Boundary Value Problems for
the Helmholtz Equation
for a Half-plane
with a Lipschitz Inclusion

Evgeny Lipachev
Kazan Federal University, 35 Kremlyovskaya ul.
Kazan, Russian Federation,
elipachev@gmail.com
http://kpfu.ru/

Abstract. This paper considers to the problems of diffraction of electromagnetic waves on a half-plane, which has a finite inclusion in the form of a Lipschitz curve. The diffraction problem formulated as boundary value problem for Helmholtz equations and boundary conditions Dirichlet or Neumann on the boundary, as well as the radiation conditions at infinity. We carry out research on these problems in generalized Sobolev spaces. We use the operators of potential type, that by their properties are analogs of the classical potentials of single and double layers. We proved the solvability of the boundary value problems of Dirichlet and Neumann. We have obtained solutions of boundary value problems in the form of operators of potential type. Boundary problems are reduced to integral equations of the second kind.

Keywords: special Lipschitz domains, Helmholtz equation, Dirichlet problem, Neumann problem, Boundary Integral Equations, operator of potential type

1 Introduction

At present, boundary problems on Lipschitz domains have been formed in a special field of research. This is caused both with applications in electrodynamics and other fields, and with the theoretical importance of these studies. An extensive bibliography on this range of issues and the most important the results are given in the articles M.S. Agranovich [12], M. Costabel [4], B. Dahlberg and C. Kenig [5], D. Jerison and C. Kenig [6,7,8], M. Mitrea and M. Taylor [9,10]. A detailed summary of the main results of the theory of boundary value problems on Lipschitz domains can be found in [1], [11], [12]. An essential part of this theory is devoted to boundary integral operators. A summary of the theory of integral operators on Lipschitz Domains is contained in article [4]. Note that this work is the most cited in this field. An important property of the Lipschitz domain is the possibility approximations by infinitely differentiable domains from
either side boundaries of this domain (see [1, 13]). Lipschitz domains also satisfy the condition of a uniform cone (see [1, 14]). These properties are used in the present paper to investigate the solvability of boundary value problems of diffraction on an unbounded boundary with a Lipschitz inclusion.

In the papers [15]–[17] we investigated the boundary value problems for the Helmholtz equation in domains with a rough smooth and piecewise smooth boundaries. These studies are based on the use of generalized potentials of single and double layers. In contrast to the classical potentials defined on closed domains, generalized potentials are considered on open curves and on domains with an infinite boundary. In the present paper this technique is extended to the case of a half-plane with a Lipschitz inclusion of finite size.

In the study of boundary value problems on Lipschitz boundaries we are introduced the operators of potential type. These operators are analogues analogues of operators of single and double layers and have properties close to those of the classical potentials of single and double layers, which makes it possible to apply, after necessary refinements, the same reasoning technique as in the classical case (e.g., see [3], [4]).

## 2 Statement of the problem

A function \( h : \mathbb{R} \to \mathbb{R} \) is called Lipschitz if there exist \( C > 0 \) such that
\[
|h(x) - h(x')| \leq C|x - x'| \quad \text{for all } x, x' \in \mathbb{R}.
\]

Let \( D \equiv D^+ = \{(x, y) \in \mathbb{R}^2 : y > h(x)\} \), where \( h(x) \) is Lipschitz function with finite support. In the terminology of publications [2, 3, 18] this domain is a special Lipschitz domain. In [5] recommender use the notation \( \Omega \) for bounded Lipschitz domains and \( D \) for special Lipschitz domains respectively.

The boundary of the domain \( D \) can be represented in the form \( \partial D = \Gamma_1 \cup \Gamma_2 \)
\[
\Gamma_1 = \{(x, h(x)) : x \not\in [0, d]\}, \quad \Gamma_2 = \{(x, 0) : x \in [0, d]\}, \quad \text{supp} \ h = [0, d].
\]

We note that this is a special case of rough boundary (e.g., see [16], [19], [20]).

Let \( D^- = \{(x, y) \in \mathbb{R}^2 : y < h(x)\} \).

We formulate the problem in the following terms. Find a function \( u(x,y) \in H^1(\Omega) \), such that
\[
\Delta u(M) + k^2 u(M) = 0, \quad M = (x, y) \in D, \quad (1)
\]
and the Dirichlet boundary condition
\[
\gamma u(P) = f(P), \quad P \in \partial D, \quad (2)
\]
or the Neumann boundary condition
\[
\gamma' u(P) = g(P), \quad P \in \partial D, \quad (3)
\]
and the radiation condition at infinity
\[
u^* = e^{ikr} O \left( \frac{1}{\sqrt{r}} \right), \quad \frac{\partial u^*}{\partial r} -iku^* = e^{ikr} O \left( \frac{1}{\sqrt{r}} \right), \quad r \to \infty, \quad (4)
\]
where \( r = \sqrt{x^2 + y^2} \) and \( u^*(x, y) = u(x, y) - \tilde{u}(x, y) \). Through \( \tilde{u}^+ \), \( \tilde{u}^- \) denoted by the solution of the diffraction problem on the half-plane \([15, 16, 17]\).

Here \( f \in H^{1/2}(\partial D), \ g \in H^{-1/2}(\partial D), \ k \in \mathbb{C} \setminus \{0\}, \ \text{Im} \ k \geq 0 \). Through \( H^t \) denoted the Sobolev spaces (e.g., see [1121]). Let \( P \in \partial D \), denote by

\[
\gamma u(P) \equiv u|_{\partial D}(P) = \lim_{M \to P, M \in \Lambda_\alpha(P)} u(M)
\]

– trace of the function \( u \) on the boundary \( \partial D \), where

\[
Q_\alpha(P) \equiv \{M \in D : |M - P| < (1 + \alpha)d(M, \partial D)\}, \quad \alpha > 0
\]

– Luzin sector. Denote by

\[
\gamma'(u)(P) \equiv \partial_n u|_{\partial D}(P) = (n(P), (\nabla u)|_{\partial D}(P)).
\]

Note that \( \gamma : H^t(D) \to H^{t-1/2}(\partial D), \ \gamma' : H^t(D) \to H^{t-3/2}(\partial D) \) (e.g., see [18, 22]).

These problems are used as a mathematical model for finding the electromagnetic field resulting from the diffraction of an electromagnetic plane wave in regions with an infinite rough boundary. From a physical point of view, the boundary value problem with condition (2) corresponds to the diffraction problem \( TE \)-polarized electromagnetic wave, and the problem with the condition (3) – the problem of diffraction \( TM \)-polarized wave (e.g., see [23]).

### 3 Uniqueness of the solution of boundary value problems

To derive Green’s formulas, the Lipschitz domain is approximated by domains with smooth boundaries. According to the results of article [13] (e.g., see [3]) for a Lipschitz domain \( D \) there is a sequence of \( C^\infty \) domains, \( D_j \subset D \), and homeomorphisms, \( A_j : \partial D \to \partial D_j \), such that \( \sup_{P \in \partial D} |A_j(P) - P| \to 0 \) as \( j \to \infty \) and \( A_j(P) \in Q_\alpha(P) \) for all \( j \) and all \( P \in \partial D \), the normal vectors \( n(A_j(P)) \) to \( D_j \), converge pointwise a.e. and in \( L_2(\partial D) \) to \( n(P) \). For a special Lipschitz domain \( D \) from the section 2 choose an approximating sequence of domains with condition

\[
\Gamma_1 = \{(x, 0) : x \in \mathbb{R} \setminus [0, d]\} \subset \partial D_j \cap \partial D.
\]

**Theorem 1.** If the condition \( \text{Im} k \geq 0 \) the boundary value problems (1), (3), (4) have no more than one solution.

**Theorem 2.** If the condition \( \text{Im} k \geq 0 \) and \( \text{Re} k \neq 0 \) the boundary value problems (1), (3), (4) have no more than one solution.

**Proof.** Let \( \{D_j\}_{j \in \mathbb{N}} \) be a system of smooth domains approximating the domain \( D \). Let \( u, v \) be two solutions of the boundary value problem and \( w = u - v \).

On the part of the boundary \( \Gamma_1 \), because \( \Gamma_1 \subset \partial D \cap \partial D_j \), we have

\[
\gamma_D|_{\Gamma_1} w = \gamma_{D_j}|_{\Gamma_1} w = \gamma|_{\Gamma_1} w = \gamma'|_{\Gamma_1} w = \partial_n w|_{\Gamma_1} = 0.
\]
Further, since
\[ \gamma_{D_j} u \to \gamma_D u, \quad \gamma_{D_j} v \to \gamma_D v, \]
we have
\[ w|_{\partial D_j} \equiv \gamma_{D_j} w = \gamma_D (u - v) \to 0, \quad j \to \infty. \]

Similarly,
\[ \partial_{\nu} w|_{\partial D_j} \equiv \gamma'_{D_j} w = \gamma'_D (u - v) \to 0, \quad j \to \infty. \]

Consider functions \( w_s \in C^\infty \) such that \( w_s \to w, \ s \to \infty \). Let \( R > d \) be a real number and \( S_R \) a circle of radius \( R \). In the bounded smooth region \( D_{j,R} = D_j \cap S_R \) we apply the second Green’s formula to the functions \( w_s \) :
\[
\int_{D_{j,R}} (w_s \Delta w_s + \overline{w_s} \Delta w_s) \, d\sigma = \int_{\partial D_{j,R}} (w_s \partial_{\nu} \overline{w_s} - \overline{w_s} \partial_{\nu} w_s) \, d\ell_P.
\]

In the limit to \( R \to \infty \), we obtain
\[
\int_{D_j} (w \Delta w + \overline{w} \Delta w) \, d\sigma = \int_{\partial D_j} (w \partial_{\nu} \overline{w} - \overline{w} \partial_{\nu} w) \, d\ell_P.
\]

Further, in the last formula we pass to the limit with respect to \( s \to \infty \) and take into account the limit relations
\[
w_s(P) \to w(P), \quad \overline{w_s}(P) \to \overline{w}(P), \quad P \in D_j,
\]
\[
w_s|_{\partial D_j} \to w|_{\partial D_j}, \quad \overline{w_s}|_{\partial D_j} \to \overline{w}|_{\partial D_j},
\]
\[
\partial_{\nu} w_s|_{\partial D_j} \to \partial_{\nu} w|_{\partial D_j}, \quad \partial_{\nu} \overline{w_s}|_{\partial D_j} \to \partial_{\nu} \overline{w}|_{\partial D_j}.
\]
As a result, we obtain
\[
\int_{D_j} (w \Delta w + \overline{w} \Delta w) \, d\sigma = \int_{\partial D_j} (w \partial_{\nu} \overline{w} - \overline{w} \partial_{\nu} w) \, d\ell_P.
\]

Now we pass to the limit with respect to \( j \to \infty \), taking into account the approximation properties of a Lipschitz domain \( D \) by smooth domains.
\[
\int_{D} (w \Delta w + \overline{w} \Delta w) \, d\sigma = \int_{\partial D} (\gamma_D w \gamma'_D \overline{w} - \gamma_D \overline{w} \gamma'_D w) \, d\ell_P. \quad (5)
\]

Because the \( \Delta w = -k^2 w, \quad \Delta \overline{w} = -\overline{k}^2 \overline{w} \), then the left-hand side of the equality \( 5 \) takes the form
\[ i4\text{Re} \text{Im} \int_{D} |w|^2 \, d\sigma. \]
As a consequence of the boundary conditions of the boundary value problem, the right-hand side of equation (5) is 0.

Since \( \text{Re}k \neq 0, \text{Im}k \geq 0 \), we have

\[
\int_D |w|^2 \, d\sigma = 0.
\]

From the last relation we conclude that \( w \equiv 0 \) is satisfied in region \( D \) and, as a consequence, we obtain \( u = v \).

4 Existence of solutions of boundary value problems

One method of solving boundary value problems of diffraction is the method of integral equations (e.g., see [24], [25]). In the classical theory, boundary value problems are considered on bounded domains with a sufficiently smooth boundary, the potentials of the single and double layers are used, as well as the technique of Green’s formulas. In the case of a deterioration of the properties of the boundary, it is required to refine the definitions of the potentials and the conditions for the applicability of the Green’s formulas. In the study of boundary value problems for the Helmholtz equation on rough boundaries, we used generalized potentials ([15]–[17]). In the case of Lipschitz boundaries, the extension of the concept of a potential is called a potential type operator. We note that in the case of Lipschitz domains the properties of operators of potential type are analogous to those of classical potentials of a single and double layers, in particular, formulas for the jump of values on the boundary.

As shown in my works [15]–[17] in the case of an rough boundary from class \( C^{(1,\nu)} \), \( \nu \in (0, 1] \), under the conditions \( \text{Im}k \geq 0, \text{Re}k \neq 0 \) the boundary value problem has a unique solution and for the solution we have the representation

\[
\begin{align*}
    u(x, y) &= \tilde{u}(x, y) + v(x, y), \\
    v(x, y) &= (W(k)\varphi)(x, y) = \int_{\partial D} \frac{\partial n_P}{\partial \nu} G_1(k; M, P)\varphi(\tau) \, ds_P \quad (6) \\
    &- \text{in the case of the problem with the condition (2) on the boundary } \partial D \text{ and} \\
    v(x, y) &= (V(k)\varphi)(x, y) = \int_{\partial D} G_2(k; M, P)\varphi(\tau) \, ds_P \quad (7) \\
    &- \text{in the case of condition (3) on the boundary. The function } \varphi(x) \text{ is a solution of the integral equation} \\
    -\pi \varphi(x) + \int_0^d \frac{\partial n_P}{\partial \nu} G_1(k; M, P)\sqrt{1 + h'^2(\tau)} \varphi(\tau) \, d\tau &= -f(M) + \tilde{u}(M) \quad (8)
\end{align*}
\]
in the case of condition (2) on the boundary and
\[ -\pi \varphi(x) + \int_0^d \partial_{n(M)} G_2(k; M, P) \sqrt{1 + h'^2(\tau)} \varphi(\tau) d\tau = -g(M) + \partial_{n(M)} \tilde{u}(M) \quad (9) \]

in the case of condition (3) on the boundary.

These formulas use functions
\[ G_m(k; M, P) = \frac{\pi i}{2} \left\{ H_0^{(1)}(kr) + (-1)^m H_0^{(1)}(kr^*) \right\}, \quad m = 1, 2, \]
where \( M = (x, h(x)), P = (\tau, h(\tau)), r = \sqrt{(x-\tau)^2 + (y-h(\tau))^2}, \]
d is the length of the irregular part of the boundary \( \partial D \).

In the case of a special Lipschitz domain \( D \), we consider the operators
\[ (V(k) \varphi)(M) = \int_{\Gamma^2} G_2(k; M, P) \varphi(\tau) d\ell_P, \quad M \in D, \quad (10) \]
\[ (W(k) \psi)(M) = \int_{\Gamma^2} \partial_{n(P)} G_1(k; M, P) \psi(\tau) d\ell_P, \quad M \notin \partial D. \quad (11) \]

Here \( n(P) \) is the unit normal vector at the point \( P \) directed to the region \( y > 0 \). This vector is defined for almost all \( P \in \partial D \).

For \( M = (x, h(x)) \in \partial D \) we define
\[ V(k) \varphi(x) = \lim_{\varepsilon \to 0} \int_{|M-P|>\varepsilon} G_2(k; M, P) \varphi(\tau) d\ell_P. \]

From the results of [13] (e.g., see [25]) the following statements.

**Lemma 1.** If \( \varphi \in L_p(\partial D), 1 < p < \infty \), then there is the direct value of the normal derivative of the operator \( (10) \)
\[ V'(k) \varphi(x) \equiv \left[ \partial_{n(M)} (V(k) \varphi) \right](x) = \lim_{\varepsilon \to 0} \int_{|M-P|>\varepsilon} \partial_{n(M)} G_2(k; M, P) \varphi(\tau) d\ell_P. \]

Here the limit is understood in the sense of convergence in \( L_p(\partial D) \) or pointwise convergence for almost all \( M \in \partial D \).

The normal derivative \( \partial_{n(M)} (V(k) \varphi) \) for almost all \( M \in \partial D \) has nontangential limits \( \partial_{n(M)} (V(k) \varphi)_{\pm} \) on the side \( D^\pm \), which are expressed by formulas
\[ \partial_{n(M)} (V(k) \varphi)_{\pm} = \pm \frac{1}{2} \varphi + V'(k) \varphi. \quad (12) \]
Lemma 2. If \( \psi \in L_p(\partial D) \), \( 1 < p < \infty \), then almost everywhere on \( \partial D \) there exists the limit

\[
W(k)\psi(x) = \lim_{\varepsilon \to 0} \frac{1}{|M-P|>\varepsilon} \int_{|M-P|>\varepsilon} \partial_n(p) G_1(k; M, P) \psi(\tau) d\ell_P.
\]  

(13)

This limit is called the direct value of the operator \( W(k)\psi \).

Lemma 3. If \( \psi \in L_p(\partial D) \), \( 1 < p < \infty \), then almost everywhere on the boundary \( \partial D \) has nontangential limits \( (W(k)\psi)_\pm \) on the side \( D^\pm \), and the following equalities hold:

\[
(W(k)\psi)_\pm = \frac{1}{2} \psi + W(k)\psi.
\]  

(14)

For the operators (10), (11), the basic potentials are satisfied (e.g., see [123]). Therefore, we can consider them analogues of the potentials of a single and a double layer. These operators are called operators of the potential type. The following propositions hold (e.g., see [137,138]):

\[
\mathcal{V}(k) : H^{t-1/2}(\partial D) \to H^{t+1/2}(\partial D),
\]

\[
W(k) : H^{t+1/2}(\partial D) \to H^{t+1/2}(\partial D), \quad \frac{1}{2} \leq t < \frac{1}{2},
\]

\[
\mathcal{V}(k) : L^2(\partial D) \to H^1(\partial D), \quad W(k) : H^1(\partial D) \to H^1(\partial D).
\]

Let \( \{D_j\}_{j \in \mathbb{N}} \) be a system of smooth domains approximating the Lipschitz domain \( D \). We denote by \( \{u_j\} \) the sequence of solutions of boundary value problems in smooth domains \( D_j \). This sequence can be adjusted so that \( u_j|_{D_k} = u_k, \quad k \leq j \).

In each smooth domain \( D_j \) we consider the boundary value problem

\[
\Delta u(M) + k^2 u(M) = 0, \quad M \in D_j,
\]

\[
u|_{\partial D_j} = u_{j+1}|_{\partial D_j}
\]

and, in addition, we require that the function \( u \) satisfy the radiation conditions [4].

Note that the value of the function \( u_{j+1} \) on \( \partial D_j \) is defined by virtue of the fact that \( D_j \subset D_{j+1} \).

For a given boundary value problem, the solvability conditions are satisfied; therefore, for each \( j \) there exists a classical solution \( u_j \) of this problem.

But, as a function of \( u_{j+1} \) in \( D_j \) also satisfies the conditions of the boundary value problem, then, by the uniqueness, we get

\[
\dot{u}_j = u_{j+1}|_{D_j}.
\]

Note also that on the general section boundary areas \( D_j \) and \( D_{j+1} \) we have

\[
\dot{u}_j|_{\partial D_j \cap \partial D_{j+1}} = u_{j+1}|_{\partial D_j \cap \partial D_{j+1}} = f|_{\partial D_j \cap \partial D_{j+1}}
\]
– in the case of the Dirichlet problem and
\[ \partial_v \bar{u}_j |_{\partial D_j \cap \partial D_{j+1}} = \partial_v u_{j+1} |_{\partial D_j \cap \partial D_{j+1}} = g |_{\partial D_j \cap \partial D_{j+1}} \]

– in the case of the Neumann problem.

For each finite set of solutions \( \{u_0, u_1, \ldots, u_j\} \) of the boundary value problems in the domains \( D_0, D_1, \ldots, D_j \) can perform such adjustment, starting from the value \( i \) and reducing to index 0. As a result, reasoning by induction, we obtain the sequence of functions \( u_0(x, y), u_1(x, y), \ldots \) satisfying the following conditions.

(i) For each \( j \in \mathbb{N} \), function \( u_j(x, y) \) at all points of the domain \( D_j \) is a solution of the Helmholtz equation.

(ii) If \( k \leq j \), then \( u_k(x, y) = u_j(x, y) |_{D_k} \).

(iii) In the case of the Dirichlet problem, \( u_j |_{\partial D_j \cap \partial D_{j+1}} = f |_{\partial D_j \cap \partial D_{j+1}} \), and

\[ \partial_v u_j |_{\partial D_j \cap \partial D_{j+1}} = g |_{\partial D_j \cap \partial D_{j+1}} \] – in the case of the Neumann problem.

(iv) In the domain \( D_j \), the function \( u_j \) can be represented as a generalized potential with a density \( \varphi_j(x) \) found as a solution of the integral equation (8) or (9).

Thus, we obtain a sequence of functions \( \varphi_j(x) \). Let us show that this sequence is fundamental in \( L_2[0, d] \). Let \( R > d \) be a real number, we define the domain \( S_R = \{(x, y) : x^2 + y^2 \geq R^2, \ y > 0\} \).

Consider in \( S_R \) the two functions \( u_i \) and \( u_j \) of this sequence and for definiteness, assume that \( i > j \).

As shown, the functions \( u_i \) and \( u_j \) coincide in domain \( D_j \), and hence in \( S_{R, i} = S_R \cap D_i \) we have

\[ u_i(x, y) - u_j(x, y) = 0, \quad i > j. \] (15)

We write the last relation, using the representation of solutions of boundary value problems in the form of potentials (8) and (7).

In the case of the Dirichlet problem, we have

\[ (u_i - u_j) |_{S_{R, i}} = \int_{\partial D_i \setminus \Gamma_i} \partial_n(P) G_1(k; M, P) \varphi_i(\tau) d\ell_P - \int_{\partial D_j \setminus \Gamma_j} \partial_n(P) G_1(k; M, P) \varphi_j(\tau) d\ell_P. \]

In the case of the Neumann problem, we have

\[ (u_i - u_j) |_{S_{R, i}} = \int_{\partial D_i \setminus \Gamma_i} G_2(k; M, P) \varphi_i(\tau) d\ell_P - \int_{\partial D_j \setminus \Gamma_j} G_2(k; M, P) \varphi_j(\tau) d\ell_P. \]

Since the function \( \varphi_i \) is defined as in domain \( D_i \) and in the domain \( D_j \) \((D_j \subset D_i \) with \( j < i \)), we can consider the potential on the boundary \( \partial D_j \) with a density \( \varphi_j \). Then the following relations hold:

\[ \int_{\partial D_i \setminus \Gamma_i} \partial_n(P) G_1(k; M, P) \varphi_i(\tau) d\ell_P - \int_{\partial D_j \setminus \Gamma_j} \partial_n(P) G_1(k; M, P) \varphi_j(\tau) d\ell_P = \]
\[
\int_{\partial D_j \setminus \Gamma_1} \partial_n (P) G_1 (k; M, P) \varphi_i (\tau) d\ell_P - \int_{\partial D_i \setminus \Gamma_1} \partial_n (P) G_1 (k; M, P) \varphi_i (\tau) d\ell_P \\
+ \int_{\partial D_j \setminus \Gamma_1} \partial_n (P) G_1 (k; M, P) (\varphi_i (\tau) - \varphi_j (\tau)) d\ell_P.
\]

Hence, from the last relation and from (15) we obtain
\[
\int_{\partial D_j \setminus \Gamma_1} \partial_n (P) G_1 (k; M, P) (\varphi_i (\tau) - \varphi_j (\tau)) d\ell_P =
\]
\[
- \int_{\partial D_i \setminus \Gamma_1} \partial_n (P) G_1 (k; M, P) \varphi_i (\tau) d\ell_P.
\]

Similarly, in the case of the Neumann problem, we have
\[
\int_{\partial D_j \setminus \Gamma_1} G_2 (k; M, P) (\varphi_i (\tau) - \varphi_j (\tau)) d\ell_P =
\]
\[
- \int_{\partial D_j \setminus \Gamma_1} G_2 (k; M, P) \varphi_i (\tau) d\ell_P.
\]

The right-hand sides of the last relations for \( j \to \infty \) tend to 0, since \( \partial D_j \) approach \( \partial D \). From these relations and the properties of approximation by smooth domains, convergence of the sequence of functions \( \{ \varphi_i \} \) follows. We denote by \( \psi^*, \varphi^* \) the limits of the sequence of densities in the case of the Dirichlet and Neumann problem, respectively.

We show that the functions \( u^* = W(k)\psi^* + \tilde{u}, \ v^* = V(k)\varphi^* + \tilde{v} \) satisfy the conditions of the boundary value problem.

The use of the trace operator \( \gamma \) to the function \( u^* \) and the operator \( \gamma' \) to the function \( v^* \), by Lemmas 1 and 3, leads to the relations
\[
\gamma u^* = - \frac{1}{2} \psi + W(k)\psi + \tilde{u}|_{\partial D}, \quad \gamma' u^* = \frac{1}{2} \varphi + V'(k)\varphi + \tilde{v}|_{\partial D}.
\]

The latter relations are understood in the sense of the nontangential limit of functions whose values on \( \partial D \) have the same values as \( f \) or \( g \) (depending on the boundary condition). Therefore, we arrive at the conclusion that the relations
\[
\gamma u^*(P) = f(P), \quad \gamma' v^*(P) = g(P), \quad P \in \partial D.
\]

As a result, we conclude that the following theorems hold.

**Theorem 3.** Under condition \( \text{Im } k \geq 0 \), the sequence \( \{ \psi_j (x) \}_{1}^{\infty} \) of the solutions of integral equations \( 6 \) converges in a space \( L_2[0, d] \) to a function \( \psi(x) \) such that the function
\[
u(M) = \tilde{u}(M) + (W(k)\psi)(M)
\]
is a solution of the boundary value problem with the Dirichlet condition on the boundary. We denote by $\tilde{u}$ the solution of the Dirichlet problem on the half-plane, and $W(k)\psi$ is the potential type operator defined by (11).

**Theorem 4.** Under conditions $\text{Im } k \geq 0$ and $\text{Re } k \neq 0$, the sequence $\{\phi_j(x)\}_{j=1}^{\infty}$ of the solutions of the integral equations (9) converges to a function $\varphi(x)$ in the space $L_2[0,d]$ such that the function

$$u(M) = \tilde{v}(M) + (V(k)\varphi)(M)$$

is a solution of the boundary value problem with the Neumann condition on the boundary. We denote by $\tilde{v}$ the solution of the Neumann boundary value problem on the half-plane, and $V(k)\varphi$ is the potential type operator defined by (10).

**Theorem 5.** There exists a unique solution $u(x,y)$ of the boundary value problems under consideration, and the representations hold

$$u = \tilde{u} + W(k) \left( (I - W(k))^{-1} f \right) \quad \text{in the case of the Dirichlet problem},$$

$$u = \tilde{v} + V(k) \left( (I - V'(k))^{-1} g \right) \quad \text{in the case of the Neumann problem}.$$

**References**

1. Agranovich, M.S.: Sobolev Spaces, Their Generalizations, and Elliptic Problems in Smooth and Lipschitz Domains. Springer International Publishing Switzerland (2015), doi: 10.1007/978-3-319-14648-5
2. Agranovich, M.S.: Spectral problems for second-order strongly elliptic systems in smooth and non-smooth domains. Russ. Math. Surv. 57 (5), pp. 847–920 (2003)
3. Agranovich, M.S., Mennicken, R.: Spectral boundary value problems for the Helmholtz equation with spectral parameter in boundary conditions on a non-smooth surface. Sb. Math. 190 (1), pp. 29–69 (1999)
4. Costabel, M.: Boundary Integral Operators on Lipschitz Domains: Elementary Results. SIAM Journal on Mathematical Analysis. 19 (3), pp. 613–626 (1988), doi: 10.1137/0519043
5. Dahlberg, B.E.J., Kenig, C.: Harmonic Analysis and Partial Differential Equations. Göteborg, Dept. of Math. Chalmers University of Technology and the University of Göteborg (1985/1996)
6. Jerison, D., Kenig, C.: The Neumann problem on Lipschitz domains. Bull. Amer. Math. Soc. 4, pp. 203–207 (1981)
7. Jerison, D., Kenig, C.: The Dirichlet problem in non-smooth domains. Ann. of Math. 113, pp. 367–382 (1981)
8. Jerison, D., Kenig, C.: The inhomogeneous Dirichlet problem in Lipschitz domains. J. Funct. Anal. 130, pp. 161–219 (1995)
9. Mitrea, M., Taylor, M.: Boundary layer methods for Lipschitz domains in Riemannian manifolds. J. Funct. Anal. 163, pp. 181–251 (1999)
10. Mitrea, M., Taylor, M.: Potential theory on Lipschitz domains in Riemannian manifolds: Sobolev–Besov space results and the Poisson problem. J. Funct. Anal. 176, pp. 1–79 (2000)
11. Grisvard, P.: Elliptic Problems in Nonsmooth Domains. SIAM (2011), doi: 10.1137/1.9781611972030
12. McLean, W.: Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, Cambridge (2000)
13. Verchota, G.: Layer potentials and regularity for the Dirichlet problem for Laplace’s equation in Lipschitz domains. J. Funct. Anal. 59, pp. 572–611 (1984), doi: 10.1016/0022-1236(84)90066-1
14. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press (1970)
15. Lipachev, E.K.: On approximate solution of boundary-valued problem of wave diffraction on domains with infinity boundaries. Russian Math. 45 (4), pp. 67–70 (2001)
16. Lipachev, E.K.: Solution of the Dirichlet Problem for the Helmholtz Equation in Domains with a Rough Boundary. Russian Math. (Iz. VUZ), 50 (9), pp. 40–46 (2006)
17. Lipachev, E.K.: Dirichlet and Neumann boundary value problems for Helmholtz equation in unbounded domains with piecewise smooth part of boundary. Kazan. Gos. Univ. Uchen. Zap. Ser. Fiz.-Mat. Nauki, 148 (3), pp. 94–108 (2006)
18. Dynkin, E.M.: Methods of the Theory of Singular Integrals: Hilbert Transform and Calderon-Zygmund Theory. In Encyclopaedia of Mathematical Sciences. Vol. 15, Commutative Harmonic Analysis I: General Survey. Classical Aspects. (V.P. Khavin, N.K. Nikolskij eds.). Springer-Verlag, Berlin, pp. 167–259 (1991)
19. Maradudin A.A. (Ed.): Light Scattering and Nanoscale Surface Roughness. Springer Science-Business Media (2007)
20. Lipachev, E.K.: Boundary value problems for the Helmholtz equation in domains with an infinite Lipschitz boundary. Tr. Lobachevskii Math. Center, 43, pp. 225–227 (2011)
21. Tartar, L.: An Introduction to Sobolev Spaces and Interpolation Spaces. Springer-Verlag Berlin Heidelberg (2007), doi: 10.1007/978-3-540-71483-5
22. Mitrea, I., Mitrea, M.: Multi-Layer Potentials and Boundary Problems: for Higher-Order Elliptic Systems in Lipschitz Domains. Springer-Verlag Berlin Heidelberg (2013)
23. Tsang, L., Kong, J.A., Ding, K.-H., Ao, C.O.: Scattering of Electromagnetic Waves. Numerical Simulations. John Wiley & Sons (2001)
24. Colton, D., Kress, R.: Integral Equation Methods in Scattering Theory. SIAM, Philadelphia (2013)
25. Mazya, V.G.: Boundary Integral Equations. In Encyclopaedia of Mathematical Sciences. Vol. 27, Analysis IV. (V.G. Mazya, S.M. Nikolskii, eds.). Springer-Verlag, Berlin, pp. 127–228 (1991)
26. Abramowitz, M., Stegun, I.A.: Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables. National Bureau of Standards. Washington (1972)