MULTIPLE SPATIOTEMPORAL COEXISTENCE STATES AND TURING-HOPF BIFURCATION IN A LOTKA-VOLterra COMPETITION SYSTEM WITH NONLOCAL DELAYS

XIANYONG CHEN AND WEIHUA JIANG*

School of Mathematics, Harbin Institute of Technology
Harbin 150001, China

(Communicated by Yuan Lou)

Abstract. We consider a two-species Lotka-Volterra competition system with both local and nonlocal intraspecific and interspecific competitions under the homogeneous Neumann condition. Firstly, we obtain conditions for the existence of Hopf, Turing, Turing-Hopf bifurcations and the necessary and sufficient condition that Turing instability occurs in the weak competition case, and find that the strength of nonlocal intraspecific competitions is the key factor for the stability of coexistence equilibrium. Secondly, we derive explicit formulas of normal forms up to order 3 by applying center manifold theory and normal form method, in which we show the difference compared with system without nonlocal terms in calculating coefficients of normal forms. Thirdly, the existence of complex spatiotemporal phenomena, such as the spatial homogeneous periodic orbit, a pair of stable spatial inhomogeneous steady states and a pair of stable spatial inhomogeneous periodic orbits, is rigorously proved by analyzing the amplitude equations. It is shown that suitably strong nonlocal intraspecific competitions and nonlocal delays can result in various coexistence states for the competition system in the weak competition case. Lastly, these complex spatiotemporal patterns are presented in the numerical results.

1. Introduction. Since the pioneering works by Lotka [23] and Volterra [35], Lotka-Volterra models have been well accepted and investigated extensively in various aspects. To describe the competition for resources, the two-species Lotka-Volterra competition systems with different factor effects have been paid much attention to, see [5, 9, 18, 24, 25] with heterogeneous environment, [8, 13, 17, 26, 28, 30, 38] with nonlocality and [5, 8, 17, 22, 31, 34, 39] with time delays.

In population dynamics, since species take time to move and it can affects the density of species in the neighborhood of their current positions, the effect interacting the past history (delay) and different locations (nonlocality) should be included in modeling process [3, 11, 14]. Thus it would better incorporate the spatial weighted functions into competition terms with delays (spatiotemporal average or nonlocal
delays) for the population models, see more examples [3, 6, 14, 15, 19] for single species models and [8, 12, 13, 17] for two species models.

Based on the modeling setting in [8, 26], we consider a Lotka-Volterra competition system with nonlocal delays as follows

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1(r_1 - a_{11}u_1 - a_{12}u_2) - \frac{b_{11}}{\pi} \int_0^\pi u_1(y, t - \tau)dy - \frac{b_{12}}{\pi} \int_0^\pi u_2(y, t - \tau)dy, \quad x \in (0, \pi), t > 0, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + u_2(r_2 - a_{21}u_1 - a_{22}u_2) - \frac{b_{21}}{\pi} \int_0^\pi u_1(y, t - \tau)dy - \frac{b_{22}}{\pi} \int_0^\pi u_2(y, t - \tau)dy, \quad x \in (0, \pi), t > 0, \\
\frac{\partial u_1}{\partial \nu} &= \frac{\partial u_2}{\partial \nu} = 0, \quad x \in \partial \Omega, t > 0, \\
u_1(x, t) &= \phi_1(x, t) \geq (\neq) 0, \quad u_2(x, t) = \phi_2(x, t) \geq (\neq) 0, \quad (x, t) \in \Omega \times [-\tau, 0],
\end{align*}
\]

(1)

where \(d_i > 0, \ r_i > 0\) are the diffusion coefficient and the intrinsic growth rate of the species \(u_i\), \(a_{ij} \geq 0, \ b_{ij} \geq 0\) are the local and nonlocal competition strength of the species \(u_j\) to the species \(u_i\) and \(\phi_i \in C([-\tau, 0]; L^2(0, \pi))\) with \(\tau > 0, \ i, j = 1, 2\). \(\partial \nu\) denote the outward normal derivative on the boundary of region \((0, \pi)\ (x = 0, \pi)\).

The kernels are chosen as spatially homogeneous incorporated with discrete delays in system (1), which can reflect that the species do not simply depend on its density at the current positions and time but on all positions in region \((0, \pi)\) and previous time \(\tau\). Both local and nonlocal intraspecific and interspecific competitions are considered and the effects of their strength are also presented to system (1).

For the two-species Lotka-Volterra competition model, the global stability of (non-)constant steady states [5, 8, 9, 13, 18, 24, 25, 26] or Hopf bifurcations induced by time delays at the (non-)constant steady states [17, 31, 34, 39] have been investigated by many researchers. In recent years, Guo and Yan [17] investigated a two-species Lotka-Volterra competition system with nonlocal delays under homogeneous Dirichlet boundary condition and the stability of spatially inhomogeneous steady-state solutions and the existence of Hopf bifurcation have been derived, which presents interesting spatiotemporal pattern formations. In [8, 26], besides the global dynamics for two constant semi-trivial and the coexistence equilibrium, the existence of spatial patterns (or Turing instability) have been rigorously proved for a two-species Lotka-Volterra competition system with both local and nonlocal intraspecific and interspecific competitions under homogeneous Neumann boundary condition. Naturally, the effects of nonlocality in competitions to the Lotka-Volterra type systems are still not completely clear and are deserved to study further. Thus we investigate the spatiotemporal pattern formations to system (1) and it shows that nonlocal competitions can lead to the occurrence of Turing-Hopf bifurcation.

As far as we know, although the existence of spatiotemporal phenomena has been studied to much extent through the Turing-Hopf bifurcation in many population dynamical models [1, 4, 7, 21, 27, 32, 33], there are few or no results in Turing-Hopf bifurcations for the competition system. Thus investigating what cause the occurrence of the Turing-Hopf bifurcation and how the spatiotemporal pattern can be formed is meaningful and worth exploring. Furthermore, the nonlocal terms can bring about difference in calculating normal forms for the codimension-two Turing-Hopf bifurcation.
In this paper, we firstly analyze the distribution of all eigenvalues for the linearized system of (1) evaluated at the coexistence equilibrium, and prove that system (1) can undergo Hopf bifurcations, Turing bifurcations and Turing-Hopf bifurcations when the nonlocal intraspecific competitions are suitably large. Moreover, the necessary and sufficient condition that the coexistence equilibrium becomes Turing instability are derived, and it can be expressed by explicit curves in $d_2 - d_1$ parameter plane. One can easily see the eigenvalue distribution for the characteristic equations associated with the linearized system in different regions of $d_2 - d_1$ parameter plane. Secondly, since center manifold theory and normal form method are efficient to study the codimension-two Turing-Hopf bifurcation, we apply the method and have found that the explicit formulas of normal forms in [20] cannot be directly used because of the existence of nonlocal terms. Thus we recalculate the terms and derive the explicit formulas of third order normal forms restricted on the center manifold at the codimension-two Turing-Hopf singularity, which shows the difference from the system without effects of nonlocal terms. Moreover, the explicit formulas only depend on parameters of original system and the calculating method can be applied in other similar systems with nonlocal terms. And via analyzing the normal form truncated to order 3 (amplitude equations), the existence of interesting and complex spatiotemporal patterns, like the spatial homogeneous periodic orbit, a pair of stable spatial inhomogeneous steady states and a pair of stable spatial inhomogeneous periodic orbits, is rigorously proved in the weak competition case, which is new phenomenon to show variety of coexistence states in the Lotka-Volterra competition system in the literature.

The rest of the paper is organized as follows. In Sect. 2, we give bifurcation analyses by analyzing distribution of all roots to the characteristic equations. In Sect. 3, by applying the results in Sect. 2, the explicit formulas of third order normal forms are derived and the existence of complex spatiotemporal patterns are proved. In Sect. 4, we summarize the results briefly and give some conjectures for the future work.

Throughout the paper, $\mathbb{N}$ is the set of all positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\Re(\cdot)$ and $\Im(\cdot)$ are functions of taking imaginary and real parts respectively. $X$ is a complexed-valued Hilbert space with $X = \{u \in W^{2,2}(0, \pi) : \frac{\partial u}{\partial \nu}(0) = \frac{\partial u}{\partial \nu}(\pi) = 0\}$ and $W^{2,2}(0, \pi)$ is a standard Sobolev space.

2. Bifurcation analysis. The well-posedness of solution for system (1) can be easily checked, see [8, 36, 37]. Following definitions in [8, 26], denote $m_{ij} = a_{ij} + b_{ij}$ the combined strength of local and nonlocal competitions of species $u_j$ to $u_i$, $i, j = 1, 2$ and $m_{11}m_{22} > m_{12}m_{21} (m_{11}m_{22} < m_{12}m_{21})$ the weak competition case (strong competition case).

For the system (1), we make the assumption

\[(H) : \frac{m_{12}}{m_{22}} < \frac{r_1}{r_2} < \frac{m_{11}}{m_{21}} \tag{2}\]

throughout the paper. From [8] or by direct calculations, the coexistence equilibrium $E^* = (u_1^*, u_2^*)$ exists and is unique, where $u_1^* = \frac{r_1 m_{22} - r_2 m_{12}}{m_{11}m_{22} - m_{12}m_{21}}$ and $u_2^* = \frac{r_2 m_{11} - r_1 m_{21}}{m_{11}m_{22} - m_{12}m_{21}}$. 
The linearized system (1) evaluated at $E^*$ takes the following form

$$
\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1^* (-a_{11} u_1 - a_{12} u_2) \\
&\quad - b_{11} \int_0^\pi u_1(y, t - \tau) dy - b_{12} \int_0^\pi u_2(y, t - \tau) dy, \quad x \in (0, \pi), t > 0, \\
\frac{\partial u_2}{\partial t} &= d_2 \Delta u_2 + u_2^* (-a_{21} u_1 - a_{22} u_2) \\
&\quad - b_{21} \int_0^\pi u_1(y, t - \tau) dy - b_{22} \int_0^\pi u_2(y, t - \tau) dy, \quad x \in (0, \pi), t > 0.
\end{align*}
$$

By choosing suitably trial solution as

$$(u_1(x, t), u_2(x, t)) = (c_1, c_2)e^{\lambda t} \cos nx, \quad n \in \mathbb{N}_0$$

and substituting it into system (3), we derive the characteristic equations of (3)

$$D(\lambda, \tau, n^2) = 0, \quad n \in \mathbb{N}_0,$$

in which

$$D(\lambda, \tau, n^2) = \begin{cases} P_0(\lambda) + P_1(\lambda)e^{-\lambda \tau} + P_2(\lambda)e^{-2\lambda \tau}, & n = 0, \\
\lambda^2 + [(d_1 + d_2)k^2 + a_{11}u_1^* + a_{22}u_2^*] \lambda + f(n^2), & n \in \mathbb{N},
\end{cases}$$

where

$$P_0(\lambda) = \lambda^2 + (a_{11}u_1^* + a_{22}u_2^*) \lambda + u_1^*u_2^*(a_{11}a_{22} - a_{12}a_{21}),$$

$$P_1(\lambda) = (b_{11}u_1^* + b_{22}u_2^*) \lambda + u_1^*u_2^*(a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12}),$$

$$P_2(\lambda) = u_1^*u_2^*(b_{11}b_{22} - b_{12}b_{21}),$$

$$f(n^2) = d_1d_2(n^2)^2 + (d_1a_{22}u_2^* + d_2a_{11}u_1^*)n^2 + u_1^*u_2^*(a_{11}a_{22} - a_{12}a_{21}).$$

Note that $D(\lambda, \tau, 0)$ can be written as the following form

$$D(\lambda, \tau, 0) = \lambda^2 + g(\lambda, \tau).$$

Evidently, $g(\lambda, \tau)$ is an analytic function in $\lambda$ and continuous in $\tau$. By (5), it is clear to see that the degree of $P_1(\lambda)$ and $P_2(\lambda)$ with respect to $\lambda$ is lower than 2. Then the following result holds.

**Theorem 2.1.** Let $D(\lambda, \tau, 0)$ be defined as (6) or (11), then the sum of multiplicities of roots of $D(\lambda, \tau, 0) = 0$ in the open right half-plane can change only if a root appears on or crosses the imaginary axis as $\tau$ varies in $[0, +\infty)$.

**Proof.** See Theorem 2.4 in [10] or Theorem 1.1 in [2] and it can also be proved by the results in [29] since zeros in the right half complex plane of $D(\lambda, \tau, 0) = 0$ are uniformly bounded.

When $\tau \geq 0$, $D(0, \tau, 0) = u_1^*u_2^*(m_{11}m_{22} - m_{12}m_{21}) > 0$ holds by (2), which implies that the assumption (A.1) in [2] is satisfied. It is also easy to verify that $P_0(\lambda)$, $P_1(\lambda)$ and $P_2(\lambda)$ have no common imaginary roots, i.e. assumption (A.2) in [2] holds.

Then we apply the geometric stability switch criterion established in [2] to investigate the distribution of all roots of $D(\lambda, \tau, 0) = 0$. 

Let $\lambda = iw$, $w > 0$ be a root of $D(\lambda, \tau, 0) = 0$. From the characteristic equation (5), separating real and imaginary parts, we have

$$
\begin{align*}
\left\{ [\Re(P_0(iw)) + \Re(P_2(iw))] \cos \omega \tau - [\Im(P_0(iw)) - \Im(P_2(iw))] \sin \omega \tau = -\Re(P_1(iw)), \\
[\Im(P_0(iw)) + \Im(P_2(iw))] \cos \omega \tau + [\Re(P_0(iw)) - \Re(P_2(iw))] \sin \omega \tau = -\Im(P_1(iw)).
\right.
\end{align*}
$$

(12)

For notational simplicity, define

$$
\begin{align*}
R(w) &= |P_0(iw)|^2 - |P_2(iw)|^2, \\
T(w) &= [\Re(P_2(iw)) - \Re(P_0(iw))]\Re(P_1(iw)) + [\Im(P_2(iw)) - \Im(P_0(iw))]\Im(P_1(iw)), \\
S(w) &= [\Im(P_0(iw)) + \Im(P_2(iw))]\Re(P_1(iw)) - [\Re(P_0(iw)) + \Re(P_2(iw))]\Im(P_1(iw)).
\end{align*}
$$

(13)

From (12), $w > 0$ must satisfies the following equations

$$
\begin{align*}
\cos \omega \tau &= \frac{T(w)}{R(w)}, \\
\sin \omega \tau &= \frac{S(w)}{R(w)}.
\end{align*}
$$

(14)

holds if $w$ belongs to the set

$$
I_w = \{ w > 0 : F(w) = 0 \},
$$

(15)

where

$$
F(w) = R^2(w) - T^2(w) - S^2(w).
$$

(16)

From (16) and (13), $F(w)$ is an even function and a polynomial function with respect to $w$, and $I_w$ is a finite set. Thus, if $I_w$ is not empty, we can take an element $w_* \in I_w$ and substitute it to (14) to derive

$$
\begin{align*}
\cos \theta(\tau) &= \frac{T(w)}{S(w)}, \\
\sin \theta(\tau) &= \frac{S(w)}{R(w)},
\end{align*}
$$

(17)

where $\theta(\tau) := w_* \tau$ and $\theta(\tau) \in [0, 2\pi]$.

Then we have

$$
\tau = \frac{\theta(\tau) + 2k\pi}{w_*} =: \tau_k(\tau), \quad k \in \mathbb{N}_0.
$$

(18)

From (14) and Lemma 2.1 of [2], $\theta(\tau) \neq 0, 2\pi$ and

$$
\theta(\tau) = \begin{cases} 
\arctan \frac{S(w)}{T(w)} & \text{if } \sin \theta(\tau) > 0, \cos \theta(\tau) > 0, \\
\frac{\pi}{2} & \text{if } \sin \theta(\tau) = 1, \cos \theta(\tau) = 0, \\
\pi + \arctan \frac{S(w)}{T(w)} & \text{if } \cos \theta(\tau) < 0, \\
\frac{3\pi}{2} & \text{if } \sin \theta(\tau) = -1, \cos \theta(\tau) = 0, \\
2\pi + \arctan \frac{S(w)}{T(w)} & \text{if } \sin \theta(\tau) < 0, \cos \theta(\tau) > 0.
\end{cases}
$$

(19)

Define the map $S_k : [0, +\infty) \to \mathbb{R}$:

$$
S_k(\tau) = \tau - \tau_k(\tau), \quad k \in \mathbb{N}_0.
$$

(20)

By (18) and (20), $S_k(\tau)$, $k \in \mathbb{N}_0$, is continuous and differentiable with respect to $\tau$.

From the analysis above, we know there must exist $\tau_k(\tau)$ such that characteristic equation $D(\lambda, \tau_k(\tau), 0) = 0$ have purely imaginary roots if $I_w$ is not empty. Then we give the following result:
Table 1. Values of system parameters

| Parameters | \( r_1 \) | \( r_2 \) | \( a_{11} \) | \( a_{12} \) | \( a_{21} \) | \( a_{22} \) | \( b_{11} \) | \( b_{12} \) | \( b_{21} \) | \( b_{22} \) |
|------------|---------|---------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| Values     | 5       | 5       | 4           | 7           | 6           | 5           | 3           | 1           | 0.5         | 4           |

**Theorem 2.2.** Assume that assumption (H) holds. \( I_w \neq \emptyset \) if

\[
(a_{11} - b_{11})(a_{22} - b_{22}) - (a_{12} - b_{12})(a_{21} - b_{21}) < 0 \tag{21}
\]

and

\[
a_{11}a_{22} - a_{12}a_{21} \neq b_{11}b_{22} - b_{12}b_{21} \tag{22}
\]

hold true.

**Proof.** Define

\[
A = a_{11}a_{22} - a_{12}a_{21}, \tag{23}
\]

\[
B = b_{11}b_{22} - b_{12}b_{21}. \tag{24}
\]

By (7), (16) and direct calculations, \( F(w) \) is an eight degree polynomial with respect to \( w \) and it can be rewrite as the following form:

\[
F(w) = w^8 + G(w) + C, \tag{25}
\]

where the highest degree of \( G(w) \) with respect to \( w \) is no more than 6 and

\[
C = (u_1^*u_2^*)^2(A - B)^2[(A + B)^2 - (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})^2] \tag{26}
\]

is the sum of all terms of \( F(w) \) independent of \( w \). Thus we have \( G(0) = 0 \) and

\[
\lim_{w \to +\infty} F(w) = +\infty. \tag{27}
\]

From (22), the sign of \( C \) is the same as the one of \( (A + B)^2 - (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})^2 \).

By direct calculations, we have

\[
(A + B)^2 - (a_{11}b_{22} + a_{22}b_{11} - a_{12}b_{21} - a_{21}b_{12})^2
= (m_{11}m_{22} - m_{12}m_{21})[(a_{11} - b_{11})(a_{22} - b_{22}) - (a_{12} - b_{12})(a_{21} - b_{21})]
\]

From assumption (H) and (21), we obtain that \( C < 0 \), which implies that

\[
F(0) < 0 \tag{28}
\]

By (27), (28) and the continuity of \( F(w) \), there must exist \( w > 0 \) such that \( F(w) = 0 \) holds. This completes the proof.

**Remark 1.** From assumption (H), we can easily verify that (21) could not be satisfied if nonlocal intraspecific competitions are weak enough, which is consistent with the results in [8] that nonlocal intraspecific competitions play an important role in resulting in the complex pattern formations to the system (1).

To show that there is no contradiction between (21), (22) and assumption (H), we take parameter values as Table 1 shows and plot the graph of \( F \) against \( w \), see Figure 1.

Since the explicit expression of \( F(w) \) is complex, we make the assumption

\[
(H_1) : F'(w) \neq 0, \quad w \in I_w \tag{29}
\]

to simplify the analysis of roots of \( D(\lambda, \tau, 0) \). Then we give the following results:
Figure 1. $w_* > 0$ is a positive root of $F(w) = 0$.

Lemma 2.3. Assume that assumptions (H) and (H1) hold. If $0 < w \in I_w$, then there must exist $\tau_w > 0$ such that

$$S_k(\tau_w) = 0, \quad k \in \mathbb{N}_0$$

and $\lambda_\pm(\tau_w) = \pm iw$ is a pair of simple conjugate pure imaginary roots of $D(\lambda, \tau, 0)$ which crosses the imaginary axis from left to right if $\delta(\tau_w) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau_w) < 0$, where

$$\delta(\tau_w) = \text{sign}\{\frac{dR(\lambda)}{d\tau}|_{\lambda=iw}\} = \text{sign}\{R(w)F'(w)\}.$$  

Moreover, all roots of $D(\lambda, \tau, 0)$ have negative real parts if $\tau < \tau_* = \min\{\tau_0(\tau_w) : w \in I_w\}$.

Proof. The first part follows from Theorem 2.1 in [2]. Since $\tau_k(\tau_w), k \in \mathbb{N}_0$, is strictly increasing with respect to $k$, we have

$$\tau_0(\tau_w) = \min_{k \in \mathbb{N}_0}\{\tau_k(\tau_w)\}.$$  

We can find the critical value $\tau_*$ by the finiteness of $I_w$. Thus the rest part holds true from Theorem 2.1.

Theorem 2.4. Assume that assumptions (H) and (H1) hold. System (1) undergoes 0-mode Hopf bifurcation near the coexistence equilibrium $E^*$ at $\tau = \tau_k(\tau_w), k \in \mathbb{N}_0$ for each $w \in I_w$ if $\delta(\tau_k(\tau_w)) \neq 0$ and $I_w \neq \emptyset$.

Next, we take diffusion coefficients $d_1$ and $d_2$ as parameters to discuss the distribution of all roots of $D(\lambda, \tau, n^2), n \in \mathbb{N}$. Since $D(\lambda, \tau, n^2)$ is independent of $\tau$, we denote $D(\lambda, \tau, n^2) = D(\lambda, n^2)$.

For convenience, in the rest of the paper, we assume that $a_{12}a_{21} - a_{11}a_{22} > 0$ holds and denote curves

$$\mathcal{L}_n : d_1 = d_1^n(d_2), \quad d_2 \in (0, d_{2,n}),$$  

$$\mathcal{S}_1 : d_1 = 0, \quad d_2 \in [d_{2,1}, +\infty),$$  

$$\mathcal{S}_{n+1} : d_1 = 0, \quad d_2 \in [d_{2,n}, d_{2,n+1})$$

(33) (34) (35)
in the $d_2 - d_1$ plane to describe the distribution of all roots of $D(\lambda, n^2)$ when $d_1$ and $d_2$ vary in the first quadrant of $d_2 - d_1$ plane, where

$$d_1^0(d_2) = \frac{u_1^*u_2^*(a_{12}a_{21} - a_{11}a_{22}) - d_2a_{11}u_1^*n^2}{(d_2n^2 + a_{22}u_2^*)n^2}$$

(36)

and

$$d_{2,n} = \frac{u_1^*u_2^*(a_{12}a_{21} - a_{11}a_{22})}{a_{11}u_1^*n^2}$$

(37)

with $n \in \mathbb{N}$.

Note that $f(n^2) = 0$ if and only if $d_1 = d_1^n(d_2)$. Evidently, if $a_{12}a_{21} - a_{11}a_{22} > 0$ holds, the definition of (33) is valid since $d_1^0(d_2), d_{2,n} > 0$, and (34) and (35) is feasible since $d_{2,n}$ is decreasing with respect to $n$. And from Remark 3.3 in [8] or assumption (H), $a_{12}a_{21} - a_{11}a_{22} > 0$ holds if the nonlocal intraspecific competitions are suitably strong, which implies the strength of nonlocal intraspecific competitions play an important role in result in pattern formations of system (1).

We also define that the curve $\mathcal{L}_n\mathcal{S}_n$ is the combination of $\mathcal{L}_n$ and $\mathcal{S}_n$ for $n \in \mathbb{N}$ and a two function sequence $\{\tilde{d}_1^n(d_2)\}_{n \in \mathbb{N}}$ with

$$\tilde{d}_1^1(d_2) = \begin{cases} d_1^0(d_2), & d_2 \in (0, d_{2,1}), \\ 0, & d_2 \in [d_{2,1}, +\infty), \end{cases}$$

(38)

$$\tilde{d}_1^{n+1}(d_2) = \begin{cases} d_1^{n+1}(d_2), & d_2 \in (0, d_{2,n+1}), \\ 0, & d_2 \in [d_{2,n+1}, d_{2,n}], \end{cases}$$

(39)

**Remark 2.** From (34)-(35) and (38)-(39), $d_3 = \tilde{d}_1^n(d_2)$ represent curve $\mathcal{L}_n\mathcal{S}_n$ in the $d_2 - d_1$ plane.

**Lemma 2.5.** Assume that $a_{12}a_{21} - a_{11}a_{22} > 0$ holds. Then $d_1^n(d_2) > \tilde{d}_1^{n+1}(d_2)$ for $d_2 \in (0, d_{2,2}]$ and $n \in \mathbb{N}$.

**Proof.** For the case that $d_2 \in [d_{2,2,1}, d_{2,2})$, we have

$$d_1^n(d_2) = a_{11}u_1^* \frac{d_{2,n} - d_2}{d_2n^2 + a_{22}u_2^*} > 0 = q_{n+1}(d_2).$$

(40)

For the case that $d_2 \in (0, d_{2,2,1})$, we only need to show $d_1^n(d_2) > \tilde{d}_1^{n+1}(d_2)$. Since $\tilde{d}_1^{n+1}(d_2) = d_1^{n+1}(d_2)$ for $d_2 \in (0, d_{2,2,1})$, $d_1^n(d_2) > \tilde{d}_1^{n+1}(d_2)$ for $d_2 \in (0, d_{2,2,1})$ is equivalent to $d_1^n(d_2) > d_1^{n+1}(d_2)$.

Note that $f(n^2) = 0$ if and only if $d_1 = d_1^n(d_2)$. And the point $P_n := (d_2, d_1^n(d_2))$ is on $\mathcal{L}_n$ in the $d_2 - d_1$ plane for fixed $d_2 \in (0, d_{2,2,1})$. Firstly, we claim that $P_n \notin \mathcal{L}_{n+1}$. By contradiction, we know $d_1^n(d_2) = d_1^{n+1}(d_2)$, that is to say, $f((n+1)^2) = 0$ when $d_1 = d_1^n(d_2)$, which contradicts that $f(n^2) = 0$ when $d_1 = d_1^n(d_2)$ since $f(n^2)$ is strictly increasing with respect to $n^2$.

Secondly, by (40) and the monotonicity of $d_{2,n}$ with respect to $n$, we have

$$\lim_{d_2 \to 0^+} d_1^n(d_2) > \lim_{d_2 \to 0^+} d_1^{n+1}(d_2)$$

(41)

and

$$\lim_{d_2 \to d_{2,2,1}} d_1^{n+1}(d_2) = 0 < d_1^n(d_2).$$

Thus by the continuity of curve $\mathcal{L}_n$, we derive that $d_1^n(d_2) > d_1^{n+1}(d_2) = \tilde{d}_1^{n+1}(d_2)$ for $d_2 \in (0, d_{2,2,1})$. This completes the proof. \qed
Proof. From Lemma 2.3 and Theorem 2.6, the proof is completed.

Remark 3. Lemma 2.5 give an completed geometry description on sequence of curves \( \{L_nS_n\}_{n \in \mathbb{N}} \) and \( \{L_n\}_{n \in \mathbb{N}} \) in the \( d_2 - d_1 \) plane. We take system parameter values as Table 1 shows and draw these curves, see Figure 2.

Thus we can derive the distribution all roots of \( D(\lambda, n^2) = 0, n \in \mathbb{N} \) as follows

**Theorem 2.6.** Assume that \( a_{12}a_{21} - a_{11}a_{22} > 0 \) holds.

1. All roots of \( D(\lambda, n^2) = 0, n \in \mathbb{N} \) are negative except \( k \) positive roots if \( (d_2, d_1) \in \Lambda_k \) with \( k \in \mathbb{N}_0 \);
2. All roots of \( D(\lambda, n^2) = 0, n \in \mathbb{N} \) are negative except a simple zero root and \( k - 1 \) positive roots when \( (d_2, d_1) \in \mathbb{L}_k \) with \( k \in \mathbb{N} \); Here, \( \mathbb{L}_k \) is defined by (33) and

\[
\Lambda_0 = \{ (d_2, d_1(d_2)) \in \mathbb{R}_+^2 : d_1(d_2) > \bar{d}_1(d_2), d_2 \in (0, +\infty) \},
\]

\[
\Lambda_k = \{ (d_2, d_1(d_2)) \in \mathbb{R}_+^2 : \bar{d}_1(d_2) < d_1(d_2) < d_k(d_2), d_2 \in (0, d_{2,k}) \}, \quad k \in \mathbb{N}.
\]

**Proof.** By Lemma 2.5, (10) and (33)-(35), (1) and (2) can be easily verified and we omit the detail here.

Following the definitions in Theorem 2.6, we derive the necessary and sufficient conditions for the occurrence of Turing instability at the coexistence equilibrium \( E^* \).

**Theorem 2.7.** Assume that assumption (H) and \( a_{12}a_{21} - a_{11}a_{22} > 0 \) hold. If delay \( \tau < \tau_{u*} \), the coexistence equilibrium \( E^* \) is locally asymptotic stable when \( (d_2, d_1) \) is in region \( \Lambda_0 \). Moreover, the coexistence equilibrium \( E^* \) will become unstable if \( (d_2, d_1) \) in region \( \Lambda_0 \) goes through \( \mathbb{L}_1 \).

**Proof.** From Lemma 2.3 and Theorem 2.6, the proof is completed.

**Remark 4.** By Theorem 2.7, \( \mathbb{L}_1 \) is called the first Turing bifurcation curve and it is sufficiently smooth but not piecewise smooth, which is different from the one derived in [21].
By combining the results of Lemmas 2.3 and 2.6, Theorem 2.4 and following the definitions above, we have the following results.

**Theorem 2.8.** Assume that assumptions $(H)$ and $(H_1)$ hold. If \( a_{12}a_{21} - a_{11}a_{22} > 0 \), then

1. system (1) undergoes \( n \)-mode Turing bifurcation at \( d_1 = d_1^n(d_2) \) near the coexistence equilibrium \( E^* \) for \( n \in \mathbb{N} \);
2. when \( I_w \) is not empty and \( \delta(\tau_k(\tau_w)) \neq 0 \), system (1) undergoes \((n,0)\)-mode Turing-Hopf bifurcation at \((\tau, d_1) = (\tau_k(\tau_w), d_1^n(d_2))\) near the coexistence equilibrium \( E^* \) for \( k \in \mathbb{N}_0 \) and \( n \in \mathbb{N} \). Moreover, when \((\tau, d_1) = (\tau_w, d_1^1(d_2))\), all roots of characteristic equations \( D(\lambda, \tau, n^2) = 0, n \in \mathbb{N}_0 \) have negative real parts except a pair of simple pure imaginary roots and a simple zero root.

3. Turing-Hopf normal forms and spatiotemporal dynamics. In this section, we assume that assumptions and conditions in Theorem 2.8 are satisfied, and calculate the normal form of \((1,0)\)-mode Turing-Hopf bifurcation at the coexistence equilibrium for system (1). Then we take a group of system parameters to show spatiotemporal dynamics of system (1).

3.1. Turing-Hopf normal forms. Introducing perturbation parameters \( \alpha = (\alpha_1, \alpha_2)^T \) and let \( d_1 = d_1^1(d_2) + \alpha_1 \) and \( \tau = \tau_w + \alpha_2 \). Then by taking the transformation \( t \rightarrow t/\tau \), translating the coexistence \( E^* \) to the origin and adding the perturbation parameters, system (1) becomes

\[
\begin{align*}
\frac{\partial u_1}{\partial t} &= (\tau_w + \alpha_2)[d_1^1(d_2) + \alpha_1] \Delta u_1 + (\tau_w + \alpha_2)(u_1 + u_1^*)[\tau_1 - a_{11}(u_1 + u_1^*)] \\
&\quad - a_{12}(u_2 + u_2^*) - \frac{b_{11}}{\pi} \int_0^\pi (u_1(y, t - 1) + u_1^*)dy - \frac{b_{12}}{\pi} \int_0^\pi (u_2(y, t - 1) + u_2^*)dy, \\
\frac{\partial u_2}{\partial t} &= (\tau_w + \alpha_2)d_2 \Delta u_2 + (\tau_w + \alpha_2)(u_2 + u_2^*)[\tau_2 - a_{21}(u_1 + u_1^*) - a_{22}(u_2 + u_2^*)] \\
&\quad - \frac{b_{21}}{\pi} \int_0^\pi (u_1(y, t - 1) + u_1^*)dy - \frac{b_{22}}{\pi} \int_0^\pi (u_2(y, t - 1) + u_2^*)dy.
\end{align*}
\]  

(44)

By similar procedures in section 2, we directly give the characteristic equations of system (44)

\[ D(\lambda, \tau_w, \alpha, n^2) = 0, \quad n \in \mathbb{N}_0. \]  

(45)

And when \( \alpha_1 = \alpha_2 = 0 \), we know that \( \lambda = 0 \) is a simple root of (45) with \( k_1 := n = 1 \) and \( \lambda = \pm iw_\tau \tau_w \) is a pair of purely imaginary roots of (45) with \( k_2 := n = 0 \).

Following the method in [20], let \( U(t) = (u_1(t), u_2(t)) \) and system (44) can be rewritten as

\[
\begin{align*}
\frac{dU(t)}{dt} &= L_0(U_t) + D_0 \Delta U(t) + \frac{1}{2}L_1(\alpha)U_t + \frac{1}{2}D_1(\alpha) \Delta U(t) \\
&\quad + \frac{1}{2}Q(U_t, U_t) + \frac{1}{3!}C(U_t, U_t, U_t) + h.o.t.,
\end{align*}
\]  

(46)
where
\[
D_0 = \tau_{w*} \begin{bmatrix} d_1(d_2) & 0 \\ 0 & d_2 \end{bmatrix}, \quad D_1(\alpha) = \begin{bmatrix} d_1(d_2)\alpha_2 + \tau_{w*}\alpha_1 & 0 \\ 0 & d_2\alpha_2 \end{bmatrix},
\]
\[
L_0\varphi = \tau_{w*} \left( -a_{11}u_1^2\varphi_1(0) - a_{12}u_1^2\varphi_2(0) \right) - a_{21}u_2^2\varphi_1(0) - a_{22}u_2^2\varphi_2(0) + \left( -\frac{b_{11}u_1^1}{\pi} \int_0^\pi \varphi_1(-1)dy - \frac{b_{12}u_1^2}{\pi} \int_0^\pi \varphi_2(-1)dy \right),
\]
\[
L_{1,1}(\alpha)\varphi = L_{1,1}(\alpha)\varphi + L_{1,2}(\alpha)\varphi,
\]
\[
Q(\varphi, \varphi) = Q^1(\varphi, \varphi) + Q^2(\varphi, \varphi), \quad C(\varphi, \varphi, \varphi) = (0, 0)^T
\]
with \( \varphi = (\varphi_1, \varphi_2)^T \in \mathcal{C} = C([-1, 0]; X^2) \) and
\[
L_{1,1}(\alpha)\varphi = 2\alpha_2 \left( -a_{11}u_1^1\varphi_1(0) - a_{12}u_1^2\varphi_2(0) \right) - a_{21}u_2^2\varphi_1(0) - a_{22}u_2^2\varphi_2(0),
\]
\[
L_{1,2}(\alpha)\varphi = 2\alpha_2 \left( -\frac{b_{11}u_1^1}{\pi} \int_0^\pi \varphi_1(-1)dy - \frac{b_{12}u_1^2}{\pi} \int_0^\pi \varphi_2(-1)dy \right),
\]
\[
Q^1(\varphi, \varphi) = \tau_{w*} \left( -2a_{11}\varphi_1^2(0) - 2a_{12}\varphi_1(0)\varphi_2(0) \right) - 2a_{21}\varphi_1(0)\varphi_2(0) - 2a_{22}\varphi_2^2(0),
\]
\[
Q^2(\varphi, \varphi) = \tau_{w*} \left( -\frac{2b_{11}u_1^1}{\pi} \int_0^\pi \varphi_1(-1)dy + \frac{2b_{12}u_1^2}{\pi} \int_0^\pi \varphi_2(-1)dy \right) + \left( -\frac{b_{21}}{\pi} \varphi_1(0) \int_0^\pi \varphi_2(-1)dy + \varphi_2(0) \int_0^\pi \varphi_1(-1)dy \right).
\]

To derive the explicit formula of third order normal forms restricted on the center manifold for system (44) and follow definitions in [20], let
\[
\phi_1(\theta) = \phi_1(0), \quad \phi_2(\theta) = \phi_2(0)e^{iw_*\tau_{w*}\theta}, \quad \theta \in [-1, 0],
\]
\[
\psi_1(s) = \psi_1(0), \quad \psi_2(s) = \psi_2(0)e^{-iw_*\tau_{w*}s}, \quad s \in [0, 1],
\]
where \( \phi_1(0), \psi_1(0), \phi_2(0), \psi_2(0) \) satisfy
\[
\Delta_{k_1}(0)\phi_1(0) = 0, \quad \psi_1(0)\Delta_{k_1}(0) = 0,
\]
\[
\Delta_{k_2}(i\omega_\tau_{w_*})\phi_1(0) = 0, \quad \psi_1(0)\Delta_{k_2}(i\omega_\tau_{w_*}) = 0,
\]
\[
(\psi_1, \phi_i)_{k_i} = 1, \quad i = 1, 2
\]
and \( \Delta_n(\cdot), (\cdot, \cdot)_n \) are defined by
\[
\Delta_n(\lambda) = \lambda I + n^2D_0 - L_0(e^{\lambda I}),
\]
\[
(\psi, \phi)_n = \psi(0)\phi(0) - \int_0^\theta \int_{-1}^0 \psi(\xi - \theta)d\eta_n(\theta)\phi(\xi)d\xi, \quad n \in \mathbb{N}_0
\]
with \( \eta_n \in BV([-1, 0], \mathbb{C}^{2\times2}) \) satisfying
\[
-n^2D_0\gamma(0) + L_{0,n}\gamma = \int_{-1}^0 d\eta_n(\theta)\gamma(\theta), \quad \gamma = (\gamma_1, \gamma_2)^T \in C([-1, 0], \mathbb{C}^2).
\]
Here

\[
L_{0,n,\gamma} = \begin{cases}
\tau_{w_*} \left( \begin{array}{l}
-a_{11} u_1^* \gamma_1(0) - a_{12} u_1^* \gamma_2(0) \\
-a_{21} u_2^* \gamma_1(0) - a_{22} u_2^* \gamma_2(0)
\end{array} \right) \\
+ \left( \begin{array}{l}
\frac{b_{11} u_1^*}{\pi} \int_0^\pi \gamma_1(-1)dy - \frac{b_{12} u_1^*}{\pi} \int_0^\pi \gamma_2(-1)dy \\
\frac{b_{21} u_2^*}{\pi} \int_0^\pi \gamma_1(-1)dy - \frac{b_{22} u_2^*}{\pi} \int_0^\pi \gamma_2(-1)dy
\end{array} \right), 
\end{cases}
\]

\(n \neq 0, \quad n \in \mathbb{N}.
\]

(61)

From (53)-(60), we obtain the characteristic eigenvectors and their dual ones with respect to \(\lambda = 0\) and \(\lambda = iw_* \tau_{w_*}\), namely,

\[
\phi_1(\theta) = \left( \begin{array}{l}
1 \\
p_1
\end{array} \right), \quad \psi_1(s) = \frac{1}{N_1} \left( \begin{array}{l}
1 \\
q_1
\end{array} \right)^T
\]

(62)

and

\[
\phi_2(\theta) = \left( \begin{array}{l}
1 \\
p_2
\end{array} \right) e^{iw_* \tau_{w_*} \theta}, \quad \psi_2(s) = \frac{1}{N_2} \left( \begin{array}{l}
1 \\
q_2
\end{array} \right)^T e^{-iw_* \tau_{w_*} s},
\]

(63)

where \(\theta \in [-1, 0], \; s \in [0, 1]\) and

\[
p_1 = - \frac{d_1^1 (d_2) k_1^2 + a_{11} u_1^*}{a_{12} u_1^*}, \quad q_1 = - \frac{d_2^1 (d_2) k_2^2 + a_{11} u_1^*}{a_{21} u_2^*},
\]

\[
p_2 = \frac{iw_* \tau_{w_*} - u_1^* (a_{11} + b_{11} e^{-iw_* \tau_{w_*}})}{u_1^* (a_{12} + b_{12} e^{-iw_* \tau_{w_*}})}, \quad q_2 = \frac{iw_* \tau_{w_*} - u_1^* (a_{11} + b_{11} e^{-iw_* \tau_{w_*}})}{u_2^* (a_{21} + b_{21} e^{-iw_* \tau_{w_*}})},
\]

\[
N_1 = 1 + p_1 q_1, \quad N_2 = 1 + p_2 q_2 - \tau_{w_*} (b_{11} u_1^* + b_{12} u_1^* + b_{21} u_2^* q_2 + b_{22} u_2^* p_2 q_2) e^{-iw_* \tau_{w_*}}.
\]

We have to emphasize that the explicit formulas for the case \(k_1 \neq 0\) and \(k_2 = 0\) (see Proposition 4.3 in [20]) cannot be directly used since the effects of nonlocal terms in system (44).

It is well known that \(-\Delta\) on domain \((0, \pi)\) subject to homogeneous Neumann boundary condition has simple eigenvalues \(\mu_n = n^2\) with corresponding normalized eigenfunctions \(\beta_n\) for \(n \in \mathbb{N}_0\), where

\[
\beta_0 = 1, \; \beta_k = \sqrt{2} \cos kx, \; k \in \mathbb{N}
\]

(64)

with

\[
< \beta_i, \beta_j > = \begin{cases}
0, & i \neq j, \\
1, & i = j
\end{cases}
\]

(65)

and \(< \cdot, \cdot >\) the standard inner product in Hilbert space \(X\), \(i, j \in \mathbb{N}_0\).
From (47), we have
\[
L_1(\alpha)(\phi_1\beta_k) = L_{1,1}(\alpha)(\phi_1)\beta_k + L_{1,2}(\alpha)(\phi_1) \int_0^\pi \beta_k \, dx,
\]
\[
L_1(\alpha)(\phi_2\beta_k) = L_{1,1}(\alpha)(\phi_2)\beta_k + L_{1,2}(\alpha)(\phi_2) \int_0^\pi \beta_k \, dx,
\]
\[
Q(\phi_1\beta_k, \phi_2\beta_k) = Q_1^1(\phi_1, \phi_2)\beta_k \beta_k + (Q_2^{1,1}(\phi_1, \phi_2)\beta_k \int_0^\pi \beta_k \, dx + Q_2^{1,2}(\phi_2, \phi_1)\beta_k \int_0^\pi \beta_k \, dx),
\]
\[
Q(\phi_1\beta_k, \phi_1\beta_k) = Q_1^1(\phi_1, \phi_1)\beta_k^2 + Q_2(\phi_1, \phi_1)\beta_k \int_0^\pi \beta_k \, dx,
\]
\[
Q(\phi_2\beta_k, \phi_2\beta_k) = Q_1^1(\phi_2, \phi_2)\beta_k^2 + Q_2(\phi_2, \phi_2)\beta_k \int_0^\pi \beta_k \, dx,
\]
where
\[
L_{1,1}(\alpha)(\phi_1) = 2\alpha_2 \left(-a_{11}u_1^\alpha\phi_1(0) - a_{12}u_1^\alpha\phi_2(0)\right),
\]
\[
L_{1,2}(\alpha)(\phi_1) = \frac{2\alpha_2}{\pi} \left(-b_{11}u_1^\alpha\phi_1(-1) - b_{12}u_1^\alpha\phi_2(-1)\right),
\]
\[
Q_1^1(\phi_1, \phi_2) = \tau_{\phi,2} \left(-2a_{11}\phi_1(0)\phi_1(0) - 2a_{12}\phi_1(0)\phi_2(0)\right),
\]
\[
Q_2(\phi_1, \phi_1) = 2\tau_{\phi,1} \left(-b_{11}\phi_1(0)\phi_1(0) - b_{12}\phi_1(0)\phi_2(0)\right),
\]
\[
Q_2^{1,1}(\phi_1, \phi_2) = \tau_{\phi,1} \left(-b_{11}\phi_1(0)\phi_2(0)\phi_1(1) - b_{12}\phi_1(0)\phi_2(0)\phi_1(1)\right),
\]
\[
Q_2^{1,2}(\phi_2, \phi_1) = \tau_{\phi,2} \left(-b_{11}\phi_2(0)\phi_1(0)\phi_1(1) - b_{12}\phi_2(0)\phi_1(0)\phi_1(1)\right),
\]
with \(\phi_{i,j}\) the j-th element of \(\phi_i, i, j = 1, 2\).

**Remark 5.** From (67), terms \(Q_2^{1,1}(\phi_1, \phi_2)\) and \(Q_2^{1,2}(\phi_2, \phi_1)\) reflect the effects of nonlocal terms, which are very different from that without nonlocal terms.

Furthermore, we obtain that
\[
Q(\phi_i\beta_k, h_q^j\beta_j) = Q_1^1(\phi_i, h_q^j)\beta_k\beta_j + Q_2^{1,1}(\phi_i, h_q^j)\beta_k \int_0^\pi \beta_j \, dx
\]
\[
+ Q_2^{2,2}(h_q^j, \phi_i) \int_0^\pi \beta_k \, dx \beta_j, \quad i = 1, 2, j \in \mathbb{N}_0, q \in \mathbb{N}_0^3,
\]
where
\[
Q_1^1(\phi_i, h_q^j) = \tau_{h, q} \left(-2a_{11}\phi_i(0)h_q^j(-1) - 2a_{12}\phi_i(0)h_q^j(-1)\right),
\]
\[
Q_2^{1,1}(\phi_i, h_q^j) = \tau_{h, q} \left(-b_{11}\phi_i(0)h_q^j(-1) - b_{12}\phi_i(0)h_q^j(-1)\right),
\]
\[
Q_2^{2,2}(h_q^j, \phi_i) = \tau_{h, q} \left(-b_{11}h_q^j(0)\phi_i(1) - b_{12}h_q^j(0)\phi_i(1)\right),
\]
with \(h_q^j\) the k-th element of \(h_q^j, k = 1, 2\). Here, \(h_q^j = \langle h_q, \beta_j \rangle\) and \(h_q\) is defined by (6.32)-(6.34) in [20].
Note that
\[
\int_0^\pi \beta_k \, dx = \begin{cases} 
\pi, & k = 0, \\
0, & k \in \mathbb{N}.
\end{cases}
\tag{70}
\]
Then for notational simplicity to calculate the coefficients of normal forms, let
\[
\begin{align*}
\tilde{L}_1(\alpha)\phi_1 = & \; L_{1,1}(\alpha)(\phi_1), \\
\tilde{L}_1(\alpha)\phi_2 = & \; L_{1,1}(\alpha)(\phi_2) + \pi L_{1,2}(\alpha)(\phi_2), \\
Q_{\phi_1\phi_2} = & \; Q^1(\phi_1, \phi_2) + \pi Q^{2,1}(\phi_1, \phi_2), \\
Q_{\phi_1\phi_1} = & \; Q^1(\phi_1, \phi_1), \\
Q_{\phi_2\phi_2} = & \; Q^1(\phi_2, \phi_2) + \pi Q^{2,1}(\phi_2, \phi_2), \\
Q_{\phi_i h_q^j} = & \; Q^1(\phi_i, h_q^j) + Q^{2,1}(\phi_i, h_q^j) \int_0^\pi \beta_k \, dx \\
& \quad + Q^{2,2}(h_q^j, \phi_i) \int_0^\pi \beta_k \, dx, \quad i = 1, 2, j \in \mathbb{N}_0, q \in \mathbb{N}_0^3.
\end{align*}
\tag{71}
\]
and (71) has same definitions if $\phi_i$ is substituted by its conjugate, $i = 1, 2$.

From \cite{20}, normal forms restricted on center manifold up to third-order for system (44) at Turing-Hopf singularity are
\[
\begin{align*}
\dot{z}_1 = & \; a_1(\alpha)z_1 + a_{200}z_1^2 + a_{011}z_1z_2 + a_{300}z_2^2 + a_{111}z_1z_2z_2 + h.o.t., \\
\dot{z}_2 = & \; i\omega_0z_2 + b_2(\alpha)z_2 + b_{110}z_1z_2 + b_{210}z_1^2z_2 + b_{021}z_2^2 + h.o.t., \\
\dot{\bar{z}}_2 = & \; -i\omega_0\bar{z}_2 + \overline{b_2(\alpha)z_2} + \overline{b_{110}z_1z_2} + \overline{b_{210}z_1^2z_2} + \overline{b_{021}z_2^2} + h.o.t..
\end{align*}
\tag{72}
\]
Following the definitions of (71), the coefficients in (72) can be calculated out by the following explicit expressions.

**Lemma 3.1.** For $k_2 = 0$, $k_1 \neq 0$ and Neumann boundary condition on spatial domain $\Omega = (0, l\pi)$, $l > 0$, the parameters $a_1(\alpha)$, $a_2(\alpha)$, $a_{200}$, $a_{011}$, $a_{300}$, $a_{111}$, $b_{110}$, $b_{210}$, $b_{021}$ in (72) are
\[
\begin{align*}
a_1(\alpha) = & \; \frac{1}{2} \psi_1(0)(\tilde{L}_1(\alpha)\phi_1 - \mu_1 D_1(\alpha)\phi_1(0)), \\
b_2(\alpha) = & \; \frac{1}{2} \psi_2(0)(\tilde{L}_1(\alpha)\phi_2 - \mu_2 D_1(\alpha)\phi_2(0)), \\
a_{200} = & \; a_{011} = b_{110} = 0, \\
a_{300} = & \; \frac{1}{4} \psi_1(0)C_{\phi_1\phi_1\phi_1} + \frac{1}{\omega_0} \psi_1(0) Re(iQ_{\phi_1\phi_1\phi_2}(0))Q_{\phi_1\phi_1} + \psi_1(0)Q_{\phi_1}(h_{200}^0 + \frac{1}{\sqrt{2}} h_{200}^{k_1}), \\
a_{111} = & \; \psi_1(0)C_{\phi_1\phi_1\phi_2} + \frac{2}{\omega_0} \psi_1(0) Re(iQ_{\phi_1\phi_2\phi_2}(0))Q_{\phi_2\phi_2} \\
& \quad + \psi_1(0)[Q_{\phi_1}(h_{011}^0 + \frac{1}{2} h_{201}^{k_1}) + Q_{\phi_2} h_{110}^{k_1} + Q_{\phi_3} h_{110}^{k_1}], \\
b_{210} = & \; \frac{1}{2} \psi_2(0)C_{\phi_1\phi_1\phi_2} + \psi_2(0)(Q_{\phi_1} h_{110}^0 + Q_{\phi_2} h_{200}^0) \\
& \quad + \frac{1}{2\omega_0} \psi_2(0) \{ 2Q_{\phi_1\phi_2} \psi_1(0)Q_{\phi_1\phi_2} + [-Q_{\phi_2\phi_2} \psi_2(0) + Q_{\phi_2\phi_2} \bar{\psi}_2(0)]Q_{\phi_1\phi_1} \}, \\
b_{021} = & \; \frac{1}{2} \psi_2(0)C_{\phi_1\phi_1\phi_2} + \psi_2(0)(Q_{\phi_1} h_{011}^0 + Q_{\phi_2} h_{020}^0) \\
& \quad + \frac{1}{4\omega_0} \psi_2(0) \{ \frac{2}{3} Q_{\phi_2\phi_2} \bar{\psi}_2(0)Q_{\phi_2\phi_2} + [-2Q_{\phi_2\phi_2} \psi_2(0) + 4Q_{\phi_2\phi_2} \bar{\psi}_2(0)]Q_{\phi_2\phi_2} \}.
\end{align*}
\]
Thus, from Lemma 2.3 and Theorems 2.6, 2.8, we know that $(1, \tau, d)$ forms of system (44) near $(1, \tau, d)$, then we perform the procedures (in subsection 3.1) of calculating third order normal forms of system (44) near $(1, \tau, d)$.

Remark 6. Although formulations in Lemma 3.1 are similar to the one of Proposition 4.3 in [20], recalculating the coefficients in (72) is necessary since it can be affected by nonlocal terms in (44) in reduction process. And for other cases (not the case $k_2 = 0, k_1 \neq 0$), explicit formulas in Lemma 3.1 may also be different. Despite all this, we provide the method to calculate explicit third-order normal forms, which can also be applied to the system with nonlocal terms similar to (1).

3.2. Spatiotemporal dynamics. In this part, based on the explicit formulas in subsection 3.1, we take concrete system parameters to show the complex spatiotemporal dynamical behaviors of system (44) near $(1, 0)$--mode Turing-Hopf singularity point.

Let $d_2 = 0.6$ and other system parameters be the same as ones in Table 1, and we obtain

$$F(w) = w^8 + 21.3843w^6 + 122.500w^4 + 87.252w^2 - 44.992.$$  \hspace{1cm} (73)

By using the mathematical computation software Maple, $F(w) = 0$ has a unique positive real root $w_\ast = 0.584588$ (see Figure 1) and $F'(w_\ast) > 0$. Obviously, we have $R(w) = w^6 + 13.2748w^2 + 3.75391 > 0$ and $I_w = \{w_\ast\}$. Thus it suffices to determine the value of $\tau_{w_\ast}$. By (18), (19) and Lemma 2.3, $\tau_{w_\ast} = \tau_0(\tau_{w_\ast}) = \frac{\theta(\tau_{w_\ast})}{w_\ast} = 1.28315$.

Next, we only have to derive $d_1^2(d_2)$. 

$$d_1^2(d_2) = \frac{u_1^*u_2^*(a_{12}a_{21} - a_{11}a_{22}) - d_2a_{11}u_1^*}{(d_2 + a_{22}u_2^*)} = 0.680628.$$ \hspace{1cm} (74)

Thus, from Lemma 2.3 and Theorems 2.6, 2.8, we know that $(\tau, d_1) = (\tau_{w_\ast}, d_1^2(d_2)) = (1.28315, 0.680628)$ is the $(1, 0)$--mode Turing-Hopf bifurcation point of system (1). Then we perform the procedures (in subsection 3.1) of calculating third order normal forms of system (44) near $(1, 0)$--mode Turing-Hopf singularity point $(\tau, d_1) =$
All constant equilibria of the (76) are reversing time bifurcation curves are respectively. and a pair of spatially inhomogeneous periodic orbits of original system (44) respectively homogeneous periodic orbit, a pair of spatially inhomogeneous steady states where

By taking cylindrical coordinate transformations $z = r, z = \rho \cos \theta + i \rho \sin \theta$ and reversing time $t$ for (75), we derive that

$$\begin{cases} \hat{r} = (0.526074\alpha_1 + 0.0261317\alpha_2)r + 4.50293\rho^3 + 7.17086r\rho^2, \\ \hat{\rho} = -0.557233\alpha_2\rho - 3.02401\sqrt{r^2}\rho + 4.21044\rho^3. \end{cases} \quad (76)$$

All constant equilibria of the (76) are

$$E_0 = (0,0),$$
$$E_1 = (0, \sqrt{0.557233\alpha_2}), \quad \alpha_2 > 0,$$
$$E_2^\pm = (\pm \sqrt{-0.526074\alpha_1 - 0.0261317\alpha_2}, 0), \quad \alpha_2 < -20.1316\alpha_1,$$
$$E_3^\pm = (\pm \sqrt{-0.245398\alpha_1 - 0.454887\alpha_2, \sqrt{0.251747\alpha_2 - 0.164801\alpha_1}}, 0.654629\alpha_1 < \alpha_2 < -0.53947\alpha_1,$$

where $E_0, E_1, E_2^\pm$ and $E_3^\pm$ of (76) correspond to the coexistence equilibrium, a spatially homogeneous periodic orbit, a pair of spatially inhomogeneous steady states and a pair of spatially inhomogeneous periodic orbits of original system (44) respectively.

From direct calculations and [16], the unfolding of (76) is case II and all pitch-fork bifurcation curves are

$$H_0 : \alpha_2 = 0,$$
$$L_1 : \alpha_2 = -20.1316\alpha_1,$$
$$T_1 : \alpha_2 = -0.53947\alpha_1,$$
$$T_2 : \alpha_2 = 0.654629\alpha_1,$$

where $H_0$ and $L_1$ are also the Hopf bifurcation and Turing bifurcation curves for the original reaction-diffusion systems respectively.

Then we give the bifurcation sets and phase portraits in Figure 3. For convenience, recording all fixed system parameters in Table 2. Therefore, we have

**Proposition 1.** Assume that system parameters are fixed as Table 2 shows. If $(\tau, d_1)$ are chosen in the neighborhood of $(\tau_w, d_1(d_2))$, then system (44) exhibits the

![Figure 3](image-url)
Table 2. Values of system parameters

| Parameters | $d_2$ | $r_1$ | $r_2$ | $a_{11}$ | $a_{12}$ | $a_{21}$ | $a_{22}$ | $b_{11}$ | $b_{12}$ | $b_{21}$ | $b_{22}$ |
|------------|-------|-------|-------|----------|----------|----------|----------|----------|----------|----------|----------|
| Values     | 0.6   | 5     | 5     | 4        | 7        | 6        | 5        | 3        | 0.5      | 4        |          |

following complex dynamics

1. When $(\tau, d_1) \in D_1$, the coexistence equilibrium $(u_1^*, u_2^*)$ is locally asymptotic stable;
2. When $(\tau, d_1) \in D_2$, a stable spatially homogeneous periodic orbit bifurcating from $(u_1^*, u_2^*)$;
3. When $(\tau, d_1) \in D_3$, a spatially homogeneous periodic orbit remains stable and a pair of unstable spatially inhomogeneous steady states occurs;
4. When $(\tau, d_1) \in D_4$, a pair of spatially inhomogeneous periodic orbits bifurcating from a pair of spatially homogeneous periodic orbits is stable, which indicates that $D_4$ is a bistable region;
5. When $(\tau, d_1) \in D_5$, a pair of spatially inhomogeneous periodic orbits remains stable and a spatially homogeneous periodic orbit disappears, which implies that $D_5$ is a bistable region;
6. When $(\tau, d_1) \in D_6$, a pair of spatially inhomogeneous periodic orbits disappears and a pair of spatial inhomogeneous steady states becomes locally asymptotic stable. Thus $D_6$ is a bistable region.

We verify the complex pattern formations near the Turing-Hopf bifurcation point $(\tau_{w^*}, d_1^*(d_2))$ in Proposition 1 by simulations, where the phenomena in regions $D_3$ and $D_5$ are omitted since similar patterns are displayed in regions $D_2$ and $D_4$, see figures 4-7.

![Figure 4](image-url)

**Figure 4.** When the initial values $u_1(t, x) = 0.45 - 0.01, u_2(t, x) = 0.27 - 0.01, t \in [-\tau, 0]$ and parameters $(\tau, d_1) = (\tau_{w^*}, d_1^*(d_2)) + (-0.01, 0.01) \in D_1$, the coexistence equilibrium is stable.
When the initial values \( u_1(t, x) = 0.45 - 0.01, u_2(t, x) = 0.27 - 0.01, t \in [-\tau, 0] \) and parameters \( (\tau, d_1) = (\tau_{w_0}, d_1^1(d_2)) + (0.01, 0.01) \in D_2 \), a spatial homogeneous periodic orbit is stable.

For \( (\tau, d_1) = (\tau_{w_0}, d_1^1(d_2)) + (0.003, -0.01) \in D_4 \), (a), (b) with initial values \( u_1(t, x) = 0.45 - 0.01 \cos x, u_2(t, x) = 0.27 - 0.01 \cos x \) and (c), (d) with initial values \( u_1(t, x) = 0.45 + 0.01 \cos x, u_2(t, x) = 0.27 + 0.01 \cos x, \) \( t \in [-\tau, 0] \). A pair of spatial inhomogeneous periodic orbits is stable, which indicates \( D_4 \) is a bistable region.

4. Conclusions and discussions. When combining the results in [8], we know that the strength of nonlocal intraspecific competitions play an important role in result in complex spatiotemporal pattern formations to the Lotka-Volterra competition system with nonlocal delays in the weak competition case, but the exclusion principle (see [13]) is still being preserved if the strength of nonlocal intraspecific competitions is sufficiently weak regardless of the delay effects or nonlocal kernels satisfied with assumptions on the spatiotemporal kernels in [8]. Moreover, the complex spatiotemporal patterns induced by Turing-Hopf bifurcations are proved in
mathematical way and verified by numerical results, which is very interesting and can be a new guide for investigating on dynamics in competition systems.

Furthermore, we believe the coexistence equilibrium $E^*$ is globally asymptotic stable if delay $\tau < \tau_{w^*}$ and $(d_2,d_1) \in \Lambda_0$ (see Figure 2) despite lack of rigorously proofs here. If the delay $\tau$ in each nonlocal term can be chosen different or the non-local competitions vary in different spatial positions for system (1), more complex spatiotemporal patterns, like spatially inhomogeneous quasi-periodic orbits may be formed.

![Figure 7](https://example.com/figure7.png)

**Figure 7.** For $(\tau,d_1) = (\tau_{w^*},d_1^*(d_2)) + (-0.01,-0.01) \in \mathcal{D}_6$, (a), (b) with initial values $u_1(t,x) = 0.45 - 0.01 \cos x$, $u_2(t,x) = 0.27 - 0.01 \cos x$ and (c), (d) with initial values $u_1(t,x) = 0.45 + 0.01 \cos x$, $u_2(t,x) = 0.27 + 0.01 \cos x$, $t \in [-\tau,0]$. A pair of spatial inhomogeneous steady states is stable, which indicates $\mathcal{D}_6$ is a bistable region.

**Acknowledgments.** We would like to thank the anonymous reviewer for their valuable comments.

**REFERENCES**

[1] Q. An and W. Jiang, Spatiotemporal attractors generated by the Turing-Hopf bifurcation in a time-delayed reaction-diffusion system, *Discrete Contin. Dyn. Syst. Ser. B*, 24 (2019), 487–510.

[2] E. Beretta and Y. Tang, Extension of a geometric stability switch criterion, *Funkc. Ekvacioj*, 46 (2003), 337–361.

[3] N. F. Britton, Spatial structures and periodic travelling waves in an integro-differential reaction-diffusion population model, *SIAM J. Appl. Math.*, 50 (1990), 1663–1688.

[4] X. Cao and W. Jiang, Turing-Hopf bifurcation and spatiotemporal patterns in a diffusive predator-prey system with Crowley-Martin functional response, *Nonlinear Anal. Real World Appl.*, 43 (2018), 428–450.
[5] S. Chen and J. Shi, Global dynamics of the diffusive Lotka-Volterra competition model with stage structure, *Calculus of Variations and Partial Differential Equations*, 59 (2020), Article number: 33.

[6] S. Chen and J. Shi, Stability and Hopf bifurcation in a diffusive logistic population model with nonlocal delay effect, *J. Differ. Equ.*, 253 (2012), 3440–3470.

[7] X. Chen and W. Jiang, Turing-Hopf bifurcation and multi-stable spatio-temporal patterns in the Lengyel-Epstein system, *Nonlinear Anal. Real World Appl.*, 49 (2019), 386–404.

[8] X. Chen, W. Jiang and S. Ruan, Global dynamics and complex patterns in Lotka-Volterra systems: The effects of both local and nonlocal intraspecific and interspecific competitions, To appear.

[9] J. Dockery, V. Hutson, K. Mischaikow and M. Pernarowski, The evolution of slow dispersal rates: A reaction diffusion model, *J. Math. Biol.*, 37 (1998), 61–83.

[10] H. I. Freedman and Y. Kuang, Stability switches in linear scalar neutral delay equations, *Funkc. Ekvacioj*, 34 (1991), 187–209.

[11] J. Furter and M. Grinfeld, Local vs. non-local interactions in population dynamics, *J. Math. Biol.*, 27 (1989), 65–80.

[12] S. A. Gourley and N. F. Britton, A predator-prey reaction-diffusion system with nonlocal effects, *J. Math. Biol.*, 34 (1996), 297–333.

[13] S. A. Gourley and S. Ruan, Convergence and travelling fronts in functional differential equations with nonlocal terms: A competition model, *SIAM J. Math. Anal.*, 35 (2003), 806–822.

[14] S. A. Gourley and J. W.-H. So, Dynamics of a food-limited population model incorporating nonlocal delays on a finite domain, *J. Math. Biol.*, 44 (2002), 49–78.

[15] S. A. Gourley, J. W.-H. So and J. Wu, Nonlocality of reaction-diffusion equations induced by delay: biological modeling and nonlinear dynamics, *J. Math. Sci.*, 124 (2004), 5119–5153.

[16] J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer-Verlag, New York, 1990.

[17] S. Guo and S. Yan, Hopf bifurcation in a diffusive Lotka-Volterra type system with nonlocal delay effect, *J. Differ. Equ.*, 260 (2016), 781–817.

[18] X. He and W.-M. Ni, Global dynamics of the Lotka-Volterra competition-diffusion system: Diffusion and spatial heterogeneity I, *Commun. Pure Appl. Math.*, 69 (2016), 981–1014.

[19] R. Hu and Y. Yuan, Spatially nonhomogeneous equilibrium in a reaction–diffusion system with distributed delay, *J. Differ. Equ.*, 250 (2011), 2779–2806.

[20] W. Jiang, Q. An and J. Shi, Formulation of the normal form of Turing-Hopf bifurcation in partial functional differential equations, *J. Differ. Equ.*, 268 (2019), 6067–6102.

[21] W. Jiang, H. Wang and X. Cao, Turing instability and Turing-Hopf bifurcation in diffusive schnakenberg systems with gene expression time delay, *J. Dyn. Differ. Equ.*, 31 (2019), 2223–2247.

[22] Y. Kuang and H. L. Smith, Convergence in Lotka-Volterra typediffusive delay systems with outwitting instantaneous negative feedbacks, *J. Austral. Math. Soc. Ser. B*, 34 (1993), 471–493.

[23] A. J. Lotka, *Elements of Physical Biology*, Williams and Wilkins, New York, 1925.

[24] Y. Lou, On the effects of migration and spatial heterogeneity on single and multiple species, *J. Differ. Equ.*, 223 (2006), 400–426.

[25] Y. Lou and P. Zhou, Evolution of dispersal in advective homogeneous environment: the effect of boundary conditions, *J. Differ. Equ.*, 259 (2015), 141–171.

[26] W. Ni, J. Shi and M. Wang, Global stability and pattern formation in a nonlocal diffusive Lotka–Volterra competition model, *J. Differ. Equ.*, 264 (2018), 6891–6932.

[27] S. Pal, S. Petrovskii, S. Ghori and M. Banerjee, Spatiotemporal pattern formation in 2d prey-predator system with nonlocal intraspecific competition, *Commun. Nonlinear Sci. Numer. Simul.*, 93 (2021), 105478, 15pp.

[28] C. V. Pao, Global asymptotic stability of Lotka-Volterra competition systems with diffusion and time delays, *Nonlinear Anal. Real World Appl.*, 5 (2004), 91–104.

[29] S. Ruan and J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10 (2003), 863–874.

[30] V. P. Shukla, Conditions for global stability of two-species population models with discrete time delay, *Bull. Math. Biol.*, 45 (1983), 793–805.

[31] Y. Song, M. Han and Y. Peng, Stability and Hopf bifurcations in a competitive Lotka-Volterra system with two delays, *Chaos Solitons Fract.*, 22 (2004), 1139–1148.
[32] Y. Song, H. Jiang, Q. Liu and Y. Yuan, Spatiotemporal dynamics of the diffusive mussel-algae model near turing-hopf bifurcation, SIAM J. Appl. Dyn. Syst., 16 (2017), 2030–2062.

[33] Y. Song, T. Zhang and Y. Peng, Turing-Hopf bifurcation in the reaction-diffusion equations and its applications, Commun. Nonlinear Sci. Numer. Simul., 33 (2016), 229–258.

[34] Y. Tang and L. Zhou, Hopf bifurcation and stability of a competition diffusion system with distributed delay, Publ. RIMS, Kyoto Univ., 41 (2005), 579–597.

[35] V. Volterra, Variazionie fluttuazioni del numero d’individui in specie animali conviventi, Mem. Acad. Licei., 2 (1926), 31–113.

[36] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

[37] Y. Yamada, Asymptotic stability for some systems of semilinear Volterra diffusion equations, J. Differ. Equ., 52 (1984), 295–326.

[38] Y. Yamada, On logistic diffusion equations with nonlocal interaction terms, Nonlinear Anal.-Theory Methods Appl., 118 (2015), 51–62.

[39] J. Zhang, W. Li and X. Yan, Bifurcation and spatiotemporal patterns in a homogeneous diffusion-competition system with delays, Int. J. Biomath., 5 (2012), 1250049, 23pp.

Received October 2020; revised November 2020.

E-mail address: xianyongchenmath@aliyun.com
E-mail address: jiangwh@hit.edu.cn