Primordial gravitational waves and curvature perturbations induced energy density perturbation

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Abstract

We study the second order scalar and density perturbations generated by the Gaussian curvature perturbations and primordial gravitational waves in the radiation-dominated era. After presenting all the possible second-order source terms, we obtain the explicit expressions of the kernel functions and the power spectra of the second order scalar perturbations. It shows that the primordial gravitational waves might affect the second order energy density perturbation \( \delta^{(2)} = \delta \rho^{(2)}/\rho^{(0)} \) significantly. The effects of the primordial gravitational waves are studied in terms of different kinds of primordial power spectra.

Keywords primordial gravitational waves · cosmological perturbation theory · primordial power spectra

1 Introduction

The cosmological perturbations which are originated from the quantum fluctuations during inflation will inevitably induce higher order perturbations. These induced higher order perturbations can also affect the evolutions of the universe \cite{1}.

The cosmological perturbations can be decomposed as scalar, vector, and tensor perturbations on account of the \( SO(3) \) symmetry of the Friedmann-Robertson-Walker (FRW) spacetime \cite{2–4}. Recently, the higher order perturbations induced by the primordial perturbations have been attracting a lot of interests because of their rich phenomenology \cite{5}. For tensor perturbation, the higher order induced tensor perturbations are known as induced gravitational waves (GWs) \cite{6–31}. For higher order induced scalar perturbations \cite{32–34}, the higher order energy density perturbation \( \delta^{(n)} = \rho^{(n)}/\rho^{(0)} \) can be calculated in terms of these scalar perturbations. And \( \delta^{(n)} \) can affect the primordial black hole (PBH) formation \cite{35, 36}, and the large-scale structure (LSS) of the Universe \cite{37, 38}. The higher order induced vector perturbations can also affect many cosmological observations \cite{3, 39–48}, such as redshift-space distortions \cite{42} and weak lensing \cite{45, 47}.

The source terms of high order induced perturbations originate from the primordial perturbations generated during the inflation. Since vector perturbations decay as \( 1/a^2 \) \cite{49}, we typically

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neglect the primordial vector perturbations. On large scales ($\geq 1\text{Mpc}$), the amplitude of the primordial scalar perturbation $A_{\zeta}$ is constrained by observations of the Cosmic Microwave Background (CMB) and large-scale structures to be about $A_{\zeta} \simeq 2^{-9}$. For the primordial tensor perturbation on large scales ($\geq 1\text{Mpc}$), the tensor-to-scalar ratio $r = A_h/A_{\zeta}$ is constrained to be less than 0.06 [49], where $A_h$ is the amplitude of primordial tensor perturbation. Therefore, when studying higher order induced perturbations on large scales ($\geq 1\text{Mpc}$), primordial tensor perturbation can be neglected compared to primordial scalar perturbation.

On small scales ($\leq 1\text{Mpc}$), the constraints of primordial scalar and tensor perturbations are significantly weaker than those on large scales [50]. Over the past few years, the primordial scalar perturbation with large amplitude on small scales have been attracting a lot of interest. It is closely related to primordial black holes and scalar induced GWs [51–56]. For the primordial tensor perturbation on small scales, its amplitude could also be much larger than it is on the large scales. The large primordial tensor perturbation on small scales can be realized by many models of early universe, such as $G^2$-inflation [57], spectator field [58], and so on [35]. Recently, the power spectra of second order tensor perturbation induced by primordial scalar and tensor perturbations with large amplitudes were studied in Ref. [17, 59–61]. They considered the log-normal primordial scalar and tensor power spectra on small scales and found that the primordial tensor perturbation has a very important effect on the second order induced tensor perturbation.

The second order induced scalar and energy density perturbations have been studied for many years [32–38, 62]. However, a complete study of the second order induced scalar perturbations has not been presented. The importance of the scalar-tensor coupling source terms $S_2 \sim \phi h$ has been neglected in previous studies. In this paper, we consider the second order energy density perturbation induced by primordial curvature and tensor perturbations during RD era systematically. The second order scalar perturbations can be generated by the scalar-scalar, scalar-tensor, and tensor-tensor coupling source terms: $S_1 \sim \phi \phi$, $S_2 \sim \phi h$, and $S_3 \sim hh$. The second order energy density perturbation $P^{(2)}_\delta$ can be calculated in terms of these second order induced scalar perturbations. The explicit expressions of second order scalar and energy density perturbations are presented in this work.

This paper is organized as follows. In Sec. 2, the second order scalar perturbations are studied. The explicit expressions of the second order power spectra are presented. In Sec. 3, we investigate the second order power spectrum in terms of the monochromatic primordial power spectra. The $P^{(2)}_\delta$ induced by log-normal primordial power spectra are studied in Sec. 4. The conclusions and discussions are summarized in Sec. 5.

2 Second order scalar perturbations

In this section, we study the equations of motion and the kernel functions of the second order scalar perturbations induced by primordial curvature and tensor perturbations in the RD era.
2.1 Equation of motion

The perturbed metric in the flat FRW spacetime with Newtonian gauge is given by

$$ds^2 = a^2 \left[ -\left( 1 + 2\phi^{(1)} + \phi^{(2)} \right) d\eta^2 + \left( 1 - 2\psi^{(1)} - \psi^{(2)} \right) \delta_{ij} + h^{(1)}_{ij} \right] dx^i dx^j,$$

where $\phi^{(n)}$ and $\psi^{(n)} (n = 1, 2)$ are first order and second order scalar perturbations, $h^{(1)}_{ij}$ is the first order tensor perturbation. Here, we neglect the first order vector perturbation since the vector modes decay as $1/a^2$ after they leaving the Hubble horizon during inflation. We use xPand package to study the perturbations of Einstein equation, xPand package can help us to obtain and simplify the equations of motion of cosmological perturbations [63]. The equations of motion of second order scalar perturbations are given by

$$\psi^{(2)} - \phi^{(2)} = -2\Delta^{-1} \left( \partial^i \Delta^{-1} \partial^j - \frac{1}{2} T^{ij} \right) S_{ij} (x, \eta),$$

$$\partial^2 \psi^{(2)} + 3H \partial_\eta \psi^{(2)} - \frac{5}{6} \Delta \psi^{(2)} + H \partial_\eta \phi^{(2)} + \frac{1}{2} \Delta \phi^{(2)} = -\frac{1}{2} T^{ij} S_{ij} (x, \eta),$$

where $T^{ij} = \delta^{ij} - \partial^i \Delta^{-1} \partial^j$ is the transverse operator. During the RD era, the conformal Hubble parameter can be expressed as $H = 1/\eta$. For convenience, we use the symbols $\phi^{(1)} \equiv \phi$ and $h^{(1)}_{ij} \equiv h_{ij}$. As shown in Fig. 1, the source term $S_{ij} (x, \eta)$ is composed of three parts $S_{ij} (x, \eta) = S_{ij,1} + S_{ij,2} + S_{ij,3}$. The explicit expressions of the source terms are given in Appendix A. Substituting

Fig. 1: The source terms $S_{ij,1}$ is composed of the first order scalar perturbation, the source terms $S_{ij,2}$ is composed of the product of the first order scalar perturbation and the first order tensor perturbation, and the source terms $S_{ij,3}$ is composed of the first order tensor perturbation.

Eq. (2) into Eq. (3), we obtain the equation of motion of second order scalar perturbation $\psi^{(2)}$ in
the RD era

\[ \frac{\partial^2}{\partial^2_\eta} \psi^{(2)} + \frac{4}{\eta} \partial_\eta \psi^{(2)} - \frac{1}{3} \Delta \psi^{(2)} \]

\[ = - \frac{1}{2} T^{ij} S_{ij} - 2 \Delta^{-1} \left( \partial^j \Delta^{-1} \partial^i - \frac{1}{2} T^{ij} \right) \left( \frac{1}{2} \Delta + \frac{1}{\eta} \partial_\eta \right) S_{ij} (x, \eta) \]

\[ = - \left( \frac{1}{2} T^{ij} - \left( \partial^j \Delta^{-1} \partial^i - \frac{1}{2} T^{ij} \right) \Delta^{-1} \right) \frac{1}{\eta} \partial_\eta \right) S_{ij} (x, \eta) \]

\[ = - \left( \partial^j \Delta^{-1} \partial^i + 2 \Delta^{-1} \left( \partial^j \Delta^{-1} \partial^i - \frac{1}{2} T^{ij} \right) \right) \frac{1}{\eta} \partial_\eta \sum_{a=1}^{3} S_{ij,a} (x, \eta) \]  

\[ \text{(4)} \]

2.2 Kernel functions

In order to solve the equation of motion of second order scalar perturbation, we rewrite Eq. (4) in momentum space as

\[ \psi^{(2)} (k, \eta) + \frac{4}{\eta} \psi^{(2)'} (k, \eta) + \frac{k^2}{3} \psi^{(2)} (k, \eta) = \sum_{a=1}^{3} S_a (k, \eta) , \]

where

\[ S_a (k, \eta) = D^a_{ij} S_{ij,a} (k, \eta) , \quad D^a_{ij} = - \left( \frac{k^i k^j}{k^2} - \frac{\delta^{ij}}{k^2} \right) \frac{k^2}{x} \partial_x \].

(6)

Here, we have defined \( x \equiv k \eta \). The explicit expressions of \( S_{ij,a} (k, \eta) (a = 1, 2, 3) \) are given in Appendix A. Substituting Eqs. (A.4)–(A.6) into Eq. (6), we obtain the expressions of \( S_a (k, \eta) \)

\[ S_1 = D^1_{ij} S_{ij,1} = \int \frac{d^3 p}{(2\pi)^3/2} f_1 (u, v, x) \frac{4}{9} \zeta_{k-p} \psi_p \]  

(7)

\[ S_2 = D^2_{ij} S_{ij,2} = \int \frac{d^3 p}{(2\pi)^3/2} f^\lambda_1 (u, v, x) \frac{2}{3} \zeta_{k-p} h^\lambda_1_p \]  

(8)

\[ S_3 = D^3_{ij} S_{ij,3} = \int \frac{d^3 p}{(2\pi)^3/2} f^\lambda_2 (u, v, x) \frac{k^2}{x} \zeta_{k-p} h^\lambda_2_p \]  

(9)

where \( \lambda_1 \) and \( \lambda_2 \) are the polarization indices, the spatial indices of \( S_{ij,a} (a = 1, 2, 3) \) are contracted.

Substituting Eqs. (7)–(9) into Eq. (5), we solve the Eq. (5) by using the Green’s function method

\[ \psi^{(2)} = \sum_{a=1}^{3} \phi_a^{(2)} (a = 1, 2, 3) \]

(10)

where

\[ \psi_1^{(2)} = \int \frac{d^3 p}{(2\pi)^3/2} f_1 (u, v, x) \frac{4}{9} \zeta_{k-p} \psi_p \]  

(11)

\[ \psi_2^{(2)} = \int \frac{d^3 p}{(2\pi)^3/2} f^\lambda_1 (u, v, x) \frac{2}{3} \zeta_{k-p} h^\lambda_1_p \]  

(12)

\[ \psi_3^{(2)} = \int \frac{d^3 p}{(2\pi)^3/2} f^\lambda_2 (u, v, x) \frac{k^2}{x} \zeta_{k-p} h^\lambda_2_p \]  

(13)
In Eqs. (11)–(13), the kernel functions \( I_a(u, v, x)(a = 1, 2, 3) \) are defined as 

\[
I_1 = \int_0^x \, d\bar{x} \left( \frac{x}{\bar{x}} \right)^2 \left\{ -\frac{x}{\sqrt{3}} \left[ j_1(x/\sqrt{3})y_1(\bar{x}/\sqrt{3}) - j_1(\bar{x}/\sqrt{3})y_1(x/\sqrt{3}) \right] \right\} f_1(u, v, \bar{x}) , \\
I_2^{\lambda_1} = \int_0^x \, d\bar{x} \left( \frac{x}{\bar{x}} \right)^2 \left\{ -\frac{x}{\sqrt{3}} \left[ j_1(x/\sqrt{3})y_1(\bar{x}/\sqrt{3}) - j_1(\bar{x}/\sqrt{3})y_1(x/\sqrt{3}) \right] \right\} f_2^{\lambda_1}(u, v, \bar{x}) , \\
I_3^{\lambda_1, \lambda_2} = \int_0^x \, d\bar{x} \left( \frac{x}{\bar{x}} \right)^2 \left\{ -\frac{x}{\sqrt{3}} \left[ j_1(x/\sqrt{3})y_1(\bar{x}/\sqrt{3}) - j_1(\bar{x}/\sqrt{3})y_1(x/\sqrt{3}) \right] \right\} f_3^{\lambda_1, \lambda_2}(u, v, \bar{x}) .
\]

In the end of this section, we calculate the kernel functions in Eq. (14). We present these three kinds of kernel functions \( I_a(u, v, x)(a = 1, 2, 3) \) as functions of \( x = k\eta \) in Fig. 2. As shown in Fig. 2, the kernel function \( I_1 \) is much larger than other kernel functions.

2.3 Initial second order perturbation

As we mentioned in Ref. [2, 32], the contributions from the initial second-order perturbation also need to be considered. More precisely, the second order scalar perturbations are composed of two parts, the second order scalar perturbations induced by the primordial perturbations and the initial second-order perturbation. The second-order curvature perturbation in the Newtonian gauge is given by [2]

\[
-\zeta^{(2)} = \psi^{(2)} + \frac{\mathcal{H}}{\rho^{(0)\nu}} \left[ \delta\rho^{(2)} - \frac{\delta\rho^{(1)v}}{\rho^{(0)\nu}}\delta\rho^{(1)} \right] - \frac{1}{4} \lambda \delta_{\rho k} + \frac{1}{4} \Delta^{-2} \partial^i \partial^j \chi_{ij} \delta\rho ,
\]

\begin{align}
\text{Fig. 2: The kernel functions } I_i \ (i = 1, 2, 3) \text{ in Eq. (14). We have set } u = v = 1.
\end{align}
where
\[
\lambda_{ij}\delta \rho \equiv -2\frac{\mathcal{H}}{\rho(0)^{\gamma}} \left[ 2\mathcal{H} \left( \frac{\delta \rho^{(1)2}}{\rho(0)^{\gamma}} - \frac{\delta \rho^{(1)\gamma}}{\rho(0)^{\gamma}} \delta \rho^{(1)} \right) \delta_{ij} - \frac{2}{\rho(0)^{2\gamma}} \partial_i \delta \rho^{(1)} \partial_j \delta \rho^{(1)} \right] + 4 \left[ -\delta \rho^{(1)} \left[ -\psi^{(1)} \delta_{ij} + \frac{1}{2} h^{(1)\gamma}_{ij} \right] + 2\mathcal{H} \left( -\psi^{(1)} \delta_{ij} + \frac{1}{2} h^{(1)\gamma}_{ij} \right) \right],
\]
(16)
\[
\lambda^k_{\delta \rho \ell} \equiv -6\frac{\mathcal{H}}{\rho(0)^{\gamma}} \left[ 2H \left( \frac{\delta \rho^{(1)2}}{\rho(0)^{\gamma}} - \frac{\delta \rho^{(1)\gamma}}{\rho(0)^{\gamma}} \delta \rho^{(1)} \right) \right] - \frac{2}{\rho(0)^{2\gamma}} \partial^k \delta \rho^{(1)} \partial_\ell \delta \rho^{(1)} + 4 \left[ -\delta \rho^{(1)} \left[ -3\psi^{(1)} + \frac{1}{2} h^{(1)k}_{k} \right] + 2\mathcal{H} \left( -3\psi^{(1)} + \frac{1}{2} h^{(1)k}_{k} \right) \right].
\]
(17)

In the superhorizon limit \((k\eta \ll 1)\), the second-order curvature perturbation can be approximated as
\[
\zeta^{(2)}(k, \eta) \simeq \left\{ -\psi^{(2)}(k, \eta) + \frac{1}{4} \delta^{(2)} - \frac{1}{8} \delta^{(1)} \psi^{(1)} + \frac{\delta^{(1)} h^{(1)k}_{k}}{4} - \frac{\delta^{(1)} \nabla \cdot \partial_j h^{(1)ij}}{4} \right\} T\psi(k\eta),
\]
(18)
where
\[
\delta^{(1)} = \frac{\delta \rho^{(1)}}{\rho(0)^{\gamma}} = -\frac{6\mathcal{H} \left( \mathcal{H} \phi^{(1)} + \psi^{(1)} \right) + 2\Delta \psi^{(1)}}{3\mathcal{H}^2} \simeq -2\phi^{(1)} = -\frac{4}{3} \zeta^{(1)}.
\]
(19)

We assume the local type non-Gaussianity here, which is parametrized as \(\zeta^{(2)} = 2a_{NL} \left( \zeta^{(1)} \right)^2\) in the superhorizon limit, and \(a_{NL} = 1\) for Gaussian perturbation \([31,32]\). Substituting the condition of the local type non-Gaussianity into Eq. (18), we obtain the contributions from the initial second-order perturbation
\[
\Psi^{(2)}(k) = 3 \left( \frac{3k^4 \dot{\kappa}}{k^4} - \frac{\dot{\kappa}^2}{k^2} \right) S_{ij} + \int \frac{d^3 p}{(2\pi)^3 / 2 \eta} \left( -\frac{4}{3} a_{NL} + \frac{28}{27} \right) \zeta_{k-p} \dot{\kappa}^\lambda_i (p) \zeta_{k-p} h^{\lambda}_{ij} + \frac{2}{9} \dot{\kappa}^\lambda_i (p) \zeta_{k-p} h^{\lambda}_{ij} + \frac{2}{9} \dot{\kappa}^\lambda_i (p) \zeta_{k-p} h^{\lambda}_{ij}.
\]
(20)

After considering the effects of the initial second-order perturbation, the second order scalar perturbation \(\Psi^{(2)}\) can be written as
\[
\Psi^{(2)}(k) = \Psi^{(2)}_{\text{in}}(k) + \sum_{a=1}^{3} \Psi^{(2)}_a, \quad (a = 1, 2, 3).
\]
(21)

We study the second order scalar and density perturbations generated by the Gaussian curvature and tensor perturbations. Therefore, we set \(a_{NL} = 1\) in this paper.

2.4 Power spectra

In this section, the power spectra of second order scalar and density perturbations are investigated. We assume that the two-point function \(\langle \zeta_{k_1} h^{\lambda}_{ij}_{k_2} \rangle = 0\) for arbitrary \(k_1\) and \(k_2\) \([17]\). Therefore, we only need to consider three kinds of four-point functions. The explicit expressions of these four-point functions are given in Appendix C. The power spectra of second order scalar perturbation is defined as
\[
\langle \Psi^{(2)}(k) \Psi^{(2)}(k') \rangle = \delta(k + k') \frac{2\pi^2}{k^3} P^{(2)}(k).
\]
(22)
Substituting Eqs. (11)–(13) into Eq. (22), we obtain

\[
\langle \psi^{(2)}(k) \psi^{(2)}(k') \rangle = \sum_{i=1}^{3} \langle \psi^{(2)}_i(k) \psi^{(2)}_i(k') \rangle ,
\]

where

\[
\langle \psi^{(2)}_1(k) \psi^{(2)}_1(k') \rangle = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3p'}{(2\pi)^{3/2}} I_1(|k-p|, p, \eta) I_1(|k'-p'|, p', \eta) \times \frac{16}{81} \langle \xi_k \xi_p \xi_{k'} \xi_{p'} \rangle ,
\]

\[
\langle \psi^{(2)}_2(k) \psi^{(2)}_2(k') \rangle = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3p'}{(2\pi)^{3/2}} I_2^{\lambda_1}(|k-p|, p, \eta) I_2^{\lambda'_1}(|k'-p'|, p', \eta) \times \frac{4}{5} \langle \xi_{k-p} \xi_{\rho} \xi_{k'-p'} \xi_{\rho'} \rangle ,
\]

\[
\langle \psi^{(2)}_3(k) \psi^{(2)}_3(k') \rangle = \int \frac{d^3p}{(2\pi)^{3/2}} \frac{d^3p'}{(2\pi)^{3/2}} I_3^{\lambda_1 \lambda_2}(|k-p|, p, \eta) I_3^{\lambda'_1 \lambda'_2}(|k'-p'|, p', \eta) \times \langle h_{k-p} \xi_{\rho} \xi_{k'-p'} \xi_{\rho'} \rangle .
\]

The corresponding Feynman diagrams of these three kinds of two-point functions are given in Fig. 3. Substituting Eqs. (C.15)–(C.17) into Eqs. (24)–(26), we obtain the explicit expressions of the power spectra

\[
P^{(2)} = P^{(2)}_1 + P^{(2)}_2 + P^{(2)}_3 ,
\]
where

\[
P_1^{(2)} = \frac{1}{2} \int_0^\infty dv \int_{[-1,1]}^{|1+v|} \frac{du}{v^2 u^2} I_1(u, v, x) P_\zeta(uk) P_\zeta(vk) \times \frac{16}{81} \left[ I_1(|k'| - p', \eta) |_{p'=p} + I_1(|k' - p'|, \eta) |_{p'=p-k} \right]_{k' \to -k},
\]

\[
P_2^{(2)} = \frac{1}{2} \int_0^\infty dv \int_{[-1,1]}^{|1+v|} \frac{du}{v^2 u^2} I_2^1(u, v, x) \delta \lambda_1 \lambda_1 P_\zeta(uk) P_h(vk) \times \frac{4}{9} \lambda_1 \lambda_1 I_1^2 (|k' - p'|, \eta) |_{p'=-p, k'=-k},
\]

\[
P_3^{(2)} = \frac{1}{2} \int_0^\infty dv \int_{[-1,1]}^{|1+v|} \frac{du}{v^2 u^2} I_3^1(u, v, x) P_h(uk) P_h(vk) \times \left[ \delta \lambda_1 \lambda_1 \delta \lambda_2 \lambda_2 \delta \lambda_3 \lambda_3 \delta (|k' - p'|, \eta) |_{p'=-p} + \delta \lambda_1 \lambda_1 \delta \lambda_2 \lambda_2 \delta \lambda_3 \lambda_3 \delta (|k' - p'|, \eta) |_{p'=-p, k'=-k} \right]_{k' \to -k}.
\]

In Eqs. (28)–(30), the substitutions of \( k' \) and \( p' \) come from the three dimensional delta functions in the Wick’s expansions of four-point functions. Since we have assumed the two-point function \( \langle \zeta_k, \zeta_{k'\phi} \rangle = 0 \), three kinds of source terms in Eqs. (A.1)–(A.3) are decoupled. More precisely, the two-point functions \( \langle \Psi_i^{(2)}(k) \Psi_j^{(2)}(k') \rangle = 0 \) if \( i \neq j \). As shown in Eqs. (28)–(30), the power spectra of second order scalar perturbation are composed of three parts, which come from the source terms \( S_1 \sim \phi \), \( S_2 \sim \phi h \), and \( S_3 \sim hh \) respectively.

The energy density perturbation can be calculated as

\[
\delta^{(2)} = \frac{\delta \rho^{(2)}}{\rho^{(0)}} = -\frac{1}{3H^2} \left( \frac{1}{4} H^2 h'_{ij} \left( h^{ij} + 8 H^2 h^{ij} \right) + 6 H^4 \left( -4 \phi^2 + \phi^{(2)} \right) + 4 H \delta \phi \delta \phi \phi \right.

+ 2 \partial_i \phi \delta \phi \partial^i \phi + H^2 \left( -2 \left( 3 \phi^2 + 8 \phi \Delta \phi + \Delta \phi \phi^{(2)} + 2 \partial_i \phi \partial^i \phi + \frac{1}{2} h^{ij} \left( -\partial_i \partial_j \phi + \Delta h_{ij} \right) \right)

\left. + \frac{1}{4} \left( 2 \partial_i h_{ij} - 3 \partial_i h_{ii} \right) \partial^j h_{ii} \right)
\]

(31)

The power spectrum of second order energy density perturbation \( P_\delta^{(2)} \) is defined by

\[
\langle \delta^{(2)}(k) \delta^{(2)}(k') \rangle = \delta (k + k') \frac{2\pi^2}{k^3} P^{(2)}_\delta,
\]

(32)

where the energy density \( \delta^{(2)} \) can be calculated in terms of Eq. (21) and Eq. (31).

3 Monochromatic primordial power spectra

As mentioned, the constraints of primordial curvature and tensor perturbations on small scales are significantly weaker than those on large scales, the tensor-to-scalar ratio \( r \) can be very large on small scales. Therefore, we start with a toy model of \( \delta \)-peak. In this case, the primordial scalar and tensor perturbations are very large on small scalar. Since we consider the second order scalar and density perturbations generated by the Gaussian curvature and tensor perturbations, we have set \( a_{NL} = 1 \) in the Eq. (20).
plot the three kinds of power spectra for second order perturbations $\Phi^{(2)}$, $\psi^{(2)}$, and $\delta^{(2)}$. The power spectra $P^{(2)}_1$, $P^{(2)}_2$, and $P^{(2)}_3$ come from the source terms $S_1 \sim \phi \phi$, $S_2 \sim \phi h$, and $S_3 \sim hh$ respectively. We have set the tensor-to-scalar ratio $r = A_h/A_\zeta = 1$, and $x_s = k_s \eta = 100$.

3.1 Monochromatic primordial power spectra with the same $k_s$

We consider the monochromatic primordial power spectra, namely

$$P_\zeta = A_\zeta k_{\zeta*} \delta (k - k_{\zeta*}) \ , \ P_h = A_h k_{h*} \delta (k - k_{h*}) \ , \ k_{\zeta*} = k_{h*} = k_*$,$$

where $k_*$ is the wavenumber at which the power spectrum has a $\delta$-function peak. In Fig. 4 we plot the three kinds of power spectra for second order perturbations $\phi^{(2)}$, $\psi^{(2)}$, and $\delta^{(2)}$. Here, we use the symbols $P^{(2)}_1$, $P^{(2)}_2$, and $P^{(2)}_3$ to represent the contributions of the source terms $S_1 \sim \phi \phi$, $S_2 \sim \phi h$ and $S_3 \sim hh$ respectively. As shown in Fig. 4 for tensor-to-scalar ratio $r = A_h/A_\zeta = 1$, the second order perturbations sourced by $S_1 \sim \phi \phi$ dominate the total power spectra of $\phi^{(2)}$, $\psi^{(2)}$, and $\delta^{(2)}$.

In order to study the effects of the large tensor-to-scalar ratio $r = A_h/A_\zeta$ on small scales, we calculate the total power spectra of $\phi^{(2)}$, $\psi^{(2)}$, and $\delta^{(2)}$ for different $r = A_h/A_\zeta$. In Fig. 5 the total power spectra of second order scalar perturbations for different $r$ are presented. For tensor-to-scalar ratio $r \ll 1$, the effects of primordial tensor perturbation become negligible, the total power spectra $P^{(2)}_\phi$, $P^{(2)}_\psi$, and $P^{(2)}_\delta$ reduce to the results in Ref. [32]. For $r \gg 1$ on small scales, the effects of the primordial tensor perturbation become obvious, the total power spectra reduce to the results in Ref. [33] [34] [35].

3.2 Monochromatic primordial power spectra with different $k_*$

The monochromatic primordial power spectra with different $k_*$ can be written as

$$P_\zeta = A_\zeta k_{\zeta*} \delta (k - k_{\zeta*}) \ , \ P_h = A_h k_{h*} \delta (k - k_{h*}) \ , \ k_{\zeta*} = k_* \neq k_{h*} .$$

In Fig. 6 we plot the three kinds of power spectra $P^{(2)}_a (a = 1, 2, 3)$ in Eq. (28)–Eq. (30) for second order energy density perturbation $\delta^{(2)} = \delta \rho^{(2)}/\rho^{(0)}$ with different $k_{h*}$. As shown in Fig. 6 for $k_{h*} \neq k_{\zeta*}$, the behaviors of the power spectra $P^{(2)}_2$ sourced by $S_2 \sim \phi h$ are different from the case of $k_{h*}/k_{\zeta*} = 1$. More precisely, for $k_{h*}/k_{\zeta*} = n > 1$, the domain of definition of $P^{(2)}_2$ is
The contributions of the source term $S$ in Eq. (A.2) are completely neglected in previous studies [32, 35, 37]. Here, we point out that the $S$ by log-normal power spectra for primordial scalar and tensor perturbations. Here, we concentrated on to consider a more realistic model, such as log-normal primordial power spectra. We consider the power spectra with $r = A_h/A_\zeta$. In this case, the contributions of the power spectra $P(2)$ sourced by $\zeta^*$ becomes a small peak near $k/k_*$. For $k_{h^*}/k_{\zeta^*} = n < 1$, the domain of definition of $P(2)$ is $[(1-n)k/k_*, (1+n)k/k_*]$. For $k_{h^*}/k_{\zeta^*} = n \ll 1$, the power spectra $P(2)$ sourced by $S_2 \sim \phi h$ becomes a large peak near $k/k_*$. As shown in Fig. 6 for $k_{h^*}/k_{\zeta^*} = 0.1$, the power spectra $P(2)$ can be larger than $P(1)$ sourced by $S_1 \sim \phi \phi$ even in the case of $r = 0.1$. Note that the contributions of the source term $S_2 \sim \phi h$ in Eq. (A.2) are completely neglected in previous studies [32, 35, 37]. Here, we point out that the contributions of the source term $S_2 \sim \phi h$ are very important for the monochromatic primordial power spectra with $k_{h^*}/k_{\zeta^*} \ll 1$.

4 Log-normal primordial power spectra

Since the monochromatic primordial power spectra have infinitesimal width, it is necessary to consider a more realistic model, such as log-normal primordial power spectra. We consider the log-normal power spectra for primordial scalar and tensor perturbations. Here, we concentrated on
the effects of the source term $S_2 \sim \phi h$ and corresponding second order power spectra $P_2^{(2)}$. The log-normal primordial power spectra are given by

$$P_\zeta = \frac{A_\zeta}{\sqrt{2\pi \sigma_\zeta^2}} \exp\left(-\frac{\ln^2 \left(\frac{k}{k \sigma_\zeta}\right)}{2\sigma_\zeta^2}\right), \quad P_h = \frac{A_h}{\sqrt{2\pi \sigma_h^2}} \exp\left(-\frac{\ln^2 \left(\frac{k}{k \sigma_h}\right)}{2\sigma_h^2}\right).$$ (35)

Here, we concentrate on the large peak of $P_2^{(2)}$ with $k_{\zeta*/k_{\zeta*}} \ll 1$. As mentioned in Sec. 3.2, for $k_{\zeta*/k_{\zeta*}} = n \ll 1$, the power spectra $P_2^{(2)}$ sourced by $S_2 \sim \phi h$ has a large peak near $k/k_*$. For the log-normal primordial power spectra, we calculate the power spectra of the second order density perturbation $\delta^{(2)}$ with $k_{\zeta*/k_{\zeta*}} = 0.1$. As shown in Fig. 8 for log-normal primordial power spectra, the peak of $P_2^{(2)}$ becomes larger during the process of $\sigma_* \to 0$. For comparison, we plot $P_1^{(2)}$ and $P_2^{(2)}$ for the second order density perturbation $\delta^{(2)}$ with tensor-to-scalar ratio $r = 0.1$ in Fig. 9. It shows that the contributions of $P_2^{(2)}$ become smaller when $\sigma_*$ increases. Namely, the effects of the source term $S_2 \sim \phi h$ become more obviously when $\sigma_*$ and $n$ become smaller.

5 Conclusions and discussions

In this paper, we systematically studied the second order density perturbations induced by primordial gravitational wave and primordial scalar perturbation. Since the constraints of primordial curvature and tensor perturbations on small scales are significantly weaker than those on large scales, we considered the large tensor-to-scalar ratio $r$ on small scales. As shown in Fig. 8 the effects of the primordial tensor perturbation become obvious for $r \gg 1$. For tensor-to-scalar ratio $r \ll 1$, the effects of primordial tensor perturbation become negligible and our results of the power spectra $P_0^{(2)}$, $P_0^{(2)}$, and $P_0^{(2)}$ reduce to the previous results in Ref. [32].

We give the explicit expressions for the power spectra of primordial scalar and tensor induced scalar and density perturbations in Eqs. (28)–(32). Specifically, for a given primordial scalar and tensor power spectra, the power spectra of the second order induced scalar and density perturbations can be calculated using these equations. In this paper, we considered the primordial power
spectra following delta and log-normal on small scales. It is essential to explore more general forms of the primordial power spectra, such as the log-normal primordial scalar power spectrum and the power-law primordial tensor power spectrum [67].

Moreover, the second order induced density perturbations offer a window to understand small-scale primordial gravitational waves and primordial curvature perturbations. More precisely, the primordial scalar and tensor power spectra can be calculated in terms of a given inflation model.

Fig. 8: The power spectra \( P^{(2)}_2 \) sourced by \( S_2 \sim \phi h \) for second order energy density perturbation \( \delta^{(2)} \) with \( r = 0.1 \). We have set \( n = k \nu_\star / k \zeta_\star = 0.1 \) and \( x_\star = k_\epsilon \eta = 100 \).

Fig. 9: The power spectra \( P^{(2)}_1 \) and \( P^{(2)}_2 \) for second order energy density perturbation \( \delta^{(2)} \) with \( r = 0.1 \) for different \( \sigma_\star \). We have set \( n = k \nu_\star / k \zeta_\star = 0.1 \) and \( x_\star = k_\epsilon \eta = 100 \).
Using Eqs. (28)–(32), we can calculate the power spectra of second order induced scalar and energy density perturbations. The induced density perturbations will affect many physical processes at small scales, such as the formation of PBH [58] and the high order GW background [59]. By observing PBHs or higher order GWs, we can constrain the power spectrum of second order induced density perturbations, thereby constraining inflationary models and the physical properties of primordial scalar and gravitational wave on small scales. Related research might be given in future work.

Appendix A: Source terms

\[
S_{ij,1}(x, \eta) = 4\phi \partial_i \partial_j \phi + \partial_i \phi \partial_j \phi - \frac{\partial_i \phi' \partial_j \phi'}{H} - \frac{\partial_i \phi \partial_j \phi'}{H} - \frac{\partial_i \phi' \partial_j \phi'}{H^2} + \delta_{ij} \left(-24H \phi \phi' \right) - 2(\phi')^2 - 4\phi \phi'' - \frac{16}{3} \phi \Delta \phi - \frac{11}{3} \partial_\eta \phi \phi' + \frac{2}{3H} \partial_\phi \phi' \phi' + \frac{1}{3H^2} \partial_\phi \phi' \phi' \phi',
\]

(A.1)

\[
S_{ij,2}(x, \eta) = -h_{ij}^\prime \phi - 2\dot{H} h_{ij}^\prime + 10H h_{ij}, \phi + 3h_{ij}, \phi'' - \phi \Delta h_{ij} - \frac{5}{3} h_{ij} \Delta \phi + h_{ij}^\prime \partial_\phi \phi + h_{ij}^\prime \partial_\phi \phi + \partial_\phi \phi \partial_\phi \phi + \partial_\phi \phi \partial_\phi \phi - \frac{1}{3} h_{ij} \partial_{\phi} \phi',
\]

(A.2)

\[
S_{ij,3}(x, \eta) = -\frac{1}{2} h_{ij}^\prime \phi - \frac{1}{2} h_{ij}^\prime \phi + \frac{1}{4} h_{bc} \partial_\phi \partial_\phi \phi + \frac{1}{4} \partial_\phi h_{bc} \partial_\phi h_{bc} + h_{ij} \partial_\phi h_{bc} + \delta_{ij} \left( \frac{5}{12} h_{bc}^\prime h_{bc}^\prime + \frac{1}{2} h_{bc}^\prime h_{bc}^\prime \right)
\]

(A.3)

where in Eqs. (A.1)–(A.3), we have defined \( \partial_\phi \phi \equiv \phi' \). The source terms in momentum space are given by

\[
S_{ij,1}(k, \eta) = -\int \frac{d^3p}{(2\pi)^{3/2}} \left( 4p_ip_j T_\phi(ux) T_\phi(vx) + ((k-p)_p)_p \left( T_\phi(ux) T_\phi(vx) - ux \frac{d}{d(ux)} T_\phi(ux) T_\phi(vx) \right) \right)
\]

\[
-ux T_\phi(ux) \frac{d}{d(ux)} T_\phi(vx) - uu x^2 \frac{d}{d(ux)} T_\phi(ux) \frac{d}{d(vx)} T_\phi(vx) \right) - \delta_{ij} \left(-24k^2 T_\phi(ux) T_\phi(vx) \right)
\]

\[
-2uk^2 T_\phi(ux) T_\phi(vx) - 4u^2 k^2 T_\phi(ux) T_\phi(vx) + \frac{16u^2 k^2}{3} T_\phi(ux) T_\phi(vx)
\]

\[
+ 11k^2 (1 - u^2 - v^2) T_\phi(ux) T_\phi(vx) - \frac{k^2 (1 - u^2 - v^2)}{3} \frac{d}{d(ux)} T_\phi(ux) T_\phi(vx)
\]

\[
- \frac{k^2 (1 - u^2 - v^2)x^2 uu}{6} T_\phi(ux) T_\phi(vx) \right) \frac{4}{5} \hat{k} \cdot \hat{p} \phi',
\]

(A.4)
In Eqs. (A.4)–(A.6), we have defined $T$ where

The polarization tensor is defined as

\[
\epsilon_{ij} = \frac{1}{2} \left( \epsilon_{i} \epsilon_{j} - \frac{1}{2} \epsilon \delta_{ij} \right)
\]

and

\[
\frac{d^3p}{(2\pi)^3/2} \left( -\frac{(k-p)^{\mu}p_{\mu}}{2} \right) \epsilon_{ij}^{\lambda} \epsilon^{\lambda \beta} \left( \frac{d^2}{d(\mu\nu)^2} T_{\mu}(\nu) - \frac{2v^2k^2}{x} \frac{d}{d(\nu)} T_{\mu}(\nu) \right)
\]

+ $3\alpha^2 k^2 \frac{d^2}{d(\nu)^2} T_{\mu}(\nu) + v^2 k^2 T_{\mu}(\nu) + \frac{5u^2 k^2}{3} T_{\mu}(\nu) T_{\nu}(\mu)$

\[
+ k^2 \left( 1 - v^2 - u^2 \right) T_{\mu}(\nu) T_{\nu}(\mu) + \frac{10uk^2}{x} \frac{d}{d(\nu)} T_{\mu}(\nu) T_{\nu}(\mu)
\]

\[
+ \delta_{ij} e^{\lambda, bc}(p)(k-p)\eta(k-p)c \left( \frac{2}{3} T_{\mu}(\nu) T_{\nu}(\mu) \right) \right) \frac{2}{3} \gamma k - p b^\lambda,
\]

(A.5)

\[
S_{ij,3}(k, \eta) = \int \frac{d^3p}{(2\pi)^3/2} \left( -\frac{(k-p)^{\mu}p_{\mu}}{2} \right) \epsilon_{ij}^{\lambda} \epsilon^{\lambda \beta} \left( \frac{d^2}{d(\mu\nu)^2} T_{\mu}(\nu) - \frac{2v^2k^2}{x} \frac{d}{d(\nu)} T_{\mu}(\nu) \right)
\]

+ $\epsilon_{ij}^{\lambda, bc}(p)(k-p)\eta(k-p)c \left( \frac{2}{3} T_{\mu}(\nu) T_{\nu}(\mu) \right) \right) \frac{2}{3} \gamma k - p b^\lambda,
\]

(A.6)

In Eqs. (A.4)–(A.6), we have defined $|k-p| = u|k| = uk$ and $p = |p| = vk$.The explicit expressions of the polarization tensor $e^{\lambda, ij}(p)$ are given in Appendix B. The first order scalar and tensor perturbations in Eqs. (A.4)–(A.6) have been written as

\[
\psi(\eta, k) = \phi(\eta, k) = \frac{2}{3} \gamma k T_\phi(k\eta), \quad h^\lambda(\eta, k) = h^\lambda_{k} T_{\mu}(k\eta),
\]

(A.7)

where $\gamma_{k}$ and $h_{k}$ are the primordial curvature and tensor perturbations respectively. The transfer functions $T_{\phi}(k\eta)$ and $T_{\mu}(k\eta)$ in the RD era are given by (32)

\[
T_{\phi}(x) = \frac{9}{x^9} \left( \frac{\sqrt{3}}{x} \sin \left( \frac{x}{\sqrt{3}} \right) - \cos \left( \frac{x}{\sqrt{3}} \right) \right), \quad T_{\mu} = \frac{\sin x}{x}.
\]

(A.8)

Appendix B: Polarization tensor

The polarization tensor is defined as

\[
e_{ij}^{\lambda}(k) = \frac{1}{\sqrt{2}} \left( e_{i}(k) e_{j}(k) + e_{j}(k) e_{i}(k) \right).
\]

(B.9)
where \((k_i/|k|, e_i(k), \bar{e}_i(k))\) is the normalized bases in three dimensional momentum space. We study the polarization tensor for a given coordinate system, namely

\[
k = (0, 0, k) \quad e_i(k) = (1, 0, 0) \quad \bar{e}_i(k) = (0, 1, 0).
\]

Then the polarization tensors \(\varepsilon_{ij}^X(k-p)\) and \(\varepsilon_{ij}^Y(p)\) can be written as

\[
\varepsilon_{ij}^X(k-p) = \frac{1}{\sqrt{2}} \left( e_i(k-p)\varepsilon_j(k-p) + \bar{e}_i(k-p)\varepsilon_j(k-p) \right),
\]

\[
\varepsilon_{ij}^Y(k-p) = \frac{1}{\sqrt{2}} \left( e_i(k-p)\varepsilon_j(k-p) + \bar{e}_i(k-p)\varepsilon_j(k-p) \right),
\]

\[
\varepsilon_{ij}^X(p) = \frac{1}{\sqrt{2}} \left( e_i(p)\varepsilon_j(p) + \bar{e}_i(p)\varepsilon_j(p) \right),
\]

\[
\varepsilon_{ij}^Y(p) = \frac{1}{\sqrt{2}} \left( e_i(p)\varepsilon_j(p) + \bar{e}_i(p)\varepsilon_j(p) \right).
\]

where

\[
k - p = k \left( -\sqrt{\frac{v^2 - 1}{4}(-u^2 + v^2 + 1)^2}, \frac{1}{2} \left( u^2 - v^2 + 1 \right) \right),
\]

\[
e_i(k-p) = \left( \frac{u^2 - v^2 + 1}{2u}, 0, \frac{\sqrt{u^4 + 2u^2 - v^2 - 1}}{2u} \right),
\]

\[
\bar{e}_i(k-p) = (0, 1, 0),
\]

\[
p = k \left( \sqrt{\frac{v^2 - 1}{4}(-u^2 + v^2 + 1)^2}, 0, \frac{1}{2} \left( -u^2 + v^2 + 1 \right) \right),
\]

\[
e_i(p) = \left( \frac{-u^2 + v^2 + 1}{2u}, 0, \frac{\sqrt{-u^4 + 2u^2(v^2 + 1) - (v^2 - 1)^2}}{2v} \right),
\]

\[
\bar{e}_i(p) = (0, 1, 0).
\]

**Appendix C: four-point function**

\[
\langle \zeta_{k-p}\zeta_{k'-p'} \rangle = \langle \zeta_{k-p}\zeta_{k'-p'} \rangle + \langle \zeta_{k-p}\zeta'_{k'-p'} \rangle
\]

\[
= \delta (k + k') \frac{(2\pi)^2}{p^2} \left( \delta (p + p') + \delta (k' + p) \right) \mathcal{P}_\zeta((k - p))\mathcal{P}_\zeta(p),
\]

\[
\langle \zeta_{k-p}\zeta_{k'-p'} \rangle = \langle \zeta_{k-p}\zeta_{k'-p'} \rangle
\]

\[
= \delta (k + k') \frac{(2\pi)^2}{p^2} \left( \delta (p + p') \mathcal{P}_\zeta((k - p))\mathcal{P}_\zeta(p) \right),
\]

\[
\langle h^\lambda_{k-p}h^\lambda_{k'-p'} \rangle = \delta (k + k') \frac{(2\pi)^2}{p^2} \left( \delta (p + p') \mathcal{P}_h((k - p))\mathcal{P}_h(p) \right)
\]

\[
\times \mathcal{P}_h(|k - p|)\mathcal{P}_h(p).
\]
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