A GEOMETRIC APPROACH TO THE SEPARABILITY OF THE NEUMANN-ROSOCHATIUS SYSTEM

CLAUDIO BARTOCCI, GREGORIO FALQUI, AND MARCO PEDRONI

Abstract. We study the separability of the Neumann-Rosochatius system on the $n$-dimensional sphere using the geometry of bi-Hamiltonian manifolds. Its well-known separation variables are recovered by means of a separability condition relating the Hamiltonian with a suitable $(1,1)$ tensor field on the sphere. This also allows us to iteratively construct the integrals of motion of the system.

1. Introduction

The Neumann system is among the widest known and best studied integrable systems in Mathematical Physics. It describes the dynamics of a point particle constrained to move on the sphere $S^n$, under the influence of a quadratic potential $V(x) = \frac{1}{2} \sum_{i=1}^{n+1} \alpha_i x_i^2$, $\alpha_i \neq \alpha_j$. In 1859, Carl Neumann [20] showed that the equations of motion of the “physical” $n = 2$ case could be solved using the Jacobi theory of separation of variables. It was noticed by Rosochatius (see [22]) that a potential given by the sum (with nonnegative weights) of the inverses of the squares of the (Cartesian) coordinates can be added without losing the separability property. The system so obtained is customarily called the Neumann-Rosochatius (NR) system.

More than one century later this separability result was generalized to the arbitrary $n$ case by Moser [19]. The starting point to solve the problem was the ingenious introduction of a special set of coordinates on $S^n$, called sphericoconical (or elliptical spherical) coordinates (already used, for $n = 2$, by Neumann). They are defined as follows: For given sets of real numbers $\alpha_1 < \alpha_1 < \cdots < \alpha_{n+1}$ and nonzero $x_1, \ldots, x_{n+1}$, the coordinates $\lambda_a(x), a = 1, \ldots, n$, are the solutions of the equation

\begin{equation}
\sum_{i=1}^{n+1} \frac{x_i^2}{\lambda - \alpha_i} = 0 .
\end{equation}

AMS Subject Classification: 37K10, 53D17, 70H20
Keywords: Bi-Hamiltonian manifolds, Separation of variables, Spectral Curves
SISSA Preprint 62/2003/FM
Later on, it was shown that the NR system could be framed within the formalism of Lax pairs and $r$-matrices (see, e.g., [1, 12]). Actually, it turns out that introducing the Lax matrix, as a function of the Cartesian coordinates $x_i, y_i, i = 1, \ldots, n+1, \frac{1}{2}\sum_{i=1}^{n+1}\frac{x_i y_i}{\lambda - \alpha_i}$,
\begin{equation}
N(\lambda) = \begin{pmatrix} -h(\lambda) + ik(\lambda) & e(\lambda) \\ f(\lambda) & h(\lambda) + ik(\lambda) \end{pmatrix},
\end{equation}
where
\begin{align*}
h(\lambda) &= \frac{1}{2}\sum_{i=1}^{n+1}\frac{x_i y_i}{\lambda - \alpha_i}, \\
k(\lambda) &= \frac{1}{2}\sum_{i=1}^{n+1}\frac{\beta_i}{\lambda - \alpha_i}, \\
f(\lambda) &= \frac{1}{2}\sum_{i=1}^{n+1}\frac{x_i^2}{\lambda - \alpha_i}, \\
e(\lambda) &= -\frac{1}{2}\left(1 + \sum_{i=1}^{n+1}\frac{y_i^2 + \beta_i^2/x_i^2}{\lambda - \alpha_i}\right),
\end{align*}
and identifying the cotangent bundle to $S^n$ with the submanifold of $\mathbb{R}^{2(n+1)}$ defined by the constraints
\begin{equation}
\sum_{i=1}^{n+1}x_i^2 = 1, \quad \sum_{i=1}^{n+1}x_i y_i = 0,
\end{equation}
the Hamilton equations of motion of the NR system acquire the form
\begin{equation}
\frac{dN(\lambda)}{dt} = [\Phi, N(\lambda)],
\end{equation}
where
\begin{equation}
\Phi = \begin{pmatrix} \sum_{i=1}^{n+1}x_i y_i & \sum_{i=1}^{n+1}(y_i^2 + \alpha_i x_i^2 \beta_i/x_i^2) \\ -\sum_{i=1}^{n+1}x_i^2 & -\sum_{i=1}^{n+1}x_i y_i \end{pmatrix}.
\end{equation}
As a consequence, the spectral invariants of $N(\lambda)$ are constants of motion. In particular, the quantities
\begin{equation}
K_i = \text{res}_{\lambda = \alpha_i} \det(N(\lambda)), \quad i = 1, \ldots, n+1
\end{equation}
(known as Uhlenbeck integrals), provide $n$ mutually commuting integrals of motion that ensure Liouville–Arnol’d integrability of the NR system, the physical Hamiltonian being given by
\begin{equation}
H_{NR} = 2\sum_{i=1}^{n+1}\alpha_i K_i + \frac{1}{2}\sum_{i>j=1}^{n+1}\beta_i \beta_j.
\end{equation}
Separation of variables is recovered in this formalism noticing that, on $S^n$, the zeroes $\{\lambda_a\}_{a=1,\ldots,n}$ of the matrix entry $f(\lambda)$ define the spheroconical coordinates, and their conjugate momenta are given (as it will be explicitly recalled in Section 4) by the values of the rational function.
$h(\lambda)$ for $\lambda = \lambda_a$. Clearly, each pair of canonical coordinates $(\lambda_a, \mu_a)$ satisfy the separated equation

\begin{equation}
\det(N(\lambda_a)) + \mu_a^2 + k^2(\lambda_a) = 0, \quad a = 1, \ldots, n.
\end{equation}

In this paper we want to provide a further geometrical interpretation of the NR system, based on the notions of bi-Hamiltonian geometry, generalizing and refining the approach described in [21]. We will follow a recently introduced set up for the theory of separation of variables for the Hamilton-Jacobi equations. In a nutshell, such a framework can be described as follows. One considers a symplectic manifold $M$ endowed with a $(1,1)$ tensor field $N$ with vanishing Nijenhuis torsion (which we will call an $\omega N$ manifold, provided that a compatibility condition between $N$ and the symplectic form is satisfied); under suitable hypotheses, $N$ selects a special subclass of canonical coordinates on $M$ (called Darboux-Nijenhuis coordinates) that have the property of diagonalizing $N$. The condition for the separability of the Hamilton-Jacobi equation associated with a Hamiltonian $H$ can be given, according to the bi-Hamiltonian theory of separation of variables, the following intrinsic formulation. One considers the distribution $D_H$ generated iteratively by the action of $N$ on the Hamiltonian vector field $X_H$ associated with $H$, and the two–form $d(N^*dH)$. Then $X_H$ is separable in the Darboux-Nijenhuis coordinates associated with $N$ if and only if

\begin{equation}
d(N^*dH)(D_H, D_H) = 0.
\end{equation}

This scheme, in its basic features, has already been considered in the literature [3 17 10 15 11] and applied to various systems (see, e.g, [9 7 3 18]); it is fair to say that, in these papers, the $\omega N$ manifold structure is fixed “a priori”, and that equation (1.7) is seen as a condition that selects those Hamiltonians which are separable in the “preassigned” Darboux-Nijenhuis coordinates.

In the present paper we will take a different logical standpoint: We will consider a given Hamiltonian $H$ (namely, the NR Hamiltonian) and look at (1.7) as an equation to determine $N$ (and hence the separation coordinates). We shall see that it is indeed possible (and, actually, easy) to solve such an equation by means of a couple of natural Ans"atze, thus arriving to induce from $H$ the separation coordinates. Also, we shall show how the iterative structure naturally associated with the (generalized) recursion relations defined by $N$ allow to recursively construct the additional integrals of motion ensuring complete integrability. Finally, we will make contact with the “Lax” approach to the separability of the Neumann-Rosochatius system showing that
the separation relations tying Darboux-Nijenhuis coordinates and these integrals are nothing but the spectral curve relations (1.6).

Obviously enough, the conditions on $N$ coming from (1.7), in their full generality, are too difficult to be solved. The couple of Ansätze which will enable us to solve them for the NR case are the following.

The first one is suggested from the fact that the phase space of the NR system is a cotangent bundle; accordingly, we will seek for a special $\omega N$ manifold structure on $T^*S^n$, defined by a $(1, 1)$ tensor $N$ induced by a suitable tensor (with zero torsion) $L$ on the base manifold $S^n$. The second one will be to use a special form of equation (1.7), that reads

\[ d(N^*dH - \frac{1}{2}\text{tr}(N) \wedge dH) = 0. \]

The plan of the paper is as follows: In Section 2 we will collect some notions of the theory of $\omega N$ manifolds, and briefly discuss the bi-Hamiltonian theorem for separation of variables. In Section 3 we will solve equations (1.8), thus showing that the geometry of $\omega N$ manifolds can be used to discover the separation variables of the Neumann-Rosochatius (NR) system. In Section 4 we will find the Stäckel separation relations and the family of commuting integrals of the system.

Acknowledgments. We wish to thank Sergio Benenti, Franco Magri, and Giovanni Rastelli for useful discussions. The results presented in Section 2 have been obtained in collaboration with Franco Magri. This work has been partially supported by the Italian M.I.U.R. under the research project Geometry of Integrable Systems.

2. The $\omega N$ Framework

In this section we wish to recall some basic properties of a special class of bi-Hamiltonian manifolds, called $\omega N$ manifolds. For a more detailed description we refer to [16, 11]. By definition, an $\omega N$ manifold is a smooth manifold $M$ endowed with a pair of compatible Poisson bivectors $P_0, P_1$ such that one of them (say, $P_0$) is nondegenerate. (Compatibility means that any linear combination of $P_0$ and $P_1$ is a Poisson bivector.) One can construct a recursion operator $N = P_1P_0^{-1}$, whose Nijenhuis torsion,

\[ T(N)(X, Y) = [NX, NY] - N([NX, Y] + [X, NY] - N[X, Y]), \]

vanishes as a straightforward consequence of the compatibility between $P_0$ and $P_1$ (see, e.g., [14]). We set $2n = \text{dim} M$ and we denote by $\omega_0$ the symplectic structure associated to $P_0$, and by $\{\cdot, \cdot\}_0,\{\cdot, \cdot\}_1$ the Poisson brackets associated, respectively, to $P_0, P_1$. 
The relevance of $\omega N$ manifolds in the theory of separable systems is mainly due to the existence, under suitable hypotheses, of a special class of canonical coordinates, that are selected by the geometric structure of the system itself.

**Definition 1.** A system of local coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ that are canonical w.r.t. the symplectic form $\omega_0$ is said to be Darboux-Nijenhuis if the matrix expressing $N$ in these coordinates is diagonal, i.e.,

$$N = \sum_{i=1}^{n} \left( \lambda_i \frac{\partial}{\partial x_i} \otimes dx_i + \nu_i \frac{\partial}{\partial y_i} \otimes dy_i \right).$$

Notice that, since $NP_0$ is antisymmetric, it follows that $\lambda_i = \nu_i$ for all $i$. In general, however, the eigenvalues $\lambda_1, \ldots, \lambda_n$ need not be distinct.

On the cotangent bundle of any differentiable manifold there is an elegant way to construct $\omega N$ structures admitting Darboux-Nijenhuis coordinates through the following procedure [13]. Let $Q$ be an $n$-dimensional manifold equipped with a type $(1, 1)$ tensor field $L$, whose Nijenhuis torsion vanishes. Let $\theta_0$ be the Liouville 1-form and $\omega_0 = d\theta_0$ the standard symplectic 2-form on $T^*Q$; the associated Poisson structure will be denoted $P_0$. By thinking of $L$ as an endomorphism of $TQ$, one can deform the Liouville 1-form to a 1-form $\theta_L$:

$$\langle \theta_L, Z \rangle_\alpha = \langle \alpha, L(\pi^*Z) \rangle_{\pi(\alpha)},$$

for any vector field $Z$ on $T^*Q$ and for any 1-form $\alpha$ on $Q$, where $\pi : T^*Q \to Q$ is the canonical projection. If we choose local coordinates $(x_1, \ldots, x_n)$ on $Q$ and set $L(X) = \sum_{i,j=1}^{n} L^i_j X^j \frac{\partial}{\partial x_i}$, we get the local expression $\theta_L = \sum_{i,j=1}^{n} L^i_j y_i dx_j$ w.r.t. the standard symplectic coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on $T^*Q$. The complete lift of $L$ is the endomorphism $N$ of $T(T^*Q)$ uniquely determined by the condition

$$d\theta_L(X, Y) = \omega_0(NX, Y),$$

for all vector fields $X, Y$ on $T^*Q$. An easy computation shows that:

$$N\left( \frac{\partial}{\partial x_k} \right) = \sum_{i} L^i_k \frac{\partial}{\partial x_i} - \sum_{i} y_i \left( \frac{\partial L^i_l}{\partial x_k} - \frac{\partial L^i_k}{\partial x_i} \right) \frac{\partial}{\partial y_i}$$

and

$$N\left( \frac{\partial}{\partial y_k} \right) = \sum_{i} L^i_k \frac{\partial}{\partial y_i}.$$
is an \(\omega N\) manifold \[13\]. The Poisson structure \(P_1\) is related to \(\omega_1 := d\theta_L\) by the formula:

\[
P_1(dF, dG) = \omega_1(\{F\}, \{G\})
\]

for all \(F, G \in C^\infty(T^*Q)\), where \(\{\cdot, \cdot\}\) are the Hamiltonian vector fields associated to \(F, G\) w.r.t. the symplectic form \(\omega_0\).

By the very definition, if \(X^{(1)}_H\) is the Hamiltonian vector field associated to \(H\) w.r.t. \(P_1\), then \(X^{(1)}_H = NX_H\).

Notice that, in general, this vector field need not be Hamiltonian or even locally Hamiltonian w.r.t. \(\omega_0\). Indeed, the 1-form \(N^*dH\) may fail to be closed (here \(N^*\) is the adjoint of the endomorphism \(N\)), and one has

\[
L_{NX_H} \omega_0 = -d(N^*dH) = L_{X_H} \omega_1.
\]

In fact, from (2.2) it follows that \((N^*dF)(Y) = -\omega_0(NX_H, Y) = -\omega_1(X_H, Y)\).

Let us now assume that \(L\) has \(n\) functionally independent eigenvalues \(\lambda_1, \ldots, \lambda_n\). Since \(L\) is torsionless, these eigenvalues determine local coordinates on \(Q\) satisfying the relations:

\[
L \frac{\partial}{\partial \lambda_i} = \lambda_i \frac{\partial}{\partial \lambda_i}.
\]

We denote by \(\mu_i\) the conjugate momentum to \(\lambda_i\); clearly, one has \(N \frac{\partial}{\partial \mu_i} = \lambda_i \frac{\partial}{\partial \mu_i}\). The coordinates \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\) are Darboux-Nijenhuis coordinates for the \(\omega N\) manifold \((T^*Q, P_0, P_1)\).

We can exploit the geometric setting of \(\omega N\) manifolds in order to find an intrinsic separability condition for a given Hamiltonian function \(H \in C^\infty(T^*Q)\). Let us suppose that the vector fields \(X_H, NX_H, \ldots, N^{n-1}X_H\) are pointwise linearly independent, so that they generate an \(n\)-dimensional distribution \(\mathcal{D}_H\). If we compute the conditions

\[
d(N^*dH)(N^iX_H, N^jX_H) = 0 \quad \text{for all } i, j = 0, 1, \ldots, n - 1
\]

in the Darboux-Nijenhuis coordinates \((\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)\), we get a system of differential equations equivalent to the Levi-Civita separability formulae \[3\] p. 208, eq. (1.230)]).

**Theorem 2.** In the above hypotheses and notations, the Darboux-Nijenhuis coordinates associated with \(L\) are separation variables for \(H\) if and only if the 2-form \(d(N^*dH)\) annihilates the distribution \(\mathcal{D}_H\).

The separability condition \[2.6\] implies that the distribution \(\mathcal{D}_H\) is integrable. So, there exist \(n\) independent local functions \(H_1, \ldots, H_n\) that are constant on the leaves of \(\mathcal{D}_H\). The distribution being invariant under the action of \(N\), the same is true for the differential ideal generated
by the \( H_i \)'s, so that the following condition holds:

\[
(2.7) \quad N^*dH_i = \sum_{k=1}^{n} F_{ik}dH_k,
\]

where \( F_{ik} \) is a matrix with distinct eigenvalues. Moreover, since \( D_H \) is Lagrangian with respect to both \( \omega_0 \) and \( \omega_1 \), we have:

\[
(2.8) \quad \{ H_i, H_j \}_0 = \{ H_i, H_j \}_1 = 0 \quad \text{for all } i, j.
\]

It follows that the functions \( H_1, \ldots, H_n \) are separable in the the Darboux-Nijenhuis coordinates singled out by \( L \).

A particular case of this state of affairs is provided by the following example, which will turn out to be of great importance in the study of the NR system. Let us consider the characteristic polynomial

\[
(2.9) \quad \det(\lambda I - L) = \lambda^n - c_1\lambda^{n-1} - c_2\lambda^{n-2} - \cdots - c_n
\]

of the endomorphism \( L \), and assume we are given a Hamiltonian \( H \) satisfying the condition

\[
(2.10) \quad d(N^*dH) = dc_1 \wedge dH.
\]

From (2.10) it follows at once that this equation is equivalent to

\[
(2.11) \quad d(LX_H\theta_L - Hdc_1) = 0.
\]

The condition (2.10) is a sufficient condition to the separability of \( H \) in the Darboux-Nijenhuis coordinates associated to \( N \), because it implies that the 2-form \( d(N^*dH) \) annihilates the distribution \( D_H \) generated by the vector fields \( X_H, NX_H, \ldots, N^{n-1}X_H \). Moreover, it can be shown that, choosing a local function \( H_2 \) such that \( dH_2 = N^*dH - c_1dH \), the 1-form \( N^*dH_2 - c_2dH \) is again closed, so that we can find a local potential \( H_3 \). By iterating this procedure, we end up with \( n \) independent local functions \( H_1 = H, \ldots, H_n \) that are constant on the leaves of \( D_H \) and satisfy the conditions:

\[
(2.12) \quad dH_{i+1} = N^*dH_i - c_idH \quad i = 2, \ldots, n - 1
\]

\[
0 = N^*dH_n - c_n dH.
\]

In this case the matrix \( F \) has the form

\[
F = \begin{pmatrix}
  c_1 & 1 & 0 & \ldots & 0 \\
  c_2 & 0 & 1 & \ddots & \vdots \\
  \vdots & \vdots & 0 & \ddots & \vdots \\
  \vdots & \vdots & \ddots & \ddots & \ddots \\
  c_n & 0 & 0 & \ldots & 0
\end{pmatrix}
\]
and the following condition is readily checked:

\[(2.13) \quad N^*dF = FdF.\]

We set \(F = S^{-1} \text{diag}(\lambda_1, \ldots, \lambda_n)S\). Then, one has \(S_{jk} = \lambda^{n-k}_j\) and, by virtue of [11, Theorem 4.2], one obtains the separability equations \(\sum_{k=1}^n H_k \lambda^{n-k}_j = U_j\). Summing up, the functions \(H_1 = H, \ldots, H_n\) are proved to be Stäckel separable in the Darboux-Nijenhuis coordinates associated to \(L\) (this means that the separation relations are affine in the \(H_k\)’s).

3. Torsionless tensors and separability of the NR system

According to the results of Section 2, to separate the NR system we seek a tensor field \(L\) of type \((1,1)\) on \(S^n\) satisfying the “strong” separability condition \((2.10)\), i.e.,

\[(3.1) \quad d(LX_a \theta_L - Hdc_1) = 0,\]

where \(\theta_L = \sum_{a,b=1}^n L^a_b p_a dq_b\) (for any set of fibered coordinates \((q_a, p_a)\)) and \(c_1 = \text{tr}L\). We have seen that the eigenvalues of such an \(L\) (if real and functionally independent) are separation variables for \(H\). The form of the constraint and of the potential suggest using on \(S^n\) the coordinates \(X_a := x_a^2\), for \(a = 1, \ldots, n.\)\footnote{In this and in the following section we use the following convention: middle indices like \(i, j, k\) run from 1 to \(n+1\), while indices like \(a, b, c\) run from 1 to \(n\).} If \(Y_a\) are the momenta conjugated to the \(X_a\) (and the point particle has unit mass), the NR Hamiltonian is given by

\[
T = 2 \sum_a X_a(1 - X_a)Y_a^2 - 4 \sum_{a<b} X_a X_b Y_a Y_b
\]

\[
V = \frac{1}{2} \sum_a \left[ (\alpha_a - \alpha_{n+1})X_a + \frac{\beta_a^2}{X_a} \right] + \frac{\beta_{n+1}^2}{2(1 - \sum_a X_a)}.
\]

Expanding in powers of the momenta, we see that condition \((3.1)\) splits into

\[(3.2) \quad d(LX_a \theta_L - Tdc_1) = 0\]

\[(3.3) \quad d(LX_a \theta_L - Vdc_1) = 0.\]

**Remark 3.** As noticed in [4], equation \((3.2)\) means that \(L\) is a symmetric conformal Killing tensor with respect to the usual Riemannian metric of \(S^n\), and implies that the torsion of \(L\) vanishes. On the other hand, equation \((3.3)\) can be written as \(d(L^*dV + c_1 dV) = 0\), which is a separability condition on the potential \(V\) appearing in the works of Benenti (see, e.g., [2]). However, since our approach applies also to...
systems which are defined on general symplectic manifolds (not necessarily cotangent bundles), or, in other words, to Hamiltonians that are not quadratic in the momenta, we will not use these results and we will solve directly equations (3.2) and (3.3). We also observe that a significant part of the “Riemannian” theory of separation of variables can be seen as a particular case of the bi-Hamiltonian approach (see [13, 8] and [3], where the Neumann system is also discussed).

We start seeking a solution \( L \) whose dependence on the coordinates \( X_a \) is affine: \( L_a^b = \sum_c A^a_{bc} X_c + B_a^b \). Let us consider, for the sake of simplicity, the case \( n = 2 \). Condition (3.2) gives

\[
A_{12}^2 = A_{22}^1 = A_{11}^2 = A_{21}^1 = B_1^2 = B_2^1 = 0 ,
\]

so that we are left with the unknowns \( A_{22}^2, B_1^1, \) and \( B_2^2 \). Now, condition (3.3) is equivalent to

\[
A_{22}^2 (\alpha_2 - \alpha_1) = (B_2^2 - B_1^1)(\alpha_3 - \alpha_2) ,
\]

which means that \( A_{22}^2 = c(\alpha_3 - \alpha_2) \) and \( B_2^2 - B_1^1 = c(\alpha_2 - \alpha_1) \) for some constant \( c \). Thus \( B_2^2 = c(\alpha_2 + d) \) and \( B_1^1 = c(\alpha_1 + d) \), where \( d \) is another constant, and the components of \( L \) are given by

\[
\begin{bmatrix}
L_1^1 & L_1^2 \\
L_2^1 & L_2^2
\end{bmatrix}
= c
\begin{bmatrix}
(\alpha_3 - \alpha_1)X_1 + \alpha_1 + d & (\alpha_3 - \alpha_2)X_1 \\
(\alpha_3 - \alpha_1)X_2 & (\alpha_3 - \alpha_2)X_2 + \alpha_2 + d
\end{bmatrix}
.
\]

Since we are interested in the coordinates given by the eigenvalues of \( L \), we can set \( c = 1 \) and \( d = 0 \) without loss of generality.

Coming back to the general case, it is not difficult to check that the 1-form

\[
\theta_L = \sum_a \alpha_a Y_a dX_a + \left( \sum_b X_b Y_b \right) \sum_a (\alpha_{n+1} - \alpha_a) dX_a ,
\]

corresponding to the \((1, 1)\) tensor field given by

(3.4) \( L_b^a = (\alpha_{n+1} - \alpha_b)X_a + \delta_b^a \alpha_a \),

satisfies conditions (3.2) and (3.3). Although these formulas define \( L \) in coordinate patches, it is not difficult to show that \( L \) is globally defined on the whole \( S^n \). Indeed, it is the restriction to \( S^n \) of the tensor field \( \hat{L} \) on \( \mathbb{R}^{n+1} \) defined as

\[
\hat{L} \frac{\partial}{\partial x_i} = \alpha_i \frac{\partial}{\partial x_i} + \frac{x_i}{r^4} \sum_{j,k} (\alpha_k - \alpha_j - \alpha_i) x_k x_j \frac{\partial}{\partial x_j}
\]
where \( r^2 = \sum_i x_i^2 \). In order to show that \( \hat{L} \) restricts to \( S^n \) it is sufficient to check that \( \hat{L}^*dr = 0 \), implying that, at every point of \( S^n \), the image of \( \hat{L} \) is (contained in) the tangent space to the sphere.

Summarizing, we have found a tensor field \( L \) satisfying the separability condition (3.1); thanks to the result of [4] referred to in Remark 3, the torsion of \( \hat{L} \) vanishes. Thus the coordinates associated with \( L \) are separated variables for the NR system. Let us explicitly check that the eigenvalues of \( L \) coincide with the spheroconical coordinates. To this end we find convenient to introduce the following notations: let \( \alpha \) and \( X \) denote the \( n \)-component vectors whose entries are, respectively,

\[
\alpha_b = \alpha_{n+1} - \alpha_b, \quad X_b = X_b, \quad b = 1, \ldots, n,
\]

and let \( A \) be the \( n \times n \) diagonal matrix of the parameters \( \alpha_a \). Then we can compactly write the matrix form (3.4) of the tensor field \( L \) as

\[
L = A + X \otimes \alpha.
\]

To compute the roots of \( \det(\lambda - L) \) we notice that the \( L \) is a rank 1 perturbation of \( A \); hence we write

\[
\lambda - L = (\lambda - A) \cdot (1 - X'(\lambda) \otimes \alpha),
\]

where \( X'(\lambda) \) is the vector with entries \( \frac{X_b}{\lambda - \alpha_b} \). Using the rank 1 Aronszajn–Weinstein formula

\[
\det(1 + x \otimes y) = 1 + \langle y, x \rangle,
\]

we arrive at

\[
\det(\lambda - L) = \prod_{a=1}^n (\lambda - \alpha_a) \left( 1 - \sum_{b=1}^n \frac{(\alpha_{n+1} - \alpha_b)X_b}{\lambda - \alpha_b} \right).
\]

Recalling the definitions \( X_b = x_b^2 \), for \( b = 1, \ldots, n \), and the constraint \( \sum_i x_i^2 = 1 \), we can by means of elementary calculations conclude that such an equation is equivalent to

\[
\det(\lambda - L) = \prod_i (\lambda - \alpha_i) \sum_i \frac{x_i^2}{\lambda - \alpha_i},
\]

that is, the eigenvalues \( \lambda_a \) of \( L \) satisfy the equations

\[
\sum_i \frac{x_i^2}{\lambda - \alpha_i} = 0, \quad \text{with} \quad \sum_i x_i^2 = 1.
\]

These are the well-known defining relations for the spheroconical (or elliptic-spherical) coordinates.
We close this section reporting, for the sake of completeness, the well-known computation of the momenta \( \mu_a \) conjugated to the \( \lambda_a \). It is easily checked that the usual rule for computing the residues gives

\[
x_i^2 = \prod_a (\alpha_i - \lambda_a) / \prod_{j \neq i} (\alpha_i - \alpha_j).
\]

Then we have that

\[
x_i \, dx_i = -\frac{1}{2} \sum_a (\alpha_i - \lambda_a) \, d\lambda_a / \prod_{j \neq i} (\alpha_i - \alpha_j),
\]

and, using again (3.10), that

\[
dx_i = \frac{1}{2} x_i \sum_a \frac{d\lambda_a}{\lambda_a - \alpha_i}.
\]

If \((x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}) \in \mathbb{R}^{2n+2} \supset TS^n \simeq T^*S^n\), then the \( \mu_a \) are given by

\[
\sum_a \mu_a \, d\lambda_a = \left( \sum_i y_i \, dx_i \right)_{|T^*S^n} = \frac{1}{2} \sum_a \left( \sum_i \frac{x_i y_i}{\lambda_a - \alpha_i} \right) \, d\lambda_a,
\]

meaning that

\[
\mu_a = \frac{1}{2} \sum_i \frac{x_i y_i}{\lambda_a - \alpha_i}.
\]

Therefore, we can conclude that the separation variables \( \lambda_a \) are the solutions of

\[
\sum_i \frac{x_i^2}{\lambda_a - \alpha_i} = 0,
\]

while the conjugated momenta are given by \( \mu_a = h(\lambda_a) \), with

\[
h(\lambda) = \frac{1}{2} \sum_i \frac{x_i y_i}{\lambda - \alpha_i}.
\]

4. Integrals of motion and Stäckel separability

In the previous section we have found a tensor field \( L \) on \( S^n \) which gives the separation coordinates of the NR system (i.e., the sphericoconical coordinates). Since \( L \) satisfies the “strong” separability condition (3.1), we know from Section 2 that:

(1) There is an iterative method for constructing \( n \) integrals of motion in involution, \((H = H_1, H_2, \ldots, H_n)\). (Of course, we have to take into account that \( T^*S^n \) is simply connected for \( n \geq 2 \).)

(2) The NR system is Stäckel-separable.
The integrals of motion are given by
\begin{equation}
\frac{dH_{a+1}}{a+1} = N^* dH_a - c_a dH, \quad a = 1, \ldots, n - 1,
\end{equation}
where \( \lambda^n - \sum_{a=1}^n c_a \lambda^{n-a} = \det(\lambda I - L) \). This defines the \( H_a \) up to additive constants. For example, in the case \( n = 2 \) one finds
\begin{align*}
H_2 &= 2(\alpha_1 Y_2^2 X_2 + \alpha_2 Y_1^2 X_1)(X_1 + X_2 - 1) - 2\alpha_3 X_1 X_2(Y_2 - Y_1)^2 \\
&\quad - \frac{1}{2} \left[ \alpha_2(\alpha_1 - \alpha_3)X_1 + \alpha_1(\alpha_2 - \alpha_3)X_2 + \frac{\alpha_2 \beta_1^2}{X_1} + \frac{\alpha_1 \beta_2^2}{X_2} \right] \\
&\quad + (\alpha_3 - \alpha_2)\beta_1^2 \frac{X_2}{X_1} + (\alpha_3 - \alpha_1)\beta_2^2 \frac{X_1}{X_2} + \beta_3^2 \frac{\alpha_2 X_1 + \alpha_1 X_2}{1 - X_1 - X_2}.
\end{align*}
Before showing that the \( H_a \) coincide with the integrals of motion known in the literature, let us consider the separability \`a la St"ackel of the NR system. It is guaranteed from the results in Section 2 that
\begin{equation}
\sum_{b=1}^n \lambda_a^{n-b} H_b = U_a(\lambda_a, \mu_a), \quad a = 1, \ldots, n,
\end{equation}
where the \( U_a \) are polynomials.

\textbf{Remark 4.} It is easy to see that the polynomials \( X_a = X_a(\lambda_1, \ldots, \lambda_n) \) and \( Y_a = Y_a(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n) \) are invariant under the exchanges \((\lambda_b, \mu_b) \leftrightarrow (\lambda_c, \mu_c)\). This entails that \( U_b = U_c \).

Next we want to compare the constants of motion defined by (4.1) with the spectral invariants of the Lax matrix (see, e.g., [12])
\begin{equation}
P(\lambda) = a(\lambda) \det N(\lambda) = \frac{1}{4} \lambda^n + \sum_{a=1}^n P_a \lambda^{n-a},
\end{equation}
where \( a(\lambda) = \prod_i (\lambda - \alpha_i) \) and the restriction to \( T^* S^n \) has been tacitly assumed. In particular, \( P_1 = \frac{1}{2} H \), where \( H \) is the NR Hamiltonian. Our strategy to prove that \( P_a = \frac{1}{2} H_a \) for all \( a \) is to show that
\begin{equation}
N^* dP(\lambda_b) = \lambda_b dP(\lambda_b), \quad b = 1, \ldots, n,
\end{equation}
which implies that the $P_a$ satisfy

$$N^a dP_a = dP_{a+1} + c_a dP_1, \quad a = 1, \ldots, n - 1,$$

because the $\lambda_b$’s are the roots of $f(\lambda_b) = 0$. Since these relations coincide with the equations (4.1) for the $H_a$, and the starting points fulfill $P_1 = \frac{1}{2} H_1$, we can conclude that

$$P_a = \frac{1}{2} H_a \quad \text{for} \ a = 1, \ldots, n. \quad (4.5)$$

To show that (4.4) holds, we recall that $\mu_b = h(\lambda_b)$ and $f(\lambda_b) = 0$, so that (4.3) entails

$$P(\lambda_b) = a(\lambda_b) \left(-h(\lambda_b)^2 - k(\lambda_b)^2 - e(\lambda_b)f(\lambda_b)\right) = -a(\lambda_b) \left(\mu_b^2 + k(\lambda_b)^2\right). \quad (4.6)$$

Then (4.4) follows from the definition of DN coordinates.

Finally, from (4.5) and (4.6) we obtain the separation relations for

$$\sum_{a=1}^{n} H_a \lambda_b^{n-a} = -\frac{1}{2} \lambda_b^{n} - 2a(\lambda_b) \left(\mu_b^2 + k(\lambda_b)^2\right),$$

i.e., the explicit form of the Stäckel vector with components $U_b$ appearing in (4.2).

References

[1] M. R. Adams, J. Harnad, J. Hurtubise, Darboux Coordinates and Liouville-Arnold Integration in Loop Algebras. Commun. Math. Phys. 155 (1993), 385-413.

[2] S. Benenti, Intrinsic characterization of the variable separation in the Hamilton-Jacobi equation. J. Math. Phys. 38 (1997), 6578–6602.

[3] M. Blaszak, Bi–Hamiltonian separable chains on Riemannian manifolds. Phys. Lett. A 243 (1998), 25–32.

[4] M. Crampin, W. Sarlet, G. Thompson, Bi-differential calculi, bi-Hamiltonian systems and conformal Killing tensors. J. Phys. A 33 (2000), 8755–8770.

[5] R. Brouzet, R. Caboz, J. Rabenivo, Two degrees of freedom bi-Hamiltonian systems, J. Phys. A. 29, (1996), 2069–2075.

[6] B. A. Dubrovin, I. M. Krichever, S. P. Novikov, Integrable Systems. I. Encyclopaedia of Mathematical Sciences, vol. 4 (Dynamical Systems IV), V. I. Arnol’d and S. P. Novikov, eds., Springer Verlag, Berlin, 1990.

[7] G. Falqui, F. Magri, M. Pedroni, Bihamiltonian geometry and separation of variables for Toda lattices. J. Nonlinear Math. Phys. 8 (2001), suppl. (Proceedings of NEEDS99), 118–127.

[8] G. Falqui, F. Magri, M. Pedroni, A bihamiltonian approach to separation of variables in mechanics. In: Proceedings of the Workshop “Nonlinearity, Integrability and all That. Twenty Years after NEEDS’79” (M. Boiti et al., eds.), World Scientific, Singapore, 2000, pp. 258–266.
[9] G. Falqui, F. Magri, M. Pedroni, J.P. Zubelli, *A Bi-Hamiltonian Theory for Stationary KdV Flows and their Separability*. Regul. Chaotic Dyn. **5** (2000), 33-52.

[10] G. Falqui, F. Magri, G. Tondo, *Bi-Hamiltonian systems and separation of variables: an example from the Boussinesq hierarchy*. Teoret. Mat. Fiz. **122** (2000), 212–230; translation in Theoret. and Math. Phys. **122** (2000), 176–192.

[11] G. Falqui, M. Pedroni, *Separation of variables for bi-Hamiltonian systems*. Math. Phys. Anal. Geom. **6** (2003), 139–179.

[12] J. Harnad, *Loop groups, R-matrices and separation of variables*. In: Integrable systems: from classical to quantum, CRM Proc. Lecture Notes, 26, Amer. Math. Soc., Providence, RI, 2000, pp. 21–54.

[13] A. Ibort, F. Magri, G. Marmo, *Bihamiltonian structures and Stäckel separability*. J. Geom. Phys. **33** (2000), 210–228.

[14] F. Magri, *Eight lectures on integrable systems*. In: Integrability of Nonlinear Systems, Lecture Notes in Physics **495**, Springer-Verlag, Berlin-Heidelberg, 1997, pp. 256-296.

[15] F. Magri, G. Falqui, M. Pedroni, *The method of Poisson pairs in the theory of nonlinear PDEs*. Lectures given at the 1999 C.I.M.E. School (Cetraro, Italy), nlin.SI/0002009 (to appear).

[16] F. Magri, T. Marsico, *Some developments of the concept of Poisson manifolds in the sense of A. Lichnerowicz*. In: Gravitation, electromagnetism, and geometrical structures (G. Ferrarese, ed.), Pitagora, Bologna, 1996, pp. 207–222.

[17] C. Morosi, G. Tondo, *Quasi–Bi–Hamiltonian systems and separability*. J. Phys. A. **30** (1997), 2799-2806.

[18] C. Morosi, G. Tondo, *The quasi-bi-Hamiltonian formulation of the Lagrange top*. J. Phys. A **35** (2002), 1741–1750.

[19] J. Moser, *Various aspects of integrable Hamiltonian systems*. In: Dynamical systems (C.I.M.E. Summer School, Bressanone, 1978), Progr. Math. **8**, Birkhäuser, Boston, Mass., 1980, pp. 233–289.

[20] C. Neumann, *De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur*. Jour. Reine Angew. Math. **56** (1859), 46–63.

[21] M. Pedroni, *Bi-Hamiltonian aspects of the separability of the Neumann system*. In: Proceedings of the NEEDS2001 Conference, Theor. Math. Phys. **133** (2002), 1720–1727.

[22] T.S. Ratiu, *The Lie algebraic interpretation of the complete integrability of the Rosochatius system*. In: Mathematical methods in hydrodynamics and integrability in dynamical systems (La Jolla, Calif., 1981), AIP Conf. Proc., 88, Amer. Inst. Physics, New York, 1982, pp. 109–115.

[23] K. Yano, S. Ishihara, *Tangent and cotangent bundles: differential geometry*. Marcel Dekker, New York, 1973.