SPANNING TREES IN A CLAW-FREE GRAPH WHOSE STEMS HAVE AT MOST $k$ BRANCH VERTECIES

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Abstract. Let $T$ be a tree, a vertex of degree one and a vertex of degree at least three is called a leaf and a branch vertex, respectively. The set of leaves of $T$ is denoted by $Leaf(T)$. The subtree $T - Leaf(T)$ of $T$ is called the stem of $T$ and denoted by $Stem(T)$. In this paper, we give two sufficient conditions for a connected claw-free graph to have a spanning tree whose stem has a bounded number of branch vertices, and those conditions are best possible. As corollaries of main results we also give some conditions to show that a connected claw-free graph has a spanning tree whose stem is a spider.

1. Introduction

In this paper, we always consider simple graphs, which have neither loops nor multiple edges. For a graph $G$, let $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of $G$, respectively. We write $|G|$ for the order of $G$ (i.e., $|G| = |V(G)|$). For a vertex $v$ of $G$, we denote by $\text{deg}_G(v)$ the degree of $v$ in $G$. For two vertices $u$ and $v$ of $G$, the distance between $u$ and $v$ in $G$ is denoted by $d_G(u, v)$.

For an integer $l \geq 2$, let $\alpha^l(G)$ denote the number defined by

$$\alpha^l(G) = \max\{|S| : S \subset V(G), d_G(x, y) \geq l \text{ for all distinct vertices } x, y \in S\}.$$ 

For an integer $k \geq 2$, we define

$$\sigma^l_k(G) = \min \left\{ \sum_{a \in S} \text{deg}_G(a) : S \subset V(G), |S| = k, d_G(x, y) \geq l \text{ for all distinct vertices } x, y \in S \right\}.$$ 

For convenience, we define $\sigma^l_k = +\infty$ if $\alpha^l(G) < k$. We note that, $\alpha^2(G)$ is often written $\alpha(G)$, which is the independent number of $G$, and $\sigma^2_k(G)$ is often written $\sigma_k(G)$, which is the minimum degree sum of $k$ independent vertices.

For a tree $T$, a vertex of degree at least three is called a branch vertex, and a tree having at most one branch vertex is called a spider. Many researchers have investigated the independent number conditions and the degree sum conditions for the existence of a spanning tree with bounded number of branch vertices or it is a spider (see [2], [3], [1] and [9] for examples). A vertex of $T$, which has degree one, is often called a leaf of $T$, and the set of leaves of $T$ is denoted by $Leaf(T)$. Many results were studied on the independent

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number conditions and the degree sum conditions for the existence of a spanning tree with bounded number of leaves (also see [1] and [9] for examples). Moreover, many analogue results for the claw-free graph are studied (see [4] and [8] for examples).

The subtree $T - \text{Leaf}(T)$ of $T$ is called the stem of $T$ and is denoted by $\text{Stem}(T)$. Recently, M. Kano and his collaborations gave an innovation by studying a spanning tree in a graph with specified stem. We introduce here some of them. Their first result is the following.

**Theorem 1.1 ([5] Kano, Tsugaki and Yan).** Let $k \geq 2$ be an integer, and $G$ be a connected graph. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then $G$ has a spanning tree whose stem has maximum degree at most $k$.

After that, the sufficient conditions for a connected graph to have a spanning tree whose stem has a few number of leaves were introduced as the following theorems.

**Theorem 1.2 ([10] Tsugaki and Zhang).** Let $G$ be a connected graph and $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then $G$ have a spanning tree whose stem has at most $k$ leaves.

**Theorem 1.3 ([6] Kano and Yan).** Let $G$ be a connected graph and $k \geq 2$ be an integer. If $\sigma_{k+1}(G) \geq |G| - k - 1$, then $G$ have a spanning tree whose stem has at most $k$ leaves.

**Theorem 1.4 ([6] Kano and Yan).** Let $G$ be a connected claw-free graph and $k \geq 2$ be an integer. If $\sigma_{k+1}(G) \geq |G| - 2k - 1$, then $G$ have a spanning tree whose stem has at most $k$ leaves.

The following theorem gives two sufficient conditions for a connected graph to have a spanning tree whose stem has a bounded number of branch vertices.

**Theorem 1.5 ([11] Yan).** Let $G$ be a connected graph and $k$ be a non-negative integer. If one of the following conditions holds, then $G$ have a spanning tree whose stem has at most $k$ branch vertices.

(i) $\alpha_4(G) \leq k + 2$.
(ii) $\sigma_{k+3}^3(G) \geq |G| - 2k - 3$.

When $k = 1$, Theorem 1.5 gives a previous result as the following.

**Theorem 1.6 ([11] Kano and Yan).** Let $G$ be a connected graph. If $\sigma_4^3(G) \geq |G| - 5$, then $G$ have a spanning tree whose stem is a spider.

**Remark 1.7.** We remark that all conditions which were mentioned above are best possible.

We are very interested in this topic. So, we would like to study the spanning tree in a graph with specified stem. The purpose of this paper is to give some sufficient conditions
for a connected claw-free graph to have a spanning tree whose stem has at most $k$ branch vertices. In particular, our main theorems are the followings.

**Theorem 1.8.** Let $G$ be a connected claw-free graph and $k$ be a non-negative integer. If $\sigma_{k+3}^4(G) \geq |G| - 2k - 5$, then $G$ has a spanning tree whose stem has at most $k$ branch vertices.

**Theorem 1.9.** Let $G$ be a connected claw-free graph and $k$ be a non-negative integer. If $\sigma_{k+3}^5(G) \geq |G| - 3k - 6$, then $G$ has a spanning tree whose stem has at most $k$ branch vertices.

Applying the main theorems with $k = 1$, we give the following.

**Theorem 1.10.** Let $G$ be a connected claw-free graph. If $\sigma_{k+3}^4(G) \geq |G| - 7$ or $\sigma_{k+3}^5(G) \geq |G| - 9$, then $G$ has a spanning tree whose stem is a spider.

2. Sharpness of Theorem 1.8 and Theorem 1.9

We first show that the condition of Theorem 1.8 is best possible. Let $m, k \geq 1$ be integers, and let $D_1, \ldots, D_{k+3}$ be disjoint copies of $K_m$ and $D = K_{k+3}$ with distinct vertices $z_1, \ldots, z_{k+3}$. Let $v_1, \ldots, v_{k+3}$ be vertices not contained in $D \cup D_1 \cup \cdots \cup D_{k+3}$. Join $z_i, v_i$ to all vertices of $D_i (1 \leq i \leq k+3)$ by edges, respectively. Let $G$ denote the resulting graph. Then $G$ is a connected claw-free graph. Setting $V = \{v_1, \ldots, v_{k+3}\}$. We are easy to see that for each set $S$ such that $S \subset V(G)$, $|S| = k+3$ and $d_G(x, y) \geq 4$ for all distinct vertices $x, y \in S$, then $S = V$ or $S = (V \setminus \{v_j\}) \cup \{y_j\}$, where $y_j \in D_j$. Then we can compute that $\sum_{a \in S} \deg_G(a) = |G| - 2k - 6$ for the first case and $\sum_{a \in S} \deg_G(a) = |G| - 2k - 5$ for the last case. So we get $\sigma_{k+3}^4(G) = |G| - 2k - 6$. On the other hand, since for any spanning tree $T$ of $G$, then there are at least $k+1$ points in the set $\{z_1, z_2, \ldots, z_{k+3}\}$ must be the branch vertices of $\text{Stem}(T)$. So $G$ has no spanning tree who stem has at most $k$ branch vertices. Therefore, the condition of Theorem 1.8 is best possible.

Now we also consider graph $G$ above with $m = 1$. So we may see that $\sigma_{k+3}^5(G) = 2 = |G| - 3k - 7$. Moreover, for every spanning tree $T$ of $G$, then there are at least $k+1$ points in the set $\{z_1, z_2, \ldots, z_{k+3}\}$ must be the branch vertices of $\text{Stem}(T)$. This shows that the condition of Theorem 1.9 is best possible.

3. Proofs of Theorem 1.8 and Theorem 1.9

Beside of giving some refinements of the proofs in [6] and [11], we will use some analogue arguments of them to prove Theorem 1.8 and Theorem 1.9.

Firstly, we recall the following useful lemma.
Lemma 3.1. Let $T$ be a tree, and let $X$ be the set of vertices of degree at least 3. Then the number of leaves in $T$ is counted as follow:

$$|\text{Leaf}(T)| = \sum_{x \in X} (\deg_T(x) - 2) + 2.$$ 

Assume that $G$ satisfies the condition in Theorem 1.8 and does not have spanning tree whose stem has at most $k$ branch vertices. We choose a tree $T$ whose stem has $k$ branch vertices in $G$ so that

(C1) $|T|$ is as large as possible.
(C2) $|\text{Leaf}(\text{Stem}(T))|$ is as small as possible subject to (C1).
(C3) $|\text{Stem}(T)|$ is as small as possible subject to (C1), (C2).

By the choice (C1), we have the following claim.

Claim 3.2. For every $v \in V(G) - V(T), N_G(v) \subseteq \text{Leaf}(T) \cup (V(G) - V(T))$.

$\text{Stem}(T)$ has $k$ branch vertices. Denote the number of leaves of $\text{Stem}(T)$ by $l$. By Lemma 3.1, $\text{Leaf}(\text{Stem}(T)) = l \geq k + 2$. Let $x_1, x_2, \ldots, x_l$ be the leaves of $\text{Stem}(T)$. Since $T$ is not a spanning tree of $G$, there exist two vertices $v \in V(G) - V(T)$ and $u \in \text{Leaf}(T)$ which are adjacent in $G$. Thus, we have the following claim.

Claim 3.3. If $u$ is adjacent with a vertex $w$ of $\text{Stem}(T)$ then $\deg_{\text{Stem}(T)}(w) = 2$.

Proof. Suppose that $u$ is adjacent with a vertex $w$ but $\deg_{\text{Stem}(T)}(w) \neq 2$.
If $\deg_{\text{Stem}(T)}(w) = 1$ then $z$ is a leaf of $\text{stem}(T)$. We consider a new tree $T_1 = T + uv$ then $\text{stem}(T_1)$ have $k$ branch vertices and $|T_1| > |T|$. This contradicts the condition (C1).
If $\deg_{\text{Stem}(T)}(w) \geq 3$ than $w$ is a branch vertex of $\text{stem}(T)$. We also consider a new tree $T_1 = T + uv$ then $\text{stem}(T_1)$ also have $k$ branch vertices and $|T_1| > |T|$. This contradicts the condition (C1). Claim 3.3 is proved. $\square$

Now, we use the properties of the claw-free graph to give the following claim.

Claim 3.4. Set $M = \{ w \in \text{Stem}(T) | \deg_{\text{Stem}(T)}(w) = 2 \}$. Then $|M| \geq 3$.

Proof. Otherwise, by Claim 3.3 we have two following cases.

Case 1. $|M| = 1$. We call $w \in M$ then $u$ is adjacent with $w$ by Claim 3.3. Let $y, t$ be two adjacent vertices of $w$ in $\text{Stem}(T)$. Here $y$ and $t$ are branch vertices, leaves or one leaf and one branch vertex. By definition of the claw-free graph, then either $uy$ or $ut$ or $yt$ is an edge in $G$. We consider a new tree $T_2$:

$$T_2 = \begin{cases} T - wu + uy + uv & \text{if } uy \in E(G), \\ T - tw + ut + uv & \text{if } ut \in E(G), \\ T - tw + ty + uv & \text{if } yt \in E(G). \end{cases}$$
Then, by $y$ is a branch vertex or a leaf of $T$, the resulting tree $T_2$ of $G$ is a tree whose stem has $k$ branch vertices and the order of the resulting tree is greater than $|T|$, which contradicts the condition (C1).

Case 2. $|M| = 2$. We call $w_1, w_2 \in M$. Without loss of generality we may assume that $u$ is adjacent with $w_1$ by Claim [3.3]. If $w_1$ is not adjacent with $w_2$ in $Stem(T)$ then by using the same arguments in case 1 we get a contradiction. On the other hand, if $w_1$ is adjacent with $w_2$ then let $y$ be another adjacent vertex of $w_1$ in $Stem(T)$. By definition of the claw-free graph, then either $uy$ or $uw_2$ or $yw_2$ is an edge in $G$. We consider a new tree

$$T_2 = \begin{cases} 
T - yw_1 + uy + uv & \text{if } uy \in E(G), \\
T - w_1 w_2 + uw_2 + uv & \text{if } uw_2 \in E(G), \\
T - w_1 w_2 + yw_2 + uv & \text{if } yw_2 \in E(G). 
\end{cases}$$

Then, by $y$ is a branch vertex or a leaf of $T$, resulting tree $T_2$ of $G$ is a tree whose stem has $k$ branch vertices and the order of the resulting tree is greater than $|T|$, which contradicts the condition (C1).

Claim [3.4] is proved.  

**Claim 3.5.** Leaf($Stem(T)$) is an independent set of $G$.

Assume that there exists two vertices $x_i$ and $x_j$ of Leaf($Stem(T)$) which are adjacent in $G$. Then add $x_i$ and $x_j$ to $T$. The resulting subgraph of $G$ includes a unique cycle, which contains an edge $e_1$ of $Stem(T)$ incident with a branch vertex. By removing the edge $e_1$, we obtain the resulting tree $T_3$ such that $Stem(T_3)$ has at most $k$ branch vertices, $|T_3| = |T|$ and $|Leaf(Stem(T_3))| \leq |Leaf(Stem(T))| - 1$. If $Stem(T)$ has $k - 1$ branch vertices, then add $uv$ to $T_3$; we obtain a tree whose stem has at most $k$ branch vertices and the order of the tree is greater than $|T|$, which contradicts the condition (C1). Otherwise, $T_3$ contradicts the condition (C2). Hence Leaf($Stem(T)$) is an independent set of $G$.

**Claim 3.6.** For every $x_i (1 \leq i \leq l)$, there exists a vertex $y_i \in Leaf(T)$ adjacent to $x_i$ and $N_{G}(y_i) \subset Leaf(T) \cup \{x_i\}$.

**Proof.** It is easy to see that for each leaf $x \in Leaf(Stem(T))$, there exists at least a vertex $y$ in $Leaf(T)$ adjacent to $x$. Now, for every leaf $y$ of $T$ adjacent to a leaf of $Stem(T)$ in $T$, $y$ is not adjacent to any vertex of $V(G) - V(T)$. Indeed, otherwise we can add an edge joining $y$ to a vertex of $V(G) - V(T)$ to $T$ then the resulting tree contradicts the condition (C1).

Suppose that for some $1 \leq i \leq l$, each leave $y_{i,j}$ of $T$ adjacent to $x_i$, is also adjacent to a vertex $z_{i,j} \in (Stem(T) - \{x_i\})$. Then for every leaf $y_{i,j}$ adjacent to $x_i$ in $T$, remove the edge $y_{i,j}x_i$ from $T$ and add the edge $y_{i,j}z_{i,j}$. Denote the resulting tree of $G$ by $T_4$. Then $T_4$ is a tree which has at most $k$ branch vertices. If $x_i$ is adjacent with a branch of $Stem(T)$,
then $Leaf(Stem(T_4)) = Leaf(Stem(T)) - \{x_i\}$, which contradicts the condition (C2). If $x_i$ is not adjacent with a branch of $Stem(T)$, then $Stem(T_4) = Stem(T) - \{x_i\}$, which contradicts the condition (C3). Therefore, the claim holds. □

**Claim 3.7.** For any two distinct vertices $y, z \in \{v, y_1, y_2, \ldots, y_l\}$, $d_G (y, z) \geq 4$.

**Proof.** First, we show that $d_G (v, y_i) \geq 4$ for every $1 \leq i \leq l$. Let $P_i$ be the shortest path connecting $v$ and $y_i$ in $G$. If all the vertices of $P_i$ between $v$ and $y_i$ are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i\}$. Then add $P_i$ to $T$ (if $P_i$ passes through $x_i$, we just add the segment of $P_i$ between $v$ and $x_i$) and remove the edges of $T$ joining $V(P_i \cap Leaf(T))$ to $V(Stem(T))$ except the edge $y_i x_i$. Then resulting tree of $G$ is a tree whose stem has at most $k$ branch vertices and the order of the resulting tree is greater than $|T|$, which contradicts the condition (C1). Then there exists a vertex $s \in V(P_i)$ with $s \in V(Stem(T)) - \{x_i\}$. Hence, by Claim 3.2 and Claim 3.6, $d_G (v, y_i) = 2$ and $d_G (s, y_i) = 2$. Therefore, $d_G (y_i, y_j) = 4$.

Next, we show that $d_G (y_i, y_j) \geq 4$ for all $1 \leq i < j \leq l$. Let $P_{ij}$ be the shortest path connecting $y_i$ and $y_j$ in $G$. We note that if $P_{ij}$ passes through $x_i$ (or $x_j$), then $y_i x_i \in E(P_{ij})$ (or $y_j x_j \in E(P_{ij})$), respectively. If all vertices of $P_{ij}$ between $y_i$ and $y_j$ are contained in $Leaf(T) \cup (V(G) - V(T)) \cup \{x_i, x_j\}$. Then add $P_{ij}$ to $T$ to remove the edges of $T$ joining $V(P_{ij} \cap Leaf(T))$ to $V(Stem(T))$ except the edges $y_i x_i$ and $y_j x_j$. Then the resulting graph of $G$ includes a unique circle, which contains an edge $e_2$ of $Stem(T)$ incident with a branch vertex. By removing the edge $e_2$, we obtain a tree $T_5$ whose stem has at most $k$ branch vertices. If $P_{ij}$ contains a vertex of $V(G) - V(T)$, then the order of $T_5$ is greater than $T$, which contradicts the condition (C1). Otherwise, $|T_5| = |T|$ and $|Leaf(Stem(T_5))| = |Leaf(Stem(T))| - 1$. This contradicts the condition (C2). Hence, $P_{ij}$ passes through a vertex $s \in Stem(T) - \{x_i, x_j\}$. Then there exists a vertex $t \in V(P_{ij})$ with $t \in V(Stem(T)) - \{x_i, x_j\}$. Hence, by Claim 3.6, $d_G (y_i, s) \geq 2$ and $d_G (s, y_j) \geq 2$. Therefore, $d_G (y_i, y_j) = d_G (y_i, s) + d_G (s, y_j) \geq 4$ for $1 \leq i < j \leq k$. □

As a corollary of Claim 3.7, we have the following claim.

**Claim 3.8.** $N_G (v) \cap N_G (y_i) = \emptyset$ and $N_G (y_i) \cap N_G (y_j) = \emptyset$ for $1 \leq i \neq j \leq l$.

Denote $Y = \{y_1, y_2, \ldots, y_l\}$. Since Claim 3.2,3.8, we have

$$N_G (v) \subseteq (V(G) - V(T) - \{v\}) \cup (N_G (v) \cap (Leaf(T) - Y)),$$

$$\bigcup_{i=1}^{k+2} N_G (y_i) \subseteq (Leaf(T) - Y - N_G (v)) \cup \{x_1, \ldots, x_{k+2}\}.$$
Using Claim 3.4, we have $|\text{Stem}(T)| \geq l + k + 3$.
Hence by setting $h = |N_G(v) \cap (\text{Leaf}(T) - Y)|$, we have

$$
\deg_G(v) + \sum_{i=1}^{k+2} \deg_G(y_i) \leq |G| - |T| - 1 + h + |\text{Leaf}(T)| - h - l + k + 2
= |G| - |\text{Stem}(T)| - l + k + 1
\leq |G| - 2l - 2 \leq |G| - 2k - 6 \text{ (by } l \geq k + 2).$$

Which contradicts the condition in Theorem 1.8.

Theorem 1.8 is proved.

Now, since the properties of the claw-free graph we have the following claim.

**Claim 3.9.** $d_G(v, y_i) \geq 5$ for all $1 \leq i \leq l$.

**Proof.** Since Claim 3.7 we have $d_G(v, y_i) \geq 4$. Assume that $d_G(v, y_i) = 4$. Let $P_i$ be the shortest path connecting $v$ and $y_i$ in $G$. Then by the proof of Claim 3.7 there exists a vertex $s \in V(P_i)$ with $s \in V(\text{Stem}(T)) - \{x_i\}$. By Claim 3.2 and 3.6 we have $d_G(v, s) \geq 2$ and $d_G(s, y_i) \geq 2$. Therefore, if $d_G(v, y_i) = d_G(v, s) + d_G(s, y_i) = 4$ then $d_G(v, y_i) = d_G(s, y_i) = 2$ and, moreover, $s$ must be in $M$ by the proof of Claim 3.3. Let $vu's$ and $y_iz_is$ be two paths in $P_i$. So $u' \in \text{Leaf}(T) \cup (V(G) - V(T))$.

If $u' \in V(G) - V(T)$ then we consider a tree $T_* = T + su'$ then $\text{Stem}(T_*)$ has $k$ branch vertices and $|T_*| > |T|$. This contradicts (C1).

If $u' \in \text{Leaf}(T)$, remove the edge of $T$ joining $u'$ and add $u's$ to $T$. Then resulting tree $T_*$ of $G$ whose stem has $k$ branch vertices, $|T_*| = |T|$, $|\text{Leaf}(\text{Stem}(T_*))| = |\text{Leaf}(\text{Stem}(T))|$ and $|\text{Stem}(T_*)| = |\text{Stem}(T)|$. Using the same arguments in the proofs of Claim 3.3 and 3.4 we can show that $\deg_{\text{Stem}(T_*)}(s) = 2$ and if $s$ is adjacent with two vertices $y, t$ in $\text{Stem}(T_*)$ then $\deg_{\text{Stem}(T_*)}(y) = \deg_{\text{Stem}(T_*)}(t) = 2$. Now, by definition of the claw-free graph $G$, then either $u't$ or $u'y$ or $ty$ is an edge in $G$. Let $p$ be a vertex of $\text{Stem}(T)$ such that $z_ip$ is in $E(T)$. We consider a new tree

$$
T_6 = \begin{cases} 
T_* - ys + u'y + u'v & \text{if } u'y \in E(G), \\
T_* - ts + u't + u'v & \text{if } u't \in E(G), \\
T_* - ts - ys - z_ip + ty + sz_i + z_iy_i + su' + u'v & \text{if } ty \in E(G), 
\end{cases}
$$

Then resulting tree $T_6$ of $G$ is a tree whose stem has $k$ branch vertices and the order of $T_6$ is greater than $|T_*| = |T|$, which contradicts the condition (C1). So $d_G(v, y_i) \geq 5$.

Fix an index $i$. Since Claim 3.2, 3.9 we have

$$
N_G(v) \subseteq (V(G) - V(T) - \{v\}) \cup (N_G(v) \cap (\text{Leaf}(T) - Y)),
N_G(y_i) \subseteq (\text{Leaf}(T) - Y - N_G(v)) \cup \{x_i\}
$$
Using Claim 3.4, we have $|\text{Stem}(T)| \geq l + k + 3$.
Hence, we have
\[
\deg_G(v) + \deg_G(y_i) \leq |G| - |T| - 1 + |\text{Leaf}(T)| - l + 1
\]
\[
= |G| - |\text{Stem}(T)| - l
\]
\[
\leq |G| - 2l - k - 3 \leq |G| - 3k - 7 \text{ (by } l \geq k + 2\text{)}.
\]
Which contradicts the condition in Theorem 1.9.

Theorem 1.9 is proved.

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