NP-completeness of 4-incidence colorability of semi-cubic graphs

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Abstract

The incidence coloring conjecture, proposed by Brualdi and Massey in 1993, states that the incidence coloring number of every graph is at most $\Delta + 2$, where $\Delta$ is the maximum degree of a graph. The conjecture was shown to be false in general by Guiduli in 1997, following the work of Algor and Alon. However, in 2005 Maydanskiy proved that the conjecture holds for any graph with $\Delta \leq 3$. It is easily deduced that the incidence coloring number of a semi-cubic graph is 4 or 5. In this paper, we show that it is already NP-complete to determine if a semi-cubic graph is 4-incidence colorable, and therefore it is NP-complete to determine if a general graph is $k$-incidence colorable.

Keywords: incidence coloring number, $k$-incidence colorable, strong-vertex coloring, semi-cubic graph, NP-complete.

AMS subject classification (2000): 05C15, 68Q17.

1 Introduction

In this paper we consider undirected, finite and simple graphs only, and use standard notations in graph theory (see [2]). Let $G = (V, E)$ be a graph, and let

$$I(G) = \{(v, e) : v \in V, e \in E, \text{ and } v \text{ is incident with } e\}$$

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be the set of all incidences of $G$. We say that two incidences $(v, e)$ and $(w, f)$ are adjacent if one of the following holds: (1) $v = w$, (2) $e = f$ and (3) the edge $vw$ equals to $e$ or $f$.

Following Shiu et al. [8] we view $G$ as a digraph by splitting each edge $uv$ into two opposite arcs $(u, v)$ and $(v, u)$. For $e = uv$, we identify the incidence $(u, e)$ with the arc $(u, v)$. By a slight abuse of notation we will refer to the incidence $(u, v)$ whenever it is convenient to do so. Two distinct incidences $(u, v)$ and $(x, y)$ are adjacent if one of the following holds: (i) $u = x$, (ii) $u = y$ and $v = x$ (iii) $v = x$. The configurations associated with (i)-(iii) are pictured in Figure 1.

We define an incidence coloring of $G$ to be a coloring of its incidences in which adjacent incidences are assigned different colors. If $\sigma : I(G) \to S$ is an incidence coloring of $G$ and $k = |S|$, then $G$ is called $k$-incidence colorable and $\sigma$ is a $k$-incidence coloring, where $S$ is a color-set. The incidence coloring number of $G$, denoted by $\chi_i(G)$, is the smallest number of colors in an incidence coloring.

This concept was first developed by Brualdi and Massey [3] in 1993. They proposed the incidence coloring conjecture (ICC), which states that for every graph $G$, $\chi_i(G) \leq \Delta + 2$, where $\Delta = \Delta(G)$ is the maximum degree of $G$. In 1997 Guiduli [5] pointed out that the ICC was solved in negative, following an example in [1]. He considered the Paley graphs of order $p$ with $p \equiv 1 \pmod{4}$. Following the analysis in [1], he showed that $\chi_i(G) \geq \Delta + \Omega(\log \Delta)$, where $\Omega = \frac{1}{8} - o(1)$. By using a tight upper bound for directed star arboricity, he obtained the upper bound $\chi_i(G) \leq \Delta + O(\log \Delta)$.

Brualdi and Massey [3] determined the incidence coloring numbers of trees, complete graphs and complete bipartite graphs. They also gave a simple bound for the incidence coloring number as follows.

**Theorem 1.1** (Brualdi and Massey [3]) For every graph $G$, $\Delta(G) + 1 \leq \chi_i(G) \leq 2\Delta(G)$.

In [8] Shiu et al. proved that $\chi_i(G) \leq 5$ for several classes of cubic (3-regular)
2-connected graphs $G$, including Hamiltonian cubic graphs. In 2005 Maydanskiy \cite{7} proved that the conjecture (ICC) holds for all graphs with $\Delta \leq 3$.

**Theorem 1.2** (Maydanskiy \cite{7}) For every graph $G$ with $\Delta(G) \leq 3$, $\chi_i(G) \leq 5$.

**Definition 1** For a graph $G$ with $\Delta = 3$, if the degree of any vertex of $G$ is 1 or 3, then the graph $G$ is called a semi-cubic graph.

Using Theorems 1.1 and 1.2, we have the following corollary.

**Corollary 1.3** The incidence coloring number of a semi-cubic graph is 4 or 5.

In this paper, we show the following result.

**Theorem 1.4** It is NP-complete to determine if a semi-cubic graph is 4-incidence colorable. Therefore, it is NP-complete to determine if a general graph is $k$-incidence colorable, or in other words, it is NP-complete to determine the incidence coloring number for a general graph.

### 2 Incidence coloring of semi-cubic graphs

In a given graph $G$, $N_G(v)$ denotes the set of vertices of $G$ adjacent to $v$, and $d_G(v) = |N_G(v)|$ is the degree of a vertex $v$ in $G$. A vertex of degree $k$ is called a $k$-vertex. We denote the set of all the incidences of the form $(v, u)$ and $(u, v)$ by $O_v$ and $I_v$, respectively, where $u$ is adjacent with $v$. Let $CO_v$ and $CI_v$ denote the set of colors assigned to $O_v$ and $I_v$ in an incidence coloring of $G$, respectively.

We define a strong vertex coloring of $G$ to be a proper vertex coloring such that for any $u, w \in N_G(v)$, $u$ and $w$ are assigned distinct colors. If $\sigma : V(G) \rightarrow S$ is a strong vertex coloring of $G$ and $k = |S|$, then $G$ is called $k$-strong-vertex colorable and $\sigma$ is a $k$-strong-vertex coloring of $G$, where $S$ is a color-set. And we say that $G$ is $k$-strong-vertex colored.

**Lemma 2.1** Given a semi-cubic graph $G$, $G$ is 4-incidence colorable if and only if $G$ is 4-strong-vertex colorable.

**Proof.** Since any two incidences in $O_v$ are adjacent, $|CO_v|$ is equal to the degree of vertex $v$ in an incidence coloring of $G$. Given a semi-cubic graph $G$, if $G$ is 4-incidence colorable and $\sigma$ is a 4-incidence coloring, then $|CI_v|$ is 1 for every
vertex \( v \) of \( G \). We can color vertex \( v \) using \( CI_v \) and obtain a vertex coloring. Since incidences \((u, v)\) and \((v, u)\) are adjacent, \( CI_v \neq CI_u \) and the vertex coloring is proper. If \( u, w \in N_G(v) \), incidences \((v, u)\) and \((v, w)\) are adjacent. Thus \( u \) and \( w \) are assigned distinct colors in the vertex coloring. So the vertex coloring is a 4-strong-vertex coloring and \( G \) is 4-strong-vertex colorable.

Suppose there exists a 4-strong-vertex coloring \( \sigma \) of \( G \), we color the incidence \((u, v)\) of \( G \) by the color \( c(v) \), where \( c(v) \) denotes the color assigned to the vertex \( v \) in \( \sigma \). If two incidences \((u, v)\) and \((x, y)\) are adjacent, then one of the following holds: (1) \( u = x, v \in N_G(u) \) and \( y \in N_G(u) \); (2) \( u = y \) and \( v = x, vy \in E(G) \); (3) \( v = x, vy \in E(G) \). From the definition of strong vertex coloring, incidences \((u, v)\) and \((x, y)\) are assigned different colors. So \( G \) is 4-incidence colorable.

3 The blocks used in the construction

For terminology and known results of NP-completeness we refer to the book [4]. The 3\( SAT \) problem is stated as follows:

3\( SAT \)

INSTANCE: Set \( U \) of variables, collection \( C \) of clauses over \( U \) such that each clause \( C_i \in C \) has \( |C_i| = 3 \).

QUESTION: Is there a truth assignment for \( U \) such that every \( C_i \in C \) is true?

It is clear that both the 4-incidence colorable problem for semi-cubic graphs and the 4-strong-vertex colorable problem for semi-cubic graphs are in the class NP. To prove their NP-completeness, we exhibit a polynomial reduction from the known NP-complete problem 3\( SAT \). Given an instance \( C \) of the problem 3\( SAT \), we will show how to construct a semi-cubic graph \( G \) of polynomial size in terms of the size of the instance \( C \) such that \( G \) is 4-strong-vertex colorable if and only if \( C \) is satisfiable, which, from Lemma 2.1 implies that \( G \) is 4-incidence colorable if and only if \( C \) is satisfiable. The semi-cubic graph \( G \) will be constructed from some pieces or “blocks” which carry out specific tasks. Information will be carried between blocks by pairs of vertices. In a 4-strong-vertex coloring of the graph \( G \), such a pair of vertices is said to represent the value \( T \) (“true”) if the vertices have the same color, and to represent \( F \) (“false”) if the vertices have distinct colors. In the following we always use \( S = \{1, 2, 3, 4\} \) to denote the set of colors.
3.1 The switch blocks

First, we give a special semi-cubic graph $H$ as shown in Figure 2. We call $H$ a Kite graph. It is easy to check that if the Kite graph $H$ is 4-strong-vertex colored, all 1-vertex $e, f, g$ have the same color.

The switch block is shown with its symbol in Figure 3. It may be checked that this block is 4-strong-vertex colorable. If this block is 4-strong-vertex colored, one of the pairs of vertices marked $a, b$ or $c, d$ must have the same color and the remaining pair of vertices must have distinct colors. $c(l)$ is equal to $c(l')$ for $l \in \{a, b, c, d\}$ in any 4-strong-vertex coloring of this block. In fact, there are only two non-equivalence ways to color the two pairs of vertices $a, b$ and $c, d$, which is shown in Figure 4.
Regarding the pair of vertices $a, b$ as the input and the pair $c, d$ as the output, the block changes a representation of $T$ to a representation of $F$, and vice versa.

### 3.2 The variable blocks

We construct the big switch block using three switch blocks by identifying the output of the first switch block with the input of the second switch block, identifying the output of the second switch block with the input of the third switch block. It is shown with its symbol in Figure 5. Regarding the pair of vertices $e, f$ as the input and the pair $p, r$ as the output of the big switch block.

We state the following lemma.

**Lemma 3.1** Given a big switch block $G$ for which the input and output are $e, f$ and $p, r$, respectively, $G$ is 4-strong-vertex colorable. And if $G$ is 4-strong-vertex colored, one of the following holds:

1. If $c(e) = c(f)$, then $c(p) \neq c(r)$ and $c(p), c(r)$ may be any two different colors in $S = \{1, 2, 3, 4\}$;

2. If $c(e) \neq c(f)$, then $c(p) = c(r)$ and $c(p)$ may be any color in $S = \{1, 2, 3, 4\}$, where $c(v)$ denotes the color assigned to the vertex $v$ of $G$.

**Proof.** Since the switch block is 4-strong-vertex colorable, it is easy to check that the big switch block $G$ is also 4-strong-vertex colorable.

If $G$ is 4-strong-vertex colored and $c(e) = c(f)$, without loss of generality, we suppose that $c(e)$ is 1. The color-set $\{c(x), c(y)\}$ of the output of the first switch block may be any two different colors in $\{2, 3, 4\}$. For the output $t, q$ of the second switch block, $c(q)$ is equal to $c(t)$ and may be any color in $S = \{1, 2, 3, 4\}$. So, $c(p) \neq c(r)$ and $c(p), c(r)$ may be any two different colors in $S = \{1, 2, 3, 4\}$.

If $G$ is 4-strong-vertex colored and $c(e) \neq c(f)$, without loss of generality, we suppose that $\{c(e), c(f)\}$ is $\{1, 2\}$. Then $c(x)$ is equal to $c(y)$ and may be any
Figure 6: The variable blocks having 1, 2, · · · , 6 output pairs of vertices respectively. More generally it is made from \( k \) big switch blocks and \( 2 \cdot \lfloor \frac{k-1}{2} \rfloor \) Kite graphs \( H \) and has \( k \) output pairs.

color in \( \{3, 4\} \). Thus \( c(q) \neq c(t) \), and \( c(q) \) and \( c(t) \) may be any two different colors in \( S = \{1, 2, 3, 4\} \). So, \( c(p) = c(r) \) and \( c(p) \) may be any color in \( S = \{1, 2, 3, 4\} \).

From Lemma 3.1 the big switch block changes a representation of \( T \) to a representation of \( F \), and vice versa.

The truth or falsity of each variable \( u_i \) will be represented by a variable block as shown in Figure 6, in which the blocks have, respectively, 1, 2, · · · , 6 pairs of output vertices. In general, the number of output pairs in the block representing \( u_i \) should be equal to the number of total appearances of \( u_i \) or \( \overline{u_i} \) among the clauses of \( C \). If \( k \) pairs of output vertices is needed, we construct a variable block which is made from \( k \) big switch blocks and \( 2 \cdot \lfloor \frac{k-1}{2} \rfloor \) Kite graphs \( H \).

From Lemma 3.1 and the construction of the big switch block, the following lemma is obvious.

**Lemma 3.2** Given a variable block \( G \), \( G \) is 4-strong-vertex colorable. And in any 4-strong-vertex coloring of \( G \), all the output pairs must represent the same value. If the output pairs represent \( T \) (“true”), then the color-set of any output pairs may be any color in \( S = \{1, 2, 3, 4\} \). If, on the other hand, the output pairs represent \( F \) (“false”), then the color-set of any output pairs may be any two different colors in \( S = \{1, 2, 3, 4\} \). ■
3.3 The clause blocks

The truth of each clause $C_j$ will be tested by a clause block as shown in Figure 7. The block is constructed from 3 switch blocks and a cycle of length 10.

The following lemma is crucial for proving our main theorem.

**Lemma 3.3** The clause block is 4-strong-vertex colorable if and only if the three input pairs of vertices do not all represent $F$.

**Proof.** Given a clause block $G$ as shown in Figure 8, we suppose that the 3 output pairs of vertices are $\{r, r'\}$, $\{w, w'\}$ and $\{z, z'\}$, respectively.

We suppose that the three input pairs of vertices all represent $F$. If $G$ is 4-strong-vertex colorable and $\sigma$ is a 4-strong-vertex coloring, then $c(r) = c(r')$, $c(w) = c(w')$ and $c(z) = c(z')$, where $c(v)$ denotes the color assigned to the vertex $v$ in $\sigma$. Without loss of generality, we suppose that $c(z)$, $c(b)$ and $c(d)$ are 1, 2 and 3, respectively. Then $c(a) = c(e) = 4$, $c(f) = 2$. Since $c(w)$ is equal to $c(w')$, $\{c(w), c(g)\} = \{c(w'), c(g)\} = \{1, 3\}$. Thus $c(h) = 4$, $c(i) = 2$. By the same token, $c(p)$ must be 4. So $c(a) = c(p) = 4$ and $pa \in E(G)$, a contradiction.

If the three input pairs of vertices do not all represent $F$, we can give a 4-strong-vertex coloring of $G$, which is shown in Figure 9. Thus $G$ is 4-strong-vertex colorable.
Figure 9: The three input pairs of vertices do not all represent $F$.

So the clause block is 4-strong-vertex colorable if and only if the three input pairs of vertices do not all represent $F$.

4 Main result

In this section, we prove the main result that it is NP-complete to determine whether the incidence coloring number of a semi-cubic graph is 4 or 5. Thus this problem has no polynomial time algorithm unless $P = NP$.

Proof of Theorem 1.4. The problem is clearly in the class NP. We exhibit a polynomial reduction from the problem 3SAT. Consider an instance $C$ of 3SAT and construct from it a semi-cubic graph $G$ as follows.

Each variable $u_i$ corresponds to a variable block $U_i$ with one output pair of vertices associated with each appearance of $u_i$ or $\overline{u_i}$ among the clauses of $C$. Each clause $C_j$ corresponds to a clause block $B_j$. Suppose literal $x_{j,k}$ in clause $C_j$ is the variable $u_i$. Then identify the $k$-th input pair of $B_j$ with the associate output pair of $U_i$. If, on the other hand, $x_{j,k}$ is $\overline{u_i}$, then insert an switch block between the $k$-th input pair of $B_j$ and the associated output pair of $U_i$. The resulting graph $G'$ has some 2-vertices. For every 2-vertex $u$, we add a pendant edge $(u, u')$. Now the resulting graph $G$ is a semi-cubic graph.

We shall show that the incidence coloring number of the semi-cubic graph $G$ is 4 if and only if there is a truth assignment to the variables which simultaneously
satisfies all the clauses in $C$.

First, we suppose that the incidence coloring number of $G$ is 4. From Lemma 2.1, $G$ is 4-strong-vertex colorable. Given a 4-strong-vertex coloring of $G$, we assign the value to each variable $u_i$ as the one that all the output pairs of the variable block corresponding to $u_i$ represent. By Lemma 3.3, the assignment to the variables simultaneously satisfies all the clauses in $C$.

Now, we suppose that there is a truth assignment to the variables which simultaneously satisfies all the clauses in $C$. For each variable $u_i$, we can 4-strong-vertex color the variable block corresponding to $u_i$ such that the value represented by the output pairs is equal to the value in the truth assignment. And by Lemma 3.2, the color-set of any output pairs may be any color or any two distinct colors in $S = \{1, 2, 3, 4\}$. So by Lemma 3.3, $G$ is 4-strong-vertex colorable. Finally, by Lemma 2.1, the incidence coloring number of $G$ is 4.

It is clear that the semi-cubic graph $G$ is constructed from $3m$ big switch blocks, $m$ clause blocks, at most $2 \cdot \lfloor \frac{3m-1}{2} \rfloor$ Kite graphs, and at most $3m$ switch blocks and some pendant edges. Thus, $|V(G)|$ is at most $521m$, and $|E(G)|$ is at most $626m$, and hence the semi-cubic graph $G$ can be constructed from $C$ polynomially in $m$, the size of $C$. The proof is now complete. $lacksquare$

**Example 1** Let $C = \{C_1, C_2\}$ an instance of the problem 3SAT and

\[
C_1 = u_1 \lor \overline{u}_2 \lor u_3,
\]

\[
C_2 = u_2 \lor u_3 \lor \overline{u}_4.
\]

By the proof of Theorem 1.4, we can construct the graph $G'$ as shown in Figure 10 and obtain the semi-cubic $G$ by adding a pendant edge to every 2-vertex of $G'$. 

![Figure 10: The graph $G'$](image-url)
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