AUTOMORPHISM SUPERGROUPS OF SUPERMANIFOLDS

DOMINIK OSTERMAYR

ABSTRACT. A classical theorem states that the group of automorphisms of a manifold $M$ preserving a $G$-structure of finite type is a Lie group. We generalize this statement to the category of cs manifolds and give some examples, some of which being generalizations of classical notions, others being particular to the super case. Notably, we have to introduce a new notion of supermanifolds which we call mixed supermanifolds.

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1. INTRODUCTION

In this article, we study geometric structures on cs manifolds and their automorphisms. Super-Riemannian structures on cs manifolds play a prominent role in the work of Zirnbauer \cite{14}. In particular, the so-called Riemannian symmetric superspaces are worth mentioning. Other instances of geometric structures on supermanifolds appeared in the context of supergravity theories \cite{13}.

By a geometric structure on a manifold $M$ we mean a reduction of the structure group of the frame bundle $L(M)$ to some closed subgroup $G \leq GL(V)$. Depending on the context, there might be additional conditions like 1-flatness. A classical theorem states that the group of automorphisms of such a $G$-structure is a Lie group provided it is of finite type. (See \cite{12} and the references therein.) This includes for instance the isometry group of a Riemannian manifold.

In this work, we study the analogous structures in the category of cs manifolds (cf. \cite{8}). First, we lay the necessary foundations for the definition of a $G$-structure. This leads naturally to the notion of mixed supermanifolds as follows. The frame bundle of an ordinary manifold locally modelled on the vector space $V$ is obtained from a cocycle $U_{ij} \to GL(V)$ by glueing. Suppose $M$ is a cs manifold (called supermanifolds in this article) which is locally modelled on the super vector space $V_{\mathbb{R}} \oplus V_{\mathbb{C}}$. Here, $V_{\mathbb{R}}$ is a real and $V_{\mathbb{C}}$ is a complex vector space. Then the analogous cocycle takes values in the mixed Lie supergroup $GL(V)$ which has as body the mixed
manifold $GL(V_0) \times GL(V_1)$. It is crucial to keep the complex analytic structure on the second factor. After having developed the basic theory of mixed supermanifolds, one can define $G$-structures, prolongations and $G$-structures of finite type along the lines of the classical definitions. Our main result concerns the functor of automorphisms of a $G$-structure of finite type that is in addition admissible. In this situation, if restricted to purely even supermanifolds, the functor is representable by a mixed Lie group and it is finite dimensional in the sense that the higher points are determined by the Lie superalgebra of infinitesimal automorphisms of the $G$-structure, which we prove to be finite dimensional. Representability can fail for two reasons here, due to the fact that the higher points of the functor of automorphisms contain all infinitesimal automorphisms of the $G$-structure. For a representable functor these are necessarily all complete and decomposable, which means that they admit a decomposition of the form $X + iY$ for two real complete vector fields. The theory of $G$-structures can be developed for real supermanifolds without need for enlarging the category. Moreover, there is no need for imposing an additional property on a $G$-structure of finite type. The only obstruction for the representability of the functor of automorphisms of finite type is the completeness of the infinitesimal automorphisms.

The paper is organized as follows. In Section 2 we introduce mixed supermanifolds. After giving the basic definitions, we give a short account on mixed Lie supergroups and principal bundles. We then show that mixed supermanifolds are the natural home for constructions such as tangent bundles and frame bundles as well as their mixed forms, the real tangent bundles and real frame bundles. In contrast to what the name suggests, mixed supermanifolds are not supermanifolds with extra structure as we show in Proposition 7.1. Moreover, we prove that, for our purposes, mixed supermanifolds cannot be avoided (Proposition 7.2).

In Section 3 we define a geometric structure to be a reduction of the real frame bundle of a mixed supermanifold and construct its prolongation. In the super context it is advisable to make the constructions in such a way that functoriality is evident. A subtlety is that the standard prolongation has to be refined to a real prolongation, which is again a geometric structure in the sense of our definition. The existence is ensured if the $G$-structure is admissible.

In Section 4 we define the functor of automorphisms of a $G$-structure. Due to functoriality, prolongation gives rise to inclusions of functors of automorphisms. Then we treat the case of a $\{1\}$-structure. We show that the underlying functor is representable and the Lie superalgebra of infinitesimal automorphisms is finite dimensional. An important ingredient is that even real vector fields possess a flow as we show in Section 7.2. Similar results on the functor of automorphisms of an admissible $G$-structure of finite type can then be deduced by embedding it into the functor of automorphisms of a $\{1\}$-structure.

Everything we have said has a direct analogue in the category of real supermanifolds, except that there are no complications caused by mixed structures and admissibility. The completeness issues remain. The analogous theorems are stated in Section 5.

Finally, in Section 6 we discuss some examples. We treat even and odd metric structures on supermanifolds and construct a canonical admissible geometric structure of finite type associated to the superization of a Riemannian spin manifold as studied in [11, 13].
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2. Recollections on supergeometry

2.1. Mixed supermanifolds. A complex super vector space is a \( \mathbb{Z}/2 \)-graded complex vector space \( V = V_0 \oplus V_1 \). A morphism is a grading preserving complex linear homomorphism. The resulting category is closed symmetric monoidal with respect to the evident notion of tensor product and inner hom objects.

A general mixed super vector space consists of the data \((V, V_R, V_C)\) where \( V \) is a complex super vector space, \( V_R \subseteq V \) is a real sub super vector space, \( V_C \subseteq V \) is a complex sub super vector such that \( V_C \subseteq V_R \) and the canonical map \( \mathbb{C} \otimes V_R/V_C \to V/V_C \) is an isomorphism. A mixed super vector space is a general mixed super vector space \((V, V_R, V_C)\) such that \((V_R)_1 = (V_C)_1 = V_1\). The class of these contains the classes of super vector spaces and complex super vector spaces as the extreme cases where \( V_C = V_1 \) and \( V_C = V_0 \), respectively. A real super vector space is a general mixed super vector space of the form \((V, V_R, 0)\). For our purposes it is not necessary to discuss the various notions of morphisms of general mixed super vector spaces at this point.

Example 2.1. One way to produce (general) mixed super vector spaces is the following. Suppose given a real sub super vector space \( V_R \) of a complex super vector space \( W \). The kernel of the induced map \( f : \mathbb{C} \otimes V_R \to W \) is of the form \( V_C = \{ i \otimes v - 1 \otimes iv \mid v \in V_C \} \cong T_C \) for a complex subspace \( V_C \subseteq W \) contained in \( V_R \). Then \((V = \text{im}(f), V_R, V_C)\) is a general mixed super vector space. Of course, conversely, given a general mixed super vector space \((V, V_R, V_C)\), \( V_C \) can be recovered from this by applying this procedure to \( V_R \to V \).

In particular, the pair \((V, V_R)\) determines \( V_C \) and the pair \((V_R, V_C)\) determines \( V \) up to isomorphism.

This leads to various notions of supermanifolds. We will first introduce the relevant notions at the level of manifolds (without grading). Consider a (purely even) mixed vector space \( V_C \subseteq V_R \subseteq V \). We denote by \( \mathcal{A}(V_R) \) the locally ringed space over \( \mathbb{C} \) given by the topological space \( V_R \) together with the sheaf \( \mathcal{O}_{V_R} \) of partially holomorphic functions, i.e. complex valued smooth functions whose differential is complex linear in the fibre \( \mathcal{A}(V_R) \times V_C \subseteq \mathcal{A}(V_R) \times V_R = TA(V_R) \).

Remark 2.2. More concretely, if we choose an isomorphism \( V \cong \mathbb{C}^n \times \mathbb{C}^m \) such that \( V_R \cong \mathbb{R}^n \times \mathbb{C}^m \) and \( V_C \cong \mathbb{C}^m \), then these are complex smooth functions \( \psi(x, z) \) on open subspaces of \( \mathbb{R}^n \times \mathbb{C}^m \) which are holomorphic in \( z \).

Definition 2.3. A mixed manifold consists of a locally ringed space \((M_0, \mathcal{O}_{M_0})\) over \( \mathbb{C} \) with second countable Hausdorff base which is locally isomorphic to \( \mathcal{A}(V_R) \) for some mixed vector space \((V, V_R, V_C)\). The subsheaf of real-valued functions is denoted by \( \mathcal{O}_{M_0, \mathbb{R}} \). The full subcategory of locally ringed spaces over \( \mathbb{C} \) with objects mixed manifolds is denoted by \( M^\mathbb{R} \).

Remark 2.4. These are precisely the smooth manifolds locally of the form \( \mathbb{R}^n \times \mathbb{C}^m \) with transition functions \( (x, z) \mapsto (\phi(x), \psi(x, z)) \), where \( \psi(x, z) \) is holomorphic in \( z \). Put differently, these are manifolds endowed with a Levi flat CR-structure (cf. [4]).
Consider now a mixed super vector space \((V, V_R, V_C)\). We denote by \(A(V_R)\) the locally ringed superspace over \(\mathbb{C}\) given by the topological space \(V_R\) together with the structure sheaf \(\mathcal{O}_{A(V_R)} \otimes_{\mathbb{C}} V^*_R\). Given a mixed super vector space \((V, V_R, V_C)\), we can forget the mixed structure and consider the mixed super vector space \((V, V, V_C)\). The associated locally ringed space will be denoted by \(A(V)\).

**Definition 2.5.** A mixed supermanifold consists of a locally ringed superspace \(M = (M_0, \mathcal{O}_M)\) over \(\mathbb{C}\) with second countable Hausdorff base which is locally isomorphic to \(A(V_R)\) for some mixed super vector space \((V, V_R, V_C)\). The full subcategory of locally ringed superspaces over \(\mathbb{C}\) with objects mixed supermanifolds is denoted by \(\mathbf{SM}^\mu\). The category \(\mathbf{SM}^\mu\) contains the full subcategories \(\mathbf{SM}\) and \(\mathbf{SM}^C\) of supermanifolds and complex supermanifolds as the extreme cases where \(V_C = V^*_1\) and \(V_C = V\), respectively.

The sheaf of nilpotent functions on a mixed (real) supermanifold \(M\) will be denoted by \(\mathcal{I}_M\). The mixed (real) supermanifold structure on \(M\) induces the structure of a mixed (real) manifold on the locally ringed space \((M_0, \mathcal{O}_M/\mathcal{I}_M)\) which we abbreviate by abuse of notation by \(M_0\). Moreover, we set \(\mathcal{O}_M := \mathcal{O}_M/\mathcal{I}_M\). Then the inclusion \(i_\bullet: \mathcal{M}^\mu \to \mathbf{SM}^\mu\) has the right adjoint \(r: \mathbf{SM}^\mu \to \mathcal{M}^\mu, M \mapsto M_0\).

Given a mixed supermanifold, we define the sheaf of real functions to be the pullback in the square of (real) supercommutative superalgebras

\[
\begin{array}{ccc}
\mathcal{O}_{M,R} & \longrightarrow & \mathcal{O}_{M_0,R} \\
\downarrow & & \downarrow \\
\mathcal{O}_M & \longrightarrow & \mathcal{O}_{M_0}.
\end{array}
\]

We will often consider a mixed supermanifold as a set-valued functor on \(\mathbf{SM}^\mu\) by the assignment \(T \mapsto \mathbf{SM}^\mu(T, M)\). Then there is a natural transformation of functors \(M \mapsto r^*i^*M = r^*M_0\) which is given by sending a map \(T \to M\) to its underlying map \(T_0 \to M_0\). The second part of the next lemma is only the first encounter of the typical reality condition enforced by a mixed structure.

**Lemma 2.6.** Consider a mixed super vector space \((V, V_R, V_C)\).

(a) There is a natural isomorphism \(\mathbf{SM}^\mu(M, A(V)) = \Gamma(\mathcal{O}_M \otimes_{\mathbb{C}} V_C)_0\).

(b) The following diagram is a pullback of functors on \(\mathbf{SM}^\mu\):

\[
\begin{array}{ccc}
A(V_R) & \longrightarrow & r^*A(V_R) \\
\downarrow & & \downarrow \\
A(V) & \longrightarrow & r^*A(V_0).
\end{array}
\]

(c) In other words, we have

\[
\mathbf{SM}^\mu(M, A(V_R)) \cong \Gamma(\mathcal{O}_{M,R,0} \otimes_{\mathbb{R}} (V_R/V_C)_0) \oplus \Gamma(\mathcal{O}_{M,0} \otimes_{\mathbb{C}} (V_C)_0) = \Gamma(\mathcal{O}_{M,1} \otimes_{\mathbb{C}} V_1) + \Gamma(\mathcal{O}_{M,0} \otimes_{\mathbb{C}} (V_C)_0).
\]

**Proof.** The proof is similar as in [7, Theorem 4.1.11].

**Corollary 2.7.** The category \(\mathbf{SM}^\mu\) admits all finite products and the full subcategory \(\mathcal{M}^\mu\) is closed under finite products in \(\mathbf{SM}^\mu\).

Let \(M_0\) be a mixed manifold. Consider the sheaf \(\mathcal{T}_{M_0}\) whose sections over \(U_0\) are complex linear derivations of \(\mathcal{O}_{M_0}|_{U_0}\) and the subsheaf \(\mathcal{T}_{M_0,R}\) of those derivations...
which restrict to derivations of $\mathcal{O}_{M,\mathbb{R}}|U_0$. Then $\mathcal{T}_{M,\mathbb{R}}$ contains a complex ideal $\mathcal{T}_{M,\mathbb{C}}$ of derivations which annihilate $\mathcal{O}_{M,\mathbb{R}}|U_0$. The quotient by this sheaf is (non-canonically) isomorphic to the sheaf of derivations of $\mathcal{O}_{M,\mathbb{R}}$.

Now, if $M$ is a mixed supermanifold, the complex tangent sheaf is the sheaf $\mathcal{T}_M$ whose sections over $U_0$ are the complex linear superderivations of $\mathcal{O}_{M}|U_0$. In analogy with the definition of the real functions, one defines the real tangent sheaf by the pullback

$$
\begin{array}{ccc}
\mathcal{T}_{M,\mathbb{R}} & \to & \mathcal{T}_{M,\mathbb{R}} \\
\downarrow & & \downarrow \\
\mathcal{T}_M & \to & \mathcal{T}_{M_\mathbb{R}}
\end{array}
$$

where the lower arrow takes a vector field to its underlying vector field.

An important point is that, although $\mathcal{T}_{M,\mathbb{R}}$ is not closed under brackets, its even part is and consists of those derivations which restrict to derivations of $\mathcal{O}_{M,\mathbb{R}}$. In analogy, one defines $\mathcal{T}_{M,\mathbb{C}} \subseteq \mathcal{T}_{M,\mathbb{R}}$ in terms of $\mathcal{T}_M$, $\mathcal{T}_{M_\mathbb{R}}$ and $\mathcal{T}_{M,\mathbb{C}}$. Then $(\mathcal{T}_{M,\mathbb{C}})_0 \subseteq (\mathcal{T}_{M,\mathbb{R}})_0$ is an ideal.

The tangent space $T_mM$ at $m \in M_0$ is the complex super vector space of complex derivations $\mathcal{O}_{M,m} \to \mathbb{C}$. This comes with a mixed structure by considering the real subspace $(T_mM)_\mathbb{R}$ consisting of those derivations which induce a derivation $\mathcal{O}_{M_\mathbb{R},m} \to \mathbb{R}$ together with its complex subspace $(T_mM)_\mathbb{C}$ of those derivations in $(T_mM)_\mathbb{R}$ which vanish on $\mathcal{O}_{M_\mathbb{R},m}$.

2.2. Mixed Lie supergroups. In this section we give a brief review on basic results concerning mixed Lie supergroups.

2.2.1. Equivalence of mixed Lie supergroups and mixed super pairs.

**Definition 2.8.** A mixed Lie supergroup is a group object in $\text{SM}^\mu$.

First we characterize mixed Lie groups, i.e. mixed Lie supergroups with trivial odd direction. For a real (resp. mixed) Lie group $G$ we will use the notation $\text{Lie}_{\mathbb{R}}(G)$ (resp. $\text{Lie}_{\mathbb{C}}(G)$) for the Lie algebra of left-invariant derivations of the sheaf of real valued smooth functions (resp. sheaf of complex functions).

We define a mixed pair to be a pair $(g_{\mathbb{C}}, G_\mathbb{sm})$ consisting of a real Lie group $G_\mathbb{sm}$ and an $\text{Ad}_{G_\mathbb{sm}}$-invariant ideal $g_{\mathbb{C}} \subseteq \text{Lie}_{\mathbb{R}}(G_\mathbb{sm})$ endowed with a complex structure which is respected by the adjoint action of $G_\mathbb{sm}$.

A morphism of such pairs is a morphism of Lie groups such that the differential at the identity respects the complex ideals.

**Lemma 2.9.** The categories of mixed Lie groups and mixed pairs are equivalent.

**Proof.** This follows from the Baker–Campbell–Hausdorff formula as in the case of complex analytic structures on Lie groups. \qed

As usual, the adjoint representation of a mixed Lie group $G$ is the differential at the identity of the conjugation action of $G$ on itself. It can be seen as a mixed morphism $G \times \mathbb{A}(g_{\mathbb{R}}) \to \mathbb{A}(g_{\mathbb{R}})$.

Now, we turn our attention to mixed Lie supergroups. A mixed super pair consists of a pair $(\mathfrak{g}, G_0)$ where $G_0$ is a mixed Lie group and $\mathfrak{g}$ is complex Lie superalgebra together with

(a) an isomorphism $\text{Lie}_{\mathbb{C}}(G_0) \cong \mathfrak{g}_0$, and
(b) an action $\sigma: G_0 \times \mathbb{A}(\mathfrak{g}) \to \mathbb{A}(\mathfrak{g})$ such that $\sigma(g)|_{\mathbb{A}(\mathfrak{g})} = \text{Ad}_{G_0}$ and the differential of $\sigma$ acts as the adjoint representation

$$d\sigma(X)(Y) = [X,Y].$$

There is an evident notion of a morphism of mixed super pairs and the following result follows along the same lines as the corresponding for real and complex Lie supergroups.

**Proposition 2.10.** The categories of mixed super pairs and mixed Lie supergroups are equivalent.

*Proof.* See [7, 7.4]. □

An important notion is the following.

**Definition 2.11.** A mixed real form of a complex Lie supergroup $G$ is a mixed Lie supergroup $G_{\mathbb{R}}$ together with a group morphism $i: G_{\mathbb{R}} \to G$ such that $i_0: (G_{\mathbb{R}})_0 \to G_0$ is the inclusion of a closed subgroup and $d_i_\ast: T_e(G_{\mathbb{R}}) \to T_e(G)$ is an isomorphism.

**Remark 2.12.** Any mixed real form $G_{\mathbb{R}} \leq G$ yields a mixed real form $(G_{\mathbb{R}})_0 \leq G_0$. Conversely, given a mixed real form $(G_0)_\mathbb{R} \leq G_0$, the pullback

$$G_{\mathbb{R}} \longrightarrow r^*(G_0)_\mathbb{R}$$

$$\downarrow$$

$$G \longrightarrow r^*G_0$$

is representable and defines a mixed real form of $G$. For that reason, we will adopt the notation $(G_{\mathbb{R}})_0 = (G_0)_\mathbb{R} = G_{0,\mathbb{R}}$.

**Example 2.13.** Finally, we come to discuss the example of linear supergroups. Let $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$ be a mixed super vector space. Then we have the complex Lie supergroup $GL(V)$ given by the complex group $GL(V_0) \times GL(V_1)$ and the Lie superalgebra $\mathfrak{gl}(V)$. An element of $GL(V)(T)$ is given by an automorphism over $T$ of the trivial vector bundle $V = T \times \mathbb{A}(V) \to T$.

Consider the subgroups of those even invertible isomorphisms of $V$ respecting $V_\mathbb{C}$ or the pair $V_\mathbb{C} \subseteq V_{\mathbb{R}}$. We will denote them by

$$GL^\mu(V)_0, \mathbb{R} \leq GL^\mu(V)_0 \leq GL(V)_0.$$

We then define the two group-valued functors $GL^\mu(V)$ and $GL^\mu(V)_\mathbb{R}$ on $\mathbf{SM}^\mu$ by the pullback

$$GL^\mu(V)_\mathbb{R} \longrightarrow r^*GL^\mu(V)_0,\mathbb{R}$$

$$\downarrow$$

$$GL(V) \longrightarrow r^*GL(V)_0,$$

where it is understood that the quantities in parentheses are only present in the latter case.

The inclusion $\text{Lie}_{\mathbb{R}}(GL^\mu(V)_0,\mathbb{R}) \subseteq \mathfrak{gl}(V)_0$ only defines a mixed structure in the cases $V_{\mathbb{C}} = V_1$ and $V_{\mathbb{C}} = V$. In this case $GL^\mu(V)_\mathbb{R}$ is representable and is a mixed real form of $GL(V)$. In general, $GL^\mu(V)_\mathbb{R}$ is not representable.
2.2.2. Actions of mixed Lie supergroups and their point functors. A left action of the mixed Lie supergroup $G$ on the mixed supermanifold $M$ is given by a unital and associative map $a : G \times M \to M$. The map $a^t$ can be made explicit in terms of two more basic objects. First, let $a$ denote the action $G_0 \times M \to G \times M \to M$. Then any $g \in G_0$ (considered as a map $g : \mathcal{A}(\{0\}) \to G$) gives a map

$$a_g : M \cong \mathcal{A}(\{0\}) \times M \xrightarrow{\cong \times M} G_0 \times M \xrightarrow{\cong} M.$$  

Secondly, the action gives rise to a Lie superalgebra antimorphism

$$\rho : g \longrightarrow \Gamma(\mathcal{I}_M), \quad X \mapsto (e \times M)^t \circ (X \otimes 1) \circ a^t$$  

(2.14)

and we have

(a) $\rho_{\phi_0}(X) = (X \otimes 1) \circ a^t$, and

(b) $\rho(g \cdot Y) = (a_g^{-1})^t \circ \rho(Y) \cdot a_g^t$.

Conversely, given an action $g : G_0 \times M \to M$ and $\rho$ satisfying (a) and (b), then one can construct an action $G \times M \to M$ (cf. [7, Propositions 8.3.3, 8.3.2]).

Now let $G$ be a mixed Lie group and $M$ a mixed supermanifold and consider an action $a^{sm} : G^{sm} \times M \to M$. This gives rise to a Lie algebra morphism $g_{\mathbb{R}} \to \Gamma(\mathcal{I}_{M,\mathbb{R}})_0$. The connection between such an action and an action of $G$ is made precise in the next lemma.

**Lemma 2.15.** The action $a^{sm}$ extends to an action $a : G \times M \to M$ if and only if $g$ fits into the following square

$$\begin{array}{ccc}
g_{\mathbb{R}} & \longrightarrow & \Gamma(\mathcal{I}_{M,\mathbb{R}})_0 \\
\downarrow & & \downarrow \\
g & \longrightarrow & \Gamma(\mathcal{I}_M)_{\bar{0}}.
\end{array}$$

the lower horizontal arrow being an antimorphism of complex Lie algebras. The extension is unique if it exists. Equivalently, the restriction of the upper horizontal arrow to $g_{\mathbb{C}}$ factors as a complex linear map through $\Gamma(\mathcal{I}_{M,\mathbb{C}})_{\bar{0}}$.

**Proof.** Uniqueness is clear since any element $X \in g$ can be written in the form $X_1 + iX_2$ for some $X_j \in g_{\mathbb{R}}$. If the extension in the diagram exists, then the differential $TG_{\mathbb{R}}^{sm} \times TM \to TM$ is complex linear on $TG_{\mathbb{C}}^{sm} \times TM \to TM$, which proves that the action extends to $G \times M$. \hfill $\Box$

Let $T$ be an arbitrary mixed supermanifold. Consider a morphism $\varphi_0 : T \to G$ and a homogeneous derivation $X : \mathcal{O}_T \to (e_{T_0})_* \mathcal{O}_T$ along $e_T : T \to * \to G$.

Given this, we construct a homogeneous derivation along $\varphi_0$ as follows:

$$\varphi_0 \cdot X : \mathcal{O}_G \xrightarrow{(\varphi_0 \times T)^t \circ (1 \otimes X) \circ a^t} (\mu_0)_* (\varphi_0 \times e_T)_{0*, \mathcal{O}_{T \times T}} \xrightarrow{\Delta^t} (\varphi_0)_* \mathcal{O}_T.$$  

Similarly, for two homogeneous derivations $X$ and $Y$ we set

$$X \cdot Y := \Delta^t \circ (\mu_0)_* ((X \otimes 1) \circ (1 \otimes Y)) \circ \mu^t.$$  

Now, suppose $G$ acts on $M$ and let $X$ and $Y$ be as above. We set

$$\rho(X) : \mathcal{O}_{T \times M} \xrightarrow{(1 \otimes X \otimes 1) \circ (\Delta \times M)^t} \mathcal{O}_{T \times T \times M} \xrightarrow{(\Delta \times M)^t} \mathcal{O}_{T \times M}.$$
Then $\rho(X)$ is the $\Theta_T$-linearization of $\rho(X) \circ p_T^*$, where $p_T: T \times M \to M$ is the projection. From the associativity of the action it follows that 

$$\rho(X \cdot Y) = (-1)^{|X||Y|} \rho(Y) \circ \rho(X).$$

Let $n \geq 0$, then $\Gamma(\mathcal{A}(\mathbb{C}^0[n]))$ is the exterior algebra on generators $\eta_i$. As usual, given a non-empty subset $I \subset \{1, \ldots, n\}$, we set $\eta' = \prod_{i \in I} \eta_i$, where we implicitly use the ordering on $I$ induced from the standard ordering on $\{1, \ldots, n\}$.

**Lemma 2.16.** Suppose $G$ is mixed and acts on the mixed supermanifold $M$.

(a) Any $\varphi \in G(\mathcal{A}(\mathbb{C}^0[n]) \times T)$ is uniquely determined by $\varphi_0 \in G(T)$ and homogeneous derivations $X_I$ along $e_T$ of degree $|I|$ and

$$\varphi^\sharp = \varphi_0^\sharp \cdot \prod_{k=1}^n \left( 1 + \sum_{k \in I \subseteq \{1, \ldots, k\}} \eta' X_I \right).$$

(b) Moreover, under this identification, the morphism $a_\varphi$, defined as the composition

$$(\mathcal{A}(\mathbb{C}^0[n]) \times T \times a) \circ ((\text{pr}_{\mathcal{A}(\mathbb{C}^0[n]) \times T}) \times \varphi) : \mathcal{A}(\mathbb{C}^0[n]) \times T \times M \longrightarrow \mathcal{A}(\mathbb{C}^0[n]) \times T \times M,$$

takes the form

$$a_\varphi^\sharp = \prod_{k=n}^1 \left( 1 + \sum_{k \in I \subseteq \{1, \ldots, k\}} \eta' \rho(X_I) \right) \cdot a_{\varphi_0}^\sharp.$$ 

**Proof.** The first part is proved by induction on $n$ and the second part then boils down to $(\mu \times M)^\sharp \circ a^\sharp = (G \times a)^\sharp \circ a^\sharp$. □

### 2.3. Mixed real forms of principal $G$-bundles.

Suppose given a mixed supermanifold $M$ and a group-valued functor $G$ on $\text{SM}^\mu$. A principal $G$-bundle is a functor $P$ on $\text{SM}^\mu$ together with a right $G$-action and a map $\pi: P \to M$ equivariant with respect to the trivial action on $M$ such that for each $m \in M_0$ there exist an open neighbourhood $U$ and equivariant isomorphisms $U \times G \to P|_U$ over $U$. This reduces to the usual definition if $G$ is representable.

Later we will need to build real forms of certain principal bundles. This will be done so with the help of the following lemma.

**Lemma 2.17.** Let $G$ be a complex Lie supergroup with mixed real form $G_{\mathbb{R}}$. Let $P \to M$ be a principal $G$-bundle over a mixed supermanifold $M$ and $P_{0,\mathbb{R}} \to P_0$ a reduction of $P_0$ to $G_{0,\mathbb{R}}$. Then the pullback

$$\begin{array}{ccc} P_{\mathbb{R}} & \longrightarrow & r^*(P_{0,\mathbb{R}}) \\
\downarrow & & \downarrow \\
P & \longrightarrow & r^*(P_0). \end{array}$$

is a principal $G_{\mathbb{R}}$-bundle.

**Proof.** We observe that $G_{\mathbb{R}}$ acts on $P_{\mathbb{R}}$ by the universal property of the pullback and the map $P_{\mathbb{R}} \to P \to M$ is equivariant with respect to this action. So we only need to show local triviality. We choose trivializations $\psi_i: U_i \times G \to P|_{U_i}$ on coordinate charts $U_i = \mathcal{A}(V_{\mathbb{R}})$ on $M$. They come with retractions $r_i: U_i \to (U_i)_0$. Without loss of generality, we may assume that $P_{0,\mathbb{R}}|_{(U_i)_0}$ is trivial, too, say by maps
\( \varphi_i : (U_i)_0 \times G_{\mathbb{R}} \rightarrow P_{0,\mathbb{R}|(U_i)_0} \). The \( \varphi_i \) induce trivializations \( \tilde{\varphi}_i : (U_i)_0 \times G_0 \rightarrow P_0|_{(U_i)_0} \) which differ from \( (\psi_i)_0 \) by maps \( g_i : (U_i)_0 \rightarrow G_0 \) in the sense that

\[
\tilde{\varphi}_i = (\psi_i)_0 \circ ((\id_{(U_i)_0})_0 \times g_i) \circ ((\id_{U_i})_0 \times G_0) : (U_i)_0 \times G_0 \rightarrow P_0|_{(U_i)_0}.
\]

Denoting by \( \tilde{\psi}_i \) the composition \( G_0 \times G \rightarrow G \times G \rightarrow G \), we now set

\[
\tilde{\psi}_i = \psi_i \circ (U_i \times \tilde{a}) \circ (U_i \times g_i \times G) \circ ((\id_{U_i})_0 \times G_0) : U_i \times G \rightarrow P|_{U_i},
\]

which is still a trivialization. Then \( (\tilde{\psi}_i)_0 = \tilde{\varphi}_i \), and the universal property of the pullback now shows that \( \tilde{\psi}_i \), restricted to \( U_i \times G_{\mathbb{R}} \), gives a trivialization of \( P_{\mathbb{R}|U_i} \). \( \square \)

### 2.4. Tangent bundles and frame bundles of mixed supermanifolds.

Suppose \( M \) is a mixed supermanifold locally modelled on the mixed super vector space \( (V, V_{\mathbb{R}}, V_{\mathbb{C}}) \). The sheaf \( \mathcal{J}_M \) is locally free on \( V \) and glueing leads to the mixed total space \( TM \rightarrow M \). If \( i : M_0 \rightarrow M \) is the canonical inclusion, then

\[
i^*TM = TM_0 \oplus TM_1
\]

for certain complex bundles \( (TM)_j \rightarrow M_0 \) (in the category of mixed manifolds). Actually, we have \( (TM)_0 = TM_0 \).

Define \( V_T = T \times \mathbb{A}(V) \rightarrow T \) to be the trivial vector bundle over \( T \) with fibre \( \mathbb{A}(V) \). There is a vector bundle of homomorphisms \( \text{Hom}(\mathcal{J}_M, TM) \rightarrow M \) and the \( T \)-points of the total space are given by squares of vector bundles

\[
\begin{array}{ccc}
V_T & \xrightarrow{\varphi} & TM \\
\downarrow & & \downarrow \\
T & \xrightarrow{f} & M.
\end{array}
\]

Equivalently, a \( T \)-point consists of a tuple \( (f, \varphi) \) consisting of a map \( f : T \rightarrow M \) and a map \( \varphi : V_T \rightarrow f^*(TM) \) of vector bundles over \( T \).

Another point of view is the following. Let \( \mathcal{J}_V^k \) be the \( (k+1) \)-truncation of the free supersymmetric algebra on \( V \):

\[
\mathcal{J}_V^k := \text{Sym}(V)/\langle V \rangle^{k+1}.
\]

Define \( \text{Spec}(\mathcal{J}_V^k) \) to be the complex super.space with reduced space a point and \( \mathcal{J}_V^k \) as algebra of functions. We set \( \mathcal{J}_V^k M := \text{Hom}(\text{Spec}(\mathcal{J}_V^k), M) \), i.e.

\[
\text{SM}^k(T, \mathcal{J}_V^k M) = \text{LRS}_C(T \times \text{Spec}(\mathcal{J}_V^k), M),
\]

where \( \text{LRS}_C \) denotes the category of locally ringed superspaces over \( C \), compare [2, 5.4]. The inclusion \( 0 \rightarrow V \) induces a map \( \text{Spec}(\mathcal{J}_V^k) \rightarrow * \) which in turn induces \( \iota_M : \mathcal{J}_V^k M \rightarrow M \) by precomposition. By inspection we have \( \mathcal{J}_V^k M = \text{Hom}(\underline{\mathcal{J}_M}^k, TM) \) over \( M \).

The frame bundle of \( M \) is the open subsuperspace of \( \text{Hom}(V_M, TM) \) characterized by

\[
L(M)(T) = \{ (f, \varphi) \in \text{Hom}(V_M, TM)(T) \mid \varphi \text{ isomorphism} \}.
\]

In terms of squares: \( (f, \varphi) \in L(M)(T) \) if and only if the associated square \( (2.18) \) is a pullback. This is a principal \( GL(V) \)-bundle over \( M \).

We have \( L(M)_0 = L(TM_0) \times_M L(TM_1) \), and thus the mixed structure of \( M \) yields subbundles

\[
L^\mu(M)_{0,\mathbb{R}} \rightarrow L^\mu(M)_{0} \rightarrow L(M)_0,
\]

\[
\text{Spec}(\mathcal{J}_V^k) \text{ is a principal}\ GL(V) \text{-bundle over } C,
\]

\[
\text{SM}^k(T, \mathcal{J}_V^k M) = \text{LRS}_C(T \times \text{Spec}(\mathcal{J}_V^k), M),
\]

\[
\text{the frame bundle of } M \text{ is the open subsuperspace of } \text{Hom}(V_M, TM) \text{ characterized by}
\]

\[
L(M)(T) = \{ (f, \varphi) \in \text{Hom}(V_M, TM)(T) \mid \varphi \text{ isomorphism} \}.
\]

In terms of squares: \( (f, \varphi) \in L(M)(T) \) if and only if the associated square \( (2.18) \) is a pullback. This is a principal \( GL(V) \)-bundle over \( M \).

We have \( L(M)_0 = L(TM_0) \times_M L(TM_1) \), and thus the mixed structure of \( M \) yields subbundles

\[
L^\mu(M)_{0,\mathbb{R}} \rightarrow L^\mu(M)_{0} \rightarrow L(M)_0,
\]
where $L^\mu(M)_0$ (resp. $L^\mu(M)_{0,\mathbb{R}}$) is the subbundle of those frames which map $V_C$ to $TM_C$ (resp. $(V_R, V_R)$ to $(TM_R, TM_C)$). By pulling back, we obtain the bundles

$$
\begin{array}{ccc}
L^\mu(M)_{(\mathbb{R})} & \xrightarrow{r^*} & r^*L^\mu(M)_{0,(\mathbb{R})} \\
\downarrow & & \downarrow \\
L(M) & \xrightarrow{r^*} & r^*L(M)_0.
\end{array}
$$

The structure group of $L^\mu(M)_{(\mathbb{R})}$ is precisely $GL^\mu(V)_{(\mathbb{R})}$, and this functor of frames is representable precisely for supermanifolds and complex supermanifolds, that is, in terms of local models $V_C \in \{V_1, V\}$.

All these principal bundles have associated bundles that fit in a square

$$
\begin{array}{ccc}
TM_R & \xrightarrow{r^*} & r^*T(M)_{0}\mathbb{R} \\
\downarrow & & \downarrow \\
TM & \xrightarrow{r^*} & r^*T(M)_0,
\end{array}
$$

which is a pullback in view of the pullback square defining $T_M, T_{M_0}$ in terms of $T_M, T_{M_0}$ and $T_{M_0,\mathbb{R}}$.

3. Geometric structures on mixed supermanifolds

We can now define the notion of a geometric structure on a mixed supermanifold. Let $G \leq GL(V)$ be a closed mixed Lie subgroup, i.e. $G^m_0 \leq GL(V)^m_0$ is closed and $G_0 \leq GL(V)_0$ is a mixed embedding.

3.1. Basic definitions.

**Definition 3.1.** A $G$-structure on $M$ is a reduction $P$ of $L^\mu(M)_{\mathbb{R}}$ to $G$. Equivalently, it is a reduction $P$ of $L(M)$ such that $P_0 \to L(M)_0$ factors through $L^\mu(M)_{0,\mathbb{R}}$.

Any $G$-structure $P$ comes with a canonical 1-form $\vartheta: TP \to \mathfrak{v}_p$. It sends a pair $(f, X) \in TP(T)$, considered as the data of a map $f = (\pi \circ f, \varphi): T \to P$ and a section $X$ of $f^*(TP)$, to the composite

$$
\begin{align*}
T & \xrightarrow{X} f^*(TP) \\
\pi^* & \xrightarrow{f^*(d\pi)} (\pi \circ f)^*(TM) \\
\vartheta & \xrightarrow{f^*\vartheta} f^*V_p \\
\vartheta & \xrightarrow{f^*\vartheta} \mathfrak{v}_p.
\end{align*}
$$

The differential of the canonical 1-form $\vartheta: TP \to \mathfrak{v}_p$ is a 2-form $d\vartheta: \Lambda^2TP \to \mathfrak{v}_p$.

**Lemma 3.2.** Let $\mathcal{Y}: P \times \mathfrak{g} \to TP$ be the restriction of the differential of the action $P \times G \to P$. For all $A: S \to \mathfrak{g}_p$, and $x: S \to TP$ with same underlying map $S \to P$ we have

$$
d\vartheta \circ (\mathcal{Y}(A) \wedge x) = -A(\vartheta(x)): S \to \mathfrak{v}_p.
$$

**Proof.** This is Proposition 4 in [9].

**Remark 3.3.** Although we will make no use of it, we remark that, in analogy with the usual definition, one can define a $G$-structure to be flat if $M$ can be covered by coordinate charts $U_i \cong V$ such that the square determined by the coordinates

$$
\begin{array}{ccc}
V_{U_i} & \xrightarrow{TM} & TM \\
U_i & \subseteq & M,
\end{array}
$$
which is contained in $L(M)(U_i)$, lies in $P(U_i)$.

3.2. Prolongation.

3.2.1. Unrestricted prolongation. Adapting the classical construction [12], we will in this subsection associate with a $G$-structure $P$ on $M$ a tower of prolongations

$$\ldots \longrightarrow P^{(k)} \longrightarrow P^{(k-1)} \longrightarrow \ldots \longrightarrow P^{(1)} \longrightarrow P^{(0)} = P \longrightarrow M,$$

where $P^{(i+1)} \to P^{(i)}$ is a reduction of $L(P^{(i)})$ to $G^{(i+1)}$. Here $G^{(0)} = G$ and $G^{(i)}$ is a vector group for all $i \geq 1$.

Remark 3.4. Given a super vector space, the associated supergroup structure on $\mathcal{A}(V)$ will be denoted by $V$. More generally, if a Lie supergroup $G$ acts linearly on a complex super vector space $V$, then the associated semi-direct product will be denoted by $G \ltimes V$ instead of $G \ltimes \mathcal{A}(V)$.

It will be convenient to introduce a name for the representation of $G$ on $V$: $\alpha: G \to GL(V)$. Applying $\mathbb{J}_V(-)$ to $G \to P \to M$ yields a principal $\mathbb{J}_V$-$G$-bundle $\mathbb{J}_V G \to \mathbb{J}_V P \to \mathbb{J}_V M$ and the usual identification $TG \cong G \ltimes \mathfrak{g}$ gives an isomorphism of groups $\mathbb{J}_V(G) \cong G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$, where $G$ acts via its adjoint representation on $\mathfrak{g}$. The bundle of horizontal frames is defined by the pullback

$$\begin{array}{ccc}
\mathcal{H} & \longrightarrow & \mathbb{J}_V P \\
\downarrow d\pi_* & & \downarrow \\
P & \longrightarrow & \mathbb{J}_V M.
\end{array}$$

Its $S$-points are the squares

$$\begin{array}{ccc}
V_S & \xrightarrow{h} & TP \\
\downarrow f & & \downarrow \\
S & \longrightarrow & P
\end{array}$$

such that the composite square

$$\begin{array}{ccc}
V_S & \longrightarrow & TP \longrightarrow TM \\
\downarrow & & \downarrow \\
S & \longrightarrow & P \longrightarrow M
\end{array}$$

lies in $P(S)$. Moreover, $\mathcal{H}$ is the total space of a principal $G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$-bundle with respect to the map $d\pi_*$. We need to construct an action of $G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$. The group $G$ acts via $\alpha$ on $\mathbb{J}_V^1(P)$ by precomposition. Together with the action of $\mathfrak{Hom}(V, \mathfrak{g}) \leq \mathbb{J}_V(G)$, this yields an action of $G \ltimes \mathfrak{g}$ on $\mathbb{J}_V^1(P)$, which restricts to an action on $\mathcal{H}$. The composition $i_P: \mathcal{H} \to \mathbb{J}_V P \to P$ is equivariant if we let act $G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$ trivially on $P$. Moreover, $d\pi_*$ is equivariant with respect to this action if we let act $G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$ on $P$ via the projection to $G$.

The canonical vertical distribution $\mathbb{V}: \mathfrak{g}_P \to TP$ gives rise to a map $\mathbb{J}_V P \to \mathbb{J}_V \oplus \mathfrak{g}_P$ and the composition $\mathcal{H} \to \mathbb{J}_V \oplus \mathfrak{g}_P$ factors through $L(P)$. Moreover, the $GL(V \oplus \mathfrak{g})$-action on $L(P)$ is seen to restrict to the action of $G \ltimes \mathfrak{Hom}(V, \mathfrak{g}) \leq GL(V \oplus \mathfrak{g})$. This identifies $i_P: \mathcal{H} \to P$ as a reduction of $L(P)$ to $G \ltimes \mathfrak{Hom}(V, \mathfrak{g})$. 

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As usual, $\mathfrak{g}^{(1)}$ is defined to be the kernel of the super-antisymmetrizer $\partial: \text{Hom}(V, \mathfrak{g}) \to \text{Hom}(V \otimes V, V)$,
\[(\partial S)(v, w) := \frac{1}{2}(S(v)(w) - (-1)^{|v||w|}S(w)(v)),\]
and the first prolongation $P^{(1)} \to P$ is obtained from $\mathcal{H} \to P$ by two successive reductions of the structure group to $\mathfrak{g}^{(1)}$ using the following lemma.

**Lemma 3.5.** Consider a short exact sequence of mixed Lie supergroups
\[1 \to H \to G \to K \to 1.\]
Let $\pi: P \to B$ be a $G$-principal bundle and assume that there is a $G$-equivariant map $f: P \to K$. Then $P/H \to B$ is a principal $K$-bundle and as such isomorphic to the trivial bundle. Moreover, the map $(\pi, f): P \to B \times K$ is a principal $H$-bundle.

**Proof.** Since any map of principal bundles is an isomorphism, it suffices to construct a $K$-equivariant map $P/H \to B \times K$ over $B$. But such a map can be constructed from the $G$-equivariant map $(\pi, f): P \to B \times K$ since $H$ acts trivially on the target. \hfill \Box

The first step is a reduction to $\text{Hom}(V, \mathfrak{g}) \leq G \times_\alpha \text{Hom}(V, \mathfrak{g})$. We have two maps $d\pi_*: \mathcal{H} \to P$ over the same map to the base $M$. Fibrewise comparison yields a map $d: \mathcal{H} \to G$. It follows now from the equivariance properties of $d\pi_*$ and $\iota_P$ that $d$ is $G \times_\alpha \text{Hom}(V, \mathfrak{g})$-equivariant if we let this group act from the right on $G$ by $g' \cdot (g', \varphi) = (g')^{-1}g$. Now we can apply Lemma 3.5 and see that $(\iota_P, d): \mathcal{H} \to P \times G$ is a principal $\text{Hom}(V, \mathfrak{g})$-bundle. Pulling back along the inclusion $P \times \{1\} \to P \times G$ yields the bundle of compatible horizontal frames $\mathcal{H}_P \to P$, a reduction of $L(P)$ to the group $\text{Hom}(V, \mathfrak{g})$. Its $S$-points consist of those squares $(f, h)$ such that $T(\pi)\circ h = f \in P(S)$.

The second reduction is a little bit more elaborate. For a section $v: T \to V_T$ and a map $f: T \to P$, we will use the shorthand $v_f := (f \times \iota(V)) \circ v: T \to V_P$.

**Lemma 3.6.** For all compatible horizontal frames $(f, h) \in \mathcal{H}(T)$ and all sections $x: T \to V_T$, we have $\vartheta(h(x)) = x_f$:
\[
\begin{array}{ccc}
T & \xrightarrow{x} & V_T \\
\downarrow \alpha_f & & \downarrow \partial \\
\downarrow & & \downarrow \\
V_P \\
\end{array}
\]

**Proof.** This follows immediately from the definition. \hfill \Box

Consider $(f, h) \in \mathcal{H}(S)$. The torsion is defined to be the composition
\[c(f, h): \Lambda^2 V_S \xrightarrow{\Lambda^2 h} \Lambda^2 TP \xrightarrow{d\partial} V_P.\]
Equivalently, it is given by a map
\[c'(f, h): S \xrightarrow{\text{Hom}(\Lambda^2 V, V)}.\]
By naturality, we obtain a map $c: \mathcal{H} \to \text{Hom}(\Lambda^2 V, V)$. Now consider two distinguished squares over $f$ with horizontal parts $h$ and $h'$. As $\mathcal{H} \to P$ is a principal $\text{Hom}(V, \mathfrak{g})$-bundle there is a unique map $S_{(f,h'),(f,h)}: V_S \to \mathfrak{g}_P$ over $f$ such that $h' = h + \nabla \circ S_{(f,h'),(f,h)}$. By adjointness this can be viewed as a map $S'_{(f,h'),(f,h)}: S \to \mathfrak{g}_P$.
\[ \text{Hom}(V, g). \] Then, by Lemmas 3.2 and 3.6 we have that for any two sections \( v, w: S \to V \),
\[
c(f, h')(v \wedge w) - c(f, h)(v \wedge w) = d\vartheta \circ h'(v) \wedge h'(w) - d\vartheta \circ h(v) \wedge h(w)
\]
\[
= d\vartheta \circ (h'(v) - h(v)) \wedge h'(w) + d\vartheta \circ h(v) \wedge (h'(w) - h(w))
\]
\[
= d\vartheta \circ (\mathcal{V} \circ S_{(f,h'),(f,h)}(v)) \wedge h'(w) + d\vartheta \circ h(v) \wedge (\mathcal{V} \circ S_{(f,h'),(f,h)}(w))
\]
\[
= -S_{(f,h'),(f,h)}(v)(\vartheta(h'(w))) - d\vartheta \circ (\mathcal{V} \circ S_{(f,h'),(f,h)}(w)) \wedge h(v)
\]
\[
= -S_{(f,h'),(f,h)}(v)(\vartheta(h'(w))) + S_{(f,h'),(f,h)}(w)(\vartheta(h(v)))
\]
\[
= -S_{(f,h'),(f,h)}(v)(w_f) + S_{(f,h'),(f,h)}(w_f)(v_f).
\]

In other words,
\[
e'(f, h') - e'(f, h) = -2\partial S'_{(f,h'),(f,h')}
\]
and if we let \( \text{Hom}(V, g) \) act on \( \text{Hom}(\Lambda^2 V, V) \) via \((-2)\partial\), then \( c: \mathcal{E}H \to \text{Hom}(\Lambda^2 V, V) \) is \( \text{Hom}(V, g) \)-equivariant. Now, we have the exact sequence
\[
0 \longrightarrow g^{(1)} \longrightarrow \text{Hom}(V, g) \longrightarrow \text{im}(\partial) \longrightarrow H^0.2(V, g) \longrightarrow 0.
\]

Consequently, any splitting \( s: \text{im}(\partial) \to \text{Hom}(\Lambda^2 V, V) \) gives rise to an equivariant map \( \mathcal{E}H \to \text{im}(\partial) \) and Lemma 3.5 applied to the short exact sequence
\[
0 \longrightarrow g^{(1)} \longrightarrow \text{Hom}(V, g) \longrightarrow \text{im}(\partial) \longrightarrow 0
\]
shows that \( \mathcal{E}H \to P \times \text{im}(\partial) \) is a principal \( g^{(1)} \)-bundle. Finally, by pulling back along \( P \times \{0\} \to P \times \text{im}(\partial) \) one obtains the first prolongation \( P^{(1)} \to P \), a reduction of \( L(P) \) to \( g^{(1)} \) which consists of those compatible horizontal frames with torsion contained in \( C := \ker(s) \).

The higher prolongations are now defined inductively: \( P^{(i+1)} := (P^{(i)})^{(1)} \). Setting \( g^{(-1)} := V \) and \( g^{(0)} := g \), we arrive at the following inductive description of \( g^{(k)} \) for \( k \geq 1 \):
\[
g^{(k)} = \{ X \in \text{Hom}(g^{(-1)}, g^{(k-1)}) | X(v)(w) = (-1)^{|v||w|} X(w)(v) \text{ for all homog. } v, w \}.
\]

By inspection, we have
\[
(g^{(1)})_{\circ} \subseteq (\text{Hom}(V, g)_{\circ})^\mu \subseteq \text{Hom}(V, g)_{\circ},
\]
i.e. any \( f \in (g^{(1)})_{\circ} \) satisfies \( f(V_C) \subseteq g_{\circ} \). This implies that \( P^{(k)} \subseteq L^\mu(P^{(k-1)}). \)

3.2.2. The real prolongation. The prolongations \( P^{(k+1)} \to P^{(k)} \) defined so far only provide reductions of \( L^\mu(P^{(k-1)}). \) To prove representability for the functor of automorphisms of a \( G \)-structure of finite type, we need to single out the real prolongation which provides a reduction of \( L^\mu(P^{(k-1)}) \). For this to be possible, we need to impose a condition on the \( G \)-structure.

To that end, consider the subspaces
\[
(\text{Hom}(V, g)_{\circ})^\mu \subseteq (\text{Hom}(V, g)_{\circ})^\mu \subseteq \text{Hom}(V, g)_{\circ}
\]
consisting of even linear maps \( f \) satisfying \( f(V_C) \subseteq g_{\circ} \) or \( f(V_R, V_C) \subseteq (g_R, g_C) \), respectively.

Recall the bundle of compatible horizontal frames with the map \( \mathcal{E}H \to P \times \text{im}(\partial) \). One readily constructs \( (\mathcal{E}H_0)_{\circ} \subseteq (\mathcal{E}H_0)^\mu \subseteq \mathcal{E}H_0 \) with structure groups
Pullback along the inclusion \( P_0 \times \{0\} \to P_0 \times \text{im}(\partial)_0 \) yields \( (P_0^{(1)})^\mu_\mathbb{R} \subseteq (P_0^{(1)})^\mu \subseteq P_0^{(1)} \) with structure groups given by the pullback

\[
\begin{array}{ccc}
(g^{(1)})_{\partial,\mathbb{R}} & \to & (\text{Hom}(V, g))^{\mu}_{(\mathbb{R})} \\
\downarrow & & \downarrow \\
(g^{(1)})_{\partial} & \to & \text{Hom}(V, g)_0,
\end{array}
\]

where once again, it is understood that the quantities in parentheses are only present for the case of \((P_0^{(1)})^\mu_\mathbb{R}\). Inductively, we obtain \((P_0^{(k)})^\mu_\mathbb{R} \subseteq (P_0^{(k)})^\mu \subseteq P_0^{(k)}\) with structure groups given by the pullback

\[
\begin{array}{ccc}
(g^{(k)})_{\partial,\mathbb{R}} & \to & (\text{Hom}(V, g^{(k-1)}))^{\mu}_{(\mathbb{R})} \\
\downarrow & & \downarrow \\
(g^{(k)})_{\partial} & \to & \text{Hom}(V, g^{(k-1)})_0.
\end{array}
\]

**Definition 3.7.** A \( G \)-structure is called \textit{admissible} if, for all \( k \geq 0 \), \((g^{(k)})_{\partial,\mathbb{R}}\) defines a mixed structure on \((g^{(k)})_{\partial}\).

Assume now that the \( G \)-structure is admissible. Since \((g^{(1)})_{\partial} \subseteq (\text{Hom}(V, g))^{\mu}_{(\mathbb{R})}\), we have that \((P_0^{(1)})^\mu = P_0^{(1)}\) and the structure group of \((P_0^{(1)})^\mu_\mathbb{R} = P_0^{(1)}\) is by definition \((g^{(1)})_{\partial,\mathbb{R}}\). Pulling back \( r^* P_0^{(1)} \to r^* P_0^{(1)} \) along \( P^{(1)} \to r^* P_0^{(1)} \) gives the functor \( P_0^{(1)} \), which is representable in view of Lemma 2.1.7 and the assumption on the \( G \)-structure. All in all, this yields the real prolongation:

\[
\cdots \to P_0^{(k)} \to P_0^{(k-1)} \to \cdots \to P_0^{(1)} \to P_0^{(0)} = P \to M.
\]

The structure group of the \( k \)th real prolongation will be denoted by \( G_0^{(k)} \).

## 4. Automorphisms of \( G \)-structures

The main object of study in this paper is the functor of automorphisms of a \( G \)-structure, which we presently define.

### 4.1. The functor of automorphisms of a \( G \)-structure

Let \( M \) be a mixed supermanifold. An automorphism \( f : S \times M \to S \times M \) over \( S \) is called an \( S \)-\textit{family of automorphisms} of \( M \). Such morphisms assemble to a functor \( \text{Diff}(M) \) given by

\[
\text{Diff}(M)(S) = \{ f : S \times M \to S \times M \mid f \text{ an } S\text{-family of automorphisms of } M \}.
\]

Moreover, for any Lie supergroup \( G \) and any principal \( G \)-bundle \( P \to M \), we let \( \text{Diff}(P)^G \subseteq \text{Diff}(P) \) be the subfunctor of equivariant automorphisms, i.e.

\[
\text{Diff}(P)^G(S) = \{ f \in \text{Diff}(P)(S) \mid f \text{ } G\text{-equivariant} \}.
\]

Note that if \( P \) is a \( G \)-structure, then inducing up from \( G \) to \( GL(V) \) gives a map \( \text{Diff}(P)^G \to \text{Diff}(L(M))^{GL(V)} \) and, moreover, the differential induces an inclusion of functors \( L(-) : \text{Diff}(M) \to \text{Diff}(L(M))^{GL(V)} \).
**Definition 4.1.** The functor of automorphisms of a \(G\)-structure \(P\) on \(M\) is defined to be the pullback

\[
\begin{array}{ccc}
\text{Aut}(P) & \rightarrow & \text{Diff}(M) \\
\downarrow & & \downarrow \\
\text{Diff}(P)^G & \rightarrow & \text{Diff}(L(M))^{GL(V)}.
\end{array}
\]

An \(S\)-point of \(\text{Aut}(P)\) is called an \(S\)-family of automorphisms of \(P\).

**Definition 4.2.** A homogeneous vector field \(\mathcal{O}_M \rightarrow (p_{S})\ast\mathcal{O}_{S\times M}\) along \(p_{S}: S \times M \to M\) is called an \(S\)-family of infinitesimal automorphisms of \(P\) if the induced vector field \(\mathcal{O}_{L(M)} \rightarrow (p_{S})\ast\mathcal{O}_{S\times L(M)}\) extends to \(\mathcal{O}_{P} \rightarrow (p_{S})\ast\mathcal{O}_{S\times P}\). For \(S = *\) this yields the Lie superalgebra \(\text{aut}(P) \subseteq \Gamma(\mathcal{J}_M)\) of infinitesimal automorphisms of \(P\). The even part has a real subalgebra defined by \(\text{aut}(P)_{0,\mathbb{R}} := \text{aut}(P)_{0} \cap \Gamma(\mathcal{J}_{M,\mathbb{R}})\).

**Remark 4.3.** There is no reason for \(\text{aut}(P)_{0,\mathbb{R}}\) to be a mixed real form or even a real form of \(\text{aut}(P)_{0}\). For instance, on a purely odd supermanifold all vector fields are real. The latter would be a necessary condition for the automorphism group to be representable by a Lie supergroup. For this reason automorphism groups of \(G\)-structures are generically mixed supermanifolds.

In analogy with Lemma 2.16 one sees that any \(\varphi \in \text{Diff}(M)(\mathbb{A}(C_0^n) \times T)\), where \(T\) is a mixed supermanifold, can be uniquely written as

\[
\varphi^\sharp = \prod_{k=n}^{1} \left(1 + \sum_{k \in I \subseteq \{1, \ldots, k\}} \eta^I X_I\right) \cdot \varphi_0
\]

where \(X_I\) are vector fields along \(p_T\) of degree \(|I|\) and \(\varphi_0 \in \text{Diff}(M)(T)\).

**Lemma 4.4.** Consider \(\varphi \in \text{Diff}(M)(\mathbb{A}(C_0^n) \times T)\). Then \(\varphi \in \text{Aut}(P)(\mathbb{A}(C_0^n) \times T)\) if and only if \(\varphi_0 \in \text{Aut}(P)(T)\) and all \(X_I\) are \(T\)-families of infinitesimal automorphisms of \(P\).

**Proof.** The condition is clearly sufficient. So, assume that \(\varphi\) is an \(\mathbb{A}(C_0^n) \times T\)-family of automorphisms of \(P\). Then \(\varphi_0\) is a \(T\)-family of such automorphisms since it is obtained by restricting along the inclusion \(T \to \mathbb{A}(C_0^n) \times T\). Now one proceeds by induction on \(n\) to show that all \(X_I\) are infinitesimal automorphisms of \(P\). \(\square\)

4.2. Prolongation of automorphisms of \(G\)-structures.

**Proposition 4.5.** Let \(P\) be an admissible \(G\)-structure on the mixed supermanifold \(M\). There is a natural inclusion of group-valued functors \(\text{Aut}(P) \to \text{Aut}(P_{\mathbb{R}}^{(1)})\).

**Proof.** This follows by repeatedly applying the universal property of the pullback in the construction of \(P_{\mathbb{R}}^{(1)}\). \(\square\)

4.3. The automorphisms of a \(\{1\}\)-structure. We now come to the issue of representability of \(\text{Aut}(P)\). Before proceeding to higher order \(G\)-structures we need to treat the simplest case \(G = \{1\}\). Then a \(G\)-structure is simply a parallelization \(\Phi: V_{\mathbb{R}}M \to TM_{\mathbb{R}}\). Such a \(\Phi\) induces an even real vector field on \(M \times \mathbb{A}(V_{\mathbb{R}})\):

\[
Z: M \times \mathbb{A}(V_{\mathbb{R}}) \longrightarrow TM_{\mathbb{R}} \times \mathbb{A}(V_{\mathbb{R}}) \longrightarrow T(M \times \mathbb{A}(V_{\mathbb{R}}))_{\mathbb{R}}
\]
and $\text{Aut}(\Phi)(S)$ consists of those automorphisms making the diagram
\[
\begin{array}{ccc}
S \times M \times \mathbb{A}(V_R) & \xrightarrow{S \times \Phi} & S \times TM_R \\
| & & | \\
S \times M \times \mathbb{A}(V_R) & \xrightarrow{S \times \Phi} & S \times TM_R
\end{array}
\]
commutative.

We first show that $i^*\text{Aut}(\Phi)$ is representable. To that end, we endow $\text{Aut}(\Phi)_0 := \text{Aut}(\Phi)(s)$ with the structure of a Lie group acting on $M$.

Recall that there is a forgetful functor sending a mixed manifold to its underlying smooth manifold. (We prove in Section 7.1 that such a functor does not exist for mixed supermanifolds.) Consider the underlying parallelization $\Phi: M_0 \times \mathbb{A}((V_R)_0) \to T(M_0)_R$ and its underlying smooth morphism $\Phi_0^{sm}: M_0^{sm} \times (V_R)_0 \to TM_0^{sm}$. In order to define a topology on $\text{Aut}(\Phi)_0$, we need the following fact.

**Lemma 4.6.** The forgetful map $\text{Aut}(\Phi)_0 \to \text{Aut}(\Phi_0)$, $s \mapsto s_0$, is injective.

**Proof.** Deferred to Section 4.5 \hfill \square

Moreover, we have that $\text{Aut}(\Phi_0) \subseteq \text{Aut}(\Phi_0^{sm})$ are precisely the elements which preserve the mixed structure on $M_0^{sm}$.

By a result of Kobayashi [12], any point $x \in M_0$ gives rise to a closed injection
\[
\text{Aut}(\Phi_0^{sm}) \to M_0^{sm}, \ s \mapsto s(x)
\]
and with this topology, $\text{Aut}(\Phi_0^{sm})$ is a Lie group such that the evaluation map $a_0^{sm}: \text{Aut}(\Phi_0^{sm}) \times M_0^{sm} \to M_0^{sm}$ is smooth [3]. This topology is the coarsest such that for all $f \in \Gamma(\mathcal{O}_{M_0^{sm}})$, the map $\text{Aut}(\Phi_0^{sm}) \to \Gamma(\mathcal{O}_{M_0^{sm}})$, $s \mapsto s^2(f)$, is continuous, where $\Gamma(\mathcal{O}_{M_0^{sm}})$ is considered as a Fréchet space with respect to the family of seminorms $|f|_{K,\partial} = \sup_K |\partial f|$, $K \subseteq M_0$ compact, $\partial$ differential operator. In this topology, $s_n \to s$ if and only if $s^2_n(f) \to s^2(f)$ in $\Gamma(\mathcal{O}_{M_0^{sm}})$ for all $f \in \Gamma(\mathcal{O}_{M_0^{sm}})$.

Being mixed is a closed condition (locally equations of the form $\partial s^2(f) = 0$ for all $f \in \mathcal{O}_M$), hence $\text{Aut}(\Phi_0) \subseteq \text{Aut}(\Phi_0^{sm})$ is closed. Then we get a Lie group $\text{Aut}(\Phi_0^{sm}) \subseteq \text{Aut}(\Phi_0)$, in view of the following lemma.

**Lemma 4.7.** The subspace $\text{Aut}(\Phi)_0 \subseteq \text{Aut}(\Phi_0)$ is closed. The topology on $\text{Aut}(\Phi)_0$ is such that $s_n \to s$ implies that for all pairs of coordinate charts $U$, $V$ such that $s_n(U) \subseteq V$ for all $n$ large enough, all the coefficients in the Taylor expansion of $s^2_n(f)$, $f \in \Gamma(\mathcal{O}_M|_V)$, with respect to the odd coordinates converge in $\mathcal{O}_{M_0^{sm}}(U_0)$.

**Proof.** Deferred to Section 4.5 \hfill \square

In particular, we have an action $a_0^\sharp: \mathcal{O}_M \to \mathcal{O}_{\text{Aut}(\Phi)_0^{sm} \times M}, \ f \mapsto (s \mapsto s^2(f))$.

**Lemma 4.8.** The map $a_0^\sharp$ extends to the action
\[
\begin{array}{ccc}
\mathcal{O}_M & \xrightarrow{a_0^\sharp} & \mathcal{O}_{\text{Aut}(\Phi)_0^{sm} \times M}, \\
& & f \mapsto (s \mapsto s^2(f)).
\end{array}
\]

**Proof.** Deferred to Section 4.5 \hfill \square
As explained in Section 7.2, even real vector fields have unique maximal flows. Using this, the action above and the description of the topology on \( \text{Aut}(\Phi)_0^{sm} \), one obtains an isomorphism

\[
\text{Lie}_R(\text{Aut}(\Phi)_0^{sm}) \cong \text{aut}(\Phi)_0^{\infty R} := \{ X \in \Gamma(T_M)_0 | [X, Z] = 0, X \text{ complete} \} \subseteq \text{aut}(\Phi).
\]

Then \( \mathbb{C} \)-linearization yields a Lie algebra morphism

\[
\mathbb{C} \otimes \text{aut}(\Phi)_{\infty R}^{\infty} \longrightarrow \Gamma(T_M)_0
\]

and the kernel is of the form

\[
\widetilde{\text{aut}}(\Phi)_0^{\infty R, C} := \{ 1 \otimes v - i \otimes v | v \in \text{aut}(\Phi)_{\infty R}^{\infty} \}
\]

for a complex invariant ideal \( \text{aut}(\Phi)_{\infty R}^{\infty} \subseteq \text{aut}(\Phi)_{\infty R}^{\infty R} \). This yields the mixed structure \( \text{Aut}(\Phi)_0 \) on \( \text{Aut}(\Phi)_0^{sm} \) and, on general grounds, the quotient

\[
(\mathbb{C} \otimes \text{aut}(\Phi)_{\infty R}^{\infty})/\widetilde{\text{aut}}(\Phi)_0^{\infty R, C} := \text{aut}(\Phi)_{\infty R}^{\infty R} \subseteq \text{aut}(\Phi)_0
\]

is the Lie algebra of left-invariant derivations of \( \mathcal{O}_{\text{Aut}(\Phi)_0} \). It is the algebra of complete decomposable infinitesimal automorphisms in the sense that any of its elements can be written as the sum \( v + iw \) of complete real vector fields \( v \) and \( w \). Moreover, with this structure \( a: \text{Aut}(\Phi)_0 \times M \to M \) is a mixed morphism, by Lemma 2.15.

Finite-dimensionality of the full algebra of infinitesimal automorphisms is ensured by the following lemma.

**Lemma 4.9.** Assume that \( M_0 \) is connected. For every \( p \in M_0 \), evaluation \( \text{aut}(\Phi) \to T_p M, X \mapsto X(p) \), is injective. If \( M_0 \) is not connected, the analogous statement holds true if one chooses one point for each connected component.

**Proof.** Deferred to Section 4.5. \( \square \)

Moreover, the conjugation action of \( \text{Aut}(\Phi)_0 \) on \( \Gamma(T_M) \) restricts to an action on \( \text{aut}(\Phi) \) and the differential of this representation is simply the restriction of the adjoint representation

\[
\text{aut}(\Phi)_{\infty R}^{\infty R} \times \text{aut}(\Phi) \longrightarrow \text{aut}(\Phi).
\]

The following result shows that \( \text{Aut}(\Phi)_0 \) has the correct topology and mixed structure.

**Proposition 4.10.** The functors \( i^* \text{Aut}(\Phi) \) and \( M^\mu(-, \text{Aut}(\Phi)_0) \) are naturally isomorphic.

**Proof.** Given a map \( T_0 \to \text{Aut}(\Phi)_0 \), the action of the group yields a map \( T_0 \times M \to T_0 \times M \). Conversely, take an element \( f: T_0 \times M \to T_0 \times M \) in \( \text{Aut}(\Phi)(T_0) \). The obvious candidate \( \tilde{f}: T_0 \to \text{Aut}(\Phi)_0 \) is a smooth mixed map since the composition

\[
T_0 \to \text{Aut}(\Phi)_0 \to M_0
\]

with evaluation at some \( m \in M_0 \) equals \( f_0(-, m_0) \) which is smooth and mixed. \( \square \)
4.4. The automorphisms of a \( G \)-structure of finite type.

**Definition 4.11.** An admissible \( G \)-structure is of finite type if there exists a \( k \geq 0 \) such that \( G^{(k+l)} \equiv \{1\} \) for all \( l \geq 0 \).

The main theorem is as follows:

**Theorem 4.12.** Suppose \( P \to M \) is an admissible \( G \)-structure of finite type. Then \( i^* \text{Aut}(P) \) is representable and its (real) Lie algebra consists of the complete real infinitesimal automorphisms of \( P \), denoted by \( \text{aut}(P)_{0,\mathbb{R}} \).

Moreover, \( \text{aut}(P) \) is finite-dimensional and the functor \( \text{Aut}(P) \) is representable if and only if \( \text{aut}(P)_{0,\mathbb{d}} = \text{aut}(P)_{0} \).

**Proof.** We choose \( k \geq 0 \) such that \( G^{(k+l)} \equiv \{1\} \) for all \( l \geq 0 \). Then we have an embedding \( \text{Aut}(P) \to \text{Aut}(P^{(k)}) \). Hence, \( \text{aut}(P) \) is finite-dimensional in view of Lemma 4.9. Let \( \Phi \) be the given parallelization of the real tangent bundle of \( P^{(k-1)} \).

We show that the inclusion
\[
\text{Aut}(P)(*) \subseteq \text{Aut}(\Phi)(*) = \text{Aut}(\Phi)_{0}^{\mathbb{m}}
\]
is closed. Recall that the topology on \( \text{Aut}(\Phi)_{0}^{\mathbb{m}} \subseteq (P^{(k-1)})_{0}^{\mathbb{m}} \) is such that \( s_n \to s \) implies that locally all \( s^i_n(f), \ f \in \Gamma(\mathcal{O}_{P^{(k-1)}}|_{V}) \), converge in the closed subspace
\[
\mathcal{O}_{P^{(k-1)}}(U_0) \cong \bigoplus_{i \in \mathbb{m}} \mathcal{O}_{(P^{(k-1)})_0}(U_0) \subseteq \bigoplus_{i \in \mathbb{m}} \mathcal{O}_{(P^{(k-1)})_0}(U_0),
\]
where the number of summands is \( 2^d \), \( d \) denoting the odd dimension of \( P^{(k-1)} \).

Now assume \( s_n \in \text{Aut}(P)(*) \) and \( s_n^{(k)} \to \bar{s} \). From the construction of the prolongation, it is clear that one obtains a diffeomorphism \( s : M \to M \) with \( k \)-th prolongation \( s^{(k)} \) equal to \( \bar{s} \). From equivariance it now follows that \( s \) is actually in \( \text{Aut}(P)(*) \).

Next, assume that the action \( \text{Aut}(P^{(i+1)})_{0}^{\mathbb{m}} \times P^{(i)}_{\mathbb{R}} \to P^{(i)}_{\mathbb{R}} \) is smooth. Restricted to \( \text{Aut}(P^{(i)})_{0}^{\mathbb{m}} \), it is pointwise equivariant, hence it is itself equivariant and thus descends to an action on \( P^{(i-1)}_{\mathbb{R}} \). This action gives the identification of the Lie algebra of \( \text{Aut}(P)_{0}^{\mathbb{m}} \) with \( \text{aut}(P)_{0,\mathbb{R}} \), and the mixed structure is now defined as in the case of \( \text{Aut}(\Phi)_{0} \).

Then the action just defined refines to an action \( \text{Aut}(P)_0 \times M \to M 
\]
by Lemma 2.13 and, using this, similar as in the situation of the automorphisms of a parallelization, one deduces that \( i^* \text{Aut}(P) \cong \text{Aut}(P)_{0} \).

Clearly, if \( \text{Aut}(P) \) is representable, then \( \text{aut}(P) \) can only consist of complete and decomposable vector fields. Conversely, if \( \text{aut}(P)_{0,\mathbb{d}} = \text{aut}(P)_{0} \), then
\[
\text{Aut}(P) = (\text{aut}(P), \text{Aut}(P)_{0})
\]
forms a mixed super pair. The action defines a map \( \text{SM}^{\mu}(-, \text{Aut}(P)) \to \text{Diff}(M) \), and in view of Lemma 2.16 it factors locally through an isomorphism to \( \text{Aut}(P) \). Hence, it factors globally as an isomorphism \( \text{SM}^{\mu}(-, \text{Aut}(P)) \cong \text{Aut}(P) \).

4.5. **Proofs of Lemmas 4.6, 4.7, 4.8, and 4.9**

**Proof of Lemma 4.6**. Let \( s \in \text{Aut}(\Phi)_{0} \) be such that \( s_0 = \text{id} \). In order to see that this implies \( s = \text{id} \), we consider, for \( k \geq 1 \), the restriction of \( s \) to the \((k-1)\)th infinitesimal neighbourhood
\[
(s^{(k-1)})^*: \mathcal{O}_M/\mathfrak{g}^k \to (s_0)_*\mathcal{O}_M/\mathfrak{g}^k.
\]
We have \((s^{(0)})^\sharp = s_0^\sharp = \text{id}\).

Now, we choose a homogeneous basis \(\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}\}\) of \(V\) and local coordinates \(\{q_1, \ldots, q_n, q_{n+1}, \ldots, q_{n+m}\}\) on an open subset \(U_0\) containing \(m \in M_0\). Here, the first \(n\) (respectively last \(m\)) entries are assumed to be even (resp. odd).

In the given basis

\[
Z_{vs} = \sum_i A_{ki} \partial_{q_i}
\]

for some even invertible matrix \(A = (A_{ki}) \in GL_{\mathcal{O}_M}(\mathcal{O}_M(U_0)^{n|m})\). The requirement for \(f\) to lie in \(\text{Aut}(\Phi)_0\) reads

\[
Jf = A^{-1} \circ f^\sharp(A)
\]

where \(Jf = (\partial_{q_i} f^\sharp(q_j))\) and we denote the natural extension of \(f^\sharp\) to matrices by the same symbol.

So assume \((f^{(k-1)})^\sharp = \text{id}\). We have

\[
Jf + \tilde{\partial}^k(U_0)^{\langle n|m\rangle \times \langle n|m\rangle} = A^{-1} \circ f^\sharp(A) + \tilde{\partial}^k(U_0)^{\langle n|m\rangle \times \langle n|m\rangle}
\]

\[
= A^{-1} \circ (f^{(k-1)})^\sharp(A) + \tilde{\partial}^k(U_0)^{\langle n|m\rangle \times \langle n|m\rangle}
\]

\[
= \text{id}_{n|m} + \tilde{\partial}^k(U_0)^{\langle n|m\rangle \times \langle n|m\rangle},
\]

and this implies \((f^{(k)})^\sharp = \text{id}\). \(\square\)

**Proof of Lemma 4.7** Let \(\{s_n\}\) be a sequence in \(\text{Aut}(\Phi)_0\) such that \(\{(s_n)_0\}\) converges to some \(\tilde{s}\). We have to show that \(\tilde{s} = s_0\) for some suitable \(s \in \text{Aut}(\Phi)_0\) and \(s_n\) converges to \(s\). Without loss of generality all \((s_n)_0\) lie in one coordinate chart (in \(\text{Aut}(\Phi)_0\)) and since \(a_0^{m}\) is smooth we may choose open subspaces \(U\) and \(V\) with coordinates \(\{p_i\}\) and \(\{q_i\}\) respectively such that every \(s_n\) restricts to a map \(U \to V\).

Let us organise the coordinates into even and odd functions \(\{p_i\} = \{x_i, \eta_j\}\), \(\{q_i\} = \{y_i, \zeta_j\}\).

In these coordinate charts the condition for \(s_n\) to lie in \(\text{Aut}(\Phi)_0\) reads

\[
J(s_n) = A \cdot s_n^\sharp(B)
\]

for certain invertible matrices \(A\) and \(B\) where \(J(s_n) = (\partial_{p_j} s_n^\sharp(q_i))\). Starting from \((s^{(0)})^\sharp := \tilde{s}^\sharp\), we inductively define \((s^{(k)})^\sharp : \mathcal{O}_M/\mathcal{J}^{k+1}(V_0) \to \mathcal{O}_M/\mathcal{J}^{k+1}(U_0)\) with reductions \(\tilde{s}\). The construction will be such that the following holds: We have \((s_n^{(k)})^\sharp(f) \to (s^{(k)})^\sharp(f)\) for all \(f \in (\mathcal{O}_M/\mathcal{J}^{k+1})(V_0)\). Here, \((\mathcal{O}_M/\mathcal{J}^{k+1})(U_0)\) is considered as a subspace of \(\bigoplus \mathcal{O}_{M^n}(U_0)\), where the number of summands is \(2^m\).

The respective lifts will be determined by the Jacobian \(J(s^{(k)})\) which naturally has values in matrices of the form

\[
\begin{pmatrix}
\mathcal{O}_M/\mathcal{J}^{k+1} & \mathcal{O}_M/\mathcal{J}^k \\
\mathcal{O}_M/\mathcal{J}^{k+1} & \mathcal{O}_M/\mathcal{J}^k
\end{pmatrix}
\]

There is a projection from \(\mathcal{O}_M/\mathcal{J}^{k+1}\)-valued matrices to such matrices. The image of a matrix \(A\) will be denoted by \(A^-\).

Assume that \(k\) is even and \((s^{(k)})^\sharp\) has been constructed such that

\[
J(s^{(k)}) = (A\, (s^{(k)})^\sharp B^{(k)})^-.
\]
First, we have to set \((s^{(k+1)})^2(q_t) = (s^{(k)})^2(q_t)\) for \(q_t\) even. The odd-odd sector of the Jacobian determines \((s^{(k+1)})^2(q_t)\) for \(q_t\) odd: In fact, it follows that

\[
\partial_{q_t}(s^{(k+1)})^2(\xi_j) = \frac{1}{n}(A^{(k+1)}(s^{(k+1)})^2 B^{(k+1)})_{ij}
\]

\[
= (A^{(k)}(s^{(k)})^2 B^{(k)})_{ij}
\]

\[
= \lim_n (A^{(k)}(s^{(k)})^2 B^{(k)})_{ij}
\]

\[
= \lim_n \partial_{q_t}(s^{(k+1)})^2(\xi_j).
\]

These derivatives fit together to give a well-defined \((s^{(k+1)})^2(\xi_j)\) since the different partial derivatives fit together, that is, for any multiindex \(I, |I| = k + 1\), with \(\eta_t, \eta_v \in I\), we have

\[
\partial_{I-\{\eta_t\}} \partial_{\eta_t}((s^{(k+1)})^2(\xi_j)) = \epsilon_{i,v} \partial_{I-\{\eta_t\}} \partial_{\eta_v}((s^{(k+1)})^2(\xi_j))
\]

since this equality holds for all \(s_n\). With this definition we have \((s^{(k+1)})^2 = \lim_n (s_n^{(k+1)})^2\), which ensures \(J(s^{(k+1)}) = (A^{(k+1)}(s^{(k+1)})^2 B^{(k+1)})^\sim\) by continuity.

If \(k\) is odd and \((s^{(k)})^2\) has been constructed in such a way that

\[
J(s^{(k)}) = (A^{(k)} \cdot (s^{(k)})^2 B^{(k)})^\sim,
\]

then one can proceed similarly. There are no changes in the pullbacks of odd coordinates and the pullbacks of the even coordinates are forced by the respective equation for the odd-even sector of the Jacobian. Again, \((s^{(k)})^2 = \lim_n (s_n^{(k)})^2\). This yields the construction of \(s|_U : O_V \to (s_0)_* O_U\). By uniqueness (Lemma 4.10), these \(s|_U\) coincide where two coordinates patches overlap, and so we obtain the desired \(s : M \to M\).

The statement concerning the topology is clear from the above considerations.

\[\Box\]

**Proof of Lemma 4.8.** Similary as in the preceding lemma, starting from \(((a')^{(0)})^2 := (a_0')^2\), we inductively construct \(((a')^{(k)})^2 : O_M/\beta^{k+1} \to (a_0')^2_0 \cdot \mathcal{O}_{\text{Aut}(\Phi)}^{\text{adm}} \times_M /\beta^{k+1}\).

First we choose some neighbourhoods \(W \subseteq \text{Aut}(\Phi)_0^{\text{adm}}\) and \(U, V \subseteq M\) given by coordinates \(p_i = \{x_i, \eta_j\}\) and \(q_t = \{y_t, \xi_j\}\) such that \(a_0'\) restricts to

\[W \times U_0 \longrightarrow V_0.\]

Then, if \(A\) and \(B\) are as in the proof above, the map \((a')^2\) to be constructed will be characterized by

\[
J^{\text{res}}(a') = A(a')^2(B).
\]

where \(J^{\text{res}}(a')\) denotes the submatrix \((\partial_{q_t}(a')^2(q_t))\) of the Jacobian. So, assume \(((a')^{(k)})^2\) is constructed such that

\[
J((a')^{(k)})^2 = (A^{(k)}((a')^{(k)})^2 B^{(k)})^\sim.
\]

Suppose first that \(k\) is even. Looking at the odd-odd sector of the Jacobian gives

\[
\partial_{q_t}(a')^{(k+1)}^2(\xi_j) = (A^{(k)}((a')^{(k)})^2 B^{(k)})_{ij}.
\]

These fit together since they do so pointwise, i.e. after specializing to any element \(s \in \text{Aut}(\Phi)^{\text{adm}}\). Moreover, the identity for the Jacobian holds true, since it holds true pointwise.

\[\Box\]
Proof of Lemma [4,7]. We follow [3] Lemma 2.4. If \( X \in \text{aut}(\Phi) \), then \( X_{V_0} := X \otimes \text{id}_{V_0} \) is a vector field on \( M \times \mathcal{A}(V_0) \) which commutes with \( Z \) (as is seen in local coordinates).

Let \( Z^0 \) be the maximal flow of the even real vector field \( Z \) (see Theorem 7.8), defined on \( V \subseteq \mathbb{R} \times M \times \mathcal{A}(V_0) \), and consider the composite \( \Theta^{Z^0} = \text{pr}_1 \circ \Theta^Z : V \to M \).

Note that \( \{1\} \times M \times \{0\} \subseteq V \), so \( \Theta^{Z^0}(1, -) \) is defined on an open neighbourhood of \( M \times \{0\} \).

We have the following: For all \( p \in M_0 \) there exists an open neighbourhood \( p \in U_0 \subseteq M_0 \) and open subspace \( V' \subseteq \mathcal{A}(V_0) \) such that for all \( q \in U_0 \) the map \( \Theta^{Z^0}(1, q, -) : V' \to M \) is a diffeomorphism onto an open subspace.

Indeed, the map \((\text{pr}_1, \Theta^{Z^0}(1, -))\) is defined on an open neighbourhood of \( M \times \{0\} \) and its differential at \((p, 0)\) is of the form
\[
\begin{pmatrix}
1 & 0 \\
* & Z
\end{pmatrix},
\]
which is invertible.

Now, assume \( \text{inj}_p^Z \circ X = X(p) = 0 \). Choose open subspaces \( U \subseteq M \) and \( V' \subseteq V \) such that \( p \in U_0 \) and \( 0 \in V' \) such that \( \varphi := \Theta^{Z^0}(1, p, -) : V' \to U \) is an isomorphism. Then
\[
\varphi^* \circ X = \text{inj}_p^Z \circ \Theta^{Z^0}(1, -)^\sharp \circ \text{pr}_1^Z \circ X \\
= \text{inj}_p^Z \circ \Theta^{Z^0}(1, -)^\sharp \circ X_V \circ \text{pr}_1^Z \\
= \text{inj}_p^Z \circ X_V \circ \Theta^{Z^0}(1, -)^\sharp \circ \text{pr}_1^Z \\
= 0,
\]
where we have used Proposition [7,10] in the third line. Since \( \varphi^Z \) is invertible, it follows that \( X = 0 \) on \( U \).

This shows that the non-empty closed set \( \{p \in M_0 | X(p) = 0\} \) is contained in the open subset \( \{p \in M_0 | X_p = 0\} \). The converse inclusion holds always, so that both subsets agree and are open and closed, hence they are all of \( M_0 \) if \( M_0 \) is connected. More generally, the argument shows that \( X(p) = 0 \) implies \( X = 0 \) on the connected component containing \( p \).

5. \( G \)-STRUCTURES OF FINITE TYPE ON REAL SUPERMANIFOLDS

Results analogous to those obtained in the mixed case hold for real supermanifolds. Their proofs are simplifications of our previous arguments, so we only briefly comment on them to provide precise statements for future reference.

A real super vector space is \( \mathbb{Z}/2 \)-graded real vector space \( V = V_0 \oplus V_\bar{1} \). The model spaces for real supermanifolds are the affine spaces \( \mathcal{A}(V) = (V_0, C^\infty_{V_0}(-) \otimes \bigwedge V_\bar{1}) \).

Definition 5.1. A real supermanifold consists of a locally ringed superspace \( M = (M_0, \mathcal{O}_M) \) over \( \mathbb{R} \) with second countable Hausdorff base that is locally isomorphic to \( \mathcal{A}(V) \) for some real super vector space \( V \). The full subcategory of locally ringed superspaces over \( \mathbb{R} \) with objects real supermanifolds is denoted by \( \mathbf{SM}_\mathbb{R} \).

Similarly as in the case of supermanifolds, a real supermanifold has a frame bundle \( L(M) \), which is a principal \( GL(V) \)-bundle. In the real category, \( GL(V) \) is a real Lie supergroup and so \( L(M) \) is an object in the category of real manifolds.
Furthermore, given a $G$-structure, i.e. a closed subgroup $G \leq GL(V)$ and a reduction $P$ of $L(M)$ to $G$, one can define the prolongation without leaving the real category.

One has a functor $i : M \to SM_{\mathbb{R}}$ and similarly as in the case of mixed supermanifolds, one obtains the following result.

**Theorem 5.2.** Suppose $P \to M$ is a $G$-structure of finite type. Then $i^*\text{Aut}(P)$ is representable and its Lie algebra consists of the complete infinitesimal automorphisms of $P$, denoted by $\text{aut}(P)^\mathbb{R}_{\circ}$. Moreover, $\text{aut}(P)$ is finite dimensional. The functor $\text{Aut}(P)$ is representable if and only if $\text{aut}(P)^\mathbb{R}_{\circ} = \text{aut}(P)^{\mathbb{R}}$.

6. **EXAMPLES OF $G$-STRUCTURES OF FINITE TYPE**

6.1. **Riemannian structures on supermanifolds.** In this section, we treat Riemannian structures on a supermanifold $M$ locally modelled on the mixed super vector space $(V, V_{\mathbb{R}}, V_{\mathbb{C}})$.

6.1.1. **Even Riemannian structures.** Consider an even non-degenerate bilinear form $J : V \otimes V \to \mathbb{C}^{1|0}$ with components $J_i : V_i \otimes V_i \to \mathbb{C}$ ($i \in \{0, 1\}$). There is a Lie supergroup $OSp(V, J)$ which represents automorphisms of the trivial vector bundle endowed with $J$:

$$OSp(V, J)(S) = \{ f \in GL(V)(S) \mid (S \times J) \circ (f \otimes f) = (S \times J) \}.$$  

**Proposition 6.1.**

(a) Reductions of $L(M)$ to $OSp(V, J)$ are in bijective correspondence with even non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \mathbb{C}^{1|0} M$.

(b) $OSp(V, J) \leq GL(V)$ is of finite type, more precisely $\mathfrak{osp}(V, J)^{(1)} = 0$.

**Proof.** Given an $OSp(V, J)$-structure on $L(M)$, one constructs a metric on $TM$ by declaring the given bases to be orthonormal. Conversely, given a metric, the orthonormal bases give rise to an $OSp(V, J)$-structure. This shows the first part. In order to show the second part, we observe that $\mathfrak{osp}(V, J)$ consists of those endomorphisms $A : V \to V$ whose homogeneous components $A_i$ satisfy $J(A_i v, w) = (-1)^{|v||w|} J(v, A_i w)$. Using a homogeneous basis $\{v_i\}$, the conditions for $T$ to lie in $\mathfrak{osp}(V, J)^{(1)}$ read $T_{jk} = (-1)^{|v_i||v_j|} T_{kj}^i$ and $T_{jk}^i = (-1)^{|v_j||v_k|} T_{kj}^i$, where we set $T_{jk}^i = J(T(v_i) v_j, v_k)$. Both together imply $T_{jk}^i = 0$. 

The underlying complex group of $OSp(V, J)$ is the product of the complex groups $O(V_0, J_0) \times Sp(V_1, J_1)$. Assume that $J_0$ restricts to a non-degenerate bilinear form $J_0 : (V_0)_{\mathbb{R}} \otimes (V_0)_{\mathbb{R}} \to \mathbb{R}$. Such a $J$ gives rise to the mixed real form $OSp(V, J)_{\mathbb{R}} \to O((V_0)_{\mathbb{R}}, J_0) \times Sp(V_1, J_1)$ with underlying group $O((V_0)_{\mathbb{R}}, J_0) \times Sp(V_1, J_1)$. Moreover, $OSp(V, J)_{\mathbb{R}} \leq GL(V)_{\mathbb{R}}$.

**Lemma 6.2.** The $OSp(V, J)_{\mathbb{R}}$-structures on $M$ are in bijective correspondence with even non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \mathbb{C}^{1|0} M$ whose restriction to $(TM)_0 \otimes (TM)_0 \subseteq i^*(TM \otimes TM)$ induce a metric $T(M_0)_{\mathbb{R}} \otimes T(M_0)_{\mathbb{R}} \to \mathbb{R}^1_{M_0}$ of the same signature as $J_0_{\mathbb{R}}$ on the underlying real manifold $M_0$.

**Proof.** This follows readily from the definition of $OSp(V, J)_{\mathbb{R}}$. 

From Theorem 4.12, we obtain the following result.
Theorem 6.3. Let $M$ be a supermanifold with an $OSp(V, J)_{\mathbb{R}}$-structure. If $M_0$ is complete and every Killing vector field is decomposable, then the isometry group functor $\text{Aut}(P)$ is representable.

Remark 6.4. In the real category the only obstruction for representability is completeness of the Killing vector fields. In this setting, an isometry group was constructed by Goertsches [11]. (The completeness condition seems to be assumed implicitly.) Our results in the real case give a rederivation of this result.

Example 6.5. The isometry group of $V$ with the $OSp(V, J)_{\mathbb{R}}$ as above is $OSp(V, J)_{\mathbb{R}} \ltimes V_{\mathbb{R}}$.

6.1.2. Odd Riemannian structures. In the super setting, there is an odd analogue of a Riemannian structure, given by an odd non-degenerate supersymmetric bilinear form $J : V \otimes V \to \mathbb{C}^{1|0}$. The Lie supergroup $P(V, J)$ is defined by the functor

$$P(V, J)(S) = \{ f \in GL(V) | (S \times J) \circ (f \otimes f) = (S \times J) \}.$$ 

Similar to the even case, one can show the following.

Proposition 6.6.

(a) The $P(V, J)$-structures on $L(M)$ and the odd non-degenerate supersymmetric maps of vector bundles $TM \otimes TM \to \mathbb{C}^{1|0}_M$ are in one-to-one correspondence.

(b) $P(V, J) \leq GL(V)$ is of finite type, more precisely $p(V, J)^{(1)} = 0$.

We have $P(V, J)_0 \cong GL(V_0)$, which comes with the mixed real form given by $GL((V_0)_{\mathbb{R}})$ and thus gives rise to $P(V, J)_R \leq GL(V)_{\mathbb{R}}$.

For any $P(V, J)$-structure $P$ on $M$, we have that $P_0 \cong L(M_0)$ and hence, it admits the real form $P_0, R \cong L(M_0)_R$. Now, one easily concludes the following.

Proposition 6.7. $P(V, J)_{\mathbb{R}}$-structures are in one-to-one correspondence with $P(V, J)$-structures.

From Theorem 4.12, we obtain the following result.

Theorem 6.8. Let $M$ be a supermanifold with a $P(V, J)_{\mathbb{R}}$-structure. If $M_0$ is complete and all infinitesimal automorphisms are decomposable, then the isometry group functor $\text{Aut}(P)$ is representable.

6.2. Superization of Riemannian spin manifolds. Let $(M_0, g_0)$ be a connected pseudo-Riemannian spin manifold endowed with a $\text{Spin}(V_0)$-structure

$$\rho(M_0) : \text{Spin}(M_0) \longrightarrow SO(M_0),$$

where we set $(V_0, \alpha) = (T_m M_0, g_m)$ for some $m \in M_0$. Choose any real or complex $Cl(V_0, \alpha)$- or $Cl(V_0, \alpha) \otimes \mathbb{C}$-module $V_1$.

The spinor bundle is the associated bundle $S = \text{Spin}(M_0) \times^{\text{Spin}(V_0)} V_1 \to M_0$, which we endow with the lift of the Levi-Civita connection. Then $TM_0 \oplus S \to M_0$ admits a reduction to $\text{Spin}(V_0) \leq GL(V_0) \times GL(V_1)$ by means of $\rho(M_0), \text{id} : \text{Spin}(M_0) \to SO(M_0) \times \text{Spin}(M_0)$.

The spinor supermanifold $M$ associated to this data is obtained by taking the exterior algebra of the dual $S^*$:

$$M = (M_0, \Gamma(-, \Lambda S^*))$$.
It is a real supermanifold or a supermanifold depending on whether $V_1$ is chosen to be real or complex. Any vector field on $M_0$ can be extended to $M$ by means of the dual connection on $S^*$, $X \mapsto \nabla_X$, and, furthermore, dual spinors can be contracted with spinors. This yields an inclusion $\iota: TM_0 \oplus \Pi S \to TM$ and hence a $Spin(V_0)$-structure $P_{Spin(V_0)} \subseteq L(M)$. 

Any $Spin(V_0)$-submodule $W \subseteq Hom(V_0, V_1)$ gives rise to a mixed Lie supergroup $Spin(V_0) \ltimes W \leq GL(V)_R$. Consequently, by inducing up, any such $W$ gives rise to a $Spin(V_0) \ltimes W$-structure on $M$:

$$P_{Spin(V_0)\ltimes W} := P_{Spin(V_0)} \times^{Spin(V_0)} (Spin(V_0) \ltimes W).$$

A particular choice is

$$W = \{ f_s: V_0 \to V_1 \mid s \in V_1, f_s(v_0) = v_0 \cdot s \}.$$

**Proposition 6.9.** For this choice of $W$, $Spin(V_0) \ltimes W \leq GL(V)$ is of finite type, provided that $\dim M \geq 3$.

**Proof.** After choosing an orthonormal basis $\{e_i\}$ of $V_0$, everything boils down to

$$e_i \cdot s_j = e_j \cdot s_i$$

for all $i$ and $j$ and certain $s_j \in V_1$, which implies $s_j = 0$ if $\dim M \geq 3$: We have $s_k = -(e_l, e_l)e_l e_k s_l$. On one hand, if $k, l$ and $j$ are such that $l \neq j$ and $l \neq k$ we have

$$s_k = -(e_l, e_l)e_l e_k s_l = -(e_j, e_l)e_l (-(e_j, e_j)e_j e_l s_j) = -(e_j, e_j)e_j e_j s_j$$

On the other hand

$$s_k = -(e_j, e_j)e_j e_k s_j.$$

So, if in addition $k \neq j$ (hence all three are different), then

$$e_j e_k s_j = 0,$$

so that we finally arrive at $s_j = 0$. \hfill \Box

**Remark 6.10.** By a theorem of Cortés et al. [1], the vector field $\iota(s)$ associated with a spinor gives rise to an infinitesimal automorphism of $P_{Spin(V_0)\ltimes W}$ if and only if $s$ is a twistor spinor, i.e. there exists a spinor $\tilde{s}$ such that for all $X$ we have $\nabla_X s = X \cdot \tilde{s}$.

7. Appendix

7.1. Non-existence of a forgetful functor $SM^c \to SM$. A mixed manifold $M$ has an underlying manifold $M^m$ which comes with a functorial map $M^m \to M$. For an affine space $M = \mathbb{A}(V, V_\mathbb{R}, V_\mathbb{C})$, the assignment is simply given by setting $M^m = \mathbb{A}(\mathbb{C} \otimes V_\mathbb{R}, V_\mathbb{R}, 0)$, and the map $M^m \to M$ is induced by the map $\mathbb{C} \otimes V_\mathbb{R} \to V$. We show that the analogous statement fails in the category of mixed supermanifolds. This is not surprising insofar as there does not even exist a forgetful functor from complex supermanifolds to supermanifolds [15]. A by-product of the argument is a proof that there is no functorial way to split even complex functions on supermanifolds into two even real functions (Proposition 2.2).
Let \((V, V_\mathbb{R}, V_\mathbb{C})\) be a mixed super vector space. The natural choice for the underlying supermanifold is given by the affine space associated with the super vector space \(u(V, V_\mathbb{R}, V_\mathbb{C}) = (\mathbb{C} \otimes (V_\mathbb{R})_0 \oplus V_1, V_\mathbb{R}, V_1)\). The natural choice for the map
\[
\epsilon_{(V, V_\mathbb{R}, V_\mathbb{C})} : A(u(V, V_\mathbb{R}, V_\mathbb{C})) \to A(V, V_\mathbb{R}, V_\mathbb{C})
\]
is induced by the \(\mathbb{C}\)-linearization of the inclusion \((V_\mathbb{R})_0 \to V_0\) and the identity on \(V_1\). Note that \(u^2 = u\). However, these natural choices do not assemble to a forgetful functor from mixed supermanifolds to supermanifolds:

**Proposition 7.1.** There is no functor \(F : SM^\mu \to SM\) such that the following two conditions hold:

(a) \(F(A(V, V_\mathbb{R}, V_\mathbb{C})) = A(u(V, V_\mathbb{R}, V_\mathbb{C}))\) and \(F(\epsilon_{(V, V_\mathbb{R}, V_\mathbb{C})}) = \text{id}_{A(u(V, V_\mathbb{R}, V_\mathbb{C}))}\).

(b) \(F|_{SM} = \text{id}_{SM}\).

**Proof.** Assume that such a functor \(F\) existed. Consider \(A(\mathbb{C})\) and \(A(\mathbb{R}^2)\) with their standard monoid structure. Then we had a commutative square
\[
A(\mathbb{R}^2) \times A(\mathbb{R}^2) \xrightarrow{\mu_{\mathbb{R}^2}} A(\mathbb{R}^2) \\
A(\mathbb{C}) \times A(\mathbb{C}) \xrightarrow{\mu_{\mathbb{C}}} A(\mathbb{C})
\]
and it would follow from the second assumption that \(F\) would take the monoid \(A(\mathbb{C})\) to the monoid \(A(\mathbb{R}^2)\).

Consider the supermanifold \(M = A(\mathbb{R}^2 \times \mathbb{C}^{0|2})\) with coordinates \((x, y, \partial_1, \partial_2)\) and consider the two maps \(\varphi_z, \varphi_{\partial_1, \partial_2} : M \to A(\mathbb{C})\) given by \(\varphi_z^\sharp(z) = x + iy\) and \(\varphi_{\partial_1, \partial_2}^\sharp(z) = \partial_1 \partial_2\), respectively. Then we have \(\varphi_z = \varepsilon_{\mathbb{C}} \circ (x, y)\), so that we would obtain \(F(\varphi_z) = F((x, y)) = (x, y)\).

For an arbitrary smooth function \(\alpha : \mathbb{R}^2 \to \mathbb{C}\) we now define \(f_\alpha : M \to M\) by
\[
f_\alpha^1(x) = x + \alpha \partial_1 \partial_2, \\
f_\alpha^2(y) = y + (1 - \alpha) \partial_1 \partial_2, \\
f_\alpha^i(\partial_i) = \partial_i.
\]
Then \(\varphi_z \circ f_\alpha = \varphi_z + \varphi_{\partial_1, \partial_2}\). However, on one hand
\[
F(\varphi_z \circ f_\alpha)^\sharp = F(f_\alpha)^\sharp \circ F(\varphi_z)^\sharp \\
= f_\alpha^\sharp \circ F(\varphi_z)^\sharp \\
= f_\alpha^\sharp \circ (x, y) \\
= (x, y) + (\alpha \varphi_{\partial_1, \partial_2}, (1 - \alpha) \varphi_{\partial_1, \partial_2})
\]
and on the other hand
\[
F(\varphi_z + \varphi_{\partial_1, \partial_2}) = F(\varphi_z) + F(\varphi_{\partial_1, \partial_2}) \\
= (x, y) + F(\varphi_{\partial_1, \partial_2}).
\]
This would imply \(F(\varphi_{\partial_1, \partial_2}) = (\alpha \varphi_{\partial_1, \partial_2}, (1 - \alpha) \varphi_{\partial_1, \partial_2})\) for arbitrary \(\alpha : \mathbb{R}^2 \to \mathbb{C}\), which is absurd. \(\square\)

Similarly, one proves the following related proposition.
Proposition 7.2. The natural transformation $\epsilon_C: \Lambda(\mathbb{R}^2) \to \Lambda(\mathbb{C})$ between functors on SM admits no section.

Proof. Assume that such a natural transformation $F$ existed. We use the notation from the previous proof. We consider again $M = \Lambda(\mathbb{R}^2 \times \mathbb{C}^{0|2})$ and the two maps $\varphi_z$ and $\varphi_{\theta_1\theta_2}$. Then $F(\varphi_z) = (x + n, y + in)$ for a nilpotent function $n$ on $M$. Defining $f_\alpha$ as previously, we have $\varphi_z \circ f_\alpha = \varphi_z + \varphi_{\theta_1\theta_2}$, and so $F(\varphi_z \circ f_\alpha)$ would be independent of $\alpha$. However, we would have

$$F(\varphi_z \circ f_\alpha) = f_\alpha^2(x + n, y + in) = (x + \alpha \theta_1 \theta_2 + n, y + (-i)(1 - \alpha)\theta_1 \theta_2 + in),$$

a contradiction. \qed

7.2. Flows of even real vector fields on mixed supermanifolds. We outline the construction of flows of vector fields on mixed supermanifolds. In this setting, only even real vector fields can be integrated. We show that they have a unique maximal flow.

Let $M$ be a mixed supermanifold and let $X$ be an even real vector field. Let $\mathcal{V} \subseteq \mathbb{R}^1 \times M$ be open such that $\{0\} \times M \subseteq \mathcal{V}$. A morphism

$$\Theta^X : \mathbb{R}^1 \times M \ni \mathcal{V} \longrightarrow M$$

is called a flow of $X$ if

(a) $\partial_t \circ \Theta^X = \Theta^X \circ \partial_t X$, and
(b) $\Theta^X|_{\{0\} \times M} = \text{id}_M$.

Following [10], an open subspace $\{0\} \times M \subseteq \mathcal{V} \subseteq \mathbb{R}^1 \times M$ such that, for all $m \in M_0$, $\mathcal{V} \cap (\mathbb{R}^1 \times \{m\})$ is an interval and a flow exists on $\mathcal{V}$ is called a flow domain.

First we show that a real vector field on a mixed manifold has a unique maximal flow. Let $M$ be a mixed manifold and $M^{sm}$ its underlying smooth manifold which comes with a map $i: M^{sm} \to M$. Then $(i^*\mathcal{T}_M), (i^*\mathcal{T}_M^C) \subseteq \mathbb{C} \otimes \mathcal{T}_{M^{sm}}$ and we have the following exact sequence:

$$0 \longrightarrow i^*\mathcal{T}_{M,C} \oplus i^*\mathcal{T}_{M,C} \longrightarrow \mathbb{C} \otimes \mathcal{T}_{M^{sm}} \longrightarrow i^*\mathcal{T}_M / i^*\mathcal{T}_M^C \longrightarrow 0. \quad (7.3)$$

In fact, locally in a neighborhood of the form $(\mathbb{C}^{n_1+n_2}, \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}, \mathbb{C}^{n_2})$, $i^*\mathcal{T}_{M,C}$ and $(i^*\mathcal{T}_{M,C})$ are spanned as $\mathcal{O}_{M^{sm}}$-modules by $\partial_{z_i}$ and $\partial_{\bar{z}_i}$ $(i \in \{n_1 + 1, \ldots, n_1 + n_2\})$, respectively.

Then we have the following observation.

Lemma 7.4. For any real vector field $X$ on $M$, there is a unique real vector field $Y$ on $M^{sm}$ such that $(\mathbb{C} \otimes Y)|_{\mathcal{O}_M} = X$.

Proof. Consider two such real vector fields $Y_1$ and $Y_2$ on $M^{sm}$. Locally on the model space defined by $(\mathbb{C}^{n_1+n_2}, \mathbb{R}^{n_1} \times \mathbb{C}^{n_2}, \mathbb{C}^{n_2})$, with coordinates $\{x = (x_1, \ldots, x_{n_1}), z = (z_1, \ldots, z_{n_2})\}$, we have

$$X = \sum_i f_i(x)\partial_{x_i} + \sum_j g_j(x, z)\partial_{z_j},$$
domains with flows $\Theta$. This follows from the above considerations by taking the union of all flow domains.

**Definition 7.9.** An even real vector field is called complete if its maximal flow is the desired vector field.

**Theorem 7.8.** Let $X$ be an even real vector field on the mixed supermanifold $M$ with underlying real vector field $X$ on $M_0$. Then there exists a unique flow map $\Theta^X : \mathcal{V} \to M$ where $\mathcal{V}$ is the maximal flow domain for $X$. Moreover, $(\Theta^X)_0$ is the maximal flow of $X$.

**Proof.** This follows from the above considerations by taking the union of all flow domains.

**Definition 7.9.** An even real vector field is called complete if its maximal flow domain $\mathcal{V}$ equals $\mathbb{R} \times M$.

The following basic properties can be proved as in the classical case.
Proposition 7.10. Suppose $X$ is an even real vector field and $Y$ is an arbitrary vector field on $M$.

(a) $\mathcal{L}_X Y := \frac{\partial}{\partial t}|_{t=0}(\Theta_t^X)^\ast \circ Y \circ (\Theta_{-t}^X)^\ast = [X,Y]$.

(b) If $[X,Y] = 0$, then $\Theta^{X^\sharp}$ and $Y$ commute.

Proof. See for instance [6, Corollary 3.7].
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Institut für theoretische Physik, Universität zu Köln, Zülpicher Str. 77, 50937 Köln, Germany
E-mail address: dosterma@math.uni-koeln.de