Hyperbolic Hamiltonian equations for general relativity

Maurice H.P.M. van Putten

Le Studium IAS, 45071 Orléans Cedex 2, Université d’Orleans, France

ABSTRACT

The 3+1 Hamiltonian formulation in the gauge $D_t N = -K$ on the lapse function fixes the direction of time associated with the trace $K$ of the extrinsic curvature tensor. The Hamiltonian equations hereby become hyperbolic. We study this new system for black hole spacetimes that are asymptotically quiescent, which introduces analyticity properties that can be exploited for numerical calculations by compactification in spherical coordinates with complex radius following a Möbius transformation. Conformal flat initial data of two black holes are hereby invariant, and correspond to a turn point in a pendulum, up for a pair of separated black holes and down for a single black hole. Here, Newton’s law appears in the relaxation of $l = 2$ deformations of semi-infinite poloidal surface elements, defined by the moment of inertia of the binary.

1. Introduction

The calculation of gravitational radiation produced in the merger of two black holes is of considerable theoretical interest. It may also provide templates of wave-forms for analysis of LIGO-Virgo data. To be of practical interest to matched filtering, the calculational methods will have to recover many wave-periods over an extended parameter range, such as in the problem of binary black hole coalescence with a range of black hole masses and spins. It poses the challenge of efficient and stable phase-accurate numerical methods. It has been appreciated that this requires an inherently stable formulation of the dynamical evolution of spacetime. Spectral methods provide the most efficient representation of functions that are periodic and everywhere analytic, and preserve accurate phase information in hyperbolic evolution.

Here, we consider the 3+1 Hamiltonian equations for the dynamical behavior of spacetime [Arnowitt, Deser & Misner 1962]. They are attractive, in describing the evolution in terms of quantities that have unambiguous geometric interpretations. However, for arbitrary choices of gauge, they are known to give rise to computational instabilities, which have thusfar prevented their application to large-scale numerical relativity.
Here, we propose a new gauge for time-evolution, which describes a correlation between the lapse function and the extrinsic curvature of the foliation of three-surfaces in spacetime. Linearized stability analysis shows that the full system of Hamiltonian equations now becomes hyperbolic with respect to arbitrary perturbations of the three-metric. Next, we focus on black hole spacetimes which are asymptotically quiescent \(^\text{[van Putten 2006]}\). This asymptotic property, if present in the initial data, is preserved for all time by causality in the equations of general relativity. It can be exploited by compactification in spherical coordinates with complex radius, \((z, \theta, \phi)\) with complex radius \(z\), \(-\infty + is < z < \infty + is\), where \(E/s\) denotes a dimensional continuation parameter for a spacetime with total mass energy \(E\). Here, \(s\) can be chosen sufficiently large, i.e., at least on the order of \(E\), to avoid singularities associated with black holes, even those that may carry spin. In this approach, we focus on the outer expansion of the Green’s function. It makes possible a three-dimensional spectral representations after application of a Möbius transformation (which preserves conformally flat initial data).

The combination of hyperbolic Hamiltonian equations and spectral representations offers a novel starting point for studying the dynamical evolution of black hole spacetimes. For illustrative purposes, we consider the conformally flat initial data of Schwarzschild black holes, as in the problem of a head-on collision of two black holes. Here, conformally flat data are special, in representing turning points corresponding to the up or down position of a pendulum. The proposed lapse function is natural choice about such turning points. To illustrate compactification by a Möbius transformation, we explicitly identify Newton’s law in terms of the evolution of the quadrupole deformations in area of poloidal flux surfaces, given by the moment of inertia \(I\) of spacetime.

2. Asymptotic wave motion

At large distances from the source region, the two wave-modes in the gravitational field appear as planar waves in the three-metric \(h_{ij}\), commonly expressed in the transverse traceless gauge as

\[
h_{ij} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 + b & c \\
0 & c & 1 - b
\end{pmatrix}, \quad b = B \sin(kz), \quad c = C \sin(kz)
\]

\(B, C\) denote the amplitude of + and × plane-waves along the \(z\)-direction with wave-number \(k\). The three-dimensional Ricci tensor associated with \(h_{ij}\) hereby assumes the following
structure. For the + waves with \((B, C) = (B, 0)\), we have

\[
R^+_{ij} = \frac{Bk^2}{2\Delta} \begin{pmatrix}
* & 0 & 0 \\
0 & -\sin kz - B \cos^2 kz + B^2 \sin^3 kz & 0 \\
0 & 0 & \sin kz - B \cos^2 kz - B^2 \sin^3 kz
\end{pmatrix},
\]

(2)

where \(b = B \sin kz\), \(c = 0\) and \(\Delta = 1 - B^2 \sin^2 kz\), and for the \(\times\) waves with \((B, C) = (0, C)\), \(b = 0\), \(c = C \sin kz\), we find, similarly,

\[
R^\times_{ij} = \frac{Ck^2}{2\Delta} \begin{pmatrix}
* & 0 & 0 \\
0 & -C \cos^2 kz & -\Delta \sin kz \\
0 & -\Delta \sin kz & -C \cos^2 kz
\end{pmatrix},
\]

(3)

where \(\Delta = 1 - C^2 \sin^2 kz\). Hence, we have the following exact algebraic identities:

\[
R^+_{22} - R^+_{33} \equiv -Bk^2 \sin(kz), \quad R^+_{23} \equiv 0,
\]

(4)

\[
R^\times_{33} - R^\times_{22} \equiv 0, \quad R^\times_{23} \equiv -\frac{C}{2}k^2 \sin(kz),
\]

(5)

where the numerical indices refer to normalized directions in the coordinate directions \((r, \theta, \phi)\).

The identities (5) show that the two polarization modes satisfy traveling wave solutions

\[
b(t, r) = B \sin \eta, \quad c(t, r) = C \sin \eta, \quad \eta = k(t - r)
\]

(6)

in the large-distance approximation (neglecting \(1/r\) curvature terms, in this section) with vanishing lapse functions \(\beta_i \equiv 0\) and constant lapse function \(N \equiv 1\), satisfying

\[
\partial_t^2 (h_{22} - h_{33}) = -2\partial_t (K_{33} - K_{22}) = 2(R_{22} - R_{33}), \quad \partial_t^2 h_{23} = -2\partial_t K_{23} = 2R_{23},
\]

(7)

where \(b(t, r) = \frac{1}{2}(h_{22} - h_{11})\), \(c(t, r) = h_{12}\), using the numerical indices to denote the normalized directions in \((r, \theta, \phi)\). The angular directions “22-33” and “23” therefore represent the \textit{hyperbolic directions}, which contain all wave-motion.

3. General 3+1 decomposition of the metric

We consider the 3+1 decomposition of the line-element in terms of \(h_{ij}\) \cite{Thorne, Price & McDonald 1986}

\[
ds^2 = -N^2 dt^2 + h_{ij} \left( dx^i + \beta^i dt \right) \left( dx^j + \beta_j dt \right).
\]

(8)
The lapse and shift functions can be seen to define the motion of observers with velocity four-vector \( u^i = (1, 0, 0, 0) \), for example, by considering their acceleration \( a^i = \frac{d}{dt} u^i = -\Gamma^i_{00} = -\frac{1}{2} g^{ii} (2 \dot{g}_{00} - \dot{g}_{00,i}) = -h^{ii} \left( \dot{\beta}_i - N \partial_t N + \beta^j D_i \beta_j \right) = \Phi^{-4} N \partial_t N \), which is determined entirely by the redshift factor \( N \) when \( \beta_i = \dot{\beta}_i = 0 \). In general, the lapse and shift functions appear in the conservation of energy and momentum, of test particles moving along geodesics. We need not impose the time-symmetric gauge condition with \( \dot{\beta}_i = 0 \). The three-metric \( h_{ij} \) satisfies the Hamiltonian evolution equations

\[
\partial_t h_{ij} = D_i \beta_j + D_j \beta_i - 2N K_{ij}, \quad N^{-1} D_i K_{ij} + 2K^m_i K_{jm} - K K_{ij} = -W_{ij}, \tag{9}
\]

where \( D_t K_{ij} = \partial_t K_{ij} - \beta^m D_m K_{ij} - K_{im} D_j \beta^m - K_{jm} D_j \beta^m \) and \( W_{ij} \) denotes the gauged three-tensor \( W_{ij} = -R_{ij} + N^{-1} D_i D_j N \).

We propose the time-evolution according to

\[
K : \begin{cases} 
D_t N & = -K, \\
D_t h_{ij} & = -2K_{ij} N, \\
D_t K_{ij} & = -D_{ij} N + K_{ij} N,
\end{cases} \tag{10}
\]

where \( D_{ij} = D_i D_j - R_{ij} \) and \( K = KK_{ij} - 2K^m_i K_{jm} \).

The gauge condition \( D_t N = -K \) is curvature-driven evolution. It is different from the product of curvature and lapse function in the harmonic slicing condition \( \partial_t N = -N^2 K \) in Eqs. (69)-(77) of Abrahams et al. (1997); see also Brown (2008) for a recent review.

In the asymptotically flat region about \( h_{ij} = \delta_{ij} \) and \( N = 1 \), we have

\[
\partial_t N = -K, \quad \partial_t^2 h_{ij} = -2R_{ij} + 2D_i D_j N, \quad \partial_t^2 K = \Delta K. \tag{11}
\]

We recall that (Wald 1984)

\[
R_{ij} = -\frac{1}{2} \Delta \delta h_{ij} + \frac{1}{2} \partial_i \partial^e \delta h_{ej} + \frac{1}{2} \partial_j \partial^e \delta h_{ei} \tag{12}
\]

where \( \delta h_{ij} = \partial h_{ij} - \frac{1}{2} \delta h \delta g \), where \( \delta h = h^{ij} \delta h_{ij} \) refers to the trace of the metric perturbations. For harmonic perturbations of the form \( h_{ij} \sim \hat{h}_{ij} e^{-i \omega t \hat{k}_i \hat{k}_j} \), we have, with conservation of momentum, \( D^i K_{ij} = D_j K_i, -i \omega N = \hat{K}, \quad \delta h_{ij} = -2i \omega^{-1} \hat{K}_{ij}, \quad k^i \hat{K}_{ij} = k_j \hat{K} \), so that

\[
\partial_i \partial^e \hat{h}_{ej} \rightarrow k_i k^e \hat{h}_{ej} - \frac{1}{2} k_i k_j \delta h = i \omega^{-1} (-2k_i k^e \hat{K}_{ej} + k_i k_j \hat{K}) = -i \omega^{-1} k_i k_j \hat{K} \tag{13}
\]

and hence

\[
\hat{R}_{ij} - \partial_i \partial_j N = \frac{1}{2} k^2 \delta h_{ij} - i \omega^{-1} k_i k_j \hat{K} + i \omega^{-1} k_i k_j \hat{K} = \frac{1}{2} k^2 h_{ij} \tag{14}
\]
We conclude that \( \partial_t h_{ij} = -2R_{ij} + 2D_iD_j\mathcal{N} \) gives rise to the dispersion relation

\[
\omega^2 = k^2
\]

(15)

for arbitrary, small amplitude metric perturbations. We conclude that the system \( K \) in (10) is asymptotically stable by virtue of the gauge choice \( D_t\mathcal{N} = -K \). This contrasts with the more common derivation of (15) in the so-called transverse traceless gauge, or harmonic coordinates—neither of these two coordinate conditions are used here.

4. Conformal flat initial data

The conformal decomposition of the Ricci tensor for \( h_{ij} = \phi^{2m}g_{ij} \) satisfies (Wald 1984)

\[
R_{ij}(\phi) - m\phi^{-1}[(n-2)D_iD_j\phi + g_{ij}\Delta\phi] + m\phi^{-2}[(1+m)(n-2)D_i\phi D_j\phi + (1-m)(n-2))g_{ij}D^p\phi D_p\phi].
\]

For \( n = 3 \) and \( m = 2 \), it reduces to (3) \( R_{ij}(g) - 2\phi^{-1}[D_iD_j\phi + g_{ij}\Delta\phi] + 2\phi^{-2}[3D_i\phi D_j\phi - g_{ij}D^p\phi D_p\phi] \), giving the familiar result

\[
R_h = \phi^{-4} \left[ R_g - 8\phi^{-1}\Delta_g\phi \right].
\]

(16)

Time-symmetric initial data around one or multiple non-rotating black holes can be conveniently described in a conformally flat representation (Lindquist 1963; Brill & Lindquist 1964; Jansen et al. 2003) on the basis of (16). It gives rise to solutions for vacuum spacetimes in terms of the Green’s function \( G(x^i, p^i) = G_p \) of flat spacetime, where \( p^i \) denotes the position of a point source.

Spacetime of a single black hole is given by the conformal factor

\[
\Phi = 1 + 2\pi mG_p = 1 + \frac{E}{2z} + \frac{mp}{2z^2}P_1(x) + \frac{I}{2z^3}P_2(x) + \cdots
\]

(17)

where \( E = m \), \( I = mp^2 \) and the expansion refers to the outer expansion of \( G_0 \) in complex spherical coordinates \( (z, x, \phi), x = \cos \theta \), by exploiting analyticity at infinity, i.e., asymptotic quiescence. The same construction applies to two black holes,

\[
\Phi = 1 + 2\pi MG_q + 2\pi m G_p = 1 + \frac{E}{2z} + \frac{Mq + mp}{2z^2}P_1(x) + \frac{I}{2z^3}P_2(x) + \cdots
\]

(18)

where \( E = M + m \) and \( I = MQ^2 + mp^2 \). Without loss of generality, we may choose \( Mq + mp = 0 \). According to the Hamiltonian evolution equations, the associated shift function, for a static foliation of spacetime, i.e., \( D_tK_{ij} = 0 \), satisfies

\[
\Delta_h\mathcal{N} + 3NR = 0.
\]

(19)
The interaction of two Schwarzschild black holes is attractive as seen in the space of the complex radial coordinate $z$ (solid curved line) or equivalently repulsive, following closure over infinity in the domain of convergence of the outer expansion of the Green’s function, i.e., in the space of the transformed variable $w = 1/z$ (dotted curved lines). For the latter, Newton’s law represents the leading order evolution of the area of the poloidal surface elements $|z| > p$, governed by the moment of inertia associated with the separation $p$ between the two black holes.
For the conformally flat data at hand, it reduces to $(\Phi^2 \partial_i N)_i = 0$, or $N = (2 - \Phi)/\Phi$.

The leading-order structure of spacetime can be studied by considering poloidal surface elements $dA = h_{\theta\theta}^{1/2} h_{zz}^{1/2} d\theta dz$ in the outer region $|z| > p$. A measure for the $l = 2$ deformations in surface elements is

$$A_2(z, t) = \int_z^\infty \int_0^\pi dA = \int_z^\infty \int_0^\pi \Phi^4 z d\theta dz + \cdots \tag{20}$$

in $z > p$, where $\cdots$ refers to time-independent divergences. It gives a geometric equivalence about $w = 0$ ($w = 1/z$) to the moment of inertia.

5. Newton’s law in the initial quadrupole evolution

The gauged tensor $W_{ij} = -R_{ij} + N^{-1} D_i D_j N$ has a regular Taylor series expansion about infinity, for the given asymptotically quiescent initial data. For the equal mass, two-hole solution with total mass-energy $E = 2M$, moment of inertia $I = 2Mp^2$, and symmetric position about the origin, the conformal factor

$$\Phi = 1 + \frac{E}{2z} + \frac{I}{2z^3} P_2(x) + \cdots \tag{21}$$

we find the large-$z$ asymptotics

$$W^*_{ij} = -\frac{EI}{2z^6} \begin{pmatrix} 2P_2 & -3zP_1 & 0 \\ -3zP_1 & -z^2[2P_2 - P_0]/(1 - x^2) & 0 \\ 0 & 0 & -z^2P_0(1 - x^2) \end{pmatrix}, \tag{22}$$

satisfying $W^* = 0$ with

$$W^* : W^* = E^2 f^2 \left( \frac{3}{2} z^{-12} - 9Ez^{-13} + \left( \frac{117}{4} E^2 - 9E^{-1} Ix^2 \right) z^{-14} \right) + O(z^{-15}) \tag{23}$$

The time-evolution of the $l = 2$ deformation of poloidal surface elements is given by

$$\dot{A}_2(z, t) = \int_0^\pi \int_z^\infty \partial_t^2 \det(h_{IJ})^{1/2} d\theta dz \tag{24}$$

where $I, J = z, \theta$, where $\det(h_{IJ}) = h_{\theta\theta} h_{zz} - (h_{z\theta})^2$. For the time-symmetric initial data, the quadratic off-diagonal terms make no initial contribution, leaving

$$\dot{A}_2(z, t) = \int_0^\pi \int_z^\infty \partial_t^2 (h_{\theta\theta}^{1/2} h_{zz}^{1/2}) d\theta dz = \frac{1}{2} \int_0^\pi \int_z^\infty (h_{\theta\theta}^{-1/2} h_{zz}^{1/2} \tilde{h}_{\theta\theta} + h_{zz}^{-1/2} h_{\theta\theta}^{1/2} \tilde{h}_{zz}) d\theta dz. \tag{25}$$
At $t = 0$, the integrand reduces in the gauge $K = 0$ with the additional time-symmetric gauge condition $\dot{\beta}_i = 0$ at $t = 0$ (a Newtonian gauge choice, as in a turning point) to

$$
\frac{1}{2} \left( h_{zz} + z^{-2} \bar{h}_{\theta\theta} \right) z = -\frac{1}{2z^2 \sin \theta} \bar{h}_{\phi\phi} = z^{-2} \sin^{-2} \theta \bar{K}_{\phi\phi} = -\frac{EI}{2z^4} + O(z^{-5}).
$$

(26)

For completely time-symmetric initial data, we thus recover Newton’s law for two equal mass black holes of mass $M$ and separation $2p$ in $\ddot{A}_2(z, t) = -\frac{1}{2} EI \dot{z}^{-2} + O(z^{-5})$, we find

$$
\ddot{p} = -\frac{M}{4p^2}
$$

(27)

for initial evolution of $A_2$ at the maximal extent $z = p$ (the lower limit of $z$) of the poloidal surface elements (in the outer expansion considered).

Thus, for a fixed $|z| > p$, it takes $t = O(z^3)$ to relax $Iz^{-3}$ by the leading-order $z^{-6}$ forcing terms in (22), when applied to the asymptotic equations. The asymptotics $Iz^{-3}$ for large $z$ hereby persists for all finite time. This observation is consistent with the a priori notion of preserving asymptotic quiescence. Here, Newton’s law (27) associated with the deformations of the poloidal surface elements $|z| > p$ represent the underlying non-local interaction.

6. Conclusions

We have shown that the 3+1 Hamiltonian equations become hyperbolic, when the evolution of the lapse function is driven by the extrinsic curvature tensor. The resulting hyperbolic system forms a starting point for three-dimensional spectral methods in numerical relativity, after compactification by a Möbius transformation for spacetimes that satisfy asymptotic quiescence, wherein Newton’s law arises in the dynamics of the quadrupole deformations of poloidal surface elements. Future work will focus on some numerical examples.

Acknowledgment. The author gratefully acknowledges stimulating discussions with A. Spallicci, M. Volkov, G. Barles, and members of the Fédération Denis Poisson. This work is supported, in part, by Le Studium IAS of the Université d’Orléans.

REFERENCES

Abrahams, A., Anderson, A., Choquet-Bruhat, Y. & York Jr., J.W., 1997, Class. Quantum Grav., A9

Arnowitt, R., Deser, R., & Misner, C.W., 1962, In Gravitation: an introduction to current research, edited by L. Witten (Wiley, New York), pp.227
Brill, D.R., & Lindquist, R.W., 1963, Phys. Rev. D., 131, 471
Brown, J.D., 2008, gr-qc/0803.0334v2
Lindquist, R.W., 1963, Phys. Rev., 4, 938
Jansen, N., Diener, P., Khokhlov, A., & Novikov, I., 2003, Class. Quant. Grav., 20, 51J
Thorne, K.S., Price, & McDonald, *The Membrane Paradigm*
van Putten, M.H.P.M., 2006, Proc. Nat. Acad. Sc., 516
Wald, R.M., 1984, *General Relativity* (Univ. Chicago Press, Chicago)