Neutron stars in the large-$N_c$ limit

Francesco Giacosa$^{(1,2)}$ and Giuseppe Pagliara$^{(3)}$

(1) Institute of Physics, Jan Kochanowski University, ul. Swietokrzyska 15, 25-406 Kielce, Poland
(2) Institute for Theoretical Physics, J. W. Goethe University, Max-von-Laue-Str. 1, 60438 Frankfurt am Main, Germany
(3) Dip. di Fisica e Scienze della Terra dell’Universita’ di Ferrara and INFN Sez. di Ferrara, Via Saragat 1, I-44100 Ferrara, Italy

We study the phase transition from dense baryonic matter to dense quark matter within the large-$N_c$ limit. By using simple constant speed of sound equations of state for the two phases, we derive the scaling with $N_c$ of the critical quark chemical potential $\mu^\text{crit}$ for this phase transition. While quark matter is strongly suppressed at large $N_c$, the phase transition at a large but finite density could nevertheless be important to determine the maximum mass of compact stars. In particular, in the range $3 \lesssim N_c \lesssim 5.5$ the quark phase would take place in compact stars and would lead to the formation of an unstable branch of hybrid stars. As a consequence, the maximum mass is restricted to the range $2.1M_\odot < M_{\text{max}} < 3M_\odot$. For larger value of $N_c$, the phase transition would occur at densities too high to be reached in the core of compact stars. However, the very requirement that it occurs (although at a very large density) translates into interesting constraints on the stiffness of the baryonic phase: its speed of sound must exceed $\sqrt{1/3}$.

PACS numbers: 25.75.Nq,26.60.Kp,11.15.Pg

I. INTRODUCTION

The so-called large-$N_c$ method, where $N_c$ stands for the number of color in Quantum Chromodynamics (QCD), is a useful nonperturbative approach to investigate strong interactions in the nonperturbative regime of QCD [1–4]. Namely, when $N_c$ is large, a series of simplifications occurs: quark-antiquark mesons and glueballs retain their masses but become very narrow, while baryon masses increase with $N_c$, due to the fact that (at least) $N_c$ quarks are needed to form a white state. The large-$N_c$ approach has been used to understand some phenomenological features such as the OZI rule and it has been applied in effective approaches of QCD in order to distinguish leading from sub-leading terms.

More recently, the large-$N_c$ method has been also used to study some features of the phase diagram of QCD [5–11]. Indeed it represents an additional possible tool, often combined with effective models, to describe (very) dense and hot matter (for a review of various approaches, see for instance Ref. [12]). In particular, in the pioneering work of Ref. [5], McLerran and Pisarski make interesting observations concerning the nature of a dense medium of hadrons: a so-called quarkyonic phase, which is confined but chirally restored and with a pressure proportional to $N_c$, is realized as an intermediate phase between a ‘standard’ baryonic phase at small density and a ‘standard’ quark-gluon phase at very high density. In a later study of the subject [6], the emergence of inhomogeneous condensations in the quarkyonic phase was found to be favorable (this is in agreement with the recent results of Refs. [13–16]).

The natural laboratory to test the properties of dense and strongly interacting matter is the core of neutron stars. In these stellar objects the central density can reach values up to ten times the nuclear matter density and it is therefore conceivable that, besides nucleons, also heavier baryons can take place, such as hyperons [17,18] and delta resonances [19], or that the phase transition to quark matter occurs. In this respect, the well-established existence of neutron stars with masses of $2M_\odot$ (thus significantly larger than the canonical value of $1.4M_\odot$) offers a unique opportunity to test the stiffness of the equation of state at high baryon densities. The question about the internal composition of these massive stars is the subject of intense theoretical and experimental/observational studies. At the moment, no firm conclusion for instance can be drawn on whether hyperons or deltas do form in such systems. While some calculations suggest that even in presence of those particles the equation of state can be stiff enough to support $2M_\odot$ [18,20], other calculations find that the threshold density for the appearance of hyperons is too large for those particles to appear even in massive compact stars [21]. Instead, other approaches find that the appearance of hyperons/delta soften too much the equation of state and that massive objects should contain quark matter, see for instance [22]. An alternative scenario, which is supported by the indication of the existence of very compact objects (which need to be confirmed by new observations, see [23]), is that two families of stars co-exist, baryonic stars and pure quark stars, see Refs. [24–26]. The possibility of the occurrence of the phase transition to quark matter inside a compact star (which would be then a hybrid star) is, similarly to the case of hyperons and deltas, quite unsettled due to uncertainties which affect both the baryonic matter equation of state and the quark matter equation of state. Different possibilities and scenarios have been analyzed, see [27–31].

Here we want to address the question about the phase transition to quark matter in compact stars in the spirit of the large-$N_c$ limit and by using the presently known constraints on the maximum mass of compact stars. The advantage of the large $N_c$ approach is that the equation of state of quark matter can be modeled by a simple prescription in which the speed of sound is constant and equal to $1/\sqrt{3}$. Of course, we will need also to establish the scaling with $N_c$ of the baryonic matter equation of state to find the critical density for the transition. We will use also in this case a prescription based on equations of state with constant speed of sound in the regime of densities larger than about two times the nuclear saturation energy den-
sity. At large $N_c$ the quark phase is clearly suppressed, as found in [3] but, as we will see, it could nevertheless play an important role in determining the maximum mass of neutron stars: a phase transition to quark matter occurring at large densities would indeed lead to unstable hybrid stars configurations. The critical density for the phase transition would correspond therefore to the central density of the neutron star with the highest possible mass.

The paper is organized as follows: in Sec. II we discuss the quark phase, the baryonic phase, and the first-order phase transition in the context of the large-$N_c$ limit. In Sec. III we study the maximum mass of neutron stars. Finally, in Sec. IV we present our conclusions.

II. EQUATIONS OF STATE AT LARGE $N_c$

The study we are going to describe is performed by using, as customary, two different models for the baryonic and the quark phase which are matched by means of a Maxwell construction. In both cases we will present simple parametrizations which however capture the main physical ingredients of the (yet unknown) equation of state.

A. Quark phase

For modeling the equation of state of the quark phase we consider a free gas of fermions with the additional contribution of a nonperturbative vacuum pressure constant. Hence, the pressure $p_q$ reads:

$$p_q = b_1 N_c \mu_q^4 - N_c^2 B$$  \hspace{1cm} (1)

where $b_1$, and $B$ are constants (independent on $N_c$ in the large-$N_c$ limit) and $\mu_q$ is the quark chemical potential. The first term stands for the kinetic contribution of quarks:

$$b_1 = \frac{N_f}{12\pi^2}.$$  \hspace{1cm} (2)

For three massless flavors one has $b_1 = 1/(4\pi^2)$: this is the case relevant for compact stars. We do not include the perturbative $\alpha_s$ corrections because they are vanishingly small in the large-$N_c$ limit. The second term represents the nonperturbative contribution provided by the vacuum pressure of QCD. This contribution is connected to the gluon condensate and therefore takes a degeneracy factor of $N_c^2$ (see the Appendix). In this way, along the temperature axis of the QCD phase diagram, one obtains a critical temperature that is large-$N_c$ independent, in agreement with basic expectations of QCD [52], with models implementing a bag constant [33, 34], as well as lattice simulations [35]. Using the dilaton potential (presented in the Appendix), one finds $\sqrt{3}B^{1/4} \sim 200$ MeV (the factor $\sqrt{3}$ is due to the normalization adopted in Eq. (1)). The value is somewhat larger than the values usually adopted within the MIT bag model [36]. However, for what concerns the large-$N_c$ limit in the context of the MIT bag model, care is needed. Namely, the MIT bag constant must be $N_c$ independent in order to reproduce the correct scaling of the masses of hadrons (see the detailed discussion in Refs. [57, 58]). For our purposes, we are interested in the thermodynamic behavior of a gas of quarks (and gluons), hence the bag is directly related to the nonperturbative QCD vacuum which scales as $N_c^2$ (see detailed discussion in the Appendix).

It is easy to show that the equation of state (1) allows for the existence of bound quark matter: the pressure indeed vanishes for

$$p_q = 0 \Leftrightarrow \mu_q = \left( \frac{B}{b_1} N_c \right)^{1/4} \propto N_c^{1/4}. \hspace{1cm} (3)$$

By using the thermodynamics relation $p_q = \mu_q n_q - \varepsilon_q$, where $n_q$ is the quark density and $\varepsilon_q$ the quark energy density, one finds that the energy per baryon for this type of bound quark matter $(\varepsilon/n)_q \propto N_c^{5/4}$ whereas its baryon density $n_b = n_q/N_c \propto N_c^{3/4}$ (see detailed discussion in the Appendix).

Notice that the energy per baryon of bound quark matter grows with $N_c$ faster than the baryon mass (which scales as $N_c$). This implies that the so-called Witten hypothesis on the absolute stability of (strange) quark matter [53] is not fulfilled in this limit. On the other hand, as shown in [8], in Walecka type models for nuclear matter, the energy per baryon of bound nuclear matter scales as $N_c$ if the $\sigma \equiv f_0(500)$ meson, responsible for the binding of nuclear matter, is interpreted as a quarkonium state. Instead, within the four-quark assignment of the $\sigma \equiv f_0(500)$ meson [16, 10, 41] bound nuclear matter ceases to exist already for $N_c = 4$, as found in Ref. [8]. In this scenario, the only bound state of strongly interacting matter would be realized in the quark phase. Yet, the corresponding phase would be metastable. This is due to the fact that, even in the absence of stable nuclear matter, a Fermi gas of nucleons sets in at $\mu_q \propto N_c^0$, while $\varepsilon_q$ grows with $n_q$ faster than the baryon mass (which scales as $N_c$). In this scenario, the only bound state of strongly interacting matter would be realized in the quark phase. Yet, the corresponding phase would be metastable. This is due to the fact that, even in the absence of stable nuclear matter, a Fermi gas of nucleons sets in at $\mu_q \propto N_c^0$ (when nuclear matter is realized, one has that $\mu_q \propto N_c^0$, hence the Fermi gas). Hence, a gas of nucleons is favored w.r.t. the formation of bound quark matter.

Moreover, even when $f_0(500)$ is predominantly a four-quark state, there is another possibility which needs to be further investigated in the future: at some large values of $N_c$ a new type of (loosely) bound nuclear matter takes place again thanks to pion exchange [42]. Namely, in the large-$N_c$ limit, the pion potential becomes a binding potential (the pion mass does not scale, but the nucleon mass increases linearly with $N_c$). In this scenario, which could not be found in Ref. [27] due to the employed mean-field approximation, nuclear matter would not exist between $N_c = 4$ up to a maximal value which needs to be determined in a future work. Also in this case, the Witten’s hypothesis is not realized in the large-$N_c$ limit.

We thus conclude that the possibility of stable quark matter does not take place at large-$N_c$ but still could be a specific feature of our “small” $N_c$ world.

B. Baryonic phase

The equation of state of baryonic matter at high density is also quite uncertain due to the intrinsic difficulties of solving the nuclear many body problem. A widely used approach is based on relativistic mean field Lagrangians (similar to the Walecka model) with parameters which are...
fixed by using the experimental constraints on symmetric nuclear matter. An updated parametrization of this class of models is the SFHo model of Ref. [13] in which also recent constraints on the symmetry energy are fulfilled. It is common to consider the results for the equation of state as computed in these type of models to be reliable up to energy densities not larger than about twice saturation density. Beyond that value the equation of state is completely unknown. We use here a simple approach that has been used in several papers, see [44–46]: we adopt the equation of state obtained within the SFHo model up to twice saturation energy density $2\varepsilon_0$ and for larger densities we use a constant speed of sound equation of state whose pressure is given as a function of the baryonic chemical potential $\mu_b$ by the relation:

$$p_b = a_1 \mu_b^\alpha - K ,$$  \hspace{1cm} (4)

where the constants $a_1$ and $K$ are fixed by matching this simple Ansatz with the SFHo model at $2\varepsilon_0$ (i.e. by requiring that at that value of the energy density the pressure and the baryon density are continuous; note, $K$ turns out to be positive). The constants $a_1$ and $K$ scale in general with $N_c$ and we need to fix such scaling behaviors in order to construct the phase transition with quark matter at large $N_c$. In order to determine $a_1 \equiv a_1(N_c)$, let us consider baryons as interacting by the exchange of conventional vector mesons with mass $m_V \propto N_c^2$ and with the dimensionless coupling constant $g_V \propto \sqrt{N_c}$ [Note, the same conclusion would be reached by any quark-antiquark mesonic exchange. We use vector mesons for definiteness and because they are known to be important for the interaction among nucleons]. The vector interaction implies that the propagator

$$\frac{g_V^2}{m_V^2} \propto N_c$$  \hspace{1cm} (5)

enters into the expressions of pressure and energy density. The next step is to notice that the constant $a_1$ must have dimension $4-\alpha$, such that the pressure $p_b$ has the dimension energy$^4$. Hence, $a_1(N_c)$ must be of the type

$$a_1(N_c) \propto \left(\frac{g_V^2}{m_V^2}\right)^{\frac{\alpha - 4}{2}} \propto N_c^{\frac{\alpha - 4}{2}} .$$  \hspace{1cm} (6)

Hence

$$a_1(N_c) = \tilde{a}_1 N_c^{\frac{\alpha - 4}{2}} ,$$  \hspace{1cm} (7)

where $\tilde{a}_1$ is a large-$N_c$ independent constant with dimension $4-\alpha$ and can be fixed by using the $N_c = 3$ SFHo equation of state as explained before. Similarly, to fix the scaling of $K$ we assume that the quark chemical potential corresponding to the zero of the baryonic pressure is $N_c$ independent [71]. That fixes

$$K = \tilde{K} N_c^{(3\alpha - 4)/2} .$$  \hspace{1cm} (8)

As a consequence, in terms of $N_c$ and $\mu_q^\alpha$ the baryonic pressure reads:

$$p_b = \tilde{a}_1 N_c^{\frac{\alpha - 4}{2}} \mu_q^\alpha - \tilde{K} N_c^{\frac{\alpha - 4}{2}} .$$  \hspace{1cm} (9)

The baryon density $n_b$ and the energy density $\varepsilon_b$, read:

$$n_b = \frac{dp_b}{d\mu_b} = a_1 \alpha \mu_b^{\alpha - 1}$$  \hspace{1cm} (10)

$$\varepsilon_b = n_b \mu_b - p_b = a_1 (\alpha - 1) \mu_b^\alpha + K$$  \hspace{1cm} (11)

Thus, the constant speed of sound is:

$$v_b = \sqrt{\frac{dp_b}{d\varepsilon_b}} = \frac{1}{\sqrt{\alpha - 1}}$$  \hspace{1cm} (12)

By imposing causality, $v_b < 1$, one obtains a first constraint on $\alpha$:

$$\alpha \geq 2 .$$  \hspace{1cm} (13)

Note, the case $\alpha = 1$ would correspond to the non-causal excluded volume prescription [52, 53] for which the baryon density saturates to a constant value when increasing the baryon chemical potential. The stiffer equation of state corresponds, in agreement with causality, to $\alpha = 2$. Quite interestingly, in this limit the pressure is proportional to $N_c$:

$$p_b = \tilde{a}_1 N_c \mu_q^2 - \tilde{K} N_c .$$  \hspace{1cm} (14)

This result is in agreement with the large-$N_c$ equation of state of nuclear matter found in Ref. [8]. In this sense, the proportionality to $N_c$ would hold in a very large range of values for the chemical potential $\mu_q$. This property is also compatible with the quarkyonic phase introduced in Ref. [6], in which a confined, but chirally restored phase with a pressure proportional to $N_c$ is realized from intermediate up to high densities.

C. Quark-hadron first-order phase transition

We now turn to the phase transition from hadronic degrees of freedom to quark degrees of freedom. The very request of the existence of this transition sets a second constraint on the value of the parameter $\alpha$. Namely, the Maxwell construction reads:

$$p_b = \tilde{a}_1 N_c \frac{3\alpha - 4}{2} \mu_q^\alpha - \tilde{K} N_c \frac{3\alpha - 4}{2} = b_1 N_c \mu_q^4 - N_c^2 B = p_q .$$  \hspace{1cm} (15)

This equation shows that at large but finite $N_c$ and in the limit of large $\mu_q$ the quark phase is favored only if $\alpha < 4$. On the other hand, larger values of $\alpha$ would imply that asymptotically the baryonic phase is the favored phase (actually, as we will show in the following, one cannot even find a physical solution of Eq. (15) if $\alpha \geq 4$). Hence, we consider the limiting case $\alpha = 4$ as being excluded; in other terms, we assume that a first order phase transition to quark matter does occur at some large but finite density. Summarizing the constraints on $\alpha$ are:

$$2 \leq \alpha < 4 .$$  \hspace{1cm} (16)

Let us now determine the critical chemical potential $\mu_q^\text{crit}$ for the phase transition to quark matter as obtained by the Maxwell construction (i.e. by imposing that the pressures of the two phases are equal at fixed quark chemical potential). It reads:
\[
\mu_q^{\text{crit}} = \left( \frac{BN_c}{b_1} \right)^{1/4} [1 + \ldots] \text{ for } 2 \leq \alpha \leq \frac{16}{7} \quad (17)
\]

and

\[
\mu_q^{\text{crit}} = \left( \frac{\tilde{a}_1}{b_1} \right)^{1/4} N_c^{3a-6} \text{ for } \frac{16}{7} < \alpha < 4. \quad (18)
\]

Note, the limit \( \alpha \to 4^- \) implies \( \mu_q^{\text{crit}} \to \infty \), in agreement with Eq. (10).

(i) For \( \frac{16}{7} < \alpha < 4 \) the critical chemical potential grows as \( \mu_q^{\text{crit}} \propto N_c^{3/4} \), therefore it is indeed possible to neglect the vacuum pressure term of the quark pressure in Eq. (15).

(ii) For \( 2 \leq \alpha \leq \frac{16}{7} \) the vacuum pressure term is important, and the algebraic solution of \( \mu_q^{\text{crit}} \) is more complicated. However, it takes place just after \( \mu_q \) exceeds \( \mu_{q,0} \propto N_c^{1/4} \) from Eq. (3) [dots in Eq. (17) refer to large-\( N_c \) suppressed terms]. In fact, just after \( \mu_{q,0} \) the quark pressure becomes positive and grows with \( N_c^2 \), while the baryonic pressures grows as \( N_c^{-\frac{2a_0-8}{7}} \). One can easily see that the baryonic pressure grows slower than the quark pressure as function of \( N_c \) only if \( \frac{2a_0-8}{7} \leq 2 \), hence for \( \alpha \leq \frac{16}{7} \).

Finally, the pressure as function of the baryon density takes the form

\[
p_b = \lambda n_b^{\frac{a}{a_1-1}} \text{ with } \lambda = \frac{1}{a_1^{a_1-1} \alpha^{-\frac{a_1-1}{a_1-1}}}. \quad (19)
\]

Notice that the proportionality constant \( \lambda \) decreases for increasing \( a_1 \) (at fixed baryon density \( n_b \)):

We now study under which conditions the phase transition occurs in a neutron star. First, we need to determine the structure of neutron stars, in particular the relation between mass and central energy density, for different values of \( \alpha \). This is easily done by solving the Tolman-Oppenheimer-Volkoff structure equation and the results are displayed in Fig. 1. It is important to remark that in the structure equation only the relation between pressure and energy density enters, therefore the maximum mass (of neutron stars) does not scale with \( N_c \). A dependence on \( N_c \) emerges only in the case in which the maximum mass is actually determined by the phase transition to quark matter as we will discuss in the following.

Fig. 1 shows that the maximum mass \( M_{\text{max}} \) increases by decreasing the value of \( \alpha \) from \( M_{\text{max}} \sim 2.1M_{\odot} \) for \( \alpha = 3.5 \) to \( M_{\text{max}} \sim 3M_{\odot} \) for \( \alpha = 2 \). The range of values

\[
2 \leq \alpha \lesssim 3.5 \quad (20)
\]

is compatible with the existence of stars with masses of \( 2M_{\odot} \). It is interesting to notice that \( \alpha = 4 \) (thus speed of sound \( \sqrt{1/3} \)) would lead to a maximum mass smaller than the observational constraints, as found in the analysis of [46]. In particular, by taking into account the uncertainties on the equation of state of baryonic matter for densities below \( 2e_0 \), the central value of the maximum mass is of about \( 1.88M_{\odot} \) [46]. We remind that in our large-\( N_c \) scheme, the requirement that the phase transition to deconfined quark matter does take place at a certain critical density translates into the condition \( \alpha < 4 \), which in turn allows to obtain masses larger than \( 2M_{\odot} \) (see the example with \( \alpha = 3.5 \)). Therefore, even if the phase transition to deconfined quark matter does not take place in compact stars, its occurrence (at a certain large density) makes the existence of neutron stars as massive as \( 2M_{\odot} \) possible. This conclusion is quite remarkable and basically model independent.

We study now whether the critical density for the phase transition can be reached in the center of neutron stars. We notice first that the stiffer the baryonic equation of state, the earlier the phase transition to quark matter. On the other hand, as one can see from Fig. 1, the central density of the maximum mass configuration decreases when reducing the value of \( \alpha \). Whether the phase transition occurs or not depends on the relative magnitude of these two opposite effects. Moreover, the critical density will increase with \( N_c \) because the larger \( N_c \) the more unfavoured the quark matter equation of state.

Let us fix first \( N_c = 3 \). By solving equation (15) and by using equation (11) (we set \( \sqrt{3B^{1/4}} = 200 \text{ MeV} \) for the present discussion, see Appendix), one can compute the critical energy density of the baryonic phase corresponding to the onset of the phase transition (i.e. the onset of the mixed phase). The full points labeled with \( N_c = 3 \) correspond to such critical density: for \( \alpha = 3.5 \) the phase transition would occur within stars with masses above \( 1.9M_{\odot} \). On the other hand, for \( \alpha = 2 \), only for masses above \( 2.1M_{\odot} \) the phase transition takes place. These possible phase transitions occur always at a quite large value of the density and the formation of quark matter (in particular the mixed phase) makes the equation of state so soft (due to the jump to a speed of sound of \( \sqrt{1/3} \) in the pure quark phase) that only unstable hybrid stars branches are obtained [48], see also [49] (case A). The onset of the phase transition would therefore correspond to the maximum mass configuration. In this respect, even if quark matter does not form in stable stellar objects, its appearance determines the maximum mass of baryonic stars. From the \( N_c = 3 \) analysis, one further reduces the range (20), namely \( \alpha \) must be smaller than about \( \alpha_{\text{max}} \approx 2.5 \) (the blue line) in order to explain the existence of \( 2M_{\odot} \) stars:

\[
2 \leq \alpha \lesssim 2.5 \text{ for } N_c = 3 . \quad (21)
\]

Moreover one should not observe stars with masses larger than about \( 2.1M_{\odot} \) (in the case in which the equation of state is the stiffest, black line). Of course, this conclusion depends on the value of the vacuum constant (the adopted value is \( B^{1/4} \approx 200 \text{ MeV} \), see Appendix A) and the value of \( N_c \), which is fixed to three in this analysis.

Clearly, a large \( N_c \) analysis with \( N_c = 3 \) is questionable (although for some specific baryonic observables even \( N_c = 3 \) is large) and the simple equation of state adopted...
for quark matter should be regarded with care. Indeed, it has been shown in the literature that perturbative corrections are responsible for a significant modification of the quark matter equation of state with respect to the simple prescription here adopted. One should therefore study the effect of increasing $N_c$ thus making perturbative corrections smaller and smaller. In Fig. 1, we indicate the values of $N_c$ for which (for each value of $\alpha$) the phase transition would occur at the center of the maximum mass configuration (see the filled dashed points). One can notice that for $\alpha = 2$, this value is close to $N_c \approx 5.5$ and decreases to $N_c \approx 3.4$ for $\alpha = 3.5$. Therefore a clear conclusion can be drawn: it is enough to fix a value of $N_c \geq 5.5$ to rule out completely the appearance of deconfined quark matter in compact stars. As a last point, to study the effect of the adopted value of the vacuum pressure constant, we have then set $N_c = 4$ and we have determined the critical mass $M_{\text{crit}}$ for the occurrence of the phase transition as a function of $\sqrt[4]{3 B}$ in the cases $\alpha = 2$ and $\alpha = 2.5$, see insert in Fig. 1. Even for the smallest value of $\sqrt[4]{3 B} \approx 165 \text{MeV}$ (hence, in the most favored case for quark matter), the phase transition would occur for masses larger than about $2.4 M_\odot$, and thus safely above $2M_\odot$. It is clear than that by just increasing $N_c$ from 3 to 4, the appearance of quark matter in compact stars becomes unlikely, unless future measurements would find compact stars with masses above $2.4 M_\odot$. Conversely, if the presence of quark matter (possibly in the form of stable strange quark matter) will be proven via other astrophysical measurements e.g. precise radii measurements \cite{24,25}, gravitational waves measurements \cite{51} and gamma-ray-bursts observations \cite{26,52,54} then the occurrence of this transition in compact stars would be a particular phenomenon of our $N_c = 3$ world, such as the existence of bound nuclear matter as remarked in \cite{8}.

IV. CONCLUSIONS

We have studied the phase transition from baryonic matter to deconfined quark matter at high density by using equations of states that, both in the quark- and in the baryonic phases, are motivated by the large-$N_c$ expansion in QCD. For the baryonic phase, we adopt (for energies above twice nuclear matter saturation energy density) a constant speed of sound equation of state in which the baryonic pressure is parametrized as follows $p_b \propto \mu_b^4$ where the parameter $\alpha$ determines the speed of sound $v_b = \frac{1}{\sqrt{(\alpha-1)}}$. For the quark-phase, we have used a free gas of quarks with the important inclusion of a gluon- (or glueball-)driven vacuum pressure constant which scales as $N_c^2$ (details are explained in Appendix A).

By imposing causality ($v_b \leq 1$) and by assuming that a first order phase transition between baryonic matter and deconfined quark matter does occur at a certain finite density, one obtains the following constraints on the values of $\alpha$: $2 \leq \alpha < 4$. In the large-$N_c$ limit, the quark-hadron phase transition has the following scaling behavior: $\mu_q^{\text{crit}} \propto N_c^{3/4}$ for $2 \leq \alpha < 16/7$ and $\mu_q^{\text{crit}} \propto N_c^{3/2}$ for $16/7 < \alpha < 4$. Hence, we continued the discussion introduced in Ref. \cite{2} at large densities.

As an application of this simple formalism, we have investigated the effect on the maximum mass of neutron stars of quark deconfinement. A first result, which completes the analysis of \cite{10}, is that the requirement of obtaining a phase transition to deconfined quark matter at high density, $\alpha < 4$, implies that the speed of sound of the baryonic matter equation of state must exceed the value of $1/\sqrt{3}$. In turn, this allows to fulfill the $2M_\odot$ limit. In this respect, the occurrence of the phase transition at high density seems to be intimately connected with the existence of massive neutron stars. A second result concerns the values of $N_c$ for which the deconfined quark phase would take place in astrophysical dense systems. While for $N_c = 3$ (our world) the phase transition limits the maximum mass to a value which is pretty close to the observed $2M_\odot$ value, already at $N_c \approx 5.5$ deconfined quark matter would not play any role in the structure of compact stars. In between, we have analyzed the case $N_c = 4$: deconfinement of quarks would start to be relevant for stars with masses $\geq 2.4 M_\odot$ (at least a candidate with a similar mass already exists \cite{52}). Finally, a promising possibility is to test the role of quark matter in determining the maximum mass of compact stars via gravitational waves observations: the threshold mass for prompt collapse in binary neutron star mergers depends strongly on the value of the maximum mass \cite{56,57}. One could imagine that if the maximum mass of neutron stars is set by the deconfinement of quarks (and thus by a sudden softening of the equation of state) the temporal evolution of the merger remnant could be qualitatively different with respect to the standard scenario in which the phase transition is not considered.

Acknowledgments: The authors thank A. Drago, E. Maksymiuk, M. Piotrowska, and A. Gazela-Zimolag for useful discussions. The work of F. G. is supported by the Polish National Science Centre NCN through the OPUS project no. 2015/17/B/ST2/01625.
Appendix A: The large-$N_c$ behavior of the vacuum pressure constant

The Yang-Mills (YM) Lagrangian for an arbitrary number of colors $N_c$ reads (see, for instance, [58]):

$$\mathcal{L}_{YM} = -\frac{1}{4} G^{a}_{\mu\nu} G^{a,\mu\nu},$$

$$G^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g f^{abc} A^{b}_{\mu} A^{c}_{\nu}, \quad (A1)$$

where $a = 1, \ldots, N_{c} - 1$ and $f_{abc}$ are the structure constants of $SU(N_{c})$. For $N_c > 1$, the YM Lagrangian contains 3-gluon and 4-gluon vertices. The YM Lagrangian invariant under space-time dilatations, $x^{\mu} \rightarrow x^{\mu} = \lambda^{-1} x^{\mu}$, however this symmetry does not survive quantization. The corresponding divergence reads:

$$\partial_{\mu} J^{\mu}_{dil} = \frac{\beta(g)}{4g} G^{a}_{\mu\nu} G^{a,\mu\nu} \neq 0, \quad \beta(g) = \frac{\partial g}{\partial \mu}, \quad (A2)$$

where the dimensionless coupling constant $g = g(\mu)$ has become an energy-dependent running coupling (µ is the energy scale at which the coupling is probed). At the one-loop level, $\beta(g) = \mu \frac{\partial g}{\partial \mu} = -bg^{3} < 0$, $b = \frac{4\pi}{16\pi}$, whose solution is $g^{2}(\mu) = \left(2b \log \frac{\mu}{\Lambda_{YM}} \right)$, where a (Landau) pole at $\Lambda_{YM}$ is realized. Numerically, $\Lambda_{YM} \simeq 250$ MeV; this number affects all hadronic processes. The fact that $\beta(g) < 0$ explains asymptotic freedom: the coupling $g(\mu)$ becomes smaller for increasing $\mu$. On the other side, for small $\mu$, the coupling $g(\mu)$ increases. A (not yet analytically proven) consequence is ‘confinement’: gluons (and quarks) are confined in white hadrons. Notice that $g$ scales as follows in the large-$N_c$ limit: $g \propto 1/\sqrt{N_c}$. This is the starting point of the study of the large-$N_c$ limit used in this work.

A purely nonperturbative consequence of the scale anomaly is the emergence of a gluon condensate. Namely, the vacuum’s expectation value of the trace anomaly does not vanish:

$$\left< \partial_{\mu} J^{\mu}_{YM,dil} \right> = -\frac{11N_{c}\alpha_{s}}{48\pi} G^{a}_{\mu\nu} G^{a,\mu\nu}, \quad (A3)$$

where $\alpha_{s} = g^{2}/4\pi$. The numerical results were obtained via lattice, see Ref. [59, 60] and refs. therein. Note, the vacuum’s expectation value scales as

$$\left< \frac{11N_{c}\alpha_{s}}{48\pi} G^{a}_{\mu\nu} G^{a,\mu\nu} \right> \propto N_{c}^{2}, \quad (A4)$$

Namely, $N_{c}\alpha_{s}$ is $N_{c}$-independent and the sum over $a$ goes from 0 to $N_{c}^{2} - 1$.

Because of confinement, in the YM vacuum glueballs are the relevant degrees of freedom [61]. The effective Lagrangian describing the trace anomaly in terms of the ground-state scalar glueball $G$ reads [62, 63]:

$$\mathcal{L}_{G} = \frac{1}{2} (\partial_{\mu} G)^{2} - V_{dil}(G),$$

$$V_{dil}(G) = \frac{m_{G}^{2}}{4 \Lambda_{G}^{2}} \left[ G^{4} \ln \left( \frac{G}{\Lambda_{G}} \right) - G^{4} \right]. \quad (A5)$$

By studying the fluctuations about the minimum, $G \rightarrow G_{0} + G$, one can see that a field with mass $m_{G}$ emerges. This particle is the famous scalar glueball. This is, according to lattice simulations [61], the lightest glueball with $m_{G} \sim 1.6 - 1.7$ GeV. The resonance $f_{0}(1710)$ is a very good candidate to describe the dilaton/glueball field $G$. Note, the mass $m_{G}$ is independent on $N_{c}$, while $\Lambda_{G}$ scales as $N_{c}$ in such a way that $G^{4}$-interaction scales as $1/N_{c}^{2}$ (this is the scaling of a four-leg glueball term [2]):

$$m_{G} \propto N_{c}^{0}, \quad \Lambda_{G} \propto N_{c}. \quad (A6)$$

The divergence of the dilatation Noether current of the dilaton field presented in the Lagrangian (A5) is:\n
$$\partial_{\mu} J^{\mu}_{dil,G} = G\partial_{\mu} V_{dil}(G) - 4G = -\frac{m_{G}^{2}}{4 \Lambda_{G}^{2}} G^{4}. \quad (A7)$$

By comparing Eqs. (A3) and (A7), one obtains ($N_{c} = 3$) (see also Ref. [67]):

$$\Lambda_{G}^{2} \simeq \frac{33}{12} \left( 0.35 \text{ GeV} \right)^{4} \simeq 0.12 \text{ GeV}^{2}, \quad (A8)$$

Finally, we notice that the YM vacuum energy reads:

$$\varepsilon_{YM} = -N_{c}^{2} B = -\frac{m_{G}^{2} \Lambda_{G}^{2}}{16}. \quad (A9)$$

This equation shows that the $\varepsilon_{YM}$ scales as $N_{c}^{2}$ (see Eq. (A6)), hence our assumption in Eq. (1) is justified. Moreover, we obtain the following numerical values for $\sqrt{3}B^{1/4} \simeq 220$ MeV. When quarks are introduced, the constant $b$ changes into $b = \frac{4N_{c} - 2N_{f}}{8\pi^{2}}$. For $N_{f} = 3$, a slight reduction of $B$ is obtained: $\sqrt{3}B^{1/4} \simeq 214$ MeV.

Strictly speaking, a negative vacuum’s energy $\varepsilon_{YM}$ of Eq. (A9) corresponds to a positive contribution $-\varepsilon_{YM}$ to the dilaton/glueball, and hence to the hadronic vacuum’s pressure. Indeed, the dilaton field with the potential in Eq. (A5) has been often introduced in chiral hadronic models [64, 65, 70]. However, it is convention to require that the hadronic matter pressure vanishes at zero baryon chemical potential [70], hence one subtract this contribution to the hadronic vacuum’s pressure and, for consistency, to the quark pressure as well. Summarizing, a negative contribution equal to $\varepsilon_{YM} = -N_{c}^{2} B$ appears in the expression for the pressure of the quark phase, see Eq. (1). Notice also that typically a chiral dilaton hadronic model predicts the value of $B$ (which represents the vacuum energy offset between the hadronic phase and the quark phase) would be density dependent. This dependence shall not change the $N_{c}$ scaling, but can change some quantitative features at finite density. We retain this possibility for a future work.
F. X. Lee, K. F. Liu and N. Mathur et al., Phys. Rev. D 73, 014516 (2006).

[62] A. A. Migdal and M. A. Shifman, Phys. Lett. B 114, 445 (1982).

[63] C. Rosenzweig, A. Salomone and J. Schechter, Phys. Rev. D 24, 2545 (1981); A. Salomone, J. Schechter and T. Tudron, Phys. Rev. D 23, 1143 (1981); C. Rosenzweig, A. Salomone and J. Schechter, Nucl. Phys. B 206, 12 (1982) [Erratum-ibid. B 207, 546 (1982)]; H. Gomm and J. Schechter, Phys. Lett. B 158, 449 (1985); R. Gomm, P. Jain, R. Johnson and J. Schechter, Phys. Rev. D 33, 801 (1986).

[64] S. Janowski, F. Giacosa and D. H. Rischke, Phys. Rev. D 90 (2014) no.11, 114005; D. Parganlija, P. Kovacs, G. Wolf, F. Giacosa and D. H. Rischke, Phys. Rev. D 87 (2013) no.1, 014011.

[65] L. -C. Gui, Y. Chen, G. Li, C. Liu, Y. -B. Liu, J. -P. Ma, Y. -B. Yang and J. -B. Zhang, Phys. Rev. Lett. 110 (2013) 021601.

[66] F. Brümmer, D. Parganlija and A. Rebhan, Phys. Rev. D 91 (2015) no.10, 106002 Erratum: [Phys. Rev. D 93 (2016) no.10, 109903].

[67] A. Drago, M. Gibilisco and C. Ratti, Nucl. Phys. A 742 (2004) 165.

[68] L. Bonanno, A. Drago and A. Lavagno, Phys. Rev. Lett. 99 (2007) 242301.

[69] L. Bonanno and A. Drago, Phys. Rev. C 79 (2009) 045801.

[70] P. Papazoglou, J. Schaffner, S. Schramm, D. Zschiesche, H. Stoecker and W. Greiner, Phys. Rev. C 55 (1997) 1499.

[71] In turn, the baryonic chemical potential corresponding to the zero of the baryonic pressure scales as $N_c$, in agreement with the results of [8] in the case of the standard quarkonium assignment for the sigma meson.

[72] Note that for the quark phase to have positive pressure at the phase transition point, $\mu_q^{\text{crit}}$ must scale at least as $N_c^{1/4}$. In turn this implies that the term proportional to $\tilde{K}$ in Eq.(15) is always sub-leading (at large $N_c$) with respect to the term proportional to $\mu_q^2$ and thus it can be neglected.