PARTIAL SUMS OF BIASED RANDOM MULTIPLICATIVE FUNCTIONS

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Abstract. Let \( \mathcal{P} \) be the set of the primes. We consider a class of random multiplicative functions \( f \) supported on the squarefree integers, such that \( \{ f(p) \}_{p \in \mathcal{P}} \) form a sequence of \( \pm 1 \) valued independent random variables with \( \mathbb{E} f(p) < 0 \), \( \forall p \in \mathcal{P} \). The function \( f \) is called strongly biased (towards classical Möbius function), if \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p} = -\infty \) a.s., and it is weakly biased if \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p} \) converges a.s. Let \( M_f(x) := \sum_{n \leq x} f(n) \). We establish a number of necessary and sufficient conditions for \( M_f(x) = o(x^{1-\alpha}) \) for some \( \alpha > 0 \), a.s., when \( f \) is strongly or weakly biased, and prove that the Riemann Hypothesis holds if and only if \( M_f(x) = o(x^{1/2+\epsilon}) \) for all \( \epsilon > 0 \) a.s., for each \( \alpha > 0 \), where \( \{ f_\alpha \}_\alpha \) is a certain family of weakly biased random multiplicative functions.

1. Introduction.

A function \( f : \mathbb{N} \to \mathbb{C} \) is called multiplicative function if \( f(1) = 1 \) and \( f(nm) = f(n)f(m) \) whenever \( n \) and \( m \) are coprime. Let \( \mathcal{P} \) be the set of the prime numbers. In this paper we consider a class of multiplicative functions \( f \) which are supported on the square-free integers, i.e. \( f(n) = 0 \) for all \( n \in \mathbb{N} \), for which \( \exists p \in \mathcal{P} \) such that \( p^2 | n \). A function \( f \) from this class is called random (binary) multiplicative function if \( \{ f(p) \}_{p \in \mathcal{P}} \) form a sequence of \( \pm 1 \) valued independent random variables.

Let \( \mu \) be the Möbius function, the multiplicative function supported on the square-free integers with \( \mu(p) = -1 \) \( \forall p \in \mathcal{P} \). We say that \( f \) is biased (towards \( \mu \)) if \( \mathbb{E} f(p) < 0 \) \( \forall p \in \mathcal{P} \). If \( f \) is biased and \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p} \) converges a.s., we say that \( f \) is weakly biased; otherwise, if \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p} = -\infty \) a.s., we say that \( f \) is strongly biased.

The Kolmogorov two series Theorem implies that if \( f \) is a biased random multiplicative function with \( \sum_{p \in \mathcal{P}} \frac{\mathbb{E} f(p)}{p} > -\infty \) (resp. \( \sum_{p \in \mathcal{P}} \frac{\mathbb{E} f(p)}{p} = -\infty \)), then \( f \) is weakly (resp. strongly) biased.

Further, for \( x \geq 1 \), we denote \( M_f(x) := \sum_{n \leq x} f(n) \). A classical result of J.E. Littlewood, [15], states that the Riemann Hypothesis (RH) holds if and only if the Merten’s function \( M_\mu(x) = o(x^{1/2+\epsilon}) \), \( \forall \epsilon > 0 \). This naturally raises a question what can be said if \( \mu \) is substituted by a weakly biased \( f \). Our first result states:

**Theorem 1.1.** Let \( \alpha > 0 \) and \( f_\alpha \) is such that \( \mathbb{E} f_\alpha(p) = -\frac{1}{p} \) \( \forall p \in \mathcal{P} \). Then the Riemann hypothesis holds if and only if \( M_{f_\alpha}(x) = o(x^{1/2+\epsilon}) \) for all \( \epsilon > 0 \) a.s., for each \( \alpha > 0 \).

Our next result is motivated by the following: Let \( f : \mathbb{N} \to \{ -1, 0, 1 \} \) and \( g : \mathbb{N} \to \{ -1, 0, 1 \} \) be multiplicative functions supported on the square-free integers, and such that \# \{ \( p \in \mathcal{P} : f(p) \neq g(p) \} < \infty \). Then for each \( c > 1/2 \), \( M_f(x) = o(x^{c+\epsilon}) \), \( \forall \epsilon > 0 \) if and only if \( M_g(x) = o(x^{c+\epsilon}) \), \( \forall \epsilon > 0 \) (see the proof of Proposition A.1 and Corollary A.1). We have:

**Theorem 1.2.** Let \( f \) be strongly biased, such that for some \( 0 < \alpha \leq 1/2 \), the series \( \sum_{p \in \mathcal{P}} \frac{1+ f(p)}{p^{1+\alpha}} \) converges for all \( \epsilon > 0 \) a.s. Then \( M_f(x) = o(x^{1-\alpha+\epsilon}) \), \( \forall \epsilon > 0 \) a.s. if and only if \( M_\mu(x) = o(x^{1-\alpha+\epsilon}) \), \( \forall \epsilon > 0 \) a.s.

The next question concerns necessary and sufficient conditions on biased \( \{ f(p) \}_{p \in \mathcal{P}} \), under which \( M_f(x) = o(x^{1-\delta}) \) for some possibly random \( 0 < \delta < 1/2 \) a.s. Let \( F(z) := \sum_{n=1}^{\infty} \frac{f(n)}{n^z}, z \in \mathbb{C}, Re(z) > 1 \), be the Dirichlet series of \( f : \mathbb{N} \to [-1, 1] \). In [13] this problem has been studied in a more
general context, where a multiplicative function \( f \) may assume values on \( \mathbb{U} = \{ z \in \mathbb{C} : |z| \leq 1 \} \), and \( \{ f(p) \}_{p \in \mathcal{P}} \) is not necessarily a random sequence. In particular, for completely multiplicative functions \( f : \mathbb{N} \to [-1,1] \), Theorem 1.6 of [13], states that if for some \( \delta \in (0,1/3) \) and \( Q \geq \exp(1/\delta) \) one has \( |M_f(x)| \leq \frac{x^{1+\delta}}{(\log x)^2} \) \( \forall x \geq Q \), then there exists \( c = c(\delta) \) and \( d = d(f) \) such that

\[
\sum_{p \leq x} f(p) \log p \ll \frac{x}{\exp(c\sqrt{\log x})} + x^{1-cd}, \quad \text{if } F(1) \neq 0
\]

\[
\sum_{p \leq x} (1 + f(p)) \log p \ll x^{1-\frac{1}{2+\delta}}, \quad \text{if } F(1) = 0,
\]

and it is also applicable to multiplicative functions supported on square-free integers. For the general account on the state of the art we refer reader to [13], and the references therein, and also to [13], Chapters II.5 and III.4, and their historical notes.

In the case of random multiplicative functions if \( M_f(x) = o(x^{1-\alpha}) \) for some \( \alpha > 0 \) a.s., then \( F(1) \neq 0 \) a.s. if \( f \) is weakly biased, and \( F(1) = 0 \) a.s. if \( f \) is strongly biased. For weakly biased \( f \) we have:

**Theorem 1.3.** Let \( f \) be weakly biased. If \( M_f(x) = o(x^{1-\delta}) \) for some possibly random \( 0 < \delta < 1/2 \) a.s., then there exists \( 0 < \alpha < 1/2 \) such that the random series \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p^{1+\alpha}} \) converges a.s.

Observe that, if \( f_\alpha \) is as in Theorem 1.1 by the Kolmogorov two series Theorem, \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p^{1+\alpha}} \) converges \( \forall \epsilon > 0 \) a.s.

For \( f \) strongly biased we prove the following:

**Theorem 1.4.** Let \( f \) be strongly biased. If for some fixed \( 0 < \alpha < 1/2 \), \( M_f(x) = o(x^{1-\alpha}) \) a.s., then the series \( \sum_{p \in \mathcal{P}} \frac{1+f(p)}{p^{1+\alpha+\epsilon}} \) converges \( \forall \epsilon > 0 \) a.s.

Observe that, for fixed \( 0 < \alpha < 1/2 \) and \( f \) strongly biased, if \( M_f(x) = o(x^{1-\alpha+\epsilon}) \) \( \forall \epsilon > 0 \) a.s., then by Theorem 1.4 \( \sum_{p \in \mathcal{P}} \frac{1+f(p)}{p^{1+\alpha+\epsilon}} \) converges \( \forall \epsilon > 0 \) a.s. Hence, by Theorem 1.2 \( M_f(x) = o(x^{1-\alpha+\epsilon}) \) \( \forall \epsilon > 0 \), and this implies that the Riemann zeta function \( \zeta \) has no zeroes in \( \{ z \in \mathbb{C} : Re(z) > 1-\alpha \} \). Thus, in the case of strongly biased \( f \), in order to provide conditions that guarantee \( M_f(x) = o(x^{1-\epsilon}) \) for some \( \epsilon > 0 \) a.s., we must assume certain half planes to be zero free regions of \( \zeta \).

Let \( \ast \) denote the Dirichlet convolution. When \( f \) is weakly biased, \( f \) can be represented as \( f = w * g \) (see Remark 4.1 and Claim 4.3), where \( w \) and \( g \) are random multiplicative functions which possibly admit zero values on primes, \( w \) is unbiased, and \( g \) is such that \( \mathbb{E}g(p) = \mathbb{E}f(p), \forall p \in \mathcal{P} \). For unbiased \( w \) with \( \mathbb{P}(w(p) = 0) \geq 0, \forall p \in \mathcal{P} \), the family \( \{ w(k) : k \in \mathbb{N} \text{ is square-free} \} \) forms a sequence of orthogonal random variables. The fact that \( w \) is unbiased implies that \( M_w(x) = o(x^{1/2+\epsilon}), \forall \epsilon > 0, \) a.s., see A.Wintner, [21], and thus, in contrast with the class of strongly biased random multiplicative functions, allows us to derive conditions which do not depend on zero free regions of \( \zeta \), and which guarantee that for weakly biased \( f \) we get that \( M_f(x) = o(x^{1-\alpha}) \) for some \( \alpha > 0 \) a.s. We are ready to state our last result:

**Theorem 1.5.** Let \( f \) be weakly biased, such that for some fixed \( 0 < \alpha < 1/2 \), \( \mathbb{E}f(p) = -\frac{\delta_p}{p^{\alpha}}, \) where \( 0 \leq \delta_p \leq 1 \). Then \( M_f(x) = o(x^{1-\alpha+\epsilon}) \) for all \( \epsilon > 0 \) a.s. If in addition we assume \( \limsup \delta_p < 1 \) and \( \sum_{p \in \mathcal{P}} \delta_p = \infty \), then \( M_f(x) \) is not \( o(x^{1-\alpha-\epsilon}) \) for any \( \epsilon > 0 \) a.s.

We conclude by mentioning that in the case of unbiased random multiplicative functions situation is far better understood. The study of partial sums \( M_f(x) \) for unbiased binary multiplicative functions \( f \) started by A.Wintner [21]. As mentioned before, he proved that \( M_f(x) = o(x^{1/2+\epsilon}) \) \( \forall \epsilon > 0 \) a.s., and that for each \( \epsilon > 0 \), \( M_f(x) \) is not \( O(x^{1/2-\epsilon}) \) a.s. Later, improvements have been
made in [8] and [9]. In a recent paper [14] it has been proved that 
$M_f(x) = O(\sqrt{x} (\log \log x)^{3/2+\epsilon})$ 
$\forall \epsilon > 0$ a.s. On the other hand, for $A > 2.5$, $M_f(x) \neq O(\sqrt{x} (\log \log x)^{-A})$ a.s. [10].

Let $\mathcal{N}(0,1)$ be the standard Gaussian distribution with unit variance. Regarding Central Limit Theorems, in [5] it has been proved that, if $y = y(x)$, $x \to \infty$ and $y \to \infty$ in a suitable way, then the distribution of $(M_f(x+y) - M_f(x))/\sqrt{\mathbb{E}|M_f(x+y) - M_f(x)|^2}$ is approximately $\mathcal{N}(0,1)$.

Let $d(n)$ denote the number of distinct prime factors of $n$, $M_f^{(k)}(x) = \sum_{n \leq x, d(n) = k} f(n)$ and $M_f^{(\leq k)}(x) = \sum_{n \leq x, d(n) \leq k} f(n)$. In [11], it has been proved that, if $k = o(\log \log x)$, then the distributions of $M_f^{(k)}(x)/\sqrt{\mathbb{E}|M_f^{(k)}(x)|^2}$ and $M_f^{(\leq k)}(x)/\sqrt{\mathbb{E}|M_f^{(\leq k)}(x)|^2}$ also are approximately $\mathcal{N}(0,1)$. Moreover, if for some $\epsilon > 0$, $k \geq \epsilon \log \log x$, then $M_f^{(\leq k)}(x)/\sqrt{\mathbb{E}|M_f^{(\leq k)}(x)|^2}$ does not approximates $\mathcal{N}(0,1)$. Consequently, $M_f(x)/\sqrt{\mathbb{E}|M_f(x)|^2}$ also does not approximates $\mathcal{N}(0,1)$. Considering $f$ a random completely multiplicative function, in [12] it has been proved that, if $k = o(\log \log \log x)$ then the distribution of $M_f^{(k)}(x)/\sqrt{\mathbb{E}|M_f^{(k)}(x)|^2}$ also is approximately $\mathcal{N}(0,1)$.\par

The paper is organized as follows. In Section 2, we set up the main notations and tools from Probability and Analytic Number Theory. In Section 3, we consider the problem of bounding convergent random Dirichlet Series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s} (Re(z) > 1/2)$ in vertical strips, where $\{X_n\}_{n \in \mathbb{N}}$ belongs to a certain class of sequences of random variables. This class includes the case in which random variables are independent. In this section we prove a key result for the proof of Theorems 1.1 and 1.2. In Section 4, we prove all the main results.

2. Preliminaries.

Notations from Probability Theory. $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a probability space. Given a set $E \in \mathcal{F}$, the random variable $\mathbb{I}_E : \Omega \to \{0,1\}$ stands for the indicator function of $E$, that is, $\mathbb{I}_E(\omega) = 1$ if $\omega \in E$ and $\mathbb{I}_E(\omega) = 0$ otherwise. Given an square integrable random variable $Y : \Omega \to \mathbb{R}$:

$$
\mathbb{E}Y := \int_\Omega Y(\omega) \mathbb{P}(d\omega),
$$

$$
\mathbb{V}Y := \mathbb{E}Y^2 - (\mathbb{E}Y)^2.
$$

Notations from Complex Analysis. A set of the form $\mathbb{H}_a := \{ z \in \mathbb{C} : Re(z) > a \}$ where $a \in \mathbb{R}$ is called half plane. Let $R_1 \subset R_2$ be two open connected sets of $\mathbb{C}$ and $h : R_1 \to \mathbb{C}$ be an analytic function. We say that $h$ has analytic extension to $R_2$ if there exists an analytic function $\overline{h} : R_2 \to \mathbb{C}$ such that for all $z \in R_1$ we have that $\overline{h}(z) = h(z)$.

Definition 2.1. Let $S \subset \mathbb{C}$. A map $f : S \times \Omega \to \mathbb{C}$ is called a random function if $\omega \in \Omega \mapsto f(s,\omega)$ is a complex valued random variable for each fixed $s \in S$, and $s \in S \mapsto f(s,\omega)$ is a function of one complex variable for each fixed $\omega \in \Omega$.

Let $f : S \times \Omega \to \mathbb{C}$ be a random function. For each fixed $\omega \in \Omega$, $f_\omega$ denotes the function $f_\omega : S \to \mathbb{C}$ given by $f_\omega(s) := f(s,\omega)$.

Definition 2.2. Let $S \subset \mathbb{C}$ be an open connected set and $f : S \times \Omega \to \mathbb{C}$ a random function. We say that $f$ is a random analytic function if the set of elements $\omega \in \Omega$, for which $f_\omega : S \to \mathbb{C}$ is analytic, contains a set $\Omega^* \in \mathcal{F}$ such that $\mathbb{P}(\Omega^*) = 1$. 

Let \((X_k)_{k \in \mathbb{N}}\) be a sequence of independent random variables such that \(\forall X_k^2 < \infty\) for all \(k\). We define
\[
\sigma_1 = \inf \left\{ \sigma \in \mathbb{R} \cup \{\infty\} : \text{the series } \sum_{k=1}^{\infty} \frac{EX_k}{k^\sigma} \text{ converges} \right\},
\]
\[
\sigma_2 = \inf \left\{ \sigma \in \mathbb{R} \cup \{\infty\} : \text{the series } \sum_{k=1}^{\infty} \frac{\mathbb{V}X_k}{k^{2\sigma}} \text{ converges} \right\}.
\]

**Proposition 2.1.** Let \((X_k)_{k \in \mathbb{N}}\) be a sequence of independent random variables and \(\sigma_1\) and \(\sigma_2\) be as in (1) and (2). Assume that \(\sigma_0 = \max\{\sigma_1, \sigma_2\} < \infty\). Then \(F : \mathbb{H}_{\sigma_0} \times \Omega \to \mathbb{C}\) given by \(F(z) := \sum_{k=1}^{\infty} \frac{X_k}{k^z}\) converges for each \(z \in \mathbb{H}_{\sigma_0}\) and it is a random analytic function.

**Proof.** Let \(\{c_k\}_{k=1}^{\infty}\) be a sequence of complex numbers and \(\sum_{k=1}^{\infty} \frac{c_k}{k^z}\) be its Dirichlet series, where \(z \in \mathbb{C}\). A classical result in the Theory of the Dirichlet series (see [1], Theorems 11.8 and 11.11) states that if the series \(\sum_{k=1}^{\infty} \frac{c_k}{k^z}\) converges for \(z_0 = \sigma_0 + it_0\) then it converges for all \(z \in \mathbb{H}_{\sigma_0}\) and also uniformly on compact subsets of this half plane. Thus the function \(z \in \mathbb{H}_{\sigma_0} \mapsto \sum_{k=1}^{\infty} \frac{c_k}{k^z}\) is analytic. The Kolmogorov two series Theorem states that if \(\{Y_k\}_{k=1}^{\infty}\) is a sequence of independent random variables such that \(\sum_{k=1}^{\infty} \frac{EY_k}{k^\sigma}\) and \(\sum_{k=1}^{\infty} \frac{\mathbb{V}Y_k}{k^{2\sigma}}\) converges then \(\sum_{k=1}^{\infty} \frac{Y_k}{k^z}\) converges a.s. Thus the assumption that \(\sigma_0 < \infty\) implies that for each \(\sigma > \sigma_0\) both series \(\sum_{k=1}^{\infty} \frac{EX_k}{k^{\sigma_0}}\) and \(\sum_{k=1}^{\infty} \frac{\mathbb{V}X_k}{k^{2\sigma_0}}\) converge. Hence by the Kolmogorov two series Theorem, for each \(\sigma_0 = \sigma_0 + k\) each event \(\Omega_k := \{F(\sigma_k) \text{ converges}\}\) has \(\mathbb{P}(\Omega_k) = 1\) as \(\Omega^* := \cap_{k=1}^{\infty} \Omega_k\). By the referred properties of convergence of a Dirichlet series we then obtain that for each \(\omega \in \Omega^*\) the Dirichlet series \(F_\omega(z)\) converges for each \(z \in \mathbb{H}_{\sigma_0}\) and uniformly in compact subsets of this half plane. We then conclude that \(F\) is a random analytic function.

**Notations from Number Theory.** In the sequel \(\mathcal{P}\) stands for the set of the prime numbers and \(p\) for a generic element of \(\mathcal{P}\). Given \(d, n \in \mathbb{N}\), \(d|n\) and \(d \nmid n\) means that \(d\) divides and that \(d\) do not divides \(n\), respectively. The set of the squarefree numbers is denoted by \(\mathcal{S} = \{k \in \mathbb{N} : p|k \Rightarrow p^2 \nmid k\}\). The Möbius function is denoted by \(\mu\) and its partial sums by \(M_p(x) := \sum_{k \leq x} \mu(k)\).

**Definition 2.3.** A random function \(f : \mathbb{N} \times \Omega \to \mathbb{C}\) is called random multiplicative function if \(f(1) = 1\),
\[
f(n) = |\mu(n)| \prod_{p|n} f(p) \quad (n \geq 2),
\]
and \(\{f(p)\}_{p \in \mathcal{P}}\) is a sequence of \(\pm 1\) independent random variables.

**Lemma 2.1.** Let \(f\) be a random multiplicative function and for each \(z \in \mathbb{H}_1\) let \(F(z) := \sum_{k=1}^{\infty} \frac{f(k)}{k^z}\). Then:

i) \(EF(z) : \mathbb{H}_1 \to \mathbb{C}\) is analytic and \(F : \mathbb{H}_1 \times \Omega\) is a random analytic function. Moreover for all \(z \in \mathbb{H}_1\), \(EF(z) \neq 0\) and \(F_\omega(z) \neq 0\) for each \(\omega \in \Omega\).

ii) There exists a random analytic function \(\theta : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C}\) such that for each \(z \in \mathbb{H}_1\)
\[
\theta(z) = \frac{F(z)}{EF(z)}.
\]

iii) The random analytic function \(\theta\) is given by
\[
\theta(z) = \exp \left( \sum_{p \in \mathcal{P}} \frac{f(p) - EF(p)}{p^2} \exp(A(z)) \right),
\]
where \(A : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C}\) is a random analytic function such that for all \(\sigma \geq \sigma_0 > \frac{1}{2}\) there exists \(C = C(\sigma_0)\) such that \(|A(\sigma + it)| \leq C\) a.s.

**Proof.** Let \(h : \mathbb{N} \to [-1, 1]\) be a multiplicative function supported on the square free integers. Let \(g_k : \mathbb{C} \to \mathbb{C}\) be given by \(g_k(z) = \frac{h(k)}{k^z}\). Then for each \(k \in \mathbb{N}\), \(g_k\) is analytic and satisfies \(|g_k(z)| \leq \frac{1}{k^z}\) where \(\sigma = Re(z)\). Thus for each \(\sigma > 1\), \(\sum_{k=1}^{\infty} g_k(z)\) is a series of complex analytic functions that
converges uniformly in the set \( \{ z \in \mathbb{C} : \text{Re}(z) \geq \sigma \} \) and hence uniformly on compact subsets of \( H_1 \). This gives that the Dirichlet series \( \sum_{k=1}^{\infty} \frac{b(k)}{k^s} \) is analytic. The same argument gives that \( z \in H_1 \mapsto \sum_{p \in \mathbb{P}} \frac{h(p)}{p^s} \) is analytic.

**Claim 2.1.** Let \( h \) be as above. Then for each \( z \in H_1 \)
\[
\sum_{k=1}^{\infty} \frac{h(k)}{k^s} = \exp \left( \sum_{p \in \mathbb{P}} \frac{h(p)}{p^s} \right) \exp(A(h, z)),
\]
where \( z \in H_{1/2} \mapsto A(h, z) \) is analytic and uniformly bounded in the set \( \{ z \in \mathbb{C} : \text{Re}(z) \geq \sigma_0 \} \) for each real \( \sigma_0 > 1/2 \).

**Proof of the claim.** The Dirichlet series \( \sum_{k=1}^{\infty} \frac{h(k)}{k^s} \) has Euler product representation (see [1], Theorem 11.6): For each \( z \in H_1 \)
\[
(5) \quad \sum_{k=1}^{\infty} \frac{h(k)}{k^s} = \prod_{p \in \mathbb{P}} \left( 1 + \frac{h(p)}{p^s} \right).
\]
Since the Taylor series \( \log(1 + x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} x^m \) converges absolutely for \( |x| < 1 \), we obtain for each real \( \sigma > 1 \) that
\[
\log \left( 1 + \frac{h(p)}{p^s} \right) = \frac{h(p)}{p^s} + A_p(\sigma) \quad \text{where} \quad A_p(\sigma) := \sum_{m=2}^{\infty} \frac{(-1)^{m+1}}{m} \frac{h(p)^s}{p^{m\sigma}}.
\]
Let \( z \in \mathbb{C} \) be such that \( \text{Re}(z) = \epsilon > 0 \). Observe that for large \( p \)
\[
|A_p(\sigma)| \leq \sum_{m=2}^{\infty} \frac{1}{p^{m\sigma}} = \frac{1}{p^\sigma (p^\sigma - 1)} \sim \frac{1}{p^\sigma}.
\]
Hence \( A_p(\sigma) \) is analytic in \( H_0 \) and there is \( C > 0 \) such that \( |A_p(\sigma)| \leq \frac{C}{p^\sigma} \quad \forall p \in \mathbb{P} \quad (\text{Re}(z) = \epsilon) \).

Since for \( \sigma > 1/2 \) the series \( \sum_{p \in \mathbb{P}} \frac{1}{p^s} \) is summable, the series of complex analytic functions \( A(h, z) := \sum_{p \in \mathbb{P}} A_p(\sigma) \) converges uniformly in the set \( \{ z \in \mathbb{C} : \text{Re}(z) \geq \sigma \} \) for each \( \sigma > 1/2 \) and hence uniformly on compact subsets of \( H_{1/2} \). This gives that \( A(h, z) \) is analytic in \( H_{1/2} \) and for each \( \sigma_0 > 1/2 \) it is uniformly bounded by some constant \( C = C(\sigma_0) \) in the set \( \{ z \in \mathbb{C} : \text{Re}(z) \geq \sigma_0 \} \).

This gives the desired properties for \( z \in H_{1/2} \mapsto A(h, z) \) and the following formula for each \( \sigma > 1 \):
\[
\log \prod_{p \in \mathbb{P}} \left( 1 + \frac{h(p)}{p^s} \right) = \sum_{p \in \mathbb{P}} \frac{h(p)}{p^s} + A(h, \sigma).
\]
This formula combined with (5) gives for each \( \sigma > 1 \) that
\[
(6) \quad \sum_{k=1}^{\infty} \frac{h(k)}{k^s} = \exp \left( \sum_{p \in \mathbb{P}} \frac{h(p)}{p^s} \right) \exp(A(h, \sigma)).
\]

Let \( F_1 \) and \( F_2 \) be two complex analytic functions defined in some open connected set \( U \subset \mathbb{C} \) such that \( F_1(z_k) = F_2(z_k) \) where \( \{z_k\}_{k=1}^{\infty} \subset U \) is a convergent sequence whose limit point is \( z \in U \).

Then \( F_1 = F_2 \) (see [1], Corollary 3.8 and 3.9). This gives that (6) holds for all \( z \in H_1 \), since both left side and right side of this equation are complex analytic functions restricted to the set \( \{ \sigma \in \mathbb{R} : \sigma > 1 \} \), finishing the proof of the claim.

**Proof of i)** For each \( \omega \in \Omega, n \mapsto f_{\omega}(n) \) is a multiplicative function supported on the square free integers. Since \( \{f(p)\}_{p \in \mathbb{P}} \) is a sequence of independent random variables, \( n \mapsto \mathbb{E} f(n) \) also is a multiplicative with the same property. Thus, by claim 2.1, for each \( \omega \in \Omega, F_\omega : H_1 \to \mathbb{C} \) and \( \mathbb{E} : H_1 \to \mathbb{C} \) are non-vanishing complex analytic functions, completing the proof of i.

**Proof of ii) and iii)** Claim 2.1 gives the following formula for each \( \omega \in \Omega \) and \( z \in H_1 \):
\[
\frac{F_\omega(z)}{\mathbb{E}(z)} = \exp \left( \sum_{p \in \mathbb{P}} \frac{f_\omega(p) - \mathbb{E} f(p)}{p^s} \right) \exp(A(f_\omega, z) - A(\mathbb{E} f, z)),
\]
where \( z \in H_{1/2} \mapsto A(f_\omega, z) \) and \( z \in H_{1/2} \mapsto A(\mathbb{E} f, z) \) are complex analytic functions which are uniformly bounded in the sets \( \{ z \in \mathbb{C} : \text{Re}(z) \geq \sigma \} \), for each \( \sigma > 1/2 \). Hence \( A : H_{1/2} \times \Omega \to \mathbb{C} \) given by \( A_\omega(z) = A(f_\omega, z) - A(\mathbb{E} f, z) \) is the desired random analytic function of iii. By Proposition
2.1 $z \in \mathbb{H}_{1/2} \mapsto \sum_{\rho \in \mathcal{P}} \frac{f(\rho) - \bar{f}(\rho)}{\rho^z}$ is a random analytic function and hence its exponential also is. This gives the desired properties of the random analytic function $\theta$. \hfill \Box

3. Bounding random Dirichlet series in vertical strips.

In this section we study the asymptotic behavior of a random Dirichlet series $\sum_{k=1}^{\infty} \frac{X_k}{k^z}$ for fixed $\sigma > 0$ and $t \to \infty$, where $\{X_k\}_{k \in \mathbb{N}}$ are centered random variables not necessarily independent. If $\{X_k\}_{k \in \mathbb{N}}$ is a sequence of independent and identically distributed random variables such that $\mathbb{P}(X_1 = -1) = \mathbb{P}(X_1 = 1) = \frac{1}{2}$, in $\mathbb{H}_F$. Carlson proved that, for each $\sigma > 1/2$ the random Dirichlet series $\sum_{k=1}^{\infty} \frac{X_k}{k^{\sigma + it}} = o(\sqrt{\log T})$ a.s. Following the same line of reasoning we prove:

**Theorem 3.1.** Let $\{X_k\}_{k \in \mathbb{N}}$ be a sequence of centered and uniformly bounded random variables. Denote for a complex $z$, $F(z) := \sum_{k=1}^{\infty} \frac{X_k}{k^z}$. Let $1/2 < \sigma_0 \leq 1$. If $\{X_k\}_{k \in \mathbb{N}}$ are either a) independent, b) a martingale difference or c) $\rho^*\text{-mixing}$, then uniformly for all $\sigma_0 \leq \sigma \leq 1$:

\begin{equation}
\begin{aligned}
F(\sigma + it) &\ll (\log t)^{\vartheta(1-\sigma)} \log \log t, \quad \text{a.s.}, \\
\end{aligned}
\end{equation}

where $\vartheta = 1$ in the case a), $\vartheta = 2$ in the case b) and $\vartheta = 3$ in the case c). Moreover, if $X_k = 0$ for all non prime $k$ then the term $\log \log t$ in (7) can be substituted by $\log \log t$.

Next we restrict ourselves to sequences $\{X_k\}_{k \in \mathbb{N}}$ which satisfy the following conditions:

i) For all $k$, $EX_k = 0$ and $|X_k| \leq C$ for some constant $C > 0$;

ii) The random series $\sum_{k=1}^{\infty} \frac{X_k}{k^{\sigma + it}}$ converges for all $z \in \mathbb{H}_{1/2}$, a.s.;

iii) There exists a constant $\gamma > 0$ and a increasing function $\lambda : [0, \infty) \to [1, \infty)$ such that $\lim_{t \to \infty} \lambda(t) = \infty$, for all $a, b \geq 0$, $\lambda(a + b) \leq e^{\gamma a} \lambda(b)$, and such that the following inequality holds for all $q > 1$ for all real numbers $\alpha_1, ..., \alpha_n$, for each $n \in \mathbb{N}$:

$$\mathbb{E} \left( \sum_{k=1}^{n} \alpha_k X_k \right)^q \leq \lambda(q) \left\{ \mathbb{E} \left( \sum_{k=1}^{n} |\alpha_k X_k|^2 \right)^{q/2} \right\}^{1/2}.$$ 

**Lemma 3.1.** Assume that $\{X_k\}_{k \in \mathbb{N}}$ satisfies conditions i)-iii) above. Let $\psi(t) := \lambda(\log t)$ and $v(\epsilon^{-1})^2 := \sum_{k=1}^{\infty} \frac{\mathbb{E}[|X_k|^2]}{k^{\sigma + \epsilon + 1}}$. Then for each $\sigma > 1/2$, uniformly for all $x \in [\sigma, 1]$:

$$\sum_{k=1}^{\infty} \frac{X_k}{k^{\sigma + it}} \ll \psi(t)^{2-2x} v(\log \psi(t))^2, \quad \text{a.s.}$$

**Proof of Lemma 3.1.** We begin the proof with the following claim:

**Claim 3.1.** Let $\{X_k\}_{k \in \mathbb{N}}$ and $v(\epsilon^{-1})$ be as in Lemma 3.1. Then there exists $D > 0$ such that for all $q > 1$, $\epsilon > 0$ and $t \in \mathbb{R}$ the following inequality holds:

$$\mathbb{E} \left( \sum_{k=1}^{n} \frac{X_k}{k^{\sigma + \epsilon + \alpha}} \right)^q \leq D^q \lambda(q)^q v(\epsilon^{-1})^q.$$

**Proof of claim 3.1.** Let $z = x + iy$ with $x = 1/2 + \epsilon/2$, where $\epsilon > 0$. Denote $|z| = \sqrt{x^2 + y^2}$. Since the random series $\sum_{k=1}^{\infty} \frac{X_k}{k^z}$ converges a.s. we obtain by Fatou’s Lemma that

$$\mathbb{E} \left( \sum_{k=1}^{n} \frac{X_k}{k^z} \right)^q \leq \liminf_{n \to \infty} \mathbb{E} \left( \sum_{k=1}^{n} \frac{X_k}{k^z} \right)^q.$$

Recall that $|a_1 + ia_2|^q \leq 2^q (|a_1|^q + |a_2|^q)$ for each $a_1, a_2 \in \mathbb{R}$. Thus taking

$$a_1 := \sum_{k=1}^{n} \frac{X_k}{k^z} \cos(y \log k), \quad a_2 := -\sum_{k=1}^{n} \frac{X_k}{k^z} \sin(y \log k),$$

For the definition of $\rho^*\text{-mixing}$ see the proof of Theorem 3.1 below.
Let \( \alpha_k = \frac{\cos(q \log k)}{k^2} \) for all \( k \geq 1 \). Observe that \( \alpha_k^2 \leq \frac{1}{k^4} \). Hence the condition iii) above implies that

\[
\mathbb{E}[a_1]^q \leq \lambda(q)^q \mathbb{E}
\left|
\sum_{k=1}^{n} \frac{X_k}{k^2}
\right|^q \leq \lambda(q)^q \mathbb{E}
\left|
\sum_{k=1}^{n} \frac{X_k^2}{k^{1+\epsilon}}
\right|^q \leq C^q \lambda(q)^qv(\epsilon^{-1})^q,
\]

where \( C > 0 \) is the constant of condition i) above. Similarly we get the same bound for \( \mathbb{E}[a_2]^q \).

Let \( F : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) be given by \( F(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^2} \). Define

\[
\Omega^* := \{ \omega \in \Omega : F_\omega(z) \text{ converges for each } z \in \mathbb{H}_{1/2} \}.
\]

By ii), \( \mathbb{P}(\Omega^*) = 1 \). Hence \( F \) is a random analytic function (see the proof of Proposition 2.1). Let \( q \geq 1, \epsilon = \epsilon(q) \in (0, 1/3], \sigma = 1/2 + \epsilon \) and \( \sigma' = 1/2 + \epsilon/2 \). Let \( R_1 \) and \( R_2 \) be the rectangles:

\[
R_1 = R_1(q, \epsilon) = [\sigma, 4/3] \times [-e^{q-2}, e^{q-2}],
\]

\[
R_2 = R_2(q, \epsilon, \omega) = [\sigma', \sigma' + 1] \times [-\tau(\omega), \tau(\omega)],
\]

where \( \tau, \tau' \in [e^{q-1}, e^{q}] \) will be chosen later. Observe that \( R_1 \subset R_2 \) and the distance from \( \partial R_1 \) to \( \partial R_2 \) equals to \( \epsilon/2 \). Decompose: \( \partial R_2 = I_1 \cup I_2 \cup I_3 \cup I_4 \), where \( I_1 \) and \( I_3 \) are the vertical lines at \( Re(s) = \sigma' + 1 \) and \( Re(s) = \sigma' \) respectively and \( I_2 \) and \( I_4 \) are the horizontal lines at \( Im(s) = \tau \) and \( Im(s) = -\tau \) respectively. For \( q \in \mathbb{N} \) and \( \omega \in \Omega^* \), define:

\[
V_j = V_j(\omega, q) = \int_{I_j} |F_\omega(s)|^q |ds|, \quad j = 1, 2, 3, 4.
\]

For all \( q \in \mathbb{N} \), \( F_\omega^q \) is analytic on \( \mathbb{H}_{1/2} \). Hence, by the Cauchy integral formula, for each \( z \in R_1 \),

\[
F_\omega^q(z) = \frac{1}{2\pi i} \int_{\partial R_2} \frac{F_\omega^q(s)}{s - z} ds = \frac{1}{2\pi i} \sum_{j=1}^{4} \int_{I_j} \frac{F_\omega^q(s)}{|s - z|} |ds| = \frac{1}{2\pi i} \sum_{j=1}^{4} V_j(\omega, q).
\]

(see [7], pg. 65). Hence:

\[
\max_{z \in R_1} |F_\omega(z)|^q \leq \frac{1}{\pi \epsilon} \sum_{j=1}^{4} V_j(\omega, q).
\]

Claim 3.2. Let \( \epsilon = \epsilon(q) := \min\{1/3, (\log(\lambda(q)))^{-1}\} \). Then there exists \( H_1 = H_1(\lambda, \nu, C) \) such that

\[
\mathbb{P}\left( \max_{z \in R_1} |F(z)| > H_1 v(\epsilon^{-1}) \lambda(q) \right) \leq \frac{1}{2^{q-1}}.
\]

Proof of the Claim 3.2 By claim 3.1 for each \( q \in \mathbb{N} \), for all \( y \in \mathbb{R} 
\]

\[
\mathbb{E}|F(\sigma' + iy)|^q \leq D^q \lambda(q)^qv(\epsilon^{-1})^q.
\]

By condition i) above

\[
|F(\sigma' + iy)|^q \leq \left( \sum_{k=1}^{\infty} \frac{|X_k|}{k^{1+\frac{q}{2}}} \right)^q \leq C^q \left( \sum_{k=1}^{\infty} \frac{\mathbb{E}[|X_k| |X_k|^q]}{k^{1+\frac{q}{2}}} \right) \leq C^q v(2)^{2q}.
\]

Let \( H = 4\epsilon \max\{D, Cv(2)^2\} \) and \( A_q \) and \( B_q \) the events

\[
A_q := \Omega^* \cap \left[ \int_{-e^{-q}}^{e^{-q}} |F(\sigma' + iy)|^q dy \geq (H \lambda(q) v(\epsilon^{-1}))^q \right],
\]

\[
B_q := \Omega^* \cap \left[ \int_{-e^{-q}}^{e^{-q}} |F(\sigma' + iy)|^q dy \leq (H \lambda(q) v(\epsilon^{-1}))^q \right].
\]
Thus we obtain by Markov’s inequality that
\[ P \left( |Z| \geq (H\lambda(q)v)q \right) \leq \frac{1}{q}e^qD^q\lambda(q)q^qv(e^{-q})^q. \]

By Fubini’s Theorem and (11):
\[ E\int_{-e^q}^{e^q} |F(\sigma' + iy)|^q dy = \int_{-e^q}^{e^q} E|F(\sigma' + iy)|^q dy \leq 2e^qD^q\lambda(q)q^qv(e^{-q})^q, \]
\[ V_{\omega}(\sigma' + iy)\]
\[ \int_{-e^q}^{e^q} |F_{\omega}(\sigma' + iy)|^q dy \leq (H\lambda(q)v(e^{-q}))^q, \]
\[ \int_{-e^q}^{e^q} |F_{\omega}(\sigma' + iy)|^q dy < (H\lambda(q)v(e^{-q}))^q. \]

The choice of \( \tau(\omega) \) and \( \tau'(\omega) \). We claim that for each \( \omega \in E_q \) we can choose \( \tau'(\omega) \) and \( \tau(\omega) \) in \([e^q-1, e^q] \) such that:
\[ V_j(\omega, q) \leq (H\lambda(q)v(e^{-q}))^q, \quad j = 1, 2, 3, 4. \]

To show the existence of \( \tau \) and \( \tau' \) as above, first we will introduce the following notation:
\[ u(y) = u_{\omega}(y) = \int_{\sigma'}^{\sigma'+1} F_{\omega}(x + iy)^q dx \quad (\omega \in E_q). \]

Let \( L = (H\lambda(q)v(e^{-q}))^q, \quad a = e^q-1 \) and \( b = e^q \). By (14) and Fubini’s Theorem:
\[ \int_a^b u(y)dy \leq \int_{-e^q}^{e^q} u(y)dy \leq L. \]

Observe that \( b - a > 1 \). Denote by \( m \) the Lebesgue measure on \( \mathbb{R} \). We claim that:
\[ m(\{y \in [a, b] : u(y) \leq L\}) > 0. \]
Indeed,
\[ m(\{y \in [a, b] : u(y) > L\}) + m(\{y \in [a, b] : u(y) \leq L\}) = b - a, \]
and since \( L > 0 \) and \( u \geq 0 \) we get:
\[ Lm(\{y \in [a, b] : u(y) > L\}) \leq \int_a^b u(y)\mathbb{I}_{|u|>L}dm(y) \int_{-e^q}^{e^q} u(y)dy \leq L. \]

Hence \( m(\{y \in [a, b] : u(y) \geq L\}) \leq 1 - b - a \). This shows that the set \( \{y \in [a, b] : u(y) \leq L\} \) is not empty and hence the existence of at least one \( \tau(\omega) \in [e^q-1, e^q] \) such that (15) is satisfied for \( j = 4 \).
A similar argument shows the existence of \( \tau'(\omega) \in [e^q-1, e^q] \) such that (15) is satisfied for \( j = 2 \). Since \( \tau, \tau' \leq e^q \), (12) and (13) gives the desired inequality for \( V_1(\omega, q) \) and \( V_3(\omega, q) \).

By condition iii) above, \( \lambda(q) \leq \lambda(0)e^{\gamma q} \). Since \( e^{-q}(q) = \max\{3, \log \lambda(q)\} \), we obtain that
\[ \theta = \sup_{q \geq 1} e^{-\frac{q}{2}}(q) < \infty. \]

Let \( H_1 = 4\theta H \). By (9) and (15) we obtain for each \( \omega \in E_q \):
\[ \max_{z \in R_1} |F_{\omega}(z)| \leq \left( \frac{1}{e(q)} \sum_{j=1}^{4} V_j \right)^{1/q} \leq \left( \frac{4(H\lambda(q)v(e^{-q}))^q}{e(q)} \right)^{1/q} \leq H_1 \lambda(q)v(e^{-q}), \]
completing the proof of the Claim. \( \square \)
Claim 3.3. Let \( \sigma = \frac{1}{2} + \frac{1}{\log \psi(t)} \). Denote \( R = R(t) := [\sigma, \frac{3}{2}] \times [-t, t] \). Then for almost all \( \omega \in \Omega \) there exists a real number \( t_0 = t_0(\omega) \) such that for all \( t \geq t_0 \):

\[
\max_{z \in R(t)} |F_{\omega}(z)| \leq H_2(1 + \log \psi(t)) \text{.}
\]

where \( H_2 = H_2(\lambda, v, C) \).

Proof of the Claim. Claim 3.2 implies that

\[
\sum_{q=1}^{\infty} \mathbb{P}\left( \max_{z \in R(t)} |F_{\omega}(z)| \geq H_2(\lambda) \right) \leq \sum_{q=0}^{\infty} \frac{1}{2^q} = 2.
\]

The Borel-Cantelli Lemma gives a set \( \Omega' \) of \( \mathbb{P}(\Omega') = 1 \) such that for each \( \omega \in \Omega' \), there exists \( q_0(\omega) \in \mathbb{N} \), such that for the following inequality holds for all integers \( q \geq q_0 \):

\[
\max_{z \in R_1(q)} |F_{\omega}(z)| \leq H_2(\lambda) \|(q^{-1})\).
\]

For \( x \geq 0 \) denote \([x]\) the integer part of \( x \). Let \( t_0(\omega) = e^{q_0+10} \). For each \( t \geq t_0 \) let \( q(t) = 3 + [\log t] \). Since \( \log t \leq [\log t] + 1 \leq q - 2 \), we get that \( t \leq e^{q-2} \) and

\[
\psi(t) = \log \lambda(\log t) \leq \log(\log t) - \log(\log(\log t)) \leq \log(\log(\log t)) = e^{-1}(q).
\]

Hence \( R(t) \subset R_1(q) \). By (17),

\[
\max_{z \in R(t)} |F_{\omega}(z)| \leq \max_{z \in R_1(q)} |F_{\omega}(z)| \leq H_2(\lambda(3 + [\log t]) \|(\log(3 + [\log t]))).
\]

Observe that \( [\log t] \leq \log t \) and \( \lambda(3 + [\log t]) \leq e^{3\gamma} \log(t) = e^{3\gamma} \psi(t). \) Also, \( (\log \lambda(3 + [\log t])) \leq \psi(3\gamma + \log \psi(t)). \) Let \( a(t) = \frac{1}{3\gamma + \log \psi(t)} \) and \( b(t) = \frac{1}{\log \psi(t)} \). Then \( \lim_{t \to \infty} \frac{b(t)}{a(t)} = 1 \) and hence

\[
b(t) - a(t) \leq 3\gamma a(t)b(t) \ll a^2(t).
\]

By Lemma A.1, \( v(3\gamma + \log \psi(t)) = v(\psi(t)) + O(1) \). Hence there exists a constant \( D_1 = D_1(v) \) such that for all large \( t \), \( v(\log \lambda(3 + [\log t])) \leq D_1 v(\log \psi(t)) \). We complete the proof of the claim by choosing \( H_2 = e^{3\gamma} D_1 H_1 \).

End of the Proof of Lemma 3.1. Let \( \sigma(t) = \frac{1}{2} + \frac{1}{\log \psi(t)} \), \( 1/2 < x \leq 1 \) and \( \Omega' \) be as in claim 3.3. In the sequel, \( \omega \in \Omega' \) is fixed and \( t_1 = t_1(\omega) \) is a large number such that (16) holds for all \( t \geq t_1 \) and \( \sigma(t) < x \). Since \( |F_{\omega}(x-i)| = |F_{\omega}(x+i)| \), it is sufficient to prove Lemma 3.1 for \( t > t_1(\omega) \). Let \( \beta = \beta(t) = \log \psi(t) \) and \( C_1, C_2, C_3 \) be concentric circles with center \( \beta + it \) and passing through the points: \( \sigma + \frac{1}{2} + it, x + it \) and \( \sigma + it \) respectively. Thus, the respective radius of \( C_1, C_2, C_3 \) are:

\[
r_1 = \beta - \sigma - \frac{1}{2} \left( \log \psi(t) \right), 
\]

\[
r_2 = \beta - x, 
\]

\[
r_3 = \beta - \sigma.
\]

Denote \( M_j = M_j(t, \omega) = \max_{z \in C_j} |F_{\omega}(z)|, j = 1, 2, 3 \). Since \( F_{\omega} \) is analytic in \( \mathbb{H}_{1/2} \), the Hadamard Three-Circles Theorem states that

\[
M_2 \leq M_1^{1-a} M_3^a,
\]

where \( a = \frac{\log \psi(t)}{\log(\psi(t))} \). In the sequel we will estimate \( M_1, M_3 \) and \( a \) separately.

Estimative for \( M_1 \). Since for all \( k \in \mathbb{N} \), \( |X_k| \leq C \),

\[
M_1 \leq \sum_{k=1}^{\infty} \frac{|X_k|}{(1 + |\beta + i(t)|)^2} \leq C v(\log \psi(t))^2.
\]

Estimative for \( M_3 \). By condition iii), the function \( \lambda \) satisfies \( \lambda(c + d) \leq e^{\gamma} \lambda(d) \) for all \( c, d \geq 0 \). In particular, \( \psi(t) = \lambda(\log t) \leq \lambda(0) \psi(t) \). Hence \( \beta(t) = \log \psi(t) \leq \log(\lambda(0) \psi(t)) = \log(\lambda(0) + \gamma \log t) \). This gives \( \psi(1 + \beta(t)) = (1 + \beta(t))/t + \log(t) \) and hence:

\[
\psi(1 + \beta(t)) = \lambda(\log(1 + \beta(t))/t + \log(t)).
\]
In particular, we obtain that \( \psi(t + \beta(t)) \ll \psi(t) \). Also we get that
\[
v(\log \psi(t + \beta(t))) \leq v(\log(1 + \beta(t)/t + \beta(t))).
\]
Since \( \beta(t)/t = o(1), \log(1 + \beta(t)/t) \sim \beta(t)/t \). By Lemma 14 we obtain that \( v(\log(1 + \beta(t)/t + \beta(t))) = v(\beta(t)) + O(1) \). These estimates combined with 16 gives:
\[
(20) \quad M_3 \ll \max_{z \in \mathbb{R}(t+\beta)} |F_c(z)| \leq H_2 \psi(t + \beta(t))v(\log(1 + \beta(t))) \ll \psi(t)v(\log \psi(t)).
\]

**Estimative for \( a(t) \).** We claim that \( a(t) = 2 - 2x + O(\beta^{-1}(t)) \). Denote \( \tau = \beta^{-1} \). Observe that
\[
r_2 \frac{1}{r_1} = 1 + \frac{1 - x + \tau}{1 - \tau(\sigma - 1/2)}, \quad r_3 \frac{1}{r_1} = 1 + \frac{\tau/2}{1 - \tau(\sigma - 1/2)}.
\]
Using that for \( \varphi \) small, \( \log(1 + \varphi) = \varphi + O(\varphi^2) \) and that \( \frac{1}{\tau}\log(1 + \gamma) = \frac{1}{\tau} + O(1) \), we obtain:
\[
a = \left( \frac{1 - x + \tau}{1 - \tau(\sigma - 1/2)} + O(\tau^2) \right) \left( \frac{2(1 - \tau(\sigma - 1/2))}{\tau} + O(1) \right) = 2(1 - x) + 2\tau + O(\tau) + O(\tau^2) = 2 - 2x + O(\tau).
\]

**Estimative for \( M_1^{1-a} \).** Let \( \tau = \frac{1}{\log v(t)} \). First observe that \( v(\epsilon^{-1})^2 \leq \zeta(1 + \epsilon) \sim \epsilon^{-1} \). Hence
\[
v(\log \psi(t)) O(\tau) = \exp \left( \log(v(\log \psi(t))) \cdot O(\tau) \right) = \exp(\log(\log \psi(t)) \cdot O(\tau)) = O(1).
\]
Recalling (19), we obtain \( M_1^{1-a} \ll v(\log \psi(t))^{1\tau - 22} \).

**Estimative for \( M_3^a \).** Since \( \psi(t) O(\tau) = O(1) \) and \( v(\log \psi(t)) O(\tau) = O(1) \), we obtain
\[
M_3^a \ll \psi(t)^{2-2x} v(\log \psi(t))^{1-2x}.
\]

**Estimative for \( F_\omega(x + it) \).** Observe that \( F_\omega(x + it) \leq M_2(t, \omega) \). Collecting the estimates above, by (18) we get
\[
M_2 \ll \psi(t)^{2-2x} v(\log \psi(t))^{2x},
\]
completing the proof. \( \square \)

**Proof of Theorem 3.2.** Assume that \( \{X_k\}_{k \in \mathbb{N}} \) satisfies condition \( i \) above. If this random variables are independent then by Proposition 2.1 it also satisfies condition \( ii \). The condition \( iii \) with \( \lambda(q) = C\sqrt{q + 1} \) for some constant \( C > 0 \) is the Marcinkiewicz-Zygmund inequality for independent random variables (see [6] pg. 366). Hence \( \psi(t) \ll \sqrt{\log t} \).

If \( \{X_k\}_{k \in \mathbb{N}} \) is a martingale difference that satisfies \( i \) above, then, \( X_1 = M_1 - M_0 \) and \( X_k = M_k - M_{k-1} \) where \( (M_n, \mathcal{F}_n)_{n \geq 0} \) is a martingale with bounded increments. Hence for any sequence of real numbers \( \{a_k\}_{k \in \mathbb{N}}, S_n := \sum_{k=1}^n a_k X_k \) also is a martingale with same filtration \( \{\mathcal{F}_n\}_{n \in \mathbb{N}} \). The condition \( iii \) with \( \lambda(q) = C(q + 1) \) for some \( C > 0 \) is the Burkhold inequality applied for \( S_n \) (see 17 pg. 499). Hence \( \psi(t) \ll \log t \). Let \( S_n(\epsilon) := \sum_{k=1}^n \frac{X_k}{\epsilon} \). For \( q = 2 \), the Burkhold inequality applied for \( S_n(\epsilon) \) gives that \( \mathbb{E}[S_n(\epsilon)]^2 \leq DA(2)c(1 + \epsilon) \) and hence that \( \sup_{n \in \mathbb{N}} \mathbb{E}[S_n(\epsilon)]^2 < \infty \). By Doob’s martingale convergence Theorem (see 17 pg. 510) we obtain the almost sure convergence of \( S_n(\epsilon) \) and hence the almost sure convergence of the Dirichlet series \( \sum_{k=1}^\infty \frac{X_k}{\epsilon^2} \) for each \( \epsilon > 0 \). The referred properties for the convergence of Dirichlet series stated in the proof of Proposition 2.1 gives that \( \{X_k\}_{k \in \mathbb{K}} \) satisfy Theorem 3.1 condition \( ii \).

Given a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) let \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) be sub-sigma algebras of \( \mathcal{F} \). For \( j = 1, 2 \), denote
\[
L^2(\mathcal{F}_j) = \{ X : \Omega \rightarrow \mathbb{R} : \mathbb{E}[|X|^2] < \infty \text{ and } X \text{ is } \mathcal{F}_j \text{- measurable} \} \text{ and } \|X\|_2 = \sqrt{\mathbb{E}[X^2]}.
\]

Let \( \rho(\mathcal{F}_1, \mathcal{F}_2) = \sup\{\mathbb{V}f_1f_2/(\|f_1\|_2\|f_2\|_2) : f_1 \in L^2(\mathcal{F}_1) \text{ and } f_2 \in L^2(\mathcal{F}_2)\}. \)
Let \( \{X_k\}_{k \in \mathbb{N}} \) be a sequence of random variables and for \( S \subset \mathbb{N} \), let \( \mathcal{F}_S \) be the sigma algebra generated by the random variables \( \{X_k\}_{k \in S} \). Define

\[
\rho^*(n) = \sup \{ \rho(\mathcal{F}_S, \mathcal{F}_T) : S, T \subset \mathbb{N} \text{ and } \min_{s \in S, t \in T} |s-t| \geq n \}.
\]

One says that the sequence \( \{X_k\}_{k \in \mathbb{N}} \) is \( \rho^* \)-mixing if \( \lim_{n \to \infty} \rho^*(n) = 0 \) (see [2] pg. 114). In particular, if \( \{X_k\}_{k \in \mathbb{N}} \) is \( \rho^* \)-mixing, then there exists \( n \in \mathbb{N} \) such that \( \rho^*(n) < 1 \). In [3], W.Bryc proved a result which implies the following: If \( \{X_k\}_{k \in \mathbb{N}} \) are centered and uniformly bounded random variables with \( \rho^*(n) < 1 \) for some large \( n \), then condition ii) is satisfied. In his proof (see Lemma 1 and 2, and Remark 4 of [3]), W.Bryc showed that condition iii) for \( q \geq 2 \) is satisfied with \( \lambda(q) = \frac{1}{1-\rho^2} \sqrt{q+1} \sim C_2(q+1)^{3/2} \), where \( \rho = \rho^*(n) \). Hence \( \psi(t) \ll (\log t)^{3/2} \).

Let \( \zeta \) be the Riemann zeta function. We recall that \( \zeta(1+\epsilon) \sim \epsilon^{-1} \). Hence \( \psi(e^{-1})^2 \ll \epsilon^{-1} \). On the other hand, if \( X_k = 0 \) for all non prime \( k \), a well known fact is that, in this case \( \psi(e^{-1}) = \log(\epsilon)^{3/2} \).

Hence for large \( t \) \( \psi(\log \psi(t)) \ll \log \log t \), and if \( X_k = 0 \) for all non prime \( k \), \( \psi(\log \psi(t))^2 \ll \log \log t \).

\[ \square \]

4. Proofs of the main results

4.1. (Theorem 1.1). Let \( R_1 \subset R_2 \) be open connected sets of \( \mathbb{C} \). An analytic function \( h : R_1 \to \mathbb{C} \) has analytic extension to \( R_2 \) if there exists an analytic function \( \tilde{h} : R_2 \to \mathbb{C} \) such that \( h(z) = \tilde{h}(z) \) for all \( z \in R_1 \). We say that a random analytic function \( \tilde{h} : R_1 \times \Omega \to \mathbb{C} \) has analytic extension to \( R_2 \) if the set of elements \( \omega \in \Omega \) for which \( \tilde{h}_\omega \) has analytic extension to \( R_2 \) contains a set \( \Omega^* \in \mathcal{F} \) such that \( \mathbb{P}(\Omega^*) = 1 \).

Proof of Theorem 1.2. Let \( \alpha \in (0, 1/2) \) and \( f_\alpha \) be the random multiplicative function such that \( \mathbb{E}(f(p)) = \frac{g(k)}{k^\alpha} \) for each prime \( p \). Denote \( F_\alpha(z) := \sum_{k=1}^{\infty} \frac{f_\alpha(k)}{k^z} \).

Claim 4.1. The half plane \( \mathbb{H}_{1/2+\alpha} \) is a zero free region for \( \zeta \) if and only if \( F_\alpha \) has analytic extension to \( \mathbb{H}_{1/2} \).

Proof of the claim. Since \( \{f(p)\}_{p \in \mathbb{P}} \) is a sequence of independent random variables, \( \mathbb{E}(f(p)) = \frac{g(k)}{k^\alpha} \). For \( z \in \mathbb{H}_1 \) we obtain that \( \mathbb{E}F_\alpha(z) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k^{\sigma+\alpha}} = \frac{1}{\zeta(\sigma+\alpha)} \). By Lemma 2.1 ii) there exists a random analytic function \( \theta : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) such that \( F_\alpha(z) = \theta(z) \frac{1}{\zeta(\sigma+\alpha)} \). By iii) of this Lemma, \( \frac{1}{\zeta(z+\alpha)} \) is analytic analytic function. Hence \( \frac{1}{\zeta(z+\alpha)} \) is analytic in \( \mathbb{H}_{1/2} \) if and only if \( F_\alpha \) has analytic extension to \( \mathbb{H}_{1/2} \). Since \( \zeta \) is analytic in \( \mathbb{C} \setminus \{1\} \) with a simple pole in \( z = 1 \), we obtain that \( \zeta \) has no zeros in the half plane \( \mathbb{H}_{1/2+\alpha} \) if and only if \( \frac{1}{\zeta(z+\alpha)} \) is analytic in \( \mathbb{H}_{1/2} \), completing the proof of the claim.

Assume \( M_{f_\alpha}(x) = o(x^{1/2+\epsilon}) \) for all \( \epsilon > 0 \) a.s.. By partial summation (Lemma A.1), the series \( F_\alpha : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) is a random analytic function. By claim 4.1 we conclude that \( \zeta \) has no zeros in the half plane \( \mathbb{H}_{1/2+\alpha} \). Thus if for each \( \alpha > 0 \), \( M_{f_\alpha}(x) = o(x^{1/2+\epsilon}) \forall \epsilon > 0 \) a.s., then \( \zeta \) has no zeros in \( \mathbb{H}_{1/2} \).

Assume RH. In [15] J.E.Littlewood proved, for fixed \( \sigma > 1/2 \), that RH implies that \( \frac{1}{\zeta(\sigma+it)} = o(t^\delta) \) for all \( \delta > 0 \). By Theorem 3.1 for fixed \( 1/2 < \sigma \leq 1 \) we have \( \theta(\sigma+it) \ll \exp(\log(t)^{1-\sigma} \log \log \log t) = o(t^\delta) \) for all \( \delta > 0 \) a.s. By claim 4.1 \( F_\alpha \) has analytic extension to \( \mathbb{H}_{1/2} \) given by \( \frac{d(\zeta)}{\zeta(\sigma+it)} \). Hence \( F_\alpha(\sigma + it) \ll t^\delta \) for all \( \delta > 0 \) a.s. We recall the following result from the theory of the Dirichlet series: Assume that \( G(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^z} \) converges absolutely \( \forall z \in \mathbb{H}_1 \), and that for some \( c < 1 \), \( G \) has analytic extension to \( \mathbb{H}_c \) given by \( \tilde{G} \). If \( \tilde{G}(\sigma+it) = o(t^\delta) \) for all \( \delta > 0 \), then \( M_{f_\alpha}(x) = o(x^{1/2+\epsilon}) \) for all \( \epsilon > 0 \) (see [18] page 134, Theorem 4). This result applied for \( F_\alpha \) completes the proof. \[ \square \]
4.2. (Theorems 1.3 and 1.5).

Proof of Theorem 1.3. Let \( \mathbb{E}f(p) = -\delta_p \) where \( 0 < \delta_p \leq 1 \). Since \( |f(p)| \leq 1 \forall p \in \mathcal{P} \), by the Kolmogorov two series Theorem, \( f \) weakly biased implies that \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p} \) converges. On the other hand, for \( \alpha \in (0, 1/2) \), the convergence of \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^\alpha} \) implies that \( \sum_{p \in \mathcal{P}} \frac{f(p)}{p^\alpha} \) converges a.s. Thus, we only need to prove that, if \( f \) is weakly biased and

\[
P(M_f(x) = o(x^{1-\epsilon}) \text{ for some } \epsilon > 0) = 1,
\]

then there exists \( \alpha > 0 \) such that the series \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^{1-\alpha}} \) converges.

Let \( F : \mathbb{H}_1 \times \Omega \to \mathbb{C}, v : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) and \( u : \mathbb{H}_1 \to \mathbb{C} \) be given by:

\[
F(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^2},
\]

\[
v(z) = \sum_{p \in \mathcal{P}} \frac{f(p) - \mathbb{E}f(p)}{p^\alpha},
\]

\[
u(z) = \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^\alpha}.
\]

By Proposition 2.1 \( v \) is a random analytic function and \( u \) is analytic. By Lemma 2.1 and claim 2.4 there exists a random analytic function \( w : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) such that for each \( z \in \mathbb{H}_1 \)

\[
F(z) = \exp(v(z) + w(z) - u(z)).
\]

Since the series \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^\alpha} \) converges, we obtain that \( \lim_{x \to 1^+} u(x) = \sum_{p \in \mathcal{P}} \frac{\delta_p}{p} < \infty \). This combined with (22) implies that \( \lim_{x \to 1^+} F(x) = 0 \), a.s. By (21) there is a set \( \Omega^* \) with \( \mathbb{P}(\Omega^*) = 1 \) such that for each \( \omega \in \Omega^* \) there exists \( \epsilon = \epsilon(\omega) > 0 \) for which \( M_{f_\omega}(x) = o(x^{1-\epsilon}) \). Hence, if \( \omega \) is a fixed element of \( \Omega^* \), Lemma 1.1 implies that the series \( F_\omega(z) = \sum_{n=1}^{\infty} \frac{f(n)}{n^2} \) converges for each \( z \in \mathbb{H}_{1-\epsilon} \) and it is an analytic function in this half plane. Thus for \( \mathbb{P} \)-almost all \( \omega \in \Omega \) we obtain an \( \epsilon = \epsilon(\omega) > 0 \) such that \( F_\omega(z) \) is analytic in \( \mathbb{H}_{1-\epsilon} \) and satisfies \( F_\omega(1) \neq 0 \). In particular for each of these \( \omega \), there exists an open ball \( B = B(\omega) \subset \mathbb{H}_{1-\epsilon} \), with positive radius and centered at \( z = 1 \) such that \( F_\omega(z) \neq 0 \) for all \( z \in \mathbb{H}_1 \cup B \). Since this random subset of \( \mathbb{C} \) is a simply connected region, \( F_\omega \) has a branch of the logarithm \( r_\omega : \mathbb{H}_1 \cup B \to \mathbb{C} \) (see [7], pg. 94-95, Corollary 6.17), i.e., \( r_\omega \) is analytic and satisfies \( F_\omega(z) = \exp(r_\omega(z)) \) for all \( z \in \mathbb{H}_1 \cup B \). This combined with (22) gives for \( \mathbb{P} \)-almost all \( \omega \) and all \( z \in \mathbb{H}_1 \)

\[
(23) \quad \lambda(z) := \exp(u(z)) = \exp(v_\omega(z) + w_\omega(z) - r_\omega(z)).
\]

In particular \( \lambda_\omega(z) := v_\omega(z) + w_\omega(z) - r_\omega(z) \) is analytic in \( \mathbb{H}_1 \cup B \) and hence it is, a.s., a branch of the logarithm for the analytic function \( \lambda : \mathbb{H}_1 \to \mathbb{C} \). A classical result from complex analysis states that there exists an integer \( k = k(\omega) \) such that for all \( z \in \mathbb{H}_1 \) and almost all \( \omega, u(z) - \lambda_\omega(z) = 2k\pi i \). That is, \( \bar{u}_\omega : \mathbb{H}_1 \cup B \to \mathbb{C} \) given by \( \bar{u}_\omega(z) = \lambda_\omega(z) + 2k\pi i \) extends \( u \) analytically to \( \mathbb{H}_1 \cup B \). Since for \( z \in \mathbb{H}_1 \), \( \bar{u}_\omega(z) = u(z) = \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^\alpha} \) is a Dirichlet series of non-negative terms that it is analytic in an open disk centered at \( z = 1 \), a classical result concerning Dirichlet series of this type (see [1] pg. 237, Theorem 11.13) implies that there is \( \alpha > 0 \) for which the series \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^{1-\alpha}} \) converges.

Proof of Theorem 1.5. We begin the proof with the following claim:

Claim 4.2. Let \( 0 < \alpha < 1/2 \). Assume that \( \mathbb{E}X_p = -\frac{\delta_p}{p} \) where \( 0 \leq \delta_p \leq 1 \), \( \limsup \delta_p = \delta < 1 \), and \( \sum_{p \in \mathcal{P}} \frac{\delta_p}{p} = \infty \). Then \( M_f(x) \) is not \( o(x^{1-\alpha-\epsilon}) \) for any \( \epsilon > 0 \), a.s.

Proof of the claim. Let \( 0 < \alpha < 1/2 \). By Lemma 2.1 there is a random analytic function \( \theta : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) such that for all \( z \in \mathbb{H}_1 \) and all \( \omega \in \Omega \)

\[
F(z) = \theta(z)\mathbb{E}F(z),
\]
where $F(z) = \sum_{k=1}^{\infty} \frac{f(k)}{k^z}$. Moreover, since $E f(p) = -\delta_p/p^\alpha$, claim 2.1 gives

$$E F(z) = \exp \left( - \sum_{p \in \mathcal{P}} \frac{\delta_p}{p^z + \alpha} \right) \exp(A(z)) \quad (z \in \mathbb{H}_1),$$

where $A : \mathbb{H}_{1/2} \to \mathbb{C}$ is analytic. Since the series $\sum_{p \in \mathcal{P}} \frac{\delta_p}{p^z + \alpha}$ converges absolutely for $z \in \mathbb{H}_{1-\alpha}$ we obtain that the function in the right side of (24) is analytic in $\mathbb{H}_{1-\alpha}$ and hence extends analytically $E F(z)$ to this half plane. Let $z \in \mathbb{H}_1$ and $\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z}$ be the Riemann zeta function. A direct application of claim 2.1 gives that

$$\zeta(z) := \exp \left( \sum_{p \in \mathcal{P}} \frac{1}{p^z} \right) \exp(B(z)),$$

where $B : \mathbb{H}_{1/2} \to \mathbb{C}$ is analytic. By combining (24) and (25):

$$\zeta(1+\epsilon)E F(1-\alpha + \epsilon) = \exp \left( \sum_{p \in \mathcal{P}} \frac{1 - \delta_p}{p^{1+\epsilon}} \right) \exp(A(1-\alpha + \epsilon) + B(1+\epsilon)).$$

Since $\limsup \delta_p = \delta < 1$ there exists $\eta > 0$ such that $1 - \delta_p \geq \eta$ for all $p$ sufficiently large. Hence

$$\sum_{p \in \mathcal{P}} \frac{1 - \delta_p}{p} = \infty.$$

This combined with (26) implies

$$\lim_{\epsilon \to 0^-} \zeta(1+\epsilon)E F(1-\alpha + \epsilon) = \infty.$$

On the other hand hypothesis $\sum_{p \in \mathcal{P}} \frac{\delta_p}{p} = \infty$ combined with (24) gives that

$$\lim_{\epsilon \to 0} E F(1-\alpha + \epsilon) = 0.$$

Hence if we assume

$$\mathbb{P}(M_f(x) = o(x^{1-\alpha-\epsilon}) \text{ for some } \epsilon > 0) = 1,$$

by Lemma A.1 we obtain for almost all $\omega \in \Omega$ an $\epsilon = \epsilon(\omega) > 0$ such that $F_\omega(z) = \sum_{k=1}^{\infty} \frac{f(k)}{k^z}$ is analytic in $\mathbb{H}_{1-\alpha-\epsilon}$. Since $E F(z) = \zeta(z) F_\omega(z)$, $E F(z)$ is analytic in a open neighborhood of $z = 1 - \alpha$. By (28), $E F(1-\alpha) = 0$ while (27) gives that this can not be an zero of an analytic function, since the Riemann zeta function has a simple pole at $z = 1$. This gives a contradiction which implies that $E F(z)$ is not analytic in $z = 1 - \alpha$, and hence that

$$\mathbb{P}(M_f(x) = o(x^{1-\alpha-\epsilon}) \text{ for some } \epsilon > 0) < 1.$$

A direct application of Corollary A.1 implies that this probability is zero.

\begin{remark}(Uniform coupling). In the sequel $(\Omega, \mathcal{F}, \mathbb{P})$ is the probability space where $\Omega$ is the set of the sequences $\omega = (\omega_p)_{p \in \mathcal{P}}$ such that $\omega_p \in [0, 1]$ for each prime $p$, $\mathcal{F}$ is the Borel sigma-algebra of $\Omega$ and $\mathbb{P}$ is the Lebesgue product measure in $\mathcal{F}$. For a random multiplicative function $f : \mathbb{N} \times \Omega \to \{-1, 0, 1\}$, we will consider that for each prime $p$, $f(p) : \Omega \to \{-1, 1\}$ is a random variable given by

$$f_\omega(p) = \mathbb{I}_{\{a_p,1\}}(\omega_p) - \mathbb{I}_{\{0,a_p\}}(\omega_p),$$

where $a_p := \mathbb{P}(f(p) = -1)$. If $f(p)$ and $g(p)$ are random variables given by (29) then

$$|\mathbb{P}(f(p) \neq g(p))| = |\mathbb{P}(f(p) = -1) - \mathbb{P}(g(p) = -1)|.$$

Let $\alpha > 0$ and assume that $\{\delta_p\}_{p \in \mathcal{P}}$ is such that $0 \leq \delta_p \leq 1 \forall p \in \mathcal{P}$. Let $f$ be a random multiplicative function such that for each prime $p$, $\{f(p)\}_{p \in \mathbb{N}}$ is given by (29) with $a_p = \frac{1}{2} + \frac{\delta_p}{2p^\alpha}$ and hence $E f(p) = -\delta_p/2p^\alpha$. Let $u, h : \mathbb{N} \times \Omega \to \{-1, 0, 1\}$ be random functions that satisfy the multiplicative property (4), and such that for each prime $p$, $u(p) = Z_p$ and $h(p) = W_p$ where

$$Z_p(\omega) := \mathbb{I}_{[0, \frac{1}{2} - \frac{\delta_p}{2p^\alpha}]}(\omega_p) + \mathbb{I}_{(\frac{1}{2} + \frac{\delta_p}{2p^\alpha}, 1]}(\omega_p),$$

$$W_p(\omega) := \mathbb{I}_{[\frac{1}{2} - \frac{\delta_p}{2p^\alpha}, \frac{1}{2} + \frac{\delta_p}{2p^\alpha}]}(\omega_p).$$
Claim 4.3. Let $\gamma = \max\{1/2, 1 - \alpha\}$. Then $\sum_{n=1}^{\infty} \frac{u(n)}{n^{\gamma+\epsilon}}$ and $\sum_{n=1}^{\infty} \frac{|h(n)|}{n^{\gamma+\epsilon}}$ converges $\forall \epsilon > 0$ a.s.

Proof of the claim. The Rademacher-Menshov Theorem [16] states that if $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of orthogonal random variables such that the series $\sum_{n=1}^{\infty} \log^2(n+1)EX_n^2$ converges and $EX_n = 0$ for all $n$, then the random series $\sum_{n=1}^{\infty} X_n$ converges a.s. If $k$ and $l$ are distinct squarefree integers, there are at least one prime $p$ such that either $p|k$ or $p|l$ while $p$ do not divide $\gcd(k, l)$, and hence $E(u(k)u(l)) = 0$. Since $|u(n)| \leq 1 \\forall n \in \mathbb{N}$, by the Rademacher-Menshov Theorem $\sum_{n=1}^{\infty} \frac{u(n)}{n^{\gamma+\epsilon}}$ converges $\forall \epsilon > 0$ a.s.

By the Kolmogorov two series Theorem, $\sum_{p \in \mathbb{P}} \frac{|h(p)|}{p^{\gamma+\epsilon}}$ converges $\forall \epsilon > 0$ a.s. since

$$\sum_{p \in \mathbb{P}} \frac{\mathbb{E}[h(p)]}{p^{\gamma+\epsilon}} \leq \sum_{p \in \mathbb{P}} \frac{1}{p^{\gamma+\epsilon+\epsilon}} < \infty,$$

$$\sum_{p \in \mathbb{P}} \frac{\mathbb{V}[h(p)]}{p^{2\gamma+2\epsilon}} \leq \sum_{p \in \mathbb{P}} \frac{1}{p^{1+2\epsilon}} < \infty.$$

We recall a classical result for a Dirichlet series of an multiplicative function $\phi : \mathbb{N} \rightarrow [-1, 1]$ which states that: If for each $p \in \mathbb{P}$, $\phi(p^m) = 0$ for all $m \geq 2$, then for each $\sigma > 0$ the series $\sum_{n=1}^{\infty} \frac{|\phi(n)|}{n^{\sigma}}$ converges if and only if the series $\sum_{p \in \mathbb{P}} \frac{\phi(p)}{p^\sigma}$ converges (see [18], pg. 106 Theorem 2). A direct application of this result for $h$ completes the proof of the claim. \hfill $\square$

The Dirichlet convolution between $u$ and $h$, denoted by $u * h$ is is given by: $(u * h)(n) := \sum_{d|n} u(d)h(n/d)$. Since for each prime $p$, $f(p) = u(p) + h(p)$ and $u(p) \cdot h(p) = 0$, we obtain that

$$u * h(p) = u(p) + h(p) = f(p),$$

$$u * h(p^m) = \sum_{k=0}^{n} u(p^k)h(p^{n-k}) = 0 \quad (n \geq 2).$$

This implies that for each prime $p$, $f(p^m) = u * h(p^m) \forall m \in \mathbb{N}$. Since the convolution between two multiplicative functions results in a multiplicative function, we conclude that $f = u * h$. A result for Dirichlet series states that if $\sum_{k=1}^{\infty} \frac{h(k)}{k^\gamma}$ converges absolutely and if $\sum_{k=1}^{\infty} \frac{u(k)}{k^\gamma}$ converges then $\sum_{k=1}^{\infty} \frac{(u+h)(k)}{k^\gamma}$ also converges (see [18], pg. 122, Notes 1.1). This combined with claim 4.3 implies that $\sum_{k=1}^{\infty} \frac{f(k)}{k^\gamma}$ converges a.s. A direct application of Kronecker’s Lemma gives that $M_f(x) = o(x^{\gamma+\epsilon})$ for all $\epsilon > 0$ a.s., completing the proof of Theorem 1.3 \hfill $\square$

4.3. (Theorems 1.2 and 1.4).

Remark 4.2. Let $0 = a_1 < a_2 < \ldots < a_{n+1} = 1$ and $I_k = [a_k, a_{k+1})$ if $1 \leq k < n$ and $I_n = [a_n, a_{n+1}]$. Let $\psi : [0, 1] \rightarrow [0, 1]$ be a bijection such that for all $1 \leq k \leq n$, $\psi : I_k \rightarrow [0, 1]$ is a translation. Then $\psi$ is called interval exchange transformation (see [20]). If $m$ denotes the Lebesgue measure on $[0, 1]$, then for each Borelian $B \subset [0, 1]$, $m(\psi^{-1}(B)) = m(B)$, i.e., an interval exchange preserves the Lebesgue measure. If $(\Omega, F, \mathbb{P})$ is the probability space introduced in the remark 4.1 and if $\psi_p : [0, 1] \rightarrow [0, 1]$ is an interval exchange transformation for all $p \in \mathbb{P}$ then $T : \Omega \rightarrow \Omega$ given by

$$T(\omega_2, \omega_3, \omega_5, \ldots) = (\psi_2(\omega_2), \psi_3(\omega_3), \psi_5(\omega_5), \ldots)$$

preserves $\mathbb{P}$, i.e., for each $B \in F$, $\mathbb{P}(T^{-1}(B)) = \mathbb{P}(B)$.

We say that a random multiplicative function $g$ supported on the squarefree integers is biased towards $|\mu|$ if $Eg(p) > 0 \forall p \in \mathbb{P}$. In the sequel, we will use the advantage of the probability space (uniform coupling) introduced in the remark 4.1 where it is defined the measure preserving transformation $T$ introduced in the remark 4.2. This will enable us to transport some properties of a biased $g$ towards $|\mu|$ to a strongly biased random multiplicative function towards $\mu$. 
Lemma 4.1. Let \( g \) be a random multiplicative function biased to \( |\mu| \). Let \( z \in \mathbb{H}_1 \) and \( G(z) := \sum_{k=1}^{\infty} \frac{g(k)}{k^z} \). If for some \( 0 < \alpha < 1/2 \) there exists a random analytic function \( G : \mathbb{H}_{1-\alpha} \times \Omega \to \mathbb{C} \) such that for all \( z \in \mathbb{H}_1 \), \( G(z) = G(z) \), then \( \sum_{p \in \mathcal{P}} \frac{g(p)}{p^{1-\alpha}} \) converges \( \forall \epsilon > 0 \) a.s.

Proof. Since the random variables \( \{g(p)\}_{p \in \mathcal{P}} \) are independent, \( k \in \mathbb{N} \mapsto E(g(k)) \) is multiplicative, supported on the square free integers and non-negative. In particular, \( E(G(z)) \) is a Dirichlet series of non-negative terms. By Lemma 2.1, for all \( z \in \mathbb{H}_1 \) there exists a non-vanishing random analytic function \( \theta : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) such that for all \( z \in \mathbb{H}_1 \), \( G(z) = E(G(z)) \theta(z) \). In particular \( \frac{1}{\theta} \) also is a random analytic function. Hence \( \Lambda : \mathbb{H}_{1-\alpha} \times \Omega \to \mathbb{C} \) given by

\[
\Lambda(z) = \frac{G(z)}{\theta(z)}
\]
is a random analytic function and satisfies \( \Lambda(z) = E(G(z)) \) for all \( z \in \mathbb{H}_1 \). In particular there exists \( \omega \in \Omega \) such that \( \Lambda_\omega : \mathbb{H}_{1-\alpha} \to \mathbb{C} \) is analytic and \( \Lambda_\omega(z) = E(G(z)) \) for all \( z \in \mathbb{H}_1 \). We recall that if a Dirichlet series of non-negative terms has analytic extension to the half plane \( \mathbb{H}_{1-\alpha} \), then actually this series converges for all \( z \) in this half plane. Hence \( \sum_{k=1}^{\infty} \frac{E(g(k))}{k^{1-\alpha}} \) converges for every \( z \in \mathbb{H}_{1-\alpha} \).

Since \( k \in \mathbb{N} \mapsto E(g(k)) \) is multiplicative and non-negative, the series \( \sum_{p \in \mathcal{P}} \frac{E(g(p))}{p^{1-\alpha}} \) converges for all \( z \in \mathbb{H}_{1-\alpha} \) (see [TS] pg. 106, Theorem 2 and remark (a)). By Proposition 2.1 we obtain that \( \sum_{p \in \mathcal{P}} \frac{g(p)}{p^{1-\alpha}} \) converges \( \forall \epsilon > 0 \) a.s. \( \square \)

Proof of Theorem 4.4. Let \( u \) and \( g \) be random multiplicative function such that \( u(p) \) and \( g(p) \) are given by [29] with

\[
P(u(p) = -1) = 1/2,
\]

\[
P(g(p) = -1) = 1/2 - P(f(p) = 1),
\]

respectively. Hence \( u \) is unbiased and

\[
E(g(p)) = 2P(f(p) = 1) = 1 + E(f(p)) > 0.
\]

Denote \( F(z) := \sum_{k=1}^{\infty} \frac{f(k)}{k^z} \), \( U(z) := \sum_{k=1}^{\infty} \frac{u(k)}{k^z} \) and \( G(z) := \sum_{k=1}^{\infty} \frac{g(k)}{k^z} \). Let \( \varphi : \mathbb{H}_1 \times \Omega \to \mathbb{C} \) and \( \psi : \mathbb{H}_1 \times \Omega \to \mathbb{C} \) be the random analytic functions

\[
\varphi(z) = \zeta(z) F(z) \text{ and } \psi(z) = U^{-1}(z) G(z),
\]

where \( U^{-1}(z) = \frac{1}{\psi(z)} \). For a random analytic function \( \lambda : \mathbb{H}_1 \times \Omega \to \mathbb{C} \) denote

\[
A_\lambda := \{ \omega \in \Omega : \lambda_\omega \text{ has analytic extension to \( \mathbb{H}_{1-\alpha} \) } \}.
\]

By Proposition 4.2, if \( \lambda_\omega : \mathbb{H}_1 \to \mathbb{C} \) is analytic for all \( \omega \in \Omega \), then \( A_\lambda \) is measurable. In particular the events \( A_F, A_G, A_U, A_U^{-1}, A_\varphi, \) and \( A_\psi \) are measurable.

Claim 4.4. \( P(A_\psi) = P(A_G) \) and under the hypothesis of Theorem 4.4 \( P(A_\varphi) = 1 \).

Proof of the claim. Observe that \( E_u(k) = 0 \) if \( k > 1 \) and \( E_u(1) = 1 \), hence \( EU(z) = 1 \). By Lemma 2.1 \( P(A_U, A_U^{-1}) = 1 \). Since \( \psi(z) = U^{-1}(z)G(z) \) and \( G(z) = \psi(z)U(z) \):

\[
P(A_G) = P(A_G \cap A_U^{-1}) \leq P(A_\psi),
\]

\[
P(A_\psi) = P(A_\psi \cap A_U) \leq P(A_G).
\]

Hence \( P(A_G) = P(A_\psi) \), completing the first statement of the claim. By hypothesis we have that \( M_f(x) = o(x^{1-\alpha}) \) a.s. A direct application of Lemma 4.1 implies that \( P(A_\psi) = 1 \). In addition, hypothesis \( E(f(p)) < 0 \) and \( \sum_{p \in \mathcal{P}} \frac{E(f(p))}{p^{1-\alpha}} = -\infty \) implies that \( \lim_{\epsilon \to 0} E(f(1 + \epsilon)) = 0 \). By applying Lemma 2.1 iii) to \( F \) we obtain that \( P(A_F \cap [F(1) = 0]) = 1 \). If \( \omega \in A_F \cap [F(1) = 0] \), then \( F_\omega : \mathbb{H}_{1-\alpha} \to \mathbb{C} \) is analytic and \( F_\omega(1) = 0 \), hence there exists an integer \( m = m(\omega) \geq 1 \) such that \( F_\omega(z) \) is analytic in \( \mathbb{H}_{1-\alpha} \) (see [4] pg. 79, Corollary 3.9). Thus \( A_F \cap [F(1) = 0] \subset A_\varphi \) since, the Riemann zeta function extends analytically to \( \mathbb{C} \setminus \{1\} \) with a simple pole at \( z = 1 \), and hence
Assume \( M(32) \) all \( \sigma > 0 \).

Kolmogorov two series Theorem, \( \sum \to a.s. \) this simple pole cancel a.s. with the zero at \( z = 1 \) of the random analytic function \( F \). Hence, \( \mathbb{P}(A_\sigma) = 1 \), completing the proof of the claim.

Let \( I_p \) and \( J_p \) be the intervals

\[
I_p := \left( \frac{1}{2} - \mathbb{P}(f(p) = 1), \frac{1}{2} \right),
\]

\[
J_p := \left( 1 - \mathbb{P}(f(p) = 1), 1 \right).
\]

Observe that \( I_p \) and \( J_p \) have the same Lebesgue measure. For each \( p \in \mathcal{P} \) let \( \psi_p : [0, 1] \to [0, 1] \) be the interval exchange transformation that exchanges only \( J_p \) and \( I_p \) such that \( \psi_p(I_p) = J_p \) and \( \psi(J_p) = I_p \). Let \( T : \Omega \to \Omega \) be the measure preserving transformation as in the remark \[1.2 \] and \( \omega^* = T(\omega) \). We claim that for each \( \omega \in \Omega \), \( \varphi_\omega = \psi_\omega^* \).

Indeed the Euler product representation (see [18] pg. 106) for \( F, U, G \) and \( \zeta \) allow us to deduce the functional equations which holds for all \( z \in \mathbb{H}_1 \):

\[
\varphi_\omega(z) = \prod_{p \in \mathcal{P}} p^z + \mathbb{I}_{I_p}(\omega_p),
\]

\[
\psi_\omega(z) = \prod_{p \in \mathcal{P}} p^z + \mathbb{I}_{J_p}(\omega_p).
\]

Since \( \mathbb{I}_{I_p}(\psi_\omega(\omega_p))) = \mathbb{I}_{J_p}(\omega_p) \) we obtain that \( \varphi_\omega(z) = \psi_\omega^*(z) \) for all \( z \in \mathbb{H}_1 \) and hence that \( T^{-1}(A_\sigma) = A_\varphi \). This combined with claim \[4.4 \] implies that

\[
\mathbb{P}(A_G) = \mathbb{P}(A_\sigma) = \mathbb{P}(T^{-1}(A_\psi)) = \mathbb{P}(A_\varphi) = 1.
\]

By Lemma \[4.1 \] we then conclude that \( \sum_{p \in \mathcal{P}} \frac{g(p)}{p^{(\log p)^{1-\sigma}}} \) converges \( \forall \epsilon > 0 \). Since \( |g(p)| \leq 1 \), By Kolmogorov two series Theorem, \( \sum_{p \in \mathcal{P}} \frac{Eg(p)}{p^{(\log p)^{1-\sigma}}} \) converges \( \forall \epsilon > 0 \). Since \( E g(p) = 1 + E f(p) \) we then conclude, by Proposition \[2.1 \] the proof of Theorem \[1.4 \].

**Proof of Theorem \[1.4 \]** Let \( \varphi, \psi, G, \) and \( U \) be as in the proof Theorem \[1.4 \]. Recall from Lemma \[2.1 \] and Theorem \[3.1 \] that, for fixed \( \sigma \in (1/2, 1] \)

\[
U^{-1}(\sigma + it) \ll \exp((\log t)^{1-\sigma} \log \log \log t) \text{ a.s.}
\]

(30)

Moreover, Lemma \[2.1 \] applied for \( G \) gives that \( G(z) = EG(z) \theta(z) (z \in \mathbb{H}_1) \), where \( \theta : \mathbb{H}_{1/2} \times \Omega \to \mathbb{C} \) is a random analytic function that satisfies for fixed \( \sigma \in (1/2, 1] \)

\[
\theta(\sigma + it) \ll \exp((\log t)^{1-\sigma} \log \log \log t) \text{ a.s.}
\]

(31)

By hypothesis we have \( \sum_{p \in \mathcal{P}} \frac{1+f(p)}{p^{(\log p)^{1-\sigma}}} \) converges \( \forall \epsilon > 0 \) a.s. Since \( |1+f(p)| \leq 2 \forall p \in \mathcal{P} \), the Kolmogorov two series Theorem gives that \( \sum_{p \in \mathcal{P}} \frac{1+E f(p)}{p^{(\log p)^{1-\sigma}}} \) converges \( \forall \epsilon > 0 \). The construction made in the proof of Theorem \[1.4 \] gives that \( EG(z) = 1 + E f(p) \) and hence that \( \sum_{p \in \mathcal{P}} \frac{E g(p)}{p^{1-\sigma}} \) is analytic in \( \mathbb{H}_{1-\sigma} \), since converges absolutely \( \forall z \in \mathbb{H}_{1-\sigma} \). This combined with claim \[2.1 \] gives that \( EG(z) \) is a non vanishing analytic function in \( \mathbb{H}_{1-\sigma} \). We conclude that \( G : \mathbb{H}_{1-\sigma} \times \Omega \to \mathbb{C} \) is a non-vanishing random analytic function that satisfies, for fixed \( \sigma \in (1/2, 1] \):

\[
G(\sigma + it) \ll \exp((\log t)^{1-\sigma} \log \log \log t).
\]

(32)

Let \( T : \Omega \to \Omega \) be the measure preserving transformation as in the proof of Theorem \[1.4 \] and \( \omega^* = T(\omega) \). The construction made in this proof gives \( \forall \omega \in \Omega \) and \( z \in \mathbb{H}_1 \) that

\[
\zeta(z) F_\omega(z) = U_{\omega^*}^{-1}(z) G_{\omega^*}(z).
\]

(33)

Assume \( M_\mu(x) = o(x^{1-\alpha+\epsilon}) \) for all \( \epsilon > 0 \). In particular \( \frac{1}{\zeta} \) is analytic in \( \mathbb{H}_{1-\alpha} \) and for each fixed \( \sigma > 1 - \alpha, 1/\zeta(\sigma + it) = o(t^\delta) \) for all \( \delta > 0 \) (see [19] pg. 336-337). Also, this implies that for almost all \( \omega \in \Omega, F_\omega \) has analytic extension to \( \mathbb{H}_{1-\alpha} \) given by

\[
F_\omega(z) = \frac{G_{\omega^*}(z)}{U_{\omega^*}(z) \zeta(z)}.
\]
By \(30\), \(32\) and the fact that \(1/\zeta(\sigma + it) = o(t^\delta)\) for all \(\delta > 0\), for each fixed \(\sigma > 1 - \alpha\), \(F(\sigma + it) \ll t^\delta \forall \delta > 0\) a.s., and hence this implies (see the proof of Theorem \(1.1\)) that \(M_f(x) = o(x^{1-\alpha + \epsilon})\) for all \(\epsilon > 0\), a.s. On the other hand, if \(M_f(x) = o(x^{1-\alpha + \epsilon}) \forall \epsilon > 0\) a.s., by Lemma \(A.1\) \(F\) has analytic extension to \(H_{1-\alpha}\). Hence there is \(\omega \in \Omega\) such that \(U_{\omega}^{-1}(z) G_{\omega^*}(z)\) is a non-vanishing analytic function in \(H_{1-\alpha}\) and \(F_{\omega}\) is analytic in this half plane. By \(33\), we obtain that \(\zeta\) and \(F_{\omega}(z)\) can not vanish in \(H_{1-\alpha} \setminus \{1\}\), since they are analytic in this set and their product is a non-vanishing analytic function in \(H_{1-\alpha}\). Hence \(\frac{1}{\zeta}\) is analytic in \(H_{1-\alpha}\) and satisfies in this half plane \(\frac{1}{\zeta(\sigma + it)} = o(t^\delta) \forall \delta > 0\), which implies that \(M_{\mu}(x) = o(x^{1-\alpha + \epsilon})\) for all \(\epsilon > 0\).

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## Appendix A.

**Theorem A.1.** Let \(f : \mathbb{N} \rightarrow [0, 1]\) be such that the Dirichlet series \(\sum_{k=1}^{\infty} \frac{f(k)}{k}\) converges absolutely for \(z \in \mathbb{H}_a\) \((a > 0)\) and that for some \(0 < c < a\)

\[
\sup_{x \geq 1} \frac{|M_f(x)|}{x^c} = C < \infty.
\]

Then the Dirichlet series \(\sum_{k=1}^{\infty} \frac{f(k)}{k^s}\) converges for all \(z \in \mathbb{H}_c\) and \(F : \mathbb{H}_c \rightarrow \mathbb{C}\) given by \(F(z) := \sum_{k=1}^{\infty} \frac{f(k)}{k^s}\) is analytic.

**Proof.** Let \(y < x\) be natural numbers. Let \(\{a_k\}_{k \in \mathbb{N}}\) and \(\{b_k\}_{k \in \mathbb{N}}\) be two sequences of real numbers and let \(\Delta a_k := a_k - a_{k-1}\). The partial summation formula states that:

\[
\sum_{k=y}^{x} a_k \Delta b_k = a_x b_x - a_y b_y - \sum_{k=y}^{x-1} b_k \Delta a_{k+1}.
\]

Let \(\delta > 0\). A direct application of \((34)\) gives that:

\[
\left| \sum_{k=y}^{x} \frac{f(k)}{k^{c+\delta}} \right| = \left| \sum_{k=y}^{x} \frac{\Delta M_f(k)}{k^{c+\delta}} \right| = \left| \frac{M_f(x)}{x^{c+\delta}} - \frac{M_f(y)}{(y-1)^{c+\delta}} - \sum_{k=y}^{x-1} \frac{M_f(k) \Delta}{(k+1)^{c+\delta}} \right|
\]

By hypothesis \(\lim_{x \rightarrow \infty} \frac{M_f(x)}{x^{c+\delta}} = 0\) and \(\lim_{y \rightarrow \infty} \frac{M_f(y)}{x^{c+\delta}} = 0\). Since \(\Delta \frac{1}{(k+1)^{c+\delta}} = -(c+\delta) \int_k^{k+1} \frac{dt}{t^{c+\delta}} \ll \frac{1}{k^{1+\delta}}\), we obtain

\[
\left| \sum_{k=y}^{x} M_f(k) \Delta \frac{1}{(k+1)^{c+\delta}} \right| \ll \sum_{k=y}^{x} \frac{1}{k^{1+\delta}} = o(y).
\]

We conclude that Dirichlet series \(F(c + \delta)\) is convergent for every \(\delta > 0\). A classical result in the Theory of the Dirichlet series (see \(11\), Theorems 11.8 and 11.11) states that if the series \(\sum_{k=1}^{\infty} \frac{f(k)}{k^s}\) converges for \(z_0 = \sigma_0 + i\theta_0\) then it converges for all \(z \in \mathbb{H}_{\sigma_0}\) and also uniformly on compact subsets of this half plane. Thus the function \(z \in \mathbb{H}_{\sigma_0} \rightarrow \sum_{k=1}^{\infty} \frac{\omega_k}{k^s}\) is analytic. \(\square\)

**Proposition A.1.** Let \(f\) be a random multiplicative function. Then for each \(c > 1/2\)

\[
E_{f,c} := \left\{ \omega \in \Omega : \sum_{k=1}^{\infty} \frac{f_{\omega}(k)}{k^c} \text{ converges} \right\}
\]

is a tail event and hence, \(\mathbb{P}(E_{f,c}) \in \{0, 1\}\).
Proof. Let \( k \in \mathbb{N} \) and \( D = D(k) := \{ p \in \mathcal{P} : p \leq k \} \). Let \( f : \mathbb{N} \to \{-1,0,1\} \) and \( h_D : \mathbb{N} \to \{-1,0,1\} \) be multiplicative functions (supported on the squarefree integers) such that
\[
h_D(p) = \begin{cases} 0, & \text{if } p \in D, \\ f(p), & \text{if } p \notin D, \end{cases}
\]

Claim A.1. The series \( \sum_{n=1}^{\infty} \frac{f(n)}{n} \) converges if and only if \( \sum_{n=1}^{\infty} \frac{h_D(n)}{n} \) converges.

Proof of the claim: Let \( u \) be a multiplicative function supported on the set of the squarefree integers such that
\[
u(p) = \begin{cases} f(p), & \text{if } p \in D, \\ 0, & \text{if } p \notin D. \end{cases}
\]
Then for all \( p \in \mathcal{P} \) \((u * h_D)(p) = u(p) + h_D(p) = f(p)\) and for every \( l \geq 2 \)
\[(u * h_D)(p^l) = \sum_{k=0}^{l} u(p^k)h_D(p^{l-k}) = u(1)h_D(p^l) + u(p)h_D(p^{l-1}) = 0,
\]
since \( u(p^m) = h_D(p^m) = 0 \) for every \( m \geq 2 \) and \( u(p)h_D(p) = 0 \). This shows that \( u * h_D(p^m) = f(p^m) \) \( \forall p \in \mathcal{P} \) and \( \forall m \geq 1 \). Since \( u \) and \( h_D \) are multiplicative, their convolution also is and hence \( f(n) = h_D * u(n) \forall n \in \mathbb{N} \). Let \( U(z) \) be the Dirichlet series of \( u \). Then \( U \) has Euler product representation
\[U(z) = \prod_{p \in D} \left( 1 + \frac{u(p)}{p^z} \right) \quad (z \in \mathbb{H}_1),
\]
and since \( D \) is finite, we obtain that the Dirichlet series \( U(z) \) converges absolutely for all \( z \in \mathbb{H}_0 \) ([IS] page 106, Theorem 2). Hence the convergence of \( \sum_{n=1}^{\infty} \frac{h_D(n)}{n} \) implies the convergence \( \sum_{n=1}^{\infty} \frac{f(n)}{n} \) ([IS] pg. 122, Notes 1.1). On the other hand, let \( u^{-1} \) be the Dirichlet inverse of \( u \), that is, \((u * u^{-1})(n) = 1_{(1)}(n)\). Then \( u^{-1} \) is multiplicative, \(|u^{-1}(n)| \leq 1 \forall n \in \mathbb{N} \) and \( \sum_{n=1}^{\infty} \frac{u^{-1}(n)}{n^z} = \frac{1}{U(z)} \) \((z \in \mathbb{H}_1)\). Moreover:
\[
\frac{1}{U(z)} = \prod_{p \in D} \left( 1 + \sum_{m=1}^{\infty} (-1)^{m} \frac{u(p)^{m}}{p^{mz}} \right).
\]
Since \( D \) is finite and
\[
\sum_{p \in D} \sum_{m=1}^{\infty} (-1)^{m} \frac{u(p)^{m}}{p^{mz}} = \sum_{p \notin D} \frac{1}{1 + \frac{u(p)}{p}},
\]
we obtain that \( \sum_{p \in D} \sum_{m=1}^{\infty} (-1)^{m} \frac{u(p)^{m}}{p^{mz}} \) converges absolutely in \( \mathbb{H}_0 \). This implies that, \( \sum_{n=1}^{\infty} \frac{u^{-1}(n)}{n^z} \) converges absolutely in \( \mathbb{H}_0 \) ([IS] page 106, Theorem 2). Since \( h_D = f * u^{-1} \) we obtain that the convergence of \( \sum_{n=1}^{\infty} \frac{h_D(n)}{n} \) implies the convergence of \( \sum_{n=1}^{\infty} \frac{f(n)}{n} \) ([IS] pg. 122, Notes 1.1), completing the proof of the claim.

Let \( \mathcal{F}_n^\infty \) be the sigma algebra generated by the random variables \( \{f(p) : p \in \mathcal{P} \text{ and } p \geq n\} \). The tail sigma algebra of \( \mathcal{F} \), denoted by \( \mathcal{F}^\ast \) is the sigma algebra
\[
\mathcal{F}^\ast = \bigcap_{n=1}^{\infty} \mathcal{F}_n^\infty.
\]
Elements of \( \mathcal{F}^\ast \) are called tail events. The Kolmogorov zero or one law states that every tail event has either probability zero or one. Recall that \( D = D(k) \) and \( E_{h_D,c} \in \mathcal{F}_k^\infty \). The claim \(\ref{A.1}\) gives that \( E_{f,c} = E_{h_D,c} \). In particular, \( E_{f,c} \in \mathcal{F}_k^\infty \forall k \in \mathbb{N} \).

\[
\text{Corollary A.1. Let } 0 < \alpha < 1/2. \text{ The following are tail events:}
\]
\[
|M_f(x)| = o(x^{1-\alpha+\epsilon}) \text{ for all } \epsilon > 0.\]
\[
|M_f(x)| = o(x^{1-\alpha-\epsilon}) \text{ for some } \epsilon > 0.
\]
Proof. Let $E_{f,\varepsilon}$ be as in Proposition A.1. Recall that if the Dirichlet series $\sum_{k=1}^{\infty} \frac{f(k)}{k^\sigma}$ converges, then it converges for all $\sigma > \sigma_0$. Thus, by Kronecker’s Lemma and by partial summation (see the proof Theorem A.1):

$$[M_f(x) = o(x^{1-\alpha+\varepsilon}) \text{ for all } \varepsilon > 0] = \bigcap_{n=1}^{\infty} E_{f,\alpha+n-1},$$

$$[M_f(x) = o(x^{1-\alpha-\varepsilon}) \text{ for some } \varepsilon > 0] = \bigcup_{n=1}^{\infty} E_{f,\alpha-n-1}. \quad \square$$

**Lemma A.1.** Let $f : \mathbb{N} \to [0, 1]$ and for $x > 0$, $L(1 + x) = \sum_{k=1}^{\infty} \frac{f(k)}{k^x}$. Let $a, b : [0, \infty) \to (0, 1]$ be such that $a(t) \leq b(t)$ for all $t$, $\lim_{t \to \infty} b(t) = 0$ and $b(t) - a(t) \ll a^2(t)$. Then as $t \to \infty$:

$$L(1 + a(t)) = L(1 + b(t)) + O(1).$$

Proof. Denote $a = a(t)$ and $b = b(t)$. Let $k \in \mathbb{N}$ and $\psi_k(x) = \exp(-x \log k)$. Hence

$$|\psi_k(a) - \psi_k(b)| \leq \int_a^b |\psi_k'(x)|dx = \log k \int_a^b |\psi_k(x)|dx \leq (b - a)\psi_k(a).$$

Let $x > 0$ and $\zeta(1 + x) = \sum_{k=1}^{\infty} \frac{1}{k^{1+x}}$. If $x > 0$ is small, a well known fact is that the Riemann zeta function is analytic on a simple pole at $z = 1$ with residue 1. Hence for $x > 0$, $\sum_{k=1}^{\infty} \frac{\log k}{k^{1+a}} = \zeta'(1 + x) \sim \frac{1}{x^2}$. This combined with the estamative for $\psi_k$ gives:

$$|L(1 + a) - L(1 + b)| = \left| \sum_{k=1}^{\infty} \left( \frac{f(k)}{k^{1+a}} - \frac{f(k)}{k^{1+b}} \right) \right|$$

$$\leq \sum_{k=1}^{\infty} \frac{f(k)}{k} |\psi_k(a) - \psi_k(b)|$$

$$\leq (b - a) \sum_{k=1}^{\infty} \frac{\log k}{k^{1+a}}$$

$$= |(b - a)\zeta'(1 + a)|$$

$$\ll \frac{(b - a)}{a^2} = O(1). \quad \square$$

### A.1. Extension of random analytic functions to half planes.

**Proposition A.2.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $f : \mathbb{H}_1 \times \Omega \to \mathbb{C}$ be a random function such that $f_\omega : \mathbb{H}_1 \to \mathbb{C}$ is analytic for all $\omega \in \Omega$. Then, for each fixed $\varepsilon < 1$, the following set is an element of $\mathcal{F}$:

$$A_{f_\omega} := \{ \omega \in \Omega : f_\omega \text{ has analytic extension to } \mathbb{H}_\varepsilon \}. \quad (35)$$

Proof. Since $f$ is a random function, for fixed $z \in \mathbb{H}_1$, the complex random variables $f(z)$ and $\{f(z + k^{-1})\}_{k \in \mathbb{N}}$ are $\mathcal{F}$-measurable. For each $\omega \in \Omega$ we have that $f_\omega : \mathbb{H}_1 \to \mathbb{C}$ is analytic. Hence, for all $z \in \mathbb{H}_1$ the limit

$$f_{\omega}^{(1)}(z) := \lim_{k \to \infty} f_\omega(z + k^{-1}) - f_\omega(z)$$

exists and it is a complex random variable measurable in $\mathcal{F}$. Hence $f^{(1)} : \mathbb{H}_1 \times \Omega \to \mathbb{C}$ is a random function which is analytic for all $\omega \in \Omega$, since for each $\omega \in \Omega$, it is the derivative of $f_\omega$. By applying these arguments inductively, we conclude that for all $n \in \mathbb{N}$, $f^{(n)} : \mathbb{H}_1 \times \Omega \to \mathbb{C}$ given by $f^{(n)}_\omega := \frac{d^n}{dz^n} f_\omega(z)$ is a random analytic function such that for each $z \in \mathbb{H}_1$, $f^{(n)}(z)$ is $\mathcal{F}$-measurable. Denote $B(a, \delta) := \{ z \in \mathbb{C} : |z - a| < \delta \}$. We recall the following result from Complex-Analysis (c.f [7], page 72, Theorem 2.8):
Claim A.2. Let $G$ be an open connected set, $h : G \to \mathbb{C}$ an analytic function, $a \in G$ and $R > 0$ such that $B(a,R) \subset G$. Then for all $z \in B(a,R)$ we have that $h(z) = \sum_{n=0}^{\infty} \frac{k^{(n)}(a)}{n!}(z-a)^n$. Moreover the radius of convergence of this power series is greater or equal to $R$. By Claim A.2, the assumption that $R_k(\omega) \geq k + 1 - c$ gives that $H_{\omega,k}$ is analytic in the open ball $B(k+1, k+1-c)$ for all $\omega \in \mathbb{H}_1 \cap B(k+1, k+1)$. Hence $G_{\omega,k}(z) := f_\omega(z)$ for all $z \in \mathbb{H}_1 \cap B(k+1, k+1-c)$. This follows from the fact that, if two analytic functions defined in a open connected set $R_1$ coincide in an open ball $B \subset R_1$, then these analytic functions coincide in all $R_1$ (see [7] Theorem 3.7, pg 78). Hence $G_{\omega,k} : \mathbb{H}_1 \cup A_k \to \mathbb{C}$ given by

$$G_{\omega,k}(z) := \begin{cases} f_\omega(z), & \text{if } z \in \mathbb{H}_1 \setminus B(k+1, k+1-c); \\ H_{\omega,k}(z), & \text{if } z \in B(k+1, k+1-c) \end{cases}$$

is an analytic extension of $f_\omega$ to $\mathbb{H}_1 \cup A_k$. Since this is an open connected set of $\mathbb{C}$, $G_{\omega,k}$ is the unique analytic function defined in $\mathbb{H}_1 \cup A_k$ that coincides with $f_\omega$ in $\mathbb{H}_1$. Observe that for each $k \in \mathbb{N}$, $\mathbb{H}_1 \cup A_k \subset \mathbb{H}_1 \cup A_{k+1}$ and these are open connected sets. Hence the unique analytic extension of $G_{\omega,k}$ to $\mathbb{H}_1 \cup A_{k+1}$ is $G_{\omega,k+1}$. This implies that for each $\omega \in \bigcap_{k \in \mathbb{N}}[R_k \geq k - c]$, $G_\omega : \mathbb{H}_c \to \mathbb{C}$ given by

$$G_\omega(z) := \begin{cases} f_\omega(z), & \text{if } z \in \mathbb{H}_1; \\ G_{\omega,k}(z), & \text{if } z \in A_k, \text{ for } k \in \mathbb{N} \end{cases}$$

is well defined and analytic. By well defined we mean that for each $z \in \mathbb{H}_c \setminus \mathbb{H}_1$, there exists $k_0$ such that $z \in A_k$ for all $k \geq k_0$ and the value $G_\omega(z) = G_{\omega,k}(z)$ does not depend on $k$. The analyticity of $G_\omega$ follows from the fact that for each fixed $z \in \mathbb{H}_c$, there is an small $\delta > 0$ and $k \in \mathbb{N}$ such that $B(z,\delta) \subset \mathbb{H}_1 \cup A_k$. Hence for all $w \in B(z,\delta)$, $G_\omega(w) = G_{\omega,k}(w)$. Since $G_{\omega,k}$ is holomorphic at $z$, $G_\omega$ also is holomorphic at $z$. Hence $G_\omega$ is holomorphic for all $z \in \mathbb{H}_c$. This completes the proof of (36) which gives that $A_f$ is the countable intersection of $\mathcal{F}$–measurable sets. □

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