UNIQUE DECOMPOSITION OF TENSOR PRODUCTS OF IRREDUCIBLE REPRESENTATIONS OF SIMPLE ALGEBRAIC GROUPS

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Abstract. We show that a tensor product of irreducible, finite dimensional representations of a simple Lie algebra over a field of characteristic zero, determines the individual constituents uniquely. This is analogous to the uniqueness of prime factorisation of natural numbers.

1. Introduction

1.1. Let \( g \) be a simple Lie algebra over \( \mathbb{C} \). The main aim of this paper is to prove the following unique factorisation of tensor products of irreducible, finite dimensional representations of \( g \):

\[
\text{Theorem 1.} \quad \text{Let } g \text{ be a simple Lie algebra over } \mathbb{C}. \text{ Let } V_1, \ldots, V_n \text{ and } W_1, \ldots, W_m \text{ be non-trivial, irreducible, finite dimensional } g\text{-modules. Assume that there is an isomorphism of the tensor products,}
\]

\[
V_1 \otimes \cdots \otimes V_n \simeq W_1 \otimes \cdots \otimes W_m,
\]

\text{as } g\text{-modules. Then } m = n, \text{ and there is a permutation } \tau \text{ of the set }
\{1, \ldots, n\}, \text{ such that}
\]

\[
V_i \simeq W_{\tau(i)},
\]

\text{as } g\text{-modules.}

The particular case which motivated the above theorem is the following corollary:

\textbf{Corollary 1.} \textit{Let } V, W \textit{ be irreducible } g\text{-modules. Assume that}
\[
\text{End}(V) \simeq \text{End}(W),
\]

\textit{as } g\text{-modules. Then } V \textit{ is either isomorphic to } W \textit{ or the dual } g\text{-module } W^*.

When \( g = \mathfrak{sl}_2 \), and the number of components is at most two, the theorem follows by comparing the highest and lowest weights that occur in the tensor product. However, this proof seems difficult to generalize.
(see Subsection 2.1). The first main step towards a proof of the theorem, is to recast the hypothesis as an equality of the corresponding products of characters of the individual representations occurring in the tensor product. A pleasant, arithmetical proof for $\mathfrak{sl}_2$ (see Proposition 4), indicates that we are on a right route. The proof in the general case depends on the fact that the Dynkin diagram of a simple Lie algebra is connected, and proceeds by induction on the rank of $\mathfrak{g}$, using the fact that any simple Lie algebra of rank $l$, has a simple subalgebra of rank $l - 1$. We analyze the restriction of the Weyl character formula of $\mathfrak{g}$ to the centralizer of the simple subalgebra, by expanding along the characters of the central $\mathfrak{gl}_1$. We compare the coefficients, which are characters of the simple subalgebra, of the highest and the second highest degrees occurring in the product. The highest degree term is again the character corresponding to a tensor product of irreducible representations. The second highest degree term is a sum of the products of irreducible characters. To understand this sum, we again argue by induction using character expansions. However, instead of leading to further complicated sums, the induction argument stabilizes, and we can formulate and prove a linear independence property of products of characters of a particular type. Combining the information obtained from the highest and the second highest degree terms occurring in the product, we obtain the theorem.

The outline of this paper is as follows: first we recall some preliminaries about representations and characters of semisimple Lie algebras. We then give the proof for $\mathfrak{sl}_2$, and also of an auxiliary result which comes up in the proof by induction. Although not needed for the proof in the general case, we present the proof for $GL_n$, since the ideas involved in the proof seem a bit more natural. Here the numerator of the Weyl-Schur character formula appears as a determinant, which can be looked upon as a polynomial function on the diagonal torus. The inductive argument arises upon expanding this function in one of the variables, the coefficients of which are given by the numerators occurring in the Weyl-Schur character formula for appropriate representations of $GL_{n-1}$. We then set up the formalism for general simple $\mathfrak{g}$, so that we can carry over the proof for $GL_n$ to the general case.

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proved a similar result for the usual degree two Asai representations, and I thank him for conveying to me his results.

2. Preliminaries

We fix the notation and recall some of the relevant aspects of the representation and structure theory of semisimple Lie algebras. We refer to [H], [S] for further details.

(1) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\Phi \subset \mathfrak{h}^*$ the roots of the pair $(\mathfrak{g}, \mathfrak{h})$.

(2) Denote by $\Phi^+ \subset \Phi$, the subset of positive roots with respect to some ordering of the root system, and by $\Delta$ a base for $\Phi^+$.

(3) Let $\Phi^* \subset \mathfrak{h}^*$, $\Phi^{++}$, $\Delta^*$ be respectively the set of coroots, positive coroots and fundamental coroots. Given a root $\alpha \in \Phi$, $\alpha^*$ will denote the corresponding coroot.

(4) Denote by $\langle ., . \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ the duality pairing. For any root $\alpha$, we have $\langle \alpha^*, \alpha \rangle \geq 2$, and the pairing takes values in integers when the arguments consist of roots and co-roots.

(5) Given a root $\alpha$, by the properties of the root system, there are reflections $s_\alpha$, $s_{\alpha^*}$ of $\mathfrak{h}^*$, $\mathfrak{h}$ respectively, defined by

$$s_\alpha(u) = u - \langle \alpha^*, u \rangle \alpha \quad \text{and} \quad s_{\alpha^*}(x) = x - \langle x, \alpha \rangle \alpha^*,$$

where $x \in \mathfrak{h}$ and $u \in \mathfrak{h}^*$.

(6) Let $W$ denote the Weyl group of the root system. The Weyl group $W$ is generated by the reflections $s_\alpha$ for $\alpha \in \Delta$, subject to the relations (see [C, Theorem 2.4.3])

$$s_\alpha^2 = 1 \quad \text{and} \quad s_\alpha s_\beta s_\alpha = s_{\alpha s_\beta(\beta)}, \ \forall \alpha, \beta \in \Phi.$$

In particular $s_\alpha$ and $s_\beta$ commute if $s_\alpha(\beta) = \beta$. There is a natural isomorphism between the Weyl groups of the root system and the dual root system, given by $\alpha \mapsto \alpha^*$ and $s_\alpha \leftrightarrow s_{\alpha^*}$, the transpose of $s_{\alpha^*}$. We identify the two actions of the Weyl group.

(7) Denote by $P \subset \mathfrak{h}^*$ the lattice of integral weights, given by

$$P = \{ \mu \in \mathfrak{h}^* \mid \mu(\alpha^*) \in \mathbb{Z}, \forall \alpha \in \Phi^* \}.$$

Dually we have a definition of the lattice of integral co-weights $P^*$.

(8) Let $P_+$ be the set of dominant, integral weights with respect to the chosen ordering, defined by

$$P_+ = \{ \lambda \in P \mid \lambda(\alpha^*) \geq 0, \forall \alpha \in \Phi^{++} \}.$$

The irreducible $\mathfrak{g}$-modules are indexed by elements in $P_+$, given by highest weight theory. To each dominant, integral weight $\lambda$, ...
we denote the corresponding irreducible \( \mathfrak{g} \)-module with highest weight \( \lambda \) by \( V_\lambda \). Let \( l = |\Delta| \) be the rank of \( \mathfrak{g} \). Index the collection of fundamental roots by \( \alpha_1, \cdots, \alpha_l \). Denote by \( \omega_1, \cdots, \omega_l \) (resp. \( \omega_1^*, \cdots, \omega_l^* \)), the set of fundamental weights (resp. fundamental co-weights) defined by

\[
\omega_i(\alpha_j^*) = \delta_{ij} \quad \text{and} \quad \omega_i^*(\alpha_j) = \delta_{ij}, \quad 1 \leq i, j \leq l.
\]

The fundamental weights form a \( \mathbb{Z} \)-basis for \( P \).

(9) Let \( l(w) \) denote the length of an element in the Weyl group, given by the least length of a word in the \( s_\alpha, \alpha \in \Delta \) defining \( w \). Let \( \epsilon(w) = (-1)^{l(w)} \) be the sign character of \( W \).

(10) **Weyl character formula.** All the representations considered will be finite dimensional. Let \( V \) be a \( \mathfrak{g} \)-module. With respect to the action of \( \mathfrak{h} \), we have a decomposition,

\[
V = \bigoplus_{\pi \in \mathfrak{h}^*} V^\pi,
\]

where \( V^\pi = \{ v \in V \mid hv = \pi(h)v, \ h \in \mathfrak{h} \} \).

The linear forms \( \pi \) for which \( V^\pi \) are non-zero belong to the weight lattice \( P \), and these are the weights of \( V \). Let \( \mathbb{Z}[P] \) denote the group algebra of \( P \), with basis indexed by \( e_\pi \) for \( \pi \in P \). The (formal) character \( \chi_V \in \mathbb{Z}[P] \) of \( V \) is defined by,

\[
\chi_V = \sum_{\pi \in P} m(\pi)e_\pi,
\]

where \( m(\pi) = \dim(V^\pi) \) is the multiplicity of \( \pi \). The character is a ring homomorphism from the Grothendieck ring \( K[\mathfrak{g}] \) defined by the representations of \( \mathfrak{g} \) to the group algebra \( \mathbb{Z}[P] \). In particular,

\[
\chi_V \otimes \chi_V' = \chi_V \chi_V'.
\]

The irreducible \( \mathfrak{g} \)-modules are indexed by elements in \( P_+ \), given by highest weight theory. To each dominant, integral weight \( \lambda \), we denote the corresponding irreducible \( \mathfrak{g} \)-module with highest weight \( \lambda \) by \( V_\lambda \), and the corresponding character by \( \chi_\lambda \). Let

\[
\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \omega_1 + \cdots + \omega_l.
\]

Define the Weyl denominator \( D \) as,

\[
D = \sum_{w \in W} \epsilon(w)e^{w \rho} \in \mathbb{Z}[P].
\]
The Weyl character formula for $V_\lambda$ is given by,

$$\chi_\lambda = \frac{1}{D} \sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}.$$ 

Let $S_\lambda = \sum_{w \in W} \epsilon(w) e^{w(\lambda + \rho)}$ denote the numerator occurring in the Weyl character formula. We have $D = S_0$.

We now recast the Main theorem. From the theory of characters, the Main theorem is equivalent to the following theorem:

**Theorem 2.** Let $g$ be a simple Lie algebra over $\mathbb{C}$. Assume that there are positive integers $n \geq m$, and non-zero dominant weights $\lambda_1, \ldots, \lambda_n$, $\mu_1, \ldots, \mu_m$ in $P_+$ satisfying,

$$(2) \quad S_{\lambda_1} \cdots S_{\lambda_n} = S_{\mu_1} \cdots S_{\mu_m} (S_0)^{n-m}.$$ 

Then $m = n$, and there is a permutation $\tau$ of the set $\{1, \ldots, n\}$, such that

$$\lambda_i = \mu_{\tau(i)}, \quad 1 \leq i \leq n.$$ 

We adopt a slight change in the notation. Assume $n \geq m$. Then (2) can be rewritten as,

$$(3) \quad S_{\lambda_1} \cdots S_{\lambda_n} = S_{\mu_1} \cdots S_{\mu_n},$$ 

where $\mu_i = 0$ for $m + 1 \leq i \leq n$.

**2.1. $\mathfrak{sl}_2$ and PRV-components.** Let $g = \mathfrak{sl}_2$. Let $V_n$ denote the irreducible representation of $\mathfrak{sl}_2$ of dimension $n + 1$, isomorphic to the symmetric $n^{th}$ power $S^n(V_1)$ of the standard representation $V_1$. Suppose we have an isomorphism of $\mathfrak{sl}_2$-modules,

$$V_{n_1} \otimes V_{n_2} \simeq V_{m_1} \otimes V_{m_2}.$$ 

For any pair of positive integers $l \geq k$, we have the decomposition,

$$V_k \otimes V_l \simeq V_{l+k} \oplus V_{l+k-2} \oplus \cdots \oplus V_{l-k}.$$ 

It follows that $n_1 + n_2 = m_1 + m_2$ by comparing the highest weights. Assuming $n_1 \geq n_2$ and $m_1 \geq m_2$, we have on comparing the lowest weights occurring in the tensor product, that $n_1 - n_2 = m_1 - m_2$. Hence the theorem follows in this special case.

It is immediate from the hypothesis of the theorem, that we have an equality of the sum of the highest weights corresponding to the irreducible modules $V_1, \ldots, V_n$ and $W_1, \ldots, W_m$ respectively. The above proof for $\mathfrak{sl}_2$ suggests the use of PRV-components: if $V_\lambda$ and $V_\mu$ are highest weight finite dimensional $g$-modules with highest weights $\lambda$ and $\mu$ respectively, and $w$ is an element of the Weyl group, then it is known that there is a Weyl group translate $\overline{\lambda + \omega w \mu}$ of the weight.
\( \lambda + w\mu \), which is dominant and such that the corresponding highest weight module \( V_{\lambda+w\mu} \) is a direct summand in the tensor product module \( V_{\lambda} \otimes V_{\mu} \) (see [SK1]). These are the generalized Parthasarathy-Ranga Rao-Varadarajan (PRV)-components. The standard PRV-component is obtained by taking \( w = w_0 \), the longest element in the Weyl group.

But the above proof for \( \mathfrak{sl}_2 \) does not generalize, as the following example for the simple Lie algebra \( \mathfrak{sp}_6 \) shows that it is not enough to consider just the standard PRV-component:

**Example 1.**

\( \mathfrak{g} = \mathfrak{sp}_6, \quad \mathfrak{h} = \mathbb{C} < e_1, e_2, e_3 >, \quad \Delta = \{e_1-e_2, e_2-e_3, 2e_3\}, \quad w_0 = -1. \)

Consider the following highest weights on \( \mathfrak{sp}_6 \):

\[
\begin{align*}
\lambda_1 &= 6e_1 + 4e_2 + 2e_3 \\
\mu_1 &= 6e_1 + 2e_2 + 2e_3 \\
\lambda_2 &= 4e_1 + 2e_2 \\
\mu_2 &= 4e_1 + 4e_2
\end{align*}
\]

Clearly \( \lambda_1 + \lambda_2 = \mu_1 + \mu_2 \). Since the Weyl group contains sign changes, we see that there exists an element of the Weyl group such that \( \lambda_1 - \lambda_2 = w(\mu_1 - \mu_2) \).

Thus we are led to consider generalized PRV-components. The problem with this approach is that although the standard PRV-component can be characterised as the component on which the Casimir acts with the smallest eigenvalue, there is no abstract characterisation of the generalized PRV-component inside the tensor product. It is not clear that a generalized PRV-component of one side of the tensor product, is also a PRV-component for the other tensor product. Although the PRV-components occur with ‘high’ multiplicity [SK2], (greater than or equal to the order of the double coset \( W_\lambda \backslash W/W_\mu \), where \( W_\lambda \) and \( W_\mu \) are the isotropy subgroups of \( \lambda \) and \( \mu \) respectively), the converse is not true. Even for \( \mathfrak{sl}_2 \), it does not seem easy to extend the above proof when the number of components involved is more than two.

3. GL(2)

The aim of this and the following section is to prove the main theorem in the context of \( GL(r) \):

**Theorem 3.** Let \( G = GL(r) \). Suppose \( V \simeq V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n} \) and \( W \simeq V_{\mu_1} \otimes \cdots \otimes V_{\mu_m} \) are tensor products of irreducible representations with non-zero highest weights \( \lambda_1, \ldots, \mu_m \). Assume that \( V \simeq W \) as \( G \)-modules. Then \( n = m \) and there is a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that for \( 1 \leq i \leq n \),

\[
V_{\lambda_i} = V_{\mu_{\tau(i)}} \otimes \det^{\alpha_i},
\]
for some integers \( \alpha_i \).

Upto twisting by a power of the determinant, we can assume that the highest weight representations \( V_\lambda \) of \( GL(r) \) are parametrised by their ‘normalized’ highest weights,

\[
\lambda = (a_1, \cdots a_r), \quad a_1 \geq a_2 \geq \cdots \geq a_r = 0,
\]

and \( a_i \) are nonnegative integers. It is enough then to show under the hypothesis of the theorem, that the normalized highest weights coincide. Let \( x = (x_1, \cdots , x_r) \) be a multivariable. We have the symmetric functions (Schur functions), defined as the quotient of two determinants,

\[
\chi_\lambda(x) = \frac{|x_1^{a_1+r-i} \cdots x_r^{a_r+r-i}|}{|x_1^{r-i} \cdots x_r^{r-i}|} = S_\lambda D,
\]

where \( S_\lambda \) denotes the determinant appearing in the numerator and \( D \) the standard Vandermonde determinant appearing in the denominator. It is known that on the set of regular diagonal matrices the Schur function \( \chi_\lambda \) is equal to the character of \( V_\lambda \). Since we have assumed \( a_n = 0 \), we have that the polynomials \( S_\lambda \) and \( x_1 \) are coprime, for any highest weight \( \lambda \).

Hence by character theory, the hypothesis of the theorem can be recast as

\[
S_{\lambda_1} \cdots S_{\lambda_n} = S_{\mu_1} \cdots S_{\mu_n},
\]

and where \( \mu_i = 0 \) for \( m + 1 \leq i \leq n \).

Write for \( 1 \leq i \leq n \),

\[
\lambda_i = (a_{i1}, a_{i2}, \cdots , a_{i(r-1)}, 0),
\]

\[
\mu_i = (b_{i1}, b_{i2}, \cdots , b_{i(r-1)}, 0).
\]

3.1. \( GL(2) \). We present now the proof of the theorem for \( GL(2) \).

**Proposition 4.** Theorem \( 3 \) is true for \( GL(2) \).

**Proof.** Specializing Equation \( 3 \) to the case of \( GL(2) \), we obtain

\[
(x_1^{a_{11}+1} - x_2^{a_{11}+1}) \cdots (x_1^{a_{n1}+1} - x_2^{a_{n1}+1}) = (x_1^{b_{11}+1} - x_2^{b_{11}+1}) \cdots (x_1^{b_{n1}+1} - x_2^{b_{n1}+1}),
\]

where for the sake of simplicity we drop one of the indices in the weights. Specialising the equation to \( x_2 = 1 \), and letting \( x = x_1 \), we obtain an equality of the product of polynomials,

\[
(x_1^{a_{11}+1} - 1) (x_1^{a_{21}+1} - 1) \cdots (x_1^{a_{n1}+1} - 1) = (x_1^{b_{11}+1} - 1)(x_1^{b_{21}+1} - 1) \cdots (x_1^{b_{n1}+1} - 1).
\]

Assume that \( a_1 = \max\{a_1, \cdots , a_n\} \) and \( b_1 = \max\{b_1, \cdots , b_n\} \). For any positive integer \( m \), let \( \zeta_m \) denote a primitive \( m \)th root of unity. The left hand side polynomial has a zero at \( x = \zeta_{a_1+1} \), and the equality
forces the right side polynomial to vanish at $\zeta_{a_1+1}$. Hence we obtain that $a_1 \leq b_1$, and by symmetry $b_1 \leq a_1$. Thus $a_1 = b_1$ and $\chi_{\lambda_1} = \chi_{\mu_1}$. Cancelling the first factor from both sides, we are left with an equality of a product of characters involving less number of factors than the equation we started with, and by induction we have proved the theorem for $GL(2)$.

\[ \square \]

Remark 1. It would be interesting to know the arithmetical properties of the varieties defined by the polynomials $S_\lambda$ for general semisimple Lie algebras $\mathfrak{g}$. It seems difficult to generalize the above arithmetical proof to general simple Lie algebras. The proof in the general case proceeds by induction on the rank, finally reducing to the case of $\mathfrak{sl}_2$.

3.2. A linear independence result. We now prove an auxiliary result for $GL(2)$, which arises in the inductive proof of Theorem 3.

Lemma 1. Let $\lambda_1, \ldots, \lambda_n$ be a set of normalized weights in $P_+$. Let $c$ be a positive integer and $\omega_1$ denote the fundamental weight. Then the set

$$\{ S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} \mid 1 \leq i \leq n \},$$

is linearly independent. In particular, suppose that there are subsets $I, J \subset \{1, \ldots, n\}$ satisfying the following:

$$\sum_{i \in I} S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = \sum_{j \in J} S_{\lambda_1} \cdots S_{\lambda_{j-1}} S_{\lambda_j + c\omega_1} S_{\lambda_{j+1}} \cdots S_{\lambda_n}. \quad (5)$$

Then there is a bijection $\theta : I \rightarrow J$, such that $\lambda_i = \lambda_{\theta(i)}$.

An equivalent statement can be made in the Grothendieck ring $K[\mathfrak{g}]$ or with characters in place of $S_{\lambda}$.

Proof. Suppose we have a relation

$$\sum_{1 \leq i \leq n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i + c\omega_1} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers $z_i$. For any index $i$, let

$$E(i) = \{ j \mid \lambda_j = \lambda_i \}.$$

To show the linear independence, we have to show that for any index $i$, we have

$$\sum_{j \in E(i)} z_j = 0.$$
Dividing by \( \prod_{i=1}^{n} S_{\lambda_i} \) on both sides and equating, we are left with the equation,

\[
\sum_{1 \leq i \leq n} z_i \frac{S_{\lambda_i + \omega_1}}{S_{\lambda_i}} = 0.
\]

Specialising \( x_1 = 1 \) and writing \( t \) instead of \( x_2 \), we obtain

\[
\sum_{1 \leq i \leq n} z_i \frac{1 - t a_i + c}{1 - t a_i} = 0.
\]

Expand now as a power series in \( t \). Consider the collection of indices \( i \), for which \( a_i \) attains the minimum value, say for \( i = 1 \). Equating the coefficient of \( t^{a_1} \), we see that \( \sum_{j \in E(1)} z_j = 0 \). Hence these terms can be removed from the relation, and we can proceed by induction to complete the proof of the lemma. \( \square \)

**Remark 2.** In retrospect, both Proposition 4 and Lemma 1 can be proved by comparing the coefficient of the second highest power of \( x_1 \) occurring on both sides of the equation (3), as in the proofs occurring in the next section. But we have included the proofs here, since it lays emphasis on the arithmetical properties of the varieties defined by these characters.

4. **Tensor products of \( GL(r) \)-modules**

We now come to the proof of Theorem 3 for arbitrary \( r \). The proof will proceed by induction on \( r \) and the maximum number of components \( n \), and we assume that the theorem is true for \( GL(s) \) with \( s < r \), and for \( GL(r) \) with number of components less than \( n \). Associated to the highest weight \( \lambda = (a_1, a_2, \cdots, a_{r-1}, 0) \) of a \( GL(r) \)-irreducible module, define

\[
\lambda' = (a_2, a_3, \cdots, a_{r-1}, 0),
\]

\[
\lambda'' = (a_1 + 1, a_3, \cdots, a_{r-1}, 0)
\]

We can rewrite

\[
\lambda'' = \lambda' + c(\lambda) \omega_1,
\]

where \( \omega_1 = (1, 0, \cdots, 0) \) is the highest weight of the standard representation of \( GL(r - 1) \), and

\[
c(\lambda) = 1 + (a_1 - a_2).
\]

Both \( \lambda' \) and \( \lambda'' \) are the highest weights of some \( GL(r - 1) \) irreducible modules. (Note: \( 0'' \neq 0 \)).
Expanding $S_\lambda$ as a polynomial in $x_1$ we obtain,
\[
S_\lambda(x_1, \ldots, x_r) = (-1)^{r+1} x_1^{a_1 + r - 1} S_{\lambda'}(x_2, \ldots, x_r) + (-1)^{r} x_1^{a_2 + r - 2} S_{\lambda''}(x_2, \ldots, x_r) + Q,
\]
where $Q$ is a polynomial whose $x_1$ degree is less than $a_2 + r - 2$. Substituting in (3), and equating the top degree term, we have an equality
\[
x_1^\sum j a_j + n(r-1) S_{\lambda'} \cdots S_{\lambda''} = x_1^\sum j b_j + n(r-1) S_{\mu'} \cdots S_{\mu''}.
\]
Since $S_\lambda$ and $x_1$ are coprime polynomials for normalized $\lambda$, we have in particular
\[
(7) \quad S_{\lambda'} \cdots S_{\lambda''} = S_{\mu'} \cdots S_{\mu''}.
\]
Hence by induction it follows that there exists a permutation $\tau'$ of the set $\{1, \ldots, n\}$ such that,
\[
\lambda'_i = \mu'_{\tau'(i)} \quad 1 \leq i \leq n.
\]

Remark 3. At this point, with a little bit of extra work, a proof of the main theorem can be given when the number of components $n$ is at most two. Substituting $x_i = t_i^{i-1}$ for $1 \leq i \leq n$, the determinants $S_\lambda$ can be evaluated as Vandermonde determinants. Arguing as in the proof of the main theorem for $GL(2)$, it can be seen that we have an equality,
\[
\{a_{11}, a_{21}\} = \{b_{11}, b_{21}\}.
\]
If we look at the constant coefficients, we obtain an equality,
\[
\{a_{11} - a_{12}, a_{21} - a_{22}\} = \{b_{11} - b_{12}, b_{21} - b_{22}\}.
\]
These two inequalities combine to prove the main theorem when the number of components is at most two.

Remark 4. The proof does not proceed by restricting to various possible $SL(2)$ mapping to $GL(r)$, and using the theorem for $SL(2)$. For instance, it is not even possible to distinguish between a representation and it’s dual by restricting to various possible $SL(2)$’s mapping to $GL(r)$. Morever we have the following example:

Example 2. Consider the following triples of highest weights on $GL(3)$:
\[
\lambda_1 = (3, 1, 0) \quad \lambda_2 = (2, 2, 0) \quad \lambda_3 = (1, 0, 0) \\
\mu_1 = (3, 2, 0) \quad \mu_2 = (2, 0, 0) \quad \mu_3 = (1, 1, 0)
\]
The calculations in the foregoing remark, give in particular that for any co-root $\alpha^*$, the sets of integers
\[
\{< \alpha^*, \lambda_i + \rho > | 1 \leq i \leq 3\} = \{< \alpha^*, \mu_i + \rho > | 1 \leq i \leq 3\}
\]
are equal, and it is not possible to differentiate the corresponding sets of highest weights. A calculation with the characters of the tensor product, indicates that the term corresponding to the second highest coefficient of $x_1$ in the corresponding product of the characters are different. This observation motivates the rest of the proof of Theorem 1.

We continue with the proof of the theorem. Let 

$$c_1 = \min \{ c(\lambda_i) \mid 1 \leq i \leq n \} \quad \text{and} \quad c_2 = \min \{ c(\mu_j) \mid 1 \leq j \leq n \}.$$  

Equating the coefficient of the second highest power of $x_1$ in Equation 3, we obtain that $c_1 = c_2 = c$. Let $I, J \subset \{1, \cdots, n\}$ denote the sets where for $i \in I$ and $j \in J$, $c(\lambda_i)$ (resp. $c(\mu_j)$) attains the minimum value. We obtain on equating the coefficient of the second highest power of $x_1$ in Equation 3:

$$\sum_{i \in I} S_{\lambda_1} \cdots S_{\lambda_i-1} S_{\lambda_i+\omega_1} S_{\lambda_i+1} \cdots S_{\lambda_n} = \sum_{j \in J} S_{\lambda_1} \cdots S_{\lambda_{j-1}} S_{\lambda_j+\omega_1} S_{\lambda_j+1} \cdots S_{\lambda_n}.$$  

Suppose there are indices $i \in I$ and $j \in J$ such that $\lambda_i = \mu_j$ and $a_i - a_{i+1} = b_j - b_{j+1} = c - 1$. It follows that $\lambda_i = \mu_j$. Cancelling these two factors from the hypothesis of Theorem 3, we can proceed by induction on the number of components to prove Theorem 3. Thus, the proof of Theorem 3 reduces now to the proof of the following key auxiliary lemma (applied to $GL(r-1)$), generalizing Lemma 1.

**Lemma 2.** Let $\lambda_1, \cdots, \lambda_n$ be a set of normalized weights in $P_+$. Let $d$ be a positive integer and $\omega_1$ denote the fundamental weight. For any index $i$, let $E(i) = \{ j \mid \lambda_j = \lambda_i \}$. Suppose we have a relation

$$\sum_{1 \leq i \leq n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_i+d\omega_1} S_{\lambda_i+1} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers $z_i$. Then for any index $i$, we have

$$\sum_{j \in E(i)} z_j = 0.$$

**Proof.** By Lemma 1, the lemma has been proved for $GL(2)$. We argue by induction on the number of components $n$ and on $r$. Let 

$$c = \min \{ c(\lambda_i) \mid 1 \leq i \leq n \}.$$  

Let $M$ be the subset of $\{1, \cdots, n\}$ consisting of those indices $i$ such that $c = 1 + a_{i_1} - a_{i_2}$. We expand both sides as a polynomial in $x_1$, and compare the coefficient of the second highest power of $x_1$. Since $d > 0$, the term $S_{\lambda_i+d\omega_1}$ in the product $S_{\lambda_1} \cdots S_{\lambda_i+d\omega_1} \cdots S_{\lambda_n}$ does not contribute
to the coefficient of the second highest degree term in $x_1$. Further we observe that $(\lambda_i + d\omega_1)' = \lambda_i'$. Hence the contribution to the coefficient of the second highest degree term in $x_1$ of $S_{\lambda_1} \cdots S_{\lambda_i} + d\omega_1 \cdots S_{\lambda_n}$ is given by,

$$\sum_{k \in M \setminus M \cap \{i\}} S_{\lambda_1'} \cdots S_{\lambda_k' + cf_1} \cdots S_{\lambda_n'},$$

where $f_1 = (1, 0, \ldots, 0)$ is a vector in $\mathbb{Z}^{r-1}$. Hence we obtain,

$$\sum_{i=1}^n z_i \sum_{k \in M \setminus M \cap \{i\}} S_{\lambda_1'} \cdots S_{\lambda_k' + cf_1} \cdots S_{\lambda_n'} = 0.$$

Fix an index $i_0 \in M$, and count the number of times $\lambda_i' + cf_1$ terms occur as a component in the product. For this, any index $i \neq i_0$ will contribute. Hence the above sum can be rewritten as,

$$(n - 1) \sum_{i \in M} z_i S_{\lambda_1'} \cdots S_{\lambda_i' + cf_1} \cdots S_{\lambda_n'} = 0.$$

Cancelling those polynomials $S_{\lambda_l'}$ for $l \notin M$, we have by induction for any index $i_0 \in M$,

$$\sum_{j \in E'(i_0)} z_j = 0,$$

where $E'(i_0) = \{ j \in M \mid \lambda_j' = \lambda_{i_0}' \}$. We have assumed that $j \in M$, since we have cancelled those extraneous terms with indices not in $M$. But since $j$ and $i_0$ belong to $M$, we conclude that $\lambda_j = \lambda_{i_0}$. Hence $E'(i_0) = E(i_0)$, and we have proved the lemma. \qed

**Remark 5.** The surprising fact is that the induction step rather than giving expressions where the character sums are spiked at more than one index, actually yields back Equation 8, which is again of the same type as the hypothesis of the lemma.

It is not clear whether there is a more general context in which the above result can be placed. For example, fix a weight $\Lambda$. Consider the collection of characters

$$\{ \prod_{i \in I} \chi_{\lambda_i} \mid I \subset P_+, \sum_{i \in I} \lambda_i = \Lambda \}.$$  

It is not true that this set of characters is linearly independent, since for a fixed $\Lambda$ the cardinality of this set grows exponentially (since it is given by the partition function), whereas the dimension of the space of homogeneous polynomials in two variables of fixed degree depends polynomially (in fact linearly) on the degree. It would be of interest to generalize and place the lemma in a more proper context.
5. Proof of the Main theorem in the general case

We now revert to the notation of Section 2. Our aim is to set up the correct formalism in the general case, so that we can carry over the inductive proof for $GL(n)$ given above. Let $\mathfrak{g}$ be a simple Lie algebra of rank greater than one. Choose a fundamental root $\alpha_1 \in \Delta$. Let $\Delta' = \Delta \setminus \{\alpha_1\}$, and let $\Phi' \subset \Phi$ be the subset of roots lying in the span of the roots generated by $\Delta'$. Let

$$\mathfrak{h}' = \sum_{\alpha \in \Delta'} \mathbb{C} \alpha^*.$$ 

It is known that $\Delta'$ is a base for the semisimple Lie algebra $\mathfrak{g}'$ defined by,

$$\mathfrak{g}' := \mathfrak{h}' \oplus \sum_{\alpha \in \Phi'} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is the weight space of $\alpha$ corresponding to the adjoint action of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}'$ is a semisimple Lie algebra of rank $l - 1$, and the roots of $(\mathfrak{g}', \mathfrak{h}')$ can be identified with $\Phi'$. A special case is when $\alpha_1$ corresponds to a corner vertex in the Dynkin diagram of $\mathfrak{g}$. In this case, $\mathfrak{g}'$ will be a simple Lie algebra.

We now want to find a suitable $\mathfrak{gl}_1$ complement to $\mathfrak{g}'$ inside $\mathfrak{g}$. This is given by the following lemma:

**Lemma 3.** $\omega_1^* \not\in \mathfrak{h}'$.

*Proof.* Suppose we can write,

$$\omega_1^* = \sum_{i=2}^{l} a_i \alpha_i^*,$$

for some complex numbers $a_i$. By definition of $\omega_1^*$, we obtain on pairing with $\alpha_j$ for $2 \leq j \leq l$, the following system of $l - 1$ linear equations, in the unknown $a_i$:

$$\sum_{i=2}^{l} \langle \alpha_i^*, \alpha_j \rangle a_i = 0.$$

But the matrix $(\langle \alpha_i^*, \alpha_j \rangle)_{2 \leq i, j \leq l}$ is the Cartan matrix of the semisimple Lie algebra $\mathfrak{g}'$, and hence is non-singular. Thus $a_i = 0$ for $i = 2, \ldots, l$, and that's a contradiction as $\omega_1^*$ is non-zero. \(\square\)

Let $W'$ denote the Weyl group of $(\mathfrak{g}', \mathfrak{h}')$, and it can be identified with the subgroup of $W$ generated by the fundamental reflections $s_\alpha$ for $\alpha \in \Delta'$. For such $\alpha$, we have,

$$\langle s_\alpha(\omega_1^*), \alpha_j \rangle = \langle \omega_1^*, \alpha_j \rangle - \langle \omega_1^*, \alpha \rangle \langle \alpha^*, \alpha_j \rangle = \delta_{1j}.$$
Hence $W'$ fixes $\omega_1^*$. Conversely, it follows from the fact that $\omega_1^*$ is orthogonal to all the roots $\alpha_2, \cdots, \alpha_l$, that any element of the Weyl group fixing $\omega_1^*$ lies in the subgroup of $W$ generated by the simple reflections $s_{\alpha_i}$, $2 \leq i \leq l$, and hence lies in $W'$ [C, Lemma 2.5.3].

Our next step is to study the restriction of the character $\chi_\lambda$ to $g' \oplus \mathfrak{gl}_1 \subset g$. Let $P'$ denote the lattice of weights of $g'$. We consider $P'$ as a subgroup of $P$, consisting of those weights which vanish when evaluated on $\omega_1^*$. Choose a natural number $m$ such that $\pi(\omega_1^*) \in \frac{1}{m}\mathbb{Z}$ for all weights $\pi \in P$. Let $Z_1$ be the subgroup, isomorphic to the integers, of linear forms on $\mathfrak{h}$, which are trivial on $\mathfrak{h}'$ and evaluated on $\omega_1^*$ lie in $\frac{1}{m}\mathbb{Z}$. We have,

$$P \subset Z_1 \oplus P',$$

and we decompose the character $\chi_\lambda$ with respect to this direct sum decomposition. Given a weight $\pi \in P$, denote by $\pi' \in P'$ it’s restriction to $\mathfrak{h}'$. Let $l_1$ be the weight in $P$, vanishing on $\mathfrak{h}'$ and taking the value 1 on $\omega_1^*$. We define $d_1(\pi) = \pi(\omega_1^*)$ as the degree of $\pi$ along $l_1$. Write any weight $\pi$ with respect to the above decomposition as,

$$\pi = d_1(\pi)l_1 + \pi'$$

so that $e^\pi = e^{d_1(\pi)l_1}e^{\pi'}$.

The numerator of the Weyl character formula decomposes as,

$$S_\lambda = \sum_{d \in \frac{1}{m}\mathbb{Z}} e^{d_1} \left( \sum_{\pi \in W_d} e^\pi e^{w(\lambda + \rho)'} \right),$$

(9)

where $W_d = \{w \in W \mid (w(\lambda + \rho))(\omega_1^*) = d\}$.

We refer to the inner sum as the coefficient of the degree $d$ component along $l_1$, or as the coefficient of $e^{d_1}$. Given a dominant integral weight $\lambda \in P_+$, define

$$a_1(\lambda) = \max\{w\lambda(\omega_1^*) \mid w \in W\},$$

$$a_2(\lambda) = \max\{w\lambda(\omega_1^*) \mid w \in W \text{ and } w\lambda(\omega_1^*) \neq a_1(\lambda)\}.$$

The formalism that we require in order to carry over the proof for $GL(n)$ to the general case, is given by the following lemma:

**Lemma 4.** Let $\lambda$ be a regular weight in $P_+$.

1. The largest value $a_1(\lambda)$ of $(w\lambda)(\omega_1^*)$ for $w \in W$, is attained precisely for $w \in W'$. In particular,

$$a_1(\lambda) = \lambda(\omega_1^*).$$
(2) The second highest value of $a_2(\lambda)$ is attained precisely for $w$ in $W's_{\alpha_1}$, and the value is given by

$$a_2(\lambda) = s_{\alpha_1}\lambda(\omega_1^*) = a_1(\lambda) - \lambda(\alpha_1^*).$$

Proof. 1) By [C, Lemma 2.5.3], we have to show that if $w\lambda(\omega_1^*)$ attains the maximum value, then $w$ fixes $\omega_1^*$. Since $\omega_1^*$ is a fundamental co-weight, we have

$$\omega_1^* - w\omega_1^* = \sum_{i=1}^l n_i\alpha_i^*,$$

for some non-negative natural numbers $n_i$. Hence,

$$\lambda(\omega_1^* - w\omega_1^*) = \sum_{i=1}^l n_i\lambda(\alpha_i^*),$$

and the latter expression is strictly positive, if some $n_i > 0$, since $\lambda$ is regular and dominant. This proves the first part.

2) We prove the second part by induction on the length $l(w)$ of $w$. The statement is clear for the fundamental reflections, which are of length one. Consider an element of the form $ws_\beta$ such that $ws_\beta\lambda(\omega_1^*) = a_2(\lambda)$, $\beta$ is a fundamental root and $l(ws_\beta) = l(w) + 1$. By [C, Lemma 2.2.1 and Theorem 2.2.2], it follows that,

$$w(\beta) \in \Phi^+.$$ We have

$$\lambda - ws_\beta\lambda = (\lambda - w\lambda) + w(\lambda - s_\beta\lambda) = (\lambda - w\lambda) + <\beta^*, \lambda > w\beta.$$ Since $w\beta \in \Phi^+$, and $<\beta^*, \lambda >$ is positive, it follows that

$$ws_\beta\lambda(\omega_1^*) \leq w\lambda(\omega_1^*),$$

with strict inequality if $<\omega_1^*, w\beta >$ is positive. Hence by induction we can assume that $w$ either belongs to $W'$, or is of the form $w_0s_{\alpha_1}$, with $w_0 \in W'$. We have to only consider the second possibility. We obtain,

$$(\lambda - w\lambda)(\omega_1^*) = (\lambda - s_{\alpha_1}\lambda)(\omega_1^*) = <\alpha_1^*, \lambda > > 0.$$ Assuming the hypothesis of the Lemma for $ws_\beta$, it follows that $ws_\beta(\omega_1^*) = w(\omega_1^*)$, and hence we obtain that $s_{\alpha_1}(\beta)$ has no $\alpha_1$ component, when we expand it as a linear combination of the fundamental roots. But

$$s_{\alpha_1}(\beta) = \beta - <\alpha_1^*, \beta > \alpha_1,$$

and it follows that $<\alpha_1^*, \beta > = 0$. From the relations defining the Weyl group, it follows that $s_{\alpha_1}$ and $s_\beta$ commute, and hence the element $ws_\beta$ is of the form $w_1s_{\alpha_1}$ for some element $w_1 \in W'$, and that concludes the proof of the lemma. $\square$
The restriction of the fundamental weights $\omega_2, \ldots, \omega_l$ to $\mathfrak{h}'$ are the fundamental weights (with the indexing set ranging from 2, \ldots, $l$, instead of 1, \ldots, $l-1$) of $\mathfrak{g}'$. In particular, the restriction $\rho'$ of $\rho$ is the sum of the fundamental weights of $\mathfrak{g}'$. For $\lambda \in P_+$, define $\lambda'' \in P'_+$ by,

(10) \hspace{1cm} \lambda'' = (s_{\alpha_1}(\lambda + \rho))' - \rho'.

We have the following corollary, giving the character expansion for the first two terms along $l_1$:

**Corollary 2.** With notation as in the character expansion given by Equation (9), we have

(11) \hspace{1cm} S_{\lambda} = e^{a_1(\lambda + \rho)l_1} S_{\chi'} - e^{a_2(\lambda + \rho)l_1} S_{\lambda''} + L(\lambda),

where $L(\lambda)$ denotes the terms of degree along $l_1$ less than the second highest degree.

**Proof.** The proof is immediate from Lemma 4 and equation 9, upon observing that $\lambda + \rho$ is regular. The second term occurs with the opposite sign, since $l(ws_{\alpha_1}) = l(w) + 1$, for $w \in W'$, and the length function of $W$ restricts to the length function of $W'$, taken with respect to $\Delta$ and $\Delta'$ respectively. \hfill \Box

We write these down explicitly in terms of the fundamental weights.

$$\lambda = n_1(\lambda)\omega_1 + \cdots + n_l(\lambda)\omega_l,$$

in terms of the fundamental weights, so that $\lambda + \rho = (n_1(\lambda) + 1)\omega_1 + \cdots + (n_l(\lambda) + 1)\omega_l$. We have,

(12) \hspace{1cm} (\lambda + \rho)' = (n_2(\lambda) + 1)\omega_2' + \cdots + (n_l(\lambda) + 1)\omega_l'.

Let $\mathfrak{g}' \simeq \bigoplus_{s \in S} \mathfrak{g}_s'$ be the decomposition of $\mathfrak{g}'$ into simple Lie algebras. For each simple component $\mathfrak{g}_s'$ of $\mathfrak{g}'$, let $\alpha_s$ be the unique simple root connected to $\alpha_1$ in the Dynkin diagram of $\mathfrak{g}$. Then

$$-<\alpha_1^*, \alpha_s> = m_{1s},$$

is positive for each $s$. This is possible, since we have assumed that $\mathfrak{g}$ is a simple Lie algebra of larger rank! A calculation yields,

(13) \hspace{1cm} s_{\alpha_1}(\lambda + \rho)' = \sum_{s \in S} (n_s(\lambda) + 1 + m_{1s}(n_1(\lambda) + 1))\omega'_s + \sum_{t \in \Delta \backslash S} n_t\omega'_t.

For example, if $\alpha_1$ is a corner root, then $\mathfrak{g}'$ is simple. Let $\alpha_2$ be the root adjacent to $\alpha_1$. In this particular case, we have

(14) \hspace{1cm} \lambda'' = \lambda' + m_{12}(n_1(\lambda) + 1)\omega'_2.
We are now in a position to prove the main theorem, the proof of which is along the same lines as the proof for $GL(n)$. We assume that we have an equality as in equation (3):

$$S_{\lambda_1} \cdots S_{\lambda_n} = S_{\mu_1} \cdots S_{\mu_n}. \quad (15)$$

We now choose a corner root $\alpha_1$, and from equation (2), we obtain,

$$\prod_{i=1}^{n} \left( e^{a_1(\lambda_i + \rho)l_1} S_{\lambda_i'} - e^{a_2(\lambda_i + \rho)l_1} S_{\lambda_i''} + L(\lambda_i) \right) \bigg|_{l_1 = 1} = \prod_{i=1}^{n} \left( e^{a_1(\mu_i + \rho)l_1} S_{\mu_i'} - e^{a_2(\mu_i + \rho)l_1} S_{\mu_i''} + L(\mu_i) \right). \quad \text{On taking products and comparing the coefficients of the topmost degree, we get,}$$

$$S_{\lambda_1'} \cdots S_{\lambda_n'} = S_{\mu_1'} \cdots S_{\mu_n'}. \quad (16)$$

By induction, we can thus assume that upto a permutation we have an equality,

$$\left( \lambda_1', \ldots, \lambda_n' \right) = \left( \mu_1', \ldots, \mu_n' \right).$$

Now we compare the term contributing to the second highest degree in the product. Let $I$ (resp. $J$) consist of those indices in the set $\{1, \cdots, n\}$, for which $n_1(\lambda_i) = \lambda_i(\alpha_1^*)$ (resp. $n_1(\mu_i)$) is minimum. By Part (2) of Lemma 4 and the character expansion as given by Corollary 2, the second highest degree along $l_1$ in the product is of degree $n_1(\lambda + \rho) = n_1(\lambda) + 1$ less than the total degree. In particular the minimum of $n_1(\lambda_i)$ and the minimum of $n_1(\mu_j)$ coincide, as we vary over the indices. We have the following equality of the second highest degree terms along $l_1$:

$$\sum_{i \in I} S_{\lambda_i'} \cdots S_{\lambda_{i-1}'} S_{\lambda_i'} S_{\lambda_{i+1}'} \cdots S_{\lambda_n'} = \sum_{j \in J} S_{\mu_j'} \cdots S_{\mu_{j-1}'} S_{\mu_j'} S_{\mu_{j+1}'} \cdots S_{\mu_n'}. \quad (17)$$

From equation (14), and equation (16), we can recast this equality as,

$$\sum_{i \in I} S_{\lambda_i'} \cdots S_{\lambda_{i-1}'} S_{\lambda_i'} + d\omega S_{\lambda_{i+1}'} \cdots S_{\lambda_n'} = \sum_{j \in J} S_{\mu_j'} \cdots S_{\mu_{j-1}'} S_{\mu_j'} + d\omega S_{\mu_{j+1}'} \cdots S_{\mu_n'}. \quad \text{where } d = m_{12}(n_1(\lambda_i) + 1) \text{ is a positive integer, since } g \text{ has been assumed to be simple.}$$

Granting Lemma 5 given below, the Main Theorem follows, since we have indices $i_0, j_0$ such that,

$$\lambda_{i_0}' = \mu_{j_0}' \quad \text{and} \quad n_1(\lambda_{i_0}) = n_1(\mu_{j_0}).$$
where the indices $i_0, j_0$ are such that the minimum of $n_1(\lambda_i)$ and $n_1(\mu_j)$ is attained. Hence we have,

$$\lambda_{i_0} = \mu_{j_0}.$$ 

Cancelling these terms from equation (3), we are left with an equality where the number of components occurring in the tensor product in the hypothesis of the Main theorem, is less than the one we started with, and an induction on the number $n$ of components in the tensor product proves the main theorem.

**Remark 6.** When the number of components is at most two, then we do not need Lemma 5. In the above equality (17), we can assume that $I \cap J$ is empty, and so can take for example $I = \{1\}$ and $J = \{2\}$, to obtain,

$$S_{\lambda'_1+d\omega_p'}S_{\lambda'_2} = S_{\lambda'_1}S_{\lambda'_2+d\omega_p'}.$$ 

By induction on the rank, assuming that the Main theorem is true with number of components at most two, we obtain the Main theorem for all $\mathfrak{g}$.

To complete the proof of the theorem, we have to state the auxiliary linear independence property, generalizing Lemma 1 and Lemma 2.

**Lemma 5.** Let $\mathfrak{g}$ be a simple Lie algebra, and let $\lambda_1, \ldots, \lambda_n$ be a set of dominant, integral weights in $P_+$. Let $d$ be a positive integer and $\omega_p$ denote a fundamental weight corresponding to the root $\alpha_p$. For any index $i$, let $E(i) = \{j \mid \lambda_j = \lambda_i\}$. Suppose we have a relation

$$\sum_{1 \leq i \leq n} z_i S_{\lambda_1} \cdots S_{\lambda_{i-1}} S_{\lambda_{i+1}+d\omega_p} S_{\lambda_{i+1}} \cdots S_{\lambda_n} = 0,$$

for some collection of complex numbers $z_i$. Then for any index $i$, we have

$$\sum_{j \in E(i)} z_i = 0.$$

**Remark 7.** Instead of $\omega_p$, we can spike up the equation with any non-zero highest weight $\lambda$, but the proof is essentially the same.

The proof of this lemma will be by induction on the rank. For simple Lie algebras not of type $D$ or $E$, and if $\omega_p$ is a fundamental weight corresponding to a corner root in the Dynkin diagram of $\mathfrak{g}$, the proof follows along the same lines as in the proof of Lemma 2 and that is sufficient to prove the main theorem in these cases. For Lie algebras of type $D$ and $E$, the proof becomes complicated, due to the fact that the root adjacent to a corner root $\alpha_1$ in the Dynkin diagram of $\mathfrak{g}$, need not be a corner root in the Dynkin diagram associated to $\Delta \setminus \{\alpha_1\}$. Before
embarking on a proof of this lemma, we will need a preliminary lemma. We say that a simple Lie algebra $\mathfrak{g}$ has Property LI, if Lemma 5 holds for $\mathfrak{g}$.

**Lemma 6.** Assume that Property LI holds for all simple Lie algebras of rank at most $l$. Let $\bigoplus_{s \in S} \mathfrak{g}_s$ be a direct sum of simple Lie algebras of $\mathfrak{g}_s$ of rank at most $l$. For each $s \in S$, assume that we are given dominant, integral weights $\lambda_{s1}, \ldots, \lambda_{sn}$ of $\mathfrak{g}_s$, a positive integer $d_s$, and a fundamental weight $\omega_s$ of $\mathfrak{g}_s$. Suppose that we have a relation,

$$\sum_{1 \leq i \leq n} z_i S_{\Lambda_1} \cdots S_{\Lambda_{i-1}} S_{\Lambda_i} S_{\Lambda_{i+1}} \cdots S_{\Lambda_n} = 0,$$

for some collection of complex numbers $z_i$, where for $1 \leq i \leq n$

$$S_{\Lambda_i} = \prod_{s \in S} S_{\lambda_{si}},$$

and

$$S_{\Lambda_i} = \prod_{s \in S} S_{\lambda_{si} + d_s \omega_s}.$$

Then for any index $i$, we have

$$\sum_{j \in E(i)} z_j = 0,$$

where $E(i) = \{ j \mid (\lambda_{sj})_{s \in S} = (\lambda_{si})_{s \in S} \}$.

**Proof.** The proof proceeds by induction on the cardinality of $S$. Consider equation (18), as an equation with respect to one of the simple Lie algebras, say $\mathfrak{g}_1$. The linear independence property reduces to the case when the number of simple Lie algebras involved is one less, and we are through by induction. □

Now we get back to the proof of Lemma 5.

**Proof.** By Lemma 1, the lemma has been proved for $GL(2)$. We argue by induction on the number of components $n$ and on the rank $l$ of $\mathfrak{g}$. We assume that the lemma has been proved for all simple Lie algebras of rank less than the rank of $\mathfrak{g}$. We use the character expansion given by Corollary 2, where we denote by $l_p$ the linear functional corresponding to $\omega_p^*$. Let

$$\mathfrak{g}' \simeq \bigoplus_{s \in S} \mathfrak{g}_s,$$

be the decomposition of $\mathfrak{g}'$ into simple Lie algebras (we are removing the fundamental root $\alpha_p$ corresponding to the fundamental weight $\omega_p$ from the Dynkin diagram). For each $s \in S$, let $\alpha_s \in \Delta$ be the root adjacent to $\alpha_p$. 
Let $M$ be the subset of $\{1, \cdots, n\}$ consisting of those indices $i$ such that $n_p(\lambda_i)$ attains the minimum. For each $s \in S$, let
\[
c_s = \min\{m_{ps}(n_p(\lambda_i) + 1) \mid i \in M\}.
\]
We expand both sides using Corollary (2), and compare the coefficient of the second highest degree along $l_p$. Since $d > 0$, the term $S_{\lambda_i + d\omega}$ in the product $S_{\lambda_1} \cdots S_{\lambda_i + d\omega} \cdots S_{\lambda_n}$ does not contribute to the coefficient of the second highest degree term in $l_p$. Further we observe that $(\lambda_i + d\omega)' = \lambda'_i$. Hence the contribution to the coefficient of the second highest degree term in $l_p$ of $S_{\lambda_1} \cdots S_{\lambda_i + d\omega} \cdots S_{\lambda_n}$ is given by,
\[
\sum_{k \in M \setminus M \cap \{i\}} S_{\lambda_k} \cdots S_{\lambda_{k-1}} S_{\lambda_k'} S_{\lambda_{k+1}} \cdots S_{\lambda_n},
\]
where $\lambda''_k$ is given by equation (13) as follows:
\[
\lambda''_k + \rho' = \sum_{s \in S}(n_s(\lambda_k) + 1 + m_{ps}(n_p(\lambda_k) + 1))\omega_s' + \sum_{t \in \Delta \setminus S} n_t\omega_t'.
\]
Since $g$ is simple and $\lambda_k + \rho$ is regular, we notice that the term $m_{ps}(n_p(\lambda_k + \rho))$ is always positive. On rearranging the sum, we obtain,
\[
\sum_{i=1}^{n} z_i \sum_{k \in M \setminus M \cap \{i\}} S_{\lambda_1} \cdots S_{\lambda_{k-1}} S_{\lambda_k'} S_{\lambda_{k+1}} \cdots S_{\lambda_n} = 0.
\]
Fix an index $i_0 \in M$, and count the number of times a given index $\lambda''_{i_0}$ occurs as a component in the product. For this, any index $i \neq i_0$ will contribute. Hence the above sum can be rewritten as,
\[
(n - 1) \sum_{i \in M} z_i S_{\lambda_1} \cdots S_{\lambda'_{i_0}} \cdots S_{\lambda_n} = 0.
\]
Cancel those polynomials $S_{\lambda_l}$ for $l \notin M$. We have by Lemma 6, for any index $i_0 \in M$,
\[
\sum_{j \in E'_{i_0}} z_j = 0,
\]
where $E'_{i_0} = \{j \in M \mid \lambda_j = \lambda_{i_0}\}$. we can assume that $j \in M$, since we have cancelled those extraneous terms with indices not in $M$. But since $j$ and $i_0$ belong to $M$ and $j$ is in $E(i_0)$, we conclude that $\lambda_j = \lambda_{i_0}$. Hence $E'_{i_0} = E(i_0)$, and we have proved the lemma.
\[\square\]

 Remark 8. The main theorem indicates the presence of an ‘irreducibility property’ for the characters of irreducible representations of simple
algebraic groups. However the naive feeling that the characters of irreducible representations are irreducible is false. This can be seen easily for \( \mathfrak{sl}_2 \). For \( GL(n) \), consider a pair of highest weights of the form, 
\[
\mu = ((n-1)a, (n-2)a, \ldots, a, 0) \quad \text{and} \quad \lambda = ((n-1)b, (n-2)b, \ldots, b, 0),
\]
for some positive integers \( k, a, b \). Then the characters can be expanded as Vandermonde determinants and we have,
\[
S_\mu = \prod_{i<j} (x^{a+1}_i - x^{a+1}_j)
\]
and
\[
S_\lambda = \prod_{i<j} (x^{b+1}_i - x^{b+1}_j).
\]
Thus we see that \( S_\mu \) divides \( S_\lambda \) if \( (a+1)|(b+1) \).

It would be of interest to give necessary and sufficient criteria on the highest weights \( \mu \) and \( \lambda \) to ensure that \( S_\mu \) divides \( S_\lambda \).

6. AN ARITHMETICAL APPLICATION

We present here an arithmetical application to recovering \( l \)-adic representations. Corollary 1 was motivated by the question of knowing the relationship between two \( l \)-adic representations given that their adjoint representations are isomorphic. On the other hand, the application to generalised Asai representations given below, was suggested by the work of D. Ramakrishnan. We refer to [R] for more details.

Let \( K \) be a global field and let \( G_K \) denote the Galois group over \( K \) of an algebraic closure \( \overline{K} \) of \( K \). Let \( F \) be a non-archimedean local field of characteristic zero. Suppose
\[
\rho_i : G_K \to GL_n(F), \quad i = 1, 2
\]
are continuous, semisimple representations of the Galois group \( G_K \) into \( GL_n(F) \), unramified outside a finite set \( S \) of places containing the archimedean places of \( K \). Given \( \rho \), let \( \chi_\rho \) denote the character of \( \rho \). For each finite place \( v \) of \( K \), we choose a place \( \overline{v} \) of \( \overline{K} \) dividing \( v \), and let \( \sigma_v \in G_K \) the corresponding Frobenius element. If \( v \) is unramified, then the value \( \chi_\rho(\sigma_v) \) depends only on \( v \) and not on the choice of \( \overline{v} \), and we will denote this value by \( \chi_\rho(\sigma_v) \).

Given a \( l \)-adic representation \( \rho \) of \( G_K \), we can construct other naturally associated \( l \)-adic representations. We consider here two such constructions: the first one, is given by the adjoint representation
\[
Ad(\rho) = \rho \otimes \rho^* : G_K \to GL_{n^2}(F),
\]
where \( \rho^* \) denotes the contragredient representation of \( \rho \).
The second construction is a generalisation of Asai representations. Let $K/k$ be a Galois extension with Galois group $G(K/k)$. Given $\rho$, we can associate the \textit{pre-Asai representation},

$$As(\rho) = \otimes_{g \in G(K/k)} \rho^g,$$

where $\rho^g(\sigma) = \rho(\tilde{g} \sigma \tilde{g}^{-1})$, $\sigma \in G_K$, and where $\tilde{g} \in G_k$ is a lift of $g \in G(K/k)$. At an unramified place $v$ of $K$, which is split completely over a place $u$ of $k$, the Asai character is given by,

$$\chi_{As(\rho)}(\sigma_v) = \prod_{v | u} \chi_{\rho}(\sigma_v).$$

Hence we get that up to isomorphism, $As(\rho)$ does not depend on the choice of the lifts $\tilde{g}$. If further $As(\rho)$ is irreducible, and $K/k$ is cyclic, then $As(\rho)$ extends to a representation of $G_k$ (called the Asai representation associated to $\rho$ when $n = 2$ and $K/k$ is quadratic).

**Theorem 5.** Let

$$\rho_i : G_K \to GL_n(F), \ i = 1, 2$$

be continuous, irreducible representations of the Galois group $G_K$ into $GL_n(F)$. Let $R$ be the representation $Ad(\rho_i)$ (adjoint case) or $As(\rho_i)$ (Asai case) associated to $\rho_i$, $i = 1, 2$.

Suppose that the set of places $v$ of $K$ not in $S$, where

$$\text{Tr}(R \circ \rho_1(\sigma_v)) = \text{Tr}(R \circ \rho_2(\sigma_v)),$$

is a set of places of positive density. Assume further that the algebraic envelope of the image of $\rho_1$ and $\rho_2$ is connected and that the derived group is absolutely almost simple. Then the following holds:

1. (Adjoint case) There is a character $\chi : G_K \to F^*$ such that $\rho_2$ is isomorphic to either $\rho_1 \otimes \chi$ or to $\rho_1^* \otimes \chi$.
2. (Asai case) There is a character $\chi : G_K \to F^*$, and an element $g \in G(K/k)$ such that $\rho_2$ is isomorphic to $\rho_1^g \otimes \chi$.

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