Analysis aerodynamics diffuser-augmented wind turbines

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Abstract. A complete and consistent, one-dimensional momentum theory is derived for the aerodynamics of slotted diffuser-augmented wind turbine (sDAWT). This theory does not require empirical relations. It results in a set of three-parameter relations for aerodynamic characteristics such as power coefficient $C_p$, rotor resistance $k$, rotor axial force coefficient $C_f$, axial induction factor $a$, wake-expansion factor $\beta$, duct axial force coefficient $C_{T, duct}$ and slot mass flux $\dot{m}_{duct}$. The theory predicts that the maximum achievable power coefficient $C_p$ of (s)DAWT’s increases monotonically with increasing $\beta$, surpassing the Betz limit of open-rotor wind turbines (ORWT’s), already for modest ($>2$) $\beta$’s. The slot of an sDAWT feeds outside air into the diffuser, which for given $\beta$ decreases the flow through the rotor and therewith $C_p$. However, the flow through the slot delays the onset of flow separation in the diffuser, increasing the maximum achievable $\beta$ and therewith the power coefficient of sDAWT’s beyond that of DAWT’s. Based on a vortex model of the (s)DAWT, an expression is derived for the velocity induced at the rotor plane by the diffuser and for the corresponding circulation of the diffuser. The derived three-parameter relations for sDAWT’s reduce to two-parameter relations for DAWT’s and the familiar one-parameter relations for ORWT’s.

1. Introduction
1.1. Background
Globally wind energy is a fast-growing energy resource. Technology development has helped the reduction in cost of wind energy. The latter is amongst others directly related to the “economics of scale”: the power generated by a wind turbine is proportional to the rotor-swept area. The introduction of offshore wind farms has therefore led to the development of turbines of more than 5 Megawatt, with rotor diameters exceeding 100 meters. Economical, technical and environmental aspects have caused wind energy to receive political support in many countries around the globe. Nevertheless, wind energy brings along disadvantages as well. The fluctuating nature of wind may lead to challenges in including large wind farms in the electricity grid. Furthermore, wind turbines may have an impact on the local environment in terms of noise, visual hindrance and casualties in wildlife. These disadvantages can be mitigated by a reduction of the size of turbines. Economically large wind turbines have to compete with fossil-energy and nuclear-energy power plants, which produce energy at about 0.05 €/kWh. However, for small wind turbines for domestic use, the costs of producing energy from wind competes with the price consumers pay at home (being approximately 0.20 €/kWh).

To enhance the power output of a small wind turbine it can be equipped with a diffuser, in the form of a duct or a shroud. The basic idea is to create a Venturi type of effect that increases the mass flow through the turbine, while at the same time tip losses are reduced. For large turbines diffuser will never be economically attractive because of high material costs. However, for small wind turbines the augmentation in power may outweigh these costs. It is against this background that the performance of such a system has been analysed in the present study. In addition, a shrouded small wind turbine. i.e. a Diffuser-Augmented Wind Turbine (DAWT) might also have improved public acceptability, because the
disadvantages such as noise and visual hindrance, may be limited. This would make small wind turbines not only acceptable in open/rural areas, but also in an urban environment. So, it might be expected that, provided their efficiency and overall performance is high, there is a market for shrouded “micro” turbines. This requirement asks for an optimal aerodynamic design as a starting point.

The concept of DAWT’s was already discussed by Betz [1] in the nineteen twenties. The concentration of wind energy requires smaller turbine diameters and lower cut-in wind speeds. Nevertheless, diffuser augmentation was considered economically non-profitable until the late nineteen seventies. At that time, driven by the oil crisis, several experiments were undertaken at the Grumman Aerospace Corporation in New York (Foreman et al. [2]; Gilbert and Foreman [3]; Foreman [4]), as well as at the Ben Gurion University in Israel (Igra [5]). The conclusions drawn by the scientific research field were that an interesting increment in power can be achieved, but that the cost of the structure of the configuration causes the DAWT to be economically unattractive. Because of this conclusion, for 20 years diffuser augmentation disappeared from the research agenda. It was in the late nineties that a company in New Zealand called Vortec Energy built the full scale DAWT (Phillips [6]), shown in figure 1. The research done by the University of Auckland, New Zealand in the years thereafter brought the subject of diffuser augmentation back into the spot lights. Recently several papers have been published on the subject of DAWT’s, both theoretical discussions (van Bussel [7]; Jamieson [8]; Lawn [9]; Werlé and Presz [10]), and experimental and computational results (Betz and Grassmann [11]; Frankovic and Vrsalovic [12]; Hansen et al. [13]; Hansen [14]; Hoffenberg and Sullivan [15]; Mansour and Meskinkhoda [16]; Ohya et al. [17]; Grassmann et al. [18]; Abe et al. [19]; Phillips et al. [20]; Venters et al. [21]).

Figure 1. Vortec-7, the 7.3m diameter diffuser augmented wind turbine by Vortec Energy Ltd, New Zealand (Phillips [6], used with permission).

1.2. Present investigation

The present paper gives a theoretical derivation of the aerodynamic performance of shrouded wind turbines (DAWT’s), based on a 1D flow model, i.e. the actuator-disc model, similar to the one used for open-rotor wind turbines (ORWT’s). The actuator disc is an infinitesimally thin porous disc often used in first-order analysis of fluid machines. The actuator disc can be thought of as a model for a rotor with an infinite number of infinitesimally thin, slightly-cambered, blades. The forces acting on the individual blades, which give rise to thrust and torque, and hence power, are not considered in detail, only actuator-disc averaged values. These values are used as design requirement for, for example, a Blade-Element-Momentum method. That method then generates the span-wise distributions of forces on the individual blades, needed for a detailed analysis of the performance and the structural design of the wind turbine. The present derivation of the theory is complete and consistent and does not rely on empirical relations between the parameters of the model. The aerodynamic performance of DAWT’s and DAWT’s with a slotted duct (sDAWT’s) is compared with that of ORWT’s.

1.3. One-dimensional flow theory

In one-dimensional flow theory the important parameters required for the design of wind turbines appear as result of the analysis. For example, one of the results is the theoretical maximum of the power drawn from the flow by ORWT’s, as well as by (s)DAWT’s. In the theory the turbine is represented by an ideal actuator disc with an actuator-disc-averaged drop in static pressure only. Azimuthal (swirl) and radial components of the velocity are not accounted for. In the analysis effects of viscosity and effects of compressibility are neglected. Therefore, the one-dimensional flow is considered incompressible and inviscid, as well as steady.
2. Analysis flow through Diffuser-Augmented Wind turbines

2.1. Control-volume formulation

The theory of 1D flow through DAWT’s and sDAWT’s is an extension of the well-known theory for ORWT’s. A famous result of this theory is that the maximum power coefficient that can be achieved by ORWT’s extracting energy from the flow is \( C_p = 16/27 \), as derived by Betz [1] already in 1919. This is known as Betz’ limit. Since the same result has been obtained by Lanchester [22] in 1915 and by Joukowski [23] in 1920, this limit is also referred to as the Betz-Lanchester-Joukowski limit, e.g. see van Kuik [24, 25].

Figure 2 shows the model of the flow through a slotted DAWT, i.e. an sDAWT, with the rotor represented as actuator disc. Note that in aircraft engineering the benefits of slotted airfoil sections have been recognized and exploited already a long time, e.g. [28]. From the results of the present analysis for sDAWT’s, the results for DAWT’s and ORWT’s are obtained by taking appropriate limits.

In the model two control volumes are utilised: (i) The stream tube passing through the edge of the rotor, i.e. the interior side of the duct, given as the dashed contour in figure 2 and; (ii) The far-field circular-cylinder, of constant cross-sectional area, that envelopes the whole wind turbine and its wake, given as the dash-dot contour in figure 2. The first control volume consists of: the stream surface passing through the edge of the actuator disc and passing along the interior side of the duct, bridging the slot; an entrance and an exit cross-flow plane. This control volume is intersected by five cross-flow planes:

(1) Plane 1: The plane far upstream, cross-sectional area of stream tube \( A_{\infty} \), with the pressure and the velocity equal to their free stream values \( p_{\infty} \) and \( V_{\infty} \), respectively.

(2) Plane 2: The plane just upstream of the actuator disc, cross-sectional area of stream tube \( A_{\text{disc}} \), with the (area-averaged) pressure and velocity equal to \( p_2 \) and \( V_2 = V_{\text{disc}} \), respectively.

(3) Plane 3: The plane just downstream of the actuator disc, cross-sectional area of stream tube \( A_{\text{disc}} \), with the (area-averaged) pressure and velocity equal to \( p_3 \) and \( V_3 = V_{\text{disc}} \), respectively.

(4) Plane 4: Inside the stream tube, between planes 3 and 5, the flow is decelerated by a ring-shaped diffuser located between planes 3 and 4. The cross-sectional area \( A_4 \) of the diffuser at its trailing edge, i.e. the exit of the diffuser, is situated in plane 4, with the (area-averaged) pressure and velocity equal to \( p_4 \) and \( V_4 \), respectively.

(5) Plane 5: The plane far downstream, cross-sectional area of stream tube \( A_5 \), where the pressure has recovered to its free-stream value \( p_5 = p_{\infty} \) and the (area-averaged) velocity equals \( V_5 \). Note that \( A_5 \) is different from the cross-sectional area \( A_4 \) of the diffuser at its trailing edge. The far-field cross-sectional area \( A_5 \) may be much larger than the exit area \( A_4 \) of the diffuser, which is due to the flow divergence induced by the flow around the diffuser that acts as a ring-shaped wing. Note that flow separation occurring on the (interior) wall of the diffuser will limit the effectiveness of the diffuser. The inclusion of a slot, or slots, in the duct has as purpose to delay the onset of this flow separation.

The second control volume consists of: a far-field circular-cylindrical surface, of constant cross-sectional area \( A_0 \), which envelopes the actuator disc and the (slotted) diffuser; an entrance and an exit cross-sectional plane. This enveloping control volume is intersected by two cross-flow planes:

(6) Plane 0: The entrance plane, at the same location as plane 1, of cross-sectional area \( A_0 \).

(7) Plane 6: The exit plane of cross-sectional area \( A_0 \) far downstream, at the same location as plane 5.
In table 1 all parameters of the model are listed. In the analysis it is assumed that the free-stream pressure \( p_\infty \), the constant density \( \rho_\infty \), the free-stream velocity \( V_\infty \), the rotor disc area \( A_{disc} \) and the area \( A_0 \) of the exit plane of the diffuser are given. The eight bold-faced parameters \((A_\infty, \ V_2 = V_3 = V_{disc}, p_2, p_3, V_4, p_4, V_5, A_5)\) are unknown. The following four quantities are also unknown: the axial force \( F \) on the actuator disc; the axial force \( F_{duct} \) on the duct; the mass flux \( \dot{m}_{duct} \) through the slot into the interior of the duct; and finally the mass flux \( \dot{m}_{side} \) out of the enveloping control volume. Therefore, the present formulation features twelve yet unknown parameters. Below, first seven equations are derived by applying the laws of conservation of mass and momentum to the two control volumes sketched in figure 2 for the sDAWT.

### Table 1. Parameters in actuator disc model for (s)DAWT's.

| Plane | 0  | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|----|
| Velocity | \( V_\infty \) | \( V_\infty \) | \( V_{disc} \) | \( V_{disc} \) | \( V_4 \) | \( V_5 \) | \( V_\infty \) |
| Pressure | \( p_\infty \) | \( p_\infty \) | \( p_2 \) | \( p_3 \) | \( p_4 \) | \( p_\infty \) | \( p_\infty \) |
| Area | \( A_0 >> A_\infty \) | \( A_\infty \) | \( A_{disc} \) | \( A_{disc} \) | \( A_4 \) | \( A_5 \) | \( A_0 >> A_5 \) |

#### 2.2. Governing equations

Consider the first control volume: the stream tube through the edge of the rotor plane, see figure 2.

Conservation of mass between plane 1 and plane 2 gives:

\[
\rho_\infty V_\infty A_\infty = \rho_\infty V_{disc} A_{disc} \tag{1a}
\]

Conservation of mass between plane 3 and plane 5 yields:

\[
\rho_\infty V_{disc} A_{disc} + \dot{m}_{duct} = \rho_\infty V_5 A_5 \tag{1b}
\]

Bernoulli’s equation for steady, incompressible, inviscid flow, neglecting effects of gravity, along a streamline inside the stream tube, passing from plane 1 to plane 2, yields:

\[
p_\infty + \frac{1}{2} \rho_\infty V_\infty^2 = p_2 + \frac{1}{2} \rho_\infty V_{disc}^2 \tag{1c}
\]

Bernoulli’s equation along a streamline, inside the stream tube, passing from plane 3 to plane 5, gives:

\[
p_3 + \frac{1}{2} \rho_\infty V_{disc}^2 = p_\infty + \frac{1}{2} \rho_\infty V_5^2 \tag{1d}
\]

Then consider the second control volume, of constant cross-sectional area \( A_0 >> A_5 \), enveloping the whole wind turbine and its wake, extending from plane 0 to plane 6. Note that the portion of plane 6 outside the stream tube, passing through the edge of the actuator disc, has pressure \( p_\infty \) and axial velocity \( V_\infty \). For the second control volume, conservation of mass gives:

\[
\rho_\infty V_\infty A_0 = \dot{m}_{side} + \rho_\infty V_5 A_5 + \rho_\infty V_\infty (A_0 - A_5) \tag{1e}
\]

For the same control volume conservation of momentum in axial direction provides, with \(-F\) the axial force exerted by the rotor on the fluid inside the control volume and \(-F_{duct}\) the axial force exerted by the duct on the fluid inside the control volume:

\[
-(p_\infty + \rho_\infty V_\infty^2)A_0 + \dot{m}_{side} V_\infty + (p_\infty + \rho_\infty V_5^2)A_5 + (p_\infty + \rho_\infty V_\infty^2)(A_0 - A_5) = -F - F_{duct} \tag{1f}
\]

The final equation is the expression for the axial force exerted by the flow on the actuator disc:

\[
F = (p_2 - p_3)A_{disc} \tag{1g}
\]

In equations (1a-g) \( V_4 \) and \( p_4 \) do not play a role, they are decoupled from the rest of variables and are solved for separately below. The solution of the seven algebraic equations (1a-g), for ten unknowns, constitutes a solution with three free parameters, to be chosen conveniently. In solutions presented in literature for ORWT’s, for which the solution is a set of expressions in terms of just one parameter, often the solution of the actuator-disc theory is expressed in terms of the axial-(dimensionless) induction factor \( a \). This factor follows from the definition:

\[
V_{disc} \equiv (1 - a) V_\infty \tag{2a}
\]

For ORWT’s \( a \in [0,1/2] \), but for (s)DAWT’s \( a \in (-\infty,1/2) \), because the duct may induce a velocity that exceeds that induced by the actuator-disc model. However, \( C_{T,disc} \) may become negative. Choices other than the axial-induction factor \( a \) are possible, such as the rotor resistance \( k \), defined as:

\[
k \equiv \frac{1}{\frac{F}{\rho_\infty V_{disc}^2 A_{disc}}} \quad \text{similarly,} \quad k_{duct} \equiv \frac{F_{duct}}{\frac{2 \rho_\infty V_{disc}^2 A_{disc}}{}} \tag{2b}
\]

denotes the dimensionless resistance factor of the duct. For ORWT’s \( k \in [0,4] \), but for (s)DAWT’s \( k \) is unbounded, as long as \( k \) is positive and such that \( C_{T,disc} > 0 \).
In literature some authors prefer to express the aerodynamic performance of wind turbines in terms of the (dimensionless) thrust coefficient $C_T$ instead of the rotor resistance factor $k$ or the axial induction factor $a$. The thrust coefficient $C_T$ is defined as:

$$C_T \equiv \frac{F}{\rho_{\infty}V_{\infty}^2A_{\text{disc}}} \quad \text{and similarly,} \quad C_{T,\text{duct}} \equiv \frac{F_{\text{duct}}}{\rho_{\infty}V_{\infty}^2A_{\text{disc}}} = k_{\text{duct}} \frac{V_{\text{duct}}^2}{V_{\infty}^2}$$

(2c)

denotes the dimensionless axial force on the duct. For ORWT’s, as well as for (s)DAWT’s, $C_T \in [0,1)$, an advantage. In the present study we have experimented with using the (dimensionless) wake-expansion factor $\beta$ as free parameter. For ORWT’s and for (s)DAWT’s this parameter is defined through

$$A_5 \equiv \beta A_{\text{disc}}.$$

(2d)

with $\beta \in (1,\infty)$. The ability to increase the wake-expansion factor during the design of a wind turbine characterizes the extent to which the turbine is able to decelerate an increasing amount of incoming air by expanding the far wake. Or in other words, to increase the mass flow through the actuator disc, therewith increasing the power coefficient. For ORWT’s and DAWT’s the parameter $\beta$ is related to the induction factor $b$ at infinity downstream, defined as $V_5 \equiv (1-b)V_\infty$. Therefore, $\beta = (1-a)/(1-b)$.

For ORWT’s the solution is a function of just one free parameter, which may be $k$, $C_T$, $a$ or $\beta$. For DAWT’s the solution is a function of two free parameters, which may be any combination of two of the parameters $k$, $C_T$, $a$, $\beta$, $k_{\text{duct}}$, or $C_{T,\text{duct}}$. For sDAWT’s a third parameter comes into play: the dimensionless mass flow through the slot, defined as:

$$M_{\text{duct}} \equiv \frac{m_{\text{duct}}}{\rho_{\infty}V_{\text{duct}}A_{\text{disc}}} \geq 0,$$

(2e)

which is the ratio of the mass flux through the slot into the stream tube and the mass flux through the actuator disc. In the solution of the governing equations $M_{\text{duct}}$ combines with the wake-expansion factor $\beta$ as:

$$\beta = \frac{\beta}{1+M_{\text{duct}}},$$

(2f)

2.3. Solution for sDAWT’s

In table 2 the solution for sDAWT’s is presented in terms of four chosen sets of three parameters, namely $(k$, $\beta$, $M_{\text{duct}})$, $(C_T$, $\beta$, $M_{\text{duct}})$, $(a$, $\beta$, $M_{\text{duct}})$ and $(C_T$, $C_{T,\text{duct}}$, $M_{\text{duct}})$. This facilitates comparison with expressions found in other studies. Table 2 includes expressions for:

$$\frac{A_5}{A_{\text{disc}}} = \frac{V_{\text{disc}}}{V_{\infty}} = \frac{p_2-p_3}{\rho_{\infty}V_{\infty}^2}, \quad \frac{p_3-p_{\infty}}{\rho_{\infty}V_{\infty}^2}, \quad \frac{A_5}{V_{\infty}}, \quad \frac{k_{\text{duct}}}{k}; \quad C_T; \quad k + k_{\text{duct}}; \quad C_T + C_{T,\text{duct}}; \quad \text{and} \quad \frac{m_{\text{side}}}{\rho_{\infty}V_{\text{duct}}A_{\text{disc}}}.$$

Furthermore, once the solution has been determined, the power coefficient $C_p$ is computed as

$$C_p \equiv \frac{\rho_{\infty}V_4A_4}{\frac{FV_{\text{disc}}}{\rho_{\infty}V_{\text{duct}}A_{\text{disc}}}} = k \frac{V_{\text{disc}}^2}{V_{\infty}^2}.$$

(3)

Furthermore, $V_4$ and $p_4$, the (area-averaged) velocity and pressure at the trailing edge of the diffuser, respectively, follow directly from $V_5 = V_{\text{disc}}$ and $p_3$, the (area-averaged) velocity and pressure at the downstream side of the actuator disc, respectively. Alternatively, $V_4$ and $p_4$ follow from $V_5$ and $p_5 = p_{\infty}$, the (area-averaged) velocity and pressure at plane 5 far downstream, respectively.

Conservation of mass and Bernoulli’s equation, both applied between plane 4 and plane 5, give:

$$\rho_{\infty}V_4A_4 = \rho_{\infty}V_5A_5 \quad \text{and} \quad p_4 + \frac{\rho_{\infty}V_4^2}{2} = p_5 + \frac{\rho_{\infty}V_5^2}{2},$$

(4a)

respectively. Defining $\alpha \equiv A_4/A_{\text{disc}} \geq 1$, it follows, e.g. using the solution given in table 2 in terms of $(k$, $\beta$, $M_{\text{duct}})$:

$$\frac{V_4}{V_{\infty}} = \frac{\beta}{\alpha \sqrt{1 + k\beta^2}} \quad \text{and} \quad \frac{p_4-p_{\infty}}{\rho_{\infty}V_{\infty}^2} = \frac{1}{\alpha \sqrt{1 + k\beta^2}} (1 - \frac{\beta^2}{\alpha^2}) < 0$$

(4b)

For other combinations of the three free parameters, $V_4/V_{\infty}$ and the corresponding pressure coefficient at plane 4 are included in table 2. From equation (4) and table 2, it follows: $0 > \frac{p_4-p_{\infty}}{\rho_{\infty}V_{\infty}^2} > \frac{p_5-p_{\infty}}{\rho_{\infty}V_{\infty}^2}$. This shows that in the diffuser the pressure increases from a sub-atmospheric pressure in plane 3 to a higher, but still sub-atmospheric pressure in plane 4, as should be the case in a diffuser, while finally, from plane 4 to the end plane 5 in the wake of the turbine, the pressure increases further to the atmospheric value $p_{\infty}$. Though the velocity and pressure in plane 4 do not play a role in the analysis of DAWT’s, the cross-sectional area of the diffuser at plane 4 does play a role when designing a diffuser that features attached flow along its interior wall.
Table 2. Solution actuator-disc model for sDAWT's.

| $k\,\beta,\,M_{duct}$ | $(C_T,\,\beta,\,M_{duct})$ | $(a,\,\beta,\,M_{duct})$ | $(C_T,\,C_T,\,M_{duct},\,M_{duct})$ |
|-------------------------|-----------------------------|-----------------------------|----------------------------------|
| $\frac{v_1}{A_{disc}}$ | $\frac{v_0}{A_{disc}}$     | $\frac{v_1}{A_{disc}}$     | $\frac{v_0}{A_{disc}}$         |
| $\frac{p_1}{\rho_0 v_0^2}$ | $\frac{p_2}{\rho_0 v_0^2}$ | $\frac{p_1}{\rho_0 v_0^2}$ | $\frac{p_2}{\rho_0 v_0^2}$     |
| $\frac{F_{duct}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{duct}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{duct}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{duct}}{\rho_0 v_0^2 A_{disc}}$ |
| $\frac{m_{duct}}{\rho_0 v_0 A_{disc}}$ | $\frac{m_{duct}}{\rho_0 v_0 A_{disc}}$ | $\frac{m_{duct}}{\rho_0 v_0 A_{disc}}$ | $\frac{m_{duct}}{\rho_0 v_0 A_{disc}}$ |
| $\frac{F}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F}{\rho_0 v_0^2 A_{disc}}$ |
| $C_{T,\,disc}$ | $C_{T,\,disc}$ | $C_{T,\,disc}$ | $C_{T,\,disc}$ |
| $\frac{F_{T,\,disc}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{T,\,disc}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{T,\,disc}}{\rho_0 v_0^2 A_{disc}}$ | $\frac{F_{T,\,disc}}{\rho_0 v_0^2 A_{disc}}$ |

Empty straight-wall diffusers feature attached flow if the (total) diffuser angle $\theta$ does not exceed a value between 11 and 18 deg. e.g. (Blevins [27]). Employing $\alpha = A_d/A_{disc} = (1 + (UR_{disc})\tan \theta)^2$, permissible $\alpha$’s for diffusers of length equal to the diameter of the rotor are in the range between 1.4 and 1.7. Ring-wing shaped diffusers are expected to have higher permissible values of $\alpha$ than simple conical diffusers.

2.4. Velocity induced by duct at actuator disc

For (s)DAWT’s the area-averaged axial velocity $V_{disc}$ can be decomposed in two parts, one part due to the actuator disc and one part due to the duct. Figure 3 shows the representation of the actuator-disc model in terms of a vortex model embedded within a uniform axial free stream with velocity $V_\infty$.

In such a model the actuator-disc consists of a circular vortex sheet carrying radial vortex lines, running from the centerline (hub) to the edge of the disc. Connected to the edge (tip) of the actuator disc is a cylindrical vortex sheet, running from the edge of the actuator disc to infinity downstream, carrying vortex lines with an azimuthal (circular) component and an axial component. These spiraling vortex
lines form the continuation of the radial vortex lines on the actuator disc. This vortex model is completed, such that it satisfies Kelvin-Helmholtz’ vortex laws, with a discrete vortex running along the axis from infinity downstream to the center of the actuator disc. For (s)DAWT’s the ring wing that forms the duct is represented by a ring vortex, in the plane of the actuator disc, running along its edge. The axial velocity induced by the azimuthal component of the vortex distribution on the semi-infinite cylindrical vortex sheet, averaged over the cross-sectional area of the stream tube, equals \( V_0 \). 

Note that, for ORWT’s, this is equivalent to stating that the induction factor, \( \alpha = 1 - V_{\text{disc}}/V_0 \), at the disc equals half the induction factor, \( \beta = 1 - V_5/V_0 \), at infinity downstream. Such a relation is identical to the result of a vortex model of the actuator disc. It is also similar to the result of Prandtl’s lifting-line theory, in which the downwash induced at the lifting line of a high-aspect-ratio wing is half the induced downwash induced at infinity downstream, Prandtl [26].

In the present 1D flow approximation, the axial velocity induced by the ring vortex, averaged over the actuator disc, \( V_r \), is apparent at the actuator disc, but negligible at plane 1, as well as at plane 5. Therefore, in the stream tube we have the axial velocity \( V_0 \) far upstream, \( V_{\text{disc}} = \frac{1}{2}(V_0 + V_5) \) at the actuator disc and \( V_5 \) far downstream. Therefore, we compute \( V_r \) from

\[
V_r \equiv \frac{V_{\text{disc}}}{V_0} - \frac{1}{2}(1 + \frac{V_5}{V_0})
\]

From the results given in table 2 we now can construct the actuator-disc-averaged axial velocity \( V_r \) induced by the ring vortex, see table 2. The velocity \( V_r \) is a linear function of \( C_{T,\text{duct}} - M_{\text{duct}}C_T \), which shows that the velocity induced by the ring vortex at the actuator disc decreases due to the presence of the slot in the duct. Then suppose that the circulation of the ring vortex equals \( \Gamma \) and the radial velocity at the ring vortex equals \( V_{\Gamma r} \), positive outward. Employing the Kutta-Joukowski theorem, the axial force \( F_{\text{duct}} \) on the ring vortex will be proportional to \( \rho_o V_r \Gamma R \), with \( R \) the radius of the ring vortex. This yields the relation between \( C_{T,\text{duct}} \) and the circulation \( \Gamma \), required to design the duct, as:

\[
C_{T,\text{duct}} = \frac{F_{\text{duct}}}{2\rho_o V_5^2 A_{\text{disc}}} \equiv \frac{2}{\rho_o V_0^2 A_{\text{disc}}} \frac{F_{\Gamma R}}{V_0} - \frac{1}{2} \frac{V_{\Gamma R} \Gamma R}{V_0} A_{\text{disc}}
\]

The radial velocity \( V_{\Gamma r} \) at the ring vortex is induced by the azimuthal component of the vortex distribution on the cylindrical vortex sheet, with the strength of the vortex distribution depending on \( C_T \), is presented elsewhere.

3. Results

This section presents results obtained by the actuator-disc model for ORWT’s, DAWT’s and sDAWT’s. Table 2 presents the three-parameter expressions for sDAWT’s. Corresponding two-parameter results for DAWT’s are obtained by setting \( M_{\text{duct}} = 0 \), which implies that \( \beta = \beta \). Then the one-parameter-results for ORWT’s are found by setting \( M_{\text{duct}} = 0 \), (i.e. \( \beta = \beta \)) and \( C_{T,\text{duct}} = 0 \). The latter implies that in the DAWT expressions in terms of \( (k,\beta) \), \( (C_T,\beta) \) and \( (a,\beta) \) the wake-expansion factor \( \beta \) is to be specified as

\[
\beta = \frac{4}{4-k}, \quad \beta = \frac{1+\sqrt{3-4k}}{2\sqrt{1+k}}, \quad \text{and} \quad \beta = \frac{1-a}{1-2a},
\]

respectively.

3.1. Results for ORWT’s

![Graphs showing power coefficient \( C_p \) as a function of thrust coefficient \( C_T \), rotor resistance \( k \), axial induction coefficient \( a \) and wake expansion factor \( \beta \).](image)

Figure 4. ORWT: Power coefficient as function of thrust coefficient \( C_T \), rotor resistance \( k \), axial induction coefficient \( a \) and wake expansion factor \( \beta \), see Table 2. Open circle: condition of maximum \( C_p \).

Figure 4 presents the power coefficient \( C_p \) as function of four different parameters. It follows that \( C_{T,Cp,max} = 8/9, k_{Cp,max} = 2, a_{Cp,max} = 1/3 \) and \( \beta_{Cp,max} = 2 \), with \( C_{p,max} = 16/27 \approx 0.593 \). This is the well-known Lanchester-Betz- Joukowski limit for ORWT’s.
It shows that up to a maximum of 59.3 percent of the kinetic energy contained in the oncoming airstream passing through the cross-flow actuator disc, with area equal to that of the rotor, can be converted into power by ORWT’s.

Figure 5 shows that to achieve maximum performance in terms of power coefficient, the rotor should be designed such that it generates a wake vortex distribution with an azimuthal component of such that the wake vortex distribution induces a (disc-averaged) axial velocity of $-V_s/3$ in the plane of the rotor and $-2V_s/3$ in the plane far downstream. Furthermore, note that for the optimal condition the area-averaged pressure coefficient $(p - p_\infty)/\rho_\infty V_s^2$ equals 0, 5/9, -1/3 and 0 in the planes 1, 2, 3 and 5, respectively. The jump in the area-averaged pressure coefficient across the actuator disc equals $(p_2 - p_1)/\rho_\infty V_s^2 \equiv C_\tau = 8/9$. At the optimal condition the pressure at the downstream side of the rotor disc has the lowest (sub-atmospheric) pressure of -1/3 in terms of the dimensionless pressure coefficient. Finally note that at the optimal condition $A_s/A_\infty = (A_s/A_{disc})/(A_s/A_{disc}) = 3$: the exit area of the stream tube is 3× the entry area.

3.2. Results for DAWT’s

Figures 6 - 9 present results for DAWT’s ($M_{duct} = 0$). Figure 6 shows the upstream cross-section of the stream tube $A_s/A_{disc}$ and the wake-expansion factor $\beta = A_s/A_{disc}$, both as iso-contours in the $(C_\tau - C_{T,duct})$ plane. This result is used to estimate a feasible range for $C_{T,duct}$, the force on the duct. It is clear that the relevant values of $C_{T,duct}$ appear to be within a triangular-like region, with low values of $C_{T,duct}$ for low values of $C_\tau$ and higher values of $C_{T,duct}$ for higher values of $C_\tau$. The region close to $C_\tau = 1.0$, where the wake-expansion factor $\beta$ increases rapidly, does not give a feasible value of $C_{T,duct}$. Note that the $A_s/A_{disc}$ iso-contours for $A_s/A_{disc} > 1.0$ all correspond to positive values of $C_{T,duct}$. However, for the $A_s/A_{disc}$ iso-contours with $A_s/A_{disc} < 1.0$, some portion of the iso-contours is associated with negative values of $C_{T,duct}$. Figure 7 shows iso-contours of the power coefficient $C_p$ (solid lines) and iso-contours of the wake-expansion factor $\beta$ (dashed lines) within the $(C_\tau , C_{T,duct})$-plane. The values selected for the power coefficient are $C_p = 0.4, 16/27 = 0.6, 0.8(0.2)2.4$, those for the wake expansion factor are $\beta = 1(1)10$. We consider figure 7 as a chart for the preliminary design of DAWT’s. The results shown in figure 7 indicate that, for a specified value of the power coefficient $C_p$, there is a range of the rotor thrust coefficients $C_T$ in which the duct thrust coefficient $C_{T,duct}$ is nearly constant, while simultaneously the wake-expansion factor $\beta$ has a modest value. For example, for $C_p = 2$, within the range $0.3 < C_\tau < 0.8$, the value of the wake expansion factor is in between $\beta = 5.2$ and $\beta = 6.0$, while $C_{T,duct}$ has a value between 1.8 and 2.0. These values are within the domain identified as feasible in figure 6, values that should be achievable in the design of DAWT’s.
Lines \( C_T + C_{T,duct} \) constant represent, to first approximation, the loading that the tower of the wind turbine should be able to accommodate. Figure 7 indicates that, for small \( C_T \), these lines run almost parallel to the iso-\( C_p \) lines.

The wake expansion factor \( \beta \) produced by DAWT’s will be strongly affected by the divergence angle of the duct, which is limited to angles that do not exhibit separated flow in the duct. A feasible design choice therefore would be to select, for a given \( C_T + C_{T,duct} \), points in the \(( C_T, C_{T,duct})\)-plane with a modest value for the wake-expansion factor \( \beta \).

It may be argued that on any iso-\( C_p \) contour, the optimum is the point at which the wake expansion factor \( \beta \) is minimal, because this will reduce the required diffuser angle of the duct, or equivalently \( A_{d}/A_{disc} \), which will make it less hard to design a flow-separation-free duct. From the expressions in table 2, it has been derived that, for this optimal design, the rotor thrust coefficient equals \( C_T = 2/3 \), i.e. a constant independent of \( C_{T,duct} \). This is shown in figure 8, together with the optimal values of \( \beta = A_{d}/A_{disc} \), \( A_{d}/A_{disc,opt} \) and \( C_p \). The latter three are linear functions of \( C_{T,duct} \). For the optimal DAWT \( V_I/V_{\infty} = 1/\sqrt{3}, \) i.e. larger than the ORWT value of 1/3. Furthermore, it follows that \( V_{disc,opt} = V_{I,opt} + \frac{3}{2}(V_{\infty} + V_3) \), with \( V_{I,opt}/V_{\infty} = \frac{3}{2} \sqrt{3} C_{T,duct} \). Finally, \( (A_{d}/A_{disc})_{opt} = (A_{d}/A_{disc})_{opt}(A_{d}/A_{disc})_{opt} = \sqrt{3} \), smaller than the ORWT value of 3.

In general, the axial velocity \( V_I \) due to the ring vortex depends linearly on the axial force \( C_{T,duct} \) on the duct and it also depends on the axial force \( C_T \) on the actuator disc. The dimensionless velocity \( V_I/V_{\infty} \) is presented in figure 9 in the form of iso-contours of \( V_I/V_{\infty} \) in the \(( C_T, C_{T,duct})\)-plane.

Figure 7. DAWT: Iso-\( C_p \) contours (solid lines, \( C_p = 0.4(0.2)2.4 \)) and iso-\( \beta \) contours (dashed lines, \( \beta = 1(1)10 \)) in \(( C_T, C_{T,duct})\) plane. The dot at \(( C_T = 8/9, C_{T,duct} = 0) \) corresponds to the optimal ORWT \( C_p = 16/27, \beta = 2) \).

Figure 8. DAWT: Optimal values for \( \beta, A_{d}/A_{disc}, C_p \) and \( C_T \) as function of \( C_{T,duct} \), with \( A = (1 + \sqrt{3})/2\sqrt{3} \).

Figure 9. DAWT: Iso-contours of velocity induced by duct ring vortex, \( V_I/V_{\infty} = 0.2(0.2)3.0 \).
3.3. Results for sDAWT’s

Table 2 indicates that the scaled power coefficient $C_p/\beta$ is maximal for $k\beta^2 = 2$, with $C_{p,\text{max}} = 2\beta/3\sqrt{3}$. This implies that the maximum value of $C_p$ increases linearly with increasing $\beta$, while at the same time $k$ decreases like $\beta^{-2}$. This suggests that also sDAWT’s feature an unbounded maximum $C_p$, however, again, only as long as the design of the diffuser is such that the flow remains attached to its interior wall. Comparing the performance of DAWT’s with that of sDAWT’s shows that since $\beta = \beta/(1 + M_{\text{duct}}) < \beta$ for fixed $\beta$, sDAWT’s feature a reduced performance. For fixed $\beta$, i.e. fixed $C_T$ and fixed $C_{T,\text{duct}}$, it follows that, for positive $M_{\text{duct}}$, less mass flow passes through the actuator disc than in case the slot is closed, so that $C_p$ decreases. However, the outside momentum added to the flow inside the diffuser, will have as result that the maximum value of $\beta$ for which the diffuser still achieves attached flow will be higher, i.e. the maximum achievable power coefficient will be higher for sDAWT’s than for DAWT’s.

$$C_p(C_T, C_{T,\text{duct}}, M_{\text{duct}}) = C_p(C_T, C_{T,\text{duct}}), 0)/(1 + M_{\text{duct}})$$

$$\beta(C_T, C_{T,\text{duct}}, M_{\text{duct}}) = \beta(C_T, C_{T,\text{duct}}, 0)$$

Figure 10 presents iso- $C_p$ contours for $M_{\text{duct}} = 0$ (solid lines), $M_{\text{duct}} = 0.1$ (dashed lines) and $M_{\text{duct}} = 0.2$ (dotted lines); iso- $\beta$ contours (dash-dotted lines, all in $(C_T, C_{T,\text{duct}})$ plane.

4. Conclusions

A complete and consistent, one-dimensional momentum theory, similar to that for open-rotor wind turbines (OWRT’s), has been derived for slotted diffuser-augmented wind turbines (sDAWT’s) without requiring empirical relations.

The result is a set of three-parameter relations for the aerodynamic performance of sDAWT’s. In the limit of zero mass flow through the slot, $M_{\text{duct}} = 0$, these relations reduce to two-parameter relations for the aerodynamic performance of DAWT’s. The one-parameter relations for ORWT’s then follow by prescribing the wake-expansion factor $\beta$ such that $C_{T,\text{duct}} = 0$.

The present theory yields that for (s)DAWT’s the maximum achievable power coefficient $C_p$ increases monotonically with increasing $\beta$, surpassing the ORWT Betz limit $C_p = 16/27$ already for modest values of this parameter. However, the design of a realistic (s)DAWT requires that the flow remains attached along the entire interior surface of the diffuser. This implies that $\beta$ should be chosen in an optimal way.

In the present study the variation of $\beta$ along iso- $C_p$ contours in the $(C_T, C_{T,\text{duct}})$ plane is determined. The optimum chosen is at the point $(C_{T,\text{opt}}, C_{T,\text{duct},\text{opt}})$ on the iso- $C_p$ contour at which $\beta$ is minimal. Then it is found that (s)DAWT’s operate optimally for a thrust coefficient of $C_{T,\text{opt}} = 2/3$, independent of $C_p$ and also independent of $M_{\text{duct}}$. At the optimal condition $\beta$, $(1 + M_{\text{duct}})(A_s/A_{\text{disc}}) = (1 + M_{\text{duct}}) C_p$ are linear functions of $C_{T,\text{duct}}$, with $A_s/A_{\text{disc}} = (1 + M_{\text{duct}})\sqrt{3}$ and $V_s/V_n = 1/\sqrt{3}$.

The problem of satisfying the attached flow condition is mitigated by utilising diffusers that feed outside air into the diffuser through slots. However, for sDAWT’s at fixed $\beta$ the power coefficient $C_p$ decreases with increasing mass flow $M_{\text{duct}}$ into the diffuser. Therefore, a balance has to be found between the increase in $C_p$ due to delay of flow separation and the decrease in $C_p$ due to mass flow through the slots. Based on a basic vorticity model of the (s)DAWT, an expression has been derived for the velocity $V_r$ induced by the diffuser at the rotor plane, as well as an estimate of the required circulation of the diffuser. At the rotor plane $V_{\text{disc}} = V_s + V_n$ with $V_s$ and $V_n$. At the optimal condition it is found to be linear in $C_{T,\text{duct}}$: $V_r/V_n = \sqrt{3}(1 + 1/\sqrt{3})C_{T,\text{duct}} - \sqrt{3}M_{\text{duct}}/(1 + M_{\text{duct}})$.
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