Polyharmonic Daubechies type wavelets in Image Processing and Astronomy, I

Ognyan Kounchev, Damyan Kalaglarsky

Abstract: We introduce a new family of multivariate wavelets which are obtained by "polyharmonic subdivision". They generalize directly the original compactly supported Daubechies wavelets.

Key words: Wavelet Analysis, Daubechies wavelet, Image Processing.

1 Introduction

We consider new multivariate polyharmonic Daubechies type wavelets which are called "polyharmonic subdivision wavelets". They have been recently introduced in the paper [5]. They are obtained by means of a procedure called "polyharmonic subdivision" which is a generalization of the classical one-dimensional subdivision scheme of Deslauriers-Dubuc [4] which is the original source for the first compactly supported wavelets of Daubechies in 1988, cf. [3]. This new family of polyharmonic wavelets is the second representative of the Polyharmonic Wavelet Analysis following the "polyspline wavelets" which have been introduced in the monograph [6].

An important feature of these newly-born wavelets is that they are a nice generalization of the one-dimensional wavelets of Daubechies: they form an orthonormal family, enjoy nice non-stationary "refinement operator" equations, and have compact filters. In addition to that they have elongated supports. Let us remind that a major drawback of the one-dimensional spline wavelets of Ch. Chui is that they do not have finite filters, and respectively, the polyspline wavelets of [6] do not have finite filters.

2 Construction of fundamental function $\Phi_m$ for exponential polynomials subdivision

The whole construction of the Daubechies type wavelets passes via the construction of the so-called fundamental function of subdivision, cf. [1]. In the present case we will work with non-stationary subdivision and we have a family of such functions $\Phi_m$ for all $m \in \mathbb{Z}$ which satisfy the refinement equations (two-scale relations) given by

$$\Phi_m(t) = \sum_{i \in \mathbb{Z}} a_i^{[m]} \Phi_{m+1}(2t - i) \quad \text{for all } t \in \mathbb{R}. \tag{1}$$

We define the non-stationary subdivision symbol by putting

$$a^{[k]}(z) := \sum_{j \in \mathbb{Z}} a_j^{[k]} z^j. \tag{2}$$
We are interested in special subdivision processes arising through the solutions of Ordinary Differential Equations. We assume that we are given a number of frequencies $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p$ and put for the frequency vector (with repetitions)

$$\Lambda = \{ \lambda_1, \lambda_2, \ldots, \lambda_N \} \cup \{ -\lambda_1, -\lambda_2, \ldots, -\lambda_N \}.$$ 

We consider the space of $C^\infty$ solutions of the ODE

$$\prod_{j=1}^{p} \left( \frac{d^2}{dt^2} - \lambda_j^2 \right) f(t) = 0.$$  

(3)

Let us recall a simple fact from ODEs: in the case of different $\lambda_j$’s the space of all $C^\infty$ solutions in (3) is spanned by the set $\{ e^{\lambda_j t} : j = 1, 2, \ldots, p \}$. In the case of $s$ coinciding indices $\lambda_i = \lambda_{i+1} = \ldots = \lambda_{i+s-1}$ we have that the solution set contains the functions $\{ t^s e^{\lambda_i t} : \ell = 0, 1, \ldots, s \}$.

Let us proceed to the construction of the subdivision symbols. We put $x_j = e^{-\lambda_j/2^{k+1}}$.

We define the following Laurent polynomial

$$d(z) := d^k(z) := \prod_{j=1}^{N} \frac{(z + x_j)(z^{-1} + x_j)}{(1 + x_j)^2}$$

and

$$P(x) := P^k(x) := \prod_{j=1}^{N} \left( 1 - \frac{4x_j}{(1 + x_j)^2} x \right).$$  

(4)

They satisfy the equality

$$d(e^{i\omega}) = P\left( \sin^2 \frac{\omega}{2} \right) \quad \text{for all } \omega \in \mathbb{R};$$  

(5)

cf. [7]. We will often drop the dependence on the upper index in $d$, $a$, $P$ and the other functions and symbols.

An important step for construction of the subdivision coefficients $a_j^{[m]}$ is the application of the Bezout theorem:

**Proposition 1** There exists a unique polynomial $Q$ with real coefficients of degree $N - 1$ such that

$$P(x) Q(x) + P(1-x) Q(1-x) = 1$$

and

$$Q(x) > 0 \quad \text{for } x \in (0, 1).$$

We define now the trigonometric polynomial $b(z) = b^k(z)$ by putting

$$b(e^{i\omega}) = Q\left( \sin^2 \frac{\omega}{2} \right).$$

We finally define the symmetric Laurent polynomial $a(z)$ by putting

$$a(z) := a^k(z) := 2 d(z) b(z) \quad \text{for } z \in \mathbb{C} \setminus \{0\}.$$  

(6)

The following proposition is important for the application of the Riesz lemma to $a(z)$ and construction of the Wavelet Analysis, cf. [7], [5].
Proposition 2 The polynomial \( a(z) \) defined in (6) satisfies
\[
a(z) = \sum_{j=-2N+1}^{2N-1} a_j z^j
\]
with \( a_j = a_{-j} = \overline{a_j} \) and
\[
a(z) \geq 0 \quad \text{for all} \quad |z| = 1.
\]
The following fundamental result shows that the symbols \( a(z) \) are the non-stationary subdivision symbols for symmetric set of frequencies \( \Lambda \), cf. [5].

Theorem 3 For every exponential polynomial, i.e. for every solution to the equation
\[
Lf(t) := \prod_{j=1}^{N} \left( \frac{d^2}{dt^2} - \lambda_j^2 \right) f(t) = 0 \quad (7)
\]
we put
\[
f_k^j = f \left( \frac{j}{2^k} \right).
\]
Then \( f \) is reproduced by means of interpolatory subdivision, i.e.
\[
f_{k+1}^{j+1} = \sum_{j=-\infty}^{\infty} a(k)_{j-2} f_k^j \quad \text{for all} \quad j' \in \mathbb{Z}
\]
\[
f_{2j+1}^j = f_k^j \quad \text{for all} \quad j \in \mathbb{Z},
\]
For every \( m \in \mathbb{Z} \) the fundamental function of subdivision \( \Phi_m(t) \) is a continuous function obtained through the subdivision process [3], where one starts from \( f_0^j = \delta_j \) for \( j \in \mathbb{Z} \) (here \( \delta_j \) is the Kronecker symbol), i.e. we put \( \Phi_m(\frac{t}{2^m}) = \delta_j \), and \( \Phi_m \) satisfies the refinement equation [4].

Having in hand the functions \( \Phi_m \) and their refinement symbols \( a|m| \) we may follow the usual scheme for construction of father and mother wavelets which has been used by Daubechies, cf. [3], [1]. The following fundamental result has been proved in [5].

Theorem 4 There exists a polynomial \( g(z) = \sum_{j \in \mathbb{Z}} g_j z^j \) such that it is the "square root" of \( 2a(z) \), i.e.
\[
a(e^{i\theta}) = \frac{1}{2} \left| g(e^{i\theta}) \right|^2 \quad (9)
\]
For every \( m \in \mathbb{Z} \) there exists a compactly supported function \( \varphi_m(t) \) which satisfies the refinement equation
\[
\varphi_m(t) = \sum_{j} g_j \varphi_{m+1}(2t-j), \quad (10)
\]
and the family \( \{ \varphi_m(t-j) \}_{j \in \mathbb{Z}} \) is orthonormal. (These are the non-stationary father wavelets.) The functions
\[
\psi_m(t) = \sum_{j \in \mathbb{Z}} (-1)^j g_{1-j} \varphi_{m+1}(2t-j)
\]
are the mother wavelets; the family \( \{ \psi_m(t-j) \}_{j \in \mathbb{Z}} \) is orthonormal and the family \( \{ \psi_m(t-j) \}_{m,j \in \mathbb{Z}} \) forms an orthonormal basis of \( L^2(\mathbb{R}) \).
2.1 The polyharmonic case

For the polyharmonic subdivision we will work with very special ODEs defined by

\[ L_\xi := (d^2/\partial t^2 - \xi^2)^N \]

which are the Fourier transform of the polyharmonic operator \( \Delta^N \). For a fixed constant \( \xi \geq 0 \) we put

\[ \Lambda := (-\xi, -\xi, ..., -\xi, \xi, \xi, ..., \xi) \in \mathbb{R}^{2N} \]  

i.e. \( \lambda_j = \xi \), for \( j = 1, 2, ..., N \). Now for fixed \( \xi \geq 0 \) and \( k \in \mathbb{Z} \) we define the polynomial

\[ d(z) := d[k]_\xi(z) := \frac{(z + x_0)^N (z - x_0)^N}{(1 + x_0)^{2N}} \quad \text{for} \quad z \in \mathbb{C}; \]

(13)

here we put \( x_0 := e^{-\xi/2^{k+1}} \). For the sake of simplicity we will very often drop the dependence on \( k \) and \( \xi \).

By (5) we have

\[ d(e^{i\omega}) = P(\sin^2 \frac{\omega}{2}) \]

where

\[ P(x) = \left( 1 - \frac{4x_0}{(1 + x_0)^2} \right)^N = (1 - \eta x)^N, \]

(14)

and we have put

\[ \eta = \eta[k]_\xi := \frac{4x_0}{(1 + x_0)^2} = \frac{2}{1 + \cosh(\xi/2^{k+1})}. \]

Then following Proposition 1 we have to find the polynomial solution \( Q \) to the equation

\[ P(x) Q(x) + Q(1 - x) P(1 - x) = 1 \]

where \( Q \) has degree \( \leq N - 1 \).

**Remark 5** Let us recall that the polynomial \( Q \) in the classical case, cf. e.g. [1], p. 195, satisfies condition

\[ (1 - y)^N Q(y) + y^N Q(1 - y) = 1. \]

The lowest degree solution polynomial \( Q \) will be called Daubechies’ polynomial and we put

\[ R_N(x) := \sum_{j=0}^{N-1} \binom{N + j - 1}{j} y^j. \]

(15)

(Note that in [3] and [1] the notation used is \( P_N \) !)

It is amazing that it is possible to solve the problem in Proposition 1 explicitly.

**Proposition 6** Let \( \Lambda = (-\xi, -\xi, ..., -\xi, \xi, \xi, ..., \xi) \in \mathbb{R}^{2N} \). Then for the corresponding polynomial \( P(x) = (1 - \eta x)^N \), the polynomial \( Q \) of degree \( N - 1 \) defined by

\[ Q(x) = Q_N^{k, \xi}(x) = (2 - \eta)^{-N} \sum_{j=0}^{N-1} \binom{N + j - 1}{j} \left( 1 - \eta (1 - x) \right)^j \]

(16)

solves the equation

\[ P(x) Q(x) + P(1 - x) Q(1 - x) = 1. \]

Hence,

\[ Q(x) = (2 - \eta)^{-N} R_N \left( \frac{1 - \eta (1 - x)}{2 - \eta} \right). \]

(18)
Hence, we find the trigonometric polynomial \( b^{[k]}(z) \) by putting
\[
  b^{[k]}(z) := b^{[k]}(ξ(e^{iω})) := Q^{[k]}(ξ) \left( \sin^2 \frac{ω}{2} \right)
\]
where we recall the notations
\[
x = \sin^2 \frac{ω}{2} = \frac{1 - \cos ω}{2} = \frac{1}{2} - \frac{z + z^{-1}}{4},
\]
Finally, we obtain the subdivision symbol \( a^{[k]}(z) \) by putting
\[
  a^{[k]}(z) := a^{[k]}(ξ) (z) := 2d^{[k]}(ξ)(z) b^{[k]}(ξ)(z).
\]

Now by Theorem 4 we find the "square root" of the symbol \( a^{[k]}(z) \). This means that we have to take separately the "square root" of the Laurent polynomials \( d^{[k]}(z) \) and \( b^{[k]}(z) \). The "square root" of \( d^{[k]}(z) \) is obvious; taking the "square root" of \( b^{[k]}(z) \) needs taking the "square root" of the polynomial \( Q \).

3 Algorithm for finding the square root of the polynomials \( Q \)

For the algorithmic aspects of taking the "square root" of the polynomial \( Q \) it will be important to describe the polynomial \( Q \) through the zeros of the Daubechies' polynomial \( R_N \) in (15).

Proposition 7 Let the zeros of the Daubechies’ polynomial (15) be \( c^D_j \), i.e.
\[
  R_N(y) = \sum_{j=0}^{N-1} \binom{N + j - 1}{j} y^j = \frac{(2N-2)!}{((N-1)!)^2} \prod_{j=1}^{N-1} (y - c^D_j).
\]

Then the polynomial \( Q \) as determined by (16) is given by
\[
  Q(x) = (2 - η)^{-2N+1} η^{N-1} \frac{(2N-2)!}{((N-1)!)^2} \prod_{j=1}^{N-1} (x - C_j),
\]
where
\[
  C_j := \frac{c^D_j (2 - η) + η - 1}{η}.
\]

By formula (18) we have the representation
\[
  d^{[k]}(z) = \left| \frac{(z + x_0)^N}{(1 + x_0)^N} \right|^2 \text{ for } z = e^{iω},
\]

hence, we take the trigonometric polynomial
\[
  M_1(z) := \frac{(z + x_0)^N}{(1 + x_0)^N}
\]
as its "square root", i.e. \( d^{[k]}(z) = |M_1(z)|^2 \) for \( |z| = 1 \). Further, we have to take care of the "square root" of the polynomial \( b^{[k]}(z) \). Thus we have to find the polynomial \( M_2 \) of degree \( ≤ N - 1 \) such that
\[
  |M_2(e^{iω})|^2 = \frac{1}{2} Q \left( \sin^2 \frac{ω}{2} \right),
\]
which may be obtained by using the roots of the Daubechies polynomials.
Remark 8 Let the polynomial $Q$ have the zeros $C_j$ as in Proposition \[7\] and let us put

$$c_j = 1 - 2C_j.$$ 

We see that $Q \left( \sin^2 \frac{\omega}{2} \right) = \tilde{Q} \left( \cos \omega \right)$ for some polynomial $\tilde{Q}$ and $c_j$ are the zeros of $\tilde{Q}$. Hence, we may apply the algorithm for the Riesz representation of $\tilde{Q}$, see e.g. \[7\], p. 197 – 198.

Thus we obtain finally for every integer $m \geq 0$ and $\xi \in \mathbb{Z}^n$ the representation

$$a^{[m],\xi}(z) = \frac{1}{2} |M_1(z) M_2(z)|^2,$$

(23)

and the family of functions

$$M(z) := M^{[m]}(z) := M^{[m],\xi}(z) := M_1(z) M_2(z)$$

(24)

represents the refinement masks for the family of scaling functions (father wavelets) \{$\phi_m(t)\}_{m\geq 0}$ for which the functions $\Phi_m$ are autocorrelation functions.

Remark 9 Note that the above factorization has been found in the special case $\xi = 0$ by Daubechies in \[3\], p. 266; the coefficients of the "square root" polynomial for $N = 2..10$ are in table 6.1 in \[3\]. A detailed discussion of more efficient methods for choosing the proper polynomial $M_2(z)$ is available in Strang-Nguyen \[8\], p. 157, in chapter 5.4 on Spectral factorization. The factorization of the Daubechies’ polynomial $R_N(y)$ is discussed in Burrus \[2\], on p. 78 and the Matlab program is $[hn,hin]=\text{daub}(N)$ in Appendix C. They work with the zeros of the polynomial $R_N$ and provide a number of manipulations for finding a more stable factorization.

Acknowledgement. The first named author was sponsored partially by the Alexander von Humboldt Foundation, and both authors were sponsored by Project DO–2-275/2008 "Astroinformatics" with Bulgarian NSF.

References

[1] Ch. Blatter, Wavelets: A Primer, A K Peters, Natick, MA, 1998.
[2] S. Burrus, R. Gopinath, H. Guo, Introduction to Wavelets and Wavelet Transforms, Prentice Hall, Englewood Cliffs, N.J., 1998.
[3] I. Daubechies, Ten lectures on wavelets, SIAM, 2002.
[4] G. Deslauriers, S. Dubuc, Symmetric iterative interpolation process, Constr. Approx., 5 (1989), 49-68.
[5] N. Dyn, O. Kounchev, D. Levin, H. Render, Polyharmonic subdivision for CAGD and multivariate Daubechies type wavelets, preprint, 2010.
[6] O. Kounchev, Multivariate polysplines: Applications to Numerical and Wavelet Analysis, Academic Press, San Diego-London, 2001.
[7] Ch. Micchelli, Interpolatory Subdivision schemes and wavelets, Jour. Approx. Theory, 86 (1996), p. 41-71.
[8] G. Strang, T. Nguyen, Wavelets and Filter Banks, Wellesley-Cambridge Press, 1996.

ABOUT THE AUTHORS

Ognyan Kounchev, Prof., Dr., Institute of Mathematics and Informatics, Bulgarian Academy of Science, tel. +359 − 2 − 9793851; kounchev@gmx.de

Damyan Kalaglarsky, Institute of Astronomy, Bulgarian Academy of Science, tel. +359 − 2 − 9793851; damyan@skyarchive.org.