VECTOR FIELDS AND DIFFERENTIAL OPERATORS: NONCOMMUTATIVE CASE.

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Abstract
A notion of Cartan pairs as an analogy of vector fields in the realm of noncommutative geometry has been proposed in [2]. In this paper we give an outline of the construction of a noncommutative analogy of the algebra of partial differential operators as well as its natural (Fock type) representation. We shall also define co-universal vector fields and covariant derivatives.

1 Introduction.

Let $A$ be an associative (not necessarily commutative) algebra with unit. A new notion of right (left) Cartan pairs over algebra $A$ has been proposed in our previous paper [2] as a noncommutative substitute of concept of vector fields. A Cartan pair consist of algebra $A$ and an $A$-bimodule $M$ equipped with a suitable right (or left) action of $M$ on $A$. As a next step we have defined a notion of right (left) dual of an $A$-bimodule in such a way that dual object is again an $A$-bimodule. Finally, it has been shown that the first order differential calculi on $A$ and right (left) $A$-Cartan pairs are dual each other. Therefore, in noncommutative settings, the concept of vector fields splits into two concepts: right and left vector fields. For modules over commutative algebras both notions of left and right vector fields coincide. In this case, above dualities restore classical dualities between one forms and vector fields which are known from the classical differential geometry on manifolds ($A = C^\infty(M)$).

The construction employs the Cartan formula

$$X(f) \equiv < X, df > \equiv i_X df .$$

(1.1)
which expresses both dualities. Our main result was that formula (1.1) allows to reconstruct the action of vector fields on "functions" (i.e. elements of the algebra) if we are given the differential. Conversely, one can find out the differential by means of action. Examples of such actions for given (noncommutative) calculi can be found in [3, 5, 10].

The Leibniz rule for a first-order differential calculus

\[ d(fg) = (df).g + f.dg \]  

remains unchanged as in the classical case: this one for vector fields (partial derivatives) has been replaced by more general axiom: action of bimodule on the algebra (see Definition 2.3 below).

Our formalism is inspired by but different from the Lie–Cartan pairs approach [8]. The last notion (with different names) has been rediscovered independently by many authors from time to time (see [9]). In particular, we have no analogue of Lie bracket. Another generalization which is based on a derivation property of vector fields (so called Lie pairs) has been introduced in [7].

Our aim here is to push further a contravariant (vector fields) formalism in the noncommutative differential geometry. The following topics will be discussed: co-universal vector fields, algebra of differential operators and covariant differentiations.

For the sake of brevity we shall develop a "right handed" version of the theory. It will become soon clear that the similar construction can be performed for left Cartan pairs (c.f [4]).

2 Preliminaries and notation.

Throughout this paper \( \mathbb{k} \) denotes some fixed unital and commutative ring. (For simplicity one can limit ourselves to the field of real or complex numbers.) Algebras are unital associative \( \mathbb{k} \)-algebras and homomorphisms are assumed to be unital. All objects considered here are \( \mathbb{k} \)-modules, all maps are assumed to be \( \mathbb{k} \)-linear maps. The tensor product \( \otimes \) unless otherwise mentioned means \( \otimes_{\mathbb{k}} \). Let \( M \) be an \((A, A)\)-bimodule (\( A \)-bimodule in short). We shall denote by dot "." the both: left and right multiplication by elements from \( A \). For example, by bimodule axioms, one has \((f.x).g = f.(x.g) = f.x.g \) for \( f, g \in A \) and \( x \in M \). In this Section we give a necessary definitions and summarize main results from [3].

Let \( M \) be an \( A \)-bimodule and let \( \text{Hom}(-, A)(M, A) \) denotes a set of all right \( A \)-module maps from \( M \) into \( A \). It is a \( \mathbb{k} \)-space. For \( X \in \text{Hom}(-, A)(M, A) \) denote as a pairing

\[ < X, m > \equiv X(m) \in A \]  

the evaluation of \( X \) on the element \( m \in M \). (see e.g. [4] p. 232 in the context of modules over commutative algebras.) Then the following formula

\[ < f.X.g, m > \equiv f < X, g.m > \]  

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where \( f, g \in A \), defines an \( A \)-bimodule structure on \( \text{Hom}_{(-,A)}(M, A) \). Therefore, the pairing (2.1) defines a bimodule map

\[
< , > : M^* \otimes_A M \to A
\]  

**Definition 2.1.** The set \( M^* = \text{Hom}_{(-,A)}(M, A) \) equipped with the bimodule structure indicated above is called a **right dual** of \( M \).

In a similar way one defines a **left dual** \( ^*M = \text{Hom}_{(A,-)}(M, A) \). These notions generalize the standard notion of duality: for modules over commutative algebra the three notions of duals; right, left and standard dual of module coincide.

**Example 2.2.** Observe that for the algebra itself \( A^* = A \) as \( A \)-bimodules. An identification is made by \( f \mapsto f(1) \). In this case \( < f, g > = fg \). Of course, in the similar way \( ^*A = A \).

By an **action** of \( M \) on \( A \) we mean a \( \mathbb{k} \)-linear mapping \( \beta \in \text{Hom}_I \mathbb{k}(M, \text{End}_I \mathbb{k}(A)) \). We shall also write \( M \ni x \mapsto x \beta \in \text{End}_I \mathbb{k}(A) \) or \( A \ni f \mapsto x \beta(f) \in A \) to denote the action \( \beta \).

**Definition 2.3.** An **\( A \)-right Cartan pair** \( (N, \rho) \) consists of an \( A \)-bimodule \( N \) and its right action \( \rho : N \to \text{End}_I \mathbb{k}(A) \), such that

\[
(f.X)\rho(g) = f X\rho(g)
\]  

and

\[
X\rho(fg) = X\rho(f)g + (X.f)\rho(g)
\]

for all \( X \in N \). Such action will be called a **right action**.

It is easy to see that \( X\rho \) annihilates scalars from \( \mathbb{k} \).

Observe that in the case of modules over commutative algebras \( X.f = f.X \) and the formulae (2.4) - (2.5) reduce to the standard Leibniz rule

\[
X\rho(fg) = X\rho(f)g + f X\rho(g)
\]

satisfied in the classical differential geometry. Therefore, \( X \) becomes a derivation of the algebra \( A \). In this sense the definition of Cartan pairs is a generalization of notion of vector fields. As we will see below the dualization of Cartan pairs leads to a differential calculus on the algebra \( A \).

**Definition 2.4.** A **first order** differential calculus \( (M, d) \) on the algebra \( A \) (or in short an **\( M \)-valued calculus on \( A ) \) consists of an \( A \)-bimodule \( M \) and a linear map \( d : A \to M \) satisfying the Leibniz rule (1.2).

The bimodule \( M \) plays the role of a bimodule of one-forms. Noncommutative differential calculi *i.e.* differential calculi on noncommutative algebras are basic objects of noncommutative differential geometry \([\mathbb{1}]\). They have been investigated by many authors (see e.g. \([3, 5, 10, 11]\)).

Let now \( (M, d) \) be a calculus on an algebra \( A \). The differential \( d \) and formula (1.1) defines an action of the right dual \( M^* \) on \( A \). This action

\[
A \ni f \mapsto X\rho(f) \doteq < X, df >
\]
will be called a right partial derivatives along the element $X \in M^*$ with respect to the calculus $(M, d)$. It appears that this action satisfies axioms of right Cartan pairs. Indeed, by (2.2)

$$(f.X)^\partial(g) = < f.X, dg > = f < X, dg > = fX^\partial(g)$$

and

$$(X.f)^\partial(g) = < X, f dg > = < X, d(fg) - df.g > = X^\partial(fg) - X^\partial(f)g.$$ 

Therefore, to each differential calculus $(M, d)$ on $A$ we can associate a unique right Cartan pair of right partial derivatives $(M^*, \partial)$ of $(M, d)$. The converse statement is also true: to each right Cartan pair $(N, \rho)$ one can associate a unique differential calculus $(N^*, \rho)$ where, $\rho : A \to \phantom{\ast} N$ is defined by formula (2.8) below. Thus we have

**Theorem 2.5.** Let $(M, d)$ be a calculus on $A$. Then $M^*$ together with an action (2.7), via the right partial derivatives, becomes the right Cartan pair $(M^*, \partial)$ on $A$. Conversely, let $(N, \rho)$ be a right Cartan pair on $A$. Then the formula

$$< X, \rho f > = X^\rho(f)$$

for each $X \in N$, determines $\rho f$ as an element of a left dual $\phantom{\ast} N$ of the bimodule $M$. The mapping $\rho : A \to \phantom{\ast} N$ defines the $\phantom{\ast} N$-valued calculus $(\phantom{\ast} N, \rho)$ on $A$.

Moreover, the module of one forms is spanned by differential if and only if the action has a trivial kernel.

In a case of reflexive bimodule a successive application of above canonical constructions give rise to the initial object [3].

3 Co-universal problem for Cartan pairs.

Our aim in this section is to associate to any algebra $A$ its co-universal (right) Cartan pair.

Let $M, N$ be two $A$–bimodules and $\alpha : M \to N$ a bimodule map between them. Consider a transpose map $\alpha^T : N^* \to M^*$ defined by the formula

$$< \alpha^T(Y), m >_M = < Y, \alpha(m) >_N$$

where $Y \in N^*$, $m \in M$. From this definition one inspects that

$$< \alpha^T(f.Y.g), m >_M = f < Y, \alpha(g.m) >_N = f < \alpha^T(Y), m.g >_M = < f.\alpha^T(Y).g, m >_M$$

Thus we have proved

**Proposition 3.1.** $\alpha^T$ is again an $A$–bimodule map.

In what follows we shall apply above Proposition in order to dualise the universal differential calculus.
Recall, that for a given algebra $A$ there exists the bimodule $\Omega^1_u(A)$, which is a kernel of the multiplication map and the differential calculus $(\Omega^1_u(A), d_u)$ with the following universal property: for every differential calculus $(M, d)$ there exists a unique bimodule map $\phi: \Omega^1_u(A) \to M$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{d_u} & \Omega^1_u(A) \\
d \searrow & & \downarrow \phi \\
& M
\end{array}
$$

is commutative. It is called the universal differential calculus on $A$.

Put $X^u(A) = (\Omega^1_u(A))^*$

**Definition 3.2.** The right Cartan pair $(X^u(A), \partial_u)$ dual to $(\Omega^1_u(A), d_u)$ is called a right co-universal Cartan pair.

Let us begin with the situation describe by the diagram (3.2). Since, $\phi: \Omega^1_u(A) \to M$ is a bimodule map then its transpose $\phi^T: M^* \to X^u(A)$ is again a bimodule map. Thus one has

$$
< X, df >_M = < X, \phi(d_u f) >_M = < \phi^T(X), d_u f >_{\Omega^1_u(A)}
$$

This means that $\phi^T(X) \in X^u(A)$ acts via $\partial_u$ on $A$.

**Theorem 3.3.** For any algebra $A$ there exists the (unique) co-universal right Cartan pair $(X^u(A), \partial_u)$. It has the following co-universal property: for an arbitrary right $A$-Cartan pair $(N, \rho)$ there exists one and only one bimodule map $\Phi: \Omega^1_u(A) \to N$ such that the diagram

$$
\begin{array}{ccc}
X^u(A) & \xrightarrow{\partial_u} & \text{End}_k(A) \\
\downarrow \Phi & & \nearrow \partial \\
N
\end{array}
$$

is commutative.

The case when the Cartan pair $(N, \rho)$ originates from the calculus $(M, d)$ has been considered above. The more general proof for an arbitrary Cartan pair involves an explicit realization of $X^u(A)$. This will be done elsewhere.

4 Algebra of differential operators.

Let $(M, \partial)$ be a (right) $A$-Cartan pair. To this data we are going to associate an algebra of (right handed, linear, partial) differential operators $D(M, \partial)$ together with its (algebraic) Fock space representation by means of operators acting on the algebra $A$. 

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More exactly, one has an algebra map $l: A \to \text{End}_k(A)$, which is an action of $A$ on $A$

$$f^l(g) = fg, \quad (fg)^l = f^l \circ g^l$$  \hspace{1cm} (4.1)

induced by the multiplication from the left. Intuitively, a differential operator is a polynomial expression in $X$ and $f$. Thus, our task is to describe the subalgebra

$$\mathcal{D}(M, \partial) \doteq \text{gen}_{\text{End}_k} \{ f^l, X^\partial | f \in A, X \in M \}$$  \hspace{1cm} (4.2)

generated in $\text{End}_k(A)$ by all endomorphism of the form $f^l, X^\partial$.

**Definition 4.1.** The algebra $\mathcal{D}(M, \partial)$ is called an algebra of a (linear, right handed) differential operators with respect to the right Cartan pair $(M, \partial)$. Its elements are called differential operators.

We shall now try to understand better an algebraic structure of $\mathcal{D}(M, \partial)$ in terms of an abstract algebra. Since $\partial: M \to \text{End}_k(A)$ is a $k$-linear map it uniquely lifts to an algebra map $\hat{\partial}: T_k M \to \text{End}_k(A)$, where $T_k M \doteq k \oplus M \oplus (M \otimes M) \cdots \doteq \bigoplus_{k \in \mathbb{N}} M^{\otimes k}$ is a tensor algebra of the $k$-module $M$.

Let $A \star T_k M$ denotes a free product of the algebras $A$ and $T_k M$. This is the algebra generated by these algebras, with no relation except for the identification of unite elements $[1]$. By the universal property of the free product $\star$, there exists a unique algebra map $\mu: A \star T_k M \to \text{End}_k(A)$, such that it extends $l$ and $\hat{\partial}$ i.e.

$$\mu|_A = l, \quad \mu|_{T_k M} = \hat{\partial}$$  \hspace{1cm} (4.3)

Thus by its very definition the algebra

$$\mathcal{D}(M, \partial) \equiv A \star T_k M / \ker \mu$$  \hspace{1cm} (4.4)

is isomorphic to the quotient algebra, where $\ker \mu$ denotes the kernel of $\mu$. The Cartan pair axioms (2.4) and (2.5) rewritten in terms of the action read

$$(f.X)^\partial = f^l \circ X^\partial$$  \hspace{1cm} (4.5)

and

$$X^\partial \circ f^l = (X,f)^\partial + (X^\partial(f))^l$$  \hspace{1cm} (4.6)

This implies that $\ker \mu$ is nontrivial and contains an ideal generated by the relations

$$\{(f.X) - f \star X , (Y,g) - Y \star g + Y^\partial(g) | f, g \in A; X, Y \in M \}$$

In the case of modules over commutative algebra (i.e. $f.X = X.f$) one sees that (4.5) - (4.6) reduce to the relations

$$[X^\partial, f^l] = (X^\partial(f))^l$$  \hspace{1cm} (4.7)

which remains the classical *Canonical Commutation Relations* (CCR – for short) between annihilation and creation operators. Here, $[a, b] \doteq a \circ b - b \circ a$ denotes
as usual the commutator in $\text{End}_k(A)$. In this sense the action of $\mathcal{D}(m, \partial)$ on $A$ bears some features of an (algebraic, oscillator type) Fock space representation: it extends the action $\partial$ of $M$ on $A$ (annihilation) from one hand and the action of $A$ on $A$ (creation) from the other. The unit $1_A \in A$ plays the role of the vacuum (or ground) state ($X^\partial(1_A) = 0$ and $f^\partial(1_A) = f$). The commutation relations between creation operators are encoded in the algebra structure of $A$. Observe that we have no a priori relations between the annihilation operators: however, they may appear in the kernel of the action $\partial : T_k M \to \text{End}_k(A)$.

Obviously, the same construction can be applied to Cartan pairs coming from differential calculi. The true, (i.e. Hilbert) Fock space representation of a $q$-deformed differential calculus on the quantum plane has been discovered in [11].

In the classical case (of differential calculus on a manifold) the definition 4.1 gives rise to the definition of the classical (linear, partial) differential operators [13]. For the case of $\mathbb{R}^n$, these operators form so called a Weyl algebra [12], which is generated by the CCR. In this way $\mathcal{D}(M, \partial)$ becomes a generalization of the Weyl algebra.

## 5 Connection versus covariant derivative.

Our aim here is to verify a possible application of Cartan pairs in a theory of connections. Provided with the notion of vector fields one can try to generalize the Koszul’s approach: the noncommutative version of covariant derivatives.

Let $M$ be an $A$-bimodule and $E$ be a left $A$-module. $M \otimes_A E$ is again a left $A$-module which can be contracted with $M^*$. More exactly one has a $k$-linear map

$$\langle \cdot, \cdot \rangle : M^* \otimes_A (M \otimes_A E) \to E$$

given by the formula

$$\langle X, m \otimes_A \xi \rangle = \langle X, m \rangle . \xi$$

(5.1)

for $X \in M^*$, $m \in M$ and $\xi \in E$.

Following [4] a left connection on $E$ with respect to the differential calculus $(M, d)$ is a linear map $\nabla : E \to M \otimes_A E$ satisfying

$$\nabla(f\xi) = f . \nabla \xi + df \otimes_A \xi$$

(5.2)

(In the original Connes approach one uses the universal differential calculus.)

Define

$$\nabla_X \xi \doteqdot \langle X, \nabla \xi \rangle$$

(5.3)

as a covariant derivative along the vector field $X$ with respect to the connection $\nabla$. Thus, the formula (5.3) defines the action

$${M^* \ni X \mapsto \nabla_X \in \text{End}_k(E)}$$
of $M$ on $E$. This action has the following properties

$$\nabla_{f,X}\xi = f.\nabla_X\xi$$

(5.4)

and

$$\nabla_X(f.\xi) = X^{\partial}(f).\xi + \nabla_{X,f}\xi$$

(5.5)

which generalize the axioms (2.4) - (2.5) of a right Cartan pair. From the other hand they may serve as an axiomatic definition of the covariant derivative and hence the connection $\nabla$.

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