Harder–Narasimhan Filtrations which are not split by the Frobenius maps

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Abstract. We will produce a smooth projective scheme $X$ over $\mathbb{Z}$, a rank 2 vector bundle $V$ on $X$ with a line subbundle $L$ having the following property. For a prime $p$, let $F_p$ be the absolute Frobenius of $X_p$, and let $L_p \subset V_p$ be the restriction of $L \subset V$. Then for almost all primes $p$, and for all $t \geq 0$, $(F^*_p)^t L_p \subset (F^*_p)^t V_p$ is a non-split Harder-Narasimhan filtration. In particular, $(F^*_p)^t V_p$ is not a direct sum of strongly semistable bundles for any $t$. This construction works for any full flag variety $G/B$, with semisimple rank of $G \geq 2$. For the construction, we will use Borel–Weil–Bott theorem in characteristic 0, and Frobenius splitting in characteristic $p$.

Keywords. Frobenius splitting; Borel–Weil–Bott theorem; strong Harder–Narasimhan Filtrations.

1. Introduction

Let $X$ be a smooth projective variety over a perfect field $k$ of characteristic $p > 0$, and $V$ be a vector bundle over $X$. Recall that in characteristic $p$, a vector bundle $W$ is called strongly semistable if $(F^n)^*(W)$ is semistable for all $n$, where $F : X \rightarrow X$ is the absolute Frobenius. If $X$ is a curve and $V$ is not strongly semistable, then for some Frobenius pullback $(F^n)^* V$ is a direct sum of strongly semistable bundles (Proposition 2.1 of [2] or Corollary 5.2 of [5]). A natural question to ask is whether this still holds for $\dim X > 1$. Biswas et al. in [1] showed that there is always a counterexample to this over any algebraically closed field of positive characteristic which is uncountable. However, we will produce a smooth projective variety over $\mathbb{Z}$ and a rank 2 vector bundle on it, which, restricted to each prime $p$ in a nonempty open subset of $\text{Spec} \, \mathbb{Z}$, constitutes a counterexample over $p$. Indeed, given any split semisimple simply connected algebraic group $G$ of semisimple rank $> 1$ over $\mathbb{Z}$, we will show that there exists a smooth projective homogeneous space $X_Z$ over $\mathbb{Z}$ and a vector bundle $V$ on $X_Z$ of rank 2 such that for each prime $p$ in a nonempty open subset of $\text{Spec} \, \mathbb{Z}$, the restriction $V \otimes \mathbb{F}_p$ as a vector bundle over $X_Z \otimes \mathbb{F}_p$ is a counterexample. We only use the Borel–Weil–Bott theorem in characteristic 0 (see [3]) and Frobenius splitting of $G/B$ in characteristic $p$ (see [6]).
Let $X_Z = G/B$, where $G$ is a split semisimple simply connected algebraic group over $\mathbb{Z}$ with semisimple rank $> 1$, and $B$ a Borel subgroup containing a split maximal torus $T$. This $X_Z$ is defined over $\mathbb{Z}$, and let $X_Q$ denote its pullback to Spec $\mathbb{Q}$. We will produce a line bundle $L_Z$ over $X_Z$ such that if $L_Q$ denotes the corresponding line bundle on $X_Z$, then $H^1(X_Q, L_Q) \neq 0$. Since the base change $\text{Spec } \mathbb{Q} \to \text{Spec } \mathbb{Z}$ is flat, we have

$$H^1(X_Z, L_Z) \otimes \mathbb{Q} = H^1(X_Q, L_Q).$$

Therefore $H^1(X_Z, L_Z) \neq 0$. By semicontinuity, $H^1(X_p, L_p) \neq 0$ where $X_p$ is the fibre over $\mathbb{Z}/p\mathbb{Z}$ for any prime $p$, since the subset of Spec $\mathbb{Z}$ where the first cohomology vanishes is a closed set and contains the generic point given by $\mathbb{Q}$. Again, since $X_Z$ is projective and thus $H^1(X_Z, L_Z)$ is a finite $\mathbb{Z}$-module, it must have a torsion free part. Take a basis of the free part, and choose one element from the basis. This element, which we will call $\theta \in H^1(X_Z, L_Z)$, remains nonzero under the natural map $H^1(X_Z, L_Z) \otimes \mathbb{F}_p \to H^1(X_p, L_p)$ for all primes $p$ lying in a nonempty open subset of Spec $\mathbb{Z}$ (because, for example, by semi-continuity, the dimension of $H^1(X_p, L_p)$ is constant on an open subset of Spec $\mathbb{Z}$, and therefore the natural map is an isomorphism on that open set). Under the identification $H^1(X_Z, L_Z) = \text{Ext}^1(\mathcal{O}_{X_Z}, L_Z)$, let the element $\theta$ denote the extension

$$0 \to L_Z \to V \to \mathcal{O} \to 0.$$

Since for each $p$, the restriction $X_p$ of our scheme $X_Z$ is Frobenius split, if $M$ is any quasicoherent sheaf, the natural map $H^1(X_p, M) \to H^1(X_p, F_p M) = H^1(X, F_p M)$ is actually an injection, where we denote by $F_p$ the absolute Frobenius of $X_p$. Therefore, the image of the element $\theta$ in $H^1(X_p, L_p)$ remains nonzero after successive application of $F_p$, and the exact sequence it stands for is thus always non-split even after successive application of $F_p^s$.

We will select a very ample line bundle $H$ on $X_Z$ in such a way that this $L$ will have degree positive, so that $V$ is not semistable at all; since the quotient of $V$ by $L_Z$ is $\mathcal{O}_{X_Z}$, we see that $0 \to L_p \to V_p$ remains the Harder–Narasimhan filtration after successive pullbacks by $F_p$. Since this is not split, this will produce our desired example. After we had put this paper on the arXiv, Adrian Langer informed us that similar examples can be constructed by blowing up suitable codimension 2 subschemes of $\mathbb{P}^2$.

2. Construction of the line bundle $L$

We will apply Borel–Weil–Bott theorem. Let $G$ be a split semisimple simply connected algebraic group over $\mathbb{Z}$, $B$ a Borel subgroup containing a split maximal torus $T$ of rank $n > 1$. We define $X_Z = G/B$. Therefore, one always has a dominant weight $\lambda_0$ with respect to a fixed basis of simple roots, and $\rho$ being the half sum of positive roots, $w$ being a length one element of the Weyl group, $s := w(\lambda_0 + \rho) - \rho$ has the property that $w(s + \rho) - \rho$ is dominant. Hence, by Borel–Weil–Bott theorem, we have the line bundle $L_Q$ defined over $X_Q$ such that $H^1(X_Q, L_Q) \neq 0$. Note that this $L_Q$ comes from a character of a split maximal torus of $G$, so that $L_Q$ is also defined over $\mathbb{Z}$ i.e. there is a line bundle $L_Z$ defined over $X_Z$ whose restriction to $X_Q$ is $L_Q$. 
3. Selection of the very ample line bundle $H$

Given any character $\lambda$ of the maximal torus $T$, we have the line bundle $L_\lambda$ on $X_Z$, and this gives an isomorphism $X^*(T) \cong \text{Pic}(X_Z)$. Then $\text{Pic}(X_Z)$ admits a basis of line bundles coming from the simple roots $\omega_1, \ldots, \omega_n$; these simple roots (and therefore the basis of $\text{Pic}(X_Z)$) has the property that a line bundle on $X_Z$ is very ample if and only if it is a strictly positive linear combination of the basis elements. Note that Kempf’s vanishing theorem (Theorem 3.1 of [4]) is independent of the particular choice of simple roots (if $\Delta$ is a base for roots, then so is $-\Delta$). Therefore the argument in §1 shows that if $L = \sum_i m_i \omega_i$, and $H^1(X_Q, L_Q) \neq 0$, all of $m_i$ are nonzero and there is $0 < r < n$ such that, by suitable reordering the indices, $m_1, \ldots, m_r$ are all positive, and the rest are negative.

Let $N_1, \ldots, N_r$ be positive integers. Then the line bundle $H = \sum_{i \leq r} N_i \omega_i + \sum_{i > r} \omega_i$ is very ample. The degree of $L$ with respect to this very ample line bundle $H$ is $L \cdot H^{d-1}$, where $d = \dim X_Z$. This can be made positive by choosing $N_i$ to be large. Indeed, all $\omega_i$ are generated by global sections, so that any monic monomial of degree $d$ will represent an effective zero cycle of nonnegative degree.

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