A multivalued logarithm on time scales

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A new definition of a multivalued logarithm on time scales is introduced for delta-differentiable functions that never vanish. This new logarithm arises naturally from the definition of the cylinder transformation that is also the wellspring of the definition of exponential functions on time scales. This definition will lead to a logarithm function on arbitrary time scales with familiar and useful properties that previous definitions in the literature lacked.

\( \sigma : T \to T \), via \( \sigma (t) = \inf \{ s \in T : s > t \} \), and the graininess function \( \mu (t) = \sigma (t) - t \); see [1–4] for more details. A recurring open problem for time scales and dynamic equations [3,4,9] has been the following [5]: On time scales, define and present the properties of a “nice” logarithm function. The aim of what follows below is to introduce on time scales a novel multivalued logarithm arising from the cylinder transformation employed in definitions of exponential functions for dynamic equations.

The development of this logarithm on general time scales will proceed as follows. In Section 2, we extend the definition of the traditional single valued cylinder transformation to a multivalued cylinder transformation. This transformation has useful properties across the circle plus (\( \oplus \)) and circle dot (\( \odot \)) operations, and is the basis for the definition of the logarithm, for non-vanishing delta-differentiable functions. In Section 3, nice properties of this new logarithm are shown to hold. Section 4 establishes a similar logarithm for the nabla case. In Section 5, the Cayley cylinder transformation is also considered, and is shown to lead to the very same logarithm. In Section 6, we give a listing of extant logarithm functions on time scales from the literature. Finally, in Section 7, we give a numerical comparison of the various logarithms on a specific time scale, and give numerous examples on various time scales illustrating the properties of the new one. For trends on time scales generally, see the recent works [1,2,7,8].

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2. A new logarithm on time scales

We begin our presentation of a new definition on general time scales of a logarithm for dynamic equations, with some motivation provided by the definition of exponential functions for dynamic equations based on the cylinder transformation. The following definition [3, Definition 2.21] (see also Hilger [9, Section 7]) is the original cylinder transformation; a modified cylinder transformation will also be examined, in Section 5.

**Definition 2.1** (Single Valued Cylinder Transformation). Fix $h > 0$, and define the cylinder transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$
\xi_h(z) = \begin{cases} 
\frac{1}{h} \log(1 + zh) & \text{for } h \neq 0 \\
\frac{1}{z} & \text{for } h = 0,
\end{cases}
$$

(2.1)

where $\mathbb{C}$ is the set of complex numbers,

$$
\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \quad \mathbb{Z}_h = \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h} \right\},
$$

(2.2)

and Log represents the principal complex logarithm function.

The following definition is [3, Definition 2.25].

**Definition 2.2** (Regressive Function). A function $p : T \to \mathbb{R}$ is regressive granted

$$
1 + \mu(\tau)p(\tau) \neq 0 \quad \text{for each } \tau \in T^k
$$

holds. We will denote via $\mathcal{R}$ the set of all rd-continuous and regressive functions $p : T \to \mathbb{R}$.

The following definition is [3, Definition 2.30].

**Definition 2.3** (Exponential Function). For functions $p \in \mathcal{R}$, the time scales exponential function is formulated via

$$
e^p(t,s) = \exp \left( \int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau \right) \quad \text{for } s, t \in T;
$$

here, $\xi_h(z)$ is the cylinder transformation given in (2.1).

We now set the foundation for offering a new definition of logarithms on time scales. This definition will be of a multivalued function, for which we need to modify the single valued cylinder function given in (2.1).

**Definition 2.4** (Multivalued Cylinder Transformation). Fix $h > 0$, and define the multivalued cylinder transformation $\zeta_h : \mathbb{C}_h \to \mathbb{C}$ by

$$
\zeta_h(z) = \begin{cases} 
\frac{1}{h} \log(1 + zh) & \text{for } h \neq 0 \\
\frac{1}{z} & \text{for } h = 0,
\end{cases}
$$

(2.3)

where the set of complex numbers is $\mathbb{C}$, the set $\mathbb{C}_h$ is given in (2.2), and log represents the multivalued complex logarithm function.

**Lemma 2.5.** Let $f, g : T \to \mathbb{C}$ be $\Delta$-differentiable functions with $f, g \neq 0$ on $T$, and let the multivalued cylinder transformation $\zeta$ be given by (2.3). Then, for fixed $\tau \in T^k$,

$$
\zeta_\mu(\tau) \left( \left( \frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g} \right)(\tau) \right) = \zeta_\mu(\tau) \left( \frac{f^\Delta(\tau)}{f(\tau)} \right) + \zeta_\mu(\tau) \left( \frac{g^\Delta(\tau)}{g(\tau)} \right).
$$

**Proof.** First, note that the useful yet simple formula $f^\sigma = \mu f^\Delta + f$ (suppressing the variable) implies

$$
\frac{(fg)^\Delta}{fg} = \frac{f^\sigma g^\Delta + f^\Delta g}{fg} = \frac{(f + \mu f^\Delta)g^\Delta}{fg} + \frac{f^\Delta}{f} = \frac{f^\Delta}{f} + \frac{g^\Delta}{g} + \mu \frac{f^\Delta g^\Delta}{fg} = \frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}.
$$

From this, we observe that for fixed $\tau \in T^k$. 


\[ \zeta_{\mu(\tau)} \left( \left( \frac{f^\Delta}{T} \oplus \frac{g^\Delta}{s} \right)(\tau) \right) = \zeta_{\mu(\tau)} \left( \frac{(fg)^\Delta(\tau)}{(fg)(\tau)} \right) = \frac{1}{\mu(\tau)} \log \left( 1 + \mu(\tau) \frac{(fg)^\Delta(\tau)}{(fg)(\tau)} \right) \text{ for } \mu(\tau) \neq 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( \frac{(fg)\theta(\tau)}{(fg)(\tau)} \right) \text{ for } \mu(\tau) = 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( \frac{f^\alpha(\tau)}{f(\tau)} \right) + \frac{1}{\mu(\tau)} \log \left( \frac{g^\alpha(\tau)}{g(\tau)} \right) \text{ for } \mu(\tau) \neq 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( \frac{(f + \mu f^\Delta)(\tau)}{f(\tau)} \right) + \frac{1}{\mu(\tau)} \log \left( \frac{(g + \mu g^\Delta)(\tau)}{g(\tau)} \right) \text{ for } \mu(\tau) = 0 \] 
\[ = \zeta_{\mu(\tau)} \left( \frac{f^\Delta(\tau)}{f(\tau)} \right) + \zeta_{\mu(\tau)} \left( \frac{g^\Delta(\tau)}{g(\tau)} \right). \]

The proof is complete. \( \square \)

**Lemma 2.6.** Let \( \alpha \in \mathbb{R} \), and let \( p : \mathbb{T} \to \mathbb{C} \) be a \( \Delta \)-differentiable function with \( p \neq 0 \) on \( \mathbb{T} \). For the multivalued cylinder transformation \( \zeta \) given by (2.3) and for fixed \( \tau \in \mathbb{T}^\kappa \),
\[ \zeta_{\mu(\tau)} \left( \left[ \alpha \odot \frac{p^\Delta}{p} \right](\tau) \right) = \alpha \zeta_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right). \]

**Proof.** Let \( \alpha \in \mathbb{R} \), and let \( p : \mathbb{T} \to \mathbb{C} \) be a \( \Delta \)-differentiable function with \( p \neq 0 \) on \( \mathbb{T} \). Then [4, Theorem 2.43] yields
\[ 1 + \mu(\alpha \odot f) = (1 + \mu f)^\alpha \]
on \( \mathbb{T}^\kappa \) for \( f = \frac{p^\Delta}{p} \). It follows that for fixed \( \tau \in \mathbb{T}^\kappa \),
\[ \zeta_{\mu(\tau)} \left( \left[ \alpha \odot \frac{p^\Delta}{p} \right](\tau) \right) \]
\[ = \frac{1}{\mu(\tau)} \log \left( 1 + \mu(\tau) \left[ \alpha \odot \frac{p^\Delta}{p} \right](\tau) \right) \text{ for } \mu(\tau) \neq 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( \frac{1 + \mu(\tau) \alpha \frac{p^\Delta(\tau)}{p(\tau)}}{\mu \frac{p^\Delta(\tau)}{p(\tau)}} \right) \text{ for } \mu(\tau) = 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( \frac{1 + \mu(\tau) \alpha \frac{p^\Delta(\tau)}{p(\tau)}}{\mu \frac{p^\Delta(\tau)}{p(\tau)}} \right) \text{ for } \mu(\tau) \neq 0 \] 
\[ = \frac{1}{\mu(\tau)} \log \left( 1 + \mu(\tau) \frac{p^\Delta(\tau)}{p(\tau)} \right) \text{ for } \mu(\tau) = 0 \] 
\[ = \alpha \frac{1}{\mu(\tau)} \log \left( 1 + \mu(\tau) \frac{p^\Delta(\tau)}{p(\tau)} \right) \text{ for } \mu(\tau) \neq 0 \] 
\[ = \alpha \zeta_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right) \text{ for } \mu(\tau) = 0 \] 
\[ = \alpha \zeta_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right). \]

The proof is complete. \( \square \)

**Definition 2.7 (Logarithm Function).** Given a \( \Delta \)-differentiable function \( p : \mathbb{T} \to \mathbb{C} \) with \( p \neq 0 \) on \( \mathbb{T} \), the multivalued logarithm function on time scales is given by
\[ \ell_{p}(t, s) = \int_{s}^{t} \zeta_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta \tau \text{ for } s, t \in \mathbb{T}, \]
where $\zeta_p(z)$ is the multivalued cylinder transformation given in (2.3). Define the principal logarithm on time scales to be

$$L_p(t, s) = \int_s^t \frac{\Delta^\mu(\tau)}{p(\tau)} \Delta \tau$$

for $s, t \in \mathbb{T}$.

where $\xi_b(z)$ is the single valued cylinder transformation given in (2.1).

**Remark 2.8.** According to this definition, if $p \equiv \text{constant}$, then $L_p(t, s) = 0$ for each $t, s \in \mathbb{T}$. Thus, this logarithm does not distinguish between either constants or constant multiples of functions. We moreover note here that even when we restrict the time scale to $\mathbb{T} = \mathbb{R}$, the dynamics along the negative and positive real line necessitate the existence of a logarithm with principal and multiple values, making a multivalued logarithm on general time scales both natural and expected, though heretofore unexplored.

3. Properties of the logarithm

Using the definition of the multivalued logarithm on time scales given above in Definition 2.7, we establish the following properties.

**Theorem 3.1.** If $p: \mathbb{T} \to \mathbb{C}$ is a $\Delta$-differentiable function with $p \neq 0$ on $\mathbb{T}$, then

$$\exp(L_p(t, s)) = e^{\frac{\Delta}{p(\tau)}}(t, s) \quad t, s \in \mathbb{T}.$$ 

In particular, if $p \in \mathbb{R}$, then

$$\exp(L_p(t, s)) = p(t, s) \quad t, s \in \mathbb{T}.$$ 

**Proof.** Presuming $p: \mathbb{T} \to \mathbb{C}$ is a $\Delta$-differentiable function with $p \neq 0$ on $\mathbb{T}$,

$$L_p(t, s) = \int_s^t \frac{\Delta^\mu(\tau)}{p(\tau)} \Delta \tau.$$ 

Now, exponentiate both sides and use the definition of $e_p(t, s)$, the exponential function. □

**Theorem 3.2** (Logarithm of Product, Quotient, & Power). Suppose $f, g, p: \mathbb{T} \to \mathbb{C}$ are $\Delta$-differentiable functions with $f, g, p \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$ and $\alpha \in \mathbb{R}$, we have the following:

(i) $\ell_{fg}(t, s) = \ell_f(t, s) + \ell_g(t, s)$,

(ii) $\ell_{f/g}(t, s) = \ell_f(t, s) - \ell_g(t, s)$,

(iii) $\ell_{p^\alpha}(t, s) = \alpha \ell_p(t, s)$.

**Proof.** Presume $f, g, p: \mathbb{T} \to \mathbb{C}$ are $\Delta$-differentiable functions with $f, g, p \neq 0$ on $\mathbb{T}$. Then, for $s, t \in \mathbb{T}$, we have via Lemma 2.5 and its proof that

$$\ell_{fg}(t, s) = \int_s^t \zeta(\mu)\left(\frac{(fg)^\Delta(\tau)}{(fg)(\tau)}\right) \Delta \tau$$

$$= \int_s^t \zeta(\mu)\left(\frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}\right)(\tau) \Delta \tau$$

$$= \int_s^t \zeta(\mu)\left(\frac{f^\Delta(\tau)}{f(\tau)}\right) \Delta \tau + \int_s^t \zeta(\mu)\left(\frac{g^\Delta(\tau)}{g(\tau)}\right) \Delta \tau$$

$$= \ell_f(t, s) + \ell_g(t, s).$$

In a similar manner,

$$\ell_{f/g}(t, s) = \int_s^t \zeta(\mu)\left(\frac{\frac{f}{g}^\Delta(\tau)}{\frac{f}{g}(\tau)}\right) \Delta \tau$$

$$= \int_s^t \zeta(\mu)\left(\frac{f^\Delta}{f} \oplus \frac{g^\Delta}{g}\right)(\tau) \Delta \tau$$

$$= \int_s^t \zeta(\mu)\left(\frac{f^\Delta(\tau)}{f(\tau)}\right) \Delta \tau - \int_s^t \zeta(\mu)\left(\frac{g^\Delta(\tau)}{g(\tau)}\right) \Delta \tau$$

$$= \ell_f(t, s) - \ell_g(t, s).$$

Let $\alpha \in \mathbb{R}$. For the multivalued cylinder transformation $\zeta$ given by (2.3) and for fixed $\tau \in \mathbb{T}^*$,

$$\zeta(\mu)\left(\frac{\alpha \circ \frac{p^\Delta}{p}(\tau)}{\frac{p^\Delta}{p}(\tau)}\right) = \alpha \zeta(\mu)\left(\frac{p^\Delta(\tau)}{p(\tau)}\right)$$


using Lemma 2.6. Moreover, by [4, Theorem 2.37], we have
\[
\left( \frac{p^{\Delta}}{p^\omega} \right) = \alpha \odot \frac{p^\Delta}{p}.
\]
Consequently,
\[
\ell_p (t, s) = \int_s^t \xi_{\mu(t)} \left( \left( \frac{p^\Delta (t)}{p^\omega (t)} \right) \Delta \tau \right.
\]
\[
= \int_s^t \xi_{\mu(t)} \left( \left( \alpha \odot \frac{p^\Delta}{p} \right) (t) \right) \Delta \tau
\]
\[
= \int_s^t \alpha \xi_{\mu(t)} \left( \left( \frac{p^\Delta (t)}{p(t)} \right) \right) \Delta \tau
\]
\[
= \alpha \ell_p (t, s).
\]
The argument proves sufficient. □

**Theorem 3.3.** Let \( p : T \to \mathbb{R} \) be a \( \Delta \)-differentiable function with \( p \neq 0 \) on \( T \). Then, for \( s, t \in T, \) we have
\[
\ell^\Delta_p (t, s) = \frac{1}{\mu(t)} \log \left( \frac{p^\omega (t)}{p^\omega (s)} \right)
\]
for \( \mu (t) \neq 0 \),
\[
\ell^\Delta_p (t, s) = \frac{\mu(t)}{\mu(t)} \log \left( 1 + \mu (t) \frac{p^\Delta (t)}{p(t)} \right)
\]
for \( \mu (t) = 0 \).

where \( \Delta \)-differentiation is with respect to \( t \).

**Proof.** Using the definition of the logarithm and \( \Delta \)-differentiating with respect to \( t \).
\[
\ell^\Delta_p (t, s) = \zeta_{\mu(t)} \left( \frac{p^\Delta (t)}{p(t)} \right)
\]
\[
= \begin{cases} 
\frac{1}{\mu(t)} \log \left( 1 + \mu(t) \frac{p^\Delta (t)}{p(t)} \right) & \text{for } \mu(t) \neq 0 \\
\frac{\mu(t)}{\mu(t)} \log \left( 1 + \mu(t) \frac{p^\Delta (t)}{p(t)} \right) & \text{for } \mu(t) = 0.
\end{cases}
\]
Now substitute \( \mu p^\Delta = p^\omega - p \). The argument proves sufficient. □

4. The nabla case

A logarithm is also possible for the nabla case.

**Definition 4.1** (Cylinder Transformation). For \( h > 0 \), define the single valued cylinder transformation \( \xi_h : \hat{\mathbb{C}}_h \to \mathbb{Z}_h \) by
\[
\hat{\xi}_h (z) = \begin{cases} \frac{-1}{h} \log (1 - zh) & \text{for } h \neq 0 \\
\frac{1}{h} & \text{for } h = 0
\end{cases}
\]
and the multivalued cylinder transformation \( \xi_h : \hat{\mathbb{C}}_h \to \mathbb{C} \) by
\[
\hat{\xi}_h (z) = \begin{cases} \frac{-1}{h} \log (1 - zh) & \text{for } h \neq 0 \\
\frac{1}{h} & \text{for } h = 0.
\end{cases}
\]

Here \( \mathbb{C} \) is the set of complex numbers, \( \mathbb{Z}_h \) is in (2.2),
\[
\hat{\mathbb{C}}_h = \left\{ z \in \mathbb{C} : z \neq \frac{1}{h} \right\},
\]
and as before, \( \log \) represents the principal complex logarithm function.

The following definition is [4, Definition 3.4].

**Definition 4.2** (Regressive Function). A function \( p : T \to \mathbb{R} \) is \( \nu \)-regressive granted
\[
1 - \nu(t)p(t) \neq 0 \quad \text{for each} \quad t \in T,
\]
holds. Let \( \hat{\mathbb{R}} \) signify the set of all ld-continuous and \( \nu \)-regressive functions \( p : T \to \mathbb{R} \).

The following definition is [4, Definition 3.10].
Definition 4.3 (Exponential Function). Let \( t, s \in \mathbb{T} \). For functions \( p \in \hat{\mathbb{R}} \), the time scales nabla exponential function is formulated via

\[
\hat{e}_p(t, s) = \exp \left( \int_s^t \xi_{\nu(\tau)}(p(\tau)) \nabla \tau \right),
\]

where \( \hat{\xi}_p(z) \) is the single valued cylinder transformation given in (4.1).

We now offer a new definition of logarithms for the nabla case on time scales.

Definition 4.4 (Logarithm Function). Given a \( \nabla \)-differentiable function \( p : \mathbb{T} \to \mathbb{R} \) with \( p \neq 0 \) on \( \mathbb{T} \), the multivalued nabla logarithm function on time scales is given by

\[
\hat{\log}_p(t, s) = \int_s^t \hat{\xi}_{\nu(\tau)} \left( \frac{p^\nu(\tau)}{p(\tau)} \right) \nabla \tau \quad \text{for} \quad s, t \in \mathbb{T},
\]

where \( \hat{\xi}_p(z) \) is the multivalued cylinder transformation given in (4.2), while the principal nabla logarithm is given by

\[
\hat{\log}_p(t, s) = \int_s^t \hat{\xi}_{\nu(\tau)} \left( \frac{p^\nu(\tau)}{p(\tau)} \right) \nabla \tau \quad \text{for} \quad s, t \in \mathbb{T},
\]

where \( \hat{\xi}_p(z) \) is the single valued nabla cylinder transformation given in (4.1).

Properties analogous to those given earlier can be established for the nabla case as well.

5. Logarithms for Cayley-exponential functions

In [6], the author introduced another time scales exponential function, dubbed the Cayley-exponential function, defined by

\[
E_p(t, s) = \exp \left( \int_s^t \Psi_{\mu(\tau)}(p(\tau)) \Delta \tau \right), \quad (5.1)
\]

where \( p : \mathbb{T} \to \mathbb{C} \) is rd-continuous and satisfies the regresivity condition \( \mu(\tau) p(\tau) \neq \pm 2 \) for all \( \tau \in \mathbb{T}^\kappa \), and the modified cylinder transformation \( \Psi \) is given by

\[
\Psi_p(z) = \frac{1}{h} \log \left( \frac{1 + \frac{1}{2}zh}{1 - \frac{1}{2}zh} \right), \quad \Psi_0(z) = z, \quad (5.2)
\]

for \( h > 0 \). Once more, Log represents the principal complex logarithm. Consider the multivalued function version of (5.2) denoted, i.e.,

\[
\psi_p(z) = \frac{1}{h} \log \left( \frac{1 + \frac{1}{2}zh}{1 - \frac{1}{2}zh} \right), \quad \psi_0(z) = z, \quad (5.3)
\]

where \( \log \) represents the multivalued complex logarithm. We introduce the following Cayley-logarithm functions on time scales.

Definition 5.1. For a \( \Delta \)-differentiable function \( p : \mathbb{T} \to \mathbb{C} \) with \( p \neq 0 \) on \( \mathbb{T} \), the multivalued Cayley-logarithm function on time scales is given by

\[
cay\log_p(t, s) = \int_s^t \psi_{\mu(\tau)} \left( \frac{2p^\Delta(\tau)}{p(\tau) + p^\Delta(\tau)} \right) \Delta \tau \quad \text{for} \quad s, t \in \mathbb{T},
\]

where \( \psi_p(z) \) is the multivalued cylinder transformation given in (5.3). Define the principal Cayley-logarithm on time scales to be

\[
\text{CayLog}_p(t, s) = \int_s^t \psi_{\mu(\tau)} \left( \frac{2p^\Delta(\tau)}{p(\tau) + p^\Delta(\tau)} \right) \Delta \tau \quad \text{for} \quad s, t \in \mathbb{T},
\]

where \( \psi_p(z) \) is the single valued cylinder transformation given in (5.2).

Lemma 5.2. The Cayley-logarithm functions are well-defined functions.

Proof. For a \( \Delta \)-differentiable function \( p : \mathbb{T} \to \mathbb{C} \) with \( p \neq 0 \) on \( \mathbb{T} \), we need to show that

\[
\mu(\tau) - \frac{2p^\Delta(\tau)}{p(\tau) + p^\Delta(\tau)} \neq \pm 2.
\]
in other words, that the regressivity condition holds. The following are equivalent:

\[
\frac{2\mu(\tau)p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)} = \pm 2 \iff \frac{p'^\lambda(\tau) - p(\tau)}{p(\tau) + p'^\lambda(\tau)} = \pm 1
\]

\[
p'^\lambda(\tau) - p(\tau) = \pm (p(\tau) + p'^\lambda(\tau)) \iff \frac{p'^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)} = p(\tau) \pm p(\tau).
\]

so that we have either \(0 = 2p(\tau)\) or \(2p'^\lambda(\tau) = 0\), both contradictions. \(\square\)

**Theorem 5.3.** For a \(\Delta\)-differentiable function \(p : \mathbb{T} \to \mathbb{C}\) with \(p \neq 0\) on \(\mathbb{T}\),

\[
\text{caylog}_p(t, s) = \ell_p(t, s) \quad \text{and} \quad \text{CayLog}_p(t, s) = L_p(t, s)
\]

(5.4)

for all \(s, t \in \mathbb{T}\).

**Proof.** Consider (5.3). For fixed \(\tau \in \mathbb{T}^\ast\) with \(\mu(\tau) \neq 0\), observe that

\[
\psi_{\mu(\tau)}\left(\frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right) = \frac{1}{\mu(\tau)} \log \left(1 + \frac{\mu(\tau)p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right)
\]

\[
= \frac{1}{\mu(\tau)} \log \left(1 + \frac{\mu(\tau)p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right)
\]

\[
= \frac{1}{\mu(\tau)} \log \left(1 + \frac{p'^\lambda(\tau)}{p(\tau)}\right)
\]

\[
= \frac{1}{\mu(\tau)} \log \left(\frac{p(\tau) + \mu(\tau)p^\lambda(\tau)}{p(\tau)}\right)
\]

\[
= \zeta_{\mu(\tau)}\left(\frac{p^\lambda(\tau)}{p(\tau)}\right)
\]

for \(\zeta_{\delta(\tau)}\) defined in (2.3). For fixed \(\tau \in \mathbb{T}^\ast\) with \(\mu(\tau) = 0\), we have \(\tau = \sigma(\tau)\) and

\[
\frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)} = \frac{p^\lambda(\tau)}{p(\tau)}.
\]

Consequently,

\[
\psi_{\mu(\tau)}\left(\frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right) = \frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)} = \frac{p^\lambda(\tau)}{p(\tau)} = \zeta_{\mu(\tau)}\left(\frac{p^\lambda(\tau)}{p(\tau)}\right).
\]

Thus, in either case, we have

\[
\psi_{\mu(\tau)}\left(\frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right) = \zeta_{\mu(\tau)}\left(\frac{p^\lambda(\tau)}{p(\tau)}\right).
\]

It follows that

\[
\text{caylog}_p(t, s) = \int_s^t \psi_{\mu(\tau)}\left(\frac{2p^\lambda(\tau)}{p(\tau) + p'^\lambda(\tau)}\right) \Delta \tau = \int_s^t \zeta_{\mu(\tau)}\left(\frac{p^\lambda(\tau)}{p(\tau)}\right) \Delta \tau = \ell_p(t, s).
\]

Similarly, we have

\[
\text{CayLog}_p(t, s) = L_p(t, s),
\]

completing the proof. \(\square\)

**Remark 5.4.** The previous theorem and proof may be generalized, as we will now show. Let \(\theta \in [0, 1]\), and set

\[
\psi^\theta(z) = \frac{1}{\theta} \log \left(\frac{1 + (1 - \theta)z}{1 - \theta z}\right), \quad \psi^\theta_0(z) = z.
\]

(5.5)

Then, for a \(\Delta\)-differentiable function \(p : \mathbb{T} \to \mathbb{C}\) with \(p \neq 0\) on \(\mathbb{T}\), and for all \(\tau \in \mathbb{T}^\ast\), we have
\[
\psi^\theta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} \right) \\
= \frac{1}{\mu(\tau)} \log \left( \frac{1 + (1 - \theta)\mu(\tau)}{1 - \theta} \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} \right) \\
= \frac{1}{\mu(\tau)} \log \left( \frac{(1 - \theta)p(\tau) + \theta p^\sigma(\tau) + (1 - \theta)\mu(\tau)p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau) - \theta \mu(\tau)p^\lambda(\tau)} \right) \\
= \frac{1}{\mu(\tau)} \log \left( \frac{p^\lambda(\tau)}{p(\tau)} \right) \\
= \frac{1}{\mu(\tau)} \log \left( \frac{p(\tau) + \mu(\tau)p^\lambda(\tau)}{p(\tau)} \right) \\
= \zeta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{p(\tau)} \right)
\]

for \( \zeta_h \) defined in (2.3). For fixed \( \tau \in \mathbb{T}^\times \) with \( \mu(\tau) = 0 \), we have \( \tau = \sigma(\tau) \) and
\[
\frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} = \frac{p^\lambda(\tau)}{p(\tau)}.
\]

As a result,
\[
\psi^\theta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} \right) = \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} = \frac{p^\lambda(\tau)}{p(\tau)} = \zeta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{p(\tau)} \right).
\]

Thus, in either case, we have
\[
\psi^\theta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} \right) = \zeta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{p(\tau)} \right)
\]
for all \( \theta \in [0, 1] \). Consequently,
\[
\log^\theta_p(t, s) := \int_s^t \psi^\theta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{(1 - \theta)p(\tau) + \theta p^\sigma(\tau)} \right) \Delta \tau = \int_s^t \zeta_{\mu(\tau)} \left( \frac{p^\lambda(\tau)}{p(\tau)} \right) \Delta \tau = \ell_p(t, s).
\]

This ends the remark.

6. Previous logarithms on time scales

As shown in previous sections, the key to arriving at useful logarithm properties is to allow for a multivalued logarithm, as exists for the \( \mathbb{T} = \mathbb{R} \) case. Here, we present the previous definitions of a logarithm on time scales, noting that they are all single valued functions. Moreover, only Definition 2.7 leads to results as given in Theorem 3.1, Theorem 3.2, Theorem 5.3, and Remark 5.4, justifying this new approach, and emphasizing the advantages of having a function satisfying familiar properties, while enconced in the more general time scales context.

The first logarithm on time scales [10] interprets the integral
\[
\int_{t_0}^t \frac{2}{\tau + \sigma(\tau)} \Delta \tau
\]
as a time scales analogue of \( \ln t \). This is understandable, because if \( \mathbb{T} = \mathbb{R} \), then \( \tau = \sigma(\tau) \), and
\[
\int_{t_0}^t \frac{2}{\tau + \sigma(\tau)} \Delta \tau = \int_{t_0}^t \frac{2}{2\tau} d\tau = \ln t - \ln t_0.
\]

A recent paper [11] applies iterates of this logarithm to Riemann–Weber-type equations.

A second approach [5, Section 3] is to view the slightly different integral
\[
\int_{t_0}^t \frac{1}{\tau + 2\mu(\tau)} \Delta \tau
\]
as the time scales version of \( \ln t \). Due to the same fact that it reduces to \( \ln t - \ln t_0 \) on \( \mathbb{T} = \mathbb{R} \), and as it is part of a solution form to a certain Euler-Cauchy dynamic equation whose differential equation analogue involves the natural logarithm.
A third approach [5, Section 4] could be to define a logarithm via

\[ L_p(t, t_0) = \int_{t_0}^{t} \frac{p^\lambda(\tau)}{p(\tau)} \Delta \tau \]

for \( \Delta \)-differentiable functions \( p : \mathbb{T} \to \mathbb{R} \). Clearly if \( p(\tau) = \tau \), then this is

\[ L_p(t, t_0) = \int_{t_0}^{t} \frac{p^\lambda(\tau)}{p(\tau)} \Delta \tau = \int_{t_0}^{t} \frac{1}{\tau} \Delta \tau, \]

a form that is similar to its continuous analogue for \( \mathbb{T} = \mathbb{R} \).

A fourth approach [12] is to take the logarithm to be given by

\[ \log_T p(t) = \frac{p^\lambda(t)}{p(t)} \]

for \( \Delta \)-differentiable functions \( p : \mathbb{T} \to \mathbb{R} \), where the time scale logarithm on \( \mathbb{R} \) does not play the role of the logarithm, clearly, but rather its derivative. The motivation here is to maintain some attractive algebraic properties of logarithms, and to serve in some sense as an inverse to the exponential function.

A fifth approach [13], only for time scales such that \( 1 \in \mathbb{T} \), is to define the natural logarithm via

\[ L^\lambda_T(t) = \int_{1}^{t} \frac{1}{\tau} \Delta \tau, \]

which hearkens back to [5, Section 4]. Here the motivation is clearly that

\[ L^\lambda_T(1) = 0, \quad L^\lambda_T(t) = \frac{1}{t}. \]

7. Numerical comparisons and examples of logarithms

Each of the definitions given in the previous section has advantages and drawbacks, and each one satisfies some of what one might wish for in a logarithm function. As shown earlier in this work, however, a multivalued logarithm on time scales with a definition based on cylinder transformations is a natural move that leads to nice properties, and has not been introduced until now. We now consider the following examples.

**Example 7.1.** In this example, we compare the values of the various logarithms on the time scale

\[ \mathbb{T} := (-\infty, -k) \cup \{-k+1, -k+2, \ldots, -1, 0, 1, \ldots, k-2, k-1\} \cup \{k, \infty\}, \quad k \in \mathbb{N}. \]

For \( p(t) = t \) on \([1, k+3]_{\mathbb{T}}\), we have the following plot and table of comparison for the logarithms on time scales mentioned in the literature to date.

| Citation | Logarithm | Value at \( t = 6 \) | Fig. 1 Color |
|----------|-----------|---------------------|-------------|
| [10]     | \( \sum_{j=1}^{k+1} \frac{2}{j+2} + \ln \left( \frac{t}{k} \right) \) | 1.75692 | blue        |
| [5, Section 3] | \( \sum_{j=1}^{k+1} \frac{1}{j+2} + \ln \left( \frac{t}{k} \right) \) | 1.13232 | orange      |
| [5, Section 4] | \( \sum_{j=1}^{k+1} \frac{1}{j} + \ln \left( \frac{t}{k} \right) \) | 2.26565 | green       |
| [12]     | \( \sum_{j=1}^{k+1} \frac{1}{j} \) | 2.26565 | green       |
| [13]     | \( \sum_{j=1}^{k+1} \frac{1}{j} \) | 2.26565 | green       |
| Definition 2.7 | \( \sum_{j=1}^{k+1} \ln \left( \frac{j+1}{j} \right) + \ln \left( \frac{1}{k} \right) \) | 1.79176 | red         |

As can be seen in the table, the new definition presented in this paper, **Definition 2.7**, leads to a unique and accurate value for this time scale. The comparison of graphs on \([1.8]_{\mathbb{T}} = [1, 2, 3, 4] \cup [5, 8] \) is given in Fig. 1.

In the rest of this section, we provide numerous examples of the new logarithm from **Definition 2.7**, for various time scales.

**Example 7.2.** For \( \mathbb{T} = \mathbb{R} \),

\[ \ell_p(t, s) = \int_s^t \zeta(t) \left( \frac{p^\lambda(\tau)}{p(\tau)} \right) \Delta \tau = \int_s^t \frac{p'(\tau)}{p(\tau)} d\tau = \log \left( \frac{p(t)}{p(s)} \right), \]
where \( \log \) represents the multivalued complex logarithm function. For \( T = h\mathbb{Z} \),

\[
\Lambda^A(\tau) = \Lambda_h \Lambda(\tau) := \frac{\Lambda(h \tau + \tau) - \Lambda(\tau)}{h}
\]

and

\[
\ell_p(t, s) = \int_s^t \zeta_{\mu(\tau)} \left( \frac{p^A(\tau)}{p(\tau)} \right) d\tau = \sum_{j=\frac{t}{h}}^{\frac{s}{h} - 1} \zeta_h \left( \frac{\Delta_h p(jh)}{p(jh)} \right) h = \sum_{j=\frac{t}{h}}^{\frac{s}{h} - 1} \log \left( 1 + \frac{h \Delta_h p(jh)}{p(jh)} \right) h
\]

\[
= \sum_{j=\frac{t}{h}}^{\frac{s}{h} - 1} \log \left( \frac{p(jh + h)}{p(jh)} \right) = \log \left( \prod_{j=\frac{t}{h}}^{\frac{s}{h} - 1} \frac{p((j + 1)h)}{p(jh)} \right) = \log \left( \frac{p(t)}{p(s)} \right).
\]

For \( T = q\mathbb{Z} \),

\[
f^A(\tau) = D_q f(\tau) := \frac{f(q \tau) - f(\tau)}{(q - 1)\tau}
\]

and

\[
\ell_p(t, s) = \int_s^t \zeta_{\mu(\tau)} \left( \frac{p^A(\tau)}{p(\tau)} \right) d\tau = \sum_{\tau \in [s, t]} \zeta_{(q-1)\tau} \left( \frac{p^A(\tau)}{p(\tau)} \right) (q - 1)\tau
\]

\[
= \sum_{\tau \in [s, t]} \frac{1}{(q - 1)\tau} \log \left( 1 + \frac{(q - 1)\tau p^A(\tau)}{p(\tau)} \right) (q - 1)\tau
\]

\[
= \sum_{\tau \in [s, t]} \log \left( \frac{p(q \tau)}{p(\tau)} \right) = \log \left( \frac{p(t)}{p(s)} \right).
\]

This ends the example.

**Example 7.3.** For real numbers \( a, b, c, d \) with \( a < b < c < d \), set \( T = [a, b] \cup [c, d] \). Assume \( p : T \to \mathbb{C} \) is differentiable with \( p \neq 0 \) on \( T \). If \( s, t \in [a, b] \) or \( s, t \in [c, d] \), then \( \mu(\tau) \equiv 0 \) for \( \tau \in [s, t] \), so that by the definition of the multivalued cylinder function (2.3),

\[
\ell_p(t, s) = \int_s^t \frac{p'(\tau)}{p(\tau)} d\tau = \log \left( \frac{p(t)}{p(s)} \right).
\]
Presume without loss of generality that $s \in [a, b]$ and $t \in [c, d]$. Then $c = \sigma(b)$, and

$$
\ell_p(t, s) = \int_s^t \xi_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta \tau
$$

$$
= \left( \int_s^b + \int_b^a + \int_a^t \right) \xi_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta \tau
$$

$$
= \log \left( \frac{p(b)}{p(s)} \right) + \log \left( \frac{p(t)}{p(\sigma(b))} \right) + \int_b^a \xi_{\mu(\tau)} \left( \frac{p^\Delta(\tau)}{p(\tau)} \right) \Delta \tau
$$

$$
= \log \left( \frac{p(b)}{p(s)} \right) + \log \left( \frac{p(t)}{p(c)} \right) + \frac{\mu(b)}{\mu(b)} \left( \frac{1}{\mu(b)} \log \left( 1 + \frac{\mu(b)p^\Delta(b)}{p(b)} \right) \right)
$$

$$
= \log \left( \frac{p(t)}{p(s)} \right) + \log \left( \frac{p(t)}{p(c)} \right) + \frac{\mu(b)}{\mu(b)} \left( \frac{1}{\mu(b)} \log \left( 1 + \frac{\mu(b)p^\Delta(b)}{p(b)} \right) \right)
$$

$$
= \log \left( \frac{p(t)}{p(s)} \right).
$$

Consequently, in all cases, we see that $\ell_p(t, s) = \log \left( \frac{p(t)}{p(s)} \right)$ on this time scale as well.

**Example 7.4.** Let $\mathbb{T} = (-\infty, -4] \cup [2, \infty)$, and $p(t) = t^3$. Let $t \geq 2$ and $s = -5$. Then

$$
\mu(-4) = \sigma(-4) - (-4) = 2 - (-4) = 6,
$$

and the principal logarithm on this time scale is

$$
L_p(t, s) = L_p(t, -5) = \int_{-5}^t \xi_{\mu(\tau)} \left( \frac{(\tau^3)^\Delta}{\tau^3} \right) \Delta \tau
$$

$$
= \left( \int_{-5}^{-4} + \int_{-4}^{-2} + \int_{-2}^{t} \right) \xi_{\mu(\tau)} \left( \frac{\sigma(\tau)^2 + \tau \sigma(\tau) + \tau^2}{\tau^3} \right) \Delta \tau
$$

$$
= 3 \left( \int_{-5}^{-4} \frac{\sigma(\tau)^2 + \tau \sigma(\tau) + \tau^2}{\tau^3} \Delta \tau \right) + \frac{\mu(-4)}{\mu(-4)} \left( \frac{2^2 - 4(2) + (-4)^2}{(-4)^3} \right)
$$

$$
= 3 \ln[-4] - \ln[-5] + \ln[t] - \ln[2] + \log \left( 1 + \frac{12}{-64} \right)
$$

$$
= 3 \ln \left( \frac{1}{5} \right) + i\pi.
$$

where Log again represents the principal complex logarithm, and ln is the natural logarithm. Again for sake of comparison, the logarithms in [10] and [5, Section 3] do not apply as they are defined exclusively in terms of $p(t) = t$, and [13] does not apply as that logarithm requires $1 \in \mathbb{T}$. If we use the logarithm in [5, Section 4] or [12], we get $3 \ln \left( \frac{1}{5} \right) - \frac{\pi}{2}$, a real-valued function, as opposed to our principal value of $3 \ln \left( \frac{1}{5} \right) + i\pi$, a complex-valued function. This example justifies our approach.

**Example 7.5.** Here is an example of Theorem 3.3. Let $t \in \mathbb{T}$ with $t \neq 0$, and set $p(t) = t$. For $s \in \mathbb{T}$, we have

$$
\ell_p^\Delta(t, s) = \begin{cases} 
\frac{1}{\mu(t)} \log \left( \frac{\sigma(t)}{t} \right) & \text{for } \mu(t) \neq 0 \\
\frac{1}{t} & \text{for } \mu(t) = 0,
\end{cases}
$$

where $\Delta$-differentiation is with respect to $t$. Thus,

$$
\ell_p^\Delta(t, s) = \begin{cases} 
\frac{1}{t} \log \left( \frac{1 + h}{t} \right) & \text{for } \mathbb{T} = \mathbb{R} \\
\log(q) & \text{for } \mathbb{T} = q\mathbb{Z},
\end{cases}
$$

where $\Delta$-differentiation is with respect to $t$. Thus,

$$
\ell_p^\Delta(t, s) = \begin{cases} 
\frac{1}{t} \log \left( \frac{1 + h}{t} \right) & \text{for } \mathbb{T} = \mathbb{R} \\
\log(q) & \text{for } \mathbb{T} = q\mathbb{Z},
\end{cases}
$$

where $\Delta$-differentiation is with respect to $t$. Thus,
where \( h > 0 \) and \( q > 1 \).

See Fig. 2 for \( h = 1 \) and \( \mathbb{T} = \mathbb{Z} \). This ends the example.

**Example 7.6.** Construct a discrete time scale with two step sizes that alternate; that is, for the two alternating step sizes \( \nu, \gamma > 0 \) with \( \gamma \neq \nu \), let

\[
\mathbb{T} := \mathbb{T}_{\gamma, \nu} = \{ 0, \gamma, (\gamma + \nu), (\gamma + \nu) + \gamma, 2(\gamma + \nu), 2(\gamma + \nu) + \gamma, 3(\gamma + \nu), \ldots \}.
\]

Then, for \( t \in \mathbb{T} \) and \( k \in \mathbb{N}_0 = \{ 0, 1, 2, \ldots \} \), we have

\[
\mu(t) = \begin{cases} 
\gamma & \text{for } t = k(\gamma + \nu), \\
\nu & \text{for } t = k(\gamma + \nu) + \gamma.
\end{cases}
\]

Set \( p(t) = t \). We claim that for \( t \in \mathbb{T}_{\gamma, \nu} \) with \( t \neq 0 \),

\[
\ell_{\nu}^\gamma(t, s) = \begin{cases} 
\frac{1}{\nu} \log \left( \frac{1 + \frac{\gamma}{t}}{1 + \frac{\nu}{t}} \right) & \text{for } t = k(\gamma + \nu) \\
\frac{1}{\nu} \log \left( \frac{1 + \frac{\nu}{t}}{1 + \frac{\gamma}{t}} \right) & \text{for } t = k(\gamma + \nu) + \gamma.
\end{cases}
\]

To verify this, note that

\[
\ell_{\nu}^\gamma(t, s) = \frac{1}{\mu(t)} \log \left( \frac{\sigma(t)}{t} \right)
\]

\[
= \begin{cases} 
\frac{1}{\gamma} \log \left( \frac{k(\gamma + \nu) + \gamma}{k(\gamma + \nu)} \right) & \text{for } t = k(\gamma + \nu) \\
\frac{1}{\nu} \log \left( \frac{(k + 1)(\gamma + \nu)}{k(\gamma + \nu) + \gamma} \right) & \text{for } t = k(\gamma + \nu) + \gamma
\end{cases}
\]

\[
= \begin{cases} 
\frac{1}{\nu} \log \left( \frac{1 + \frac{\gamma}{t}}{1 + \frac{\nu}{t}} \right) & \text{for } t = k(\gamma + \nu) \\
\frac{1}{\nu} \log \left( \frac{1 + \frac{\nu}{t}}{1 + \frac{\gamma}{t}} \right) & \text{for } t = k(\gamma + \nu) + \gamma.
\end{cases}
\]

This ends the example.

**Remark 7.7.** The first three examples, given above, suggest that this new logarithm may be a kind of exact discretization, in other words, that, by definition it yields the usual logarithm function restricted to the given time scale. This remains an open question for more intricate and general time scales.

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