On fundamental solutions for multidimensional Helmholtz equation with three singular coefficients
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The main result of the present paper is the construction of fundamental solutions for a class of multidimensional elliptic equations with three singular coefficients, which could be expressed in terms of a confluent hypergeometric function of four variables. In addition, the order of the singularity is determined and the properties of the found fundamental solutions that are necessary for solving boundary value problems for degenerate elliptic equations of second order are found.

Key words: multidimensional elliptic equation with three singular coefficients, fundamental solutions, confluent hypergeometric functions of four variables.

1. Introduction

It is known that fundamental solutions have an essential role in studying partial differential equations. Formulation and solving of many local and non-local boundary value problems are based on these solutions. Moreover, fundamental solutions appear as potentials, for instance, as simple-layer and double-layer potentials in the theory of potentials.

The explicit form of fundamental solutions gives a possibility to study the considered equation in detail. For example, in the works of Barros-Neto and Gelfand [1], fundamental solutions for Tricomi operator, relative to an arbitrary point in the plane were explicitly calculated. We also mention Leray’s work [12], which it was described as a general method, based upon the theory of analytic functions of several complex variables, for finding fundamental solutions for a class of hyperbolic linear differential operators with analytic coefficients. Among other results in this direction, we note a work by Itagaki [10], where 3D high-order fundamental solutions for a modified Helmholtz equation were found. The found solutions can be applied to some boundary value problems [5,11,14,15].

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In this article, at first we shall introduce one confluent hypergeometric function of four variables. Furthermore, by means of the introduced hypergeometric function we construct fundamental solutions of the equation (1.1) in an explicit form. For studying the properties of the fundamental solutions, the introduced confluent hypergeometric function is expanded in products of hypergeometric functions of Gauss. With the help of the obtained expansion it is proved that the constructed fundamental solutions of equation (1.1) have a singularity of order $1/r^{p-2}$ at $r \to 0$.

2. About one confluent hypergeometric function

In [3], also [16], p.74, (4b) a hypergeometric function of many variables of the form

$$H_{n,p}(a,b_1,\ldots,b_n,c_{p+1},\ldots,c_n;d_1,\ldots,d_p;x_1,x_2,\ldots,x_n)$$

is considered, where $(a)_m = \Gamma(a+m)/\Gamma(a)$ is the Pochhammer symbol, $m$ is an integer number, $a$ is a complex number, and $a \neq 0,-1,-2,\ldots$, if the Pochhammer symbol $(a)_m$ is on the denominator.

The hypergeometric function (2.1) in the four variables case looks like

$$H_{4,3}(a,b_1,b_2,b_3,b_4,c_4;d_1,d_2,d_3;x,y,z,t) = \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k+l}(b_1)_{m+n+k}(b_2)_{n+l+k}(b_3)_{k+l} (c_4)_l}{(d_1)_m(d_2)_n(d_3)_k(d_4)_l} x^m y^n z^k t^l,$$
where $|x| + |y| + |z| < 1$, $|t| < 1/(1 + |x| + |y| + |z|)$.

From the hypergeometric function (2.2) we shall define the following confluent hypergeometric function

$$H^0_{4,3}(a,b_1,b_2,b_3; d_1,d_2,d_3; x,y,z,t) = \lim_{\varepsilon \to 0} H_{4,3}(a,b_1,b_2,b_3, \frac{1}{\varepsilon} - \varepsilon, d_1,d_2,d_3; x,y,z,\varepsilon^2 t).$$

At the determination of the hypergeometric function $H^0_{4,3}(a,b_1,b_2,b_3; d_1,d_2,d_3; x,y,z,t)$ the equality $\lim_{\varepsilon \to 0} (1/\varepsilon)_n \cdot \varepsilon^n = 1$ ($n$ is a natural number) has been used. The found confluent hypergeometric function has the following form

$$H^0_{4,3}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k+t}(b_1)_{m}(b_2)_{n}(b_3)_k}{(d_1)_m(d_2)_n(d_3)_k m! n! k! l!} x^m y^n z^k t^l, |x| + |y| + |z| < 1. \quad (2.3)$$

It has the following integral representation:

$$H^0_{4,3}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \prod_{i=1}^{3} \left[ \frac{\Gamma(d_i)}{\Gamma(b_i)\Gamma(d_i - b_i)} \right] \int_0^1 \int_0^1 x_1^{b_i-1} y_1^{b_i-1} z_1^{b_i-1} (1 - x_1)^{d_i-b_i-1} \times (1 - y_1)^{d_i-b_i-1} (1 - z_1)^{d_i-b_i-1} (1 - x_1 - y_1 - z_1)^{-a} F_1(1 - a; -(1 - x_1 - y_1 - z_1)t) dx_1 dy_1 dz_1,$$

where $\Re d_i > \Re b_i > 0, i = 1, 2, 3$.

Using the formula of derivation

$$\frac{\partial^{i+j+k+l}}{\partial x^i \partial y^j \partial z^k \partial t^l} H^0_{4,3}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \frac{(a)_{i+j+k+l}(b_1)_{i}(b_2)_{j}(b_3)_{k}}{(d_1)_i(d_2)_j(d_3)_k} \times H^0_{4,3}(a + i + j + k - l, b_1 + i, b_2 + j, b_3 + k; d_1 + i, d_2 + j, d_3 + k; x, y, z, t)$$

it is easy to show that the confluent hypergeometric function in (2.3) satisfies the system of hypergeometric equations

\[
\begin{cases}
  x(1-x)\omega_{xx} - xy\omega_{xy} - x\omega_{x} + [d_1 - (a + b_1 + 1)x]\omega_x - b_1 y\omega_y - b_1 z\omega_z + b_1 \omega_t - a b_1 \omega = 0 \\
  y(1-y)\omega_{yy} - yz\omega_{yz} + y\omega_{y} + [d_2 - (a + b_2 + 1)y]\omega_y - b_2 x\omega_x + b_2 \omega_t - a b_2 \omega = 0 \\
  z(1-z)\omega_{zz} - x\omega_{xz} - y\omega_{yz} + z\omega_{z} + [d_3 - (a + b_3 + 1)z]\omega_z - b_3 x\omega_x - b_3 \omega_t - a b_3 \omega = 0 \\
  \omega_{tt} - x\omega_{xt} + y\omega_{yt} - z\omega_{zt} + (1-a)\omega_t + \omega = 0,
\end{cases}
\]  
\quad (2.4)

where

$$\omega(x,y,z,t) = H^0_{4,3}(a,b_1,b_2,b_3; d_1,d_2,d_3; x,y,z,t).$$

Having substituted $\omega(x,y,z,t) = x^y y^z x^{\mu} t^\delta$ in the system of hypergeometric equations (2.4), it is possible to be convinced that for the values which are in the following table

| $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\omega_5$ | $\omega_6$ | $\omega_7$ | $\omega_8$ |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| $\tau$    | 0         | 1 - $d_1$ | 0         | 1 - $d_1$ | 1 - $d_1$ | 0         | 1 - $d_1$ |
| $\nu$     | 0         | 0         | 1 - $d_2$ | 0         | 1 - $d_2$ | 0         | 1 - $d_2$ |
| $\mu$     | 0         | 0         | 0         | 1 - $d_3$ | 0         | 1 - $d_3$ | 1 - $d_3$ |
| $\delta$  | 0         | 0         | 0         | 0         | 0         | 0         | 0         |

Consequently, the system (2.4) has 8 linearly independent solutions

$$\omega_1(x,y,z,t) = H^0_{4,3}(a,b_1,b_2,b_3; d_1,d_2,d_3; x,y,z,t), \quad (2.5)$$

$$\omega_2(x,y,z,t) = x^{1-d_1} H^0_{4,3}(a+1-d_1,b_1+1-d_1,b_2,b_3; 2-d_1,d_2,d_3; x,y,z,t), \quad (2.6)$$

$$\omega_3(x,y,z,t) = y^{1-d_2} H^0_{4,3}(a+1-d_2,b_1,b_2+1-d_2,b_3; 2-d_2,d_3; x,y,z,t), \quad (2.7)$$

$$\omega_4(x,y,z,t) = z^{1-d_3} H^0_{4,3}(a+1-d_3,b_1,b_2,b_3+1-d_3; 2-d_3,d_3; x,y,z,t), \quad (2.8)$$

2
\[ \omega_5(x, y, z, t) = x^{1-d_1} y^{1-d_2} H_{4,3}^0(a + 2 - d_1 - d_2, b_1 + 1 - d_1, b_2 + 1 - d_2, b_3; 2 - d_1, 2 - d_2, d_3; x, y, z, t), \]  
(2.9)

\[ \omega_6(x, y, z, t) = x^{1-d_1} y^{1-d_2} H_{4,3}^0(a + 2 - d_1 - d_3, b_1 + 1 - d_1, b_2 + 1 - d_3; 2 - d_1, 2 - d_2, d_3; x, y, z, t), \]  
(2.10)

\[ \omega_7(x, y, z, t) = y^{1-d_2} z^{1-d_3} H_{4,3}^0(a + 2 - d_2 - d_3, b_1 + 1 - d_2, b_2 + 1 - d_3; d_1, 2 - d_2, 2 - d_3; x, y, z, t), \]  
(2.11)

\[ \omega_8(x, y, z, t) = x^{1-d_1} y^{1-d_2} z^{1-d_3} \times H_{4,3}^0(a + 3 - d_1 - d_2 - d_3, b_1 + 1 - d_1, b_2 + 1 - d_2, b_3 + 1 - d_3; 2 - d_1, 2 - d_2, 2 - d_3; x, y, z, t). \]  
(2.12)

For further studying the properties, the expansion in products of hypergeometric functions of Gauss is required for the confluent hypergeometric function \( H_{4,3}^0(a, b_1, b_2; d_1, d_2, d_3; x, y, z, t) \). For this purpose we shall consider the expression

\[ H_{4,3}^0(a, b_1, b_2; d_1, d_2, d_3; x, y, z, t) = \sum_{l=0}^{\infty} \frac{(-1)^l \cdot t_l}{(1 - a)_l} F_A^{(3)}(a - l, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z), \]  
(2.13)

where

\[ F_A^{(3)}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) = \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+n+k} (b_1)_{n+k} (b_2)_{n+k} (b_3)_{m+n}}{(d_1)_m (d_2)_n (d_3)_k m! n! k!} x^m y^n z^k, \]  
(2.14)

Following the work [2] in [8,9] for the function in (2.14) the following expansion is found

\[ F_A^{(3)}(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) = \sum_{m,n,k=0}^{\infty} \frac{(a)_{m+n+k} (b_1)_{n+k} (b_2)_{m+k} (b_3)_{m+n}}{(d_1)_m (d_2)_n (d_3)_k m! n! k!} x^m y^n z^k \times \sum_{l=0}^{\infty} \frac{(-1)^l \cdot t_l}{(1 - a)_l} F_A^{(3)}(a - l, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z), \]  
(2.15)

where

\[ F(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} x^m. \]

Considering expansion (2.15), from the identity (2.13) we find

\[ H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k-l} (b_1)_{n+k} (b_2)_{m+k} (b_3)_{m+n}}{(d_1)_l (d_2)_m (d_3)_k m! n! k! l!} \times \sum_{l=0}^{\infty} \frac{(-1)^l \cdot t_l}{(1 - a)_l} F_A^{(3)}(a - l, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z) \times F(a + m + k - l, b_2 + m + k; d_2 + m + k; y) F(a + m + n + l, b_3 + m + n; d_3 + m + n; z), \]  
(2.16)

By virtue of the formula [4]

\[ F(a, b; c; x) = (1 - x)^{-b} F\left(c - a, b; c; \frac{x}{x - 1}\right), \]

we get expansion (2.16)

\[ H_{4,3}^0(a, b_1, b_2, b_3; d_1, d_2, d_3; x, y, z, t) = (1 - x)^{-b_1} (1 - y)^{-b_2} (1 - z)^{-b_3} \times \sum_{m,n,k,l=0}^{\infty} \frac{(a)_{m+n+k-l} (b_1)_{n+k} (b_2)_{m+k} (b_3)_{m+n}}{(d_1)_l (d_2)_m (d_3)_k m! n! k! l!} \times \left(\frac{z}{1 - z}\right)^{m+n} \times \left(\frac{y}{1 - y}\right)^{m+k} \times \left(\frac{x}{x - 1}\right)^{m+k} \times F\left(d_2 - a + l, b_2 + m + k; d_2 + m + k; \frac{y}{1 - y}\right) F\left(d_3 - a + l, b_3 + m + n; d_3 + m + n; \frac{z}{1 - z}\right). \]  
(2.17)

Expansion (2.17) will be used for studying properties of the fundamental solutions.

3. Fundamental solutions
Consider equation (1.1) in $R^3_+$. Let $x := (x_1, \ldots, x_p)$ be any point and $x_0 := (x_{01}, \ldots, x_{0p})$ be any fixed point of $R^3_+$. We search for a solution of (1.1) as follows:
\[ u(x, x_0) = P(r)w(\sigma), \tag{3.1} \]
where
\[ \sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4), \quad r^2 = \sum_{i=1}^{p} (x_i - x_{0i})^2, \quad r_k^2 = (x_k + x_{0k})^2 + \sum_{i=1, i \neq k}^{p} (x_i - x_{0i})^2, \quad P(r) = (r^2)^{-\alpha}, \]
\[ \alpha = \alpha_1 + \alpha_2 + \alpha_3 - 1 + \frac{p}{2}, \quad \sigma_k = \frac{r^2 - r_k^2}{r^2}, \quad k = 1, 2, 3; \quad \sigma_4 = \frac{1}{4} \lambda^2 r^2. \]

We calculate all necessary derivatives and substitute them into equation (1.1):
\[ \sum_{m=1}^{4} A_m \omega_m \sigma_m m + \sum_{m=1}^{3} \sum_{m=n+1}^{3} B_{m,n} \omega_m \sigma_m n + \sum_{m=1}^{3} C_m \omega_m \sigma_m + \sum_{m=1}^{4} D_m \omega_m + E \omega = 0, \tag{3.2} \]
where
\[ A_k = -\frac{4P(r)}{r^2} \frac{x_k}{x_{0k}} \sigma_k (1 - \sigma_k), \quad C_k = \frac{4P(r)}{r^2} \frac{x_{0k}}{x_k} \sigma_k \sigma_4 + \frac{\lambda^2}{2} P(r) \sigma_k, \]
\[ B_{k,l} = \frac{4P(r)}{r^2} \left( \frac{x_{0k}}{x_k} + \frac{x_{0l}}{x_l} \right) \sigma_k \sigma_l, \quad k \neq l, \quad l = 1, 2, 3, \]
\[ D_k = -\frac{4P(r)}{r^2} \left\{ -\sigma_k \sum_{m=1}^{3} \frac{x_{0m}}{x_m} \alpha_m + \frac{x_{0k}}{x_k} \left[ 2\alpha_k - \alpha_4 \right] \right\}, \quad A_4 = \lambda^2 P(r) \sigma_4, \]
\[ D_4 = \frac{4P(r)}{r^2} \sigma_4 \sum_{m=1}^{3} \frac{x_{0m}}{x_m} \alpha_m + \lambda^2 P(r) \alpha_4, \quad E = \frac{4\alpha P(r)}{r^2} \sum_{m=1}^{3} \frac{x_{0m}}{x_m} - \lambda^2 P(r). \]

Using the above given representations of coefficients we simplify equation (3.2) and obtain the following system of equations:
\[ \begin{aligned}
\sigma_1 (1 - \sigma_1) \omega_{1\sigma_1} &- \sigma_1 \sigma_2 \omega_{1\sigma_2} - \sigma_1 \sigma_3 \omega_{1\sigma_3} + \sigma_4 \omega_{1\sigma_4} + \\
+ [2\alpha_1 - (\alpha + \alpha_1 + 1) \alpha_1] \omega_{1\sigma_1} &- \alpha_1 \sigma_2 \omega_{1\sigma_2} - \alpha_1 \sigma_3 \omega_{1\sigma_3} + \alpha_1 \sigma_4 \omega_{1\sigma_4} = 0 \\
\sigma_2 (1 - \sigma_2) \omega_{2\sigma_2} &- \sigma_2 \sigma_3 \omega_{2\sigma_3} - \sigma_2 \sigma_4 \omega_{2\sigma_4} + \\
+ [2\alpha_2 - (\alpha + \alpha_2 + 1) \alpha_2] \omega_{2\sigma_2} &- \alpha_2 \sigma_3 \omega_{2\sigma_3} + \alpha_2 \sigma_4 \omega_{2\sigma_4} = 0 \\
\sigma_3 (1 - \sigma_3) \omega_{3\sigma_3} &- \sigma_3 \sigma_4 \omega_{3\sigma_4} - \\
+ [2\alpha_3 - (\alpha + \alpha_3 + 1) \alpha_3] \omega_{3\sigma_3} &- \alpha_3 \sigma_4 \omega_{3\sigma_4} = 0 \\
\sigma_4 (1 - \sigma_4) \omega_{4\sigma_4} &- \alpha_1 \omega_{1\sigma_4} - \alpha_2 \omega_{2\sigma_4} - \alpha_3 \omega_{3\sigma_4} + (1 - \alpha) \omega_{4\sigma_4} = 0.
\end{aligned} \tag{3.3} \]

Considering the solutions (2.5)-(2.12) of the system (2.4), we define the solutions $\omega_i(\sigma), i = 1, \ldots, 8$ of the system (3.3) and substituting those found solutions into the expression (3.1), we get some fundamental solutions of the equation (1.1)
\[ q_1(x, x_0) = k_1 (r^2)^{-\alpha} H^{0}_{4,3}(\alpha, \alpha_1, \alpha_2, \alpha_3; 2a_1, 2a_2, 2a_3; \sigma), \tag{3.4} \]
\[ q_2(x, x_0) = k_2 (r^2)^{2a_1 - a_0 - 1} (x_{1x_0})^{1-2a_0} H^{0}_{4,3}(1 + \alpha - 2a_1, 1 - \alpha_1, \alpha_2, \alpha_3; 2 - 2a_1, 2a_2, 2a_3; \sigma), \tag{3.5} \]
\[ q_3(x, x_0) = k_3 (r^2)^{2a_2 - a_1 - 1} (x_{2x_0})^{1-2a_2} H^{0}_{4,3}(1 + \alpha - 2a_1, 1 - \alpha_1, \alpha_2, 2a_3; 2a_1, 2a_2, 2a_3; \sigma), \tag{3.6} \]
\[ q_4(x, x_0) = k_4 (r^2)^{2a_3 - a_2 - 1} (x_{3x_0})^{1-2a_3} H^{0}_{4,3}(1 + \alpha - 2a_1, 1 - \alpha_1, 2a_3, 1 - \alpha_2, 2a_1, 2a_2, 2a_3; \sigma), \tag{3.7} \]
\[ q_5(x, x_0) = k_5 (r^2)^{2a_1 + 2a_2 - a_0 - 2} (x_{1x_0})^{1-2a_0} (x_{2x_0})^{1-2a_2} \]
\[ \times H^{0}_{4,3}(2 + \alpha - 2a_1 - 2a_2, 1 - \alpha_1, 1 - \alpha_2, 2 - 2a_1, 2 - 2a_2, 2a_3; \sigma), \tag{3.8} \]
\[ q_6(x, x_0) = k_6 (r^2)^{2a_1 + 2a_3 - a_0 - 2} (x_{1x_0})^{1-2a_0} (x_{3x_0})^{1-2a_3} \]
\[ \times H^{0}_{4,3}(2 + \alpha - 2a_1 - 2a_3, 1 - \alpha_1, \alpha_2, 1 - \alpha_3; 2 - 2a_1, 2a_2, 2 - 2a_3; \sigma), \tag{3.9} \]
\[ q_7(x, x_0) = k_7 \left( r^2 \right)^{2q_2 + 2q_3 - \alpha - 2} (x_2 x_0^2)^{1 - 2q_2} (x_3 x_0^3)^{1 - 2q_3} \]
\[ \times H_4^0 (2 + \alpha - 2a_2 - 2a_3, a_1, 1 - a_2, 1 - a_3; 2a_2, 2 - 2a_2, 2 - 2a_3; \sigma), \]  
\[ (3.10) \]
\[ q_8(x, x_0) = k_8 \left( r^2 \right)^{a_1 + 2a_2 + 2a_3 - \alpha - 3} (x_1 x_0)^{1 - 2a_1} (x_2 x_0^2)^{1 - 2a_2} (x_3 x_0^3)^{1 - 2a_3} \]
\[ \times H_4^0 (3 + \alpha - 2a_1 - 2a_2 - 2a_3, 1 - a_1, 1 - a_2, 1 - a_3; 2 - 2a_1, 2 - 2a_2, 2 - 2a_3; \sigma), \]  
\[ (3.11) \]
where \( k_1, ..., k_8 \) are constants which will be determined at solving boundary value problems for equation (1.1).

4. Singularity properties of fundamental solutions

Let us show that the found solutions (3.4)-(3.11) have a singularity. We choose a solution \( q_1(x, x_0) \). For this aim we use the expansion (2.17) for the confluent hypergeometric function (2.3). As a result, solution (3.4) can be written as follows

\[ q_1(x, x_0) = \frac{k_1}{r_1} \left( \frac{r_2}{r_1} \right)^{-\alpha_1} \left( \frac{r_3}{r_1} \right)^{-\alpha_2} f \left( r_2, r_1^2, r_2^2, r_3^2 \right), \]

where

\[
 f \left( r^2, r_1^2, r_2^2, r_3^2 \right) = \sum_{m,n,k,l=0}^{\infty} \frac{(-1)^k (r_1)^k (r_2)^m (r_3)^l}{(2a_1)_{m+k} (2a_2)_{m+k} (2a_3)_{m+n}} \frac{1}{n! m! k! l!} \left( 1 - \frac{r^2}{r_1^2} \right)^{n+k} \times \left( 1 - \frac{r^2}{r_2^2} \right)^{m+n} \left( 1 - \frac{r^2}{r_3^2} \right)^{k+l} \frac{2^l}{l!} F \left( 2a_1 - \alpha + l, a_1 + n + 1; 2a_1 + n + k; 1 - \frac{r^2}{r_1^2} \right) \times F \left( 2a_2 - \alpha + l, a_2 + m + k; 2a_2 + m + k; 1 - \frac{r^2}{r_2^2} \right) \frac{2^m}{m!} F \left( 2a_3 - \alpha + l, a_3 + m + n; 2a_3 + m + n; 1 - \frac{r^2}{r_3^2} \right). \]  
\[ (4.1) \]

Following the work [7], it is easy to show that

\[ f \left( 0, r_1^2, r_2^2, r_3^2 \right) = C, \ C = \text{constant}. \]  
\[ (4.2) \]

Expressions (4.1) and (4.2) give us the possibility to conclude that the solution \( q_1(x, x_0) \) reduces to infinity of the order \( r^{2-p} \) at \( r \to 0 \). Similarly it is possible to be convinced that solutions \( q_i(x, x_0), \ i = 2, 3, ..., 8 \) also reduce to infinity of the order \( r^{2-p} \) when \( r \to 0 \).

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