Ball generated property of direct sums of Banach spaces

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Abstract. A Banach space $X$ is said to have the ball generated property (BGP) if every closed, bounded, convex subset of $X$ can be written as an intersection of finite unions of closed balls. In [1] S. Basu proved that the BGP is stable under (infinite) $c_0$- and $\ell^p$-sums for $1 < p < \infty$. We will show here that for any absolute, normalised norm $\|\cdot\|_E$ on $\mathbb{R}^2$ satisfying a certain smoothness condition the direct sum $X \oplus E Y$ of two Banach spaces $X$ and $Y$ with respect to $\|\cdot\|_E$ enjoys the BGP whenever $X$ and $Y$ have the BGP.

1 Introduction

Let $X$ be a real Banach space. For $x \in X$ and $r > 0$ we denote by $B_r(x)$ the closed ball with center $x$ and radius $r$. The closed unit ball $B_1(0)$ is simply denoted by $B_X$, while $S_X$ stands for the unit sphere. Finally, $X^*$ denotes the dual space of $X$.

$X$ is said to have the ball generated property (BGP) if every closed, bounded, convex subset $C \subseteq X$ is ball generated, i.e. it can be written as an intersection of finite unions of closed balls, formally: there exists $\mathcal{A} \subseteq \mathcal{B}$ such that $\bigcap \mathcal{A} = C$, where

$$\mathcal{B} := \left\{ \bigcup_{i=1}^{n} B_{r_i}(x_i) : n \in \mathbb{N}, r_1, \ldots, r_n > 0, x_1, \ldots, x_n \in X \right\}.$$

The ball topology $b_X$ is defined to be the coarsest topology on $X$ with respect to which every ball $B_r(x)$ is closed. A basis for $b_X$ is given by $\{X \setminus B : B \in \mathcal{B}\} \cup \{X\}$, where $\mathcal{B}$ is as above. Obviously, $X$ has BGP if and only if every closed, bounded, convex subset of $X$ is also closed with respect to $b_X$.

Ball generated sets and the ball topology were introduced by Godefroy and Kalton in [5] but the notions implicitly appeared before in [4]. By [5, Theorem 8.1], every weakly compact subset of a Banach space is ball...
generated. In particular, every reflexive space has the BGP. \( c_0 \) is an example of a nonreflexive space with BGP (see for instance the more general result [1, Theorem 4] on \( c_0 \)-sums). A standard example for a Banach space which fails to have the BGP is \( \ell^1 \) (see the remark at the end of [4]).

We now list some easy remarks on the ball topology (see [5, p.197]; some of them may be used later without further notice):

(i) For every \( y \in X \), the map \( x \mapsto x + y \) is continuous with respect to \( b_X \).

(ii) For every \( \lambda > 0 \), the map \( x \mapsto \lambda x \) is continuous with respect to \( b_X \).

(iii) \( b_X \) is not a Hausdorff topology, but it is a \( T_1 \)-topology (i.e. singletons are closed).

It follows from [5, Theorem 8.3] that \( X \) has the BGP if and only if the ball topology and the weak topology coincide on \( B_X \). For further information on the ball topology, the BGP and related notions, the reader is referred to [1, 3, 5–7] and references therein.

In the paper [1] by S. Basu many stability results for the BGP are established, in particular, for any family \((X_i)_{i \in I}\) of Banach spaces and any \( p \in (1, \infty) \), the \( \ell^p \)-sum \( \bigoplus_{i \in I} X_i \) has BGP if and only if each \( X_i \) has BGP ([1, Theorem 7]). An analogous result holds for \( c_0 \)-sums ([1, Theorem 4]).

In this paper we will study the BGP for direct sums of two spaces only, but with respect to more general norms. We start by recalling the necessary definitions: a norm \( \| \cdot \|_E \) on \( \mathbb{R}^2 \) is called absolute if \( \|(a, b)\|_E = \|(a|, b)\|_E \) for all \((a, b) \in \mathbb{R}^2\), and it is called normalised if \( \|(1, 0)\|_E = \|(0, 1)\|_E = 1 \). We write \( E \) for the normed space \( (\mathbb{R}^2, \| \cdot \|_E) \). For example, the standard \( p \)-norm \( \| \cdot \|_p \) is an absolute, normalised norm for any \( p \in [1, \infty] \). Some important properties of absolute, normalised norms are listed below (see [2, p. 36, Lemma 1 and 2]):

\[
\begin{align*}
(\text{i}) & \quad \|(a, b)\|_\infty \leq \|(a, b)\|_E \leq \|(a, b)\|_1 \quad \forall (a, b) \in \mathbb{R}^2, \\
(\text{ii}) & \quad |a| \leq |c|, \ |b| \leq |d| \Rightarrow \|(a, b)\|_E \leq \|(c, d)\|_E, \\
(\text{iii}) & \quad |a| < |c|, \ |b| < |d| \Rightarrow \|(a, b)\|_E < \|(c, d)\|_E.
\end{align*}
\]

For two Banach spaces \( X \) and \( Y \), their direct sum \( X \oplus_E Y \) with respect to \( \| \cdot \|_E \) is defined as the space \( X \times Y \) endowed with the norm \( \|(x, y)\|_E := \|(x|, y)\|_E \) for \( x \in X \) and \( y \in Y \). This is again a Banach space and convergence in \( X \oplus_E Y \) is equivalent to coordinatewise convergence. For \( \| \cdot \|_E = \| \cdot \|_p \), one obtains the usual \( p \)-direct sum of Banach spaces.

We are going to prove that \( X \oplus_E Y \) has the BGP if \( X \) and \( Y \) have the BGP and the norm \( \| \cdot \|_E \) is Gâteaux-differentiable at \((1, 0)\) and \((0, 1)\). To do so, we will use a description of absolute, normalised norms by the boundary curve of their unit ball, which will be discussed in the next section.
2 Boundary curves of unit balls of absolute norms

The following Proposition is quite probably well-known (moreover, its assertion is intuitively clear) but since the author was not able to find a reference, a formal proof is included here for the readers’ convenience.

**Proposition 2.1.** Let \( \| \cdot \|_E \) be an absolute, normalised norm on \( \mathbb{R}^2 \). Then for every \( x \in (-1,1) \) there exists exactly one \( y \in (0,1) \) such that \( \|(x,y)\|_E = 1 \).

**Proof.** Let \( x \in (-1,1) \). Since the function \( t \mapsto \|(x,t)\|_E \) is continuous with \( \lim_{t \to -\infty} \|(x,t)\|_E = \infty \) and \( \|(x,0)\|_E = |x| < 1 \), it follows that there exists \( y > 0 \) such that \( \|(x,y)\|_E = 1 \). We also have \( y \leq \|(x,y)\|_E = 1 \).

Now we prove the uniqueness assertion. By symmetry it suffices to consider the case \( x \geq 0 \). Suppose there exist \( 0 < y_1 < y_2 \leq 1 \) such that \( \|(x,y_1)\|_E = \|(x,y_2)\|_E = 1 \). Let \( 0 < \lambda < 1 - y_1/y_2 \). It follows that \( z := (x,y_2) + \lambda((1,0) - (x,y_2)) = (x + \lambda(1-x), y_2(1-\lambda)) \) still lies in \( B_E \).

But \( x + \lambda(1-x) > x \) and \( y_2(1-\lambda) > y_1 \), thus by property (iii) of absolute norms listed in the introduction we must have \( \|z\|_E > \|(x,y_1)\|_E = 1 \), which is a contradiction. \( \square \)

We denote by \( f_E \) the function from \((-1,1)\) to \((0,1)\) which assigns to each \( x \in (-1,1) \) the corresponding value \( y \) given by Proposition 2.1. Thus \( \|(x,f_E(x))\|_E = 1 \) for every \( x \in (-1,1) \). The function \( f_E \) will be called the upper boundary curve of the unit ball \( B_E \).

The following properties of \( f_E \) are easily verified: \( f_E \) is a concave (and hence continuous), even function on \((-1,1)\) with \( f_E(0) = 1 \). Further, \( f_E \) is increasing on \((-1,0)\) and decreasing on \([0,1)\). In particular, the limits \( \lim_{x \uparrow 1} f_E(x) \) and \( \lim_{x \downarrow -1} f_E(x) \) exist. Thus we may extend \( f_E \) to a continuous function from \([-1,1]\) to \([0,1]\), which will be again denoted by \( f_E \).

It is possible to characterise properties of the norm \( \|\cdot\|_E \) by corresponding properties of the function \( f_E \). As examples we state below characterisations of strict convexity and strict monotonicity. Once again, this is probably well-known and so the (anyway easy) proofs are omitted, but let us first recall the definitions.

A Banach space \( X \) is strictly convex if \( \|x + y\| = 2 \) and \( \|x\| = \|y\| = 1 \) implies \( x = y \).

An absolute, normalised norm \( \|\cdot\|_E \) on \( \mathbb{R}^2 \) is said to be strictly monotone if the following holds: whenever \( a, b, c, d \in \mathbb{R} \) with \( |a| \leq |c| \) and \( |b| \leq |d| \) and one these inequalities is strict, then \( \|(a,b)\|_E < \|(c,d)\|_E \).

**Proposition 2.2.** Let \( \|\cdot\|_E \) be an absolute, normalised norm on \( \mathbb{R}^2 \).

The space \( E := (\mathbb{R}^2, \|\cdot\|_E) \) is strictly convex if and only if \( f_E \) is strictly concave\(^1\) on \((-1,1)\) and \( f_E(1) = 0 \).

\(^1\)This means \( f_E(\lambda x + (1-\lambda)y) > \lambda f_E(x) + (1-\lambda)f_E(y) \) for all \( \lambda \in (0,1) \) and all \( x, y \in (-1,1) \) with \( x \neq y \).
The norm \( \| \cdot \|_E \) is strictly monotone if and only if \( f_E \) is strictly decreasing on \([0, 1)\) and \( f_E(1) = 0 \).

Next we would like to study the smoothness of \( \| \cdot \|_E \) in terms of differentiability of \( f_E \). This, too, is quite probably known, but the author could not find a reference. Since these results are important for our main result on sums of spaces with the BGP, we will provide them here with complete proofs.

First recall that, since \( f_E \) is concave on \((-1, 1)\), it possesses left and right derivatives \( f'_E- \) and \( f'_E+ \) on \((-1, 1)\) which are decreasing and satisfy \( f'_E- \leq f'_E+ \). Moreover, for every \( x_0 \in (-1, 1) \) and \( a \in \mathbb{R} \) we have

\[
f_E(x) \leq f_E(x_0) + a(x-x_0) \quad \forall x \in (-1, 1) \iff f'_E+(x_0) \leq a \leq f'_E-(x_0). \tag{2.1}
\]

Also, \( f_E \) is differentiable at \( x_0 \in (-1, 1) \) if and only if \( f'_E+ \) is continuous at \( x_0 \) and if and only if \( f'_E- \) is continuous at \( x_0 \). All this follows immediately from the corresponding well-known facts for convex functions, see for example [8, p.113ff].

For \( x \in [-1, 1] \), we will denote by \( S_E(x) \) the set of support functionals at \((x, f_E(x))\), i.e. \( S_E(x) := \{ g \in E^* : \|g\|_{E^*} = 1 = g(x, f(x)) \} \).

**Proposition 2.3.** Let \( x_0 \in (-1, 1) \). For all \( a \in [f'_E+(x_0), f'_E-(x_0)] \) we have \( f_E(x_0) \geq ax_0 + 1 \) and \( S_E(x_0) \) consists exactly of the functionals \( g \) of the form

\[
g(x, y) = \frac{ax - y}{ax_0 - f_E(x_0)} \tag{2.2}
\]

for some \( a \in [f'_E+(x_0), f'_E-(x_0)] \).

**Proof.** Let \( a \in [f'_E+(x_0), f'_E-(x_0)] \). By (2.1) we have \( f_E(x_0) - ax_0 \geq f_E(0) = 1 \).

If \( g \) is defined by (2.2) then it follows from (2.1) that \( g(x, f_E(x)) \leq 1 \) for all \( x \in (-1, 1) \). From this it is easy to deduce that \( g(x, y) \leq 1 \) for all points \((x, y)\) of norm 1, thus \( \|g\|_{E^*} \leq 1 \). Moreover, \( g(x_0, f_E(x_0)) = 1 \), so \( g \in S_E(x_0) \).

Conversely, suppose that \( g \) is a functional belonging to \( S_E(x_0) \). It is of the form \( g(x, y) = Ax + By \) for constants \( A \) and \( B \). We then have

\[
Ax + Bf_E(x) \leq 1 \quad \forall x \in (-1, 1) \quad \text{and} \quad Ax_0 + Bf_E(x_0) = 1. \tag{2.3}
\]

We first prove that \( B > 0 \). If \( B \leq 0 \), then (2.3) implies \( Ax_0 \geq 1 \). In the case \( x_0 > 0 \) we would obtain, by (2.3), \( 1 \geq Ax - Bf_E(x) \geq Ax \geq x/x_0 \) for all \( x \in (0, 1) \), which is a contradiction. A similar argument works for \( x_0 < 0 \). So we must have \( B > 0 \) and hence it follows from (2.3) that

\[
f_E(x) \leq \frac{1}{B} - \frac{A}{B}x \quad \forall x \in (-1, 1).
\]
Since $Ax_0 + Bf_E(x_0) = 1$ we conclude
\[ f_E(x) \leq f_E(x_0) - \frac{A}{B}(x - x_0) \quad \forall x \in (-1, 1). \]

Now (2.1) implies that $a := -A/B$ lies in $[f'_{E+}(x_0), f'_{E-}(x_0)]$.

From $Ax_0 + Bf_E(x_0) = 1$ we obtain $B = 1/(f_E(x_0) - ax_0)$ and hence $A = a/(ax_0 - f_E(x_0))$. Thus $g$ is of the form (2.2).

As is well-known, the norm of a Banach space is Gâteaux-differentiable at a point of norm one if and only if this point has a unique support functional, which is then the Gâteaux-derivative of the norm at this point. Thus the following is an immediate corollary of Proposition 2.3.

**Corollary 2.4.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ and $x_0 \in (-1, 1)$. The norm $\|\cdot\|_E$ is Gâteaux-differentiable at $(x_0, f_E(x_0))$ if and only if $f_E$ is differentiable at $x_0$. In this case, the Gâteaux-derivative of $\|\cdot\|_E$ is given by
\[ (x, y) \mapsto \frac{f'_E(x_0)x - y}{f'_E(x_0)x_0 - f_E(x_0)}. \]

It remains to characterise the support functionals at the end points $(-1, f_E(-1))$ and $(1, f_E(1))$. This requires to distinguish a number of cases. We will state the result below for completeness, but skip the proof (once again, it should be already known).

**Proposition 2.5.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$. Let $a := \inf_{x \in (0, 1)} f'_{E-}(x) \in [-\infty, 0]$.

For $A, B \in \mathbb{R}$ denote by $g_{A,B}$ the functional given by $g_{A,B}(x, y) = Ax + By$. The following holds:

(i) If $f_E(1) > 0$ then $\|\cdot\|_E$ is Gâteaux-differentiable at each point $(1, b)$ with $b \in (-f_E(1), f_E(1))$ and the Gâteaux-derivative at each such point is $g_{1,b}$.

(ii) $f_E(1) = 1$ if and only if $a = 0$ if and only if $\|\cdot\|_E = \|\cdot\|_\infty$. In that case $S_E(1) = \{g_{A,B} : A, B \geq 0 \text{ and } A + B = 1\}$.

(iii) If $a = -\infty$ then $\|\cdot\|_E$ is Gâteaux-differentiable at $(1, f_E(1))$ with $S_E(1) = \{g_{1,0}\}$.

(iv) If $f_E(1) > 0$ and $-\infty < a < 0$, then $g_{A,B} \in S_E(1)$ if and only if $(A, B) = \left(\frac{c}{c - f_E(1)}, \frac{1}{c - f_E(1)}\right)$ for some $c \in (-\infty, a]$ or $(A, B) = (1, 0)$.

(v) If $f_E(1) = 0$ and $-\infty < a < 0$, then $g_{A,B} \in S_E(1)$ if and only if $(A, B) = (1, \pm \frac{1}{c})$ for some $c \in (-\infty, a]$ or $(A, B) = (1, 0)$.
By symmetry arguments, an analogous characterisation holds for the left endpoint \((-1, f_E(-1))\). Let us also remark that similar characterisations of support functionals of absolute, normalised norms (on $\mathbb{C}^2$ even) can be found for example in [2, p.38, Lemma 4]. These characterisations do not use the function $f_E$, but rather the function $\psi$ given by $\psi(t) = \|(1 - t, t)\|_E$ for $t \in [0, 1]$.

3 Direct sums of spaces with the BGP

We start with the following analogue of [1, Lemma 5].

**Lemma 3.1.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ with the following property:

$$\forall \varepsilon > 0 \exists s_0 > \varepsilon \forall s \geq s_0 \|(1, s - \varepsilon)\|_E < s.$$  \hspace{1cm} (3.1)

Let $X,Y$ be Banach spaces and $Z := X \oplus_E Y$. Let $((x_i, y_i))_{i \in I}$ be a net in $B_Z$ which is convergent to 0 in the ball topology $b_Z$. Then $(y_i)_{i \in I}$ converges to 0 in the topology $b_Y$.

Likewise, if $\|\cdot\|_E$ satisfies

$$\forall \varepsilon > 0 \exists s_0 > \varepsilon \forall s \geq s_0 \|(s - \varepsilon, 1)\|_E < s,$$ \hspace{1cm} (3.2)

one can conclude that $(x_i)_{i \in I}$ converges to 0 with respect to $b_X$.

**Proof.** The proof is also analogous to that of [1, Lemma 5]. We suppose that $y_i \not\to 0$ with respect to $b_Y$. Then, by passing to a subnet if necessary, we may assume that there are $y \in Y$ and $r > 0$ such that $y_i \in B_r(y)$ for all $i \in I$ and $0 \in Y \setminus B_r(y)$, i.e. $\|y\| > r$.

Put $\varepsilon := \|y\| - r$. By (3.1) we can find $s = \max\{\varepsilon, \|y\|\}$ such that $t := \|(1, s - \varepsilon)\|_E < s$.

Now if $u \in B_X$ and $v \in B_{s-\varepsilon}(sy/\|y\|)$, then by the monotonicity of $\|\cdot\|_E$,

$$\|(u, v) - (0, sy/\|y\|)\|_E = \|(\|u\|, \|v - sy/\|y\||)\|_E \leq \|(1, s - \varepsilon)\|_E = t,$$

in other words: $B_X \times B_{s-\varepsilon}(sy/\|y\|) \subseteq B_t((0, sy/\|y\|))$.

But for $w \in B_r(y)$ we have

$$\|w - sy/\|y\|| \leq \|w - y\| + \|y - sy/\|y\|| \leq r + s - \|y\| = s - \varepsilon,$$

thus $B_r(y) \subseteq B_{s-\varepsilon}(sy/\|y\|)$.

Altogether it follows that $(x_i, y_i) \in B_t((0, sy/\|y\|))$ for every $i \in I$. But $0 \not\in B_t((0, sy/\|y\|))$, since $t < s$. So the complement of $B_t((0, sy/\|y\|))$ is a $b_2$-neighbourhood of 0 not containing any of the points $(x_i, y_i)$. With this contradiction the proof is finished. \qed
As mentioned in the introduction, $X$ has the BGP if and only if the ball topology and the weak topology of $X$ coincide on $B_X ([5, \text{Theorem 8.3}])$. Thus we can, as in [1], derive the following stability result.

**Corollary 3.2.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ satisfying both (3.1) and (3.2). Let $X$ and $Y$ be Banach spaces with the BGP. Then $X \oplus_E Y$ also has the BGP.

**Proof.** It follows from Lemma 3.1 that for every bounded net $((x_i, y_i))_{i \in I}$ in $X \oplus_E Y$ which is convergent to some point $(x, y)$ in the ball topology we also have $x_i \to x$ and $y_i \to y$ in the respective ball topologies of $X$ and $Y$. Since $X$ and $Y$ have the BGP, it follows that these nets also converge in the weak topology of $X$ resp. $Y$, which in turn implies $(x_i, y_i) \to (x, y)$ in the weak topology of $X \oplus_E Y$. Thus $X \oplus_E Y$ has the BGP. □

It remains to determine which absolute norms satisfy the conditions (3.1) and (3.2). As it turns out, (3.1) resp. (3.2) is equivalent to the Gâteaux-differentiability of $\|\cdot\|_E$ at $(0, 1)$ resp. $(1, 0)$. To prove this we will use the description of the norm by its upper boundary curve $f_E$ from the previous section and the following version of the mean value theorem for one-sided derivatives (see for instance [9, p.204] or [10, p.358] for an even more general statement).

**Theorem 3.3.** Let $I$ be an interval and $f : I \to \mathbb{R}$ a continuous function. Let $J$ be another interval. Suppose that the right derivative $f'_+(x)$ exists and lies in $J$ for all but at most countably many interior points from $I$. Then

$$\frac{f(b) - f(a)}{b - a} \in J \quad \forall a, b \in I \text{ with } a \neq b.$$ 

An analogous statement holds for the left derivative.

**Proposition 3.4.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$. $\|\cdot\|_E$ is Gâteaux-differentiable at $(0, 1)$ resp. $(1, 0)$ if and only if (3.1) resp. (3.2) holds.

**Proof.** We only prove the statement for $(0, 1)$, the other case follows from this one by considering instead of $\|\cdot\|_E$ the norm given by $\|(x, y)\|_F := \|(y, x)\|_E$.

Assume first that $\|\cdot\|_E$ is Gâteaux-differentiable at $(0, 1)$. By Corollary 2.4 the function $f_E$ is differentiable at 0 and the Gâteaux-derivative of $\|\cdot\|_E$ at $(0, 1)$ is given by

$$(x, y) \mapsto -f'_E(0)x + y.$$ 

But this Gâteaux-derivative must be the projection onto the second coordinate, thus $f'_E(0) = 0$. 

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For each real number $s > 0$ we define $f_s(x) := sf_E(x/s)$ for $x \in (-s, s)$. The functions $f_s$ are continuous and differentiable from the right with $f'_s(x) = f'_{E+}(x/s)$.

Let $\varepsilon > 0$. Since $f'_{E+}$ is continuous at 0 (cf. the remarks preceding Proposition 2.3) we can find $\delta \in (0, 1)$ such that $|f'_{E+}(x)| < \varepsilon$ for every $x \in (-\delta, \delta)$. Let $s_0 > \max\{\varepsilon, 1/\delta\}$ and $s \geq s_0$. Then $|f'_{E+}(x)| < \varepsilon$ for all $x \in (0, 1)$ and thus by Theorem 3.3 $|f_s(1) - f_s(0)| < \varepsilon$, hence $f_s(1) > s - \varepsilon$. This implies $\|f(1, s - \varepsilon)\|_E < s$, for otherwise we would have $s = \|(1, f_s(1))\|_E \geq \|(1, s - \varepsilon)\|_E \geq s$, which would mean $f_E(1/s) = 1 - \varepsilon/s$ and thus we would obtain the contradiction $f_s(1) = s - \varepsilon$. This completes one direction of the proof.

To prove the converse we assume that (3.1) holds but $\|\cdot\|_E$ is not Gâteaux-differentiable at $(0, 1)$. Then by Corollary 2.4, the function $f_E$ is not differentiable at 0. Since $f_E$ is increasing on $(-1, 0]$ we have $a := f'_{E-}(0) \geq 0$ and because $f_E$ is even we have $f'_{E+}(0) = -a$. Hence $a > 0$ and by (2.1) $f_E(x) \leq f_E(0) + f'_{E+}(0)x = 1 - ax$ for all $x \in (-1, 1)$.

If we define $f_s$ as above it follows that

$$f_s(x) \leq s - ax \quad \forall x \in (-s, s), \forall s > 0. \tag{3.3}$$

By (3.1) we can choose $s > \max\{1, a\}$ such that $\|(1, s - a)\|_E < s$. Then by (3.3) $f_s(1) \leq s - a$ and hence $s = \|(1, f_s(1))\|_E \leq \|(1, s - a)\|_E < s$. This contradiction finishes the proof. \hfill \square

Putting Corollary 3.2 and Proposition 3.4 together we obtain the final result.

**Corollary 3.5.** Let $\|\cdot\|_E$ be an absolute, normalised norm on $\mathbb{R}^2$ which is Gâteaux-differentiable at $(0, 1)$ and $(1, 0)$. Let $X$ and $Y$ be Banach spaces with the BGP. Then $X \oplus_E Y$ also has the BGP.

This result contains in particular the case of $p$-sums for $1 < p \leq \infty$ that—as we mentioned in the introduction—was already treated in [1] (even for infinite sums). As was also mentioned in [1], the BGP cannot be stable under infinite $\ell^p$-sums (since $\ell^p$ itself does not have the BGP), but it is open whether $X \oplus_1 Y$ has the BGP whenever $X$ and $Y$ have it.

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