Eigenmoments for Multifragmentation

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Abstract: Linear rate equations are used to describe the cascading decay of an initial heavy cluster into fragments. Using a procedure inspired by the similar, but continuous case of jet fragmentation in QCD, this discretized process may be analyzed into eigenmodes, corresponding to moments of the distribution of multiplicities. The orders of these moments are usually noninteger numbers. The resulting analysis can be made time independent and is applicable to various phenomenological multifragmentation processes, in which case it leads to new approximate finite-size scaling relations for the spectrum of fragments.

In this work we consider binary fragmentation processes where any fragment with mass number $k$ breaks into fragments with mass numbers $j$ and $k - j$, $j = 1, 2... k - 1$, with a probability $w_{jk}$ per unit of time. It is assumed that $w_{jk}$ is time independent. By definition, $w_{jk}$ is symmetric if $j$ is replaced by $k - j$, naturally. Let $N_j(t)$ be the multiplicity of fragment $j$ at time $t$ in a process initiated from the decay of a cluster $A$, namely $N_j(0) = \delta_{jA}$. The model under study is described by the following set of linear, first order differential equations,

$$\frac{dN_j}{dt} = -c_j N_j + \sum_{k=j+1}^{A} w_{jk} N_k, \ j = 1,...A, \quad (1)$$

with

$$c_j = \sum_{\ell=1}^{j-1} \frac{w_{\ell j}}{2}. \quad (2)$$

With components $N_j, \ j = 1,...A$, for a column vector $|N>$, the system, Eqs.(1), obviously boils down to $d|N>/dt = \mathcal{W}|N>$ with a triangular matrix $\mathcal{W}$. For the sake of clarity we show here $\mathcal{W}$ when $A = 4$,

$$\mathcal{W} = \begin{pmatrix} 0 & w_{12} & w_{13} & w_{14} \\ 0 & -w_{12}/2 & w_{23} = w_{13} & w_{24} \\ 0 & 0 & -(w_{13} + w_{23})/2 & w_{34} = w_{14} \\ 0 & 0 & 0 & -(w_{14} + w_{24} + w_{34})/2 \end{pmatrix}. \quad (3)$$
The general solution of Eqs.(1) is obviously a sum of exponentials whose rates of decay in time are the diagonal matrix elements $-c_j$, the trivial eigenvalues of the triangular $W$.
(We notice that any increase of the dimension $A$ leaves intact the preexisting eigenvalues and only adds new ones.)

This matrix $W$ has a remarkable property, namely a fixed left (row-like) eigenstate $M_1$, whose components $M_{1j} = j$, $j = 1, \ldots A$, do not depend on the $w_{tk}$'s. This comes from the symmetries of $W$ demanded by the conservation of the total mass $M_1 = \sum_{j=1}^{A} j N_j$, with $dM_1/dt = \langle M_1 | W | N \rangle$. The corresponding eigenvalue is, naturally, $-c_1 = 0$. Moreover, it is clear that the other eigenvalues, $-c_\lambda$, $\lambda = 2, \ldots A$, induce a triangular matrix of left† eigenstates $M_\lambda$, namely the components $M_{\lambda j}$ vanish when $j < \lambda$.

The higher $\lambda$, the more the corresponding (bra-like) eigenstate is only probing heavy fragments in the multiplicity distribution $N$. This is reminiscent of a hierarchy of moments $M_q = \sum_{j=1}^{A} j^q N_j$, when the exponent $q$ increases. Even though moments do not make a strictly triangular rearrangement of the information contained in $N$, they represent a natural continuation of the first eigenvector $M_1$, and this letter will show that there is a practical connection between moments and eigenvectors. Indeed, besides triangularity, there is another argument indicating the interest of moments in the solution of rate equations like Eqs.(1). In field theory models of jet fragmentation at high energy, similar evolution equations appear as solutions of perturbative QCD; they are continuous, since the quantity which fragments is the energy-momentum of quarks and gluons, before their transformation into the observed hadrons. These equations are well known as Gribov-Lipatov-Altarelli-Parisi (GLAP) equations[1]. As a function of the fraction $x$ of energy-momentum, the equations for the fragmentation function $N(x)$ can be exactly diagonalized by moments $M_q = \int dx x^q N(x)$. In this case, all values of $q \geq 1$ are admitted, as a consequence of the continuous character of the equations. Our task, in this letter, is to examine the effect of discretization on this field-theoretical result. Note that such a discretization have been introduced long ago[2], but only for a numerical approximation of the continuous equations.
In the following, we will consider “triangularly redefined” moments

\[ M_\lambda = \sum_{j=\lambda}^{A} j^{q(\lambda)} N_j, \]  

where the set of exponents \( q(\lambda), \lambda = 2, \ldots A, \) may contain nonintegers. The time derivative of such a redefined moment is

\[ \frac{dM_\lambda}{dt} = \sum_{j=\lambda}^{A} j^{q(\lambda)} \left( \sum_{k=j+1}^{A} w_{jk} N_k - c_j N_j \right) = \sum_{k=\lambda}^{A} k^{q} N_k d(\lambda, k), \]  

with

\[ d(\lambda, k) = \left( \sum_{j=\lambda}^{k} \left( \frac{j}{k} \right)^q w_{jk} - \frac{1}{2} \sum_{j=1}^{k} w_{jk} \right), \]  

where we have used an interchange of indices \( j \) and \( k \) in the double summation, and also the convention that diagonal rates \( w_{kk} \) are identically vanishing. The double summation is sketched on Fig.1, displaying the weight matrix \( w_{jk} \).

It turns out that, in practice, there exists special values \( q(\lambda) \) for which the coefficients \( d(\lambda, k) \) happen to be, at least approximately, independent of \( k \) when \( k \) is large enough. Then, they can be factored out in formula (5) and they appear as eigenvalues for the corresponding eigenvector \( M_\lambda \). Moreover, simultaneously, since we know the exact eigenvalues (see Eq.(2)), one gets \( d(\lambda, k) \simeq -c_\lambda \). This will be shown both analytically in a “continuous” limit (i.e. large matrices) and numerically for various models with finite sizes \( A \). In fact one is led to consider first the discrete models which admit the field-theoretical type of equations in the continuous limit - let us call them the \textit{scale-invariant case}, since no dependence on the matrix size appears explicitly, and then the more general situation.

1. \textit{The scale-invariant case}

When \( A \) is large, \( \lambda \) finite, and \( k \) large but smaller than \( A \), then the ratio \( x = j/k \) can be considered as a continuous label, \( 0 \leq x \leq 1 \), in Eq.(6). Moreover, with positive values of \( q \), an extension of the first summation in Eq.(6), \( \sum_{j=\lambda}^{k} \), into a summation \( \sum_{j=1}^{k} \) brings a weak contribution from \( 0 < x < \lambda/k \). From QCD, where \( x \) corresponds to the fragmentation of momentum and a scale-invariant property of the transition weights is
valid, one may consider a large class of models setting
\[ w_{jk} = \varphi(j/k)/k = \varphi(x)/k, \tag{7} \]
where \( \varphi \) is any suitable function of the scaling variable \( x \). More precisely, because of the symmetry necessary for \( w_{jk} \), a large class of legitimate models correspond to \( w_{jk} = [f(j/k) + f(1 - j/k)]/(2k) = [f(x) + f(1 - x)]/(2k) \). Hence, for large values of \( k \), both summations in Eq.(6) amount to the discretization of an integral
\[
d(\lambda) = \int_0^1 dx \frac{[f(x) + f(1 - x)](x^q - 1/2)}{2} = \int_0^1 dx \frac{[f(x) + f(1 - x)][x^q + (1 - x)^q - 1]}{4}, \tag{8} \]
where \( dx \) replaces \( 1/k \). It will be noticed that \( d(\lambda) \) does not depend any more on \( k \). It still depends on \( \lambda \) via the exponent \( q(\lambda) \), naturally. Note also that the “splitting” function \( f \) can be general, even with some singularity at both ends of the integration domain, provided the integral itself converges.

There may also be a continuous limit for \( c_\lambda \) if \( \lambda \) becomes large. Indeed, according to Eq.(2), \( c_\lambda \rightarrow \int_0^1 dx \varphi(x)/2 \), if \( 1/\lambda \) amounts to \( dx \) and if this integral converges. In such a case, the spectrum of \( W \) accumulates into a quasi degeneracy. However, it is important to realize that the convergence of the \( c_j \)’s is not required. On the contrary, it is quite possible that the limiting continuous model does not exist, leading to an infinite hierarchy of \( q(\lambda) \). We shall meet such cases later on. Moreover, for low values of the label \( \lambda \), this continuous limit is not in order.

We now notice that \( \varphi(x) \) is a semi-positive definite function since \( w_{jk} \), a transition rate, cannot become a negative number. Hence \( d(\lambda) \) is a monotonically decreasing function of \( q \). It vanishes for \( q = 1 \), as expected from the conservation of \( M_1 \). According to Eq.(5), the time evolution of a triangular moment \( M_\lambda \) becomes very simple if \( d(\lambda) \) can be identified with the eigenvalue, \(-c_\lambda\). Hence, for each \( \lambda \), we consider the exponent \( q(\lambda) \) which is the unique solution of the consistency equation
\[
\int_0^1 dx [f(x) + f(1 - x)][1 - x^{q(\lambda)} - (1 - x)^{q(\lambda)}] = 4c_\lambda. \tag{9} \]
The discrete set of solutions \( q_\lambda \) of this equation, when the integer label \( \lambda \) runs from 1 to \( A \), define the “eigenmoments” of the theory, namely those moments whose time
evolution is (almost) proportional to just one exponential $exp(-c\lambda t)$, rather than a mixture of such exponentials. It will be noticed that, since the sequence of coefficients $c_\lambda$ increases monotonically, the sequence of solutions $q(\lambda)$ is also a monotonically increasing sequence, starting from $q(1) = 1$ with $c_1 = 0$.

Set temporarily $f(x) = 1/x^{\beta}$, with $\beta = 1$. This case is reminiscent of the QCD evolution equations for gluons whose kernel contains the same singularity at small $x$ [1]. Then $w_{jk} = [1/j + 1/(k-j)]/2$, and one finds easily that $c_2 = 1/2$, and that the corresponding solution of Eq.(9) is $q(2) = 2$. One also finds that $c_3 = 3/4$, and that the corresponding solution of Eq.(9) is $q(3) = 3$. More generally, one finds for Eq.(9) the solution $q(\lambda) = \lambda$. This definitely suggests that integer moments form an infinite sequence of eigenmoments for that choice of $f$, $f(x) = 1/x$. In agreement with this hint, ones finds easily from Eq.(6) that the sequence of coefficients $d(2,20) = -0.475$, $d(2,21) = -0.476$, $d(2,49) = -0.4898$, $d(2,50) = -0.4900$,... converges towards $-c_2$. Just to give another example, the sequence $d(5,20) = -0.980$, $d(5,21) = -0.983$, $d(5,49) = -1.016$, $d(5,50) = -1.017$,... converges towards $-c_5 = -1.042$. And so on for all the moments $M_\lambda$, which thus generate excellent approximations to eigenvectors when the exponents $q(\lambda)$ are just integers.

This argument is independent from the normalization of $f$, since Eq.(9) is homogeneous with respect to trivial multiplications of $f$ by an overall constant. It may be interesting to note that the diverging sequence of values for $q(\lambda)$ might be related to the divergence of the eigenvalue sequence at infinite $\lambda$.

A similar result can be observed if $\beta \neq 1$, but now the solutions of Eq.(9) do not correspond to integer exponents. For instance, with $\beta = -0.5$, one finds $c_2 = 0.177$, $q(2) \simeq 3.13$, $c_3 = 0.232$, $q(3) \simeq 5.26$, $c_4 = 0.259$, $q(4) \simeq 7.4$, $c_5 = 0.275$, $q(5) \simeq 9.6$,... The convergence of the coefficients $d$ towards the eigenvalues is still surprisingly good. For instance $d(2,20) = -0.176$, and $d(5,20) = -0.275$.

This discussion (for the scale invariant case) is illustrated on Figs.2. Here, rather than asking whether a moment may behave like an eigenvector, we consider the reverse question: given an eigenvector $M_\lambda$, does it happen that the components $M_{\lambda j}$ induce effective moments $M_{q(\lambda)}$? Namely, is there an exponent $q(\lambda)$ compatible with $M_{\lambda j} \propto$
$j^q(\lambda), j > \lambda$? In Figs.2, for instance, the components of the second ($\lambda = 2$) and fourth ($\lambda = 4$) eigenvectors are displayed as functions of the fragment size. It will be stressed that they are almost linear in a Log-Log plot in a large interval starting from the maximal chosen value $A = 30$. These figures, Figs.2, show the structure of moments depending on the parameter $\beta$.

In table I we show the comparison between the actual eigenvalues $c_\lambda$, (for $\lambda = 2$ and 4) with those obtained through Eq.(9) after the determination of the effective values $q(\lambda)$ from Figs.(2). The agreement is pretty good, except perhaps for $\beta \simeq 2$, which is at the borderline of convergence of the integral in Eq.(9). It can be thus claimed that various choices of $\varphi$ in Eq.(7) make it possible to find eigenmoments via the continuous limit and solutions of Eq.(9). It will be noticed that this continuous limit is closely linked to the denominator $k$ in Eq.(7), since this denominator induces the needed measure $dx$, independently of the overall scale given by $A$.

2. Scale-dependent cases

In a more general situation, $w_{jk}$ is not compatible with Eq.(7). This will in general lead to an introduction of the overall scale $A$ in the problem, and corresponds to cases where the scale invariance of the weights is not preserved. Let us illustrate this by the following instance:

$$w_{jk} = [(j/k)^{-\beta} + (1 - j/k)^{-\beta}]/(2k^\alpha),$$  \hspace{1cm} (10)

where $\alpha$ may be different from 1.

To be specific, but as an example of more general value, we display on Fig.3 the Log-Log plot $\mathcal{M}_{\lambda j}$ versus $j$, for $\lambda = 2, A = 50, \beta = 0$ and various values of $\alpha$. From this figure, one realizes on this simple example that the diagonalization by eigenmoments is obtained for $\alpha \leq 1$, while for $\alpha > 1$ there is a clear distortion of eigenvectors with respect to moments. It is interesting to interpret this phenomenon analytically by inspecting the modification which occurs with the choice of Eq.(10) for the weights. Instead of Eqs.(5-6), one finds the following,

$$\frac{dM_\lambda}{dt} = \sum_{k=\lambda}^A k^{q-\alpha+1} N_k d(\lambda, k), \quad d(\lambda, k) = \left(\sum_{j=\lambda}^k \left(\frac{j}{k}\right)^q \tilde{w}_{jk} - \frac{1}{2} \sum_{j=1}^k \tilde{w}_{jk}\right),$$  \hspace{1cm} (11)
where $\bar{w}$ corresponds to *rescaled* weights with $\alpha = 1$. Following the previous discussion, based on the existence of a continuous limit $d(\lambda, k) \simeq d(\lambda)$, one is led to the approximate consistency equation:

$$-d(\lambda) \simeq A^{\alpha-1} c(\lambda),$$

(12)

where the renormalisation factor $A^{\alpha-1}$ takes care of the initial values of the moments. The occurrence of the $A$–dependent factor is the signal of the lack of scale invariance of the fragmentation dynamics when $\alpha \neq 1$. Note that Eq.(12) uses the assumption that the eigenvalues do not change substantially between $q$ and $q - 1 + \alpha$. It can only be an approximation.

In Table II, we display the different values obtained for the $q(2)$ for various values of $\alpha$, obtained by the fitted slopes at the origin for the curves obtained in Fig.3. We compare them to those obtained from Eq.(12) when the input are the actual exact eigenvalues $c_2$. The agreement is here also quite satisfactory, except in the region when $\alpha > 1$. One might associate this phenomenon to the well-known\cite{3} fact that a shattering transition takes place at finite time, the conservation of mass being broken in the continuum limit. A special study of this case is in order for the future.

The class of models which can be analyzed by eigenmoments is thus larger than the class described by Eq.(7). It must be noted, however, that the “eigenorders” $q(\lambda)$ are not universal, but clearly model-dependent.

As an application of the properties of eigenmoments, let us consider the problem of 3-dimensional bond percolation in a finite-size square lattice. This model seems to give a successful description of nuclear multifragmentation, when a heavy ion receives enough excitation energy to form a highly unstable state and decays into several fragments\cite{4}. The statistics of fragment numbers and sizes seem to follow predictions of a percolation model in which each lattice site is populated by a nucleon, and the percolation parameter $p$, namely the survival probability for bonds, varies between 0 and 1. There is no obvious time scale in this model, hence a comparison of its predictions with those of linear rate equations models requires the use of eigenmoments in order to obtain an intrinsic time scale from the evolution of such eigenmoments.
For this purpose, we remark that, if they are identified as eigenmoments, the $M_q$'s are linked by linear relations in Log-Log plots, and their explicit time dependence disappears. We are thus led to display in the same way the moments obtained from the percolation model, choosing for instance $M_2$ for reference, see Fig.4. Different moments are displayed (with $q = 1, 1.5, 2, 3, 4, 5$) and show the interesting feature of a quasi-linear dependence for the values $q = 3, 4, 5$, given the fact that for $q = 1$ (mass conservation) and $q = 2$ (reference scale) the linear dependence is fixed. It is clear from this figure that the quasi-linear form is obtained between $p = 1$ and $p = p_c$, where $p_c$ is the critical value above which, in the continuous limit, an infinite percolation cluster is formed. Indeed, the figure shows the dominant contribution of the cluster of largest mass to the averaged moments. This largest cluster is, for finite size problems, the representative of the infinite cluster when $p \geq p_c$.

Notice that the moments implied by the rate equations are the full moments, including the largest fragment, while in usual analyses of percolation models\cite{4}, scaling properties are investigated with moments modified by the subtraction of the largest cluster. Moreover, in such traditional analyses of percolation, the reference time scale is generally given by the moment $M_0$ or a similar variable related to the multiplicity of fragments. The comparison and compatibility of our approach with such analyses is an open problem of some interest\cite{6}.

In conclusion, from this first study on the percolation model, we obtain a hint that linear rate equations could provide a time dependent description of multifragmentation. But it is difficult at this stage to obtain informations on the set of eigenorders $q(\lambda)$ which could be associated with percolation. The existence of scaling relations between moments can be proven in the vicinity of $p_c$ for percolation through finite-size scaling\cite{4}. Our result is compatible with this and, furthermore, involves the whole region $1 > p > p_c$. In the representation provided by rate equations, we have obtained scaling relations valid for short time scales, while the previous general results, see for instance Ref.\cite{7}, involve long time scales only. An open problem is to connect both analyses for a general system of equations.

Acknowledgments R.P. thanks Xavier Campi for his patient explanations on nuclear multifragmentation and for providing the authors with the suitable percolation program.
Thanks are due to Gérard Auger, Brahim Elattari, Pierre Grangé, Hubert Krivine, Eric Plagnol and Jean Richert for fruitful discussions.

†Footnote The right-hand-side, ket-like eigenstates define also a triangular matrix, apparently unrelated to the matrix of bra eigenstates, except for trivial biorthogonality relations. Up to now we have been unable to find a practical use of these ket-like eigenvectors.

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Figure captions

Fig.1: Discrete rate equations: Double Summation
The double summation range for truncated moments $M_\lambda$ is represented by the hatched triangle. White dots: diagonal weights $w_{jj}$ are zero. Black dots: non-diagonal weights $w_{jk}$. The interchange of indices $j$ and $k$ in the description of the hatched triangle leads to Eqs.(5-6).

Fig.2: Eigenvectors: scale-invariant case ($\alpha = 1$)
The components of the rate equation eigenvectors $M_{\lambda j}$ are displayed as functions of $j$ in a Log-Log plot. The chosen weights correspond to Eq.(10) with $\alpha = 1$, and different values of $\beta$. The curves correspond to a smooth interpolation (dark line), resp. extrapolation (dashed line), of the exact eigencomponents for a system of size $A = 30$. Fig 2-a: $\lambda = 2$ ; Fig 2-b: $\lambda = 4$. The curves are used for the determination of the effective values $q(2), q(4)$, see Table I.

Fig.3: Eigenvectors: scale-dependent cases ($\beta = 0$)
Same as Figure 2-a but for weights following Eq.(10) with $\beta = 0$ and different values of $\alpha$. The curves give the eigenorders listed in Table II.

Fig.4 Percolation analyzed with the “$M_2$ time scale”
Relative strengths of moments $M_q$, $q = 0, 1, 1.5, 2, 3, 4, 5$, as functions of $M_2$ in a Log-Log plot. Data taken from 3-d bond percolation on a $6*6*6$ lattice. The corresponding values of the bond survival probability $p$ are shown on the horizontal axis. Its critical value is $p_c = .25$. Full lines: moments. Dashed lines: contributions of the largest cluster. Dashed-dotted line: the reference moment $M_2$. Notice that a linear behaviour is approximately obtained for $0 \leq p \lesssim p_c$ and $q = 3, 4, 5$.

Table Captions

Table I: For $\alpha = 1$, $\lambda = 2, 4$ and different values of $\beta$, the effective exponents $q(\lambda)$ from Figs.2-a,b and the corresponding values for $-d(\lambda)$, see Eq.(9). The latter are compared with the exact eigenvalues $c_\lambda$.

Table II: For $\beta = 0$, $\lambda = 2$ and different values of $\alpha$, comparison of the measured effective eigenorders (obtained from Fig.3) with those predicted from Eq.(9).
\[
\beta \quad q(2) \quad -d(2) \quad c_2 \quad q(4) \quad -d(4) \quad c_4
\]

| $\beta$ | $q(2)$ | $-d(2)$ | $c_2$ | $q(4)$ | $-d(4)$ | $c_4$ |
|--------|--------|--------|------|--------|--------|------|
| -2     | 2.4    | .061   | .062 | 6.1    | .11    | .11  |
| -1.5   | 2.7    | .09    | .09  | 6.5    | .14    | .14  |
| -.5    | 3.1    | .18    | .18  | 7.4    | .26    | .26  |
| 0      | 3      | .25    | .25  | 7.1    | .38    | .38  |
| .5     | 2.6    | .36    | .35  | 5.8    | .57    | .57  |
| 1      | 2      | .51    | .50  | 4.2    | .94    | .92  |
| 1.5    | 1.5    | .79    | .71  | 2.8    | 1.8    | 1.6  |
| 2      | 1.1    | .85    | 1    | 1.7    | 4.6    | 2.7  |

Table I

\[
\alpha \quad q(2)_{\text{measured}} \quad q(2)_{\text{predicted}}
\]

| $\alpha$ | $q(2)_{\text{measured}}$ | $q(2)_{\text{predicted}}$ |
|----------|--------------------------|--------------------------|
| 0        | 1.03                     | 1.04                     |
| .5       | 1.17                     | 1.22                     |
| .6       | 1.26                     | 1.33                     |
| .8       | 1.64                     | 1.7                      |
| 1        | 3.0                      | 3                        |
| 1.1      | 5.7                      | 5.2                      |
| 1.2      | 20                       | 14                       |

Table II