RAPID EXPONENTIAL STABILIZATION BY BOUNDARY STATE FEEDBACK FOR A CLASS OF COUPLED NONLINEAR ODE AND $1 - d$ HEAT DIFFUSION EQUATION

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Abstract. In this paper, we solve the problem of rapid exponential stabilization for coupled nonlinear ordinary differential equation (ODE) and $1 - d$ unstable linear heat diffusion. The control acts at a boundary of the heat domain and the heat equation enters in the ODE by Dirichlet connection. We show that the infinite dimensional backstepping transformation introduced recently for stabilization of coupled linear ODE-PDE can deal with a nonlinear ODE and obtain a global stabilization result. Our result is innovative and no similar result can be found in the literature as it combines the three following factors, i) nonlinear term in the ODE subsystem, ii) unstable PDE subsystem and iii) mixed boundary condition. Not only this, the techniques used in this work can provide answers to fundamental questions, such as the stabilization of coupled systems where both subsystems may contain nonlinear terms.

1. Introduction. Intensive research efforts on the control of ODE-PDE coupled systems have been carried out in recent decades. These research activities are due to the fact that many practical engineering processes are modeled by coupled ODE-PDE systems, as for example plants controlled by the heat generated by a chemical reaction or biological fermentation [25], road traffic [6], gas pipelines [22], power converters connected to transmission lines [7], oil drilling [17] and many other applications.

The controllability problem of coupled PDE-ODE systems has been discussed in [23] and [26]. Many problems of stabilization for a class of linear coupled PDE-ODE have been solved [22], [21], [20], [11], [19], [27], to name just a few. Some nonlinear extensions are studied in [24], [1], [5] where the non linearity is assumed to be globally Lipschitz, and in [2], [8], [9], [4] for general nonlinear ODE. A new result on rapid exponential stabilization for coupled systems is established recently in [10].

Recently, in [3] and under some restriction on the length of the heat domain, a stabilization result is obtained for a class of nonlinear ODE coupled with stable heat

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equation. This work is a continuation in the same direction which aims at stabilizing a general coupled nonlinear ODE with heat equation (stable or unstable) such as the one investigated by [2] for the coupled nonlinear ODE and wave equation. Noting that the stabilization problem of coupled ODE-heat is more complicated than the one of coupled ODE-wave due to the reversibility of the conservative wave equation. To be more precise, in this paper, we solve the rapid exponential stabilization problem for a class of coupled nonlinear ODE and unstable reaction-diffusion PDE systems. To the best of our knowledge, there is no result establishing rapid stabilization of coupled ODE-PDE systems. Our main novelty in our considered coupled systems is combination of the three ingredients i) the ODE subsystem is nonlinear, ii) the PDE subsystem is unstable and iii) mixed boundary condition.

The rapid stabilization result is established by combining a high gain linear state feedback for the nonlinear ODE system and the powerful backstepping method introduced recently for coupled ODE-PDE systems [12] and [20]. The goal behind employing a high gain feedback for the ODE subsystem is the attenuation of the additive nonlinear term which assumed to be bounded by linear growth condition. This combination is made possible by careful selection of an appropriate of a suitable upper bound of the kernel norm of the backstepping transformation (see Lemma 4).

The structure of the paper is as follows. In the next section, the problem statement is presented, a two steps backstepping transformation is given and a state feedback is proposed. Section 3 is devoted to prove the well posedness of the closed loop system. In section 4, we establish the main result of the paper consisting in the rapid exponential stabilization of a class of coupled ODE-PDE system. Finally, a conclusion is formulated in section 5.

2. Problem setting. Before going on, we introduce some notations that will be used in this paper. Along this article, the $L^2$-norm of a function $\theta \in L^2(0,L)$ is denoted as

$$\|\theta\| = \sqrt{<\theta,\theta>} = \left(\int_0^L \theta^2(y)dy\right)^{\frac{1}{2}}$$

where $<,>$ stands for the standard inner product of $L^2(0,L)$ and $L$ is a positive constant. Also, we consider the Hilbert spaces $H^1(0,L)$ and $H^2(0,L)$ which are the usual Sobolev spaces on the open interval $(0,L)$. Moreover, we introduce the Hilbert space $\mathcal{H} = \mathbb{R}^n \times L^2(0,L)$ endowed with its canonical norm $\|(X,\theta)\| = |X|^2 + \|\theta\|^2$, where $|X|$ is the Euclidean norm of $\mathbb{R}^n$. In addition, we introduce the Hilbert space $\mathcal{H}_1 = \mathbb{R}^n \times \mathbf{H}$, where $\mathbf{H} = \{\theta \in H^2(0,L), \theta'(0) = \theta(L) = 0\}$. The subscript (') denotes the usual derivation with respect to the variable under consideration. The null $n \times m$-matrix is denoted by $0_{n \times m}$ while $I_n$ is the identity matrix of $\mathbb{R}^{n \times n}$. Also, for any $n \times n$ matrix $M$, $|M|$ stands for the matrix norm induced by Euclidean vector norm on $\mathbb{R}^n$. We recall that

$$|M|^2 = \sup_{x \in \mathbb{R}^n} \{|Mx|^2, \ |x| = 1\} = \lambda_{\max}(M^TM),$$

where $\lambda_{\max}(M^TM)$ is the largest eigenvalue of the symmetric matrix $M^TM$, in which we have confused the Euclidean norm of a vector and its subordinate matrix norm. We remind that for any matrices $M$ and $N$ with appropriate dimensions, we have $|MN| \leq |M| |N|$. We will often use the following standards inequalities.

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Lemma 2.1. For all $\mu > 0$, $(a, b) \in \mathbb{R}^2$ and for all $u \in H^1(0, L)$ such that $z(L) = 0$ (or $u(0) = 0$), the following inequalities hold

- Cauchy-Schwartz inequality: $2ab \leq \mu a^2 + \mu^{-1}b^2$,
- Agmon’s inequality: $\max_{x \in [0, L]} |u(x)|^2 \leq 2\|u\|\|u'\|$, 
- Poincaré inequality: $\|u\|^2 \leq 4L^2\|u'\|^2$.

2.1. Problem formulation. In this paper, we are concerned with global rapid exponential stabilization of general class of coupled nonlinear ODE and an unstable reaction diffusion heat equation in one dimension

\[
\dot{X}(t) = AX(t) + B\theta(0, t) + f(X(t)), \\
\epsilon \theta_x(x, t) = \theta_{xx}(x, t) + \lambda \theta(x, t), \ x \in (0, L), \\
\theta_x(0, t) = \alpha \theta(0, t) - CX(t), \\
\theta(L, t) = U(t),
\]

where $(X(t), \theta(\cdot, t)) \in \mathcal{H}$ is the state and $U(t) \in \mathbb{R}$ is the control input of the coupled system. And, $\alpha$ is a scalar constant, $\epsilon, \lambda$ are two positive constants and $L > 0$ is the length of the heat domain $(0, L)$. While, the nonlinear function $f(X)$ is supposed to be locally Lipschitz on $\mathbb{R}^n$ and $f(0) = 0$. The matrix $C$ is in $\mathbb{R}^{1 \times n}$ and the matrices $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times 1}$ given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}. 
\]

From the physical point of view, equation (2) is simply the heat reaction-diffusion equation, where $\epsilon^{-1}$ is called a diffusion coefficient and $\lambda \epsilon^{-1}$ is the influence parameter. In this paper, $\epsilon$ is a small positive parameter that will be designed later. It is well known that when $\lambda$ is negative, the subsystem (2) with boundary conditions (3) and (4) with $C = 0$ is stabilizable and the control design is straightforward. For that reason, we assume that $\lambda$ is positive. Furthermore, and as pointed out in the recent paper [14], the coupled boundary condition (3) comes from the Fourier’s law of heat exchange, where $\theta_x(0, t)$ is the speed of heat exchange, $\alpha$ is the Fourier constant and $\theta(0, t) - \tilde{C}X(t)$ is the difference of temperatures at $x = 0$, where $\tilde{C} = \alpha^{-1}C$.

To the best of our knowledge, the class of system considered is not addressed in the works dealing with ODE-PDE coupled systems. To be more precise, the coupled system combines the following three features, i) the ODE subsystem (1) contains a nonlinear part, ii) the PDE subsystem (2) is unstable, and, iii) a mixed boundary condition (3). Notably, the presence of a nonlinear term in the ODE subsystem (1) adds complexity to the boundary feedback design. Despite all these challenges, we prove that the backstepping design method combined with high gain feedback design are able to solve our problem of rapid stabilization under the following classical restriction on nonlinear function $f(X) = [f_1(X), \cdots, f_n(X)]^T$. 
Assumption 1. The function $f(\cdot)$ is locally Lipschitz and it satisfies the linear growth condition

$$|f_i(X)| \leq c \sum_{k=1}^{i} |X_k|,$$

(6)

for all $i = 1, \cdots, n$ and all $X = [X_1, \cdots, X_n]^T \in \mathbb{R}^n$, where $c$ is some known positive constant.

Domination (6) is called linear growth domination and is widely known in the control theory community specializing in the stabilization of uncertain nonlinear systems [15] and [13]. Other more complicated domination can be found in the literature. The domination is intended to avoid blow-up in finite time for the ODE subsystem.

In what follows, we introduce the control objective that details the notion of rapid state feedback stabilization addressed in this paper.

Control objective: For any prefixed positive decay rate $\sigma$, design a state feedback $U(t)$, such that the closed loop system admits solutions in $H$ defined in $[0, +\infty)$ and satisfying

$$\|(X(t), \theta(\cdot, t))\|_{\mathcal{H}} \leq c_1 \|(X(0), \theta(\cdot, 0))\|_{\mathcal{H}} e^{-\sigma t}$$

(7)

for all $(X(0), \theta(\cdot, t)) \in \mathcal{H}$, and all $t \geq 0$, where $c_1$ is a positive constant independent from the initial condition.

Remark 1. System (1)-(4) can be viewed as a particular class of the the following general class

$$\dot{X}(t) = g(X(t), \theta(0, t)),$$

(8)

$$\theta_t(x, t) = \theta_{xx}(x, t) + \lambda \theta(x, t), \quad x \in (0, L),$$

(9)

$$\theta_x(0, t) = h(X(t), \theta(0, t)),$$

(10)

$$\theta(L, t) = U(t),$$

(11)

where $g(\cdot, \cdot)$ and $h(\cdot, \cdot)$ are two smooth nonlinear functions. A minimal assumption for solving the stabilization problem of the coupled system (8)-(10) is the one assumed in [2] (where the nonlinear ODE is coupled with wave equation) stating that for the system

$$\dot{X}(t) = g(X(t), v(t) + \omega(t))$$

there exists a nonlinear state feedback $v(t) = \kappa(X(t))$ for which the closed loop is input-to-state stable (ISS) respecting the signal $\omega(t)$. Moreover, in [2] the decomposition of the conservative wave equation as the connection of two transport equations was been crucial for solving the stabilization problem. However, in the case of the nonlinear ODE-heat coupling, the problem is completely different and an original idea must be implemented to overcome the effects of the heat equation. This paper and [3] can be viewed as initiatives to solve the general case by adopting the backstepping design method for coupled ODE-PDE under the assumption that nonlinear ODE has a stabilizing linear state feedback. To address this stabilization problem, it is natural to start by considering the well-known special form $g(X, v) = AX + Bv + f(X)$ where $f(X)$ satisfies the linear growth condition (6).

In this case, the nonlinear term $f(X)$ is dominated by a linear term which can therefore be stabilized by a high gain linear feedback. By applying the backstepping transformation, the nonlinearity will be transported to the heat equation and it is difficult to manage such nonlinearity if its behavior is not linear at infinity. As we will see, despite the simplicity of the nonlinearity $f(X)$, the way in which the
nonlinear part is treated in the heat subsystem is not at all trivial (see (53) and Lemma 4.2).

**Remark 2.** It is worth noting that the choice of matrices $A$ and $B$ in subsystem (1) is not restrictive. Indeed, given a more general subsystem

$$\dot{X}(t) = \dot{A}X(t) + \dot{B}u(t) + f(X(t))$$

with $u \in \mathbb{R}$ (single input), the standard assumption for achieving global stabilization based only on the matrices $\dot{A}$ and $\dot{B}$ is the controllability property of the pair $(\dot{A}, \dot{B})$. In this setting, there exists a linear change of coordinates rendering the matrices $A$ and $B$ in the forms (5). For more details see [3].

2.2. **Backstepping transformations.** To achieve the control objective, we design a state feedback by using backstepping method for coupled ODE-PDE systems initiated by [21] and [12]. We break up the backstepping control design in two steps. The first transformation has become classical to deal with the stabilization problem of coupled ODE-heat systems and it aims to appear a nominal stabilizing term in the ODE subsystem (1) and to cancel the all boundary conditions in (3). While in the second step, a second backstepping transformation is performed to design a control that eliminates the term instability term $\lambda \theta(x,t)$ in the heat subsystem (2). We mention that in the paper [3], only one transformation was performed since there is no instability term in the heat equation.

To begin with, let $r \geq 1$ be a constant scalar gain and $D_r$ the following $n \times n$ diagonal matrix

$$D_r = \text{diag} \left(1, \frac{1}{r}, \cdots, \frac{1}{r^{n-1}}\right).$$

In view of the controllability property of the pair $(A, B)$, it is possible to select a $1 \times n$ matrix $K$ such that $A + BK$ is Hurwitz. Hereafter, we introduce the two steps of the backstepping transformations converting the coupled system (1)-(4) to a suitable target system.

2.2.1. **First backstepping transformation.** Consider the backstepping transformation $(X, \theta) \rightarrow (Y, w)$ defined as

$$Y(t) = D_r X(t),$$

$$w(x,t) = \theta(x,t) - \int_0^x q(x,y)\theta(y,t)dy - H(x)D_r X(t),$$

where the kernels $q(x,y) \in \mathbb{R}$ and $H(x) \in \mathbb{R}^{1 \times n}$ are defined in the triangle $T = \{(x, y) \in \mathbb{R}^2, 0 \leq y \leq x \leq L\}$ and in the closed interval $[0, L]$, respectively. The expressions of $H(x)$ and $q(x,y)$ are

$$H(x) = e^{-\alpha x} \left[ r^n K, \alpha r^n K - CD_r^{-1}, -3CD_r^{-1} + r^n K (\alpha A - (\lambda + 3\alpha^2)I_n) \right] e^{x\Lambda} \left[ I_n \atop 0_{n \times n} \right]$$

$$+ e^{-\alpha x} \left[ 0_{1 \times n}, 0_{1 \times n}, -\alpha CD_r^{-1} \right] \int_0^x e^{(x-z)\Lambda}dz \left[ I_n \atop 0_{n \times n} \right],$$

$$q(x,y) = e^{-\alpha(x-y)} \left( \alpha - \epsilon r^{1-n} \int_0^{x-y} e^{\alpha z} H(z)Bdz \right),$$

for $\epsilon > 0$.
with $\Lambda$ being

$$
\Lambda = \begin{bmatrix}
0_{n \times n} & 0_{n \times n} & cr^{1-n}BCD^{-1}r
I_n & 0_{n \times n} & crA\big((\lambda + \alpha^2)\big)I_n
0_{n \times n} & I_n & 2\alpha I_n
\end{bmatrix}.
$$

(16)

The outcome of the first backstepping transformation is outlined in the following proposition, whose elements of proof are set out in the appendix at the end of the manuscript.

**Proposition 1.** The backstepping transformation (12)-(16) converts the coupled system (1)-(4) into the following coupled system

$$
\dot{Y}(t) = r(A + BK)Y(t) + r^{1-n}Bw(0, t) + f_r(Y(t)),
$$

(17)

$$
\epsilon w_t(x, t) = w_{xx}(x, t) + \lambda w(x, t) - \epsilon H(x) f_r(Y(t)), \; x \in (0, L),
$$

(18)

$$
w_x(0, t) = 0,
$$

(19)

$$
w(L, t) = V(t),
$$

(20)

where $f_r(Y) = D_r f(D_r^{-1}Y)$ and $V(t)$ is a new control input defined by

$$
V(t) = U(t) - \int_0^L q(L, y) \theta(y, t) dy - H(L)Y(t).
$$

(21)

As mentioned above, the main purpose of the first transformation is to raise in the ODE subsystem (17) the stabilizing term $r(A + BK)Y$ and to cancel the mixed boundary term $\alpha \theta(0, t) - CX(t)$.

2.2.2. Second backstepping transformation. In the second step of the backstepping design, we convert $(Y, w)$ to $(Y, \omega)$, where

$$
\omega(x, t) = w(x, t) - \int_0^x q_1(x, y) w(y, t) dy,
$$

(22)

with

$$
q_1(x, y) = -\lambda x I_1 \left( \frac{\sqrt{\lambda(x^2 - y^2)}}{\sqrt{\lambda(x^2 - y^2)}} \right),
$$

(23)

for all $(x, y) \in T = \{(x, y) \in [0, L]^2, \; 0 \leq y \leq x \leq L\}$, where $I_1$ denotes the modified Bessel function of the first kind. Noting that the function $q_1(., .)$ is continuous on $(x, x)$, for all $x \in [0, L]$ since that the function $z \rightarrow I_1(z)/z$ can be expressed by power series as follows

$$
\frac{I_1(z)}{z} = \sum_{k=0}^{+\infty} \frac{z^{2k}}{2^{2k+1}k!(k+1)!}
$$

In the following, we state the result of the second backstepping transformation, where its proof is presented in the Appendix.

**Proposition 2.** The transformation (22)-(23) converts the intermediate target system (17)-(20) to a final target system

$$
\dot{Y}(t) = r(A + BK)Y(t) + r^{1-n}B\omega(0, t) + f_r(Y(t)),
$$

(24)

$$
\epsilon \omega_t(x, t) = \omega_{xx}(x, t) - \epsilon \tilde{H}(x) f_r(Y(t)), \; x \in (0, L),
$$

(25)

$$\omega_x(0, t) = 0,
$$

(26)

$$\omega(L, t) = 0,
$$

(27)
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for the state feedback control

\[ V(t) = \int_0^L q_1(L, y) w(y, t) dy, \tag{28} \]

and the kernel \( \tilde{H}(x) \) is given by

\[ \tilde{H}(x) = H(x) + \int_0^x q_1(x, y) H(y) dy. \tag{29} \]

Thus using Proposition 1, Proposition 2 and equation (21), the state feedback \( U(t) \) for the original coupled system (1)-(4) is obtained as

\[ U(t) = \int_0^L \bar{q}(L, y) \theta(y, t) dy - \bar{H}(L) D_r X(t), \tag{30} \]

where

\[ \bar{q}(L, y) = q(L, y) + q_1(L, y) - \int_y^L q_1(s, y) ds, \tag{31} \]

\[ \bar{H}(L) = H(L) - \int_0^L q_1(L, y) H(y) dy. \tag{32} \]

In the next section, we will show in the following section that system (1)-(4) in closed loop with the state feedback (30) is well posed.

3. Well posedness. In the following, we prove that for all initial condition in the Hilbert space \( \mathcal{H} \), system (1)-(4) in closed loop with state feedback (30) has a unique local mild solution. Let us first note that the two backstepping transformations are invertible. More than that, the inverse of the first backstepping transformation (12)-(13) is given in [14] and is defined as

\[ X(t) = D_r^{-1} Y(t), \tag{33} \]

\[ \theta(t) = w_{xx}(x, t) - \int_0^x p(x, y) w(y, t) dy - R(x) D_r^{-1} Y(t), \tag{34} \]

for some kernels \( p(x, y) \) and \( R(x) \) defined on the triangle \( T \) and the interval \( (0, L) \), respectively. We emphasize that our first transformation is closely similar to the one introduced by [14] in that the only difference is the presence of the invertible matrix \( D_r \), which does not affect the results of continuity and invertibility of the transformation. In addition, it is well known [18] that the inverse of the final backstepping transformation (22) has the following form

\[ w(x, t) = \omega(x, t) - \int_0^x p_1(x, y) \omega(y, t) dy, \tag{35} \]

for some kernel \( p_1(x, y) \) defined on the triangle \( T \). Furthermore, it is shown in [14] and [18] that the two transformations (12)-(13) and (22) are continuous, invertible and its inverse are continuous. Consequently, the coupled closed-loop system (1)-(4) with state feedback (30) is well posed (or exponentially stable) if and only if the final target system (24)-(27) is well posed (or exponentially stable) as well. In the following, we prove that the final target system (24)-(27) is well posed in the sense stated above.
Theorem 3.1. There exist two positive constants \( r_0^* \) and \( \epsilon_0^* \), such that for all \( r \geq r_0^* \) and all \( 0 < \epsilon < \epsilon_0^* \), system (24)-(27) is well posed for any initial condition \((Y_0, \omega_0) \in \mathcal{H}\). More precisely, for all initial condition \((Y_0, \omega_0) \in \mathcal{H}\), there exists a unique local mild solution of the target system (24)-(27) defined on a maximal interval \([0, T_{\text{max}})\), for some positive time \( T_{\text{max}} \), and it satisfies
\[
(Y(t), \omega(\cdot, t)) \in C([0, T_{\text{max}}), \mathcal{H}).
\]
Moreover, if the initial condition \((Y_0, \omega_0)\) is in \( \mathcal{H}_1 \), then the corresponding local mild solution is a classical solution and it satisfies
\[
(Y(t), \omega(\cdot, t)) \in C^1([0, T_{\text{max}}), \mathcal{H}) \cap C([0, T_{\text{max}}), \mathcal{H}_1).
\]

Proof. Without losing generality, we can pick an \( n \times 1 \) matrix \( K \), so that \( A + BK \) is Hurwitz and there exists a positive definite matrix \( P = P^T > 0 \), such that
\[
(A + BK)^TP + P(A + BK) \leq -I.
\]

We endowed the Hilbert space \( \mathcal{H} \) with the new inner product
\[
\langle (Z_1, w_1), (Z_2, w_2) \rangle_1 = Z_1^T P Z_2 + < w_1, w_2 >,
\]
for all \((Z_1, w_1), (Z_2, w_2) \in \mathcal{H}\). It is straightforward that the canonical inner product \( \langle \cdot, \cdot \rangle \) and the new inner product \( \langle \cdot, \cdot \rangle_1 \) defined (37) are equivalents due to the fact that the matrix \( P \) are symmetric and positive definite. Let consider the operator \( A : \mathcal{H}_1 \rightarrow \mathcal{H} \) defined as
\[
A \mathcal{Y} = (r(A + BK)Y + r^{1-n}B \omega(0), \epsilon^{-1} \omega''),
\]
where \( \mathcal{Y} = (Y, \omega) \). Consider the locally Lipschitz nonlinear function \( F : \mathcal{H} \rightarrow \mathcal{H} \) defined by
\[
F(\mathcal{Y}) = (f_r(Y), -\tilde{H}(x)f_r(Y)).
\]
Subsequently, the current final target system (24)-(25) can be reformulated in a compact form as
\[
\dot{Y}(t) = A \mathcal{Y}(t) + F(\mathcal{Y}(t)).
\]
In the following, we will show that the operator \( A \) generates an infinitesimal \( C_0 \)-semigroup on the Hilbert space \( \mathcal{H} \). In order to achieve this, we prove the operator \( A \) is \( m \)-dissipative. This will be done in two steps. We show first, that \( A \) is dissipative, and second, it is maximal. Let’s start by showing that \( A \) is dissipative. As a starting point, we observe that
\[
\langle A \mathcal{Y}, \mathcal{Y} \rangle_1 = rY^T (A + BK)^TP Y + r^{1-n}Y^TPB \omega(0) + \epsilon^{-1} \int_0^L \omega(y)\omega''(y)dy. \tag{40}
\]
for all \( \mathcal{Y} = (Y, \omega) \in \mathcal{H}_1 \). Integrating by parts the integral in the right hand side in (40) and using (36), we obtain
\[
\langle A \mathcal{Y}, \mathcal{Y} \rangle_1 \leq -\frac{r}{2} \| Y \|^2 - \epsilon^{-1} \| \omega' \|^2 + r^{1-n}Y^TPB \omega(0). \tag{41}
\]
Using Cauchy-Schwartz, Agmon and Poincaré inequalities, we get
\[
2Y^TPB \omega(0) \leq \lambda_{\text{max}}^2(P) \| Y \|^2 + \omega^2(0),
\]
\[
\leq \lambda_{\text{max}}^2(P) \| Y \|^2 + 4L \| \omega' \|^2,
\]
where \( \lambda_{\text{max}}(P) \) is the largest eigenvalue of the matrix \( P \). Then, from (41) we get
\[
\langle A \mathcal{Y}, \mathcal{Y} \rangle_1 \leq -\frac{1}{2}(r - \lambda_{\text{max}}^2(P)) \| Y \|^2 - (\epsilon^{-1} - 2L) \| \omega' \|^2 \leq 0, \tag{42}
\]
for parameters \( r \) and \( \epsilon \) satisfying respectively, \( r \geq r_0' = \max\{1, \lambda_{\text{max}}^2(P)\} \) and \( \epsilon \in (0, \epsilon_0') \), where \( \epsilon_0' = (2L)^{-1} \). So, the operator \( A \) is dissipative provided that \( r \geq r_0' \) and \( \epsilon \in (0, \epsilon_0') \).

Now, let’s move to show that the operator \( A \) is maximal. In accordance with the classical results, to show that the dissipative operator \( A \) is maximal, we will be content to verify that \( \gamma I - A \) is onto for a certain \( \gamma > 0 \). Given \( Z = (Z, \psi) \in \mathcal{H} \), we are looking for a \( Y = (Y, \omega) \in \mathcal{H}_1 \) verifying \( (\gamma I - A)Y = Z \), which is equivalent to

\[
-r(A + BK - \gamma I_n)Y - r^{1-n}B\omega(0) = Z, \quad (43)
\]

\[
(\gamma I - \epsilon^{-1}\partial_{xx}^2)\omega = \psi. \quad (44)
\]

Let’s define the operator \( \mathcal{L} = \gamma I - \epsilon^{-1}\partial_{xx} : \mathcal{D}(\mathcal{L}) \to L^2(0, L) \) with domain \( \mathcal{D}(\mathcal{L}) = \{\omega \in H^2(0, L), \, \omega'(0) = \omega(L) = 0\} \). Clearly, the operator \( \mathcal{L} \) is positive definite for all positive constants \( \gamma \) and \( \epsilon \). Then, \( \mathcal{L} \) is invertible and as a result the equation \( (44) \) has a unique solution \( \omega = \mathcal{L}^{-1}\psi \in \mathcal{D}(\mathcal{L}) \). Since \( \omega \) is in \( \mathcal{D}(\mathcal{L}) \), it follows that \( \omega(0) \) exists and is well defined. Moreover, since the matrix \( A + BK \) is Hurwitz, then the matrix \( A + BK - \gamma I_n \) is invertible for all \( \gamma > 0 \). Thus, equation \( (43) \) has a unique solution \( Y = -r^{-1}(A + BK - \gamma I_n)^{-1}(Z + r^{1-n}B\omega(0)) \). Hence, for all positive constant \( \gamma \), the operator \( \gamma I - A \) is onto. Thus, the operator \( A \) is maximal dissipative and consequently it generates an infinitesimal \( C_0 \)-semigroup on \( \mathcal{H} \).

On the other hand, due to the local Lipschitz property of the function \( F(Z) \), it follows that for all initial condition \( Y_0 = (Y_0, \omega_0) \in \mathcal{H} \), system \( (39) \) has a unique local mild solution \( Y(t) = (Y(t), \omega(\cdot, t)) \) defined in a maximal interval \( [0, T_{\text{max}}] \), for some positive time \( T_{\text{max}} \) \([16] \) (see Theorem 1.4 page 185-186). Moreover, if the initial condition \( Y_0 = (Y_0, \omega_0) \) is in \( \mathcal{D}(A) = \mathcal{H}_1 \), then the corresponding mild solution is a classical one. In addition to this, we recall the classical result which state that if the mild local solution \( Y(t) = (Y(t), \omega(\cdot, t)) \) is bounded on the maximal interval \( [0, T_{\text{max}}] \), then the solution is complete, i.e. \( T_{\text{max}} = +\infty \). Thus, the proof of the well posedness result is finished.

4. Rapid state feedback stabilization. Now, we can state our main result. We prove that, by tuning the parameters \( r, \epsilon \) and with kernels \( q(x, y), H(x) \) and \( q_1(x, y) \) given by \( (15), (14) \) and \( (23) \), respectively, the state feedback \( (30) \) solves the global rapid exponential stabilization problem for the coupled ODE-PDE system \( (1)-(4) \). This statement is stated in the following Theorem.

**Theorem 4.1.** Suppose that Assumption 1 is fulfilled by the nonlinear function \( f(X) \). Then, for any prescribed decay rate \( \tau \), there exist positive constants \( \bar{r}_\tau > r_\tau > 0 \) and \( \epsilon_\tau > 0 \), such that for all \( r \in (\bar{r}_\tau, r_\tau), \epsilon \in (0, \epsilon_\tau) \) and for all initial condition \( (X_0, \theta_0) \in \mathcal{H}_1 \), the coupled ODE-PDE system \( (1)-(4) \) in closed loop with the state feedback \( (30) \) has unique maximal classical solution \( (X(t), \theta(\cdot, t)) \in \mathcal{H} \) defined on \( [0, +\infty) \) and satisfying

\[
\| (X(t), \theta(\cdot, t)) \|_{\mathcal{H}} \leq c_1 \| (X_0, \theta_0) \|_{\mathcal{H}} e^{-\tau t}, \quad (45)
\]

for all \( t \geq 0 \), where \( c_1 \) is a positive constant independent from the initial condition.

Before proceeding to the proof of the theorem, it is interesting to make the following synthesis about the restriction taken on the parameter \( \epsilon \) to guarantee the rapid stabilization of the nonlinear coupled system \( (1)-(4) \).
Remark 3. An important observation is that, to achieve simple exponential stability for the coupled system (1)-(4), ϵ must be selected sufficiently small. This restriction has reasonable reason. In fact, the eigenvalues of the unbounded operator $A$ defined in (38) are given by
\[ \lambda_k = -\epsilon^{-1} \left( \frac{(2k + 1)\pi}{2L} \right)^2, \quad k = 0, 1, \cdots. \]

Then, the largest eigenvalue of the operator $A$ is
\[ \lambda_0 = -\epsilon^{-1} \left( \frac{\pi}{2L} \right)^2. \]

On the other hand, the nonlinear term $F(Y)$ of the final target system written in a compact form (39) satisfies the domination
\[ \|F(Y)\| \leq c \|Y\|, \]
where $c$ is a positive fixed constant. It is quite clear that if the parameter $\epsilon$ is not chosen small enough, then the coefficient $c$ must be small (therefore the nonlinearity $F(Y)$ is small enough) to avoid that the eigenvalues in the linearized of the nonlinear target system (24)-(27) have eigenvalues with positive real part. In consequence, since the parameter $c$ is fixed, we are not able to take $\epsilon$ arbitrarily. Indeed, the inequality
\[ -\epsilon^{-1} \left( \frac{\pi}{2L} \right)^2 + c < 0, \]
cannot be satisfied for all positive $\epsilon$. Thus, for stabilization of coupled system, we can not chose $\epsilon$ arbitrarily. More than that, we conjecture that it is not possible to exponentially rapidly stabilize the coupled system (1)-(4), with the presence of any nonlinear term satisfying Assumption 1, by boundary linear feedback for any $\epsilon$.

This being said, we will now move on to the proof of the theorem.

Proof. Let $r \geq 1$ be a positive constant. Let $(Y_0, \omega_0) \in \mathcal{H}_1 = D(A)$ an initial condition and $(Y(t), \omega(\cdot, t))$ be the corresponding local classical solution of the final target system (24)-(27) defined on a maximal interval $[0, T_{\text{max}})$, where $T_{\text{max}} > 0$. Let consider the Lyapunov function
\[ V(Y(t), \omega(\cdot, t)) = Y^T(t)PY(t) + \frac{1}{2}\|\omega(\cdot, t)\|^2. \]
where $P = P^T$ is a positive definite matrix meeting the following matrix inequality
\[ P(A + BK) + (A + BK)^TP \leq -I. \]
(47)
Since $V$ is quadratic, it follows that there are two positive constants $\nu_1$ and $\nu_2$, such that
\[ \nu_1 \left( |Y|^2 + \|\omega\|^2 \right) \leq V(Y, \omega) \leq \nu_2 \left( |Y|^2 + \|\omega\|^2 \right), \]
for all $(Y, \omega) \in \mathcal{H}$. In the sequel, the value of the Lyapunov function $V(Y, \omega)$ on the trajectory $(Y(t), \omega(\cdot, t))$ will be denoted simply by $V(t)$ instead of $V(Y(t), \omega(\cdot, t))$. Differentiating $V(t)$ with respect to the time $t$ along the solutions of the final target system and integrating by parts, it follows
\[ \dot{V}(t) \leq -r|Y|^2 + 2r^{1-n}Y^T PB\omega(0, t) + 2Y^TPf_r(Y) - \epsilon^{-1} \|\omega(\cdot, t)\|^2 \\
- \int_0^L \omega(y, t)\bar{H}(y)f_r(Y)dy. \]
(49)
We are going to estimate the right hand side of (49) term by term. To start with, let build an upper bound of $2Y^TPf_r(Y)$ using Assumption 1. Let $f_r(Y) = [f_{r,1}(Y), \cdots, f_{r,n}(Y)]^T$. In view of Assumption 1 and using the fact that $r \geq 1$, it becomes that for all $i = 1, \cdots, n$, we have
\[
|f_{r,i}(Y)| = \frac{1}{r^{i-1}} |f_i(Y_1, rY_2, \cdots, r^{n-1}Y_n)|,
\]
\[
\leq \frac{c}{r^{i-1}} \sum_{j=1}^{i} r^{j-1} |Y_j|,
\]
\[
\leq c \sum_{j=1}^{i} |Y_j|,
\]
\[
\leq c \left( \sum_{j=1}^{i} |Y_j|^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^{i} 1^2 \right)^{\frac{1}{2}},
\]
\[
\leq c \sqrt{n} |Y|.
\]
Then, it yields $|f_r(Y)|^2 = \sum_{i=1}^{n} (f_{r,i}(Y))^2 \leq \sum_{i=1}^{n} c^2 n |Y|^2 \leq c^2 n^2 |Y|^2$. So, it follows that
\[
2Y^TPf_r(Y) \leq 2nc\lambda_{\text{max}}(P)|Y|^2.
\]
(50)
Using successively Cauchy-Schwartz, Agmon’s and Poincaré inequalities, it yields
\[
2Y^TPB\omega(0, t) \leq \lambda_{\text{max}}(P)|Y|^2 + 4L\|\omega_x(\cdot, t)\|^2,
\]
(51)
\[
- \int_0^L \omega(y, t) \dot{H}(y) f_r(Y) dy \leq \frac{1}{2} n^2 c^2 |Y|^2 + \frac{1}{2} \|\dot{H}\|^2 \|\omega_x(\cdot, t)\|^2,
\]
\[
\leq \frac{1}{2} n^2 c^2 |Y|^2 + 2L^2 \|\dot{H}\|^2 \|\omega_x(\cdot, t)\|^2.
\]
(52)
Thus, from (49) we get
\[
\dot{V}(t) \leq - \left( r - 2nc\lambda_{\text{max}}(P) - \lambda_{\text{max}}^2(P) - \frac{1}{2} n^2 c^2 \right) |Y|^2
\]
\[
- \left( c^{-1} - 4L - 2L^2 \|\dot{H}\|^2 \right) \|\omega_x(\cdot, t)\|^2.
\]
(53)
In view of inequality (53), it is simple to see that if we select $r$ sufficiently large and $\epsilon$ sufficiently small, we can recover the rapid exponential stability. Unfortunately, this is not easy to do because the term $\|\dot{H}\|^2$ depends on both parameters $r$ and $\epsilon$. An additional effort must be made in order to succeed in bounding $\|\dot{H}\|^2$ independently of the parameter $\epsilon$.

It should be noted that in the previous literature, the main reason for the lack of research results on the rapid exponential stabilization of coupled ODE-PDE systems is that the interconnection terms after the backstepping transformation have not been analyzed in depth. To overcome this obstacle, we are going to introduce a suitable upper bound $h_{\text{max}}$ of $\|\dot{H}\|^2$ independent from the parameter $\epsilon$. To start with, and by examining the inequality (53), let’s notice that a necessary condition for establishing exponential stability is that $\epsilon$ must satisfies $\epsilon < (4L)^{-1}$. We point out that such suitable upper bound is used to design $\epsilon^*$ as large as possible, for which we have : for all $\epsilon \in (0, \epsilon^*)$, the final target system (24)-(27) (or equivalently the coupled system (1)-(4)) can be stabilized exponentially. The proof of the following Lemma is given in appendix at the end of the manuscript.
Lemma 4.2. If $\epsilon < (4L)^{-1}$, then $\|\tilde{H}\| \leq h_{\text{max}}(r)$, where

$$h_{\text{max}}^2(r) = 2(1 + \|q_{1}\|^2) \frac{d_1 + d_2}{d_3} (e^{d_3 r} - 1),$$

and

$$d_1 = 2 r^{2n} |K|^2 ((4L)^{-1} r + \lambda + \alpha^2)^2 + 2 \alpha^2 |C D_{r}^{-1}|^2 L^2 e^{2L \alpha_{\text{max}}},$$

$$d_2 = 2 (r^{2n} |K|^2 + |\alpha r^n K - C D_{r}^{-1}|^2 + 18 \alpha^2 |C D_{r}^{-1}|^2),$$

$$d_3 = 2 (\alpha_{\text{max}} - \alpha),$$

$$\Lambda_{\text{max}}^2 \leq 2^{2(n-1)} \left( \sqrt{1 + (\lambda + \alpha^2)^2 + 4 \alpha^2} + (4L)^{-1} \sqrt{2} \max\{|C|, 1\} \right)^2.$$  

First, from (23), we can see that $\|q_{1}\|$ is independent from the parameters $r$ and $\epsilon$. Then, it follows that $h_{\text{max}}(r) = Q(r)e^{kr^{(n-1)/2}}$, where $Q(r)$ is polynomial function and $k$ is some positive constant. Moreover $Q(r)$ and $k$ are independent from $\epsilon$. In consequence $h_{\text{max}}(r)$ is an increasing function with respect to the variable $r$ on the interval $[r_0, +\infty)$, for some constant $r_0 \geq 1$. Let $\tau$ a positive constant and consider

$$r_{1}^* = \max\{1, r_0, 2n c \lambda_{\text{max}} + \Lambda_{\text{max}}^2 + \frac{1}{2} n^2 c^2\},$$

$$\epsilon_{1}^* = (4L + 2L^2 h_{\text{max}}^2(r_{1}^* + 2\tau \nu_2 + r_0))^{-1},$$

where $r_0$ is any positive constant. In the following, we will design $\bar{\tau}_{\epsilon}, \bar{\tau}_{\epsilon}$ and $\epsilon_{\tau}$, such that any trajectory of the target system (24)-(27) decreases exponentially in time with a decay rate greater than $\tau$ (i.e. inequality (45) is satisfied), as soon as $r \in [\bar{\tau}_{\epsilon}, \bar{\tau}_{\epsilon}]$ and $\epsilon \in (0, \epsilon_{\tau})$. Let's pick $\bar{\tau}_{\epsilon} = r_1^* + 2\tau \nu_2, \bar{\tau}_{\epsilon} = \bar{\tau}_{\epsilon} + r_0$ and $\epsilon_{\tau} = (\epsilon_{1}^*)^{-1} + 8 \tau \nu_2 L^2$ and then select $r \in [\bar{\tau}_{\epsilon}, \bar{\tau}_{\epsilon}]$ and $\epsilon \in (0, \epsilon_{\tau})$. It is simple to see that

$$\epsilon^{-1} \geq \epsilon_{1}^{-1} \geq (\epsilon_{1}^*)^{-1} + 8 \tau \nu_2 L^2 
\geq 4L + 2L^2 h_{\text{max}}^2(\bar{\tau}_{\epsilon}) + 8 \tau \nu_2 L^2
\geq 4L + 2L^2 h_{\text{max}}^2(r) + 8 \tau \nu_2 L^2,$$

since $h_{\text{max}}(r)$ is increasing. Then, from (53) and by Poincaré inequality, it yields

$$\dot{\mathcal{V}}(t) \leq -2 \tau \nu_2 |Y(t)|^2 - 8 \tau \nu_2 L^2 \|\omega_{r}(\cdot, t)\|^2,$$

$$\leq -2 \tau \nu_2 \left(|Y(t)|^2 + \|\omega(\cdot, t)\|^2\right),$$

$$\leq -2 \tau \mathcal{V}(t).$$

Integrating (61), it yields $\mathcal{V}(t) \leq \mathcal{V}(0)e^{2\tau t}$, or equivalently

$$\|\mathcal{V}(t)\|_{\mathcal{H}} \leq \frac{\nu_2}{\nu_1} \|\mathcal{V}(0, \omega_{0})\|_{\mathcal{H}} e^{2\tau t},$$

for all $t \in [0, T_{\text{max}}]$. So, the local solution $(Y(t), \omega(\cdot, t))$ of the final target system (24)-(27) is bounded on the maximal interval $[0, T_{\text{max}}]$. Then, $T_{\text{max}} = +\infty$ and the solution $(Y(t), \omega(\cdot, t))$ converges exponentially to the origin with decay rate at least $\tau$.

Besides, on the other hand, by using the two backstepping transformations (12), (13), (22) and its inverses (33), (34) and (35), it is simple to obtain the following...
inequalities

\[ \|\omega\|^2 \leq a_1 \|\theta\|^2 + b_1 |X|^2, \quad |Y|^2 \leq |X|^2, \quad (63) \]
\[ \|\theta\|^2 \leq a_2 \|\omega\|^2 + b_2 |Y|^2, \quad |X|^2 \leq r^{2(n-1)} |Y|^2, \quad (64) \]

where \(a_1, b_1, a_2\) and \(b_2\) are positive constants that depend on the kernels \(q, H\) and \(q_1\). Then, using (63) and (64), from (62), we obtain (45). Thus, the proof of theorem is achieved.

5. Conclusion. In this paper, boundary state feedback that guaranteed global rapid exponential stabilization for a class of coupled nonlinear ODE and \(1-d\) unstable linear heat diffusion equation is given with mixed boundary condition. The design procedure is based on infinite dimensional backstepping method for coupled ODE-PDE systems combined with high gain feedback for finite dimensional system. The difficulties encountered for the rapid stabilization of this nonlinear system are mainly the treatment of the nonlinear part as well as the adjustment of the high gain control parameters. An important direction in future work will be to use the technique developed in this paper for the rapid exponential stabilization of a nonlinear ODE coupled with the following nonlinear \(1-d\) heat equation

\[ \epsilon \theta_t(x,t) = \theta_{xx}(x,t) + g(\theta(x,t)), \quad x \in (0, L), \]

where \(g(\cdot)\) is some nonlinear scalar function satisfying some linear growth rate condition.

6. Appendices.

6.1. Proof of proposition 1. By differentiating \(w(x,t)\) defined in (13) with respect to \(x\), it follows

\[ w_x = \theta_x - q(x,x)\theta - \int_0^x q_x(x,y)\theta(y,t)dy - H'(x)D_r X(t), \quad (65) \]
\[ w_{xx} = \theta_{xx} - \left((q(x,x))' + q_x(x,x)\right)\theta - q(x,x)\theta_t - \int_0^x q_{xx}(x,y)\theta(y,t)dy - H''(x)D_r X(t). \quad (66) \]

Let’s derive \(w(x,t)\) with respect to \(t\) and perform an integration by parts, that gives

\[ \epsilon w_t = \epsilon \theta_t - \epsilon \int_0^x q(x,y)\theta_t(y,t)dy - \epsilon H(x)D_r \dot{X}(t), \]
\[ = \theta_{xx} + \lambda \theta - \int_0^x q(x,y)(\theta_{xx}(y,t) + \lambda \theta(y,t))dy \]
\[ - \epsilon H(x)D_r \left(AX(t) + B\theta(0,t) + f(X(t))\right), \]
\[ = \theta_{xx} + (\lambda + q_y(x,x))\theta - q(x,x)\theta_t + q(x,0)\theta_x(0,t) - q_y(x,0)\theta(0,t) \]
\[ - \int_0^x (q_{yy}(y,t) + \lambda q(y,t))\theta(y,t)dy - \epsilon H(x)D_r \left(AX(t) + B\theta(0,t)\right) \]
\[ - \epsilon H(x)D_r f(X(t)), \]
\[ = \theta_{xx} + (\lambda + q_y(x,x))\theta - (q_y(x,0) - \alpha q(x,0) + \epsilon H(x)D_r B)\theta(0,t) \]
\[ - q(x,x)\theta_x - \int_0^x (q_{yy}(y,t) + \lambda q(y,t))\theta(y,t)dy \]
\[ + (q(x,0)CD_r^{-1} - \epsilon r H(x)A)Y(t) - \epsilon H(x)D_r f(X(t)). \quad (67) \]
We have used $D_r AD_r^{-1} = rA$. From (66), (67) and (13), we get
\[
eqw - w_{xx} - \lambda w = \frac{2}{2} (q(x, x))'\theta - (q_y(x, 0) - \alpha q(x, 0) + \epsilon H(x) D_r B) \theta(0,t) + \epsilon H(x) D_r f(X(t)) - \int_0^x \left( q_{yy}(y, t) - q_{xx}(y, t) \right) \theta(y,t) dy + \left( H''(x) - H(x)(e r A - \lambda I) + q(x, 0) CD_r^{-1} \right) Y(t).
\]
(68)

On the other hand, setting $x = 0$ in the equations (13) and (65), we get
\[
w(0, t) = \theta(0, t) - H(0) Y(t),
\]
(69)
\[
w_x(0, t) = (\alpha - q(0, 0)) \theta(0, t) - (C + H'(0) D_r) X(t)
\]
(70)

Since we want to have $w_x(0, t) = 0$ in the intermediate target system, it results that $q(0, 0) = \alpha$ and $H'(0) = -CD_r^{-1}$. Moreover, using (1) and (69), the time derivative of $Y(t)$ defined in (12) satisfies
\[
\dot{Y}(t) = r (A + r^{-n} BH(0)) Y(t) + r^{-n} B w(0, t) + f_r(Y(t)),
\]
(71)

where $f_r(Y) = D_r f(D_r^{-1} Y)$. In order to reach the intermediate target system (17)-(20), we select the kernels $q(x, y)$ and $H(x)$ as follows
\[
H''(x) - H(x)(e r A - \lambda n I) = -q(x, 0) CD_r^{-1},
\]
(72)
\[
H'(0) = -CD_r^{-1},
\]
(73)
\[
H(0) = r^n K,
\]
(74)
\[
q_{yy}(x, y) - q_{xx}(x, y) = 0,
\]
(75)
\[
q_y(x, 0) = \alpha q(x, 0) - e r^{1-n} H(x) B,
\]
(76)
\[
q(x, x) = \alpha,
\]
(77)

for all $0 \leq y \leq x \leq L$. It should be noted that we have used the fact $(q(x, x))' = 0$, which implies that $q(x, x) = q(0, 0) = \alpha$, for all $x \in [0, L]$.

On the other hand, we underline that the nonlinear terms $-\epsilon H(x) D_r f(X(t))$ and $f_r(Y(t))$ cannot be canceled from (68) and (71), respectively. Consequently, by using a well known backstepping transformation as (12)-(13), any target system must be strongly interconnected, where the $Y$-dynamic and $w$-dynamic contain respectively the nonlinear terms $f_r(Y(t))$ and $-\epsilon H(x)f_r(Y(t))$. The solutions $H(x)$ and $q(x, y)$ of (72)-(77), are calculated explicitly as follows. First, classical computations give the solution $q(x, y)$ of PDE (75)-(77)
\[
q(x, y) = e^{-\alpha(x-y)} \left( \alpha - e r^{1-n} \int_0^x e^{\alpha z} H(z) B dz \right).
\]
(78)

Then, we move to compute $H(x)$. Plugging (78) in (72), it yields
\[
H''(x) - H(x)(e r A - \lambda n I) + e^{-\alpha x} \left( \alpha - e r^{1-n} \int_0^x e^{\alpha z} H(z) B dz \right) CD_r^{-1} = 0.
\]
(79)

Let’s denote by $\tilde{h}(x) = e^{\alpha x} H(x)$, $x \in [0, L]$. From (79), it follows
\[
\tilde{h}''(x) - 2\alpha \tilde{h}'(x) - \tilde{h}(x)(e r A - (\lambda + \alpha^2) I_n)
\]
\[
+ \left( \alpha - e r^{1-n} \int_0^x \tilde{h}(z) B dz \right) CD_r^{-1} = 0.
\]
(80)

Differentiating (80) with respect to $x$, we obtain the following linear ODE
\[
\tilde{h}''(x) - 2\alpha \tilde{h}'(x) - \tilde{h}'(x)(e r A - (\lambda + \alpha^2) I_n) - e r^{1-n} \tilde{h}(x) B C D_r^{-1} + \alpha CD_r^{-1} = 0.
\]
(81)
where \( \widehat{H}'(0) = -3 \alpha CD_r^{-1} + r^n K (\epsilon r A - (\lambda + 3 \alpha^2) I_n) \). Let \( \mathbb{H}(x) = [\widehat{H}(x), \widehat{H}'(x), \widehat{H}''(x)] \). Then, (81) can be written compactly as
\[
\mathbb{H}'(x) = \mathbb{H}(x) \Lambda + \Gamma,
\]
where the \( 3n \times 3n \)-matrix \( \Lambda \) is defined in (16) and \( \Gamma = [0_{1 \times n} \ 0_{1 \times n} - \alpha CD_r^{-1}] \). Integrating equation (82), we get
\[
\mathbb{H}(x) = \mathbb{H}(0) e^{\alpha \Lambda} + \Gamma \int_0^x e^{(x-y)\Lambda} dy.
\]
So, the kernel \( H(x) \) is readily found as (14). From (78), we deduce that the kernel \( q(x, y) \) is given by (15).

6.2. **Proof of proposition 2.** Differentiating (22) twice with respect \( x \), it results in
\[
\omega_{xx} = w_{xx} - ((q_1(x, x)')' + q_{1x}(x, x)) w - q_1(x, x) w_x - \int_0^x q_{1xx}(x, y) w(y, t) dy. \tag{83}
\]
In addition, by differentiating (22) with respect the time \( t \) and then integrating by parts, it follows
\[
\epsilon \omega_t = w_{xx} + (\lambda + q_{1y}(x, x)) w - q_1(x, x) w_x - \int_0^x q_{1yy}(x, y) + \lambda q_1(x, y) w(y, t) dy
- q_{1y}(x, 0) w(0, t) - \epsilon \hat{H}(x) f_r(Y), \tag{84}
\]
where \( \hat{H}(x) \) is defined in (29). Then, we get
\[
\epsilon \omega_t - \omega_{xx} = (\lambda + 2(q_1(x, x)')') w - q_{1y}(x, 0) w(0, t)
- \epsilon \hat{H}(x) f_r(Y) - \int_0^x (q_{1yy} - q_{1xx} + \lambda q) w(y, t) dy. \tag{85}
\]
In view of (24)-(27), we design \( q_1(x, y) \) such that
\[
q_{1xx}(x, y) - q_{1yy}(x, y) = \lambda q_1(x, y), \tag{86}
q_{1y}(x, 0) = 0, \tag{87}
q_1(x, x) = -\frac{\lambda}{2} x, \tag{88}
\]
The solution of (86)-(88) was found explicitly in [18] (see Theorem 10 page 2197) as given in (23).

6.3. **Proof of lemma 4.2.** Using (29), (14) and (23), and applying Cauchy-Schwartz inequality, the following holds
\[
\|\hat{H}\|^2 \leq 2\|H\|^2 (1 + \|q_1\|^2_{L^2(0, L)})^2, \tag{89}
\]
Since \( \|q_1\|_{L^2(0, L)}^2 \) is independent of \( \epsilon \) and \( r \), designing an upper bound of \( \|\hat{H}\| \) which is independent of \( \epsilon \) is equivalent to bounded \( \|H\| \) independently of \( \epsilon \).

First, we have
\[
|H(x)|^2 \leq 2 \left( \alpha^2 CD_r^{-1} |x|^2 \int_0^x e^{-zA} dz \right)^2 e^{2z\|\Lambda\|} + 2 \left( r^n K|K|^2 + |\alpha r^n K| \right)^2 e^{2z\|\Lambda\|},
\]
\[
- CD_r^{-1} |x|^2 + | - 3\alpha CD_r^{-1} + r^n K (\epsilon r A - (\lambda + \alpha^2) I_n)|^2 e^{2z\|\Lambda\|}.
\]
It is simple to get,
\[ \int_0^x e^{-z^2} dz \leq \int_0^x |e^{-z^2}| dz \leq \int_0^x e^{z^2} dz \leq Le^{L^2}. \]
Then, from (90), we obtain
\[ |H(x)|^2 \leq (d_1(\epsilon) + d_2)d_{d_3(\epsilon)x}, \quad (90) \]
where
\[ d_1(\epsilon) = 2\epsilon^2 n|K|^2 |erA - (\lambda + \alpha^2)I_n|^2 + 2\alpha^2|CD^{-1}_r|^2 L^2 e^{2L^2}, \]
\[ d_2(\epsilon) = 2(\epsilon |\Lambda| - \alpha), \]
and \( d_2 \) is given in (56). We underline that only the constant \( d_2 \) is independent from the parameter \( \epsilon \). Consequently, in estimation (90), only expressions \( |erA - (\lambda + \alpha^2)I_n|^2 \) in \( d(\epsilon) \) and \( |\Lambda| \) depend simultaneously on the design parameter \( \epsilon \).
We begin by treating the expression \( |erA - (\lambda + \alpha^2)I_n|^2 \). Applying triangular inequality, it yields
\[ |erA - (\lambda + \alpha^2)I_n| \leq \epsilon |\Lambda| + (\lambda + \alpha^2)|I_n|, \]
\[ \leq \epsilon \sqrt{\lambda_{\text{max}}(AA^T) + (\lambda + \alpha^2)}, \]
\[ \leq \epsilon + (\lambda + \alpha^2). \]
Now, we compute an upper bound \( \Lambda_{\text{max}} \) for \( |\Lambda| \). First, we observe that \( \Lambda = \text{diag}(D_r, D_r, D_r)\tilde{\Lambda}\text{diag}(D_r, D_r, D_r)^{-1} \), where
\[ \tilde{\Lambda} = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & \epsilon BC \\ I_n & 0_{n \times n} & \epsilon A - (\lambda + \alpha^2)I_n \\ 0_{n \times n} & I_n & 2\alpha I_n \end{bmatrix} \]
Consequently, \( |\Lambda| \leq |\text{diag}(D_r, D_r, D_r)| |\tilde{\Lambda}| |\text{diag}(D_r, D_r, D_r)^{-1}| \leq r^{n-1}|\tilde{\Lambda}|. \) To compute \( |\tilde{\Lambda}| \), let observe that
\[ \tilde{\Lambda} = \tilde{\Lambda}_1 + \epsilon \tilde{\Lambda}_2, \]
where
\[ \tilde{\Lambda}_1 = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \\ I_n & 0_{n \times n} & -(\lambda + \alpha^2)I_n \\ 0_{n \times n} & I_n & 2\alpha I_n \end{bmatrix}, \tilde{\Lambda}_2 = \begin{bmatrix} 0_{n \times n} & 0_{n \times n} & BC \\ 0_{n \times n} & 0_{n \times n} & A \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix} \]
We have
\[ \tilde{\Lambda}_1\tilde{\Lambda}_1^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & (1 + (\lambda + \alpha^2)^2)I_n & -2\alpha(\lambda + \alpha^2)I_n \\ 0 & -2\alpha(\lambda + \alpha^2)I_n & (1 + 4\alpha^2)I_n \end{bmatrix}. \]
Any eigenvalue \( \lambda_1 \) of the symmetric matrix \( \tilde{\Lambda}_1\tilde{\Lambda}_1^T \) satisfy the following characteristic equation : \( \lambda_1 = 0 \) or
\[ (1 + (\lambda + \alpha^2)^2 - \lambda_1) (1 + 4\alpha^2 - \lambda_1) = 4\alpha^2(\lambda + \alpha^2)^2. \]
which gives \( \lambda_1 = 1 \) or \( \lambda_1 = 1 + (\lambda + \alpha^2)^2 + 4\alpha^2. \) Thus,
\[ |\tilde{\Lambda}_1|^2 = \lambda_{\text{max}}(\tilde{\Lambda}_1\tilde{\Lambda}_1^T) = 1 + (\lambda + \alpha^2)^2 + 4\alpha^2. \]
To compute \( |\tilde{\Lambda}_2| \), let determine the maximum eigenvalue of the symmetric matrix
\[ \tilde{\Lambda}_2\tilde{\Lambda}_2^T = \begin{bmatrix} BC(BC)^T & BCA^T & 0_{n \times n} \\ A(BC)^T & AA^T & 0_{n \times n} \\ 0_{n \times n} & 0_{n \times n} & 0_{n \times n} \end{bmatrix}. \]
Moreover, we have

$$BC(BC)^T = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 & \cdots & 0 \end{bmatrix}, \quad BCA^T = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & 0 \\ c_2 & \cdots & c_n \end{bmatrix},$$

and $AA^T = \text{diag}(1, \cdots, 1, 0)$. Hence, it is easy to see that 0 is an eigenvalue of $\tilde{\Lambda}_2^2$ of order 2. The rest of eigenvalues are eigenvalue of the matrix

$$M = \begin{bmatrix} |C|^2 & c_2 & \cdots & c_n \\ c_2 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ c_n & 0 & \cdots & 0 & 1 \end{bmatrix}.$$  

We have

$$X^T M X = |C|^2 x_1^2 + x_2^2 + \cdots + x_n^2 + 2c_2 x_1 x_2 + \cdots + 2c_n x_1 x_n,$$

for all $X = [x_1 \cdots x_n]^T \in \mathbb{R}^n$. Using the inequalities

$$-c_i^2 x_1^2 - x_i^2 \leq 2c_i x_1 x_i \leq c_i^2 x_1^2 + x_i^2,$$

for all $i = 2, \cdots, n$, it follows that

$$c_i^2 x_1^2 \leq X^T M X \leq 2|C|^2 x_1^2 + 2x_2^2 + \cdots + 2x_n^2. \tag{91}$$

From the left hand side of the inequality (91), we deduce that the matrix $M$ is positive, while the right hand side implies that $\lambda_{\text{max}}(M) \leq \max\{2|C|^2, 2\}$. Then, we get

$$|\tilde{\Lambda}_2|^2 = \lambda_{\text{max}}(M) \leq \max\{2|C|^2, 2\}.$$  

Then,

$$|\hat{A}| \leq |\tilde{\Lambda}_2| + \epsilon |\tilde{\Lambda}_2|,$$

$$\leq \sqrt{1 + (\lambda + \alpha^2)^2 + 4\alpha^2} + \epsilon \sqrt{2} \max\{|C|, 1\}.  

Since $\epsilon < (4L)^{-1}$, we get $|A| \leq \Lambda_{\text{max}}$ where $\Lambda_{\text{max}}$ is given in (58). As consequence, $d_1(\epsilon) \leq \bar{d}_1$ and $d_2(\epsilon) \leq \bar{d}_3$, where $\bar{d}_1$ and $\bar{d}_3$ are given in (55) and (57), respectively. Finally, integrating (90), it yields

$$\|H\|^2 \leq \frac{\bar{d}_1 + d_2}{d_3} (e^{\alpha L} - 1). \tag{92}$$

Thus, from (92) and (89), it yields (54). Then, the proof of Lemma 4.2 is achieved.

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