Optimal reinsurance contracts under the reinsurer’s risk constraint with VaR risk measures

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Abstract. Most researchers in insurance field have not been considering the reinsurer’s risk. Theoretically, the quantity of reinsurer’s risk is not limited. However, in the real-world insurance industry, reinsurer always put a limit on coverage, otherwise the reinsurer will experience a heavy financial burden when the insurer bears an unexpected large loss. In this paper, we review the optimal reinsurance problem under VaR risk measures framework when the limitation for reinsurer’s risk exposure are limited by a constant. By analyzing the VaR of the insurer’s total risk exposure, optimality can be fulfilled and end up in obtaining the expected reinsurance policy. Through the study, it is reveal that the stop-loss reinsurance with a policy limit is constantly the optimal reinsurance policy under VaR risk measures. Determination of optimal quantity of ceded risk depends on the confidence level, the safety loading and the limitation of the constant. A numerical analysis on company claims data is carried out in order to find the optimal ceded loss function.

1. Introduction

Theoretically, the risk borne by an insurance company as insurer can be very large, which is adjusted to the amount of risk transferred from the insured. But in reality, insurers have limitations in bearing these risks. Therefore, if there is an unexpectedly large claim, it will be a financial burden for the insurer. The limitations of the insurer have led to the presence of a reinsurance that aims to help insurer share the huge risks.

Reinsurance can help insurer meet their solvency requirements, by providing additional assurance capacity needed to accept individual risks and types of business that should not be intolerable. Just as an insurance company charges a premium to policyholders, the reinsurance company also charges the premium to the insurer for the transfer of risk carried out, referred to as the reinsurance premium.

An insurer should overcome trade-off between the risk retained and the premium paid to the reinsurer [1]. In the study of optimal reinsurance problems, reinsurance contract design is a very important issue among insurance industry competitive player in the society.

The optimality of a reinsurance contract based on both the chosen optimization criterion and the premium principle [1]. Many criteria have been proposed for the optimality of reinsurance contract such as by Lu et.al [1], Bernard and Tian [2], Cai and Tan [3], Chi and Tan [4].

2. Problem Formulation

In this paper, the loss transferred by the insurance company to reinsurance is referred to as ceded loss function, denoted by \( I(X) \). \( I(X) \) is a loss for a reinsurance company so that the retention of the insurance company (denoted by \( R_r(X) \)) is the difference from the loss that occurs and the loss transferred to the reinsurance. However, the reinsurance company also charges the reinsurance
premium to the insurance company denoted by $\pi_i(X)$, so that the total risk that must be borne by the insurance company is the sum of reinsurance premiums and retention from insurance companies,

$$T_i(X) = R_i(X) + \pi_i(X)$$

(1)

As noted in Cheung et al [5], any possible reinsurance policy should fulfill the following characteristics: (1) the greater the loss that will be borne by insurance company, the greater the loss that will be transferred to the reinsurance company, and (2) the rate of loss growth that will be received by a reinsurance company must not be greater than the rate of loss growth that occurs. Mathematically, the ceded loss function for certain value of $X = x$, $I(x)$ is assumed to satisfy the following:

(a) $I(0) = 0$ and $I$ is nondecreasing,
(b) $I(x_2) - I(x_1) \leq x_2 - x_1$ for any $0 \leq x_1 \leq x_2$

which means that $I(x)$ is Lipschitz continuous function. These conditions imply that $0 \leq I(x) \leq x$ for any $x \geq 0$, which is a basic assumption in the study of optimal reinsurance. As noted in Lu et. al [1], the conditions above are called incentive compatible constraints. The set of all functions $I(x)$ that fulfill above properties is expressed as $\mathcal{K}$, where $\mathcal{K} = \{I(x), x \geq 0, x \in \mathbb{R}\}$. In this paper, the risk measurement of total risk using Value-at-Risk.

**Definition 1.** Let $X$ denote a loss random variable. The Value-at-Risk of $X$ at the 100p% level, denoted $VaR_p(X)$ or $\pi_p$, is the 100p percentile (or quantile) of the distribution of $X$. [6]

For continuous distributions, we can simply write $VaR_p(X)$ for random variable $X$ as the value of $\pi_p$ satisfying

$$\Pr(X > \pi_p) = 1 - p$$

There are three properties that applicable for Value-at-Risk. Let $Y$ denote a loss random variable and $c$ is any scalar , then

**Proposition 1.** $VaR$ is translation invariance $VaR_{a}(Y + c) = VaR_{a}(Y) + c$ [7]

**Proposition 2.** $VaR$ is positively homogeneity $VaR_{a}(cY) = c.VaR_{a}(Y)$ [7]

**Proposition 3.** $VaR$ is transformed random variable $VaR_{a}(g(Y)) = g(VaR_{a}(Y))$, where $g$ is a continuous and nondecreasing function. [7]

In this paper, we review the optimal reinsurance problem under VaR risk measures when reinsurer’s risk exposure are limited by a constant. We will examine that the ceded loss function is constrained a constant $L > 0$, i.e., $I(x) \leq L$. The constant $L$ reflect the reinsurers risk tolerance. In particular, we attempt to look for an optimal reinsurance policy from the following type of feasible sets of ceded loss functions:

$$\mathcal{G} = \{I(x) \in \mathcal{K} | I(x) \leq L\}$$

In this paper, the reinsurance premium is determined by the common expectation premium principle: $\pi_i(X) = (1 + \theta)E[I(X)]$, where $\theta > 0$ is the safety loading factor as stated by Lu et. al [1]. Now our optimal reinsurance models can be formulated as:

$$\min_{I \in \mathcal{G}}\{VaR_p(T_i(X))\} = VaR_p(T_{\pi^*}(X))$$

Where $I^*$ is used to denote the optimal ceded loss function. To facilitate the discussion below, we introduce the notation $\theta^* = \frac{\theta}{1+\theta}$. 


3. Optimal Reinsurance Under VaR

In this section, we will derive the optimal solution for the optimization problem above. By applying the property of translation invariance (proposition 1), positively homogeneity (proposition 2) and transformed random variable of VaR (proposition 3), the objective $VaR_p(T_1(X))$ can be represented as

$$VaR_p(X) - I\left(VaR_p(X)\right) + (1 + \theta)E[I(X)]$$

Ceded loss function that we can use and being as a familiar form is stop-loss reinsurance ceded loss function. Stop-loss reinsurance is part of the $G$. The class of this non-negative functions $g(x,D)$ defined on $[0,\infty)$ and

$$g(x,D) = \begin{cases} 0, & 0 \leq x < D \\ x - D, & D \leq x < VaR_\alpha(X) \\ VaR_\alpha(X) - D, & x \geq VaR_\alpha(X) \end{cases}$$ (2)

or $g(x,D) = (x - D)_+ - (x - VaR_\alpha(X))_+$. While the insurance retain interval is holding between $\max[0, VaR_\alpha(X) - L] \leq D \leq VaR_\alpha(X)$. The ceded loss function with the stop-loss reinsurance form is denoted as $\eta$.

**Lemma 1.** For any confidence level $p \in (p_0, 1)$ and $I(x) \in G$ there always exists a function $g(x,D) \in \eta$ such that $VaR_\alpha\left(T_1(X)\right) \leq VaR_p(T_1(X)).$ [1]

**Proof.**

For any $I(x) \in G$, let $g(x,e) = (x - e)_+ - (x - VaR_p(X))_+$, with $e = VaR_p(X) - I\left(VaR_p(X)\right)$. $g(x,e) \in \eta$ and $g\left(VaR_p(X), e\right) = I\left(VaR_p(X)\right)$. For any $x \in [x, VaR_p(X)]$, by using the Lipschitz continuity of $I(x)$, we have $I(VaR_p(X)) - I(x) \leq VaR_p(X) - x$ which implies $I(x) \geq x + I(VaR_p(X)) - VaR_p(X)) = g(x,e)$. By noticing that $g(x,e) \leq I(x)$, for all $x \in [0,\infty)$ and we can conclude that $g(x,e) \leq I(x) \leq L$ for all $x \in [0,\infty)$. $g(x,e) \in \eta$. Through the optimization problem we know that $g(x,e)$ is the desired function. ■

According to Lu et. al [1], we can achieve the optimal ceded loss function with stop-loss reinsurance for the optimization we have concerned through Theorem 1 below.

**Theorem 1.** For a given confidence level $p \in (p_0, 1)$:

i. If $p \leq \theta^*$,

$$\min_{I(x) \in G} \{VaR_p(T_1(X))\} = VaR_p(X)$$

and the minimum of $VaR_p(T_1(X))$ is attained at $I^*(x) = 0$.

ii. If $p > \theta^*$ dan $L > VaR_p(X) - VaR_{\theta^*}(X)$,

$$\min_{I(x) \in G} \{VaR_p(T_1(X)) = VaR_p(X) + (1 + \theta) \int_{VaR_p(X)}^{VaR_{\theta^*}(X)} S_\alpha(x)dx$$

and the minimum of $VaR_p(T_1(X))$ is attained at $I^*(x) = (x - VaR_{\theta^*}(X))_+ - (x - VaR_p(X))_+$.

iii. If $p > \theta^*$ dan $L \leq VaR_p(X) - VaR_{\theta^*}(X)$, then

$$\min_{I(x) \in G} \{VaR_p(T_1(X)) = VaR_p(X) - L + (1 + \theta) \int_{VaR_p(X)}^{VaR_{\theta^*}(X)-L} S_\alpha(x)dx$$

and the minimum of $VaR_p(T_1(X))$ is attained at $I^*(x) = (x - (VaR_p(X) - L))_+ - (x - VaR_p(X))_+$.

We have got the desired optimal ceded loss function for the reinsurance risk exposure through the Theorem 1.
4. Implementation in Finding Optimal Ceded Loss Function for Panin Premier Maxilinked Product

In this paper, 384 data claims for Panin Premier Maxilinked will be used. Panin Premier Maxilinked is an insurance product that give a medical benefit for the policyholder if they admit a failure. This data is obtained in the 2015.

| Statistic            | Value   | Percentile | Value |
|----------------------|---------|------------|-------|
| Sample Size          | 384     |            | 7500  |
| Range                | 7.7186E+7 | 5%         | 90000 |
| Mean                 | 3.9432E+6 | 10%        | 1.2000E+5 |
| Variance             | 5.3973E+13 | 25%(Q1)   | 3.0000E+5 |
| Std.Deviation        | 7.3466E+6 | 50%(Median)| 1.5000E+6 |
| Coef. Of Variation   | 1.8631  | 75%(Q3)   | 5.0000E+6 |
| Std.Error            | 3.7491E+5 | 90%       | 9.4196E+7 |
| Skewness             | 5.1417  | 95%       | 1.6090E+7 |
| Excess Kurtosis      | 37.266  | Max       | 7.719E+7 |

From table 1, Panin Premier Maxilinked data claims is very variative, which is from 7,500 until 77 million in rupiahs. It can be indicate through the table 1 that the range for this data claims is very large and a big standard deviation number contribute for a not repetition claims in 2015. Figure 1 is the plot for the 384 data claims.

By using EasyFit software, Panin Premier Maxilinked medical product is fit for a Weibull distribution with three parameters $\alpha = 0.64614$, $\beta = 2.785,400$ and $\gamma = 7500$. This indicate that the probability to happen a small claim is bigger than the probability to happen a big claims. It is concluded because of the Weibull distribution characteristics.

The next step is finding the optimal ceded loss function for Panin Premier Maxilinked medical benefit product. Through the Theorem 1, we will choose the confidence level $p = 0.95$ and $\theta = 4$ cause $\theta^* = \frac{4}{1+4} = 0.8$ while $L = 5$ million. By choosing $p = 0.95 > \theta^* = 0.8$ and the value $\text{VaR}_{0.95}(X) = 15,225,181.5$, $\text{VaR}_{0.9}(X) = 5,825,147.4$, then $\text{VaR}_{0.95}(X) - \text{VaR}_{0.9}(X) = 15,225,181.5 - 5,825,147.4 = 9,400,034.1$. Next, $L = 5,000,000 < \text{VaR}_{0.95}(X) - \text{VaR}_{0.9}(X) = 9,400,034.1$ and $\text{VaR}_{0.95}(X) - L = 15,225,181.5 - 5,000,000 = 10,225,181.5$. Then our optimal ceded loss function will have a form $I^*(x) = (x - 10,225,181.5)_+ - (x - 15,225,181.5)_+$ or

$$I^*(x) = \begin{cases} 
0, & 0 < x < 1.02E + 7 \\
 x - 10,225,181.5, & 1.02E + 7 \leq x < 1.52E + 7 \\
5,000,000, & x \geq 1.52E + 7 
\end{cases}$$

with reinsurance premium $\pi^*_r = (1 + \theta)E[I(X)] = (1 + 4) \times (355,130.09) = 1,775,650.46$. 

![Figure 1. Plot of Claim Amount of Panin Premier Maxilinked Product](image-url)
5. Conclusion
In this work, we study the optimal reinsurance problem under VaR risk measures and under the constraint that the reinsurer’s risk exposure has an upper bound. It is shown that the stop-loss reinsurance form is the best-ceded loss function under the optimization and through the Panin Premier Maxilinked data claims, we have achieved the optimal ceded loss function that desired.

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