PAPER

Demonstrating non-Abelian braiding of surface code defects in a five qubit experiment

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Abstract
Currently, the mainstream approach to quantum computing is through surface codes. One way to store and manipulate quantum information with these to create defects in the codes which can be moved and used as if they were particles. Specifically, they simulate the behaviour of exotic particles known as Majoranas, which are a kind of non-Abelian anyon. By exchanging these particles, important gates for quantum computation can be implemented. Here we investigate the simplest possible exchange operation for two surface code Majoranas. This is found to act non-trivially on only five qubits. The system is then truncated to these five qubits, so that the exchange process can be run on the IBM 5Q processor. The results demonstrate the expected effect of the exchange. This paper has been written in a style that should hopefully be accessible to both professional and amateur scientists.

Surface codes are the most well-known starting point for fault-tolerant quantum computation [1]. One way to store and manipulate information in these is to engineer certain kinds of defects [2, 3]. These can be moved around and manipulated in much the same way as particles. However, their restriction to the two-dimensional (2D) structure of the surface codes allows them to exhibit some of the exotic behaviour possible for particles in 2D universes.

The important features of these particles are their effects when fused and braided. These refer to the processes of combining particles and of exchanging the positions of particles, respectively. It has been predicted that the fusion rules for these defects identical to so-called Majorana modes or Ising anyons. Their braiding forms a projective representation of the braid group, but is otherwise also identical to that of Majoranas. We will therefore refer to them simply as Majoranas from henceforth.

In this paper we present an experiment performed on a small patch of surface code. The patch is nevertheless large enough for these defects to be introduced and even for pair of them to exchange positions. We implement this and show that the effects of the process are consistent with the braiding of Majoranas, as predicted.

Anyons

There are many types of particle in the Universe, but when exchanged, their properties place them into just two categories: bosons and fermions, neither of which are very complex. This is because a loop around a point in three spatial dimensions can be continuously deformed to a loop that is not around a point. One can simply pick the loop up and place it elsewhere. As such, all topological properties of one particle moving in a loop around another must be trivial. There are only two ways to achieve this: bosonic and fermionic exchange behaviour.

In a 2D universe, this no longer holds. There is no longer an ’up’ to move the loop through, and so it is stuck around the point. The braiding of particles may therefore have non-trivial properties. This would allow almost any type of exchange. Such particles are therefore known collectively as anyons [4]. For a simple introduction, see [5].

Though we do not live in 2D universe, we can make two–2D physical systems. This allows us to find phases of matter in which anyons arise as quasi-particles or other localised features of the system than can be moved and otherwise manipulated in the same manner as particles [4, 6].
Majoranas

The most interesting families of anyons are the non-Abelian anyons. One example is the Ising anyon model. This holds two types of particle: one known as a Majorana, and the other a fermion denoted by $\psi$ [4].

The fermion type $\psi$ is its own antiparticle. Combining two of these will always result in annihilation.

The Majoranas are their own antiparticle, so a pair of them can be created from vacuum. When combined, these would annihilate back to vacuum. However, it is also possible to obtain a pair of Majoranas from the decay of a $\psi$. These would recreate the fermion if brought back together.

These two hypothetical pairs of Majoranas are completely indistinguishable. Their memory of whether to annihilate or form a fermion is not stored in any locally accessible feature. Instead it is stored non-locally through quantum entanglement in the underlying physical system.

This non-locality makes these particles an attractive proposition for storing quantum information in a quantum computer. By associating a pair that annihilate with a bit value 0, and a pair that form a $\psi$ with bit value 1, they can be used to store a bit (or qubit). By keeping the Majoranas well separated, it would take a large and concerted effort for errors to affect the bit (by moving the particles together to read out the value, for instance).

As such, the information will have an inherent robustness against errors, as long as error correction is performed to detect the errors before they build up [7-9].

Now consider two pairs of Majoranas created from vacuum. If one Majorana from one pair is combined with one from the other, there is no shared history to determine the result. The pair will therefore randomly choose to either annihilate or become a $\psi$.

Since everything initially came from vacuum, to vacuum they must return. Combining the remaining pair of Majoranas must then always yield the same result as for the first pair, such than any $\psi$ formed by one will be joined by its antiparticle from the other. It is this process, and the resulting correlation, that is the key to the experiment we will perform.

The exchange of Majoranas, which is simply swapping the positions of two of the anyons, also allows for many interesting effects. For one thing, and as we will show experimentally, it can be used to move particles to be combined with strangers, and so exhibit the effect described above as well as many more complicated versions.

Also, suppose we take again the two pairs of Majoranas created from vacuum. One Majorana from one pair is moved around a Majorana from the other. Combining these two pairs will now result in both forming a $\psi$, and hence flipping the bits that they are storing. This is certainly a non-trivial effect that bosons and fermions could only dream of. It demonstrates the power of non-Abelian anyons to not only store information, but to process it too.

Note that the fault-tolerance described above applies only when the Majoranas can be kept well separated, and when error correction is performed. The small nature of our experimental set-up prevents the possibility of either. As will be seen later, this means that our results will not be free of noise of the effects of noise.

For another approach towards exchanging Majoranas, as well as a summary of all other experimental progress towards Majoranas in a variety of physical systems, see [10, 11] and references therein.

Qubits

Qubits are the quantum analogue of bits. Just as a bit can be either 0 or 1, a qubit can be either $|0\rangle$ or $|1\rangle$. But it can also be any quantum superposition of the two, such as

\[ |\psi\rangle = a |0\rangle + b |1\rangle, \]

for any arbitrary complex numbers $a$ and $b$ [12]. For a simple introduction, see [13].

The states $|0\rangle$ and $|1\rangle$ are referred to as the Z basis of the qubit. If we measure to see whether the qubit is $|0\rangle$ or $|1\rangle$, this is called a Z measurement.

Two particular examples of possible superpositions are the states $|+\rangle$ and $|-\rangle$, which are known as the X basis states,

\[ |\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle + \pm |1\rangle). \]

These are orthogonal states, and as different from each other as $|0\rangle$ is from $|1\rangle$. This means we can also measure whether a qubit is $|+\rangle$ or $|-\rangle$, known as an X measurement. Similarly we may also define so-called Y basis states, and a Y measurement.

The manipulation of qubits can be understood using the circuit model. In this work, we will be considering the same circuits as used in the IBM Quantum Experience, and so refer to that for an introduction [14].

The most notable operation applied in these circuits is the CNOT. This acts on two qubits, with one called the ‘control’ and the other the ‘target’. The effect is to add the Z basis value of the control to that of the target.
However, since the target is restricted to the two states 0 and 1, only the parity of the sum is retained. The target is therefore left in the state 0 if the sum was even, and 1 if the sum was odd. If the target was initially in the $|0\rangle$ state, the effect is to copy the $Z$ basis state of the control to the target. The CNOT gate is a reversible equivalent of the XOR gate used in electronic circuits.

In the IBM 5Q processor, there are five qubits, labelled $Q_0$ to $Q_5$. The only CNOTs that can be applied are those with $Q_2$ as the target. In order to allow more flexibility in circuit design, we can implement a CNOT with $Q_2$ as control by applying the Hadamard gate, $H$, on both control and target before and after the standard CNOT is applied.

Matching code

The grid shown in figure 1 is an example of a matching code [15], which is a type of quantum error correcting code [16] that is based on the honeycomb lattice model [4]. It is a variant of the surface code [1].

In this code, there is a qubit located at each vertex, edge and hexagon. The vertex qubits are those that are truly part of the code. The edge and hexagon qubits are there to make measurements of the vertex qubits.

We associate a measurement with each edge. Each of these measures a collective property of the two qubits on the vertices connected by the edge.

For the vertical edges, we have a so-called ZZ measurement. There are two possible outcomes, which correspond to whether the state of the qubits differ in the $Z$ basis. If both qubits are $|0\rangle$, or both are $|1\rangle$, the measurement outputs 0 to show there is no difference. The same will happen for a superposition of both being $|0\rangle$ and both being $|1\rangle$, and the measurement will not disturb this superposition. On the other hand, if one qubit is $|0\rangle$ and the other is $|1\rangle$, the measurement will output 1 to show that there is a difference.

The other two edges correspond to XX and YY measurements. These are the same as a ZZ measurement, except that they compares whether the states are the same (output 0) or different (output 1) in the $X$ and $Y$ basis, respectively. The circuits that implement these measurements are shown in figure 2.

Measurements that share a vertex qubit do not commute. This means that making a measurement of one will affect the outcome of the other. For example, suppose we prepare a state such that the ZZ measurement on the light blue edge of figure 1 will give the outcome 0 with certainty. We then measure the XX shown in light red. Subsequently measuring ZZ will then randomly output 0 or 1, due to the effect of the non-commuting XX. Further explanation of this can be found in [17].
The process shown in figure 3 exchanges two Majoranas. The process of this exchange was introduced in [15], but is also explained in the caption of the figure.

The implementation of the exchange acts non-trivially only on three vertex qubits and two edge qubits. We may therefore truncate the lattice to this small area, and use a five qubit process to implement the exchange.

The truncated system is shown in figure 4. In common with other truncated surface code experiments [19–21], not all stabilisers have full support on the truncated area. They must therefore be reduced to the part with support on this area. Most notably, the three vertical links incident on the area only have one vertex within it. The ZZ measurements required for the full system then become single Z measurements. The fermionic occupancy of these three links therefore corresponds exactly to the Z basis state of the three vertex qubits.
Accordingly, the initial state is simply that for which all vertex qubits are \(|0\rangle\). The initial state of the edge qubits, as always, should also be \(|0\rangle\).

The most straightforward implementation of this process is shown in figure 5. Here the two edge qubits are used to make the required XX and YY measurements.

This circuit cannot be run on the IBM 5Q processor due to its restricted topology. Because of this we must use a single qubit as the intermediary for both the XX and ZZ measurements. In order for the results of the two measurements to be distinguished, the result of the YY measurement is copied onto an otherwise unused qubit using a CNOT.

Figure 3. In (a) we see the code with ZZ stabilisers shown in blue. The corresponding pairs of Majoranas can be associated with the vertices. These two pairs are shown explicitly for two ZZ stabilisers shown in purple and teal. (b) To move the green Majorana on vertex \(v_2\) we add the adjacent YY measurement to the stabiliser. Since this does not commute with the ZZ measurement above, it is also removed from the stabiliser. The YY measurement binds the Majoranas on vertices \(v_1\) and \(v_2\) into a well-defined fermionic mode, whereas that on \(v_4\) becomes unbound. The Majorana initially at \(v_4\) is therefore effectively teleported to \(v_1\) [15, 18]. (c) An XX measurement is similarly used to teleport the purple Majorana initially at \(v_5\) to \(v_1\). (d) Finally the initial ZZ measured, and so returned to the stabiliser. This moves the teal Majorana from \(v_2\) to \(v_4\), completing the exchange.

Figure 4. System used for the exchange is reduced to the five qubits on which the process acts non-trivially. The ZZ measurements are reduced to single \(Z\) measurements on the vertex qubits. The pairs of Majoranas that residing on these edges therefore now reside fully on the vertices. The \(Z\) measurements determine whether they would combine to annihilate or form a fermion, just as the ZZ measurements did previously. Other edge operators remain unchanged.

Figure 5. The ideal circuit first performs the YY measurement of \(v_1\) and \(v_2\) using \(e_1\), and then the XX of \(v_2\) and \(v_3\) using \(e_2\). The \(Z\) measurement of \(v_2\) is then performed. Afterwards, \(Z\) measurements are made on \(v_1\) and \(v_3\) to verify the expected correlations.
It is also important to apply the circuit as quickly as possible, and to assign the qubits to their tasks based on the noise levels of their entangling gates and also their lifetimes. Doing so results in the final circuit as shown in figure 6.

**Results**

The circuit of figure 6 was implemented using the IBM 5Q processor, using the interface provided by the IBM quantum experience. The success of the circuit is categorised by the correlation function

\[ C = P(00) + P(11) - P(01) - P(10). \]

Here \( P(01) \) is the probability that the measurement of \( v_1 \) yields 0 and \( v_3 \) yields 1, both in the case of no stray fermions. The theoretical prediction for the ideal case is of perfect correlations, corresponding to \( C = 1 \). In contrast \( C = 0 \) would correspond to a lack of correlations and \( C = -1 \) to perfect anticorrelation.

A simulation of the circuit under noiseless conditions showed exactly the result expected: \( C = 1 \). A simulation under realistic conditions yields \( C = 0.454 \).

Running the circuit for 24576 shots on the IBM 5Q processor yields \( C = 0.530 \). This is in good agreement with the theoretical prediction made under realistic conditions. It is also well above zero, and so clearly demonstrates the expected correlations.

One possible problem with this result is that decay to the \( |0\) state on \( v_1 \) and \( v_3 \) could lead to a false positive. To test this, tomography on the state is performed. Specifically, one can consider the Majoranas to encode a single logical qubit. The exchange has the effect of rotating this qubit to a \( Y \) basis state. The logical \( X \) operator corresponds to the product of the \( XX \) and \( YY \) link operators, and so to an \( YZX \) operator on \( v_1, v_2 \) and \( v_3 \). Respectively. Since the logical \( Z \) is simply a single \( Z \) on \( v_1 \) or \( v_3 \), the \( Y \) will be \( XZX \) or \( YZY \) respectively.

Since these three qubit measurements are performed at readout, superpositions in the system need not be preserved. We may therefore do three single qubit measurements and calculate the corresponding correlators afterwards. This simply requires additional rotations on \( v_1 \) and \( v_3 \) of figure 6 to access the correct basis. The results for each are obtained from 8192 shots.

With these results, we can reconstruct the density matrix of the resulting state and compute its fidelity to the required \( Y \) basis state. It is found that this fidelity is 70.6% when using \( YZY \). The closest point on the surface of the Bloch sphere has a fidelity of 94.0% to the required state. This is therefore a very encouraging result.

When using \( XZX \) the state obtained has a fidelity of only 54.6%, and that of the closest point on the Bloch sphere is 67.4%. The poor result in this case is most likely due to the longer time needed to measure the qubit \( v_1 = Q_0 \), which has the lowest lifetime of all the qubits.

All data can be found in [22].

**Conclusion**

The exchange of two Majoranas causes the state of two Majorana pairs with definite fusion result to become one with indefinite but correlated results. In this work we showed that this could be realised on a surface code based architecture using only five qubits. The experiment was then performed using the IBM 5Q processor. The results obtained demonstrate the expected correlations.

With higher fidelity qubits, this work could be extended by applying two exchanges. The effect of a full braid of one Majorana around another could then be demonstrated, as well as the effect of undoing the first exchange...
by one in the opposite direction. The orthogonality of the results in these two cases will provide an even stark demonstration of the braiding of non-Abelian anyons.

It would also be interesting to see this experiment reproduced with spin qubits. This could use the process proposed in [23] or the system introduced in [24].

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