Framed 4-Valent Graphs: Euler Tours, Gauss Circuits and Rotating Circuits

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Abstract

In the present paper we give an explicit formula which allows us immediately to describe a unique Gauss circuit on a framed 4-valent graph (a graph with a structure of opposite edges) from an arbitrary Euler tour on the graph whenever the Gauss circuit exists. This formula only depends on the adjacency matrix of an Euler tour and also tells us whether there exists a Gauss tour on a framed 4-valent graph or not. It turns out that the results are also valid for all symmetric matrices (not just realisable by a chord diagram).

1 Introduction

In the present paper, we consider finite connected framed 4-graphs, i.e. 4-valent graphs with a pairing of the (half)edges emanating from each vertex. The two (half)edges belonging to the same pair are said to be opposite at this vertex. An example of a framed 4-graph is a diagram of a virtual link where classical crossings play the role of vertices, and virtual crossings are just intersection points of images of different edges, and edges forming overpass or underpass are opposite to each other. Graphs obtained from links in such a way are called projections of links.

Each framed 4-graph $G$ has an Euler tour $U$, i.e. a continuous map from the circle $S^1$ onto $U$ and this map is bijective outside the vertices of $G$ and has exactly two inverse images at each vertex of $G$. In each vertex we have two possibilities of running edges: we move from an

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(half)edge to the opposite to it (half)edge; we move from an (half)edge
to a non-opposite to it (half)edge. There are two special sorts of Euler
tours on a given framed 4-graph $G$: only the first possibility occurs in
each vertex of an Euler tour, and only the second possibility occurs in
each vertex of an Euler tour. It is not difficult to see that Euler tours
of the first sort do not always exist and if they do then there exists
only a unique one (the Gauss circuit), but Euler tours of the second
sort (rotating circuits) exist on any connected framed 4-graphs and the
number of them for a 4-graph is larger than one, see [IM1, IM2, Ma1].
If we consider a projection of a knot, i.e. a link with one component,
then it has the Gauss circuit, but a projection of a link with larger
than 1 components does not have a Gauss circuit.

In low-dimensional topology both approaches, the Gauss circuit
approach and the rotating circuit approach, are very widely used.
The Gauss circuit approach is applied in knot theory, namely in the
construction of finite-type invariants, Vassiliev invariants [BN, GPV,
[CDL], and in the planarity problem of immersed curves, [CE1, CE2,
RR]. However, for detecting planarity of a framed 4-graph it is more
convenient to use the rotating circuit approach, see [Ma1, Ma2]. The
criterion of the planarity of an immersed curve, which is the framed
4-graph, is formulated very easy: an immersed curve is planar if and
only if the chord diagram obtained from a rotating circuit is a $d$-
diagram, i.e. the set of all the chords can be split into two sets and
the chords from one set do not intersect each other, see [Ma3]. If we
want to generalise the planarity problem to the problem of finding
the minimum genus of a closed surface which a given curve can be
immersed in the rotating circuit approach is also more useful. There
are criteria giving us the answer to the question what is the minimum
genus for a given curve, see [Ma1].

Since there are many rotating circuits corresponding to the same
Gauss circuit many properties of the Gauss circuit can be read out
of any of these rotating circuits no matter which one is considered.
Consequently, these properties do not depend on the particular choice
of a rotating circuit. For instance, if one of rotating circuits has a $d$-diagram then the other ones do the same. Thus it is necessary to
obtain an easy formula allowing us to get the Gauss circuit from a
rotating circuit and vice versa. Of course having a framed 4-graph
we can answer the question whether there is or not a Gauss circuit
on it and find the Gauss circuit whenever it exists by just traveling
along our 4-graph. But this method does not reflect explicit relations
between topology and combinatorics of Euler tours if we have a 4-
graph with many vertices. In the present paper we give an explicit
formula which depends only on the adjacency matrix of an Euler tour:
taking any Euler tour and constructing its adjacency matrix we can
find the adjacency matrix of the Gauss circuit. It turns out that
the given formula is also valid for all symmetric matrices (not just
realisable by a chord diagram). Investigating this formula we can get
some interesting facts of symmetric matrices.

The present paper is organised as follows.

We first give main definitions concerning 4-graphs, framed 4-graphs,
Euler tours, and preliminary results about them.

In the second section we investigate the question of the existence
of a Gauss circuit in terms of its adjacency matrix, and in the third
section an explicit formula connecting adjacency matrices of distinct
Euler tours is given. Using this formula we can easy get the adjacency
matrix of the Gauss circuit.

In the fourth section we generalise the results of the preceding
sections to the case of all symmetric matrices not only matrices realised
by chord diagrams.

2 Main Definitions and Preliminary Re-
sults

During the whole article by graph we mean a connected finite graph,
possibly, having loops and/or multiple edges.

2.1 4-Valent Graphs and Euler Tours

Let $G$ be a graph with the set of vertices $V(G)$ and the set of edges
$E(G)$. We think of an edge as an equivalence class of the two half-
edges. We say that a vertex $v \in V(G)$ has the degree $k$ if $v$ is incident
to $k$ half-edges. A graph whose vertices have the same degree $k$ is
called $k$-valent or a $k$-graph. The free loop, i.e. the graph without
vertices, is also considered as $k$-valent graph for any $k$.

Let $H$ be a connected 4-graph on the set of vertices $V(H)$ and let $U$
be an Euler tour of $H$, i.e. a tour while traveling along it we run each
degree exactly one time. For every vertex $v \in V(H)$ there are precisely
two closed paths $P_v$ and $Q_v$ on $U$ having no common edges, starting
and ending at $v$ and having no internal vertex equal to $v$. There exists
precisely one Euler tour distinct from $U$ also connecting the paths $P_v$ and $Q_v$ (if we run along $U$ in some direction then in the new Euler tour we run along $P_v$ according to the orientation of $U$, and run along $Q_v$ according to the reverse orientation of $U$). Let us denote by $U \ast v$ the new Euler tour obtained from $U$. The transformation $U \mapsto U \ast v$ has been introduced by Kotzig [Ko] who called it a $k$-transformation. He proved the following statement.

**Proposition 2.1.** Any two Euler tours of a 4-graph are related by a sequence of $k$-transformations.

Let $w = x_1x_2 \ldots x_{k-1}x_k$ be a word, i.e. a sequence of letters from some finite alphabet $X$. The mirror image of $w$ is $\bar{w} = x_kx_{k-1} \ldots x_2x_1$. We will consider the class of words where each word from this class is either a cyclic permutation $w_i = x_ix_{i+1} \ldots x_kx_1 \ldots x_{i-1}$, $1 \leq i \leq k$, of $x_1 \ldots x_k$ or the mirror image of a cyclic permutation $w_i$. We denote this class by $(x_1 \ldots x_k)$ and we call this class a cyclic word.

**Definition 2.1.** A word is called a double occurrence word if each its letter occurs twice in it.

Let $X$ be a finite set. Let $m$ be a double occurrence cyclic word over $X$, i.e. a class of words. Then $m$ has a chord representation, which is constructed by placing successively the letters of $m$ around a circle $S^1$, choosing a point of $S^1$ near each occurrence of a letter and joining by a chord each pair of points corresponding to the two occurrences of the same letter. It is not difficult to see that we get the one-to-one correspondence between the set of double occurrence cyclic words and the set of chord diagrams.

**Example.** Consider $m = (abacdbcd)$. The word $m$ has the chord representation depicted in Fig. [1]

Define the operation $\ast$ on words which will correspond to the $k$-transformation. Let $m = (vAvB)$ where $A$, $B$ are subwords of $m$, and letters belong to some finite alphabet. Then we define $m \ast v = (v\bar{A}vB)$, $\bar{A}$ is the mirror image of $A$. In Fig.[2] the transformation $m \mapsto m \ast v$ is depicted for chord diagrams (dashed arcs of chord diagrams contain the ends of all the chords distinct from $v$). Mostly for each transformation on a chord diagram we assume that only a fixed fragment of the chord diagram is being operated on. The pieces of the chord diagram not containing chords participating in this transformation are depicted by dashed arcs.
Figure 1: A Chord Representation of \((abacdbcd)\)

Figure 2: The Operation \(*\) on Chord Diagrams
Let $U$ be an Euler tour of a connected 4-graph $H$ with the set of vertices $V(H) = \{v_1, \ldots, v_n\}$, which is also considered as an alphabet. When traveling along $U$ we meet each vertex twice. Let us denote by $m(U)$ the (cyclic) word over $V(H)$ that equals (up to cyclic equivalence) the sequence of the vertices that are successively met along $U$. It is obvious that in the obtained word each vertex appears precisely twice then Euler tours are encoded by double occurrence cyclic words. It follows from the definition that $m(U \ast v) = m(U) \ast v$ and if we have a double occurrence cyclic word $m$ or a chord diagram we can construct the 4-graph having an Euler tour $U$ such that $m(U) = m$. We just contract each pair of vertices of the chord diagram labeled by a same letter (a chord) into a single vertex and identify the new vertex with this letter.

### 2.2 Framed 4-Valent Graphs and Euler Tours

In the present subsection we will consider 4-graphs with the additional structure.

**Definition 2.2.** A 4-graph is called **framed** if for every vertex the four emanating half-edges are split into two pairs of (formally) opposite edges. The edges from one pair are called **opposite to each other**.

Let $H$ be a framed 4-graph and $U$ be an Euler tour on $H$. Construct the framed double occurrence cyclic word $m(U)$ (the framed chord diagram) corresponding to $U$. In each vertex $v$ of $H$ we have the following three possibilities of running along $U$ through $v$:

1. we pass from a half-edge to the opposite to it half-edge, see Fig. 3 a). The vertex $v$ is called a **Gaussian vertex for $U$** and the chord corresponding to this vertex is also called a **Gaussian chord**;

2. we pass from a half-edge to a non-opposite to it half-edge, and the orientations of opposite edges are different, see Fig. 3 b). The vertex $v$ is called a **non-Gaussian vertex for $U$ with framing 0** and the chord corresponding to this vertex is also called a **non-Gaussian chord with framing 0**;

3. we pass from a half-edge to a non-opposite to it half-edge, and the opposite edges have the same orientation, see Fig. 3 c). The vertex $v$ is called a **non-Gaussian vertex for $U$ with framing 1**
Figure 3: Passing through the Vertex

and the chord corresponding to this vertex is also called a non-Gaussian chord with framing 1.

**Definition 2.3.** An Euler tour having only Gaussian vertices is called a Gauss circuit. An Euler tour having only non-Gaussian vertices is called a rotating circuit (see [IM1, IM2, Ma1, Ma2]).

When running along an Euler tour $U$ on a 4-graph $H$ we meet each vertex of $H$ twice. Now we are ready to construct the framed double occurrence cyclic word $m(U)$ corresponding to $U$. Words will be constructed over the alphabet $X = V(H) \cup V(H)^{-1} \cup V(H)^G$, where $V(H)^{-1}$ is the set of letters $v^{-1}$ for each $v \in V(H)$ and $V(H)^G$ is the set of letters $v^G$ for each $v \in V(H)$. Vertices (of $H$) of the first type will be denoted in $m(U)$ by the same symbols as in $V(H)$ but with the superscripts $G$, i.e. by letters from $V(H)^G$. For example, $m(U) = (Av^G Bv^G)$ if $v$ is the vertex of the first type. Vertices (of $H$) of the second type will be denoted by the same symbols as in $V(H)$ but with the superscripts $\{\pm 1\}$, and the superscripts are the same for both occurrences of the same letter, i.e. by letters from $V(H) \cup V(H)^{-1}$. For example, $m(U) = (Av Bv)$ (in practice, the superscripts $+1$ are omitted) or $m(U) = (Av^{-1} Bv^{-1})$ if $v$ is the vertex of the second type (we will not make a difference between these words, i.e. we consider an equivalence class). Vertices of the third type will be denoted by the same symbols as in $V(H)$ but with the superscripts $\{\pm 1\}$, and the superscripts are different, i.e. by letters from $V(H) \cup V(H)^{-1}$. For example, $m(U) = (Av Bv^{-1})$ or $m(U) = (Av^{-1} Bv)$ if $v$ is the vertex of the third type (we will not make a difference between these words, i.e. we consider an equivalence class).

Depicting a double occurrence cyclic word by a chord diagram we will use thick chords for vertices with framing 0, dashed chords for
vertices with framing 1, and chords with the label $G$ for Gaussian vertices.

**Example.** Consider the framed word $m = (ab^{-1}acdG^{-1}eb^{-1}c^{-1}e)$. We have: $d$ is a Gaussian letter, $a, b$ are non-Gaussian letters with framing 0 and $c, e$ are non-Gaussian letters with framing 1. The corresponding framed chord diagram is depicted in Fig. 4.

Let $V$ be a finite set. Having a framed double occurrence cyclic word (a framed chord diagram) $m$ over $V \cup V^{-1} \cup V^G$ we can construct the framed 4-graph having an Euler tour $U$ such that the framed word $m(U)$ coincides with $m$. We construct the 4-graph and then define the type of each vertex.

**Remark 2.1.** When we consider framed double occurrence cyclic words over an alphabet it is only important for us the positions of the same letters but not its symbols, see [Tu].

Let us define the framed star operation on the set of framed double occurrence words. We denote this operation by the symbol $\ast$.

**Remark 2.2.** We use the same notations as for double occurrence cyclic words. Further we will consider only framed double occurrence cyclic words and the same notations will not cause confusion.

Firstly, we construct the operation $\overline{w}$, where $m$ is an arbitrary subword (not necessarily a double occurrence word) of a framed double
Figure 5: The Framed Star Operation

occurrence cyclic word. Let \( w = x_1^{\varepsilon_1} \ldots x_k^{\varepsilon_k} \). Then \( \overline{w} = x_k^{\varepsilon_k} \ldots x_1^{\varepsilon_1} \), here \( \overline{x_i^{\varepsilon_i}} = x_i^{\varepsilon_i} \) if \( \varepsilon_i = g \), \( \overline{a_i^{\varepsilon_i}} = a_i^{-\varepsilon_i} \) if \( \varepsilon_i = \pm 1 \). Further, \( m = (a^\varepsilon m_1 a^\varepsilon' m_2) \) is a double occurrence cyclic word. We have \( m * a = (a m_1 m_2) \) if \( \varepsilon = \varepsilon' = g \) (Fig. 5 a)); \( m * a = (a m_1 m_2 a^\varepsilon m_2) \) if \( \varepsilon = \varepsilon' \neq g \) (Fig. 5 a)); \( m * a = (a m_1 m_2 a^{-1}) \) if \( \varepsilon = -\varepsilon' \) (Fig. 5 b)), i.e. applying the framed star to a Gaussian letter it is transformed to the non-Gaussian letter with framing 0, applying to a non-Gaussian letter with framing 0 it is transformed to the Gaussian letter and applying to a non-Gaussian letter with framing 1 it is transformed to the non-Gaussian letter with the same framing 1.

**Statement 2.1** ([IM1, IM2, Ma1]). *Any two framed letters obtained from a framed 4-graph are related to each other by a sequence of the framed star operations.*

The following corollary immediately follows from Statement 2.1

We just apply the framed star operations to Gaussian letters.
Corollary 2.1. Every framed 4-graph has a rotating circuit.

Remark 2.3. It is not difficult to prove Corollary 2.1 by using other methods, but we want to get used to the framed star operation.

It is obvious that there are many rotating circuits and that not every framed 4-graph has a Gauss circuit (if it has a Gauss circuit then this tour is unique). The next section tells us whether there exists a Gauss circuit or not, and how to get it whenever it exists. The following theorem tells us how two rotating circuits are related.

Statement 2.2 ([IM1, IM2, Ma1]). Any two rotating circuits given by framed double occurrence cyclic words are related by a sequence of the following two operations: the framed star operation applying to a non-Gaussian letter with framing 1, and ((m * a) * b) * a, here m is a framed double occurrence cyclic word, a, b are non-Gaussian letters with framing 0 and they alternate in m, i.e. m = (a...b...a...b).

3 Gauss Circuits

In this section we give an explicit formula which allows us immediately to describe a unique Gauss circuit on a framed 4-graph from an arbitrary Euler tour on the graph whenever the Gauss circuit exists.

3.1 The Existence of a Gauss Circuit

We shall need two notions for establishing a criterion of the existence of a Gauss circuit: the adjacency matrix of a framed double occurrence cyclic word (a framed chord diagram) and a surgery along chords.

Definition 3.1. Let D be a framed chord diagram. We call two chords linked if the ends of the first chord belong to the different connected components of S^1 without the second chord’s ends. Using the language of framed double occurrence cyclic words we call two letters a, b alternate if we meet them alternatively (a^α1 ... b^β1 ... a^α2 ... b^β2 ...) when reading the word cyclically.

Remark 3.1. If we draw the chords of a chord diagram inside the circle then linked chords intersect each other.

Definition 3.2. The adjacency matrix of a chord diagram D with enumerated n chords is n × n matrix A(D) = (a_ij) over Z_2 defined by
1. \( a_{ii} \) is the framing of the chord with the number \( i \), i.e. either \( G \) or \( \pm 1 \);
2. \( a_{ij} = 1, \ i \neq j \) if and only if the chords with the numbers \( i \) and \( j \) are linked;
3. \( a_{ij} = 0, \ i \neq j \), if and only if \( i \) and \( j \) unlinked.

**Example.** Let \( D \) be the framed chord diagram depicted in Fig. 4. Enumerate all the chords of \( D \): the chord \( \text{aa} \) has the number 1, the chord \( b \) has the number 2 etc. Then

\[
A(D) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & G & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

Assume we are given a chord diagram \( D \) with all the chords having framing \( \pm 1 \) (with no Gaussian chord).

**Definition 3.3.** Define the *surgery along a set of chords* as follows. For every chord having the framing 0 (resp., 1), we draw a parallel (resp., intersecting) chord near it and remove the arc of the circle between adjacent ends of the chords as in Fig. 6. By a small perturbation, the picture in \( \mathbb{R}^2 \) is transformed into a one-manifold in \( \mathbb{R}^3 \). This manifold \( M(D) \) is the *result of surgery*, see Fig. 7.

Surprisingly, the number of connected components of \( M(D) \) can be determined from the adjacency matrix \( A(D) \) of \( D \).

**Theorem 3.1** ([CL], [Sob], [BG], [Tr]). Let \( D \) be a chord diagram with all the chords having the framing \( \pm 1 \). Then the number of connected components of \( M(D) \) equals \( \text{corank} A(D) + 1 \), where \( A(G) \) is the adjacency matrix of \( D \) over \( \mathbb{Z}_2 \) and \( \text{corank} \) is calculated over \( \mathbb{Z}_2 \).
Remark 3.2. All the matrices are considered over \( \mathbb{Z}_2 \) and we will not indicate it explicitly.

By using Theorem 3.1 we can formulate a criterion of the existence of a Gauss circuit.

Let \( D \) be a framed chord diagram with the adjacency matrix \( A(D) \). Construct the matrix \( \hat{A}(D) \) by deleting the rows and columns of \( A(D) \) corresponding to Gaussian chords.

**Theorem 3.2 ([IM1], [IM2]).** Let \( H \) be a framed 4-graph and \( U \) be an Euler tour of \( H \). Then \( H \) has a Gauss circuit if and only if \( \text{corank}(\hat{A}(D) + E) = 0 \), here \( D \) is a framed chord diagram constructed from \( U \) and \( E \) is the identity matrix.

**Proof.** The proof immediately follows from Fig. 8. In order to get a Gauss circuit, i.e. the tour while traveling along it we pass from \( e_3 \) to \( e_1 \), we have to delete all the Gaussian chords, to replace all the chords having the framing 1 with intersecting chords and all the chords having the framing 0 with parallel chords.

\[ \square \]

3.2 The Gauss Circuit

Let \( H \) be a framed 4-graph having the Gauss circuit and let \( U \) be an Euler tour of \( H \). By using Corollary 2.1 we can assume that \( m(U) \) (the corresponding chord diagram \( D \)) has no Gaussian vertices (\( U \) is a rotating circuit). The main result of the whole paper is the following theorem.

**Theorem 3.3.** The adjacency matrix of the Gauss circuit is equal to \( (A(D) + E)^{-1} \) (over \( \mathbb{Z}_2 \)) up to diagonal elements.

**Proof.** Let \( V(H) = \{v_1, \ldots, v_n\} \). It is not difficult to show that the following two operations applied to \( D \) decrease the number of non-Gaussian chords:
1. the framed star operation applying to a non-Gaussian chord having the framing 0;

2. \(((m \ast a) \ast b) \ast a\), here \(m\) is a framed double occurrence word, \(a, b\) are non-Gaussian letters (chords) having the framing 1 and they alternate in \(m\) (are linked).

We call these operations decreasing operations. The decreasing operations change an Euler tour \(U\) on a 4-graph, and the new Euler tour has the number of non-Gaussian vertices smaller than \(U\) has, see Fig. 5, a).

Let \(D\) be a framed chord diagram and let \(A(D)\) be its adjacency matrix. Let us apply decreasing operations. Without loss of generality we may assume that the decreasing operations are applied to the chords having the first numbers in our numeration. Then the first decreasing operation is

\[
A(D) = \begin{pmatrix}
0 & 0^\top & 1^\top \\
0 & A_0 & A_1 \\
1 & A_1^\top & A_2
\end{pmatrix} \leadsto A(D') = \begin{pmatrix}
G & 0^\top & 1^\top \\
0 & A_0 & A_1 \\
1 & A_1^\top & A_2 + (1)
\end{pmatrix},
\]

Figure 8: The Structure of Framed Chord Diagram
Figure 9: The Decreasing Operation
and the second one is

\[ A(D) = \begin{pmatrix}
1 & 1 & 0^T & 1^T & 0^T & 1^T \\
1 & 1 & 0^T & 0^T & 1^T & 1^T \\
0 & 0 & A_0 & A_1 & A_2 & A_3 \\
1 & 0 & A_1^T & A_4 & A_5 & A_6 \\
0 & 1 & A_2^T & A_5^T & A_7 & A_8 \\
1 & 1 & A_3^T & A_6^T & A_8 & A_9 \\
\end{pmatrix} \]

\[ \sim A(D') = \begin{pmatrix}
G & 1 & 0^T & 0^T & 1^T & 1^T \\
1 & G & 0^T & 1^T & 0^T & 1^T \\
0 & 0 & A_0 & A_1 & A_2 & A_3 \\
0 & 1 & A_1^T & A_4 & A_5 + (1) & A_6 + (1) \\
1 & 0 & A_2^T & A_5^T + (1) & A_7 & A_8 + (1) \\
1 & 1 & A_3^T & A_6^T + (1) & A_8 + (1) & A_9 \\
\end{pmatrix}, \]

where bold characters 0 and 1 indicate column vectors with all entries the same 0 and 1, respectively, and \( A_i \) are matrices.

We will successively apply these operations to \( D \). On the next goal is to show that after applying these two decreasing operations to the framed chord diagram \( D \) having no Gaussian vertices we will get the framed chord diagram with the adjacency matrix \((A(D) + E)^{-1}\) up to diagonal elements.

To get the matrix \((A(D) + E)^{-1}\) we will perform elementary manipulations with rows of \( B(D) = A(D) + E \), \( \det(A(D) + E) = 1 \). Let us construct the matrix \((A(D) + E)[E]\) with the size \( n \times 2n \). We denote by \( \hat{M}_{ij...k} \) be the matrix obtained from \( M \) by deleting \( i, j, \ldots, k \)-th rows and \( i, j, \ldots, k \)-th columns.

The induction base. As \( \det B(D) = 1 \) then either there is a diagonal element equal to 1 or there are two diagonal elements with the numbers \( i \) and \( j \) such that \( b_{ii} = b_{ij} = 0 \), \( b_{ij} = b_{ji} = 1 \).

In the first case, without loss of generality assume that \( b_{11} = 1 \). Then after performing elementary manipulations with \( B(D) \) using the first row we get

\[ B(D) = A(D) + E = \begin{pmatrix}
1 & 0^T & 1^T \\
0 & A_0 + E & A_1 \\
1 & A_1^T & A_2 + E \\
\end{pmatrix} \]

\[ \sim B'(D) = \begin{pmatrix}
1 & 0^T & 1^T \\
0 & A_0 + E & A_1 \\
0 & A_1^T & A_2 + E + (1) \\
\end{pmatrix} \]
and

\[
(B(D)|E) \sim (B'(D)|E')
\]

\[
= \begin{pmatrix}
1 & 0^\top & 1^\top & 0^\top & 0^\top \\
0 & A_0 + E & A_1 & 0 & E \\
0 & A_1^\top & A_2 + E + (1) & 1 & 0 & E
\end{pmatrix}.
\]

After performing the first decreasing operation to $D$ the chord corresponding to $v_1$ becomes a Gaussian chord and the adjacencies of non-Gaussian chords are defined by matrix $\hat{B}'_1(D)$ and the other adjacencies are defined by the first column of $E'$ (up to diagonal elements).

In the second case, we may assume without loss of generality $b_{11} = b_{22} = 0$, $b_{12} = b_{21} = 1$. Then after performing elementary manipulations applied to the first two rows of $B(D)$, we get

\[
B(D) = A(D) + E = \begin{pmatrix}
0 & 1 &^\top & 1^\top & 0^\top \\
1 & 0 & 0^\top & 1^\top & 1^\top \\
0 & 0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & 1 & A_1^\top & A_4 + E & A_5 & A_6 \\
0 & 1 & A_2^\top & A_5^\top & A_7 + E & A_8 \\
1 & 1 & A_3^\top & A_6^\top & A_8^\top & A_9 + E
\end{pmatrix}
\]

\[
\sim B'(D) = \begin{pmatrix}
1 & 0 & 0^\top & 1^\top & 0^\top \\
0 & 1 & 0^\top & 1^\top & 1^\top \\
0 & 0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & 0 & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\
0 & 0 & A_2^\top & A_5^\top + (1) & A_7 + E & A_8 + (1) \\
0 & 0 & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 + E
\end{pmatrix}.
\]

and

\[
(B(D)|E) \sim (B'(D)|E')
\]

\[
= \begin{pmatrix}
1 & 0 & 0^\top & 1^\top & 0^\top \\
0 & 1 & 1^\top & 0^\top & 1^\top \\
0 & 0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & 0 & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\
0 & 0 & A_2^\top & A_5^\top + (1) & A_7 + E & A_8 + (1) \\
0 & 0 & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 + E
\end{pmatrix}.
\]
After performing the second decreasing operation to $D$ the chords corresponding to $v_1$ and $v_2$ become Gaussian chords and the adjacencies of the non-Gaussian chords are defined by matrix $\hat{B}'_{12}(D)$ and the other adjacencies are defined by the first two columns of $E'$.

The induction step. Let us suppose that we have performed $k$ decreasing operations. After these operations the matrix $(B(D)|E)$ is transformed into a matrix

$$
(B'(D)|E') = \begin{pmatrix}
E & C & F & 0 \\
0 & R & S & E
\end{pmatrix}
$$

and $F$ is a $l \times l$ matrix, $S$ is a symmetric matrix. Then the new framed chord diagram contains $l$ Gaussian chords, and the adjacencies of non-Gaussian chords are defined by $R$ and the other adjacencies are defined by first $l$ rows of $E'$. As $\det B'(D) = 1$ then $\det R = 1$, and in the matrix $R$ there is either a diagonal element equal to 1 or there are numbers $p$ and $q$ such that $r_{pp} = r_{qq} = 0$, $r_{pq} = r_{qp} = 1$.

Let us consider the first case. Without loss of generality we may assume that $r_{11} = 1$. In this case we apply the first decreasing operation. We will get

$$
(B'(D)|E') = \begin{pmatrix}
E & C & F & 0 \\
0 & R & S & E
\end{pmatrix}
= \begin{pmatrix}
E & C_1 & C_2 & C_3 & F & 0 & 0 & 0 \\
0 & 1 & 0^\top & 0^\top & S_1 & 0^\top & 0^\top & 0^\top \\
0 & 0 & R_1 & R_2 & S_2 & 0 & E & 0 \\
0 & 1 & R_2^\top & R_3 & S_3 & 0 & 0 & E
\end{pmatrix}
\sim
\begin{pmatrix}
E & C_2^\prime & C_3^\prime & F_1^\prime & F_2^\prime & 0 & 0 \\
0 & 1 & 0^\top & 0^\top & S_1^\prime & 0^\top & 0^\top \\
0 & 0 & R_1 & R_2 & S_2^\prime & 0 & E & 0 \\
0 & 0 & R_2^\top & R_3 + (1) & S_3^\prime & 1 & 0 & E
\end{pmatrix}
= \begin{pmatrix}
E & C' & F' & 0 \\
0 & R' & S' & E
\end{pmatrix} = (B''(D)|E''),
$$
where $F'$ is a $(l + 1) \times (l + 1)$ matrix, $D'$ is a symmetric matrix. The number of Gaussian vertices is $l + 1$, and the adjacencies of non-Gaussian vertices are defined by $R'$ and the other adjacencies are defined by first $l$ rows of $E''$. The second case is consider analogously to the first one.

We end up with the matrix

$$
( E \mid (A(D) + E)^{-1} )
$$

and the framed chord diagram having only Gaussian vertices. The adjacency matrix of this chord diagram is $(A(D) + E)^{-1}$ up to diagonal elements. We have proved Theorem for non-diagonal vertices. But we know that all the diagonal elements is $G$.

**Remark 3.3.** Let $U_1$ and $U_2$ be two rotating circuits, and let $D_1$ and $D_2$ be its framed chord diagram such that $\det(A(D_i) + E) = 1$. Then the matrices $(A(D_1) + E)^{-1}$ and $(A(D_2) + E)^{-1}$ coincide only up to diagonal elements.

**Example.** Consider the framed 4-graph having 4 vertices $v_i$, Fig. 10. Let $U_1$ and $U_2$ be two rotating circuits given by the framed double occurrence cyclic words $m(U_1) = (v_1v_4v_2v_1^{-1}v_2v_3v_4v_3^{-1})$ and $m(U_2) = (v_1v_4v_3v_4v_2v_3v_1v_2^{-1})$, respectively. Then

$$
A(m(U_1)) = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}, \quad A(m(U_2)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
$$

We get

$$
(A(m(U_1)) + E)^{-1} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},
$$

$$
(A(m(U_2)) + E)^{-1} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$

and

$$
A = \begin{pmatrix} G & 0 & 1 & 1 \\ 0 & G & 1 & 1 \\ 1 & 1 & G & 1 \\ 1 & 1 & 1 & G \end{pmatrix}
$$
Figure 10: The Decreasing Operation

is the adjacency matrix of Gauss circuit given by \((v_1v_4v_3v_1v_2v_4v_3v_2)\).

**Definition 3.4.** A chord diagram is called a \(d\)-diagram if its set of chords can be split into two sets such that the chords from one set do not intersect each other and all the chords are non-Gaussian having framing 0.

The next Corollary immediately follows from the criterion of the planarity of an immersed curve and from atom theory, see [Fom, Ma3].

**Corollary 3.1 (V.O.Manturov).** If \(D\) is a chord diagram, all the chords of \(D\) are non-Gaussian having framing 0 and \(\det(A+E) = 1\). Then for any \(n\) numbers \(\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_2\) such that \(\det((A+E)^{-1} + \text{diag}(\lambda_1, \ldots, \lambda_n)) = 1\) the matrix \(((A+E)^{-1} + \text{diag}(\lambda_1, \ldots, \lambda_n))^{-1}\) has 1 on the diagonal. Moreover, if \(D\) is a \(d\)-diagram then the matrix \(((A+E)^{-1} + \text{diag}(\lambda_1, \ldots, \lambda_n))^{-1}\) is the adjacency matrix of a \(d\)-diagram. Here \(\text{diag}(\lambda_1, \ldots, \lambda_n)\) is the diagonal matrix with \(\lambda_1, \ldots, \lambda_n\) on its diagonal.

### 4 Adjacency Matrices

It is well known that there are symmetric matrices over \(\mathbb{Z}_2\) which cannot be realised by chord diagrams, see [Bou], and adjacency matrices which can be realised by different chord diagrams. In Theorem 3.3 we have just used elementary manipulations and adjacency matrices.
It turns out that Theorem 3.3 and Corollary 3.1 can be reformulated for arbitrary symmetric matrices.

In this section all the matrices are over $\mathbb{Z}_2$ with the diagonal elements equal to $\pm 1$. Consider the following operation over the set of symmetric matrices. Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix. Let us fix an arbitrary element $a_{kk} = 1$ and construct the matrix $\text{Loc}(A, k) = (\tilde{a}_{ij})$, here $\tilde{a}_{pq} = a_{pq} + 1$, $p, q \neq k$, if both $a_{pk} = 1$ and $a_{kq} = 1$, and $\tilde{a}_{pq} = a_{pq}$ otherwise. We call the transformation $A \mapsto \text{Loc}(A, k)$ a local complementation of the matrix $A$ at the element $a_{kk}$ (this operation is analogous to the framed star).

It is not difficult to show that the matrices $\text{Loc}(\text{Loc}(\text{Loc}(A, i), j), i)$ and $\text{Loc}(\text{Loc}(\text{Loc}(A, j), i), j)$ obtained from a matrix $A$ coincide up to the diagonal elements with the numbers $i, j$.

**Definition 4.1.** Let $A$ be a symmetric matrix with $a_{ii} = a_{jj} = 0$, $a_{ij} = a_{ji} = 1$. A pivot operation is the transformation $A \mapsto \tilde{A}$, where the diagonal elements of $\tilde{A}$ coincide with the diagonal elements of $A$ and the other elements of $\tilde{A}$ coincide with the corresponding elements of $\text{Loc}(\text{Loc}(\text{Loc}(A, i), j), i, j)$.

Let $\text{Sym}(n, \mathbb{Z}_2)$ be the set of all symmetric $n \times n$ matrices over $\mathbb{Z}_2$. Consider two equivalence relations on $\text{Sym}(n, \mathbb{Z}_2)$. The first relation is: two matrices $A$ and $B$ are said to be equivalent up to diagonal, denote this equivalence relation by $A \sim_D B$, if $A$ and $B$ coincide up to diagonal elements. The second equivalence is defined as follows: two matrices $A$ and $B$ are said to be obtained from each other by changing circuit, denote the second relation by $A \sim_C B$, if $A$ and $B$ are related by a sequence of local complementations and pivot operations.

**Lemma 4.1** ([IM1], [IM2]). If $\det(A + E) = 1$ and $B \sim_C A$ then $\det(B + E) = 1$, here $E$ is the identity matrix.

Let $\text{Sym}_+(n, \mathbb{Z}_2) \subset \text{Sym}(n, \mathbb{Z}_2)$ be the subset of the set of symmetric matrices consisting of matrices $A$ with $\det(A + E) = 1$.

**Corollary 4.1.** The relation $\sim_C$ is also the equivalence relation on $\text{Sym}_+(n, \mathbb{Z}_2)$.

Consider the two set

$\mathcal{L}(n) = \text{Sym}(n, \mathbb{Z}_2)/ \sim_D$ and $\mathcal{G}(n) = \text{Sym}_+(n, \mathbb{Z}_2)/ \sim_C$.
Lemma 4.2. Every element of $\mathcal{L}(n)$ has a representative with the determinant equal to 1.

Proof. Let us prove this lemma by induction on the size of a matrix.

The induction base. For $n = 1$ the claim of the lemma is evident.

The induction step. Assume the statement of the lemma holds for $n-1$ and let $A$ be a $n \times n$ matrix. By the induction hypothesis, we can assume $\det A^{11} = 1$, $A^{ij}$ is the algebraic complement to $a_{ij}$. Then either

$$\det A = a_{11}A^{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = a_{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = 1$$

or

$$\det \tilde{A} = (a_{11} + 1)A^{11} + \sum_{j=2}^{n} a_{1j}A^{1j} = a_{11} + 1 + \sum_{j=2}^{n} a_{1j}A^{1j} = 1,$$

where the matrix $\tilde{A}$ is different from $A$ only by the element $\tilde{a}_{11}$.

Lemma 4.3. Let $B$ and $\tilde{B}$ be two matrices over $\mathbb{Z}_2$ with $\det B = \det \tilde{B} = 1$, and let $B$ and $\tilde{B}$ coincide up to one element on the diagonal. Then the matrices $B^{-1} + E$ and $\tilde{B}^{-1} + E$ are related by a local complementation.

Proof. Without loss of generality we may assume that $B = (b_{ij})$, $\tilde{B} = (\tilde{b}_{ij})$ are $n \times n$ matrices and $b_{nn} = b_{nn} + 1 = 0$, $\tilde{b}_{ij} = b_{ij}$, $i \neq n$ or $j \neq n$. We will perform elementary manipulations with rows of $B$ and $\tilde{B}$ to get two identity matrices. Then we will apply these elementary manipulations to two identity matrix to get the inverse matrices.

Using the equality $\det B = \det \tilde{B} = 1$ we have

$$1 = \det \tilde{B} = \det B + \det \tilde{B}_n^n = 1 + \det \tilde{B}_n^n, \quad \det \tilde{B}_n^n = 0,$$

$$\text{rank } B = n, \quad \text{rank } \tilde{B}_n^n = n - 2,$$

here $\tilde{B}_n^n$ is the matrix obtained from $B$ by deleting the $n$-th row and $n$-th column. As $\tilde{B}_n^n$ is a symmetric matrix, without loss of generality we may assume that $\det C = 1$, here $C$ is the matrix obtained from $\tilde{B}_n^n$ by deleting the $(n-1)$-th row and $(n-1)$-th column.

Performing elementary manipulations with rows of $B$ and $\tilde{B}$ (the first $(n-1)$ rows of $\tilde{B}$ are the same as the ones of $B$), we get

$$B \sim \begin{pmatrix} E & u & v \\ \top^0 & 0 & 1 \\ a^\top & 1 & 0 \\ \top^0 & 1 & 1 \end{pmatrix} \begin{pmatrix} F & 0 & 0 \\ \top^0 & 1 & 0 \\ b^\top & 0 & 1 \end{pmatrix},$$
\[ \tilde{B} \sim \begin{pmatrix} E & u & v & F & 0 & 0 \\ 0 & 0 & 1 & a^T & 1 & 0 \\ 0 & 1 & 0 & b^T & 0 & 1 \end{pmatrix}, \]

here \( F \) is a \((n - 2) \times (n - 2)\)-matrix, \( a, b, u, v \) are \((n - 2)\)-column vectors. Further, performing elementary manipulations with rows, we have

\[ \tilde{B} \sim \begin{pmatrix} E & u & v & F & 0 & 0 \\ 0^T & 0 & 1 & a^T & 1 & 0 \\ 0^T & 1 & 0 & b^T & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} E & u & v & F & 0 & 0 \\ 0^T & 1 & 0 & b^T & 0 & 1 \\ 0^T & 0 & 1 & a^T & 1 & 0 \end{pmatrix}, \]

i.e. \( u = a, v = b \) (the inverse matrix to a symmetric matrix is symmetric),

\[ B \sim \begin{pmatrix} E & u & v & F & 0 & 0 \\ 0 & 0 & 1 & a^T & 1 & 0 \\ 0^T & 1 & 1 & b^T & 0 & 1 \end{pmatrix}, \]

It is not difficult to see that \( F_3 \) is obtained from \( F_2 \) by adding \( a^T \) to the rows of \( F_2 \) corresponding to the rows of \( B^{-1} \) which have the last element equal to 1. Therefore, the matrix \( B^{-1} + E \) is obtained from \( \tilde{B}^{-1} + E \) by the local complementation at the element corresponding to \( \tilde{b}_{nn} \).

**Lemma 4.4.** Let \( B \) and \( \tilde{B} \) be two matrices over \( \mathbb{Z}_2 \) with \( \det B = \det \tilde{B} = 1 \), and let \( B \) and \( \tilde{B} \) coincide up to two elements on the diagonal with numbers \( i \) and \( j \). Suppose that \( \det \tilde{B}_i^j = \det \tilde{B}_j^i = 1 \), here \( \tilde{B}_k^k \) is the matrix obtained from \( B \) by deleting the \( k \)-th row and \( k \)-th
column. Then the matrices $B^{-1} + E$ and $\tilde{B}^{-1} + E$ are related by a pivot operation.

Proof. Without loss of generality, we may assume that $B = (b_{ij})$, $\tilde{B} = (\tilde{b}_{ij})$ are $n \times n$ matrices and $\tilde{b}_{(n-1)(n-1)} = b_{(n-1)(n-1)} + 1$, $\tilde{b}_{nn} = b_{nn} + 1$, $\tilde{b}_{ij} = b_{ij}$ for $(i, j) \neq (n - 1, n - 1), (n, n)$. We will perform elementary manipulations with rows of $B$ and $\tilde{B}$ to get two identity matrices. Then we will apply these manipulations to two identity matrices to get the inverse matrices.

By using the equality $\det B = \det \tilde{B} = 1$, we have

$$1 = \det \tilde{B} = \det B + \det \tilde{B}_{n-1} + \det \tilde{B}_n + \det \tilde{B}^{(n-1)n}_n = 1 + \det \tilde{B}^{(n-1)n}_{(n-1)n},$$

$$\det \tilde{B}^{(n-1)n}_{(n-1)n} = 0, \quad \text{rank}\tilde{B}^{n-1}_n = \text{rank}\tilde{B}^n_n = n - 1, \quad \text{rank}\tilde{B}^{(n-1)n}_n = n - 3,$$

here $\tilde{B}^{(n-1)n}_n$ is the matrix obtained from $\tilde{B}^n_n$ by deleting the $(n - 1)$-th row and $(n - 1)$-th column. As $\tilde{B}^{(n-1)n}_n$ is a symmetric matrix, without loss of generality we may assume that $\det C = 1$, here $C$ is the matrix obtained from $\tilde{B}^{(n-1)n}_n$ by deleting the $(n - 2)$-th row and $(n - 2)$-th column. It is not difficult to show that the matrices obtained from $\tilde{B}$ by deleting the $n$-th row, $n$-th column and the $(n - 1)$-th row, the $(n - 1)$-th column, respectively, are both nondegenerate.

Performing elementary manipulations with rows of $B$ and $\tilde{B}$ (the first $(n - 2)$ rows of $\tilde{B}$ are the same as the ones of $B$), we get

$$B \sim \begin{pmatrix} E & u & v & w \\ 0^T & 0 & 1 & 1 \\ 0^T & 1 & 1 & l \\ 0^T & 1 & l & 0 \end{pmatrix},$$

$$\tilde{B} \sim \begin{pmatrix} E & u & v & w \\ 0^T & 0 & 1 & 1 \\ 0^T & 1 & 0 & l \\ 0^T & 1 & l & 1 \end{pmatrix},$$

here $F$ is a $(n - 3) \times (n - 3)$-matrix, $a, b, c, u, v, w$ are $(n - 3)$-column vectors, and $l \in \{0, 1\}$. Further, performing elementary manipulations with rows, we have

$$B \sim \begin{pmatrix} E & u & v & w \\ 0^T & 0 & 1 & 1 \\ 0^T & 1 & 1 & l \\ 0^T & 1 & l & 0 \end{pmatrix}.$$
\[
\begin{pmatrix}
E & u & v & w \\
0^\top & 1 & 0 & 0 \\
0^\top & 0 & 1 & 0 \\
0^\top & 0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
F \\
l a^\top + l b^\top + (1 + l) c^\top \\
l a^\top + b^\top + c^\top \\
(1 + l) a^\top + b^\top + c^\top \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
l & l & 1 + l \\
l & 1 & 1 \\
1 + l & 1 & 1 \\
\end{pmatrix}
\sim \begin{pmatrix}
E \\
0^\top \\
0^\top \\
0^\top \\
\end{pmatrix}
\begin{pmatrix}
B^{-1} \\
\end{pmatrix},
\]

where

\[
B^{-1} =
\begin{pmatrix}
F_1 \\
l(a + b + c) + c \\
l a + b + c \\
(l + 1) a + b + c \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
l & l & 1 + l \\
l & 1 & 1 \\
1 + l & 1 & 1 \\
\end{pmatrix},
\]

and

\[
\begin{pmatrix}
E & u & v & w \\
0^\top & 0 & 1 & 1 \\
0^\top & 0 & 1 & 0 \\
0^\top & 1 & l & 1 \\
\end{pmatrix}
\begin{pmatrix}
F \\
l(a^\top + b^\top + c^\top) + b^\top \\
(l + 1) a^\top + b^\top + c^\top \\
l a^\top + b^\top + c^\top \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
l & 1 + l & l \\
1 + l & 1 & 1 \\
l & 1 & 1 \\
\end{pmatrix}
\sim \begin{pmatrix}
E \\
0^\top \\
0^\top \\
0^\top \\
\end{pmatrix}
\begin{pmatrix}
\bar{B}^{-1} \\
\end{pmatrix},
\]

here

\[
\bar{B}^{-1} =
\begin{pmatrix}
F_2 \\
l(a^\top + b^\top + c^\top) + b^\top \\
(l + 1) a^\top + b^\top + c^\top \\
l a^\top + b^\top + c^\top \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 \\
l & 1 + l & l \\
1 + l & 1 & 1 \\
l & 1 & 1 \\
\end{pmatrix}.\]

Let us investigate the matrices $F_1$ and $F_2$. We have 4 cases.

a) If a row of the matrix $B^{-1}$ has the last two elements equal to 0 then the corresponding row of $B^{-1}$ has also the last two elements equal to 0. We have two options: either the rows of $F_1$ and $F_2$ are obtained from $F$ by adding $l a^\top + b^\top + c^\top$ and $l(a + b + c)$ to the corresponding row of $F$ or they are equal to the corresponding row of $F$. In both cases, we have the equality of rows of $B^{-1}$ and $\bar{B}^{-1}$ having the last two elements equal to 0.
b) If a row of the matrix $B^{-1}$ has the last two elements equal to $1$ then the corresponding row of $\tilde{B}^{-1}$ has also the last two elements equal to $1$. We have two options: either the rows of $F_1$ and $F_2$ are obtained from $F$ by adding $la^T + b^T + c^T$ or by adding $(1+l)a^T + b^T + c^T$ to the corresponding row of $F$. In both cases, we have the equality of rows of $B^{-1}$ and $\tilde{B}^{-1}$ having the last two elements equal to $1$.

c) If a row of the matrix $B^{-1}$ has the penultimate element equal to $0$ and the last one is $1$ then the corresponding row of $\tilde{B}^{-1}$ has the penultimate element equal to $1$ and the last one is $0$. We have two options: either the rows of $F_1$ and $F_2$ are obtained from $F$ by adding $la^T + (1+l)c^T$ and $l(la^T + b^T + c^T)$ for $F_1$ and $l(a^T + b^T + c^T) + b^T$ and $l((1+l)a^T + b^T + c^T)$ for $F_2$ or by adding $la^T + b^T + (1+l)c^T$, $(1+l)(la^T + b^T + c^T)$ and $(1+l)(la^T + b^T + c^T)$ for $F_1$ and $l(a^T + b^T + c^T) + b^T$, $(1+l)(la^T + b^T + c^T)$ and $la^T + b^T + c^T$ for $F_2$ to the corresponding row of $F$. In both cases, the sum of the rows of $B^{-1}$ and $\tilde{B}^{-1}$ is $la^T + b^T + c^T$.

d) If a row of the matrix $B^{-1}$ has has the penultimate element equal to $1$ and the last one is $0$ then the corresponding row of $\tilde{B}^{-1}$ has the penultimate element equal to $0$ and the last one is $1$. We have two options: either the rows of $F_1$ and $F_2$ are obtained from $F$ by adding $la^T + lb^T + (1+l)c^T$ and $l(la^T + b^T + c^T)$ for $F_1$ and $l(a^T + b^T + c^T) + b^T$ and $(1+l)(la^T + b^T + c^T)$ for $F_2$ or by adding $la^T + lb^T + (1+l)c^T$, $l(la^T + b^T + c^T)$ and $(1+l)a^T + b^T + c^T$ for $F_1$ and $l(a^T + b^T + c^T) + b^T$, $l((1+l)a^T + b^T + c^T)$ and $la^T + b^T + c^T$ for $F_2$ to the corresponding row of $F$. In both cases, the sum of the rows of $B^{-1}$ and $\tilde{B}^{-1}$ is $(1+l)a^T + b^T + c^T$.

Therefore, the matrices $B^{-1} + E$ and $\tilde{B}^{-1} + E$ are related by a pivot operation.

\textbf{Theorem 4.1.} 1. The map $\chi: \mathfrak{S}(n) \to \mathcal{L}(n)$ given by the formula
\[ \chi[A]_C = [(A + E)^{-1}]_D \] is well defined.

2. There exists the inverse map $\chi^{-1}: \mathcal{L}(n) \to \mathfrak{S}(n)$.

\textbf{Proof.} Let $E_{ij}$ be the matrix with $1$ on the whole diagonal and the element in the intersection of the $i$-th row and $j$-th column is $1$, the others are $0$.

1) Let $A \sim_C \tilde{A}$.

If $A$ and $\tilde{A}$ are related by a pivot operation for the first two ele-
ments. Then

\[
B = A + E = \begin{pmatrix}
1 & 1 & 0^\top & 1^\top & 0^\top & 1^\top \\
1 & 1 & 0^\top & 0^\top & 1^\top & 1^\top \\
0 & 0 & A_0 + E & A_1 & A_2 & A_3 \\
1 & 0 & A_1^\top & A_4 + E & A_5 & A_6 \\
0 & 1 & A_2^\top & A_5^\top & A_7 + E & A_8 \\
1 & 1 & A_3^\top & A_6^\top & A_8^\top & A_9 + E \\
\end{pmatrix}
\]

\[
\tilde{B} = \tilde{A} + E = \begin{pmatrix}
1 & 1 & 0^\top & 0^\top & 1^\top & 1^\top \\
1 & 1 & 0^\top & 1^\top & 0^\top & 1^\top \\
0 & 0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & 1 & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\
1 & 0 & A_2^\top & A_5^\top + (1) & A_7 + E & A_8 + (1) \\
1 & 1 & A_3^\top & A_6^\top + (1) & A_8^\top + (1) & A_9 + E \\
\end{pmatrix}
\]

\[
= B E_{1k_1} \cdots E_{1k_p} E_{2(k_p+1)} \cdots E_{2k_q} E_{1(k_q+1)} \cdots E_{1n} \\
\cdot E_{2(k_q+1)} \cdots E_{2n} E_{12} E_{21} E_{12} = BM,
\]

Here \(k_1 > \ldots, k_p\) are the numbers of those columns which have 1 in the first row and 0 in the second row, \(k_p + 1, \ldots, k_q\) are the numbers of those columns which have 0 in the first row and 1 in the second row, and \(k_q + 1, \ldots, n\) are the numbers of those columns which have 1 in the first two rows.

We get \(\tilde{B}^{-1} = M^{-1} B^{-1}\). The last matrix is obtained from \(B^{-1}\) by adding rows to the first and second rows of it. As matrices \(\tilde{B}^{-1}\) and \(B^{-1}\) are symmetric then \(\tilde{B}^{-1}\) might differ from \(B^{-1}\) only by the four elements located in the first two rows and columns. So we have to prove the equality of \(b^{12} = \tilde{b}^{12}\), \(B^{-1} = (\tilde{b}^{12})\), \(\tilde{B}^{-1} = (\tilde{\tilde{b}}^{12})\). We have

\[
b^{12} = \det \begin{pmatrix}
1 & 0^\top & 0^\top & 1^\top & 1^\top \\
0 & A_0 + E & A_1 & A_2 & A_3 \\
1 & A_1^\top & A_4 + E & A_5 & A_6 \\
0 & A_2^\top & A_5^\top & A_7 + E & A_8 \\
1 & A_3^\top & A_6^\top & A_8^\top & A_9 + E \\
\end{pmatrix}
\]

\[
= \det \begin{pmatrix}
1 & 0^\top & 0^\top & 1^\top & 1^\top \\
0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\
0 & A_2^\top & A_5^\top & A_7 + E & A_8 \\
0 & A_3^\top & A_6^\top & A_8^\top + (1) & A_9 + E + (1) \\
\end{pmatrix}
\]
\[
\begin{align*}
\tilde{b}^{12} &= \det \begin{pmatrix}
1 & 0^\top & 1^\top & 0^\top & 1^\top \\
0 & A_0 + E & A_1 & A_2 & A_3 \\
0 & A_1^\top & A_4 + E & A_5 + (1) & A_6 + (1) \\
0 & A_2^\top & A_5^\top & A_7 + E & A_8 \\
0 & A_3^\top & A_6^\top & A_8^\top + (1) & A_9 + E + (1)
\end{pmatrix} = b^{12}.
\end{align*}
\]

We have proven $B^{-1} \sim_D \tilde{B}^{-1}$.

If $A$ and $\tilde{A}$ are related by the local complementation for the first element, then

\[
B = A + E = \begin{pmatrix}
0 & 0^\top & 1^\top \\
0 & A_0 + E & A_1 \\
1 & A_1^\top & A_2 + (1) + E
\end{pmatrix}
\]

\[
\tilde{B} = \begin{pmatrix}
0 & 0^\top & 1^\top \\
0 & A_0 + E & A_1 \\
1 & A_1^\top & A_2 + (1) + E
\end{pmatrix} = (A(G_1) + E)E_{1m}E_{1(m+1)} \ldots E_{1n},
\]

here the numbers $m, m+1, \ldots, n$ correspond to the numbers of columns containing 1 in the first row.

We get $\tilde{B}^{-1} = E_{1n} \ldots E_{1m}B^{-1}$. The matrix $\tilde{B}^{-1}$ is obtained from $B^{-1}$ by adding the rows with numbers from $m$ to $n$ to the first row of it. As matrices $\tilde{B}^{-1}$ and $B^{-1}$ are symmetric then $\tilde{B}^{-1}$ might differ from $B^{-1}$ only by the first diagonal element. So we have proven $B^{-1} \sim_D \tilde{B}^{-1}$.

If $A$ and $\tilde{A}$ are related by pivot operations and local complementations then consequently by applying two preceding cases we get $(A + E)^{-1} \sim_D (\tilde{A} + E)^{-1}$. 

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2) If $B \sim_D \tilde{B}$ and $\det B = \det \tilde{B} = 1$ then, by using Lemmas 4.3 and 4.4, $B^{-1} + E \sim_C \tilde{B}^{-1} + E$. By using Lemma 4.2 we see that there exists some $B$ with $\det B = 1$ in each class $[C]_D$. So we can define the inverse map $\chi^{-1}: \mathfrak{L}(n) \to \mathfrak{G}(n)$ by $\chi^{-1}([C]_D) = [B^{-1} + E]_C$.

**Remark 4.1.** The map $\chi$ gives rise to an isomorphism between the set of looped interlacement graphs modulo the Reidemeister moves, see [TZ], and graph-knots, see [IM1, IM2]. We shall address this question in a sequel of the present paper.

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