Analyticity of extremizers to the Airy–Strichartz inequality

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Abstract

We prove that there exists an extremal function to the Airy–Strichartz inequality

$$\|e^{-t\partial_x^3}f\|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R})} \leq C\|f\|_{L^2(\mathbb{R})},$$

using the linear profile decomposition. Furthermore we show that, if \(f\) is an extremizer, then \(f\) is extremely fast decaying in Fourier space and so \(f\) can be extended to be an entire function on the whole complex domain. The rapid decay of the Fourier transform of extremizers is established with a bootstrap argument which relies on a refined bilinear Airy–Strichartz estimate and a weighted Strichartz inequality.

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1. Introduction

It is well known that the (generalized) Korteweg–de Vries equations (KdV or gKdV) are good approximations to the evolution of waves on shallow water surface \([14, 29, 30]\):

$$\partial_t u + \partial_x^3 u \pm \partial_x (u^p) = 0$$

(1)

for \(p \geq 2\). The linear form is the Airy equation

$$\partial_t u + \partial_x^3 u = 0$$

(2)

In general, for an initial data \(u(0) = f\) the solution \(e^{-t\partial_x^3}f\) to the Airy equation can be expressed as

$$e^{-t\partial_x^3}f(x) := (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ixk + itk^3} \hat{f}(k) \, dk,$$

(3)

The linear Strichartz inequality for (2) asserts that

$$\|D^{\alpha} e^{-t\partial_x^3}f\|_{L^q_t L^r_x} \leq C\|f\|_{L^2},$$

(4)

for \(-\alpha + 3/q + 1/r = \frac{1}{2}\) and \(-\frac{1}{2} < \alpha \leq 1/q\) (see [21, Theorem 2.1]), and some \(C > 0\). When \(\alpha = 1/q\), the inequality above is called ‘endpoint’ while ‘nonendpoint’ for \(\alpha < 1/q\). It plays an important role in establishing local or global wellposedness theory for the Cauchy problem of
In this paper, we study the following symmetrical Strichartz inequality:

$$\|e^{-t\partial_x^2}f\|_{L^8_t L^8_x(\mathbb{R} \times \mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}, \quad (5)$$

and consider ‘extremizers’ for (5): the existence of extremizers and characterization of some of their properties.

To begin with, we denote the optimal constant for (5) by \(A\):

$$A := \sup \{\|e^{-t\partial_x^2}f\|_{L^8_t L^8_x(\mathbb{R} \times \mathbb{R})} : \|f\|_2 = 1\}. \quad (6)$$

A simple argument, together with (4) shows that \(A < \infty\); see the proof of Theorem 2.4.

**Definition 1.1.** A function \(f \in L^2\) is said to be an extremizer for (5) if \(f\) is not equal to the zero function almost everywhere and

$$\|e^{-t\partial_x^2}f\|_{L^8_t L^8_x(\mathbb{R} \times \mathbb{R})} = A \|f\|_{L^2(\mathbb{R})}. \quad (7)$$

The first result is the following theorem.

**Theorem 1.2.** There exists an extremal function \(f \in L^2\) for the Airy Strichartz inequality (5).

This theorem is proved in Section 3. The proof makes use of the linear profile decomposition for the Airy evolution operator \(e^{-t\partial_x^2}\) acting on a bounded sequence of \(\{f_n\} \in L^2\), which we develop in Section 2 based on the previous result in [31]. In [32], the profile decomposition for the Schrödinger equation developed in [2] was used to prove the existence of extremizers to the Strichartz inequality for the Schrödinger equation in higher dimensions. The profile decomposition can be viewed as a manifestation of the idea of ‘concentration-compactness’; see Lions [24–27].

**Remark 1.3.** Theorem 1.2 is different from that of [31] where a dichotomy result is obtained on the existence of extremizers to the Strichartz inequality \(\|e^{-t\partial_x^2}D^{1/6}f\|_{L^6_t L^{6/5}_x(\mathbb{R} \times \mathbb{R})} \leq C \|f\|_{L^2(\mathbb{R})}\), which is the symmetric ‘endpoint’ Strichartz inequality; in other words, for this Strichartz inequality, either an extremizer exists or a sequence of modulated Gaussians approximates to the extremizer. The dichotomy is due to the presence of highly oscillatory terms in the refined profile decomposition; see Theorem 2.3. Another instance of a dichotomy result on extremizers to a Strichartz-type inequality is in [20]. The presence of highly oscillatory terms in the profile decomposition is not a problem for the existence of extremizers if the equation is invariant under boosts, that is, shifts in momentum (or Fourier) space, which is the case for the Schrödinger and wave equations. The Airy equation (2) is, however, not invariant under shifts in momentum space. Hence to obtain the existence of maximizers for (5), we need a profile decomposition which avoids highly oscillatory terms, which is done in Theorem 2.4.

Extremizers to the Strichartz inequality for the Schrödinger equation and the wave equation have been studied intensively recently. For the Strichartz inequality for the Schrödinger equation, Kunze [23] proved the existence of extremizers to the one-dimensional Strichartz inequality by establishing that any nonnegative extremizing sequence converges strongly to an extremizer in \(L^2\) up to the natural symmetries of the inequality. In the lower-dimensional case,
the existence of extremizers was shown by Foschi [17] and Hundertmark and Zharnitsky [19]: Gaussians are extremizers, which are unique up to the natural symmetries of the inequality. Later works devoted to the study of the Strichartz inequality for the Schrödinger equation with different emphases include [3, 7, 10]. To the best of our knowledge, we remark that all the previous known methods do not seem to be adapted directly to finding the explicit form of ‘extremizers’ to (5) in our setting. For extremizers to the Strichartz inequality for the wave equation, see [5, 17].

Closely related to the Strichartz inequality for the Schrödinger equations, Christ and Shao [11, 12] studied ‘extremizers’ to an adjoint Fourier restriction inequality for the sphere, namely the Tomas–Stein inequality $L^2(S^2) \to L^4(\mathbb{R}^3)$ for the two-dimensional sphere $S^2$. Although the Strichartz inequality for the Schrödinger equation can be viewed as an adjoint Fourier restriction inequality for the paraboloid, the situation for the sphere is different from the paraboloid case due to the nonlocal property and the lack of scaling symmetry of the adjoint Fourier restriction operator: $L^2(S^2) \to L^4(\mathbb{R}^3)$. They were able to show that there exists an extremal by proving that any extremizing sequence of nonnegative functions in $L^2(S^2)$ has a strongly convergent subsequence and establish some characterizations of extremals; it is an endpoint result. For existence of extremizers for a family of nonendpoint Fourier restriction operators; see [16]. For existence and characterization of quasi-extremizers and extremizers to the convolution inequality with the surface measure on the paraboloid or the sphere; see [8, 9, 13, 34].

Next we turn to the characterization of the extremizers to (5) from studying the corresponding generalized Euler–Lagrange equation:

$$\omega f = \int e^{it\partial_x^2} [|e^{-it\partial_x^2} f|^6 e^{-it\partial_x^2} f] \, dt,$$

where $\omega$ is a Lagrange multiplier, which for extremizers $f$ is given by $\omega = A^8 \|f\|_6^2$ where $A$ is the optimal constant defined in (6). The Euler–Lagrange equation (8) can be established by a standard variational argument. Traditionally, once the existence of an extremizer has been shown its properties are deduced from studying the associated Euler–Lagrange equation. Note that in our case (8) is a highly nonlinear and nonlocal equation, which makes this a rather nontrivial task. Nevertheless, the following strong regularity result for extremizers holds.

**Theorem 1.4.** For any extremizer $f$ to the Airy–Strichartz inequality (5) there exists $\mu_0 > 0$ such that

$$k \mapsto e^{\mu_0 |k|^3} \hat{f}(k) \in L^2,$$

where $\hat{f}$ is the Fourier transform of $f$. In particular, $f$ can be extended to be an entire function on the complex plane.

The proof of this theorem is based on a bootstrap argument, which relies on a refined bilinear Strichartz inequality for the Airy operator $e^{-it\partial_x^2} f$, and a weighted Strichartz inequality. The argument uses some ideas similar to Erdogan, Hundertmark and Lee [15], which in turn is based in part on [18]. In [15], it is shown that solutions to the dispersion managed nonlinear Schrödinger equation in the case of zero residual dispersion are exponentially fast decaying not only in the Fourier space but also in the spatial space. The fact that [15] also establishes decay in the spatial space is essentially due to the fact that the linear Schrödinger operator $e^{it\Delta}$ involved enjoys an identity

$$e^{it\Delta} f(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\xi + it|\xi|^2} \hat{f}(\xi) \, d\xi = C t^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4it} f(y) \, dy \quad \text{for some } C > 0,$$

for some $C > 0$. (10)
which enables one to obtain the decay in the spatial space from that on the Fourier side. There is no such identity for the Airy operator and thus our Theorem 1.4 gives decay only in Fourier space. On the other hand, the decay given by Theorem 1.4 is much more rapid than even Gaussian decay.

The organization of the paper is as follows. In Section 2, we establish the linear profile decomposition. In Section 3, we show the existence of extremizers to the Airy–Strichartz inequality \( L^2 \to L^8_{t,x} \). In Section 4, we show that any solution to the generalized Euler–Lagrange equation, which includes the extremizer as a special case, obeys a bound of the form (9) and can be extended to be analytic on the complex plane. It is proved by assuming an important bootstrap lemma, which we establish in Section 5.

The notation \( A \lesssim B \) denotes that there exists a constant \( C > 0 \) such that \( A \leq CB \).

2. The linear profile decomposition

We will use the linear profile decomposition for the Airy evolution operator \( e^{-t\partial_x^3} \) for \( L^2 \) initial data to prove the existence of extremizers for (5). Roughly speaking, the linear profile decomposition is to investigate the general structure of solutions \( \{e^{-t\partial_x^3} f_n\} \) for bounded \( \{f_n\} \in L^2 \), and aims to compensate for the loss of compactness of the solution operator caused by the symmetries of the equation; see, for instance [24]. For a sequence \( \{e^{-t\partial_x^3} f_n\} \), it is expected to be written as a superposition of concentrating waves, ‘profiles’ plus a negligible remainder term; the interaction of the profiles is small; see the precise statements in Theorems 2.3 and 2.4. The profile decomposition for the nonlinear wave and Schrödinger equation, and the gKdV equation have been developed in [1, 2, 6, 22, 28, 31]. To prepare for the linear profile decomposition theorem for the Airy evolution operator in the Strichartz norm \( \|u\|_{L^8_{t,x}} \) needed in this paper, we recall two definitions from [31].

**Definition 2.1.** For any phase \( \theta \in \mathbb{R}/2\pi \mathbb{Z} \), position \( x_0 \in \mathbb{R} \) and scaling parameter \( h_0 > 0 \), we define the unitary transform \( g_{\theta,x_0,h_0} : L^2 \to L^2 \) by the formula

\[
[g_{\theta,x_0,h_0}] f(x) := \frac{1}{h_0^{1/2}} e^{i\theta} f \left( \frac{x - x_0}{h_0} \right).
\]

We let \( G \) be the collection of such transformations. It is easy to see that \( G \) is a group which preserves the \( L^2 \) norm.

**Definition 2.2.** For \( j \neq k \), two sequences \( \Gamma^j_n := (h^j_n, \xi^j_n, x^j_n, t^j_n)_{n \geq 1} \) and \( \Gamma^k_n := (h^k_n, \xi^k_n, x^k_n, t^k_n)_{n \geq 1} \) in \( (0, \infty) \times \mathbb{R}^3 \) are orthogonal if there holds:

\[
\text{either } \limsup_{n \to \infty} \left( \frac{h^j_n}{h^k_n} + \frac{h^k_n}{h^j_n} + h^j_n |\xi^j_n - \xi^k_n| \right) = \infty, \quad (11)
\]

\[
\text{or } (h^j_n, \xi^j_n) = (h^k_n, \xi^k_n) \text{ and }
\limsup_{n \to \infty} \left( \frac{|h^j_n - t^j_n|}{h^j_n} + \frac{3|t^j_n - t^k_n| |\xi^j_n|}{h^j_n} + \frac{|x^j_n - x^k_n| + 3(t^j_n - t^k_n)(|\xi^j_n|^2)}{h^j_n} \right) = \infty. \quad (12)
\]

Let \( D^\alpha, \alpha \in \mathbb{R} \), be the fractional derivative operator defined in terms of the Fourier multiplier, \( \mathcal{F} D^\alpha f = |\xi|^\alpha \hat{f} \). We state the following linear profile decomposition in the Strichartz norm \( \|D^{1/6} \|_{L^8_{t,x}} \) from [31].
THEOREM 2.3. Let \((f_n)_{n \geq 1}, f_n : \mathbb{R} \to \mathbb{C}\) be a sequence of functions satisfying \(\|f_n\|_2 \leq 1\). Then up to a subsequence, there exists a sequence of \(L^2\) functions \((\phi^j)_{j \geq 1} : \mathbb{R} \to \mathbb{C}\) and a family of pairwise orthogonal sequences \(\Gamma^j = (h^j_n, \xi^j_n, x^j_n, t^j_n) \in (0, \infty) \times \mathbb{R}^3\) such that, for any \(l \geq 1\), there exists an \(L^2\) function \(w^j_n : \mathbb{R} \to \mathbb{C}\) satisfying
\[
f_n = \sum_{1 \leq j \leq l} e^{t^j_n \partial^2_x} g^j_n(\phi^j) + w^j_n,
\]
where \(g^j_n := g_{0,x^j_n,h^j_n} \in G\) and
\[
\lim_{l \to \infty} \limsup_{n \to \infty} \|D^{1/6} e^{-i \partial_x^3} w^j_n\|_{L^8_{t,x}} = 0.
\]
Moreover, for every \(l \geq 1\),
\[
\lim_{n \to \infty} \left\|f_n\right\|_2^2 - \left(\sum_{j=1}^{l} \left\|\phi^j\right\|_2^2 + \left\|w^j_n\right\|_2^2\right) = 0.
\]

As a consequence of this theorem, we can develop a linear profile decomposition in the Airy–Strichartz norm \(\|\cdot\|_{L^8_{t,x}}\), where the highly oscillatory terms \(e^{i\sigma h^j_n \xi^j_n \phi^j(x)}\) with \(|h^j_n \xi^j_n| \to \infty\) disappear.

THEOREM 2.4. Let \((f_n)_{n \geq 1}, f_n : \mathbb{R} \to \mathbb{C}\) be a sequence of functions satisfying \(\|f_n\|_2 \leq 1\). Then up to a subsequence, there exists a sequence of \(L^2\) functions \((\phi^j)_{j \geq 1} : \mathbb{R} \to \mathbb{C}\) and a family of parameters \(\Gamma^j = (h^j_n, x^j_n, t^j_n) \in (0, \infty) \times \mathbb{R}^2\) such that, for any \(l \geq 1\), there exists an \(L^2\) function \(w^j_n : \mathbb{R} \to \mathbb{C}\) satisfying
\[
f_n = \sum_{1 \leq j \leq l} e^{t^j_n \partial^2_x} g^j_n(\phi^j) + w^j_n,
\]
where \(g^j_n := g_{0,x^j_n,h^j_n} \in G\) and
\[
\lim_{n \to \infty} \left\|e^{-i \partial_x^3} w^j_n\right\|_{L^8_{t,x}} = 0,
\]
and for \(j \neq k\),
\[
\lim_{n \to \infty} \left(\frac{h^k_n}{h^k_n} + \frac{h^k_n}{h^k_n} + \frac{|t^k_n - t^j_n|}{h^k_n(h^j_n)^3} + \frac{|x^k_n - x^j_n|}{h^k_n} \right) = \infty.
\]
Moreover, we have two orthogonality results: for every \(l \geq 1\),
\[
\lim_{n \to \infty} \left\|f_n\right\|_2^2 - \left(\sum_{j=1}^{l} \left\|\phi^j\right\|_2^2 + \left\|w^j_n\right\|_2^2\right) = 0.
\]
\[
\lim_{n \to \infty} \left\|\sum_{1 \leq j \leq l} e^{-(t^j_n - t^j_n) \partial^2_x} g^j_n(\phi^j)\right\|_{L^8_{t,x}}^8 - \sum_{1 \leq j \leq l} \left\|e^{-i \partial_x^3} \phi^j\right\|_{L^8_{t,x}}^8 = 0.
\]

REMARK 2.5. By (18) we have
\[
\sum_{j=1}^{l} \left\|\phi^j\right\|_2^2 \leq \limsup_{n \to \infty} \left(\sum_{j=1}^{l} \left\|\phi^j\right\|_2^2 + \left\|w^j_n\right\|_2^2\right) \leq \limsup_{n \to \infty} \left\|f_n\right\|_2^2 \leq 1
\]
for any \(l \in \mathbb{N}\). Hence \(\sum_{j=1}^{\infty} \left\|\phi^j\right\|_2^2 \leq 1\).
Proof. This argument consists of three steps. We first see that the error term $w_n^j$ still converges to zero in this new Strichartz norm $\| \cdot \|_{L^8_{t,x}}$. Indeed, by the Sobolev embedding,

$$\| e^{-it\partial_x^3} u_0 \|_{L^8_{t,x}} \lesssim C \| D^{1/6} e^{-it\partial_x^3} u_0 \|_{L^6_{t,x}};$$

so an application of (14) yields that

$$\limsup_{n \to \infty} \limsup_{l \to \infty} \| e^{-it\partial_x^3} w_n^l \|_{L^8_{t,x}} = 0.$$  

Secondly we claim that, for $1 \leq j \leq l$, when $\lim_{n \to \infty} h_n^j \xi_n^j = \infty,$

$$\lim_{n \to \infty} \| e^{-(t-t_n^j)\partial_x^3} g_n^j [ e^{i(\cdot) h_n^j \xi_n^j \phi^j} ] \|_{L^8_{t,x}} = 0. \quad (20)$$

It shows that the highly oscillatory terms can be reorganized into the error term. To show (20), by using the symmetries, we reduce to proving that, for $\phi \in L^2$,

$$\lim_{N \to \infty} \| e^{-it\partial_x^3} [ e^{i(jN)\phi} ] \|_{L^8_{t,x}} = 0. \quad (21)$$

By a density argument, we may assume $\phi \in S$, the set of Schwartz functions, and that $\phi$ has the compact Fourier support, say, in $(-1,1)$. Write

$$e^{-it\partial_x^3} [ e^{i(jN)\phi} ](x) = e^{ixN} e^{i3N t} \int e^{i(x+3t \xi N^2) + i3N t \xi^2 + it \xi^3 \dot{\phi}(\xi)} d\xi.$$  

Setting $x' := x + 3t N^2$ and $t' := 3N t$, we have,

$$\lim_{N \to \infty} \| e^{-it\partial_x^3} [ e^{i(jN)\phi} ] \|_{L^8_{t,x}} = c N^{-1/8} \left\| \int e^{ix' \xi + it' \xi^2 + i\xi^3 \dot{\phi}} d\xi \right\|_{L^8_{t',x'}}.$$  

for some $c > 0$. Then the dominated convergence theorem yields

$$\lim_{N \to \infty} \left\| \int e^{ix \xi + it \xi^2 + i(\xi^3/3N) \dot{\phi}} d\xi \right\|_{L^8_{t',x'}} = \| e^{-it\partial_x^3} \phi \|_{L^8_{t,x}}.$$  

Here $e^{-it\partial_x^3}$ denotes the Schrödinger evolution operator defined via

$$e^{-it\partial_x^3} f(x) := \int e^{ix \xi + it \xi^2} \hat{f}(\xi) d\xi.$$  

Indeed,

$$\int e^{ix' \xi + it' \xi^2 + i(\xi^3/3N) \dot{\phi}} d\xi \longrightarrow e^{-it' \partial_x^3} \phi(x'),$$

almost everywhere, and by using [33, Corollary, p. 334] or integration by parts,

$$\int e^{ix' \xi + it' \xi^2 + i(\xi^3/3N) \dot{\phi}} d\xi \leq C \phi B(t', x').$$

for $n$ large enough but still uniform in $n$. Here

$$B(t', x') = \begin{cases} (1 + |t'|)^{-1/2} \leq C[(1 + |x'|)(1 + |t'|)]^{-1/4} & \text{for } |x'| \leq 6 |t'|, \\ (1 + |x'|)^{-1} \leq C[(1 + |x'|)(1 + |t'|)]^{-1/2} & \text{for } |x'| > 6 |t'|. \end{cases}$$

It is easy to observe that $B \in L^8_{t',x'}$. Then (21) follows immediately.

Finally we claim that, for $j \neq k$,

$$\lim_{n \to \infty} \| e^{-(t-t_n^j)\partial_x^3} g_n^j [ e^{-it\partial_x^3} g_n^k (\phi^k) ] \|_{L^8_{t,x}} = 0.$$  

This is a consequence of the orthogonality condition (17), whose proof is a special case of Lemma 2.7 below. The remaining conclusions in Theorem 2.4 follow from Theorem 2.3 accordingly.  

\[ \square \]
Remark 2.6. A linear profile decomposition for all nonendpoint Airy Strichartz inequalities can be established by using the first two observations in the previous lemma and Lemma 2.7. The statement is similar to Theorem 2.4 and so we omit the details.

Lemma 2.7. When \(-\alpha + 3/q + 1/r = \frac{1}{2}, -1/2 < \alpha < \frac{1}{2}\). Then for \(j \neq k\),
\[
\lim_{n \to \infty} \|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}} = 0
\]
provided that \(\{(h_n^j, x_n^j, t_n^j)\}\) and \(\{(h_n^k, x_n^k, t_n^k)\}\) satisfies the orthogonality condition in (17).

Proof. We will prove (22) by studying (17) case by case.
Case I. Assume \(\lim sup_{n \to \infty} (h_n^j/h_n^k) + h_n^j/h_n^k = \infty\). For any \(R > 0\), we define
\[
\Omega_n^j(\alpha) = \left\{ (t, x) : \frac{|x-x_n^j|}{h_n^j} + \frac{|t-t_n^j|}{(h_n^j)^3} \leq R \right\},
\]
\[
\Omega_n^k(\alpha) = \left\{ (t, x) : \frac{|x-x_n^k|}{h_n^k} + \frac{|t-t_n^k|}{(h_n^k)^3} \leq R \right\},
\]
\[
(\Omega_n^j)^c : = \mathbb{R}^2 \setminus \Omega_n^j(\alpha), \quad (\Omega_n^k)^c : = \mathbb{R}^2 \setminus \Omega_n^k(\alpha).
\]
By using Hölder’s inequality and the Strichartz inequality followed by a change of variables, we have
\[
\|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}((\Omega_n^j)^c)}
\]
\[
\leq C \|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j)\|_{L_t^{q/2}L_x^{r/2}((\Omega_n^j)^c)} \|e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}((\Omega_n^k)^c)}
\]
\[
\leq C \|\phi^j\|_2 \|e^{-(t-t_n^j)\partial_x^2} D^\alpha(\phi^j)\|_{L_t^{q/2}L_x^{r/2}(|t|+|t| \geq R)}.
\]
The latter integral converges to zero when \(R\) goes to infinity from the dominated convergence theorem. So we can choose a sufficiently large \(R > 0\) such that
\[
\|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}((\Omega_n^j)^c)}
\]
as small as we want. Likewise for \(\|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}((\Omega_n^k)^c)}\). So by fixing a large \(R\), we may restrict our attention onto \(\Omega_n^j \cap \Omega_n^k\). We aim to show that the integral on \(\Omega_n^j \cap \Omega_n^k\) converges to zero when \(n\) goes to infinity. Indeed, by using trivial \(L_{t,x}\) bounds on \(e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j)\) and \(e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\), we see that
\[
\|e^{-(t-t_n^j)\partial_x^2} D^\alpha g_n^j(\phi^j) e^{-(t-t_n^k)\partial_x^2} D^\alpha g_n^k(\phi^k)\|_{L_t^{q/2}L_x^{r/2}(\Omega_n^j \cap \Omega_n^k)}
\]
\[
\leq C (h_n^j h_n^k)^{1/2-\alpha} \min \{ (h_n^j)^{6/q+2/r}, (h_n^k)^{6/q+2/r} \}
\]
\[
\leq C \min \left\{ \left( \frac{h_n^j}{h_n^k} \right)^{1/2+\alpha}, \left( \frac{h_n^k}{h_n^j} \right)^{1/2+\alpha} \right\} \to 0
\]
as \(n\) goes to infinity. Note that \(C > 0\) depends on \(R, \|\widehat{\phi^j}\|_{L^1}\), and \(\|\widehat{\phi^k}\|_{L^1}\). Thus (22) is obtained, which completes the proof of Case I.

Case II. In this case, \(\lim sup_{n \to \infty} (h_n^j/h_n^k) + h_n^j/h_n^k \in (0, \infty)\). Up to passing to a subsequence, we may assume that \(\lim_{n \to \infty} (h_n^j/h_n^k) = c\) for some \(c > 0\). Without loss of generality, we may assume that \(c = 1\). To simplify it further, we may assume that \(h_n^j = h_n^k\) for all \(n\). The general case can be proved similarly by the argument below. Now we aim to prove (22) under the
assumption that
\[
\lim_{n \to \infty} \frac{|x^j_n - x^k_n|}{h^*_n} + \frac{|t^j_n - t^k_n|}{(h^*_n)^3} = \infty.
\]
We change variables \(x' = (x - x^k_n)/h^*_n\) and \(t' = (t - t^k_n)/(h^*_n)^3\) and see that we need to show that
\[
\left\| e^{-t'(t + (t^j_n - t^k_n)/(h^*_n)^3)} (D^\alpha \phi^j) \left( x' + \frac{x^k_n - x^j_n}{h^*_n} \right) e^{-t' \partial_x^3} (D^\alpha \phi^k)(x') \right\|_{L^q_t L^r_x} \to 0
\]
as \(n \to \infty\). We define
\[
\Omega^k(R) := \{(t, x) : |t'| + |x'| \leq R\},
\]
\[
\Omega^j_i(R) := \{(t, x) : \left| x' + \frac{x^k_n - x^j_n}{h^*_n} \right| + \left| t' + t^j_n - t^k_n \right| \leq R\}.
\]
As proving Case I, we may reduce to the domain \(\Omega^k \cap \Omega^j_i\). While for this case, we observe that, for any fixed large \(R > 0\),
\[
|\Omega^k \cap \Omega^j_i| \to 0 \quad \text{as} \quad n \to \infty.
\]
This, together with the \(L^\infty_t\) bounds, proves Case II. Therefore the proof of Lemma 2.7 is complete.

Remark 2.8. With this Lemma 2.7, we have the following orthogonality result: for \((\alpha, q, r)\) defined as in Lemma 2.7 and \(l \geq 1\),
\[
\limsup_{n \to \infty} \left\| D^\alpha \sum_{j=1}^l e^{-(t - t^j_n) \partial_x^3} g^j_n \phi^j \right\|_q ^q \leq \sum_{j=1}^l \limsup_{n \to \infty} \| D^\alpha e^{-(t - t^j_n) \partial_x^3} g^j_n \phi^j \|_{L^q_t L^r_x},
\]
for \(q \leq r\); while for \(r \leq q\),
\[
\limsup_{n \to \infty} \left\| D^\alpha \sum_{j=1}^l e^{-(t - t^j_n) \partial_x^3} g^j_n \phi^j \right\|_r ^r \leq \sum_{j=1}^l \limsup_{n \to \infty} \| D^\alpha e^{-(t - t^j_n) \partial_x^3} g^j_n \phi^j \|_{L^q_t L^r_x}.
\]
See [32] for a similar proof.

3. Existence of extremizers

In this section, we apply the linear profile decomposition Theorem 2.4 to prove the existence of extremizers for (5).

Proof. Choose an extremizing sequence \((f_n)_{n \geq 1}\) such that
\[
\|f_n\|_2 = 1, \quad \lim_{n \to \infty} \| e^{-t \partial_x^3} f_n \|_{L^8_{t,x}} = A.
\]
By applying the linear profile decomposition in Theorem 2.4, we see that there is a sequence of profiles \(\phi^j\) and errors \(w^j_n\) such that for all \(l \in \mathbb{N}\), up to a subsequence (in \(n\),
\[
f_n = \sum_{1 \leq j \leq l} e^{t l \partial_x^3} g^j_n(\phi^j) + w^j_n.
\]
Moreover,

\[
A^8 = \lim_{n \to \infty} \| e^{-t \partial_x^3} f_n \|_{L^8_t L^8_x}^8 = \lim_{l \to \infty} \lim_{n \to \infty} \left\| \sum_{j=1}^l e^{-(t-t_{n,l}) \partial_x^3} g_n^j (\phi^j) \right\|_{L^8_t L^8_x}^8
\]

\[
= \sum_{j=1}^\infty \| e^{-t \partial_x^3} \phi^j \|_{L^8_t L^8_x}^8 \leq A^8 \sum_{j=1}^\infty \| \phi^j \|_{2 \times 4}^2 \leq A^8 \left( \sum_{j=1}^\infty \| \phi^j \|_2^2 \right) \leq A^8,
\]

where the second equality follows from (16), the third equality from (19), the first inequality from the definition of \( A \), and the last inequality from \( \sum_j \| \phi^j \|_2^2 \leq 1 \); see Remark 2.5.

Thus the equal signs at the beginning and at the end force all the signs in this chain to be equal. Hence, we have

\[
1 = \left( \sum_{j=1}^\infty \| \phi^j \|_{2 \times 4}^2 \right)^{1/4} \leq \sum_{j=1}^\infty \| \phi^j \|_2^2 \leq 1.
\]

Thus

\[
\left( \sum_{j=1}^\infty \| \phi^j \|_{2 \times 4}^2 \right)^{1/4} = \sum_{j=1}^\infty \| \phi^j \|_2^2,
\]

which in turn implies that there is exactly one \( j \) remaining. Without loss of generality, we may assume that

\[
\phi^j = 0 \quad \text{for } j \geq 2.
\]

Thus, \( \phi^1 \) is an extremizer as desired. \( \square \)

**Remark 3.1.** Combining this argument with the orthogonality in Remark 2.8, the existence of extremizers for any nonendpoint Strichartz inequality can be obtained similarly. We omit the details here.

### 4. Analyticity of extremizers

In this section, we establish that any extremizer \( f \) to (5) enjoys an exponential decay in the Fourier space, Theorem 1.4, from which the property of analyticity of extremizers follows easily. We begin with a bilinear Airy–Strichartz estimate in the spirit of [4, Lemma 111].

**Lemma 4.1 (Bilinear Airy estimates).** Suppose \( \text{Supp} \hat{f}_1 \subset \{ \xi : |\xi| \leq N_1 \} \) and \( \text{Supp} \hat{f}_2 \subset \{ \xi : N_2 \leq |\xi| \leq 2N_2 \} \), and \( N_1 \ll N_2 \). Then

\[
\| e^{-t \partial_x^3} f_1 e^{-t \partial_x^3} f_2 \|_{L^4_t L^4_x} \leq C \left( \frac{N_1}{N_2} \right)^{1/4} \| f_1 \|_2 \| f_2 \|_2.
\]

where the constant \( C > 0 \) is independent of \( N_1 \) and \( N_2 \).

**Proof.** We observe that

\[
\| e^{-t \partial_x^3} f_1 e^{-t \partial_x^3} f_2 \|_{L^4_t L^4_x} = \left\| \int e^{ix(\xi_1 + \xi_2) + it(\xi_1^3 + \xi_2^3)} \hat{f}_1 (\xi_1) \hat{f}_2 (\xi_2) \, d\xi_1 \, d\xi_2 \right\|_{L^4_t L^4_x}.
\]

We restrict the region to \( \{(\xi_1, \xi_2) : \xi_1, \xi_2 \geq 0\} \) and change variables \( a := \xi_1 + \xi_2 \) and \( b := \xi_1^3 + \xi_2^3 \); then we see that the Jacobian \( J \sim N_2^2 \) since \( N_1 \ll N_2 \). We apply the Hausdorff–Young
inequality and changes of variables to see that (24) is bounded by
\[
\lesssim \left( \int J^{-1/3} |\hat{f}_{1}\hat{f}_{2}|^{4/3} d\xi_{1} d\xi_{2} \right)^{3/4} \\
\lesssim |J|^{-1/4} \|f_{1}\|_{2} N_{1}^{1/4} \|f_{2}\|_{2} N_{2}^{1/4} \\
\lesssim \left( \frac{N_{1}}{N_{2}} \right)^{1/4} \|f_{1}\|_{2} \|f_{2}\|_{2}.
\]

Corollary 4.2. If \( \text{Supp} \hat{f}_{1} \subset \{ |\xi| \leq s \} \) and \( \text{Supp} \hat{f}_{2} \subset \{ |\xi| \geq L s \} \) for some \( s > 1 \) and \( L \gg 1 \), then
\[
\|e^{-t\partial_{x}^{2}}f_{1} e^{-t\partial_{x}^{2}}f_{2}\|_{L_{t,x}^{1}} \lesssim CL^{-1/4} \|f_{1}\|_{2} \|f_{2}\|_{2}.
\] (25)
where the constant \( C > 0 \) is independent of \( L \).

Proof. Let \( P_{k} \) denote the Littlewood–Paley projection operator to the frequency \( \{2^{k} \leq |\xi| \leq 2^{k+1} \} \) for any \( k \in \mathbb{Z} \). We dyadically decompose \( f_{2} = \sum_{k: 2^{k+1} \geq L s} P_{k} f_{2} \). Then by the triangle inequality and Lemma 4.1,
\[
\|e^{-t\partial_{x}^{2}}f_{1} e^{-t\partial_{x}^{2}}f_{2}\|_{L_{t,x}^{1}} \lesssim \sum_{k: 2^{k+1} \geq L s} \|e^{-t\partial_{x}^{2}}f_{1} e^{-t\partial_{x}^{2}}P_{k} f_{2}\|_{L_{t,x}^{1}} \\
\lesssim \sum_{k: 2^{k+1} \geq L s} \left( \frac{s}{2^{k}} \right)^{1/4} \|f_{1}\|_{L^{2}} \|P_{k} f_{2}\|_{L^{2}} \\
\lesssim \|f_{1}\|_{L^{2}} s^{1/4} \sum_{k: 2^{k+1} \geq L s} 2^{-k/4} \|P_{k} f_{2}\|_{L^{2}} \\
\lesssim \|f_{1}\|_{L^{2}} s^{1/4} \left( \sum_{k: 2^{k+1} \geq L s} 2^{-k/2} \right)^{1/2} \left( \sum_{k} \|P_{k} f_{2}\|_{L^{2}}^{2} \right)^{1/2} \\
\lesssim \|f_{1}\|_{L^{2}} s^{1/4} (L s)^{-1/4} \|f_{2}\|_{L^{2}} \lesssim L^{-1/4} \|f_{1}\|_{L^{2}} \|f_{2}\|_{L^{2}}.
\] (26)
This completes the proof of Corollary 4.2.

We define an 8-linear form,
\[
Q(f_{1}, \ldots, f_{8}) := \int \int \prod_{i=1}^{4} (e^{-t\partial_{x}^{2}} f_{i}) \prod_{m=5}^{8} (e^{-t\partial_{x}^{2}} f_{m}) dt dx,
\] (27)
where \( f_{i} \in L^{2}, 1 \leq i \leq 8 \). By the Airy–Strichartz inequality (5),
\[
|Q| \lesssim \prod_{i=1}^{8} \|f_{i}\|_{2}^{8}.
\] (28)
Inspired by the Euler–Lagrange equation (8), we define the notion of weak solutions.

Definition 4.3. A function \( f \in L^{2} \) is said to be a weak solution to the Euler–Lagrange equation (8) if it satisfies the following integral equation:
\[
\omega(g, f) = Q(g, f, \ldots, f) \quad \forall g \in L^{2}
\] (29)
for some \( \omega > 0 \). Here \( \langle \cdot, \cdot \rangle \) is the inner product in \( L^{2} \) defined by \( \langle g, f \rangle = \int_{\mathbb{R}} \overline{g} f dx \).
Remark 4.4. In view of the Euler–Lagrange equation (8), we see that, any extremizer \( f \) to the Airy–Strichartz inequality (5) is actually a weak solution, as any solution \( f \) of (8) satisfies
\[
\omega(g, f) = Q(g, f, \ldots, f) \quad \text{with} \quad \omega = A^8\|f\|_2^6.
\]

Now we list some additional notation and observations that are used in the following sections: Set
\[
a(\eta) := \sum_{l=1}^{4} \eta_l^3 - \sum_{m=5}^{8} \eta_m^3, \quad \quad \quad b(\eta) := \sum_{l=1}^{4} \eta_l - \sum_{m=5}^{8} \eta_m,
\]
\[
M(h_1, \ldots, h_8) := \int_{\mathbb{R}^8} \sum_{j=1}^{8} |h_j(\eta_j)| \delta(a(\eta)) \delta(b(\eta)) \, d\eta,
\]
where \( \delta \) denotes the Dirac mass. Then using the Fourier transform to represent \( e^{-t\partial_x^3} f \) and doing the \( t \) and \( x \) integrals in the definition of \( Q \), using \( (2\pi)^{-1} \int e^{isx} \, dr = \delta(s) \) as distributions, we rewrite \( Q \) as
\[
Q(f_1, \ldots, f_8) = (2\pi)^{-3} \int_{\mathbb{R}^8} \prod_{l=1}^{4} \tilde{f}_l(\eta) \prod_{m=5}^{8} \tilde{f}_m(\eta_m) \delta(a(\eta)) \delta(b(\eta)) \, d\eta.
\]
Then it is not hard to see that
\[
Q(f_1, \ldots, f_8) \leq (2\pi)^{-3} M(|\tilde{f}_1|, \ldots, |\tilde{f}_8|),
\]
\[
M(h_1, \ldots, h_8) = (2\pi)^3 Q(|h_1|^\vee, \ldots, |h_8|^\vee),
\]
where \( f^\vee(x) := (2\pi)^{-1/2} \int \xi e^{ix\xi} \tilde{f}(\xi) \, d\xi \) is the inverse Fourier transform.

Now we define a weighted version of \( M \), for any function \( F : \mathbb{R} \rightarrow \mathbb{R} \),
\[
M_F(h_1, \ldots, h_8) := \int_{\mathbb{R}^8} e^{F(\eta_1) - \sum_{j=2}^{8} F(\eta_j)} \prod_{j=1}^{8} |h_j(\eta_j)| \delta(a(\eta)) \delta(b(\eta)) \, d\eta.
\]
Then
\[
M(e^F h_1, e^{-F} h_2, e^{-F} h_3, e^{-F} h_4, e^{-F} h_5, e^{-F} h_6, e^{-F} h_7, e^{-F} h_8) = M_F(h_1, \ldots, h_8).
\]
We define, for \( \mu > 0, \varepsilon > 0 \),
\[
F_{\mu, \varepsilon}(k) := \frac{\mu |k|^3}{1 + \varepsilon |k|^3}.
\]

Proposition 4.5. For \( F_{\mu, \varepsilon} \) defined as above, we have
\[
M_{F_{\mu, \varepsilon}}(h_1, \ldots, h_8) \leq M(h_1, \ldots, h_8)
\]
for all \( \mu, \varepsilon \geq 0 \).

Proof. We see that the claim (40) reduces to proving
\[
F_{\mu, \varepsilon}(\eta) \leq \sum_{l=2}^{8} F_{\mu, \varepsilon}(\eta_l) \quad \text{when} \quad a(\eta) = b(\eta) = 0
\]
since then \( e^{F_{\mu, \varepsilon}(\eta_l) - \sum_{i=2}^{8} F_{\mu, \varepsilon}(\eta)} \leq e^0 = 1 \). In fact, we only need \( a(\eta) = 0 \) for this to hold.
Since \( a(\eta) = 0 \) implies \( \eta_1^3 = \sum_{l=2}^{8} (-1)^l \eta_l^3 \),

\[
F_{\mu,\varepsilon}(\eta_1) = \mu \frac{|\eta_1|^3}{1 + \varepsilon|\eta_1|^3} = \mu \frac{|\sum_{l=2}^{8} (-1)^l \eta_l^3|}{1 + \varepsilon|\sum_{l=2}^{8} (-1)^l \eta_l^3|} \leq \mu \frac{\sum_{l=2}^{8} |\eta_l|^3}{1 + \varepsilon\sum_{l=2}^{8} |\eta_l|^3} = \sum_{l=2}^{8} \frac{\mu|\eta_l|^3}{1 + \varepsilon\sum_{l=2}^{8} |\eta_l|^3} \leq \sum_{l=2}^{8} F_{\mu,\varepsilon}(\eta_l),
\]

where we have used the fact that \( t \mapsto t/(1 + \varepsilon t) \) is increasing on \([0, \infty)\).

**Remark 4.6.** From the proof, we can easily see that Proposition 4.5 remains true if \( F_{\mu,\varepsilon} \) is replaced by \( \hat{F}_{\mu,\varepsilon} \) where \( \hat{F}(k) = \tilde{F}(|k|^3) \) with \( \tilde{F} \) increasing and \( \tilde{F}(a + b) \leq \tilde{F}(a) + \tilde{F}(b) \) for \( a, b \geq 0 \).

Thus, Proposition 4.5 holds for a much larger class of functions than the one given in (39). However, for our goal of proving Theorem 1.4, the class of functions in (39) is the one we need.

Combining (28), (36) and Corollary 4.2 with Proposition 4.5 and Parseval’s identity, we can easily deduce the following:

**Corollary 4.7.** There exist a constant \( C > 0 \) such that for \( F_{\mu,\varepsilon} \) defined as above and all \( \mu, \varepsilon \geq 0 \)

\[
M_{F_{\mu,\varepsilon}}(h_1, \ldots, h_8) \leq C \prod_{j=1}^{8} \|h_j\|_2
\]

for all \( h_j \in L^2, j = 1, \ldots, 8 \). Moreover

\[
M_{F_{\mu,\varepsilon}}(h_1, \ldots, h_8) \leq C L^{-1/4} \prod_{j=1}^{8} \|h_j\|_2
\]

provided that there exists at least one \( h_j \) supported on \([-s, s]\) and another \( h_k \) supported on \([-Ls, Ls]^c\) where \( L \gg 1 \) and \( s \geq 1 \).

The following proposition is the key to the proof of Theorem 1.4. Let \( F_{\mu,\varepsilon} \) be defined as above for some \( \varepsilon > 0, \mu > 0 \). Let \( s > 1 \), we set

\[
\hat{f}_> := \hat{f} 1_{[-s^2, s^2]} \quad \text{and} \quad \|\hat{f}\|_{\mu, s, \varepsilon} := \|e^{F_{\mu,\varepsilon}} \hat{f}_>\|_2,
\]

where \( 1_\Omega \) denotes the indicator function of the set \( \Omega \).

**Proposition 4.8.** If \( \hat{f} \) is a weak solution to the Euler–Lagrange equation (8) as defined in (29) with \( \|\hat{f}\|_2 = 1 \), then for \( \mu = s^{-6} \) with \( s \gg 1 \), there exists a constant \( C > 0 \) such that

\[
\omega \|\hat{f}\|_{s^{-6}, s, \varepsilon} \leq o_1(1) \|\hat{f}\|_{s^{-6}, s, \varepsilon} + C \sum_{l=2}^{7} \|\hat{f}\|_{s^{-6}, s, \varepsilon} + o_2(1),
\]

where \( o_i(1) \to 0 \) uniformly in \( \varepsilon > 0 \) as \( s \to \infty \), \( i = 1, 2 \); the constant \( C > 0 \) is independent of \( \varepsilon \) and \( s \).

Let us postpone the proof of this proposition to the next section and finish the proof of Theorem 1.4.
Proof of Theorem 1.4. Given Proposition 4.8, the proof is similar to the proof of exponential decay of dispersion management solitons given in [15]. We set
\[ G(v) := \frac{\omega}{2} v - C \sum_{l=2}^{7} \theta^l \] for \( v \geq 0 \).

Then in (45), choosing \( \varepsilon = 1 \), and \( s \) large enough so that \( \alpha_1(1) \leq \omega/2 \), we have
\[ G(\| \hat{f} \|_s^{-6}, s, 1) \leq \alpha_2(1). \] (46)

We observe that the graph of \( G \) is concave in \([0, \infty)\) and intersects the \( x \)-axis only at two points: \( v = 0 \) and \( v = x_0 \) for some \( x_0 > 0 \). Let \( v_0, v_1 > 0 \) so that \( G(v_0) = G(v_1) = G_{\max}/2 \), where \( G_{\max} = \max\{G(v) : v \geq 0\} \). Again we take \( s \) to be large enough such that \( \alpha_2(1) \leq G_{\max}/2 \). Then we have a dichotomy,
\[ \text{either } \| \hat{f} \|_s^{-6}, s, 1 \leq v_0 \text{ or } \| \hat{f} \|_s^{-6}, s, 1 \geq v_1. \] (47)

However, the second choice is impossible if \( s \) is chosen to be large, because by definition
\[ F_{s^{-6}, 1}(k) = \frac{s^{-6}|k|^3}{1 + |k|^2} < s^{-6} \leq 1, \]
which yields
\[ \| \hat{f} \|_s^{-6}, s, 1 = \| e^{F_{s^{-6}, 1}} \hat{f} \|_2 \leq e^{s^{-6}} \| \hat{f} \|_{[-s^2, s^2], \varepsilon} \|_2 \rightarrow 0 \quad \text{as } s \rightarrow \infty. \]

Now we fix such a large \( s > 0 \) and consider the function \( \varepsilon \mapsto \| \hat{f} \|_s^{-6}, s, \varepsilon \), which is continuous by the dominated convergence theorem for \( \varepsilon > 0 \). Again by (45),
\[ G(\| \hat{f} \|_s^{-6}, s, \varepsilon) \leq G_{\max}/2 \] (48)
for all \( \varepsilon > 0 \). Hence by continuity, we must have that \( \| \hat{f} \|_s^{-6}, s, \varepsilon \) is in the same connected component of \( G^{-1}(0, G_{\max}/2) = [0, v_0] \cup [v_1, \infty) \). On the other hand, since we already know that \( \| \hat{f} \|_s^{-6}, s, 1 \in [0, v_0] \), we deduce that
\[ \| \hat{f} \|_s^{-6}, s, \varepsilon \in [0, v_0] \quad \forall \varepsilon > 0. \] (49)

This implies, by the monotone convergence theorem,
\[ \| \hat{f} \|_s^{-6}, s, 0 = \lim_{\varepsilon \rightarrow 0} \| \hat{f} \|_s^{-6}, s, \varepsilon \leq v_0. \] (50)

In other words,
\[ e^{s^{-6}k^3} \hat{f} \in L^2 \] (51)
and since \( e^{s^{-6}k^3} \) is bounded on \([-s^2, s^2]\) this yields
\[ k \mapsto e^{s^{-6}|k|^3} \hat{f}(k) \in L^2. \] (52)

Let \( \mu_0 = s^{-6} \) for this \( s > 0 \). Then the super Gaussian decay in Theorem 1.4 is established.

We are left with proving that \( f \) is an entire function on the complex plane \( \mathbb{C} \). Indeed, by the Cauchy–Schwarz inequality, for any \( \mu \in \mathbb{R} \), we have
\[ e^{\mu|k|} \hat{f}(k) = e^{\mu|k|-\mu_0|k|^3} e^{\mu_0|k|} \hat{f}(k) \in L^1(\mathbb{R}). \] (53)

Then for any \( z \in \mathbb{C} \), we can always choose \( \mu > |z| \) such that
\[ f(z) = (2\pi)^{-1/2} \int e^{izk} \hat{f}(k) \, dk = (2\pi)^{-1/2} \int e^{izk-\mu|k|} e^{\mu|k|} \hat{f}(k) \, dk. \] (54)

Since the first factor \( e^{izk-\mu|k|} \) is bounded and the second factor is in \( L^1 \) by (53), \( f \) is an entire function.

\[ \square \]

It remains to prove Proposition 4.8, which we carry out in the next section.
5. The bootstrap argument

In this section, we prove Proposition 4.8, for which we only have the definition of weak solutions in (29) and the definition of $Q$ at our disposal. We set $F = F_{\mu, \varepsilon}$ for $F_{\mu, \varepsilon}$ defined in (39) and define $f_\geq$, $h$ and $h_\geq$ by

$$f_\geq = \hat{f}_1_{[-s^2, s^2]}, \quad h(k) = e^{F(k)} \hat{f}(k), \quad h_\geq(k) := e^{F(k)} \hat{f}_\geq(k).$$ (55)

**Proof of Proposition 4.8.** We use $g = e^{2F(P)} f_\geq$ with $P = -i\partial_x$ in (29). Using that the operator $e^{F(P)}$ is simply multiplying with $e^{2F(k)}$ in Fourier space, the representation (34) of $Q$ and $h_\geq$ for the inverse Fourier transform of $h$, one sees

$$\omega \|e^{F} \hat{f}_\geq\|_2^2 = \omega \langle e^{F} \hat{f}_\geq(k), e^{F} \hat{f}_\geq(k) \rangle = \omega \langle e^{2F} \hat{f}_\geq, \hat{f} \rangle = \omega \langle e^{2F(P)} f_\geq, f \rangle$$
$$= Q(e^{2F(P)} f_\geq, f, f, f, f, f) = Q((e^{F} h_\geq)\wedge, f, f, f, f, f)$$
$$= Q((e^{F} h_\wedge)\wedge, (e^{-F} h)\wedge, (e^{-F} h)\wedge, (e^{-F} h)\wedge, (e^{-F} h)\wedge, (e^{-F} h)\wedge)$$
$$=: Q_F.$$ (56)

Then by (35),

$$|Q_F| \leq CM_F(h_\geq, h, h, h, h, h, h) \leq CM(h_\geq, h, h, h, h, h, h),$$ (57)

where the last inequality follows from Proposition 4.5. Continuing (57), we split $h$ and use that the operator $M$ is sublinear in each component,

$$M(h_\geq, h, h, h, h, h, h) \leq M(h_\geq, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge)$$
$$+ \sum_{j_2, \ldots, j_n \in \{>, <, \}, \text{at least one } j_1 = >} M(h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8}) := A + B.$$ (58)

We split further $h_\wedge = h_\ll + h_\wedge$, where the low frequency part $h_\ll := h_{1[-s, s]}$ and the median frequency part $h_\wedge := h_{1[-s^2, s^2]\setminus[-s, s]}$.

We estimate $A$ by using the bilinear Airy–Strichartz estimate in Lemma 4.1:

$$A = M(h_\geq, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge)$$
$$\leq M(h_\geq, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge) + M(h_\geq, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge, h_\wedge)$$
$$\leq s^{-1/4} \|h_\geq\|_2 \|h_\wedge\|_2 \|h_\wedge\|_2 \|h_\wedge\|_2 \|h_\wedge\|_2$$
$$= \|h_\wedge\|_2 (s^{-1/4} \|h_\wedge\|_2 + \|h_\wedge\|_2) \|h_\wedge\|_2.$$ (59)

Recalling that $\|f\|_2 = 1$, then

$$\|h_\wedge\|_2 = \|e^{F_{\mu, s}} \hat{f}_\ll\|_2 \leq \|e^{\mu k^3} \hat{f}_\ll\|_2 \leq e^{\mu s^6} \|f\|_2,$$

$$\|h_\wedge\|_2 = \|e^{F_{\mu, s}} \hat{f}_\ll\|_2 \leq e^{\mu s^3} \|f\|_2,$$

$$\|h_\wedge\|_2 = \|e^{F_{\mu, s}} \hat{f}_\wedge\|_2 \leq e^{\mu s^6} \|f_\wedge\|_2.$$ (60)

we obtain

$$A \leq C \|h_\wedge\|_2 (s^{-1/4} e^{\mu s^3 - \mu s^6} + \|f_\wedge\|_2) e^{7s^6}.$$ (61)

Now we turn to estimate $B$:

$$B \leq \sum_{j_2, \ldots, j_n \in \{>, <, \}, \text{exactly one } j_1 = >} M(h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8})$$
$$+ \sum_{j_2, \ldots, j_n \in \{>, <, \}, \text{two or more } j_1 = >} M(h_{j_2}, h_{j_3}, h_{j_4}, h_{j_5}, h_{j_6}, h_{j_7}, h_{j_8}) := B_1 + B_2.$$ (62)
For $B_2$,  
\begin{equation}
B_2 \lesssim \|h>\|_2 \sum_{l=2}^8 \|h_{\lessgtr l}\|_2 \lesssim \|h>\|_2 \left( \sum_{l=2}^7 \|h_{\lessgtr l}\|_2^{7-l} \|h>\|_2^l \right) \lesssim \|h>\|_2 e^{\sqrt{s}/6} \sum_{l=2}^7 \|h>\|_2^l, \tag{63}
\end{equation}

where we have used that $\|h_{\lessgtr l}\|_2 \lesssim e^{\mu s_6} \|f_{\lessgtr l}\|_2 \lesssim e^{\mu s_6}$.

For $B_1$, we split one of the $h_{\lessgtr}$ into $h_{\lessgtr} = h_{\lessgtr} + h_{\lessgtr}$ and then use the sublinearity of $M$,  
\begin{equation}
B_1 \lesssim \|h>\|_2 (s^{-1/4} \|h_{\lessgtr}\|_2 + \|h_{\lessgtr}\|_2) \|h_{\lessgtr}\|_2 \\
\lesssim \|h>\|_2 s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\mu s_6} \|h_{\lessgtr}\|_2^5 \\
\lesssim \|h>\|_2 s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\mu s_6} . \tag{64}
\end{equation}

Thus, we conclude that  
\begin{equation}
B \leq B_1 + B_2 \lesssim \|h>\|_2^2 (s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\sqrt{s}/6} + e^{\sqrt{s}/6} \|h>\|_2 \sum_{l=2}^7 \|h>\|_2^l. \tag{65}
\end{equation}

Therefore from (56)–(58), (61) and (65), we have  
\begin{equation}
\omega \|\hat{h}>\|_2^2 \lesssim \|h>\|_2 (s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\sqrt{s}/6} \\
+ \|h>\|_2 s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\sqrt{s}/6} + e^{\sqrt{s}/6} \|h>\|_2 \sum_{l=2}^7 \|h>\|_2^l. \tag{66}
\end{equation}

Cancelling one $\|\hat{h}>\|_2$ on both sides, we see that  
\begin{equation}
\omega \|\hat{h}>\|_2 \lesssim (s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\sqrt{s}/6} \\
+ \|h>\|_2 s^{-1/4} e^{\mu s_6} - \mu s_6 + \|f_{\lessgtr}\|_2) e^{\sqrt{s}/6} + e^{\sqrt{s}/6} \sum_{l=2}^7 \|h>\|_2^l. \tag{67}
\end{equation}

Since $\|f_{\lessgtr}\|_2 = \|\hat{f}_{[-s^2, s^2]} \|_{[-s, s]} \|_2 \lesssim \|\hat{f}_{[-s^2, s^2]} \|_{[-s, s]} = o(1)$ as $s \to \infty$, and $e^{\sqrt{s}/6} = e^6$ if taking $\mu = s^{-6}$, we conclude that  
\begin{equation}
\omega \|h>\|_2 \leq o_1(1) \|h>\|_2 + C \sum_{l=2}^7 \|h>\|_2^l + o_2(1). \tag{68}
\end{equation}

Therefore, the proof of Proposition 4.8 is complete. \hfill \Box

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