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Completeness of roots elements of linear operators in Banach spaces and application

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ABSTRACT

In this paper the general spectral properties of linear operators in Banach spaces are studied. We find sufficient conditions on structure of Banach spaces and resolvent properties that guarantee completeness of roots elements of Schatten class operators. This approach generalizes the well known result for operators in Hilbert spaces. In application, the boundary value problems for the abstract equation of second order with variable coefficients are studied. The principal part of the appropriate differential operator is not self-adjoint. The discreteness of spectrum and completeness of root elements of this operator are obtained.

Key Words: Uniformly convex Banach spaces; Abstract functions; Schatten class of operators; Completeness of root elements; Separable boundary value problems; Differential-operator equations;

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One of the fundamental results on spectral theory of operators is the completeness of roots elements of Schatten class operators in Hilbert spaces:

Theorem [8, Theorem XI. 9.29]. Assume:

(1) $H$ is a Hilbert space and $A$ is an operator in $C_p(H)$, for some $p \in (1, \infty)$;

(2) $\gamma_1, \gamma_2, ..., \gamma_s$ is non-overlapping, differentiable arcs in the complex plane starting at the origin. Suppose that each of the $s$ regions into which the plane is divided by these arcs is contained in an angular sector of opening less then $\frac{\pi}{p}$.

Moreover, $m > 0$ is an integer so that the resolvent of $A$ satisfies the inequality $\|R(\lambda, A)\| = O(|\lambda|^{-m})$ as $\lambda \to 0$ along any of the arcs $\gamma_i$. Then the $spA$ contains the subspace $A_mH$.

The main aim of the present paper is the generalization of the above important theorem [8, Theorem XI. 9.29] for Banach spaces. The spectral properties of linear operators in Banach spaces is a subject which is not much investigated. The related effort, indeed requires new tools of modern analysis and operator theory. Nevertheless, the results in this field have numerous applications in pure differential, pseudo differential and functional-differential equations. For this reason, it was very important to have general result about spectral properties of linear operators in Banach spaces. The articles [2], [6] and [15] are devoted to this question in Banach spaces. In this paper, we disclose different sufficient condition for completeness of roots elements of linear operators. We consider the class of Banach spaces which satisfy some given conditions, but by virtue of Remark1, our class of operators are wider than the class of operators considered in [2], [6] and [15]. Also, in [6] the extra condition is assumed to
be nonempty of spectrum of these class of operators. Moreover, our method of proofs are different from proofs provided in the cited references.

We find sufficient conditions on structure of Banach spaces which allow to define the trace of operators and its properties. Also, we get Carleman estimate of quasi nuclear operators (QNOs) and its spectral properties. In application we consider nonlocal boundary value problem (BVP) for the second order differential-operator equation (DOE) with top variable coefficients

\[ L u = a(x) u^{(2)}(x) + B(x) u^{(1)}(x) + A(x) u(x) = f(x), \quad x \in (0, 1), \]

where \( a_k \) are complex-valued functions, \( A(x), B(x) \) are linear operators in a Banach space \( E \) and \( f \) is a \( E \)-valued function. The principal part of the associate differential operator is not self-adjoint. We prove that, the spectrum of the associated differential operator is discrete and the system of roots elements are complete in \( E \)-valued weighted \( L^p \) spaces. Note that, differential-operator equations (DOEs) have been studied extensively by many researchers (see [1, 3], [7, 9, 11, 13, 14], [16-26] and the references therein).

We start by giving the notations and definitions to be used in this paper. Let \( \gamma = \gamma(x) \) be a positive measurable weighted function on the region \( \Omega \subset \mathbb{R}^n \). Let \( L^p_{\gamma}(\Omega; E) \) denote the space of all strongly measurable \( E \)-valued functions that are defined on \( \Omega \) with the norm

\[ \|f\|_{p,\gamma} = \|f\|_{L^p_{\gamma}(\Omega; E)} = \left( \int_{\Omega} \|f(x)\|^p_E \gamma(x) \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \]

The weight \( \gamma(x) \) we will consider satisfy an \( A_p \) condition. i.e., \( \gamma(x) \in A_p \), \( p \in (1, \infty) \) if there is a positive constant \( C \) such that

\[ \left( \frac{1}{|Q|} \int_Q \gamma(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \gamma^{\frac{1}{p-1}} \, dx \right) \leq C \]

for all balls \( Q \subset \mathbb{R}^n \).

For \( \gamma(x) \equiv 1 \) the space \( L^p_{\gamma}(\Omega; E) \) will be denoted by \( L^p(\Omega; E) \). The Banach space \( E \) is said to be a \( \zeta \)-convex space (see e.g. [4]) if there exists a symmetric real-valued function \( \zeta(u, v) \) on \( E \times E \) which is convex with respect to each of the variables, and satisfies the conditions

\[ \zeta(0, 0) > 0, \quad \zeta(u, v) \leq \|u + v\| \text{ for } \|u\| = \|v\| = 1. \]

The Banach space \( E \) is called an \( UMD \)-space if the Hilbert operator \( (Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{f(y)}{x-y} \, dy \) is bounded in \( L^p(-\infty, \infty, E) \), \( p \in (1, \infty) \) (see e.g. [4]).

\( UMD \) spaces include e.g. \( L_p, l_p \) spaces and Lorentz spaces \( L_{pq}, p, q \in (1, \infty) \). It is shown [4] that the Banach space \( E \) is \( UMD \) if only if this space is a \( \zeta \)-convex space.

Let \( \mathbb{C} \) be the set of complex numbers and
$S_{\varphi} = \{ \lambda : \lambda \in \mathbb{C}, |\arg \lambda| \leq \varphi \} \cup \{0\}, 0 \leq \varphi < \pi.$

Let $E_1$ and $E_2$ be two Banach spaces. $B(E_1, E_2)$ denotes the space of bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

A linear operator $A$ is said to be positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and

$$\left\| (A + \lambda I)^{-1} \right\|_{B(E)} \leq M (1 + |\lambda|)^{-1}$$

with $\lambda \in S_{\varphi}, \varphi \in (0, \pi]$, $I$ is an identity operator in $E$. Sometimes instead of $A + \lambda I$ will be written $A + \lambda$ and it will be denoted by $A_{\lambda}$. Let $E(A)$ denote $D(A)$ with the graphical norm.

A set $W \subset B(E_1, E_2)$ is called $R$-bounded (see e.g. [9]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in W$ and $u_1, u_2, ..., u_m \in E_1, m \in N$

$$\int_{0}^{1} \left\| \sum_{j=1}^{m} r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_{0}^{1} \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$.

The positive operator $A$ is said to be an $R$-positive in a Banach space $E$ if there exists $\varphi \in [0, \pi)$ such that the set $\{A(A + \xi I)^{-1} : \xi \in S_{\varphi}\}$ is $R$-bounded.

A linear operator $A = A(x), x \in [a, b]$ is said to be uniformly positive in a Banach space $E$, if $D(A(x))$ dense in $E$ and does not depend on $x$ and there is a constant $M > 0$ such that

$$\left\| (A(x) + \lambda I)^{-1} \right\|_{B(E)} \leq M (1 + |\lambda|)^{-1}$$

for all $\lambda \in S_{\varphi}, x \in [a, b]$ and some $\varphi \in [0, \pi)$.

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embeds into $E$.

Let $W_{\gamma}^{p, \gamma}(0, 1; E_0, E)$ denote a space of all functions $u \in L_{p, \gamma}(0, 1; E_0)$ possess the generalized derivatives $u^{(2)} \in L_{p, \gamma}(0, 1; E)$ with the norm

$$\|u\|_{W_{\gamma}^{p, \gamma}(0, 1; E_0, E)} = \|u\|_{L_{p, \gamma}(0, 1; E_0)} + \|u^{(2)}\|_{L_{p, \gamma}(0, 1; E)} < \infty.$$

$Sp A$ denote the closure of the linear span of the roots elements of the operator $A$.

Let $E$ be a Banach space and $E^*$ denotes its dual. For $u \in E, f \in E^*$ let $\langle u, f \rangle$ denote the value of $f$ for $u$, i.e. $\langle u, f \rangle = f(u)$. Suppose $\{e_j, f_j\}$, $j = 1, 2, ...$ is a biorthonormal basis systems in $E \times E^*$, i.e.

$$\{e_j\} \subset E, \{f_j\} \subset E^*, \langle e_j, f_i \rangle = \delta_{ij}, \|e_j\|_E = 1, \|f_i\|_{E^*} = 1, i, j = 1, 2, ...$$
For \( u \in E, f \in E^\ast \) let \( \alpha_j = \langle u, f_j \rangle \) and \( \beta_j = \langle e_j, f \rangle \) denote the Fourier coefficients of \( u \) and \( f \) with respect to systems \( \{e_j\} \subset E \) and \( \{f_j\} \subset E^\ast \), respectively.

**Definition 1.** A separable Banach space with base is said to be the space satisfying the \( B \)-condition, if there are a positive constant \( C \) and a \( p \in (1, \infty) \) such that

\[
\langle u, f \rangle = \sum_{j=1}^{\infty} \alpha_j \beta_j, \quad \|u\|_E^p \leq C \sum_{j=1}^{\infty} |\alpha_j|^p < \infty
\]

for all biorthonormal basis systems \( \{e_j, f_j\}, j = 1, 2, ... \) in \( E \times E^\ast \).

The Hilbert spaces satisfies this condition for \( p = 2 \). For examples \( L_p \) and \( l_p \) spaces, \( p \in (1, \infty) \) satisfies the \( B \)-condition. Note that, all uniformly convex Banach spaces with base satisfies the \( B \)-condition (see \([10, \S\ 6, \text{p. 75}], \text{Theorem} 1\)).

**Definition 2.** A bounded linear operator \( A \) is said to be a quasi nuclear operator (QNO) of order \( p \) if there is a \( p \in (1, \infty) \) such that

\[
\|A\|^p_{\sigma_p(E)} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |< Ae_i, f_j >|^p < \infty.
\]

The collection of such operators will be denoted by \( \sigma_p(E) \).

Let \( s_j(A) \) denote the approximation numbers of the operator \( A \) (see e.g. \([24, \S\ 1.16.1]\)). Let

\[
C_p(E) = \left\{ A : A \in \sigma_\infty(E), \sum_{j=1}^{\infty} s_j^p(A) < \infty, \ 1 \leq p < \infty \right\}.
\]

**Remark 1.** Let \( H \) be a Hilbert space and \( A \) be a compact operator in \( H \). Then \( s_j(A) = \lambda_j (A^*A)^{\frac{1}{2}} \), where \( \lambda_1, \lambda_2, ... \) are eigenvalues of non negative self adjoint operator \( T = (A^*A)^{\frac{1}{2}} \), arranged in decreasing order and repeated according to multiplicity. \( \{s_j(A)\} \) are called the characteristic numbers of the operator \( A \). By Corollary 7 in \([8, \text{Corollary XI. 9.1}]\), if \( A \in C_p(H), p \in (0, \infty) \), then the Weyl type inequality is true:

\[
\sum_{j=1}^{\infty} |s_j(A)|^p \leq \sum_{j=1}^{\infty} s_j^p(A).
\]  \( (1) \)

By choosing \( E = H, A \in C_p(H) \) and by putting \( f_j = e_j, j = 1, 2, ... \) in Definition 2, where \( e_j \) are orthonormal eigenvectors of the operator \( A \), by (1) we obtain

\[
\|A\|^p_{\sigma_p(H)} = \sum_{i=1}^{\infty} |< Ae_i, e_j >|^p = \sum_{j=1}^{\infty} |\lambda_j(A)|^p \leq \sum_{j=1}^{\infty} s_j^p(A) = \|A\|^p_{\sigma_p(H)} < \infty.
\]

It implies that \( C_p(H) \subset \sigma_p(H) \). The embedding \( C_p(E) \subset \sigma_p(E) \) can also be shown for the Banach spaces \( E \) satisfying the \( B \)-condition. Thus, let \( E \) be
a Banach space satisfying the $B$-condition and $A \in C_p(E)$ such that $\{e_j\}$ is an
eigen system of the operator $A$ corresponding to the eigen values $\{\lambda_j\}$ of the $A$.
So, for the appropriate biorthonormal system $\{e_j, f_j\}$, $j = 1, 2, \ldots$ in $E \times E^*$ we get

$$< Ae_i, f_j > = | < \lambda_i e_i, f_j > | = | \lambda_i | .$$

Then, by virtue of Weyl type inequality in Banach spaces \cite[p. 85]{12} we have

$$\|A\|_{\sigma_p(E)}^p = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \| < Ae_i, f_j > \|^p = \sum_{i=1}^{\infty} \| \lambda_i (A) \|^p \leq \sum_{i=1}^{\infty} s_i^p(A) = \|A\|_{C_p(E)}^p < \infty. $$

Since all $A \in C_p(E)$ can be approximated by sequences of finite dimensional operators in the Banach spaces $E$ with basis, the embedding $C_p(E) \subset \sigma_p(E)$
is shown for all $A \in C_p(E)$.

Let us firstly, point out some properties of the set $\sigma_p(E)$.

**Corollary 1.** Let $E$ be a Banach space satisfying the $B$-condition and $A \in \sigma_p(E)$ for a $p \in (1, \infty)$. Suppose $\{e_j, f_j\}$, $j = 1, 2, \ldots$ is a biorthonormal basis system in $E \times E^*$, then there is a positive constant $C$ so that

$$\|A\|_p \geq C \left( \sum_{i=1}^{\infty} \| Ae_i \|^p_E \right)^{\frac{1}{p}} .$$

**Proof.** Really, by virtue of B-condition we have

$$\|A\|_p^p = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \| < Ae_i, f_j > \|^p \geq C \sum_{i=1}^{\infty} \| Ae_i \|^p_E .$$

It is implies the assertion.

$U$ is an unitary operator in $E$ if $U$ and $U^{-1}$ are bounded in $E$ and $\|Ux\|_E = \|x\|_E$, $\|U^* g\|_{E^*} = \|g\|_{E^*}$ for all $x \in E$ and $g \in E^*$. Moreover if $\{e_j, f_j\}$, $j = 1, 2, \ldots$ is a biorthonormal basis system in $E \times E^*$, then $\{Ue_j, (U^{-1})^* f_j\}$ and $\{U^{-1}e_j, U^* f_i\}$ are also biorthonormal basis systems in $E \times E^*$.

**Lemma 1.** Let $E$ be a Banach space satisfying the $B$-condition. The $\sigma_p(E)$ norms, for a fixed $p \in (1, \infty)$ with respect to the different biorthonormal basis systems used in its definition, are equivalent. If $A \in \sigma_p(E)$ and $U$ is a unitary operator in $E$, then $U^{-1}AU \in \sigma_p(E)$ and there are positive constants $C$, $C_1$ and $C_2$ such that:

(a)  $$\|A\|_{B(E)} \leq C \|A\|_{\sigma_p(E)} , \quad \|A\|_{\sigma_p(E)} = \|A^*\|_{\sigma_p(E^*)} .$$

(b)  $$C_1 \|A\|_{\sigma_p(E)} \leq \|U^{-1}AU\|_{\sigma_p(E)} \leq C_2 \|A\|_{\sigma_p(E)} .$$

**Proof.** Suppose $\{e_j, f_j\}$ and $\{v_j, g_j\}$, $j = 1, 2, \ldots$ are two biorthonormal basis systems in $E \times E^*$. Then there is a unitary operator $U$ such that $v_j = U e_j$.
and \( g_i = (U^{-1})^* f_i \). I.e, there are a system of numbers \( \{a_{jk}\}, \{b_{ik}\} \) such that
\[
e_j = \sum_{k=1}^{\infty} a_{jk} \upsilon_k, \quad f_i = \sum_{m=1}^{\infty} b_{im} g_m, \quad \text{where}
\]
\[
a_{jk} = \langle e_j, g_k \rangle \quad \text{and} \quad b_{ik} = \langle f_i, \upsilon_k \rangle .
\]

Let \( \|A\|_{1,p} \) and \( \|A\|_{2,p} \) denote \( \sigma_p \) norms of the operator \( A \) with respect to first and second basis systems, respectively. Substituting the above equality in the expression \( \|A\|_{1,p}^p \) and by using the linearity properties of \( A \) and \( f_i \) we have
\[
\|A\|_{1,p}^p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} |\langle A e_j, f_i \rangle|^p = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |a_{jk}|^p \sum_{m=1}^{\infty} |b_{im}|^p |\langle \upsilon_k, g_m \rangle|^p .
\]

By virtue of \( B \)-condition, \( \sum_{j=1}^{\infty} |a_{jk}|^p \leq C \), for all \( k \) and \( \sum_{i=1}^{\infty} |b_{im}|^p \leq C \) for all \( m \).

Then we get from the above
\[
\|A\|_{1,p}^p \leq C \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} |\langle \upsilon_k, g_m \rangle|^p = C \|A\|_{2,p}^p .
\]

In a similar way, we get
\[
\|A\|_{2,p}^p \leq C \|A\|_{1,p}^p .
\]

This implies that \( \sigma_p (E) \) norms are independent of the biorthonormal basis systems.

Let \( \{e_j, f_j\}, j = 1, 2, ... \) be a biorthonormal basis system in \( E \times E^* \). By using Definition 2 it is seen that
\[
\|A\|^p = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle A e_i, f_j \rangle|^p = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle e_i, A^* f_j \rangle|^p = \|A^*\|^p .
\]

The assertion (b) is obtained from the equivalence of \( \sigma_p (E) \) norms with respect to different basis systems. Really, if \( U \) is a uniter operator in \( E \), then \( \{U e_i, (U^{-1})^* f_j\} \) is a biorthonormal system in \( E \times E^* \). So, we have
\[
\|U^{-1} A U\|^p = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle U^{-1} A U e_i, f_j \rangle|^p
\]
\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |\langle U e_i, (U^{-1})^* f_j \rangle|^p \leq C_2 \|A\|^p .
\]

In a similar way we get
\[
\|A\|^p \leq C_1 \|U^{-1} A U\|^p .
\]
These two inequalities imply the assumption (b).

Finally, if \( \varepsilon > 0 \) let \( u_0 \) be an element of unit norm such that

\[
\|A\|_{B(E)}^p \leq \|Au_0\|_{E}^p + \varepsilon.
\]

Then, by definition of \( \sigma_p(E) \) and by Corollary 1 we get

\[
\|A\|_{B(E)} \leq C \|A\|_{\sigma_p(E)}.
\]

**Remark 2.** The basis equivalence of \( \sigma_p(E) \) norms, for a fixed \( p \in (1, \infty) \), mean that, there are the positive constants \( C_1, C_2 \) such that \( \|A\|_{i,p}, i = 1, 2 \) norms with respect to different two biorthonormal basis systems satisfy the relation

\[
C_1 \|A\|_{1,p}^p \leq \|A\|_{2,p}^p \leq C_2 \|A\|_{1,p}^p.
\]

The independence of \( \sigma_p(E) \) norms of basis systems are valid when \( E \) is a Hilbert space.

In a similar way as in [8, Theorem XI. 6.4-7, ] we have

**Theorem A1.** Let \( E \) be a Banach space satisfying the \( B \)-condition. Then, the set \( \sigma_p(E), p \in (1, \infty) \) is a Banach space under \( \sigma_p(E) \) norm.

**Theorem A2.** Let \( E \) be a Banach space satisfying the \( B \)-condition. Then, every \( A \in \sigma_p(E), p \in (1, \infty) \) is a compact operator in \( E \) and is a limit in \( \sigma_p(E) \) norm of a sequence of operators with finite dimensional range.

**Theorem A3.** Let \( E \) be a Banach space satisfying the \( B \)-condition. If \( A \in \sigma_p(E) \) for a \( p \in (1, \infty) \) and \( F \) is a single-valued analytic function on its spectrum which vanishes at zero, then \( F(A) \in \sigma_p(E) \) and the map \( A \to F(A) \) is continuous in \( \sigma_p(E) \). Furthermore, if \( \{F_n \} \) is a sequence of such functions having as common domain a neighborhood \( N \) of the spectrum of \( A \) and if \( F_n(\lambda) \to F(\lambda) \) uniformly for \( \lambda \in N \), then \( F_n(A) \to F(A) \) in \( \sigma_p(E) \).

**Lemma 2.** Let \( E \) be a Banach space satisfying the \( B \)-condition and \( A \in \sigma_p(E), B \in \sigma_q(E) \) for a \( p, q \in (1, \infty) \), where \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose \( \{e_j, f_j \}, j = 1, 2, \ldots \) is a biorthonormal basis system in \( E \times E^* \), then the series \( \sum_{i=1}^{\infty} < Ae_i, B^* f_i > \) converges absolutely to a limit which is independent of the basis. Moreover,

\[
\sum_{i=1}^{\infty} < Ae_i, B^* f_i > = \sum_{i=1}^{\infty} < Be_i, A^* f_i >.
\]

**Proof.** By Hölder inequality we have

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |< Ae_i, f_j >| < e_i, B^* f_j > \leq \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |< Ae_i, f_j>|^p \right\}^{\frac{1}{p}} \cdot \left\{ \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |< e_i, B^* f_j>|^q \right\}^{\frac{1}{q}} = \|A\|_{\sigma_p(E)} \|B^*\|_{\sigma_q(E)}.
\]
Thus the double series \( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle A e_i, f_j \rangle < e_i, B^* f_j \rangle \) converges absolutely, and hence the corresponding iterated series exists and are equal. Moreover, by \( B \)-condition, there is another biorthonormal basis system \( \{e'_j, f'_j\}, j = 1, 2, \ldots \) in \( E \times E^* \) such that

\[
\sum_{i=1}^{\infty} < A e_i, B^* f_i >= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} < A e_i, f'_j > < e'_i, B f_j > \tag{2}
\]

\[
= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} < e_i, A^* f'_j > < B e'_i, f_j > = \sum_{j=1}^{\infty} < B e'_i, A^* f'_j > .
\]

From (2) we obtain

\[
\sum_{i=1}^{\infty} < A e_i, B^* f_i >= \sum_{i=1}^{\infty} < B e_i, A^* f_i > .
\]

Hence, this expression is symmetric in \( A \) and \( B \). By using (2) we obtain the independence of the limit from the basis systems.

**Definition 3.** Let \( E \) be a Banach space satisfying the \( B \)-condition and

\[ A \in \sigma_p(E), \ B \in \sigma_q(E) \text{ for a } p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1. \]

Suppose \( \{e_i, f_i\}, i = 1, 2, \ldots \) is a biorthonormal basis system in \( E \times E^* \), then the trace of \( (A, B) \) is defined to be as:

\[ Tr(A, B) = \sum_{i=1}^{\infty} < A e_i, B^* f_i > . \]

**Corollary 2.** Let \( A \in \sigma_p(E), B \in \sigma_q(E) \text{ for a } p, q \in (1, \infty), \frac{1}{p} + \frac{1}{q} = 1, \) then the trace is a symmetric bilinear function and

\[
Tr(A, B) \leq \|A\|_{\sigma_p(E)} \|B^*\|_{\sigma_q(E)} . \tag{3}
\]

**Proof.** The symmetry of the trace function were proved during the proof of Lemma 2. Moreover, by (2) we get

\[
\sum_{i=1}^{\infty} < A e_i, B^* f_i >= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} < A e_i, f'_j > < e'_i, B f_j > .
\]

So by Holder inequality and \( B \)-condition we have

\[
\sum_{i=1}^{\infty} < A e_i, B^* f_i > \leq \left( \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |< A e_i, f'_j >|^p \right)^{\frac{1}{p}}
\]
This relation implies the assertion.

In a similar manner as [8, Lemma XI. 6.20 ] we have

**Lemma A.** Let $E$ be a Banach space satisfying the B-condition. Suppose $A \in \sigma_p (E)$ for a $p \in (1, \infty)$ having a finite dimensional range. Let $N (A)$ be the null space of $A$, and let $P$ be the orthogonal projection onto a finite dimensional subspace of $E$ containing $|N (A)|$. Then:

(a) the spectra of the operators $A$ and $PA$ coincide;

(b) For a single valued analytic function $F$ on spectrum of $A$ with $F (0) = 0$, the following hold

$$F (PA) = PF (A), F (A) = F (A) P;$$

(c) $Tr (F (A), A) = Tr (PF (A), PA)$ and $Tr (F (PA), PA)$ coincide with the trace of the restriction of the operator $PA (A)$ to the finite dimensional space $PE$.

**Proof.** The (a) and (b) parts are proving by using the spectral properties of compact operators and operator calculus as in [8, Lemma XI. 6.20 ]. Let $\{e_j, f_j\}$, $j = 1, 2, ...$ is a biorthonormal basis system in $E \times E^*$. Since $PE$ is finite dimensional we may suppose that there is a number $d$ such that finite set $\{e_j\}$, $j = 1, 2, .., d$ is a basis for $PE$, and the sub set $\{e_j\}$, $j = d + 1, d + 2, ...$ is a basis for $(I - P) E$. Then, since $A = AP$, we have $A^* = PA^*$ and

$$Tr (F (PA), PA) = Tr (PF (A), PA) = \sum_{j=1}^{\infty} < PF (A) e_j, (PA)^* f_j >$$

$$= \sum_{j=1}^{d} < F (A) e_j, A^* f_j > = \sum_{j=1}^{d} < PAF (A) e_j, f_j > = Tr [PAF (A) \mid PE].$$

Since $F (A) E = F (A)$, we have $F (A) (I - P) = 0$, $F (A) e_j = 0$ for $j = d + 1, d + 2, ...$, and it follows from the above that

$$Tr (F (A), A) = \sum_{j=1}^{\infty} < F (A) e_j, A^* f_j > = \sum_{j=1}^{d} < F (A) e_j, A^* f_j >$$

$$= Tr (F (PA), PA)$$

which implies the (c) part.

In a similar way as [8, Lemmas XI. 6.21- 6.23 ] we obtain, respectively.

**Lemma A.** Let $\lambda$ and $z$ be complex numbers with $\lambda z \neq 1$ and let

$$F (\lambda, z) = z^{-1} [\log (1 - \lambda z) + \lambda z].$$
Let $E$ be a Banach space satisfying the $B$-condition and $A \in \sigma_p(E)$ for a $p \in (1, \infty)$ whose spectrum does not include the number $\lambda^{-1}$. Suppose $\{e_j, f_j\},$ $j = 1, 2, \ldots$ is a biorthonormal basis system in $E \times E^*$. Then for any finite subsets $\{e_j\}, \{f_j\},$ $j = 1, 2, \ldots, d$ the following inequality holds:

$$\exp [Tr (F (\lambda, A), A)] \leq \exp \left\{ \frac{1}{p} \sum_{j=1}^{d} |\lambda A e_j|^p \right\} \prod_{j=1}^{d} \left[ 1 - 2 \text{Re} < \lambda A e_j, f_j > + \|\lambda A e_j\|^p \right]^{\frac{1}{p}}.$$

**Lemma A.3.** For any positive $\varepsilon$ we have

$$\lim_{|\lambda| \to \infty} e^{-\varepsilon |\lambda|^p} \exp [Tr (F (\lambda, A), A)] = 0.$$

In a similar way as [8, Theorem XI. 6.24] we have

**Theorem A.4.** Let $E$ be a Banach space satisfying the $B$-condition. Assume $N \in \sigma_p(E)$ for a $p \in (1, \infty)$ is a quasi-nilpotent operator. Then $Tr (N, N) = 0$.

We are now in a position to obtain results in infinite dimensional Banach spaces by using of key finite dimensional results. By this aim by following [8, Theorem XI. 6.24] we obtain

**Theorem 1.** Let $E$ be a Banach space satisfying the $B$-condition. Suppose $A \in \sigma_p(E)$ for a $p \in (1, \infty)$ and $\lambda_1, \lambda_2, \ldots$ are its eigenvalues repeated according to multiplicities. If $F$ and $g$ are functions analytic in a neighborhood of the spectrum of $A$ with $F(0) = 0, g(0) = 0$, then $F(A), g(A) \in \sigma_p(E)$, and

$$Tr (F (A), g (A)) = \sum_{i=1}^{\infty} F (\lambda_i) g (\lambda_i),$$

where the series on the right hand side is absolutely convergent.

**Proof.** At first, by reasoning as the beginning of the proof [8, Theorem XI. 6.25], we get

$$\sum_{i=1}^{\infty} |F (\lambda_i)|^p < \infty, \sum_{i=1}^{\infty} |F (\lambda_i) g (\lambda_i)| < \infty.$$

Let $P_i = P (\lambda_i; A)$ denote the projection operators defined in [8, \& 11.3] i.e.

$$P_i E = E_i, \dim E_i < \infty, i = 1, 2, \ldots$$

Let $G_1$ be the closure of the subspace $\sum_{i=1}^{\infty} P_i E$ and $G_2$ be the orthocomplement of the $G_1$, i.e.

$$G_2 = \{f \in E^* : < u, f > = 0, u \in G_1 \}.$$

Suppose $\{e_j, f_j\}, j = 1, 2, \ldots$ is a biorthonormal basis system in $E \times E^*$. Assume $\{e_j\}$ so that the sub system$\{e_j\}, j = 1, 2, \ldots n_1$ is a basis for $E_1$, $\{e_j\}, j = 1, 2, \ldots n_2$ is a basis for $E_2$, $\{e_j\}, j = 1, 2, \ldots n_3$ is a basis for $E_3$ and so on.
1, 2, ..., \(n_2\) is a basis for \(E_2\), etc. Let \(\{\Psi_k\}\) be a sub system of \(\{f_j\} \subset E^*\) which is a basis for \(G_2\). Then by Definition 3 and Theorem A_3 we get

\[
\text{Tr} (F(A), g(A)) = \sum_{j=1}^{\infty} \langle F(A)e_j, (g(A))^*f_j \rangle + \sum_{k=1}^{\infty} \langle F(A)e_k, (g(A))^*\Psi_k \rangle.
\]

By Theorems A_1-A_2 and Lemma A_1 we have

\[
\sum_{j=1}^{\infty} \langle F(A)e_j, (g(A))^*f_j \rangle = \lim_{j \to \infty} \sum_{j=1}^{n_j} \langle F(A)e_j, (g(A))^*f_j \rangle
\]

\[
= \lim_{j \to \infty} \text{Tr} (gF(A), AP_j) = \sum_{i=1}^{\infty} F(\lambda_i) g(\lambda_i).
\]

Now it is sufficient to show the equality

\[
\sum_{k=1}^{\infty} \langle F(A)e_k, (g(A))^*\Psi_k \rangle = 0.
\]

By Lemma 2 we have

\[
\langle F(A)e_k, (g(A))^*\Psi_k \rangle = \langle g(A)e_k, (F(A))^*\Psi_k \rangle.
\]

So, the validity of (3) is a consequence of the validity of the following equations

\[
\sum_{k=1}^{\infty} \langle F(A)e_k, (F(A))^*\Psi_k \rangle = 0, \quad \sum_{k=1}^{\infty} \langle g(A)e_k, (g(A))^*\Psi_k \rangle = 0,
\]

\[
\sum_{k=1}^{\infty} \langle (F + g)Ae_k, (F + g)(A)^*\Psi_k \rangle = 0.
\]

All these equations being of the same form. So it is sufficient to show one of them. Let us prove the first of them.

By [8, Theorem \(\lor\) 11.3.20], \(G_1\) is mapped into itself by \(F(A)\). Thus \(G_2\) is mapped into itself by \((F(A))^*\). Let

\[
F(A)^* | G_2 = S.
\]

Then by Theorem A_3, Lemma 1, and Definition 2 we get \(S \in \sigma_p(E)\) and

\[
\langle PF(A)u, v \rangle = \langle F(A)u, v \rangle = \langle u, F(A)^*v \rangle, \quad u, v \in G_2,
\]

where \(P\) denoted the projection of \(E\) on \(G_2\). Thus \(PF(A)|G_2 = S^*\). Hence (4) is equivalent to the assertion

\[
\text{Tr} (S, S) = 0.
\]

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It follows from Theorem A_4 that to prove (5), it sufficient to show that S is quasi-nilpotent. If this is not so, then by [8, Theorem 11.4.5], there exists a non-zero complex number μ and a non-zero element u ∈ G_2 such that Su = μu. Thus, by [8, Theorem 11.4.5] again, P(μ, F(A)^∗) G_2 ≠ {0}. By definitions [8, 11.3.9, 3.17] and by [8, Lemma 1 \vee 2.10], it is seen that

\[ P(μ, F(A)^∗) = (P(μ, F(A)))^∗. \]

Hence, according to [8, Theorem 11.3.20], there is a non-zero complex number ν such that P(ν, A^∗)G_2 ≠ {0}. However, since \(<G_2, P(ν, A^∗)E> = 0\) for ν ≠ 0, by definition we have a contradiction which proves the present theorem.

In a similar way as [8, Theorem XI. 6.26] we have

**Theorem A_5.** Assume E is a Banach space satisfying the B-condition. Let A ∈ σ_p(E) for a p ∈ (1, ∞) and let λ_1, λ_2, ..., λ_n be its eigenvalues repeated according to multiplicities. Then the infinite product \(φ_{λ}(A) = \prod_{i=1}^{∞} (1 - \frac{λ}{λ_i}) e^{\frac{λ}{λ_i}}\) converges and defines a function analytic for λ ≠ 0. For each fixed λ ≠ 0 and \(φ_{λ}(A)\) is a continuous complex valued function on the Banach space of \(σ_p(E)\).

Now we can state the following Carleman theorem in Banach spaces.

**Theorem 2.** Let E be a Banach space satisfying the B-condition. Let A ∈ σ_p(E) for a p ∈ (1, ∞). If λ is in the resolvent set of the operator A, then

\[ \|φ_{λ}(A)(λ - A)^{-1}\|_{B(E)} ≤ |λ| \exp \left\{ \frac{1}{2} \left( 1 + \frac{∥A∥^p}{|λ|^2} \right) \right\}. \]

**Proof.** It follows from Theorem A_5 and [8, Lemma 11.6.1], that it is sufficient to consider the case in which A has a finite dimensional range \(R(A)\).

Let \(N(A) = \{u \in E : Au = 0\}\). Then \(E/N(A)\) is mapped by A in a one-to-one fashion into \(R(A)\). Thus \(E/N(A)\) is the finite dimensional space. Let V be a one dimensional subspace of \(N(A)\), \(V_1 = E/N(A) + R(A) + V\) and \(V_2 = E/V_1\). Then \(AV_2 = 0\), and \(AV \subset V_1\). Put \(A_1 = A|V_1\). Then it is easy to see that

\[ ∥A_1∥_{σ_p(E)} = ∥A∥_{σ_p(E)}, \ σ(A_1) = σ(A), \ φ_{λ}(A_1) = φ_{λ}(A). \]

Moreover, if \(u_i \in V_i, \ i = 1, 2\), then

\( (λ - A)^{-1}(u_1 + u_2) = (λ - A_1)^{-1}u_1 + λ^{-1}u_2. \)

Thus

\[ ∥(λ - A)^{-1}∥ = \max \left\{ ∥\lambda^{-1}∥, ∥(λ - A_1)^{-1}∥ \right\}. \]

On the other hand, we have

\[ ∥(λ - A_1)^{-1}∥ ≥ ∥\lambda^{-1}∥. \]

Really, if we suppose \(∥(λ - A_1)^{-1}∥ < ∥\lambda^{-1}∥\), then [8, Lemma 11.6.1] imply that \(A_1\) had an inverse which is impossible since the eigenvectors in V belong to its domain V_1. Thus

\[ ∥(λ - A)^{-1}∥ = ∥(λ - A_1)^{-1}∥. \]
Consequently, the present theorem follows immediately from [8, Theorem XI. 15].

Theorem 2 implies the following

**Corollary 3.** Let $E$ be a Banach space satisfying the $B$-condition. Let $N$ be a quasi-nilpotent operator in $\sigma_p (E)$ for a $p \in (1, \infty)$. Then for every $\lambda \neq 0$ we have

$$\| (\lambda - N)^{-1} \| \leq |\lambda| \exp \left\{ M \left( 1 + \left\| \frac{N}{\lambda} \right\|^p \right) \right\}, M > 0.$$\

Now we are in a position to prove the main theorem.

**Theorem 3.** Assume:

1. $E$ is a Banach space satisfying the $B$-condition and $A$ is an operator in $\sigma_p (E)$ for a $p \in (1, \infty)$;
2. $\gamma_1, \gamma_2, ..., \gamma_s$ is non overlapping, differentiable arcs in the complex plane starting at the origin. Suppose that each of the $s$ regions into which the plane is divided by these arcs is contained in an angular sector of opening less than $\frac{\pi}{s}$;
3. $m > 0$ is an integer so that the resolvent of $A$ satisfies the inequality $\| R (\lambda, A) \| = O \left( |\lambda|^{-m} \right)$ as $\lambda \rightarrow 0$ along any of the arcs $\gamma_i$.

Then the subspace $spA$ contains the subspace $A^m E$.

**Proof.** By the Hahn-Banach theorem it suffices to prove that every element $f \in E^*$ satisfying the condition $< u, f > = 0$ for $u \in spA$ also has $< A^m u, f > = 0$ for all $u \in E$. Let $f$ be such element. By theorem [8, Theorem \textcircled{1} 4.5], the function $f (\lambda) = \lambda^m R (\lambda, A^*) f$ is analytic everywhere in the plane except at $\lambda = 0$ and at an isolated set of points $\lambda_k \rightarrow \infty$, and at the points $\lambda_k$ the function $f (\lambda)$ may have a pole. For $\lambda \neq \lambda_k$ and $\lambda$ in the neighborhood of $\lambda_k$ we have

$$f (\lambda) = \lambda^m P (\lambda_k, A^*) R (\lambda, A^*) f + \lambda^m R (\lambda, A^*) (I - P (\lambda_k, A^*)) f =$$

$$= \lambda^m P (\lambda_k, A^*)^* R (\overline{\lambda}, A)^* f + f_1 (\lambda).$$

By virtue of [8, Theorem \textcircled{1} 11.3.20] and [8, Lemma \textcircled{1} 11.3.2] the function $f_1 (\lambda)$ is analytic at $\lambda = \lambda_k$. It will now be shown that the function $f_2 (\lambda) = \lambda^m P (\lambda_k, A)^* R (\overline{\lambda}, A)^* f$ vanishes which will prove that $f (\lambda)$ is analytic at all the points $\lambda = \lambda_k$, so that $f (\lambda)$ can only fail to be analytic at the point $\lambda = 0$. Really, note that

$$< u, f_2 (\lambda) > = < u, \lambda^m P (\lambda_k, A)^* R (\lambda, A)^* f >$$

$$= \lambda^m < P (\lambda_k, A) R (\overline{\lambda}, A) u, f > .$$

(7)

It follows from [8, Theorem \textcircled{1} 11.4.5] that

$$P (\lambda_k, A) R (\overline{\lambda}, A) u \in spA.$$\

Since $f \in (spA)^\perp$, (6) implies that $< u, f_2 (\lambda) > = 0$ for every $u \in E$ and thus $f_2 (\lambda) = 0$. Therefore $\lambda^m R (\lambda, A^*) f$ is analytic everywhere in the plane.
except at the origin. If the function $f$ is analytic at the origin then by reasoning as in [8, Theorem XI. 6.29] and by Liouville’s theorem we obtain the assertion. So the proof rests upon the assertion that the function $f(\lambda)$ is analytic at $\lambda = 0$. By using the Corollary 3, in a similar way as [8, Theorem XI. 6.29], we get that

$$\|R(\lambda, A)\| = O\left(\exp \left\{ M \left(1 + \frac{\|N\|}{\lambda}\right)^p \right\}\right), \quad M > 0$$

as $\lambda \to 0$. Then by virtue of Phragmen-Lindelöf theorem we obtain that the function $f$ is analytic at the origin.

By using Theorem 3, in a similar way as [8, Corollary XI. 6.31] we have

Corollary 4. Suppose (1) and (2) condition of Theorem 3 hold and resolvent of $A$ satisfies the inequality $\|R(\lambda, A)\| = O\left(|\lambda|^{-1}\right)$ as $\lambda \to \infty$ along any of the arcs $\gamma_i$. Then the subspace $\text{sp}A$ contains the subspace $E$.

**Proof:** It is sufficient to show that joint span of the range $R(A)$ and the null space $N(A)$ is the entire space $E$. Let $\{\lambda_n\}$ be a sequence of complex numbers converging to zero along one of the arcs $\gamma_i$ and let $u$ be an arbitrary element from $E$. By assumptions, the sequence $\{\lambda_nR(\lambda_n, A)\}$ is bounded. Since $E$ is reflexive, then this sequence is weakly convergent to an element $\upsilon$. The proof will be completed by showing that $Au = 0$ and $u - \upsilon \in \overline{N(A)}$. Then, by reasoning as in the proof of [8, Corollary XI. 6.30] we obtain the assertion.

By using Theorem 3, in a similar way as [8, Corollary XI. 6.31] we have

Corollary 5. Suppose:

1. $E$ is a Banach space satisfying the $B$-condition;
2. $A$ is a densely defined unbounded operator in $E$, with the property that for some $\lambda$ in the resolvent, the operator $R(\lambda, A)$ is of class $\sigma_p(E)$ for a $p \in (1, \infty)$;
3. $\gamma_1, \gamma_2, \ldots, \gamma_s$ is non overlapping, differentiable arcs in the complex plane having a limiting direction at infinity, and such that no adjacent pair of arcs form an angle as great as $\pi$ at infinity;
4. the resolvent of $A$ satisfies the inequality $\|R(\lambda, A)\| = O\left(|\lambda|^{-1}\right)$ as $\lambda \to \infty$ along any of the arcs $\gamma_i$.

Then the subspace $\text{sp}A$ contains the entire space $E$.

**Spectral properties of abstract elliptic operators**

Consider the nonlocal BVP for differential operator equation

$$(L + \lambda)u = a(x)u^{(2)}(x) + B(x)u^{(1)}(x) + A\lambda(x)u(x) = f(x), \quad x \in (0, 1) \quad (8)$$

$$L_ku = \sum_{i=0}^{m_k} \alpha_{ki}u^{(i)}(0) + \beta_{ki}u^{(i)}(1) + \sum_{j=1}^{N_k} \delta_{kij}u^{(i)}(x_{kj}) = 0, \quad k = 1, 2, \quad (9)$$
where \( A_\lambda = A + \lambda \), \( A = A(x) \), \( B = B(x) \) are linear operators in a Banach space \( E \), \( a = a(x) \) is a complex valued function, \( \alpha_{k_1}, \beta_{k_1}, \delta_{k_1} \) are complex numbers, \( x_{k_1} \in (0, 1) \) and \( \lambda \) is a spectral parameter. Let us denote \( \alpha_{km_1} \) and \( \beta_{km_1} \) by \( \alpha_k \) and \( \beta_k \), respectively. Let \( \omega_1 = \omega_1(x), \omega_2 = \omega_2(x) \) be roots of the characteristic equation \( a(x)\omega^2 + 1 = 0 \) and

\[
\eta = \eta(x) = \begin{pmatrix} (-\omega_1)^{m_1} \alpha_1 & \beta_1 \omega_1^{m_1} \\ (-\omega_2)^{m_2} \alpha_2 & \beta_2 \omega_2^{m_2} \end{pmatrix}.
\]

Function \( u \in W^2_{p, \gamma}(0, 1; E(A), E), L_ku = 0 \) satisfying the equation (7) a.e. on \((0, 1)\) is said to be solution of the problem (7) – (8).

We say that the problem (7) – (8) is \( L_{p, \gamma} \)-separable, if for all \( f \in L_{p, \gamma}(0, 1; E) \) there exists a unique solution \( u \in W^2_{p, \gamma}(0, 1; E(A), E) \) of the problem (7) – (8) and there exists a positive constant \( C \) such that the coercive estimate holds

\[
\|u(2)\|_{L_{p, \gamma}(0, 1; E)} + \|Au\|_{L_{p, \gamma}(0, 1; E)} \leq C \|f\|_{L_{p, \gamma}(0, 1; E)}.
\]

Let \( Q \) denote the operator generated by BVP (7) – (8) i.e.

\[
D(Q) = W^2_{p, \gamma}(0, 1; E(A), E), L_k, \quad Qu = au^{(2)} + Au + Bu^{(1)}.
\]

Let \( I(E(A), E) \) denote the embedding operator from \( E(A) \) to \( E \).

**Condition 1.** Let the following conditions be satisfied:

(1) \( E \) is an uniformly convex Banach space space with base and \( \gamma \in A_p, p \in (1, \infty); \)

(2) \( A \) is an \( R \)-positive in \( E \) with \( \varphi \in [0, \pi], A(x)A^{-1}(\bar{x}) \in C(\{0, 1\}; B(E)), \)

\( \bar{x} \in (0, 1) \) and \( BA(\frac{1}{2} - \mu) \in L_\infty(0, 1; B(E)) \) for \( 0 \leq \mu < \frac{1}{2}; \)

(3) \( -a \in S(\varphi_1) \cap \mathbb{C}/\mathbb{R}, a \neq 0, \eta(x) \neq 0, 0 \leq \varphi_1 < \pi, \lambda \in S(\varphi_2), \)

\( \varphi_1 + \varphi_2 < \varphi; \)

Let \( I = I(W^2_{p, \gamma}(0, 1; E(A), E), L_{p, \gamma}(0, 1; E)) \) denote the embedding operator

\[
W^2_{p, \gamma}(0, 1; E(A), E) \rightarrow L_{p, \gamma}(0, 1; E).
\]

In a similar way as in [19, Theorem 3] we obtain

**Theorem A**. Suppose the Condition1 holds. Then the problem (7) – (8) for \( f \in L_{p, \gamma}(0, 1; E), |\arg \lambda| \leq \varphi \) and sufficiently large \( |\lambda| \) has a unique solution \( u \in W^2_{p, \gamma}(0, 1; E(A), E) \) and the coercive uniform estimate holds

\[
\sum_{i=0}^{2} |\lambda|^{1-\frac{j}{p}} \|u(0)\|_{L_{p, \gamma}(0, 1; E)} + \|Au\|_{L_{p, \gamma}(0, 1; E)} \leq M \|f\|_{L_{p, \gamma}(0, 1; E)}.
\]

Moreover from [3] we have:

**Theorem A**. Let \( E \) be Banach spaces with base. Suppose the operator \( A \) is positive in \( E \) and \( A^{-1} \in \sigma_{\infty}(E) \). Assume that

\[
0 \leq \gamma < p - 1, 1 < p < \infty,
\]

\[
s_j(I(E(A), E)) \sim j^{-\frac{1}{p}}, \text{ for some } \nu > 0, j = 1, 2, ..., \infty.
\]
Then the embedding $W_{p,\gamma}^2(0,1;E(A),E) \subset L_{p,\gamma}(0,1;E)$ is compact and

$$s_j \left( I \left( W_{p,\gamma}^2(0,1;E(A),E), L_{p,\gamma}(0,1;E) \right) \right) \sim j^{-\frac{3}{2p}}.$$

**Remark 3.** Really, Theorems A$_6$ and A$_7$ are proven under condition that $E$ is an $\zeta$-convex Banach space. Since all uniformly convex space is a $\zeta$-convex space i.e. is an UMD space, by applying [3] we get the assertions.

By applying Theorem 3 and Theorems A$_6$, A$_7$ we obtain

**Theorem 4.** Suppose the Condition1 holds and

$$s_j \left( I(\xi), E(A) \right) \sim j^{-\frac{1}{2}}, \text{ for some } \nu > 0, \ j = 1, 2, ..., \infty;$$

Then:

(a) spectrum of the operator $Q$ is discrete;

(b) $s_j \left( (Q + \lambda)^{-1}(L_{p,\gamma}(0,1;E)) \right) \sim j^{-\frac{3}{2p}}.$ \hfill (11)

(c) if $\varphi \leq \frac{\pi}{2q}, \ q > \nu + \frac{1}{2}$ then the system of root functions of differential operator $Q$ is complete in $L_{p,\gamma}(0,1;E)$.

**Proof.** By virtue Theorem A$_1$, there exists a resolvent operator $(Q + \lambda)^{-1}$ which is bounded from $L_{p,\gamma}(0,1;E)$ to $W_{p,\gamma}^2(0,1;E(A),E)$. Moreover, by virtue of Theorem A$_3$ the embedding operator $I \left( W_{p,\gamma}^2(0,1;E(A),E), L_{p,\gamma}(0,1;E) \right)$ is compact and

$$s_j \left( I \left( W_{p,\gamma}^2(0,1;E(A),E), L_{p,\gamma}(0,1;E) \right) \right) \sim j^{-\frac{3}{p}}.$$ \hfill (12)

Since

$$(Q + \lambda)^{-1}(L_{p,\gamma}(0,1;E)) = (Q + \lambda)^{-1}(L_{p,\gamma}(0,1;E),W_{p,\gamma}^2(0,1;E(A),E))$$

$$I \left( W_{p,\gamma}^2(0,1;E(A),E), L_{p,\gamma}(0,1;E) \right)$$

then from relations (11) and (12) we obtain the assertions (a) and (b). Moreover, the estimate (9) and the relation (11) implies that operator $Q$ is positive in $L_{p,\gamma}(0,1;E)$ and

$$(Q + \lambda)^{-1} \in \tilde{\sigma}_q(L_{p,\gamma}(0,1;E)), \ \text{for } q > \nu + \frac{1}{2} \ \text{and } \lambda \in S(\varphi).$$

By virtue of Remark1, the above estimate implies

$$(Q + \lambda)^{-1} \in \sigma_q(L_{p,\gamma}(0,1;E)), \ q > \nu + \frac{1}{2}.$$ \hfill (13)

Then in view of the estimate (9), the relation (13) and by Theorem 3 we obtain the assertion (b).

Consider the following nonlocal BVP for degenerate DOE

$$(L + \lambda)u = a(x)u^{[2]}(x) + B(x)u^{[1]}(x) + A_\lambda(x)u(x) = f(x), \ x \in (0,1) \hfill (14)$$
\[
L_k u = \sum_{i=0}^{m_k} \alpha_{ki} u[i] (0) + \beta_{ki} u[i] (1) + \sum_{j=1}^{N_k} \delta_{kji} u[i] (x_{kj}) = 0, k = 1, 2,
\]
where
\[
u[i] = \left( x^\gamma \frac{d}{dx} \right)^i u.
\]

Let \( O \) denote the operator generated by problem (14) and
\[
W^{[2]}_p (0, 1; E_0, E) = \{ u \in L_p (0, 1; E_0) , u[2] \in L_p (0, 1; E) ,
\]

\[
\| u \|_{W^{[2]}_p (0, 1; E_0, E)} = \| u \|_{L_p (0, 1; E)} + \| u[2] \|_{L_p (0, 1; E)} < \infty.
\]

Theorem 4 implies the following result:

**Result 1.** Suppose all conditions of Theorem 4 are satisfies. Then the assertions (a), (b) and (c) of Theorem 4 are hold for the operator \( O \) in \( L_p (0, 1; E) \).

Really, under the substitution
\[
y = \int_0^x z^{-\gamma} dz
\]
the spaces \( L_p (0, 1; E) , W^{[2]}_p (0, 1; E (A), E) \) are mapped isomorphically onto spaces \( L_{p, \tilde{\gamma}} (0, b; E) , W^{[2]}_{p, \tilde{\gamma}} (0, b; E (A), E) \), respectively, where \( \tilde{\gamma} = \frac{1 - \gamma}{1 - \gamma} \).

Moreover, under this substitution the problem (14) is transformed into a non degenerate problem (7) – (8).

**References**

1. Amann H., Linear and quasi-linear equations,1, Birkhauser, 1995.

2. Agranovich M. S., Spectral Boundary Value Problems in Lipschitz Domains or Strongly Elliptic Systems in Banach Spaces \( H_p^\sigma \) and \( B_p^\sigma \), Functional Analysis and its Applications, 42 (4), (2008), 249-267.

3. Agarwal, R. P, Bohner, R., Shakhmurov, V. B, Maximal regular boundary value problems in Banach-valued weighted spaces, Bound. Value Probl., 1 (2005), 9-42.

4. Burkholder D. L., A geometrical conditions that implies the existence certain singular integral of Banach space-valued functions, Proc. conf. Harmonic analysis in honor of Antonu Zigmund, Chicago, 1981, Wads Worth, Belmont, (1983), 270-286.

5. Bourgain J., Some remarks on Banach spaces in which martingale differ- ences are unconditional, Arkiv Math. 21 (1983), 163-168.
6. Burgoyne J., Denseness of the generalized eigenvectors of a discrete operator in a Banach space, J. Operator Theory, 33 (1995), 279–297.

7. Dore G. and Yakubov S., Semigroup estimates and non coercive boundary value problems, Semigroup Form 60 (2000), 93-121.

8. Dunford N., Schwartz J. T., Linear operators. Parts 2. Spectral theory, Interscience, New York, 1963.

9. Denk R., Hieber M., Prüss J., $R$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (2003), n.788.

10. Diestel, J., Geometry of Banach spaces-selected topics, Springer-Verlag, Berlin-Heidelberg-New-York, 1975.

11. Gorbachuk V. I. and Gorbachuk M. L., Boundary value problems for differential-operator equations, Naukova Dumka, Kiev, 1984.

12. König, H., Eigenvalue Distribution of Compact Operators, Operator Theory: Advances and Applications, vol. 16, Birkhauser, Basel etc., 1986.

13. Krein S. G., Linear differential equations in Banach space, Providence, 1982.

14. Lunardi A., Analytic semigroups and optimal regularity in parabolic problems, Birkhauser, 2003.

15. Markus A. S., Some criteria for the completeness of a system of root vectors of a linear operator in a Banach space, Mat. Sb., 70 (112):4 (1966), 526–561.

16. Sobolevskii P. E., Inequalities coerciveness for abstract parabolic equations, Dokl. Akad. Nauk. SSSR, (1964), 57(1), 27-40.

17. Shahmurov R., Solution of the Dirichlet and Neumann problems for a modified Helmholtz equation in Besov spaces on an annuals, J. Differential Equations, 249(3) (2010), 526-550.

18. Shakhmurov V. B., Imbedding theorems and their applications to degenerate equations, Differential equations, 24 (4), (1988), 475-482.

19. Shakhmurov V. B., Coercive boundary value problems for regular degenerate differential-operator equations, J. Math. Anal. Appl., 292 (2), (2004), 605-620.

20. Shakhmurov V. B., Embedding theorems and maximal regular differential operator equations in Banach-valued function spaces, J. Inequal. Appl., 2(4)(2005), 329-345, 2(4), (2005), 329-345.
21. Shakhmurov V. B., Separable anisotropic differential operators and applications, J. Math. Anal. Appl. 2006, 327(2), 1182-1201.

22. Shakhmurov V. B., Embedding and maximal regular differential operators in Banach-valued weighted spaces, Acta. Math. Sin., (Engl. Ser.), (2012), 28 (9), 1883-1896.

23. Shahmurov R., On strong solutions of a Robin problem modeling heat conduction in materials with corroded boundary, Nonlinear Anal. Real World Appl., 13(1) (2011), 441-451.

24. Triebel H., Interpolation theory. Function spaces. Differential operators., North-Holland, Amsterdam, 1978.

25. Yakubov S., Completeness of root functions of regular differential operators, Longman, Scientific and Technical, New York, 1994.

26. Yakubov S and Yakubov Ya., Differential-operator equations. Ordinary and Partial Differential equations, Chapman and Hall /CRC, Boca Raton, 2000.