CLASSIFICATION OF SOLUTIONS TO EQUATIONS INVOLVING HIGHER-ORDER FRACTIONAL LAPLACIAN

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Abstract. In this paper, we are concerned with the following equation involving higher-order fractional Laplacian
\[
\begin{cases}
(-\Delta)^{p+\frac{\alpha}{2}} u(x) = u_+^\gamma, & x \in \mathbb{R}^n, \\
\int_{\mathbb{R}^n} u_+^\gamma dx < +\infty,
\end{cases}
\]
where \(p \geq 1\) is an integer, \(0 < \alpha < 2\), \(n > 2p + \alpha\) and \(\gamma \in \left(1, \frac{n}{n-2p-\alpha}\right)\). We establish an integral representation formula for any nonconstant classical solution satisfying certain growth at infinity. From this we prove that these solutions are radially symmetric about some point in \(\mathbb{R}^n\) and monotone decreasing in the radial direction via method of moving planes in integral forms.

1. Introduction

In this paper, we study the following higher-order fractional equation
\[
\begin{cases}
(-\Delta)^{p+\frac{\alpha}{2}} u(x) = u_+^\gamma, & x \in \mathbb{R}^n, \\
\int_{\mathbb{R}^n} u_+^\gamma dx < +\infty,
\end{cases}
\]
where \(\gamma \in \left(1, \frac{n}{n-2p-\alpha}\right)\), \(n > 2p + \alpha\), \(0 < \alpha < 2\), \(p \geq 1\) is an integer, \(u_+ = \max\{u, 0\}\) and the higher-order fractional Laplacian is defined by \((-\Delta)^{p+\frac{\alpha}{2}} := (-\Delta)^p (-\Delta)^{\frac{\alpha}{2}}\). It is well-known that the integral form of fractional Laplacian \((-\Delta)^{\frac{\alpha}{2}}\) \((0 < \alpha < 2)\) is
\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = C_{n,\alpha} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+\alpha}} dy,
\]
where the constant \(C_{n,\alpha} = \frac{2^\alpha \Gamma(n/2)}{\pi^{n/2} \Gamma(-\frac{\alpha}{2})}\).

Denote
\[
\mathcal{L}_\alpha(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < +\infty \right\}.
\]
It is known that \((-\Delta)^{\frac{\alpha}{2}} u\) is well defined for \(u \in C_{\text{loc}}^{[\alpha], [\alpha] + \varepsilon} \cap \mathcal{L}_\alpha(\mathbb{R}^n)\), where \([\alpha]\) denotes the integer part of \(\alpha\), \(\{\alpha\} := \alpha - [\alpha]\) and any \(\varepsilon > 0\). In order to guarantee that \((-\Delta)^{\frac{\alpha}{2}} u \in C^{2p}(\mathbb{R}^n)\), we have to assume \(u \in C_{\text{loc}}^{2p + [\alpha], [\alpha] + \varepsilon} \cap \mathcal{L}_\alpha(\mathbb{R}^n)\) (see \([14], [19]\)\), and hence \(u\) is a classical solution of the equation in \((1)\) in the sense that \((-\Delta)^{p+\frac{\alpha}{2}} u\) is point-wise well-defined in the whole \(\mathbb{R}^n\).
For decades, many researchers are interested in the classification of solutions to semi-linear elliptic equations. We now recall some relevant results. The following conformally invariant equations

\[( -\Delta )^\beta u = u^{n+\alpha} \quad \text{in } \mathbb{R}^n, \quad 0 < \beta < n \]

have been extensively studied (see [2, 3, 7, 10, 12, 13, 18] and the reference therein). In case \( \beta = 2 \), [2] becomes the well-known Yamabe equation. Cafferelli, Gidas and Spruck [2] (see also [6]) classified all positive solutions of Yamabe equation. For general \( 0 < \beta < n \), Chen, Li and Ou in [8] (see also [18]) classified all positive \( L^{\infty}_{loc} \) solutions to the equivalent integral equation of [2] by the method of moving planes in integral forms. For \( 0 < \beta < 2 \), Chen, Li and Li in [7] developed a direct method of moving planes for fractional Laplace equation and complete the classification of all nonnegative solutions. When \( \beta = 3 \) is an odd integer, Dai and Qin in [10] derived the classification of nonnegative classical solutions to (2) with the assumption \( \int_{\mathbb{R}^n} u^{\frac{n+\alpha}{n-3}} \, dx < +\infty \). Recently, Cao, Dai and Qin in [3] extended the result of [10] to general case \( 0 < \beta < n \). Their classification results completely improved the results in [10] without assumption on integrability. Precisely they proved the super poly-harmonic properties for nonnegative solutions by making full use of the the Poisson representation formula for \( (-\Delta)^\beta (0 < \alpha < 2) \) and developing some new integral estimates on the outer-spherical average \( \int_{R^n} \frac{K^\gamma}{r(r^2-R^2)^\gamma} \, dr \) and iteration techniques. Based on super poly-harmonic properties, they established integral representation formula, Liouville’s Theorem and classification of classical solutions to higher-order equations involving fractional Laplacian.

For [1], classification of nonconstant solutions have been established for some particular \( p, \alpha \) and \( \gamma \). For \( p = 1, \alpha = 0 \) and \( \gamma = \frac{n}{n-2} \), Wang and Ye [17] classified all nonconstant solutions of [1]. Suzuki and Takahashi [15] extended the results of [17] from the exponent \( \frac{n}{n-2} \) to more general exponent. Precisely, they considered the problem

\[-\Delta v = v^\gamma_+ \quad \text{in } \mathbb{R}^n, \quad n > 2, \quad \int_{\mathbb{R}^n} v_+^{\frac{n(\gamma-1)}{n}} \, dx < +\infty, \]

where \( \gamma \in (1, \frac{n+2}{n-2}) \). For the case \( p = 2, \alpha = 0 \) and \( \gamma \in (1, \frac{n}{n-3}] \), Chammakhi, Harrabi and Selmi [4] completed the classification of all sign-changing solutions of [1]. The first, second and fourth authors [11] extended the result of [4] to the corresponding polyharmonic equation, namely for the case of any integer \( p \) larger than 1. Particularly for the case \( p = 2 \), they obtain the same results for more general exponent \( \gamma \in (1, \frac{n+2}{n-4}) \). They also considered nonconstant solutions of the following fractional equation (namely \( p = 0 \) corresponds to the equation in [1])

\[-\Delta^\gamma_+ v = v^\gamma_+ \quad \text{in } \mathbb{R}^n, \quad \alpha \in (0, 2), \quad n > 2, \quad \int_{\mathbb{R}^n} v_+^{\frac{n(\gamma-1)}{n}} \, dx < +\infty, \]

where \( \gamma \in (1, \frac{n+\alpha}{n-\alpha}) \), and obtained the corresponding classification results.

To classify nonconstant solutions of [1] for general \( 0 < \alpha < 2 \), \( p \geq 1 \) and \( \gamma \in (1, \frac{n-2p-\alpha}{n-2p}) \), we need to establish the following super poly-harmonic properties.

**Theorem 1.** Assume \( \gamma \in (1, \frac{n}{n-2p-\alpha}) \), \( n > 2p+\alpha \), \( 0 < \alpha < 2 \) and \( p \geq 1 \) is an integer. Suppose \( u \) is a nonconstant classical solution to [4] satisfying \( u(x) = o(|x|^\frac{\alpha}{\alpha}) \). Then for every \( i = 0, 1, \ldots, p-1 \),

\[-\Delta^\gamma_+ u(x) \geq 0, \quad \forall \, x \in \mathbb{R}^n.\]
As well-known, establishing equivalent integral equation of (1.1) is beneficial to classifying solutions to higher order equations. As a consequence of super poly-harmonic properties for equation (1), we can derive the equivalence between the differential equation (1) and the following integral equation

\[
    u(x) = \int_{\mathbb{R}^n} \frac{R_{2p+\alpha,n}}{|x-y|^{n-2p-\alpha}} u(y)^{\gamma} dy + C_0,
\]

where \( C_0 < 0 \) and the Riesz potential’s constant \( R_{\mu,n} = \frac{\Gamma(\frac{n-\mu}{2})}{\pi^{\frac{n}{2}} 2^{\mu} \Gamma(\frac{\mu}{2})} \) for \( 0 < \mu < n \) (see [16]).

**Theorem 2.** Under the assumptions of Theorem 1, suppose \( u \) is a nonconstant classical solution to (1) satisfying \( u(x) = o(|x|^\alpha + \frac{4}{3}) \). Then \( u \) is a solution to integral equation (5), and vice versa. Moreover, the support of \( u_+ \) is compact.

Apparently, for (1), any non-positive constant is its solution and any positive constant is not its solution. Hence we only consider nonconstant solutions. Actually, there is no positive solutions to (1) from the nonexistence of positive solutions in the subcritical exponent case of Theorem 1.9 in [3]. All negative nonconstant solutions of (1) can be ruled out by the Liouville theorem for fractional poly-harmonic functions (see Theorem 1.3 in [3]). Consequently any nonconstant solutions of (1) are sign-changing solutions. From this observation and Theorem 2 the following classification results can be obtained by applying the method of moving planes in integral forms.

**Theorem 3.** Under the assumptions of Theorem 1, suppose \( u \) is a sign-changing classical solution to equation (1) satisfying \( u(x) = o(|x|^\alpha + \frac{4}{3}) \) at infinity, then \( u \) is symmetric about some point \( x_0 \in \mathbb{R}^n \) and \( \frac{\partial u}{\partial r} < 0 \), where \( r = |x - x_0| \).

Throughout this paper, we will use \( C \) to denote a positive constant which may change from line to line and even within the same line.

**2. Preliminaries**

The purpose of this section is to introduce several useful properties which will be crucial to the forthcoming sections.

**Proposition 1.** [1] Let \( R > 0, x_0 \in \mathbb{R}^n, h \in C^{\alpha+\epsilon}(B_R(x_0)) \cap C(\overline{B_R(x_0)}), 0 < \alpha < 2 \) and let

\[
    u(x) := \begin{cases}
        \int_{B_R(x_0)} G^\alpha_R(x,y) h(y) dy, & x \in B_R(x_0), \\
        0, & x \in \mathbb{R}^n \setminus B_R(x_0).
    \end{cases}
\]

Then \( u \) is the unique point-wise continuous solution of the following problem

\[
    \begin{cases}
        (-\Delta)^{\frac{\alpha}{2}} u(x) = h(x), & x \in B_R(x_0), \\
        u(x) = 0, & x \in \mathbb{R}^n \setminus B_R(x_0).
    \end{cases}
\]

The Green function is defined by

\[
    G^\alpha_R(x,y) := \frac{C_{n,\alpha}}{|x-y|^{n-\alpha}} \int_0^{|x-y|^{\alpha}} \frac{b^{\frac{\alpha}{2}-1}}{(1 + b)^{\frac{\alpha}{2}}} db, \quad x, y \in B_R(x_0)
\]
with \( s_R = \frac{|x-y|^2}{R^2} \), \( t_R = \left( 1 - \frac{|x-x_0|^2}{R^2} \right) \left( 1 - \frac{|y-x_0|^2}{R^2} \right) \), and \( G^n_R(x,y) = 0 \) if \( x \) or \( y \) \( \in \mathbb{R}^n \setminus B_R(x_0) \), where

\[
C_{n,\alpha} = \frac{\Gamma\left( \frac{n}{\alpha} \right)}{2^n \pi^{\frac{n}{2}} \Gamma^2\left( \frac{\alpha}{2} \right)} \text{ if } n \neq \alpha, \quad C_{1,1} = \frac{1}{\pi} \text{ if } n = \alpha = 1.
\]

**Proposition 2.** \([2]\) Let \( R > 0 \), \( x_0 \in \mathbb{R}^n \), \( g \in L_\alpha(\mathbb{R}^n) \cap C(\mathbb{R}^n) \), \( 0 < \alpha < 2 \) and let

\[
u_\alpha(x) := \left\{ \begin{array}{ll}
\int_{\mathbb{R}^n \setminus B_R(x_0)} P^n_R(x,y)g(y)dy, & x \in B_R(x_0), \\
g(x), & x \in \mathbb{R}^n \setminus B_R(x_0).
\end{array} \right.
\]

Then \( \nu_\alpha \) is the unique point-wise continuous solution of the following problem

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = 0, & x \in B_R(x_0), \\
u(x) = g(x), & x \in \mathbb{R}^n \setminus B_R(x_0).
\end{cases}
\]

The Poisson kernel \( P^n_R \) is defined by

\[
P^n_R(x,y) := \frac{\Gamma\left( \frac{n}{\alpha} \right)}{\pi^{\frac{n}{2} + 1}} \sin \frac{\pi \alpha}{2} \left( \frac{R^2 - |x-x_0|^2}{|y-x_0|^2 - R^2} \right)^{\frac{\alpha}{2}} \frac{1}{|x-y|^n}.
\]

**Proposition 3.** \([2]\) Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \). Assume that \( u \in L_\alpha \cap C^{1,1}_{\text{loc}}(\Omega) \) and is lower semi-continuous on \( \Omega \). If

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) \geq 0, & x \in \Omega, \\
u(x) \geq 0, & x \in \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

then \( u(x) \geq 0 \) in \( \Omega \). Moreover, if \( u = 0 \) at some point in \( \Omega \), then \( u = 0 \) a.e. in \( \mathbb{R}^n \). These conclusions also holds for unbounded domain \( \Omega \), if we further assume that \( \liminf_{|x| \to +\infty} u(x) \geq 0 \).

3. **Proof of Theorem 1**

In this section we will complete the proof of Theorem 1.

**Proof** We borrow the idea in \([3]\) to prove this theorem. Denote \( v_i := (-\Delta)^{i+\frac{\alpha}{2}} u(x) \) for \( i = 0,1,\ldots,p-1 \). It follows from \([1]\) that

\[
\begin{cases}
(-\Delta)^{\frac{\alpha}{2}} u(x) = v_0, & x \in \mathbb{R}^n, \\
n - \Delta v_0 = v_1, & x \in \mathbb{R}^n, \\
\cdots \cdots \\
n - \Delta u_{p-1} = v_p, & x \in \mathbb{R}^n.
\end{cases}
\]

Assume that Theorem 1 is not true, then there must exists a largest integer \( 0 \leq k \leq p-1 \) and a point \( x_0 \in \mathbb{R}^n \) such that

\[
v_k(x_0) = (-\Delta)^{k+\frac{\alpha}{2}} u(x_0) < 0.
\]

For any \( r > 0 \), define

\[
g(r,x_0) := \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} g(x)d\sigma,
\]

where \( |\partial B_r(x_0)| \) denotes the area of the sphere \( \partial B_r(x_0) \). For simplicity, we denote \( g(r,x_0) \) as \( g(r) \).
First, we will illustrate that \( 0 \leq k \leq p - 1 \) is even by contradiction. Assume \( k \) is odd. From the well-known property \( \Delta u(r) = \frac{1}{r^{n-1}} (r^{n-1} \bar{u}'(r))' \) and (6), we have
\[
\bar{v}_k(r) \leq \bar{v}_k(0) := -c_0 < 0, \quad \forall r > 0.
\]
Simple calculation shows that
\[
\bar{v}_{k-1}(r) \geq \bar{v}_{k-1}(0) + \frac{c_0}{2n} r^2, \quad \forall r > 0,
\]
and
\[
\bar{v}_{k-2}(r) \leq \bar{v}_{k-2}(0) - \frac{\bar{v}_{k-1}(0)}{2n} r^2 - \frac{c_0}{8n(n+2)} r^4, \quad \forall r > 0.
\]
Repeating the above argument, we derive
\[
\bar{v}_0(r) \geq \bar{v}_0(0) + c_1 r^2 + c_2 r^4 + \cdots + c_k r^{2k}, \quad \forall r > 0,
\]
where \( c_k > 0 \). From (7), we deduce that there exists a \( r_0 \) large enough such that
\[
(8) \quad \bar{v}_0(r) \geq \frac{1}{2} c_k r^{2k} \quad \forall r > r_0.
\]
The first equation in (6), Proposition 1 and Proposition 2 imply that for arbitrary \( R > 0 \),
\[
u(x) = \int_{B_R(x_0)} G_R^2(x, y) v_0(y) dy + \int_{|y-x_0|>R} P_R^2(x, y) u(y) dy, \quad \forall x \in B_R(x_0).
\]
Therefore, we have
\[
\int_{B_R(x_0)} \frac{C_{n, \alpha}}{|x - y|^n - \alpha} \left( \int_0^R \frac{b^{\frac{n-\alpha-1}{2}}}{(1+b)^{\frac{n}{2}}} db \right) v_0(y) dy + C_{n, \alpha} \int_{|y-x_0|>R} \frac{R^{\alpha}}{|y-x_0|^n} u(y) dy
\]
\[
= I + II.
\]
Observe that \( 0 < r \leq \frac{R}{2} \) implies \( 3 \leq \frac{R^2}{r^2} - 1 < +\infty \). Thus
\[
\int_0^3 \frac{b^{\frac{\alpha-1}{2}}}{(1+b)^{\frac{n}{2}}} db \leq \int_0^{\frac{R^2}{r^2} - 1} \frac{b^{\frac{\alpha-1}{2}}}{(1+b)^{\frac{n}{2}}} db \leq \int_0^{+\infty} \frac{b^{\frac{\alpha-1}{2}}}{(1+b)^{\frac{n}{2}}} db.
\]
From (8)-(10), we conclude that for any \( R > 2r_0 \) there exist \( c_i > 0 \) \((i = 1, \ldots, 6)\) independent of \( R \) such that
\[
I = C_{n, \alpha} |\partial B_1| \int_0^R r^{\alpha-1} \left( \int_0^{\frac{R^2}{r^2} - 1} \frac{b^{\frac{\alpha-1}{2}}}{(1+b)^{\frac{n}{2}}} db \right) \bar{v}_0(y) dy
\]
\[
\geq c_1 \int_{r_0}^{R} r^{\alpha-1} \bar{v}_0(r) dr - c_2 \int_0^{r_0} r^{\alpha-1} |\bar{v}_0(r)| dr
\]
\[
\geq c_3 \int_{r_0}^{R} r^{2k+\alpha-1} dr - c_4
\]
\[
\geq c_5 R^{2k+\alpha} - c_6.
\]
Due to \( u(x) = o(|x|^{\alpha - \frac{4}{n}}) \) at infinity, we derive that for sufficiently large \( R \)

\[
C_{n, \alpha}' \int_{R < |y-x_0| < R + R^{1 - \frac{4}{n}}} \frac{R^\alpha}{(|y-x_0|^2 - R^2)^{\frac{\alpha}{n}}} \frac{u(y)}{|y-x_0|^n} \, dy
\]

\[
= CR^\alpha \int_{R}^{R + R^{1 - \frac{4}{n}}} \frac{\bar{u}(r)}{r^{(r^2 - R^2)^{\frac{\alpha}{2}}}} \, dr
\]

\[
\leq \frac{2 - \alpha}{2 \frac{2}{2} + \frac{4}{n} + \alpha} c_5 R^\alpha \int_{R}^{R + R^{1 - \frac{4}{n}}} \frac{r^{\alpha + \frac{4}{2}}}{r^{(r^2 - R^2)^{\frac{\alpha}{2}}}} \, dr
\]

\[
\leq \frac{2 - \alpha}{8} c_5 R^{\alpha + \frac{4}{n} - 1} \int_{R}^{R + R^{1 - \frac{4}{n}}} (r - R)^{-\frac{\alpha}{2}} \, dr
\]

\[
\leq \frac{c_5}{4} R^{2+\alpha}.
\]

(12)

Note that if \(|y-x_0| \geq R + R^{1 - \frac{4}{n}}\), one has

\[
\frac{R^2}{|y-x_0|^2} \leq \frac{R^2}{(R + R^{1 - \frac{4}{n}})^2} \leq \frac{R^2}{R^2 + R^{2 - \frac{4}{n}}},
\]

which gives that

\[
|y-x_0|^2 - R^2 \geq \frac{2R^{2 - \frac{4}{n}}}{R^2 + R^{2 - \frac{4}{n}}} |y-x_0|^2.
\]

Obviously for large \( R \), \(|y-x_0| \geq \frac{|y|}{2}\) holds if \(|y-x_0| \geq R + R^{1 - \frac{4}{n}}\). Therefore for \( R \) large enough, we have

\[
C_{n, \alpha}' \int_{R^n \setminus B_{R + R^{1 - \frac{4}{n}}} (x_0)} \frac{R^\alpha}{(|y-x_0|^2 - R^2)^{\frac{\alpha}{n}}} \frac{u(y)}{|y-x_0|^n} \, dy
\]

\[
\leq CR^\alpha \int_{R^n \setminus B_{R + R^{1 - \frac{4}{n}}} (x_0)} \frac{(R^2 + 2R^{2 - \frac{4}{n}})^{\frac{\alpha}{2}} |u(y)|}{(R^{2 - \frac{4}{n}})^{\frac{\alpha}{2}} |y|^{n+\alpha}} \, dy
\]

\[
\leq CR^{2\alpha-(\alpha-2)} \int_{R^n \setminus B_{R + R^{1 - \frac{4}{n}}} (x_0)} \frac{|u(y)|}{|y|^{n+\alpha}} \, dy
\]

\[
\leq \frac{c_5}{4} R^{2+\alpha},
\]

where the last inequality holds due to \( u \in \mathcal{L}_{\alpha}(\mathbb{R}^n) \). Combining (12) with (13), we conclude

(14)

\[ \Pi \leq \frac{c_5}{2} R^{\alpha+2}. \]

From (9), (11) and (14), we derive that for \( R \) large enough,

(15)

\[ + \infty > u(x_0) \geq c_5 R^{2k+\alpha} - c_6 - \frac{c_5}{2} R^{2+\alpha}. \]

Letting \( R \to +\infty \) in (15), we get immediately a contradiction. Thus, \( k \) must be even.

Next, we will show that \( k = 0 \). Otherwise, suppose that \( 2 \leq k \leq p - 1 \) is even, using the same arguments as in deriving (7), we deduce

(16)

\[ \bar{v}_0(r) \leq \bar{v}_0(0) - c_1 r^2 - c_2 r^4 - \cdots - c_k r^{2k}, \quad \forall r > 0, \]
where \( c_k > 0 \). From (16), we infer that there exists \( r_1 \) large enough such that

\[
\bar{v}_0(r) \leq -\frac{1}{2} c_k r^{2k} \quad \forall r > r_1.
\]

Note that if \( \frac{R}{2} < r < R \), then \( 0 < \frac{R^2}{r^2} - 1 < 3 \), and hence

\[
\int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq \int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq C \left( \frac{R^2}{r^2} - 1 \right)^{\frac{\alpha}{2}}.
\]

From (9), (10), (17) and (18), we get for any \( R > r_1 \),

\[
\int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq \int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq C \left( \frac{R^2}{r^2} - 1 \right)^{\frac{\alpha}{2}}.
\]

From \( \int_k |\partial B_k| R^\alpha \tilde{u}(r) dr = -C_n, \alpha \int_0^R r^{\alpha-1} \left( \int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \right) \bar{v}_0(r) dr + u(x_0) \geq C \int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq \int_0^{\frac{R^2}{r^2} - 1} \frac{db}{(1 + b) \frac{r^2}{R^2}} \geq C \left( \frac{R^2}{r^2} - 1 \right)^{\frac{\alpha}{2}} dr + u(x_0)
\]

where \( C \) and \( \bar{C} \) are independent of \( R \). Thus there exists a \( r_2 > 2r_1 \) large enough such that

\[
\int_0^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)\frac{\alpha}{2}} \geq CR^{2k+\alpha}, \quad \forall R > r_2.
\]

Due to \( u \in L_\alpha(\mathbb{R}^n) \), we obtain

\[
\int_1^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r^{1+\alpha}} dr = C \int_{|x-x_0|>1} \frac{|u(x)|}{|x-x_0|^{n+\alpha}} dx < +\infty.
\]

Therefore for any \( \delta > 0 \),

\[
\int_1^{\infty} \frac{1}{R^{1+\alpha+\delta}} \int_R^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)\frac{\alpha}{2}} dr dR = \int_1^{\infty} \frac{|\bar{u}(r)|}{r} \int_1^{\infty} \frac{1}{R^{1+\delta} (r^2 - R^2)\frac{\alpha}{2}} dR dr\leq C \int_1^{\infty} \frac{|\bar{u}(r)|}{r^{1+\alpha}} \int_1^{\infty} \frac{1}{R^{1+\delta} (r^2 - R^2)\frac{\alpha}{2}} dR dr + C \int_1^{\infty} \frac{|\bar{u}(r)|}{r^{1+\delta}} \int_1^{\infty} \frac{1}{R^{1+\delta} (r^2 - R^2)\frac{\alpha}{2}} dR dr
\]

which leads to a contradiction with (19). Thus \( k = 0 \).

From \( k = 0 \), we conclude that

\[
\bar{v}_0(r) \leq \bar{v}_0(0) := -c_0 < 0, \quad \forall r > 0.
\]

Thus for any \( R > 0 \), (9), (10), (18) and (21) yield that

\[
|\partial B_k| \int_R^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)^\frac{\alpha}{2}} dr \geq C \int_0^{\frac{R^2}{r^2} - 1} r^{\alpha-1} dr + C \int_0^{\frac{R^2}{r^2} - 1} \left( \frac{R^2}{r^2} - 1 \right)^{\frac{\alpha}{2}} dr + u(x_0) \geq CR^\alpha - \bar{C},
\]
where positive constants $C$ and $\tilde{C}$ are independent of $R$. Thus there exists $R_0$ large enough such that
\begin{equation}
\int_{R}^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)\frac{\alpha}{2}} dr \geq CR^\alpha, \quad \forall R > R_0.
\end{equation}
Obviously
\begin{equation}
\int_{N}^{\infty} \frac{u(r)}{r^{1+\alpha}} = C \int_{|x-x_0|>N} \frac{|u(x)|}{|x-x_0|^{n+\alpha}} dx = o_N(1)
\end{equation}
as $N \to +\infty$ due to $u \in L^\alpha(\mathbb{R}^n)$. Therefore
\begin{equation}
\int_{2R}^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)\frac{\alpha}{2}} dr \leq CR^\alpha \int_{2R}^{\infty} \frac{|\bar{u}(r)|}{r^{1+\alpha}} \leq o_R(1)R^\alpha,
\end{equation}
as $R \to +\infty$. It follows from (22) and (23) that there exists $R_1 \geq R_0$ large enough such that
\begin{equation}
\int_{2R_1}^{\infty} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)\frac{\alpha}{2}} dr \geq C \frac{R^\alpha}{2}, \quad \forall R > R_1.
\end{equation}
On the other hand, from (20), we derive that
\begin{align*}
\int_{1}^{\infty} \frac{1}{R^{1+\alpha}} \int_{R}^{2R} \frac{R^\alpha |\bar{u}(r)|}{r(r^2 - R^2)^{\frac{\alpha}{2}}} dr dR &\leq \int_{1}^{\infty} \frac{|\bar{u}(r)|}{r} \int_{r}^{R_1} \frac{1}{R(r^2 - R^2)^{\frac{\alpha}{2}}} dR dr \\
&\leq C \int_{1}^{\infty} \frac{|\bar{u}(r)|}{r^{2+\frac{\alpha}{2}}} \int_{r}^{R_1} \frac{1}{(r-R)^{\frac{\alpha}{2}}} dR dr \\
&\leq C \int_{1}^{\infty} \frac{|\bar{u}(r)|}{r^{1+\alpha}} < +\infty,
\end{align*}
which contradicts with (24). Hence Theorem 1 is proved. \hfill \Box

4. Proof of Theorem 2

To complete the proof of Theorem 2 we need to establish the following lemmas.

**Lemma 1.** Under the assumptions of Theorem 1, suppose $u$ is a nonconstant classical solution to \eqref{1.1} satisfying $u(x) = o(|x|^\alpha + \frac{1}{\alpha})$ at infinity. Then $(-\Delta)^{\frac{n}{2}}u$ satisfies the following integral equation
\begin{equation}
(-\Delta)^{\frac{n}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2p,n}(x, y)u^\gamma(y)dy}{|x-y|^{n-2p}} + C_p,
\end{equation}
where $C_p$ is a nonnegative constant.

**Proof.** Firstly, we prove that
\begin{equation}
(-\Delta)^{p-1+\frac{n}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}(x, y)u^\gamma(y)dy}{|x-y|^{n-2}} \quad \forall x \in \mathbb{R}^n.
\end{equation}
To this end, for arbitrary $R > 0$, denote $f_1(x) = u^\gamma_+(x)$ and
\begin{equation}
v^R_1(x) := \int_{B_R(x_0)} G^\gamma_{R}(x, y)f_1(y)dy,
\end{equation}
where the Green’s function for $-\Delta$ on $B_R(0)$ is given by

$$G_R^2(x, y) = R_{2n}\left[\frac{1}{|x-y|^{n-2}} - \frac{1}{(|x| \cdot |\frac{R}{|x|} - \frac{y}{R}|)^{n-2}}\right], \quad x, y \in B_R(0),$$

and $G_R^2(x, y) = 0$ if $x$ or $y \in \mathbb{R}^n \setminus B_R(0)$. Then we derive that $(-\Delta)^{p-\frac{n}{2}} u - v_1^R$ is harmonic in $B_R(0)$ and continuous up to $\partial B_R(0)$. Hence $v_1^R(x) \in C^2(B_R(0)) \cap C(\mathbb{R}^n)$ and satisfies

$$\left\{\begin{array}{ll}
-\Delta v_1^R(x) = u_1^R, & x \in B_R(0), \\
v_1^R(x) = 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{array}\right.$$

(27)

Set $w_1^R(x) := (-\Delta)^{p-\frac{n}{2}} u - v_1^R$. Combining Theorem 1 with (27), we obtain $w_1^R(x) \in C^2(B_R(0)) \cap C(\mathbb{R}^n)$ and satisfies

$$\left\{\begin{array}{ll}
-\Delta w_1^R(x) = 0, & x \in B_R(0), \\
w_1^R(x) \geq 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{array}\right.$$

The maximum principle implies that

$$w_1^R(x) = (-\Delta)^{p-\frac{n}{2}} u(x) - v_1^R(x) \geq 0, \quad \forall x \in \mathbb{R}^n.$$  
(28)

Now for each fixed $x \in \mathbb{R}^n$, letting $R \to +\infty$ in (28), we have

$$(-\Delta)^{p-\frac{n}{2}} u(x) \geq \int_{\mathbb{R}^n} \frac{R_{2n}}{|x-y|^{n-2}} f_1(y)dy =: v_1(x) \geq 0.$$  
(29)

Since $v_1 - v_1^R$ is smooth in $B_R(0)$ for any $R > 0$, one can obtain that $v_1 \in C^2(\mathbb{R}^n)$ satisfies

$$-\Delta v_1(x) = u_1^R(x), \quad \forall x \in \mathbb{R}^n.$$  
(30)

Set $w_1(x) := (-\Delta)^{p-\frac{n}{2}} u(x) - v_1(x)$. From 1, (29) and (30), we have that $w_1(x) \in C^2(\mathbb{R}^n)$ and satisfies

$$\left\{\begin{array}{ll}
-\Delta w_1(x) = 0, & x \in \mathbb{R}^n, \\
w_1(x) \geq 0, & x \in \mathbb{R}^n.
\end{array}\right.$$

Then by the Liouville Theorem for harmonic function, we can infer that

$$w_1(x) = (-\Delta)^{p-\frac{n}{2}} u(x) - v_1(x) \equiv C_1 \geq 0.$$  

Therefore

$$(-\Delta)^{p-\frac{n}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2n}}{|x-y|^{n-2}} u_1^R(y)dy + C_1 =: f_2(x) \geq C_1 \geq 0.$$  
(31)

Next, for arbitrary $R > 0$, denote

$$v_2^R(x) := \int_{B_R(x_0)} G_R^2(x, y) f_2(y)dy.$$  

Then we have

$$\left\{\begin{array}{ll}
-\Delta v_2^R(x) = f_2(x), & x \in B_R(0), \\
v_2^R(x) = 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{array}\right.$$  
(32)
Set $w_2^R(x) := (−Δ)^{p−2+\frac{α}{2}}u − v^R_2$. By Theorem 1, (31) and (32), we obtain
\[
\begin{cases}
−Δw_2^R(x) = 0, & x \in B_R(0), \\
w_2^R(x) ≥ 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
\]

The maximum principle implies that for any $R > 0$
\begin{align}
(33)
\quad w_2^R(x) = (−Δ)^{p−2+\frac{α}{2}}u(x) − v_2^R(x) ≥ 0, \quad \forall x \in \mathbb{R}^n.
\end{align}

Now for each fixed $x \in \mathbb{R}^n$, letting $R → +∞$ in (33), we have
\begin{align}
(34)
\quad (−Δ)^{p−2+\frac{α}{2}}u(x) ≥ \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}f_2(y)dy =: v_2(x) ≥ 0.
\end{align}

Taking $x = 0$ in (33) and using (31), we get
\[
\int_{\mathbb{R}^n} \frac{C_1}{|y|^{n−2}}dy ≤ \int_{\mathbb{R}^n} \frac{f_2(y)}{|y|^{n−2}}dy < +∞.
\]

It follows immediately that $C_1 = 0$, and hence we shows that (26) holds, that is
\begin{align}
(35)
\quad (−Δ)^{p−1+\frac{α}{2}}u(x) = f_2(y) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}u_+(y)dy, \quad \forall x \in \mathbb{R}^n.
\end{align}

One can easily observe that $v_2$ is a solution of
\begin{align}
(36)
\quad −Δv_2(x) = f_2(x), \quad \forall x ∈ \mathbb{R}^n.
\end{align}

Set $w_2(x) := (−Δ)^{p−2+\frac{α}{2}}u(x) − v_2(x)$. Then it solves
\[
\begin{cases}
−Δw_2(x) = 0, & x \in \mathbb{R}^n, \\
w_2(x) ≥ 0, & x \in \mathbb{R}^n.
\end{cases}
\]

Applying the Liouville Theorem for harmonic functions again, we can infer that
\[
w_2(x) = (−Δ)^{p−2+\frac{α}{2}}u(x) − v_2(x) ≡ C_2 ≥ 0.
\]

Therefore, we deduce that
\begin{align}
(37)
\quad (−Δ)^{p−2+\frac{α}{2}}u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}f_2(y)dy + C_2 =: f_3(x) ≥ C_2 ≥ 0.
\end{align}

By the same methods as above, we can prove that $C_2 = 0$, and hence
\[
(−Δ)^{p−2+\frac{α}{2}}u(x) = f_3(y) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}f_2(y)dy, \quad \forall x ∈ \mathbb{R}^n.
\]

Define
\begin{align}
(38)
\quad f_{k+1}(x) := \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}f_k(y)dy
\end{align}

for $k = 1, 2, \ldots, p$. Repeating the above argument we can derive that
\begin{align}
(39)
\quad (−Δ)^{p−k+\frac{α}{2}}u(x) = \int_{\mathbb{R}^n} \frac{R_{2,n}}{|x − y|^{n−2}}f_k(y)dy, \quad \forall x ∈ \mathbb{R}^n.
\]

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In particular, it follows from (38)-(41) and Fubini’s theorem that

**Lemma 2.** Under the assumptions of Theorem 1, if

Thus the lemma holds.

Owing to the properties of the Riesz potential, for any \( \alpha_1, \alpha_2 \in (0, n) \) such that \( \alpha_1 + \alpha_2 \in (0, n) \), it is easy to know that

\[
\int_{\mathbb{R}^n} \frac{R_{\alpha_1, n}}{|x-y|^{n-\alpha_1}} \cdot \frac{R_{\alpha_2, n}}{|y-z|^{n-\alpha_2}} dy = \frac{R_{\alpha_1+\alpha_2, n}}{|x-z|^{n-(\alpha_1+\alpha_2)}}.
\]

In particular, it follows from (38)-(41) and Fubini’s theorem that

\[
(-\Delta)^{p-k+\frac{\alpha}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2k,n}}{|x-y|^{n-2k}} u_+^\gamma(y) dy, \quad \forall x \in \mathbb{R}^n
\]

for \( k = 1, 2, \ldots, p-1 \) and

\[
(-\Delta)^{\frac{\alpha}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2p,n}}{|x-y|^{n-2p}} u_+^\gamma(y) dy + C_p, \quad \forall x \in \mathbb{R}^n.
\]

Thus the lemma holds.

**Lemma 2.** Under the assumptions of Theorem 1, if \( u \in C^2_{\text{loc}}(\mathbb{R}^n) \) satisfies \( u_+^\gamma \in L^1(\mathbb{R}^n) \) and \( \|u_+^\gamma\|_{L^1(\mathbb{R}^n)} \leq \epsilon \), then \( \Delta u \in L^\infty(\mathbb{R}^n) \).

**Proof.** From (25), (38) and the formula of fundamental solution for \((-\Delta)^{\frac{\alpha}{2}}\), we obtain

\[
-\Delta u(x) = (-\Delta)^{1-\frac{\alpha}{2}} \left( \int_{\mathbb{R}^n} \frac{R_{2p,n}}{|x-y|^{n-2p}} u_+^\gamma(y) dy + C_p \right)
\]

\[
= (-\Delta)^{1-\frac{\alpha}{2}} \left( \int_{\mathbb{R}^n} u_+^\gamma(y) \int_{\mathbb{R}^n} \frac{R_{2-\alpha,n}}{|x-z|^{n-2+\alpha}} \frac{R_{2p-2+\alpha,n}}{|z-y|^{n-2p+2-\alpha}} dz dy \right)
\]

\[
= (-\Delta)^{1-\frac{\alpha}{2}} \left( \int_{\mathbb{R}^n} \frac{R_{2-\alpha,n}}{|x-z|^{n-2+\alpha}} \int_{\mathbb{R}^n} \frac{R_{2p-2+\alpha,n}}{|z-y|^{n-2p+2-\alpha}} u_+^\gamma(y) dy dz \right)
\]

\[
= R_{2p-2+\alpha,n} \int_{\mathbb{R}^n} \frac{u_+^\gamma(y)}{|x-y|^{n-2p+2-\alpha}} dy.
\]

Then analogous to Lemma 2.3 in [11], we can immediately infer that \( \Delta u \in L^\infty(\mathbb{R}^n) \). Indeed, from (42), we derive that

\[
\Delta u(x) + R_{2p-2+\alpha,n} \int_{\mathbb{R}^n} \frac{u_+^\gamma(y)}{|x-y|^{n-2p+2-\alpha}} dy = 0.
\]

Denote \( \beta := \int_{\mathbb{R}^n} u_+^\gamma(y) dy \). For any \( x_0 \in \mathbb{R}^n \), consider the solution \( h_1 \) of the boundary value problem

\[
\begin{cases}
-\Delta h_1(x) = R_{2p-2+\alpha,n} \int_{B_R(x_0)} \frac{u_+^\gamma(z)}{|z-y|^{n-2p+2-\alpha}} dz \quad \text{in } B_R(x_0), \\
h_1 = 0 \quad \text{on } \partial B_R(x_0),
\end{cases}
\]

where \( R \) will be determined later. Set

\[
v_1(x) = C_{n,p,\alpha} \int_{B_R(x_0)} \frac{u_+^\gamma(y)}{|x-y|^{n-2p+2-\alpha}} dy \quad \text{for any } x \in B_R(x_0),
\]

for \( k = 1, 2, \ldots, p-1 \) and

\[
(-\Delta)^{p-k+\frac{\alpha}{2}} u(x) = \int_{\mathbb{R}^n} \frac{R_{2k,n}}{|x-y|^{n-2k}} u_+^\gamma(y) dy, \quad \forall x \in \mathbb{R}^n
\]
where $C_{n,p,\alpha} = \frac{R_{2p-2+\alpha,n}}{(n-2p-\alpha)(2p-2+\alpha)}$. It’s obvious that $v_1(x) \geq 0$ in $B_R(x_0)$ and

$$-\Delta v_1(x) = R_{2p-2+\alpha,n} \int_{B_R(x_0)} \frac{u^\gamma_+(y)}{|x-y|^{n-2p+2-\alpha}} dy.$$  \hspace{1cm} (45)

The comparison principle allows us to conclude that

$$|h_1(x)| \leq v_1(x), \quad x \in B_R(x_0).$$

Therefore there exists $t > 1$ and a constant $C > 0$ independent of $x_0$ such that

$$\int_{B_R(x_0)} |h_1(x)|^{t\gamma} dx \leq C.$$  \hspace{1cm} (46)

Indeed

$$\int_{B_R(x_0)} |h_1(x)|^{t\gamma} dx \leq \int_{B_R(x_0)} |v_1(x)|^{t\gamma} dx = \int_{B_R(x_0)} C_{n,p,\alpha} \int_{B_R(x_0)} \frac{u^\gamma_+(y)}{|x-y|^{n-2p-\alpha}} dy \frac{t\gamma}{dx}.$$  \hspace{1cm} (47)

Owing to $\gamma \in (1, \frac{n}{n-2p-\alpha})$, we know that there exists $t > 1$ such that $t\gamma < \frac{n}{n-2p-\alpha}$. Denote $d\mu = u^\gamma_+ dy/\int_{B_R(x_0)} u^\gamma_+(y) dy$, then the assumption $u^\gamma_+(y) \in L^1(\mathbb{R}^n)$ and Jensen’s inequality imply that

$$\int_{B_R(x_0)} \left( \int_{B_R(x_0)} \frac{u^\gamma_+(y)}{|x-y|^{n-2p-\alpha}} dy \right)^\gamma dx \leq C \int_{B_R(x_0)} \left( \int_{B_R(x_0)} \frac{1}{|x-y|^{n-2p-\alpha}} d\mu \right)^\gamma dx$$

$$\leq C \int_{B_R(x_0)} \int_{B_R(x_0)} \frac{1}{|x-y|^{(n-2p-\alpha)t\gamma}} dx d\mu \leq C,$$  \hspace{1cm} (48)

where the final inequality used the fact that $t\gamma(n-2p-\alpha) < n$. From this and (47), we obtain (49).

Now, we consider the function $q(x) := u(x) - h_1(x)$ in the smaller ball $B_{R-1}(x_0)$. First we observe that

$$\Delta q(x) = \Delta u(x) - \Delta h_1(x) = -R_{2p-2+\alpha,n} \int_{\mathbb{R}^n \setminus B_R(x_0)} \frac{u^\gamma_+(y)}{|x-y|^{n-2p+2-\alpha}} dy.$$  \hspace{1cm} (49)

If $x \in B_{R-1}(x_0)$ and $y \in \mathbb{R}^n \setminus B_R(x_0)$, then $|x-y| \geq 1$. Therefore we have

$$0 \leq -\Delta q(x) \leq R_{2p-2+\alpha,n} \beta.$$  \hspace{1cm} (50)

Hence, it follows from weak Harnack principle that

$$\sup_{B_{R-2}(x_0)} q(x) \leq C [\|q_+\|_{L^\gamma(B_{R-1}(x_0))} + \|\Delta q\|_{L^\infty(B_{R-1}(x_0))}].$$  \hspace{1cm} (51)

(51) shows that the second term of the right hand side of the above inequality is bounded independent of $x_0$. By using (46) and $u^\gamma_+ \in L^1(\mathbb{R}^n)$, we obtain that

$$\int_{B_{R-1}(x_0)} (q_+(x))^{\gamma} dx \leq C,$$
where $C$ independent of $x_0$. Therefore, it follows that $u(y) = q(y) + h_1(y) \leq C + |h_1(y)|$ in the smaller ball $B_{R-2}(x_0)$. This and \(13\) yield the estimate

$$\int_{B_{R-2}(x_0)} u_+^{2\gamma}(y) dy \leq C.$$

Now consider the solution $h_2$ of the boundary value problem

$$\begin{cases}
-\Delta h_2(x) = R_{2p-2+\alpha,n} \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p+\alpha}} dy & \text{in } B_{R-2}(x_0), \\
h_2 = 0 & \text{on } \partial B_{R-2}(x_0).
\end{cases}$$

Set

$$v_2(x) = C_{n,p,\alpha} \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy$$

for any $x \in B_{R-2}(x_0)$, and we have that $|h_2(x)| \leq v_2(x)$ in $B_{R-2}(x_0)$. Simple computation shows that

$$\int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy \leq C \left( \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy \right)^{\frac{1}{2}} \left( \int_{B_{R-2}(x_0)} \frac{1}{|x-y|^{n-2p-\alpha}} dy \right)^{\frac{1}{2}} \leq C \left( \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy \right)^{\frac{1}{2}}.$$

Hence

$$\int_{B_{R-2}(x_0)} |h_2(x)|^{2\gamma} dx \leq \int_{B_{R-2}(x_0)} |v_2(x)|^{2\gamma} dx$$

$$= C \int_{B_{R-2}(x_0)} \left( \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy \right)^{2\gamma} dx \leq C \int_{B_{R-2}(x_0)} \left( \int_{B_{R-2}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy \right)^{2\gamma} dx.$$

Let $d\mu = u_+^{\gamma} dy / \int_{B_{R-2}(x_0)} u_+^{\gamma}(y) dy$. Therefore \(51\), \(53\) and Jensen’s inequality imply that there exists a constant $C > 0$ independent of $x_0$ such that

$$\int_{B_{R-2}(x_0)} |h_2(x)|^{2\gamma} dx \leq C.$$

Similar argument as the above, we can obtain that there exists a constant $C > 0$ independent of $x_0$ such that

$$\int_{B_{R-4}(x_0)} u_+^{2\gamma}(y) dy \leq C.$$

Let $h_3$ be the solution of the equation

$$\begin{cases}
-\Delta h_3(x) = R_{2p-2+\alpha,n} \int_{B_{R-4}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p+\alpha}} dy & \text{in } B_{R-4}(x_0), \\
h_3 = 0 & \text{on } \partial B_{R-4}(x_0),
\end{cases}$$

and

$$v_3(x) = C_{n,p,\alpha} \int_{B_{R-4}(x_0)} \frac{u_+^{\gamma}(y)}{|x-y|^{n-2p-\alpha}} dy$$

for all $x \in B_{R-4}(x_0)$. 

\[13\]
Similar argument as before yields that
\[
\int_{B_{R-k}(x_0)} u_{+}^\gamma(y)dy \leq C.
\]
Repeating the process, we derive that for any \( k \in \mathbb{N} \) satisfying \( R - 2k > 0 \),
\[
(56) \quad \int_{B_{R-2k}(x_0)} u_{+}^{t^k\gamma}(y)dy \leq C.
\]

Choose \( k > 0 \) large enough such that \( \frac{1}{t^k} + \frac{1}{s} = 1 \) where \( s \) satisfies \( s(n - 2p + 2 - \alpha) < n \). For such \( k \), we choose \( R \) large enough such that \( R - 2k > 2 \). From (53), we have
\[
|\Delta u(x_0)| = R_{2p-2+\alpha,n} \int_{\mathbb{R}^n \setminus B_{R-2k}(x_0)} \frac{u_+^\gamma(y)}{|x_0 - y|^{n-2p+2+\alpha}}dy + R_{2p-2+\alpha,n} \int_{B_{R-2k}(x_0)} \frac{u_+^\gamma(y)}{|x_0 - y|^{n-2p+2+\alpha}}dy
\leq \frac{R_{2p-2+\alpha,n}}{4} \int_{\mathbb{R}^n} u_+^\gamma(y)dy + C \left( \int_{B_{R-2k}(x_0)} u_+^{t^k\gamma}(y)dy \right)^{\frac{1}{t^k}} \left( \int_{B_{R-2k}(x_0)} \frac{1}{|x_0 - y|^{(n-2p+2+\alpha)s}}dy \right)^{\frac{1}{s}},
\]
where the final inequality used the fact that \( (n-2p+2+\alpha)s < n \). \( \square \) and the integral constraint in (11) show that \( |\Delta u(x_0)| \leq C \), where \( C > 0 \) is independent of \( x_0 \). Therefore, we finish the proof of this lemma.

**Lemma 3.** If \( \gamma \in (1, \frac{n}{n-2p+\alpha}) \), \( n > 2p + \alpha \), \( p \geq 1 \) is an integer and \( u \) is a solution of (11) satisfying \( u(x) = o(|x|^{\alpha+\frac{n}{2}}) \) at infinity, then there exists a constant \( M > 0 \) such that \( \sup_{\mathbb{R}^n} u \leq M \).

**Proof** From Lemma 2, we have that there exists \( A > 0 \) such that \( |\Delta u| \leq A \) in \( \mathbb{R}^n \). Denote \( h(x) = -\Delta u(x) \). Given \( x_0 \in \mathbb{R}^n \), let \( u_1 \) be the solution of
\[
(57) \quad \begin{cases} 
-\Delta v = h, & \text{in } B_1(x_0), \\
v = 0, & \text{on } \partial B_1(x_0).
\end{cases}
\]
It follows from the elliptic theory that \( |u_1| \leq C \), where \( C > 0 \) independent of \( x_0 \). Denote \( u_2 = u - u_1 \), then \((u_2)_+ \leq u_+ + |u_1|\). Since \( |u_1| \leq C \) in \( B_1(x_0) \) and \( \int_{\mathbb{R}^n} u_+^\gamma(x)dx < +\infty \), we derive
\[
\int_{B_1(x_0)} (u_2)_+^\gamma(x) \leq C.
\]
Note that \( \Delta u_2 = 0 \) in \( B_1(x_0) \). For the subharmonic function \((u_2)_+\), we have
\[
\|(u_2)_+\|_{L^\infty(B_1/2(x_0))} \leq C \int_{B_1(x_0)} (u_2)_+^\gamma(x)dx \leq C \left( \int_{B_1(x_0)} (u_2)_+^\gamma(x)dx \right)^{\frac{1}{\gamma}} \leq C,
\]
where \( C \) is independent of \( x_0 \). Recalling that \( u = u_1 + u_2 \) and the arbitrariness of \( x_0 \), we derive that there exists \( M > 0 \) independent of \( x_0 \) such that \( u_+(x) \leq M \). Hence, \( \sup_{\mathbb{R}^n} u \leq M \). \( \square \)

Set
\[
\zeta(x) = -\int_{\mathbb{R}^n} \frac{R_{2p+\alpha,n}}{|x-y|^{n-2p+\alpha}} u_+^\gamma(y)dy.
\]
Then it is easy to obtain some asymptotic behaviors about $\zeta(x)$ at infinity as follows.

**Lemma 4.** $\zeta(x)$ satisfies

$$\lim_{|x| \to +\infty} (\Delta)^i \zeta(x)|x|^{n-2p-\alpha+2i} = a_i, \quad i = 0, 1, \ldots, p - 1,$$

where $a_0 = -R_{2p+\alpha,n} \int_{\mathbb{R}^n} u_+^\gamma(y)dy$ and $a_{i+1} = a_i(n - 2p - \alpha + 2i)(2p + \alpha - 2i - 2), \quad i = 0, 1, \ldots, p - 2$.

Moreover,

$$\lim_{|x| \to +\infty} (\Delta)^{i+\frac{n}{2}} \zeta(x)|x|^{n-2p+2i} = b_i, \quad i = 0, 1, \ldots, p - 1,$$

where $b_0 = -R_{2p,n} \int_{\mathbb{R}^n} u_+^\gamma(y)dy$ and $b_{i+1} = b_i(n - 2p + 2i)(2p - 2i - 2), \quad i = 0, 1, \ldots, p - 2$.

We are now ready to complete the proof of the equivalence between (1) and (5), that is proof of Theorem 2.

**Proof of Theorem 2** From Lemma 1, we derive (25) holds. Next we will show that $C_p = 0$.

For arbitrary $R > 0$, let

$$v_{p+1}(x) := \int_{B_R(x_0)} G_R^\alpha(x, y)(f_{p+1}(y) + C_p)dy,$$

where $G_R^\alpha(x, y)$ is the Green’s function for $(-\Delta)^\frac{n}{2}$ with $0 < \alpha < 2$ on $B_R(0)$. Then, we can get

$$\begin{cases}
(\Delta)^{\frac{n}{2}} v_{p+1}^R(x) = f_{p+1}(x) + C_p, & x \in B_R(0), \\
v_{p+1}^R(x) = 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
$$

(58)

Denote $w_{p+1}^R(\mathbb{R}^n) := M - u(x) + v_{p+1}^R(x)$. From (25), (58) and Lemma 3 we have

$$\begin{cases}
(\Delta)^{\frac{n}{2}} w_{p+1}^R(x) = 0, & x \in B_R(0), \\
w_{p+1}^R(x) \geq 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
$$

By maximum principle, we can deduce that for any $R > 0$

$$w_{p+1}^R(x) = M - u(x) + v_{p+1}^R(x) \geq 0, \quad \forall x \in \mathbb{R}^n.
$$

(59)

Now for each fixed $x \in \mathbb{R}^n$, letting $R \to +\infty$ in (59), we have

$$M - u(x) \geq -\int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|x - y|^{n-\alpha}}(f_{p+1}(y) + C_p)dy =: -v_{p+1}(x).
$$

(60)

Thus as $R \to +\infty$, we have

$$\begin{cases}
(\Delta)^{\frac{n}{2}}(M - u(x) + v_{p+1}(x)) = 0, & x \in B_R(0), \\
M - u(x) + v_{p+1}(x) \geq 0, & x \in \mathbb{R}^n \setminus B_R(0).
\end{cases}
$$

By Liouville Theorem, we obtain

$$M - u(x) + v_{p+1}(x) \equiv C \geq 0.
$$

That is,

$$u(x) = \int_{\mathbb{R}^n} \frac{R_{\alpha,n}}{|x - y|^{n-\alpha}}(f_{p+1}(y) + C_p)dy + C.
$$

(61)
Taking $x = 0$ in (61), we get
\[
\int_{\mathbb{R}^n} \frac{C_p}{|y|^{n-\alpha}} dy \leq \int_{\mathbb{R}^n} \frac{f_{p+1}(y) + C_p}{|y|^{n-\alpha}} dy < +\infty.
\]
It follows immediately that $C_p = 0$. Thus, from (33), (11) and (61), we derive that (5) holds. Moreover, we assert that $C_0 < 0$. Indeed, if $C_0 \geq 0$, we have $u(x) \geq 0$ in $\mathbb{R}^n$, which is impossible from $\gamma \in (1, \frac{n+2p+\alpha}{n-2p-\alpha})$ and the results of Theorem 1.9 in [3]. Hence (5) holds.

From Lemma 4 and the fact $C_0 < 0$, we obtain that the support of $u_+$ is compact. Meanwhile, it is obvious that if $u$ is a solution of (5), then it satisfies equation (1). The proof of Theorem 2 is completed. \qed

5. Proof of Theorem 3

In this section, we verify Theorem 3 by taking advantage of the method of moving planes in integral forms.

Proof To complete the proof of Theorem 3, it’s enough to show that $\zeta$ is symmetric about some point $x_0 \in \mathbb{R}^n$ and $\frac{\partial \kappa}{\partial r} > 0$ where $r = |x - x_0|$. From Lemmas 2-4, we obtain that $\lim_{|x| \to \infty} \zeta(x) = 0$ and
\[
\zeta(x) = -R_{2p+\alpha,n} \int_{\mathbb{R}^n} \frac{(C_0 - \zeta)^{\gamma}_+(y)}{|x-y|^{n-2p-\alpha}} dy, \quad x \in \mathbb{R}^n.
\]
For $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, we define $T_\lambda = \{x \in \mathbb{R}^n | x_1 = \lambda\}$, $\Sigma_\lambda = \{x \in \mathbb{R}^n | x_1 < \lambda\}$, $x^\lambda = (2\lambda - x_1, x_2, ..., x_n)$ and $\zeta_\lambda(x) = \zeta(2\lambda - x_1, x_2, ..., x_n) = \zeta(x^\lambda)$. Set $w_\lambda(x) = \zeta(x) - \zeta_\lambda(x)$. It’s obvious that
\[
\zeta_\lambda(x) = -R_{2p+\alpha,n} \int_{\mathbb{R}^n} \frac{(C_0 - \zeta_\lambda)^{\gamma}_+(y)}{|x-y|^{n-2p-\alpha}} dy, \quad x \in \mathbb{R}^n.
\]
From this and (62), we have
\[
\zeta_\lambda(x) - \zeta(x) = R_{2p+\alpha,n} \int_{\Sigma_\lambda} \left( \frac{1}{|x-y|^{n-2p-\alpha}} - \frac{1}{|x-y^\lambda|^{n-2p-\alpha}} \right) ((C_0 - \zeta)^{\gamma}_+(y) - (C_0 - \zeta_\lambda)^{\gamma}_+(y)) dy.
\]
**Step 1:** We claim that for $\lambda$ sufficiently negative,
\[
w_\lambda(x) > 0, \quad x \in \Sigma_\lambda.
\]
Due to $\lim_{|x| \to \infty} \zeta(x) = 0$ and $C_0 < 0$, we have for $\lambda$ sufficiently negative
\[
(C_0 - \zeta)^{\gamma}_+ - (C_0 - \zeta_\lambda)^{\gamma}_+ = -(C_0 - \zeta_\lambda)^{\gamma}_+ \leq 0, \quad x \in \Sigma_\lambda.
\]
From this, (63) and the fact that $u$ is a sign-changing classical solution of (11), we have $\zeta_\lambda(x) - \zeta(x) < 0$ for any $x \in \Sigma_\lambda$. Thus (61) holds.

**Step 2:** Step 1 provides a starting point, from which we can now move the plane $T_\lambda$ to the right as long as (61) holds to its limiting position. Define
\[
\lambda_0 = \sup\{\lambda | w_\mu(x) > 0, \forall x \in \Sigma_\mu, \mu \leq \lambda\}.
\]
It’s obviously that $\lambda_0 < +\infty$ and
\[ w_{\lambda_0}(x) \geq 0, \; x \in \Sigma_{\lambda_0}. \]
We will show that $w_{\lambda_0}(x) \equiv 0$ for $x \in \Sigma_{\lambda_0}$.

Otherwise if $w_{\lambda_0} \geq 0$ and $w_{\lambda_0} \neq 0$, we must have

\[ w_{\lambda_0}(x) > 0, \; x \in \Sigma_{\lambda_0}, \]
where (65) follows from (63). We can derive that there exists $R$ large enough such that
\[ (C_0 - \zeta)_+ \equiv 0, \; x \in \mathbb{R}^n \setminus B_R(0), \]
due to $\lim_{|x| \to \infty} \zeta(x) = 0$ and $C_0 < 0$. Fixing this $R$, we have there exists constant $\delta > 0$ and $c > 0$

\[ w_{\lambda_0}(x) \geq c, \; x \in \Sigma_{\lambda_0-\delta} \cap B_R(0). \]
Therefore by the continuity of $w_{\lambda}$ in $\lambda$ there exists $\varepsilon > 0$ and $\varepsilon < \delta$ such that for all $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$, we have
\[ w_{\lambda}(x) \geq 0, \; x \in \Sigma_{\lambda_0-\delta} \cap B_R(0). \]
We will show that for sufficiently small $0 < \varepsilon < \delta$ and any $\lambda \in [\lambda_0, \lambda_0 + \varepsilon)$

\[ w_{\lambda}(x) \geq 0, \; x \in \Sigma_{\lambda}, \]
which contradicts with the definition of $\lambda_0$. Therefore we must have $w_{\lambda_0} \equiv 0$. Define
\[ \Sigma^-_\lambda = \{ x \in \Sigma_{\lambda} | w_{\lambda}(x) < 0 \}. \]
Next we claim that $\Sigma^-_\lambda$ must be measure zero.

For $y \in \Sigma^-_\lambda$, we can obtain that
\[ (C_0 - \zeta)_+(y) - (C_0 - \zeta_{\lambda})_+(y) \leq \gamma(C_0 - \zeta)_+^{-1}(y)|w_{\lambda}(y)|. \]
Thus for $x \in \Sigma_{\lambda}$,

\[ \zeta_{\lambda}(x) - \zeta(x) \leq R_{2p_\alpha,n} \int_{\Sigma_{\lambda}} \left( \frac{1}{|x - y|^{n-2p_\alpha}} - \frac{1}{|x - y^\lambda|^{n-2p_\alpha}} \right) \left( (C_0 - \zeta)_+^{-1}(y) - (C_0 - \zeta_{\lambda})_+^{-1}(y) \right) dy \]
\[ \leq R_{2p_\alpha,n} \int_{\Sigma_{\lambda}} \left( \frac{1}{|x - y|^{n-2p_\alpha}} - \frac{1}{|x - y^\lambda|^{n-2p_\alpha}} \right) \gamma(C_0 - \zeta(y))_+^{-1}|w_{\lambda}(y)| dy. \]
Applying Hardy-Littlewood-Sobolev inequality [10] and Hölder inequality to [70] we obtain that
\[ \| \zeta_{\lambda}(x) - \zeta(x) \|_{L^{n-2p_\alpha}(\Sigma_{\lambda})} \leq C \left( \int_{\Sigma_{\lambda}} \left( (C_0 - \zeta)_+^{-1}(y)|w_{\lambda}(y)| \right)^{\frac{n}{2p_\alpha + \alpha}} dy \right)^{\frac{2p_\alpha + \alpha}{n}} \]
\[ \leq C \left( \int_{\Sigma_{\lambda}} \left( (C_0 - \zeta)_+^{-1}(y) \right)^{\frac{n}{2p_\alpha + \alpha}} dy \right)^{\frac{2p_\alpha + \alpha}{n}} \left( \int_{\Sigma_{\lambda}} |w_{\lambda}(y)|^{\frac{n-2p_\alpha}{n}} dy \right)^{\frac{n-2p_\alpha}{2n}}. \]
Recall that \( \Sigma^- \subset ((\Sigma_\lambda \setminus \Sigma_{\lambda_0-\delta}) \cap B_R) \cup (\Sigma_\lambda \setminus B_R) \) and \(-\zeta\) is bounded above, we can choose \( \delta \) sufficiently small such that
\[
C \left( \int_{\Sigma^- \cap B_R} \left( (C_0 - \zeta)^{\gamma-1} (y) \right)^{\frac{2p}{n}} dy \right)^{\frac{2p}{n}} \leq \frac{1}{2}.
\]
From this and (66), we have
\[
C \left( \int_{\Sigma^-} \left( (C_0 - \zeta)^{\gamma-1} (y) \right)^{\frac{2p}{n}} dy \right)^{\frac{2p}{n}} \leq \frac{1}{2}.
\]
Now (71) implies that \( \|w_\lambda\|_{L^{\frac{n}{2p+2\alpha},\frac{2p}{n}}(\Sigma^-)} = 0 \) and therefore \( \Sigma^- \) must be measure zero.

This verifies (68). Thus we must have \( w_{\lambda_0} \equiv 0 \).

**Step 3**: We show that \( \frac{\partial \zeta}{\partial x_1} < 0 \) for \( x \in \Sigma_{\lambda_0} \).

In fact, from the definition of \( \lambda_0 \) we have for any \( \lambda < \lambda_0 \),

\[
(72) \quad w_\lambda(x) > 0, \quad x \in \Sigma_\lambda.
\]

Simple calculation gives that for any \( x \in T_\lambda \) with \( \lambda < \lambda_0 \),
\[
\zeta_{x_1}(x) = R_{2p+\alpha,n}(n-2p-\alpha) \int_{\mathbb{R}^n} \frac{(C_0 - \zeta)^{\gamma} (y)(x_1 - y_1)}{|x - y|^{n-2p+1-\alpha}} dy
\]
\[
= R_{2p+\alpha,n}(n-2p-\alpha) \int_{\Sigma_\lambda} \frac{(C_0 - \zeta)^{\gamma} (y) - (C_0 - \zeta_\lambda)^{\gamma} (y)}{|x - y|^{n-2p+1}} (x_1 - y_1) dy
\]
\[< 0,
\]
where the last inequality follows from (72). Thus the claim holds.

Since the problem is invariant with respect to rotation, we can take any direction as the \( x_1 \) direction. Hence we have that \( \zeta \) is radially symmetric about some \( x_0 \in \mathbb{R}^n \) and \( \frac{\partial \zeta}{\partial r} > 0 \) where
\[r = |x - x_0|.
\]

Actually, we may also prove Theorem 3 by applying moving plane method to the function \((-\Delta)^{p-1+\gamma} u\), after asymptotic behaviors at infinity of this function and its first-order derivatives are established.

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