PAPER

Directly comparing entanglement-enhancing non-Gaussian operations

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Abstract

Single-photon-level non-Gaussian operations—photon addition, photon subtraction, and their coherent superposition—are powerful tools with which to increase entanglement in continuous-variable optical states. Although such operations are typically not deterministic, there may be other advantages such as noiseless manipulation. Therefore, to fully account for the efficacy of a particular non-Gaussian operation in a practical scenario, we develop figures of merit which trade off the advantages of such protocols against their success probability. Specifically, we define ‘entanglement enhancement rate’ as the increase in entanglement per trial of a generic non-Gaussian operation on a two-mode squeezed vacuum (TMSV) state. We consider states generated by photon-subtraction, photon-addition and a coherent superposition of subtraction and addition. We compare each strategy when applied to one or both modes of a TMSV state, and also in the presence of channel losses prior to the operation. In many cases, additional properties are analytically calculable, including excess noise arising from the operation and the fidelity of the resulting states to particular Gaussian and non-Gaussian states. Finally, by incorporating loss, we derive optimal interaction parameters for each non-Gaussian operation which maximize the effectiveness of the particular protocol under investigation.

1. Introduction

Entanglement is an important resource upon which many quantum communication protocols are based. However, it is a fragile resource susceptible to decoherence—in an optical setting this is mainly induced by loss. Therefore, the ability to increase the amount of entanglement available is a useful strategy to overcome that which may be lost during transmission. In the continuous-variable (CV) domain, one important entangled resource state is the two-mode squeezed vacuum (TMSV), a state which may be created deterministically by nonlinear optical processes. The TMSV is an example of a Gaussian state, whereby the first and second moments of its Wigner function uniquely describe the state. Although such states can be generated on-demand, Gaussian states require at least one non-Gaussian state and/or operation in order to increase their entanglement [1]. Since the nonlineairities necessary for non-Gaussian operations—at least 3rd order—are so small at the single-photon level, such nonlinearities must normally be induced by measurement [2].

Single-photon-level non-Gaussian operations such as photon subtraction [3–15], photon addition [4, 12, 16, 17] and a coherent superposition of addition and subtraction [18–20], here termed photon replacement, are powerful tools which, under certain conditions, can be used to increase entanglement. In the case of photon subtraction, the entanglement-enhancing capability has been experimentally demonstrated [21–23].

In all cases, there is a fundamental trade-off between success probability of each operation and the resultant increase in entanglement. In addition to investigating the increase in entanglement, previous analyses of these processes have adopted different figures of merit to compare each protocol, for example teleportation fidelity [3–5, 9, 10], log-negativity [6, 8, 11, 14, 17, 20] or entanglement entropy [12, 19]. However, the probabilistic nature of these operations, and the implications thereof, have been largely overlooked. Indeed, in the case of
both subtraction and addition, we will show that these figures of merit are maximized when the probability of a successful event is precisely zero. By contrast, the photon replacement operation maximizes entanglement at non zero probability. However, photon replacement, and indeed photon addition, requires additional ancillary photons, the production of which is not deterministic. Strictly speaking, one must also account for the probability associated with the production of photons when considering the probability of a success of the protocol.

Furthermore, analyses up to now consider that failure modes can be freely discarded without consequence. This may not be the case for all protocols; the corollary of accounting for success must therefore be to properly account for the failure modes and any negative effects these may have on the protocol.

A number of factors influence the efficacy of practical entanglement enhancement protocols. One might be interested in, for example:

- the total entanglement generated;
- the probability of success of a protocol;
- the rate at which entanglement is increased;
- the ‘quality’ of the states generated, i.e. their suitability for a particular task, e.g. teleportation fidelity [24];
- the resource costs of running each protocol;
- the tolerance of loss.

If entanglement-based quantum communication is to become widespread, at least some of such factors should be considered collectively and, where possible, quantified.

In this paper, we investigate the probabilities associated with photon subtraction, photon addition and photon replacement (the coherent superposition of subtraction and addition) on a TMSV state, in order to provide a practical comparison between them. In section 2 we will briefly review each operation. In section 3, we will define figures of merit accounting for the success probabilities with which each operation can be compared. We also calculate further figures of merit which include a measure of Gaussianity based on the quantum relative entropy (QRE) [25, 26] induced by the non-Gaussian operation, as well as fidelity to certain Gaussian and non-Gaussian states. In section 4 we will include an analysis of loss in each case. Finally, we will conclude in section 5.

### 2. Non-Gaussian operations

The starting point for all operations is the two-mode squeezed state

$$|\psi_{\text{TMSV}}\rangle = \sum_{n=0}^{\infty} c_n |n, n\rangle = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n |n, n\rangle,$$

which is characterized by the squeezing parameter $\lambda$. The normalized coefficients of the Schmidt decomposition, $c_n$, are the probability amplitudes of the state existing in the state $|n, n\rangle$. These $c_n$ may be modified by partial measurement of the state.

Our metric of the entanglement contained in such a state is the log-negativity $E_N$, the logarithm of the sum of the negative eigenvalues of the partially transposed density matrix [27]. For pure states written in the Schmidt form, i.e. $|\psi\rangle = \sum_{n=0}^{\infty} c_n |n, n\rangle$, it is given by

$$E_N = 2 \log_2 \left( \sum_{n=0}^{\infty} |c_n|^2 \right),$$

where $c_n$ are the normalized Schmidt coefficients. Thus a TMSV has entanglement $E_N = \log_2 \left( \frac{1 + \lambda}{1 - \lambda} \right)$, which is the benchmark from which all changes in entanglement resulting from the non-Gaussian operations will be measured. In the remainder of this section, we will review each operation and the form and properties of the resulting non-Gaussian states.

A schematic for all the operations is shown in figure 1, whereby each case is governed by different ancilla Fock states $|n\rangle$, $|n'\rangle$ and perfect photon-number projective measurements $|m\rangle \langle m|$, $|m'\rangle \langle m'|$.

The key parameter in each of these processes—one which is under direct control of the experimentalist—is the beam splitter transmissivity $T$, $T'$. The value of $T$, $T'$ not only dictates the resulting state $|\psi\rangle$ but also the probability of the desired projective measurement outcome.
2.1. Photon subtraction

Photon subtraction \([3, 4]\) is perhaps the most straightforward non-Gaussian conditional operation to carry out \([7, 28]\).

In symmetric photon subtraction, we consider the case where one photon is subtracted from each mode; that is, \(\ket{n} = \ket{n'} = \ket{0}\) and \(\ket{m}\langle m| = \ket{m'}\langle m'| = \ket{1}\langle 1|\) in figure 1. The resulting state is

\[
\psi = \sum_{\lambda, \lambda'} \rho_{\lambda, \lambda'} (1 - T^2) \lambda \lambda' (1 + \lambda^2 T^4) \ket{\lambda, \lambda'}.
\]

where \(T = T'\) is the reflectivity of the subtracting beam splitters and \(N_{\text{sub}}\) is a normalization constant. Writing \(N_{\text{sub}(2)}\) in this form relates this constant to the probability of subtracting a photon \(P_{\text{sub}(2)} = N_{\text{sub}(2)}^{-2}\), i.e.

\[
P_{\text{sub}(2)} = \frac{(1 - T^2) \lambda^2 (1 - \lambda^2)(1 + \lambda^2 T^4)^3}{(1 - \lambda^2 T^4)^3}.
\]

The log-negativity of this state, calculated following equation (2), is given by

\[
E_N(\rho_{\text{sub}(2)}) = \log_2 \left[ \frac{(1 + T^2 \lambda)^3}{(1 + T^2 \lambda^2)(1 - T^2 \lambda)} \right]
\]

For a particular range of \(T\) and \(\lambda\), the log-negativity is greater than that of the initial TMSV, as we will show in section 3. Increasing entanglement by photon subtraction has been experimentally demonstrated \([22, 23]\).

2.1.1. Single-mode photon subtraction

In asymmetric photon subtraction, we consider the case in which photons are subtracted from one mode only, and the other mode is left unimpeded. In this case, the ancilla states remain \(\ket{n} = \ket{n'} = \ket{0}\), as does one of the projective measurements, e.g. \(\ket{m}\langle m| = \ket{1}\langle 1|\) as in the two-mode case; but no subtraction occurs on the other mode, i.e. \(T' = 1\) and \(\ket{m'}\langle m'| = 0\).

The state that results following this operation is

\[
\psi_{\text{sub}(1)} = N_{\text{sub}(1)} \sum_{n=0}^{\infty} \sqrt{1 - \lambda^2} \lambda^{n+1} \lambda' \lambda' \sqrt{1 - T^2} \sqrt{1 - T^2} \sqrt{n + 1} \ket{n, n + 1},
\]

where \(T\) is the transmissivity of the beam splitter in subtracted mode. This event has a probability \(P_{\text{sub}(1)} = N_{\text{sub}(1)}^{-2}\) given by

\[
P_{\text{sub}(1)} = \frac{(1 - \lambda^2)(1 - T^2) \lambda^2}{(1 - T^2 \lambda^2)^2}.
\]

The state given in equation (5), although pure, is not of the form \(\rho = \sum_{n=0}^{\infty} C_n \ket{n, n}\), therefore we cannot use equation (2) to derive an analytic form for the log-negativity of the state. However, we have previously shown how an exact form for the log-negativity can be found \([14]\). In the results that follow, the log-negativity was calculated numerically.
2.2. Photon addition

Symmetric photon addition consists of single photons in the ancilla modes $|n\rangle = |n\rangle = |1\rangle$ and vacuum projection $|m\rangle = |m\rangle = |0\rangle = 0$ in the detected mode. This creates the state

$$\rho_{\text{add}} = N_{\text{add}}(2) \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^{n-1} n T^2 (n-1) \left( 1 - T^2 \right) |n, n\rangle,$$

where again $T = T'$ is the transmissivity of both beam splitters in figure 1 and $N_{\text{add}}^2 = P_{\text{add}}$ is the probability of photon addition, given explicitly by

$$P_{\text{add}} = \frac{(1 - T^2)^2 \left( 1 - \lambda^2 \right) \left( 1 + \lambda^2 T^4 \right)}{\left( 1 - \lambda^2 T^4 \right)^2}.$$  

For fixed transmissivity $T$, this is always greater than $P_{\text{sub}}$ by a factor of $\lambda^{-2}$.

The log-negativity of this state $E_N(\rho_{\text{add}})$ is identical to the subtracted case:

$$E_N(\rho_{\text{add}}) = \log \left[ \frac{\left( 1 + T^2 \lambda \right)^3}{\left( 1 + T^4 \lambda^2 \right) \left( 1 - T^2 \lambda \right)} \right].$$

Again, for a range of $T$, $\lambda$, entanglement may be increased from the initial value.

2.2.1. Single-mode photon addition

Similarly, asymmetric photon addition comprises a single ancilla photon added to just one mode, e.g. $|n\rangle = |1\rangle$, $|m\rangle = |0\rangle$ and $T = 1$. This operation creates the state

$$\rho_{\text{add}} = N_{\text{add}}(1) \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} \lambda^n \sqrt{1 - T^2} \sqrt{n + 1} \left| n + 1 , n \right\rangle,$$

with probability

$$P_{\text{add}} = \frac{(1 - \lambda^2) \left( 1 - T^2 \right)}{\left( 1 - \lambda^2 T^2 \right)^2}.$$  

Again, this probability is a factor of $\lambda^{-2}$ greater than single-mode photon subtraction.

Regarding the experimental feasibility of photon addition, it is immediately apparent that it is more involved than subtraction. In the model described above, one requires a reservoir of single photons to add to the correct mode, and vacuum detection to herald success. Despite considerable effort, on-demand single-photon sources—particularly sources of photons mode-matched to squeezed states—have not yet been demonstrated. Furthermore, vacuum projection onto the correct mode is experimentally challenging, and is also yet to be shown experimentally. This is largely because extremely high-efficiency detectors are required, although such detectors [29] are beginning to be used [30, 31]. To overcome both of these problems, experimental schemes typically use seeded parametric down-conversion to herald photons added to the mode of interest [32–35]. This approach requires a much more detailed model of the nonlinear interaction producing the added photons, which is beyond the scope of the present work.

2.3. Photon replacement

Photon replacement [18, 19] is a coherent superposition of the subtraction and addition operations, leaving the same total number of photons in the beam after the operation as before it. In a state engineering context, this operation has also been termed “quantum catalysis” [36, 37]. A photon-replaced state is generated by interfering a mode with a single-photon ancilla state, and projecting one output of the interference onto a single-photon state. In general, acting on a single mode, the photon replacement operation enacts the operation

$$|\psi\rangle = \sum_n c_n |n\rangle \rightarrow |\psi'\rangle = N' \sum_n c_n T^{n-1} \left[ T^2 - n \left( 1 - T^2 \right) \right] |n\rangle,$$

where $N'$ is a normalization constant depending on $T$.

In figure 1, single-mode replacement, $\text{rep}(1)$, corresponds to $n = m = 1$, $0 < T < 1$, while two-mode replacement, $\text{rep}(2)$, shown explicitly in figure 2, corresponds to $n = m = n' = m'$, $0 < T, T' < 1$.

Unlike photon subtraction and addition, this operation conserves photon number in the resultant mode, leading to some interesting features. For example, one replacement operation on each mode (two-mode photon...
replacement) for which the transmissivities of the beam splitters are equal, is entirely equivalent to two iterations of single photon replacement on the same mode, as can be seen from equation (12). The one- and two-mode photon-replaced states, \( \varsigma_{\text{rep}(1)} \) and \( \varsigma_{\text{rep}(2)} \) are given by

\[
\sum_{n=0}^{\infty} \psi_{\lambda} (\lambda_1, \lambda_2, n) |n, n\rangle = \sum_{n=0}^{\infty} e^{-\frac{1}{2} \lambda_1^2 T^2} e^{-\frac{1}{2} \lambda_2^2 T^2} |n, n\rangle,
\]

and the probability of the conditioning measurement in each case is

\[
P_{\text{rep}(1)} = \frac{(1 - \lambda^2) \left( \lambda^2 + T^2 \left[ 1 + (T^2 - 4) \lambda^2 + \lambda^4 \right] \right)}{(1 - T^2)^2},
\]

\[
P_{\text{rep}(2)} = \frac{(1 - \lambda) (1 + \lambda)}{(1 - T^2)^2} \left[ T^4 + \left( 1 - 8T^2 + 24T^4 - 32T^6 + 11T^8 \right) \lambda^2 
+ T^4 \left( 11 - 56T^2 + 96T^4 - 56T^6 + 11T^8 \right) \lambda^4 
+ T^8 \left( 11 - 32T^2 + 24T^4 - 8T^6 + T^8 \right) \lambda^6 + T^{12} \lambda^8 \right].
\]

Unlike the single-mode subtracted and added states, the log-negativity of the single-mode replaced state is analytically tractable, since it is of the form required for equation (2) (i.e. it remains a number-correlated state). For the single-mode replaced state \( |\psi_{\text{rep}(1)}\rangle \), given \( 0 < T < 1 \), the only positive \( c_n \) is \( c_0 \). All other \( c_{n>0} \) are negative for the range \( 0 < T < \sqrt{n}/n + 1 \). On the other hand, calculating the entanglement in the case where photons are replaced in both modes \( |\psi_{\text{rep}(2)}\rangle \) requires no such manipulation since the Schmidt coefficients are always positive. For each case, the entanglement is given by

\[
E_N \left( \hat{\rho}_{\text{rep}(1)} \right) = \log_2 \left[ \frac{(1 + T^2 \lambda^2) \left( \lambda + T \left[ 1 + 2T \lambda (T \lambda - 2) \right] \lambda \right)^2}{(1 - T^2) \left( \lambda^2 + T^2 \left[ 1 + (T^2 - 4) \lambda^2 + \lambda^4 \right] \right)^2} \right],
\]

\[
E_N \left( \hat{\rho}_{\text{rep}(2)} \right) = \log_2 \left[ \frac{1 + T^2 \lambda^2}{1 - T^2} \left( \lambda + T^2 \left[ 1 + \lambda \left( -4 + T^2 + \lambda \right) \right] \right)^2 \times \left[ \frac{1}{1 - T^2} \right] \left[ T^4 + \left( 1 - 8T^2 + 24T^4 - 32T^6 + 11T^8 \right) \lambda^2 
+ T^4 \left( 11 - 56T^2 + 96T^4 - 56T^6 + 11T^8 \right) \lambda^4 
+ T^8 \left( 11 - 32T^2 + 24T^4 - 8T^6 + T^8 \right) \lambda^6 + T^{12} \lambda^8 \right]^{-1} \right],
\]

whereby the expression for \( E_N \left( \hat{\rho}_{\text{rep}(1)} \right) \) is valid for \( 0 < T < 1/\sqrt{2} \).

\footnote{Two-mode replacement is not, however, equivalent to replacing two photons in a single operation, i.e. using a two-photon ancilla Fock state with two-photon projective measurement.}
3. Figures of merit

Equipped with the expressions for the entanglement and success probabilities of each of the entanglement-enhancing non-Gaussian operations, we are now in a position to compare and contrast them in an operational sense. Based on the considerations of a realistic entanglement-enhancement protocol above, any number of figures of merit can be constructed. We suggest several and show how each entanglement-enhancing non-Gaussian operation performs.

3.1. Total entanglement

Perhaps the most straightforward question to ask is how much entanglement exists after a successful measurement-induce non-Gaussian operation. This is shown in figure 3; as a function of beam splitter transmissivity $T$ and initial squeezing $\lambda$, the total entanglement resulting from single- and two-mode operations are shown in figures 3(a) and (b), respectively. It is apparent from figure 3(c)—a cross-section of figures 3(a) and (b) at $\lambda = 0.1$—that photon replacement, both from one and two modes, produces more entanglement than subtraction/addition at low initial squeezing, the most experimentally accessible region. For subtraction and addition, the entanglement dependence on these parameters are equal, and approach the maximum value as $T \to 1$ for all $\lambda$. The replacement schemes have maxima that depend on $\lambda$: at $\lambda = 0.1$ these occur at $T = 0.1$ and $T = 0.27$ for single- and double-photon replacement, respectively.

3.2. Success probability

As with all nondeterministic operations, success probability is an important quantity. In figure 4 we show the probability for each scheme as a function of the initial squeezing $\lambda$ and beam splitter transmissivity $T$.

It is clear that protocols that involve introducing an extra photon are more probable; all but photon subtraction have probabilities greater than $10^{-2}$ for some set of parameters. However, this requires a source of single-photon ancilla states that are readily accessible.

3.2.1. Probability of ancilla photon generation

While a range of technologies exist which generate single photons [39], true on-demand single-photon states are not yet available. Furthermore, for the schemes described above to be successful, photons from the TMSV state must be indistinguishable from the ancilla photons taking part in the operation (i.e. photons added or replaced). One approach is to use heralded single-photon generation from an identical source of TMSV states. In this case, the probability of generating a heralded single photon is $P_{\text{herald}} = (1 - \lambda^2)\lambda^2$. It is therefore reasonable to incorporate this probability in the success probability of each protocol when considering their practical application.

In the case of single photon replacement on one mode of a TMSV, an additional factor $P_{\text{herald}}$ is required. For double replacement, two photons are required, therefore the additional factor is $P_{\text{herald}}^2$. The same is true for photon addition on both modes.

It is interesting to note from figure 4 that the probability of single-mode addition, with the probability of ancilla photon generation included, is identical to the probability of single-mode photon subtraction. In turn, this probability is of similar magnitude to single-mode replacement, indicating some level of ‘conservation of difficulty’.

3.3. Combining entanglement and probability

To consider the effects of both final entanglement and success probability, we can use the beam splitter transmissivity $T$ to parametrically plot these quantities as a function of initial squeezing $\lambda$. This is shown in figure 5. Figures 5(a) and (b) show that subtraction and addition out-perform replacement as the initial squeezing increases. However, at low squeezing, the replacement operations are optimal. This is particularly clear from figure 5(c): for initial squeezing $\lambda = 0.1$, single-mode replacement produces the most entanglement at the highest probability. Note also that in this particular figure of merit, photon subtraction never outperforms photon addition.

3.3.1. Entanglement rate

Perhaps most interesting from figures 3 and 4 is that the replacement schemes produce the highest entanglement at reasonable probabilities. In subtraction and addition, the highest increase in entanglement occurs when $T = 1$, for which the probability is precisely zero. However, the optimal transmissivities for single and double replacement are, for $\lambda = 0.1$, $T = 0.1$ and $T = 0.27$ respectively, corresponding to success probabilities 0.02 and 0.01, respectively.
To account for this property, we can define a new figure of merit, the entanglement rate $E_r(\hat{\rho})$, by simply multiplying entanglement and probability together, that is

$$E_r(\hat{\rho}) = E_N(\hat{\rho})P(\hat{\rho}),$$

where $\hat{\rho}_{\text{init}}$, $\hat{\rho}$ are the initial and final states, respectively. This gives the average entanglement per trial of the protocol, and is shown for the four protocols in figure 6. Here again, it is clear that protocols that introduce a single photon give a higher entanglement rate.

### 3.3.2. Entanglement difference

In addition to the absolute level of entanglement obtained following a successful protocol, and the rate at which this is achieved, one may also be interested in the absolute increase in entanglement. We define this as the entanglement difference, $\Delta E_N(\hat{\rho})$, which is calculated.
3.3.3. Enhancement rate

From the expression for the entanglement difference, we can calculate the average increase per trial, which we define as the enhancement rate $\Delta E_r$, where

$$\Delta E_N \left( \hat{\rho} \right) = E_N \left( \hat{\rho} \right) - E_N \left( \hat{\rho}_{\text{init}} \right),$$

and shown in figure 7. Again, this plot confirms that at experimentally accessible values of initial squeezing $\lambda$, both single- and double-mode photon replacement yields the highest increase in entanglement.

$$\Delta E_r \left( \hat{\rho} \right) = P \left( \hat{\rho} \right) \Delta E_N \left( \hat{\rho} \right) = P \left( \hat{\rho} \right) \left( E_N \left( \hat{\rho} \right) - E_N \left( \hat{\rho}_{\text{init}} \right) \right).$$

Figure 4. Success probability of (a) single-mode non-Gaussian operations and (b) two-mode operations. (c) Probability as a function of beam splitter transmissivity $T$ at $\lambda = 0.1$. The plots show the probability of successful entanglement generation for various transmissivities, with different colors representing different operations. The figure confirms that at experimental values of initial squeezing $\lambda$, single- and double-mode photon replacement yields the highest increase in entanglement.
As it is a difference in log-negativities multiplied by a probability (necessarily dimensionless), this has units of ebits. This is shown in figure 8. Again, this confirms that single-mode photon replacement performs better than the other protocols.

By accounting for success probability, the enhancement rate is a clear operational measure of the efficacy of an entanglement-enhancing non-Gaussian operation. Figure 8 clearly shows that there exists an optimal transmissivity $T$—an experimentally tunable parameter—which maximizes this quantity. This is shown in figure 9(a), as a function of initial squeezing $\lambda$.
When comparing the different strategies, it is sensible to compare each at its optimal set of controllable parameters. Therefore, having extracted the optimal $T$ in order to maximize enhancement rate—i.e. the $T$ which produces the highest increase in entanglement most often—we can plot the resulting optimal enhancement rate as a function of initial squeezing. This is shown in figure 9(b). It is clear that different strategies are best in different regimes of squeezing: single-mode photon replacement is best for $\lambda \lesssim 0.375$, whereas single-mode photon addition is best for $\lambda \gtrsim 0.375$. Note that neither replacement operation increases entanglement for $\lambda \geq 0.5$.

This analysis presumes on-demand generation of the necessary ancilla photons for each operation. However, when we include the probability of generating these single photons, the situation changes, as shown in figure 9(c). Single-mode photon replacement is still optimal at initial squeezing below $\lambda \lesssim 0.35$, after which single-mode subtraction or addition becomes optimal.
3.4. Utility

In terms of log-negativity, all entangled states are equal, but some are more equal than others. That is to say, our states have more entanglement than the two-mode squeezed state with which we started (at least for some range of parameters), but are they any more useful? Utility in this sense depends entirely on the protocol to be employed. To that end, we investigate a number of figures of merit relating to tasks that require entanglement and calculate them for the four states. To obtain analytic results, we include only those non-Gaussian operations which generate photon-number correlated states, i.e. those states of the form $\psi = \sum_n c_n |n, n\rangle$ (single-mode subtraction and addition are omitted).

3.4.1. Gaussianity

One of the principal motivations behind CV entanglement distillation is to generate Gaussian states with higher entanglement than the initial two-mode squeezed states. To quantify the Gaussianity of our state, we use the QRE proposed in [25] (see also [26, 40]). This is defined as the difference in entropy between the state of interest $\rho$ and the closest Gaussian state $\rho'$—that is, a state fully described by the same mean and covariance matrix.

The QRE $\delta(\rho)$ of a general state $\rho$ is calculated by

$$\delta(\rho) = S(\rho') - S(\rho),$$

where $S(\rho) = \text{Tr}(\rho \log(\rho))$ is the von Neumann entropy of the state $\rho$. Note that for pure states, the von Neumann entropy is zero, therefore the QRE is given by $S(\rho')$. Using the method outlined in [26], we find that the QRE for two-mode number-correlated pure states of the form $\psi = \sum_n c_n |n, n\rangle$ is

![Figure 7. Entanglement difference following each non-Gaussian operation. (a) Difference as a function of interaction parameters beam splitter transmissivity $T$ and initial squeezing $\lambda$; (b) difference as a function of $T$ at $\lambda = 0.1$. Note that this 2D line-out corresponds directly to figure 3(c).](image-url)
\[
\delta(p) = \left(1 + \sqrt{x}\right) \log\left[\frac{1}{2} \left(1 + \sqrt{x}\right)\right] \\
\quad - \left(\sqrt{x} - 1\right) \log\left[\frac{1}{2} \left(\sqrt{x} - 1\right)\right],
\]

where \(x = (1 - 2\langle n\rangle)^2 = 4\langle a^+a\rangle^2\) and expressions for \(\langle n\rangle\) and \(\langle a^+a\rangle\) for each state are given in appendix A.1.

The QRE for each state is plotted in figure 10. This shows that subtraction returns the state closest to a Gaussian state, whereas the photon-added state is highly non-Gaussian, exhibiting significant QRE. The QRE produced by the photon replacement schemes depends strongly on the parameters.

3.4.2. Gaussian state overlap: TMSV
Related to the Gaussianity of the state is its fidelity to a two-mode squeezed state of equivalent entanglement. That is, given a state with entanglement $E_N$, what is its fidelity to a TMSV exhibiting the same log-negativity. We therefore define an effective squeezing parameter $\lambda_{\text{eff}}$, which is the squeezing parameter of the TMSV with the same entanglement as the non-Gaussian state. This is calculated by

$$\lambda_{\text{eff}} = \frac{E_N - 1}{E_N + 1},$$

Equation (21)
such that the fidelity between the non-Gaussian state $|\psi_{NG}\rangle$ and the effective TMSV $|\psi_{TMSV}\rangle$ is readily calculated

$$\left|\left\langle \psi_{TMSV} | \psi_{NG} \right\rangle \right|^2 = \left( \sqrt{1 - \lambda_{\text{eff}}^2} \sum_n \lambda_{\text{eff}} c_n \right)^2,$$

(22)

where $c_n$ are the normalized Schmidt coefficients of the non-Gaussian state $|\psi_{NG}\rangle$. This is shown in figure 11.

The subtracted state is very close to a TMSV, while the added state exhibits almost no overlap. Both photon-replaced states vary smoothly between low and high overlap in the parameter region corresponding to increasing entanglement from the initial two-mode squeezed state.

A related measure is the 'squeezed vacuum affinity,' introduced in [9]. This metric maximizes the overlap between a non-Gaussian state and a TMSV state, with the vacuum squeezing as a free parameter. The two measures are not equivalent: the entanglement of a non-Gaussian state may well differ from the entanglement present in the closest Gaussian state (i.e. the Gaussian state the minimizes the overlap).

3.4.3. Non-Gaussian state overlap: Bell state

Another relevant set of important non-Gaussian entangled states is the set of Bell states. Number-correlated states of the kind in which we are interested can approximate the Bell state

$$|\psi_{\text{Bell}}\rangle = 1/\sqrt{2} \left( |0, 0\rangle + e^{i\phi} |1, 1\rangle \right),$$

(23)

where $\phi$ is a free parameter. The accuracy of the approximation is given by their overlap $\langle \psi_{\text{Bell}} | \psi_{NG} \rangle$.

In this case, as shown in figure 12, neither the subtracted nor added states exhibit significant overlap with the Bell state ($<0.7$ for $\lambda = 0.1$). By contrast, the double mode replaced state has $99.95\%$ overlap with a Bell state of phase $\phi = 0$ at $T = 0.27$, $\lambda = 0.1$, while the single-photon replaced state exhibits $99.98\%$ overlap with the bell state of phase $\phi = \pi$ for $T = 0.098$ when $\lambda = 0.1$. However, the subtracted state is very close to a TMSV, while the added state exhibits almost no overlap.
Interestingly, these are exactly the parameters at which entanglement is maximized, shown in figure 3(c).

3.5. Teleportation

A typical motivation behind distributing entanglement is to facilitate a quantum teleportation protocol [24, 41–45]. Teleportation is a quantum communication task which comprises mapping quantum states from one mode to another, without the modes themselves interacting. The fidelity with which this is achieved is a metric for success of the protocol. Indeed, early theoretical work used the teleportation fidelity as the principal figure of merit for entanglement-enhancement strategies [3–5].

Teleportation fidelities following the non-Gaussian operations described have been compared (independently) by Lee et al [19]. Here, we expand on their results by probing the range of parameters that enhance teleportation for each operation.

3.5.1. Teleportation fidelity

The average teleportation fidelity $F$ can be calculated by [46]

$$F = \int d^2\beta |\langle \psi | \hat{T}(\beta) | \psi \rangle|^2,$$

where $\hat{T}(\beta)$ is known as the transfer operator, $|\psi\rangle$ is the state to be teleported and the integral is over all phase space. For ancillary entangled states of the form $\sum_{n=-\infty}^{\infty} c_n |n\rangle |n\rangle$, Cochrane et al [4] showed that the transfer operator may be written

$$\hat{T}(\beta) = \frac{1}{\sqrt{\mathcal{P}}} \sum_{n=0}^{\infty} c_n \hat{D}(g\beta) |n\rangle |n\rangle \hat{D}(-g\beta),$$

where $\hat{D}(g\beta)$ is a displacement $\beta$ in phase space given a gain $g$ of the teleporter. For the purposes of this paper, we set $g = 1$ since we do not consider gain.
When teleporting a coherent state, $|\psi\rangle = |\alpha\rangle$, the average fidelity is independent of $\alpha$. In this case, the expression for average teleportation fidelity $F$ is

$$F = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dp \left[ \sum_{n=0}^{\infty} \frac{(x^2 + p^2)^n}{n!} e^{-x^2 - p^2} \right]^2. \quad (26)$$

For a two-mode squeezed state, the teleportation fidelity is given by

$$F_{TMSV} = \frac{1 + \lambda}{2}. \quad (27)$$

Following each of the non-Gaussian operations, the teleportation fidelities of the resulting states are given by

$$F_{\text{sub}(2)} = \frac{(1 + T^2\lambda)^3}{4(1 + T^4\lambda^2)}, \quad (28a)$$

$$F_{\text{add}(2)} = \frac{(1 + T^2\lambda)^3}{4(1 + T^4\lambda^2)}, \quad (28b)$$

$$F_{\text{rep}(1)} = \frac{(1 + T^2\lambda)^3 \left\{ 2T^2 - 2T(1 + T^2)\lambda + 2T^4 + 2T^2 \left( 1 - T^2 \right) + \left( 1 - T^2 \right)^2 \lambda^2 \right\}}{4 \left\{ \lambda^2 + T^2 \left[ 1 + \left( T^2 - 4 \right)\lambda^2 + \lambda^4 \right] \right\}}. \quad (28c)$$

Note that $F_{\text{rep}(2)}$ is too lengthy to display adequately here; its expression is given in appendix B.
We plot these fidelities in figure 13. From these plots it is apparent that not all entanglement-enhancing non-Gaussian operations increase the teleportation fidelity. Indeed, there is no region for which photon addition or single-mode photon replacement are better than the original TMSV state in terms of teleportation fidelity, even though they exhibit more entanglement. Therefore, if the purpose of the protocol is to increase teleportation fidelity, then two-mode subtraction or two-mode photon replacement is the only option.

3.5.2. Quantum bounds

Ralph and Lam [43] showed that for teleportation to be unambiguously quantum, the average fidelity must exceed $\bar{F} = 2/3$ (see also Grosshans and Grangier [47]). In figure 14, we show where a TMSV, subtracted state and double-mode photon-replaced state exceed this bound as a function of initial squeezing $\lambda$. Note that for the subtracted state we have assumed $T = 1$ (i.e. an operation producing the highest entanglement but with vanishing success probability), and for the photon-replaced state, $T$ is optimized to maximize the fidelity for each $\lambda$.

The different states reach the Ralph–Lam bound for different $\lambda$: double-mode photon replacement at $\lambda = 0.12$ (and $T = 0.33$), photon subtraction at $\lambda = 0.18$ and the original TMSV at $\lambda = 1/3$. Therefore, replacement appears to be a good strategy for teleportation at low $\lambda$ (it is also far more probable in this parameter range, as per figure 4).

3.5.3. Entanglement utility

The final property we will consider is the efficiency with which entanglement in a state is used in a teleportation protocol. That is to say, given a fixed amount of entanglement, which state produces the highest teleportation fidelity? This is shown in figure 15. Both subtraction and double replacement produce states that use entanglement similarly efficiently as a TMSV. However, given the initial squeezing required to generate the final
entanglement, in each case, the two-mode replacement operation appears to use the initial entanglement more efficiently.

4. Loss

We now consider the case where losses in the channel prior to the non-Gaussian operation play a role. The schematic in this case is shown in figure 16, whereby additional beam splitters of transmissivities $\eta$, $\eta'$ introduce loss modes. In the case of photon subtraction, losses at various stages of the protocol can be modelled as a single channel efficiency parameter [14]. While this is not generally true for all non-Gaussian operations, we consider photon losses prior to any non-Gaussian operation as the principle source of loss in a realistic scenario.

While allowing different channel efficiencies on each mode is more general (and studied for photon subtraction [14]), we limit ourselves to the case of symmetric loss in each channel, i.e. that $\eta = \eta'$. Furthermore, we do not consider losses at any other point in the protocol, for example in the heralding of the ancilla states required in the addition and replacement operations.

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2 This is a reasonable approach since the advent of high-efficiency number resolving detectors [29].

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Figure 14. Teleportation fidelity as a function of $\lambda$ following different non-Gaussian operations: two-mode subtraction (purple) and two-mode replacement (orange). The fidelity of the original TMSV is shown in dashed grey. Fidelities greater than $2/3$ (blue shaded region) require state reproduction that is unambiguously quantum [43], while fidelities above $1/2$ require entanglement during the teleportation protocol [44].

Figure 15. Teleportation fidelity as a function of entanglement following different non-Gaussian operations: single-mode replacement (red); two-mode subtraction (purple) and addition (gold), two-mode replacement (orange).
Figure 16. Model of non-Gaussian operations incorporating channel losses $\eta$, $\eta'$ in each mode.

Figure 17. (a) Optimal beam splitter transmissivity $T$, given initial squeezing $\lambda$ and channel efficiency $\eta$. (b) Resulting enhancement rate given optimal $T$ for a given initial squeezing $\lambda$ and channel efficiency $\eta$ for each non-Gaussian operation: single-mode photon replacement (red), single-mode photon subtraction (blue) and single-mode photon addition (green), given on-demand generation of ancilla photons; (c) Resulting optimal enhancement rate incorporating ancilla photon generation probabilities. Note that the two-mode strategies all perform worse than their single-mode counterparts, the results of which are thus hidden.
4.1. Entanglement behaviour under loss

In section 3.3.3 we identified enhancement rate $\Delta E_r \left( \bar{\rho} \right)$ as a useful figure of merit to compare probabilistic entanglement enhancing operations. In the absence of loss, enhancement rate depends on the initial squeezing $\lambda$ and beam splitter transmissivity $T$, and we found the optimal $T$ which maximized $\Delta E_r \left( \bar{\rho} \right)$ for a given $\lambda$.

When accounting for loss, we can again find the optimal transmissivity which maximizes $\Delta E_r \left( \bar{\rho} \right)$ for a given initial squeezing $\lambda$ and channel efficiency $\eta$. This, along with the enhancement rate it generates, is shown in figure 17.

It is clear from figure 17 that losses are highly detrimental to the entanglement following the non-Gaussian operations. For a large set of parameters (channel efficiency $\eta$ and initial squeezing $\lambda$), it is clear that single-mode replacement is the optimal strategy to increase entanglement, yielding the highest gain in entanglement per trial. However, as shown in figure 15, this strategy cannot increase the teleportation fidelity. A more thorough analysis would investigate the effect of loss on the particular figure of merit of interest—sufficient to say loss is expected to be detrimental in all cases.

5. Conclusion

We have considered three classes of non-Gaussian entanglement enhancing operations: photon subtraction, photon addition and photon replacement. We considered these operations on one or both modes of a two-mode squeezed state, and in the presence of equal channel losses. All of these operations are non-deterministic, and we derived a number of figures of merit to incorporate both the success probabilities of each protocol and any increase in entanglement. In all cases, one key parameter over which the experimentalist has control is the beam splitter transmissivity, which we optimized in all cases to produce the highest increase in entanglement most often. It turns out that for a wide variety of channel parameters and initial squeezing, single-mode photon replacement is the best strategy, even when the probability of generating the required ancilla photon is included in the analysis. More generally, our analysis highlights the importance of probability in non-deterministic operations.

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Appendix A. Calculating two-mode quadrature expectation values and variances

The two-mode superposition quadratures are

$$X = \frac{1}{2^2} \left( \hat{a}_1^\dagger + \hat{a}_1 + \hat{a}_2^\dagger + \hat{a}_2 \right), \quad (A1a)$$

$$P = \frac{i}{2^2} \left( \hat{a}_1^\dagger - \hat{a}_1 + \hat{a}_2^\dagger - \hat{a}_2 \right). \quad (A1b)$$

The variance in the two-mode $X$ quadrature is

$$\left\langle \left( \Delta X \right)^2 \right\rangle = \left\langle X^2 \right\rangle, \quad (A2)$$

since $\left\langle \psi \left| \hat{a} \right| \psi \right\rangle = \left\langle \psi \left| a^\dagger \right| \psi \right\rangle = 0$. The expansion of $\left\langle X^2 \right\rangle$ contains 16 terms For a number correlated state of the form

$$\left| \psi \right\rangle = \sum_{n=0}^\infty c_n \left| n, n \right\rangle, \quad (A3)$$

the non-zero terms in this expansion are of the form $\left\langle a_i^\dagger a_j^\dagger \right\rangle_{i \neq j}$, $\left\langle a_i^\dagger a_i \right\rangle \delta_{ij}$, $i, j \in 1, 2$, and their complex conjugates. This yields eight terms:
\[
\langle \hat{\chi}^2 \rangle = \frac{1}{8} \left( \langle a_1^\dagger \hat{a}_1 \rangle + \langle a_2^\dagger a_2 \rangle + \langle \hat{a}_1 a_1^\dagger \rangle + \langle \hat{a}_2 a_2^\dagger \rangle \right) \\
+ \langle a_2^\dagger a_1^\dagger \rangle + \langle a_2^\dagger a_2 \rangle + \langle \hat{a}_2 \hat{a}_1^\dagger \rangle + a_2 a_2^\dagger \right) \\
= \frac{1}{8} \left( 4 \langle n \rangle + 2 \left( \langle a_1^\dagger a_2^\dagger \rangle + \langle \hat{a}_1 \hat{a}_2^\dagger \rangle \right) + 2 \right),
\]

where we have used \( \langle a_1^\dagger a_1 \rangle = \langle a_2^\dagger a_2 \rangle = \langle n \rangle \) and the commutator \([\hat{a}_i, a_j^\dagger] = \delta_{ij}\). These quantities are straightforward to calculate:

\[
\langle n \rangle = \sum_{n=0}^{\infty} |c_n|^2 n, \tag{A5a}
\]

\[
\langle a_1^\dagger a_2^\dagger \rangle = \sum_{n=0}^{\infty} c_n c_{n+1} (n + 1), \tag{A5b}
\]

\[
\langle \hat{a}_1 \hat{a}_2 \rangle = \sum_{n=0}^{\infty} c_n^* c_{n+1} (n + 1). \tag{A5c}
\]

If the coefficients \( c_n \) are real, then \( \langle a_1^\dagger a_2^\dagger \rangle = \langle \hat{a}_1 \hat{a}_2 \rangle \), such that

\[
\langle \hat{\chi}^2 \rangle = \frac{1}{4} \left( 2 \langle n \rangle + \langle \hat{a}_1 \hat{a}_2 \rangle \right) + 1 \tag{A6a}
\]

\[
= \frac{1}{4} \left( 2 \left( \sum_{n=0}^{\infty} n c_n^2 + \sum_{n=0}^{\infty} c_n c_{n+1} (n + 1) \right) + 1 \right). \tag{A6b}
\]

Similarly for \( \langle (\Delta \hat{\chi})^2 \rangle = \langle \hat{\chi}^2 \rangle \):

\[
\langle \hat{\chi}^2 \rangle = \frac{1}{4} \left( 2 \langle n \rangle - \langle \hat{a}_1 \hat{a}_2 \rangle \right) + 1 \tag{A7a}
\]

\[
= \frac{1}{4} \left( 2 \left( \sum_{n=0}^{\infty} n c_n^2 - \sum_{n=0}^{\infty} c_n c_{n+1} (n + 1) \right) + 1 \right). \tag{A7b}
\]

A.1. Properties of common states

Using the relations above, we calculate the mean photon number \( \bar{n} = \langle n \rangle \), two-mode ladder operator expectations values \( \langle a_1^\dagger a_2^\dagger \rangle = \langle \hat{a}_1 \hat{a}_2 \rangle \), and two-mode quadrature variances \( \langle \Delta \hat{\chi} \rangle, \langle \Delta \hat{P} \rangle \) for the states generated by two-mode photon subtraction, addition and replacement, and also single-mode photon replacement, since these operations preserve the number correlations of the state.

A.1.1. Two-mode squeezed states \( |\psi_{\text{TMSV}} \rangle \)

\[
\langle n \rangle = \frac{\lambda^2}{1 - \lambda^2}, \tag{A8a}
\]

\[
\langle a_1^\dagger a_2^\dagger \rangle = \langle \hat{a}_1 \hat{a}_2 \rangle = \frac{\lambda}{1 - \lambda^2}, \tag{A8b}
\]

\[
\langle (\Delta \hat{\chi})^2 \rangle = \frac{1}{4} \left( 1 + \frac{1}{1 - \lambda^2} \right), \tag{A8c}
\]

\[
\langle (\Delta \hat{P})^2 \rangle = \frac{1}{4} \left( 1 - \frac{1}{1 + \lambda^2} \right). \tag{A8d}
\]

The TMSV is clearly a minimum uncertainty state since \( \sqrt{\langle \hat{\chi}^2 \rangle \langle \hat{\chi}^2 \rangle} = 1/4 \).

A.1.2. Photon subtracted state \( |\psi_{\text{sub}} \rangle \)

\[
\langle n \rangle = \frac{2 T^4 \lambda^2 \left( 2 + T^4 \lambda^2 \right)}{1 - T^4 \lambda^2}, \tag{A9a}
\]
\[ \langle a_1^+ a_2^+ \rangle = \frac{2\lambda}{1 - T^2}, \quad \text{(A9b)} \]

\[ \left\langle \Delta \hat{\lambda} \right\rangle^2 = \frac{1}{4} \left( 1 + 3T^2 \lambda + 5T^2 \lambda^2 + 3T^2 \lambda^3 \right), \quad \text{(A9c)} \]

\[ \left\langle \Delta \hat{\rho} \right\rangle^2 = \frac{1}{4} \left( 1 - 3T^2 \lambda - 5T^2 \lambda^2 - 3T^2 \lambda^3 \right), \quad \text{(A9d)} \]

\[ \Delta \text{Var} = \frac{T^2 \lambda^2}{\sqrt{2} (1 + T^2 \lambda^2)}. \quad \text{(A9e)} \]

A.1.3. Photon added state \( |\psi_{\text{add}}\rangle \).

\[ \langle n \rangle = 1 + 4T^4 \lambda^2 + T^2 \lambda^4, \quad \text{(A10a)} \]

\[ \langle a_1^+ a_2^+ \rangle = \frac{2T^2 \lambda}{1 - T^2 \lambda^4}; \quad \text{(A10b)} \]

\[ \left\langle \Delta \hat{\lambda} \right\rangle^2 = \frac{1}{4} \left( 3 + 5T^2 \lambda + 3T^2 \lambda^2 + T^2 \lambda^3 \right), \quad \text{(A10c)} \]

\[ \left\langle \Delta \hat{\rho} \right\rangle^2 = \frac{1}{4} \left( 3 - 5T^2 \lambda + 3T^2 \lambda^2 - T^2 \lambda^3 \right), \quad \text{(A10d)} \]

\[ \Delta \text{Var} = \frac{1}{\sqrt{2} (1 + T^2 \lambda^2)}. \quad \text{(A10e)} \]

A.1.4. Single-mode photon replaced state \( |\psi_{\text{rep}(1)}\rangle \).

\[ \langle n \rangle = \frac{\lambda^2 \left( 1 + T^2 \left( 2 - 4 \lambda^2 + 4 \right) + T^2 \left( 4 + 2 \left( 4 - 4 \lambda^2 + \lambda^2 \right) \right) \right)}{(1 - T^2 \lambda^4) \left( 1 - 4 \lambda^2 + \lambda^4 \right)}, \quad \text{(A11a)} \]

\[ \langle a_1^+ a_2^+ \rangle = \frac{T^2 \lambda^2 + T^2 \lambda^2 (4 - 4 \lambda^2 + 2T \left( 1 - 5 \lambda^2 + \lambda^4 \right) - 1)}{(1 - T^2 \lambda^4) \left( 1 + \left( T^2 - 4 \right) \lambda^2 + \lambda^4 \right)}, \quad \text{(A11b)} \]

\[ \left\langle \Delta \hat{\lambda} \right\rangle^2 = \frac{1}{4} \left( \frac{1 + T \lambda}{1 - T \lambda} \left( \frac{T^2 - 2T \left( 1 - T \lambda \right) \lambda + (3 - 8T^2 + 3T^4) \lambda^2 + 2T \left( 1 - T \lambda \right) \lambda^3 + T^2 \lambda^4 \right) \right), \quad \text{(A11c)} \]

\[ \left\langle \Delta \hat{\rho} \right\rangle^2 = \frac{1}{4} \left( \frac{(1 + T \lambda)}{(1 - T \lambda)} \left( \frac{T^2 + 2T \left( 1 - T \lambda \right) \lambda + (3 - 8T^2 + 3T^4) \lambda^2 - 2T \left( 1 - T \lambda \right) \lambda^3 + T^2 \lambda^4 \right) \right), \quad \text{(A11d)} \]

\[ \Delta \text{Var} = \frac{\lambda^2}{\sqrt{2} \left( 1 + \left( T^2 - 4 \right) \lambda^2 + \lambda^4 \right)}. \quad \text{(A11e)} \]

A.1.5. Double-mode photon replaced state \( |\psi_{\text{rep}(2)}\rangle \). Although analytically tractable, the expressions are too lengthy to reproduce here.

Appendix B. Teleportation fidelity of a two-mode replaced state

The expression for the teleportation fidelity of a two-mode replaced state, after teleporting the coherent state \( |\alpha \rangle \), is:
$\mathcal{F}_{\text{rep}(2)} = \left(1 + T^2 \lambda \right)^5 \left\{ \left(1 - T^2\right)^4 \lambda^2 + 2T^{12} \lambda^4 \right. \\
+ 4T^{10} \lambda \left[ \left(1 - T^2\right) \lambda - 2 \right] \\
+ T^8 \left[ 2 - 4 \left(1 - T^2\right) \lambda + 2 \left(1 - T^2\right)^3 \lambda^3 \right] \\
+ \left(1 - T^2\right)^4 \lambda^2 \left[ 2 + \left(1 - T^2\right) \lambda \left( \{1 - T^2\} \lambda - 4 \right) \right] \\
+ 4T^6 \lambda \left[ 3 + \left(1 - T^2\right) \lambda \left( \{1 - T^2\} \lambda - 3 \right) \right] \\
+ 2T^6 \lambda \left[ \left(1 - T^2\right) \lambda - 1 \right] \\
\times \left[ 4 + \left(1 - T^2\right) \lambda \left( \{1 - T^2\} \lambda - 2 \right) \right] \\
\times \left\{ 4 \left[ T^4 + \left(1 - 8T^2 + 24T^4 - 32T^6 + 11T^8\right) \lambda^2 \right] \\
+ T^4 \left(1 - 56T^2 + 96T^4 - 56T^6 + 11T^8\right) \lambda^3 \right. \\
+ T^8 \left(1 - 32T^2 + 24T^4 - 8T^6 + T^8\right) \lambda^6 \right. \\
+ \left. \left. T^4 \lambda^2 \right] \right\}^{-1} \\
\text{(B1a)}$

**Appendix C. Lossy states**

Analytic forms for the states which undergo loss can be written as follows:

\[ \rho_{\text{TMSV}} = \left(1 - \lambda^2\right)^n \sum_{n=0}^{\infty} \sum_{l_f=0}^{\min(n,m)} \frac{n! \eta! \left(\lambda \eta\right)^{n+m} \left(1 - \eta^2\right)^{l_f}}{\left(n!\sqrt{(n-l)!} (n-l)! \sqrt{n! (n-l)!} \sqrt{(n-l)!} \sqrt{(n-l)!} \right)} |1 - l, n - l, n - l, n - l|, \]

\[ \rho_{\text{sub}(1)} = N_{\text{sub}(1)} \left(1 - \lambda^2\right)^n \sum_{n=0}^{\infty} \sum_{l_f=0}^{\min(n,m)} \frac{n! \eta! \left(\lambda \eta\right)^{n+m} \lambda^2 \lambda \eta^2 \left(1 - \eta^2\right)^{l_f} T^{-2l} \left(1 - T^2\right)}{\left(n!\sqrt{(n-l)!} (n-l)! \sqrt{n! (n-l)!} \sqrt{(n-l)!} \sqrt{(n-l)!} \right)} |1 - l, n - l, n - l, n - l | \\
\times \left[ n - l, n + 1 - l | \right] \\
\rho_{\text{sub}(2)} = N_{\text{sub}(2)} \left(1 - \lambda^2\right)^n \sum_{n=0}^{\infty} \sum_{l_f=0}^{\min(n,m)} \frac{n! \eta! \left(\lambda \eta\right)^{n+m} \left(\lambda \eta \left(1 - T^2\right)\right)^{l_f} \left(1 - T^2\right)^{n+m}}{\left(n!\sqrt{(n-l)!} (n-l)! \sqrt{n! (n-l)!} \sqrt{(n-l)!} \sqrt{(n-l)!} \right)} |1 - l, n - l, n - l, n - l | \\
\times \left[ n - l, n + 1 - l | \right] \\
\rho_{\text{add}(1)} = N_{\text{add}(1)} \left(1 - \lambda^2\right)^n \sum_{n=0}^{\infty} \sum_{l_f=0}^{\min(n,m)} \frac{n! \eta! \left(\lambda \eta\right)^{n+m} \left(1 - T^2\right)^{l_f} \left(1 - T^2\right)}{\left(n!\sqrt{(n-l)!} (n-l)! \sqrt{n! (n-l)!} \sqrt{(n-l)!} \sqrt{(n-l)!} \right)^2} |1 - l, n - l, n - l, n - l | \\
\times \left[ n - l, n + 1 - l | \right] \\
\text{(C1a)
\[ \rho_{\text{add}(2)} = \mathcal{N}_{\text{add}(2)} \left( 1 - \lambda^2 \right) \sum_{n, \eta = 0}^{\infty} \sum_{l, l' = 0}^{\min(n, \eta, l, l')} \frac{n! \eta! \left( \lambda \eta T^2 \right)^{n+l} \left( 1 - \eta \right)^{l+l'} \left( 1 - T^2 \right)^2 \sqrt{(n-l+1)(n-l'+1)(\bar{n}-l+1)(\bar{n}-l'+1)}}{l! \eta! \sqrt{(n-l)!(n-l')!(\bar{n}-l)!(\bar{n}-l')!}} \times [n-l+1, n-l'+1](\bar{n}-l+1, \bar{n}-l'+1), \]  

\[ \rho_{\text{rep}(1)} = \mathcal{N}_{\text{rep}(1)} \left( 1 - \lambda^2 \right) \sum_{n, \eta = 0}^{\infty} \sum_{l, l' = 0}^{\min(n, \eta, l, l')} \frac{n! \eta! \left( \lambda \eta T^2 \right)^{n+l} \left( 1 - \eta \right)^{l+l'} \left( 1 - T^2 \right)^{2(l+1)}}{l! \eta! \sqrt{(n-l)!(n-l')!(\bar{n}-l)!(\bar{n}-l')!}} \times \left[ T^2 - (n-l)(1-T^2) \left( \frac{T^2 - (n-l)(1-T^2)}{T^2 - (n-l)(1-T^2)} \right) \right] \times [n-l, n-l'](\bar{n}-l, \bar{n}-l'), \]  

\[ \rho_{\text{rep}(2)} = \mathcal{N}_{\text{rep}(2)} \left( 1 - \lambda^2 \right) \sum_{n, \eta = 0}^{\infty} \sum_{l, l' = 0}^{\min(n, \eta, l, l')} \frac{n! \eta! \left( \lambda \eta T^2 \right)^{n+l} \left( 1 - \eta \right)^{l+l'} \left( 1 - T^2 \right)^4}{l! \eta! \sqrt{(n-l)!(n-l')!(\bar{n}-l)!(\bar{n}-l')!}} \times \left[ T^2 - (n-l)(1-T^2) \left( \frac{T^2 - (n-l)(1-T^2)}{T^2 - (n-l)(1-T^2)} \right) \right] \times [n-l, n-l'](\bar{n}-l, \bar{n}-l'), \]  

where the normalization constants \( \mathcal{N} \) are related to the protocol success probability by \( \mathcal{N} = P^{-1} \). These may be written explicitly in each case:

\[ P_{\text{sub}(1)} = \left( 1 - \lambda^2 \right) \left( 1 - T^2 \right)^3 \eta \lambda^2 \left( 1 - (1 - (1 - T^2) \eta) \lambda^2 \right)^2, \]  

\[ P_{\text{sub}(2)} = \left( 1 - \lambda^2 \right) \left( 1 - T^2 \right)^3 \eta^2 \lambda^2 \left( 1 + (1 - (1 - T^2) \eta) \lambda^2 \right) \left( 1 - (1 - (1 - T^2) \eta) \lambda^2 \right)^2, \]  

\[ P_{\text{add}(1)} = \left( 1 - \lambda^2 \right) \left( 1 - T^2 \right)^2 \left( 1 - (1 - \eta) \lambda^2 \right), \]  

\[ P_{\text{add}(2)} = \left( 1 - \lambda^2 \right) \left( 1 - T^2 \right)^2 \left( 1 - (1 - \eta) \lambda^2 \right) \left( 1 - 2 \eta \left( 4 - 2T^2(1-\eta) - 2\eta + T^2 \eta \right) \lambda^2 \right) + (1 - \eta) \lambda^2 \left( 1 - \eta \left( 1 - T^2 \right)^2 \lambda^2 \right) \left( 1 - (1 - (1 - T^2) \eta) \lambda^2 \right)^2, \]  

\[ P_{\text{rep}(1)} = \left( 1 - \lambda^2 \right) \left( T^2 + (\eta + T^2 \eta - 2T^2(1+\eta)) \lambda^2 + (T^2 - (1 - T^2) \eta + (1 - T^2) \eta) \lambda^4 \right) \left( 1 - (1 - (1 - T^2) \eta) \lambda^2 \right)^4, \]  

\[ P_{\text{rep}(2)} = \left( 1 - \lambda^2 \right) \left( T^2 + (\eta + T^2 \eta - 2T^2(1+\eta)) \lambda^2 + (T^2 - (1 - T^2) \eta + (1 - T^2) \eta) \lambda^4 \right) \left( 1 - (1 - (1 - T^2) \eta) \lambda^2 \right)^4, \]
To calculate the entanglement and associated figures of merit in each case, it is beneficial to write each state in block-diagonal form [11]. By making the substitutions $\lambda \rightarrow \lambda + i$, $\eta \rightarrow \eta + i$, $\lambda \eta \rightarrow \lambda - \eta$, $\lambda - \eta \rightarrow \lambda + \eta$, and $\lambda + \eta \rightarrow \lambda - \eta + i$, the state may be written

\[
\rho = \mathcal{N} \sum_{K=0}^{\infty} \sum_{i=0}^{K} C^{(K)}_{ij} |i\rangle \langle j| \otimes |K - i\rangle \langle K - j|,
\]

where $C^{(K)}_{ij}$ are the elements of the $K$th block matrix of dimension $K + 1$. For each state in equation (C1), these transformations yield

\[
C^{(K)}_{ij}_{\text{init}} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K (i + l)! (j + l)!,
\]

\[
C^{(K)}_{ij}_{\text{sub}(1)} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K (i+j)! (1 - T^2)^2 (i + l + 1)! (j + l + 1)!,
\]

\[
C^{(K)}_{ij}_{\text{sub}(2)} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K (i+j)! (1 - T^2)^2 (i + l + 1)! (j + l + 1)!,
\]

\[
C^{(K)}_{ij}_{\text{add}(1)} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K T^{i+j+2l} (1 - T^2)^2 (i + l)! (j + l)! (K + 1)! ((K - i)! (K - j)!)!
\]

\[
C^{(K)}_{ij}_{\text{add}(2)} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K T^{i+j+2l} (1 - T^2)^2 (i + l + 1)! (j + l)! (K + 1)! ((K - i)! (K - j)!)!
\]

\[
C^{(K)}_{ij}_{\text{rep}(1)} = \sum_{l=0}^{\infty} \frac{\lambda (1 - \eta)^{i+j+2l}}{1 - \eta} \left( \frac{\eta}{1 - \eta} \right)^K T^{i+j+2l} (1 - T^2)^2 (i + l)! (j + l)! (K + 1)! ((K - i)! (K - j)!)!
\]
where, in all cases, \( l_0 = \max(0, K - 1 - j) \).

Following Zhang and van Loock [11], if the density matrix of the state is both symmetric and centro-symmetric in block diagonal form, the entanglement may be calculated by

\[
E_N(\rho) = \log_2 \left\{ 1 - \lambda^2 \sum_{K=0}^{\infty} \frac{\rho}{K} \right\},
\]

where \( f^{(K)} \) is the anti-identity matrix of dimension \( K \). Using the proof in [11], we find that while the states following two-mode operations are indeed persymmetric, the states following single-mode operations are not, therefore an analytic form for the entanglement cannot be found with this method. In this case we calculate the relevant figures of merit numerically.

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