ON THE X-RAY TRANSFORM OF PLANAR SYMMETRIC TENSORS

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ABSTRACT. In this article we characterize the range of the attenuated and non-attenuated X-ray transform of compactly supported symmetric tensor fields in the Euclidean plane. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. INTRODUCTION

We consider here the problem of the range characterization of (non)-attenuated X-ray transform of a real valued symmetric $m$-tensors in a strictly convex bounded domain in the Euclidean plane. As the X-ray and Radon transform [38] for planar functions (0-tensors) differ merely by the way lines are parameterized, the $m = 0$ case is the classical Radon transform [38], for which the range characterization has been long established independently by Gelfand and Graev [13], Helgason [14], and Ludwig [22]. Models in the presence of attenuation have also been considered in the homogeneous case [21,2], and in the non-homogeneous case in the breakthrough works [3,32,33], and subsequently [28,6,5,17,25]. The references here are by no means exhaustive.

The interest in the range characterization problem in the 0-tensors case stems out from their applications to data enhancement in medical imaging methods such as Single Photon Emission Computed Tomography or Positron Emission Computed Tomography [27,12]. The X-ray transform of 1-tensors (Doppler transform [29,46]) appears in the investigation of velocity distribution in a flow [7], in ultrasound tomography [47,44], and also in non-invasive industrial measurements for reconstructing the velocity of a moving fluid [30,31]. The X-ray transform of second order tensors arises as the linearization of the boundary rigidity problem [46]. The case of tensor fields of rank four describes the perturbation of travel times of compressional waves propagating in slightly anisotropic elastic media [46, Chapters 6,7]. Thus, due to the various applications the range characterization problem has been a continuing subject of research.

Unlike the scalar case, the X-ray transform of tensor fields has a non-zero kernel, and the null-space becomes larger as the order of the tensor field increases. For tensors of order $m \geq 1$, it is easy to check that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors, and it is possible to reconstruct uniquely (without additional information of moment ray transforms [46]) only the solenoidal part of a tensor field. The non-injectivity of the X-ray transform makes the range characterization problem even more interesting.

For the attenuating media in planar domains, interesting enough, the 1-tensor field can be recovered in the regions of positive absorption as shown in [18,5,48,40], without using some additional data information [45,9,23]. It is due to a surprising fact that the two-dimensional attenuated Doppler transform with positive attenuation is injective while the non-attenuated Doppler transform is not.

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The systematic study of tensor tomography in non-Euclidean spaces originated in [46]. On simple Riemannian surfaces, the range characterization of the geodesic $X$-ray of compactly supported 0 and 1-tensors has been established in terms of the scattering relation in [37], and the results were extended in [4, 11, 20] to symmetric tensors of arbitrary order. Explicit inversion approaches in the Euclidean case have been proposed in [17, 10, 24]. In the attenuating media, tensor tomography was solved for the cases $m = 0, 1$ in [43]. Inversion for the attenuated $X$-ray transform for solenoidal tensors of rank two and higher can be found in [35], with a range characterization in [36, 25, 4].

The original characterization in [13, 14, 22] was extended to arbitrary symmetric $m$-tensors in [34]; see [10] for a partial survey on the tensor tomography in the Euclidean plane. The connection between the Euclidean version of the characterization in [37] and the characterization in [13, 14, 22] was established in [24]. Recently, in [41] the connection between the range characterization result in [39] and the original range characterization in [13, 14, 22] has been established.

In here we build on the results in [39, 40, 42], and extends them to symmetric tensor fields of any arbitrary order. In particular, the range characterization therein are given in terms of the Bukhgeim-Hilbert transform [39] (the Hilbert-like transform associated with $A$-analytic maps in the sense of Bukhgeim [8]). The characterization in here can be viewed as an explicit description of the scattering relation in [35, 36] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible $m$-tensors yielding identical $X$-ray data; see (43) and (69) for the non-attenuated case and (94) and (122) for the attenuated case.

This article is organized as follows: All the details establishing notations and basic properties of symmetric tensor fields needed here are in Section 2. In Section 3 we briefly recall existing results on $A$-analytic maps that are used in the proofs. In Section 4 and Section 5 we provide range characterization of symmetric tensor field $f$ of even order, respectively, odd order in the non-attenuated case. In Section 6 and Section 7 we provide range characterization of symmetric tensor field $f$ of even order, respectively, odd order in the attenuated case.

2. Preliminaries

Given an integer $m \geq 0$, let $T^m(\mathbb{R}^2)$ denote the space of all real-valued covariant tensor fields of rank $m$:

$$f(x^1, x^2) = f_{i_1 \cdots i_m}(x^1, x^2) dx^{i_1} \otimes dx^{i_2} \otimes \cdots \otimes dx^{i_m}, \quad i_1, \cdots, i_m \in \{1, 2\},$$

where $\otimes$ is the tensor product, $f_{i_1 \cdots i_m}$ are the components of tensor field $f$ in the Cartesian basis $(x^1, x^2)$, and where by repeating superscripts and subscripts in a monomial a summation from 1 to 2 is meant.

We denote by $S^m(\mathbb{R}^2)$ the space of symmetric covariant tensor fields of rank $m$ on $\mathbb{R}^2$. Let $\sigma : T^m(\mathbb{R}^2) \rightarrow S^m(\mathbb{R}^2)$ be the canonical projection (symmetrization) defined by $(\sigma f)_{i_1 \cdots i_m} = \frac{1}{m!} \sum_{\pi \in \Pi_m} f_{i_{\pi(1)} \cdots i_{\pi(m)}}$, where the summation is over the group $\Pi_m$ of all permutations of the set $\{1, \cdots, m\}$.

A planar covariant symmetric tensor field of rank $m$ has $m + 1$ independent component, which we denote by

$$\tilde{f}_k := f_{\underbrace{1 \cdots 1}_{m-k} \underbrace{2 \cdots 2}_k}, \quad (k = 0, \cdots, m),$$
in connection with this, a symmetric tensor \( f = (f_{i_1 \cdots i_m}, \ i_1, \cdots, i_m = 1, 2) \) of rank \( m \) will be given by a pseudovector of size \( m + 1 \)

\[
f = (\tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_{m-1}, \tilde{f}_m).
\]

We identify the plane \( \mathbb{R}^2 \) by the complex plane \( \mathbb{C} \), \( z^1 \equiv z = x^1 + 1x^2 \), \( z^2 \equiv \bar{z} = x^1 - 1x^2 \). We consider the Cauchy-Riemann operators

\[
\frac{\partial}{\partial z^1} \equiv \frac{\partial}{\partial z} := \frac{1}{2} \left( \frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2} \right), \quad \frac{\partial}{\partial z^2} \equiv \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^2} \right),
\]

and the inverse relation by

\[
\frac{\partial}{\partial x^1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial x^2} = \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}.
\]

Let \( f = (f_{i_1 \cdots i_m}(x^1, x^2), \ i_1, \cdots, i_m = 1, 2) \) be real valued symmetric \( m \)-tensor field in Cartesian coordinates \((x^1, x^2)\), then in complex coordinates \((z^1, z^2)\) it will have new components \((F_{i_1 \cdots i_m}(z, \bar{z}))\), which are formally expressed by the covariant tensor law:

\[
F_{i_1 \cdots i_m}(z, \bar{z}) = \frac{\partial x^{s_1}}{\partial z^1} \cdots \frac{\partial x^{s_m}}{\partial z^m} f_{i_1 \cdots i_m}(x^1, x^2), \quad \text{and}
\]

\[
f_{i_1 \cdots i_m}(x^1, x^2) = \frac{\partial z^1}{\partial x^{i_1}} \cdots \frac{\partial z^m}{\partial x^{i_m}} F_{i_1 \cdots i_m}(z, \bar{z}),
\]

where the Jacobian matrix has the form

\[
J := \begin{pmatrix}
\frac{\partial x^1}{\partial z^1} & \frac{\partial x^1}{\partial \bar{z}} \\
\frac{\partial x^2}{\partial z^1} & \frac{\partial x^2}{\partial \bar{z}}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad J^{-1} = \begin{pmatrix}
\frac{\partial z^1}{\partial x^1} & \frac{\partial z^1}{\partial x^2} \\
\frac{\partial z^2}{\partial x^1} & \frac{\partial z^2}{\partial x^2}
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\]

Adopting the notation in \([17]\), we shall write the transformations \((4)\) as

\[
f = \{f_{i_1 \cdots i_m}(x^1, x^2)\} \quad \mapsto \quad F = \{F_{i_1 \cdots i_m}(z, \bar{z})\}, \quad \text{and}
\]

\[
F = \{F_{i_1 \cdots i_m}(z, \bar{z})\} \quad \mapsto \quad f = \{f_{i_1 \cdots i_m}(x^1, x^2)\}.
\]

A symmetric tensor \( F \) of rank \( m \), obtained from the real symmetric tensor \( f \) by passing to complex variables, we also define a pseudovector \((F_0, F_1, \cdots, F_{m-1}, F_m)\) with components

\[
F_k = F_{\underbrace{1 \cdots 1}_m \underbrace{2 \cdots 2}_k}, \quad k = 0, \cdots, m,
\]

and subject to the conditions

\[
F_k = \overline{F}_{m-k}, \quad k = 0, \cdots, m.
\]

Taking into account the tensor law \((4)\), we obtain formulas relating the components of pseudovectors in \((2)\) and pseudovectors in \((6)\):

\[
F_k = \frac{(-1)^{m-k} m-k}{2^m} \sum_{q=0}^{m-k} \sum_{p=0}^{k} \binom{m-k}{q} \binom{k}{p} 1^{k-p+q} \tilde{f}_{p+q}, \quad k = 0, 1, \cdots, m,
\]

\[
\tilde{f}_k = i^k \sum_{q=0}^{m-k} \sum_{p=0}^{k} \binom{m-k}{q} \binom{k}{p} (-1)^{k-p} F_{p+q}, \quad k = 0, 1, \cdots, m.
\]

In Cartesian coordinates covariant and contravariant components are the same, and thus contravariant components of the tensor field \( f \) coincide with its corresponding covariant components, \( \tilde{f}_{i_1 \cdots i_m} = f^{i_1 \cdots i_m} \). The dot product on \( S^m(\mathbb{R}^2) \) induced by the Euclidean metric is defined by

\[
(f, h) := f_{i_1 \cdots i_m} h^{i_1 \cdots i_m}.
\]
Note that if $f_1 \mapsto F_1$ and $f_2 \mapsto F_2$, then the pointwise inner product of tensors is invariant:

\begin{equation}
\langle f_1, f_2 \rangle = \langle F_1, F_2 \rangle.
\end{equation}

For $\theta = (\theta^1, \theta^2) = (\cos \theta, \sin \theta) \in S^1$, we denote by $\theta^m$ the tensor product $\theta^m := \theta \otimes \theta \otimes \cdots \otimes \theta$ and $\theta^m$ will be an $m$-contravariant tensor in Cartesian coordinates. According to the tensor law for contravariant components its representation in complex coordinates will look like

$$
\theta \mapsto \Theta, \quad \Theta^k = \frac{\partial z^k}{\partial x^s} \theta^s, \quad \Theta = (\Theta^1, \Theta^2) = (e^{i\theta}, e^{-i\theta}),
$$

and $\Theta^m := \Theta \otimes \Theta \otimes \cdots \otimes \Theta$ be an $m$-contravariant tensor, and we also have $\theta^m \mapsto \Theta^m$. Using (11), we get

\begin{equation}
\langle f, \theta^m \rangle = \langle F, \Theta^m \rangle = \sum_{k=0}^{m} \binom{m}{k} F_k e^{i\theta(m-k)} e^{-i\theta k} = \sum_{k=0}^{m} \binom{m}{k} F_k e^{i(m-2k)\theta}
\end{equation}

\begin{equation}
= \begin{cases}
\sum_{k=0}^{q} f_{-2k} e^{i(2k)\theta} + \sum_{k=1}^{q} f_{2k} e^{-i(2k)\theta}, & \text{(if } m = 2q, \ q \geq 0), \\[1.5ex]
\sum_{k=0}^{q} f_{-(2k+1)} e^{i(2k+1)\theta} + f_{2k+1} e^{-i(2k+1)\theta}, & \text{(if } m = 2q + 1, \ q \geq 0),
\end{cases}
\end{equation}

where

\begin{equation}
f_{-2k} = \binom{2q}{q-k} F_{q-k}, \quad 0 \leq k \leq q, \ q \geq 0, \quad (q = \frac{m}{2}, \ m \text{ even}),
\end{equation}

\begin{equation}
f_{-(2k+1)} = \binom{2q+1}{q-k} F_{q-k}, \quad 0 \leq k \leq q, \ q \geq 0, \quad (q = \frac{m-1}{2}, \ m \text{ odd}),
\end{equation}

and $f_n = f_{-n}$ and $F_n = F_{m-n}$, for $0 \leq n \leq m$.

Let $f$ be a real valued symmetric $m$-tensor, with integrable components of compact support in $\mathbb{R}^2$, and $a \in L^1(\mathbb{R}^2)$ a real valued function. The attenuated X-ray transform of $f$ is given by

\begin{equation}
X_a f(x, \theta) := \int_{-\infty}^{\infty} \langle f(x + t\theta), \theta^m \rangle \exp \left\{-\int_{t}^{\infty} a(x + s\theta)ds \right\} dt,
\end{equation}

where $x \in \mathbb{R}^2, \ \theta \in S^1$, and $\langle \cdot, \cdot \rangle$ is the inner product in (10). For the non attenuated case ($a \equiv 0$), we use the notation $Xf$.

In here, we consider the tensor field $f$ be defined on a strongly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary $\Gamma$; further regularity and the order of vanishing will be specified in the theorems. In the statements below we use the notations in [46]:

$$C^\mu(S^m; \Omega) = \{ f = (f_{i_1\ldots i_m}) \in S^m(\Omega) : f_{i_1\ldots i_m} \in C^\mu(\Omega) \}$$

$0 < \mu < 1$, for the space of real valued, symmetric tensor fields of order $m$ with locally Hölder continuous components. Similarly, $L^1(S^m; \Omega)$ denotes the tensor fields of order $m$ with integrable components.

For any $(x, \theta) \in \overline{\Omega} \times S^1$, let $\tau(x, \theta)$ be length of the chord passing through $x$ in the direction of $\theta$. Let also consider the incoming ($-$), respectively outgoing ($+$) submanifolds of the unit bundle
restricted to the boundary

\[ \Gamma_\pm := \{(x, \theta) \in \Gamma \times S^1 : \pm \theta \cdot \nu(x) > 0\}, \]

and the variety

\[ \Gamma_0 := \{(x, \theta) \in \Gamma \times S^1 : \theta \cdot \nu(x) = 0\}, \]

where \( \nu(x) \) denotes outer normal.

The \( \alpha \)-attenuated X-ray transform of \( f \) is realized as a function on \( \Gamma_+ \) by

\[ X_\alpha f(x, \theta) = \int_0^\infty \langle f(x + t\theta), \theta^m \rangle e^{-\int_0^t a(x(s\theta))ds} dt, (x, \theta) \in \Gamma_+. \]

We approach the range characterization via the well-known connection with the transport model as follows: The boundary value problem

\[
\begin{align*}
\theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) &= \langle f(x), \theta^m \rangle, \quad (x, \theta) \in \Omega \times S^1, \\
u|_{\Gamma_-} &= 0,
\end{align*}
\]

has a unique solution in \( \Omega \times S^1 \) and

\[ u|_{\Gamma_+}(x, \theta) = X_\alpha f(x, \theta), \quad (x, \theta) \in \Gamma_. \]

The range characterization is given in terms of the trace

\[ g := u|_{\Gamma_+ \times S^1} = \begin{cases} X_\alpha f, & \text{on } \Gamma_+, \\
0, & \text{on } \Gamma_- \cup \Gamma_0. \end{cases} \]

We note that from (12), the expression \( \langle f, \theta^m \rangle \) in the transport equation (19a) is represented in the Fourier decomposition in \( \theta \) as in terms of the following Fourier modes:

\[
\langle f, \theta^m \rangle = \begin{cases} f_0 + f_{\pm 2}e^{\mp 2\theta} + f_{\pm 4}e^{\mp 4\theta} + \cdots + f_{\pm m}e^{\mp m\theta} & (m \text{ even}), \\
 f_{\pm 1}e^{\mp \theta} + f_{\pm 3}e^{\mp 3\theta} + \cdots + f_{\pm m}e^{\mp m\theta} & (m \text{ odd}). \end{cases}
\]

### 3. Ingredients from A-analytic theory

In this section we briefly introduce the properties of A-analytic maps needed later. For \( 0 < \mu < 1, p = 1, 2 \), we consider the Banach spaces:

\[
\begin{align*}
l_\infty^{1,p}(\Gamma) := \left\{ g = \langle g_0, g_{-1}, g_{-2}, \ldots \rangle : \|g\|_{l_\infty^{1,p}(\Gamma)} := \sup_{\xi \in \Gamma} \sum_{j=0}^\infty \langle j \rangle^p |g_{-j}(\xi)| < \infty \right\}, \\
C^\mu(\Gamma; l_1) := \left\{ g = \langle g_0, g_{-1}, g_{-2}, \ldots \rangle : \sup_{\xi \in \Gamma} \|g(\xi)\|_{l_1} + \sup_{\xi, \eta \in \Gamma_{\xi \neq \eta}} \frac{\|g(\xi) - g(\eta)\|_{l_1}}{|\xi - \eta|^\mu} < \infty \right\}, \\
Y_\mu(\Gamma) := \left\{ g : g \in l_\infty^{1,2}(\Gamma) \text{ and } \sup_{\xi, \eta \in \Gamma_{\xi \neq \eta}} \sum_{j=0}^\infty \langle j \rangle \frac{|g_{-j}(\xi) - g_{-j}(\eta)|}{|\xi - \eta|^\mu} < \infty \right\},
\end{align*}
\]

where \( l_\infty(, l_1) \) is the space of bounded (, respectively summable) sequences, and for brevity, we use the notation \( \langle j \rangle = (1 + |j|^2)^{1/2} \). Similarly, we consider \( C^\mu(\overline{\Omega}; l_1) \), and \( C^\mu(\overline{\Omega}; l_\infty) \).
A sequence valued map \( \Omega \ni z \mapsto \mathbf{v}(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), \ldots \rangle \) in \( C(\overline{\Omega}; l_\infty) \cap C^1(\Omega; l_\infty) \) is called \( L^k \)-analytic (in the sense of Bukhgeim), \( k = 1, 2, \) if
\[
(23) \quad \overline{\mathbf{v}}(z) + L^k \partial \mathbf{v}(z) = 0, \quad z \in \Omega,
\] where \( L \) is the left shift operator \( L(\langle v_0, v_{-1}, v_{-2}, \ldots \rangle) = \langle v_{-1}, v_{-2}, \ldots \rangle \), and \( L^2 = L \circ L \).

Bukhgeim’s original theory in [8] shows that solutions of (23), satisfy a Cauchy-like integral formula,
\[
(24) \quad \mathbf{v}(z) = B[\mathbf{v}|_\Gamma](z), \quad z \in \Omega,
\] where \( B \) is the Bukhgeim-Cauchy operator acting on \( \mathbf{v}|_\Gamma \). We use the formula in [12], where \( B \) is defined component-wise for \( n \geq 0 \) by
\[
(25) \quad (Bg)_n(z) := \frac{1}{2\pi i} \int_\Gamma \frac{g_n(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{\pi i} \int_\Gamma \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - z} \right\} \sum_{j=1}^{\infty} g_{n-j}(\zeta) \left( \frac{\zeta - \overline{z}}{\zeta - z} \right)^j, \quad z \in \Omega.
\]

The following regularity result in [39, Proposition 4.1] is needed.

**Proposition 3.1.** [39, Proposition 4.1] Let \( \mu > 1/2 \) and \( g = \langle g_0, g_{-1}, g_{-2}, \ldots \rangle \) be the sequence valued map of non-positive Fourier modes of \( g \).

(i) If \( g \in \mathcal{C}^\mu(\Gamma; C^1([S^1])) \), then \( g \in l_{1,1}^1(\Gamma) \cap C^\mu(\Gamma; l_1) \).

(ii) If \( g \in \mathcal{C}^\mu(\Gamma; C^1([S^1])) \cap C(\Gamma; C^2([S^1])) \), then \( g \in Y_\mu(\Gamma) \).

Similar to the analytic maps, the traces of \( L \)-analytic maps on the boundary must satisfy some constraints, which can be expressed in terms of a corresponding Hilbert-like transform introduced in [39]. More precisely, the Bukhgeim-Hilbert transform \( \mathcal{H} \) acting on \( g \),
\[
(26) \quad \Gamma \ni z \mapsto (\mathcal{H}g)(z) = \langle (\mathcal{H}g)_0(z), (\mathcal{H}g)_{-1}(z), (\mathcal{H}g)_{-2}(z), \ldots \rangle
\] is defined component-wise for \( n \geq 0 \) by
\[
(27) \quad (\mathcal{H}g)_n(z) = \frac{1}{\pi} \int_\Gamma \frac{g_n(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{\pi} \int_\Gamma \left\{ \frac{d\zeta}{\zeta - z} - \frac{d\overline{\zeta}}{\overline{\zeta} - z} \right\} \sum_{j=1}^{\infty} g_{n-j}(\zeta) \left( \frac{\zeta - \overline{z}}{\zeta - z} \right)^j, \quad z \in \Gamma,
\]
and we refer to [39] for its mapping properties.

Note that the Bukhgeim-Cauchy integral formula in (25) above is restated in terms of \( L \)-analytic maps as opposed to \( L^2 \)-analytic as in [39]. The only change is the index relabeling. In particular, the index \( g_{n-j} \) will change to \( g_{n-2j} \) therein to account for \( L^2 \)-analytic. Moreover, the same index relabelling in the Bukhgeim-Hilbert transform formula (27) is made to account for the difference between \( L \)-analytic and \( L^2 \)-analytic.

The following result recalls the necessary and sufficient conditions for a sufficiently regular map to be the boundary value of an \( L^k \)-analytic function, \( k = 1, 2, \)

**Theorem 3.1.** Let \( 0 < \mu < 1 \), and \( k = 1, 2 \). Let \( B \) be the Bukhgeim-Cauchy operator in (25).

Let \( g = \langle g_0, g_{-1}, g_{-2}, \ldots \rangle \in Y_\mu(\Gamma) \) for \( \mu > 1/2 \) be defined on the boundary \( \Gamma \), and let \( \mathcal{H} \) be the Bukhgeim-Hilbert transform acting on \( g \) as in (27).

(i) If \( g \) is the boundary value of an \( L^k \)-analytic function, then \( \mathcal{H}g \in C^\mu(\Gamma; l_1) \) and satisfies
\[
(28) \quad (I + i\mathcal{H})g = 0.
\]
(ii) If $g$ satisfies (28), then there exists an $L^k$-analytic function $v := Bg \in C^{1,\mu}(\Omega; l_1) \cap C^{\mu}(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty)$, such that

$$
(29) \quad v|_r = g.
$$

For the proof of Theorem 3.1 we refer to [39, Theorem 3.2, Corollary 4.1, and Proposition 4.2] and [40, Proposition 2.3].

Another ingredient, in addition to $L^2$-analytic maps, consists in the one-to-one relation between solutions $u := \langle u_0, u_{-1}, u_{-2}, \ldots \rangle$ satisfying

$$
(30) \quad \partial u_{-n}(z) + \partial u_{-n-2}(z) + a(z)u_{-n-1}(z) = 0, \quad z \in \Omega, \ n \geq 0,
$$

and the $L^2$-analytic map $v = \langle v_0, v_{-1}, v_{-2}, \ldots \rangle$ satisfying

$$
(31) \quad \partial v_{-n}(z) + \partial v_{-n-2}(z) = 0, \quad z \in \Omega, \ n \geq 0;
$$

via a special function $h$, see [42, Lemma 4.2] for details. The function $h$ is defined as

$$
(32) \quad h(z, \theta) := Da(z, \theta) - \frac{1}{2} (I - iH) Ra(z \cdot \theta^\perp, \theta^\perp),
$$

where $\theta^\perp$ is the counter-clockwise rotation of $\theta$ by $\pi/2$, $Ra(s, \theta^\perp) = \int_{-\infty}^{\infty} a(s\theta^\perp + t\theta) \, dt$ is the Radon transform in $\mathbb{R}^2$ of the attenuation $a$, $Da(z, \theta) = \int_{-\infty}^{\infty} a(z + t\theta) \, dt$ is the divergent beam transform of the attenuation $a$, and $H h(s) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(t)}{s - t} \, dt$ is the classical Hilbert transform [26], taken in the first variable and evaluated at $s = z \cdot \theta^\perp$. The function $h$ appeared first in [27] and enjoys the crucial property of having vanishing negative Fourier modes yielding the expansions

$$
(33) \quad e^{-h}(z, \theta) := \sum_{k=0}^{\infty} \alpha_k(z) e^{ik\theta}, \quad e^{h}(z, \theta) := \sum_{k=0}^{\infty} \beta_k(z) e^{ik\theta}, \quad (z, \theta) \in \overline{\Omega} \times \mathbb{S}^1.
$$

Using the Fourier coefficients of $e^\pm h$, define the integrating operators $e^{\pm G} u$ component-wise for each $n \leq 0$, by

$$
(34) \quad (e^{-G} u)_n = (\alpha \ast u)_n = \sum_{k=0}^{\infty} \alpha_k u_{n-k}, \quad \text{and} \quad (e^{G} u)_n = (\beta \ast u)_n = \sum_{k=0}^{\infty} \beta_k u_{n-k},
$$

where $\alpha$ and $\beta$ is given by

$$
\overline{\Omega} \ni z \mapsto \alpha(z) := \langle \alpha_0(z), \alpha_1(z), \alpha_2(z), \ldots \rangle, \quad \overline{\Omega} \ni z \mapsto \beta(z) := \langle \beta_0(z), \beta_1(z), \beta_2(z), \ldots \rangle.
$$

Note that $e^{\pm G}$ can also be written in terms of left translation operator as

$$
(35) \quad e^{-G} u = \sum_{k=0}^{\infty} \alpha_k L^k u, \quad \text{and} \quad e^{G} u = \sum_{k=0}^{\infty} \beta_k L^k u,
$$

where $L^k$ is the $k$-th composition of left translation operator. It is important to note that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$. We refer [42, Lemma 4.1] for the properties of $h$, and we restate the following result [39, Proposition 5.2] to incorporate the operators $e^{\pm G}$ notation used in here.
Proposition 3.2. [39, Proposition 5.2] Let \( a \in C^{1,\mu}(\Omega), \mu > 1/2 \). Then \( \alpha, \partial \alpha, \beta, \partial \beta \in l^1_{C^1} (\Omega) \), and the operators

\[
(i) \ e^{\pm G} : C^\mu(\Omega; l_\infty) \rightarrow C^\mu(\Omega; l_\infty);
(ii) \ e^{\pm G} : C^\mu(\Omega; l_1) \rightarrow C^\mu(\Omega; l_1);
(iii) \ e^{\pm G} : Y_\mu(\Gamma) \rightarrow Y_\mu(\Gamma).
\]

Lemma 3.1. [40, Lemma 4.2] Let \( a \in C^{1,\mu}(\Omega), \mu > 1/2 \), and \( e^{\pm G} \) be operators as defined in (34).

(i) If \( u \in C^1(\Omega, l_1) \) solves \( \overline{\partial} u + L^2 \partial u + a L u = 0 \), then \( v = e^{-G} u \in C^1(\Omega, l_1) \) solves \( \overline{\partial} v + L^2 \partial^2 v = 0 \).

(ii) Conversely, if \( v \in C^1(\Omega, l_1) \) solves \( \overline{\partial} v + L^2 \partial^2 v = 0 \), then \( u = e^G v \in C^1(\Omega, l_1) \) solves \( \overline{\partial} u + L^2 \partial u + a L u = 0 \).

4. EVEN ORDER \( m \)-TENSOR - NON-ATTENUATED CASE

We establish necessary and sufficient conditions for a sufficiently smooth function on \( \Gamma \times S^1 \) to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field \( f \) of even order \( m = 2q, q \geq 0 \). In this non-attenuated case, the transport equation (19a) becomes

\[
\theta \cdot \nabla u(x, \theta) = \sum_{k=-q}^{q} f_{2k}(x) e^{-i(2k)\theta}, \quad x \in \Omega,
\]

where \( f_{2k} \) defined in (13), and \( f_{2k} = \overline{f_{-2k}} \), for \( -q \leq k \leq q, q \geq 0 \). Note that \( f_0 \) is real-valued while other modes are complex conjugates.

For \( z = x_1 + i x_2 \in \Omega \), the advection operator \( \theta \cdot \nabla \) in complex notation becomes \( e^{-i \theta} \overline{\partial} + e^{i \theta} \partial \), where \( \theta = (\cos \theta, \sin \theta) \), and \( \overline{\partial}, \partial \) are the Cauchy-Riemann operators in (3).

If \( \sum_{n \in \mathbb{Z}} u_n(z) e^{i n \theta} \) is the Fourier series expansion in the angular variable \( \theta \) of a solution \( u \) of (37), then, provided some sufficient decay (to be specified later) of \( u_n \) to allow regrouping, the equation (37) reduces to the system:

\[
\begin{align*}
(38) \quad & \overline{\partial} u_{-2(n-1)}(z) + \partial u_{-(2n+1)}(z) = f_{2n}(z), \quad 0 \leq n \leq q, q \geq 0, \\
(39) \quad & \overline{\partial} u_{-(2n-1)}(z) + \partial u_{-(2n+1)}(z) = 0, \quad n \geq q + 1, q \geq 0, \\
(40) \quad & \overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) = 0, \quad n \geq 0.
\end{align*}
\]

Recall that the trace \( u|_{\Gamma^+} = g \) as in (21), with \( g = Xf \) on \( \Gamma^+ \) and \( g = 0 \) on \( \Gamma^- \cup \Gamma_0 \).

The range characterization is given in terms of the Fourier modes of \( g \) in the angular variables:

\[
g(\zeta, \theta) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{i n \theta}, \quad \zeta \in \Gamma.
\]

Since the trace \( g \) is also real valued, its Fourier modes will satisfy \( g_{-n} = g_n \), for \( n \geq 0 \).

From the non-positive Fourier modes, we built the sequences

\[
\text{g}^{\text{even}} := \langle g_0, g_{-2}, g_{-4}, \ldots \rangle, \quad \text{and} \quad \text{g}^{\text{odd}} := \langle g_{-1}, g_{-3}, g_{-5}, \ldots \rangle.
\]

From the negative odd modes starting from mode \( (2q+1) \), we built the sequence

\[
L^q \text{g}^{\text{odd}} := \langle g_{-(2q+1)}, g_{-(2q+3)}, g_{-(2q+5)}, \ldots \rangle, \quad q \geq 0,
\]

where \( L^q \) is the \( q \)-th composition of left translation operator.
We characterize next the non-attenuated X-ray data \( g \) in terms of the Bukhgeim-Hilbert Transform \( \mathcal{H} \) in (27). We will construct the solution \( u \) of the transport equation (37), whose trace matches the boundary data \( g \), and also construct the right hand side of the (37). The construction of solution \( u \) is in terms of its Fourier modes in the angular variable. We first construct the non-positive Fourier modes and then the positive Fourier modes are constructed by conjugation. For even \( m = 2q \), \( q \geq 1 \), apart from \( q \) many Fourier modes \( u_{-1}, u_{-3}, \cdots, u_{-(2q-1)} \), all non-positive Fourier modes are defined by Bukhgeim-Cauchy integral formula (25) using boundary data. Other than having the traces \( u_{-(2j-1)}|_{\Gamma} = g_{-(2j-1)}, \ 1 \leq j \leq q, \ q \geq 1 \), on the boundary, the \( q \) many Fourier modes \( u_{-(2j-1)}, \ 1 \leq j \leq q, \ q \geq 1 \), are unconstrained. They are chosen arbitrarily from the class \( \Psi_g^{\text{even}} \) of functions of cardinality \( q = \frac{m}{2} \) with prescribed trace on the boundary \( \Gamma \) defined as

\[
\Psi_g^{\text{even}} := \left\{ (\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}) \in \left( C^{1,\mu}(\overline{\Omega}; \mathbb{C}) \right)^q, 2\mu > 1 : \psi_{-(2j-1)}|_{\Gamma} = g_{-(2j-1)}, \ 1 \leq j \leq q, \ q \geq 1 \right\}.
\]

(43)

**Remark 4.1.** In the 0-tensor case (\( m = 0 \)), there is no class, and the characterization of the X-ray data \( g \) is in terms of the Fourier modes \( g \).

**Theorem 4.1** (Range characterization for even order tensors). (i) Let \( f \in C^{1,\mu}(S^m; \Omega), \ \mu > 1/2, \) be a real-valued symmetric tensor field of even order \( m = 2q, \ q \geq 0 \), and

\[
g = Xf \text{ on } \Gamma_+ \text{ and } g = 0 \text{ on } \Gamma_+ \cup \Gamma_0.
\]

Then \( g^{\text{even}}, g^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1) \) satisfy

\[
[I + 1\mathcal{H}]g^{\text{even}} = 0,
\]

(44)

\[
[I + 1\mathcal{H}]L_g^{\infty}g^{\text{odd}} = 0,
\]

(45)

where \( g^{\text{even}}, g^{\text{odd}} \) are sequences in (41), and \( \mathcal{H} \) is the Bukhgeim-Hilbert operator in (27).

(ii) Let \( g \in C^\mu(\Gamma; C^{1,\mu}(S^1)) \cap C(\Gamma; C^{2,\mu}(S^1)) \) be real valued with \( g|_{\Gamma_+ \cup \Gamma_0} = 0 \). For \( q = 0 \), if the corresponding sequences \( g^{\text{even}}, g^{\text{odd}} \in Y_\mu(\Gamma) \) satisfies (44) and (45), then there is a unique real valued symmetric 0-tensor \( f \) such that \( g|_{\Gamma_+} = Xf \). Moreover, for \( q \geq 1 \), if \( g^{\text{even}}, g^{\text{odd}} \in Y_\mu(\Gamma) \) satisfies (44) and (45), and for each element \( (\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}) \in \Psi_g^{\text{even}} \), then there is a unique real valued symmetric m-tensors \( f_\psi \in C^\mu(S^m; \Omega) \) such that \( g|_{\Gamma_+} = Xf_\psi \).

**Proof.** (i) **Necessity:** Let \( f = (f_i, \cdots, f_m) \in C^{1,\mu}_0(S^m; \Omega) \). Since all components \( f_i, \cdots, f_m \in C^{1,\mu}_0(\Omega) \) are compactly supported inside \( \Omega \), then for any point at the boundary there is a cone of lines which do not meet the support. Thus \( g \equiv 0 \) in the neighborhood of the variety \( \Gamma_0 \) which yields \( g \in C^{1,\mu}(\Gamma \times S^1) \). Moreover, \( g \) is the trace on \( \Gamma \times S^1 \) of a solution \( u \in C^{1,\mu}(\overline{\Omega} \times S^1) \) of the transport equation (37). By [39] Proposition 4.1 \( g^{\text{even}}, g^{\text{odd}} \in l_{\infty}^{1,1}(\Gamma) \cap C^\mu(\Gamma; l_1) \).

If \( u \) solves (37) then its Fourier modes satisfy (38), (39), and (40). Since the negative even Fourier modes \( u_{2n} \), for \( n \leq 0 \), satisfies the system (40), then the sequence valued map

\[
\Omega \ni z \mapsto u^{\text{even}}(z) := \langle u_0(z), u_{-2}(z), u_{-4}(z), \cdots \rangle
\]

is \( L \)-analytic in \( \Omega \) and the necessity part in Theorem 3.1 yields the condition (44).

The equation (39) for negative odd Fourier modes starting from negative \( 2q + 1 \) mode, yield that the sequence valued map

\[
z \mapsto \langle u_{-(2q+1)}, u_{-(2q+3)}, u_{-(2q+5)}, \cdots \rangle
\]

is \( L \)-analytic in \( \Omega \) and the necessity part in Theorem 3.1 gives the condition (45).
(ii) **Sufficiency:** Let \( g \in C^\mu(\Gamma; C^{1,\mu}(S^1)) \cap C(\Gamma; C^{2,\mu}(S^1)) \) be real valued with \( g|_{r_\pm r_0} = 0 \). Since \( g \) is real valued, its Fourier modes in the angular variable occurs in conjugates\(^{(46)}\)

\[
   g_{-n}(\zeta) = \overline{g}_n(\zeta), \quad \text{for } n \geq 0, \, \zeta \in \Gamma.
\]

Let the corresponding sequences \( g^{\text{even}} \) satisfying \(^{(44)}\) and \( g^{\text{odd}} \) satisfying \(^{(45)}\). By Proposition \(^{(3.1)}\), \( g^{\text{even}}, g^{\text{odd}} \in Y_\mu(\Gamma) \).

Let \( m = 2q, \, q \geq 0 \), be an even integer. To prove the sufficiency we will construct a real valued symmetric \( m \)-tensor \( f \) in \( \Omega \) and a real valued function \( u \in C^1(\Omega \times S^1) \cap C(\overline{\Omega} \times S^1) \) such that \( u|_{\Gamma \times S^1} = g \) and \( u \) solves \(^{(37)}\) in \( \Omega \). The construction of such \( u \) is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1:** The construction of even modes \( u_{2n} \) for \( n \in \mathbb{Z} \).

Apply the Bukhgeim-Cauchy Integral operator \(^{(25)}\) to construct the negative even Fourier modes:

\[
   \langle u_0(z), u_{-2}(z), u_{-4}(z), u_{-6}(z), \ldots \rangle := Bg^{\text{even}}(z), \quad z \in \Omega.
\]

By Theorem \(^{(3.1)}\) the sequence valued map

\[
   z \mapsto \langle u_0(z), u_{-2}(z), u_{-4}(z), \ldots \rangle \in C^{1,\mu}(\Omega) \cap C^\mu(\overline{\Omega}; l_1),
\]

is \( L \)-analytic in \( \Omega \), thus the equations

\[
   \overline{\partial}u_{-2n} + \partial u_{-2n-2} = 0,
\]

are satisfied for all \( n \geq 0 \). Moreover, the hypothesis \(^{(44)}\) and the sufficiency part of Theorem \(^{(3.1)}\) yields that they extend continuously to \( \Gamma \) and \( u_{-2n}|_{\Gamma} = g_{-2n}, \forall n \geq 0 \).

Construct the positive even Fourier modes by conjugation: \( u_{2n} := \overline{u}_{-2n} \), for all \( n \geq 1 \).

By conjugating \(^{(48)}\) we note that the positive even Fourier modes also satisfy

\[
   \overline{\partial}u_{2n+2} + \partial u_{2n} = 0, \quad n \geq 0.
\]

Moreover, by reality of \( g \) in \(^{(46)}\) they extend continuously to \( \Gamma \) and

\[
   u_{2n}|_{\Gamma} = \overline{u}_{-2n}|_{\Gamma} = \overline{g}_{-2n} = g_{2n}, \quad n \geq 1.
\]

Thus, as a summary from above equations, we have shown that the even modes \( u_{2n} \) satisfy

\[
   \overline{\partial}u_{2n} + \partial u_{2n-2} = 0, \quad \text{and} \quad u_{2n}|_{\Gamma} = g_{2n}, \quad \forall n \in \mathbb{Z}.
\]

**Step 2:** The construction of odd modes \( u_{2n-1} \), for \( |n| \geq q, \, q \geq 0 \).

Apply the Bukhgeim-Cauchy Integral operator \(^{(25)}\) to construct the other odd negative modes:

\[
   \langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), \ldots \rangle := B_{l_{-q}} g^{\text{odd}}(z), \quad z \in \Omega.
\]

By Theorem \(^{(3.1)}\) the sequence valued map

\[
   z \mapsto \langle u_{-(2q+1)}(z), u_{-(2q+3)}(z), u_{-(2q+5)}(z), \ldots \rangle \in C^{1,\mu}(\Omega) \cap C^\mu(\overline{\Omega}; l_1),
\]

is \( L \)-analytic in \( \Omega \), thus the equations

\[
   \overline{\partial}u_{-(2n+1)} + \partial u_{-(2n+3)} = 0,
\]

are satisfied for all \( n \geq q, \, q \geq 0 \). Moreover, the hypothesis \(^{(45)}\) and the sufficiency part of Theorem \(^{(3.1)}\) yields that they extend continuously to \( \Gamma \) and

\[
   u_{-(2n+1)}|_{\Gamma} = g_{-(2n+1)}, \quad \forall n \geq q, \, q \geq 0.
\]

Construct the positive odd Fourier modes by conjugation: \( u_{2n+1} := \overline{u}_{-(2n+1)} \), for all \( n \geq q, \, q \geq 0 \).

By conjugating \(^{(51)}\) we note that the positive odd Fourier modes also satisfy

\[
   \overline{\partial}u_{2n+3} + \partial u_{2n+1} = 0, \quad \forall n \geq q, \, q \geq 0.
\]
Moreover, by (46) they extend continuously to $\Gamma$ and

$$u_{2n+1}|_{\Gamma} = \mathcal{T}_{-1}(2n+1)|_{\Gamma} = \mathcal{T}_{-1}(2n+1) = g_{2n+1}, \quad n \geq q, q \geq 0.$$  

**Step 3: The construction of the tensor field $f$ in the $q = 0$ case.** In the case of the 0-tensor, $f = f_0$, and $f_0$ is uniquely determined from the odd Fourier mode $u_{-1}$ in (50), by

$$f_0 := 2 \Re \partial u_{-1}, \quad (\text{for } q = 0 \text{ case}).$$  

We consider next the case $q \geq 1$ of tensors of order 2 or higher. In this case the construction of the tensor field $f_{\Psi}$ is in terms of the Fourier mode $u_{-(2q+1)}$ in (50) and the class $\Psi_{\text{even}}^q$ in (43).

**Step 4: The construction of odd modes $u_{-(2n-1)}, \text{for } 1 \leq n \leq q, q \geq 1.$**

Recall the non-uniqueness class $\Psi_{\text{even}}^q$ in (43).

For $(\psi_{-1}, \psi_{-3}, \ldots, \psi_{-(2q-1)}) \in \Psi_{\text{even}}^q$ arbitrary, define the modes $u_{\pm 1}, u_{\pm 3}, \ldots, u_{\pm (2q-1)}$ in $\Omega$ by

$$u_{-(2n-1)} := \psi_{-(2n-1)} \quad \text{and} \quad u_{2n-1} := \mathcal{T}_{-1}(2n-1), \quad 1 \leq n \leq q, q \geq 1.$$  

By the definition of the class (43), and the reality of $g$ in (46), we have

$$u_{-(2n-1)}|_{\Gamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\Gamma} = \mathcal{T}_{-1}(2n-1) = g_{2n-1}, \quad 1 \leq n \leq q, q \geq 1.$$  

**Step 5: The construction of the tensor field $f_{\Psi}$ whose X-ray data is $g$.**

The components of the $m$-tensor $f_{\Psi}$ are defined via the one-to-one correspondence between the pseudovectors $\langle f_0, f_1, \ldots, f_m \rangle$ and the functions $\{f_{2n} : -q \leq n \leq q\}$ as follows.

For $q \geq 1$, we define $f_{2q}^\Psi$ by using $\psi_{-(2q-1)}$ from the non-uniqueness class (43), and Fourier mode $u_{-(2q+1)}$ from the Bukhgeim-Cauchy formula (50). Then, define $\{f_{2n} : 0 \leq n \leq q-1\}$ solely from the information in the non-uniqueness class. Finally, define $\{f_{-2n} : 1 \leq n \leq q\}$ by conjugation.

$$f_{2q} := \mathcal{T}_{-1}(2q-1) + \partial u_{-(2q+1)}, \quad q \geq 1,$$

$$f_{2n} := \mathcal{T}_{-1}(2n-1) + \partial \psi_{-(2n+1)}, \quad 1 \leq n \leq q-1, q \geq 2,$$

$$f_0 := 2 \Re \partial \psi_{-1}, \quad \text{and} \quad q \geq 1,$$

$$f_{-2n} := \mathcal{T}_{2n}, \quad 1 \leq n \leq q, q \geq 1.$$  

By construction, $f_{2n} \in C^\mu(\Omega)$, for $-q \leq n \leq q$, as $\psi_{-1}, \ldots, \psi_{-(2q+1)} \in C^{1,\mu}(\Omega)$. We use these Fourier modes $f_0, f_{\pm 2}, f_{\pm 4}, \ldots, f_{\pm 2q}$ for $q \geq 1$, and equations (13), (7) and (9) to construct the pseudovectors $\langle \tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_m \rangle$, and thus the $m$-tensor field $f_{\Psi} \in C^\mu(S^m; \Omega)$.

In order to show $g|_{\Gamma} = Xf_{\Psi}$ for $q \geq 1$, with $f_{\Psi}$ being constructed as in (58), we define the real valued function $u$ via its Fourier modes for $q \geq 1$,

$$u(z, \theta) = \sum_{n=\infty}^{\infty} u_{2n} e^{i2n\theta} + \sum_{|n| \geq q} u_{2n+1} e^{i(2n+1)\theta} + \sum_{n=1}^{q} \psi_{-(2n-1)} e^{-i(2n-1)\theta} + \sum_{n=1}^{q} \mathcal{T}_{-1}(2n-1) e^{i(2n-1)\theta}.$$  

Since $g \in C^\mu(\Gamma; C^{1,\mu}(\Omega)) \cap C(\Gamma; C^{2,\mu}(\Omega))$, we use Proposition 3.1 (ii) and [39, Proposition 4.1 (iii)] to conclude that $u$ defined in (59) belongs to $C^{1,\mu}(\Omega \times S^1) \cap C^{\mu}(\Omega \times S^1)$. Using (49), (52), (54), (57), and definition of $(\psi_{-1}, \psi_{-3}, \ldots, \psi_{-(2q-1)}) \in \Psi_{\text{even}}^q$ for $q \geq 1$, the trace $u(\cdot, \theta)$ in (59) extends to the boundary,

$$u(\cdot, \theta)|_{\Gamma} = g(\cdot, \theta).$$
Assume that justified, and $u$ satisfy (57):

$$
\theta \cdot \nabla u = \partial \psi^{-1} + \partial \psi^{-1} + \sum_{n=1}^{q-1} \left( \psi^{-2(n-1)} + \psi^{-2(n+1)} \right)c^{-(2n)\theta} + \sum_{n=1}^{q-1} \left( \psi^{-2(n+1)} + \psi^{-2(n-1)} \right)c^{i(2n)\theta}
$$

$$
+ e^{-\theta}(\psi^{-2(2q-1)} + \psi^{-2(2q+1)}) + e^{i\theta}(\psi^{-2(q-1)} + \psi^{-2(q+1)})
$$

$$
= \sum_{n=-q}^{q} f_{2n}(z)e^{-\theta} = \langle f, \theta \rangle,
$$

where the cancellation uses equations (49), (51), (53), (56), and the second equality uses the definition of $f_{2k}$’s in (58).

\[\square\]

5. Odd order $m$-tensor - non-attenuated case

In this section we establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times S^1$ to be the non-attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $f$ of odd order $m = 2q + 1$, $q \geq 0$.

In the non-attenuated odd $m$-tensor case, the transport equation (19a) becomes

$$
\theta \cdot \nabla u(z, \theta) = \sum_{n=0}^{q} \left( f_{2n+1}(z)e^{-\theta} + f_{2n+1}(z)e^{\theta} \right), \quad (z, \theta) \in \Omega \times S^1,
$$

where $f_{2n+1}$ defined in (14), and $f_{2n+1} = f_{-2n-1}$, for $0 \leq n \leq q$, $q \geq 0$.

If $\sum_{n \in \mathbb{Z}} u_n(z)e^{in\theta}$ is the Fourier series expansion in the angular variable $\theta$ of a solution $u$ of (60), then, by identifying the Fourier modes of the same order, the equation (60) reduces to the system:

$$
\partial u_{-2n}(z) + \partial u_{-2(n+2)}(z) = f_{2n+1}(z), \quad 0 \leq n \leq q, \quad q \geq 0,
$$

$$
\partial u_{-2n}(z) + \partial u_{-2(n+2)}(z) = 0, \quad n \geq q + 1, \quad q \geq 0,
$$

$$
\partial u_{-2(n+1)}(z) + \partial u_{-(2n+1)}(z) = 0, \quad n \geq 0.
$$

In the odd $m$-tensor case, the even and odd Fourier modes of $u$ plays a different role, unlike the even $m$-tensor case in the previous section. To emphasize this difference we separate the non-positive even modes $u^{\text{even}} := \langle u_0, u_{-2}, u_{-4}, \ldots \rangle$, and negative odd modes $u^{\text{odd}} := \langle u_{-1}, u_{-3}, \ldots \rangle$, and note that if $\langle u_0(z), u_{-1}(z), u_{-2}(z), \ldots \rangle$ is $L^2$-analytic, then $u^{\text{even}}, u^{\text{odd}}$ are $L$-analytic.

Let us consider the sequence $\{u^{2k-1}\}_{k \geq 1} \subset C(\Omega; l_\infty) \cap C^1(\Omega; l_\infty)$ given by

$$
u_{2k-1} := \langle u_{2k-1}, u_{2k-3}, \ldots, u_1, u_{-1}, u_{-3}, u_{-5}, \ldots \rangle, \quad k \geq 1,
$$

obtained by augmenting the sequence of negative odd indices $\langle u_{-1}, u_{-3}, u_{-5}, \ldots \rangle$ by $k$ many terms in the order $u_{2k-1}, u_{2k-3}, \ldots, u_1$.

One of the ingredients in our characterization of the odd $m$-tensor is the following simple property of $L$-analytic maps, shown in [39, Lemma 2.6].

**Lemma 5.1.** [39, Lemma 2.6] Let $\{u^{2k-1}\}_{k \geq 1}$ be the sequence of $L$-analytic maps defined in (64). Assume that

$$
u_{2k-1}|_f = \nu_{-(2k-1)}|_f, \quad \forall k \geq 1.
$$


Then, for each $k \geq 1$,
\begin{equation}
\begin{multlined}
    u_{2k-1}(z) = u_{-(2k-1)}(z), \quad z \in \Omega.
\end{multlined}
\end{equation}

The range characterization of data $g$ will be given in terms of its Fourier modes:
\begin{equation}
    g(\zeta, \theta) = \sum_{n=-\infty}^{\infty} g_n(\zeta) e^{in\theta}, \quad \zeta \in \Gamma.
\end{equation}
Since the trace $g$ is also real valued, its Fourier modes will satisfy $g_n = \overline{g_n}$, for $n \geq 0$. From the non-positive even modes, we build the sequence
\begin{equation}
    g_{\text{even}} := \{g_0, g_{-2}, g_{-4}, g_{-6}, \ldots\}.
\end{equation}
For each $k \geq 1$, we use the odd modes $\{g_{-1}, g_{-3}, g_{-5}, \ldots\}$ to build the sequence
\begin{equation}
    g^{2k-1} := \{g_{2k-1}, g_{2k-3}, \ldots, g_1\}
\end{equation}
by augmenting the negative odd indices by $k$-many terms in the order $g_{2k-1}, g_{2k-3}, \ldots, g_1$.

Similar to the non-attenuated even $m$-tensor case before, we will construct the solution $u$ of the transport equation (60), whose trace matches the boundary data $g$, and also construct the right hand side of the (60). The construction of solution $u$ is in terms of its Fourier modes in the angular variable. Except for non-positive modes $u_0, u_{-2}, \ldots, u_{-2q}$, all non-positive modes are defined by Bukhgeim-Cauchy integral formula in (25) using boundary data. Other than having the traces $u_{-2j}\big|_{\Gamma} = g_{-2j}, \; 0 \leq j \leq q, \; q \geq 0$, on the boundary, the $q + 1$ many Fourier modes $u_{-2j}, \; 0 \leq j \leq q, \; q \geq 0$, are unconstrained. They are chosen arbitrarily from the class of functions
\begin{equation}
    \Psi_{g}^{\text{odd}} := \{\{\psi_0, \psi_{-2}, \ldots, \psi_{-2j}\} \in C^{1,\mu}(\overline{\Omega}; \mathbb{R}) \times (C^{1,\mu}(\overline{\Omega}; \mathbb{C}))^q : 2\mu > 1 :
    \begin{cases}
        \psi_{-2j}\big|_{\Gamma} = g_{-2j}, \; 0 \leq j \leq q, \; q \geq 0
    \end{cases}\}
\end{equation}

\noindent **Remark 5.1.** In the 1-tensor case ($m = 1$), only Fourier mode $u_0$ be an arbitrary function in $C^1(\Omega) \cap C(\overline{\Omega})$ with $u_0\big|_{\Gamma} = g_0$. The arbitrariness of $u_0$ characterizes the non-uniqueness (up to the gradient field of a function which vanishes at the boundary) in the reconstruction of a vector field from its Doppler data.

**Theorem 5.1** (Range characterization for odd tensors.). Let $f \in C^{1,\mu}(\overline{S^m}; \Omega)$, $\mu > 1/2$, be a real-valued symmetric tensor field of odd order $m = 2q + 1$, $q \geq 0$, and
\begin{equation}
    g = Xf \text{ on } \Gamma_+, \text{ and } g = 0 \text{ on } \Gamma_+ \cup \Gamma_0.
\end{equation}
Then $g_{\text{even}}, g^{2k-1} \in L^{1,1}_\infty(\Gamma) \cap C^\mu(\Gamma; l_1)$ for $k \geq 1$, and satisfy
\begin{align}
    [I + i\mathcal{H}] L^{\frac{m+1}{2}} g_{\text{even}} &= 0, \quad (70) \\
    [I + i\mathcal{H}] g^{2k-1} &= 0, \quad \forall k \geq 1, \quad (71)
\end{align}
where $g_{\text{even}}$ is the sequence in (41), $g^{2k-1}$ for $k \geq 1$ is the sequence in (68), and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in (27).

(ii) Let $g \in C^\mu(\Gamma; C^{1,\mu}(\overline{S^1})) \cap C(\Gamma; C^{2,\mu}(\overline{S^1}))$ be real valued with $g\big|_{\Gamma_+ \cup \Gamma_0} = 0$. If the corresponding sequence $g_{\text{even}} \in Y_\mu(\Gamma)$ satisfies (70), $g^{2k-1} \in Y_\mu(\Gamma)$ for $k \geq 1$, satisfies (71), and for each element $(\psi_0, \psi_{-2}, \ldots, \psi_{-2j}) \in \Psi_{g}^{\text{odd}}$, then there is a unique real valued symmetric $m$-tensor $f_\psi \in C^\mu(\overline{S^m}; \Omega)$ such that $g\big|_{\Gamma_+} = Xf_\psi$. 

Proof. (i) Necessity: Let \( f = (f_{(i_1,\ldots,i_m)}) \in C^{1,H_0}(S^m;\Omega) \). Since all components \( f_{(i_1,\ldots,i_m)} \in C^{H_0}(\Omega) \), \( Xf \in C^{1,H_0}(\Gamma_+) \), and, thus, the solution \( u \) to the transport equation \( (60) \) is in \( C^{1,H_0}(\Omega \times S^1) \). Moreover, its trace \( g = u|_{\Gamma \times S^1} \in C^{1,H_0}(\Gamma \times S^1) \). By Proposition 4.1 \( g^{\text{even}}, g^{2k-1} \in L^1_\infty(\Gamma) \cap C^H(\Gamma;I_1) \) for all \( k \geq 1 \).

If \( u \) solves \( (60) \) then its Fourier modes satisfy \( (61), (62), \) and \( (63) \). Since the negative even Fourier modes \( u_{-2n} \) for \( n \geq \frac{m+2}{2} \), satisfies the system \( (62) \), then the sequence valued map
\[
\Omega \ni z \mapsto u(z) := (u_1(z), u_{-1}(z), u_{-3}(z), \ldots)
\]
is \( L \)-analytic in \( \Omega \) and the necessity part in Theorem 3.1 yields the condition \( (70) \).

The system \( (63) \) yield that the sequence valued map
\[
\Omega \ni z \mapsto u^1(z) := (u_1(z), u_{-1}(z), u_{-3}(z), \ldots)
\]
is \( L \)-analytic in \( \Omega \) with the trace satisfying \( u_{2k-1}|_{\Gamma} = g_{2k-1} \), for all \( k \leq 1 \).
By Theorem 3.1 necessity part, the sequence \( g^1 = (g_1, g_{-1}, g_{-3}, \ldots) \) must satisfy
\[
[I + iH]g^1 = 0.
\]
Recall that \( u \) is real valued so that its Fourier modes occur in conjugates \( u_n = \overline{u_{-n}} \) for all \( n \geq 0 \). Consider now the equation \( (63) \) for \( n = 1 \) and take its conjugate to yield
\[
(72)
\]
Equation \( (72) \) together with \( (63) \) yield that the sequence valued map
\[
\Omega \ni z \mapsto u^3(z) := (u_3(z), u_1(z), u_{-1}(z), u_{-3}(z), \ldots)
\]
is \( L \)-analytic in \( \Omega \) with the trace satisfying \( u_{2k-1}|_{\Gamma} = g_{2k-1} \) for all \( k \leq 2 \).
By the necessity part in Theorem 3.1 it must be that \( g^3 = (g_3, g_1, g_{-1}, g_{-3}, \ldots) \) satisfies
\[
[I + iH]g^3 = 0.
\]
Inductively, the argument above holds for any odd index \( 2k - 1 \) to yield that the sequence
\[
\Omega \ni z \mapsto u^{2k-1}(z) := (u_{2k-1}(z), u_{2k-3}(z), \ldots, u_1(z), u_{-1}(z), u_{-3}(z), \ldots)
\]
is \( L \)-analytic in \( \Omega \). Then, again by the necessity part in Theorem 3.1 its trace \( u^{2k-1}|_{\Gamma} = g^{2k-1} \) must satisfy the condition \( (71) \):
\[
[I + iH]g^{2k-1} = 0, \quad \text{for all} \ k \geq 1.
\]
(ii) Sufficiency: Let \( g \in C^H(\Gamma; C^{1,H_0}(S^1)) \cap C(\Gamma; C^{2,H_0}(S^1)) \) be real valued with \( g|_{\Gamma \cup \Gamma_0} = 0 \).
Since \( g \) is real valued, its Fourier modes in the angular variable occurs in conjugates
\[
(73)
\]
Let the corresponding sequences \( g^{\text{even}} \) satisfying \( (44) \) and \( g^{\text{odd}} \) satisfying \( (45) \). By Proposition 3.1, \( g^{\text{even}}, g^{\text{odd}} \in Y_{\mu}(\Gamma) \).

Let \( m = 2q + 1, q \geq 0, \) be an odd integer. To prove the sufficiency we will construct a real valued symmetric \( m \)-tensor \( f \) in \( \Omega \) and a real valued function \( u \in C^1(\Omega \times S^1) \cap C(\Omega \times S^1) \) such that \( u|_{\Gamma \times S^1} = g \) and \( u \) solves \( (60) \) in \( \Omega \). The construction of such \( u \) is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1:** The construction of even modes \( u_{2n} \) for \( |n| \geq 2q + 1, q \geq 0 \).
Apply the Bukhgeim-Cauchy integral formula \( (25) \) to construct the negative even Fourier modes:
\[
(74)
\]
By Theorem 3.1, the sequence valued map
\[ \Omega \ni z \mapsto \langle u_{-2(q+1)}(z), u_{-2(q+2)}(z), u_{-2(q+3)}(z), \ldots \rangle \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1), \]
is $L$-analytic in $\Omega$, thus the equations
\[ (75) \quad \overline{\partial}u_{-2n} + \partial u_{-(2n+2)} = 0, \]
are satisfied for all $n \geq q + 1$, $q \geq 0$. Moreover, the hypothesis (70) and the sufficiency part of Theorem 3.1 yields that they extend continuously to $\Gamma$ and
\[ (76) \quad u_{-2n}|_\Gamma = g_{-2n}, \quad n \geq q + 1, \quad q \geq 0. \]

Construct the positive even Fourier modes by conjugation: $u_{2n} := \overline{u_{-2n}}$, for all $n \geq q + 1$, $q \geq 0$.

By conjugating (75) we note that the positive even Fourier modes also satisfy
\[ (77) \quad \overline{\partial}u_{2n+2} + \partial u_{2n} = 0, \quad n \geq q + 1, \quad q \geq 0. \]
Moreover, by reality of $g$ in (73), they extend continuously to $\Gamma$ and
\[ (78) \quad u_{2n}|_\Gamma = \overline{u_{-2n}}|_\Gamma = g_{-2n} = g_{2n}, \quad n \geq q + 1, \quad q \geq 0. \]

**Step 2: The construction of even modes $u_{2n}$, for $|n| \leq 2q$, $q \geq 0$.**

Recall the non-uniqueness class $\Psi_g^{\text{odd}}$ in (69).

For $(\psi_0, \psi_{-2}, \ldots, \psi_{-2q}) \in \Psi_g^{\text{odd}}$ arbitrary, define the modes $u_0, u_{\pm 2}, u_{\pm 4}, \ldots, u_{\pm 2q}$ in $\Omega$ by
\[ (79) \quad u_{-2n} := \psi_{-2n}, \quad \text{and} \quad u_{2n} := \overline{\psi_{-2n}}, \quad 0 \leq n \leq q. \]

By the definition of the class (69), and reality of $g$ in (73), we have
\[ (80) \quad u_{2n}|_\Gamma = \overline{g_{-2n}} = g_{2n}, \quad 0 \leq n \leq q. \]

**Step 3: The construction of negative modes $u_{2n-1}$ for $n \in \mathbb{Z}$.**

Use the Bukhgeim-Cauchy Integral formula (25) to construct the negative odd Fourier modes:
\[ (81) \quad \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), \ldots \rangle := Bg^{\text{odd}}(z), \quad z \in \Omega. \]

By Theorem 3.1 the sequence valued map
\[ \Omega \ni z \mapsto \langle u_{-1}(z), u_{-3}(z), u_{-5}(z), \ldots \rangle \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1), \]
is $L$-analytic in $\Omega$, thus the equations
\[ (82) \quad \overline{\partial}u_{-2n-1} + \partial u_{-2n-3} = 0, \]
are satisfied for all $n \geq 0$.

Note that $Lg^1 = g^{\text{odd}}$. By hypothesis (71), $[I + i\mathcal{H}]g^1 = 0$. Since $\mathcal{H}$ commutes with the left translation $L$, then
\[ 0 = L[I + i\mathcal{H}]g^1 = [I + i\mathcal{H}]Lg^1 = [I + i\mathcal{H}]g^{\text{odd}}. \]

By applying Theorem 3.1 sufficiency part, we have that each $u_{2n-1}$ extends continuously to $\Gamma$:
\[ u_{2n-1}|_\Gamma = g_{-2n-1}, \quad n \geq 1. \]

If we were to define the positive odd index modes by conjugating the negative ones (as we did for the non-attenuated even tensor case) it would not be clear why the equation (63) for $n = 0$:
\[ \overline{\partial}u_1 + \partial u_{-1} = 0, \]
should hold. To solve this problem we will define the positive odd modes by using the Bukhgeim-Cauchy integral formula (25) inductively.
Let \( u^1 = \langle u_1, u_{-1}, u_{-3}, \cdots \rangle \) be the \( L \)-analytic map defined by
\begin{equation}
(83) \quad u^1 := Bg^1.
\end{equation}
The hypothesis (71) for \( k = 1 \),
\[ [I + i\mathcal{H}]g^1 = 0, \]
allows us to apply the sufficiency part of Theorem 3.1 to yield that \( u^1 \) extends continuously to \( \Gamma \) and has trace \( g^1 \) on \( \Gamma \). However, \( Lu^1 = u^{\text{odd}} \) is also \( L \)-analytic with the same trace \( g^{\text{odd}} \) as \( u^{\text{odd}} \). By the uniqueness of \( L \)-analytic maps with the given trace we must have the equality
\[ \langle u^1_{-1}, u^1_{-3}, \cdots \rangle = \langle u_{-1}, u_{-3}, \cdots \rangle. \]
In other words the formula (83) constructs only one new function \( u_1 \) and recovers the previously defined negative odd functions \( u_{-1}, u_{-3}, \cdots \). In particular \( u^1 = \langle u_1, u_{-1}, u_{-3}, \cdots \rangle \) is \( L \)-analytic, and the equation \( \partial \overline{u}_1 + \partial u_{-1} = 0 \) holds in \( \Omega \). We stress here that, at this stage, we do not know that \( u_1 \) is the complex conjugate of \( u_{-1} \).

Inductively, for \( k \geq 1 \), the formula
\begin{equation}
(84) \quad u^{k-1} = \langle u_{2k-1}, u_{2k-3}, \ldots, u_1, u_{-1}, \cdots \rangle := Bg^{2k-1}
\end{equation}
defines a sequence \( \{u^{k-1}\}_{k \geq 1} \) of \( L \)-analytic maps with \( u^{k-1}|_\Gamma = g^{2k-1} \). By the uniqueness of \( L \)-analytic maps with the given trace, a similar reasoning as above shows
\[ Lu^{k-1} = u^{k-3}, \quad \forall k \geq 2. \]
In particular for all \( k \geq 1 \), the sequence
\[ \{u^{k-1}\}_{k \geq 1} \]
is \( L \)-analytic. Note that the sequence \( \{u^{k-1}\}_{k \geq 1} \) constructed above satisfies the hypotheses of the Lemma 5.1 and therefore for each \( k \geq 1 \),
\begin{equation}
(85) \quad u_{2k-1}(z) = \overline{\psi}_{-(2k-1)}(z), \quad z \in \Omega.
\end{equation}
We stress here that the identities (85) need the hypothesis (71) for all \( k \geq 1 \), cannot be inferred directly from the Bukhgeim-Cauchy integral formula (25) for finitely many \( k \)'s.

We have shown that
\begin{equation}
\overline{\partial}u_{2n-1} + \partial u_{2n-3} = 0, \quad \text{and} \quad u_{2n-1}|_\Gamma = g_{2n-1}, \quad \forall n \in \mathbb{Z}.
\end{equation}

**Step 4: The construction of the tensor field \( f_\psi \) whose X-ray data is \( g \).**

The components of the \( m \)-tensor \( f_\psi \) are defined via the one-to-one correspondence between the pseudovectors \( \langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle \) and the functions \( \{f_{\pm n}^{\pm m+1} : 0 \leq n \leq q \} \) as follows.

For \( q \geq 0 \), we define \( f_{2q+1} \) by using \( \psi_{-2q} \) from the non-uniqueness class in (69), and Fourier mode \( u_{-(2q+2)} \) from the Bukhgeim-Cauchy formula (74). Then, define \( \{f_{2n+1} : 0 \leq n \leq q-1 \} \) solely from the information in the non-uniqueness class. Finally, define \( \{f_{-(2n+1)} : 0 \leq n \leq q \} \) by conjugation.

\begin{align*}
f_{2q+1} & := \overline{\partial}\psi_{-2q} + \partial u_{-(2q+2)}, \quad q \geq 0, \\
f_{2n+1} & := \overline{\partial}\psi_{-2n} + \partial \psi_{-(2n+2)}, \quad 0 \leq n \leq q-1, \quad q \geq 1, \quad \text{and}
\end{align*}
\begin{align*}
f_{-(2n+1)} & := \overline{f}_{2n+1}, \quad 0 \leq n \leq q, \quad q \geq 0.
\end{align*}
By construction, \( f_{\pm(2n+1)} \in C^\mu(\Omega) \), for \( 0 \leq n \leq q \), as \( \psi_0, \psi_{-2}, \ldots, \psi_{-2q} \in C^{1, \mu}(\Omega) \). We use these Fourier modes \( f_{\pm1}, f_{\pm3}, \ldots, f_{\pm m} \) for \( m = 2q + 1 \), \( q \geq 0 \), and equations (14), (7) and (9) to construct the pseudovectors \( \{ \tilde{f}_0, \tilde{f}_1, \ldots, \tilde{f}_m \} \), and thus the \( m \)-tensor field \( f_\Psi \in C^\mu(S^m; \Omega) \).

In order to show \( g|_{\Gamma_\alpha} = Xf_\Psi \) with \( f_\Psi \) being constructed from pseudovectors via Fourier modes as in (87) from class \( \Psi^{\text{odd}} \), we define the real valued function \( u \) via its Fourier modes

\[
(88) \quad u(z, \theta) := \sum_{n=-\infty}^{\infty} u_{2n-1}(z)e^{i(2n-1)\theta} + \sum_{|n| \geq 1} u_{2n}(z)e^{2n\theta} + \sum_{n=0}^{q} \psi_{-2n}(z)e^{-i2n\theta} + \sum_{n=0}^{q} \tilde{\psi}_{-2n}(z) e^{i2n\theta}.
\]

Since \( g \in C^\mu(\Gamma; C^{1, \mu}(\mathbb{S}^1)) \cap C(\Gamma; C^{2, \mu}(\mathbb{S}^1)) \), we use Proposition 3.1 (ii) and [39, Proposition 4.1 (iii)] to conclude that \( u \) defined in (88) belongs to \( C^{1, \mu}(\Omega \times \mathbb{S}^1) \cap C(\overline{\Omega} \times \mathbb{S}^1) \).

Using (76), (78), (80), (86), and element \((\psi_0, \psi_{-2}, \ldots, \psi_{-2q}) \in \Psi^{\text{odd}}_g \), the \( u(\cdot, \theta) \) in (88) extends to the boundary

\[
(92) \quad u(\cdot, \theta)|_{\Gamma} = g(\cdot, \theta),
\]

Since \( u \in C^{1, \mu}(\Omega \times \mathbb{S}^1) \cap C^{\mu}(\overline{\Omega} \times \mathbb{S}^1) \), then the term by term differentiation in (88) is now justified, satisfying the transport equation (60):

\[
\theta \cdot \nabla u = 2 \mathbb{R} \left\langle \left( \partial \psi_{-2q} + \partial u_{-(2q+2)} \right) e^{i(2q+1)\theta} \right\rangle + 2 \mathbb{R} \left\langle \sum_{n=0}^{q-1} \left( \partial \psi_{-2n} + \partial \psi_{-(2n+2)} \right) e^{i(2n+1)\theta} \right\rangle
\]

\[
= \sum_{n=0}^{q} \left( f_{2n+1} e^{-i(2n+1)\theta} + f_{-(2n+1)} e^{i(2n+1)\theta} \right) = \langle f, \theta^{2q+1} \rangle,
\]

where the cancellation uses equations (75), (77), (80), and the second equality uses the definition of \( f_{2k+1}\)’s in (87).

\[\square\]

6. Even order \( m \)-tensor - attenuated case

Let \( a \in C^{2, \mu}(\overline{\Omega}) \), \( \mu > 1/2 \), with \( \min \Omega > 0 \). We now establish necessary and sufficient conditions for a sufficiently smooth function on \( \Gamma \times \mathbb{S}^1 \) to be the attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field \( f \) of even order \( m = 2q \), \( q \geq 0 \). In this case \( a \neq 0 \), the transport equation (19a) becomes

\[
(89) \quad \theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \sum_{k=0}^{q} f_{-2k} e^{i(2k)\theta} + \sum_{k=1}^{q} f_{2k} e^{-i(2k)\theta},
\]

where \( f_{2k} \) defined in (13), and \( f_{2k} = \overline{f_{-2k}} \), for \(-q \leq k \leq q \), \( q \geq 0 \).

If \( \sum_{n \in \mathbb{Z}} u_n(z) e^{in\theta} \) is the Fourier series expansion in the angular variable \( \theta \) of a solution \( u \) of (89), then by identifying the Fourier coefficients of the same order, equation (89) reduces to the system:

\[
(90) \quad \overline{\partial} u_{-(2n+1)}(z) + \partial u_{-(2n+1)}(z) + a u_{-2n}(z) = f_{2n}(z), \quad 0 \leq n \leq q, \ q \geq 0,
\]

\[
(91) \quad \overline{\partial} u_{-2n}(z) + \partial u_{-(2n+2)}(z) + a u_{-2n-1}(z) = 0, \quad 0 \leq n \leq q - 1, \ q \geq 1,
\]

\[
(92) \quad \overline{\partial} u_{-n}(z) + \partial u_{-(n+2)}(z) + a u_{-(n+1)}(z) = 0, \quad n \geq 2q, \ q \geq 0.
\]
Recall that the trace \( u|_{\Gamma_+ \times \mathbb{R}^1} := g \) as in (21), with \( q = X_a f \) on \( \Gamma_+ \) and \( g = 0 \) on \( \Gamma_- \cup \Gamma_0 \).

We expand the attenuated X-ray data \( g \) in terms of its Fourier modes in the angular variables:

\[
g(\zeta, \theta) = \sum_{n=-\infty}^{\infty} g_n(\zeta)e^{in\theta}, \quad \zeta \in \Gamma.
\]

Since the trace \( g \) is also real valued, its Fourier modes will satisfy \( g_{-n} = \overline{g_n} \), for \( n \geq 0 \). From the negative modes, we built the sequence \( g := (g_0, g_{-1}, g_{-2}, g_{-3}, \ldots) \). From the special function \( h \) defined in (32) and the data \( g \), we built the sequence

\[
g_h := e^{-G}g := \langle \gamma_0, \gamma_1, \gamma_2, \gamma_3, \ldots \rangle,
\]

where \( e^{\pm G} \) as defined in (34). From the negative even, respectively, negative odd Fourier modes, we built the sequences

\[
\Psi^\text{even}_a := \langle \gamma_0, \gamma_2, \gamma_4, \ldots \rangle, \quad \text{and} \quad \Psi^\text{odd}_a := \langle \gamma_1, \gamma_3, \gamma_5, \ldots \rangle.
\]

Next we characterize the attenuated X-ray data \( g \) in terms of its Fourier modes \( \langle g_0, g_{-1}, g_{-2}, \cdots g_{-(m-1)} \rangle \), and the Fourier modes

\[
L^m g_h := L^m e^{-G}g := \langle \gamma_{-m}, \gamma_{-(m+1)}, \gamma_{-(m+2)}, \ldots \rangle.
\]

Similar to the non-attenuated case as before, we construct simultaneously the right hand side of the transport equation (89) together with the solution \( u \) via its Fourier modes. For \( m = 2q, q \geq 1 \), apart from modes \( u_0, u_{-1}, u_{-2}, \cdots u_{-(2q-1)} \), all Fourier modes are constructed uniquely from the data \( L^{2q}g_h \). The modes \( u_0, u_{-2}, u_{-4}, \cdots u_{-(2q-2)} \) will be chosen arbitrarily from the class \( \Psi_{a,g} \) of cardinality \( q = \frac{m}{2} \) with prescribed trace and gradient on the boundary \( \Gamma \) defined as

\[
\Psi^\text{even}_{a,g} := \left\{ \left( \psi_0, \psi_{-2}, \cdots, \psi_{-(2q-1)} \right) \in C^2(\overline{\Omega}; \mathbb{R}) \times \left( C^2(\overline{\Omega}; \mathbb{C}) \right) \right\}^q:
\]

\[
\begin{align*}
\psi_{-2j}|_\Gamma &= g_{-2j}, \quad 0 \leq j \leq q-1, \quad q \geq 1, \\
\overline{\partial} \psi_{-(2q-1)}|_\Gamma &= -\partial(e^G B e^{-G}g)_{-2q}|_\Gamma - a|_\Gamma g_{-(2q-1)}, \quad q \geq 1, \\
\overline{\partial} \psi_{-2j}|_\Gamma &= -\overline{\partial} \psi_{-(2j+2)}|_\Gamma - a|_\Gamma g_{-(2j+1)}, \quad 0 \leq j \leq q-2, \quad q \geq 2
\end{align*}
\]

where \( B \) be the Bukheim-Cauchy operator in (25), and the operators \( e^{\pm G} \) as defined in (34).

**Remark 6.1.** In the 2-tensor case \( m = 2 \), apart from zeroth mode \( u_0 \) and negative one mode \( u_{-1} \), all Fourier modes are constructed uniquely from the data \( L^2g_h \). The mode \( u_0 \) will be chosen arbitrarily from the class \( \Psi_{a,g}^m \). We rewrite the above class \( \Psi_{a,g}^m \) explicitly for \( m = 2 \), as

\[
\Psi_{a,g}^m := \left\{ \psi_0 \in C^2(\overline{\Omega}; \mathbb{R}) : \psi_0|_\Gamma = g_0, \quad \overline{\partial} \psi_0|_\Gamma = -\partial(e^G B e^{-G}g)_{-2}|_\Gamma - a|_\Gamma g_{-1} \right\}.
\]

In the 0-tensor case \( m = 0 \), there is no class, and the characterization of the attenuated X-ray data \( g \) is in terms of the Fourier modes \( g_h := e^{-G}g \).

Next, we characterize the range for even \( m = 2q, q \geq 0 \), in the attenuated case.
Theorem 6.1 (Range characterization for even order tensors). Let $a \in C^{2,\mu}(\overline{\Omega})$, $\mu > 1/2$ with $\min a > 0$. (i) Let $f \in C^{1,\mu}(S^n; \Omega)$, be a real-valued symmetric tensor field of even order $m = 2q$, $q \geq 0$, and $g = X_a f$ on $\Gamma_+$ and $g = 0$ on $\Gamma_0 \cup \Gamma_0$. Then $g^\text{even}_h, g^\text{odd}_h \in L_{\infty}^2(\Gamma) \cap C^\mu(\Gamma; l_1)$ satisfy
\begin{equation}
[I + i\mathcal{H}] L^m T g^\text{even}_h = 0, \quad [I + i\mathcal{H}] L^m T g^\text{odd}_h = 0.
\end{equation}
where $g^\text{even}_h, g^\text{odd}_h$ are sequences in $\mathcal{W}_3$, and $\mathcal{H}$ is the Bukhgeim-Hilbert operator in (27).

(ii) Let $g \in C^\mu(\Gamma; C^{1,\mu}(S^1)) \cap C(\Gamma; C^{2,\mu}(S^1))$ be real valued with $g|_{\Gamma_0 \cup \Gamma_0} = 0$. For $q = 0$, if the corresponding sequences $g^\text{even}_h, g^\text{odd}_h \in Y^\mu(\Gamma)$ satisfies (96), then there is a unique real valued symmetric 0-tensor $f$ such that $g|_{\Gamma_+} = X_a f$. Moreover, for $q \geq 1$, if $g^\text{even}_h, g^\text{odd}_h \in Y^\mu(\Gamma)$ satisfies (96), and for each element $(\psi_0, \psi_{-2}, \cdots, \psi_{-2(q-1)}) \in \Psi^\text{even}_{a,g}$, there is then a unique real valued symmetric $m$-tensor $f_\psi \in C(S^m; \Omega)$ such that $g|_{\Gamma_+} = X_a f_\psi$.

Proof. (i) Necessity: Let $f = (f_{i_1 \cdots i_m}) \in C^{1,\mu}_0(S^n; \Omega)$. Since all components $f_{i_1 \cdots i_m} \in C^{1,\mu}_0(\Omega)$ are compactly supported inside $\Omega$, for any point at the boundary there is a cone of lines which do not meet the support. Thus $g \equiv 0$ in the neighborhood of the variety $\Gamma_0$ which yields $g \in C^{1,\mu}(\Gamma \times S^1)$. Moreover, $g$ is the trace on $\Gamma \times S^1$ of a solution $u \in C^{1,\mu}(\overline{\Omega} \times S^1)$ of the transport equation (89). By Proposition 3.1(i) and Proposition 3.2, $g^\text{even}_h = e^{-G} g \in L_{\infty}^2(\Gamma) \cap C^\mu(\Gamma; l_1)$.

If $u$ solves (89) then its Fourier modes satisfy (90), (91) and (92). In particular, the sequence valued map $\mathbf{u} := \langle u_0, u_{-1}, u_{-2}, \cdots \rangle$, satisfies $\partial L^m u + L^2 \partial L^m u + a L^m u = 0$.

Let $v := e^{-G} L^m u$, then by Lemma 3.1 and the fact that the operators $e^{\pm G}$ commute with the left translation, $[e^{\pm G}, L] = 0$, the sequence $v = L^m e^{-G} u$ solves $\partial v + L^2 \partial v = 0$, i.e. $v$ is $L^2$ analytic. Thus, the negative even subsequence $\langle v_0, v_{-2}, \cdots \rangle$, and negative odd subsequence $\langle v_1, v_{-3}, \cdots \rangle$, respectively, are $L$ analytic, with traces $L^2 g^\text{even}_h$, respectively, $L^2 g^\text{odd}_h$. The necessity part in Theorem 3.1 yields (96):
\begin{equation}
[I + i\mathcal{H}] L^m T g^\text{even}_h = 0, \quad [I + i\mathcal{H}] L^m T g^\text{odd}_h = 0.
\end{equation}
This proves part (i) of the theorem.

(ii) Sufficiency: Let $g \in C^\mu(\Gamma; C^{1,\mu}(S^1)) \cap C(\Gamma; C^{2,\mu}(S^1))$ be real valued with $g|_{\Gamma_0 \cup \Gamma_0} = 0$. Let the corresponding sequences $g^\text{even}_h, g^\text{odd}_h$ as in (93) satisfying (96). By Proposition 3.1(ii) and Proposition 3.2(iii), we have $g^\text{even}_h, g^\text{odd}_h \in Y^\mu(\Gamma)$.

Let $m = 2q$, $q \geq 0$, be an even integer. To prove the sufficiency we will construct a real valued symmetric $m$-tensor $f$ in $\Omega$ and a real valued function $u \in C^{1}(\Omega \times S^1) \cap C^\mu(\Omega \times S^1)$ such that $u|_{\Gamma \times S^1} = g$ and $u$ solves (89) in $\Omega$. The construction of such $u$ is in terms of its Fourier modes in the angular variable and it is done in several steps.

Step 1: The construction of modes $u_m$ for $|n| \geq 2q$, $q \geq 0$.

Use the Bukhgeim-Cauchy Integral formula (25) to define the $L$-analytic maps
\begin{align*}
v^\text{even}(z) &= \langle v_0(z), v_{-2}(z), v_{-4}(z), \cdots \rangle := BL^q g^\text{even}_h(z), \quad z \in \Omega, \\
v^\text{odd}(z) &= \langle v_{-1}(z), v_{-3}(z), v_{-5}(z), \cdots \rangle := BL^q g^\text{odd}_h(z), \quad z \in \Omega.
\end{align*}
By intertwining the above $L$-analytic maps, define also the $L^2$-analytic map
\begin{equation}
v(z) := \langle v_0(z), v_{-1}(z), v_{-2}(z), v_{-3}(z), \cdots \rangle, \quad z \in \Omega.
\end{equation}
By Theorem 3.1(ii),
\begin{equation}v, v^\text{even}, v^\text{odd} \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\overline{\Omega}; l_1) \cap C^2(\Omega; l_\infty).
\end{equation}
Moreover, since \( g^\text{even}_h, g^\text{odd}_h \) satisfy the hypothesis (96), by Theorem 3.1 sufficiency part, we have

\[
v^\text{even}|_r = L^q g^\text{even}_h \quad \text{and} \quad v^\text{odd}|_r = L^q g^\text{odd}_h.
\]

In particular, \( v \) is \( L^2 \)-analytic map with trace:

\[
v|_r = L^{2q} g_h = L^{2q} e^{-G} g,
\]

where \( g_h \) is formed by intertwining \( g^\text{even}_h \) and \( g^\text{odd}_h \).

Define the sequence valued map

\[
\Omega \ni z \mapsto L^{2q}(\mathbf{u})(z) = (u_{-2q}(z), u_{-2q-1}(z), u_{-2q-2}(z), \cdots) := e^G \mathbf{v}(z),
\]

where the operator \( e^G \) as defined in (34). Since convolution preserves \( l_1 \), by Proposition 3.2

\[
L^{2q} u \in C^{1,\mu}(\Omega; l_1) \cap C^\mu(\Omega; l_1).
\]

Moreover, since \( v \in C^2(\Omega; l_\infty) \) as in (97), we also conclude from convolution that \( L^{2q} u \in C^2(\Omega; l_\infty) \).

As \( v \) is \( L^2 \) analytic, by Lemma 3.1 \( L^{2q} u \) satisfies

\[
\overline{\partial} L^{2q} u + L^2 \partial L^{2q} u + aL^{2q+1} u = 0,
\]

which in component form is written as:

\[
\overline{\partial} u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad n \geq 2q, \ q \geq 0.
\]

The trace satisfy

\[
L^{2q} u|_r = e^G v|_r = e^G L^{2q} e^{-G} g = L^{2q} g,
\]

where the second equality follows from (96) and in the last equality we use the fact that the operators \( e^{\pm G} \) commute with the left translation, \([e^{\pm G}, \partial] = 0\).

Construct the positive Fourier modes by conjugation: \( u_n := \overline{u_{-n}}, \) for all \( n \geq 2q, \ q \geq 0 \). Moreover using (102), the traces \( u_n|_r \) for each \( n \geq 2q, \ q \geq 0 \), satisfy

\[
u_n|_r = \overline{u_{-n}}|_r = \overline{g_{-n}} = g_n, \quad n \geq 2q, \ q \geq 0.
\]

By conjugating (101) we note that the positive Fourier modes also satisfy

\[
\overline{\partial} u_{n+2} + \partial u_n + au_{n+1} = 0, \quad n \geq 2q, \ q \geq 0.
\]

**Step 2: The construction of the tensor field \( f \) in the \( q = 0 \) case.**

In the case of the 0-tensor, \( f = f_0 \), and \( f_0 \) is uniquely determined from the odd Fourier mode \( u_{-1} \), and the zeroth mode \( u_0 \) in (99), by

\[
f := 2 \Re \overline{\partial} u_{-1} + au_0, \quad (f \text{ for } q = 0 \text{ case}).
\]

We consider next the case \( m = 2q, q \geq 1 \) of tensors of order 2 or higher. In this case the construction of the tensor field \( f_q \) is in terms of the mode \( u_{-2q} \) in (99) and the class \( \Psi^\text{even}_{a,g} \) in (94).

**Step 3: The construction of modes \( u_n \) for \( \lvert n \rvert \leq 2q - 1 \) \( q \geq 1 \).**

Recall that \( a \in C^{2,\mu}(\overline{\Omega}), \mu > 1/2 \) with \( \min a > 0 \), and the non-uniqueness class \( \Psi^\text{even}_{a,g} \) in (94).

For \( (\psi_0, \psi_2, \cdots, \psi_{2(q-1)}) \in \Psi^\text{even}_{a,g} \) arbitrary, define the modes \( u_0, u_{\pm 2}, \cdots, u_{\pm (2(q-1))} \) in \( \Omega \) by

\[
u_{-2j} := \psi_{-2j}, \quad \text{and} \quad u_{2j} := \psi_{2j}, \quad 0 \leq j \leq q - 1, \ q \geq 1.
\]

Using the mode \( u_{-2q} \) from (99) and \( \psi_{-2(q-1)} \), define the modes \( u_{\pm (2q-1)} \) by

\[
u_{-(2q-1)} := \frac{\overline{\partial} \psi_{-2(q-1)} + \partial u_{-2q}}{a}, \quad \text{and} \quad u_{2q-1} := \overline{\psi}_{-(2q-1)}, \text{ for all } q \geq 1.
\]
As \( \psi_0 \in C^2(\Omega; \mathbb{R}) \) and \( \psi_{-(2j+2)} \in C^2(\Omega; \mathbb{C}) \), for \( 0 \leq j \leq q - 2, \ q \geq 2 \), define modes

\begin{equation}
    u_{-(2j+1)} := -\frac{\partial \psi_{-2j} + \partial \psi_{-(2j+2)}}{a}, \quad \text{and} \quad u_{2j+1} := \overline{u_{-(2j+1)}}, \quad \text{for all} \ 0 \leq j \leq q - 2, \ q \geq 2.
\end{equation}

By the construction in (106), (107), and (108):

\begin{equation}
    u_{-2j} \in C^2(\Omega; l_{\infty}), \quad \text{for} \quad 0 \leq j \leq q - 1, \ q \geq 1,
\end{equation}

\begin{equation}
    u_{-(2j+1)} \in C^1(\Omega; l_{\infty}), \quad \text{for} \quad 0 \leq j \leq q - 1, \ q \geq 1, \quad \text{and}
\end{equation}

\[ \overline{\partial u_{-2j} + \partial u_{-(2j+2)}} + a u_{-(2j+1)} = 0, \quad \text{for} \quad 0 \leq j \leq q - 1, \ q \geq 1, \]

are satisfied. Moreover, by conjugating the last equation in (109) yields

\begin{equation}
    \partial u_{2j} + \overline{\partial u_{(2j+2)}} + a u_{(2j+1)} = 0, \quad \text{for} \quad 0 \leq j \leq q - 1, \ q \geq 1.
\end{equation}

By the definition of the class (94), and reality of \( g \), we have the trace of \( u_{-2j} \) in (106) satisfies

\begin{equation}
    u_{-2j} \mid_{\Gamma} = g_{-2j}, \quad \text{and} \quad u_{2j} \mid_{\Gamma} = \overline{g_{-2j}} = g_{2j}, \quad 0 \leq j \leq q - 1, \ q \geq 1.
\end{equation}

We check next that the trace of \( u_{-(2j+1)} \) is \( g_{-(2j+1)} \) for \( 0 \leq j \leq q - 2, \ q \geq 2 \):

\begin{equation}
    u_{-(2j+1)} \mid_{\Gamma} = -\frac{\overline{\partial \psi_{-2j} + \partial \psi_{-(2j+2)}}}{a} \mid_{\Gamma} = g_{-(2j+1)},
\end{equation}

where the last equality uses the condition in class (94). Similar calculation to (112) for mode \( u_{-(2q-1)} \) give the trace

\begin{equation}
    u_{-(2q-1)} \mid_{\Gamma} = -\frac{\overline{\partial \psi_{-(2q-1)} + \partial u_{-2q}}}{a} \mid_{\Gamma} = g_{-(2q-1)}.
\end{equation}

Thus, from (111) - (113), we have the traces:

\begin{equation}
    u_n \mid_{\Gamma} = g_n, \quad \forall |n| \leq 2q - 1.
\end{equation}

**Step 4: The construction of the tensor field \( f_\psi \) whose attenuated X-ray data is \( g \).**

The components of the \( m \)-tensor \( f_\psi \) are defined via the one-to-one correspondence between the pseudovectors \( \langle \tilde{f}_0, \tilde{f}_1, \cdots, \tilde{f}_m \rangle \) and the functions \( \{ f_{2n} : -q \leq n \leq q \} \) as follows.

We define first \( f_{2q} \) by using \( \psi_{-(2q-1)} \) from the non-uniqueness class, and Fourier modes \( u_{-2q}, u_{-(2q+1)} \in C^2(\Omega; l_{\infty}) \) from (99). Then, next define \( f_{2q-2} \) by using \( \psi_{-(2q-1)}, \psi_{-(2q-2)} \) from the non-uniqueness class, and Fourier mode \( u_{-2q} \) from (99). Then, define \( \{ f_{2n} : 0 \leq n \leq q - 2 \} \) solely from the information in the non-uniqueness class. Finally, define \( \{ f_{-2n} : 1 \leq n \leq q \} \) by conjugation.
Equations (13), (7) and (9) to construct pseudovectors as in (115) from class $\Psi$.

By construction, $f_{2n} \in C(\Omega)$, for $0 \leq n \leq q$, $q \geq 1$, as $\psi_{-2n} \in C^2(\Omega; l_\infty)$, for $0 \leq n \leq q - 1$, from (94). Note that $f_{2n}$ satisfy (90). We use these Fourier modes $\langle f_0, f_{\pm 1}, f_{\pm 2}, \ldots, f_{\pm m} \rangle$ and equations (13), (7) and (9) to construct pseudovectors $\langle f_0, f_1, \ldots, f_m \rangle$, and thus $m$-tensor field $f_\psi \in C(S^m; \Omega)$.

In order to show $g|_{\Gamma} = X_a f_\psi$, with $f_\psi$ being constructed from pseudovectors via Fourier modes as in (115) from class $\Psi_{a,g}$, we define the real valued function $u$ via its Fourier modes

\[
\begin{aligned}
\frac{\partial \psi_{-2(q-1)} + \partial u_{-2q}}{a} \quad &+ \partial u_{-(2q+1)} + a u_{-2q}, \quad q \geq 1, \\
\frac{\partial \psi_{-2(q-2)} + \partial \psi_{-2(q-1)}}{a} \quad &- \partial \left( \frac{\partial \psi_{-2(q-1)} + \partial u_{-2q}}{a} \right) + a \psi_{-2(q-1)}, \quad q \geq 2, \\
\frac{\partial \psi_{-2(n-1)} + \partial \psi_{-2n}}{a} \quad &- \partial \left( \frac{\partial \psi_{-2n} + \partial \psi_{-2(n+1)}}{a} \right) + a \psi_{-2n}, \quad 1 \leq n \leq q - 2, \quad q \geq 3,
\end{aligned}
\]

\[
\begin{aligned}
f_0 := \begin{cases} 
-2 \Re \partial \left( \frac{\partial \psi_0 + \partial u_{-2}}{a} \right) + a \psi_0, & q = 1, \\
-2 \Re \partial \left( \frac{\partial \psi_0 + \partial \psi_{-2}}{a} \right) + a \psi_0, & q \geq 2,
\end{cases}
\end{aligned}
\]

\[
f_{-2n} := f_{2n}, \quad 0 \leq n \leq q, \quad q \geq 1,
\]

By construction, $f_{2n} \in C(\Omega)$, for $0 \leq n \leq q$, $q \geq 1$, as $\psi_{-2n} \in C^2(\Omega; l_\infty)$, for $0 \leq n \leq q - 1$, from (94). Note that $f_{2n}$ satisfy (90). We use these Fourier modes $\langle f_0, f_{\pm 1}, f_{\pm 2}, \ldots, f_{\pm m} \rangle$ and equations (13), (7) and (9) to construct pseudovectors $\langle f_0, f_1, \ldots, f_m \rangle$, and thus $m$-tensor field $f_\psi \in C(S^m; \Omega)$.

In order to show $g|_{\Gamma} = X_a f_\psi$, with $f_\psi$ being constructed from pseudovectors via Fourier modes as in (115) from class $\Psi_{a,g}$, we define the real valued function $u$ via its Fourier modes

\[
\begin{aligned}
\sum_{|n| \geq 2q} u_n(z) e^{i n \theta} + 2 \Re \left( \frac{-\partial \psi_{-2(q-1)} + \partial u_{-2q}}{a} \right) e^{-i(2q-1)\theta}
+ 2 \Re \left\{ \sum_{n=0}^{q-1} \psi_{-2n}(z) e^{-i(2n)\theta} \right\} + 2 \Re \left\{ \sum_{n=0}^{q-2} \frac{-\partial \psi_{-2j} + \partial \psi_{-(2j+2)}}{a} \right\} e^{-i(2n+1)\theta}
\end{aligned}
\]

and check that it has the trace $g$ on $\Gamma$ and satisfies the transport equation (89).

Since $g \in C^\mu(\Gamma; C^{1,\mu}(S^1)) \cap C(\Gamma; C^{2,\mu}(S^1))$, we use Proposition 3.1(ii) and [39, Proposition 4.1(iii)] to conclude that $u$ defined in (116) belongs to $C^{1,\mu}(\Omega \times S^1) \cap C(\Omega \times S^1)$. In particular $u(\cdot, \theta) = (\cos \theta, \sin \theta)$ extends to the boundary and its trace satisfies

\[
u(\cdot, \theta)|_{\Gamma} = \sum_{|n| \geq 2q} u_n|_{\Gamma} e^{i n \theta} + \sum_{|n| \leq 2q-1} u_n|_{\Gamma} e^{i n \theta} = \sum_{|n| \geq 2q} g_n e^{i n \theta} + \sum_{|n| \leq 2q-1} g_n e^{i n \theta} = g(\cdot, \theta),
\]

where in the second equality above we use (98), (103) and (114).

Since $u \in C^{1,\mu}(\Omega \times S^1) \cap C^{\mu}(\Omega \times S^1)$, then using (101), (104), (107), (109), (110), and the definition of $f_{2n}$ for $-q \leq n \leq q$, $q \geq 1$, in (115), the real valued $u$ defined in (116) satisfies the transport equation (89):

\[
\theta \cdot \nabla u + au = \langle f_\psi, \theta^{2q} \rangle, \quad q \geq 1.
\]
7. Odd order m-tensor - attenuated case

In this section, we establish necessary and sufficient conditions for a sufficiently smooth function on $\Gamma \times S^1$ to be the attenuated X-ray data of some sufficiently smooth real valued symmetric tensor field $f$ of odd order $m = 2q + 1$, $q \geq 0$.

In this case $a \neq 0$, the transport equation becomes

$$\theta \cdot \nabla u(x, \theta) + a(x)u(x, \theta) = \sum_{n=0}^{q} (f_{2n+1}(x)e^{-i(2n+1)\theta} + f_{-(2n+1)}(x)e^{i(2n+1)\theta}), \quad x \in \Omega,$$

where $f_{2n+1} = f_{-(2n+1)}$, $0 \leq n \leq q$, $q \geq 0$.

If $\sum_{n \in Z} u_n(z)e^{m\theta}$ is the Fourier series expansion in the angular variable $\theta$ of a solution $u$ of (117), then by identifying the Fourier coefficients of the same order, the equation (117) reduces to the system:

$$\Gamma(118) \quad \overline{\partial}u_{-n}(z) + \partial u_{-(n+2)}(z) + au_{-(n+1)}(z) = f_{n+1}(z), \quad 0 \leq n \leq q, \quad q \geq 0,$$

$$\Gamma(119) \quad \overline{\partial}u_{-(n+1)}(z) + \partial u_{-(n+2)}(z) + au_{-(n+1)}(z) = 0, \quad 0 \leq n \leq q, \quad q \geq 0,$$

$$\Gamma(120) \quad \overline{\partial}u_{-n}(z) + \partial u_{-(n-2)}(z) + au_{-(n-1)}(z) = 0, \quad n \geq 2q + 1, \quad q \geq 0,$$

Recall that the trace $u|_{R \times S^1} := g$ as in (21), with $g = X_a f$ on $\Gamma_+$ and $g = 0$ on $\Gamma_- \cup \Gamma_0$.

We expand the attenuated X-ray data $g$ in terms of its Fourier modes in the angular variables:

$$g(\zeta, \theta) = \sum_{n=\infty}^{\infty} g_n(\zeta)e^{i\theta n}, \quad \text{for } \zeta \in \Gamma. \quad \text{From the non-positive modes of } g, \text{ we built the sequences }$$

$$g^e := \langle g_0, g_{-1}, g_{-2}, \ldots \rangle, \text{ and } g^{G} := e^{-G}g := \langle \gamma_0, \gamma_{-1}, \gamma_{-2}, \ldots \rangle, \text{ where } e^{\pm G} \text{ as defined in (34). From the non-positive even, respectively, negative odd Fourier modes, we built the sequences }$$

$$g^{even}_n = \langle \gamma_0, \gamma_{-2}, \gamma_{-4}, \ldots \rangle, \quad \text{and } g^{odd}_n = \langle \gamma_{-1}, \gamma_{-3}, \gamma_{-5}, \ldots \rangle.$$

Next we characterize the attenuated X-ray data $g$ in terms of its $m$ many modes $g_0, g_{-1}, \cdots g_{-(m-1)}$, and the Fourier modes $L^m g^e := L^m e^{-G}g := \langle \gamma_{-m}, \gamma_{-(m+1)}, \gamma_{-(m+2)}, \ldots \rangle$.

As before we construct simultaneously the right hand side of the transport equation (117) together with the solution $u$. Construction of $u$ via its Fourier modes. We first construct the negative modes and then the positive modes are constructed by conjugation. For $m = 2q + 1$ (odd integer), $q \geq 1$, the modes will be chosen arbitrarily from the class $\psi_{a,g}^{odd}$ of cardinality $q = \frac{m-1}{2}$ with prescribed trace and gradient on the boundary $\Gamma$ defined as

$$\psi_{a,g}^{odd} := \left\{ \psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)} \right\} \in \left( C^2(\Omega; \mathbb{C}) \right)^q :$$

$$\Gamma(122) \quad \overline{\partial}\psi_{-(2j-1)}|_\Gamma = -\partial(e^G B e^{-G} g)|_{-(2q+1)}|_\Gamma - a|_\Gamma g_{-2q}, \quad q \geq 1,$$

$$\Gamma(122) \quad \overline{\partial}\psi_{-(2j-1)}|_\Gamma = -\partial\psi_{-(2j+1)}|_\Gamma - a|_\Gamma g_{-2j}, \quad 1 \leq j \leq q - 1, \quad q \geq 2,$$

$$\Gamma(122) \quad 2 \left( \Re e \partial\psi_{-1} \right) = -a|_\Gamma g_0,$$

where $B$ be the Bukhgeim-Cauchy operator in (25), and the operators $e^{\pm G}$ as defined in (34).
Remark 7.1. In the 1-tensor case \((q = 0)\), there is no class, and the feature of the attenuated X-ray data \(g\) is in terms of its zero-th mode \(g_0 = \oint g(\cdot, \theta) \, d\theta\) and negative Fourier modes of \(g_h := e^{-G}g\).

**Theorem 7.1** (Range characterization for odd order tensors). Let \(a \in C^{2, \mu}({\overline{\Omega}})\), \(\mu > 1/2\) with \(\min \alpha > 0\), and \(m = 2q + 1, \ q \geq 0\). (i) Let \(f \in C^{1, \mu}_0(S^m; \Omega)\) be a real-valued symmetric \(m\)-tensor field of odd order and

\[
g = X_a f \quad \text{on } \Gamma_+ \quad \text{and} \quad g = 0 \quad \text{on } \Gamma_- \cup \Gamma_0.
\]

Then \(g_{\text{even}}\), \(g_{\text{odd}}\) \(\in l_{1,1}^f(\Gamma) \cap C^\mu(\Gamma; l_1)\) satisfy

\[
[I + i\mathcal{H}]L^{m+1} g_{\text{even}} = 0, \quad [I + i\mathcal{H}]L^{m-1} g_{\text{odd}} = 0, \quad \text{for} \quad q \geq 0,
\]

where \(g_{\text{even}}, g_{\text{odd}}\) are sequences in \((121)\). Additionally, in \(q = 0\) case, for each \(\zeta \in \Gamma\), the zero-th Fourier mode \(g_0\) of \(g\) satisfy

\[
g_0(\zeta) = \lim_{\Omega_\zeta \rightarrow \zeta \in \Gamma} \frac{-2 \Re e \partial (e^G B g_h)_{-1}(z)}{a(z)}, \quad \text{for} \quad q = 0,
\]

where \(B\) be the Bukhgeim-Cauchy operator in \((25)\), and the operators \(e^{\pm G}\) as defined in \((34)\).

(ii) Let \(g \in C^\mu(\Gamma; C^{1, \mu}(S^1)) \cap C(\Gamma; C^{2, \mu}(S^1))\) be real valued with \(g|_{\Gamma_- \cup \Gamma_0} = 0\). For \(q = 0\), if the corresponding sequences \(g_{\text{even}}, g_{\text{odd}} \in Y_\mu(\Gamma)\) satisfies \((123)\), and \(g_0\) satisfies \((124)\), then there exists a unique real valued vector field \((1\text{-tensor})\) \(f \in C(S^m; \Omega)\) such that \(g|_{\Gamma_+} = X_a f\). Moreover, for \(q \geq 1\), if \(g_{\text{even}}, g_{\text{odd}} \in Y_\mu(\Gamma)\) satisfies \((123)\), and for each element \(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)} \in \Psi_{a, \mu}\), then there is a unique real valued symmetric \(m\)-tensor \(f_\psi \in C(S^m; \Omega)\) such that \(g|_{\Gamma_+} = X_a f_\psi\).

**Proof.** (i) **Necessity:** Let \(f = (f_{1, \cdots, m}) \in C^{1, \mu}_0(S^m; \Omega)\). Since all components \(f_{1, \cdots, m} \in C^{1, \mu}(\Omega)\), \(X_a f \in C^{1, \mu}(\Gamma_+)\), and, thus, the solution \(u\) to the transport equation \((117)\) is in \(C^{1, \mu}(\overline{\Omega} \times S^1)\). Moreover, its trace \(g = u|_{\Gamma \times S^1} \in C(\Gamma \times S^1)\). By Proposition 3.1 (i) and Proposition 3.2, \(g_h = e^{-G}g \in l_{1,1}^f(\Gamma) \cap C^\mu(\Gamma; l_1)\).

If \(u\) solves \((117)\) then its Fourier modes satisfies \((118), (119)\) and \((120)\). In particular, the sequence valued map \(u = \langle u_0, u_{-1}, u_{-2}, \ldots \rangle\) satisfy \(\overline{\partial} L^m u + L^2 \partial L^m u + a L^{m+1} u = 0\).

Let \(v := e^{-G}L^m u\), then by Lemma 3.1 and the fact that the operators \(e^{\pm G}\) commute with the left translation, \([e^{\pm G}, L] = 0\), the sequence \(v = L^m e^{-G} u\) solves \(\overline{\partial} v + L^2 \partial v = 0\), i.e \(v\) is \(L^2\) analytic. The non-positive even and negative odd subsequence \(\langle v_{0}, v_{-3}, \cdots \rangle\), respectively, are \(L\) analytic, with traces \(L^m e^{-G} g_{\text{even}}, \) respectively, \(L^m e^{-G} g_{\text{odd}}\). The necessity part in Theorem 3.1 yields \((123)\):

\[
[I + i\mathcal{H}]L^{m+1} g_{\text{even}} = 0, \quad [I + i\mathcal{H}]L^{m-1} g_{\text{odd}} = 0, \quad \text{for} \quad m = 2q + 1, \ q \geq 0.
\]

Additionally, in the \(q = 0\) case, the Fourier modes \(u_0, u_{-1}, u_1\) of \(u\) solve \((119)\) for \(n = 0\). Since \(a > 0\) in \(\Omega\), we have

\[
u_0(z) = \lim_{\Omega_\zeta \rightarrow \zeta \in \Gamma} \frac{-2 \Re e \partial u_{-1}(z)}{a(z)}, \quad z \in \Omega.
\]

Since the left hand side of \((125)\) is continuous all the way to the boundary, so is the right hand side. Moreover, the limit below exists and in the \(q = 0\) case, we have

\[
g_0(z_0) = \lim_{\Omega_\zeta \rightarrow \zeta \in \Gamma} u_0(z) = \lim_{\Omega_\zeta \rightarrow \zeta \in \Gamma} -2 \Re e \partial u_{-1}(z) = \frac{-2 \Re e \partial u_{-1}(z)}{a(z)},
\]

thus \((124)\) holds. This proves part (i) of the theorem.
(ii) **Sufficiency:** Let \( g \in C^\mu (\Gamma; C^1.S(\Omega)) \cap C(\Gamma; C^2.S(\Omega)) \) be real valued with \( g|_{\Gamma \cap \Gamma_0} = 0 \). Let the corresponding sequences \( g^\text{even}_h, g^\text{odd}_h \) as in (121) satisfying (123). By Proposition 3.1(ii) and Proposition 3.2(iii), \( g^\text{even}_h, g^\text{odd}_h \in Y^\mu_h (\Gamma) \).

Let \( m = 2q + 1, q \geq 0 \), be an odd integer. To prove the sufficiency we will construct a real valued symmetric \( m \)-tensor \( f \) in \( \Omega \) and a real valued function \( u \in C^1(\Omega \times S^1) \cap C(\Omega \times S^1) \) such that \( u|_{\Gamma \times S^1} = g \) and \( u \) solves (117) in \( \Omega \). The construction of such \( u \) is in terms of its Fourier modes in the angular variable and it is done in several steps.

**Step 1:** The construction of modes \( u_n \) for \(|n| \geq 2q + 1, q \geq 0\).

Use the Bukhgeim-Cauchy Integral formula (25) to define the \( L \)-analytic maps
\[
\begin{align*}
v^\text{even}(z) &= \langle v_0(z), v_{-2}(z), v_{-4}(z), \ldots \rangle := \mathcal{B}L^{q+1}g^\text{even}_h(z), \quad z \in \Omega, \\
v^\text{odd}(z) &= \langle v_{-1}(z), v_{-3}(z), v_{-5}(z), \ldots \rangle := \mathcal{B}L^qg^\text{odd}_h(z), \quad z \in \Omega.
\end{align*}
\]

By intertwining let also define \( L^2 \)-analytic map
\[
v(z) = \langle v_0(z), v_{-1}(z), v_{-2}(z), v_{-3}(z), \ldots \rangle, \quad z \in \Omega.
\]

By Theorem 3.1(ii),
\[
v^\text{even}, v^\text{odd}, v \in C^1(\Omega; l_1) \cap C^2(\Omega; l_1) \cap C^2(\Omega; l_\infty).
\]

Moreover, since \( g^\text{even}_h, g^\text{odd}_h \) satisfy the hypothesis (96), by Theorem 3.1 sufficiency part, we have
\[
v^\text{even}|_{\Gamma} = L^{q+1}g^\text{even}_h \quad \text{and} \quad v^\text{odd}|_{\Gamma} = L^q g^\text{odd}_h, \quad q \geq 0.
\]

In particular, \( v \) is \( L^2 \)-analytic with trace:
\[
v|_{\Gamma} = L^{2q+1}g^\text{even}_h = L^{2q+1}e^{-G}g, \quad q \geq 0,
\]

where \( g_h \) is formed by intertwining \( g^\text{even}_h \) and \( g^\text{odd}_h \).

For \( q \geq 0 \), define the sequence valued map
\[
\Omega \ni z \mapsto L^{2q+1}u(z) = \langle u_{-(2q+1)}(z), u_{-(2q+2)}(z), u_{-(2q+3)}(z), \ldots \rangle := e^{\mathcal{G}}v(z).
\]

By Proposition 3.2, \( L^{2q+1}u \in C^1(\Omega; l_1) \cap C^2(\Omega; l_1) \). Moreover, since \( v \in C^2(\Omega; l_\infty) \) as in (126), we also conclude from convolution that \( L^{2q+1}u \in C^2(\Omega; l_\infty) \). Thus,
\[
L^{2q+1}u \in C^1(\Omega; l_1) \cap C^2(\Omega; l_1) \cap C^2(\Omega; l_\infty).
\]

As \( v \) is \( L^2 \) analytic, by Lemma 3.1, \( L^{2q+1}u \) satisfies \( \partial L^{2q+1}u + L^2 \partial L^{2q+1}u + aL^{2q+2}u = 0 \), for \( q \geq 0 \), which in component form is written as:
\[
\partial u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad n \geq 2q + 1, \quad q \geq 0.
\]

The trace satisfy
\[
L^{2q+1}u|_{\Gamma} = e^{\mathcal{G}}v|_{\Gamma} = e^{\mathcal{G}}L^{2q+1}e^{-\mathcal{G}}g = L^{2q+1}g, \quad q \geq 0,
\]

where the second equality follows from (127) and in the last equality we use \( [e^{\pm \mathcal{G}}, L] = 0 \).

Construct the positive Fourier modes by conjugation: \( u_n := \overline{u_{-n}} \), for all \( n \geq 2q + 1, q \geq 0 \). Moreover using (131), and the reality of \( g \), the traces \( u_n|_{\Gamma} \) satisfy
\[
u_n|_{\Gamma} = \overline{u_{-n}}|_{\Gamma} = \overline{g_{-n}} = g_n, \quad n \geq 2q + 1, \quad q \geq 0.
\]

By conjugating (130), and from (131) and (132), we thus have the Fourier modes satisfy
\[
\partial u_{-n} + \partial u_{-n-2} + au_{-n-1} = 0, \quad \text{and} \quad u_n|_{\Gamma} = g_n, \quad \forall |n| \geq 2q + 1, \quad q \geq 0.
\]

**Step 2:** The construction of 1-tensor (\( q = 0 \) case).
Since $a > 0$ in $\Omega$, we can define $u_0$ (in $q = 0$ case) by using the Fourier mode $u_{-1}$ from (128):

\begin{equation}
(134) \quad u_0(z) := -\frac{2 \text{Re} \partial u_{-1}(z)}{a(z)}, \quad z \in \Omega, \quad (\text{for } q = 0 \text{ case}).
\end{equation}

Note that $u_0$ satisfy (133) for $n = -1$. In particular $\overline{\partial u_1} + \partial u_{-1} + au_0 = 0$ holds.

From (124), $u_0$ defined above extends continuously to the boundary $\Gamma$ and

\[ u_0|_{\Gamma} = g_0, \quad (\text{for } q = 0 \text{ case}). \]

Moreover, since $u_{-1} \in C^2(\Omega)$ as shown in (129) and $a \in C^2(\Omega)$ we get $u_0 \in C^1(\Omega)$.

Using the Fourier modes $u_{-1}, u_{-2}$ from (128) and $u_0$ as in (134), we next define the real valued vector field $\mathbf{f} \in C(\Omega; \mathbb{R}^2)$ (for $q = 0$ case) by

\begin{equation}
(135) \quad \mathbf{f} = \langle 2 \text{Re} f_1, 2 \text{Im} f_1 \rangle, \quad \text{where } f_1 := \overline{\partial u_0} + \partial u_{-2} + au_{-1}.
\end{equation}

We consider next the case $q \geq 1$ of tensors of order 3 or higher. In this case the construction of the tensor field $\mathbf{f}_q$ is in terms of the Fourier modes $u_{-(2q+1)}, u_{-(2q+2)}$ in (128) and the class $\Psi^\text{odd}_{a,g}$ as in (122).

**Step 3: The construction of modes $u_n$ for $|n| \leq 2q, q \geq 1$.**

Recall the non-uniqueness class $\Psi^\text{odd}_{a,g}$ as in (122).

For $(\psi_{-1}, \psi_{-3}, \cdots, \psi_{-(2q-1)}) \in \Psi^\text{odd}_{a,g}$ arbitrary, firstly define the odd modes

\begin{equation}
(136) \quad u_{-(2n-1)} := \psi_{-(2n-1)}, \quad \text{and} \quad u_{2n-1} := \overline{\psi_{-(2n-1)}}, \quad 1 \leq n \leq q, \quad q \geq 1.
\end{equation}

Secondly, by using $\psi_{-1}, \psi_{-(2q-1)}$ and the mode $u_{-(2q+1)}$ from (128), we define the modes

\begin{equation}
(137) \quad u_0 := -\frac{2 \text{Re} \partial \psi_{-1}}{a},
\end{equation}

\begin{equation}
(138) \quad u_{-2q} := -\frac{\overline{\partial \psi_{-(2q-1)}} + \partial u_{-(2q+1)}}{a}, \quad \text{and} \quad u_{2q} := \overline{u_{-2q}} \quad \text{for } q \geq 1.
\end{equation}

Lastly, by using $\psi_{-(2n-1)} \in C^2(\Omega; \mathbb{C})$, for $1 \leq n \leq q-1, \quad q \geq 2$, we define the even modes

\begin{equation}
(139) \quad u_{-2n} := -\frac{\overline{\partial \psi_{-(2n-1)}} + \partial \psi_{-(2n+1)}}{a}, \quad 1 \leq n \leq q-1, \quad q \geq 2, \quad \text{and}
\end{equation}

\begin{equation}
\begin{split}
\quad u_{2n} := \overline{u}_{-2n}, \quad 1 \leq n \leq q-1, \quad q \geq 2.
\end{split}
\end{equation}

By the construction in (137), (138), and (139), we have

\begin{equation}
(140) \quad u_{-(2n-1)} \in C^2(\Omega; l_\infty), \quad \text{for } 1 \leq n \leq q, \quad q \geq 1,
\end{equation}

\begin{equation}
(140) \quad u_{-2n} \in C^1(\Omega; l_\infty), \quad \text{for } 0 \leq n \leq q, \quad q \geq 1, \quad \text{and}
\end{equation}

\begin{equation}
\overline{\partial u_{-(2n-1)}} + \partial u_{-(2n+1)} + au_{-2n} = 0, \quad \text{for } 0 \leq n \leq q, \quad q \geq 1,
\end{equation}

is satisfied. Moreover, by conjugating the last equation in (140), we have the Fourier modes satisfy

\begin{equation}
(141) \quad \overline{\partial u_{-(2n-1)}} + \partial u_{-(2n+1)} + au_{-2n} = 0, \quad \text{for } |n| \leq q, \quad q \geq 1.
\end{equation}

By the class (122), and reality of $g$, we have the trace of $u_{-(2n-1)}$ in (136) satisfy

\begin{equation}
(142) \quad u_{-(2n-1)}|_{\Gamma} = g_{-(2n-1)}, \quad \text{and} \quad u_{2n-1}|_{\Gamma} = \overline{g}_{-(2n-1)} = g_{2n-1}, \quad 1 \leq n \leq q, \quad q \geq 1.
\end{equation}

We check next that the trace of $u_{-2n}$ is $g_{-2n}$ for $1 \leq n \leq q-1, \quad q \geq 2$:

\begin{equation}
(143) \quad u_{-2n}|_{\Gamma} = -\frac{\overline{\partial \psi_{-(2n-1)}} + \partial \psi_{-(2n+1)}}{a} \bigg|_{\Gamma} = g_{-2n},
\end{equation}

\begin{equation}
\overline{\partial \psi_{-(2n-1)}} + \partial \psi_{-(2n+1)}
\end{equation}
where the last equality uses the condition in class \((122)\). Similar calculation to \((143)\) for mode \(u_0\) in \((137)\), and mode \(u_{-2q}\) in \((138)\), give the trace

\[
(u_0)_r = g_0, \quad \text{and} \quad (u_{-2q})_r = g_{-2q}, \quad q \geq 1.
\]

Thus, from \((142)\), \((143)\) and \((144)\), we have the traces:

\[
(u_n)_r = g_n, \quad \forall |n| \leq 2q, \quad q \geq 1.
\]

**Step 4:** The construction of the tensor field \(f_\Psi\) whose attenuated X-ray data is \(g\).

The components of the \(m\)-tensor \(f_\Psi\) are defined via the one-to-one correspondence between the pseudovectors \(|\vec{f}_0, \vec{f}_1, \ldots, \vec{f}_m|\) and the functions \(|\{f_{\pm(2n+1)}: 0 \leq n \leq q\}\) as follows.

We first define \(f_{2q+1}\) by using \(\psi_{-(2q-1)}\) from the non-uniqueness class, and the Fourier modes \(u_{-(2q+1)}, u_{-(2q+2)}\) in \((128)\). Next, define \(f_{2q-1}\) by using \(\psi_{-(2q-1)}, \psi_{-(2q-3)}\) from the non-uniqueness class, and Fourier mode \(u_{-(2q+1)}\) in \((128)\). Then, define \(|\{f_{2n+1} : 0 \leq n \leq q - 2\}|\) solely from the information in the non-uniqueness class. Finally, define \(|\{f_{-(2n+1)} : 0 \leq n \leq q\}|\) by conjugation.

\[
f_{2q+1} := -\vec{\partial} \left( \frac{\partial \psi_{-(2q-1)} + \partial u_{-(2q+1)}}{a} \right) + \partial u_{-(2q+2)} + au_{-(2q+1)}, \quad q \geq 1,
\]

\[
f_{2q-1} := -\vec{\partial} \left( \frac{\partial \psi_{-(2q-3)} + \partial \psi_{-(2q-1)}}{a} \right) - \partial \left( \frac{\partial \psi_{-(2q-1)} + \partial u_{-(2q+1)}}{a} \right) + a \psi_{-(2q-1)}, \quad q \geq 2,
\]

\[
f_{2n+1} := -\vec{\partial} \left( \frac{\partial \psi_{-(2n-1)} + \partial \psi_{-(2n+1)}}{a} \right) - \partial \left( \frac{\partial \psi_{-(2n+1)} + \partial \psi_{-(2n+3)}}{a} \right) + a \psi_{-(2n+1)}, \quad 1 \leq n \leq q - 2,
\]

\[
f_1 := \begin{cases} 
-2 \vec{\partial} \left( \frac{\text{Re } \partial \psi_{-1}}{a} \right) - \partial \left( \frac{\partial \psi_{-1} + \partial u_{-3}}{a} \right) + a \psi_{-1}, & q = 1, \\
-2 \vec{\partial} \left( \frac{\text{Re } \partial \psi_{-1}}{a} \right) - \partial \left( \frac{\partial \psi_{-1} + \partial \psi_{-3}}{a} \right) + a \psi_{-1}, & q \geq 2,
\end{cases}
\]

\[
f_{-(2n+1)} := f_{2n+1}, \quad 0 \leq n \leq q, \quad q \geq 1,
\]

By construction, \(f_{2n+1} \in C(\Omega)\) for \(0 \leq n \leq q, \quad q \geq 1\), as \(u_{-(2q+1)} \in C^2(\Omega; L_\infty)\) from \((129)\), and \(\psi_{-(2n-1)} \in C^2(\Omega; L_\infty)\), for \(1 \leq n \leq q - 1, \quad q \geq 1\), from \((122)\). We use these \(m + 1\) Fourier modes \(|\{f_{\pm1}, f_{\pm2}, \ldots, f_{\pm m}\}|\), and equations \((14)\), \((7)\) and \((9)\) to construct the pseudovectors \(|\vec{f}_0, \vec{f}_1, \ldots, \vec{f}_m|\), and thus the \(m\)-tensor field \(f_\Psi \in C(S^m; \Omega)\).

Define the real valued function \(u\) via its Fourier modes

\[
u(z, \theta) := \sum_{|n| \geq 2q+1} u_n(z)e^{in\theta} + 2 \text{Re} \left\{ \sum_{n=1}^{q} \psi_{-(2n-1)}(z)e^{-i(2n-1)\theta} \right\} + \frac{-2 \text{Re } \partial \psi_{-1}(z)}{a} + 2 \text{Re} \left\{ \sum_{n=1}^{q-1} u_{-2n}e^{-i(2n\theta)} \right\}.
\]

(147)

Using \((133)\) and \((145)\), and definition of \(|\psi_{-1}, \psi_{-3}, \ldots, \psi_{-(2q-1)}|\) \(\in \Psi_{a,g}^{\text{odd}}\) for \(q \geq 1\), the trace \(u(\cdot, \theta)\) in \((147)\) extends to the boundary, and its trace satisfy \((u(\cdot, \theta)|_r = g(\cdot, \theta))\).

Moreover, by using \((133)\), \((141)\) and the definition of \(f_{2n-1}\) for \(|n| \leq q, \quad q \geq 1\) in \((146)\), the real valued \(u\) defined in \((147)\) satisfies the transport equation \((117)\):

\[
\theta \cdot \nabla u + au = (f_\Psi, \theta^{2q+1}), \quad q \geq 1.
\]
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