Levy processes: long time behavior and convolution-type form of the Ito representation of the infinitesimal generator

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Abstract

In the present paper we show that the Levy-Ito representation of the infinitesimal generator $L$ for Levy processes $X_t$ can be written in a convolution-type form. Using the obtained convolution form we have constructed the quasi-potential operator $B$. We denote by $p(t, \Delta)$ the probability that a sample of the process $X_t$ remains inside the domain $\Delta$ for $0 \leq \tau \leq t$ (ruin problem). With the help of the operator $B$ we find a new formula for $p(t, \Delta)$. This formula allows us to obtain long time behavior of $p(t, \Delta)$.

1 Introduction

Let us introduce the notion of the Levy processes.

Definition 1.1 A stochastic process $\{X_t: t \geq 0\}$ is called Levy process, if the following conditions are fulfilled:
1. Almost surely $X_0 = 0$, i.e. $P(X_0 = 0) = 1$.
One says that an event happens almost surely (a.s.) if it happens with probability one.

2. For any $0 < t_1 < t_2 < \ldots < t_n < \infty$ the random variables $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_4}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent (independent increments).

(To call the increments of the process $X_t$ independent means that increments $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_4}, \ldots, X_{t_n} - X_{t_{n-1}}$ are mutually (not just pairwise) independent.)

3. For any $s < t$ the distributions of $X_t - X_s$ and $X_{t-s}$ are equal (stationary increments).

4. Process $X_t$ is almost surely right continuous with left limits.

Then Levy-Khinchine formula gives (see [2], [18])

$$\mu(z, t) = E\{\exp[izX_t]\} = \exp[-t\lambda(z)], \quad t \geq 0,$$

where

$$\lambda(z) = \frac{1}{2}Az^2 - i\gamma z - \int_{-\infty}^{\infty} (e^{ixz} - 1 - ixz1_{|x|<1}) \nu(dx).$$

Here $A \geq 0$, $\gamma = \bar{\gamma}$, $z = \bar{z}$ and $\nu(dx)$ is a measure on the axis $(-\infty, \infty)$ satisfying the conditions

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} \nu(dx) < \infty.$$

The Levy-Khinchine formula is determined by the Levy-Khinchine triplet $(A, \gamma, \nu(dx))$.

By $P_t(x_0, \Delta)$ we denote the probability $P(X_t \in \Delta)$ when $P(X_0 = x_0) = 1$ and $\Delta \in \mathcal{R}$. The transition operator $P_t$ is defined by the formula

$$P_t f(x) = \int_{-\infty}^{\infty} P_t(x, dy)f(y).$$

Let $C_0$ be the Banach space of continuous functions $f(x)$, satisfying the condition $\lim_{|x| \to \infty} f(x) = 0, \quad |x| \to \infty$ with the norm $||f|| = \sup_x |f(x)|$. We denote by $C_0^n$ the set of $f(x) \in C_0$ such that $f^{(k)}(x) \in C_0,$ \hspace{1em} $(1 \leq k \leq n)$. It is known that [18]

$$P_t f \in C_0,$$

if $f(x) \in C_0^n$.

Now we formulate the following important result (see [18]).
**Theorem 1.2 (Levy-Ito decomposition.)** The family of the operators $P_t \quad (t \geq 0)$ defined by the Levy process $X_t$ is a strongly continuous semigroup on $C_0$ with the norm $||P_t|| = 1$. Let $L$ be its infinitesimal generator. Then

$$Lf = \frac{1}{2} A \frac{d^2 f}{dx^2} + \gamma \frac{df}{dx} + \int_{-\infty}^{\infty} (f(x + y) - f(x) - y \frac{df}{dx} 1_{|y| < 1}) \nu(dy), \quad (1.6)$$

where $f \in C_0^2$.

Slightly changing the Sato classification [18] we introduce the following definition:

**Definition 1.3** We say that a Levy process $X_t$ generated by $(A, \nu, \gamma)$ has type I if

$$A = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \nu(dx) < \infty, \quad (1.7)$$

and $X_t$ has type II if

$$A \neq 0 \quad \text{or} \quad \int_{-\infty}^{\infty} \nu(dx) = \infty. \quad (1.8)$$

**Remark 1.4** The introduced type I coincides with the type $A$ in the Sato classification. The introduced type II coincides with the union of the types $B$ and $C$ in the Sato classification.

The properties of these two types of the Levy processes are quiet different.

2. In the present paper we show that the Levy-Ito representation of the generator $L$ can be written in the convolution form

$$Lf = \frac{d}{dx} S \frac{d}{dx} f, \quad (1.9)$$

where the operator $S$ is defined by the relation

$$Sf = \frac{1}{2} Af + \int_{-\infty}^{\infty} k(y - x) f(y) dy. \quad (1.10)$$

We note that for arbitrary $a \quad (0 < a < \infty)$ the inequality

$$\int_{-a}^{a} |k(t)| dt < \infty \quad (1.11)$$
is true. Formulas (1.9) and (1.10) were proved before in our works [16] under some additional conditions. In the present paper we omit these additional conditions and prove these formulas for the general case. The representation of $L$ in form (1.9) is convenient as the operator $L$ is expressed with the help of the classic differential and convolution operators. Assuming that $X_t$ belongs to the class II we have constructed the quasi-potential operator $B$. This operator $B$ is linear and bounded in the space of the continuous functions. We denote by $p(t, \Delta)$ the probability that a sample of the process $X_t$ remains inside the domain $\Delta$ for $0 \leq \tau \leq t$ (ruin problem). With the help of the operator $B$ we find a new formula for $p(t, \Delta)$. This formula allows us to obtain the long time behavior of $p(t, \Delta)$. Namely, we have proved the following asymptotic formula

$$p(t, \Delta) = e^{-t/\lambda_1}[c_1 + o(1)], \quad c_1 > 0, \; \lambda_1 > 0, \quad t \to +\infty. \quad (1.12)$$

Let $T_\Delta$ be the time during which $X_t$ remains in the domain $\Delta$ before it leaves the domain $\Delta$ for the first time. It is easy to see that

$$p(t, \Delta) = P(T_\Delta > t). \quad (1.13)$$

In Sections 1 and 2 we often follow the presentation from [17].

Remark 1.5 In the next paper we plan to apply the convolution representation (1.9) to the Levy processes $X_t$ of the type I.

2 Convolution-type form of infinitesimal generator

1. By $C(a)$ we denote the set of functions $f(x) \in C_0$ which have the following property:

$$f(x) = 0, \quad x \notin [-a, a] \quad (2.1)$$

i.e. the function $f(x)$ is equal to zero in the neighborhood of $x = \infty$. We note, that parameter $a$ can be different for different $f$. We introduce the functions

$$\mu_-(x) = \int_{-\infty}^{x} \nu(dx), \quad x < 0, \quad (2.2)$$

i.e. the function $f(x)$ is equal to zero in the neighborhood of $x = \infty$. We note, that parameter $a$ can be different for different $f$.

We introduce the functions

$$\mu_-(x) = \int_{-\infty}^{x} \nu(dx), \quad x < 0, \quad (2.2)$$
\[ \mu_+(x) = -\int_x^\infty \nu(dx), \quad x > 0, \quad (2.3) \]

where the functions \( \mu_-(x) \) and \( \mu_+(x) \) are monotonically increasing and right continuous on the half-axis \((-\infty, 0] \) and \([0, \infty) \) respectively. We note that

\[
\mu_+(x) \to 0, \quad x \to +\infty; \quad \mu_-(x) \to 0, \quad x \to -\infty, \quad (2.4)
\]

\[
\mu_-(x) \geq 0, \quad x < 0; \quad \mu_+(x) \leq 0, \quad x > 0. \quad (2.5)
\]

In view of (1.3) the integrals in the right sides of (2.2) and (2.3) are convergent.

**Theorem 2.1** The following relations

\[
\varepsilon^2 \mu_{\pm}(\pm \varepsilon) \to 0, \quad \varepsilon \to +0, \quad (2.6)
\]

\[
\int_{-a}^0 x \mu_-(x) dx < \infty, \quad -\int_0^a x \mu_+(x) dx < \infty, \quad 0 < a < \infty \quad (2.7)
\]

are true.

**Proof.** According to (1.3) we have

\[
0 \leq \int_{-a}^{-\varepsilon} x^2 d\mu_-(x) \leq M, \quad (2.8)
\]

where \( M \) does not depend from \( \varepsilon \). Integrating by parts the integral of (2.8) we obtain:

\[
\int_{-a}^{-\varepsilon} x^2 d\mu_-(x) = \varepsilon^2 \mu_-(\varepsilon) - a^2 \mu_-(a) - 2 \int_{-a}^{-\varepsilon} x \mu_-(x) dx \leq M, \quad (2.9)
\]

The function \( -\int_{-a}^{-\varepsilon} x \mu_-(x) dx \) of \( \varepsilon \) is monotonic increasing. In view of (2.9) this function is bounded. Hence we have

\[
\lim_{\varepsilon \to +0} \int_{-a}^{-\varepsilon} x \mu_-(x) dx = \int_{-a}^0 x \mu_-(x) dx \quad (2.10)
\]

It follows from (2.9) and (2.10) that

\[
\lim_{\varepsilon \to +0} \varepsilon^2 \mu_-(\varepsilon) = m, \quad m \geq 0. \quad (2.11)
\]

Using (2.10) and (2.11) we have \( m = 0 \). Thus, relations (2.9) and (2.10) are proved for \( \mu_-(x) \). In the same way relations (2.9) and (2.10) can be proved for \( \mu_+(x) \). □
2. Let us introduce the functions

\[ k_-(x) = \int_{-1}^{x} \mu_-(t) \, dt, \quad -\infty \leq x < 0, \quad (2.12) \]

\[ k_+(x) = -\int_{x}^{1} \mu_+(t) \, dt, \quad 0 < x \leq +\infty. \quad (2.13) \]

In view of (2.5) the integrals on the right-hand sides of (2.12) and (2.13) are absolutely convergent. From (2.12) and (2.13) we obtain the assertions:

**Theorem 2.2**

1. The function \( k_-(x) \) is continuous, monotonically increasing on the \((-\infty, 0)\) and

\[ k_-(x) \geq 0, \quad -1 \leq x < 0. \quad (2.14) \]

2. The function \( k_+(x) \) is continuous, monotonically decreasing on the \((0, +\infty)\) and

\[ k_+(x) \geq 0, \quad 0 < x \leq 1. \quad (2.15) \]

Further we need the following result:

**Theorem 2.3**

The relations

\[ \varepsilon k_-(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow +0; \quad \varepsilon k_+(\varepsilon) \rightarrow 0, \quad \varepsilon \rightarrow +0; \quad (2.16) \]

\[ \int_{-1}^{0} k_-(x) \, dx < \infty; \quad \int_{0}^{1} k_+(x) \, dx < \infty \quad (2.17) \]

are valid.

**Proof.** According to (2.5) we have

\[ 0 \leq -\int_{-1}^{-\varepsilon} x \mu_-(x) \, dx \leq M, \quad (2.18) \]

where \( M \) does not depend from \( \varepsilon \). Integrating by parts the integral of (2.18) we obtain:

\[ -\int_{-1}^{-\varepsilon} x \mu_-(x) \, dx = \varepsilon k_-(\varepsilon) - k_-(\varepsilon) - k_-(1) + \int_{-1}^{-\varepsilon} k_-(x) \, dx \leq M, \quad (2.19) \]

The function \( \int_{-1}^{-\varepsilon} k_-(x) \, dx \) of \( \varepsilon \) is monotonic increasing. This function is bounded (see (2.19)). Hence we have

\[ \lim_{\varepsilon \rightarrow +0} \int_{-1}^{-\varepsilon} k_-(x) \, dx = \int_{-1}^{0} k_-(x) \, dx \quad (2.20) \]
It follows from (2.19) and (2.20) that
\[
\lim_{\varepsilon \to +0} \varepsilon k_-(\varepsilon) = p, \quad p \geq 0.
\] (2.21)

Using (2.20) and (2.21) we have \( p = 0 \). Thus, the first relations (i.e., the
relations for \( k_-(x) \)) in (2.16) and (2.17) are proved. The second relations
(i.e., the relations for \( k_+(x) \)) in (2.16) and (2.17) can be proved in the same
way. □

3. We use the following notation
\[
J(f) = J_1(f) + J_2(f),
\] (2.22)
where
\[
J_1(f) = \frac{d}{dx} \int_{-\infty}^{x} f'(y) k_-(y - x) dy, \quad f(x) \in C(a),
\] (2.23)
\[
J_2(f) = \frac{d}{dx} \int_{x}^{\infty} f'(y) k_+(y - x) dy, \quad f(x) \in C(a).
\] (2.24)

Lemma 2.4 The operator \( J(f) \) defined by (2.22) can be represented in the
form
\[
J(f) = \int_{-\infty}^{\infty} [f(y + x) - f(x) - y \frac{df(x)}{dx} 1_{|y| \leq 1}] \mu(dy) + \Gamma f'(x),
\] (2.25)
where \( \Gamma = \Gamma f \) and \( f(x) \in C(a) \).

Proof. From (2.24) we obtain the relation
\[
J_1(f) = - \int_{x-1}^{x} [f'(y) - f'(x)] k'_-(y - x) dy - \int_{-a}^{x-1} f'(y) k'_-(y - x) dy.
\] (2.26)
By proving (2.26) we used relations (2.13) and equality
\[
\int_{x-1}^{x} k_-(y - x) dy = \int_{-a}^{0} k_-(v) dv.
\] (2.27)
We introduce the notations
\[
P_1(x, y) = f(y) - f(x) - (y - x) f'(x), \quad P_2(x, y) = f(y) - f(x).
\] (2.28)
Using notations (2.28) we represent (2.26) in the form

\[ J_1(f) = - \int_{-1}^{0} \frac{\partial}{\partial y} P_1(x, y+x) \mu_-(y) dy - \int_{-a-x}^{-1} \frac{\partial}{\partial y} P_2(x, y+x) \mu_-(y) dy, \quad (2.29) \]

Integrating by parts the integrals of (2.29) we deduce that

\[ J_1(f) = f'(x) \gamma_1 + \int_{-1}^{0} P_1(x, y+x) \mu_-(y) dy + \int_{-a-x}^{-1} P_2(x, y+x) \mu_-(y) dy + P_2(x, -a) \mu_-(a), \quad (2.30) \]

where \( \gamma_1 = k'_-(1) \). It follows from (1.3) that the integrals in (2.30) are absolutely convergent. Passing to the limit in (2.30), when \( a \to + \infty \), and taking into account (2.17), (2.27) we have

\[ J_1(f) = \int_{-\infty}^{x} \left[ f(y+x) - f(x) - y \frac{df(x)}{dx} \right] 1_{|y| \leq 1} \mu_-(y) + \gamma_1 f'(x). \quad (2.31) \]

In the same way it can be proved that

\[ J_2(f) = \int_{x}^{\infty} \left[ f(y+x) - f(x) - y \frac{df(x)}{dx} \right] 1_{|y| \leq 1} \mu_+(y) + \gamma_2 f'(x), \quad (2.32) \]

where \( \gamma_2 = k'_+(1) \). The relation (2.26) follows directly from (2.31) and (2.32). Here \( \Gamma = \gamma_1 + \gamma_2 \). The lemma is proved. \( \square \)

**Remark 2.5** The operator \( L_0 f = \frac{d}{dx} f \) can be represented in form (1.9), (1.10), where

\[ S_0 f = \int_{-\infty}^{\infty} p_0(x-y)f(y)dy, \quad (2.33) \]

\[ p_0(x) = \frac{1}{2} \text{sign}(x). \quad (2.34) \]

From Lemmas 2.4 and Remark 2.5 we deduce the following assertion.

**Theorem 2.6** The infinitesimal generator \( L \) has a convolution-type form (1.9), (1.10).

**Example 2.7** Let us consider the Poisson process.
In this case we have
\[ \lambda(z) = -(e^{iz} - 1) \]  
(2.35)

According to (1.2) and (2.35) the relations
\[ \mu_-(x) = 0, \; x < 0; \; \mu_+(x) = \begin{cases} -1 & \text{if } 0 < x < 1; \\ 0 & \text{if } x \geq 1 \end{cases} \]  
(2.36)

are true. Using (2.8), (2.9) and (2.36) we obtain
\[ k_-(x) = 0, \; x < 0; \; k_+(x) = \begin{cases} 1 - x & \text{if } 0 < x < 1; \\ 0 & \text{if } x \geq 1 \end{cases} \]  
(2.37)

Hence the operator \( L \) for the Poisson process has the following convolution form:
\[ Lf = \frac{d}{dx} \int_x^{x+1} (1 - y + x)f'(y)dy. \]  
(2.38)

Formula (2.38) coincides with Levy-Ito formula:
\[ Lf = -f'(x) + f(x + 1) - f(x). \]  
(2.39)

3 The Probability of the Levy process (type II) remaining within the given domain

1. We remind, that the definition of the Levy processes and the definition of the type II are given in the section 1.

Let us denote by \( \Delta \) the set of segments \([a_k, b_k]\) such that
\[ a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n, \; 1 \leq k \leq n. \]  
In many theoretical and applied problems it is important to estimate the quantity
\[ p(t, \Delta) = P(X_\tau \in \Delta; 0 \leq \tau \leq t), \]  
(3.1)
i.e. the probability that a sample of the process \( X_\tau \) remains inside \( \Delta \) for \( 0 \leq \tau \leq t \) (ruin problem).

**Condition 3.1** Further we consider only the Levy processes of type II and assume, that \( \Delta \) belongs to the support of \( X_t \), \( t > 0 \) (see section 6).

According to Condition 3.1 \( \Delta \in \mathbb{R} \) if either condition 1) or condition 2) of Theorem 6.3 is fulfilled. If condition 3) of Theorem 6.3 is fulfilled and \( \gamma = 0, \)
then either $\Delta \in [0, \infty)$ or $\Delta \in (-\infty, 0]$.  
We denote by $F_0(x, t)$ the distribution function of Levy process $X_t$, i.e.
\begin{align}
F_0(x, t) &= P(X_t \leq x). \tag{3.2}
\end{align}

We need the following statement (see [18])

**Theorem 3.1** The distribution function $F_0(x, t)$ is continuous with respect to $x$ if and only if the Levy process belongs to type II.

We introduce the sequence of functions
\begin{align}
F_{n+1}(x, t) &= \int_0^t \int_{-\infty}^{\infty} F_0(x - \xi, t - \tau)V(\xi)d\xi F_n(\xi, \tau)d\tau, \tag{3.3}
\end{align}

where the function $V(x)$ is defined by relations $V(x) = 1$ when $x \notin \Delta$ and $V(x) = 0$ when $x \in \Delta$. In the right side of (3.3) we use Stieltjes integration. It follows from (1.1) that
\begin{align}
\mu(z, t) &= \mu(z, t - \tau)\mu(z, \tau). \tag{3.4}
\end{align}

Due to (3.4) and convolution formula for Stieltjes-Fourier transform (see [3], Ch.4) the relation
\begin{align}
F_0(x, t) &= \int_{-\infty}^{\infty} F_0(x - \xi, t - \tau)d\xi F_0(\xi, \tau) \tag{3.5}
\end{align}
is true. Using (3.3) and (3.5) we have
\begin{align}
0 \leq d_x F_n(x, t) \leq t^n d_x F_0(x, t)/n!, \text{ if } dx > 0. \tag{3.6}
\end{align}

Relation (3.6) implies that
\begin{align}
0 \leq F_n(x, t) \leq t^n F_0(x, t)/n!. \tag{3.7}
\end{align}

Hence the series
\begin{align}
F(x, t, u) = \sum_{n=0}^{\infty} (-1)^n u^n F_n(x, t) \tag{3.8}
\end{align}
converges. The probabilistic meaning of $F(x, t, u)$ is defined by the relation (see [10], Ch.4):
\begin{align}
E\{\exp[-u \int_0^t V(X_\tau)d\tau], c_1 < X_t < c_2\} = F(c_2, t, u) - F(c_1, t, u). \tag{3.9}
\end{align}
The inequality $V(x) \geq 0$ and relation (3.9) imply that the function $F(x, t, u)$ monotonically decreases with respect to the variable "u" and monotonically increases with respect to the variable "x". Hence, the following formula

$$0 \leq F(x, t, u) \leq F(x, t, 0) = F_0(x, t) \quad (3.10)$$

is true. In view of (3.2) and (3.10) the Laplace transform

$$\Psi(x, s, u) = \int_0^{\infty} e^{-st}F(x, t, u)dt, \quad s > 0. \quad (3.11)$$

has meaning. According to (3.3) the function $F(x, t, u)$ is the solution of the equation

$$F(x, t, u) + u\int_0^{t} \int_{-\infty}^{\infty} F_0(x - \xi, t - \tau)V(\xi)d\xi F(\xi, \tau, u)d\tau = F_0(x, t) \quad (3.12)$$

Taking from both parts of (3.12) the Laplace transform bearing in mind (3.11) and using the convolution property (see [3], Ch.4) we obtain

$$\Psi(x, s, u) + u\int_{-\infty}^{\infty} \Psi_0(x - \xi, s)V(\xi)d\xi \Psi(\xi, s, u) = \Psi_0(x, s), \quad (3.13)$$

where

$$\Psi_0(x, s) = \int_0^{\infty} e^{-st}F_0(x, t)dt. \quad (3.14)$$

It follows from (3.11) and (3.14) that

$$\int_{-\infty}^{\infty} e^{ixp}d_x\Psi_0(x, s) = \frac{1}{s + \lambda(p)}. \quad (3.15)$$

According to (3.13) and (3.14) we have

$$\int_{-\infty}^{\infty} e^{ixp}[s + \lambda(p) + uV(x)]d_x\Psi(x, s, u) = 1. \quad (3.16)$$

Now we introduce the function

$$h(p) = \frac{1}{2\pi} \int_{\Delta} e^{-ixp}f(x)dx, \quad (3.17)$$
where the function $f(x)$ belongs to $C_{\Delta}$. Multiplying both parts of (3.16) by $h(p)$ and integrating them with respect to $p$ ($-\infty < p < \infty$) we deduce the equality
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ixp}[s + \lambda(p)]h(p)dx\Psi(x, s, u)dp = f(0).
\] (3.18)
We have used the relations
\[
V(x)f(x) = 0, \quad -\infty < x < \infty,
\] (3.19)
\[
\frac{1}{2\pi} \lim_{N \to \infty} \int_{N}^{N} \int_{-N}^{N} e^{-ixp}f(x)dxdp = f(0), \quad N \to \infty.
\] (3.20)
Since the function $F(x, t, u)$ monotonically decreases with respect to "$u$" this is also true for the function $\Psi(x, s, u)$ (see (3.11)). Hence there exist the limits
\[
F_{\infty}(x, s) = \lim F(x, s, u), \quad \Psi_{\infty}(x, s) = \lim \Psi(x, s, u), \quad u \to \infty.
\] (3.21)
It follows from (3.8) that
\[
p(t, \Delta) = P(X_{\tau} \in \Delta, 0 < \tau < t) = \int_{\Delta} dx F_{\infty}(x, t).
\] (3.22)
Hence we have
\[
\int_{0}^{\infty} e^{-st}p(t, \Delta)dt = \int_{\Delta} dx \Psi_{\infty}(x, s).
\] (3.23)
Using relations (1.2) and (1.6) we deduce that
\[
\lambda(z) \int_{-\infty}^{\infty} e^{-it\xi} f(x)dx = - \int_{-\infty}^{\infty} e^{-it\xi}[Lf(x)]d\xi.
\] (3.24)
2. By $C_{\Delta}$ we denote the set of functions $g(x)$ on $L^2(\Delta)$ such that
\[
g(a_k) = g(b_k) = g'(a_k) = g'(b_k) = 0, \quad 1 \leq k \leq n, \quad g''(x) \in L^p(\Delta), \quad p > 1.
\] (3.25)
We introduce the operator $P_{\Delta}$ by relation $P_{\Delta}f(x) = f(x)$ if $x \in \Delta$ and $P_{\Delta}f(x) = 0$ if $x \not\in \Delta$. 

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Definition 3.2 The operator
\[ L_\Delta = P_\Delta LP_\Delta = \frac{d}{dx}S_\Delta \frac{d}{dx}, \text{ where } S_\Delta = P_\Delta SP_\Delta. \] (3.26)
is called a truncated generator.

Relations (3.18), (3.21) and (3.24) imply the following assertion.

Theorem 3.3 Let \( X_t \) be the Levy process. If the corresponding distribution function \( F(x,t) \) is continuous with respect to \( x \) then the relation
\[ \int_\Delta (sI - L_\Delta)fdx \Psi_\infty(x,s) = f(0) \] (3.27)
is true.

Remark 3.4 For the symmetric stable processes equality (3.27) was deduced by M.Kac [10]. For the Levy processes with continuous density the equality (3.27) was obtained in the book [10]. Now we deduced equality (3.27) for the Levy processes with continuous distribution.

3. Relations (3.23) implies the following assertion

Proposition 3.5 The function \( \Psi_\infty(x,s) \) for all \( s > 0 \) is monotonically increasing and continuous with respect to \( x \).

The behavior of \( \Psi_\infty(x,s) \) when \( s = 0 \) we shall consider separately, using the following Hengartner and Thedorescu result ( [8], see [18] too):

Theorem 3.6 Let \( X_t \) be a Levy process. Then for any finite interval \( K \) the estimation
\[ P(X_t \in K) = 0(t^{-1/2}) \text{ as } t \rightarrow \infty \] (3.28)
is valid.

Hence, we have the assertion

Theorem 3.7 Let \( X_t \) be a Levy process. Then for any integer \( n > 0 \) the estimation
\[ p(t, \Delta) = 0(t^{-n/2}) \text{ as } t \rightarrow \infty \] (3.29)
is valid.
Proof. Let the interval $K$ be such, that $\Delta \in K$. Then the inequality
\[
P(X_t \in \Delta) \leq P(X_t \in K)
\] (3.30)
holds. According to Levy processes properties (independent and stationary increments) and (3.30) the following inequality
\[
p(t, \Delta) \leq P^n(X_t/n \in K) = 0(t^{-n/2}) \quad as \quad t \to \infty
\] (3.31)
is true. □

It follows from (3.9) that
\[
\int_0^\infty e^{-st} p(t, \delta) dt = \int_\delta d_x \Psi_\infty(x, s),
\] (3.32)
where $\delta$ is a set of segments which belong to $\Delta$ and
\[
p(t, \delta) = P(X_t \in \delta; 0 \leq \tau \leq t).
\] (3.33)
If $\delta = \Delta$, then formula (3.32) coincides with formula (3.23) We need the following partial case of (3.32):
\[
\int_0^\infty p(t, \delta) dt = \int_\delta d_x \Psi_\infty(x, 0),
\] (3.34)
According to (3.31) the integral in the left side of (3.34) exists. Let us prove the following statement.

**Proposition 3.8** The function $\Psi_\infty(x, 0)$ is strictly monotonically increasing and continuous in the domain $\Delta$.

**Proof.** In view of (3.34) and Condition 3.1 the function $\Psi_\infty(x, 0)$ is strictly monotonically increasing. To prove that $\Psi_\infty(x, 0)$ is continuous we introduce the function
\[
q(t, \delta) = P(X_t \in \delta).
\] (3.35)
It is obvious that
\[
p(t, \delta) \leq q(t, \delta).
\] (3.36)
We consider $\delta = [x_1, x_2]$. As $F(x, t)$ is continuous function with respect to $x$ the relations
\[
\lim_{x_2 \to x_1} q(t, \delta) = 0, \quad \lim_{x_2 \to x_1} p(t, \delta) = 0
\] (3.37)
are valid. Formulas (3.31), (3.34) and (3.37) imply that the function $\Psi_\infty(x, 0)$ is continuous. The proposition is proved.

**Remark 3.9** Formulas (3.32) and (3.34) will play in the next sections an essential role.
4 Quasi-potential

1. We remind that the domain $\Delta$ is the set of segments $[a_k, b_k]$, where $a_1 < b_1 < a_2 < b_2 < ... < a_n < b_n$, $1 \leq k \leq n$.

We denote by $D_{\Delta}$ the space of the continuous functions $g(x)$ on the domain $\Delta$ such that

$$g(a_k) = g(b_k) = 0,$$

$1 \leq k \leq n$.

The norm in $D_{\Delta}$ is defined by the relation $||f|| = \sup_{x \in \Delta} |f(x)|$.

**Definition 4.1** The operator $B$ with the definition domain $D_{\Delta}$ is called a quasi-potential if the following relations

$$- L_{\Delta}Bf = f, \quad f \in D_{\Delta} \quad (4.1)$$

$$- BL_{\Delta}g = g, \quad g \in C_{\Delta} \quad (4.2)$$

are true.

**Remark 4.2** In a number of cases (see the next section) we need relations (4.1) and (4.2). In these cases we can use the quasi-potential $B$, which is often simpler than the corresponding potential $Q$.

**Theorem 4.3** Let the considered Levy process $X_t$ belong to the type II. Then the corresponding quasi-potential $B$ is bounded in the space $D_{\Delta}$, has the form

$$Bf = \int_{\Delta} f(y)d_y\Phi(x,y), \quad f \in D_{\Delta}, \quad (4.3)$$

where the real-valued function $\Phi(x,y)$ is continuous with respect to $x$ and $y$, monotonically increasing with respect to $y$.

Before proving Theorem 4.3, we investigate some properties of the operator $B$, which is defined by (4.3). The operator $B$ maps the space $D_{\Delta}$ into himself. We remind the following definitions.

**Definition 4.4** The total variation of a complex-valued function $g$, defined on $\Delta$ is the quantity

$$V_{\Delta}(g) = \sup_P \sum_{i=0}^{n_P-1} |g(x_{i+1}) - g(x_i)|,$$

where the supremum is taken over the set of all partitions $P = (x_0, x_1, ..., x_{n_P})$ of the $\Delta$. 

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Definition 4.5. A complex-valued function $g$ on the $\Delta$ is said to be of bounded variation (BV function) on the $\Delta$ if its total variation is finite.

By $D_\Delta^*$ we denote the conjugate space to $D_\Delta$. It is well-known that the space $D_\Delta^*$ consists from functions $g(x)$ with a bounded total variation $V_\Delta(g)$. The norm in $D_\Delta^*$ is defined by the relation $||g|| = V_\Delta(g)$, the functional in $D_\Delta$ is defined by the relation

$$ (f,g)_\Delta = \int_\Delta f(x) \overline{g(x)}, \quad f \in D_\Delta, \quad g \in D_\Delta^*. \quad (4.4) $$

Hence, the conjugate operator $B^*$ maps the space $D_\Delta^*$ into himself and has the form

$$ B^*g = \int_\Delta \Phi(y,x) dg(y). \quad (4.5) $$

Let us consider an arbitrary inner point $y_0$ from the domain $\Delta$. We introduce the new domain $\Delta(y_0) = \Delta - y_0$. We denote the corresponding truncated generator by $L_\Delta(y_0)$, the corresponding quasi-potential by $B(y_0)$, the corresponding kernel by $\Phi(x,y,y_0)$, the corresponding $\Psi$ function by $\Psi_\infty(x,y,y_0)$

If $y_0 = 0$, then according to (4.5) we have $\Phi(0,x,0) = B(0)^* \sigma(x)$, where $\sigma(x) = -1/2$, when $x < 0$ and $\sigma(x) = 1/2$, when $x > 0$. The next assertion follows directly from (3.27).

Lemma 4.6 If $y_0 = 0$ belongs to the inner part of $\Delta$ and $\Phi(0,x,0) = \Psi_\infty(x,0,0)$, then

$$ -(L_\Delta(0)f, B(0)^* \sigma)_\Delta = f(0). \quad (4.6) $$

Now we shall reduce the general case to the case $y_0 = 0$. We introduce the operator

$$ Uf = g(x), \quad g(x) = f(x - y_0), \quad x \in \Delta, \quad (4.7) $$

which maps the space $D_{\Delta(y_0)}$ onto $D_\Delta$. Using formulas (2.1) and (2.2) we deduce that

$$ L_\Delta = UL_\Delta(y_0)U^{-1}. \quad (4.8) $$

Hence the equality

$$ B = UB(y_0)U^{-1} \quad (4.9) $$

is valid. The last equality can be written in the terms of the kernels

$$ \Phi(x,y) = \Phi(x - y_0, y - y_0, y_0). \quad (4.10) $$
Relation (3.27) in the case $\Delta(y_0)$ takes the form
\[
\int_{\Delta(y_0)} -L\Delta(y_0) f \, dx \Psi_\infty(x, 0, y_0) = f(0).
\] (4.11)

According to (4.11) Lemma 4.6 can be written in the following form:

**Lemma 4.7** If $\Phi(0, x, y_0) = \Psi_\infty(x, 0, y_0)$, then
\[
- (L\Delta(y_0)f, B(y_0)^*\sigma)_{\Delta(y_0)} = f(0).
\] (4.12)

In view of (4.18) and (4.19) equality (4.12) can be rewritten in the form
\[
(-L\Delta(0)g, B(0)^*\sigma(x - y_0))_{\Delta} = g(y_0),
\] (4.13)
where
\[
g(x) = f(x + y_0).
\] (4.14)

Using (4.12) we define the kernel $\Phi(x, y)$ of the operator $B$ by the relation
\[
\Phi(x, y) = \Psi_\infty(y - x, 0, x),
\] (4.15)

According to Proposition 3.8 and (4.15) we have the assertion

**Proposition 4.8** The function $\Phi(x, y)$ is continuous with respect to $x$ and $y$ and monotonically increasing with respect to $y$.

Relation (4.15) implies the equality
\[
- BL\Delta g = g, \quad g \in C_\Delta.
\] (4.16)

2. **Sectorial properties**

We shall need the following Pringsheim’s result.

**Theorem 4.9** (see [23], Ch.1) Let $f(t)$ be non-increasing function over $(0, \infty)$ and integrable on any finite interval $(0, \ell)$. If $f(t) \to 0$ when $t \to \infty$, then for any positive $x$ we have
\[
\frac{1}{2}[f(x + 0) + f(x - 0)] = \frac{2}{\pi} \int_0^\infty \cos xu \left[ \int_0^\infty f(t) \cos t \, dt \right] du,
\] (4.17)
\[
\frac{1}{2}[f(x + 0) + f(x - 0)] = \frac{2}{\pi} \int_0^\infty \sin xu \left[ \int_0^\infty f(t) \sin t \, dt \right] du.
\] (4.18)
We choose such the functions $k_-(x)$ and $k_+(x)$ that
\[ k_-(a) = k_+(a) = 0, a > 0. \] (4.19)

In this case we have
\[ k_-(x) = \int_{-a}^{x} \mu_-(t) dt, -\infty \leq x < 0, \] (4.20)
\[ k_+(x) = -\int_{x}^{a} \mu_+(t) dt, 0 < x \leq +\infty. \] (4.21)

Using the integration by parts and taking into account (2.7) and (2.16) we deduce the assertion.

**Proposition 4.10** Let conditions (4.19) - (4.21) be fulfilled. Then the relation
\[ \int_{-a}^{a} k(t) \cos xt dt = -[k'_+(a) - k'_-(a)] \frac{1 - \cos xa}{x^2} + \int_{-a}^{a} \frac{1 - \cos xt}{x^2} d\mu(t), \] (4.22)
is true. Here $\mu(t) = \mu_-(t)$ when $t < 0$ and $\mu(t) = \mu_+(t)$ when $t > 0$.

The following relations
\[ k'_-(a) = \mu_-(a) \geq 0, k'_+(a) = \mu_+(a) \leq 0 \] (4.23)
are valid. It follows from (4.23) and Theorem 4.9 that the kernel $k(x)$ admits the representation
\[ k(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} m(t)e^{ixt} dt, \] (4.24)
where
\[ \text{Re}[m(t)] = -[k'_+(a) - k'_-(a)] \frac{1 - \cos xa}{x^2} + \int_{-a}^{a} \frac{1 - \cos xt}{x^2} d\mu(t) \geq 0. \] (4.25)

Further we use the following notions.

**Definition 4.11** The bounded operator $S_\Delta$ in the space $L^2(\Delta)$ is called sectorial if
\[ (S_\Delta f, f) \neq 0, \quad f \neq 0 \] (4.26)
and
\[ -\frac{\pi}{2} \beta \leq \arg(S_\Delta f, f) \leq \frac{\pi}{2} \beta, \quad 0 < \beta \leq 1. \] (4.27)
Definition 4.12 The sectorial operator $S$ is called a strongly sectorial if for some $\beta < 1$ relation (4.27) is valid.

We note that we use two definitions for scalar product
\[(f,g) = \int_{\Delta} f(x)g(x)dx\] and \[(f,g)_{\Delta} = \int_{\Delta} f(x)dg(x)\].

3. Let the equality $\Delta = [-a,a]$ holds. Then due to (4.24) the relation
\[(S_{\Delta}f,f) = \int_{-\infty}^{\infty} m(u)|\int_{\Delta} f(t)e^{itu}dt|^{2}du\] (4.28)
is valid. If $0 \in \Delta$, then the Levy measure $\nu(\Delta) > 0$. In this case the entire function $m_1(x) = \Re m(x)$ is equal to zero only in finite number of points on every finite interval. In view of (4.28) we have $(S_{\Delta}f,f) \neq 0$ and $\Re(S_{\Delta}f,f) > 0$, when $||f|| \neq 0$. Thus, we have obtained the assertion.

Proposition 4.13 If the Levy process $X_t$ belongs to the type II and $0$ belongs to $\Delta$, then
\[-\pi/2 < \arg(S_{\Delta}f,f) < \pi/2. \quad f(t) \in L^2(\Delta), \ f \neq 0.\] (4.29)

Remark 4.14 We have deduced Proposition 4.13 for the case, when $\Delta = [-a,a]$. It is easy to repeat this proof for an arbitrary $\Delta$.

4. We denote by $G$ the numerical range of $S_{\Delta}$. Let us consider the point $\alpha$, which belongs to the boundary of $G$. There exists a sequence of functions $f_n \in L^2(\Delta), ||f_n|| = 1$, which converges in weakly sense to $f$ and $(S_{\Delta}f_n,f_n) \rightarrow = \alpha$. In view of (4.3) the operator $S_{\Delta}$ is compact. So, $(S_{\Delta}f_n,f_n) \rightarrow (S_{\Delta}f,f) = \alpha$. The last relation implies:

Proposition 4.15 Let relations (4.29) be true. If $\alpha$ belongs to the closed set $G$ then either $\alpha = 0$, or $\Re \alpha > 0$.

Now we can prove the following important in our theory result.

Theorem 4.16 The operator $S_{\Delta}$ is strongly sectorial.

Proof. The assertion of the theorem in the case $A > 0$ follows directly from (4.29) and $\Re(S_{\Delta}f,f) \geq A/2, ||f|| = 1$. According to theorem Toeplitz-Hausdorff (see [5]) the numerical range set $G$ is convex. Using this fact and Proposition 4.15 we obtain the assertion of the theorem in the case $A = 0$. □
Remark 4.17 Theorem 4.16 under additional conditions was obtained before in our works (see [10]).

We note that operators $S_{\Delta}$ plays an important role not only in Levy processes theory (see [15] and [16]).

5. According to (4.16) we have

$$-L_{\Delta}B L_{\Delta} g = L_{\Delta} g, \quad g \in C_{\Delta}. \tag{4.30}$$

Using relations (3.25) we can represent the operator $L_{\Delta}$ in the form

$$L_{\Delta} g = S_{\Delta} g'', \quad g \in C_{\Delta}. \tag{4.31}$$

It follows from Theorem 4.16 and equality (4.31), that the range $L_{\Delta}$ is dense in $D_{\Delta}$. Hence, it follows from (4.30) the statement.

Lemma 4.18 If the kernel $\Phi(x, y)$ satisfies the relation (4.15), then

$$-L_{\Delta} B f = f, \quad f \in D_{\Delta}. \tag{4.32}$$

Now we can prove the Theorem 4.3.

Proof of Theorem 4.3.
The function $\Phi(x, y)$, defined by (4.15), is bounded,continuous and monotonically increasing with respect to $y$. Hence, the corresponding operator $B$ is bounded in the space $D_{\Delta}$. It follows from (4.16) and (4.32) that the constructed operator $B$ is quasi-potential. The theorem is proved.

6. We need the following result.

Proposition 4.19 The operator $B$ is strongly sectorial.

Proof. Let the function $g(x)$ satisfies conditions (3.25). Then the relation

$$(-L_{\Delta} g, g) = (S_{\Delta} g', g') \tag{4.33}$$

holds. Equalities (1.1) and (4.33) imply that

$$(f, B f) = (S_{\Delta} g', g'), \quad g = B f. \tag{4.34}$$

Inequality

$$(B f, f) \neq 0, \text{ if } f \neq 0 \tag{4.35}$$

follows from relation (4.33). The operator $S_{\Delta}$ is strongly sectorial. Hence, according to (4.33) the operator $B$ is strongly sectorial too. □

6. Now we shall find the relation between $\Psi_{\infty}(x, s)$ and $\Phi(0, x)$. 20
Theorem 4.20  Let the considered Levy process belong to the type II. Then in the space $D_\Delta$ there is one and only one function

$$\Psi(x, s) = (I + sB^*)^{-1}\Phi(0, x), \quad (4.36)$$

which satisfies relation $(3.27)$.

Proof. In view of (4.2) we have

$$-BL_\Delta f = f, \quad f \in C_\Delta. \quad (4.37)$$

Relations $(4.36)$ and $(4.37)$ imply that

$$((sI - L_\Delta)f, \psi(x, s))_\Delta = -((I + sB)L_\Delta f, \psi)_\Delta = -(L_\Delta f, \Phi(0, x))_\Delta. \quad (4.38)$$

Since $\Phi(0, x) = B^*\sigma(x)$, then according to $(4.36)$ and $(4.38)$ relation $(3.27)$ is true.

Let us suppose that in $L(\Delta)$ there is another function $\Psi_1(x, s)$ satisfying $(3.27)$. Then the equality

$$((sI - L_\Delta)f, \phi(x, s))_\Delta = 0, \quad \phi = \Psi - \Psi_1 \quad (4.39)$$

is valid. We write relation $(4.39)$ in the form

$$(L_\Delta f, (I + sB^*)\phi)_\Delta = 0. \quad (4.40)$$

The range of $L_\Delta$ is dense in $D_\Delta$. Hence in view of $(4.40)$ we have $\phi = 0$. The theorem is proved. □

It follows from relations $(3.27)$ and $(4.36)$ that

$$\Psi_\infty(x, s) = (I + sB^*)^{-1}\Phi(0, x), \quad (4.41)$$

Remark 4.21  We stress an important fact: Operators $B$ and $B^*$ are bounded in the spaces $D_\Delta$ and $D^*_\Delta$ respectively.

7.

Definition 4.22  The Levy process $X_t$ with the triplet $(A, \gamma, \nu)$ is symmetric (see [18]) if $\gamma = 0$ and the function $\nu(x)$ is odd.
According to (2.2), (2.3) and (2.13), (2.14) in the symmetric case the following equality \( k(-x) = k + (x), \ x > 0 \) is valid. Taking into account (3.26) we obtain the assertion:

**Proposition 4.23** Let \( X_t \) be the symmetric Levy process. Then the corresponding operators \( L_\Delta \) and \( B \) are self-adjoint in the Hilbert space \( L^2(\Delta) \) with the inner product

\[
(f, g) = \int_\Delta f(x)\overline{g(x)} \, dx.
\]  

(4.42)

So, in the symmetric case the operator \( B \) is strongly sectorial and self-adjoint. Hence we have

**Corollary 4.24** Let \( X_t \) be the symmetric Levy process. The spectrum of the corresponding operator \( B \) is real and non-negative.

## 5 Long time behavior

1. We apply the following Krein-Rutman theorem ([12], section 6):

**Theorem 5.1** If a linear compact operator \( B \) leaving invariant a cone \( K \), has a point of the spectrum different from zero, then it has a positive eigenvalue \( \lambda_1 \) not less in modulus than any other eigenvalues \( \lambda_k \), \( k > 1 \). To this eigenvalue \( \lambda_1 \) corresponds at least one eigenvector \( g_1 \in K, (Bg_1 = \lambda_1 g_1) \) of the operator \( B \) and at least one eigenvector \( h_1 \in K^*, (B^* h_1 = \lambda_1 h_1) \) of the operator \( B^* \).

We remark that in our case the cone \( K \) consists of non-negative continuous real functions \( g(x) \in D_\Delta \) and the cone \( K^* \) consists of monotonically increasing bounded functions \( h(x) \in D_{\Delta}^* \). In this section we investigate the asymptotic behavior \( p(t, \Delta) \) when \( t \to \infty \).

The spectrum \( (\lambda_k, k > 1) \) of the operator \( B \) is situated in the sector

\[
-\frac{\pi}{2} \beta \leq \arg z \leq \frac{\pi}{2} \beta, \quad 0 \leq \beta < 1, \quad |z| \leq \lambda_1.
\]  

(5.1)

We introduce the domain \( D_\epsilon \): 

\[
-\frac{\pi}{2}(\beta + \epsilon) \leq \arg z \leq \frac{\pi}{2}(\beta + \epsilon), \quad |z - (1/2)\lambda_1| < (1/2)(\lambda_1 - \epsilon),
\]  

(5.2)
where $0 < \epsilon < 1 - \beta$, $0 < r < \lambda_1$. If $z$ belongs to the domain $D_\epsilon$ then the relation

$$\text{Re}(1/z) > 1/\lambda_1$$

(5.3)

holds. Indeed, relation (5.3) is equivalent to inequality

$$(x - \lambda_1/2)^2 + y^2 < \lambda_1^2/4, \ x = \Re z, \ y = \Im z, \ i.e. \ |z - \lambda_1/2| < \lambda_1/2.$$ 

We take so small $r$ that the the circle

$$|z - (1/2)(1/2)| = (1/2)(1/2 - r).$$

(5.4)

has the points $z_1$ and $z_2 = \overline{z_1}$ of the intersections with half-lines $\arg z = \pi/2(\beta + \epsilon)$ and $\arg z = -\pi/2(\beta + \epsilon)$. We denote the boundary of domain $D_\epsilon$ by $\Gamma_\epsilon$. We stress that $\Gamma_\epsilon$ contains only part of circle (5.4), which is situated between the points $z_1$ and $z_2$. Without loss of generality we may assume that the points of spectrum $\lambda_k \neq 0$ do not belong to $\Gamma_\epsilon$. Now we formulate the main result of this section.

**Theorem 5.2** Let Levy process $X_t$ have type II and let the corresponding quasi-potential $B$ satisfy the following conditions:

I. Operator $B$ is compact in the Banach space $D_\Delta$. II. There exists such constant $M > 0$ that

$$\left|((B - zI)^{-1}, \Phi(0, x))_\Delta\right| \leq M/|z|, \ z \in \Gamma_\epsilon.$$ 

(5.5)

Then we have

1). The eigenfunction $g_1(x)$ of the operator $B$ is continuous and $g_1(x) > 0$, where $x$ are the inner points of $\Delta$.

2). The eigenfunction $h_1(x)$ of the operator $B^*$ is absolutely continuous and strictly monotonic.

3). The asymptotic equality

$$p(t, \Delta) = e^{-t/\lambda_1} [q(t) + o(1)], \ t \to +\infty$$

(5.6)

is true. The function $q(t)$ has the form

$$q(t) = c_1 + \sum_{k=2}^m c_k e^{\beta t \nu_k} \geq 0, \ c_1 > 0,$$

(5.7)

where $\nu_k$ are real.
Proof. The operator $B$ has a point of the spectrum different from zero (see Theorem 6.6). Properties 1) and 2) follows directly from Krein-Rutman Theorem 5.1. Hence we have

$$
(g_1, h_1)_\Delta = \int_\Delta g_1(x)dh_1(x) > 0. \quad (5.8)
$$

Using (3.23) we obtain the equality

$$
p(t, \Delta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{iyt}, \Psi_\infty(x, iy))_\Delta dy, \quad t > 0. \quad (5.9)
$$

Changing the variable $z = iy$ and taking into account (4.36) we rewrite (5.9) in the form

$$
p(t, \Delta) = \frac{1}{2i\pi} \int_{i\infty}^{-i\infty} (e^{-t/z}, (zI - B^*)^{-1}\Phi(0, x))_\Delta \frac{dz}{z}, \quad t > 0. \quad (5.10)
$$

As the operator $B$ is compact, only a finite number of eigenvalues $\lambda_k, 1 < k \leq m$ of this operator does not belong to the domain $D_\epsilon$. We deduce from formula (5.10) the relation

$$
p(t, \Delta) = \sum_{k=1}^{m} \sum_{j=0}^{n_k-1} e^{-t/\lambda_k} t^j c_{k,j} + J, \quad (5.11)
$$

where $n_k$ is the index of the eigenvalue $\lambda_k$,

$$
J = -\frac{1}{2i\pi} \int_{T_{\epsilon}} \frac{1}{z} e^{-t/z}(1, (B^* - zI)^{-1}\Phi(0, x))_\Delta dz. \quad (5.12)
$$

We remind that the index of the eigenvalue $\lambda_k$ is defined as the dimension of the largest Jordan block associated to that eigenvalue. We note that

$$
n_1 = 1. \quad (5.13)
$$

Indeed, if $n_1 > 1$ then there exists such a function $f_1$ that

$$
Bf_1 = \lambda_1 f_1 + g_1. \quad (5.14)
$$

In this case the relations

$$
(Bf_1, h_1)_\Delta = \lambda_1 (f_1, h_1)_\Delta + (g_1, h_1)_\Delta = \lambda_1 (f_1, h_1)_\Delta \quad (5.15)
$$
are true. Hence \((g_1, h_1)_\Delta = 0\). The last relation contradicts (5.8). It proves equality (5.13).

Relation (4.13) implies that

\[
\Phi(0, x) \in D^*_\Delta. \tag{5.16}
\]

Among the numbers \(\lambda_k\) we choose the ones for which \(\text{Re}(1/\lambda_k)\), \((1 \leq k \leq m)\) has the smallest value \(\delta\). Among the obtained numbers we choose \(\mu_k\), \((1 \leq k \leq \ell)\) the indexes \(n_k\) of which have the largest value \(n\). We deduce from (5.11), (5.12) and (5.5) that

\[
p(t, \Delta) = e^{t\delta}t^n\left[\sum_{k=1}^\ell e^{-t/\mu_k}c_k + o(1)\right], \quad t \to \infty. \tag{5.17}
\]

We note that the function

\[
Q(t) = \sum_{k=1}^\ell e^{it\text{Im}(\mu_k^{-1})}c_k \tag{5.18}
\]

is almost periodic (see [13]). Hence in view of (5.17) and the inequality \(p(t, \Delta) > 0\), \((t \geq 0)\) the following relation

\[
Q(t) \geq 0, \quad -\infty < t < \infty \tag{5.19}
\]

is valid.

First we assume that at least one of the inequalities

\[
\delta < \lambda_1^{-1}, \quad n > 1 \tag{5.20}
\]

is true. Using (5.20) and the inequality

\[
\lambda_1 \geq |\lambda_k|, \quad k = 2, 3, ...
\]

we have

\[
\text{Im}\mu_j^{-1} \neq 0, \quad 1 \leq j \leq \ell. \tag{5.22}
\]

It follows from (5.18) that

\[
c_j = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T Q(t)e^{-it(\text{Im}\mu_j^{-1})}dt, \quad T \to \infty. \tag{5.23}
\]

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In view of (5.19) the relations
\[ |c_j| \leq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} Q(t) dt = 0, \quad T \to \infty, \] 
(5.24)
are valid, i.e. \( c_j = 0, \quad 1 \leq j \leq \ell. \) This means that relations (5.18) are not true. Hence the equalities
\[ \delta = \lambda_1^{-1}, \quad n = 1 \] 
(5.25)
are true. From (5.25) we get the asymptotic equality
\[ p(t, \Delta) = e^{-t/\lambda_1} [q(t) + o(1)] \quad t \to \infty, \] 
(5.26)
where the function \( q(t) \) is defined by relation (5.7) and
\[ c_k = g_k(0) \int_{\Delta} dh_k(x), \quad \nu_k = \text{Im}(\mu^{-1}), \; k > 1, \] 
(5.27)
\[ c_1 = g_1(0) \int_{\Delta} dh_1(x) > 0. \] 
(5.28)
Here \( g_k(x) \) are the eigenfunctions of the operator \( B \) corresponding to the eigenvalues \( \lambda_k \), and \( h_k(x) \) are the eigenfunctions of the operator \( B^* \) corresponding to the eigenvalues \( \overline{\lambda_k} \). The following conditions are fulfilled
\[ (g_k, h_k) = \int_{\Delta} g_k(x) dh_k(x) = 1, \] 
(5.29)
\[ (g_k, h_\ell) = \int_{\Delta} g_k(x) dh_\ell(x) = 0, \quad k \neq \ell. \] 
(5.30)
Using the almost periodicity of the function \( q(t) \) we deduce from (5.17) the inequality \( q(t) \geq 0 \). The theorem is proved. \( \square \)

2. Now we shall consider the important case when
\[ \text{rank} \lambda_1 = 1. \] 
(5.31)
We remain that \( \text{rank} \) of an eigenvalue is defined as the number of linearly independent eigenvectors with that eigenvalue, i.e. \( \text{rank} \) of an eigenvalue coincides with the geometric multiplicity of this eigenvalue.
Theorem 5.3 Let the conditions of Theorem 5.2 be fulfilled. In the case \(5.31\) the following relation
\[
p(t, \Delta) = e^{-t/\lambda_1}[c_1 + o(1)], \quad c_1 > 0, \quad t \to +\infty \tag{5.32}
\]
is true.

Proof In view of \(5.6\) we have
\[
\lim \frac{1}{T} \int_0^T q(t) dt \geq |\lim \frac{1}{T} \int_0^T q(t)e^{-it(\Im \mu_j^{-1})} dt|, \quad T \to \infty, \tag{5.33}
\]
i.e.
\[
g_1(0) \int_{\Delta_1} dh_1(x) \geq |g_j(0) \int_{\Delta_1} dh_j(x)|. \tag{5.34}
\]
In the same way we can prove that
\[
g_1(x_0) \int_{\Delta_1} dh_1(x) \geq |g_j(x_0) \int_{\Delta_1} dh_j(x)|, \tag{5.35}
\]
where
\[
x_0 \in \Delta_1 \in \Delta. \tag{5.36}
\]
It follows from \(5.35\) that
\[
g_1(x_0)h_1(x) \geq |g_j(x_0)h_j(x)|. \tag{5.37}
\]
We introduce the normalization conditions
\[
g_1(x_0) = g_j(x_0), \quad h_1(x_0) = h_j(x_0) = 0. \tag{5.38}
\]
Due to \(5.35\) and \(5.37\) the inequalities
\[
\int_{\Delta_1} dh_1(x) \geq |\int_{\Delta_1} dh_j(x)|, \tag{5.39}
\]
\[
h_1(x) \geq |h_j(x)| \tag{5.40}
\]
are true. The equality sign in \(5.37\) and \(5.38\) can be only if
\[
h_j(x) = |h_j(x)|e^{i\alpha}. \tag{5.41}
\]
It is possible only in the case when \( j = 1 \). Hence there exists such a point \( x_1 \) that
\[
h_1(x_1) > |h_j(x_1)|
\]
(5.42)

Thus we have
\[
1 = \int_{\Delta_1} g_1(x)dh_1(x) > \int_{\Delta_1} g_j(x)dh_j(x) = 1,
\]
(5.43)

where \( x_1 \in \Delta_1 \). The received contradiction (5.43) means that \( j = 1 \). Now the assertion of the theorem follows directly from (5.2).

**Corollary 5.1** Let conditions of Theorem 5.2 be fulfilled. If \( \text{rank}\lambda_1 = 1 \) and \( x_0 \in \Delta_1 \in \Delta \) then the asymptotic equality
\[
p(x_0, \Delta_1, t, \Delta) = e^{-t/\lambda_1}g_1(x_0) \int_{\Delta_1} h_1(x)dx[1 + o(1)], \quad t \to +\infty
\]
(5.44)
is true.

According to Theorem 5.2 and the relation \( 0 < \text{Re}(1/\lambda_k) \leq 1/\lambda_1 \) the following assertion is true.

**Corollary 5.2** Let the conditions of Theorem 5.2 be fulfilled. Then all the eigenvalues \( \lambda_j \) of \( B \) belong to the disk
\[
|z - (1/2)\lambda_1| \leq (1/2)\lambda_1.
\]
(5.45)

All the eigenvalues \( \lambda_j \) of \( B \) which belong to the boundary of disc (5.45) have the indexes \( n_j = 1 \).

3. We consider again the Levy process \( X_t \) of the type II. Now we do not suppose that the quasi-potential operator \( B \) is compact. As we proved before the operator \( B \) is bounded in the Banach space \( D_\Delta \).

**Theorem 5.4** Let the considered Levy process \( X_t \) have the type II and let the spectrum of the corresponding operator \( B \) belong to the intersection of the discs \( |\lambda| \leq \rho \) and \( |z - (1/2)\rho| \leq (1/2)\rho \). If the condition (5.5) is fulfilled, then we have
\[
p(t, \Delta) = O(e^{-t/(\rho + \varepsilon)}), \quad \varepsilon > 0, \quad t \to \infty,
\]
(5.46)
4. Now we assume that the function $\Phi(x, y)$ is absolutely continuous with respect to $y$. Hence, the corresponding operator $B$ has the form

$$Bf = \int_\Delta f(y)K(x, y)dy, \quad K(x, y) = \frac{\partial}{\partial y}\Phi(x, y),$$

(5.47)

where for all $x \in \Delta$ the inequality

$$\int_\Delta |K(x, y)|dy < \infty$$

(5.48)

holds. In addition we assume that for all $x \in \Delta$ the following relation

$$\lim_{h \to 0} \int_\Delta |K(x + h, y) - K(x, y)|dy = 0$$

(5.49)

is fulfilled. Then the operator $B$ is compact in the space $D_\Delta$. Here we use the well-known Radon’s theorem [14]:

**Theorem 5.5** The operator $B$ of the form (5.47) is compact in the space $D_\Delta$ if and only if the relations (5.48) and (5.49) are fulfilled.

5. We can formulate the condition (5.5) of Theorem 5.2 in the terms of the kernel $\Phi(x, y)$. We suppose that there exists such positive, monotonically decreasing function $r(s)$ that

$$|K(x, y)| \leq r(|x - y|), \quad \int_0^b r^2(s)ds < \infty, \quad b > 0.$$  

(5.50)

It follows from (5.50) that the operator $B$ is bounded in the Hilbert space $L^2(\Delta)$. The numerical range $W(B)$ of the operator $B$ in the Hilbert space $L^2(\Delta)$ is the set $W(B) = \{(Bf, f), ||f|| = 1\}$. The closure of the convex hull of $W(B)$ is situated in the sector (5.1). Hence, the estimation

$$||(B - zI)^{-1}1|| \leq M/|z|, \quad z \in \Gamma_\epsilon$$

(5.51)

is valid (see [21]). It follows from (5.50) and (5.51) that

$$||(B - zI)^{-1}1, K(0, x)|| \leq M/|z|, \quad z \in \Gamma_\epsilon$$

(5.52)

So, we have proved the following statement.

**Theorem 5.6** Let relations (5.49) and (5.50) be fulfilled. Then the inequality (5.5) holds.
6 Appendix

1. Support.
Here we remind the following definitions (see [18]).

**Definition 6.1** For any measure $\rho$ on $\mathbb{R}$ its support $S(\rho)$ is defined to be the set of $x \in \mathbb{R}$ such that $\rho(G) > 0$ for any interval $G$ containing $x$.

The support $S(\rho)$ is closed set.

**Definition 6.2** For any random variable $X$ on $\mathbb{R}$ the support of the corresponding distribution $F(x)$ is called the support of $X$ and denoted by $S(X)$.

Now we can formulate the well-known H. Tucker’s theorem ([24]).

**Theorem 6.3** Let $X_t$ be a Levy process on $\mathbb{R}$ with triplet $(A, \nu, \gamma)$.

1) If either $A > 0$ or
   \[ \int_{-\infty}^{\infty} |x| d\nu = \infty, \]  
   then $S(X_t) = \mathbb{R}$.

2) If $0 \in S(\nu)$, $S(\nu) \cap (0, \infty) \neq \emptyset$, $S(\nu) \cap (-\infty, 0) \neq \emptyset$, then $S(X_t) = \mathbb{R}$.

3) Suppose that $A = 0$, $0 \in S(\nu)$ and
   \[ \int_{-\infty}^{\infty} |x| d\nu < \infty. \]  
   If $S(\nu) \in [0, \infty)$, then $S(X_t) = [t\gamma, \infty)$.
   If $S(\nu) \in (-\infty, 0]$, then $S(X_t) = (-\infty, t\gamma]$.

2. Continuity.
Let us consider the Levy process $X_t$ of the type II. In this case the corresponding distribution $F(x, t)$ is continuous with respect to $x$ (see Theorem 1.3). We use the representation (see (1.1)):

\[ \int_{-\infty}^{\infty} e^{ixz} dF(x, t) = \mu(z, t). \]  

Hence, the equality

\[ \int_{-\infty}^{\infty} e^{ixz} dF(x, t) = \mu(z, t). \]  

holds ([25], Ch.XVI.) Now we shall prove the following fact.
Theorem 6.4 Let $X_t$ be a Levy process of type II. Then the corresponding distribution function $F(x,t)$ is jointly continuous with respect to $t > 0$ and $x \in \mathbb{R}$.

Proof. Using (6.4) and the dominated convergence theorem we deduce that the left side of (6.4) is continuous with respect to $t > 0$ and $x \in \mathbb{R}$. Hence, the function

$$G(x,t) = F(x,t) - F(-x,t)$$

(6.5)

is continuous with respect to $t > 0$. Suppose, that $X_t$ satisfies condition 3) of Theorem 6.3 and $\gamma = 0$. In this case we have either $F(-x,t) = 0$, $x > 0$ or $F(x,t) = 0$, $x > 0$. As the function $G(x,t)$ is continuous with respect to $t > 0$ the function $F(x,t)$ is continuous with respect to $t > 0$ too. Now let us consider an arbitrary Levy process $X_t$ of type II. We can represent $X_t$ in the following form

$$X_t = X_t^{(1)} + X_t^{(2)},$$

(6.6)

where Levy processes $X_t^{(1)}$ and $X_t^{(2)}$ are independent and the process $X_t^{(2)}$ satisfies condition 3) of Theorem 6.3. The processes $X_t^{(1)}$ and $X_t^{(2)}$ have the following Levy triplets $(A, \gamma, \nu_1)$ and $(0, 0, \nu_2)$, where

$$\nu(x) = \nu_1(x) + \nu_2(x), \int_{-\infty}^{\infty} d\nu_2(x) = \infty.$$ 

(6.7)

The distribution functions of $X_t$, $X_t^{(1)}$ and $X_t^{(2)}$ are denoted by $F(x,t)$, $F_1(x,t)$ and $F_2(x,t)$ respectively. The convolution formula for Stieltjes-Fourier transform ([3], Ch.4) gives

$$F(x,t) = \int_{-\infty}^{\infty} F_2(x - \xi, t - \tau) d\xi F_1(\xi, \tau)$$

(6.8)

Since the process $X_t^{(2)}$ is of type II and satisfies the condition 3) of Theorem 6.3, the distribution $F_2(x,t)$ is continuous with respect to $x$ and with respect to $t$. The assertion of the theorem follows from this fact and relation (6.8) \(\square\)

Example 6.5 We consider the case when

$$\int_{0}^{\infty} d\nu(x) = \infty.$$ 

(6.9)
We shall explain the method of constructing Levy measures $\nu_1(x)$ and $\nu_2(x)$. We take
\[
\nu_1(x) = \nu(x) - \nu_2(x) \tag{6.10}
\]
\[
\nu_2(x) = 0 \ (x < 0); \ d\nu_2(x) = x\nu(x) \ (0 \leq x \leq 1); \ d\nu_2(x) = d\nu(x) \ (x > 1). \tag{6.11}
\]
It is easy to see that the corresponding Levy processes $X_t^{(1)}$ and $X_t^{(2)}$ satisfy (6.6), are independent and the process $X_t^{(2)}$ satisfies condition 3).

3. Non-zero points of the spectrum.

**Theorem 6.6** Let Levy process $X_t$ have type II and let the corresponding quasi-potential $B$ is compact in the Banach space $D_\Delta$. Then the operator $B$ has a point of the spectrum different from zero.

**Proof.** Let a continuous, non-negative function $u(x)$ in the domain $\Delta$ be such that $||u(x)|| = 1$ and
\[
u(x) = 0, \ x \in [a_k, a_k + \epsilon_k][b_k - \epsilon_k, b_k], \ 0 < \epsilon_k < (b_k - a_k)/2, \ 1 \leq k \leq n. \tag{6.12}
\]
The function $v(x) = Bu(x)$ will be continuous in the domain $\Delta$ and $v(x) > 0$, where $x$ are inner points of $\Delta$. Hence there exists such $c > 0$ that
\[
v(x) \leq cu(x). \tag{6.13}
\]
The assertion of the theorem follows directly from (6.13) and Krein-Rutman result (see [12], theorem 6.2). □

4. Weakly singular operators

The operator
\[
Bf = \int_\Delta K(x, y)f(y)dy \tag{6.14}
\]
is weakly singular if the kernel $K(x, y)$ satisfies the inequality
\[
|K(x, y)| \leq M|x - y|^\beta, \ 0 < \beta < 1, \ M = \text{constant.} \tag{6.15}
\]

**Proposition 6.7** The weakly singular operator $B$ is bounded and compact in the spaces $L^p(\Delta), 1 \leq p \leq \infty$. 

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The operator $B$ is bounded in the spaces $L^p(\Delta), 1 \leq p \leq \infty$ (see [15], p.24). A sufficiently high iterate of the kernel $K(x, y)$ is bounded. Hence the operator $B$ is compact in the spaces $L^p(\Delta), 1 \leq p \leq \infty$. □

In particular, the weakly singular operator $B$ is bounded and compact in the Hilbert space $L^2(\Delta)$. Repeating the arguments of the Theorem 5.2 in the case $L^2(\Delta)$, we obtain the assertion.

**Theorem 6.8** Let Levy process $X_t$ have type II and let the corresponding quasi-potential $B$ is weakly singular. Then we have

1) $K(x, y) = \frac{\partial}{\partial y} \Phi(x, y)$. \hspace{1cm} (6.16)

2) The eigenfunction $g_1(x)$ of the operator $B$ is continuous and $g_1(x) > 0$, where $x$ are the inner points of $\Delta$.

3) The eigenfunction $H_1(x) = h'_1(x) \in L^2(\Delta)$ is almost everywhere positive.

4) The equalities (5.6) and (5.7) are valid.

**Proof.** Now we consider the operator $B$ in the Hilbert space $L^2(\Delta)$. We note that the operator $B$ has the same eigenvalues $\lambda_j$ and the same eigenvectors $g_j$ in the spaces $L^2(\Delta)$ and $D_\Delta$. The operator $B^*$ has the same eigenvalues $\lambda_j$ in the spaces $L^2(\Delta)$ and $D_\Delta^*$. The corresponding eigenvectors $H_j(x)$ and $h_j(x)$ are connected by the relation

$$H_j(x) = h'_j(x). \hspace{1cm} (6.17)$$

We denote by $W(B)$ the numerical range of $B$. The closure of the convex hull of $W(B)$ is situated in the sector $[5,2]$. Hence the estimation

$$||(B - zI)^{-1}|| \leq \frac{M}{|z|}, \quad z \in \Gamma_\varepsilon \hspace{1cm} (6.18)$$

is valid [21]. The contour $\Gamma_\varepsilon$ is defined in the section 5. If $\beta < 1/2$ then $K(0, x) \in L^2(\Delta)$. According to (6.18) we have

$$|((B - zI)^{-1}1, K(0, x))| \leq \frac{M}{|z|}, \quad z \in \Gamma_\varepsilon \hspace{1cm} (6.19)$$

All conditions of Theorem 5.2 are fulfilled. So, in the case $\beta < 1/2$ the theorem is proved.

Let us consider the case when $1/2 \leq \beta 1/2 < 1$. In this case we have

$$(B^*)^m K(0, x) \in L(\Delta), \quad m = 0, 1, 2, \ldots \hspace{1cm} (6.20)$$

□
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