A NOTE ON $\sigma$-POINT AND NONTANGENTIAL CONVERGENCE

JAYANTA SARKAR

Abstract. In this article, we generalize a theorem of Victor L. Shapiro concerning nontangential convergence of the Poisson integral of a $L^p$-function. We introduce the notion of $\sigma$-points of a locally finite measure and consider a wide class of convolution kernels. We show that convolution integrals of a measure have nontangential limits at $\sigma$-points of the measure. We also investigate the relationship between $\sigma$-point and the notion of the strong derivative introduced by Ramey and Ullrich. In one dimension, these two notions are the same.

1. Introduction

In this article, by a measure $\mu$ we will always mean a complex Borel measure or a signed Borel measure such that the total variation $|\mu|$ is locally finite, that is, $|\mu|(K)$ is finite for all compact sets $K$. If $\mu(E)$ is nonnegative for all Borel measurable sets $E$ then $\mu$ will be called a positive measure. The notion of Lebesgue point of a measure was defined by Saeki [3] which we recall. A point $x_0 \in \mathbb{R}^n$ is called a Lebesgue point of a measure $\mu$ on $\mathbb{R}^n$ if there exists $L \in \mathbb{C}$ such that

\begin{equation}
\lim_{r \to 0} \frac{|\mu - L m(B(x_0, r))|}{m(B(0, r))} = 0,
\end{equation}

where $B(x_0, r)$ denotes the open ball of radius $r$ with center at $x_0$ with respect to the Euclidean metric and $m$ denotes the Lebesgue measure of $\mathbb{R}^n$. In this case, the symmetric derivative of $\mu$ at $x_0$,

\begin{equation}
D_{sym} \mu(x_0) := \lim_{r \to 0} \frac{\mu(B(x_0, r))}{m(B(0, r))}
\end{equation}

exists and is equal to $L$. The set of all of Lebesgue points of a measure $\mu$ is called the Lebesgue set of $\mu$. It is not very hard to see that the Lebesgue set of a measure $\mu$ includes almost all (with respect to the Lebesgue measure) points of $\mathbb{R}^n$ (see Proposition 2.3). Given a measure $\mu$ on $\mathbb{R}^n$, its Poisson integral $P\mu$ on the upper half space $\mathbb{R}_+^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ is defined by the convolution

$P\mu(x,t) = \int_{\mathbb{R}^n} P(x - \xi, t) \, d\mu(\xi),$
whenever the integral exists. Here, the kernel \( P(x, t) \) is the usual Poisson kernel of \( \mathbb{R}^{n+1}_+ \) given by the formula
\[
P(x, t) = c_n \frac{t}{(t^2 + \|x\|^2)^\frac{n+1}{2}}, \quad c_n = \pi^{-(n+1)/2} \Gamma \left( \frac{n + 1}{2} \right).
\]

It is known that if the integral above exists for some \((x_0, t_0) \in \mathbb{R}^{n+1}_+\) then it exists for all points in \( \mathbb{R}^{n+1}_+ \) and defines a harmonic function in \( \mathbb{R}^{n+1}_+ \). In this article, we will be concerned with the nontangential convergence of \( P \mu \) or of more general convolution integrals. For \( x_0 \in \mathbb{R}^n \) and \( \alpha > 0 \), we define the conical region \( S(x_0, \alpha) \) with vertex at \( x_0 \) and aperture \( \alpha \) by
\[
S(x_0, \alpha) = \{ (x, t) \in \mathbb{R}^{n+1}_+ : \|x - x_0\| < \alpha t \}.
\]

**Definition 1.1.** A function \( u \) defined on \( \mathbb{R}^{n+1}_+ \) or on a strip \( \mathbb{R}^n \times (0, t_0) \) for some \( t_0 > 0 \), is said to have nontangential limit \( L \in \mathbb{C} \) at \( x_0 \in \mathbb{R}^n \) if, for every \( \alpha > 0 \),
\[
\lim_{(x, t) \to (x_0, 0)} u(x, t) = L.
\]

It is a classical result that if \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \) then the Poisson integral \( Pf \) of \( f \) has nontangential limit \( f(x_0) \) at each Lebesgue point \( x_0 \) of \( f \) (see [8, Theorem 3.16]). In [3], Saeki generalized this result for more general class of kernels as well as for measures instead of \( L^p \)-functions (see Theorem 2.5). A natural question arises that what happens to the nontangential convergence at non-Lebesgue points. To answer this question, Shapiro [6] introduced the notion of \( \sigma \)-point of a locally integrable function.

**Definition 1.2.** A point \( x_0 \in \mathbb{R}^n \) is called a \( \sigma \)-point of a locally integrable function \( f \) on \( \mathbb{R}^n \) provided the following holds: for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
\left| \int_{B(x_0, r)} (f(\xi) - f(x_0)) \, dm(\xi) \right| < \epsilon (\|x - x_0\| + r)^n,
\]
whenever \( \|x - x_0\| < \delta \) and \( r < \delta \).

The set of all \( \sigma \)-points of \( f \) is called the \( \sigma \)-set of \( f \). As observed by Shapiro, the Lebesgue set of a locally integrable function is contained in the \( \sigma \)-set of the function [6, P.3182]. This containment is strict for some functions. In fact, Shapiro constructed a function \( f \in L^p(\mathbb{R}^2) \), \( 1 \leq p \leq \infty \) such that \( 0 \) is a \( \sigma \)-point of \( f \) but not a Lebesgue point of \( f \) (see [6, Section 3]). Our main aim in this article is to generalize the following result of Shapiro [6, Theorem 1] for measures.

**Theorem 1.3.** Let \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \). If \( x_0 \in \mathbb{R}^n \) is a \( \sigma \)-point of \( f \), then \( Pf \) has nontangential limit \( f(x_0) \) at \( x_0 \).

In the next section, we will define the notion of \( \sigma \)-point of a measure on \( \mathbb{R}^n \) and prove a generalization of Theorem 1.3. We will consider a wide class of kernels which includes the Poisson kernel and Gauss-Weierstrass kernel and discuss the nontangential behavior of convolutions of these kernels with measures.
It is worth mentioning that Ramey-Ullrich [11] had also discussed the nontangential behavior of $P\mu$ by considering the strong derivative of a positive measure $\mu$. We will discuss the result of Ramey-Ullrich and the relationship between the strong derivative and $\sigma$-point in the last section.

2. NONTANGENTIAL CONVERGENCE OF CONVOLUTION INTEGRALS

Let us start by defining the notion of $\sigma$-point of a measure.

**Definition 2.1.** Let $\mu$ be a measure on $\mathbb{R}^n$. A point $x_0 \in \mathbb{R}^n$ is called a $\sigma$-point of $\mu$ if there exists $L \in \mathbb{C}$ satisfying the following: for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$|(\mu - Lm)(B(x, r))| < \epsilon(\|x - x_0\| + r)^n,$$

whenever $\|x - x_0\| < \delta$ and $r < \delta$. In this case, we will denote $D\sigma\mu(x_0) = L$.

**Remark 2.2.** Every Lebesgue point of the measure $\mu$ is a $\sigma$-point of $\mu$. Moreover, $D\sigma\mu(x_0) = D_{sym}\mu(x_0)$, whenever $x_0$ is a Lebesgue point of $\mu$. To see this, we take a Lebesgue point $x_0 \in \mathbb{R}^n$ of $\mu$. We fix $\epsilon > 0$. By the definition of Lebesgue point, there exists $\delta > 0$ such that

$$|\mu - Lm|(B(x_0, r)) < \frac{\epsilon}{m(B(0, 1))}m(B(x_0, r)) = \epsilon r^n,$$

whenever $0 < r < \delta$, where $L = D_{sym}\mu(x_0)$. This implies that

$$|(\mu - Lm)(B(x, r))| \leq |\mu - Lm|(B(x, r)) \leq |\mu - Lm|(B(x_0, r + \|x - x_0\|)) < \epsilon(\|x - x_0\| + r)^n,$$

whenever $0 < r + \|x - x_0\| < \delta$. This shows that $x_0$ is a $\sigma$-point of $\mu$ and $D\sigma\mu(x_0) = D_{sym}\mu(x_0)$.

We have already mentioned in the introduction that the converse is not true.

The following proposition shows that almost every points of $\mathbb{R}^n$ is a Lebesgue point and hence $\sigma$-point of a measure.

**Proposition 2.3.** The Lebesgue set of a measure $\mu$ on $\mathbb{R}^n$ includes almost all (with respect to the Lebesgue measure) points of $\mathbb{R}^n$.

**Proof.** Let $d\mu = f dm + d\mu_s$ be the Radon-Nikodym decomposition of $\mu$ with respect to $m$, where $f \in L^1_{loc}(\mathbb{R}^n)$ and $\mu_s \perp m$ (see [2, P.121-123]). If $L_f$ denotes the Lebesgue set of $f$, then we know that [8, P.12]

$$m(\mathbb{R}^n \setminus L_f) = 0.$$

We observe that $|\mu_s| \perp m$ and hence by [2, Theorem 7.13],

$$m(\mathbb{R}^n \setminus A) = 0, \quad \text{where} \quad A = \{x \in \mathbb{R}^n \mid D_{sym}|\mu_s|(x) = 0\}.$$

Consequently, $\mathbb{R}^n \setminus (L_f \cap A)$ is of Lebesgue measure zero. Now, for $x_0 \in L_f \cap A$ and any $r > 0$,

$$\frac{|\mu - f(x_0)m|(B(x_0, r))}{m(B(0, r))} \leq \frac{1}{m(B(0, r))} \int_{B(x_0, r)} |f(x) - f(x_0)| dm(x) + \frac{|\mu_s|(B(x_0, r))}{m(B(0, r))}.$$
Since \( x_0 \in L_f \cap A \), each summand on the right hand side of the inequality above goes to zero as \( r \to 0 \). This proves our assertion. □

Let \( \phi : \mathbb{R}^n \to [0, \infty) \) be radial and radially decreasing measurable function, that is,
\[
\phi(x) = \phi(y), \quad \text{if } \|x\| = \|y\|
\]
\[
\phi(x) \geq \phi(y), \quad \text{if } \|x\| < \|y\|,
\]
with
\[
\int_{\mathbb{R}^n} \phi(x) \, dm(x) = 1.
\]

For \( t > 0 \), we consider the usual approximate identity
\[
\phi_t(x) = t^{-n} \phi \left( \frac{x}{t} \right), \quad x \in \mathbb{R}^n.
\]

Given a measure \( \mu \), we define the convolution integral \( \phi[\mu] \) by
\[
(2.1) \quad \phi[\mu](x, t) = \mu * \phi_t(x) = \int_{\mathbb{R}^n} \phi_t(x - \xi) \, d\mu(\xi),
\]
whenever the integral exists for \( (x, t) \in \mathbb{R}^{n+1}_+ \).

**Remark 2.4.** It was proved in [3, Remark 1.4] that if \( \mu \) is a measure on \( \mathbb{R}^n \) and \( \phi \) is a nonnegative, radially decreasing measurable function on \( \mathbb{R}^n \) then the finiteness of \( |\mu| * \phi_{t_0}(x_0) \) implies the finiteness of \( |\mu| * \phi_t(x) \) for all \( (x, t) \in \mathbb{R}^n \times (0, t_0) \). Note also that if \( |\mu|(\mathbb{R}^n) \) is finite then \( \mu * \phi_t(x) \) is well defined for all \( (x, t) \in \mathbb{R}^{n+1}_+ \).

In addition to the above, in some of our results we will also assume that \( \phi \) is strictly positive and satisfies the following comparison condition [3, P.134].

\[
(2.2) \quad \sup \left\{ \frac{\phi_t(x)}{\phi(x)} \mid t \in (0, 1), \|x\| > 1 \right\} < \infty.
\]

It is easy to see that \( P(x, 1) \) and the Gaussian
\[
(2.3) \quad w(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{\|x\|^2}{4}}, \quad x \in \mathbb{R}^n
\]
satisfy the comparison condition (2.2) [5, Example 2.2]. The following generalization of the nontangential convergence of the Poisson integral \( Pf \) was proved by Saeki [3, Theorem 1.5].

**Theorem 2.5.** Suppose \( \phi : \mathbb{R}^n \to (0, \infty) \) satisfies the following conditions:

1. \( \phi \) is radial, radially decreasing measurable function with \( \|\phi\|_{L^1(\mathbb{R}^n)} = 1 \).
2. \( \phi \) satisfies the condition (2.2).

Suppose \( \mu \) is a measure on \( \mathbb{R}^n \) such that \( |\mu| * \phi_{t_0}(x_1) \) is finite for some \( t_0 > 0 \) and \( x_1 \in \mathbb{R}^n \). Then the convolution integral \( \phi[\mu] \) has nontangential limit \( D_{sym}\mu(x_0) \) at each Lebesgue point \( x_0 \) of \( \mu \).
Remark 2.6. It was shown in [3, Remark 1.6] that the theorem above fails in the absence of condition (2.2). The assumption that $x_0 \in \mathbb{R}^n$ is a Lebesgue point is so strong that one can prove the nontangential convergence of $\psi[\mu]$ (defined analogously to (2.1)) at $x_0$ for any measurable function $\psi : \mathbb{R}^n \to \mathbb{C}$ (not necessarily radial, radially decreasing) such that $|\psi(x)| \leq \phi(x)$, for all $x \in \mathbb{R}^n$.

Our main interest in this paper is to prove a generalization of Theorem 1.3 for convolution integral of the form (2.1). Our first lemma shows that condition (2.2) can be used to reduce matters to the case of a measure $\mu$ such that $|\mu|(\mathbb{R}^n) < \infty$.

Lemma 2.7. Suppose $\phi$ is as in Theorem 2.5. If $\mu$ is a measure such that $|\mu| \ast \phi_{t_0}(0)$ is finite for some $t_0 \in (0, \infty)$, then for all $\alpha > 0$,

$$\lim_{(x,t) \to (0,0)} \mu \ast \phi_t(x) = \lim_{(x,t) \to (0,0)} \tilde{\mu} \ast \phi_t(x),$$

where $\tilde{\mu}$ is the restriction of $\mu$ on the closed ball $B(0, t_0)$. Moreover, if zero is a $\sigma$-point of $\mu$ then zero is also a $\sigma$-point of $\tilde{\mu}$ and vice versa. In both the cases,

$$D_\sigma \mu(0) = D_\sigma \tilde{\mu}(0).$$

Proof. In view of Remark 2.4, without loss of generality we assume that $t_0 < 1$. We write for $0 < t < t_0$, $x \in \mathbb{R}^n$,

$$\mu \ast \phi_t(x) = \tilde{\mu} \ast \phi_t(x) + \int_{\{\xi \in \mathbb{R}^n : \|\xi\| > t_0\}} \phi_t(x - \xi) \, d\mu(\xi).$$

Since $\phi$ is a radial function, we will write for the sake of simplicity $\phi(r) = \phi(\xi)$, whenever $r = \|\xi\|$. For any $r \in (0, \infty)$, we have

$$\int_{r/2 \leq \|\xi\| \leq r} \phi(\xi) \, dm(\xi) \geq \omega_{n-1} \phi(r) \int_{r/2}^r s^{n-1} \, ds = A_n r^n \phi(r),$$

where $\omega_{n-1}$ is the surface area of the unit sphere $S^{n-1}$ and $A_n$ is a positive constant which depends only on the dimension. Since $\phi$ is an integrable function, the integral on the left hand side converges to zero as $r$ goes to zero and infinity. Hence, it follows that

$$\lim_{\|\xi\| \to 0} \|\xi\|^n \phi(\xi) = \lim_{\|\xi\| \to \infty} \|\xi\|^n \phi(\xi) = 0.$$  

We denote the integral appearing on the right-hand side of (2.5) by $I(x, t)$. We fix $\alpha > 0$. We observe that for $0 < t < \min\{\frac{1}{2}, \frac{t_0}{2\alpha}\}$,

$$\|x - \xi\| \geq \|\xi\| - \|x\| \geq \|\xi\| - \frac{\|\xi\|}{2} = \frac{\|\xi\|}{2},$$

$$\|x - \xi\| \geq \|\xi\| - \|x\| \geq \|\xi\| - \frac{\|\xi\|}{2} = \frac{\|\xi\|}{2},$$
whenever \( \|\xi\| > t_0 \) and \((x, t) \in S(0, \alpha)\). Therefore, using the fact that \( \phi \) is radially decreasing, we obtain for \((x, t) \in S(0, \alpha) \cap (\mathbb{R}^n \times (0, \min \{\frac{1}{2}, \frac{t_0}{2\alpha}\}))\),

\[
|I(x, tt_0)| = (tt_0)^{-n} \left| \int_{\{\xi \in \mathbb{R}^n : \|\xi\| > t_0\}} \phi\left(\frac{x - \xi}{tt_0}\right) d\mu(\xi) \right| \\
\leq (tt_0)^{-n} \int_{\{\xi \in \mathbb{R}^n : \|\xi\| > t_0\}} \phi\left(\frac{x - \xi}{tt_0}\right) d|\mu|(\xi) \\
\leq (tt_0)^{-n} \int_{\{\xi \in \mathbb{R}^n : \|\xi\| > t_0\}} \phi\left(\frac{\xi}{2tt_0}\right) d|\mu|(\xi)
\]

(2.7)

\[
= \int_{\{\xi \in \mathbb{R}^n : \|\xi\| > t_0\}} \left( \frac{\|\xi\|}{tt_0} \right)^n \phi\left(\frac{\xi}{2tt_0}\right) \phi_{t_0}(\xi) d|\mu|(\xi)
\]

From (2.6) we get that

\[
\lim_{t \to 0} \left( \frac{\|\xi\|}{tt_0} \right)^n \phi\left(\frac{\xi}{2tt_0}\right) = 0,
\]

for each fixed \( \xi \in \mathbb{R}^n \). On the other hand, by the comparison condition (2.2), there exists some positive constant \( C \) such that

\[
\left( \frac{\|\xi\|}{tt_0} \right)^n \phi\left(\frac{\xi}{2tt_0}\right) \leq 2^n \frac{\phi_{2t}(\xi)}{\phi\left(\frac{\xi}{t_0}\right)} \leq C,
\]

for \( \|\xi\| > t_0 \) and \( 0 < t < 1/2 \). Since \( |\mu| * \phi_{t_0}(0) < \infty \), that is, \( \phi_{t_0} \in L^1(\mathbb{R}^n, d|\mu|) \), by the dominated convergence theorem, it follows from (2.7) that

\[
\lim_{(x,t) \to (0,0)} \int_{(x,t) \in S(0,\alpha)} \phi_t(x - \xi) d\mu(\xi) = \lim_{(x,t) \to (0,0)} I(x,t) = \lim_{(x,t) \to (0,0)} I(x,tt_0^{-1}t_0) = 0,
\]

as \( 0 < t_0 < 1 \). This proves (2.4). Suppose that \( D_{\sigma}\mu(0) = L \). We take \( \epsilon > 0 \). Then there exists \( 0 < \delta < t_0/2 \) such that

\[
|\langle \mu - Lm \rangle(B(x, r))| < \epsilon(\|x\| + r)^n,
\]

whenever \( \|x\| < \delta \) and \( r < \delta \). But for \( \|x\| < \delta \) and \( r < \delta \), we observe that

\[
B(x, r) \subset B(0, 2\delta) \subset B(0, t_0).
\]

Using this observation and the definition of \( \tilde{\mu} \) in the last inequality, we get that

\[
|\langle \tilde{\mu} - Lm \rangle(B(x, r))| < \epsilon(\|x\| + r)^n,
\]

whenever \( \|x\| < \delta \) and \( r < \delta \). This shows that

\[
D_{\sigma}\tilde{\mu}(0) = L = D_{\sigma}\mu(0).
\]
Proof of the converse implication is similar. □

Before proceed to our next lemma, we recall that a real valued function $f$ on a topological space $X$ is said to be lower semicontinuous if $\{x \in X : f(x) > s\}$ is open for every real number $s$ \cite{P.37}.

**Lemma 2.8.** Assume that $\phi : \mathbb{R}^n \to [0, \infty)$ is a radial, radially decreasing, integrable function. If $\phi$ is lower semicontinuous then, for every $t \in (0, \phi(0))$

$$B_t = \{x \in \mathbb{R}^n : \phi(x) > t\},$$

is an open ball centred at zero with some finite radius $\theta(t)$ (say).

**Proof.** Since $\phi$ is integrable, there exists $x_0 \in \mathbb{R}^n$ such that $\phi(x_0) \leq t$. For any $x \in B_t$,

$$\phi(x) > t \geq \phi(x_0).$$

As $\phi$ is radially decreasing, the inequality above implies that $\|x\| < \|x_0\|$ and hence $B_t$ is bounded. Therefore,

$$\theta(t) := \sup\{r > 0 : \overline{B(0, r)} \subset B_t\} < \infty.$$ We claim that $\overline{B(0, \|x\|)} \subset B_t$ for all $x \in B_t$. Consequently,

$$B(0, \theta(t)) \subset B_t \subset \overline{B(0, \theta(t))},$$

We shall show that

$$B_t = B(0, \theta(t)) \quad \text{or} \quad B_t = \overline{B(0, \theta(t))}.$$ Suppose there exists $x \in B_t \setminus B(0, \theta(t))$. Then by (2.8), $\|x\| = \theta(t)$ and hence by radiality of $\phi$, $B_t = \overline{B(0, \theta(t))}$. Since $\phi$ is lower semicontinuous, $B_t = B(0, \theta(t))$. □

**Remark 2.9.** It follows from the proof that if $\phi$ is not a lower semicontinuous function then $B_t$ may turn out to be a closed ball centred at origin. This can be seen from the following example. Define $\phi : \mathbb{R}^n \to (0, \infty)$ by

$$\phi(x) = \begin{cases} e^{-\|x\|}, & \|x\| \leq 1 \\ e^{-2\|x\|}, & \|x\| > 1. \end{cases}$$

Then for any $t \in (e^{-2}, e^{-1})$, $B_t = \overline{B(0, 1)}$.

We are now ready to present our main result which generalizes Theorem 1.3.

**Theorem 2.10.** Suppose $\phi$ and $\mu$ be as in Theorem 2.5. Further assume that $\phi$ is lower semicontinuous. If $x_0 \in \mathbb{R}^n$ is a $\sigma$-point of $\mu$ with $D_\sigma \mu(x_0) = L \in \mathbb{C}$, then $\phi[\mu]$ has nontangential limit $L$ at $x_0$. 
Therefore, there exists a positive constant \( r \) (2.10) Combining above two inequalities, we obtain Using finiteness of the total variation of \( \mu \).

For each \( 0 < t < \phi \), we can assume \( x_0 = 0 \). Indeed, we consider the translated measure \( \mu_0 = \tau_{-x_0}\mu \), where

\[
\tau_{-x_0}\mu(E) = \mu(E + x_0),
\]

for all Borel subsets \( E \subset \mathbb{R}^n \). Using translation invariance of the Lebesgue measure it follows that

\[
(\mu_0 - Lm)(B(x,r)) = (\mu - Lm)(B(x + x_0, r)).
\]

We fix \( \epsilon > 0 \). Since \( x_0 \) is a \( \sigma \)-point of \( \mu \) with \( D_\phi \mu(x_0) = L \), the equality above implies that there exists \( \delta > 0 \) such that

\[
\|\mu_0 - Lm\|B(x,r)\| < \epsilon (\|x\| + r)^n, \quad \text{whenever} \quad \|x\| < \delta, \quad r < \delta.
\]

This shows that 0 is a \( \sigma \)-point of \( \mu_0 \) with \( D_\phi \mu_0(0) = L \). As translation commutes with convolution, it also follows that

\[
(2.9) \quad \mu_0 * \phi_r(x) = (\tau_{-x_0} \mu * \phi_r)(x) = \tau_{-x_0} (\mu * \phi_r)(x) = \mu * \phi_r(x + x_0),
\]

for any \((x,t) \in \mathbb{R}^n \times (0, t_0)\). We fix an arbitrary positive number \( \alpha \). As \((x,t) \in S(0, \alpha)\) if and only if \((x + x, t) \in S(x_0, \alpha)\), one infers from (2.9) that

\[
\lim_{(x,t) \to (0,0)} \phi[\mu_0](x,t) = \lim_{(x,t) \to (0,0)} \phi[\mu](\xi,t).
\]

Hence, it suffices to prove the theorem under the assumption that \( x_0 = 0 \). Applying Lemma 2.7 we can restrict \( \mu \) on \( \overline{B}(0, t_0) \), if necessary, to assume that \( |\mu|(\mathbb{R}^n) < \infty \).

Since \( D_\phi \mu = L \),

\[
\lim_{r \to 0+} \frac{\mu(B(0,r))}{m(B(0,r))} = L.
\]

Therefore, there exists a positive constant \( r_0 \) such that

\[
\frac{|\mu(B(0,r))|}{m(B(0,r))} < L + 1, \quad \text{for all} \quad r < r_0.
\]

Using finiteness of the total variation of \( \mu \), we get that

\[
\frac{|\mu(B(0,r))|}{m(B(0,r))} \leq \frac{|\mu|(B(0,r))}{m(B(0,r))} \leq \frac{|\mu|(\mathbb{R}^n)}{m(B(0,r_0))}, \quad \text{for all} \quad r \geq r_0.
\]

Combining above two inequalities, we obtain

\[
(2.10) \quad M \mu(0) := \sup_{r > 0} \frac{|\mu(B(0,r))|}{m(B(0,r))} < \infty.
\]

For each \( 0 < t < \phi(0) \), we define

\[
B_t = \{x \in \mathbb{R}^n : \phi(x) > t\}.
\]

By Lemma 2.8, \( B_t \) is an open ball with centre at 0 and radius \( \theta(t) \). Clearly, \( \theta \) is a monotonically decreasing function in \((0, \phi(0))\) and hence measurable. We also note that for any \( r \in (0, \infty) \) and \( x \in \mathbb{R}^n \),

\[
\left\{ \xi \in \mathbb{R}^n : \phi \left( \frac{x - \xi}{r} \right) > t \right\}
\]
is an open ball with centre at \( x \) and radius \( r \theta(t) \). Let \( \{(x_k, t_k)\}_{k=1}^\infty \) be a sequence in \( S(0, \alpha) \) converging to \((0, 0)\). Without loss of generality we assume that \( t_k \in (0, t_0) \) for all \( k \). As \( \int_{\mathbb{R}^n} \phi(x) \, dm(x) = 1 \), we can write

\[
\mu \ast \phi_{t_k}(x_k) - L = t_k^{-n} \int_{\mathbb{R}^n} \phi \left( \frac{x_k - \xi}{t_k} \right) \, d\mu(\xi) -Lt_k^{-n} \int_{\mathbb{R}^n} \phi \left( \frac{x_k - \xi}{t_k} \right) \, dm(\xi)
\]

\[
= t_k^{-n} \int_{\mathbb{R}^n} \phi \left( \frac{x_k - \xi}{t_k} \right) \, d(\mu - Lm)(\xi)
\]

\[
= t_k^{-n} \int_{\mathbb{R}^n} \int_0^\phi \left( \frac{x_k - \xi}{t_k} \right) \, ds \, d(\mu - Lm)(\xi).
\]

As \( |\mu - Lm| \ast \phi_{t}(x) \) is finite for all \((x, t) \in \mathbb{R}^n \times (0, t_0)\), applying Fubini’s theorem on the right hand side of the last equality, we obtain

\[
\mu \ast \phi_{t_k}(x_k) - L = t_k^{-n} \int_0^{\phi(0)} (\mu - Lm) \left( \{ \xi \in \mathbb{R}^n : \phi \left( \frac{x_k - \xi}{t_k} \right) > s \} \right) \, ds
\]

\[
= \int_0^{\phi(0)} (\mu - Lm) (B(x_k, t_k \theta(s))) \left( \frac{\|x_k\| + t_k \theta(s)}{t_k} \right)^n \, ds.
\]

(2.11)

Since \( D_\sigma \mu(0) = L \),

\[
\lim_{(x, r) \to (0, 0)} \frac{(\mu - Lm)(B(x, r))}{(\|x\| + r)^n} = 0.
\]

Therefore, for each \( s \in (0, \phi(0)) \), integrand on the right hand side of (2.11) has limit zero as \( k \to \infty \) because \( \|x_k\|/t_k < \alpha \), for all \( k \). Moreover, using (2.10), the integrand is bounded by the function

\[
s \mapsto m(B(0, 1))(M \mu(0) + L)(\theta(s) + \alpha)^n, \quad s \in (0, \phi(0)).
\]

In order to apply the dominated convergence theorem on the right hand side of (2.11), we need to show that this function is integrable in \((0, \phi(0))\). For this, it is enough to show that the function \( s \mapsto \theta(s)^n \) is integrable in \((0, \phi(0))\). Using a well-known formula involving distribution functions \[2, \text{ Theorem 8.16}\], we observe that

\[
\int_{\mathbb{R}^n} \phi(x) \, dm(x) = \int_0^{\phi(0)} m \left( \{ x \in \mathbb{R}^n : \phi(x) > s \} \right) \, ds
\]

\[
= \int_0^{\phi(0)} m(B_s) \, ds
\]

\[
= m(B(0, 1)) \int_0^{\phi(0)} \theta(s)^n \, ds.
\]

Hence, applying the dominated convergence theorem on the right hand side of (2.11) we obtain

\[
\lim_{k \to \infty} \mu \ast \phi_{t_k}(x_k) = L.
\]

This completes the proof. \(\square\)
Shapiro also considered nontangential limits of Gauss-Weierstrass integral of a $L^p$-function [6, Theorem 2]. We recall that the Gauss-Weierstrass kernel or the heat kernel of $\mathbb{R}^{n+1}_+$ is given by

$$W(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}, \quad (x,t) \in \mathbb{R}^{n+1}_+. $$

The Gauss-Weierstrass integral of a measure $\mu$ is given by the convolution

$$W\mu(x,t) = \int_{\mathbb{R}^n} W(x-y,t) \, d\mu(y), \quad x \in \mathbb{R}^n, \quad t \in (0, \infty),$$

whenever the above integral exists. Recalling (2.3), we observe that

$$\|W\mu\| := \sup_{x \in \mathbb{R}^n, t \in (0, \infty)} \left( \int_{\mathbb{R}^n} W(x-y,t) \, d\mu(y) \right).$$

As an easy corollary of Theorem 2.10, we get the following generalization of the above mentioned theorem of Shapiro.

**Corollary 2.11.** Suppose $\mu$ is a measure on $\mathbb{R}^n$ such that $W\mu|(x_1,t_0)$ is finite for some $x_1 \in \mathbb{R}^n$ and $t_0 > 0$. If $x_0 \in \mathbb{R}^n$ is a $\sigma$-point of $\mu$ with $D_\sigma \mu(x_0) = L \in \mathbb{C}$, then the Gauss-Weierstrass integral $W\mu$ has nontangential limit $L$ at $x_0$.

**Proof.** We fix an arbitrary positive number $\alpha$. We have already mentioned that $w$ satisfies the comparison condition (2.2). Moreover, $\|w\|_{L^1(\mathbb{R}^n)} = 1$ (see [8, P.9]). Thus, $w$ satisfies all the hypothesis of Theorem 2.10. Hence, in view of (2.12), Theorem 2.10 gives

$$\lim_{(x,t) \to (x_0,0)} \frac{W\mu(x,t) - L}{\sqrt{\alpha t}} = 0.$$

Note that

$$S(x_0, \alpha) \cap \{(x,t) \in \mathbb{R}^{n+1}_+ | t < \frac{1}{\alpha} \} \subset \{(x,t) \in \mathbb{R}^{n+1}_+ | \|x-x_0\| < \sqrt{\alpha t}, \quad t < \frac{1}{\alpha} \}.$$

Using this set containment relation together with the equation above, we conclude that $W\mu$ has nontangential limit $L$ at $x_0$. \qed

We can drop the comparison condition (2.2) in Theorem 2.10 by imposing some growth condition on $\mu$. More precisely, we have the following.

**Theorem 2.12.** Let $\phi : \mathbb{R}^n \to [0, \infty)$ be radial, radially decreasing, lower semicontinuous function with $\|\phi\|_{L^1(\mathbb{R}^n)} = 1$. Suppose $\mu$ is a measure on $\mathbb{R}^n$ such that

$$|\mu|(B(0,r)) = O(r^\alpha), \quad as \quad r \to \infty,$$

and that $\mu * \phi_{t_0}(x_1)$ is finite for some $x_1 \in \mathbb{R}^n$ and $t_0 \in (0, \infty)$. If $x_0 \in \mathbb{R}^n$ is a $\sigma$-point of $\mu$ with $D_\sigma \mu(x_0) = L \in \mathbb{C}$, then $\phi[\mu]$ has nontangential limit $L$ at $x_0$.

**Proof.** Without loss of generality, we assume that $x_0 = 0$. We will use the same notation as in the proof of Theorem 2.10. From the proof of Theorem 2.10, we observe that it suffices to prove that $M\mu(0) < \infty$ and then the the rest of the arguments remains same. As $D_\sigma \mu(0) = L$, it follows that $D_{sym} \mu(0) = L$ and hence there exists a positive constant $r_0$ such that

$$\frac{|\mu(B(0,r))|}{m(B(0,r))} < L + 1, \quad for \quad all \quad r \leq r_0.$$
Using (2.13), we get two positive constants $M_0$ and $R_0$ such that
\[ \frac{|\mu(B(0,r))|}{m(B(0,r))} < M_0, \text{ for all } r \geq R_0. \]
Finally, for all $r \in (r_0, R_0)$
\[ \frac{|\mu(B(0,r))|}{m(B(0,r))} \leq \frac{|\mu|(B(0,R_0))}{m(B(0,r_0))} \]
From the last three inequalities and the fact that $|\mu|$ is locally finite, we conclude that
\[ M\mu(0) = \sup_{r>0} \frac{|\mu(B(x_0,r))|}{m(B(0,r))} < \infty. \]
□

Remark 2.13. We can drop the assumption that $\phi$ is lower semicontinuous from Theorem 2.10 and Theorem 2.12 in the following two special cases.

i) $x_0$ is a Lebesgue point of $\mu$.

ii) $\mu$ is absolutely continuous with respect to the Lebesgue measure $m$.

3. $\sigma$-POINT AND STRONG DERIVATIVE

In this section, we will discuss the relationship between $\sigma$-point of a measure and the notion of strong derivative of a measure introduced by Ramey-Ullrich [1]. We recall the definition of strong derivative of a measure.

Definition 3.1. Given a measure $\mu$ on $\mathbb{R}^n$, we say that $\mu$ has strong derivative $L \in \mathbb{C}$ at $x_0 \in \mathbb{R}^n$ if
\[ \lim_{r \to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L \]
holds for every open ball $B \subset \mathbb{R}^n$. Here, $rB = \{rx \mid x \in B\}$, $r > 0$. The strong derivative of $\mu$ at $x_0$, if it exists, is denoted by $D\mu(x_0)$. Note that $rB(\xi, s) = B(r\xi, rs)$.

Proposition 3.2. Let $\mu$ be a measure on $\mathbb{R}^n$. If $x_0 \in \mathbb{R}^n$ is a $\sigma$-point of $\mu$ with $D_\sigma\mu(x_0) = L \in \mathbb{C}$, then the strong derivative of $\mu$ at $x_0$ exists and is equal to $L$.

Proof. We take a ball $B = B(x, s)$ in $\mathbb{R}^n$ and fix $\epsilon > 0$. As $x_0$ is a $\sigma$-point of $\mu$, there exists $\delta > 0$ such that
\[ |(\mu - Lm)(x_0 + rB)| = |(\mu - Lm)(B(x_0 + rx, rs))| < \epsilon(|rx| + rs)^n, \]
whenever $|rx| < \delta$ and $rs < \delta$. This implies that
\[ \left| \frac{\mu(x_0 + rB)}{m(rB)} - L \right| < \epsilon \left( \frac{(|x| + s)^n}{m(B(0,s))} \right), \text{ whenever } |rx| < \delta \text{ and } rs < \delta. \]
Taking $r_0 = \min\{\frac{\delta}{|x|+1}, \frac{\delta}{s}\}$, it follows that the last inequality holds for all $r < r_0$. This completes the proof. □
Remark 3.3. In [11] Theorem 2.2, among other things, Ramey-Ullrich proved that if \( \mu \) is a positive measure on \( \mathbb{R}^n \) with well-defined Poisson integral \( P\mu \) then the strong derivative of \( \mu \) at \( x_0 \in \mathbb{R}^n \) is \( L \in [0, \infty) \) if and only if \( P\mu \) have nontangential limit \( L \) at \( x_0 \). In view of Proposition 3.2 we can deduce Theorem 2.10 for \( \phi = P(., 1) \) and \( \mu \) positive from the result of Ramey-Ullrich.

The converse of Proposition 3.2 is true in one dimension. If \( \mu \) is a locally finite signed measure on \( \mathbb{R} \) then there is a function \( f : \mathbb{R} \to \mathbb{R} \) of bounded variation such that the positive and negative parts of \( f \) are right continuous and

\[
\mu ((a, b]) = f(b) - f(a), \quad a, b \in \mathbb{R}, \ a < b.
\]

For more discussion on this see [7, P.281-284].

**Proposition 3.4.** Suppose that \( \mu \) and \( f \) as above and \( x_0 \in \mathbb{R} \).

i) The function \( f \) is differentiable at \( x_0 \) if and only if \( x_0 \) is a \( \sigma \)-point of \( \mu \). In this case, \( f'(x_0) = D_\sigma\mu(x_0) \).

ii) The function \( f \) is differentiable at \( x_0 \) if and only if the strong derivative \( \mu \) at \( x_0 \) exists. In this case, \( f'(x_0) = D\mu(x_0) \).

**Proof.** We first prove i). Suppose \( f \) is differentiable at \( x_0 \) and \( f'(x_0) = L \in \mathbb{R} \). Fix \( \epsilon > 0 \) and choose \( \delta > 0 \) such that

\[
(3.1) \quad \left| \frac{f(x_0 + h) - f(x_0)}{h} - L \right| < \epsilon, \quad \text{whenever } |h| < \delta.
\]

For \( x \in \mathbb{R}, \ r > 0 \) with \( |(x - x_0) + r| < \delta \) and \( |(x - x_0) - r| < \delta \), we have

\[
|((\mu - Lm)((x - r, x + r))| = |f(x + r) - f(x) - 2rL| = \left| \frac{f(x_0 + x - x_0 + r) - f(x_0)}{x - x_0 + r} \times (x - x_0 + r) - (x - x_0 + r)L + (x - x_0 - r)L \right|
\]

\[
\leq |x - x_0 + r| \left| \frac{f(x_0 + x - x_0 + r) - f(x_0)}{x - x_0 + r} \right| - \left| \frac{f(x_0 + x - x_0 - r) - f(x_0)}{x - x_0 - r} \right| \times (x - x_0 - r)
\]

\[
< |x - x_0 + r| \epsilon + |x - x_0 - r| \epsilon \quad \text{(by (5.1))}
\]

This implies that

\[
|((\mu - Lm)(B(x, r))| < 2\epsilon(|x - x_0| + r), \quad \text{whenever } |x - x_0| < \delta/2, \ r < \delta/2.
\]

Thus, \( x_0 \) is a \( \sigma \)-point of \( \mu \) with \( D_\sigma\mu(x_0) = L \).

Conversely, we assume that \( x_0 \) is a \( \sigma \)-point of \( \mu \) with \( D_\sigma\mu(x_0) = L \in \mathbb{R} \) and fix \( \epsilon > 0 \). Then there exists \( \delta > 0 \) such that

\[
(3.2) \quad |((\mu - Lm)((x - r, x + r))| < \epsilon(|x - x_0| + r), \quad \text{whenever } |x - x_0| < \delta, \ r < \delta.
\]
Taking \( x = x_0 + r \) with \( r > 0 \) in (3.2), we obtain
\[
|\mu((x_0, x_0 + 2r)) - 2rL| = |f(x_0 + 2r) - f(x_0) - 2rL| = 2r \left| \frac{f(x_0 + 2r) - f(x_0)}{2r} - L \right| < \epsilon,
\]
whenever \( r < \delta \). This shows that \( f'(x_0+) = L \). Similarly, by taking \( x = x_0 - r \) with \( r > 0 \) in (3.2), we get that \( f'(x_0-) = L \).

The statement \( ii \) can be proved by arguing in a similar fashion. We refer the reader to [4, Remark 2.6 (2)] where it was proved under the assumption that \( f \) is monotonically increasing.

\( \square \)

Considering real and imaginary parts of a measure, if necessary, we obtain the following corollary.

**Corollary 3.5.** Suppose \( \mu \) is a measure on \( \mathbb{R} \) and \( x_0 \in \mathbb{R} \). Then \( x_0 \) is a \( \sigma \)-point of \( \mu \) if and only if \( \mu \) has strong derivative at \( x_0 \). Moreover, \( D_\sigma \mu(x_0) = D\mu(x_0) \).

**Remark 3.6.**

i) It is not known to us whether for a measure \( \mu \) on \( \mathbb{R}^n \), the \( \sigma \)-set of \( \mu \) coincides with the set of points at which the strong derivative of \( \mu \) exists, if \( n > 1 \). It would be surprising if it is true in higher dimensions. A heuristic reasoning behind this is the following observation. Suppose \( \mu \) is a measure on \( \mathbb{R}^n \). If \( 0 \) is a \( \sigma \)-point of \( \mu \) then
\[
(\mu - D_\sigma(0)m)(B(x, r)) \rightarrow 0, \quad (x, r) \rightarrow (0, 0).
\]
On the other hand, existence of strong derivative at \( 0 \) only ensures
\[
(\mu - D\mu(0)m)(B(x, r)) \rightarrow 0, \quad (x, r) \rightarrow (0, 0),
\]
along the rays of the form \( \{(rx_0, rt_0) \mid r > 0\} \), where \((x_0, t_0) \in \mathbb{R}^{n+1}_+ \).

ii) Suppose \( \mu \) and \( \phi \) as in Theorem 2.10 and \( n = 1 \). If \( D\mu(x_0) = L \) then it follows from Proposition 3.4 and Theorem 2.10 that \( \phi[\mu] \) converges nontangentially to \( L \). It is not known whether the same is true for dimension \( n > 1 \). However, the following theorem shows that a weaker version of convergence for \( \phi[\mu] \) holds at the points where the strong derivative \( D\mu \) exists.

**Theorem 3.7.** Let \( \phi \) and \( \mu \) be as in Theorem 2.10. Suppose \( \mu \) has strong derivative \( L \in \mathbb{C} \) at \( x_0 \in \mathbb{R}^n \). Then \( \phi[\mu](x, t) \) has limit \( L \) as \((x, t) \rightarrow (x_0, 0)\) along each ray through \((x_0, 0)\) in \( \mathbb{R}^{n+1}_+ \). In other words,
\[
\lim_{r \to 0} \phi[\mu](x_0 + r\xi, r\eta) = L, \quad \text{for each fixed } (\xi, \eta) \in \mathbb{R}^{n+1}_+.
\]

**Proof.** Without loss of generality, we can assume \( x_0 = 0 \). Let \( \tilde{\mu} \) be the restriction of \( \mu \) on the ball \( B(0, t_0) \). If \( B(y, \tau) \) is any given ball, then for all \( 0 < r < t_0(\tau + ||y||)^{-1} \), it follows that \( rB(y, \tau) \) is contained in \( B(0, t_0) \). This in turn implies that \( D\mu(0) \) and \( D\tilde{\mu}(0) \) are equal. Thus, in view of Lemma 2.7, without loss of generality, we can assume that \( |\mu|(\mathbb{R}^n) \) is finite. We will use the same notation as in the proof of Theorem 2.10.

Since \( D\mu(0) \) is equal to \( L \), it follows that \( D_{\text{sym}}\mu(0) \) is also equal to \( L \) and hence \( M\mu(0) \) is finite (see the argument preceding (2.10)). We take \((\xi, \eta) \in \mathbb{R}^{n+1}_+ \) and a sequence \( \{r_k\} \)
of positive numbers converging to zero. Substituting $x_k = r_k \xi$, $t_k = r_k \eta$ in equation (2.11), we obtain

\begin{equation}
(3.3) \quad \phi[\mu](r_k \xi, r_k \eta) - L = \int_0^{\phi(0)} \frac{(\mu - Lm) \left(B(r_k \xi, r_k \eta \theta(s))\right)}{(r_k \eta \theta(s))^n} \theta(s)^n \, ds.
\end{equation}

As $D\mu(0) = L$, using the definition of strong derivative, we observe that for each fixed $s \in (0, \phi(0))$

\begin{equation}
(3.4) \quad \lim_{k \to \infty} \frac{(\mu - Lm) \left(B(r_k \xi, r_k \eta \theta(s))\right)}{(r_k \eta \theta(s))^n} = \lim_{k \to \infty} \left( \frac{\mu(r_k B(\xi, \eta \theta(s)))}{m(r_k B(\xi, \eta \theta(s)))} - L \right) c'_n = 0,
\end{equation}

where $c'_n = m(B(0,1))$. The integrand on the right hand side of (3.3) is bounded by the function

\[ s \mapsto m(B(0,1))(M\mu(0) + L)\theta(s)^n, \quad s \in (0, \phi(0)). \]

We have seen in the proof of Theorem 2.10 that this function is integrable in $(0,\phi(0))$. In view of (3.4), we can now apply dominated convergence theorem on the right-hand side of (3.3) to complete the proof. \qed

We show by an example that the existence of limit of $\phi[\mu]$ along every ray through $(x_0,0)$ may not imply the existence of the strong derivative at $x_0$.

**Example 3.8.** Consider the measure $d\mu = \chi_{[0,1]} \, dm$ on $\mathbb{R}$. Then $D_{sym} \mu(0)$ is $1/2$ but the strong derivative $D\mu$ does not exist at the origin (see [4, Remark 2.5]). Taking $\phi = P(\cdot, 1)$, we see that

\[ \phi[\mu](x, t) = \frac{1}{\pi} \int_0^1 \frac{t}{t^2 + (x - \xi)^2} \, d\mu(\xi) = \frac{1}{\pi} \left( \arctan \frac{1 - x}{t} + \arctan \frac{x}{t} \right), \quad (x, t) \in \mathbb{R}^{n+1}. \]

Therefore, for each fixed $(\xi_0, t_0) \in \mathbb{R}^{n+1}_+$ we have

\[ \lim_{r \to 0} \phi[\mu](r \xi_0, rt_0) = \lim_{r \to 0} \frac{1}{\pi} \left( \arctan \frac{1 - r \xi_0}{rt_0} + \arctan \frac{r \xi_0}{rt_0} \right) = \frac{1}{\pi} \left( \frac{\pi}{2} + \arctan \frac{\xi_0}{t_0} \right). \]

This shows that $\phi[\mu]$ has limit along every ray through the origin but the limit depends on the ray.

**Acknowledgements**

The author would like to thank Swagato K. Ray for many useful discussions during the course of this work. The author is supported by a research fellowship from Indian Statistical Institute.

**References**

[1] Ramey, Wade; Ullrich, David *On the behavior of harmonic functions near a boundary point*. Trans. Amer. Math. Soc. 305, no. 1, 207–220 (1988).

[2] Rudin, Walter *Real and complex analysis*. Third edition. McGraw-Hill Book Co., New York (1987).

[3] Saeki, S. *On Fatou-type theorems for non-radial kernels*. Math. Scand. 78, no. 1, 133–160 (1996).

[4] Sarkar, J. *On parabolic convergence of positive solutions of the heat equation*, arXiv:2012.11234 (2020).
[5] Sarkar, J. On Pointwise converse of Fatou’s theorem for Euclidean and Real hyperbolic spaces, arXiv:2012.01824 (2020).

[6] Shapiro, Victor L. Poisson integrals and nontangential limits. Proc. Amer. Math. Soc. 134, no. 11, 3181–3189 (2006).

[7] Stein, Elias M.; Shakarchi, Rami Real analysis. Measure theory, integration, and Hilbert spaces. Princeton Lectures in Analysis, 3. Princeton University Press, Princeton, NJ (2005).

[8] Stein, Elias M.; Weiss, Guido Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J. (1971).

STAT-MATH UNIT, INDIAN STATISTICAL INSTITUTE, 203, B.T. ROAD, KOLKATA-700108, INDIA
Email address: jayantasarkarmath@gmail.com