A note on blow-up for Nakao’s type problem with nonlinearities of derivative type

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Abstract

In the present note, we investigate blow-up for a class of semilinear hyperbolic coupled system in $\mathbb{R}^n$ with $n \geq 1$, which is part of the so-called Nakao’s type problem weakly coupled a semilinear damped wave equation with a semilinear wave equation with nonlinearities of derivative type. By constructing suitable time-dependent functionals and employing iteration method for unbounded multiplier with slicing procedure, the results of blow-up and upper bound estimates for the lifespan of energy solutions are derived. Particularly, the blow-up result for one dimensional case is optimal.

Keywords: Semilinear hyperbolic system, wave equation, damped wave equation, blow-up, lifespan estimates.

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1 Introduction

The problem of critical curve, which describes the threshold condition between global (in time) existence of small data weak solutions and blow-up of small data weak solutions, of the power exponents for the weakly coupled system of wave equations and damped wave equations was proposed by Professor Mitsuhiro Nakao, Emeritus of Kyushu University (see, for instance, [14, 21]), namely,

\[
\begin{cases}
  u_{tt} - \Delta u + u_t = f_1(v, v_t), & x \in \mathbb{R}^n, \ t > 0, \\
  v_{tt} - \Delta v = f_2(u, u_t), & x \in \mathbb{R}^n, \ t > 0, \\
  (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{cases}
\]

(1.1)

where the nonlinearities on the right-hand sides are given by the mixture of the power type and the derivative type nonlinearities

\[
  f_1(v, v_t) := d_1|v|^{p_1} + d_2|v_t|^{p_2}, \\
  f_2(u, u_t) := d_3|u|^{q_1} + d_4|u_t|^{q_2},
\]

carrying some nonnegative constants $d_1, \ldots, d_4$ and $p_1, p_2, q_1, q_2 > 1$. Here, to guarantee the hyperbolic coupled system (1.1) being a nonlinear problem, we have to restrict ourselves that all coefficients $d_1 \ldots d_4$ will not be zero simultaneously. Roughly speaking, the main difficulty to treat Nakao’s type problem is to understand varying degrees of influence from damped wave equations and wave equations. It is well-known that decay properties and diffusion phenomenon hold in

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damped wave equations due to the frictional damping $u_t$. However, these effects disappear in wave equations and Huygens’ principle is valid in wave equations, which make huge differences of the treatments between semilinear wave equations and semilinear damped wave equations. In other words, Nakao’s type problem bridges a connection between semilinear wave equations and semilinear damped wave equations in a weakly coupled sense. We emphasize that the critical condition for (1.1) is still an open problem for $n \geq 2$.

In recent years, some blow-up results for Nakao’s type problem with power nonlinearities

$$
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v &= |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) &= (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{align*}
$$

(1.2)

that is the special case of the hyperbolic coupled system (1.1) with $d_1 = d_3 = 1$, $d_2 = d_4 = 0$ and $p_1 = p$, $q_1 = q$, have been derived in [21, 4]. With the aim of guaranteeing existence of local (in time) solutions, let us generally assume $p, q \leq n/(n - 2)$ if $n \geq 3$ in the discussion of this paragraph. Firstly, by using a test function method, the author of [21] demonstrated blow-up of local (in time) weak solutions with suitable assumption on initial data providing that

$$
\max \left\{ \frac{q/2 + 1}{pq - 1} + \frac{1}{2}, \frac{q + 1}{pq - 1}, \frac{p + 1}{pq - 1} \right\} \geq \frac{n}{2}.
$$

(1.3)

The condition (1.3) is optimal in $n = 1$ since it is equivalent to $1 < p, q < \infty$. Later, the authors of [4] employed iteration method associated with slicing procedure to improve the blow-up condition (1.3) for $n \geq 2$. Precisely, the result in [21] partially improved for $2 \leq n \leq 3$ and completely improved for $n \geq 4$ such that if

$$
\max \left\{ \frac{q/2 + 1}{pq - 1}, \frac{2 + p^{-1}}{pq - 1}, \frac{1/2 + p}{pq - 1} - \frac{1}{2} \right\} > \frac{n - 1}{2},
$$

(1.4)

then every local (in time) energy solution blows up in finite time. In other words, the authors of [4] observed that Nakao’s type problem with power nonlinearities is hyperbolic-like rather than parabolic-like due to fact that the component $(2 + p^{-1})/(pq - 1)$ plays an dominant role when $n \geq 3$. This effect comes from the semilinear wave equations. More detail explanations of parabolic-like versus hyperbolic-like are referred interested readers to Section 2.1 in [4]. Therefore, an interesting and viable problem is to ask the situation of nonlinearities of derivative type, i.e. the hyperbolic coupled system (1.1) carrying $d_1 = d_3 = 0$, $d_2 = d_4 = 1$ and $p_2 = p$, $q_2 = q$. We will give a possible answer from the blow-up point of view that Nakao’s type problem with nonlinearities of derivative type still could be hyperbolic-like model.

In this note, we study blow-up of solutions and lifespan estimates from the above for Nakao’s type problem with derivative type nonlinearities, namely,

$$
\begin{align*}
    u_{tt} - \Delta u + u_t &= |v|^p, & x \in \mathbb{R}^n, \ t > 0, \\
    v_{tt} - \Delta v &= |u|^q, & x \in \mathbb{R}^n, \ t > 0, \\
    (u, u_t, v, v_t)(0, x) &= \varepsilon(u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n,
\end{align*}
$$

(1.5)

where $p, q > 1$ and $\varepsilon$ is a positive parameter describing the size of initial data. As we will show in Theorem 1.1, the blow-up condition of Nakao’s type problem (1.5) is strongly related to the
According to the previous study [5], we expect that the global (in time) solution of the last system, which is also the critical exponent for the single semilinear wave equation with nonlinearity

\[ \frac{u_{tt} - \Delta u = |v_t|^p,}{v_{tt} - \Delta v = |u_t|^q,} \]

are strongly related to Nakao’s type problem (1.5). Concerning the weakly coupled system of derivative type as follows:

\[ (u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), \quad x \in \mathbb{R}^n, \]

One may see the validity of the Glassey exponent in [10, 19, 13, 18, 17, 1, 7, 20, 24, 8, 12]. Next, we refer to the related works [6, 22, 11, 9, 15]. Taking consideration of the case when \( p = q \), the critical exponent is given by the so-called Glassey exponent

\[ \alpha_{GW}(p, q) := \frac{\max\{p, q\} + 1}{pq - 1} = \frac{n - 1}{2}. \]

Particularly, under certain integral sign assumptions for initial data, if \( \alpha_{GW}(p, q) \geq (n - 1)/2 \), then every local (in time) solution \((u, v)\) blows up in finite time. Considering the critical curve (1.7), we refer to the related works [6, 22, 11, 9, 15]. Taking consideration of the case when \( p = q \), the critical exponent is given by the so-called Glassey exponent

\[ p_{Gla}(n) := \begin{cases} \infty & \text{if } n = 1, \\ \frac{n + 1}{n - 1} & \text{if } n \geq 2, \end{cases} \]

which is also the critical exponent for the single semilinear wave equation with nonlinearity \(|u_t|^p\). One may see the validity of the Glassey exponent in [10, 19, 13, 18, 17, 1, 7, 20, 24, 8, 12]. Next, we turn to the weakly coupled system of semilinear classical damped wave equations with nonlinearity of derivative type as follows:

\[ \begin{align*}
&u_{tt} - \Delta u + u_t = |v_t|^p, & x \in \mathbb{R}^n, \ t > 0, \\
v_{tt} - \Delta v + v_t = |u_t|^p, & x \in \mathbb{R}^n, \ t > 0, \\
(u, u_t, v, v_t)(0, x) = (u_0, u_1, v_0, v_1)(x), & x \in \mathbb{R}^n.
\end{align*} \]

According to the previous study [5], we expect that the global (in time) solution of the last system uniquely exists for any \( 1 < p, q < \infty \) and any \( n \geq 1 \). Namely, the solution does not blow up for any dimensions. This effect also appears in the wave equations with scale-invariant damping of the effective case (see, Theorem 2.2 of [16] by letting parameters \( \mu_1, \mu_2 \to \infty \)). Thus, the consideration of Nakao’s type problem (1.5) is reasonable.

Let us first introduce a suitable definition of energy solutions of Nakao’s type problem (1.5).

**Definition 1.1.** Let \((u_0, u_1, v_0, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\). One may say that \((u, v)\) is an energy solution of Nakao’s type problem (1.5) on \([0, T]\) if

\[ u \in C((0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \quad \text{and} \quad u_t \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n), \]

\[ v \in C((0, T), H^1(\mathbb{R}^n)) \cap C^1([0, T], L^2(\mathbb{R}^n)) \quad \text{and} \quad v_t \in L^p_{\text{loc}}([0, T) \times \mathbb{R}^n), \]

where \( p \) and \( q \) are numbers such that \( \frac{1}{p} + \frac{1}{q} = 1 \).
satisfies \((u, v)(0, \cdot) = (u_0, v_0)\) in \(H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)\) and the following integral relations:

\[
\int_0^t \int_{\mathbb{R}^n} (-u_t(s, x) \phi(s, x) + u_t(s, x) \phi(s, x) + \nabla u(s, x) \cdot \nabla \phi(s, x)) dx ds + \int_{\mathbb{R}^n} u_t(t, x) \phi(t, x) dx - \int_{\mathbb{R}^n} u_1(x) \phi(0, x) dx = \int_0^t \int_{\mathbb{R}^n} |v_t(s, x)|^p \phi(s, x) dx ds
\]

and

\[
\int_0^t \int_{\mathbb{R}^n} (-v_t(s, x) \psi(s, x) + \nabla v(s, x) \cdot \nabla \psi(s, x)) dx ds + \int_{\mathbb{R}^n} v_t(t, x) \psi(t, x) dx - \int_{\mathbb{R}^n} v_1(x) \psi(0, x) dx = \int_0^t \int_{\mathbb{R}^n} |u_t(s, x)|^q \psi(s, x) dx ds
\]

for any test functions \(\phi, \psi \in C_0^\infty([0, T) \times \mathbb{R}^n)\) and any \(t \in (0, T)\).

**Remark 1.1.** Actually, similarly to treatments in [4], by applying further steps of integration by parts in (1.9) as well as (1.10), respectively, and taking \(t \to T\), we may claim that \((u, v)\) fulfills the definition of weak solutions of Nakao’s type problem (1.5).

From Banach’s fixed point theorem and Duhamel’s principle associated with some estimates of solutions of the corresponding linear Cauchy problem to (1.5), one may derive local (in time) existence of weak solutions with compact support localized in a ball with radius \(R + t\) of Nakao’s type problem (1.5) with compact supported data in a ball with radius \(R\) if \(p, q > 1\) for \(n = 1, 2\), and \(1 < p, q \leq n/(n - 2)\) for \(n \geq 3\).

Let us state the blow-up result for Nakao’s type problem (1.5).

**Theorem 1.1.** Let us consider the exponents \(p, q > 1\) such that

\[
pq < \begin{cases} \infty & \text{if } n = 1, \\ p_G(n) & \text{if } n \geq 2. \end{cases}
\]

Furthermore, let \((u_0, v_0, u_1, v_1) \in (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)) \times (H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n))\) are nonnegative and compactly supported functions with supports contained in \(B_R\) for some \(R > 0\) such that \(u_1, v_1\) are not identically zero. Let \((u, v)\) be the local (in time) energy solution of Nakao’s type problem (1.5) according to Definition 1.1 with lifespans \(T = T(\varepsilon)\). Then, these solutions satisfy

\[
supp u, supp v \subset \{(t, x) \in [0, T) \times \mathbb{R}^n : |x| \leq R + t\}. \tag{1.12}
\]

Moreover, there exists a positive constant \(\varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, p, q, n, R)\) such that for any \(\varepsilon \in (0, \varepsilon_0]\) the energy solution \((u, v)\) blows up in finite time. In addition, the upper bound estimate for the lifespans

\[
T(\varepsilon) \leq C\varepsilon^{-2(pq-1)/(-(n-1)pq+1)}
\]

holds, where \(C > 0\) is a constant independent of \(\varepsilon\).

**Remark 1.2.** Due to the condition \(pq < (n + 1)/(n - 1)\) for \(n \geq 2\), it is trivial that \(p, q \leq n/(n - 2)\) for \(n \geq 3\). In other words, under the condition of exponents \(p\) and \(q\) in Theorem 1.1, the weak solution having compact support in \(B_{R+t}\) of Nakao’s type problem (1.5) locally (in time) exists.
Remark 1.3. In the one dimensional case, the energy solution of Nakao’s type problem (1.5) blows up for all $1 < p, q < \infty$, which means that the result is optimal. However, for the high dimensional cases when $n \geq 2$, the critical curve in the $p-q$ plane for Nakao’s type problem (1.5) is open.

To end this section, let us give some remarks on the blow-up conditions for energy solutions of Nakao’s problem (1.5) with respect to the exponents $p, q$ for $n = 2$ and $n \geq 3$, respectively.

![Figure 1: Blow-up conditions in the $p-q$ plane](image)

According to Figure 1, we may observe that

$$\left\{(p,q) : pq < p_{\text{Gla}}(n) = \frac{n+1}{n-1}\right\} \subset \left\{(p,q) : \alpha_W(p,q) = \frac{\max\{p,q\} + 1}{pq - 1} < \frac{n-1}{2}\right\}$$

for any $n \geq 2$. Again, $\alpha_W(p,q) = (n-1)/2$ is the critical curve in the $p-q$ plane for the weak coupled system (1.6). This effect is caused by the influence of friction $u_t$ on the first equation of (1.5). For the reason of the blow-up condition $pq < p_{\text{Gla}}(n)$, where $p_{\text{Gla}}(n)$ is the critical exponent for semilinear wave equation with derivative type nonlinearity, we feel that Nakao’s type problem (1.5) is of hyperbolic-like rather than parabolic-like.

Notation: We give some notations to be used in this paper. We write $f \lesssim g$ when there exists a positive constant $C$ such that $f \leq Cg$. We denote $\lceil r \rceil := \min\{C \in \mathbb{Z} : r \leq C\}$ as the ceiling function. Moreover, $B_R$ denotes the ball around the origin with radius $R$ in $\mathbb{R}^n$.

2 Proof of Theorem 1.1 via an iteration argument

2.1 Iteration frame

In order to apply an iteration argument in the proof, we should derive integral inequalities for some suitable time-dependent functions. To begin with, let us introduce the eigenfunction $\Phi = \Phi(x)$ of
the Laplace operator in \( n \)-dimensions Euclidean space such that

\[
\Phi(x) := e^x + e^{-x} \quad \text{if } n = 1,
\]

\[
\Phi(x) := \int_{\mathbb{S}^{n-1}} e^{x \cdot \omega} \, d\sigma \quad \text{if } n \geq 2,
\]

where \( \mathbb{S}^{n-1} \) is the \( n-1 \) dimensional sphere. This test function \( \Phi \) has been introduced in the pioneering paper [23]. It fulfills the property \( \Delta \Phi = \Phi \) and the asymptotic behavior

\[
\Phi(x) \sim |x|^{-\frac{n+1}{2}} e^x \quad \text{as } |x| \to \infty.
\]

(2.1)

Moreover, we define the test function \( \Psi = \Psi(t,x) \) with separate variables such that \( \Psi(t,x) := e^{-t} \Phi(x) \). Clearly, the function \( \Psi \) is a special solution of the homogeneous wave equation \( \Psi_{tt} - \Delta \Psi = 0 \). By using asymptotic behavior (2.1), it immediately yields the estimate

\[
\int_{B_{R+t}} \Psi(t,x) \, dx \leq C_1(R+t)^{\frac{n+1}{2}}
\]

(2.2)

for any \( t \geq 0 \), where \( C_1 \) is a positive constant. The previous estimate (2.2) was shown in [12].

To construct the iteration frame, it is necessary for us to introduce suitable functionals with respect to \( u_t \) and \( v_t \) due to the derivative type nonlinearities of the hyperbolic system (1.5). With the aid of the above test function \( \Psi \), we may define new time-dependent functionals \( F_1 = F_1(t) \) and \( F_2 = F_2(t) \) such that

\[
F_1(t) := \int_0^t \int_{\mathbb{R}^n} u_t(s,x)\Psi(s,x) \, dx \, ds \quad \text{and} \quad F_2(t) := \int_{\mathbb{R}^n} v_t(t,x)\Psi(t,x) \, dx.
\]

Here, we should emphasize that \( F_1(t) \) has the similar form to \( F_2(t) \), which is beneficial to process the iteration procedure later.

Due to the fact that \( u, v \) are supported in a forward cone \( \{(s,x) \in [0,t] \times \mathbb{R}^n : |x| \leq R+s\} \), we can apply the definition of energy solution \((u,v)\) with \( \Psi \) to be the test function in (1.9) and (1.10).

For one thing, by using integration by parts in (1.9) with \( \phi(t,x) = \Psi(t,x) \), we have

\[
\int_0^t \int_{\mathbb{R}^n} u_t(s,x)\Psi(s,x) \, dx \, ds + \int_{\mathbb{R}^n} (u_t(t,x)\Psi(t,x) + u(t,x)\Psi(t,x)) \, dx
\]

\[
= \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\Phi(x) \, dx + \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x) \, dx \, ds,
\]

which can also be rewritten by

\[
F_1'(t) + F_1(t) + \int_{\mathbb{R}^n} u(t,x)\Psi(t,x) \, dx
\]

\[
= \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\Phi(x) \, dx + \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x) \, dx \, ds.
\]

(2.3)

Taking time-derivative in the above equality and using \( \Psi_t(t,x) = -\Psi(t,x) \) brings

\[
F_1''(t) + 2F_1'(t) - \int_{\mathbb{R}^n} u(t,x)\Psi(t,x) \, dx = \int_{\mathbb{R}^n} |v_t(t,x)|^p \Psi(t,x) \, dx.
\]

(2.4)
Adding up (2.3) and (2.4), one may derive
\[ F''_1(t) + 3F'_1(t) + F_1(t) = \varepsilon \int_{\mathbb{R}^n} (u_0(x) + u_1(x))\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x)dxds 
+ \int_{\mathbb{R}^n} |v_t(t,x)|^p \Psi(t,x)dx. \] 
(2.5)

For another thing, we employ once integration by parts in (1.10) with \( \psi(t,x) = \Psi(t,x) \) to get
\[ F_2(t) + \int_{\mathbb{R}^n} v(t,x)\Psi(t,x)dx = \varepsilon \int_{\mathbb{R}^n} (u_0(x) + v_1(x))\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^q \Psi(s,x)dxds. \]
Similarly to the treatment of \( F_1(t) \), we differentiate the last equality with respect to \( t \), which implies
\[ F''_2(t) + F'_2(t) - \int_{\mathbb{R}^n} v(t,x)\Psi(t,x)dx = \int_{\mathbb{R}^n} |u_t(t,x)|^q \Psi(t,x)dx. \]
Summarizing the derived equations, one has
\[ F''_2(t) + 2F'_2(t) = \varepsilon \int_{\mathbb{R}^n} (v_0(x) + v_1(x))\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^q \Psi(s,x)dxds 
+ \int_{\mathbb{R}^n} |u_t(t,x)|^q \Psi(t,x)dx. \] 
(2.6)

With the aim of constructing the iteration frame, we need to transfer (2.5) and (2.6) to suitable integral inequalities, respectively. Let us consider (2.5) initially. We now define a time-dependent functional:
\[ G_1(t) := F'_1(t) + \frac{3 + \sqrt{5}}{2} F_1(t) - \varepsilon \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx - \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x)dxds. \]
Then, it is obvious from (2.5) that
\[ G'_1(t) + \frac{3 - \sqrt{5}}{2} G_1(t) = \varepsilon \int_{\mathbb{R}^n} u_0(x)\Phi(x)dx + \frac{(\sqrt{5} - 1)\varepsilon}{2} \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx 
+ \frac{\sqrt{5} - 1}{2} \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x)dxds. \]
According to the nonnegative hypothesis on initial data \( u_0 \) and \( u_1 \), we are able to conclude
\[ e^{-\frac{3 - \sqrt{5}}{2}t} \left( e^{\frac{3 - \sqrt{5}}{2}t} G_1(t) \right)' = G'_1(t) + \frac{3 - \sqrt{5}}{2} G_1(t) \geq 0, \]
which results that
\[ G_1(t) \geq e^{-\frac{3 - \sqrt{5}}{2}t} G_1(0) = 0. \]
For this reason, we obtain
\[ e^{-\frac{3 + \sqrt{5}}{2}t} \left( e^{\frac{3 + \sqrt{5}}{2}t} F_1(t) \right)' \geq \varepsilon \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx + \int_0^t \int_{\mathbb{R}^n} |v_t(s,x)|^p \Psi(s,x)dxds. \] 
(2.7)

By ignoring the nonnegative nonlinear integral term on the right-hand side, multiplying the previous equality by \( e^{\frac{3 + \sqrt{5}}{2}t} \) and integrating the resultant over \([0,t]\), we arrive at
\[ F_1(t) \geq e^{\frac{3 + \sqrt{5}}{2}t} F_1(0) + \frac{2\varepsilon}{3 + \sqrt{5}} \left( 1 - e^{\frac{3 + \sqrt{5}}{2}t} \right) \int_{\mathbb{R}^n} u_1(x)\Phi(x)dx \geq C_2\varepsilon > 0 \] 
(2.8)
for any $t \geq 1$, where $C_2$ is a suitably positive constant depending on $u_1$. Here, we used nontrivial assumption on $u_1$ and $F_1(0) = 0$. What’s more, by omitting the term containing initial data $u_1$ in (2.7) we find that

$$F_1(t) \geq \int_0^t e^{\frac{\alpha}{2}(\tau-t)} \int_0^\tau |v_t(s,x)|^p \Psi(s,x)dx d\tau$$

$$\geq \int_0^t e^{\frac{\alpha}{2}(\tau-t)} \int_0^\tau |F_2(s)|^p \left( \int_{|x| \leq R+s} \Psi(s,x)dx \right)^{-\frac{(p-1)}{2}} ds d\tau$$

$$\geq C_1^{-\frac{p}{2}} \int_0^t e^{\frac{\alpha}{2}(\tau-t)} \int_0^\tau (R+s)^{\frac{(n-1)(q-1)}{2}} |F_2(s)|^p ds d\tau,$$

(2.9)

where we used the support condition for the wave model and the estimate (2.2).

Next, we treat (2.6) by constructing another time-dependent functional such that

$$G_2(t) := F_2(t) - \frac{\varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x)dx - \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^q \Psi(s,x)dx ds.$$

(2.10)

In other words, in the light of (2.6) we find

$$e^{-2t} \left( e^{2t} G_2(t) \right)' = \varepsilon \int_{\mathbb{R}^n} v_0(x) \Phi(x)dx + \frac{1}{2} \int_{\mathbb{R}^n} |u_t(t,x)|^q \Psi(t,x)dx \geq 0,$$

where the nonnegativity of $v_0$ was applied. It immediately conduces to

$$G_2(t) \geq e^{-2t} G_2(0) = \frac{e^{-2t} \varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x)dx \geq 0$$

from the nonnegativity of $v_1$. Consequently, the nontrivial assumption on $v_1$ associated with the relation (2.10) shows

$$F_2(t) \geq \frac{\varepsilon}{2} \int_{\mathbb{R}^n} v_1(x) \Phi(x)dx = C_3 \varepsilon > 0,$$

(2.11)

with a positive constant $C_3$ depending on $v_1$, and

$$F_2(t) \geq \frac{1}{2} \int_0^t \int_{\mathbb{R}^n} |u_t(s,x)|^q \Psi(s,x)dx ds$$

$$\geq \frac{C_1^{1-q}}{2} \int_0^t (R+s)^{\frac{(n-1)(q-1)}{2}} |F_1'(s)|^q ds$$

$$\geq \frac{C_1^{1-q}}{2} \left( \int_0^t (R+s)^{\frac{n-1}{2}} ds \right)^{\frac{(n-1)(q-1)}{2}} \left( \int_0^t |F_1'(s)| ds \right)^q$$

$$\geq C_4 (R + t)^{\frac{(n-1)(q-1)}{2}} |F_1(t)|^q,$$

(2.12)

with a positive constant $C_4$, where we utilized Hölder’s inequality associated with (2.2) again and

$$|F_1(t)| = |F_1(t) - F_1(0)| = \left| \int_0^t F_1'(s) ds \right| \leq \int_0^t |F_1'(s)| ds.$$
We remark that the above constants are nonnegative. On one hand, we combine (2.11) with (2.9) to deduce
\[ F_1(t) \geq C_1^{1-p} C_3^p \varepsilon^p \int_0^t \int_0^\tau (R + s)^{-\frac{(n-1)(p-1)}{2}} ds d\tau \]
\[ \geq C_1^{1-p} C_3^p \varepsilon^p (R + t)^{-\frac{(n-1)(p-1)}{2}} \int_{t/2}^t e^{\frac{3+\sqrt{5}}{2}(\tau-t)} d\tau \]
\[ \geq C_1^{1-p} C_3^p \frac{3 + \sqrt{5}}{3 + \sqrt{5}} \varepsilon^p (R + t)^{-\frac{(n-1)(p-1)}{2}} \left(1 - e^{-\frac{3+\sqrt{5}}{4}t}\right) \]
\[ \geq C_1^{1-p} C_3^p \frac{3 + \sqrt{5}}{3 + \sqrt{5}} \left(1 - e^{-1 - \frac{3+\sqrt{5}}{4}}\right) \varepsilon^p (R + t)^{-\frac{(n-1)(p-1)}{2}} (t - L_1) \]
for any \( t \geq L_1 := 1 + 4/(3 + \sqrt{5}) \), which provides first lower bound estimates for \( F_1(t) \). On the other hand, we summarize (2.8) and (2.12). It results
\[ F_2(t) \geq C_2^q C_4 \varepsilon^q (R + t)^{-\frac{(n+1)(q-p)}{2}} \]
for any \( t \geq L_1 > 1 \). The suitable choice of \( L_1 \) is concerned about the slicing procedure dealing with the unbounded multiplier in the next subsection.

All in all, we derived first lower bound estimates as follows:
\[ F_1(t) \geq D_1(R + t)^{-a_1} (t - L_1)^{b_1}, \]
\[ F_2(t) \geq Q_1(R + t)^{-a_1} (t - L_1)^{b_1}, \]
for any \( t \geq L_1 \), where the multiplicative constants are given by
\[ D_1 := \frac{C_1^{1-p} C_3^p}{3 + \sqrt{5}} \left(1 - e^{-1 - \frac{3+\sqrt{5}}{4}}\right) \varepsilon^p, \quad Q_1 := C_2^q C_4 \varepsilon^q, \]
and the exponents are represented by
\[ a_1 := \frac{(n-1)(p-1)}{2}, \quad a_1 := \frac{(n+1)(q-1)}{2}, \quad b_1 := 1, \quad b_1 := 0. \]
We remark that the above constants are nonnegative.

### 2.2 Iteration argument

In this part, we will derive sequences of lower bound estimates for the functionals \( F_1(t) \) and \( F_2(t) \) by using some derived inequalities in the last subsection. To be specific, the following lower bounds will be proved:
\[ F_1(t) \geq D_j(R + t)^{-a_j} (t - L_j)^{b_j}, \]
\[ F_2(t) \geq Q_j(R + t)^{-a_j} (t - L_j)^{b_j}, \]
for any \( t \geq L_j \), where \( \{D_j\}_{j \geq 1}, \{Q_j\}_{j \geq 1}, \{a_j\}_{j \geq 1}, \{\alpha_j\}_{j \geq 1}, \{\beta_j\}_{j \geq 1} \) and \( \{b_j\}_{j \geq 1} \) are sequences of nonnegative real numbers that will be determined later in the iteration procedure. Motivated by the recent papers [2, 3], we may define a crucial sequence \( \{L_j\}_{j \geq 1} \) of the partial products of the convergent infinite product
\[ \prod_{k=1}^{\infty} \ell_k \text{ with } \ell_k := 1 + \frac{4}{3 + \sqrt{5}} (pq)^{-\frac{k-1}{2}} \text{ for any } k \geq 1, \]
that is,\[
L_j := \prod_{k=1}^{j} \ell_k \quad \text{for any } j \geq 1. \tag{2.18}
\]

Here, we recall that \( L_1 = \ell_1 = 1 + 4/(3 + \sqrt{5}) \). Essentially, thanks to the ratio test carrying

\[
\lim_{k \to \infty} \frac{\ln \ell_{k+1}}{\ln \ell_k} = \lim_{k \to \infty} \frac{\ell_k}{\ell_{k+1}(pq)^{1/2}} = (pq)^{-1/2} < 1,
\]

we claim that the infinite product\[
\prod_{k=1}^{\infty} \ell_k = \exp \left( \sum_{k=1}^{\infty} \ln \ell_k \right)
\]
is convergent. Furthermore, the desired estimates (2.15) and (2.16) for \( j = 1 \) are given in (2.13) and (2.14), respectively.

As a consequence, with the aim of demonstrating (2.15) and (2.16), we just need to procure the induction step with the aim of proving (2.15) and (2.16). In other words, by assuming that (2.15) and (2.16) hold for \( j \), one oughts to prove them being valid for \( j+1 \). Let us first substitute (2.16) into (2.9), which leads to

\[
F_1(t) \geq C_1^{1-p}Q_j^p \int_0^t e^{3\sqrt{5}(\tau-t)} \int_0^\tau (R+s)^{-\frac{(n-1)(p-1)}{2}} - a_j p (s - L_j)^bpdsd\tau
\]

\[
\geq C_1^{1-p}Q_j^p(R+t)^{\frac{(n-1)(p-1)}{2}} - a_j p \int_t^t e^{3\sqrt{5}(\tau-t)} \int_0^\tau (s - L_j)^bpdsd\tau
\]

\[
\geq C_1^{1-p}Q_j^p(R+t)^{\frac{(n-1)(p-1)}{2}} - a_j p \int_t^t e^{3\sqrt{5}(\tau-t)} (\tau - L_j)^bp^{p+1}d\tau.
\]

In view of \( t \geq L_{j+1} = L_j\ell_{j+1} \), i.e. \( L_j \leq t/\ell_{j+1} \), we may instantly shrink the interval \([L_j, t]\) into \([t/\ell_{j+1}, t]\) so that

\[
F_1(t) \geq \frac{C_1^{1-p}Q_j^p}{b_j p + 1}(R+t)^{\frac{(n-1)(p-1)}{2}} - a_j p \int_t^t e^{3\sqrt{5}(\tau-t)} (\tau - L_j)^bp^{p+1}d\tau
\]

\[
\geq \frac{2C_1^{1-p}Q_j^p}{(3 + \sqrt{5})(b_j p + 1)\ell_{j+1}^{b_j p+1}} \left( 1 - e^{3\sqrt{5}(1/\ell_{j+1}-1)t} \right) (R+t)^{\frac{(n-1)(p-1)}{2}} - a_j p (t - L_{j+1})^{b_j p+1}.
\]

By considering \( t \geq L_{j+1} \geq \ell_{j+1} \) with the formula of \( \ell_{j+1} \), one observes

\[
1 - e^{3\sqrt{5}(1/\ell_{j+1}-1)t} \geq 1 - e^{3\sqrt{5}(\ell_{j+1}-1)} \geq \frac{3 + \sqrt{5}}{2} (\ell_{j+1} - 1) \left( 1 - \frac{3 + \sqrt{5}}{4} (\ell_{j+1} - 1) \right)
\]

\[
\geq 2(pq)^{-\frac{1}{2}} \left( 1 - (pq)^{-\frac{1}{2}} \right) \geq 2 \left( (pq)^{1/2} - 1 \right) (pq)^{-j} > 0
\]

for any \( j \geq 1 \). In conclusion, it yields

\[
F_1(t) \geq \frac{4C_1^{1-p} \left( (pq)^{1/2} - 1 \right) (pq)^{-j}Q_j^p}{(3 + \sqrt{5})(b_j p + 1)\ell_{j+1}^{b_j p+1}} (R+t)^{\frac{(n-1)(p-1)}{2}} - a_j p (t - L_{j+1})^{b_j p+1}
\]
for any $t \geq L_{j+1}$. Then, the combination of (2.12) as well as (2.15) shows

$$F_2(t) \geq C_4 D_j^q (R + t)^{-\frac{(n+1)(q-1)}{2} - \alpha_j q (t - L_{j+1})^\beta q}$$

for any $t \geq L_{j+1}$, where we used the fact that $L_j \leq L_j \ell_{j+1} = L_{j+1}$ carrying $\ell_{j+1} > 1$.

In other words, (2.15) and (2.16) are valid supposing that $\alpha_j \beta_j + \alpha_j = \beta_j q + 1$.

### 2.3 Upper bound estimates for the lifespan

In the last subsection, we derive a sequence of lower bound estimates for $F_1(t)$ and $F_3(t)$, respectively. In the forthcoming part, we will demonstrate that the $j$-dependent lower bounds for the functionals $F_1(t)$ and $F_2(t)$ blow up as $j \to \infty$. At the same time, the blow-up result and upper bound estimates for the lifespan stated in Theorem 1.1 will be concluded.

We will begin with the explicit formulas for the sequences $\alpha_j, \beta_j, a_j, b_j$, which devote to estimates for the multiplicative constants $D_j$ and $Q_j$.

Particularly, concerning the formulas of $\alpha_j$ and $a_j$, we need to discuss the case when $j$ is an odd integer only, which is sufficient for our proof. Taking account of the relation between $\alpha_j$ and $a_j$, we may get for odd number $j$ that

$$\alpha_j = \frac{(n-1)(p-1)}{2} + a_{j-1}p = \frac{(n+1)pq - 2p - (n-1)}{2} + \alpha_{j-2}pq$$

$$= \frac{2(n+1)pq - 2p - (n-1)}{2} \sum_{k=0}^{(j-3)/2} (pq)^{k} + \alpha_1(pq)^{\frac{j-1}{2}}$$

$$= \left( \alpha_1 + \frac{(n+1)pq - 2p - (n-1)}{2(pq-1)} \right)(pq)^{\frac{j-1}{2}} - \frac{(n+1)pq - 2p - (n-1)}{2(pq-1)},$$

and similarly,

$$a_j = \frac{(n+1)(q-1)}{2} + a_{j-1}q = \frac{(n-1)pq + 2q - (n+1)}{2} + a_{j-2}pq$$

$$= \frac{2(n-1)pq + 2q - (n+1)}{2} \sum_{k=0}^{(j-3)/2} (pq)^{k} + a_1(pq)^{\frac{j-1}{2}}$$

$$= \left( a_1 + \frac{(n-1)pq + 2q - (n+1)}{2(pq-1)} \right)(pq)^{\frac{j-1}{2}} - \frac{(n-1)pq + 2q - (n+1)}{2(pq-1)}.$$

Furthermore, by the definition of $\beta_j$ and $b_j$, one derives for odd number $j$ that

$$\beta_j = 1 + b_{j-1} = 1 + \beta_{j-2}pq = \sum_{k=0}^{(j-3)/2} (pq)^{k} + \beta_1(pq)^{\frac{j-1}{2}} = \left( \beta_1 + \frac{1}{pq-1} \right)(pq)^{\frac{j-1}{2}} - \frac{1}{pq-1},$$

$$b_j = \beta_{j-1}q = q + b_{j-2}pq = q \sum_{k=0}^{(j-1)/2} (pq)^{k} + b_1(pq)^{\frac{j-1}{2}} = \left( b_1 + \frac{q}{pq-1} \right)(pq)^{\frac{j-1}{2}} - \frac{q}{pq-1}.$$
For the even number $j$, which means that $j-1$ is an odd number, we make use of the previous two equalities to arrive at

$$\beta_j = 1 + b_{j-1}p = q^{-1} \left( b_1 + \frac{q}{pq - 1} \right) (pq)^{\frac{j}{2}} - \frac{1}{pq - 1},$$

$$b_j = \beta_{j-1}q = p^{-1} \left( \beta_1 + \frac{1}{pq - 1} \right) (pq)^{\frac{j}{2}} - \frac{q}{pq - 1}.$$ 

For this reason, it holds

$$\beta_j \leq B_0(pq)^{\frac{j}{2}} \text{ and } b_j \leq B_1(pq)^{\frac{j}{2}}$$

for any $j \geq 1$, where $B_0 = B_0(p, q, n)$ and $B_1 = B_1(p, q, n)$ are positive constants independent of $j$. Before estimating the constants $D_j$ and $Q_j$ from the below, we apply L'Hôpital's rule to show

$$\lim_{j \to \infty} \ell_j^{b_{j-1}p+1} = \lim_{j \to \infty} \ell_j^{\beta_j} \leq \lim_{j \to \infty} \exp \left( B_0(pq)^{\frac{j}{2}} \ln \left( 1 + \frac{4}{3 + \sqrt{5}} (pq)^{-\frac{j-1}{2}} \right) \right)$$

$$= \exp \left( \frac{4B_0}{3 + \sqrt{5}} (pq)^{\frac{j}{2}} \right) > 0$$

so that there exists a suitable constant satisfying $1/\ell_j^{b_{j-1}p+1} \geq M > 0$ for any $j \geq 1$. As a result, the next iterated relations for the lower bounds come:

$$D_j = \frac{4C_1^{1-p} ((pq)^{1/2} - 1) (pq)^{-j+1}}{(3 + \sqrt{5})(b_{j-1}p + 1) \ell_j^{b_{j-1}p+1} Q_{j-1}^p} \geq \frac{4C_1^{1-p} ((pq)^{1/2} - 1) M}{(3 + \sqrt{5})B_0} (pq)^{-\frac{j+1}{2}} Q_{j-1}^p$$

$$\geq \frac{4C_1^{1-p} C_4 ((pq)^{1/2} - 1) M}{(3 + \sqrt{5})B_0} (pq)^{-\frac{j+1}{2}} D_{j-2}^{pq} =: E_0(pq)^{-\frac{j+1}{2}} D_{j-2}^{pq}, \quad (2.19)$$

and simultaneously,

$$Q_j = C_4 D_{j-1}^{\frac{j}{2}} \geq \frac{4a C_1^{(1-p)q} C_4 ((pq)^{1/2} - 1) q M^{q/2}}{(3 + \sqrt{5})q B_0^q} (pq)^{-\frac{j+1}{2}} q Q_{j-2}^{pq} =: E_1(pq)^{-\frac{j+1}{2}} Q_{j-2}^{pq}, \quad (2.20)$$

with suitable constants $E_0 = E_0(p, q, n) > 0$ and $E_1 = E_1(p, q, n) > 0$ independent of $j$. Considering (2.19) with an odd number $j$, we take the logarithmic on the both sides to deduce

$$\log D_j \geq pq \log D_{j-2} - \left( \frac{3}{2} j - 1 \right) \log(pq) + \log E_0$$

$$\geq (pq)^{\frac{j-1}{2}} \log D_1 - \frac{3}{2} \log(pq) \sum_{k=1}^{(j-1)/2} \left( (j + 2 - 2k)(pq)^{k-1} \right)$$

$$+ (\log(pq) + \log E_0) \sum_{k=1}^{(j-1)/2} (pq)^{k-1}$$

$$= (pq)^{\frac{j-1}{2}} \left( \log D_1 + \frac{\log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_0}{pq - 1} \right)$$

$$+ \frac{\log(pq)}{pq - 1} \left( \frac{3}{2} \left( \frac{2pq}{pq - 1} + j \right) - 1 \right) - \frac{\log E_0}{pq - 1}.$$
where we used the next formula in the last line of the chain inequalities:

$$
\sum_{k=1}^{(j-1)/2} ((j + 2 - 2k)(pq)^{k-1}) = \frac{1}{pq - 1} \left( \frac{2pq}{pq - 1} \left( \frac{3}{2}(pq)^{\frac{j-1}{2}} - \frac{1}{2}(pq)^{\frac{j-3}{2}} - 1 \right) \right).
$$

Thus, for all nonnegative odd numbers \( j \) satisfying

$$
j \geq j_0 := \left\lceil \frac{2}{3} + \frac{2 \log E_0}{3 \log(pq)} - \frac{2pq}{pq - 1} \right\rceil,
$$

we conclude

$$\log D_j \geq (pq)^{\frac{j-1}{2}} \left( \log D_1 + \frac{\log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_0}{pq - 1} \right)$$

$$= (pq)^{\frac{j-1}{2}} \log \left( D_1(pq)^{(1-7pq)/(2(pq-1)^2)} E_0^{1/(pq-1)} \right) = (pq)^{\frac{j-1}{2}} \log(E_2\varepsilon^q)
$$

for a suitable positive constant \( E_2 = E_2(p, q, n) \). By the same way of calculation, we may illustrate

$$\log Q_j \geq pq \log Q_{j-2} - \left( \frac{3}{2}jq - q \right) \log(pq) + \log E_1$$

$$\geq (pq)^{\frac{j-1}{2}} \log Q_1 - \frac{3}{2}q \log(pq) \sum_{k=1}^{(j-1)/2} ((j + 2 - 2k)(pq)^{k-1})$$

$$+ (q \log(pq) + \log E_1) \sum_{k=1}^{(j-1)/2} (pq)^{k-1}$$

$$= (pq)^{\frac{j-1}{2}} \left( \log Q_1 + \frac{q \log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_1}{pq - 1} \right)$$

$$+ \frac{\log(pq)}{pq - 1} \left( \frac{3q}{2} \left( \frac{2pq}{pq - 1} + j \right) - q \right) - \frac{\log E_1}{pq - 1}.$$  

Consequently, for all nonnegative odd numbers \( j \) fulfilling

$$j \geq j_1 := \left\lceil \frac{2}{3} + \frac{2 \log E_1}{3q \log(pq)} - \frac{2pq}{pq - 1} \right\rceil,$$

we conclude

$$\log Q_j \geq (pq)^{\frac{j-1}{2}} \left( \log Q_1 + \frac{q \log(pq)}{2(pq - 1)^2} (1 - 7pq) + \frac{\log E_1}{pq - 1} \right)$$

$$= (pq)^{\frac{j-1}{2}} \log \left( Q_1(pq)^{q(1-7pq)/(2(pq-1)^2)} E_1^{1/(pq-1)} \right) = (pq)^{\frac{j-1}{2}} \log(E_3\varepsilon^q)
$$

for a suitable positive constant \( E_3 = E_3(p, q, n) \).

Let us now denote

$$L := \lim_{j \to \infty} L_j = \prod_{j=1}^{\infty} \ell_j > 1.$$

Due to \( \ell_j > 1 \) the sequence \( \{L_j\}_{j \geq 1} \) is converging to \( L \) as \( j \to \infty \). Namely, the relations (2.15) and (2.16) hold for any odd number \( j \geq 1 \) and any \( t \geq L \).
Let us now consider an odd number \( j \) such that \( j \geq \max\{j_0, j_1\} \). The estimate (2.15) can be estimated by

\[
F_1(t) \geq \exp \left( (pq)^{\frac{n-1}{2}} \log(E_2 \varepsilon^p) \right) (R + t)^{-\alpha_j} (t - L)^{\beta_j}
\geq \exp \left( (pq)^{\frac{n-1}{2}} \log(E_2 \varepsilon^p) - \left( \alpha_1 + \frac{1}{2} \frac{pq - 2p - (n-1)}{pq - 1} \right) \log(R + t) + \left( \beta_1 + \frac{1}{pq - 1} \right) \log(t - L) \right)
\times (R + t) \frac{(n+1)pq - 2p - (n-1)}{2(pq - 1)} (t - L)^{-\frac{1}{pq - 1}}
\]

for any odd number \( j \geq \max\{j_0, j_1\} \) and any \( t \geq L \). Choosing \( t \geq \max\{R, 2L\} \), since \( R + t \leq 2t \) and \( t - L \geq t/2 \), the functional \( F_1(t) \) can be estimated by the following way:

\[
F_1(t) \geq \exp \left( (pq)^{\frac{n-1}{2}} \log \left( E_2 \varepsilon^p \frac{n-1}{2} \frac{(n+1)pq - 2p - (n-1)}{pq - 1} \frac{pq - 1}{pq - 1} t - \frac{pq - 2p - (n-1)}{2(pq - 1)} \frac{n+1}{pq - 1} \frac{pq - 1}{pq - 1} \right) \right)
\times (R + t) \frac{(n+1)pq - 2p - (n-1)}{2(pq - 1)} (t - L)^{-\frac{1}{pq - 1}}
\]  

(2.21)

for any odd number \( j \geq \max\{j_0, j_1\} \). The exponent of \( t \), in the previous one, can be represented as follows:

\[
-\frac{(n-1)(p-1)}{2} - \frac{(n+1)pq - 2p - (n-1)}{2(pq - 1)} + \frac{pq}{pq - 1}
= \frac{p(-(n-1)pq + n + 1)}{2(pq - 1)} =: pT_1(p, q, n).
\]

By our assumption that \( pq < (n+1)/(n-1) \) for any \( n \geq 2 \) and \( p, q > 1 \) for any \( n \geq 1 \), the power of \( t \) in the exponential term of (2.21) is positive.

In a similar way to the above, we may deduce the lower bound estimate for an odd number \( j \) fulfilling \( j \geq \max\{j_0, j_1\} \)

\[
F_2(t) \geq \exp \left( (pq)^{\frac{n-1}{2}} \log \left( E_2 \varepsilon^p \frac{n-1}{2} \frac{(n+1)pq + 2p - (n+1)}{pq - 1} \frac{pq - 1}{pq - 1} t - \frac{pq - 2p - (n+1)}{2(pq - 1)} \frac{n+1}{pq - 1} \frac{pq - 1}{pq - 1} \right) \right)
\times (R + t) \frac{(n+1)pq + 2p - (n+1)}{2(pq - 1)} (t - L)^{-\frac{q}{pq - 1}}
\]

Thus, the power of \( t \) in the exponential term can be represented by

\[
-\frac{(n+1)(q-1)}{2} - \frac{(n-1)pq + 2p - (n+1)}{2(pq - 1)} + \frac{q}{pq - 1}
= \frac{q(-(n+1)pq + 2p + n + 1)}{2(pq - 1)} =: qT_2(p, q, n).
\]

By assuming \( (n+1)pq - 2p - (n+1) < 0 \), the power for \( t \) in the exponential term of (2.22) is positive. We should emphasize that

\[
\{(p, q) : (n+1)pq - 2p - (n+1) < 0\} \subseteq \{(p, q) : (n-1)pq < (n+1)\}
\]

for all \( p, q > 1 \) and \( n \geq 1 \). To put it differently, the condition \( (n-1)pq < n + 1 \) is sufficient to guarantee the positivity of the power for \( t \) in the exponential term of (2.22).

Eventually, for studying upper bound estimates for the lifespan, we now should introduce \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, p, q, n, R) > 0 \) such that

\[
\left( E_{2}^{-\frac{(n+1)(p-1)}{2} + \frac{(n+1)pq - 2p - (n-1)}{2pq - 1} + \frac{pq}{pq - 1}} \right)^{1/pT_1(p, q, n)} := E_4 \geq \varepsilon_0^{1/T_1(p, q, n)}.
\]
Hence, for \( \varepsilon \in (0, \varepsilon_0] \) and for \( t > E_3\varepsilon^{-1/T_1(p,q,n)} \) as well as \( t \geq \max\{R, 2L\} \), letting \( j \to \infty \) in (2.21), we claim that the lower bound for the functional \( F_1(t) \) blows up. By the same way, in the case when \( T_2(p,q,n) > 0 \), then we also can find a positive constant \( \varepsilon_0 = \varepsilon_0(u_0, u_1, v_0, v_1, p, q, n, R) > 0 \) such that

\[
\left( E_3^{-1} \frac{(n+1)(q-1)}{2} + \frac{(n-1)p+2q-(n+1)}{2(q-1)} + \frac{q}{p+1} \right)^{1/(qT_2(p,q,n))} := E_5 \geq \varepsilon_0^{1/T_2(p,q,n)}.
\]

For \( \varepsilon \in (0, \varepsilon_0] \) and \( t > E_5\varepsilon^{-1/T_2(p,q,n)} \) carrying \( t \geq \max\{R, 2L\} \), letting \( j \to \infty \) in (2.22), we may immediately show that the lower bound for the functional \( F_2(t) \) blows up. In conclusion, these statements proved that the energy solution \((u, v)\) is not defined globally in time and, simultaneously, the lifespan of this local (in time) solution \((u, v)\) can be estimated by

\[
T(\varepsilon) \leq C\varepsilon^{-1/ \max\{T_1(p,q,n), T_2(p,q,n)\}} = C\varepsilon^{-1/T_1(p,q,n)},
\]

where we used \( T_1(p,q,n) > T_2(p,q,n) \) for any \( n \geq 1 \). The proof of the theorem is complete.

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