THE ERDŐS-SELFRIDGE AND THE SCHINZEL-TIJDEMAN THEOREMS HOLD IN $PA^-$

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Abstract. We show that “The product of consecutive integers is never a power” and several results by Schinzel and Tijdeman on the solutions of the equation $y^m = P(x)$, for $m > 1$, $y > 1$, and $P(x)$ a polynomial with rational coefficients and with at least two distinct zeros, hold in a weak fragment of Peano Arithmetic, $PA^-$, which lacks any kind of induction, and whose models are the positive cones of discretely ordered rings.

1. Introduction

In [1], P. Erdős and J. L. Selfridge settled a 150 year old conjecture proving that

\[(n + 1) \cdots (n + k) = x^l\]

has no solution in integers with $k \geq 2$, $l \geq 2$, and $n \geq 0$.

One year later, A. Schinzel and R. Tijdeman [5] have proved a series of, in a sense, more general statements, such as: If a polynomial $P(x)$ with rational coefficients has at least two simple zeros, then the equation

\[P(x) = y^m\]

has only finitely many integer solutions $m, x, y$, with $x \geq 0$, $m > 2$, $y > 1$, and these solutions can be found effectively.

The purpose of this note is to show that, when the Erdős-Selfridge theorem and the Schinzel-Tijdeman results are formalized without the use of the exponential function, they are true in a theory significantly weaker than Peano Arithmetic ($PA^{-}$), in which there is no form of an induction axiom.

In the case of Erdős-Selfridge theorem, this means that it is thought of as the collection of all statements (where, for $u \geq 1$, we denote by $\overline{u}$ the term $(\ldots((1 + 1) + 1)+\ldots)+1$, in which there are $u$ many 1s, and we let $\overline{0}$ be 0 itself; the terms $\overline{n}$ will be referred to as numerals)

\[(\Phi_{k,t}) (\forall n)(\forall x) (n + 1) \cdots (n + k) \neq x^t.\]

Each statement $\Phi_{k,t}$, for $k \geq 2$ and $t \geq 2$ holds in $PA^{-}$.

Very few interesting number-theoretic results are known to hold without any form of induction (see [3]). That results as deep as the Erdős-Selfridge theorem and the Schinzel-Tijdeman theorem and its corollaries [5] are among them comes as a surprise.
2. The axioms of PA−

PA−, a theory expressed in a language with +, ·, 1, 0, < as primitive notions, has an axiom system consisting of 15 axioms. We will reproduce them here from [2, pp. 16-18] for the reader’s convenience, and we will omit the universal quantifiers for all universal axioms.

A 1. \((x + y) + z = x + (y + z)\)
A 2. \(x + y = y + x\)
A 3. \((x \cdot y) \cdot z = x \cdot (y \cdot z)\)
A 4. \(x \cdot y = y \cdot x\)
A 5. \(x \cdot (y + z) = x \cdot y + x \cdot z\)
A 6. \(x + 0 = x \land x \cdot 0 = 0\)
A 7. \(x \cdot 1 = x\)
A 8. \((x < y \land y < z) \rightarrow x < z\)
A 9. \(\neg x < x\)
A 10. \(x < y \lor x = y \lor y < x\)
A 11. \(x < y \rightarrow x + z < y + z\)
A 12. \((0 < z \land x < y) \rightarrow x \cdot z < y \cdot z\)
A 13. \((\forall x)(\forall y)(\exists z) x < y \rightarrow x + z = y\)
A 14. \(0 < 1 \land (x > 0 \rightarrow (x > 1 \lor x = 1))\)
A 15. \(x > 0 \lor x = 0\)

Let PA− be the theory axiomatized by A1-A15.

3. The main lemmas

Let \(\mathcal{M}\) be an arbitrary model of PA−. We know that \(\mathcal{M}\) must contain a copy of \(\mathbb{N}\), and it may contain other elements as well, called nonstandard numbers. If \(\mathcal{M}\) is \(\mathbb{N}\) then the result is true by the Erdős-Selfridge theorem. Suppose \(\mathcal{M}\) does contain nonstandard numbers. We want to show that \(\mathbb{N}\) cannot have solutions with \(n\) (and thus \(x\)) nonstandard. Let \(R_{\mathcal{M}}\) denote the ordered ring with \(\mathcal{M}\) as its positive cone, let \(Q_{2\mathcal{M}}\) denote its field of fractions, and let \(\overline{Q}_{2\mathcal{M}}\) denote the real closure of \(Q_{2\mathcal{M}}\) (whose existence does not require AC, as shown in [3]; we thus denote by \(\overline{Q}\) the field of real algebraic numbers). Let \(F_{2\mathcal{M}}\) denote the ring of finite elements of \(\overline{Q}_{2\mathcal{M}}\), i.e. \(F_{2\mathcal{M}} = \{ x \in \overline{Q}_{2\mathcal{M}} \mid (\exists n \in \mathbb{N}) |x| < n \}\), and let \(P_{2\mathcal{M}}\) denote the ideal of infinitely small elements of \(\overline{Q}_{2\mathcal{M}}\), i.e. \(P_{2\mathcal{M}} = \{ x \in \overline{Q}_{2\mathcal{M}} \mid (\forall n \in \mathbb{N} \setminus \{0\}) |x| < \frac{1}{n} \}\). We write \(a \ll b\), for \(a, b \in \overline{Q}_{2\mathcal{M}}\), whenever \(\frac{a}{b} \in P_{2\mathcal{M}}\).

Lemma 3.1. If \(n \in \mathcal{M} \setminus \mathbb{N}\), \(x < n^\alpha\), with \(\alpha \in \mathbb{Q}\), and

\[
\sum_{i=1}^{m} d_i x^n^i = \sum_{i=0}^{s} c_i n^i,
\]
with $c_i, d_i \in \mathbb{Q}$, $d_1 \neq 0$, $c_\alpha \neq 0$, with $\beta_1 \geq \alpha(i-1) + \beta_i$, for all $i \geq 2$, then

$$\tag{5} x = \frac{c_0}{d_1} n^{s-\beta_1} + y,$$

with $y \ll n^{s-\beta_1}$.

**Proof.** By factoring $x n^{\beta_1}$ on the left hand side of (4), and $n^s$ on its right hand side, we get

$$\tag{6} x n^{\beta_1} \left( d_1 + \sum_{i=2}^{m} d_i x^{i-1} n^{\beta_i - \beta_1} \right) = n^s (c_\alpha + \sum_{i=0}^{s-1} c_i n^{i-s}).$$

Notice that each term in $\sum_{i=2}^{m} d_i x^{i-1} n^{\beta_i - \beta_1}$ is infinitely small, for $x^{i-1} \ll n^{\alpha(i-1)}$, thus $d_i x^{i-1} n^{\beta_i - \beta_1} \ll n^{\alpha(i-1) + (\beta_i - \beta_1)}$, so, since $\alpha(i-1) + (\beta_i - \beta_1) \leq 0$, we have $d_i x^{i-1} n^{\beta_i - \beta_1} \ll 1$, so $\sum_{i=2}^{m} d_i x^{i-1} n^{\beta_i - \beta_1} \ll 1$, which is another way of stating that $\sum_{i=2}^{m} d_i x^{i-1} n^{\beta_i - \beta_1} \in P \mathfrak{M}$. Let $\epsilon = \sum_{i=2}^{m} d_i x^{i-1} n^{\beta_i - \beta_1}$. That $\sum_{i=1}^{s-1} c_i n^{i-s}$ is infinitely small is plain, given that $n \in \mathfrak{M} \setminus \mathbb{N}$. Let $\epsilon' = \sum_{i=0}^{s-1} c_i n^{i-s}$. Thus (6) can be rewritten as $x n^{\beta_1} (d_1 + \epsilon) = n^s (c_\alpha + \epsilon')$, with $\epsilon, \epsilon' \in P \mathfrak{M}$, i.e.

$$x = n^{s-\beta_1} c_\alpha + \epsilon' + \epsilon,$$

and, since $\frac{c_\alpha + \epsilon'}{d_1 + \epsilon} = \frac{c_\alpha + \epsilon'}{c_\alpha + \epsilon} \cdot \frac{d_1}{d_1 + \epsilon} = \frac{c_\alpha}{d_1} (1 + \epsilon''),$ where $\epsilon'' \in P \mathfrak{M}$, this proves (5) holds.

We now turn to the proof of

**Lemma 3.2.** If $n \in \mathfrak{M} \setminus \mathbb{N}$, $P(x) = \sum_{i=0}^{k} c_i x^i$ is a polynomial of degree $k \geq 2$ with non-negative rational coefficients $c_i$, with $c_0 > 0$, and, for some $x \in \mathfrak{M}$,

$$\tag{7} P(n) = x^l,$$

then

$$\tag{8} P(n) = \sum_{i=1}^{l_0} c_i n^{\alpha_i} l,$$

with $\alpha_i \in \mathbb{Q}$, $c_i \in \mathbb{Q}$, $c_i \neq 0 \alpha_i > \alpha_{i+1}$ for all $i \leq i_0 - 1$, $\alpha_{i_0} \leq 0$.

**Proof.** Let us assume now that the equation (7) holds in $\mathfrak{M}$, with $n$, and thus $x$, non-standard. By (7), we must have $x^l > c_k n^k$. We thus have $x > c_k n^k$, where by $u$ we have denoted $\frac{u}{n}$ (with $u$ being an element of $\mathbb{Q} \mathfrak{M}$, the inequality $x > c_k n^k$ holds in $\mathbb{Q} \mathfrak{M}$). Thus there exists some $x_1 \in \mathbb{Q} \mathfrak{M}$, $x_1 > 0$, such that

$$\tag{9} x = c_k n^k + x_1.$$

Raising both sides to the power $l$, and using (7), we get

$$\tag{10} x^l = c_k n^{kl} + \sum_{i=1}^{l} \binom{l}{i} (c_k n^{kl})^{l-i} x_1^i.$$
Let \( x_1 = \lambda n^u \). Then \( x^l = c_k n^k + \lambda \alpha_x n^k + \sum_{i=2}^{l-1} \left( \binom{l-1}{i} \right) c_k n^{i-1} x_1^i \), so \( x^l > c_k n^k + \lambda \alpha_x n^k \), and thus, using (7), \( c_k n^k + \sum_{i=0}^{k-1} c_i n^i > c_k n^k + \lambda \alpha_x n^k \), so, with \( \varphi(k-1) = \max \{ i : i \leq k - 1, c_i \neq 0 \} \)

\[
\begin{align*}
(11) \\
n^\varphi(k-1) (c_{\varphi(k-1)} + \sum_{i=0}^{\varphi(k-1)-1} c_i n^{i-\varphi(k-1)}) > \lambda \alpha_x n^k,
\end{align*}
\]

where \( \sum_{i=0}^{\varphi(k-1)-1} c_i n^{i-\varphi(k-1)} \) is 0 in case \( \varphi(k-1) = 0 \).

With \( \epsilon_1 = \sum_{i=0}^{\varphi(k-1)-1} c_i n^{i-\varphi(k-1)} \), we have \( \epsilon_1 \in \mathbb{P}_{2n} \), and (11) becomes

\[
(12) \quad c_{\varphi(k-1)} + \epsilon_1 > \lambda \alpha_x n,
\]

so

\[
(13) \quad \lambda < \frac{c_{\varphi(k-1)} + \epsilon_1}{\epsilon_x n},
\]

thus \( \lambda \in \mathbb{P}_{2n} \). We thus have \( x_1 \ll n^u \) in (9). Setting the left hand side of (7) equal to the right hand side of (10), canceling \( c_k n^k \) from both sides, we get

\[
(14) \quad \sum_{i=0}^{\varphi(k-1)} c_i n^i = \sum_{i=1}^{l} \left( \binom{l-1}{i} \right) c_k n^{i-1} x_1^i,
\]

and bearing in mind that \( x_1 \ll n^u \), we notice that the conditions of Lemma 3.1 are satisfied, and thus that

\[
(15) \quad x_1 = \frac{c_{\varphi(k-1)}}{\epsilon_x n^{\varphi(k-1)-u(l-1)}} n^{\varphi(k-1)-u(l-1)} + x_2,
\]

with \( x_2 \ll n^{\varphi(k-1)-u(l-1)} \). Plugging in the right hand side of (15) for \( x_1 \) in (14) we will be able to cancel the terms with the highest power of \( n \) (in this case \( n^{\varphi(k-1)} \)) from both sides, and the remaining equation satisfies once more the conditions of Lemma 3.1 and we continue this way to obtain a sequence \( x_1, x_2, \ldots \) with at most \( k \) terms, such that at every step we get an equation of the form \( x_i = e_1 n^{\alpha_1} + x_{i+1} \), with \( \alpha_1 \in \mathbb{Q} \) and \( e_1 \in \mathbb{Q} \), and \( x_{i+1} \ll n^{\alpha_1} \), and at least one term on the left hand side of (14) cancels out with its counterpart on the right hand side, so there is an \( i_0 \leq k \), such that, when plugging in \( x_{i_0} = e_{i_0} n^{\alpha_{i_0}} + x_{i_0+1} \) for \( x_{i_0} \) in what remained at that stage of (14), and canceling the terms equal to one of the terms on the left hand side of (14), there is nothing left on the left hand side. Notice that \( \alpha_{i_0} \) must be \( \leq 0 \), for if \( \alpha_{i_0} > 0 \), then the free term on the left hand side of (7), \( c_0 \), could not have canceled out with a term on the right hand side (as a sum of positive powers of \( n \), multiplied by real algebraic coefficients, can never be equal to a non-zero rational number, given that such a sum is either infinitely large or 0).

Combining (9), (15), and all \( x_i = e_i n^{\alpha_i} + x_{i+1} \), for \( i \) up to and including \( i_0 \), and plugging in the value of \( x \) thus obtained in (7), we get
(16) \[ \sum_{i=0}^{k} c_i n^i = \left( \sum_{i=1}^{i_0} e_i n^{\alpha_i} + x_{i_0+1} \right)^l, \]

with \( \alpha_i \in \mathbb{Q}, e_i \in \mathbb{Q}, e_i \neq 0, \alpha_i > \alpha_{i+1} \) for all \( i \leq i_0 - 1, \alpha_{i_0} \leq 0, x_{i_0+1} \ll n^{\alpha_{i_0}} \).

We also know that, after multiplying out the right hand side of (16) and canceling equal terms on both sides, we are left with 0 on the left hand side, and on the right hand side with

(17) \[ \sum_{i=1}^{s} t_i n^{\beta_i} + \sum_{i=1}^{l-1} n^{\gamma_i} x_{i_0+1} + x_{i_0+1}^l, \]

where every term in \( \sum_{i=1}^{s} t_i n^{\beta_i} \) and \( \sum_{i=1}^{l-1} n^{\gamma_i} x_{i_0+1} \) — by being a product of \( l \) many factors, at least one of which is \( \gg x_{i_0+1} \), and all of which are \( \geq x_{i_0+1} \) — is \( \gg x_{i_0+1}^l \), so, since we have

(18) \[ x_{i_0+1}^l = - \left( \sum_{i=1}^{s} t_i n^{\beta_i} + \sum_{i=1}^{l-1} n^{\gamma_i} x_{i_0+1} \right), \]

and, if \( -(\sum_{i=1}^{s} t_i n^{\beta_i} + \sum_{i=1}^{l-1} n^{\gamma_i} x_{i_0+1}) \neq 0 \), we have \( x_{i_0+1} \ll -(\sum_{i=1}^{s} t_i n^{\beta_i} + \sum_{i=1}^{l-1} n^{\gamma_i} x_{i_0+1}) \), we must have \( x_{i_0+1} = 0 \). With this (16) becomes (8) with \( \alpha_i \in \mathbb{Q}, e_i \in \mathbb{Q}, e_i \neq 0, \alpha_i > \alpha_{i+1} \) for all \( i \leq i_0 - 1, \alpha_{i_0} \leq 0 \).

**Remark on Lemma 3.2** If \( c_k = 1 \) in Lemma 3.2, then the coefficients \( e_i \) in (8) are all in \( \mathbb{Q} \). This can be easily seen by following the proof of Lemma 3.2 with \( c_k = 1 \) and by noticing that there is no more \( l \)-th roots of the coefficient \( c_k \) involved, and all coefficients turn out to be fractions with rational numerators and denominators.

4. The Erdős-Selfridge theorem holds in \( \text{PA}^- \)

Our aim is to show that

**Theorem 4.1.** \( \text{PA}^- \vdash \Phi_{k,l} \) for all \( k, l \) with \( k \geq 2, l \geq 2 \).

**Proof.** Let \( \mathfrak{M} \) be an arbitrary model of \( \text{PA}^- \). By Lemma 3.2 we must have (8) holding for \( P(n) = (n+1) \cdot \ldots \cdot (n+k) \). We now look at the possible values of \( \alpha_{i_0} \) occurring in (8).

If \( \alpha_{i_0} < 0 \), then, if we multiplied both sides of (8) by \( n^{-i_0} \), we would have on the left hand side a sum of positive powers of \( n \), whereas on the right hand side a sum of terms consisting, with one exception, of rational coefficients multiplied with positive powers of \( n \) (that is, summands of the form \( cn^\gamma \)), and one term being 1. If we move all the terms \( cn^\gamma \) from the right hand side to the left hand side, we have an equation in which the left hand side is a sum of positive powers of \( n \) followed by a sum of terms of the form \( -cn^\gamma \), with \( \gamma > 0 \), and on the right hand side a 1. The left hand side is either infinitely large (i.e., in \( \mathfrak{M} \setminus \mathbb{N} \)) or is 0, so it cannot be 1.

If \( \alpha_{i_0} = 0 \), then the only element of \( F_{\mathfrak{M}} \) on the right hand side of (8) is \( c_{i_0}^l \), all the other terms being products of rational numbers and positive powers of \( n \). Thus, if we get the numerals and the positive powers of \( n \) from (8) together, we get
where the $e_i \in \mathbb{Q}$ (by the remark on Lemma 3.2) and the $\gamma_i > 0$. Since the right hand side is either in $2\mathbb{R} \setminus \mathbb{N}$ or is 0, we must have

$$k! = e_i^l$$

with $e_i^l \in \mathbb{Q}$. Since this means that $e_i^l \in \mathbb{N}$, and we know (by the Erdős-Selfridge theorem with $n = 0$ and $k \geq 2$, $l \geq 2$, or by a simple application of Cebyshev’s Theorem, stating that there is a prime between $n$ and $2n$, that leads us to the conclusion that the greatest prime $\leq k$ occurs in $k!$ at the power 1) that (1) cannot have solutions, it follows that (20) has no solutions, and so (8), and with it (16), has no solutions. □

5. The Schinzel-Tijdeman theorem and corollaries hold in PA$^-$

We will now prove that PA$^-$ proves the main theorem and the two corollaries of the paper by A. Schinzel and R. Tijdeman [5], namely

$T_P$. If a polynomial $P(x)$ with rational coefficients has at least two distinct zeros, then there is an effectively computable constant $c(P)$, such that for all natural numbers $m \geq c(P)$, the equation

$$y^m = P(x), \ x, y \text{ non-negative integers, } y > 1$$

has no solutions.

$C1_P$. If a polynomial $P(x)$ with rational coefficients has at least two simple zeros, then the equation (21) has only finitely many integer solutions $m, x, y$, with $x \geq 0$, $m > 2$, $y > 1$, and these solutions can be found effectively.

$C2_P$. If a polynomial $P(x)$ with rational coefficients has at least three simple zeros, then the equation (21) has only finitely many integer solutions $m, x, y$, with $x \geq 0$, $m > 1$, $y > 1$, and these solutions can be found effectively.

By this we mean that, for any fixed polynomial $P(x)$ satisfying the conditions of $T_P$, there is a positive integer $c(P)$ depending only on $P$, such that for any natural number $m \geq c(P)$ we have

$$PA^- \vdash y > 1 \rightarrow y^m \neq P(x).$$

Also, in case $P(x)$ is a polynomial satisfying the conditions of $C1_P$ and $m > 2$, or of $C2_P$ and $m > 1$, then there is a computable finite set $M$ (possibly empty) of positive natural numbers (all greater than 2, respectively greater than 1), as well as a computable set $S$ of solutions $(x, y)$ of (21) and we have for every $m \in M$,

$$PA^- \vdash (y^m = P(x) \land y > 1) \rightarrow (\bigvee_{(a, b) \in S} (x = a \land y = b)),$$

and, for every $m \notin M$ (if $M = \emptyset$, then for all natural numbers $m > 1$, respectively $m > 2$),
(24) \[ PA^- \vdash y > 1 \rightarrow y^m \neq P(x). \]

It is worth mentioning that in \[22\], \[23\], and \[24\], since fractions do not exist in \(PA^-\), the formal statements consist of \(dy^m = dP(x)\) respectively \(dy^m \neq dP(x)\) (rather than \(y^m = P(x)\) respectively \(y^m \neq P(x)\)), where \(d\) is the lowest common denominator of all coefficients of \(P(x)\).

To see that \[22\], \[23\], \[24\] hold, notice that, in the case of \(T_P\), all we have to show is that \[24\] has no nonstandard solution. To show this, there is no loss in generality in assuming that \(P(x)\) has non-negative coefficients and that its free term is positive. For, suppose \(P(x) = \sum_{i=0}^{k} c_i x^i\) is a polynomial with rational coefficients, whose leading coefficient is positive (which it certainly needs to be if \(y > 1\)). Let \(N\) denote its greatest positive integer zero, if it has any such zero, and \(N = 0\), otherwise. Let \(I\) be the set of indices \(i\) in \(\{1, \ldots, k\}\) for which \(c_i < 0\), where the \(c_i\) denote the coefficients of \(P(x)\). \(C = -d \sum_{i \in I} c_i\) (in case \(I = \emptyset\) we set \(C = 0\)), where \(d\) denotes the least common denominator of the coefficients \(c_0, \ldots, c_k\), and let \(N_1 = \max\{N + 1, -C + 1\}\). Then \(R(x) = P(x + N_1)\) is a polynomial all of whose coefficients are non-negative, with positive free term, and to any nonstandard solution \((x, y)\) of \[2\] there corresponds a nonstandard solution \((x + N_1, y)\) of \[2\] with \(R\) instead of \(P\), so if \[2\] with \(R\) instead of \(P\) has no nonstandard solution, \[2\] cannot have a nonstandard solution either.

We assume from now on that \(P(x)\) has no negative coefficients and that its free term is positive.

Suppose \(n \in \mathbb{N} \setminus \mathbb{N}\) were a solution of \[2\]. By Lemma 3.2 we get \[5\]. We can thus conclude that our \(n\) would have to satisfy

(25) \[ P(n) = (\sum_{i=1}^{i_0} e_i n^{\alpha_i})^m, \]

with \(e_i \in \mathbb{Q}, e_i \neq 0, \alpha_i = \frac{m_i}{m},\) where \(m_i \in \mathbb{Z}, \alpha_i > \alpha_{i+1}\) for all \(i \leq i_0 - 1, \alpha_{i_0} \leq 0\).

Let \(x_1, x_2\) be two distinct zeros of \(P\) in the algebraic closure of \(Q_{\mathbb{R}}\). One of them, say \(x_1\), must be different from zero. Let \(X = n^\frac{1}{m}\). Then \[26\] becomes

(26) \[ P(X^m) = (\sum_{i=1}^{i_0} e_i X^{m_i})^m, \]

and, since any set of different powers of an infinitely large \(X \in Q_{\mathbb{R}}\) is linearly independent over \(\mathbb{Q}\), the equation \[26\] must be a polynomial identity in \(X\). Let \(z_1, \ldots, z_m\) be the different solutions of \(X^m = x_1\). Since \(x_1\) is a zero of \(P(x)\), we have \(P(X^m) = (X^m - x_1)Q(X^m) = (X - z_1) \ldots (X - z_m)Q(X^m)\), for some polynomial \(Q(x)\) of degree at least 1. Since \[26\] is a polynomial identity, we must have the factors \((X - z_1), \ldots, (X - z_m)\) on its right hand side as well, i.e.

(27) \[ (X - z_1) \ldots (X - z_m)Q(X^m) = ((X - z_1) \ldots (X - z_m)R(X))^m, \]
where \( R(x) \) is a polynomial of degree at least 1. Now the degree on the left hand side of (27) is \((\deg P)m\) whereas the degree on the right hand side of (27) is \(\geq m^2 + m\), thus \((\deg P)m \geq m^2 + m\), so \(\deg P \geq m + 1\), i.e. \(m \leq \deg P - 1\). If we now define \(c'(P) = \max\{c(P), \deg P - 1\}\), where \(c(P)\) is the constant from the Schinzel-Tijdeman theorem in \(\mathbb{N}\), we notice that Theorem 22 holds with this value \(c'(P)\) of the constant instead of \(c(P)\). This proves \(T_P\).

To see that \(C_1^P\) and \(C_2^P\) hold in \(PA^-\), notice that the existence of one simple zero \(x_1 \neq 0\) of \(P(x)\) renders the identity (27) impossible for \(m > 1\), as \(Q(x)\) is not allowed to have \(x_1\) as a zero, and thus, for some \(i \in \{1, \ldots, m\}\), \(X - z_i\) does not divide \(Q(X^m)\).

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