Exponential Approximation of Band-limited Signals from Nonuniform Sampling

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Abstract

Reconstructing a band-limited function from its finite sample data is a fundamental task in signal analysis. A simple Gaussian or hyper-Gaussian regularized Shannon sampling series has been proved to be able to achieve exponential convergence for uniform sampling. In this paper, we prove that exponential approximation can also be attained for general nonuniform sampling. The analysis is based on the residue theorem to represent the truncated error by a contour integral. Several concrete examples of nonuniform sampling with exponential convergence will be presented.

1 Introduction

In this paper, we will use the following notations. For $\sigma > 0$, we denote by $B_\sigma$ the set of all entire functions of exponential type at most $\sigma$. In other words, $B_\sigma$ consists of functions $f$ that are analytic in the whole complex plane and satisfy

$$\limsup_{r \to \infty} \frac{\log \max_{|z|=r} |f(z)|}{r} \leq \sigma.$$ 

For $1 \leq p \leq \infty$, the Bernstein space $B_p^\sigma$ is the set of all $f \in B_\sigma$ whose restrictions to the real axis belong to $L^p(\mathbb{R})$. The norm is defined as that of $L^p(\mathbb{R})$. Functions in the Bernstein spaces $B_p^\sigma$ are band-limited in the sense that they have a Fourier transform with compact support in $[-\sigma, \sigma]$, which follows from the Paley-Wiener theorem [14]. By Plancherel-Pólya theorem, we have

$$B_p^\sigma \subset B_2^\sigma \subset B_\infty^\sigma \subset C(\mathbb{R})$$

for all $1 \leq p \leq q \leq \infty$. It is well-known that a signal $f$ is of finite energy and band-limited to $[-\sigma, \sigma]$ if and only if $f$ is the restriction to $\mathbb{R}$ of a $B_2^\sigma$ function.

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The classical Whittaker-Kotelnikov-Shannon sampling theorem \cite{16} states that we can reconstruct every $f \in B_\pi^2$ from its infinite sampling data $\{f(n) : n \in \mathbb{Z}\}$ by the cardinal series:

$$f(x) = \sum_{n=-\infty}^{\infty} f(n) \text{sinc}(x - n) = \sum_{n=-\infty}^{\infty} f(n) \frac{\sin \pi(x - n)}{\pi(x - n)},$$

(1.1)

where the series converges absolutely and uniformly on $\mathbb{R}$.

For practical reason, we can only sum up finite sample data near the point $x$ to approximate $f(x)$. Thus, one may consider the truncation of the cardinal series. However, due to the slow decayness of the sinc function, truncating the cardinal series leads to a convergence rate of order $O(N^{-1/2})$ when $f \in B_\pi^2$. Furthermore, this truncated series is the optimal algorithm for $B_\pi^2$ in the worst case scenario \cite{10}. Thus, in order to have a fast convergence, we should consider the oversampling case $f \in B_\sigma^2$ with $\sigma < \pi$. There are several papers, for example \cite{12,13,15,9,17}, considering the convergence rate of regularized sampling series of the form

$$\sum_{n=-N}^{N} f(n) \text{sinc}(x - n)G_N(x - n).$$

In \cite{12,13}, Qian considered the case $G_N$ are Gaussian functions and he used Fourier analysis method to establish the convergence rate $O(N^{1/2}\exp(-\frac{\pi-\sigma}{2}N))$ for $f \in B_\sigma^2$ with $\sigma < \pi$. While the authors of \cite{15} deduced the convergence rate $O(N^{-1/2}\exp(-\frac{\pi-\sigma}{2}N))$ for the same regularized sampling series and more general $f$ by complex method. In \cite{17}, the convergence rate $O(N^{-1/2}\exp(-(\pi-\sigma)\mu_mN))$ was obtained for the case $G_N(x) = e^{-c(N,m)x^2}$ are hyper-Gaussian and $f \in B_\sigma^2$, where $m$ is a positive integer and

$$\mu_m = \frac{1}{2m} \left( (2m - 1) \sin \frac{\pi}{4m - 2} \right) \frac{2m - 1}{2m}.$$

Note that all these papers are concerned about the uniform sampling points $\Lambda = \mathbb{Z}$.

In this paper, we generalize the results in uniform sampling to more general nonuniform sampling sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$. More precisely, we use the complex method in \cite{15} to analyze the convergence rate of the regularized nonuniform sampling series

$$A_N^G f(z) = \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\lambda,n}(z)G_N(z - \lambda_n), \quad f \in B_\sigma^\infty$$

where $\sigma < \pi$, $\varphi_{\lambda,n}$ is defined as (2.6) and $G_N$ are Gaussian or hyper-Gaussian. We will show that several kinds of nonuniform sampling series have linear convergence.

## 2 Regularized nonuniform sampling series

We begin with some notation and assumption on the sampling sequence $\Lambda$. For simplicity, throughout this paper, we consider the real sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ and suppose that
\(\lambda_n < \lambda_{n+1}\) for all \(n \in \mathbb{Z}\), \(\lambda_0 = 0\) and \(\Lambda\) is separated, that is,
\[
\delta_\Lambda := \inf_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| > 0.
\] (2.2)

By Weierstrass factorization theorem, there exist entire functions \(\varphi_\Lambda\) whose zeros are exactly the points in \(\Lambda\). Since
\[
\sum_{n \neq 0} \frac{1}{|\lambda_n|^2} \leq \sum_{n \neq 0} \frac{1}{\delta_\Lambda^2 n^2} < \infty,
\]
one of these entire functions is the canonical product
\[
\varphi_\Lambda(z) = z \prod_{n \neq 0} \left(1 - \frac{z}{\lambda_n}\right) e^{z/\lambda_n}.
\] (2.3)

If we further have \(\lim_{r \to \infty} \sum_{0 < |\lambda_n| < r} \frac{1}{\lambda_n} < \infty\), then
\[
\varphi_\Lambda(z) = z \lim_{r \to \infty} \prod_{0 < |\lambda_n| < r} \left(1 - \frac{z}{\lambda_n}\right)
\] (2.4)
is an entire function have the desired property. The function \(\varphi_\Lambda\) is called the generating function of \(\Lambda\). Based on \(\varphi_\Lambda\), we define
\[
\phi_\Lambda(z) = |\varphi_\Lambda(z)| e^{-\pi |\text{Im} z|},
\] (2.5)
which will be used later.

We now turn to the sampling series and choose a generating function \(\varphi_\Lambda\) for the sampling sequence \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\). Since \(\varphi_\Lambda(z)\) is an entire function with zeros exactly at the points \(\lambda_n, n \in \mathbb{Z}\), we can define for every \(n \in \mathbb{Z}\)
\[
\varphi_{\Lambda,n}(z) = \frac{\varphi_\Lambda(z)}{\varphi_\Lambda(\lambda_n)(z - \lambda_n)}.
\] (2.6)
These are entire functions which solve the interpolation problem \(\varphi_{\Lambda,n}(\lambda_k) = \delta_{n,k}\) and are independent of the choice of generating function \(\varphi_\Lambda\). Following the idea of classical Lagrange interpolation, we define
\[
A_N f(z) = \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\Lambda,n}(z).
\] (2.7)
Together with the idea of regularization, we study the series
\[
A_N^G f(z) = \sum_{n=-N}^{N} f(\lambda_n) \varphi_{\Lambda,n}(z) G_N(z - \lambda_n), \quad f \in B^\infty_{\sigma},
\] (2.8)
where $\sigma < \pi$ and $G_N(z)$ are entire functions with $G_N(0) = 1$. Note that if $\Lambda = \mathbb{Z}$, then $\varphi_{\Lambda}(z) = \sin \pi z$, $\varphi_{\Lambda,n}(z) = \text{sinc} (z - n)$ and series (2.7) is the cardinal series (1.1).

Next, we will use complex analysis method to estimate the error $f(z) - A_N^G f(z)$. The idea is that we can represent the error by a contour integral. More specifically, let $\mathcal{L}_N$ be the positively oriented rectangle with vertices at $T_N^+ + i S_N^+$, where

$$
\lambda_N < T_N^+ < \lambda_{N+1}, \quad \lambda_{-N} < T_N^- < \lambda_N, \quad S_N^+ > 0, \quad S_N^- < 0.
$$

Then by the residue theorem, for $z = x + iy \notin \Lambda$ with $T_N^- < x < T_N^+$ and $S_N^- < y < S_N^+$, we can write the error as

$$
f(z) - A_N^G f(z) = \frac{\varphi_{\Lambda}(z)}{2\pi i} \int_{\mathcal{L}_N} f(\zeta) G_N(z - \zeta) \varphi_{\Lambda}(\zeta) (\zeta - z) d\zeta, \quad (2.9)
$$

since $G_N(z)$ is an entire function and $G_N(0) = 1$. Now, denote by $I_{\text{hor}}^\pm$, the contributions to the last integral coming from the two horizontal parts of $\mathcal{L}_N$, where + and − refer to the upper and the lower line segment, respectively. Similarly, denote by $I_{\text{ver}}^\pm$ the contributions coming from the two vertical parts of $\mathcal{L}_N$, where + and − refer to the right and the left line segment, respectively. Then

$$
f(z) - A_N^G f(z) = \frac{\varphi_{\Lambda}(z)}{2\pi i} (I_{\text{hor}}^+(z) + I_{\text{hor}}^-(z) + I_{\text{ver}}^+(z) + I_{\text{ver}}^-(z)). \quad (2.10)
$$

Since $|f(t + is)| \leq \|f\|_{\infty} e^{\sigma|s|}$ for $f \in B_\sigma^\infty$ (by Phragmén-Lindelöf principle) and $\varphi_{\Lambda}(z) = |\varphi_{\Lambda}(z)| e^{-\pi |1mz|} > 0$ on $\mathcal{L}_N$, we can estimate $I_{\text{hor}}^\pm$ and $I_{\text{ver}}^\pm$ as follows

$$
|I_{\text{hor}}^\pm(x + iy)| = \left| \int_{T_N^-}^{T_N^+} \frac{f(t + iS_N^+) G_N(x + iy - t - iS_N^+)}{\varphi_{\Lambda}(t + iS_N^+)(t + iS_N^+ - x - iy)} dt \right|
\leq \int_{T_N^-}^{T_N^+} \frac{\|f\|_{\infty} e^{\sigma|S_N^+|} |G_N(x - t + iy - iS_N^+)|}{\min_{\zeta \in \mathcal{L}_N} \phi_{\Lambda}(\zeta) e^{\pi|S_N^+|}} |t - x + iS_N^- - iy| dt
\leq \frac{\|f\|_{\infty}}{\min_{\zeta \in \mathcal{L}_N} \phi_{\Lambda}(\zeta)} e^{-(\pi - \sigma)|S_N^+|} \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - iS_N^+)| dt,
$$

$$
|I_{\text{ver}}^\pm(x + iy)| = \left| \int_{S_N^-}^{S_N^+} \frac{f(T_N^+ + is) G_N(x + iy - T_N^+ - is)}{\varphi_{\Lambda}(T_N^+ + is)(T_N^+ + is - x - iy)} ds \right|
\leq \int_{S_N^-}^{S_N^+} \frac{\|f\|_{\infty} e^{\sigma|s|} |G_N(x - T_N^+ + iy - is)|}{\min_{\zeta \in \mathcal{L}_N} \phi_{\Lambda}(\zeta) e^{\pi|s|}} |T_N^+ - x + is - iy| ds
\leq \frac{\|f\|_{\infty}}{\min_{\zeta \in \mathcal{L}_N} \phi_{\Lambda}(\zeta)} e^{-(\pi - \sigma)|s|} \int_{S_N^-}^{S_N^+} |G_N(x - T_N^+ + iy - is)| ds.
$$

Note that the choice of $\varphi_{\Lambda}$ do not change the series (2.8), but it may change the above estimates of $|I_{\text{hor}}^\pm(z)|$ and $|I_{\text{ver}}^\pm(z)|$, since we need a lower estimate of $\min_{\zeta \in \mathcal{L}_N} \phi_{\Lambda}(\zeta)$. 

4
3 Gaussian regularization

When \(G_N(z)\) are Gaussian functions \(G_N(z) = e^{-r(N)z^2}\), we can give more explicitly estimates of \(|I_{\text{hor}}^+(z)|\) and \(|I_{\text{ver}}^+(z)|\) so that we can estimate the error \(f(z) - A_N^G f(z)\) as follows.

**Theorem 3.1** Suppose \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) is separated, and for every \(\lambda_N < T^+_N < \lambda_{N+1}, \lambda_{N-1} < T^-_N < \lambda_N\), denote \(N_* = \min\{\lambda_N - T^-_N, T^+_N - \lambda_N\}\). Define the series \(A_N^G f\) as \((2.8)\), where \(G_N(z) = e^{-r^2z^2} + r^2N_*y\), then for \(N > 1\) and \(z = x + iy\) satisfying \(\lambda_N < x < \lambda_1\) and \(|y| < N_*\), we have

\[
|f(z) - A_N^G f(z)| \leq C_N(y) \frac{\|f\|_\infty |\varphi_A(z)|}{\pi \min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} e^{-\frac{z^2}{2N_*}},
\]

(3.11)

for every \(f \in B_\sigma^\infty\), \(0 < \sigma < \pi\), where \(\mathcal{L}_N\) is the rectangle with vertices at \(T^+_N + i(y \pm N_*)\) and

\[
C_N(y) = \sqrt{\frac{2\pi}{(\pi - \sigma)N_*}} \cosh((\pi - \sigma)y) + \frac{4}{(\pi - \sigma)N_*} (1 - (y/N_*)^2).
\]

**Proof:** For every \(z = x + iy\) satisfying the condition of the theorem, we choose \(S^+_N = y \pm N_*\), then \(|S^+_N| = N_* \pm y\), \(|S^-_N| = N_*\). Hence,

\[
|I^+_{\text{hor}}(z)| \leq \frac{\|f\|_\infty}{\min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} |S^+_N - y| e^{-\frac{(\pi - \sigma)|y|}{2N_*}} \int_{T^-_N}^{T^+_N} |G_N(x - t + iy - iS^+_N)| dt
\]

\[
\leq \frac{\|f\|_\infty e^{-\min_{\xi \in \mathcal{L}_N} (\pi - \sigma)|y|}}{\min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} e^{r^2N_*^2} \int_{-\infty}^{\infty} e^{-r^2(x - t)^2} dt
\]

\[
= \sqrt{\frac{2\pi}{(\pi - \sigma)N_*}} \min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)| N_* e^{-\frac{r^2N_*^2}{2}}.
\]

By the choice of \(N_*\), we have \(|T^+_N - x| \geq N_*\). Then,

\[
|I^+_{\text{ver}}(z)| \leq \frac{\|f\|_\infty}{\min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} |T^+_N - x| \int_{-\infty}^{S^+_N} e^{-\frac{(\pi - \sigma)|y|}{2N_*}} |G_N(x - T^+_N + iy - is)| ds
\]

\[
\leq \frac{\|f\|_\infty}{\min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} e^{-r^2N_*^2} \int_{-\infty}^{S^+_N} e^{2(y - s)^2 - (\pi - \sigma)|y|} ds
\]

\[
= \frac{\|f\|_\infty e^{-\frac{r^2N_*^2}{2}}}{\min_{\xi \in \mathcal{L}_N} |\varphi_A(\xi)|} N_* e^{r^2s^2 - (\pi - \sigma)|s + y|} ds.
\]

To estimate the last integral, we use the convexity of parabolas,

\[
r^2s^2 - (\pi - \sigma)|s + y| \leq \begin{cases} \frac{\pi - \sigma}{2} [y + (1 - y/N_*)s], & -N_* \leq s \leq -y, \\ -\frac{\pi - \sigma}{2} [y + (1 + y/N_*)s], & -y \leq s \leq N_*.
\end{cases}
\]
With these majorants, we obtain

\[ \int_{-N_*}^{-y} e^{2s^2-(\pi-\sigma)s+y} ds \leq \frac{2e^{(\pi-\sigma)y^2/(2N_*)}}{(\pi-\sigma)(1-y/N_*)} \]

and

\[ \int_{-y}^{N_*} e^{2s^2-(\pi-\sigma)s+y} ds \leq \frac{2e^{(\pi-\sigma)y^2/(2N_*)}}{(\pi-\sigma)(1+y/N_*)}. \]

As a consequence,

\[ |T_{\text{res.}}^{\pm}(z)| \leq \frac{4\|f\|_\infty}{(\pi-\sigma)\min_{z \in \mathcal{L}_N} \phi_\Lambda(\zeta)} \frac{e^{(\pi-\sigma)y^2/(2N_*)}}{1-(y/N_*)^2} e^{-\frac{y}{2}N_*}. \]

Combining these inequalities and using equality (2.10), we have (3.11). \( \square \)

Note that theorem 3.1 is a generalization of the result in [15] and we only use the fact that \( \Lambda \) is separated. In view of the proof, the choice of \( N_* \) is only used to get the lower estimate of \(|T_N^\pm - x|\). Thus, every \( N_* \leq \min\{\lambda_{-1} - T_N^-, T_N^+ - \lambda_1\} \) will work. So, in practice, we can simply let \( T_N^+ = \frac{\lambda_N + \lambda_{N-1}}{2}, \quad T_N^- = \frac{\lambda_N + \lambda_{N-1}}{2} \) and use the lower estimate of \(|\lambda_N|, |\lambda_{-N}|\) to choose \( N_* \), and then use \( N_* \) to calculate \( G_N(z) \) and the estimate (3.11).

In order to get a convergence rate of the regularized nonuniform sampling series, \( \Lambda \) need to be "dense enough" so that it is an oversampling for \( B^\infty_\sigma \). In view of theorem 3.1 it is natural to pose conditions on the function \( \varphi_\Lambda \).

**Corollary 3.2** Under the condition of theorem 3.1 and let \( T_N^+ = \frac{\lambda_N + \lambda_{N-1}}{2}, \quad T_N^- = \frac{\lambda_N + \lambda_{N-1}}{2} \). If there exists \( 0 < \delta < \bar{\delta}/2 \) such that

\[ |\varphi_\Lambda(z)| \geq C|z|^{-p}e^{\pi|\text{Im }z|} \text{ whenever } \text{dist}(z, \Lambda) > \delta \]

(3.12)

for some constants \( C > 0 \) and \( p \geq 0 \). Then

\[ |f(x) - A_N^{\varphi} f(x)| \leq \left( \sqrt{\frac{2\pi}{\pi - \sigma}} + \frac{4}{(\pi - \sigma)\sqrt{N_*}} \right) \frac{\|f\|_\infty|\varphi_\Lambda(x)|\bar{N}^p}{C\pi\sqrt{N_*}} e^{-\frac{\pi}{2} \bar{N}^p} \]

for every \( \lambda_{-1} < x < \lambda_1 \) and \( f \in B^\infty_\sigma \) with \( \sigma < \pi \), where \( \bar{N} = \sqrt{\max\{|T_N^+|^2, |T_N^-|^2\} + N_*^2} \).

**Proof:** By hypothesis, \( \phi_\Lambda(\zeta) \geq C|\bar{N}|^{-p} \) for every \( \zeta \in \mathcal{L}_N \). \( \square \)

If we further know the growth of \(|\lambda_N|\) and \(|\lambda_{-N}|\), then we can estimate \( N_* \) and \( \bar{N} \) and have a more explicit estimate of \(|f(x) - A_N^{\varphi} f(x)|\). We give some examples in section 5.

The condition (3.12) can be seen as a requirement of the density of \( \Lambda \). For instance, if \( \Lambda = a\mathbb{Z} \) for some \( a > 1 \), then \( \varphi_\Lambda(z) = \sin(\frac{\pi}{a}z) \) satisfying \(|\varphi_\Lambda(z)| \leq e^{\pi|\text{Im }z|} \). Thus, it cannot satisfy condition (3.12) for any \( p \). Actually, in this case, \( \Lambda \) is an under-sampling for \( B^\infty_\sigma \) with \( \sigma > \frac{\pi}{a} \), so that there is no hope to reconstruct \( f \in B^\infty_\sigma \) from its sample data on \( \Lambda \).
4 Hyper-Gaussian regularization

In this section, we consider the nonuniform sampling series (2.8) with \( G_N(z) = e^{-r_m(N)z^{2m}} \), where \( m > 1 \) is an integer and \( r_m(N) \) will be chosen later. We will use the Laplace’s method (see [11]) to estimate \( |I_{hor}| \).

**Lemma 4.1 (Laplace’s method)** Let \( f \) be a twice differentiable real-valued function on a finite interval \([a, b]\). Assume \( c \in (a, b) \) is the only maximum point of \( f \) on \([a, b] \), \( f'(c) = 0 \), and \( f''(c) < 0 \). Then

\[
\int_a^b e^{Nf(t)} dt = e^{Nf(c)} \left( \sqrt{\frac{2\pi}{-f''(c)N}} + o\left(\frac{1}{\sqrt{N}}\right) \right), \quad N \to +\infty.
\]

We firstly use the Laplace’s method to estimate an integral which will be used later.

**Lemma 4.2** Let \( m > 1 \) be an integer and \( h_m(t) = -\text{Re}(t + i)^{2m} \), then

\[
\int_0^\infty e^{Nh_m(t)} dt = e^{N(\sin \frac{\pi}{4m-2})^{1-2m}} \left( \sqrt{\frac{\pi}{m(2m-1)}} \left( \sin \frac{\pi}{4m-2} \right)^{m-\frac{3}{2}} \frac{1}{\sqrt{N}} + o\left(\frac{1}{\sqrt{N}}\right) \right)
\]

as \( N \to \infty \). Consequently, there exists a constant \( A_m \) such that for all \( N > 0 \),

\[
\int_0^\infty e^{Nh_m(t)} dt \leq \frac{A_m}{\sqrt{N}} e^{N(\sin \frac{\pi}{4m-2})^{1-2m}}.
\]

**Proof:** We first find the extrema of \( h_m \) on \([0, \infty)\). Since

\[ h'_m(t) = -2m \text{Re}(t + i)^{2m-1}, \]

the critical points of \( f \) are \( t_k \in [0, \infty) \) such that

\[ \arg(t_k + i) = \frac{\pi + 2k\pi}{4m-2}, \quad 0 \leq k \leq m - 1. \]

We can calculate the value at \( t_k \) as follow

\[ h_m(t_k) = -\left( \sin \frac{\pi + 2k\pi}{4m-2} \right)^{-2m} \cos \left( 2m \frac{\pi + 2k\pi}{4m-2} \right) = (-1)^k \left( \sin \frac{\pi + 2k\pi}{4m-2} \right)^{1-2m}. \]

Observe that \( t_0 = \cot \frac{\pi}{4m-2} \) is the only maximum point of \( h_m \) on \([0, \infty)\) and

\[ h''_m(t_0) = -2m(2m-1) \left( \sin \frac{\pi}{4m-2} \right)^{3-2m} < 0. \]

On the other hand, since

\[ \lim_{t \to +\infty} \frac{h_m(t)}{t} = -\infty, \]
there exists some \( a > t_0 \) such that
\[
\int_a^\infty e^{Nh_m(t)} dt \leq \int_a^\infty e^{-Nt} dt \leq \frac{1}{N}.
\]

By lemma 4.1,
\[
\int_0^a e^{Nh_m(t)} dt = e^{N(\sin \frac{\pi}{4m} - \frac{1}{2})} \left( \frac{1}{\sqrt{N}} + o \left( \frac{1}{\sqrt{N}} \right) \right)
\]
as \( N \to \infty \). Combining last two estimates prove the lemma.

Similar to the Laplace’s method, we will use the following lemma to estimate \(|I_{ver}^\pm|\).

**Lemma 4.3** Let \( f \) be a continuous real-valued function on a finite interval \([0, b]\). Assume that \( f(t) < f(0) \) for \( t \in (0, b] \) and \( (f(t) - f(0))/t \to -k \) when \( t \to 0 \) with \( k > 0 \). Then
\[
\int_0^b e^{Nf(t)} dt = e^{Nf(0)} \left( \frac{1}{kN} + o \left( \frac{1}{N} \right) \right), \quad N \to +\infty.
\]

**Proof:** Without loss of generality, we suppose that \( f(0) = 0 \). Then for any \( \epsilon > 0 \), the maximum of \( f(t) \) when \( t \geq \epsilon \) is negative so that we can write
\[
\int_0^b e^{Nf(t)} dt = \int_0^\epsilon e^{Nf(t)} dt + O(e^{-D\epsilon N})
\]
for some \( D_\epsilon > 0 \) depending on \( \epsilon \). Now, we expanse \( f(t) \) around \( t = 0 \):
\[
f(t) = -kt + o(t).
\]
For any two real numbers \( t_1, t_2 \) we have the inequality
\[
|e^{t_1} - e^{t_2}| \leq |t_1 - t_2|e^{t_3}, \quad t_3 = \max(t_1, t_2).
\]
Applying this with \( t_1 = Nf(t), \ t_2 = -Nkt \) for \( t \in [0, \epsilon] \), we can take \( t_3 \leq -Nkt/2 \) by taking \( \epsilon \) small enough. With this choice of \( \epsilon \) we can write
\[
|e^{Nf(t)} - e^{-Nkt}| \leq No(t)e^{-Nkt/2}, \quad 0 \leq t \leq \epsilon.
\]
Then,
\[
\int_0^\epsilon |e^{Nf(t)} - e^{-Nkt}| dt = o \left( \frac{1}{N} \right).
\]
Consequently,
\[
\left| \int_0^\epsilon e^{Nf(t)} dt - \frac{1}{kN} \right| = \left| \int_0^\epsilon e^{Nf(t)} dt - \int_0^\infty e^{-Nkt} dt \right|
\]
\[
\leq \int_0^\epsilon |e^{Nf(t)} - e^{-Nkt}| dt + \int_\epsilon^\infty e^{-Nkt} dt
\]
\[
= o \left( \frac{1}{N} \right) + \frac{1}{kN} e^{-Nk\epsilon},
\]
which prove the lemma. □

Now, we are ready to use similar arguments in theorem 3.1 to deduce an estimate in the Hyper-Gaussian regularization case. Note that in the following theorem, we use an extra assumption sup\(|\lambda_N - \lambda_{-N}| < \infty\), which is not needed in theorem 3.1.

**Theorem 4.4** Suppose \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) is separated and sup\(|\lambda_N + \lambda_{-N}| < \infty\), and for every \(\lambda_N < T_N^+ < \lambda_{N+1}, \lambda_{-N-1} < T_N^- < \lambda_{-N}\), denote \(N_* = \min\{\lambda_{-1} - T_N^-, T_N^- - \lambda_1\}\). Define the series \(A_N^G f\) as (2.8), where \(G_N(z) = e^{-r_m N z^{2m}}\) with the integer \(m > 1\), \(r_m N = \mu_m N^{1-2m}\) and

\[
\mu_m = \frac{2m - 1}{2m} (\pi - \sigma) b_m, \quad b_m = (2m - 1)^{-\frac{1}{2m}} \left(\sin \frac{\pi}{4m - 2}\right)^{\frac{2m - 1}{2m}},
\]

then for \(N > 1\) and \(z = x + iy\) satisfying \(\lambda_{-1} < x < \lambda_1\) and \(|y| < b_m N_*\), there exists a constant \(C_m\) depending on \(m, \sigma\) and \(\Lambda\) such that

\[
|f(z) - A_N^G f(z)| \leq C_m \left\| f \right\|_\infty |\varphi_\Lambda(z)| e^{(\pi - \sigma)|y|} \min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) \sqrt{N_*} e^{-\mu_m N_*},
\]

for every \(f \in B_\sigma^\infty\), \(0 < \sigma < \pi\), where \(\mathcal{L}_N\) is the rectangle with vertices at \(T_N^\pm + i(y \pm b_m N_*)\).

**Proof:** For every \(z = x + iy\) satisfying the condition of the theorem, we choose \(S_N^\pm = y \pm b_m N_*\), then \(|S_N^\pm| = b_m N_* \pm y\), \(|S_N^\pm - y| = b_m N_*\). Observing that \(\Re (at \pm ia)^{2m} = |a|^{2m} \Re (t \pm i)^{2m}\) for every \(a, t \in \mathbb{R}\), we have

\[
|I_{hor}^\pm(z)| \leq \frac{\left\| f \right\|_\infty}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) |S_N^\pm - y|} e^{-(\pi - \sigma)|y|} \int_{T_N^-}^{T_N^+} |G_N(x - t + iy - is_N^\pm)| dt
\]

\[
= \frac{\left\| f \right\|_\infty}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) |S_N^\pm - y|} e^{-(\pi - \sigma)|y|} \int_{T_N^-}^{T_N^+} e^{-r_m N \Re (x - t + iy - is_N^\pm)^{2m}} dt
\]

\[
\leq \frac{2\left\| f \right\|_\infty}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) |S_N^\pm - y|} e^{-(\pi - \sigma)|y|} \int_0^\infty e^{-r_m N |S_N^\pm - y|^{2m} \Re (t + i)^{2m}} dt
\]

\[
= \frac{2\left\| f \right\|_\infty e^{(\pi - \sigma)|y|}}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta)} e^{-(\pi - \sigma)b_m N_*} \int_0^\infty e^{-\mu_m b_m^2 N_* \Re (t + i)^{2m}} dt.
\]

By lemma 4.2 there exists a constant \(A_m\) such that

\[
|I_{hor}^\pm(z)| \leq \frac{2A_m \left\| f \right\|_\infty e^{(\pi - \sigma)|y|}}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) \sqrt{\mu_m b_m^2 N_* N_*}} e^{-(\pi - \sigma)b_m N_* + \mu_m b_m^2 N_* \sin (\frac{\pi}{4m - 2})^{1-2m}}
\]

\[
= \frac{2A_m \left\| f \right\|_\infty e^{(\pi - \sigma)|y|}}{\min_{\zeta \in \mathcal{L}_N} \phi_\Lambda(\zeta) \sqrt{\mu_m b_m^2 N_* N_*}} e^{-\mu N_*}.
\]
On the other hand,

$$
|I_{\text{ver}}^\pm(z)| \leq \frac{\|f\|_{\infty}}{\min_{\zeta \in L_N} \phi_\Lambda(\zeta)} \int_{S_N^+} e^{-(\pi - \sigma)|s|} |G_N(x - T_N^\pm + iy - is)| ds
$$

$$
= \frac{\|f\|_{\infty}}{\min_{\zeta \in L_N} \phi_\Lambda(\zeta)} \int_{S_N^+} e^{-(\pi - \sigma)|s|} -\sigma_m, N \Re(x - T_N^\pm + iy - is)^{2m} ds
$$

$$
\leq 2\|f\|_{\infty} e^{(\pi - \sigma)|y|} \int_0^{b_m N_s} e^{-(\pi - \sigma)|s|} -\sigma_m, N \Re(T_N^\pm + x + is)^{2m} ds
$$

$$
= 2\|f\|_{\infty} e^{(\pi - \sigma)|y|} \int_0^{b_m N_s} e^{-(\pi - \sigma)|s|} -\sigma_m, N \Re(T_N^\pm + x + is)^{2m} Re(1 + is)^{2m} ds
$$

$$
\leq 2\|f\|_{\infty} e^{(\pi - \sigma)|y|} \int_0^{b_m N_s} e^{-(\pi - \sigma)|s|} -\sigma_m, N \Re(T_N^\pm + x + is)^{2m} Re(1 + is)^{2m} ds.
$$

Since we assume that sup $\lambda_N + \lambda_{-N} < \infty$, we can decompose $|T_N^\pm - x| = N_s + k_N^\pm(x)$ for some bounded $k_N^\pm(x)$, therefore

$$
-\sigma_m, N |T_N^\pm - x|^{2m} = -\mu_m N_s - 2m k_N^\pm(x) + O(N_s^{-1}), \quad N_s \to \infty.
$$

Thus, there exists a constants $B_m$ such that

$$
|I_{\text{ver}}^\pm(z)| \leq \frac{2B_m\|f\|_{\infty} e^{(\pi - \sigma)|y|}}{\min_{\zeta \in L_N} \phi_\Lambda(\zeta)} \int_0^{b_m N_s} e^{N_s h_m(s)} ds,
$$

where $h_m(s) = -(\pi - \sigma)s - \mu_m \Re(1 + is)^{2m}$. If $h_m(s) < h_m(0) = -\mu_m$ for all $s \in (0, b_m]$, then by lemma 4.3 there exists a constant $B'_m$ such that

$$
|I_{\text{ver}}^\pm(z)| \leq \frac{2B'_m\|f\|_{\infty} e^{(\pi - \sigma)|y|}}{(\pi - \sigma) \min_{\zeta \in L_N} \phi_\Lambda(\zeta) N_s} e^{-\mu_m N_s}.
$$

To prove the theorem, it remains to show that $h_m(s) < h_m(0)$ for $s \in (0, b_m]$. Let $s = \tan \beta$, $\beta \in [0, \pi/2)$, then we can calculate its derivative:

$$
h'_m(s) = \sigma - \pi - (-1)^m 2m \mu_m \Re(s + i)^{2m-1} = \sigma - \pi + 2m \mu_m (\cos \beta)^{1-2m} \sin(2m - 1) \beta.
$$

Observe that $h'_m(s)$ is negative around $s = 0$ and increases on $[0, \tan \frac{\pi}{4m-2}]$. Since

$$
(2m - 1) \sin \frac{\pi}{4m-2} \geq 1 > \left(1 - \sin^2 \frac{\pi}{4m-2}\right)^{m} = \left(\cos \frac{\pi}{4m-2}\right)^{2m}, \quad \text{(4.13)}
$$

we have $b_m < \tan \frac{\pi}{4m-2}$. So the maximum point of $h_m(s)$ on $[0, b_m]$ is $s = 0$ or $s = b_m$. Therefore, we need to show that $h_m(b_m) < h_m(0) = -\mu_m$, which is equivalent to

$$
\Re(1 + ib_m)^{2m} + \frac{1}{2m - 1} > 0.
$$
By the definition of $b_m$, this inequality is

$$\text{Re} \left( \left( (2m - 1) \sin \frac{\pi}{4m - 2} \right)^{\frac{1}{2m}} + i \sin \frac{\pi}{4m - 2} \right)^{2m} > -\sin \frac{\pi}{4m - 2}.$$ 

We consider the function

$$F_m(t) = \text{Re} \left( t + i \sin \frac{\pi}{4m - 2} \right)^{2m}, \quad t \geq 0$$

and observe that $F_m(\cos(\frac{\pi}{4m - 2})) = \text{Re} e^{i \frac{2m}{4m - 2} \pi} = -\sin \frac{\pi}{4m - 2}$ and

$$F'_m(t) = 2m \text{Re} \left( t + i \sin \frac{\pi}{4m - 2} \right)^{2m-1} = 2m \left( \sin \frac{\pi}{4m - 2} \right)^{2m-1} \text{Re} (\cot \theta + i)^{2m-1},$$

where $\cot \theta = t/\sin(\frac{\pi}{4m - 2})$. As a consequence, when $t > \cos(\frac{\pi}{4m - 2})$, $F'_m(t) > 0$. Using inequality (4.13), we have $F_m((2m - 1) \sin(\frac{\pi}{4m - 2})^{1/2m}) > F_m(\cos(\frac{\pi}{4m - 2}))$, which is the inequality we want to prove. \qed

We remark that $\mu_m$ is monotonically decreasing as $m$ increases with $\mu_1 = \frac{\pi - \sigma}{2}$, which is the same exponent in theorem 3.1. Thus, judging by the exponential term, the Gaussian regularizer is the best among hyper-Gaussian regularizer.

Similar to the corollary 3.2, we have the following corollary.

**Corollary 4.5** Under the condition of theorem 4.4 and let $T^+_N = \frac{\lambda_N + \lambda_{N+1}}{2}$, $T^-_N = \frac{\lambda_N - \lambda_{N-1}}{2}$.

If there exists $0 < \delta < \delta_\Lambda/2$ such that

$$|\varphi_\Lambda(z)| \geq C |z|^{-p} e^{\pi |\text{Im} z|} \quad \text{whenever} \quad \text{dist} (z, \Lambda) > \delta$$

for some constants $C > 0$ and $p \geq 0$. Then

$$|f(x) - A_N f(x)| \leq C_m \frac{\|f\|_\infty |\varphi_\Lambda(x)| \tilde{N}^p}{C \sqrt{N_*}} e^{-\mu_m N_*},$$

for every $\lambda_1 < x < \lambda_1$ and $f \in B^\infty_\sigma$ with $\sigma < \pi$, where $\tilde{N} = \sqrt{\max\{|T^+_N|^2, |T^-_N|^2\} + N_*^2}$.

## 5 Examples

In this section, we provide several examples of Nonuniform sampling and prove that the corresponding regularized sampling sequences have linear convergence.
5.1 Uniform sampling sequence

The fundamental example is \( \Lambda = \mathbb{Z} \). In this case, \( \varphi_{\Lambda}(z) = \sin(\pi z) \) and

\[
A_{N}^{G}f(z) = \sum_{n=-N}^{N} f(n) \text{sinc}(x-n)G_{N}(x-n).
\]

We can choose \( T_{N}^{+} = -T_{N}^{-} = N + 1/2 \), \( N_{*} = N - 1/2 \), then \( \phi_{\Lambda} \geq 1/2 \) on the the rectangle \( \mathcal{L}_{N} \). Therefore, if \( -1 < x < 1 \) and \( G_{N}(x) = \exp(-\frac{\pi - \sigma}{2N-1}x^{2}) \), then

\[
|f(x) - A_{N}^{G}f(x)| \leq \left( \sqrt{\frac{2\pi}{\pi - \sigma}} + \frac{4}{(\pi - \sigma)\sqrt{N - \frac{1}{2}}} \right) \frac{2\|f\|_{\infty}|\sin(\pi x)|}{\pi \sqrt{N - \frac{1}{2}}} e^{-\frac{\pi}{2}(N - \frac{1}{4})}
\]

\[
= O(N^{-\frac{1}{2}}e^{-\frac{\pi}{2}(N - \frac{1}{4})})
\]

for every \( f \in B_{\sigma}^{\infty} \) with \( 0 < \sigma < \pi \). If \( G_{N}(x) = \exp(-\mu_{m}(N - 1/2)^{1/2}x^{2}) \), we have

\[
|f(x) - A_{N}^{G}f(x)| \leq C_{m}\frac{\|f\|_{\infty}|\sin(\pi x)|}{\sqrt{N - 1/2}} e^{-\mu_{m}(N - 1/2)} = O(N^{-1/2}e^{-\mu_{m}N}).
\]

Note that these results are already established in [12, 13, 15, 9, 17].

5.2 Zeros of a sine-type function

Definition 5.1 (Sine-type Function) An entire function \( \varphi \) of exponential type \( \pi \) is said to be a sine-type function if it has simple and separated zeros and there exist positive constants \( A, B, H \) such that

\[
Ae^{\pi|y|} \leq |\varphi(x + iy)| \leq Be^{\pi|y|} \quad \text{for all } x \in \mathbb{R} \text{ and } |y| \geq H.
\]

The zeros of a sine-type function lie in a horizontal strip and if we enumerate them in increasing order of their real parts then \( \Lambda \) satisfies (2.2) and

\[
\sup_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_{n}| < \infty.
\]

Moreover, for each \( \epsilon > 0 \), there exist constants \( M_{1} \) and \( M_{2} \) such that

\[
0 < M_{1} < |\varphi(z)|e^{-\pi|\text{Im}z|} < M_{2} < \infty, \quad \text{dist}(z, \Lambda) > \epsilon. \quad (5.14)
\]

Any sine-type function \( \varphi \) can be determined from its zero set \( \Lambda \) by (2.21). If \( \Lambda \subset \mathbb{R} \) is the zeros of a sine-type function, then \( \Lambda \) is a complete interpolating sequence for \( B_{\sigma}^{2} \). Consequently, it has uniform density: for every \( x \in \mathbb{R} \),

\[
D(\Lambda) = \lim_{r \to \infty} \frac{\#(\Lambda \cap [x, x + r])}{r} = 1.
\]
See [18, 3, 8, 5] for more details.

Here is a simple way to construct sine-type functions: For any function $g$ that can be represented as

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} h(\xi) e^{iz\xi} d\xi, \quad z \in \mathbb{C}$$

for some $h \in L^1[-\sigma, \sigma]$, we can define the sine wave crossings of $g$ as

$$\varphi_g(z) = A \sin(\pi z) - g(z), \quad z \in \mathbb{C}$$

with a constant $A > \|h\|_{L^1}$. These functions are all sine-type functions. Moreover, the zeros of $\varphi_g$ are all real and simple if $g$ is real on the real axis [1, 2, 4]. Therefore, given any function in $B^2_\pi$, we can construct sine-type function in this way. Note that the zeros of the sine-type function $\varphi(z) = \sin(\pi z)$ is the uniform sampling sequence $\Lambda = \mathbb{Z}$.

Now, suppose that $\Lambda \subset \mathbb{R}$ is the zeros of a sine-type function $\varphi_\Lambda$, then we know that

$$\delta_\Lambda := \inf_{n \in \mathbb{Z}} |\lambda_{n+1} - \lambda_n| > 0.$$ 

If we choose a $\epsilon < \delta_\Lambda/2$ in (5.14), and

$$T^+_N = \frac{\lambda_N + \lambda_{N+1}}{2}, \quad T^-_N = \frac{\lambda_{-N} + \lambda_{-N-1}}{2}, \quad S^+_N > \epsilon, \quad S^-_N < -\epsilon,$$

then by (5.14), $\phi_\Lambda > M_1$ on the rectangle $\mathcal{L}_N$. The density $D(\Lambda) = 1$ implies that

$$\lim_{N \to \infty} \frac{N}{|T^+_N|} = 1.$$ 

Thus, for every $0 < \eta < 1$, $|T^+_N| > \eta N$ for sufficiently large $N$. Therefore, for $\lambda_{-1} < x < \lambda_1$ and $f \in B^\infty_\pi$, if $G_N(x) = \exp(-\frac{x^2}{2\eta N})$, then

$$|f(x) - A^G_N f(x)| = O(N^{-\frac{1}{2}} e^{-\frac{\pi^2}{\eta^2} \eta N}) \quad \forall 0 < \eta < 1.$$ 

If $\sup_N |\lambda_N + \lambda_{-N}| < \infty$ and $G_N(x) = \exp(-\mu_m(\eta N)^{1-2m} x^{2m})$, we have

$$|f(x) - A^G_N f(x)| = O(N^{-\frac{1}{2}} e^{-\mu_m \eta N}) \quad \forall 0 < \eta < 1.$$ 

5.3 Perturbed uniform sequence

**Definition 5.2 (Perturbed Uniform Sequence)** The sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of real number is called a perturbed uniform sequence with $L \geq 0$ if

$$|\lambda_n - n| \leq L \quad (n \in \mathbb{Z}).$$

Suppose that $\Lambda$ is a perturbed uniform sequence with $L$. If $L < \frac{1}{2}$, the celebrated Kadets $\frac{1}{4}$ theorem (see [8]) shows that it is a complete interpolating sequence for $B^2_\pi$. However, when $L \geq \frac{1}{4}$, $\Lambda$ may not be a complete interpolating sequence. Nevertheless, the following lemma proved in [6] provides an estimate of the generating function $\varphi_\Lambda(z)$ of $\Lambda$ when $L < \frac{1}{2}$. 

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Lemma 5.3 Suppose that \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) is a perturbed uniform sequence with \( L < \frac{1}{2} \) and \( \lambda_0 = 0 \). Then the generating function \( \varphi_\Lambda(z) \) of \( \Lambda \) defined by (2.4) is an entire function, and there are constants \( C_1, C_2 \) such that for all \( z = |z|e^{i\theta} \in \mathbb{C} \) with \( |z| \) large,

\[
C_1 H_1(z) H_2(L, z) \leq |\varphi_\Lambda(z)| \leq C_2 H_1(z) H_2(-L, z),
\]

where

\[
H_1(z) := e^{\pi |\text{Im} z|} \prod_{k=N(z)}^{N(z)+2} |\lambda_k - z|, \quad |\text{Im} z| > 1, \quad \text{Im} z \leq 1 \quad \text{and} \quad \text{Re} z > 0,
\]

\[
\prod_{k=-N(z)-2}^{-N(z)} |\lambda_k - z|, \quad |\text{Im} z| \leq 1 \quad \text{and} \quad \text{Re} z < 0,
\]

\[
H_2(d, z) := \begin{cases} 
|z|^{-4d}, & 0 \leq |\sin \theta| \leq \sin(\pi/(2|z|)), \\
|z|^{-2d}|\sin \theta|^{2d}, & \sin(\pi/(2|z|)) < |\sin \theta| \leq 1,
\end{cases}
\]

\((N(z) \text{ is a suitable index and } d = L, -L \text{ respectively}).

If \( \Lambda \) is a perturbed uniform sequence with \( L < \frac{1}{2} \), then \( \delta_\Lambda \geq 1 - 2L \). For every \( \epsilon < \frac{1}{2} - L \), we have \( |\lambda_k - z| > \epsilon \) whenever \( \text{dist}(z, \Lambda) > \epsilon \). By lemma 5.3 there exists a constant \( C \) such that

\[
\phi_\Lambda(z) = |\varphi_\Lambda(z)|e^{-\pi |\text{Im} z|} \geq C|z|^{-4L} \quad \text{dist}(z, \Lambda) > \epsilon.
\]

Thus, if we choose \( T_N^z = -T_N^z = N + \frac{1}{2} \), then \( \phi_\Lambda(z) \geq C|z|^{-4L} \) on the rectangle \( \mathcal{L}_N \) and \( N - \frac{1}{2} - L \leq N_\epsilon \leq N - \frac{1}{2} + L \). Therefore, if \( \lambda_{-1} < x < \lambda_1 \) and \( G_N(x) = \exp(-\frac{\pi}{2N-2} x^2) \), then

\[
|f(x) - A_N^G f(x)| = O(N^{4L}e^{-\frac{\pi x^2}{2N-2}}).
\]

If \( G_N(x) = \exp(-\mu m (N-1)^{1-2m} x^{2m}) \), we have

\[
|f(x) - A_N^G f(x)| = O(N^{4L}e^{-\mu m N}).
\]

Note that when \( L = 0 \), the estimate reduce to the case \( \Lambda = \mathbb{Z} \).

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