Four loop anomalous dimension of the second moment of the non-singlet twist-2 operator in QCD

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Abstract

We present the result of a calculation for the first even moment of the non-singlet four-loop anomalous dimension of Wilson twist-2 operators in QCD with full color and flavor structures.
Calculation of anomalous dimensions of the Wilson twist-2 operators is one of the part of operator product expansion for the structure functions in the framework of perturbative Quantum Chromodynamics (QCD). At the present time such calculations are performed up to three-loop order \[1, 2, 3, 4, 5, 6, 7\], while other part of operator product expansion, the coefficient functions, which are known in the same order \[8, 9, 10, 11\], demand the four-loop anomalous dimensions.

In this paper we present the result of calculations for the first even moment of the non-singlet anomalous dimension of Wilson twist-2 operators at fourth order in perturbative QCD. Similar result can be found in Ref. \[12\], but our result contains full color and flavor structures and the calculations are performed with a different method\[1\].

The moments of the structure functions \(F_k\) are expressed through the parameters of the following operator product expansion of the \(T\)-product of electromagnetic currents:

\[
T_{\mu\nu} = i \int d^4 z \, e^{iqz} T \{ J_\mu(z) J_\nu(0) \} = \sum \left[ \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) q_{\mu_1} q_{\mu_2} C_{L,N}^{a,1}(Q^2/\mu^2, a_s) \right. \\
- \left. \left( g_{\mu_1 \nu_1} g_{\mu_2 \nu_2} q^2 - g_{\mu_1 \nu_1} q_{\mu_2} q_{\nu_2} - g_{\nu_1 \nu_2} q_{\mu_1 \mu_2} - g_{\mu_1 \nu_2} q_{\mu_2} - g_{\nu_1 \mu_2} q_{\mu_1} q_{\nu_2} \right) C_{2,N}^{a,2}(Q^2/\mu^2, a_s) \right] \\
\times q_{\mu_3} \ldots q_{\mu_N} \left( \frac{1}{Q^2} \right)^N \mathcal{O}_{a,\{\mu_1 \ldots \mu_N\}} + \text{singlet contributions} + \text{higher twists}, \tag{1}
\]

where it is usual to use the following notation

\[ a_s = \frac{g^2}{16\pi^2} = \frac{\alpha_s}{4\pi} \tag{2} \]

for the QCD strong coupling constant. The sum in eq. (1) runs over the standard set of the spin-\(N\), twist-2 irreducible (i.e. symmetrical and traceless in indices \(\mu_1 \ldots \mu_N\)) flavor non-singlet quark operators:

\[
\mathcal{O}_{a,\{\mu_1 \ldots \mu_N\}} = \bar{\psi} \lambda^a \gamma_\mu \gamma_\nu \mathcal{D}^{\mu_2} \ldots \mathcal{D}^{\mu_N} \psi, \quad a = 1, 2, \ldots, 8, \tag{3}
\]

where \(\mathcal{D}^{\mu_j}\) are the covariant derivatives, \(\lambda^a\) are the generators of the flavor group \(SU(n_f)\) and \(C_{k,N}(Q^2/\mu^2, a_s)\) are the corresponding coefficient functions.

The non-singlet moments of the structure functions \(F_k\) are expressed through operator product expansion (1) in the following form:

\[
M_{k,N} = \int d^4 x N^{-2} F_{k,c}^{en}(x, Q^2) = \sum_a C_{k,N}^{a}(Q^2/\mu^2, a_s) \left[ A_{N,\text{proton}}(\mu^2) - A_{N,\text{neutron}}(\mu^2) \right] \tag{4}
\]

\[ ^1\text{Note, that there is all-loop prediction for the } \mathcal{O}(1/N_f) \text{ contribution to the non-singlet anomalous dimension of twist-2 operators in QCD [13].} \]
where \( A_{N,\text{nucleon}} \) is the spin-averaged nucleon matrix elements of the operator:

\[
\langle p, \text{nucleon} | \mathcal{O}^{\mu_1...\mu_N} | \text{nucleon}, p \rangle = p^{\mu_1} \ldots p^{\mu_N} A^a_{N,\text{nucleon}}(\mu^2).
\]

(5)

Application of the renormalization group technique gives for the coefficient functions the following standard expression:

\[
C^a_{k,N}\left( \frac{Q^2}{\mu^2}, a_s(\mu^2) \right) = C^a_{k,N}(1, a_s(Q^2)) \exp \left(- \int_{a_s(\mu^2)}^{a_s(Q^2)} da' \frac{\gamma(a')}{\beta(a')} \right).
\]

(6)

The anomalous dimensions \( \gamma_N \) in eq. (6) are defined as

\[
\gamma_N(a_s) = \mu \frac{d}{d \mu^2} \log Z_N = \sum_{n=0}^{\infty} \gamma_n^{(n)} a_s^{(n+1)}.
\]

(7)

and renormalized operators and bare ones are connected as follows:

\[
(\mathcal{O}^{\mu_1...\mu_N})_R = (Z_N)^{-1} (\mathcal{O}^{\mu_1...\mu_N})_B.
\]

(8)

The four-loop approximation for the \( \beta \)-function in QCD in the MS-scheme was obtained in Refs. [14, 15]:

\[
\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^4 - \beta_2 a_s^6 - \beta_3 a_s^8 + O(a_s^6),
\]

(9)

\[
\beta_0 = 11 - \frac{2}{3} n_f,
\]

(10)

\[
\beta_1 = 102 - \frac{38}{3} n_f,
\]

(11)

\[
\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2,
\]

(12)

\[
\beta_3 = \left( \frac{149753}{6} + 3564 \zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) n_f + \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) n_f^2 + \frac{1093}{729} n_f^3.
\]

(13)

The perturbative expansion in \( a_s \) for the coefficient functions is

\[
C_{k,N}(1, a_s) = B^{(0)}_{k,N} + B^{(1)}_{k,N} a_s + B^{(2)}_{k,N} a_s^2 + B^{(3)}_{k,N} a_s^3 + O(a_s^4),
\]

(14)

where the CallanGross relation gives \( B^{(0)}_{L,N} = 0 \) for all \( N \) and the standard deep inelastic normalization [8] of the coefficient functions implies \( B^{(0)}_{2,N} = 1 \).

The next-next-next-to-leading (NNNL) approximations for the non-singlet moments after renormalization group improvement are

\[
M_{2,N}(Q^2) = a_s^{\gamma^{(0)}/\beta_0} \left( B^{(0)}_{2,N} + B^{(1)}_{2,N} a_s + B^{(2)}_{2,N} a_s^2 + B^{(3)}_{2,N} a_s^3 \right) E(a_s) A_N(\mu^2).
\]

(15)
with

\[ E(a_s) = 1 + \frac{a_s^2}{\beta_0} E_1 + \frac{a_s^2}{\beta_0^2} \left[ \frac{E_2^2}{2} - E_1 \beta_0 + E_2 \beta_0^2 \right] + \frac{a_s^3}{\beta_0^3} \left[ \frac{E_3^3}{6} - E_1^2 \beta_0 + E_1 \left( \beta_1 - \beta_0 \beta_2 \right) \beta_0^2 + E_2 \beta_0^3 - E_2 \beta_1 \beta_0^3 + E_3 \beta_0^4 \right] \]  

where \( E_i = \gamma^{(i)}_N \beta_0 - \gamma^{(0)}_N \beta_i \) and for the NNNL approximation one should keep the four leading orders in the product of the power series for the coefficient functions with the series \( E(a_s) \). So, for the NNNL approximation to the non-singlet moments \( M_{2,N} \) we need to known the 3-loop coefficients \( B^{(3)}_{2,N} \) for the coefficient functions \( C_{2,N}(1, a_s) \) and the 4-loop coefficients \( \gamma^{(3)}_N \) for the anomalous dimensions \( \gamma_N(a_s) \).

For the first even moment \( (N = 2) \) non-singlet operator (3) has the following form

\[ O^{\mu \nu}_{NS} = \bar{\psi} \lambda^a \sigma^\mu \rho \psi + \bar{\psi} \lambda^a \sigma^\nu \rho \psi - \frac{2}{D} g^\mu \nu \bar{\psi} \lambda^a \sigma^\rho \rho \psi, \]  

where \( D = 4 - 2\epsilon \) is space-time dimension.

The calculation of the anomalous dimension of such operators can be performed in a usual way through the computation of the Green’s function with the operator insertion, which have the following general form in momentum space (see [16]):

\[ G_{\mu \nu}^{\alpha \beta}(p) = \langle \psi(p) \left[ O^{\alpha \beta \mu \nu}(0) \right] \bar{\psi}(-p) \rangle = \sum_{\alpha \beta}^{(1)} \left( \gamma^\mu p^\nu + \gamma^\nu p^\mu - \frac{2}{D} \delta^\alpha \beta \right) + \sum_{\alpha \beta}^{(2)} \left( p^\mu p^\nu p^\omega - \frac{p^2}{D} \delta^\alpha \beta \right), \]

where \( p \) is the momentum flowing through the external quark legs. To determine different components we use the following projectors (see [16])

\[ P_{\mu \nu}^{(1)} = \frac{1}{8(D-1)} \left[ \text{tr} \left( \gamma_{\mu \nu} + \gamma_{\nu \mu} - \frac{2}{D} \delta g_{\mu \nu} \right) \right] - 2 \text{tr} \left( p_{\mu \nu} p_{\mu \nu} - \frac{p^2}{D} \delta g_{\mu \nu} \right) \]  

\[ P_{\mu \nu}^{(2)} = \frac{-1}{4(D-1)} \left[ \text{tr} \left( \gamma_{\mu \nu} + \gamma_{\nu \mu} - \frac{2}{D} \delta g_{\mu \nu} \right) \right] - (D + 2) \text{tr} \left( p_{\mu \nu} p_{\mu \nu} - \frac{p^2}{D} \delta g_{\mu \nu} \right), \]

Really, to find the anomalous dimension of the operator \( O_{NS} \) we should compute only \( \sum_{\alpha \beta}^{(1)} \). A total number of four-loop diagrams is 12816. As in our previous work [17, 18, 19, 20, 21] all calculations were performed with FORM [22], using FORM package COLOR [23] for evaluation of the color traces. For the dealing with a huge number of diagrams we use a program DIANA [24], which call QGRAF [25] to generate all diagrams. For evaluation of Feynman integrals we used the method from Refs. [26, 27] and our own implementation of the Laporta’s algorithm [28] in the form of the MATHEMATICA package BAMBA with the master integrals from Ref. [29].

3
For the renormalization we need the three-loop renormalization constant for the operator insertion \( g \bar{\psi} \lambda^a \gamma^{(\mu, A^\nu)} \psi \) with two quarks and one gluon legs, which can be obtained order by order in a usual way from the renormalization of operator \( O^a, \{ \mu \nu \} \) as

\[
Z_{\bar{\psi} \lambda^a \gamma^{(\mu, A^\nu)} \psi} = Z_{O^a, \{ \mu \nu \}}^1 Z_{A}^1 Z_{g}^1 Z_{\psi},
\]

where \( Z_{A} \), \( Z_{g} \) and \( Z_{\psi} \) are the renormalization constants for gluon field \( A^\mu \), coupling constant and quark correspondingly.

Our final result is

\[
\gamma_{NS}^{4-\text{loop}}(2) = a_s \frac{8}{3} C_F + a_s^2 \left[ \frac{376}{27} C_F C_A - \frac{128}{27} C_F n_f T_F - \frac{112}{27} C_F^2 \right] + a_s^3 \left[ C_F^3 \left( \frac{128}{3} \zeta_3 - \frac{560}{243} \right) + C_F^2 T_F n_f \left( \frac{128}{3} \zeta_3 - \frac{6824}{243} \right) + C_F^2 C_A \left( -64 \zeta_3 - \frac{8528}{243} \right) - \frac{896}{243} C_F T_F n_f^2 + C_F C_A T_F n_f \left( -\frac{128}{3} \zeta_3 - \frac{6256}{243} \right) + C_F C_A^2 \left( \frac{64 \zeta_3 + 20920}{243} \right) \right] + a_s^4 \left[ C_F^4 \left( \frac{10880}{81} \zeta_3 - \frac{1280}{3} \zeta_5 + \frac{194392}{2187} \right) + C_F^3 T_F n_f \left( -\frac{5056}{81} \zeta_3 + \frac{256}{3} \zeta_4 - \frac{1280}{3} \zeta_5 + \frac{381824}{2187} \right) + C_F^3 C_A \left( \frac{31040}{81} \zeta_3 - \frac{704}{3} \zeta_4 + \frac{1280}{3} \zeta_5 + \frac{238676}{2187} \right) + C_F^2 T_F^2 n_f^2 \left( -\frac{512}{3} \zeta_3 + \frac{256}{3} \zeta_4 + \frac{99776}{2187} \right) + C_F^2 C_A T_F n_f \left( \frac{25856}{27} \zeta_3 - \frac{1088}{3} \zeta_4 + \frac{640}{9} \zeta_5 - \frac{355496}{2187} \right) + C_F^2 C_A^2 \left( -\frac{25744}{27} \zeta_3 + 352 \zeta_4 + \frac{4480}{9} \zeta_5 - \frac{1626064}{2187} \right) + C_F T_F^3 n_f^3 \left( \frac{1024}{81} \zeta_3 - \frac{8192}{2187} \right) + C_F C_A T_F^2 n_f^2 \left( \frac{512}{3} \zeta_3 - \frac{256}{3} \zeta_4 + \frac{25400}{729} \right) + C_F C_A^2 T_F n_f \left( -\frac{8080}{9} \zeta_3 + \frac{832}{3} \zeta_4 + \frac{8960}{27} \zeta_5 - \frac{106036}{243} \right) + C_F C_A^3 \left( \frac{34936}{81} \zeta_3 - \frac{352}{3} \zeta_4 - \frac{12160}{27} \zeta_5 + \frac{1734130}{2187} \right) + \frac{32}{9} d_F^{abcd} d_A^{abcd} \left( \frac{62 \zeta_3 + 160 \zeta_5 - 23}{N_F} + n_f \frac{128}{9} d_F^{abcd} d_A^{abcd} \left( \frac{16 \zeta_3 - 40 \zeta_5 + 13}{N_F} \right) \right) \right],
\]
where (see Ref. [15])

\[
\frac{d_{abcd}d_{abcd}}{d_{F}} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2},
\]

(23)

\[
\frac{d_{abcd}d_{abcd}}{d_{A}} = \frac{N_c(N_c^2 + 6)}{8},
\]

(24)

\[
\frac{d_{A}d_{abcd}}{d_{A}} = \frac{N_c^2(N_c^2 + 36)}{24},
\]

(25)

and for the color group \(SU(N_c)\) the more simple Casimir operators are:

\[
T_F = \frac{1}{2}, \quad C_F = \frac{N_c^2 - 1}{2N_c}, \quad C_A = N_c, \quad N_A = N_c^2 - 1.
\]

(26)

The obtained result coincides with the existing result for the first even moment of the four-loop non-singlet anomalous dimension from Ref. [12] if we substitute the explicit expression for all Casimir operators for QCD with three active quarks (i.e. for the gauge group \(SU(3)\) with \(n_f = 3\)). Moreover, the part of our result, which is proportional to \((n_f)^{i-1}a_s^i\), coincide with the prediction from Ref. [13], while the non-planar part was calculated by us in Ref. [21].

The substitution of the colour factors with \(N_c = 3\) into eq. (22) gives the following result for QCD case

\[
\gamma_{NS}^{\text{4-loop}}(2) = +\frac{32a_s}{9} + a_s^2\left(\frac{11744}{243} - \frac{256n_f}{81}\right) + a_s^3\left(-\frac{896n_f^2}{729} + n_f\left(-\frac{1280\zeta_3}{27} - \frac{167200}{2187}\right) + \frac{1280\zeta_3}{81} + \frac{5514208}{6561}\right)
+ a_s^4\left(\frac{26606864\zeta_3}{6561} - \frac{7040\zeta_4}{27} - \frac{1249280\zeta_5}{243} + \frac{3100369144}{177147}\right)
+ n_f\left(-\frac{6322976\zeta_3}{2187} + \frac{64640\zeta_4}{81} + \frac{14720\zeta_5}{9} - \frac{167219672}{59049}\right)
+ n_f^2\left(\frac{2560\zeta_3}{27} - \frac{1280\zeta_4}{27} + \frac{1084904}{19683}\right) + n_f^3\left(\frac{512\zeta_3}{243} - \frac{4096}{6561}\right).
\]

(27)

In conclusion we give the explicit results for the different number of active quarks:

\[
\gamma_{NS}^{\text{4-loop}}(2, n_f = 3) = 3.55556a_s + 38.84774a_s^2 + 448.07162a_s^3 + 6532.13656a_s^4,
\]

(28)

\[
\gamma_{NS}^{\text{4-loop}}(2, n_f = 4) = 3.55556a_s + 35.68724a_s^2 + 306.02989a_s^3 + 3679.66906a_s^4,
\]

(29)

\[
\gamma_{NS}^{\text{4-loop}}(2, n_f = 5) = 3.55556a_s + 32.52675a_s^2 + 161.53001a_s^3 + 1108.56696a_s^4,
\]

(30)

\[
\gamma_{NS}^{\text{4-loop}}(2, n_f = 6) = 3.55556a_s + 29.36626a_s^2 + 14.571953a_s^3 - 1169.71912a_s^4.
\]

(31)
It is interesting to compare our result with the predictions \cite{30}, coming from the Pade resummation, which for $\gamma^{(3)}_{NS}(2, n_f = 4)$ gives 2629 or 2557 depending on the resummations procedure. Note, that in four-loop order new colour structures (23)-(25) appear, which can disimprove resummation. So, we give below our result for $\gamma^{(3)}_{NS}(2)$ with the contributions from different colour structures:

\begin{align}
\gamma^{(3)}_{NS}(2, n_f = 3) &= 4626.76262 + 1932.76417 d_{44}^{FA} - 27.39022 d_{44}^{FF}, \\
\gamma^{(3)}_{NS}(2, n_f = 4) &= 1783.42519 + 1932.76417 d_{44}^{FA} - 36.52029 d_{44}^{FF}, \\
\gamma^{(3)}_{NS}(2, n_f = 5) &= -778.54684 + 1932.76417 d_{44}^{FA} - 45.65037 d_{44}^{FF}, \\
\gamma^{(3)}_{NS}(2, n_f = 6) &= -3047.70285 + 1932.76417 d_{44}^{FA} - 54.78044 d_{44}^{FF},
\end{align}

where $d_{44}^{FF}$ and $d_{44}^{FA}$ are the contributions coming from (23) and (24) correspondingly.

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