Collisionless and Decentralized Formation Control for Strings

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Abstract

A decentralized feedback controller for multi-agent systems, inspired by vehicle platooning, is proposed. The closed-loop resulting from the decentralized control action has three distinctive features: the generation of collision-free trajectories, flocking of the system towards a consensus state in velocity, and asymptotic convergence to a prescribed pattern of distances between agents. For each feature, a rigorous dynamical analysis is provided, yielding a characterization of the set of parameters and initial configurations where collision avoidance, flocking, and pattern formation is guaranteed. Numerical tests assess the theoretical results presented.

Keywords: Multi-agent systems, decentralized control, nonlinear control, consensus and pattern formation control, autonomous vehicles.

1. Introduction

Multi-agent systems (MAS) have proven to be a versatile framework for studying diverse scalability problems in Science and Engineering, such as dynamic networks [23], autonomous vehicles [4], collective behaviour of humans or animals [29, 30], and many others. Mathematically, MAS are often modelled as large-scale dynamical systems where each agent can be considered as a subset of states, updated via interaction forces such as attraction, repulsion, alignment, etc., [18, 14] or through the optimization of a pay-off function in a control/game framework [21, 20].

In this work, we approach the study of MAS from a control viewpoint. We study a class of sparsely interconnected agents in one dimension, interacting through nonlinear couplings and a decentralized control law. The elementary building block of our approach is the celebrated Cucker-Smale model for consensus dynamics [14], which corresponds to a MAS where each agent is endowed with second-order nonlinear dynamics for velocity alignment, and where the influence of neighbouring agents decays with distance. The Cucker-Smale model and variants can represent the physical motion of agents on the real line, inspired by autonomous vehicle formations in platooning with a nearest-neighbour interaction scheme [28, 31]. The couplings to be studied are motivated by the more general setting of the Cucker-Smale dynamics in arbitrary dimension. The original Cucker-Smale dynamics considers full network connectivity in the agent interactions, generating flocking dynamics capable of exhibiting emergent consensus behavior, that is, agents that may reach a common velocity in steady state, without the action of external forces. This framework has been extended in several directions, being most notable the inclusion of forcing terms and control [13, 17, 7, 19], optimal control [3, 6, 2] formation control [24, 25, 11], leadership [15] and collision-avoidance capabilities [12, 1, 5, 10], the latter being an increasingly sought after property of formation control schemes for autonomous fleets of vehicles.

The main contribution of this paper is to propose a nearest neighbour interaction of agents on the real line which exhibits emergent consensus and collision avoidance under the action of a simple decentralized control law. The proposed feedback enforces a desired inter-agent distance in steady state and is inspired by formation control in
vehicular platoons [28, 26]. Such a model has the potential to achieve the goals of platooning applications, that is, the coordinated, scalable, secure and efficient travel of automated vehicles [16], while at the same time offering flocking, pattern formation, and collision avoidance features from the non-linear dynamics. Moreover, we provide collision-avoidance guarantees that, from a safety viewpoint, do not rely on traditional concepts in platooning such as string stability. For the derivation of collision-avoidance results we consider the framework developed in [8, 9, 22], which uses singular interaction kernels that blow-up whenever two agents are located at the same position. We modify this setup to also consider agents with a volume by including a threshold inter-agent distance where the kernel becomes singular.

Our main result is a rigorous characterization of flocking, collision-avoidance, and platooning behaviour for the proposed nonlinear model, in terms of the initial configuration of the system, interaction and control law parameters. Cucker-Smale models usually consider full state measurements of every agent, available to every agent, which reflects a free interaction between them giving rise to self-organization properties (see for example the recent work [32] where the full interaction Cucker-Smale model is characterised on the real line). We show that these properties are still present when a highly sparse nearest neighbour interaction is considered in the 1D case.

The remainder of the paper is structured as follows. In Section 2 we present the proposed model to be studied, a Cucker-Smale model with nearest neighbor singular interactions and a decentralized feedback control. We also define here a total energy functional $E(x, v)$ for the model and show that it is not increasing in time. In Section 3 we give the results ensuring the collision-avoidance behaviour of the controlled system, and Section 4 includes a flocking estimate showing that the velocity alignment between individuals and the inter-agent distances are uniformly bounded in time. In Section 5 we present the main formation control result. We provide different numerical experiments that illustrate our theoretical results in Section 6 along with some concluding remarks.

2. Problem description and preliminary results

We consider a string of $N$ agents each characterized by a pair $(x_i(t), v_i(t))$ in $\mathbb{R}^2$ evolving in time $t$ through second-order dynamics of the form

$$\begin{align*}
\frac{dx_i(t)}{dt} &= v_i(t), \quad i = 1, \ldots, N, \quad t > 0, \\
\frac{dv_i(t)}{dt} &= I_i(t) + u_i(t),
\end{align*}$$

subject to initial data

$$(x_i(0), v_i(0)) = (x^0_i, v^0_i) \quad \text{for} \quad i = 1, \ldots, N. \quad (2)$$

Here, the term $I_i$ describes nonlocal velocity interactions between individuals which are weighted by a singular

Figure 1: Diagram at a particular instant of three consecutive agents for the considered MAS. The singular interactions, providing barriers to the agents, are indicated with the radii $\delta_i$ of the semicircles.
Lemma 1. Let \( u_t \) be a smooth solution to the system \( (1) \) on the time interval \([0, T]\). Then:

(i) the mean velocity is conserved in time:
\[ \nu_c(t) := \frac{1}{N} \sum_{i=1}^{N} \nu_i(t) = \nu_c(0). \]

(ii) the total energy is not increasing in time:
\[ \frac{d}{dt} E(x(t), \nu(t)) + D(x(t), \nu(t)) = 0. \]

Proof. (i) A straightforward computation yields
\[ \sum_{i=1}^{N} I_i = \sum_{i=1}^{N} u_i = 0. \]

communication function \( \psi(r) : \mathbb{R}^+ \to \mathbb{R} \),
\[
\begin{align*}
I_1 &= \psi(|x_2 - x_1 - \delta_1|)(v_2 - v_1), \\
I_k &= \psi(|x_k - x_{k-1} - \delta_{k-1}|)(v_{k-1} - v_k) + \psi(|x_{k+1} - x_k - \delta_k|)(v_{k+1} - v_k), \\
I_N &= \psi(|x_N - x_{N-1} - \delta_{N-1}|)(v_{N-1} - v_N),
\end{align*}
\]

where the parameters \( \delta_i > 0 \) are fixed. This interaction term induces consensus in the velocities of the agents while preventing collisions. The second term \( u_t \) serves as a decentralized feedback control depending on weight function \( \phi(r) : \mathbb{R}^+ \to \mathbb{R} \) and is given by
\[
\begin{align*}
u_1 &= -\phi(|x_1 - x_2 - z_1|^2)(x_1 - x_2 - z_1), \\
u_k &= \phi(|x_{k-1} - x_k - z_{k-1}|^2)(x_{k-1} - x_k - z_{k-1}) - \phi(|x_k - x_{k+1} - z_k|^2)(x_k - x_{k+1} - z_k), \\
u_N &= \phi(|x_{N-1} - x_N - z_{N-1}|^2)(x_{N-1} - x_N - z_{N-1}).
\end{align*}
\]

This feedback also depends on a vector of relative distances \( z := (z_1, \ldots, z_{N-1}) \in \mathbb{R}^{N-1} \). The objective of this control law is to induce the formation of a string pattern characterized by \( z \). The complete setting is depicted in Figure 1. For the sake of clarity, the weight functions \( \psi \) and \( \phi \) are chosen as
\[
\psi(r) = \frac{1}{r^\alpha}, \quad \phi(r) = \frac{1}{(1 + r)^\beta}, \quad \alpha, \beta > 0.
\]

However, the results we will state in the forthcoming sections can extended to the case in which \( \phi \) is bounded and Lipschitz continuous. We will establish conditions under which the string \( (1) \) converge to a consensus state with a prescribed formation while avoiding collisions between agents. For this, we begin by stating an a priori energy estimate which will be significantly used for estimating consensus emergence. We first define the total energy functional
\[ E(x, v) := E_1(v) + E_2(x) = \frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 + \frac{1}{2} \sum_{i=2}^{N} \int_0^{|x_{i-1} - x_i - z_{i-1}|^2} \phi(r) \, dr, \]
and its dissipation rate
\[ D(x, v) := \sum_{i=2}^{N} \psi(|x_i - x_{i-1} - \delta_{i-1}|)(v_i - v_{i-1})^2. \]
This implies
\[ \frac{d}{dt} v_i(t) = 0, \quad \text{i.e.} \quad v_i(t) = v_i(0) \tag{4} \]
for \( t \in [0, T] \).

(ii) Let us first begin with the estimate for the kinetic energy:
\[
\frac{1}{4N} \frac{d}{dt} \sum_{i,j=1}^{N} (v_i - v_j)^2 = \frac{1}{2N} \sum_{i,j=1}^{N} (v_i - v_j) \left( \frac{dv_j}{dt} - \frac{dv_i}{dt} \right) = \frac{1}{2N} \sum_{i,j=1}^{N} \left( \frac{dv_i}{dt} + v_j \frac{dv_j}{dt} - v_i \frac{dv_i}{dt} \right) 
\]
\[ = \sum_{i=1}^{N} v_i \frac{dv_i}{dt} = \sum_{i=1}^{N} v_i (I_i + u_i), \tag{5} \]
where we used (4). Here, we use the same idea of [11, Lemma 3.1] to obtain
\[ \sum_{i=1}^{N} v_i u_i = - \frac{1}{2N} \sum_{i=2}^{N} \int_{0}^{x_{i-1} - x_{N-1}} \phi(r) dr. \tag{6} \]

On the other hand, we estimate the term with \( I_i \) as
\[
\sum_{i=2}^{N-1} v_i I_i = \sum_{i=2}^{N-1} v_i (\psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_i) + \psi(|x_i - x_{i+1}| - \delta_i)(v_{i+1} - v_i)) 
\]
\[ = \sum_{i=2}^{N-1} v_i (\psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_i)) + \sum_{i=3}^{N-1} v_{i-1} (\psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_{i-1})) 
\]
\[ = v_2 \psi(|x_2 - x_1| - \delta_1)(v_1 - v_2) - \sum_{i=3}^{N-1} \psi(|x_i - x_{i-1}| - \delta_i)(v_{i-1} - v_{i-2}) + v_{N-1} \psi(|x_N - x_{N-1}| - \delta_{N-1})(v_N - v_{N-1}) 
\]
\[ = - v_2 I_1 - v_{N-1} I_N - \sum_{i=2}^{N-1} \psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_{i-1})^2. \]

This asserts
\[ \sum_{i=1}^{N} v_i I_i = v_1 I_1 + \sum_{i=2}^{N-1} v_i I_i + v_N I_N = -(v_2 - v_1) I_1 - \sum_{i=2}^{N-1} \psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_{i-1})^2 - (v_{N-1} - v_N) I_N \]
\[ = - \sum_{i=2}^{N} \psi(|x_i - x_{i-1}| - \delta_i)(v_i - v_{i-1})^2. \tag{7} \]

Combining the estimates (5), (7), and (6), we conclude the desired result. \( \square \)

3. Global and local existence of solutions

In this section, we show that system (1), under certain parametric and initial conditions, exhibits a non-collisional behaviour, which together with Cauchy-Lipschitz theory, subsequently provides global-in-time existence and uniqueness of smooth solutions to the system (1)-(2). We present two results regarding existence of non-collisional trajectories. The first theorem requires a prescribed ordering for the initial datum \( x_0 \) and the power \( \alpha \) in (3) to be \( \alpha \geq 1 \). The second result requires \( \alpha \geq 2 \), but the initial ordering assumption is removed from the string. A third result characterizes a pathological case, where a 2-agent string blows up in finite time.

**Theorem 1.** Suppose that \( \alpha \geq 1 \) and the initial configuration \( x_0 \) satisfies \( x_{i+1}^0 > x_i^0 + \delta_i \) for all \( i = 1, \ldots, N-1 \). Then, there exists the global unique smooth solution to the system (1)-(2) satisfying \( x_{i+1}(t) > x_i(t) + \delta_i \) for all \( i = 1, \ldots, N-1 \) and all \( t > 0 \).
Proof. We first notice that $\psi(x_{i+1} - x_i - \delta_i)$ is regular as long as $x_{i+1} > x_i + \delta_i$, and thus there exists a unique smooth solution to the system (1). For a fixed $T \in (0, \infty)$, let us assume that there is $t_* \in (0, T]$ where the smoothness of solutions breaks down for the first time, i.e., there is an index $\ell$ such that

$$x_{\ell+1}(t) - x_{\ell}(t) > \delta_{\ell} \text{ for } t \in (0, t_*) \quad \text{and} \quad \lim_{t \to t_*^-} x_{\ell+1}(t) - x_{\ell}(t) = \delta_{\ell}. \tag{8}$$

We denote by $[\ell]$ the set of such indices and set $i_* = \min[\ell]$. We first claim $i_* \geq 2$. If $i_* = 1$, then for $t \in (0, t_*)$ we estimate

$$\frac{d}{dt} \Psi(x_2 - x_1 - \delta_1) = \psi(x_2 - x_1 - \delta_1)(v_2 - v_1) = I_1 = \frac{d}{dt}(v_1 - v_i) - u_1,$$

where we used (4) and $\Psi$ is the primitive of $\psi$, i.e.,

$$\Psi(r) = \begin{cases} \ln(r) & \text{for } \alpha = 1, \\ \frac{1}{1-\alpha} r^{1-\alpha} & \text{for } \alpha > 1 \end{cases}$$

From this, we deduce that

$$\Psi(x_2(t) - x_1(t) - \delta_1) = \Psi(x_2^0 - x_1^0 - \delta_1) = (v_1(t) - v_i(t)) - (v_1^0 - v_i(0)) - \int_0^t u_1(s) \, ds \tag{9}$$

for $t \in [0, t_*)$. On the other hand, by Hölder’s inequality we find

$$|v_1(t) - v_i(t)| \leq \frac{1}{N} \sum_{k=1}^N |v_k(t) - v_k(t)| \leq \frac{1}{N} \sum_{k=1}^N (v_1(t) - v_k(t))^2$$

and

$$|u_1(s)| \leq \|\phi\|_{L^1} |x_2 - x_1 - z| \leq \|\phi\|_{L^1} \left(|z| + |x_2^0 - x_1^0| + \int_0^s |v_2(\tau) - v_1(\tau)| \, d\tau\right).$$

These observations together with the energy estimate in Lemma 1 imply that the right hand side of (9) is bounded on the time interval $(0, t_*)$, and subsequently $t \mapsto \Psi(x_2(t) - x_1(t) - \delta_1)$ is bounded on the time interval $[0, t_*]$. This is a contradiction to (8) and thus the claim follows. By the definition of $i_*$, there exists a constant $c_{i_*} > 0$ such that

$$x_{i_*}(t) - x_{i_*-1}(t) - \delta_{i_*-1} > c_{i_*} \tag{10}$$

for all $t \in (0, t_*)$. Similarly as above, we now estimate

$$\frac{d}{dt} \Psi(x_{i_*+1} - x_{i_*} - \delta_{i_*}) = \psi(x_{i_*+1} - x_{i_*} - \delta_{i_*})(v_{i_*+1} - v_{i_*})$$

$$= I_{i_*} + \psi(x_{i_*} - x_{i_*-1} - \delta_{i_*-1})(v_{i_*} - v_{i_*-1})$$

$$= \frac{d}{dt}(v_{i_*} - v_{i_*}) + \psi(x_{i_*} - x_{i_*-1} - \delta_{i_*-1})(v_{i_*} - v_{i_*-1}) - u_{i_*},$$

and thus

$$\Psi(x_{i_*+1}(t) - x_{i_*}(t) - \delta_{i_*}) = \Psi(x_{i_*+1}^0 - x_{i_*}^0 - \delta_{i_*}) + (v_{i_*}(t) - v_{i_*}(t)) - (v_{i_*}^0 - v_{i_*}(0))$$

$$+ \int_0^t \psi(x_{i_*}(s) - x_{i_*-1}(s) - \delta_{i_*-1})(v_{i_*}(s) - v_{i_*-1}(s)) \, ds - \int_0^t u_{i_*}(s) \, ds \tag{11}$$

for $t \in (0, t_*)$. Here the boundedness of the second and fourth terms can be obtained by using almost the same argument as above. We also use (10) to obtain

$$|\psi(x_{i_*}(s) - x_{i_*-1}(s) - \delta_{i_*-1})(v_{i_*}(s) - v_{i_*-1}(s))| \leq c_{i_*}^2 \alpha 4 NE_1(v(t)) \leq c_{i_*}^2 \alpha 4 NE(\lambda^0, v^0).$$
Hence, the right hand side of \( |x^0_i - x^0_{i+1}| \) is bounded on the time interval \( [0, t) \), so is the left hand side. This leads to a contradiction and thus the unique smooth solution can be actually exists up to an arbitrary finite time \( T > 0 \). This completes the proof.

We next present the second existence theorem whose proof is based on the energy estimate. For this, we first introduce a function \( L^{\alpha-2}_\alpha \) with \( \alpha \geq 2 \) given by

\[
L^{\alpha-2}_\alpha(t) = \begin{cases} 
\sum_{i=1}^{N-1} \|x_i(t) - x_{i+1}(t)\|^{-(\alpha-2)} & \text{for } \alpha > 2, \\
\sum_{i=1}^{N-1} \log(\|x_i(t) - x_{i+1}(t)\| - \delta_i) & \text{for } \alpha = 2.
\end{cases}
\]

Note that \( |L^{\alpha-2}_\alpha(t)| < \infty \) for \( t \in [0, T] \) for some \( \alpha \geq 2 \) if and only if the distances between agents \( x_i(t) \) and \( x_{i+1}(t) \) are strictly greater than \( \delta_i \) for all \( i = 1, \ldots, N-1 \) and \( t \in [0, T] \).

**Theorem 2.** Suppose that \( \alpha \geq 2 \) and that the initial configuration \( x_0 \) satisfies

\[
|x_i^0 - x_{i+1}^0| > \delta_i
\]

for all \( i = 1, \ldots, N-1 \). Then, there exists the global unique smooth solution to the system (1)-(2) where the distances between agents satisfy \( |x_i(t) - x_{i+1}(t)| > \delta_i \) for all \( i = 1, \ldots, N-1 \) and all \( t > 0 \).

**Proof.** We first introduce the maximal life-span \( T_0 = T(x^0) \) of the initial datum \( x^0 \):

\[
T_0 := \sup \{ s \in \mathbb{R}_+ : \exists \text{ solution } (x(t), v(t)) \text{ for the system (1)-(2) in a time-interval } [0, s] \}
\]

By the assumption, \( T_0 > 0 \). We then claim \( T_0 = \infty \). First, note that it follows from Lemma 1 that

\[
\sum_{i=1}^{N-1} \int_0^s \frac{(v_{i+1}(s) - v_i(s))^2}{(x_i(s) - x_{i+1}(s))^p} ds \leq E(x^0, v^0).
\]

(12)

Let us prove the above claim by dealing with two cases separately: \( \alpha = 2 \) and \( \alpha > 2 \).

(i) \( \alpha = 2 \): A straightforward computation gives

\[
\frac{\partial}{\partial t} \sum_{i=1}^{N-1} \log(\|x_i(t) - x_{i+1}(t)\| - \delta_i) = \sum_{i=1}^{N-1} \frac{(x_i(t) - x_j(t)) \cdot (v_i(t) - v_j(t))}{[x_i(t) - x_j(t)](\|x_i(t) - x_{i+1}(t)\| - \delta_i)} \leq \sum_{i=1}^{N-1} \frac{|v_i(t) - v_j(t)|}{\|x_i(t) - x_{i+1}(t)\| - \delta_i}
\]

for \( t \in [0, T_0) \). This yields

\[
\sum_{i=1}^{N-1} \log(\|x_i(t) - x_{i+1}(t)\| - \delta_i) \leq \sum_{i=1}^{N-1} \log(\|x_i^0 - x_{i+1}^0\| - \delta_i) + \sum_{i=1}^{N-1} \int_0^s \frac{|v_i(s) - v_{i+1}(s)|}{\|x_i(s) - x_{i+1}(s)\| - \delta_i} ds.
\]

On the other hand, by using the Hölder inequality and (12), we estimate

\[
\sum_{i=1}^{N-1} \int_0^s \frac{|v_i(s) - v_{i+1}(s)|}{\|x_i(s) - x_{i+1}(s)\| - \delta_i} ds \leq \sqrt{N-1} \left( \int_0^s \frac{|v_i(s) - v_{i+1}(s)|^2}{(\|x_i(s) - x_{i+1}(s)\| - \delta_i)^2} ds \right)^{1/2} \leq \sqrt{(N-1)E(x^0, v^0)}.
\]

Thus, we obtain

\[
\sum_{i=1}^{N-1} \log(\|x_i(t) - x_{i+1}(t)\| - \delta_i) \leq \sum_{i=1}^{N-1} \log(\|x_i^0 - x_{i+1}^0\| - \delta_i) + \sqrt{(N-1)E(x^0, v^0)},
\]

(13)
for \( t \in [0, T_0) \).

(ii) \( \alpha > 2 \): Taking the time derivative to \( L_0^{n-2} \), we get (omitting the time arguments)

\[
\frac{dL_0^{n-2}}{dt} = -(\alpha - 2) \sum_{i=1}^{N-1} \left( \frac{|x_i - x_{i+1}| - \delta_i)^{n-2}}{|x_i - x_{i+1}|} \right) \left( x_i - x_{i+1}, v_i - v_{i+1} \right)
\]

\[
\leq C \sum_{i=1}^{N-1} \left( \frac{|x_i - x_{i+1}| - \delta_i)^{n-2}}{|x_i - x_{i+1}|} \right) \left( v_i - v_{i+1} \right)
\]

\[
\leq C \sum_{i=1}^{N-1} \frac{1}{(x_i - x_{i+1})} + C \sum_{i=1}^{N-1} \left( |v_i - v_{i+1}| \right)^2
\]

\[
= CL_0^{n-2}(t) + C \sum_{i=1}^{N-1} \left( |v_i - v_{i+1}| \right)^2
\]

for \( t \in [0, T_0) \), where we used Young’s inequality. Applying Gronwall’s inequality to the above, we have

\[
L_0^{n-2}(t) \leq L_0^{n-2}(0)e^{Ct} + Ce^{Ct} \sum_{i=1}^{N-1} \int_0^t \frac{|v_i(s) - v_{i+1}(s)|^2}{(x_i(s) - x_{i+1}(s))} \frac{d|x_i(s) - x_{i+1}(s)|}{|x_i(s) - x_{i+1}(s)|} ds \leq e^{Ct} \left( L_0^{n-2}(0) + CE(0, v^0) \right),
\]

for \( t \in [0, T_0) \), due to (12). Since the right hand sides of (13) and (14) are uniformly bounded in the time interval \([0, T_0)\), the life-span \( T_0 \) should be infinity, i.e., \( T_0 = \infty \). This completes the proof. \( \square \)

**Remark 1.** In Theorem 2 it is crucially used the fact that the system is posed in one dimension. However, Theorem 2 can also deal with higher dimensional problems, see [59].

We conclude this section with a negative result characterizing a pathological configuration with 2 agents where the system blows up in finite time.

**Theorem 3.** Let \( \alpha \in (0, 1) \) and \( N = 2 \). Furthermore, we assume that \( \delta_1, z_1 \), and the initial data \( ((x_i^0, v_i^0))_{i=1}^2 \) satisfy \( \delta_1 + z_1 \geq 0, x_2^0 > x_1^0 + \delta_1 \), and

\[
v_1^0 - v_2^0 = \frac{2}{1 - \alpha} (x_1^0 - x_1^0 - \delta_1)^{1-\alpha}.
\]

Then, the smoothness of solutions to the system (11)-(12) breaks down in finite time.

**Proof.** For the proof, it suffices to show that there exists a finite time \( t_* \) such that \( x_1(t_*) + \delta_1 = x_2(t_*) \). For notational simplicity, we set \( x : = x_2 - x_1 \) and \( v : = v_2 - v_1 \). Then we easily find that \( x \) and \( v \) satisfy

\[
\begin{align*}
\frac{dx(t)}{dt} &= v(t), \\
\frac{dv(t)}{dt} &= -2(I_1(t) + u_1(t)) = -2\psi(x(t) - \delta_1)v - 2\phi((x(t) + z_1)^2(x(t) + z_1)).
\end{align*}
\]

Note that the smooth solutions exist as long as \( x(t) > \delta_1 \), and this and the assumption \( \delta_1 + z_1 \geq 0 \) imply \( x(t) + z_1 \geq 0 \). Since \( \phi \geq 0 \), this implies that

\[
\frac{dv(t)}{dt} \leq -2\psi(x(t) - \delta_1)v = -2 \frac{d}{dt} \Psi(x(t) - \delta_1).
\]

Here \( \Psi \) is the primitive of \( \psi \), i.e.

\[
\Psi(r) = \frac{1}{1 - \alpha} r^{1-\alpha}.
\]

We then solve the above differential inequality to get

\[
\frac{d(x(t) - \delta_1)}{dt} = v(t) \leq -2\Psi(x(t) - \delta_1) = -2 \frac{2}{1 - \alpha} (x(t) - \delta_1)^{1-\alpha}
\]

(16)
due to (15). We notice that the above differential inequality is sub-linear, and thus there exists \( t_* \) such that 
\[ x(t_*) - \delta_1 = 0. \] Indeed, we obtain from (15) that
\[ (x(t) - \delta_1)^2 \leq (x^0 - \delta_1)^2 - \frac{2\alpha}{1 - \alpha} t. \]

Hence we have
\[ t_* \leq \frac{(x^0 - \delta_1)^2}{2\alpha} (1 - \alpha), \]
thus completing the proof.

4. Time-asymptotic behavior

Having characterized the well-posedness of the trajectories of the system (1), we now turn our attention to the study of flocking emergence within the controlled string. In a flocking configuration, all agents travel with the same constant velocity, and as a direct consequence the distance between agents remain constant. We provide a rigorous asymptotic flocking estimate for the system (1).

**Theorem 4.** Suppose that either assumptions of Theorems 1 or 2 hold. Furthermore, we assume the asymptotic flocking estimate for the system (1).

**Proof.** From Theorems 1 or 2 it follows existence and uniqueness of a smooth solution globally in time. (Uniform-in-time boundedness): It follows from the energy estimate in Lemma [1] that
\[ E_2(x(t)) \leq E(x^0, v^0), \]
i.e.
\[ \frac{1}{2N} \sum_{i,j=1}^{N} |v^0_i - v^0_j|^2 + \frac{1}{2N} \sum_{i,j=1}^{N} \int_{0}^{\infty} \phi(r) dr \]
\[ \leq \frac{1}{2N} \sum_{i,j=1}^{N} |v^0_i - v^0_j|^2 \text{ for } t \geq 0. \] (18)

On the other hand, under our main assumptions, we can find some constant \( \rho > 0 \) such that
\[ \frac{1}{2N} \sum_{i,j=1}^{N} |v^0_i - v^0_j|^2 + \frac{1}{2N} \sum_{i,j=1}^{N} \int_{0}^{\infty} \phi(r) dr \leq \int_{0}^{\rho^2} \phi(r) dr. \]

This together with (18) yields
\[ 0 \leq \frac{1}{2N} \sum_{i,j=1}^{N} |v^0_i - v^0_j|^2 \leq \int_{0}^{\rho^2} \phi(r) dr \]
for all \( i = 2, \ldots, N \). This implies that
\[ |x_{i-1}(t) - x_i(t) - z_{i-1}| \leq \rho \text{ for } i = 2, \ldots, N. \] (19)

For any \( i < j \), by telescoping and the triangle inequality we estimate
\[ |x_i - x_j| = \left| \sum_{\ell=i}^{j-1} (x_{\ell} - x_{\ell+1}) \right| \leq \sum_{\ell=i}^{j-1} |x_{\ell} - x_{\ell+1}| \leq \sum_{\ell=i}^{j-1} |x_{\ell} - x_{\ell+1} - z_{\ell}| + \sum_{\ell=i}^{j-1} |z_{\ell}|. \]
and thus
\[ |x_i - x_j| \leq |j - i| \rho + \sum_{t=i}^{j-1} |z_t| \leq (N - 1) \rho + \sum_{i=1}^{N-1} |z_i| < \infty, \]
given the boundedness of distances between agents at all times.

**Velocity alignment behavior**: From the bound above, we find
\[ |x_i - x_{i-1} - \delta_{i-1} \leq |x_i - x_{i-1} - z_{i-1} - \delta_{i-1} \leq \rho + |z_{i-1}| + \delta_{i-1} \leq \rho + \max_{i=1,\ldots,N-1} (|z_i| + \delta_i). \]

Since \( \psi \) is monotonically decreasing, we obtain
\[ \psi_m := \min_{2 \leq i \leq N} \psi(|x_i - x_{i-1} - \delta_{i-1}) \geq \psi \left( \rho + \max_{i=1,\ldots,N-1} (|z_i| + \delta_i) \right) > 0. \]
This implies that the dissipation rate \( D \) is bounded from below by
\[ D(x(t), v(t)) = \sum_{i=2}^{N} \psi(|x_i - x_{i-1} - \delta_{i-1}|)(v_{i-1} - v_i)^2 \geq \psi_m \sum_{i=2}^{N} (v_{i-1} - v_i)^2. \]
Then, by Lemma we get
\[ \sum_{i=2}^{N} \int_0^\infty (v_{i-1}(t) - v_i(t))^2 \, dt < \infty, \]
and subsequently, this leads to
\[ \int_0^\infty E_i(v(t)) \, dt = \frac{1}{4N} \sum_{i=1}^{N} \int_0^\infty |v_i(t) - v_j(t)|^2 \, dt < \infty, \]
Indeed, by telescoping, for any \( i < j \)
\[ |v_i - v_j| \leq \sum_{\ell=i}^{j-1} |v_{\ell} - v_{\ell+1}| \leq \sqrt{|i - j|} \left( \sum_{\ell=i}^{j-1} |v_{\ell} - v_{\ell+1}|^2 \right), \]
and thus
\[ \sum_{i,j=1}^{N} |v_i - v_j|^2 \leq c_N \sum_{i,j=1}^{N} |v_{i-1} - v_j|^2, \quad \text{where} \quad c_N := \sum_{i,j=1}^{N} |i - j|. \]
Moreover, we also find
\[ \left| \sum_{i=1}^{N} v_i(t) \right| \leq \left| \sum_{i=1}^{N} \phi(x_i - x_i - z_{i-1})((x_{i-1} - x_i - z_{i-1}, v_{i-1} - v_i) \right| \leq \rho \sum_{i=2}^{N} |v_{i-1} - v_i| \leq C \sqrt{E(x^0, v^0)}, \]
where \( C \) is independent of \( t \) and we used
\[ \max_{1 \leq i, j \leq N} |v_i(t) - v_j(t)| \leq \sqrt{\sum_{i,j=1}^{N} |v_i(t) - v_j(t)|^2} \leq 2 \sqrt{N} \left| E(x^0, v^0) \right|. \]
Furthermore, note that
\[ \frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 = \int_0^\infty (-D(x(s), v(s))) \, ds + \sum_{i=1}^{N} \int_0^\infty v_i(s)u_i(s) \, ds + \frac{1}{4N} \sum_{i,j=1}^{N} |v_i^0 - v_j^0|^2. \]
The dissipation rate $D$ is integrable and thus the first term on the right side of the above equality is absolutely continuous. Regarding the second term, its time-derivative is uniformly bounded in time, see \((21)\), from where it follows that it is Lipschitz continuous. This implies that the $E_1(v(t))$ is the sum of an absolutely continuous function and a Lipschitz continuous function. Thus, we obtain that $E_1(v(t))$ is uniformly continuous. Since $E_1(v(t))$ is also integrable, $E_1(v(t)) \to 0$ as $t \to \infty$. This completes the proof.

**Remark 2.** If $\beta \leq 1$, then $\phi$ is not integrable, thus the left hand side of \((17)\) becomes infinity. This implies that the assumption \((17)\) automatically holds. On the other hand, if $\beta > 1$, we obtain
\[
\int_0^\infty \phi(r) \, dr = \int_0^\infty \frac{1}{(1 + r)^\beta} \, dr = \frac{1}{\beta - 1}.
\]
and thus \((17)\) can be rewritten as
\[
E(x_0, v_0) < \frac{1}{2(\beta - 1)}.
\]

From these equivalences, it becomes evident that the fulfilment of the flocking condition depends only on two parameters of the model, namely, the number of agents $N$ in the string and the control interaction constant $\beta > 1$, which regulates the strength of the control action. The constant $a$ does not play a role on the condition. Having fixed a number of agents and $\beta$, flocking solely depends on the cohesiveness of the initial configuration.

**Remark 3.** If we define $\Phi$ by the primitive of $\phi$, then it is clear that $r \mapsto \Phi(r)$ is strictly increasing, and thus the constant $\rho$ appeared in \((19)\) can be expressed by
\[
\rho = \sqrt{\Phi^{-1}\left(\frac{1}{2N} \sum_{i,j=1}^N [v_i^0 - v_j^0]^2 + \Phi[(x_{i-1}^0 - x_i^0 - z_{i-1})^2]\right)}.
\]

## 5. Exponential emergence of pattern formation and velocity alignment

In this section, we conclude our characterization of the string trajectories by studying the exponential emergence of pattern formation and velocity alignment behavior under additional assumption on the solutions. We first provide an auxiliary result, a modification of Young’s inequality, which can be proved by a similar argument as in \[11\] Lemma 6.1. We thus omit its proof here.

**Lemma 2.** Let $a_1, \ldots, a_{N-1}$ be a set of vectors in $\mathbb{R}^d$ and $b_1, \ldots, b_{N-1}$ be a set of positive scalars. Then
\[
- \sum_{i=1}^{N-1} b_i |a_i|^2 + \sum_{i=1}^{N-2} b_i (a_i, a_{i+1}) \leq -\varepsilon_0 \sum_{i=1}^{N-1} b_i |a_i|^2,
\]
where $\varepsilon_0 \in (0, 1)$ is a sufficiently small number.

We now state our main result, which provides non-collisional behavior, flocking, and an exponential decay estimate towards the string configuration encoded in the relative distance vector $z$.

**Proposition 1.** Suppose that the assumptions of Theorem 4 are satisfied. Furthermore, we assume that
\[
\inf_{t \geq 0} \min_{i \leq N-1} (|x_i(t) - x_{i+1}(t)| - \delta_i) > 0.
\]

Then, we have
\[
\max_{i=1,\ldots,N} |x_i(t) - x_{i-1}(t) - z_{i-1}| + \max_{i,j=1,\ldots,N} |v_i(t) - v_j(t)| \to 0
\]
extponentially fast as $t \to \infty$. 

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Proof. We first notice that the energy estimate in Lemma 1 only provides a dissipation rate for the velocity. In order to have a complete exponential decay estimate, it is required to obtain the dissipation rate associated to the positions. For this, we consider the following quantity:

$$\sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(v_i - v_{i+1}).$$

Note that the total energy is bounded from below and above by

$$\frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 + \phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 \leq E(x,v) \leq \frac{1}{4N} \sum_{i,j=1}^{N} |v_i - v_j|^2 + \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2,$$

where we used

$$\phi_m := \min_{s \in [0,\gamma]} \phi(s) \leq \phi(r) \leq 1.$$ 

This shows that a modified total energy $E_\gamma$, defined as

$$E_\gamma(x,v) := \gamma E(x,v) + \sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(v_i - v_{i+1}),$$

has similar upper and lower bound estimates as the one for $E(x,v)$ when $\gamma > 0$ large enough. Indeed, for $\gamma > \sqrt{2N/\phi_m}$, we readily find

$$E_\gamma(x,v) \geq \frac{\gamma \phi_m}{2} \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 + \left( \frac{\gamma}{4N} - \frac{1}{2\gamma \phi_m} \right) \sum_{i,j=1}^{N} |v_i - v_j|^2$$

$$\geq c_\gamma \left( \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 + \sum_{i,j=1}^{N} |v_i - v_j|^2 \right),$$

where $c_\gamma > 0$ is given by

$$c_\gamma := \min \left( \frac{\gamma \phi_m}{2}, \frac{\gamma}{4N} - \frac{1}{2\gamma \phi_m} \right).$$

The upper bound on $E_\gamma$ can be easily obtained. On the other hand, it follows from (1) that

$$\frac{d}{dt} \sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(v_i - v_{i+1}) = \sum_{i=1}^{N-1} (v_i - v_{i+1})^2 + \sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(I_i - I_{i+1}) + \sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(u_i - u_{i+1}).$$

(26)

Since

$$|I_i| \leq \psi_M |v_2 - v_1|, \quad |I_N| \leq \psi_M |v_{N-1} - v_N|, \quad \text{and} \quad |I_i| \leq \psi_M (|v_{i-1} - v_i| + |v_i - v_{i+1}|)$$

for $i = 2, \ldots, N-1$, we easily find

$$\sum_{i=1}^{N-1} |I_i - I_{i+1}|^2 \leq 2 \left( |I_1|^2 + 2 \sum_{i=2}^{N-1} |I_i|^2 + |I_N|^2 \right) \leq 16 \psi_M^2 \sum_{i=1}^{N-1} |v_i - v_{i+1}|^2,$$

where $\psi_M := \sup_{r \in [0,\infty)} \psi(r) < \infty$, which can be defined by the assumption (24). Thus, by using Young’s inequality we have

$$\sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(I_i - I_{i+1}) \leq \epsilon_1 \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 + \frac{4\psi_M^2}{\epsilon_1} \sum_{i=1}^{N-1} |v_i - v_{i+1}|^2,$$

(27)
where \( \epsilon_1 \) will be determined later. Regarding the term with \( u_i \), we estimate
\[
\sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(u_i - u_{i+1}) = -2 \sum_{i=1}^{N-1} \phi(|x_i - x_{i+1} - z_i|^2)|x_i - x_{i+1} - z_i|^2 \\
+ 2 \sum_{i=1}^{N-2} \phi(|x_i - x_{i+1} - z_i|^2)(x_i - x_{i+1} - z_i)(x_{i+1} - x_{i+2} - z_{i+1}).
\]

We then use Lemma 2 with \( b_i = 2\phi(|x_i - x_{i+1} - z_i|^2) \) and \( a_i = x_i - x_{i+1} - z_i \) to get
\[
\sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(u_i - u_{i+1}) \leq -2\epsilon_0 \sum_{i=1}^{N-1} \phi(|x_i - x_{i+1} - z_i|^2)|x_i - x_{i+1} - z_i|^2 \\
\leq -2\epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2,
\]
where \( \epsilon_0 \) is given in Lemma 2. This together with (26), (27), and choosing \( \epsilon_1 = \epsilon_0\phi_m \) implies
\[
\frac{d}{dt} \sum_{i=1}^{N-1} (x_i - x_{i+1} - z_i)(v_i - v_{i+1}) \leq -\epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 + \frac{4\psi^3_0}{\epsilon_0\phi_m} \sum_{i=1}^{N-1} |v_i - v_{i+1}|^2.
\]
Thus the modified total energy \( E_\gamma \) satisfies
\[
\frac{d}{dt} E_\gamma(x, v) \leq -\gamma\psi_m - \frac{4\psi^3_0}{\epsilon_0\phi_m} \sum_{i=1}^{N-1} |v_i - v_{i+1}|^2 - \epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2.
\]
By taking \( \gamma > \max(\{4\psi^3_0/(\epsilon_0\phi_m), \sqrt{2N/\phi_m}\}) \) and using (26) and (25), we further estimate
\[
\frac{d}{dt} E_\gamma(x, v) \leq -\frac{1}{c\gamma} \left( \gamma\psi_m - \frac{4\psi^3_0}{\epsilon_0\phi_m} \sum_{i=1}^{N} |v_i - v_j|^2 - \epsilon_0\phi_m \sum_{i=1}^{N-1} |x_i - x_{i+1} - z_i|^2 \right) \\
\leq -\frac{1}{c\gamma} \min \left\{ \frac{1}{cN} \left( \gamma\psi_m - \frac{4\psi^3_0}{\epsilon_0\phi_m}, \epsilon_0\phi_m \right), \phi_m \right\} E_\gamma(x, v).
\]
Applying the Grönwall’s lemma to the above gives the exponential decay of the modified total energy \( E_\gamma \). Moreover, the relation (25) concludes the desired result.

**Remark 4.** The a priori assumption (24) imposes some constraints on \( z_i \). For instance, if we fix the order for the initial positions as \( x^0_{i+1} < x^0_i \) for \( i = 1, \ldots, N-1 \), then by Theorem 2 we have \( x_i(t) + \delta_i < x_{i+1}(t) \) for all \( t \geq 0 \). This implies that in order to have the time-asymptotic pattern formation \( z_i \) and \( \delta_i \) should satisfy \( z_i > \delta_i \) for all \( i = 1, \ldots, N-1 \).

On the other hand, if we assume
\[
z_{i-1} > \rho + \delta_{i-1}, \quad i = 2, \ldots, N,
\]
where \( \rho \) is given as in (23), then
\[
|x_i(t) - x_{i-1}(t)| - \delta_{i-1} \geq z_{i-1} - \rho - \delta_{i-1} > 0
\]
for all \( t \geq 0 \) and all \( i = 2, \ldots, N \). Indeed, it follows from (19) that
\[
|x_i(t) - x_{i-1}(t)| = |x_i(t) - x_{i-1}(t) - z_{i-1} + z_{i} - z_{i-1}| \geq z_{i-1} - \rho,
\]
thus subtracting \( \delta_{i-1} \) from the both sides gives the desired result.
6. Numerical examples

In the following, we further elucidate the applicability of our results through two numerical experiments illustrating string flocking to a pattern formation, as well as energy evolution.

6.1. Regular collision-less behavior of the interconnected system

For simplicity in the visualization, we will first consider a collection of \( N = 5 \) agents. We will assume that the agents are in the desired ordering, that is, the final configuration of the agents does not require a collision to occur. For the model parameters we select \( \alpha = 2.1, \beta = 0.8 \) and \( \delta = 2 \). We will consider two cases for the agents initial conditions: 1) the agents are initially at rest and located at non-collided positions, 2) the agents have different initial velocities with \( v_c(0) = 0 \) and are close to each other but not collided. In both cases the desired inter-agent spacings are given by \( z_i = \delta + 2 \) (note that \( \delta \) is added to avoid configurations that are collided in steady state). In such a scenario, the agents should reposition themselves reaching the consensus velocity of zero and the desired pattern. Figure 2 illustrates both cases. As predicted by Theorems 2 and 4, and given that \( \alpha > 2 \) and \( \beta < 1 \) with no initial collisions, we have that the agents do not collide as time progresses, even when they are initially almost touching. Moreover, the agents reach the desired inter-agent spacings.

Now, for the same parameters as before, Figure 3 illustrates the behaviour of the multi-agent system when \( v_c(0) = -0.2 \). We can observe that the statement of \( 1 \) is satisfied over the time evolution of the dynamics, that is, the total energy of the system is non-increasing and the dissipation rate is entirely determined by the interaction term (see Figure 3 bottom-left. Some agents are very close to the interaction limit determined by \( \delta = 2 \), at \( t = 0[\text{sec}] \), which is highlighted by Figure 3 top-right. The agents 2 and 3 then approach each other for a short period of time and around \( t = 10[\text{sec}] \) all the agents begin to spread out to reach the desired formation, with an average velocity of \( -0.2 \) in steady state.

6.2. Flocking and formation acquisition for \( \beta > 1 \)

We now consider \( N = 10 \) agents with \( \alpha = 2.2 \) and \( \delta = 0 \). For the same initial conditions with \( v_c(0) = 0 \), we consider two cases: 1) \( \beta = 4.1 \) and \( \beta = 1.025 \), which are presented in Figs. 4 and 5 respectively. It can be seen that the parameter \( \beta \) influences the value of the initial energies, as noted earlier. However, the kinetic energy is positive and the same in both cases, that is, the agents are not initially moving with the same velocity. For \( \beta = 4.1 \), condition (17) of Theorem 4 is not satisfied, as the value \( 0.5/(4.1 - 1) \approx 0.1613 \) is less than the initial energy of the system. We can see in Fig. 4 top-right that the system does not achieve flocking nor the desired formation. Some errors \( x_i(t) - x_{i+1}(t) - z_i \)
diverge and we can appreciate clustering. On the other hand, for $\beta = 1.025$, condition (17) is met, and we have that the system does achieve the desired formation with all the errors $|x_i(t) - x_{i+1}(t) - z| \rightarrow 0$ as the system evolves. Note that the uncontrolled system is plotted at the bottom-right corner in both cases for comparison, that is, with $u_i(t) = 0$ for all $t, i$.

Conclusions

We have presented a control system for platooning composed by a string of agents interacting under nonlinear singular dynamics and a decentralized feedback law. The resulting closed-loop exhibits important features for platooning control, namely, collision-avoidance, velocity flocking, and asymptotic pattern formation. The derivation of rigorous energy estimates allow the characterization of conditions under which the aforementioned features are guaranteed. Energy estimates are governed by: the number of agents in the string, the strength of the control interaction term expressed through the parameter $\beta$ in (3), and the cohesiveness of the initial configuration. In particular, the dependence with respect to the number of agents is a relevant topic of interest for future research. Although our results are asymptotic, we have observed the transient behaviour of the control system and it exhibits similarities to linear-time invariant platooning, namely, slow transients as the number of agents increases. The energy analysis we presented can be extended to study mean field dynamics arising when $N \rightarrow \infty$ and the system is characterized by an agent density function [27]. Although the applicability of the mean field framework seems inadequate from a safety viewpoint as collision-avoidance is an eminently microscopic phenomenon, it can be a powerful mathematical method to further understand the large-scale structure of the control system.

Acknowledgements

YPC has been supported by NRF grant (No. 2017R1C1B2012918) and Yonsei University Research Fund of 2020-22-0505. DK was supported by a public grant as part of the Investissements d’avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH, and by the UK Engineering and Physical Sciences Research Council (EPSRC) grants EP/V04771X/1, EP/T024429/1, and EP/V025899/1.
Figure 4: 10 particles on the line with zero average velocity when $\beta = 4.1$. Left-Top: Positions over time of the particles when the control is used; Left-Bottom: Energy decomposition and flocking condition for $\beta > 1$; Right-Top: Errors from the desired formation $x_i - x_{i+1} - z_i$; Right-Bottom: Positions over time of the particles when the control is not used. Flocking does not occur, although collisions are still avoided.

Figure 5: 10 particles on the line with zero average velocity when $\beta = 1.025$. Left-Top: Positions over time of the particles when the control is used; Left-Bottom: Energy decomposition and flocking condition for $\beta > 1$; Right-Top: Errors from the desired formation $x_i - x_{i+1} - z_i$; Right-Bottom: Positions over time of the particles when the control is not used. As predicted by Theorem 4, given that the initial total energy satisfies (22), no collisions occur and the desired formation is achieved in steady state.
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