MOORE MACHINES DUALITY

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Abstract. We present a simple algorithm to find the Moore machine with the minimum number of states equivalent to a given one.

1. Introduction

It is known that there is a unique, up to isomorphism, deterministic finite automaton with a minimum number of states which recognizes the same language as a given one. As explained in [7] (pages 29–30), the states of this minimal automaton can be taken as the classes modulo an equivalence relation on the input words.

Several algorithms to minimize a deterministic finite automaton has been proposed [4, 5, 8]. According to [2] these algorithms fall within three categories. Some of them apply as well to non deterministic finite automata and to Moore machines.

In Section 2.5 we describe an algorithm to minimize a Moore machine, and establish its consistency. This algorithm lies in the same family as Brzozowski’s one [5]. But its description as well as the proof of its correctness are very simple and do not require much apparatus. Furthermore, this gives another proof, simple and elementary, of the existence and uniqueness of a minimum automaton (see Section 3). In Section 4 we extend these results to morphisms of free monoids.

2. Moore machine and duality

Definition 1. A Moore machine is a 6-tuple $\mathcal{M} = (Q, \Sigma, \Delta, \delta, \lambda, i)$ where

- $Q$ is a finite set, the set of states,
- $\Sigma$ is a finite set, the input alphabet,
- $\Delta$ is a finite set, the output alphabet,
- $\delta$ is a mapping from $Q \times \Sigma$ to $Q$, called the transition function,
- $\lambda \in \Delta^Q$ is a mapping from $Q$ to $\Delta$,
- $i \in Q$ is the initial state.

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2.1. Actions of words on $Q$. The set of words on the alphabet $\Sigma$, including the empty word $\epsilon$, is denoted by $\Sigma^*$. The concatenation of the words $u$ and $v$ is simply written $uv$.

The transition function $\delta$ can be extended in two ways as a mapping from $Q \times \Sigma^*$ to $Q$.

**Right action.** We define, by recursion on the length of $w$ a mapping, written $(a, w) \mapsto a \cdot w$, from $Q \times \Sigma^*$ to $Q$:
- for $a \in Q$, $a \cdot \epsilon = a$,
- for $a \in Q$ and $j \in \Sigma$, $a \cdot j = \delta(a, j)$,
- $a \cdot wj = (a \cdot w) \cdot j$ for $a \in Q$, $w \in \Sigma^*$, and $j \in \Sigma$.

**Left action.** In the same way we define a mapping, written $(w, a) \mapsto w \cdot a$, from $\Sigma^* \times Q$ to $Q$:
- for $a \in Q$, $\epsilon \cdot a = a$,
- for $a \in Q$ and $j \in \Sigma$, $j \cdot a = \delta(a, j)$,
- $jw \cdot a = j \cdot (w \cdot a)$ for $a \in Q$, $w \in \Sigma^*$, and $j \in \Sigma$.

2.2. Right and left machines. We can feed $M$ with a word in $\Sigma^*$ on the right or on the left and get an output: $M \cdot w = \lambda(i \cdot w)$ and $w \cdot M = \lambda(w \cdot i)$.

If a state does not belong to the set $\{i \cdot w : w \in \Sigma^*\}$, it is useless and can be removed. From now on we assume that $G = \{i \cdot w : w \in \Sigma^*\}$.

Of course, since $w \cdot M = M \cdot \overline{w}$ (where $\overline{w}$ stands for the mirror word of $w$) it would be enough to consider right actions only. But using both actions will prove convenient in what follows.

We say that two Moore machines $M$ and $M'$, with the same $\Sigma$ and $\Delta$, are equivalent if, for all $w \in \Sigma^*$, $M \cdot w = M' \cdot w$.

Let $M$ and $M'$ be two Moore machines. If there exists a bijection $\xi$ from $Q$ onto $Q'$ such that, for all $a \in Q$ and $j \in \Sigma$, $\delta'(\xi(a), j) = \xi(\delta(a), j)$ and $\lambda'(\xi(a)) = \lambda(a)$, we say that these two machines are isomorphic.

2.3. Duals.

**Actions of words on functions.** If $f$ is a function from $Q$ to some set and $w \in \Sigma^*$ we define two functions $w \cdot f$ and $f \cdot w$:

$$\text{for all } a \in Q, \quad (w \cdot f)(a) = f(a \cdot w) \quad \text{and} \quad (f \cdot w)(a) = f(w \cdot a)$$

Then $((vw) \cdot f)(a) = f(a \cdot (vw)) = (w \cdot f)(a \cdot v) = (v \cdot (w \cdot f))(a)$. So,

$$(vw) \cdot f = v \cdot (w \cdot f)$$

and similarly

$$f \cdot (vw) = (f \cdot v) \cdot w$$
Right dual. This is the Moore machine $\mathcal{M}_* = (Q_*, \Sigma, \Delta_*, \lambda_*, i_*)$ defined as follows:

- $Q_* = \{ w \cdot \lambda : w \in \Sigma^* \} \subset \Delta^Q_*$
- $\delta_*(f, j) = j \cdot f$
- $\lambda_* : Q_* \rightarrow \Delta$ so defined: $\lambda_*(f) = f(i)$
- $i_* = \lambda$.

Then, for all $w \in \Sigma^*$, $\lambda_*(w \cdot i_*) = (w \cdot i_*)(i) = (w \cdot \lambda)(i) = \lambda(i \cdot w)$, which means

$$w \cdot \mathcal{M}_* = \mathcal{M} \cdot w$$

Left dual. We define the machine $\mathcal{M}^\ast = (\ast Q, \Sigma, \Delta_\ast, \delta_\ast, \lambda_\ast, i_\ast)$ in the same way:

- $\ast Q = \{ \lambda \cdot w : w \in \Sigma^* \} \subset \Delta^Q_\ast$
- $\delta_\ast(f, j) = f \cdot j$
- $\lambda_\ast : Q_\ast \rightarrow \Delta$ so defined: $\lambda_\ast(f) = f(i)$
- $i_\ast = \lambda$.

Then, for all $w \in \Sigma^*$, $\lambda_\ast(i \cdot w) = (i \cdot w)(i) = (\lambda \cdot w)(i) = \lambda(w \cdot i)$, which means

$$\ast \mathcal{M} \cdot w = w \cdot \mathcal{M}$$

2.4. Bidual. To avoid cumbersome notation, set $\mathcal{M}^\sharp = \ast(\mathcal{M}_*)$. Then both $\mathcal{M}$ and $\mathcal{M}^\sharp$ are right machines and are equivalent in that sense that they give the same outputs.

There is a natural mapping $\tau$ from $Q$ into $\Delta^Q_\ast$: let $a \in Q$ and $f \in Q_\ast$, then $f \in \Delta^Q$, so $f(a) \in \Delta$; set $\tau_a(f) = f(a)$. Then $\tau : a \mapsto \tau_a$ is a mapping from $Q$ to $\Delta^Q_\ast$.

We have $\lambda_\ast(w \cdot f) = \lambda_\ast(w \cdot f)(i) = (w \cdot f)(i) = \tau_{i \cdot w}(f)$ for all $w \in \Sigma^*$ and $f \in Q_\ast$. So

$$\lambda_\ast \cdot w = \tau_{i \cdot w} \quad {\text{for all}} \quad w \in \Sigma^*.$$ 

Since $Q^\sharp = \{ \lambda_\ast \cdot w : w \in \Sigma^* \} \subset \Delta^Q_\ast$, then $\tau$ maps $Q$ onto $Q^\sharp$, and $\text{card } Q^\sharp \leq \text{card } Q$.

One would like to know when $\text{card } Q^\sharp = \text{card } Q$, that is when $\tau$ is one-to-one. Let $a \in Q$ and $b \in Q$. Then

$$\tau_a = \tau_b \iff \forall f \in Q_\ast, f(a) = f(b)$$

$$\iff \forall w \in \Sigma^*, (w \cdot \lambda)(a) = (w \cdot \lambda)(b)$$

$$\iff \forall w \in \Sigma^*, \lambda(a \cdot w) = \lambda(b \cdot w).$$

Also, one can consider the mapping $\tau_\ast$ from $Q_\ast$ to $\Delta^Q_\ast$ defined in the same way as $\tau$ is:

$$\tau_{\ast, a}(f) = f(a) \quad \text{for} \quad a \in Q_\ast \ast \text{ and } f \in Q^\sharp.$$
Let \( a_* \in Q_* \) and \( b_* \in Q_* \). Then

\[
\tau_{*,a_*} = \tau_{*,b_*} \iff\ \forall f \in Q^?, f(a_*) = f(b_*)
\]

\[
\iff\ \forall w \in \Sigma^*, (\lambda_*w)(a_*) = (\lambda_*w)(b_*)
\]

\[
\iff\ \forall w \in \Sigma^*, \lambda_*(w \cdot a_*) = \lambda_*(w \cdot b_*)
\]

\[
\iff\ \forall w \in \Sigma^*, (w \cdot a_*)(i) = (w \cdot b_*)(i)
\]

\[
\iff\ \forall w \in \Sigma^*, a_*(i \cdot w) = b_*(i \cdot w)
\]

\[
\iff\ \forall a \in Q, a_*(a) = b_*(a)
\]

\[
\iff\ a_* = b_*.
\]

This means that \( \tau_* \) is one-to-one and that \( \mathcal{M}_* \) is equal to its bidual. In the same way \( \mathcal{M}_* \) is equal to its bidual. Therefore \( \mathcal{M}^? \) is equal to its bidual.

2.5. The algorithm. We illustrate these constructions with the following example.

\[
\begin{array}{c}
0
\end{array}
\begin{array}{c}
\text{i}
\end{array}
\begin{array}{c}
1
\end{array}
\begin{array}{c}
\text{a}
\end{array}
\begin{array}{c}
0
\end{array}
\begin{array}{c}
b
\end{array}
\begin{array}{c}
0
\end{array}
\]

Here \( Q = \{i, a, b\} \), \( \Sigma = \{0, 1\} \), and \( \Delta = \{0, 1\} \). The transition and the output function are defined by the above tables. Also, as it is customary, this machine is described by the above diagram.

Now we describe an algorithm on this example. We consider a stack whose members are elements of \( \Delta^Q \). We start with the stack whose \( \lambda \) is the only element. We say that an element \( \alpha \) of the stack is happy if \( 0 \cdot \alpha \) and \( 1 \cdot \alpha \) are both elements of the stack. To add elements to the stack proceed as follows: starting from the bottom take the first unhappy element, say \( \alpha \), put \( 0 \cdot \alpha \) on the top, set \( \delta_* (\alpha, 0) = 0 \cdot \alpha \), and do the same with \( 1 \cdot \alpha \). Repeat until all the elements are happy. Then \( Q_* \) is the set of elements of the stack and the transition function is \( \delta_* \).

The following table is the result of this process. A column whose first element is \( w \cdot \lambda \) contains two sub-columns; the left one contains the states \( i \cdot w, a \cdot w, \) and \( b \cdot w \), the right one contains \( w \cdot \lambda \), i.e., \( \lambda(i \cdot w), \lambda(a \cdot w), \) and \( \lambda(b \cdot w) \). The last line is just a renaming of the states. This is illustrated by the diagram and the tables below.

| \( \lambda \) | 0-\( \lambda \) | 1-\( \lambda \) | 00-\( \lambda \) | 10-\( \lambda \) | 01-\( \lambda \) | 11-\( \lambda \) | 001-\( \lambda \) | 101-\( \lambda \) |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| i         | i         | 0         | 1         | i         | 0         | b         | 0         | a         |
| a         | 1         | b         | 0         | i         | 0         | a         | 1         | 0         |
| b         | 0         | b         | 0         | a         | 1         | b         | 0         | i         |
| t         | u         | v         | u         | u         | w         | t         | w         | w         |
The following is the computation of the bidual (the states $u$ and $w$ have been omitted in the next table because they would not have brought any information).

| $\lambda_*$ | $\lambda_*0$ | $\lambda_*1$ | $\lambda_*10$ | $\lambda_*11$ |
|------------|-------------|-------------|-------------|-------------|
| $t$        | $u$         | $v$         | $u$         | $t$         |
| $u$        | $u$         | $u$         | $u$         | $0$         |
| $v$        | $w$         | $t$         | $v$         | $1$         |
| $w$        | $w$         | $w$         | $w$         | $1$         |

As expected, the bidual has less states as the original machine.

3. Minimal Moore machine

Here we give another proof of the existence and uniqueness of the minimal machine.

3.1. Product of machines. Let $\mathcal{M}_1 = (Q_1, \Sigma, \Delta_1, \delta_1, \lambda_1, i_1)$ and $\mathcal{M}_2 = (Q_2, \Sigma, \Delta_2, \delta_2, \lambda_2, i_2)$ be two Moore machines, $\Delta$ a finite set, and $\gamma$ a mapping from $\Delta_1 \times \Delta_2$ to $\Delta$. We define a new machine

$\mathcal{M}_1 \otimes \gamma \mathcal{M}_2 = (Q, \Sigma, \Delta, \delta, \lambda \circ (\lambda_1 \times \lambda_2), (i_1, i_2))$

by setting

$Q = \{(i_1 \cdot w, i_2 \cdot w) : w \in \Sigma^*\}$ and $\delta((a_1, a_2), j) = (\delta_1(a_1, j), \delta_2(a_2, j))$.

3.2. Minimal Moore machine. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two equivalent right machines such that each one is isomorphic to its bidual.

Let $p_1$ and $p_2$ be the projections of $Q_1 \times Q_2$ onto $Q_1$ and $Q_2$. Consider the machines $\mathcal{B}_k = \mathcal{M}_1 \otimes_{p_k} \mathcal{M}_2$, for $k = 1, 2$.

For $w \in \sigma^*$, $k \in \{1, 2\}$, and $(a_1, a_2) \in Q_1 \times Q_2$ we have

$$((w \cdot (p_k \circ (\lambda_1 \times \lambda_2))) (a_1, a_2) = p_k \circ (\lambda_1 \times \lambda_2) (a_1 \cdot w, a_2 \cdot w)$$

$$= \lambda_k(a_k \cdot w) = (w \cdot \lambda_k)(a_k).$$

This means $w \cdot (p_k \circ (\lambda_1 \times \lambda_2)) = (w \cdot \lambda_k) \circ p_k$, which implies $\mathcal{B}_k = \mathcal{M}_{k*}$. Therefore $\mathcal{B}_k = \mathcal{M}_k = \mathcal{M}_{k*}$.
Because $\mathcal{M}_1$ and $\mathcal{M}_2$ are equivalent we have $\lambda_1(a_1) = \lambda_2(a_2)$ for all $(a_1, a_2) \in Q$ (i.e., $a_1 = i_1 \cdot w$ and $a_2 = i_2 \cdot w$ for some $w \in \Sigma^*$). Therefore $\mathcal{B}_1 = \mathcal{B}_2$.

As a conclusion, for any Moore machine, there is a unique (up to isomorphism) simplest equivalent machine which is the bidual. One could notice that this algorithm provides a normal form for a Moore machine.

4. **Substitutions**

Consider a 5-tuple $\mathcal{I} = (Q, \Delta, \sigma, \lambda, i)$ where
- $Q$ and $\Delta$ are finite sets,
- $\sigma$ is an endomorphism of the free monoid $Q^*$ generated by $Q$,
- $\lambda$ is a mapping from $Q$ to $\Delta$,
- $i \in Q$.

The endomorphism $\sigma$, also called substitution, is determined by the images of the generators, i.e., by the words $\{\sigma(a)\}_{a \in Q}$. We adopt the following notation: if $w$ is a word on some alphabet, $|w|$ stands for its length (i.e., the number of letters it is made of), and $w_j$ stands for its $(j+1)$-th letter from the left (the letters are numbered $0, 1, 2, \ldots$). Let $q = \max_{a \in Q} |\sigma(a)|$.

When $\delta(i, 0) = i$, the words $\left(\sigma^n(i)\right)_{n \geq 0}$ have a limit $\sigma^\infty(i)$ in $Q^N$. Such sequences as $\lambda(\sigma^\infty(i))$, called projections of fixed points of substitutions, have been studied in many contexts, namely for their algebraic, dynamic, and combinatoric properties [1, 6, 3].

**Constant length substitutions.** This is the case when $|\sigma(a)| = q$ for all $a \in Q$. We then set $\Sigma = \{0, 1, \ldots, q - 1\}$ and $\delta(a, j) = \sigma(a)_j$ for $a \in Q$ and $0 \leq j < q$. So we have a Moore machine:

$$\mathcal{M} = (Q, \Sigma, \Delta, \delta, \lambda, i).$$

In this context if $k \geq 0$ and $n = n_0 + n_1q + \ldots + n_{k-1}q^{k-1}$ (the base $q$ expansion of $n$) it is easy to see that the letter of index $n$ in the word $\sigma^k(a)$ is $n_0n_1 \ldots n_{k-1} \cdot a$, i.e.,

$$\left(\sigma^k(a)\right)_n = n_0n_1 \ldots n_{k-1} \cdot a. \quad (1)$$

This is the connection between Moore machines and substitutions. The bidual of the left Moore machine $\mathcal{M}$ defines a substitution on a smaller alphabet equivalent to the original one.

**Non constant length substitutions.** Let $\omega$ be a symbol not in $Q$. Set $Q' = Q \cup \{\omega\}$ and $\Sigma = \{0, 1, \ldots, q - 1\}$. For $0 \leq j < q$ set $\delta(\omega, j) = \omega$, and for $a \in Q$ set $\delta(a, j) = (\sigma(a)\omega^\nu)_j$, where $\nu = q - |\sigma(a)|$.

We just have defined a finite automaton $\mathcal{A} = (Q', \Sigma, \delta, i, Q)$, where $i$ is the initial state and $Q$ is the set of final states. Let $\mathcal{L}$ be the language recognized by $\mathcal{A}$ considered as a left automaton: $\mathcal{L} = \{w \in \Sigma^* : \$
If we define $\lambda(\omega)$ arbitrarily, we also have a Moore machine, $M = (Q', \Sigma, \Delta, \delta, \lambda, i)$. Let $\mathcal{L}' = \mathcal{L} \setminus \{\epsilon\}$. Observe that, if $w \in \mathcal{L}'$, the path leading to $w \cdot i$ does not include any $\omega$.

If $w \in \mathcal{L}'$, set $\varphi(w) = \sum_{0 \leq j < |w|} w_j q^j$. Then the order on $\mathbb{N}$ induces a well-order on $\mathcal{L}'$: $v \prec w \iff \varphi(v) < \varphi(w)$. So, we can define $\psi(n)$ to be the $n$-th element of $\mathcal{L}'$ (starting from 0). Then the counterpart of (1) is

\[
\left(\sigma^k(i)\right)_j = \psi(j) \cdot i
\]

for $k \geq 0$ and $0 \leq j < |\sigma^k(i)|$.

As previously, the bidual of the left Moore machine $M$ defines a substitution on a smaller alphabet equivalent to the original one.

Nevertheless, observe that instead of appending some $\omega$’s one could have padded the $\sigma(a)$’s up to size $q$ by inserting $\omega$’s in various places. Then, of course, the language $\mathcal{L}$ would be different, but the above analysis still holds.

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