FRAMES FOR SPACES OF PALEY-WIENER FUNCTIONS ON Riemannian Manifolds

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ABSTRACT. It is shown that Paley-Wiener functions on Riemannian manifolds of bounded geometry can be reconstructed in a stable way from some countable sets of their inner products with certain distributions of compact support. A reconstruction method in terms of frames is given which is a generalization of the classical result of Duffin-Schaeffer about exponential frames on intervals. All results are specified in the case of the two-dimensional hyperbolic space in its Poincare upper half-plane realization.

1. Introduction

A function \( f \in L_2(R) \) is called \( \omega \)-bandlimited if its \( L_2 \)-Fourier transform

\[
\hat{f}(t) = \int_{-\infty}^{+\infty} f(x)e^{-2\pi ixt}dx
\]

has support in \([-\omega, \omega]\).

The Paley-Wiener theorem states that \( f \in L_2(R) \) is \( \omega \)-bandlimited if and only if \( f \) is an entire function of exponential type not exceeding \( 2\pi\omega \). \( \omega \)-bandlimited functions form the Paley-Wiener class \( PW_\omega \) and often called Paley-Wiener functions.

The classical sampling theorem says, that if \( f \) is \( \omega \)-bandlimited then \( f \) is completely determined by its values at points \( j/2\omega, j \in \mathbb{Z} \), and can be reconstructed in a stable way from the samples \( f(j/2\omega) \), i.e.

\[
f(x) = \sum_{j \in \mathbb{Z}} f\left(\frac{j}{2\omega}\right) \sin(2\pi\omega(x - j/2\omega)) \frac{1}{2\pi\omega(x - j/2\omega)},
\]

where convergence is understood in the \( L_2 \)-sense. Moreover, the following equality between "continuous" and "discrete" norms holds true

\[
\left(\int_{-\infty}^{+\infty} |f(x)|^2dx\right)^{1/2} = \left(\frac{1}{2\omega} \sum_{j \in \mathbb{Z}} |f(j/2\omega)|^2\right)^{1/2}.
\]

This equality follows from the fact that the functions \( e^{2\pi it(j/2\omega)} \) form an orthonormal basis in \( L_2[-\omega, \omega] \).

The formulas (1.1) and (1.2) involve regularly spaced points \( j/2\omega, j \in \mathbb{Z} \). If one would like to consider irregular sampling at a sequence of points \( \{x_j\} \) and still have

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a stable reconstruction from the samples \( f(x_j) \) then instead of equality (1.2) the following Plancherel-Polya inequality should hold true

\[
C_1 \sum_{j \in \mathbb{Z}} |f(x_j)|^2 \leq \int_{-\infty}^{+\infty} |f(x)|^2 \, dx \leq C_2 \sum_{j \in \mathbb{Z}} |f(x_j)|^2.
\]

Such inequalities are also known as the frame inequalities.

Since the support of the Fourier transform \( \hat{f} \) is in \([-\omega, \omega]\) this inequality can be written in the following form

\[
C_1 \sum_{j \in \mathbb{Z}} \left| \int_{-\omega}^{+\omega} \hat{f}(t) e^{2\pi i tx_j} \, dt \right|^2 \leq \int_{-\omega}^{+\omega} |\hat{f}(t)|^2 \, dt \leq C_2 \sum_{j \in \mathbb{Z}} \left| \int_{-\omega}^{+\omega} \hat{f}(t) e^{2\pi i tx_j} \, dt \right|^2,
\]

that means that the functions \( \{e^{2\pi i tx_j}\} \) form a kind of a basis (not necessary orthogonal), which is called a frame in the space \( L_2[-\omega, \omega]\).

There is a remarkable result of Duffin and Schaeffer [4], that the inequalities (1.3) imply existence of a dual frame \( \{\theta_j\} \) which consists of \( \omega \)-bandlimited functions such that any \( \omega \)-bandlimited function can be reconstructed according to the following formula

\[
f(x) = \sum_{j \in \mathbb{Z}} f(x_j) \theta_j(x),
\]

which is a generalization of the formula (1.1).

From this point of view the irregular sampling was considered in the classical paper of Duffin and Schaeffer [4] in which they show that for the so-called uniformly dense sequences of scalars \( \{x_j\}, x_j \in \mathbb{R} \), the exponentials \( \{e^{ix_j t}\} \) form frames in appropriate spaces \( L_2[-\omega, \omega]\). In fact it was a far going development of some ideas of Paley and Wiener [17] about irregular sampling. The theory of irregular sampling was very active for many years [1], [2], [13], and it is still active now [16], [14].

The goal of the present article is to construct certain frames in spaces of Paley-Wiener functions on Riemannian manifolds and to show existence of reconstruction formulas of the type (1.5). Our main Theorem 3.2 is a generalization of the Duffin-Schaeffer result to the case of a Riemannian manifold of bounded geometry and the Theorem 4.4 is a specification of this general result to the case of the two-dimensional hyperbolic space in its upper-half plane realization.

The notion of Paley-Wiener functions on manifolds were introduced in the papers [18]–[26]. A subspace of Paley-Wiener functions \( PW_\omega(M) \) on a Riemannian manifold \( M \) of bounded geometry consist of all \( L_2(M) \) functions whose image in the spectral representation of the Laplace-Beltrami operator \( \Delta \) has support in the interval \([0, \omega]\).

In [24], [26] a version of the Paley-Wiener theorem was shown and an irregular sampling Theorem on manifolds was proven. In the paper [25] a similar theory was developed in the context of a general Hilbert space. The cases of \( \mathbb{R}^d \) and subelliptic cases on stratified Lie groups were considered in [21], [22], [23]. In all these situations the reconstruction algorithms were based one the notion of variational splines on manifolds and in Hilbert spaces. In the paper [27] an iterative reconstruction algorithm was introduced.

The results of the present paper are different from our previous results in the sense that
1) we use the notion of a frame to introduce a method of reconstruction of Paley-Wiener functions on manifolds;
2) we consider a kind of the ”derivative sampling” which means that we reconstruct functions from certain sets of weighted average values of $(1 + \Delta)^k f, k \in \mathbb{N}$.

Note that interesting results in a similar direction on locally compact groups were obtained recently by H. Führ [8] and H. Führ and K. Gröchenig [9].

In the case of the hyperbolic space we consider also reconstruction from pure ”derivatives” $\Delta^k f$, for any fixed natural number $k > 0$. It is interesting to note that such result would be impossible to obtain on a general manifold. Moreover, this result does not hold true even in the case of $\mathbb{R}^d$. The reason why a Paley-Wiener function $f$ on the hyperbolic plane can be reconstructed from its pure derivative $\Delta^k f$ is, that the Laplace-Beltrami operator on the hyperbolic plane has bounded inverse. In this sense the situation on the hyperbolic plane is even ”better” than on the $\mathbb{R}^d$. Note, that situation similar to the situation on the hyperbolic plane takes place on a general non-compact symmetric space.

Some partial results in these directions will appear in [27].

In the next section 2 some preliminary information on the subject is given. The role of frames is explained in the section 3. An example of the hyperbolic plane in its Poincare upper half-plane realization is given in the section 4.

2. Plancherel-Polya inequalities for Paley-Wiener functions on Riemannian manifolds of bounded geometry

Let $M$, $\dim M = d$, be a connected $C^\infty$-smooth Riemannian manifold with a $(2,0)$ metric tensor $g$ that defines an inner product on every tangent space $T_x(M), x \in M$. The corresponding Riemannian distance $d$ on $M$ is the function $d : M \times M \to R_+ \cup \{0\}$, which is defined as

$$d(x,y) = \inf_a \int_a^b \sqrt{g(\frac{dx}{dt}, \frac{dx}{dt})} dt,$$

where $\inf$ is taken over all $C^1$-curves $\alpha : [a, b] \to M, \alpha(a) = x, \alpha(b) = y$.

Let $exp_x : T_x(M) \to M$, be the exponential geodesic map i.e. $exp_x(u) = \gamma(1), u \in T_x(M)$, where $\gamma(t)$ is the geodesic starting at $x$ with the initial vector $u$ : $\gamma(0) = x, \frac{d\gamma(0)}{dt} = u$. If the $inj(M) > 0$ is the injectivity radius of $M$ then the exponential map is a diffeomorphism of a ball of radius $\rho < inj(M)$ in the tangent space $T_x(M)$ onto the ball $B(x, \rho)$. For every choice of an orthonormal (with respect to the inner product defined by $g$) basis of $T_x(M)$ the exponential map $exp$ defines a coordinate system on $B(x, \rho)$ which is called geodesic.

Throughout the paper we will consider only geodesic coordinate systems.

We make the following assumptions about $M$:

1) the injectivity radius $inj(M)$ is positive;
2) for any $\rho \leq inj(M)$, and for every two canonical coordinate systems $\vartheta_x : T_x(M) \to B(x, \rho), \vartheta_y : T_y(M) \to B(x, \rho)$, the following holds true

$$\sup_{x \in B(x, \rho)} \sup_{y \in B(y, \rho)} |\vartheta_x^{-1}\vartheta_y| \leq C(\rho, k);$$
3) the Ricci curvature $Ric$ satisfies (as a form) the inequality

$$Ric \geq -kg, \ k \geq 0.$$  

The Riemannian measure on $M$ is given in any coordinate system by

$$d\mu = \sqrt{\det(g_{ij})}dx,$$

where the $\{g_{ij}\}$ are the components of the tensor $g$ in a local coordinate system, and $dx$ is the Lebesgue’s measure in $R^d$.

The following Lemma was proved in the paper [26].

**Lemma 2.1.** For any Riemannian manifold of bounded geometry $M$ there exists a natural $N_M$ such that for any sufficiently small $\rho > 0$ there exists a set of points $\{x_i\}$ such that

1) balls $B(x_i, \rho/4)$ are disjoint,
2) balls $B(x_i, \rho/2)$ form a cover of $M$,
3) multiplicity of the cover by balls $B(x_i, \rho)$ is not greater $N_M$.

A set $\{x_i\}$ with such properties will be called a $\rho$-lattice and will be denoted $Z(x_i, \rho, N_M)$.

Let $K \subset B(x_0, \rho/2)$ be a compact subset and $\mu$ be a positive measure on $K$. We will always assume that the total measure of $K$ is finite, i.e.

$$0 < |K| = \int_K d\mu < \infty.$$

We consider the following distribution on $C_0^\infty(B(x_0, \rho))$,

$$\Phi(\varphi) = \int_K \varphi d\mu,$$

where $\varphi \in C_0^\infty(B(x_0, \rho))$. As a compactly supported distribution of order zero it has a unique continuous extension to the space $C^\infty(B(x_0, \rho))$.

Some examples of such distributions which are of particular interest to us are the following.

1) Delta functionals. In this case $K = \{x\}, x \in B(x_0, \rho/2)$, measure $d\mu$ is any positive number $\mu$ and $\Phi(f) = \mu \delta_x(f) = \mu f(x)$.

2) Finite or infinite sequences of delta functions $\delta_j, x_j \in B(x_0, \rho/2)$, with corresponding weights $\mu_j$. In this case $K = \{x_j\}$ and

$$\Phi(f) = \sum_j \mu_j \delta_{x_j}(f),$$

where we assume the following

$$0 < |K| = \sum_j |\mu_j| < \infty, K = \{x_j\}.$$  

3) $K$ is a smooth submanifold in $B(x_0, \rho/2)$ of any codimension and $d\mu$ is its "surface" measure.

4) $K$ is the closure of $B(x_0, \rho/2)$ and $d\mu$ is the restriction of the Riemannian measure $dx$ on $M$.  

We chose a lattice $Z(x_i, \rho, N_M)$ and in every ball $B(x_i, \rho/2)$ we consider a distribution $\Phi_i$ of type (2.2) with support $K_i \subset B(x_i, \rho/2)$.

We say that a family $\Phi = \{\Phi_j\}$ is uniformly bounded, if there exists a positive constant $C_\Phi$ such that
\[ |K_j| \leq C_\Phi \]
for all $j$.

We will also say that a family $\Phi = \{\Phi_j\}$ is separated from zero if there exists a constant $c_\Phi > 0$ such that
\[ |K_j| \geq c_\Phi \]
for all $j$ where $|K_j| = \int_{K_j} \mu_j$.

To construct Sobolev spaces $W^k_p(M), 1 \leq p \leq \infty, k \in \mathbb{N}$, we consider a lattice $Z(y_\nu, \lambda, N_M), \lambda < \text{inj} M$. We remained that this assumption means in particular that the multiplicity of the covers $\{B(y_\nu, \lambda)\}$ and $\{B(y_\nu, \lambda/2)\}$ is not greater than $N_M$.

We construct a partition of unity $\varphi_\nu$ that subordinate to the family $\{B(y_\nu, \lambda/2)\}$ and has the following properties.

i) $\varphi_\nu \in C_0^\infty B(y_\nu, \lambda/2)$,

ii) $\sup_x \sup_{|\alpha| \leq k} |\varphi_\nu^{(\alpha)}(x)| \leq C(k)$, where $C(k)$ is independent on $\nu$ for every $k$ in geodesic coordinates.

We introduce Sobolev space $W^k_p(M), k \in \mathbb{N}$, as the completion of $C_0^\infty(M)$ with respect to the norm
\[ \|f\|_{W^k_p(M)} = \left( \sum_\nu \|\varphi_\nu f\|_{W^k_p(B(y_\nu, \lambda/2))}^p \right)^{1/p}, k \in \mathbb{N}. \]

We will use the notation $H^k(M)$ for the space $W^k_2(M)$.

Let $\Delta$ be the Laplace-Beltrami operator on a Riemannian manifold $M, \dim M = d$, with metric tensor $g$. In any local coordinate system
\[ \Delta f = \sum_{m,k} \frac{1}{\sqrt{\det(g_{ij})}} \partial_m \left( \sqrt{\det(g_{ij})} g^{mk} \partial_k f \right) \]
where $\det(g_{ij})$ is the determinant of the matrix $(g_{ij})$.

It is known that $\Delta$ is a self-adjoint positive definite operator in the corresponding space $L_2(M, dx)$, where $dx$ is the Riemannian measure. The regularity Theorem for the Laplace-Beltrami operator $\Delta$ states that domains of the powers $\Delta^{k/2}, k \in \mathbb{N}$, coincide with the Sobolev spaces $H^k(M), k \in \mathbb{N}$, and the norm (2.5) is equivalent to the graph norm $\|f\| + \|\Delta^{k/2} f\|$.

We consider the positive square root $\Delta^{1/2}$ from the positive definite self-adjoint operator $\Delta$. According to the spectral theory [3] for a selfadjoint positive definite operator $\Delta^{1/2}$ in a Hilbert space $L_2(M)$ there exist a direct integral of Hilbert spaces $X = \int X(\lambda) d\mu(\lambda)$ and a unitary operator $F$ from $L_2(M)$ onto $X$, which transforms domain of $\Delta^{\mu/2}, \mu \geq 0$, onto $X_\mu = \{x \in X | \lambda^\mu x \in X\}$ with norm
\[
\|x(\lambda)\|_{X,\mu} = \left( \int_0^\infty \lambda^{2\mu} \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2}
\]

besides \( F(\Delta^{\mu/2} f) = \lambda^{\mu/2} (Ff) \), if \( f \) belongs to the domain of \( \Delta^{\mu/2} \). As known, \( X \) is the set of all \( m \)-measurable functions \( \lambda \to x(\lambda) \in X(\lambda) \), for which the norm

\[
\|x\|_X = \left( \int_0^\infty \|x(\lambda)\|_{X(\lambda)}^2 dm(\lambda) \right)^{1/2}
\]

is finite.

We will say that a function \( f \) from \( L_2(M) \) belong to the Paley-Wiener space \( PW_\omega \) if its "Fourier transform" \( F f \) has support in \([0, \omega]\).

The next two theorems for an abstract selfadjoint operator in a Hilbert space can be found in [24], [25].

**Theorem 2.2.** Let the \( D(\Delta^k), k \in \mathbb{N} \), be the domain of the operator \( \Delta^k \) and \( D^\infty = \bigcap_{k \in \mathbb{N}} D(\Delta^k) \). The following holds true:

a) the set \( \bigcup_{\omega > 0} PW_\omega(M) \subset D^\infty \) is dense in \( L_2(M) \);

b) for every \( \omega > 0 \) the set \( PW_\omega(M) \) is a linear closed subspace in \( L_2(M) \).

Using the spectral resolution of identity \( P_t \) of the operator \( \Delta \) we define the unitary group of operators by the formula

\[
e^{it\Delta} f = \int_0^\infty e^{it\Delta} dP_t f, f \in L_2(M).
\]

The next theorem can be considered as a form of the Paley-Wiener theorem.

**Theorem 2.3.** The following conditions are equivalent:

1) \( f \in PW_\omega(M) \);

2) for all \( s \geq 0 \) the following Bernstein inequality holds true

\[
\|\Delta^s f\| \leq \omega^{2s}\|f\|;
\]

3) for every \( g \in L_2(M) \) the scalar-valued function of the real variable \( t \in \mathbb{R}^1 \)

\[
\langle e^{it\Delta} f, g \rangle = \int_M e^{it\Delta} f \overline{g} dx
\]

is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \omega \);

4) the abstract-valued function \( e^{it\Delta} f \) is bounded on the real line and has an extension to the complex plane as an entire function of the exponential type \( \omega \).

The following Theorem follows from the Sobolev embedding theorems and from a well known result of Nelson [15] about analytic vectors.

**Theorem 2.4.** The following continuous embeddings hold true

\[
PW_\omega(M) \subset W^{m}_p(M),
\]

where \( W^{m}_p(M) \), is the Sobolev space on \( M \), \( p > 2 \), and in particular

\[
PW_\omega(M) \subset C^k_b(M),
\]

where \( C^k_b(M) \) is the space of \( k \)-differentiable bounded functions on \( M \) and \( k > d/2 \).
If, in addition, the manifold $M$ is real-analytic then every Paley-Wiener function is real-analytic.

In our paper [25] the following generalization of the Plancherel-Polya inequality was proved.

**Theorem 2.5.** For any given $C_\Phi > 0, c_\Phi > 0, m = 0, 1, 2, ..., $ there exist positive constants $C, c_1, c_2,$ such that for every $\omega > 0$, every $\rho$-lattice $Z(x_i, \rho, N_M)$ with $0 < \rho < (C\omega)^{-1}$, every family of distributions $\{\Phi_i\}$ of the form (2.2) with properties (2.3), (2.4) and every $f \in PW_{\omega}(M)$ the following inequalities hold true

$$c_1 \left( \sum_j |\Phi_j(f)|^2 \right)^{1/2} \leq \rho^{-d/2} \|f\|_{L_2(M)} \leq c_2 \left( \sum_j |\Phi_j(f)|^2 \right)^{1/2}.$$  \hspace{1cm} (2.9)

In the case of Euclidean space when $\Phi_i = \delta_{x_i}$ and $\{x_i\}$ is the regular lattice the above inequality represents the so-called Plancherel-Polya inequality.

We prove the following extension of the Theorem 2.5.

**Theorem 2.6.** For any given $C_\Phi > 0, c_\Phi > 0, m = 0, 1, 2, ..., $ there exist positive constants $C, c_1, c_2,$ such that for every $\omega > 0$, every $\rho$-lattice $Z(x_i, \rho, N_M)$ with $0 < \rho < (C\omega)^{-1}$, every family of distributions $\{\Phi_i\}$ of the form (2.2) with properties (2.3), (2.4) and every $f \in PW_{\omega}(M)$ the following inequalities hold true

$$c_1 \left( \sum_j |(1 + \Delta)^n f_j|^2 \right)^{1/2} \leq \rho^{-d/2} \|f\|_{L_2(M)} \leq c_2 \left( \sum_j |(1 + \Delta)^n f_j|^2 \right)^{1/2}.$$ \hspace{1cm} (2.10)

This implies in particular that on the space $PW_{\omega}(M)$ the $L_2(M)$ norm is equivalent to the norm of the Sobolev space $H^m(M)$ for every $m = 0, 1, 2, ....$

**Proof.** Indeed, since the space $PW_{\omega}(M)$ is invariant under the Laplace-Beltrami operator we have according to the Theorem 2.5

$$\rho^{-d/2} \| (1 + \Delta)^n f \|_{L_2(M)} \leq c_2(n) \left( \sum_j |(1 + \Delta)^n f_j|^2 \right)^{1/2}.$$ \hspace{1cm} (2.11)

But, because the operator $(1 + \Delta)^n$ has bounded inverse

$$\|f\|_{L_2(M)} = \|(1 + \Delta)^{-n}(1 + \Delta)^n f\|_{L_2(M)} \leq c(k) \|(1 + \Delta)^n f\|_{L_2(M)}.$$ \hspace{1cm} (2.12)

These two inequalities imply the second part of the inequality (2.10).

By the same inequality (2.9) we have

$$c_1 \left( \sum_j |(1 + \Delta)^n f_j|^2 \right)^{1/2} \leq \rho^{-d/2} \|(1 + \Delta)^n f\|_{L_2(M)},$$

and then the Bernstein inequality for $f$ gives the left side of the inequality (2.10). \hspace{1cm} $\Box$
3. Uniqueness, stability and reconstruction in terms of frames

We consider the set of distributions of the form

\[ \Phi_j^{(n)} = (1 + \Delta)^n \Phi_j, n \in \mathbb{N} \cup \{0\}, \]

where \( n \in \mathbb{N} \cup \{0\} \), is a fixed number and

\[ (1 + \Delta)^n \Phi_j(f) = \Phi_j((1 + \Delta)^n f) = \int_{K_j} (1 + \Delta)^n f \, d\mu_j. \]

We say that a set of functionals \( \Phi_j^{(n)} = \{ \Phi_j^{(n)} \} \) is a uniqueness set for \( PW_\omega(M) \), if every \( f \in PW_\omega(M) \) is uniquely determined by its values \( \{ \Phi_j^{(n)}(f) \} \).

For any such set \( \Phi_j^{(n)} \) and any \( \omega > 0 \) the notation \( l_2^\omega(\Phi_j^{(n)}) \) will be used for a linear subspace of all sequences \( \{ v_j \} \) in \( l_2 \) for which there exists a function \( f \) in \( PW_\omega(M) \) such that

\[ \Phi_j^{(n)}(f) = v_j. \]

In general \( l_2^\omega(\Phi_j^{(n)}) \neq l_2 \).

**Definition 3.1.** A linear reconstruction method \( R \) for a set \( \Phi_j^{(n)} = \{ \Phi_j^{(n)} \} \) is a linear operator

\[ R : l_2^\omega(\Phi_j^{(n)}) \rightarrow PW_\omega(M) \]

such that

\[ R : \{ \Phi_j^{(n)}(f) \} \rightarrow f. \]

The reconstruction method is said to be stable, if it is continuous in the topologies induced respectively by \( l_2 \) and \( L_2(M) \).

**Theorem 3.2.** For the given \( n \in \mathbb{Z}, C_\Phi > 0, c_\Phi > 0 \), there exists a constant \( C > 0 \) such that for any \( \omega > 0 \), any \( \rho < (C\omega)^{-1} \), any lattice \( Z(x_j, \rho, N_M) \), and any family of distributions \( \Phi = \{ \Phi_j \} \) of the type (2.2) that satisfy (2.3) and (2.4) with the given \( C_\Phi, c_\Phi \), every function \( f \in PW_\omega(M) \) is uniquely defined by the set of samples \( \{ \Phi_j((1 + \Delta)^n f) \}_{j \in \mathbb{Z}} \), in other words, if for a \( \{ v_j \} \in l_2 \), there exists a function \( f \in PW_\omega(M) \) such that \( \Phi_j((1 + \Delta)^n f) = v_j \) for all \( j \), then such function is unique.

Moreover, any reconstruction method from the set of samples \( \{ \Phi_j^{(n)}(f) \}_{j \in \mathbb{Z}} \) is stable.

The proof of the Theorem is an immediate consequence of the Plancherel-Polya inequalities (2.10).

Next, using the idea and the method of Duffin and Schaeffer [4] we are going to describe a stable method of reconstruction of a function \( f \in PW_\omega(M) \) from the samples

\[ \{ \Phi_j((1 + \Delta)^n f) \}_{j \in \mathbb{Z}} \in l_2 \]

**Theorem 3.3.** For the given \( n \in \mathbb{Z}, C_\Phi > 0, c_\Phi > 0 \), there exists a constant \( C > 0 \) such that for any \( \omega > 0 \), any \( \rho < (C\omega)^{-1} \), any lattice \( Z(x_j, \rho, N_M) \), and any family of distributions \( \Phi = \{ \Phi_j \} \) of the type (2.2) that satisfy (2.3) and (2.4) with the given \( C_\Phi, c_\Phi \), the following statement holds true:
there exists a frame \( \{ \Theta_j^{(n)} \} \) in the space \( PW_\omega(M) \) such that every \( \omega \)-band limited function \( f \in PW_\omega(M) \) can be reconstructed from a set of samples \( \{ \Phi_j ((1 + \Delta)^{(n)}(f)) \} \in l_2 \) by using the formula
\[
(3.1) \quad f = \sum_j \Phi_j (\Phi_j f) \Theta_j^{(n)}.
\]

Proof. For the functional
\[
f \to \Phi_j (\Phi_j f),
\]
defined on the space \( PW_\omega(M) \) we will use the notation \( \Phi_j^{(n)} \). The definition of the functionals \( \Phi_j \) and the Bernstein inequality for functions from the space \( PW_\omega(M) \) imply that every such functional is continuous on \( PW_\omega(M) \). By the Riesz theorem there are functions \( \phi_j^{(n)} \in PW_\omega(M) \) such, that for every \( f \in PW_\omega(M) \)
\[
\Phi_j^{(n)}(f) = \langle \phi_j^{(n)}, f \rangle = \int_M \phi_j^{(n)} f
\]
Moreover, the inequalities (2.10) show that the set of functionals \( \{ \Phi_j^{(n)} \} \) is a frame in the space \( PW_\omega(M) \).

The next goal is to show that the so-called frame operator
\[
(3.2) \quad Ff = \sum_j \langle \phi_j^{(n)}, f \rangle \phi_j^{(n)}, \quad f \in PW_\omega(M),
\]
is an automorphism of the space \( PW_\omega(M) \) onto itself and
\[
\|F\| \leq c_2, \quad \|F^{-1}\| \leq c_1^{-1},
\]
where \( c_1, c_2 \) are from (2.10). Let us introduce the operator
\[
F_j : PW_\omega(M) \to PW_\omega(M),
\]
which is given by the formula
\[
F_j f = \sum_{j \leq J} \langle f, \phi_j^{(n)} \rangle \phi_j^{(n)}, \quad f \in PW_\omega(M).
\]
By the frame inequalities (2.10) and the Holder inequality we have
\[
\|F_{j_1} f - F_{j_2} f\|^2 \leq \sup_{\|h\|=1} \sum_{j_1 < j \leq J_2} \left| \sum_{j_1 < j \leq J_2} \langle f, \phi_j^{(n)} \rangle \langle \phi_j^{(n)}, h \rangle \right|^2 \leq c_2 \sum_{j_1 < j \leq J_2} \left| \langle f, \phi_j^{(n)} \rangle \right|^2.
\]
By the same frame inequality the right side goes to zero when \( J_1, J_2 \) go to infinity. Thus the limit
\[
\lim_{j \to \infty} F_j f = Ff, \quad f \in PW_\omega(M),
\]
does exists. Next,
\[
\|Ff\|^2 = \sup_{\|h\|=1} \left| \sum_j \langle f, \phi_j^{(n)} \rangle \langle \phi_j^{(n)}, h \rangle \right|^2 \leq \sup_{\|h\|=1} c_2^2 \|f\|^2 \|h\|^2 = c_2^2 \|f\|^2,
\]
which shows that the operator \( F \) is continuous.

Now, the frame inequalities (2.10) imply that
\[
c_1 I \leq F \leq c_2 I,
\]
where $I$ is the identity operator. Thus, we have

$$c_2^{-1} F \leq I, \quad c_2^{-1} F^{-1} \geq c_1 c_2^{-1} I,$$

and then

$$0 \leq I - c_2^{-1} F \leq I - c_1 c_2^{-1} I = (c_2 - c_1) c_2^{-1} I.$$ 

It implies

$$\|I - c_2^{-1} F\| \leq \|(c_2 - c_1) c_2^{-1} I\| \leq (c_2 - c_1) c_2^{-1} < 1.$$ 

Thus, it shows that the operator $(c_2^{-1} F)^{-1}$ and consequently the operator $F^{-1}$ are well defined bounded operators and because

$$F^{-1} = c_2^{-1} (C_2^{-1} F)^{-1} = c_2^{-1} \sum_{m=0}^{\infty} (I - c_2^{-1} F)^m,$$

it gives us the desired estimate $\|F^{-1}\| \leq c_1^{-1}$.

We obtain

$$f = F^{-1} F f = F^{-1} \lim_{J \rightarrow \infty} \sum_{j \leq J} \langle f, \phi_j^{(n)} \rangle \phi_j^{(n)} = \sum_j \langle f, \phi_j^{(n)} \rangle \Theta_j^{(n)},$$

where

$$\Theta_j^{(n)} = F^{-1} \phi_j^{(n)},$$

gives a dual frame $\Theta_j^{(n)}$ in the space $PW_\omega(M)$.

The Theorem is proven. 

\[ \square \]

4. The Poincare Upper Half-plane

As an illustration of our results we will consider the hyperbolic plane in its upper half-plane realization.

Let $G = SL(2, \mathbb{R})$ be the special linear group of all $2 \times 2$ real matrices with determinant 1 and $K = SO(2)$ is the group of all rotations of $\mathbb{R}^2$. The factor $\mathbb{H} = G/K$ is known as the 2-dimensional hyperbolic space and can be described in many different ways. In the present paper we consider the realization of $\mathbb{H}$ which is called Poincare upper half-plane (see [10], [11], [30]).

As a Riemannian manifold $\mathbb{H}$ is identified with the regular upper half-plane of the complex plane

$$\mathbb{H} = \{x + iy | x, y \in \mathbb{R}, y > 0\}$$

with a new Riemannian metric

$$ds^2 = y^{-2}(dx^2 + dy^2)$$

and corresponding Riemannian measure

$$d\mu = y^{-2} dxdy.$$ 

If we define the action of $\sigma \in G$ on $z \in \mathbb{H}$ as a fractional linear transformation

$$\sigma \cdot z = (az + b)/(cz + d),$$

then the metric $ds^2$ and the measure $d\mu$ are invariant under the action of $G$ on $\mathbb{H}$. The point $i = \sqrt{-1} \in \mathbb{H}$ is invariant for all $\sigma \in K$. The Haar measure $dg$ on $G$ can be normalizes in a way that the following important formula holds true

$$\int_\mathbb{H} f(z) y^{-2} dxdy = \int_G f(g \cdot i) dg.$$
In the corresponding space of square integrable functions $L_2(G)$ with the inner product
\[ < f, h > = \int_G f \overline{h} y^{-2} dx dy \]
we consider the Laplace-Beltrami operator
\[ \Delta = y^2 (\partial_x^2 + \partial_y^2) \]
of the metric $ds^2$.

It is known that $\Delta$ as an operator in $L_2(H) = L_2(H, d\mu)$ which initially defined on $C_0^\infty(H)$ has a self-adjoint closure in $L_2(H)$.

Moreover, if $f$ and $\Delta f$ belong to $L_2(H)$, then
\[ < \Delta f, f > \leq -\frac{1}{4} \| f \|^2, \]
where $\| f \|$ means the $L_2(H)$ norm of $f$.

The Helgason transform of $f$ for $s \in \mathbb{C}, \varphi \in (0, 2\pi]$, is defined by the formula
\[ \hat{f}(s, \varphi) = \int_H f(z) \text{Im}(k_\varphi z)^s y^{-2} dx dy, \]
where $k_\varphi \in SO(2)$ is the rotation of $\mathbb{R}^2$ by angle $\varphi$.

We have the following inversion formula for all $f \in C_0^\infty(H)$
\[ f(z) = (8\pi^2)^{-1} \int_{t \in \mathbb{R}} \int_0^{2\pi} \hat{f}(it + 1/2, \varphi) \text{Im}(k_\varphi z)^it + 1/2 t \tanh \pi t d\varphi dt. \]

The Plancherel Theorem states that a such defined map $f \to \hat{f}$ can be extended to an isometry of $L_2(H)$ with respect to invariant measure $d\mu$ onto $L_2(\mathbb{R} \times (0, 2\pi])$ with respect to the measure
\[ \frac{1}{8\pi^2} t \tanh \pi t dt d\varphi. \]

If $f$ is a function on $H$ and $\varphi$ is a $K = SO(2)$-invariant function on $H$ their convolution is defined by the formula
\[ f \ast \varphi(g \cdot i) = \int_{SL(2, \mathbb{R})} f(gu^{-1} \cdot i) \varphi(u) du, \]
where $du$ is the Haar measure on $SL(2, \mathbb{R})$. It is known, that for the Helgason-Fourier transform the following formula holds true
\[ \hat{f} \ast \varphi = \hat{f} \cdot \hat{\varphi}. \]

The following formula holds true
\[ (4.1) \quad \Delta \hat{f} = -\left( t^2 + \frac{1}{4} \right) \hat{f}. \]
and our Theorem 2.3 takes the following form.

**Theorem 4.1.** A function $f \in L_2(H)$ belongs to the space $PW_\omega(H)$ if and only if for every $\sigma \in \mathbb{R}$ the following holds true
\[ (4.2) \quad \| \Delta^\sigma f \| \leq \left( \omega^2 + \frac{1}{4} \right)^\sigma \| f \|. \]
As it follows from the Theorem 2.4 if $f \in PW_\omega(H)$ then for every $\sigma \geq 0$ the function $\Delta^\sigma f$ belongs to $C^\infty(H)$ and is bounded on $H$.

Moreover functions from $PW_\omega(H)$ are not just infinitely differentiable on $H$ but they are real analytic functions on the upper half-plane.

Indeed, every $f \in PW_\omega(H)$ is an analytic vector of the elliptic differential operator $\Delta$ which means [15] that the series
$$\sum_{k=0}^{\infty} \frac{\|\Delta^k f\|}{k!} s^k \leq e(\omega^2 + \frac{1}{4}) s \|f\|$$
is convergent for any $s > 0$. By the famous result of Nelson [15] it implies that the function $f$ is real analytic on $H$. It shows in particular that the support of the function $f$ is the entire half-plane $H$.

This fact can be treated as a form of the uncertainty principle on $H$:

If the support of the Helgason-Fourier transform $\hat{f}$ of a function $f \in L_2(H, y^{-2}dxdy)$ is contained in a set $(-\omega, \omega) \times (0, 2\pi]$, then the support of $f$ is the entire half-plane $H$.

By applying the Theorem 2.3 we obtain the following result.

**Theorem 4.2.** A function $f$ is an $\omega$-band-limited signal, if for any $g \in L_2(H, d\mu)$ the complex valued function
$$t \rightarrow < e^{it\Delta} f, d\mu > = \int_H e^{it\Delta} f(z) \overline{g(z)} y^{-2} dxdy$$
of the real variable $t$ is an entire function of exponential type $\omega^2 + \frac{1}{4}$ which is bounded on the real line.

It is known that the one parameter group of operators $e^{it\Delta}$ acts on functions by the following formula
$$e^{it\Delta} f(z) = f * G_t,$$
where
$$G_t(ke^{-r}i) = (4\pi)^{-1} \int_{\eta \in \mathbb{R}} e^{-i(\eta^2 + \frac{1}{4})t} P_{\eta - 1/2}(\cosh r) \eta \tan\pi \eta d\eta,$$
here $k \in SO(2)$, $r$ is the geodesic distance, $ke^{-r}i$ is representation of points of $H$ in the geodesic polar coordinate system on $H$, and $P_{\eta - 1/2}$ is the associated Legendre function.

The last Theorem can be reformulated in the following terms.

**Theorem 4.3.** A function $f \in L_2(H, d\mu)$ is $\omega$-band-limited if and only if for every $g \in L_2(H, d\mu)$ function
$$t \rightarrow < f * G_t, g > = \int_H f * G_t d\mu$$
is an entire function of the exponential type $\omega^2 + \frac{1}{4}$ which is bounded on the real line.

We are going to describe our reconstruction algorithm using the language of frames.

Note that according to the inversion formula for the Helgason-Fourier transform we have
\[ \Phi_j f(z) = (8\pi^2)^{-1} \int_{t \in \mathbb{R}} \int_0^{2\pi} \hat{f}(it + 1/2, \varphi) \Phi_j \left( \text{Im}(k \varphi z) \right)^{it+1/2} t \tanh \pi t d\varphi dt. \]

which means that the Helgason-Fourier transform of a distribution \( \Phi_j \) is given by the formula

\[ \hat{\Phi}_j = \Phi_j \left( \text{Im}(k \varphi z) \right)^{it+1/2} \]

and

\[ \Delta^n \Phi_j = \left( \omega^2 + \frac{1}{4} \right)^n \Phi_j \left( \text{Im}(k \varphi z) \right)^{it+1/2} \]

In the case of Poincaré upper half-plane it is not difficult to formulate our main Theorem 3.2. Moreover as the formula 4.1 shows the Laplacian \( \Delta \) in the space \( L_2(\mathbb{H}, y^{-2} dxdy) \) has bounded inverse which allows reconstruction from pure derivatives \( \Delta^n f \). Namely

**Theorem 4.4.** Given two constants \( 0 < c_\Phi \leq C_\Phi \) and an \( n \in \mathbb{N} \cup \{0\} \), there exists a constant \( c = c(C_\Phi, c_\Phi, N, n) > 0 \) such that for any \( \omega > 0 \), any \((\rho, N)\)-lattice \( \mathbb{Z} (x_j, \rho, N) \) with

\[ 0 < \rho < c \left( \omega^2 + \frac{1}{4} \right)^{-1/2}, \]

and every family of distributions \( \Phi = \{ \Phi_j \} \) that satisfy (2.2)-(2.4) with the given \( c_\Phi, C_\Phi \), the following statements hold true.

1) The set of analytic functions \( \{ \Delta^n \Phi_j \} \) is a frame in the space

\[ L_2 \left( [\omega, \omega] \times (0, 2\pi], \frac{1}{8\pi^2} t \tanh \pi t d\varphi \right). \]

2) There exists a frame \( \{ \Theta_j^{(n)} \} \) in the space \( PW_\omega(\mathbb{H}) \) such that every \( \omega \)-band limited function \( f \in PW_\omega(\mathbb{H}) \) can be reconstructed from a set of samples \( \{ \Phi_j (\Delta^n f) \} \) by using the formula

\[ f = \sum_j \Phi_j (\Delta^n f) \Theta_j^{(n)}. \]

When \( n = 0 \) and every \( \Phi_j \) is a Dirac measure \( \delta_{x_j} \) at a point \( x_j \in \mathbb{H} \), the Theorem can be considered as an analog of the Duffin-Schaeffer result about exponential frames, since it means that Fourier transforms of the measures \( \delta_{x_j} \) form a frame on the Fourier transform side.

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