Arbitrary-order optical differentiation in reflection by sequence of first-order differentiators

Nikita V. Golovastikov1,2, Dmitry A. Bykov1,2, Leonid L. Doskolovich1,2

1 Image Processing Systems Institute – Branch of the Federal Scientific Research Centre “Crystallography and Photonics” of Russian Academy of Sciences, Samara, Russia
2 Samara National Research University, Samara, Russia

Email: nikita.golovastikov@gmail.com

Abstract. We propose a simple analytical method for designing arbitrary-order optical differentiators operating in reflection by successively arranging several first-order optical differentiators. The approach is based on the scattering matrix analysis, does not involve numerical optimization and is valid for differentiators of arbitrary configurations. The numerical simulation results demonstrate high-quality differentiation of a picosecond pulse for up to the fourth order. The proposed method can find application in the design of analogue optical computing and optical information processing systems.

1. Introduction
Analogue optical differentiation is a basic signal processing operation essential for high-speed analogue optical computing and photonic signal processing. Design of an arbitrary-order optical differentiator is a complicated task, and previously proposed approaches are often closely tied to a particular diffraction structure configuration and involve heavy numerical optimization [1-4]. As a result, there is demand for a simple universal approach based around a first-order differentiator.

While high-order optical differentiation in transmission is relatively straightforward and can be done with a succession of first-order differentiators [5], this cannot be done for differentiation in reflection. Previously it was shown that an $N$th order optical pulse differentiation in reflection can be achieved with a uniform fiber Bragg grating having $N \pi$-phase shifts properly located along its profile [1]. The approach proposed in [1] involves numerically solving a nonlinear multi-variable equation and is valid only for fiber Bragg gratings with small variations of the refraction coefficients.

In this work we propose a simple analytical approach to designing an arbitrary-order optical differentiator operating in reflection by successively arranging several first-order optical differentiators. This approach is not restricted to a specific differentiator geometry.

2. Pulse differentiation
The $N$th order derivative of an optical pulse envelope can be achieved as a result of the signal propagation through a linear system with the transfer function $H_{\text{diff}}(\omega - \omega_0) = \left(-i(\omega - \omega_0)\right)^N$, where $\omega_0$ is the carrier frequency [1-4]. Thus, to perform this operation it is sufficient to provide a diffraction structure with the reflection spectrum $R(\omega) = H_{\text{diff}}(\omega - \omega_0)$ in the vicinity of $\omega = \omega_0$. 

Content from this work may be used under the terms of the Creative Commons Attribution 3.0 licence. Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.
A scattering matrix of a diffraction structure $S$ relates the complex amplitudes of plane waves incident on the structure from the superstrate $I_1$ and substrate $I_2$ and the complex amplitudes of the reflected $R$ and transmitted $T$ waves [6]. Provided the structure in consideration supports only zeroth diffraction orders, the $S$–matrix has the form:

$$
\begin{bmatrix}
T \\
R
\end{bmatrix} = S \begin{bmatrix}
I_1 \\
I_2
\end{bmatrix}, \quad S = \begin{bmatrix}
T & R \\
R & T
\end{bmatrix},
$$

(1)

with $R$ and $T$ being the complex reflection and transmission coefficients. At a given angle of incidence these coefficients are functions of frequency $\omega$. Let the reflection coefficient $R$ become zero at a certain frequency value $\omega_0$, so that the Taylor expansion of the coefficients $R$ and $T$ are as follows:

$$
R(\omega) = r(\omega - \omega_0) + O((\omega - \omega_0)^2),
$$

$$
T(\omega) = e^{i\phi} + t(\omega - \omega_0) + O((\omega - \omega_0)^2),
$$

(2)

where $r = \partial R/\partial \omega|_{\omega=\omega_0}$, $t = \partial T/\partial \omega|_{\omega=\omega_0}$. Let us denote this structure $S$-matrix by $S_1$.

Consider a structure composed of two identical diffraction structures with $S$–matrixes $S_1$, separated by a dielectric layer of the thickness $l_1$. Following [6], the resulting $S$-matrix $S_2$ is defined by:

$$
S_2 = S_1 \otimes L_{l_1}(l_1) \otimes S_1,
$$

(3)

where $L_{l_1}(l_1) = \exp\{i\psi(l_1)\}E$, $\psi(l_1) = k_0 n \sqrt{n^2 - k^2}$ – phase incursion of a plane wave with frequency $\omega_0$, corresponding to the layer with thickness $l_1$ and refraction index $n$, $E$ – identity matrix, $\otimes$ – the Redheffer star product. Substituting (2) in (3) and deriving the reflection and transmission coefficients $R_2$ and $T_2$ of the composite structure, one should notice that once the thickness of the phase-shift region $l_1$ satisfies the condition:

$$
\psi(l_1) = \pi/2 - \phi + \pi m, \quad m \in \mathbb{Z},
$$

(4)

the reflection coefficient $R_2(\omega)$ possesses a second-order real zero:

$$
R_2 = -2r \cdot e^{-i\phi} (\omega - \omega_0)^2 + O((\omega - \omega_0)^3),
$$

$$
T_2 = e^{i\phi} + 2it(\omega - \omega_0) + O((\omega - \omega_0)^2).
$$

(5)

Generalizing this approach, it can be proven by induction that two identical diffraction structures, the reflection coefficients of which possess a real zero of the order $N$ $(R_N(\omega) \approx (\omega - \omega_0)^N)$, separated by a dielectric layer with the phase shift

$$
\psi_N(l_N) = \pi N/2 - \phi + \pi m, \quad m \in \mathbb{Z}
$$

(6)

will have the cumulative reflection coefficient possessing a real zero of the order $N+1$ $(R_{N+1}(\omega) \approx (\omega - \omega_0)^{N+1})$. This means that one can easily design an arbitrary-order optical differentiator in reflection by placing several identical optical differentiators sequentially. Note that this approach holds true regardless of the physical nature of the reflection zero.

The same approach can be utilized to design arbitrary-order spatial differentiators with the only difference that one will consider the reflection coefficient $R(k_x)$ as a function of the in-plane wave component $k_x$ [7].

3. Numerical simulation

As an example, we consider a previously proposed optical differentiator based on a W-type waveguide [8]. This is a horizontally symmetrical three-layer dielectric structure composed of a high-index central (core) layer (refractive index $n_{core}$, thickness $h_{core}$), surrounded by two identical
low-index cladding layers (refractive index \(n_{cl} < n_{core}\), thicknesses \(h_{cl}\)) (figure 1a). If refractive index of the surrounding medium \(n_{out}\) is greater than that of the cladding layers \((n_{out} > n_{cl})\), the reflection coefficient of the considered structure will vanish at certain values of frequency and angle of incidence, which enables the differentiation of the incident optical pulse. This resonant feature is associated with the excitation of a leaky mode localized at the central layer and described by the dispersion relation of the so-called W-type waveguide [9].

**Figure 1.** Geometry of the W-structure differentiator (a). The refractive index profile of the W-structure differentiator (b). Composite structure for second-order differentiation (c).

For numerical simulation we consider a W-structure with the following parameters: \(n_{core} = 2.2698\) (TiO\(_2\)), \(n_{cl} = 1.457\) (SiO\(_2\)), \(n_{out} = 1.7786\) (SF11), \(h_{core} = 40\) nm, \(h_{cl} = 700\) nm (figures 1a-1b). Its reflection spectrum \(R(\omega)\) has linear form with distinct minimum at the wavelength \(\lambda = 670\) nm (frequency \(\omega_0 = 2.811 \times 10^{15}\) s\(^{-1}\)) at the angle of incidence \(\theta_0 = 60.7^\circ\) (figure 2a). To confirm the resonant nature of the observed minimum, we calculated the complex frequency of the eigenmode for the structure \(\omega_0 = 2.811 \times 10^{15} - 4.099 \times 10^4\) i s\(^{-1}\) as a pole of the reflection coefficient \(R(\omega)\). The reflection spectra of the composed structures for second-, third- and fourth-order differentiation are shown in figures 2b-2d. Note that the reflection coefficients are approximated by polynomial functions of the appropriate order to high precision. The spectra in figure 2 were calculated using the rigorous coupled-wave analysis method (RCWA) [10] for the case of TE polarization of the incident wave.

**Figure 2.** Absolute value (solid black line) and phase (dashed black line, right vertical axis) of the reflection coefficients of the first- (a), second- (b), third- (c) and fourth- (d) order differentiators and their polynomial approximations of the corresponding order (dotted red line).
The possibility of performing optical differentiation was confirmed by numerically simulating the diffraction of an optical pulse with the Gaussian envelope $v(t) = \exp\left(-t^2/\sigma^2\right)$, $\sigma = 6$ ps with central frequency $\omega_0 = 2.811 \times 10^{15}$ s$^{-1}$ by the designed differentiating structures (figures 3a-3d). The normalized spectrum of the incident pulse is indicated by the dotted lines in figures 2a-2d. The normalized root mean square deviation of the reflected pulse envelopes from the analytical derivatives (disregarding the reflected pulse delay) increases from 1% (figure 3a) to 14% (figure 3d) which indicates high differentiation accuracy.

![Figure 3](image_url)

**Figure 3.** Absolute value of the reflected pulse envelope (solid black curve) and of the exact derivative (dashed red curve), and the incident pulse envelope (dotted black curve, right vertical axis) for first- (a), second- (b), third- (c) and fourth- (d) order differentiators.

4. Conclusion

A simple analytical approach for designing arbitrary-order optical differentiators in reflection is proposed. It is shown that a succession of two $N$th-order optical differentiators can perform $(N+1)$th-order differentiation of the incident pulse envelope. The approach provides a simple analytical expression for the distance between the differentiators and holds true for differentiators of any kind, as well as for spatial differentiation of an optical beam profile. The obtained results may find application in analog optical computing systems design.

Acknowledgments

This work was funded by the Russian Science Foundation grant no. 19-19-00514 (analytical investigation of the scattering matrix), by the Russian Federation Ministry of Science and Higher Education under state contract with the "Crystallography and Photonics" Research Center of the RAS under agreement 007-Г3/3363/26 (software development for electromagnetic wave diffraction simulation), and by the Russian Foundation for Basic Research grants 18-37-20038 and 16-29-11683 (numerical simulation of the optical pulse differentiation).

References

[1] Kulishov M, Azaña J 2007 Optics Express 15 6152
[2] Kulishov M, Krčmařík D, Slavík R 2007 Optics Letters 32 2978
[3] Ashrafi R, Asghari M A, Azaña J 2011 IEEE Photonics Journal 3 353
[4] Liu X, Shu X 2017 Journal of Lightwave Technology 35 2926
[5] Bykov D A, Doskolovich L L, Soifer V A 2012 Journal of Experimental and Theoretical Physics 114 724
[6] Li L 1996 Journal of the Optical Society of America A 13 1024
[7] Doskolovich L L, Bykov D A, Bezus E A, Soifer V A 2014 Optics Letters 39 1278
[8] Golovastikov N V, Doskolovich L L, Bezus E A, Bykov D A, Soifer V A 2018 Journal of Experimental and Theoretical Physics 127 202
[9] Hu J, Menyuk C R 2009 Advances in Optics and Photonics 1, 58
[10] Moharam M G, Pommet D A, Grann E B, Gaylord T K 1995 Journal of the Optical Society of America A 12 1077