Asymptotic Expansion for the Functional of Markovian Evolution in $\mathbb{R}^d$ in the Circuit of Diffusion Approximation

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Abstract

Is studied asymptotic expansion for solution of singularly perturbed equation for functional of Markovian evolution in $\mathbb{R}^d$. The view of regular and singular parts of solution is found.

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1 Introduction

The problems of asymptotic expansion for solutions of PDE and PDE systems were studied by many authors. A lot of references could be found in [5]. There are studied, as a rule, border problems and the small parameter is at the higher derivative by $t$.

For example, in [3](p. 155) is studied the system of first order equations with the small parameter by $t$ and $x$ that corresponds the telegraph equation.

In this work we study asymptotic expansion for solution of singularly perturbed equation for functional of Markovian evolution in $\mathbb{R}^d$.

Let $x \in \mathbb{R}^d$, $\xi(s)$ - an ergodic Markovian process in the set $E = \{1, \ldots, N\}$ with the intensity matrix $Q = \{q_{ij}, i, j = 1, \ldots, N\}$.

The probability of being in the $i$ - th state longer then $t$ is $P\{\theta_i > t\} = e^{-q_it}$, where $q_i = \sum_{j \neq i} q_{ij}$. 
Let \( a(i) = (a_1(i), \ldots, a_d(i)) \) - vector-function on \( E \). We regard a vector-function as a corresponding vector-column.

Put matrix \( A = \{a_k(i), k = 1, d, i = 1, N\} \).

Let us study evolution

\[
x^\varepsilon(t) = x + \varepsilon^{-1} \int_0^t a(\xi(s/\varepsilon^2))ds =
\]

\[
x + \varepsilon \int_0^t a(\xi(s))ds.
\]

It’s well-known [6], that the functionals of evolution, determined by a test-function \( f(x) \in C^\infty(\mathbb{R}^d) \) (here \( f \) is integrable on \( \mathbb{R}^d \) and has equal components \( f(x) = (f(x), \ldots, f(x)) \)):

\[
u_{i}^\varepsilon(x, t) = E_i f(x^\varepsilon(t)), i = 1, N
\]

(here \( i \) is a start state of \( \xi(s) \)) satisfy the system of Kolmogorov backward differential equations:

\[
\frac{\partial}{\partial t} u_{i}^\varepsilon(x, t) = \varepsilon^{-2} Q u_{i}^\varepsilon(x, t) + \varepsilon^{-1} A\nabla u_{i}^\varepsilon(x, t),
\]

where \( u_{i}^\varepsilon(x, t) = (u_{i}^\varepsilon(x, t), \ldots, u_{n}^\varepsilon(x, t)), A\nabla = diag([a(i), \nabla], i = 1, N), \nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)\).

As an example we’ll describe a well-known model, where an equation of type (1) appears.

**Example 1.1:** In the works [6, 7] functionals of the view

\[
u_{i}(x, t) = E_i f(x + v \int_0^t \tau_{\xi(s)}ds), i = 0, n
\]

were studied. Here \( \xi(u) \) - Poisson process with parameter \( \lambda \), \( \xi(0) = 0 \), \( v \) - velocity of particle’s motion, \( \tau_i, i = 0, n \) - vectors that determine the directions of motion. The systems of Kolmogorov backward differential equations were received for the functionals \( u_i(x, t), i = 0, n \) in case of cyclic and uniform change of motion directions.

In a matrix form we have:

\[
\frac{\partial}{\partial t} u_{i}^\varepsilon(x, t) = [\lambda Q + v A\nabla] u_{i}^\varepsilon(x, t),
\]

where \( u_{i}^\varepsilon(x, t) = (u_{i}^\varepsilon(x, t), \ldots, u_{n}^\varepsilon(x, t)), A\nabla = diag([\tau_i, \nabla], i = 0, n], Q = \begin{bmatrix} q_{ii}, i, j = 0, n \end{bmatrix}, \lambda = \begin{bmatrix} q_{ii}, i, j = 0, n \end{bmatrix} \) Here \( q_{ii} = -1, q_{i,i+1} = 1, q_{ij} = 0, j \neq i, j \neq i + 1 \) in case of cyclic change of directions, and \( q_{ii} = -1, q_{i,j} = 1/n, i \neq j \) in case of uniform change.

If we put in (2) \( v = \varepsilon^{-1}, \lambda = \varepsilon^{-2} \), where \( \varepsilon \) - is a small parameter, we’ll have a singularly perturbed equation of type (1):

\[
\frac{\partial}{\partial t} u_{i}^\varepsilon(x, t) = [\varepsilon^{-2} Q + \varepsilon^{-1} A\nabla] u_{i}^\varepsilon(x, t).
\]

Initial condition \( u_{i}^\varepsilon(x, 0) = f(x) := (f(x), \ldots, f(x)) \).
Equations of type (1) were also studied in the works \[2, 3\], partially in \[3\] for the distribution of absorption time of Markov chain with continuous time that depends on small parameter \(\varepsilon\) the following equation was received:

\[
e^\varepsilon \frac{d}{dx} u^\varepsilon(x) = (Q - \varepsilon G) u^\varepsilon(x), Q = P - I.
\]

Asymptotic expansion of its solution was found there.

In this work we study system (1) with the second order singularity. This problem has interesting probabilistic sense: hyperbolic equation of high degree, corresponding system (2) (see \[7\]) becomes parabolic equation of Wiener process in hydrodynamic limit, when \(\varepsilon \to 0\). The fact that solution of (2) in hydrodynamic limit tends to the functional of Wiener process is well-known and studied, for example, in \[4\].

To find asymptotic expansion of the solution of (1) we use the method proposed in \[8\]. The solution consists two parts - regular terms and singular terms - which are determined by different equations. Asymptotic expansion lets not only determine the terms of asymptotic, but to see the velocity of convergence in hydrodynamic limit.

Besides, when studying this problem, we improved the algorithm of asymptotic expansion. Partially, the initial conditions for the regular terms of asymptotic are determined without the use of singular terms, i.e. the regular part of the solution may be found by a separate recursive algorithm; scalar part of the regular term is found and without the use of singular terms. These and other improvements of the algorithm are pointed later.

2 Asymptotic expansion of the solution

Let \(P(t) = e^{Qt} = (p_{ij}(t); i, j = 1, N)\).

Put \(\pi_j = \lim_{t \to \infty} p_{ij}(t)\) and \(-R_0 = \int_0^\infty (p_{ij}(t) - \pi_j) dt; i, j = 1, N\) = \(r_{ij}; i, j = 1, N\).

Let \(\Pi\) be a projecting matrix on the null-space \(N_Q\) of the matrix \(Q\). For any vector \(g\) we have \(\Pi g = \hat{g} 1\), where \(\hat{g} = \sum_{i=1}^N g_i \pi_i, 1 = (1, \ldots, 1)\). Then for the matrix \(Q\) the following correlation is true \(\Pi Q \Pi = 0\) (see \[4\], chapter 3).

Let the matrix \(A\) satisfy balance condition:

\[
\Pi A \Pi = 0.
\]

We put:

\[
R_0 A \nabla = \{r_{ij}(a(j), \nabla), i, j = 1, N\} = \left\{ \sum_{k=1}^d r_{ij} a_k(j) \frac{\partial}{\partial x_k}, i, j = 1, N \right\},
\]

\[
A \nabla R_0 = \{(a(i), \nabla)r_{ij}, i, j = 1, N\} = \left\{ \sum_{k=1}^d a_k(i) r_{ij} \frac{\partial}{\partial x_k}, i, j = 1, N \right\},
\]

\[
A \nabla R_0 A \nabla = \{(a(i), \nabla)r_{ij}(a(j), \nabla), i, j = 1, N\} = \left\{ \sum_{k=1}^d \sum_{l=1}^d a_k(i) r_{ij} a_l(j) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}, i, j = 1, N \right\},
\]

3
exp_0(Qt) := e^{Qt} - \Pi,
\tilde{a}_{kl} = \sum_{i,j=1}^{N} \pi_i a_k(i)r_{ij}a_l(j)\pi_j,

here, following [6], we need the condition:
\tilde{a}_{kl} > 0.

**Theorem 2.1.** The solution of equation (1) with initial condition \( u^\varepsilon(x, 0) = \overline{f}(x) \), where \( \overline{f}(x) \in C^\infty(R^d) \) and integrable on \( R^d \) has asymptotic expansion

\[ u^\varepsilon(x, t) = u^{(0)}(x, t) + \sum_{n=1}^{\infty} \varepsilon^n \left( u^{(n)}(x, t) + v^{(n)}(x, t/\varepsilon^2) \right). \]  

(3)

Regular terms of the expansion are: \( u^{(0)}(x, t) \) - the solution of equation

\[ \frac{\partial}{\partial t} u^{(0)}(x, t) = \sum_{k,l=1}^{d} \tilde{a}_{kl} \frac{\partial^2 u^{(0)}(x, t)}{\partial x_k \partial x_l} \]  

(4)

with initial condition \( u^{(0)}(x, 0) = \overline{f}(x) \),

\[ u^{(1)}(x, t) = R_0 A\nabla u^{(0)}(x, t) = \left[ \sum_{k=1}^{d} \sum_{j=1}^{N} r_{ij} a_k(j) \frac{\partial u^{(0)}(x, t)}{\partial x_k} \right], i = 1, N \]

for \( k \geq 2 \):

\[ u^{(k)}(x, t) = R_0 \left[ \frac{\partial}{\partial t} u^{(k-2)}(x, t) - A\nabla u^{(k-1)}(x, t) \right] + c^{(k)}(t) := \]

\[ = R_0 \Phi \left[ u^{(k-2)}(x, t), u^{(k-1)}(x, t) \right] + c^{(k)}(t), \]

where

\[ c^{(k)}(t) \in NQ, c^{(k)}(t) = c^{(k)}(0) + \int_{0}^{t} \tilde{L}_k c^{(0)}(s) ds, \]

here

\[ c^{(0)}(t) = u^{(0)}(x, t), L_0 = \left\{ \sum_{k=1}^{d} \sum_{l=1}^{d} a_k(i) r_{ij} a_l(j) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}, i, j = 1, N \right\} \]

\[ \tilde{L}_k = \Pi L_k \Pi, L_k = (-1)^{k+1} (A\nabla R_0)^k L_0, k \geq 1, \]

\[ L_0 = \left\{ \frac{\partial}{\partial t} - \sum_{k=1}^{d} \sum_{l=1}^{d} a_k(i) r_{ij} a_l(j) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l}, i, j = 1, N \right\}. \]
The singular terms of the expansion have the view:

\[ v^{(1)}(x, t) = \exp_0(Qt)A\nabla f(x), \]

for \( k > 1 \):

\[ v^{(k)}(x, t) = \exp_0(Qt)v^{(k)}(x, 0) + \int_0^t \exp_0(Q(t-s))A\nabla v^{(k-1)}(x, s)ds - \]

\[ \Pi \int_t^\infty A\nabla v^{(k-1)}(x, s)ds. \]

Initial conditions:

\[ c^{(0)}(0) = f(x), \]

\[ u^{(1)}(0) = R_0A\nabla f(x), v^{(1)}(0) = -\frac{1}{2}A\nabla f(x), \]

for \( k > 1 \):

\[ v^{(k)}(0) = \Phi \left[ u^{(k-2)}(0), u^{(k-1)}(0) \right], \]

\[ c^{(k)}(0) = -A\nabla v^{(k-1)}(x, 0), \]

where \( \bar{v}^{(1)}(x, 0) = -R_0A\nabla f(x), \)

\[ \bar{v}^{(k)}(0) = R_0\Phi \left[ u^{(k-2)}(0), u^{(k-1)}(0) \right] + R_0A\nabla \bar{v}^{(k-1)}(x, 0) + \]

\[ \Pi A\nabla \left( \bar{v}^{(k-1)}(x, \lambda) \right)'|_{\lambda=0}, \]

\[ (\bar{v}^{(k)}(x, \lambda))'_\lambda|_{\lambda=0} = R_0^2\Phi \left[ u^{(k-2)}(0), u^{(k-1)}(0) \right] + R_0^2Q_1\bar{v}^{(k-1)}(x, 0) + \]

\[ R_0A\nabla \left( \bar{v}^{(k-1)}(x, \lambda) \right)'|_{\lambda=0}. \]

Remark 2.2. The initial conditions for the regular terms of asymptotic are determined without the use of singular terms, i.e. the regular part of the solution may be found by a separate recursive algorithm (comp. with [3]).

Remark 2.3. In case of evolution described in Example 1 equation (4) has the view:

\[ \frac{\partial}{\partial t} u^{(0)}(x, t) = \frac{1}{(n+1)^2} \Delta u^{(0)}(x, t) \]

with initial condition \( u^{(0)}(x, 0) = f(x) \).

Solution of this problem in the class of integrable and infinitely differentiable functions of exponential growth is:

\[ u^{(0)}(x, t) = (2\pi t)^{-\frac{n}{2}} \frac{1}{(n+1)^2} \int_{\mathbb{R}^n} e^{-(n+1)^2 \frac{<(x-y), (x-y)>}{4t}} f(y) dy. \]
Proof of Theorem 2.1: Let us substitute the solution \( u^\varepsilon(x,t) \) in the view (3) to the equation (1) and equal the terms at \( \varepsilon \) degrees. We'll have the system for the regular terms of asymptotic:

\[
\begin{aligned}
&Qu^{(0)} = 0 \\
&Qu^{(1)} + A\nabla u^{(0)} = 0 \\
&Qu^{(k)} = \frac{\partial}{\partial t}u^{(k-2)} - A\nabla u^{(k-1)}, k \geq 2
\end{aligned}
\tag{5}
\]

and for the singular terms:

\[
\begin{aligned}
&\frac{\partial}{\partial t}v^{(1)} = Qv^{(1)} \\
&\frac{\partial}{\partial t}v^{(k)} - Qv^{(k)} = A\nabla v^{(k-1)}, k > 1.
\end{aligned}
\tag{6}
\]

From (5) we have: \( u^{(0)} \in N_Q, u^{(1)} = R_0A\nabla u^{(0)} + c^{(1)}(t) \). For \( u^{(2)} \) we receive: \( Qu^{(2)} = \frac{\partial}{\partial t}u^{(0)} - A\nabla u^{(1)} = \frac{\partial}{\partial t}u^{(0)} - A\nabla R_0A\nabla u^{(0)} = \frac{\partial}{\partial t}u^{(0)} - L_0u^{(0)} \).

The solvability condition for \( u^{(2)} \) has the view:

\[
\Pi Q\Pi u^{(2)} = 0 = \frac{\partial}{\partial t}u^{(0)} - \Pi L_0\Pi u^{(0)}.
\]

So, we have equation (4) for \( u^{(0)}(x,t) \).

We note that in [3] solvability condition is written for the equation that contains the terms \( u^{(0)}(x,t) \) and \( u^{(1)}(x,t) \). In this work we have to express \( u^{(1)}(x,t) \) through \( u^{(0)}(x,t) \) and only then to write down solvability condition for the equation that contains the terms \( u^{(0)}(x,t) \) and \( u^{(2)}(x,t) \).

For \( u^{(1)} \) we have:

\[
u^{(1)} = R_0A\nabla u^{(0)} + c^{(1)}(t).
\]

Using the last equation from (5) we receive:

\[
u^{(k)}(x,t) = R_0\left[ \frac{\partial}{\partial t}u^{(k-2)}(x,t) - A\nabla u^{(k-1)}(x,t) \right] + c^{(k)}(t) := = R_0\Phi \left[ u^{(k-2)}(x,t), u^{(k-1)}(x,t) \right] + c^{(k)}(t),
\]

where \( c^{(k)}(t) \in N_Q \).

To find \( c^{(k)}(t) \) we’ll use the fact that \( u^{(0)} \in N_Q \). Let us put \( c^{(0)}(t) = u^{(0)}(x,t) \). From the equation \( Qu^{(2)} = \frac{\partial}{\partial t}c^{(0)}(t) - L_0c^{(0)}(t) = L_0c^{(0)}(t) \) we have

\[
u^{(2)} = R_0L_0c^{(0)}(t).
\]

For \( u^{(3)} \):

\[
Qu^{(3)} = \frac{\partial}{\partial t}c^{(1)}(t) - A\nabla u^{(2)} = (c^{(1)}(t))' - A\nabla R_0L_0c^{(0)}(t) = L_1c^{(0)}(t).
\]

From the solvability condition \( \Pi Q\Pi u^{(3)} = 0 = \frac{\partial}{\partial t}c^{(1)}(t) - \Pi A\nabla R_0L_0\Pi c^{(0)}(t) = (c^{(1)}(t))' - L_1c^{(0)}(t) \) we find:

\[
c^{(1)}(t) = c^{(1)}(0) + \int_0^t L_1c^{(0)}(s)ds,
\]
and \( u^{(3)} = R_0 \mathcal{L}_1 c^{(0)}(t) \), where \( \mathcal{L}_1 = (-L_1) c^{(0)}(t) \), as soon as \( R_0 \mathcal{L}_1 = 0 \).

By induction:

\[
c^{(k)}(t) = c^{(k)}(0) + \int_0^t \hat{L}_k c^{(0)}(s) ds,
\]

where \( \hat{L}_k = \Pi L_k \Pi, L_k = (-1)^{k+1}(A \nabla R_0) L_0, \mathcal{L}_0 = \frac{\partial}{\partial t} - L_0, k \geq 2 \).

In contrast to [3], where the equations for \( c^{(k)}(t) \) were found, in this work we may find \( c^{(k)}(t) \) explicitly through \( c^{(0)}(t) \).

For the singular terms we have from (6):

\[
v^{(1)}(x, t) = \exp_0(Q t) v^{(1)}(x, 0).
\]

Here we should note that the ordinary solution \( v^{(1)}(x, t) = \exp_0(Q t) v^{(1)}(x, 0) \) is corrected by the term \(-\Pi v^{(1)}(x, 0)\) in order to receive the following limit \( \lim_{t \to \infty} v^{(1)}(x, t) = 0 \). This limit is true for all singular terms due to uniform ergodicity of switching Markovian process.

For the homogenous part of the second equation of the system we have the following solution:

\[
v^{(k)}(x, t) = \exp_0(Q t) v^{(k)}(x, 0).
\]

But as soon as the equation is not homogenous the corresponding solution should be

\[
v^{(k)}(x, t) = \exp_0(Q t) v^{(k)}(x, 0) + \int_0^t \exp_0(Q(t-s)) A \nabla v^{(k-1)}(x, s) ds.
\]

But here we should again correct the solution, in order to receive the limit \( \lim_{t \to \infty} v^{(k)}(x, t) = 0 \), by the term \(-\Pi \int_0^\infty A \nabla v^{(k-1)}(x, s) ds\).

And so the solution is:

\[
v^{(k)}(x, t) = \exp_0(Q t) v^{(k)}(x, 0) + \int_0^t \exp_0(Q(t-s)) A \nabla v^{(k-1)}(x, s) ds - \\
\Pi \int_0^\infty A \nabla v^{(k-1)}(x, s) ds.
\]

We should finally find the initial conditions for the regular and singular terms.

We put \( c^{(0)}(t) = u^{(0)}(x, t) \), so \( c^{(0)}(0) = u^{(0)}(x, 0) = \overline{f}(x) \).

From the initial condition for the solution \( u^k(x, 0) = u^{(0)}(x, 0) = (f(x), \ldots, f(x)) \), we have \( u^{(k)}(x, 0) + v^{(k)}(x, 0) = 0, k \geq 1 \). Let us rewrite this equation for the null-space \( N_Q \) of matrix \( Q \):

\[
\Pi u^{(k)}(x, 0) + \Pi v^{(k)}(x, 0) = 0, k \geq 1,
\]

(7)

and the space of values \( R_Q \):

\[
(I - \Pi) u^{(k)}(x, 0) + (I - \Pi) v^{(k)}(x, 0) = 0, k \geq 1.
\]

(8)
Lemma 2.4.

As we proved for $k > 1$:

$$u^{(k)}(x, 0) = R_0\Phi[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)] + c^{(k)}(0) =$$

$$= (I - \Pi)\Phi[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)] + \Pi c^{(k)}(0),$$

$$v^{(k)}(x, 0) = (I - \Pi)v^{(k)}(x, 0) - \Pi \int_0^\infty A\nabla v^{(k-1)}(x, s)ds.$$

Functions $v^{(k-1)}(x, s), u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)$ are known from the previous steps of induction. So, we’ve found $\Pi v^{(k)}(x, 0)$ in (7) and $(I - \Pi)u^{(k)}(x, 0)$ in (8).

Now we may use the correlations (7), (8) to find the unknown initial conditions:

$$c^{(k)}(0) = -\int_0^\infty A\nabla v^{(k-1)}(x, s)ds,$$

$$v^{(k)}(x, 0) = \Phi[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)].$$

In [3] an analogical correlation was found for $c^{(k)}(0)$. To find $c^{(k)}(0)$ explicitly and without the use of singular terms we’ll find Laplace transform for the singular term. The following lemma is true.

**Lemma 2.4. Laplace transform for the singular term of asymptotic expansion**

$$\bar{v}^{(k)}(x, \lambda) = \int_0^\infty e^{-\lambda s}v^{(k)}(x, s)ds$$

has the view:

$$\bar{v}^{(1)}(x, \lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}[-R_0A\nabla \bar{f}(x)],$$

$$\bar{v}^{(k)}(x, \lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] +$$

$$(\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}A\nabla \bar{v}^{(k-1)}(x, \lambda) + \frac{1}{\lambda}\Pi A\nabla [\bar{v}^{(k-1)}(x, \lambda) - \bar{v}^{(k-1)}(x, 0)],$$

where

$$\bar{v}^{(1)}(x, 0) = -R_0A\nabla \bar{f}(x),$$

$$\left(\bar{v}^{(1)}(x, \lambda)\right)'_{\lambda=0} = -R_0^2A\nabla \Pi \bar{f}(x),$$

$$\bar{v}^{(k)}(x, 0) = R_0\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] + R_0A\nabla \bar{v}^{(k-1)}(x, 0) +$$

$$\Pi A\nabla (\bar{v}^{(k-1)}(x, \lambda))'_{\lambda=0},$$

$$\left(\bar{v}^{(k)}(x, \lambda)\right)'_{\lambda=0} = R_0^2\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] + R_0^2Q_1\bar{v}^{(k-1)}(x, 0) +$$

$$R_0A\nabla (\bar{v}^{(k-1)}(x, \lambda))'_{\lambda=0}.$$
Proof.

\[
\tilde{v}^{(1)}(x, \lambda) = \int_0^\infty e^{-\lambda s}v^{(1)}(x, s)ds = \int_0^\infty e^{-\lambda s}[e^{Qs} - \Pi]dsv^{(1)}(x, 0) = \\
= (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}[-A\nabla \overline{f}(x)],
\]

where the correlation for the resolvent was found in [4].

\[
\tilde{v}^{(1)}(x, 0) = -R_0A\nabla \overline{f}(x),
\]

\[
(\tilde{v}^{(1)}(x, \lambda))^\prime|_{\lambda=0} = \lim_{\lambda \to 0} \frac{R(\lambda) - R_0}{\lambda}[-A\nabla \overline{f}(x)] = -R_0^2A\nabla \overline{f}(x).
\]

For the next terms we have:

\[
\tilde{v}^{(k)}(x, \lambda) = (\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] + \\
(\lambda - \Pi + (R_0 + \Pi)^{-1})^{-1}A\nabla \overline{v}^{(k-1)}(x, \lambda) + \frac{1}{\lambda} \Pi A\nabla [\tilde{v}^{(k-1)}(x, \lambda) - \overline{v}^{(k-1)}(x, 0)],
\]

here the last term was found using the following correlation:

\[
\int_0^\infty e^{-\lambda s}A\nabla v^{(k-1)}(x, \tau)d\tau ds = \int_0^{\tau} e^{-\lambda s}A\nabla v^{(k-1)}(x, \tau)d\tau ds = \\
= \int_0^\infty \left(-\frac{1}{\lambda}\right)(e^{-\lambda s} - 1)A\nabla v^{(k-1)}(x, \tau)d\tau = \frac{1}{\lambda}A\nabla [\tilde{v}^{(k-1)}(x, \lambda) - \overline{v}^{(k-1)}(x, 0)].
\]

So,

\[
\tilde{v}^{(k)}(x, 0) = R_0\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] + R_0A\nabla \overline{v}^{(k-1)}(x, 0) + \Pi A\nabla [\tilde{v}^{(k-1)}(x, \lambda)]^\prime|_{\lambda=0},
\]

\[
(\tilde{v}^{(k)}(x, \lambda))^\prime|_{\lambda=0} = R_0^2\Phi\left[u^{(k-2)}(x, 0), u^{(k-1)}(x, 0)\right] + R_0^2Q_1 \overline{v}^{(k-1)}(x, 0) + \\
R_0A\nabla [\tilde{v}^{(k-1)}(x, \lambda)]^\prime|_{\lambda=0} - \lim_{\lambda \to 0} \left\{ \frac{1}{\lambda^2} \Pi A\nabla [\tilde{v}^{(k-1)}(x, \lambda) - \overline{v}^{(k-1)}(x, 0)] - \\
\frac{1}{\lambda} \Pi A\nabla [\tilde{v}^{(k-1)}(x, \lambda)]^\prime \right\},
\]

where the last limit tends to 0.

Lemma is proved.

So, the obvious view of the initial condition for the \(c^{(k)}(t)\) is:

\[
\overline{c}^{(k)}(0) = -A\nabla \overline{v}^{(k-1)}(x, 0).
\]

Theorem is proved.
3 Estimate of the remainder

Let function $f(x, i)$ in the definition of the functional $u^\varepsilon(x, t)$ belongs to Banach space of twice continuously differentiable by $x$ functions $C^2(R^d \times E)$.

Let us write (1) in the view

$$\tilde{u}^\varepsilon(x, t) = u^\varepsilon(x, t) - u_0^\varepsilon(x, t)$$

(9)

where $u_0^\varepsilon(x, t) = u^{(0)}(x, t) + \varepsilon(u^{(1)}(x, t) + v^{(1)}(x, t)) + \varepsilon^2(u^{(2)}(x, t) + v^{(2)}(x, t))$, and the explicit view of the functions $u^{(i)}(x, t), v^{(j)}(x, t), i = 0, 2, j = 1, 2$ is given in Theorem 2.1.

By theorem from [4] in Banach space $C^2(R^d \times E)$ for the generator of Markovian evolution $L^\varepsilon = \varepsilon^{-2}Q + \varepsilon^{-1}A\nabla$, exists bounded inverse operator $(L^\varepsilon)^{-1} = \varepsilon^2[Q + \varepsilon A\nabla]^{-1}$.

Let us substitute the function (9) into equation (1):

$$\frac{d}{dt} \tilde{u}^\varepsilon - L^\varepsilon \tilde{u}^\varepsilon = \frac{d}{dt} u_0^\varepsilon - L^\varepsilon u_0^\varepsilon := \varepsilon w^\varepsilon.$$ (10)

Here $\varepsilon w^\varepsilon = \varepsilon [\frac{d}{dt}((u^{(1)} + v^{(1)}) + \varepsilon(u^{(2)} + v^{(2)})) - (\varepsilon^{-1}Q(u^{(1)} + v^{(1)}) + Q(u^{(2)} + v^{(2)}) + A\nabla(u^{(1)} + v^{(1)}) + \varepsilon A\nabla(u^{(2)} + v^{(2)))].$

The initial condition has the order $\varepsilon$, so we may write it in the view:

$$\tilde{u}^\varepsilon(0) = \varepsilon \tilde{u}^\varepsilon(0).$$

Let $L^\varepsilon f(x, i) = E[f(x^\varepsilon(t), \xi^\varepsilon(t/\varepsilon^2))|x^\varepsilon(0) = x, \xi^\varepsilon(0) = i]$ be the semigroup corresponding to the operator $L^\varepsilon$.

**Theorem 3.1.** The following estimate is true for the remainder (9) of the solution of equation (1):

$$||\tilde{u}^\varepsilon(t)|| \leq \varepsilon||\tilde{u}^\varepsilon(0)|| \exp\{\varepsilon L||w^\varepsilon||\},$$

where $L = 2||L^\varepsilon||^{-1}$.

**Proof:** The solution of equation (10) is:

$$\tilde{u}^\varepsilon_2(t) = \varepsilon[L^\varepsilon \tilde{u}^\varepsilon(0) + \int_0^t L^\varepsilon_2 w^\varepsilon(s) ds].$$

For the semigroup we have $L^\varepsilon_2 = I + L^\varepsilon \int_0^t L_2^\varepsilon ds$, so $\int_0^t L^\varepsilon_2 ds = (L^\varepsilon)^{-1}(L^\varepsilon - I)$. Using Gronwall-Bellman inequality [1], we receive

$$||\tilde{u}^\varepsilon(t)|| \leq \varepsilon L^\varepsilon_2 ||\tilde{u}^\varepsilon(0)|| \exp\{\varepsilon \int_0^t L^\varepsilon_2 w^\varepsilon(t - s) ds\} \leq \varepsilon L^\varepsilon_2 ||\tilde{u}^\varepsilon(0)|| \exp\{\varepsilon L||w^\varepsilon||\},$$

where $L = 2||L^\varepsilon||^{-1}$.

Theorem is proved.
Remark 3.2. For the remainder of asymptotic expansion (3) of the view

\[ \tilde{u}_{N+1}^\varepsilon(x,t) := u^\varepsilon(x,t) - u_{N+1}^\varepsilon(x,t), \]

where \( u_{N+1}^\varepsilon(x,t) = u^{(0)}(x,t) + \sum_{n=1}^{N+1} \varepsilon^n (u^{(n)}(x,t) + v^{(n)}(x,t)) \) we have analogical estimate:

\[ ||\tilde{u}_{N+1}^\varepsilon(t)|| \leq \varepsilon^N ||\tilde{u}(0)|| \exp\{\varepsilon^N L ||w_N^\varepsilon||\}, \]

where \( \frac{d}{dt} u_{N+1}^\varepsilon - L^\varepsilon u_{N+1}^\varepsilon := \varepsilon^N w_N^\varepsilon. \)

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References

[1] Bainov D., Simeonov P. Integral inequalities and applications, Kluver Acad. Publ., Dordrecht, (1992), 316p.

[2] Korolyuk V.S., Boundary layer in asymptotic analysis for random walks, Theory of Stochastic Processes 1-2, 25-36 (1998).

[3] Koroljuk V.S., Penev I.P., Turbin A.F., Asymptotic expansion for the distribution of absorption time of Markov chain, Cybernetics 4, 133-135 (1973), (in Russian).

[4] Koroljuk V.S., Turbin A.F. Mathematical foundation of state lumping of large systems, Kluver Acad. Press, Amsterdam, (1990), 280p.

[5] Markush I.I. Development of asymptotic methods in the theory of differential equations, Uzhgorod, (1975), 224 p. (in Ukrainian).

[6] Pinsky M. Lectures on random evolutions, World Scientific, Singapore, (1991), 136 p.

[7] Samoilenko I.V., Markovian random evolution in \( \mathbb{R}^n \), Rand. Operat. and Stoc. Equat. 2, 139-160, (2001).

[8] Vasiljeva A.B., Butuzov V.F. Asymptotic methods in the theory of singular perturbations, Vyschaja shkola, Moscow, (1990), 208 p. (in Russian).