When the Riemann Hypothesis might be false

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Abstract Robin criterion states that the Riemann Hypothesis is true if and only if the inequality \( \sigma(n) < e^\gamma \times n \times \log \log n \) holds for all natural numbers \( n > 5040 \), where \( \sigma(n) \) is the sum-of-divisors function and \( \gamma \approx 0.57721 \) is the Euler-Mascheroni constant. Let \( q_1 = 2, q_2 = 3, \ldots, q_m \) denote the first \( m \) consecutive primes, then an integer of the form \( \prod_{i=1}^{m} q_i^{a_i} \) with \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) is called an Hardy-Ramanujan integer. If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers \( n > 5040 \) such that Robin inequality does not hold and \( n < (4.48311)^m \times N_m \), where \( N_m = \prod_{i=1}^{m} q_i \) is the primorial number of order \( m \).

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

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1 Introduction

In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part \( \frac{1}{2} \) [4]. As usual \( \sigma(n) \) is the sum-of-divisors function of \( n \) [2]:

\[
\sigma(n) = \sum_{d \mid n} d
\]

where \( d \mid n \) means the integer \( d \) divides to \( n \) and \( d \nmid n \) means the integer \( d \) does not divide to \( n \). Define \( f(n) \) to be \( \frac{\sigma(n)}{n} \). Say Robins\((n)\) holds provided

\[
f(n) < e^\gamma \times \log \log n.
\]

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The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

**Theorem 1.1** If the Riemann Hypothesis is false, then there are infinitely many natural numbers $n > 5040$ such that Robins$(n)$ does not hold [4].

We recall that an integer $n$ is said to be square free if for every prime divisor $q$ of $n$ we have $q^2 \nmid n$ [2]. Robins$(n)$ holds for all natural numbers $n > 5040$ that are square free [2]. In addition, we show that Robins$(n)$ holds for some $n > 5040$ when $\frac{\pi^2}{6} \times \log \log n' \leq \log \log n$ such that $n'$ is the square free kernel of the natural number $n$. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first $m$ consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [2].

Based on the theorem 1.1, we know this result:

**Theorem 1.2** If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robins$(n)$ does not hold and $n < (4.48311)^m \times N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order $m$.

## 2 A Central Lemma

These are known results:

**Lemma 2.1** [2]. For $n > 1$:

$$f(n) < \prod_{q \mid n} \frac{q}{q - 1}. \quad (2.1)$$

**Lemma 2.2** [3].

$$\prod_{k=1}^{\infty} \left(1 - \frac{1}{q_k^2}\right) = \zeta(2) = \frac{\pi^2}{6}. \quad (2.2)$$

The following is a key lemma. It gives an upper bound on $f(n)$ that holds for all natural numbers $n$. The bound is too weak to prove Robins$(n)$ directly, but is critical because it holds for all natural numbers $n$. Further the bound only uses the primes that divide $n$ and not how many times they divide $n$.

**Lemma 2.3** Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}. \quad (2.3)$$

**Proof** We use that lemma 2.1:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}. $$
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Now for $q > 1$,

$$\frac{1}{1 - \frac{1}{q^2}} = \frac{q^2}{q^2 - 1}.$$  

So

$$\frac{1}{1 - \frac{1}{q^2}} \times \frac{q + 1}{q} = \frac{q^2}{q^2 - 1} \times \frac{q + 1}{q}$$

$$= \frac{q}{q - 1}.$$  

Then by lemma 2.2,

$$\prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} < \zeta(2) = \frac{\pi^2}{6}.$$  

Putting this together yields the proof:

$$f(n) < \prod_{i=1}^{m} \frac{q_i}{q_i - 1}$$

$$\leq \prod_{i=1}^{m} \frac{1}{1 - \frac{1}{q_i^2}} \times \frac{q_i + 1}{q_i}$$

$$< \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$  

3 A Particular Case

We can easily prove that Robins$(n)$ is true for certain kind of numbers:

**Lemma 3.1** Robins$(n)$ holds for $n > 5040$ when $q \leq 5$, where $q$ is the largest prime divisor of $n$.

**Proof** Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^\gamma \times \log \log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^\gamma \times \log \log n$$

according to the lemma 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^\gamma \times \log \log (5040) \approx 3.81.$$  

However, we know for $n > 5040$

$$e^\gamma \times \log \log (5040) < e^\gamma \times \log \log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$.  

4 Helpful Lemmas

For every prime number \( p_n > 2 \), we define the sequence \( Y_n = \frac{e^{\frac{1}{2 \log(p_n)}}}{(1 - \frac{1}{\log(p_n)})} \).

**Lemma 4.1** For every prime number \( p_n > 2 \), the sequence \( Y_n \) is strictly decreasing.

**Proof** For every real value \( x \geq 3 \), we state the function

\[
f(x) = \frac{e^{\frac{1}{x \log(x)}}}{(1 - \frac{1}{\log(x)})}
\]

which is equivalent to

\[
f(x) = g(x) \times h(u)
\]

where \( g(x) = e^{\frac{1}{x \log(x)}} \) and \( h(u) = \frac{u}{u^2 - 1} \) for \( u = \log(x) \). We know that \( g(x) \) decreases as \( x \geq 3 \) increases, Moreover, we note that \( h(u) \) decreases as \( u > 1 \) increases where \( u = \log(x) > 1 \) for \( x \geq 3 \). In conclusion, we can see that the function \( f(x) \) is monotonically decreasing for every real value \( x \geq 3 \) and therefore, the sequence \( Y_n \) is monotonically decreasing as well. In addition, \( Y_n \) is essentially a strictly decreasing sequence, since there is not any natural number \( n > 1 \) such that \( Y_n = Y_{n+1} \).

In mathematics, the Chebyshev function \( \theta(x) \) is given by

\[
\theta(x) = \sum_{p \leq x} \log p
\]

where \( p \leq x \) means all the prime numbers \( p \) that are less than or equal to \( x \).

**Lemma 4.2** [5]. For \( x \geq 41 \):

\[
\theta(x) > (1 - \frac{1}{\log(x)}) \times x.
\]

Besides, we know that

**Lemma 4.3** [5]. For \( x \geq 286 \):

\[
\prod_{q \leq x} q - 1 < e^\gamma \times (\log x + \frac{1}{2 \times \log(x)}).
\]

We will prove another important inequality:

**Lemma 4.4** Let \( q_1, q_2, \ldots, q_m \) denote the first \( m \) consecutive primes such that \( q_1 < q_2 < \cdots < q_m \) and \( q_m > 286 \). Then

\[
\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times (Y_m \times \theta(q_m)).
\]
**Proof** From the theorem 4.2, we know that

\[ \theta(q_m) > (1 - \frac{1}{\log(q_m)}) \times q_m. \]

In this way, we can show that

\[ \log (Y_m \times \theta(q_m)) > \log \left( Y_m \times (1 - \frac{1}{\log(q_m)}) \times q_m \right) \]
\[ = \log q_m + \log \left( Y_m \times (1 - \frac{1}{\log(q_m)}) \right). \]

We know that

\[ \log \left( Y_m \times (1 - \frac{1}{\log(q_m)}) \right) = \log \left( e^{\frac{1}{2}\log(q_m)} \times (1 - \frac{1}{\log(q_m)}) \right) \]
\[ = \log \left( e^{\frac{1}{2}\log(q_m)} \right) \]
\[ = \frac{1}{2} \times \log(q_m). \]

Consequently, we obtain that

\[ \log q_m + \log \left( Y_m \times (1 - \frac{1}{\log(q_m)}) \right) \geq \left( \log q_m + \frac{1}{2 \times \log(q_m)} \right). \]

Due to the theorem 4.3, we prove that

\[ \prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^{\gamma} \times \log q_m + \frac{1}{2 \times \log(q_m)} < e^{\gamma} \times \log (Y_m \times \theta(q_m)) \]
when \( q_m > 286. \)

### 5 Proof of Main Theorems

The next theorem implies that Robins(n) holds for a wide range of natural numbers \( n > 5040. \)

**Theorem 5.1** Let \( \frac{\pi^2}{6} \times \log \log n' \leq \log \log n \) for some \( n > 5040 \) such that \( n' \) is the square free kernel of the natural number \( n. \) Then Robins(n) holds.

**Proof** Let \( n' \) be the square free kernel of the natural number \( n. \) Let \( n' \) be the product of the distinct primes \( q_1, \ldots, q_m. \) By assumption we have that

\[ \frac{\pi^2}{6} \times \log \log n' \leq \log \log n. \]
For all square free \( n' \leq 5040 \), \( \text{Robins}(n') \) holds if and only if \( n' \not\in \{2, 3, 5, 6, 10, 30\} \) [2]. However, \( \text{Robins}(n) \) holds for all natural numbers \( n > 5040 \) when \( n' \in \{2, 3, 5, 6, 10, 15, 30\} \) due to the lemma 3.1. When \( n' > 5040 \), we know that \( \text{Robins}(n') \) holds and so

\[
f(n') < e^\gamma \times \log \log n'.
\]

By the previous lemma 2.3:

\[
f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.
\]

Suppose by way of contradiction that \( \text{Robins}(n) \) fails. Then

\[
f(n) \geq e^\gamma \times \log \log n.
\]

We claim that

\[
\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n.
\]

Since otherwise we would have a contradiction. This shows that

\[
\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^\gamma \times \log \log n'.
\]

Thus

\[
\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^\gamma \times \log \log n',
\]

and

\[
\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),
\]

This is a contradiction since \( f(n') \) is equal to

\[
\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}.
\]

**Theorem 5.2** If the Riemann Hypothesis is false, then there are infinitely many Hardy-Ramanujan integers \( n > 5040 \) such that \( \text{Robins}(n) \) does not hold and \( n < (4.48311)^m \times N_m \), where \( N_m = \prod_{i=1}^{m} q_i \) is the primorial number of order \( m \).

**Proof** Let \( \prod_{i=1}^{m} q_i^{a_i} \) be the representation of some natural number \( n > 5040 \) as a product of primes \( q_1 < \cdots < q_m \) with natural numbers as exponents \( a_1, \ldots, a_m \). The primes \( q_1 < \cdots < q_m \) must be the first \( m \) consecutive primes and \( a_1 \geq a_2 \geq \cdots \geq a_m \geq 0 \) since the natural number \( n > 5040 \) could be an Hardy-Ramanujan integer. We assume that \( \text{Robins}(n) \) does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as \( n > 5040 \) when the Riemann Hypothesis is false according to the theorem 1.2. From the lemma 4.4, we know that

\[
\prod_{i=1}^{m} \frac{q_i}{q_i - 1} < e^\gamma \times \log (Y_m \times \theta(q_m)) = e^\gamma \times \log \log (N_m^Y).
\]
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when \( q_m > 286 \). In this way, if \( \text{Robins}(n) \) does not hold, then \( n < N_m^{Y_m} \) since by the lemma 2.1 we have that

\[
f(n) < \prod_{l=1}^{m} \frac{q_l}{q_l - 1}.
\]

That is the same as \( n < N_m^{Y_m-1} \times N_m \). We can check that \( q_m^{Y_m-1} \) is monotonically decreasing for all primes \( q_m > 286 \) due to the lemma 4.1. Certainly, the function

\[
g(x) = x \left( \frac{1}{1 - \log(x)} - 1 \right)
\]

complies that its derivative is lesser than zero for all real numbers \( x > 286 \). Indeed, a function \( g(x) \) of a real variable \( x \) is monotonically decreasing in some interval if the derivative of \( g(x) \) is lesser than zero and the function \( g(x) \) is continuous over that interval [1]. We know that \( q_m \) could comply with \( q_m \geq 1000000! \) for infinitely many Hardy-Ramanujan integers \( n > 5040 \) such that \( \text{Robins}(n) \) does not hold, where \((\ldots)!\) is the factorial function. Certainly, if \( q_m \) would have an upper bound by some positive value, then there would not be infinitely many natural numbers \( n > 5040 \) which are an Hardy-Ramanujan integer and \( \text{Robins}(n) \) does not hold because of the theorem 5.1. Consequently, it is enough to show that

\[
q_m^{Y_m-1} \leq g(1000000!) < 4.48311
\]

for all primes \( q_m \geq 1000000! \). Moreover, we would obtain that

\[
q_m^{Y_m-1} > q_j^{Y_m-1}
\]

for every integer \( 1 \leq j < m \). Finally, we can state that \( n < (4.48311)^m \times N_m \) since \( N_m^{Y_m-1} < (4.48311)^m \) when \( n > 5040 \) could be any of the infinitely many Hardy-Ramanujan integers such that \( \text{Robins}(n) \) does not hold and \( q_m \geq 1000000! \).

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References

1. Anderson, G., Vamanamurthy, M., Vuorinen, M.: Monotonicity Rules in Calculus. The American Mathematical Monthly 113(9), 805–816 (2006). DOI 10.1080/00029890.2006.11920367
2. Choie, Y ., Lichiardopol, N., Moree, P., Solé, P.: On Robin’s criterion for the Riemann hypothesis. Journal de Théorie des Nombres de Bordeaux 19(2), 357–372 (2007). DOI doi:10.5802/jtnb.591
3. Edwards, H.M.: Riemann’s Zeta Function. Dover Publications (2001)
4. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. pures appli 63(2), 187–213 (1984)
5. Rosser, J.B., Schoenfeld, L.: Approximate Formulas for Some Functions of Prime Numbers. Illinois Journal of Mathematics 6(1), 64–94 (1962). DOI doi:10.1215/ijm/1255631807