Uncertainty relations for several observables via the Clifford algebras

V V Dodonov
Institute of Physics and International Center for Physics, University of Brasilia, P.O. Box 04455, Brasilia 70919-970, Federal District, Brazil
E-mail: vdodonov@fis.unb.br

Abstract. New sum and product uncertainty relations, containing variances of up to five observables, but not containing explicitly their covariances, are derived. New inequalities for three observables, especially for the angular momentum and spin-1/2 operators, are also presented.

1. Introduction
Uncertainty relations for several observables were derived for the first time by Robertson [1]. His scheme is as follows. Consider $N$ arbitrary operators $\hat{z}_1, \hat{z}_2, \ldots, \hat{z}_N$, acting on vectors $|\psi\rangle$ in the corresponding Hilbert space. Let us construct the operator $\hat{f} = \sum_{j=1}^{N} \alpha_j \delta \hat{z}_j$, where $\alpha_j$ are arbitrary complex numbers and $\delta \hat{z} \equiv \hat{z} - \langle \hat{z} \rangle$. The inequalities, which can be interpreted as generalized uncertainty relations, are the consequences of the fundamental inequality $\langle \psi | \hat{f}^{\dagger} \hat{f} | \psi \rangle \geq 0$, that must be satisfied for any quantum state $|\psi\rangle$ (the symbol $\hat{f}^{\dagger}$ means the Hermitian conjugated operator). In the explicit form, this inequality is the condition of positive semi-definiteness of the quadratic form $\alpha^*_j F_{jm} \alpha_m$ (with the summation over repeated indexes), whose coefficients $F_{jm} = \langle \psi | \delta \hat{z}_j^{\dagger} \delta \hat{z}_m | \psi \rangle$ form the Hermitian matrix $F = \| F_{jm} \|$. One has only to use the known conditions of the positive semi-definiteness of Hermitian matrices to write down the explicit inequalities for the elements of matrix $F$. Many of them can be found, e.g., in review [2] and more recent publications [3–15].

If all operators $\hat{z}_j$ are Hermitian, then it is convenient to split matrix $F$ as $F = X + iY$, where $X$ and $Y$ are real symmetric and antisymmetric matrices, consisting of the elements $X_{mn} = \frac{1}{2} \langle \psi | \{ \delta \hat{z}_m, \delta \hat{z}_n \} | \psi \rangle$ and $Y_{mn} = \frac{1}{2i} \langle \psi | [\delta \hat{z}_m, \delta \hat{z}_n] | \psi \rangle$. The symbols $\{ , \}$ and $[ , ]$ mean the anticommutator and commutator, respectively. The fundamental inequality ensuring the positive semi-definiteness of matrix $F$ is

$$\det F = \det \| X + iY \| \geq 0. \quad (1)$$

Two simple inequalities proven by Robertson have the form

$$X_{11}X_{22} \ldots X_{NN} \geq \det Y, \quad (2)$$

$$\det X \geq \det Y, \quad (3)$$
In the case of \( N = 2 \), inequality (2) becomes the Robertson uncertainty relation [16]
\[
X_{11}X_{22} \geq |Y_{12}|^2,
\]
whereas (3) turns into the Robertson–Schrödinger inequality [17,18]
\[
X_{11}X_{22} - X_{12}^2 \geq |Y_{12}|^2.
\]

Unfortunately, inequalities (1) and (3) are rather complicated for \( N > 2 \) observables, because they contain, in addition to \( N \) variances \( X_{kk} \) and \( N(N-1)/2 \) mean values of commutators \( Y_{jk} \), numerous sums and products of various combinations of \( N(N-1)/2 \) covariances \( X_{jk} \) with \( j \neq k \). For example, if \( N = 4 \), then \( \det X \) contains 17 different products of covariances [2,19], in addition to 6 different products of mean values of commutators in \( \det Y \). Moreover, inequalities (2) and (3) are totally useless if \( N \) is an odd number, as soon as \( \det Y = 0 \) in this case.

The aim of this article is to present the general scheme enabling one to get rid of covariances for an arbitrary value of \( N \) and to provide the explicit inequalities for five observables. In addition, some new results (missed in the recent paper [20]) are found for \( N = 3 \) and \( N = 4 \).

2. General scheme

One can get rid of all \( N(N-1)/2 \) covariances, using the generalization of the scheme first proposed in [21] for the special case of \( N = 3 \). Suppose that we know \( N \) Hermitian \( M \times M \) anticommuting matrices \( R_k \) satisfying the relations of the Clifford algebra
\[
R_jR_k + R_kR_j = 2I_M \delta_{jk},
\]
where \( I_M \) is the \( M \times M \) unit matrix. Consider the operator \( \hat{f} = \sum_{k=1}^{N} \xi_k \delta \hat{z}_k R_k \), where \( \xi_k \) are arbitrary real coefficients and \( \hat{z}_k \) arbitrary Hermitian operators. It acts in the extended Hilbert space of states \( |\Psi\rangle = |\psi\rangle \otimes |\chi\rangle \), where \( |\chi\rangle \) is an auxiliary \( M \)-dimensional vector. Then the condition \( \langle \Psi | \hat{f}^\dagger \hat{f} | \Psi \rangle \geq 0 \) can be written as the condition of positive semi-definiteness of the Hermitian \( M \times M \) matrix
\[
F = gI_M + i \sum_{j<k} R_jR_ky_{jk},
\]
\[
g = \sum_{k=1}^{N} \xi_k^2 X_{kk}, \quad y_{jk} = 2\xi_j \xi_k Y_{jk} = -y_{kj}.
\]

The covariances \( X_{jk} \) with \( j \neq k \) go out due to the anti-commutation relations (6).

The special cases of \( N = 3 \) (when matrices \( R_j \) are three Pauli’s \( 2 \times 2 \) matrices) and \( N = 4 \) (four Dirac’s \( 4 \times 4 \) matrices) were studied in [20]. The main aim of the present article is to obtain explicit inequalities in the case of \( N = 5 \).

3. Sum inequalities for variances of five observables

Remember that besides four Dirac’s matrices \( \gamma_1, \gamma_2, \gamma_3 \) and \( \gamma_4 \), there exists the fifth matrix \( \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \) which also satisfies relations (6). Then five \( 4 \times 4 \) matrices \( R_j \) can be chosen as follows (here \( k = 1, 2, 3 \), and \( \sigma_k \) are the standard \( 2 \times 2 \) Pauli matrices):
\[
R_k = \left| \begin{array}{cc} 0 & \sigma_k \\ \sigma_k & 0 \end{array} \right|, \quad R_4 = \left| \begin{array}{cc} I_2 & 0 \\ 0 & -I_2 \end{array} \right|, \quad R_5 = \left| \begin{array}{cc} 0 & -iI_2 \\ iI_2 & 0 \end{array} \right|.
\]

Then we have the following matrix products:
\[
R_jR_k = i\varepsilon_{jkl} \left| \begin{array}{cc} \sigma_l & 0 \\ 0 & \sigma_l \end{array} \right|, \quad R_kR_4 = \left| \begin{array}{cc} 0 & -\sigma_k \\ \sigma_k & 0 \end{array} \right|,
\]
\[ R_k R_5 = \begin{vmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{vmatrix}, \quad R_4 R_5 = \begin{vmatrix} 0 & -iI_2 \\ -iI_2 & 0 \end{vmatrix}, \]

where \( j, k, l = 1, 2, 3 \), and \( \epsilon_{jkl} \) is totally antisymmetric tensor with \( \epsilon_{123} = 1 \). In this case, \( 4 \times 4 \) matrix (7) takes the form

\[
F = \begin{vmatrix} g - y_{12} - y_{35} & x_3^* - x_5^* & y_{45} - iy_{34} & y_{42} - iy_{14} \\ x_3 - x_5 & g + y_{12} + y_{35} & y_{24} - iy_{14} & iy_{34} + y_{45} \\ iy_{34} + y_{45} & y_{24} + iy_{14} & g - y_{12} + y_{35} & x_3^* + x_5^* \\ y_{42} + iy_{14} & y_{45} - iy_{34} & x_3 + x_5 & g + y_{12} - y_{35} \end{vmatrix},
\]

where \( x_3 = y_{32} + iy_{13} \) and \( x_5 = y_{15} + iy_{25} \).

The non-negativeness of diagonal elements of matrix (10) is the consequence of the Robertson inequality (4). Other conditions of the positive semi-definiteness of matrix (10) are non-negative values of six principal minors of the second order and four principal minors of the third order. Each of these ten inequalities separately is strongly asymmetric with respect to all indexes. But the sum of four third order inequalities yields the simple symmetric inequality

\[
g^2 \geq v, \quad v = \sum_{j<k} y_{jk}^2.
\]

The sum of six second order inequalities results in a weaker inequality \( g^2 \geq v/3 \). Remember that \( g \) is the sum of five terms and \( v \) is the sum of ten terms for \( N = 5 \).

Note that inequality (11) must hold for arbitrary sets of five real parameters \( \xi_k \). In particular, taking all \( \xi_k = 1 \) (this means that the dimensions of operators \( \hat{z}_k \) should be made equal by means of some scaling transformations), we get the nice inequality

\[
\sum_{k=1}^{5} X_{kk} \geq 2 \left( \sum_{j<k} Y_{jk}^2 \right)^{1/2}.
\]

Inequality (11) can be strengthened, if one considers the most complete condition of positive semi-definiteness of matrix \( F \), namely \( \det F \geq 0 \). After some algebra, it can be written in the following compact form:

\[
\det F = (g^2 - v)^2 - 4u^2 \geq 0, \quad u^2 = \sum_{k=1}^{5} u_k^2,
\]

\[
u_1 = y_{23}y_{45} + y_{34}y_{25} + y_{42}y_{35}, \quad u_2 = y_{13}y_{45} + y_{34}y_{15} + y_{41}y_{35},
\]

\[
u_3 = y_{12}y_{45} + y_{24}y_{15} + y_{41}y_{25}, \quad u_4 = y_{12}y_{35} + y_{23}y_{15} + y_{31}y_{25},
\]

\[
u_5 = y_{12}y_{34} + y_{23}y_{14} + y_{31}y_{24}.
\]

In view of (11), inequality (13) can be written in a simpler form

\[
g^2 \geq v + 2|u|.
\]

Taking \( \xi_5 = 0 \) we arrive at the inequality for four observables derived in [20]. In this case \( u = u_5 \) and \( v \) contains six terms. If in addition \( \xi_4 = 0 \), then inequality (17) for three observables has \( u = 0 \) [20] (and \( v \) contains three terms only).
4. Product inequalities

4.1. The simplest inequalities

To find the inequality for the “uncertainty product” \( \Pi_N = \left( \prod_{k=1}^{N} X_{kk} \right)^{1/2} \), let us put \( \xi_j^2 = X_{jj}^{-1} \) in (11). Then we get for \( N = 3 \) the inequality [20]

\[
\Pi_3^2 \geq 4W_3/9, \quad W_3 = X_{11}Y_{23}^2 + X_{22}Y_{13}^2 + X_{33}Y_{12}^2. \tag{18}
\]

But the sum \( W_3 \) can be interpreted as function \( g \) in (8) with the coefficients \( \xi_1 = Y_{23}, \xi_2 = Y_{13} \) and \( \xi_3 = Y_{12} \). Applying inequality (11) with these coefficients, we obtain the new inequality

\[
X_{11}X_{22}X_{33} \geq (4/3)^{3/2} |Y_{12}Y_{13}Y_{23}|. \tag{19}
\]

Its special case for \(|Y_{12}| = |Y_{13}| = |Y_{23}| \) was found in [20]. Note that (19) is stronger (by 50\%) than the immediate consequence of the product of three Robertson’s inequalities (4)

\[
X_{11}X_{22}X_{33} \geq |Y_{12}Y_{13}Y_{23}|. \tag{20}
\]

This fact was noticed for some specific set of three observables in [22].

For \( N = 4 \), the same choice \( \xi_j^2 = X_{jj}^{-1} \) leads to the inequality \( \Pi_4^2 \geq W_4/4 \) with

\[
W_4 = Y_{12}^2X_{33}X_{44} + Y_{13}^2X_{22}X_{44} + Y_{14}^2X_{33}X_{22} + Y_{23}^2X_{11}X_{44} + Y_{24}^2X_{33}X_{11} + Y_{34}^2X_{11}X_{22}. \]

Applying (4) to each pair product \( X_{jj}X_{kk} \), we get the inequality

\[
X_{11}X_{22}X_{33}X_{44} \geq \frac{1}{2} \left( Y_{12}^2Y_{34}^2 + Y_{13}^2Y_{24}^2 + Y_{14}^2Y_{23}^2 \right) \equiv B_4. \tag{21}
\]

Note that the direct application of (4) to three different products of two pairs of variances yields a weaker inequality with the coefficient 1/3 instead of 1/2 in the right-hand side of (21).

For \( N = 5 \), the same scheme yields \( \Pi_5^2 \geq 4W_5/25 \) with

\[
W_5 = X_{55}W_4 + Y_{15}^2X_{22}X_{33}X_{44} + Y_{25}^2X_{11}X_{33}X_{44} + Y_{35}^2X_{22}X_{11}X_{44} + Y_{45}^2X_{33}X_{22}X_{11}. \]

Applying inequality (19) to each triple product of variances, we arrive at the inequality

\[
X_{11}X_{22}X_{33}X_{44}X_{55} \geq \frac{32}{75\sqrt{3}} A_5 \equiv B_5, \tag{22}
\]

where \( A_5 \) is the sum of ten terms:

\[
A_5 = Y_{12}^2 |Y_{34}Y_{35}Y_{45}| + Y_{13}^2 |Y_{24}Y_{25}Y_{45}| + Y_{14}^2 |Y_{34}Y_{25}Y_{45}| + Y_{23}^2 |Y_{34}Y_{25}Y_{35}| + Y_{24}^2 |Y_{34}Y_{15}Y_{45}| + Y_{25}^2 |Y_{14}Y_{15}Y_{45}| + Y_{24}^2 |Y_{13}Y_{35}Y_{15}| + Y_{25}^2 |Y_{34}Y_{13}Y_{14}| + Y_{24}^2 |Y_{12}Y_{15}Y_{25}| + Y_{25}^2 |Y_{14}Y_{12}Y_{24}| + Y_{23}^2 |Y_{12}Y_{13}Y_{23}|. \]

4.2. More precise inequalities

Note however that (21) and (22) are not the strongest inequalities for \( N = 4 \) and \( N = 5 \). This can be seen in the simple example of two independent pairs of canonically conjugated variables: \((x, p_x)\) and \((y, p_y)\). Then the product \( \Pi_4^2 \) obviously must be greater than \((h/2)^4\), while the right-hand side of (21) is twice smaller. To obtain more strong lower bounds for the products of variances one should start from inequality (17) instead of (11). Then taking again \( \xi_j^2 = X_{jj}^{-1} \) one can write (17) for \( N = 4 \) as

\[
16\Pi_4^2 - 8A_4 - 4W_4 \geq 0, \quad A_4 = |Y_{12}Y_{34} + Y_{23}Y_{14} + Y_{31}Y_{24}|. \tag{23}
\]
Since $W_4 \geq 4B_4$, the consequence of (23) is the required inequality [20]
\[
4\Pi_4 \geq \Lambda_{4*} + \sqrt{\Lambda_{4*}^2 + 16B_4}
\]
(the second solution, giving an upper bound for $\Pi$, is unphysical). One can check that the right-hand side of (24) yields the correct lower boundary for the product $\Pi_4$ for two independent pairs of canonically conjugated variables.

The same scheme for $N = 5$ leads to the inequality
\[
25\Pi_5^2 - 8|\Lambda|\Pi - 4W_5 \geq 0,
\]
(25)
\[
\Lambda^2 = \sum_{k=1}^{5} X_{kk}U_k^2.
\]
(26)
The coefficients $U_k$ are given by formulas (14)-(16) with the replacement $y_{jk} \rightarrow Y_{jk}$. To get rid of the variances $X_{kk}$ in the expression for $\Lambda$, we notice again that the right-hand side of (26) has the form (8) with $\xi_k = U_k$. Then, using (17) we can write
\[
|\Lambda| \geq \Lambda_{5*} = \left(4 \sum_{j<k} (U_jU_kY_{jk})^2 + 8\sqrt{3}|U_1U_2U_3U_4U_5|\right)^{1/4}.
\]
(27)
Resolving inequality (25) with respect to variable $\Pi$ with the aid of the relation $W_5 \geq 25B_5/4$, we arrive at the new inequality
\[
25\Pi_5 \geq 4\Lambda_{5*} + \sqrt{16\Lambda_{5*}^2 + 625B_5}.
\]
(28)

5. Examples
5.1. Five observables
It is not easy to find impressive special cases for $N = 5$. One of ‘natural’ examples could be the set of five Hermitian operators
\[
\hat{z}_1 = \delta \hat{x}, \quad \hat{z}_2 = \delta \hat{p}, \quad \hat{z}_3 = (\delta \hat{x})^2, \quad \hat{z}_4 = (\delta \hat{p})^2, \quad \hat{z}_5 = (\delta \hat{x}\delta \hat{p} + \delta \hat{p}\delta \hat{x})/2.
\]
Then the coefficients $Y_{jk}$ are as follows:
\[
Y_{15} = h\langle \delta \hat{x} \rangle/2 = 0, \quad Y_{25} = -h\langle \delta \hat{p} \rangle/2 = 0, \quad Y_{14} = h\langle \delta \hat{p} \rangle = 0, \quad Y_{23} = -h\langle \delta \hat{x} \rangle = 0,
\]
\[
Y_{12} = h/2, \quad Y_{13} = Y_{24} = 0, \quad Y_{34} = 2h\sigma_{xp}, \quad Y_{35} = h\sigma_{xx}, \quad Y_{45} = -h\sigma_{pp}.
\]
Therefore, we have four nonzero combinations of these coefficients entering relation (28):
\[
\mathcal{A}_5 = h^5\sigma_{xx}\sigma_{pp}|\sigma_{xp}|, \quad U_3 = -h^2\sigma_{pp}/2, \quad U_4 = h^2\sigma_{xx}/2, \quad U_5 = h^2\sigma_{xp},
\]
so that $\Lambda_{5*} = 3^{1/4}h^{5/2}|\sigma_{xx}\sigma_{pp}\sigma_{xp}|^{1/2}$. Unfortunately, both $\mathcal{A}_5$ and $\Lambda_{5*}$ are proportional to $|\sigma_{xp}|$, so that inequality (28) becomes useless for any state with $\sigma_{xp} = 0$, in particular for any state with a real wave function.

Another example is the set of ‘regular polygons’ in the phase space, defined as [15]
\[
z_j = x \cos(j\phi) + p \sin(j\phi), \quad \phi = 2\pi/N, \quad j = 1, 2, \ldots, N
\]
(29)
where $x$ and $p$ are scaled variables with the same dimensionality, satisfying the canonical commutation relation $[\hat{x}, \hat{p}] = i\hbar$. In this case, $Y_{kj} = \hbar \sin(\phi) \phi / 2$. The following product inequality was obtained for operators (29) for any natural number $N$ in [15]:

$$\Pi_3^N \geq (\hbar/2)^N. \quad (30)$$

For $N = 3$ this inequality coincides with (19): $\Pi_3^3 \geq (\hbar/2)^3$. Also (30) coincides with (21) and (24) for $N = 4$ (in this special case $\Lambda_{4s} = 0$). But the situation is different for $N = 5$ with $\phi = 2\pi/5$, $\cos(\phi) = (\sqrt{5} - 1)/4$ and $\sin(\phi) = [(5 + \sqrt{5})/8]^{1/2}$. Then $\Lambda_{5s} = 0$ and

$$B_5 = \left[\left(20 + \sqrt{80}\right)/54\right]^{1/2} (\hbar/2)^5 \approx 0.73(\hbar/2)^5.$$

Therefore, (22) and (28) are weaker than (30) for the variables (29) with $N = 5$.

5.2. Three observables: spin-1/2 components

A natural example of three observables is the set of three angular momentum operators $\hat{L}_1$, $\hat{L}_2$ and $\hat{L}_3$. Then inequality (19) can be written as

$$\frac{L_{11}L_{22}L_{33}}{L_1L_2L_3} \geq \frac{\hbar^3}{3\sqrt{3}}, \quad L_k \equiv \langle \hat{L}_k \rangle, \quad L_{kk} \equiv \langle \hat{L}_k^2 \rangle - L_k^2. \quad (31)$$

The simplest way to verify inequality (31) is to consider the special case of three spin-1/2 operators $\hat{L}_k = (\hbar/2)\sigma_k$, where $\sigma_k$ are the Pauli matrices ($k = 1, 2, 3$). Then (31) takes the form

$$\frac{(1 - \langle \sigma_1 \rangle^2)(1 - \langle \sigma_2 \rangle^2)(1 - \langle \sigma_3 \rangle^2)}{\langle \sigma_1 \rangle \langle \sigma_2 \rangle \langle \sigma_3 \rangle} \geq (4/3)^{3/2}. \quad (32)$$

Let us consider the normalized spinor $|\psi\rangle = \left(\sqrt{1 - \rho^2}, \rho e^{i\phi}\right)$ with $\rho \geq 0$. Then

$$\langle \sigma_1 \rangle = 2\rho \cos(\phi) \sqrt{1 - \rho^2}, \quad \langle \sigma_2 \rangle = 2\rho \sin(\phi) \sqrt{1 - \rho^2}, \quad \langle \sigma_3 \rangle = 1 - 2\rho^2.$$

For $\rho \to 0$, the left-hand side of (32) equals $[\cos(\phi) \sin(\phi)]^{-1}$, which is bigger than 2. The equality in (32) is observed for the ‘totally symmetric state’ with $\phi = \pi/4$ and $\rho^2 = (\sqrt{3} - 1)/(2\sqrt{3})$, when $\langle \sigma_1 \rangle = \langle \sigma_2 \rangle = \langle \sigma_3 \rangle = 1/\sqrt{3}$. This example explains the origin of the ‘weird’ number $(4/3)^{3/2}$ in the inequalities (19) and (32).

6. Discussion

The main results of this paper are the new inequalities (17) and (28) for five observables, and (19), (31)-(32) for three ones. How to generalize them for $N > 5$ observables? According to the general scheme presented in section 2, one needs $N$ anticommuting matrices $R_j$ satisfying the Clifford algebra relations (6). The main technical problem is the dimension of such matrices. According to [23–25], this dimension is $2^n \times 2^n$ for $N = 2n$ and $N = 2n + 1$. For example, one needs matrices $8 \times 8$ for $N = 6, 7$, matrices $16 \times 16$ for $N = 8, 9$, and so on. Although the structure of such matrices $R_j$ is rather simple [26], the analysis of the positivity conditions for the corresponding matrix (7) is a formidable task. Probably, this can be done with the aid of some computer algebra programs, although the usefulness of such generalizations can be questioned, as was demonstrated in examples of section 5.1.

However, one simple inequality deserves attention, namely inequality (12). It can be written in the nice matrix form

$$[\text{Tr}(X)]^2 \geq 2\text{Tr}(Y \hat{Y}), \quad (33)$$
where $\tilde{Y}$ is the transposed matrix. This inequality was proved for $N = 3$ and $N = 4$ in [20]. The generalization of (12) to the case of $N$ observables was found in [15] under the restriction that these observables are arbitrary linear combinations of the canonical coordinate and momentum operators. Therefore, it would be interesting to prove (33) for any dimension $N > 5$ and arbitrary Hermitian operators. Although it will not be the most precise relation [as inequality (17) shows for $N = 5$], it looks so nice that it seems worth trying to find the general proof.

Acknowledgment

A partial support of the Brazilian funding agency Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) is acknowledged.

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