BLOW-UP OF SOLUTIONS TO THE PATLAK-KELLER-SEGEL EQUATION IN DIMENSION $\nu \geq 2$

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Abstract. We prove a blow-up criterion for the solutions to the $\nu$-dimensional Patlak-Keller-Segel equation in the whole space. The condition is new in dimension three and higher. In dimension two it is exactly Dolbeault’s and Perthame’s blow-up condition, i.e., blow-up occurs, if the total mass exceeds $8\pi$.

1. Introduction

It is well-known that Patlak-Keller-Segel type equations describing chemotaxis (Patlak [11] and Keller and Segel [7, 8]) allow for both diffusion and aggregation phenomena: depending on the initial data, the solution might exist globally in time or blow up in finite time. Since Jäger’s and Luckhaus’ [6] pioneering work the analysis of this system proliferated. It is too ambitious to mention all the important results in this short article, instead we refer to the review of Wang et al [14].

We will focus on the blow-up of solutions of the following system

$$\begin{align*}
\partial_t \rho - \nabla \cdot (\nabla \rho - \rho \nabla c) &= 0, \\
-\Delta c &= \rho - Z\delta, \quad x \in \mathbb{R}^\nu, \quad t \geq 0, \\
\rho(x, 0) &= \rho_0(x) \geq 0,
\end{align*}$$

(1)

where $\delta$ is the Dirac delta function at the origin. The parameter $Z \in \mathbb{R}$ represents the strength of the external source. The source is repulsive, if $Z$ is positive, and attractive, if $Z$ is negative.

For sake of simplicity, we write $M_\mu(t) := \int_{\mathbb{R}^\nu} |x|^\mu \rho(x, t)dx$, $\mu = 0, 1, \cdots$ for the moments of a solution $\rho$ at time $t$. The system (1) conserves the total mass, i.e., for all times $t$ for which the solution exists

$$M_0(t) = M_0(0) =: M_0.$$

For $\nu = 2$ and $Z = 0$, Dolbeault and Perthame [4] and – later in greater detail – Blanchet et al [1] showed that, if the mass $M_0$ exceeds $8\pi$, then any classical solution blows up in finite time, while, if $M_0 < 8\pi$, then a classical solution exists globally, since the diffusion dominates the aggregation which follows from the logarithmic Hardy-Littlewood-Sobolev inequality.

Also for $\nu = 2$, Wolansky and Espejo [15], show that adding a repulsive point source will slow down the blow-up compared to $Z = 0$, where the blow-up occurs for masses that exceed $8\pi$, while adding an attractive point source will enhance blow-up. This generalizes [4].

In higher dimensions, i.e., $\nu > 2$, it is known (Perthame [12, Chapter 6], Chapter 6) that there exists a constant $c$ such that, if the initial datum fulfills $\|\rho_0\|_{L^{\nu/2}} \leq c$, 
then a solution exists globally. On the other hand there exists a (small) constant $c$

(2) $\mathfrak{M}_2 < c\mathfrak{M}_0^{\frac{1}{\nu - 1}}$, 

then all classical solutions blow up in finite time. However, the results are much weaker than the corresponding version in dimension two. This paper is a contribution to fill this gap.

A tool often used to show blow-up is the time evolution of the second moment $\mathfrak{M}_2(t)$: if one can show that the time derivative is strictly negative, then the finite time blow-up is proved. However, the second moment is – in some sense – only natural in dimension two.

The blow-up of solutions has also been studied for other variants of the Patlak-Keller-Segel system. An example is $\partial_t \rho - \nabla \cdot (\nabla \rho^m - \rho \nabla c) = 0$, where a porous media type degenerate diffusion is included. Sugiyama [13] and Blanchet et al [11] obtained both existence and blow-up results for $m = 2 - \frac{2}{\nu}$, and Chen et al [2] for $m = \frac{2\nu}{\nu + 2}$. Furthermore, Chen and Wang [3] obtained sharp results on the occurrence of the blow-up and existence for $m$ in between $2 - \frac{2}{\nu}$ and $\frac{2\nu}{\nu + 2}$. In particular they were able to identify very large classes of initial conditions for which they can predict blow-up in finite time respectively existence of the solutions for different $\alpha$.

In this paper, we treat the $\nu$-dimensional system with a point source of strength $Z$. We have the following sufficient condition for blow-up:

**Theorem 1.** For $\nu \geq 2$, assume that the initial datum $\rho_0$ satisfies

(3) $\mathfrak{M}_2^{\frac{\nu - 2}{\nu - 1}}(0) < \frac{1}{(\nu - 1)2^{\nu}||S^{\nu - 1}||} \mathfrak{M}_0^{\frac{2}{\nu - 1}} + \frac{Z}{2(\nu - 1)||S^{\nu - 1}||} \mathfrak{M}_0^{\frac{1}{\nu - 1}}$, 

where $||S^{\nu - 1}||$ is the volume of $\nu - 1$-dimensional sphere. Then there is no classical solution that exists for all times, i.e., there exists a finite $T > 0$ such that $\lim_{t \to T^-} ||\rho(\cdot, t)||_{\infty} = +\infty$.

This has two immediate consequences:

**No point source:** Without an external point source, i.e., $Z = 0$, the condition (3) is a blow-up criterion for the multi-dimensional parabolic-elliptic Patlak-Keller-Segel system, i.e., if, initially

$$\mathfrak{M}_2^{\frac{\nu - 2}{\nu - 1}}(0) < \frac{1}{(\nu - 1)2^{\nu}||S^{\nu - 1}||} \mathfrak{M}_0^{\frac{2}{\nu - 1}},$$

then the solution blows up in finite time. This is actually the condition we are looking for, since for $\nu = 2$, it is exactly Dolbeault’s and Perthame’s condition [11]

$$1 < \frac{1}{8\pi \mathfrak{M}_0}.$$  

**With point source and $\nu = 2$:** In dimension two, the condition becomes

$$1 + \frac{Z}{4\pi} < \frac{1}{8\pi \mathfrak{M}_0},$$

which is exactly Wolansky’s and Espejo’s blow-up condition [13].

**Remark.** It is a interesting question whether the reverse inequality, i.e.,

$$\mathfrak{M}_2^{\frac{\nu - 2}{\nu - 1}}(0) > \frac{1}{(\nu - 1)2^{\nu}||S^{\nu - 1}||} \mathfrak{M}_0^{\frac{2}{\nu - 1}} + \frac{Z}{2(\nu - 1)||S^{\nu - 1}||} \mathfrak{M}_0^{\frac{1}{\nu - 1}},$$

then a solution exists globally. On the other hand there exists a (small) constant $c$ such that, if the initial datum fulfills
would already imply existence of a classical solution. It has an affirmative answer in dimension two but is open in higher dimensions.

2. Proof of the main result

Proof. By using the fundamental solution $\Phi$ of the Poisson equation, the system (1) can be rewritten into the following form, for $\nu \geq 2$,

\[ \partial_t \rho = \Delta \rho - \nabla [\rho \mathcal{K} * (\rho - Z\delta)], \quad x \in \mathbb{R}^\nu, \]

where

\[ \mathcal{K} = \nabla \Phi = -\frac{1}{|\mathbb{S}^{\nu-1}|} \frac{x}{|x|^{\nu}}. \]

Multiplication of (1) by $|x|^{\nu}$ and integration gives

\[
\frac{d}{dt} \int_{\mathbb{R}^\nu} |x|^{\nu} \rho(x) dx \leq 2\nu (\nu - 1) \int_{\mathbb{R}^\nu} |x|^{\nu-2} \rho(x) dx + \frac{\nu Z}{|\mathbb{S}^{\nu-1}|} \int_{\mathbb{R}^\nu} \rho(x) dx
\]

\[
- \frac{\nu}{|\mathbb{S}^{\nu-1}|} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} |x|^{\nu-1} \frac{x - y}{|x - y|^\nu} \rho(x) \rho(y) dy dx dy
\]

\[ = 2\nu (\nu - 1) \int_{\mathbb{R}^\nu} |x|^{\nu-2} \rho(x) dx + \frac{\nu Z}{|\mathbb{S}^{\nu-1}|} \int_{\mathbb{R}^\nu} \rho(x) dx
\]

\[
- \frac{\nu}{2|\mathbb{S}^{\nu-1}|} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} \left( |x|^{\nu-1} \frac{x}{|x|} - |y|^{\nu-1} \frac{y}{|y|} \right) \rho(x) \rho(y) dy dx =: V
\]

We estimate $V$ from below. To this end we set $r := |x|$, $s := |y|$, and $u := x \cdot y /(|x||y|)$. Thus

\[
V = \frac{r^\nu - (r^{\nu-1} s + rs^{\nu-1}) u + s^\nu}{(r^2 + s^2 - 2rsu)u^{\nu/2}}
\]

\[ = \frac{(r/s)^{\nu/2} + (s/r)^{\nu/2} - [(r/s)^{\nu/2-1} + (s/r)^{\nu/2-1}]u}{(r/s + s/r - 2u)^{\nu/2}}. \]

The right hand side of (3) is monotone increasing in $u$. To see this we first write $\tau := r/s$ and remark that the derivative of $V$ with respect to $u$ is up to an irrelevant non-negative factor

\[
\geq (2 - \nu) \left( \tau^{\frac{\nu}{2} - 1} + \tau^{-\left(\frac{\nu}{2} - 1\right)} \right) + \nu \left( \tau^{\frac{\nu}{2}} + \tau^{-\frac{\nu}{2}} \right) \geq 0
\]

where we have used that $\tau^\alpha + \tau^{-\alpha}$ is monotone increasing in $\alpha$ for positive $\alpha$ and $\tau$. Thus

\[
V \geq \frac{(r/s)^{\nu/2} + (s/r)^{\nu/2} - [(r/s)^{\nu/2-1} + (s/r)^{\nu/2-1}]u}{(r/s + s/r + 2u)^{\nu/2}} \geq \min_{\tau \in [0,1]} f(\tau)
\]

\[
= \frac{(r/s)^{\frac{\nu}{2}} + (s/r)^{\frac{\nu}{2}} - [(r/s)^{\frac{\nu}{2}-1} + (s/r)^{\frac{\nu}{2}-1}]}{(\sqrt{r/s} + \sqrt{s/r})^{\nu}} \geq \min_{\tau \in [0,1]} f(\tau)
\]
with
\[ f(\tau) = \frac{1 + \tau^\nu + \tau + \tau^\nu - 1}{(1 + \tau)^\nu} = \frac{1 + \tau^\nu - 1}{(1 + \tau)^\nu - 1}. \]

Next we prove that \( f \) is decreasing and the minimum achieved at \( \tau = 1. \) In fact,
\[ f'(\tau) = \frac{(\nu - 1)\tau^{\nu - 2}(1 + \tau)^{\nu - 1} - (\nu - 1)(1 + \tau)^{\nu - 2}(1 + \tau^{\nu - 1})}{(1 + \tau)^{2(\nu - 1)}} \]
\[ = -\frac{\nu - 1}{(1 + \tau)^{\nu - 2}} - 1 \leq 0. \]

Thus
\[ V \geq f(1) = 2^{2-\nu} \]
and therefore
\[ \frac{d}{dt} \mathcal{M}_\nu \leq 2\nu(\nu - 1)\mathcal{M}_{\nu - 2} \frac{\nu^{2-\nu}}{|S^{\nu - 1}|} \mathcal{M}_0^2 + \frac{\nu Z}{|S^{\nu - 1}|} \mathcal{M}_0. \]

Estimating by Hölder’s inequality yields
\[ \mathcal{M}_\nu \leq 2\nu(\nu - 1)\mathcal{M}_{\nu - 2} \mathcal{M}_0^{\frac{\nu - 2}{\nu}} - \frac{\nu^{2-\nu}}{|S^{\nu - 1}|} \mathcal{M}_0^2 + \frac{\nu Z}{|S^{\nu - 1}|} \mathcal{M}_0. \]

In particular, we have a shrinking \( \nu \)-th moment, if the initial moments fulfill
\[ \mathcal{M}_{\nu - 2} (0) < \frac{1}{(\nu - 1)^2 |S^{\nu - 1}|} \mathcal{M}_0^{2 - \frac{2}{\nu}} - \frac{Z}{2(\nu - 1)} |S^{\nu - 1}| \mathcal{M}_0^{1 - \frac{2}{\nu}}, \]

However, a shrinking \( \nu \)-th moment implies blow-up. \( \square \)

At this point we would like to remark that the strategy of the proof, namely multiplication with an appropriate power based on a dimensional analysis, has been previously used in the context of effective quantum models and dates back – at least – to an unpublished observation of Benguria to bound the excess charge of atoms. Later, Lieb [10] extended the argument to the quantum case; Lenzmann and Lewin [9] used it in the time dependent setting.

In conclusion, we would like to point that our blow-up condition (3) implies (2).

To see this, we first note that \( \mathcal{M}_2 \leq \mathcal{M}_0^{\frac{\nu - 2}{\nu}} \mathcal{M}_\nu^{\frac{\nu}{\nu - 2}} \) by interpolation. Estimating the right hand side by using (3) gives (2) with the extra bonus of a definite constant instead of an uncontrolled one. Finally, note that this is only necessary when \( \nu > 2 \), since Inequality (2) is an empty statement in dimension two whereas Inequality (3) remains meaningful.

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