The Right Complexity Measure in Locally Private Estimation:  
It is not the Fisher Information  

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June 2018  

Abstract  
We identify fundamental tradeoffs between statistical utility and privacy under local models of privacy in which data is kept private even from the statistician, providing instance-specific bounds for private estimation and learning problems by developing the local minimax risk. In contrast to approaches based on worst-case (minimax) error, which are conservative, this allows us to evaluate the difficulty of individual problem instances and delineate the possibilities for adaptation in private estimation and inference. Our main results show that the local modulus of continuity of the estimand with respect to the variation distance—as opposed to the Hellinger distance central to classical statistics—characterizes rates of convergence under locally private estimation for many notions of privacy, including differential privacy and its relaxations. As consequences of these results, we identify an alternative to the Fisher information for private estimation, giving a more nuanced understanding of the challenges of adaptivity and optimality, and provide new minimax bounds for high-dimensional estimation showing that even interactive locally private procedures suffer poor performance under weak notions of privacy.

1 Introduction  
The increasing collection of data at large scale—medical records, location information from cell phones, internet browsing history—points to the importance of a deeper understanding of the tradeoffs inherent between privacy and the utility of using the data collected. Classical mechanisms for preserving privacy, such as permutation, small noise addition, releasing only mean information, or basic anonymization are insufficient, and notable privacy compromises with genomic data [30] and movie rating information [39] have caused the NIH to temporarily stop releasing genetic information and Netflix to cancel a proposed competition for predicting movie ratings. Balancing the tension between utility and the risk of disclosure of sensitive information is thus essential.

In response to these challenges, researchers in the statistics, databases, and computer science communities have studied differential privacy [54, 25, 24, 23, 27, 21] as a formalization of disclosure risk limitation, providing strong privacy guarantees. This literature discusses two notions of privacy: local privacy, in which data is privatized before it is even shared with a data collector, and central privacy, where a centralized curator maintains the sample and guarantees that any information it releases is appropriately private. The local model is stronger, and consequently it is more challenging to develop statistically efficient algorithms. Yet the strong privacy protections local privacy provides encourage its adoption. Whether for ease of compliance with regulatory strictures, for example with European Union privacy rules [48]; for reasons of transparency and belief in the importance of privacy; or to avoid the risks proximate to holding sensitive data, like hacking or subpoena risk, because private data never leaves an individual’s device in the clear; major technology companies have adopted local differential privacy protections in their data collection and machine learning tools. Apple provides local differential privacy in many of its iPhone systems [3], and Google has built systems supplying central and local differential privacy [47, 1]. The broad impact of privacy protections in billions of devices suggest we should carefully understand the fundamental limitations and possibilities of learning with local notions of privacy.
To address this challenge, we study the local minimax complexity of estimation and learning under local notions of privacy. Worst-case notions of complexity may be too stringent for statistical practice [21], and in real-world use, we wish to understand how difficult the actual problem we have is, and whether we can adapt to this problem difficulty, so that our procedures more efficiently solve easy problems as opposed to being tuned to worst-case notions of difficulty. Our adoption of local minimax complexity is thus driven by three desiderata: we seek fundamental limits on estimation and learning that (i) are instance specific, applying to the particular problem at hand, (ii) are uniformly attainable, in that there exist adaptive procedures to achieve the instance-specific difficulty, and (iii) have super-efficiency limitations, so that if a procedure achieves better behavior than the lower bounds suggest is possible, there should be problem instances in which the procedure must have substantially worse behavior. In this paper, we provide a characterization of the difficulty of estimation of one-dimensional quantities under local privacy that satisfies these desiderata.

The celebrated Le Cam–Hajek local asymptotic minimax theory [26, 32, 34, 50, 35] cleanly delineates efficient from inefficient estimators in classical statistics and highlights the importance of local notions of optimality (making Fisher information bounds rigorous). As an example encapsulating the differences between global and local minimax complexity, consider the one-dimensional logistic regression problem of predicting $y \in \{\pm 1\}$ from $x \in \mathbb{R}$, with

$$p_\theta(y \mid x) = (1 + \exp(-yx\theta))^{-1},$$

where—taking our motivation from applications of machine learning [28, 3, 1]—we wish only to accurately estimate $p(y \mid x)$. This problem is easier the larger $|\theta|$ is, and a calculation shows that the maximum likelihood estimator has misclassification error decreasing exponentially in $|\theta|$; a fully minimax analysis provides a lower bound at $\theta = 0$, or random guessing, with convergence lower bound $1/\sqrt{n}$ independent of $\theta$. For most applications of statistical learning, we hope our model substantially outperforms random guessing, so such (global worst-case) analyses are of limited utility in the design of (near-) optimal procedures. To that end, any practicable theory of optimal private estimation should encapsulate a local notion of problem difficulty.

1.1 Contributions, outline, and related work

Our development of instance-specific (local) notions of problem complexity under privacy constraints allows us to more precisely quantify the statistical price of privacy. Identifying the tension here is of course of substantial interest, and Duchi, Jordan, and Wainwright [21, 20] develop a set of statistical and information-theoretic tools for understanding the minimax risk in locally differentially private settings, providing the point of departure for our work. To understand their and our coming approach, let us formally define our setting.

We have i.i.d. data $X_1, \ldots, X_n$ drawn according to a distribution $P$ on a space $\mathcal{X}$. Instead of observing the original sample $\{X_i\}$, however, the statistician or learner sees only privatized data $Z_1, \ldots, Z_n$, where the data $Z_i$ is drawn from a Markov kernel $Q(\cdot \mid X_i)$ conditional on $X_i$ (following information-theoretic parlance, we often call $Q$ the privacy channel [13]; in the privacy literature $Q$ is the mechanism [24]). In full generality, we allow the channel to be sequentially interactive [21], meaning that at observation $i$, the channel may depend on the previous (private) observations $Z_1, \ldots, Z_{i-1}$. That is, we have

$$Z_i \mid X_i = x, Z_1, \ldots, Z_{i-1} \sim Q(\cdot \mid x, Z_{1:i-1}).$$

This notion of interactivity is important for procedures, such as stochastic gradient methods [21] or the one-step-corrected estimators we develop in the sequel, which modify the mechanism after some number of observations to more accurately perform inference.

The statistical problems we consider are, abstractly, as follows. Let $\mathcal{P}$ be a family of distributions, and let $\theta : \mathcal{P} \to \Theta$ be a parameter belonging to a parameter space $\Theta$ we wish to estimate,
where $\theta(P)$ denotes the estimand. Let $L : \Theta \times \mathcal{P} \to \mathbb{R}_+$ be a loss function measuring the loss of an estimated value $\hat{\theta}$ for the distribution $P$, where we assume that $L(\theta(P), P) = 0$ for all distributions $P$. As an example, we may consider the mean $\theta(P) = \mathbb{E}_P[X] \in \mathbb{R}$ and squared error $L(\theta, P) = (\theta - \mathbb{E}_P[X])^2 = (\theta - \mathbb{E}_P[X])^2$. Let $Q$ be a collection of appropriately private channels, for example, $\varepsilon$-differentially private channels (which we define in the sequel). The \textit{private minimax risk} \cite{21} is

$$
\mathcal{M}_n(L, \mathcal{P}, Q) := \inf_{\hat{\theta}, Q \in Q} \sup_{P \in \mathcal{P}} \mathbb{E}_{Q \circ P} \left[ L(\hat{\theta}(Z_1, \ldots, Z_n), P) \right]
$$

where $Q \circ P$ denotes the marginal $X_i \sim P$ and $Z_i$ drawn conditionally \eqref{eq:4}. Duchi et al. \cite{21} provide upper and lower bounds on this quantity when $Q$ is the collection of $\varepsilon$-locally differentially private channels, developing strong data processing inequalities to quantify the costs of privacy.

The worst-case nature of the formulation \eqref{eq:3} may suggest lower bounds that are too pessimistic for practice, and does not allow a characterization of problem-specific difficulty, which is important for a deeper understanding of adaptive and optimal procedures. Accordingly, we adopt a \textit{local minimax approach}, which builds out of the classical statistical literature on hardest one-dimensional alternatives that begins with Stein \cite{44, 4, 16, 17, 18, 9, 11}. In the same setting as the above, we look for the hardest alternative distribution $P_1 \in \mathcal{P}$.

To situate our contributions, let us first consider the non-private variant of the minimax complexity \eqref{eq:3}, when $Q = \{\text{id}\}$ (the identity mapping), and we use the squared error loss $L_{sq}(\theta, P) = (\theta - \mathbb{E}_P[X])^2$. Let us first consider a classical setting, in which we wish to estimate a linear function $v^T \theta$ of a parameter $\theta$ in a parametric family $\mathcal{P} = \{P_0\}_{\theta \in \Theta}$ with Fisher information matrix $I_{\theta}$. The Fisher information bound \cite{35} for the parameter $\theta_0$ is

$$
\mathcal{M}_n^{\text{loc}}(P_{\theta_0}, L_{sq}, \mathcal{P}, \{\text{id}\}) \asymp \frac{1}{n} \mathbb{E} \left[ (v^T Z)^2 \right] \text{ for } Z \sim \mathcal{N}(0, I_{\theta_0}^{-1}).
$$

More generally, if we wish to estimate a functional $\theta(P) \in \mathbb{R}$ of a distribution $P$, Donoho and Liu \cite{16, 17, 18} show how the \textit{modulus of continuity} takes the place of the classical information bound. Again considering the squared error, define the modulus of continuity of $\theta(\cdot)$ over $\mathcal{P}$ with respect to Hellinger distance by

$$
\omega_{\text{hel}}(\delta; \mathcal{P}) := \sup_{P_0, P_1 \in \mathcal{P}} \{ (\theta(P_0) - \theta(P_1))^2 \mid P_0, P_1 \in \mathcal{P}, d_{\text{hel}}(P_0, P_1) \leq \delta \}
$$

where $d_{\text{hel}}^2(P_0, P_1) = \frac{1}{2} \int (\sqrt{dP_0} - \sqrt{dP_1})^2$. Then under mild regularity conditions,

$$
\mathcal{M}_n(L_{sq}, \mathcal{P}, \{\text{id}\}) \asymp \omega_{\text{hel}}(n^{-1/2}; \mathcal{P}),
$$

which highlights that separation in Hellinger distance precisely governs problem difficulty in non-private classical statistical problems. In the local minimax case, similar characterizations via a local modulus of continuity are available in some problems, including estimation of the value of a convex function \cite{9} and stochastic optimization \cite{11}.

In contrast, the work of Duchi et al. \cite{21, 20} suggests that for $\varepsilon$-locally differentially private estimation, we should replace the Hellinger distance by \textit{variation distance}. In the case of higher-dimensional problems, there are additional dimension-dependent penalties in estimation that local
differential privacy makes unavoidable, at least in a minimax sense [21]. In work independent of and contemporaneous to our own, Rohde and Steinberger [43] build off of [21] to show that (non-local) minimax rates of convergence under $\varepsilon$-local differential privacy are frequently governed by a modulus of continuity (4), except that the variation distance $\|P_0 - P_1\|_{TV} = \sup_A |P_0(A) - P_1(A)|$ replaces the Hellinger distance $d_{hel}$. Rohde and Steinberger also exhibit a mechanism that is minimax optimal for “nearly” linear functionals based on randomized response [54, 43, Sec. 4]. Thus, locally differentially private procedures give rise to a different geometry than classical statistical problems.

Now we are in a position for a high-level description of our results. Our results apply in a variety of locally private estimation settings, whose definitions we formalize in Section 2, but all of them consist of weakenings of $\varepsilon$-differential privacy (including concentrated and Rényi-differential privacy [24, 22, 8, 38]). We provide a precise characterization of the local minimax complexity (3) in these settings. If we define the local modulus of continuity (for the squared error) at $P_0$ by

$$\omega_{TV}(\delta; P_0, \mathcal{P}) := \sup_{P \in \mathcal{P}} \{ (\theta(P_0) - \theta(P))^2 \mid \|P - P_0\|_{TV} \leq \delta \},$$

then a consequence of our Theorem 1 is that for the squared loss and family $Q_\varepsilon$ of $\varepsilon$-locally private channels,

$$\mathfrak{m}_n^{loc}(P_0, L_{sq}, \mathcal{P}, Q_\varepsilon) \asymp \omega_{TV}( (n\varepsilon^2)^{-1/2}; P_0, \mathcal{P} ).$$

We provide this characterization in more detail and for general losses in Section 3. Moreover, we show a super-efficiency result that any procedure that achieves risk better than the local minimax complexity at a distribution $P_0$ must suffer higher risk at another distribution $P_1$, so that this characterization does indeed satisfy our desiderata of an instance-specific complexity measure.

The departure of these risk bounds from the typical Hellinger-based moduli of continuity (4) has consequences for locally private estimation and adaptivity of estimators, which we address via examples in Section 4. For instance, instead of the classical Fisher information, an alternative we term the $L_1$-information characterizes the complexity of locally private estimation: the classical Fisher information bounds are unobtainable. A challenging consequence of these results is that, for some parametric models (including Bernoulli estimation and binomial logistic regression), the local complexity (3) is independent of the underlying parameter: problems that are easy (in the classical Fisher information sense) are never easy under local privacy constraints. Our proofs, building off of those of Duchi et al. [21], rely on novel Markov contraction inequalities for divergence measures, which strengthen classical strong data processing inequalities [12, 14].

Developing adaptive procedures uniformly achieving the instance-specific local minimax risk (3) is challenging, but we show that such optimal design is possible in a number of cases in Section 4, including well- and mis-specified exponential family models, using an extension of classical one-step corrected estimators. We compare these locally optimal procedures with the minimax optimal procedures Duchi et al. [21] propose on a protein expression-prediction problem in Section 6; the experimental results suggests that the local minimax perspective indeed outperforms the global minimax procedures, however, the costs of privacy are still nontrivial.

Lastly, because we consider weaker notions of privacy, one might ask whether it is possible to improve the minimax bounds that Duchi et al. [21, 20] develop. Unfortunately, this appears impossible (see Section 5). Duchi et al. show only that non-interactive privacy mechanisms (i.e. the channel $Q$ may depend only on $X_i$ and not the past observations) must suffer poor performance in high dimensions under stringent differential privacy constraints. Our results show that this is unavoidable, even with weaker notions of privacy and allowing interactive mechanisms. We provide some additional discussion and perspective in the closing of the paper in Section 7.
2 Definitions of privacy

Our starting point is a formalization of our notions of local privacy. With the notion (1) of sequentially interactive channels, where the ith private observation is drawn conditionally on the past as \( Z_i \mid X_i = x, Z_1, \ldots, Z_{i-1} \sim Q(.) \mid x, Z_{1:i-1} \), we consider several notions of privacy, going from the strongest to the weakest. First is local differential privacy, which Warner \[54\] first proposed (implicitly) in his 1965 work on survey sampling, then explicitly defined by Evfimievski et al. \[25\] and Dwork et al. \[24\].

**Definition 1.** The channel \( Q \) is \( \epsilon \)-locally differentially private if for all \( i \in \mathbb{N}, x, x' \in \mathcal{X} \), and \( z_{1:i-1} \in \mathcal{Z}^{i-1} \), we have

\[
\sup_{A \in \sigma(Z)} \frac{Q(A \mid x, z_{1:i-1})}{Q(A \mid x', z_{1:i-1})} \leq e^\epsilon.
\]

The channel \( Q \) is non-interactive if \( Q(A \mid x, z_{1:i-1}) = Q(A \mid x) \) for all \( z_{1:i-1} \in \mathcal{Z}^{i-1} \) and \( A \in \sigma(Z) \).

Duchi et al. \[21\] consider this notion of privacy, developing its consequences for minimax optimal estimation. It is a satisfying definition from a privacy point of view, and an equivalent view is that an adversary knowing the data is either \( x \) or \( x' \) cannot accurately test, even conditional on the output \( Z \), whether the generating data was \( x \) or \( x' \). To mitigate the consequent difficulties for estimation and learning with differentially private procedures, researchers have proposed a number of weakenings of Definition 1, which we also consider.

To that end, a second notion of privacy, which Dwork and Rothblum \[22\] propose and Bun and Steinke \[8\] develop reposes on Rényi-divergences. For an \( \alpha \geq 1 \), the Rényi-divergence of order \( \alpha \) is

\[
D_\alpha (P\|Q) := \frac{1}{\alpha - 1} \log \int \left( \frac{dP}{dQ} \right)^\alpha dQ,
\]

where for \( \alpha = 1 \) one takes the downward limit as \( \alpha \downarrow 1 \), yielding \( D_1 (P\|Q) = D_{kl} (P\|Q) \). We then have the following definition.

**Definition 2.** The channel \( Q \) is \((\kappa, \rho)\)-zero-concentrated locally differentially private \((zCDP)\) if for all \( \alpha, x, x' \in \mathcal{X} \), and \( z_{1:i-1} \in \mathcal{Z} \), we have

\[
D_\alpha \left( Q(.) \mid x, z_{1:i-1} \right) \| Q(.) \mid x', z_{1:i-1} \right) \leq \kappa + \rho \alpha.
\]

An equivalent definition is that the log likelihood ratio \( \log \frac{dQ(Z \mid x, z_{1:i-1})}{dQ(Z \mid x', z_{1:i-1})} \) has sub-Gaussian tails, that is, for \( Z \sim Q(.) \mid x', z_{1:i-1} \), we have

\[
L := \log \frac{dQ(Z \mid x, z_{1:i-1})}{dQ(Z \mid x', z_{1:i-1})} \quad \text{satisfies} \quad \mathbb{E} \left[ \exp \left( \lambda L \right) \right] \leq \exp \left( \rho \lambda^2 + \lambda (\rho + \kappa) \right)
\]

for all \( \lambda \geq 0 \) (and \( \mathbb{E} [\exp (L) ] = 1 \)). Mironov \[38\] proposes a natural relaxation of Definition 2, suggesting that we require it hold only for a single fixed \( \alpha > 1 \). This yields

**Definition 3.** The channel \( Q \) is \((\alpha, \epsilon)\)-Rényi locally differentially private if for all \( x, x' \in \mathcal{X} \), and \( z_{1:i-1} \in \mathcal{Z} \), we have

\[
D_\alpha \left( Q(.) \mid x, z_{1:i-1} \right) \| Q(.) \mid x', z_{1:i-1} \right) \leq \epsilon.
\]
Perhaps the most salient point in Definition 3 is the choice $\alpha = 2$, which will be important in our subsequent analysis. Consider a prior distribution on two points $x, x'$, represented by $\pi(x) \in [0, 1]$ and $\pi(x') = 1 - \pi(x)$, and then consider the posterior $\pi(x | Z)$ and $\pi(x' | Z)$ after observing the private quantity $Z \sim Q(\cdot | x)$. Then $(\alpha, \varepsilon)$-Rényi privacy is equivalent [38, Sec. VII] to the condition that the prior and posterior odds ratios of $x$ against $x'$ do not change much in expectation:

$$
E \left[ \frac{\pi(x | Z)}{\pi(x')} / \frac{\pi(x)}{\pi(x')} \mid x \right] \leq e^\varepsilon
$$

for all two-point priors $\pi$, where the expectation is taken over $Z | x$. (For $\varepsilon$-differential privacy, the inequality holds for all $Z$ without expectation). Because Rényi divergences are monotonic in $\alpha$ (cf. [51, Thm. 3]), any channel that is $(\alpha, \varepsilon)$-Rényi private is also $(\alpha', \varepsilon)$-Rényi private for $\alpha' \leq \alpha$.

Our final notion of privacy is based on $f$-divergences, which is related to Definition 3. Recall for a convex function $f : \mathbb{R}_+ \rightarrow \mathbb{R} \cup \{+\infty\}$ with $f(1) = 0$, the $f$-divergence between distributions $P$ and $Q$ is

$$
D_f (P\|Q) := \int f \left( \frac{dP}{dQ} \right) dQ,
$$

which is non-negative and strictly positive when $P \neq Q$ and $f$ is strictly convex at the point 1. We consider $f$-divergences parameterized by $k \in [1, \infty)$ of the form

$$
f_k(t) := |t - 1|^k.
$$

**Definition 4.** For $k \in [1, \infty)$, the channel $Q$ is $\varepsilon$-$f_k$-divergence locally private if for all $i \in \mathbb{N}$, $x, x' \in \mathcal{X}$, and $z_{1:i-1} \in \mathcal{Z}$, we have

$$
D_{f_k} (Q(\cdot | x, z_{1:i-1}) \| Q(\cdot | x', z_{1:i-1})) \leq \varepsilon^k.
$$

When $k = 2$, this is the familiar $\chi^2$-divergence [37, 46], and it is equivalent to ($2, \log(1 + \varepsilon^2)$)-Rényi differential privacy. We describe this special situation as $\varepsilon^2$-$\chi^2$-privacy.

The definitions provide varying levels of privacy. It is immediate that if a channel is $\varepsilon$-differentially private, then it is $(\varepsilon^2 - 1)$-$f_k$-divergence locally private. For $\varepsilon \leq 1$, this implies $(\varepsilon - 1)\varepsilon$-$f_k$-divergence local privacy. It is also clear that Definition 1 is stronger than 2, which is stronger than 3. We can quantify this as well: $\varepsilon$-differential privacy implies $(0, \frac{1}{2}\varepsilon^2)$-zero-concentrated differential privacy. For $k = 2$, we also find that if the channel $Q$ is $(\kappa, \rho)$-zCDP, then it is immediate that it satisfies $\varepsilon^2$-$\chi^2$-divergence privacy with $\varepsilon^2 = e^{\kappa + 2\rho} - 1$, where we take $\alpha = 2$ in the definition of the Rényi-divergence. Our results all apply for $\chi^2$-private channels, so that $\chi^2$-privacy (Definition 3 with $\alpha = 2$ or Definition 4 with $k = 2$) implies strong lower bounds on estimation.

# 3 Local minimax complexity and private estimation

We turn to our main goal of establishing localized minimax complexities for locally private estimation. To that end, we begin in Section 3.1 by defining the *modulus of continuity* of estimands, showing how it provides a tight lower bound on localized complexity for private estimation. Section 3.2 continues the development of Section 3.1 by establishing a super-efficiency result, showing that any estimator achieving lower risk than our localized modulus of continuity for some distribution $P_0$ must be inefficient on other distributions $P$. In Section 3.3, we present the main technical tools that underlie our results, providing new strong data-processing inequalities showing precisely how locally private channels degrade the information in statistical problems.
3.1 The modulus of continuity and local minimax complexities

Recall our setting, where we wish to estimate a parameter \( \theta(P) \) of a distribution \( P \in \mathcal{P} \), a collection of possible distributions, and we measure performance of an estimand \( \theta \) via a loss \( L : \Theta \times \mathcal{P} \to \mathbb{R}_+ \) satisfying \( L(\theta(P), P) = 0 \). We define the “distance” between distributions \( P_0 \) and \( P_1 \) for the loss \( L \) by

\[
d_L(P_0, P_1) := \inf_{\theta \in \Theta} \{ L(\theta, P_0) + L(\theta, P_1) \},
\]

which is always non-negative. As an example, if \( \theta \) is 1-dimensional and we use the squared error \( L(\theta, P) = \frac{1}{2}(\theta - \theta(P))^2 \),

\[
d_L(P_0, P_1) = \frac{1}{4}(\theta(P_0) - \theta(P_1))^2.
\]

More generally, for any symmetric convex function \( \Phi : \mathbb{R}^d \to \mathbb{R}_+ \), if \( L(\theta, P) = \Phi(\theta - \theta(P)) \),

\[
d_L(P_0, P_1) = 2\Phi\left( \frac{1}{2}(\theta_0 - \theta_1) \right)
\]

where \( \theta_a = \theta(P_a) \). A similar result holds for general losses; if \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \) is non-decreasing and we measure the parameter error \( L(\theta, P) = \Phi(\|\theta - \theta(P)\|_2) \), then

\[
2\Phi\left( \frac{1}{2}\|\theta_0 - \theta_1\|_2 \right) \geq d_L(P_0, P_1)
\]

\[
= \inf_{\lambda \in [0,1]} \{ \Phi(\lambda\|\theta_0 - \theta_1\|_2) + \Phi((1 - \lambda)\|\theta_0 - \theta_1\|_2) \} \geq \Phi\left( \frac{1}{2}\|\theta_0 - \theta_1\|_2 \right),
\]

as it is no loss of generality to assume that \( \theta \in [\theta_0, \theta_1] \) in the definition of the distance.

For a family of distributions \( \mathcal{P} \), the modulus of continuity associated with the loss \( L \) at the distribution \( P_0 \) is

\[
\omega_L(\delta; P_0, \mathcal{P}) := \sup_{P \in \mathcal{P}} \{ d_L(P, P_0) \mid \|P - P_0\|_{TV} \leq \delta \}. \tag{8}
\]

As we shall see, this modulus of continuity fairly precisely characterizes the difficulty of locally private estimation of functionals. The key in this definition is that the modulus of continuity is defined with respect to variation distance. This is in contrast to classical results on optimal estimation, where the more familiar modulus of continuity with respect to Hellinger distance characterizes problem difficulty. Indeed, Le Cam’s theory of quadratic mean differentiability, contiguity, local asymptotic normality, and local alternatives for testing all reposes on closeness in Hellinger distance [34, 35, 42, 50], which justifies the use of Fisher Information in classical statistical problems. In nonparametric problems, as we mentioned briefly in the introduction, the modulus of continuity of the parameter \( \theta(P) \) with respect to Hellinger distance also characterizes minimax rates for estimation of functionals of distributions [4, 16, 17] (at least in a global minimax sense), and in some instances, it governs local minimax guarantees as well [9]. These results all correspond to replacing the variation distance \( \|\cdot\|_{TV} \) in definition (8) with the Hellinger distance between \( P \) and \( P_0 \). As we illustrate, the difference between the classical Hellinger-based modulus of continuity and ours (8) leads to to substantially different behavior for private and non-private estimation problems.

With this, we come to our first main result, which lower bounds the local minimax risk using the modulus (8). We defer the proof to Section 3.4.3, using our results on strong data-processing inequalities to come to prove it.

**Theorem 1.** Let \( \mathcal{Q} \) be the collection of \( \varepsilon \cdot \chi^2 \)-locally private channels. Then for any distribution \( P_0 \), we have

\[
\mathcal{M}^{loc}_{\mathcal{Q}}(P_0, L, \mathcal{P}, \mathcal{Q}) \geq \frac{1}{8} \omega_L \left( \frac{1}{2\varepsilon} \sqrt{e^{\frac{1}{2\varepsilon}} - 1}; P_0, \mathcal{P} \right).
\]
Noting that the modulus of continuity is increasing in its first argument and that $e^x - 1 \geq x$ for all $x$, we have the simplified lower bound

$$
\mathcal{M}_{\text{loc}}^n(P_0, L, P, Q) \geq \frac{1}{8} \omega_L \left( \frac{1}{\sqrt{8n\varepsilon^2}}; P_0, P \right).
$$

In (nearly) simultaneous independent work, Rohde and Steinberger [43] provide a global minimax lower bound, akin to (2), using a global modulus of continuity with respect to variation distance, extending [16, 17, 18] to the private case. Our focus here is on instance-specific bounds, with the hope that we may calculate practically useful quantities akin to classical information bounds [50, 35].

Achievability in the theorem is a somewhat more delicate argument; demonstrating procedures that achieve the lower bound uniformly is typically nontrivial. With that said, under two reasonable conditions on our loss, distance, and growth of the modulus of continuity, we can show a converse to Theorem 1, showing that the modulus $\omega_L$ indeed describes the local minimax complexity to within numerical constants.

**Condition C.1** (Reverse triangle inequality). There exists $\gamma < \infty$ such that for $\theta_a = \theta(P_a)$,

$$
L(\theta_1, P_0) + L(\theta_0, P_1) \leq \gamma d_L(P_0, P_1).
$$

In the case that the loss is based on the error $L(\theta, P) = \Phi(||\theta - \theta(P)||)$ for $\Phi \geq 0$ nondecreasing, the inequality (7) shows that Condition C.1 holds whenever $\Phi(2t) \leq C\Phi(t)$ for all $t \geq 0$. In addition, we sometimes use the following condition on the modulus of continuity.

**Condition C.2** (Polynomial growth). At the distribution $P_0$, there exist constants $\alpha, \beta < \infty$ such that for all $c \geq 1$

$$
\omega_L(c\delta; P_0, P) \leq (\beta c)^{\alpha} \omega_L(\delta; P_0, P).
$$

Condition C.2 is similar to the typical Hölder-type continuity properties assumed on the modulus of continuity for estimation problems [16, 17]. We give examples satisfying Condition C.2 presently.

The conditions yield the following converse to Theorem 1, which shows that the modulus of continuity characterizes the local minimax complexity. See Appendix A.2 for a proof of the result.

**Proposition 1.** Let Conditions C.1 and C.2 on $L$ and $P$ hold. Let $\varepsilon \geq 0$ and $\delta_\varepsilon = \frac{8^\varepsilon - 1}{2}$, and let $Q$ be the collection of non-interactive $\varepsilon$-differentially private channels (Definition 1). Then

$$
\mathcal{M}_{\text{loc}}^n(P_0, L, P, Q) \leq \gamma \beta^{\alpha} e^{\frac{1}{2}[\log \frac{2}{\gamma} - 1]} \omega_L \left( \frac{\sqrt{2}}{\delta_\varepsilon \sqrt{n}}; P_0, P \right).
$$

The proposition as written is a bit unwieldy, so we unpack it slightly. For $\varepsilon \leq \frac{7}{4}$, we have $\delta_\varepsilon \geq \frac{\varepsilon}{5}$, so that for a constant $c$ that may depend on $\alpha, \beta$, and $\gamma$, for each $P_1 \in P$ there exists a non-interactive $\varepsilon$-differentially private channel $Q$ and estimator $\hat{\theta}$ such that

$$
\max_{P \in \{P_0, P_1\}} \mathbb{E}_{P, Q}[L(\hat{\theta}(Z_{1:n}), P)] \leq c \cdot \omega_L \left( \frac{5\sqrt{2}}{\sqrt{n\varepsilon^2}}; P_0, P \right).
$$

This matches the lower bound in Theorem 1 up to a numerical constant.

We briefly discuss one example satisfying Condition C.2 to demonstrate that we typically expect it to hold, so that the modulus of continuity characterizes the local minimax rates. We also
show that that (potentially mis-specified) exponential family models also satisfy Condition C.2 in Section 4.4, Lemma 3.

**Example 1** (Modulus of continuity in (nonparametric) mean estimation): Consider loss \( L(\theta, P) = \Phi(\|\theta - \theta(P)\|_2) \) where \( \Phi \) is nondecreasing. Inequality (7) implies that using the shorthand
\[
\omega_2(\delta) := \sup_{P \in \mathcal{P}} \{\|\theta_0 - \theta(P)\|_2 \mid \|P - P_0\|_{TV} \leq \delta\},
\]
we have
\[
\Phi\left(\frac{1}{2} \omega_2(\delta)\right) \leq \omega_L(\delta; P_0, \mathcal{P}) \leq 2\Phi\left(\frac{1}{2} \omega_2(\delta)\right),
\]
and assuming that the function \( \Phi \) itself satisfies \( \Phi(2t) \leq C\Phi(t) \) for \( t \geq 0 \), to show \( \omega_L \) satisfies Condition C.2, it is sufficient to show that \( \omega_2(\delta) \) satisfies Condition C.2.

Now consider the problem of estimating \( \theta(P) = \mathbb{E}_P[X] \), where the unknown distribution \( P \) belongs to \( \mathcal{P} = \{P : \text{supp } P \subset \mathcal{X}\} \) for some compact set \( \mathcal{X} \). Denote \( \theta_0 = \mathbb{E}_{P_0}[x] \). We claim the following upper and lower bounds on \( \omega_2(\delta) \):
\[
\delta \cdot \sup_{x \in \mathcal{X}} \|x - \theta_0\|_2 \leq \omega_2(\delta) \leq 2\delta \cdot \sup_{x \in \mathcal{X}} \|x - \theta_0\|_2,
\]
which of course combine to imply Condition C.2. To see the lower bound (9), for any \( x \in \mathcal{X} \), define \( P_x = (1 - \delta)P_0 + \delta \cdot 1_x \), where \( 1_x \) denotes a point mass at \( x \). Then \( \|P_x - P_0\|_{TV} \leq \delta \) for all \( x \in \mathcal{X} \), so
\[
\omega_2(\delta) \geq \sup_{x} \|\theta_0 - \theta_{P_x}\|_2 = \delta \cdot \sup_{x \in \mathcal{X}} \|x - \theta_0\|_2.
\]
The upper bound (9) is similarly straightforward: for all \( P \in \mathcal{P} \), we have
\[
\|\theta(P) - \theta_0\|_2 = \left\| \int (x - \theta_0)(dP(x) - dP_0(x)) \right\|_2 \leq 2 \sup_{x \in \mathcal{X}} \|x - \theta_0\|_2 \|P - P_0\|_{TV}
\]
by the triangle inequality, which is our desired result. ◇

### 3.2 Super-efficiency

To demonstrate that the local modulus of continuity is indeed the “correct” lower bound on estimation, we consider the third of the desiderata for a strong lower bound that we identify in the introduction: a super-efficiency result. We provide this via a constrained risk inequality [6, 19].

Our result applies in the typical setting in which the loss is \( L(\theta, P) := \Phi(\|\theta - \theta(P)\|_2) \) for some increasing function \( \Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \), and we use the shorthand \( R(\hat{\theta}, \theta, P) := \mathbb{E}_P[\Phi(\|\hat{\theta}(Z) - \theta\|_2)] \) for the risk (expected loss) of the estimator \( \hat{\theta} \) under the distribution \( P \). The starting point for our development is an inequality extending Brown and Low [6, Thm. 1] showing that if \( \hat{\theta} \) has small risk for a parameter \( \theta \) under a distribution \( P_0 \), then its risk under a distribution \( P_1 \) close to \( P_0 \) may be large (see also [45, Thm. 6]). In the lemma and the remainder of this section, for measures \( P_0 \) and \( P_1 \) we define the \( \chi^2 \)-affinity
\[
\rho(P_0\|P_1) := D_{\chi^2}(P_0\|P_1) + 1 = \mathbb{E}_{P_1} \left[ \frac{dP_0}{dP_1} \right]^2 = \mathbb{E}_{P_0} \left[ \frac{dP_0}{dP_1} \right],
\]
which measures the similarity between distributions \( P_0 \) and \( P_1 \). With these definitions, we have the following constrained risk inequality.

**Lemma 1** ([19], Theorem 1). Let \( \theta_0 = \theta(P_0), \theta_1 = \theta(P_1) \), and define \( \Delta = \Phi(\|\theta_0 - \theta_1\|_2) \). If the estimator \( \hat{\theta} \) satisfies \( R(\hat{\theta}, \theta_0, P_0) \leq \delta \) for some \( \delta \geq 0 \), then
\[
R(\hat{\theta}, \theta_1, P_1) \geq \left[ \Delta^{1/2} - (\rho(P_0\|P_0) \cdot \delta)^{1/2} \right]^2.
\]
The goal is then to provide upper bounds on the Dobrushin condition \[15\]. For inequalities, and in the mixing properites of Markov chains under so-called strong mixing conditions, such as the Dobrushin condition \[15\], where one studies strong contractive properties have been important in the study of information channels \[12, 14\], where one studies strong contractions on the space of probability measures. Such contractive properties have been important in the study of information channels \[12, 14\], where one studies strong data processing inequalities, and in the mixing properties of Markov chains under so-called strong mixing conditions, such as the Dobrushin condition \[15\]. For a \(a \in (0, 1)\), define the marginal distributions

\[
M_a(S) := \int Q(S \mid x) dP_a(x).
\]

The goal is then to provide upper bounds on the f-divergence \(D_f(M_0 \mid M_1)\) in terms of the channel \(Q\); the standard data-processing inequality \[13, 37\] guarantees \(D_f(M_0 \mid M_1) \leq D_f(P_0 \mid P_1)\). Dobrushin’s celebrated ergodic coefficient \(\alpha(Q) := 1 - \sup_{x, x'} \| Q(\cdot \mid x) - Q(\cdot \mid x') \|_{TV} \) guarantees that for any f-divergence (see \[12, 14\]),

\[
D_f(M_0 \mid M_1) \leq \sup_{x, x'} \| Q(\cdot \mid x) - Q(\cdot \mid x') \|_{TV} D_f(P_0 \mid P_1).
\]  

Thus, as long as the Dobrushin coefficient is strictly positive, one obtains a strong data processing inequality. In our case, our privacy guarantees provide a stronger condition than the positivity of the Dobrushin coefficient. Consequently, we are able to provide substantially stronger data processing inequalities: we can even show that it is possible to modify the underlying f-divergence.

Thus, we reconsider the notions of privacy based on divergences between the channels \(Q(\cdot \mid x)\) and \(Q(\cdot \mid x')\). We have the following proposition, which provides a strong data processing inequality for all channels satisfying the divergence-based notion of privacy (Definition 4).
Proposition 3. Let \( f_k(t) = |t - 1|^k \) for some \( k > 1 \), and let \( P_0 \) and \( P_1 \) be arbitrary distributions on a common space \( \mathcal{X} \). Let \( Q \) be a Markov kernel from \( \mathcal{X} \) to \( \mathcal{Z} \) satisfying
\[
D_{f_k}(Q(\cdot | x)\|Q(\cdot | x')) \leq \varepsilon^k
\]
for all \( x, x' \in \mathcal{X} \) and \( M_\alpha(\cdot) = \int Q(\cdot | x)dP_\alpha(x) \). Then
\[
D_{f_k}(M_0\|M_1) \leq (2\varepsilon)^k \|P_0 - P_1\|_{TV}^k.
\]

Jensen’s inequality implies that \( 2^k \|P_0 - P_1\|_{TV}^k \leq D_{f_k}(P_0\|P_1) \), so Proposition 3 provides a stronger guarantee than the classical bound (11) for the specific divergence associated with \( f_k(t) = |t - 1|^k \). Because \( \|P_0 - P_1\|_{TV} \leq 1 \) for all \( P_0, P_1 \), it is possible that the \( f_k \)-divergence is infinite, while the marginals are much closer together. It is this transfer from power divergence to variation distance, that is, \( f_k \) to \( f_1(t) = |t - 1| \), that allows us to prove the strong localized lower bounds depending on variation distance such as Theorem 1.

As a corollary of Proposition 3, we may parallel the proof of [21, Theorem 1] to obtain a tensorization result. In this context, the most important divergence for us is the \( \chi^2 \) divergence, which corresponds to the case \( k = 2 \) in Proposition 3, that is, \( f(t) = (t - 1)^2 \), which also corresponds to Rényi differential privacy with \( \alpha = 2 \) (Def. 3) with a guarantee that prior and posterior odds of discovery do not change much (Eq. (5)). Recall our formulation (1), in which the channel \( Q(\cdot) \) may be defined sequentially as as \( Q(\cdot | x, z_{1:i-1}) \), and let
\[
Q^n(S | x_{1:n}) := \int_{z_{1:n} \in S} \prod_{i=1}^n dQ(z_i | x_i, z_{1:i-1}).
\]

Now, let \( P_\alpha, \alpha = 0, 1 \) be product distributions on \( \mathcal{X} \), where we say that the distribution of \( X_i \) either follows \( P_{0,i} \) or \( P_{1,i} \), and define \( M_\alpha^n(\cdot) = \int Q^n(\cdot | x_{1:n})dP_\alpha(x_{1:n}) \), noting that \( dP_\alpha(x_{1:n}) = \prod_{i=1}^n dP_{\alpha,i}(x_i) \) as \( P_\alpha \) is a product distribution. We have the following corollary.

**Corollary 1.** Let \( Q \) be a sequentially interactive channel satisfying \( \varepsilon^2 \chi^2 \)-divergence privacy, that is, \( D_{\chi^2}(Q(\cdot | x, z_{1;i})\|Q(\cdot | x', z_{1;i})) \leq \varepsilon^2 \) for all \( x, x' \in \mathcal{X} \) and \( z_{1;i} \in \mathcal{Z}^i \). Then
\[
D_{\chi^2}(M_0^n\|M_1^n) \leq \prod_{i=1}^n \left( 1 + 4\varepsilon^2 \|P_{0,i} - P_{1,i}\|_{TV}^2 \right) - 1.
\]

See Section 3.4.2 for a proof. An immediate consequence of Corollary 1 and the fact [46, Lemma 2.7] that \( D_{kl}(P_0\|P_1) \leq \log(1 + D_{\chi^2}(P_0\|P_1)) \) yields
\[
D_{kl}(M_0^n\|M_1^n) \leq \sum_{i=1}^n \log \left( 1 + 4\varepsilon^2 \|P_{0,i} - P_{1,i}\|_{TV}^2 \right) \leq 4\varepsilon^2 \sum_{i=1}^n \|P_{0,i} - P_{1,i}\|_{TV}^2. \tag{12}
\]

The tensorization (12) is the key to our results, as we see in the later sections.

### 3.4 Proofs

We collect the proofs of our main results in this section, as they are reasonably brief and (we hope) elucidating. We begin with the key contraction inequality in Proposition 3, as it underlies all subsequent results.
3.4.1 Proof of Proposition 3

Let \( p_0 \) and \( p_1 \) be the densities of \( P_0, P_1 \) with respect to some base measure \( \mu \) dominating \( P_0, P_1 \). Without loss of generality, we may assume that \( Z \) is finite, as all \( f \)-divergences are approximable by finite partitions \([49]\); we let \( m_a \) denote the associated p.m.f. For \( k > 1 \), the function \( t \mapsto t^{1-k} \) is convex on \( \mathbb{R}_+ \). Thus, applying Jensen’s inequality, we may bound \( D_{f_k} (M_0 \| M_1) \) by

\[
D_{f_k} (M_0 \| M_1) = \sum_z \frac{|m_0(z) - m_1(z)|^k}{m_1(z)^{k-1}} \leq \sum_z \left( \int \frac{|m_0(z) - m_1(z)|^k}{q(z \mid x_0)^{k-1}} p_1(x_0) d\mu(x_0) \right) \leq \int \left( \sum_z \frac{|m_0(z) - m_1(z)|^k}{q(z \mid x_0)^{k-1}} \right) p_1(x_0) d\mu(x_0). \tag{13}
\]

It thus suffices to upper bound \( W(x_0) \). To do so, we rewrite \( m_0(z) - m_1(z) \) as

\[
m_0(z) - m_1(z) = \int q(z \mid x) (dP_0(x) - dP_1(x)) = \int (q(z \mid x) - q(z \mid x_0)) (dP_0(x) - dP_1(x)),
\]

where we have used that \( \int (dP_0 - dP_1) = 0 \). Now define the function

\[
\Delta(z \mid x, x_0) := q(z \mid x) - q(z \mid x_0) \quad q(z \mid x_0)^{1-1/k}.
\]

By Minkowski’s integral inequality, we have the upper bound

\[
W(x_0)^{1/k} = \left( \sum_z \left( \int \Delta(z \mid x, x_0) (p_0(x) - p_1(x)) d\mu(x) \right)^k \right)^{1/k} \tag{14}
\]

\[
\leq \int \left( \sum_z |\Delta(z \mid x, x_0) (p_0(x) - p_1(x))|^k \right)^{1/k} d\mu(x) = \int \left( \sum_z |\Delta(z \mid x, x_0)|^k \right)^{1/k} |dP_0(x) - dP_1(x)|.
\]

Now we compute the inner summation: we have that

\[
\sum_z |\Delta(z \mid x, x_0)|^k = \sum_z \left| \frac{q(z \mid x)}{q(z \mid x_0)} - 1 \right|^k q(z \mid x_0) = D_{f_k} (Q(\cdot \mid x) \| Q(\cdot \mid x_0)).
\]

Substituting this into our upper bound (14) on \( W(x_0) \), we obtain that

\[
W(x_0) \leq \sup_{x \in X} D_{f_k} (Q(\cdot \mid x) \| Q(\cdot \mid x_0)) 2^k \|P_0 - P_1\|_{TV}^k,
\]

as \( \int |dP_0 - dP_1| = 2 \|P_0 - P_1\|_{TV} \). Substitute this upper bound into inequality (13) to obtain the proposition.

3.4.2 Proof of Corollary 1

We use an inductive argument. The base case in which \( n = 1 \) follows immediately by Proposition 3. Now, suppose that Corollary 1 holds at \( n - 1 \); we will show that the claim holds for \( n \in \mathbb{N} \). We use the shorthand \( m_a(Z_{1:k}) \) for the density of the measure \( M_a \), \( a \in \{0, 1\} \) and \( k \in \mathbb{N} \), which we may assume exists w.l.o.g. Then, by definition of \( \chi^2 \)-divergence, we have,

\[
D_{\chi^2} (M_0^n \| M_1^n) + 1 = \mathbb{E}_{M_1} \left[ \frac{m_0^n(Z_{1:n})}{m_1^n(Z_{1:n})} \right] = \mathbb{E}_{M_1} \left[ \frac{m_0^n(Z_{1:n})}{m_1^n(Z_{1:n})} \right] \mathbb{E}_{M_1} \left[ \frac{m_0^n(Z_{1:n})}{m_1^n(Z_{1:n})} \right] \mathbb{E}_{M_1} \left[ \frac{m_0^n(Z_{1:n})}{m_1^n(Z_{1:n})} \right] .
\]
Noting that the kth marginal distributions $M_{a,k}(\cdot \mid z_{1:k-1}) = \int Q(\cdot \mid x, z_{1:k-1}) dP_{a,i}(x)$ for $a \in \{0, 1\}$, we see that for any $z_{1:n-1} \in \mathbb{Z}^{n-1}$,
\[
\mathbb{E}_{M_i} \left[ \frac{m_0^2(Z_n \mid z_{1:n-1})}{m_1^2(Z_n \mid z_{1:n-1})} \mid z_{1:n-1} \right] = 1 + D_{\chi^2}(M_{0,n}(\cdot \mid z_{1:n-1}) \| M_{1,n}(\cdot \mid z_{1:n-1}))
\leq 1 + 4\varepsilon^2 \|P_{0,n}(\cdot \mid z_{1:n-1}) - P_{1,n}(\cdot \mid z_{1:n-1})\|_{TV}^2
= 1 + 4\varepsilon^2 \|P_{0,n} - P_{1,n}\|_{TV}^2,
\]
where the inequality is Proposition 3 and the final equality follows because $X_n$ is independent of $Z_{1:n-1}$. This yields the inductive step and completes the proof once we recall the inductive hypothesis and that $\mathbb{E}_{M_i} \left[ \frac{m_0^2(Z_{1:n-1})}{m_1^2(Z_{1:n-1})} \right] = D_{\chi^2}(M_{0,n}^1 \| M_{1,n}^1) + 1$.

### 3.4.3 Proof of Theorem 1

We follow the typical reduction of estimation to testing, common in the literature on lower bounds [2, 21, 46, 56]. By definition of the “distance” $d_L$, we have the mutual exclusion, true for any $\theta$, that
\[
L(\theta, P_0) \leq \frac{1}{2} d_L(P_0, P_1) \implies L(\theta, P_1) \geq \frac{1}{2} d_L(P_0, P_1). \tag{15}
\]
Let $M_0^n$ and $M_1^n$ be the marginal probabilities over observations $Z_{1:n}$ under $P_0$ and $P_1$ for a channel $Q \in \mathcal{Q}$. Using Markov’s inequality, we have for any estimator $\hat{\theta}$ based on $Z_{1:n}$ and any $\delta \geq 0$ that
\[
\mathbb{E}_{M_0^n} \left[ L(\hat{\theta}, P_0) \right] + \mathbb{E}_{M_1^n} \left[ L(\hat{\theta}, P_1) \right] \geq \delta \left[ M_0^n(L(\hat{\theta}, P_0) \geq \delta) + M_1^n(L(\hat{\theta}, P_1) \geq \delta) \right]
= \delta \left[ 1 - M_0^n(L(\hat{\theta}, P_0) < \delta) + M_1^n(L(\hat{\theta}, P_1) \geq \delta) \right].
\]
Setting $\delta = \delta_{01} := \frac{1}{4} d_L(P_0, P_1)$ and using the implication (15), we obtain
\[
\mathbb{E}_{M_0^n} \left[ L(\hat{\theta}, P_0) \right] + \mathbb{E}_{M_1^n} \left[ L(\hat{\theta}, P_1) \right] \geq \delta_{01} \left[ 1 - M_0^n(L(\hat{\theta}, P_0) < \delta) + M_1^n(L(\hat{\theta}, P_1) \geq \delta) \right]
\geq \delta_{01} \left[ 1 - M_0^n(L(\hat{\theta}, P_1) < \delta) + M_1^n(L(\hat{\theta}, P_1) \geq \delta) \right]
\geq \delta_{01} \left[ 1 - \|M_0^n - M_1^n\|_{TV} \right], \tag{16}
\]
where in the last step we used the definition of the variation distance.

Now we make use of the contraction inequality of Corollary 1 and its consequence (12) for KL-divergences. By Pinsker’s inequality and the corollary, we have
\[
2 \|M_0^n - M_1^n\|_{TV}^2 \leq D_{kl}(M_0^n \| M_1^n) \leq \log(1 + D_{\chi^2}(M_0^n \| M_1^n)) \leq n \log \left( 1 + 4\varepsilon^2 \|P_0 - P_1\|_{TV}^2 \right).
\]
Substituting this into our preceding lower bound (16) and using that $\hat{\theta}$ is arbitrary and $\delta_{01} = \frac{1}{2} d_L(P_0, P_1)$, we have that for any distributions $P_0$ and $P_1$,
\[
\inf_{\theta} \inf_{Q \in \mathcal{Q}} \sup_{P \in \{P_0, P_1\}} \mathbb{E}_P \left[ L(\hat{\theta}, P) \right] \geq \frac{1}{4} d_L(P_0, P_1) \left[ 1 - \sqrt{n} \log \left( 1 + 4\varepsilon^2 \|P_0 - P_1\|_{TV}^2 \right) \right].
\]
Now, for any $\delta \geq 0$, if $\frac{n}{2} \log(1 + 4\varepsilon^2\delta^2) \leq \frac{1}{4}$, or equivalently, $\delta^2 \leq \frac{1}{4 \varepsilon^2}(\exp(\frac{1}{2\varepsilon}) - 1)$, then $1 - \sqrt{\frac{n}{2} \log(1 + 4\varepsilon^2\delta^2)} \geq \frac{1}{2}$. Applying this to the bracketed term in the preceding display, we obtain
\[
\mathbb{M}^{loc}_n(P_0, L, P, \mathcal{Q}) \geq \frac{1}{8} \sup_{P \in \mathcal{P}} \left\{ d_L(P_0, P_1) \mid \|P_0 - P_1\|_{TV}^2 \leq \frac{1}{4 \varepsilon^2} \left[ e^{\frac{1}{2\varepsilon}} - 1 \right] \right\}
= \frac{1}{8} \omega_L \left( \frac{1}{2\varepsilon} \sqrt{e^{\frac{1}{2\varepsilon}} - 1} ; P_0, P \right).
\]

13
3.4.4 Proof of Proposition 2

For shorthand let $R_a(\hat{\theta}) = R(\hat{\theta}, \theta_a, M^n_\theta)$ denote the risk under the marginal $M^n_\theta$. By Lemma 1, for any distributions $P_0$ and $P_1$, we have

$$R_1(\hat{\theta}) \geq \left[ \Phi \left( \frac{1}{2} \| \theta_0 - \theta_1 \|_2 \right) - \left( \rho \left( M^n_\theta | M^n_0 \right) R(\hat{\theta}, M^n_0) \right)^{1/2} \right]^2,$$

and by Corollary 1 we have

$$\rho \left( M^n_\theta | M^n_0 \right) \leq \left( 1 + 4 \varepsilon^2 \| P_0 - P_1 \|_{TV}^2 \right)^n \leq \exp \left( 4n \varepsilon^2 \| P_0 - P_1 \|_{TV}^2 \right).$$

For $t \in [0, 1]$, let $\mathcal{P}_t$ be the collection of distributions

$$\mathcal{P}_t := \left\{ P \in \mathcal{P} \mid \| P_0 - P_1 \|_{TV}^2 \leq \frac{t \log \frac{1}{n}}{4n \varepsilon^2} \right\},$$

so that under the conditions of the proposition, any distribution $P_1 \in \mathcal{P}_t$ satisfies

$$R_1(\hat{\theta}) \geq \left[ \Phi \left( \frac{1}{2} \| \theta_0 - \theta_1 \|_2 \right) - \eta \frac{(1-t)}{2} \omega_L \left( \left( 4n \varepsilon^2 \right)^{-1/2} ; P_0 \right)^{1/2} \right]^2. \quad (17)$$

By inequality (7), $2 \Phi \left( \frac{1}{2} \| \theta_0 - \theta(P) \|_2 \right) \geq d_L(P_0, P_1)$. Thus, inequality (17) implies that for all $t \in [0, 1]$, there exists $P_1 \in \mathcal{P}_t$ such that

$$R(\hat{\theta}, M^n_\theta) \geq \left[ \frac{1}{2} - \eta \frac{(1-t)}{2} \right] \left( \sqrt{t \log \frac{1}{n}} ; P_0 \right)^{1/2} - \eta \frac{(1-t)}{2} \omega_L \left( \left( 4n \varepsilon^2 \right)^{-1/2} ; P_0 \right)^{1/2} \right]^2.$$

Because $\delta \mapsto \omega_L(\delta)$ is non-decreasing, if $t \in [0, 1]$ we may choose $P_1 \in \mathcal{P}_t$ such that

$$R(\hat{\theta}, M^n_\theta) \geq \left[ \frac{1}{2} - \eta \frac{(1-t)}{2} \right] \omega_L \left( \sqrt{t \log \frac{1}{n}} ; P_0 \right) \cdot (18)$$

Lastly, we lower bound the modulus of continuity at $P_0$ by a modulus at $P_1$. We claim that under Condition C.1, for all $\delta > 0$, if $\| P_0 - P_1 \|_{TV} \leq \delta$ then

$$\omega_L(2\delta; P_0) \geq \gamma^{-1} \omega_L(\delta; P_1). \quad (19)$$

Deferring the proof of this claim, note that by taking $\delta^2 = t \log \frac{1}{n}/(16n \varepsilon^2)$ in inequality (19), Eq. (18) implies that there exists $P_1 \in \mathcal{P}_t$ such that

$$R(\hat{\theta}, M^n_\theta) \geq \left[ \frac{1}{2} - \eta \frac{(1-t)}{2} \right] \omega_L(2\delta; P_0) \geq \gamma^{-1} \left[ \frac{1}{2} - \eta \frac{(1-t)}{2} \right] \omega_L \left( \frac{1}{4} \sqrt{t \log \frac{1}{n}} ; P_0 \right) \cdot$$

Let us return to the claim (19). For distributions $P_0, P_1, P_2$ with associated parameters $\theta_a = \theta(P_a)$, we use that $L(\theta, P) = \Phi(\| \theta - \theta(P) \|_2)$ to obtain

$$d_L(P_0, P_2) \leq \Phi(\| \theta_1 - \theta_0 \|_2) + \Phi(\| \theta_1 - \theta_2 \|_2) \leq \frac{\gamma}{2} d_L(P_0, P_1) + \frac{\gamma}{2} d_L(P_1, P_2).$$
by Condition C.1. Then for any $\delta \geq 0$ and $P_1$ with $\|P_1 - P_0\|_{TV} \leq \delta$, we have

$$\sup_{\|P - P_0\|_{TV} \leq 2\delta} d_L(P_0, P) \geq \sup_{\|P - P_0\|_{TV} \leq \delta} d_L(P_0, P)$$

$$\geq \sup_{\|P - P_1\|_{TV} \leq \delta} \left\{2\gamma^{-1} d_L(P_1, P) - d_L(P_0, P_1)\right\} \geq 2\gamma^{-1} \omega_L(\delta; P_1) - \omega_L(\delta; P_0).$$

Rearranging, we have for any distribution $P_1$ such that $\|P_0 - P_1\|_{TV} \leq \delta$,

$$2\omega_L(2\delta; P_0) \geq \omega_L(\delta; P_0) + \omega_L(2\delta; P_0) \geq 2\gamma^{-1} \omega_L(\delta; P_1),$$

which is inequality (19).

4 Examples

The ansatz of finding a locally most difficult problem via the modulus of continuity gives an approach to lower bounds that leads to non-standard behavior for a number of classical and not-so-classical problems. In this section, we investigate examples to illustrate the consequences of assuming local privacy, showing how it leads to a different geometry of lower bounds than classical cases. Our first step is to provide a private analogue of the Fisher Information (Sec. 4.1), showing in particular that Fisher Information no longer governs the complexity of estimation. We use this to prove lower bounds for estimation in Bernoulli and logistic models (Sec. 4.2), showing that even in one dimension there are substantial consequences to (locally) private estimation. In the final two sections within this section, we develop a methodology based on Fisher scoring and one-step corrected estimators to adaptively achieve our local minimax bounds for exponential families with and without mis-specification.

4.1 Private analogues of the Fisher Information

Our first set of results builds off of Theorem 1 by performing asymptotic approximations to the variation distance for regular enough parametric families of distributions. By considering classical families, we can more easily relate the modulus of continuity-based lower bounds to classical results, such as the Hájek-Le-Cam local asymptotic minimax theorem. One major consequence of our results is that, under the notions of locally private estimation we consider, the Fisher information is not the right notion of complexity and difficulty in estimation, but a precise analogy is possible.

We begin by considering parametric families of distributions that are parameterized in a way reminiscent of the classical quadratic mean differentiability conditions of Le Cam [50, Ch. 7]. Define the collection $\mathcal{P} = \{P_\theta\}_{\theta \in \Theta}$, parameterized by $\theta \in \mathbb{R}^d$, all dominated by a measure $\mu$ (at least for $\theta$ in a neighborhood of some fixed $\theta_0 \in \text{int} \Theta$), with densities $p_\theta = dP_\theta/d\mu$. We say that $\mathcal{P}$ is $L^1$-differentiable at $\theta_0$ with score function $\ell_{\theta_0} : \mathcal{X} \to \mathbb{R}^d$ if

$$2 \|P_{\theta_0 + h} - P_{\theta_0}\|_{TV} = \int_{\mathcal{X}} \left| h^T \ell_{\theta_0}(x) \right| p_{\theta_0}(x) d\mu(x) + o(||h||)$$

(20)

as $h \to 0$. An evidently sufficient condition for this to hold is that

$$\int |p_{\theta_0 + h} - p_{\theta_0} - h^T \ell_{\theta_0} p_{\theta_0}| d\mu = o(||h||),$$

which makes clear the appropriate differentiability notion. Recall that a family of distributions is quadratic mean differentiable (QMD) if

$$\int \left( \sqrt{p_{\theta_0 + h}} - \sqrt{p_{\theta_0}} - \frac{1}{2} h^T \ell_{\theta_0} \sqrt{p_{\theta_0}} \right)^2 d\mu = o(||h||^2),$$

(21)
which is the analogue of definition (20) for the Hellinger distance. Most “classical” families (e.g., exponential families) of distributions are QMD with the familiar score function \( \ell_\theta(x) = \nabla_\theta \log p_\theta(x) \) (cf. [36, 50]). For QMD families, \( L^1 \)-differentiability is automatic.

**Lemma 2.** Let the family \( \mathcal{P} := \{ P_\theta \}_{\theta \in \Theta} \) be QMD (21) at the point \( \theta_0 \). Then \( \mathcal{P} \) is \( L^1 \)-differentiable at \( \theta_0 \) with identical score \( \ell_\theta \) to the QMD case.

As the proof of Lemma 2 is a more or less standard exercise, we defer it to Appendix A.1.

In the classical case of quadratic-mean-differentiable families (21), the Fisher information matrix is \( I_\theta = \mathbb{E}_{P_\theta} [\ell_\theta \ell_\theta^T] \), and we have

\[
d^2_{\text{hel}}(P_{\theta+h}, P_\theta) = \frac{1}{8} h^T I_\theta h + o(\|h\|^2).
\]

Written differently, if we define the Mahalanobis norm \( \|h\|_Q = \sqrt{h^T Q h} \) for a matrix \( Q \succeq 0 \), then the Fisher information is the unique matrix \( I_\theta \) such that \( d_{\text{hel}}(P_{\theta+h}, P_\theta) = \frac{1}{2\sqrt{2}} \|h\|_I + o(\|h\|) \). By analogy with this notion of Fisher information, we define the \( L^1 \)-information as the (semi)norm

\[
J_{\theta_0} : \mathbb{R}^d \to \mathbb{R}_+, \quad J_{\theta_0}(h) := \int |h^T \ell_{\theta_0}(x)| \, dP_{\theta_0}(x), \tag{22}
\]

which is the unique (semi)norm \( \|\cdot\|_g \) for which \( \|P_{\theta+h} - P_\theta\|_{\text{TV}} = \frac{1}{2} \|h\|_g + o(\|h\|) \).

With these definitions, we can establish information lower bounds for \( L^1 \)-differentiable families. We consider a somewhat general case in which we wish to estimate the value \( \psi(\theta) \) of a functional \( \psi : \Theta \to \mathbb{R} \), where \( \psi \) is continuously differentiable in a neighborhood of \( \theta_0 \). We measure our error by a nondecreasing loss \( \Phi : \mathbb{R}_+ \to \mathbb{R}_+ \), where \( \Phi(0) = 0 \) and \( \Phi'(\theta) = \Phi'(\psi(\theta) - \psi(\theta_0)) \). Before stating the proposition, we also recall the definition of the dual norm \( \|\cdot\|_* \) to a norm \( \|\cdot\| \), defined by \( \|v\|_* = \sup_{\|h\| \leq 1} v^T h \). Let \( J_\theta^* \) denote the dual norm to \( J_\theta \). In the classical case of the Fisher information, where the norm \( \|h\| = \sqrt{h^T I_{\theta_0} h} \) is Euclidean, we have dual norm \( \|h\|_* = \sqrt{h^T I_{\theta_0}^{-1} h} \), that is, the usual inverse Fisher information. In the case of \( L^1 \)-information, because the norm is no longer quadratic, such explicit formulae are no longer possible.

**Proposition 4.** Let \( \mathcal{P} = \{ P_\theta \}_{\theta \in \Theta} \) be \( L^1 \)-differentiable at \( \theta_0 \) with score \( \ell_{\theta_0} \), and assume that the classical Fisher information \( I_{\theta_0} = \mathbb{E}_{\theta_0} [\ell_{\theta_0} \ell_{\theta_0}^T] > 0 \). Let \( Q_\epsilon \) be the family of \( \epsilon^2 \times \chi^2 \)-private, sequentially interactive channels. Then

\[
\mathcal{M}^\text{loc}(P_{\theta_0}, L, \mathcal{P}, Q_\epsilon) \geq \frac{1 - o(1)}{16\sqrt{2}} \cdot \Phi \left( \frac{1}{2\sqrt{2n\epsilon^2}} J_{\theta_0}^* (\nabla \psi(\theta_0)) \right).
\]

**Proof.** We apply Theorem 1. We have that

\[
\mathcal{M}^\text{loc}(P_{\theta_0}, L, Q) \geq \frac{1}{8} \omega_L \left( \frac{1}{\sqrt{8n\epsilon^2}}; P_{\theta_0} \right). \tag{23}
\]

Now, we evaluate \( \omega_L(\delta; P_{\theta_0}) \) for small \( \delta > 0 \). Note that \( d_L(P_{\theta_0+h}, P_{\theta_0}) \geq \Phi(\frac{1}{2} |\psi(\theta_0 + h) - \psi(\theta_0)|) \) by the calculation (7), so that

\[
\omega_L(\delta; P_{\theta_0}) \geq \sup_h \left\{ \Phi \left( \frac{1}{2} |\psi(\theta_0 + h) - \psi(\theta_0)| \right) \right\} \|P_{\theta_0+h} - P_{\theta_0}\|_{\text{TV}} \leq \delta
\]

\[
\geq \sup_h \left\{ \Phi \left( \frac{1}{2} |\nabla \psi(\theta_0)^T h + o(\|h\|)| \right) \right\} J_{\theta_0}(h) + o(\|h\|) \leq 2\delta.
\]
The assumption that the score \( \hat{\ell}_{\theta_0} \) has positive definite second moment matrix guarantees that 
\( J_{\theta_0}(h) > 0 \) for all \( h \neq 0 \), and as \( J_{\theta_0} \) is homogeneous, we see that for \( \delta \downarrow 0 \), we have 
\[
\omega_L(\delta; P_{\theta_0}) \geq \sup_h \left\{ \Phi \left( \frac{1}{2} |\nabla \psi(\theta_0)^T h| + o(\delta) \right) \mid J_{\theta_0}(h) \leq 2\delta(1 + o(1)) \right\} = \Phi \left( (1 - o(1))\delta J_{\theta_0}^*(\nabla \psi(\theta_0)) \right) .
\]
Substituting this into inequality (23) and setting \( \delta = \frac{1}{\sqrt{8n\varepsilon^2}} \) gives the proposition.

To understand the proposition more clearly, let us compare with the classical Fisher information bounds. In this case, the analogous lower bound is 
\[
\Phi \left( \sqrt{\frac{1}{n} \nabla \psi(\theta_0)^T I_{\theta_0}^{-1} \nabla \psi(\theta_0)} \right) ,
\]
which is the familiar local minimax complexity for estimating a one-dimensional functional [50, Ch. 7]. In the case that \( \Phi \) is the squared error, \( \Phi(t) = t^2 \), for example, the non-private lower bound becomes \( \frac{1}{n} \nabla \psi(\theta_0)^T I_{\theta_0}^{-1} \nabla \psi(\theta_0) \). In the one-dimensional case, the lower bounds become somewhat cleaner to state and are more easily comparable to Fisher information bounds. To that end, consider direct estimation of a real-valued parameter \( \theta_0 \).

**Corollary 2.** Let the conditions of Proposition 4 hold, but specialize \( \Phi(t) = t^2 \wedge 1 \) to the truncated squared error. Assume \( \Theta \subset \mathbb{R} \), and let \( \psi(\theta) = \theta \) be the identity map. Then there exists a numerical constant \( C > 0 \) such that
\[
\mathfrak{m}^\text{loc}_{n} (P_{\theta_0}, L, \mathcal{P}, Q_\varepsilon) \geq C \frac{1}{n \varepsilon^2} \cdot \frac{1}{E_{\theta_0} [\| \ell_{\theta_0} \|^2]} \wedge 1.
\]
Because \( E_{\theta_0} [\| \ell_{\theta_0} \|^2] \leq \frac{2}{E_{\theta_0} [\ell_{\theta_0}^2]} \), the \( L^1 \)-information is always smaller than the Fisher information. In some cases, as we shall see, it can be much smaller.

## 4.2 Logistic regression and Bernoulli estimation

Let us consider the local minimax complexity metrics and \( L^1 \)-information in the context of two problems for which the results are particularly evocative and simple to describe: estimating the parameter of a Bernoulli random variable and estimation in a 1-dimensional logistic regression. The problems are related, so we treat them simultaneously.

### 4.2.1 Private Bernoulli estimation

For the Bernoulli case, we start by letting the distribution \( P_0 \) be that \( X_i \overset{iid}{\sim} \text{Bernoulli}(p_0) \), that is, \( P_0(X_i = 1) = p_0 \). In this case, the sample mean \( \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \) achieves mean-squared error \( \mathbb{E}[(\bar{X}_n - p_0)^2] = \frac{\mathbb{E}(1 - p_0)}{n} \), so that for \( p_0 \) near 0 or 1, the problem is easy. In the private case, however, the difficulty of the problem is independent of the parameter \( p_0 \). Indeed, let \( P \) be the Bernoulli\((p)\) distribution, which satisfies \( \| P - P_0 \|_{TV} = |p - p_0| \). Theorem 1 then yields the following corollary.

**Corollary 3.** There exists a numerical constant \( c > 0 \) such that the following holds. Let \( P_0 \) be Bernoulli\((p_0)\), \( \mathcal{P} = \{ \text{Bernoulli}(p) \}_{p \in [0,1]} \) the family of Bernoulli distributions, and \( L(\theta, P_p) = (\theta - p)^2 \) be the squared error. Then for the collection \( Q_\varepsilon \) of \( \varepsilon^2 \cdot \chi^2 \)-private channels,
\[
\mathfrak{m}^\text{loc}_{n} (P_0, L, \mathcal{P}, Q_\varepsilon) \geq c \frac{1}{n \varepsilon^2}.
\]
Corollary 3 shows that under our notions of privacy, it is impossible to adapt to problem difficulty. The lower bound in Corollary 3 is tight to within numerical constants for \( \varepsilon \) not too large (even under \( \varepsilon \)-differential privacy); the classical randomized response estimator [54, 21] achieves the risk.
4.2.2 Private 1-dimensional logistic regression

A similar result to Corollary 3 holds for logistic regression problems that may be more striking, which is relevant for modern uses of private estimation, such as learning a classifier from (privately shared) user data [47, 1, 3]. In this case, the lower bound and gap between private and non-private estimation is more striking. To see this, let $P_0$ be the distribution on pairs $(x, y) \in \{-1, 1\}^2$ such that $x$ is uniform and

$$P_0(y \mid x) = \frac{1}{1 + e^{-\theta_0 x}},$$

where $\theta_0 > 0$. The Fisher information for the parameter $\theta$ in this model is $I^{-1}_\theta = (1 + e^\theta)(1 + e^{-\theta})$, so that given an i.i.d. sample $(X_i, Y_i) \sim P_0$, the maximum likelihood estimator $\hat{\theta}_n^{\text{ml}}$ satisfies

$$\sqrt{n}(\hat{\theta}_n^{\text{ml}} - \theta_0) \xrightarrow{d} N \left(0, 2 + e^{\theta_0} + e^{-\theta_0}\right). \tag{24}$$

The asymptotics (24) are not the entire story. In many applications of logistic regression, especially in machine learning [28], one wishes to construct a classifier with low classification risk or to provide good confidence estimates $p(y \mid x)$ of a label $y$ given covariates $x$. We expect in such situations that large parameter values $\theta_0$ should make the problem easier; this is the case. To make this concrete, consider the absolute error in the conditional probability $p_0(y \mid x)$, a natural error metric for classification or confidence accuracy: for a logistic distribution $P_0$ parameterized by $\theta_0$, we define the loss

$$L_{\text{pred}}(\theta, P_0) := \mathbb{E}_{P_0} \left[|p_0(Y \mid X) - p_0(Y \mid X)|\right],$$

where $p_0(y \mid x) = \frac{1}{1 + e^{-\theta_0 x}}$. By the delta method and convergence (24), setting $\phi(t) = \frac{1}{1 + e^t}$, we have

$$\sqrt{n} \cdot L_{\text{pred}}(\hat{\theta}_n^{\text{ml}}, P_0) \xrightarrow{d} \frac{1}{\sqrt{2 + e^{\theta_0} + e^{-\theta_0}}} |W| \quad \text{where } W \sim N(0, 1),$$

and because $L_{\text{pred}}$ is bounded in $[0, 1]$, we have

$$\mathbb{E}_{P_0} \left[L_{\text{pred}}(\hat{\theta}_n^{\text{ml}}, P_0)\right] = \frac{\sqrt{2/\pi}}{\sqrt{2 + e^{\theta_0} + e^{-\theta_0}}} \cdot \frac{1}{\sqrt{n}}(1 + o(1)). \tag{25}$$

The asymptotic value of the loss $L_{\text{pred}}$, normalized by $\sqrt{n}$, scales as $e^{-|\theta_0|/2}$, so that the problem is easier when the parameter $\theta_0$ is large. In the private case, large parameters yield no such easy classification problems, and there is an exponential gap (in the parameter $\theta$) between the prediction risk of private and non-private estimators.

**Corollary 4.** There exists a numerical constant $c > 0$ such that the following holds. Let $\mathcal{P}_{\log}$ be the family of 1-parameter logistic distributions on pairs $(x, y) \in \{-1, 1\}^2$ and let $Q_\varepsilon$ be the collection of $\varepsilon^2$-$\chi^2$-private channels. Then the local minimax prediction error satisfies

$$\mathfrak{m}_{n}^{\text{loc}}(P_0, L_{\text{pred}}, \mathcal{P}_{\log}, Q_\varepsilon) \geq c \min \left\{ \frac{1}{\sqrt{n} \varepsilon^2}, \frac{1}{1 + e^{\theta_0}} \right\}.$$  

**Proof.** Without loss of generality, we assume that $\theta_0 > 0$. The variation distance between logistic distributions $P_\theta$ and $P_{\theta_0}$ for $\theta = \theta_0 + \Delta \in \mathbb{R}$ is

$$\|P_\theta - P_{\theta_0}\|_{TV} = \left| \frac{e^\theta - e^{\theta_0}}{1 + e^\theta + e^{\theta_0} + e^{\theta+\theta_0}} \right| = \left| \frac{e^{\theta_0}}{1 + e^{\theta_0+\Delta} + e^{\theta_0} + e^{2\theta_0+\Delta}} \right| |e^\Delta - 1| \leq e^{-\theta_0} |1 - e^{-\Delta}|.$$
For any $\delta > 0$, to have $\|P_{\theta_0 + \Delta} - P_{\theta_0}\|_{TV} \leq \delta$, then, it suffices to have

$$e^{-\theta_0}(1 - e^{-\Delta}) \leq \delta \text{ or } 0 \leq \Delta \leq -\log \left[1 - e^{\theta_0}\right].$$

(26)

Now we evaluate $d_L(P_0, P_1)$, the separation $P_0$ and $P_1$ for the loss $L_{pred}$. Denote $\phi(t) = 1/(1 + e^t)$. For distributions $P_d$ with parameters $\theta$, $L_{pred}(\theta, P_0) = \phi(-\theta_0)|\phi(-\theta) - \phi(-\theta_0)| + \phi(\theta_0)|\phi(\theta) - \phi(\theta_0)|$, and thus $d_L(P_0, P_1)$ satisfies

$$d_L(P_0, P_1) = \inf_{\theta} \{L_{pred}(\theta, P_0) + L_{pred}(\theta, P_1)\} = |\phi(-\theta_0) - \phi(-\theta_1)| = |\phi(\theta_0) - \phi(\theta_1)|,$$

where we have used that $\phi(\theta) + \phi(-\theta) = 1$ for all $\theta$. For $\delta > 0$, then, using the sufficient condition (26) for $\|P_{\theta_0 + \Delta} - P_{\theta_0}\|_{TV} \leq \delta$, the choice $\Delta = -\log \left[1 - e^{\theta_0}\right]$ yields that whenever $\delta < e^{-\theta_0},$

$$\omega_{L_{pred}}(P_{\theta_0}, \delta, P_{\log}) \geq d_L(P_{\theta_0}, P_{\theta_0 + \Delta}) = \frac{\delta}{(1 + e^{-\theta_0})(1 + e^{-\theta_0}(1 - \delta e^{\theta_0}))} \geq \frac{1}{(1 + e^{-\theta_0})^2} \cdot \delta.$$

For $\delta \geq e^{-\theta_0}$, we have $\omega_{L_{pred}}(P_{\theta_0}, \delta) \geq (1 + e^{\theta_0})^{-1}$. Substituting the choice $\delta = 1/\sqrt{8n\epsilon^2}$ as in Theorem 1 gives the desired lower bound. \hfill \Box

### 4.3 Uniform achievability in one-parameter exponential families

Corollary 2 shows a lower bound of $(n\epsilon^2 J_{\theta_0}^2)^{-1}$, where $J_{\theta_0} = E_{\theta_0}[|\ell_{\theta_0}|]$, for the estimation of a single parameter in a suitably smooth family of distributions. One of our three desiderata for a “good” lower bound is uniform achievability, that is, the existence of an estimator that uniformly achieves the instance-specific lower bound. In this section, we develop a locally private estimation scheme that does this for single parameter exponential family models, and show how a methodology based on Fisher scoring can adaptively attain our local minimax lower bounds.

Let $\mathcal{P} = \{P_{\theta}\}_{\theta \in \Theta}$ be a one parameter exponential family, so that for a base measure $\mu$ on $X$, each distribution $P_{\theta}$ has density

$$p_{\theta}(x) := \frac{dP_{\theta}}{d\mu}(x) = \exp(\theta T(x) - A(\theta)),$$

where $T(x)$ is the sufficient statistic and $A(\theta) = \log \int e^{\theta T(x)} d\mu(x)$ is the log partition function. It is well known (cf. [5, 36, Ch. 2.7]) that $A$ satisfies $A'(\theta) = E_{\theta}[T(X)]$ and $A''(\theta) = \text{Var}_{\theta}(T(X))$. In this case, the $L^1$-information (22) is the mean absolute deviation

$$J_{\theta} = E_{\theta}[|T(X) - A(\theta)|] = E_{\theta}[|T(X) - E_{\theta}[T(X)]|].$$

We now provide a procedure asymptotically achieving mean square error scaling as $(n\epsilon^2 J_{\theta}^2)^{-1}$, which Corollary 2 shows is optimal. Our starting point is the observation that for a one-parameter exponential family, the functional $\theta \mapsto P_{\theta}(T(X) \geq t)$ is strictly increasing in $\theta$ for any fixed $t \in \text{supp}\{T(X)\}$ [36, Lemma 3.4.2]. A natural idea is thus to first estimate $P_{\theta}(T(X) \geq t)$ and then invert this value to give good estimate of $\theta$. To make this (near) optimal, we develop a two-sample procedure, where in with the first we estimate $t \approx E[T(X)]$ and then use the second sample to approximate $P_{\theta}(T(X) \geq t)$ for this particular $t$, which we invert.

With this motivation, we now formally define our $\epsilon$-differentially private one-step corrected estimator. Define the function $\Psi : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ by

$$\Psi(t, \theta) := P_{\theta}(T(X) \geq t) = \int 1\{T(x) \geq t\} \exp(\theta T(x) - A(\theta)) d\mu(x).$$

(27)
The private two stage algorithm we develop splits a total sample of size 2n in half. In the first stage, the algorithm uses the first half of the sample to construct a crude estimate $\hat{T}_n$ of the value $A'(\theta) = \mathbb{E}_{\theta}[T]$; we require only that $\hat{T}_n$ be consistent (we may use Duchi et al.’s \varepsilon-differentially private mean estimators, which provide consistent estimates of $\mathbb{E}[T(X)]$ so long as $\mathbb{E}[|T(X)|^k] < \infty$ for some $k > 1$ [21, Corollary 1].) In the second stage, the algorithm uses the crude estimate $\hat{T}_n$ and the second half of the sample in a randomized response procedure as follows: we construct $V_i$ and the private $Z_i$ as

$$V_i = 1\{T(X_i) \geq \hat{T}_n\}, \quad Z_i = \frac{e^\varepsilon + 1}{e^\varepsilon - 1} \left\{ \frac{V_i}{1 - V_i} \right\} - \frac{1}{e^\varepsilon + 1}.$$ 

By inspection, this is \varepsilon-differentially-private and $\mathbb{E}[Z_i \mid V_i] = V_i$. Now, define the inverse function

$$H(p, t) := \inf \{ \theta \in \mathbb{R} \mid P_\theta(T(X) \geq t) \geq p \} = \inf \{ \theta \in \mathbb{R} \mid \Psi(t, \theta) \geq p \}.$$ 

Setting $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i$, our final \varepsilon-differentially private estimator is

$$\hat{\theta}_n = H(\bar{Z}_n, \hat{T}_n). \quad (28)$$

We then have the following convergence result, which shows that the estimator (28) (asymptotically) has risk within a constant factor of the local minimax bounds. The proof is somewhat involved, so we defer it to Appendix A.3.

**Proposition 5.** Assume that $\text{Var}_{\theta_0}(T(X)) > 0$ and $\hat{T}_n \overset{P}{\to} t_0 := \mathbb{E}_{\theta_0}[T(X)]$. Define $\delta_\varepsilon^2 = \frac{e^\varepsilon}{(e^\varepsilon-1)^2}$.

Then there exist random variables $G_n = \Psi(\hat{T}_n, \theta_0) \in [0, 1]$, $\mathcal{E}_{n,1}$, and $\mathcal{E}_{n,2}$ such that under $P_{\theta_0}$, the estimator (28) satisfies

$$\sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) = 2J_{\theta_0}^{-1}(\mathcal{E}_{n,1} + \mathcal{E}_{n,2}) + o_P(1)$$

where

$$\left( \frac{\mathcal{E}_{n,1}}{G_n(1-G_n)} \right) \overset{d}{\to} \mathcal{N}(0, \text{diag}(\delta_\varepsilon^{-2}, 1)). \quad (29)$$

The complexity of the statement arises because the distribution of $T(X)$ may be discontinuous, including at $\mathbb{E}_{\theta_0}[T(X)]$, necessitating the construction of the random variables $\mathcal{E}_{n,1}, \mathcal{E}_{n,2}$, and $G_n$ to demonstrate a limit distribution.

### 4.4 Mis-specified models and multi-parameter exponential families

While Section 4.3 provides a procedure that achieves the optimal behavior for parametric exponential families, it relies strongly on the model’s correctness and its single-dimensionality. In this section, we consider the situation in which we wish to estimate a functional of an exponential family model that may be mis-specified. To describe the results, we first review some of the basic properties of exponential families. Let $\{P_\theta\}_{\theta \in \Theta}$ be a d-parameter exponential family with densities $p_\theta(x) = \exp(\theta^T x - A(\theta))$ with respect to some base measure, where for simplicity we assume that the exponential family is regular and minimal, meaning that dom $A, \nabla^2 A(\theta) = \text{Cov}_\theta(X) \succ 0$ for all $\theta \in \text{dom } A$, and the log partition function $A(\theta)$ is analytic on the interior of its domain [36, Thm. 2.7.1]. We record here a few standard facts on the associated convex analysis (for more, see the books [5, 53, 29]). First, recall the conjugate function $A^*(x) := \sup_\theta \{ \theta^T x - A(\theta) \}$. Standard convex analysis results [29, Ch. X] give that

$$\nabla A^*(x) = \theta_x \text{ for the unique } \theta_x \text{ such that } \mathbb{E}_{\theta_x}[X] = x. \quad (30)$$
In addition, $\nabla A^*$ is continuously differentiable, one-to-one, and
\[
dom A^* \supset \text{Range}(\nabla A(\cdot)) = \{ E_\theta[X] \mid \theta \in \dom A \}.
\]
Moreover, by the inverse function theorem, we also have that on the interior of $\dom A^*$,
\[
\nabla^2 A^*(x) = (\nabla^2 A(\theta_x))^{-1} = \text{Cov}_{\theta_x}(X)^{-1} \quad \text{for the unique } \theta_x \text{ s.t. } E_{\theta_x}[X] = x.
\]
(31)
The uniqueness follows because $\nabla A^*$ is one-to-one, a consequence of the minimality of the exponential family and that $\nabla^2 A(\theta) > 0$. For a distribution $P$ with mean $E_P[X]$, so long as the mean belongs to the range of $\nabla A(\theta) = E_\theta[X]$ under the exponential family model as $\theta$ varies, the minimizer of the log loss $\ell_\theta(x) = -\log p_\theta(x)$ is
\[
\hat{\theta}(P) := \arg\min_{\theta} E_P[\ell_\theta(X)] = \nabla A^*(E_P[X]).
\]
Mis-specified exponential families are are sufficiently regular—as we discuss after Theorem 1 and Proposition 1—to guarantee the polynomial growth condition C.2, so that the modulus of continuity $\omega_L$ characterizes the local minimax complexity. The following lemma shows that this is the case for the $\ell_2$ modulus of continuity, and the extension to general losses of the form $L(\theta, P) = \Phi(\|\theta - \theta(P)\|_2)$ from this case is immediate and exactly as in Example 1.

Lemma 3. Let \{\P_0\} be an exponential family model as above and $\P = \{P \mid \text{supp } P \subset X\}$, where $X \subset \mathbb{R}^d$ is compact and $\dom A^* \supset X$. Then
\[
\omega_2(\delta) := \sup \\{ \|\theta(P) - \theta(P_0)\|_2 \mid P \in \P, \|P - P_0\|_{\text{TV}} \leq \delta \}
\]
\[
= \sup \{ \|\nabla A^*(E_P[X]) - \nabla A^*(E_{P_0}[X])\|_2 \mid P \in \P, \|P - P_0\|_{\text{TV}} \leq \delta \}
\]
satisfies growth condition C.2 with $\alpha = 1$ and some $\beta < \infty$.
(See Appendix 4.4.3 for the somewhat technical proof.)

With these preliminaries and basic results in place, we describe our estimation setting. We consider estimation of functionals $\psi : \mathbb{R}^d \rightarrow \mathbb{R}$ of the parameters $\theta$ of the form $\psi(\theta)$, where we assume that $\psi$ is differentiable. We measure the loss of an estimated value $\hat{\psi}$ by
\[
L(\hat{\psi}, P) = \Phi(\hat{\psi} - \psi(\theta(P))),
\]
where the loss function $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ is assumed to be convex and symmetric about zero. In the next two sections, we show local lower bounds on estimation under loss $L$ and develop a near-optimal estimator.

4.4.1 A lower bound on estimation
In our mis-specified exponential family setting, we have the following local minimax lower bound.

Proposition 6. Let $\P$ be a family of distributions on $X$ such that the collection of means $\{E_P[X]\}_{P \in \P}$ is bounded with $\{E_P[X]\}_{P \in \P} \subset \text{int } \dom A^*$. Let $\Q_\varepsilon$ denote the collection of all $\varepsilon^2$-$\chi^2$-private sequentially interactive channels. Then
\[
\mathcal{M}^\text{loc}_n(P_0, L, \P, \Q) \geq \frac{1}{4} \sup_{P \in \P} \Phi \left( \frac{(\nabla \psi(\theta_0)^T \nabla^2 A(\theta_0)^{-1} (E_{P_0}[X] - E_P[X])}{2\sqrt{8\varepsilon^2}} + O \left( \frac{1}{n\varepsilon^2} \right) \right).
\]
The proof is similar to our previous results, so we defer it to Section 4.4.3. Let us instead give a corollary. Assume that $\Phi(t) = t^2$ is the squared error and additionally that the set $\mathcal{P}$ consists of all distributions supported on the norm ball $\{x \in \mathbb{R}^d \mid \|x\| \leq r\}$. Then we have

**Corollary 5.** Let the conditions of Proposition 6 and the preceding paragraph hold. Then there exists a numerical constant $c > 0$ such that

$$
\mathfrak{m}_{\text{loc}}^{\text{loc}}(P_0, L, \mathcal{P}, Q_\epsilon) \geq c \frac{r^2 \|\nabla^2 A(\theta_0)\|_2^2}{n \epsilon^2} \mathcal{N}(1) + O\left(\frac{1}{n^2 \epsilon^4}\right).
$$

Before we turn to estimation in the private case, we compare Proposition 6 and Corollary 5 to the non-private case. In this case, a simple estimator is to take the sample mean $\hat{\mu}_n = \frac{1}{n} \sum_{i=1}^n X_i$ and then set $\hat{\theta}_n = \nabla A^*(\hat{\mu}_n)$. Letting $\theta_0 = \nabla A^*(\mathbb{E}_P[X])$, then classical Taylor expansion arguments and the delta-method [50, Chs. 3–5] yield

$$
\sqrt{n}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathcal{N}(0, \nabla^2 A(\theta_0)^{-1} \mathbb{V} P(X) \nabla^2 A(\theta_0)^{-1})
$$

and

$$
\sqrt{n}(\hat{\psi}_n - \psi(\theta_0)) \xrightarrow{d} \mathcal{N}(0, \nabla \psi(\theta_0)^T \nabla^2 A(\theta_0)^{-1} \mathbb{V} P(X) \nabla^2 A(\theta_0)^{-1} \nabla \psi(\theta_0)).
$$

The lower bound in Corollary 5 is always worse than this classical limit. In this sense, the private lower bounds exhibit the lack of adaptivity that is common in this paper: the local lower bounds show that in the private case, no estimator can adapt to “easy” problems where the covariance $\mathbb{V} P(X)$ is small.

### 4.4.2 An optimal one-step procedure

An optimal procedure for functionals of (possibly) mis-specified exponential family models has strong similarities to classical one-step estimation procedures [e.g. 50, Ch. 5.7]. To motivate the approach, let us assume we have a “good enough” estimate $\tilde{\mu}_n$ of $\mu_0 := \mathbb{E}_P[X]$. Then we observe that if $\tilde{\theta}_n = \nabla A^*(\tilde{\mu}_n)$, we have

$$
\psi(\theta_0) = \psi(\tilde{\theta}_n) + \nabla \psi(\tilde{\theta}_n)^T (\theta_0 - \tilde{\theta}_n) + O(\|\theta_0 - \tilde{\theta}_n\|^2)
$$

$$
= \psi(\tilde{\theta}_n) + \nabla \psi(\tilde{\theta}_n)^T (\nabla A^*(\mu_0) - \nabla A^*(\tilde{\mu}_n)) + O(\|\mu_0 - \tilde{\mu}_n\|^2)
$$

$$
= \psi(\tilde{\theta}_n) + \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} (\mu_0 - \tilde{\mu}_n) + O(\|\mu_0 - \tilde{\mu}_n\|^2),
$$

where each equality freely uses the duality relationships (30) and (31). In this case, if $\tilde{\mu}_n - \mu_0 = \sigma P(n^{-1/4})$ and we have an estimator $T_n$ satisfying

$$
\sqrt{n} \left( T_n - \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} \mu_0 \right) \xrightarrow{d} \mathcal{N}(0, \sigma^2),
$$

then the estimator

$$
\hat{\nu}_n := \psi(\tilde{\theta}_n) + T_n - \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} \tilde{\mu}_n
$$

satisfies $\sqrt{n}(\hat{\nu}_n - \psi(\theta_0)) \xrightarrow{d} \mathcal{N}(0, \sigma^2)$ by Slutsky’s theorems.

We now exhibit such an estimator. To avoid some of the difficulties associated with estimation from unbounded data [21], we assume the data $X \subset \mathbb{R}^d$ are contained in a norm ball $\{x \in \mathbb{R}^d \mid \|x\| \leq 1\}$. Let us split the sample of size $n$ into two sets of size $n_1 = \lceil n^{2/3} \rceil$ and $n_2 = n - n_1$. For the first set, let $Z_i$ be any $\epsilon$-locally differentially private estimate of $X_i$ satisfying $\mathbb{E}[Z_i \mid X_i] = X_i$ and $\mathbb{E}[\|Z_i\|^2] < \infty$, so that the $Z_i$ are i.i.d.; for example, $X_i + W_i$ for a random vector of appropriately
large Laplace noise suffices [24, 21]. Define \( \tilde{\mu}_n = \frac{1}{n} \sum_{i=1}^{n} Z_i \), in which case \( \tilde{\mu}_n - \mu_0 = O_P(n^{-1/3}) \), and let \( \tilde{\theta}_n = \nabla A^*(\tilde{\mu}_n) \). Now, for \( i = n_1 + 1, \ldots, n \), define the \( \varepsilon \)-differentially private quantity

\[
Z_i := \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} X_i + \frac{\| \nabla^2 A(\tilde{\theta}_n)^{-1} \nabla \psi(\tilde{\theta}_n) \|_F}{\varepsilon} W_i \quad \text{where} \quad W_i \overset{iid}{\sim} \text{Laplace}(1).
\]

Letting \( \bar{X}_{n_2} = \frac{1}{n_2} \sum_{i=n_1+1}^{n} X_i \) and similarly for \( \bar{W}_{n_2} \) and \( \bar{Z}_{n_2} \), we find that

\[
\sqrt{n} \left[ \bar{Z}_{n_2} - \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} \mu_0 \right]
\]

\[
= \sqrt{n} \left[ \nabla \psi(\tilde{\theta}_n)^T \nabla^2 A(\tilde{\theta}_n)^{-1} (\bar{X}_{n_2} - \mu_0) + \frac{\| \nabla^2 A(\tilde{\theta}_n)^{-1} \nabla \psi(\tilde{\theta}_n) \|_F}{\varepsilon} \bar{W}_{n_2} \right] \overset{d}{\to} N(0, \sigma^2(P, \psi, \varepsilon))
\]

by Slutsky’s theorem, where for \( \theta_0 = \nabla A^*([P(X)]) \) we define

\[
\sigma^2(P, \psi, \varepsilon) := \nabla \psi(\theta_0)^T \nabla^2 A(\theta_0)^{-1} \text{Cov}_P(X) \nabla^2 A(\theta_0)^{-1} \nabla \psi(\theta_0) + \frac{2}{\varepsilon^2} \| \nabla^2 A(\theta_0)^{-1} \psi(\theta_0) \|^2.
\]  

(33)

Summarizing, we have the following proposition, which shows that the two-step estimator (32) is (asymptotically) locally minimax optimal.

**Proposition 7.** Let \( \hat{\psi}_n \) be the estimator (32) with the choices \( T_n = Z_{n_2} \) and \( \tilde{\theta}_n = \nabla A^*(\tilde{\mu}_n) \) as above, and let \( \theta_0 = \nabla A^*([P(X)]) = \arg \min \mathbb{E}[\ell_0(X)] \) and \( \sigma^2(P, \psi, \varepsilon) \) be as in (33). Then

\[
\sqrt{n}(\hat{\psi}_n - \psi(\theta_0)) \overset{d}{\to} N\left(0, \sigma^2(P, \psi, \varepsilon)\right).
\]

### 4.4.3 Proof of Proposition 6

Let \( P_t \in \mathcal{P} \) be a distribution with mean \( x = \mathbb{E}_{P_t}[X] \), and for \( t \in [0, 1] \), define \( P_t = (1-t)P_0 + tP_1 \). Then \( \|P_0 - P_t\|_{TV} \leq t \). Let us first consider the distance \( d_L \), which for any distribution \( P \) satisfies

\[
d_L(P_0, P) = 2\Phi \left( \frac{1}{2}(\psi(\theta(P_0)) - \psi(\theta(P))) \right),
\]

by a calculation identical to that for the convex case (6). Now, with our choice of \( P_t \), let us consider the value \( \theta(P_t) \), which evidently satisfies

\[
\theta(P_0) - \theta(P_t) = \nabla A^*([P_0[X])] - \nabla A^*([P_t[X] + t([P_0[X] - x])] = t\nabla^2 A^*([P_0[X]])([P_0[X] - x]) + O(t^2).
\]

Recalling Eq. (31) and using the shorthand \( \theta_t = \nabla A^*([P_t[X]]) \), we have

\[
\theta_0 - \theta_t = t\nabla^2 A(\theta_0)^{-1}([P_0[X] - x]) + O(t^2).
\]

Because \( \psi \) is smooth by assumption, this yields

\[
\psi(\theta_0) - \psi(\theta_t) = \nabla \psi(\theta_0)^T (\theta_0 - \theta_t) + O(||\theta_0 - \theta_0||^2) = t\psi(\theta_0)^T \nabla^2 A(\theta_0)^{-1}([P_0[X] - x]) + O(t^2).
\]

As a consequence of these derivations, that \( \psi \) is smooth by assumption, and that convex functions are locally Lipschitz, the modulus of continuity of \( L \) at \( P_0 \) has lower bound

\[
\omega_L(t; P_0, \mathcal{P}) \geq 2\Phi \left( \frac{1}{2} \nabla \psi(\theta_0)^T \nabla^2 A(\theta_0)^{-1}([P_0[X] - x]) + O(t^2) \right)
\]

as \( t \to 0 \). Substituting \( t = \frac{1}{\sqrt{n}e^2} \), and applying Theorem 1 gives the result.
5 Generalized Le Cam’s method and the failure of high-dimensional estimation

Our localized minimax complexity results essentially characterize the difficulty of locally private estimation—under many accepted notions of privacy—for functionals of distributions. It is important to investigate more complex inferential and estimation problems, including in high dimensional settings, and we complement our local minimax results by studying a few such problems here. Duchi, Jordan, and Wainwright [21, Sec. 4] develop sophisticated contraction inequalities for divergences to study local \( \varepsilon \)-differentially private estimation in high dimensions. Their results suggest that local differential privacy is too strong a notion to allow high-dimensional estimation, but their results apply only to non-interactive channels. Thus, we might hope that by weakening the notion of local privacy (to, say, Rényi-differential privacy, Def. 3) or allowing (sequential) interaction, we mitigate this curse of dimensionality. This hope is misplaced.

In this case, a number of tools are available, but to keep our presentation tighter, we focus on the generalized version of Le Cam’s method [33, 56]. We begin with a result essentially due to Yu [56, Lemma 1] (see also [52]), known as the generalized Le Cam’s method. Let \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) be two collections of distributions on a space \( X \). We say that the sets \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are \( \delta \)-separated if

\[
d_L(P_0, P_1) \geq \delta \quad \text{for all } P_0 \in \mathcal{P}_0, P_1 \in \mathcal{P}_1.
\]

We then have the following lemma.

**Lemma 4** (Le Cam’s method). Let \( \mathcal{P} \) be a collection of probability distributions. Let \( \mathcal{P}_0 \subset \mathcal{P} \) and \( \mathcal{P}_1 \subset \mathcal{P} \) be \( \delta \)-separated for the loss \( L \). For any estimator \( \hat{\theta} \),

\[
\sup_{P \in \mathcal{P}} \mathbb{E}_P [L(\hat{\theta}, P)] \geq \frac{\delta}{2} \sup_{P_i \in \text{Conv}(\mathcal{P}_i)} (1 - \|P_0 - P_1\|_{TV}).
\]

See Appendix A.5 for the proof of Lemma 4, which we include for completeness.

In our context of private estimation, we use a slight reformulation and simplification of the lemma to prove our results. Consider a (sequentially interactive) private channel \( Q \) taking input data from a space \( X \) and outputting \( Z \in Z \). Now, we consider any estimation problem in which the set of possible distributions \( \mathcal{P} \) contains a collection \( \{P_v\}_{v \in V} \subset \mathcal{P} \) of distributions on \( X \), indexed by \( v \in V \), as well as a distribution \( P_0 \in \mathcal{P} \). For each of these distributions, we have i.i.d. observations \( X_i \), that is, samples from the product with density

\[
d_P^n(x_{1:n}) = \prod_{i=1}^n d_P(x_i).
\]

We define the marginal distributions \( M^n_v(\cdot) := \int Q(\cdot | x_{1:n}) dP^n_v(x_{1:n}) \) and \( \overline{M}^n := \frac{1}{|V|} \sum_{v \in V} M^n_v \). Then an immediate consequence of Le Cam’s method is the following private analogue:

**Lemma 5** (Private generalized Le Cam’s method). Let the conditions above hold. For any estimator \( \hat{\theta} \) based on the privatized observations \( Z_1, \ldots, Z_n \) drawn from the channel \( Q \),

\[
\sup_{P \in \mathcal{P}} \mathbb{E}_{Q,P} [L(\hat{\theta}(Z_1, \ldots, Z_n), P)] \geq \frac{1}{2} \min_{v \in V} d_L(P_0, P_v) \cdot (1 - \|M^n_0 - \overline{M}^n\|_{TV}).
\]

(34)

Based on Lemma 5, our approach to proving minimax bounds is roughly a two-step process, which follows the classical approaches to “local” minimax bounds [56, 55, 21]. First, we choose a
collection of distributions \( \{ P_v \} \) and base distribution \( P_0 \) that are well-separated for our loss \( L \), that is, \( \min_v d_L(P_0, P_v) > 0 \). If we can then show that the variation distance \( \| M^n_0 - \overline{M}^n \|_{TV} \leq \frac{1}{2} \), or is otherwise upper bounded by a constant less than 1 for all appropriately private channels \( Q \), then we obtain the minimax lower bound \( \frac{1}{2} \min_v d_L(P_0, P_v) \) using inequality (34). Often, we scale the separation \( d_L(P_0, P_v) \) by a parameter \( \delta > 0 \) to choose the optimal (worst-case) tradeoff.

With this rough outline in mind, the most important step is to control the variation distance between \( M^n_0 \) and \( \overline{M}^n \), and a variational quantity bounds this distance. For each \( v \in \mathcal{V} \), define the linear functional
\[
\varphi_v(f) := \int f(x) (dP_0(x) - dP_v(x)) \cdot 1.
\]
We then define the following measures, which we parameterize to apply for different variants of privacy of the channel \( Q \), by a power \( r \in [1, \infty] \) as follows:
\[
C_r(\{ P_v \}_{v \in \mathcal{V}}) := \inf_{\text{supp} P^* \subset \mathcal{X}} \sup_f \left\{ \frac{1}{|\mathcal{V}|} \sum_{v \in \mathcal{V}} \varphi_v(f)^2 \mid \| f \|_{L^r(P^*)} \leq 1 \right\},
\]
where the infimum is taken over all distributions \( P^* \) supported on \( \mathcal{X} \).

In their study of locally private estimation, Duchi et al. [21] also consider a similar quantity governing private estimation and mutual information; their Theorem 2 shows that for a single observation of a privatized random variable (the case \( n = 1 \)), one obtains a result similar to ours in the \( \varepsilon \)-differentially private case. However, their results do not extend to interactive channels, that is, the important and more general scenario in which the private random variables \( Z_1, \ldots, Z_{t-1} \) may influence the choice of the channel used to privatize observation \( Z_t \). This interactivity, as we have seen, is often useful for building optimal estimators. With the definition (35), we have the following tensorization and contraction result, which we prove in Appendix A.6.

**Theorem 2.** Let the channel \( Q \) be \( \varepsilon \)-differentially private. Then for any distribution \( P \) on \( \mathcal{X} \),
\[
D_{KL} (M^n_0 || \overline{M}^n) \leq \frac{n(e^{\varepsilon/2} - e^{-\varepsilon/2})^2}{4} \cdot C_{\infty}(\{ P_v \}_{v \in \mathcal{V}}) \cdot \min \left\{ e^\varepsilon, \max_{v \in \mathcal{V}} \| dP/dP_v \|_{\infty} \right\}.
\]
If the channel \( Q \) is \( \varepsilon^2 \cdot \chi^2 \)-private, then for any distribution \( P \) on \( \mathcal{X} \),
\[
D_{KL} (M^n_0 || \overline{M}^n) \leq ne^{\varepsilon^2} \cdot C_2(\{ P_v \}_{v \in \mathcal{V}}) \cdot \max_{v \in \mathcal{V}} \| dP_v/dP \|_{\infty}.
\]

In the remainder of this section, we apply Theorem 2 to high-dimensional estimation problems, deriving new and stronger results showing that—even if we allow interactive channels or relaxed \( \varepsilon^2 \cdot \chi^2 \) privacy—local privacy precludes high-dimensional estimation.

### 5.1 High-dimensional mean estimation

There has been substantial recent interest in estimation problems in which the nominal dimension \( d \) of the estimand is much larger than the sample size \( n \), but some underlying latent structure—such as sparsity—makes consistent estimation possible [7, 41]. The simplest version of this problem is sparse high-dimensional mean estimation. Duchi et al. [21, Corollary 5] consider this problem as well, but one of the weaknesses of their paper is that they do not allow sequentially interactive channels in high-dimensional settings; there are subtle difficulties in the application of Fano’s method that our approach via the generalized Le Cam’s method circumvents.

We consider the class of distributions with \( s \)-sparse means supported on the radius 1 box in \( \mathbb{R}^d \),
\[
\mathcal{P}_s := \left\{ \text{distributions } P \text{ supported on } [-1, 1]^d \text{ s.t. } \| \mathbb{E}_P [X] \|_0 \leq s \right\}.
\]
In the non-private case, an \( \ell_1 \)-regularized (soft-thresholded mean) estimator achieves \( \mathbb{E}[\|\hat{\theta}_n - \theta\|_2^2] \lesssim \frac{s \log(d/s)}{n} \) for the \( s \)-sparse case [31, 7, 52]. In the private case, the problem is much more difficult. To make this concrete, let us consider estimation of some linear function \( \psi : \mathbb{R}^d \to \mathbb{R}^k \) of the mean \( \mathbb{E}_P[X] \) under a symmetric convex loss \( \Phi : \mathbb{R}^k \to \mathbb{R}_+ \) with \( \Phi(0) = 0 \). Then for \( P \in \mathcal{P}_s \), we have loss \( L(\theta, P) := \Phi(\theta - \psi(\mathbb{E}_P[X])) \). With this choice of loss function, we have the following result, where we recall that \( e_j \) denotes the \( j \)th standard basis vector.

**Proposition 8.** Let \( L \) be the parameter-based error \( L(\theta, P) = \Phi(\theta - \psi(\mathbb{E}_P[X])) \) above and let \( Q_\varepsilon \) denote the collection of \( \varepsilon^2 \)-\( \chi^2 \)-private and sequentially interactive channels. Then

\[
\mathbb{M}_n(\theta(\mathcal{P}_1), L, Q_\varepsilon) \geq \frac{1}{2} \min_{j \in [d]} \Phi\left( \sqrt{\frac{d}{4n\varepsilon^2}} \land 1 \right) \psi(e_j).
\]

To demonstrate the technique to develop this result from the divergence bounds in Theorem 2, we provide the proof in Section 5.3.

A few consequences of Proposition 8 are illustrative. If we use the mean-squared error \( \Phi(\theta) = \frac{1}{2} \|\theta\|_2^2 \), then this result says that the minimax risk for estimation of a 1-sparse mean scales at least as \( \frac{d}{n\varepsilon^2} \land 1 \), so that estimation when \( d \gtrsim n/\varepsilon^2 \) is effectively impossible, even for channels satisfying weaker definitions of privacy or \( \varepsilon \)-differential privacy and allowing interactivity. This contrasts the non-private case, where non-asymptotic rates of \( \frac{\log d}{n} \) are possible [7, 41]. Even estimating linear functionals is challenging: suppose that we wish to estimate the sum \( \sum_{j=1}^d \theta_j \), so that we consider \( \Phi(\theta) = \|1^T \theta\| \) and loss \( L(\theta, P) = \|1^T(\theta - \mathbb{E}_P[X])\| \). In this case, results on \( \ell_1 \)-consistency of sparse estimators (e.g. [41, Corollary 2] or [31]) yield that in the non-private case, a soft-thresholded sample mean estimator obtains \( \mathbb{E}[\|\hat{\theta} - \mathbb{E}[X]\|_1] \lesssim \sqrt{\log d/n} \). Thus, in the non-private case, we have \( \mathbb{E}[\|1^T(\hat{\theta} - \mathbb{E}[X])\|] \leq \|1\|_{\infty} \mathbb{E}[\|\hat{\theta} - \mathbb{E}[X]\|_1] \lesssim \sqrt{\log d/n} \). In contrast, in our private case,

\[
\sup_{P \in \mathcal{P}_1} \mathbb{E}_{P,Q} \left[ \|\hat{\theta}(Z_{1:n}) - 1^T \mathbb{E}_P[X]\| \right] \geq \frac{1}{2} \left( \sqrt{\frac{d}{4n\varepsilon^2}} \land 1 \right)
\]

for all \( \varepsilon^2 \)-\( \chi^2 \)-private channels \( Q \) and any estimator \( \hat{\theta} \) of \( \sum_j \mathbb{E}[X_j] \).

### 5.2 High-dimensional sparse logistic regression

The lower bounds for the 1-sparse mean estimation problem in Section 5.1 are illustrative of the difficulties one must encounter when performing locally private inference: under the notions of privacy we use, at least in a minimax sense, there must be additional dimension dependence. As we develop in Section 4, the dependence of estimation methods on the parameters of underlying problem at hand also causes difficulties; Section 4.2.2 shows this in the case of logistic regression. A similar difficulty arises in high-dimensional problems, which we demonstrate here.

Let \( \theta_0 \in \mathbb{R} \) be a fixed base parameter, and consider the following family of \( d \)-dimensional 1-sparse logistic models on pairs \( (x, y) \in \{-1, 1\}^d \times \{-1, 1\} \):

\[
\mathcal{P}_{\text{log}, \theta_0, d} := \left\{ p_\theta \mid p_\theta(y \mid x) = \frac{e^{y(\theta^T x + \theta_0)}}{1 + e^{y(\theta^T x + \theta_0)}}, X \sim \text{Uniform}(\{-1, 1\}^d), \|\theta\|_2 \leq 1, \|\theta\|_0 \leq 1 \right\}.
\]

Thus, there is the “null” distribution with conditional distributions \( p(y \mid x) = \frac{1}{1 + e^{-y\theta_0}} \), or a fixed bias parameter \( \theta_0 \), while the parameter to be estimated is an at most 1-sparse vector \( \theta \in \mathbb{R}^d \) with \( \|\theta\|_2 \leq 1 \). For this class, we have the following lower bound, which applies to convex losses \( \Phi \) for estimating linear functions \( \psi \) of \( \theta \) as in Proposition 8 (See Appendix A.7 for a proof.)
Proposition 9. Let $\mathcal{P}_{\log, \theta_0, d}$ denote the family of $1$-sparse logistic regression models, and let $\mathcal{Q}_\varepsilon$ denote the collection of $\varepsilon^2$-$\chi^2$-private and sequentially interactive channels. Define

$$\delta_n^2 := \min \left\{ \frac{e^{2\theta_0 d}}{64n\varepsilon^2}, \frac{e^{\theta_0}}{8(1 - e^{-\theta_0})\sqrt{n}\varepsilon^2}, 1 \right\}.$$ 

Then

$$\mathcal{M}_n (\mathcal{P}_{\log, \theta_0, d}, L, \mathcal{Q}_\varepsilon) \geq \min_{j=1, \ldots, d} \frac{1}{2} \Phi (\delta_n \psi(e_j)).$$

As a particular example, the $\ell_2$-error satisfies $\mathbb{E} [\| \hat{\theta} - \theta \|_2^2] \gtrsim \min \{ \frac{e^{2\theta_0 d}}{n\varepsilon^2}, \frac{e^{\theta_0}}{(1 - e^{-\theta_0})\sqrt{n}\varepsilon^2}, 1 \}$. The contrast with the non-private case is striking. A careful tracking of constants in Negahban et al. [40, 41, Sec. 4.4] shows the following result. Consider the empirical logistic loss

$$L_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \log (1 + \exp (-Y_i(\theta_0 + (X_i, \theta)))),$$

and for $\Delta \in \mathbb{R}^d$ let $D_{L_n}(\Delta, \theta^*) = L_n(\theta^* + \Delta) - L_n(\theta^*) - \langle \nabla L_n(\theta^*), \Delta \rangle$ be the first-order error in $L_n$, which is non-negative. Then if $(X_i, Y_i) \overset{iid}{\sim} P_{\theta^*} \in \mathcal{P}_{\log, \theta_0, d}$, there are numerical constants $c_1, c_2$ such that with high probability, the restricted strong convexity condition

$$D_{L_n}(\Delta, \theta^*) \geq c_1 e^{-2\theta_0} \| \Delta \|_2^2 - c_2 \log d \frac{d}{n} \| \Delta \|_1^2$$

for all $\| \Delta \|_2 \leq 1$ (38)

holds. Negahban et al. [41, Thm. 1] show that for the non-private $\ell_1$-regularized estimator $\hat{\theta}_{\lambda_n} := \arg\min_{\theta} \{ L_n(\theta) + \lambda_n \| \theta \|_1 \}$, if $2 \| \nabla L_n(\theta^*) \|_\infty \leq \lambda_n$ and $\frac{n \log d}{\log \lambda_n} \geq e^{-2\theta_0}$, then $\| \hat{\theta}_{\lambda_n} - \theta^* \|_2 \leq c e^{2\theta_0} \lambda_n^2$ for a numerical constant $c < \infty$. In the case of 1-sparse logistic regression, an argument with Bernstein’s inequality immediately yields that $\| \nabla L_n(\theta^*) \|_\infty \leq C \frac{1}{1 + e^{\theta_0}} \sqrt{n^{-1} \log d}$ with high probability; thus, the choice $\lambda_n = C e^{-\theta_0} \sqrt{n^{-1} \log d}$ yields an estimator that w.h.p. achieves

$$\| \hat{\theta}_{\lambda_n} - \theta^* \|_2 \leq c e^{\theta_0} \frac{\log d}{n}.$$ 

An argument similar to our derivation of the lower bounds in Corollary 4 shows that for the prediction loss $| p_0(y \mid x) - p_{\theta^*}(y \mid x)|$, we must have a minimax lower bound of at least $\sqrt{d/n\varepsilon^2}$ for any fixed bias term $\theta_0$—the problem never gets easier, and the dimension dependence is unavoidable.

5.3 Proof of Proposition 8

As we outline following the statement (Lemma 5) of Le Cam’s method, the proof proceeds in two phases: first, we choose a well-separated collection of distributions $P_v$, scaled by some $\delta \geq 0$ to be chosen. We then upper bound the variation distance between their (private) mixtures using Theorem 2. Fix $\delta \geq 0$, which we will optimize later. Define the base (null) distribution $P_0$ to be uniform on $\{-1, 1\}^d$, so that $\theta_0 := \mathbb{E}_{P_0}[X] = 0$, with p.m.f. $p_0(x) = \frac{1}{2^d}$. We let the collection $\mathcal{V} = \{ \pm e_j \}_{j=1}^d$ be the collection of the standard basis vectors and their negatives, and for each $v \in \mathcal{V}$ we define $P_v$ to be the slightly tilted distribution with p.m.f.

$$p_v(x) = \prod_{j=1}^{d} \frac{1 + \delta v_j x_j}{2} = \frac{1}{2^d} (1 + \delta v^T x) \text{ for } x \in \{ \pm 1 \}^d.$$
For each of these, by inspection we have $P_v \in \mathcal{P}_1$ and $E_{P_v}[X] = \delta v$, which yields the separation condition that

$$d_L(P_0, P_v) \geq \inf_{\theta \in \mathbb{R}^k} \{\Phi(\theta + \psi(\delta v)) + \Phi(\theta)\} = 2\Phi(\psi(\delta v)/2).$$

(39)

Following our standard approach, we now upper bound the complexity measure $C_2(\{P_v\}_{v \in V})$. Indeed, we may take $P^*$ in the definition (35) to be $P_0$, in which case we obtain

$$C_2(\{P_v\}_{v \in V}) \leq \sup_{f: \mathcal{P}_0 f^2 \leq 1} \frac{1}{|V|} \sum_{v \in V} \left( \int f(x)(dP_0(x) - dP_v(x)) \right)^2$$

$$= \sup_{f: \mathcal{P}_0 f^2 \leq 1} \frac{\delta^2}{2d} \frac{1}{4^d} \sum_{v \in V} \sum_{x_1 \in X} \sum_{x_2 \in X} f(x_1)f(x_2)(x_1^Tv)(x_2^Tv)$$

$$= \sup_{f: \mathcal{P}_0 f^2 \leq 1} \frac{\delta^2}{d} \sum_{x_1, x_2} f(x_1)f(x_2)x_1^Tx_2$$

$$= \frac{\delta^2}{d} \sup_{f: \mathcal{P}_0 f^2 \leq 1} \|E_{P_0}[f(X)X]\|_2^2,$$

where in line (i) we have used that $\sum_{v \in V} v^Tv = 2I_{d \times d}$. To bound the final quantity, note that $\|a\|_2^2 = \sup_{\|v\|_2 \leq 1} \langle v, a \rangle^2$ for any vector $a$, and thus, by Cauchy–Schwarz,

$$\sup_{f: \mathcal{P}_0 f^2 \leq 1} \|E_{P_0}[f(X)X]\|_2^2 = \sup_{f: \mathcal{P}_0 f^2 \leq 1, \|v\|_2 \leq 1} E_{P_0}[f(X)v^TX]^2$$

$$\leq \sup_{f: \mathcal{P}_0 f^2 \leq 1, \|v\|_2 \leq 1} E_{P_0}[f(X)^2]E_{P_0}[(v^TX)^2] = 1,$$

where we have used that $E[XX^T] = I_{d \times d}$.

As a consequence, we have the upper bound

$$C_2(\{P_v\}_{v \in V}) \leq \frac{\delta^2}{d}.$$

We may substitute this upper bound on the complexity into Theorem 2, choosing $P = P_0$ in the theorem so that $\|dP_v/dP_0\|_{TV} = 1 + \delta$, whence we obtain that for our choices of $P_v$ and $P_0$, for any sequentially interactive channel $Q$, we have

$$2 \left\| M_0^n - \overline{M}^n \right\|_{TV}^2 \leq D_{KL}(M_0^n \| \overline{M}^n) \leq \frac{n\varepsilon^2}{d} \delta^2(1 + \delta).$$

Now, if we solve this last quantity so that $\frac{n\varepsilon^2}{d} \delta^2(1 + \delta) \leq \frac{1}{2}$, we see that the choice $\delta^2 = \min\{\frac{d}{4n\varepsilon^2}, 1\}$ guarantees that $\delta^2(1 + \delta) \leq \frac{d}{2n\varepsilon^2}$. Thus, using the separation lower bound (39) and Lemma 5, we obtain

$$\mathfrak{M}_n \geq \min_{v \in V} \Phi(\psi(\delta v)/2) \left( 1 - \sqrt{\frac{n\varepsilon^2}{2d}} \delta^2(1 + \delta) \right) \geq \min_{j=1, \ldots, d} \frac{1}{2} \Phi \left( \min \left\{ \sqrt{\frac{d}{4n\varepsilon^2}}, 1 \right\} \psi(e_j) \right).$$

6 Experimental investigation: generalized linear modeling

In this section, we perform experiments investigating the behavior of our proposed locally optimal estimators, comparing their performance both to non-private estimators and to minimax optimal
estimators developed by Duchi et al. [21] for locally private estimation. In our experiments, we consider fitting a generalized linear model for a variable \( Y \) conditioned on \( X \), where the model has the form

\[
p_\theta(y \mid x) = \exp\left(T(x, y)^T \theta - A(\theta \mid x)\right),
\]

where \( A(\theta \mid x) = \int e^{T(x,y)^T \theta} \mu(y) \) for some base measure \( \mu \) and \( T : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^d \) is the sufficient statistic. We give a slight elaboration of the techniques of Duchi et al. [21, Sec. 5.2] for generalized linear models. In our context, we wish to model \( P(Y \mid X) \) using the GLM model (40), which may be mis-specified as in Section 4.4, but where we assume that the base distribution on \( X \) is known. This assumption is strong, but may (approximately) hold in practice; in biological applications, for example, we may have a large collection of covariate data and wish to estimate the conditional distribution of \( Y \mid X \) for a new outcome \( Y \) [e.g. 10].

For a distribution \( P \) on the pair \((X,Y)\), let \( P_\theta \) denote the marginal over \( X \), which we assume is fixed and known, \( P_{y|x} \) be the conditional distribution over \( Y \) given \( X \), and \( P = P_{y|x} P_\theta \) for shorthand. Define the population risk using the log loss \( \ell_\theta(y \mid x) = -\log p_\theta(y \mid x) \), by

\[
R_P(\theta) = E_P[\ell_\theta(Y \mid X)] = E_P[-T(X,Y)^T \theta + E_{P_\theta}[A(\theta \mid X)] = -E_P[T(X,Y)^T \theta + A_{P_\theta}(\theta)],
\]

where we use the shorthand \( A_{P_\theta}(\theta) := E_{P_\theta}[A(\theta \mid X)] \). Now, let \( \mathcal{P}_y \) be a collection of conditional distributions of \( Y \) given \( X \), and for \( P_{y|x} \in \mathcal{P}_y \), we define

\[
\theta(P_{y|x}) := \arg\min_{\theta} R_{P_{y|x}}(\theta) = \nabla A_{P_\theta}(E_{P_{y|x}}[T(X,Y)]),
\]

in complete analogy to the general exponential family case in Section 4.4.

In our experiments, we study estimation of the linear functional \( v^T \theta \), where the loss for an estimator \( \hat{\psi} \) is

\[
L(\hat{\psi}, P_{y|x}) := \Phi\left(||\hat{\psi} - v^T \theta(P_{y|x})||\right),
\]

where \( \Phi : \mathbb{R} \to \mathbb{R}_+ \) is nondecreasing. As motivation, consider the problem of testing whether a covariate \( X_j \) is relevant to a binary outcome \( Y \in \{-1,1\} \). In this case, a logistic GLM model (40) has the form \( p_\theta(y \mid x) = \exp(yx^T \theta) / (1 + \exp(yx^T \theta)) \), and using \( v = e_j \), the standard basis vector, estimating \( v^T \theta \) corresponds to asking whether \( \theta_j \leq 0 \), while controlling for the other covariates.

### 6.1 Optimal estimation for linear functionals of GLM parameters

Extending the results of Section 4.4 is nearly immediate for our situation. Let us assume for this that the range of the sufficient statistic \( T(x,y) \) is contained in a norm ball \( \{ t \in \mathbb{R}^d \mid ||t|| \leq 1 \} \). Then Proposition 6, applied to the loss \( L(\hat{\psi}, P_{y|x}) = \Phi(\hat{\psi} - v^T \theta(P_{y|x})) \), yields the following lower bound.

**Corollary 6.** Let \( \mathcal{P}_y \) be a collection of conditional distributions on \( Y \mid X \), \( P_0 \in \mathcal{P}_y \), and \( Q_\varepsilon \) be the collection of \( \varepsilon^2 \cdot \chi^2 \)-sequentially interactive private channels. Then for numerical constants \( c_0, c_1 > 0 \) and all large enough \( n \),

\[
\mathcal{M}_n^{loc}(P_0, L, \mathcal{P}_y, Q_\varepsilon) \geq c_0 \sup_{P_{y|x} \in \mathcal{P}_y} \Phi\left(c_1 \left| v^T \nabla^2 A_{P_\theta}(\theta_0)^{-1} \left( \mathbb{E}_{P_{y|x}}[T(X,Y)] - \mathbb{E}_{P_{y|x}} T(X,Y) \right) \right| \frac{\sqrt{n\varepsilon^2}}{\varepsilon} \right).
\]

If the set \( \mathcal{P}_y \) and distribution \( P_\theta \) are such that \( \{ \mathbb{E}_{P_{y|x}} T \mid P_{y|x} \in \mathcal{P}_y \} \supset \{ t \in \mathbb{R}^d \mid ||t|| \leq r \} \), then we have the simplified lower bound

\[
\mathcal{M}_n^{loc}(P_0, L, \mathcal{P}_y, Q_\varepsilon) \geq c_0 \Phi\left(c_1 \left| \nabla^2 A_{P_\theta}(\theta_0)^{-1} v \right| \frac{\sqrt{n\varepsilon^2}}{\varepsilon} \right).
\]
An optimal estimator is similar to that we describe in Section 4.4. Consider a non-private sample \( \{(X_i, Y_i)\}_{i=1}^{n_1} \), and split it into samples of size \( n_1 = \lceil n^{2/3} \rceil \) and \( n_2 = n - n_1 \). As in Section 4.4, for \( i = 1, \ldots, n_1 \), let \( Z_i \) be any \( \varepsilon \)-locally differentially private estimate of \( T(X_i, Y_i) \) with \( \mathbb{E}[Z_i \mid X_i, Y_i] = T(X_i, Y_i) \) and \( \mathbb{E}[\|Z_i\|^2] < \infty \), and define \( \tilde{\mu}_n = \frac{1}{n} \sum_{i=1}^{n_1} Z_i \) and \( \tilde{\theta}_n = \nabla A_{P_k}^\top(\tilde{\mu}_n) = \text{argmin}_\theta \{-\tilde{\mu}_n^\top \theta + A_{P_k}(\theta)\} \). Then, for \( i = n_1 + 1, \ldots, n \), let
\[
Z_i = v^T \nabla^2 A_{P_k}(\tilde{\theta}_n)^{-1} T(X_i, Y_i) + \frac{r \|\nabla^2 A_{P_k}(\tilde{\theta}_n)^{-1} v\|_*}{\varepsilon} W_i \quad \text{where } W_i \overset{iid}{\sim} \text{Laplace}(1),
\]
where we recall that \( \nabla^2 A_{P_k}(\theta) = \mathbb{E}_P[\text{Cov}_P(T(X, Y) \mid X)] \) for \( P \) the GLM model (40). The \( Z_i \) are evidently \( \varepsilon \)-differentially private, and we then define the private estimator
\[
\hat{\psi}_n := Z_{n_2} + v^T \left( \tilde{\theta}_n - \nabla^2 A_{P_k}(\tilde{\theta}_n)^{-1} \tilde{\mu}_n \right). \tag{42}
\]
An identical analysis to that we use to prove Proposition 7 then gives the following corollary, in which we recall the Mahalanobis norm \( \|x\|_C^2 = x^T C x \).

**Corollary 7.** Let \( \hat{\psi}_n \) be the estimator (42) and \( \theta_0 = \nabla A_{P_k}^\top(\mathbb{E}_P[T(X, Y)]) \) = argmin \( \theta \) \( R_P(\theta) \). Then
\[
\sqrt{n}(\hat{\psi}_n - v^T \theta_0) \overset{d}{\sim} \mathcal{N} \left( 0, \|\nabla^2 A_{P_k}(\theta_0)^{-1} v\|_*^2 \right)
\]
\[
\left( \mathbb{E}_{Cov(T(X, Y))} + \frac{2}{\varepsilon^2} \|\nabla^2 A_{P_k}(\theta_0)^{-1} v\|_*^2 \right).
\]
In this case, when \( \varepsilon \) is large, the estimator becomes nearly efficient—the local minimax variance is precisely the first term in the variance of Corollary 7.

### 6.2 Flow cytometry experiments

In this section, we investigate the performance of our locally private estimators on a flow-cytometry dataset for predicting protein expression [28, Ch. 17]. We compare our local optimal one-step estimators against minimax optimal (parameter) estimators that Duchi et al. [21] develop. The flow-cytometry dataset contains measurements of the expression levels of \( d = 11 \) proteins on \( n = 7466 \) cells, and the goal is to understand the network structure linking the proteins: how does the expression level of protein \( j \) depend on the remaining proteins. The raw data is heavy-tailed and skewed, so we perform an inverse tangent transformation so that each expression level \( x_{ij} \mapsto \tan^{-1}(x_{ij}) \). We treat the data as a matrix \( X \in \mathbb{R}^{n \times d} \) and then consider the problem of predicting column \( i \) of \( X \) from the remaining columns. To compare the methods and to guarantee existence of a ground truth in our experiments, we treat \( X \) as the **full population**, so that each experiment consists of sampling rows of \( X \) with replacement.

Let \( x \in \mathbb{R}^d \) denote a row of \( X \). For each \( i \in [d] \), we wish to predict whether \( x_i \) based on \( x_{-i} \in \mathbb{R}^{d-1} \), the remaining covariates, and we use a logistic regression model to perform the slightly simpler task of predicting \( y = \text{sign}(x_i) \). That is, for each \( i \) we model
\[
\log \frac{P_\theta(\text{sign}(x_i) = 1 \mid x_{-i})}{P_\theta(\text{sign}(x_i) = -1 \mid x_{-i})} = \theta^T x_{-i} + \theta_{\text{bias}},
\]
so that \( T(x_{-i}, y) = y[x_{-i}^T, 1]^T \) and \( A(\theta \mid x_{-i}) = \log(e^{\theta^T x_{-i} + \theta_{\text{bias}}} + e^{-\theta^T x_{-i} - \theta_{\text{bias}}}) \), where \( y = \text{sign}(x_i) \) is the sign of the expression level of protein \( i \). For each \( i \in \{1, \ldots, d\} \), we let \( \theta_{\text{ml}}^i \in \mathbb{R}^d \) be the parameter (including the bias) maximizing the likelihood for this logistic model of predicting \( x_i \) using the full data \( X \).

We perform multiple experiments, where each is as follows. We sample \( N \) rows of \( X \) uniformly (with replacement), and we perform two private procedures (and one non-private procedure) on the sampled data \( X_{\text{new}} \in \mathbb{R}^{N \times d} \). We vary the privacy parameter in \( \varepsilon \in \{1, 4\} \).
(i) The first procedure is the minimax optimal stochastic gradient procedure of Duchi et al. [21, Secs. 4.2.3 & 5.2]. In brief, this procedure begins from \( \theta^0 = 0 \), and at iteration \( k \) draws a pair \((x, y)\) uniformly at random, then uses a carefully designed \( \varepsilon \)-locally private version \( Z^k \) of \( T = T(x, y) \) with the property that \( \mathbb{E}[Z | x, y] = T(x, y) \) and \( \sup_k \mathbb{E}[\|Z^k\|^2] < \infty \), updating

\[
\theta^{k+1} = \theta^k - \eta_k \left( \nabla A_{F_k}(\theta^k) - Z^k \right),
\]

where \( \eta_k > 0 \) is a stepsize sequence. (We use optimal the \( \ell_\infty \) sampling mechanism [21, Sec. 4.2.3] to construct \( Z_i \).) We use stepsizes \( \eta_k = 1/(20\sqrt{k}) \), which gave optimal performance over many choices of stepsize and power \( k^{-\beta} \). We perform \( N \) steps of this stochastic gradient method, yielding estimator \( \tilde{\theta}^{(\text{ml})} \) for prediction of protein \( i \) from the others.

(ii) The second procedure is the one-step corrected estimator (42). To construct the initial \( \tilde{\theta}_n \), we again use Duchi et al.’s \( \ell_\infty \) sampling mechanism to construct an approximate estimate \( \bar{\mu}_n = \frac{1}{n} \sum_{i=1}^n Z_i \) and let \( \tilde{\theta}_{\text{init}} = \bar{\theta}_n \). For each coordinate \( i = 1, \ldots, d \), we then construct \( \tilde{\theta}^{(\text{os})} \) precisely as in Eq. (42), using \( v = e_1, \ldots, e_d \).

(iii) The final procedure is the non-private maximum likelihood estimator based on the resampled data of size \( N \).

We perform each of these three-part tests \( T = 100 \) times, where within each test, each method uses an identical sample (the samples are of course independent across tests).

We give summaries of our results in Figure 1 and Table 1. In Figure 1, we show histograms of the errors across all coordinates of \( \tilde{\theta}^{(i)}_{\text{ml}} \), \( i = 1, \ldots, d \), and all \( T = 100 \) tests, of the three procedures: the minimax stochastic gradient procedure [21], our one-step correction, and the maximum likelihood estimator. For each, we use a sample of size \( N = 10n \), though results are similar for sample sizes \( N = 4n, 6n \) and \( 8n \). In the figures, we see that the non-private estimator is quite concentrated in its errors around the “population” solution based on the data \( X \) (we truncate the top of the plot). In the case that we have “high” privacy (\( \varepsilon = 1 \), the left of Fig. 1), we see that the one-step estimator has errors more concentrated around zero than the minimax estimator, though the two have comparable performance. In the slightly lower privacy regime, corresponding to \( \varepsilon = 4 \), the one-step-corrected estimator has much better performance. The non-private classical minimax estimator substantially outperforms it, but the one-step-corrected estimator still has much tighter concentration of its errors than does the minimax procedure.

In Table 1, we compare the performance of the one-step estimator with other possible estimators more directly. For the estimators \( \tilde{\theta}_{\text{init}}, \tilde{\theta}^{(i)}_{\text{sg}}, \) and \( \tilde{\theta}^{(i)}_{\text{os}} \) of the true parameter \( \theta^{(i)} \), we count the number of experiments (of \( T \)) and parameters \( j = 1, \ldots, d \) for which

\[
|\tilde{\theta}^{(i)}_{\text{os}}|_j - |\theta^{(i)}_{\text{ml}}|_j < |\tilde{\theta}_{\text{init}}|_j - |\theta^{(i)}_{\text{ml}}|_j \quad \text{and} \quad |\tilde{\theta}^{(i)}_{\text{sg}}|_j - |\theta^{(i)}_{\text{ml}}|_j < |\tilde{\theta}^{(i)}_{\text{sg}}|_j - |\theta^{(i)}_{\text{ml}}|_j ,
\]

that is, the number of experiments in which the one-step estimator provides a better estimate than its initializer or the minimax stochastic gradient-based procedure. Table 1 shows these results, displaying the proportion of experiments in which the one-step method has higher accuracy than the other procedures for sample sizes of \( N = 2n \) and \( 10n \) and privacy levels \( \varepsilon \in \{1, 4\} \). The table shows that the one-step estimator does typically provide better performance than the other methods—though this comes with some caveats. When the privacy level is high, meaning \( \varepsilon = 1 \), the performance between the methods is more similar, as it is at smaller sample sizes. An explanation that we believe plausible is that in the case of the small sample sizes, the initializer is
Figure 1. Histogram of errors across all experiments in estimation of $v^T \theta_{ml}$, for $v = e_1, \ldots, e_d$ and $i = 1, \ldots, d$, in the logistic regression model. Left: privacy level $\varepsilon = 1$, right: privacy level $\varepsilon = 4$.

| Sample size  | $N = 2n$ | $N = 10n$ |
|--------------|----------|----------|
| Privacy $\varepsilon$ | $\varepsilon = 1$ | $\varepsilon = 4$ | $\varepsilon = 1$ | $\varepsilon = 4$ |
| vs. initializer | 0.501 | 0.82 | 0.808 | 0.851 |
| vs. minimax (stochastic gradient) | 0.321 | 0.677 | 0.659 | 0.79 |

Table 1. Frequency with which the one-step estimator outperforms initialization and minimax (stochastic-gradient-based) estimator over $T = 100$ tests, all coordinates $j$ of the parameter and proteins $i = 1, \ldots, d$ for the flow-cytometry data.

Inaccurate enough that the one-step correction has a poor Hessian estimate, so that it becomes a weak estimator. In the low privacy regime, the full minimax procedure [21] adds more noise than is necessary, as it privatizes the entire statistic $xy$ in each iteration—a necessity because it iteratively builds the estimates $\hat{\theta}_{sg}$—thus causing an increase in sample complexity over the local minimax estimator, which need not explicitly estimate $\theta$.

In summary, a one-step correction—which we demonstrate is locally minimax optimal—typically outperforms alternative (non-optimal) approaches, though the benefits become apparent only in reasonably large sample regimes. This type of behavior may be acceptable, however, in scenarios in which we actually wish to apply local privacy. As our results and those of Duchi et al. [21] make clear, there are substantial costs to local privacy protections, and so they may only make sense for situations (such as web-scale data) with very large sample sizes.

7 Discussion

By the careful construction of locally optimal and adaptive estimators, as well as our local minimax lower bounds, we believe results in this paper indicate more precisely the challenges associated with locally private estimation. To illustrate this, let us reconsider the estimation of a linear functional $v^T \theta$ in a classical statistical problem. Indeed, let $\{P_0\}$ be a family with Fisher information matrices $\{I_\theta\}$ and a score function $\ell_\theta: X \to \mathbb{R}^d$. Then a classical estimators $\hat{\theta}_n$ of the parameter $\theta_0$ is
efficient [50, Sec. 8.9] (among regular estimators) if and only if

$$\hat{\theta}_n - \theta_0 = \frac{1}{n} \sum_{i=1}^{n} -I_{\theta_0}^{-1} \ell_{\theta_0}(X_i) + o_P(1/\sqrt{n}),$$

and an efficient estimator \(\hat{\psi}_n\) of \(v^T \theta\) satisfies \(\hat{\psi}_n = v^T \theta_0 - n^{-1} \sum_{i=1}^{n} v^T I_{\theta_0}^{-1} \ell_{\theta_0}(X_i) + o_P(n^{-1/2}).\) In contrast, in the private case, our locally minimax optimal estimators (recall Sections 4.3 and 4.4) have the asymptotic form

$$\hat{\psi}_{\text{priv},n} = v^T \theta_0 - v^T \left( \frac{1}{n} \sum_{i=1}^{n} I_{\theta_0}^{-1} \ell_{\theta_0}(X_i) \right) + \frac{1}{n} \sum_{i=1}^{n} W_i + o_P(1/\sqrt{n}),$$

where the random variables \(W_i\) must add noise of a magnitude scaling as \(\frac{1}{\epsilon} \sup_{x} |v^T I_{\theta_0}^{-1} \ell_{\theta_0}(x)|\), because otherwise it is possible to distinguish examples for which \(v^T I_{\theta_0}^{-1} \ell_{\theta_0}(X_i)\) is large from those for which it has small magnitude. This enforced lack of distinguishability of "easy" problems (those for which the scaled score \(I_{\theta_0}^{-1} \ell_{\theta_0}(X_i)\) is typically small) from "hard" problems (for which it is large) is a feature of local privacy schemes, and it helps to explain the difficulty of estimation.

We thus believe it prudent to more carefully explore feasible definitions of privacy, especially in local senses. Regulatory decisions and protection against malfeasance may require less stringent notions of privacy than pure differential privacy, but local notions of privacy—where no sensitive non-privatized data leaves the hands of a sample participant—are desirable. The asymptotic expansions above suggest a notion of privacy that allows some type of relative noise addition, to preserve the easiness of "easy" problems, owill help. Perhaps large values of \(\epsilon\), at least for high-dimensional problems, may still provide acceptable privacy protection, at least in concert with centralized privacy guarantees. We look forward to continuing study of these fundamental limitations and acceptable tradeoffs between data utility and protection of study participants.

A Technical appendices

A.1 Proof of Lemma 2

By the triangle inequality, we have

$$\int |p_{\theta_0 + h} - p_{\theta_0} - h^T \ell_{\theta_0} p_{\theta_0}| d\mu$$

$$\leq \int |p_{\theta_0 + h} - p_{\theta_0} - \frac{1}{2} h^T \ell_{\theta_0} \sqrt{p_{\theta_0}} (\sqrt{p_{\theta_0 + h}} + \sqrt{p_{\theta_0}}) d\mu + \int \frac{1}{2} h^T \ell_{\theta_0} \sqrt{p_{\theta_0}} (\sqrt{p_{\theta_0 + h}} - \sqrt{p_{\theta_0}}) d\mu.$$

We show that each of the integral terms \(I_1\) and \(I_2\) are both \(o(||h||)\) as \(h \to 0\). By algebraic manipulation and the Cauchy–Schwarz inequality,

$$I_1(h; \theta_0) = \int \left| \sqrt{p_{\theta_0 + h}} + \sqrt{p_{\theta_0}} \right| \left| \sqrt{p_{\theta_0 + h}} - \sqrt{p_{\theta_0}} - \frac{1}{2} h^T \ell_{\theta_0} \sqrt{p_{\theta_0}} \right| d\mu$$

$$\leq \left( \int \left| \sqrt{p_{\theta_0 + h}} + \sqrt{p_{\theta_0}} \right|^2 d\mu \right)^{\frac{1}{2}} \cdot \left( \int \left| \sqrt{p_{\theta_0 + h}} - \sqrt{p_{\theta_0}} - \frac{1}{2} h^T \ell_{\theta_0} \sqrt{p_{\theta_0}} \right|^2 d\mu \right)^{\frac{1}{2}}.$$
Jensen’s inequality gives $\int |\sqrt{p_{0,h} + \sqrt{p_{0,h}}}^2| d\mu \leq 2 \int (p_{0,h} + p_{0,h}) d\mu = 2$. The assumption that $\mathcal{P}$ is QMD at $\theta_0$ immediately yields $I_1(h; \theta_0) = O(||h||)$. To bound $I_2$, we again apply the Cauchy–Schwarz inequality, obtaining

$$2I_2(h; \theta_0) \leq \left( \int |h^T h_0 \sqrt{\mathcal{P}} d\mu \right)^{\frac{1}{2}} \left( \int |h^T h_0 \sqrt{\mathcal{P}} d\mu \right)^{\frac{1}{2}}$$

Since $\mathcal{P}$ is QMD at $\theta_0$, we have $\int |h^T h_0 \sqrt{\mathcal{P}} d\mu = \int |h^T h_0 \sqrt{\mathcal{P}} d\mu + o(||h||^2) = O(||h||^2)$ (see [50, Ch. 7.2]). Thus $I_2(h; \theta_0) = O(||h||^2)$, giving the lemma.

### A.2 Proof of Proposition 1

Let $P_0$ and $P_1$ be distributions on $\mathcal{X}$, each with densities $p_0, p_1$ according to some base measure $\mu$. Let $\theta_0 = \theta(P_0)$, and consider the problem of privately collecting observations and deciding whether $\theta = \theta_0$ or $\theta = \theta_1$. We define a randomized-response based estimator for this problem using a simple hypothesis test.

Define the “acceptance” set

$$A := \{ x \in \mathcal{X} \mid p_0(x) > p_1(x) \}.$$ 

Then we have $P_0(A) - P_1(A) = \|P_0 - P_1\|_{TV}$ by a calculation. Now, consider the following estimator: for each $X_i$, define

$$T_i = 1\{ X_i \in A \} \text{ and } Z_i \mid \{ T_i = t \} = \begin{cases} 1 & \text{with probability } (e^\varepsilon + 1)^{-1}(e^\varepsilon t + 1 - t) \\ 0 & \text{with probability } (e^{-\varepsilon} + 1)^{-1}(e^{-\varepsilon} t + 1 - t) \end{cases}$$

Then the channel $Q(\cdot \mid X_i)$ for $Z_i \mid X_i$ is $\varepsilon$-differentially-private by inspection, and setting $\delta_\varepsilon = \frac{e^\varepsilon}{1+e^\varepsilon} - \frac{1}{2}$, we have

$$E_0[Z_i] = \frac{1 + \delta_\varepsilon}{2} P_0(A) + \frac{1 - \delta_\varepsilon}{2} P_0(A^c) = \frac{1 - \delta_\varepsilon}{2} + \delta_\varepsilon P_0(A) \text{ and } E_1[Z_i] = \frac{1 - \delta_\varepsilon}{2} + \delta_\varepsilon P_1(A)$$

while $Z_i \in \{0, 1\}$. Now, define the statistic

$$K_n := \frac{1}{\delta_\varepsilon} \left( \frac{1}{n} \sum_{i=1}^n Z_i - \frac{1 - \delta_\varepsilon}{2} \right),$$

so that $E_0[K_n] = P_0(A)$ and $E_1[K_n] = P_1(A)$. We define our estimator to be

$$\hat{\theta} := \begin{cases} \theta_0 & \text{if } K_n \geq \frac{P_0(A) + P_1(A)}{2} \\ \theta_1 & \text{if } K_n < \frac{P_0(A) + P_1(A)}{2}. \end{cases}$$

We now analyze the performance of our randomized choice $\hat{\theta}$. By construction of the acceptance set $A$, note that

$$\frac{P_0(A) + P_1(A)}{2} = P_0(A) + \frac{P_1(A) - P_0(A)}{2} = P_0(A) - \frac{1}{2} \|P_1 - P_0\|_{TV} = P_1(A) + \frac{1}{2} \|P_1 - P_0\|_{TV}.$$ 

By Hoeffding’s inequality, we thus have

$$\max \left\{ P_0 \left( K_n \leq \frac{P_0(A) + P_1(A)}{2} \right), P_1 \left( K_n \geq \frac{P_0(A) + P_1(A)}{2} \right) \right\} \leq \exp \left( -\frac{n \delta_\varepsilon^2 \|P_0 - P_1\|_{TV}^2}{2} \right).$$

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In particular, we have
\[
\mathbb{E}_0[\ell(\hat{\theta}, P_0)] + \mathbb{E}_1[\ell(\hat{\theta}, P_1)] \leq [\ell(\theta_1, P_0) + \ell(\theta_0, P_1)] \exp \left( -\frac{n\delta^2}{2} \frac{\|P_0 - P_1\|^2_{TV}}{2} \right).
\]

Using the reverse triangle condition C.1 on the distance function’s growth, we obtain
\[
\mathbb{E}_0[\ell(\hat{\theta}, P_0)] + \mathbb{E}_1[\ell(\hat{\theta}, P_1)] \leq \gamma d_L(P_0, P_1) \exp \left( -\frac{n\delta^2}{2} \frac{\|P_0 - P_1\|^2_{TV}}{2} \right) 
\leq \gamma \sup_{P_1 \in P} d_L(P_0, P_1) \exp \left( -\frac{n\delta^2}{2} \frac{\|P_0 - P_1\|^2_{TV}}{2} \right) = \gamma \sup_{r \geq 0} \left\{ \omega_L(r; P_0) \exp \left( -\frac{n\delta^2 r^2}{2} \right) \right\}. \tag{43}
\]

The bound (43) is the key inequality. Let us substitute \( \tau^2 = \frac{nr^2\delta^2}{2} \), or \( r = \sqrt{2\tau \frac{\delta}{\delta \sqrt{n}}} \) in the expression, which yields
\[
\mathbb{E}_0[\ell(\hat{\theta}, P_0)] + \mathbb{E}_1[\ell(\hat{\theta}, P_1)] \leq \gamma \sup_{\tau \geq 0} \left\{ \omega_L \left( \sqrt{2\tau \frac{\delta}{\delta \sqrt{n}}}; P_0 \right) e^{-\tau^2} \right\}.
\]

For all \( \tau \leq 1 \), this gives the result; otherwise, we use the growth condition on the modulus of continuity to obtain
\[
\mathbb{E}_0[\ell(\hat{\theta}, P_0)] + \mathbb{E}_1[\ell(\hat{\theta}, P_1)] \leq \gamma \omega_L \left( \sqrt{2\tau \frac{\delta}{\delta \sqrt{n}}}; P_0 \right) \beta^\alpha \sup_{\tau \geq 1} \tau^\alpha e^{-\tau^2}
\]

Noting that \( \sup_{\tau \geq 0} \tau^\alpha e^{-\tau^2} = (\alpha/2)^{\alpha/2} e^{-\alpha/2} \) gives the result.

### A.3 Proof of Proposition 5

We require one additional piece of notation before we begin the proof. Let \( W_i = Z_i - V_i \) be the error in the private version of the quantity \( V_i \), so that \( \mathbb{E}[W_i | V_i] = 0 \), and
\[
W_i = \begin{cases} 
\frac{2}{e^\varepsilon - 1} V_i - \frac{1}{e^\varepsilon - 1} & \text{w.p. } \frac{e^\varepsilon}{e^\varepsilon + 1} \\
\frac{2e^\varepsilon}{e^\varepsilon + 1} V_i + \frac{1}{e^\varepsilon + 1} & \text{w.p. } \frac{e^\varepsilon}{e^\varepsilon + 1}.
\end{cases}
\]

Recall our definitions of \( V_i = 1\{T(X_i) \geq \hat{T}_n\} \) and \( Z_i \) as the privatized version of \( V_i \). Letting \( \mathbb{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i \), and similarly for \( \mathbb{V}_n \) and \( \mathbb{W}_n \), recall also the definition of the random variable \( G_n := \Psi(\hat{T}_n, \theta_0) = P_{\theta_0}(T(X) \geq \hat{T}_n) \). By mimicking the delta method, we will show that
\[
\sqrt{n}(\hat{\theta}_n - \theta_0) = 2J_{\theta_0}^{-1} \cdot \sqrt{n}(\mathbb{V}_n - G_n + \mathbb{W}_n) + o_P(1). \tag{44}
\]

Deferring the proof of the expansion (44), let us show how it implies the proposition.

First, with our definition of the \( W_i \), we have
\[
\text{Var}(W_i | V_i) = \mathbb{E}[W_i^2 | V_i] = \frac{e^\varepsilon}{(e^\varepsilon - 1)^2} = \delta^2.
\]
so that $W_n = \frac{1}{n} \sum_{i=1}^{n} W_i$ satisfies $\sqrt{n} W_n \xrightarrow{d} N(0, \delta^2)$ by the Lindeberg CLT. Thus, assuming the expansion (44), it remains to show the weak convergence result

$$\frac{\sqrt{n} (\tilde{V}_n - G_n)}{G_n(1 - G_n)} \xrightarrow{d} N(0, 1).$$

(45)

where $G_n = \Psi(\tilde{T}_n, \theta_0)$. By definition, the $\{X_i\}_{i=1}^n$ are independent of $\tilde{T}_n$, and hence

$$E[V_i | \tilde{T}_n] = \Psi(\tilde{T}_n, \theta_0) = G_n \quad \text{and} \quad \text{Var}(V_i | \tilde{T}_n) = \Psi(\tilde{T}_n, \theta_0)(1 - \Psi(\tilde{T}_n, \theta_0)) = G_n(1 - G_n).$$

The third central moments of the $V_i$ conditional on $\tilde{T}_n$ have the bound

$$E \left[ \left| V_i - E[V_i | \tilde{T}_n] \right|^3 | \tilde{T}_n \right] \leq \Psi(\tilde{T}_n, \theta_0)(1 - \Psi(\tilde{T}_n, \theta_0)) = G_n(1 - G_n).$$

Thus, we may apply the Berry-Esseen Theorem [36, Thm 11.2.7] to obtain

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sqrt{n} (\tilde{V}_n - G_n)}{G_n(1 - G_n)} \leq t \right) - \Phi(t) \right| \leq \mathbb{E} \left[ \sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sqrt{n} (\tilde{V}_n - G_n)}{G_n(1 - G_n)} \leq t \right) - \Phi(t) \right] \right] \leq \mathbb{E}[U_n]$$

Jensens’s inequality then implies

$$\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{\sqrt{n} (\tilde{V}_n - G_n)}{G_n(1 - G_n)} \leq t \right) - \Phi(t) \right| \leq \mathbb{E}[U_n].$$

To show the convergence (45), it is thus sufficient to show that $\mathbb{E}[U_n] \to 0$ as $n \uparrow \infty$. To that end, the following lemma on the behavior of $\Psi(t, \theta) = P_{\theta}(T(X) \geq t)$ is useful.

**Lemma 6.** Let $t_0 = E_{\theta_0}[T(X)]$ and assume that $\text{Var}_{\theta_0}(T(X)) > 0$. Then there exist $\epsilon > 0$ and $c \in (0, \frac{1}{2})$ such that if $t \in [t_0 + \epsilon]$ and $\theta \in [\theta_0 + \epsilon]$, then $\Psi(t, \theta) \in [c, 1 - c]$.  

**Proof.** By the dominated convergence theorem and our assumption that $\text{Var}_{\theta_0}(T(X)) > 0$, where $t_0 = E_{\theta_0}[T(X)]$, we have

$$\liminf_{t \uparrow t_0} \Psi(t, \theta_0) = P_{\theta_0}(T(X) \geq t_0) \in (0, 1) \quad \text{and} \quad \limsup_{t \downarrow t_0} \Psi(t, \theta_0) = P_{\theta_0}(T(X) > t_0) \in (0, 1).$$

The fact that $t \mapsto \Psi(t, \theta_0)$ is non-increasing implies that for some $\epsilon_1 > 0, c \in (0, \frac{1}{2})$, we have $\Psi(t, \theta_0) \in [2c, 1 - 2c]$ for $t \in [t_0 - \epsilon_1, t_0 + \epsilon_1]$. Fix this $\epsilon_1$ and $c$. By [36, Thm 2.7.1], we know that any $t \in \mathbb{R}$, the function $\theta \mapsto \Psi(t, \theta)$ is continuous and non-decreasing. Thus for any $\epsilon_2 > 0$, we have

$$\Psi(t_0 + \epsilon_1, \theta_0 - \epsilon_2) \leq \Psi(t, \theta) \leq \Psi(t_0 - \epsilon_1, \theta_0 + \epsilon_2) \quad \text{for} \quad (t, \theta) \in [t_0 \pm \epsilon_1] \times [\theta_0 \pm \epsilon_2].$$

Using the continuity of $\theta \mapsto \Psi(t, \theta)$, we may choose $\epsilon_2 > 0$ small enough that

$$\Psi(t, \theta) \in [c, 1 - c] \quad \text{for} \quad (t, \theta) \in \{t_0 - \epsilon_1, t_0 + \epsilon_1\} \times \{\theta_0 - \epsilon_2, \theta_0 + \epsilon_2\}.$$

The lemma follows by taking $\epsilon = \epsilon_1 \wedge \epsilon_2$. \hfill \Box

As $\text{Var}_{\theta_0}(T(X)) > 0$ by assumption, Lemma 6 and the fact that $\tilde{T}_n \xrightarrow{p} t_0$ imply

$$G_n := \Psi(\tilde{T}_n, \theta_0) = P_{\theta_0}(T(X) \geq \tilde{T}_n) \in [c + o_P(1), 1 - c + o_P(1)].$$

(46)

The bounds (46) imply that $G_n \geq c(1 - c) + o_P(1)$, so $U_n \xrightarrow{p} 0$. By construction $|U_n| \leq 2$ for all $n$, so the bounded convergence theorem implies $E[U_n] \to 0$, which was what we required to show the weak convergence result (45). The joint convergence in the proposition follows because $W_n$ and $\tilde{V}_n - G_n$ are conditionally uncorrelated.

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The delta method expansion. We now return to demonstrate the claim (44). For \( p \in [0, 1] \), recall the definition (28) of the function \( H \), and define
\[
H_n(p) := H(p, \hat{T}_n) = \inf \left\{ \theta \in \mathbb{R} \mid P_0(T(X) \geq \hat{T}_n) \geq p \right\},
\]
where the value is \(-\infty \) or \(+\infty \) for \( p \) below or above the range of \( \theta \mapsto P_0(T(X) \geq \hat{T}_n) \), respectively. Then \( \hat{\theta}_n = H_n(\mathbb{E}_n) \) by construction (28). We would like to apply Taylor’s theorem and the inverse function theorem to \( \hat{\theta}_n - \theta_0 = H_n(\mathbb{E}_n) - \theta_0 \), but this requires a few additional steps.

By the inverse function theorem, \( p \mapsto H_n(p) \) is \( C^\infty \) on the interval \((\inf_{\theta} \Psi(\hat{T}_n, \theta), \sup_{\theta} \Psi(\hat{T}_n, \theta))\), and letting
\[
\hat{\Psi}_\theta(t, \theta) = \frac{\partial}{\partial \theta} \Psi(t, \theta) = E_\theta[1\{T(X) \geq t\} (T(X) - A'(\theta))]
\]
be the derivative of \( P_0(T(X) \geq t) \) with respect to \( \theta \), we have \( H'_n(p) = \hat{\Psi}_\theta(\hat{T}_n, H_n(p))^{-1} \) whenever \( p \) is interior to the range of \( \theta \mapsto P_0(T(X) \geq \hat{T}_n) \). To show that \( \mathbb{E}_n \) is (typically) in this range, we require a bit of analysis on \( \hat{\Psi}_\theta \).

**Lemma 7.** The function \( (t, \theta) \mapsto \hat{\Psi}_\theta(t, \theta) = E_\theta[1\{T(X) \geq t\} (T(X) - A'(\theta))] \) is continuous at \((t_0, \theta_0)\), where \( t_0 = E_{\theta_0}[T(X)] = A'(\theta_0) \).

To avoid disrupting the flow, we defer the proof to Section A.3.1. Now, we have that \( \hat{\Psi}_\theta(t_0, \theta_0) = \frac{1}{2} E_{\theta_0}[|T(X) - t_0|] > 0 \), so Lemma 7 implies there exists \( \epsilon > 0 \) such that
\[
\inf_{|t-t_0| \leq \epsilon, |\theta-\theta_0| \leq \epsilon} \hat{\Psi}_\theta(t, \theta) \geq c > 0
\]
for some constant \( c \). Thus, we obtain that
\[
\mathbb{P}\left( \mathbb{E}_n \notin \text{Range}(\Psi(\hat{T}_n, \cdot)) \right) \leq \mathbb{P}\left( \mathbb{E}_n \notin \text{Range}(\Psi(\hat{T}_n, \cdot)), \hat{T}_n \in [t_0 \pm \epsilon] \right) + \mathbb{P}\left( \hat{T}_n \notin [t_0 \pm \epsilon] \right) \\
\leq \mathbb{P}\left( \mathbb{E}_n \notin [\Psi(\hat{T}_n, \theta_0) \pm c\epsilon] \right) + o(1) \rightarrow 0, \tag{49}
\]
where inequality \((i)\) follows because \( \text{Range}(\Psi(t, \cdot)) \supseteq [\Psi(\theta_0) \pm c\epsilon] \) for all \( t \) such that \(|t-t_0| \leq \epsilon\) by condition \((48)\), and the final convergence because \( \mathbb{E}_n - \Psi(\hat{T}_n, \theta_0) \overset{p}{\to} 0 \) and \( \hat{T}_n \) is consistent for \( t_0 \).

We recall also that for any fixed \( t, \theta \mapsto \Psi(t, \theta) \) is analytic on the interior of the natural parameter space and strictly increasing at all \( \theta \) for which \( \Psi(t, \theta) \in (0, 1) \) (cf. [36, Thm. 2.7.1, Thm. 3.4.1]). Thus,
\[
H_n(\Psi(\hat{T}_n, \theta)) = \theta \quad \text{whenever} \quad \Psi(\hat{T}_n, \theta) \in (0, 1).
\]
As \( G_n = \Psi(\hat{T}_n, \theta_0) \in [c + o_P(1), 1 - c + o_P(1)] \) by definition \((46)\) of \( G_n \), we obtain
\[
\mathbb{P}\left( H_n(\Psi(\hat{T}_n, \theta_0)) \neq \theta_0 \right) \rightarrow 0.
\]
Now, by the differentiability of \( H_n \) on the interior of its domain (i.e. the range of \( \Psi(\hat{T}_n, \cdot) \)), we use the convergence \((49)\) and Taylor’s intermediate value theorem to obtain that for some \( p_n \) between \( \mathbb{E}_n \) and \( \Psi(\hat{T}_n, \theta_0) \), we have
\[
\sqrt{n}(\hat{\theta}_n - \theta_0) = \sqrt{n}(\hat{\theta}_n - H_n(\Psi(\hat{T}_n, \theta_0))) + o_P(1) \\
= H'_n(p_n)\sqrt{n}\left( \mathbb{E}_n - \Psi(\hat{T}_n, \theta_0) \right) + o_P(1) \\
= \hat{\Psi}_\theta(\hat{T}_n, H_n(p_n))^{-1}\sqrt{n}\left( \mathbb{E}_n - \Psi(\hat{T}_n, \theta_0) \right) + o_P(1) \tag{50}
\]
as \( p_n \in \text{int dom } H_n \) with high probability by (49).

It remains to show that \( H_n(p_n) \xrightarrow{P} \theta_0 \). To see this, note that whenever \( \hat{T}_n \in [t_0 \pm \epsilon] \), the growth condition (48) implies that

\[
\Psi(\hat{T}_n, \theta_0 + \epsilon) = P_{\theta_0 + \epsilon}(T(X) \geq \hat{T}_n) \geq P_{\theta_0}(T(X) \geq \hat{T}_n) + c\epsilon = \Psi(\hat{T}_n, \theta_0) + c\epsilon
\]

\[
\Psi(\hat{T}_n, \theta_0 - \epsilon) = P_{\theta_0 - \epsilon}(T(X) \geq \hat{T}_n) \leq P_{\theta_0}(T(X) \geq \hat{T}_n) - c\epsilon = \Psi(\hat{T}_n, \theta_0) - c\epsilon,
\]

and thus

\[
P(|H_n(p_n) - \theta_0| \geq \epsilon) \leq P(\mathbb{Z}_n - \Psi(\hat{T}_n, \theta_0) | \geq c\epsilon) + P(|\hat{T}_n - t_0| \geq \epsilon) \to 0.
\]

We have the convergence \( \hat{\Psi}_\theta(\hat{T}_n, H_n(p_n)) \xrightarrow{P} \frac{1}{2}\mathbb{E}_\theta[|T(X) - A'(\theta_0)|] = \frac{1}{2}J_{\theta_0} \) by the continuous mapping theorem, and Slutsky’s theorem applied to Eq. (50) gives the delta-method expansion (44).

### A.3.1 Proof of Lemma 7

We have

\[
\hat{\Psi}_\theta(t_0, \theta_0) - \hat{\Psi}_\theta(t, \theta) = \mathbb{E}_\theta_0[1\{T(X) \geq t_0\} (T(X) - A'(\theta_0))] - \mathbb{E}_\theta[1\{T(X) \geq t\} (T(X) - A'(\theta))]
\]

\[
= \mathbb{E}_\theta_0 [(T(X) - t_0)_+] - \mathbb{E}_\theta[1\{T(X) \geq t\} (T(X) - t + t - A'(\theta))]
\]

\[
= \mathbb{E}_\theta_0 [(T(X) - t_0)_+] - \mathbb{E}_\theta [(T(X) - t)_+] + P_\theta(T(X) \geq t)(t - A'(\theta))
\]

where step (i) follows because \( t_0 = A'(\theta_0) = \mathbb{E}_\theta_0[T(X)] \), while the inclusion (ii) is a consequence of the 1-Lipschitz continuity of \( t \mapsto |t|_+ \). Now we use the standard facts that \( A(\theta) \) is analytic in \( \theta \) and that \( \theta \mapsto \mathbb{E}_\theta[f(X)] \) is continuous for any \( f \) (cf. [36, Thm. 2.7.1]) to see that for any \( \epsilon > 0 \), we can choose \( \delta > 0 \) such that \( |t - t_0| \leq \delta \) and \( |\theta - \theta_0| \leq \delta \) imply

\[
|t - t_0| \leq \epsilon, \quad |t - A'(\theta)| \leq \epsilon, \quad \text{and} \quad \mathbb{E}_\theta_0 [(T(X) - t_0)_+] - \mathbb{E}_\theta [(T(X) - t_0)_+] \leq \epsilon.
\]

This gives the result.

### A.4 Proof of Lemma 3

Indeed, for \( P \in \mathcal{P} \) let \( \mu(P) = \mathbb{E}_P[X] \) and \( \mu_0 = \mathbb{E}_{P_0}[X] \), and define the diameter \( \mathcal{D} := \text{diam}(\mathcal{X}) = \sup\{\|x_1 - x_2\|_2 \mid x_1, x_2 \in \mathcal{X}\} \). As \( A \) and \( A^* \) are \( C^\infty \), if we define the remainder \( R(\mu; \mu') = \nabla A^*(\mu) - \nabla A^*(\mu') - \nabla^2 A^*(\mu') (\mu - \mu') \), then there exists \( G < \infty \) such that

\[
|R(\mu; \mu')| \leq \frac{1}{2}G \|\mu - \mu'\|_2 \quad \text{for all } \mu, \mu' \in \mathcal{X}.
\]

Now, define the constant

\[
K := \sup_{x \in \mathcal{X}} \|\nabla^2 A^*(\mu_0)(x - \mu_0)\|_2,
\]

which is positive whenever \( \mathcal{X} \) has at least two elements, as \( \nabla^2 A^* > 0 \) (recall Eq. (31)). If \( K = 0 \) then the example becomes trivial as \( \text{card}(\mathcal{X}) \leq 1 \). We claim that

\[
\delta K - 2G^2\mathcal{D}^2 \delta^2 \leq \omega_2(\delta) \leq 2\delta K + 2G^2\mathcal{D}^2 \delta^2
\]

(51)

A Taylor approximation is useful for this result. Define the linearized modulus

\[
\omega_2^{\text{lin}}(\delta) = \sup_{P \in \mathcal{P}} \left\{ \|\nabla^2 A^*(\mu_0)(\mu(P) - \mu_0)\|_2 : \|P - P_0\|_{TV} \leq \delta \right\}
\]

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Then the remainder guarantee $R(\mu, \mu') \leq \frac{1}{2} G \|\mu - \mu'\|_2^2$ implies
\[
\left| \omega_2(\delta) - \omega_2^{\text{lin}}(\delta) \right| \leq \frac{1}{2} G \cdot \{\|\mu - \mu_0\|_2^2 \mid \|P - P_0\|_{TV} \leq \delta \}^{(i)} \leq 2G\delta^2K^2.
\]
where inequality (i) follows from Eq. (9). The bounds on $\delta K \leq \omega_2^{\text{lin}}(\delta) \leq 2\delta K$ are immediate consequences of inequality (9), yielding the bounds (51).

We now show how inequalities (51) imply the polynomial growth condition C.2 on $\omega_2$, that is, that for some constant $\beta < \infty$,
\[
\sup_{c \geq 1, \delta > 0} \frac{\omega_2(c\delta)}{c \cdot \omega_2(\delta)} \leq \beta. \quad (52)
\]
Define $\delta_0 = \frac{K}{4G^2D^2}$. We consider three cases. In the first, when $\delta \geq \delta_0$, we have for $c \geq 1$ that
\[
\frac{\omega_2(c\delta)}{c \cdot \omega_2(\delta)} \leq \frac{\omega_2(1)}{\omega_2(\delta)} \leq \frac{2K + 2G^2D^2}{\delta_0 K - 2G^2D^2\delta_0^2} = \frac{32(K + G^2D^2)G^2D^2}{K^2}.
\]
In the case that $\delta \leq \delta_0$ and $c\delta \leq 1$, we have $\delta K - 2G^2D^2\delta^2 \geq \delta K/2$, and
\[
\frac{\omega_2(c\delta)}{c \cdot \omega_2(\delta)} \leq \frac{2c\delta K + 2G^2D^2(c\delta)^2}{c\delta K - 2cG^2D^2\delta^2} \leq \frac{4c\delta K + G^2D^2(c\delta)^2}{c\delta K} \leq 4 + 4\frac{G^2D^2}{K},
\]
and in the final case that $c\delta \geq 1$, we have $\frac{1}{c} \leq \delta$ and so
\[
\frac{\omega_2(c\delta)}{c \cdot \omega_2(\delta)} \leq \frac{\delta \omega_2(1)}{\omega_2(\delta)} \leq \frac{2K + 2G^2D^2}{\delta K/2} = 4 + 4\frac{G^2D^2}{K}.
\]
These cases combined yield inequality (52) for large enough $\beta$, and thus Condition C.2 holds.

### A.5 Proof of Lemma 4

Let us define the minimal loss function
\[
L_{i,0}(\hat{\theta}) := \inf_{P \in \mathcal{P}_1} L(\theta, P).
\]
Then by the definition of the quantity $d_L$ and assumption that $d_L(P_0, P_1) \geq \delta$ for all $P_0 \in \mathcal{P}_0, P_1 \in \mathcal{P}_1$, we immediately see that $L_{0,0}(\hat{\theta}) + L_{1,0}(\hat{\theta}) \geq \delta$ for all $\theta$. For any measure $\pi$ on $\mathcal{P}$, define $P_\pi = \int Pd\pi(P)$ to be the mixture distribution according to $\pi$. Then for any measures $\pi_0$ on $\mathcal{P}_0$ and $\pi_1$ on $\mathcal{P}_1$, we have
\[
2 \sup_{P \in \mathcal{P}} \mathbb{E}_P[L(\hat{\theta}, P)] \geq \int \mathbb{E}_P[\ell(\hat{\theta}, P)]d\pi_0(P) + \int \mathbb{E}_P[\ell(\hat{\theta}, P)]d\pi_1(P) \geq \int \mathbb{E}_P[L_{0,0}(\hat{\theta})]d\pi_0(P) + \int \mathbb{E}_P[L_{1,0}(\hat{\theta})]d\pi_1(P) = \mathbb{E}_{P_{\pi_0}}[L_{0,0}(\hat{\theta})] + \mathbb{E}_{P_{\pi_1}}[L_{1,0}(\hat{\theta})].
\]
Using the standard variational equality $1 - \|P_0 - P_1\|_{TV} = \inf_{f \geq 1} \int f(dP_0 + dP_1)$, the fact othath $L_{0,0} + L_{1,0} \geq \delta$ implies
\[
2 \sup_{P \in \mathcal{P}} \mathbb{E}_P[L(\hat{\theta}, P)] \geq \delta \inf_{f \geq 1} \{\mathbb{E}_{P_{\pi_0}}[f] + \mathbb{E}_{P_{\pi_1}}[f]\} = \delta (1 - \|P_{\pi_0} - P_{\pi_1}\|_{TV}),
\]
which is the desired result.
A.6 Proof of Theorem 2

We begin with the arguments common to each result, then specializing to prove the two inequalities claimed in the theorem. By the convexity and tensorization properties of the KL-divergence, respectively, we have

\[
D_{\text{kl}} \left( M_0^n \parallel \overline{M}^n \right) \leq \frac{1}{|V|} \sum_{v \in V} D_{\text{kl}} \left( M_0^n \parallel M_v^n \right)
\]

\[
= \frac{1}{|V|} \sum_{v \in V} \sum_{i=1}^n \int D_{\text{kl}} \left( M_0^i(\cdot | z_{1:i-1}) \parallel M_v^i(\cdot | z_{1:i-1}) \right) dM_0(z_{1:i-1})
\]

\[
= \sum_{i=1}^n \left[ \frac{1}{|V|} \sum_{v \in V} D_{\text{kl}} \left( M_0^i(\cdot | z_{1:i-1}) \parallel M_v^i(\cdot | z_{1:i-1}) \right) \right] dM_0(z_{1:i-1}), \quad (53)
\]

where \( M^i \) denotes the distribution over \( Z_i \). We consider Without loss of generality, we may assume that the \( Z_i \) are finitely supported (as all \( f \)-divergences can be approximated by finitely supported distributions [13]), and we let \( m \) and \( q \) denote the p.m.f.s of \( M \) and \( Q \), respectively. Then, as \( X_i \) is independent of \( Z_{1:i-1} \) for all \( i \in \mathbb{N} \), we obtain that

\[
m_v(z_i | z_{1:i-1}) = \int q(z_i | x_i, z_{1:i-1}) dP_v(x_i | z_{1:i-1}) = \int q(z_i | x_i, z_{1:i-1}) dP_v(x_i).
\]

Returning to expression (53) and using that \( D_{\text{kl}}(P \parallel Q) \leq \log(1 + D_{\chi^2}(P \parallel Q)) \) for any \( P \) and \( Q \) (see [46, Lemma 2.7]), we obtain

\[
\frac{1}{|V|} \sum_{v \in V} D_{\text{kl}} \left( M_0^i(\cdot | z_{1:i-1}) \parallel M_v^i(\cdot | z_{1:i-1}) \right) \leq \frac{1}{|V|} \sum_{v \in V} D_{\chi^2}(M_0^i(\cdot | z_{1:i-1}) \parallel M_v^i(\cdot | z_{1:i-1}))
\]

Thus, abstracting the entire history as \( \hat{z} \), the theorem reduces to proving upper bounds on the quantity

\[
\frac{1}{|V|} \sum_{v \in V} D_{\chi^2}(M_0^i(\cdot | z_{1:i-1}) \parallel M_v^i(\cdot | z_{1:i-1})) = \frac{1}{|V|} \sum_{v \in V} \sum_{z} \frac{(m_0(z | \hat{z}) - m_v(z | \hat{z}))^2}{m_v(z | \hat{z})}
\]

\[
= \frac{1}{|V|} \sum_{v \in V} \sum_{z} \frac{\left( \int q(z | x, \hat{z}) dP_0(x) - dP_v(x) \right)^2}{\int q(z | x, \hat{z}) dP_v(x)}. \quad (54)
\]

We provide upper bounds for the quantity (54) in the two cases of the theorem, that is, when the channel \( Q \) is \( \varepsilon \)-differentially private and when it is \( \varepsilon^2 \chi^2 \)-private.

**The differentially private case** We begin with the case in which \( q(z | x, \hat{z}) / q(z | x', \hat{z}) \leq e^\varepsilon \) for all \( x, x', z, \) and \( \hat{z} \). Let \( P \) be any distribution on \( \mathcal{X} \), which we are free to choose, and define \( m(z | \hat{z}) = \int q(z | x, \hat{z}) dP(x) \). In this case, we note that \( \int m(z | \hat{z}) (dP_0 - dP_v)(x) = 0 \) for any \( v \in V \), so that the quantity (54) is equal to

\[
\frac{1}{|V|} \sum_{v \in V} \sum_{z} \left( \int \frac{q(z | x, \hat{z}) - m(z | \hat{z})}{m_v(z | \hat{z})} (dP_0(x) - dP_v(x)) \right)^2 m_v(z | \hat{z})
\]

\[
= \sum_{z} \frac{1}{|V|} \sum_{v \in V} \left( \int \frac{q(z | x, \hat{z}) - m(z | \hat{z})}{m(z | \hat{z})} (dP_0(x) - dP_v(x)) \right)^2 m(z | \hat{z}) m_v(z | \hat{z})
\]

\[
\leq \left[ \sum_{z} \frac{1}{|V|} \sum_{v \in V} \left( \int \frac{q(z | x, \hat{z}) - m(z | \hat{z})}{m(z | \hat{z})} (dP_0(x) - dP_v(x)) \right)^2 m(z | \hat{z}) \right] \cdot \max_{v', z'} m_v'(z' | \hat{z}).
\]
Let $C = \max_{v,z} \frac{m_v(z | \tilde{z})}{m_v(z | \tilde{z})}$, which is evidently bounded by $e^\varepsilon$. For any $z, \tilde{z},$ and $x$, we have

$$\frac{q(z \mid x, \tilde{z}) - m(z \mid \tilde{z})}{m(z \mid \tilde{z})} \in [e^{-\varepsilon} - 1, 1 - e^{-\varepsilon}]$$

so we may choose the constant $c = \frac{1}{2}(e^\varepsilon + e^{-\varepsilon}) - 1$ to see that inequality (54) has the further upper bound

$$C \cdot \sum \frac{1}{|V|} \sum \left( \int \frac{q(z \mid x, \tilde{z}) - m(z \mid \tilde{z})}{m(z \mid \tilde{z})} - c \right) (dP_0(x) - dP_v(x))^2 m(z \mid \tilde{z})$$

$$\leq \frac{C}{4} (e^{\varepsilon/2} - e^{-\varepsilon/2})^2 \sup_{\|f\|_\infty \leq 1} \frac{1}{|V|} \sum \left( \int f(x)(dP_0(x) - dP_v(x))^2 m(z \mid \tilde{z}) \right)$$

$$= \frac{C}{4} (e^{\varepsilon/2} - e^{-\varepsilon/2})^2 C_{\chi^2}(\{P_v \mid \sum\} \in V).$$

Recalling that for our choice of $C = \max_{v,z} \frac{m(z | \tilde{z})}{m_v(z | \tilde{z})}$, we have $C \leq e^\varepsilon$ and $C \leq \max_v \sup_x \frac{dP_0(x)}{dP_v(x)}$, we obtain the desired result (36) for the differentially private case.

**The $\chi^2$-private case**  In the second case, when the channel is $\chi^2$-private, we require a slightly different argument. We begin by using the convexity of the function $t \mapsto 1/t$ and, as in the proof of the differentially private case, that $\int c(dP_0 - dP_v) = 0$, to obtain

$$\frac{(m_v(z \mid \tilde{z}) - m_0(z \mid \tilde{z}))^2}{m_v(z \mid \tilde{z})} = \inf_{x_0} \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x_0, \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z} \| P_0(x') \int q(z \mid x', \tilde{z})dP_0(x') \right)}$$

$$\leq \int \inf_{x_0} \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x_0, \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z})} \int q(z \mid x', \tilde{z})dP_0(x') \right)$$

$$\leq \int \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x', \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z})} \right) dP_0(x').$$

Now, let $P$ be an arbitrary distribution on $X$. As a consequence of the preceding display, we may upper bound the average (54), using the shorthand $\tilde{z} = z_{1:i-1}$ for the history $z_{1:i-1}$, by

$$\frac{1}{|V|} \sum_{v \in V} D_{\chi^2}(M_0(\cdot \mid \tilde{z}) \| M_v(\cdot \mid \tilde{z}))$$

$$\leq \sum_{q} \frac{1}{|V|} \sum_{v \in V} \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x', \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z})} \int q(z \mid x', \tilde{z})dP_0(x') \right)$$

$$\leq \sup_{v,x} \frac{dP_v(x)}{dP(x)} \left[ \int \sum_{q} \frac{1}{|V|} \sum_{v \in V} \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x', \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z})} \right) dP_0(x') \right].$$

Rearranging, the final expression is equal to

$$\max_v \left( \frac{dP_v}{dP} \right)_{\infty} \cdot \left[ \int \sum_{q} \frac{1}{|V|} \sum_{v \in V} \left( \frac{\int (q(z \mid x, \tilde{z}) - q(z \mid x', \tilde{z}))(dP_0(x) - dP_v(x))^2}{q(z \mid x', \tilde{z})} \right) q(z \mid x', \tilde{z})dP(x') \right]$$

$$= \max_v \left( \frac{dP_v}{dP} \right)_{\infty} \cdot \left[ \int \sum_{q} \frac{1}{|V|} \sum_{v \in V} \left( \int (\Delta(z \mid x, x', \tilde{z})(dP_0(x) - dP_v(x))^2 q(z \mid x', \tilde{z})dP(x') \right) \right],$$

(55)
where in the equality (55) we defined the quantity

$$\Delta(z \mid x, x', \bar{z}) := \frac{q(z \mid x, \bar{z})}{q(z \mid x', \bar{z})} - 1.$$  

For any distribution $P^*$ supported on $X$, we can further upper bound the innermost summation over $v \in V$ in expression (55) by

$$\frac{1}{|V|} \sum_{v \in V} \left( \int \Delta(z \mid x, x', \bar{z})(dP_0(x) - dP_v(x)) \right)^2 \leq \max_v \left\{ \frac{1}{|V|} \sum_{v \in V} \left( \int f(x)(dP_0(x) - dP_v(x)) \right)^2 \mid \int f(x)^2 dP^*(x) \leq \int \Delta(z \mid x, x', \bar{z})^2 dP^*(x) \right\}.$$  

Recall the definition (35) of $C_2$, and assume without loss of generality that the supremum in the quantity is attained by $P^*$. In this case, we then obtain

$$\frac{1}{|V|} \sum_{v \in V} \left( \int \Delta(z \mid x, x', \bar{z})(dP_0(x) - dP_v(x)) \right)^2 \leq C_2(\{P_v\}_{v \in V}) \cdot \int \Delta(z \mid x, x', \bar{z})^2 dP^*(x).$$  

Substituting this into the upper bound (55) and applying Fubini’s theorem, we obtain

$$\frac{1}{|V|} \sum_{v \in V} \left( D_{X^2}(M_0(\cdot \mid \bar{z}) \| M_v(\cdot \mid \bar{z})) \right) \leq \max_v \left\{ \frac{dP_v}{dP} \right\}_\infty C_2(\{P_v\}_{v \in V}) \int \left[ \sum_z \Delta(z \mid x, x', \bar{z})^2 q(z \mid x', \bar{z}) \right] dP(x') dP^*(x) = \max_v \left\{ \frac{dP_v}{dP} \right\}_\infty C_2(\{P_v\}_{v \in V}) \int \left[ D_{X^2}(Q(\cdot \mid x, \bar{z}) \| Q(\cdot \mid x', \bar{z})) \right] dP(x') dP^*(x).$$  

Of course, by assumption we know that $D_{X^2}(Q(\cdot \mid x, \bar{z}) \| Q(\cdot \mid x', \bar{z})) \leq \epsilon^2$, which gives the result (37) by way of equality (54).

### A.7 Proof of Proposition 9

As in our proof of Proposition 8 (Section 5.3) we construct distributions for which the parameters are well-separated but for which the complexity measure $C_2$ is reasonably small. With this in mind, let $\theta_0 \geq 0$ without loss of generality, and consider the base distribution $P_0$ defined by the conditional p.m.f.

$$p_0(y \mid x) = \frac{1}{1 + \exp(-y\theta_0)}$$  

where we assume $X \sim \text{Uniform}([-1,1]^d)$. Now we construct the well-separated packing set by, as in the proof of Proposition 8, setting $\mathcal{V} = \{\pm e_j\}_{j=1}^d$ and for a parameter $\delta \in [0,1]$ to be chosen, we set

$$p_v(y \mid x) = \frac{1}{1 + \exp(-y(\theta_0 + \delta v^T x))};$$  

letting $X \sim \text{Uniform}([-1,1]^d)$ as in the case of $P_0$. With this setting, we have $\theta_v = \theta(P_v) = \delta v$.

Our starting point is the following lemma, which controls the complexity measure $C_2$ for this class. (See Appendix A.7.1 for a proof.)
Lemma 8. Let $P_0$ and $P_v$ be as above, and define
\[ \alpha = \frac{e^{\theta_0}}{e^{\theta_0} + 1} - \frac{e^{\theta_0}}{e^{\theta_0} + e^{-\delta}} \quad \text{and} \quad \beta = \frac{e^{\theta_0}}{e^{\theta_0} + 1} - \frac{e^{\theta_0}}{e^{\theta_0} + e^\delta}. \tag{56} \]

Then
\[ C_2(\{P_v\}_{v \in \mathcal{V}}) \leq 2 \max \left\{ \frac{(\alpha - \beta)^2}{d}, (\alpha + \beta)^2 \right\}. \]

Inspecting this quantity, we obtain that for $\theta_0 \geq 0$, we have
\[ 0 \leq \alpha + \beta = \frac{e^{\theta_0}(e^{\theta_0} - 1)}{1 + e^{\theta_0} - e^{\theta_0} + e^{\theta_0} + e^{\theta_0^2} + e^{2\theta_0} - e^{2\theta_0} + e^{2\theta_0^2} + e^{3\theta_0}} \leq e^{-\theta_0}(e^{\theta_0} - 1), \]
\[ \text{while} \]
\[ 0 \leq \beta - \alpha = \frac{e^{\theta_0}(e^{\theta_0} - e^{-\delta})}{e^{\theta_0} + e^{\theta_0} + e^{\theta_0^2} + e^{2\theta_0} - e^{2\theta_0} + e^{2\theta_0^2} + e^{3\theta_0}} \leq e^{-\theta_0}(e^{\theta_0} - e^{-\delta}). \]

For $0 \leq \delta \leq 1$, we thus obtain that $0 \leq \alpha + \beta \leq 2e^{-\theta_0}(1 - e^{-\theta_0})\delta^2$ and $0 \leq \beta - \alpha \leq 2e^{-\theta_0}\delta$, so that in this regime, we have the complexity bound
\[ C_2(\{P_v\}_{v \in \mathcal{V}}) \leq \frac{8\delta^2}{e^{-\theta_0}} \max \left\{ d^{-1}, (1 - e^{-\theta_0})^2 \delta^2 \right\}. \tag{57} \]

Using the upper bound (57), we can substitute into Theorem 2, choosing $P = P_0$ in the theorem so that $\|dP_v/dP_0\|_\infty \leq 1 + e^{\delta} \leq 4$ for $\delta \leq 1$; we thus obtain
\[ \left\| M_0^n - \overline{M}^n \right\|_{TV}^2 \leq \frac{1}{2} D_{kl} \left( M_0^n || \overline{M}^n \right) \leq \frac{16n\varepsilon^2}{e^{2\theta_0}} \max \left\{ d^{-1}, (1 - e^{-\theta_0})^2 \delta^2 \right\} \delta^2. \]

If we solve for $\delta^2$ to obtain $\left\| M_0^n - \overline{M}^n \right\|_{TV} \leq \frac{1}{2}$, we see that the choice
\[ \delta_n^2 := \min \left\{ \frac{de^{2\theta_0}}{64n\varepsilon^2}, \frac{e^{\theta_0}}{8(1 - e^{-\theta_0})^2 \sqrt{n}\varepsilon^2}, 1 \right\} \]
is sufficient to guarantee that $\left\| M_0^n - \overline{M}^n \right\|_{TV} \leq \frac{1}{2}$. Using Lemma 5 and the separation lower bound (39), we obtain the minimax lower bound
\[ \mathcal{M}_n \geq \min_{v \in \mathcal{V}} \frac{1}{2} \Phi(\delta_n \psi(v)/2) = \min_{j=1,\ldots,d} \frac{1}{2} \Phi(\delta_n \psi(e_j)). \]

A.7.1 Proof of Lemma 8

We take $P^*$ in the definition (35) to be uniform on $(x, y) \in \{-1, 1\}^d \times \{-1, 1\}$. We then have
\[ C_2(\{P_v\}) \leq \sup_{f: P^* f^2 \leq 1} \frac{1}{4^d \cdot d} \sum_{v \in \mathcal{V}} \left( \sum_{x \in \{\pm 1\}^d} \sum_{y \in \{\pm 1\}} f(x, y)(p_0(y \mid x) - p_v(y \mid x)) \right)^2 \]
\[ \leq \sup_{f: P^* f^2 \leq 1} \frac{1}{4^d \cdot d} \sum_{v \in \mathcal{V}} \left( \sum_{x \in \{\pm 1\}^d} f(x)(p_0(1 \mid x) - p_v(1 \mid x)) \right)^2 + \cdots \]
\[ \sup_{f: P^* f^2 \leq 1} \frac{1}{4^d \cdot d} \sum_{v \in \mathcal{V}} \left( \sum_{x \in \{\pm 1\}^d} f(x)(p_0(-1 \mid x) - p_v(-1 \mid x)) \right)^2, \tag{58} \]
where we have used that \((a + b)^2 \leq 2a^2 + 2b^2\). Fixing \(f\) temporarily, we consider the terms inside the squares, recalling that \(v \in \{\pm e_j\}_{j=1}^d\). For a fixed \(v \in \{\pm e_j\}\), because \(x\) is uniform, we have

\[
\sum_{x\in\{\pm 1\}^d} f(x)(p_0(1 \mid x) - p_v(1 \mid x))
\]

\[
= \sum_{x:v_jx_j=1} f(x) \left[ \frac{e_{\theta_0}}{1 + e_{\theta_0}} - \frac{e_{\theta_0}}{e_{\theta_0} + e^{-\beta}} \right] + \sum_{x:v_jx_j=-1} f(x) \left[ \frac{e_{\theta_0}}{1 + e_{\theta_0}} - \frac{e_{\theta_0}}{e_{\theta_0} + e^{\beta}} \right]
\]

\[
= \sum_{x:v_jx_j=1} f(x)\alpha + \sum_{x:v_jx_j=-1} f(x)\beta = \sum_x f(x)\alpha T x (\alpha - \beta) + \sum_x f(x)\alpha + \beta \frac{1}{2},
\]

where \(\alpha\) and \(\beta\) are as defined in expression (56). A similar calculation yields that

\[
\sum_x f(x) (p_0(-1 \mid x) - p_v(-1 \mid x)) = - \sum_{x:v_jx_j=1} f(x)\alpha - \sum_{x:v_jx_j=-1} f(x)\beta
\]

\[
= \sum_x f(x)\beta T x (\beta - \alpha) - \sum_x f(x)\alpha + \beta \frac{1}{2}.
\]

Returning to expression (58), we thus obtain

\[
C_2(\{P_v\}) \leq \sup_{f: \mathbb{E}[f(X)T] \leq 1} \frac{1}{2 \cdot 4^d} \frac{1}{d} \sum_{v \in V} \left( \sum_x f(x)vT x (\alpha - \beta) + \sum_x f(x) (\beta + \alpha) \right)^2
\]

\[
\leq \sup_{f: \mathbb{E}[f(X)T] \leq 1} \frac{1}{d} \left( \sum_{v \in V} (\alpha - \beta)^2 \mathbb{E}[f(X)XTV] + (\alpha + \beta)^2 \mathbb{E}[f(X)^2] \right)
\]

where \(X \sim \text{Uniform}\{\pm 1\}^d\). Finally, we use that \(\sum_{v \in V} vvT = 2I_{d \times d}\) to obtain that

\[
C_2(\{P_v\}) \leq \frac{2}{d} \sup_{f: \mathbb{E}[f(X)T] \leq 1} \left( (\alpha - \beta)^2 \mathbb{E}[f(X)X] \right) + \frac{1}{d} (\alpha + \beta)^2 \mathbb{E}[f(X)^2].
\]

As the last step, we apply an argument analogous to that in the proof of Proposition 8 to bound the expectations involving \(f(X)\). Indeed, for any \(a, b \geq 0\) and \(\mathbb{E}[f(X)^2] = 1\), we have

\[
a^2 \mathbb{E}[f(X)X]_2^2 + b^2 \mathbb{E}[f(X)^2] = \sup_{\|w\|_2 \leq 1} \mathbb{E} f(X) \left[ \begin{array}{c} aX \\ b \end{array} \right] w^T \left[ \begin{array}{c} a^2 f \\ 0 \end{array} \right] w = \max\{a^2, b^2\}.
\]

Returning to the preceding bound on \(C_2\), we find that

\[
C_2(\{P_v\}) \leq 2 \max \left\{ \frac{(\alpha - \beta)^2}{d}, (\alpha + \beta)^2 \right\}.
\]

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