Symplectic Induction, Prequantum Induction, and Prequantum Multiplicities

Tudor S. Ratiu* and François Ziegler†

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Abstract

Frobenius reciprocity asserts that induction from a subgroup and restriction to it are adjoint functors in categories of unitary G-modules. In the 1980s, Guillemin and Sternberg established a parallel property of Hamiltonian G-spaces, which (as we show) unfortunately fails to mirror the situation where more than one G-module “quantizes” a given Hamiltonian G-space. This paper offers evidence that the situation is remedied by working in the category of prequantum G-spaces, where this ambiguity disappears; there, we define induction and multiplicity spaces, and establish Frobenius reciprocity as well as the “induction in stages” property.

Introduction

Beyond the mere parametrization of irreducible unitary representations by coadjoint orbits originating in the work of Borel-Weil and Kirillov [S54, K62], there exists a certain well-known parallelism between representation theory and the symplectic theory of Hamiltonian G-spaces. To capture it with precision, papers like [K78, W78, G82, G83] introduced purely symplectic constructions meant to mirror operations such as $\text{Ind}_H^G$ (inducing a representation from a subgroup) or $\text{Hom}_G$ (forming the space of intertwining operators between two representations). In that setting, one of course expects basic properties like induction in stages or Frobenius reciprocity to hold in symplectic geometry. A first goal of this paper is to spell out their proofs (§2, §3), fulfilling promises made in [Z96, p. 9] and [M07, p. 105].

A second goal of the paper is to point out that, while these constructions fit their purpose when the correspondence from representations to coadjoint orbits is one-to-one, as in the Borel-Weil theory for compact groups or the Kirillov-Bernat theory for exponential groups [K62, F15], they fall short when it is many-to-one, as in the Auslander-Kostant theory for solvable groups [A71, S94]. To remedy this, we propose new versions of both constructions in the category of prequantum G-spaces (§5, §6) and establish the stages and Frobenius properties in that setting (§7, §8). Finally, we illustrate the need for our prequantized versions by what we believe is the simplest example (§4, §9).

*School of Mathematical Sciences and Ministry of Education Key Lab on Scientific and Engineering Computing, Shanghai Jiao Tong University, Shanghai 200240, China and Section de Mathématiques, Université de Genève and Ecole Polytechnique Fédérale de Lausanne, Switzerland. ratiu@sjtu.edu.cn, tudor.ratiu@epfl.ch
†Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460-8093, USA. fziegler@georgiasouthern.edu

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Notation and conventions

We use a concise notation for the translation of tangent and cotangent vectors to a Lie group: for fixed $g, q \in G$,

\begin{equation}
\begin{array}{c}
T_g G \rightarrow T_{gq} G \\
v \mapsto gv,
\end{array}
\quad
\begin{array}{c}
T^*_g G \rightarrow T^*_{gq} G \\
p \mapsto gp
\end{array}
\end{equation}

will denote the derivative of $q \mapsto gq$, respectively its contragredient, i.e., $\langle gp, v \rangle = \langle p, g^{-1}v \rangle$. Likewise, we define $vg$ and $pg$ with $\langle pg, v \rangle = \langle p, vg^{-1} \rangle$.

By a Hamiltonian G-space we mean the triple $(X, \omega, \Phi)$ of a manifold $X$ on which $G$ acts, a $G$-invariant symplectic form $\omega$ on it, and a $G$-equivariant momentum map $\Phi : X \rightarrow g^*$. We identify spaces $X_1, X_2$ which are isomorphic, i.e., related by a $G$-equivariant diffeomorphism which transforms $\omega_1$ into $\omega_2$ and $\Phi_1$ into $\Phi_2$. If several are in play, we also use subscripts like $\omega_X, \Phi_X$. We recall two cardinal properties of the momentum map:

\begin{equation}
\begin{array}{c}
\mathrm{(a)} \ \mathrm{Ker}(D\Phi(x)) = g(x)^\omega \\
\mathrm{(b)} \ \mathrm{Im}(D\Phi(x)) = \text{ann}(g_x).
\end{array}
\end{equation}

The first is the orthogonal relative to $\omega$ of the tangent space $g(x)$ to the orbit $G(x)$, $x \in X$; the second is the annihilator in $g^*$ of the stabilizer Lie subalgebra $g_x \subset g$.

1 Symplectic Induction

Given a closed subgroup $H \subset G$ and a Hamiltonian H-space $(Y, \omega_Y, \Psi)$, [K78] constructs an induced Hamiltonian G-space as follows. Let $\sigma$ denote the canonical 1-form on $T^*G$ given by $\sigma(\delta p) = \langle p, \delta q \rangle$, where $\delta p \in T_p(T^*G)$, $\delta q = \pi_*(\delta p) \in T_{N(p)}G$, and $\pi : T^*G \rightarrow G$ is the canonical projection. Endow $N := T^*G \times Y$ with the symplectic form $\omega := d\sigma + \omega_Y$ and the $G \times H$-action $(g, h)(p, y) = (gph^{-1}, h(y))$, where $h(y)$ denotes the action of $h \in H$ on $y \in Y$. This action admits the equivariant momentum map $\Phi \times \psi : N \rightarrow g^* \times h^*$,

\begin{equation}
\begin{cases}
\Phi(p, y) = pq^{-1} \\
\psi(p, y) = \Psi(y) - q^{-1}p|_h
\end{cases}
\quad (p \in T^*_q G).
\end{equation}

The induced manifold is, by definition, the Marsden-Weinstein reduced space of $N$ at $0 \in h^*$, i.e.

\begin{equation}
\dim \text{Ind}_H^G Y := N//H = \psi^{-1}(0)/H.
\end{equation}

In more detail: the action of $H$ is free and proper (because it is free and proper on the factor $T^*G$, where it is the right action of $H$ regarded as a subgroup of the group $T^*G$ [B72, §III.1.6]); so $\psi$ is a submersion (0.2b), $\psi^{-1}(0)$ is a submanifold, and (1.2) is a manifold; moreover $\omega|_{\psi^{-1}(0)}$ degenerates exactly along the $H$-orbits (0.2a), so it is the pull-back of a uniquely defined symplectic form, $\omega^\mathbb{N//H}$, on the quotient. Furthermore, the $G$-action commutes with the $H$-action and preserves $\psi^{-1}(0)$, and its momentum map $\Phi$ is constant on $H$-orbits. Passing to the quotient, we obtain the required $G$-action on $\text{Ind}_H^G Y$ and momentum map $\Phi_{\mathbb{N//H}} : \text{Ind}_H^G Y \rightarrow g^*$. Note that since $\psi$ is a submersion and $H$ acts freely, (1.2) has dimension equal to $\dim(N) - 2 \dim(H)$, i.e.

\begin{equation}
\dim(\text{Ind}_H^G Y) = 2 \dim(G/H) + \dim(Y).
\end{equation}

2 Symplectic Induction in Stages

(2.1) Theorem (Stages). If $H \subset K \subset G$ are closed subgroups of the Lie group G, then

\begin{equation}
\text{Ind}_H^G \text{Ind}_K^H Y = \text{Ind}_K^G Y.
\end{equation}
Figure 1: Construction of the isomorphism $t$.

Proof. Let $(N, \omega, \phi \times \psi)$ be as in §1 and consider $M = T^*G \times T^*K \times Y$ with 2-form $\omega_M = d\sigma_{T^*G} + d\sigma_{T^*K} + \omega_Y$ and $G \times K \times H$-action

$$(g, k, h)(p, \tilde{p}, y) = (g pk^{-1}, k \tilde{p} h^{-1}, h(y)).$$

This admits the equivariant momentum map $\phi \times \tilde{\phi} \times \psi : M \to g^* \times t^* \times \mathfrak{h}^*$:

$$(2.3) \begin{align*}
\phi(p, \tilde{p}, y) &= p q^{-1} \\
\tilde{\phi}(p, \tilde{p}, y) &= \tilde{p} q^{-1} - q^{-1} p |_{t} \\
\psi(p, \tilde{p}, y) &= \Psi(y) - \tilde{q}^{-1} \tilde{p} |_{\mathfrak{h}}
\end{align*}$$

for $(p, \tilde{p}) \in T^*_qG \times T^*_qK$. Define $r : M \to N$ by $r(p, \tilde{p}, y) = (p \tilde{q}, y)$ and consider the commutative diagram in Fig. 1, where we have written $j_1, j_2, j_3$ and $\pi_1, \pi_2, \pi_3$ for the inclusion and projection maps involved in constructing the reduced spaces $M//H = T^*G \times \text{Ind}_H^K Y$, $(M//H)//K = \text{Ind}_H^G \text{Ind}_K^H Y$, and $N//H = \text{Ind}_H^G Y$; also $j_1, \pi_1$ are the obvious inclusion and restriction, and $\phi_{M//H}$ is the momentum map for the residual $K$-action on $M//H$. The map $r \circ j_1 \circ j$ satisfies

$$(2.4) \begin{align*}
\psi((r \circ j_1 \circ j)(p, \tilde{p}, y)) &= \psi(p \tilde{q}, y) \\
&= \Psi(y) - (q \tilde{q})^{-1} p \tilde{q} |_{\mathfrak{h}} \\
&= \Psi(y) - \tilde{q}^{-1} (q^{-1} p) |_{t} |_{\mathfrak{h}} \\
&= \Psi(y) - \tilde{q}^{-1} \tilde{p} |_{\mathfrak{h}} \quad \text{since } \tilde{\phi}(p, \tilde{p}, y) = 0 \\
&= 0 \quad \text{since } \psi(p, \tilde{p}, y) = 0.
\end{align*}$$

So $r \circ j_1 \circ j$ takes values in $\psi^{-1}(0)$, i.e., there is a map $s$ as indicated in Fig. 1. Moreover, $s$ is onto since one verifies that $(p, y) \mapsto (p, (q^{-1} p) |_{t}, y)$ provides a right inverse. The map $s$ is equivariant relative to the $G \times K \times H$-action on $(\phi \times \tilde{\phi})^{-1}(0)$ and the $G \times H$-action on $\psi^{-1}(0)$:

$$(2.5) \begin{align*}
s((g, k, h)(p, \tilde{p}, y)) &= r(g pk^{-1}, k \tilde{p} h^{-1}, h(y)) \\
&= (gp \tilde{q} h^{-1}, h(y)) \\
&= (g, h)(p \tilde{q}, y) \\
&= (g, h)(s(p, \tilde{p}, y)).
\end{align*}$$
Hence \( s \) descends to a \( G \)-equivariant surjection \( t \) as indicated in Fig. 1. Furthermore, one checks without trouble that the fibers of \( s \) are precisely the \( K \)-orbits in its domain. As \( \pi_2 \circ \pi \) collapses these orbits to points, it follows that \( t \) is bijective, hence a diffeomorphism by [B67, 5.9.6]. The relation \( \phi = \tau^* \Phi \) implies that \( t \) relates the momentum maps for \( G \): \( \Phi_{(M/H)/K} = t^* \Phi_{N/H} \), so there only remains to see that \( \omega_{(M/H)/K} = t^* \omega_{N/H} \). To this end we compute

\[
(s^* j_1^* \circ \tau^* G)(\delta p, \delta \tilde{p}, \delta q) = \tau^* G(\delta [\tilde{p} \tilde{q}])
= \langle \tilde{p} \tilde{q}, \delta q \rangle
= \langle p, \delta q \rangle + \langle q^{-1} p, [\delta \tilde{q}] \tilde{q}^{-1} \rangle \quad \text{since} \quad \delta [\tilde{q} \tilde{q}] = [\delta q] \tilde{q} + q [\delta \tilde{q}]
= \langle p, \tilde{q}, \delta \tilde{q} \rangle \quad \text{since} \quad \tilde{\phi}(p, \tilde{p}, y) = 0
= \tau^* G(\delta p) + \tau^* G(\delta \tilde{p}).
\]

(2.6)

Taking exterior derivatives and adding \( \omega_Y \) we obtain \( s^* j_2^* \omega_N = j^* j_1^* \omega_M \) or, equivalently (by commutativity of the diagram and definition of the reduced 2-forms), \( \pi^* \pi_2^* t^* \omega_{N/H} = \pi^* \pi_2^* \omega_{(M/H)/K} \). Since \( \pi_2 \circ \pi \) is a submersion, we are done. \( \Box \)

## 3 Symplectic Frobenius Reciprocity

It is quite rare for an induced Hamiltonian \( G \)-space to be homogeneous or a fortiori a coadjoint orbit (by which we mean that its momentum map is 1-1 onto an orbit). In fact we have the following, where \( \Phi_{N/H} \) is the momentum map for the induced space (1.2).

### (3.1) Proposition

Let \((Y, \omega_Y, \Psi)\) be a Hamiltonian \( H \)-space.

(a) A coadjoint orbit \( O \) of \( G \) intersects \( \text{Im}(\Phi_{N/H}) \Leftrightarrow \text{Im}(\Psi) \).

(b) If \( \text{Ind}^G_H Y \) is homogeneous under \( G \), then \( Y \) is homogeneous under \( H \).

(c) If \( \text{Ind}^G_H Y \) is a coadjoint orbit of \( G \), then \( Y \) is a coadjoint orbit of \( H \).

**Proof.** (a): This re-expresses \( \text{Im}(\Phi_{N/H}) = \phi_j(\tilde{\Phi}^{-1}(0)) \) (1.1). (b): Assume \( G \) is transitive on \( \text{Ind}^G_H Y \) and let \( y_1, y_2 \in Y \). Pick \( m_1 \in \mathfrak{g}^* \) such that \( \Psi(y_1) = m_1 |_{\mathfrak{h}} \). Then the \( H \)-orbits \( x_i = H(m_i, y_i) \) are points in (1.2). So transitivity says that \( x_1 = g(x_2) \), i.e.

\[
(m_1, y_1) = (gm_2 h^{-1}, h(y_2)) \quad \text{for some} \quad h \in H.
\]

In particular \( y_1 = h(y_2) \), as claimed. (c): Assume further that \( \Phi_{N/H} \) is injective and suppose \( \Psi(y_1) = \Psi(y_2) \). Then we can pick \( m_1 = m_2 \) above. Since \( \Phi_{N/H}(x_i) = m_i \) it follows, by injectivity, that \( x_1 = x_2 \), i.e., we have (3.2) with \( g = e \). But then \( h = e \) and hence \( y_1 = y_2 \), as claimed. \( \Box \)

If \( Y \) is a coadjoint orbit, (3.1a) says that \( \text{Ind}^G_H Y \) “involves” just those orbits \( O \) whose projection in \( \mathfrak{h}^* \) contains \( Y \). Guillemin and Sternberg [G82, §6] proposed to measure the “multiplicity” of this involvement by the (possibly empty) space \( \text{Hom}_G(O, \text{Ind}^G_H Y) \), where we write suggestively

\[
\text{Hom}_G(X_1, X_2) := (X_1^* \times X_2)/G,
\]

i.e., the Marsden-Weinstein reduction of \( X_1^* \times X_2 \) at \( 0 \in \mathfrak{g}^* \); here \( X_1^* \) is the Hamiltonian \( G \)-space \( X_1 \) with its 2-form and momentum map replaced by their negatives, and we regard (3.3) simply as a set. Then (3.1a) can be refined by the following analog of Frobenius’s theorem [B88, III.6.2], already found in [G83, Thm 2.2] when both \( X \) and \( Y \) are coadjoint orbits.

### (3.4) Theorem (Frobenius reciprocity)

If \( X \) is a Hamiltonian \( G \)-space and \( Y \) a Hamiltonian \( H \)-space, then

\[
\text{Hom}_G(X, \text{Ind}^G_H Y) = \text{Hom}_H(\text{Res}^G_H X, Y).
\]

4
(3.5) Remarks. Here $\text{Res}^G_H X$ means $X$ regarded as a Hamiltonian $H$-space, and “−” means only a natural bijection as sets. We believe (but haven’t proved) that both sides are automatically isomorphic as diffeological spaces with diffeological 2-forms as discussed in [S85, §2.5], [I13, §6.38], [K16].

Note also that, by the symmetry of (3.3), we may equally write Frobenius reciprocity in the form $\text{Hom}_G(\text{Ind}_H^G Y, Z) = \text{Hom}_H(Y, \text{Res}_H^G Z)$.

Proof. For bookkeeping reasons, soon to become clear, rename $G$ also as $K$. Consider the spaces $N = X^- \times Y$ with $H$-action $h(x, y) = (h(x), h(y))$, and $M = X^- \times T^* K \times Y$ with $K \times H$-action $(k, h)(x, \bar{p}, y) = (k(x), k\bar{p}h^{-1}, h(y))$. Their equivariant momentum maps are $\bar{\psi} : N \to \mathfrak{h}^*$,

$$\psi(x, y) = \Psi(y) - \Phi(x)|_{\mathfrak{h}}$$

and $\bar{\psi} \times \bar{\psi} : M \to \mathfrak{k}^* \times \mathfrak{h}^*$,

$$\psi(x, y) = \Psi(y) - \Phi(x)|_{\mathfrak{h}}$$

and $\bar{\psi} \times \bar{\psi} : M \to \mathfrak{k}^* \times \mathfrak{h}^*$,

$$\begin{align*}
\bar{\psi}(x, \bar{p}, y) &= \bar{\psi}(y) - \Phi(x)|_{\mathfrak{h}} \\
\bar{\psi}(x, \bar{p}, y) &= \Psi(y) - \bar{\psi}(x)|_{\mathfrak{h}} \quad (\bar{p} \in T^*_q K)
\end{align*}$$

where $\Phi$ and $\Psi$ are the equivariant momentum maps of $X$ and $Y$, respectively. Defining $r : M \to N$ by $r(x, \bar{p}, y) = (\tilde{q}^{-1}(x), y)$ now sets us up for a proof using the same previous diagram (Fig. 1). Indeed, we have again this time

$$\begin{align*}
\bar{\psi}((r \circ j_1 \circ j)(x, \bar{p}, y)) &= \psi(\tilde{q}^{-1}(x), y) \\
&= \Psi(y) - \Phi(\tilde{q}^{-1}(x)|_{\mathfrak{h}}) \\
&= \psi(y) - \bar{\psi}^{-1}(x)|_{\mathfrak{h}} \\
&= 0 \quad \text{since } \bar{\psi}(x, \bar{p}, y) = 0
\end{align*}$$

so there is a map $s$ as indicated in Fig. 1. Again, $s$ is onto since $(x, y) \mapsto (x, \Phi(x), y)$ provides a right inverse, and $s$ is equivariant relative to the $K \times H$-action on $(\bar{\psi}^{-1}(0)$ and the $H$-action on $\psi^{-1}(0)$:

$$s((k, h)(x, \bar{p}, y)) = r(k(x), k\bar{p}h^{-1}, h(y))$$

$$= ((k\tilde{q}h^{-1})^{-1}(k(x)), h(y))$$

$$= (h(\tilde{q}^{-1}(x)), h(y))$$

$$= h(s(x, \bar{p}, y)).$$

So the fibers of $s$ are again the $K$-orbits and $s$ descends again to a bijection $t$ as required and indicated in Fig. 1. \qed

4 An Example

A basic shortcoming in the analogy of (3.4) with representation theory is that it cannot mirror cases where more than one representation “quantizes” a given Hamiltonian $G$-space or $H$-space. To make this assertion precise, we restrict attention to the case where $X$ is a coadjoint orbit of a type I solvable Lie group $G$, endowed with its Kirillov-Kostant-Souriau 2-form $\omega_X$. In that setting “quantization” is well-defined by the theory of Auslander and Kostant [A71] where, we recall:

- $X$ has irreducible unitary representations attached to it iff the de Rham cohomology class $[\omega_X]$ belongs to $H^2(X, \mathbb{Z})$ (in particular if $\omega_X$ is exact).

- If so, and if $G$ is simply connected, then $X$ has as many representations attached to it as there are homomorphisms (a.k.a. characters) from the fundamental group $\pi_1(X)$ to the circle group $T$. 


This can be summed up by saying that the unitary dual $\hat{G}$ is parametrized by prequantized coadjoint orbits in the sense of Section 5 below. (One should beware that this can fail beyond the solvable context [R82, T83]; the minimal nilpotent coadjoint orbit of $\text{SL}_3(\mathbb{R})$ has four prequantum bundles but only three representations attached to it.) The simplest example where (3.4) falls short is then as follows. In the solvable group $G'$ of all upper triangular matrices of the form

$$g' = \begin{pmatrix} e^{ia} & 0 & b \\ 1 & e & f \\ 1 & a & 1 \end{pmatrix}, \quad a, e, f \in \mathbb{R}, \quad b \in \mathbb{C},$$

write $G$ for the normal subgroup in which $e = 0$ and $H$ for the subgroup of $G$ in which $a \in 2\pi\mathbb{Z}$. Identify $g'^*$ with $\mathbb{R} \times \mathbb{C} \times \mathbb{R}^2$ by writing $(p, q, s, t)$ for the value at the identity of the 1-form

$$pda + \text{Re}(qdb) - sde - tdf.$$

The coadjoint action of $G'$ leaves the hyperplane $t = 1$ invariant and acts there by

$$g' \begin{pmatrix} p \\ q \\ s \end{pmatrix} = \begin{pmatrix} p + e + \text{Re}(\overline{t}bge^{ia}) \\ qe^{ia} \\ s + a \end{pmatrix}.$$ 

Likewise, identify $g^*$ with triples $(p, q, t)$ and $h^*$ with pairs $(q, t)$ so that the projections $g'^* \rightarrow g^* \rightarrow h^*$ become $(p, q, s, t) \mapsto (p, q, t) \mapsto (q, t)$ and the coadjoint actions are by appropriate restrictions of (4.3). Then the coadjoint orbit $X' = G'(\tilde{c})$ of $\tilde{c} = (0, 1, 0, 1)$ projects onto the orbit $X = G(\tilde{c}|_h)$ and is its universal covering:

$$X' = \{(p, e^{it}, s, 1) : (p, s) \in \mathbb{R}^2\}, \quad \omega_{X'} = dp \wedge ds = d(pds),$$

$$X = \{(p, q, 1) : (p, q) \in \mathbb{R} \times T\}, \quad \omega_{X} = dp \wedge \frac{dq}{iq} = d\left(p, \frac{dq}{iq}\right).$$

So there is a single representation attached to $X'$, and a circle worth of representations attached to $X$. Moreover, one checks (or finds by [Z14] applied to the normal subgroup $H^0$) that

$$X = \text{Ind}_{H^0}^G \{\tilde{c}|_h\}, \quad \text{and likewise} \quad X' = \text{Ind}_{H'}^G \{\tilde{c}|_h\}.$$

where $H'$ is the normal subgroup of $G'$ in which $a = 0$. So symplectic Frobenius reciprocity (3.4) gives the relation

$$\text{Hom}_G(X, \text{Res}^G_{H'} X') = \text{Hom}_{H'}(\{\tilde{c}|_h\}, \text{Res}^G_{H'} X')$$

$$(X' \rightarrow h^*)^{-1}(\tilde{c}|_h)/H = \{\text{a point}\}$$

between $X$ and the restriction of $X'$. But this fact is of little use for representation theory, as it fails to predict into which of the representations attached to $X$ the representation attached to $X'$ will split (when restricted to $G$). As one knows, this should be fixed by working instead with prequantum spaces in the sense of the next section.

## 5 Prequantum G-spaces

Following [S70], we call prequantum manifold a manifold $\tilde{X}$ with a contact 1-form $\varpi$ whose Reeb vector field generates a circle group action. We recall that $\varpi$ contact means that $\text{Ker}(d\varpi)$ is 1-dimensional.
and transverse to \( \Ker(\sigma) \); its Reeb vector field, \( i \), on \( \tilde{X} \) is defined by

\[(5.1) \quad i(\tilde{z}) \in \Ker(d\sigma) \quad \text{and} \quad \sigma(i(\tilde{z})) = 1 \quad \forall \tilde{z} \in \tilde{X}.\]

Then \( (\tilde{X}, d\sigma) \) is a presymplectic manifold whose null leaves are the orbits of the circle group \( T = U(1) \) acting on \( X \) and \( d\sigma \) descends to a symplectic form \( \omega \) on the leaf space \( \tilde{X} = X/T \). If a Lie group \( G \) acts on \( \tilde{X} \) and preserves \( \sigma \), then it commutes with \( T \) and the equivariant momentum map \( \Phi : \tilde{X} \to \mathfrak{g}^* \),

\[(5.2) \quad \langle \Phi(\tilde{z}), Z \rangle = \sigma(Z(\tilde{z})),\]

descends to a momentum map \( \Phi : X \to \mathfrak{g}^* \), making \( (X, \omega, \Phi) \) a Hamiltonian G-space prequantized by the prequantum G-space \( (\tilde{X}, \sigma) \).

We do not distinguish between two spaces \( \tilde{X}_1, \tilde{X}_2 \) which are isomorphic, i.e., related by a G-equivariant diffeomorphism which transforms \( \sigma_1 \) into \( \sigma_2 \). (If several are in play, we may also use subscripts like \( \sigma_X, i_X, \Phi_X \), etc.) We recall three basic constructions in the prequantum category:

\[\begin{align*}
\text{(5.3) Prequantum dual.} & \quad (\text{[S70, 18.47].}) \text{ We write } \tilde{X}^- \text{ for the G-space equal to } \tilde{X} \text{ but with opposite } 1\text{-form } -\sigma \text{ (and consequently opposite Reeb field and } T\text{-action). It prequantizes the dual G-space } (X^-, -\omega, -\Phi). \\
\text{(5.4) Prequantum product.} & \quad (\text{[S70, 18.52].}) \text{ If } \tilde{X}_1 \text{ and } \tilde{X}_2 \text{ are prequantum G-spaces, then } \tilde{X}_1 \times \tilde{X}_2 \text{ (with diagonal G-action) is a } \mathbb{T}^2\text{-space in which the action of the anti-diagonal } \Delta = \{(z^{-1}, z) : z \in T\} \text{ has as its orbits the characteristic leaves of the 1-form } \sigma_1 + \sigma_2. \text{ Hence this descends to the quotient } \tilde{X}_1 \boxtimes \tilde{X}_2 := (\tilde{X}_1 \times \tilde{X}_2)/\Delta \text{ as a 1-form making it a prequantization of the symplectic product } \tilde{X}_1 \times \tilde{X}_2. \text{ In view of (5.3), the } \Delta\text{-action on } \tilde{X}_1^{-} \times \tilde{X}_2^{+} \text{ is } z(\tilde{z}_1, \tilde{z}_2) = (z(\tilde{z}_1), z(\tilde{z}_2)). \\
\text{(5.5) Prequantum reduction.} & \quad (\text{[L01, Thm 2].}) \text{ Assume } G \text{ acts freely and properly on } \tilde{X}, \text{ and consider the level } L := \Phi^{-1}(0). \text{ By the very definition (5.2) of } \Phi \text{ and its being a momentum map, we have } g(\tilde{z}) \subset \Ker(\sigma|_{\tilde{L}}) \cap \Ker(d\sigma|_{\tilde{L}}). \text{ Since } \sigma|_{\tilde{L}} \text{ is also } G\text{-invariant, it follows (see [S70, 5.21]) that it descends to a contact 1-form on the quotient } \tilde{X}/G := \Phi^{-1}(0)/G. \text{ This prequantizes the symplectic reduction } \tilde{X}/G = \tilde{\Phi}^{-1}(0)/G. \] \]

## 6 Prequantum Induction

Given a closed subgroup \( H \subset G \) and a prequantum H-space \( (\tilde{Y}, \sigma_{\tilde{Y}}) \) whose momentum map (5.2) we denote \( \Psi \), we propose to construct an induced prequantum G-space \( \text{Ind}_H^G \tilde{Y} \) as follows. Consider the prequantum \( (G \times H)\)-space \( \tilde{N} = T^*G \times \tilde{Y} \) with 1-form \( \sigma_{T^*G} + \sigma_{\tilde{Y}} \) and action \((g, h)(p, \tilde{y}) = (gp^{-1}, h(\tilde{y}))\). This action has the equivariant momentum map \( \phi \times \psi : \tilde{N} \to \mathfrak{g}^* \times \mathfrak{h}^* \),

\[(6.1) \begin{cases} \phi(p, \tilde{y}) = pq^{-1} \\ \psi(p, \tilde{y}) = \Psi(\tilde{y}) - q^{-1}p|_{\mathfrak{h}} \end{cases} \quad (p \in T^*_\tilde{y}G).\]

The same arguments as with (1.2), then, show that

\[(6.2) \quad \text{Ind}_H^G \tilde{Y} := \tilde{N}/H = \psi^{-1}(0)/H \]

(prequantum reduction (5.5)) is naturally a prequantum G-space which prequantizes the symplectically induced manifold (1.2).

## 7 Prequantum Induction in Stages

\[\begin{align*}
\text{(7.1) Theorem (Stages).} & \quad \text{If } H \subset K \subset G \text{ are closed subgroups of the Lie group } G, \text{ then} \\
\text{Ind}_K^G \text{Ind}_H^K \tilde{Y} = \text{Ind}_H^G \tilde{Y}. \]
\]
Proof. The proof is *mutatis mutandis* the same as for (2.1), only simpler. We just switch to working with restrictions and push-forwards of the 1-form \(\sigma(\delta p, \delta \bar{p}, \delta \tilde{y}) = \langle p, \delta q \rangle + \langle \bar{p}, \delta \bar{q} \rangle + \sigma_Y(\delta \tilde{y})\) on \(M = T^*G \times T^*K \times \tilde{Y}\) instead of the 2-form \(\omega\) on \(M\).

## 8 Prequantum Frobenius Reciprocity

The three constructions (5.3–5.5) put together furnish us with a notion of the *intertwiner space* of two prequantum \(G\)-spaces,

(8.1) \[\text{Hom}_G(\tilde{X}_1, \tilde{X}_2) := (\tilde{X}_1^{-} \boxtimes \tilde{X}_2)/G.\]

Freeness and properness of the last \(G\)-action are not assumed and we again regard (8.1) as just a set.

(8.2) **Theorem (Frobenius reciprocity).** If \(\tilde{X}\) is a prequantum \(G\)-space and \(\tilde{Y}\) a prequantum \(H\)-space, then

\[\text{Hom}_G(\tilde{X}, \text{Ind}^G_H \tilde{Y}) = \text{Hom}_H(\text{Res}^G_H \tilde{X}, \tilde{Y}).\]

Proof. With \(\Delta\) as in (5.4), define \(\tilde{\tau}\) in the following commutative diagram by

\[\tilde{\tau}(\tilde{x}, p, \tilde{y}) = (q^{-1}(\tilde{x}), \tilde{y}),\]

where \(p \in T^*_qG:\)

(8.3) \[\begin{array}{ccc}
\tilde{M} := \tilde{X}^- \times T^*G \times \tilde{Y} & \xrightarrow{\tilde{\tau}} & \tilde{N} := \tilde{X}^- \times \tilde{Y} \\
\mod \Delta & & \mod \Delta \\
\tilde{M} := \tilde{X}^- \boxtimes T^*G \times \tilde{Y} & \xrightarrow{\tilde{\tau}} & \tilde{N} := \tilde{X}^- \boxtimes \tilde{Y} \\
\mod (T^2/\Delta) & & \mod (T^2/\Delta) \\
M := X^- \times T^*G \times Y & \xrightarrow{\tau} & N := X^- \times Y.
\end{array}\]

Then \(\tilde{\tau}\) descends, as indicated, to a map \(\tilde{\tau}\) and a map \(\tau\) which is the one in our proof of (3.4). Now each floor of this diagram supports a horizontal copy of Fig. 1 giving rise to the appropriate tilded versions of \(s\) and \(t\); a straightforward diagram chase checks that \(\tilde{\tau}: (M/\!/H)/G \to N/\!/H\) is the required bijection.

## 9 An Example (Reprise)

Recall the coadjoint orbits \(X \cong R \times T\) and \(X' \cong R^2\) of (4.4). Referring to [S70, 18.117, 18.133, 18.134] as well as [K70, Thm 5.1.1] and performing direct verifications, one finds:

- There is a unique prequantum \(G'\)-space prequantizing \(X'\), namely

(9.1) \[\tilde{X}' = X' \times T \ni (p, s, z) \quad \text{with} \quad \sigma' = pds + \frac{dz}{iz}\]

with \(G'\)-action (uniquely lifted from (4.3) to preserve \(\sigma\))

(9.2) \[g' \begin{pmatrix} p \\ s \\ z \end{pmatrix} = \begin{pmatrix} p + e + \text{Re}(\tilde{d} e^{(s + a)}) \\ s + a \\ ze^{\text{Re}(\tilde{e}^{(s + a)} - ze - f)} \end{pmatrix}.\]
• There is a circle worth of inequivalent prequantum G-spaces over X, namely all

\begin{equation}
\tilde{X}_\lambda = X \times T \ni (p, q, z) \quad \text{with} \quad \sigma_\lambda = (p + \lambda) \frac{dq}{iq} + \frac{dz}{iz}
\end{equation}

where $\lambda_1, \lambda_2 \in \mathbb{R}$ give equivalent prequantizations iff they differ by an integer (cf. [A59, K06]), and the G-action on $\tilde{X}_\lambda$ (uniquely lifted from that on X to preserve $\sigma_\lambda$) reads

\begin{equation}
g \begin{pmatrix} p \\ q \\ z \end{pmatrix} = \begin{pmatrix} p + \text{Re}(ibqe^{ia}) \\ bqe^{ia} \\ ze^{[\text{Re}(bqe^{ia}) - \lambda a - f]} \end{pmatrix}.
\end{equation}

Moreover, from (4.5) and the sentence following (6.2) one deduces without trouble that

\begin{equation}
\tilde{X}_\lambda = \text{Ind}_{G}^{H} T_\lambda \quad \text{and} \quad \tilde{X}' = \text{Ind}_{G'}^{H'} T'
\end{equation}

where $T_\lambda$ is the unit circle on which $H$ acts by the character $\chi_\lambda(h) = e^{-ibae^{\text{Re}(b)-f}}$, and $T'$ is the unit circle on which $H'$ acts by the character $\chi'(h') = e^{ibae^{\text{Re}(b)-f}}$ (notation (4.1)). Consequently (8.2) gives

\begin{equation}
\text{Hom}_G((\tilde{X}_\lambda, \text{Res}_G^{G'} \tilde{X}')) = \text{Hom}_H(T_\lambda, \text{Res}_H^{H'} \tilde{X}')
\end{equation}

as one easily computes from (8.1, 9.2). This replaces (4.6) and “predicts” that once restricted to $G$, the irreducible representation $\text{Ind}_{G'}^{G} \chi'$ (attached to $\tilde{X}'$ by Auslander-Kostant [A71]) splits into the direct integral over $\lambda \in \mathbb{R}/\mathbb{Z}$ of the irreducible representations $\text{Ind}_{H}^{G} \chi_\lambda$ (attached to $\tilde{X}_\lambda$) with multiplicity 1; this prediction is correct and can be checked directly.

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