Already in the 1970s there were attempts to present a set of ground rules, sometimes referred to as a theory of gravitation theories, which theories of gravity should satisfy in order to be considered viable in principle and, therefore, interesting enough to deserve further investigation. From this perspective, an alternative title of the present paper could be “why are we still unable to write a guide on how to propose viable alternatives to general relativity?”. Attempting to answer this question, it is argued here that earlier efforts to turn qualitative statements, such as the Einstein Equivalence Principle, into quantitative ones, such as the metric postulates, stand on rather shaky grounds — probably contrary to popular belief — as they appear to depend strongly on particular representations of the theory. This includes ambiguities in the identification of matter and gravitational fields, dependence of frequently used definitions, such as those of the stress-energy tensor or classical vacuum, on the choice of variables, etc. Various examples are discussed and possible approaches to this problem are pointed out. In the course of this study, several common misconceptions related to the various forms of the Equivalence Principle, the use of conformal frames and equivalence between theories are clarified.

Keywords: gravitation theory; equivalence principle; metric postulates; Jordan and Einstein frames.

1. Introduction

The axiomatic formulation of general relativity (or gravitational theories in general) seems to resemble the myth of the Holy Grail. Serious attempts have been made to find it and everybody seems to be interested in it, but nobody actually knows where to look for it. Of course, one could ask how useful a collection of axioms could be for a theory, like general relativity, for which we already know the field equations or the action. Indeed, knowledge of any of the latter suffices to fully describe the dynamics of the theory, at least at the classical level, hence making the absence of
an axiomatic formulation less needed, at least for practical purposes.

Nonetheless, there are numerous reasons why one would like to explore the conceptual basis of a theory. On the theoretical side, one could mention that a set of axioms could help us understand the theory in depth and provide a better insight in finding solutions to long standing problems, the most prominent of which being to propose a theory of quantum gravity (for example it could help in determining the fundamental classical properties one should expect to recover and recognize which of them could break down at the quantum level).

Furthermore, if emergent gravity scenarios are considered (i.e., scenarios in which the metric and the affine-connection are collective variables and general relativity would be a sort of hydrodynamics emergent from more fundamental constituents) such a set of axioms could provide a much needed guidance in reconstructing the microscopic system at the origin of classical gravitation, for example by constraining its microscopic properties so to reproduce the emergent physical features encoded in these axioms.

There could be important benefits at the purely experimental level as well. Past experience taught us that experiments test principles and not theories (for example weak equivalence principle tests, such as the gravitational shift ones [1] were initially erroneously regarded as tests of general relativity). So one would want to know exactly which principles/axioms to test in order to discriminate at least among classes of gravitational theories.

Finally, nowadays we have a number of alternative theories of gravity. How can we characterise the way in which they differ from general relativity, group them, or obtain some insight into which of them are preferable with respect to others? Even if we are far from a coherent and strict axiomatic formulation, at least a set of principles, or what is sometimes called a theory of gravitation theories (i.e., a “meta–theory of gravitation”), would definitely prove useful to this end, allowing us not only to “catalogue” presently known theories but also to build new ones by just relaxing or adding some fundamental principles/axioms. Last but not least, having such a meta–theory would come up extremely handy given, for example, the current revival of alternative theories of gravitation as a possible explanation of nowadays puzzling cosmological observations (e.g., the dark energy problem). Indeed, having such a theory would greatly help in interpreting the experimental results and their implications for discriminating among alternative theories of gravitation.

Considering all of the above, it is quite disappointing that no real progress has been made in this direction in the last thirty years. Indeed, it seems that the scientific community has somewhat given up on such an ambitious task given that the latest serious attempts date back to the 70’s, even though the subject of alternative theories of gravity has been quite an active one throughout this time. We feel that it is important to understand the practical reasons for this lack of progress if we wish to step forward in this research and go beyond the trial-and-error approach that is mostly used in modified gravity. Hopefully, this exploration will also give, as a byproduct, interesting clarifications relevant to certain common misconceptions.
(weak equivalence principle, equivalence of theories, etc.) and maybe even serve as a motivational point of reference for future work.

In Sections 2 and 3 we formulate the problem in general terms by analysing the several formulations of the Equivalence Principle and their implementations via the so called metric postulates. In Section 4 we distinguish between theories and their representations. Concrete examples are given in the following sections, including scalar-tensor theories in Section 5, the metric and Palatini versions of $f(R)$ gravity in Section 6, and the Einstein-Cartan-Sciama-Kibble theory in Section 7. Finally, Section 8 contains a discussion of the results and our conclusions.

In what follows purely classical physics will be considered. The issue of the compatibility between the Equivalence Principle(s) and quantum mechanics, although rich in facets and consequences is beyond the scope of this work. The metric is taken to have signature $(-, +, +, +)$, we define $\kappa = 8\pi G$ where $G$ is Newton’s constant and $c$ the speed of light in vacuum, units where $G = c = 1$ are used, and the basic notation of Wald’s book is adopted.

2. Equivalence Principle(s)

In creating a meta–theory of gravitation one immediately faces the daunting task to provide some sufficiently general criteria for characterizing the physical features of possible alternative theories. As already mentioned, providing a strict axiomatic formulation is hardly an easy goal, but one could hope to give at least some set of physical viability principles, even if the latter are not necessarily at the level of axioms. It is clear that in order to be useful such statements need to be formulated in a theory-independent way and should be amenable to experimental tests so that we could select at least among classes of gravitational theories by suitable observations/experiments. The best example in this direction has been so far the Equivalence Principle in its various versions. Let us restate them here and then pass to analyze their physical implications:

Weak Equivalence Principle (WEP): If an uncharged test body is placed at an initial event in spacetime and given an initial velocity there, then its subsequent trajectory will be independent of its internal structure and composition.

Einstein Equivalence Principle (EEP): (i) WEP is valid, (ii) the outcome of any local non-gravitational test experiment is independent of the velocity of the freely falling apparatus (Local Lorentz Invariance or LLI) and (iii) the outcome of any local non-gravitational test experiment is independent of where and when in the universe it is performed (Local Position Invariance or LPI).

Strong Equivalence Principle (SEP): (i) WEP is valid for self-gravitating bodies as well as for test bodies, (ii) the outcome of any local test experiment is independent

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Footnote: There has been, however, an attempt towards an axiomatic formulation of gravitational theories from a more mathematically-minded point of view.
of the velocity of the freely falling apparatus (Local Lorentz Invariance or LLI) and
(iii) the outcome of any local test experiment is independent of where and when in
the universe it is performed (Local Position Invariance or LPI).

It is important to stress that the WEP only says that there exist some preferred
trajectories, the free fall trajectories, that test particles will follow and these curves
are the same independently of the mass and internal composition of the particles
that follow them (universality of free fall). WEP does not imply, by itself, that
there exist a metric, geodesics, etc. — this comes about only through the EEP
by combining the WEP with requirements (ii) and (iii) (see Will’s book
for a
detailed discussion). The same is true for the covariance of the field equations. As
far as the SEP is concerned, the main thrusts extending the validity of the WEP
to self-gravitating bodies and the applicability of LLI and LPI to gravitational
experiments, in contrast to the EEP. There are experimental tests of all the EPs
but the stringent ones are for the WEP and the EEP.

Let us stress that there are at least three subtle points in relation to the use and
meaning of the EP formulations, the first one concerning the relation between the
SEP and general relativity. While there are claims that SEP holds only for general relativity,
no proof of this statement has been given so far. Indeed, it would be
a crucial step forward to pinpoint a one-to-one association between GR and the
SEP but it is difficult to relate directly and uniquely a qualitative statement, such
as the SEP, to a quantitative one, namely Einstein’s equations. The second subtle
point is the reference to test particles in all the EP formulations. Apart from the
obvious limitation of restricting attention to particles and ignoring classical fields
(such as, e.g., the electromagnetic one), apparently no true test particles exist,
hence the question is how do we know how “small” a particle should be in order
to be considered a test particle (i.e., its gravitational field can be neglected)? The
answer is likely to be theory-dependent (see e.g., Geroch and Jang
and references
therein for the case of general relativity), so there is no guarantee that a theory
cannot be put together in which the WEP is valid in principle but, in practice,
experiments would show a violation because, within the framework of the theory,
a “small” particle is not close enough to a test particle. Of course, such a theory
would not be viable but this would not be obvious when we refer only to the WEP
from a theoretical perspective (e.g., calculate free fall trajectories and compare with
geodesics). A third subtlety, on which we shall come back later, is related to the fact
that sometimes the same theory appears to evidently satisfy or not some version of
the EP depending on which variables are used in describing it, an example being
the Jordan versus the Einstein frame in scalar-tensor theories of gravity.

Taking all of the above into consideration, it seems that the main problem with
all forms of the equivalence principle is that they are of little practical value. As
principles they are by definition qualitative and not quantitative. However, quan-
titative statements are what is needed in practice. An attempt to overcome this
difficulty was indeed made by Thorne and Will
and is embodied by the so called
metric theories postulates.

3. Metric Postulates

The metric postulates can be stated in the following way:\textsuperscript{14}

(1) there exists a metric $g_{\mu\nu}$ (second rank non degenerate tensor).

(2) $\nabla_\mu T^{\mu\nu} = 0$, where $\nabla_\mu$ is the covariant derivative defined with the Levi-Civita connection of this metric and $T_{\mu\nu}$ is the stress-energy tensor of non-gravitational (matter) fields.

Theories that satisfy the metric postulates are often called \textit{metric theories}. Let us first see how these postulates come about starting from the EEP and how they encapsulate its validity. The EEP adds two more statements to the WEP: Local Lorentz Invariance and Local Position invariance. A freely falling observer carries a local frame in which test bodies have unaccelerated motions. According to the requirements of LLI the outcomes of non-gravitational experiments are independent of the velocity of the freely falling frame and therefore, if two such frames located at the same event $P$ have different velocities, this should not affect the predictions of identical non-gravitational experiments. Local Position Invariance requires that the above should hold for all spacetime points. Therefore, roughly speaking, in local freely falling frames the theory should reduce to special relativity.

This implies that there should be (at least one) second rank tensor fields which reduce, in the local freely falling frame, to metrics conformal to the Minkowski one. The freedom of having an arbitrary conformal factor is due to the fact that the EEP does not forbid a conformal rescaling in order to arrive to special-relativistic expressions of the physical laws in the local freely falling frame. Note, however, that while one could think to allow each specific matter field to be coupled to a different one of these conformally related second rank tensors, the conformal factors relating these tensors can at most differ by a multiplication constant if the couplings to different matter fields are to be turned to constants under a conformal rescaling as the LPI requires (this highlights the relation between LPI and varying coupling constants). We can then conclude that rescaling coupling constants and performing a conformal transformation leads to a metric $g_{\mu\nu}$ which, in every freely falling local frame, reduces (locally) to the Minkowski metric $\eta_{\mu\nu}$\textsuperscript{14}.

It should be stressed that all conformal metrics $\phi g_{\mu\nu}$, $\phi$ being the conformal factor, can be used to write down the equations or the action of the theory. $g_{\mu\nu}$ is only special in the following sense: Since at each event $P$ there exist local frames called local Lorentz frames, one can find suitable coordinates in which at $P$

$$g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O} \left( \sum_\alpha |x^\alpha - x^\alpha (P)|^2 \right),$$

This does not exclude the possibility of having a second metric tensor in the theory as long as this metric does not couple to the matter (this case leading to theories of the bi-metric kind).
and \( \partial g_{\mu\nu}/\partial x^\alpha = 0 \). In local Lorentz frames the geodesics of the metric \( g_{\mu\nu} \) are straight lines. Free fall trajectories are straight lines in a local freely falling frame. Identifying the two frames we realize that the geodesics of \( g_{\mu\nu} \) coincide with free fall trajectories. Put in other words, the EEP requires the existence of a family of conformal metrics, one of which should have geodesics which coincide with free fall trajectories. On the other hand, geodesic motion for test particles can be derived from the condition \( \nabla_\mu T^{\mu\nu} = 0 \), when \( \nabla_\mu \) is the covariant derivative defined by the Levi-Civita connection of the metric, whose geodesics the test particles have to follow.

Appealing as they may seem, however, the metric postulates lack clarity. As pointed out also by the very authors of the paper introducing them, any metric theory can perfectly well be given a representation that appears to violate the metric postulates (recall, for instance, that \( g_{\mu\nu} \) is a member of a family of conformal metrics and that there is no \textit{a priori} reason why this metric should be used to write down the field equations). See also Anderson for an earlier criticism of the need for a metric and, indirectly, of the metric postulates. One of the goals of this paper is to elaborate on the problems mentioned above, as well as on other prominent ambiguities stated below and trace their roots.

### 3.1. What is precisely the definition of stress-energy tensor?

In order to answer this question one could refer to an action. This is a significant restriction to begin with, since it would add to the EEP the prerequisite that a reasonable theory has to come from an action. Even so, such an assumption would not solve the problem: one could claim that \( T_{\mu\nu} \equiv -(2/\sqrt{-g})\delta S_M/\delta g^{\mu\nu} \) but then how is the matter action \( S_M \) defined? Claiming that it is the action from which the field equations for matter are derived is not sufficient since it does not provide any insight about the presence of the gravitational fields in \( S_M \). Invoking a minimal coupling argument, on the other hand, is strongly theory-dependent (which coupling is really minimal in a theory with extra fields or, say, an independent connection?). Furthermore, whether a matter field couples minimally or non-minimally to gravity or to matter should be decided by experiments. Since a non-minimal coupling could be present and evade experimental detection (as proposed in string theories), it seems prudent to allow for it in the action or the theory.

Setting actions aside and resorting to the correspondence with the stress-energy tensor of special relativity does not help either. There is always more than one tensor that one can construct, which will reduce to the special-relativistic stress-energy tensor when gravity is “switched off”. Moreover it is not clear what “switched off” exactly means when extra fields describing gravity (scalar or vector) are present in the theory together with the metric tensor.

Finally, mixing the two tentative definitions described above makes the situation even worse: one can easily imagine theories in which \( T_{\mu\nu} = -(2/\sqrt{-g})\delta S_M/\delta g^{\mu\nu} \) does not reduce to the special-relativistic stress-energy tensor in some limit. Are
these theories necessarily non-metric? This point highlights also another important
question: are the metric postulates a necessary or a sufficient condition for the
validity of the EEP? Concrete examples are provided in sections 5, 6 and 7.

3.2. What does “non-gravitational fields” mean?
There is no precise definition of “gravitational” and “non-gravitational” field. One
could say that a field non-minimally coupled to the metric is gravitational whereas
the rest are matter fields. This definition does not appear to be rigorous or sufficient
and it is shown in the following that it strongly depends on the perspective and the
terminology one chooses.

Consider, for example, a scalar field $\phi$ non-minimally coupled to the Ricci cur-
vature in $\lambda \phi^4$ theory, as described by the action

$$ S = \int d^4x \sqrt{-g} \left[ \left( \frac{1}{2\kappa} - \xi \phi^2 \right) R - \frac{1}{2} \nabla^\mu \phi \nabla_\mu \phi - \lambda \phi^4 \right]. $$

If one begins with a classical scalar field minimally coupled to the curvature (i.e.,
$\xi = 0$) in the potential $\lambda \phi^4$ and quantizes it, one finds that first loop corrections
prescribe a non-minimal coupling term (i.e., $\xi \neq 0$) if the theory is to be renormaliz-
able, thus obtaining the “improved energy-momentum tensor" of Callan, Coleman,
and Jackiw (see also Chernikov and Tagirov). Does quantization change the
character of this scalar field from “non-gravitational” to “gravitational”? Formally,
the resulting theory is a scalar-tensor theory according to every definition of such
theories that one finds in the literature, but many authors consider $\phi$ a
non-gravitational field, and certainly this is the point of view of Callan et al.
(in which $\phi$ is regarded as a matter field being quantized) and of most particle
physicists.

4. Theories and representations
As it will be demonstrated later on, many misconceptions arise when a theory
is identified with one of its representations and other representations are implic-
itly treated as different theories. Even though this might seem too abstract, to
avoid confusion, one would like to provide precise definitions of the words “theory”
and “representation”. This, however, is not trivial. For the term “theory”, even if
one looks at a popular internet dictionary, a number of possible definitions can be
found:

1. An unproven conjecture.
2. An expectation of what should happen, barring unforeseen circumstances.
3. A coherent statement or set of statements that attempts to explain observed
   phenomena.
4. A logical structure that enables one to deduce the possible results of every
   experiment that falls within its purview.
(5) A field of study attempting to exhaustively describe a particular class of constructs.

(6) A set of axioms together with all statements derivable from them.

It is apparent that definitions (i) and (ii) are not applicable to physical theories. On the other hand, (iii) and (iv) seem to be complementary statements describing the use of the word “theory” in natural sciences, whereas (v) and (vi) appear to have a mathematical and logical basis respectively. In a loose sense, a more complete definition for the word “theory” in the context of physics would probably come from a combination of (iv) and (vi), in order to combine the reference to experiments in (iv) and the mathematical rigidity of (vi). An attempt towards this direction could be:

**Definition 1.** Physical Theory: A coherent logical structure, preferably expressed through a set of axioms together with all statements derivable from them, plus a set of rules for their physical interpretation, that enable one to deduce and interpret the possible results of every experiment that falls within its purview.

Note that no reference is made as to whether there is an agreement between the predictions of the theory and the actual experiments. This is a further step which should be considered in order to check how successful a theory is in describing the physical world. There could be criteria according to which the theory is successful or not according to how large a class of observations is explained by it and to the level of accuracy obtained (see for example Hawking’s book). Additionally, one could consider simplicity as a merit and characterize a theory according to the number of assumptions that it is based on (Ockham’s razor). However, all the above should not be included in the definition of the word “theory” itself.

Physical theories should acquire a mathematical representation. This requires the introduction of physical variables (functions or fields) with which the axioms can be encoded in mathematical relations. We attempt to give a definition:

**Definition 2.** Representation (of a theory): A finite collection of equations interrelating the physical variables which are used to describe the elements of a theory and assimilate its axioms.

The reference to equations can be restrictive as one may claim that in many cases a theory could be represented fully by an action. At the same time it is obvious that a representation of a theory is far from being unique. Therefore, one might prefer
to modify the above definition as follows:

**Definition 3.** Representation (of a theory): A non-unique choice of physical variables between which, in a prescribed way, one can form inter-relational expressions that assimilate the axioms of the theory and can be used in order to deduce derivable statements.

It is worth stressing here that when choosing a representation for a theory it is essential to provide also a set of rules for the physical interpretation of the variables involved in it. This is needed for formulating the axioms (i.e., the physical statements) of the theory in terms of these variables. It should also be noted that these rules come as extra information not *a priori* contained in the mathematical formalism. Furthermore, once they are consistently used to interpret the variables of the latter, they would allow to consistently predict the outcome of experiments in any alternative representation (we shall come back to this point and discuss an example later on in section 5.2).

All of the above definitions are, of course, tentative or even naive ones and others can be found that are more precise and comprehensive. However, they are good enough to make the following point: the arbitrariness that inevitably exists in choosing the physical variables is bound to affect the representation. More specifically, it will affect the clarity with which the axioms or principles of the theory appear in every representation. Therefore, there will be representations in which it will be obvious that a certain principle is satisfied and others in which it will be more intricate to see that. However, it is clear that the theory is one and the same and that the axioms or principles are independent of the representation.

5. Scalar-tensor gravity

In order to make the discussion of the previous sections clearer, let us use scalar-tensor theories of gravitation as an example. As in most current theories, scalar-tensor theory was not originally introduced as a collection of axioms but directly through a representation. More precisely, this class of theories is described by the action

\[ S = S^{(g)} + S^{(m)} \left[ e^{2\alpha(\phi)} g_{\mu\nu}, \psi^{(m)} \right] , \tag{3} \]

where

\[ S^{(g)} = \int d^4x \sqrt{-g} \left[ \frac{A(\phi)}{16\pi G} R - \frac{B(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right] . \tag{4} \]

In order to write this action we have used the notation of Flanagan\(^{27}\) (see also Shapiro and Takata\(^{28}\)). Note that this is not the most conventional notation found in the literature, as some of the unspecified functions \(A, B, V,\) and \(\alpha\) can be fixed without loss of generality, *i.e.*, without choosing a theory within the class. However, this would come at the expense of fixing the representation, which is exactly what...
we intend to analyse here. Therefore, this notation is indeed the most convenient for our purposes.

Let us first see how this action comes about from first principles. As discussed in section 3 following Will’s book\cite{12} one can argue that the EEP can only be satisfied if there exists some metric and the matter fields are coupled to it not necessarily minimally but through a non-constant scalar, i.e., they can be coupled to a quantity $\phi g_{\mu \nu}$, where $\phi$ is some scalar. However, this coupling should be universal in the sense that all fields should couple to $\phi$ in the same way. So the most general form of the matter action will have a dependence on $\phi g_{\mu \nu}$. Of course, one can always choose to write $\phi$ as $e^{2\alpha(\phi)}$, where $\phi$ is a dynamical field.

Now, the rest of the action should depend on $\phi$, the metric and their derivatives. No real principle leads directly to the action above. However, one could impose that the resulting field equations should be of second order both in the metric and the scalar and utilize diffeomorphism invariance arguments to arrive to this action. Then, (4) is the most general scalar-tensor action that one can write, once no fields other than $\phi$ and the metric are considered, and no other couplings than a non-minimal coupling of the scalar to the curvature is allowed.

We now return to the role of the four yet to be defined functions $A(\phi)$, $B(\phi)$, $V(\phi)$, $\alpha(\phi)$ and examine whether there are redundancies. As already said, the action (4) describes a class of theories, not a single theory. Specifying some of the four functions will pin down a specific theory within that class. However, one can already see that this action is formally invariant under arbitrary conformal transformations $\tilde{g}_{\mu \nu} = \Omega^2(\phi)g_{\mu \nu}$. In fact, it can be recast in its initial form by simply redefining the undetermined functions $A(\phi)$, $B(\phi)$, $V(\phi)$, $\alpha(\phi)$ after the conformal transformation. This implies that one can set any one of the functions $A(\phi)$, $B(\phi)$, $V(\phi)$ and $e^{2\alpha(\phi)}$ to a (non-vanishing) constant through a suitable choice of $\Omega(\phi)$. Additionally, the scalar field $\phi$ can be redefined conveniently, in order to set yet another of these functions to a constant. Therefore, we conclude that setting two of these functions to a constant (or just unity) is merely a choice of representation and has nothing to do with the theory. Actually, it does not even select a theory within the class.

This has a precise physical meaning: it demonstrates our ability to choose our clocks and rods at will\cite{30}. One could decide not to allow that in a theory (irrespective of how natural that would be). Therefore, it constitutes a very basic physical assumption or even an axiom.

Let us now turn our attention to the matter fields $\psi^{(m)}$: the way we have written the action implies that we have already chosen a representation for them. However, it should be clear that we could always redefine the matter fields at will. For example one could set $\tilde{\psi} = \Omega^s \psi^{(m)}$ where $s$ is a conveniently selected conformal weight\cite{10} so

\footnote{This is not the case in supergravity and string theories, in which gravivector and graviscalar fields can couple differently to particles with different quark content\cite{29}.}
that, after a conformal transformation, the matter action will be

\[ S^{(m)} = S^{(m)} \left[ \tilde{g}_{\mu\nu}, \tilde{\psi} \right]. \]  

(5)

The tilde is used in order to distinguish the physical variable in the two representations. We can now make use of the previously discussed freedom to fix two of the four functions of the field at will and set \( A = B = 1 \). The action (4) will then take the same form as that of general relativity with a scalar field minimally coupled to gravity.

However this theory is not general relativity since now \( \tilde{\psi} = \tilde{\psi}(\phi) \) which essentially means that we have allowed the masses of elementary particles and the coupling constants to vary with \( \phi \) and consequently with their location in spacetime. From a physical perspective, this is translated into our ability to choose whether it will be our clocks and rods that are unchanged in time and space or instead the outcome of our measurements (which, remember, are always dimensionless constants or dimensionless ratios, as, even the measurement of a dimensional quantity such as \( e.g. \), a mass, is nothing more than a comparison with a fixed unit of the same dimensions). We will return to this issue again in section 5.2.

To summarize, we can practically choose two of the four functions in the action (4) without specifying the theory. In addition, we can fix even a third function at the expense of allowing the matter fields \( \psi^{(m)} \) to depend explicitly on \( \phi \), which leads to varying fundamental units. Once any of these two options is chosen, the representation is completely fixed and any further fixing of the remaining function or functions leads to a specific theory within the class. On the other hand, by choosing any two functions and allowing for redefinitions of the metric and the scalar field, it is possible to fully specify the theory and still leave the representation completely arbitrary.

However, it is now obvious that each representation might display different characteristics of the theory and care should be taken in order not to be misled into representation-biased conclusions, exactly as it happens in different coordinate systems. This highlights the importance of distinguishing between different theories and different representations.

This situation is very similar to a gauge theory in which one must be careful to derive only gauge-independent results. Every gauge is an admissible “representation” of the theory, but only gauge-invariant quantities should be computed for comparison with experiment. In the case of scalar-tensor gravity however, it is not clear what a “gauge” is and how to identify the analog of “gauge-independent” quantities.

5.1. Alternative theories and alternative representations: Jordan and Einstein frames

Let us now go one step further and pick up specific scalar-tensor theories. With \( \psi^{(m)} \) representing the matter fields and by choosing \( \alpha = 0 \) and \( A(\phi) = \phi \) we fully
fix the representation. Let us now suppose that all other functions are known. The action takes the form

\[ S = S^{(g)} + S^{(m)} \left[ g_{\mu\nu}, \psi^{(m)} \right], \]  

(6)

where

\[ S^{(g)} = \int d^4x \sqrt{-g} \left[ \frac{\phi}{16\pi G} R - \frac{B(\phi)}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right], \]  

(7)

and it is apparent that \( T_{\mu\nu} \equiv -\left(2/\sqrt{-g}\right)\delta S^{(m)}/\delta g^{\mu\nu} \) is divergence-free with respect to the metric \( g_{\mu\nu} \) and, therefore, the metric postulates are satisfied.

Now let us take instead a representation where \( A = B = 1 \). Then the action (4) takes the form

\[ S = S^{(g)} + S^{(m)} \left[ e^{2\tilde{\alpha}(\phi)} g_{\mu\nu}, \psi^{(m)} \right], \]  

(8)

with

\[ S^{(g)} = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{16\pi G} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \tilde{V}(\phi) \right]. \]  

(9)

As we have argued, for any (non-pathological) choice of \( B \) and \( V \) in the action (6) there exist some conformal factor \( \Omega(\phi) \) relating \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \) and some suitable redefinition of the scalar \( \phi \) to the scalar \( \tilde{\phi} \), which brings action (6) to the form of action (8), therefore relating \( B \) and \( V \) with \( \tilde{V} \) and \( \tilde{\alpha} \). Actions (4) and (8) are just different representations of the same theory after all, assuming that \( B \) and \( V \) or \( \tilde{V} \) and \( \tilde{\alpha} \) are known.

According to most frequently used terminology, the first representation is called the Jordan frame and the second the Einstein frame and the way we have just introduced them should make it crystal clear that they are just alternative, but physically equivalent, representations, of the same theory. (Furthermore, infinitely many conformal frames are possible, corresponding to the freedom of choosing the conformal factor.)

Let us note however that if one defines the stress-energy tensor in the Einstein frame as \( \tilde{T}_{\mu\nu} = -(2/\sqrt{-\tilde{g}})\delta S^{(m)}/\delta \tilde{g}^{\mu\nu} \) one can show that it is not divergence-free with respect to the Levi-Civita connection of the metric \( \tilde{g}_{\mu\nu} \). In fact the transformation property of the matter stress-energy tensor under the conformal transformation \( g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu} \) is \( \tilde{T}_{\mu\nu} = \Omega^2 T_{\mu\nu} \), where the appropriate conformal weight in four spacetime dimensions is \( s = -6 \). Then, the Jordan frame covariant conservation equation \( \nabla^\beta T_{\alpha\beta} = 0 \) is mapped into the Einstein frame equation

\[ \tilde{\nabla}_\alpha \tilde{T}^{\alpha\beta} = -\tilde{T} \frac{\tilde{g}^{\alpha\beta} \tilde{\nabla}_\alpha \Omega}{\Omega}, \]  

(10)

which highlights the fact that the Einstein frame energy-momentum tensor of matter is not covariantly conserved, unless it describes conformally invariant matter with vanishing trace \( T \), which of course is not the general case.
In summary, we see that while the actions (6) and (8) are just different representations of the same theory, the metric postulates and the EEP are obviously satisfied in terms of the variables of the Jordan frame, whereas, at least judging naively from eq. (10), one could be led to the conclusion that the EEP is not satisfied by the variables of the Einstein frame representation. However this is obviously paradoxical as we have seen that the general form of the scalar-tensor action (3) can be derived from the EEP.

The point is that an experiment is not sensitive to the representation, and hence in the case of the action (3) it will not show any violation of the EEP. The EEP will not be violated in any chosen representation of the theory. A common misconception is that people speak about violation of the EEP or the WEP in the Einstein frame simply implying that $\tilde{g}_{\mu\nu}$ is not the metric whose geodesics coincide with free fall trajectories. Even though this is correct, it does not imply a violation of the WEP or the EEP simply because all that these principles require is that there exist some metric whose geodesics coincide with free fall trajectories, and indeed we have one, namely $g_{\mu\nu}$, the metric tensor of the Jordan frame. Whether or not one chooses to represent the theory with respect to this metric is simply irrelevant.

To go one step further let us study free fall trajectories in the Einstein frame. By considering a dust fluid with stress-energy tensor $\tilde{T}^{\alpha\beta} = \tilde{\rho} \tilde{u}^{\alpha} \tilde{u}^{\beta}$, eq. (10) becomes

$$\tilde{\nabla}^{\alpha} (\tilde{\rho} \tilde{u}^{\alpha} \tilde{u}^{\beta}) = \tilde{\rho} \tilde{\nabla}^{\beta} \phi. \tag{11}$$

By projecting this equation onto the 3-space orthogonal to $\tilde{u}^{\mu}$ by means of the operator $\tilde{h}_{\mu\nu}$ defined by $\tilde{g}_{\mu\nu} = -\tilde{u}_{\mu} \tilde{u}_{\nu} + \tilde{h}_{\mu\nu}$ and satisfying $\tilde{h}_{\alpha\beta} \tilde{u}^{\beta} = 0$, one obtains

$$\tilde{a}^{\gamma} = \tilde{h}_{\beta}^{\gamma} \tilde{u}^{\alpha} \tilde{\nabla}_{\alpha} \tilde{u}^{\beta} = \tilde{h}_{\beta}^{\gamma} \tilde{u}^{\alpha} \frac{\partial_{\alpha} \Omega(\phi)}{\Omega(\phi)}. \tag{12}$$

The term on the right hand side of eq. (11), which would have been zero if the latter was the standard geodesic equation, can be seen as due to the gradient of the scalar field $\phi$, or as due to the variation of the particle mass $\tilde{m} = \Omega^{-1} m$ along its trajectory, or as due to the variation of the Einstein frame unit of mass $\tilde{m}_u = \Omega^{-1} m_u$ (where $m_u$ is the constant unit of mass in the Jordan frame) with the spacetime point — see Farahi and Nadeau for an extensive discussion.

Massive particles in the Einstein frame are always subject to a force proportional to $\nabla^{\mu} \phi$, hence there are no massive test particles in this representation of the theory. From this perspective, the formulation of EEP “(massive) test particles follow (timelike) geodesics” is not satisfied nor violated: it is simply empty. Clearly, the popular formulation of the EEP in terms of the metric postulates is representation-dependent.

The metric $g_{\mu\nu}$ has in this sense a distinguished status with respect to any other conformal metric, such as $\tilde{g}_{\mu\nu}$. However, it is a matter of taste and sometimes misleading to call a representation physical or not. The fact that it is better highlighted in the Jordan frame that the theory under discussion satisfies the EEP does not
make this frame preferable, in the same sense that the Local Lorentz coordinate frame is not a preferred one. The Einstein frame is much more suitable for other applications, e.g., finding new exact solutions by using mappings from the Einstein conformal frame, or the computation of the spectrum of density perturbations during inflation in the early universe.

Let us now concentrate on the ambiguities related to the metric postulates mentioned in sections 3.1 and 3.2. One should be already convinced that these postulates should be generalized to include the phrase “there exists a representation in which” (Thorne and Will themselves comment on the dependence of their postulates on the representation). But apart from that, there are additional problems. For example, in the Jordan frame \( \phi \) couples explicitly to the Ricci scalar. One could, therefore, say that \( \phi \) is a gravitational field and not a matter field. In the Einstein frame, however, \( \phi \) is not coupled to the Ricci scalar—it is actually minimally coupled to gravity and non-minimally coupled to matter. Can then one consider it a matter field? If this is the case then maybe one should define the stress-energy tensor differently than before and include the \( \phi \) terms in the action, i.e., define

\[
\tilde{S}^{(m)} = \int d^4x \sqrt{-\tilde{g}} \left[ -\frac{1}{2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \tilde{V}(\tilde{\phi}) \right] + S^{(m)} \left[ e^{2\tilde{\alpha}(\tilde{\phi})} \tilde{g}_{\mu\nu}, \psi^{(m)} \right]
\] (13)

and

\[
\tilde{T}_{\mu\nu} = -(2/\sqrt{-\tilde{g}}) \delta \tilde{S}^{(m)}/\delta \tilde{g}^{\mu\nu}.
\] (14)

In this case though, \( \tilde{T}_{\mu\nu} \) will indeed be divergence-free with respect to \( \tilde{g}_{\mu\nu} \) ! The easiest way to see that is to consider the field equations that one derives from the action (8) through a variation with respect to \( \tilde{g}_{\mu\nu} \) and once the redefinitions of eqs. (8) and (14) are taken into account. These are

\[
\tilde{G}_{\mu\nu} = \kappa \tilde{T}_{\mu\nu},
\] (15)

where \( \tilde{G}_{\mu\nu} \) is the Einstein tensor of the metric \( \tilde{g}_{\mu\nu} \). The contracted Bianchi identity \( \tilde{\nabla}_\mu \tilde{G}^{\mu\nu} = 0 \) directly implies that \( \tilde{\nabla}_\mu \tilde{T}^{\mu\nu} = 0 \).

Does that solve the problem, and the fact that it was not apparent that the EEP is not violated in the Einstein frame was just due to a wrong choice of the stress-energy tensor? Unfortunately, this is not the case. First of all \( \tilde{g}^{\mu\nu} \) is still not the metric whose geodesics coincide with free fall trajectories, as it has been shown. Secondly, \( \tilde{T}_{\mu\nu} \) has the following form

\[
\tilde{T}_{\mu\nu} = \tilde{\nabla}_\mu \tilde{\phi} \tilde{\nabla}_\nu \tilde{\phi} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{\nabla}_\sigma \tilde{\phi} \tilde{\nabla}^\sigma \tilde{\phi} - \tilde{g}_{\mu\nu} \tilde{V}(\tilde{\phi}) + \tilde{T}_{\mu\nu},
\] (16)

with \( \tilde{T}_{\mu\nu} \) depending on \( \tilde{\phi} \) as well, and it will not reduce to the special-relativistic stress-energy tensor for the matter field \( \psi^{(m)} \) if \( \tilde{g}_{\mu\nu} \) is taken to be flat. The same is true for the action \( \tilde{S}^{(m)} \). Both of these features are due to the fact that \( \tilde{T}_{\mu\nu} \) includes a non-minimal coupling between the matter fields \( \psi^{(m)} \) and the scalar \( \phi \). Actually,
setting $\tilde{g}_{\mu \nu}$ equal to the Minkowski metric does not correspond to choosing the Local Lorentz frame: that would be the one in which $g_{\mu \nu}$ is flat to second order (see section 3).

The moral is that one can find quantities that indeed formally satisfy the metric postulates but these quantities are not necessarily physically meaningful. There are great ambiguities as mentioned before, in defining the stress-energy tensor or in judging whether a field is gravitational or just a matter field that practically makes the metric postulates useless outside of a specific representation (and how does one know, in general, when given an action, if it is in this representation, i.e., if the quantities of this representation are the ones to be used directly to check the validity of the metric postulates or a representation change is due before doing so?).

5.2. Matter or geometry? An ambiguity

We already saw that treating $\phi$ as a matter field merely because it is minimally coupled to gravity and including it in the stress-energy tensor did not help in clarifying the ambiguities of the metric postulates. Since, however, this did not answer the question of whether a field should be considered of gravitational (“geometric”) or of non-gravitational (“matter”) nature, let us try to get some further insight.

Consider again, as an example, scalar-tensor gravity. Choosing, $A(\phi) = 8\pi G \phi$ and $\alpha$ to be a constant, the action (4) can be written as

$$S = \int d^4x \sqrt{-g} \left[ \frac{\phi R}{2} - \frac{B(\phi)}{2} g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right. + \alpha \psi \mathcal{L}(\phi) \left( g_{\mu \nu}, \phi^{(m)} \right) \right],$$

(17)

where $\alpha_\psi$ is the coupling constant between gravity and the specific matter field $\psi^{(m)}$ described by the Lagrangian density $\mathcal{L}(\phi)$. This representation is the Jordan frame and it is no different than that of the action (5), apart from the fact that we have not specified the value of the coupling constant $\alpha_\psi$ to be 1.

It is common practice to say that the Brans-Dicke scalar field $\phi$ is gravitational, i.e., it describes gravity together with the metric $g_{\mu \nu}$, indeed, $1/\phi$ plays the role of a (variable) gravitational coupling. However, this interpretation only holds in the Jordan frame. As, discussed earlier, the conformal transformation to the Einstein frame $g_{\alpha \beta} \rightarrow \tilde{g}_{\alpha \beta} = \Omega^2 g_{\alpha \beta}$ with $\Omega = \sqrt{G\phi}$, together with the scalar field redefinition

$$d\tilde{\phi} = \sqrt{\frac{2\omega(\phi) + 3}{16\pi G}} \frac{d\phi}{\phi},$$

(18)

casts the action into the form

$$S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{\tilde{R}}{2} - \frac{1}{2} \tilde{g}^{\mu \nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \tilde{V}(\tilde{\phi}) + \tilde{\alpha}_\psi \mathcal{L}(\tilde{\phi}) \right],$$

(19)
where

$$\tilde{V}(\tilde{\phi}) = \frac{V[\phi(\tilde{\phi})]}{\phi^2(\tilde{\phi})}$$  \hspace{1cm} (20)$$

and

$$\tilde{\alpha}_\psi(\tilde{\phi}) = \frac{\alpha_\psi}{\phi^2(\tilde{\phi})}.$$  \hspace{1cm} (21)$$

The “new” scalar field $\tilde{\phi}$ is now minimally coupled to the Einstein frame Ricci scalar $\tilde{R}$ and has canonical kinetic energy: \textit{a priori}, nothing forbids to interpret $\tilde{\phi}$ as being a “matter field”. The only memory of its gravitational origin as seen from the Jordan frame is in the fact that now $\tilde{\phi}$ couples non-minimally to matter, as described by the varying coupling $\tilde{\alpha}_\psi(\tilde{\phi})$. But, by itself, this coupling only describes an interaction between $\tilde{\phi}$ and the “true” matter field $\psi(m)$. One could, for example, take $\psi(m)$ as the Maxwell field and consider an axion field that couples explicitly to it, obtaining an action similar to (19) and being unable to discriminate between this axion and a putative “geometrical” field on the base of its non-minimal coupling. Even worse, this “anomalous” coupling of $\tilde{\phi}$ to matter is lost if one considers only the gravitational sector of the theory by dropping $L(\psi)$ from the discussion. This is the situation, for example, if the scalar $\tilde{\phi}$ is supposed to dominate the dynamics of an early, inflationary, universe or of a late, quintessence-dominated, universe.

More generally, the distinction between gravity and matter (“gravitational” versus “non-gravitational”) becomes blurred in any change of representation involving a conformal transformation of the metric $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$. The transformation property of the Ricci tensor is

$$\tilde{R}_{\alpha\beta} = R_{\alpha\beta} - 2 \nabla_\alpha \nabla_\beta (\ln \Omega) - g_{\alpha\beta} g^{\gamma\delta} \nabla_\gamma \nabla_\delta (\ln \Omega)$$

$$+ 2 (\nabla_\alpha \ln \Omega) (\nabla_\beta \ln \Omega) - 2 g_{\alpha\beta} g^{\gamma\delta} (\nabla_\gamma \ln \Omega) (\nabla_\delta \ln \Omega).$$  \hspace{1cm} (22)$$

A vacuum solution \textit{(i.e., one with $R_{\alpha\beta} = 0$)} in the Jordan frame is mapped into a non-vacuum solution ($\tilde{R}_{\alpha\beta} \neq 0$) in the Einstein frame by the conformal transformation. The conformal factor $\Omega$, a purely “geometrical” field in the Jordan frame is now playing the role of a form of “matter” in the Einstein frame.

A possible way of keeping track of the gravitational nature of $\Omega$ is by remembering that the Einstein frame units of time, length, and mass are not constant but scale according to $\tilde{t}_a = \Omega t_a$, $\tilde{l}_a = \Omega l_a$, and $\tilde{m}_a = \Omega^{-1} m_a$, respectively (where $t_a$, $l_a$, and $m_a$ are the corresponding constant units in the Jordan frame). However, one would not know this prescription by looking only at the Einstein frame action (19) unless the prescription for the units is made part of the theory \textit{(i.e., carrying extra information with respect to the one given by the action!)}. In practice,

\footnote{This point was also stressed by Einstein.}
even when the action (19) is explicitly obtained from the Jordan frame representation, the variation of units with $\Omega$ (and therefore with the spacetime point) is most often forgotten in the literature \cite{footnote} hence leading to the study of a different theory with respect to that expressed by the action (17).

Going back to the distinction between material and gravitational fields, an alternative possibility to distinguish between “matter” and “geometry” would seem to arise by labeling as “matter fields” only those described by a stress-energy tensor that satisfies some energy condition. In fact, a conformally transformed field that originates from Jordan frame geometry does not, in general, satisfy any energy condition. The “effective stress-energy tensor” of the field $\Omega$ derived from eq. (22) does not have the canonical structure quadratic in the first derivatives of the field but contains instead terms that are linear in the second derivatives. Because of this structure, the stress-energy tensor $\Omega$ violates the energy conditions. While it would seem that labeling as “matter fields” those that satisfy the weak or null energy condition could eliminate the ambiguity, this is not the case. As we have previously seen, one can always redefine the scalar field in such a way that it is minimally coupled to gravity and has canonical kinetic energy (this is precisely the purpose of the field redefinition (18)). Keeping track of the transformation of units in what amounts to a full specification of the representation adopted (action plus information on how the units scale with the scalar field) could help making the property of satisfying energy conditions frame-invariant, but at the cost of extra “structure” in defining a given theory.

As a conclusion, the concept of vacuum versus non-vacuum, or of “matter field” versus “gravitational field” is representation-dependent. One might be prepared to accept a priori and without any real physical justification that one representation should be chosen in which the fields are to be characterized as gravitational or non-gravitational and might be willing to carry this extra “baggage” in any other representation in the way described above. Even so, a solution to the problem which would be as tidy as one would like, is still not provided.

These considerations, as well those discussed at the end of section 5.1, elucidate a more general point: it is not only the mathematical formalism associated with a theory that is important, but the theory must also include a set of rules to interpret physically the mathematical laws. As an example from the classical mechanics of point particles, consider two coupled harmonic oscillators described by the Lagrangian

$$L = \frac{\dot{q}_1^2}{2} + \frac{\dot{q}_2^2}{2} - \frac{q_1^2}{2} - \frac{q_2^2}{2} + \alpha q_1 q_2.$$  \hfill (23)

A different representation of this physical system is obtained by using normal coordinates $Q_1 (q_1, q_2), Q_2 (q_1, q_2)$, in terms of which the Lagrangian (23) becomes

$$L = \frac{\dot{Q}_1^2}{2} + \frac{\dot{Q}_2^2}{2} - \frac{Q_1^2}{2} - \frac{Q_2^2}{2}.$$  \hfill (24)

Taken at face value, this Lagrangian describes a different physical system, but we
know that the mathematical expression is not all there is to the theory: the interpretation of \( q_1 \) and \( q_2 \) as the degrees of freedom of the two original oscillators prevents viewing \( Q_1 \) and \( Q_2 \) as the physically measurable quantities. In addition to the equations of motion, a set of interpretive rules constitutes a fundamental part of a theory. Without such rules it is not only impossible to connect the results derived through the mathematical formalism to a physical phenomenology but one would not even be able to distinguish alternative theories from alternative representations of the same theory. Note however, that once the interpretative rules are assigned to the variables in a given representation they do allow to predict the outcome of experiments in any other given representation of the theory (if consistently applied), hence assuring the physical equivalence of the possible representations.

While the above comments hold in general for any physical theory, it must however be stressed that gravitation theories are one of those cases in which the problem is more acute. In fact, while the physical interpretation of the variables is clear in simple systems, such as the example of the two coupled oscillators discussed above, the physical content of complex theories (like quantum mechanics or gravitation theories) is far less intuitive. Indeed, for what regards gravity, what we actually know more about is the phenomenology of the system instead of the system itself. Therefore, it is often difficult, or even arbitrary, to formulate explicit interpretive rules, which should nevertheless be provided in order to completely specify the theory.

6. \( f(R) \) gravity

To highlight even more the ambiguity of whether a field is a gravitational or matter field, as well as to demonstrate how the problems discussed here can actually go beyond representations that just involve conformal redefinitions of the metric, let us utilise one further example: that of \( f(R) \) gravity (see also \( \text{[2]} \) for a recent review and references). These gravity theories have received considerable attention lately, as they lead to interesting cosmological phenomenology which may be able to account for dark energy\( ^{[4]} \). The action of the theory is

\[
S_{\text{met}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(R) + S_M(g_{\mu\nu}, \psi). \tag{25}
\]

Variation with respect to the metric gives\( ^{[3]} \)

\[
f'(R)R_{\mu\nu} - \frac{1}{2} f(R)g_{\mu\nu} - \nabla_\mu \nabla_\nu f'(R) + g_{\mu\nu} \Box f' = \kappa T_{\mu\nu}, \tag{26}
\]

where a prime denotes differentiation with respect to the argument. These are fourth order partial differential equations for the metric.

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\( ^{[4]} \)Several concerns have been expressed about the viability of these theories, and in many cases there is still open debate.\( ^{[2]} \) However, viability is irrelevant here as we are using these theories as examples to make a completely different point.
One could also choose to consider the connection $\Gamma_{\mu\nu}^\lambda$ as an independent quantity and construct the Riemann tensor and the Ricci tensor accordingly. The Ricci tensor, $R_{\mu\nu}$, is constructed using only the connections and the scalar curvature $\mathcal{R}$ is then defined to be the contraction of $R_{\mu\nu}$ with the metric, i.e., $\mathcal{R} \equiv g^{\mu\nu}R_{\mu\nu}$. The action will then be formally the same but $R$ will be replaced with $\mathcal{R}$, i.e.,

$$S_{\text{pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} f(\mathcal{R}) + S_M(g_{\mu\nu}, \psi).$$ (27)

Notice that the matter action is chosen not to depend on the independent connection. Were the matter action allowed to depend on the independent connection, the resulting theory would be a metric-affine theory of gravity. Independent (Palatini) variations with respect to the metric and the connection lead to the two field equations

$$f'(\mathcal{R})R_{(\mu\nu)} - \frac{1}{2} f(\mathcal{R})g_{\mu\nu} = \kappa T_{\mu\nu},$$ (28)

$$\nabla^\lambda (\sqrt{-g} f'(\mathcal{R})g^{\mu\nu}) = 0,$$ (29)

one for the metric and one for the independent connection. $\nabla_\mu$ denotes the covariant derivative defined with the independent connection $\Gamma_{\mu\nu}^\lambda$. Theories described by the action (27) are called $f(\mathcal{R})$ theories of gravity in the Palatini formalism or simply Palatini $f(\mathcal{R})$ theories of gravity.

Now, consider the action (25) of metric $f(R)$ gravity. One can introduce a new field $\chi$ and write a dynamically equivalent action:

$$S_{\text{met}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [f(\chi) + f'(\chi)(R - \chi)] + S_M(g_{\mu\nu}, \psi).$$ (30)

Variation with respect to $\chi$ leads to the equation $\chi = R$ if $f''(\chi) \neq 0$, which reproduces action (25). Redefining the field $\chi$ by $\phi = f'(\chi)$ and setting

$$V(\phi) = \chi(\phi)\phi - f(\chi(\phi)),$$ (31)

the action takes the form

$$S_{\text{met}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi R - V(\phi)] + S_M(g_{\mu\nu}, \psi).$$ (32)

This is the action of a Brans–Dicke theory with Brans–Dicke parameter $\omega_0 = 0$, or the specific choice of $A = \phi$, $B = 0$, $\alpha = 0$ when one refers to the action (41) (fixing both the theory and the representation). So, metric $f(R)$ theories, as has been observed long ago, are fully equivalent to a class of Brans–Dicke theories with vanishing kinetic term.

Similar things can be said for the action (27) for Palatini $f(R)$ gravity. Introducing the scalar field $\chi$ as before and redefining it by using $\phi$, the action takes the form:

$$S_{\text{pal}} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} [\phi R - V(\phi)] + S_M(g_{\mu\nu}, \psi).$$ (33)
Even though the gravitational part of this action is formally the same as that of action (32), this action is not a subcase of action (4) as $R$ is not the Ricci scalar of the metric $g_{\mu\nu}$. However, eq. (29) implies that the connections are the Levi-Civita connections of the metric $h_{\mu\nu} = f'(R)g_{\mu\nu}$. Using the definition of $\phi$ we can write $h_{\mu\nu} = \phi g_{\mu\nu}$. Then we can express $R$ in terms of $R$ and $\phi$:

$$R = R + \frac{3}{2\phi^2} \nabla_\mu \phi \nabla^\mu \phi - \frac{3}{\phi} \Box \phi.$$  \hspace{1cm} (34)

Substituting in the action (33) yields

$$S_{pal} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[ \phi R + \frac{3}{2\phi} \nabla_\mu \phi \nabla^\mu \phi - V(\phi) \right] + S_M(g_{\mu\nu}, \psi),$$  \hspace{1cm} (35)

where we have neglected a total divergence. The matter action has now no dependence on the independent connection $\Gamma^{\lambda}_{\mu\nu}$. Therefore, this is indeed the action of a Brans-Dicke theory with Brans-Dicke parameter $\omega_0 = -3/2$, or, in terms of the action (4), it corresponds to the choice $A = \phi$, $B = -3/2$, $\alpha = 0$.

Notice that the general representation used in the action (4) is actually not as general as one may expect, as we have just shown that theories that are indeed described by this action under suitable choices of the parameters, can even acquire completely different, non-conformal, representations. One can, in principle, add at will auxiliary fields, such as the scalar field $\chi$ used above, in order to change the representation of a theory and these fields need not necessarily be scalar fields. Therefore, all of the problems described so far in this paper are not specific to conformal representations. In this $f(R)$ representation the scalar $\phi$ is not even there, so how one can decide if it is a gravitational or matter field? For the case of metric $f(R)$ gravity, the scalar field was eliminated without introducing any other field, and the metric became the only field describing gravity. On the other hand, in the Palatini formalism the outcome is even more surprising if one considers that the scalar field was replaced with an independent connection, which, theoretically speaking, could have forty degrees of freedom assuming that it is symmetric, and in practice it has only one!

7. Einstein-Cartan-Sciama-Kibble theory

Our final example is Einstein-Cartan-Sciama-Kibble theory. In this theory, one starts with a metric and an independent connection which is not symmetric but has zero non-metricity. We will avoid to present here extensive calculations and details. Instead, we address the reader to the thorough review of Hehl et al. (see also Shapiro for a review on torsion and its quantum aspects). What we would like to focus on is the fact that, as the theory has an independent connection, one usually arrives to the field equations through independent variations with respect to the metric and the connection. Additionally, since the matter action depends on both the metric and the connection, its variation will lead to two objects describing the matter fields: the stress-energy tensor $T_{\mu\nu}$, which is the product of the variation
of the matter action with respect to the metric as usual, and the hypermomentum $\Delta^\lambda_{\mu\nu}$ which comes out of the variation of the matter action with respect to the independent connection.

In this theory $T_{\mu\nu}$ is not divergence-free with respect to either the covariant derivative defined with the Levi-Civita connection, or with respect to the one defined with respect to the independent connection. On the other hand, it also does not reduce to the special-relativistic stress-energy tensor in the suitable limit. However, it can be shown, that a suitable, non-trivial, combination of $T_{\mu\nu}$ and $\Delta^\lambda_{\mu\nu}$ can lead to a tensor that indeed has the latter property. What is more, a third connection can be defined, which leads to a covariant derivative with respect to which this tensor is divergence-free. That is sufficient to guarantee that the EEP is satisfied. Does this make Einstein-Cartan a metric theory? And how useful are the metric postulates to discuss violations of the EEP if, in order to show that they are satisfied, one will have already shown geodesic motion or LLI on the way?

8. Discussion and conclusions

We have attempted to shed some light on the difference between different theories and different representations of the same theory, as well as to reveal the important role played by a representation in our understanding of a theory. To this end, several examples which hopefully highlight this issue have been presented. It has been argued that certain conclusions about a theory which may be drawn in a straightforward manner in one representation, might require serious effort when a different representation is used and vice-versa. Additionally, care should be taken as certain representations may be completely inconvenient or even misleading for specific applications.

It is worth commenting at this point, that the literature is seriously biased towards particular representations. Additionally, this bias is not always a result of the convenience of certain representations in a specific application, but many times it is a mere outcome of habit. It is common, for instance, to bring alternative theories of gravity in a general-relativity-like representation due to its familiar form, even if this might be misleading when it comes to the deeper understanding of the theory.

So far, this seemingly inevitable representation-dependent formulation of our gravitational theories has already been the cause of several misconceptions. What is more, one can very easily recognise a representation bias in the definition of commonly used quantities, such as the stress-energy tensor. Notions such as vacuum and the possibility of distinguishing between gravitational fields and matter fields are also representation-dependent. This is often overlooked due to the fact that one is very much accustomed to the representation-dependent definition given in the literature. On the other hand, representation-free definitions do not exist.

Note, that even though the relevant literature focuses almost completely on conformal frames, the problems discussed here are not restricted to conformal rep-
resentations. Even if conformally invariant theories were considered, nothing forbids the existence of other non-conformal representations of these theories under which the action or the field equations will, of course, not be invariant. These might be implying that creating conformally invariant theories is not the answer to this issue. After all, even though measurable quantities are always dimensionless ratios and are, consequently, conformally invariant, matter is not generically conformally invariant and, therefore, neither can (classical) physics be conformally invariant, at least when its laws are written in terms of the fields representing this matter.

The issue discussed here seems to have its roots in a more fundamental problem: the fact that in order to describe a theory in mathematical terms, a non-unique set of variables has to be chosen. Such a set will always correspond to just one of the possible representations of the theory. Therefore, even though abstract statements such as the EEP are representation-independent, attempts to turn such statements into quantitative mathematical relations that are of practical use, such as the metric postulate, turn out to be severely representation-dependent. Moreover, the EP, although representation-independent, appears to be of little practical use and this is true even if we confine ourselves to the realm of classical physics. The mathematical approach towards an axiomatic formulation mentioned earlier may eventually turn out to be more convenient in this regard.

The analogy between a choice of a representation and a choice of a coordinate system is practically unavoidable. Indeed, consider classical mechanics: one can choose a set of coordinates in order to write down an action describing some system. However, such an action can be written in a coordinate invariant way. In classical field theory one has to choose a set of fields — a representation — in order to write down the action. From a certain viewpoint, these fields are considered to be generalized coordinates. Therefore, one could expect that there should be some representation-independent way to describe the theory. However, up to this point no real progress has been made on this issue.

The representation dependence of quantitative statements acts in such a way that, instead of merely selecting for us viable theories, they actually predispose us to choose theories which, in a specific representation, naively appear more physically meaningful than others irrespectively of whether this is indeed the case. The same problem is bound to appear when one attempts to generalise a theory and at the same time is biased towards a specific representation, as certain generalisations might falsely appear as more “physical” than others in this representation. This effectively answers the question why most of our current theories of gravitation eventually turn out to be just different representations of the same theory or a class of theories. Scalar-tensor theories and theories which include higher order curvature invariants, such as $f(R)$ gravity or fourth order gravity, are typical examples.

Even though this discussion might at some level appear to be purely philosophical, the practical implication of representation dependence should not be underestimated. For instance, how can we formulate theories that relate matter/energy and gravity if we do not have a clear distinction between the two, or if we cannot even
conclude whether such a distinction should be made? Should we then aim to drop any statement based on a sharp separation between matter and gravity sectors?

In conclusion, even though significant progress has been made on the front of gravitation theories, one cannot help but notice that it is still unclear how to relate principles and experiments in order to form simple theoretical viability criteria expressed in a mathematical way. Our inability to enunciate these criteria, as well as several of our very basic definitions, in a representation-invariant way seem to have played a crucial role in this lack of progress. However, this seems to be a critical obstacle to overcome, if we want to go beyond a trial-and-error approach when it comes to gravitational theories.

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