Scaling of finite size effect of $\alpha$-Rényi entropy in disjointed intervals under dilation

Long Xiong,1 Shunyao Zhang,1 Guang-Can Guo,1,2,3 and Ming Gong1,2,3,∗

1CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei, 230026, China
2Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
3CAS Center For Excellence in Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China

(Dated: March 22, 2022)

The $\alpha$-Rényi entropy in the gapless models have been obtained by the conformal field theory, which is exact in the thermodynamic limit. However, the calculation of its finite size effect (FSE) is challenging. So far only the FSE in a single interval in the XX model has been understood and the FSE in the other models and in the other conditions are totally unknown. Here we report the FSE of this entropy in disjointed intervals $A = \cup_i A_i$ under a uniform dilation $\lambda A$ in the XY model, showing of a universal scaling law as

$$\Delta^\alpha_A = \Delta^\alpha_A \lambda^{-\eta} B(A,\lambda),$$

where $|B(A,\lambda)| \leq 1$ is a bounded function and $\eta = \min(2, 2/\alpha)$ when $\alpha < 10$. We verify this relation in the phase boundaries of the XY model, in which the different central charges correspond to the physics of free Fermion and free Boson models. We find that in the disjointed intervals, two FSEs, termed as extrinsic FSE and intrinsic FSE, are required to fully account for the FSE of the entropy.

Physically, we find that only the edge modes of the correlation matrix localized at the open ends $\partial A$ have contribution to the total entropy and its FSE. Our results provide some incisive insight into the entanglement entropy in the many-body systems.

I. INTRODUCTION

Entanglement has played a more and more important role in quantum information and many-body physics. A large number of investigations have shown that the ground state of the gapped and gapless phases will have totally different entanglement entropies. For a regime $A$ (see Fig. 1 (a)), we can denote the reduced density matrix as $\rho_A$, then the Shannon entropy can be calculated using $S_A = -\text{Tr}(\rho_A \ln \rho_A)$. In the gapped phase, its entropy satisfies the area law [1–5].

$$S_A = \tilde{\alpha} \partial A - \tilde{\gamma} \sim L^{d-1}. \quad (1)$$

However, in the gapless phase, it satisfies a different area law with logarithmic correlation as [6, 7]

$$S_A \sim L^{d-1} \ln L. \quad (2)$$

In the above two equations, $L$ is the system size and $d$ is its system dimension. When $d = 1$, it yields the Logarithm divergence of the entropy with the increasing of system size (see below). The similar features may also be found for their low lying excited states [8–11]. By generalizing this concept in terms of $\alpha$-Rényi entropy, one find that in the one dimensional gapless phase [12, 13]

$$S^\alpha_A = \frac{1}{1 - \alpha} \log_2 \text{Tr} \rho^\alpha_A = \frac{c + \bar{c}}{12(1 + \alpha)} \log_2 L + s_0^\alpha + \Delta^\alpha_A, \quad (3)$$

where $c$ and $\bar{c}$ are the holomorphic and antiholomorphic central charges respectively [14, 15], $s_0$ is a non-universal constant and $\Delta^\alpha_A$ is its finite size effect (FSE), satisfying

$$\lim_{L \to \infty} \Delta^\alpha_A = 0, \quad (4)$$

by its definition. The expression of $\alpha$-Rényi entropy has been examined numerically in some of the solvable models [13, 16–22], which can be more rigorously obtained by the conformal field theory (CFT) [14, 23–25]. Since the gapped phase and gapless phases have totally different entanglement properties, these features are used to diagnose the phase transitions in some of the many-body models [26–32].

The scaling laws of the FSE in Eq. 4, which in the gapped and gapless phases should exhibit totally different behaviors, are the major concern of this manuscript. To date, it has been rarely investigated. In the XX model with free fermions [13, 19, 20, 33], it has been calculated

FIG. 1. (a) Entanglement entropy in a single interval $A$; and (b) Entanglement entropy in two disjointed intervals $A_1$ and $A_2$ separated by an interval $B_1$. The configuration in (b) can be generalized to disjointed intervals; see results in Fig. 5 and Fig. 6. The correlation between different intervals will be calculated by the correlation functions $\mathcal{G}(x)$.
using the Jin-Korepin (JK) approach [13, 34], yielding a extremely complicated polynomial of the length $L$ with exponents $\eta = 2$ and $\eta = 2n/\alpha$ for $n \in \mathbb{Z}^+$ [34–38] (see discussion in section III). However, the FSE in the other models or in disjointed intervals are unknown (see Fig. 1 (b)), which are also challenging to be calculated by the JK approach [39–43]. Great endeavor has been made trying to explore this FSE in disjointed intervals [44–46], and failed to find some universal scaling behaviors in them.

This work aims to explore the scaling law of the above FSE in multiple intervals (see Fig. 1 (b)) in free Fermion ($c = \bar{c} = 1$) and Boson ($c = \bar{c} = 1/2$) models, in which the correlators exhibit some kind of scaling laws under uniform dilation, such as $\langle \phi(x)\phi(y) \rangle = \lambda^{-\nu/2} \langle \phi(x)\phi(y) \rangle$. This feature can give rise to scaling law in $\Delta_{\lambda}^\alpha$ if it is a function of these correlators, which has not yet been unveiled in the previous literature. Let us denote $A = \cup_i A_i$ to be the jointed structure of some disjointed intervals $A_i$ and $\lambda A$ (with $\lambda \in \mathbb{Z}^+$) denotes its uniform dilation. The set of open ends are denoted as $\partial A = \cup_i \partial A_i$. The key result of this work in the large size limit can be formulated as

$$\Delta_{\lambda}^\alpha = \Delta_{A}^\alpha \lambda^{-\eta} B(A, \lambda),$$

where $|B(A, \lambda)| \leq 1$ is a bounded function and $\eta = \min(2, 2/\alpha)$ when $\alpha < 10$. We find that only the edge modes of the correlation matrix with wave functions localized near the open ends contribute to the Rényi entropy and its FSE. We confirm Eq. 5 in both free Fermion and free Boson models. Our results may shed new insight into the FSE of Rényi entropy in multiple intervals in the other many-body systems.

This work is organized as following. In section II, we present the XY model, in which the properties of the correlation function is discussed in details. These correlation functions are essential for the scaling laws of the entanglement entropy. We will show that the correlation functions in the gapped and gapless phases are totally different. In section III, we will discuss the major results by JK. In section IV, two different FSEs are defined in disjointed intervals, and their features in two and three intervals are discussed. At the end of this section, the results in the gapped phases, which are trivial, will also be briefly discussed. In section V, we conclude our results. In section A, we show that the entropy defined in this way is well-defined.

II. XY SPIN CHAIN

We illustrated the above conclusion using the following exact solvable one dimensional XY spin chain

$$H = \sum_i (\frac{1+\gamma}{2}) s_i^x s_{i+1}^x + (\frac{1-\gamma}{2}) s_i^y s_{i+1}^y + h s_i^z,$$

where $s_i^\alpha (\alpha = x, y, z)$ are Pauli matrices and $h$ is the transverse Zeeman field. After a Jordan-Wigner trans-

![FIG. 2. (a) Phase diagram of the transverse XY model. The thick lines correspond to the gapless phase with $c = \bar{c} = 1$ for free Fermions and $c = \bar{c} = 1/2$ for free Bosons. (b) Insulator phases with $\gamma = 0$ and $|h| > 1$ and gapless phase with Fermi points $\cos(k_F) = h$ when $|h| < 1$. (c) The special points with $\gamma = \pm 1$ and $h = 0$. In the Majorana Fermion representation, this model is decoupled into paired Majorana fermions (represented by the hemicycles), with $\alpha_1$ and $\alpha_{2L}$ unpaired, giving rise to degenerate zero modes. This special case has been studied by Kitaev [47]. The insulator phases for $|h| > 1$ and $\gamma = 0$ in (b) and the Kitaev points in (c) have zero range correlation with $G(x) = 0$.]

formation by assigning Fermion operators $c_i$ and $c_i^\dagger$ to each site, it is mapped to a free Fermion model as

$$H = -\sum_i c_i^\dagger c_{i+1} + \gamma c_i^\dagger c_{i+1} + h.c. + h(1 - 2c_i^\dagger c_i),$$

with excitation gap

$$\epsilon_k = \sqrt{(\cos(k) - h)^2 + \gamma^2 \sin(k)^2}.$$
Wave Function of Matrix $K(x)$

![Figure 3](image)

FIG. 3. $K(x)$ in the gapped phase and gapless phase. In the gapped phase, we have chosen $\gamma = 0.4$ and $h = 0.5$; while in the gapless phase, we used $\gamma = 0.0$ and $h = 0.5$. Only values at $x \in \mathbb{Z}$ are plotted.

![Figure 4](image)

FIG. 4. Wave functions of the correlation matrix $\Gamma$ in a single interval with $L = 100$. (a) and (b) show the wave function in the gapless phase ($\gamma = 1.36$ and $h = 1.0$) for $\nu_{01}$, $\nu_{10}$ and $\nu_{12}$ (using $\nu_{100+i} = -\nu_{100-i}$ from the particle-hole symmetry of $\Gamma$); and (c) and (d) show the results for $\gamma = 1.36$ and $h = 1.5$ in the gapped phase for $\nu_{01}$, $\nu_{10}$, and $\nu_{12}$. Here $\Gamma$ is a $2L \times 2L$ matrix, thus we have in totally $2L = 200$ eigenvalues.

Based on the Majorana operators $\alpha_{2L-1} = (\prod_{m<1} \sigma_m^z) \sigma_m^\alpha$ and $\alpha_{2L} = (\prod_{m<1} \sigma_m^z) \sigma_m^\beta$. Here, $\Gamma$ is a skew matrix with entries given by

$$\Gamma_{i,i+x} = -i(\langle \alpha_i \alpha_{i+x} \rangle - \delta_{x0}) = \left( \begin{array}{cc} 0 & \mathcal{G}(x) \\ -\mathcal{G}(x) & 0 \end{array} \right),$$

where $\gamma = 2\pi L$,

$$\mathcal{G}(x) = \frac{\gamma \sin(k) \sin(kx) - e_k \cos(kx)}{2\pi e_k} dk,$$  (11)

with $e_k = h - \cos(k)$. This integral determines all the properties of the Rényi entropy. It has a number of salient features [20]. In the gapless phases, it decays algebraically as

$$\mathcal{G}(x) = \frac{K(x)}{x},$$  (12)

where $K(x)$ is a bounded oscillating function. Specifically, we find that:

(I) For free Fermions with $\gamma = 0$ and $|h| \leq 1$, we have

$$K(x) = \frac{(2\sin(x \arccos(|h|)) - \sin(\pi x))}{\pi}.$$  (13)

When $h = \pm 1$, the spectra is gapless with quadratic dispersion as $E_k \propto k^2$ and $\mathcal{G}(x) = 0$, which violate conformal symmetry. In the fully gapped phase with $|h| > 1$, we always have $\mathcal{G}(x) = 0$ for the reason of a vacuum state or a fully filled state (see Fig. 2 (b)), hence $\Gamma$ is always equal to zero (see Eq. 10).

(II) For free Bosons with $|h| = 1$ and $\gamma \neq 0$, we have

$$K(x) \simeq -2\text{sign}(\gamma)(1 - \cos(\pi x)),$$  (14)

which is long-range correlated. The above two $K(x)$ are bounded functions, that is, $|K(x)| \leq C$ for some positive constant $C$; and $\mathcal{G}(x)$ decays according to $1/x$ besides their oscillating behaviors. This long-range correlator is essential for the logarithm relation of the entanglement entropy, as shown in Eq. 2 and Eq. 3.

(III) In the gapped phases, $\mathcal{G}(x)$ is short-range correlated with an exponential decaying behavior. Based on Eq. 11, we can even show at the Kitaev points with $h = 0$ and $\gamma = \pm 1$ (see Fig. 2 (c)), $\mathcal{G}(x)$ is zero range correlated since

$$K(x) = \sin(\pi x) = 0, \quad x \in \mathbb{Z},$$  (15)

which can be understood that in these two points, $\alpha_{2L}$ and $\alpha_{2L+1}$ are paired, leaving only $\alpha_1$ and $\alpha_{2L}$ to be the dangling operators left out from the Hamiltonian. In the gapped phases with finite energy gap, by expanding $e_k = a + bk^2$, where $a = |h| \pm 1$ and $b = (|h| \pm 1 + \gamma^2)/2|h| \pm 1|$ for the energy gap at $h = 0$ (−) or $\pi$ (+), we obtain

$$\mathcal{G}(x) \sim e^{-|x|/\xi},$$  (16)

where the decay length $\xi = |\gamma|/(\sqrt{2}|h| \pm 1)$. It implies of the area law [49], as shown in Eq. 1. In Fig. 3, we plot the results of $K(x)$ in the gapped and gapless phases, showing of excellent agreement with the above analysis. In the gapped phase, $K(x)$ will always vanished at large $x \in \mathbb{Z}$.

Eq. 9 is essential to calculate the density matrix of $\rho_A$ and its eigenvalues numerically. Noticed that $W$ and $\Gamma$ are real skew matrices, and can be solved by an orthogonal matrix, then we have $\rho_A = \prod_l G_l \rho_l$, with $\rho_l = \text{diag}(1-\nu_l, 1+\nu_l)$, with eigenvalues as $\lambda = \prod_{s=1,2}^{1+\nu_s/2}$. By definition we have [50]

$$S_A^{\alpha} = \frac{1}{1-\alpha} \sum_l \log_2((1+\nu_l/2)^\alpha + (1-\nu_l/2)^\alpha),$$  (17)

for the area law [49], as shown in Eq. 1.
The FSE of this entropy is defined as the difference between the exact (numerical) Rényi entropy and the prediction from CFT [35, 51–53]. In a single interval with $\gamma = 0$ for free Fermions, $\Delta^\alpha_A$ was obtained by the JK approach [13]. To the leading term [34, 35]

$$
\Delta^\alpha_A = \frac{A_1}{(2L|\sin(k_F)|)^{\alpha}} + \frac{A_2}{(2L|\sin(k_F)|)^{\alpha}} + O(L^{-\frac{2\alpha}{\pi}}),
$$
where

$$A_1 = \frac{12(3\alpha^2 - 7) + (49 - \alpha^2)\sin^2(k_F)}{(285\alpha^3)}(1 + \alpha),$$

and

$$A_2 = \frac{2Q\cos(2k_FL)}{(1 - \alpha)},$$

which is reduced to the Shannon entropy when $\alpha \to 1$, with $S_A = -\sum_{l=1}^{L}(\log_2 (\frac{1+2l}{2}))$. Furthermore, when $\alpha$ is large enough, saturation of the Rényi entropy happens, with $S_A^\infty = -\sum_{l=0}^{\infty} \log_2 (\frac{1+2l}{2})$. The eigenvalues and eigenvectors of the Hermite operator $i\tilde{\Gamma}$ in Fig. 4 show that only the modes localized at the edges have contribution to the Rényi entropy, while the extended modes with $\nu_l \to \pm 1$ will not. This is expected, since for the extended modes with vanished amplitudes at the open ends $\partial A$, the coupling between regime $A$ with its complement $\bar{A}$ is vanishing small; however for the localized edge modes, the coupling is strong. This is also the essential origin of the area law quoted above.
with \( Q = \Gamma(1/2 + 1/(2\alpha))^2 / \Gamma(1/2 - 1/(2\alpha))^2 \), and \( k_F = \arccos(|h|) \) is the Fermi momentum. The next leading terms correspond to \( n > 1 \). Note that the second term oscillate periodically with spatial period \( d = \pi / k_F \), reflecting the coupling between the scatterings near the two Fermi momenta \( \pm k_F \). This result shows that the first term is irrelevant when \( \alpha > 1 \) with \( B = \cos(2k_F L) \), and the second term is irrelevant when \( \alpha < 1 \) with \( B = 1 \), while both terms are important near the Shannon entropy with \( \alpha \sim 1 \). Thus \( \eta = \min(2, 2/\alpha) \). Our data in Fig. 5 (a) - (b) show excellent agreement with this prediction. One should be noticed that \( A_1 \) and \( A_2 \) may have similar amplitudes but opposite signs near \( \alpha \sim 1 \) with a proper choice of \( k_F \), which may yield strong cancellation between them, thus it can not be fitted well using Eq. 5 at the regime with \( \alpha \sim 1 \) in Fig. 5 (b).

The following section will generalize the results in Eq. 18 to disjointed intervals, showing of great similarity between them. In the disjointed intervals, the FSE is much more complicated, and two FSEs — the extrinsic and intrinsic FSEs — should be defined, all of which have similar scaling laws under uniform dilation, including the basic feature of the bounded function \( B(L) \).

IV. TWO FSES IN DISJOINTED INTERVALS

To characterize the FSE in multiple intervals, we need to define two different FSEs, beside of \( \Delta^\alpha \) in a single interval. To this end, we first consider the FSE in two intervals \( A_1, A_2 \) separated by \( B_1 \) (see configuration in Fig. 1 (b)), which reads as

\[
S^\alpha_{A_1, A_2} = S^\alpha_{A_1, B_1} + S^\alpha_{B_1, A_2} + \Delta^\alpha_{A_1B_1A_2} + \Delta^\alpha_{B_1A_2} + \Delta^\alpha_{A_1B_1A_2} + \Delta^\alpha_{A_2}.
\]

This definition is well-defined (see appendix A). The right hand side contains the entropies in all possible single intervals, while the left hand side is the entropy of two disjointed intervals [45, 54–56]. Here we have introduced \( \Delta^\alpha_{ij} \) to accounts for the difference between the left and right hand sides, making it to be an exact identity. When the sizes of \( A_1, B_1 \) and \( A_2 \) approaches infinity, we expect that

\[
\lim_{L_{A_1, A_2, B_1} \to \infty} \Delta^\alpha_{A_1B_1A_2} = 0.
\]

Since it is an extrinsic effect to force the above identity, it is termed as the extrinsic FSE of the disjointed intervals. Meanwhile, we can write

\[
S^\alpha_{A_1, A_2} = S^\alpha_{A_1, A_2} + \Delta^\alpha_{A_1A_2}
\]

using the definition in Eq. 3, where \( S^\alpha_{A_1, A_2} \) is the entropy from CFT that can be found in Refs. [52 and 57]. This FSE is an intrinsic effect, not related to the identity above, it is termed as the intrinsic FSE of the disjointed intervals. By definition, we also expect this FSE is vanished when the intervals and its separation approach infinity.

The expression of \( S^\alpha_{A_1, A_2} \) may also be inferred from Eq. 21, assuming of negligible FSE and \( S^\alpha_A \) in a single interval given by Eq. 3. Thus we have an identity between all FSEs as

\[
\Delta^\alpha_{A_1A_2} = \Delta^\alpha_{A_1B_1A_2} - \Delta^\alpha_{A_1B_1} - \Delta^\alpha_{B_1A_2} + \Delta^\alpha_{A_1B_1} + \Delta^\alpha_{A_1A_2} + \Delta^\alpha_{B_1A_2} + \Delta^\alpha_{B_1A_2} + \Delta^\alpha_{A_1B_1A_2} + \Delta^\alpha_{A_1A_2}.
\]

From the fact that CFT is analytically exact when the system size is large enough and that the correlation between the two intervals decreases with the increasing of separation, we expect all \( \Delta^\alpha \) in Eq. 24 approach zero at large separation according to \( \Delta^\alpha \propto 1/L\eta \). This limit can be used to extract the non-universal constant of \( s_0 \) in Eq. 3.

In the gapless phase, all these FSEs are in the same order of magnitude, thus all of them are important; they reflect the FSE of \( A \) from different aspects.

This definition can be easily generalized to three (or many) disjointed intervals using the finding from the CFT that

\[
S^\alpha_{A_1, A_2, A_3} = S^\alpha_{A_1, B_1, B_2, B_3} - S^\alpha_{A_1, B_1} - S^\alpha_{B_1, A_2, B_2} - S^\alpha_{B_1, B_2, A_2} + \cdots + S^\alpha_{B_1} + S^\alpha_{A_3} + S^\alpha_{B_1} + S^\alpha_{B_2} + S^\alpha_{B_3} + \Delta^\alpha_1 + \Delta^\alpha_2 + \Delta^\alpha_3,
\]

where \( \Delta^\alpha_{A_1A_2A_3} \) is the intrinsic FSE of the three intervals. We can find an identity between all FSEs exactly the same as Eq. 21. Similarly, we define

\[
S^\alpha_{A_1A_2A_3} = S^\alpha_{A_1A_2} + \Delta^\alpha_{A_1A_2A_3},
\]

where \( \Delta^\alpha_{A_1A_2A_3} \) is the intrinsic FSE of the three intervals. We can find an identity between all FSEs exactly the same as Eq. 24. Thus we see that the FSE can be fully characterized by these two FSEs in many intervals. In our numerical simulation, we can extract these two FSEs, and discuss its effect under uniform dilation.

We present the data for these \( \Delta^\alpha \) in Fig. 5 (c) - (f) under uniform dilation for free Fermions with \( c = \bar{c} = 1 \) (see Fig. 2 (a)). We find a strong oscillation of \( \Delta^\alpha_\lambda \) for \( \alpha > 1 \) in (c) - (f), which is consistent with Eq. 18 with a somewhat modified \( A_1 \) and \( A_2 \); unfortunately, these values can not be determined analytically. We find that when \( \alpha < 1 \), the \( A_1 \) term is always relevant with \( B = 1 \); while when \( \alpha > 1 \), the \( A_2 \) term is relevant with \( |B(A, \lambda)| \) being a complicated yet non-analytical bounded function. These observations yield the major conclusion of Eq. 5. In Fig. 5 (b), we present the fitted exponent \( \alpha \) as a function of \( \alpha \) for all these \( \Delta^\alpha \), all of which falls to the same expression \( \alpha = \min(2, 2/\alpha) \) when \( \alpha < 10 \). When \( \alpha \sim 1 \), it may not be well fitted using Eq. 5 for the same reason of cancellation in Eq. 18. Moreover, in Figs. 5 (b), saturation of entropy happens when \( \alpha > 10 \), at which the exponent \( \alpha \) will deviate from \( 2/\alpha \) from the unspecified high-order terms \( L^{-\lambda n/\alpha} \) \( (n > 1) \) in Eq. 18; see [13, 34, 35].
These results can also be found for free Bosons with \( c = \bar{c} = 1/2 \) in Fig. 6. However, the correlator \( G(x) \) oscillates with period \( k_F = \pi \), thus the oscillation of the FSE from \( A_2 \) is disappeared and \( B(A, \lambda) = 1 \) for all disjointed intervals \( A \). As a result, for all the FSEs in the one, two and three disjointed intervals, all the FSEs decay monotonically with the increasing of dilation ratio \( \lambda \), following the claim of Eq. 5. We also show that when the dilation is non-uniform, this general relation is failed. In Fig. 6 (b), the summarized \( \eta \) are also the same as that in Fig. 5 (b), showing of the similar cancellation effect prescribed by Eq. 18 even in multiple intervals, though their analytical expressions are impossible.

Finally, we briefly discuss the FSE in the gapped phases. We find that all the \( \Delta^\alpha_A \) decay exponentially under dilation in the multiple intervals, with \( S^\alpha_A \) satisfies the area law [49]. In these phases, the correlator \( G(x) \) also decays exponentially as a function of \( x \). From Fig. 4 (c) and (d), we show that only the edge modes of \( \partial A \) contribute to \( S^\alpha_A \). Due to the short-range correlation from \( G(x) \), the FSEs will be quickly disappeared following \( \Delta^\alpha_A \sim e^{-x/\xi} \), where \( x \) is the minimal separation between the open ends in \( \partial A \). This result is trivial, thus is not presented in this manuscript. For this reason, \( \Delta^\alpha_A \) in multiple intervals also exhibit different kinds of scaling laws in the gapped and gapless phases, which can be used for the diagnostication of phase transitions [26–32].

V. CONCLUSION

To conclude, we examine the FSE of \( \alpha \)-Rényi entropy in the free Fermion and free Boson models in the XY model, which exhibit the same scaling law during uniform dilation that \( \Delta^\alpha_{A,A} = \lambda^{-\eta} \Delta^\alpha_{B}(A, \lambda) \). We find that the regime \( \alpha < 1 \) and \( \alpha > 1 \) are described by different relevant terms, thus exhibit different scaling behaviors. When \( \alpha \) is not large enough, we find \( \eta = \min(2, 2/\alpha) \). For the Shannon entropy, we thus have \( \eta = 2 \) exactly. From the correlation matrix \( i^\prime \), we find that only the edge modes localized at the open ends \( \partial A \) contribute to the \( \alpha \)-Rényi entropy as well as its FSE. Our results in multiple intervals provide some incisive insight into the entanglement entropy in the many-body system, in which the analytical calculation is scarcely possible. Since this FSE is different in the gapped and gapless phases, the FSE of the disjointed intervals can also be used to characterize this difference and their phase transitions.

Appendix A: The well-defined entropy \( S_A \)

The formula of entropy in disjointed intervals (see for example in Eq. 21 for two intervals) depends only on the separation between \( A_1 \) and \( A_2 \), which is independent of the other part of the infinity system. The same conclusion is for more complicated structures. To understand this, let us assume a ring geometry in Fig. 7, in which \( A_1 \) and \( A_2 \) are separated by either \( B_1 \) or \( B_1 \). We assume that their sizes are large enough, then the FSEs are negligible. We have two different methods — using \( A_1 - B_1 - A_2 \) and \( A_1 - B_2 - A_2 \) — to accounts for the entropy of \( A_1 \) and \( A_2 \), that is

\[
S^\alpha_{A_1, A_2} = S^\alpha_{A_1, B_1, A_2} - S^\alpha_{A_1, B_1} - S^\alpha_{B_1, A_2} + S^\alpha_{A_1} + S^\alpha_{A_2} + S^\alpha_{B_1},
\]  

(A1)

for \( i = 1 \) and \( i = 2 \). Using the fact that \( S_A = S_{\bar{A}} \) by definition, we have \( S^\alpha_{A_1, B_1, A_2} = S^\alpha_{B_2}, S^\alpha_{A_1, B_2, A_2} = S^\alpha_{B_1}, S^\alpha_{A_1, B_2} = S^\alpha_{B_1, A_2}, S^\alpha_{A_1, B_1} = S^\alpha_{B_2, A_2} \), and we can show directly that the above two calculations will yield the same result. This method can be generalized to much more complicated disjointed structures. For this reason, we also expect well-defined FSE of these entropies.

The above result can also be understood intuitively using the following way. Let us assume that \( S^\alpha_{A_1, A_2} = x_1 S^\alpha_{A_1, B_1, A_2} + x_2 S^\alpha_{A_1, B_2, A_2} + x_3 S^\alpha_{B_1, A_2} + x_4 S^\alpha_{A_1} + x_5 S^\alpha_{A_2} + x_6 S^\alpha_{B_1} \), where \( x_i \) are undetermined coefficients. This definition should satisfy some basic features. (1) The above two calculations should yield the same result; (2) When the separation \( B_1 \) and \( B_2 \) are much larger than the sizes of \( A_1 \) and \( A_2 \), we will recover the limit that \( S^\alpha_{A_1, A_2} = S^\alpha_{A_1} + S^\alpha_{A_2} \); (3) This expression has well defined symmetry, that is, \( S^\alpha_{A_1, A_2} = S^\alpha_{A_2, A_1} \). These three constraints will yield uniquely the above entropy in two disjointed intervals.

Finally, if we assume that the entropy of two intervals is correct even when \( A_2 \) is a union of two disjointed intervals. Then we assume \( A_2 \rightarrow A_2 \cup A_3 \), where \( A_2 \) and \( A_3 \) are separated by \( B_2 \). In this way, we will find that the right-hand side of Eq. 21 is made by disjointed intervals. For instance, \( S^\alpha_{A_1, B_1, (A_2 A_3)} \) is the total entropy of \( A_1 \cup B_1 \cup A_2 \) and \( A_3 \), which can be calculated, again, using Eq. 21. Collecting all these results will yield Eq. 25. In this way, we can derive the expression of entropy in many disjointed intervals. In the large size limit, the same expression can be found by CFT. This result suggest that

\[
S^\alpha_{\partial A} = S^\alpha_A + \frac{c + \bar{c}}{12(1 + \alpha)} k \ln \lambda + \Delta^\alpha_{A,A},
\]

(A2)
for $k$ disjointed intervals.

**Acknowledgments:** This work is supported by the National Key Research and Development Program in China (Grants No. 2017YFA0304504 and No. 2017YFA0304103) and the National Natural Science Foundation of China (NSFC) with No. 11774328.

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