New upper bound for the cardinalities of \(s\)-distance sets on the unit sphere

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Abstract

We have the Fisher type inequality and the linear programming bound as upper bounds for the cardinalities of \(s\)-distance sets on \(S^{d-1}\). In this paper, we give a new upper bound for the cardinalities of \(s\)-distance sets on \(S^{d-1}\) for any \(s\). This upper bound improves the Fisher type inequality and is useful for \(s\)-distance sets which are not applicable to the linear programming bound.

1 Introduction

Let \(X\) be a finite set on \(S^{d-1}\). We define

\[ A(X) := \{(x, y) \mid x, y \in X, x \neq y\}, \]

where \((\ast, \ast)\) is the standard inner product. \(X\) is called an \(s\)-distance set, if the number of Euclidean distances between any distinct vectors in \(X\) is \(s\), that is, \(|A(X)| = s\). The following upper bound is well known.

**Theorem 1.1** (Fisher type inequality [2]).

1. Let \(X\) be an \(s\)-distance set on \(S^{d-1}\). Then,
   \[ |X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}. \]

2. Let \(X\) be an antipodal \(s\)-distance set on \(S^{d-1}\). Then,
   \[ |X| \leq 2\binom{d+s-2}{s-1}. \]

\(X\) is said to be antipodal, if \(-X = \{-x \mid x \in X\} \subset X\). We prepare the Gegenbauer polynomial.

**Definition 1.1.** Gegenbauer polynomials are a set of orthogonal polynomials \(\{G^{(d)}_l(t) \mid l = 1, 2, \ldots\}\) of one variable \(t\). For each \(l\), \(G^{(d)}_l(t)\) is a polynomial of degree \(l\) and defined in the following manner.

1. \(G^{(d)}_0(t) \equiv 1, G^{(d)}_1(t) = dt.\)

2. \(tG^{(d)}_l(t) = \lambda_{l+1}G^{(d)}_{l+1}(t) + (1 - \lambda_{l-1})G^{(d)}_{l-1}(t)\) for \(l \geq 1\), where \(\lambda_l = \frac{l}{d+2l-2}\.\)

The following upper bound for the cardinalities of \(s\)-distance sets is well known.
Theorem 1.2 (Linear programming bound [2]). Let $X$ be an $s$-distance set on $S^{d-1}$. We define the polynomial $F_X(t) := \prod_{\alpha \in A(X)} (t - \alpha)$ for $X$. We have the Gegenbauer expansion

$$F_X(t) = \prod_{\alpha \in A(X)} (t - \alpha) = \sum_{k=0}^{s} f_k G_{\alpha}^{(d)}(t),$$

where $f_k$ are real numbers. If $f_0 > 0$ and $f_i \geq 0$ for all $1 \leq i \leq s$, then

$$|X| \leq \frac{F_X(1)}{f_0}.$$

This upper bound is very useful when $A(X)$ is given. However, if there exists $f_i$ which is a negative number, then we have no useful upper bound for the cardinalities of $s$-distance sets. In this paper, we give useful upper bounds for that case.

Let $\text{Harm}_l(\mathbb{R}^d)$ denote the linear space of all harmonic homogeneous polynomials of degree $l$, in $d$ variables. Let $h_l$ denote the dimension of $\text{Harm}_l(\mathbb{R}^d)$. The following are the main theorems in this paper.

Theorem 1.3. Let $X$ be an $s$-distance set on $S^{d-1}$. We define the polynomial $F_X(t)$ of degree $s$:

$$F_X(t) := \prod_{\alpha \in A(X)} (t - \alpha) = \sum_{i=0}^{s} f_i G_{\alpha}^{(d)}(t),$$

where $f_i$ are real numbers. Then,

$$|X| \leq \sum_{i \text{ with } f_i > 0} h_i,$$

(1.1)

where the summation runs through $0 \leq i \leq s$ satisfying $f_i > 0$.

Theorem 1.4 (Antipodal case). Let $X$ be an antipodal $s$-distance set on $S^{d-1}$. There exists $Y$ such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. We define the polynomial $F_Y(t)$ of degree $s - 1$:

$$F_Y(t) := \prod_{\alpha \in A(Y)} (t - \alpha) = \sum_{i=0}^{s-1} f_i G_{\alpha}^{(d)}(t),$$

where $f_i$ are real number and $f_i = 0$ for $i \equiv s \mod 2$. Then,

$$|X| \leq 2 \sum_{i \text{ with } f_i > 0} h_i,$$

(1.2)

where the summation runs through $0 \leq i \leq s$ satisfying $f_i > 0$.

If $f_i > 0$ for all $0 \leq i \leq s$ (antipodal case: $f_i > 0$ for all $i \equiv s - 1 \mod 2$), then this upper bound is the same as Fisher type inequality. Therefore, this upper bound is better than the Fisher type inequality.
2 Proof of main theorems

First, we prepare the two results to prove the main theorems. Let \( \{\varphi_{l,k}\}_{1 \leq k \leq h_l} \) be an orthonormal basis of \( \text{Harm}_l(\mathbb{R}^d) \) with respect to \( \langle \ast, \ast \rangle \), where \( \langle f, g \rangle := \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) g(x) d\sigma(x) \).

**Theorem 2.1** (Addition formula [2][1]). For any \( x, y \) on \( S^{d-1} \), we have
\[
\sum_{k=1}^{h_l} \varphi_{l,k}(x) \varphi_{l,k}(y) = G^{(d)}_l((x, y)).
\]

The following lemma is elementary.

**Lemma 2.1.** Let \( M \) be a symmetric matrix in \( M_n(\mathbb{R}) \) and \( N \) be an \( m \times n \) matrix. \( ^tN \) means the transpose matrix of \( N \). \( D_{u,v} \) is an \( m \times m \) diagonal matrix such that the number of positive entries is \( u \) and the number of negative entries is \( v \). If the equality \( NM^t = D_{u,v} \) holds, then the number of the positive (resp. negative) eigenvalues of \( M \) is bounded below by \( u \) (resp. \( v \)).

**Proof of Theorem 2.3.** Let \( X := \{x_1, x_2, \ldots, x_{|X|}\} \) be an \( s \)-distance set on \( S^{d-1} \). Let \( \{\varphi_{l,k}\}_{1 \leq k \leq h_l} \) be an orthonormal basis of \( \text{Harm}_l(\mathbb{R}^d) \). \( H_l \) is the characteristic matrix that is indexed by \( X \) and an orthonormal basis \( \{\varphi_{l,k}\}_{1 \leq k \leq h_l} \) and whose entry is defined by \( H_l(x_i, \varphi_{l,j}) := \varphi_{l,j}(x_i) \). Then,
\[
[f_0H_0, f_1H_1, \ldots, f_sH_s] \quad \text{and} \quad [H_0, H_1, \ldots, H_s]
\]
are \( |X| \times \sum_{i=0}^s h_i \) matrices. We have the Gegenbauer expansion \( F_X(t) = \prod_{\alpha \in A(X)} \frac{t-\alpha}{1-\alpha} = \sum_{i=0}^s f_i G^{(d)}_i(t) \). By the addition formula,
\[
I_{|X|} = [f_0H_0, f_1H_1, \ldots, f_sH_s] = [H_0, H_1, \ldots, H_s] \text{Diag} \begin{bmatrix} f_0 & f_1 & \cdots & f_s \end{bmatrix},
\]
where \( I_{|X|} \) is the identity matrix of degree \( |X| \), \( \text{Diag}[*] \) means a diagonal matrix and the number of entries \( f_i \) is \( h_i \). By Lemma 2.1,
\[
|X| \leq \sum_{i \text{ with } f_i > 0} h_i.
\]

We can prove the antipodal case by using the same method as above proof.
3 Examples

In this section, we introduce some examples which attain the upper bound in the main theorems.

3.1 The case $s = 1$, $f_0 > 0$ and $f_1 \leq 0$

**Corollary 3.1.** Let $X$ be a 1-distance set and $A(X) = \{\alpha\}$. Then, $F_X(t) := t - \alpha = \sum_{i=0}^{1} f_i G_i^d(t)$ where $f_0 = 1/d$ and $f_1 = -\alpha$. If $\alpha \geq 0$, then

$$|X| \leq h_1 = d.$$

Clearly, a $(d - 1)$-dimensional regular simplex with a nonnegative inner product on $S^{d-1}$ attains this upper bound.

3.2 The case $s = 2$, $f_0 > 0$, $f_1 \leq 0$ and $f_2 > 0$

**Corollary 3.2.** Let $X$ be a 2-distance set and $A(X) = \{\alpha, \beta\}$. Then, $F_X(t) := (t - \alpha)(t - \beta) = \sum_{i=0}^{2} f_i G_i^d(t)$ where $f_0 = \alpha\beta + 1/d$, $f_1 = -(\alpha + \beta)/d$ and $f_2 = 2/(d(d + 2))$. If $\alpha + \beta \geq 0$, then

$$|X| \leq h_0 + h_2 = \left(\frac{d + 1}{2}\right).$$

Musin proved this corollary by using a polynomial method in [3]. The following examples attain this upper bound.

**Example 3.1.** Let $U_d$ be a $d$-dimensional regular simplex. We define

$$X := \left\{ \frac{x + y}{2} \left| x, y \in U_d, x \neq y \right. \right\}$$

for $d \geq 7$. Then, $X$ is a 2-distance set on $S^{d-1}$, $|X| = \left(\frac{d + 1}{2}\right)$, $f_0 > 0$, $f_1 \leq 0$ and $f_2 > 0$.

3.3 Examples from tight spherical $(2s - 1)$-designs

**Corollary 3.3.** Let $X$ be an $s$-distance set on $S^{d-1}$. We have the Gegenbauer expansion $F_X(t) = \prod_{\alpha \in A(X)} (t - \alpha) = \sum_{i=0}^{s} f_i G_i^d(t)$. If $f_i > 0$ for all $i \equiv s \mod 2$ and $f_i \leq 0$ for all $i \equiv s - 1 \mod 2$, then

$$|X| \leq \sum_{i=0}^{\lfloor s/2 \rfloor} h_{s-2i} = \left(\frac{d + s - 2}{s - 1}\right).$$

The following examples attain above upper bound.

**Example 3.2.** Let $X$ be a tight spherical $(2s - 1)$-design, that is, $X$ is an antipodal $s$-distance set which attains the Fisher type inequality [2]. There exist a subset $Y$ such that $X = Y \cup (-Y)$ and $|X| = 2|Y|$. $Y$ is an $(s - 1)$-distance set and $F_Y(t) := \sum_{i=0}^{s-1} f_i G_i^d(t)$. Then, $f_i = 0$ for all $i \equiv s - 2 \mod 2$ and $f_i > 0$ for all $i \equiv s - 1 \mod 2$ and $|Y| = \left(\frac{d + s - 3}{s - 2}\right)$. 
References

[1] Ei. Bannai and Et. Bannai, Algebraic Combinatorics on Spheres, Springer, Tokyo, 1999 (in Japanese).

[2] P. Delsarte, J.M. Goethals and J.J. Seidel: Spherical Codes and Designs, Geom. Dedicata 6 (1977), No. 3, 363–388.

[3] O.R. Musin: On spherical two-distance sets, preprint.