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Left-invariant almost para-complex structures on six-dimensional nilpotent Lie groups

Abstract

There are five six-dimensional nilpotent Lie groups $G$, which do not admit neither symplectic, nor complex structures and, therefore, can be neither almost pseudo-Kähler, nor almost Hermitian. In this paper, we study precisely these Lie groups. The aim of the paper is to define new left-invariant geometric structures on the Lie groups under consideration that compensate, in some sense, the absence of symplectic and complex structures. Weakening the closedness requirement of left-invariant 2-forms $\omega$ on the Lie groups, non-degenerated 2-forms $\omega$ are obtained, whose exterior differential $d\omega$ is also non-degenerated in Hitchin sense [7]. Therefore, the Hitchin’s operator $K_{d\omega}$ is defined for the 3-form $d\omega$. It is shown that $K_{d\omega}$ defines an almost complex or almost para-complex structure for $G$ and the pair $(\omega, d\omega)$ defines pseudo-Riemannian metrics of signature (2.4) or (3.3), for which the Ricci operator is diagonal and has two eigenvalues differing in sign.

1 Introduction

Left-invariant Kähler structure on Lie group $G$ is a triple $(g, \omega, J)$ consisting of a left-invariant Riemannian metric $g$, left-invariant symplectic form $\omega$ and orthogonal left-invariant complex structure $J$, where $g(X, Y) = \omega(X, JY)$ for any left-invariant vectors fields $X$ and $Y$ on $G$. Therefore, such a structure on $G$ can be given by a pair $(\omega, J)$, where $\omega$ is a symplectic form, and $J$ is a complex structure compatible with $\omega$, that is, such that $\omega(JX, JY) = \omega(X, Y)$. If $\omega(X, JX) > 0$, $\forall X > 0$, it is Kähler metrics. If the positivity condition is not satisfied, then $g(X, Y) = \omega(X, JY)$ is a pseudo-Riemannian metric and then $(g, \omega, J)$ is called a pseudo-Kähler structure on the Lie group $G$. Classification of real six-dimensional nilpotent Lie algebras admitting invariant complex structures were obtained in [9]. Classification of symplectic structures on six-dimensional nilpotent Lie algebras was obtained in [6]. Out of 34 classes of isomorphic simply connected six-dimensional nilpotent Lie groups, only 26 admit left-invariant symplectic structures. Condition of existence of left-invariant positively definite metric on Lie group $G$ applies restrictions to the structure of its Lie algebra $g$. For example, it was shown in [2] that such a Lie algebra can not be nilpotent except for the abelian case. Although nilpotent Lie groups and nilmanifolds (except for torus) do not admit Kähler left-invariant metrics, on such manifolds left-invariant pseudo-Riemannian Kähler metrics may exist. It was shown in [5] that 14 classes of symplectic six-dimensional nilpotent Lie groups admit compatible complex structures and, therefore, define pseudo-Kähler metrics. A more complete study of the properties of the curvature of such pseudo-Kähler and almost pseudo-Kähler structures was carried out in [10][11]. In [4], it was shown that on a nilpotent Lie algebra of dimension up to six, Einstein metrics are Ricci-flat.

As mentioned before, 26 out of 34 classes of six-dimensional nilpotent Lie groups admit left-invariant symplectic structures. Out of last 8 classes of non-symplectic Lie groups, 5 Lie
groups $G_i$ do not also admit complex structures [9], their Lie algebras $\mathfrak{g}_i$ are shown below:

\[
\begin{align*}
\mathfrak{g}_1 &= (0,0,12,13,14+23,34-25), \\
\mathfrak{g}_2 &= (0,0,12,13,14,34-25), \\
\mathfrak{g}_3 &= (0,0,0,12,13,14+35), \\
\mathfrak{g}_4 &= (0,0,0,12,23,14+35), \\
\mathfrak{g}_5 &= (0,0,0,0,12,15+34).
\end{align*}
\]

In this paper we study precisely these Lie groups. The aim of the paper is to define new left-invariant geometric structures on the Lie groups under consideration that compensate, in some sense, the failure of symplectic and complex structures. It is shown that on all such Lie groups $G_i$ any left-invariant closed 2-form $\omega$ is degenerated. There are natural ways to weaken the closedness requirement of $\omega$ to preserve non-degeneracy $\omega$, in ways that 3-form $d\omega$ is also non-degenerated and property $\omega \wedge d\omega = 0$ is satisfied. Hitchin’s operator $K_{d\omega}$ corresponding to non-degenerated 3-form $d\omega$, can define either almost complex structure, or almost para-complex, depending on the chosen $\omega$. Associated metric $g(X,Y) = \omega(X,J(Y))$ is pseudo-Riemannian of signatures $(3,3)$ or $(2,4)$. The structural group is reduced to $SL(3,\mathbb{R})$ in case of signature $(3,3)$ and to $SU(1,2)$ in case of signature $(2,4)$. On the groups $G_2 - G_5$, these metrics have non-zero scalar curvature and the Ricci operator with two distinct sign eigenvalues. Pseudo-Riemannian metrics on the group $G_1$ are not Ricci-flat, but the scalar curvature may be zero for some values of the parameters. An explicit form of these metrics is presented. As a result, we obtain a compatible couple $(\omega,\Omega)$, where $\Omega = d\omega$. We present an explicit form of pseudo almost Hermitian half-flat and para-complex half-flat structures.

For any nilpotent Lie group $G$ with rational structure constants there exists a discrete subgroup $\Gamma$ such that $M = \Gamma \setminus G$ is a compact manifold called a nilmanifold. Therefore, all the results hold for the corresponding six-dimensional compact nilmanifolds.

All calculations were made in the Maple system according to the usual formulas for the geometry of left-invariant structures.

2 Preliminaries

Let $G$ be a real Lie group of dimension $m$ and $\mathfrak{g}$ be its Lie algebra. Lower central series of Lie algebra $\mathfrak{g}$ is decreasing sequence of ideals $C^0\mathfrak{g}, C^1\mathfrak{g}, \ldots$, being defined inductively: $C^0\mathfrak{g} = \mathfrak{g}$, $C^{k+1}\mathfrak{g} = [\mathfrak{g}, C^k\mathfrak{g}]$. Lie algebra $\mathfrak{g}$ is called nilpotent, if $C^k\mathfrak{g} = 0$ for some $k$. In this case, the minimum length of lower central series is called class (or step) of nilpotency. In other words, the Lie algebra class is equal to $s$, if $C^s\mathfrak{g} = 0$ and $C^{s-1}\mathfrak{g} \neq 0$. In this case, $C^{s-1}\mathfrak{g}$ lies in the center $Z(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The increasing central sequence $\{\mathfrak{g}_i\}$ was defined for nilpotent $s$-step Lie algebra,

\[
\mathfrak{g}_0 = 0 \subset \mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{s-1} \subset \mathfrak{g}_s = \mathfrak{g},
\]

where the ideals $\mathfrak{g}_i$ were defined inductively by the rule:

\[
\mathfrak{g}_i = \{X \in \mathfrak{g} | [X, \mathfrak{g}] \subseteq \mathfrak{g}_{i-1}\}, \quad l \geq 1.
\]

Particularly, $\mathfrak{g}_1$ is the center of Lie algebra. One can see from this sequence that nilpotency property is equivalent to existence of basis $\{e_1, \ldots, e_m\}$ of the Lie algebra $\mathfrak{g}$, for which

\[
[e_i, e_j] = \sum_{k>i,j} C^k_{ij} e_k, \quad 1 \leq i < j \leq m.
\]
Nilpotency is also equivalent to the existence of basis \( \{e^1, \ldots, e^m\} \) of left-invariant 1-forms on \( G \) such that
\[
de e^i \in \Lambda^2 \{e^1, \ldots, e^{i-1}\}, \quad 1 \leq i \leq m,
\]
where the right side is considered to be zero for \( i = 1 \). As is known, the exterior differential of a left-invariant 1-form is expressed through the structural constants of a Lie algebra by the formula \( 8 \):
\[
de e^k = - \sum_{i<j} C_{ij}^k e^i \wedge e^j,
\]
where \( \{e^1, \ldots, e^m\} \) is the dual basis in \( g^* \) to \( \{e_1, \ldots, e_m\} \). Therefore the structure of Lie algebra is given either by specifying nonzero Lie brackets or by differentials of basis left-invariant 1-forms. The Lie algebra \( g \) is often defined as an \( m \)-tuple based on a sequence of differentials \((0, \ 0, \ de^3, \ 0, \ de^3)\) of basis 1-forms, in the notation \( i\ j \) is used instead of \( e^{ij} = -e^i \wedge e^j \). For example, notation \((0, 0, 0, 0, 12, 14 + 23)\) denotes Lie algebra with structural equations:
\[
de e^1 = de^2 = de^3 = 0, \ de^4 = 0, \ de^5 = -e^1 \wedge e^2 \text{ and } de^6 = -e^1 \wedge e^4 = -e^2 \wedge e^3.
\]
Equivalently, the basis \( \{e_1, \ldots, e_6\} \) of \( g \) satisfies \( e_1, e_2 = e_5, \ \{e_1, e_4\} = [e_2, e_3] = e_6 \).

Left-invariant symplectic structure on Lie group \( G \) is a left-invariant closed 2-form \( \omega \) of the maximal rank. It is given by 2-form \( \omega \) of the maximal rank on Lie algebra \( g \). Closedness of the 2-form is equivalent to condition
\[
\omega([X, Y], Z) - \omega([X, Z], Y) + \omega([Y, Z], X) = 0, \quad \forall X, Y, Z \in g.
\]
In this case, Lie algebra \( g \) and group \( G \) will be called symplectic ones.

Left-invariant almost complex structure on Lie group \( G \) is left-invariant field of endomorphisms \( J : TG \rightarrow TG \) of tangent bundle \( TG \), having the property \( J^2 = -Id \). Since \( J \) is defined by linear operator \( J \) on Lie algebra \( g = T_eG \), we will say that \( J \) is a left-invariant almost complex structure on Lie algebra \( g \). In order for the almost complex structure \( J \) to define a complex structure on the Lie group \( G \), it is necessary and sufficient (according to the Newlander-Nirenberg theorem \( 8 \)) that the Nijenhuis tensor vanishes:
\[
[JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] = 0, \quad \forall X, Y \in g.
\]
For the left-invariant complex structure on Lie group \( G \) left shifts \( L_a : G \rightarrow G, \ a \in G \) are holomorphic.

Left-invariant Kähler structure on Lie group \( G \) is a triple \((g, \omega, J)\) consisting of a left-invariant Riemannian metric \( g \), left-invariant symplectic form \( \omega \) and orthogonal left-invariant complex structure \( J \), where \( g(X, Y) = \omega(X, JY), \ \forall X, Y \in g \). Therefore, such a structure on Lie group \( G \) can be specified by a couple \((\omega, J)\), where \( \omega \) is a symplectic form, and \( J \) is a complex structure being compatible with \( \omega \), i.e. such that \( \omega(JX, JY) = \omega(X, Y), \ \forall X, Y \in g \). If \( \omega(X, JY) > 0, \ \forall X \neq 0 \), then it is Kähler metric \( g(X, Y) = \omega(X, JY) \). But if the positivity condition is not fulfilled, then \( g(X, Y) \) is pseudo-Riemannian metric and then \( (g, \omega, J) \) is called pseudo-Kähler structure on Lie group \( G \). In further, (pseudo)Kähler structure will be specified by pair \((\omega, J)\) of compatible left-invariant complex and symplectic structures. It follows from left-invariance that (pseudo)Kähler structure \((g, \omega, J)\) can be given by the values \( J, \omega \) and \( g \) on Lie algebra \( g \) of the Lie group \( G \). In this case \((g, \omega, J, g)\) is called pseudo-Kähler Lie algebra.

Almost para-complex structure on \(2n\)-dimensional manifold \( M \) is a field \( P \) of endomorphisms of the tangent bundle \( TM \) such that \( P^2 = Id \), where ranks of eigen-distributions \( T^\pm M := \ker(Id \pm P) \) are equal. Almost para-complex structure \( P \) is called integrable if distributions
$T^\pm M$ are involutive. In this case, $P$ is called *para-complex structure*. A manifold $M$ supplied by (almost) para-complex structure $P$, is called (almost) para-complex manifold. The Nijenhuis tensor $N_P$ of almost para-complex structure $P$ is defined by equation

$$N_P(X, Y) = [X, Y] + [PX, PY] - P[PX, Y] - P[X, PY],$$

for all vector fields $X, Y$ on $M$. As in the case with complex structure, para-complex one $P$ is integrable if and only if $N_P = 0$.

Para-Kähler manifold can be defined as pseudo-Riemannian manifold $(M, g)$ with skew-symmetric para-complex structure $P$, that is parallel with respect to the Levi-Civita connection. If $(g, P)$ is a para-Kähler structure on $M$, then $\omega = g \cdot P$ is symplectic structure, and eigen-distributions $T^\pm M$, corresponding to eigen-values $\pm 1$ of field $P$, represent two integrable $\omega$-Lagrangian distributions. Therefore the para-Kähler structure can be identified with bi-Lagrangian structure $(\omega, T^\pm M)$, where $\omega$ is a symplectic structure, and $T^\pm M$ are the integrable Lagrangian distributions. In [1] presents a review of the theory para-complex structures, and the invariant para-complex and para-Kähler structures on homogeneous spaces of semi-simple Lie groups are considered in detail. It is shown that every invariant para-Kähler structure $P$ on $M = G/H$ defines a unique para-Kähler Einstein structure $(g, P)$ with given non-zero scalar curvature.

Since the 2-form $\omega$ is not closed, it is possible to consider the 3-form $d\omega$. In [7] Hitchin had defined the concept of non-degeneracy (stability) for 3-forms $\Omega$ and built a linear operator $K_\Omega$, whose square is proportional to identity operator $Id$. Recall the basic Hitchin’s constructions.

Let $V$ be a six-dimensional real vector space, $\mu$ be a volume form on $V$, and $\Lambda^3 V^*$ be the 20-dimensional linear space of skew-symmetric 3-forms on $V$. We shall take interior product $\iota_X \Omega \in \Lambda^2 V^*$ for the form $\Omega \in \Lambda^3 V^*$ and vector $X \in V$. Then $\iota_X \Omega \wedge \Omega \in \Lambda^5 V^*$. Natural pairing by the exterior product $V^* \otimes \Lambda^5 V^* \to \Lambda^6 V^* \cong \mathbb{R} \mu$ defines the isomorphism $A: \Lambda^5 V^* \to V$. Using $\Lambda^5 V^* \cong V$ we define linear map $K_\Omega: V \to V$ as

$$K_\Omega(X) = A(\iota_X \Omega \wedge \Omega).$$

In other words, $\iota_{K_\Omega(X) \mu} = \iota_X \Omega \wedge \Omega$. Define $\lambda(\Omega) \in \mathbb{R}$ as a trace of the square of $K_\Omega$:

$$\lambda(\Omega) = \frac{1}{6} \text{tr} K_\Omega^2.$$ 

The form $\Omega$ is called non-degenerated (or stable) if $\lambda(\Omega) \neq 0$.

It is shown in [7] that if $\lambda(\Omega) \neq 0$, then

- $\lambda(\Omega) > 0$ if and only if $\Omega = \alpha + \beta$, where $\alpha, \beta$ are real decomposable 3-forms and $\alpha \wedge \beta \neq 0$;
- $\lambda(\Omega) < 0$ if and only if $\Omega = \alpha + \overline{\alpha}$, where $\alpha \in \Lambda^3 (V^* \otimes \mathbb{C})$ is complex decomposable 3-form and $\alpha \wedge \overline{\alpha} \neq 0$.

It follows that if $\Omega$ is real and $\lambda(\Omega) > 0$, then it lies in $GL(V)$-orbit of form $\varphi = \theta^1 \wedge \theta^2 \wedge \theta^3 + \theta^4 \wedge \theta^5 \wedge \theta^6$ for some basis $\theta^1, \ldots, \theta^6$ in $V^*$, and if $\lambda(\Omega) < 0$, then it lies in orbit of form $\varphi = \alpha + \overline{\alpha}$, where $\alpha = (\theta^1 + i\theta^2) \wedge (\theta^3 + i\theta^4) \wedge (\theta^5 + i\theta^6)$.

Then the real 20-dimensional vector space $\Lambda^3 (V^*)$ contains invariant quadratic hypersurface $\lambda(\Omega) = 0$ dividing the $\Lambda^3 (V^*)$ to 2 open sets: $\lambda(\Omega) > 0$ and $\lambda(\Omega) < 0$. The component of the unit of the stabilizer of the 3-form lying in the first set is conjugate to the group $SL(3, \mathbb{R}) \times$
$SL(3, \mathbb{R})$, and in the other case to the group $SL(3, \mathbb{C})$. The linear transformation of $K_\Omega$ has \cite{7} the following properties: $\text{tr} K_\Omega = 0$ and $K_\Omega^2 = \lambda(\Omega) \text{Id}$. In the case $\lambda(\Omega) < 0$, the real 3-form $\Omega$ defines the structure $J_\Omega$ of complex vector space on real vector space $V$ as follows:

$$J_\Omega \frac{1}{\sqrt{-\lambda(\Omega)}} K_\Omega.$$

But if $\lambda(\Omega) > 0$, 3-form $\Omega$ defines the para-complex structure $J_\Omega$, i.e. $J_\Omega^2 = 1$, $J_\Omega \neq 1$ on real vector space $V$ by similar formula:

$$J_\Omega \frac{1}{\sqrt{\lambda(\Omega)}} K_\Omega.$$

Recall that the structure of almost a product is called para-complex, if eigen-subspaces have the same dimension.

The elements of $GL(V)$-orbits of 3-form $\Omega$, corresponding to $\lambda(\Omega) > 0$, have stabilizer $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ in $GL^+(V)$ and $J_\Omega$ is para-complex structure, i.e. $J_\Omega^2 = 1$, $J_\Omega \neq 1$. The elements of orbit corresponding to $\lambda(\Omega) < 0$, have stabilizer $SL(3, \mathbb{C})$ in $GL^+(V)$ and $J_\Omega$ is almost complex structure, i.e. $J_\Omega^2 = -1$. In both cases, dual to $\Omega$ form is defined by formula $\tilde{\Omega} = J_\Omega^2 \Omega$. If $\lambda(\Omega) > 0$ and $\Omega = \alpha + \beta$, then $\tilde{\Omega} = \alpha - \beta$. But if $\lambda(\Omega) < 0$ and $\Omega = \alpha + \beta$, then $\tilde{\Omega} = i(\beta - \alpha)$. Note that $\Omega \wedge = -\Omega$ in both cases and $J_\Omega = -\varepsilon J_\Omega$, where $\varepsilon$ is the sign of $\lambda(\Omega)$. The additional 3-form $\tilde{\Omega}$ has a defining property: if $\lambda(\Omega) > 0$, then $\Psi = \Omega + \tilde{\Omega}$ is decomposable, and if $\lambda(\Omega) < 0$, then complex form $\Psi = \Omega + i\tilde{\Omega}$ is decomposable.

The pair $(\omega, \Omega) \in \Lambda^2(V^*) \times \Lambda^3(V^*)$ of non-degenerated forms is called compatible if $\omega \wedge \Omega = 0$ (or, equivalently, $\tilde{\Omega} \wedge \omega = 0$), and it is called normalized, if $\tilde{\Omega} \wedge \Omega = 2\omega^3/3$.

Each compatible pair $(\omega, \Omega)$ uniquely defines $\varepsilon$-complex structure $J_\omega$ (i.e. $J_\omega^2 = \varepsilon$), scalar product $g_{(\omega, \Omega)}(X,Y) = \omega(X, J_\omega Y)$ (signatures $(3,3)$ for $\varepsilon = 1$ and signatures $(2,4)$ or $(4,2)$ for $\varepsilon = -1$), and (para-)complex volume form $\Psi = \Omega + i\varepsilon \tilde{\Omega}$ of type $(3,0)$ with respect to $J_\omega$ (where $i\varepsilon$ is a complex or para-complex imaginary unit). In addition, the stabilizer of $(\omega, \Omega)$ pair is $SU(p,q)$ for $\varepsilon = -1$ and $SL(3, \mathbb{R}) \subset SO(3,3)$ for $\varepsilon = 1$. Therefore, $(\omega, \Omega)$ pair for $\varepsilon = -1$ defines pseudo almost Hermitian structure. But if $\varepsilon = 1$, it defines almost para-Hermitian structure. Such structures are also called special almost $\varepsilon$-Hermitian.

Also recall that $SU(3)$ structure on real six-dimensional almost Hermitian manifold $(M, g, J, \omega)$ is specified by $(3,0)$-form $\Psi$. Almost Hermitian 6-manifold is called half-flat \cite{3} if it admits a reduction to $SU(3)$, for which $d\Re(\Psi) = 0$ and $\omega \wedge d\omega = 0$.

In the case of pseudo-Riemannian manifold, each compatible pair $(\omega, \Omega)$ uniquely defines the reduction to $SU(1,2)$ for $\varepsilon = -1$ and to $SL(3, \mathbb{R}) \subset SO(3,3)$ for $\varepsilon = 1$. Therefore, 6-manifold with the $(\omega, \Omega)$ pair possessing the properties $d\Omega = 0$ and $\omega \wedge d\omega = 0$, will be called half-flat pseudo almost Hermitian if it admits the reduction to $SU(1,2)$ or half-flat almost para-complex if it admits the reduction to $SL(3, \mathbb{R}) \subset SO(3,3)$.

**Remark.** In this work, we assume that exterior product and exterior differential are defined without normalizing constant. In particular, then $dx \wedge dy = dx \otimes dy - dy \otimes dx$ and $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$. Let $\nabla$ be the Levi-Civita connectivity corresponding to left-invariant (pseudo)Riemannian metric $g$. It is defined by six-membered formula \cite{3}, which becomes the following form for left-invariant vector fields $X,Y,Z$ on Lie group: $2g(\nabla_X Y, Z) = g([X,Y], Z) + g([Z,X], Y) + g(X, [Z,Y])$. If $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is a curvature tensor, then Ricci tensor $\text{Ric}(X,Y)$ for (pseudo)Riemannian metric $g$ is defined as a construction of a curvature tensor over the first and fourth (upper) indices.
3 Singular Lie groups

In this section Lie groups that do not admit neither symplectic, nor left-invariant complex structures will be considered. Such Lie groups will be called **singular**. It will be shown that they admit non-degenerated left-invariant 2-forms, whose exterior differentials are non-degenerated. In addition, they admit almost para-complex structures and pseudo-Riemannian metrics of signature \((3.3)\), which have a diagonal Ricci operator with two eigenvalues, differing by the sign.

3.1 Lie group \(G_1\)

Singular group \(G_1\) that does not admit neither symplectic, nor complex structures. Non-zero commutation relations:

\[
[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_2, e_3] = e_5, [e_3, e_4] = e_6, [e_2, e_5] = -e_6.
\]

Lie algebra \(\mathfrak{g}\) has ideals: \(C^1\mathfrak{g} = D^1\mathfrak{g} = \mathbb{R}\{e_3, e_4, e_5, e_6\}, C^2\mathfrak{g} = \mathbb{R}\{e_4, e_5, e_6\}, C^3\mathfrak{g} = \mathbb{R}\{e_5, e_6\}, C^4\mathfrak{g} = Z = \mathbb{R}e_6\) is the center of Lie algebra. Filiform Lie algebra. Does not admit half-flat structures [3].

Let \(\omega = a_{ij}e_i \wedge e_j\) be any left-invariant 2-form. For the general form \(\omega\) Hitchin’s operator \(K_\omega\) has quite complicated form and the following function \(\lambda(d\omega)\):

\[
\lambda = 4 \left( a_{16}a_{56} + 4a_{35}a_{56}^2 + 4a_{26}a_{56} - 4a_{36}a_{46} \right) a_{56} + a_{46}^2.
\]

Thus, in general, 3-form \(d\omega\) is non-degenerated. It is easy to see that 2-form \(\omega\) is closed if and only if \(a_{16} = a_{26} = a_{36} = a_{35} = a_{45} = a_{46} = a_{56} = 0\) and \(a_{34} = -a_{25}, a_{24} = a_{15}\). However, such 2-form \(\omega\) is degenerated. There are several natural ways to weaken the closedness requirement of the 2-form \(\omega\), so as not to lose the nondegeneracy of \(\omega\) and \(d\omega\).

3.1.1 Option 1.

In that case we will not suppose that coefficients \(a_{46}\) and \(a_{56}\), which essentially occur in the expression for function \(\lambda(d\omega)\), are non-zero. Moreover, we will suppose that \(a_{56} \neq 0\). Then the property \(\omega \wedge d\omega = 0\) is fulfilled under condition \(a_{15} = 0, a_{25} = 0\) and \(a_{12}a_{56} = a_{13}a_{46}, a_{23} = -a_{14}\). 2-form \(\omega\) is non-degenerated under condition that \(a_{14}a_{56} \neq 0\) and \(\omega\) and \(d\omega\) become

\[
\omega = e^1 \wedge (a_{13}a_{46}/a_{56} e^2 + a_{13} e^3 + a_{14} e^4) - a_{14} e^2 \wedge e^3 + a_{46} e^4 \wedge e^5 + a_{56} e^5 \wedge e^6,
\]

\[
d\omega = -a_{46} e^{136} + a_{46} e^{245} - a_{56} e^{146} - a_{56} e^{236} + a_{56} e^{345}.
\]

The function \(\lambda(d\omega)\) of the Hitchin’s operator [7] for 3-form \(d\omega\) becomes \(\lambda = a_{46}^2\). The Hitchin’s operator \(K_{d\omega}\) has a matrix

\[
K_{d\omega} = \begin{bmatrix}
-a_{46}^2 & -2 a_{46} a_{56} & -2 a_{56}^2 & 0 & 0 & 0 \\
0 & a_{46}^2 & 2 a_{46} a_{56} & 2 a_{56}^2 & 0 & 0 \\
0 & 0 & -a_{46}^2 & -2 a_{46} a_{56} & 0 & 0 \\
0 & 0 & 0 & a_{46}^2 & 0 & 0 \\
0 & 0 & 0 & 0 & a_{46}^2 & 2 a_{56}^2 \\
0 & 0 & 0 & 0 & 0 & -a_{46}^2
\end{bmatrix}
\]
Determine the operator $P = K_{d\omega}/a_{46}^2$. It defines left-invariant almost para-complex structure, $P^2 = Id$, having the property $\omega(PX, PY) = -\omega(X, Y)$. Eigen-subspaces $E^{\pm}$ related to the eigen-values $\pm 1$ of operator $P$ are generated by the following vectors:

$$E^+ = \{ a^3_{50} e_1 - a_{56} a^2_{46} e_3 + a^3_{46} e_4, -a_{56} e_1 + a_{46} e_2, e_5 \},$$

$$E^- = \{ e_1, -a_{56} e_2 + a_{46} e_3, -a^2_{56} e_5 + a^2_{46} e_6 \}.$$

It is easy to see that they are not closed relative to Lie bracket, so $P$ defines non-integrable almost para-complex structure.

Define pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$ of signature $(3, 3)$. It is given by:

$$g = \begin{bmatrix}
0 & a_{13} a_{14} / a_{56} & a_{13} & a_{14} & 0 & 0 \\
a_{13} a_{14} / a_{56} & 2 a_{13} & 2 a_{13} a_{56} a_{14} / a_{46} & 2 a_{14} a_{56} / a_{46} & 0 & 0 \\
a_{13} & 2 a_{13} a_{56} a_{14} / a_{46} & 2 a_{56} (a_{13} a_{56} a_{14}) / a_{46}^2 & 2 a_{14} a_{56} / a_{46}^2 & 0 & 0 \\
a_{14} & 2 a_{14} a_{56} / a_{46} & 2 a_{14} a_{56}^2 / a_{46}^2 & 0 & 0 & -a_{46} \\
0 & 0 & 0 & 0 & 0 & -a_{56} \\
0 & 0 & 0 & -a_{46} & -a_{56} & -2 a_{56} a_{46}^2 / a_{46}^2
\end{bmatrix}.$$  

Direct calculations of curvature tensor in Maple system show that this metric is not Einsteinian and that it has scalar curvature

$$R = \frac{8 a_{13} a_{56}^7 - 8 a_{14} a_{46} a_{56}^6 - a_{46}^8}{a_{14} a_{46}^2 a_{56}}.$$  

For $a_{13} := (a_{46}^8 + 8 a_{56} a_{46} a_{14})/(8 a_{56}^2)$ we get an example of a pseudo-Riemannian metric with zero scalar curvature and non-zero Ricci tensor.

3.1.2 Option 2.

Take the 2-form $\omega$ in the view $\omega = \omega_0 + \omega_C$, where $\omega_0$ is a closed 2-form and $\omega_C$ is a non-degenerated 2-form on the ideality $C^2 g = \mathbb{R}\{e_4, e_5, e_6\}$. We require that the 2-form $\omega$ should have the property $\omega \wedge d\omega = 0$:

$$a_{25} = 0, a_{15} = 0, a_{12} a_{56} = a_{13} a_{46}, a_{56} a_{23} + a_{14} a_{56} = 0.$$  

Then $\omega$ is non-degenerated under condition $a_{14} a_{56} \neq 0$. The $\omega$ and $d\omega$ take the view:

$$\omega = e^1 \wedge (a_{13} a_{46} / a_{56} e^2 + a_{13} e^3 + a_{14} e^4) - a_{14} e^2 \wedge e^3 + a_{45} e^4 \wedge e^5 + a_{46} e^4 \wedge e^6 + a_{56} e^5 \wedge e^6,$$

$$d\omega = a_{45} e^{234} - a_{45} e^{135} - a_{46} e^{136} + a_{46} e^{245} - a_{56} e^{146} - a_{56} e^{236} + a_{56} e^{345}.$$  

In this case, $\lambda(d\omega)$ function is expressed by the $\lambda = a_{46}^4 - 4 a_{46} a_{45} a_{56}^2$ and it can take both positive and negative values.
3.1.3 Case 1
The function $\lambda(d\omega)$ takes the value $-1$ when $a_{45} = (a_{46}^4 + 1)/(4a_{46}a_{56}^2)$. Then the operator $J = K_{d\omega}$ defines almost complex structure compatible with $\omega$ and has the form:

$$J = \begin{bmatrix}
-a_{46}^2 & -2a_{46}a_{56} & -2a_{56}^2 & 0 & 0 & 0 \\
\frac{1+a_{46}^4}{2a_{46}a_{56}} & a_{46}^2 & 2a_{46}a_{56} & 2a_{56}^2 & 0 & 0 \\
0 & 0 & -a_{46}^2 & -2a_{46}a_{56} & 0 & 0 \\
0 & 0 & \frac{1+a_{46}^4}{2a_{46}a_{56}} & a_{46}^2 & 0 & 0 \\
0 & 0 & -\frac{1+a_{46}^4}{2a_{56}^2} & -\frac{1+a_{46}^4}{2a_{46}a_{56}} & a_{46}^2 & 2a_{56}^2 \\
0 & 0 & \frac{(1+a_{46}^4)^2}{8a_{46}^2a_{56}^4} & 0 & -\frac{1+a_{46}^4}{2a_{56}^2} & -a_{46}^2
\end{bmatrix}$$

Specify the associated pseudo-Riemannian metric by formula $g(X, Y) = \omega(X, JY)$ of signature $(2,4)$. Direct calculations of curvature tensor in Maple system show that this metric is not Einsteinian and that it has scalar curvature

$$R = \frac{8a_{56}^7a_{13} - 8a_{56}^6a_{46}a_{14} - 1}{a_{14}^2a_{56}}$$

For $a_{13} := (1 + 8a_{56}^6a_{46}a_{14})/(8a_{56}^7)$ we get an example of a pseudo-Riemannian metric with zero scalar curvature and non-zero Ricci tensor.

3.1.4 Case 2
The function $\lambda(d\omega)$ takes the value $+1$ when $a_{45} = (a_{46}^4 - 1)/(4a_{46}a_{56}^2)$. Then the operator $P = K_{d\omega}$ defines almost para-complex structure compatible with $\omega$ and $P$ has the same matrix, as the above almost complex structure $J$ has, where it is necessary to substitute $(a_{46}^4 - 1)$ instead of $(a_{46}^4 + 1)$. The associated metric $g(X, Y) = \omega(X, PY)$ is pseudo-Riemannian of signature $(3,3)$; it is not the Einsteinian one and has the same scalar curvature as in the first case.

Conclusions. Any left-invariant closed 2-form $\omega$ on Lie group $G_1$ is degenerated. There are several ways to weaken the closedness requirement of $\omega$ to preserve non-degeneracy $\omega$, in ways that 3-form $d\omega$ is non-degenerated and the property $\omega \wedge d\omega = 0$ is fulfilled. Hitchin’s operator $K_{d\omega}$ corresponding to non-degenerated 3-form $d\omega$, can define either almost complex structure, or almost para-complex, depending on the chosen $\omega$. Associated metric $g(X, Y) = \omega(X, J\omega Y)$ is pseudo-Riemannian of signature $(2,4)$ or $(3,3)$. As a result, we have obtained a compatible pair $(\omega, \Omega)$, where $\Omega = d\omega$. Therefore, the properties $d\Omega = 0$ and $\omega \wedge d\omega = 0$ are fulfilled in an obvious way. The $(3,0)$-form has a view of $\Psi = d\omega + i_\xi d\omega$, where $i_\xi$ is a complex or para-complex unit. Thus, half-flat pseudo almost Hermitian and half-flat para-complex structures were naturally defined on Lie group $G_1$.

3.2 Lie group $G_2$
Singular group $G_2$ that does not admit neither symplectic, nor complex structures. Commutation relations

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5, [e_3, e_4] = e_6, [e_2, e_5] = -e_6.$$
Lie algebra $\mathfrak{g}$ has ideals: $C^1 \mathfrak{g} = D^1 \mathfrak{g} = \mathbb{R}\{e_3, e_4, e_5, e_6\}$, $C^2 \mathfrak{g} = \mathbb{R}\{e_4, e_5, e_6\}$, $C^3 \mathfrak{g} = \mathbb{R}\{e_5, e_6\}$, $C^4 \mathfrak{g} = \mathbb{Z} \mathbb{R}\{e_6\}$ is the center of Lie algebra. Filiform Lie algebra. Does not admit half-flat structures [3].

Let $\omega = a_{ij} e_i \wedge e_j$ be any left-invariant non-degenerated 2-form. For such a generic 2-form the square of the Hitchin’s operator [7] for 3-form $d\omega$ has a diagonality: $K_{d\omega} = (a_{46}^2 - 2a_{36}a_{56})^2 Id$. Therefore, 3-form $d\omega$ is non-degenerated if $a_{46}^2 - 2a_{36}a_{56} \neq 0$. The 2-form $\omega$ is closed only in the case when it has the form:

$$\omega = e^1 \wedge (a_{12} e^2 + a_{13} e^3 + a_{14} e^4 + a_{15} e^5) + e^2 \wedge (a_{23} e^3 - a_{34} e^5) + a_{34} e^3 \wedge e^4.$$  

Such 2-form $\omega$ is non-degenerated. In order to preserve the non-degeneracy of the $\omega$ and $d\omega$ at the minimal weakening of closedness property of $\omega$, two variants are possible: $a_{46} \neq 0$, or $a_{36} \neq 0$ and $a_{56} \neq 0$. However, if $a_{56} \neq 0$, then simple calculations show that the property $\omega \wedge d\omega = 0$ is incompatible with the non-degeneracy $\omega$.

Therefore, consider a case when $a_{46} \neq 0$. Then $K_{d\omega} = a_{46}^4 Id$. In addition, $\omega \wedge d\omega = 0$ under condition that $a_{13} = 0$ and $a_{34} = 0$. Then the 2-form $\omega$ is non-degenerated under condition $a_{23}a_{15}a_{46} \neq 0$, and the $\omega$ and $d\omega$ take the view:

$$\omega = e^1 \wedge (a_{12} e^2 + a_{14} e^4 + a_{15} e^5) + a_{23} e^2 \wedge e^3 + a_{46} e^4 \wedge e^6,$$

$$d\omega = a_{46}(-e^{136} + e^{245}).$$

The operator $K_{d\omega}$ for the 3-form $d\omega$ has the diagonal form: $K_{d\omega} = \text{diag}\{-a_{36}^2, a_{46}^2, -a_{46}^2, a_{46}^2, a_{46}^2, -a_{46}^2\}$. Define the operator $P = K_{d\omega}/a_{46}^2$. It defines left-invariant almost para-complex structure, $P^2 = Id$, having the property $\omega(PX, PY) = -\omega(X, Y)$. Eigen-subspaces $E^\pm$ related to the eigen-values $\pm 1$ of operator $P$ are generated by the following vectors:

$$E^+ = \{e_2, e_4, e_5\}, E^- = \{e_1, e_3, e_6\}.$$  

It is easy to see that they are not closed relative to Lie bracket, so $P$ defines non-integrable almost a para-complex structure.

Define the pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$. It has a signature $(3,3)$ and it is given by:

$$g = \begin{bmatrix}
0 & a_{12} & 0 & a_{14} & a_{15} & 0 \\
-a_{12} & 0 & -a_{23} & 0 & 0 & 0 \\
0 & a_{23} & 0 & 0 & 0 & 0 \\
-a_{14} & 0 & 0 & 0 & 0 & -a_{46} \\
a_{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{46} & 0 & 0
\end{bmatrix}$$

Direct calculations in the Maple system show that for $a_{14} = 0$, this metric has a diagonal Ricci operator with two eigenvalues differing by the sign:

$$RIC = \frac{a_{46}}{2a_{15}a_{23}} \text{diag}\{-1, -1, -1, +1, -1, +1\}.$$
3.3 Lie group $G_3$

Singular group $G_3$ that does not admit neither symplectic, nor complex structures. Commutation relations

$$[e_1, e_2] = e_4, [e_1, e_3] = e_5, [e_1, e_4] = e_6, [e_3, e_5] = e_6.$$ 

Lie algebra $\mathfrak{g}$ has ideals: $C^1\mathfrak{g} = D\mathfrak{g} = \mathbb{R}\{e_4, e_5, e_6\}$, $C^2\mathfrak{g} = \mathbb{R}\{e_6\} = Z$ is the center of Lie algebra. Does admit half-flat structure [3].

Let $\omega = a_{ij}e_i \wedge e_j$ be any 2-form. The Hitchin’s operator $K_{d\omega}$ for generic 2-form $\omega$ has a quite complicated view. Moreover, $K^2_{d\omega} = a^4_{16}Id$. For $\lambda = a^4_{16} \neq 0$ the 3-form $d\omega$ is non-degenerated. The operator $P = K_{d\omega}/a^2_{16}$ defines the left-invariant almost para-complex structure on $\mathfrak{g}$. The property is fulfilled under the following conditions:

$$a_{12} a_{46} - a_{14} a_{26} - a_{23} a_{56} + a_{24} a_{16} + a_{25} a_{36} - a_{35} a_{26} = 0,$$

$$a_{25} a_{46} - a_{24} a_{56} - a_{26} a_{45} = 0, \quad a_{35} a_{46} - a_{36} a_{45} - a_{34} a_{56} = 0.$$

It is easy to see that the 2-form $\omega$ is closed only if

$$\omega = e^1 \wedge (a_{12}e^2 + a_{13}e^3 + a_{14}e^4 + a_{15}e^5) + e^2 \wedge a_{23}e^3 + a_{24}e^4 + a_{25}e^5) + e^3 \wedge (a_{25}e^4 + a_{35}e^5).$$

In order to preserve the non-degeneracy of the $\omega$ and $d\omega$ at the minimal weakening of closedness property of $\omega$, consider the case when $a_{46} \neq 0$. The $\omega \wedge d\omega = 0$ property is fulfilled under the condition $a_{12} = 0, a_{25} = 0, a_{35} = 0$. Thus, 2-form $\omega$ is non-degenerated if $a_{15}a_{23}a_{46} \neq 0$ and then we obtain:

$$\omega = e^1 \wedge (a_{13}e^3 + a_{14}e^4 + a_{15}e^5) + e^2 \wedge (a_{23}e^3 + a_{24}e^4) + a_{46} e^4 \wedge e^6,$$

$$d\omega = -a_{46}(e^{126} + e^{345}).$$

The operator $K_{d\omega}$ for the 3-form $d\omega$ has the diagonal view, $K_{d\omega} = \text{diag}\{-a^2_{46}, -a^2_{16}, a^2_{16}, a^2_{16}, a^2_{46}, -a^2_{46}\}$. Define the operator $P = K_{d\omega}/a^2_{16}$. It defines left-invariant almost para-complex structure, $P^2 = Id$, having the property $\omega(PX, PY) = -\omega(X, Y)$. Eigen-subspaces $E^\pm$ related to the eigen-values $\pm 1$ of operator $P$ are generated by the following vectors:

$$E^+ = \{e_3, e_4, e_5\}, \quad E^- = \{e_1, e_2, e_6\}.$$ 

It is easy to see that they are not closed relative to Lie bracket, so $P$ sets non-integrable almost a para-complex structure.

Define the pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$ of signature (3,3). It is given by:

$$g = \begin{bmatrix}
0 & 0 & a_{13} & a_{14} & a_{15} & 0 \\
0 & 0 & a_{23} & a_{24} & 0 & 0 \\
a_{13} & a_{23} & 0 & 0 & 0 & 0 \\
a_{14} & a_{24} & 0 & 0 & 0 & -a_{46} \\
a_{15} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{46} & 0 & 0
\end{bmatrix}$$

Direct calculations in the Maple system show that for $a_{14} = a_{24} = 0$, this metric has a diagonal Ricci operator with two eigenvalues differing by the sign:

$$RIC = \frac{a_{46}}{2a_{15}a_{23}} \text{diag}\{-1, -1, -1, +1, -1, +1\}.$$
3.4 Lie group $G_4$

Singular group that does not admit neither symplectic, nor complex structures. Commutation relations:

$$[e_1, e_2] = e_4, [e_2, e_3] = e_5, [e_1, e_4] = e_6, [e_3, e_5] = e_6.$$  

Lie algebra $\mathfrak{g}$ has ideals: $C^1\mathfrak{g} = D^1\mathfrak{g} = \mathbb{R}\{e_4, e_5, e_6\}$, $C^2\mathfrak{g} = \mathbb{R}\{e_6\} = \mathbb{Z}$ is the center of Lie algebra. Does admit half-flat structure [3].

Let $\omega = e_{ij}e_i \wedge e_j$ be any 2-form. The Hitchin’s operator $K_{d\omega}$ for generic 2-form $\omega$ has a quite complicated form. Moreover, $K_{d\omega}^2 = (a_{46}^2 - a_{56}^2)^2 Id$. The $\omega \wedge d\omega = 0$ property is fulfilled under the following conditions:

$$-a_{12} a_{46} + a_{14} a_{26} - a_{16} a_{24} + a_{23} a_{56} - a_{25} a_{36} + a_{26} a_{35} = 0,$$

$$-a_{14} a_{56} + a_{15} a_{46} - a_{16} a_{45} = 0, \quad a_{34} a_{56} - a_{35} a_{46} + a_{36} a_{45} = 0.$$  

It is easy to see that the 2-form $\omega$ is closed only if

$$\omega = e^1 \wedge (a_{12} e^2 + a_{13} e^3 + a_{14} e^4 + a_{15} e^5) + e^2 \wedge (a_{23} e^3 + a_{24} e^4 + a_{25} e^5) + e^3 \wedge (-a_{15} e^4 + a_{35} e^5).$$

In order to preserve the non-degeneracy of the $\omega$ and $d\omega$ at the minimal weakening of closedness property of $\omega$, consider the case when $a_{46} \neq 0$ and $a_{56} \neq 0$. The $\omega \wedge d\omega = 0$ property is fulfilled under the following conditions:

$$-a_{12} a_{46} + a_{23} a_{56} = 0, \quad -a_{14} a_{56} + a_{15} a_{46} = 0, \quad -a_{15} a_{56} - a_{35} a_{46} = 0$$

for generic 2-form $\omega$ has the same view $\lambda = (a_{46}^2 - a_{56}^2)^2$. And operator $K_{d\omega}$ is given by:

$$K_{d\omega} = (-a_{46}^2 - a_{56}^2) e_1 \otimes e^1 + (-a_{46}^2 + a_{56}^2) e_2 \otimes e^2 + (a_{46}^2 + a_{56}^2) e_3 \otimes e^3 + (a_{46}^2 - a_{56}^2) e_4 \otimes e^4 +$$

$$+(a_{46}^2 - a_{56}^2) e_5 \otimes e^5 + (-a_{46}^2 + a_{56}^2) e_6 \otimes e^6 + 2a_{46} a_{56} e_1 \otimes e^3 - 2a_{46} a_{56} e_3 \otimes e^1.$$  

Define the operator $P = K_{d\omega}/|a_{46}^2 - a_{56}^2|$. It defines left-invariant almost para-complex structure, having the property $\omega(PX, PY) = -\omega(X, Y)$. Eigen-subspaces related to eigen-values $(a_{46}^2 - a_{56}^2)/|a_{46}^2 - a_{56}^2|$ and $-(a_{46}^2 - a_{56}^2)/|a_{46}^2 - a_{56}^2|$ of the operator $P$ are generated by the following vectors:

$$E_1 = \{a_{56} e_1 + a_{46} e_3, e_4, e_5 \}, \quad E_2 = \{a_{46} e_1 + a_{56} e_3, e_2, e_6 \}.$$  

It is easy to see that they are not closed relative to Lie bracket, so $P$ sets non-integrable almost a para-complex structure.

Define the pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$ of signature $(3,3)$. Direct calculations in the Maple system show that this metric has a non-diagonal Ricci operator and the following scalar curvature

$$R = \frac{|a_{46}^2 - a_{56}^2|}{a_{13}(a_{24} a_{56} - a_{25} a_{46})^2}.$$  

In particular case, when one of the arguments $a_{46}$ or $a_{56}$ is equal to zero, the situation becomes much simpler. For example, let $a_{56} = 0$. The property $\omega \wedge d\omega = 0$ is fulfilled under
the following conditions:  \( a_{12} = 0, a_{15} = 0, a_{35} = 0 \). Then 2-form \( \omega \) is non-degenerated under the condition \( a_{13}a_{25}a_{46} \neq 0 \), and we obtain:

\[
\omega = e^1 \wedge (a_{13}e^3 + a_{14}e^4) + e^2 \wedge (a_{23}e^3 + a_{24}e^4 + a_{25}e^5) + a_{46}e^4 \wedge e^6,
\]

\[
d\omega = -a_{46}e^{126} - a_{46}e^{345},
\]

\[
K_{d\omega} = \text{diag}\{a_{46}^2, -a_{46}^2, a_{46}^2, a_{46}^2, a_{46}^2, -a_{46}^2\}.
\]

Define the operator \( P = K_{d\omega}/a_{46}^2 \). It specifies almost para-complex structure, \( P^2 = \text{Id} \), possessing the property \( \omega(PX, PY) = -\omega(X, Y) \). Eigen-subspaces related to the eigen-values \( \pm 1 \) of operator \( P \) are generated by the following vectors:

\[
E^+ = \{e_3, e_4, e_5\}, \quad E^- = \{e_1, e_2, e_6\}.
\]

It is easy to see that they are not closed relative to Lie bracket, so \( P \) sets non-integrable almost a para-complex structure.

The pseudo-Riemannian metric \( g(X, Y) = \omega(X, PY) \) of signature (3,3) is given by

\[
g = \begin{bmatrix}
0 & 0 & a_{13} & a_{14} & 0 & 0 \\
0 & 0 & a_{23} & a_{24} & a_{25} & 0 \\
a_{13} & a_{23} & 0 & 0 & 0 & 0 \\
a_{14} & a_{24} & 0 & 0 & 0 & -a_{46} \\
0 & a_{25} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -a_{46} & 0 & 0
\end{bmatrix}
\]

Direct calculations in the Maple system show that for \( a_{14} = a_{24} = 0 \), this metric has a diagonal Ricci operator with two eigenvalues differing by the sign:

\[
Ric = \frac{a_{46}}{2a_{25}a_{23}}\text{diag}\{-1, -1, -1, 1, 1, 1\}.
\]

### 3.5 Lie group \( G_5 \)

Singular group that does not admit neither symplectic, nor complex structures. Commutation relations:

\[
[e_1, e_2] = e_5, [e_1, e_5] = e_6, [e_3, e_4] = e_6.
\]

Lie algebra \( g \) has ideals: \( C^1g = D^1g = \mathbb{R}\{e_5, e_6\}, C^2g = \mathbb{R}\{e_6\} = Z \) is the center of Lie algebra. Does admit half-flat structure \([3]\).

Let \( \omega = a_{ij}e_i \wedge e_j \) be any 2-form. The Hitchin’s operator \( K_{d\omega} \) for generic 2-form \( \omega \) is given by a quite complicated form. Moreover, \( K_{d\omega}^2 = a_{56}^4\text{Id} \). The \( \omega \wedge d\omega = 0 \) property is fulfilled under the following conditions:

\[
a_{34}a_{56} + a_{35}a_{46} - a_{36}a_{45} = 0,
\]

\[
a_{12}a_{56} - a_{15}a_{26} + a_{16}a_{25} - a_{23}a_{46} + a_{24}a_{36} - a_{26}a_{34} = 0.
\]

It is easy to see that the 2-form \( \omega \) is closed only if

\[
\omega = e^1 \wedge (a_{12}e^2 + a_{13}e^3 + a_{14}e^4 + a_{15}e^5) + e^2 \wedge (a_{23}e^3 + a_{24}e^4 + a_{25}e^5) + a_{34}e^3 \wedge e^4.
\]
Such 2-form $\omega$ is non-degenerated. In order to preserve the non-degeneracy of the $\omega$ and $d\omega$ at the minimal weakening of closedness property of $\omega$, consider the case when $a_{56} \neq 0$. Then the property $\omega \land d\omega = 0$ is fulfilled under the following conditions: $a_{34} = 0$ and $a_{12} = 0$. The 2-form $\omega$ is non-degenerated under the condition $a_{56}(a_{13}a_{24} - a_{14}a_{23}) \neq 0$, and the following formulas occur:

$$\omega = e^1 \land (a_{13}e^3 + a_{14}e^4 + a_{15}e^5) + e^2 \land (a_{23}e^3 + a_{24}e^4 + a_{25}e^5) + a_{56} \ e^5 \land e^6,$$

$$d\omega = -a_{56} \ e^{126} + a_{56} \ e^{345},$$

$$K_{d\omega} = a_{56}^2 \cdot \text{diag}\{+1, +1, -1, -1, -1, +1\}.$$

Define the operator $P = K_{d\omega}/a_{56}^2$. It defines left-invariant almost para-complex structure $P$, having the property $\omega(PX, PY) = -\omega(X, Y)$. Eigen-subspaces related to the eigen-values $\pm 1$ of operator $P$ are generated by the following vectors:

$$E^+ = \{e_1, e_2, e_6\}, \quad E^- = \{e_3, e_4, e_5\}.$$

It is easy to see that they are not closed relative to Lie bracket, so $P$ sets non-integrable almost a para-complex structure.

The pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$ of signature $(3,3)$ is given by

$$g = \begin{bmatrix}
0 & 0 & -a_{13} & -a_{14} & -a_{15} & 0 \\
0 & 0 & -a_{23} & -a_{24} & -a_{25} & 0 \\
-a_{13} & -a_{23} & 0 & 0 & 0 & 0 \\
-a_{14} & -a_{24} & 0 & 0 & 0 & 0 \\
-a_{15} & -a_{25} & 0 & 0 & 0 & a_{56} \\
0 & 0 & 0 & 0 & a_{56} & 0
\end{bmatrix}.$$

Direct calculations in the Maple system show that for $a_{15} = a_{25} = 0$, this metric has a diagonal Ricci operator with two eigenvalues differing by the sign:

$$RIC = \frac{a_{56}}{2a_{13}a_{24} - 2a_{14}a_{23}} \cdot \text{diag}\{-1, -1, -1, -1, +1, +1\}.$$

**Conclusions.** Any left-invariant closed 2-form $\omega$ on Lie groups $G_2 - G_5$ is degenerated. When the closedness requirement of $\omega$ is weakened in order to preserve the non-degeneracy of $\omega$ and $d\omega$ and of the property $\omega \land d\omega = 0$, the Hitchin’s operator $K_{d\omega}$ corresponding to $d\omega$, defines almost para-complex structure $P$. Pseudo-Riemannian metric $g(X, Y) = \omega(X, PY)$ depending on 5 to 7 arguments is of signature $(3,3)$ and has a diagonal Ricci operator with two eigenvalues differing by the sign. We have obtained a compatible pair $(\omega, \Omega)$, where $\Omega = d\omega$. Therefore, the properties $d\Omega = 0$ and $\omega \land d\omega = 0$ were fulfilled in an obvious way. The para-complex $(3,0)$-form is given by $\Psi = d\omega + i_e d\omega$, where $i_e$ is a para-complex unit. Thus, multiparametric families of almost para-complex half-flat structures were naturally defined on the Lie groups $G_2 - G_5$ and corresponding nilmanifolds. Their structural group is reduced to $SL(3, \mathbb{R}) \subset SO(3, 3)$. 
4 Formulas for evaluations

We now present the formulas which were used for the evaluations (on Maple) of the Nijenhuis tensor and curvature tensor of the associated metrics. Let $e_1, \ldots, e_{2n}$ be a basis of the Lie algebra $g$ and $C^k_{ij}$ a structure constant of the Lie algebra in this base:

$$[e_i, e_j] = \sum_{k=1}^{2n} C^k_{ij} e_k,$$

(1)

1. Connection components. These are the components $\Gamma^k_{ij}$ in the formula $\nabla_{e_i} e_j = \Gamma^k_{ij} e_k$. For left-invariant vector fields we have: $2g(\nabla_X Y, Z) = g([X, Y], Z) + g([Z, X], Y) - g([Y, Z], X)$. For the basis vectors we have:

$$2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) + g([e_k, e_i], e_j) + g(e_i, [e_k, e_j]),$$

$$2g_{lk} \Gamma^l_{ij} = g_{pk} C^p_{ij} + g_{pj} C^p_{ki} + g_{ip} C^p_{kj},$$

$$\Gamma^m_{ij} = \frac{1}{2} g^{kn} \left( g_{pk} C^p_{ij} + g_{pj} C^p_{ki} + g_{ip} C^p_{kj} \right).$$

(2)

Maple code:

> Ginv:=inverse(G):
> Gamma[i,j,n]:=(1/2)*(sum(Ginv[k,n]*(sum(C[i,j,p]*G[p,k], ’p’=1..6)), ’k’=1..6)):

2. Curvature tensor. The formula is: $R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$. For the basis vectors we have: $R(e_i, e_j) e_k = R^s_{ijk} e_s$,

$$R(e_i, e_j) e_k = \nabla_{e_i} \nabla_{e_j} e_k - \nabla_{e_j} \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k.$$

Therefore:

$$R^s_{ijk} = \Gamma^s_{ip} \Gamma^p_{jk} - \Gamma^s_{jp} \Gamma^p_{ik} - C^p_{ij} \Gamma^s_{pk}.$$

(3)

Maple code:

> Riem[i,j,k,s]:=simplify(sum(Gamma[i,p,s]*Gamma[j,k,p], ’p’=1..6));
> -Gamma[j,p,s]*Gamma[i,k,p] -C[i,j,p]*Gamma[p,k,s], ’p’=1..6));

3. Ricci tensor and scalar curvature. The Ricci tensor of a metric $g$ is the tensor $Ric$ obtained by convolution the tensor $R^s_{ijk}$ by the first and the upper indices:

$$Ric_{ij} = R^k_{kiij}.$$

(4)

Maple code:

> Ric[n,m]:=sum(Riem[i,n,m,i], ’i’=1..6);
> R:=simplify(sum(sum(Ginv[i,j]*Ric[i,j], ’i’=1..6), ’j’=1..6));

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