6C^2k\sigma \log T}{\Delta_{\min}} + \frac{2kC\zeta(3)(1 + \sqrt{3\log(T)/2})}{\eta} \quad \text{(from Theorem 1)} \]

\[ \leq \eta T + \frac{6C^2k\sigma \log T}{\eta} + \frac{2kC\zeta(3)(1 + \sqrt{3\log(T)/2})}{\eta}. \quad \text{(As } \Delta_{S_t} \geq \eta) \]

Choose \( \eta = C \left( \frac{6k\sigma \log T}{T} \right)^{1/2} \) to get the desired upper bound. \( \Box \)

**Theorem 3.** The instance-independent sleeping regret of CS-UCB when the reward function satisfies Lipschitz condition (Properties 1 and 2) is given by

\[ \mathcal{R}_{CS-UCB}(T) \leq C(1 + \sqrt{3\log(T)/2}) \]

where \( \lambda = (1 + \sqrt{3\log(T)/2}) \).

**Proof.** Let the regret of selecting super-arm \( S_t \) at round \( t \) be, \( \Delta_{S_t} := \gamma \cdot \text{OPT}_{A_t} - R_{S_t}(\mu) \), where \( \text{OPT}_{A_t} := R_{S_t}^* = \max_{S \subseteq A_t} R_S \). From Theorem 1 it is easy to see that the said instance-independent upper bound holds if \( \Delta_{\min} \geq \left( \frac{T}{\log(T)} \right)^{1/3} \). Hence, without loss of generality let \( \Delta_{\min} < \left( \frac{T}{\log(T)} \right)^{-1/3} \). Further, let \( \eta \geq \left( \frac{T}{\log(T)} \right)^{-1/3} \) be a constant. We decompose the regret at any time \( t \) into two parts, where the per round regret is at most \( \eta \) and larger than \( \eta \). From Observation 2, observe that \( \Delta_{S_t} \leq C \left( 1 + \sqrt{3\log(T)/2} \right) \).

Let \( \lambda = \left( 1 + \sqrt{3\log(T)/2} \right) \). Thus, instance-independent sleeping regret of CS-UCB,

\[ \mathcal{R}_{CS-UCB}(T) = \mathbb{E} \left[ \sum_{t=1}^{T} (\gamma \cdot \beta \cdot R_{S_t^*} - R_{S_t}) \right] \]

\[ = \left[ \sum_{t=1}^{T} \mathbb{P} \{ S_t \in S_B(A_t) | B_t \} \Delta_{S_t} \mathbb{1} \{ \Delta_{S_t} < \eta \} + \sum_{t=1}^{T} \mathbb{P} \{ S_t \in S_B(A_t) | B_t \} \Delta_{S_t} \mathbb{1} \{ \Delta_{S_t} \geq \eta \} \right] \cdot \beta \]
\[
\eta T + \sum_{t=1}^{T} \mathbb{P}\left\{ S_t \in S_B(A_t) \cup \Delta_{S_t} \geq \eta |B_t| \right\} \Delta_{S_t}
\]
\[= \eta T + \sum_{t \in T_c \cup T_u} \mathbb{P}\left\{ S_t \in S_B(A_t) \cup \Delta_{S_t} \geq \eta |B_t| \right\} \Delta_{S_t} + \sum_{t \in D} \mathbb{P}\left\{ S_t \in S_B(A_t) \cup \Delta_{S_t} \geq \eta |B_t| \right\} \Delta_{S_t}
\]
\[\leq \eta T + 2\zeta(3) kC \left( 1 + \frac{3 \log(T)}{2} \right) + C \left( 1 + \frac{3 \log(T)}{2} \right) \sum_{t \in D} \mathbb{P}\{\Delta_{S_t} \geq \eta\}
\]

(Observation 2)

Choose \( \eta = C \left( \frac{6 \log(T)}{T} \right)^{1/3} \) to get the following sleeping regret:

\[
\mathcal{R}_{\text{CS-UCB}}(T) \leq C(1 + \lambda) \cdot \sqrt[3]{6kT^2 \log(T) + 2k\lambda \zeta(3)}.
\]

**Theorem 4.** The expected sleeping regret incurred by CS-UCB when the reward function satisfies bounded smoothness condition (Properties 1 and 3), is upper bounded by

\[
\mathcal{R}_{\text{CS-UCB}}(T) \leq \left[ \frac{6 \log(T)}{(f^{-1}(\Delta_{\text{min}}))^2} \right] + 2\zeta(3) \right] k \cdot \Delta_{\text{max}}.
\]

**Proof.** Following the similar 3 step proof of Theorem 1. We choose with \( t_e = \frac{6 \log(t)}{(f^{-1}(\Delta_{\text{min}}))} \) and \( \varepsilon_t = \frac{3 \log(T)}{2\left| A_t \right|} \) and divide the time instants into sets \( T_e \) and \( T_u \) as described in Section 4. Step 1 and 2 is proved as Lemma 4 and Lemma 5. Observe that Step 3 follows trivially as for in Theorem 1.

**Lemma 4.** Let \( t \in T_e \), when the reward function satisfies monotonicity and Lipschitz smoothness property then \( \mathbb{P}\{S_t \in S_B(A_t) | B_t\} \leq 2|A_t|t^{-3} \).

**Proof of the lemma.** Let \( t_e := \frac{6 \log(t)}{(f^{-1}(\Delta_{\text{min}}))} \) and \( \varepsilon_t := \sqrt{\frac{3 \log(T)}{2\left| A_t \right|}} \). We have

\[
\mathbb{P}\{S_t \in S_B(A_t)\} = \mathbb{P}\{\exists i \in A_t : \hat{\mu}_{i,t} - \mu_i \geq \varepsilon_t, S_t \in S_B(A_t) | B_t\} + \mathbb{P}\{\forall i \in A_t : \hat{\mu}_{i,t} - \mu_i < \varepsilon_t, S_t \in S_B(A_t) | B_t\}. 
\]

We first prove an upper bound on the first term on the right side of the above expression. We have, for all the arms \( i \) in \( A_t \),

\[\mathbb{P}\{|\hat{\mu}_{i,t} - \mu_i| \geq \varepsilon_t\} \leq 2e^{-2N_{i,t} \varepsilon_t^2} \quad \text{(from Hoeffding’s inequality)} \]

\[= 2e^{-N_{i,t} \frac{3 \log(t)}{t}} \leq 2t^{-3}. \quad \text{(as } N_{i,t} \geq \ell_t)\]

Using union bound we get the following upper bound on the first term

\[
\mathbb{P}\{\exists i \in A_t : |\hat{\mu}_{i,t} - \mu_i| \geq \varepsilon_t, S_t \in S_B(A_t)\} \leq \mathbb{P}\{\exists i \in A_t : |\hat{\mu}_{i,t} - \mu_i| \geq \varepsilon_t\} \leq 2|A_t|t^{-3}. \quad \text{(11)}
\]
Next, we bound the second term. From Equation 4 and the bounded smoothness property (Property 3), for any $S_t' \subseteq A_t$, we have $|R_{S_t'}(\mu_t) - R_{S_t'}(\mu)| < f(2\epsilon_t)$. In particular, for the selected super-arm $S_t$ we have,

$$|R_{S_t}(\mu_t) - R_{S_t}(\mu)| < f(2\epsilon_t). \quad (12)$$

This implies,

$$R_{S_t}(\mu) + \Delta_{\min} = R_{S_t}(\mu) + f(2\epsilon_t) \quad \text{(As } f(2\epsilon_t) = \Delta_{\min})$$

$$> R_{S_t}(\mu_t) \quad \text{(from Eq.12)}$$

$$\geq \gamma \cdot R_{S_t'}(\mu_t) \quad \text{(As } S_t \text{ is optimal super-arm for } \mu_t)$$

$$\geq \gamma \cdot R_{S_t'}(\mu) = \gamma \cdot \OPT_{A_t}, \quad \text{(from the monotonicity property)}$$

Hence, we have $\Delta_{\min} > \gamma \cdot \OPT_{A_t} - R_{S_t}(\mu)$. This contradicts the definition of $\Delta_{\min}$ and hence we have that $\mathbb{P}\{\forall i \in A_t : |\mu_{i,t} - \mu_i| < \epsilon_t, S_t \subseteq S_B(A_t)|B_t\} = \mathbb{P}\{\forall i \in A_t : |\mu_{i,t} - \mu_i| < \epsilon_t\} = 0$. This, Eq. 10 and Eq. 11 completes the proof of the lemma.

**Lemma 5.** For given $t$, if $\forall i \in S_t, N_{i,t} \geq \ell_t$ is true and reward function satisfies monotonicity and bounded smoothness property then

$$\mathbb{P}\{S_t \in S_B(A_t)|B_t\} \leq \frac{2|S_t|}{\ell^3}.$$ 

**Proof.** Let $\ell_t := \frac{6 \log(t)}{f^{-1}(\Delta_{\min})^2}$ and $\epsilon_t := \sqrt{\frac{3 \log(t)}{2\ell_t}}$. Consider $t \in E$, where $E = \{t \in T_a | \forall j \in S_t, N_{j,t} \geq \ell_t\}$, i.e., at each $t \in E$ the each arm in super-arm are saturated. We have,

$$R_{S_t}(\mu_t) \geq R_S(\mu_t), \quad \forall S \in 2^{A_t}. \quad (13)$$

Let $S_t^* = \arg\max_{S \in A_t} R_S(\mu)$ be an optimal super-arm for given available arms $A_t$ at time $t$. For all $j \in S_t$ we have $N_{j,t} > \ell_t$. Hence from Observation 1, with probability atleast $1 - \frac{2|S_t|}{t^3}$ we have,

$$\max_{j \in S_t} |\mu_{j,t} - \mu_j| \leq 2\epsilon_t. \quad (14)$$

This implies,

$$|R_{S_t}(\mu_t) - R_{S_t}(\mu)| < \Delta_{\min} \quad \text{(Property 3, } \ell_t \text{ and } N_{i,t} \geq \ell_t)$$

$$\implies R_{S_t}(S_t) - R_{S_t}(\mu) < \Delta_{\min} \leq \Delta_{\min}(A_t)$$

$$R_{S_t'}(\mu_t) - R_{S_t}(\mu) < \Delta_{\min}(A_t) \quad \text{(As, } R_{S_t}(\mu_t) \geq R_{S_t'}(\mu_t))$$

$$\implies R_{S_t}(\mu) > \max_{S \in S_B(A_t)} R_S(\mu). \quad \text{(by definition of } \Delta_{\min}(A_t) \text{ and monotonicity property)}$$

Hence, we have $\mathbb{P}\{S_t \notin S_B(A_t)|B_t\} \geq 1 - \frac{2|S_t|}{t^3}$, which implies that $\mathbb{P}\{S_t \in S_B(A_t)|B_t\} \leq \frac{2|S_t|}{t^3}$ for all $t \in E$.

**Putting everything together:**

With this, the upper bound on sleeping regret of CS-UCB under Bounded smoothness setting is

$$\mathcal{R}_{CS-UCB}(T) \leq \beta \cdot \Delta_{\max} \left[ \sum_{t \in T_e} \mathbb{P}\{S_t \in S_B(A_t)|B_t\} + \sum_{t \in E} \mathbb{P}\{S_t \in S_B(A_t)|B_t\} + \sum_{t \in D} \mathbb{P}\{S_t \in S_B(A_t)|B_t\} \right]$$
\[
\leq \beta \cdot \left[ \sum_{t=1}^{T} 2 \frac{\Delta_{\text{max}}}{t^3} |S_t| + \Delta_{\text{max}} |D| \right] \quad \text{(From Lemma 2, Lemma 4, and Lemma 5)}
\]

\[
\leq \left[ 2\zeta(3)k\Delta_{\text{max}} + k\ell_T \Delta_{\text{max}} \right] \cdot \beta = \left[ \frac{6C^2 \log(T)}{(f^{-1}(\Delta_{\text{min}}))^2} + 2\zeta(3) \right] \beta \cdot k \cdot \Delta_{\text{max}}.
\]