NOTES ON FORMAL DEFORMATIONS OF ABELIAN CATEGORIES

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Abstract. In these notes we provide the foundation for the deformation theoretic parts of arXiv:0807.375 and arXiv:math/0102005.

1. Introduction

In these notes we provide the foundation for the deformation theoretic parts of [16, 17]. In [17] we construct non-commutative analogues of quadrics and in [16] we define non-commutative \( \mathbb{P}^1 \)-bundles over commutative varieties. A notable special case of the latter are non-commutative analogues of Hirzebruch surfaces.

Indeed [16] contains a proof that any formal deformation of a Hirzebruch surface (in a suitable sense) is given by a non-commutative Hirzebruch surface. Similarly the original (privately circulated) version of [17] contains a proof that any formal deformation of a quadric is a non-commutative quadric (see [15, §11.2] for a sketch). I deleted this proof when I first put the paper on the arXiv (8 years after it was written) since I was unhappy with the deformation theoretic setup that was used.

Meanwhile a satisfactory infinitesimal deformation theory for abelian categories has been developed in [11, 12]. In the noetherian setting (which is sufficient for the applications we have in mind) the passage from the infinitesimal context to the formal context is an application of Jouanolou’s results in [10]. Nonetheless Jouanolou’s exposé is written for a different purpose so some translation is necessary. After several (not very satisfactory) attempts to rewrite the deformation theoretic parts of [16, 17] using Jouanolou’s language of “AR-J-adic systems” I decided that it was better to write a self contained paper on formal deformations of noetherian abelian categories, which resulted in the current paper. On the purely mathematical level there is very little originality in what we will do. Besides Jouanolou’s exposé we have also borrowed from [3] (which basically discusses trivial deformations) and [8, §5] (which discusses formal schemes). On the expository level we deviate from the aforementioned references by systematically using Pro-objects instead of adic objects. Pro-objects form a co-Grothendieck category so in particular they have very well behaved inverse limits.

We now give a more detailed exposition of our setup. Let \( R \) be a commutative noetherian ring and let \( J \) be an ideal in \( R \).

Definition 1.1. Let \( \mathcal{C} \) be a noetherian \( R \)-linear abelian category. The completion \( \widehat{\mathcal{C}} \) of \( \mathcal{C} \) is the full subcategory of \( \text{Pro}(\mathcal{C}) \) consisting of the pro-objects \( M \) over \( \mathcal{C} \) such

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that $M/MJ^n \in \mathcal{C}$ for all $n$ and such that the canonical map $M \to \text{proj lim}_n M/MJ^n$ is an isomorphism.

We reproduce Jouanolou’s proof (recast in our language) that $\hat{\mathcal{C}}$ is a noetherian abelian category (see Proposition 2.2.5 below).

There is an obvious exact functor $\Phi : \mathcal{C} \to \hat{\mathcal{C}} : M \mapsto \text{proj lim}_n M/MJ^n$ and we say that $\mathcal{C}$ is complete if this functor is an equivalence. Roughly speaking an $(R, J)$-deformation of an $R/J$-linear abelian category $\mathcal{E}$ will be a complete $R$-linear category $\mathcal{F}$ together with an equivalence $\mathcal{E} \cong \mathcal{F}_{R/J}$ where $\mathcal{F}_{R/J}$ is the full subcategory of $\mathcal{F}$ consisting of objects annihilated by $J$. To make this definition work one has to impose certain flatness conditions. See §2.1 and §3 for more details.

So to understand formal deformations of abelian categories we have to understand completion. We first note that completion extends to functors. We show that if $(T^i)_i : \mathcal{C} \to \mathcal{D}$ is a $\partial$-functor between noetherian abelian $R$-linear categories then this functor extends to a $\partial$-functor $(\hat{T}^i)_i : \hat{\mathcal{C}} \to \hat{\mathcal{D}}$ (see Theorem 2.3.1). This is a slight improvement over [10] as Jouanolou imposes some extra conditions on $T^i$ which seem to be superfluous.

We use the good behaviour of $\partial$-functors to study $\text{Ext}$-groups. Let $\mathcal{C}_t$ be the full subcategory of $\mathcal{C}$ consisting of the objects which are annihilated by some power of $J$. Then we define the completed $\text{Ext}$-groups between objects $M, N \in \hat{\mathcal{C}}$ as follows

\begin{equation}
\text{Ext}_C^i(M, N) = \text{Ext}_{\text{Pro}(\mathcal{C}_t)}^i(M, N)
\end{equation}

An alternative point of view to this definition is that we consider the full subcategory $D_c(\mathcal{C})$ of $D(\text{Pro}(\mathcal{C}_t))$ of complexes whose cohomology lies in $\hat{\mathcal{C}}$. Thus $D_c(\mathcal{C})$ has a $t$-structure whose heart is $\hat{\mathcal{C}}$. Then the completed $\text{Ext}$-groups for $M, N \in \hat{\mathcal{C}}$ may be reinterpreted as

\begin{equation}
\text{Ext}_{\mathcal{C}}^i(M, N) = \text{Hom}_{D_c(\mathcal{C})}(M, N[n])
\end{equation}

In the case that $\mathcal{C}$ is the category of torsion $l$-adic constructible sheaves it would be interesting to compare this derived category to the standard derived category of $l$-adic sheaves [4, 5, 7].

Obviously $\text{Ext}_{\mathcal{C}}^i(-, -)$ is a $\partial$-functor in both arguments but apart from this we don’t have anything to say about it. However in the event that $\mathcal{C}_{R/J}$ (the objects in $\mathcal{C}$ annihilated by $J$) has finitely generated $\text{Ext}$-groups over $R/J$ and $\mathcal{C}$ is “formally flat” (see §2.4) then we have the expected formula

\begin{equation}
\text{Ext}_C^i(M, N) = \text{proj lim}_k \text{inj lim}_l \text{Ext}_{\mathcal{C}_{R/J}}^i(M/MJ^l, N/NJ^k)
\end{equation}

An important theorem in algebraic geometry is Grothendieck’s existence theorem [8]. This theorem extends to the current setting (see also [3]). In §4 we introduce the notion of a strongly ample sequence. By definition a sequence $(O(n))_{n \in \mathbb{Z}}$ of objects in a noetherian abelian category $\mathcal{E}$ is strongly ample if the following conditions hold

(A1) For all $M \in \mathcal{E}$ and for all $n$ there is an epimorphism $\oplus_{i=1}^t O(-n_i) \to M$ with $n_i \geq n$.

(A2) For all $M \in \mathcal{E}$ and for all $i > 0$ one has $\text{Ext}_C^i(O(-n), M) = 0$ for $n \gg 0$.

\footnote{We could have used ample sequences [14] but to have good behaviour of higher $\text{Ext}$-groups it is convenient to use a slightly stronger notion.}
A strongly ample sequence \((O(n))_{n \in \mathbb{Z}}\) in \(\mathcal{E}\) is ample in the sense of [14]. Hence using the methods of [2] or [14] one obtains \(\mathcal{E} \cong \text{gr}(A)/f.l.\) if \(\mathcal{E}\) is Hom-finite, where \(A\) is the noetherian \(\mathbb{Z}\)-algebra \(\oplus_{ij} \text{Hom}_S(O(-j), O(-i))\).

The following is our version of Grothendieck’s existence theorem.

**Proposition 1.2.** (see Proposition 4.1) Assume that \(R\) is \(J\)-adically complete. Let \(\mathcal{E}\) be an Ext-finite \(R\)-linear noetherian category with a strongly ample sequence \((O(n))_n\). Then \(\mathcal{E}\) is complete and furthermore if \(\mathcal{E}\) is flat (see §2.1) we have for \(M, N \in \mathcal{E}\):

\[
\text{Ext}_E^i(M, N) = \text{Ext}_E^i(M, N)
\]

The property for a sequence to be strongly ample lifts well under deformations.

**Theorem 1.3.** (an extract of Theorem 4.2) Let \(\mathcal{D}\) be an \(R\)-deformation of an Ext-finite flat \(R/J\)-linear noetherian abelian category \(\mathcal{C}\) and \((O(n))_n\) be a sequence of \(R\)-flat objects in \(\mathcal{D}\). Then \((O(n)/O(n).J)_n\) is strongly ample in \(\mathcal{C}\) if and only if \((O(n))_n\) is strongly ample in \(\mathcal{D}\).

Many algebraic varieties (e.g. Del Pezzo surfaces) have a strongly ample sequence consisting of exceptional objects. Such a sequence can then be lifted to any deformation (see §5). This idea is basically due to Bondal and Polishchuk and is described explicitly in [15, §11.2]. It was used to define non-commutative quadrics in [17] and indirectly in the classification of non-commutative Hirzebruch surfaces in [16]. See also the recent paper [6].

Let us also mention that a very complete treatment of deformations of algebraic varieties as ringed spaces (including their derived categories) over \(k[[t]]\) has been given in [9].

2. Completion of abelian categories

2.1. Base extension. We recall briefly some notions from [12]. Throughout \(R\) will be a commutative noetherian ring and \(\text{mod}(R)\) is its category of finitely generated modules.

Let \(\mathcal{C}\) be an \(R\)-linear abelian category. Then we have bifunctors \(- \otimes_R - : \mathcal{C} \times \text{mod}(R) \to \mathcal{C}\), \(\text{Hom}_R(-, -) : \text{mod}(R) \circ \times \mathcal{C} \to \mathcal{C}\) defined in the usual way. These functors may be derived in their \(\text{mod}(R)\)-argument to yield bi-delta-functors \(\text{Tor}_i^R(-, -), \text{Ext}_R^i(-, -)\). An object \(M \in \mathcal{C}\) is \(R\)-flat if \(M \otimes_R -\) is an exact functor, or equivalently if \(\text{Tor}_i^R(M, -) = 0\) for \(i > 0\).

By definition (see [12, §3]) \(\mathcal{C}\) is \(R\)-flat if \(\text{Tor}_i^R\) or equivalently \(\text{Ext}_R^i\) is effaceable in its \(\mathcal{C}\)-argument for \(i > 0\). This implies that \(\text{Tor}_i^R\) and \(\text{Ext}_R^i\) are universal \(\partial\)-functors in both arguments.

If \(f : R \to S\) is a morphism of commutative rings and \(\mathcal{C}\) is an \(R\)-linear abelian category then \(\mathcal{C}_S\) denotes the (abelian) category of objects in \(\mathcal{C}\) equipped with an \(S\)-action. We usually refer to objects in \(\mathcal{C}_S\) as \((S, \mathcal{C})\)-objects and if \(S\) is graded then we also talk about graded \((S, \mathcal{C})\)-objects. If \(f\) is surjective then \(\mathcal{C}_S\) identifies with the full subcategory of \(\mathcal{C}\) given by the objects annihilated by \(\ker f\). If \(R\) is noetherian and \(S\) is module finite over \(R\) then the inclusion functor \(\mathcal{C}_S \to \mathcal{C}\) has right and left adjoints given respectively by \(\text{Hom}_R(S, -)\) and \(- \otimes_R S\).
2.2. Completion of noetherian abelian categories. Below we refer to a pair
\((R, J)\) where \(R\) is a commutative noetherian ring and \(J \subset R\) is an ideal as a \(J\)-adic
noetherian ring. Below \((R, J)\) is a \(J\)-adic noetherian ring. We put \(R_n = R/J^n\) and
we denote the \(J\)-adic completion of \(R\) by \(\hat{R}\). This is also a noetherian ring. Using
a slight abuse of notation we denote the extended ideal \(J\hat{R}\) by \(J\).

Recall that an abelian category \(\mathcal{C}\) is said to be noetherian if it is essentially small
and all objects are noetherian. Below \(\mathcal{C}\) is an \(R\)-linear noetherian category.

If \(\mathcal{D}\) is an essentially small abelian category then the category \(\text{Pro}(\mathcal{D})\) of pro-
objects over \(\mathcal{D}\) is the category whose objects are filtered inverse systems \((M_\alpha)\) and
whose Hom-sets are given by

\[ \text{Hom}_{\text{Pro}(\mathcal{D})}((M_\alpha)_\alpha, (N_\beta)_\beta) = \text{proj lim}_{\alpha} \text{lim}_{\beta} \text{Hom}_{\mathcal{D}}(M_\alpha, N_\beta) \]

In other words if we identify \(\mathcal{D}\) with the one-object inverse systems in \(\text{Pro}(\mathcal{D})\) then
\((M_\alpha)_\alpha = \text{proj lim}_{\alpha} M_\alpha\) in \(\text{Pro}(\mathcal{D})\).

**Lemma 2.2.1.** [1, §I.8] Assume that \(\mathcal{D}\) is an essentially small abelian category.
Then \(\text{Pro}(\mathcal{D})^{\text{opp}}\) is a Grothendieck category and in particular \(\text{Pro}(\mathcal{D})\) has exact
filtered inverse limits and enough projectives. The natural functor \(\mathcal{D} \to \text{Pro}(\mathcal{D})\)
is fully faithful exact and its essential image is closed under extensions. If \(D \in \mathcal{D}\) then
\(\text{Hom}_{\text{Pro}(\mathcal{D})}(\ldots, D)\) sends inverse limits to direct limits (in other words \(D\) is co-finitely
presented).

**Definition 2.2.2.** The completion \(\widehat{\mathcal{C}}\) of \(\mathcal{C}\) is the full subcategory of \(\text{Pro}(\mathcal{C})\) consisting
of the objects \(M\) such that \(M/MJ^n \in \mathcal{C}\) for all \(n\) and such that the canonical map
\(M \to \text{proj lim}_n M/MJ^n\) is an isomorphism.

It is easy to see that \(\widehat{\mathcal{C}}\) is a \(\hat{R}\)-linear category. To study objects in \(\widehat{\mathcal{C}}\) we need
to consider filtrations. By definition a filtration on an object \(M\) of an \(R\)-linear
category \(\mathcal{D}\) is a descending chain of subobjects \(M = \cdots \supset F_0 M \supset F_1 M \supset \cdots\)

The associated graded objects \(\text{gr}_F M\) is the \(\mathbb{Z}\)-graded object over \(\mathcal{D}\) defined by the
formal direct sum \(\bigoplus_n F_n M/ F_{n+1} M\). By \(F_j\) we denote the \(J\)-adic filtration. I.e.
\(F_j M = MJ^i\) for \(i \geq 0\) and \(F_j M = M\) for \(i \leq 0\).

We say that the filtration \(F\) is adapted to \(J\) if \((F_i M)J \subset F_{i+1} M\) (see [10, §4.2]).
In that case \(\text{gr}_F M\) is a graded \((\text{gr}_J R, \mathcal{D}_{R,J})\)-object.

**Lemma 2.2.3.** If \(M\) is a noetherian object in an \(R\)-linear abelian category \(\mathcal{D}\) and \(S\)
is a positively graded noetherian \(R\)-algebra such that \(S_0 = R\). Then \(M \otimes_R S\) is a
noetherian graded \((S, \mathcal{D})\)-object.

**Proof.** This follows from a variant of Hilbert’s basis theorem. See e.g. [10, Thm 5.1.4] and [10, Lemma 4.2.4].

**Lemma 2.2.4.** (compare with [10, Thm 4.2.6]) Assume that \(K \in \text{Pro}(\mathcal{C})\) is equipped
with a \(J\)-adapted filtration \(F\) such that

1. \(\text{proj lim}_n K/F_n K = K\).
2. \(\text{gr}_F K\) is a noetherian graded \((\text{gr}_J R, \mathcal{C}_{R,J})\)-object.

Then \(K \in \widehat{\mathcal{C}}\).

**Proof.** We follow somewhat the idea of [10, Lemma 4.2.7]. For any \(r \geq 0\) define
\(F_i^{(r)} K = K J_i^{r-i} \cap F_{i-1} K + F_i K\) (with \(J_i^{r-i} = R\) for \(r \geq i\)). Then we have
\(F_i K \subset F_i^{(r)} K \subset F_{i+1} K\) and \((F_i^{(r)} K)J \subset F_i^{(r)} K \subset F_i K\). In other words
\(\text{gr}_F^{(r)} K \overset{\text{def}}{=} \cdots\)
\( \bigoplus_{i=1}^{r} K/F_{i+1}K \) is an ascending chain of graded \((\text{gr}_J, \mathcal{F}_{J+1})\)-subobjects of \(\text{gr}_F K\) which must be stationary. Thus there is an \( r \) such that for all \( i \)

\[
KJ^{i-r} \cap F_{i-1}K + \ F_i K = KJ^{i-r-1} \cap F_{i-1}K + F_i K = \cdots = F_{i-1}K
\]

and in particular \( F_{i-1}K \subseteq KJ^{i-r} + F_i K \). Iterating this inclusion and renumbering we get that there exists an \( r \) such that

\[
F_i K \subseteq KJ^{i-r} + F_j K
\]

for all \( j \geq i \). Fix \( i \) and choose generators \( f_1, \ldots, f_p \) for \( J^{i-r} \). Then we get diagrams for \( j \geq i \)

\[
\begin{array}{ccccccccc}
0 & & K/F_j K & & & & K/(F_j K + J^{i-r} K) & & 0 \\
& & & & & & & & \\
K & & \downarrow & & K/F_i K & & & & \\
& & & & & & & & \\
& & (K/F_j K)^p (f_i) & & K/F_j K & & K/(F_j K + J^{i-r} K) & & 0 \\
& & & & & & & & \\
& & K & & \downarrow & & K/F_i K & & \\
\end{array}
\]

(2.2)

Using exactness of filtered inverse limits we get from (2.2)

\[
\begin{array}{ccccccccc}
0 & & K & & \text{proj lim}_j K/(F_j K + J^{i-r} K) & & 0 \\
& & & & & & & & \\
K^p (f_i) & & K & & \text{proj lim}_j K/(F_j K + J^{i-r} K) & & 0 \\
& & & & & & & & \\
& & K & & \downarrow & & K/F_i K & & \\
\end{array}
\]

and hence from the upper exact sequence we obtain

\[
K/KJ^{i-r} = \text{proj lim}_j K/(F_j K + J^{i-r} K)
\]

In other words the identity map \( K \to K \) induces a map \( K/F_i K \to K/KJ^{i-r} \) which yields \( F_i K \subseteq KJ^{i-r} \).

The fact that \( \text{gr}_F K \) is noetherian implies easily that it has left bounded grading. Since \( C \) is closed under extensions inside \( \text{Pro}(C) \) it follows that \( K/F_i K \in C \) for all \( i \). Furthermore since \( C \) is an abelian subcategory of \( \text{Pro}(C) \) it is also closed under \(- \otimes_R M \) for \( M \in \text{mod}(R) \).

Hence \( K/KJ^i = (K/F_{i+r}K) \otimes_R R/J^i \in C \). Furthermore since the \( J \)-adic filtration and the \( F \)-filtration are cofinal we also get \( \text{proj lim}_n K/KJ^n = \text{proj lim}_n K/F_n K = K \). This shows that indeed \( K \in \hat{C} \). \( \square \)

**Proposition 2.2.5.** (compare with [10, Thm 5.2.3]) \( \hat{C} \) is a noetherian abelian subcategory of \( \text{Pro}(C) \).

**Proof.** We first prove that \( \hat{C} \) is an abelian subcategory of \( \text{Pro}(C) \). It is obviously closed under cokernels (using the exactness of \( \text{proj lim} \) and right exactness of \( - \otimes_R R/J^n \)) so we must prove it is closed under kernels.

Let

\[
0 \to K \to M \to N
\]
be an exact sequence in \( \text{Pro}(C) \) with \( M, N \in \hat{C} \). We must prove \( K \in \hat{C} \). Put \( F_n K \overset{\text{def}}{=} M/J^n \cap K \supseteq K J^n \). This is a filtration on \( K \) which is adapted to \( J \). Furthermore we have exact sequences
\[
0 \to K/F_n K \to M/MJ^n \to N/NJ^n
\]
By exactness of filtered inverse limits we deduce \( K = \text{proj lim}_n K/F_n K \). Furthermore we obtain exact sequences
\[
0 \to \text{gr}_F K \to \text{gr}_{F_j} M \to \text{gr}_{F_j} N
\]
Since \( M/MJ \in C \) it follows from Lemma 2.2.3 that \( \text{gr}_{F_j} M \) is a noetherian graded \((\text{gr}_R, C_{R/J})\) object. Hence \( \text{gr}_F K \) is also a noetherian graded \((\text{gr}_J, C_{R/J})\)-object. By Lemma 2.2.4 we conclude \( K \in \hat{C} \).

It remains to show that \( \hat{C} \) is noetherian. Since any object \( M \) in \( \hat{C} \) satisfies
\[
M = \text{proj lim}_n M/J^n M \quad \text{and the category of } \mathbb{Z}\text{-indexed inverse systems over } C \text{ is essentially small it follows that } \hat{C} \text{ is essentially small as well. Thus it remains to show that any } M \in \hat{C} \text{ is noetherian.}
\]

Let \( N \hookrightarrow M \) be a subobject of \( M \) in \( \hat{C} \). Put \( F_n N = N \cap M J^n \). Then \( N/F_n N \) is the image of \( N/NJ^n \rightarrow M/MJ^n \) and so it lies in \( \hat{C} \). Furthermore taking the inverse limits of the maps
\[
N/NJ^n \longrightarrow N/F_n N \longrightarrow M/MJ^n
\]
and using exactness of filtered inverse limits we get \( N = \text{proj lim}_n N/F_n N \). Now assume that we have inclusions \( N_1 \subseteq N_2 \subseteq M \) in \( \hat{C} \) such that if we equip \( N_1 \), \( N_2 \) with the filtrations induced from the \( J \)-adic filtration on \( M \) then the map \( \text{gr}_F N_1 \rightarrow \text{gr}_F N_2 \) is an isomorphism. We claim that then necessarily \( N_1 = N_2 \). Indeed from the five lemma we obtain \( N_1/F_1 N_1 = N_2/F_1 N_2 \). It then suffices to take inverse limits.

Now let \( M^{(r)} \subseteq M \) be an ascending chain of subobjects and equip them with the filtrations induced from the \( J \)-adic filtration on \( M \). As indicated above \( \text{gr}_{F_j} M \) is a noetherian graded object over \( (\text{gr}_J, C_{R/J}) \) and hence the chain \( (\text{gr}_F M^{(r)}) \) is stationary. By the discussion in the previous paragraph the chain \( (M^{(r)})_r \) is stationary as well.

\[
\tag{2.3}
0 \to K/F_n K \to M/MJ^n \to N/NJ^n
\]

We may compare our definition of \( \hat{C} \) with the notion of \( J \)-adic inverse systems.

**Definition 2.2.6.** (see [10, §3.1]) Let \( C \) be an \( R \)-linear noetherian abelian category. The category of \( J \)-adic inverse systems \( \hat{C} \) over \( C \) is defined as the full subcategory of inverse systems \((M_n, \phi_n)\) over \( C \) such that \( M_n J^n = 0 \) and such that the transition maps \( \phi_n : M_n \to M_{n-1} \) induce isomorphisms \( M_n/M_{n}J^{n-1} \rightarrow M_{n-1} \).

**Proposition 2.2.7.** The functor
\[
\Sigma : \hat{C} \to \hat{C} : M \mapsto (M/MJ^n)_n
\]
is an equivalence of categories. Its inverse is given by
\[
\Psi : \hat{C} \to \hat{C} : (N_n)_n \mapsto \text{proj lim}_n N_n
\]

**Proof.** We first show that \( \Psi \) is well defined. Let \((N_n)_n \in \hat{C}\) and let \( N \) be its inverse limit in \( \text{Pro}(C) \). Using exactness of filtered inverse limits in \( \text{Pro}(C) \) we get
\[
N/NJ^1 = (\text{proj lim}_n N_n) \otimes_R J^1 = \text{proj lim}_n (N_n/N_nJ^1) = N
\]
Thus we have indeed $N = \projlim N_i = \projlim_i N/NJ^i$. From this reasoning we also get $\Psi(N_i)_i = (N_i)_i$.

The fact that $\Psi \Sigma$ is the identity is by definition.

The following easy result motivates the definition of $\hat{C}$.

**Proposition 2.2.8.** One has $\mod(R)^\sim = \mod(\hat{R})$.

**Proof.** In the proof we must distinguish between inverse limits in $\Mod(R)$ and $\Pro(\mod(R))$. Therefore we will temporarily denote the latter by $\proprojlim$.

Let $\proprojlim R/J^n$ be the object of $\mod(R)^\sim$ given by $\proprojlim R/J^n$. Its endomorphism ring is equal to $\projlim_i R/J^i = \hat{R}$. It suffices to prove that $\proprojlim R$ is a projective generator of $\mod(R)^\sim$.

We first show that $\proprojlim R$ is projective. Let $M \in \mod(R)^\sim$. Then

$$\Hom(\proprojlim R, M) = \proprojlim \Hom(R/J^n, M/MJ^n)$$

$$= \proprojlim M/MJ^n$$

Hence we must prove that $M \mapsto \proprojlim M/MJ^n$ is exact. Now let

$$0 \to K \to M \to N \to 0$$

be an exact sequence in $\mod(R)^\sim$. By Lemma 2.2.9 below we have that

$$0 \to K/KJ^n \to M/MJ^n \to N/NJ^n \to 0$$

is exact up to essentially zero systems. From this one easily deduces that its inverse limit is exact.

Now we prove that $\proprojlim R$ is a generator. Let $M$ be a object of $\mod(R)^\sim$. Choose $R/J$-generators for $M/JM \in \mod(R/J)$ and lift those to $R/J^n$-generators for $M/MJ^n \in \mod(R/J^n)$. By Nakayama we get compatible epimorphisms $(R/J^n)^t \to M/MJ^n$ for some fixed $t$.

Taking inverse limits we obtain an epimorphism $\proprojlim R \to M$ and we are done.

The following lemma was used.

**Lemma 2.2.9.** Assume that $\mathcal{D}$ is an $R$-linear abelian category and $M \subset N$ is an inclusion of noetherian objects in $\mathcal{D}$. Then these objects satisfy the Artin-Rees condition in the sense that there exists an $r$ such that for all $i$ we have $NJ^{n+r} \cap M \subset MJ^n$.

**Proof.** This is proved in the standard way. Let $\check{R} = \bigoplus_{n \geq 0} J^n$ be the Rees ring of $R$.

The graded ring $\check{R}$ is finitely generated over $R$ and it follows from Lemma 2.2.3 that $\bigoplus_i N.J^i$ is a noetherian graded object over $(\check{R}, \mathcal{D})$. Hence so is $C = \bigoplus_i N.J^i \cap M$.

Inside $C$ we have an ascending chain of subobjects $C^{(r)} = \bigoplus_i N.J^i \cap MJ^{i-r}$ (with $J^p = R$ for $p \leq 0$) which must be stationary. Hence for a certain $r$ we have for any $i$: $N.J^i \cap MJ^{i-r} = N.J^i \cap MJ^{i-r-1} = \cdots = N.J^i \cap M$. Putting $i = n + r$ yields $MJ^n \supset N.J^{n+r} \cap MJ^n = N.J^{n+r} \cap M$.

There is a canonical functor

$$\Phi : C \to \hat{C} : M \mapsto \proprojlim_n (M/MJ^n)$$
Proposition 2.2.10. The functor $\Phi$ introduced above is exact. It induces an equivalence

$$\mathcal{C}_{R/J^n} \cong (\hat{\mathcal{C}})_{R/J^n}$$

Proof. Exactness is a consequence of Lemma 2.2.9. It is similar to the proof of exactness of $M \mapsto \text{proj lim}_n M/MJ^n$ in the proof of Proposition 2.2.8.

The second statement is a tautology when written out formally. □

Lemma 2.2.11. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a functor between $R$-linear noetherian abelian categories which induces equivalences $\Phi_n : \mathcal{C}_{R/J^n} \to \mathcal{D}_{R/J^n}$. Then

$$\hat{\Phi} : \hat{\mathcal{C}} \to \hat{\mathcal{D}} : M \mapsto \text{proj lim}_n \Phi_n(M/MJ^n)$$

is an equivalence.

Proof. One easily checks that $\hat{\Phi}$ is well defined and that its inverse is given by $\hat{\Phi}^{-1}(N) = \text{proj lim}_n \Phi_n^{-1}(N/NJ^n)$. □

Definition 2.2.12. A noetherian $R$-linear abelian category $\mathcal{C}$ is complete if the functor $\Phi : \mathcal{C} \to \hat{\mathcal{C}}$ is an equivalence.

Proposition 2.2.13. $\hat{\mathcal{C}}$ is complete.

Proof. This follows from the second statement of Proposition 2.2.10 combined with Lemma 2.2.11. □

For completeness let us recall the following result

Lemma 2.2.14. (Nakayama) Let $M \in \hat{\mathcal{C}}$ be such that $MJ = M$. Then $M = 0$.

Proof. We have $M = \text{proj lim}_n M/MJ^n = 0$ since $M = MJ = MJ^2 = \cdots$. □

2.3. Functors. Now we consider functors. Let $T = (T_i)_{i \geq 1}$ be a $\partial$-functor between $R$-linear noetherian abelian categories $\mathcal{C}$ and $\mathcal{D}$. We extend $T$ to a $\partial$-functor $\hat{T}$ commuting with filtered inverse limits between $\text{Pro}(\mathcal{C})$ and $\text{Pro}(\mathcal{D})$.

In this section we prove the following strengthening of [10, Prop. 5.3.1].

Theorem 2.3.1. The functor $\hat{T}$ sends $\hat{\mathcal{C}}$ to $\hat{\mathcal{D}}$.

Proof. This is a variant of [10, Prop. 5.3.1]. For the convenience of the reader we adapt the proof in loc. cit. to our setting.

We need some rudiments from the foundation of the theory of spectral sequences. In its abstract form a spectral sequence over an abelian category $\mathcal{E}$ is a sequence of complexes $E = (E_r^i, d_r)_{r \geq 1}$ together with isomorphisms $H^*(E_r^i, d) \cong (E_{r+1}^*0)$. If the terms of the complexes $E_r^i$ carry a grading then we assume that $d_r$ is homogeneous. Note that this setup is shifted with respect to the usual indexing of spectral sequences. This is more convenient for filtered objects.

Starting from spectral sequence $E$ we may construct subobjects

$$0 = B_r^1 \subset \cdots \subset B_r^n \subset \cdots \subset Z_r^n \subset \cdots \subset Z_r^1 = E_1^n$$

with $E_r^i = Z_r^i/B_r^i$. The subobjects $B_r^n, Z_r^n$ are constructed recursively using the following exact sequences.

$$0 \to Z_r^n/B_r^n \to Z_r^n/B_r^n \xrightarrow{d_r^n} Z_r^{n+1}/B_r^{n+1} \to Z_r^{n+1}/B_r^{n+1} \to 0$$
If \( Z^n = \text{proj lim}_k Z^n_k \), \( B^n \) exist then we say that \( E \) converges to \( E^n_\infty = Z^n_\infty / B^n_\infty \). The graded object \( E^n_\infty \) is called the limit of the spectral sequence.

The spectral sequence is said to degenerate at \( E^n_\infty \) if \( d^n_r = 0 \) for \( r \geq r_0 \). In that case it follows from (2.5) that \( B^n_\infty = B^n_r \), \( Z^n_\infty = Z^n_r \) for \( r \geq r_0 \) and thus \( E^n_\infty \) exists and is equal to \( E^n_\infty \).

If \( (T^n)_n : E \to F \) is a \( \partial \)-functor between abelian categories and \( X \in E \) is an object equipped with a descending filtration \( F_k+1 X \subset F_k X \) indexed by \( Z \) then the method of exact couples yields a spectral sequence starting with \( E^n_1 = T^n(\text{gr}_F X) \).

Here \( T^n(\text{gr}_F X) \) is viewed as a \( Z \)-graded object over \( F \). The expressions for \( Z^n_\infty \) and \( B^n_\infty \) are

\[
Z^n_r = \bigoplus_k \ker(T^n(F_k X/F_{k+1} X) \to T^{n+1}(F_{k+1} X/F_{k+r} X))
\]

\[
B^n_r = \bigoplus_k \text{im}(T^{n-1}(F_{k-r+1} X/F_k X) \to T^n(F_k X/F_{k+1} X))
\]

We now make a number of hypotheses.

(1) \( E, F \) are complete with exact filtered limits.

(2) \( (T^n)_n \) commutes with filtered limits.

(3) We have \( X = F_k X \) for \( k \ll 0 \).

(4) \( X \) is complete. I.e. \( X = \text{proj lim}_k X/F_k X \).

We note that limits and colimits on graded objects can be computed degreewise. Hence \( Z^n_\infty \) exists and is equal to

\[
Z^n_\infty = \bigoplus_k \ker(T^n(F_k X/F_{k+1} X) \to T^{n+1}(F_{k+1} X/F_{k+r} X))
\]

Similarly \( B^n_\infty \) exists and is equal to

\[
B^n_\infty = \bigoplus_k \text{im}(T^{n-1}(X/F_k X) \to T^n(F_k X/F_{k+1} X))
\]

It is now well-known and an easy verification that

\[
\frac{Z^n_\infty}{B^n_\infty} = \bigoplus_k \frac{\text{im}(T^n(F_k X) \to T^n X)}{\text{im}(T^n(F_k X) \to T^n X)}
\]

In other words if we equip \( T^n X \) with the filtration \( F_k(T^n X) = \text{im}(T^n(F_k X) \to T^n X) \) then \( E^n_\infty \) is complete for this filtration. Indeed

\[
\text{proj lim}_k T^n X/F_k T^n X = \text{proj lim}_k \text{coker}(T^n(F_k X) \to T^n X)
\]

\[
= \text{coker}(T^n(\text{proj lim}_k F_k X) \to T^n X)
\]

\[
= T^n X
\]

Now revert to the notations in the statement of the proposition. We apply the previous discussion with \( E = \text{Pro}(C) \), \( F = \text{Pro}(D) \) and \( X = M \). We equip \( M \) with the \( J \)-adic filtration. By the above discussion we get a spectral sequence \( E \) with \( E^n_1 = T^n(\text{gr}_F M) \) which converges to \( \text{gr}_F \hat{T}^n(M) \). The terms occuring in this spectral sequence are graded \( (\text{gr}_J R, D_{R/J}) \) objects. The limit is a priori only a \( (\text{gr}_J R, (\text{Pro} D)_{R/J}) \)-object.
By Lemma 2.2.3 \( \text{gr}_J M \) is a noetherian \((\text{gr}_J \mathcal{R}, \mathcal{C}_{\mathcal{R}/J})\) object. Hence by Lemma 2.3.2 below \( T^n(\text{gr}_J M) \) is a noetherian graded \((\text{gr}_J \mathcal{R}, \mathcal{D}_{\mathcal{R}/J})\)-object. Hence the ascending chain \( B^n_r \) must be stationary. By (2.5) we obtain \( d^n_r = 0 \) for \( r \gg 0 \). Hence \( E \) degenerates at \( E^n_r \) for \( r \gg 0 \). It follows that \( E_\infty^n = \text{gr}_F T^n(M) \) is a noetherian graded \((\text{gr}_J \mathcal{R}, \mathcal{D}_{\mathcal{R}/J})\)-object.

Since we had already shown that \( \hat{T}_n(M) \) is complete we conclude by Lemma 2.2.4.

**Lemma 2.3.2.** Let \( S_0 \) be a commutative noetherian ring and let \( S \) be a finitely generated positively commutative graded \( S_0 \)-algebra whose part of degree zero is \( S_0 \). Let \( (T^i) \), be a \( \partial \) functor between noetherian abelian categories \( \mathcal{E}, \mathcal{F} \). Then for any noetherian graded \((S, \mathcal{E})\)-object \( N \) and for any \( i \) we have that \( T^i(N) \) is a noetherian graded \((S, \mathcal{F})\)-object.

**Proof.** We perform induction on the minimal number of generators \( d \) of \( S \) as \( S_0 \)-algebra. If \( d = 0 \) then \( S = S_0 \) and hence \( N \) is concentrated in a finite number of degrees. In this case \( T^i(N) \) is obviously noetherian.

Now assume \( d > 0 \) and pick a homogeneous generator \( t \) of \( S \) over \( S_0 \) of strictly positive degree \( f \). Then \( N \) can be written as an extension

\[
0 \to N' \to N \to N'' \to 0
\]

where \( N' \) is annihilated by some power of \( t \) and \( N'' \) is \( t \)-torsion free. The object \( N' \) can itself be written as a repeated extension of objects annihilated by \( t \). Thus it suffices to treat the cases where \( N \) is annihilated by \( t \) and where \( N \) is \( t \)-torsion free.

If \( N \) is annihilated by \( t \) then \( T^i(N) \) is noetherian by induction (since \( N \) is now an \( S/tS \)-module and \( S/tS \) has one generator less than \( S \)). Hence we assume that \( N \) is \( t \)-torsion free. From the exact sequence

\[
0 \to N' \xrightarrow{\alpha} N \to N/Nt \to 0
\]

we obtain an injection

\[
T^i(N)/T^i(N)t \hookrightarrow T^i(N/Nt)
\]

By induction \( T^i(N/Nt) \) is noetherian and hence so is \( T^i(N)/T^i(N)t \). From this one easily obtains that \( T^i(N) \) itself is noetherian.

2.4. **Formal flatness.** The notations \( R, J, \mathcal{C} \) are as above. For use below it would be convenient to assume that \( \mathcal{C} \) if flat. Unfortunately even if \( \mathcal{C} \) is \( R \)-flat then there seems to be no a priori reason for \( \hat{\mathcal{C}} \) to be flat (although we do not know an explicit counter example). To work around this issue we make the following definition

**Definition 2.4.1.** The \( R \)-linear category \( \mathcal{C} \) is formally flat if the categories \( \mathcal{C}_{R/J^n} \) are \( R/J^n \)-flat for all \( n \).

Since \( \hat{\mathcal{C}}_{R/J^n} = \mathcal{C}_{R/J^n} \) it immediately follows that if \( \mathcal{C} \) is formally flat then so is \( \hat{\mathcal{C}} \). The following proposition yields a different characterization of formal flatness.

**Proposition 2.4.2.** Let \( \mathcal{C}_{\text{t}} \) be the full subcategory of objects in \( \mathcal{C} \) that are annihilated by some power of \( J \). This is naturally an \( R \)-linear category. Then \( \mathcal{C} \) is formally flat if and only if \( \mathcal{C}_{\text{t}} \) is flat.
Proof. Assume first that $C$ is formally flat. Take $M \in C$, $N \in \text{mod}(R)$. We must prove that $\text{Tor}_i^R(M, N)$ is effaceable in its first argument in $C$ for $i > 0$. We assume $M \in C_{R/J^n}$.

By dimension shifting in $N$ we may reduce to the case $i = 1$. Take an exact sequence in $\text{mod}(R)$

$$0 \to N' \to P \to N \to 0$$

with $P$ projective. Then we get an exact sequence

$$(2.6) \quad 0 \to \text{Tor}_1^R(M, N) \to M \otimes_R N' \to M \otimes_R P \to M \otimes_R N \to 0$$

Let $F$ be the filtration on $N'$ induced from the $J$-adic filtration on $P$. From

$$0 \to N'/F_l N' \to P/PJ^l \to N/NJ^l \to 0$$

we obtain an exact sequence (for $l \geq n$)

$$0 \to \text{Tor}_1^R(M, N/NJ^l) \to M \otimes_R N'/F_l N' \to M \otimes_R P/PJ^l \to M \otimes_R N/NJ^l \to 0$$

Combining these two sequences we see that there is a map

$$(2.7) \quad \text{Tor}_1^R(M, N) \to \text{Tor}_1^{R/J^l}(M, N/NJ^l)$$

natural in $M$ (for $l \geq n$).

By the Artin-Rees condition (Lemma 2.2.9) we may take an $l$ such that $F_l N' \subset N'J^n$. Then from (2.6) we obtain an exact sequence

$$0 \to \text{Tor}_1^R(M, N) \to M \otimes_{R/J^l} N'/F_l N' \to M \otimes_{R/J^l} P/PJ^l \to M \otimes_{R/J^l} N/NJ^l \to 0$$

and thus we have deduced that for $l$ large (2.7) is an isomorphism

$$(2.8) \quad \text{Tor}_1^R(M, N) = \text{Tor}_1^{R/J^l}(M, N/NJ^l)$$

Since $C_{R/J^l}$ is flat $\text{Tor}_1^{R/J^l}(M, N/NJ^l)$ is effaceable in its first argument in $C_{R/J^l}$ by an epimorphism $M' \to M$. Thus we get a commutative diagram

$$\begin{array}{ccc}
\text{Tor}_1^R(M', N) & \longrightarrow & \text{Tor}_1^{R/J^l}(M', N/NJ^l) \\
\downarrow & & \downarrow \\
\text{Tor}_1^R(M, N) & \longrightarrow & \text{Tor}_1^{R/J^l}(M, N/NJ^l)
\end{array}$$

with the right most map being zero. It follows that the left most map is also zero. Thus $\text{Tor}_1^R(M, N)$ is effaceable in $M$ in $C$ and hence $C$ is flat.

Conversely assume $C$ is flat. Since $C_{L,R/J^n} = C_{R/J^n}$ and flatness is stable under base change ([12, Prop. 4.8]) we conclude that $C_{R/J^n}$ is flat. □

By Lemma 2.2.11 the inclusion of abelian categories $C \to \hat{C}$ yields an equivalence

$$\hat{C} \cong \hat{C}$$

Hence when $C$ is formally flat we may always reduce to the case that $C$ is flat.

Flatness on the level of objects does not present any pitfalls as the following proposition shows.

**Proposition 2.4.3.** Let $M \in \hat{C}$. Then $M$ is $R$-flat if and only if $M/MJ^n$ is $R/J^n$-flat for all $n$. 

Proof. We consider the non-obvious direction. Assume that $M \in \hat{C}$ is such that all $M/MJ^n$ are flat. We need to prove that $M \otimes_R -$ is exact.

Consider an exact sequence in $\text{mod}(R)$.

$$0 \to K \to L \to N \to 0$$

We have to show that

$$0 \to M \otimes_R K \to M \otimes_R L \to M \otimes_R N \to 0$$

is exact. After tensoring with $R/J^n$ is is sufficient to show that

$$0 \to M \otimes_R K/KJ^n \to M \otimes_R L/LJ^n \to M \otimes_R N/NJ^n \to 0$$

is exact up to essentially zero systems. This is the same sequence as

$$0 \to M/MJ^n \otimes_R K/KJ^n \to M/MJ^n \otimes_R L/LJ^n \to M/MJ^n \otimes_R N/NJ^n \to 0$$

Hence by flatness of $M/MJ^n$ it is sufficient that

$$0 \to K/KJ^n \to L/LJ^n \to N/NJ^n \to 0$$

is exact up to essentially zero systems. This follows from the Artin-Rees condition (see Lemma 2.2.9). □

2.5. Ext-groups. Now we discuss Ext-groups, by which we always mean Yoneda Ext-groups. We keep the notations from the previous section. Thus $(R, J)$ is a noetherian $J$-adic ring and $C$ is an $R$-linear noetherian abelian category.

We make the following definition

Definition 2.5.1. Let $M, N \in \hat{C}$. Then the completed Ext-groups between $M,N$ are defined by

$$\text{Ext}^i_{\hat{C}}(M, N) = \text{Ext}^i_{\text{pro}C_t}(M, N)$$

It is clear that $\text{Ext}^i_{\hat{C}}(M, N)$ is a $\partial$-functor in both arguments. Apart from this nice property we don’t know if completed Ext-groups are meaningful objects in general. To get better control we will assume that $\hat{C}$ is formally flat and we impose an additional finiteness condition

Definition 2.5.2. An $R$-linear abelian category $D$ is Ext-finite if for all objects $M, N \in D$ we have that $\text{Ext}^i_D(M, N)$ is a finitely generated $R$-module for all $i$.

The following will be the main result of this section

Proposition 2.5.3. Assume that $C$ is formally flat and that $C_{R/J}$ is Ext-finite. Then $\text{Ext}^i_{\hat{C}}(M, N) \in \text{mod}(\hat{R})$ for $M, N \in \hat{C}$ and furthermore

$$\text{Ext}^i_{\hat{C}}(M, N) = \text{proj lim} \text{Ext}^i_{\hat{C}}(M, N/NJ^k)$$

$$= \text{proj lim} \text{inj lim} \text{Ext}^i_{C_{R/J}}(M/MJ^l, N/NJ^k)$$

If $M$ is in addition $R$-flat then

$$\text{Ext}^i_{\hat{C}}(M, N) = \text{proj lim} \text{Ext}^i_{C_{R/J}}(M/MJ^k, N/NJ^k)$$

The proof is a series of lemmas.

Lemma 2.5.4. Assume that $C$ is formally flat and that $C_{R/J}$ is $R/J$-Ext finite then $C_{R/J^n}$ is $R/J^n$-Ext-finite for all $n$. 

Lemma 2.5.6. Assume that $\mathcal{C}$ is formally flat and that $\mathcal{C}_{R/J}$ is Ext-finite. Let $M \in \mathcal{C}$ and $N \in \mathcal{C}_{R/J^n}$. Then $\text{Ext}^i_{\text{Pro}(\mathcal{C}_l)}(M, N)$ is a finitely generated $R/J^n$-module for all $i$. Furthermore we have

\begin{equation}
\text{Ext}^i_{\text{Pro}(\mathcal{C}_l)}(M, N) = \text{Ext}^i_{\mathcal{C}_{R/J^n}}(M/M^i, N)
\end{equation}

and

\begin{equation}
\text{Ext}^i_{\text{Pro}(\mathcal{C}_l)}(M, N) = \text{Ext}^i_{\mathcal{C}_{R/J^n}}(M, N)
\end{equation}

Finally if $M$ is flat over $R$ then

\begin{equation}
\text{Ext}^i_{\text{Pro}(\mathcal{C}_l)}(M, N) = \text{Ext}^i_{\mathcal{C}_{R/J^n}}(M/M^i, N)
\end{equation}

Proof. If $\mathcal{C}_l$ is flat then so is $\text{Pro}(\mathcal{C}_l)$ (see [12, Prop. 3.6] for the dual statement). Now we use the spectral sequence (for $l \geq n$)

\begin{equation}
E^{pq}_2(l) : \text{Ext}^p_{\text{Pro}(\mathcal{C}_l)}(\text{Tor}^R_q(M, R/J^l), N) \Rightarrow \text{Ext}^{p+q}_{\text{Pro}(\mathcal{C}_l)}(M, N)
\end{equation}

which may derived in a similar way as [12, Prop. 4.7] (the existence depends on flatness of $\text{Pro}(\mathcal{C}_l)$). The formation of Pro-objects commutes with certain base changes (see [12, Prop. 4.5] for the dual statement) and in particular $\text{Pro}(\mathcal{C}_{R/J}) = \text{Pro}(\mathcal{C}_{R/J^l})$.

Since $\text{Tor}^R_q(M, R/J^l)$ lies both in $\mathcal{C}$ and is annihilated by $J^l$ it lies in $\mathcal{C}_{R/J^n}$. For an object $K \in \mathcal{C}_{R/J}$ we have $\text{Ext}^i_{\text{Pro}(\mathcal{C}_{R/J^n})}(K, N) = \text{Ext}^i_{\mathcal{C}_{R/J^n}}(K, N)$ (see [12, Prop. 2.14]). Hence the spectral sequence (2.15) becomes

\begin{equation}
E^{pq}_2(l) : \text{Ext}^p_{\mathcal{C}_{R/J^n}}(\text{Tor}^R_q(M, R/J^l), N) \Rightarrow \text{Ext}^{p+q}_{\text{Pro}(\mathcal{C}_l)}(M, N)
\end{equation}

To prove finite generation we put $l = n$ and invoke Lemma 2.5.4.

To prove (2.12) we note that for $l \leq l'$ there are maps of spectral sequence $E(l) \to E(l')$ which are given by the compositions

$\text{Ext}^p_{\mathcal{C}_{R/J^n}}(\text{Tor}^R_q(M, R/J^l), N) \to \text{Ext}^p_{\mathcal{C}_{R/J^n}}(\text{Tor}^R_q(M, R/J^{l'}), N) \to \text{Ext}^p_{\mathcal{C}_{R/J^n}}(\text{Tor}^R_q(M, R/J^l), N)$

It follows from Lemma 2.5.6 below that $E^{pq}_2(l) \to E^{pq}_2(l')$ is zero for $q > 0$ and $l \ll l'$ (taking into account that $M$ is a noetherian object in $\mathcal{C}$). Taking a direct limit over $l$ of (2.16) we find that indeed

$\text{inj lim}_{l} \text{Ext}^p_{\mathcal{C}_{R/J^n}}(M/M^l, N) = \text{Ext}^{p+q}_{\text{Pro}(\mathcal{C}_l)}(M, N)$

The claim (2.13) follows from Lemma 2.5.7 below together with (2.12). The claim (2.14) follows from the degeneration of the spectral sequence (2.16).

We have used the next two lemmas.

Lemma 2.5.6. Let $\mathcal{D}$ be an $R$-linear abelian category and assume that $M \in \mathcal{D}$ is a noetherian object. Then $\text{Tor}^R_i(M, R/J^n)$ is an essentially zero system for $i > 0$.
Proof. We first replace $\mathcal{D}$ with the smallest abelian subcategory of $\mathcal{D}$ containing $M$. This is a noetherian abelian category. Then we replace $\mathcal{D}$ by its category of Ind-objects. Then $\mathcal{D}$ becomes a locally noetherian Grothendieck category and $M$ is still a noetherian object in $\mathcal{D}$. In particular $\text{Ext}^i_{\mathcal{D}}(M, -)$ commutes with filtered colimits.

As a Grothendieck category $\mathcal{D}$ has enough injectives. Hence we have to show that for an arbitrary injective object $E \in \mathcal{D}$ we have

$$\text{inj lim}_n \text{Hom}_{\mathcal{D}}(\text{Tor}_R^i(M, R/J^n), E) = 0$$

Now we have

$$\text{inj lim}_n \text{Hom}_{\mathcal{D}}(\text{Tor}_R^i(M, R/J^n), E) = \text{inj lim}_n \text{Ext}^i_{\mathcal{D}}(M, \text{Hom}_R(R/J^n, E))$$

Thus we have to show that $F = \text{inj lim}_n \text{Hom}_R(R/J^n, E)$ is injective. In a locally noetherian Grothendieck category we can test this on inclusions of noetherian objects. Hence let $K \hookrightarrow M$ be such an inclusion. We need to prove that $\text{inj lim}_n \text{Hom}(M/MJ^n, E) \rightarrow \text{inj lim}_n \text{Hom}(K/KJ^n, E)$ is an epimorphism, or equivalently that the kernel of $K/KJ^n \rightarrow M/MJ^n$ is an essentially zero system. This follows from the Artin-Rees condition (Lemma 2.2.9). \hfill \Box

Lemma 2.5.7. Let $\mathcal{D}$ be a noetherian $R$-linear category and assume that $N \in \mathcal{D}$ is annihilated by $J^n$. Let $M \in \mathcal{D}$. Then we have

$$\text{Ext}^i_{\mathcal{D}}(M, N) = \text{inj lim}_t \text{Ext}^i_{\mathcal{D}_{R/J^t}}(M/MJ^t, N)$$

Proof. From the Artin-Rees condition (Lemma 2.2.9) we know that $(- \otimes_R R/J^t)_t$ is exact up to essentially zero systems. From this we obtain in the usual way that $M \mapsto \text{inj lim}_t \text{Ext}^i_{\mathcal{D}_{R/J^t}}(M/MJ^t, N)$ is a $\partial$-functor with values in $R$-modules. To show that this $\partial$-functor coincides with $\text{Ext}^i_{\mathcal{D}}(M, N)$ it is sufficient to prove this for $i = 0$ and to show that any element of $\text{inj lim}_t \text{Ext}^i_{\mathcal{D}_{R/J^t}}(M/MJ^t, N)$ is effaceable for $i > 0$. The case $i = 0$ is trivial so assume that $a \in \text{Ext}^i_{\mathcal{D}_{R/J^t}}(M/MJ^t, N)$ represents an element $\bar{a}$ of $\text{inj lim}_i \text{Ext}^i_{\mathcal{D}_{R/J^t}}(M/MJ^t, N)$ for $i > 0$.

There exists an epimorphism $T \rightarrow M/MJ^t$ which effaces $a$ for some $T \in \mathcal{D}_{R/J^t}$. Let $M'$ be the pullback of $T$ for the map $M \rightarrow M/MJ^t$. Then the epimorphism $M'/M'J^t \rightarrow M/MJ^t$ factors through $T$ and hence effaces $a$. This means that $M' \rightarrow M$ effaces $\bar{a}$

Lemma 2.5.8. Assume that $\mathcal{C}$ is formally flat and that $\mathcal{C}_{R/J^t}$ is Ext-finite and let $M, N \in \mathcal{C}_t$. Then

$$\text{proj lim}_n \text{Ext}^i_{\text{Pro}(\mathcal{C})}(M, N/NJ^n) \in \text{mod}(R)^{-} \quad (\cong \text{mod}(\hat{R}))$$

Proof. This follows by applying Theorem 2.3.1 to the $\partial$-functor

$$\text{Ext}^i_{\text{Pro}(\mathcal{C})}(M, -)_t : \mathcal{C}_t \rightarrow \text{mod}(R)_t$$

To construct this functor we use Lemma 2.5.5. \hfill \Box
Proof of Proposition 2.5.3. Let \( P_* \) be a projective resolution of \( M \) in \( Pro(\mathcal{C}) \). Since \( \mathcal{C} \) is flat by Proposition 2.4.2 the \( P_m \) are \( R \)-flat (see [12, Prop. 3.4] for the dual version). We compute

\[
\text{Ext}^i_{Pro(\mathcal{C})}(M, N) = H^i(\text{Hom}_{Pro(\mathcal{C})}(P_*, \text{proj lim}_n N/NJ^n))
\]

We need to exchange \( H^i \) and \( \text{proj lim}_n \). This is possible if the terms of the inverse system of complexes \( \text{Hom}_{Pro(\mathcal{C})}(P_*, N/NJ^n) \) as well as its cohomology satisfy the Mittag-Leffler condition. For the terms this follows from the projectivity of \( P_m \). For the cohomology which is equal to \( \text{Ext}^i_{Pro(\mathcal{C})}(M, N/NJ^n) \) we invoke Lemma 2.5.8 together with Lemma 2.5.9 below.

\[\text{Assuming this we now obtain}\]

\[
\text{Ext}^i_{Pro(\mathcal{C})}(M, N) = \text{proj lim}_n H^i(\text{Hom}_{Pro(\mathcal{C})}(P_*, N/NJ^n))
\]

This implies (2.9) via (2.13). Furthermore we obtain (2.10) via (2.14). Finally we obtain (2.11) from (2.13). \qed

The following lemma was used.

Lemma 2.5.9. Let \((U_n)_n\) be an inverse system in an \( R \)-linear noetherian abelian category \( \mathcal{D} \) such that \( U_nJ^n = 0 \) and such that the pro-object \((U_n)_n\) lies in \( \hat{\mathcal{D}} \). Then \((U_n)_n\) satisfies the Mittag-Leffler condition.

Proof. Let \( U \) be the pro-object in \((U_n)_n\). Define \( C_n, K_n \in \mathcal{D} \) as the kernel and cokernel of the natural maps.

\[
U/UJ^n \to U_n
\]

Taking inverse limits in \( \text{Pro(mod}(R)) \) we see that \((U_n)_n\) and \((C_n)_n\) are zero pro-objects, or equivalently they are essentially zero systems. From one easily deduces that \((U_n)_n\) satisfies the Mittag-Leffler condition. \qed

2.6. The complete derived category. We use the same notations as above. In particular \((R, J)\) is a \( J \)-adic noetherian ring and \( \mathcal{C} \) is an \( R \)-linear noetherian abelian category. We define \( D_\mathcal{C}(\mathcal{C}) \) as the full subcategory of \( D(Pro(\mathcal{C})) \) of complexes whose cohomology lies in \( \hat{\mathcal{C}} \). Thus \( D_\mathcal{C}(\mathcal{C}) \) has a \( t \)-structure whose heart is \( \hat{\mathcal{C}} \). Then the completed Ext-groups for \( M, N \in \hat{\mathcal{C}} \) may be reinterpreted as

\[
\text{Ext}\mathcal{C}(M, N) = \text{Hom}_{D_\mathcal{C}(\mathcal{C})}(M, N[n])
\]

In the case that \( \mathcal{C} \) is the category of torsion \( l \)-adic constructible sheaves it would be interesting to compare this derived category to the standard derived category of \( l \)-adic sheaves [4, 5, 7].

3. Formal deformations of abelian categories

Let \( R \) be a noetherian \( J \)-adic ring and \( \mathcal{C} \) a flat \( R/J \)-linear noetherian abelian category. Then we define an \( R \)-deformation of \( \mathcal{C} \) to be a formally flat complete \( R \)-linear abelian category \( \mathcal{D} \) together with an equivalence \( \mathcal{D}_{R/J} \cong \mathcal{C} \). It follows from the above discussion that \( \mathcal{D} \) is specified up to equivalence by specifying the flat
$R/J^n$-linear categories $\mathcal{D}_n = \mathcal{D}_{R/J^n}$ together with the equivalences (isomorphisms in this case) $\mathcal{D}_m = \mathcal{D}_{n,R/J^m}$ for $n \geq m$ and $\mathcal{D}_1 \cong \mathcal{C}$.

Denote by $\text{Def}_R(\mathcal{C})$ the class of $R$-deformations of $\mathcal{C}$. This is a 2-groupoid. The observations in the previous paragraph may be used to construct a 2-equivalence

$$\text{Def}_R(\mathcal{C}) \cong \text{3-proj lim } \text{Def}_{R/J^n}(\mathcal{C})$$

We leave it to the interested reader to formalize this statement. It will not be used in this form.

4. Ampleness

We define what we mean by a strongly ample sequence. This is stronger than strictly necessary but easier to work with.

Let $\mathcal{E}$ an noetherian abelian category. For us a sequence $(O(n))_{n \in \mathbb{Z}}$ of objects in $\mathcal{E}$ is strongly ample if the following conditions hold

(A1) For all $M \in \mathcal{E}$ and for all $n$ there is an epimorphism $\oplus_{i=1}^n O(-n_i) \to M$ with $n_i \geq n$.

(A2) For all $M \in \mathcal{E}$ and for all $i > 0$ one has $\text{Ext}_i^\mathcal{E}(O(n), M) = 0$ for $n \gg 0$.

A strongly ample sequence $(O(n))_{n \in \mathbb{Z}}$ in $\mathcal{E}$ is ample in the sense of [14]. Hence using the methods of [2] or [14] one obtains $\mathcal{E} \cong \text{qgr}(A)$ if $\mathcal{E}$ is Hom-finite, where $A$ is the noetherian $\mathbb{Z}$-algebra $\oplus_{ij} \text{Hom}_\mathcal{E}(O(-j), O(-i))$.

Below we fix a complete noetherian $J$-adic ring $R$. The following is a version of Grothendieck’s existence theorem [8].

**Proposition 4.1.** Let $\mathcal{E}$ be an Ext-finite $R$-linear noetherian category with a strongly ample sequence $(O(n))_{n \in \mathbb{Z}}$. Then $\mathcal{E}$ is complete and furthermore if $\mathcal{E}$ is flat we have for $M, N \in \mathcal{E}$:

$$(4.1) \quad \text{Ext}_i^\mathcal{E}(M, N) = \text{Ext}_i^\mathcal{E}(M, N)$$

**Proof.**

**Step 1.** We first claim that $\mathcal{E}$ satisfies Nakayama’s lemma. This would follow from Lemma 2.2.14 once we knew $\mathcal{E}$ is complete but we are not there yet.

Let $M \in \mathcal{E}$ me such that $MJ = M$. Pick generators for $a_1, \ldots, a_n$ for $J$ and consider the corresponding epimorphisms

$$M \oplus_n (a_i) \to M \to 0$$

Applying $\text{Hom}_\mathcal{E}(O(-m), -)$ for $m$ large we get an epimorphism

$$\text{Hom}_\mathcal{E}(O(-m), M) \oplus_n (a_i) \to \text{Hom}_\mathcal{E}(O(-m), M) \to 0$$

and thus by Nakayama’s lemma for $R$ and Ext-finiteness

$$\text{Hom}_\mathcal{E}(O(-m), M) = 0$$

for large $m$. It then follows from (A1) that $M = 0$.

**Step 2.** Let $M$ be an object in $\mathcal{E}$. We claim that for $i > 0$ and for large $m$ we have

$$(4.2) \quad \text{Ext}_i^\mathcal{E}(O(-m), MJ^n) = 0$$

for all $n$ and furthermore for large $m$ we also have

$$(4.3) \quad \text{Hom}_\mathcal{E}(O(-m), MJ^n) = \text{Hom}_\mathcal{E}(O(-m), M)J^n$$
for all \( n \).

Put \( \tilde{R} = R \oplus J \oplus J^2 \cdots \). According to Lemma 2.2.3 \( \tilde{M} = M \oplus MJ \oplus MJ^2 \cdots \) is a noetherian graded \((\tilde{R}, \mathcal{E})\)-object.

Now we follow a similar inductive method as in the proof of Lemma 2.3.2. We only give a sketch. Put \( W = \tilde{M} \). We need to prove \( \text{Ext}^i_\mathcal{E}(O(-m), W) = 0 \) for large \( m \). We deduce this from the noetherian property of \( W \). Let \( d \) be the minimal number of generators of \( \tilde{R} \) over \( R \) (i.e. the number of generators of the \( R \)-ideal \( J \)). If \( d = 0 \) there is nothing to prove. If \( d > 0 \) then we pick a homogeneous generator \( t \) of strictly positive degree in \( \tilde{R} \) and we reduce to the cases \( Wt = 0 \) and \( W \) is \( t \)-torsion free. The first case is clear by induction and in the second case we obtain (also by induction) \( \text{Ext}^i_\mathcal{E}(O(-m), W) = \text{Ext}^i_\mathcal{E}(O(-m), W) \) for large \( m \). The conclusion now follows from the fact that \( t \) has strictly positive degree.

To prove (4.3) we have to prove that \( \text{Hom}_\mathcal{E}(O(-m), W) \) is generated in degree zero for \( m \) large. We use induction based on the noetherian property of \( W \) and the fact that \( W \) is generated in degree zero. If \( d = 0 \) then \( W \) must be concentrated in degree zero and there is nothing to prove. If \( d \neq 0 \) then we pick \( t \) as above. Using Ext-vanishing (which is already proved) we obtain for \( m \) large an exact sequence

\[
\text{Hom}_\mathcal{E}(O(-m), W) \xrightarrow{\times t} \text{Hom}_\mathcal{E}(O(-m), W) \to \text{Hom}_\mathcal{E}(O(-m), W/Wt) \to 0
\]

We now finish by induction and the fact that \( t \) has strictly positive degree.

**Step 3.** Now we show that the functor

\[
\Phi : \mathcal{E} \to \tilde{\mathcal{E}}
\]

is an equivalence. If \( M, N \in \mathcal{E} \) then since \( \Phi(M) \) is the pro-object \( M/MJ^m \) and similarly for \( N \) we deduce from (2.1) we have

\[
\text{Hom}_\mathcal{E}(\Phi(M), \Phi(N)) = \proj\lim_{k} \inj\lim_{l} \text{Hom}_{\mathcal{E}_{R/J^k}}(M/MJ^l, N/NJ^k) = \proj\lim_{k} \text{Hom}_\mathcal{E}(M, N/NJ^k)
\]

Thus we have to show that the natural map

\[
(4.4) \quad \text{Hom}_\mathcal{E}(M, N) \to \proj\lim_{n} \text{Hom}_\mathcal{E}(M, N/NJ^n)
\]

is an isomorphism. Using left exactness of \( \proj\lim \) and (A1) we immediately reduce to \( M = O(-m) \) where \( m \) may be chosen arbitrarily large.

We first observe that (4.4) must be a monomorphism. Suppose on the contrary that there is an \( f : M \to N \) such that \( K = \text{im} f \subseteq NJ^n \) for all \( n \). Then by Lemma 2.2.9 we have \( K = KJ \). Hence \( K = 0 \) by Step 1 which implies \( f = 0 \).

Now we prove that (4.4) is an epimorphism when \( M = O(-m) \) for \( m \) large. Suppose we are given a compatible system of maps \( f_n : M \to N/NJ^n \). By (4.2) the \( f_n \) may be lifted to maps \( f'_n : M \to N \) such that \( \text{im}(f'_n - f'_{n+1}) \subseteq NJ^n \). Using (4.2) this implies \( f'_n - f'_{n+1} \in \text{Hom}_\mathcal{E}(M, N)J^n \). Since \( \text{Hom}_\mathcal{E}(M, N) \in \text{mod}(R) \) the limit \( f = \lim_n f'_n \) exists. It has the property that the image of \( f \) in \( \text{Hom}_\mathcal{E}(M, N/NJ^n) \) is equal to \( f_n \). This proves what we want.

**Step 4.** Finally we prove (4.1). To show that \( \text{Ext} = \text{Ext} \) in \( \mathcal{E} \) it is sufficient to show that \( \text{Hom} = \text{Hom} \) and furthermore that \( \text{Ext} \) is effaceable in its first argument. The fact that \( \text{Hom} = \text{Hom} \) is the fact that (4.4) is an isomorphism. So let us show that \( \text{Ext} \) is effaceable.
Let $N$ be an object in $\mathcal{E}$. If $m$ is large and $i > 0$ then it follows from (4.2) that $\text{Ext}^i(\mathcal{O}(-m), N/NJ^n) = 0$ for all $n$. Hence $\text{Ext}^i(\mathcal{O}(-m), N) = 0$ by (2.9). Now let $M \in \mathcal{E}$. To efface $\text{Ext}^i(M, N)$ we take an epimorphism $\bigoplus_{i=1}^t \mathcal{O}(-n_i) \to M$ with the $n_i$ sufficiently large. This finishes the proof. □

Now we fix an Ext-finite flat $R/J$-linear noetherian abelian category $\mathcal{C}$ and an $R$-deformation $\mathcal{D}$ of $\mathcal{C}$.

**Theorem 4.2.** Let $\mathcal{O}(n)_n$ be a sequence of $R$-flat objects in $\mathcal{D}$. Then

1. $(\mathcal{O}(n)/\mathcal{O}(n)J)_n$ is strongly ample in $\mathcal{C}$ if and only if $\mathcal{O}(n)_n$ is strongly ample in $\mathcal{D}$.
2. If $(\mathcal{O}(n)/\mathcal{O}(n)J)_n$ is strongly ample then $\mathcal{D}$ is flat (instead of just formally flat).
3. $\mathcal{D}$ is Ext-finite as $R$-linear category.

**Proof.** Property (2) follows immediately from (A1) and (1). To prove (1) we note that since the $\mathcal{O}(n)$ are flat we have

\[(4.5) \quad \text{Ext}^i(\mathcal{O}(n), M) = \text{Ext}^i(\mathcal{O}(n)/\mathcal{O}(n)J, M)\]

for $M \in \mathcal{C}$. From this it is easy to see that if $\mathcal{O}(n)_n$ is strongly ample in $\mathcal{C}$ then $(\mathcal{O}(n)/\mathcal{O}(n)J)_n$ is strongly ample in $\mathcal{C}$. So our main task is to prove the converse.

So assume that $(\mathcal{O}(n)/\mathcal{O}(n)J)_n$ is strongly ample. We will first show that $\mathcal{O}(n)_n$ is strongly ample if we replace Ext by $\text{Ext}$ in the definition of strongly ample.

Let $M$ be a noetherian object in $\mathcal{D}$. Put $S = R/J \oplus J^2 \oplus \cdots$. According to Lemma 2.2.3 $N = M/MJ \oplus MJ/MJ^2 \oplus \cdots$ is a noetherian graded $(S, C)$-object.

We claim that for $m$ large we have

\[(4.6) \quad \text{Ext}^i_D(\mathcal{O}(-m), MJ^n/MJ^{n+1}) = 0 \quad \text{for all } n\]

This is again proved using a similar inductive method as in the proof of Lemma 2.3.2 and Proposition 4.1 (Step 2). We leave the proof to the reader.

In particular we find that for $m$ large one has

\[\text{Ext}^i_D(\mathcal{O}(-m), M/MJ^n) = 0 \quad \text{for all } n\]

Taking inverse limits and using Proposition 2.5.3 we find $\text{Ext}^i_D(\mathcal{O}(-m), M) = 0$ for all $i > 0$. Hence this proves (A2) for $\text{Ext}$.

We now prove (A1) (which does not involve any Ext). We first find $m_0$ such that (4.6) holds for $i = 1$ and $m \geq m_0$. Using (A1) for $\mathcal{C}$ we find that there is an epimorphism $F \overset{\text{def}}{=} \bigoplus_{i=1}^t \mathcal{O}(-n_i) \to M/MJ$ with $n_i \geq m$. We may lift this map to a compatible series of maps $F \to M/MJ^n$. Taking the inverse limit yields a map $F \to M$. By Nakayama’s lemma (see Lemma 2.2.14) it follows that this must be an epimorphism.

So we have proved ampleness with $\text{Ext}$ replacing Ext. Now we claim that in fact $\text{Ext} = \text{Ext}$ in $\mathcal{D}$. This is proved in the same way as Step 4 of the proof of Proposition 4.1 (using condition (A1) which was already proved).

Property (3) follows from Proposition 2.5.3. □

5. **Lifting and base change**

The usual lifting results for infinitesimal deformations generalize without difficulty to formal deformations. As usual $R$ is a $J$-adic noetherian ring and we assume...
that \( D \) is an \( R \)-deformation of a noetherian Ext-finite \( R/J \)-linear flat abelian category \( C \). To simplify the notation we assume \( C = D_{R/J} \).

**Proposition 5.1.** Let \( M \in C \) be a flat object such that \( \text{Ext}^i_C(M, M \otimes_{R/J} J^n/J^{n+1}) = 0 \) for \( i = 1, 2 \) and \( n \geq 1 \). Then there exists a unique \( R \)-flat object (up to non-unique isomorphism) \( \overline{M} \in D \) such that \( \overline{M}/\overline{M}J \cong M \).

**Proof.** This follows in a straightforward way from the infinitesimal lifting criterion for objects (see [13, Theorem A] for the dual version). \( \square \)

**Proposition 5.2.** Let \( \overline{M}, \overline{N} \in D \) be flat objects and put \( \overline{M}/\overline{M}J = M, \overline{N}/\overline{N}J = N \). Assume that for all \( X \) in \( \text{mod}(R/J) \) we have \( \text{Ext}^i_C(M, N \otimes_{R/J} X) = 0 \) for a certain \( i > 0 \). Then we have \( \text{Ext}^i_D(\overline{M}, \overline{N} \otimes_R X) = 0 \) for all \( X \in \text{mod}(R) \).

**Proof.** This follows in a straightforward way from [12, Prop. 6.13]. \( \square \)

**Proposition 5.3.** Let \( \overline{M}, \overline{N} \in D \) be flat objects and put \( \overline{M}/\overline{M}J = M, \overline{N}/\overline{N}J = N \). Assume that for all \( X \) in \( \text{mod}(R/J) \) we have \( \text{Ext}^i_C(M, N \otimes_{R/J} X) = 0 \). Then \( \text{Hom}_D(\overline{M}, \overline{N}) \) is \( R \)-flat and furthermore for all \( X \) in \( \text{mod}(R) \) we have \( \text{Hom}_D(\overline{M}, \overline{N} \otimes_R X) = \text{Hom}_D(\overline{M}, \overline{N}) \otimes_R X \).

**Proof.** This is routine. Choose a short exact sequence

\[
0 \rightarrow Y \rightarrow P \rightarrow X \rightarrow 0
\]

with \( P \) a finitely generate projective. We then get an exact sequence

\[
0 \rightarrow \text{Hom}_D(\overline{M}, \overline{N} \otimes_R Y) \rightarrow \text{Hom}_D(\overline{M}, \overline{N} \otimes_R P) \rightarrow \text{Hom}_D(\overline{M}, \overline{N} \otimes_R X) \rightarrow \text{Ext}_D^1(\overline{M}, \overline{N} \otimes_R Y)
\]

By the Proposition 5.2 and the hypotheses we get \( \text{Ext}_D^1(\overline{M}, \overline{N} \otimes_R Y) = 0 \). We then obtain a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_D(\overline{M}, \overline{N} \otimes_R Y) \\
\alpha_Y & & \downarrow \\
\text{Hom}_D(\overline{M}, \overline{N} \otimes_R Y) & \longrightarrow & \text{Hom}_D(\overline{M}, \overline{N} \otimes_R P) & \longrightarrow & \text{Hom}_D(\overline{M}, \overline{N} \otimes_R X) & \longrightarrow & 0 \\
\alpha_X & & \downarrow & & \downarrow & & \downarrow \\
\text{Hom}_D(\overline{M}, \overline{N} \otimes_R Y) & \longrightarrow & \text{Hom}_D(\overline{M}, \overline{N} \otimes_R P) & \longrightarrow & \text{Hom}_D(\overline{M}, \overline{N} \otimes_R X) & \longrightarrow & 0
\end{array}
\]

We obtain that \( \alpha_X \) is an epimorphism for all \( X \). But then \( \alpha_Y \) is an epimorphism from which we then deduce that \( \alpha_X \) is an isomorphism. But then \( \alpha_Y \) is an isomorphism and hence the lower right exact sequence is in fact exact. Thus \( \text{Hom}_R(\overline{M}, \overline{N}) \) is \( R \)-flat, finishing the proof. \( \square \)

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