Approximation of the Axisymmetric Elasticity Equations with Weak Symmetry

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Abstract

In this article we consider the linear elasticity problem in an axisymmetric three dimensional domain, with data which are axisymmetric and have zero angular component. The weak formulation of the three dimensional problem reduces to a two dimensional problem on the meridian domain, involving weighted integrals. The problem is formulated in a mixed method framework with both the stress and displacement treated as unknowns. The symmetry condition for the stress tensor is weakly imposed. Well posedness of the continuous weak formulation and its discretization are shown. Two approximation spaces are discussed and corresponding numerical computations presented.

Key words. axisymmetric elasticity problem, well posedness, mixed finite element method

AMS Mathematics subject classifications. 35Q72, 65N30, 65N12

1 Introduction

During the past twenty years, a number of papers have emerged in the numerical analysis literature investigating three-dimensional axisymmetric problems. This class of problem has attracted attention because a three-dimensional axisymmetric problem can be reduced to a two-dimensional problem when cylindrical coordinates are used (see Figure 1.1). It is well recognized that the computational effort required to solve a two-dimensional problem is significantly less than the computational effort needed to solve a three-dimensional problem.

For axisymmetric problems, Mercier and Raugel [24] undertook one of the first finite element analyses of these problems. In [11], Bernardi, Dauge and Maday studied the axisymmetric formulation of a number of standard problems (including Laplace, Stokes and Maxwell equations), and introduced tools for analyzing axisymmetric spectral methods. Assous, Ciarlet, et al. investigated the numerical approximation of the axisymmetric solution of the static and time dependent Maxwell equations in [6, 7]. Following these papers, a number of studies analyzing different axisymmetric...
problems appeared. Notably, a computational framework for the axisymmetric Poisson equation was developed by Ciarlet, Jung et al. in [14], and a computational framework for div-curl systems was presented by Copeland, Gopalakrishnan, and Pasciak in [15]. More recently, [26] Oh used finite element exterior calculus techniques to study the axisymmetric Hodge Laplacian problem. For axisymmetric fluid dynamics problems, In [10], Bermúdez, Reales, et al. used axisymmetry to reduce the dimension of an eddy current model, and in [1] Anaya, Mora et al. developed a computational framework for axisymmetric Brinkman flows. The axisymmetric Stokes and Darcy problems have been studied in [8, 18, 19, 23, 31]. A coupled axisymmetric Stokes-Darcy problem was investigated by Ervin in [16].

The finite element approximation of the linear elasticity problem has been extensively studied (see [12] for a detailed discussion). For many years, the only known stable finite elements for the mixed method formulation, involving the stress and displacement, used macro-elements in which the stress tensor was approximated on a finer mesh than the displacement vector [3, 22, 30]. In [5] Arnold and Winther developed a stable pair of piecewise polynomials with respect to a single triangulation. These elements, however, carry a significant computational cost since the lowest order representation uses 24 degrees of freedom per triangle.

The major difficulty to creating a stable finite element scheme for the mixed formulation of the linear elasticity problem is in enforcing the symmetry of the stress tensor, which represents the law of conservation of angular momentum. To avoid enforcing symmetry in the stress tensor strongly, a Lagrangian multiplier can be used to weakly enforces symmetry in the stress tensor [2, 4, 20, 25, 28, 29].

The form of differential operators expressed in cylindrical coordinates (e.g. the addition of a $\frac{1}{r}$ term) is an important reason why the numerical analysis for the finite element approximation to the axisymmetric linear elasticity problem is challenging. A consequence of this radial scaling is that the gradient and divergence operators do not map polynomial spaces to polynomial spaces. This feature makes the construction of suitable inf-sup stable finite element approximation spaces more difficult than in the Cartesian setting.

Following, in Sections 2-4 notation and needed preliminary results are introduced. A continuous weak formulation for the axisymmetric linear elasticity problem is presented in Section 5 and shown to be well posed. Then, in Section 6, the corresponding discrete weak formulation is analyzed, and sufficient conditions for its well posedness established in terms of the existence of a suitable bounded projection operator. Shown in Section 7 is the existence of projection operators for two well known approximation spaces which, together with the assumption of boundedness of the projection,
establishes the approximation spaces are inf-sup stable. An error analysis is given in Section 8. The numerical computations presented in Section 9 support the derived theoretical results. Some concluding remarks are given in Section 10.

2 Notation

In this section we introduce the notation used below. Bold Greek letters (e.g. $\sigma$) represent vectors, while bold Greek letters with an underline (e.g. $\underline{\sigma}$) denote tensors. For English letters, bold lowercase letters (e.g. $p$) denote vectors, while bold uppercase letters (e.g. $P$) denote tensors. Matrices are represented with capital, non-bold letters (e.g. $A$). Additionally, $\mathbb{M}^n$ denote the space of $n \times n$ dimensional real matrices, $\mathbb{S}^n$ denote the space of $n \times n$ dimensional real symmetric matrices and $\mathbb{K}^n$ denote the space of $n \times n$ dimensional real skew-symmetric matrices.

The space of piecewise polynomials of degree less than or equal to $k$ on a partition, $\mathcal{T}_h$, of a domain is denoted as $P_k(\mathcal{T}_h)$. The polynomials of degree less than or equal to $k$ on a specific domain $\mathcal{T}$, or on an element $T \in \mathcal{T}_h$, are notated by $P_k(T)$. When referencing a vector or tensor space of polynomials, the notation $(P_k(T))^n$ and $(P_k(T))^{n \times n}$ is used, respectively.

The symmetric gradient operator, $\varepsilon$ applied to a vector $u$, is given by

$$
\varepsilon(u)_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).
$$

For $\sigma_i$ denoting the $i$ row of $\sigma$, the vector $\nabla \cdot \sigma$ is given by

$$
(\nabla \cdot \sigma)_i = \nabla \cdot \sigma_i.
$$

The trace operator, $\text{tr}$, is defined as

$$
\text{tr}(\sigma) = \sum_{i=1}^{n} \sigma_{ii}.
$$

The skew-symmetric part of a tensor $\sigma$ is defined as

$$
as(\sigma) = \frac{1}{2}(\sigma - \sigma^t)\n$$

where $\sigma^t$ is the transpose of $\sigma$. Furthermore, in two dimensions, $as(\sigma)$ can be identified with a scalar value $q \in \mathbb{R}$ and the following operator

$$
as(\sigma) = S^2(q) = \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix} \text{ where } q = \frac{1}{2}(\sigma_{12} - \sigma_{21}).
$$

For vectors $a = (a_1, a_2)^t$ and $b = (b_1, b_2)^t$,

$$
\nabla \times a = \begin{pmatrix} -\partial_y a_1 \\ \partial_x a_1 \end{pmatrix}, \quad a \otimes b = \begin{pmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{pmatrix}.
$$
If \( \mathbf{w} = (w_1, w_2)^t \) and \( \mathbf{v} = (v_1, v_2)^t \) are vectors, then the two-dimensional wedge product is
\[
\mathbf{w} \wedge \mathbf{v} = w_1v_2 - w_2v_1.
\]

For a tensor \( \mathbf{T} \) and vector \( \mathbf{v} \), the wedge product is
\[
(\mathbf{T} \wedge \mathbf{v}) = \begin{pmatrix}
\tau_{11}v_2 - \tau_{21}v_1 \\
\tau_{12}v_2 - \tau_{22}v_1
\end{pmatrix}.
\]

For \( \mathbf{x} = (x_1, x_2)^t \), \( \mathbf{x}^\perp \) is defined as, \( \mathbf{x}^\perp = (x_2, -x_1)^t \).

To distinguish between inner product and bilinear forms defined in Cartesian coordinates from those defined in cylindrical coordinates, a \( c \) subscript is attached to all Cartesian inner products and bilinear forms.

### 3 Variational Formulation

As a starting point for the derivation of our weak formulation for the axisymmetric problem, we begin with the weak (mixed) formulation for the elasticity problem, subject to a weakly enforced symmetry condition for the stress.

For \( \sigma \) denoting the stress tensor, \( \mathbf{u} \) is the displacement, \( \bar{\Omega} \subset \mathbb{R}^3 \), \( \mu \) and \( \lambda \) Lamé constants, the modeling equations of linear elasticity, subject to a fixed boundary, are given by
\[
\mathcal{A}\sigma = \epsilon(\mathbf{u}), \quad \nabla \cdot \sigma = \mathbf{f} \quad \text{in} \quad \bar{\Omega},
\]
\[
\text{subject to} \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial \bar{\Omega}. \quad (3.1)
\]

In (3.1) the compliance tensor \( \mathcal{A} : \mathbb{S}^{n \times n} \rightarrow \mathbb{S}^{n \times n} \) is a bounded, symmetric positive definite operator that, for isotropic materials, takes the form
\[
\mathcal{A}\sigma = \frac{1}{2\mu} \left( \sigma - \frac{\lambda}{2\mu + m\lambda} \text{tr}(\sigma))I \right). \quad (3.3)
\]

In order to describe the weak formulation we introduce the following function spaces.

\[
\begin{align*}
L^2(\bar{\Omega}) &= \{ v : \int_{\bar{\Omega}} v^2 \, d\bar{\Omega} < \infty \}, \\
L^2(\bar{\Omega}) &= \{ v : v_i \in L^2(\bar{\Omega}) \text{ for } i = 1, \cdots, n \}, \\
\mathbf{H}^1(\bar{\Omega}) &= \{ v : v_i \in L^2(\bar{\Omega}), \nabla v_i \in L^2(\bar{\Omega}) \text{ for } i = 1, \cdots, n \}, \\
L^2(\bar{\Omega}; \mathbb{M}^n) &= \{ \sigma \in \mathbb{M}^n : \sigma_{ij} \in L^2(\bar{\Omega}) \text{ for } i, j = 1, \cdots, n \}, \\
L^2(\bar{\Omega}; \mathbb{K}^n) &= \{ \sigma \in \mathbb{K}^n : \sigma_{ij} \in L^2(\bar{\Omega}) \text{ for } i, j = 1, \cdots, n \}, \\
\mathbf{H}^1(\bar{\Omega}, \mathbb{M}^n) &= \{ \sigma \in L^2(\bar{\Omega}, \mathbb{M}^n) : \nabla \sigma_{ij} \in L^2(\bar{\Omega}) \text{ for } i, j = 1, \cdots, n \}, \quad \text{and} \\
\mathbf{H}(\text{div}, \bar{\Omega}, \mathbb{M}^n) &= \{ \sigma \in L^2(\bar{\Omega}, \mathbb{M}^n) : \nabla \cdot \sigma \in L^2(\bar{\Omega}) \}.
\end{align*}
\]
Letting $X = \mathbb{H}(\text{div}, \Omega, \mathbb{M}_n)$, $Q = L^2(\Omega)$, and $W = L^2(\partial \Omega; \mathbb{R}^n)$. Then, the weak formulation is given by [2, 20, 25, 28, 29]: Given $f \in Q$, determine $(\sigma, u, \rho) \in X \times Q \times W$ such that, for all $(\tau, v, \xi) \in X \times Q \times W$

\[
\int_{\Omega} (A\sigma : \tau + \nabla \cdot \tau \cdot u + \tau : \rho) \, d\Omega = 0
\]  
(3.4)

\[
\int_{\Omega} \nabla \cdot \sigma \cdot v \, d\Omega = \int_{\Omega} f \cdot v \, d\Omega
\]  
(3.5)

\[
\int_{\Omega} \sigma : \xi \, d\Omega = 0.
\]  
(3.6)

With the inner products,

\[
a_c(\cdot, \cdot) : X \times X \to \mathbb{R}, \quad a_c(\sigma, \tau) := \int_{\partial \Omega} A\sigma : \tau \, d\Omega,
\]  
(3.7)

\[
b_c(\cdot, \cdot) : Q \times X \to \mathbb{R}, \quad b_c(u, \tau) := \int_{\partial \Omega} (\nabla \cdot \tau) \cdot u \, d\Omega,
\]  
(3.7)

\[
c_c(\cdot, \cdot) : W \times X \to \mathbb{R}, \quad c_c(\rho, \tau) := \int_{\partial \Omega} \rho : \tau \, d\Omega.
\]  
(3.7)

and taking $A_c(\sigma, \tau) = a_c(\sigma, \tau)$ and $B_c(\tau, (u, \rho)) = b_c(u, \tau) + c_c(\rho, \tau)$, (3.4) – (3.6) can be rewritten in the familiar saddle-point formulation: Given $f \in Q$, determine $(\sigma, (u, \rho)) \in X \times (Q \times W)$ such that, for all $(\tau, v, \xi) \in X \times (Q \times W)$

\[
A(\sigma, \tau) + B(\tau, (u, \rho)) = 0
\]

\[
B(\sigma, (v, \xi)) = (f, v).
\]  
(3.8)

For a detailed analysis of (3.4) – (3.6), see [12].

4 Axisymmetric Function Spaces

When the three dimensional axisymmetric linear elasticity problem is expressed in cylindrical coordinates, it can be expressed as a decoupled meridian and azimuthal problem. Changing the coordinate system from Cartesian to cylindrical, however, alters the algebraic form of differential operators and requires a new set of function spaces and notation. In this section, we introduce the key changes needed to present and discuss the meridian axisymmetric linear elasticity problem. Appendix C provides additional details on cylindrical coordinates and the procedure for decoupling the axisymmetric problem.

For axisymmetric vectors $u = u_re_r + u_z e_z = (u_r, u_z)^t$, we define the gradient operators $\nabla$ and $\nabla_{\text{axi}}$ as

\[
\nabla u = \begin{pmatrix}
\frac{\partial u_r}{\partial r} & \frac{\partial u_r}{\partial z} \\
\frac{\partial u_z}{\partial r} & \frac{\partial u_z}{\partial z}
\end{pmatrix}, \quad \text{and} \quad \nabla_{\text{axi}} u = \begin{pmatrix}
\frac{\partial u_r}{\partial r} & 0 & \frac{\partial u_r}{\partial z} \\
0 & \frac{1}{r} u_r & 0 \\
\frac{\partial u_z}{\partial r} & 0 & \frac{\partial u_z}{\partial z}
\end{pmatrix}.
\]  
(4.1)
Note that it is necessary to represent the gradient and axisymmetric gradient as tensors with different sizes because the non-constant nature of the cylindrical coordinate unit vectors creates additional terms in axisymmetric derivatives. However, in order to express the meridian problem using a two-dimensional formulation, we represent the tensor \( \nabla_{\text{axi}}u \) as an ordered pair made up of a tensor and a scalar function. That is
\[
\nabla_{\text{axi}} u = (\nabla u, \frac{1}{r}u_r). \tag{4.2}
\]

Next, for the axisymmetric vector \( u = (u_r, u_z)^t \), the divergence operators \( \nabla \cdot \) and \( \nabla_{\text{axi}} \cdot \) are defined as
\[
\nabla \cdot u = \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z}, \text{ and } \nabla_{\text{axi}} \cdot u = 1 \frac{\partial (r u_r)}{\partial r} + \frac{\partial u_z}{\partial z} = \nabla_{rz} \cdot u + \frac{1}{r}u_r. \tag{4.3}
\]

As alluded to in (4.2), the stress tensor that appears in the meridian problem can be represented as \((\sigma, \sigma)\), where \( \sigma \) denotes an \( M^2 \) tensor function and \( \sigma \) represents a scalar function. The divergence of the meridian stress tensor is
\[
\nabla_{\text{axi}} \cdot (\sigma, \sigma) = \begin{pmatrix} \nabla_{\text{axi}} \cdot \sigma_1 - \frac{1}{r}\sigma \\ \nabla_{\text{axi}} \cdot \sigma_2 \end{pmatrix}.
\]

At times, the axisymmetric divergence operator will also be applied to an \( M^2 \) tensor function \( \sigma \), in which case
\[
\nabla_{\text{axi}} \cdot \sigma = \nabla_{\text{axi}} \cdot (\sigma, 0) = \begin{pmatrix} \nabla_{\text{axi}} \cdot \sigma_1 \\ \nabla_{\text{axi}} \cdot \sigma_2 \end{pmatrix}.
\]

Note that for the skew symmetric component of \((\sigma, \sigma)\) we have
\[
as((\sigma, \sigma)) = as(\sigma) = S^2(q), \text{ where } q = \frac{1}{2}(\sigma_{12} - \sigma_{21}).
\]

The curl of an axisymmetric scalar function \( p \) is denoted by \( \nabla_{\text{ac}} \) and is defined as
\[
\nabla_{\text{ac}} p = \begin{pmatrix} \frac{\partial p}{\partial z} - \frac{1}{r} \frac{\partial (r p)}{\partial r} \end{pmatrix}. \tag{4.4}
\]

Note that \( \nabla_{\text{ac}} \) returns a row-vector. For a vector function \( p = (p_r, p_z)^t \) we have
\[
\nabla_{\text{ac}} p = \begin{pmatrix} \nabla_{\text{ac}} p_r \\ \nabla_{\text{ac}} p_z \end{pmatrix}. \tag{4.5}
\]

In addition to the divergence and curl, the cylindrical coordinate inner product also takes a different form from the Cartesian inner product. Consider the change of variables for a Cartesian function \( \tilde{p} \in L^2(\tilde{\Omega}) \) into cylindrical coordinates
\[
\int_\Omega \tilde{p}^2 \, d\tilde{\Omega} = \int_\Omega \int_{\theta=0}^{2\pi} \int_0^p p^2 r \, d\theta \, dr \, dz. \tag{4.6}
\]
Notice the $r = r(x)$ scaling in the measure. In the axisymmetric setting, $p \equiv p(r, z)$ and the $\theta$ integral can be computed to give a factor of $2\pi$. As this term is a constant factor in all such integrals arising, we omit it. To distinguish the cylindrical coordinate inner product from the Cartesian inner product, we use the following notation

$$(p, q) = \int_\Omega p \, q \, r \, dr \, dz.$$ 

To account for this scaling in the inner product, we introduce the following function spaces

$$_aL^2(\Omega) = \{ v : \int_\Omega v^2 r^a \, dr \, dz < \infty \},$$

$$_aL^2(\Omega) = \{ v \in \mathbb{R}^n : v_i \in _aL^2(\Omega) \text{ for } i = 1, \ldots, n \},$$

$$_aL^2(\Omega; \mathbb{M}^n) = \{ \sigma \in \mathbb{M}^n : \sigma_{ij} \in _aL^2(\Omega) \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n \},$$

and

$$_aL^2(\Omega, \mathbb{K}^n) = \{ \sigma \in \mathbb{K}^n : \sigma_{ij} \in _aL^2(\Omega) \text{ for } i = 1, \ldots, n \text{ and } j = 1, \ldots, n \}.$$ 

The norms associated with these $_aL^2$ spaces are

$$\|v\|^2_{_aL^2(\Omega)} = \int_\Omega v^2 r^a \, dr \, dz,$$

$$\|\sigma\|^2_{_aL^2(\Omega; \mathbb{M}^n)} = \sum_{i=1}^n \sum_{j=1}^n \|\sigma_{ij}\|^2_{_aL^2(\Omega)} \text{ and } \|\sigma\|^2_{_aL^2(\Omega, \mathbb{K}^n)} = \sum_{i=1}^n \sum_{j=1}^n \|\sigma_{ij}\|^2_{_aL^2(\Omega)}.$$ 

In addition to the $_aL^2$ spaces, the elasticity problem requires divergence spaces for the stress tensors. These spaces are

$$_aH(\text{div}_{\text{axi}}, \Omega) = \{ v \in _aL^2(\Omega) : \nabla_{\text{axi}} \cdot v \in _aL^2(\Omega) \},$$

$$_aH(\text{div}_{\text{axi}}, \Omega; \mathbb{M}^n) = \{ \sigma \in _aL^2(\Omega; \mathbb{M}^n) : \nabla_{\text{axi}} \cdot \sigma \in _aL^2(\Omega) \},$$

and

$$_aH(\text{div}_{\text{axi}}, \Omega; \mathbb{K}^n) = \{ \sigma \in _aL^2(\Omega; \mathbb{K}^n) : \nabla_{\text{axi}} \cdot \sigma \in _aL^2(\Omega) \}.$$ 

with norms

$$\|v\|^2_{_aH(\text{div}_{\text{axi}}, \Omega)} = \|\nabla_{\text{axi}} \cdot v\|^2_{_aL^2(\Omega)} + \|v\|^2_{_aL^2(\Omega)},$$

$$\|\sigma\|^2_{_aH(\text{div}_{\text{axi}}, \Omega; \mathbb{M}^n)} = \|\nabla_{\text{axi}} \cdot \sigma\|^2_{_aL^2(\Omega)} + \|\sigma\|^2_{_aL^2(\Omega; \mathbb{M}^n)},$$

and

$$\|\sigma\|^2_{_aH(\text{div}_{\text{axi}}, \Omega; \mathbb{K}^n)} = \|\nabla_{\text{axi}} \cdot \sigma\|^2_{_aL^2(\Omega)} + \|\sigma\|^2_{_aL^2(\Omega; \mathbb{K}^n)}.$$ 

For $\zeta$ a nonnegative integer and $v$ a $\zeta$ times differentiable function, let

$$\nabla^\zeta v = \left[ \frac{\partial^\zeta v}{(\partial r)^\zeta}, \frac{\partial^\zeta v}{(\partial r)^{\zeta-1}(\partial z)}, \cdots, \frac{\partial^\zeta v}{(\partial r)(\partial z)^{\zeta-1}}, \frac{\partial^\zeta v}{(\partial z)^\zeta} \right].$$ 

Then,

$$_aH^k(\Omega) = \{ v \in _aL^2(\Omega) : \nabla^\zeta v \in _aL^2(\Omega) \text{ for all } \zeta \leq k \},$$

$$_aH^k(\Omega) = \{ v \in _aL^2(\Omega) : \nabla^\zeta v_i \in _aL^2(\Omega) \text{ for all } \zeta \leq k \text{ and } i = 1, 2, \ldots, n \},$$

and

$$_aH^k(\Omega; \mathbb{M}^n) = \{ \sigma \in _aL^2(\Omega; \mathbb{M}^n) : \nabla^\zeta \sigma_{ij} \in _aL^2(\Omega) \text{ for all } \zeta \leq k \text{ and } i, j = 1, 2, \ldots, n \}.$$
with norms
\[ \|v\|_{H^k(\Omega)}^2 = \|v\|_{L^2(\Omega)}^2 + \sum_{\zeta=1}^k \|\nabla^\zeta v\|_{L^2(\Omega)}^2, \]
\[ \|\mathbf{v}\|_{H^k(\Omega)}^2 = \|\mathbf{v}\|_{L^2(\Omega)}^2 + \sum_{i=1}^n \sum_{\zeta=1}^k \|\nabla^\zeta v_i\|_{L^2(\Omega)}^2, \]
\[ \|\sigma\|_{H^k(\Omega, M^n)}^2 = \|\sigma\|_{L^2(\Omega, M^n)}^2 + \sum_{i=1}^n \sum_{j=1}^n \sum_{\zeta=1}^k \|\nabla^\zeta \sigma_{ij}\|_{L^2(\Omega)}^2. \]

Next we consider some subtle details related to function spaces containing axisymmetric derivative terms. To begin, using (4.1),
\[ \|\nabla_{axi} v\|_{L^2(\Omega)}^2 = \int_\Omega \nabla_{axi} v : \nabla_{axi} v \, d\Omega = \int_\Omega \nabla v : \nabla v \, d\Omega + \int_\Omega \frac{1}{r} v_r^2 \, d\Omega. \]

Therefore, in order that \( \|\nabla_{axi} v\|_{L^2(\Omega)} < \infty \), it is necessary for \( v_r \in H^1(\Omega) \) and \( v_r \in -L^2(\Omega) \). To denote this important subspace of \( H^1(\Omega) \), we define
\[ 1V^1(\Omega) = \{v \in H^1(\Omega) : v \in -L^2(\Omega)\} \]
with associated norm
\[ \|v\|_{1V^1(\Omega)} = \left( \|v\|_{-L^2(\Omega)}^2 + \|v\|_{H^1(\Omega)}^2 \right)^{1/2}. \]

Also, we introduce
\[ 1VH^1(\Omega) = \{v = (v_r, v_z)^t : v_r \in V^1(\Omega), v_z \in H^1(\Omega)\} \]
with associated norm
\[ \|v\|_{1VH^1(\Omega)} = \left( \|v_r\|_{V^1(\Omega)}^2 + \|v_z\|_{H^1(\Omega)}^2 \right)^{1/2}. \]

It is also important to observe, that unlike in the Cartesian setting, \( \mathbf{VH}^1(\Omega) \not\subset \mathbf{H}(\text{div}_{axi}, \Omega) \).

When referencing a function space whose functions have a vanishing trace along the boundary segment \( \Gamma \), we use a zero subscript, i.e.,
\[ 1H^1_0(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}. \]

Note that \( \Gamma \) here does not include the rotation axis portion of the boundary of \( \Omega \) as illustrated in Figure 1.1.

As eluded to above, in transforming from \( \tilde{\Omega} \) to \( \Omega \), we have the following relationships.

**Lemma 4.1.** [8, Proposition 1] The space of axisymmetric vector fields in \( H^1(\tilde{\Omega})^3 \) with zero angular component is isomorphic to \( 1VH^1(\Omega) \).

**Lemma 4.2.** The space of axisymmetric tensors in \( \mathbf{H}^1(\tilde{\Omega}, M^3) \) with zero azimuthal components is isomorphic to
\[ \left\{ \tau = \begin{pmatrix} \tau_{rr} & 0 & \tau_{rz} \\ 0 & \tau_{\theta\theta} & 0 \\ \tau_{rz} & 0 & \tau_{zz} \end{pmatrix} : \tau_{rr}, \tau_{\theta\theta}, \tau_{zz} \in H^1(\Omega), \tau_{rz} \in V^1(\Omega) \right\}. \]

(4.7)
Proof. The representation of an axisymmetric tensor \( \mathbf{\tau} \), in cylindrical coordinates with zero azimuthal components is given by the tensor in (4.7). Let,

\[
D\tau_r = \begin{pmatrix}
\frac{\partial}{\partial r} \tau_{rr} & 0 & \frac{1}{r} \left( \tau_{rr} - \tau_{\theta\theta} \right) \\
0 & \frac{1}{r} & 0 \\
\frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) & 0 & \frac{\partial}{\partial z} \tau_{rr}
\end{pmatrix},
\]

\[
D\tau_{\theta} = \begin{pmatrix}
0 & \frac{1}{r} & 0 \\
\frac{1}{r} & 0 & \frac{1}{r} \tau_{rz} \\
0 & \frac{1}{r} \tau_{rz} & 0
\end{pmatrix},
\]

and

\[
D\tau_z = \begin{pmatrix}
\frac{\partial}{\partial r} \tau_{zr} & 0 & \frac{1}{r} \tau_{zr} \\
0 & \frac{1}{r} & 0 \\
\frac{1}{r} \tau_{zr} & 0 & \frac{\partial}{\partial z} \tau_{zz}
\end{pmatrix}.
\]

Then,

\[
\|\mathbf{\tau}\|_{H^1(\tilde{\Omega})}^2 = \int_{\tilde{\Omega}} \|\mathbf{\tau}(x,y,z)\|^2 d\tilde{\Omega} + \int_{\tilde{\Omega}} \|D\mathbf{\tau}(x,y,z)\|^2 d\tilde{\Omega}
\]

\[
= 2\pi \int_{\Omega} \left( \tau_{rr}^2 + \tau_{rz}^2 + \frac{\tau_{\theta\theta}^2 + \tau_{zr}^2 + \tau_{zz}^2}{r^2} \right) r \, dr \, dz + 2\pi \int_{\Omega} (D\tau_r : D\tau_r + D\tau_{\theta} : D\tau_{\theta} + D\tau_z : D\tau_z) \, r \, dr \, dz.
\]

Hence \( \|\mathbf{\tau}\|_{H^1(\tilde{\Omega})}^2 < \infty \) implies, \( \tau_{rr}, \tau_{\theta\theta}, \tau_{zz} \in H^1(\Omega) \), \( \tau_{rz}, \tau_{zr} \in V^1(\Omega) \), \( \tau_{rr} - \tau_{\theta\theta} \in -L^2(\Omega) \).

Reversing the argument establishes the isomorphism between the spaces. \( \blacksquare \)

In the discussions that follow, we take \( U = L^2(\Omega), Q = L^2(\Omega) \). As the meridian stress tensor is made up of a tensor and scalar component, we introduce the space \( \Sigma(\Omega) \) defined by

\[
\Sigma(\Omega) = \{ (\mathbf{\sigma}, \sigma) \in L^2(\Omega, M^2) \times L^2(\Omega) : \nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma) \in L^2(\Omega) \}.
\]

Associated with \( \Sigma(\Omega) \) we have the norm

\[
\|(\mathbf{\sigma}, \sigma)\|_{\Sigma(\Omega)} = \left( \|\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma)\|^2_{L^2(\Omega)} + \|\mathbf{\sigma}\|_{L^2(\Omega, M^2)}^2 + \|\sigma\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}.
\]

Additionally, we define \( S(\Omega) \subset \Sigma(\Omega) \) by

\[
S(\Omega) = \{ (\mathbf{\sigma}, 0) \in \Sigma(\Omega) : \mathbf{\sigma} = \begin{pmatrix} w^t \\ z^t \end{pmatrix} ; \ w, z \in V^{H^1}(\Omega) \},
\]

with norm

\[
\|(\mathbf{\sigma}, 0)\|_{S(\Omega)} = \left( \|\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, 0)\|^2_{L^2(\Omega)} + \|w\|_{V^{H^1}(\Omega)}^2 + \|z\|_{V^{H^1}(\Omega)}^2 \right)^{\frac{1}{2}}.
\] (4.8)

For convenience, when the context is clear, \( \Sigma(\Omega) \) and \( S(\Omega) \) will be denoted as \( \Sigma \) and \( S \).
5 Axisymmetric Variational Formulation

In this section we present the variational form of the axisymmetric meridian problem. This problem has many similarities with the elasticity problem in the Cartesian setting, however, new terms are introduced into the bilinear forms as a consequence of the change of variable from Cartesian to cylindrical coordinates. Details of the derivation can be found in Appendix C.

Analogous to (3.7), define the bilinear forms

\[ \tilde{a}(\cdot, \cdot) : \Sigma \times \Sigma \to \mathbb{R}, \quad \tilde{a}((\mathbf{\sigma}, \sigma), (\mathbf{\tau}, \tau)) = (A\mathbf{\sigma}, \mathbf{\tau}) + (A\sigma, \tau) - \frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda} ((\sigma, \text{tr}(\mathbf{\tau})) + (\text{tr}(\mathbf{\sigma}), \tau)), \]  

(5.1)

\[ \tilde{b}(\cdot, \cdot) : \Sigma \times U \to \mathbb{R}, \quad \tilde{b}((\mathbf{\tau}, \tau), \mathbf{u}) = (\nabla_{\text{axi}} \cdot \mathbf{\tau}, \mathbf{u}) - (\frac{\tau}{r}, u_r), \]  

(5.2)

\[ \tilde{c}(\cdot, \cdot) : \Sigma \times Q \to \mathbb{R}, \quad \tilde{c}((\mathbf{\sigma}, \sigma), p) = (\mathbf{\sigma}, \mathbf{S}^2(p)), \]  

(5.3)

where the operator \( A \) applied to the scalar function \( \sigma_{yy} \) is given by (3.3) for \( m = 1 \).

The axisymmetric meridian problem with weak symmetry is then: Given \( f \in L^2(\Omega) \), find \((\mathbf{\sigma}, \sigma), \mathbf{u}, p) \in \Sigma \times U \times Q \) such that for all \((\mathbf{\tau}, \tau), \mathbf{v}, q) \in \Sigma \times U \times Q \)

\[ \tilde{a}((\mathbf{\sigma}, \sigma), (\mathbf{\tau}, \tau)) + \tilde{b}((\mathbf{\tau}, \tau), \mathbf{u}) + \tilde{c}((\mathbf{\tau}, \tau), p) = 0 \]  

(5.4)

\[ \tilde{b}((\mathbf{\sigma}, \sigma), \mathbf{v}) = (f, \mathbf{v}) \]  

(5.5)

\[ \tilde{c}((\mathbf{\sigma}, \sigma), q) = 0. \]  

(5.6)

Of interest is to develop discrete inf-sup stable elements for the approximation of (5.4)-(5.6). In cylindrical coordinates, the divergence operator does not map polynomial spaces into polynomial spaces, so some of the standard techniques for verifying inf-sup stability cannot be used. Thus, to help establish a variational formulation for which stable triples of finite elements may be verified to satisfy the discrete inf-sup condition, we make two modifications to (5.4)-(5.6).

First, we add a grad-div stabilization term to \( \tilde{a}(\cdot, \cdot) \) and define a new bilinear form \( a(\cdot, \cdot) : \Sigma \times \Sigma \to \mathbb{R} \)

\[ a((\mathbf{\sigma}, \sigma), (\mathbf{\tau}, \tau)) = \tilde{a}((\mathbf{\sigma}, \sigma), (\mathbf{\tau}, \tau)) + \gamma (\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma), \nabla_{\text{axi}} \cdot (\mathbf{\tau}, \tau)) \]  

(5.7)

where \( \gamma \) is the grad-div stabilization term. This stabilization term ensures that \( a((\cdot, \cdot), (\cdot, \cdot)) \) is coercive in the \( \| \cdot \|_{\Sigma} \) norm. Unless specified otherwise, we take \( \gamma = 1 \).

Recall from (3.1) that in cylindrical coordinates, \( \nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma) = f \). Therefore, to account for the grad-div stabilization term in the constituent equation, \((f, \nabla_{\text{axi}} \cdot (\mathbf{\tau}, \tau)) \) must also be added to the right hand side of (5.4).

For the second modification, recall that \( \tilde{c}((\mathbf{\sigma}, \sigma), q) = (\mathbf{\sigma}, \mathbf{S}^2(q)) \), and let \( \mathbf{x} = (r, z)^t \). As described in Lemma 6.1 below,

\[ \int_{\Omega} \sigma : \mathbf{S}^2(q) \, r \, d\Omega = -\int_{\Omega} (\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma) \wedge \mathbf{x}) \cdot q \, r \, d\Omega \]  

\[ + \int_{\partial\Omega} (\mathbf{\sigma} \cdot \mathbf{n}) \cdot \mathbf{x} \cdot q \, r \, ds + \int_{\partial\Omega} \sigma : (\mathbf{x} \otimes \nabla q) \, q \, r \, d\Omega - \int_{\Omega} \frac{1}{r} \sigma z q \, r \, d\Omega, \]  

(5.8)
or equivalently
\[
\int_{\Omega} \sigma : S^2(q) \, r \, d\Omega + \int_{\Omega} (\nabla \cdot (\sigma, \sigma) \wedge x) \, q \, r \, d\Omega \\
= \int_{\partial \Omega} (\sigma \cdot n) \cdot x^\perp q \, r \, ds - \int_{\Omega} \sigma : (x^\perp \otimes \nabla q) \, r \, d\Omega - \int_{\Omega} \sigma \cdot z \, q \, d\Omega.
\] (5.9)

In terms of establishing stable approximation elements via the construction of a suitable projection (see Theorem 6.1) it is more convenient to use equation (5.9) than (5.8). To introduce (5.9) into the weak form, we add \( \int_{\Omega} (\nabla \cdot (\sigma, \sigma) \wedge x) \, q \, r \, d\Omega \) to both sides of (5.6) giving
\[
\tilde{c}((\sigma, \sigma), q) + \int_{\Omega} (\nabla \cdot (\sigma, \sigma) \wedge x) \, q \, r \, d\Omega = \int_{\Omega} (f \wedge x) \, q \, r \, d\Omega,
\] (5.10)
where we have used the relationship \( \nabla \cdot (\sigma, \sigma) = f \) on the right hand side. To represent the left hand side of (5.10), we define a new bilinear form \( c(\cdot, \cdot) : \Sigma \times Q \to \mathbb{R} \) as
\[
c((\sigma, \sigma), q) := \tilde{c}((\sigma, \sigma), q) + (\nabla \cdot (\sigma, \sigma) \wedge x, q)
\]
\[= (\sigma, S^2(q)) + (\nabla \cdot (\sigma, \sigma) \wedge x, q).\] (5.11)

Therefore, (5.6) becomes
\[
c((\sigma, \sigma), q) = (f \wedge x, q).
\]

For notational consistency in the new formulation, we let \( b(\cdot, \cdot) = \tilde{b}(\cdot, \cdot) \).

To maintain the saddle point structure of the variational formulation with the bilinear form \( c(\cdot, \cdot) \), we need to add and subtract \( (\nabla \cdot (\tau, \tau) \wedge x, p) \) to the left hand side of (5.4). To understand the affect of this modification on the formulation, first observe that
\[
\nabla \cdot (\tau, \tau) \wedge x = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial r} + \frac{\partial \tau_{21}}{\partial z} + \frac{1}{r} (\tau_{11} - \tau) \\ \frac{\partial \tau_{21}}{\partial r} + \frac{\partial \tau_{22}}{\partial z} + \frac{1}{r} \tau_{21} \end{pmatrix} \wedge \begin{pmatrix} r \\ z \end{pmatrix}
\]
\[= z \left( \frac{\partial \tau_{11}}{\partial r} + \frac{\partial \tau_{12}}{\partial z} + \frac{1}{r} (\tau_{11} - \tau) \right) - r \left( \frac{\partial \tau_{21}}{\partial r} + \frac{\partial \tau_{22}}{\partial z} + \frac{1}{r} \tau_{21} \right) = (\nabla \cdot (\tau, \tau)) \cdot x^\perp.
\]

Therefore,
\[
((\nabla \cdot (\tau, \tau)) \wedge x, p) = \int_{\Omega} \nabla \cdot (\tau, \tau) \wedge x \, p \, r \, d\Omega = \int_{\Omega} ((\nabla \cdot (\tau, \tau)) \cdot x^\perp \wedge x^\perp \, p \, r \, d\Omega
\]
\[= b((\tau, \tau), x^\perp p).
\] (5.12)

This shows that \( ((\nabla \cdot (\tau, \tau)) \wedge x, p) \) can be expressed as \( b((\tau, \tau), x^\perp p) \). As a result, the negative part of \( ((\nabla \cdot (\tau, \tau) \wedge x, p) \) that is used to balance the constituent equation enters into the expression as part of the bilinear form \( b(\cdot, \cdot) \). That is,
\[
b((\tau, \tau), u) - b((\tau, \tau), x^\perp p) = b((\tau, \tau), u - x^\perp p).
\] (5.13)
Lemma 5.1. The operator \( a(\cdot, \cdot) \) defined in (5.7) is bounded. That is,
\[
a((\mathbf{\sigma}, \mathbf{\sigma}), (\mathbf{\tau}, \mathbf{\tau})) \leq C \|(\mathbf{\sigma}, \mathbf{\tau})\|_{\Sigma} \|\mathbf{\tau}\|_{\Sigma}
\]
for some \( C > 0 \) and all \( (\mathbf{\sigma}, \mathbf{\sigma}), (\mathbf{\tau}, \mathbf{\tau}) \in \Sigma \).

Proof. Using the Cauchy-Schwarz inequality,
\[
a((\mathbf{\sigma}, \mathbf{\sigma}), (\mathbf{\tau}, \mathbf{\tau})) = \frac{1}{2\mu} (\mathbf{\sigma}, \mathbf{\tau}) + \frac{1}{2\mu} (\sigma, \tau) - \frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda} (\text{tr}(\mathbf{\sigma}) + \sigma, \text{tr}(\mathbf{\tau}) + \tau)
\]
\[
+ (\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \mathbf{\tau})), (\nabla_{\text{axi}} \cdot (\mathbf{\tau}, \mathbf{\tau}))
\]
\[
\leq \frac{1}{2\mu} \left( \|\mathbf{\sigma}\|_{L^2(\Omega)} \|\mathbf{\tau}\|_{L^2(\Omega)} + \|\sigma\|_{L^2(\Omega)} \|\tau\|_{L^2(\Omega)} \right)
\]
\[
+ \|\nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \mathbf{\tau})\|_{L^2(\Omega)} \|\nabla_{\text{axi}} \cdot (\mathbf{\tau}, \mathbf{\tau})\|_{L^2(\Omega)}
\]
\[
+ \frac{\lambda}{2\mu} \frac{1}{2\mu + 3\lambda} \|\text{tr}(\mathbf{\sigma}) + \sigma\|_{L^2(\Omega)} \|\text{tr}(\mathbf{\tau}) + \tau\|_{L^2(\Omega)}
\]
\[
\leq C \left( \|(\mathbf{\sigma}, \mathbf{\tau})\|_{\Sigma} \|(\mathbf{\tau}, \mathbf{\tau})\|_{\Sigma} + \|\text{tr}(\mathbf{\sigma}) + \sigma\|_{L^2(\Omega)} \|\text{tr}(\mathbf{\tau}) + \tau\|_{L^2(\Omega)} \right).
\]

Further, for any \( (\mathbf{\sigma}, \mathbf{\sigma}) \in \Sigma \),
\[
\|\text{tr}(\mathbf{\sigma}) + \sigma\|_{L^2(\Omega)} = \int_{\Omega} (\text{tr}(\mathbf{\sigma}) + \sigma)^2 d\Omega \leq C \|(\mathbf{\sigma}, \mathbf{\sigma}) + (\sigma, \sigma)\|
\]
\[
\leq C \|(\mathbf{\sigma}, \mathbf{\tau})\|_{L^2(\Omega; M^2 \times \mathbb{R}^1)} \leq C \|(\mathbf{\sigma}, \mathbf{\sigma})\|_{\Sigma}.
\]

Combining (5.19) and (5.20) yields (5.18). □
Lemma 5.2. The operator $a(.,.)$ defined in (5.7) is coercive. That is,
\[
a((\sigma, \sigma), (\sigma, \sigma)) \geq c\|((\sigma, \sigma))\|_{\Sigma}^2 \quad \text{where} \quad c = \min\left\{\frac{1}{2\mu}, \frac{1}{2\mu + 3\lambda}, 1\right\}.
\]
\[\text{(5.21)}\]

Proof. We begin with the observation that
\[
a((\sigma, \sigma), (\sigma, \sigma)) = (A\sigma, \sigma) + (A\sigma, \sigma) - \frac{1}{2\mu} \frac{\lambda}{2 + 3\lambda} (\sigma, \tr(\sigma)) + (\tr(\sigma), \sigma)
+ (\nabla_{\text{axi}} \cdot (\sigma, \sigma), \nabla_{\text{axi}} \cdot (\sigma, \sigma))
= \frac{1}{2\mu} (\sigma, \sigma) + \frac{1}{2\mu} (\sigma, \sigma) - \frac{1}{2\mu} \frac{\lambda}{2 + 3\lambda} (\tr(\sigma) + \sigma, \tr(\sigma) + \sigma)
+ (\nabla_{\text{axi}} \cdot (\sigma, \sigma), \nabla_{\text{axi}} \cdot (\sigma, \sigma)).
\]
\[\text{(5.22)}\]

Next we must incorporate the $(\tr(\sigma) + \sigma, \tr(\sigma) + \sigma)$ term into (5.22) in a way that will allow us to obtain the $\Sigma$ norm. To do so, we start by adding the inequalities
\[
\sigma_{11}^2 + \sigma_{22}^2 \geq 2\sigma_{11}\sigma_{22}, \quad \sigma_{11}^2 + \sigma_{22}^2 \geq 2\sigma_{22}\sigma_1, \quad \sigma_{11}^2 + \sigma_2^2 \geq 2\sigma_{11}\sigma
\]
to get that $2(\sigma_{11}^2 + \sigma_{22}^2 + \sigma^2) \geq 2(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma + \sigma_{11}\sigma)$. Adding additional positive terms to the left-hand side of this inequality gives
\[
2(\sigma_{11}^2 + \sigma_{22}^2 + \sigma^2) + 3(\sigma_{12}^2 + \sigma_{21}^2) \geq 2(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma + \sigma_{11}\sigma).
\]

Since $(\tr(\sigma) + \sigma)^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma^2 + 2(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma + \sigma_{11}\sigma)$ and $\mu, \lambda > 0$,
\[
\frac{1}{2\mu} \frac{\lambda}{2 + 3\lambda} (\tr(\sigma) + \sigma, \tr(\sigma) + \sigma) \leq \frac{1}{2\mu} \frac{3\lambda}{2 + 3\lambda} \int_\Omega (\sigma_{11}^2 + \sigma_{12}^2 + \sigma_{21}^2 + \sigma_{22}^2 + \sigma^2) r \, d\Omega
= \frac{1}{2\mu} \frac{3\lambda}{2 + 3\lambda} (\sigma, \sigma) + \frac{1}{2\mu} \frac{3\lambda}{2 + 3\lambda} (\sigma, \sigma).
\]
\[\text{(5.23)}\]

Combining (5.22) and (5.23)
\[
a((\sigma, \sigma), (\sigma, \sigma)) \geq \frac{1}{2\mu} \frac{2\mu}{2 + 3\lambda} ((\sigma, \sigma) + (\sigma, \sigma)) + (\nabla_{\text{axi}} \cdot (\sigma, \sigma), \nabla_{\text{axi}} \cdot (\sigma, \sigma))
= \frac{1}{2\mu} \frac{2\mu}{2 + 3\lambda} \|((\sigma, \sigma))\|^2_{L^2(\Omega; M^2 \times \mathbb{R}^3)} + \|\nabla_{\text{axi}} \cdot (\sigma, \sigma)\|^2_{L^2(\Omega)}
\geq c\|((\sigma, \sigma))\|^2_{\Sigma}.
\]
\[\square\]

5.1.1 Satisfying the continuous inf-sup condition (5.17)

To establish the inf-sup condition (5.17), we follow a similar two step argument as used in [12] for the planar elasticity problem. In Step 1 $a(\tau_1, \tau_1)$ is found such that, for $\nu, p$ given, $b((\tau_1, \tau_1), \nu) = \|\nu\|^2_U$. Then, in Step 2 $(\tau_2, \tau_2)$ is constructed to handle the $c(.,.)$ term, while satisfying $b((\tau_2, \tau_2), \nu) = 0$. The following lemma is useful in the construction of $(\tau_2, \tau_2)$. 

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Lemma 5.3. Given $\beta \in L^2(\Omega)$ there exist $(\tau, \tau) \in \Sigma$ such that

$$\nabla_{axi} \cdot (\tau, \tau) = 0, \quad \text{as} (\tau) = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix}, \quad \text{and} \quad \| (\tau, \tau) \| \Sigma \leq C \| \beta \|_{L^2(\Omega)}. \tag{5.24}$$

Proof. From (5.24),

$$\frac{1}{2}(\tau_{12} - \tau_{21}) = \beta, \quad \Rightarrow \quad \tau_{21} = \tau_{12} - 2\beta, \tag{5.25}$$

$$\partial_r \tau_{11} + \frac{1}{r} \tau_{11} + \partial_z \tau_{12} - \frac{1}{r} \tau = 0, \tag{5.26}$$

$$\partial_r \tau_{21} + \frac{1}{r} \tau_{21} + \partial_z \tau_{12} = 0. \tag{5.27}$$

Substituting (5.25) into (5.27), then multiplying (5.26) and (5.27) through by $r$ and simplifying we obtain

$$\partial_r (r \tau_{11}) + \partial_z (r \tau_{12}) - \tau = 0, \tag{5.28}$$

$$\partial_r (r \tau_{12}) + \partial_z (r \tau_{22}) = \partial_r (2r \beta). \tag{5.29}$$

Integrating (5.28) with respect to $z$, and (5.29) with respect to $r$, yields for arbitrary $f_1$ and $f_2$,

$$\int^z \partial_r (r \tau_{11}) \, dz + r \tau_{12} - \int^z \tau \, dz = f_1(r), \tag{5.30}$$

$$r \tau_{12} + \int^r \partial_z (r \tau_{22}) \, dr = 2r \beta + f_2(z). \tag{5.31}$$

Interchanging the order of integration and differentiation, and then subtracting (5.30) from (5.31), yields

$$\partial_r \left( -\int^z (r \tau_{11}) \, dz \right) + \partial_z \left( \int^r (r \tau_{22}) \, dr \right) + \int^z \tau \, dz = 2r \beta + f_2(z) - f_1(r).$$

Dividing through by $r$, choosing $f_1 = f_2 = 0$, and rearranging we have

$$\frac{1}{r} \partial_r \left( r(-\int^z \tau_{11} \, dz) \right) + \partial_z \left( \frac{1}{r} \int^r (r \tau_{22}) \, dr \right) - \frac{1}{r} \left( -\int^z \tau \, dz \right) = 2\beta. \tag{5.32}$$

Let $\sigma_{rr} = (-\int^z \tau_{11} \, dz), \quad \sigma_{rz} = \frac{1}{r} \int^r (r \tau_{22}) \, dr, \quad \text{and} \quad \sigma_{\theta\theta} = -\int^z \tau \, dz. \tag{5.33}$

Then, (5.32) can be embedded in the meridian problem

$$\nabla_{axi} \cdot \begin{bmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{rz} & 0 & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} 2\beta \end{bmatrix} \quad \text{in} \quad \Omega. \tag{5.34}$$

Lifting (5.34) from $\Omega$ to $\tilde{\Omega}$ we obtain an axisymmetric elasticity problem in $\tilde{\Omega}$, $\nabla \cdot \sigma = f$ (see (3.1)), with $f \in L^2(\tilde{\Omega})$. From [21], we have that, for $\tilde{\Omega}$ a bounded polyhedral domain, $\sigma \in H^1(\tilde{\Omega}, M^3)$. Additionally, $\| \sigma \|_{H^1(\Omega, M^3)} \leq C \| f \|_{L^2(\Omega)} \leq C \| \beta \|_{L^2(\Omega)}$. 

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Then, using Lemma 4.2 we have that
\[ \sigma_{rr}, \sigma_{\theta\theta} \in H^1(\Omega), \sigma_{rz} \in V^1(\Omega), \text{ and } (\sigma_{rr} - \sigma_{\theta\theta}) \in L^2(\Omega). \quad (5.35) \]

From (5.33) and (5.35),
\[ \tau = -\partial_z \sigma_{\theta\theta}. \text{ As } \sigma_{\theta\theta} \in H^1(\Omega), \text{ then } \tau \in L^2(\Omega). \]
\[ \tau_{11} = -\partial_z \sigma_{rr}. \text{ As } \sigma_{rr} \in H^1(\Omega), \text{ then } \tau_{11} \in L^2(\Omega). \]
\[ \tau_{22} = \frac{1}{r} \sigma_{rz} + \partial_r \sigma_{rz}. \text{ As } \sigma_{rz} \in V^1(\Omega), \text{ then } \tau_{22} \in L^2(\Omega). \]

Also, it follows that
\[ \|\tau\|_{1, L^2(\Omega)} + \|\tau_{11}\|_{1, L^2(\Omega)} + \|\tau_{22}\|_{1, L^2(\Omega)} \leq C \|\sigma\|_{H^1(\Omega, M^3)} \leq C \|\beta\|_{1, L^2(\Omega)}. \quad (5.36) \]

Next, from (5.31) and (5.35),
\[ \tau_{12} = 2\beta - \frac{1}{r} \int r \partial_z (r \tau_{22}) \, dr = 2\beta - \partial_z \left( \frac{1}{r} \int r \tau_{22} \, dr \right) \]
\[ = 2\beta - \partial_z \sigma_{rz}. \]
\[ \Rightarrow \tau_{12} \in L^2(\Omega), \text{ with } \|\tau_{12}\|_{1, L^2(\Omega)} \leq C \|\beta\|_{1, L^2(\Omega)}. \quad (5.37) \]

Also, from (5.25) and (5.37),
\[ \tau_{21} = \tau_{12} - 2\beta \in L^2(\Omega), \text{ with } \|\tau_{21}\|_{1, L^2(\Omega)} \leq C \|\beta\|_{1, L^2(\Omega)}. \quad (5.38) \]

Finally, we confirm that (5.26) and (5.27) are satisfied.
\[ \partial_r \tau_{11} + \frac{1}{r} \tau_{11} + \partial_z \tau_{12} - \frac{1}{r} \tau = \frac{1}{r} \partial_r (r \tau_{11}) + \partial_z \tau_{12} - \frac{1}{r} \tau \]
\[ = \frac{1}{r} \partial_r (r \partial_z (-\sigma_{rr})) + \partial_z (2\beta - \partial_z \sigma_{rz}) - \frac{1}{r} \left( -\partial_z \sigma_{\theta\theta} \right) \]
\[ = -\partial_z \left( \frac{1}{r} \partial_r (r \sigma_{rr}) \right) + \partial_z \sigma_{rz} - \frac{1}{r} \sigma_{\theta\theta} - 2\beta \]
\[ = -\partial_z (0) = 0 \text{ (from (5.34))}. \]

Also,
\[ \partial_r \tau_{21} + \frac{1}{r} \tau_{21} + \partial_z \tau_{22} = \frac{1}{r} \partial_r (r \tau_{21}) + \partial_z \tau_{12} \]
\[ = \frac{1}{r} \partial_r (r (-\partial_z \sigma_{rz})) + \partial_z \left( \frac{1}{r} \partial_r (r \sigma_{rz}) \right) \]
\[ = -\partial_z \left( \frac{1}{r} \partial_r (r \sigma_{rz}) \right) + \partial_z \left( \frac{1}{r} \partial_r (r \sigma_{rz}) \right) = 0. \]

This completes the proof. □
The following lemma established the inf-sup condition (5.17).

**Lemma 5.4.** For any \( \mathbf{v} \in U \) and \( p \in Q \), there exists a \( C > 0 \) and a \((\mathbf{r}, \tau) \in \Sigma\) such that

\[
\begin{align*}
b((\mathbf{r}, \tau), \mathbf{v}) + c((\mathbf{r}, \tau), p) &= \|\nabla \mathbf{v}\|^2_U + \|p\|^2_Q, \\
with \quad \|(\mathbf{r}, \tau)\|_{\Sigma} &\leq C(\|\mathbf{v}\|_U + \|p\|_Q).
\end{align*}
\]

**Proof.** Let \( \mathbf{v} = (v_1, v_2)^t \in U \) and \( p \in Q \) be given. Then, there exist vectors \( \mathbf{w}, \mathbf{z} \in _1 \mathbf{VH}^1(\Omega) \) such that

\[
\nabla_{\text{axi}} \cdot \mathbf{w} = v_1 \quad \text{and} \quad \nabla_{\text{axi}} \cdot \mathbf{z} = v_2
\]

where \( \|\nabla_{\text{axi}} \cdot \mathbf{w}\|_{L^2(\Omega)} + \|\mathbf{w}\|_{_1 \mathbf{VH}^1(\Omega)} \leq C \|v_1\|_{L^2(\Omega)} \) and \( \|\nabla_{\text{axi}} \cdot \mathbf{z}\|_{L^2(\Omega)} + \|\mathbf{z}\|_{_1 \mathbf{VH}^1(\Omega)} \leq C \|v_2\|_{L^2(\Omega)} \).

To compute the vectors \( \mathbf{w} \) and \( \mathbf{z} \), one can map the axisymmetric scalar functions \( v_1 \) and \( v_2 \) into 3D Cartesian space and solve scalar Laplace equations to obtain functions \( t_1 \) and \( t_2 \). The gradient functions \( \mathbf{w} = \nabla_{(x,y,z)} t_1 \) and \( \mathbf{z} = \nabla_{(x,y,z)} t_2 \) are then computed. Finally, using Lemma 4.1, mapping \( \mathbf{w} \) and \( \mathbf{z} \) from \( \bar{\Omega} \) to \( \Omega \), we obtain \( \mathbf{w} \) and \( \mathbf{z} \).

Using \( \mathbf{w} \) and \( \mathbf{z} \), we then construct a matrix \( \mathbf{r}^1 \), where

\[
\mathbf{r}^1 = \begin{pmatrix} \mathbf{w}^t \\ \mathbf{z}^t \end{pmatrix}
\]

Thus, taking \((\mathbf{r}^1, \tau^1) = (\mathbf{r}^1, 0)\) one has that

\[
\nabla_{\text{axi}} \cdot (\mathbf{r}^1, \tau^1) = \mathbf{v}, \quad \text{hence} \quad b((\mathbf{r}^1, \tau^1), \mathbf{v}) = \|\mathbf{v}\|^2_U,
\]

and \( \|(\mathbf{r}^1, \tau^1)\|_{\Sigma} \leq \|(\mathbf{r}^1, 0)\|_{\Sigma} \leq \|v_1\|_U \leq C \|\mathbf{v}\|_U \leq C(\|\mathbf{v}\|_U + \|p\|_Q) \).

To build \((\mathbf{r}^2, \tau^2) \in \Sigma\), we first choose \( \theta, \gamma \in _1 L^2(\Omega) \) such that

\[
S^2(\theta) = aS(\mathbf{r}^1), \quad \text{and} \quad \gamma = \frac{1}{2}(v_1 z - v_2 r) = \frac{1}{2}(\nabla_{\text{axi}} \cdot \mathbf{r}^1 \wedge \mathbf{x}).
\]

Next, set \( \beta = (\gamma + \frac{1}{2} (v_1 z - v_2 r)) \in _1 L^2(\Omega) \). Note that

\[
\|\theta\|^2_{_1 L^2(\Omega)} = \int_{\Omega} (\tau_{12} - \tau_{21})^2 \quad 2 \int_{\Omega} (\tau_{11}^2 + \tau_{12}^2 + \tau_{21}^2 + \tau_{22}^2) \quad 2 \int_{\Omega} (\mathbf{r}^1, \tau^1\|_{\Sigma} \leq \|\mathbf{v}\|^2_U + \|p\|_Q\
\]

Also,

\[
\|\gamma\|^2_{_1 L^2(\Omega)} = \int_{\Omega} \left(\frac{1}{4}(v_1 z - v_2 r)^2 \quad \frac{1}{2} \int_{\Omega} (v_1^2 z^2 + v_2^2 r^2) \quad C \int_{\Omega} \mathbf{v} \cdot \mathbf{v} \quad C \|\mathbf{v}\|^2_U.
\]

Therefore,

\[
\beta \quad \|\beta\|_{0, \mathbf{VH}^1(\Omega)} \leq C(\|\mathbf{v}\|_U + \|p\|_Q).
\]
Next, \((\mathbf{t}^2, \tau^2) \in \Sigma\) is constructed using Lemma 5.3, with \(\beta \to -\beta\). For such a \((\mathbf{t}^2, \tau^2)\), it follows that
\[
\nabla_{\text{axi}} \cdot (\mathbf{t}^2, \tau^2) = 0, \quad \text{hence} \quad b((\mathbf{t}^2, \tau^2), \mathbf{v}) = 0, \tag{5.47}
\]
as\(\mathbf{t}^2\)\n\[
\text{as}(\mathbf{t}^2) = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \quad (\text{as}(\mathbf{t}^2), \text{S}^2(p)) = (\text{S}^2(-\beta), \text{S}^2(p)), \tag{5.48}
\]
and, \(\|(\mathbf{t}^2, \tau^2)\|_{\Sigma} \leq C\|\beta\|_{L^2(\Omega)} \leq C(\|\mathbf{v}\|_U + \|p\|_Q). \tag{5.49}\)

As a result, for \((\mathbf{t}, \tau) = (\mathbf{t}^1, \tau^1) + (\mathbf{t}^2, \tau^2)\) we have using (5.43) and (5.47)
\[
b((\mathbf{t}, \tau), \mathbf{v}) = b((\mathbf{t}^1, \tau^1), \mathbf{v}) + b((\mathbf{t}^2, \tau^2), \mathbf{v}) = \|\mathbf{v}\|_U^2, \tag{5.50}
\]
and
\[
c((\mathbf{t}, \tau), p) = c((\mathbf{t}^1, \tau^1), p) + c((\mathbf{t}^2, \tau^2), p)
\]
\[
= (\mathbf{t}^1, \text{S}^2(p)) + (\nabla_{\text{axi}} \cdot (\mathbf{t}^1, \tau^1) \wedge \mathbf{x}, p) + (\mathbf{t}^2, \text{S}^2(p)) + (\nabla_{\text{axi}} \cdot (\mathbf{t}^2, \tau^2) \wedge \mathbf{x}, p)
\]
\[
= (\mathbf{t}^1, \text{S}^2(p)) + (\nabla_{\text{axi}} \cdot \mathbf{t}^1 \wedge \mathbf{x}, p) + (\mathbf{t}^2, \text{S}^2(p)), \quad \text{(using } \nabla_{\text{axi}} \cdot (\mathbf{t}^2, \tau^2) = 0) \]
\[
= (\text{as}(\mathbf{t}^1), \text{S}^2(p)) + 2(\gamma, p) + (\text{as}(\mathbf{t}^2), \text{S}^2(p)), \quad \text{(using} \tag{5.45})
\]
\[
= (\text{S}^2(\theta), \text{S}^2(p)) + (\text{S}^2(\gamma), \text{S}^2(p)) + (\text{S}^2(-\beta), \text{S}^2(p)), \quad \text{(using} \tag{5.48})
\]
\[
= (\text{S}^2(\frac{1}{2}p), \text{S}^2(p))
\]
\[
= \|p\|_Q^2. \tag{5.51}
\]

Thus, from (5.50) and (5.51), \((\mathbf{t}, \tau)\) satisfies (5.39), and using (5.44) and (5.49),
\[
\|(\mathbf{t}, \tau)\|_{\Sigma} \leq \|(\mathbf{t}^1, \tau^1)\|_{\Sigma} + \|(\mathbf{t}^2, \tau^2)\|_{\Sigma} \leq C(\|\mathbf{v}\|_Q + \|p\|_Q). \tag{5.52}
\]

\section{Discrete Axisymmetric Variational Formulation}

In this section we present the setting for the approximation of (5.14)-(5.16). We begin by introducing the approximation spaces used:
\[
\Sigma_h := \Sigma_{h,\mathbf{t}} \times \Sigma_{h,\tau} = \{ (\mathbf{t}_h, \tau_h) : \mathbf{t}_h \in \Sigma_{h,\mathbf{t}}, \tau_h \in \Sigma_{h,\tau} \} \subset \Sigma, \quad U_h \subset U, \quad \text{and} \quad Q_h \subset Q . \tag{6.1}
\]

Additionally, we assume that there exists a piecewise polynomial space \((\Theta_h)^2\) such that \(((\Theta_h)^2, Q_h)\) is a stable axisymmetric Stokes pair satisfying: \textit{Given } \beta \in L^2(\Omega), \text{ there exists } \mathbf{w}_h = (w_{h1}, w_{h2}) \in (\Theta_h)^2 \text{ such that}
\[
(\nabla_{\text{axi}} \cdot \mathbf{w}_h, q_h) = (\beta, q_h), \text{ for all } q_h \in Q_h, \tag{6.2}
\]
\[\quad\text{and } \|\mathbf{w}_h\|_{1, VH^4(\Omega)} \leq C\|\beta\|_{L^2(\Omega)}. \tag{6.3}\]
Remark: As \( w_h \in (\Theta_h)^2 \), a piecewise polynomial space, from (6.3) we also have the bound
\[
\left( \sum_{T \in T_h} \left\| \frac{\partial^2 w_h}{\partial z^2} \right\|^2_{L^2(T)} \right)^{1/2} \leq C \left( \sum_{T \in T_h} \left\| w_h \right\|^2_{L^2(T)} \right)^{1/2} \leq C \left( \sum_{T \in T_h} \left\| \nabla w_h \right\|^2_{L^2(T)} \right)^{1/2} \leq C \left\| \beta \right\|_{L^2(\Omega)}.
\] (6.4)

The discrete axisymmetric meridian problem with weak symmetry is then: Given \( f \in L^2(\Omega) \) find \((\sigma_h, \tau_h), w_h, p_h) \in \Sigma_h \times U_h \times Q_h \) such that for all \((\sigma, \tau), v_h, q_h) \in \Sigma_h \times U_h \times Q_h\)
\[
a((\sigma_h, \tau_h), (\sigma, \tau)) + b((\sigma_h, \tau_h), w_h) + c((\sigma_h, \tau_h), p_h) = (f, \nabla_{axi} \cdot (\tau, \tau))
\] (6.5)
\[
b((\sigma_h, \tau_h), v_h) = (f, v_h)
\] (6.6)
\[
c((\sigma_h, \tau_h), q_h) = (f \wedge x, q_h).
\] (6.7)

6.1 Well posedness of the discrete variational formulation (6.5)-(6.7)

Analogous to the continuous formulation, the well posedness of (6.5)-(6.7) relies on the boundedness and coercivity of \( a(\cdot, \cdot) \) on \( \Sigma_h \times \Sigma_h \) and that \( \Sigma_h \times (U_h \times Q_h) \) satisfy the inf-sup condition
\[
\inf_{v_h \in U_h, p_h \in Q_h} \sup_{(\sigma, \tau) \in \Sigma_h} b((\sigma, \tau), v_h) + c((\sigma, \tau), p_h) \geq C \|(\tau, \tau)\|_{\Sigma} (\|v_h\|_U + \|p_h\|_Q).
\] (6.8)

To establish (6.8) we use Fortin’s Lemma [13]. Given \( u_h \in U_h \subset U, p_h \in Q_h \subset Q \) we determine, as in the proof of Lemma 5.4, a \((\tau, \tau) = (\tau^1, \tau^1) + (\tau^2, \tau^2)\) such that the continuous inf-sup condition is satisfied. Then, using a suitably defined projection (see (6.14)-(6.16)), we obtain \((\tau, \tau) \in \Sigma_h\) satisfying (6.8).

Helpful in this discussion is to define the restriction of the operators \( b(\cdot, \cdot) \) and \( c(\cdot, \cdot) \) to \( T \in T_h\):
\[
b((\tau, \tau), u)_T = (\nabla_{axi} \cdot \tau, u)_T - (\frac{\tau}{r}, u_r)_T
\] (6.9)
\[
c((\tau, \tau), p)_T = (a(s(\tau), S^2(p)) + ((\nabla_{axi} \cdot (\tau, \tau)) \wedge x, p)_T.
\] (6.10)

Next, we present the following identity for the operator \( c(\cdot, \cdot)_T\).

Lemma 6.1. For \( T \in T_h\),
\[
c((\tau, \tau), p)_T = \int_{\partial T} (\tau \cdot n) \cdot x^\perp r \, ds - \int_T \tau : (x^\perp \otimes \nabla p) \, r \, dT - \int_T \tau \cdot z \, p \, dT.
\] (6.11)

Proof. See Section A in the appendix. ■

Theorem 6.1. Assume \( \Sigma_h, U_h, Q_h \) satisfy (6.1). Let
\[
S_{\Theta_h} = \left\{ (\tau, \tau) : \tau = \begin{pmatrix} \frac{\partial w_{h1}}{\partial z} & -1 \frac{\partial}{\partial r} (r \, w_{h1}) - \frac{\partial w_{h2}}{\partial z} \\ 0 & 0 \end{pmatrix}, \tau = (r \, \frac{\partial^2 w_{h2}}{\partial z^2}), w_h = (w_{h1}, w_{h2})^t \in (\Theta_h)^2 \right\}
\]
If there exists a mapping \( \Pi_h = \Pi_h \times \pi_h : (S + S_{\Theta_h}) \to \Sigma_h \) such that:

\[
\| \Pi_h \times \pi_h (\mathbf{t}, \tau) \|_{\Sigma} \leq C \| (\mathbf{t}, \tau) \|_S, \quad \forall (\mathbf{t}, \tau) \in S, \tag{6.12}
\]

\[
\| \Pi_h \times \pi_h (\mathbf{t}, \tau) \|_{\Sigma} \leq C \| w_h \|_{\mathcal{V}H^1(\Omega)}, \quad \forall (\mathbf{t}, \tau) \in S_{\Theta_h}, \tag{6.13}
\]

and for all \( T \in T_h \)

\[
\int_T (\mathbf{t} - \Pi_h \mathbf{t}) : (\nabla u_h + x^{\perp} \otimes \nabla q_h) r \, dT = 0, \quad \forall u_h \in U_h, \forall q_h \in Q_h, \tag{6.14}
\]

\[
\int_{\ell} ((\mathbf{t} - \Pi_h \mathbf{t}) \cdot n_{K}) \cdot (u_h + x^{\perp} q_h) r \, ds = 0, \quad \forall \text{edges } \ell, \forall u_h \in U_h, \forall q_h \in Q_h, \tag{6.15}
\]

\[
\int_T \frac{1}{r} (\tau - \pi_h \tau) \sigma \, dT = 0, \quad \forall \sigma \in \{z q_h : q_h \in Q_h\} \cup \{v_{h1} : (v_{h1}, 0) \in U_h\}, \tag{6.16}
\]

then \( \Sigma_h \times (U_h \times Q_h) \) are inf-sup stable.

**Proof.** The approach to this proof is similar to that used in [12]. Let \( \mathbf{v}_h = (v_{h1}, v_{h2})^T \in U_h \subset L^2(\Omega, \mathbb{R}^2) \) and \( p_h \in Q_h \subset L^2(\Omega) \).

Recall from Lemma 5.4 that for \( \mathbf{v}_h \) given, there exists \((\mathbf{t}^1, 0) \in S(\Omega) \subset \Sigma(\Omega)\) such that

\[
\nabla_{axi} : (\mathbf{t}^1, 0) = \mathbf{v}_h \quad \text{and} \quad \| (\mathbf{t}^1, 0) \|_S \leq C (\| v_h \|_U + \| p_h \|_Q). \tag{6.17}
\]

Also, from (5.43), \( b((\mathbf{t}^1, 0), \mathbf{v}_h) = (\mathbf{v}_h, \mathbf{v}_h) \), hence

\[
b((\mathbf{t}^1, 0) - \Pi_h (\mathbf{t}^1, 0), \mathbf{v}_h) = b((\mathbf{t}^1 - \Pi_h \mathbf{t}^1, 0), \mathbf{v}_h)
\]

\[
= \sum_{T \in T_h} (\nabla_{axi} : (\mathbf{t}^1 - \Pi_h \mathbf{t}^1), \mathbf{v}_h)_T + (0, v_{h1})_T
\]

\[
= 0 \quad \text{(using (6.14)-(6.15) with } q_h = 0). \tag{6.18}
\]

Furthermore, from (6.12) and (6.17),

\[
\| \Pi_h (\mathbf{t}^1, 0) \|_{\Sigma} \leq C \| (\mathbf{t}^1, 0) \|_S \leq C (\| v_h \|_U + \| p_h \|_Q). \tag{6.19}
\]

Next, combining (6.11) with (6.14)-(6.15) (with \( u_h = 0 \)),

\[
c((\mathbf{t}^1, 0) - \Pi_h (\mathbf{t}^1, 0), p_h) = \sum_{T \in T_h} c((\mathbf{t}^1 - \Pi_h \mathbf{t}^1, 0), p_h)_T
\]

\[
= \sum_{T \in T_h} \left( \int_{\partial T} ((\mathbf{t}^1 - \Pi_h \mathbf{t}^1) \cdot n_T) \cdot x^{\perp} p_h r \, ds - \int_T (\mathbf{t}^1 - \Pi_h \mathbf{t}^1) : (x^{\perp} \otimes \nabla p_h) r \, dT \right.
\]

\[
- \int_T \left. \frac{1}{r} z p_h r \, dT \right) = 0. \tag{6.20}
\]

The \((\mathbf{t}^2, \tau^2)\) used in establishing the continuous inf-sup condition is not sufficiently regular in order to construct a suitable projection. To circumvent this problem we use (6.2) to determine a suitable replacement for \((\mathbf{t}^2, \tau^2)\), namely \((\mathbf{t}^{2}_h, \tau^{2}_h)\), and then use a projection of \((\mathbf{t}^{2}_h, \tau^{2}_h)\) to help satisfy (6.8).
Using the assumptions that \((\Theta_h)^2 \times Q_h\) is a stable axisymmetric Stokes pair, we obtain \(w_h \in (\Theta_h)^2\) satisfying (6.2)-(6.4).

Define

\[
\tau^2_h = 2 \left( \frac{\partial w_{h1}}{\partial z} - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial w_{h1}}{\partial z} \right) - \frac{\partial w_{h2}}{\partial z} \right) \quad \text{and} \quad \tau^2_h = 2 \frac{\partial^2 w_{h2}}{\partial z^2}.
\]

Then, (cf. (5.47)-(5.49))

\[
\nabla_{\text{axi}} \cdot (\tau^2_h, \tau^2_h) = 0, \quad \text{hence} \quad b((\tau^2_h, \tau^2_h), v_h) = 0, \quad (6.21)
\]

and \((\text{as}(\tau^2_h), S^2(p_h)) = (S^2(-\beta), S^2(p_h))\) (using (6.2)).

Also, from (6.3), (6.4), and that \(\|\beta\|_{L^2(\Omega)} \leq C (\|v_h\|_U + p_h\|Q\|),\)

\[
\|(\tau^2_h, \tau^2_h)\|_{\Sigma} \leq C (\|v_h\|_U + p_h\|Q\|).
\]

Now, for \(\Pi_h(\tau^2_h, \tau^2_h),\) proceeding as in (6.18); using (6.14)-(6.15) (with \(q_h = 0\)), and (6.16)

\[
b((\tau^2_h, \tau^2_h) - \Pi_h(\tau^2_h, \tau^2_h), v_h) = b((\tau^2_h - \Pi_h(\tau^2_h, \tau^2_h) - \pi_h(\tau^2_h)), v_h)
\]

\[
= \sum_{T \in T_h} \left( \nabla_{\text{axi}} \cdot (\tau^2_h - \Pi_h(\tau^2_h), v_h) \right)_T + \left( \frac{1}{r} (\tau^2_h - \pi_h(\tau^2_h), v_h) \right)_T = 0.
\]

Also, as in (6.20), and using (6.16),

\[
c((\tau^2_h, \tau^2_h) - \Pi_h(\tau^2_h, \tau^2_h), p_h) = \sum_{T \in T_h} c((\tau^2_h - \Pi_h(\tau^2_h), (\tau^2_h - \pi_h(\tau^2_h)), p_h)_T
\]

\[
= \sum_{T \in T_h} \left( \int_{\partial T} ((\tau^2_h - \Pi_h(\tau^2_h)) \cdot n_T) \cdot x^1 p_h \, r \, ds - \int_T (\tau^2_h - \Pi_h(\tau^2_h)) : (x^1 \otimes \nabla p_h) \, r \, dT \right)
\]

\[
- \int_T ((\tau^2_h - \pi_h(\tau^2_h)) \cdot (p_h) \, dT = 0.
\]

Finally, with \((\tau_h, \tau_h) = (\Pi_h(\tau^1, \tau^1), \Pi_h(\tau^2, \tau^2))\),

\[
\sup_{(\sigma_h, \sigma_h) \in \Sigma_h} \frac{b((\sigma_h, \sigma_h), v_h) + c((\sigma_h, \sigma_h), p_h)}{\|(\sigma_h, \sigma_h)\|_{\Sigma} (\|v_h\|_U + p_h\|Q\|)} \geq \frac{b((\tau_h, \tau_h), v_h) + c((\tau_h, \tau_h), p_h)}{\|(\tau_h, \tau_h)\|_{\Sigma} (\|v_h\|_U + p_h\|Q\|)}
\]

\[
\geq C \frac{b((\tau^1, \tau^1), v_h) + c((\tau^1, \tau^1), p_h) + 0 + c((\tau^2, \tau^2), p_h)}{\|(\tau^1, \tau^1)\|_S + \|(\tau^2, \tau^2)\|_{\Sigma} (\|v_h\|_U + p_h\|Q\|)}
\]

\[
\geq C \frac{\|v_h\|_U^2 + p_h\|Q\|}{\|v_h\|_U + p_h\|Q\| (\|v_h\|_U + p_h\|Q\|)}
\]

\[
\geq C.
\]

Throughout the remainder of this document, we will denote the space \((\Sigma + S_{\text{axi}})\) as \(\Sigma^S\). Additionally, we denote the tensor and scalar components of \(\Sigma^S\) as \(\Sigma^S_{\text{axi}}\) and \(\Sigma^S_{\text{axi}}\), i.e., \(\Sigma^S = \Sigma^S_{\text{axi}} \times \Sigma^S_{\text{axi}}\).
7 Approximation spaces satisfying (6.14),(6.15)

In this section we investigate approximation spaces for $\Sigma$, $U_h$, and $Q_h$ such that there exists a projection operator $\Pi_h(\tau, \tau)$ satisfying (6.14)-(6.16).

7.1 $\Sigma_h = (\text{BDM}_1(\mathcal{T}_h))^2 \times P_1(\mathcal{T}_h)$, $U_h = (P_0(\mathcal{T}_h))^2$, and $Q_h = P_0(\mathcal{T}_h)$

In this section we show that for the choice of spaces $\Sigma_{h,\sigma} = (\text{BDM}_1(\mathcal{T}_h))^2$, $\Sigma_{h,\tau} = P_1(\mathcal{T}_h)$, $U_h = (P_0(\mathcal{T}_h))^2$, and $Q_h = P_0(\mathcal{T}_h)$ there exists a projection operator, $\Pi_h(\tau, \tau)$, satisfying (6.14)-(6.16).

Lemma 7.1. Let $T \in \mathcal{T}_h$. The mappings $\Pi_h : \Sigma_{h,\sigma}^S(T) \to (P_1(T))^4$ and $\pi_h : \Sigma_{h,\tau}^S(T) \to P_1(T)$ given by

$$\int_\ell (\tau - \Pi_h \tau) \cdot n_k \cdot p_1 r \, ds = 0 \text{ for all edges } \ell \in \partial T \text{ and } p_1 \in (P_1(\ell))^2 \quad (7.1)$$

$$\int_T \frac{1}{r} (\tau - \pi_h \tau) \cdot p_1 r \, dT = 0 \text{ for all } p_1 \in P_1(T) \quad (7.2)$$

are well defined. Hence the spaces

$$\Sigma_{h,\sigma} = (\text{BDM}_1(\mathcal{T}_h))^2 \quad \Sigma_{h,\tau} = P_1(\mathcal{T}_h) \quad U_h = (P_0(\mathcal{T}_h))^2 \quad Q_h = P_0(\mathcal{T}_h) \quad (7.3)$$

satisfy (6.14)-(6.16).

Proof. Observe that $\pi_h$ is the well defined $L^2$ projection.

Next we show that $\Pi_h$ is well defined. Note that $\Pi_h \tau \in (P_1(T))^4 = (\text{BDM}_1(T))^2$ has 12 degrees of freedom, and $(P_1(\ell))^2$ has 4 degrees of freedom per edge. Thus the number of unknowns in $\Pi_h \tau$ is equal to the number of constraints in (7.1). It follows that if $\tau = 0$ implies that $\Pi_h \tau = 0$, then the projection $\Pi_h$ is well defined.

Consider a single row of the tensor projection (7.1). In this case, for $\tau = (\tau_1, \tau_2)^t$ the projection (7.1) takes the form

$$\int_\ell (\tau_s - \Pi_h \tau_s) \cdot n_k \cdot p_1 r \, ds = 0 \text{ for } p_1 \in P_1(\ell), \ s = 1, 2. \quad (7.4)$$

Next, observe that the function $\Pi_h \tau_s \cdot n_k \cdot p_1 r$ is a cubic polynomial. Recalling that a degree $n$ Gauss quadrature rule integrates polynomials of degree $2n - 1$ exactly, we select two Gauss quadrature points \( \{ q_{i}^{\ell_k} \}_{i=1}^{2} \) on each edge $\ell_k$ for $k = 1, 2, 3$.

For $\ell_k \in \partial K$, define a basis for $P_1(\ell_k)$ so that

$$p_1^{\ell_k}(x) = \begin{cases} 1 & \text{if } x = q_1^{\ell_k} \\ 0 & \text{if } x = q_2^{\ell_k} \end{cases} \quad \text{and} \quad p_2^{\ell_k}(x) = \begin{cases} 0 & \text{if } x = q_1^{\ell_k} \\ 1 & \text{if } x = q_2^{\ell_k} \end{cases}. \quad (7.5)$$

Let $\{ \phi_i^{\ell_k} \}$ be a basis for $\text{BDM}_1(T)$ [17] such that

$$(\phi_i^{\ell_m} \cdot n)(q_j^{\ell_n}) = \delta_{(i,j),(\ell_m,\ell_n)} \text{ for } i, j = 1, 2 \text{ and } m, n = 1, 2, 3.$$
Note that the normal component of the basis functions satisfy a Lagrangian property at the boundary quadrature points. Since \( \Pi_h \tau_s \in \text{BDM}_1(T) \), it can be written as

\[
\Pi_h \tau_s = \sum_{k=1}^{3} \sum_{i=1}^{2} \alpha_i^k \phi_i^k.
\]

With \( \tau_s = 0 \), taking the basis function \( p_1^k \) for \( k \in \partial K \) and using (7.1) and Gaussian quadrature gives

\[
0 = \int_{\ell_k} \Pi_h \tau_s \cdot n \, p_1 \; r \; ds = \sum_{j=1}^{2} (\Pi_h \tau_s \cdot n) (q_j^k) \cdot p_1^k(q_j^k) \; r(q_j^k) \; w(q_j^k) = \alpha_1^k \cdot p_1^k(q_1^k) \; r(q_1^k)w(q_1^k) + \alpha_2^k \cdot p_1^k(q_2^k) \; r(q_2^k)w(q_2^k) = \alpha_1^k r(q_1^k)w(q_1^k).
\]

In the case where \( r(q_1^k) \neq 0 \), this implies \( \alpha_1^k = 0 \). If, however, \( r(q_1^k) = 0 \), then \( \alpha_1^k = \beta_1^k \) must be zero, otherwise, the normal stress along the axis of symmetry will be non-zero implying that the solution is not axisymmetric. A similar argument can be used to show that the other \( \alpha_1 \) terms are also zero. Hence the vector projection from (7.4) is well defined.

To extend the vector projection from (7.4) to the tensor projection given in (7.1), we extend the basis for \( P_1(\ell_k) \) from (7.5) to \((P_1(\ell_k))^2\) by using

\[
(P_1(\ell_k))^2 = \text{span} \left\{ \begin{pmatrix} p_1^k \\ 0 \end{pmatrix}, \begin{pmatrix} p_2^k \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ p_1^k \end{pmatrix}, \begin{pmatrix} 0 \\ p_2^k \end{pmatrix} \right\}.
\]

With this basis, the arguments presented above for the vector case can be applied to each row of (7.1) to show that \( \Pi_h \) is well defined.

Lastly, we verify that the spaces given in (7.3) satisfy the conditions outlined in (6.14)-(6.16). Since gradients of the piecewise constant spaces \( U_h \) and \( Q_h \) are zero on each element \( T \), (6.14) is trivially satisfied. Next, observe that the test space of (7.1) includes all \( p_1 \in (P_1(\ell_k))^2 \) for \( k = 1, 2, 3 \), while (6.15) only requires that the projection is satisfied on a subspace of \((P_1(\ell_k))^2\). Finally, since \( P_0(T) \subset P_1(T) \), (7.2) ensures that (6.16) is satisfied.

### 7.2 \( \Sigma_h = (\text{BDM}_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h), U_h = (P_1(\mathcal{T}_h))^2, \text{ and } Q_h = P_1(\mathcal{T}_h) \)

In this section we show that for the choice of spaces \( \Sigma_{h,\sigma} = (\text{BDM}_2(\mathcal{T}_h))^2 \), \( \Sigma_{h,\sigma} = P_2(\mathcal{T}_h), \) \( U_h = (P_1(\mathcal{T}_h))^2 \), and \( Q_h = P_1(\mathcal{T}_h) \) there exists a projection operator, \( \Pi_h(\tau, \tau) \), satisfying (6.14)-(6.16).

**Lemma 7.2.** Let \( T \in \mathcal{T}_h \). The projection operators \( \Pi_h : \Sigma^S_{\sigma}(T) \rightarrow (P_2(T))^4 \) and \( \pi_h : \Sigma^S_{\sigma}(T) \rightarrow \)
$P_2(T)$ given by

$$
\int_T (\tau - \Pi_h \tau) : (\mathbf{p}_0 + x^\perp \otimes \mathbf{p}_0) \, r \, dT = 0 \quad \forall \mathbf{p}_0 \in (P_0(T))^{2 \times 2} \quad \forall \mathbf{p}_0 \in (P_0(T))^2 \tag{7.6}
$$

$$
\int_\ell (\tau - \Pi_h \tau) \cdot n_k \cdot p_2 \, r \, ds = 0 \quad \forall \text{edges } \ell \quad \forall p_2 \in (P_2(\ell))^2 \tag{7.7}
$$

$$
\int_T \frac{1}{r} (\tau - \pi_{h_T}) p_2 \, r \, dT = 0 \quad \text{for all } p_2 \in P_2(T) \tag{7.8}
$$

are well defined. Hence the spaces $\Sigma_{h,\sigma} = (\text{BDM}_2(\mathcal{T}_h))^2$, $\Sigma_{h,\sigma} = P_2(\mathcal{T}_h)$, $U_h = (P_1)^2(\mathcal{T}_h)$, $Q_h = P_1(\mathcal{T}_h)$ satisfy (6.14)-(6.16).

**Proof.** Observe that $\pi_h$ is the well defined $L^2$ projection.

Next we show that $\Pi_h$ is well defined. First observe that the number of constraints defined by $\Pi_h$, 24, is the same as number of degrees of freedom in $(P_2(T))^4 = (\text{BDM}_2(T))^2$. We verify that the projection is injective by showing that

$$
\int_T \Pi_h \tau : (\mathbf{p}_0 + x^\perp \otimes \mathbf{p}_0) \, r \, dT = 0 \quad \forall \mathbf{p}_0 \in (P_0(T))^{2 \times 2} \quad \forall \mathbf{p}_0 \in (P_0(T))^2 \tag{7.10}
$$

$$
\int_\ell \Pi_h \tau \cdot n_k \cdot p_2 \, r \, ds = 0 \quad \forall \text{edges } \ell \quad \forall p_2 \in (P_2(\ell))^2 \tag{7.11}
$$

has the unique solution $\Pi_h \tau = 0$.

We can represent $\Pi_h \tau$ in terms of the basis for $(\text{BDM}_2(\mathcal{T}))^2$, where $\text{BDM}_2(\mathcal{T})$ is the reference element representation presented in [17, Section 4.2]. This $\text{BDM}_2(\mathcal{T})$ basis is expressed in terms of edge and interior element functions. Using equation (7.11) with three Gauss quadrature points and an argument analogous to that used in the proof of Lemma 7.1, it follows that all 18 of the $\text{BDM}_2(\mathcal{T})$ edge basis functions must equal zero.

Therefore, the only possible non-zero basis functions on $\mathcal{T}$ are the interior element functions

$$
\phi_1 = \frac{\sqrt{2}}{(g_2 - g_1)} (1 - \xi - \eta) \left( \frac{g_2 \xi}{(g_2 - 1) \eta} \right) \quad \phi_2 = \frac{1}{(g_2 - g_1)} \xi \left( \frac{g_2 \xi + \eta - g_2}{(g_2 - 1) \eta} \right) \quad \phi_3 = \frac{1}{(g_2 - g_1)} \eta \left( \frac{(g_2 - 1) \xi}{\xi + g_2 \eta - g_2} \right) \tag{7.12}
$$

where $g_1 = 1/2 - \sqrt{3}/6$ and $g_2 = 1/2 + \sqrt{3}/6$ are the Gaussian quadrature points on $[0, 1]$. Thus, $\Pi_h \tau$, the representation of $\Pi_h \tau$ on $\mathcal{T}$, must have the form

$\Pi_h \tau = \left( \begin{array}{c} \phi_1^t \\ \phi_2^t \\ \phi_3^t \end{array} \right)$

where

$$
\phi_1^t = \alpha_1 \phi_1^t + \alpha_2 \phi_2^t + \alpha_3 \phi_3^t \quad \text{and} \quad \phi_2^t = \beta_1 \phi_1^t + \beta_2 \phi_2^t + \beta_3 \phi_3^t.
$$

It remains to show that $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3$. To do so, we consider the matrix representation of equation (7.6). The functions in (7.12) can be used as the six trial basis functions of (7.6), while
the test space of (7.6) has dimension 6, and is spanned by the functions

\[ \psi_i = \begin{pmatrix} \delta_{i1} & \delta_{i2} \\ \delta_{i3} & \delta_{i4} \end{pmatrix} + \begin{pmatrix} \eta \delta_{i5} & \eta \delta_{i6} \\ -\xi \delta_{i5} & -\xi \delta_{i6} \end{pmatrix} \] 

for \( i = 1, \ldots, 6 \) and \( \delta_{ij} \in \mathbb{R} \) for \( i, j = 1, 2, \ldots, 6 \). \hspace{1cm} (7.13)

Taking \( \psi_i \) as the test function for row \( i \), the resulting matrix representation of equation (7.6) is presented in (7.14) where \( I(\cdot) \) is defined in (B.1).

To illustrate how the elements of (7.14) are calculated, we consider the first row of (7.14). From (B.1), Lemma B.2 and (B.4), the entries of the first row are

\[
\begin{align*}
I(g_2(1 - \xi - \eta)\xi) &= \int_{\bar{T}} g_2(1 - \xi - \eta) \xi (r^*_1 \xi + r^*_2 \eta + 1) \, d\bar{T} \\
&= g_2 \left[ \int_{\bar{T}} (\xi - \xi^2 - \eta \xi) (r^*_1 \xi + r^*_2 \eta + 1) \, d\bar{T} \right] \\
&= g_2 \left( \frac{1}{4!} (2r^*_1 + r^*_2 + 4) - \frac{2}{5!} (3r^*_1 + r^*_2 + 4) - \frac{1}{5!} (2r^*_1 + 2r^*_2 + 5) \right) \\
&= g_2 \left( \frac{1}{5!} (2r^*_1 + r^*_2 + 5) \right) \\
I(\xi(g_2 \xi - g_2)) &= \int_{\bar{T}} (g_2^2 \xi^2 + \xi \eta - g_2^2 \xi) (r^*_1 \xi + r^*_2 \eta + 1) \, d\bar{T} \\
&= \left( \frac{2g_2^2}{5!} (3r^*_1 + r^*_2 + 4) + \frac{1}{5!} (2r^*_1 + 2r^*_2 + 5) - \frac{g_2^2}{4!} (2r^*_1 + r^*_2 + 4) \right) \\
&= 2(1 - 2g_2) r^*_1 + (2 - 3g_2) r^*_2 + 5(1 - 2g_2) \\
I((g_2 - 1)\eta \xi) &= \int_{\bar{T}} (g_2 - 1) \eta \xi (r^*_1 \xi + r^*_2 \eta + 1) \, d\bar{T} \\
&= (g_2 - 1) \left[ \frac{1}{5!} (2r^*_1 + 2r^*_2 + 5) \right]
\end{align*}
\]

with the remaining columns equaling zero. A similar procedure can be used to find the remaining terms in the system. The complete entries of the matrix expressed in terms of the coordinates of the triangle \( T \) are shown in (7.15) which we denote \( M_T \).

Taking the determinate of (7.15) yields

\[
|M_T| = \frac{1}{36} (r^*_1 + r^*_2 + 3)(2r^*_1 + r^*_2 + 5)(r^*_1 + 2r^*_2 + 5) \\
(2r^*_1 + 2r^*_2 + 5)((r^*_1)^2 + 4r^*_1 r^*_2 + (r^*_2)^2 + 10r^*_1 + 10r^*_2 + 15).
\]

Since \( r^*_1, r^*_2 \geq 0 \), it follows that \( |M_T| > 0 \) implying that the matrix representation of the projection operator is full rank. Therefore \( \Pi_{h-T} = \{0\} \) is the unique solution. Hence \( \Pi_{h} \) is well defined.

Finally, we verify that the spaces given in (7.9) satisfy (6.14)-(6.16). Observe that for \( U_h = (P_1)^2 \) and \( Q_h = P_1 \) the test space of (6.14) is the set

\[
\left\{ \left( \begin{array}{ll} \delta_1 + z \delta_5 & \delta_2 + z \delta_6 \\ \delta_3 - r \delta_5 & \delta_4 - r \delta_6 \end{array} \right) \mid \forall \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6 \in \mathbb{R} \right\}.
\]
which is the same as the test space described in (7.6). Furthermore, Theorem 6.1 requires that (6.15) is satisfied on a subset of

$$\left\{ \begin{pmatrix} s_1 + s_2 s + s_3 s^2 \\ s_4 + s_5 s + s_6 s^2 \end{pmatrix} \mid \forall s_1, s_2, s_3, s_4, s_5, s_6 \in \mathbb{R} \right\}$$

for all $\ell$. Since the boundary integral (7.7) is satisfied for all quadratic polynomials on all $\ell$, this condition is also satisfied. Lastly, for $U_h(T) = (P_1(T))^2$, $Q_h(T) = P_1(T)$, the test functions in (6.16) are a subset of the test functions in (7.8).
\[
\begin{pmatrix}
I(g_2(1 - \xi - \eta) \xi) & I(\xi(g_2 \xi + \eta - g_2)) & I((g_2 - 1)\eta) \\
I((g_2 - 1)(1 - \xi - \eta) \eta) & I((g_2 - 1)\eta) & I(\xi(g_2 \eta - g_2)) \\
0 & 0 & I((g_2 - 1)(1 - \xi - \eta) \eta)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & I((g_2 - 1)(1 - \xi - \eta) \eta) \\
I((1 - \xi - \eta)(g_2 \eta - g_2)\eta) & I((g_2 - 1)\eta) & I(\xi(g_2 \xi + \eta - g_2))
\end{pmatrix} = (7.14)
\]

\[
\begin{pmatrix}
g_2(2r_1^2 + r_2^2 + 3) & 2(1 - 2g_2)r_2^1 + (2 - 3g_2)r_2^2 + 5(1 - 2g_2) \\
(g_2 - 1)(r_1^2 + 2r_2^2 + 5) & (g_2 - 1)(2r_2^1 + 2r_2^2 + 5) & (2 - 3g_2)r_2^1 + 2(1 - 2g_2)r_2^2 + 5(1 - 2g_2)
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & I((g_2 - 1)(1 - \xi - \eta) \eta) \\
I((1 - \xi - \eta)(g_2 \eta - g_2)\eta) & I((g_2 - 1)\eta) & I(\xi(g_2 \xi + \eta - g_2))
\end{pmatrix} = (7.15)
\]
8 Error Analysis

In this section, for \( \Sigma_h \times U_h \times Q_h \) satisfying the inf-sup condition

\[
\inf_{w_h \in U_h, p_h \in Q_h} \sup_{(\sigma_h, p_h) \in \Sigma_h} \frac{b((\sigma_h, \sigma_h), w_h) + c((\sigma_h, \sigma_h), p_h)}{\|w_h\|_U + \|p_h\|_Q} \geq \beta > 0,
\]

we present an error analysis for the solution to the discrete linear elasticity problem (6.5)-(6.7). For notational compactness, we let

\[
B((\sigma_h, \sigma_h), (v_h, p_h)) = b((\sigma_h, \sigma_h), v_h) + c((\sigma_h, \sigma_h), p_h).
\]

Recall that operator \( a(\cdot, \cdot) : \Sigma_h \times \Sigma_h \to \mathbb{R} \) as defined in (5.7) is continuous and coercive (see Lemmas 5.1 and 5.2). That is,

\[
a((\sigma, \sigma), (\sigma, \sigma)) \geq \gamma > 0 \quad \text{for all} \quad (\sigma, \sigma) \in \Sigma_h,
\]

\[
a((\sigma, \sigma), (\tau, \tau)) \leq \alpha \|\sigma\|_{\Sigma} \|\tau\|_{\Sigma}
\]

for some \( \alpha > 0 \) and all \((\sigma, \sigma), (\tau, \tau) \in \Sigma \). We also note that \( B((\cdot, \cdot), (\cdot, \cdot)) \) is continuous since

\[
B((\sigma, \sigma), (v, q)) = b((\sigma, \sigma), v) + c((\sigma, \sigma), q) = (v, \nabla_{\text{axi}} \cdot \sigma) - (v_r, \sigma_r) + (\sigma, \nabla^2 q) + (\nabla_{\text{axi}} \cdot (\sigma, \sigma) \wedge x, q)
\]

\[
\leq C_1 \|v\|_{L^2(\Omega)} \|\sigma\|_{\Sigma} + C_2 \|q\|_{L^2(\Omega)} \|\sigma\|_{\Sigma}
\]

\[
\leq \beta \|\sigma\|_{\Sigma} (\|v\|_U + \|q\|_Q)
\]

for all \((\sigma, \sigma) \in \Sigma, v \in U \) and \( q \in Q \) where \( C_1, C_2, \beta > 0 \).

The discrete null space of the operator \( B((\cdot, \cdot), (\cdot, \cdot)) \) is defined as

\[
Z_h = \{(\tau_h, \tau_h) \in \Sigma_h : B((\tau_h, \tau_h), (v_h, q_h) = 0 \text{ for all} \ v_h \in U_h \text{ and} \ q_h \in Q_h\}.
\]

Since \( B((\tau_h, \tau_h), (v_h, q_h)) = 0 \) only holds on the discrete subspaces \( U_h \) and \( Q_h \), \( Z_h \not\subset Z \). This observation motivates the following theorem which bounds the error \( \sigma_h \) in terms of the spaces \( U_h, Q_h \) and \( Z_h \).

**Theorem 8.1.** Let \((\sigma, \sigma), w, p) \) solve (5.14)-(5.16) and \((\sigma_h, \sigma_h) \) solve (6.5)-(6.7). If \( \Sigma_h \subset \Sigma \), \( U_h \subset U \), \( Q_h \subset Q \), and \( Z_h \) is defined as in (8.6), then

\[
\|\sigma - \sigma_h, \sigma - \sigma_h\|_{\Sigma} \leq C \left( \inf_{(\tau_h, \tau_h) \in \Sigma_h} \|\sigma - \tau_h, \sigma - \tau_h\|_{\Sigma} + \inf_{v_h \in U_h} \|w - v_h\|_U + \inf_{q_h \in Q_h} \|p - q_h\|_Q \right),
\]

where \( C > 0 \) is independent of \( h \).

**Proof.** Let \((\sigma_h, \sigma_h) \in Z_h \) be the unique solution to

\[
a((\sigma_h, \sigma_h), (\tau_h, \tau_h)) = (f, \nabla_{\text{axi}} \cdot (\tau_h, \tau_h)) \text{ for all} \ (\tau_h, \tau_h) \in Z_h,
\]

\[
(8.7)
\]
as ensured by the Lax-Milgram Theorem (provided that \( f \) lives in the dual space of \( H(div_{\Omega}; \mathbb{R}^2) \)).

To develop an error bound, for \((\sigma_h, \sigma_h)\), we must compare it with the true solution \((\sigma, \sigma)\). Noting again that \( Z_h \not\subset Z \), from (5.14)-(5.16) the true solution \((\sigma, \sigma), (w, p)\) satisfies

\[
a((\sigma, \sigma), (\xi_h, \xi)) = (f, \nabla_{\Omega} \cdot (\xi_h, \xi)) - B((\xi_h, \xi_h), (w, p)) \text{ for all } (\xi_h, \xi) \in \Sigma_h. \tag{8.8}
\]

Subtracting (8.7) from (8.8)

\[
a((\sigma - \sigma_h, \sigma - \sigma_h), (\xi_h, \xi)) = -B((\xi_h, \xi_h), (w, p)) \text{ for all } (\xi_h, \xi) \in Z_h.
\]

From (8.6) it then follows that for all \((\xi_h, \xi) \in Z_h, v_h \in U_h, q_h \in Q_h\)

\[
a((\sigma - \sigma_h, \sigma - \sigma_h), (\xi_h, \xi)) = -B((\xi_h, \xi_h), (w, p)) + B((\xi_h, \xi_h), (v_h, q_h)). \tag{8.9}
\]

Next, adding and subtracting \((\tau_h, \tau_h) \in Z_h\) in \(a(\cdot, \cdot)\), (8.9) becomes

\[
a((\tau_h - \sigma_h, \tau_h - \sigma_h), (\xi_h, \xi)) = -a((\sigma - \tau_h, \sigma - \tau_h), (\xi_h, \xi)) - B((\xi_h, \xi_h), (w - v_h, p - q_h)).
\]

Choosing \((\xi_h, \xi) = (\tau_h - \sigma_h, \tau_h - \sigma_h) \in Z_h\), and using the continuity and coercivity of \(a(\cdot, \cdot)\) (described in (8.4), (8.3)) and the continuity of \(B((\cdot, \cdot), (\cdot, \cdot))\) (described in (8.5)) we obtain

\[
0 < \gamma \| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|^2_{\Sigma} \leq \alpha \| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|_{\Sigma} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} + \beta \| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|_{\Sigma} \| (w - v_h) U + \| p - q_h \| Q \|.
\]

Dividing through by \(\gamma \| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|_{\Sigma}\) gives

\[
\| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|_{\Sigma} \leq \frac{\alpha}{\gamma} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} + \frac{\beta}{\gamma} (\| w - v_h \| U + \| p - q_h \| Q). \tag{8.10}
\]

Next, applying the triangle inequality, for an arbitrary element \((\tau_h, \tau_h) \in \Sigma_h,\)

\[
\| (\sigma - \sigma_h, \sigma - \sigma_h) \|_{\Sigma} \leq \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} + \| (\tau_h - \sigma_h, \tau_h - \sigma_h) \|_{\Sigma}. \tag{8.11}
\]

Since \((\tau_h, \tau_h) \in \Sigma_h, v_h \in U_h\) and \(q_h \in Q_h\) are arbitrary, combining (8.10) and (8.11) we get

\[
\| (\sigma - \sigma_h, \sigma - \sigma_h) \|_{\Sigma} \leq \left(1 + \frac{\alpha}{\gamma} \right) \inf_{(\tau_h, \tau_h) \in Z_h} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} + \frac{\beta}{\gamma} \left( \inf_{v_h \in U_h} \| w - v_h \| U + \inf_{q_h \in Q_h} \| p - q_h \| Q \right). \tag{8.12}
\]

In order to lift the approximation of \((\sigma - \tau_h, \sigma - \tau_h)\) from the infimum over \(Z_h\) to the infimum over \(\Sigma_h\), we use the inf-sup condition (8.1). A equivalent property to the spaces \(\Sigma_h \times U_h \times W_h\) satisfying (8.1) is the existence of a projection \(\Pi_h : \Sigma \rightarrow \Sigma_h\) satisfying

\[
B(((\tau, \tau) - \Pi_h(\tau, \tau)), (v_h, q_h)) = 0 \text{ for all } (v_h, q_h) \in U_h \times Q_h
\]

and

\[
\| \Pi_h(\tau, \tau) \|_{\Sigma} \leq C_{\Pi} ((\tau, \tau) \|_{\Sigma}.
\]
where $C_{II} > 0$ is a constant that is independent of $h$.

Let $(\xi_h, \xi_h) \in \Sigma_h$, and introduce $(\rho_h, \rho_h) \in \Sigma_h$ satisfying

$$(\rho_h, \rho_h) = \Pi_h \left( \sigma - \xi_h, \sigma - \xi_h \right) \text{ where } \| (\rho_h, \rho_h) \|_{\Sigma} \leq C_{II} \| (\sigma - \xi_h, \sigma - \xi_h) \|_{\Sigma}.$$  

Taking $(\tau_h, \tau_h) = (\xi_h + \rho_h, \xi_h + \rho_h)$

$$B((\tau_h, \tau_h), (w_h, q_h)) = B((\xi_h, \xi_h), (v_h, q_h)) + B((\rho_h, \rho_h), (v_h, q_h))$$

$$= B((\xi_h, \xi_h), (v_h, q_h)) + B((\sigma, \sigma), (v_h, q_h)) - B((\xi_h, \xi_h), (v_h, q_h))$$

$$= B((\sigma, \sigma), (v_h, q_h)) = 0,$$

which implies that $(\tau_h, \tau_h) \in Z_h$.

Next, using $(\tau_h, \tau_h) = (\xi_h + \rho_h, \xi_h + \rho_h)$

$$\| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} \leq \| (\sigma - \xi_h, \sigma - \xi_h) \|_{\Sigma} + \| (\rho_h, \rho_h) \|_{\Sigma} \leq (1 + C_{II}) \| (\sigma - \xi_h, \sigma - \xi_h) \|_{\Sigma} .$$

Finally, taking infima over the appropriate spaces on the left and right sides gives the result

$$\inf_{\left( \tau_h, \tau_h \right) \in Z_h} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} \leq (1 + C_{II}) \inf_{(\xi_h, \xi_h) \in \Sigma_h} \| (\sigma - \xi_h, \sigma - \xi_h) \|_{\Sigma} . \quad (8.13)$$

Combining (8.12) and (8.13) we obtain

$$\| (\sigma - \sigma_h, \sigma - \sigma_h) \|_{\Sigma} \leq C \left( \inf_{\left( \tau_h, \tau_h \right) \in \Sigma_h} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} \right. \nonumber$$

$$\left. + \inf_{v_h \in V_h} \| w - v_h \|_U + \inf_{q_h \in Q_h} \| p - q_h \|_Q \right) . \quad (8.14)$$

With error bounds for the stress space established, the following theorem establishes error bounds for the displacement and skew-symmetry approximations.

**Theorem 8.2.** For $((\sigma, \sigma), (w, p))$ satisfying (5.14)-(5.16) and $((\sigma_h, \sigma_h), (w_h, p_h))$ satisfying (6.5)-(6.7) there exists $C > 0$, independent of $h$, such that

$$\| w - w_h \|_U + \| p - p_h \|_Q$$

$$\leq C \left( \inf_{\left( \tau_h, \tau_h \right) \in \Sigma_h} \| (\sigma - \tau_h, \sigma - \tau_h) \|_{\Sigma} + \inf_{v_h \in V_h} \| w - v_h \|_U + \inf_{q_h \in Q_h} \| p - q_h \|_Q \right) . \quad (8.15)$$

**Proof.** Subtracting equations (6.5) from (5.14) gives

$$B((\xi_h, \xi_h), (w - w_h, p - p_h)) = -a((\sigma - \sigma_h, \sigma - \sigma_h), (\xi_h, \xi_h))$$

for all $(\xi_h, \xi_h) \in \Sigma_h$. 

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For any \( v_h \in U_h \) and \( q_h \in Q_h \), the inf-sup condition (8.1) gives

\[
\beta \left( \| w_h - v_h \|_U + \| p_h - q_h \|_Q \right) \leq \sup_{(\xi_h, \xi_h) \in \Sigma_h} \frac{|B((\xi_h, \xi_h), (w_h - v_h, p_h - q_h))|}{\| (\xi_h, \xi_h) \|_\Sigma} \\
\leq \sup_{(\xi_h, \xi_h) \in \Sigma_h} \left( \frac{|B((\xi_h, \xi_h), (w_h - w, p - p))|}{\| (\xi_h, \xi_h) \|_\Sigma} + \frac{|B((\xi_h, \xi_h), (w - v_h, p - q_h))|}{\| (\xi_h, \xi_h) \|_\Sigma} \right) \\
\leq \max\{\alpha, \beta\} \left( \| (\sigma - \sigma_h, \sigma - \sigma_h) \|_\Sigma + \| w - v_h \|_U + \| p - q_h \|_Q \right)
\]

where in the last step we have used the continuity of \( a(\cdot, \cdot) \) and \( B(\cdot, \cdot) \).

Combining (8.16) with the triangle inequality gives

\[
\| w - w_h \|_U + \| p - p_h \|_Q \leq \| w - v_h \|_U + \| v_h - w_h \|_U + \| p - q_h \|_Q + \| q_h - p_h \|_Q \\
\leq C \left( \| (\sigma - \sigma_h, \sigma - \sigma_h) \|_\Sigma + \| w - v_h \|_U + \| p - q_h \|_Q \right).
\]

As \( v_h \in U_h \) and \( q_h \in Q_h \) are arbitrary, (8.15) follows from (8.17) and (8.14).

Combining Theorems 8.1 and 8.2 we have the following.

**Corollary 8.1.** Let \( ((\sigma, \sigma), w, p) \in \Sigma \times U \times Q \) be the solution of (5.14)-(5.16) and \( ((\sigma_h, \sigma_h), w_h, p_h) \in \Sigma_h \times U_h \times Q_h \) the solution of (6.5)-(6.7), then

\[
\| (\sigma - \sigma_h, \sigma - \sigma_h) \|_\Sigma + \| w - w_h \|_U + \| p - p_h \|_Q \\
\leq C \left( \inf_{(\tau_h, \tau_h) \in \Sigma_h} \| (\sigma - \tau_h, \sigma - \tau_h) \|_\Sigma + \inf_{v_h \in U_h} \| w - v_h \|_U + \inf_{q_h \in Q_h} \| p - q_h \|_Q \right).
\]

Using Corollary 8.1, and additional smoothness assumptions, we can now form an error bound in terms of the mesh parameter \( h \). First observe that for the axisymmetric BDM\(_k\) interpolation operator \( \tilde{\rho}_h : 1H^1(\Omega) \rightarrow BDM_k(\mathcal{T}_h) \) as defined in [18], if \( u \in 1H^{k+1}(\Omega) \), then for some \( C > 0 \),

\[
\| u - \tilde{\rho}_h(u) \|_{L^2(\Omega)} \leq C h^{k+1} \| u \|_{H^{k+1}(\Omega)}.
\]

In addition, if \( \nabla_{axi} \cdot u \in 1H^k(\Omega) \) where \( \left( \Sigma_{T \in \mathcal{T}_h} | \nabla_{axi} \cdot \tilde{\rho}_h(u) |_{H^{k+1}(T)}^2 \right)^{1/2} \leq C_1 \), then for some \( C > 0 \),

\[
\| \nabla_{axi} \cdot u \|_{L^2(\Omega)} \leq C h^k.
\]

Combining the results and assumptions of (8.18) and (8.19), if \( u \in 1H^{k+1}(\Omega) \) and \( \nabla_{axi} \cdot u \in 1H^k(\Omega) \) where \( \left( \Sigma_{T \in \mathcal{T}_h} | \nabla_{axi} \cdot \tilde{\rho}_h(u) |_{H^{k+1}(T)}^2 \right)^{1/2} \leq C_1 \), then there exists \( C > 0 \) such that

\[
\| u - \tilde{\rho}_h u \|_{H(\text{div}, \Omega)} \leq C h^k.
\]

Under analogous assumptions, this result can be extended to the tensor case, where \( \tilde{\rho}_h : 1H^1(\Omega) \rightarrow (BDM_k(\mathcal{T}_h))^2 \) represents the BDM\(_k\) interpolation operator applied to the rows of a tensor so that

\[
\| \sigma - \tilde{\rho}_h \sigma \|_{H(\text{div}, \Omega)} \leq C h^k.
\]

(8.20)
Next we present a result from [8] which bounds the Clément operator \( \Lambda_h^k \). The Clément operator \( \Lambda_h^k \) maps \( 1L^2(\Omega) \) into the space of degree \( k \) Lagrangian finite elements on the mesh \( T_h \). Indeed, as stated in Corollary 2 of Theorem 1 in [8], for \( v \in 1H^{k+1}(\Omega) \), there exists a \( C \) independent of \( h \) such that

\[
\|v - \Lambda_h^k v\|_{1L^2(\Omega)} \leq Ch^{k+1}\|v\|_{1H^{k+1}(\Omega)}.
\]  

(8.21)

As with the BDM interpolation \( \tilde{p}_h \), the bound for \( \Lambda_h^k \) can be extended to vector and tensor functions. The following corollary gives the error bound in terms of the mesh parameter \( h \).

**Corollary 8.2.** Assume that \( \Pi_h \) of Lemma 7.1 or 7.2 satisfies (6.12)-(6.13). If \( (\sigma, \sigma, w, p) \in 1H^k(\Omega) \times 1L^2(\Omega) \times 1H^k(\Omega) \) solves (5.14)-(5.16) and \( (\sigma_h, \sigma_h, w_h, p_h) \in (\text{BDM})^2(\Omega) \times P_k(T_h) \times (P_{k-1}(T_h))^2 \) is the solution of (6.5)-(6.7) for \( k = 1, 2 \), then

\[
\|\sigma - \sigma_h, \sigma - \sigma_h\|_{\Sigma} + \|w - w_h\|_{U} + \|p - p_h\|_{Q} \leq C h^k.
\]

(8.22)

**Proof.** From Corollary 8.1,

\[
\|\sigma - \sigma_h, \sigma - \sigma_h\|_{\Sigma} + \|u - u_h\|_{U} + \|p - p_h\|_{Q}
\]  

\[
\leq C \left( \inf_{(\tau_h, \tau_h) \in \Sigma_h} \|\sigma - \tau_h, \sigma - \tau_h\|_{\Sigma} + \inf_{v_h \in U_h} \|u - v_h\|_{U} + \inf_{q_h \in Q_h} \|p - q_h\|_{Q} \right).
\]

(8.23)

The BDM error bounds from (8.20), (8.21) gives

\[
\inf_{(\tau_h, \tau_h) \in \Sigma_h \times S_h} \|\sigma - \tau_h, \sigma - \tau_h\|_{\Sigma} \leq C h^k.
\]

(8.24)

In addition, using a vector generalization of (8.21)

\[
\inf_{v_h \in U_h} \|u - v_h\|_{U} \leq \|u - \Lambda_h^{k-1} u\|_{1L^2(\Omega)} \leq C h^k \|u\|_{1H^k(\Omega)} ,
\]

(8.25)

and

\[
\inf_{q_h \in Q_h} \|\Sigma^2(p - q_h)\|_{Q} \leq C_1 \|p - \Lambda_h^k p\|_{1L^2(\Omega)} \leq C h^k \|p\|_{1H^k(\Omega)}.
\]

(8.26)

Combining (8.23), (8.24), (8.25) and (8.26) gives the result. □

To conclude this section, we establish an error bound for the true displacement \( u \). At this point, error bounds have been established in terms of the pseudo displacement variable \( w \). Recall from Section 5, however, that \( w = u - x^p \).

**Corollary 8.3.** Let \( (\sigma, \sigma, w, p) \in \Sigma \times U \times Q \) be the solution of (5.14)-(5.16) and \( (\sigma_h, \sigma_h, w_h, p_h) \in \Sigma_h \times U_h \times Q_h \) the solution of (6.5)-(6.7). Furthermore, let \( u = w + x^p \) denote the true displacement, and \( u_h = w_h + x^{p_h} \) denote the discrete approximation to the true displacement. There exists a \( C > 0 \) independent of \( h \), such that

\[
\|u - u_h\|_{U} \leq C \left( \inf_{(\tau_h, \tau_h) \in \Sigma_h} \|\sigma - \tau_h, \sigma - \tau_h\|_{\Sigma} + \inf_{v_h \in U_h} \|w - v_h\|_{U} + \inf_{q_h \in Q_h} \|p - q_h\|_{Q} \right).
\]

(8.27)
Proof. For a bounded domain $\Omega$, observe that
\[
\|(u^h - (p^h))\|_U \leq C_{x^\perp} \|p - p_h\|_Q,
\]
where the constant $C_{x^\perp} > 0$ is independent of $h$. Therefore, using Theorem 8.2 we have that
\[
\|u - u_h\|_U = \|(w - w_h) + x^\perp(p - p_h)\|_U \leq \|w - w_h\|_U + \|x^\perp(p - p_h)\|_U
\]
\[
\leq C\left(\inf_{(T_h, \tau_h) \in \Sigma_h} \|(\mathbf{\sigma} - T_h, \sigma - \tau_h)\|_\Sigma + \inf_{\nu_h \in U_h} \|u - \nu_h\|_U
\right)
\]
\[+ \inf_{q_h \in Q_h} \|p - q_h\|_Q\).
\]

\section{Numerical Experiments}

In this section we present numerical experiments to investigate our theoretical results. With $\Omega = (0, 1) \times (0, 1)$, the displacement solution is taken to be
\[
u(r, z) = \begin{pmatrix} 4r^3(1 - r)z(1 - z) \\ -4r^3(1 - r)z(1 - z) \end{pmatrix},
\]
(9.1)

Using $\mu = \frac{1}{2}$ and $\lambda = 1$, the true symmetric stress tensor is
\[
\mathbf{\sigma} = \begin{pmatrix} 4r^2(-2r^2 + r^2 + 9r^2z - 7rz - r - 7z^2 + 7z) \\ 2r^2(2r^2 - r^2 - 4rz + 2r + 3z^2 - 3z) \end{pmatrix},
\]
(9.2)
\[
\sigma = 4r^2(-2r^2 + r^2 + 6rz - 4rz - r - 5z^2 + 5z)
\]
and the divergence of the stress tensor is
\[
\nabla_{axi} \cdot (\mathbf{\sigma}, \sigma) = \begin{pmatrix} 2r(2r^3 - 24r^2z + 10r^2 + 60rz^2 - 42rz - 9r - 32z^2 + 32z) \\ -2r(8r^3 + r^2(7 - 30z) + 4r(4z^2 + 2z - 3) - 9(z - 1)z) \end{pmatrix},
\]
(9.4)

The solution has been chosen to be consistent with homogenous Dirichlet conditions while having a sufficiently high order polynomial degree to investigate the orders of convergence.

Presented in Table 9.1-9.2 are results of the simulation with the grad-div parameter (see (5.7)) $\gamma = 1$. We note that the convergence rate for the displacement reflects the true displacement, $\|u - u_h\|_U$.

Computations were performed using the approximation elements ($(\text{BDM}_1(T_h))^2 \times P_1(T_h)) \times (P_0(T_h))^2 \times P_0(T_h)$ (shown in Table 9.1), and ($(\text{BDM}_2(T_h))^2 \times P_2(T_h)) \times (P_1(T_h))^2 \times P_1(T_h)$ (shown in Table 9.2).

The computational results are consistent with the theoretically predicted results from Corollaries 8.2 and 8.3.
Table 9.1: Convergence rates for \((\text{BDM}_1(T_h))^2 \times P_1(T_h)) \times (P_0(T_h))^2 \times P_0(T_h)\) finite elements with grad-div stabilization parameter \(\gamma = 1\).

| \(h\) | \(\|[(\sigma, \sigma) - (\sigma_h, \sigma_h)]\|_{\Sigma}\) | Cvg. Rate | \(\|u - u_h\|_U\) | Cvg. Rate | \(\|\delta (\sigma - \sigma_h)\|_Q\) | Cvg. Rate |
|---|---|---|---|---|---|
| \(1/4\) | 1.273E+00 | 1.0 | 2.908E-02 | 1.0 | 1.912E-01 | 1.1 |
| \(1/6\) | 8.444E-01 | 1.0 | 1.911E-02 | 1.1 | 1.200E-01 | 1.1 |
| \(1/8\) | 6.308E-01 | 1.0 | 1.410E-02 | 1.0 | 8.636E-02 | 1.1 |
| \(1/10\) | 5.034E-01 | 1.0 | 1.115E-02 | 1.0 | 6.727E-02 | 1.1 |
| \(1/12\) | 4.189E-01 | – | 9.227E-03 | – | 5.508E-02 | – |
| Pred. | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 | 1.0 |

Table 9.2: Convergence rates for \((\text{BDM}_2(T_h))^2 \times P_2(T_h)) \times (P_1(T_h))^2 \times P_1(T_h)\) finite elements with grad-div stabilization parameter \(\gamma = 1\).

| \(h\) | \(\|[(\sigma, \sigma) - (\sigma_h, \sigma_h)]\|_{\Sigma}\) | Cvg. Rate | \(\|u - u_h\|_U\) | Cvg. Rate | \(\|\delta (\sigma - \sigma_h)\|_Q\) | Cvg. Rate |
|---|---|---|---|---|---|---|
| \(1/4\) | 6.797E-02 | 2.0 | 8.381E-03 | 1.9 | 1.602E-02 | 2.1 |
| \(1/6\) | 3.061E-02 | 2.0 | 3.915E-03 | 1.9 | 6.753E-03 | 2.1 |
| \(1/8\) | 1.730E-02 | 2.0 | 2.238E-03 | 2.0 | 3.647E-03 | 2.1 |
| \(1/10\) | 1.109E-02 | 2.0 | 1.442E-03 | 2.0 | 2.264E-03 | 2.1 |
| \(1/12\) | 7.711E-03 | – | 1.005E-03 | – | 1.536E-03 | – |
| Pred. | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 |

10 Conclusion

We have developed a computational framework for the axisymmetric linear elasticity problem with weak symmetry. Provided the projection bounds (6.12)-(6.13) are satisfied, Lemmas 7.1 and 7.2 establish that the finite element spaces \((\text{BDM}_1(T_h))^2 \times P_1(T_h)) \times (P_0(T_h))^2 \times P_0(T_h)\) and \((\text{BDM}_2(T_h))^2 \times P_2(T_h)) \times (P_1(T_h))^2 \times P_1(T_h)\) are inf-sup stable, resulting in approximations satisfying the error bounds stated in Corollary 8.2. Computational presented in Section 9 support these results.

It is an open question if for \(k \geq 3\), \((\text{BDM}_k(T_h))^2 \times P_k(T_h)) \times (P_{k-1}(T_h))^2 \times P_{k-1}(T_h)\) form an inf-sup stable set of approximation spaces for this problem.

In the Cartesian setting, the spaces \((\text{BDM}_k(T_h))^2 \times (P_{k-1}(T_h))^2 \times P_{k-1}(T_h)\) form an inf-sup stable set of approximation spaces for the linear elasticity problem with weak symmetry [12]. Therefore, it is reasonable to conjecture that \((\text{BDM}_k(T_h))^2 \times P_k(T_h)) \times (P_{k-1}(T_h))^2 \times P_{k-1}(T_h)\) are inf-sup stable for the axisymmetric problem. To test this conjecture, Table 10.3 present convergence results for \((\text{BDM}_3(T_h))^2 \times P_3(T_h)) \times (P_2(T_h))^2 \times P_2(T_h)\) for the Example described in Section 9. For this example the approximations converge with convergence rate \(O(h^k) = O(h^3)\).

Acknowledgement: The authors thankfully acknowledge helpful discussions with Professors
Table 10.3: Convergence rates for \((\text{BDM}_3(\mathcal{T}_h))^2 \times P_3(\mathcal{T}_h)) \times (P_2(\mathcal{T}_h))^2 \times P_2(\mathcal{T}_h)\) finite elements with \(\gamma = 1\).

| \(h\) | \(\|\sigma - (\sigma_h, \sigma_h)\|_2\) Cvg. Rate | \(\|\mathbf{u} - \mathbf{u}_h\|_U\) Cvg. Rate | \(\|\text{as}(\sigma - \sigma_h)\|_Q\) Cvg. Rate |
|-------|--------------------------------|--------------------------------|--------------------------------|
| \(1/4\) | 1.155E-02 | 3.0 | 1.359E-03 | 2.9 | 1.454E-03 | 3.1 |
| \(1/6\) | 3.459E-02 | 3.0 | 4.233E-04 | 2.9 | 4.192E-04 | 3.1 |
| \(1/8\) | 1.465E-02 | 3.0 | 1.816E-04 | 3.0 | 1.733E-04 | 3.1 |
| \(1/10\) | 7.517E-04 | 3.0 | 9.370E-05 | 3.0 | 8.742E-05 | 3.1 |
| \(1/12\) | 4.356E-04 | – | 5.445E-05 | – | 5.002E-05 | – |

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A Proof of Lemma 6.1

Let \(\mathbf{x} = (r, z)\). Then

\[
\nabla_{\text{axi}} \cdot (\tau \wedge \mathbf{x}) = \nabla_{\text{axi}} \cdot \begin{pmatrix} \tau_{11} z - \tau_{21} r \\ \tau_{12} z - \tau_{22} r \end{pmatrix} = \frac{\partial}{\partial r} (\tau_{11} z - \tau_{21} r) + \frac{\partial}{\partial z} (\tau_{12} z - \tau_{22} r) + \frac{1}{r} (\tau_{11} z - \tau_{21} r) = z \frac{\partial \tau_{11}}{\partial r} - \tau_{21} - r \frac{\partial \tau_{21}}{\partial z} + \tau_{12} + z \frac{\partial \tau_{12}}{\partial z} - r \frac{\partial \tau_{22}}{\partial z} + \frac{z}{r} \tau_{11} - \tau_{21} = z \left( \frac{\partial \tau_{11}}{\partial r} + \frac{\partial \tau_{12}}{\partial z} + \frac{1}{r} \tau_{11} \right) - r \left( \frac{\partial \tau_{21}}{\partial r} + \frac{\partial \tau_{22}}{\partial z} + \frac{1}{r} \tau_{21} \right) + \tau_{12} - \tau_{21} = (\nabla_{\text{axi}} \cdot \begin{pmatrix} \tau_{11} \\ \tau_{12} \\ \tau_{21} \\ \tau_{22} \end{pmatrix}) \wedge \begin{pmatrix} r \\ z \end{pmatrix} + \begin{pmatrix} \tau_{11} \\ \tau_{12} \\ \tau_{21} \\ \tau_{22} \end{pmatrix} : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\nabla_{\text{axi}} \cdot \tau) \wedge \mathbf{x} + \tau : \mathbb{P}. \tag{A.1}
\]

Therefore,

\[
\nabla_{\text{axi}} \cdot (\tau \wedge \mathbf{x}) - \frac{z}{r} \tau = \nabla_{\text{axi}} \cdot (\tau, \tau) \wedge \mathbf{x} + \tau : \mathbb{P}.
\]

Next we multiply the left and right hand sides of (A.1) by \(p r\) and integrate over \(T\) to yield

\[
\int_T \nabla_{\text{axi}} \cdot (\tau \wedge \mathbf{x}) \, p r \, dT = \int_T z \tau \, p \, dT = \int_T (\nabla_{\text{axi}} \cdot (\tau, \tau)) \wedge \mathbf{x} \, p \, r \, dT + \int_T \tau : \mathbb{P} \, p \, r \, dT = c((\tau, \tau), p)_T + (as(\tau, \tau), S(p))_T. \tag{A.2}
\]
Note that we have used the relationship $\mathbb{T} \circ p = \alpha(p, \tau) : \mathcal{S}^2(p)$. Next, applying integration by
parts to the first term on the left-hand side of (A.2) gives
\[
\int_T \nabla_{axi} \cdot (\mathbb{T} \wedge x) p r \, dT = \int_T \nabla \cdot (r \mathbb{T} \wedge x) \, p \, dT
\]
\[
= \int_{\partial T} (\mathbb{T} \wedge x) \cdot n \, p r \, ds - \int_T (\mathbb{T} \wedge x) \cdot \nabla p \, r \, dT. \tag{A.3}
\]
Then combining (A.2), and (A.3) yields
\[
c(\mathbb{T}, p) = \int_{\partial T} (\mathbb{T} \wedge x) \cdot n \, p r \, ds - \int_T (\mathbb{T} \wedge x) \cdot \nabla p \, r \, dT - \int_T \tau z p \, dT.
\]
Finally, since
\[
(\mathbb{T} \wedge x) \cdot \nabla p = \mathbb{T} : (x^\perp \otimes \nabla p) \quad \text{and} \quad (\mathbb{T} \wedge x) \cdot n = (\mathbb{T} \cdot n) \cdot x^\perp,
\]
we have
\[
c((\mathbb{T}, \tau), p)_T = \int_{\partial T} (\mathbb{T} \cdot n) \cdot x^\perp \, p r \, ds - \int_T \mathbb{T} : (x^\perp \otimes \nabla p) \, r \, dT - \int_T \tau z p \, dT.
\]

\section*{B \ Computations on the reference triangle $\hat{T}$}

The reference triangle $\hat{T}$ is defined as the triangle with vertices $(0, 0), (1, 0)$ and $(0, 1)$.

![Figure B.1: Reference Triangle](image)

Every triangle $T \in \mathcal{T}_h$ has three coordinates $(r_0, z_0)$, $(r_1, z_1)$ and $(r_2, z_2)$, which we assume are always labeled in a counter-clockwise manner such that $r_0 \leq r_1, r_2$. Further, an affine mapping $F_T$ from the reference triangle $\hat{T}$ (see Figure (B.1)) to the physical domain $T \in \mathcal{T}_h$ exists and takes the form
\[
\begin{pmatrix} r \\ z \end{pmatrix} = \begin{pmatrix} r_1 - r_0 & r_2 - r_0 \\ z_1 - z_0 & z_2 - z_0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} r_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} r_{10} & r_{20} \\ z_{10} & z_{20} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} r_0 \\ z_0 \end{pmatrix}.
\]
Observe that we have used the notational short hand $r_i - r_j = r_{ij}$, and $z_i - z_j = z_{ij}$. Associated with each affine mapping $F_T$ is the determinant of the Jacobian matrix $|J_T| = |r_{10} z_{20} - z_{10} r_{20}|$.

Provided that the triangulation $T_h$ is regular, every affine map $F_T$ can be expressed as

$$
(r) = \begin{pmatrix} r_{10} & r_{20} \\ z_{10} & z_{20} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} r_0 \\ z_0 \end{pmatrix},
$$

where $a_{\text{min}} \leq c_{10}, c_{20}, d_{10}, d_{20} \leq a_{\text{max}}$. Furthermore, the determinant of the Jacobian is $|J_T| = h^2 (c_{10} d_{20} - d_{10} c_{20}) = h^2 J_D$ where $0 < j_{d_{\text{min}}} \leq J_D \leq j_{d_{\text{max}}}$, for $j_{d_{\text{min}}}, j_{d_{\text{max}}} \in \mathbb{R}^+$. For every regular triangulation $T_h$ of an axisymmetric domain $\Omega$ with symmetry axis $\Gamma_0$, each triangle $T \in T_h$ can be categorized as one of three types:

- **Type I:** $\partial T \cap \Gamma_0 = e^*$ where $e^*$ denotes an entire edge,
- **Type II:** $\partial T \cap \Gamma_0 = P_0$ where $P_0$ is a single point,
- **Type III:** $\partial T \cap \Gamma_0 = \emptyset$.

For each type of triangle, we can be more specific about the form of the affine mapping $F_T$. In the following, $\hat{r}$ and $\hat{z}$ represent the mapping of the variables $r$ and $z$ on the physical element $T$ to the reference triangle $\hat{T}$ as functions of $\xi$ and $\eta$.

If $T$ is Type I, then

$$
\hat{r} = h \ c_{10} \ \xi \\
\hat{z} = (z_0 + h \ d_{10} \xi + h \ d_{20} \eta)
$$

and

$$
J_T = h \begin{pmatrix} c_{10} & 0 \\ d_{10} & d_{20} \end{pmatrix} = h \tilde{J}_T.
$$

Since $c_{20} = 0$, it must be the case that $c_{10} > 0$ to ensure that $T$ is well defined.

If $T$ is Type II, then

$$
\hat{r} = h \ (c_{10} \xi + c_{20} \eta) \\
\hat{z} = (z_0 + h \ d_{10} \xi + h \ d_{20} \eta)
$$

and

$$
J_T = h \begin{pmatrix} c_{10} & c_{20} \\ d_{10} & d_{20} \end{pmatrix} = h \tilde{J}_T.
$$

In addition, since only one node lies on the symmetry axis, $c_{10}, c_{20} > 0$.

Finally, if $T$ is Type III, then

$$
\hat{r} = (r_0 + h \ c_{10} \xi + h \ c_{20} \eta) \\
\hat{z} = (z_0 + h \ d_{10} \xi + h \ d_{20} \eta)
$$
and

\[ J_T = h \begin{pmatrix} c_{10} & c_{20} \\ d_{10} & d_{20} \end{pmatrix} = h \tilde{J}_T \]

where \( c_{10}, c_{20} \geq 0 \) and \( c_{10} + c_{20} > 0 \).

In many cases, it is more convenient to work on the reference triangle \( \hat{T} \) than the physical domain \( T \). However, it is important to recall that when mapping vector functions in \( \mathbf{H}(\nabla_{axi}, \mathbb{M}^2) \) between \( T \) and \( \hat{T} \), it is necessary to preserve normal components. Therefore, rather than using a standard affine mapping, we must use the contravariant Piola transformation \([17, 9]\). Let \( J_T \) be the Jacobian matrix associated with the affine mapping \( F_T : \hat{T} \rightarrow T \), then the Piola mapping of the function \( \hat{q} \) (defined on the reference triangle) is

\[ P(\hat{q})(x) := \frac{1}{|J_T|} J_T \hat{q}(\hat{x}), \text{ where } x = F(\hat{x}). \]

The following lemma describes some useful properties of the Piola map as it relates to the integration of \( \mathbf{H}(\nabla_{axi}, \mathbb{M}^2) \) functions.

**Lemma B.1.** Let \( \hat{\tau}, \hat{\sigma} \in \mathbf{H}(\nabla_{axi}, \hat{T}; \mathbb{M}^2) \) and \( \hat{v} \in \mathbf{L}^2(\hat{T}) \), and let \( \tau = P(\hat{\tau}), \sigma = P(\hat{\sigma}) \), and \( v = \hat{v} \circ F^{-1} \)

\[
\begin{align*}
\int_T \tau : \sigma \, r \, dT &= \int_{\hat{T}} J_T \hat{\tau}^T : J_T \hat{\sigma}^T \frac{1}{|J_T|} \hat{r} \, d\hat{T} \\
\int_{\partial T} (\tau \cdot n) \cdot v \, ds &= \int_{\partial \hat{T}} (\hat{\tau} \cdot n) \cdot \hat{v} \, ds 
\end{align*}
\]

Additional details and proofs can be found in \([17, 13]\).

As a result of using polynomials as the discrete finite element approximation spaces, many of the integrals that appear in the finite element formulation have a similar structure. The next lemma introduces an analytical solution for a common class of integrals that appear in the discrete finite element formulation of the axisymmetric linear elasticity problem.

To begin, for convenience of notation, if \( r_0 > 0 \), let

\[ \hat{r} = \frac{r_1 - r_0}{r_0} \xi + \frac{r_2 - r_0}{r_0} \eta + 1 = r_1^* \xi + r_2^* \eta + 1 \]

while if \( r_0 = 0 \), then

\[ \hat{r} = r_1 \xi + r_2 \eta. \]

Since we assume that the coordinates of \( T \) are labeled such that \( r_0 \leq r_1, r_2 \), it follows that \( r_1^*, r_2^* \geq 0 \). Thus, if we are calculating the integral of a function \( f(r, z) \) on \( T \) using the reference element \( \hat{T} \),

\[
\int_T f(r, z) \, r \, dT = \begin{cases} 
\int_{\hat{T}} \hat{f}(\xi, \eta) \left( r_1^* \xi + r_2^* \eta + 1 \right) \, d\hat{T} = r_0 \int_{\hat{T}} \hat{f}(\xi, \eta) \, d\hat{T} = I(\hat{f}(\xi, \eta)) \text{ if } r_0 > 0 \\
\int_{\hat{T}} \hat{f}(\xi, \eta) \left( r_1 \xi + r_2 \eta \right) \, d\hat{T} = I(\hat{f}(\xi, \eta)) \text{ if } r_0 = 0.
\end{cases}
\]

where \( d\hat{T} = |J_T| \, d\xi \, d\eta. \)
Lemma B.2. For integers $s \geq 0$ and $t \geq 0$,

\[
\int_0^1 \int_0^{1-x} \xi^s \eta^t (r_1 \xi + r_2 \eta + 1) \, d\eta \, d\xi = \frac{s! \, t!}{(s + t + 2)!} \left[ \frac{r_1 (s + 1) + r_2 (t + 1)}{(s + t + 3)} + 1 \right]
\]

and

\[
\int_0^1 \int_0^{1-x} \xi^s \eta^t (r_1 \xi + r_2 \eta) \, d\eta \, d\xi = \frac{s! \, t!}{(s + t + 2)!} \left[ \frac{r_1 (s + 1) + r_2 (t + 1)}{(s + t + 3)} \right]
\]

Proof. First, for $\Gamma(\cdot)$ denoting the gamma function, note that

\[
\int_0^1 \xi^s \eta^t \, d\eta = \frac{\Gamma(s + 1)\Gamma(t + 2)}{\Gamma(s + t + 3)} = \frac{s! \, t!}{(s + t + 2)!}.
\]

Therefore

\[
\int_0^1 \int_0^{1-x} \xi^s \eta^t (r_1 \xi + r_2 \eta + 1) \, d\eta \, d\xi
\]

\[
= r_1 \int_0^1 \int_0^{1-x} \xi^{s+1} \eta^t \, d\eta \, d\xi + r_2 \int_0^1 \int_0^{1-x} \xi^s \eta^{t+1} \, d\eta \, d\xi + \int_0^1 \int_0^{1-x} \xi^s \eta^t \, d\eta \, d\xi
\]

\[
= r_1 \frac{(s + 1)! \, t!}{(s + t + 3)!} + r_2 \frac{s! \, (t + 1)!}{(s + t + 3)!} + \frac{s! \, t!}{(s + t + 2)!}
\]

\[
= \frac{s! \, t!}{(s + t + 2)!} \left( r_1 \left( \frac{(s + 1)}{(s + t + 3)} \right) + r_2 \left( \frac{(t + 1)}{(s + t + 3)} + 1 \right) \right).
\]

which verifies (B.2). Removing the +1 from $(r_1 \xi + r_2 \eta + 1)$ yields (B.3). \]

Some useful integrals computed using Lemma B.2 for $r_0 > 0$ are given below

\[
\int_F \eta \, \hat{d}T = \frac{1}{4!} [r_1^* + 2r_2^* + 4] \quad \int_F \xi \, \hat{d}T = \frac{1}{4!} [2r_1^* + r_2^* + 4]
\]

\[
\int_F \eta^2 \, \hat{d}T = \frac{2}{5!} [r_1^* + 3r_2^* + 5] \quad \int_F \eta \xi \, \hat{d}T = \frac{1}{5!} [2r_1^* + 2r_2 + 5]
\]

\[
\int_F \xi^2 \, \hat{d}T = \frac{2}{5!} [3r_1^* + r_2^* + 5] \quad \int_F \eta^3 \, \hat{d}T = \frac{3!}{6!} [r_1^* + 4r_2^* + 6]
\]

\[
\int_F \xi \eta \, \hat{d}T = \frac{2}{6!} [2r_1^* + 3r_2 + 6] \quad \int_F \xi^2 \eta \, \hat{d}T = \frac{2}{6!} [3r_1^* + 2r_2 + 6]
\]

\[
\int_F \xi^3 \, \hat{d}T = \frac{3!}{6!} [4r_1 + r_2 + 6].
\]
C Modeling equations for axisymmetric linear elasticity

In this Appendix, we illustrate how using a change of variable from Cartesian to cylindrical coordinates, the axisymmetric linear elasticity problem can be expressed as the decoupled meridian and azimuthal problems. Recall that cylindrical coordinates form a triple \((r, \theta, z)\) where \(r\) is the radial distance, \(\theta\) is the azimuthal coordinate and \(z\) is the vertical coordinate. In this section, let \(\bar{\Omega}\) denote a three dimensional axisymmetric domain, \(\Omega\) represent an \((r, z)\) cross section of \(\bar{\Omega}\) and \(\Omega_\theta\) denotes the domain of the \(\theta\) angle.

C.1 Cylindrical Coordinate Operators and Function Spaces

First we define the differential forms and inner products that arise in cylindrical coordinates. To begin, the cylindrical coordinate unit vectors are denoted \(e_r, e_\theta\) and \(e_z\). Expressed in terms of Cartesian unit vectors,

\[
\begin{align*}
e_r &= \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}, & e_\theta &= \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix}, & e_z &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}
\]

One can note from these equations that the cylindrical coordinate unit vectors vary in space. Moreover, unless otherwise specified, we assume tensors and vectors are represented in terms of the cylindrical coordinates unit vectors. That is,

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = \phi_1 e_r + \phi_2 e_\theta + \phi_3 e_z
\]

and

\[
\begin{pmatrix}
\phi_{rr} & \phi_{r\theta} & \phi_{r\theta} \\
\phi_{r\theta} & \phi_{\theta\theta} & \phi_{\theta z} \\
\phi_{r\theta} & \phi_{\theta z} & \phi_{zz}
\end{pmatrix} = \phi_{rr} e_{rr} + \phi_{r\theta} e_{r\theta} + \phi_{r\theta} e_{r\theta} + \phi_{\theta\theta} e_{\theta\theta} + \phi_{\theta\theta} e_{\theta\theta} + \phi_{\theta z} e_{\theta z}
\]

\[+ \phi_{zr} e_{zr} + \phi_{z\theta} e_{z\theta} + \phi_{zz} e_{zz},\]

where \(e_{ij} = e_i \otimes e_j\).

As a result of the spatially varying unit vectors, differential operators in cylindrical coordinates have a different algebraic form than in Cartesian coordinates. These operators are not derived here, but details can be found in many sources including [27].

We use two forms of notation for differential operators in cylindrical coordinates: \(\nabla_{cyl}\) and \(\nabla_{axi}\). The first denotes the complete cylindrical coordinate operator, while the second represents the cylindrical coordinate operator applied to an axisymmetric function (recall that \(\frac{\partial u}{\partial \theta} = 0\) if \(u\) is axisymmetric).

The cylindrical coordinate del operator is

\[
\nabla_{cyl} = e_r \frac{\partial}{\partial r} + e_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + e_z \frac{\partial}{\partial z}.
\]
Applied to the scalar function $f$, this gives the gradient operators

$$\nabla_{\text{cyl}} f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_\theta + \frac{\partial f}{\partial z} e_z \quad \text{and} \quad \nabla_{\text{axi}} f = \frac{\partial f}{\partial r} e_r + \frac{\partial f}{\partial z} e_z.$$  

For a vector function $\mathbf{u} = (u_r, u_\theta, u_z)^t$, the gradient tensor is

$$\nabla_{\text{cyl}} \mathbf{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r} \frac{\partial u_r}{\partial \theta} & \frac{1}{r} \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} & \frac{1}{r} \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r} \frac{\partial u_z}{\partial \theta} & \frac{1}{r} \frac{\partial u_z}{\partial z} \end{pmatrix} \quad \text{and} \quad \nabla_{\text{axi}} \mathbf{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} & \frac{-u_\theta}{r} & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{u_r}{r} & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{u_z}{r} & \frac{\partial u_z}{\partial z} \end{pmatrix}. \quad (C.1)$$

For a vector function $\mathbf{u} = (u_r, u_z)^t$, we also define the gradient operator $\nabla$ such that

$$\nabla \mathbf{u} = \begin{pmatrix} \frac{\partial u_r}{\partial r} \\ \frac{\partial u_r}{\partial \theta} \\ \frac{\partial u_z}{\partial r} \\ \frac{\partial u_z}{\partial \theta} \end{pmatrix}.$$  

The divergence operator applied to $\mathbf{u} = (u_r, u_\theta, u_z)^t$ gives

$$\nabla_{\text{cyl}} \cdot \mathbf{u} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \quad \text{and} \quad \nabla_{\text{axi}} \cdot \mathbf{u} = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z}. \quad (C.2)$$

The divergence of an $M^3$ tensor $\mathbf{\sigma}$ is,

$$\nabla_{\text{cyl}} \cdot \mathbf{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{\theta\theta} + \sigma_{rr}) \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{z\theta}}{\partial \theta} + \frac{1}{r} (\sigma_{rr} + \sigma_{z\z}) \end{pmatrix} \quad \text{and} \quad \nabla_{\text{axi}} \cdot \mathbf{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{rr}}{\partial r} + \frac{\partial \sigma_{rz}}{\partial z} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) \\ \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\sigma_{z\theta}}{\partial z} + \frac{1}{r} (\sigma_{\theta\theta} + \sigma_{rr}) \\ \frac{\partial \sigma_{rz}}{\partial r} + \frac{\partial \sigma_{z\z}}{\partial z} + \frac{1}{r} (\sigma_{rr} + \sigma_{z\z}) \end{pmatrix}. $$

C.2 Meridian and Azimuthal Subspaces

Next, we assume all functions are axisymmetric and define the meridian and azimuthal subspaces for tensor and vector functions. In addition, we specify the action of the differential operators introduced in Section C.1 on the meridian and azimuthal subspaces.
Note that \( \alpha \) the meridian and azimuthal subspaces for the displacement space \( \Omega ; M^3 \) are

\[
\alpha H_M(\nabla \text{axi}, \hat{\Omega} ; M^3) = \left\{ \sigma \in \alpha H(\nabla \text{axi}, \hat{\Omega} ; M^3) : \sigma = \begin{pmatrix} \sigma_{rr} & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{pmatrix} \right\},
\]

\[
\alpha H_A(\nabla \text{axi}, \hat{\Omega} ; M^3) = \left\{ \sigma \in \alpha H(\nabla \text{axi}, \hat{\Omega} ; M^3) : \sigma = \begin{pmatrix} 0 & 0 & \sigma_{z\theta} \\ \sigma_{r\theta} & 0 & 0 \\ 0 & \sigma_{z\theta} & 0 \end{pmatrix} \right\}.
\]

and for tensors in \( H_M(\nabla \text{axi}, \hat{\Omega} ; K^3) \),

\[
a_H(\nabla \text{axi}, \hat{\Omega} ; K^3) = \left\{ \sigma \in a_H(\nabla \text{axi}, \hat{\Omega} ; K^3) : \sigma = \begin{pmatrix} 0 & 0 & \sigma_{z\theta} \\ \sigma_{r\theta} & 0 & 0 \\ 0 & \sigma_{z\theta} & 0 \end{pmatrix} \right\}.
\]

For \( \sigma \in H_M(\nabla \text{axi}, \hat{\Omega} ; M^3) \),

\[
\nabla \text{axi} \cdot \sigma = \left( \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} (\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\partial \sigma_{rz}}{\partial z} \right) e_r + \left( \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r} \sigma_{rr} + \frac{\partial \sigma_{zz}}{\partial z} \right) e_z,
\]

(C.3)

and for \( \sigma \in H_A(\nabla \text{axi}, \hat{\Omega} ; M^3) \),

\[
\nabla \text{axi} \cdot \sigma = \left( \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta\theta}}{\partial z} + \frac{1}{r} (\sigma_{r\theta} + \sigma_{\theta r}) \right) e_{\theta}.
\]

The meridian and azimuthal subspaces for the displacement space \( L^2(\hat{\Omega}) \) are

\[
L^2_M(\hat{\Omega}) = \left\{ u : \begin{pmatrix} u_r \\ u_z \end{pmatrix} \in L^2(\hat{\Omega}) \right\} \quad \text{and} \quad L^2_A(\hat{\Omega}) = \left\{ u : \begin{pmatrix} 0 \\ u_{\theta} \end{pmatrix} \in L^2(\hat{\Omega}) \right\}.
\]

For \( u_M \in L^2_M(\hat{\Omega}) \) and \( u_A \in L^2_A(\hat{\Omega}) \), the cylindrical gradient operator (C.1) has the form

\[
\nabla \text{axi} u_M = \begin{pmatrix} \frac{\partial u_r}{\partial r} & 0 & \frac{\partial u_r}{\partial z} \\ 0 & u_r & 0 \\ \frac{\partial u_z}{\partial r} & 0 & \frac{\partial u_z}{\partial z} \end{pmatrix} \quad \text{and} \quad \nabla \text{axi} u_A = \begin{pmatrix} 0 & -\frac{u_{\theta}}{r} & 0 \\ \frac{\partial u_{\theta}}{\partial r} & 0 & \frac{\partial u_{\theta}}{\partial z} \end{pmatrix}.
\]

For \( u_M \in L^2_M(\hat{\Omega}) \) and \( u_A \in L^2_A(\hat{\Omega}) \), the divergence operator (C.2) has the form

\[
\nabla \text{axi} \cdot u_M = \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} \quad \text{and} \quad \nabla \text{axi} \cdot u_A = 0.
\]
Because of axisymmetry, the $\theta$ variable does not appear in the meridian or azimuthal subspaces. Therefore, for functions $p, q \in L^2(\hat{\Omega})$, we define the axisymmetric cylindrical coordinate inner product as

$$(p, q) = \frac{1}{2\pi} \int_\Omega \int_{\theta=0}^{2\pi} p \, q \, r \, d\theta \, dr \, dz = \int_\Omega p \, q \, r \, dr \, dz.$$  

When working with the meridian and azimuthal problems, it is helpful to use the following reduced dimensional representations of the meridian and azimuthal subspaces. To begin, elements $u \in L^2_M(\hat{\Omega}; \mathbb{R}^3)$, can be represented as $\mathbb{R}^2$ vectors

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \rightarrow \begin{pmatrix} u_r \\ u_\theta \end{pmatrix} \in L^2(\Omega; \mathbb{R}^2).$$

Elements of $H^1(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)$ can be represented as an $\mathbb{M}^2$ tensor and a scalar function

$$\begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{zr} & \sigma_{zz} \end{pmatrix} \in L^2(\Omega; \mathbb{M}^2) \quad \text{and} \quad \sigma_{\theta\theta} \in L^2(\Omega),$$

where $\nabla \cdot \cdot \left( \begin{pmatrix} \sigma_{rr} & \sigma_{r\theta} \\ \sigma_{zr} & \sigma_{zz} \end{pmatrix}, \sigma_{\theta\theta} \right) \in L^2(\Omega)$.

To specify that the reduced form notation is being used, elements $u \in L^2_M(\hat{\Omega}; \mathbb{R}^3)$ are denoted $u_M$. Further, the reduced form of $\sigma \in H^1(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)$ is the pair $(\sigma_M, \sigma_{\theta\theta})$ where $\sigma_M$ is a tensor component and $\sigma_{\theta\theta}$ is a scalar component of $\sigma$. Moreover, $\nabla \cdot \cdot (\sigma_M, \sigma_{\theta\theta}) = \nabla \cdot \cdot \sigma$ as defined in (C.3).

Elements of $u \in L^2_A(\hat{\Omega}; \mathbb{R}^3)$ can be identified with scalar functions

$$\begin{pmatrix} 0 \\ u_\theta \\ 0 \end{pmatrix} \rightarrow u_\theta \in L^2(\Omega)$$

and elements of $H^1(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)$, can written as a $\mathbb{M}^2$ tensors

$$\begin{pmatrix} 0 & \sigma_{r\theta} & 0 \\ \sigma_{r\theta} & 0 & \sigma_{rz} \\ 0 & \sigma_{rz} & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{r\theta} & \sigma_{\theta\theta} \\ \sigma_{rz} & \sigma_{zz} \end{pmatrix} \in H^1(\nabla \cdot \cdot, \Omega; \mathbb{M}^2).$$

To indicate the reduced form is being used, for $u \in L^2_A(\hat{\Omega}; \mathbb{R}^3)$, the reduced form will be expressed simply as the scalar function $u_\theta$. Further, the reduced form of $\sigma \in H^1_A(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)$, is denoted $\sigma_A$.

Norms in reduced form are inherited from the norms of the original space. For example, taking $\sigma \in H^1_M(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)$,

$$\|\sigma\|^2_{H^1_M(\nabla \cdot \cdot, \hat{\Omega}; \mathbb{M}^3)} = \|(\sigma_M, \sigma_{\theta\theta})\|^2_{H^1_M(\nabla \cdot \cdot, \Omega)} = ||\nabla \cdot \cdot (\sigma_M, \sigma_{\theta\theta})||^2_{L^2(\Omega)} + ||(\sigma_M, \sigma_{\theta\theta})||^2_{L^2(\Omega)}.$$
In the following, we take
\[
\Sigma = \{ (\mathbf{\sigma}, \sigma) \in 1L^2(\Omega, M^2) \times 1L^2(\Omega) : \nabla_{\text{axi}} \cdot (\mathbf{\sigma}, \sigma) \in 1L^2(\Omega) \},
\]
\[
\Sigma = 1H(\nabla_{\text{axi}}, \Omega, M^2),
\]
\[
U = 1L^2(\Omega), \text{ and } Q = 1L^2(\Omega).
\]

### C.3 Axisymmetric Weak Form

At this point, we are ready to define the weak form of the meridian and azimuthal problems. First we note that the strong form of the axisymmetric linear elasticity problem (3.1) is
\[
A\mathbf{\sigma} - \frac{1}{2}(\nabla_{\text{axi}}u + (\nabla_{\text{axi}}u)^t) = \mathbf{0} \text{ in } \tilde{\Omega}
\]
\[
\nabla_{\text{axi}} \cdot \mathbf{\sigma} = \mathbf{f} \text{ in } \tilde{\Omega}
\]
where we assume the clamped boundary condition, \( u = 0 \) on \( \partial \tilde{\Omega} \).

An axisymmetric solution to (C.4) and (C.5) can be expressed in terms of the orthogonal subspaces \( H_A(\nabla_{\text{axi}} \cdot, \tilde{\Omega}, M^3) \) and \( H_M(\nabla_{\text{axi}} \cdot, \tilde{\Omega}, M^3) \), and \( 1L^2_A(\tilde{\Omega}; \mathbb{R}^3) \) and \( 1L^2_M(\tilde{\Omega}; \mathbb{R}^3) \).

### C.3.1 Meridian problem

The first step to derive the meridian problem is to multiply (C.4) with a test function \( \mathbf{\tau} \in H_M(\nabla_{\text{axi}} \cdot, \tilde{\Omega}, M^3) \) and integrate. For \( \mathbf{\sigma} \in H_M(\nabla_{\text{axi}} \cdot, \tilde{\Omega}, M^3) \), \( A\mathbf{\sigma} \) has the form (recall the operator \( A(3.3) \) for \( m = 3 \))
\[
A\mathbf{\sigma} = \frac{1}{2\mu} \begin{pmatrix} \sigma_{rr} - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\mathbf{\sigma}) & 0 & \sigma_{rz} \\ 0 & \sigma_{\theta\theta} - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\mathbf{\sigma}) & 0 \\ \sigma_{zr} & 0 & \sigma_{zz} - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\mathbf{\sigma}) \end{pmatrix}.
\]

Therefore, for \( \mathbf{\tau} \in H_M(\nabla_{\text{axi}} \cdot, \tilde{\Omega}, M^3) \),
\[
A\mathbf{\sigma} : \mathbf{\tau} = \frac{1}{2\mu} (\sigma_{rr} \tau_{rr} + \sigma_{\theta\theta} \tau_{\theta\theta} + \sigma_{zz} \tau_{zz} + \sigma_{rz} \tau_{rz} + \sigma_{zr} \tau_{zr} - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\mathbf{\sigma}) \text{tr}(\mathbf{\tau})).
\]

Using reduced form notation,
\[
A\mathbf{\sigma} : \mathbf{\tau} = \frac{1}{2\mu} (\sigma_{rr} \tau_{rr} + \sigma_{\theta\theta} \tau_{\theta\theta} + \sigma_{zz} \tau_{zz} + \sigma_{rz} \tau_{rz} + \sigma_{zr} \tau_{zr} - \frac{\lambda}{2\mu + 3\lambda} \text{tr}(\mathbf{\sigma}) \text{tr}(\mathbf{\tau})).
\]
\[
= A\mathbf{\sigma_M} : \mathbf{\tau} + A\sigma_{\theta\theta} \tau_{\theta\theta} - \frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda} (\sigma_{\theta\theta} \text{tr}(\mathbf{\tau_M})) + \text{tr}(\mathbf{\sigma_M}) \tau_{\theta\theta}),
\]

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For \( C.7 \) and \( C.9 \) we define the bilinear form \( b \):

\[
\begin{align*}
\forall u \in \mathbb{C}^2, (\tau, \tau \in \mathbb{C}^2) & \rightarrow \mathbb{R}, \\
\left( A_{\mathbb{C}^2} \sigma_{\mathbb{C}^2}, \tau \right) + (A_{\mathbb{C}^2} \sigma_{\mathbb{C}^2}, \tau) & - \frac{1}{2\mu} \frac{\lambda}{2\mu + 3\lambda} \left[ (\sigma_{\mathbb{C}^2}, \text{tr}(\tau)) + (\text{tr}(\sigma_{\mathbb{C}^2}), \tau) \right].
\end{align*}
\]

Observe that for \( u \in L^2M(\Omega) \) and \( (\tau, \tau) \in \Sigma \),

\[
- \int_\Omega \nabla_{\text{axi}} u : \tau \, d\Omega = - \int_\Omega \left( \frac{\partial u_r}{\partial r} \frac{\partial u_r}{\partial z} + \frac{\partial u_r}{\partial r} \frac{\partial u_z}{\partial z} \right) : (\tau_{rr} \quad \tau_{rz} \quad \tau_{rz} \quad \tau_{zz}) \, r \, d\Omega - \int_\Omega \frac{u_r}{r} \tau_{\theta\theta} \, r \, d\Omega
\]

\[
= - \int_\Omega \nabla u_M : \tau \, d\Omega - \int_\Omega \frac{u_r}{r} \tau_{\theta\theta} \, r \, d\Omega. 
\]

Next, we apply integration by parts to the expression

\[
- \int_\Omega \nabla u_M : \tau \, d\Omega = - \int_\Omega \frac{u_r}{r} \, (\tau_M)_{1,1} \cdot n \, r \, d\Omega + \int_\Omega \frac{u_r}{r} \nabla \cdot (r(\tau_M)_{1,1}) \, d\Omega
\]

\[
= - \int_\Omega \frac{u_r}{r} \, (\tau_M)_{1,1} \cdot n \, r \, d\Omega + \int_\Omega \frac{u_r}{r} \nabla \cdot (r(\tau_M)_{2,2}) \, d\Omega. 
\]

As we are integrating over the domain \( \Omega \), the boundary \( \partial \Omega \) is comprised of two parts. The first corresponds to the boundary of the entire three dimensional domain \( \partial \Omega \) upon which clamped displacement condition \( u_M = 0 \) is enforced. The second part of the boundary \( \Gamma_0 \) corresponds to the symmetry axis along which \( u_r = 0 \), and we assume that \( \tau_M \cdot n = 0 \). Therefore, all of the boundary integrals in \( C.8 \) vanish so that

\[
- \int_\Omega \nabla u_M : \tau \, d\Omega = \int_\Omega \frac{u_r}{r} \nabla \cdot (r(\tau_M)_{1,1}) \, d\Omega + \int_\Omega \frac{u_r}{r} \nabla \cdot (r(\tau_M)_{2,2}) \, d\Omega
\]

\[
= \int_\Omega u_M : \nabla_{\text{axi}} \cdot (\tau_M) \, r \, d\Omega. 
\]

Thus from \( C.7 \) and \( C.9 \) we define the bilinear form \( b_M \):

\[
\forall (\sigma, \sigma \in \mathbb{C}^2, (\tau, \tau \in \mathbb{C}^2) \rightarrow \mathbb{R}, \]

\[
b_M((\tau, \tau), u_M) = (u_M, \nabla_{\text{axi}} \cdot \tau) - (u_r, \frac{\tau_{\theta\theta}}{r}).
\]

For \( (\sigma, \sigma) \in \Sigma \), multiplying the left hand side of \( C.5 \) with a test function \( v \in L^2M(\Omega) \) gives

\[
(\nabla_{\text{axi}} \cdot \sigma) \cdot v = (\partial_r \sigma_{rr} + \frac{1}{r}(\sigma_{rr} - \sigma) + \partial_z \sigma_{rz} + \frac{1}{r}(\sigma_{rz} + \sigma_{zz}) v_r + (\partial_r \sigma_{zz} + \frac{1}{r}(\sigma_{zz} + \partial_z \sigma_{zz}) v_z
\]

\[
= (\nabla_{\text{axi}} \cdot \sigma_M) \cdot v + \frac{1}{r} \sigma_{\theta\theta} v_r.
\]

From integrating this expression we define the bilinear form

\[
b_M((\sigma, \sigma), v_M) = ((\nabla_{\text{axi}} \cdot \sigma_M), v_M) - (v_r, \frac{\sigma_{\theta\theta}}{r}).
\]
Finally, multiplying the right hand side of (C.5) with a test function $v \in L^2_M(\Omega)$ and integrating, defines the linear functional $(f_M, v_M)$.

The meridian problem can now be defined as: Given $f_M \in L^2_M(\Omega)$, find $((\sigma_M, \sigma_{\theta\theta}), u_M) \in \Sigma \times U$ such that for all $((\tau_M, \tau_{\theta\theta}), v_M) \in \Sigma \times U$

$$a_M((\sigma_M, \sigma_{\theta\theta}), (\tau_M, \tau_{\theta\theta})) + b_M((\tau_M, \tau_{\theta\theta}), u_M) = 0$$

$$b_M((\sigma_M, \sigma_{\theta\theta}), v_M) = (f_M, v_M).$$

For the weak symmetry constraint (recall (3.6)), we define the bilinear form $c_M(\cdot, \cdot) : \Sigma \to \mathbb{R}$

$$c_M((\sigma_M, \sigma_{\theta\theta}), p) = (\rho_M, p).$$

The meridian problem with weak symmetry is: Given $f_M \in L^2_M(\Omega)$, find $((\sigma_M, \sigma_{\theta\theta}), u_M, p) \in \Sigma \times U \times Q$ such that for all $((\tau_M, \tau_{\theta\theta}), v_M, q) \in \Sigma \times U \times Q$

$$a_M((\sigma_M, \sigma_{\theta\theta}), (\tau_M, \tau_{\theta\theta})) + b_M((\tau_M, \tau_{\theta\theta}), u_M) + c_M((\tau_M, \tau_{\theta\theta}), p) = 0$$

$$b_M((\sigma_M, \sigma_{\theta\theta}), v_M) = (f, v_M)$$

$$c_M((\sigma_M, \sigma_{\theta\theta}), q) = 0.$$

### C.3.2 Azimuthal Problem

Finally we consider the azimuthal problem. Recall that for $\sigma \in H_A(\nabla_{axi}^\Omega; \tilde{\Omega}; M^3)$, $A\sigma$ has the form

$$A\sigma = \frac{1}{2\mu} \begin{pmatrix} 0 & \sigma_r \theta & 0 \\ \sigma_\theta r & 0 & \sigma_\theta z \\ 0 & \sigma_z \theta & 0 \end{pmatrix}.$$

Thus, for all $\tau \in H_A(\nabla_{axi}^\Omega; \tilde{\Omega}; M^3)$, the first term of (C.4) is

$$A\sigma : \tau = \frac{1}{2\mu}(\sigma_r \theta \tau_r + \sigma_\theta \tau_r r + \sigma_\theta \tau_z z + \sigma_z \theta \tau_z), \quad (C.10)$$

and using reduced form notation, $A\sigma : \tau = \frac{1}{2\mu}A\sigma : A\tau$. Integrating (C.10) defines the bilinear form $a_A(\cdot, \cdot) : \Sigma \times \Sigma \to \mathbb{R}$,

$$a_A(\sigma_A, \tau_A) = \frac{1}{2\mu}(\sigma_A, A\tau_A).$$

For the second term in (C.4), taking $u \in L^2_A(\tilde{\Omega})$ and $\tau \in H_A(\nabla_{axi}^\Omega; \tilde{\Omega}; M^3)$,

$$-\int_{\Omega} \nabla_{axi} u \cdot \tau \, r \, d\Omega = -\int_{\Omega} \begin{pmatrix} 0 & -u_\theta \\ u_\theta r & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & \tau_r \theta & 0 \\ \tau_\theta r & 0 & \tau_\theta z \\ 0 & \tau_z \theta & 0 \end{pmatrix} \begin{pmatrix} r \, d\Omega \\ 0 \\ 0 \end{pmatrix}$$

$$= -\int_{\Omega} \nabla u_\theta \cdot \tau_\theta r \, d\Omega + \int_{\Omega} \tau_r \theta \, u_\theta r \, d\Omega.$$
Integrating the first term by parts

\[- \int_{\Omega} \nabla u_\theta \cdot \frac{\tau_\theta r}{\tau_\theta z} r \, d\Omega = - \int_{\partial \Omega} u_\theta \frac{\tau_\theta r}{\tau_\theta z} \cdot n \, \partial \Omega + \int_{\Omega} u_\theta \nabla \cdot \left( r \frac{\tau_\theta r}{\tau_\theta z} \right) \, d\Omega.\]

Since \( \mathbf{\tau} \cdot n = 0 \) on \( \partial \Omega \), it follows that

\[- \int_{\Omega} \nabla_{\text{axi}} u \mathbf{\tau} r \, d\Omega = \int_{\Omega} u \cdot (\nabla_{\text{axi}} \cdot \mathbf{\tau}) \mathbf{r} \, d\Omega.\]

Using the reduced form notation,

\[\int_{\Omega} u \cdot (\nabla_{\text{axi}} \cdot \mathbf{\tau}) \mathbf{r} \, d\Omega = \int_{\Omega} u_\theta \left( \partial_r \tau_{r\theta} + \partial_z \tau_{z\theta} + \frac{1}{r} (\tau_{r\theta} + \tau_{\theta r}) \right) \mathbf{r} \, d\Omega \]

\[= \int_{\Omega} u_\theta \nabla_{\text{axi}} \cdot \mathbf{\tau}_A \mathbf{r} \, d\Omega.\]

This defines the bilinear form \( b_A(.,.) : Q \times \Sigma \to \mathbb{R} \) as

\[b_A(u_\theta, \mathbf{\tau}_A) = (u_\theta, \nabla_{\text{axi}} \cdot \mathbf{\tau}_A).\]

For \( \mathbf{\sigma} \in H_A(\nabla \cdot, \bar{\Omega}, M^3) \), multiplying the left hand side of (C.5) with a test function \( v \in_1 L^2_A(\bar{\Omega}) \) and integrating over \( \Omega \) gives

\[\int_{\Omega} \left( \nabla_{\text{axi}} \cdot \mathbf{\sigma} \right) \cdot v \, r \, d\Omega = \int_{\Omega} \left( \partial_r \sigma_{r\theta} + \partial_z \sigma_{z\theta} + \frac{1}{r} (\sigma_{r\theta} + \sigma_{\theta r}) \right) v_\theta \, r \, d\Omega \]

\[= b_A(v_\theta, \mathbf{\sigma}_A).\]

Therefore, the weak form of the azimuthal problem can be defined as: Given \( f_\theta \in_1 L^2(\Omega) \), find \( (\mathbf{\sigma}_A, u_A) \in \Sigma \times Q \) such that for all \( (\mathbf{\tau}_A, v) \in \Sigma \times Q \)

\[a_A(\mathbf{\sigma}_A, \mathbf{\tau}_A) + b_A(u, \mathbf{\tau}_A) = 0 \]

\[b_A(v, \mathbf{\sigma}_A) = (f_\theta, v).\]

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