Degenerate and Inverted Hierarchical Models of Majorana Neutrinos from See-saw Mechanism

N. Nimai Singh† and Mahadev Patgiri‡

† Department of Physics, Gauhati University, Guwahati-781014, India
‡ Department of Physics, Cotton College, Guwahati-781001, India

Abstract

In order to explain the recently reported result on the double beta decay \((0\nu\beta\beta)\) experiment, together with the earlier established data on LMA MSW solar and atmospheric neutrino oscillations, we generate the textures of the nearly degenerate as well as the inverted hierarchical models of the left-handed Majorana neutrino mass matrices within the framework of the see-saw mechanism in a model independent way. The leptonic mixings are generated from the texture of the right-handed Majorana mass matrices while keeping the Dirac neutrino mass matrices in diagonal forms.

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1 Introduction

The recently reported experimental result on double beta decay ($0\nu\beta\beta$) shows the possible evidence for the non-zero Majorana mass of the electron neutrino in the range of $m_{ee} = (0.05 - 0.86)$ eV at 95% C.L. with the best value of $m_{ee} = 0.4$ eV[1]. There are certain important implications of this result. Firstly, it rules out all models predicting the Dirac neutrino masses, leaving only the option for the Majorana type of neutrinos. It also allows the lepton number violating processes such as leptogenesis in a natural way. Secondly, this result together with the earlier experimental data on the atmospheric[2] and the solar[3] neutrino oscillations, allows only the degenerate and the inverted hierarchical solutions for the three generation left-handed Majorana neutrinos[1,4].

In the above context, it is important to construct theoretical models which can predict the degenerate and inverted hierarchical patterns of the Majorana neutrino mass matrices within the framework of the grand unified theories(GUTs) with or without supersymmetry[4,5]. In this short paper we attempt to generate the degenerate as well as the inverted hierarchical pattern of the left-handed Majorana neutrino mass matrices using the seesaw formula in a model independent way. This is, in fact, a continuation of our earlier work[6] where the neutrino mixings are provided from the texture of the right-handed Majorana mass matrix $M_{RR}$, while keeping the Dirac neutrino mass matrix $m_{LR}$ in the diagonal form. We had taken the Dirac neutrino mass matrix $m_{LR}$ as either the charged lepton mass matrix ($m_{LR} = \tan\beta m_l$ referred to as case(i)) or the up-quark mass matrix ($m_{LR} = m_{up}$ referred to as case(ii))[7]. While referring to the earlier paper[6] for details, the model successfully generated both the hierarchical and the inverted hierarchical (having opposite sign mass eigenvalues) neutrino mass matrices as a result of the proper choice of the parameters in texture of $M_{RR}$. In section 2 we present the generation of the degenerate as well as inverted hierarchical neutrino mass matrices using the seesaw formula, and their predictions on mass eigenvalues and mixing angles. Section 3 is devoted to summary and conclusion.
2 Neutrino mass matrices from See-saw formula

The left-handed Majorana neutrino mass matrix $m_{LL}$ is given by the celebrated see-saw formula \[8\],

$$m_{LL} = -m_{LR}M^{-1}_{RR}m_{LR}^T$$  \hspace{1cm} (1)

where $m_{LR}$ is the Dirac neutrino mass matrix in the left-right (LR) convention\[9\]. The leptonic (MNS) mixing matrix is now given by $V_{MNS} = V_{\nu L}^T$ where $m_{LL}^{\text{diag}} = V_{\nu L} m_{LL} V_{\nu L}^T$. Here both $m_{LR}$ and the charged lepton mass matrix $m_l$ are taken as diagonal, whereas the right-handed Majorana neutrino mass matrix $M_{RR}$ as non-diagonal. Using the see-saw formula (1) we generate both patterns of $m_{LL}$ viz., (I) nearly degenerate and (II) inverted hierarchical neutrino mass matrices with same sign mass eigenvalues. We concentrate here only on the cases which have bimaximal mixings with (+) entries in $m_{LL}$ listed in Table-I.

Table-I: Zeroth order neutrino mass matrices with texture zeros allowed by the result on $0\nu\beta\beta$ decay. The (+) entries in the matrices correspond to the LMA MSW solution with bimaximal mixings whereas (-) correspond to SMA MSW solution with single maximal mixing[4,10].

| Type | $m_{LL}$ | $m_{LL}^{\text{diag}}$ |
|------|----------|-----------------|
| I(A) | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{pmatrix}$ | $m_0$ | $\text{Diag}(1,\pm 1,\mp 1)m_0$ |
| I(B) | $\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ | $m_0$ | $\text{Diag}(\pm 1,1,1)m_0$ |
| II(A) | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm \frac{1}{2} & \pm \frac{1}{2} \\ 0 & \pm \frac{1}{2} & \pm \frac{1}{2} \end{pmatrix}$ | $m_0$ | $\text{Diag}(1,\pm 1,0)m_0$ |
| II(B) | $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ | $m_0$ | $\text{Diag}(1,-1,0)m_0$ |
I(A). Nearly degenerate mass matrix with opposite sign mass eigenvalues:

The most common texture for the nearly degenerate mass matrix $m_{LL}$ with opposite mass eigenvalues is represented by

$$m_{LL} = \begin{pmatrix}
(1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_1 \\
-\delta_1 & -\delta_2 & (1 - \delta_2) \\
-\delta_1 & (1 - \delta_2) & -\delta_2
\end{pmatrix} m_0 \quad (2)$$

where $m_0$ controls the overall magnitude of the masses of the neutrinos whereas $\delta_1$ and $\delta_2$ give the desired splittings for solar and atmospheric data. When $\delta_1 = \delta_2 = 0$, Eq.(2) reduces to the zeroth order mass matrix of the Type I(A) in Table-I, with no splittings[5,10].

The diagonalisation of $m_{LL}$ in Eq.(2) leads to the following eigenvalues and mixing angles [see Appendix]:

- $m_{\nu_1} \simeq (1 - 2\delta_2 - (\sqrt{3} + 1)\delta_1)m_0$,
- $m_{\nu_2} \simeq (1 - 2\delta_2 + (\sqrt{3} - 1)\delta_1)m_0$,
- $m_{\nu_3} \simeq -m_0$,
- $\sin^2 2\theta_{12} = \frac{2}{3}$, $\sin^2 2\theta_{23} = 1$, $\sin^2 2\theta_{13} = 0$.

For the choice[5] of the values of the parameters $m_0 = 0.4 \text{ eV}$, $\delta_1 = 3.6 \times 10^{-5}$, $\delta_2 = 3.9 \times 10^{-3}$, Eq.(2) leads to the following numerical predictions

**Mixing angles**:  
$\sin^2 2\theta_{12} = 0.67$, $\sin^2 2\theta_{23} \approx 1.0$, $|V_{e3}| = 1.5 \times 10^{-14}$,

**Mass eigenvalues**:  
$m_{\nu_i} = (0.39684, 0.396892, -0.4) \text{ eV}$, $i = 1, 2, 3$; $\Delta m^2_{12} = 4.13 \times 10^{-5}\text{eV}^2$ and $\Delta m^2_{23} = 2.48 \times 10^{-3}\text{eV}^2$.

The prediction on solar mixing angle is consistent with the LMA MSW solution[3].

**Case (i) where $m_{LR} = tan\beta m_i$[6,7]**

The degenerate mass matrix $m_{LL}$ in Eq.(2) is now generated through see-saw formula[8] in Eq.(1) for the following choices of $m_{LR}$ and $M_{RR}$ re-
respectively:
\[ m_{LR} = \tan \beta \begin{pmatrix} \lambda^6 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_\tau, \tag{3} \]

and
\[ M_{RR} = \begin{pmatrix} (1 + 2\delta_1 + 2\delta_2)\lambda^{12} & \delta_1\lambda^8 & \delta_1\lambda^6 \\ \delta_1\lambda^8 & \delta_2\lambda^4 & (1 + \delta_2)\lambda^2 \\ \delta_1\lambda^6 & (1 + \delta_2)\lambda^2 & \delta_2 \end{pmatrix} v_R. \tag{4} \]

The inverse of \( M_{RR} \) has a simple form for \( \delta_2 > \delta_1 \) and \( \delta_1, \delta_2 \ll 1, \)
\[ M_{RR}^{-1} = \begin{pmatrix} (1 - 2\delta_1 - 2\delta_2)\lambda^{-12} & -\delta_1\lambda^{-8} & -\delta_1\lambda^{-6} \\ -\delta_1\lambda^{-8} & -\delta_2\lambda^{-4} & (1 - \delta_2)\lambda^{-2} \\ -\delta_1\lambda^{-6} & (1 - \delta_2)\lambda^{-2} & -\delta_2 \end{pmatrix} (v_R^{-1}). \]

The expression for \( m_0 \) in Eq.(2) is worked out as \( m_0 = m_t^2\tan^2 \beta/v_R. \) For input values of \( m_0 = 0.4 \text{eV}, \tan \beta = 40, m_\tau = 1.7 \) GeV, and \( \lambda = 0.22, \) the see-saw scale is calculated as \( v_R \approx 10^{13} \) GeV. This in turn gives the masses of the three right-handed Majorana neutrinos after the diagonalization of \( M_{RR}: \)
\[ |M_{RR}^{diag}| = (4.67 \times 10^{11}, 1.296 \times 10^5, 5.06 \times 10^4) \text{ GeV}. \]

**Case(ii) where \( m_{LR} = m_{up}[6,7] \)**

The texture of \( m_{LL} \) in Eq.(2) is again realised through the see-saw formula (1) with the following textures of \( m_{LR} \) and \( M_{RR}: \)
\[ m_{LR} = \begin{pmatrix} \lambda^8 & 0 & 0 \\ 0 & \lambda^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_t, \tag{5} \]

and
\[ M_{RR} = \begin{pmatrix} (1 + 2\delta_1 + 2\delta_2)\lambda^{16} & \delta_1\lambda^{12} & \delta_1\lambda^8 \\ \delta_1\lambda^{12} & \delta_2\lambda^8 & (1 + \delta_2)\lambda^4 \\ \delta_1\lambda^8 & (1 + \delta_2)\lambda^4 & \delta_2 \end{pmatrix} v_R. \tag{6} \]

We have \( m_0 = m_t^2/v_R \) in Eq.(2), and with the input values \( m_0 = 0.4 \text{eV}, m_t = 200 \) GeV, we obtain \( v_R = 10^{14} \) GeV and the mass eigenvalues of the right-handed Majorana neutrinos: \[ |M_{RR}^{diag}| = (1.105 \times 10^{11}, 3.035 \times 10^3, 5.005 \times 10^{11}) \text{ GeV}. \]
I(B). Nearly degenerate mass matrix with same sign mass eigenvalues:

We propose another form of the nearly degenerate mass matrix,

\[ m_{LL} = \begin{pmatrix}
(1 - 2\delta_1 - 2\delta_2) & -\delta_1 & -\delta_2 \\
-\delta_2 & (1 - \delta_2) & -\delta_2 \\
-\delta_2 & -\delta_2 & (1 - \delta_2)
\end{pmatrix} m_0 \]  

(7)

The diagonalisation of \( m_{LL} \) in Eq.(7) leads to [see Appendix]

\[ m_{\nu_1} \simeq (1 - 2\delta_2 - (\sqrt{3} + 1)\delta_1)m_0, \]
\[ m_{\nu_2} \simeq (1 - 2\delta_2 + (\sqrt{3} - 1)\delta_1)m_0, \]
\[ m_{\nu_3} \simeq m_0, \]
\[ \sin^2 2\theta_{12} = \frac{2}{3}, \sin^2 2\theta_{23} = 1, \sin^2 2\theta_{13} = 0. \]

The numerical solution leads to \( m_{\nu_i} = (0.39684, 0.396892, 0.4) \) eV, \( i = 1, 2, 3 \) for the same choices of the input values of \( \delta_{1,2} \) and \( m_0 \) as in Eq.(2). Further, the predictions on the three mixing angles are the same as in Eq.(2). When \( \delta_1 = \delta_2 = 0 \), it reduces to the type I(B) in the Table-I.

This form of the mass matrix(7) can be realised in the see-saw mechanism(1) using the following textures of \( m_{LR} \) and \( M_{RR} \):

Case(i) where \( m_{LR} = \tan \beta m_l \)

Here \( m_{LR} \) is given in Eq.(3) and the right-handed neutrino mass matrix takes the form

\[ M_{RR} = \begin{pmatrix}
(1 + 2\delta_1 + 2\delta_2)\lambda^{12} & \delta_1\lambda^8 & \delta_1\lambda^6 \\
\delta_1\lambda^8 & (1 + \delta_2)\lambda^4 & \delta_2\lambda^2 \\
\delta_1\lambda^6 & \delta_2\lambda^2 & (1 + \delta_2)
\end{pmatrix} v_R \]  

(8)

The inverse of \( M_{RR} \) has a simple form for \( \delta_2 > \delta_1 \) and \( \delta_1, \delta_2 << 1 \),

\[ M_{RR}^{-1} = \begin{pmatrix}
(1 - 2\delta_1 - 2\delta_2)\lambda^{-12} & -\delta_1\lambda^{-8} & -\delta_1\lambda^{-6} \\
-\delta_1\lambda^{-8} & (1 - \delta_2)\lambda^{-4} & \delta_2\lambda^{-2} \\
-\delta_1\lambda^{-6} & -\delta_2\lambda^{-2} & (1 - \delta_2)
\end{pmatrix} (v_R^{-1}) \]

The expression for \( m_0 \) in Eq.(7) is again worked out as \( m_0 = m_\tau^2 \tan^2 \beta / v_R \).
Case (ii) where $m_{LR} = m_{up}$

The texture of $m_{LR}$ is given in Eq. (5), and the texture of $M_{RR}$ has the form

$$
M_{RR} = \begin{pmatrix}
(1 + 2\delta_1 + 2\delta_2)\lambda^{16} & \delta_1\lambda^{12} & \delta_1\lambda^8 \\
\delta_1\lambda^{12} & (1 + \delta_2)\lambda^8 & \delta_2\lambda^4 \\
\delta_1\lambda^8 & \delta_2\lambda^4 & (1 + \delta_2)
\end{pmatrix} v_R
$$

(9)

and $m_0 = m^2_t/v_R$.

II (A). Inverted hierarchical mass matrix with same sign mass eigenvalues:

The most general form of the inverted hierarchical mass matrix $m_{LL}$ with the same sign mass eigenvalues can be expressed as

$$
m_{LL} = \begin{pmatrix}
(1 - 2\epsilon) & -\epsilon & -\epsilon \\
-\epsilon & a & (a - \eta) \\
-\epsilon & (a - \eta) & a
\end{pmatrix} m'_0
$$

(10)

where $a = 0.5$ and $m'_0$ is the overall factor for the masses of the neutrinos. The parameters $\epsilon$ and $\eta$ give the desired splittings for solar and atmospheric data.

The diagonalisation of $m_{LL}$ in Eq. (10) leads to the following eigenvalues and mixing angles [see Appendix]:

\begin{align*}
m_{\nu_1} & \simeq (1 - (\sqrt{3} + 1)\epsilon - \frac{2}{3} + \frac{\sqrt{\epsilon}}{6})m'_0, \\
m_{\nu_2} & \simeq (1 + (\sqrt{3} - 1)\epsilon - \frac{2}{3} - \frac{\sqrt{\epsilon}}{6})m'_0, \\
m_{\nu_3} & \simeq \eta m'_0,
\end{align*}

and mixing angles:

$$
\sin^2 2\theta_{12} = \frac{2}{3}, \quad \sin^2 2\theta_{23} = 1, \quad \sin^2 2\theta_{13} = 0.
$$

When $\epsilon = \eta = 0$, Eq. (10) reduces to the zeroth order mass matrix of the type II(A) in Table-I, with no solar splitting [4,10]. For solution of the LMA MSW solar data and atmospheric neutrino oscillation, we have the choice of the parameters $m'_0 = 0.05$ eV, $\epsilon = 0.002$ and $\eta = 0.0001$ leading to the following predictions:

**Mixing angles:**

$$
\sin^2 2\theta_{12} = 0.67, \quad \sin^2 2\theta_{23} \approx 1.0, \quad |V_{e3}| = 3.04 \times 10^{-13},
$$
Mass eigenvalues:

\[ m_{\nu_i} = (0.05007, 0.04973, 0.000005) \text{ eV}, \ i = 1, 2, 3; \text{ leading to } \Delta m^2_{12} = 3.393 \times 10^{-5} \text{eV}^2 \text{ and } \Delta m^2_{23} = 2.47 \times 10^{-3} \text{eV}^2. \]

Here the neutrino mass eigenvalues are of the same sign and this differs from the other inverted hierarchical mass matrix having the following form\[11,\]

\[ m_{LL} = \begin{pmatrix} \epsilon & 1 & 1 \\ 1 & \delta_1 & \delta_2 \\ 1 & \delta_2 & \delta_1 \end{pmatrix} m'_0, \epsilon, \delta_1, \delta_2 << 1 \tag{11} \]

which gives opposite sign mass eigenvalues. For \( \delta_1, \delta_2, \epsilon = 0, \) it leads to the type II(B) in Table-I. This structure has been successfully generated within this model\[6,\], and the radiative correction has been found to be weak\[12,13,\]. We shall not address again this model here. Instead, we concentrate on the generation of the texture of \( m_{LL} \) given in Eq.(10) from the see-saw mechanism (1).

Case(i) when \( m_{LR} = \tan\beta m_l \)

The inverted hierarchical mass matrix \( m_{LL} \) in Eq.(10) can now be realised with the choice of \( m_{LR} = \tan\beta m_l \) given in Eq.(3) and \( M_{RR} \) of the following form

\[ M_{RR} = \begin{pmatrix} 2a\eta(1+2\epsilon)\lambda^{12} & \eta\epsilon\lambda^8 & \eta\epsilon\lambda^6 \\ \eta\epsilon\lambda^8 & a\lambda^4 & -(a-\eta)\lambda^2 \\ \eta\epsilon\lambda^6 & -(a-\eta)\lambda^2 & a \end{pmatrix} \frac{v_R}{2a\eta} \tag{12} \]

where \( M^{-1}_{RR} \) has a simple form for \( \epsilon > \eta \) and \( \epsilon, \eta << 1, \)

\[ M^{-1}_{RR} = \begin{pmatrix} (1-2\epsilon)\lambda^{-12} & -\epsilon\lambda^{-8} & -\epsilon\lambda^{-6} \\ -\epsilon\lambda^{-8} & a\lambda^{-4} & (a-\eta)\lambda^{-2} \\ -\epsilon\lambda^{-6} & (a-\eta)\lambda^{-2} & a \end{pmatrix} (v^{-1}_R) \]

The expression for \( m'_0 \) in Eq.(10) is given by \( m'_0 = m^2_e\tan^2\beta/v_R. \) For input values of \( m'_0 = 0.05 \text{ eV}, \tan\beta = 5, m_e = 1.7 \text{ GeV}, \) we obtain \( v_R = 1.445 \times 10^{12} \text{ GeV which leads to } |M_{RR}^{diag}| = (9.742 \times 10^3, 2.831 \times 10^4, 7.24 \times 10^{15}) \text{ GeV}. \)

Case(ii) when \( m_{LR} = m_{up} \)

The texture of \( m_{LL} \) in Eq.(10) is again realised using \( m_{LR} = m_{up} \) given in
Eq. (5) and the texture of $M_{RR}$:

$$M_{RR} = \begin{pmatrix}
2a\eta(1 + 2\epsilon)\lambda^{16} & \eta\epsilon\lambda^{12} & -\eta\epsilon\lambda^{8} \\
\eta\epsilon\lambda^{12} & a\lambda^{8} & -(a - \eta)\lambda^{4} \\
\eta\epsilon\lambda^{8} & -(a - \eta)\lambda^{4} & a
\end{pmatrix} \frac{v_R}{2a\eta} \quad (13)$$

which leads to $m'_0 = m_t^2/v_R$ in Eq. (10). Using the input values $m'_0 = 0.05\text{eV}$, $m_t = 200\text{GeV}$, we have $v_R = 8 \times 10^{14} \text{ GeV}$ and $|M_{RR}^{\text{diag}}| = (2.4 \times 10^4, 4 \times 10^{18}, 2.4 \times 10^9) \text{ GeV}$ where the mass of the heaviest right-handed Majorana neutrino lies above the GUT scale but below the Planck scale[10].

In the above calculation we have confined to the (+) entries of $m_{LL}$ in Table-I and these are consistent with the LMA MSW solution of the solar neutrino problem. The (-) entries in the texture of $m_{LL}$ in Table-I which generally lead to SMA MSW solar neutrino oscillation solution, can also be worked out in this model under the same framework with the proper choices of (±) signs before the elements of the $M_{RR}$.

A few comments on the stability condition under radiative corrections are in order. The nearly degenerate mass matrices $m_{LL}$ in Eqs. (2) and (7) are found to be unstable under radiative corection in minimal supersymmetric standard model. The inverted hierarchical mass matrix with the same mass eigenvalues given in Eq. (10) is also found to be unstable under radiative correction. However, the inverted hierarchical mass matrix given in Eq. (11) with opposite sign mass eigenvalues is stable under radiative corectio n [12,13].

## 3 Summary

In summary, we generate the textures of the nearly degenerate as well as the inverted hierarchical left-handed Majorana neutrino mass matrices from the see-saw formula using the diagonal form of the Dirac mass matrix and non-diagonal form of the right-handed Majorana neutrino mass matrix. The predictions on lepton mixing angles $\sin^22\theta_{12} \approx 0.67$, $\sin^22\theta_{23} \approx 1.0$ and $|V_{e3}| \approx 0$ are in excellent agreement with the experimental values. This is also true for the predictions of $\Delta m^2_{12}$ and $\Delta m^2_{23}$ which are necessary for the $0\nu\beta\beta$ decays, LMA MSW solar oscillation and atmospheric oscillation data. In all cases the masses of the right-handed Majorana neutrinos are above the weak scale. An interesting observation is that the prediction on the solar mixing angle is large but not maximal without any extra fine-tuning. Though the present analysis is a model independent one, it may be a useful guide
for building models under the framework of grand unified theories with the extended flavour U(1) symmetry.

Appendix

Diagonalisation of the mass matrix $m_{LL}$

Although the diagonalisation of $m_{LL}$ is trivial[5], we find it convenient to follow the general result given in Ref.[15]. The neutrino mass matrix $m_{LL}$ of the general form

$$m_{LL} = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

where $c = -t_{23}b$ and $t_{23} = \frac{\sin \theta_{23}}{\cos \theta_{23}}$, $f = d + (t_{23}^{-1} - t_{23})e$ can be diagonalised by $V_{MNS}$

$$V_{MNS} = \begin{pmatrix} \cos \theta_{12} & \sin \theta_{12} & 0 \\ -\cos \theta_{23} \sin \theta_{12} & \cos \theta_{23} \sin \theta_{12} & \sin \theta_{23} \\ \sin \theta_{23} \sin \theta_{12} & -\sin \theta_{23} \cos \theta_{12} & \cos \theta_{23} \end{pmatrix}$$

where we have taken $\sin \theta_{13} = 0$. This mixing MNS matrix transforms $|\nu_i>$ with the masses $(m_{\nu_1}, m_{\nu_2}, m_{\nu_3})$ into $|\nu_f>$ via $|\nu_f> = V_{MNS}|\nu_i>$, $f = e, \mu, \tau$ and $i = 1, 2, 3$. The mass eigenvalues and $\theta_{12}$ are calculated as

$$m_{\nu_1} = a - \frac{1}{2} \sqrt{\frac{b^2 + c^2}{2}} (x + \eta \sqrt{x^2 + 8}), m_{\nu_2} = (\eta \rightarrow -\eta \text{ in } m_{\nu_1}),$$

$$m_{\nu_3} = d + t_{23}^{-2} (d - a + x \sqrt{\frac{b^2 + c^2}{2}}), \sin^2 2\theta_{12} = \frac{8}{8 + x^2}$$

with $x = (a - d + t_{23}e) / (\sqrt{\frac{b^2 + c^2}{2}})$ where $|m_{\nu_1}| < |m_{\nu_2}|$ is always maintained by adjusting the sign of $\eta(= \pm 1)$.

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