GLOBAL DYNAMICS IN A TWO-SPECIES CHEMOTAXIS-COMPETITION SYSTEM WITH TWO SIGNALS

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Abstract. In this paper, we consider a chemotaxis-competition system of parabolic-elliptic-parabolic-elliptic type
\[
\begin{align*}
  u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u - a_1 w), \quad x \in \Omega, \quad t > 0, \\
  0 &= \Delta v - v + w, \quad x \in \Omega, \quad t > 0, \\
  w_t &= \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w (1 - a_2 u - w), \quad x \in \Omega, \quad t > 0, \\
  0 &= \Delta z - z + u, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]
with homogeneous Neumann boundary conditions in an arbitrary smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, where $\chi_i, \mu_i$ and $a_i$ ($i = 1, 2$) are positive constants. It is shown that for any positive parameters $\chi_i, \mu_i, a_i$ ($i = 1, 2$) and any suitably regular initial data $(u_0, w_0)$, this system possesses a global bounded classical solution provided that $\frac{\mu_i}{\mu_i}$ are small. Moreover, when $a_1, a_2 \in (0, 1)$ and the parameters $\mu_1$ and $\mu_2$ are sufficiently large, it is proved that the global solution $(u, v, w, z)$ of this system exponentially approaches to the steady state $(\frac{1-a_1}{1-a_1-a_2}, \frac{1-a_2}{1-a_1-a_2}, \frac{1-a_1}{1-a_1-a_2}, \frac{1-a_1}{1-a_1-a_2})$ in the norm of $L^\infty(\Omega)$ as $t \to \infty$. If $a_1 \geq 1 > a_2 > 0$ and $\mu_2$ is sufficiently large, the solution of the system converges to the constant stationary solution $(0, 1, 1, 0)$ as time tends to infinity, and the convergence rates can be calculated accurately.

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1. **Introduction.** In many biological processes, such as pattern formation, embryo development, tumor invasion etc, the cells move towards the higher concentration of the chemical substance when they plunge into hunger, this phenomena is referred to as chemotaxis. A well-known mathematical model for one-species and one-stimuli chemotaxis model was first proposed by Keller-Segel in [14] as follows:

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), \\
  \tau v_t &= \Delta v - \chi u + \nabla \cdot (u \nabla v),
\end{align*}
\]

where \(\tau \in \{0, 1\}, \chi > 0\) is called chemotactic sensitivity, \(u(x, t)\) denotes the cell density and \(v(x, t)\) represents the chemical concentration. The classical Keller-Segel system has been intensively studied. Such as in one dimensional case, the solutions of (1.1) are uniformly bounded-in-time (see [32]). However, in the higher-dimensional case \(n \geq 2\), the solutions to (1) can blow up in finite time (see e.g. [9, 27, 40]). In [33, 38, 41], it is shown that the logistic growth term can prevent the occurrence of blow-up. To prevent any chemotactic collapse in (1), the chemotaxis system with growth terms

\[
\begin{align*}
  u_t &= \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), \\
  \tau v_t &= \Delta v - v + u,
\end{align*}
\]

was studied (see e.g. [38, 41, 42, 16, 43, 45]), where \(\chi > 0, \tau \in \{0, 1\}\).

After the pioneering works above, the following one-species and two-stimuli chemotaxis model, which is a generalized problem of Keller-segel system

\[
\begin{align*}
  u_t &= \Delta u - \chi v \nabla \cdot (u \nabla w) + \xi \nabla \cdot (u \nabla w), \\
  \tau v_t &= \Delta v - \beta v + \alpha u, \\
  \tau w_t &= \Delta w - \delta w + \gamma u,
\end{align*}
\]

was studied by several authors (see e.g. [4, 10, 13, 20, 21, 22, 36]), where the parameters \(\alpha, \beta, \chi, \xi, \gamma, \delta > 0\) and \(\tau \in \{0, 1\}\).

Moreover, in the two-species and one-stimuli phenomena, there exists a competition between two species, so a two-species chemotaxis system with competitive kinetics

\[
\begin{align*}
  u_t &= \Delta u - \chi_1 v \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\
  v_t &= \Delta v - \chi_2 v \nabla \cdot (u \nabla w) + \mu_2 u(1 - v - a_2 u), \\
  \tau w_t &= \Delta w - \delta w + b_1 u + b_2 v,
\end{align*}
\]

with some \(\delta, \chi_i, \mu_i, a_i, b_i (i = 1, 2) > 0\) and \(\tau \in \{0, 1\}\), was proposed by Tello-Winkler [37] and was studied (see [1, 19, 24, 25, 26, 28, 29, 46]). In the case \(\tau = 0\), when \(a_1, a_2 \in (0, 1)\), the global boundedness and large time behavior of solutions were obtained under suitable assumptions on the parameters (see [1, 2]), moreover, when \(a_1 > 1, a_2 \in (0, 1)\) and \(\frac{a_2}{\chi_1}, \frac{a_1}{\chi_2}\) are sufficiently large, the global solution \((u, v, w)\) of (4) converges to \((0, 1, \frac{a_2}{\chi_2})\) as \(t \to \infty\) (see [34]). In the case \(\tau = 1\), when \(n \geq 2\), Lin et al. [19] established the global existence of classical solutions provided that \(\Omega\) is convex. Later on by means of the construction of suitable energy functionals, Bai et al. [1] showed the large time behavior of bounded solution for the two cases \(a_1, a_2 \in (0, 1)\) and \(a_1 \geq 1, a_2 \in (0, 1)\) when \(n = 2\). Recently, Mizukami [25] extended these results to more particular situations (see also [18]). When there is no competition (i.e. \(a_1 = a_2 = 0\)), Negreanu et al. [30, 31] established the global existence and global behavior for small diffusion rates, where the smallness was later removed in [26]. We also refer to [7, 12, 39] for the research on steady states and refer to [11, 17, 47] for nonlinear chemotaxis sensitivity functions.
Recently, the two-species and two-stimuli chemotaxis system
\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\
    0 &= \Delta v - v + w, & x \in \Omega, \ t > 0, \\
    w_t &= \Delta w - \chi_2 \nabla \cdot (w \nabla z), & x \in \Omega, \ t > 0, \\
    0 &= \Delta z - z + u, & x \in \Omega, \ t > 0,
\end{cases}
\end{align*}
\]
was studied by Tao and Winkler [35], they studied the global boundedness and finite-time blow-up for (5) when \( \chi_1, \chi_2 \in \{-1, 1\} \). Especially, when \( n = 2, \chi_1 = \chi_2 = 1 \) (i.e. attraction-attraction case), and the initial data \((u_0, w_0) \in (C^0(\Omega))^2\) satisfies
\[
\max \left\{ \int_\Omega u_0 dx, \int_\Omega w_0 dx \right\} < \frac{4}{C_{GN}},
\]
where \( C_{GN} > 0 \) is a constant, system (5) possesses a unique global bounded classical solution. Recently, Zheng [48] generalized the results of [35] to the quasilinear cases. Later, in the case of \( n = 2 \), Zheng and Mu [49] removed the condition of small initial masses under the Lotka-Volterra-type competition. Recently, Zheng et al. [50] studied the persistence property of global bounded solutions for a fully parabolic two-species chemotaxis system with two signals. Moreover, Zheng et al. [51] studied the boundedness and stabilization of global classical solutions for (5) with the Lotka-Volterra-type weak competition (i.e. \( a_1, a_2 \in (0, 1) \)), however, the large time behavior of global solutions is still open under the strong competition (i.e. \( a_1 \geq 1 > a_2 > 0 \)).

In this paper, we consider the two-species and two-stimuli chemotaxis system
\[
\begin{align*}
\begin{cases}
    u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u (1 - u - a_1 w), & x \in \Omega, \ t > 0, \\
    0 &= \Delta v - v + w, & x \in \Omega, \ t > 0, \\
    w_t &= \Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w (1 - a_2 u - w), & x \in \Omega, \ t > 0, \\
    0 &= \Delta z - z + u, & x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \Omega, \ t > 0, \\
    u(x, 0) = u_0(x), w(x, 0) = w_0(x), & x \in \Omega, \ t > 0,
\end{cases}
\end{align*}
\]
where \( \Omega \subset R^n (n \geq 2) \) is a bounded domain with smooth boundary \( \partial \Omega \), \( \frac{\partial}{\partial \nu} \) represents differentiation with respect to the outward normal on \( \partial \Omega \), and the parameters \( \chi_i, \mu_i \) and \( a_i (i=1,2) \) are positive. Here \( u \) and \( w \) denote the density of two species, \( v \) and \( z \) stand for the concentration of the chemical substance.

However, there are only few results for the multi-species and multi-stimuli chemotaxis model. In this paper, we will show that for any positive parameters \( \chi_i, \mu_i \), \( a_i (i=1,2) \) and any suitably regular initial data \((u_0, w_0)\), this system possesses a global bounded classical solution provided that \( \frac{\Delta}{\mu_i} \) are small. Moreover, when \( a_1, a_2 \in (0, 1) \) and the parameters \( \mu_1 \) and \( \mu_2 \) are sufficiently large, it is proved that the global solution \((u, v, w, z)\) of this system exponentially approaches to the steady state \( \left( \frac{1-a_1}{1-a_1+a_2}, \frac{1-a_2}{1-a_1+a_2}, \frac{1-a_2}{1-a_1+a_2}, \frac{1-a_1}{1-a_1+a_2} \right) \) in the norm of \( L^\infty(\Omega) \) as \( t \to \infty \); while \( a_1 \geq 1 > a_2 > 0 \) and \( \mu_2 \) is sufficiently large, the solution of the system converges to the constant stationary solution \((0, 1, 1, 0)\) as time tends to infinity, and the convergence rates can be calculated accurately.

To establish the global dynamical properties, let us assume that the initial data \( u_0 \) and \( w_0 \) satisfy
\[
\begin{align*}
\begin{cases}
    0 \leq u_0 \in C^0(\Omega) \text{ and } u_0 \neq 0, \\
    0 \leq w_0 \in C^0(\Omega) \text{ and } w_0 \neq 0.
\end{cases}
\end{align*}
\]
Under these assumptions, our results in this paper are stated as follows. Firstly, we give the global boundedness of solutions to (7) for all positive parameters $\chi_i, \mu_i$ and $a_i, i = 1, 2$.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smoothly bounded domain, and let $\chi_i > 0$, $\mu_i > 0$ and $a_i > 0$, $i = 1, 2$. Suppose that $\chi_1, \chi_2, \mu_1$ and $\mu_2$ satisfy

$$\frac{\chi_1}{\mu_1} < \frac{a_1 n}{n - 2} \quad \text{and} \quad \frac{\chi_2}{\mu_2} < \frac{a_2 n}{n - 2}.$$  \hspace{1cm} (9)

Then for any $(u_0, w_0)$ fulfilling (8) with some $q > n$, and for any $a_1 \geq 0$ and $a_2 \geq 0$, problem (7) possesses a unique global classical solution $u, v, w, z$, which is uniformly bounded in the sense that

$$\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) \|_{W^{1,q}(\Omega)} + \| w(\cdot, t) \|_{L^\infty(\Omega)} + \| z(\cdot, t) \|_{W^{1,q}(\Omega)} \leq C$$

for all $t \geq 0$, with some constant $C > 0$ that is independent of $t$. And the solutions $u, v, w, z$ are the Hölder continuous functions, i.e. there exist $\sigma \in (0, 1)$ and $K > 0$ such that

$$\| u(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| v(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| w(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| z(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} \leq K$$

for all $t \geq 1$.

Next, we shall discuss the asymptotic behavior of global solutions to (7) in the weak competition case $a_1, a_2 \in (0, 1)$.

**Theorem 1.2.** Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smoothly bounded domain, $\chi_i > 0$, $\mu_i > 0$, $a_i \in (0, 1), i = 1, 2$, and the initial data $(u_0, w_0)$ satisfies (8). Assume that there exists a global classical solution $(u, v, w, z)$ of (7) satisfying

$$\| u(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| v(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| w(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} + \| z(\cdot, t) \|_{C^{\sigma}([0, t] \times [t, t+1])} \leq K$$

for all $t > 0$, with some $K > 0$ and $\sigma \in (0, 1)$. Furthermore, suppose that there exist $k_1, k_2 \in (0, 1)$ such that

$$k_1 k_2 > a_1 a_2$$

and

$$\mu_1 > \frac{a_2 \chi_1^2 w^*}{16 a_1 (1 - k_2)}$$

as well as

$$\mu_2 > \frac{a_1 \chi_2^2 w^*}{16 a_2 (1 - k_1)}.$$  \hspace{1cm} (11)

Then there exist $C_1 > 0$ and $m > 0$ such that

$$\| u(\cdot, t) - u^* \|_{L^\infty(\Omega)} + \| v(\cdot, t) - v^* \|_{L^\infty(\Omega)} + \| w(\cdot, t) - w^* \|_{L^\infty(\Omega)} + \| z(\cdot, t) - z^* \|_{L^\infty(\Omega)} \leq C_1 e^{-mt}$$

for all $t > 0$, where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}, \quad w^* := \frac{1 - a_2}{1 - a_1 a_2}, \quad z^* := \frac{1 - a_1}{1 - a_1 a_2}.$$  \hspace{1cm} (13)

Finally, we consider the large time behavior of global solutions to (7) in the strong competition case $a_1 \geq 1 > a_2 > 0$. 
Theorem 1.3. Let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a smoothly bounded domain, and let \( \chi_i > 0, \mu_i > 0 \) \( (i = 1, 2) \) and \( a_i \geq 1 > a_2 > 0 \). Assume that there exists a global classical solution \((u,v,w,z)\) of (7) satisfying

\[
\begin{align*}
\| u'(\cdot,t) \|_{C^1(\bar{\Omega} \times [t,t+1])} + \| v'(\cdot,t) \|_{C^1(\bar{\Omega} \times [t,t+1])} \\
+ \| w'(\cdot,t) \|_{C^1(\bar{\Omega} \times [t,t+1])} + \| z'(\cdot,t) \|_{C^1(\bar{\Omega} \times [t,t+1])} \leq K
\end{align*}
\]

for all \( t > 0 \), with some \( K > 0 \) and \( \sigma 
\)

Furthermore, suppose that there exist \( l \in (0,1) \) such that

\[
l > a_1 a_2 \quad \quad \quad \quad \quad \quad \quad \quad \quad (14)
\]

and

\[
\mu_2 > \frac{a_1 \chi_2^2}{16 a_2 (1-l)^2} \quad \quad \quad \quad \quad \quad (15)
\]

Then \((u,v,w,z)\) has the following properties:

1. If \( a_1 > 1 \), then there exist \( C_2 > 0 \) and \( m > 0 \) satisfying

\[
\| u(\cdot,t) \|_{L^\infty(\Omega)} + \| v(\cdot,t) - 1 \|_{L^\infty(\Omega)} + \| w(\cdot,t) - 1 \|_{L^\infty(\Omega)} + \| z(\cdot,t) \|_{L^\infty(\Omega)} \leq C_2 e^{-mt}
\]

for all \( t > 0 \).

2. If \( a_1 = 1 \), then there exist \( C_3 > 0 \) and \( m > 0 \) such that

\[
\begin{align*}
\| u(\cdot,t) \|_{L^\infty(\Omega)} &+ \| v(\cdot,t) - 1 \|_{L^\infty(\Omega)} + \| w(\cdot,t) - 1 \|_{L^\infty(\Omega)} \\
+ \| z(\cdot,t) \|_{L^\infty(\Omega)} &\leq C_3 (t + 1)^{-m}
\end{align*}
\]

for all \( t > 0 \).

**Remark 1.** In the previous reference [51], the boundedness and stabilization of global solutions to (7) were only derived under the case \( a_1, a_2 \in (0,1) \). In this paper, we further study the conditions which assert global existence of classical bounded solutions for all \( a_1, a_2 \in (0,\infty) \), which covers the case that \( a_1, a_2 \in [1,\infty) \). Moreover, the convergence rates can be calculated accurately under the cases \( a_1, a_2 \in (0,1) \) and \( a_1 \geq 1 > a_2 > 0 \), respectively. Hence, the results of this paper improve the previous works.

**Remark 2.** In both Theorems 1.2 and 1.3, we only give the convergence rates of global solutions for (7) under the cases \( a_1, a_2 \in (0,1) \) and \( a_1 \geq 1 > a_2 > 0 \), respectively. For the case \( a_2 \geq 1 > a_1 > 0 \), it is easy to derive the convergence rates by the same method used in the proof of Theorem 1.3. Thus, we don’t discuss this condition here. However, for the case \( a_1, a_2 \in [1,\infty) \), there is still an open problem about stabilization of global bounded solutions.

The paper is organized as follows. In Section 2, we prove global existence and boundedness by extending a method in [23] by establishing the \( L^p \)-estimate for \( u \) and \( w \) with \( p > \frac{n}{n-2} \). In Section 3, inspired by the method in [23, 1], we construct some energy-type functionals to obtain the convergence properties of solutions to (7) in the cases \( a_1, a_2 \in (0,1) \) and \( a_1 \geq 1 > a_2 > 0 \) respectively.

2. Global existence and boundedness. In this section, we first prove the local existence of solutions to (7) by means of a straightforward fixed point argument and the strong maximum principle which are adapted in [49].

**Lemma 2.1.** Let \( \chi_i > 0, \mu_i > 0 \), \( a_i > 0 \) \( (i = 1,2) \) and \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a smoothly bounded domain. Then for any \((u_0,w_0)\) satisfying (8), there exist a
maximal $T_{\text{max}} \in (0, \infty]$ and a unique nonnegative function $(u, v, w, z) \in C^0(\overline{\Omega} \times [0, T_{\text{max}}]) \cap C^2(\Omega \times (0, T_{\text{max}}))$ that solve (7) classically. Moreover,

either $T_{\text{max}} = \infty$, or $||u(., t)||_{L^\infty(\Omega)} + ||w(., t)||_{L^\infty(\Omega)} \to \infty$ as $t \to T_{\text{max}}$.

In order to derive an estimate for $(u, w)$ in $L^\infty(\Omega)$, we next estimate the bounds for $||u||_{L^p(\Omega)}$ and $||w||_{L^p(\Omega)}$ with some $p > \frac{n}{2}$.

**Lemma 2.2.** Suppose that (8) and (9) are satisfied. Then for any solution $(u, v, w, z)$ of (7) and $p \in I_1 \cap I_2$, there exists $C(p) > 0$ such that

$$||u(., t)||_{L^p(\Omega)} + ||w(., t)||_{L^p(\Omega)} \leq C(p) \text{ for all } t \in (0, T_{\text{max}}),$$

where

$$I_1 := \left(\frac{n}{2} \chi_1 (\chi_1 - a_1 \mu_1)^+\right), \quad I_2 := \left(\frac{n}{2} \chi_2 (\chi_2 - a_2 \mu_2)^+\right).$$

**Proof.** Noting from the condition (9) to obtain that $I_1 \cap I_2 \neq \emptyset$, we fix $p \in I_1 \cap I_2$. Testing the first equation in (7) by $u^{p-1}$ and using the second equation in (7), we see that

$$\frac{d}{dt} \int_{\Omega} u^p = \int_{\Omega} \Delta u \cdot u^{p-1} - \chi_1 \int_{\Omega} (\nabla u \nabla v + u \Delta v) u^{p-1} + \mu_1 \int_{\Omega} u^p (1 - u - a_1 w)$$

$$= -(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + (p-1) \chi_1 \int_{\Omega} u^{p-1} \nabla u \nabla v + \mu_1 \int_{\Omega} u^p (1 - u - a_1 w)$$

$$= -(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{(p-1) \chi_1}{p} \int_{\Omega} u^p \Delta v + \mu_1 \int_{\Omega} u^p (1 - u - a_1 w)$$

$$= -(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{(p-1) \chi_1}{p} \int_{\Omega} u^p (w - v) + \mu_1 \int_{\Omega} u^p (1 - u - a_1 w).$$

By the positivity of $u$ and $v$, we obtain that

$$\frac{d}{dt} \int_{\Omega} u^p \leq \mu_1 \int_{\Omega} u^p - \mu_1 \int_{\Omega} u^{p+1} - \frac{p-1}{p} \chi_1 \int_{\Omega} u^p w.$$  

From the condition $p \in I_1 \cap I_2$, we infer that

$$a_1 \mu_1 - \frac{p-1}{p} \chi_1 > 0.$$  

By the Hölder inequality, we can estimate

$$\int_{\Omega} u^p \leq |\Omega|^{\frac{p}{p+1}} (\int_{\Omega} u^{p+1})^{\frac{p}{p+1}}.$$  

Thus there exists $\rho_1 > 0$ satisfying

$$\frac{d}{dt} \int_{\Omega} u^p \leq -\rho_1 (\int_{\Omega} u^p)^{\frac{p+1}{p}} + \mu_1 \int_{\Omega} u^p.$$  

This yields

$$||u(., t)||_{L^p} \leq \min \left\{ \frac{\mu_1}{\rho_1} \right\} \text{ for all } t \in (0, T_{\text{max}}).$$

Similarly, we can obtain the $L^p$-estimate for $w$

$$||w(., t)||_{L^p} \leq \min \left\{ \frac{\mu_2}{\rho_2} \right\} \text{ for all } t \in (0, T_{\text{max}}),$$

where $\rho_2 > 0$. This readily yields (16). \qed
Next by using Lemma 2.2 and the semigroup estimates, we can obtain the following lemma.

**Lemma 2.3.** Under the assumptions (8) and (9). Then for any solution $(u, v, w, z)$ of (7), there exists $C > 0$ such that

$$\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) \|_{W^{1,s}(\Omega)} + \| w(\cdot, t) \|_{L^\infty(\Omega)} + \| z(\cdot, t) \|_{W^{1,s}(\Omega)} \leq C$$

(17)

for all $t \geq 0$. Furthermore, there exist some $K > 0$ and $\sigma \in (0, 1)$ satisfying

$$\| u(\cdot, t) \|_{C^{\sigma}((\bar{\Omega} \times [t, t+1]))} + \| v(\cdot, t) \|_{C^{\sigma}((\bar{\Omega} \times [t, t+1]))} + \| w(\cdot, t) \|_{C^{\sigma}((\bar{\Omega} \times [t, t+1]))} + \| z(\cdot, t) \|_{C^{\sigma}((\bar{\Omega} \times [t, t+1]))} \leq K$$

(18)

for all $t \geq 1$.

**Proof.** With $I_1, I_2$ defined in Lemma 2.2, we fix $p \in I_1 \cap I_2 \cap (0, n)$. Based on Lemma 2.2, we can find $C_1, C_2 > 0$ such that

$$\| u(\cdot, t) \|_{L^p(\Omega)} \leq C_1$$

for all $t \in (0, T_{max})$, (19)

and

$$\| w(\cdot, t) \|_{L^p(\Omega)} \leq C_2$$

for all $t \in (0, T_{max})$. (20)

By the standard elliptic regularity argument (see Theorem 19.1 in [5]), there exist $C_\epsilon, C_\epsilon'$ such that

$$\| v \|_{W^{2,\epsilon}(\Omega)} \leq C_\epsilon(q) \| w \|_{L^q(\Omega)}$$

for all $t \in (0, T_{max})$ and $q \in (1, \infty)$, (21)

and

$$\| z \|_{W^{2,\epsilon}(\Omega)} \leq C_\epsilon'(q) \| u \|_{L^q(\Omega)}$$

for all $t \in (0, T_{max})$ and $q \in (1, \infty)$. (22)

By the Sobolev embedding theorem

$$W^{2,p} \hookrightarrow W^{1, \frac{np}{n-p}}.$$ (23)

According to (19)-(23), we can obtain

$$\| \nabla v \|_{L^{\frac{np}{n-p}}(\Omega)} \leq C_1''$$

for all $t \in (0, T_{max})$, (24)

and

$$\| \nabla z \|_{L^{\frac{np}{n-p}}(\Omega)} \leq C_2''$$

for all $t \in (0, T_{max})$. (25)

To establish the $L^\infty$-estimate for $u$, we choose $r > n$ such that

$$p > \frac{nr}{n + r}.$$ (26)

We take $\sigma > 1$, $\sigma' := \frac{\sigma}{\sigma - 1}$ satisfying

$$r \sigma' < \frac{np}{n - p}.$$ (27)

Then $\frac{1}{\sigma} < 1 - \frac{r(n-p)}{np}$. Next we define

$$\Lambda(T) := \sup_{t \in (0, T)} \| u(\cdot, t) \|_{L^\infty(\Omega)}$$

for all $T \in (0, T_{max})$. (28)
which is obviously finite. Next to estimate $\Lambda(T)$, fix $t \in (0, T)$ and let $t_0 = \max\{0, t - 1\}$, and use the variation of constants representation, we obtain

$$u(\cdot, t) = e^{(t-t_0)\Delta}u(t_0) - \chi_1 \int_{t_0}^{t} e^{(t-s)\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) \, ds$$

$$+ \mu_1 \int_{t_0}^{t} e^{(t-s)\Delta} u(\cdot, s)(1 - u(\cdot, s) - a_1 w(\cdot, s)) \, ds$$

$$= \phi_1(\cdot, t) + \phi_2(\cdot, t) + \phi_3(\cdot, t).$$

When $t \leq 1$, due to the fact that the Neumann semigroup is order preserving in $\Omega$, we see that

$$\| \phi_1(\cdot, t) \|_{L^\infty(\Omega)} \leq \| u_0 \|_{L^\infty(\Omega)} \leq d_1$$

for all $t \in (0, 1] \cap (0, T)$. (29)

with $d_1 > 0$ due to $u_0 \in C^0(\Omega)$. When $t > 1$, we apply semigroup estimates (see Lemma 1.3 in [44]) to find $d_2 > 0$ such that

$$\| \phi_1(\cdot, t) \|_{L^\infty(\Omega)} \leq d_2$$

for all $t \in (1, T).$ (30)

with $C_1$ given by (19). Since by the smoothing property of $(e^{t\Delta})_{t \geq 0}$ (see Lemma 3.3 in [6]), we can find $C_3$ such that

$$\| \phi_2(\cdot, t) \|_{L^\infty(\Omega)} \leq C_3 \sup_{t \in (0, T)} \| u(\cdot, t) \|_{L^\infty(\Omega)} \int_0^1 \zeta^{-\frac{1}{4}} d\zeta.$$

And by $r' \leq \frac{np}{n-p}$, (19) and (24), we can use Hölder's inequality to obtain

$$\| u(\cdot, t) \|_{L^\infty(\Omega)} \| \nabla v(\cdot, t) \|_{L^{r'}(\Omega)}$$

$$\leq \| u(\cdot, t) \|_{L^p(\Omega)} \| \nabla v(\cdot, t) \|_{L^{\frac{np}{n-p}}(\Omega)}$$

$$\leq C_4 \Lambda(T)^{1 - \frac{p}{n}}$$

with some $C_4 > 0$. Noting that $n < r$ and $\Lambda(T)$ is finite, there exists $C_5 > 0$ such that

$$\| \phi_3(\cdot, t) \|_{L^\infty(\Omega)} \leq C_5$$

for all $t \in (0, T)$. (31)

As for the $\phi_3$, we note that

$$\mu_1 u(1 - u - a_1 w) \leq -\mu_1 (u - \frac{1 + \mu_1}{2\mu_1})^2 + \frac{(1 + \mu_1)^2}{4\mu_1} \leq \frac{(1 + \mu_1)^2}{4\mu_1},$$

thus by the maximal principle, we can obtain that there exists $C_6 > 0$ such that

$$\| \phi_3(\cdot, t) \|_{L^\infty(\Omega)} \leq C_6$$

for all $t \in (0, T).$ (32)

Since $t \in (0, T)$ is arbitrary, $C_1, \ldots, C_5$ are independent of $t$, then we can infer from (29)-(32) that there exists $C_7 > 0$ such that

$$\Lambda(T) \leq C_7 (1 + \Lambda(T)^{1 - \frac{p}{n}}),$$

(33)

since $C_7 > 0$ is independent of $T$, and by $1 - \frac{p}{n} > 0$, we can obtain that

$$\Lambda(T) \leq C_8$$

for all $T \in (0, T_{\max})$ (34)

with some $C_8 > 0$. Thus we obtain $T_{\max} = \infty$ in view of Lemma 2.1. Similarly, we can verify the $L^\infty$-estimate for $w$. Next by (21) and (22), we can obtain

$$\| v \|_{W^{1,1}(\Omega)} \leq C_9,$$

$$\| z \|_{W^{1,1}(\Omega)} \leq C_9$$

for all $t \in (0, T_{\max})$ (35)
with some $C_9 > 0$. Furthermore, by the known regularity argument (see Proposition 2.3 in [3]), we can find some $K > 0$ and $\sigma \in (0, 1)$ satisfying
\[
\|u(\cdot, t)\|_{C^\sigma(\Omega) \times [t, t+1]} + \|v(\cdot, t)\|_{C^\sigma(\Omega) \times [t, t+1]} + \|w(\cdot, t)\|_{C^\sigma(\Omega) \times [t, t+1]} + \|z(\cdot, t)\|_{C^\sigma(\Omega) \times [t, t+1]} \leq K
\]
for all $t \geq 1$, which implies (18).

**Proof of Theorem 1.1.** The assert of Theorem 1.1 is an immediate consequence of Lemma 2.3.

3. **Asymptotic behavior.** In order to derive the asymptotic behavior of the solutions to (7), it is essential to assume that the solutions of (7) satisfy the conditions in Theorem 1.1, and next we will recall the following important lemmas for the proofs of Theorems 1.2 and 1.3.

**Lemma 3.1.** Let $f(x, t) \in C^0(\bar{\Omega} \times [0, \infty))$ satisfy that there exist constant $C^* > 0$ and $\sigma^* > 0$ such that
\[
\|f(x, t)\|_{C^{\sigma^*}(\bar{\Omega} \times [t, t+1])} \leq C^* \quad \text{for all } t \geq 1.
\]
Assume that there exists some positive constant $M^*$ such that
\[
\int_0^\infty \int_{\Omega} (f(x, t) - M^*)^2 dx dt < \infty.
\]
Then $f(\cdot, t) \to M^*$ in $C^0(\bar{\Omega})$ as $t \to \infty$.

**Proof.** This lemma is a straightforward result from lemma 4.6 in [8].

Next, we generalize Lemma 3.3 in [23] to our model and obtain the following lemma.

**Lemma 3.2.** Let $(u, v, w, z)$ be a solution to (7). Assume that $g : [0, \infty) \to \mathbb{R}$ is a decreasing function satisfies that
\[
\|u(\cdot, t) - u^*\|_{L^2(\Omega)} + \|w(\cdot, t) - w^*\|_{L^2(\Omega)} \leq g(t)
\]
for all $t \geq 1$, then there exists $\varrho > 0$ such that
\[
\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} + \|z(\cdot, t) - z^*\|_{L^\infty(\Omega)} \leq \varrho g^\frac{1}{\alpha + \beta}(t - 1)
\]
for all $t \geq 1$, $m \in \mathbb{N}^+$. 

**Proof.** By the Hölder inequality, we can obtain
\[
\|f\|_{L^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)}^{1 - \frac{2}{p}} \|f\|_{L^2(\Omega)}^{\frac{2}{p}}
\]
where $p > 2$ and $f \in L^\infty(\Omega)$. From the boundedness of $u, w$ and (40), we can obtain that
\[
\|u(\cdot, t) - u^*\|_{L^p(\Omega)} + \|w(\cdot, t) - w^*\|_{L^p(\Omega)} \leq C_1(p)g^\frac{2}{p}(t) \quad \text{for all } t > 0.
\]
with some $C_1(p) > 0$. It follows from (21) and (22) that
\[
\|v(\cdot, t) - v^*\|_{W^{2,2m+2}(\Omega)} + \|z(\cdot, t) - z^*\|_{W^{2,2m+2}(\Omega)} \leq C_2^* (2m + 2) g^\frac{1}{\alpha + \beta}(t)
\]
for all $t > 0$, where $C_2^* (2m + 2)$ is a positive constant. Thus we can infer that
\[
\|\nabla v(\cdot, t)\|_{L^{2m+2}(\Omega)} + \|\nabla z(\cdot, t)\|_{L^{2m+2}(\Omega)} \leq C_2 g^\frac{1}{\alpha + \beta}(t) \quad \text{for all } t > 0
\]
with some $C_2 > 0$. Then from (43) and the downward monotonicity of $g$, there exists $\rho_1 > 0$ such that
\[
\left( \int_{t-1}^t \int_{\Omega} |\nabla v(x, \tau)|^2 \, dx \, d\tau \right)^{\frac{1}{m+2}} \leq \rho_1 g^{\frac{1}{m+2}}(t-1) \quad \text{for all } t > 1.
\] (44)

Inspired by the proof of Lemma 3.6 in [1], we use the variation-of-constants formula to the first equation in (7) to obtain that
\[
\| u(\cdot, t) - u_* \|_{L^\infty(\Omega)}
\leq \| e^{A}(u(\cdot, t_0) - u_*) \|_{L^\infty(\Omega)} + \chi_1 \int_{t-1}^t \| e^{(t-s)A} \nabla \cdot (u(\cdot, s)\nabla v(\cdot, s)) \|_{L^\infty(\Omega)} \, ds
\]
\[
+ \mu_1 \int_{t-1}^t \| e^{(t-s)A} u(\cdot, s)(1 - u(\cdot, s) - a_1 w(\cdot, s)) \|_{L^\infty(\Omega)} \, ds
\]
\[
:= J_1 + J_2 + J_3 \quad \text{for all } t > 1,
\] (45)

where $t_0 = t - 1$. Due to the smoothing properties of the heat semigroup, Hölder inequality and (41), we see that
\[
J_1 \leq \rho_2 (t - (t - 1))^{-\frac{1}{4}} \| u(\cdot, t_0) - u_* \|_{L^{2m}(\Omega)}
\]
\[
\leq \rho_2' \| u(\cdot, t_0) - u_* \|_{L^{2m+2}(\Omega)}
\]
\[
\leq \rho_3 g^{\frac{1}{m+1}}(t-1),
\] (46)

where $\rho_2$, $\rho_2'$ and $\rho_3$ are positive constants. Also, applying the $L^p - L^q$ estimates for $(e^{sA})_{s \geq 0}$, invoking the Hölder inequality and (44), there exist positive constants $\rho_4$ and $\rho_5$ such that
\[
J_2 \leq \rho_4 \int_{t-1}^t (t - s)^{-\frac{1}{4} - \frac{1}{m+2}} \| u(\cdot, s)\nabla v(\cdot, s) \|_{L^{2m+2}(\Omega)} \, ds
\]
\[
\leq \rho_4 \left( \int_{t-1}^t (t - s)^{-\frac{3m+2}{m+2}} \, ds \right)^{\frac{2m+1}{m+2}} \times \left( \int_{t-1}^t \| u(\cdot, s)\nabla v(\cdot, s) \|_{L^{2m+2}(\Omega)}^2 \, ds \right)^{\frac{1}{m+2}}
\]
\[
\leq \rho_5 \| u \|_{L^\infty(\Omega \times (0,\infty))} \| \nabla v \|_{L^{\infty}(\Omega \times (1,\infty))} \times \left( \int_{t-1}^t \int_{\Omega} |\nabla v(x, s)|^2 \, dx \, ds \right)^{\frac{m}{m+2}}
\]
\[
\leq \rho_6 g^{\frac{1}{m+1}}(t-1),
\] (47)

where $\rho_6 = \rho_5 \| u \|_{L^\infty(\Omega \times (0,\infty))} \| \nabla v \|_{L^{\infty}(\Omega \times (1,\infty))}$ is finite thanks to Theorem 1.1 and Lemma 2.3. As for $J_3$, firstly, we note that
\[
u(1 - u - a_1 w) = \begin{cases} (u - u^*) (1 - u - a_1 w) & \text{if } a_1 \geq 1, \\ u((u^* - u) + a_1 (w^* - w)) & \text{if } a_1 < 1, \end{cases}
\] (48)

hence, there exists a $\rho_7 > 0$ such that
\[
\| u(\cdot, s)(1 - u(\cdot, s) - a_1 w(\cdot, s)) \|_{L^{2m}(\Omega)} \leq \| u(\cdot, s)(1 - u(\cdot, s) - a_1 w(\cdot, s)) \|_{L^{2m+2}(\Omega)}
\]
\[
\leq \rho_7 g^{\frac{1}{m+1}}(t-1),
\] (49)

for all $s \in (t-1, t)$. 

Then applying the $L^p - L^q$ estimate and (49) to establish that
\[ J_3 \leq \rho_8 \int_{t-1}^{t} (t-s)^{-\frac{p}{q}} \parallel u(\cdot,s)(1-u(\cdot,s) - a_1 w(\cdot,s)) \parallel_{L^\infty(\Omega)} ds \]
\[ \leq \rho_9 g \frac{1}{a_1} (t-1), \]
with $\rho_8 > 0, \rho_9 > 0$.

Therefore a combination of (46), (47) and (50), we can obtain that
\[ \parallel u(\cdot,t) - u^* \parallel_{L^\infty(\Omega)} \leq \rho_1 g \frac{1}{a_1} (t-1) \] for all $t > 1$.

Similarly, the estimate for $\parallel w - w^* \parallel_{L^\infty(\Omega)}$ can be obtained. Thus there exists $C_3$ such that
\[ \parallel u(\cdot,t) - u^* \parallel_{L^\infty(\Omega)} + \parallel w(\cdot,t) - w^* \parallel_{L^\infty(\Omega)} \leq C_3 g \frac{1}{a_1} (t-1) \] for all $t > 1$.

Since $(u^*,v^*,w^*,z^*)$ satisfies
\[ v^* = w^*, z^* = u^*, \]
and by using the maximal principle to
\[ -\Delta(v - v^*) + (v - v^*) = w - w^* and -\Delta(z - z^*) + (z - z^*) = u - u^*, \]
we can obtain
\[ \parallel v(\cdot,t) - v^* \parallel_{L^\infty(\Omega)} \leq C_4 \parallel w - w^* \parallel_{L^\infty(\Omega)}, \parallel z(\cdot,t) - z^* \parallel_{L^\infty(\Omega)} \leq C_4 \parallel u - u^* \parallel_{L^\infty(\Omega)}, \]
where $C_4$ and $C_4'$ are some positive constants. Here (52) enables us to see that
\[ \parallel v(\cdot,t) - v^* \parallel_{L^\infty(\Omega)} + \parallel z(\cdot,t) - z^* \parallel_{L^\infty(\Omega)} \leq C_5 g \frac{1}{a_1} (t-1) \] for all $t > 1$,
which concludes the proof of this lemma.

\[ \square \]

3.1. The case of $a_1, a_2 \in (0,1)$. In this section, we turn our attention to the asymptotic behaviour of solutions in (7) when $a_1, a_2 \in (0,1)$, the parameter $\chi_i, \mu, i = 1,2$ satisfy (11) and (12), with $(u^*,v^*,w^*,z^*)$ as given by (13), the solution of (7) converges to the steady state exponentially. The cornerstone of our approach is based on the following energy inequality
\[ \frac{d}{dt} H_1(u,v,w,z) \leq -\delta \left\{ \int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 + \int_\Omega (w-w^*)^2 + \int_\Omega (z-z^*)^2 \right\} \]
for all $t > 0$, where $\delta$ is a positive constant. Inspired by lemma 3.4 in [23], we define $H_1(u,v,w,z)$ by
\[ H_1(u,v,w,z) := \int_\Omega \left\{ a_2 \mu_2 (u-u^* - u^* \log \frac{u}{u^*}) + a_1 \mu_1 (w-w^* - w^* \log \frac{w}{w^*}) \right\} \]
for all $t > 0$.

**Lemma 3.3.** Let $a_1, a_2 \in (0,1)$ and assume that (10)-(12) are satisfied. Let $(u,v,w,z)$ be a global bounded classical solution of (7), which is under the same assumptions in Theorem 1.2. Then there exists a constant $\delta > 0$ such that
\[ \frac{d}{dt} H_1(u,v,w,z) \leq -\delta \left\{ \int_\Omega (u-u^*)^2 + \int_\Omega (v-v^*)^2 + \int_\Omega (w-w^*)^2 + \int_\Omega (z-z^*)^2 \right\} \]
for all \( t > 0 \), and \( H_1(u, v, w, z) \) is a nonnegative function. Moreover, there exists \( M > 0 \) satisfying
\[
\int_0^\infty \int_\Omega (u - u^*)^2 + \int_0^\infty \int_\Omega (v - v^*)^2 + \int_0^\infty \int_\Omega (w - w^*)^2 + \int_0^\infty \int_\Omega (z - z^*)^2 \leq M.
\] (58)

**Proof.** With \( H_1(u, v, w, z) \) as defined by (57), it follows from a straightforward computation that
\[
\frac{d}{dt} H_1(u, v, w, z)
= a_2\mu_2 \int_\Omega (1 - \frac{u^*}{u}) |\Delta u - \chi_1 \nabla \cdot (u \nabla v) + \mu_1 u(1 - u - a_1 w) |
+ a_1\mu_1 \int_\Omega (1 - \frac{w^*}{w}) |\Delta w - \chi_2 \nabla \cdot (w \nabla z) + \mu_2 w(1 - w - a_2 u) |
\leq a_2\mu_2 \mu_2 \int_\Omega (u - u^*)(1 - u - a_1 w) + a_1\mu_1 \mu_2 \int_\Omega (w - w^*)(1 - w - a_2 u)
- a_2\mu_2 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} - a_1\mu_1 w^* \int_\Omega \frac{|\nabla w|^2}{w^2} + a_2\mu_2 \chi_1 u^* \int_\Omega \frac{\nabla u \nabla v}{u}
+ a_1\mu_1 \chi_2 w^* \int_\Omega \frac{\nabla w \nabla z}{w}
\] (59)

By Young’s inequality, we obtain
\[
a_2\mu_2 \chi_1 u^* \int_\Omega \frac{\nabla u \nabla v}{u} \leq a_2\mu_2 u^* \int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{a_2\mu_2 \chi_1^2 u^*}{4} \int_\Omega |\nabla v|^2
\] (60)

and
\[
a_1\mu_1 \chi_2 w^* \int_\Omega \frac{\nabla w \nabla z}{w} \leq a_1\mu_1 w^* \int_\Omega \frac{|\nabla w|^2}{w^2} + \frac{a_1\mu_1 \chi_2^2 w^*}{4} \int_\Omega |\nabla z|^2.
\] (61)

Together with (59), (60) and (61), we derive
\[
\frac{d}{dt} H_1(u, v, w, z)
\leq a_2\mu_1 \mu_2 \int_\Omega (u - u^*)(1 - u - a_1 w) + a_1\mu_1 \mu_2 \int_\Omega (w - w^*)(1 - w - a_2 u)
+ \frac{a_2\mu_2 \chi_1^2 u^*}{4} \int_\Omega |\nabla v|^2 + \frac{a_1\mu_1 \chi_2^2 w^*}{4} \int_\Omega |\nabla z|^2
\leq a_2\mu_1 \mu_2 \int_\Omega (u - u^*)(u^* - u + a_1 w^* - a_1 w) + \frac{a_2\mu_2 \chi_1^2 u^*}{4} \int_\Omega |\nabla v|^2
+ a_1\mu_1 \mu_2 \int_\Omega (w - w^*)((w^* - w + a_2 w^* - a_2 w) + \frac{a_1\mu_1 \chi_2^2 w^*}{4} \int_\Omega |\nabla z|^2
= -a_2 k_1 \mu_1 \mu_2 \int_\Omega (u - u^*)^2 - a_2(1 - k_1) \mu_1 \mu_2 \int_\Omega (u - u^*)^2
- a_1 k_2 \mu_1 \mu_2 \int_\Omega (w - w^*)^2 - a_1(1 - k_2) \mu_1 \mu_2 \int_\Omega (w - w^*)^2
- 2a_1 a_2 \mu_1 \mu_2 \int_\Omega (u - u^*)(w - w^*) + \frac{a_1\mu_1 \chi_2^2 w^*}{4} \int_\Omega |\nabla z|^2
\] (62)
where \( k_1, k_2 \in (0,1) \) and \( k_1k_2 > a_1a_2 \). Noting from \( v^* = w^* \) and the second equation in (7) that

\[
\int_{\Omega} |\nabla v|^2 = -\int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)(v - v^*),
\]

solving for \( \delta \) similarly, we can obtain

\[
\int_{\Omega} |\nabla z|^2 = -\int_{\Omega} (z - z^*)^2 + \int_{\Omega} (u - u^*)(z - z^*).
\]

Next, by (62), (63), and (64), we establish

\[
\frac{d}{dt} H_1(u, v, w, z) \leq -a_2k_1\mu_1\mu_2 \int_{\Omega} (u - u^*)^2 - a_2(1 - k_1)\mu_1\mu_2 \int_{\Omega} (u - u^*)^2
- a_1k_2\mu_1\mu_2 \int_{\Omega} (w - w^*)^2 - a_1(1 - k_2)\mu_1\mu_2 \int_{\Omega} (w - w^*)^2
- 2a_1a_2\mu_1\mu_2 \int_{\Omega} (u - u^*)(w - w^*)
- \frac{a_1\mu_1^2w_*^2}{4} \int_{\Omega} (z - z^*)^2 + \frac{a_1\mu_1^2w_*^2}{4} \int_{\Omega} (u - u^*)(z - z^*)
- \frac{a_2\mu_2^2u_*^2}{4} \int_{\Omega} (v - v^*)^2 + \frac{a_2\mu_2^2u_*^2}{4} \int_{\Omega} (w - w^*)(v - v^*).
\]

According to the conditions (10)-(12), we can choose a constant \( \delta > 0 \) satisfying

\[
\delta \leq \min \left\{ \frac{a_1a_2\mu_1\mu_2[k_1k_2 - a_1a_2]}{a_2k_1 + a_1k_2}, \frac{a_2\mu_2^2u_*^2[16a_1\mu_1(1 - k_2) - a_2w_*^2\chi_1^2]}{16[a_1w_*^2\chi_2^2 + 4a_2(1 - k_1)]}, \right. \\
\left. \frac{a_1\mu_1^2w_*^2[16a_2\mu_2(1 - k_1) - a_1w_*^2\chi_2^2]}{16[a_1w_*^2\chi_2^2 + 4a_2(1 - k_1)]} \right\},
\]

then we can obtain

\[
\frac{d}{dt} H_1(u, v, w, z) \leq -\delta \left\{ \int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 + \int_{\Omega} (w - w^*)^2 + \int_{\Omega} (z - z^*)^2 \right\}
- (a_2k_1\mu_1\mu_2 - \delta) \int_{\Omega} (u - u^*)^2 - (a_2(1 - k_1)\mu_1\mu_2 - \delta) \int_{\Omega} (u - u^*)^2
- (a_1k_2\mu_1\mu_2 - \delta) \int_{\Omega} (w - w^*)^2 - (a_1(1 - k_2)\mu_1\mu_2 - \delta) \int_{\Omega} (w - w^*)^2
- 2a_1a_2\mu_1\mu_2 \int_{\Omega} (u - u^*)(w - w^*)
- \left( \frac{a_1\mu_1^2w_*^2}{4} - \delta \right) \int_{\Omega} (z - z^*)^2 + \frac{a_1\mu_1^2w_*^2}{4} \int_{\Omega} (u - u^*)(z - z^*)
- \left( \frac{a_2\mu_2^2u_*^2}{4} - \delta \right) \int_{\Omega} (v - v^*)^2 + \frac{a_2\mu_2^2u_*^2}{4} \int_{\Omega} (w - w^*)(v - v^*).
\]

Thanks to (66), we establish

\[
(2a_1a_2\mu_1\mu_2)^2 - 4(a_2k_1\mu_1\mu_2 - \delta)(a_1k_2\mu_1\mu_2 - \delta) \leq 0,
\]
of Lemma 3.3, we should obtain the classical solution of (7), which is under the same assumptions in Theorem 1.2. Then

\[
\frac{a_2}{4} \left( \frac{a_2 u^*}{2} \right)^2 - 4(a_1 - k_2)u_0 - \delta \left( \frac{a_2 u^*}{2} \right) \leq 0,
\]

(69)

as well as

\[
\frac{a_1}{4} \left( \frac{a_1 w^*}{2} \right)^2 - 4(a_2 - k_1)u_0 - \delta \left( \frac{a_1 w^*}{2} \right) \leq 0.
\]

(70)

By (68), (69) and (70), we can obtain

\[
-(a_2 k_1 + \delta) \int_\Omega (u - u^*)^2 - (a_1 k_2 + \delta) \int_\Omega (w - w^*)^2
\]

\[
-2a_1 a_2 \int_\Omega (u - u^*)(w - w^*) \leq 0,
\]

(71)

and

\[
-(a_2 k_1 + \delta) \int_\Omega (v - v^*)^2 + \frac{a_2}{4} \left( \frac{a_2 u^*}{2} \right) \int_\Omega (w - w^*)(v - v^*)
\]

\[
-(a_1 k_2 + \delta) \int_\Omega (w - w^*)^2 \leq 0
\]

(72)

as well as

\[
-(a_2 k_1 + \delta) \int_\Omega (z - z^*)^2 + \frac{a_1}{4} \left( \frac{a_1 w^*}{2} \right) \int_\Omega (u - u^*)(z - z^*)
\]

\[
-(a_2 k_1 + \delta) \int_\Omega (w - w^*)^2 \leq 0.
\]

(73)

Therefore by a combination of (67), (71), (72) and (73), we can obtain

\[
\frac{d}{dt} H_1(u, v, w, z) \leq -\delta \left\{ \int_\Omega (u - u^*)^2 + \int_\Omega (v - v^*)^2 + \int_\Omega (w - w^*)^2 + \int_\Omega (z - z^*)^2 \right\}
\]

(74)

for all \( t > 0 \). Then from the Taylor formula \( H_1(u, v, w, z) \) is a nonnegative function for \( t > 0 \) (see Lemma 3.2 in [1]). Finally, integrating the above inequality over \( (0, \infty) \), which yields (58).

\[\square\]

**Lemma 3.4.** Assume that (10)-(12) are satisfied. Let \((u, v, w, z)\) be a global bounded classical solution of (7), which is under the same assumptions in Theorem 1.2. Then

\[
\| u(\cdot, t) - u^* \|_{L^\infty(\Omega)} + \| v(\cdot, t) - v^* \|_{L^\infty(\Omega)} + \| w(\cdot, t) - w^* \|_{L^\infty(\Omega)} + \| z(\cdot, t) - z^* \|_{L^\infty(\Omega)} \to 0 \quad \text{as} \quad t \to \infty.
\]

(75)

**Proof.** A combination of Lemma 3.1 and Lemma 3.3 implies this lemma. \(\square\)

Next, in order to establish the convergence rates for the solution of (7), in view of Lemma 3.3, we should obtain the \(L^2\)-convergence rate for the solution.

**Lemma 3.5.** Assume that (10)-(12) are satisfied. Then there exist \( C > 0 \) and \( m > 0 \) such that

\[
\| u(\cdot, t) - u^* \|_{L^2(\Omega)} + \| v(\cdot, t) - v^* \|_{L^2(\Omega)} \leq C e^{-mt} \quad \text{for all} \quad t > 0.
\]

(76)

**Proof.** According to L’Hôpital’s rule, by Lemma 3.4 and a similar argument in the proof of Lemma 3.7 in [1], there exists some \( t_0 > 0 \) such that

\[
C_1 \left\{ \| u(\cdot, t) - u^* \|_{L^2(\Omega)}^2 + \| v(\cdot, t) - v^* \|_{L^2(\Omega)}^2 \right\} \leq H_1(t)
\]

\[
\leq C_2 \left\{ \| u(\cdot, t) - u^* \|_{L^2(\Omega)}^2 + \| v(\cdot, t) - v^* \|_{L^2(\Omega)}^2 \right\}
\]

(77)
for all \( t > t_0 \) with \( C_1, C_2 > 0 \), where \( H_1(t) \) is given by (57). Then it follows from (3.3) that there exist \( C_3 > 0 \) such that

\[
\frac{d}{dt} H_1(t) \leq -C_3 H_1(t),
\]

which implies

\[
H_1(t) \leq C_4 e^{-C_3 t} \quad \text{for all } t > 0
\]

with \( C_4 > 0 \). Finally, by (77) and (79), we derive

\[
\| u(\cdot, t) - u^* \|_{L^2(\Omega)}^2 + \| w(\cdot, t) - w^* \|_{L^2(\Omega)}^2 \leq C_5 e^{-C_3 t} \quad \text{for all } t > 0
\]

with \( C_5 > 0 \).

**Proof of Theorem 1.2.** Lemma 3.2 and Lemma 3.5 directly show Theorem 1.2.

3.2. The case of \( a_1 \geq 1 > a_2 > 0 \). In this case, by using the similar approach in section 3.1, we will obtain that the first species becomes extinct asymptotically when \( \mu_2 \) satisfies (15). The key ingredients are base on the functional \( H_2(u, v, w, z) \) defined as follows

\[
H_2(u, v, w, z) := a_2 \mu_2 \int_\Omega u + a_1 \mu_1 \int_\Omega (w - \log w - 1) \quad \text{for all } t > 0.
\]

Thanks to (14), we can obtain a similar inequality as Lemma 3.3 in the following lemma.

**Lemma 3.6.** Assume that (14) and (15) are satisfied. Then there exist a constant \( \xi > 0 \) such that

\[
\frac{d}{dt} H_2(t) \leq -\xi \left\{ \int_\Omega u^2 + \int_\Omega (w - 1)^2 + \int_\Omega z^2 \right\}
\]

for all \( t > 0 \), and \( H_2(t) = H_2(u, v, w, z) \) is a nonnegative function. Furthermore, there exists some positive constant \( M' \) satisfying

\[
\int_0^\infty \int_\Omega u^2 + \int_0^\infty \int_\Omega (w - 1)^2 + \int_0^\infty \int_\Omega z^2 \leq M'.
\]

**Proof.** By the first equation and third equation in (7), we obtain

\[
\frac{d}{dt} H_2(t) = a_2 \mu_2 \int_\Omega u_t + a_1 \mu_1 \int_\Omega (w_t - \frac{w_1}{w})
\]

\[
= -a_2 \mu_1 \mu_2 \int_\Omega u^2 - 2a_1 a_2 \mu_1 \mu_2 \int_\Omega u(w - 1) - (a_1 - 1) a_2 \mu_1 \mu_2 \int_\Omega u
\]

\[
- a_1 \mu_1 \int_\Omega \frac{1}{w^2} |\nabla w|^2 + a_1 \mu_1 \mu_2 \int_\Omega \frac{1}{w} \nabla w \nabla z - a_1 \mu_1 \mu_2 \int_\Omega (w - 1)^2.
\]

By Young’s inequality, we have

\[
a_1 \mu_1 \mu_2 \int_\Omega \frac{1}{w} \nabla w \nabla z \leq a_1 \mu_1 \int_\Omega \frac{|\nabla w|^2}{w^2} + \frac{a_1 \mu_1 \mu_2^2}{4} \int_\Omega |\nabla z|^2.
\]

Multiplying the fourth equation in (7) with \( z \), we derive

\[
\int_\Omega |\nabla z|^2 = - \int_\Omega z^2 + \int_\Omega z u.
\]
By (14) and (15), we can choose a constant \( \xi > 0 \) such that
\[
\xi \leq \min \left\{ \frac{a_2a_1\mu_1\mu_2 - a_1^2\mu_2^2}{a_2l + a_1}, \frac{16a_1a_2(1-l)\mu_1\chi_2^2\mu_2 - a_1^2\mu_1\chi_2^2}{16|4a_2(1-l)\mu_2 + a_1\chi_2^2|} \right\},
\]
due to the fact that \( a_1 \geq 1 \), and a combination of (84), (85) and (86), then we establish
\[
\frac{d}{dt} H_2(t)
\leq -a_2\mu_2 a_1 l \int_\Omega u^2 - 2a_1 a_2 \mu_1 \mu_2 \int_\Omega u(w-1) - (a_1 - 1)a_2 \mu_1 a_2 \int_\Omega u
\]
\[
+ \frac{a_1 \mu_1^2 \chi_2^2}{4} \int_\Omega |\nabla z|^2 - a_1 \mu_1 a_2 \int_\Omega (w-1)^2
\]
\[
\leq -(\mu_1 a_2 l - \xi) \int_\Omega u^2 - [a_2 \mu_1 (1-l) - \xi] \int_\Omega u^2 - 2a_1 a_2 \mu_1 \mu_2 \int_\Omega u(w-1)
\]
\[
- (\frac{a_1 \mu_1^2 \chi_2^2}{4} - \xi) \int_\Omega |z|^2 + \frac{a_1 \mu_1^2 \chi_2^2}{4} \int_\Omega uz - (a_1 \mu_1 a_2 - \xi) \int_\Omega (w-1)^2
\]
\[-\xi \int_\Omega u^2 - \xi \int_\Omega (w-1)^2 - \xi \int_\Omega z^2.
\]
thanks to (87), we obtain
\[
(2a_1 a_2 \mu_1 a_2)^2 - 4(\mu_1 a_2 l - \xi)(\mu_1 a_2 a_1 - \xi) \leq 0
\]
and
\[
(\frac{a_1 \mu_1^2 \chi_2^2}{4})^2 - (a_1 \mu_1 l - 4\xi)(a_2 \mu_1 a_2 (1-l) - \xi) \leq 0.
\]
by (89) and (90), we obtain
\[
-(\mu_1 a_2 l - \xi) \int_\Omega u^2 - 2a_1 a_2 \mu_1 \mu_2 \int_\Omega u(w-1) - (\mu_1 a_2 a_1 - \xi) \int_\Omega (w-1)^2 \leq 0
\]
and
\[
-(a_2 \mu_1 a_2 (1-l) - \xi) \int_\Omega u^2 - (\frac{a_1 \mu_1 a_2 (1-l)}{4} - \xi) \int_\Omega |z|^2 + \frac{a_1 \mu_1 l}{4} \int_\Omega uz \leq 0,
\]
which yields
\[
\frac{d}{dt} H_2(t) \leq -\xi \int_\Omega u^2 - \xi \int_\Omega (w-1)^2 - \xi \int_\Omega z^2
\]
for all \( t > 0 \). Then by Lemma 3.4 in [1], we can see that \( H_2(u, v, w, z) \) is a nonnegative function for \( t > 0 \). Finally, integrating the above inequality over \( (0, \infty) \), which yields (83).

Lemma 3.7. Assume that (14)-(15) are satisfied. Let \( (u, v, w, z) \) be a global bounded classical solution of (7), which is under the same assumptions in Theorem 1.3. Then
\[
\| u(\cdot, t) \|_{L^\infty(\Omega)} + \| v(\cdot, t) - 1 \|_{L^\infty(\Omega)} + \| w(\cdot, t) - 1 \|_{L^\infty(\Omega)}
\]
\[
+ \| z(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty.
\]
Proof. A combination of Lemma 3.1 and Lemma 3.6 implies that
\[ \| u(\cdot, t) \|_{L^\infty(\Omega)} + \| w(\cdot, t) - 1 \|_{L^\infty(\Omega)} + \| z(\cdot, t) \|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty. \] (95)
Invoking the standard \( L^p - L^q \) estimates for the Neumann heat semigroup (see [15]) applied the second equation in (7), we establish
\[ \| v(\cdot, t) - 1 \|_{L^\infty(\Omega)} \to 0 \text{ as } t \to \infty, \] (96)
which completes the proof of this lemma. \qed

Next we shall establish the the convergence rates for the solution of (7) when \( a_1 \geq 1 > a_2 > 0 \) by using the similar method above, we should obtain the \( L^2 \)-convergence rate for the solution.

Lemma 3.8. Let \( a_1 > 1 > a_2 > 0 \). Assume that (14)-(15) are satisfied. Let \((u, v, w, z)\) be a global bounded classical solution of (7), which satisfies the same assumptions in Theorem 1.3. Then there exist \( C > 0 \) and \( m_1 > 0 \) such that
\[ \| u(\cdot, t) \|_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \leq C e^{-m_1 t} \text{ for all } t > 0. \] (97)
Proof. By using Lemma 3.4 and a similar argument in the proof of Lemma 4.3 in [25], there exists some \( t_0 > 0 \) such that
\[ C_1 \left\{ \int_{\Omega} u(\cdot, t) + \| u(\cdot, t) \|_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|_{L^2(\Omega)}^2 \right\} \leq H_2(t) \]
\[ \leq C_2 \left\{ \int_{\Omega} u(\cdot, t) + \| u(\cdot, t) \|_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|_{L^2(\Omega)}^2 \right\} \text{ for all } t > t_0 \] (98)
with \( C_1, C_2 > 0 \), where \( H_2(t) \) is given by (81). Then by (82) we can derive that there exists \( C_3 > 0 \) such that
\[ \frac{d}{dt} H_2(t) \leq -C_3 H_2(t), \] (99)
which yields
\[ H_2(t) \leq C_4 e^{-C_3 t} \text{ for all } t > 0 \] (100)
with \( C_4 > 0 \). Finally, by (98) and (100), we can obtain
\[ \| u(\cdot, t) \|_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \leq C_5 e^{-C_3 t} \text{ for all } t > 0 \] (101)
with \( C_5 > 0 \), which concludes the proof of this lemma. \qed

Lemma 3.9. Let \( a_1 = 1 > a_2 > 0 \). Assume that (14)-(15) are satisfied. Let \((u, v, w, z)\) be a global bounded classical solution of (7), which is under the same assumptions in Theorem 1.3. Then there exist \( C' > 0 \) such that
\[ \| u(\cdot, t) \|_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \leq \frac{C'}{\sqrt{t+2}} \text{ for all } t > 0. \] (102)
Proof. By using lemma 3.4 and a similar argument in the proof of Lemma 3.7 in [1], there exists some \( t'_0 > 0 \) such that
\[ C'_1 \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \leq \int_\Omega (w - \log w - 1) \leq C'_2 \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \text{ for all } t > t'_0 \] (103)
with $C_1', C_2' > 0$, then by the definition of $H_2(t)$, there exists $C_3 > 0$ such that
\[
H_2(t) = a_2 \mu_2 \int_{\Omega} u + a_1 \mu_1 \int_{\Omega} (w - \log w - 1) \\
\leq a_2 \mu_2 \int_{\Omega} u + a_1 \mu_1 C_2' \| w(\cdot, t) - 1 \|^2_{L^2(\Omega)} \\
\leq C_3' \| u \|_{L^2(\Omega)} + C_3' \| w(\cdot, t) - 1 \|_{L^2(\Omega)} \\
\leq \sqrt{2} C_3' \left( \| u \|^2_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|^2_{L^2(\Omega)} \right) \frac{1}{2} \text{ for all } t > t'_0.
\]
According to (82), we establish
\[
\frac{d}{dt} H_2(t) \leq -C_4' H_2^2(t) \text{ for all } t > t'_0
\]
with $C_4' > 0$. Finally, thanks to (103), (105) and the boundedness of $u$, we can find some $C_5', C_6' > 0$ such that
\[
\| u(\cdot, t) \|^2_{L^2(\Omega)} + \| w(\cdot, t) - 1 \|^2_{L^2(\Omega)} \leq C_4' H_2(t) \leq \frac{C_6'}{t + 2} \text{ for all } t > t'_0,
\]
which yields this lemma.

Proof of Theorem 1.3. A combination of Lemma 3.2, Lemma 3.8 and Lemma 3.9 yields the results in Theorem 1.3., Thus we complete our proof.

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