ANALYTICAL SOLUTIONS OF SKYRME MODEL

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ABSTRACT. Exact analytic solutions of the kink soliton equation obtained in a recent interesting study of the classical Skyrme model defined on a simple spherically symmetric background are presented. By a variational method, the existence of spherically symmetric monopole solutions are proved. In particular, all finite-energy kink solitons must be Bogomool'nyi–Prasad–Sommerfield are showed. Moreover, together with numerical analysis, we can clearly see the validity of our theoretical results.

1. Introduction. Skyrme’s theory [21] is an important model in particle physics. A extraordinary property of Skyrme term is that excitations around Skyrme solitons may represent Fermionic degrees of freedom that could describe nucleons [2, 22]. The Skyrme model is therefore one of the most important nonlinear field theories in nuclear and high-energy physics. Moreover, the Skyrmions are the important objects which appear in many branches of physics such as astrophysics [19], condensed-matter physics [12], nematic liquids [11] and magnetics structures [18].

However, it is difficult to obtain exact solutions in nonlinear Skyrme models, due to their highly nonlinear characters. Therefore one often adopts a certain ansatz to make the field equations more tractable. Among others, the best known one for Skyrme models is the hedgehog ansatz for spherically symmetric systems, which reduces the field equations to a single scalar equation. In those studies, one mostly relies on numerical analyses, until very recently no exact analytic solutions of the Skyrme model with non-trivial topological charges were known. One of the reasons is that the Skyrme-BPS bound [3, 20] on the energy cannot be saturated for non-trivial spherically symmetric configurations [16, 17]. Although some partial progress has been developed for an existence theory based on nonlinear functional analysis [7, 10, 13, 14, 15]. In an interesting study of Canfora, et al [7] developed
from some recent earlier work [1, 4, 5, 6], a static ansatz describing a spherically symmetric Skyrmion in the background [6]. The second-order governing equation takes a complicated form but it may be solved by a much simpler first-order equation whose solutions interpolate adjacent minima of the one-dimensional Skyrme energy density. The importance of the work of Canfora, et al [7] is that it makes an explicit construction of kink-like solutions in the classical Skyrme model possible, which hints our research.

In the present paper, exact spherically symmetric solutions of the Skyrme model with both a non-trivial winding number and a finite soliton mass (topological charge) are studied. We will prove that the first-order equation of Canfora, et al and the second-order equation of the one-dimensional kink energy of Canfora, et al are actually equivalent under the finite-energy condition. Therefore we obtain the sharp properties of all finite-energy solutions of the Skyrme model within the Canfora, et al kink ansatz [7].

Let \( \mu, \nu \) denote the Minkowski spacetime indices with metric \( g^{\mu\nu} = \text{diag}\{-1, 1, 1, 1\} \). The action of the \( SU(2) \) Skyrme system in four dimensional space-time is

\[
\Gamma_{\text{Skyrme}} = \frac{K}{4} \int d^4x \sqrt{-g} \text{Tr}\left( R^\mu R_\mu + \frac{\lambda}{8} F^\mu_\nu F^\nu_\mu \right),
\]

where \( K, \lambda > 0 \) are the coupling parameters, \( R_\mu = U^{-1} \nabla_\mu U = R^i_\mu t_i \) for an \( SU(2) \)-valued map \( U \), \( t_i(i = 1, 2, 3) \) are generators of the Lie algebra of \( SU(2) \), and \( F^\mu_\nu = [R_\mu, R_\nu] \). The first term of the Skyrme action (1) is mandatory to describe points while the second is the only covariant term leading to second order field equations in time which supports the existence of Skyrmions in four dimensions. The \( SU(2) \)-valued scalar \( U(x^\mu) \) is represented through the expression

\[
U(x^\mu) = Y^0(x^\mu)1 + Y^i(x^\mu) t_i, \quad (Y^0)^2 + Y^i Y_i = 1,
\]

where 1 denotes the \( 2 \times 2 \) identity matrix and \( Y^0, Y^i \) are scalar functions over spacetime. Thus, the hedgehog ansatz describing a spherically symmetric Skyrmion can be written in terms of the unit vectors

\[
Y^0 = \cos \alpha, \quad Y^i = \hat{n}^i \sin \alpha, \quad \alpha = \alpha(x),
\]

and \( \hat{n}^1 = \sin \theta \cos \phi, \hat{n}^2 = \sin \theta \sin \phi \) and \( \hat{n}^3 = \cos \theta \). So, the Skyrme field equations of the action (1) (or the associated Euler–Lagrange equations) are

\[
\nabla^\mu R_\mu + \frac{\lambda}{4} \nabla^\mu [R^\nu, F_{\mu\nu}] = 0
\]

We will supply variational method analogous to that of [8, 9, 23, 24], for the Skyrme field equations, which is different from the analytic method in [7]. Our study here may provide insight into some of those more complicated problem.

The rest of this paper is organized as follows. In section 2, we prove the existence of the Skyrme model solutions by a variational method. Furthermore, we obtain the sharp properties of an energy-minimizing solution. In section 3, we show that the solutions of the second-order equations also satisfies the reduced self-dual equations, establishing the equivalence of the Skyrme field equations and the BPS monopoles equations under the radial symmetry assumption. Moreover, we demonstrate a few concrete numerical examples to illustrate the effectiveness of our results.
2. Existence of variational solution. In this section, we first consider the existence of the Skyrme model solution. The reduced one-dimensional energy functional (1) is as follows

$$E(\alpha) = 2\pi K R_0^2 \int_{-\infty}^{\infty} \left[ (\alpha')^2 + \frac{2\sin^2 \alpha}{R_0^2} + \frac{\lambda}{R_0^2} \left(2(\alpha')^2 + \frac{\sin^2 \alpha}{R_0^2}\right)\right] dx,$$  \hspace{1cm} (5)

where $R_0 > 0$ is much larger than the radius of the proton. It is clear that the Euler-Lagrange equations of (5) is

$$\left(1 + \frac{2\lambda}{R_0^2} \sin^2 \alpha\right) \alpha'' - \frac{\sin(2\alpha)}{R_0^2} \left(1 - \lambda \left((\alpha')^2 - \frac{\sin^2 \alpha}{R_0^2}\right)\right) = 0,$$  \hspace{1cm} (6)

where $\alpha' = \frac{d\alpha}{dx}$. We note that for the ansatz (3), the Skyrme field equations (4) reduce to the single ordinary differential equation for the Skyrmion profile $\alpha$ (6).

The reduced energy density is

$$H(\alpha) = \frac{K}{2} \left((\alpha')^2 + \frac{2\sin^2 \alpha}{R_0^2} + \frac{\lambda}{R_0^2} \left(2(\alpha')^2 + \frac{\sin^2 \alpha}{R_0^2}\right)\right),$$  \hspace{1cm} (7)

In view of finite-energy condition, we impose the boundary conditions

$$\alpha(-\infty) = 0, \; \alpha(\infty) = \pi.$$  \hspace{1cm} (8)

The winding number for the hedgehog ansatz in (3) is

$$W = \frac{\pi}{2} \int_{\alpha(x_1)}^{\alpha(x_2)} \sin^2 \alpha d\alpha,$$

where $(x_1, x_2)$ correspond to the limits in the range of spatial direction, that can be taken as $(-\infty, \infty)$. This winding number takes integer values $n$, for boundary conditions (8)

$$\alpha(\infty) - \alpha(-\infty) = n\pi, \; 1 = n \in \mathbb{Z}.$$  \hspace{1cm} (9)

We need to find solutions to equation (6) subject to the boundary conditions (8), which amounts to solving a two-point boundary value problem which seems difficult. We will solve the problem by the variational method.

The admissible space $A$ is defined by

$$A = \{\alpha | \alpha \text{ is absolutely continuous on every compact subinterval of } (-\infty, \infty) \}$$

so that it satisfies the boundary condition (8) and $E(\alpha) < \infty$.  \hspace{1cm} (10)

With above preparation, we can state our main results as follows.

**Theorem 2.1.** The minimization problem

$$\min\{E(\alpha) | \alpha \in A\}$$

has a solution.

**Proof.** Consider

$$E_m = \inf\{E(\alpha) | \alpha \in A\}.$$  \hspace{1cm} (11)

For $\alpha_n \in A$, we can take a minimizing sequence $\{\alpha_n\}$. Without loss of generality, we may assume that:

$$E(\alpha_n) \leq E_m + 1, \; n = 1, 2, \cdots$$  \hspace{1cm} (12)

Besides, the form of the energy $E$ given in (5) indicates that we may assume

$$0 \leq \alpha_n(x) \leq \pi, \; x \in (-\infty, \infty).$$  \hspace{1cm} (13)
We deduce from (15) and (16) that the energy-minimizing solution of Theorem 2.2.

Otherwise we may modify the sequence to fulfill the above inequality meanwhile without enlarging the energy.

We may get that the sequence \( \{\alpha_n\} \) is bounded in \( W^{1,2}(-N, N) \) for any integer \( N \geq 2 \). Using weak compactness, we may assume that \( \alpha_n \) (in fact, a subsequence in it) is weakly convergent in \( W^{1,2}(-N, N) \). Applying a diagonal subsequence argument, we may assume there is a \( \alpha \in W^{1,2}_{loc}(-\infty, \infty) \) so that

\[
\lim_{n \to \infty} \alpha_n = \alpha, \tag{14}
\]

weakly in \( W^{1,2}(-N, N) \) and strongly in \( C[-N, N] \) for any \( N = 2, 3, \ldots \). Consequently we see that \( \alpha \) is absolutely continuous in any compact subinterval of \( (-\infty, \infty) \) and obeys (13) as well. Furthermore, in order to show that \( \alpha \in \mathcal{A} \), we need to have \( \alpha(-\infty) = 0 \) and \( \alpha(\infty) = \pi \). In fact, \( 0 \leq \alpha(x) \leq \pi \), we may suppose that there is an \( -\infty < x_1 \) such that \( \alpha(x) \leq \frac{\pi}{100} \) when \( x \in (-\infty, x_1) \). Therefore we may get \( \alpha^2(x) \leq C_1 \sin^2 \alpha(x) \), where \( C_1 \) is the appropriate constant. Then

\[
\int_{-\infty}^{x_1} \alpha^2(x)dx \leq \frac{C_1 R_0^2}{2} \int_{-\infty}^{\infty} \frac{2 \sin^2 \alpha(x)}{R_0^2} dx \leq \frac{C_1}{4\pi K}(E_m + 1) < \infty. \tag{15}
\]

Meanwhile, we have

\[
\int_{-\infty}^{x_1} (\alpha')^2 dx \leq C_1 \int_{-\infty}^{\infty} (\alpha')^2 dx \leq \frac{1}{2\pi K R_0^2}(E_m + 1) < \infty. \tag{16}
\]

We deduce from (15) and (16) that \( \alpha \in W^{1,2}(-\infty, x_1) \) which immediately leads to \( \alpha(-\infty) = 0 \).

Similarly, there is a \( 0 < x_2 < \infty \) such that \( \frac{99\pi}{100} \leq \alpha(x) \leq \pi \) when \( x \in (x_2, \infty) \). Therefore we may get \( 0 < \pi - \alpha \leq \frac{\pi}{100} \) and \( (\pi - \alpha)^2 \leq C_2 \sin^2(\pi - \alpha) \), where \( C_2 \) is the appropriate constant. Then we know that \( \pi - \alpha \in W^{1,2}(x_2, \infty) \) which immediately leads to \( \alpha(\infty) = \pi \).

Let

\[
E(\alpha) = 4\pi R_0^2 \int_{-\infty}^{\infty} \mathcal{H}(\alpha) dx, \tag{17}
\]

where \( \mathcal{H}(\alpha) \) denoting the energy density defined in (7). Using the weak lower semi-continuity property of the functional we obtain the inequality

\[
4\pi R_0^2 \int_a^b \liminf_{n \to \infty} \mathcal{H}(\alpha_n) dx = 4\pi R_0^2 \int_a^b \mathcal{H}(\alpha) dx \leq \liminf_{n \to \infty} E(\alpha_n) = E_m, \tag{18}
\]

for \( -\infty < a < b < \infty \). Letting \( a \to -\infty \) and \( b \to \infty \) in (18), we have

\[
E(\alpha) = 4\pi R_0^2 \int_{-\infty}^{\infty} \mathcal{H}(\alpha) dx \leq E_m \leq E(\alpha). \tag{19}
\]

Thus we see that \( \alpha \) fulfills the complete boundary conditions (8). Therefore \( \alpha \in \mathcal{A} \), and (19) allows us to obtain \( E(\alpha) = E_m \). That is, \( \alpha \) is found to be a solution of (11). As a consequence, \( \alpha \) is a finite-energy solution of (6) and (8). The proof of theorem (2.1) is now completed.

Next, we will establish some sharp properties of energy-minimizing solution.

**Theorem 2.2.** The energy-minimizing solution of (6) and (8) \( \alpha \) satisfies:

(i) \( 0 < \alpha(x) < \pi \), for all \( x \in (-\infty, \infty) \);

(ii) \( \alpha(x) \) is strictly increasing;

(iii) \( \alpha(x) \) has the property of convex function as \( |x| \to \infty \);
(iv) $\alpha(x)$ holds the following asymptotic estimates

$$
\alpha(x) = O(e^{x\sqrt{b_1(1-\epsilon)}}), \quad \text{as } x \to -\infty, \quad (20)
$$

$$
\alpha(x) = \pi - O(e^{-x\sqrt{b_2(1-\epsilon)}}), \quad \text{as } x \to \infty, \quad (21)
$$

where $b_1 = \frac{1}{R^2 + 2\alpha}, \ b_2 = \frac{2}{R^2 + 2\alpha}$ and $\epsilon > 0$ is small enough.

**Proof.** (i) Let $\alpha$ be the energy-minimizing solution of (6)–(8), satisfying $0 \leq \alpha(x) \leq \pi, \ x \in (-\infty, \infty)$.

If there is a point $x_0 \in (-\infty, \infty)$ such that $\alpha(x_0) = 0$, then $\alpha'(x_0) = 0$ since $x_0$ is a minimum point for the function $\alpha$. Applying the uniqueness theorem of the initial value problem of an ordinary differential equation, we have $\alpha(x) = 0$ for all $x \in (-\infty, \infty)$, which contradicts the fact $\alpha(\infty) = \pi$. This proves

$$
\alpha(x) > 0, \ \forall \ x \in (-\infty, \infty). \quad (22)
$$

On the other hand, we can show that

$$
\alpha(x) < \pi, \ \forall \ x \in (-\infty, \infty). \quad (23)
$$

Indeed, if there is a point $x_0 > 0$ such that $\alpha(x_0) = \pi$, then $x_0$ is a maximum point of $\alpha$ and we have $\alpha''(x_0) < 0$. Inserting these into (6), we arrive at a contradiction. Thus (23) is valid.

(ii) We first prove that $\alpha(x)$ is a nondecreasing function.

Suppose that there is a point $-\infty < x_1$ such that $\alpha'(x) < 0$. Of course, we already have $\alpha(x) > 0$ due to (22). Since $\alpha(-\infty) = 0$, we may deduce the existence of a pair of points, $-\infty < x_2 < x_3 < \infty$ such that

$$
\alpha(x) \geq \alpha(x_3), \ x \in [x_2, x_3], \ \alpha(x_2) = \alpha(x_3), \ \alpha(x) > \alpha(x_3) \text{ for some } x \in (x_2, x_3). \quad (24)
$$

Now define

$$
\tilde{\alpha}(x) = \begin{cases} 
\alpha(x), & x \in [x_2, x_3], \\
\alpha(x_3), & x \in [x_2, x_3], 
\end{cases} \quad (25)
$$

then $\tilde{\alpha} \in \mathcal{A}$ and $E(\tilde{\alpha}) < E(\alpha)$, which is false.

Next, we will also show that $\alpha(x)$ strictly increase. Otherwise we have $-\infty < x_1 < x_2 < \infty$ so that $\alpha(x_1) = \alpha(x_2)$ which implies $\alpha(x) = \alpha(x_1)$ for all $x \in (x_1, x_2)$. This is impossible in view of (6).

(iii) We now consider the convexity of $\alpha(x)$ as $|x| \to \infty$. From (6), we have

$$
\left(1 + \frac{2\lambda}{R^2} \sin^2 \alpha \right) \alpha'' = \frac{\sin(2\alpha)}{R^2} \left(1 + \frac{\lambda \sin^2 \alpha}{R^2} - \lambda \alpha'(x)^2 \right). \quad (26)
$$

Moreover, from the above discussion, we know that $0 < \alpha(x) < \pi, \ \alpha'(x) > 0$ and $\lim_{|x| \to \infty} \alpha'(x) = 0$. Assuming $\alpha'(x)$ attains its nontrivial maximum at $x_0$, then for any $x \in (-\infty, \infty)$, we have $\alpha'(x) \leq \alpha'(x_0)$ and $\alpha''(x_0) = 0$. Hence we can get from (26)

$$
\frac{\sin(2\alpha(x_0))}{R^2} \left(1 + \frac{\lambda \sin^2 \alpha(x_0)}{R^2} - \lambda \alpha'(x_0)^2 \right) = 0,
$$

then

$$
\sin(2\alpha(x_0)) = 0, \ \alpha(x_0) = \frac{\pi}{2}. \quad (27)
$$
Or we have
\[(\alpha'(x_0))^2 = \frac{1}{\lambda} + \frac{\sin^2 \alpha(x_0)}{R_0^2} \leq \frac{1}{\lambda} + \frac{1}{R_0^2},\]
\[
\alpha'_{max} = \sqrt{\frac{1}{\lambda} + \frac{1}{R_0^2}} \text{ if and only if } \alpha(x_0) = \frac{\pi}{2}.
\]
Thus, when \(\alpha' < \alpha'_{max}\), there holds
\[
1 + \lambda \sin^2 \frac{\alpha}{R_0^2} - \lambda (\alpha')^2 > 0. \tag{29}
\]
Then, when \(x < x_0\), i.e., \(x \in (-\infty, x_0)\), we can get \(\alpha(x) < \frac{\pi}{2}\) and \(\sin(2\alpha) > 0\). In view of (29), we see that the right-hand side of (26) is positive. Hence, \(\alpha'' > 0\).

On the other hand, when \(x > x_0\), i.e., \(x \in (x_0, \infty)\), we obtain \(\frac{\pi}{2} < \alpha(x) < \pi\) and \(\sin(2\alpha) < 0\). At this time, the right-hand side of (26) is negative. Therefore we have \(\alpha'' < 0\).

Summarizing the above results, we can see that the energy-minimizing solution \(\alpha(x)\) is convex as \(x \to -\infty\) and is concave as \(x \to \infty\).

(iv) We finally derive the sharp asymptotic estimates of \(\alpha(x)\).

We first consider the decay estimate of \(\alpha(x)\) as \(x \to -\infty\). Recall the boundary condition (8), we have \(\lim_{x \to -\infty} \alpha(x) = 0\). As a result, we can get
\[
\lim_{x \to -\infty} \alpha'(x) = \lim_{x \to -\infty} \frac{\alpha(x)}{x} = 0. \tag{30}
\]
Then for any given small enough \(\epsilon > 0\), there exists sufficient large \(x_\epsilon > 0\) such that \(x < -x_\epsilon\), we have \(\alpha' < \epsilon, \sin \alpha < \epsilon\) and \(\sin 2\alpha > 2(1-\epsilon)\alpha\). Choosing a comparison function
\[
\beta_1(x) = C e^\varphi \sqrt{b_1(1-\epsilon)}, \tag{31}
\]
where \(b_1 = \frac{1}{R_0^2 + 2\lambda} > 0\), \(\epsilon > 0\) small enough and \(C > 0\) will be determined later. We have
\[
(\alpha - \beta_1)^{'''} = \frac{1}{1 + \frac{1}{R_0^2} \sin^2 \alpha} \cdot \frac{\sin(2\alpha)}{R_0^2} \left(1 + \frac{\lambda \sin^2 \alpha}{R_0^2} - \lambda (\alpha')^2\right) - \beta_1^{'''}
\geq \frac{2(1-\epsilon)}{R_0^2 + 2\lambda \alpha} \alpha - \frac{\lambda \sin(2\alpha)}{R_0^2 + 2\lambda \alpha} (\alpha')^2 - \beta_1^{'''}
\geq \frac{2(1-\epsilon)}{R_0^2 + 2\lambda \alpha^2} (\alpha - \beta_1) - \frac{2\lambda \alpha}{R_0^2} (\alpha - \beta_1)'
+ \beta_1 \left(\frac{2(1-\epsilon)}{R_0^2 + 2\lambda \alpha^2} - \frac{2\lambda \alpha}{R_0^2} \sqrt{b_1(1-\epsilon)} - b_1(1-\epsilon)\right). \tag{32}
\]
Since \(b_1 = \frac{1}{R_0^2 + 2\lambda}\), then we can get
\[
\frac{2(1-\epsilon)}{R_0^2 + 2\lambda \alpha^2} - \frac{2\lambda \alpha}{R_0^2} \sqrt{b_1(1-\epsilon)} - b_1(1-\epsilon) > 0.
\]
Hence, for any \(x < -x_\epsilon\), we obtain
\[
(\alpha - \beta_1)^{'''} \geq \frac{2(1-\epsilon)}{R_0^2 + 2\lambda \alpha^2} (\alpha - \beta_1) - \frac{2\lambda \alpha}{R_0^2} (\alpha - \beta_1)'. \tag{33}
\]
Meanwhile, for the given \(-x_\epsilon\), the value of \(\alpha(-x_\epsilon)\) is determined. Therefore, we can select suitable \(C > 0\) such that \((\alpha - \beta_1)(-x_\epsilon) \leq 0\). Then we claim that
The solution of BPS monopoles equations. To see this fact, we may proceed using a contradiction argument. If there is a point \( x_1 < -x_e \) such that \( (\alpha - \beta_1)(x_1) > 0 \), we may assume without loss of generality that \( (\alpha - \beta_1)(x) \) attains its local maximum at \( x_1 \). Hence \( (\alpha - \beta_1)'(x_1) = 0 \) and \( (\alpha - \beta_1)''(x_1) < 0 \). Inserting these into (33), we arrive at a contradiction. Thus \( \alpha(x) \leq \beta_1(x) \) for any \( x < -x_e \). This immediately implies (20).

In what follows, we consider the asymptotic estimate of \( \alpha(x) \) as \( x \to \infty \). From (8), we know \( \lim_{x \to \infty} \alpha(x) = \pi \). Letting \( \gamma(x) = \pi - \alpha(x) > 0 \), then we have \( \lim_{x \to \infty} \gamma(x) = 0 \) and \( \gamma'(x) < 0 \). We also have \( \lim_{x \to \infty} \gamma'(x) = \lim_{x \to \infty} \frac{\gamma(x)}{x} = 0 \). So for any sufficiently small \( \epsilon > 0 \), there exists sufficient large \( x_\epsilon > 0 \) such that \( x > x_\epsilon \), we have \( \gamma' < \epsilon, \sin \gamma < \epsilon \) and \( \sin 2\gamma > 2(1 - \epsilon)\gamma \).

A direct calculation, we can obtain \( \gamma(x) \) satisfying

\[
(1 + 2\frac{\lambda}{R_0^2} \sin^2 \gamma)\gamma'' = \frac{\sin(2\gamma)}{R_0^2} \left( 1 - \lambda(\gamma')^2 + \frac{\lambda \sin^2 \gamma}{R_0^2} \right).
\]

Similarly, we choose a comparison function

\[
\beta_2(x) = Ce^{-x\sqrt{b_2(1-\epsilon)}} ,
\]

where \( b_2 = \frac{2}{R_0^2 + 2\lambda} \), \( \epsilon > 0 \) is small enough and \( C > 0 \) is a suitable constant. Then for any given \( \epsilon > 0 \) sufficiently small, there exists large enough \( x_\epsilon > 0 \) such that \( x > x_\epsilon \), we can get

\[
(\gamma - \beta_2)'' \geq \frac{2(1-\epsilon)}{R_0^2 + 2\lambda x^2} (\gamma - \beta_2) - \frac{2\lambda x^2}{R_0^2} (\gamma - \beta_2)'
\]

for \( \forall x > x_\epsilon \).

Setting \( \Phi = \gamma - \beta_2 \), then we can choose \( C > 0 \) large enough such that \( \Phi(x_\epsilon) = \gamma(x_\epsilon) - \beta_2(x_\epsilon) \leq 0 \) for the fixed \( x_\epsilon > 0 \). Thus

\[
\Phi''(x) \geq \frac{2(1-\epsilon)}{R_0^2 + 2\lambda x^2} \Phi(x) - \frac{2\lambda x^2}{R_0^2} \Phi'(x),
\]

\[
\Phi(\infty) = 0, \quad \Phi(x_\epsilon) \leq 0 .
\]

Similar discussions with above, we have

\[
\Phi(x) \leq 0, \quad \forall x > x_\epsilon .
\]

Thus for any \( x > x_\epsilon, \gamma(x) \leq \beta_2(x) \), i.e., \( \gamma(x) = O(e^{-x\sqrt{b_2(1-\epsilon)}}) \) as \( x \to \infty \). Then (21) immediately follows.

Then we complete the proof of theorem (2.2).

3. The solution of BPS monopoles equations. We next turn to the energy \( E(\alpha) \), the classical equations coincide with the equations of the BPS monopole. In fact, one is to find solutions to (6) subject to the boundary condition (8), which amounts to solving a two-point boundary value problem which seems difficult. Fortunately, Canfora, Correa, and Zanelli [7] identify a nontrivial kink charge \( Q \) given by

\[
Q = \int_{-\infty}^{\infty} \sqrt{32\pi KR_0^2} \left[ \frac{\sin^2 \alpha}{R_0^2} (1 + \frac{2\lambda}{R_0^2} \sin^2 \alpha)(1 + \frac{\lambda}{2R_0^2} \sin^2 \alpha) \right]^\frac{1}{2} \alpha' dx,
\]
so that they obtain through a BPS trick \[3, 20\] the following expression for the total energy functional (5)

\[ E(\alpha) = 2\pi KR_0^2 \int_{-\infty}^{\infty} \left[ (\alpha')^2 + 2 \frac{\sin^2 \alpha}{R_0^2} + \lambda \left( \frac{\sin^2 \alpha}{R_0^2} (2(\alpha')^2 + \sin^2 \alpha) \right) \right] dx \]

\[ = 2\pi KR_0^2 \int_{-\infty}^{\infty} \left( (1 + 2 \frac{\lambda}{R_0^2} \sin^2 \alpha) \alpha' \pm \left( \frac{2 \sin^2 \alpha}{R_0^2} (1 + \frac{\lambda}{2R_0^2} \sin^2 \alpha) \right) \right) dx \pm Q \geq |Q|, \quad (40) \]

for \( |Q| = \mp Q \), which leads them to arrive at the conclusion that the energy lower bound in (40) is attained when \( \alpha \) satisfies the first-order equation

\[ \alpha' \pm \left( \frac{2 \sin^2 \alpha}{R_0^2} (1 + \frac{\lambda}{2R_0^2} \sin^2 \alpha) \right)^{\frac{1}{2}} = 0. \quad (41) \]

Equation in (41) are actually the self-dual BPS monopoles equations (6).

**Theorem 3.1.** In the context of finite-energy solutions satisfying the boundary condition (8), the Euler-Lagrange equation (6) of the Skyrme soliton energy (5) and the BPS monopoles equation (41) are equivalent.

**Proof.** It is straightforward to check that (41) implies (6). It will take some effort, however, to show that the converse is also true. In other words, we shall prove that any finite-energy solution of (6) satisfies (41) as well. That is, the first-order equation (41) and the second-order equation (6) are actually equivalent.

Equation (6) can be rewritten as

\[ \alpha'' = \frac{1}{R_0^2} \sin(2\alpha)(1 + \frac{\lambda}{R_0^2} \sin^2 \alpha - \lambda(\alpha')^2) \]

(42)

Let \( \alpha \) be a finite-energy solution of (42) and set

\[ P_{\pm} = \sqrt{1 + 2 \frac{\lambda}{R_0^2} \sin^2 \alpha \alpha' \pm \frac{\sin \alpha}{R_0} \sqrt{2 + \frac{\lambda}{R_0^2} \sin^2 \alpha}}. \quad (43) \]

Then, in view of (42), we have

\[ P_+ P_- = (1 + 2 \frac{\lambda}{R_0^2} \sin^2 \alpha)(\alpha')^2 - \frac{\sin \alpha}{R_0} \sqrt{2 + \frac{\lambda}{R_0^2} \sin^2 \alpha}, \]

\[ (P_+ P_-)' = 0, \quad x \in (-\infty, \infty), \]

\[ (P_{\pm})' = \pm \frac{2 \cos \alpha}{R_0} \sqrt{1 + \frac{\lambda}{R_0^2} \sin^2 \alpha} \frac{1}{\sqrt{(2 + \frac{\lambda}{R_0^2} \sin^2 \alpha)(1 + \frac{2\lambda}{R_0^2} \sin^2 \alpha)}} P_{\pm}. \quad (45) \]

Therefore, if there is some \( x_0 \in (-\infty, \infty) \) such that \( P_+(x_0) = 0 \) or \( P_-(x_0) = 0 \), then applying the uniqueness theorem for the initial value problem of an ordinary differential equation we obtain \( P_+ \equiv 0 \) or \( P_- \equiv 0 \), which implies \( \alpha \) must satisfy one of the equations stated in (41).

By the finite-energy condition (5), we have

\[ \int_{-\infty}^{\infty} P_{\pm}^2(x) dx \leq 2 \int_{-\infty}^{\infty} \left[ (1 + 2 \frac{\lambda}{R_0^2} \sin^2 \alpha)(\alpha')^2 + \frac{\sin^2 \alpha}{R_0^2} (2 + \frac{\lambda}{R_0^2} \sin^2 \alpha) \right] dx \]

\[ < \infty. \quad (46) \]
Inserting (46) into (45), we also have
\[ \int_{-\infty}^{\infty} (P_\pm'(x))^2 \, dx < \infty. \] (47)
Combining (46) and (47), we conclude with
\[ \lim_{|x| \to \infty} P_\pm(x) = 0. \] (48)
In view of (44) and (48), we deduce
\[ (P_+, P_-)(x) = 0, \quad x \in (-\infty, \infty) \] (49)
That is, for any \( x \in (-\infty, \infty) \), either \( P_+(x) = 0 \) or \( P_-(x) = 0 \), which establishes \( P_\pm \equiv 0 \) as anticipated, as argued earlier. Thus, the theorem is proved. \( \square \)

In the following, we give a few numerical examples visually to observe the above theoretical results.

**Figure 1.** The graph for \( \alpha(x) \) over \((-15, 15)\).

Now we consider the first-order equation (41) with the form of negative sign. Choose above boundary conditions over a finite range, e.g., \((-L, L)\). We shall fix \( R_0 = 2 \) and coupling parameter \( \lambda = 0.1 \). Take the mesh step size \( \delta h = 0.1 \), then divide the interval \((-L, L)\) uniformly into \( 2M \) subinterval where \( M = L/h \) and denote the points on the subintervals as \( x_i \) where \( x_i = hi \ (i = 0, \pm 1, \pm 2, \ldots, \pm M) \). The value of the solution of (41) at the point \( x_i \) will be denoted by \( \alpha_i \). We will use the classical four order Runge–Kutta method to implement our computation.

We first compute the solution in \((-L, L)\) with \( L = 8 \). The value of \( \alpha_M \), i.e., the approximate value of \( \alpha(L) \) is 3.0909 which is away from the desired value of the
second boundary condition in (8). We then take $L = 10$, the computation result of the solution at the right endpoint is 3.1386. Although this value is very close to the expected value of the boundary, the approximate value of solution $\alpha(x)$ did not exhibit a stable trend near the right endpoint.

Now we let $L = 15$, Fig.1 presents the curve of $\alpha(x)$. Computation results show that the approximate value of $\alpha(x)$ are all 3.1416 when $x$ near the right endpoint. Within the allowable error range, which implies the agreement with the theoretical results. Similar conclusion can be obtained when $L \geq 15$ by further computer experiments.

We have solved numerically equation (41). We can clearly see many of the properties of energy-minimizing solution $\alpha(x)$ from the graph. Furthermore, the numerical examples which also demonstrate the effectiveness of our theory results.

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