Ground states of Nicolai and $\mathbb{Z}_2$ Nicolai models

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Abstract
We derive explicit recursions for the ground state degeneracy generating functions of the one-dimensional Nicolai model and $\mathbb{Z}_2$ Nicolai model. Both are examples of lattice models with $\mathcal{N}=2$ supersymmetry. The relations that we obtain for the $\mathbb{Z}_2$ model were numerically predicted by Sannomiya, Katsura, and Nakayama.

Keywords: supersymmetry, lattice model, homology, quantum mechanics

1. Introduction

More than forty years ago, Hermann Nicolai proposed a lattice model featuring $\mathcal{N}=2$ supersymmetry [1]. It was an early example of a realization of $\mathcal{N}=2$ supersymmetric quantum mechanics with an underlying spatial lattice structure. Supersymmetric quantum mechanics arises as soon as a quantum mechanical hamiltonian $H$ can be written as

$$H = \{Q, Q^\dagger\}$$

with supercharges $Q$ and $Q^\dagger$ satisfying $Q^2 = (Q^\dagger)^2 = 0$. An easy consequence is that both supercharges commute with $H$,

$$[Q, H] = 0, \quad [Q^\dagger, H] = 0.$$

This algebraic structure implies that eigenstates of the hamiltonian, satisfying $H|\psi\rangle = E|\psi\rangle$, can be organized into doublets $\{|\psi\rangle, Q^\dagger|\psi\rangle\}$ and singlets. The latter are annihilated by both supercharges $Q$ and $Q^\dagger$ and have energy eigenvalue $E = 0$.

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The supercharges of the Nicolai model act in the Hilbert space of spin-less fermions on a 1D lattice. On a general lattice (or graph) \( \Lambda \), spin-less fermions are described by creation and annihilation operators \( c_i, c_i^\dagger \), satisfying
\[
\{ c_i, c_j^\dagger \} = \delta_{ij}, \quad i,j \in \Lambda.
\]
The supercharge of the Nicolai model is given in equation (1) below.

1.1. Supersymmetric lattice models

In later years, many other supersymmetric lattice models for spin-less fermions have been proposed and studied.

**M\(_k\) models—integrable CFT and QFT [2, 3]**: In these models, defined on 1D lattices, it is assumed that at the most \( k \) fermions can occupy adjacent lattice sites. It has been found that these models can be tuned to be critical, and that their critical behavior is captured by the \( k \)th model of \( \mathcal{N} = 2 \) supersymmetric conformal field theory (CFT). Particular off-critical deformations, obtained by staggering parameters in the supercharges, lead to integrable \( \mathcal{N} = 2 \) supersymmetric massive quantum field theories (QFT) with superspace superpotential given by Chebyshev polynomials [4].

**M\(_1\) model on ladders and 2D lattices—superfrustration**: The M\(_1\) model can be defined on any graph \( \Lambda \). While for 1D chains the number of supersymmetric groundstates never exceeds 2, the number on 2D lattices tends to be exponential in the size (perimeter or area) of the lattice [5, 6].

**Models with \( Q \) cubic in \( c_i \)**: These include the \( \mathbb{Z}_2 \) Nicolai model (with \( g = 0 \)) [7] and \( \mathcal{N} = 2 \) supersymmetric SYK models [8].

**Supersymmetric model of coupled fermion chains (FS model) [9]**: In these models the supercharge \( Q \) transports a particle from one chain to the other. To guarantee the fermionic nature of the supercharge, the particles on the individual chains are viewed as semions with fermion number \( \pm 1/2 \).

**A particle-hole symmetric version of the 1D M\(_1\) model [10]**: Surprisingly, this model turns out to be equivalent to the FS model—an explicit unitary map was presented in [11].

A common feature to many of these models are large degeneracies of \( E = 0 \) supersymmetric groundstates. In many cases it has been established that their number is exponential in the size of the system, meaning that the ground state entropy is extensive. It has been suggested [12, 13] that such situations lead to a breaking of ergodicity and to a phenomenology similar to that of many-body localization (MBL).

1.2. Counting supersymmetric groundstates

Clearly, an important step in the analysis of supersymmetric lattice models is to understand the number and the nature of their supersymmetric ground states. This problem turns out to be quite hard in general.

For the M\(_k\) models in 1D a detailed understanding has been reached, thanks to integrability by Bethe Ansatz and to connections with supersymmetric CFT and (integrable) QFT [3, 4]. In the coupled chain model [9] the supersymmetric ground states are understood as tightly bound interchain pairs. In the equivalent particle-hole symmetric M\(_1\) model the ground states have been analyzed with the help of a Bethe Ansatz [10].
The ground state counting problem for the $M_1$ model on 2D lattices is understood in special cases [5, 6, 14, 15] but remains an open problem in general [16]. An important observation [6] is that, from a mathematical perspective, the number of supersymmetric ground states is the dimension of the homology of $Q$. Recall that for an operator $Q: V \to V$ acting on a vector space $V$ such that $Q^2 = 0$, the homology is defined as the quotient $H(V, Q) := \ker Q / \im Q$. Typically one considers a graded vector space $V = \oplus_{i \in \mathbb{Z}} V_i$ with $Q$ typically of degree $-1$, that is, $Q(V_i) \subseteq V_{i-1}$ (note that in principle one can consider $Q$ with some negative degree other than $-1$, for instance below we use $Q$ of degree $-3$), and then the homology is also a graded vector space $H_*(V, Q) := \oplus_{i \in \mathbb{Z}} H_i(V, Q)$, where $H_i(V, Q) := (V_i \cap \ker Q) / (V_i \cap \im Q)$. With this, the counting problem can be cast in strict mathematical terms, and mathematical tools for computing homologies can be employed.

In [6, 15] the ground state counting problem for the $M_1$ model on a variety of 1D and 2D lattices was solved by employing the ‘tic-tac-toe’ lemma for double complexes. Here we employ a different method based on the homological perturbation lemma, which we present in section 2.3.

For the Nicolai model, numerical analysis revealed ground state degeneracies for small system size, but the systematics behind these numbers remained unclear. The paper [17] zoomed in a subset of all supersymmetric ground states, the so-called classical ground states. For the $\mathbb{Z}_2$ Nicolai model a recursion for the ground state degeneracy generating function was presented in [7]. In this letter we prove this conjectured relation and similarly establish a recursion for the generating function of the original Nicolai model. Note that in order to rigorously present the derivation of the recursion relations for the generating functions we have to switch to a quite formal language in the rest of the text.

2. Homological computations

In this section we formulate the problem of the computation of the ground state degeneracy generating functions for the one dimensional Nicolai and $\mathbb{Z}_2$ Nicolai models in formal purely mathematical terms and derive recursions for these functions.

By $\mathcal{H}_n$, $n \geq 0$, we denote the free graded commutative associative algebra generated by the degree 1 elements $c_i^\dagger$, $i = 1, \ldots, n$. Its dimension is $2^n$. The operator $c_i: \mathcal{H}_n \to \mathcal{H}_n$, $i = 1, \ldots, n$, acts as the derivative with respect to $c_i^\dagger$. The operators $c_i$ and the operators of multiplication by $c_i^\dagger$, $i = 1, \ldots, n$, are the same as the annihilation and the creation operators given in the introduction, respectively. Abusing notation, we denote by $c_i^\dagger$ also the operator of multiplication by $c_i^\dagger$.

By $\langle \cdot \rangle$ we denote the linear span of the vectors. Thus, for instance, $\langle c_1^\dagger \rangle$ is a one-dimensional graded vector space concentrated in degree 1, and the homogeneous components of the graded vector space $\mathcal{H}_n$ can be represented as $\langle \mathcal{H}_n \rangle_\ell = \langle \prod_{i \in J} c_i^\dagger \rangle_{|J| = \ell}$, $\ell = 0, 1, 2, \ldots, n$.

2.1. Nicolai model

The state space of the Nicolai model is $\mathcal{H}_{2m+1}$, $m \geq 1$. We consider the operator $Q: \mathcal{H}_{2m+1} \to \mathcal{H}_{2m+1}$ of degree $-1$ defined as

$$Q := \sum_{i=1}^m c_{2i-1} c_{2i}^\dagger c_{2i+1}.$$

(1)

Obviously, $Q^2 = 0$, so we can compute its homology. The ground state degeneracy generating function of this model, $P_{2m+1}(z)$ is the Poincaré polynomial of the homology of $Q$, that is,
Recall that the notation $H_i(\mathcal{H}_{2m+1}, Q)$ here means the $i$th homogeneous component of the graded vector space $H_*(\mathcal{H}_{2m+1}, Q)$.

**Theorem 1.1.** The polynomials $P_{2m+1}(z)$, $m \geq 3$, are determined by the recursion

$$P_{2m+1}(z) = (1 + z^2)P_{2m-1}(z) + (z + 2z^2 + z^3)P_{2m-3}(z)$$

with the initial values given by

$$P_3(z) = 1 + 2z + 2z^2 + z^3 \quad \text{and} \quad P_5(z) = 1 + 3z + 6z^2 + 6z^3 + 3z^4 + z^5.$$  

**Corollary 1.2.** The total number of the ground states, $a_{2m+1} := P_{2m+1}(1)$, satisfies the recursion

$$a_{2m+1} = 2a_{2m-1} + 4a_{2m-3}$$

with the initial values given by $a_3 = 6$ and $a_5 = 20$.

### 2.2. $\mathbb{Z}_2$ Nicolai model

The state space of the $\mathbb{Z}_2$ Nicolai model is $\mathcal{H}_n$, $n \geq 3$. We consider the operator $Q^{\mathbb{Z}_2}: \mathcal{H}_n \rightarrow \mathcal{H}_n$ of degree $-3$ defined as

$$Q^{\mathbb{Z}_2} := \sum_{i=1}^{n-2} c_i c_{i+1} c_{i+2}. \quad (2)$$

Though $Q^{\mathbb{Z}_2}$ is not of degree $-1$, we still have $(Q^{\mathbb{Z}_2})^2 = 0$, so we can compute its homology. The ground state degeneracy generating function of this model, $P^{\mathbb{Z}_2}_n(z)$, is the Poincaré polynomial of the homology of $Q^{\mathbb{Z}_2}$, that is,

$$P^{\mathbb{Z}_2}_n(z) := \sum_{i=0}^{n} \dim H_i(\mathcal{H}_n, Q^{\mathbb{Z}_2}) z^i.$$  

**Theorem 2.1.** The polynomials $P^{\mathbb{Z}_2}_n(z)$, $n \geq 3$, can be determined by the recursion

$$P^{\mathbb{Z}_2}_0(z) = 1, \quad P^{\mathbb{Z}_2}_1(z) := 1 + z \quad \text{and} \quad P^{\mathbb{Z}_2}_2(z) = 1 + 2z + z^2.$$  

**Corollary 2.2 (Conjecture of Sannomiya, Katsura, and Nakayama [7]).** The total number of ground states, $a^{\mathbb{Z}_2}_n := P^{\mathbb{Z}_2}_n(1)$, satisfies the recursion

$$a^{\mathbb{Z}_2}_n = 2a^{\mathbb{Z}_2}_{n-2} + 2a^{\mathbb{Z}_2}_{n-3}$$

with the initial values given by $a^{\mathbb{Z}_2}_0 = 1, a^{\mathbb{Z}_2}_1 = 2$, and $a^{\mathbb{Z}_2}_2 = 4$.  

Remark 2.3. Note that in principle \( P^\mathbb{Z}(z) \) and \( a_n := P_n(1) \), \( n = 0, 1, 2 \), are formal initial values that do not fit in the \( \mathbb{Z}_2 \)-Nicolai model, since we have a requirement \( n \geq 3 \) there. However, one might extend the definition of \( \mathbb{Z}_2 \)-Nicolai to an arbitrary \( n \geq 0 \), where the formula for the supercharge would be still given by equation (2), which means \( Q^\mathbb{Z} = 0 \) for \( n = 0, 1, 2 \). In this case the homology \( H_n(\mathcal{H}_n, Q^\mathbb{Z}) \) is isomorphic to \( \mathcal{H}_n \) as a graded vector space, \( n = 0, 1, 2 \). So, in this case the Poincaré polynomials are given by \( P_n(z) := \sum_{i=0}^n \dim H_i(\mathcal{H}_n, 0) z^i = \sum_{i=0}^n \dim(\mathcal{H}_n) z^i, \ n = 0, 1, 2 \), which are exactly the formal initial values that we use above.

Remark 2.4 (Witten index). We would like also to comment on the Witten index computed in [7, table II]. Often the Witten index is defined to be the Euler characteristic of the chain complex, and then it is not interesting in this case, since for \( n \geq 1 \) the Euler characteristic is equal to \( P_n(-1) = 0 \). However, in [7] a much more creative definition is used. In our terms, it is given by
\[
W_n := \frac{1}{2} \left( |P_n(r) + P_n(r^{-1}) + P_n(-1)| + |r P_n(r) + r^{-1} P_n(r^{-1}) + (-1) P_n(-1)| \right.
\]
\[
\left. + |r^2 P_n(r) + r^{-2} P_n(r^{-1}) + (-1)^2 P_n(-1)| \right).
\]
where \( r = \exp(i\pi/3) \). Though theorem 2.1 does not imply any nice recursion for the Witten index defined this way, it is easy to use the recursion for the Poincaré polynomials in order to reproduce the last line in [7, table II].

2.3. Homological perturbation lemma

The main technical tool that we use in the proofs of theorems 1.1 and 2.1 is a version of the so-called homological perturbation lemma. Consider a graded vector space \( \mathcal{H} \) with two commuting differentials of degree \(-1\), \( d_1 \) and \( d_2 \). Assume that we have chosen a deformation retract data connecting the differential graded spaces \( \mathcal{H} \) with the differential \( d_1 \) and its homology, that is, the space \( H_*(\mathcal{H}, d_1) \) with the zero differential. By a deformation retract data we mean that we have an operator \( h: \mathcal{H} \rightarrow \mathcal{H} \) of degree \( 1 \) and two quasi-isomorphisms (the chain maps that induce the isomorphisms on the homology level) \( i: H_*(\mathcal{H}, d_1) \rightarrow \mathcal{H} \) and \( p: \mathcal{H} \rightarrow H_*(\mathcal{H}, d_1) \), which can be arranged in a diagram
\[
(H_*(\mathcal{H}, d_1), 0) \xrightarrow{p} (\mathcal{H}, d_1) \xleftarrow{i} h
\]
such that
\[
pi = \text{Id}_{H_*(\mathcal{H}, d_1)} \quad \text{and} \quad ip = \text{Id}_{\mathcal{H}} + d_1 h + h d_1
\]
(note that the condition \( pi = \text{Id}_{H_*(\mathcal{H}, d_1)} \) just follows from the assumption that both \( p \) and \( i \) are quasi-isomorphisms, so we just included it here for completeness), and the operator \( 1 - d_2 h \) is invertible (for instance, there can be an additional gradation that guarantees invertibility, as in the case of bicomplex).

Lemma 3.1 ([18]). Under these assumptions we have the following isomorphism of graded vector spaces:
\[
H_*(\mathcal{H}, d_1 + d_2) \cong H_*(\mathcal{H}, d_1), p(1 - d_2 h)^{-1} d_2 i).
\]
This lemma has many much stronger versions and refinements, but this form is exactly what we use in this paper.

One more definition that will be useful below is the suspension of a chain complex.

**Definition 3.2.** Let \( (\mathcal{H}, Q) \) be a chain complex. For any \( k \in \mathbb{Z} \) the chain complex \( (\mathcal{H}[k], Q[k]) \) is defined by formally adding \( k \) to the gradation, and the differential \( Q \) is twisted by the sign \((-1)^k\). On the level of Poincaré polynomials the effect of suspension corresponds to multiplication by \( z^k \).

**2.4. Proof of theorem 1.1**

We define the differentials \( d_1 := c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \) and \( d_2 := Q - d_1 \). We have:

\[
H_*(\mathcal{H}_{2m+1}, d_1) \cong \langle 1, c_{2m-1}^1, c_{2m+1}^1, c_{2m}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2}.
\]

The choice of the representatives of the homology classes of \( d_1 \) here defines a natural quasi-isomorphism \( i: H_*(\mathcal{H}_{2m+1}, d_1) \to H_{2m+1} \). Define the map \( h: H_{2m+1} \to H_{2m+1} \) as \( h := c_{2m+1}^1c_{2m}^1c_{2m-1}^1 \), and the map \( p: H_{2m+1} \to \langle 1, c_{2m-1}^1, c_{2m+1}^1, c_{2m}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2} \subset H_{2m+1} \)

as the projection with the kernel \( \langle c_{2m}^1, c_{2m-1}^1c_{2m+1}^1 \rangle \cdot H_{2m-2} \).

The maps \( h, i \), and \( p \) satisfy all conditions of lemma 3.1. In particular, the inverse of \( 1 - dh \) is \( 1 + d h \) (it is straightforward to check that \( dh h = 0 \)). Thus we have:

\[
H_*(\mathcal{H}^2_{2m+1}, Q) \cong H_*(\langle 1, c_{2m-1}^1, c_{2m+1}^1, c_{2m}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2}, pd_2i + pd_2pd_2i)
\]

(note that the operator \( i \) here acts tautologically, but we keep it for the sake of notation).

Observe that the only summand in \( d_2 \) that can affect the generators \( c_{2m-1}^1, c_{2m}^1, c_{2m+1}^1 \) in \( H_{2m+1} \) is \( c_{2m-3}c_{2m-2}^1c_{2m-1}^1 \). The image of \( c_{2m-3}c_{2m-2}^1c_{2m-1}^1 \) is a subspace of

\[
\langle c_{2m-2}^1c_{2m-2}^1c_{2m}^1, c_{2m-2}^1c_{2m-2}^1c_{2m+1}^1 \rangle \cdot H_{2m-4}.
\]

therefore, the images of \( h d_2 i \) and \( d_2 h d_2 i \) are subspaces of \( \langle c_{2m+1}^1c_{2m-1}^1c_{2m-2}^1 \rangle \cdot H_{2m-4} \), which lies in the kernel of the projection \( p \). So, the term \( pd_2 h d_2 i \) acts trivially on the homology of \( d_1 \).

Therefore,

\[
H_*(\mathcal{H}^2_{2m+1}, Q) \cong H_*\bigg( \langle 1, c_{2m-1}^1, c_{2m+1}^1, c_{2m}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2}, pd_2 i \bigg).
\]

We split the space \( \langle 1, c_{2m-1}^1, c_{2m+1}^1, c_{2m}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2} \) into the eigenspaces of the action of \( pd_2 i \) as

\[
\langle 1, c_{2m-1}^1 \rangle \cdot H_{2m-2} \oplus \langle c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2} \oplus \langle c_{2m+1}^1 \rangle \cdot H_{2m-2} \oplus \langle c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2} \oplus \langle c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2}.
\]

From the definition of \( d_2 \) we have:

\[
(\langle 1, c_{2m-1}^1 \rangle \cdot H_{2m-2}, pd_2 i) \cong (H_{2m-1}, Q)
\]

\[
(\langle c_{2m}^1c_{2m+1}^1, c_{2m-1}^1c_{2m}^1c_{2m+1}^1 \rangle \cdot H_{2m-2}, pd_2 i) \cong (H_{2m-1}[2], Q[2]).
\]

As for the last two summands, we note that \( pc_{2m-3}c_{2m-2}c_{2m-1}^1 \) acts trivially on \( \langle c_{2m+1}^1 \rangle \cdot H_{2m-2} \) and \( \langle c_{2m-1}^1c_{2m}^1 \rangle \cdot H_{2m-2} \). Indeed, in the first case it is sufficient to observe
that \( e_{2m-1} \) anticommutes with \( c_{2m+1} \) and vanishes on \( H_{2m-2} \) (we can ignore \( i \) as it acts tautologically). In the second case, the image of \( e_{2m-3}, e_{2m-2} c_{2m-1} \) applied to \( \langle c_{2m-1}^\dagger c_{2m}^\dagger \rangle \cdot H_{2m-2} \) lies in \( \langle c_{2m}^\dagger \rangle \cdot H_{2m-2} \subset \ker p \). Therefore,

\[
\left( \langle c_{2m+1}^\dagger \rangle \cdot H_{2m-2} \cdot pd_{2}\right) i \cong \left( \langle c_{2m+1}^\dagger \rangle \cdot c_{2m-2} c_{2m+1} + c_{2m-2} c_{2m+1} \right) \cdot H_{2m-2},
\]

\[
\cong (H_{2m-3}[1], Q^{[1]}) \oplus (H_{2m-3}[2], Q^{[2]});
\]

\[
\left( \langle c_{2m-1}^\dagger c_{2m}^\dagger \rangle \cdot H_{2m-2} \cdot pd_{2}\right) i \cong \left( \langle c_{2m-1}^\dagger c_{2m}^\dagger \rangle \cdot c_{2m-3} c_{2m} + c_{2m-3} c_{2m} \right) \cdot H_{2m-2},
\]

\[
\cong (H_{2m-3}[2], Q^{[2]}) \oplus (H_{2m-3}[3], Q^{[3]}).
\]

Thus, we prove that \( H_\bullet(H_{2m+1}, Q) \) is isomorphic to the homology of the direct sum of complexes

\[
(H_{2m-1}, Q) \oplus (H_{2m-3}[1], Q^{[1]}) \oplus (H_{2m-3}[2], Q^{[2]}) \oplus (H_{2m-3}[3], Q^{[3]}).
\]

Therefore, the Poincaré polynomial \( P_{2m+1}(z) \) of the homology of \( H_{2m+1}, Q \) is equal to

\[
P_{2m+1}(z) + \bar{z} P_{2m-1}(z) + z P_{2m-3}(z) + \bar{z}^2 P_{2m-2}(z) + z^2 P_{2m-3}(z) + \bar{z}^2 P_{2m-3}(z) + z^2 P_{2m-3}(z).
\]

This completes the proof of the recursion.

We still have to compute \( P_1(z) \) and \( P_3(z) \). Note that the recursion that we obtained above also works for the case \( m = 2 \), which is formally not defined, since we only defined \( P_{2k+1} \) for \( k \geq 1 \) for the Nicolai model. However, we can extend the definition for \( m = 1 \), with the supercharge \( Q = 0 \) on \( H_1 \). Then the recursion for the Poincaré polynomials derived above can also be applied in the case \( m = 2 \).

For \( m = 0 \), we have: \( H_\bullet(H_1, Q) \cong H_1 = \{1, c_1^\dagger\} \). Therefore, \( P_1(z) = 1 + z \). For \( m = 1 \), \( H_\bullet(H_1, Q) \cong \{1, c_1^\dagger, c_2^\dagger, c_2^\dagger c_1^\dagger, c_2^\dagger c_3^\dagger\} \), so \( P_3(z) = 1 + 2z + 2z^2 + z^3 \). Applying the recursion for the Poincaré polynomials, we have:

\[
P_3(z) = (1 + z^2)(1 + 2z + 2z^2 + z^3) + (1 + 2z^2 + z^3)(1 + z) = 1 + 3z + 2z^2 + 6z^3 + 3z^4 + z^5.
\]

This completes the proof of theorem 1.1.

2.5. Proof of theorem 2.1

In this case the operator \( Q^{\exists z} \) is not of degree \(-1\), so we have two ways how to turn this situation into a standard one: either we can just redevelop the homological perturbation lemma for the operators of degree \(-3\) (then the homotopy contraction operator \( h \) should be of degree \(+3\)), or we can split the space \( H_n \) into the direct sum of three subspaces, depending on the remainder \( \mod 3 \) of the degree, and redefine the degree for each of them as the quotient of division with remainder by 3 of the original degree. These two approaches are equivalent, and we choose the first one since the ground state degeneracy generating function is formulated in terms of the original degrees. So, we use lemma 3.1 with a new convention that the degree of \( h \) is equal to 3.

We define the differentials \( d_1 := c_{n-2} c_{n-1}^\dagger c_n + d_2 := Q^{\exists z} - d_1 \). We have:

\[
H_\bullet(H_n, d_1) \cong \langle c_{n-2}^\dagger, c_{n-1}^\dagger, c_n^\dagger, c_{n-2} c_{n-1}^\dagger, c_{n-2} c_n^\dagger, c_{n-2} c_n^\dagger c_{n-2}^\dagger \rangle \cdot H_{n-3},
\]

and this isomorphism defines a natural map \( i: H_\bullet(H_n, d_1) \rightarrow H_n \). Define the map \( h: H_n \rightarrow H_n \) as \( h := c_n^\dagger c_{n+1}^\dagger c_{n-2} \), and the map
as the projection with kernel \( \langle 1, c_n^\dagger c_{n-1}^\dagger, c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger c_{n-3}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \subset \mathcal{H}_n \)

The maps \( h, i, \) and \( p \) satisfy all conditions of lemma 3.1, and since \( d^2 h = 0 \), the inverse of \( 1 - d^2 h \) is equal to \( 1 + d^2 h \). Thus we have:

\[
\mathcal{H}_n(\mathcal{H}_m, Q^{\mathcal{Z}}) \cong \mathcal{H}_n(\langle c_{n-2}^\dagger c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger c_{n-3}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i + pd_2 d_2 i)
\]

(note that the operator \( i \) here act tautologically, but we keep it for the sake of notation).

Observe that the only two summands in \( d_2 \) that can affect the generators \( c_n^\dagger \) and \( c_n^\dagger \) in \( \mathcal{H}_n \) are \( c_n^\dagger c_{n-3} c_{n-2} + c_n^\dagger c_{n-3} c_{n-1} \). The image of \( (c_n^\dagger - 4c_n^\dagger c_{n-2} + c_{n-3} c_{n-2} c_{n-1})i \) is a subspace of

\[
\mathcal{H}_{n-4} \oplus \langle c_n^\dagger, c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-5}.
\]

so the images of \( hd_2 i \) and \( d_2 hd_2 i \) are subspaces of \( \langle c_{n-2}^\dagger c_{n-3} c_n^\dagger \rangle \cdot \mathcal{H}_{n-4} \), which lies in the kernel of the projection \( p \). So, the term \( pd_2 d_2 i \) acts trivially on the homology of \( d_i \). Therefore,

\[
\mathcal{H}_n(\mathcal{H}_m, Q^{\mathcal{Z}}) \cong \mathcal{H}_n(\langle c_{n-2}^\dagger c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger c_{n-3}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i).
\]

We split the space \( \langle c_{n-2}^\dagger c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger c_{n-3}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \) into a direct sum of subspaces invariant under the action of \( pd_2 i \) as

\[
\langle c_n^\dagger, c_{n-2}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \oplus \langle c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-3} \oplus \langle c_{n-2}^\dagger c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-3} \oplus \langle c_{n-1}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3}.
\]

On the first two spaces the operator \( p(c_n^\dagger - 3c_{n-2}c_{n-1})i \) acts trivially. Then, since \( \langle c_n^\dagger, c_{n-2}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \cong \langle c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \) and \( \langle c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-3} \cong \langle c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-2} \), we have

\[
\langle c_n^\dagger, c_{n-2}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i) \cong \langle \mathcal{H}_{n-2}[1], (Q^{\mathcal{Z}})[1] \rangle,
\]

\[
\langle c_{n-1}^\dagger, c_{n-2}^\dagger c_{n-1}^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i) \cong \langle \mathcal{H}_{n-2}[1], (Q^{\mathcal{Z}})[1] \rangle.
\]

Note that on \( \langle c_{n-2}^\dagger \rangle \cdot \mathcal{H}_{n-3} \) and \( \langle c_{n-1}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \) the operator \( p(c_n^\dagger - 4c_{n-2}c_{n-1})i \) acts trivially. Indeed, in the first case we observe that the image of \( \langle c_{n-2}^\dagger \rangle \cdot \mathcal{H}_{n-3} \) under \( c_n^\dagger - 4c_{n-2}c_{n-1} \) is a subspace of \( \mathcal{H}_{n-3} \subset \ker p \). In the second case, the operator \( c_{n-2}^\dagger \) vanishes identically on \( \langle c_{n-1}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3} \). Therefore,

\[
\langle c_{n-2}^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i) \cong \langle \mathcal{H}_{n-3}[1], (Q^{\mathcal{Z}})[1] \rangle,
\]

\[
\langle c_{n-1}^\dagger c_n^\dagger \rangle \cdot \mathcal{H}_{n-3}, pd_2 i) \cong \langle \mathcal{H}_{n-3}[2], (Q^{\mathcal{Z}})[2] \rangle.
\]

Thus we prove that \( \mathcal{H}_n(\mathcal{H}_m, Q^{\mathcal{Z}}) \) is isomorphic to the homology of

\[
(\mathcal{H}_{n-2}[1], (Q^{\mathcal{Z}})[1]) \oplus (\mathcal{H}_{n-3}[1], (Q^{\mathcal{Z}})[1]) \oplus (\mathcal{H}_{n-3}[1], (Q^{\mathcal{Z}})[1]) \oplus (\mathcal{H}_{n-3}[2], (Q^{\mathcal{Z}})[2]).
\]

Therefore, the Poincaré polynomial \( P^{\mathcal{Z}}(z) \) of the homology of \( \mathcal{H}_n, Q^{\mathcal{Z}} \) is equal to

\[
z^{\mathcal{H}_{n-2}}(z) + z^{\mathcal{H}_{n-3}}(z) + z^{\mathcal{H}_{n-3}}(z) + z^{\mathcal{H}_{n-3}}(z).
\]

This completes the proof of the recursion for the Poincaré polynomials. Note that this recursion works for any \( n \geq 3 \) once we extend the \( \mathbb{Z}_2 \)-Nicolai model for \( n = 0 \) as suggested in remark 2.3. Then we can use as the initial values the Poincaré polynomials \( P_0^{\mathcal{Z}} \), \( P_1^{\mathcal{Z}} \), and \( P_2^{\mathcal{Z}} \), which are just the Poincaré polynomials of the graded vector spaces \( \mathcal{H}_0, \mathcal{H}_1, \) and \( \mathcal{H}_2 \), respectively. This completes the proof of theorem 2.1.
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