Phase Transitions of the $k$-Majority Dynamics in a Biased Communication Model

Emilio Cruciani
Inria, I3S, UCA, CNRS
Sophia Antipolis, France
emilio.cruciani@inria.fr

Hlafo Alfie Mimun
Matteo Quattropani
LUISS Guido Carli
Rome, Italy

Sara Rizzo
Gran Sasso Science Institute
L’Aquila, Italy
sara.rizzo@gssi.it

{hmimun,mquattropani}@luiss.it

ABSTRACT

Consider a graph where each of the $n$ nodes is in one of two possible states, say $\mathcal{R}$ or $\mathcal{B}$. Here, we analyze the synchronous $k$-majority dynamics, where nodes sample $k$ neighbors uniformly at random with replacement and adopt the majority binary state among the nodes in the sample (potential ties are broken uniformly at random). This class of dynamics generalizes other well-known dynamics, e.g., voter and 3-majority, which have been studied in the literature as distributed algorithms for consensus.

We consider a biased communication model: whenever nodes sample a neighbor they see, w.l.o.g., state $\mathcal{B}$ with some probability $p$, regardless of the state of the sampled node, and its true state with probability $1-p$. Such a communication model allows to reason about the robustness of a consensus protocol when communication channels between nodes are noisy. Differently from previous works where specific graph topologies—typically characterized by good expansion properties—are considered, our analysis only requires the graphs to be sufficiently dense, i.e., to have minimum degree $\omega(\log n)$, without any further topological assumption.

In this setting we prove two phase transition phenomena, both occurring asymptotically almost surely, depending on the bias $p$ and on the initial unbalance toward state $\mathcal{B}$. More in detail, we prove that for every $k \geq 3$ there exists a $p_k^*$ such that if $p > p_k^*$ the process reaches in $O(1)$ rounds a $\mathcal{B}$-almost-consensus, i.e., a configuration where a fraction $1-\gamma$ of the volume is in state $\mathcal{B}$, for any arbitrarily-small positive constant $\gamma$. On the other hand, if $p < p_k^*$, we look at random initial configurations in which every node is in state $\mathcal{B}$ with probability $1-q$ independently of the others. We prove that there exists a constant $q_{p,k}^*$ such that if $q < q_{p,k}^*$ then a $\mathcal{B}$-almost-consensus is still reached in $O(1)$ rounds, while, if $q > q_{p,k}^*$ the process spends $n^{\omega(1)}$ rounds in a metastable phase where the fraction of volume in state $\mathcal{B}$ is around a constant value depending only on $p$ and $k$.

Finally we also investigate, in such a biased setting, the differences and similarities between $k$-majority and other closely-related dynamics, namely voter and deterministic majority.

CCS CONCEPTS

• Theory of computation → Random walks and Markov chains; Distributed algorithms; • Mathematics of computing → Probabilistic algorithms; Markov processes.

KEYWORDS

Biased Communication, Consensus, Majority Dynamics, Markov Chains, Metastability

ACM Reference Format:
Emilio Cruciani, Hlafo Alfie Mimun, Matteo Quattropani, and Sara Rizzo. 2021. Phase Transitions of the $k$-Majority Dynamics in a Biased Communication Model. In International Conference on Distributed Computing and Networking 2021 (ICDCN ’21), January 5–8, 2021, Nara, Japan. ACM, New York, NY, USA, 10 pages. https://doi.org/10.1145/3427796.3427811

1 INTRODUCTION

Designing distributed algorithms that let the nodes of a graph reach a consensus, i.e., a configuration of states where all the nodes agree on the same state, is a fundamental problem in distributed computing and multi-agent systems. Consensus algorithms are used in protocols for other tasks, such as leader election and atomic broadcast, and in real-world applications such as clock synchronization tasks and blockchains.

Recently there has been a growing interest in the analysis of simple local dynamics as distributed algorithms for the consensus problem [8, 14, 16, 22, 30, 31, 41], inspired by simple mechanisms studied in statistical mechanics for interacting particle systems [38]. In this scenario, nodes are anonymous (i.e., they do not have distinct IDs) and they have a state that evolves over time according to some common local interaction with their neighbors.

The first dynamics investigated with this goal has been the voter [31], where nodes copy the state of a random neighbor. The authors prove that the dynamics reaches a consensus on state $\sigma$ with probability proportional to the volume of nodes initially in state $\sigma$ in $O(n^3 \log n)$ rounds, regardless of the graph structure. Recently the upper bound has been improved to $O(n^3)$ [34], which is shown to be tight. However, the dynamics is slow in reaching consensus, e.g., it needs $\Omega(n)$ rounds in the complete graph despite the extremely good connectivity properties of the topology. Therefore, simple generalizations of voter have been considered in order to achieve a faster distributed algorithm for consensus. One of the directions has been that of considering more than a single neighbor in the sample. For example, in the 3-majority dynamics, where each node samples 3 neighbors with replacement and updates its state to the most frequent state among those in the sample, the time needed to reach a consensus on the complete graph lowering from $\Omega(n)$ to
We formally present the communication model in Section 3 and we account to extremely dense ones.

In Section 5.2. Clearly in the slow convergence regime, where a different initial configuration with some of the nodes already in state $\mathcal{B}$ could be enough to change the behavior of the dynamics. In this more general scenario, we prove a second phase transition phenomenon.

Informal description of Theorem 5.7.

Consider the $k$-MAJORITY dynamics in the communication model with bias $p$ and where initially each node is in state $\mathcal{B}$ with probability $1 - q$, independently of the others. For every $k \geq 3$ there exists a constant $q_{p,k}^*$ such that:

- If $q < q_{p,k}^*$, then a $\mathcal{B}$-almost-consensus is reached in $O(1)$ rounds, a.a.s.
- If $q > q_{p,k}^*$, then a $\mathcal{B}$-almost-consensus is reached in $O(n\log n)$ rounds, a.a.s.

Note that the notation $q_{p,k}^*$ is different from the one appearing in Section 5.2 in order to make clearer this introductory exposition.

In Section 4 we prove essentially the same result on directed infinite trees. In that case the phase transition occurs on the probability of the root to be in the state not promoted by the bias as the number of rounds goes to infinity. The result in this “toy model” is presented before the others and used as a guideline along the paper: the intuitions behind the proof are the same as those of the previously presented results, but the calculations are much less technically involved.

In Section 6, we characterize the behavior of VOTER and of DETERMINISTIC MAJORITY, where nodes simply update to the majority state in their neighborhood, under the same biased framework. For the former we observe no phase transition but only a quick convergence to a $\mathcal{B}$-almost-consensus in $O(1)$ rounds, asymptotically
almost surely (Proposition 6.1). The latter, instead, exhibits again a
sharp transition, as that described before for \( k\)-MAJORITY, on the
critical value \( p^* = \frac{1}{k} \) (Proposition 6.2). We also discuss, in Proposition 6.3, differences and similarities between this dynamics and \( k\)-MAJORITY for large values of \( k \).

From a high level perspective, our results show that adding a bias to \( k\)-MAJORITY affects the dynamics in a non-trivial way. In particular, the arise of a metastable phase makes the framework suitable to design distributed algorithms to recover planted partitions in networks [10, 20, 44]. In this direction, we discuss such a potential application of the mathematical framework presented in this paper in Section 7.

Some proofs are not reported in the main text due to space limitations, but they can be found in the full version of the paper, which is available here: https://arxiv.org/abs/2007.15306.

2 RELATED WORK

Dynamics for consensus. Simple mathematical model of interaction between nodes in a network have been studied since the first half of the 20th century in statistical mechanics, e.g., to model interacting particle systems or ferromagnetism phenomena [38]. The simplest dynamics of interaction between the nodes involve local majority-based changes of states, e.g., as in the voter model [23, 33] or in the majority dynamics [37]. A substantial line of research has been devoted to study the use of such simple dynamics as light-weight distributed algorithms to solve complex tasks, mirroring the behavior of complex systems from which they take inspiration. Here we are interested in discussing some of the contributions among the large body of work on dynamics for consensus. The reader is deferred to [7] for a more detailed survey on the topic. All dynamics taken into consideration share a common communication model, where nodes can pull information from some fixed number of neighbors before updating their state.

As discussed in Section 1, voter is the first—and arguably the simplest—dynamics considered for consensus [31]. The 2-CHOICES dynamics is a variation of voter in which nodes sample two random neighbors and update their states to the majority among two, breaking ties toward their own states. The dynamics has been studied with opinions on \( d\)-regular and expander graphs [16], proving that, given a sufficient initial unbalance between the two opinions, a consensus on the initial majority is reached within a polylogarithmic number of rounds, with high probability. Such results have been later improved in [17], relaxing the assumptions on graph’s expansion, and generalized to the case of multiple opinions [18, 25]. More recently, the 2-CHOICES dynamics has been analyzed on networks with a core-periphery structure [19], where, depending on the initialization, it exhibits a phase transition phenomenon.

In the 3-MAJORITY dynamics, nodes update their color to that of the majority among the state of 3 randomly sampled neighbors. On the complete graph and with \( h \) possible opinions, the process converges to a plurality consensus in \( O(\min(h, \sqrt{n}/\log n) \cdot \log n) \) with high probability, if the initial unbalance between the plurality color and the second one is large enough [8]. In [12] unconditional lower and upper bounds for 2-CHOS and 3-MAJORITY on the complete graph are provided, whenever the number of initial colors is large. The scenario in which an adversary can modify some of the \( h \) opinions, again for 2-CHOICES and/or 3-MAJORITY, is considered in [9, 22, 30, with the best result proving convergence to a valid consensus in \( O(h \log n) \) rounds, with high probability, even if the adversary can control \( \sqrt{h(n)} \) nodes. The 3-MAJORITY dynamics has been recently analyzed also on non complete topologies [35]. The authors consider a random initialization in which every node is red with probability \( 1/2 + \delta \) and blue otherwise and graphs with minimum degree \( d = \Omega(n^{1/\log \log n}) \). Their result implies, e.g., a consensus on state red in \( O(\log \log n) \) rounds, w.h.p., if \( \delta = \Omega(1/\log \log n) \).

To the best of our knowledge, \( k\)-MAJORITY has not been extensively studied for generic \( k \). Among the few works that consider it, in [1, 2] such a dynamics is analyzed for \( k \geq 5 \) on preferential attachment graphs, on the configuration model, and on Erdős–Rényi graphs. In both works the authors show that, given an initial unbalance toward one of the two possible colors, the process converges to the initial majority within \( O(\log \log n) \) steps, with high probability. In [45] a new model is proposed, which contains majority rules as special cases. In particular, for \( k\)-MAJORITY with odd \( k \) and in a binary state setting, the convergence time on expander graphs is proved to be \( O(\log n/\log h) \) rounds for \( k = o(n/\log n) \).

In the deterministic majorities, every node updates its state according to the majority state of its neighborhood as a whole, loosing the random interaction, which is a fundamental feature of the dynamics previously discussed. This deterministic protocol has been extensively studied in the literature; we mention, for example, its analysis on expander graphs [41, 47], random regular graphs [29], and Erdős–Rényi random graphs [11, 47].

Biased communication models in opinion dynamics. A different perspective coming from other disciplines, such as economics and sociology, is that of considering interaction models between nodes of a network as models of opinion diffusion. The main models, e.g., DeGroot [21] and Friedkin–Johnsen [28], are based on averaging dynamics, i.e., nodes move toward the average opinion seen in their neighborhoods. Nevertheless, also the dynamics previously discussed can be framed in the modeling of opinion dynamics (see, e.g., [3, 5, 40, 42, 46]).

Some opinion dynamics have been considered with biased communication models, specially in asynchronous case. The binary deterministic majorities have been considered in a setting where nodes have a fixed private opinion and, when active, announce a public opinion as the majority opinion in their neighborhood, but ties are broken toward their private belief. Such a process has been proved to converge to the initial private majority whenever the graphs are sufficiently sparse and expansive [26] or preferential attachment trees [6]. The binary deterministic majorities, as well as the binary voter dynamics, have also been analyzed in asynchronous models presenting different forms of bias [4, 43]. In [43], if the network prefers, say, opinion \( a \) instead of \( b \), every node holding opinion \( b \) updates more frequently than the others; this particular feature is modeled by allowing nodes in state \( b \) to revise their opinion at all points of a Poisson process with rate \( q_b > q_a \). In [4] the bias is defined toward one of the two possible opinions: nodes have a fixed probability \( a \) of updating their state to such an opinion, independently of the dynamics.
A biased version of voter has also been studied in a synchronous model in [13]: the nodes, after selecting a random neighbor, have a probability of copying its state that depends on the state itself.

3 NOTATION AND COMPUTATIONAL MODEL

Let $G = (V, E)$ be a simple graph with $V = \{1, \ldots, n\}$. For each node $u \in V$, let $N_u := \{v \in V : (u, v) \in E\}$ be the neighborhood of $u$. In the following, we focus only on sufficiently dense graphs, i.e., graphs where every node $u \in V$ has degree $\delta_u := |N_u| = \omega(\log n)$. We use the Bachmann–Landau notation (i.e., $\omega$, $\Omega$, $\Theta$, $O$, $o$) to describe the limiting behavior of functions depending on $n$. We denote the volume of a set of nodes $T \subseteq V$ as $\text{vol}(T) := \sum_{u \in T} \delta_u$.

We consider a process on $G$ that evolves in discrete, synchronous rounds, where, in every round $t \in \mathbb{N}_0$, every node $u \in V$ has a binary state $x_u^{(t)} \in \{\mathcal{R}, \mathcal{B}\}$ that can change over time according to a function of the states of its neighbors; we denote the configuration of the system at round $t$, i.e., the vector of states of the nodes of $G$, as $\mathbf{x}^{(t)} \in \{\mathcal{R}, \mathcal{B}\}^n$; we define $\mathcal{R}^{(t)} := \{u \in V : x_u^{(t)} = \mathcal{R}\}$ and $\mathcal{B}^{(t)} := \{u \in V : x_u^{(t)} = \mathcal{B}\}$.

We let the nodes of the network run the $(k, p, \mathcal{B})$-majority dynamics (formally introduced in Definition 3.1), a slight modification of the well-known $k$-majority dynamics where the communication between nodes is biased (through a parameter $p$) toward one of the two possible states, w.l.o.g. $\mathcal{B}$, as if nodes were communicating across Z-channels and thus having noise affecting the information sent [39]. Recall that in the $(k, \mathcal{B})$-majority dynamics, in each round, every node samples $k$ neighbors uniformly at random and with replacement; then it updates its state to the state held in the previous round by the majority of the neighbors in the sample; ties are broken uniformly at random. Differently, in the $(k, p, \mathcal{B})$-majority dynamics, whenever node $u$ samples a neighbor $v$, the state of $v$ seen by $u$ could be altered, i.e., $v$ is seen in state $\mathcal{B}$, with probability $p$, regardless of its actual state.

We denote by $P = P^{(n)}$ the law of the dynamics on $G$. We usually drop the dependence on $n$ when it is clear from the context. We use the notation $\mathbbm{1}_{\{A\}}$ for the indicator variable of the event $A$, i.e., $\mathbbm{1}_{\{A\}} = 1$ if $A$ holds and $\mathbbm{1}_{\{A\}} = 0$ otherwise.

Formally, $(k, p, \mathcal{B})$-majority can be described as follows. Let $p \in [0, 1]$ be the parameter that models the bias toward state $\mathcal{B}$; let $k \in \mathbb{N}$ be the size of the sampling. For each round $t$, let $S_u^{(t)}$ be the multi-set of neighbors sampled by node $u$ in round $t$. For each sampled node $v \in S_u^{(t)}$, we call $\tilde{x}_v^{(t)}(u)$ the state in which node $u$ sees $v$ after the effect of the bias, i.e., $\tilde{x}_v^{(t)}(u) = \mathcal{B}$ with probability $p$, independently of the state of $v$, and $\tilde{x}_v^{(t)}(u) = \mathcal{R}$ otherwise; formally

$$P\left(\tilde{x}_v^{(t)}(u) = \mathcal{B} \mid x_v^{(t)} = \tilde{x}\right) = \mathbbm{1}_{\{x_v^{(t)} = \tilde{x}\}} \cdot p + \mathbbm{1}_{\{x_v^{(t)} \neq \mathcal{B}\}}.$$ 

We define $\tilde{\mathcal{R}}^{(t)} := \{v \in S_u^{(t)} : \tilde{x}_v^{(t)}(u) = \mathcal{R}\}$ and $\tilde{\mathcal{B}}^{(t)} := \{v \in S_u^{(t)} : \tilde{x}_v^{(t)}(u) = \mathcal{B}\}$ as the sets that node $u$ sees respectively in state $\mathcal{R}$ and in state $\mathcal{B}$ after the effect of the bias $p$.

Definition 3.1 ($(k, p, \mathcal{B})$-majority dynamics). Let $p \in [0, 1]$, $k \in \mathbb{N}$. Starting from an initial configuration $x^{(0)}$, at each round $t$ every node $u \in V$ decides its state for the next round as

$$x_u^{(t+1)} = \begin{cases} \mathcal{R} & \text{if } |\tilde{\mathcal{R}}^{(t)}| > |\tilde{\mathcal{B}}^{(t)}|, \\ \mathcal{R} \text{ or } \mathcal{B} \text{ with probability } 1/2 & \text{if } |\tilde{\mathcal{R}}^{(t)}| = |\tilde{\mathcal{B}}^{(t)}|, \\ \mathcal{B} & \text{if } |\tilde{\mathcal{R}}^{(t)}| < |\tilde{\mathcal{B}}^{(t)}|. \end{cases}$$

Note that the $(k, p, \mathcal{B})$-majority dynamics is a Markov Chain, since the configuration $x^{(t)}$ in a round $t > 0$ depends only on the configuration $x^{(t-1)}$ at the previous round. Moreover, when $p > 0$, it has a single absorbing state, in which $x_u^{(t)} = \mathcal{B}$ for every $u \in V$, i.e., eventually all nodes reach a consensus on state $\mathcal{B}$. Indeed, being the configuration space finite for every choice of $n$, it is easy to prove that the probability of jumping to such a configuration is positive when $p$ is positive, e.g., giving a simple lower bound of $p^{ka}$ to the event that all nodes see all their neighbors in state $\mathcal{B}$ in any given round. Moreover, such a configuration is the unique absorbing state because it is the only in which, once there, the nodes cannot change their state to $\mathcal{R}$ having no chance to see a neighbor in state $\mathcal{R}$. Notice also that our dynamics is monotone and hence the time needed to reach the $\mathcal{B}$-consensus decreases if we increase the number of vertices in state $\mathcal{B}$ in the current configuration. For this reason, in Section 5.1 we consider the scenario in which all nodes are initially in state $\mathcal{R}$. Later, in Section 5.2, we analyze the general scenario in which nodes are initially in state $\mathcal{R}$, independently, with probability $q$. We examine these scenarios in order to understand the effect of the bias and of the initial imbalance toward state $\mathcal{R}$ on the process. In particular, we study the time needed by the process to reach a $\mathcal{B}$-almost-consensus, i.e., a configuration in which most of the nodes are in state $\mathcal{B}$, that we define as follows.

Definition 3.2 ($\mathcal{B}$-almost-consensus). For any constant $\gamma \in (0, 1)$, consider the stopping time

$$\tau_\gamma := \inf \left\{t \geq 0 \mid \frac{\text{vol}(\mathcal{B}^{(t)})}{\text{vol}(V)} > 1 - \gamma \right\}.$$ 

We say that a process on $G$ reaches a $\mathcal{B}$-almost-consensus within $O(f(n))$ rounds if for any constant $\gamma > 0$ there exists some constant $c = c(\gamma)$ such that

$$\mathbb{P}(\tau_\gamma \leq cf(n)) = 1 - o(1).$$

Note that the choice of volume instead of number of nodes is equivalent in case the underlying graph is regular.

In the following sections, we say that an event $E_n$ holds asymptotically almost surely (a.a.s., in short) if $\mathbb{P}(E_n) = 1 - o(1)$. In this sense, our results only hold for large $n$. We also use the notation $\text{Bin}(n, p)$ to indicate a random variable sampled from the Binomial distribution of parameters $n$ (number of trials) and $p$ (probability of success).

4 A TOY MODEL: INFINITE TREES

We start the analysis with a "toy model", i.e., a directed infinite complete balanced k-ary tree $T = (V, E)$ with edges oriented toward the children. We call $v_0 \in V$ the root of $T$. For each node $u \in V$ we define the binary state $x_u^{(t)} \in \{\mathcal{R}, \mathcal{B}\}$ that can change over time as described in Definition 3.1, but with the only difference that...
$S_u \equiv S_u^{(t)}$ will be the set of the $k$ children of node $u$, independently of $t$; this difference in the dynamics will allow us to analyze the behavior of the processes using the same techniques. Hence, fixed the bias $p \in [0, 1]$, for each $u \in S_u$ we can define the binary random variable $x_u^{(t)}(u)$ and at each round $t$ we can construct the sets $R^{(t)}_u$ and $B^{(t)}_u$ as discussed in Section 3.

Suppose that $k$ is odd. We study the evolution of the probability of the event “$v_0 \in R^{(t)}_u$”, i.e., the root of $T$ is in state $R$ at round $t$. The result in the forthcoming Theorem 4.3 is based on the analysis of the function $F_{p,k}$, described by the following definition, that represents the evolution of the event under analysis.

**Definition 4.1 (Function $F_{p,k}$).** Let $h \in \mathbb{N}$ and let $k := 2h + 1$. Let $p \in [0, 1]$. We define the function $F_{p,k} : [0, 1] \rightarrow [0, 1]$ as

$$F_{p,k}(x) := \mathbb{P}\left(\text{Bin}(k, (1-p)x) \geq \frac{k + 1}{2}\right).$$

In particular, we will use the following facts about $F_{p,k}$, which are depicted in Figure 1.

**Lemma 4.2.** For every finite odd $k \geq 3$, there exists $p_k^* \in [\frac{1}{2}, \frac{3}{4})$ such that:

- if $p < p_k^*$, there exist $\phi_{p,k}^- > \phi_{p,k}^+$ such that $F_{p,k}(x) = x$ has solutions $0, \phi_{p,k}^-$ and $\phi_{p,k}^+$;
- if $p = p_k^*$, there is an $\phi_{p,k} \in (\frac{1}{2(1-p)^3}, \frac{1}{1-p})$ such that $F_{p,k}(x) = x$ has only solutions $0, \phi_{p,k}$;
- if $p > p_k^*$, then $F_{p,k}(x) = x$ has $0$ as unique solution.

Moreover the sequence $(p_k^*)_k$ is increasing.

Note that it is not possible to give a closed formula of $p_k^*$ for generic $k$ because it is the root of a polynomial of degree $k$; numerical approximations can be computed for any given $k$. However, as proved in Lemma 4.2, $p_k^*$ monotonically increases with $k$, starting from $\frac{3}{8}$ (for $k = 3$) and up to $\frac{3}{4}$ (its limit value as $k \rightarrow \infty$).

We now prove the following result, which will be a guideline for the whole paper.

**Theorem 4.3.** Consider the $(k, p, \mathcal{B})$-majority dynamics on $T$, where at round $t = 0$ each vertex of the tree is in state $R$ with probability $q$ or in state $B$ with probability $1 - q$, independently of the others. Then,

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(v_0 \in R^{(t)}\right) = \begin{cases} \phi_{p,k}^- & \text{if } p < p_k^* \text{ and } q > \phi_{p,k}^+; \\ 0 & \text{if } p < p_k^* \text{ and } q < \phi_{p,k}^-; \\ 0 & \text{if } p > p_k^*. \end{cases}$$

(1)

**Proof.** Define $q_t := \mathbb{P}\left(v_0 \in R^{(t)}\right)$ and observe that, since we are working on a complete balanced $k$-ary infinite tree, we have

$$\mathbb{P}\left(v_0 \in R^{(t)}\right) = \mathbb{P}\left(u \in R^{(t)}\right) \text{ for all } u \in V \text{ and for all } t \in \mathbb{N}_0.$$

Note also that, for any siblings $v, w \in V$, the events “$v \in R^{(t)}$” and “$w \in R^{(t)}$” are independent at any round $t \in \mathbb{N}_0$. Hence the random variables in the family $\{1_{\{v \in R^{(t)}\}} : v \in V\}$ are i.i.d. and

$$\mathbb{P}\left(1_{\{v \in R^{(t)}\}} = 1\right) = q_t.$$

Let us now compute $q_{t+1} = \mathbb{P}\left(v_0 \in R^{(t+1)}\right)$. We have:

$$q_{t+1} = \mathbb{P}\left(\sum_{w \in S_v} \mathbb{I}_{w \in R^{(t)}} \mathbb{I}_{x_w^{(t)}(v_0) = x_w^{(t)}} \geq \frac{k + 1}{2}\right).$$

(2)

Note that the random variables $\{1_{x_w^{(t)}(v_0) = x_w^{(t)}} : w \in S_v\}$ are i.i.d. Bernoulli random variables of parameter $p$. Moreover the family of random variables $\{1_{x_w^{(t)}(v_0) = x_w^{(t)}} : w \in S_v\}$ is independent of the family $\{1_{w \in R^{(t)}} : w \in S_v\}$.

Then $\mathbb{P}\left(\sum_{w \in S_v} \mathbb{I}_{w \in R^{(t)}} \mathbb{I}_{x_w^{(t)}(v_0) = x_w^{(t)}} \geq \frac{k + 1}{2}\right)$ is a family of i.i.d. Bernoulli random variables of parameter $(1 - q_p)q_t$ and hence

$$\sum_{w \in S_v} \mathbb{I}_{w \in R^{(t)}} \mathbb{I}_{x_w^{(t)}(v_0) = x_w^{(t)}} \geq \mathbb{P}\left(|S_v|, (1 - q_p)q_t\right) = \mathbb{P}\left(k, (1 - p)q_t\right).$$

Thus, for every $t \geq 0$, we can write

$$q_{t+1} = \mathbb{P}\left(\sum_{w \in S_v} \mathbb{I}_{w \in R^{(t)}} \mathbb{I}_{x_w^{(t)}(v_0) = x_w^{(t)}} \geq \frac{k + 1}{2}\right).$$

(3)

By Definition 4.1, the sequence described in Eq. (3) can be rewritten as

$$q_{t+1} = \begin{cases} F_{p,k}(q) & \text{if } t = 0, \\ F_{p,k}(q_t) & \text{if } t \geq 1. \end{cases}$$

Hence, the limit behavior in Eq. (1) for the sequence $(q_t)_t$ follows from Lemma 4.2.

**5 PHASE TRANSITIONS**

In this section we exploit the results in Section 4 to prove that $(k, p, \mathcal{B})$-majority behaves on dense graphs similarly to what we observed in the previous section on infinite trees.

We start by setting the ground for Theorem 5.1, introducing the required notation. For every node $u \in V$ and for every round $t$ we define the fraction of neighbors of $u$ in state $R$ as $\phi_u^{(t)} := |R_u^{(t)}| / |N_u|$. Similarly we let $\phi_{\text{max}}^{(t)} := \max_{v \in V} \phi_v^{(t)}$ denote the maximum fraction of neighbors in state $R$ at round $t$ over the nodes.

Given any configuration $x^{(t)} = \hat{x}$, we have that, for every $u \in V$, the expected fraction of neighbors of $u$ in state $R$ at round $t + 1$ is

$$E\left[\phi_u^{(t+1)} \mid x^{(t)} = \hat{x}\right] = \frac{1}{\delta_u} \sum_{v \in N_u} \mathbb{P}\left(|R_v^{(t)}| \geq \frac{k + 1}{2} \mid x^{(t)} = \hat{x}\right)$$

$$= \frac{1}{\delta_u} \sum_{v \in N_u} F_{p,k}(\phi_v^{(t)}).$$

(4)

Note that in the last equality we applied Definition 4.1, and used the fact that, given $x^{(t)}$, the random variable $|R_u^{(t)}| = \text{Bin}\left(k, (1 - p)\phi_v^{(t)}\right)$, for every $u \in V$.

**5.1 On the Bias in the Communication $p$**

The main result of this work shows the existence of a critical value $p_k^*$ (see Lemma 4.2) for the bias $p$ when the nodes of a dense graph execute $(k, p, \mathcal{B})$-majority starting from an initial configuration where all nodes are in state $R$. Roughly speaking, on the one hand Theorem 5.1 states that, if the bias is smaller than $p_k^*$, a superpolynomial number of rounds is needed to reach a $\mathcal{B}$-almost-consensus:
we will show that the system will remain trapped in a metastable phase, in which the volume of nodes in state $\mathcal{R}$ is some constant fraction of the total, for every polynomial number of rounds. On the other hand, if the bias is larger than $p_k^*$, a constant number of rounds suffices to reach a $B$-almost-consensus.

**Theorem 5.1.** Consider a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that $\min_{v \in V} \delta_v = o(\log n)$. For every fixed $n$, consider the $(k, p, B)$-majority dynamics, with $p \in [0, 1]$ and $k \geq 3$, starting from the initial configuration $x^{(0)}$ where $x^{(0)}_u = \mathcal{R}$ for every $u \in V$. If $|p - p_k^*| > c$ for some constant $c > 0$, then:

1. Slow convergence: if $p < p_k^*$, then for every constant $\gamma > 0$ there exists a $T = T(\gamma)$ s.t. $\forall k > 0$

   $$P\left( \forall t \in [T, n^K], \frac{\text{vol}(R(t))}{\text{vol}(V)} \in [\phi_{p,k}^+, -\phi_{p,k} + \gamma] \right) = 1 - o(1),$$

   where $\phi_{p,k}^+$ is the largest fixed point of $F_{p,k}$ (see Lemma 4.2).

2. Fast convergence: if $p > p_k^*$, then for every constant $\gamma > 0$ there exists a $T = T(\gamma)$ s.t.

   $$P\left( \exists t < T \text{ s.t. } \frac{\text{vol}(R(t))}{\text{vol}(V)} > 1 - \gamma \right) = 1 - o(1).$$

In the rest of the section we assume $k$ is odd and we prove the theorem for this case only. We later show in Proposition 5.11 that this assumption is not necessary. In fact, the behavior of $(k, p, B)$-majority is equivalent to that of $(k + 1, p, B)$-majority for every odd $k$.

We split the proof into two parts. Theorem 5.1 then follows from Corollaries 5.3 and 5.6.

**Slow convergence.** The proof for the slow convergence regime directly exploits the connection with the infinite tree model described in Section 4. In Proposition 5.2 we show that for every round $t \in \text{poly}(n)$ all the nodes in $G$ have a fraction of neighbors in state $\mathcal{R}$ which is asymptotically equal to $q_1$, i.e., the probability that the root $v_0$ of the infinite tree $T$ is in state $\mathcal{R}$ at round $t$.

**Proposition 5.2.** Consider $p = p_k^* - c$, for some $c > 0$, so that the quantity $\phi_{p,k}$ in Lemma 4.2 is well defined. Consider the $(k, p, B)$-majority starting from the initial configuration in which each vertex is $\mathcal{R}$. Then, for all $\gamma > 0$ and for all $k > 0$

$$P\left( \forall t \leq n^K, \forall u \in V, \phi_{p,k}^{(t)} \in [q_1 - \gamma, q_1 + \gamma] \right) = 1 - o(1).$$

where the sequence $(q_t)_{t \geq 0}$ is defined recursively as in Section 4, by choosing $q = 1$.

The result in Proposition 5.2 is discussed and proved for a more general class of initial conditions in the full version of the paper. The core of the proof lies in the contraction property of $F_{p,k}$. In fact one can check that $F_{p,k} < 1$ in the interval $[p_k^*, 1]$. The proof of Proposition 5.2 amounts to show that this fact is sufficient to ensure a bilateral concentration. Moreover, given the density assumption, the concentration can be pushed to hold uniformly in the vertices at any polynomial round.

By Proposition 5.2 the fraction of $\mathcal{R}$ neighbors of a node concentrates around $q_1$, i.e., the probability that the root of the infinite tree is in state $\mathcal{R}$ at round $t$. Then, we use Proposition 5.2 to show that a metastable phase—in which the fraction of volume in state $\mathcal{R}$ is approximately $\phi_{p,k}^*$—is attained in constant time. Moreover, an additional superpolynomial number of rounds is necessary to reach a $B$-almost-consensus.

**Corollary 5.3.** Assume $p = p_k^* - c$ for some constant $c > 0$ and that the initial configuration is such that all nodes are in state $\mathcal{R}$. Then for all constant $\gamma > 0$ there exists a sufficiently large $T = T(\gamma)$ such that $\forall k > 0$

$$P\left( \forall t \in [T, n^K], \frac{\text{vol}(R(t))}{\text{vol}(V)} \in [\phi_{p,k}^+, -\phi_{p,k} + \gamma] \right) = 1 - o(1).$$
By taking a union bound of Eq. (8) with respect to Chernoff Bound [24, Exercise 1.1], we get
\[ \Pr \left( \forall t \in [T, n^K], \forall u \in V, \phi_u^{(t)} \in [\phi_{p, k}^+, \gamma, \phi_{p, k}^- + \gamma] \right) = 1 - o(1). \]

At this point it is sufficient to rewrite the volume of nodes in state R as follows
\[ \text{vol}(R^{(t)}) = \sum_{u \in R^{(t)}} \delta_u = \sum_{u \in V} (\delta_u - |B_u^{(t')}|) = \sum_{u \in V} \delta_u \phi_u^{(t')} \quad (5) \]
Therefore, it follows from Proposition 5.2 that, for all \( K > 0, \)
\[ (\phi_{p, k}^+ - \gamma) \text{vol}(V) \leq \text{vol}(R^{(t)}) \leq (\phi_{p, k}^+ + \gamma) \text{vol}(V), \]
for all \( u \) and all \( t \in [T, n^K]. \)

**Fast convergence.** In the fast convergence regime, for every node of the graph, we can upper bound the expected fraction of neighbors in state \( R. \) This fact is formalized in the following lemma.

**Lemma 5.4.** Let \( p = p_k^* + c, \) for some \( c > 0. \) There exists some \( c = \epsilon(p, k) > 0 \) such that for every \( u \in V \) it holds that
\[ \mathbb{E} \left[ \phi_u^{(t+1)} \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right] \leq (1 - \epsilon) \phi_u^{(t)} \max. \quad (6) \]

Lemma 5.4 essentially tells us that the fraction of neighbors of \( u \) in state \( R, \) maximized over \( u \in V, \) is a supermartingale. We will use this fact to prove the following proposition.

**Proposition 5.5.** Fix \( p = p_k^* + c, \) for some \( c > 0. \) For all \( \gamma > 0 \) there exists some \( T = T(\gamma) \) such that
\[ \Pr \left( \exists t < T \text{ s.t. } \phi_u^{(t)} \leq \gamma \right) = 1 - o(1). \quad (7) \]

**Proof.** For any \( u \in V, \) by multiplying both sides of Eq. (6) by \( \delta_u \) we get that for all \( \gamma > 0, \)
\[ \mathbb{E} \left[ |R_u^{(t+1)}| \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right] \leq (1 - \epsilon) \max \phi_u^{(t)} \delta_u \leq (1 - \epsilon) \max \phi_u^{(t)} \gamma \delta_u. \]

We aim at bounding the quantity \( \phi_u^{(t+1)}, \) namely
\[ \Pr \left[ \phi_u^{(t+1)} > (1 - \epsilon^2) \max \phi_u^{(t)} \gamma \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right] = \Pr \left[ \left| R_u^{(t+1)} \right| > (1 - \epsilon^2) \max \phi_u^{(t)} \gamma \delta_u \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right] = \Pr \left[ \left| R_u^{(t+1)} \right| > (1 + \epsilon)(1 - \epsilon) \max \phi_u^{(t)} \gamma \delta_u \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right]. \]

Note that, given the configuration at round \( t, \) \( |R_u^{(t+1)}| \) is a Binomial random variable and, thus, by applying a multiplicative form of the Chernoff Bound [24, Exercise 1.1], we get
\[ \Pr \left( \phi_u^{(t+1)} > (1 - \epsilon^2) \max \phi_u^{(t)} \gamma \bigg| \mathbf{x}^{(t)} = \mathbf{x} \right) \leq \exp \left[ -\epsilon^2 (1 - \epsilon) \max \phi_u^{(t)} \gamma \delta_u \right]. \]

Since \( \delta_u = \omega(n^{\epsilon n}) \) by hypothesis and \( \max \gamma, \phi_u^{(t)} \gamma \delta_u \) regardless of \( \phi_u^{(t)} \), by taking a union bound over \( u \in V \) and integrating over the conditioning we get
\[ \Pr \left( \phi_u^{(t+1)} > (1 - \epsilon^2) \max \phi_u^{(t)} \gamma \right) \leq n \cdot e^{-c(\log n)}. \quad (8) \]

Define the event \( \mathcal{F} := \{ \forall t < T, \phi_u^{(t)} \leq (1 - \epsilon^2) \max \phi_u^{(t)} \gamma \}. \)
By taking a union bound of Eq. (8) with respect to \( t < T \) we have
\[ \Pr (\mathcal{F}) \geq 1 - T \cdot n \cdot e^{-c(\log n)}. \]
Then we make use of the next proposition, whose proof follows such that for some fixed $k$ configuration coincides at the first order. Then, at that finite time, we can apply the generalized argument of Proposition 5.2 and conclude the proof as in Corollary 5.3.

Fast convergence. For the fast convergence regime, the proof is similar to the proof in the second half of Section 5.1. Analogously to what has been done in Lemma 5.4, we first upper bound the expected fraction of neighbors in state $R$, for each node of the graph.

**Lemma 5.9.** Consider any $x^{(t)}$ such that $\phi^{(t)}_{\max} \leq \phi_{p,k}^c - \eta$, for some $\eta > 0$. Then, there exists some constant $\epsilon' = \epsilon'(p,k,\eta) > 0$ such that

$$E\left[\phi^{(t+1)}_1 \mid x^{(t)} = \hat{x}\right] \leq (1 - \epsilon')\phi^{(t)}_{\max}.$$ 

Then we make use of the next proposition, whose proof follows the same flow as that one of Proposition 5.5, and the conclusion follows by the same argument used in Corollary 5.6.

**Proposition 5.10.** Let $p = p_k^* - c$, for some $c > 0$. Consider $(k,p,B)$-majority starting from the initial configuration in which each vertex is in state $R$, independently of the others, with probability $q = \phi_{p,k}^c - \gamma$ for some $\gamma > 0$. Then, for all $\gamma > 0$ there exists some $T = T(\gamma)$ such that

$$P \left( \exists t \leq T \text{ s.t. } \phi^{(t)}_{\max} \leq \gamma \right) = 1 - o(1).$$

### 5.3 On Even Values of the Sample Size $k$

In the previous section we assumed samples of odd size $k$, thus avoiding potential ties. In this section, we close the gap showing the equivalence between the Markov chains $(2h+1,p,B)$-majority and $(2h+2,p,B)$-majority. The result in Proposition 5.11 is essentially a special case of [27, Appendix B]. However, it has been obtained independently and with a simplified proof, which is available in the full version.

**Proposition 5.11.** Let $G = (V,E)$ be a graph with binary state configuration $x^{(t)}$ in round $t$. Consider the $(k,p,B)$-majority dynamics, for some fixed $k \in \mathbb{N}$, $p \in [0,1]$. Pick a node $u \in V$ and define the event $E_k: \text{“node } u \text{ does not update its state to } B \text{ at round } t + 1\text{”,}$ or equivalently $E_k := (x^{(t+1)}_u \neq B)$, where $k$ is that of $(k,p,B)$-majority. It holds that $P \left( E_{2h+1} \mid x^{(t)} = \hat{x} \right) = P \left( E_{2h+2} \mid x^{(t)} = \hat{x} \right)$, for every $h \in \mathbb{N}_0$.

**Remark 1.** Consider a graph with any initial configuration $x^{(0)}$. Fix any $p \in [0,1]$. Proposition 5.11 implies that, for every $h \in \mathbb{N}_0$, the $(2h+1,p,B)$-majority and the $(2h+2,p,B)$-majority follow the same law and, thus, their evolution is the same.

### 6 LIMIT CASES: VOTER AND DETERMINISTIC MAJORITY DYNAMICS

In this section we analyze the two limit cases of $(k,p,B)$-majority, considering the case $k = 1$ as well as the case in which $k$ is large. In particular, $1$-majority is equivalent to voter, i.e., nodes copy the state of a randomly sampled neighbor. In Section 6.1 we analyze its behavior on the biased communication model described in Section 3. On the other hand, for large values of $k$, one might expect a similar behavior to that of deterministic majority, in which nodes update their state to that supported by the majority of nodes in their entire neighborhood. In Section 6.2 we investigate the relation between the two dynamics in our biased communication model.

#### 6.1 Voter Dynamics

Differently from the general case of Theorem 5.1 where $k \geq 3$, no phase transition is observed for $(1,p,B)$-majority, due to the linearity of the dynamics. Moreover, the effect of the bias $p$ has a strong impact on its behavior. While the standard binary voter dynamics needs $\Omega(n)$ rounds to reach a consensus on one of the two states, e.g., on the complete graph, we show in Proposition 6.1 that in the biased communication model a $B$-almost-consensus is reached in $O(1)$ rounds, a.a.s., regardless of the initial configuration. In other words, in a constant number of rounds any potential majority is subverted to a majority on the state promoted by the bias. In the complete graph, for example, this would imply a convergence to a consensus in $O(\log n)$ rounds, i.e., exponentially faster than in the classic scenario with no bias.

**Proposition 6.1.** Consider a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that $\min_{u \in V} \delta_u = o(\log n)$. For any fixed $n$, consider the $(1,p,B)$-majority dynamics with any initial configuration $x^{(0)}$ and $p > 0$ constant. The process reaches a $B$-almost-consensus within $O(1)$ rounds, a.a.s.

The proof is essentially the same as that one of the fast convergence regime in Theorem 5.1.

#### 6.2 Deterministic Majority Dynamics

As mentioned earlier, one might expect that as $k$ grows $(k,p,B)$-majority would behave similarly to deterministic majority in the biased communication model, which we denote with $(p,B)$-deterministic majority. We make this link rigorous in Propositions 6.2 and 6.3. In particular, we show that, if the graph satisfies the density assumption $\min_{u \in V} \delta_u = o(\log n)$, the $(p,B)$-deterministic majority has a sharp phase transition at $p = \frac{1}{2}$, that is the limit as $k \to \infty$ of the critical value $p_k^*$.

For the sake of readability, we restrict the analysis of this section to the case in which the initial configuration in which every vertex is in state $R$.

**Proposition 6.2.** Consider a sequence of graphs $(G_n)_{n \in \mathbb{N}}$ such that $\min_{u \in V} \delta_u = o(\log n)$. For any fixed $n$, consider the $(p,B)$-deterministic majority dynamics. Then:

- if $p = \frac{1}{2} + c$ for some $c > 0$, then

$$P \left( \text{vol} \left( \phi^{(1)} \right) = \text{vol} \left( V \right) \right) = 1 - o(1);$$

\[ \text{The specific graph topology allows to use our arguments until a complete consensus.} \]
We point out that the same behavior of Proposition 6.2 can be achieved in a weaker sense—for \((k, p, B)\)-majority, for all sufficiently large values of \(k\). For the sake of clarity, in the next proposition we will use the notation \(P_k\) to denote the law of \((k, p, B)\)-majority for a given value of \(k\).

Proposition 6.3. Consider a sequence of graphs \((G_n)_{n \in \mathbb{N}}\) such that \(\min_{n \in \mathbb{N}} \Delta_n = o(\log n)\). For any fixed \(n\), consider the \((k, p, B)\)-majority dynamics. Let \(p - 1/2 = c\) for some \(c > 0\). Then for all \(\gamma > 0\) there exists \(H = H(c, \gamma)\) such that:

- \(p > 1/2\) then, \(\forall k > H, \quad P_k(\sum_{i=0}^{n^K} \frac{\text{vol}(B_i)}{\text{vol}(V)} \geq 1 - \gamma) = 1 - o(1)\); (10)

- \(p < 1/2\) then, \(\forall k > 0\) and \(\forall k > H, \quad P_k(\forall t \in [0, n^K], \frac{\text{vol}(R(t))}{\text{vol}(V)} \geq 1 - \gamma) = 1 - o(1)\). (11)

We point out that the same behavior of Proposition 6.2 can be achieved for \((k, p, B)\)-majority without any topological assumption, by just letting \(k\) grow with \(n\) as \(k = \omega(\log n)\).

7 DISCUSSION AND OUTLOOK

The biased communication model described in Section 3 and analyzed throughout the paper for the \(k\)-majority dynamics allows us to reason about the strength of consensus against adversarial nodes with a simple framework already presented in [19]. Indeed, the framework described in Section 3 describes a biased communication channel shared by the two nodes is subject to a form of adversarial noise. Our results, despite considering a very simple model for the communication noise, shed light on the relation between the amount of noise (the communication bias \(p\)) and the amount of information shared by nodes (the number \(k\) of neighbor each node pulls information from in order to update its state) providing a complete characterization of the behavior of \(k\)-majority in sufficiently dense graphs. We believe such an analysis to be important to understand the robustness of the considered class of dynamics in solving the consensus problem when communication among nodes is affected by noise.

Moreover, the arise of a metastable phase makes the framework suitable to design distributed algorithms for community detection (a.k.a. graph clustering) based on the \(k\)-majority dynamics. In particular, this is possible for a class of graphs known as volume-regular graphs, recently introduced in [10] and strictly related to ordinary lumpable Markov Chains [36]. Notable examples of volume-regular graphs are those sampled from the regular Stochastic Block Model [32], where the nodes are partitioned into several clusters. As motivating example, consider a volume-regular graph \(G = (V, E)\) where the vertex set \(V\) is partitioned into two clusters \(V_1, V_2\). Since \(G\) is volume-regular, it has the property that, for every pair of nodes in each cluster, their fraction of neighborhood toward the other cluster equals some constant \(z\). Let \(G\) run the \(k\)-majority and suppose that the two clusters reach a local almost-consensus on different states, say nodes in \(V_1\) agree on \(R\) and nodes in \(V_2\) on \(B\). The local evolution of such a process inside, e.g., \(V_1\) can be described by the \((k, z, B)\)-majority dynamics run by the subgraph induced by \(V_1\): the effect of noise, with probability \(p = z\), mimics the fact that a node in \(V_1\) is sampling a neighbor in \(V_2\) in the worst-case scenario in which nodes in \(V_2\) never change color. If \(G\) is such that \(z = p < p^*_c\), Theorem 5.1 implies that \(V_1\) remains in an almost-consensus configuration, a.a.s. Since the same reasoning can be done for \(V_2\), it follows that the graph would stay in a configuration that highlights its clustered structure for \(n^{o(1)}\) rounds, hence making \(k\)-majority suitable for the design of a distributed community detection protocol.

As already noted earlier, a stronger assumption on the minimum degree, i.e., considering graphs with minimum degree \(\Omega(n)\), would be sufficient to prove a full consensus on the state \(B\) promoted by the bias, in \(O(\log n)\) rounds, in all the scenarios in which we prove a \(B\)-almost-consensus. Graphs satisfying such assumption are not necessarily graphs with strong expansion properties, e.g., consider two equal-sized cliques connected by a single edge. It remains unclear whether it is possible to prove similar results for sparser topologies and which are the minimum topological assumptions that are necessary to prove a full consensus. More precisely, it would be interesting to see whether our results can be sharpened by assuming a particular topology as, e.g., an Erdős–Rényi random graph \(G(n, p)\) with \(p = \frac{\omega(\log n)}{n}\).

From the statistical physics perspective, it would also be of interest the analysis of the critical case \(p = p^*_c\) on some particular topologies. In particular, it would be notable if precise asymptotics on the convergence time could be obtained in the critical regime without any topological assumption on the underlying graph. Finally, possible research directions that could lead to non-obvious conclusions are that of applying our biased framework to other dynamics or to consider more than two states.

REFERENCES

[1] Mohammed Amin Abdullah, Michel Bode, and Nikolaos Fountoulakis. 2015. Local Majority Dynamics on Preferential Attachment Graphs. In Algorithms and Models for the Web Graph - 12th International Workshop, WAW 2015, Eindhoven, The Netherlands, Dec 10–11, 2015 (Lecture Notes in Computer Science, Vol. 9479) Springer, 95–106. https://doi.org/10.1007/978-3-319-26784-5_8

[2] Mohammed Amin Abdullah and Moez Draief. 2015. Global majority consensus by local majority polling on graphs of a given degree sequence. Discret. Appl. Math. 180 (2015), 1–10. https://doi.org/10.1016/j.dam.2014.07.026

[3] Gideon Amir, Rangel Baldasso, and Nissan Benin. 2019. Majority dynamics and the median process: connections, convergence and some new conjectures. arXiv preprint arXiv:1911.08613 (2019).

[4] Aris Anagnostopoulos, Luca Becchetti, Emilio Cruciani, Francesco Pasquale, and Sara Rizzó. 2020. Biased Opinion Dynamics: When the Devil is in the Details. In Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI 2020. ijcai.org, 53–59. https://doi.org/10.24963/ijcai2020/8

[5] Vincenzo Auletta, Diodato Ferraioli, and Gianluigi Greco. 2018. Reasoning about Consensus when Opinions Diffuse through Majority Dynamics. In Proc. of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018, Jul 13–19, 2018, Stockholm, Sweden. 49–55. https://doi.org/10.24963/ijcai.2018/7

154
