DIFFERENTIAL TWISTED STRING AND FIVEBRANE STRUCTURES

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Abstract. In the effective background field theory of string theory, the Green-Schwarz anomaly cancellation mechanism plays a key role. Here we reinterpret it and its magnetic dual version in terms of, differential twisted String- and differential twisted Fivebrane-structures that generalize the notion of Spin-structures and Spin-lifting gerbes and their differential refinement to smooth Spin-connections. We show that all these structures can be encoded in terms of nonabelian cohomology and twisted nonabelian cohomology and differential twisted nonabelian cohomology, extending the differential generalized abelian cohomology as developed by Hopkins and Singer and shown by Freed to formalize the global description of anomaly cancellation problems in higher gauge theories arising in string theory. We demonstrate that the Green-Schwarz mechanism for the $H_3$-field, as well as its magnetic dual version for the $H_7$-field define cocycles in differential twisted nonabelian cohomology that may be called, respectively, differential twisted Spin-$n$-, String-$n$- and Fivebrane-$n$- structures on target space, where the twist in each case is provided by the obstruction to lifting the classifying map of the gauge bundle through a higher connected cover of $U(n)$ or $O(n)$. We work out the (nonabelian) $L_\infty$-algebra ($L_\infty$-algebroid) valued differential form data provided by the differential refinements of these twisted cocycles and demonstrate that this reproduces locally the differential form data with the twisted Bianchi identities as known from the string theory literature. The treatment for M-theory leads to new models for the C-field and its dual in differential nonabelian cohomology.

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References
1. Introduction

String theory and M-theory involve various higher gauge-fields, which are locally given by differential form fields of higher degree and which are globally modeled by higher bundles with connection (higher gerbes with connection, higher differential characters) [33] (cf. [70]). Some of these entities arise in terms of lifts through various connected covers of orthogonal or unitary groups. For example, an orientation of a manifold $M$ can be given by a lifting of the classifying map $\chi : M \to BO$ for the tangent or frame bundle of $M$ to a map $\omega : M \to BSO$. In turn, a spin structure on $M$ can be given by a further lifting $sp : M \to BSpin$. The existence of a Spin structure is an anomaly cancellation condition for fermionic particles propagating on $M$.

The spaces $BO$, $BSO$ and $BSpin$ are the first steps in the Whitehead tower of $BO$. The next step above $BSpin$ is known as $BString$, with String the topological group known as the String group.

Originally Killingback [50] defined a string structure on $M$ as a lift of the transgressed map $L M \to BLSpin$ on loop space through the Kac-Moody central extension $BLSpin(n)$. This cancels an anomaly of the heterotic superstring on $M$. Later it was realized that this is captured down on $M$ by a lift of $sp : M \to BSpin(n)$ to $str : M \to BString(n)$ [78]. A further lift $fiv : M \to BFivebrane(n)$ through the next step in the Whitehead tower of $BO(n)$ is similarly related to anomaly cancellation for the NS-Fivebrane on $M$ and accordingly the corresponding space is called $BFivebrane(n)$ [69].

While anomalies are cancelled by these lifts of maps of topological spaces, the dynamics of these systems is controlled by smooth refinements of such maps. This is well understood for the first steps: the topological groups $O(n)$, $SO(n)$ and $Spin(n)$ naturally carry Lie group structures and the differential refinement of $X \to BSpin(n)$ is well known to be given by a differential nonabelian $Spin(n)$-cocycle, namely a smooth $Spin(n)$-principal bundle with connection.

However, the higher connected topological groups $String(n)$ and $Fivebrane(n)$ cannot be finite dimensional Lie groups and are not known to admit any infinite-dimensional Lie group structure. But $String(n)$ does have a natural incarnation as a smooth 2-group [9] [10] [11] [12], a higher categorical (see [56] for all matters...
of higher category theory needed here) version of a Lie group. Similarly, Fivebrane\((n)\) does naturally exist as a smooth 6-group. Generally, every topological space \(BO(k)\) in the Whitehead tower of \(BO\) naturally has a smooth incarnation as a smooth \(\infty\)-groupoid: an \((\infty,1)\)-categorical sheaf \([50]\) \(\text{\(\infty\text{-stack}\)}\) on a site of smooth test spaces. As such, these spaces have \(L_\infty\)-algebras as their infinitesimal approximation in the same way that \(SO(n)\) has a Lie algebra associated with it. Therefore after passing to the smooth \(\infty\)-groupoid incarnation of the objects in the Whitehead tower of \(BO\) there is a chance of obtaining differential refinements of String\((n)\)- and Fivebrane\((n)\)-structures etc. that are expressed in terms of higher smooth bundles with smooth \(L_\infty\)-algebra valued connection forms on them \([68]\).

The general refinement of cohomology classes to differential cohomology classes for the case of abelian (Eilenberg-Steenrod-type) generalized cohomology theories has been discussed by Hopkins and Singer \([42]\) and shown by Freed \([33]\) to encode various differential (and twisted) structures in String theory. However, the cohomological structures that appear in the Freed-Witten \([36]\) and in the Green-Schwarz anomaly cancellation mechanism \([38]\), as well as in the magnetic dual Green-Schwarz mechanism \([69]\), themselves originate from and are controlled by nonabelian structures, namely the \(O(n)\)-principal bundle underlying the tangent bundle of spacetime and the \(U(n)\)-principal bundle underlying the gauge bundle on spacetime, as well as their lifts to the higher connected structure groups: a map \(X \rightarrow BSpin(n)\) (smooth or not) gives a cocycle in nonabelian cohomology, and so do its lifts \(X \rightarrow BString(n)\) etc.

Therefore here we describe a theory of (twisted) differential nonabelian cohomology, that builds on \([10]\) \([72]\) \([73]\) \([75]\) \([68]\) and is discussed in more detail in \([70]\). We show that the Freed-Witten and the Green-Schwarz mechanisms, as well as the magnetic dual Green-Schwarz mechanism, define differential twisted nonabelian cocycles that may be interpreted as differential twisted Spin\(^c\)-, String- and Fivebrane-structures, respectively. Equivalent anomalies differ by a coboundary, so that they are given by cohomologous nontrivial cocycles. Equivalence classes of anomalies are captured by the relevant cohomology. We thus have a refinement of the treatment in \([69]\) to the twisted, smooth and differential cases.

In particular, the various abelian background fields appearing in the theory, such as the Kalb-Ramond field and the supergravity 3-form field, are unified into a natural coherent structure with the nonabelian background fields – the spin- and gauge-connections – with which they interact. For instance, the relations between the abelian and the nonabelian differential forms that govern the Green-Schwarz mechanism \([38]\) are realized here as a (twisted) Bianchi identity of a single nonabelian \(L_\infty\)-algebra valued connection on a twisted String\((n)\)-principal 2-bundle.

A similar structure appears in M-theory (for which the above string theory is essentially a boundary) and, in fact, we get a model for the M-theory degree three \(C\)-field in nonabelian cohomology, extending previous models (cf. \([28]\)). Our formalism also provides a model for the dual of the \(C\)-field in degree eight.

Aspects of such differential twisted nonabelian cohomology in low degree had been indicated in \([4]\) in the language of twisted nonabelian bundle gerbes with connection. This is further developed in \([45]\), where a 2-bundle gerbe realization of the twisted String\((n)\)-structures we discuss is presented. The formalism that we give in section \([2]\) is meant to provide the fully general picture of such differential twisted nonabelian cohomology in an elegant, albeit somewhat abstract, language. Its more concrete realization in terms of \(L_\infty\)-algebra valued Cartan-Ehresmann connection forms, as introduced in \([68]\), is derived from the abstract formalism in \([2,4,7]\) and described further in section \([4]\) and the explicit derivation of the twisted Bianchi identities of \(L_\infty\)-algebra connections corresponding to the Green-Schwarz mechanism and its magnetic dual is in section \([5]\).

Summary. In this paper we achieve the following goals:

(1) generalize String- and higher structures to the twisted case;
(2) provide differential cohomology versions of these twisted structures;
(3) provide a model for the M-theory \(C\)-field;
(4) provide a model for the dual of the M-theory \(C\)-field.
Figure 2. Abelian versus nonabelian cohomology. Since the groups String(n) as well as Fivebrane(n) are shifted central extension of nonabelian groups, cohomology with coefficients in these groups has nonabelian and abelian components. This appears as abelian cohomology twisted by nonabelian cocycles in a certain way. The Green-Schwarz mechanism implies that two classes in ordinary abelian cohomology, namely in degree four differential integral cohomology, coincide. But these classes are particularly obstruction classes to String-lifts in nonabelian cohomology. The middle part of Figure 1, labelled “abelian cohomology”, identifies the cocycle representative in $H^4(X,\mathbb{Z})$ and the coboundary between them, but does not specify where these cocycles come from. The outer part of the diagram, labelled “nonabelian cohomology” does specify the object whose class is the one identified by the middle part. We have adopted cohomological language in the greatest generality, namely homotopy classes of maps $X \to A$ for given $A$, often with an algebraic structure. Thus we refer to such maps as cocycles. This is elaborated in Section 2.1. The situation becomes more pronounced when this setup is refined to differential nonabelian cohomology, discussed in section 5.

The first two are purely mathematical results that we hope will be of independent interest in developing higher (algebraic, geometric, topological, categorical) phenomena. The third and fourth are applications to M-theory which we hope will add to a better understanding of structures appearing in that theory, which in turn is hoped to result in identification of yet more rich mathematical structures within it. From a mathematics point of view, they serve as interesting concrete examples of the formalism we have developed in the first two points.

Examples. Section 3.1 discusses how the anomaly cancellation mechanisms in String-theory can be understood topologically in terms of twisted higher structures given by twisted nonabelian topological cocycles. The physical examples of most relevance here arise in various anomaly cancellations in String theory. The Freed-Witten condition in type IIA string theory says that the third integral Stiefel-Whitney class $W_3$ of a D-brane $Q$ has to be trivial relative to the Neveu-Schwarz field $H_3|Q$ restricted to the D-brane, in that the two classes agree: $W_3 = [H_3|Q]$.

Higher versions of this example are the Green-Schwarz mechanism and its magnetic dual version. Recall the notion of String structures from [69] as maps from a space $X$ to $B\text{String}(n)$, the 3-connected cover of $B\text{Spin}(n)$. In [51] the notion of twist for a String structure was considered: a space $X$ can have a twisted String structure without having a String structure, i.e. the fractional Pontrjagin class $\frac{1}{2}p_1(TX)$ of the tangent bundle can be nonzero while the modified class $\frac{1}{2}p_1(TX) + [\beta] = 0$, where $\beta : X \to K(\mathbb{Z}, 4)$ is a fixed twist for the String structure. The Green-Schwarz mechanism in String theory may be understood as defining a twisted String-structure on target space, with twist given in terms of a classifying map for the gauge bundle.
Since a String structure is refined by a Fivebrane structure in analogy to how a String structure itself refines a Spin structure, it is natural to consider twists of Fivebrane structures in the above sense. In this paper we give a definition of twisted Fivebrane structures and show that the dual Green-Schwarz mechanism in heterotic String theory reviewed in detail in [83] provides an example. Hence the twisted Fivebrane conditions do in fact appear in string theory and M-theory and they correspond, as we will see, to anomaly cancelation conditions for the heterotic fivebrane [30] [55] and for the M-fivebrane [88] [90] [34] [28], respectively. We discuss these two cases in section 3.2 and section 3.3 in terms of topological cocycles (maps to the appropriate classifying spaces) and describe in section 5.3 and 5.5 their differential refinements.

2. Cohomology: nonabelian, twisted and differential

This section indicates the general theory and tools that we employ to describe differential twisted nonabelian cocycles. A detailed formal development is given in [70]; here we are content with giving an overview sufficient to put our main constructions in perspective.

2.1. Nonabelian cohomology as homotopy theory. Ordinary cohomology in degree 1 of a ‘nice’ topological space $X$ with values in a possibly nonabelian group $G$ can be defined as the pointed set of homotopy classes of maps of topological spaces from $X$ into the classifying space $BG$. Alternatively, we could adopt an axiomatic approach and discuss contravariant functors from a class of spaces to sets or groups, etc. E.H. Brown in [14] showed that such functors to sets satisfying two axioms (Wedge and Mayer-Vietoris) were representable, that is, given such a functor $H$ there was a space $A_H$ such that $H$ was naturally isomorphic to $[A_H, A_H] := \pi_0Maps(A_H, A_H)$, the set of homotopy classes of maps to $A_H$. If $H$ takes values in the category of groups, then Adams [3] showed $A_H$ is something like a homotopy associative H-space. Earlier Brown [13] had shown that in the presence of an additional axiom (Suspension) for a generalized Eilenberg-Steenrod cohomology theory, there were representing spaces that formed an $\Omega$-spectrum.

More generally, nonabelian cohomology is used to describe functors to sets which are so representable by general topological spaces. Given a topological space $X$ and a topological space $A$, the cohomology of $X$ with coefficients in $A$ is just $H(X, A) = \pi_0\text{Top}(X, A)$. The fact that this is a reasonable definition depends only on the property that Top is naturally an $(\infty, 1)$-topos [56]: a category that has between any two objects an $\infty$-groupoid of maps, homotopies between maps, homotopies between homotopies, etc, and which shares crucial structural properties with Top.

As we will discuss below, the discussion of associated and principal $\infty$-bundles classified by such cocycles rests only on the validity of the analog of Giraud’s axioms for ordinary toposes in an $(\infty, 1)$-topos: in particular the fact that every groupoid object in an $(\infty, 1)$-topos is effective. Therefore for $H$ any other $(\infty, 1)$-topos it makes good sense to regard for $X, A \in H$ any two objects in $H$ the set

$$H(X, A) := \pi_0H(X, A)$$

as the $A$-valued cohomology of $X$. In more detail

- objects of $H(X, A)$ are $A$-valued cocycles on $X$;
- morphisms in $H(X, A)$ are coboundaries between these cocycles;
- equivalence classes in $H(X, A)$ are cohomology classes,
- so that the homotopy hom-set $H(X, A) := \pi_0H(X, A)$ is the $A$-valued cohomology set of $X$. It is a group if $A$ has a group structure.

Remark 2.1. When unwrapping this statement and restricting to abelian coefficients $A$, it becomes a classical fact known in sheaf cohomology theory: by [56] every hypercomplete $(\infty, 1)$-topos of $\infty$-stacks is presented by a model category of simplicial sheaves as developed by Brown, Joyal and Jardine [47]. Abelian sheaf cohomology is just a special case of this general notion of cohomology in this context for the case that the coefficient object $A$ happens to be in the image of the Dold-Kan map of an abelian chain complex of sheaves. For the purpose of our discussion of differential twisted structures, we will consider a site $C$ of smooth test spaces – such as a subcategory of smooth manifolds or a subcategory of smooth loci [61] – and then take $H$ to be the hypercomplete $(\infty, 1)$-topos of $\infty$-stacks on $C$. Objects in such $H$ are smooth $\infty$-groupoids and
cohomology in \( H \) is a kind of smooth cohomology that classified smooth principal \( \infty \)-bundles. The reader may safely assume, as mentioned earlier, that we can manipulate generalized smooth spaces just as we would ordinary topological spaces.

Inside such \( H \) we then set up a notion of differential cohomology, further below, that describes smooth connections and parallel transport in such smooth principal \( \infty \)-bundles. Recall that by the very nature of \((\infty,1)\)-toposes little harm is done by thinking of their objects as just topological spaces most of the time. The formalism at work in the background takes care of the fact that these spaces are actually richer than topological spaces. Henceforth \( H \) shall denote an arbitrary such \((\infty,1)\)-topos, unless otherwise mentioned.

2.2. Principal and associated \( \infty \)-bundles. Since familiar homotopical constructions such as homotopy limits and colimits all exist in \( H \), we have all the notions derived from these. In particular for \( A \in H \) any object equipped with a point \( pt_A : \ast \to A \) its loop space object \( \Omega A \) is the homotopy pullback

\[
\begin{array}{ccc}
\Omega A & \to & \ast \\
\downarrow & & \downarrow \\
\ast & \to & A
\end{array}
\]

Furthermore, for \( G \) any object, we say an object \( BG \) is a delooping of \( G \) if it has an essentially unique point \( \ast \to BG \) and \( G \simeq \Omega BG \). If a delooping for \( G \) exists, we call \( G \) once deloopable and we call it an \( \infty \)-group. More precisely, the simplicial Čech nerve of the morphism \( \ast \to BG \)

\[
\begin{array}{ccc}
\ast \times_{BG} \ast \times_{BG} \ast \times_{BG} \ast & = & \ast \times G \times G \to \ast
\end{array}
\]

is a group object in \( H \), in that it is a groupoid object in an \((\infty,1)\)-topos in the sense of \([56]\) with the terminal object in degree 0. By Lurie’s generalization (page 437 of \([56]\) and the later discussion of Giraud’s axioms) of the classical result \([76]\) in \( H = \text{Top} \) we have that every (possibly \( A_\infty \)) group object in \( H \) is equivalent to one of this form.

By the general reasoning of nonabelian cohomology discussed above, a cocycle for nonabelian \( G \)-cohomology on \( X \in H \) is just a morphism \( g : X \to BG \) in \( H \). To this is canonically associated its homotopy fiber

\[
\begin{array}{ccc}
P & \to & \ast \\
\downarrow & & \downarrow \\
X & \to & BG
\end{array}
\]

and we claim that \( P \to X \) canonically extends to the structure of a groupoid object in \( H \) that exhibits the action of \( G \) on \( P \) in that it is a groupoid object over \( G \): it fits naturally into a diagram

\[
\begin{array}{ccc}
P \times_X P & \to & P \times G \times G \to G \\
\downarrow & & \downarrow \\
P \times_X P & \to & P \times G \to G \\
\downarrow & & \downarrow \\
P & = & P \\
\downarrow & & \downarrow \\
X & = & X \\
\downarrow & & \downarrow \\
X & \to & BG
\end{array}
\]

We call \( P \to X \) the \( G \)-principal \( \infty \)-bundle classified by \( g : X \to BG \).
For ordinary principal bundles the following terminology is standard, which applies immediately to the above ∞-categorical situation, too:

- the morphism \( P \times_X P \to P \times G \) is the division map;
- the fact that the division map is an equivalence is the principality condition on the action;
- the image \( \rho : P \times G \to P \) of the projection to the second factor \( p_2 : P \times_X P \to P \) under this equivalence is the action of \( G \) on \( P \).

The above shows how every cocycle \( g : X \to BG \) induces a map \( P \to X \) equipped with a \( G \)-action. Conversely, we may define a \( G \)-action on an object \( V \) to be a groupoid object in \( H \) sitting over \( G \).

2.2.1. Associated ∞-bundles. Let \( G \) be a group object in \( H \). An action of \( G \) on another object \( V \in H \) is a groupoid object \( V//G \to G \) over \( G \), meaning a morphism of simplicial diagrams

\[
\begin{array}{ccc}
V \times G \times G & \to & G \times G \\
| & | & | \\
V \times G & \to & G \\
| & | & | \\
V & \to & * \\
\end{array}
\]

such that the corresponding diagram of effective epimorphisms

\[
\begin{array}{ccc}
V & \to & * \\
| & | & | \\
V//G := \lim \to V \times G^{\times n} & \to & BG \\
\end{array}
\]

(recalling [56] that by the axioms of an ∞-stack (∞,1)-topos every groupoid object is effective) is a homotopy pullback. Conversely, this means that an action of \( G \) on \( V \) is a fibration sequence \( V \to V//G \to BG \). We call the object \( V//G \) with this structure of a groupoid object the action groupoid of \( G \) acting on \( V \). In \( H = Top \), \( V//G \) is usually thought of as the Borel construction \( EG \times_G V \).

For \( V \xrightarrow{i_\rho} V//G \xrightarrow{p_\rho} BG \), a fibration sequence encoding an action \( \rho \) of the ∞-group \( G \) on a space \( V \) as above and for \( g : X \to BG \) a \( BG \)-cocycle on a space \( X \), we call the fibration \( E \to X \) obtained as the homotopy pullback

\[
\begin{array}{ccc}
E & \to & V//G \\
| & p_E & | \\
X & \xrightarrow{g} & BG \\
\end{array}
\]

the \( V \)-bundle \( \rho \)-associated to the \( G \)-principal bundle \( P_g \to X \) classified by \( g \). A (homotopy) section \( \sigma \) of such an \( E \to X \) is a homotopy commutative diagram

\[
\begin{array}{ccc}
E & \to & V//G \\
| & p_E & | \\
X & \xrightarrow{\sigma} & X \\
\end{array}
\]
By the universal property of the homotopy pullback which defines \( E \), this is equivalent to a lift of the cocycle \( g \) through the morphism \( p_\rho \), i.e. a homotopy commutative diagram

\[
\begin{array}{ccc}
V/\!/G & \xrightarrow{\sigma} & X \\
\downarrow & & \downarrow g \\
\ast & \xrightarrow{\rho} & BG
\end{array}
\]

where by abuse of notation we denote the lift by the same symbol as the section it corresponds to. As we shall see in the next section, this means that we can identify the collection \( \Gamma(E) \) of sections \( \sigma \) of the bundle \( E \to X \) which is \( \rho \)-associated to the bundle \( P \to X \) classified by \( g \in H(X, BG) \) as the \([g]\)-twisted \( V/\!/G \)-cohomology on \( X \):

\[
\Gamma(E) := H_{[g]}(X, V/\!/G) \to H(X, BG).
\]

When comparing the notion of sections with the general discussion of twisted cohomology in the next section, notice that if the fibration sequence \( V \to V/\!/G \to BG \) through which the section is a lift extends one more step to the right as \( BG \to C \), then \( k \circ g \) is the obstruction for \( E \to X \) to admit a section. But in applications one is often interested in associated bundles which always admit at least the trivial section \( X \to X/\!/G \), such as (higher) vector bundles.

2.2.2. Local (semi)trivialization. The statement that every principal \( \infty \)-bundle \( P \to X \) becomes trivializable when pulled back to its own total space is just another way to read the defining homotopy pullback square:

\[
\begin{array}{ccc}
P & \xrightarrow{g} & X \\
\downarrow & & \downarrow \\
\ast & \xrightarrow{\rho} & BG
\end{array}
\]

The pullback of the classifying map \( g \) of \( P \) to the total space of \( P \) is just the composite classifying map \( P \to X \to BG \) and that this is trivializable is the statement that the above square commutes up to homotopy. A relative version of this statement is useful when the coefficient object \( BG \) is an extension, in that it sits in a fibration sequence

\[
\begin{array}{ccc}
BK & \longrightarrow & BG \\
\downarrow & & \downarrow \\
B \ast & \longrightarrow & BH
\end{array}
\]

This is in particular these case for \( G = \text{String}(n) \) and \( G = \text{Fivebrane}(n) \) which are shifted central extensions

\[
\begin{array}{ccc}
B^2U(1) & \longrightarrow & B\text{String}(n) \\
\downarrow & & \downarrow \\
B\ast & \longrightarrow & B\text{Spin}(n)
\end{array}
\]

and

\[
\begin{array}{ccc}
B^6U(1) & \longrightarrow & BFivebrane(n) \\
\downarrow & & \downarrow \\
B\ast & \longrightarrow & B\text{String}(n)
\end{array}
\]

In such a case consider the composite homotopy pullback diagram

\[
\begin{array}{ccc}
Q & \longrightarrow & BK \\
\downarrow & & \downarrow \\
X & \longrightarrow & BG
\end{array}
\]

where the right square is the homotopy pullback exhibiting the fibration sequence of coefficient objects, and where \( Q \) is by definition the homotopy pullback in the left square. Since homotopy pullback squares compose
to homotopy pullback squares, also the total rectangle here is a homotopy pullback and hence exhibits $Q$ as the $H$-principal $\infty$-bundle underlying the $G$-principal $\infty$-bundle that is classified by $g : X \to BG$.

This means that for $G$ an extension as above, $G$-principal $\infty$-bundles may be encoded as $K$-principal $\infty$-bundles on the total space of their underlying $H$-principal $\infty$-bundles. This we may call a local semi-trivialization of the $G$-principal $\infty$-bundle: instead of pulling it back to its total space where it trivializes, we just pull it back to the total space of its underlying $H$-principal bundle where it doesn’t necessarily trivialize but does reduce to a $K$-principal $\infty$-bundle.

This construction is, more or less implicitly, the way by which one sees $\text{String}(n)$-bundles encoded in the literature in terms of abelian gerbes on total spaces of Spin-bundles: for the fibration sequence $B^3 U(1) \to B \text{String}(n) \to B \text{Spin}(n)$ the above says that a $\text{String}(n)$-principal bundle is encoded in a $B\text{U}(1)$-principal bundle on the total space $Q$ of the underlying $\text{Spin}(n)$-principal bundle.

One can in fact say more: the $K$-principal $\infty$-bundle on the total space $Q$ remembers that it came from a local semi-trivialization of a $G$-principal $\infty$-bundle in that its restriction to the $K$-fibers of $Q$ is classified by the cocycle that controls the extension $K \to G \to H$. To see this, consider the diagram

$$
\begin{array}{ccc}
G & \longrightarrow & P \\
\downarrow & & \downarrow \\
H \simeq Q_x & \longrightarrow & Q \\
\downarrow & & \downarrow \\
* & \longrightarrow & BK \\
\downarrow & & \downarrow \\
x & \longrightarrow & X \\
\downarrow & & \downarrow \\
* & \longrightarrow & BG \\
\downarrow & & \downarrow \\
& \longrightarrow & BH
\end{array}
$$

where $x : * \to X$ is any point in $X$ and where every single square and hence all composite rectangles are homotopy pullbacks. This says that the restriction of the $K$-cocycle $Q \to BK$ to the fiber $Q_x \simeq H$ of $Q$ over $x$ is the cocycle that classifies the extension $G \to H$.

In the example of $\text{String}(n)$-principal bundles this says that a $\text{String}(n)$-bundle may be characterized by a $B\text{U}(1)$-principal $\infty$-bundle ($\simeq$ a $U(1)$-bundle gerbe) on the total space of the underlying $\text{Spin}(n)$-principal bundle, which has the special property that restricted to each $\text{Spin}(n)$-fiber its class is the generator of $H^3(\text{Spin}, \mathbb{Z})$. This description of $\text{String}(n)$-bundles by abelian gerbes on total spaces of $\text{Spin}(n)$-bundles is the approach taken in [83]. All this is just yet another aspect of the general mechanism of twisted nonabelian cohomology discussed in the next section. More details are given in [70].

2.3. Twisted (nonabelian) cohomology. In the general context of cohomology in $H$ as above, we describe a general notion of twisted (nonabelian) cohomology that subsumes the nonabelian twisted String- and Fivebrane structures discussed here as well as the ordinary notion of twisted abelian (Eilenberg-Steenrod-type) cohomology. Moreover, our definition of non-flat differential cohomology in section 2.2.1 will be conceived as curvature-twisted flat differential cohomology. This way the differential twisted String- and Fivebrane-structures in section 5.1 are realized as cocycles in bi-twisted cohomology, as described there: one twist being the topological twist, the other being the non-vanishing curvature. All these twists are on the same general footing described here.

2.3.1. Introduction. The now standard example of twisted cohomology is twisted K-theory: let $A$ be a degree-0 space in a K-theory spectrum, i.e. for instance $A = \mathbb{Z} \times BU$ or $A = Fred(H)$, the space of Fredholm operators on a separable Hilbert space $H$. There is a canonical action on this space of the projective unitary group $G = PU(H)$ of $H$. Since $PU(H)$ has the homotopy type of an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, a $PU(H)$-principal bundle $P \to X$ defines a class $c \in H^3(X, \mathbb{Z})$ in ordinary integral cohomology.

The twisted K-theory (in degree 0) of $X$ with that class as its twist is the set of homotopy classes of sections $X \to P \times_{PU(H)} Fred(H)$ of the associated bundle. This generalizes straightforwardly to the case
that $A$ is an infinite loopspace with a (topological) group $G$ acting on it. The for a specified $G$-principal bundle $P \to X$ one says that the collection of homotopy classes of sections $X \to P \times_G A$ (where $P \times_G A \to X$ is the associated bundle) is the twisted $A$-cohomology of $X$ with the twist specified by the class of $P$.

**Remark.** If $A$ is (the degree 0-space of) a spectrum, the associated bundle $P \times_G A$ is in general no longer itself a spectrum: twisted abelian cohomology is not an example of generalized (Eilenberg-Steenrod) cohomology. To stay within the spectrum point of view, May and Sigurdsson suggested [60] that twisted abelian cohomology should instead be formalized in terms of parameterized homotopy theory, where one thinks of $P \times_G A$ as a parameterized family of spectra. But $P \times_G A$ is in any case a coefficient object in nonabelian cohomology.

By the above discussion, the action of a group $G$ on an object $A$ is entirely encoded in the corresponding action groupoid fibration sequence

$$A \to A//G \to BG$$

In that case, the object $A//G$ is traditionally modeled in terms of the Borel construction and written $A_G = A//G \simeq EG \times_G A$. This is the $A$-bundle associated to the universal $G$-principal bundle. Moreover, one can see, as described in detail below, that for a given $G$-principal bundle $P \to X$ that is classified by an element $[c] \in \pi_0 \text{Top}(X,BG)$, otherwise known as $H^1(X,G)$, the set of homotopy classes of sections $X \to P \times_G A$ is the set of connected components of the homotopy pullback

$$
\text{Top}_c(X,A) \to \text{Top}(X,A//G) \to \text{Top}(X,BG).
$$

This suggests a general notion of twisted cohomology in any context $H$ and for twists more general than given by a group action: For

- any fibration sequence $A \to B \to C$ in $H$
- and for any $B$-cocycle $c \in H(X,B)$

it makes sense to say that the connected components $H^*_c(X,A) := \pi_0 H_c(X,A)$ in the homotopy pullback

$$
H_c(X,A) \to \text{Top}(X,BG) \to \text{Top}(X,C).
$$

are the $[c]$-twisted $A$-cohomology classes of $X$.

2.3.2. **Obstruction theory.** It is helpful to think of this in terms of the obstruction problem in cohomology: let $A \to B \to C$ be a fibration sequence in $H$, i.e. a sequence such that the square

$$
\begin{array}{ccc}
A & \to & * \\
\downarrow & & \downarrow \\
B & \to & C
\end{array}
$$

is a homotopy pullback square, with $*$ denoting the generalized point/trivial object. Since the $\infty$-groupoid-valued hom in an $(\infty,1)$-category is exact with respect to homotopy limits, it follows that for every object
X, there is fibration sequence of cocycle ∞-groupoids

\[
\begin{array}{ccc}
\mathcal{H}(X, A) & \to & * \\
\downarrow & & \downarrow \text{const}_* \\
\mathcal{H}(X, B) & \to & \mathcal{H}(X, C)
\end{array}
\]

This may be read as:
- the obstruction to lifting a \(B\)-cocycle to an \(A\)-cocycle is its image in \(C\)-cohomology (all with respect to the given fibration sequence)

But it also says:
- \(A\)-cocycles are, up to equivalence, precisely those \(B\)-cocycles whose class in \(C\)-cohomology is the trivial class (given by the trivial cocycle \(\text{const}_*\)).

This motivates the following definition

**Definition 2.2.** For
- \(A \to B \to C\) a fibration sequence in \(\mathcal{H}\);
- \(X \in \mathcal{H}\) an object of \(\mathcal{H}\);
- and \(c \in \mathcal{H}(X, C)\) a \(C\)-cocycle on \(X\)

the \(c\)-twisted \(A\)-cohomology of \(X\) is the the set of equivalence classes

\[
\mathcal{H}_c(X, A) := \pi_0 \mathcal{H}_c(X, A)
\]

of the \(\infty\)-groupoid \(\mathcal{H}_c(X, A)\) that is defined as the homotopy pullback

\[
\begin{array}{ccc}
\mathcal{H}_c(X, A) & \to & * \\
\downarrow & & \downarrow c \to * \\
\mathcal{H}(X, B) & \to & \mathcal{H}(X, C)
\end{array}
\]

Notice well that, compared to the previous fibration sequence arising in the obstruction problem, the homotopy limit in the above definition replaces the trivial cocycle \(\text{const}_*\) by the prescribed \(C\)-cocycle \(c\).

More generally, let \(\mathcal{H}(X, C) \to \mathcal{H}(X, C)\) be a section of the projection \(\mathcal{H}(X, A) \to \pi_0 \mathcal{H}(X, A) =: \mathcal{H}(X, A)\) which picks one representative \(C\)-cocycle on \(X\) in each cohomology class. Then the **total \(C\)-twisted \(A\)-cohomology** defined by the fibration sequence \(A \to B \to C\) is the set of connected components \(H_{tw}(X, A) := \pi_0 \mathcal{H}_{tw}(X, A)\) of the homotopy pullback

\[
\begin{array}{ccc}
\mathcal{H}_{tw}(X, A) & \to & \mathcal{H}(X, C) \\
\downarrow & & \downarrow \\
\mathcal{H}(X, B) & \to & \mathcal{H}(X, C)
\end{array}
\]

Notice that this is a slight abuse of notation, which however shouldn’t be harmful: the twisted cohomology \(H_{tw}(X, A)\) does depend on the choice of fibration sequence that defines it. We choose to suppress this in the notation, as in applications the fibration sequence will always be understood. This total \(C\)-twisted cohomology comes hence naturally with two projections

\[
\begin{array}{ccc}
\mathcal{H}_{tw}(X, A) & \to & \mathcal{H}(X, C) \\
\downarrow u & & \downarrow \text{tw} \\
\mathcal{H}(X, B) & \to & \mathcal{H}(X, C)
\end{array}
\]

For \([\lambda] \in H_{tw}(X, A)\) a twisted cocycle
- \(u[\lambda]\) is the underlying \(B\)-cocycle;
• tw[λ] is the C-valued twist.

2.3.3. In terms of sections. To see that the above indeed reproduces the description in terms of sections of associated bundles, consider the $A$-bundle $A//G \simeq EG \times_G A \to BG$ associated to the universal $G$-principal $\infty$-bundle following section 2.2. Then then $A$-bundle $E = P \times_G A$ associated to a $G$-principal $\infty$-bundle $P$ classified by a morphism $g : X \to BG$ is the homotopy pullback

$$
E \simeq P \times_G A \longrightarrow A//G \simeq EG \times_G A .
$$

And again it is precisely the universal property of the homotopy pullback that asserts that sections $X \to P$ of this bundle are in bijection, up to homotopy, with those maps $X \to A//G$ whose projection to $X \to BG$ reproduces the prescribed twist $g$. In summary we have:

**Proposition.** The connected components of $H[c](X, A)$ are in bijection with the homotopy classes of sections of the $A$-bundle $P \to X$ associated to the fibration classified by $c : \pi_0 \Gamma(P) \simeq H[c](X, A)$.

2.3.4. Local semi-trivializations as twisted cohomology. In the light of this general notion of twisted cohomology, reconsider the notion of local semi-trivializations from section 2.2.2. We had seen that for $BK \to BG \to BH$ a given fibration sequence in $\mathbf{H}$, $G$-principal $\infty$-bundles on $X$ may be characterized by $K$-principal $\infty$-bundles on the underlying $H$-principal $\infty$-bundle $Q \to X$. With the above language of twisted cohomology, this means conversely, with the cocycle $c : X \to BH$ of the underlying $H$-principal $\infty$-bundle fixed, the $K$-principal bundles on $Q$ which characterize $G$-principal bundles on $X$ are cocycles in the $[c]$-twisted $K$-cohomology $H[c](X, BK)$ on $X$.

Typically in applications – notably in the cases $G = \text{Spin}, \text{String}, \text{Fivebrane}$, the $\infty$-group $K$ is abelian (in these examples it is $\mathbb{Z}_2$, $BU(1)$ and $B^3U(1)$, respectively), so that in these cases the above says that twisted abelian $K$-cohomology characterizes nonabelian $G$-cohomology.

We may summarize this by the following **Principle:**

- nonabelian cohomology may disguise as twisted abelian cohomology;
- conversely: twisted higher abelian cohomology is really a special case of nonabelian cohomology.

2.4. Differential nonabelian cohomology. We now refine to differential nonabelian cohomology by describing a theory that

- generalizes the notion of differential cohomology from Eilenberg-Steenrod-type (“abelian” or “stable”) cohomology to nonabelian cohomology;
- generalizes the notion of connection on a bundle together with the notion of parallel transport from principal bundles to higher gerbes and principal $\infty$-bundles.

**Remark 2.3.** This is based on a definition of differential nonabelian cohomology that works in great generality in any $(\infty, 1)$ – topos whose underlying topos is a lined topos: a topos equipped with a line object that induces an interval object which in turn induces a notion of paths. In low categorical dimension this reproduces the description in [10, 72, 74]. From this one obtains differential nonabelian cocycles that are encoded by Ehresmann $\infty$-connections, as we introduce below. If $\mathbf{H}$ is a smooth $(\infty, 1)$-topos (in the sense of smooth toposes in synthetic differential geometry), it admits a notion of $\infty$-Lie theory and these Ehresmann $\infty$-connections refine to Cartan-Ehresmann $\infty$-connections expressed in terms of $\infty$-Lie algebroid valued differential forms on the total space of a principal $\infty$-bundle. These are the structures introduced and studied in [63]. Below we recall them and describe their conceptual origin.

Throughout this subsection, $\mathbf{H}$ will always denote a smooth $(\infty, 1)$-topos. While it is well known that differential abelian cohomology models - and was largely motivated by the description of - abelian gauge fields in quantum field theory, many natural examples are in fact differential nonabelian cocycles, such as the differential refinements of String- and Fivebrane-structures that we are interested in here, but also...
for instance the entire field content of supergravity theories in the first order D’Auria-Fré-formulation of supergravity [27, 68]. Using all this, in section 5 we exhibit classes of cocycles in differential nonabelian cohomology arising from obstruction problems in twisted cohomology that capture the anomaly cancellation Green-Schwarz mechanisms in quantum field theory.

The following sections give an overview. The details will be described elsewhere [70].

2.4.1. Idea. The most general notion of cohomology $H(X, A)$ of an object $X$ with coefficients in an object $A$ supposes that $X$ and $A$ are both objects of $H$ and is defined by

\begin{equation}
H(X, A) := \text{Ho}_H(X, A) := \pi_0 H(X, A),
\end{equation}

where $\text{Ho}_H$ is the homotopy category of $H$, whose hom-sets are the connected components of the $\infty$-groupoid $H(X, A)$ of maps, homotopies between maps, homotopies between homotopies etc., from $X$ to $A$. See [56] for these notions. Since $A$ could be but is not required to be a (connective) spectrum, this is more general than what is called generalized (Eilenberg-Steenrod) cohomology: both in that

- $A$ need not be abelian or an $\infty$-loop space. It may in particular be an arbitrary homotopy $n$-type for any $0 \leq n \leq \infty$. In particular $A$ may be an arbitrary $\infty$-groupoid, possibly with extra structure, such as the smooth structure of an $\infty$-Lie groupoid.
- $X$ and $A$ may have more structure than just topological spaces, for instance they may have smooth structure in that they are parameterized over smooth test spaces is a suitable way. Such parameterized objects are called $\infty$-stacks – for instance smooth $\infty$-stacks.

This general notion is sometimes called nonabelian cohomology. A standard example of an object in parameterized nonablian cohomology is a nonabelian gerbe: being a stack (with extra properties) it is parameterized over the site it lives on, usually taken to be that of open subsets of the base space. More generally nonabelian cohomology classifies principal $\infty$-bundles, and principal $\infty$-groupoid bundles.

For ordinary abelian (Eilenberg-Steenrod-type) cohomology, there is a well known prescription for how to refine that to differential cohomology. Differential cohomology is to cohomology as fiber bundles are to bundles with connection.

2.4.2. Overview of the theory. We list the basic steps of definitions, constructions and theorems along which the theory proceeds.

2.4.3. The path $\infty$-groupoid. In order to extract differential cohomology in the context given by some $H$ we need to have a notion of parallel transport along paths in the objects of $H$. This is encoded by assigning to each object $X$ its path $\infty$-groupoid $\Pi(X)$. Morphisms $\Pi(X) \to A$ may be thought of as encoding flat $A$-valued parallel transport on $X$ or equivalently $A$-valued local systems on $X$. This assignment has a right adjoint

\begin{equation}
H \leftarrow H : \text{flat}.
\end{equation}

2.4.4. Flat differential cohomology. In the presence of a notion of path $\infty$-groupoid we take flat differential $A$-valued cohomology to be the cohomology with coefficients in an object $A_{\text{flat}}$ in the image of this right adjoint, and write

\begin{equation}
H_{\text{flat}}(X, A) := H(\Pi(X), A) \cong H(X, A_{\text{flat}}).
\end{equation}

There is a natural morphism $X \to \Pi(X)$ that includes each object as the collection of 0-dimensional paths into its path $\infty$-groupoid. This induces correspondingly a natural morphism of coefficient objects $A_{\text{flat}} \to A$. Lifting an $A$-cocycle $X \to A$ through this morphism to a flat differential $A$-cocycle means equipping it with a flat connection.
2.4.5. Differential cohomology with curvature classes. We identify general non-flat differential nonabelian cocycles with the obstruction to the existence of the lift through $A_{\text{flat}} \rightarrow A$ from bare $A$-cohomology to flat differential $A$-cohomology. There are two flavors of this obstruction theory whose applicability depends on whether $A$ is at least once deloopable or more generally nonabelian. We survey the deloopable case for simplicity: when $A$ is once deloopable, the morphism $A_{\text{flat}} \rightarrow A$ fits into a fibration sequence

$$A_{\text{flat}} \rightarrow A \rightarrow BA_{dR}$$

where $BA_{dR}$ is the coefficient for nonabelian de Rham cohomology with coefficients in $BA$: a cocycle $X \rightarrow BA_{dR}$ is a flat differential $BA$-cocycle whose underlying ordinary $A$-cocycle is trivial.

This being a fibration sequence means that the obstruction to lifting an $A$-cocycle $X \rightarrow A$ to a flat differential $A$-cocycle is the $BA_{dR}$-cocycle given by the composite map $X \rightarrow A \rightarrow BA_{dR}$. The class of this $BA$-cocycle we call the curvature characteristic class of the original $A$-cocycle. If this obstruction does not vanish but is given by some fixed curvature characteristic class $P$, there is a general notion of twisted cohomology that encodes the $P$-twisted-flat (that is: general) differential cocycles. This curvature-twisted flat differential nonabelian cohomology is finally our definition of differential nonabelian cohomology: For $A$ a (once deloopable) object in $H$ and for $[P] \in H(X, BA_{dR})$ a fixed curvature characteristic class, the differential $A$-cocycles with curvature characteristic $P$ are the elements in the the homotopy pullback

$$H_{[P]}(X, A) \rightarrow \ast \rightarrow H(X, A) \rightarrow H(X, BA_{dR})$$

One can show that for $X$ an ordinary smooth space the objects in $H_{[P]}(X, A)$ correspond to diagrams in the model $\text{SPSh}(\mathcal{C})_{\text{loc}}$ of $H$ of simplicial presheaves $[47].$

where $Y \xrightarrow{\sim} X$ is a given hypercover (cofibrant replacement). Here the interpretation of each of the horizontal layers is as indicated, as shown by the next step.

2.4.6. Ehresmann $\infty$-connection. It can be shown that the above differential cocycles constitute an $\infty$-groupoidal version of the notion of Ehresmann connection. This is achieved by noticing that every cocycle $X \rightarrow A$ trivializes on the total space $P \rightarrow X$ of the principal $\Omega A \infty$-bundle $P$ that it classifies. On $P$, we have the vertical path $\infty$-groupoid $\Pi_{\text{vert}}(P)$ of fiberwise paths. It can be shown that every differential cocycle encoded by a diagram as above (with $A$ fibrant in the given model structure on simplicial presheaves) gives rise to a diagram

$$\Pi_{\text{vert}}(P) \rightarrow A \rightarrow \ast \rightarrow \Pi(P) \rightarrow \ast \rightarrow \Pi(X) \rightarrow \ast$$

where $Y \xrightarrow{\sim} X$ is a given hypercover (cofibrant replacement). Here the interpretation of each of the horizontal layers is as indicated, as shown by the next step.
where each horizontal morphism is now a cocycle in nonabelian de Rham cohomology.

A principal $\infty$-bundle $P$ equipped with such nonabelian de Rham cocycle data we call an *Ehresmann $\infty$-connection* as it generalizes the notion of Ehresmann connection on ordinary principal bundles. Its expression in terms of $\infty$-Lie algebroid valued differential forms is given by the next step.

2.4.7. *Cartan-Ehresmann $\infty$-connection.* The above is still a generalization of the notion of Ehresmann connection to an ambient context $H$ that need not necessarily have an ordinary notion of differential forms. To obtain that we assume for the following that $H$ is actually a *smooth $(\infty,1)$-topos*, an analog of an ordinary smooth topos in the sense of synthetic differential geometry [61]. This is for instance obtained by taking $H$ to be the $(\infty,1)$-category of $\infty$-stacks on a site of smooth loci [61].

In that context we have a notion of $\infty$-*Lie theory*, which allows us to replace in the above diagram all $\infty$-Lie groupoids $A$ appearing with their sub-$\infty$-groupoids of *morphisms of infinitesimal extension*: the $\infty$-Lie algebroid $a = \text{Lie}(A)$. In particular the $\infty$-Lie algebroid of the path $\infty$-groupoid $\Pi(X)$ is the $\infty$-groupoid $\Pi^\text{inf}(X)$ of *infinitesimal* paths in any dimension. For $X$ an ordinary manifold, this is known as the *tangent Lie algebroid* of $X$. Then every parallell transport morphism $\text{tra} : \Pi(X) \to A$ factors as

\[
\begin{array}{ccc}
\Pi^\text{inf}(X) & \xrightarrow{\omega} & a \\
\downarrow & & \downarrow \\
\Pi(X) & \xrightarrow{\text{tra}} & A
\end{array}
\]

and the top horizontal morphism is a *collection* of flat $\infty$-Lie algebroid valued differential forms.

An $\infty$-Lie algebroid of the form $g := \text{Lie}(BG)$ is an $L_\infty$-algebra, a higher generalization of a Lie algebra. Just as every Lie algebra comes with its Chevalley-Eilenberg dg-algebra, so does every $L_\infty$-algebra and every $\infty$-Lie algebroid. It can be shown that if the object $X \in H$ is presented by a simplicial presheaf, that is a simplicial manifold $X_\bullet$, then the Chevalley-Eilenberg algebra of $\Pi^\text{inf}(X)$ is weakly equivalent to the *simplicial de Rham complex* $\Omega^\bullet(X)$ [32] – the total complex of the double complex

\[
\begin{array}{c}
\Omega^p(X_{q+1}) \\
\xrightarrow{d_{\text{deR}}} \\
\sum_i (-1)^i \delta_i^*
\end{array}
\hspace{1cm}
\begin{array}{c}
\xrightarrow{d_{\text{deR}}} \\
\sum_i (-1)^i \delta_i^*
\end{array}
\Omega^{p+1}(X_{q+1})
\]

coming from the de Rham differential and the differential given by alternating sums of pullback along face maps:

\[
(2.15) \quad \text{CE}(\Pi^\text{inf}(X)) \simeq \Omega^\bullet(X).
\]

Using this we shall write in general $\Omega^\bullet(X)$ for the Chevalley-Eilenberg algebra of $\Pi^\text{inf}(X)$ for any object $X$ and speak of the de Rham complex on $X$. 
This way, by first restricting along $\Pi^\infty(X) \hookrightarrow \Pi(X)$ to infinitesimal (higher) paths and then passing to Chevalley-Eilenberg dg-algebras of $L_\infty$-algebroids, the above diagram characterizing an Ehresmann $\infty$-connection translates into a diagram of graded commutative dg-algebras

$$
\begin{align*}
\Omega^*_{vert}(P) &\xrightarrow{A_{vert}} \text{CE}(a) & \text{flat vertical differential forms on fibers} \\
\Omega^*(P) &\xrightarrow{(A,F_a)} W(a) & \text{first Ehresmann condition} \\
\Omega^*(X) &\xrightarrow{P(F_a)\text{ inv}(a)} \text{curvature characteristic forms on base space}
\end{align*}
$$

Here $W(a)$ is the Weil algebra of $a$. Each horizontal morphism here represents a collection of $L_\infty$-algebra (or even $\infty$-Lie algebroid)-valued differential forms, as described in detail in [68]. This we call a Cartan-Ehresmann $\infty$-connection on the $\infty$-bundle $P$. These are the $L_\infty$-algebra connections that have been introduced and discussed in [68].

3. Twisted String- and Fivebrane structures in String theory

With the general formalism of twisted nonabelian cohomology set up, we now describe its applications in String theory.

3.1. Twisted structures: anomalies in cohomological terms. In this section we consider target space anomalies in string theory described in terms of (twisted) ordinary cohomology. Later in section 5 we extend the discussion to differential twisted cohomology and exhibit the differential form refinement of the discussion here.

Three kinds of anomalies in string theory. The following are expressed bilingually in the languages of physicists and of algebraic topologists. All of the examples involve a manifold $X$ (or $M$), its tangent bundle $TX$, and possibly a ‘gauge bundle’ $E$ on $X$. Recall that a Spin structure on an oriented $n$-manifold $M$ can be interpreted as a lifting of the classifying map $\chi : M \to BSO(n)$ of $TM$ to $BSpin(n)$. The existence of such a lift requires the vanishing of the Stiefel-Whitney class $w_2 \in H^2(M, \mathbb{Z}_2)$.

1. The Freed-Witten anomaly: This is a global worldsheet anomaly of type II string theory in the presence of D-branes and a nontrivial $H_3$-field. The statement for the cancellation of the anomaly is that “a D-brane can wrap a cycle $Q \to X$ in a ten-dimensional spacetime $X$” only if

$$
W_3(Q) + [H_3]|_Q = 0 \in H^3(X^{10}, \mathbb{Z})
$$

(3.1)

where $W_3(Q)$ is the third integral Stiefel-Whitney class of $TQ$. (Note that in the physics literature, ‘D-brane’ is sometimes used interchangeably with ‘cycle.’) We shall see later that the cancellation is encoded in the $c$-twisted cohomology where $c$ is the class $[H_3]|_Q$. When $H_3|_Q$ is trivial in cohomology, i.e. $H_3|_Q = dB_2$, the Freed-Witten condition states that the D-brane must be Spin$^c$.

The vanishing of $W_3$ allows the existence of a Spin$^c$-structure. Hence here $H_3|_Q$ is sometimes referred to as a twist of the Spin$^c$-structure, in the sense of a Spin$^c$-structure relative to $H_3|_Q$. Diagrammatically, this means that there is a coboundary/homotopy $W_3(Q) \xrightarrow{\eta} H_3|_Q$

$$
\begin{align*}
Q &\xrightarrow{f} BSpin(n) \\
H_3|_Q &\xrightarrow{W_3} K(\mathbb{Z}, 3)
\end{align*}
$$

All spaces involved here can be taken to be ordinary topological spaces, all morphisms ordinary continuous maps between these and all 2-morphisms ordinary homotopies between those as long as one considers just...
the topological classes. On the other hand, the analogous discussion for the differential refinement of this situation requires all spaces here to be replaced with generalized smooth spaces.

2. The Green-Schwarz anomaly: This is an anomaly in heterotic and type I string theory, i.e. a string theory coupled to a gauge theory, with an $H_3$-field. The cancellation of the anomaly is via the Green-Schwarz anomaly cancellation mechanism [38] which amounts to canceling a gravitational anomaly, coming from the coupling of fermions to the gravity part of the action, with a gauge anomaly, coming from the coupling of fermions to the gauge field in the gauge bundle $E \to X$. The process requires the following condition to hold

$$p_1(TX) - ch_2(E) = 0 \in H^4(X; \mathbb{Z}).$$

This formula in cohomology is trivialized by $H_3$, i.e. at the level of differential forms, the expression with representatives in place of classes is equal to $dH_3$. Mathematically (cf. [33]), the above two contributions correspond to the Pfaffian line bundle $\text{Pfaff}$ and an electric charge line bundle $\mathcal{L}_e$, and the statement is that the anomaly line bundle $\text{Pfaff} \otimes \mathcal{L}_e$ needs to be trivialized. The local (global) anomaly is the curvature (holonomy) of this line bundle. As for Freed-Witten, the Green-Schwarz anomaly is encoded in the $c$-twisted cohomology where $c$ is the class $ch_2(E)$.

3. The dual Green-Schwarz anomaly: This is also an anomaly in heterotic and type I string theory, but now with an $H_7$-field in the dual formulation of the theory [22]. The cancellation of the anomaly is via the dual of the above Green-Schwarz anomaly cancellation mechanism [67] [37] [64]. The process requires the following condition to hold

$$\frac{1}{48}p_2(X) - ch_4(E) + \frac{1}{48}p_1(X)ch_2(E) - \frac{1}{64}p_1(X)^2 = 0 \in H^8(X; \mathbb{Z}).$$

This formula in cohomology is trivialized by $H_7$, i.e. at the level of differential forms, the expression with representatives in place of classes is equal to $dH_7$. Mathematically (cf. [33]), the statement is that the anomaly line bundle $\text{Pfaff} \otimes \mathcal{L}_m$, where $\mathcal{L}_m$ is the magnetic charge line bundle, needs to be trivialized. As above, the trivialization is encoded in the $H_7$-twisted cohomology.

The vanishing of the first fractional Pontrjagin class $\frac{1}{2}p_1(X)$ of a Spin-manifold $X$ is also known as the condition for $X$ for admitting a String structure [50], i.e. a lifting of the structure group of the tangent bundle from Spin($n$) to String($n$). (Notice that in homotopy theory and in physics the class $\frac{1}{2}p_1 \in H^4(X, \mathbb{Z})$, which is well-defined on a Spin manifold, is sometimes called $\lambda$.) We continue this pattern, showing that each anomaly cancellation corresponds to a specific twisted structure. Namely, in Freed-Witten we will see a twisted Spin structure, in Green-Schwarz a twisted String structure and in dual Green-Schwarz a twisted Fivebrane structure.

We now consider twisted structures and make the following definitions, which originate in [54] and which we have already interpreted in section 2 in terms of twisted nonabelian cohomology.

**Definition 3.1.** An $\alpha$-twisted String structure (or a String structure relative to $\alpha$) on a Spin manifold $M$ with classifying map $f : M \to B\text{Spin}(n)$ is a cocycle $\alpha : M \to K(\mathbb{Z}, 4)$ and a homotopy $\eta$:

$$\begin{array}{ccc}
M & \xrightarrow{f} & B\text{Spin}(n) \\
\downarrow{\alpha} & \nearrow{\eta} & \downarrow{\frac{1}{2}p_1} \\
& K(\mathbb{Z}, 4) & 
\end{array}$$

If $\alpha$ is trivial (e.g. factors through a point) then this reduces to an ordinary String-structure. Analogously for twisted Fivebrane-structures:
Definition 3.2. An $\alpha$-twisted Fivebrane structure (or a Fivebrane structure relative to $\alpha$) on a String manifold $M$ with classifying map $f: M \to B\text{String}(n)$ is a cocycle $\alpha: M \to K(\mathbb{Z}, 8)$ and a homotopy $\eta$:

\begin{equation}
\begin{array}{c}
M \\
\downarrow \alpha \\
K(\mathbb{Z}, 8)
\end{array}
\xrightarrow{\eta} \begin{array}{c}
B\text{String}(n)
\end{array}
\end{equation}

(3.6)

If $\alpha$ is trivial (e.g. factors through a point) then this reduces to an ordinary Fivebrane-structure.

**Notation.** We fix once and for all connections $\omega$ and $A$ on the Spin bundles and the gauge bundles respectively. The corresponding curvatures are $F_\omega$ and $F_A$, respectively. We will use these to give differential form representatives of the corresponding characteristic classes. We will use the convention of writing a class with argument the curvature form to indicate the differential form representative of the class, written with argument the corresponding bundle. For instance, $p_i(TM) = p_i(M)$ means the cohomology class while $p_i(F_\omega)$ will mean the differential 4-$i$-form representative.

3.2. **Twisted string structures. Relative trivialization on branes.** As the example of the twisted Spin$^c$-structures, discussed in the introduction, already indicates, in string theory such structures usually arise on branes $M$ (in our purposes, but for more detailed discussion, see e.g. [21]) sitting in an ambient space $X$, $\iota: M \to X$, and the twist is by the restriction $\alpha := \beta|_M := \iota^* \beta$:

\begin{equation}
\begin{array}{c}
X \\
\downarrow \beta \\
K(\mathbb{Z}, n)
\end{array}
\xrightarrow{\iota = \iota^* \beta} \begin{array}{c}
M \\
\downarrow \iota
\end{array}
\end{equation}

(3.7)

of a class of the ambient space to the brane. Since this special case of twisted structures is important in applications, we state it as separate definition:

**Definition 3.3.** A $\beta$-twisted String structure on a brane $\iota: M \to X$ with Spin structure classifying map $f: M \to B\text{Spin}(n)$ is a cocycle $\beta: X \to K(\mathbb{Z}, 4)$ and a homotopy $\eta$:

\begin{equation}
\begin{array}{c}
M \\
\downarrow \iota \\
X \\
\downarrow \beta
\end{array}
\xrightarrow{\eta} \begin{array}{c}
B\text{Spin}(n)
\end{array}
\xrightarrow{\eta} \begin{array}{c}
\frac{1}{2} p_1
\end{array}
\xrightarrow{\iota = \iota^* \beta} \begin{array}{c}
X \\
\downarrow \beta
\end{array}
\xrightarrow{\iota = \iota^* \beta} \begin{array}{c}
K(\mathbb{Z}, n)
\end{array}
\end{equation}

(3.8)

This is essentially the definition also given in [84]. This situation arises with $X$ being the 11-dimensional M-theory target space and $M = \partial X$, its 10-dimensional boundary, being the target for the heterotic string.

3.2.1. **A refinement for further divisibility of $\frac{1}{2} p_1$.** We have given above the definition of a twisted String structure, essentially following [34], see definitions [33] and [33]. In version 1 of the eprint [34], it was also anticipated [3] that this is related to Witten’s quantization condition. In fact, it is essentially the same, because of the extra factor of $\frac{1}{2}$ in front of $\frac{1}{2} p_1$ in Witten’s formula in [87]. In this section we exhibit a space in which $\frac{1}{2} p_1 = \frac{1}{2} \lambda$ is an obstruction, thus obtaining the flux quantization condition in M-theory exactly, as well as providing a further example from string theory.

\footnote{in version 1 of the eprint.}
In order to characterize \( \frac{1}{4} p_1 \), we consider a Spin structure on a space \( Y \) and consider the following diagram

\[
\begin{array}{ccccccc}
Y & \xrightarrow{x} & BO(4) & \xrightarrow{\times 2} & K(\mathbb{Z}, 4) \\
\downarrow & & \downarrow & & \downarrow \\
(BO)(4) & = & BSpin & \xrightarrow{\frac{1}{4} p_1} & K(\mathbb{Z}, 4) & \xrightarrow{w_4} & K(\mathbb{Z}_2, 4) \\
\downarrow & & \downarrow & & \downarrow \\
(BO)(2) & = & BSO & \xrightarrow{w_4} & K(\mathbb{Z}_2, 4),
\end{array}
\]

where \( x \) is our class \( \frac{1}{4} p_1 \) which naturally lives not in \( BO(4) = BSpin \) but rather in the desired space \( BO(4) \). The above diagram specifies \( BO(4) \). Thus, we have

**Observation 1.** The class \( \frac{1}{4} p_1 \) is the obstruction to lifting an \( BO(4) \) bundle, where \( BO(4) = BO(\frac{1}{4} p_1) \) is defined by diagram (3.9), to a String bundle.

This observation is analogous to proposition 2 in [69] for the Fivebrane case, where there we were considering the comparison of \( \frac{1}{4} p_2 \) to the obstruction to Fivebrane structure given by \( \frac{1}{6} p_2 \).

### 3.2.2. The Green-Schwarz anomaly and the M-theory C-field.

**Example I: The Green-Schwarz formula.** We consider the first setting where twisted String structures make an appearance. Anomaly cancelation in heterotic string theory is governed by the Green-Schwarz mechanism [38]. Consider a ten-dimensional Spin manifold \( M \), on which there is also a vector bundle \( E \) with rank 16 structure group \( G \), which is either \( E_8 \times E_8 \) or \( \text{Spin}(32)/\mathbb{Z}_2 \). The bundle \( E \) is part of the data of a (super)Yang-Mills (SYM) theory and has characteristic classes built out of the curvature \( F \). Both the tangent bundle \( TM \) and the gauge vector bundle a priori have degree four classes \( \lambda(M) = \frac{1}{2} p_1(TM) \) and \( \lambda(E) = \frac{1}{2} p_1(E) \), coming from pullbacks from \( B\text{Spin}(10) \) and \( BG \), respectively. The anomaly cancelation condition is given by

\[
\frac{1}{2} p_1(M) - \frac{1}{2} p_1(E) = 0.
\]

Inspecting this formula we can immediately identify it as a twisted String structure with a twist given by \( -\lambda(E) = -\frac{1}{2} p_1(E) \). Therefore we immediately have

**Proposition 1.** The Green-Schwarz anomaly cancelation condition defines a twisted String structure.

**Example II: (Heterotic) M-theory.** We next consider the second setting where twisted String structures appear. The low energy limit of M-theory is eleven-dimensional supergravity (Sugra). The dimensional reduction of the latter corresponds to ten-dimensional supergravity, which in turn is the low energy limit of superstring theory. If the process of taking a boundary is done carefully, one can actually recover also the coupling to Yang-Mills theory. Then taking a high energy limit leads to heterotic string theory. This is the subject of heterotic M-theory, and the process is depicted in this diagram

\[
\begin{array}{ccccccc}
M - \text{theory} & \xrightarrow{\text{low energy}} & D = 11 \text{ Supergravity} \\
\xrightarrow{\partial_2} & \xrightarrow{\ell_1} & D = 10 \text{ Supergravity + SYM} \\
\downarrow & & \downarrow \\
\text{Heterotic String} & \xrightarrow{\text{low energy}} & D = 10 \text{ Supergravity + SYM} \\
\xrightarrow{\ell_2} & \xrightarrow{\partial_1} & 
\end{array}
\]

In [43], Horava and Witten carefully studied the map \( \partial_1 \) and gave arguments on how to extend towards the strong coupling limit, i.e. going along \( \ell_1^{-1} \) and \( \ell_2^{-1} \). The result is a modification of the usual Green-Schwarz cancelation condition for \( G = E_8 \times E_8 \)

\[
\frac{1}{4} p_1(M) - \frac{1}{2} p_1(E) = 0.
\]
The extension of this to the eleven-dimensional bulk, i.e. roughly towards the upper left corner of diagram (3.11) leads, by certain locality arguments, to the analogous condition to (3.12) but now for the eleven-dimensional spacetime $Y$ 

\[(3.13) \quad \frac{1}{4}p_1(Y) - \frac{1}{2}p_1(E) = 0,\]

where, with an obvious abuse of notation, $E$ in equation (3.12) is the restriction of $E$ in equation (3.13).

From (3.12), (3.13), and proposition 1 we get

**Proposition 2.** The anomaly cancelation condition in heterotic M-theory and the flux quantization condition in M-theory each define a twisted String structure on $BO(4) = BO(\frac{1}{2}p_1)$.

### 3.3. Twisted fivebrane structures.

In section 3.2 we interpreted the conditions on degree four classes in heterotic string theory and in M-theory as obstructions to twisted String structures. On the other hand, in [69] we showed that the dual fields give rise to Fivebrane structures, provided some additional terms are set to zero. In this section we show that these dual fields give rise to twisted Fivebrane structures. In doing so, we also remedy some of the caveats raised in [69].

We consider the obvious generalization of definition 3.3 to the Fivebrane case. Recall [69] that we have $\frac{1}{2}p_2 : B\text{String} \to K(\mathbb{Z}, 8)$ as the classifying map of the principal $K(\mathbb{Z}, 7)$-bundle $B\text{Fivebrane} \to B\text{String}$, represented by the generator of $H^8(B\text{String}, \mathbb{Z})$.

**Definition 3.4.** A $\beta$-twisted Fivebrane structure on a brane $\iota : M \to X$ with String structure classifying map $f : M \to B\text{String}(n)$ is a cocycle $\beta : X \to K(\mathbb{Z}, 8)$ and a homotopy $\eta$:

\[(3.14) \quad \begin{array}{ccc}
M & \xrightarrow{f} & B\text{String}(n) \\
\iota \downarrow & & \downarrow \frac{1}{2}p_2 \\
X & \xleftarrow{\eta} & K(\mathbb{Z}, 8)
\end{array}\]

The above definition reduces to the usual definition of untwisted String structure of a space $M$ upon setting $X$ to a point; setting $\beta$ to zero follows from setting $X$ to a point.

**Remarks.**

1. Two $\beta$-twisted Fivebrane structures $\eta$ and $\eta'$ on $M$ are called equivalent if there is a homotopy between $\eta$ and $\eta'$.
2. From the definition, given a String manifold $M$ and a space $X$ with a Fivebrane twisting $\beta : X \to K(\mathbb{Z}, 8)$, then $M$ admits a $\beta$-twisted Fivebrane structure if and only if there is a continuous map $\iota : M \to X$ such that

\[(3.15) \quad \frac{1}{6}p_2(M) + \iota^*([\beta]) = 0\]

in $H^8(M, \mathbb{Z})$.
3. If $\iota^*([\beta]) + \frac{1}{6}p_2(M) = 0$, then the set of equivalence classes of $\beta$-twisted Fivebrane structures on $M$ are in one-to-one correspondence with elements in $H^7(M, \mathbb{Z})$.

#### 3.3.1. A refinement for further divisibility of $\frac{1}{2}p_2$.

We first consider the case of heterotic string theory. We start at the level of differential forms and then refine to the integral case. The condition

\[(3.16) \quad dH_3 = \frac{1}{2}p_1(F_\omega)\]

from the Green-Schwarz anomaly cancelation condition [38] in heterotic string theory on $M$, in the absence of gauge bundles, i.e. for $E$ the trivial vector bundle, says that $\frac{1}{2}p_1(M)$ is exact. Since $[dH_3] = 0$, this
implies that $M$ lifts to $B\text{String}$ and the set of lifts is labeled by $H_3$. Similarly, the condition

$$dH_7 = \frac{1}{48}p_2(F_\omega)$$

appears in two theories: In type IIA string theory with String structure and trivial Ramond-Ramond fields, and in heterotic string theory with String structure and with a trivial gauge bundle. This condition (3.17), being a triviality condition on $\frac{1}{48}p_2(M)$, implies that $M$ lifts to $B\text{Fivebrane}$ and the set of lifts is labeled by $H_7$. What we are interested in is the case where the fractional Pontrjagin classes are not trivialized, but are rather shifted by a nontrivial class, which we interpret as a twist.

**Remark.** The two equations (3.16) and (3.17) can be thought of as expressions in differential integral cohomology, with $\hat{H}_3$ and $\hat{H}_7$ the differential cochains for the Neveu-Schwarz field and its dual, and $\frac{1}{2}p_1(M)$ and $\frac{1}{48}p_2(M)$ the differential cocycles of the heterotic fivebrane magnetic charge and heterotic string electric charge, respectively. In fact, in the heterotic theory, the fields $H_3$ and $H_7$ should be thought of as being in differential K-theory for the heterotic string.

Recall that in [68] [69] the classes encountered in the anomaly expressions do not involve quite the obstruction $\frac{1}{6}p_2$, but rather involve $\frac{1}{48}p_2$. The extra division by 8 was explained in [69], where it was interpreted as living in a space $F$ rather than on $B\text{String}$ and the corresponding maps were given. Here we change the notation suggestively and we label the space as follows: $F\langle 8 \rangle = B\text{String}$ $F = BO\langle \frac{1}{48}p_2 \rangle$. Then we have the following definition.

**Definition 3.5.** A $\beta$-twisted $F\langle 8 \rangle$-structure is defined by a homotopy $\eta$ in the diagram

$$
\begin{array}{ccc}
M & \overset{\nu}{\longrightarrow} & F\langle 8 \rangle \\
\downarrow \eta & & \downarrow \frac{1}{48}p_2 \\
X & \overset{\beta}{\longrightarrow} & K(\mathbb{Z}, 8)
\end{array}
$$

The obstruction in this case that would replace (3.15) is given by the following.

**Observation 2.** The condition for a twisted Fivebrane structure obtained by lifting an $F\langle 8 \rangle$ structure is given by

$$\frac{1}{48}p_2(M) + \iota^*([\beta]) = 0.$$  

**Another description of the fractional classes.** We will use the path space of the Eilenberg-MacLane spaces to provide an alternative, but related, description of the fractional obstructions. The path space $PK(\mathbb{Z}, m)$ is a contractible space, and so it has trivial homotopy groups. Then, from the long exact sequence on homotopy of the path fibration

$$\Omega K(\mathbb{Z}, m) \longrightarrow PK(\mathbb{Z}, m) \longrightarrow K(\mathbb{Z}, m)$$

we get that $\pi_{i-1}(\Omega K(\mathbb{Z}, m)) \simeq \pi_i(K(\mathbb{Z}, m))$, so that $\Omega K(\mathbb{Z}, m)$ is the Eilenberg-MacLane space $K(\mathbb{Z}, m-1)$. As in [52], denote, for $m \equiv 0 \mod 4$, by $B_{d,m} \to BO(m)$ the pullback of the fibration (3.20) via a map $\phi : BO(m) \to K(\mathbb{Z}, m)$ such that the induced map $\pi_* : \pi_m(BO(m)) \cong \mathbb{Z} \to \pi_m(K(\mathbb{Z}, m)) = \mathbb{Z}$ is multiplication by $d$. This determines $\phi$ up to homotopy. The long exact sequence on homotopy of the fibration

$$B_{d,m} \longrightarrow BO(m) \overset{\phi}{\longrightarrow} K(\mathbb{Z}, m)$$

shows that the induced map

$$\pi_m(B_{d,m}) \longrightarrow \pi_m(BO(m)) \cong \mathbb{Z}$$

is multiplication by $d$.

**Remarks**
1. For the String structure, we have \( m = 4 \). Then we have the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & BO(4) \\
\downarrow & & \downarrow \\
B_{d,4} & \rightarrow & PK(\mathbb{Z}, 4) \\
\downarrow & & \downarrow \\
& & K(\mathbb{Z}, 3)
\end{array}
\]

Various fractions of the String structure correspond to various choices of \( d \). In the examples we saw that \( d = 2 \) was special.

2. For the Fivebrane structure, we have \( m = 8 \). Then we have the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & BO(8) \\
\downarrow & & \downarrow \\
B_{d,8} & \rightarrow & PK(\mathbb{Z}, 8) \\
\downarrow & & \downarrow \\
& & K(\mathbb{Z}, 8)
\end{array}
\]

In the examples in this case, the value \( d = 8 \) play a special role.

3.3.2. The dual Green-Schwarz anomaly and the dual M-theory C-field. We have defined (see section 3.3.1) the notion of twisted Fivebrane structures and \( F^{(9)} \) structures. In this section we show that such structures appear in String theory and M-theory and they are in fact conditions for cancelation of anomalies. We will consider the dual version of the Green-Schwarz mechanism and the dual field in M-theory. Note that one of the two main examples in [69] was type IIA string theory. In that theory the one-loop term on a String manifold is given simply by \( \frac{1}{48} p_2 \), i.e. without a twist. Therefore, type IIA string theory does not need the twisted structure we define in this paper.

Example III: The dual Green-Schwarz formula. We now consider ten-dimensional heterotic and type I string theories, whose low energy limit is type I supergravity theory coupled to superYang-Mills theory with structure group \( E_8 \times E_8 \) or \( \text{Spin}(32)/\mathbb{Z}_2 \). In [69] the main example of a Fivebrane structure came from the dual formulation [67] [37] of the Green-Schwarz anomaly cancelation mechanism [38], using the dual \( H \)-field \( H^7 \) of [22]. The expression is given by

\[
dH^7 = 2\pi \left[ \text{ch}_4(F_A) - \frac{1}{48} p_1(F_\omega) \text{ch}_2(F_A) + \frac{1}{64} p_1(F_\omega)^2 - \frac{1}{48} p_2(F_\omega) \right].
\]

In order to define a Fivebrane structure, we assume we already have a String structure, so we require \( \frac{1}{2} p_1(TM) = 0 \). Then the expression (3.25) becomes

\[
dH^7 = 2\pi \left[ \text{ch}_4(F_A) - \frac{1}{48} p_2(F_\omega) \right].
\]

In [69] we had to find ways to get rid of the extra terms to isolate the non-decomposable terms. In the twisted formalism in this paper we see that the presence of such terms amounts to a part of the twist and that it does not matter how many terms we have as long as they have the same total degree and hence provide a map to \( K(\mathbb{Z}, 8) \). Indeed, if we can define

\[
[\beta] := -\text{ch}_4(E) : M \xrightarrow{1} K(\mathbb{Q}, 8),
\]

i.e. require factorization

\[
\begin{array}{ccc}
\text{M} & \xrightarrow{[\beta]} & K(\mathbb{Q}, 8) \\
& \downarrow & \downarrow \\
& K(\mathbb{Z}, 8) & 
\end{array}
\]

then we can reinterpret expression (3.26) as \( \frac{1}{48} p_2(TM) + [\beta] = 0 \), since \( [dH^7] = 0 \), the cohomology class of an exact form.
We discuss the validity of the map in (3.27). The Chern character is in general not an integral expression, but rather
\[
ch : K^0(X) \to H^{even}(X; \mathbb{Q}).
\]
One way out of this is to first define a rational version of the twist, for which the map in (3.27) is replaced by a map from \(M\) to the rational Eilenberg-MacLane space
\[
[\beta] := -ch_4(E) : M \to K(\mathbb{Q}, 8),
\]
which gives that indeed \(ch_4(E)\) is in general in \([M, K(\mathbb{Q}, 8)] = H^8(M, \mathbb{Q})\). Hence

**Definition 3.6.** A rational Fivebrane twist on \(M\) is a map from \(M\) to \(K(\mathbb{Q}, 8)\), i.e. an element of \(H^8(M, \mathbb{Q})\).

However, we can also give conditions under which the map in (3.27) is valid. The degree four Chern character is given by
\[
ch_4 = \frac{1}{24} (c_4^4 - 4c_1^2c_2 + 4c_1c_3 + 2c_2^2 - 4c_4),
\]
The Chern classes are integral classes and so the Chern character is a priori integral up to a factor of 24. We describe this as follows. The Chern character is not integral in \(BU\) but it will be integral in some lift, say \(BU\langle 8 \rangle\), of \(BU\). Then we ask: when can we lift to this new space? This is given in terms of the following diagram
\[
\begin{array}{cccccc}
M & \xrightarrow{ch_4} & K(\mathbb{Q}, 7) & \xrightarrow{ch_4} & K(\mathbb{Q}, 8) & \xrightarrow{24ch_4} K(\mathbb{Q}, 8) \\
& \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& BU & \xrightarrow{f} & BU & \xrightarrow{\times 24} & BU & \xrightarrow{\times 24} \end{array}
\]
The right-most factor \(K(\mathbb{Q}, 8)\) represents the obstruction: there is a class \(k\) in \(H^8(M; \mathbb{Z}_{24})\) which measures this obstruction. The top-most factor \(K(\mathbb{Q}, 7)\) represents the different labeling of lifts \(f\) to the new space \(BU\langle 8 \rangle\). If we take connected covers of \(BU\) rather than \(BU\) itself in the diagram, then we have that the space \(BU\langle 8 \rangle\) is isomorphic to another space in which \(\frac{1}{8}c_4\), instead of \(ch_4\), is integral. The relevance of the unitary groups here is because they provide the adjoint representation for our structure groups and this is the representation relevant for Yang-Mills theory. For \(E_8\), the adjoint representation is \(ad : E_8 \to SU(248)\), so that the adjoint representation of \(G = E_8 \times E_8\) is
\[
(ad, ad) : E_8 \times E_8 \to SU(248) \times SU(248) \to SU(496)
\]
Note that the above general discussion can be simplified. For both structure groups \(E_8 \times E_8\) and \(Spin(32)/\mathbb{Z}_2\) we have \(c_1(E) = 0\), so that in this case
\[
ch_4(E) = \frac{1}{12} (c_2(E)^2 - 2c_4(E))
\]
We now consider two cases. First, that, in addition, \(c_2(E) = 0\). In this case, the formula for the Chern character \(ch_4(E)\) further simplifies to
\[
ch_4(E) = -\frac{1}{6} c_4(E).
\]
Here what we have really done is lifted the unitary group to its connected cover \(BU(8)\). Indeed let us consider the result from [75] where the mod \(p\) (\(p\) an odd prime) cohomology of the connective cover \(BU(2n)\) was calculated. From that result and the result of Stong [79] for \(p = 2\), the following divisibility result was
deduced for all primes \( p \) in \([75]\). Let \( c_k \in H^{2k}(BU; \mathbb{Z}) \) be the universal Chern class in \( BU \), then the Chern class \( r^*_n(c_k) \) in \( BU(2n) \) where \( r_n : BU(2n) \to BU \) be the canonical projection is divisible by \([75]\).

\[
(3.36) \quad \prod_p p^q
\]

where \( q \) is the least integer part of \( \frac{(n-1)-\sigma_p(k-1)}{p-1} \), with \( \sigma_p(n) = \sum a_i \) the sum of the coefficients in the unique decomposition of the integer \( n \) as \( n = a_0 + a_p + \cdots + a_k p^k \), with \( a_i < p \). Applying this result for \( n = 4 \), \( p = 2, 3 \), and using \( \sigma_2(3) = 2, \sigma_3(3) = 1 \), we get that \( r^*_4(c_4) \) is divisible by

\[
(3.37) \quad 2^{2-\sigma_2(3)} \cdot 3^{3-\sigma_3(3)} = 6.
\]

We will give an example where this occurs and where the expression \([6,35]\) is integral.

**Example.** Consider a complex vector bundle \( E \) on the eight-sphere \( S^8 \). For ten-manifold we can simply take \( S^8 \times \mathbb{R}^2 \) for example. The index of the Dirac operator on \( S^8 \times \mathbb{R} \) coupled to the vector bundle \( E \) is given by the evaluation of the twisted \( \hat{A} \)-genus \( \hat{A}(S^8, E) := \left( ch(E) \cdot \tilde{A}(S^8) \right) [S^8] \) on the fundamental class \([S^8]\) of \( S^8 \)

\[
(3.38) \quad \text{Index}_{DE} = \left\langle \hat{A}(TS^8) \cdot ch(E) \right\rangle = \text{ch}(E)[S^8],
\]

as \( \hat{A}(TS^8) = 1 \), since spheres have stably trivial tangent bundles. Since \( S^8 \) is a Spin manifold, the index should be an integer. This then gives the requirement

\[
(3.39) \quad \text{ch}_4(E)[S^8] = -\frac{1}{6} \text{c}_4(E)[S^8] \in \mathbb{Z}.
\]

Recall that we have refined the Fivebrane structure and its twisted version to include the division by 8 in definition \([6,35]\). Given equation \((3.19)\), the discussion leading to \((3.37)\) then proves the following

**Proposition 3.** The right hand side of the formula for the dual Green-Schwarz anomaly cancelation condition on a String 10-manifold \( M \) is the image in rational cohomology of the sum of integral classes representing the obstruction to defining a twisted Fivebrane structure, with the integral twist given by \( \text{ch}_4(E) \), the fourth Chern character of the gauge bundle \( E \), which itself is lifted to \( BU(8) \).

We can actually view the above result as providing a characterization of when the dual Green-Schwarz cocycle is integral. We have a sufficient result that this is so when the bundles are lifted from the String case to the Fivebrane case.

In section \([5]\) we will give a more complete result which takes into account the differential refinements we discussed in section \([4]\).

**Remarks 3.7.** 1. We can define complex- String and Fivebrane structures, as is implicitly done in \([39]\), as the lifts of maps to \( BU(4) \) to maps to \( BU(8) \) and to \( BU(10) \), respectively. These can also be twisted leading to twisted complex- String and Fivebrane structures, in a similar way as in the real case. The twist in proposition \([3]\) is an example of a twisted complex Fivebrane structure for the complex vector bundle corresponding to the gauge bundle.

2. In proposition \([3]\) and the discussion around it, we took the point of view that the natural bundle (i.e. the lift of the tangent bundle leading to Spin then String and so on) is the one that is being twisted by the gauge bundle. Of course we could have taken another point of view where the natural bundle acts as a twist for the corresponding gauge bundle. However, we prefer the first point of view here because the natural bundles seem to be, in a sense, more intrinsic and hence should come first in the order of giving structures.

**Description in terms of nonabelian cohomology.** In what follows we put the above discussion in the context of the discussion of nonabelian cohomology of section \([2]\). Later in section \([5]\) we consider the differential case.
We consider bi-twisted cohomology in the sense of definition 5.1 with respect to the two fibration sequences

\[
\begin{array}{cccc}
\text{BU}(10) & \longrightarrow & \ast \\
\downarrow & & \\
\text{BU}(8) & \longrightarrow & \text{B}^8\mathbb{Z} \\
\end{array}
\quad
\begin{array}{cccc}
\text{BFivebrane} & \longrightarrow & \ast \\
\downarrow & & \\
\text{BString}^F & \downarrow_{\frac{1}{2}p_2} & \text{B}^8\mathbb{Z} \\
\end{array}
\quad
\begin{array}{cccc}
\text{BString} & \downarrow_{\frac{1}{2}p_2} & \text{B}^8\mathbb{Z} \\
\end{array}
\]

and relate it to the condition known as the \textit{dual Green-Schwarz anomaly cancellation mechanism} in dual magnetic heterotic string theory. Before proceeding, notice that above we have shown

**Lemma 3.8.** The pullback of the cohomology class \(\frac{1}{6}c_4 : \text{BU} \to \text{B}^8\mathbb{R}\) to the universal \(7\)-connected cover \(\text{BU}(8)\) of \(\text{BU}\) is integral:

\[
\begin{array}{cccc}
\text{BU}(8) & \longrightarrow & \text{BU} \\
\downarrow_{\frac{1}{2}c_4} & & \\
\text{B}^8\mathbb{Z} & \longrightarrow & \text{B}^8\mathbb{Z} \\
\end{array}
\]

**Definition 3.9** (Gauge-twisted Fivebrane structure). On a space \(X\) let \(E \in H(X, \text{BU})\) be a cocycle to be called the (class of the) gauge bundle which has a lift \(\tilde{E} \in H(X, \text{BU}(8))\). By lemma 3.8, this implies that its class \(\frac{1}{6}c_4(E)\) is integral. Then in the sense of definition 5.1 and definition 5.1, we say that the space of gauge twisted Fivebrane-structures \(H^{[E]}(X, \text{BFivebrane})\) on \(X\) with gauge twist \(E\) is the \(\text{BFivebrane}^F - \text{BU}(10)\)-bitwisted cohomology whose \(\text{BU}(10)\)-twist is \([\text{ch}_4(E)]\), i.e. the homotopy pullback

\[
\begin{array}{cccc}
H^{[E]}(X, \text{BFivebrane}) & \longrightarrow & \ast \\
\downarrow & & \\
H(X, \text{BFivebrane} \times_{\text{B}^8\mathbb{Z}} \text{BU}(10)) & \longrightarrow & H(X, \text{BU}(8)) & \longrightarrow & H(X, \text{BU}) \\
\downarrow & & \downarrow_{\frac{1}{2}c_4} & & \downarrow_{\frac{1}{2}c_4} \\
H(X, \text{BString}^F) & \downarrow_{\frac{1}{2}p_2} & H(X, \text{B}^8\mathbb{Z}) & \longrightarrow & H(X, \text{B}^8\mathbb{R}) \\
\downarrow & & \downarrow_{\times 8} & & \\
H(X, \text{BString}) & \downarrow_{\frac{1}{2}p_2} & H(X, \text{B}^8\mathbb{Z}) \\
\end{array}
\]

**Definition 3.10** (Dual Green-Schwarz anomaly cancellation). For \(X\) an oriented space and \(E \to X\) a complex vector bundle on \(X\), the \textit{dual Green-Schwarz anomaly cancellation condition} is the requirement that the following equation holds in \(H^8(X, \mathbb{R})\):

\[
\frac{1}{48}p_2(X) - \text{ch}_4(E) + \frac{1}{48}p_1(X)\text{ch}_2(E) - \frac{1}{64}p_1(X)^2 = 0.
\]

**Proposition 4** (Dual Green-Schwarz and twisted Fivebrane-structure). If \(X\) has a \(\text{BString}^F\)-structure and \(E\) has a complex String-structure in that we have lifts of classifying maps of bundles

\[
\begin{array}{cccc}
\text{BString}^F(n) & \longrightarrow & \ast \\
\downarrow & & \\
X & \longrightarrow & \text{BSO}(n) \\
\end{array}
\quad
\begin{array}{cccc}
\text{BU}(8) & \longrightarrow & \ast \\
\downarrow & & \\
X & \longrightarrow & \text{B}^8\mathbb{Z} \\
\end{array}
\]

then the dual Green-Schwarz anomaly cancellation condition from definition 3.10 is equivalent to the condition that \(X\) has an \([E]\)-twisted Fivebrane-structure lifting \(\tilde{g}_{TX}\).
Proof. By the assumption of a String$^F$-structure the class $\frac{1}{2}p_1(X)$ vanishes. Therefore the mixed terms $p_1(X)(\cdots)$ in the dual Green-Schwarz condition vanish. Similarly, using the assumption of complex String-structure one finds that $\text{ch}_4(E) = \frac{1}{2}c_4(E)$ as in the discussion leading to equation (3.37). It follows that the dual Green-Schwarz condition says in this case that the outer diagram in

![Diagram](image)

commutes up to homotopy. By definition of $\text{BFivebrane} \times \text{B}^8\text{Z} \rightarrow \text{BU}(10)$, this is the case precisely if $\hat{g}_{TX}$ and $\hat{g}_E$ have a common lift given by the dashed morphism in the above diagram. This lift is by definition the $[E]$-twisted Fivebrane-structure lifting $\hat{g}_{TX}$.

Example II: The dual field in M-theory. Now we consider M-theory, via its low energy limit, namely eleven-dimensional supergravity. The M-theory $C$-field $C_3$ is a degree three ‘potential’ whose curvature form is the degree four field strength $G_4$. Its dual is obtained in the following way. The equation of motion for $C_3$ is obtained from varying the action

\begin{equation}
S(C_3) = \int_Y \left[ \frac{1}{6} G_4 \wedge *G_4 + \frac{1}{6} G_4 \wedge G_4 \wedge C_3 - I_8 \wedge C_3 \right]
\end{equation}

on an eleven-dimensional manifold $Y$ to obtain

\begin{equation}
d*G_4 = -\frac{1}{2} G_4 \wedge G_4 + I_8.
\end{equation}

Here $I_8$ is the one-loop term [82] [81] given in terms of the Pontrjagin classes of the tangent bundle $TY$ to $Y$

\begin{equation}
I_8 = \frac{p_2(TY) - \frac{1}{2}(\frac{1}{2}p_1(TY))^2}{48},
\end{equation}

and $*$ is the Hodge duality operator in eleven dimensions.

The integral lift of (3.41) leads to a class defined in [28]

\begin{equation}
[G_8] = \left[ \frac{1}{2} G_4^2 - I_8 \right]
\end{equation}

\begin{equation}
= \frac{1}{2} a(a - \lambda) + \frac{7\lambda^2 - p_2}{48},
\end{equation}

where $\lambda = \frac{1}{2}p_1$, and $a$ is the degree four class of an $E_8$ bundle coming from Witten’s shifted quantization condition for $G_4$ [87]

\begin{equation}
[G_4] = a - \frac{1}{2} \lambda = a - \frac{1}{4}p_1.
\end{equation}

In [80] Witten interpreted the vanishing of a certain torsion class $\theta$ on the M-fivebrane worldvolume as a necessary condition for the decoupling of the fivebrane from the ambient space ( “the bulk”). Hence the vanishing of $\theta$ meant that the fivebrane can have a well-defined partition function. Consider the embedding $\iota : W \hookrightarrow Y$ of the fivebrane with six-dimensional worldvolume $W$ into eleven-dimensional spacetime $Y$. Consider the ten-dimensional unit sphere bundle $\pi : X \rightarrow W$ of $W$ with fiber $S^5$ associated to the normal bundle $N \rightarrow W$ of the embedding $\iota$. Then it was shown in [28] that the integration of $G_8$ over the fiber of $X$ gives exactly the torsion class $\theta$ on the fivebrane worldvolume

\begin{equation}
\theta = \pi_*(G_8) \in H^4(W; \mathbb{Z}).
\end{equation}
Therefore, the vanishing of $G_8$ is a necessary condition for the existence of a non-zero partition function \[28\].

We now proceed with the interpretation. Since we have Fivebrane structures in mind, we assume that $Y$ already admits a String structure, i.e. that $\frac{1}{2}p_1(Y) = 0$. Then, from (3.43) we see that the class $G_8(Y)$ simplifies to

\begin{equation}
G_8(Y) = \frac{1}{2}a^2 - \frac{1}{48}p_2(Y).
\end{equation}

The class $a$ is an integral class of an $E_8$ bundle and hence defines a map to $K(\mathbb{Z}, 4)$. Then the square of $a$ defines a map to $K(\mathbb{Z}, 8)$, and hence defines a twist for us. As we also have the class $\frac{1}{18}p_2$, then we have a twist for the modified fivebrane structure. Thus we have the following.

**Proposition 5.** The integral class in M-theory dual to $G_4$ defines an obstruction to a twisted Fivebrane structure lifted from an $F^{(8)}$-structure.

This is the obstruction to having a well-defined partition function for the M-fivebrane.

**Necessity of the Fivebrane condition?** The Fivebrane condition is stronger than simply the requirement that the one-loop term $I_8$ to vanish. For the former we require the obstructions $\frac{1}{2}p_1$ and $\frac{1}{3}p_2$ vanish separately, whereas for the latter we only require the combination to vanish. This has been studied in \[46\] \[45\] \[85\]. For instance, following \[85\], a Riemannian 8-dimensional spin manifold $M^8$ is said to be doubly supersymmetric if and only if the tangent bundle $TM^8$ and the spinor bundles $\Delta_+ M^8$ and $\Delta_- M^8$ are associated with a principal $G$-fiber bundle such that there exist $G$-structure and String cobordism $\Omega_2(M^8)$ which defines a map to $K(\mathbb{Z}, 4)$. Then, from (3.43) we see that the class $\chi(M^8) = 16 \bar{A}(M^8)$. In particular, $\chi(M^8) \equiv 0 \mod 16$. One example is $PSU(3)$-structure for which

\begin{align}
\begin{aligned}
w_1 = w_2 &= 0, & e &= 0, & 4p_2 &= p_1^2,
\end{aligned}
\end{align}

where $e$ is the Euler class. Then this implies for the signature $sgn(M^8) = 16 \bar{A}(M^8)$. In particular, $sgn(M^8) \equiv 0 \mod 16$. One example is $PSU(3)$-structure for which

\begin{align}
\begin{aligned}
w_i &= 0 \ (i \neq 4), & w_4^2 &= 0, \\
e &= 0, & p_1^2 &= 4p_2.
\end{aligned}
\end{align}

In particular, all Stiefel-Whitney numbers vanish.

A second example is a differentiable 8-fold $M^8$ with an odd topological generalized Spin(7)-structure for which

\begin{equation}
\chi(M^8) = 0, \quad p_1(M^8)^2 - 4p_2(M^8) = 0.
\end{equation}

The 7-sphere admits a Spin structure and therefore admits a generalized $G_2$-structure. The tangent bundle of the 8-sphere is stably trivial and therefore all the Pontrjagin classes vanish. Since the Euler class is non-trivial, there exists a generalized $Spin(7)$-structure on an 8-sphere. However, equation (3.50) is automatically satisfied for manifolds of the form $M^8 = S^1 \times N^7$ with $N^7 Spin$.

**(Twisted) Fivebrane cobordism.** Recall that Spin cobordism $\Omega^{spin} = \Omega^{(4)}$ refers to cobordism of spaces equipped with Spin structure and String cobordism $\Omega^{String} = \Omega^{(8)}$ refers to cobordism of spaces equipped with a String structure. For spaces $X$ with Fivebrane structure we can also define Fivebrane cobordism $\Omega^{Fivebrane} = \Omega^{(9)}(X)$ in a similar manner. Given a manifold $X$ with a twisting $\beta: X \to K(\mathbb{Z}, 8)$, one can form a cobordism category, in analogy to the String case \[83\], called the $\beta$-twisted Fivebrane cobordism over $(X, \beta)$, whose objects are compact smooth String manifolds over $X$ with a $\beta$-twisted Fivebrane structure. We call the corresponding cobordism group $\Omega^{Fivebrane}(X, \beta)$ the $\beta$-twisted Fivebrane cobordism group of $X$.

4. $L_\infty$-Connections

Before passing from the topological twisted structures in the previous section to their differential refinements in the next section, it is worthwhile to introduce, review and discuss some of the $\infty$-Lie algebraic structures that appear in the description of the respective $L_\infty$-algebra valued differential form connection
data. The standard discussion of connections and curvatures is in terms of the Lie algebra of the structure group, e.g. $\text{SO}(n)$ and $\text{Spin}(n)$. But $\text{String}(n)$ and $\text{Fivebrane}(n)$ cannot be realized as Lie groups (certainly not as finite dimensional ones and no ordinary infinite dimension model is known), such realizations for connected covers are not available. However, as defined homotopy theoretically, they do have a natural incarnation as smooth $\infty$-groupoids [40] [9] and have corresponding $L_\infty$-algebras, which is adequate for defining generalizations of connections and curvatures. We ended section 2 by indicating a derivation and recalling the definition of the Cartan-Ehresmann $\infty$-connections introduced in [08] that do achieve this.

After recalling or introducing details of $L_\infty$-algebroids and $L_\infty$-connections, we introduce the $L_\infty$-algebraic analogs of the constructions in section 2.2 representations of $L_\infty$-algebras and associated $L_\infty$-connections. Following the discussion of the relation between sections of associated bundles and twisted cohomology at the end of section 2.2 this will provide a useful supplementary perspective on the constructions and computations in section 5 below.

4.1. Review of $L_\infty$-algebras. The tools we employ are varied, so we provide in this section a review of the essential $L_\infty$-algebra notions that we need. All $L_\infty$-algebras will be of finite type, i.e. finite-dimensional in each degree. By “quasi-free” DGCAs we mean those that are free as GCAs (Graded Commutative Algebras).

4.1.1. $L_\infty$-algebras and $L_\infty$-algebroids. Lie algebras are defined as structures on vector spaces and $L_\infty$-algebras as structures on graded vector spaces. Both generalize to modules over commutative algebras, for example, over $A := \mathcal{C}^\infty(X)$, the algebra of smooth functions over a manifold $X$. These modules are called $L_\infty$-algebras and $L_\infty$-algebroids.

Remark. Grading conventions can be a nuisance when dealing with differential graded algebras. Here we shall take the grading convention of the de Rham complex as fundamental and choose our conventions such that the Chevalley-Eilenberg algebra of the tangent Lie algebroid $TX$ of a smooth space $X$ coincides with the de Rham complex $\Omega^*(X)$ with the correct grading. This implies that Chevalley-Eilenberg algebras of general $L_\infty$-algebroids over $\mathcal{C}^\infty(X)$ are taken to be $\mathbb{N}$-graded and hence the $L_\infty$-algebras themselves which are dual to them are taken to be $\mathbb{Z}$-graded and concentrated in non-positive degree. To make the pattern more obvious, we will say that a $\mathbb{Z}$-graded complex concentrated in non-positive degree is $\mathbb{N}$-graded.

Definition 4.1. A (degreewise finite rank) $L_\infty$-algebroid $(X, \mathfrak{g})$ is a smooth space $X$ and a $\mathbb{N}$-graded cochain complex $\mathfrak{g}$ of degreewise finite rank $A := \mathcal{C}^\infty(X)$-modules together with a degree $+1$ derivation
\begin{equation}
  d : \wedge_A \mathfrak{g}^* \to \wedge_A \mathfrak{g}^*,
\end{equation}
linear over the ground field (not necessarily over $A$) on the free (over $A$) graded-symmetric algebra generated from the $\mathbb{N}$-graded dual $\mathfrak{g}^*$ (over $A$), such that $d^2 = 0$. The quasi-free (over $A$) differential graded-commutative algebra
\begin{equation}
  \text{CE}_A(\mathfrak{g}) := (\wedge_A \mathfrak{g}^*, d)
\end{equation}
defined this way we call the Chevalley-Eilenberg algebra of the $L_\infty$-algebroid $(A, \mathfrak{g})$.

Remark 4.2. (types of $L_\infty$-algebroids). We have the following special cases:
\begin{itemize}
  \item For $X = \text{pt}$ and $\mathfrak{g}$ concentrated in degree 0 we have $\text{CE}(\mathfrak{g}) = (\wedge^\bullet \mathfrak{g}, d_\mathfrak{g})$ where $\wedge^\bullet \mathfrak{g}$ is the Grassmann algebra on the vector space $\mathfrak{g}$ in degree 1 and $d_\mathfrak{g}$ is the Chevalley-Eilenberg differential uniquely corresponding to the structure of a Lie algebra on $\mathfrak{g}$.
  \item For $X = \text{pt}$ and $\mathfrak{g}$ in arbitrary (non-positive) degree we have an arbitrary $L_\infty$-algebra (of finite type).
  \item For arbitrary $X$ and $\mathfrak{g}$ concentrated in degree 0 (being finitely generated and projective as a module over $\mathcal{C}^\infty(X)$) this is equivalent to the usual definition of Lie algebroids as vector bundles $E \to X$ with anchor map $\rho : E \to TX$: we have $\mathfrak{g} = \Gamma(E)$ and the anchor is encoded as $d_\mathfrak{g}|_{\mathcal{C}^\infty(X)} : f \mapsto \rho^\flat(f)$.
  \item If $\mathfrak{g}$ is concentrated in degree 0 and $-(n - 1)$, then it is called 2-stage in homotopy theory.
\end{itemize}
• If \( \mathfrak{g} \) is concentrated in degrees 0 through \(-(n - 1)\), then we speak of a Lie \( n \)-algebra, (for instance the Lie 2-algebras in \([8]\) or the various Lie \( n \)-algebras in \([43]\).
• If \( X = \text{pt} \) and \( d = d_{\mathfrak{g}} : \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \mathfrak{g}^* \wedge \mathfrak{g}^* \), then \( \mathfrak{g} \) is a dg-Lie algebra with the component \( \mathfrak{g}^* \rightarrow \mathfrak{g}^* \) the dual to the differential on the chain complex underlying \( \mathfrak{g} \) and the component \( \mathfrak{g}^* \rightarrow \mathfrak{g}^* \wedge \mathfrak{g}^* \) the dual of the Lie bracket.

More generally, if \( A \) is any commutative associative algebra, we may speak of an \( \infty \)-Lie-Rinehart algebra \([65] \ [44]\) which we may think of as an \( \infty \)-Lie algebroid over a noncommutative base space. This case will however not concern us here.

4.1.2. \( L_\infty \)-algebra valued differential forms and twisting forms. Recall, for instance from \([68]\) and following H. Cartan \([24]\) for ordinary Lie algebras, that, for \( \mathfrak{g} \) any \( L_\infty \)-algebra, differential form data on a space \( X \) with values in \( \mathfrak{g} \) is a GCA morphism (not necessarily respecting the differentials) from \( \text{CE}(\mathfrak{g}) \) into forms on \( X \):

\[
\Omega^*(X, \mathfrak{g}) := \text{Hom}_{\text{GCA}}(\text{CE}(\mathfrak{g}), \Omega^*(X)).
\]

Later we wish to work entirely within homomorphisms of differential graded algebras. This is accomplished by passing from the Chevalley-Eilenberg algebra \( \text{CE}(\mathfrak{g}) \) to the Weil algebra \( W(\mathfrak{g}) \), which can be defined as the DGCA which is universal with the property that \( \Omega^*(X, \mathfrak{g}) \) is isomorphic, up to homotopy, to \( \text{Hom}_{\text{DGCA}}(W(\mathfrak{g}), \Omega^*(X)) \). More constructively, over the point,

**Definition 4.3.** \( W(\mathfrak{g}) := (\wedge^\bullet \mathfrak{g}^* \otimes \wedge^\bullet s\mathfrak{g}^*, D) \), where \( s \) denotes shift in degree by 1 and where \( D \) is given on generators by

\[
D|_{\mathfrak{g}^* \otimes s\mathfrak{g}^*} := d_C E \otimes 1 + s \circ d_C E \circ s^{-1} + s.\]

**Remark 4.4.** The Weil algebra can also be regarded as the Chevalley-Eilenberg algebra \( \text{CE}(\mathfrak{g} \oplus s\mathfrak{g}) \), the semidirect product of \( \mathfrak{g} \) with its adjoint action on \( s\mathfrak{g} \) with trivial \( L_\infty \)-structure and \( s \) as internal differential. We come back to that in a moment when we discuss representations of \( \infty \)-Lie algebroids.

The space of GCA homomorphisms is a subspace of the space of linear maps of graded vector spaces from \( \text{CE}(\mathfrak{g}) \) to \( \Omega^*(\mathfrak{g}) \) and, since \( \text{CE}(\mathfrak{g}) \) is freely generated as a GCA and of finite type, this is isomorphic to the space of grading preserving homomorphisms

\[
\text{Hom}_{\text{Vect}[\mathbb{Z}]}(\mathfrak{g}^*, \Omega^*(X))
\]

of linear grading-preserving maps from the graded vector space \( \mathfrak{g}^* \) of dual generators to \( \Omega^*(X) \), with \( \mathfrak{g}^* \) still regarded as being in positive degree. By the usual relation in \( \text{Vect}[\mathbb{Z}] \) for \( \mathfrak{g} \) of finite type, this is isomorphic to the space of elements of total degree degree 1 in forms tensored with \( \mathfrak{g} \):

\[
\Omega^*(X, \mathfrak{g}) \cong (\Omega^*(X) \otimes \mathfrak{g})_0.
\]

(Recall that \( \mathfrak{g} \) is \(-\mathbb{N}\)-graded, i.e. in non-positive degree by definition.)

If instead we consider the corresponding DGCA homomorphisms from \( \text{CE}(\mathfrak{g}) \) into forms, we find that respecting the differentials implies having what we can call flat \( L_\infty \)-algebra valued forms

\[
\Omega^*_{\text{flat}}(X, \mathfrak{g}) := \text{Hom}_{\text{DGCA}}(\text{CE}(\mathfrak{g}), \Omega^*(X)).
\]

The inclusion

\[
\Omega^*_{\text{flat}}(X, \mathfrak{g}) \hookrightarrow \Omega^*(X) \otimes \mathfrak{g}
\]

realizes flat \( L_\infty \)-algebra valued forms as elements \( A \in \Omega^*(X) \otimes \mathfrak{g} \) of forms of total degree 0 with the special property that they satisfy a flatness constraint of the form

\[
dA + \partial A + [A \wedge A] + [A \wedge A \wedge A] + \cdots = 0,
\]
where $d$ and $\wedge$ are the operations in $A \in \Omega^\bullet(X) \otimes \mathfrak{g}$ and where $[\cdot, \cdot, \cdots]$ are the $n$-ary brackets in the $L_\infty$-algebra and $\partial$ is the differential in the chain complex $\mathfrak{g}$. For $\mathfrak{g}$ a dg-Lie algebra, only the binary bracket is present and $A$ is an ordinary Maurer-Cartan element:

$\begin{equation}
DA + [A \wedge A] = 0,
\end{equation}$

where $D = d + \partial$.

This equation of course has a long and honorable history in various guises. When the algebra is that of differential forms on a Lie group, it is called the Maurer-Cartan equation. In deformation theory, it is the integrability equation. In mathematical physics, especially in the Batalin-Vilkovisky formalism, it is known as the Master Equation. At present, the name Maurer-Cartan equation seems to have taken over in all these disciplines.

**Remark 4.5.** There is an obvious well-known generalization of the above where the DGCA $\Omega^\bullet(X)$ is replaced by any other DGCA $(A, d_A)$. Then by the above reasoning DGCA homomorphisms

$\begin{equation}
(A, d_A) \longrightarrow \text{CE}(\mathfrak{g})
\end{equation}$

correspond to certain “flat” elements $\tau$ of degree 1 in the tensor product $\tau \in A \otimes \mathfrak{g}$, where the flatness condition is again

$\begin{equation}
d_A \tau + d_\mathfrak{g} \tau + [\tau \wedge \tau] + \cdots.
\end{equation}$

See for instance definition 3.1 in [39]. This provides an example of a twisted tensor product:

$\begin{equation}
(A \hat{\otimes} \mathfrak{g}, d_A \otimes 1 + 1 \otimes d_\mathfrak{g} + \tau \wedge),
\end{equation}$

the untwisted differential $d$ being $d_A \otimes 1 + 1 \otimes d_\mathfrak{g}$. The twist provided by $\tau$ can be considered as a twisting cochain $\tau : C \rightarrow L$ where $C$ is a dg coalgebra such that $A := \text{Hom}(C, \mathbb{C})$ is the dg algebra dual to $C$. Without assuming flatness, a similar element in $A \hat{\otimes} \mathfrak{g}$ is what K.T. Chen calls a connection [22]. Chen saw that his condition for flatness becomes that of a twisting cochain. For our purposes (see section 4.2), particularly important examples are given by extensions of Lie algebras by abelian Lie algebras and their Chevalley-Eilenberg complexes. There the twisted differential $d_{\mathfrak{g}_u}$ can be written as $d_\mathfrak{g} \otimes 1 + \mu \partial$.

4.1.3. $L_\infty$-algebra connections. The definition of $L_\infty$-algebraic connections from [68] is a generalization of ordinary connections on ordinary principal bundles as follows. For $\mathfrak{g}$ a Lie algebra of a Lie group $G$ and $\pi : P \rightarrow X$ a principal $G$-bundle, an Ehresmann connection on $P$ is a $\mathfrak{g}$-valued 1-form on $P$, $A \in \Omega^1(P, \mathfrak{g},)$ which satisfies two conditions:

1. **First Ehresmann condition:** $A$ restricts to the canonical flat $\mathfrak{g}$-valued 1-form on the fibers.
2. **Second Ehresmann condition:** $A$ is equivariant with respect to the $G$-action on $P$.

H. Cartan observed that this could be expressed in terms of a morphism of graded-commutative algebras on which there is the action of a Lie group (though only the action of the Lie algebra $\mathfrak{g}$ is necessary). For each vector $x \in \mathfrak{g}$, there are derivations called ‘infinitesimal transformation’ $\mathcal{L}(x)$ (today usually known as the Lie derivative) and ‘interior product’ $i(x)$ satisfying the relations:

1. $\mathcal{L}$ is a Lie morphism
2. $i([x, y]) = \mathcal{L}(x)i(y) - i(y)\mathcal{L}(x)$
3. $i(x)d + di(x)$

A Cartan connection $\Omega^\bullet(P) \longrightarrow \text{CE}(\mathfrak{g})$ is then defined as respecting the operations $i(x)$ and $\mathcal{L}(x)$ for all $x \in \mathfrak{g}$, but not necessarily respecting $d$. In formulas, for $x \in \mathfrak{g}$ and $a \in \text{CE}(\mathfrak{g})$

1. **First Cartan-Ehresmann condition:** $i(x)A(a) = A(i(x)a)$
2. **Second Cartan-Ehresmann condition:** $\mathcal{L}(x)A(a) = A(\mathcal{L}(x)a)$.

If we extend $A$ to a morphism not just of graded-commutative algebras, but to a morphism of differential graded commutative algebras $\Omega^\bullet(P) \longrightarrow W(\mathfrak{g})$, then we can express these two conditions in terms of diagrams as follows. Let $\Omega^\bullet_{\text{vert}}(P)$ denote the *quotient* of $\Omega^\bullet(P)$ by the image $\pi^*\Omega^\bullet(X)$. 

The first Cartan-Ehresmann condition says that the following square of DGCA morphisms commutes
\[
\begin{array}{ccc}
\Omega^\bullet (P) & \xrightarrow{A_{\text{vert}}} & CE(g) \\
\downarrow && \downarrow \\
\Omega^\bullet (P) & \leftarrow & W(g).
\end{array}
\] (4.13)

The relevance of the second Cartan condition is that it ensures that plugging the curvature of the 1-form \(A\) into an invariant polynomial of the Lie algebra yields a basic form on \(P\) which comes from pulling back a form on \(X\). This is equivalent to saying that the following square of DGCA morphisms commutes:
\[
\begin{array}{ccc}
\Omega^\bullet (P) & \leftarrow & W(g) \\
\downarrow && \downarrow \\
\Omega^\bullet (X) & \leftarrow & \{P_i\} \text{inv}(g).
\end{array}
\] (4.14)

Here \(\{P_i\}\) denotes the set of images in \(\Omega^\bullet (X)\) of the generators of the algebra \(\text{inv}(g)\) of invariant polynomials: the diagram says that these are, under the Chern-Weil homomorphism, the characteristic forms obtained from the curvature \(F_A\) of the connection form \(A\) corresponding to the indecomposable invariant polynomials on \(g\).

The advantage of these two diagrams is that they have an immediate generalization from Lie algebras to arbitrary \(L_\infty\)-algebras, which is the content of the following definition. In particular, the second diagram allows us to generalize characteristic forms of \(L_\infty\)-algebra valued forms without having to deal with equivariance on total spaces of higher bundles, which is a delicate issue: in this approach equivariance is not mentioned but instead the crucial consequence of equivariance, the descent of characteristic forms down to a base space, is encoded in a definition.

**Definition 4.6** \((L_\infty\text{ Cartan-Ehresmann connection})\). For \(g\) the \(L_\infty\)-algebra of an \(\infty\)-group \(G\), for \(\pi : P \to X\) a \(G\)-principal \(\infty\)-bundle, we say that a pair of commutative diagrams
\[
\begin{array}{ccc}
\Omega^\bullet (P) & \xrightarrow{A_{\text{vert}}} & CE(g) \\
\downarrow && \downarrow \\
\Omega^\bullet (P) & \leftarrow & W(g) \\
\downarrow && \downarrow \\
\Omega^\bullet (X) & \leftarrow & \{P_i\} \text{inv}(g)
\end{array}
\] (4.15)

is a Cartan-Ehresmann \(\infty\)-connection on \(P\).

The invariant polynomials on \(g\) are invariant with respect to the adjoint \(L_\infty\)-action of \(g\) on itself. Often we just say “\(g\)-connection” or even just “connection” for such objects.

### 4.2. String-like Lie \(n\)-algebras

The main applications of our general theory are specific examples of \(L_\infty\)-algebras: the String Lie 2-algebra \(g_\mu\) and its generalization to higher String-like extensions, especially the Fivebrane Lie 6-algebra considered in [68, 69]. There is a straightforward generalization of the String algebra in which \(\mu\) is of arbitrary odd degree. The String-like extensions were originally considered in [8]. In terms of the DGCA language of definition 4.1 they read as follows:

**Definition 4.7** (String-like extensions). For \(g\) an ordinary Lie algebra and \(\mu\) a Lie algebra cocycle of degree \((n + 1)\), the String-like extension \(g_\mu\) is the Lie \(n\)-algebra determined by its Chevalley-Eilenberg algebra as
\[
\begin{align}
\text{CE}(g_\mu) := \left( \Lambda^\bullet \left( g_1^* \oplus \langle b \rangle \right), d|_{g^*} = d_g, db = \mu \right).
\end{align}
\] (4.16)
The differential $\mu$ have the 3-cocycle $\Omega^3 S^2 \mu$. This is a shifted central extension of $L_\infty$-algebras $b^{n-1}u(1) \to g_\mu \to g$. 

2. The differential $d_{g_\mu}$ is a twisted differential

\[ d_{g_\mu} = d_g + \mu \land \frac{\partial}{\partial b} \]

of the kind which we will interpret in terms of twisting cochains in section 3.3. and, essentially equivalently, in terms of representations of $L_\infty$-algebras as described in proposition 7 below.

3. The finite dimensional but weak Lie 2-algebra $g_{\mu_3}$ is equivalent to the strict but infinite-dimensional Lie 2-algebra $(\Omega g \to P g)$.

4. $g_{\mu_3}$ integrates in various ways to the String Lie 2-group as discussed specifically in [9, 40, 70].

Topologically, using rational homotopy theory, one can interpret the qDGCA with which we are dealing, as models for the DGCA of differential forms on certain spaces. Recall the following basic facts from rational homotopy theory:

- In general, the cohomology of a given qDGCA represents the real cohomology of a space. If the given qDGCA is minimal, i.e. if there are no linear terms in the differential, then the homology of the space of generators is isomorphic to the dual Hom($\pi$, $\mathbb{R}$) of the homotopy groups of the space. A generator of degree $n$ represents a basis element of $\pi_n \otimes \mathbb{R}$.
- For $2n + 1$ odd, the DGCA $CE(b^{2n-1}u(1)) = (\land^n (b), d = 0)$ represents the $2n + 1$-sphere $S^{2n+1}$, whose only non-torsion homotopy group is $\pi_{2n+1}(S^{2n+1})$.
- For $2n$ even, the DGCA $CE(b^{2n-1}u(1)) = (\land^n (b), d = 0)$ represents not the $2n$-sphere $S^{2n}$, but the loop space, $\Omega S^{2n+1}$ whose only non-torsion homotopy group is $\pi_{2n}(\Omega S^{2n+1})$.
- The $2n$-sphere $S^{2n}$ is instead represented by the DGCA $(\land^n (b) \oplus c), db = 0, dc = b \land b)$, where

the second generator $c$ is such that it trivializes the unwanted cocycles $b \land b, b \land b \land b$ etc, so that the only remaining nontrivial cocycle is $b$ itself. Notice that indeed for $2n$, the non-torsion homotopy groups of the $2n$-sphere are $\pi_{2n}(S^{2n})$ and $\pi_{4n-1}(S^{2n})$.

Thus the string-like extension $g_\mu$ can be realized in terms of the differential forms of a fibration

\[ \Omega S^{2n+1} \to \hat{G} \to G \]

where $g$ is the Lie algebra of the semisimple Lie group $G$. Moreover, at least up to homotopy, this can also be realized as a fibration

\[ \hat{G} \to G \to S^{2n+1} \]

In fact, in the sense of real or rational homotopy, $G$ has the homotopy type of a product of odd dimensional spheres. Since $\mu$ is indecomposable, it is represented by one of the spheres; in other words, this second fibration splits and we have:

**Proposition 6.** For $g$ a semisimple Lie algebra, the cohomology of $CE(g_\mu)$ is that of $CE(g)$ modulo the class of $\mu$:

\[ H^\bullet(CE(g_\mu)) \simeq H^\bullet(CE(g))/[\mu] . \]
Further, we can handle such cycles together. For example, for $g = \mathfrak{so}(n)$ we have the 3-cocycle $\mu_3$ and 7-cocycle $\mu_7$. We call $(g_\mu)_\mu_7$ the Fivebrane Lie $6$-algebra. This will be used in section 3.3.

### 4.3. Associated $L_\infty$-connections

To discuss the twisted structures that are of use to us in the context of $L_\infty$-connections, we need the following concepts in addition to the material covered in [68] and reviewed in section 4.1.

#### 4.3.1. $L_\infty$-algebra representations on cochain complexes

**Remark 4.8.** Cochain complexes in non-positive degree are sometimes referred to as $\infty$-vector spaces.

**Definition 4.9** (representations of $L_\infty$-algebroids). A representation of an $L_\infty$-algebroid $(A, g)$ on a cochain complex $V$ of finite rank $A$-modules is an $L_\infty$-algebroid $(A, g, V)_\rho$ whose Chevalley-Eilenberg algebra $CE_\rho(g, V)$ is an extension of $CE(g)$ by $\wedge_A^* V^*$

$$\wedge_A^* V^* \xrightarrow{\partial} CE_{A, \rho}(g, V) \xrightarrow{\rho} CE_A(g)$$

where $CE_\rho(g, V) = (\wedge_A^* (g^* \oplus V^*), d)$.

This means that the differential is

$$d|_{g^*} = d_g$$

$$d|_{V^*} = d_V + d_\rho,$$

where

$$d_\rho : V^* \rightarrow g^* \wedge_A (\wedge_A^* (g^* \oplus V^*))$$

encodes the action of $g$ on $V$.

**Remarks.**

1. In roughly this latter form, the definition appears in [17], where it is called a superconnection. Indeed, in cases where the $L_\infty$-algebra in question is similar to a tangent Lie algebroid of some space, its representations behave like (flat) connections on that space. Of more relevance to our present purposes are the representations of $L_\infty$-algebras in full generality. In the work of [1] for the special case of 1-Lie algebroids such representations are called representations up to homotopy. There is a related notion of representations up to strong homotopy, referring to a coherent set of higher homotopies.

2. Notice that the definition can also be phrased as follows. An $L_\infty$-algebroid $(A, g, V)_\rho$ is an $L_\infty$-algebroid $(A, g \oplus V)$ of a special form: $V$ is an abelian subalgebra and an ideal and $(A, g)$ is an $L_\infty$-subalgebra, just as not all ordinary Lie algebra structures on $g \oplus V$ come from $V$ as a representation of $g$.

3. One might expect that a representation of an $L_\infty$-algebra $g$ should be an $L_\infty$-morphism to an $L_\infty$-algebra $\text{end}(V)$ of endomorphisms of the complex $V$. In fact, for $V$ an ordinary (or graded or differential graded) vector space, $\text{end}(V)$ is an ordinary (respectively graded or differential graded) Lie algebra. For $V$ a cochain complex, there is the definition of $L_\infty$-actions – sh-representations [77] – by Lada and Markl [54] in coalgebra language. The above definition captures that but retains the DGCA perspective on representations.

The following example shows that that ordinary Lie algebra representations are a special case of definition 4.9.

**Example:** ordinary Lie representations. Let $g$ be a Lie algebra with basis $\{t_a\}$ and dual basis $\{t^a\}_{\text{deg}=+1}$ of $g^*$. Let $V$ be a vector space with basis $\{v_i\}$ and let $\rho : g \oplus V \rightarrow V$ be an ordinary Lie representation of $g$ on $V$ with components $\{\rho^i_{ja}\}$. Then

$$CE_\rho(g, V) = (\wedge^*(g^* \oplus V^*), d_\rho)$$
with the differential given by the dual of the representation map $d^*_\rho|_{V^*} := \rho^*$ is the corresponding qDGCA. In terms of components relative to the chosen basis this reads

\begin{equation}
(4.25) \quad d^*_\rho : v_i \mapsto \rho^i a v_j \wedge t^a
\end{equation}

This is nothing but the standard Chevalley-Eilenberg complex of the $g$-module $V$, but expressed in terms of bases. It can be regarded as a DGCA instead of just as a complex by thinking of $V$ as a trivial/Abelian Lie algebra.

**Representation in terms of Lie algebras of endomorphisms.**

Let $V$ be a finite-dimensional vector space and $\text{end}(V)$ its Lie algebra of endomorphisms. Then a representation is a linear map $\rho : g \otimes V \to V$ which satisfies

\begin{equation}
(4.26) \quad \rho([X,Y]) = \rho(X) \circ \rho(Y) \pm \rho(Y) \circ \rho(X)
\end{equation}

for $X,Y \in g$. To see how this is a special case of the above general definition, choose a basis $\{v_i\}$ of $V$ inducing a basis of $\text{end}(V)$ is $\{\omega^i\}$ with dual basis $\{\omega^i\}_{\text{deg}=1}$ and Chevalley-Eilenberg differential

\begin{equation}
(4.27) \quad d\omega^i = - \sum_k \omega^k \wedge \omega^k.
\end{equation}

Given any $L_\infty$-algebra $g$ with basis $\{e^a, \cdots\}$ for $g^*$, a morphism $\rho : g \to \text{end}(V)$ is a DGCA morphism $\rho^* : \text{CE}(\text{end}(V)) \to \text{CE}(g)$ given on basis elements by $\rho^* : \omega^i \mapsto \rho^i a t^a$ and satisfying

\begin{equation}
(4.28) \quad \rho^i a (d_g t^a) = -\rho^i k b \rho^k j c b^j b^c.
\end{equation}

It can be directly checked that the data encoded in $\rho^*$ is equivalent to the twisted differential on $\Lambda^*(V^* \oplus g^*)$ given by $d_{\rho} : v^i \mapsto \rho^i a v^j \wedge t^a$ since its nilpotency requires that

\begin{equation}
(4.29) \quad d_{\rho} d_{\rho} : v^i \mapsto \rho^i k b \rho^k j c t^c \wedge t^b + \rho^i a (d_g t^a),
\end{equation}

which vanishes by (4.28).

**The adjoint representation.** For $g$ any $L_\infty$-algebra, there is a representation of $g$ on itself given by the adjoint representation.

**Definition 4.10** (adjoint representation for $L_\infty$-algebras). Let $g$ be any $L_\infty$-algebra so that, by our conventions $g^*$ is concentrated in positive degree. Let $V_g$ be the underlying cochain complex of $g$ shifted down by one such that it is concentrated in non-positive degree.

\begin{equation}
(4.30) \quad \text{CE}_{\text{ad}}(g,g) := \left(\Lambda^*(g^* \oplus V_g^*), d_{\rho}\right)
\end{equation}

with $d_{\rho}\big|_{V_g} = \sigma^{-1} \circ d_g \circ \sigma$, where $\sigma : V_g^* \to g^*$ is the canonical isomorphism of cochain complexes which shifts degrees up by one, $\sigma^{-1}$ is its inverse and both are extended as graded derivations to $\Lambda^*(g^* \oplus V_g^*)$.

One checks that $(d_{\rho})^2 = 0$ by noticing that while $\sigma$ and $\sigma^{-1}$ are not inverses as graded derivations, they satisfy $\sigma \circ \sigma^{-1}|_{\Lambda^* g^*} = \text{id}$.

**Remarks 4.11.** 1. In terms of higher brackets as in [53] the adjoint representation is given by

\begin{equation}
(4.31) \quad \rho : X_1 \otimes \cdots \otimes X_n \otimes Y \mapsto [X_1, \cdots, X_n, Y]
\end{equation}

for $X_i, i = 1, \cdots, n$ and $Y$ in $g$.

2. Notice that the construction of the adjoint representation of the $L_\infty$-algebra $g$ essentially analogous to the construction of the Weil algebra $W(g)$, only that here the shift operation is down in degree, where for the Weil algebra it goes up in degree.
Example: ordinary adjoint representation. Let $\mathfrak{g}$ be an ordinary Lie algebra with basis $\{t_a\}$ and structure constants $\{C^a_{bc}\}$. Write $\{\tilde{t}_a\}_{\text{deg}=+1}$ for the corresponding dual basis elements and $\{\chi^a\}_{\text{deg}=0}$ for the corresponding basis elements of $\mathfrak{g}^*$. Then we have $d_\rho \chi^a = \sigma^{-1}(d_\mathfrak{g} \tilde{t}^a) = \sigma^{-1}(\sum_b C^a_{bc} \tilde{t}^b \land \tilde{t}^c) = C^a_{bc} \tilde{t}^b \chi^c$.

Definition 4.12 (extended standard representation of $b^{2k}u(1)$). Define the extended representation by its CE-algebra as

\[
\text{CE}_\rho(b^{2k}u(1), \bigoplus_r \mathbb{R}[2kr]) := \left(\bigwedge \left(\bigoplus_r \langle v_{2kr} \rangle \oplus \langle h \rangle_{\text{deg}=2k+1} \right), d\right)
\]

with $d : v_{2kr} \mapsto v_{2k(r-1)} \land h$ and $d : h \mapsto 0$.

Notice that also the “twisted” Chevalley-Eilenberg algebras arising from the String-like extensions in definition 4.7 involve examples of representations:

Proposition 7. CE$(\mathfrak{g}_\mu)$ from definition 4.7 is a representation of $\mathfrak{g}$ on the shifted 1-dimensional vector space $\mathbb{R}[\mu]$ such that

\[
(A^\bullet(b), d = 0) \quad \text{CE}(\mathfrak{g}_\mu) = (A^\bullet(\mathfrak{g}^* \oplus \langle b \rangle), d) \quad \text{CE}(\mathfrak{g})
\]

Proof. This obviously satisfies the axioms of a representation.

Remark 4.13. This is of course just another way of saying that $\mathfrak{g}_\mu$ is entirely governed by the Lie algebra cocycle $\mu$. It is well known from the theory of higher groups that such cocycles can be regarded as higher representations on shifted vector spaces (see [11] and references within).

4.3.2. Sections, covariant derivatives, and morphisms of $L_\infty$-connections. Here we give the $L_\infty$-algebraic version of the constructions in section 2.2.

Definition 4.14 (sections and covariant derivatives). Let $\mathfrak{g}$ be an $L_\infty$-algebra, let $(\rho, \mathfrak{g}, V)$ be a representation of $\mathfrak{g}$ and consider the principal $\mathfrak{g}$-Cartan-Ehresmann $\infty$-connection $\Omega^\bullet(\mathfrak{g})$. Then a section of the $\rho$-associated connection is an extension of this diagram through the extension defining the representation in that it is a choice of the dotted arrows in
Here
\bullet s is the section itself, the image of V in Ω∗(Y).

\text{JIM: words missing? how can a section be the image? what is Y?}
\bullet \nabla_A s is its covariant derivative.

**Example: ordinary vector bundles.** Let g be an ordinary Lie algebra with Lie group G, let V be a vector space (a chain complex concentrated in degree 0) and ρ an ordinary representation of g on V, let Y := P a principal G-bundle and (A, F_A) an ordinary Cartan-Ehresmann connection on P. Then the dotted morphism in

\[
\begin{array}{ccc}
\Omega^\bullet(P) & \xrightarrow{A_{\text{vert}}} & \text{CE}(g) \\
\rho(g, V) & \xleftarrow{(s, A_{\text{vert}})} & \text{CE}_\rho(g, V)
\end{array}
\]

is dual to a V-valued function on the total space of the bundle (not on base space!) s : P → V, which is covariantly constant along the fibers in that the covariant derivative

\[
\nabla_A s := ds + (\rho \circ A)s
\]

vanishes when evaluated on vertical vectors, where (ρ ∘ A)s denotes the action of A on the section s using the representation ρ. This means that s descends to a section of the associated vector bundle P ×_G V. The covariant derivative 1-form \nabla_A s of the section s is one component of the extension in the middle part of our diagram

\[
\begin{array}{ccc}
\Omega^\bullet(P) & \xrightarrow{A_{\text{vert}}} & \text{W}(g) \\
W(g) & \xleftarrow{(s, \nabla_A s, A_{\text{FA}})} & \Omega^\bullet(P)
\end{array}
\]

The equation

\[
\nabla_A \nabla_A s = (\rho \circ F_A) \wedge s
\]

is the Bianchi identity for \nabla_A s. If s is everywhere non-vanishing, this says that the curvature F_A of our bundle is covariantly exact on P. In the case that g = u(1) it follows that F_A is an exact 2-form on P and the choice of the non-vanishing section amounts to a trivialization of the bundle.

**Opposite L\infty-algebras.** We would like to describe morphisms of L\infty-connections following the description of morphisms between vector bundles E_1 → E_2 in terms of a section of the tensor product bundle E_1^* ⊗ E_2. If E_1 is a G-associated bundle and E_2 a G'-associated bundle, then E_1^* ⊗ E_2 is a G^op × G'-associated bundle, for G^op the group G equipped with the opposite product g_1 ·^op g_2 := g_2 · g_1. On the level of Lie algebras passing to the opposite corresponds to change the Lie bracket by a sign. The following definition generalizes this from Lie algebras to L\infty-algebras.

**Definition 4.15** (opposite L\infty-algebra). For g any L\infty-algebra with Chevalley-Eilenberg algebra \text{CE}(g) = (\Lambda^\bullet g^*, d_g) we define the opposite L\infty-algebra \text{g}^\text{op} to have the same underlying vector space CE(g) = (\Lambda^\bullet g^*, d_{g^\text{op}}) but the differential of the Chevalley-Eilenberg algebra is d_{g^\text{op}} := (-1)^{N+1} \circ d_g, where N is the operator which counts word length in the free graded algebra \Lambda^\bullet g^*.

This implies that the structure constants of g^\text{op} are those of g equipped with a sign if they have an even number of input arguments. There is a canonical morphism of L\infty-algebras (hence of their CE DGCAs) CE(g) → CE(g^\text{op}), which sends each generator to its negative.

**Definition 4.16.** We say a morphism from a g_1-connection to a g_2-connection is a representation ρ of g_1^\text{op} ⊕ g_2 and a section of the g_1^\text{op} ⊕ g_2-connection canonically induced by the given g_1-connection and g_2-connection.
Example 4.17. Morphisms of $b^{n-1}u(1)$-connections and related physics. Let $g_1 = g_2 = b^{(n-1)}u(1)$.

The standard representation of $b^{(n-1)}u(1)$ naturally extends to a representation of $b^{(n-1)}u(1) \oplus b^{(n-1)}u(1)^{op}$. Let $\sigma$ be the shift operator which shifts the degree up by 1. The Weil-algebra $W_\rho(b^{n-1}u(1), V)$ of $W_\rho(b^{n-1}u(1), V)$ looks as follows:

$$W_\rho(h^{n-1}u(1), V) = (\wedge^k \langle v_0 \rangle \oplus \langle \sigma v_0 \rangle \oplus \langle v_{n-1} \rangle \oplus \langle \sigma v_{n-1} \rangle \oplus \langle h \rangle \oplus \langle \sigma h \rangle, d)$$

with

$$dv_0 = \sigma v_{n-1}$$
$$d\sigma v_0 = 0$$
$$dv_{n-1} = v_0 \wedge h + \sigma v_{n-1}$$
$$d\sigma v_{n-1} = -\sigma v_0 \wedge h - v_0 \wedge \sigma h.$$

The above standard representation of $b^{n-1}u(1)$ has a straightforward generalization for the case that $n$ is odd. The case $n$ even does not occur because the differentials do not square to zero. The connection itself is

$$\Omega^s(P) \xrightarrow{(A_2, F_2)-(A_1, F_1)} W(b^{n-1}u(1)^{op} \oplus b^{n-1}u(1)),$$

given by forms $H_1, H_2 \in \Omega^n(P)$ being the images of the generator $\rho$ and $dH_1, dH_2 \in \Omega^{n+1}(P)$ the images of the generator $\sigma b$, for $b^{n-1}u(1)$ and its opposite, respectively. As we extend this morphism through the twisted DGCA of the standard representation definition $\Omega^s(P)$

$$W_\rho(g_1^{op} \oplus g_2, V)$$

we pick up forms which are the images of the other generators appearing in equations (4.39):

$$v_0 \mapsto s_0 \in \Omega^0(P)$$
$$\sigma v_0 \mapsto ds_0 \in \Omega^1(P)$$
$$v_{n-1} \mapsto s_{n-1} \in \Omega^{n-1}(P)$$
$$\sigma v_{n-1} \mapsto \nabla s_{n-1} := ds_{n-1} + s_0 \wedge (H_2 - H_1).$$

For instance, in the case of the Green-Schwarz mechanism we have the two $b^2u(1)$ Chern-Simons 3-connections as described in $\text{KS}$ with 3-form connections $H_1 = \text{CS}(\omega_{g_2(n)})$ and $H_2 = \text{CS}(A_{g_2})$. The above section of the difference of these two connections then is to be interpreted itself a twisted 2-connection with connection 2-form $\sigma_2$ and curvature 3-form $H_3 := \nabla \sigma_2$ which satisfies the twisted Bianchi identity

$$dH_3 = \langle F_\omega \wedge F_\omega \rangle - \langle F_A \wedge F_A \rangle.$$

This, and its magnetic dual version, is discussed in more detail below.

Another example is given by the degree two $F_2$ and the degree zero component $F_0$ of the Ramond-Ramond (RR) fields in type IIA string theory. Consider the case $n = 3$ so that $g_1 = g_2 = b^2u(1)$. The curvature $F_2$ is twisted by the Neveu-Schwarz field $H_3$

$$dF_2 + H_3 \wedge F_0 = 0,$$

where $F_0$, also known as the cosmological constant in this theory, satisfies $dF_0 = 0$. We thus have the identification of $\sigma_2$ with $F_2$ and $\sigma_0$ with $F_0$. Equation (4.44) then says that $F_2$ is covariantly constant with respect to $\nabla$. 
4.3.3. **Twisted $L_\infty$-connections.** Let $g$ be some $L_\infty$-algebra. In [68] we had discussed that the obstruction to lifting a $g$-connection (see definition 4.15) through a String-like central extension

\[
0 \rightarrow b^{n-1}u(1) \rightarrow g_\mu \rightarrow g \rightarrow 0
\]

is the $b^n u(1)$-connection obtained by canonically completing this diagram to the right as shown in figure 3. The obstruction is given by starting from the top-rightmost entry in the big square in figure 3 and continuing all the way horizontally to the left.

![Diagram](image)

**Figure 3.** Obstructing $b^n u(1)$ $(n+1)$-connections and “twisted” $g_\mu$ $n$-connections are two aspects of the same mechanism: the $(n+1)$-connection is the obstruction to “untwisting” the $n$-connection. The $n$-connection is “twisted by” the $(n+1)$-connection. There may be many non-equivalent twisted $n$-connections corresponding to the same twisting $(n+1)$-connection. We can understand these as forming a collection of $n$-sections of the $(n+1)$-connection.

The construction crucially involves first forming the lift of the $g$-connection to a $(b^{n-1}u(1) \hookrightarrow g_\mu)$-connection, where $(b^{n-1}u(1) \hookrightarrow g_\mu)$ is the “weak cokernel” or “homotopy quotient” of the injection of $b^{n-1}u(1)$ into $g_\mu$. This lift through the homotopy quotient always exists, since the homotopy quotient is in fact equivalent to just $g$. But performing the lift to the homotopy quotient also extracts the failure of the underlying attempted lift to $g_\mu$ proper. This failure may be projected out under

\[
(b^{n-1}u(1) \hookrightarrow g_\mu) \twoheadrightarrow b^n u(1)
\]

to yield the $b^n u(1)$-connection which obstructs the lift. It is the morphism denoted $f^{-1}$ in figure 3 which picks up the information about the twist/obstruction. This was constructed in proposition 40 of [68]. However, the
(b^{n-1}u(1) \otimes g_\mu)-connection itself deserves to be considered in its own right: this is just the $L_\infty$-connection version of “twisted bundles” or “gerbe modules”. In particular, the obstruction problem can also be read the other way round: given a $b^n u(1)$-bundle, we may ask for which $g$-bundles it is the obstruction to lifting these to a $g_\mu$-bundle. In string theory, this is actually usually the more natural point of view:

- given the Kalb-Ramond background field (a $b u(1)$-connection) pulled back to the worldvolume of a D-brane, the “twisted $U(H)$-bundles” corresponding to it are the “Chan-Paton bundles” supported on that D-brane;
- given the supergravity 3-form field (a $b^2 u(1)$-connection) pulled back to the end-of-the-world 9-branes, the “twisted $B U(1)$-2-bundle” corresponding to it is the Kalb-Ramond field, with the twist giving the failure of its 3-form curvature to close $dH_3 = G_4$.

5. Differential twisted structures in String theory

We now use the tools from section 4 to explicitly derive the $L_\infty$-algebra valued local differential form data of differential twisted String($n$)- and Fivebrane($n$) cocycles, i.e. the local $L_\infty$-connection data of twisted String($n$)- and Fivebrane($n$)-principal $\infty$-bundles with connection. This will explicitly derive the higher form fields known in string theory together with their familiar twisted Bianchi identities.

Note that this section ignores the normalization prefactors in front of de Rham representatives of classes, as they do not matter here.

5.1. Differential twisted cohomology. Differential twisted cohomology is the pairing of the notions of twisted cohomology with differential cohomology. Recalling that

- a cocycle in differential cohomology is to the underlying bare cocycle as a connection on an $\infty$-bundle is to a principal $\infty$-bundle;
- a cocycle in twisted cohomology is to an ordinary cocycle as a twisted $\infty$-bundle is to a principal $\infty$-bundle,

We expect cocycles in differential twisted cohomology to classify twisted principal $\infty$-bundles with connection. The definition of twisted differential cohomology is obvious when we remember that for $A \to B \to C$ a fibration sequence and $c : X \to C$ a given twisting cocycle, a $c$-twisted $A$-cocycle is nothing but a $B$-cocycle that satisfied a certain constraint. Therefore twisted differential $A$-cocycles are just differential $B$-cocycles satisfying a corresponding constraint.

More formally, we can restate this using the fact that differential nonabelian cohomology itself is defined here as a kind of twisted cohomology, where the twist is given by curvature characteristic classes. Therefore differential twisted cohomology may be understood as an example of bi-twisted cohomology.

**Definition 5.1.** Consider two fibration sequences $A \to B \to C$ and $A' \to B \to C'$ with the same object $B$ in the middle. Then given a $C$- and a $C'$-cocycle $c : X \to C$ and $c' : X \to C'$, respectively, we say that the set of connected components

$$H_{[c,c']}(X, A \times_B A') := \pi_0 H_{[c,c']}(X, A \times_B A')$$

of the homotopy pullback

$$H_{[c,c']}(X, A \times_B A') \xrightarrow{\partial} H(X, B) \xrightarrow{c \times c'} H(X, C \times C')$$

is the $[c, c']$-bitwisted cohomology of $X$.

Twisted differential cohomology is an example of this as follows:
Definition 5.2. Consider a fibration sequence $A \to B \to C$. As before, assume first for simplicity that $B$ is once deloopable (semi-abelian). Then according to our discussion of differential cohomology it sits also in the fibration sequence

$$B_{\text{flat}} \to B \to B_{\text{dR}}.$$ 

Then given a $C$-cocycle $c : X \to C$ and a curvature characteristic $P : X \to B_{\text{dR}}$, we say that the differential $[c]$-twisted cohomology with curvature class $[P]$ is the $[c,P]$-bitwisted cohomology $H_{[c,P]}(X, A \times B_{\text{flat}})$ of $X$.

Our main statement here is about the examples of this given by twisted differential String($n$)- and Fivebrane($n$)-structures. In our chosen smooth context $\mathcal{H}$ (recall section 2.1), we have smooth versions of the String($n$)- and the Fivebrane($n$) fibration sequence

\[ (5.1) \quad B \text{String}(n) \to B \text{Spin}(n) \to B^2 U(1) \]

and

\[ (5.2) \quad B \text{Fivebrane}(n) \to B \text{String}(n) \to B^7 U(1). \]

For given twisting classes

\[ (5.3) \quad \frac{1}{2} p_1(TX) : X \to B \text{Spin}(n) \to B^3 U(1) \]

and

\[ (5.4) \quad \frac{1}{6} p_1(TX) : X \to B \text{String}(n) \to B^7 U(1) \]

we determine the local differential form data and twisted Bianchi identity of differential refinements of the corresponding twisted String($n$)- and twisted Fivebrane($n$)-cocycles, respectively. In the following we discuss the

Theorem 5.3.

- The local differential form data of a twisted String($n$)-bundle with connection is that known from the Green-Schwarz mechanism.
- The local differential form data of a twisted Fivebrane($n$)-bundle with connection is that of the dual Green-Schwarz mechanism.

This is described in the following subsections. The crucial point to notice is that the twisting morphism

\[ (5.5) \quad \frac{1}{2} p_1 : B \text{String}(n) \to B^3 U(1) \]

may be modeled by the span of crossed complexes

\[ (1 \to 1 \to \text{Spin}(n) \to *) \xrightarrow{\sim} (U(1) \to \hat{\Omega} \text{Spin}(n) \to P\text{Spin}(n) \to 1) \xrightarrow{\hat{p}_1} (U(1) \to 1 \to 1 \to *). \]

The corresponding morphism of $L_\infty$-algebras is

\[ (5.6) \quad so(n) \xrightarrow{\sim} (bu(1) \to \text{string}(n)) \to b^2 u(1). \]

Therefore for $\hat{g}_{tw}$ a cocyle for a smooth $\hat{p}_1(TX)$-twisted String($n$)-principal $\infty$-bundle $P \to X$ is one fitting into a diagram

$$ \begin{array}{ccc} P & \xrightarrow{g} & B \text{Spin}(n) & \xrightarrow{\hat{p}_1} & B^3 U(1) \\ \downarrow \cong & & & & \downarrow \cong \\ \hat{g}_{tw} & & & & \\ \end{array} $$
The corresponding Cartan-Ehresmann $\infty$-connection is given by the diagram

$$\begin{array}{c}
\Omega^\bullet_{vert}(P) \xleftarrow{A^\text{vert}} \text{CE}(bu(1) \rightarrow \text{string}(n)) \\
\Omega^\bullet(P) \xrightarrow{(A,F_A)} W(bu(1) \rightarrow \text{string}(n)) \\
\Omega^\bullet(X) \xleftarrow{P(F_A)} \text{inv}(bu(1) \rightarrow \text{string}(n))
\end{array}$$

such that the corresponding twist is the given one. According to the above triangular diagram, this correct projection of the twist is ensured by inserting this diagram into the bigger one that exhibits the twisting differential cocycles. This big diagram is displayed in the following and then its content is analyzed by seeing which dg-algebra morphisms it encodes and what differential form data that amounts to.

The discussion for the Fivebrane($n$)-case then is entirely analogous. We now describe and analyze these diagrams in more detail.

5.2. Twisted $u(n)$ 1-connections. As a warmup for the following two sections, we describe twisted 1-bundles with connection in terms of their $L_\infty$-algebraic formulation and rederive in this language the familiar fact that their Chern character is closed in $H_3$-twisted de Rham cohomology, where $H_3$ is the curvature 3-form of the twisting 2-bundle.

In [4] twisted bundles and twisted gerbes are conceived of in terms of local transition data, using a nonabelian variant of Deligne-cohomology notation. Twisted bundles appear in the middle of section 3, while twisted gerbes are described in section 4 of that paper. It is not hard to see that their equation in between equations (55) and (56) expresses the idea which we emphasize here: that twisted $n$-bundles are potentially twisted lifts, i.e. obstructions to lifts, through $b^{n-1}u(1)$-extensions.

Consider the extension of Lie algebras

$$0 \rightarrow u(1) \rightarrow u(k) \rightarrow \text{pu}(k) \rightarrow 0$$

where $\text{pu}(k)$ denotes the Lie algebra of the projective unitary group $PU(k)$. It is the same as the Lie algebra of $SU(k)$, but we write $\text{pu}(k)$ to remind us that we would like to integrate to $PU(k)$ eventually. $PU(k)$-bundles and the corresponding twisted $U(k)$-bundles model the Chan-Paton bundles on D-branes and give classes in twisted K-theory [89] [49] [20] [19].

The weak quotient Lie 2-algebra ($u(1) \rightarrow u(k)$). We describe in detail the Lie 2-algebra arising as the weak (or homotopy) quotient of $u(k)$ by $u(1)$. Let, as usual, $\{t^0, t^a\}$ be a basis of $u(k)^*$ regarded as being in degree 1, with $t^0$ dual to the generator of the center = $u(1)$. Let $\{C^a_{\, bc}\}$ be the structure constants of $u(k)$ in that basis. Then the Chevalley-Eilenberg DGCA of ($u(1) \rightarrow u(k)$) is

$$\text{CE}(u(1) \rightarrow u(k)) = \left(\bigwedge^\bullet (u(k)^* \oplus \langle b \rangle)\right)$$

with the differential defined on the generators as

$$\begin{align*}
\text{d}t^0 &= -b \\
\text{d}t^a &= -\frac{1}{2}C^a_{\, bc} t^b \wedge t^c \\
\text{d}b &= 0.
\end{align*}$$

(5.9)
The Weil algebra is
\[(5.10) \quad W(u(1) \to u(k)) = \left( \bigwedge^\bullet (u(k)^* \oplus \langle a \rangle \oplus u(k)^* \oplus \langle c \rangle ) \right) \]
with differential given by
\[(5.11) \quad dt^0 = -b + r^0, \quad dt^a = -\frac{1}{2} C^a_{bc} t^b \wedge t^c + r^a, \quad db = -c,\]
where \(\{r^0, r^a\} = \sigma\{t^0, t^a\}\) is the induced basis on \(u(k)^*\) in degree 2. Finally, the algebra of invariant polynomials is
\[(5.12) \quad W(u(1) \to u(k))_{\text{basic}} = \left( \bigwedge^\bullet (\langle c \rangle \oplus \langle r^0 \rangle \oplus \langle c_2 \rangle \oplus \langle c_3 \rangle \oplus \cdots ) \right), \quad d\]
where the differential vanishes on all the \(c_i\) and on \(c\) and satisfies \(dr^0 = c\). Under the inclusion
\[(5.13) \quad W(u(1) \to u(k))_{\text{basic}}, \quad c \mapsto c, \quad r^0 \mapsto r^0 \quad \text{and} \quad c_i \mapsto \text{the corresponding Chern polynomial forms} \quad c_2 \mapsto (c_2)_{ab} r^a \wedge r^b, \quad c_3 \mapsto (c_3)_{abc} r^a \wedge r^b \wedge r^c, \ldots \]
this \(r_0\) is the non-closed invariant polynomial which will give rise to twisted de Rham cohomology. By the general principle, a twisted \(u(k)\)-connection now is a Cartan-Ehresmann connection with structure Lie 2-algebra \(g = (u(1) \to u(k)):\)
\[(5.14) \quad \Omega^\bullet_{\text{vert}}(P) \xleftarrow{A_{\text{vert}}} \text{CE}(u(1) \to u(k)) \]
\[(5.15) \quad \Omega^\bullet(P) \xleftarrow{(A,F_A)} W(u(1) \to u(k)) \quad \Omega^\bullet(\tilde{X}) \xleftarrow{\{P_i\}} \text{inv}(u(1) \to u(k)) \quad F_A = (F^0 = dA^0 + B, \quad F^a = dA^a + [A \wedge A]^a) \quad dF^0 = H_3\]
is given by a DGCA homomorphisms \(W(u(1) \to u(k))_{\text{basic}} \to \Omega^\bullet(\tilde{X})\). This is a collection consisting of a closed 3-form \(c \mapsto H_3 \in \Omega^3_{\text{closed}}(X)\), a 2-form \(r^0 \mapsto u \in \Omega^2(X)\), and a series of closed even forms coming from the \(c_i\). The Chern character of this connection for the product \(U(n) = SU(n) \times U(1)\) is as usual the combination
\[(5.15) \quad \text{ch}(F_A) := \text{tr} \exp(F + c_1) = e^{c_1} \text{tr} \exp(F) \quad \text{of the} \ c_i. \quad \text{The only difference to an ordinary} \ u(k)\text{-connection is that now no longer are all of the} \ c_i \text{closed, but that} \ dc_1 = H_3. \quad \text{Hence} \quad \text{dch} = H_3 \wedge c, \quad \text{which says that} \ \text{ch is closed in} \ H_3\text{-twisted de Rham cohomology.}\]

The Chern character of a twisted \(u(k)\)-connection lives in \(H_3\)-twisted (periodic) de Rham cohomology
\[(5.17) \quad \text{ch}(F_A) \in H_{\text{dR}}^\bullet(X, H_3). \quad \text{Here periodic twisted de Rham cohomology is the cohomology of the} \ \mathbb{Z}_2\text{-graded complex} \ \Omega_{\text{even}}^\bullet(X) \otimes \Omega_{\text{odd}}^\bullet(X), \quad \text{equipped with the differential} \ d_{H_3} = d + H_3\wedge.\]
Interpretation in terms of sections of 3-connections. We can reinterpret the twisted cohomology part of the situation in terms of sections of associated 3-connections as a generalization of the mechanism in the example below definition 4.13. Let \( g := b_2^1 u(1) \). The extended standard representation of \( b_2^1 u(1) \) from definition 4.12 comes with a Weil algebra given by the obvious generalization of that defined in equations 4.39. Interpret the closed globally defined 3-form as a flat definition 4.12 comes with a Weil algebra given by the obvious generalization of that defined in equations 4.39. Interpret the closed globally defined 3-form as a flat 3-connection

\[
(5.18) \quad \Omega^*(P) \xrightarrow{(H_3, dH_3=0)} W(b_2^1 u(1)^{op} \oplus b^{n-1} u(1))
\]

and consider a section of this connection via the extended standard representation in definition 4.12 of \( b_2^1 u(1) \).

As we extend the connection morphism through the twisted DGCA of the extended standard representation

\[
(5.19) \quad W_\rho(b_2^1 u(1), \oplus_r \mathbb{R}[2r])
\]

we pick up forms which are the images of the other generators appearing in equations (4.39): Let \( v_{2r} \mapsto c_{2r} \in \Omega^{2r}(P) \)

\[
(5.20) \quad v_{2r} \quad \mapsto \quad c_{2r} \in \Omega^{2r}(P)
\]

\[
(5.21) \quad \sigma v_{2r} \quad \mapsto \quad \nabla c_{2r} := d_{H_3} c_{2r} := dc_{2r} + H_3 \wedge c_{2(r-1)}
\]

The \( H_3 \)-twisted de Rham differential now appears as the covariant derivative of the associated cochain complex which is associated via the extended standard representation of \( b_2^1 u(1) \) to the \( L_\infty \)-connection obtained from interpreting the globally defined 3-form \( H_3 \) as the connection 3-form on a trivial flat 3-bundle. This means that we are interpreting the \( d + H_3 \) twist at 2 different but closely related levels:

- the twisted 1-connection which is a morphism into the 2-connection has \((d + H_3)\)-closed Chern character.
- Regarding the untwisted \( H_3 \) 2-connection as itself being twisted, but by the trivial twist given by a flat 3-connection the interpretation of \( H_3 \) changes from that of a 3-form curvature to a 3-form connection. The covariant derivative of this 3-form connection with respect to the extended standard representation of \( b_2^1 u(1) \) is again \( \nabla = d + H_3 \).

5.3. Twisted string\((n)\) 2-connections. Let \( X \) be a smooth space (an object in \( \mathbf{H} \)) with Spin structure given by a cocycle \( g \in H(X, B\text{Spin}) \) and hence with fractional first Pontrjagin class

\[
(5.22) \quad \frac{1}{2} p_1(X) : X \xrightarrow{g} B\text{String} \xrightarrow{\frac{1}{2} p_1} U(1).
\]

We now work out the differential form data, according to section 4.5.1 carried by a cocycle in differential \( \frac{1}{2} p_1(X) \)-twisted BString-cohomology \( q \in H_{[\frac{1}{2} p_1(X)]}(X, B\text{String}) \), i.e. the connection and curvature data and the twisted Bianchi identity of a twisted String-2-bundle. We demonstrate that this twisted Bianchi identity is a relation between differential forms as appearing in the Green-Schwarz mechanism.

For that purpose let \( p : P \longrightarrow X \) be the total space of the twisted String-2-bundle concretely realized as the pullback

\[
(5.23) \quad \begin{array}{c}
\begin{array}{c}
P \\
\downarrow \\
\end{array}
\end{array} \xrightarrow{\text{E}(BU(1) \to \text{String})} \xrightarrow{\text{g}_{\text{tw}}} \begin{array}{c}
\begin{array}{c}
B(\text{BU}(1) \to \text{String}) \\
\downarrow \\
\text{BSpin}
\end{array}
\end{array}
\]

for some twisted lift \( g_{\text{tw}} \) of \( g \).
On this cover $P \to X$, the computation is essentially a special case of the general description of higher Chern-Simons connections in section 7 of [68]: there we computed the differential data of the obstruction to a differential String-lift, here we fix the obstruction and compute the nature of the cocycles twisted by it. The reader can without essential loss think of $P$ as an ordinary manifold and of $\Omega^\bullet(P)$ as ordinary differential forms on this manifold.

In yet another equivalent formulation of this situation, we describe the covariant derivative and its Bianchi identity of a section of a bundle associated with respect to the representation of $B^2 U(1)$ induced by the canonical $L_\infty$-algebra inclusion

$$CE(bu(1) \hookrightarrow g_\mu) \longrightarrow CE(b^2u(1))$$

(5.24)

to a $B^2U(1)$-3-bundle with local connection 3-form $C_3 \in \Omega^3(P)$ and with curvature 4-form $G_4 \in \Omega^4_{\text{closed}}(X)$, By the discussion in section 4.3.2 this is a choice of dashed morphisms in the diagram

It may be helpful to recall what each of the terms in this diagram means. The following diagram is a labeled map for the above one.
Chevalley-Eilenberg algebra of structure $L_\infty$-algebra for twisted String 2-bundle

- flat nonabelian differential forms on fibers of total space
- or equivalently section of 2-gerbe / line 3-bundle

Weil algebra of structure $L_\infty$-algebra for twisted String 2-bundle

- connection and curvature on twisted String 2-bundle
- or equivalently section with covariant derivative of 2-gerbe / line 3-bundle

characteristic forms of twisted String 2-bundle

- invariant polynomials on structure $L_\infty$-algebra of twisted String 2-bundle

connection and curvature on 2-gerbe / line 3-bundle

vertical differential forms on total space

- flat abelian differential forms on fibers

Weil algebra of structure $L_\infty$-algebra of 2-gerbe / line 3-bundle

invariant polynomials on structure $L_\infty$-algebra of 2-gerbe / line 3-bundle

characteristic forms on 2-gerbe / line 3-bundle

differential forms on total space

$p^*$

differential forms on base space
Now, chasing the generators of the graded-commutative algebras through this diagram and recording the condition imposed by the respect of the morphisms of DGCAs for differentials, one finds that in components the commutativity of this diagram encodes the following differential form data and the following relations on that.
Here, as usual, $P \in W(\mathfrak{g})$ is the invariant polynomial on $\mathfrak{g}$ in transgression with with the cocycle $\mu \in CE(\mathfrak{g})$. With $\{t^a\}$ a fixed chosen basis of $\mathfrak{g}^*$ in degree 1 and $\{r^a\}$ the corresponding basis in degree 2, we have $P = P_{ab}t^a \wedge t^b$ and $\mu = \mu_{abcd}t^a \wedge t^b \wedge t^c$ and $c_8 = F_{ab} \wedge r^a + \frac{1}{6} \mu_{abcd} t^a \wedge t^b \wedge t^c$. We have

| curvature | $H_3 := dB + C_3 - CS(A, F_A)$ |
| Bianchi identity | $dH_3 = G_4 - (F_A \wedge F_A)$ |

In [68] this situation was considered from a different perspective for the special case $B = 0$ and $\nabla B = 0$. There the dashed morphism was obtained as a twisted lift of a $\mathfrak{g}$-connection to a $\mathfrak{g}_\mu$-connection and the $b^2 u(1)$-connection appeared as the corresponding obstruction. Here now the perspective is switched: the $b^2 u(1)$-connection is prescribed and the choice of dashed morphisms is a choice of twisted $\mathfrak{g}_\mu$-connections with prescribed twist $G_4$.

The covariant derivative 3-form $\nabla B$ of the twisted $\mathfrak{g}_\mu$-connection, which we denote by $H_3$, measures the difference between the prescribed $b^2 u(1)$-connection and the twist of the chosen twisted $\mathfrak{g}_\mu$-connection. The Bianchi identity

\[(5.25)\quad dH_3 = G_4 - P(F_A)\]

which appears in the middle on the left says that this difference has to vanish in cohomology, as one expects. Indeed, this is the structure of the differential forms in the Green-Schwarz mechanism.

5.4. **A model for the M-theory C-field.** Our formalism allows for (a generalization of) three points of view regarding the description of the M-theory C-field. These are

1. as a shifted differential 2-character. This views the $E_8$ class $a$ as somewhat more ‘basic’ and then $\frac{1}{2} \lambda$ is a shift leading to a shifted differential 2-character [28].
2. as a twisted string structure. This takes $\frac{1}{2} \lambda$ as the more ‘basic’ for which the $E_8$ class $a$ acts as a twist.
3. a more democratic point of view of both classes as twists for degree four cohomology. This is the bi-twisted point of view.

The description of the M-theory C-field is very closely related to that of the fields in heterotic string theory discussed in the previous section. In fact, one way of deriving the quantization condition (3.14) of $G_4$ is by comparing [37] to the heterotic theory on the boundary [43]. The condition in the latter is a trivialization of the cohomology class $\text{ch}_2(E) - p_1(X)$ on a ten-manifold $X$. As we saw above this is equated at the level of forms to $dH_3$. The condition in M-theory on a Spin manifold $Y$ is a trivialization of the cohomology class $[G_4] + \frac{1}{2} p_1 - a$, where $a$ is the class of the $E_8$ bundle. At the level of forms, this is equated to $dC_3$.

We can already see the close similarity in the mathematical structures between the two quantization conditions. We will use this to provide a model for the C-field in twisted nonabelian differential cohomology using the case of the heterotic string from the previous section. We see that the changes we need to make to the diagrams in the previous section are simply

1. Replace $G_4$ by $G_4$.
2. Replace $H_3$ by $C_3$.
3. Replace $dB_2$ by $c_3$.
4. Add the term $\langle F_\omega \wedge F_\omega \rangle$.

From this we conclude that the **C-field in M-theory is a cocycle in the total differential twisted cohomology**

\[(5.26)\quad \hat{H}_{\text{tw}}(X, B\text{String} \times BU(8)),\]

using the notation from section 2.3.

5.5. **Twisted fivebrane(n) 6-connections.** Now we consider the connection on a twisted Fivebrane-bundle obtained from a twisted lift of a Spin-bundle. The discussion is entirely analogous to that in the previous section, only that now, more differential forms enter the picture.
Suppose a $\mathfrak{so}(n)$ connection is given and we are asking for a lift to a $\text{fivebrane}(n) \simeq (\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection. We discussed the obstruction for that in [68]. By the general discussion in section 4.3.2, if the obstruction does not vanish, we still get a twisted $\text{fivebrane}(n)$-connection, namely a connection with structure $L_\infty$-algebra being

\[(5.27) \quad (b^5u(1) \hookrightarrow (bu(1) \hookrightarrow (g_{\mu_3})_{\mu_7})).\]

The twisted Bianchi identity in this case is nothing but the dual Green-Schwarz formula [69] in terms of differential forms.

To see this, consider a section of a $b^2u(1) \oplus b^5u(1)$-connection given by a pair consisting of a connection 3- and 7-form $(C_3, C_7) \in \Omega^3(X) \times \Omega^7(X)$ with curvature 4- and 8-form $(G_4, G_8) \in \Omega^4_{\text{closed}}(X) \times \Omega^8_{\text{closed}}(X)$ with respect to the canonical inclusion

\[(5.28) \quad \text{CE}((bu(1) \oplus b^5u(1)) \hookrightarrow (\mathfrak{so}(n)_{\mu_3})_{\mu_7}) \rightarrow \text{CE}(b^2u(1) \oplus b^5u(1)) \].

Again by the discussion in section 4.3.2 this is a choice of dashed morphisms in the diagram with the appropriate changes in the various entries.
Here is again the interpretation of the terms in this diagram:

- **Chevalley-Eilenberg algebra of structure $L_\infty$-algebra for twisted Fivebrane 6-bundle**
- **flat nonabelian differential forms on the fibers or equivalently section of 7-bundle**
- **vertical differential forms on the total space**
- **Weil algebra of structure $L_\infty$-algebra for twisted Fivebrane 6-bundle**
- **differential forms on the total space**
- **connection and curvature on twisted Fivebrane 6-bundle or equivalently section with covariant derivative 7-bundle**
- **Weil algebra of structure $L_\infty$-algebra of line 7-bundle**
- **connection and curvature on 7-bundle**
- **differential forms on the total space**
- **characteristic forms of twisted Fivebrane 6-bundle**
- **forms on base space**
- **characteristic forms on 7-bundle**
- **p^* invariant polynomials on structure $L_\infty$-algebra of twisted Fivebrane 6-bundle**
- **invariant polynomials on 7-bundle**
By again chasing elements through the diagram one finds the following data:

\[
\begin{align*}
  dt^a &= -\frac{1}{2}C^a_{bc}t^b \wedge t^c \\
  db_2 &= \mu_3 - k_3 \\
  db_6 &= \mu_7 - k_7 \\
  dk_3 &= 0 \\
  dk_7 &= 0
\end{align*}
\]

\[
\begin{align*}
  t^a &\mapsto A^a_{\text{vert}} \\
  b_2 &\mapsto (B_2)_{\text{vert}} \\
  b_6 &\mapsto (B_6)_{\text{vert}} \\
  k_3 &\mapsto (C_3)_{\text{vert}} \\
  k_7 &\mapsto (C_7)_{\text{vert}}
\end{align*}
\]

\[
\begin{align*}
  F_{A_{\text{vert}}} &= 0 \\
  d(B_2)_{\text{vert}} &= \mu_3(A_{\text{vert}}) - (C_3)_{\text{vert}} \\
  d(B_6)_{\text{vert}} &= \mu_7(A_{\text{vert}}) - (C_7)_{\text{vert}} \\
  d(C_3)_{\text{vert}} &= 0 \\
  d(C_7)_{\text{vert}} &= 0
\end{align*}
\]

\[
\begin{align*}
  H_3 &= \nabla B_2 = dB_2 + C_3 - CS_3(A, F_A) \\
  H_7 &= \nabla B_6 = dB_6 + C_7 - CS_7(A, F_A) \\
  dH_3 &= \mathcal{G}_4 - (F_A \wedge F_A) \\
  dH_7 &= \mathcal{G}_8 - (F_A \wedge F_A \wedge F_A) \\
  d\mathcal{G}_4 &= 0 \\
  d\mathcal{G}_8 &= 0
\end{align*}
\]

\[
\begin{align*}
  dP_4 &\mapsto (F_A \wedge F_A) \\
  dP_8 &\mapsto (F_A \wedge F_A \wedge F_A)
\end{align*}
\]
As usual, $P_4, P_8 \in W(g)$ are the invariant polynomials on $g$ in transgression with with the cocycles $\mu_3, \mu_7 \in CE(g)$. The covariant derivative 7-form $\nabla B_6$ of the twisted $(\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection which we denote by $H_7$ measures the difference between the prescribed $b^6u(1)$-connection and the twist of the chosen twisted $(\mathfrak{so}(n)_{\mu_3})_{\mu_7}$-connection. The Bianchi-identity
\begin{equation}
(5.29) \quad dH_7 = G_8 - P_8(F_A)
\end{equation}
which appears in the middle on the left says that this difference has to vanish in cohomology, as one expects. This is the differential form data of the dual Green-Schwarz mechanism \cite{69}.

5.6. A model for the M-theory dual C-field. Similarly to the case of the C-field, our formalism allows for (a generalization of) three points of view regarding the description dual of of the M-theory C-field. These are

1. as a shifted differential 6-character. This views the $E_8$ degree 8 class $\frac{1}{2}a^2$ as somewhat more 'basic' and then $\frac{1}{48}p_2$ is a shift leading to a shifted differential 6-character. This is a generalization of the case in \cite{28} to the degree eight case.
2. as a twisted Fivebrane structure. This takes $\frac{1}{48}p_2$ as the more ‘basic’ for which the $E_8$ class $\frac{1}{2}a^2$ acts as a twist.
3. we can also give a more democratic point of view by viewing both classes as twists for degree eight cohomology cohomology. This is the bi-twisted point of view.

In this section we provide a model for the dual $G_8$ of the C-field in an analogous way that we did for the case of the C-field itself in section 5.4. Here again we notice the similarity in structure between the dual $H_7$ of the $H_3$ field in ten dimensions and the dual $G_8$ of $G_4$ in eleven dimensions. $H_7$ provides a trivialization of the dual of the Green-Schwarz anomaly formula, while $G_8$ is itself part of the sum of cohomology class, and hence, at the level of differential forms, it is itself trivialized rather than acting as trivialization. Hence, as in the degree four case, we have an extra term $dC_7$ that acts as a trivialization. In fact $C_7$ is the right hand side of the equation of motion for $G_4$.

We can again see the close similarity in the mathematical structures between the two quantization conditions (see equation 3.46). We will use this to provide a model for the dual of the C-field in twisted nonabelian differential cohomology using the case of the dual heterotic string from previous section. We see that the changes we need to make to the diagrams in the previous section are simply

1. Replace $G_8$ by $G_8$.
2. Replace $H_7$ by $C_7$.
3. Replace $dB_6$ by $c_7$.
4. Add the term $\langle F_\omega \wedge F_\omega \wedge F_\omega \wedge F_\omega \rangle$.

From this we conclude, again, that the dual C-field in M-theory is a cocycle in the total twisted differential cohomology \begin{equation}
(5.30) \quad \bar{H}_{tw}(X, BFivebrane \times BU(10))
\end{equation}

using the notation from section 2.3.

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