ON CONTINUOUS SPECTRUM OF MAGNETIC SCHRÖDINGER OPERATORS ON PERIODIC DISCRETE GRAPHS

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ABSTRACT. We consider Schrödinger operators with periodic electric and magnetic potentials on periodic discrete graphs. The spectrum of such operators consists of an absolutely continuous (a.c.) part (a union of a finite number of non-degenerate bands) and a finite number of eigenvalues of infinite multiplicity. We prove the following results: 1) the a.c. spectrum of the magnetic Schrödinger operators is empty for specific graphs and magnetic fields; 2) we obtain necessary and sufficient conditions under which the a.c. spectrum of the magnetic Schrödinger operators is empty; 3) the spectrum of the magnetic Schrödinger operator with each magnetic potential $t\alpha$, where $t$ is a coupling constant, has an a.c. component for all except finitely many $t$ from any bounded interval.

1. INTRODUCTION AND MAIN RESULTS

There are many physical phenomena described by periodic Schrödinger operators. In general, the spectral analysis of a periodic operator $H$ is based on the so-called Floquet decomposition, where this operator has a representation as a direct integral of a family of fiber operators $H(k)$ with discrete spectra, see e.g., [12]. The fiber operator $H(k)$ depends on the so-called quasimomentum $k$ belonging to the torus $T^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$. Its eigenvalues $\lambda_j(k)$, $j = 1, 2, \ldots$, arranged in non-decreasing order, depend on $k$ continuously. The spectrum of the operator $H$ is a union of spectral bands $\sigma_j$, arising as the ranges of the band functions $\lambda_j(\cdot)$, i.e., $\sigma_j = \lambda_j(T^d)$. If some function $\lambda_j(k) = \Lambda_j = \text{const}$ on an open domain of $T^d$, then $\Lambda_j$ is an eigenvalue of $H$ of infinite multiplicity. We call $\{\Lambda_j\}$ a flat band or a degenerate band. All other bands are non-degenerate. The union of all flat bands is the flat spectrum of $H$ and the union of all non-degenerate bands is the a.c. spectrum of $H$. One of important problems is to describe the spectrum: when the operator $H$ has the flat spectrum and when it does not. There are a lot of results about the spectrum of periodic differential operators, see, e.g., [1, 2, 13, 14].

There are some results about the spectrum of discrete Schrödinger operators with periodic electric potentials on periodic graphs, see [6, 9, 11]. It is shown in [11] that the first band of these operators is non-degenerate. Thus, the spectrum of the Schrödinger operators always has an a.c. component. Some classes of periodic graphs on which the discrete Laplacian always has only the a.c. spectrum are described in [6].

We consider discrete Schrödinger operators with periodic magnetic and electric potentials on periodic graphs. This case and even the case of magnetic Laplacians are much more complicated compared to the non-magnetic one. We do not know results about the flat spectrum of the discrete magnetic Schrödinger operators. Our goal is to show that, in contrast
to the non-magnetic case, the a.c. spectrum of the magnetic Schrödinger operators may be empty (for specific graphs and magnetic potentials) and to determine necessary and sufficient conditions for absence of the a.c. spectrum. Moreover, we show that absence of the a.c. spectrum is a quite rare situation, i.e., the spectrum of the magnetic Schrödinger operator with a periodic magnetic potential $t\alpha$, where $t \in \mathbb{R}$ is a coupling constant, has an a.c. component for all except finitely many $t$ from any bounded interval.

1.1. Magnetic Schrödinger operators on periodic graphs. Let $G = (V, \mathcal{E})$ be a connected infinite graph, possibly having loops and multiple edges and embedded into the space $\mathbb{R}^d$. Here $V$ is the set of its vertices and $\mathcal{E}$ is the set of its unoriented edges. Considering each edge in $\mathcal{E}$ to have two orientations, we introduce the set $A$ of all oriented edges. An edge starting at a vertex $u$ and ending at a vertex $v$ from $V$ will be denoted as the ordered pair $(u, v) \in A$ and is said to be incident to the vertices. Let $e = (v, u)$ be the inverse edge of $e = (u, v) \in A$. We define the degree $\kappa_v$ of the vertex $v \in V$ as the number of all edges from $A$, starting at $v$.

Let $\Gamma$ be a lattice of rank $d$ in $\mathbb{R}^d$ with a basis $\{a_1, \ldots, a_d\}$, i.e.,

$$\Gamma = \{a : a = \sum_{s=1}^d n_s a_s, \ n_s \in \mathbb{Z}, \ s \in \mathbb{N}_d\}, \ N_d = \{1, \ldots, d\},$$

and let

$$\Omega = \{ x \in \mathbb{R}^d : x = \sum_{s=1}^d t_s a_s, \ 0 \leq t_s < 1, \ s \in \mathbb{N}_d\}$$

(1.1)

be the fundamental cell of the lattice $\Gamma$. We define the equivalence relation on $\mathbb{R}^d$:

$$x \equiv y \ (\text{mod} \ \Gamma) \iff x - y \in \Gamma \ \forall x, y \in \mathbb{R}^d.$$

We consider locally finite $\Gamma$-periodic graphs $G$, i.e., graphs satisfying the following conditions:

- $G = G + a$ for any $a \in \Gamma$ and the quotient graph $G_\Gamma = G/\Gamma$ is finite.

The basis $a_1, \ldots, a_d$ of the lattice $\Gamma$ is called the periods of $G$. We call the quotient graph $G_\Gamma = G/\Gamma$ the fundamental graph of the periodic graph $G$. The fundamental graph $G_\Gamma$ is a graph on the $d$-dimensional torus $\mathbb{R}^d/\Gamma$. The graph $G_\Gamma = (V_\Gamma, \mathcal{E}_\Gamma)$ has the vertex set $V_\Gamma = V/\Gamma$, the set $\mathcal{E}_\Gamma = \mathcal{E}/\Gamma$ of unoriented edges and the set $\mathcal{A}_\Gamma = \mathcal{A}/\Gamma$ of oriented edges which are finite.

Let $\ell^2(V)$ be the Hilbert space of all square summable functions $f : V \to \mathbb{C}$ equipped with the norm

$$\|f\|_{\ell^2(V)}^2 = \sum_{v \in V} |f(v)|^2 < \infty.$$ 

We consider the magnetic Schrödinger operators $H_\alpha$ acting on $\ell^2(V)$ and given by

$$H_\alpha = \Delta_\alpha + Q,$$ 

(1.2)

where $Q$ is an electric potential and $\Delta_\alpha$ is the magnetic Laplacian having the form

$$(\Delta_\alpha f)(v) = \sum_{e = (v, u) \in \mathcal{A}} (f(v) - e^{i\alpha(e)} f(u)), \quad f \in \ell^2(V), \quad v \in V,$$ 

(1.3)

$\alpha : \mathcal{A} \to \mathbb{R}$ is a magnetic vector potential on $G$, i.e., it satisfies

$$\alpha(e) = -\alpha(e), \quad \forall e \in \mathcal{A}.$$ 

(1.4)
The sum in (1.3) is taken over all edges from $\mathcal{A}$ starting at the vertex $v$. We assume that the magnetic potential $\alpha$ and the electric potential $Q$ are $\Gamma$-periodic, i.e., they satisfy

$$\alpha(e + a) = \alpha(e), \quad Q(v + a) = Q(v), \quad \forall (v, e, a) \in \mathcal{V} \times \mathcal{A} \times \Gamma.$$  

(1.5)

It is known that the magnetic Schrödinger operator $H_\alpha$ is a bounded self-adjoint operator on $\ell^2(\mathcal{V})$ (see, e.g., [3]).

1.2. Spectrum of the magnetic Schrödinger operator. The magnetic Schrödinger operator $H_\alpha$ on $\ell^2(\mathcal{V})$ has the decomposition into a constant fiber direct integral given by

$$\mathcal{H} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \ell^2(\mathcal{V}_e) \, dk, \quad U H_\alpha U^{-1} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} H_\alpha(k) \, dk,$$

(1.6)

$\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, for some unitary operator $U : \ell^2(\mathcal{V}) \to \mathcal{H}$. Here $\ell^2(\mathcal{V}_e) = \mathbb{C}^\nu$ is the fiber space, $\nu = \# \mathcal{V}_e$. The parameter $k \in \mathbb{T}^d$ is called the quasimomentum. The precise expression of the fiber operator $H_\alpha(k)$ is given by (2.15) – (2.16). Note that $H_\alpha(0)$ is the magnetic Schrödinger operator on $\mathcal{G}_\alpha$. Each fiber operator $H_\alpha(k), k \in \mathbb{T}^d$, has $\nu$ real eigenvalues $\lambda_{\alpha,j}(k)$, $j \in \mathbb{N}_\nu$, which are labeled in non-decreasing order (counting multiplicities) by

$$\lambda_{\alpha,1}(k) \leq \lambda_{\alpha,2}(k) \leq \ldots \leq \lambda_{\alpha,\nu}(k), \quad \forall k \in \mathbb{T}^d.$$  

(1.7)

Each $\lambda_{\alpha,j}(\cdot), j \in \mathbb{N}_\nu$, is a real and piecewise analytic function on the torus $\mathbb{T}^d$ and creates the spectral band $\sigma_j(H_\alpha)$ given by

$$\sigma_j(H_\alpha) = [\lambda_{\alpha,j}^-, \lambda_{\alpha,j}^+] \cup \lambda_{\alpha,j}(\mathbb{T}^d).$$

(1.8)

Note that if $\lambda_{\alpha,j}(\cdot) = \Lambda_j = \text{const}$ on some subset of $\mathbb{T}^d$ of positive Lebesgue measure, then the operator $H_\alpha$ on $\mathcal{G}$ has the eigenvalue $\Lambda_j$ of infinite multiplicity. We call $\{\Lambda_j\}$ a flat band. Thus, the spectrum of the operator $H_\alpha$ on the periodic graph $\mathcal{G}$ is given by

$$\sigma(H_\alpha) = \bigcup_{k \in \mathbb{T}^d} \sigma(H_\alpha(k)) = \bigcup_{j=1}^\nu \sigma_j(H_\alpha) = \sigma_{ac}(H_\alpha) \cup \sigma_{fl}(H_\alpha),$$

(1.9)

where $\sigma_{ac}(H_\alpha)$ is the a.c. spectrum, which is a union of non-degenerate bands, and $\sigma_{fl}(H_\alpha)$ is the flat spectrum which is the set of all flat bands (eigenvalues of infinite multiplicity).

It is known (see, e.g., Proposition 4.2 in [3]) that $\lambda_\nu$ is an eigenvalue of infinite multiplicity of the operator $H_\alpha$ if and only if $\lambda_\nu$ is an eigenvalue of the operator $H_\alpha(k)$ for all $k \in \mathbb{T}^d$. It gives another labeling of the eigenvalues of $H_\alpha(k)$. If the operator $H_\alpha$ has $r \geq 0$ eigenvalues of infinite multiplicity

$$\mu_1 \leq \mu_2 \leq \ldots \leq \mu_r, \quad r \leq \nu,$$

(1.10)

then we can separate constant eigenvalues $\mu_j(\cdot) = \mu_j = \text{const}$, of $H_\alpha(\cdot), j \in \mathbb{N}_r$. All other eigenvalues $\mu_j(k), r < j \leq \nu,$ of $H_\alpha(k)$ can be enumerated in non-decreasing order by

$$\mu_{r+1}(k) \leq \mu_{r+2}(k) \leq \ldots \leq \mu_\nu(k), \quad \forall k \in \mathbb{T}^d,$$

(1.11)

counting multiplicities. We define the spectral bands $\mathcal{B}_j(H_\alpha) = [\mu_j^-, \mu_j^+] = \mu_j(\mathbb{T}^d), j \in \mathbb{N}_\nu,$ where each band $\mathcal{B}_j(H_\alpha), r < j \leq \nu,$ is non-degenerate, i.e., $\mu_j^- < \mu_j^+$. Thus, the number of non-degenerate spectral bands of the operator $H_\alpha$ is $\nu - r$. 


1.3. Main results. Recall that the a.c. spectrum of the Schrödinger operator \( H_0 \) with the magnetic potential \( \alpha = 0 \) is not empty, see [11]. The following simple examples show that this is not true for the magnetic Schrödinger operator \( H_\alpha \), i.e., the a.c. spectrum of \( H_\alpha \) on \( \mathcal{G} \) is empty for specific periodic graphs and magnetic potentials \( \alpha \). In the first example we construct the magnetic Schrödinger operators \( H_\alpha \) when their fiber operators \( H_\alpha(k) \) do not depend on the quasimomentum \( k \in \mathbb{T}^d \). It is clear that in this case the spectrum of \( H_\alpha \) is flat, since all band functions are constants.

**Example 1.1.** Let each oriented edge of a periodic graph \( \mathcal{G} \) have multiplicity 2, and let a periodic magnetic vector potential \( \alpha \) on \( \mathcal{G} \) satisfy

\[
|\alpha(e) - \alpha(\tilde{e})| = \pi \quad \text{for each pair of multiple edges } (e, \tilde{e}) \in \mathcal{A}^2.
\]  

Then for any periodic electric potential \( Q \) the magnetic Schrödinger operator \( H_\alpha = \Delta_\alpha + Q \) has the decomposition (1.11), where its fiber operator has the form \( H_\alpha(k) = \text{diag} \left( \alpha_v + Q(v) \right)_{v \in \mathcal{V}_s} \), and \( \alpha_v \) is the degree of the vertex \( v \in \mathcal{V}_s \). In particular, the spectrum of \( H_\alpha \) is flat and is given by \( \{ \alpha_v + Q(v) \}_{v \in \mathcal{V}_s} \).

The second example shows that the condition \( H_\alpha(\cdot) = \text{const} \) is not necessary for absence of the a.c. spectrum of \( H_\alpha \).

\[\begin{aligned}
\mathcal{G} & \quad \text{a)} \quad \mathcal{G}_s & \quad \text{b)}
\end{aligned}\]

**Figure 1.** a) The periodic graph \( \mathcal{G} = (\mathbb{Z}, \mathcal{E}) \); b) the fundamental graph \( \mathcal{G}_s = \mathcal{G}/(2\mathbb{Z}) \).

**Example 1.2.** We consider the periodic graph \( \mathcal{G} = (\mathbb{Z}, \mathcal{E}) \), where the edge set is given by

\[\mathcal{E} = \{e_{n,1} = (n, n + 1) \text{ for all } n \in \mathbb{Z}\} \cup \{e_{n,2} = (n, n + 1) \text{ for all } n \in \mathbb{Z}\}\]

(see Fig[7]b). Let \( H_\alpha = \Delta_\alpha + Q \) be the Schrödinger operator on \( \mathcal{G} \), where a 2-periodic electric potential \( Q \) and a periodic magnetic potential \( \alpha \) are given by

\[
\begin{cases}
Q(0) = 0 \\
Q(1) \in \mathbb{R}
\end{cases}, \quad \alpha(e_{n,s}) = \begin{cases}
\alpha_o, \quad &\text{if } s = 1 \text{ and } n \text{ is even} \\
\pi + \alpha_o, \quad &\text{if } s = 2 \text{ and } n \text{ is even}, \quad n \in \mathbb{Z}, \quad s = 1, 2,
0, \quad &\text{if } s = 1 \text{ and } n \text{ is odd}
\end{cases}
\]  

(1.13)

and \( \alpha_o \in \mathbb{R} \). Then the fiber operator \( H_\alpha(k) \) depends on \( k \in \mathbb{T}^d \), but the spectrum of \( H_\alpha \) is flat and is given by

\[
\lambda_s = 3 + \frac{1}{2} (Q(1) + (-1)^s \sqrt{Q^2(1) + 4}), \quad s = 1, 2.
\]  

(1.14)

The proof of Examples [11, 12] is given in Subsection 2.4.

**Remark.** Below in Theorem 2.5 we determine necessary and sufficient conditions when the spectrum of the magnetic Schrödinger operator \( H_\alpha \) is flat, i.e., the a.c. spectrum of \( H_\alpha \) is empty. In order to present this result we need more definitions.
Examples [1,2] show that all spectral bands of the magnetic Schrödinger operator $H_\alpha$ may be flat. In the following theorem we prove that this is a quite rare situation.

**Theorem 1.3.** Let $H_{t\alpha} = \Delta_{t\alpha} + Q$ be the magnetic Schrödinger operator defined by (1.2) – (1.3) with a periodic magnetic potential $t\alpha$ and a periodic electric potential $Q$ on a periodic graph $G$, where $t$ is a coupling constant. Then the a.c. spectrum of $H_{t\alpha}$ is not empty for all except finitely many $t \in [0, 1]$.

**Remarks.**
1) It is known [1, 13] that the spectrum of the Schrödinger operator with periodic electric and magnetic potentials on $L^2(\mathbb{R}^d)$ is a.c. and the proof is based on the Thomas’ approach. We do not use this argument. In order to prove Theorem 1.3 we determine trace formulas for the magnetic Schrödinger operator (see Theorem 2.4) as Fourier series and use classical function theory. Moreover, as an unperturbed case we use the result from [11] that the first spectral band of the Schrödinger operator without magnetic field is not flat.

2) Theorem 1.3 also holds true for weighted magnetic Laplace and Schrödinger operators, in particular, for the normalized ones. The proof repeats the proof of Theorem 1.3.

1.4. **Historical review.** There are a lot papers about the spectrum of periodic differential operators, see articles [1, 2, 4, 13, 14] and the references therein. The first result in this direction was obtained by Thomas [14]. He proved that each band function of Schrödinger operators with periodic potentials on $L^2(\mathbb{R}^d)$ is not flat. Later on this approach was used in many papers [1, 2, 4, 13]. Danilov [2] proved that the spectrum of the Dirac operator with periodic potentials on $\mathbb{R}^d$ is a.c. Hempel and Herbst [4] proved that the spectrum of magnetic Laplacians with small periodic magnetic vector potentials in $\mathbb{R}^d$ is a.c. Birman and Suslina [1] (for the case $d = 2$), and Sobolev [13] (for $d \geq 2$) proved that the spectrum of the Schrödinger operator with periodic electric and magnetic potentials on $L^2(\mathbb{R}^d)$ is a.c.

In the discrete settings the situation is quite different. The spectrum of the discrete Schrödinger operator with periodic electric and magnetic potentials on periodic graphs consists of a finite number of bands. Some of them may be degenerate. Thus, the spectrum of the discrete Schrödinger operators has an a.c. component which is a union of all non-degenerate bands and a flat component which is the set of all degenerate bands (eigenvalues of infinite multiplicity). In [11] it was proved that the first band of the discrete Schrödinger operators with periodic electric potentials on periodic graphs is always non-degenerate, i.e., the a.c. spectrum is not empty. It was also shown that all except two bands of the Schrödinger operators may be degenerate (see Proposition 7.2 in [9]) and the number of the flat bands may be arbitrary. We do not know results about the flat spectrum of the discrete magnetic Schrödinger operators.

2. **Proof**

2.1. **Edge indices.** For each $x \in \mathbb{R}^d$ we introduce the vector $x_\mathbb{A} \in \mathbb{R}^d$ by

$$x_\mathbb{A} = (x_1, \ldots, x_d), \quad \text{where} \quad x = \sum_{s=1}^d x_s a_s,$$

(2.1)
i.e., $x_\mathbb{A}$ is the coordinate vector of $x$ with respect to the basis $\mathbb{A} = \{a_1, \ldots, a_d\}$ of the lattice $\Gamma$.

For any vertex $v \in \mathcal{V}$ of a $\Gamma$-periodic graph $\mathcal{G}$ the following unique representation holds true:

$$v = v_0 + [v], \quad \text{where} \quad v_0 \in \Omega, \quad [v] \in \Gamma,$$

(2.2)
\( \Omega \) is a fundamental cell of the lattice \( \Gamma \) defined by \([1,1]\). In other words, each vertex \( v \) can be obtained from a vertex \( v_0 \in \Omega \) by a shift by a vector \([v] \in \Gamma \). For any oriented edge \( e = (u, v) \in A \) we define the edge index \( \tau(e) \) as the vector of the lattice \( \mathbb{Z}^d \) given by

\[
\tau(e) = [v]_\Lambda - [u]_\Lambda \in \mathbb{Z}^d,
\]

(2.3)

where \([v] \in \Gamma \) is defined by \((2.2)\) and the vector \([v]_\Lambda \in \mathbb{Z}^d \) is given by \((2.1)\).

The edge indices \( \tau(e) \) and the magnetic potential \( \alpha \) are \( \Gamma \)-periodic, i.e., they satisfy

\[
\tau(e + a) = \tau(e), \quad \alpha(e + a) = \alpha(e), \quad \forall (e, a) \in A \times \Gamma.
\]

(2.4)

On the set \( A \) of all oriented edges of the \( \Gamma \)-periodic graph \( G \) we define the surjection

\[
f : A \to A_\ast = A / \Gamma,
\]

(2.5)

which maps each \( e \in A \) to its equivalence class \( e_\ast = f(e) \) which is an oriented edge of the fundamental graph \( G_\ast \). The identities \((2.4)\) and the mapping \( f \) allow us to define uniquely

- the magnetic potential \( \alpha \) on edges of the fundamental graph \( G_\ast = (\mathcal{V}_\ast, A_\ast) \) which is induced by the magnetic potential \( \alpha \):

\[
\alpha(e_\ast) = \alpha(e) \quad \text{for some } e \in A \quad \text{such that } \ e_\ast = f(e), \quad e_\ast \in A_\ast;
\]

(2.6)

- indices of the fundamental graph edges which are induced by the indices of the periodic graph edges:

\[
\tau(e_\ast) = \tau(e) \quad \text{for some } e \in A \quad \text{such that } \ e_\ast = f(e), \quad e_\ast \in A_\ast.
\]

(2.7)

2.2. Cycle indices and magnetic fluxes. A path \( p \) in a graph \( G = (\mathcal{V}, A) \) is a sequence of consecutive edges

\[
p = (e_1, e_2, \ldots, e_n),
\]

(2.8)

where \( e_s = (v_{s-1}, v_s) \in A, s = 1, \ldots, n, \) for some vertices \( v_0, v_1, \ldots, v_n \in \mathcal{V} \). The vertices \( v_0 \) and \( v_n \) are called the initial and terminal vertices of the path \( p \), respectively. If \( v_0 = v_n \), then the path \( p \) is called a cycle. The number \( n \) of edges in a cycle \( c \) is called the length of \( c \) and is denoted by \(|c|\), i.e., \(|c| = n\).

Remark. A path \( p \) is uniquely defined by the sequence of it’s oriented edges \( (e_1, e_2, \ldots, e_n) \). The sequence of it’s vertices \( (v_0, v_1, \ldots, v_n) \) does not uniquely define \( p \), since multiple edges are allowed in the graph \( G \).

Let \( \mathcal{C} \) be the sets of all cycles of the fundamental graph \( G_\ast \). For any cycle \( c \in \mathcal{C} \) we define

- the cycle index \( \tau(c) \in \mathbb{Z}^d \) by

\[
\tau(c) = \sum_{e \in c} \tau(e), \quad c \in \mathcal{C},
\]

(2.9)

- the flux \( \alpha(c) \in [-\pi, \pi] \) of the magnetic potential \( \alpha \) through the cycle \( c \) by

\[
\alpha(c) = \left( \sum_{e \in c} \alpha(e) \right) \mod 2\pi, \quad c \in \mathcal{C}.
\]

(2.10)

Note that we consider fluxes of the magnetic potential \( \alpha \) modulo \( 2\pi \), since the magnetic potential \( \alpha \) appears in the Laplacian \( \Delta_\alpha \) via the factor \( e^{i\alpha(e)} \), \( e \in A \). For each cycle \( c = \)
(e₁, . . . , eₙ) we define a cycle c = (eᵣ, . . . , e₁). From the definition of indices and fluxes it follows that
\[ \tau(e) = -\tau(e), \quad \alpha(e) = -\alpha(e), \quad \forall e \in A; \]
\[ \tau(c) = -\tau(c), \quad \alpha(c) = -\alpha(c), \quad \forall c \in C. \] (2.11)

2.3. Direct integral. We introduce the Hilbert space, i.e., a constant fiber direct integral,
\[ \mathcal{H} = L² \left( \mathbb{T}^d, \frac{dk}{(2\pi)^d}, \ell²(V_*) \right) = \int_{\mathbb{T}^d} \ell²(V_*) \frac{dk}{(2\pi)^d}, \quad \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d, \] (2.12)
equipped with the norm \( \|g\|²_{\mathcal{H}} = \int_{\mathbb{T}^d} \|g(k, \cdot)\|²_{\ell²(V_*)} \frac{dk}{(2\pi)^d} \), where the function \( g(k, \cdot) \in \ell²(V_*) \) for almost all \( k \in \mathbb{T}^d \). We identify the vertices of the fundamental graph \( G_* = (V_*, E_*) \) with the vertices from the fundamental cell \( \Omega \). The unitary Gelfand transform \( U : \ell²(V) \to \mathcal{H} \) is given by
\[ (Uf)(k, v) = \sum_{m=(m₁, . . . , mₜ) \in \mathbb{Z}^d} e^{-i\langle m, k \rangle} f(v + m₁a₁ + \ldots + mₜaₜ), \quad (k, v) \in \mathbb{T}^d \times V_*, \] (2.13)
where \( \{a₁, . . . , aₜ\} \) is the basis of the lattice \( \Gamma \), and \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \). We recall Theorem 3.1 from [10].

**Theorem 2.1.** The magnetic Schrödinger operator \( H_\alpha = \Delta_\alpha + Q \) on \( \ell²(V) \) has the following decomposition into a constant fiber direct integral
\[ UH_\alpha U^{-1} = \int_{\mathbb{T}^d} H_\alpha(k) \frac{dk}{(2\pi)^d}, \] (2.14)
where \( U : \ell²(V) \to \mathcal{H} \) is the unitary Gelfand transform defined by (2.13), and the fiber magnetic Schrödinger operator \( H_\alpha(k) \) on \( \ell²(V_*) \) has the form
\[ H_\alpha(k) = \Delta_\alpha(k) + Q, \quad \forall k \in \mathbb{T}^d. \] (2.15)
Here \( Q \) is the electric potential on \( \ell²(V_*) \) and \( \Delta_\alpha(k) \) is the fiber magnetic Laplacian given by
\[ (\Delta_\alpha(k)f)(v) = \sum_{e=(u, v) \in A} \left( f(v) - e^{i\langle \alpha(e)+\tau(e), k \rangle} f(u) \right), \quad f \in \ell²(V_*), \quad v \in V_*, \] (2.16)
where \( \tau(e) \) is the index of the edge \( e \in A \) defined by (2.3), (2.7).

From Theorem 2.1 it follows that the fiber magnetic Schrödinger operator \( H_\alpha(k) \) is the \( \nu \times \nu \) matrix defined by (2.15) – (2.16) and in the standard orthonormal basis of \( \ell²(V_*) = \mathbb{C}^\nu \), \( \nu = \#V_*, \) is given by
\[ H_\alpha(k) = q - A_\alpha(k), \quad q = \text{diag}(q_v)_{v \in V_*}, \quad q_v = \kappa_v + Q(v), \] (2.17)
where \( \kappa_v \) is the degree of the vertex \( v \), and the matrix \( A_\alpha(k) = (A_{\alpha,vv}(k))_{u,v \in V_*} \) has the form
\[ A_{\alpha,vv}(k) = \sum_{e=(u,v) \in A} e^{-i\langle \alpha(e)+\tau(e), k \rangle}. \] (2.18)

**Lemma 2.2.** Let \( H_\alpha(k) \) be the fiber operator given by (2.17), (2.18). Then \( \lambda_* \) is an eigenvalue of infinite multiplicity of the magnetic Schrödinger operator \( H_\alpha \) if and only if \( \lambda_* \) is an eigenvalue of \( H_\alpha(k) \) for all \( k \in \mathbb{T}^d \).
Proof. The proof is quite standard. But for the reader’s convenience we repeat it. Let \( \lambda_s \) be an eigenvalue of infinite multiplicity of \( H_\alpha \). Then \( \det (\lambda_s I_\nu - H_\alpha (k)) = 0 \) for all \( k \in \mathcal{B} \), where \( \mathcal{B} \) is a subset of \( \mathbb{T}^d \) of positive Lebesgue measure, and \( I_\nu \) is the identity \( \nu \times \nu \) matrix. The function \( f(k) = \det (\lambda_s I_\nu - H_\alpha (k)) \) is real analytic in \( k \in \mathbb{T}^d \) (moreover, it is an entire function of \( k \in \mathbb{C}^d \)). Then \( f(k) = 0 \) for all \( k \in \mathbb{T}^d \), i.e., \( \lambda_s \) is an eigenvalue of \( H_\alpha (k) \) for all \( k \in \mathbb{T}^d \). The converse is obvious. \( \blacksquare \)

2.4. Proof of Examples. We prove Examples 1.1 and 1.2 where we construct magnetic Schrödinger operators with the empty a.c. spectrum.

Proof of Example 1.1. Let each oriented edge of a periodic graph \( \mathcal{G} \) have multiplicity 2, and let a magnetic vector potential \( \alpha \) on \( \mathcal{G} \) satisfy \( \{1.12\} \). Then for each edge \( e = (u, v) \in \mathcal{A}_e \) of the fundamental graph \( \mathcal{G}_e = (\mathcal{V}_e, \mathcal{A}_e) \) there exists an edge \( \tilde{e} = (u, v) \in \mathcal{A}_e \) such that

\[
\tau(\tilde{e}) = \tau(e), \quad \text{and} \quad |\alpha(e) - \alpha(\tilde{e})| = \pi.
\]

Thus, the matrix \( A_0(k) = (A_0,uv(k))_{u,v \in \mathcal{V}_e} \) given by \( \{2.18\} \) has the form

\[
A_0,uv(k) = \sum_{e = (u,v) \in \mathcal{A}_e} e^{-i(\alpha(e) + \tau(e), k)} = \frac{1}{2} \sum_{e = (u,v) \in \mathcal{A}_e} \left( e^{-i(\alpha(e) + \tau(e), k)} + e^{-i(\alpha(\tilde{e}) + \tau(\tilde{e}), k)} \right) = \frac{1}{2} \sum_{e = (u,v) \in \mathcal{A}_e} e^{-i\tau(e), k} (e^{-i\alpha(e)} + e^{-i\alpha(\tilde{e})}) = 0.
\]

This and \( \{2.17\} \) yield that the fiber magnetic Schrödinger operator \( H_\alpha (k) \) is diagonal and has the form \( H_\alpha (k) = \text{diag} \left( \chi_\nu + Q(v) \right)_{v \in \mathcal{V}_e} \). Thus, the spectrum of \( H_\alpha \) on \( \mathcal{G} \) is flat and is given by \( \{ \chi_\nu + Q(v) \}_{v \in \mathcal{V}_e} \). \( \blacksquare \)

Proof of Example 1.2. The fundamental graph \( \mathcal{G}_e = \mathcal{G}/(2\mathbb{Z}) \) consists of two vertices \( v_0, v_1 \) and three multiple edges \( e_1, e_2, e_3 \) connecting these vertices (Fig. 1b) with indices

\[
\tau(e_1) = \tau(e_2) = 0, \quad \tau(e_3) = 1,
\]

and their inverse edges. The magnetic potential \( \alpha \) on edges of \( \mathcal{G}_e \) is given by

\[
\alpha(e_1) = \alpha_0, \quad \alpha(e_2) = \pi + \alpha_0, \quad \alpha(e_3) = 0,
\]

for some \( \alpha_0 \in \mathbb{R} \). Then the fiber magnetic Schrödinger operator \( H_\alpha (k) \), \( k \in \mathbb{T} \), given by \( \{2.17\} \)–\( \{2.18\} \), on \( \mathcal{G}_e \) has the form

\[
H_\alpha (k) = \left( \begin{array}{cc} 3 & -e^{-i\alpha_0} - e^{i(\pi + \alpha_0)} - e^{-i\pi} \\ -e^{i\alpha_0} - e^{i(\pi + \alpha_0)} - e^{-i\pi} & 3 + Q(1) \end{array} \right) = \left( \begin{array}{cc} 3 & -e^{-ik} \\ -e^{ik} & 3 + Q(1) \end{array} \right),
\]

where \( Q(0) = 0 \). Thus, the eigenvalues of \( H_\alpha (k) \) are given by \( \{1.14\} \). \( \blacksquare \)

2.5. Trace formulas. In order to formulate trace formulas for the fiber magnetic Schrödinger operator \( H_\alpha (k) \), we need some modifications of the fundamental graph \( \mathcal{G}_e \). We add a loop \( e_v \) with index \( \tau(e_v) = 0 \) and the magnetic potential \( \alpha(e_v) = 0 \) at each vertex \( v \) of the fundamental graph \( \mathcal{G}_e = (\mathcal{V}_e, \mathcal{A}_e) \) and consider the modified fundamental graph \( \tilde{\mathcal{G}}_e = (\mathcal{V}_e, \tilde{\mathcal{A}}_e) \), where

\[
\tilde{\mathcal{A}}_e = \mathcal{A}_e \cup \{ e_v \}_{v \in \mathcal{V}_e}.
\]

We denote by \( \tilde{\mathcal{C}} \) the set of all cycles on \( \tilde{\mathcal{G}}_e \). For each cycle \( c \in \tilde{\mathcal{C}} \) we define the weight

\[
\omega(c) = \omega(e_1) \ldots \omega(e_n), \quad \text{where} \quad c = (e_1, \ldots, e_n) \in \tilde{\mathcal{C}},
\]
and $\omega(e)$ is defined by

$$\omega(e) = \begin{cases} -1, & \text{if } e \in A, \\ q_v, & \text{if } e = e_v, \end{cases}, \quad q_v = \alpha_v + Q(v). \tag{2.21}$$

The next lemma shows that the operator $H_\alpha(k)$ can be considered as a fiber weighted magnetic operator on the modified fundamental graph $\tilde{G}_s = (V_s, \tilde{A}_s)$.

**Lemma 2.3.** The fiber magnetic Schrödinger operator $H_\alpha(k) = (H_{\alpha,u,v}(k))_{u,v \in V_s}$ given by (2.17), (2.18) satisfies

$$H_{\alpha,u,v}(k) = \sum_{e = (u,v) \in \tilde{A}_s} \omega(e)e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} \quad \forall u,v \in V_s, \quad \forall k \in \mathbb{T}^d, \tag{2.22}$$

where $\tilde{A}_s$ is the set of all edges of the modified fundamental graph $\tilde{G}_s$ defined by (2.17); $\omega(e)$ is given by (2.21), and $\tau(e)$ is the index of the edge $e \in A_s$ defined by (2.3), (2.7).

**Proof.** Let $u,v \in V_s$. If $u \neq v$, then, using (2.21) and (2.17), (2.18), we have

$$\sum_{e = (u,v) \in \tilde{A}_s} \omega(e)e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} = -\sum_{e = (u,v) \in A_s} e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} = H_{\alpha,u,v}(k).$$

Similarly, if $u = v$, then we obtain

$$\sum_{e = (v,v) \in \tilde{A}_s} \omega(e)e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} = \omega(e_v) - \sum_{e = (v,v) \in A_s} e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} = q_v - \sum_{e = (v,v) \in A_s} e^{-i(\alpha(e) + \langle \tau(e), k \rangle)} = H_{\alpha,vv}(k).$$

Thus, the identity (2.22) has been proved. □

Let $\tilde{C}_{n,m}$ be the set of all cycles from $\tilde{C}$ of length $n$ and with index $m \in \mathbb{Z}^d$:

$$\tilde{C}_{n,m} = \{ c \in \tilde{C} : |c| = n \text{ and } \tau(c) = m \}, \tag{2.23}$$

where $|c|$ is the length of the cycle $c$, and $\tau(c)$ is the index of $c$ defined by (2.9).

In the following theorem we determine all Fourier coefficients of $\text{Tr} H_\alpha^n(k)$ as functions of $k \in \mathbb{T}^d$. This is a crucial point for our consideration.

**Theorem 2.4.** Let $H_\alpha(k)$, $k \in \mathbb{T}^d$, be the fiber magnetic Schrödinger operator defined by (2.15) – (2.16). Then for each $n \in \mathbb{N}$ the trace of $H_\alpha^n(k)$ has the following Fourier series representation

$$\text{Tr} H_\alpha^n(k) = \sum_{m \in \mathbb{Z}^d} h_{\alpha,n,m}e^{-i(m,k)},$$

$$h_{\alpha,n,m} = \sum_{c \in \tilde{C}_{n,m}} \omega(c)e^{-i\alpha(c)}, \quad \text{supp } h_{\alpha,n,m} \subset \{ m \in \mathbb{Z}^d : ||m|| \leq n \tau_+ \}, \tag{2.24}$$

where $\tau_+ = \max_{e \in A_s} ||\tau(e)||$, $\tau(e)$ is the index of the edge $e \in A_s$, and $|| \cdot ||$ is the standard norm in $\mathbb{R}^d$. Here $\tilde{C}_{n,m}$ is defined by (2.23); $\alpha(c)$ is the flux of the magnetic potential $\alpha$ through the cycle $c$ defined by (2.9), and $\omega(c)$ is given by (2.21).
where the coefficients $h$ for $2\pi$ Thus, we have

Proof. Using (2.22), for each $n \in \mathbb{N}$ we obtain

\[
\text{Tr } H_\alpha^n(k) = \sum_{v_1, \ldots, v_n \in \mathbb{V}} H_{\alpha,v_1v_2}(k)H_{\alpha,v_2v_3}(k) \ldots H_{\alpha,v_{n-1}v_n}(k)H_{\alpha,v_nv_1}(k)
\]

\[
= \sum_{v_1, \ldots, v_n \in \mathbb{V}} \sum_{c \in c} \omega(e_1)\omega(e_2) \ldots \omega(e_n)e^{-i(\alpha(e_1)+\alpha(e_2)+\ldots+\alpha(e_n)+\tau(e_1)+\tau(e_2)+\ldots+\tau(e_n),k)}
\]

\[
= \sum_{c \in \tilde{C}_n} \omega(c)e^{-i(\alpha(c)+\tau(c),k)}, \quad \text{where } e_j = (v_j, v_{j+1}), \ j \in \mathbb{N}_n, \ v_{n+1} = v_1,
\]

and $\tilde{C}_n$ is the set of all cycles of length $n$ on $\tilde{G}_\alpha$. Thus, we have the finite Fourier series for $2\pi \mathbb{Z}^d$-periodic function $\text{Tr } H_\alpha^n(k)$, since $\tau(c) \in \mathbb{Z}^d$. We rewrite this Fourier series in the standard form

\[
\text{Tr } H_\alpha^n(k) = \sum_{c \in \tilde{C}_n} \omega(c)e^{-i(\alpha(c)+\tau(c),k)} = \sum_{m \in \mathbb{Z}^d} \sum_{c \in \tilde{C}_n,m} \omega(c)e^{-i\alpha(c)}e^{-im,k}
\]

\[
= \sum_{m \in \mathbb{Z}^d} e^{-im,k} \sum_{c \in \tilde{C}_n,m} \omega(c)e^{-i\alpha(c)} = \sum_{m \in \mathbb{Z}^d} h_{\alpha,n,m}e^{-im,k}, \quad (2.25)
\]

where the coefficients $h_{\alpha,n,m}$ have the form

\[
h_{\alpha,n,m} = \sum_{c \in \tilde{C}_n,m} \omega(c)e^{-i\alpha(c)}. \quad (2.26)
\]

By the definition of the cycle index (2.9), for each cycle $c$ of length $n$ we have

\[
\|\tau(c)\| \leq \sum_{c \in c} \|\tau(c)\| \leq n\tau_+, \quad \text{where } \tau_+ = \max_{c \in A_\alpha} \|\tau(c)\|.
\]

Thus, $h_{\alpha,n,m} = 0$ for all $m \in \mathbb{Z}^d$ such that $\|m\| > n\tau_+$. \quad \blacksquare

2.6. Proof of the main Theorems. We present necessary and sufficient conditions under which the spectrum of the magnetic Schrödinger operators is flat.

Theorem 2.5. Let $H_\alpha = \Delta_\alpha + Q$ be the magnetic Schrödinger operator defined by (1.3) – (1.5) with a periodic magnetic potential $\alpha$ and a periodic electric potential $Q$ on a periodic graph $G$. Then the following statements are equivalent:

(i) The spectrum of $H_\alpha$ is flat.

(ii) For each $n \in \mathbb{N}$, the trace of $H_\alpha^n(k)$ does not depend on $k \in \mathbb{T}^d$.

(iii) The Fourier coefficients $h_{\alpha,n,m}$ of $\text{Tr } H_\alpha^n(k)$ for all $(n, m) \in \mathbb{N}_n \times (\mathbb{Z}^d \setminus \{0\})$ satisfy

\[
h_{\alpha,n,m} = \sum_{e \in \tilde{C}_n,m} \omega(e)e^{-i\alpha(e)} = 0. \quad (2.27)
\]
(iv) For any \( n \in \mathbb{N}_\nu \) the following identities hold true

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \text{Tr}^2 H^n_\alpha(k) dk = |h_{\alpha,n,0}|^2, \quad \text{where} \quad h_{\alpha,n,0} = \sum_{c \in \mathcal{C}_{n,0}} \omega(c)e^{-i\alpha(c)}. \quad (2.28)
\]

Here \( \widetilde{C}_{n,m} \) is defined by (2.22); \( \alpha(c) \) is the flux of the magnetic potential \( \alpha \) through the cycle \( c \) defined by (2.9), and \( \omega(c) \) is given by (2.20).

**Remark.** Theorem 2.5 also determines the necessary and sufficient conditions under which the spectrum of the magnetic Schrödinger operators has an a.c. component.

**Proof.** The determinant of \( (\lambda I_\nu - H_\alpha(k)) \) has the decomposition

\[
\begin{align*}
\det(\lambda I_\nu - H_\alpha(k)) &= \prod_{j=1}^{\nu} (\lambda - \lambda_{\alpha,j}(k)) = \lambda^\nu + \xi_1\lambda^{\nu-1} + \xi_2\lambda^{\nu-2} + \ldots + \xi_{\nu-1}\lambda + \xi_\nu, \quad (2.29)
\end{align*}
\]

where the coefficients \( \xi_j \) are given by (see, e.g., p. 87–88 in [3])

\[
\xi_n = -\frac{1}{n} \left( T_n + \sum_{j=1}^{n-1} T_{n-j} \xi_j \right), \quad T_n = \text{Tr} H^n_\alpha(k), \quad n \in \mathbb{N}_\nu, \quad (2.30)
\]

and, in particular, \( \xi_1 = -T_1, \xi_2 = -\frac{1}{2}(T_2 - T_1^2), \ldots \).

(i) \( \iff \) (ii). Let the spectrum of \( \tilde{H}_\alpha \) be flat. Then all band functions \( \lambda_{\alpha,j}(\cdot), j \in \mathbb{N}_\nu, \) are constant, and, consequently, \( \text{Tr} H^n_\alpha(k) = \sum_{j=1}^{\nu} \lambda_{\alpha,j}(k) \) does not depend on \( k \) for each \( n \in \mathbb{N}_\nu \).

Conversely, let \( \text{Tr} H^n_\alpha(k) \) does not depend on \( k \) for each \( n \in \mathbb{N}_\nu \). From this and (2.29), (2.30) it follows that the determinant \( \det(\lambda I_\nu - H_\alpha(k)) \) does not depend on \( k \). Then all band functions \( \lambda_{\alpha,j}(\cdot), j \in \mathbb{N}_\nu, \) are constant and all spectral bands \( \sigma_j(H_\alpha) \) are degenerate.

(ii) \( \Rightarrow \) (iii). Let \( \text{Tr} H^n_\alpha(k) \) do not depend on \( k \) for each \( n \in \mathbb{N}_\nu \). Then, using the Fourier series (2.24), we obtain

\[
h_{\alpha,n,m} = \sum_{c \in \mathcal{C}_{n,m}} \omega(c)e^{-i\alpha(c)} = 0, \quad \forall (n, m) \in \mathbb{N}_\nu \times (\mathbb{Z}^d \setminus \{0\}).
\]

(iii) \( \Rightarrow \) (ii). Let the condition (2.27) hold true. Then, by Theorem 2.4 we obtain

\[
\text{Tr} H^n_\alpha(k) = h_{\alpha,n,0} = \sum_{c \in \mathcal{C}_{n,0}} \omega(c)e^{-i\alpha(c)}, \quad \forall n \in \mathbb{N}_\nu, \quad (2.31)
\]

i.e., the traces \( \text{Tr} H^n_\alpha(k), n \in \mathbb{N}_\nu \), do not depend on \( k \).

(iii) \( \iff \) (iv). Using the Parseval’s identity for the Fourier series (2.24), we obtain

\[
\frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \text{Tr}^2 H^n_\alpha(k) dk = \sum_{m \in \mathbb{Z}^d} |h_{\alpha,n,m}|^2, \quad \forall n \in \mathbb{N}.
\]

Thus, the condition (2.27) and (2.28) are equivalent. \( \blacksquare \)

We prove Theorem 1.3 about the flat spectrum of the magnetic Schrödinger operators.

**Proof of Theorem 1.3.** The first spectral band \( \sigma_1(H_0) \) of the Schrödinger operator \( H_0 \) without magnetic field is non-degenerate (see Theorem 2.1.ii in [11]). Then the a.c. spectrum of \( H_0 \) is not empty. Then, by Theorem 2.5 there exists \( (n, m) \in \mathbb{N}_\nu \times (\mathbb{Z}^d \setminus \{0\}) \) such that
the Fourier coefficient \( h_{0,n,m} \neq 0 \), where \( h_{0,n,m} \) is defined by (2.24). For this \((n, m)\) and for the magnetic potential \( t\alpha \) we define the function

\[
f(t) := h_{t\alpha,n,m} = \sum_{c \in \tilde{C}_{n,m}} \omega(c) e^{-it\alpha(c)}, \quad t \in \mathbb{R},
\]

and note that the sum is finite. This function has an analytic extension to the whole complex plane. If \( f = \text{const} \), then we obtain \( f = f(0) = h_{0,n,m} \neq 0 \) for such specific \((n, m)\). Then Theorem 2.5 yields that the a.c. spectrum of the operator \( H_{t\alpha} \) is not empty for all \( t \in \mathbb{R} \).

If \( f \neq \text{const} \), then \( f \) has a finite number of zeros on any bounded interval. Then Theorem 2.5 yields that the a.c. spectrum of \( H_{t\alpha} \) is not empty for all except finitely many \( t \in [0,1] \).

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