Tate’s algorithm and F-theory

Sheldon Katz\textsuperscript{1}, David R. Morrison\textsuperscript{2,3,4}, Sakura Schäfer-Nameki\textsuperscript{5,6}, and James Sully\textsuperscript{3}

\textsuperscript{1} Department of Mathematics  
University of Illinois at Urbana-Champaign  
1409 West Green Street, Urbana, IL 61801, USA  

\textsuperscript{2} Department of Mathematics  
University of California, Santa Barbara, CA 93106, USA  

\textsuperscript{3} Department of Physics  
University of California, Santa Barbara, CA 93106, USA  

\textsuperscript{4} Institute for the Physics and Mathematics of the Universe, University of Tokyo  
5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan  

\textsuperscript{5} Kavli Institute for Theoretical Physics  
University of California, Santa Barbara, CA 93106, USA  

\textsuperscript{6} Department of Mathematics, King’s College  
University of London, The Strand, WC2R 2LS London, England

Abstract

The “Tate forms” for elliptically fibered Calabi-Yau manifolds are reconsidered in order to determine their general validity. We point out that there were some implicit assumptions made in the original derivation of these “Tate forms” from the Tate algorithm. By a careful analysis of the Tate algorithm itself, we deduce that the “Tate forms” (without any further divisibility assumptions) do not hold in some instances and have to be replaced by a new type of ansatz. Furthermore, we give examples in which the existence of a “Tate form” can be globally obstructed, i.e., the change of coordinates does not extend globally to sections of the entire base of the elliptic fibration. These results have implications both for model-building and for the exploration of the landscape of F-theory vacua.
## Contents

1 Introduction .................................................................................. 3

2 Normal forms for Weierstrass equations ........................................... 5

3 Global obstructions to Tate form: An example .................................. 7

4 The $I_m$ case ................................................................................. 8
   4.1 Step 1 ................................................................................. 9
   4.2 Step 2 ................................................................................. 9
   4.3 Step 2 without monodromy ...................................................... 10
   4.4 Step 3 ................................................................................. 10
   4.5 Step 4 ................................................................................. 11
   4.6 Step 4 without monodromy ...................................................... 12
   4.7 Induction ........................................................................... 13
   4.8 Case without monodromy ....................................................... 13
   4.9 Outliers ............................................................................. 15
   4.10 Summary for $I_m$ ................................................................. 16

5 The $II$, $III$, and $IV$ cases ............................................................. 17

6 The $I_m^*$ case .............................................................................. 17
   6.1 Step 1 .............................................................................. 18
   6.2 Step 2 .............................................................................. 18
   6.3 Step 3 .............................................................................. 18
   6.4 Induction ........................................................................... 19
   6.5 Summary of the $I_m^*$ case .................................................... 21

7 The $IV^*$, $III^*$, and $II^*$ cases ...................................................... 21

8 Conclusions .................................................................................. 22

A “Tate forms” from [4] ..................................................................... 22

B Lemmas using unique factorization .................................................. 24

C Global obstructions for $SU(5)$ ......................................................... 27
1 Introduction

F-theory offers an important complement to traditional, perturbative compactifications of string theory. F-theory utilizes the mathematical theory of elliptic fibrations to produce compactifications of type IIB string theory which fully exploit and manifest the SL(2, Z) S-duality of that theory. In particular, F-theory compactifications generically have no weak coupling regime and must be studied non-perturbatively.

Originally proposed as 8D effective theories [1], F-theory models were soon extended to 6D [2, 3, 4] and 4D [5], and extensively studied in the late 1990’s. (See [6, 7] for reviews which include discussions of F-theory.) The past several years have seen a significant revival of the study of F-theory models in 4D, beginning with [8, 9, 10] and reviewed in [11]. Moreover, F-theory has played an important role in the study of the “landscape” of string compactifications, both in four dimensions [12, 13, 14] and in recent work in six dimensions [15, 16, 17, 18].

Since its very beginning, F-theory has been closely tied to the mathematics of elliptic fibrations, a subject which got its start with work of Kodaira [19] nearly fifty years ago. Kodaira’s original classification of one-parameter families of elliptic curves was extended to a number-theoretic setting by Néron [20], and was analyzed in an algorithmic way by Tate [21], whose approach can be applied in both the number theory and algebraic geometry contexts.

The Kodaira classification and the Tate algorithm have become important tools in analyzing F-theory models. One of the early papers to use the Tate algorithm for this purpose [4] derived a collection of ansätze which can be used for constructing F-theory models with a specified gauge group. (It has become common to refer to models in one of the forms given in [4] as being in “Tate form”.)

Recent work to construct supersymmetric grand unified theories (GUTs) in F-theory have relied heavily on the Tate form of the singularity, in particular in the context of Higgs bundles and spectral cover constructions for these models [22, 23]. In this context, we consider an elliptically fibered Calabi-Yau fourfold, with singularity over a divisor in the base. The singularity type determines the gauge group realized on this codimension one locus and the “Tate form” is a very useful starting point for analysing the dynamics of this gauge theory. For instance, in the case of SU(5), the “Tate form” is precisely an unfolded $E_8$ singularity, which from the point of view of the gauge theory has the interpretation that the SU(5) results from Higgsing an $E_8$ gauge theory. It is therefore interesting, also from a physics point of view, to know whether the “Tate form” can be achieved in general (which in the case of SU(5) is indeed possible) and whether it holds up globally (we give some discussion in appendix C).

Unfortunately, in the derivation of the “Tate form” ansätze in [4] the assumptions made were not spelled out very clearly[1]. We re-analyze the validity of these “Tate forms” in this paper.

Our main conclusion is that, with the exception of one new ansatz which must be introduced for certain Sp groups, the “Tate forms” of [4] hold up fairly well, as long as

(1) we avoid certain matter representations (such as the 2-symmetric representation of $SU(m)$)

---

[1] Potential problems were pointed out, for example, in footnote 2 of [24].
which are associated with singularities in components of the discriminant locus of the elliptic fibration \[25\].

(2) we avoid gauge groups \(SU(m)\) with \(6 \leq m \leq 9\), \(Sp(n)\) with \(n = 3, 4\), and \(SO(\ell)\) with \(\ell = 13, 14\).

These restrictions will not seem too confining for those constructing explicit GUT models using F-theory, since no change is needed for the widely-studied GUT groups \(SU(5)\), \(SO(10)\), or \(E_6\). (The “dangerous” matter representations are also avoided in standard GUT scenarios.) For those wishing to explore the entire landscape of F-theory vacua, however, our results are a signal that more work must be done before truly systematic studies can be made. In fact, some of that work has been completed and is being reported on in a companion paper to this one \[26\].

The Tate algorithm proceeds by a sequence of coordinate changes adapted to the geometry of the fibration. We wish to emphasize that the coordinate changes necessary to write the fibration in Tate form may only be defined locally on the base. The main result of the paper is that, by a careful analysis of the algorithm, we find that there are local coordinate changes which bring the fibration into either “Tate form” or a new ansatz which we introduce in this paper (except for classical groups of certain rank, or cases which involve certain exotic matter representations).

Another result of this paper is the construction of a simple explicit example for which we demonstrate that the required coordinate change cannot be defined globally. This aspect of Tate’s algorithm has not been taken into account in the recent constructions of GUT models via F-theory, and it seems likely that the added flexibility provided by treating Tate’s algorithm only locally will allow the construction of new models. We leave such constructions for future work.

The outline of our paper is as follows. In section 2, we fix our notation for the Weierstrass form and introduce the first coordinate change used in Tate’s algorithm. In section 3 we show that the coordinate changes leading to Tate form may fail to be defined globally, requiring different coordinate changes in different local coordinate charts. In sections 4-7 we discuss in turn all singularity types and spell out in detail the steps that are required to achieve Tate form. That is, we go through Tate’s algorithm carefully, paying close attention to issues that arise in codimension two and greater. We determine normal forms for local equations for the classical groups \(SU(m)\), \(Sp(n)\), and \(SO(\ell)\) of sufficiently large rank by using an induction argument for the \(I_m\) and \(J_m^*\) cases. Sections 4.10 and 6.5 give concise summaries of our results for these cases. In section 8, we state our conclusions.

For completeness, we reproduce the main table with the “Tate form” ansätze from \[4\] in Appendix A. We also have Appendix B on unique factorization, and Appendix C where we spell out some sufficient conditions for global obstructions for \(SU(5)\).
2 Normal forms for Weierstrass equations

We consider an elliptically fibered Calabi–Yau manifold $Y$ (of arbitrary dimension) with Weierstrass equation

$$y^2 = x^3 + fx + g,$$

(2.1)

and let $B$ denote the base of the fibration. Each local factor $G$ of the nonabelian part of the gauge group of the corresponding F-theory model is associated with a divisor $S = S_G \subset B$, over which a singularity is located which enhances the gauge group. We assume that each such divisor $S$ is nonsingular, which implies that on any sufficiently small (Zariski) open set of $S$, the coordinate ring is a unique factorization domain [27]. We will use the unique factorization property repeatedly in our analysis.

Our focus is on one chosen divisor $S$, but our analysis can be applied to any gauge-symmetry-enhancing divisor on $B$ (as long as it is nonsingular). Note, however, that at an intersection point between two or more such divisors, the coordinate changes dictated by Tate’s algorithm for each of the divisors may be different.

If we restrict to a sufficiently small Zariski open set $U \subset B$, that is, a sufficiently small set $U$ whose complement $W = B - U$ is defined by polynomial equations, the restriction $S|_U$ will have a local defining equation of the form $\{z = 0\}$, and we can expand the Weierstrass coefficients $f$ and $g$ as power series in $z$

$$f = \sum_i f_i z^i, \quad g = \sum_i g_i z^i.$$  

(2.2)

The coefficients in this expansion are algebraic functions on $U$ but they may have poles on $W = B - U$; different expressions may be needed for different open sets. Moreover, the leading non-zero coefficients, when restricted to $S$, are well-defined, but the higher terms in the sequence may not be well-defined. We will comment on these issues further when they arise in our computation.

The discriminant of the elliptic fibration is

$$\Delta = 4f^3 + 27g^2,$$

(2.3)

and Kodaira’s analysis [19] (the results of which are reproduced in Table 1) determines the general singularity type along $S$ in terms of the orders of vanishing of $f$, $g$, and $\Delta$. To determine the local contribution $G$ to the gauge group, one must also use the part of Tate’s analysis [21] which specifies the monodromy of the exceptional curves along $S$. The various possibilities for each Kodaira type are exhibited in the final column of Table 1. We will discuss the conditions on the equation which determine the monodromy (and local gauge group) when we come to them in the algorithm.

---

2Two compact Lie groups are said to be locally isomorphic when they have isomorphic Lie algebras. The gauge group of these theories is built from various “local” factors (determined by the corresponding Lie algebra) by forming the product group from the factors and then taking a quotient by a finite group, if necessary.
Expanding $\Delta$ in $z$, the leading terms are

$$\Delta = (4f_0^3 + 27g_0^2) + (12f_1f_0^2 + 54g_0g_1)z + O(z^2).$$ (2.4)

As described in Appendix B because $S$ is nonsingular, we can find a function $u_0$ (possibly after shrinking $U$) such that

$$f_0 = -\frac{1}{3}u_0^2 + O(z), \quad g_0 = \frac{2}{27}u_0^3 + O(z).$$ (2.5)

We replace $f_0$ by $-\frac{1}{3}u_0^2$ and $g_0$ by $\frac{2}{27}u_0^3$, modifying the higher coefficients as necessary. Now following Tate, we make a change of coordinates

$$(x, y) \mapsto (x + \frac{1}{3}u_0, y)$$ (2.6)

which transforms the defining equation to

$$y^2 = x^3 + u_0x^2 + (f_1z + f_2z^2 + \cdots)x + (g_1 + \frac{1}{3}u_0f_1)z + (g_2 + \frac{1}{3}u_0f_2)z^2 + (g_3 + \frac{1}{3}u_0f_3)z^3 + \cdots$$ (2.7)

As can be seen in Appendix A, this is “Tate form” for type $I_1$. To simplify later formulas, we set $\tilde{g}_j = g_j + \frac{1}{3}u_0f_j$, and write the equation in the form

$$y^2 = x^3 + u_0x^2 + (f_1z + f_2z^2 + \cdots)x + (\tilde{g}_1z + \tilde{g}_2z^2 + \tilde{g}_3z^3 + \cdots)$$ (2.8)

More precisely, here and in the rest of this paper, by “Tate form” we mean the more compact version given in Table 3, where an entry “$\infty$” means that a coefficient has been set to zero.

| $I_0$ | $I_1$ | $I_2$ | $I_m, m \geq 1$ | $II$ | $III$ | $IV$ | $IV^*$ | $III^*$ | $II^*$ | non-minimal |
|-------|-------|-------|----------------|------|-------|------|-------|--------|-------|-------------|
| $\text{ord}_S(f)$ | $\geq 0$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\text{ord}_S(g)$ | $\geq 0$ | 0 | 1 | 1 | 1 | 1 | 1 | 2 | 2 | 3 |
| $\text{ord}_S(\Delta)$ | 0 | $m$ | $A_{m-1}$ | $A_1$ | 2 | 6 | $m + 6$ | $m + 5$ | 5 | $m + 6$ |
| singularity | none | $A_1$ | $Sp(\left[\frac{m}{2}\right])$ or $SU(m)$ | $SU(2)$ | $SU(2)$ | $SU(3)$ | $SO(2m + 7)$ or $SO(2m + 8)$ | $E_7$ | $E_8$ | non-canonical |
| local gauge group factor | – | – | $SU(2)$ | – | – | – | – | – | – | – |

Table 1: Kodaira’s classification of singular fibers and gauge groups
3 Global obstructions to Tate form: An example

In this section we give an example which shows that the key coordinate change used in the preceding section to bring a Weierstrass equation into a Tate-type normal form may only be possible locally.

For simplicity we consider a two-dimensional base \( \mathbb{P}^1 \times \mathbb{P}^1 \). Line bundles on this variety are denoted by \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b) \), labeled by their bi-degree \((a, b)\). As will we see, the important property for the example is that \( H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)) \neq \{0\} \). We use \([x_0, x_1]\) and \([y_0, y_1]\) as homogeneous coordinates on the two \( \mathbb{P}^1 \) factors.

Consider a fibration determined by Weierstrass coefficients \( f = \frac{1}{3}a(x)b(x)y_0^3y_1^4 \) and \( g = -\frac{2}{27}a(x)b(x)^2y_0^3y_1^9 \), where \( a(x, x_1) \) and \( b(x, x_1) \) are homogeneous polynomials of degree 4. The discriminant is given by

\[
\Delta = 4f^3 + 27g^2 = \frac{4}{27}a(x)^2b(x)^3y_0^6y_1^{12}(a(x)y_0^6 + b(x)y_1^6)
\]

Let \( S = \{z = 0\} \) be the divisor with equation \( z = a(x)y_0^6 + b(x)y_1^6 \).

By the standard Tate procedure outlined in the preceding section we can find functions

\[
\begin{align*}
   u_0 &= -\frac{9}{2} \frac{g}{f} = \frac{b(x)y_0^5}{y_0} \\
   u_1 &= \frac{2}{3} \frac{f^2}{g} = -\frac{a(x)y_0^5}{y_1}.
\end{align*}
\]

defined respectively on the open sets \( U_0 = \{y_0 \neq 0\} \) and \( U_1 = \{y_1 \neq 0\} \) such that on each \( U_i \cap S \) we have \( f = -(1/3)u_i^2 + O(z) \) and \( g = (2/27)u_i^3 + O(z) \). Note that on \((U_0 \cap U_1)|_S\) we have that

\[
\left. u_0|_{U_0 \cap U_1} - u_1|_{U_0 \cap U_1} = \frac{(b(x)y_0^6 + a(x)y_1^6)}{y_0y_1}\right|_{U_0 \cap U_1},
\]

which implies that \( u_0|_{U_0 \cap U_1 \cap S} = u_1|_{U_0 \cap U_1 \cap S} \). We define \( \tilde{u} := u_i|_S \).

We will show there does not exist any global section \( u \) such that \( u|_{U_0} \equiv u_0 \) modulo \( z \) and \( u|_{U_1} \equiv u_1 \) modulo \( z \). Our argument relies on the exact sequence of sheaves

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2) \rightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4) \rightarrow \mathcal{O}_S(4, 4) \rightarrow 0,
\]

where the first map is multiplication by the equation \( z \) of \( S \). Let \( \mathcal{U} = \{U_0, U_1\} \) be the open cover of the base defined by our open sets. We have \( \tilde{u} \in H^0(\mathcal{U}, \mathcal{O}_S(4, 4)) \), where here and in the sequel the notation \( H^i(\mathcal{U}, \ldots) \) emphasizes that we are computing Čech cohomology for the cover \( \mathcal{U} \).

Let us compute the image of \( \tilde{u} \) under the coboundary map

\[
H^0(\mathcal{U}, \mathcal{O}_S(4, 4)) \rightarrow H^1(\mathcal{U}, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)).
\]
To compute the image, we use the function \( u_j \) as a lift of \( \bar{u}|_{U_j \cap S} \) to \( U_j \). The difference \( u_0 - u_1 \) then maps to zero when restricted to \( S \), and so is in the image of multiplication by \( z \). We compute the pre-image using (3.1):

\[
\left. \frac{u_0 - u_1}{z} \right|_{U_0 \cap U_1} = \left. \frac{1}{y_0 y_1} \right|_{U_0 \cap U_1}
\]

so under the coboundary map, \( \bar{u} \) maps to

\[
\left( \frac{1}{y_0 y_1} \right) \in H^1(U, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)).
\]

But it is easy to see that this is a non-trivial class in that cohomology group: in fact, it generates it.

Thus, since \( \bar{u} \) has a nonzero image in \( H^1(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)) \), it cannot lie in the image of the map

\[
H^0(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4)) \to H^0(\mathcal{O}_S(4, 4)).
\]

That is, there is no global section \( u \) of \( \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(4, 4) \) which restricts to \( u_0 \) modulo \( z \) on \( U_0 \) and restricts to \( u_1 \) modulo \( z \) on \( U_1 \). Thus, our coordinate change cannot be made globally.

We have chosen a particularly simple \( f \) and \( g \) for expository purposes; the construction is much more general. In fact, our example is too simplistic: the resulting Calabi-Yau has physically unacceptable singularities over the lines \( y_0 = 0 \) and \( y_1 = 0 \) in the base. However, we can alter the equations of \( f \) and \( g \) by adding generic multiples of \( z \). This will remove the singularities but will not alter the nonvanishing element of \( H^1(U, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, -2)) \) computed above.

Note that this example works equally well at higher order in the vanishing of the discriminant. For example, in the case \( I_2 \), we can use

\[
f \equiv \frac{1}{3} a(x) b(x) y_0^4 y_1^4 + y_1 f_1 z
\]

\[
g \equiv -\frac{2}{27} a(x) b(x)^2 y_0^3 y_1^9 + y_0^3 \left( \frac{1}{3} a(x) f_1 y_0^2 + \frac{1}{27} a(x) b(x) y_1^3 \right) z,
\]

as Weierstrass coefficients, in which case we face the same non-existence of the required global section.

In appendix C we repeat this analysis for the case of \( SU(5) \) singularities, which are particularly interesting for \( SU(5) \) GUT model-building. Rather than present an explicit example, we derive one set of sufficient conditions to construct a globally obstructed \( SU(5) \) fibration. There are surely many other ways to construct analogous examples.

## 4 The \( I_m \) case

We return to a general base manifold, and go through Tate’s algorithm step by step. At each step, Tate’s algorithm specifies a coordinate change to be made, and in the original formulation these coordinate changes can involve rational functions on the base (with denominators
allowed). Our goal here is to find a version of the algorithm in which the coordinate changes can be made without using functions with poles on the base.

We first treat the case of $I_m$. According to Kodaira, to be in the $I_m$ branch of the classification we must have $z \not| u_0$. As noted in section 2 when $m = 1$ the normal form (2.8) coincides with “Tate form” for $I_1$.

4.1 Step 1

For an equation of the form (2.8), the leading terms of the discriminant can be written

$$
\Delta = 4u_0^3 \tilde{g}_1 z + O(z^2). 
$$

Assuming that $m \geq 2$ so that $z^2 | \Delta$, we have

$$
\tilde{g}_1 = 0 + O(z).
$$

Thus, we may absorb $\tilde{g}_1$ into $\tilde{g}_2$ by adjusting the coefficients, and can assume that $\tilde{g}_1$ has been set to 0. In this case, the defining equation becomes

$$
y^2 = x^3 + u_0 x^2 + (f_1 z + f_2 z^2 + \cdots)x + (\tilde{g}_2 z^2 + \tilde{g}_3 z^3 + \cdots). 
$$

This is “Tate form” for type $I_2$.

4.2 Step 2

At the next order the discriminant is

$$
\Delta = u_0^2 (4u_0 \tilde{g}_2 - f_1^2) z^2 + O(z^3). 
$$

As explained in Appendix 3, if the leading term vanishes (i.e., $m \geq 3$) then we can find functions $s_0$ and $\mu$ such that $\mu | s$ is square-free, and a function $t_1$, such that

$$
u_0 = \frac{1}{4} \mu s_0^2 + O(z), \quad f_1 = \frac{1}{2} \mu s_0 t_1 + O(z). 
$$

The monodromy condition for $I_m$ is tested by asking whether $u_0 | s$ has a square root or not: this determines whether the local gauge group factor is $SU(m)$ or $Sp(\lfloor \frac{m}{2} \rfloor)$. With our notation, this amounts to asking whether $\mu | s$ has any zeros or not. We will assume that $\mu$ has been chosen so that $\mu \equiv 1$ if $\mu | s$ has no zeros, and this will be our criterion for monodromy. That is, in the $I_m$ case the local factor $G$ of the gauge group will be $SU(m)$ if $\mu \equiv 1$, and will be $Sp(\lfloor \frac{m}{2} \rfloor)$ if $\mu | s$ has zeros.

Now we can solve for $\tilde{g}_2$ as

$$
\tilde{g}_2 = \frac{1}{4} \mu t_1^2 + O(z). 
$$

We replace $u_0$, $f_1$, and $\tilde{g}_2$ by $\frac{1}{4} \mu s_0^2$, $\frac{1}{2} \mu s_0 t_1$, and $\frac{1}{4} \mu t_1^2$ respectively, and adjust the other coefficients accordingly.
In order to put this into Tate form, we would like to make the substitution $x \mapsto x - t_1 z / s_0$. However, we cannot do this near zeros of $s_0$.

Thus, the “Tate form” for $I_3$ which was described in [4] cannot be achieved. We introduce a new ansatz for $I_3$ (and will eventually extend this to all $I_{2n+1}$ cases):

$$y^2 = x^3 + \frac{1}{4} \mu s_0^2 x^2 + (\frac{1}{2} \mu s_0 t_1 z + f_2 z^2 + f_3 z^3 + \cdots) x + (\frac{1}{4} \mu t_1^2 z^2 + \tilde{g}_3 z^3 + \tilde{g}_4 z^4 + \cdots)$$  \hspace{1cm} (4.7)

Unlike the “Tate forms” presented in [4], this ansatz cannot be described purely in terms of the vanishing of certain coefficients in an expansion, but involves a particular relationship among leading terms in the expansion of the coefficients of $x^2$, $x^1$, and $x^0$ in the equation.

### 4.3 Step 2 Without Monodromy

In the case of $I_3$ with no monodromy (i.e., the case of $SU(3)$), we have $\mu \equiv 1$, and there is a change of coordinates which puts this into Tate form:

$$(x, y) \mapsto (x, y + \frac{1}{2} s_0 x + \frac{1}{2} t_1 z),$$  \hspace{1cm} (4.8)

which yields the equation

$$y^2 + s_0 xy + t_1 z y = x^3 + (f_2 z^2 + f_3 z^3 + \cdots) x + (\tilde{g}_3 z^3 + \tilde{g}_4 z^4 + \cdots).$$  \hspace{1cm} (4.9)

This is “Tate form” for $SU(3)$ (i.e., $I_3$ with no monodromy).

### 4.4 Step 3

The discriminant at the next order is

$$\Delta = \frac{1}{16} \mu^3 s_0^3 \left( s_0^3 \tilde{g}_3 - s_0^2 t_1 f_2 - t_1^3 \right) z^3 + O(z^4).$$  \hspace{1cm} (4.10)

We now assume in addition that $z^4 | \Delta$, which can be achieved by

$$t_1 = -\frac{1}{3} s_0 u_1 + O(z), \quad \tilde{g}_3 = -\frac{1}{3} u_1 f_2 - \frac{1}{27} u_1^3 + O(z),$$  \hspace{1cm} (4.11)

using Lemma 1 in appendix B. We replace $t_1$ and $\tilde{g}_3$ by $-\frac{1}{3} s_0 u_1$ and $-\frac{1}{3} u_1 f_2 - \frac{1}{27} u_1^3$ respectively in (4.7), and adjust the other coefficients accordingly. Our equation becomes

$$y^2 = x^3 + \frac{1}{4} \mu s_0^2 x^2 + (\frac{1}{6} \mu s_0^2 u_1 z + f_2 z^2 + f_3 z^3 + \cdots) x + \frac{1}{36} \mu s_0^2 u_1^2 z^2 + (\frac{1}{3} u_1 f_2 - \frac{1}{27} u_1^3) z^3 + \tilde{g}_4 z^4 + \cdots.$$

(4.12)

This can be simplified with the change of coordinates $(x, y) \mapsto (x + \frac{1}{3} u_1 z, y)$ which yields:

$$y^2 = x^3 + (u_0 + u_1 z + \cdots) x^2 + (f_2 + \frac{1}{3} u_1^2 z^2 + f_3 z^3 + f_4 z^4 + \cdots) x + (\tilde{g}_4 z^4 + \tilde{g}_5 z^5 + \cdots).$$  \hspace{1cm} (4.13)
where \( u_0 = \frac{1}{4}\mu s_0^2 \) as above, and \( \hat{g}_j = \tilde{g}_j + \frac{1}{3}u_1 f_{j-1} \). Let \( \hat{f}_2 = f_2 + \frac{1}{3}u_1^2 \). This is “Tate form” for \( I_4 \).

In the case without monodromy we again have \( \mu = 1 \) and there is a coordinate change

\[
(x, y) \mapsto \left( x, y + \frac{1}{2}s_0 x \right)
\]

which puts the equation into the “Tate form” for \( SU(4) \)

\[
y^2 + s_0 xy = x^3 + (u_1 z + \cdots )x^2 + \left( \hat{f}_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots \right) x + (\hat{g}_4 z^4 + \hat{g}_5 z^5 + \cdots )
\]

In summary for \( I_4 \), the “Tate form” can be achieved with and without monodromy, yielding either \( Sp(2) \) or \( SU(4) \) gauge groups, respectively.

4.5 Step 4

Finally, the discriminant at order \( z^4 \) is

\[
\Delta = \frac{1}{16} \mu^2 s_0^4 \left( \mu s_0^2 \hat{g}_4 - \hat{f}_2^2 \right) z^4 + O \left( z^5 \right)
\]

If \( z^5 | \Delta \), as in step 2, this can be solved as follows. Since \( \mu | s \) is square-free, we must have \( (\mu s_0 ) | s \) dividing \( \hat{f}_2^2 | s \). That is, there exists a function \( t_2 \) (possibly after shrinking \( U \)) such that

\[
\hat{f}_2 = \frac{1}{2} \mu s_0 t_2 + O(z)
\]

Then to satisfy the discriminant condition, we simply need

\[
\hat{g}_4 = \frac{1}{4} \mu t_2^2 + O(z)
\]

We replace \( \hat{f}_2 \) and \( \hat{g}_4 \) by \( \frac{1}{2} \mu s_0 t_2 \) and \( \frac{1}{4} \mu t_2^2 \) respectively, and adjust the other coefficients accordingly. The equation becomes

\[
y^2 = x^3 + \left( \frac{1}{4} \mu s_0^2 + u_1 z \right)x^2 + \left( \frac{1}{2} \mu s_0 t_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots \right) x + \left( \frac{1}{4} \mu t_2^2 z^4 + \hat{g}_5 z^5 + \cdots \right)
\]

This is our new ansatz for \( I_5 \), and it again involves a particular relationship among leading terms in the expansion of the coefficients of \( x^2 \), \( x^1 \), and \( x^0 \) in the equation. (The “Tate form” for \( I_5 \) without monodromy would require the substitution \( x \mapsto x - t_2 z^2 / s_0 \), which is not possible near zeros of \( s_0 \).)
4.6 Step 4 without monodromy

However, for $I_5$ with no monodromy (i.e., $SU(5)$), we can again achieve Tate form by a different change of variables. Since there is no monodromy, $\mu \equiv 1$ and we can make the change of variables

$$(x, y) \mapsto (x, y + \frac{1}{2}s_0x + \frac{1}{2}t_2z^2),$$

which yields the equation

$$y^2 + s_0xy + t_2z^2y = x^3 + u_1zx^2 + (f_3z^3 + f_4z^4 + \ldots)x + (g_5z^5 + \ldots)$$

This is “Tate form” for $SU(5)$.

4.7 Induction

In subsection 4.4, we showed how to obtain the Tate form up to $I_4$ (making no further assumptions about monodromy). To extend this to higher $I_m$ singularities we will now make an inductive argument, starting with a Tate form for an $I_{2n}$ singularity with $n \geq 2$, with monodromy. It will be useful to write this in terms of

$$y^2 = x^3 + ux^2 + vx + w,$$

or expanded in terms of $z$

$$y^2 = x^3 + (u_0 + u_1z + u_2z^2 + \ldots + u_{n-1}z^{n-1})x^2 + (v_nz^n + v_{n+1}z^{n+1} + \ldots)x + (w_{2n}z^{2n} + w_{2n+1}z^{2n+1} + \ldots),$$

assuming that the expansion of $u$ contains no terms divisible by $z^n$, and also assuming

$$z^n \mid v \text{ and } z^{2n} \mid w.$$ 

Since we are assuming the Kodaira type is $I_m$ for some $m \geq 2n$, we should have $z \nmid u$. As in the earlier analysis, we assume that $u_0$ takes the form $\mu s_0^2$ with $\mu \mid s$ square-free (and $\mu \equiv 1$ when there is no monodromy).

To relate this to the Weierstrass form used earlier (2.1) we complete the cube to

$$y^2 = \left(x + \frac{1}{3}u\right)^3 + \left(-\frac{1}{3}u^2 + v\right)\left(x + \frac{1}{3}u\right) + \left(\frac{2}{27}u^3 - \frac{1}{3}uv + w\right),$$

identifying

$$f = -\frac{1}{3}u^2 + v, \quad g = \frac{2}{27}u^3 - \frac{1}{3}uv + w.$$

\[3\text{From these formulas we see that if } z \text{ had divided } u, \text{ we would not be in Kodaira type } I_m.\]
It follows that the discriminant is
\[
\Delta = 4 \left( -\frac{1}{3} u^2 + v \right)^3 + 27 \left( \frac{2}{27} u^3 - \frac{1}{3} uv + w \right)^2
\]
\[= 4u^3w - u^2v^2 - 18uvw + 4v^3 + 27w^2. \quad (4.27)\]

The known order of vanishing of each of these terms is $2n$, $2n$, $3n$, $3n$, $4n$. Since we are assuming $n \geq 2$
\[
\Delta = u^2(4uw - v^2) + O(z^{2n+2}). \quad (4.29)
\]

The type of condition we are now going to use is a condition which holds in codimension two on the base. We already have a divisor $z = 0$ which is codimension one, and we have been considering quantities like $u_i$, $v_i$ or $w_i$ which might have zeros at a subvariety of $z = 0$, i.e., in codimension two. For any subvariety $\Sigma \subset \{ z = 0 \}$, we can ask about the multiplicity along $\Sigma$ of $f$, $g$, and $\Delta$ and apply Kodaira’s classification to determine the generic singularity type along $\Sigma$. If those multiplicities satisfy
\[
\text{mult}_\Sigma(f) \geq 4, \quad \text{mult}_\Sigma(g) \geq 6, \quad \text{mult}_\Sigma(\Delta) \geq 12
\]
(i.e., we are in the “non-minimal” part of Kodaira’s classification in codimension two), then we can blow up $\Sigma$ and still have a Calabi–Yau total space of the fibration; this implies that the low-energy spectrum has peculiar things such as light tensors and, massless strings. Thus, we will exclude such elliptic fibrations from consideration.

To use this condition here, we assume that $m$, the actual order of vanishing of the discriminant, is at least 10. (Our current value of $n$ is related to this by $m \geq 2n$.) In that case, if the multiplicity of $f$ exceeds 2 and/or the multiplicity of $g$ exceeds 3, then we are not in either of the $I_m$ or the $I_{m-6}$ cases, so we must be in one of the exceptional or non-minimal cases. However, the multiplicity of $\Delta$ necessarily increases along some codimension two subvariety $\Sigma \subset \Delta$, so $\Delta$ has multiplicity strictly greater than 10 along $\Sigma$. Therefore the model would be non-minimal in codimension two, by Kodaira’s classification in Table 1. Since we are assuming that this doesn’t happen, the multiplicity of $f$ is at most 2 and the multiplicity of $g$ is at most 3. This implies that the multiplicity of $u$ is at most 1, and that is the condition we will actually use.

So, assuming our equation is in the form of eq. (4.23), we expand $\Delta$ as follows:
\[
\Delta = u_0^2(4u_0w_{2n} - v_n^2)z^{2n} + \left( u_0^2(4u_0w_{2n+1} + 4u_1w_{2n} - 2v_nv_{n+1}) + 2u_0u_1(4u_0w_{2n} - v_n^2) \right)z^{2n+1} + O(z^{2n+2}). \quad (4.30)
\]

We first assume that $z^{2n+1}$ divides $\Delta$. Then $(4u_0w_{2n} - v_n^2)|_{z=0}$ must be identically zero. Writing $u_0 = \frac{1}{7}\mu s_0^2$ with $\mu|_S$ square-free, we see that $\mu s_0$ must divide $v_n$ modulo $z$. That is, there exists a function $t_n$ such that $v_n = \frac{1}{7}\mu s_0 t_n + O(z)$ and it then follows that $w_{2n} = \frac{1}{7}\mu t_n^2 + O(z)$.
We replace $v_n$ and $w_{2n}$ by $\frac{1}{2}u_0 t_n$ and $\frac{1}{3}u_1 t_n^2$, respectively, and adjust the other coefficients accordingly resulting in
\[
\begin{align*}
y^2 &= x^3 + x^2 \left( \frac{1}{4}u_0 s_0^2 + u_1 z + u_2 z^2 + \cdots + u_{n-1} z^{n-1} \right) \\
&\quad + x \left( \frac{1}{2}u_0 t_n z^n + v_{n+1} z^{n+1} + \cdots \right) + \left( \frac{1}{4}u_1 t_n^2 z^{2n} + w_{2n+1} z^{2n+1} + \cdots \right). \tag{4.33}
\end{align*}
\]
We have achieved our new ansatz for $I_{2n+1}$.

As we have seen before, however, with this form of the equation and no further divisibility assumptions, we are unable to make a change of coordinates which would put the equation into “Tate form” (with monodromy). If there is no monodromy, the situation is different and we will return to that case later.

Now, as a second inductive step, we assume also that $z^{2n+2}$ divides $\Delta$. This time,
\[
(4u_0 w_{2n+1} + 4u_1 w_{2n} - 2v_n v_{n+1})|S
\]
must be identically zero (since we already know that $(4u_0 w_{2n} - v_n^2)|S$ is identically zero). We can rewrite that as
\[
\mu \left( s_0^2 (w_{2n+1}|S) + t_n^2 (u_1|S) - s_0 t_n (v_{n+1}|S) \right) = 0. \tag{4.34}
\]
Let $\gamma$ be the greatest common divisor of $s_0|S$ and $t_n|S$. Then any irreducible factor $\phi$ of $(s_0|S)/\gamma$ must divide $u_1|S$. If $\phi$ is not a unit in the coordinate ring of $S$, choose an irreducible component $\Sigma$ of $\{\phi = 0\} \subset S$. Then $u_0$ has multiplicity at least two along $\Sigma$ and $u_1$ has multiplicity at least one along $\Sigma$, so $u = u_0 + u_1 z + O(z^2)$ has multiplicity at least two along $\Sigma$. But this contradicts our hypothesis!

Thus, there are no non-trivial factors of $(s_0|S)/\gamma$, which implies that $s_0|S$ divides $t_n|S$. That is, there exists a function $u_n$ such that $t_n = -\frac{1}{3}s_0 u_n + O(z)$ (possibly after shrinking $U$). We can then solve (4.33) by
\[
w_{2n+1} = -\frac{1}{3}u_n v_{n+1} + \frac{1}{9}u_1 u_n^2 + O(z). \tag{4.35}
\]
We replace $t_n$ and $w_{2n+1}$ by $-\frac{1}{3}s_0 u_n$ and $-\frac{1}{3}s_0 u_{n+1} - \frac{1}{9}u_1 u_n^2$, respectively, and make the corresponding adjustments to the other coefficients.

Now when we make a change of coordinates
\[
x \mapsto x + \frac{1}{3}u_n z^n, \tag{4.36}
\]
the equation takes the form
\[
y^2 = x^3 + (u_0 + u_1 z + \cdots + u_{n-1} z^{n-1} + u_n z^n) x^2 + (\widetilde{v}_{n+1} z^{n+1} + \cdots) x + (\widetilde{w}_{2n+2} z^{2n+2} + \cdots) \tag{4.37}
\]
where again $u_0 = \mu s_0^2/4$ and
\[
\begin{align*}
\widetilde{v}_{n+1} &= v_{n+1} + \frac{2}{3} u_1 u_n, \\
\widetilde{w}_{2n+2} &= w_{2n+2} + \frac{1}{9} u_2 u_n^2. \tag{4.38}
\end{align*}
\]
(Note that \(u_0, \ldots, u_{n-1}\) are unchanged by this change of coordinates.) Thus, we have achieved the same form of the equation but with \(n\) replaced by \(n+1\), i.e., \(I_{2n+2}\) “Tate form” with monodromy, and the inductive step is verified.

### 4.8 Case without monodromy

Our induction argument has established that for \(I_m\) with \(m \geq 10\), there is always a locally defined change of coordinates which puts the equation into “Tate form with monodromy” when \(m\) is even, and into the form of our new ansatz (which replaces “Tate form with monodromy” from [4]) when \(m\) is odd.

For models without monodromy, there is a further coordinate change which can be made which puts these equations into “Tate form without monodromy”. Recall that we detect the lack of monodromy by the condition \(\mu \equiv 1\).

If \(m = 2n\) is even, we can apply the coordinate change

\[
(x, y) \mapsto \left( x, y + \frac{1}{2} s_0 x \right)
\]

(4.39)

to (4.23) (bearing in mind that \(u_0 = \frac{1}{4} \mu s_0^2 = \frac{1}{4} s_0^2\)) to obtain

\[
y^2 + s_0 x y = x^3 + (u_1 z + u_2 z^2 + \cdots + u_{n-1} z^{n-1}) x^2 + (v_n z^n + v_{n+1} z^{n+1} + \cdots) x + \left( w_{2n} z^{2n} + w_{2n+1} z^{2n+1} + w_{2n+2} z^{2n+2} + \cdots \right) .
\]

(4.40)

This is “Tate form without monodromy” for \(I_{2n}\).

On the other hand, if \(m = 2n + 1\) is odd, we can apply the coordinate change

\[
(x, y) \mapsto \left( x, y + \frac{1}{2} s_0 x + \frac{1}{2} t_n z^n \right)
\]

(4.41)

to (4.33) to obtain

\[
y^2 + s_0 x y + t_n z^n y = x^3 + (u_1 z + \cdots + u_{n-1} z^{n-1}) x^2 + (v_{n+1} z^{n+1} + \cdots) x + \left( w_{2n+1} z^{2n+1} + \cdots \right) .
\]

(4.42)

This is the “Tate form without monodromy” for \(I_{2n+1}\).

### 4.9 Outliers

Our inductive proof shows that for \(I_m\) with \(m \geq 10\), there is a coordinate change (on a sufficiently small open set \(U\)) which either puts the equation into “Tate form” or into the form given by our new ansatz. We also showed this explicitly for \(I_m\) with \(m \leq 5\). What about the intermediate cases: the cases \(I_6, \ldots, I_9\)?

We do not have any general results about these cases to report on here. However, there are some examples already in the literature which show that at least sometimes, neither Tate form nor our new ansatz can be achieved by a coordinate change. (Although these examples appear
in the literature, it does not seem to have been observed that they cannot be put into “Tate form.”

In [28], the Cartan deformation of an $E_6$ singularity to $A_5$ was shown to be

$$y^2 = x^3 + \frac{81 t^4}{64} x^2 - \frac{9 t^2 z^2}{8} x + \frac{z^4}{4}. \quad (4.43)$$

This already satifies our new ansatz for $I_5$, and since there is no monodromy, it can also be put into “Tate form” for $I_5$ by the coordinate change $(x, y) \mapsto (x, y - \frac{9 t^2 x + \frac{1}{2} z^2})$, giving an equation

$$y^2 - \frac{9}{4} t^2 x y + z^2 y = x^3. \quad (4.44)$$

However, this singularity has type $I_6$, so let us attempt to follow the algorithm in this case. We have $s_0 = -\frac{9}{4} t^2$ and $t_2 = 1$ so at the next step we would define $u_2 = -3 t_2 / s_0$ and make the coordinate change $x \mapsto x + \frac{9}{4} t^2 z$. But since $u_2 = \frac{4}{3} t^{-2}$, this cannot be done.

Similarly, for $E_7$ deformed to $A_6$, [28] found

$$y^2 = \tilde{x}^3 + (729 t^6 + 63 t^2 z) \tilde{x}^2 + (13122 t^8 z + 243 t^4 z^2 - 16 z^3) \tilde{x} + (59049 t^{10} z^2 - 2187 t^6 z^3). \quad (4.45)$$

The first steps in Tate’s algorithm are accomplished by the coordinate change $\tilde{x} = x - 9 t^2 z$, which leaves us with the equation

$$y^2 = x^3 + (729 t^6 + 36 t^2 z) x^2 + (-648 t^4 z^2 - 16 z^3) x + 144 z^4 t^2. \quad (4.46)$$

Again, this satisfies our new ansatz for $I_5$, although since there is no monodromy, it can also be put into “Tate form” for $I_5$ by the coordinate change: $(x, y) \mapsto (x, y - 27 t^3 x + 12 t z^2)$, giving an equation

$$y^2 - 54 t^3 x y + 24 t z^2 y = x^3 + 36 t z x^2 - 16 t^3 x. \quad (4.47)$$

This singularity has type $I_7$, so we again attempt to follow the algorithm. We have $s_0 = -54 t^3$ and $t_2 = 24 t$, so that $u_2 = -3 t_2 / s_0 = \frac{4}{3} t^{-2}$ is ill-defined. As in the previous case, this obstructs us from carrying out the algorithm to put the equation into $I_6$ or $I_7$ form.

### 4.10 Summary for $I_m$

In summary, for the $I_m$ case we have shown that without further assumptions, the following forms for the elliptic fibration can be achieved:

| Type                      | Form                                                      |
|----------------------------|-----------------------------------------------------------|
| $I_2, I_4, I_{2n}, n > 5$ with/without monodromy | “Tate form” can be achieved.                               |
| $I_3, I_5, I_{2n+1}, n > 5$ with monodromy       | “Tate form” not possible near zeros of $s_0$ in $S$. New ansatz (4.33) can be achieved. |
| $I_3, I_5, I_{2n+1}, n > 5$ without monodromy    | “Tate form” can be achieved.                               |
| $I_6, I_7, I_8, I_9$     | Neither “Tate form” nor new ansatz can be achieved.        |
5 The $II$, $III$, and $IV$ cases

For Kodaira fibers of types $II$, $III$, and $IV$, we revert to Weierstrass form as our starting point. In these cases the Kodaira criterion is very straightforward, and can be applied immediately.

1. To obtain Kodaira type $II$ (which has no enhanced gauge symmetry), we need $z \mid f_0$ and $z \mid g_0$. We may absorb these coefficients into $f_1$ and $g_1$, respectively, and find an equation of the form

\[ y^2 = x^3 + (f_1z + f_2z^2 + \cdots)x + (g_1z + g_2z^2 + \cdots). \]  

(5.1)

This is “Tate form” for type $II$, and has type $II$ provided that $z \not\mid g_1$.

2. To obtain Kodaira type $III$ (which has $SU(2)$ local gauge symmetry), we need in addition $z \mid g_1$. We may thus absorb $g_1$ into $g_2$ and find an equation of the form

\[ y^2 = x^3 + (f_1z + f_2z^2 + \cdots)x + (g_2z^2 + g_3z^3 + \cdots). \]  

(5.2)

This is “Tate form” for type $III$, and has type $III$ provided that $z \not\mid f_1$.

3. To obtain Kodaira type $IV$ (which has either $SU(3)$ or $Sp(1)$ local gauge symmetry), we need in addition $z \mid f_1$. We may thus absorb $f_1$ into $f_2$ and find an equation of the form

\[ y^2 = x^3 + (f_2z^2 + f_3z^3 + \cdots)x + (g_2z^2 + g_3z^3 + \cdots). \]  

(5.3)

This is “Tate form” for type $IV$, and has type $IV$ provided that $z \not\mid g_2$.

The gauge symmetry is determined by the monodromy, which according to Tate [21], depends on whether or not $g_2|_S$ is a square. If $g_2|_S$ is not a square, then there is monodromy and the local gauge symmetry is $Sp(1)$. If $g_2|_S$ is a square, then there is no monodromy and the local gauge symmetry is $SU(3)$.

6 The $I_m^*$ case

For fibers of type $I_m^*$, we once again start in Weierstrass form. Kodaira tells us that to have type $I_0^*$, we must have $z^2 \mid f$, $z^3 \mid g$, and the order of $\Delta$ along $S$ must be exactly 6.

\[ y^2 = x^3 + (f_2z^2 + f_3z^3 + \cdots)x + (g_3z^3 + g_4z^4 + \cdots) \]  

(6.1)

This is “Tate form” for type $I_0^*$. The monodromy in this case is quite subtle, but involves analyzing the branching behavior of the cubic polynomial

\[ x^3 + \frac{f}{z^2}x + \frac{g}{z^3}. \]  

(6.2)
6.1 Step 1
For $I^*_m$ with $m > 0$, the condition is slightly different: the orders of $f$, $g$, and $\Delta$ must be exactly 2, 3 and $m + 6$. We write the leading terms in the discriminant as

$$\Delta = (4f_2^3 + 27g_3^2)z^6 + O(z^7).$$

and note that the first term must vanish whenever $z^7 \mid \Delta$. In this case, by an argument in appendix $B$ there exists a function $u_1$ (possibly after shrinking $U$) such that

$$f_2 = -\frac{1}{3}u_1^2 + O(z), \quad g_3 = \frac{2}{27}u_1^3 + O(z).$$

(6.4)

We replace $f_2$ by $-\frac{1}{3}u_1^2$ and $g_3$ by $\frac{2}{27}u_1^3$, modifying the higher coefficients as necessary. Now following Tate, we make a change of coordinates

$$(x, y) \mapsto \left( x + \frac{1}{3}u_1 z, y \right),$$

(6.5)

which transforms the defining equation to

$$y^2 = x^3 + u_1 z x^2 + (f_3 z^3 + f_4 z^4 + \cdots) x + (g_4 + \frac{1}{3}u_1 f_3) z^4 + (g_5 + \frac{1}{3}u_1 f_4) z^5 + \cdots.$$  

(6.6)

This is “Tate form” for type $I^*_1$. To simplify later formulas, we set $\tilde{g}_j = g_j + \frac{1}{3}u_1 f_{j-1}$, and write the equation in the form

$$y^2 = x^3 + u_1 z x^2 + (f_3 z^3 + f_4 z^4 + \cdots) x + \tilde{g}_4 z^4 + \tilde{g}_5 z^5 + \cdots.$$  

(6.7)

6.2 Step 2
At the next order the discriminant is

$$\Delta = 4u_1^6 \tilde{g}_4 z^7 + O(z^8).$$

(6.8)

The condition that the discriminant vanishes to the next order is

$$\tilde{g}_4 = 0.$$  

(6.9)

The fibration then takes the form

$$y^2 = x^3 + u_1 z x^2 + (f_3 z^3 + f_4 z^4 + \cdots) x + (\tilde{g}_5 z^5 + \tilde{g}_6 z^6 + \cdots).$$  

(6.10)

This is “Tate form” for $I^*_2$. 

18
6.3 Step 3

At the next order the discriminant is
\[ \Delta = u_1^2 \left( 4u_1 \tilde{g}_5 - f_3^2 \right) z^8 + O(z^9). \] (6.11)

Analogously to the case $I_n$, we can find functions $s_0$ and $\mu_1$ such that $\mu_1|_{S}$ is square-free and a function $t_2$ such that
\[ u_1 = \frac{1}{4} \mu_1 s_0^2, \quad f_3 = \frac{1}{2} \mu_1 s_0 t_2. \] (6.12)

We can then solve for $\tilde{g}_5$ as
\[ \tilde{g}_5 = \frac{1}{4} \mu_1 t_2^2 \] (6.13)
giving a fibration of the form
\[ y^2 = x^3 + \frac{1}{4} \mu_1 s_0^2 \tilde{g}_5 x^2 + \left( \frac{1}{2} \mu_1 s_0 t_2 z^3 + f_4 z^4 + \cdots \right) x + \left( \frac{1}{4} \mu_1 t_2^2 z^5 + \tilde{g}_6 z^6 + \cdots \right). \] (6.14)

The necessary coordinate change to put this in Tate form for type $I^*_3$,
\[ (x, y) \to \left( x - \frac{t_2 z^2}{s_0}, y \right) \] (6.15)
does not exist near zeros of $s_0$ along $S$.

6.4 Induction

We now set up an induction which assumes that the orders of vanishing of $f$ and $g$ are precisely 2 and 3, respectively, and that $z^k | \Delta$ for some $k < m$. Our induction will assume a certain form for the equation (to be described shortly), and proceed to derive the corresponding form for $k + 1$. Our equations will all have the general form
\[ y^2 = x^3 + u x^2 + v x + w \] (6.16)
with $z | u$, $z^3 | v$ and $z^4 | w$. We let $u = \tilde{u} z$. Note that we cannot have $z | \tilde{u}$, or else we would be in a different branch of Kodaira’s classification (as in that case $f$ and $g$ would vanish to order greater than 2 and 3, respectively).

As an initial hypothesis we assume that $m \geq 4$ so that $\Delta$ vanishes to order at least 10. The reason is similar to the $I_m$ case: under this hypothesis, $\tilde{u}|_{S}$ can have no zeros. For if there were any zero of $\tilde{u}|_{S}$ then the multiplicities of $f$, $g$, and $\Delta$ at such a point would be at least $(3, 4, 11)$, and that forces the point into the “non-minimal” part of Kodaira’s classification, with the consequent massless tensors, light strings, etc.

Now to our induction. We start by assuming that we have achieved “Tate form” for some $I^*_k$ with $k < m$ and will show how to increase $k$. If $k = 2n - 1$ is odd, we assume by inductive hypothesis that the expansion of $u = \tilde{u} z = u_1 z + u_2 z^2 + \cdots + u_n z^n$ has no term divisible by
$z^{n+1}$ and that the “Tate form” for $I^*_2 n - 1$ holds (ignoring the monodromy condition): namely, that $z^{n+2} \mid v$ and $z^{2n+2} \mid w$. Note that the form achieved in (6.7) is exactly of this type for $n = 1$, under the simple assumption that $z^7 \mid \Delta$. Thus, our induction has a place to begin.

Under this assumption, there is only one contribution to the leading term in the discriminant (4.28):

$$
\Delta = 4u_1^3 w_{2n+2} z^{2n+5} + O(z^{2n+6}).
$$

(6.17)

Since we are assuming that $k < m$, the leading term must vanish, that is, $z \mid w_{2n+2}$ or equivalently $z^{2n+3} \mid w$ (since $\widehat{u} \mid S$ is not identically zero). We can thus absorb $w_{2n+2}$ into $w_{2n+3}$ by adjusting the latter, after which we have achieved the conditions $z^{n+2} \mid v$ and $z^{2n+3} \mid w$ when $z^{2n+6} \mid \Delta$. This will take to be our corresponding inductive hypothesis when $k = 2n$ is even.

(This is “Tate form” for $I^*_2 n$, ignoring the monodromy condition.)

As the second step in the induction, we now assume we are in that for $m$. This time, the leading contribution to the discriminant contains two terms:

$$
\Delta = u_1^2 (4u_1 w_{2n+3} - v_{n+2}^2) z^{2n+6} + O(z^{2n+7}).
$$

(6.18)

Since again $k < m$ by our inductive assumption, we must have

$$
(4u_1 w_{2n+3} - v_{n+2}^2) \mid S \equiv 0.
$$

(6.19)

Since $u_1 \mid S$ has no zeros by our initial hypothesis, we may find a function $u_{n+1}$ (possibly after shrinking $U$) such that

$$
v_{n+2} = -\frac{2}{3} u_1 u_{n+1} + O(z).
$$

(6.20)

Then in order to have the vanishing specified in (6.19) we must also have

$$
w_{2n+3} = \frac{1}{9} u_1 u_{n+1}^2 + O(z).
$$

(6.21)

Replace $v_{n+2}$ by $-\frac{2}{3} u_1 u_{n+1}$ and $w_{2n+3}$ by $\frac{1}{9} u_1 u_{n+1}^2$, and adjust the other coefficients accordingly. Now we can make a change of coordinates $(x, y) \mapsto (x + \frac{1}{3} u_{n+1} z^{n+1}, y)$. This adds $u_{n+1} z^{n+1}$ to the coefficient of $x^2$, and increases the order of vanishing of $v$ and $w$ by one each. That is, we have $z^{2n+7} \mid \Delta$ while $z^{n+3} \mid v$ and $z^{2n+4} \mid w$. This reproduces our inductive hypotheses for $k = 2n + 1$, so our induction argument is complete.

The final remark is about the monodromy in $I^*_{m}$ cases. According to Tate’s original algorithm, for $I^*_2 n - 1$ the test for monodromy is whether $w_{2n+2} \mid S$ has a square root. (This distinguishes between $SO(4n+5)$ when there is no square root, and $SO(4n+6)$ when there is a square root.) Similarly, for $I^*_2 n$ the test for monodromy is whether $(4u_1 w_{2n+3} - v_{n+2}^2) \mid S$ has a square root. (This distinguishes between $SO(4n+7)$ when there is no square root, and $SO(4n+8)$ when there is a square root.)
6.5 Summary of the $I^*_m$ case

We have thus shown that we can always write a Weierstrass fibration in “Tate form” for types $I^*_1, I^*_2,$ and $I^*_m$ for $m \geq 4$. For the case $I^*_3$, where “Tate form” is not always achievable, we can find simple examples in the literature that exhibit this behaviour. In particular [29], the unfolding of $E_8$ to $D_7$ is described by the fibration

$$y^2 = x^3 + t^2 z^2 x + 2 t^3 t^2 x^2 + z^5.$$  \hspace{1cm} (6.22)

This is already in “Tate form” for $I^*_2$ with $u_1 = t^2$, $v_3 = 2 t$, and $w_5 = 1$. However, the generic singularity is of type $I^*_3$; if we attempt to follow the algorithm, then the next coordinate change involves $u_3 = -3 v_3 / 2 u_1 = -3 t^{-1}$ which is ill-defined. Thus, this example cannot be put into “Tate form” for $I^*_3$.

7 The $IV^*$, $III^*$, and $II^*$ cases

For Kodaira fibers of types $IV^*$, $III^*$, and $II^*$, we again revert to Weierstrass form as our starting point. In these cases the Kodaira criterion is very straightforward, and can be applied immediately.

1. To obtain Kodaira type $IV^*$ (which has either $E_6$ or $F_4$ local gauge symmetry), we need $z^3 \mid f$ and $z^4 \mid g$. We may thus choose our expansion to take the form

$$y^2 = x^3 + (f_3 z^3 + f_4 z^4 + \cdots) x + (g_4 z^4 + g_5 z^5 + \cdots).$$  \hspace{1cm} (7.1)

This is “Tate form” for type $IV^*$, and has type $IV^*$ provided that $z \not \mid g_4$. There is also a monodromy question in this case, measured by whether $g_4 \mid S$ is a square or not. If $g_4 \mid S$ is not a square, there is monodromy and the local gauge symmetry is $F_4$. If $g_4 \mid S$ is a square, there is no monodromy and the local gauge symmetry is $E_6$.

2. To obtain Kodaira type $III^*$ (which has $E_7$ local gauge symmetry), we need in addition $z \mid g_4$. We may thus absorb $g_4$ into $g_5$ and find an equation of the form

$$y^2 = x^3 + (f_3 z^3 + f_4 z^4 + \cdots) x + (g_5 z^5 + g_6 z^6 + \cdots).$$  \hspace{1cm} (7.2)

This is “Tate form” for type $III^*$, and has type $III^*$ provided that $z \not \mid f_3$.

3. To obtain Kodaira type $II^*$ (which has $E_8$ local gauge symmetry), we need in addition $z \mid f_3$. We may thus absorb $f_3$ into $f_4$ and find an equation of the form

$$y^2 = x^3 + (f_4 z^4 + f_5 z^5 + \cdots) x + (g_5 z^5 + g_6 z^6 + \cdots).$$  \hspace{1cm} (7.3)

This is “Tate form” for type $II^*$, and has type $II^*$ provided that $z \not \mid g_5$.

4. Finally, if $z \mid g_5$ we have reached the “non-minimal” line on Kodaira’s table; algorithmically, we should alter our Weierstrass model to one of lower degree and repeat the algorithm.
8 Conclusions

In this paper we re-analyzed the “Tate forms” that were introduced in [4] in the light of their general validity and found that, except in a few instances, the Tate algorithm can be carried through without many modifications. Specifically we found that, except for $I_6$, $I_7$, $I_8$, $I_9$ and $I_3^*$, the “Tate forms” can be achieved for the cases $I_n$ without monodromy and for $I_m^*$. For $I_{2n+1}$ with monodromy we can generically (i.e. without making any further assumptions about divisibility of the sections) only achieve the new form (4.33). The main obstruction stems from changes of variables that are necessary in the algorithm, but which may involve poles on the divisor over which the singularity resides.

Furthermore, we demonstrated with a simple example that Tate forms may not hold globally over the entire base. For the example in section 3 with an $I_2$ singularity and in appendix C with $SU(5)$ there are global obstructions to achieving Tate form. For the case of $SU(5)$ we only provided one possible recipe for constructing a globally-obstructed Tate form. It would be interesting to find explicit geometries realizing these (or analogous) criteria. These geometries could, in particular, be relevant for GUT model building with $SU(5)$ gauge group by giving more freedom in their construction.

Acknowledgements

We thank A. Grassi and W. Taylor for discussions. SK and DRM thank the Simons Workshop in Mathematics and Physics for hospitality at the inception of this project, and IPMU, University of Tokyo, for hospitality at a later stage. SSN thanks the Caltech theory group for their generous hospitality, and JS thanks IPMU, University of Tokyo, for hospitality. This research was partially supported by the National Science Foundation under grants DMS-1007414, DMS-05-55678, and PHY05-51164, and by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan.

A “Tate forms” from [4]

We start with an equation in the general form

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6. \tag{A.1}$$

(In the main body of this paper, $s$ and $t$ were used on the left side of the equation in place of $a_1$ and $a_3$, while $u$, $v$ and $w$ were used on the right side of the equation in place of $a_2$, $a_4$, and $a_6$.) Table 2, which is reproduced from the first part of Table 2 in [4], gives special forms of (A.1) which lead to enhanced gauge symmetry—these have come to be called “Tate forms,” but, as emphasized in the body of this paper, they serve as convenient ansätze which do not always apply. One piece of notation in this Table needs explanation: a superscript of $^s$ on the Kodaira symbol indicates no monodromy, while a superscript of $^{ns}$ or $^{ss}$ indicates monodromy.
(For \( I_0^* \), the only case in which \( ^{ss} \) appears, there are two types of monodromy and the notation distinguishes between them.)

We modified Table 2 of \([4]\) by changing \( k \) to \( n \) to match the notation of this paper, and by correcting and completing the “group” column in the table, according to the more precise conclusions about gauge groups which were found some years later in \([30]\).

| type | group | \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( a_6 \) | \( \Delta \) |
|------|-------|-------|-------|-------|-------|-------|-------|
| \( I_0 \) | —     | 0     | 0     | 0     | 0     | 0     | 0     |
| \( I_1 \) | —     | 0     | 0     | 1     | 1     | 1     | 1     |
| \( I_2 \) | \( SU(2) \) | 0     | 0     | 1     | 1     | 2     | 2     |
| \( I_3^{ns} \) | \( Sp(1) \) | 0     | 0     | 2     | 2     | 3     | 3     |
| \( I_3^s \) | \( SU(3) \) | 0     | 1     | 1     | 2     | 3     | 3     |
| \( I_{2n}^{ns} \) | \( Sp(n) \) | 0     | 0     | \( n \) | \( n \) | 2\( n \) | 2\( n \) |
| \( I_{2n}^s \) | \( SU(2n) \) | 0     | 1     | \( n \) | \( n \) | 2\( n \) | 2\( n \) |
| \( I_{2n+1}^{ns} \) | \( Sp(n) \) | 0     | 0     | \( n+1 \) | \( n+1 \) | 2\( n+1 \) | 2\( n+1 \) |
| \( I_{2n+1}^s \) | \( SU(2n+1) \) | 0     | 1     | \( n+1 \) | \( n+1 \) | 2\( n+1 \) | 2\( n+1 \) |
| \( \bar{I} \) | —     | 1     | 1     | 1     | 1     | 1     | 1     |
| \( III \) | \( SU(2) \) | 1     | 1     | 1     | 1     | 2     | 2     |
| \( IV^{ns} \) | \( Sp(1) \) | 1     | 1     | 1     | 2     | 2     | 4     |
| \( IV^s \) | \( SU(3) \) | 1     | 1     | 1     | 2     | 3     | 4     |
| \( I_{0}^{ns} \) | \( G_2 \) | 1     | 1     | 2     | 2     | 3     | 6     |
| \( I_{0}^{s} \) | \( SO(7) \) | 1     | 1     | 2     | 2     | 4     | 6     |
| \( I_{1}^{ns} \) | \( SO(8)^* \) | 1     | 1     | 2     | 2     | 4     | 6     |
| \( I_{1}^{s} \) | \( SO(9) \) | 1     | 1     | 2     | 3     | 4     | 7     |
| \( I_{2}^{ns} \) | \( SO(10) \) | 1     | 1     | 2     | 3     | 5     | 7     |
| \( I_{2}^{s} \) | \( SO(11) \) | 1     | 1     | 3     | 3     | 5     | 8     |
| \( I_{3}^{ns} \) | \( SO(12)^* \) | 1     | 1     | 3     | 3     | 5     | 8     |
| \( I_{3}^{s} \) | \( SO(4n+1) \) | 1     | 1     | \( n+1 \) | \( n+1 \) | 2\( n+3 \) |
| \( I_{4}^{ns} \) | \( SO(4n+2) \) | 1     | 1     | \( n+1 \) | \( 2n+1 \) | 2\( n+3 \) |
| \( I_{4}^{s} \) | \( SO(4n+3) \) | 1     | 1     | \( n+1 \) | \( 2n+1 \) | 2\( n+4 \) |
| \( I_{5}^{ns} \) | \( SO(4n+4)^* \) | 1     | 1     | \( n+1 \) | \( 2n+1 \) | 2\( n+4 \) |
| \( IV^{ns} \) | \( F_4 \) | 1     | 2     | 2     | 3     | 4     | 8     |
| \( IV^{s} \) | \( E_6 \) | 1     | 2     | 2     | 3     | 5     | 8     |
| \( III^{*} \) | \( E_7 \) | 1     | 2     | 3     | 3     | 5     | 9     |
| \( II^{*} \) | \( E_8 \) | 1     | 2     | 3     | 4     | 5     | 10    |
| non-min | —     | 1     | 2     | 3     | 4     | 6     | 12    |

Table 2: “Tate forms” (from \([4]\))

The special forms are specified by declaring various coefficients \( a_i \) to vanish along the discriminant component \( \{ z = 0 \} \) to various orders (at a minimum): the orders are specified in Table 2. By finding the lowest row in the table for which the vanishing conditions are satisfied, we determine the Kodaira fiber type and monodromy. In addition, the asterisks next to \( SO(4n+4) \) indicate that one more condition must be fulfilled in those cases: for \( SO(8) \) we must have that

\[
\frac{a_2^2 - 4a_4}{z^4} \bigg|_{z=0}
\]
is a square, whereas for $SO(4n + 4)$ with $n \geq 3$ we need that

$$\frac{a_4^2 - 4a_2a_6}{z^{2k+2}} \Big|_{z=0}$$

is a square.

Now in general, to pass from (A.1) to Weierstrass form involves completing the square of the left hand side of (A.1), and then completing the cube on the right hand side. For the forms specified in Table 2, usually part of this completing the square and/or cube can be done without disturbing the vanishing conditions. This gives a more compact version of the “Tate form” in each case, in which some of the coefficients in (A.1) are suppressed altogether. The results of this operation are displayed in Table 3, in which an entry “∞” indicates that a coefficient is to be set to zero. The same extra condition for $SO(4n + 4)$ must be applied as in the original form.

As mentioned earlier, it is these compact versions of Tate forms which we have referred to repeatedly in the body of the paper.

**B Lemmas using unique factorization**

In this appendix we prove the lemmas that were used in the text in implementing Tate’s algorithm.

We keep the notation in the main text: $B$ is the smooth base, $S$ a smooth divisor over which enhancement occurs, and $U \subset B$ an affine open set. By smoothness, the rings of algebraic functions on $U$ or $U \cap S$ are unique factorization domains. In these UFDs, the units are just the nowhere vanishing functions.

Recall that we are identifying the restricted leading coefficients $f_0|_S$ and $g_0|_S$ with well-defined as functions on $S \cap U$. (More generally, they will be well-defined sections of line bundles on $S$.)

We use the notation $X = Y + O(z)$ to indicate that $X - Y \in \mathcal{I}_S$, since $z$ is a local defining equation for $S$.

We will routinely extend functions on $U \cap S$ to functions on $U$. If we were only dealing with regular functions, this would be automatic since the coordinate ring of $U \cap S$ is a quotient of the ring of regular functions on $U$. In the more general situation of algebraic functions, we may have to shrink $U$ to keep the functions single-valued.

**Lemma 1** If $(4f_0^3 + 27g_0^2)|_S = 0$ then possibly after shrinking $U$, there exists a function $u_0$ on $U$ such that $f_0|_S = -\frac{1}{3}u_0^2|_S$ and $g_0|_S = \frac{2}{27}u_0^3|_S$, i.e., $f_0 = -\frac{1}{3}u_0^2 + O(z)$ and $g_0 = \frac{2}{27}u_0^3 + O(z)$.

**Proof:** We factor the restrictions of $f_0$ and $g_0$ into irreducibles

$$f_0|_S = \prod_{i=1}^m f_i^{\alpha_i}, \quad g_0|_S = \prod_{j=1}^n g_j^{\beta_j}, \quad (B.1)$$
| type     | group      | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_6$ | $\Delta$ |
|----------|------------|-------|-------|-------|-------|-------|----------|
| $I_0$    | —          | $\infty$ | $\infty$ | $\infty$ | 0 | 0 | 0 |
| $I_1$    | —          | $\infty$ | 0 | $\infty$ | 1 | 1 | 1 |
| $I_2$    | $SU(2)$   | $\infty$ | 0 | $\infty$ | 1 | 2 | 2 |
| $I_3^{ns}$ | $Sp(1)$     | $\infty$ | 0 | $\infty$ | 2 | 3 | 3 |
| $I_3$    | $SU(3)$   | $0$ | $\infty$ | 1 | 2 | 3 | 3 |
| $I_4^{ns}$ | $Sp(n)$  | $\infty$ | 0 | $\infty$ | $n$ | 2$n$ | 2$n$ |
| $I_2$    | $SU(2n)$ | 0 | 1 | $\infty$ | $n$ | 2$n$ | 2$n$ |
| $I_3^{ns}$ | $Sp(n)$ | $\infty$ | 0 | $\infty$ | $n+1$ | 2$n+1$ | 2$n+1$ |
| $I_4^{ns}$ | $SU(2n)$ | 0 | 1 | $n$ | $n+1$ | 2$n+1$ | 2$n+1$ |
| $I_5^I$  | —          | $\infty$ | $\infty$ | $\infty$ | 1 | 1 | 2 |
| $I_3^I$  | $SU(2)$   | $\infty$ | $\infty$ | $\infty$ | 1 | 2 | 3 |
| $I_4^{ns}$ | $Sp(1)$  | $\infty$ | $\infty$ | 2 | 2 | 4 |
| $I_5^I$  | $SU(3)$   | $\infty$ | 1 | $\infty$ | 2 | 3 | 4 |
| $I_3^{ns}$ | $G_2$     | $\infty$ | $\infty$ | 2 | 3 | 6 |
| $I_3^{ss}$ | $SO(7)$ | $\infty$ | 1 | $\infty$ | 2 | 4 | 6 |
| $I_5^I$  | $SO(8)^*$ | $\infty$ | 1 | $\infty$ | 2 | 4 | 6 |
| $I_3^{ns}$ | $SO(9)$  | $\infty$ | 1 | $\infty$ | 3 | 4 | 7 |
| $I_5^I$  | $SO(10)$ | $\infty$ | 1 | 2 | 3 | 5 | 7 |
| $I_3^{ns}$ | $SO(11)$ | $\infty$ | 1 | $\infty$ | 3 | 5 | 8 |
| $I_5^I$  | $SO(12)^*$ | $\infty$ | 1 | $\infty$ | 3 | 5 | 8 |
| $I_3^{ns}$ | $SO(4n+1)$ | $\infty$ | 1 | $\infty$ | $n+1$ | 2$n$ | 2$n+3$ |
| $I_3^{ns}$ | $SO(4n+2)$ | $\infty$ | 1 | $n$ | $n+1$ | 2$n+1$ | 2$n+3$ |
| $I_3^{ns}$ | $SO(4n+3)$ | $\infty$ | 1 | $\infty$ | $n+1$ | 2$n+1$ | 2$n+4$ |
| $I_3^{ns}$ | $SO(4n+4)^*$ | $\infty$ | 1 | $\infty$ | $n+1$ | 2$n+1$ | 2$n+4$ |
| $I_4^{ns}$ | $E_4$     | $\infty$ | $\infty$ | 3 | 4 | 8 |
| $I_4^{ns}$ | $E_6$     | $\infty$ | $\infty$ | 2 | 3 | 5 | 8 |
| $I_4^{ns}$ | $E_7$     | $\infty$ | $\infty$ | 3 | 5 | 9 |
| $I_4^{ns}$ | $E_8$     | $\infty$ | $\infty$ | 4 | 5 | 10 |
| non-min  | —          | $\infty$ | $\infty$ | $\infty$ | 4 | 6 | 12 |

Table 3: “Tate forms” (compact version)

unique up to ordering and multiplication by units. From $(4f_0^3 + 27g_0^2)|S = 0$ and unique factorization, we see that $m = n$, and that after reordering the $f_i$ and $g_i$ if necessary that the $f_i$ and $g_i$ are equal up to multiplication by a unit. We conclude that there are integers $\gamma_i$ such that for all $i$ we have $\alpha_i = 2\gamma_i$ and $\beta_i = 3\gamma_i$.

We put

$$v_0 = c \prod_{i=1}^{m} f_i^{\gamma_i} \tag{B.2}$$

for a constant $c$ and demand that $f_0|S = -\frac{1}{3}v_0^2$ and $g_0|S = \frac{2}{27}v_0^3$. The first condition requires $c^2 = -3$ and the second condition fixes the choice of the square root to determine $c$. We now let $u_0$ be any function on $U$ restricting to $v_0$ on $S$ (shrinking $U$ if necessary) and we are done.

**Lemma 2** Given a function $u_0$ whose restriction to $S$ is not identically zero, then possibly after
shrinking $U$ there exist functions $s_0$ and $\mu$ such that $\mu|_S$ is square-free, and $u_0 = \frac{1}{4}\mu s_0^2 + O(z)$.

**Proof:** We write

$$u_0|_S = \prod_{i=1}^{m} u_i^{\alpha_i}$$

(B.3)

Without loss of generality we may suppose that $\alpha_i$ is odd for $i \leq k$ and even for $k+1 \leq i \leq m$. We put $\alpha_i = 2\beta_i + 1$ for $i \leq k$ and $\alpha_i = 2\beta_i$ for $k+1 \leq i \leq m$. Then we put

$$t_0 = \prod_{i=1}^{m} u_i^{\beta_i} \quad \nu = 4 \prod_{i=1}^{k} u_i,$$

(B.4)

so that $\nu$ is square-free and $u_0|_S = \frac{1}{4}\nu t_0^2$.

Now let $s_0$ and $\mu$ be any functions on $U$ restricting to $t_0$ and $\nu$ respectively on $S$ (shrinking $U$ if necessary), so that $u_0 = \frac{1}{4}\mu s_0^2 + O(z)$ and we are done.

**Lemma 3** If $(4\mu s_0^2 \tilde{g}_2 - f_1^2)|_S = 0$ and $\mu|_S$ is square-free, then possibly after shrinking $U$ there exists a function $t_1$ such that $f_1 = \frac{1}{2}\mu s_0 t_1 + O(z)$.

**Proof:** Since $\mu|_S$ is square-free, we have a factorization

$$\mu|_S = \prod_{i=1}^{n} \mu_i$$

(B.5)

with distinct factors. We also factor

$$\tilde{g}_2|_S = \prod_{j=1}^{m} h_j^{\beta_j}$$

(B.6)

Since the exponents in $(4\mu s_0^2 \tilde{g}_2)|_S$ are even, we can reorder the $h_j$ if necessary to achieve $h_j = \mu_j$ up to a unit and $\beta_j$ odd for $j \leq n$, and $\beta_j$ even for $j > n$. As in the proofs of the earlier lemmas, we can easily write down a function $s_1$ on $U \cap S$ such that

$$f_1|_S = \frac{1}{2} (\mu s_0)|_s s_1.$$  

(B.7)

We then extend $s_1$ to a function $t_1$ on $U$ (shrinking $U$ if necessary) and we are done.

**Lemma 4** If

$$(s_0^3 \tilde{g}_3 - s_0^2 t_1 f_2 - t_1^3)|_S,$$

then possibly after shrinking $U$, there exists a function $u_1$ on $U$ such that $t_1 = -\frac{1}{3} s_0 u_1 + O(z)$.

**Proof:** We argue that $s_0|_S$ divides $t_1|_S$ by induction on the number of irreducible factors of $s_0|_S$. If there are none, we are done. If there is an irreducible factor $\alpha$, then it divides the first two terms in eq. (B.8) so it must divide $t_1^2|_S$ and hence $t_1|_S$. This implies that $\alpha^3$ divides all three terms in eq. (B.8); dividing by $\alpha^3$ reduces the number of irreducible factors of $s_0|_S$ and by induction we are finished.
C Global obstructions for $SU(5)$

In section 3 we gave an explicit example of an $I_2$ Weierstrass fibration that could not be written globally in Tate-type normal form. Although it seems clear that the same type of obstruction can appear at higher order, it would be nice to have an analogous example for $SU(5)$. Unfortunately it is not easy to mechanically construct such an example. Instead, we simply give a list of sufficient criteria for such an example to exist. Note that we have not proven that it is possible to satisfy all of the following conditions simultaneously.

Consider a base $B$ with an effective anti-canonical divisor $-K$ and an effective divisor $S$ on $B$ such that $-12K - 5S$ is effective. As the discriminant $\Delta$ is a section of $-12K$, this allows us to build a fibration that vanishes to fifth order on $S$. In the following, we will make frequent use of the exact sequence of sheaves

$$0 \to \mathcal{O}_B(-nK - (m+1)S) \xrightarrow{-z} \mathcal{O}_B(-nK - mS) \to \mathcal{O}_S(-nK - mS) \to 0 \quad (C.1)$$

for various integers $m,n$, where $\cdot z$ is multiplication by the defining section of $S$ on $B$.

As a first step, we ask that $H^1(\mathcal{O}_B(-K - S)) \neq 0$ and $H^1(\mathcal{O}_B(-K)) = 0$. This guarantees that we can find a non-trivial class $\sigma \in H^1(\mathcal{O}_B(-K - S))$ and a section $\tilde{s}_0 \in H^0(\mathcal{O}_S(-K))$ such that $\delta \tilde{s}_0 = \sigma$. Since $\tilde{s}_0$ has a non-trivial image, it cannot itself be the image of a global section on $B$ under restriction to $S$.

We define $\tilde{f}_0, \tilde{g}_0$ on $S$ in terms of $\tilde{s}_0$ as we discussed in section 4: $\tilde{f}_0 = -1/48 \tilde{s}_0^4$ and $\tilde{g}_0 = 1/216 \tilde{s}_0^6$. If we further assume that $H^1(\mathcal{O}_B(-4K - S)) = 0$ and $H^1(\mathcal{O}_B(-6K - S)) = 0$ then exactness determines that $\tilde{f}_0$ and $\tilde{g}_0$ lift to respective global sections $f_0, g_0$ on $B$.

As in section 3, we have so far constructed an $I_1$ fiber over $S$, but here have avoided non-trivial monodromy (by taking $\tilde{u}_0 = 1/4 \tilde{s}_0^2$) with the aim of constructing $SU(n)$ fibers. It remains to give conditions for the higher order of vanishing. By construction we have

$$\Delta^{(0)} := 4f_0^3 + 27g_0^2 = \delta_1 z \quad (C.2)$$

(recall $z$ is the defining section of $S$). Now define

$$f^{(1)} = f_0 + f_1 z$$
$$g^{(1)} = g_0 + g_1 z \quad (C.3)$$

with some putative sections $f_1, g_1$. Correspondingly, define $\Delta^{(1)} := 4f^{(1)}_0 + 27g^{(1)}_0$. Vanishing at next order is the condition $\Delta^{(1)} = \delta_2 z^2$. This requires

$$\delta_1 + 12f_1f_0^2 + 54g_0g_1 = 0 + O(z) \quad (C.4)$$

Restricting to $S$ this is equivalent to

$$\tilde{g}_1 = -\tilde{f}_1 \tilde{u}_0 - \frac{1}{4} \tilde{\delta}_1 \tilde{u}_0^3, \quad (C.5)$$
which has a solution provided $\tilde{\delta}_1/\tilde{u}_0^3$ has no poles over $S$. Then if we require $H^1(\mathcal{O}_B(-6K - 2S)) = 0$ we must have that $\tilde{g}_1$ lifts to a section $g_1$ of $H^0(\mathcal{O}(-6K - S))$ on $B$. Finding such sections we have constructed an $SU(2)$ example given by $f = f^{(1)}$ and $g = g^{(1)}$.

The general procedure follows similarly. We iteratively define

$$
\Delta^{(i-1)} = 4f^{(i-1)3} + 27g^{(i-1)2} = \delta_iz^i, \quad f^{(i)} = f^{(i-1)} + f_iz^i, \quad g^{(i)} = g^{(i-1)} + g_iz^i. \quad (C.6)
$$

Vanishing of $\Delta^{(i)}$ at order $i + 1$ requires, over $S$, that

$$
\tilde{g}_i = -f_iz_0 - \frac{1}{4}\tilde{\delta}_i/\tilde{u}_0^3. \quad (C.7)
$$

Moreover, $\tilde{g}_i$ is well-defined and lifts to some global $g_i$ on $B$ provided $\tilde{\delta}_i/\tilde{u}_0^3$ has no poles over $S$ and $H^1(\mathcal{O}_B(-6K - (i + 1)S)) = 0$. For an $SU(5)$ example we must successfully complete this iterative procedure for $i = 1, 2, 3, 4$ with final result the fibration $f = f^{(4)}$ and $g = g^{(4)}$.

References

[1] C. Vafa, Evidence for F-theory, Nucl. Phys. B 469 (1996) 403–418, arXiv:hep-th/9602022

[2] D. R. Morrison and C. Vafa, Compactifications of F-theory on Calabi–Yau threefolds, I, Nucl. Phys. B 473 (1996) 74–92, arXiv:hep-th/9602114

[3] ______, Compactifications of F-theory on Calabi–Yau threefolds, II, Nucl. Phys. B 476 (1996) 437–469, arXiv:hep-th/9603161

[4] M. Bershadsky, K. Intriligator, S. Kachru, D. R. Morrison, V. Sadov, and C. Vafa, Geometric singularities and enhanced gauge symmetries, Nucl. Phys. B 481 (1996) 215–252, arXiv:hep-th/9605200

[5] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, and C. Vafa, F-theory, geometric engineering and N = 1 dualities, Nucl. Phys. B 505 (1997) 153–164, arXiv:hep-th/9612052

[6] D. R. Morrison, TASI lectures on compactification and duality, Strings, Branes, and Gravity, TASI 99 (J. Harvey, S. Kachru, and E. Silverstein, eds.), World Scientific, 2001, pp. 653–719, arXiv:hep-th/0411120

[7] F. Denef, Les Houches lectures on constructing string vacua, arXiv:0803.1194 [hep-th].

[8] R. Donagi and M. Wijnholt, Model building with F-theory, arXiv:0802.2969 [hep-th].

[9] C. Beasley, J. J. Heckman, and C. Vafa, GUTs and exceptional branes in F-theory - I, JHEP 01 (2009) 058, arXiv:0802.3391 [hep-th].
[10] H. Hayashi, R. Tatar, Y. Toda, T. Watari, and M. Yamazaki, New aspects of heterotic–F theory duality, Nucl. Phys. B 806 (2009) 224–299, arXiv:0805.1057 [hep-th].

[11] T. Weigand, Lectures on F-theory compactifications and model building, Class. Quant. Grav. 27 (2010) 214004, arXiv:1009.3497 [hep-th].

[12] M. Graña, Flux compactifications in string theory: A comprehensive review, Phys. Rept. 423 (2006) 91–158, arXiv:hep-th/0509003.

[13] M. R. Douglas and S. Kachru, Flux compactification, Rev. Mod. Phys. 79 (2007) 733–796, arXiv:hep-th/0610102.

[14] F. Denef, M. R. Douglas, and S. Kachru, Physics of string flux compactifications, Ann. Rev. Nucl. Part. Sci. 57 (2007) 119–144, arXiv:hep-th/0701050.

[15] V. Kumar and W. Taylor, A bound on 6D N=1 supergravities, JHEP 12 (2009) 050, arXiv:0910.1586 [hep-th].

[16] V. Kumar, D. R. Morrison, and W. Taylor, Mapping 6D N = 1 supergravities to F-theory, JHEP 02 (2010) 099, arXiv:0911.3393 [hep-th].

[17] _____, Global aspects of the space of 6D N = 1 supergravities, JHEP 11 (2010) 118, arXiv:1008.1062 [hep-th].

[18] W. Taylor, TASI Lectures on Supergravity and String Vacua in Various Dimensions, arXiv:1104.2051 [hep-th].

[19] K. Kodaira, On compact analytic surfaces. II, III, Ann. of Math. (2) 77 (1963) 563–626, 78 (1963) 1–40.

[20] A. Néron, Modèles minimaux des variétés abéliennes sur les corps locaux et globaux, Inst. Hautes Études Sci. Publ. Math. 21 (1964) 5–128.

[21] J. Tate, Algorithm for determining the type of a singular fiber in an elliptic pencil, Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), Lecture Notes in Math., vol. 476, Springer, Berlin, 1975, pp. 33–52.

[22] R. Donagi and M. Wijnholt, Higgs bundles and UV completion in F-theory, arXiv:0904.1218 [hep-th].

[23] J. Marsano, N. Saulina, and S. Schafer-Nameki, Monodromies, fluxes, and compact three-generation F-theory GUTs, JHEP 08 (2009) 046, arXiv:0906.4672 [hep-th].

[24] O. J. Ganor, D. R. Morrison, and N. Seiberg, Branes, Calabi–Yau spaces, and toroidal compactification of the N=1 six-dimensional E8 theory, Nucl. Phys. B 487 (1997) 93–127, arXiv:hep-th/9610251.
[25] V. Sadov, *Generalized Green–Schwarz mechanism in F-theory*, Phys. Lett. B 388 (1996) 45–50, arXiv:hep-th/9606008.

[26] D. R. Morrison and W. Taylor, *Matter and singularities*, to appear.

[27] M. Auslander and D. A. Buchsbaum, *Unique factorization in regular local rings*, Proc. Nat. Acad. Sci. U.S.A. 45 (1959) 733–734.

[28] S. Katz and C. Vafa, *Matter from geometry*, Nucl. Phys. B 497 (1997) 146–154, arXiv:hep-th/9606086.

[29] A. Grassi and D. R. Morrison, *Anomalies and the Euler characteristic of elliptic Calabi–Yau threefolds*, to appear.

[30] P. S. Aspinwall, S. Katz, and D. R. Morrison, *Lie groups, Calabi–Yau threefolds, and F-theory*, Adv. Theor. Math. Phys. 4 (2000) 95–126, arXiv:hep-th/0002012.