RANDOM MATRICES AND SUPERSYMMETRY IN DISORDERED SYSTEMS

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Abstract
It is described how one comes to the Wigner-Dyson random matrix theory (RMT) starting from a model of a disordered metal. The lectures start with a historical introduction where basic ideas of the RMT and theory of disordered metals are reviewed. This part is followed by an introduction into supermathematics (mathematics operating with both commuting and anticommuting variables). The main ideas of the supersymmetry method are given and basic formulae are derived. Both level-level correlations and fluctuations of amplitudes of wave functions are discussed. It is shown how one can both obtain known formulae of the RMT and go beyond. In the last part some recent progress in the further development of the method and possible perspectives are discussed.

Keywords: Random matrices, disordered systems, supersymmetry, non-linear sigma-model.

Introduction
Wigner-Dyson Theory
According to basic principles of quantum mechanics the energy spectrum of a particle in a limited volume is discrete. The precise values of the energy depend on the boundary conditions and the interactions in system. In many cases these quantities can be calculated with a certain accuracy. However, often the interactions are so complicated that calculations for the single levels become impossible. On the other hand, the complexity of the interactions can lead to the idea of a statistical description in which information about separate levels is neglected and only averaged quantities are studied. Density of states, energy level and
wave functions correlations are quantities that can be studied in the statistical approach. Sometimes, it is sufficient to study the average of, e.g., the density of states or its variance. In other cases one may be interested in a full statistical description that can be achieved calculating distribution functions. Studying the level statistics is, to some extent, analogous to the statistical study of the motion of atoms and molecules, which is the subject of statistical physics.

The idea of the statistical description of the energy levels was first proposed by Wigner [1] for study of highly excited nuclear levels in complex nuclei. In such nuclei a large number of particles interact in an unknown way and the main assumption was that the interactions were equally probable. Of course, in order to specify the meaning of the words “equally probable” one had to formulate a statistical hypothesis in terms of a probability distribution that would play the role of the Gibbs distribution.

This was done in Ref. [1] in the following way. Choosing a complete set of eigenfunctions as a basis, one represents the Hamiltonian $H$ as a matrix with matrix elements $H_{mn}$. The matrix elements $H_{mn}$ are assumed random with a certain probability distribution. It is clear that the distribution should not depend of the basis chosen, which implies an invariant form of the distribution function. In the language of the random matrices $H_{mn}$ the corresponding distribution function can contain only $Tr f (H)$, where $f$ is a function.

The first statistical theory [1] was based on a Gaussian distribution. According to the Gaussian statistical hypothesis a physical system having $N$ quantum states has the statistical weight $D(H)$

$$D(H) = A \exp \left[ -\sum_{m,n=1}^{N} \frac{|H_{mn}|^2}{2a^2} \right] = A \exp \left[ -\frac{Tr H^2}{2a^2} \right]$$

(1)

In Eq. (1) the parameter $a$ is a cutoff excluding strong interaction, $H$ is a random $N \times N$ matrix, and $A$ is a normalization coefficient.

It is important to emphasize that the weight $D(H)$ is rather arbitrary and other forms for the distribution functions can be suggested. Of course, the dependence of the mean energy level spacing $\Delta(\varepsilon)$

$$\Delta^{-1}(\varepsilon) = \langle tr(\delta(\varepsilon - H)) \rangle_D$$

(2)

on the energy $\varepsilon$ is different for different distributions $D(H)$. (In Eq. (2) the symbol $\langle \ldots \rangle_D$ stands for the averaging with the distribution $D(H)$). For example, the distribution function $D(H)$ for the Gaussian distribution, Eq. (1), has the form of a semicircle (Wigner semicircle law) (for a review, see, e.g. Ref.[2]).
What is more interesting, level correlations described, for example, by the level-level correlation function $R(\omega)$

$$R(\omega) = \Delta^2(\varepsilon) \langle Tr \delta(\varepsilon - H) Tr \delta(\varepsilon - \omega - H) \rangle$$

prove to be universal in the limit $N \to \infty$ provided the energy $\omega$ is much smaller than the characteristic scale of the variation of $\Delta(\varepsilon)$. For the Wigner semicircle law, the latter condition means that the energy $\varepsilon$ is not close to the points $\varepsilon_0, -\varepsilon_0$, where the quantity $\Delta^{-1}(\varepsilon)$ proportional to the average density of states turns to zero.

In the absence of magnetic interactions violating the time reversal symmetry, the wave functions and matrix elements $H_{mn}$ in Eq. (1) can be chosen real. In this case the statistical properties of the systems are described by real symmetric random matrices. The ensemble of real symmetric matrices with the Gaussian distribution is often called the Gaussian orthogonal ensemble (GOE).

If the magnetic interactions are present in the system, the time reversal and spin-rotation symmetry is violated and the wave functions are no longer real. This means that one should deal with general Hermitian matrices without any additional symmetry and integrate over the matrix elements $H_{mn}$ using only the constraint $H_{mn} = (H_{nm})^\ast$. This system is called the Gaussian unitary ensemble (GUE).

The third possible type of the symmetry arises when the system is time-reversal invariant but does not have central symmetry. In this case it is also impossible to make all the matrix elements real. Nevertheless an additional symmetry exists in this case. According to the Kramers theorem all levels of the system remain doubly degenerate and every eigenvalue of the matrix $H$ must appear twice. Matrices consisting of real quaternions $H_{mn}$ of the form

$$\begin{pmatrix}
p_{mn} & q_{mn} \\
-q_{mn}^\ast & p_{mn}^\ast
\end{pmatrix}$$

and satisfying the condition $H_{mn} = (H_{nm})^+$ have this property. The corresponding ensemble is called the Gaussian symplectic ensemble (GSE).

Somewhat different distribution functions were introduced later by Dyson [3] who suggested characterizing the system not by its Hamiltonian but by an unitary $N \times N$ matrix $S$ whose elements give the transition probabilities between states, where, again, $N$ is the number of the levels. This matrix is related to the Hamiltonian $H$ of the system in a complicated way that is not specified in the theory. According to the Dyson hypothesis the correlation properties of $n$ successive energy levels of the system ($n \ll N$) are statistically equivalent to those of $n$
successive angles provided all the unitary matrices $S$ have equal probabilities. Again, depending on the symmetry, one can distinguish among the three different ensembles. The corresponding ensembles are called Circular ensembles.

As concerns the complex nuclei, the orthogonal ensembles are most relevant for their description because in order to change the level statistics one needs, e.g. huge magnetic fields that hardly exist. However, the other two ensembles have been under intensive discussions for problems of mesoscopic physics (for a review, see [4]). Moreover, it has been realized that one might formulate additional ensembles of e.g. chiral matrices relevant for studying properties of models for QCD (for a review, see the lecture by Jac Verbaarschot [5]). It has been proven later that in total 10 different symmetry classes exist [6]. Most of the new ensembles may be relevant to different disordered mesoscopic systems due the presence of, e.g., superconductivity or additional symmetries of the lattice.

I do not plan to discuss in these lectures the non-standard symmetry classes and restrict myself by the Wigner-Dyson (WD) statistics. It is relevant to say that calculation of the level-level correlation function, Eq. (3), starting from the Gaussian or Circular ensembles is not a simple task. The conventional method of the evaluation is using orthogonal polynomials [2]. The procedure is not difficult for the unitary ensembles but one has to put a considerable effort to perform the calculations for the orthogonal and symplectic ones. As it has been mentioned, the final results are universal in the limit $N \to \infty$ and can be written for the orthogonal, unitary and symplectic ensembles in the form

$$ R_{\text{orth}}(\omega) = 1 - \frac{\sin^2 x}{x^2} - \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_1^\infty \frac{\sin xt}{t} dt $$

$$ R_{\text{unit}}(\omega) = 1 - \frac{\sin^2 x}{x^2} $$

$$ R_{\text{sym}}(\omega) = 1 - \frac{\sin^2 x}{x^2} + \frac{d}{dx} \left( \frac{\sin x}{x} \right) \int_0^1 \frac{\sin xt}{t} dt $$

where $x = \pi \omega / \Delta$, and $\Delta$ is the mean level spacing.

The functions $R(\omega)$, Eqs. (4, 5, 6), tend to 1 in the limit $\omega \to \infty$, which means that the correlations are lost in this limit. In the opposite limit $x \to 0$ they turn to zero as $x^\beta$, where $\beta = 1, 2, 3$ for the orthogonal, unitary and symplectic ensembles, respectively. This means that the probability of finding a level at the distance $\omega$ from another level decays at small $\omega$. The effect is know as “level repulsion”. It is important that Eqs. (4-6) describe the level-level correlation function, Eq. (3), for both the Gaussian and Circular ensembles.
Random Matrices and Supersymmetry

Small disordered particles

Nowadays the relevance of the random matrix theory (RMT) to mesoscopic physics is almost evident and it is the starting point of many works on transport in quantum dots, electromagnetic response of metallic grains, etc. However, it took quite a long time before the ideas of the RMT penetrated from nuclear to condensed matter physics.

There were several reasons for this slow development. First, until the end of 60’s of the last century most of the objects studied in condensed matter physics were macroscopic and the discreteness of the energy levels could be neglected. Therefore the question about the level statistics was not so interesting. Second, from the theoretical point of view it was not clear at all how one could come to the Wigner-Dyson level statistics starting from the Schrödinger equation. It was clear that making perturbation theory in interaction or disorder did not lead to the anything that would resemble Eqs. (4-6).

As soon as experimentalists started investigation of granular materials (which happened in 60’s), a theory that would describe small metal systems became necessary. Such materials consist of small metal particles (grains) with the diameter down to $10 - 100\text{Å}$. These grains can be covered by an insulator and therefore be well isolated from each other. A schematic view of the pattern can be found in Fig.1.

It is clear that one can speak now about discrete levels and study their statistics. Of course, one needs low temperatures in order to prevent inelastic processes smearing the levels but this is not so difficult (temperatures $< 1K$ can be sufficient).

In practice, the form of the grains can be not very regular, they can contain defects and impurities and therefore the energy spectrum strongly fluctuates from grain to grain. All this true even if one neglects
the electron-electron interaction. So, one naturally comes to the idea to describe the levels statistically.

The first work on the application of the RMT to small metallic grains was done by Gorkov and Eliashberg (GE) [7]. These authors studied the electromagnetic response of the system of the grains and therefore they needed an information about the level-level correlation in a single grain. As the RMT is purely phenomenological and nothing is assumed about the origin of the randomness, this theory was taken by GE to describe the correlations. Starting from the explicit form of the level-level correlation function $R(\omega)$, Eqs. (4-6), they calculated the desired physical quantity.

GE identified correctly physical situations when the three symmetry classes might be used. In the absence of any magnetic and spin orbit interactions they suggested to use the orthogonal ensemble. If a magnetic field is applied or there are magnetic impurities in the grains, the unitary ensemble should be applicable. If there are no magnetic interactions but spin-orbital impurities are present, the grains should be described by the symplectic ensemble.

Being the first application of the RMT in solid state physics, the paper [7] remained the only application during the next 17 years. This is not surprising because using calculational schemes existed in that time no indication in non-trivial level correlations could be seen. Let us discuss this point in more details.

Studying a disordered system and neglecting electron-electron interactions one can start with the following Hamiltonian

$$H = \varepsilon (\hat{\mathbf{p}}) + U (\mathbf{r})$$  \hspace{1cm} (7)

where $\varepsilon (\hat{\mathbf{p}})$ is the operator of the kinetic energy and calculate the Green function $G_\varepsilon = (\varepsilon - H)^{-1}$ performing an expansion in the disorder potential $U (\mathbf{r})$. Usually, it is assumed that the $U (\mathbf{r})$ is random and its fluctuations are Gaussian with

$$\langle U (\mathbf{r}) \rangle = 0, \quad \langle U (\mathbf{r}) U (\mathbf{r}') \rangle = \frac{1}{2\pi \nu \tau} \delta (\mathbf{r} - \mathbf{r}')$$  \hspace{1cm} (8)

where $\nu$ is the density of states and $\tau$ is the mean scattering time.

Making the perturbation theory in the random potential $U (\mathbf{r})$ and averaging over this potential can be done using the “cross technique” [8]. Only diagrams without intersections of the impurity lines are important in the standard approximation of the weak disorder. For the one particle Green function the typical diagrams are represented in Fig.2.
As a result, one obtains for the averaged Green function $\langle G \rangle$ the following expression

$$G = \frac{1}{\varepsilon - \varepsilon(p) \pm i/2\tau}$$

where “+” corresponds to the retarded and “−” to the advanced Green functions.

Standard calculations based on a summation of “ladder diagrams” lead in this case to the classical Drude formula for, e.g., the conductivity $\sigma(\omega)$

$$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}, \quad \sigma_0 = 2e^2\nu D_0$$

where $D_0 = v_0^2\tau/3$ is the classical diffusion coefficient, $\omega$ is the frequency and $e$ is the electron charge.

So long as the grain size remained larger than the atomic distances, no deviations from this formula could be found and therefore the question about the applicability of the RMT remained open. It is relevant to mention works by Kubo performed in approximately the same time. In Ref. [9] he argued that even very small irregularities (with size of the order of atomic distances) of the shape of the metallic grains must lead to lifting of all degeneracies of eigenstates that are present in ideally spherical particles. This lead him to the conclusion that the mean energy level spacing $\Delta$ is inversely proportional to the volume $V$ of the particle

$$\Delta = (\nu V)^{-1}$$

where $\nu$ is density of states at the Fermi surface of the metal. Later Kubo suggested [10] that the spacing distribution had to follow the Poisson law. The latter differs essentially from the Wigner-Dyson RMT by the
Figure 3. Cooperon: a singular correction to conductivity

absence of the level repulsion. The second work by Kubo, Ref.[10], was published several years after the work by Gorkov and Eliashberg, Ref. [7], and this shows that the applicability of the Wigner-Dyson statistics to the small metal particles was far from being established in that time.

The situation started to change only at the end of 70’s with the new developments in the theory of Anderson localization. In the publication [11] a new scaling idea was put forward for description of disordered samples of an arbitrary dimensionality. The most unusual was a prediction that two dimensional initially metallic samples could not remain metals in the presence of an arbitrary weak disorder and had to acquire insulating properties. Again, using Eq. (10) it was not clear why something had to happen in two dimension (the localization in one-dimensional chains had been proven before and it was clear that the ladder diagrams summation leading to Eq. (10) was not sufficient for that case).

Trying to understand how something unusual could happen in two dimensions Gorkov, Larkin and Khmelnitskii [12] investigated more complicated diagrams and found that a certain class of diagrams could lead to a divergence in any dimensionality $d \leq 2$. These are “fan” diagrams with the maximal number of crossings. They are represented in Fig. 3 and their sum is a new effective mode that is usually called “cooperon”. This mode has a form of a diffusion propagator and its contribution to the conductivity can be written in the form

$$\sigma (\omega) = \sigma_0 \left(1 - \frac{1}{\pi \nu} \int \frac{1}{D_0 k^2 - i\omega} \frac{d^d k}{(2\pi)^d}\right)$$  \hspace{1cm} (12)

It is clear from Eq. (12) that in the limit of low frequencies $\omega \rightarrow 0$ the second term in the brackets in Eq. (12) diverges in any dimension $d \leq 2$. This means that in a disordered film the quantum correction to
the classical conductivity diverges and this signals (but, of course, does not prove) the possibility of the localization.

The scaling theory of the localization and the discovery of the new diffusion modes was revolutionary in the theory of disordered systems but how can these findings be related to the Wigner-Dyson theory?

Actually, Eq. (12) is the key to understanding that there can be something beyond the classical Eq. (10) in small metal particles. One needs only realizing that the case of small metal grains corresponds to the zero dimensionality of the integral in Eq. (12). Due to the finite size the values of the momentum $k_a$ are quantized such that $k_a = 2\pi n_a/L_a$, where $n_a = 0, \pm 1, \pm 2, \pm 3, \ldots$, $a = (x,y,z)$, and $L_a$ is the size of the grain in the $a$-direction.

At low frequencies $\omega \ll D_0/L^2$ the most important contribution in the integral in Eq. (12) comes from the zero harmonics with $k = 0$ and one can see that this contribution strongly diverges when $\omega \to 0$. Moreover, the quantum correction is proportional to $\Delta/\omega$, where $\Delta$ is given by Eq. (11) and this is what one can expect from Eqs. (4-6).

So, Eq. (12) really signals that something nontrivial can happen in metal grains and the WD theory is not excluded. The diffusion modes play a prominent role and it seems, at first glance, that one should merely write proper diagrams and sum their contribution. However, even if this were possible this would hardly correspond to Eq. (4-6). The problem is that the expansion in terms of $\Delta/\omega$ cannot take into account the oscillating part in Eqs. (4-6) even in principle. Summing the diagrams one can hope to reproduce only non-oscillating asymptotics of these equations. This means that another approach has to be developed.

The possibility to demonstrate that the energy level and wave functions statistics can really be described by the RMT came first with the development of the supersymmetry approach [13, 14]. This method is based on a representation of Green functions of a disordered metal in terms of an integral over both commuting and anticommuting variables. Singling out excitations with the lowest energy (diffusion modes) one can reduce calculations to a supermatrix non-linear model. The supersymmetry method allowed to prove for the first time that the level-level correlation function for disordered particles is really described by Eqs. (4-6). Later, using the supersymmetry technique Verbaarschot, Weidenmüller and Zirnbauer [15] have derived Eqs. (4-6) starting directly from the Gaussian ensembles, Eq. (1).

When deriving the non-linear $\sigma$-model for metallic particles it was very important that they contained disorder. However, it is not the necessary condition. Several years later Bohigas, Gianonni and Schmidt [16] conjectured that the RMT should describe correctly spectral properties
of quantum systems which are chaotic in their classical limit. In particular, the Wigner-Dyson statistics had to be observed in clean metal particles (quantum billiards) provided their shape was such that classical motion would be chaotic. Their hypothesis was made on the basis of extensive numerics. For a review of the subsequent activity in these fields the book [17] is a good reference.

Historically, the description of disordered systems with a non-linear $\sigma$-model has been suggested by Wegner [18] using the replica method and integration over conventional complex variables. In the first work there were problems with convergence of functional integrals and therefore the replica approach was further developed in the publications [19] and [20]. The $\sigma$-model of Ref. [20] was obtained by the integration over conventional variables and, as a result, the group of the matrices $Q$ was non-compact. In contrast, the starting point of Ref. [19] was a representation of Green functions in terms of integrals over anticommuting (Grassmann) variables and this lead to a compact group of the matrices $Q$.

Although both the replica and supersymmetry approach are equivalent when doing the perturbation theory in the diffusion modes, the latter method is much more efficient for non-perturbative calculations like the study of the level-level correlations. This had become clear shortly after the works [19, 20] were finished and this drawback of the replica approach motivated the development of the supersymmetry one. It should be noticed that recently the oscillating behavior of the level-level correlation function $R(\omega)$ has been obtained [21–23] using the compact replica $\sigma$-model of Ref. [19]. However, the procedures used in these references are considerably more complicated than the calculations by the supersymmetry method and the limit of low frequencies $\omega \lesssim \Delta$ is still hardly achievable.

It is fair to say that the replica approach allows including electron-electron interactions in a comparatively easy way [24, 25] and this has been the main motivation in the attempts [21–23] to obtain non-perturbative results within the replica technique. At the same time, it was believed for a long time that an inclusion of the electron-electron interaction into the supersymmetry scheme was impossible. However, this is not quite so and, at least, very strong interaction can be incorporated in the supermatrix $\sigma$-model [26].

It follows from this discussion that the supersymmetry approach is better suitable for making connections with the RMT and therefore the present lectures contain discussions based on this method only. It is possible neither review here all works made in this direction nor present all details of the calculations. For a more detailed information see the book
Random Matrices and Supersymmetry

[4] and more recent reviews [27–31]. It is relevant to mention here that the word *supersymmetry* has appeared in the condensed matter physics in the publication by Parisi and Sourlas [32], who discovered a complex symmetry in a model describing ferromagnets in a random magnetic field. They used a concept of superspace including both commuting and anticommuting variables.

Two other related directions of the use of the RMT are reviewed at this school by Boris Altshuler (Quantum Chaos) and Jac Verbaarschot (QCD).

In the next sections I want to present the main ideas of the supersymmetry approach and show how it can be used for both the level correlations and wave functions statistics. It will be shown how to obtain the Wigner-Dyson statistics and how to go beyond it. A new development concerning a generalization of the supersymmetric $\sigma$-model to more complicated situations will be outlined in the last section.

1. **Supersymmetry method.**

**Supermathematics**

The supersymmetry method is based on the use of the so called Grassmann variables $\chi_i$, $i = 1, 2, ..., n$ (the elements of the Grassmann algebra) that are introduced in a completely formal way. These are abstract objects but in many cases abstract mathematical constructions drastically influence the development of physics. For example, nobody can dispute the usefulness of complex numbers for physics, but what is the physical meaning of $\sqrt{-1}$? Here I want to remind the reader basic formulae concerning definitions and operations with objects containing combinations of the Grassmann variables and conventional numbers (supermathematics).

The Grassmann variables are some mathematical objects obeying the following anticommutation rules [33]

$$\{\chi_i, \chi_j\} = \chi_i \chi_j + \chi_j \chi_i = 0$$  \hspace{1cm} (13)

for any $1 \leq i, j \leq n$.

The anticommutation rules, Eq. (13) hold in particular for $i = j$ and we see that the square of an arbitrary variable $\chi_i$ is zero

$$\chi_i^2 = 0$$  \hspace{1cm} (14)

For any anticommuting variable $\chi$ one can introduce its “complex conjugate) $\chi^*$. It is assumed by the definition that $(\chi^*)^* = -\chi$, such that the “square of the modulus is “real”

$$(\chi_i^* \chi_i)^* = -\chi_i \chi_i^* = \chi_i^* \chi_i$$  \hspace{1cm} (15)
The anticommuting variables $\chi_i$, Eq. (13-15), remained not very useful until Berezin introduced integrals over these variables. The integrals are nothing more than formal symbols introduced as follows

$$\int d\chi_i = \int d\chi_i^* = 0, \quad \int \chi_i d\chi_i = \int \chi_i^* d\chi_i^* = 1$$  (16)

It is implied that the “differentials” $d\chi_i$, $d\chi_i^*$ anticommute with each other and with the variables $\chi_i, \chi_i^*$:

$$\{d\chi_i, d\chi_j\} = \{d\chi_i, d\chi_j^*\} = \{d\chi_i^*, d\chi_j\} = \{d\chi_i^*, d\chi_j^*\} = 0$$  (17)

The definition, Eq. (16), is sufficient for introducing integrals of an arbitrary function. If such a function depends only on one variable $\chi_i$ it must be linear in $\chi_i$ because already $\chi_i^2 = 0$. Assuming that the integral of a sum of two functions equals the sum of the integrals we calculate the integral of the sum with Eq. (16). The repeated integrals are implied by integrals over several variables. This enables us to calculate the integral of a function of an arbitrary number of variables.

The most important for the development of the supersymmetry method are Gaussian integrals. The direct integration shows that the following relation is fulfilled

$$I = \int \exp \left( -\chi^+A\chi \right) \prod_{i=1}^n d\chi_i^* d\chi_i = \text{Det} A$$  (18)

where $A$ is an $n \times n$ Hermitian matrix and

$$\chi = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{pmatrix}, \quad \chi^+ = \begin{pmatrix} \chi_1^* & \chi_2^* & \cdots & \chi_n^* \end{pmatrix}$$  (19)

Eq. (18) differs from the corresponding equation for the commuting variables by giving det $A$ instead $(\text{det} A)^{-1}$. This remarkable difference is the basis of the supersymmetry method presented in these lectures. In addition to Eq. (18) one can write one more useful integral

$$I_2 = \frac{\int \chi_i \chi_i^* \exp \left( -\chi^+A\chi \right) \prod_{i=1}^n d\chi_i^* d\chi_i}{\int \exp \left( -\chi^+A\chi \right) \prod_{i=1}^n d\chi_i^* d\chi_i} = (A^{-1})_{ik}$$  (20)

In contrast to the integral $I$, Eq. (18), the integral $I_2$, Eq. (20), is completely similar to the corresponding integral over conventional numbers. Eq. (20) can be proven by the differentiation of $\ln I$ in $A_{ki}$. 
The next step is the introduction of supervectors and supermatrices. An \( n + m \) component supervector is introduced as

\[
\Phi = \left( \begin{array}{c} \chi \\ S \end{array} \right)
\]

(21)

where the \( n \)-component vector \( \chi \) is defined in Eq. (19). The \( m \)-component vector \( S \) has a similar form

\[
\begin{pmatrix}
S_1 \\
S_2 \\
\vdots \\
S_m
\end{pmatrix}
\]

(22)

but its components are conventional complex numbers.

In analogy with conventional vectors one can introduce the Hermitian conjugation

\[
\Phi^+ = (\Phi^T)^* 
\]

(23)

and the scalar product

\[
\Phi^{i^+} \Phi^j = \sum_{\alpha=1}^n \chi_{\alpha}^{i^*} \chi_{\alpha}^j + \sum_{\alpha=1}^m S_{\alpha}^{i^*} S_{\alpha}^j
\]

(24)

A linear transformation \( F \) in the space of the supervectors converts a supervector \( \Phi \) into another supervector \( \tilde{\Phi} \)

\[
\tilde{\Phi} = F \Phi
\]

(25)

Of course, the supervector \( \tilde{\Phi} \) must have the same structure, Eq. (21), as the supervector \( \Phi \). This imposes a restriction on the structure of the supermatrix \( F \) corresponding to the linear transformation \( F \): it has to be of the form

\[
F = \begin{pmatrix}
a & \sigma \\
\rho & b
\end{pmatrix}
\]

(26)

In Eq. (26) \( a \) and \( b \) are \( n \times n \) and \( m \times m \) matrices containing only commuting variables, \( \sigma \) and \( \rho \) are \( n \times m \) and \( m \times n \) matrices consisting of anticommuting ones. Matrices having the structure, Eq. (26) can be called supermatrices.

Two supermatrices \( F \) and \( G \) of the rank \( (m+n) \times (n+m) \) are assumed to multiply according to the conventional rules

\[
(FG)_{ik} = \sum_{l=1}^{m+n} F_{il} G_{lk}
\]

(27)
and one can see that \( FG \) is a supermatrix of the same form. In order to define the supertranspose \( F^T \) of the supermatrix \( F \) one should use the notion of the scalar product of two supervectors, Eq. (24). Again, by analogy with the conventional definition the supermatrix \( F^T \) is introduced as
\[
(\Phi_1^T F^T \Phi_2) = (F \Phi_1)^T \Phi_2
\]
(28)

The transpose of a conventional matrix is obtained by transposing its indexes. This is not as simple for the supermatrices. Writing out the scalar product on both sides of Eq. (28) explicitly and using the anticommutation relation, Eq. (13), one can see that the supermatrix \( F^T \) is equal to
\[
F^T = \begin{pmatrix}
a^T & -\rho^T \\
\sigma^T & b^T
\end{pmatrix}
\]
(29)

where \( a^T, b^T, \sigma^T, \) and \( \rho^T \) stand for the conventional transposition of the matrices \( a, b, \sigma, \) and \( \rho \).

Using the scalar product, Eq. (28) one obtains immediately
\[
(F_1 F_2)^T = F_2^T F_1^T
\]
(30)

The Hermitian conjugate \( F^+ \) of the matrix \( F \) can be defined in a standard way
\[
F^+ = (F^T)^* 
\]
(31)

Combining Eqs. (15, 30, 31) one can obtain
\[
(F_1 F_2)^+ = F_2^+ F_1^+ \quad \text{and} \quad (F^+)^+ = F 
\]
(32)

The latter equality shows that the operation of the Hermitian conjugation is inverse to itself. The same is not generally true for the transposition
\[
(F^T)^T \neq F
\]
(33)

A very important operation in the theory of conventional matrices is taking the trace of a matrix. If one takes the trace of a product of several matrices it is invariant under cyclic permutations of the matrices. However, due to the presence of anticommuting elements a proper operation for the supermatrices should be defined in a different way. The supertrace \( STrF \) of matrix of the form, Eq.(26) is defined as
\[
STrF = Tra - Trb
\]
(34)

where the symbol \( Tr \) stands for the conventional trace.

Although somewhat strange, the definition, Eq. (34) is very useful because it is this operation that provides the invariance under the cyclic
permutations. We obtain for arbitrary supermatrices $F_i$ of the form, Eq. (26)

$$STr F_1 F_2 = STr F_2 F_1$$

(35)

and

$$STr (F_1 F_2...F_n) = STr (F_n F_1 F_2...F_{n-1})$$

(36)

In addition to the supertrace it is convenient to introduce a superdeterminant of the supermatrix $F$

$$\ln SDet F = STr \ln F$$

(37)

The superdeterminant $SDet F$ can also be written as

$$SDet F = Det (a - \sigma b^{-1}) Det b^{-1}$$

(38)

The connection between the superdeterminant and supertrace enables us to prove immediately the multiplicity of the superdeterminant

$$SDet (F_1 F_2) = (SDet F_1) (SDet F_2)$$

(39)

The rules of the operations with the supervectors and supermatrices are very convenient because they are similar to those of conventional linear algebra. In fact, one can manipulate superobjects in exactly the same way as conventional objects. This simplifies calculations with quantities containing both types of variables considerably.

Now we can write Gaussian integrals over supervectors that generalize the integrals over conventional complex numbers or Grassmann variable. A direct calculation shows that

$$I^s = \int \exp \left( -\Phi^+ F \Phi \right) d\Phi^* d\Phi = SDet F,$$

(40)

$$d\Phi^* d\Phi = \pi^{-m} \prod_{i=1}^n d\chi^*_i d\chi_i \prod_{k=1}^m dS^*_i dS_i$$

and

$$I^s_2 = \frac{\int \Phi_i \Phi^*_k \exp \left( -\Phi^+ F \Phi \right) d\Phi^* d\Phi}{\int \exp \left( -\Phi^+ F \Phi \right) d\Phi^* d\Phi} = (F^{-1})_{ik}$$

(41)

The formulae written in this subsection give complete information about integrals over the Grassmann variables, supervectors, and supermatrices. This information will be directly used for constructing the supersymmetry method.
Physical quantities as integrals over supervectors.  
Averaging over disorder.

Eq. (41) enables us to express physical quantities in terms of functional integrals over supervectors. The form of the integrals that will be obtained is such that averaging over disorder can be performed the beginning of all calculations.

I start with the Schrödinger equation for electrons without any electron-electron interactions but in a presence of an external potential containing both regular and irregular parts. The regular part can describe potential walls and other features of the system whereas the irregular part $H_1$ of the Hamiltonian stands for disorder. The Schrödinger equation takes the form

$$H\phi_k = \varepsilon_k\phi_k, \ H = H_0 + H_1, \langle H_1 \rangle = 0 \quad (42)$$

where $\phi_k$ and $\varepsilon_k$ are eigenfunctions and eigenvalues, respectively. The angular brackets $\langle ... \rangle$ stand for the averaging over disorder.

The most important physical quantities can be expressed in terms of retarded $G^R_{\varepsilon}$ and advanced $G^A_{\varepsilon}$ Green functions of the Schrödinger equation. Using the spectral expansion the Green functions $G^R_{\varepsilon,A}$ can be written in the form

$$G^{R,A}_{\varepsilon}(r,r') = \sum_k \frac{\phi_k(r)\phi^*_k(r')}{\varepsilon - \varepsilon_k \pm i\delta} = \sum_k G^{R,A}_{\varepsilon k} \phi_k(r)\phi^*_k(r') \quad (43)$$

These functions satisfy the equation

$$(\varepsilon - H)G^{R,A}_{\varepsilon}(r,r') = \delta (r - r') \quad (44)$$

The average density of states $\rho(\varepsilon)$ (this quantity is proportional to $\Delta^{-1}$, Eq. (2)) can be written as

$$\langle \rho(\varepsilon,r) \rangle = \left\langle \sum_k \frac{\phi_k^*(r)\phi_k(r)}{\varepsilon - \varepsilon_k} \delta (\varepsilon - \varepsilon_k) \right\rangle \quad (45)$$

$$= \frac{1}{\pi} \langle \text{Im} G^A_{\varepsilon}(r,r) \rangle \quad (46)$$

whereas the level-level correlation function $R(\omega)$ takes the form

$$R(\omega) = \left( \frac{\Delta}{\pi} \right)^2 \left\langle \sum_{k,m} \text{Im} G^A_{k,\varepsilon-\omega} \text{Im} G^A_{m,\varepsilon} \right\rangle \quad (47)$$

We see from Eqs. (45, 47) that, as soon as we are able to average the Green functions or their products over the disorder, the quantities...
of interest are found. However, this cannot be done directly using Eqs. (45, 47) and we need another representation for the Green functions. Of course, one can do perturbation theory in the disorder potential but, as it has been already discussed, such an approach can hardly help in obtaining the Wigner-Dyson statistics.

What will be done now is writing the Green functions in a form that would be suitable for averaging over the disorder in the very beginning. This can be conveniently done with the integrals over the supervectors. I want to present here the main scheme only. All necessary details can be found in the book [4]. As the main interest is to calculate the level-level correlation function $R(\omega)$, Eq. (47), all formulae will be written for this case. Calculating the density of states, Eq. (45), is a simpler and less interesting task.

Let us introduce 8-component supervectors $\psi$ consisting of 4-component supervectors $\psi^1$ and $\psi^2$ such that

$$
\psi^m = \left( \begin{array}{c} \theta^m \\ v^m \end{array} \right), \quad \theta^m = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \lambda^* \cr \chi^\dagger \end{array} \right), \quad v^m = \frac{1}{\sqrt{2}} \left( \begin{array}{c} S^* \\ S \end{array} \right),
$$

$m = 1, 2$.

For the supervectors $\psi$ of the form of Eq. (48) one can define, in addition to transposition and Hermitian conjugation, the operation of the “charge conjugation”

$$
\bar{\psi} = (C \psi)^T, \quad \bar{\psi}^m = \left( \begin{array}{c} \bar{\theta}^m \\ \bar{v}^m \end{array} \right)
$$

In Eq. (49), $T$ stands for transposition, and $C$ is the supermatrix of the form

$$
C^{mn} = \Lambda^{mn} \left( \begin{array}{cc} c_1 & 0 \\ 0 & c_2 \end{array} \right)
$$

where $\Lambda$ is the diagonal supermatrix

$$
\Lambda = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)
$$

with $1$ the unity $4 \times 4$ unity matrix.

The matrices $c_1$ and $c_2$ have the form

$$
c_1 = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right), \quad c_2 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)
$$

The function $R(\omega)$ is determined by products $G^A G^R$, $G^R G^R$ and $G^A G^A$. The last two products are not interesting because the average of their products is equal for weak disorder to the product of their average. When
calculating these quantities the diffusion modes discussed in the Introduction do not appear. Therefore let us concentrate on the calculation of the product $G^A G^R$. Using Eqs. 40, 41) we can write this quantity as

$$G^A (r, r') G^R (r', r') = \left( \frac{\Delta}{\pi} \right)^2 \int \psi^1 (r) \bar{\psi}^1 (r) \psi^2 (r') \bar{\psi}^2 (r') \exp (-L) D\psi$$

(52)

where the Lagrangian $L$ has the form

$$L = i \int \bar{\psi} (r) \left( -\tilde{H}_0 - U (r) - \frac{1}{2} (\omega + i\delta) \Lambda \right) \psi (r) \, dr$$

(53)

where $U (r)$ the impurity potential. The operator $\tilde{H}_0$ equals

$$\tilde{H}_0 = \varepsilon (-i\nabla_r) - \varepsilon + \frac{\omega}{2}$$

(54)

where $\varepsilon (-i\nabla_r)$ is the spectrum.

I would like to draw attention at this point that the weight denominator is absent in Eq. (52), which contrasts the analogous integrals, Eqs. (20,41). Actually, this is a consequence of the fact that $F$ in Eq. (41) is taken as unity in the space of the $2 \times 2$ supermatrices when writing Eq. (52). In other words, the weight denominator is absent due to the difference of the results of the Gaussian integration over the anticommuting variables $\chi$, Eq. (18), (one obtains $\text{Det} A$) and the corresponding formula for the integration over conventional numbers that gives $(\text{Det} A)^{-1}$.

The possibility to write the Green function in the form of Eq. (52) without the weight denominator is the reason why the integration over the supervectors is used. As the weight denominator is absent one can immediately average over impurities in Eq. (52).

Let us assume for simplicity that the distribution of the random potential $U (r)$ in Eq. (53) is specified by Eq. (8). Then, the averaging over the random potential is simple and one obtains again Eq. (52) but now the Lagrangian $L$ should be written as

$$L = \int \left[ -i\bar{\psi} \tilde{H}_0 \psi + \frac{1}{4\pi\nu\tau} (\bar{\psi} \psi)^2 - \frac{i (\omega + i\delta)}{2} \bar{\psi} A \psi \right] \, dr$$

(55)

Eq. (55) shows that we have reduced the initial disordered problem to a regular model with an $\psi^4$ interaction. Of course, it does not help to solve it exactly but now we can use approximations well developed in theory of interacting particles.

The Lagrangian $L$ in Eq. (55) is similar to those studied in field theory. Let us remark that at $\omega = 0$ the Lagrangian is invariant under rotations.
of the supervectors in the superspace because it depends on the square of “length” only. The frequency $\omega$ violates this symmetry and, if one uses an analogy with spin models, plays the role of an “external field”. The violation of supersymmetry by the frequency is due to the fact that Eq. (55) is written for the product $G^R G^A$ (the presence of the matrix $\Lambda$ is a direct consequence of this). The symmetry would not be violated in the corresponding integrals for $\langle G^R G^R \rangle$ and $\langle G^A G^A \rangle$. The averaging of the simpler Lagrangian corresponding to the density of states (one averaged Green function) results also in a model with the interaction $\psi^4$, but the supersymmetry in this case is not violated. The diffusion modes discussed previously exist only as a result of the violation of the supersymmetry (Goldstone modes).

**Spontaneous breaking of the symmetry and Goldstone modes. Non-linear $\sigma$-model.**

It is clearly not possible to calculate any correlation function with the Lagrangian $L$, Eq. (55), exactly (except for the case of the one-dimensional chain or the Bethe lattice, where one can write recurrence equations. Further calculations will be performed in the limit of large mean free times $\tau$, which correspond to a weak interaction in the Lagrangian $L$. However, the use of the standard perturbation theory as we have seen is impossible even in this limit because of the existence of the diffusion modes and so one should try to use non-perturbative approaches.

One of the standard approaches used for such a type of theories is the mean field approximation. According to this scheme one simplifies the $\psi^4$ interaction replacing pairs $\psi \psi$ by their averages. For the interaction $\psi^4$ there can be six different pairings, which can be written as follows

$$L_{\text{int}} = \frac{1}{4\pi \nu \tau} \int (\bar{\psi}\psi)^2 \, d\mathbf{r} \to L_1 + L_2 + L_3,$$

$$L_1 = \frac{1}{4\pi \nu \tau} \sum_{\alpha,\beta} \int 2 \langle \bar{\psi}_\alpha \psi_\alpha \rangle_{\text{eff}} \bar{\psi}_\beta \psi_\beta \, d\mathbf{r},$$

$$L_2 = \frac{1}{4\pi \nu \tau} \sum_{\alpha,\beta} \int 2 \bar{\psi}_\alpha \langle \psi_\alpha \bar{\psi}_\beta \rangle_{\text{eff}} \psi_\beta \, d\mathbf{r},$$

$$L_3 = \frac{1}{4\pi \nu \tau} \sum_{\alpha,\beta} \int \left( \langle \bar{\psi}_\alpha \bar{\psi}_\beta \rangle_{\text{eff}} \bar{\psi}_\beta \psi_\alpha + \bar{\psi}_\alpha \bar{\psi}_\beta \langle \psi_\beta \psi_\alpha \rangle \right) \, d\mathbf{r}$$
In Eqs.(56), the symbol

\[ \langle \ldots \rangle_{\text{eff}} = \int \langle \ldots \rangle \exp(-L_{\text{eff}}) D\psi \]

stands for the functional averaging with the effective Lagrangian \( L_{\text{eff}} = L_0 + L_1 + L_2 + L_3 \), where \( L_0 \) is the quadratic part of the Lagrangian \( L \) in Eq. (55), and \( \alpha \) and \( \beta \) stand for the components of the supervectors \( \bar{\psi} \) and \( \psi \).

In fact the terms in \( L_2 \) and \( L_3 \) are equal to each other. The average \( \langle \bar{\psi}_\alpha \psi_\beta \rangle \) renormalizes the energy \( \varepsilon \) and is not important. The averages in \( L_2 \) and \( L_3 \) can be both commuting and anticommuting variables, depending on the subscripts \( \alpha \) and \( \beta \). The final Lagrangian \( L_{\text{eff}} \) takes the form

\[ L_{\text{eff}} = \int \left[ -i \bar{\psi} \left( \tilde{H} + \frac{1}{2} (\omega + i\delta) \Lambda + \frac{iQ}{2\tau} \right) \psi \right] dr \]  

with the \( 8 \times 8 \) supermatrix \( Q \) satisfying the following self-consistency equation

\[ Q = \frac{2}{\pi\nu} \langle \bar{\psi} \psi \rangle_{\text{eff}} \]  

Calculating the Gaussian integral in Eq. (58) with the help of Eq.(41) we obtain

\[ Q = \frac{1}{\pi\nu} \int g_0(p) \frac{d^d p}{(2\pi)^d} \]

\[ g_0(p) = i \left( \varepsilon(p) + \frac{\omega + i\delta}{2} - \varepsilon + \frac{1}{2} (\omega + i\delta) \Lambda + \frac{iQ}{2\tau} \right)^{-1} \]

The integral over the momenta \( p \) in Eq. (60) has both a real and an imaginary part. As concerns the imaginary part the main contribution comes from the region \( |\varepsilon(p) - \varepsilon| \gg \tau^{-1}, \omega \) and, therefore, is proportional to the unit matrix \( 1 \). This contribution leads to a small renormalization of the energy \( \varepsilon \). Assuming that this energy has already been renormalized we can forget about the imaginary part of \( Q \) and concentrate on the real part. The main contribution to the real part of Eq. (60) comes from the region \( |\varepsilon(p) - \varepsilon| \sim \tau^{-1} \ll \varepsilon \). Introducing the variable \( \xi = \varepsilon(p) - \varepsilon \) we can rewrite Eq. (60) in the form

\[ Q = \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \xi + \frac{1}{2} (\omega + i\delta) \Lambda + \frac{iQ}{2\tau} \right)^{-1} d\xi \]

Eq. (61) determines the contribution to the real part of \( Q \) only and has at \( \omega \neq 0 \) one evident solution

\[ Q = \Lambda, \ \omega \neq 0 \]
However, putting $\omega = 0$ we see that any $Q$ of the form

$$Q = V \Lambda \tilde{V}$$  \hspace{1cm} (63)

where $V$ is an arbitrary unitary supermatrix, $V \tilde{V} = 1$, satisfies Eq. (61). Here the symbol of conjugation “$-$” for an arbitrary matrix $A$ means the following

$$\tilde{A} = C A^T C^T$$  \hspace{1cm} (64)

with $C$ from Eqs. (49, 51). The supermatrix $Q$ is self conjugate, $Q = \tilde{Q}$.

The degeneracy of the ground state, Eq. (63), leads to the existence of the low-lying Goldstone modes, and their contribution to physical quantities should be taken into account properly. These Goldstone modes are just the diffusion modes discussed in the Introduction. In the language of spin models the degeneracy of the solution, Eq. (63) is equivalent to the degeneracy due to an arbitrary spin direction.

Substituting the mean field solution, Eq. (63), into the effective Lagrangian into Eq. (57) we obtain zero, which shows that this approximation is not sufficient. In order to describe a contribution of the diffusion modes we have to take into account slow variations of the supermatrix $Q$ in space. This can be done assuming that the supermatrix $V$ is a slow function of the coordinates $r$ and expanding in the gradients of $V$ (or, equivalently, $Q$).

As a result of the expansion in the gradients and small frequencies one can obtain a functional $F[Q]$ describing slow variations of the supermatrix $Q$ in space

$$F[Q] = \frac{\pi \nu}{8} S Tr \int \left[ D_0 (\nabla Q)^2 + 2i (\omega + i \delta) \Lambda Q \right] dr$$  \hspace{1cm} (65)

where $D_0$ is the classical diffusion coefficient, Eq. (7), and the supermatrix $Q$ is described by Eq. (63). The free energy functional, Eq. (65), has the form of a non-linear $\sigma$-model.

In order to calculate, e.g. the level-level correlation function $R(\omega)$, Eq. (47), one should express it in terms of a functional integral over the supermatrix $Q$. As a result, this function takes the form

$$R(\omega) = \frac{1}{2} - \frac{1}{2V^2} \text{Re} \int Q_{11}^{11} (r) Q_{22}^{11} (r') \exp \left(-F[Q] \right) DQ dr dr'$$  \hspace{1cm} (66)

where $V$ is the volume of the system. In Eq. (66) the superscripts of $Q$ enumerate “retarded-advanced blocks”, the subscripts relate to the matrix elements within these blocks.

Eqs. (65, 66) show that the calculation of physical quantities for disordered systems can be reduced to study of a supermatrix non-linear
σ-model. There is a variety of physical problems that can be solved by considering this model in different dimensionalities. To some extent, the problem of the level statistics in a limited volume is the simplest one because it corresponds to the zero-dimensional σ-model (no dependence of $Q$ on the coordinates).

Adding magnetic or spin orbit interactions results in additional “external fields” partially breaking the symmetry. If these interactions are not very weak they simply cut some degrees of freedom. As a result, one comes again to Eq. (65) but with a reduced symmetry of the supermatrix $Q$.

There can be three different types of the symmetries:

1. In the absence of both magnetic and spin-orbital interactions the system is invariant under time reversal and spatial inversion. It will be called “orthogonal” because, as we will see, one comes in small particles to the Wigner-Dyson statistic for the orthogonal ensemble.

2. In the presence of magnetic interactions the time reversal symmetry is violated and this will be called unitary ensemble.

3. In the absence of magnetic interactions but in the presence of spin-orbital ones, one obtains the symplectic ensemble.

These three ensembles lead to quite different results in different situations and each of them should be considered separately.

**Level statistics in a limited volume.**

Eq. (63) specifies a general form for the $8 \times 8$ supermatrix $Q$. In other words, it obeys the constraint

$$Q^2 = 1$$

(67)

However, this constraint is not sufficient to determine unambiguously the precise structure of $Q$ because Eq. (67) could correspond both to rotations on a sphere and on a hyperboloid. It turns out that the symmetry of the supermatrix $Q$ is more complex than those corresponding to these two possibilities. The supermatrix $Q$ consists of two parts: one describing rotations on the sphere and the other on the hyperboloid, such that the group of rotations of $Q$ is a mixture of compact and non-compact groups of rotations [4]. These two parts are glued by anticommuting elements. We can describe the explicit form of the supermatrix $Q$ writing it in the form

$$Q = UQ_0\hat{U},$$

(68)

where

$$U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$$

(69)
and $\bar{u}u = 1$, $v\bar{v} = 1$. All Grassmann variables are included in the $4 \times 4$ supermatrices $u$ and $v$. These matrices contain also some phases. Their form is simple and not very important for our discussion. More details can be found in Ref. [4].

The form of $Q_0$ is more interesting and can be written as

$$Q_0 = \begin{pmatrix} \cos \hat{\theta} & i \sin \hat{\theta} \\ -i \sin \hat{\theta} & -\cos \hat{\theta} \end{pmatrix}, \quad \hat{\theta} = \begin{pmatrix} \hat{\theta}_{11} & 0 \\ 0 & \hat{\theta}_{22} \end{pmatrix} \quad (70)$$

The $2 \times 2$ matrices are different for different classes of the symmetry.

We see that the symmetry of $Q_0$ corresponds to a group rotations on both the sphere and the hyperboloid. The explicit form of the matrices $\hat{\theta}_{11}$ and $\hat{\theta}_{22}$ can be expressed as follows

$$\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}, \quad \hat{\theta}_{22} = i \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix} \quad (71)$$

with $0 < \theta < \pi$, $\theta_1 > 0$, $\theta_2 > 0$ for the orthogonal ensemble,

$$\hat{\theta}_{11} = \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix}, \quad \hat{\theta}_{22} = i \begin{pmatrix} \theta_1 & 0 \\ 0 & \theta_1 \end{pmatrix} \quad (72)$$

with $0 < \theta < \pi$, $\theta_1 > 0$ for the unitary ensemble, and

$$\hat{\theta}_{11} = \begin{pmatrix} \theta_1 & \theta_2 \\ \theta_2 & \theta_1 \end{pmatrix}, \quad \hat{\theta}_{22} = i \begin{pmatrix} \theta & 0 \\ 0 & \theta \end{pmatrix} \quad (73)$$

for the symplectic one.

Eqs. (65,66, 70-73) specify the non-linear supermatrix $\sigma$-model. What remains to do for the level-level correlation function, Eq. (66), is to calculate the functional integral over $Q$. Many of the physical quantities can also be expressed in a form of a correlation function of the supermatrices $Q$ with the free energy functional $F[Q]$, Eq. (65).

Let us consider now the level-level correlation function in a limited volume. The functional integral, Eqs. (65, 66) can be considerably simplified in the limit of small frequencies. In a finite volume one can expand the supermatrix $Q$ in Fourier series. Then, it is not difficult to understand from Eq. (65) that for $\omega \ll D_0/L^2$, where $L$ is the sample size, only the zero space harmonics is essential. In this case one can integrate over the supermatrices $Q$ not varying in space and the integral for the function $R(\omega)$, Eq. (66), becomes just a definite integral over several variables. This integral takes the following form

$$R(\omega) = \frac{1}{2} - \frac{1}{2} \text{Re} \int Q_{11} Q_{11}^{22} \exp (-F_0[Q]) \, dQ \quad (74)$$
where $F_0 [Q]$ takes the form

$$F_0 [Q] = \frac{i\pi (\omega + i\delta)}{4\Delta} STr (\Lambda Q)$$  \hspace{1cm} (75)$$

One can say that Eqs. (74, 75) determine a zero dimensional non-linear $\sigma$-model. In general, it is clear that the dimensionality of the $\sigma$-model is determined at not very high temperatures by the geometry of the sample. For example, the one dimensional $\sigma$-model describes a long wire of a finite thickness.

In order to calculate the integral over $Q$ in Eq. (74) it is very convenient to use Eqs. (70, 73). We see immediately that the supermatrix $U$, Eq. (69), drops out from $F_0 [Q]$ entering the pre-exponential in Eq. (74) only. This allows us to integrate first over the elements of $U$, and reduce the integral to the variables $\hat{\theta}_{11}, \hat{\theta}_{22}$ only. The integration over the Grassmann variables is, according to Eq. (16), a very simple task. Actually, one has to calculate also Jacobians arising when changing from the integration over $Q$ to the integration over the “eigenvalues” $\lambda, \lambda_1, \lambda_2$. As a result, one comes to the following integrals for all three ensembles

$$R_{\text{orth}} (\omega) = 1 + \text{Re} \int_{1}^{\infty} \int_{1}^{\infty} \int_{-1}^{1} \frac{(\lambda_1 \lambda_2 - \lambda)^2 (1 - \lambda^2)}{\left(\lambda^2 + \lambda_1^2 + \lambda_2^2 - 2 \lambda_1 \lambda_2 - 1\right)^2} \times \exp \left[i (x + i\delta) (\lambda_1 \lambda_2 - \lambda)\right] d\lambda_1 d\lambda_2 d\lambda \hspace{1cm} (76)$$

$$R_{\text{unit}} (\omega) = 1 + \frac{1}{2} \text{Re} \int_{1}^{\infty} \int_{-1}^{1} \exp \left[i (x + i\delta) (\lambda_1 - \lambda)\right] d\lambda d\lambda_1 \hspace{1cm} (77)$$

$$R_{\text{sympl}} (\omega) = 1 + \text{Re} \int_{1}^{\infty} \int_{0}^{1} \int_{-1}^{1} \frac{(\lambda - \lambda_1 \lambda_2)^2 (\lambda^2 - 1)}{\left(\lambda^2 + \lambda_1^2 + \lambda_2^2 - 2 \lambda_1 \lambda_2 \lambda - 1\right)^2} \times \exp \left[i (x + i\delta) (\lambda - \lambda_1 \lambda_2)\right] d\lambda_1 d\lambda_2 d\lambda \hspace{1cm} (78)$$

We see that the integrals over the supermatrix $Q$ can be reduced to integrals over the “eigenvalues” $\lambda, \lambda_1, \lambda_2$. Depending on the ensemble one obtains twofold or threefold integrals. Calculation of such integrals is a much simpler task than calculation of integrals over a large number $N$ of variables encountered in RMT[2]. Although the reduction to the integrals over the eigenvalues has been carried out for the level-level correlation function only, the corresponding manipulations for studying other physical quantities are the same. The only thing that remains to be done when calculating different physical quantities is to write a proper pre-exponential and carry out integration over the elements of
supermatrices $u$ and $v$ entering the pre-exponential only. Then one obtains integrals over the variables $\lambda, \lambda_1, \lambda_2$ analogous to those in Eqs. (76, 77, 78).

The integration over $\lambda$ and $\lambda_1$ in Eq. (77) for the unitary ensemble is simple. At first glance, the integrals for the orthogonal and symplectic ensembles look terrifying. However, they can be calculated by Fourier transforming the integrals from the frequencies $\omega$ to the real time $t$. As a result, one comes to the Wigner-Dyson formulae, Eqs. (4-6), which demonstrates that the level-level correlation function for a small metal particle is really the same as that given by the RMT. This is how one proves the relevance of the RMT for disordered systems [13]. Actually, the model of the disordered metal was the first microscopic model for which the Wigner-Dyson statistics had been proven.

Of course, this is possible for not very high frequencies $\omega \ll D_0/L^2$. In the opposite limit $\omega \gg D_0/L^2$, the situation is no longer zero dimensional but one can do perturbation theory in diffusion modes. This calculation was done in Ref. [34].

One can also come to the zero dimensional $\sigma$-model starting from the Gaussian distribution for the random matrices, Eq. (1), in the limit of large $N$. This was done in the review, Ref. [15]. Therefore, the supersymmetry method can be considered as an alternative to the method of the orthogonal polynomials[2] in RMT.

Using the non-linear $\sigma$-model, Eq. (65), one can consider thick disordered wires. This corresponds to the one dimensional $\sigma$-model. In this case one can use a transfer matrix technique that allows to reduce calculation of a one dimensional functional integral to solving of an effective “Schrödinger equation”. The exact solution found for this model [35] proves localization of all states (vanishing of the conductivity) for any arbitrarily weak disorder. In the language of random matrices this quasi-one-dimensional model corresponds to a model of random banded matrices [36].

The exact solution can also be found for a model with disordered grains connected in a such a way that they constitute a Bethe lattice. For this model, using recursion relations one can write a non-linear integral equation[37]. It was demonstrated that within the model on the Bethe lattice one could have a metal-insulator transition with a very unusual critical behavior. The model on the Bethe lattice has been shown to be equivalent to models of certain sparse random matrices [38]. Sparse matrices are relevant for description of diluted spin models, some combinatorial optimization problems [39] and other interesting systems.

Properties of two dimensional disordered metals can be studied using a renormalization group scheme [18–20, 13, 14]
2. Wave functions fluctuations in a finite volume. Multifractality.

General definitions.

In this Section statistics of wave functions is discussed. Investigation of wave functions is complimentary to study of energy levels. In the language of random matrices, energy levels correspond to eigenvalues of the matrices whereas the wave functions relate to eigenvectors.

Study of wave functions has become popular in condensed matter physics not long ago with the development of mesoscopic physics. One can study, e.g., electron tunneling through so called quantum dots, which are actually small quantum wells. At low temperatures one can reach a resonance tunnelling regime when the electron tunnel via one resonance level. In this case the tunnelling amplitude is very sensitive to the wave function of the resonance state.

Wave functions can also be measured in optical and acoustic resonators where they are electromagnetic or sound waves, respectively.

We start with the standard Schrödinger equation

\[ H\phi_\alpha (r) = \varepsilon_\alpha \phi_\alpha (r) \] (79)

that determines the eigenenergies \( \varepsilon_\alpha \) and eigenfunctions \( \phi_\alpha (r) \).

We assume that a finite volume \( V \) is considered, such that the spectrum of the energies \( \varepsilon_\alpha \) is discrete.

The full statistics of the wave functions \( \phi_\alpha (r) \) at a given point \( r \) can be described by the following distribution function \( f \)

\[ f (t) = \Delta \left\langle \sum_\alpha \delta \left( t - |\phi_\alpha (r)|^2 \right) \delta \left( \varepsilon - \varepsilon_\alpha \right) \right\rangle \] (80)

The function \( f (t) \), Eq. (80), gives the probability that the square of the absolute value of the wave function (intensity) at the point \( r \) and energy \( \varepsilon \) is equal to \( t \). The distribution function \( f (t) \) and the wave functions \( \phi_\alpha (r) \) are assumed to be properly normalized such that

\[ t_0 = 1, \ t_1 = V^{-1} \] (81)

where \( t_n (V) \) are coefficients of the so called inverse participation ratio

\[ t_n = \int_0^\infty t^n f (t) dt = \Delta \left\langle \sum_\alpha |\phi_\alpha (r)|^{2n} \delta \left( \varepsilon - \varepsilon_\alpha \right) \right\rangle \] (82)

These coefficients indicate very sensitively the degree of localization of states through their dependence \( t_n (V) \) on the volume of the system. In
a pure metal or a ballistic chaotic box where the wave functions extend over the whole system one has

\[ t_n \propto V^{-n} \] (83)

If disorder makes the localization length \( L_c \), at which the typical wave functions decay, is much shorter than the sample size \( L \sim V^{1/d} \), the coefficients \( t_n \) are insensitive to \( L \). However, a very interesting information about the development of localization can be gained through an analysis of \( t_n (V) \) for small samples with \( L < L_c \).

As soon as the localization length \( L_c \) exceeds the sample size, any length scale disappears and, in the language of the coefficients \( t_n \), this is described as

\[ V t_n \propto L^{-\tau(n)}, \quad \tau (n) = (n - 1) d^* (n) \] (84)

where \( d^*(n) \) may differ from the physical dimension \( d \) of the system and be a function of \( n \). This function gives the values of the fractal dimensions \( d^*(n) \) for each \( n \).

If the behavior of the wave function is described by Eq. (83), the fractal dimension \( d^* \) coincides with the physical dimension \( d \). We will see that the fractal dimension of a system obeying the RMT coincides with the physical dimension. In such a situation, although the amplitude fluctuations are possible, they are not very strong.

Once we assume that the envelope of a typical wave function at a length scale shorter than \( L_c \) obeys a power law \( \phi (r) \propto r^{-\mu} \) with a single fixed exponent \( \mu < d/2 \), the set of the coefficients \( t_n \) reveals \( d^* = d - 2\mu \) different from \( d \) but the same for all \( n > d/(2\mu) \). This is when one speaks of fractal behavior with the fractal dimension \( d^* \).

If \( d^*(n) \) is not a constant, that signals a more sophisticated structure of the wave functions. They can be imagined as splashes of multiply interfering waves at different scales and with various amplitudes, and possibly, self-similarity characterized by a relation between the amplitude \( t \) of the local splash of the wave function and the exponent \( \mu (t) \) of the envelope of its extended power law tail.

We will see below that the multifractality of the wave functions of two dimensional weakly disordered conductors is the most general property of these systems as soon as the sample size \( L \) does not exceed the localization length \( L_c \) [40].

**Porter-Thomas distribution.**

Before starting more complicated calculations I would like to present here what is known about the distribution of wave functions from nuclear
physics (see, e.g. Refs. [41, 42]) where it was studied for description of level width fluctuations in neutron scattering. The wave function fluctuations are obtained there again from the RMT.

To start the calculation one should choose an arbitrary basis of eigenfunctions $\rho_m (r)$ and expand the function $\phi_n (r)$ in this basis

$$\phi_n (r) = V^{-1/2} \sum_{m=1}^{\infty} a_{nm} \rho_m (r)$$

where $V$ is the volume. It is convenient to truncate the basis to a finite $N$-dimensional set and take the limit $N \to \infty$, as is usually done in the RMT. The main statistical hypothesis is that all coefficients $a_{mn}$ are uniformly distributed. The only restriction on the coefficients $\{a_{mn}\}$ is imposed by normalization of the wave functions, and the probability density $P(\{a_{mn}\})$ can be written as

$$\tilde{P}(\{a_{mn}\}) = \frac{2}{\Omega_N} \delta \left( \sum_{m=1}^{N} |a_{mn}|^2 - 1 \right)$$

where $\Omega_n$ is the solid angle in $N$ dimensions. Because of the truncation of the basis the condition of completeness of the basis $\{\rho_m\}$ should be written in the form

$$\sum_{m=1}^{\infty} \rho_m^2 (r) \equiv |\vec{\eta}|^2 = N$$

where $\vec{\eta}$ is an $N$-dimensional vector with components $\{\rho_m\}$. The distribution function of the intensities $W (v)$ at the point $r$ is introduced as

$$W (v) = \frac{2}{\Omega_N} \int \delta \left( v - |\vec{a}\vec{\eta}|^2 \right) \delta \left( |\vec{a}|^2 - 1 \right) d\vec{a}$$

where the vector $\vec{a}$ is an $N$-dimensional vector with components $\{a_{mn}\}$. The distribution function $W (v)$, Eq. (88) is defined in such a way that

$$\int W (v) dv = 1$$

In the unitary ensemble, one should integrate over complex vectors $\vec{a}$. Integrating first over the component of the vector $\vec{a}$ parallel to the vector $\vec{\eta}$ and using Eq. (87) one obtains

$$W (v) = \frac{2\pi}{N\Omega_N} \int \delta \left( |\vec{a}_\perp|^2 - \left( 1 - \frac{v}{N} \right) \right) d\vec{a}$$
where $\vec{a}_\perp$ is the component perpendicular to $\vec{\eta}$. The remaining integration in Eq. (90) can be carried out easily. Taking the limit $N \to \infty$ one obtains for the unitary ensemble a simple formula

$$W(v) = \exp(-v) \quad (91)$$

Computing the integral in Eq. (88) for real vectors $\vec{a}$ and $\vec{\eta}$ one can obtain the distribution function for the orthogonal ensemble

$$W(v) = \frac{1}{\sqrt{2\pi v}} \exp \left(-\frac{v^2}{2}\right) \quad (92)$$

The functions $W(v)$, Eqs. (91,92) satisfy the normalization conditions, Eq. (89). The distribution functions $W(v)$ are universal and do not depend on details of models for disorder. The amplitudes $v$ are related to $t$ from the previous section as $v = Vt$. The functions $W$ and $f$ are related to each other accordingly. Eqs. (91, 92) are usually referred to as the Porter-Thomas distribution.

**Non-linear $\sigma$-model and the statistics of wave functions.**

Now we concentrate on the calculation of the distribution function $f(t)$, Eq. (80). At first glance, this task does not seem easy. In the previous section we were able to reduce the level-level correlation function to a functional integral over $8 \times 8$ supermatrices $Q$. It became possible because the level-level correlation function $R(\omega)$, Eq. (3), could be expressed in terms of the product of two Green functions, Eq. (47). Actually, the size $8 \times 8$ of the supermatrices is determined by the fact that only two Green functions are needed.

So, in order to calculate the distribution function $f(t)$, Eq. (80), we have to make two necessary steps:

1. To express Eq. (80) in terms of the Green functions.

2. To express products of the Green functions in terms of an integral over supervectors $\psi$. If we really want to make explicit calculations the supervectors $\psi$ should not have too many components.

It turns out that both the steps are possible and the necessary number of the components of the supervector $\psi$ is just 8 for the orthogonal ensemble and 4 for the unitary one.

The step 1 is done introducing Green functions $G^{R,A}_{\varepsilon,\gamma}$ for a system with smeared levels

$$G^{R,A}_{\varepsilon,\gamma}(r,r') = \sum_{\alpha} \frac{\phi_{\alpha}(r) \phi^*_{\alpha}(r')}{\varepsilon - \varepsilon_\alpha \pm \frac{i\gamma}{2}} \quad (93)$$
Then, Eq. (80) can be written as

$$f(t) = \Delta \left\langle \lim_{\gamma \to 0} \sum_{\alpha} \delta \left( t - \frac{i\gamma}{2} G^{R}_{\varepsilon\gamma}(r, r) \right) \delta(\varepsilon - \varepsilon_{\alpha}) \right\rangle$$

(94)

$$= \frac{\Delta}{2\pi} \lim_{\gamma \to 0} \lim_{\beta \to 0} \left\langle \int \delta \left( t - \frac{i\gamma}{2} G^{R}_{\varepsilon\gamma}(r, r) \right) \left( G^{A}_{\varepsilon\beta}(r', r') - G^{R}_{\varepsilon\beta}(r', r') \right) \right\rangle \, dr'$$

In Eq. (94) one should first take the limit $\beta \to 0$ and then $\gamma \to 0$. Since the distribution function $f(t)$ is represented in terms of a function of only two Green functions at two points $r$ and $r'$ one can express it in terms of an integral over 8-component supervectors $\psi$ using the Wick theorem. The corresponding Lagrangian is the same as the one for the level-level correlation function, Eq. (55), provided one replaces the frequency $\omega$ by the level width $\gamma$.

The derivation of the corresponding $\sigma$-model is standard and one comes to the following expression for the distribution function $f(t)$

$$f(t) = \lim_{\gamma \to 0} \int \int STr \left( \pi^{(1)}_{b}(Q)(r) \right) \delta \left( t - \frac{\pi \mu \gamma}{4} STr \left( \pi^{(2)}_{b}(Q)(r_o) \right) \right)$$

$$\times (-F[Q]) \, DQ \frac{dr}{4V}$$

(95)

where the free energy functional $F[Q]$ has the form

$$F = \frac{\pi \mu}{8} \int STr \left[ D_0 (\nabla Q)^2 - \gamma \Lambda Q (r) \right]$$

(96)

and $r_o$ is the “observation point”. When the system can be described by the zero dimensional $\sigma$- model the distribution function $f(t)$ does not depend on $r_o$. However, beyond the 0D approximation, this function can also be a function of the coordinates. The matrices $\pi^{(1,2)}_{b}$ in Eq. (95) select from the supermatrix $Q$ its boson-boson sector and have the form

$$\pi^{(1)}_{b} = \begin{pmatrix} \pi_b & 0 \\ 0 & 0 \end{pmatrix}, \quad \pi^{(2)}_{b} = \begin{pmatrix} 0 & 0 \\ 0 & \pi_b \end{pmatrix}, \quad \pi_b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

(97)

As we have discussed previously, the $\sigma$-model is noncompact, Eqs. (70-73). Therefore, in order to avoid divergent integrals over the variables $\theta_{22}$, we must calculate the integrals keeping $\gamma$ finite. The limit $\gamma \to 0$ can be taken only at the end of the calculations. However, it is not very convenient to keep an additional free parameter, and it is better to get rid of the parameter $\gamma$ at an earlier stage. This can be done by
integrating over the zero space harmonics of \( Q \) in the very beginning of the calculations.

To carry out this procedure one should represent the supermatrix \( Q (r) \) in the form of Eq. (63) and change the variables of integration \( V (r) \) to \( \tilde{V} (r) \) as \( V (r) = V (r_o) \tilde{V} (r) \). This leads to supermatrices \( \tilde{Q} : Q (r) = V (r_o) \tilde{Q} (r) \tilde{V} (r_o) \). In terms of the new variables \( \tilde{V} (r) \) and \( \tilde{Q} (r) \) the gradient term in Eq. (96) preserves its form, but now the condition

\[
V (r_o) = 1, \quad \tilde{Q} (r_o) = \Lambda \tag{98}
\]

has to be fulfilled. Changing the variables of the integration for all points \( r \neq r_o \) from \( Q (r) \) to \( \tilde{Q} (r) \) one obtains a new free energy functional that does not contain \( V (r_o) \) or \( Q (r_o) \). These variables enter only the pre-exponential and the term with \( \gamma \), and, hence, the integral over \( V (r_o) \) can be computed without making approximations. The result of the integration contains only the variables \( \tilde{Q} (r) \) with the boundary condition, Eq. (98). This means that the reduced \( \sigma \)-model obtained in this way operates only with relative variations of the field \( Q \) with respect to its value at the observation point.

The limit \( \gamma \to 0 \) simplifies the computation because the main contribution to the integral over the variable \( \theta_{1o} \) entering the parametrization, Eqs. (70-73), is from \( \cosh \theta_{1o} \sim 1/\gamma \) (for simplicity we are considering the unitary ensemble but the final results are similar for all ensembles). After standard manipulations one can express the distribution function \( f (t) \) in the form

\[
f (t) = \frac{1}{\tilde{V}} \frac{d^2 \Phi (t)}{dt^2}, \quad \Phi (t) = \int_{\tilde{Q}(r_o) = \Lambda} \exp \left( -\tilde{F} \left[ \tilde{Q}, t \right] \right) D\tilde{Q} (r) \tag{99}
\]

where the free energy \( \tilde{F} [Q, t] \) has the following form

\[
\tilde{F} [Q, t] = \frac{1}{8} \int ST r \left[ \pi \nu D_0 \left( \nabla \tilde{Q} \right)^2 - 2t \Lambda \Pi \tilde{Q} \right] dr \tag{100}
\]

The matrix \( \Pi \) selects from \( \tilde{Q} \) its noncompact “boson-boson” sector.

If \( t \) is not very large one can take into account the zero space harmonics of \( Q \) only. Taking into account Eq. (98) we can just put everywhere \( Q = \Lambda \), which leads us immediately to the Porter-Thomas distribution, Eq. (91) (and Eq. (92) for the orthogonal ensemble).

Nonetheless, this would be only an approximate procedure because the value \( \tilde{Q} (r) = \Lambda \) does not correspond to the minimum of the functional \( \tilde{F} \) when \( t \neq 0 \). An equation for the minimum can be found by taking into account the noncompact variable \( \theta_1 \) under the condition at the boundary
and at the observation point
\[ \mathbf{n} \nabla \theta_1 = 0, \quad \theta_1 (\mathbf{r}_o) = 0 \quad (101) \]

where \( \mathbf{n} \) is the unit vector at the boundary and perpendicular to it.

Using Eq. (101) one writes the equation for the extremum solution \( \theta_t \) in the form
\[ \Delta r \theta_t = -\frac{t}{\pi \nu D_0} \exp (-\theta_t (\mathbf{r})) \quad (102) \]

where \( \Delta r \) is the Laplacian. The solution \( \theta_t (\mathbf{r}) \) of Eq. (102) has to be substituted into the energy functional \( \tilde{F} \), Eq. (100), which takes the form
\[ F_t = \frac{1}{2} \int \left[ \pi \nu D_0 (\nabla \theta_t)^2 + 2t \exp (-\theta_t) \right] d\mathbf{r} \quad (103) \]

It is remarkable that Eq. (102) for the non-trivial vacuum of the reduced \( \sigma \)-model in two dimensions is exactly the Liouville equation known in the conformal theory of 2D quantum gravity [43, 44]. Within this model one has to calculate the functional integral over all \( \theta \) with the free energy functional determined by Eq. (103). Although this can lead to helpful analogies [45], results that might be anticipated in this way can be used only as intermediate asymptotics.

Eq. (102) for the minimum looks very similar to a saddle point equation derived in Ref. [46] when considering the problem of long-living current relaxation. However, the non-linear term is different, which leads to different solutions.

The most interesting is the solution of Eq. (102) in two dimensions where it can be found exactly. However, the exact solution is somewhat cumbersome and I write here its asymptotics at distances \( r \) much smaller than the sample size \( L \) (but exceeding the mean free path \( l \))
\[ \exp (-\theta_t) \approx (l/r)^{2\mu} \quad (104) \]

where \( \mu \) is a parameter depending on disorder.

With the same accuracy, the free energy of the vacuum state can be approximated by
\[ F_t \approx 4\pi^2 \nu D_0 \left\{ \mu + \mu^2 \ln (L/l) \right\} \quad (105) \]

The parameter \( \mu \) can be determined from the following equation
\[ \mu \approx \frac{z (T)}{2 \ln (L/l)}, \quad \text{where} \quad ze^z = T \equiv \frac{tV \ln (L/l)}{2\pi^2 \nu D_0} \quad (106) \]

In principle, Eqs. (104-106) determine the distribution function \( f(t) \) for arbitrary \( t \) (\( \mu \) must remain small, though). However, analytical expressions can be written only in the limiting cases \( T \ll 1 \) and \( T \gg 1 \). In
these limits, the distribution function \( f(t) \), Eq. (80), can be written as [40]

\[
f(t) = AV \begin{cases} 
\exp \left( -Vt \left[ 1 - \frac{T}{T} + \ldots \right] \right), & T \ll 1 \\
\exp \left( -\frac{\pi^2 \nu D_0}{\ln(L/l)} \ln^2 T \right), & T \gg 1
\end{cases}
\]

where \( A \) is a normalization constant.

We see from Eq. (107) that at small values of the amplitudes, such that \( T \ll 1 \), the distribution function \( f(t) \) agrees with the Porter-Thomas distribution, Eq. (91), thus proving the latter for disordered systems. In this limit one can make expansion in \( T \).

At large \( t \) \((T \gg 1)\) the function \( f(t) \) has log-normal asymptotics that is strikingly similar to the asymptotics of the distribution function of the local density of states or conductances discovered by Altshuler, Kravtsov and Lerner [48] who came to this result considering renormalization of terms high gradients in the \( \sigma \)-model. Even the numerical coefficients in the exponentials are the same, although, of course, the logarithms contain different variables. It appears that the log-normal form is really universal. The slower decay of the distribution function \( f(t) \) at large \( t \) is due to localization effect. Unfortunately, until now it is not clear how the growth of the high gradient terms is related to the existence of the non-trivial vacuum considered in this Section.

As concerns the coefficients of the inverse participation ratio \( t_n \), Eq. (82), they show the multifractal behavior, Eq. (84). Using Eqs. (102,106) one comes to the following expression for the fractal dimension \( d^*(n) \)

\[
d^*(n) = 2 - n (4\pi^2 \nu D_0)^{-1}
\]

We see that even for a weak disorder the fractal dimension \( d^*(n) \) strongly deviates from 2. Of course, it cannot become negative and this determines the region of the applicability of Eq. (108).

3. Recent and possible future developments.

In the preceding sections it was demonstrated how one can come to the random matrix theory starting from a model of a disordered metallic particle. This became possible using the supersymmetry method. Actually, to the best of my knowledge, the model of a disordered metal was the first microscopic model for which the Wigner-Dyson statistics was confirmed.

Starting from the first works [7, 13] where the relevance of the Wigner-Dyson theory to the disordered systems was suggested and proven, a huge number of problems have been attacked using these ideas. Calculations were performed either assuming that RMT was applicable for
the description of small particles (they are often called “quantum dots”) and using methods of the RMT [2] or making direct computation starting from a disordered metal and applying the non-linear supermatrix $\sigma$-model. Reviewing all these application within several lectures is impossible even though several related topics are considered in this volume by Boris Altshuler and Jac Verbaarschot. At this point I can only refer again to recent reviews [4, 6, 17, 27–31, 49] and apologize in case if some references are missing here.

The selection of topics of the present lectures was motivated mainly by the desire to give a feeling of how to calculate within the supersymmetry method both level and wave function correlations. We have seen that one could obtain results that agreed in a certain region of parameters with the predictions of the RMT and, at the same time, go beyond the RMT.

Essential conditions for the derivation of the $\sigma$-model were the absence of the electron-electron interaction and a sufficiently high concentration of impurities. For the problem of the level statistics, the latter condition corresponds to the case when the mean free path $l$ is much smaller than the sample size. In other words, an electron can scatter many times on the impurities in the bulk before it reaches a boundary of the sample.

At the same time, the RMT was initially suggested for description of complex nuclei, where disorder is absent but interactions are strong. A natural question that can be asked is: Can one prove the relevance of the RMT for clean or/and interacting systems? Clearly, the supermatrix $\sigma$-model discussed in the previous sections is not applicable in these situations and one needs a generalization of this method.

It seems that really new ideas are necessary in order to achieve this goal. Nevertheless, first steps towards constructing more general schemes have been done and I want to present here the main ideas of the new approaches.

**Supersymmetry with interaction.**

From the beginning of the use of the supersymmetry method it was clear that the method could be applicable for non-interacting particles only. The method is based on the result of the Gaussian integration, Eq. (18), that gives $\text{Det} A$ instead of the usual $(\text{Det} A)^{-1}$. Introducing an interaction results in non-quadratic terms in the Lagrangian. Therefore the trick with writing Green functions in terms of a Gaussian integral without a weight denominator does not work anymore. This is the reason why, in contrast to the replica approach where a proper $\sigma$-model
has been derived long ago [24, 25], introducing an interaction into the supersymmetry scheme was believed to be impossible.

To some extent, it is true and it is not clear how to include the interaction into the supersymmetry exactly. However, a weak interaction can really be included without considerable difficulties [26]. The initial electron model with the interaction is not supersymmetric and cannot be made supersymmetric by a transformation. This is why one cannot get rid off the weight denominator.

The main idea of Ref. [26] is to replace approximately the initial electron model by an effective supersymmetric model. This is possible for any disorder in the limit of a weak interaction. The effective model takes into account the most important (Hartree-Fock type) diagrams for any fixed configuration of impurities. Then, the derivation of the proper $\sigma$-model is quite standard.

The resulting non-linear $\sigma$-model resembles very much the replica model of Finkelstein[24, 25] but contains supermatrices and does not have replica indices. The supermatrices $Q$ contain, in addition to those for the non-interacting systems, indices for Matsubara frequencies. One should also write properly spin indices. As a result, the supermatrix non-linear $\sigma$-model for electron systems with interaction takes the form (unitary ensemble)

\[
F = \frac{\pi \nu}{4} \int \mathrm{d} r \operatorname{Str} \left[ D(\nabla Q)^2 - 4EQ \right] + \frac{\pi \nu}{4} \int \mathrm{d} r \left[ \Gamma_2 Q \gamma_2 Q - \Gamma_1 Q \gamma_1 Q \right]
\]

where $\gamma_1$ and $\gamma_2$ are certain operators acting on the supermatrices $Q$, and $\Gamma_1$ and $\Gamma_2$ are scattering amplitudes characterizing the interaction (they are different from those obtained for the replica $\sigma$-model [24, 25]. As usual, one has the constraint $Q^2 = 1$ but now the product of two supermatrices includes summation also summation over the Matsubara frequencies.

Using the sigma-model Eq. (109) renormalization group equations of Refs. ([24, 25]) have been reproduced in the first order in the interaction constants and there is a hope to use this model for non-perturbative calculations.

**Method of quasiclassical Green functions.**

Although the interaction is included in Eq. (109), it is written for systems with considerably high concentration of short range impurities. Studying problems for clean systems or systems with a long range disorder one needs a different scheme. Statistical properties of clean chaotic
systems are covered in this school by Boris Altshuler and I do not review them here. Instead, I want to concentrate on calculational schemes.

The saddle point equation (61) is not a good approximation for clean systems and systems with long range disorder and we cannot follow the same way as the one used for diffusive models. The method that I want to present now is based on using quasiclassical Green functions.

Introducing an $8 \times 8$ matrix function $G(r, r')$ as

$$G(r, r') = 2 \langle \psi(r) \bar{\psi}(r') \rangle \psi$$

we can write in a standard way the following equation for this function

$$\left[ H_{0r} + U(r) + \frac{\omega + i \delta}{2} + i J(r) \right] G(r, r') = i \delta(r - r')$$

(111)

where the subscript $r$ of $H_{0r}$ means that the operator acts on $r$. The notations are the same as in Eqs. (53, 54), and $J(r)$ is a source term that allows to extract more complicated correlation functions.

Conjugating Eq. (111) we obtain another equation for the matrix $G(r, r')$ with the operator $H_{0r'}$ acting on its second variable

$$G(r, r') \left[ H_{0r'} + U(r') + \frac{\omega + i \delta}{2} + i J(r') \right] = i \delta(r - r')$$

(112)

Until now no approximations have been done and Eqs. (111, 112) are exact. Now we can use the assumption that the potential $U(r)$ changes slowly on the wavelength $\lambda_F$. If the mean free path $l$ for the scattering on the random potential exceeds $\lambda_F$ the Green function varies as a function of $r - r'$ at distances of the order of $\lambda_F$ but, at the same time, is a slow function of $(r + r')/2$. The Fourier transform $G_p((r + r')/2)$ of $G(r, r')$ respective to $r - r'$ has a sharp maximum near the Fermi surface.

In order to cancel large terms we subtract Eq. (112) from Eq. (111). Using the assumption that the potential $U(r)$ is smooth and expanding it in gradients we obtain in the lowest order

$$\left[ -\frac{i p \nabla R}{m} + i \nabla R U(R) \frac{\partial}{\partial p} \right] G_p(R) + \frac{\omega + i \delta}{2} [\Lambda, G_p(R)] + i [J(R), G_p(R)] = 0$$

(113)

where $R = (r + r')/2$ and $[\cdot]$ stands for the commutator. When deriving Eq. (113), not only the potential $U(r)$ but also the function $J(r)$ was assumed to be smooth.

The dependence of the Green function $G_p(R)$ on $|p|$ is more sharp than on other variables. In order to avoid this sharp dependence we
integrate Eq. (113) over $|p|$. Of course, this procedure makes a sense for very large samples when the level discreteness can be neglected.

The most interesting contribution in the integral over $|p|$ comes from the vicinity of the Fermi-surface. A contribution given by momenta considerably different from $p_F$ is proportional to the unity matrix and drops out from Eq. (113).

Introducing the function $g_n (r)$

$$g_n (r) = \frac{1}{\pi} \int G_{pn} (r) d\xi, \quad \xi = \frac{p^2 - p_F^2}{2m}$$

(114)

where $n$ is a unit vector pointing a direction on the Fermi surface, we obtain the final quasiclassical equation

$$\left( v_F n \nabla - p_F^{-1} \nabla_U (r) \partial_n \right) g_n (r) + \frac{i (\omega + i\delta)}{2} \left[ \Lambda, g_n (r) \right] - [J, g_n] = 0$$

(115)

where

$$\partial_n = \nabla_n - n, \quad \nabla_n = -[n \times [n \times \frac{\partial}{\partial n}]]$$

Eq. (115) should be complemented by a boundary condition at the surface of the sample. Considering a closed sample we assume that the current across the border is equal to zero. This leads the boundary condition at the surface

$$g_n (r)|_{surface} = g_{-n} (r)|_{surface}$$

(116)

where $n_\perp$ is the component of the vector $n$ perpendicular to the surface.

Eq. (115) is similar to an Eilenberger equation written long ago in superconductivity theory [50]. As in the theory of superconductivity, the solution for the Eq.(115) satisfies the condition $g_n^2 (r) = 1$. Eq. (115) is written for a non-averaged potential $U (r)$ and it is valid also in the absence of the long range potential.

The quasiclassical equation, Eq. (115), has been written first by Muzykantskii and Khmelnitskii, Ref. [51] who guessed a functional for which Eq. (115) is just a condition for an extremum. Then, they proceeded to work with this functional without estimating fluctuations near this minimum.

It came later as a surprise that, actually, one could write the proper solution of Eq. (115) in terms of a functional integral over supermatrices exactly[52]. The exact solution for Eq. (115) can be written as
In Eq. (117), the partition function $Z[J(r)]$ is

$$Z[J] = \int_{Q_n^2=1} \exp \left( -\frac{\pi\nu}{2} \Phi[J(Q_n(r))] \right) DQ_n,$$

and the integration is performed over the self-conjugate supermatrices $Q_n = \bar{Q}_n(r)$ satisfying the following relation

$$Q_n^2(r) = 1$$

We see that the quasiclassical Green function $g_n(r)$ can be written in the form of a functional integral over supermatrices $Q_n(r)$ depending both on the coordinates $r$ and the direction of the momentum $n$ and satisfying the constraint, Eq. (119). The first term in the free energy functional is written in terms of the supermatrices $T_n$ rather than $Q_n$. However, it can be written in a form of a Wess-Zumino-Novikov-Witten term containing the supermatrices $Q$ only[51]. Writing this term one should introduce an additional coordinate varying at the interval $[0,1]$.

The model described by the functional $\Phi$, Eqs. (117), is usually referred to as a “ballistic $\sigma$-model”. The partition function $Z[J]$, Eq. (118), is unity at $J=0$ due to the supersymmetry and this allows us to average (if necessary) over the smooth potential $U(r)$.

The method of quasiclassical Green functions suggested here for a static external potential does not seem to be restricted by the non-interacting case. There are indications that it can be generalized to describe clean interacting systems. Of course, in this case one should write the quasiclassical equations in time representation because the interaction mixes states with different energies. Study of interacting systems with this method may be a very interesting direction of research.

As concerns attempts to prove the Wigner-Dyson statistics for clean non-interacting systems one can try to start with the functional $\Phi$, Eq. (117). At first glance, we should simply restrict ourselves with the integration over $Q$ depending neither on the coordinates, nor on the momenta. Then, the functional $\Phi$ would contain only the last term and we
would have the zero dimensional $\sigma$-model, which leads immediately to the WD statistics.

However, a very important question is whether one averages over the potential $U(\mathbf{r})$ or puts $U(\mathbf{r}) = 0$ and averages over the spectrum. In the former case one gets after averaging over $U(\mathbf{r})$ an additional term in functional $\Phi$ quadratic in gradients. This term leads eventually to a suppression of non-zero harmonics and one can really obtain the zero-dimensional $\sigma$-model (see [52] and references therein).

The situation with $U(\mathbf{r}) = 0$ and averaging over the energy is more interesting. Everything depends on whether the system is classically integrable or chaotic. It is just the situation for which the authors of Ref. [16] made their hypothesis.

It turns out that within the model with the functional $\Phi$, Eq. (117), and $U(\mathbf{r}) = 0$ one cannot come to the zero-dimensional $\sigma$-model. There is a common consensus that a “regularizer” (see e.g. [53]) containing something like square of gradients in coordinates or momenta is necessary in the correct ballistic $\sigma$-model. Aleiner and Larkin [54] argued that in order to come to the zero dimensional $\sigma$-model one had to take into account diffraction, which is clearly absent in the ballistic $\sigma$-model, Eq. (117). They did not manage to include the diffraction in their calculational scheme microscopically and mimiced it by introducing artificial quantum impurities that would correspond to the potential $U(\mathbf{r})$ in Eq. (117). This allowed them to come to the zero dimensional $\sigma$-model, confirm the WD statistics and calculate corrections to it. It is worth emphasizing that the effective potential $U(\mathbf{r})$ was very weak such that the computation was done in the ballistic regime.

As concerns the real physical diffraction, it cannot be directly included in the $\sigma$-model using the quasiclassical scheme and a more sophisticated approach is necessary.

**Beyond the quasiclassics**

We have seen that the solution of the equations for the Green functions, Eqs. (111, 112), can be written in the quasiclassical approximation in terms of the functional integral over $8 \times 8$ supermatrices $Q_n(\mathbf{r})$. For certain problems this approximation is not sufficient and the natural question is: can we do better than that and find a solution for the Green functions in terms of a functional integral over supermatrices valid at all distances including those of the order of the electron wavelength $\lambda_F$?

This attempt has been undertaken recently in Ref. [55], where an integral of such a type was suggested for a solution of Eq. (111). The idea is rather close to the one known in field theory where it is called
bosonization [56]. The final expressions obtained in Ref. ([56]) are rather complicated and this method, to the best of my knowledge, has not evolved into an efficient calculational tool.

However, the supersymmetric form of the Green functions considered here seems to promise more and the derivation is rather simple. I follow here a simpler derivation of Ref. [57].

What I want to show now is that the Green function, Eq. (111), can be represented exactly as an integral over supermatrices \( Q(\mathbf{r}, \mathbf{r}') \) depending on two coordinates \( \mathbf{r} \) and \( \mathbf{r}' \)

\[
G(\mathbf{r}, \mathbf{r}') = Z^{-1}[J] \int Q(\mathbf{r}, \mathbf{r}') \exp(-\Phi[Q]) dQ
\]  

(120)

where \( Z[J] \) is a new partition function

\[
Z[J] = \int \exp(-\Phi[Q]) dQ
\]  

(121)

and the functional \( \Phi[Q] \) has the form

\[
\Phi[Q] = \frac{i}{2} \text{Str} \int \left( H_{0r} + U(\mathbf{r}) + \frac{\omega + i\delta}{2} \Lambda \right) \times \delta(\mathbf{r} - \mathbf{r}') Q(\mathbf{r}, \mathbf{r}') d\mathbf{r} d\mathbf{r}' \\
+ \frac{1}{2} \text{Str} \ln Q - \frac{1}{2} \text{Str} \int J(\mathbf{r}, \mathbf{r}') Q(\mathbf{r}', \mathbf{r}) d\mathbf{r} d\mathbf{r}'
\]  

(122)

where \( J(\mathbf{r}, \mathbf{r}') \) is a source term. The structure of the supermatrix \( Q(\mathbf{r}, \mathbf{r}') \) in the integral Eq. (120) should be the same as that of the Green function \( G(\mathbf{r}, \mathbf{r}') \), i.e. be the same as of the product \( \psi(\mathbf{r}) \bar{\psi}(\mathbf{r}') \). In particular, this means that \( Q(\mathbf{r}, \mathbf{r}') \) is self-conjugated

\[
\bar{Q}(\mathbf{r}, \mathbf{r}') \equiv C Q^T(\mathbf{r}', \mathbf{r}) C^T = Q(\mathbf{r}, \mathbf{r}')
\]  

(123)

In order to prove Eq. (120) we write the following identity

\[
-2iZ^{-1}[J] \int \left[ \int \frac{\delta \exp\left(-\frac{1}{2}\text{Str} \ln Q\right)}{\delta Q(\mathbf{r}'', \mathbf{r})} Q(\mathbf{r}'', \mathbf{r}') d\mathbf{r}'' \right] \times \\
\exp\left(-\frac{i}{2} \text{Str} \left[ \tilde{H}_{0r} + \frac{\omega + i\delta}{2} \Lambda + iJ \right] \right) Q(\mathbf{r}, \mathbf{r}') dQ = \\
i\delta(\mathbf{r} - \mathbf{r}')
\]  

(124)

and integrate over \( Q \) by parts. The derivative \( \delta/\delta Q \) should act now on both \( Q \) and the exponential. At this point, the supersymmetry plays
a crucial role. Differentiating first the supermatrix $Q$ we obtain the supermatrix product $(\delta/\delta Q)\bar{Q}$. As the number of the anticommuting variables in the sum over the matrix elements is equal to the number of the boson ones and the derivatives have the opposite signs, this matrix product vanishes. Differentiating the exponential only we come to the following equation

$$Z^{-1}[J] \int dr'' \left( \hat{H}_{0r} + U(r) \frac{\omega + i\delta}{2} \Lambda + iJ \right)(r, r'') \times \int Q(r'', r') \exp(-\Phi[Q]) DQ = i\delta(r - r') \quad (125)$$

Eq. (125) proves immediately that the integral Eq.(120) does satisfy Eq.(111) and we have really the alternative representation of the Green function in terms of an integral over the supermatrices $Q$.

Making the Fourier transform $Q(r', r'')$ in the difference $r' - r''$ (Wigner transformation) one can express the functional $\Phi[Q]$ in terms of the variables $Q_p(r)$, where $p$ is the momentum and $r$ is the center of mass $(r' + r'')/2$. Then the free energy functional $\Phi$, Eq. (122) can be written as

$$\Phi[Q] = \frac{i}{2} \int \text{Str} \left[ \mathcal{H}_J(x) * Q(x) - i \ln Q(x) \right] dx \quad (126)$$

where $x = (r, p)$ is the coordinate in the phase space,

$$\mathcal{H}_J(x) = H_0(p) + U(r) + \frac{\omega + i\delta}{2} + iJ(p, r) \quad (127)$$

is classical Hamilton function.

The product $*$ of two matrices $A(x)$ and $B(x)$ is defined by Moyal formula

$$A(x) * B(x) = A(x)e^{\frac{1}{2}\left(\nabla_r \nabla_p - \nabla_r \nabla_p\right)}B(x).$$

The scheme of calculations using the Wigner representations and the star product “$*$” is known as Weyl symbol calculus [58]. This method is convenient for quasiclassical expansions.

If the potential $U(r)$ is smooth, one can simplify Eq. (126) and come again to Eq. (117). This procedure is described in Ref. [55]. The functional $\Phi$, Eq. (126) has a form of the Lagrangian of a non-commutative field theory [58]. The method suggested here can naturally be called “superbosonization”.

At the end I have to warn that Eqs. (120-122, 126) are not complete yet because nothing has been said about the contour of integration over
the supermatrices $Q$. This was not very important in Refs. ([55, 57]) where a saddle point approximation was used. However, generally, this question requires a more careful study and this is a subject of a current work. In case if the difficulties are overcome, the superbosonization can become very useful in the theory of random matrices for more general models than the one described by the conventional Eq. (1).

4. Summary

In these lectures I tried to achieve two goals:

1. The random matrix theory is to a large extent a phenomenological theory. Therefore, it is very important to have examples when it can be obtained starting from a microscopic model. The model of the disordered metals considered here is the first one for which the relevance of the RMT has been proven. The proof became possible with the help of the supermatrix non-linear $\sigma$-model first derived for other purposes.

2. Having presented the derivation of the $\sigma$-model I demonstrated how one obtains the Wigner-Dyson statistics from its zero-dimensional version. However, in many situations the supersymmetry method allows to go beyond the Wigner-Dyson model and obtain completely different results like the log-normal distribution of the amplitudes of the wave functions of Section 2. Of course, as soon as one has a Hamiltonian one comes to random matrices. However, generally it is not clear how to write the distribution function for these matrices in each particular situation and the supersymmetry method can play an important role for investigation of microscopic models.

I wanted by no means to oppose the RMT and the supersymmetry method to each other. They can be considered as complimentary methods, although with a considerable overlap. We see at this school that the random matrices find more and more applications in many fields of physics, which is an exciting development. I believe, in many cases the supersymmetry technique can also be useful in these new applications and one should keep in mind a possibility of using this scheme.

Acknowledgments

This work was supported by the Transregio 12 "Symmetries and Universality in Mesoscopic Systems" of German Research Society".
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