A hybridizable discontinuous Galerkin method for the indefinite time-harmonic Maxwell equations

Gang Chen* Jintao Cui† Haijun Wu‡ Liwei Xu§

Abstract

In this paper, we aim to develop a hybridizable discontinuous Galerkin (HDG) method for the indefinite time-harmonic Maxwell equations with the perfectly conducting boundary in the three-dimensional space. First, we derive the wavenumber explicit regularity result, which plays an important role in the error analysis for the HDG method. Second, we prove a discrete inf-sup condition which holds for all positive mesh size \( h \), for all wavenumber \( k \), and for general domain \( \Omega \). Then, we establish the optimal order error estimates of the
1 Introduction

Let $\Omega$ be a bounded simply-connected Lipschitz polyhedron in $\mathbb{R}^3$ with a connected boundary $\Gamma := \partial \Omega$. We consider the following lossless case of the time-harmonic Maxwell equations with the perfectly conducting boundary condition in a mixed form [45]:

Find the electric field $u$ and the Lagrange multiplier $p$ such that

$$\nabla \times \nabla \times u - k^2 u + (k^2 + 1) \nabla p = f \quad \text{in } \Omega,$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega,$$

$$n \times u = 0 \quad \text{on } \Gamma,$$

$$p = 0 \quad \text{on } \Gamma.$$

Here, $n$ is the outward normal unit vector to the boundary $\Gamma$, $f \in [L^2(\Omega)]^3$ is a given external source field, $k := \omega \sqrt{\varepsilon_0 \mu_0}$ is a real wavenumber, where $\omega > 0$ is a given temporal frequency, and $\varepsilon_0$ and $\mu_0$ are the electric permittivity and the magnetic permeability of the free space, respectively. Note that in the special case here (relative electric permittivity of the medium equals one and perfect conducting boundary condition), the real and imaginary parts are decoupled, and thus we assume that $u$, $p$ and $f$ are real.

The numerical solution of the indefinite time-harmonic Maxwell equations suffers from the following two challenges. First, on a non-convex domain, the solution of Maxwell equations is only in $[H^s(\Omega)]^3$ with $s \in (1/2, 1)$. A direct application of continuous finite element methods will result in a discrete solution that convergences to a function that is not a solution of the Maxwell equations. Second, the quality of discrete numerical solutions to the Maxwell equation depends significantly on the wavenumber $k$.

Different methods are applied to solve the electromagnetic models, including boundary integral methods [4, 23, 41], boundary element methods [3, 5], and finite element methods. The finite element method was the most popular computational technique for solving the time-harmonic Maxwell equation. In particular, finite element methods using $H(\text{curl}; \Omega)$-conforming edge elements have been studied in vast literatures for (1) and its reduced problem.
where $\nabla \cdot f = 0$, see [39, 40, 31, 38, 51, 50]. Moreover, preconditioners for finite element methods solving the indefinite Maxwell equations were investigated in [1, 47, 28, 29] and the references therein. Since the late 1970s, the discontinuous Galerkin (DG) methods have become increasingly popular due to its attractive features, including preserving local conservation of physical quantities, their flexible in meshing, easy of design and implementation, their suitable in parallel computation, and easy of use within an $hp$-adaptive strategy. DG methods for solving the time-harmonic Maxwell equations with zero wavenumber were first developed in [44, 35]. Later, interior penalty discontinuous Galerkin (IPDG) methods for the indefinite Maxwell equations was studied in [45, 46]. Since there was no wavenumber explicit regularity result available for the time-harmonic Maxwell equation with the perfect conducting boundary condition (1c), the constants in stability results and error estimates of the IPDG methods in [45] are highly dependent on the wavenumber. In [25, 33], the authors proposed and analyzed DG methods for the indefinite Maxwell equations with the impedance boundary condition, and derived the wavenumber explicit convergence results. We would like to remark that there are no research on the error estimates with explicit wavenumber dependence for the indefinite Maxwell equations with the perfect conducting boundary. We should also mention that in [7, 6, 8, 48], the DG methods for the spurious Maxwell modes were considered.

In recent years, the hybridizable discontinuous Galerkin (HDG) method, a “new” type of DG methods, has been successfully applied to solve various types of differential equations, see [18, 10, 11, 21, 27, 17, 13] and many other references. The HDG method retains the advantages of standard DG methods and can significantly reduce the number of degrees of freedom, therefore, allowing for a substantial reduction in the computational cost. The first work [42] that applies HDG methods to solve the indefinite time-harmonic Maxwell equations appears in 2011. In that paper, two HDG schemes are introduced and numerical results are reported to illustrate the performance of the proposed schemes. The convergence analysis is not given therein. Recently, two HDG methods for the time-harmonic Maxwell equations with zero wavenumber are proposed and analyzed in [15, 14, 12], where the a priori and a posteriori error estimates are derived. The HDG methods are also studied in [24, 37] for the indefinite time-harmonic Maxwell equations with the impedance boundary condition. The error estimates are derived where the constants depend explicitly on the wavenumber. The convergence analysis therein is based on the regularity results of Maxwell equations developed in
In this paper, we propose a new HDG method for the indefinite time-harmonic Maxwell equations (1) with the perfect conducting boundary condition. We first derive the wavenumber explicit regularity result of the Maxwell equations, that is: there exists a regularity index \( s \in (1/2, 1] \) dependent on \( \Omega \), such that \( u \in H^s(\text{curl}; \Omega) \) and
\[
(k + 1)\|u\|_s + \|\nabla \times u\|_s \leq C \mathcal{M}_k\|f\|_0,
\]
where \( \mathcal{M}_k \) is defined as
\[
\mathcal{M}_k = \sup_{\lambda_i \in E_\lambda} \frac{|\lambda_i + k^2|}{|\lambda_i - k^2|}.
\]
and \( E_\lambda \) is the set of all eigenvalues of the corresponding eigenvalue problems. The above regularity result is not yet available in the literature. Then based on the new regularity result, we establish the error estimates for the proposed HDG method, where the constants are independent of the wavenumber:
\[
\|u_h - u\|_0 \leq C (\mathcal{M}_k h^{s^*} + \mathcal{M}_k^2 h^2) \|f\|_0,
\]
providing \( \Omega \) is convex and \( \mathcal{M}_k k^2 h^{s^*} \leq C_0 \). To the best of our knowledge, such convergence result is also the first of its kind in the numerical study of the indefinite time-harmonic Maxwell equations.

The rest of this paper is organized as follows. In section 2, we give a regularity result of the indefinite time-harmonic Maxwell equations. In section 3 and 4, we propose a new HDG method and establish its well-posedness. In section 5, we develop the convergence analysis of the HDG method based on the regularity and stability results. In section 6, numerical experiments are performed to verify the theoretical results.

Throughout this paper, we use \( C \) to denote a positive constant independent of mesh size and the wavenumber \( k \), not necessarily the same at its each occurrence. For convenience we use the shorthand notation \( a \lesssim b \) and \( a \gtrsim b \) for the inequality \( a \leq Cb \) and \( b \leq Ca \). \( a \simeq b \) stands for \( a \lesssim b \) and \( a \gtrsim b \).

### 2 The wavenumber explicit regularity

For any bounded Lipschitz domain \( \Lambda \subset \mathbb{R}^s \) (s = 2, 3), let \( H^m(\Lambda) \) and \( H^m_0(\Lambda) \) denote the usual \( m^{th} \)-order Sobolev spaces on \( \Lambda \), and \( \| \cdot \|_{m, \Lambda}, | \cdot |_{m, \Lambda} \) denote
the norm and semi-norm on these spaces. We use \((\cdot, \cdot)_{m, \Lambda}\) to denote the inner product of \(H^m(\Lambda)\), with \((\cdot, \cdot)_{\Lambda} := (\cdot, \cdot)_{0, \Lambda}\). We use \(v|_{\partial \Lambda}\) to denote the trace of \(v\) on \(\partial \Lambda\). When \(\Lambda = \Omega\), we denote by \(\| \cdot \|_m := \| \cdot \|_{m, \Omega}, | \cdot |_m := | \cdot |_{m, \Omega}, (\cdot, \cdot) := (\cdot, \cdot)_\Omega\). In particular, for a surface \(F\) and a curve \(E\) in \(\mathbb{R}^3\) we use \((\cdot, \cdot)_F\) and \(\langle \langle \cdot, \cdot \rangle \rangle_E\) to denote the \(L^2\) inner products on \(F\) and \(E\), respectively. The bold face fonts will be used for vector (or tensor) analogues of the Sobolev spaces along with vector-valued (or tensor-valued) functions. Define the spaces

\[
H(\text{curl}; \Omega) := \{ v \in [L^2(\Omega)]^3 : \nabla \times v \in [L^2(\Omega)]^3 \}, \\
H^s(\text{curl}; \Omega) := \{ v \in [H^s(\Omega)]^3 : \nabla \times v \in [H^s(\Omega)]^3 \} \text{ with } s \geq 0, \\
H_0(\text{curl}; \Omega) := \{ v \in H(\text{curl}; \Omega) : n \times v|_\Gamma = 0 \}, \\
H(\text{div}; \Omega) := \{ v \in [L^2(\Omega)]^3 : \nabla \cdot v \in L^2(\Omega) \}, \\
H_0(\text{div}; \Omega) := \{ v \in H(\text{div}; \Omega) : n \cdot v|_\Gamma = 0 \}, \\
H(\text{div}^0; \Omega) := \{ v \in H(\text{div}; \Omega) : \nabla \cdot v = 0 \}, \\
X := H(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \\
X^N := H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega), \\
X^N_0 := H_0(\text{curl}; \Omega) \cap H(\text{div}^0; \Omega), \\
X_T := H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega).
\]

Lemma 2.1 (Helmholtz decomposition [38]). For any \(v \in L^2(\Omega)\), there exist functions \(\phi \in H^1(\Omega)\) and \(\psi \in H^1_0(\Omega)\) such that

\[
v = z + \nabla \psi, \quad \nabla \cdot z = 0, \quad (2)
\]

and

\[
\|z\|_0 + \|\nabla \psi\|_0 \leq C\|v\|_0. \quad (3)
\]

We define the bilinear form:

\[
a^\pm(u, v) = (\nabla \times u, \nabla \times v) \pm k^2(u, v). \quad (4)
\]

By testing the first equation of (1) with functions \(v \in X^N_0\), it is easy to check that the solution \(u\) of (1) is also the solution of the weak problem:

Find \(u \in X_{N, 0}\) such that

\[
a^- (u, v) = (f, v) \text{ for all } v \in X_{N, 0}. \quad (5)
\]

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Similarly, by testing the first equation of (1) with $\nabla q$ where $q \in H^1_0(\Omega)$, we observe that the solution $p$ of (1) is also the solution of the weak problem: Find $p \in H^1_0(\Omega)$ such that

$$(k^2 + 1)(\nabla p, \nabla q) = (f, \nabla q) \text{ for all } q \in H^1_0(\Omega).$$

(6)

Introduce the following auxiliary problem: find $\tilde{u} \in X_{N,0}$ such that

$$a^+ (\tilde{u}, v) = (f, v) \text{ for all } v \in X_{N,0}.$$ 

(7)

Define the solution operator $K_k : [L^2(\Omega)]^3 \mapsto X_{N,0}$ as: for any $w \in [L^2(\Omega)]^3$, find $K_k w \in X_{N,0}$ such that

$$a^+ (K_k w, v) = -2k^2 (w, v) \text{ for all } v \in X_{N,0}.$$ 

(8)

Let $u \in X_{N,0}$ be the weak solution to (1) (i.e., the solution to (5)), then it is obvious that

$$a^+ ((I + K_k) u, v) = a^+ (\tilde{u}, v),$$

which leads to the following relation:

$$(I + K_k) u = \tilde{u}. \quad (9)$$

We recall the classical estimation for vector potential $v \in X$ in the following lemma.

**Lemma 2.2** (cf. [26, Proposition 7.4]). Let $v \in X$, $\Gamma_s \cap \Gamma_\nu = \emptyset$ and $\Gamma_s \cup \Gamma_\nu = \Gamma$, then there exists a constant such that

$$\|v\|_0 \leq C (\|\nabla \times v\|_0 + \|\nabla \cdot v\|_0 + \|n \times v\|_{0, \Gamma_s} + \|n \cdot v\|_{0, \Gamma_\nu}).$$

The stability results of problems (7) and (8) are established in the next two lemmas.

**Lemma 2.3.** The problem (7) has a unique solution satisfying the following estimate:

$$(k^2 + 1) \|\tilde{u}\|_0 + (k + 1) \|\nabla \times \tilde{u}\|_0 \leq C \|f\|_0.$$ 

(10)
Proof. By taking $\Gamma = \Gamma_s$ in Lemma 2.2, we have that $\|v\|_0 \leq C\|\nabla \times v\|_0$ for any $v \in X_{N,0}$. Therefore the bilinear form $a^+$ is continuous and coercive under the norm $((k^2 + 1)\|v\|_0^2 + \|\nabla \times v\|_0^2)^{\frac{1}{2}}$. By the Lax-Milgram lemma, (7) attains a unique solution $\tilde{u}$, and there holds

$$(k^2 + 1)\|	ilde{u}\|_0^2 + \|
abla \times \tilde{u}\|_0^2 \leq C\|f\|_0\|	ilde{u}\|_0 \leq \frac{1}{2}(k^2 + 1)\|	ilde{u}\|_0^2 + C(k^2 + 1)^{-1}\|f\|_0^2,$$

which implies (10).

\[\square\]

Lemma 2.4. There hold

\begin{enumerate}[(i)]
    \item For a given $w \in [L^2(\Omega)]^3$, the problem (8) has a unique solution $K_k w$ satisfying the following stability estimate:

    $$(k + 1)\|K_k w\|_0 + \|\nabla \times K_k w\|_0 \leq Ck\|w\|_0; \quad (11)$$

    \item $K_k$ is self-adjoint and compact operator on $[L^2(\Omega)]^3$;

    \item $X_{N,0}$ admits a countably infinite orthonormal basis $\{u_i\}$ of eigenvectors of $K_k$, with corresponding eigenvalues $\{\mu_i\} \subset \mathbb{R}$ satisfying $\mu_i \to 0$.
\end{enumerate}

Proof. It is clear that (i) is a consequence of Lemma 2.3 with $f = -2k^2 w$.

(ii) follows directly from the definition (8) of $K_k$ and the compact embedding of $X_{N,0}$ in $[L^2(\Omega)]^3$ (cf. [38, Page 87, Theorem 4.7]).

(iii) follows from (ii), the spectral theory of compact self-adjoint operator on Hilbert space (cf. [30, Page 60, Theorem 6.21]), and the fact that the orthogonal complement of the kernel of $K_k$ is $X_{N,0}$, which may be proved by the definition (8) of $K_k$. We omitted the details. This completes the proof of the lemma.

\[\square\]

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ be the nonzero eigenvalues of the Maxwell operator $\nabla \times \nabla \times$ on $H_0(\text{curl}; \Omega)$ and $u_i \in H_0(\text{curl}; \Omega)$ be the corresponding eigenfunctions:

$$\begin{align*}
    (\nabla \times u_i, \nabla \times v) = & \lambda_i(u_i, v) \quad \forall v \in H_0(\text{curl}; \Omega), \quad (12a) \\
    \|u_i\|_0 = & 1. \quad (12b)
\end{align*}$$

Lemma 2.5. The eigenvalues of $K_k$ consists of $\mu_i := \frac{-2k^2}{\lambda_i + k^2}$, $i = 1, 2, \cdots$, with corresponding eigenfunctions $u_i$. 

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Proof. First we note that $u_i \in X_{N,0}$ since $\lambda_i \neq 0$. It follows from the Helmholtz decomposition lemma 2.1 that (12) is equivalent to the following eigenvalue problem: $(\lambda, u_i) \in \mathbb{R} \times X_{N,0}$ such that
\begin{align}
(\nabla \times u_i, \nabla \times v) &= \lambda_i (u_i, v) \quad \forall v \in X_{N,0}, \quad (13a)
\|u_i\|_0 &= 1. \quad (13b)
\end{align}

Clearly, (13a) is equivalent to
\begin{align}
a^+(u_i, v) &= (\lambda_i + k^2)(u_i, v) \quad \forall v \in X_{N,0},
\end{align}
the proof of the lemma following by using the definition of $K_k$ and some simple calculations. \hfill \Box

The well-posedness of (5) is given in the next lemma.

**Lemma 2.6.** Suppose $k^2$ is not a Maxwell eigenvalue of (12), then problem (5) has a unique solution. Moreover, the inverse of $I + K_k$ exists, and
\begin{equation}
\|(I + K_k)^{-1}w\|_0 \leq \mathcal{M}_k \|w\|_0 \quad \forall w \in X_{N,0},
\end{equation}
where
\begin{equation}
\mathcal{M}_k := \sup_{i=1,2,\cdots} \left| \frac{\lambda_i + k^2}{\lambda_i - k^2} \right|.
\end{equation}

Proof. From Lemma 2.6, the eigenvalues is given by $\frac{\lambda_i - k^2}{\lambda_i + k^2}, i = 1, 2, \cdots$, which are all nonzero. Therefore, $I + K_k$ is invertible and (14) follows from Lemma 2.4 (iii) and the $L^2$-orthogonality of the basis $\{u_i, i = 1, 2, \cdots\}$. Then the well-posedness of (5) follows by using (9). \hfill \Box

**Remark 2.1.** Let us take a close look at the constant $\mathcal{M}_k$. First, it could be arbitrary large if $k^2$ approaches to any nonzero Maxwell eigenvalue. Next we illustrate the lower bound of $\mathcal{M}_k$ by considering the case when $\Omega$ is a convex polyhedron. Similar to [19, Theorem 4.1], the nonzero Maxwell eigenvalues are also eigenvalues of the Laplace operator with Neumann boundary condition, whose $n^\text{th}$ eigenvalue $\hat{\lambda}_n$ behaves asymptotically as $\hat{\lambda}_n \sim \hat{c} n^{\frac{2}{3}}$ where $\hat{c}$ is a constant depending only on the domain $\Omega$ (see e.g.[16, 49]). Therefore if the wave number $k$ is sufficient large and is located $(\hat{\lambda}_n, \hat{\lambda}_{n+1})$ for some large $n$,
\begin{equation}
\mathcal{M}_k = \sup_{i=n,n+1} \left| \frac{\lambda_i + k^2}{\lambda_i - k^2} \right| \geq \frac{n^{\frac{2}{3}}}{(n + 1)^{\frac{2}{3}} - n^{\frac{2}{3}}} \geq n \geq k^3.
\end{equation}
In the rest of section, we derive stability and regularity results for the indefinite time-harmonic Maxwell’s equations (1).

**Lemma 2.7.** (1) has a unique weak solution \((u, p)\), and the following stability estimate holds

\[
(k^2 + 1)\|u\|_0 + (k + 1)\|\nabla \times u\|_0 \leq CM_k\|f\|_0.
\]  

**Proof.** By combining (9), (10), (14), we get

\[
(k^2 + 1)\|u\|_0 = (k^2 + 1)\|(I + K_k)^{-1}u\|_0 \leq M_k(k^2 + 1)\|\tilde{u}\|_0 \leq CM_k\|f\|_0.
\]

It follows from (9), (10), (11) and (17) that

\[
(k^2 + 1)\|\nabla \times u\|_0 \leq (k^2 + 1)\|\nabla \times K_k u\|_0 + (k + 1)\|\nabla \times \tilde{u}\|_0 \\
\leq C(k^2 + 1)\|u\|_0 + C\|f\|_0 \\
\leq CM_k\|f\|_0,
\]

which together with (17) implies that (16) holds. \(\square\)

The following embedding theory is a useful tool in the analysis of Maxwell equations.

**Lemma 2.8** (cf. [2, Proposition 3.7]). If the domain \(\Omega\) is a Lipschitz polyhedron, then \(X_T(\Omega)\) and \(X_N(\Omega)\) are continuously embedded in \([H^s(\Omega)]^3\) for some real number \(s \in (1/2, 1]\).

Finally, we give the wavenumber explicit regularity result of (1).

**Theorem 2.1** (Regularity). Let \((u, p)\) be the solution of (1), then there exists a regularity index \(s \in (1/2, 1]\) dependent on \(\Omega\), such that \(u \in H^s(\text{curl}; \Omega)\) and

\[
(k + 1)\|u\|_s + \|\nabla \times u\|_s \leq CM_k\|f\|_0.
\]

If in addition \(f \in H(\text{div}; \Omega)\), it holds that \(p \in H^{s+1}(\Omega)\) and

\[
(k^2 + 1)\|p\|_{1+s} \leq C\|\nabla \cdot f\|_0.
\]

In particular, \(s = 1\) if \(\Omega\) is convex. If furthermore \(f \in H(\text{div}; \Omega)\) and \(\Omega\) is convex, there exists some regularity index \(s^* \in (1, 2]\) dependent on \(\Omega\), such that \(u \in H^{s^*}(\Omega)\)

\[
\|u\|_{s^*} \leq CM_k\|f\|_0.
\]
Proof. Let \( u \) be the solution of (1). Note that \( \mathbf{n} \cdot (\nabla \times \mathbf{u}) = 0 \) since \( \mathbf{n} \times \mathbf{u} = 0 \).

Hence by lemma 2.8, there exists a real number \( s_1 \in (1/2, 1] \) such that

\[
\| u \|_{s_1} \leq C (\| u \|_0 + \| \nabla \times \mathbf{u} \|_0), \\
\| \nabla \times \mathbf{u} \|_{s_1} \leq C (\| \nabla \times \mathbf{u} \|_0 + \| \nabla \times \nabla \times \mathbf{u} \|_0).
\]

We apply \( \nabla \cdot \) on (1a), and combine (1b) to get

\[
(k^2 + 1)\Delta p = \nabla \cdot f, \quad \text{in } \Omega, \quad (20a) \\
p = 0, \quad \text{on } \partial \Omega. \quad (20b)
\]

Since \( \Omega \) is a Lipschitz polyhedron, by the standard elliptic regularity results in [22], we obtain the regularity result for (20): there exists a real number \( s_2 \in (1/2, 1] \) such that

\[
\| (k^2 + 1)p \|_{1+s_2} \leq C\| \nabla \cdot f \|_0.
\]

Therefore the first two inequalities hold with \( s = \min(s_1, s_2) \). The last inequality may be derived by using the regularity result in [20, §4] and (16).

This completes the proof of the theorem. \( \square \)

Remark 2.2. In [45], it has been proved that

\[
\| u \|_s + \| \nabla \times \mathbf{u} \|_s \leq C_{\text{reg}}\| f \|_0,
\]

where \( C_{\text{reg}} \) dependents on \( k \). Here, we give explicitly the result that how \( C_{\text{reg}} \) dependent on \( k \).

3 An HDG method

By introducing \( \mathbf{r} = \nabla \times \mathbf{u} \), we can rewrite (1) as:

Find \(( \mathbf{r}, \mathbf{u}, p \) that satisfies

\[
\mathbf{r} - \nabla \times \mathbf{u} = 0 \quad \text{in } \Omega, \quad (21a) \\
\nabla \times \mathbf{r} - k^2 \mathbf{u} + (k^2 + 1)\nabla p = f \quad \text{in } \Omega, \quad (21b) \\
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (21c) \\
\mathbf{n} \times \mathbf{u} = 0 \quad \text{on } \Gamma, \quad (21d) \\
p = 0 \quad \text{on } \Gamma. \quad (21e)
\]
Let $\mathcal{T}_h = \bigcup \{T\}$ be a shape-regular partition of the domain $\Omega$ consisting of arbitrary polyhedra. For any $T \in \mathcal{T}_h$, let $h_T$ be the infimum of the diameters of spheres containing $T$ and denote the mesh size $h := \max_{T \in \mathcal{T}_h} h_T$. Let $\mathcal{F}_h = \bigcup \{F\}$ be the union of all faces of $T \in \mathcal{T}_h$; let $\mathcal{F}_h^I$ and $\mathcal{F}_h^B$ be the set of interior faces and boundary faces, respectively. We denote by $h_F$ the diameter of smallest circle containing face $F$. Moreover, we define the mesh-size function $h$ as

$$h(x) := \begin{cases} h_F, & \text{for } x \in F \in \mathcal{F}_h, \\ h_T, & \text{for } x \in \text{the interior of } T \in \mathcal{T}_h. \end{cases}$$  \hspace{1cm} (22)$$

For any $T \in \mathcal{T}_h$, we denote by $n_T$ the unit outward normal vector to $\partial T$. We extend the definition of $n$ to the boundary of elements by letting $n_{\partial T} = n_T$. Note that $n$ is double valued on interior faces with opposite directions. For any interior face $F = \partial T \cap \partial T' \in \mathcal{F}_h^I$ shared by element $T$ and element $T'$ and any piecewise function $\phi$, we define the jump of $\phi$ on $F$ as

$$[\phi]|_F := \phi|_T - \phi|_{T'}.$$  

On a boundary face $F = \partial K \cap \partial \Omega$, we set $[\phi]|_F := \phi$. For $u, v \in L^2(\partial \mathcal{T}_h)$, we define the following inner product and norm

$$\langle u, v \rangle_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \langle u, v \rangle_{\partial T}, \quad \|v\|^2_{0, \partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \|v\|^2_{0, \partial T}.$$  

Broken curl, divergent and gradient operators with respect to mesh partition $\mathcal{T}_h$ are donated by $\nabla_h \times$, $\nabla_h \cdot$ and $\nabla_h$, respectively.

For an integer $\ell \geq 0$, $\mathbb{P}_\ell(\Lambda)$ denotes the set of all polynomials defined on $\Lambda$ with degree no greater than $\ell$. For any integer $\ell \geq 1$ and $m \in \{\ell - 1, \ell\}$, we introduce the following finite dimensional spaces:

$$\mathbf{R}_h := \{s_h \in [L^2(\Omega)]^3 : s_h|_T \in [\mathbb{P}_m(T)]^3, \ \forall T \in \mathcal{T}_h\},$$

$$\mathbf{U}_h := \{v_h \in [L^2(\Omega)]^3 : v_h|_T \in [\mathbb{P}_\ell(T)]^3, \ \forall T \in \mathcal{T}_h\},$$

$$\hat{\mathbf{U}}_h := \{\hat{v}_h \in [L^2(\mathcal{F}_h)]^3 : \hat{v}_h|_F \in [\mathbb{P}_\ell(F)]^3, \ \forall F \in \mathcal{F}_h, \hat{v}_h \cdot n|_{\mathcal{F}_h} = 0, n \times \hat{v}_h|_{\Gamma} = 0\},$$

$$M_h := \{q_h \in L^2(\Omega) : q_h|_T \in \mathbb{P}_\ell(T), \ \forall T \in \mathcal{T}_h\},$$

$$\hat{M}_h := \{\hat{q}_h \in L^2(\mathcal{F}_h) : \hat{q}_h|_F \in \mathbb{P}_\ell(F), \ \forall F \in \mathcal{F}_h, \hat{q}_h|_{\Gamma} = 0\}.$$  

The HDG method for (1) reads as follows.
Find an approximation \((r_h, u_h, \tilde{u}_h, p_h, \tilde{p}_h) \in R_h \times U_h \times \hat{U}_h \times M_h \times \hat{M}_h\) such that

\[
\begin{align*}
(r_h, s_h) - (u_h, \nabla \times s_h) - (n \times \tilde{u}_h, s_h)|_{\partial \Omega} &= 0, \\
(r_h, \nabla \times v_h) + (n \times \tilde{r}_h, v_h)|_{\partial \Omega} - (k^2 + 1)(p_h, \nabla \cdot v_h) \\
&+ (k^2 + 1)(\tilde{p}_h, n \cdot v_h)|_{\partial \Omega} - k^2(u_h, v_h) = (f, v_h), \\
-(k^2 + 1)(u_h, \nabla q_h) + (k^2 + 1)(n \cdot \tilde{u}_h, q_h)|_{\partial \Omega} &= 0, \\
\langle n \times \tilde{r}_h, \tilde{v}_h \rangle_{\partial \Omega} &= 0, \\
\langle n \cdot \tilde{u}_h, q_h \rangle_{\partial \Omega} &= 0,
\end{align*}
\]  
(23a)  
(23b)  
(23c)  
(23d)  
(23e)

for any \((s_h, v_h, \tilde{v}_h, q_h, \tilde{q}_h) \in R_h \times U_h \times \hat{U}_h \times M_h \times \hat{M}_h\), where the numerical fluxes are defined as

\[
\begin{align*}
n \times \tilde{r}_h &= n \times r_h + h^{-1}n \times (u_h - \tilde{u}_h) \times n, \quad \text{on } \partial T, \forall T \in T_h; \\
n \cdot \tilde{u}_h &= n \cdot u_h + h^{-1}(p_h - \tilde{p}_h), \quad \text{on } \partial T, \forall T \in T_h.
\end{align*}
\]  
(24a)  
(24b)

**Remark 3.1.** The above HDG method is different from it in \([42]\) in the following two aspects: the stabilization parameters in \([42]\) are \(O(1)\) and the stabilization parameters here are \(O(h^{-1})\); the scheme in \([42]\) used \(\ell^th\) polynomials for all variables and we allow \((\ell - 1)^{th}\) polynomials for the approximation of \(r\).

By using (24a)–(24b) and (23d)–(23e) to eliminate \(n \times \tilde{r}_h\) and \(n \cdot \tilde{u}_h\) in (23a)–(23c) and using integration by parts, we get the following saddle point system:

Find \((r_h, u_h, \tilde{u}_h, p_h, \tilde{p}_h) \in R_h \times U_h \times \hat{U}_h \times M_h \times \hat{M}_h\) such that

\[
\begin{align*}
a_h(r_h, s_h) + b_h(u_h, \tilde{u}_h; s_h) &= 0, \\
b_h(v_h, \tilde{v}_h; r_h) + c_h(p_h, \tilde{p}_h; v_h) - s_h^n(u_h, \tilde{u}_h; v_h, \tilde{v}_h) + k^2(u_h, v_h) &= -(f, v_h), \\
c_h(q_h, \tilde{q}_h; u_h) + s_h^p(p_h, \tilde{p}_h; q_h, \tilde{q}_h) &= 0,
\end{align*}
\]  
(25a)  
(25b)  
(25c)

for all \((s_h, v_h, \tilde{v}_h, q_h, \tilde{q}_h) \in R_h \times U_h \times \hat{U}_h \times M_h \times \hat{M}_h\). Here the bilinear forms
$a_h, b_h, s_h^u$ and $s_h^p$ are defined by

$$
a_h(r,s) = (r,s),
b_h(u,\hat{u};s) = - (\nabla_h \times u, s) + \langle n \times (u - \hat{u}), s \rangle_{\partial T_h},
c_h(q,\hat{q};v) = -(k^2 + 1)(v, \nabla_h q) + (k^2 + 1)\langle n \cdot v, q - \hat{q} \rangle_{\partial T_h},
s_h^u(u,\hat{u};v,\hat{v}) = \langle h^{-1} n \times (u - \hat{u}), n \times (v - \hat{v}) \rangle_{\partial T_h},
s_h^p(p,\hat{p};q,\hat{q}) = (k^2 + 1)\langle h^{-1}(p - \hat{p}), q - \hat{q} \rangle_{\partial T_h}.
$$

To simplify the notation, we introduce the spaces

$$
\Sigma := \prod_{T \in T_h} H^s(T) \times \prod_{T \in T_h} H^s(T) \cap H(\text{curl}, T) \cap H(\text{div}, T) \tag{26}
$$

$$
\times \prod_{F \in F_h} L^2(F) \times \prod_{T \in T_h} H^1(T) \times \prod_{F \in F_h} L^2(F),
$$

$$
\Sigma_h := R_h \times U_h \times \hat{U}_h \times M_h \times \hat{M}_h. \tag{27}
$$

Clearly $\Sigma_h \subset \Sigma$. Introduce the following bilinear forms on $\Sigma \times \Sigma$. Given

$$
\sigma = (r, u, \hat{u}, p, \hat{p}), \quad \tau := (s, v, \hat{v}, q, \hat{q}) \in \Sigma,
$$

let

$$
B^h_k(\sigma, \tau) := a_h(r,s) + b_h(u,\hat{u};s)
+ b_h(v,\hat{v};r) + c_h(p,\hat{p};v) + c_h(q,\hat{q};u) + k^2(u,v)
- s_h^u(u,\hat{u};v,\hat{v}) + s_h^p(p,\hat{p};q,\hat{q})
= (r,s) - (\nabla_h \times u, s) + \langle n \times (u - \hat{u}), s \rangle_{\partial T_h}
- (\nabla_h \times v, r) + \langle n \times (v - \hat{v}), r \rangle_{\partial T_h}
- (k^2 + 1)(v, \nabla_h p) - \langle n \cdot v, p - \hat{p} \rangle_{\partial T_h}
- (k^2 + 1)(u, \nabla_h q) - \langle n \cdot u, q - \hat{q} \rangle_{\partial T_h} + k^2(u,v)
- \langle h^{-1} n \times (u - \hat{u}), n \times (v - \hat{v}) \rangle_{\partial T_h} + (k^2 + 1)\langle h^{-1}(p - \hat{p}), q - \hat{q} \rangle_{\partial T_h}. \tag{28}
$$

Denote by

$$
F_h(\tau) := - (f,v). \tag{29}
$$

The HDG method (25) can be rewritten in the following compact form.
Find $\mathbf{\sigma}_h = (\mathbf{r}_h, \mathbf{u}_h, \mathbf{v}_h, p_h, \phi_h) \in \Sigma_h$ such that
\begin{equation}
\mathcal{B}_h^-(\mathbf{\sigma}_h, \mathbf{\tau}_h) = \mathcal{F}_h(\mathbf{\tau}_h) \quad \text{for all } \mathbf{\tau}_h \in \Sigma_h.
\end{equation}

By doing integration by parts, it is easy to verify that the following orthogonality property holds for the HDG scheme (30).

**Lemma 3.1** (Orthogonality). Let $(\mathbf{r}, \mathbf{u}, p)$ and $\mathbf{\sigma}_h \in \Sigma_h$ be the solutions of (21) and (30), respectively. Then we have
\begin{equation}
\mathcal{B}_h^-(\mathbf{\sigma} - \mathbf{\sigma}_h, \mathbf{\tau}_h) = 0 \quad \text{for all } \mathbf{\tau}_h \in \Sigma_h.
\end{equation}
where $\mathbf{\sigma} = (\mathbf{r}, \mathbf{u}, \mathbf{u}|_{F_h}, p, p|_{F_h})$ and $|F_h$ denotes the restriction of a function to a union of faces in $F_h$.

We introduce the following mesh-dependent norm and seminorms.
\begin{align}
||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{\text{curl}}^2 &= ||| \nabla \times \mathbf{v} |||^2_{0} + ||| \mathbf{h}^{-1/2} \mathbf{n} \times (\mathbf{v} - \mathbf{\tilde{v}}) |||^2_{0, \partial T_h}, \\
||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{\text{div}}^2 &= ||| \mathbf{h} \nabla \cdot \mathbf{v} |||^2_{0} + ||| \mathbf{h}^{1/2} [\mathbf{n} \cdot \mathbf{v}] |||^2_{0, \partial F_h}, \\
||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{U}^2 &= ||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{\text{curl}}^2 + (k^2 + 1) ||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{\text{div}}^2, \\
|| (q, \mathbf{\tilde{q}}) |||_P^2 &= (k^2 + 1) ||| \nabla q |||^2_{0} + (k^2 + 1) ||| \mathbf{h}^{-1/2} (q - \mathbf{\tilde{q}}) |||^2_{0, \partial T_h}, \\
|| \mathbf{\tau} |||_{\Sigma_h}^2 &= ||| \mathbf{s} |||^2_{0} + ||| (\mathbf{v}, \mathbf{\tilde{v}}) |||_{U}^2 + ||| (q, \mathbf{\tilde{q}}) |||_P^2 + k^2 ||| \mathbf{v} |||^2_{0, \partial T_h} \\
&\quad + (k^2 + 1) \left( ||| \mathbf{h} \nabla \cdot \mathbf{v} |||^2_{0} + ||| \mathbf{h}^{1/2} [\mathbf{n} \cdot \mathbf{v}] |||^2_{0, \partial F_h} \\
&\quad + ||| \nabla \mathbf{h} q |||^2_{0} + ||| \mathbf{h}^{-1/2} (q - \mathbf{\tilde{q}}) |||^2_{0, \partial T_h} \right) + k^2 ||| \mathbf{v} |||^2_{0}.
\end{align}

where $\mathbf{\tau} = (\mathbf{s}, \mathbf{v}, \mathbf{\tilde{v}}, q, \mathbf{\tilde{q}})$.

## 4 Elliptic projection

In this section, we derive the error estimate of the following elliptic project based on the bilinear form $\mathcal{B}_h^+=\mathcal{B}_h$, which will be used to analyze the proposed HDG method: Given $\mathbf{\sigma} := (\mathbf{r}, \mathbf{u}, \mathbf{u}|_{F_h}, p, p|_{F_h})$, find $\mathbf{P}_h \mathbf{\sigma} \in \Sigma_h$ such that
\begin{equation}
\mathcal{B}_h^+(\mathbf{P}_h \mathbf{\sigma}, \mathbf{\tau}_h) = \mathcal{B}_h^+(\mathbf{\sigma}, \mathbf{\tau}_h) \quad \forall \mathbf{\tau}_h \in \Sigma_h.
\end{equation}
4.1 Approximation errors

In this subsection, we consider approximation properties of the discrete space $\Sigma_h$.

For any $T \in \mathcal{T}_h$, $F \in \mathcal{F}_h$ and any integer $j \geq 0$, let $\Pi^0_j : L^2(T) \to \mathbb{P}_j(T)$ and $\Pi^0_j : L^2(F) \to \mathbb{P}_j(F)$ be the usual $L^2$ projection operators. The following stability and error estimates are standard.

**Lemma 4.1.** For any $T \in \mathcal{T}_h$ and $F \in \mathcal{F}_h$ and $j \geq 0$, it holds

\[
\| v - \Pi^0_j v \|_{0,T} \leq C h^{s} | v |_{s,T}, \quad \forall v \in H^s(T),
\]

\[
\| v - \Pi^0_j v \|_{0,\partial T} \leq C h^{s-1/2} | v |_{s,T}, \quad \forall v \in H^s(T),
\]

\[
\| v - \Pi^0_j v \|_{0,\partial T} \leq C h^{s-1} | v |_{s,T}, \quad \forall v \in H^s(T),
\]

\[
\| \Pi^0_j v \|_{0,T} \leq \| v \|_{0,T}, \quad \forall v \in L^2(T),
\]

\[
\| \Pi^0_j v \|_{0,F} \leq \| v \|_{0,F}, \quad \forall v \in L^2(F),
\]

where $s \in (1/2, j + 1)$.

Next, we recall the error estimate results for the interpolation operator $\mathcal{P}_\ell^{\text{curl}} : H^s(\text{curl}; T) \to [\mathbb{P}_\ell(T)]^3$ for the Nédélec element of second type (see [40]).

**Lemma 4.2** (cf. [40, 1, 38]). There hold for $t \in (1/2, \ell]$ and $t^* \in (1, \ell + 1)$,

\[
\| v - \mathcal{P}_\ell^{\text{curl}} v \|_{0,T} \leq C h^{t^*} | v |_{t^*,T},
\]

(34a)

\[
\| v - \mathcal{P}_\ell^{\text{curl}} v \|_{0,T} \leq C h^t (| v |_{t,T} + h | \nabla \times v |_{t,T}),
\]

(34b)

\[
\| \nabla \times (v - \mathcal{P}_\ell^{\text{curl}} v) \|_{0,T} \leq C h^t | \nabla \times v |_{t,T}.
\]

(34c)

Next we recall two lemmas which present two interpolation operators of Osward type [43]. The first one says that every discontinuous piecewise polynomials in $M_h$ has a good $H^1$-conforming approximation (see, e.g., [43, 9, 12, 36, 52]).

**Lemma 4.3.** There exists an interpolation operator $\Pi^e_\ell : M_h \to M_h \cap H^1_0(\Omega)$ such that

\[
\| \nabla q_h - \nabla \Pi^e_\ell q_h \|_0 \leq C \| h^{-1/2} [q_h] \|_{\mathcal{F}_h} \quad \forall q_h \in M_h.
\]

Note that the superscript $^e$ stands for “conforming”. The second one says that every discontinuous piecewise polynomials in $[\mathcal{P}_\ell(\mathcal{T}_h)]^3$ has a good $H(\text{curl})$-conforming approximation.
Lemma 4.4 (cf. [34, Proposition 4.5]). There is an interpolation $\Pi_h^{\text{curl},c}$ from $[P_\ell(T_h)]^3$ to $[P_\ell(T_h)]^3 \cap H_0(\text{curl}; \Omega)$ such that for all $v_h \in [P_\ell(T_h)]^3$, we have the following approximation properties

$$\|\Pi_h^{\text{curl},c} v_h - v_h\|_0 \leq C\|h^{1/2} n \times [v_h]\|_{0,F_h},$$

$$\|\nabla \times \Pi_h^{\text{curl},c} v_h - \nabla \times v_h\|_0 \leq C\|h^{-1/2} n \times [v_h]\|_{0,F_h},$$

with a constant $C > 0$ independent of mesh size.

The following lemma says that every discrete function in $H_0(\text{curl}; \Omega) \cap U_h$ has a discrete Helmholtz decomposition and the discrete divergence free part in the decomposition has a good “continuous” approximation. (see, e.g., [31, Theorem 4.1 and Lemma 4.5], [38, §7.2.1])

Lemma 4.5. For any $w_h \in H_0(\text{curl}; \Omega) \cap U_h$, there exist $z_h \in H_0(\text{curl}; \Omega) \cap U_h$ and $q_h \in H_0^1(\Omega) \cap M_h$ such that

$$w_h = z_h + \nabla \xi_h, \quad (z_h, \nabla q_h) = 0 \quad \forall q_h \in H_0^1(\Omega) \cap M_h.$$  \hspace{1cm} (35)

Moreover there exist $\Theta \in H_0(\text{curl}; \Omega) \cap H(\text{div}; \Omega)$ and a constant $s \in (1/2, 1]$ determined by $\Omega$, such that

$$\nabla \times \Theta = \nabla \times w_h, \quad \|\Theta\|_s \leq C\|\nabla \times w_h\|_0, \quad \text{and} \quad \|z_h - \Theta\|_0 \lesssim h^s\|\nabla \times w_h\|_0.$$ \hspace{1cm} (36)

Next we consider the approximation properties of the discrete space $\Sigma_h$. Given $r, u, p$, let $\sigma := (r, u, u|_{T_h}, p, p|_{T_h})$ and define its approximation in $\Sigma_h$ by

$$I_h \sigma := (I_{\text{curl}}^\sigma r, P_\ell^{\text{curl}} u, n \times P_\ell^{\text{curl}} u \times n|_{T_h}, I_{\text{div}}^\sigma p, I_{\text{dev}}^\sigma (p|_{T_h})).$$

The following lemma gives the error estimate of $I_h$ in the norm $\|\cdot\|_{\Sigma_h}$.

Lemma 4.6. Assume that $(r, u, p) \in [H^t(\Omega)]^3 \times [H^t(\text{curl}; \Omega)]^3 \times H^{t+1}(\Omega)$ with $t \in (1/2, \ell)$. Then there holds

$$\|\sigma - I_h \sigma\|_{\Sigma_h} \leq C h^t \left( |r|_t + (k+1) |u|_t + (1 + kh) |\nabla \times u|_t + (k+1) |p|_{t+1} \right).$$

Proof. From the definition (32e) of $I_h$ we have

$$\|\sigma - I_h \sigma\|_{\Sigma_h}^2 = \left( \|r - I_{\text{curl}}^\sigma r\|_0^2 + \|\nabla \times (u - P_\ell^{\text{curl}} u)\|_0^2 + k^2 \|u - P_\ell^{\text{curl}} u\|_0^2 \right)$$

$$+ (k^2 + 1) \left( \|h \nabla h \cdot (u - P_\ell^{\text{curl}} u)\|_0^2 + \|h^{1/2} [n \cdot (u - P_\ell^{\text{curl}} u)]\|_{0,F_h}^2 \right)$$

$$+ (k^2 + 1) \left( \|\nabla h (p - I_{\text{div}}^\sigma p)\|_0^2 + \|h^{-1/2}(I_{\text{dev}}^\sigma (p - I_{\text{div}}^\sigma p))\|_{0,\partial T_h}^2 \right)$$

$$= I^2 + II^2 + III^2.$$ \hspace{1cm} (37)
Next we estimate the three terms \( I, II, III \). First, from Lemma 4.1 and Lemma 4.2, we have
\[
I \lesssim h^t |r|_t + (1 + kh)h^t |\nabla \times u|_t + kh^t |u|_t.
\] (38)
Secondly, from the trace inequality, the inverse inequality, Lemma 4.1, and Lemma 4.2, we conclude that
\[
II \lesssim (k + 1) \left( \| h \nabla_h \cdot (u - \Pi^0_t u) \|_0 + \| h^{1/2} |n \cdot (u - \Pi^0_t u)\|_{0, F^t} \right)
+ \| h \nabla_h \cdot (\Pi^0_t u - \mathcal{P}^\text{curl}_t u) \|_0 + \| h^{1/2} |n \cdot (\Pi^0_t u - \mathcal{P}^\text{curl}_t u)\|_{0, F^t} \right)
\lesssim (k + 1) (h^t |u|_t + \| \Pi^0_t u - \mathcal{P}^\text{curl}_t u \|_0)
\lesssim (k + 1) (h^t |u|_t + h^{t+1} |\nabla \times u|_t).
\] (39)
Thirdly, It follows from Lemma 4.1 that
\[
III \lesssim (k + 1) h^t |p|_{l+1}.
\] (40)
Then the proof of the theorem follows by plugging (38)–(40) into (37). \( \square \)

The following lemma show that \( \sigma - \mathcal{I}_h \sigma \) satisfies an approximate Galerkin orthogonality with respect to the bilinear form \( B^+_h \).

**Lemma 4.7.** Assume that \( (r, u, p) \in [H^t(\Omega)]^{3\times 3} \times [H^t(\text{curl}, \Omega)]^3 \times H^{t+1}(\Omega) \) with \( t \in (1/2, \ell] \). Let \( \sigma = (r, u, u|_{\partial F^t_h}, p, p|_{\partial F^t_h}) \). Then
\[
|B^+_h(\sigma - \mathcal{I}_h \sigma, \tau)| \leq C h^t (|r|_t + (k + 1)|u|_t + (1 + kh)|\nabla \times u|_t \)
+ (k + 1)|p|_{l+1}) \| \tau \|_{\Sigma_h}, \ \forall \tau \in \Sigma.
\] (41)

**Proof.** For any \( \tau = (s, v, \tilde{v}, q, \tilde{q}) \in \Sigma \), it follows from the definition of \( B^+_h \) in (28), integration by parts, and the identity \( \langle n \cdot v, p - \Pi^0_t p \rangle_{\partial F_h} = \langle \| n \cdot v \|, p - \Pi^0_t p \rangle_{\partial F^t_h} \) that
\[
B^+_h(\sigma - \mathcal{I}_h \sigma, \tau) \quad (r - \Pi^0_m r, s) - (\nabla_h \times (u - \mathcal{P}^\text{curl}_t u), s) - k^2 (u - \mathcal{P}^\text{curl}_t u, v)
- (\nabla_h \cdot v, r - \Pi^0_m r) + \langle n \times (v - \tilde{v}), r - \Pi^0_m r \rangle_{\partial F_h}
+ (k^2 + 1)(\| n \cdot v \|, p - \Pi^0_t p) - \langle \| n \cdot v \|, p - \Pi^0_t p \rangle_{\partial F^t_h}
- (k^2 + 1)(u - \mathcal{P}^\text{curl}_t u, \nabla_h q) - \langle n \cdot (u - \mathcal{P}^\text{curl}_t u), q - \tilde{q} \rangle_{\partial F_h}
+ (k^2 + 1)\langle h^{-1}(-\Pi^0_t p + \Pi^0_t p), q - \tilde{q} \rangle_{\partial F_h}.
\]
Therefore, from the Cauchy-Schwarz inequality, (37), and (32e), we conclude that
\[ |\mathcal{B}_h^+(\sigma - \mathcal{I}_h \sigma, \tau)| \lesssim (\|\sigma - \mathcal{I}_h \sigma\|_{\Sigma_h} + \|h^{1/2}(r - \Pi^0_m r)\|_{\partial T_h} + \|(u - \mathcal{P}_\ell^{\text{curl}} u)\|_0) \|\tau\|_{\Sigma_h} \]
which together with Lemmas 4.1, 4.2, and 4.6 completes the proof of the lemma.

4.2 Discrete inf-sup condition
In this subsection we show that \( \mathcal{B}_h^+ \) satisfies a discrete inf-sup condition.

The following theorem derive a discrete inf-sup condition for \( \mathcal{B}_h^+ \).

**Theorem 4.1** (Discrete inf-sup condition). For all \( \sigma_h \in \Sigma_h \), the bilinear form \( \mathcal{B}_h^+ (\cdot, \cdot) \) defined in (28) satisfies
\[
\sup_{0 \neq \tau_h \in \Sigma_h} \frac{\mathcal{B}_h^+(\sigma_h, \tau_h)}{\|\tau_h\|_{\Sigma_h}} = \sup_{0 \neq \tau_h \in \Sigma_h} \frac{\mathcal{B}_h^+(\tau_h, \sigma_h)}{\|\tau_h\|_{\Sigma_h}} \geq \beta \|\sigma_h\|_{\Sigma_h},
\]
where \( \beta \) is a constant independent of \( k \) and \( h \).

**Proof.** The proof is divided into five steps.

**Step one:**
Let \( \tau_h^1 = (r_h, -u_h, -\hat{u}_h, p_h, \hat{p}_h) \in \Sigma_h \). By (32e) and (28), we have
\[
\|\tau_h^1\|_{\Sigma_h} = \|\sigma_h\|_{\Sigma_h},
\]
and
\[
\mathcal{B}_h^+(\sigma_h, \tau_h^1) = \|r_h\|_0^2 + \|h^{-1/2} n \times (u_h - \hat{u}_h)\|_{0, \partial T_h}^2 + (k^2 + 1)\|h^{-1/2}(p_h - \hat{p}_h)\|_{0, \partial T_h}^2 + k^2\|u_h\|_0^2.
\]

**Step two:**
Let \( \tau_h^2 = (-\nabla_h \times u_h, 0, 0, 0, 0) \in \Sigma_h \). By (32e) and (28), we have
\[
\|\tau_h^2\|_{\Sigma_h} = \|\nabla_h \times u_h\|_0 \leq \|\sigma_h\|_{\Sigma_h},
\]
\[ B_h^+(\sigma_h, \tau_h^2) = -(r_h, \nabla_h \times u_h) + \|\nabla_h \times u_h\|^2_0 - \langle \nabla_h \times u_h, n \times (u_h - \widehat{u}_h) \rangle_{\partial \Omega_h} \]
\[
\geq \frac{1}{2}\|\nabla_h \times u_h\|^2_0 - C_1\|r_h\|^2_0 - C_2\|\mathbf{h}^{-1/2}n \times (u_h - \widehat{u}_h)\|^2_{0,\partial \Omega_h},
\]

(46)

where we have used the inverse trace inequality \(\|\mathbf{h}^{1/2} \nabla_h \times u_h\|_{0,\partial \Omega_h} \lesssim \|\nabla_h \times u_h\|_0\) and the Young’s inequality to derive the last inequality.

**Step three:**
Let \(w_h = \mathbf{h}^2 \nabla_h \cdot u_h\) on \(\Gamma_h\) and let \(\widehat{w}_h = -\mathbf{h}[n \cdot u_h]\) on \(\Gamma^I_h\) and \(\widehat{w}_h = 0\) on \(\Gamma\). Let \(\tau_h^3 = (0, 0, 0, w_h, \widehat{w}_h) \in \Sigma_h\). By (32e) and inverse inequality, we have

\[
\|\tau_h^3\|^2_{\Sigma_h} = (k^2 + 1)\left(\|\nabla_w w_h\|^2_0 + \|\mathbf{h}^{-1/2}(w_h - \widehat{w}_h)\|^2_{0,\partial \Omega_h}\right)
\leq C(k^2 + 1)\|\mathbf{h} \nabla_h \cdot u_h\|^2_0 + (k^2 + 1)\|\mathbf{h}^{1/2}[n \cdot u_h]\|^2_{0,\Gamma^I_h}
\leq C\|\sigma_h\|^2_{\Sigma_h}.
\]

(47)

By (28) and \(-(u_h, \nabla_h w_h) + \langle n \cdot u_h, w_h - \widehat{w}_h \rangle_{\partial \Omega_h} = (\nabla_h \cdot u_h, w_h) - \langle n \cdot u_h, \widehat{w}_h \rangle_{\partial \Omega_h}\), we have

\[
B_h^+(\sigma_h, \tau_h^3) = (k^2 + 1)\left(\|\mathbf{h} \nabla_h \cdot u_h\|^2_0 + \|\mathbf{h}^{1/2}[n \cdot u_h]\|^2_{0,\Gamma^I_h}\right)
+ (k^2 + 1)\|\mathbf{h}^{-1/2}(p_h - \widehat{p}_h), w_h - \widehat{w}_h\|_{\partial \Omega_h}
\geq \frac{k^2 + 1}{2}\left(\|\mathbf{h} \nabla_h \cdot u_h\|^2_0 + \|\mathbf{h}^{1/2}[n \cdot u_h]\|^2_{0,\Gamma^I_h}\right)
- C_3(k^2 + 1)\|\mathbf{h}^{-1/2}(p_h - \widehat{p}_h)\|^2_{0,\Gamma^I_h}.
\]

(48)

**Step four:**
Let \(\tau_h^4 = (0, -\nabla \Pi_h^c p_h, -n \times \nabla \Pi_h^c p_h \times n, 0, 0) \in \Sigma_h\). By (32e), inverse inequality, and Lemma 4.3, we have

\[
\|\tau_h^4\|^2_{\Sigma_h} = (k^2 + 1)\left(\|v_T \nabla \Pi_h^c p_h\|^2_0 + \|\mathbf{h}^{1/2}[\nabla \Pi_h^c p_h \cdot n]\|_0,\Gamma^I_h\right) + k^2\|\nabla \Pi_h^c p_h\|^2_0
\leq C(k^2 + 1)\|\nabla \Pi_h^c p_h\|^2_0 \leq C(k^2 + 1)\left(\|\nabla p_h\|^2_0 + \|\mathbf{h}^{-1/2}[p_h]\|_0,\Gamma_h\right)
= C(k^2 + 1)\left(\|\nabla p_h\|^2_0 + \|\mathbf{h}^{-1/2}[p_h - \widehat{p}_h]\|_0,\Gamma_h\right)
\leq C\|\sigma_h\|^2_{\Sigma_h}.
\]

(49)
It follows from (28), Lemma 4.3, Cauhy-Schwarz’s inequality, and Young’s inequality that
\[ B_h^+(\sigma_h, \tau_h^4) = (k^2 + 1)((\nabla \Pi_c^\ell p_h, \nabla h p_h) - \langle n \cdot \nabla \Pi_c^\ell p_h, p_h - \hat{p}_h \rangle_{\partial T_h}) + k^2(u_h, \nabla \Pi_c^\ell p_h) \]
\[ = (k^2 + 1)((\nabla h p_h, \nabla h p_h) + (\nabla \Pi_c^\ell p_h - \nabla h p_h, \nabla h p_h)) \]
\[ + (k^2 + 1)\langle n \cdot \nabla \Pi_c^\ell p_h, \hat{p}_h - p_h \rangle_{\partial T_h} + k^2(u_h, \nabla \Pi_c^\ell p_h) \]
\[ \geq \frac{k^2 + 1}{2} \|\nabla h p_h\|_0^2 - C_4(k^2 + 1)\|h^{-1/2}(p_h - \hat{p}_h)\|_{0,\partial T_h}^2 - C_5 k^2 \|u_h\|_0^2 \]  
(50)

Step five:
Let \( \tau_h = \alpha \tau_1 + \tau_2 + \tau_3 + \tau_4 \) with \( \alpha = \max(C_1, C_2, C_5, C_3 + C_4) + 1/2. \) By (43), (45), (47) and (49), we can get
\[ \|\tau_h\|_{\Sigma_h} \leq C\|\sigma_h\|_{\Sigma_h}. \]
Moreover, by combining (44), (46), (48) and (50), and (32e), we have
\[ B_h^+(\sigma_h, \tau_h) \geq \frac{1}{2}\|\sigma_h\|_{\Sigma_h}^2. \]
The last two inequalities together lead to
\[ B_h^+(\sigma_h, \tau_h) \geq C\|\sigma_h\|_{\Sigma_h}\|\tau_h\|_{\Sigma_h}, \]
which implies (42). This completes the proof of Theorem 4.1.

4.3 Error estimates of the elliptic projection

**Theorem 4.2.** Assume that \((r, u, p) \in [H^t(\Omega)]^{3 \times 3} \times [H^t(\text{curl}, \Omega)]^3 \times H^{t+1}(\Omega)\) with \(t \in (1/2, \ell]. \) Let \( \sigma := (r, u, u|_{\mathcal{F}_h}, p|_{\mathcal{F}_h}) \) and let \( \mathcal{P}_h \sigma = (r_h^p, u_h^p, \hat{u}_h^p, p_h^p, \hat{p}_h^p) \)
be it elliptic projection defined in (33). Then
\[ \|\sigma - \mathcal{P}_h \sigma\|_{\Sigma_h} \lesssim h^t(|r|_t + (k + 1)|u|_t + (1 + kh)|\nabla \times u|_t + (k + 1)|p|_{t+1}), \]  
(51a)
\[ \|u - u_h^p\|_0 \lesssim \|u - \mathcal{P}_c^\text{curl} u\|_0 + (k + 1 + kh) \times \]
\[ h^{t+s}(|r|_t + (k + 1)|u|_t + (1 + kh)|\nabla \times u|_t + (k + 1)|p|_{t+1}). \]  
(51b)
Proof. From the inf-sup condition of Theorem 4.1 and the definition of the elliptic projection (33), we have

\[ \left\| \mathcal{I}_h \sigma - P_h \sigma \right\|_{\Sigma_h} \lesssim \sup_{0 \neq \tau_h \in \Sigma_h} \frac{B_h^+ (\mathcal{I}_h \sigma - P_h \sigma, \tau_h)}{\| \tau_h \|_{\Sigma_h}} \]

which implies (51a) by using Lemmas 4.7 and 4.6, and the triangle inequality.

It remains to prove (51b). Denote by \( \eta := u - u_h^P \). We have the following decomposition:

\[ \| \eta \|_0^2 = (u - P^\text{curl}_\ell u, \eta) + (\Pi^\text{curl}_{\ell, c} u_h^P - u_h^P, \eta) + (w_h - \Theta, \eta) + (\Theta, \eta) \quad (52) \]
\[ w_h := P^\text{curl}_\ell u - \Pi^\text{curl}_{\ell, c} u_h^P, \]

where \( \Theta \) is defined in Lemma 4.5. Let \( w_h \) be decomposed as (35) in Lemma 4.5. For any \( q_h \in H^1_0(\Omega) \cap M_h \), by taking \( \tau_h = (0, 0, 0, \nabla q_h, 0) \) in (33) and using the definition (28) of \( B_h^+ \), we conclude that \( (u_h^P, \nabla q_h) = 0 \), that is, \( u_h^P \) is discrete divergence free. Noting that \( \text{div}\ u = 0 \), we have

\[ (w_h - \Theta, \eta) = (z_h + \nabla \xi_h - \Theta, \eta) = (z_h - \Theta, \eta). \]

It follows from Lemmas 4.4 and 4.5, the triangle Inequality, and (32c) that

\[ (\Pi^\text{curl}_{\ell, c} u_h^P - u_h^P, \eta) + (w_h - \Theta, \eta) \]
\[ \lesssim (\| h^{1/2} n \times [u_h^P] \|_{0, \mathcal{F}_h} + h^s \| \nabla \times w_h \|_0) \| \eta \|_0 \]
\[ \lesssim (\| h^{1/2} n \times [u_h^P] \|_{0, \mathcal{F}_h} + h^s \| \nabla_h \times (P^\text{curl}_{\ell} u - u_h^P) \|_0) \| \eta \|_0 \]
\[ \lesssim h^s (\| \sigma - P_h \sigma \|_{\mathcal{S}_h} + \| \nabla \times (P^\text{curl}_{\ell} u - u) \|_0) \| \eta \|_0. \quad (53) \]

Introduce the following dual problem: Find \((r^d, u^d, p^d)\) such that

\[ r^d - \nabla \times u^d = 0 \quad \text{in } \Omega, \quad (54a) \]
\[ \nabla \times r^d + k^2 u^d + (k^2 + 1) \nabla p^d = \Theta \quad \text{in } \Omega, \quad (54b) \]
\[ \nabla \cdot u^d = 0 \quad \text{in } \Omega, \quad (54c) \]
\[ n \times u^d = 0 \quad \text{on } \Gamma, \quad (54d) \]
\[ p^d = 0 \quad \text{on } \Gamma. \quad (54e) \]
Note that the above dual problem is positive definite since the sign before $k^2$ in the second equation is positive (cf. (21b)). Similar to Theorem 2.1, we have the following regularity estimate of problem (54):

$$\|r^d\|_s + \|u^d\|_s \lesssim \|\Theta\|_0, \quad p^d = 0,$$

where the regularity index $s \in (1/2, 1]$ depends on $\Omega$. Denote by $\sigma^d = (r^d, u^d, u^d|_{F_h}, p^d, p^d|_{F_h})$. By a parallel derivation as in §3, we conclude that $\sigma^d$ satisfies the following variational formulation

$$B_h^+(\sigma^d, \tau) = -(\Theta, v), \quad \forall \tau = (s, v, \hat{v}, q, \hat{q}) \in \Sigma.$$

By taking $\tau = \sigma - P_h \sigma$ and using (33) and Lemma 4.7, we have

$$\begin{align*}
(\Theta, \eta) &= B_h^+(\sigma^d, P_h \sigma - \sigma) = B_h^+(\sigma^d - I_h \sigma^d, P_h \sigma - \sigma) \\
&\lesssim h^s((1 + kh)\|r\|_s + (k + 1)\|u\|_s)\|\sigma - P_h \sigma\|_{\Sigma_h} \\
&\lesssim (k + 1 + kh)h^s\|\Theta\|_0\|\sigma - P_h \sigma\|_{\Sigma_h}. \quad (55)
\end{align*}$$

On the other hand, from Lemmas 4.4 and 4.5 and $w_h = P_{\ell} \text{curl} u - u + \eta + u_P^{\ell} - \Pi_{\ell} \text{curl} c u_P^{\ell}$ we have

$$\begin{align*}
\|\Theta\|_0 &\leq \|\Theta - z_h\|_0 + \|z_h\|_0 \lesssim h^s\|\nabla \times w_h\|_0 + \|w_h\|_0 \\
&\lesssim h^s(\|\nabla \times (u - P_{\ell} \text{curl} u)\|_0 + \|\sigma - P_h \sigma\|_{\Sigma_h}) + \|u - P_{\ell} \text{curl} u\|_0 + \|\eta\|_0 \quad (56)
\end{align*}$$

By combining (52), (53), (55), and (56) we obtain

$$\|\eta\|_0 \lesssim \|u - P_{\ell} \text{curl} u\|_0 + (k + 1 + kh)h^s(\|\sigma - P_h \sigma\|_{\Sigma_h} + \|\nabla \times (u - P_{\ell} \text{curl} u)\|_0)$$

which together with (51a) and (34c) implies (51b). This completes the proof of the theorem.

5 Error estimates of the HDG methods

In this section, we derive error estimates for the HDG method (23) (or (30)) by using a modified duality argument.

We first show that the error of the HDG solution $\sigma_h$ in the norm $\|\cdot\|_{\Sigma_h}$ can be bounded by the interpolation error and the $L^2$ error $\|u_h - u\|_0$. 

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Lemma 5.1. Let \((r, u, p)\) and \(\sigma_h \in \Sigma_h\) be the solutions to (21) and (30), respectively. Then we have the following estimate

\[
\|\sigma_h - \sigma\|_{\Sigma_h} \lesssim h^t ((k + 1)|u|_t + (1 + kh)|\nabla \times u|_t + (k + 1)|p|_{t+1})
\]

\[
+ k\|u_h - u\|_0,
\]

where \(\sigma = (r, u, u|_{\mathcal{F}_h}, p, p|_{\mathcal{F}_h})\).

Proof. It follows from the discrete inf-sup condition in Theorem 4.1, (28), and the orthogonality in Lemma 5.1.

Finally, we give the error estimates of the proposed HDG method in the following theorem.

Theorem 5.1. Let \((r, u, p)\) and \(\sigma_h = (r_h, u_h, \hat{u}_h, p_h, \hat{p}_h) \in \Sigma_h\) be the solutions of (1) and (30), respectively. Suppose \((r, u, p) \in [H^t(\Omega)]^3 \times [H^t(\Omega)]^3 \times H^{t+1}(\Omega)\) with \(t \in (1/2, l]\).

(i) There exists a constant \(C_0 > 0\) independent of \(k\) and \(h\) such that if \(M_k h^s \leq C_0\), we have

\[
\|r_h - r\|_0 \leq Ch^t ((k + 1)|u|_t + |r|_t + (k + 1)|p|_{t+1}),
\]

\[
\|u_h - u\|_0 \leq Ch^t |u|_t + CM_k h^{t+s}(|u|_t + |r|_t + (k + 1)|p|_{t+1}),
\]

\[
\|\nabla_h p_h - \nabla p\|_0 \leq Ch^t (|u|_t + (k + 1)^{-1}|r|_t + |p|_{t+1}).
\]
(ii) If in addition \( u \in [H^{s+}(\Omega)]^3 \), it holds
\[
\|u_h - u\|_0 \leq Ch^{s+}\|u\|_{t,s} + C\mathcal{M}_kh^{t+s}((k+1)\|u\|_t + |r|_t + (k+1)|p|_{t+1}).
\] (59)

(iii) If \( \Omega \) is convex, there exists a constant \( C_0 > 0 \) independent of \( k \) and \( h \) such that if \( \mathcal{M}_kh^2s^* \leq C_0 \), we have
\[
\|r_h - r\|_0 \leq C(h^t + \mathcal{M}_kh^{t+1})((k+1)|u|_t + |r|_t + (k+1)|p|_{t+1}),
\] (60a)
\[
\|u_h - u\|_0 \leq C h^t|u|_t + C\mathcal{M}_kh^{t+1}((k+1)|u|_t + |r|_t + (k+1)|p|_{t+1}),
\] (60b)
\[
\|\nabla_h p_h - \nabla p\|_0 \leq C(h^t + \mathcal{M}_kh^{t+1})(|u|_t + (k+1)^{-1}|r|_t + |p|_{t+1}).
\] (60c)

Proof. It suffices to prove the \( L^2 \) error estimates since the estimates (58a) and (58c) are direct consequences of (32d), (32e), and (58b). For simplicity, denote by \( \xi := u_h - u \) and \( C_{u,p} := (k+1)|\nabla u|_t + |\nabla \times u|_t + (k+1)|p|_{t+1} \). (??) implies that the following estimate holds if \( kh \lesssim 1 \).
\[
\|\sigma_h - \sigma\|_{\Sigma_h} \lesssim h^tC_{u,p} + k\|\xi\|_0.
\] (61)

Similar to the proof of (51b), we decompose the square of the \( L^2 \) error as
\[
\|\xi\|_0^2 = (\mathcal{P}_c^\ell u - u, \xi) + (u_h - \Pi_c^\ell u_h, \xi) + (w_h - \Theta, \xi) + (\Theta, \xi)
\] (62)
where the \( \Theta \) is defined as in Lemma 4.5. Let \( w_h \) be decomposed as (35) in Lemma 4.5. For any \( q_h \in H_0^1(\Omega) \cap M_h \), from (23c), we conclude that \((u_h, \nabla q_h) = 0\). Noting that \( \text{div} \, u = 0 \), we have
\[
(w_h - \Theta, \xi) = (z_h + \nabla \xi_h - \Theta, \xi) = (z_h - \Theta, \xi).
\]

Similar to (53), we have
\[
(u_h - \Pi_c^\ell u_h, \xi) + (w_h - \Theta, \xi) \\
\lesssim h^s(\|\sigma - \sigma_h\|_{\Sigma_h} + \|\nabla \times (\mathcal{P}_c^\ell u - u)\|_0)\|\xi\|_0 \\
\lesssim h^s(h^tC_{u,p} + k\|\xi\|_0)\|\xi\|_0,
\] (63)
where we have used (61) and Lemma 4.2 to derive the last Inequality.
Introduce the following duality problem: Find \((r^d, u^d, p^d)\) such that
\[
\begin{align*}
\nabla \times r^d &= 0 \quad \text{in } \Omega, \\
\nabla \times r^d - k^2 u^d + (k^2 + 1) \nabla p^d &= 0 \quad \text{in } \Omega, \\
\nabla \cdot u^d &= 0 \quad \text{in } \Omega, \\
n \times u^d &= 0 \quad \text{on } \Gamma, \\
p^d &= 0 \quad \text{on } \Gamma.
\end{align*}
\]

By Theorem 2.1, we have the following regularity estimate of problem (64):
\[
\|r^d\|^s + (k+1) \|u^d\|^s \lesssim \mathcal{M}_k \|\Theta\|_0, \quad p^d = 0,
\]
and
\[
\|u^d\|^s^\ast \lesssim \mathcal{M}_k \|\Theta\|_0 \quad \text{if } \Omega \text{ is convex.}
\]
where the regularity indices \(s \in (1/2, \sigma]\) and \(s^\ast > 1\) depend on \(\Omega\). By combining Lemma 4.2 and (65)–(66) we have
\[
\|u^d - \mathcal{P}_\ell \text{curl } \u^d\|_0 \lesssim \mathcal{M}_k D_{k,h} \|\Theta\|_0
\]
where
\[
D_{k,h} := \begin{cases} h^s & \text{if } \Omega \text{ is convex,} \\ (k+1)^{-1} h^s & \text{otherwise.} \end{cases}
\]
It is easy to check that \(\sigma^d : (r^d, u^d, u^d|_\Sigma, p^d, p^d|_\Sigma)\) satisfies the following variational formulation.
\[
\mathcal{B}_h^- (\sigma^d, \tau) = - (\Theta, \upsilon) \quad \forall \tau \in \Sigma.
\]
Let \(u^d_{hp}\) be the second component of \(\mathcal{P}_h \sigma^d\). Then by Lemmas 4.6 and 4.7, the regularity estimate (65), the orthogonality in Lemma 3.1, (67), and the fact that \(p^d = 0\), we have
\[
(\Theta, \xi) = \mathcal{B}_h^- (\sigma^d, \sigma - \sigma_h) = \mathcal{B}_h^- (\sigma^d - \mathcal{P}_h \sigma^d, \sigma - \sigma_h) \\
= \mathcal{B}_h^+ (\sigma^d - \mathcal{P}_h \sigma^d, \sigma - \sigma_h) + 2k^2 (\|u^d - u^d_{hp}\|_0)^2 \|\xi\|_0 \\
\lesssim \mathcal{M}_k (h^s (\|r^d\|^s + (k+1) \|u^d\|^s) \|\sigma - \mathcal{I}_h \sigma\|_{\Sigma_h} \\
+ k^2 (\|u^d - \mathcal{P}_\ell \text{curl } u^d\|_0 + h^{2s} (\|r^d\|^s + (k+1) \|u^d\|^s)) \|\xi\|_0 \\
\lesssim \mathcal{M}_k (h^s \|\sigma - \mathcal{I}_h \sigma\|_{\Sigma_h} + k^2 (D_{k,h} + h^{2s}) \|\xi\|_0) \\
\lesssim \mathcal{M}_k (h^{s+t} C_{u,p} + k^2 (D_{k,h} + h^{2s}) \|\xi\|_0),
\]
(70)
On the other hand, from Lemmas 4.2, 4.4, and 4.5, \( w_h = \Pi_{\ell}^{\text{curl},c} u_h - u_h + \xi + u - \mathcal{P}_{\ell}^{\text{curl}} u \), (61), and the triangle inequality, we have

\[
\|\Theta\|_0 \leq \|\Theta - z_h\|_0 + \|z_h\|_0 \lesssim h^s \|\nabla \times w_h\|_0 + \|w_h\|_0
\]

\[
\lesssim h^s \left( \|\nabla \times (\Pi_{\ell}^{\text{curl},c} u_h - u_h)\|_0 + \|\nabla \times \xi\|_0 + \|\nabla \times (u - \mathcal{P}_{\ell}^{\text{curl}} u)\|_0 \right)
\]

\[
+ \|\Pi_{\ell}^{\text{curl},c} u_h - u_h\|_0 + \|\xi\|_0 + \|\mathcal{P}_{\ell}^{\text{curl}} u - u\|_0
\]

\[
\lesssim h^{s+t} C_{u,p} + \|\mathcal{P}_{\ell}^{\text{curl}} u - u\|_0 + (1 + kh^s) \|\xi\|_0.
\]

Combining (62), (63), (70), (71), and the Young’s inequality gives

\[
\|\xi\|_0 \lesssim (1 + M_k (k^2 D_{k,h} + k^2 h^{2s})) \|\mathcal{P}_{\ell}^{\text{curl}} u - u\|_0
\]

\[
+ M_k h^{s+t} (1 + kh^s + k^2 D_{k,h} + k^2 h^{2s}) C_{u,p}
\]

\[
+ (kh^s + M_k (1 + kh^s) (k^2 D_{k,h} + k^2 h^{2s})) \|\xi\|_0.
\]

Therefore, if \( M_k k^2 D_{k,h} \) is sufficiently small,

\[
\|u_h - u\|_0 \lesssim \|\mathcal{P}_{\ell}^{\text{curl}} u - u\|_0 + M_k h^{s+t} C_{u,p},
\]

which implies (58b) and (59). This completes the proof of the theorem. \( \square \)

As a consequence of the above theorem, we have the following well-posedness of the proposed HDG method.

**Corollary 5.1.** Under the conditions of Theorem 5.1, the HDG scheme (30) has a unique solution \( \sigma_h \in \Sigma_h \).

In view of Theorem 2.1 and Theorem 5.1, we can obtain the following error estimates for the linear HDG method on convex domain.

**Corollary 5.2.** Suppose \( \Omega \) is convex, \( \ell = 1 \), and \( \nabla \cdot f = 0 \). Then there exists a constant \( C_0 > 0 \) independent of \( k \) and \( h \) such that if \( M_k k^2 h^{2s} \leq C_0 \), the following error estimates hold.

\[
\|r_h - r\|_0 \leq C(M_k h + M_k k^2 h^{2s} + M_k^2 k h^2) \|f\|_0,
\]

\[
\|u_h - u\|_0 \leq C(M_k k^2 h^s + M_k^2 h^2) \|f\|_0.
\]
6 Numerical experiments

The numerical tests are programmed in C++. When implementing of the HDG scheme, all interior unknowns \( r_h, u_h \) and \( p_h \) are eliminated, and the only global unknowns in the resulting system are \( \hat{u}_h \) and \( \hat{p}_h \). After solving the global system, the \( r_h, u_h \) and \( p_h \) are recovered locally inside each element. The solvers for the linear systems are chosen as GMRES and SparseLU. Let \( T_h \) be a uniform simplex decomposition of \( \Omega \), we denote by \( h \) the smallest length of the edge in decomposition \( T_h \).

Let \( \Omega = [0, 1]^3 \). We take to following exact solution \( u \) and \( p \), and compute the functions \( r \) and \( f \) accordingly.

\[
\begin{align*}
    u_1 &= 200(x - x^2)^2(2y^3 - 3y^2 + y)(2z^3 - 3z^2 + z), \\
    u_2 &= -100(y - y^2)^2(2z^3 - 3z^2 + z)(2x^3 - 3x^2 + x), \\
    u_2 &= -100(z - z^2)^2(2x^3 - 3x^2 + x)(2y^3 - 3y^2 + y), \\
    p &= (x^2 - x)(y^2 - y)(z^2 - y)(k^2 + 1)^{-1}.
\end{align*}
\]

Though we require \( \ell \geq 1 \) in the error analysis, the numerical results for \( \ell = 0, 1, 2 \) are all presented to illustrate the performance of the proposed HDG method. We take \( k = 1 \) and \( k = 8 \) and report the errors in Table 6 and 6, respectively. It can be observed that: when \( \ell = m = 0 \), the convergence rates are nearly zero for all variables; when \( \ell \geq 1 \), the convergence orders are as predicted by Corollary 5.2, provided that \( h \) is small enough; in particular, when \( \ell = m = 2 \), the convergence results are better than the previous prediction for variables \( r \) and \( p \).

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| $\ell$ and $m$ | $h^{-1}$ | $\|r_h - r\|_0$ | Error | Rate | Error | Rate | $\|u_h - u\|_0$ | $\|\nabla_h p_h - \nabla p\|_0$ | Error | Rate | Error | Rate | DOF |
|---------------|---------|----------------|------|------|------|------|----------------|----------------|------|------|------|------|-----|
| $\ell = 0$    | 2       | 3.7906E-01     | 9.5130E-02 | 1.6667E-02 | 1.6667E-02 | 0.00 | 152064 |
| $m = 0$       | 4       | 2.5031E-01     | 4.1590E-02 | 1.19 | 1.6667E-02 | 0.00 | 2592 |
|               | 8       | 1.4666E-01     | 2.1067E-02 | 0.98 | 1.6667E-02 | 0.00 | 19584 |
|               | 16      | 9.1599E-02     | 1.6154E-02 | 0.38 | 1.6667E-02 | 0.00 | 152064 |
|               | 32      | 6.9852E-02     | 1.5301E-02 | 0.08 | 1.6667E-02 | 0.00 | 1198080 |

| $\ell = 1$    | 2       | 3.6969E-01     | 2.1000E-01 | 1.1601E-01 | 1080 |
| $m = 0$       | 4       | 2.7883E-01     | 8.4783E-02 | 1.31 | 1.1252E-01 | 0.04 | 7776 |
|               | 8       | 1.5310E-01     | 5.7498E-02 | 0.97 | 5.7498E-02 | 0.97 | 58752 |
|               | 16      | 7.7765E-02     | 3.0455E-03 | 3.00 | 3.0455E-03 | 3.00 | 456192 |
|               | 32      | 6.9852E-02     | 1.5301E-02 | 0.08 | 1.5301E-02 | 0.08 | 456192 |

| $\ell = 2$    | 2       | 1.1268E-01     | 1.7747E-02 | 4.6996E-02 | 2160 |
| $m = 2$       | 4       | 2.0568E-02     | 2.5278E-03 | 2.81 | 9.8661E-03 | 2.25 | 15552 |
|               | 8       | 3.0455E-03     | 3.0921E-04 | 3.03 | 3.0921E-04 | 3.03 | 117504 |
|               | 16      | 1.5005E+02     | 6.4286E+01 | 4.2702E+00 | 1198080 |
|               | 32      | 2.1350E+00     | 6.2434E+01 | 6.0256E+02 | 2160 |

| $\ell = 1$    | 2       | 1.5005E+02     | 6.4286E+01 | 4.2702E+00 | 1198080 |
| $m = 2$       | 4       | 2.1350E+00     | 6.2434E+01 | 6.0256E+02 | 2160 |
|               | 8       | 3.0455E-03     | 3.0921E-04 | 3.03 | 3.0921E-04 | 3.03 | 117504 |
|               | 16      | 1.5005E+02     | 6.4286E+01 | 4.2702E+00 | 1198080 |

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