$K \rightarrow 2\pi$ Decay in the Nambu-Jona-Lasinio Model

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Abstract

We study the $K \rightarrow 2\pi$ decays using the $U_L(3) \times U_R(3)$ version of the Nambu-Jona-Lasinio model with the effective $\Delta S = 1$ nonleptonic weak interaction. The $\Delta I = \frac{3}{2}$ amplitude is in reasonable agreement with experimental data. On the other hand, the calculated $\Delta I = \frac{1}{2}$ amplitudes strongly depend on the mass of the low-lying scalar-isoscalar $\sigma$ meson, and therefore give a strong constraint on the parameters of the model.

I. INTRODUCTION

The $\Delta I = \frac{1}{2}$ rule in the $K \rightarrow 2\pi$ decays is one of the interesting problems of the low-energy hadron physics. Since all the particle in the initial and final states in these decay processes are the octet pseudoscalar mesons, which are considered as the Nambu-Goldstone (NG) bosons associated with the spontaneous breaking of chiral symmetry, one can use all the known properties of the NG bosons in the analyses of the processes.

One of the widely used frameworks to study the low-energy properties of the NG bosons is the chiral perturbation theory (ChPT). In ChPT, we introduce an effective lagrangian
constructed in terms of nonlinearly transforming NG boson fields and treat the explicit breaking of the chiral symmetry and the external momenta of the NG bosons perturbatively.

In order to study hadronic weak interactions at low energy, an effective weak interaction lagrangian is commonly introduced. For $K \to 2\pi$ decays, we use $\Delta S = 1$ nonleptonic weak interaction lagrangian $L_W$, which has been derived from the standard theory by integrating out heavy degrees of freedom. The renormalization group equation with one-loop QCD corrections is solved to give $L_W$ at a hadronic mass scale $\mu$ in terms of local four quark operators [4–8].

Using $L_W$, the $K \to 2\pi$ decays have been studied in the ChPT [7,9]. In this approach, the four-quark operators in $L_W$ have to be mapped on to the operators of the NG bosons. Thus, one should examine whether this treatment is justified. It should be also noted that the $K \to 2\pi$ decay amplitudes are expected to be sensitive to the explicit breaking of the chiral symmetry because the matrix elements vanish in the $SU(3)$ limit. In ChPT, this symmetry breaking is represented by the finite masses of the NG bosons. It is, however, controversial whether the ChPT can qualitatively handle the explicit symmetry breaking in the strangeness sector [10]. Therefore it seems interesting to study the $K \to 2\pi$ decays in an effective theory with the symmetry breaking in a direct form.

In order to investigate these two points, it may be a good idea to use a chiral effective quark model such as the Nambu-Jona-Lasinio model [11]. If the quark fields in the chiral effective quark model are identified with those in $L_W$, there is no ambiguity of using $L_W$. Another good point of the chiral effective quark model is that the explicit breaking of the chiral symmetry can be introduced naturally by the current quark mass term.

A few years ago, Morozumi et al. [12] derived a chiral weak lagrangian by performing the bosonization of the $L_W$ using the NJL model as a guide and applied it to the $K \to 2\pi$ decays. They compared their results with those in ChPT and found that the $\Delta I = \frac{1}{2}$ amplitude receives additional enhancement and thus the long-standing $\Delta I = \frac{1}{2}$ enhancement problem is solved. However in ref. [12] the derivative expansion was used and the current $s$-quark mass was treated as a free parameter. Furthermore they calculated only the leading
contributions in the $\frac{1}{N_C}$ expansion.

We present fuller calculations of the $K \to 2\pi$ decays in a $U_L(3) \times U_R(3)$ NJL model combined with the effective $\Delta S = 1$ nonleptonic weak interaction lagrangian $L_W$ in the Feynman diagram approach. Here the low-lying meson states are calculated by solving the Bethe-Salpeter type equations in the ladder approximation without using the derivative expansion. The model parameters, such as the quark masses, are chosen so as to be consistent with the observed NG boson properties. We calculate not only the leading terms but also the next order terms in the $\frac{1}{N_C}$ expansion of the $K \to 2\pi$ decay amplitudes.

Although the NJL model does not confine quarks, it is considered that the NG bosons are strongly bound and therefore the processes containing only the NG bosons, such as $K \to 2\pi$ transitions, can be described by this model fairly well. It is also known that the low energy theorems of the NG bosons are satisfied in this model.

II. EXTENDED NAMBU-JONA-LASINIO MODEL

We work with the NJL model lagrangian density extended to $U_L(3) \times U_R(3)$ case \cite{10}:

\[ L_{NJL} = \bar{\psi}(i\partial_\mu \gamma^\mu - \hat{m})\psi + \frac{G_S}{2} \sum_{a=0}^{8} \left[ (\bar{\psi}\lambda^a \psi)^2 + (\bar{\psi}\lambda^a i\gamma_5 \psi)^2 \right]. \] (1)

Here the quark field $\psi$ is a column vector in color, flavor and Dirac spaces and $\lambda^a$ is the $U(3)$ generator in flavor space. The lagrangian $L_{NJL}$ incorporates the current quark mass matrix $\hat{m} = \text{diag}(m_u, m_d, m_s)$ which breaks the chiral $U_L(3) \times U_R(3)$ invariance explicitly.

Quark condensates and constituent quark masses are self-consistently determined by the gap equations. We assume the isospin symmetry, i.e., $m_u = m_d$. The pseudoscalar channel quark-antiquark scattering amplitudes are then calculated in the ladder approximation. From the pole position of the scattering amplitude, the pion mass $m_\pi$ and the kaon mass $m_K$ are determined. The pion-quark coupling constant $g_\pi$ and the kaon-quark coupling constant $g_K$ are determined by the residues of the scattering amplitudes at the pion and kaon poles respectively. The pion decay constant $f_\pi$ and the kaon decay constant $f_K$ are also determined by calculating the quark-antiquark one-loop graphs.
III. EFFECTIVE NONLEPTONIC WEAK INTERACTION

The effective $\Delta S = 1$ nonleptonic weak interaction lagrangian density we have used is

$$\mathcal{L}_W = \frac{G_F}{\sqrt{2}} \sum_{r=1, r\neq 4}^6 \left( C_r^c \xi_c + C_r^t \xi_t \right) Q_r$$

(2)

with $\xi_i \equiv V_{id} V_{is}^*$ and $V$ is the Kobayashi-Maskawa matrix. The four-quark operators $Q_r$ are

$$Q_1 = (\bar{s}_\alpha d_\alpha)_{V-A} (\bar{u}_\beta u_\beta)_{V-A}, \quad Q_2 = (\bar{s}_\alpha d_\beta)_{V-A} (\bar{u}_\beta u_\alpha)_{V-A},$$

$$Q_3 = (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V-A}, \quad Q_5 = (\bar{s}_\alpha d_\alpha)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\beta)_{V+A},$$

$$Q_6 = (\bar{s}_\alpha d_\beta)_{V-A} \sum_{q=u,d,s} (\bar{q}_\beta q_\alpha)_{V+A},$$

(3)

where $\alpha$ and $\beta$ are indices in the color space. $Q_1$ and $Q_2$ have both the $\Delta I = \frac{1}{2}$ and $\Delta I = \frac{3}{2}$ components while $Q_3$, $Q_5$, $Q_6$ and $Q_2 - Q_1$ have only the $\Delta I = \frac{1}{2}$ component. As for the Wilson coefficients $C_r^c$ and $C_r^t$, we take the results at the energy scale $\mu = 0.244\text{GeV}$ with the top quark mass $m_t = 200\text{GeV}$ in ref. [8]. The numerical values are $C_1 = -0.216$, $C_2 = 0.346$, $C_3 = -7.12 \times 10^{-3}$, $C_5 = 3.89 \times 10^{-3}$ and $C_6 = -0.021$ for $C_r \equiv C_r^c \xi_c + C_r^t \xi_t$.

IV. $K \rightarrow 2\pi$ DECAY AMPLITUDES

The invariant amplitude for the $K \rightarrow 2\pi$ decay $\mathcal{T}_{K \rightarrow 2\pi}$ is defined by

$$\langle \pi(k_1)\pi(k_2) | K(q) \rangle = (2\pi)^4 \delta^4(k_1 + k_2 - q) \mathcal{T}_{K \rightarrow \pi\pi}.$$  (4)

By calculating the Feynman graphs shown in fig. 1, we obtain the following results.

$$\mathcal{T}_{K^+ \rightarrow \pi^+\pi^0} = \frac{G_F}{\sqrt{2}} \left( C_1 + C_2 \right) \frac{1}{\sqrt{2}} \left( 1 + \frac{1}{N_c} \right) X,$$

(5)

$$\mathcal{T}_{K^0 \rightarrow \pi^+\pi^-} = \frac{G_F}{\sqrt{2}} \left( \frac{C_1}{N_c} X + C_2 X + \frac{C_3}{N_c} X + \frac{C_5}{N_c} Y + C_6 Y \right),$$

(6)

$$\mathcal{T}_{K^0 \rightarrow \pi^0\pi^0} = \frac{G_F}{\sqrt{2}} \left( -\frac{C_1}{N_c} X - C_2 X + \frac{C_3}{N_c} X + \frac{C_5}{N_c} Y + C_6 Y \right).$$

(7)

Here $X$ and $Y = (Y_1 + Y_2 + Y_3)$ are
\[
X = -2\sqrt{2}g_\pi^2g_K \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^u(p)\gamma^\mu\gamma_5 S_F^d(p-k_1)\gamma_5 \right] \\
\times \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^s(p-k_1)\gamma_\mu S_F^u(p)\gamma_5 S_F^d(p+k_2)\gamma_5 \right],
\]
\[
Y_1 = -4\sqrt{2}g_\pi^2g_K \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^u(p)\gamma_5 S_F^d(p+k_2)\gamma_5 \right] \\
\times \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^s(p-k_1)S_F^u(p)\gamma_5 S_F^d(p+k_2)\gamma_5 \right],
\]
\[
Y_2 = -4\sqrt{2}g_\pi^2g_K \left(1 + T_\sigma(q^2)\right) \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^s(p-q)\gamma_5 S_F^d(p)\gamma_5 \right] \\
\times \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^d(p+q)S_F^d(p)\gamma_5 S_F^u(p+k_1)\gamma_5 \right],
\]
\[
Y_3 = -4\sqrt{2}g_\pi^2g_K \int \frac{d^4p}{(2\pi)^4} \left\{ Tr^{(c,D)} \left[ S_F^d(p)\right] - Tr^{(c,D)} \left[ S_F^s(p)\right] \right\} \\
\times \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^s(p)\gamma_5 S_F^d(p)\gamma_5 S_F^u(p+k_1)\gamma_5 S_F^d(p+q)\gamma_5 \right],
\]

where \( Tr^{(c,D)} \) means trace in color and Dirac spaces. The effect of the \( \bar{q}q \)-collective \( \sigma \) meson in the intermediate state is expressed by

\[
T_\sigma(q^2) = \frac{G_S I(q^2)}{1 - G_S I(q^2)},
\]

with

\[
I(q^2) = i \int \frac{d^4p}{(2\pi)^4} Tr^{(c,f,D)}[S_F(p)\lambda^u S_F(p+q)\lambda^u],
\]

where \( Tr^{(c,f,D)} \) means trace in color, flavor and Dirac spaces and \( \lambda^u \equiv \sqrt{\frac{2}{3}}\lambda^0 + \sqrt{\frac{1}{3}}\lambda^8 \) is a matrix in flavor space. Fig. 1 (e) does not contribute because the integral

\[
q_\mu \int \frac{d^4p}{(2\pi)^4} Tr^{(c,D)} \left[ S_F^u(p+q)\gamma^\mu S_F^u(p)\gamma_5 S_F^d(p+k_1)\gamma_5 \right]
\]

vanishes in the isospin symmetry limit.

**V. NUMERICAL RESULTS**

The experimental values of the invariant amplitudes for the \( K \rightarrow 2\pi \) decays are as follows [13].
\[ |T_{K^+ \rightarrow \pi^+\pi^0}| = 1.83 \times 10^{-8}\text{GeV}, \]
\[ |T_{K^0 \rightarrow \pi^+\pi^-}| = 2.76 \times 10^{-7}\text{GeV}, \]
\[ |T_{K^0 \rightarrow \pi^0\pi^0}| = 2.63 \times 10^{-7}\text{GeV}. \] 

(14)

Here the \( K^+ \rightarrow \pi^+\pi^0 \) decay is a pure \( \Delta I = \frac{3}{2} \) decay and the other two decays have both \( \Delta I = \frac{1}{2} \) and \( \Delta I = \frac{3}{2} \) components.

In our theoretical calculations, the parameters of the NJL model are the current quark masses \( m_u = m_d, m_s \), the coupling constant \( G_S \) and the covariant cutoff \( \Lambda \). First, we take \( m_u = m_d \) as a free parameter and study the \( m_u \) dependence of the \( K \rightarrow 2\pi \) decay amplitudes. Other parameters \( m_s, G_S \) and \( \Lambda \) are determined so as to reproduce the observed values of \( m_{\pi}, m_K \) and \( f_{\pi} \). The calculated results of \( |T_{K^+ \rightarrow \pi^+\pi^0}|, |T_{K^0 \rightarrow \pi^+\pi^-}| \) and \( |T_{K^0 \rightarrow \pi^0\pi^0}| \) are shown in fig. 2. As can be seen from fig. 2, the \( \Delta I = \frac{3}{2} \) amplitude has little dependence on \( m_u \), while \( \Delta I = \frac{1}{2} \) amplitudes strongly depend on \( m_u \). This strong dependence is attributed to the \( T_{\sigma} \) term in \( Y_2 \), eq. (10). In fig. 3 we show the calculated constituent quark masses \( M_u \) and \( M_s \) as functions of the current u-quark mass \( m_u \). As a result of our fitting procedure, the constituent quark masses almost linearly increase when the current u-quark mass increases. The quark-antiquark scattering amplitude in the scalar-isoscalar channel has the pole of the \( \sigma \) meson just above the constituent quark-antiquark threshold in the NJL model [14]. Therefore \( T_{\sigma}(q^2 = m_K^2) \) quickly increases when \( m_{\sigma} (\simeq 2M_u) \) becomes close to \( m_K \) by changing \( m_u \).

If we fit \( m_u \) so as to reproduce the \( \Delta I = \frac{1}{2} \) amplitudes reasonably, then we obtain the NJL model parameters, \( m_u = m_d = 7.1\text{MeV}, m_s = 178.2\text{MeV}, \Lambda = 0.844\text{GeV} \) and \( \frac{\Lambda^2 N_c}{4\pi} G_S = 0.66 \). The calculated constituent quark masses are \( M_u = M_d = 286.2\text{MeV} \) and \( M_s = 529.3\text{MeV} \) and the meson-quark coupling constants are \( g_{\pi} = 3.023 \) and \( g_K = 3.254 \). We have used \( m_{\pi}, m_K \) and \( f_{\pi} \) as inputs, so \( f_K \) is the prediction and our result is \( f_K = 101\text{MeV} \), which is about 11% smaller than the observed value. The ratio of the current s-quark mass to the current u,d-quark mass is \( m_s/m_u = 25.1 \), which agrees well with \( m_s/\hat{m} = 25 \pm 2.5, (\hat{m} = \frac{1}{2}(m_u + m_d)) \) derived from ChPT [15].

The calculated results of the \( K \rightarrow 2\pi \) decay amplitudes in the above parameter set are
\[ |T_{K^+ \to \pi^+ \pi^0}| = 2.52 \ (1.89) \times 10^{-8}\text{GeV}, \]
\[ |T_{K^0 \to \pi^+ \pi^-}| = 2.65 \ (2.95) \times 10^{-7}\text{GeV}, \]
\[ |T_{K^0 \to \pi^0 \pi^0}| = 2.30 \ (2.68) \times 10^{-7}\text{GeV}. \quad (15) \]

Here the numbers in parentheses are the results without including \( \frac{1}{N_C} \) terms. The \( \frac{1}{N_C} \) terms enhance the \( |T_{K^+ \to \pi^+ \pi^0}| \) and reduce the \( |T_{K^0 \to \pi^+ \pi^-}| \) and \( |T_{K^0 \to \pi^0 \pi^0}| \). This is opposite to the nonlinear sigma model case. It is possible that the full QCD contains other \( \frac{1}{N_C} \) corrections which are not taken into account in the NJL model. The \( \Delta I = \frac{3}{2} \) amplitude is our prediction and it is about 38% bigger than the experimental value. In ref. \([12]\), \( K \to 2\pi \) decay amplitudes have been calculated in the leading order of the derivative expansion without the \( \frac{1}{N_C} \) terms and their result is \( |T_{K^+ \to \pi^+ \pi^0}| = 2.5 \times 10^{-8} \) GeV. So the higher order terms of the derivative expansion improve the agreement to the experimental values.

The contributions from each diagram are as follows. \( X = 2.49 \times 10^{-2}\text{GeV}^3, \ Y_1 = 0.274\text{GeV}^3, \ Y_2 = -1.564\text{GeV}^3 \) and \( Y_3 = -4.1 \times 10^{-4}\text{GeV}^3 \). The contribution of \( Y_3 \) type is not included in the ChPT and the bosonization approach in ref. \([12]\). As far as we know, this is the first estimation of this type of the contribution, however our numerical result shows that its contribution is negligible.

The main origin of the enhancement of the \( \Delta I = \frac{1}{2} \) amplitudes is the existence of the low-lying scalar-isoscalar \( \sigma \) meson, which is not experimentally confirmed. It is argued that the \( \sigma \) meson has a very large \( \sigma \to 2\pi \) decay width and thus is not observed. In our calculation, the \( \sigma \to 2\pi \) decay width is not taken into account although the \( \sigma \) meson has a small (50MeV \( \sim \) 100MeV) \( \sigma \to q\bar{q} \) decay width. We estimate the effect of the strong \( \sigma \to 2\pi \) decay on the \( K \to 2\pi \) transitions by using the following scalar-isoscalar form factor

\[ \tilde{T}_\sigma(q^2) = \frac{g_\sigma}{(2M_u - i\frac{\Gamma}{2})^2 - q^2} \quad (16) \]

instead of \( T_\sigma(q^2) \) in eq. (12). Here \( \Gamma \) is the parameter which represents the \( \sigma \to 2\pi \) decay width and \( g_\sigma \) is determined so as to reproduce our results given in eq. (15) in the case of \( \Gamma = 0 \). If one takes \( \Gamma = 500\text{MeV} \), then the amplitudes become \( |T_{K^0 \to \pi^+ \pi^-}| = 8.59 \times 10^{-8}\text{GeV} \).
and $|T_{K^0\to\pi^0\pi^0}| = 6.52 \times 10^{-8}$ GeV, which are about $\frac{1}{3}$ of the original values. Even in this case, if the model parameters are refitted again, one can reproduce the $K \to 2\pi$ decay amplitudes reasonably by taking $m_\sigma$ close to $m_K$.

VI. CONCLUDING REMARKS

Using a three-flavor Nambu-Jona-Lasinio (NJL) model with the low-energy effective $\Delta S = 1$ nonleptonic weak interaction, we have studied the $K \to 2\pi$ decays. As suggested in ref. [12], we have found that the existence of the $\sigma$ meson with the mass close to $m_K$ enhances the $\Delta I = \frac{1}{2}$ amplitudes and all the $K \to 2\pi$ decay amplitudes are reproduced reasonably well. The role of the $\sigma$ meson in the low-energy hadron physics is not established so well. In the $K \to 2\pi$ decays, the weak interaction gives rise to the flavor change $s \to d$ and it is one of the reason why the $\sigma$ plays an important role in the processes. It may be useful to study the $K \to 2\pi$ decays and the $\pi\pi$ scattering processes simultaneously by including the two pion intermediate states in the analysis since the $\sigma$ meson plays an important role in the $\pi\pi$ scattering processes too. It is left as a further study.

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Figure Captions

Fig. 1  Feynman diagrams for the $K \rightarrow 2\pi$ decays. The dotted lines correspond to the $\bar{q}q$-collective mesons in the NJL model while the solid lines to the constituent quark propagators. The hatched ellipses represent the effective $\Delta S = 1$ nonleptonic weak interactions. The diagrams (a), (b), (c) and (d) correspond to the $X$, $Y_1$, $Y_2$ and $Y_3$ contributions respectively.

Fig. 2  Dependence of the calculated $K \rightarrow 2\pi$ decay amplitudes on the current u-quark mass. The solid line represents $10 \times T_{K^+ \rightarrow \pi^+ \pi^0}$ and the dashed line corresponds to $T_{K^0 \rightarrow \pi^+ \pi^-}$ while the dash-dotted line to $T_{K^0 \rightarrow \pi^0 \pi^0}$.

Fig. 3  Dependence of the calculated constituent u-quark mass $M_u$ and the constituent s-quark mass $M_s$ on the current u-quark mass $m_u$. 

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Fig. 1

(a)

(b)

(c)
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Fig. 2
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Fig. 3

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig3}
\caption{Graph showing the relationship between mu meson mass ($m_\mu$) and some other quantities.}
\end{figure}