Harmonic Hermitian Structures on Riemannian Manifolds with Skew Torsion

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Abstract. We find geometric conditions on a four-dimensional Hermitian manifold endowed with a metric connection with totally skew-symmetric torsion under which the complex structure is a harmonic map from the manifold into its twistor space considered with a natural family of Riemannian metrics defined by means of the metric and the given connection on the base manifold.

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1. Introduction

Recall that an almost complex structure on a Riemannian manifold is called almost Hermitian or compatible if it is an orthogonal endomorphism of the tangent bundle of the manifold. It is well known that if a Riemannian manifold \((N,h)\) admits an almost Hermitian structure, it possesses many such structures. In Refs. \([6,9]\), for example, this is shown by considering an almost Hermitian structure on \((N,h)\) as a section of the twistor bundle \(\pi : Z \to N\) whose fibre at a point \(p \in N\) consists of all \(g\)-orthogonal complex structures \(I_p : TN \to T_pN\) \((I_p^2 = -Id)\) on the tangent space of \(N\) at \(p\). Thus, it is natural to look for “reasonable” criteria that distinguish some of the almost Hermitian structures on a Riemannian manifold. Since we shall consider the almost Hermitian structures on \((N,h)\) as sections of the bundle \(Z\), let us note that the fibre of this bundle is the compact Hermitian symmetric space \(O(2m)/U(m)\), \(2m = \text{dim} M\), and its standard metric \(G = -\frac{1}{2}\text{Trace} (I_1 \circ I_2)\) is Kähler–Einstein. The Levi-Civita connection of \((N,h)\) gives rise to a splitting \(TZ = \mathcal{H} \oplus \mathcal{V}\) of the tangent bundle of \(Z\) into horizontal and vertical parts. This decomposition allows one to define a 1-parameter family of Riemannian metrics \(h_t = \pi^*h + tG\), \(t > 0\), for which the projection map \(\pi : (Z,h_t) \to (N,h)\)

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is a Riemannian submersion with totally geodesic fibres. In the terminology of [2, Definition 9.7], the family $h_t$ is the canonical variation of the metric $h$. Motivated by harmonic map theory, Wood [21,22] has suggested to consider as “optimal” those almost Hermitian structures $J : (N, h) \to (\mathcal{Z}, h_1)$ that are critical points of the energy functional under variations through sections of $\mathcal{Z}$. In general, these critical points are not harmonic maps, but, by analogy, they are referred to as “harmonic almost complex structures” in Ref. [22]; they are also called “harmonic sections” in Ref. [21], a term which seems more appropriate in the context of this paper. Forgetting the bundle structure of $\mathcal{Z}$, we can consider the almost Hermitian structures that are points of the energy functional under variations through arbitrary smooth maps $N \to \mathcal{Z}$, not just sections. These structures are genuine harmonic maps from $(N, h)$ into $(\mathcal{Z}, h_t)$, and we refer to Ref. [10] for basic facts about harmonic maps. This point of view is taken in Ref. [9] where the problem of when an almost Hermitian structure on a Riemannian four-manifold is a harmonic map from the manifold into its twistor space is discussed.

Every metric connection $D$ on $(N, h)$ yields a decomposition of $T\mathcal{Z}$ into horizontal and vertical subbundles, and having such a connection we can define the corresponding family $h_t$ of Riemannian metrics on the twistor space $\mathcal{Z}$. Of special interest are metric connections with (totally) skew-symmetric torsion which have many applications in various areas of mathematics and physics. From the point of view of twistor theory, it is worth noting that the Atiyah–Hitchin–Singer [1] almost complex structures on the twistor space of an oriented four-dimensional Riemannian manifold defined by means of the Levi-Civita connection and by a metric connection with skew-symmetric torsion have the same integrability condition, see, for example, Ref. [7]. Now, recall that a smooth map $f : (N, h) \to (N', h')$ of Riemannian manifolds is harmonic if and only if the trace of its second fundamental form $II_f(\nabla, \nabla')$ defined by means of the Levi-Civita connections $\nabla$ of $(N, h)$ and $\nabla'$ of $(N', h')$ vanishes. If $D$ and $D'$ are metric connections with skew-symmetric torsion a simple calculation shows that the trace of the second fundamental form $II_f(D, D')$ defined by means of $D$ and $D'$ coincides with $Trace II_f(\nabla, \nabla')$. Thus, the metric connections with skew-symmetric torsion can be used for studying harmonic maps between Riemannian manifolds endowed with additional structures that are preserved by such connections.

In the present paper, we consider the twistor space of a four-dimensional Riemannian manifold $(M, g)$ endowed with a metric connection $D$ with skew-symmetric torsion and, by means of this connection, we define the metrics $h_t$. We find the conditions under which a Hermitian structure $J$ on $(M, g)$ is a harmonic map $J : (M, g) \to (\mathcal{Z}, h_t)$. In particular, if $D$ is the Bismut–Strominger connection [3,19], the unique metric connection with skew-torsion preserving the complex structure, then the map $J$ is always harmonic. Two examples illustrating the obtained result are given, one of them showing that, for a fixed metric, there can be many metric connections with skew-torsion for which $J$ is a harmonic map.
2. Harmonic Maps Between Riemannian Manifolds Endowed with Metric Connection with Skew Torsion

Let \((N, h)\), \((N', h')\) be Riemannian manifolds and \(f : N \rightarrow N'\) a smooth map. If \(f^*TN'\) is the pull-back bundle of the bundle \(TN'\) under the map \(f\), we can consider the differential \(f^* : TN \rightarrow TN'\) as a section of the bundle \(\text{Hom}(TN, f^*TN')\). Suppose we are given connections \(D\) and \(D'\) on \(TN\) and \(TN'\), respectively. Denote by \(D'^*\) the connection on \(f^*TN'\) induced by the connection \(D'\) of \(TN'\). The connections \(D\) and \(D'^*\) give rise to a connection \(\tilde{D}\) on the bundle \(\text{Hom}(TN, f^*TN')\). Define a bilinear form on \(N\) setting

\[ II_f(D, D')(X, Y) = (\tilde{D}X f^*)(Y), \quad X, Y \in TN. \]

If \(D\) and \(D'\) are torsion-free, this form is symmetric. Recall that the map \(f\) is said to be harmonic if for the Levi-Civita connections \(\nabla\) and \(\nabla'\) of \(TN\) and \(TN'\),

\[ \text{Trace}_h II_f(\nabla, \nabla') = 0. \]

Now, suppose that \(D\) and \(D'\) are metric connections with (totally) skew-symmetric torsions \(T\) and \(T'\), i.e., the trilinear form

\[ T(X, Y, Z) = h(T(X, Y), Z), \quad X, Y, Z \in TN \]

is skew-symmetric, and similarly for \(T'(X, Y, Z) = h'(T'(X, Y), Z)\). Recall that, on a Riemannian manifold, there is a unique metric connection with a given torsion; for an explicit formula, see, for example, [12, Sec. 3.5, formula (14)]. Since the torsion 3-forms \(T\) and \(T'\) are skew-symmetric,

\[ D_X Y = \nabla_X Y + \frac{1}{2} T(X, Y), \quad D'_X Y = \nabla'_X Y + \frac{1}{2} T'(X, Y). \] (1)

**Lemma 1.** Let \(f : (N, h) \rightarrow (N', h')\) be a smooth map of Riemannian manifolds endowed with metric connections \(D\) and \(D'\) with skew-symmetric torsions \(T\) and \(T'\). Denote the Levi-Civita connections of \((N, h)\) and \((N, h')\) by \(\nabla\) and \(\nabla'\). Then

\[ \text{Trace}_h II_f(D, D') = \text{Trace}_h II_f(\nabla, \nabla'). \]

**Proof.** Take a point \(p \in N\) and let \(E_1, \ldots, E_n\) and \(E'_1, \ldots, E'_{n'}\) be frames of \(TN\) and \(TN'\) defined on neighbourhoods \(U\) of \(p\) and \(U'\) of \(p' = f(p)\) such that \(f(U) \subset U'\). Choose the frame \(E_1, \ldots, E_n\) to be \(h\)-orthonormal. Then \(f \circ E_i = \sum_{\alpha=1}^{n'} \lambda_{i\alpha} E'_\alpha \circ f\) where \(\lambda_{i\alpha}\) are smooth functions on \(U\). Under this notation,
\[
\text{Trace}_h II_f(D,D') = \sum_i [D'_E_i(f_* \circ E_i) - f_*(D_E_i E_i)] \\
= \sum_i \sum_\alpha [E_i(\lambda_{i\alpha})E'_\alpha \circ f + \sum_\beta \lambda_{i\alpha} \lambda_{i\beta} (D'_{E'_\beta} E'_\alpha) \circ f] - \sum_i f_*(D_E_i E_i) \\
= \sum_i \sum_\alpha [E_i(\lambda_{i\alpha})E'_\alpha \circ f + \sum_\beta \lambda_{i\alpha} \lambda_{i\beta} (\nabla'_{E'_\beta} E'_\alpha) \circ f] \\
+ \frac{1}{2} \sum_i T'(f_*(E_i), f_*(E_i)) - \sum_i f_*(\nabla E_i E_i) - \frac{1}{2} \sum_i T(E_i, E_i) \\
= \text{Trace}_h II_f(\nabla, \nabla').
\]

Corollary 1. A map \( f : N \to N' \) is harmonic if and only if
\[
\text{Trace}_h II_f(D,D') = 0.
\]

3. Basics on Twistor Spaces

We recall first some basic facts about twistor spaces following [6, 7].

Let \((M, g)\) be an oriented (connected) Riemannian manifold of dimension four. The metric \(g\) induces a metric on the bundle of two-vectors \(\pi: \Lambda^2 TM \to M\) by the formula
\[
g(v_1 \wedge v_2, v_3 \wedge v_4) = \frac{1}{2} \det [g(v_i, v_j)]
\]
(2)

(the choice of the factor 1/2 is explained below).

Let \(\ast : \Lambda^k TM \to \Lambda^{4-k} TM, k = 0, \ldots, 4\), be the Hodge star operator. Its restriction to \(\Lambda^2 TM\) is an involution, thus we have the orthogonal decomposition
\[
\Lambda^2 TM = \Lambda_-^2 TM \oplus \Lambda_+^2 TM,
\]
where \(\Lambda_{\pm}^2 TM\) are the subbundles of \(\Lambda^2 TM\) corresponding to the \((\pm 1)\)-eigenvalues of the operator \(\ast\).

Let \((E_1, E_2, E_3, E_4)\) be a local oriented orthonormal frame of \(TM\). Set
\[
s_1^\pm = E_1 \wedge E_2 \pm E_3 \wedge E_4, \quad s_2^\pm = E_1 \wedge E_3 \pm E_4 \wedge E_2, \quad s_3^\pm = E_1 \wedge E_4 \pm E_2 \wedge E_3.
\]
(3)

Then \((s_1^\pm, s_2^\pm, s_3^\pm)\) is a local orthonormal frame of \(\Lambda_{\pm}^2 TM\). This frame defines an orientation on \(\Lambda_{\pm}^2 TM\) which does not depend on the choice of the frame \((E_1, E_2, E_3, E_4)\) (see, for example, [6]). We call this orientation “canonical”.

For every \(a \in \Lambda^2 TM\), define a skew-symmetric endomorphism of \(T_{\pi(a)} M\) by
\[
g(K_a X, Y) = 2g(a, X \wedge Y), \quad X, Y \in T_{\pi(a)} M.
\]
(4)
Denoting by $G$ the standard metric $-\frac{1}{2} Trace PQ$ on the space of skew-symmetric endomorphisms, we have $G(K_a, K_b) = 2g(a, b)$ for $a, b \in \Lambda^2 TM$. If $a \in \Lambda^2 TM$ is of unit length, then $K_a$ is a complex structure on the vector space $T_{\pi(a)}M$ compatible with the metric $g$, i.e., $g$-orthogonal. Conversely, the 2-vector $a$ dual to one half of the fundamental 2-form of such a complex structure is a unit vector in $\Lambda^2 TM$. Therefore the unit sphere bundle $Z$ of $\Lambda^2 TM$ parametrizes the complex structures on the tangent spaces of $M$ compatible with the metric $g$ (so, the factor $1/2$ in (2) is chosen to have spheres with radius 1). This bundle is called the twistor space of the Riemannian manifold $(M, g)$. Since $M$ is oriented, the manifold $Z$ has two connected components $Z_\pm$ called the positive and the negative twistor spaces of $(M, g)$. These are the unit sphere subbundles of $\Lambda^2_\pm TM$. The bundle $Z_\pm \to M$ parametrizes the complex structures on the tangent spaces of $M$ compatible with the metric and $\pm$ the orientation via the correspondence $Z_\pm \ni \sigma \to K_\sigma$. Note that changing the orientation of $M$ interchanges the roles of $\Lambda^2_- TM$ and $\Lambda^2_+ TM$ and, respectively, of $Z_-$ and $Z_+$.

The vertical space $V_\sigma = \{V \in T_\sigma Z_\pm : \pi_* V = 0\}$ of the bundle $\pi : Z_\pm \to M$ at a point $\sigma$ is the tangent space to the fibre of $Z_\pm$ through $\sigma$. Thus, considering $T_\sigma Z_\pm$ as a subspace of $T_\sigma(\Lambda^2_\pm TM)$, $V_\sigma$ is the orthogonal complement of $\mathbb{R}\sigma$ in $\Lambda^2_\pm T_{\pi(\sigma)}M$.

Let $D$ be a metric connection on $(M, g)$. The induced connection on $\Lambda^2 TM$ will also be denoted by $D$. If $(s^+_1, s^+_2, s^+_3)$ is the orthonormal frame of $\Lambda^2_+ TM$ defined by means of an oriented orthonormal frame $(E_1, ..., E_4)$ of $TM$ via (3), we have $g(D_X s^+_i, s^-_j) = g(D_X s^-_i, s^+_j) = 0$ and $g(D_X s^+_i, s^+_j) = -g(D_X s^-_i, s^-_j)$ for every $X \in TM$ and every $i, j = 1, 2, 3$. Therefore the connection $D$ preserves the bundles $\Lambda^2_\pm TM$ and induces a metric connection on each of these bundles denoted again by $D$. Let $\sigma \in Z_\pm$, and let $s$ be a local section of $Z_\pm$ such that $s(p) = \sigma$ where $p = \pi(\sigma)$. Considering $s$ as a section of $\Lambda^2_\pm TM$, we have $D_X s \perp s(p)$, i.e., $D_X s \in V_\sigma$ for every $X \in T_p M$ since $s$ has a constant length. Moreover, $X^b_\sigma = s, X - D_X s \in T_p Z_\pm$ is the horizontal lift of $X$ at $\sigma$ with respect the connection $D$ on $\Lambda^2_\pm TM$. Thus, the horizontal distribution of $\Lambda^2_\pm TM$ with respect to $D$ is tangent to the twistor space $Z_\pm$. In this way, we have the decomposition $TZ_\pm = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $Z_\pm$ into horizontal and vertical components. The horizontal and vertical parts of a tangent vector $A \in TZ_\pm$ will be denoted by $\mathcal{H}A$ and $\mathcal{V}A$, respectively.

**Convention.** In what follow, we shall freely identify the bundle $\Lambda^2 TM$ with the bundle $A(TM)$ of $g$-skew-symmetric endomorphism of $TM$ by means of the isomorphism $a \to K_a$ defined by (4).

Using the basis (3), it is easy to check that if $a, b \in \Lambda^2_\pm T_p M$, the isomorphism $\Lambda^2 TM \cong A(TM)$ sends $a \times b$ to $\pm \frac{1}{2}[K_a, K_b]$. In the case when $a \in \Lambda^2_+ T_p M$, $b \in \Lambda^2_- T_p M$, the endomorphisms $K_a$ and $K_b$ of $T_p M$ commute. If $a, b \in \Lambda_\pm T_p M$,

$$K_a \circ K_b = -g(a, b)Id \pm K_{a \times b}.$$
In particular, $K_a$ and $K_b$ anti-commute if and only if $a$ and $b$ are orthogonal.

For every $t > 0$, define a Riemannian metric $h_t = h_t^D$ on $Z_+$ by

$$h_t(X^h_\sigma + V, Y^h_\sigma + W) = g(X, Y) + tg(V, W)$$

for $\sigma \in Z_+, X, Y \in T_{\pi(\sigma)}M, V, W \in V_\sigma$.

**Notation.** We set $Z = Z_+$. The sections $s_i^+$ of $\Lambda^2_+ TM$ defined via (3) will be denoted by $s_i$, $i = 1, 2, 3$. The Levi-Civita connection of the metric $h_t$ on $Z$ will be denote by $\tilde{D}$.

By the Vilms theorem (see, for example, [2, Theorem 9.59]), the projection map $\pi : (Z, h_t) \rightarrow (M, g)$ is a Riemannian submersion with totally geodesic fibres. This can also be proved by an easy direct computation as is shown below.

**Convention.** For the curvature tensor we adopt the following definition:

$$R^D(X, Y) = D_{[X,Y]} - [DX, DY].$$

The curvature tensor of the induced connection on $\Lambda^2_+ TM$ will again be denoted by $R^D$. The curvature operator $R^D : \Lambda^2 TM \rightarrow \Lambda^2 TM$ is defined by $g(R^D(X \wedge Y), Z \wedge U) = g(R^D(X, Y)Z, U)$.

The following formula is easy to be checked.

**Lemma 2.** If $a, b \in \Lambda^2_+ T_p M$, then

$$g(R^D(X, Y)a, b) = g(R^D(X \wedge Y), a \times b)$$

for every $X, Y \in T_p M$.

Let $(\mathcal{N}, x_1, ..., x_4)$ be a local coordinate system of $M$ and let $(E_1, ..., E_4)$ be an oriented orthonormal frame of $TM$ on $\mathcal{N}$. If $(s_1, s_2, s_3)$ is the local frame of $\Lambda^2_+ TM$ defined by (3), then $\tilde{x}_\alpha = x_\alpha \circ \pi, y_j(\sigma) = g(\tau, (s_j \circ \pi)(\sigma))$, $1 \leq \alpha \leq 4, 1 \leq j \leq 3$, are local coordinates of $\Lambda^2_+ TM$ on $\pi^{-1}(\mathcal{N})$.

The horizontal lift $X^h$ on $\pi^{-1}(\mathcal{N})$ of a vector field

$$X = \sum_{\alpha=1}^4 X^\alpha \frac{\partial}{\partial x_\alpha}$$

is given by

$$X^h = \sum_{\alpha=1}^4 (X^\alpha \circ \pi) \frac{\partial}{\partial \tilde{x}_\alpha} - \sum_{j,k=1}^3 y_j(g(DX s_j, s_k) \circ \pi) \frac{\partial}{\partial y_k}. \quad (5)$$

Let $\sigma \in Z$. Under the standard identification $T_\sigma(\Lambda^2_+ T\pi(\sigma)M) \approx \Lambda^2_+ T\pi(\sigma)M$, formula (5) implies the well-known identity

$$[X^h, Y^h]_\sigma = [X, Y]_\sigma^h + R^D_p(X, Y)_\sigma, \quad p = \pi(\sigma). \quad (6)$$

For a fixed $\sigma \in Z$, take an oriented orthonormal frame $E_1, ..., E_4$ such that $(E_3)_p = K_\sigma(E_2)_p, (E_4)_p = K_\sigma(E_1)_p$, $p = \pi(\sigma)$, so $s_3(p) = \sigma$, where $s_3$ is defined by means of $E_1, ..., E_4$ via (3). Define coordinates $(\tilde{x}_\alpha, y_j)$ as above by means of this frame and a coordinate system of $M$ at $p$. Set
\[ V_1 = (1 - y_2^2)^{-1/2} \left( y_3 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_3} \right), \]
\[ V_2 = (1 - y_2^2)^{-1/2} \left( -y_1 y_2 \frac{\partial}{\partial y_1} + (1 - y_2^2) \frac{\partial}{\partial y_2} - y_2 y_3 \frac{\partial}{\partial y_3} \right). \]  

Then \( V_1, V_2 \) is a \( g \)-orthonormal frame of vertical vector fields in a neighbourhood of the point \( \sigma \) such that \( (V_1)_\sigma = s_1(p), (V_2)_\sigma = s_2(p) \). An easy computation using \( (5) \) gives
\[ [X^h, V_1]_\sigma = g(D_X s_1, s_2) s_2(p), \quad [X^h, V_2]_\sigma = -g(D_X s_1, s_2) s_1(p). \]  

Then the Koszul formula for the Levi-Civita connection implies that the vectors \( \tilde{D}_X V_i \) are orthogonal to every horizontal vector \( X^h \), hence they are vertical vectors. It follows that the fibers of the bundle \( \pi : (Z, h_t) \to (M, g) \) are totally geodesic submanifolds.

**Notation.** We denote the Levi-Civita connection of \( (M, g) \) by \( \nabla \).

The Koszul formula, identity \( (6) \) and the fact that the fibres of the twistor bundle are totally geodesic submanifolds imply the following formulas.

**Lemma 3** ([8]). If \( X, Y \) are vector fields on \( M \) and \( V \) is a vertical vector field on \( Z \), then
\[ (\tilde{D}_X Y^h)_\sigma = (\nabla_X Y)^h_\sigma + \frac{1}{2} R^D_{\pi(\sigma)}(X, Y)_\sigma, \quad \sigma \in Z. \]  

Also, \( \tilde{D}_V X^h = \mathcal{H}\tilde{D}_X V \), where \( \mathcal{H} \) means “the horizontal component”. Moreover,
\[ h_t(\tilde{D}_V X^h, Y^h)_\sigma = -\frac{1}{2} g(R^D_{\pi(\sigma)}(X, Y)_\sigma, V), \quad \sigma \in Z. \]

**4. The Second Fundamental Form of an Almost Hermitian Structure as a Map into the Twistor Space**

Let \( (M, g, J) \) be an almost Hermitian manifold of dimension four. Define a section \( \mathcal{J} \) of \( \Lambda^2 TM \) by
\[ g(\mathcal{J}, X \wedge Y) = \frac{1}{2} g(JX, Y), \quad X, Y \in TM. \]

Note that the section \( 2\mathcal{J} \) is dual to the fundamental 2-form of \( (M, g, J) \). Consider \( M \) with the orientation induced by the almost complex structure \( J \). Then \( \mathcal{J} \) takes its values in the (positive) twistor space \( Z \) of the Riemannian manifold \( (M, g) \).

**Notation.** Denote by \( \tilde{D}^* \) the connection on the pull-back bundle \( \mathcal{J}^* TZ \) induced by the Levi-Civita connection \( \tilde{D} \) of \( (Z, h_t) \). The connections \( D \) on \( TM \) and \( \tilde{D}^* \) on \( \mathcal{J}^* TZ \) induce a connection \( \tilde{D} \) on the bundle \( \text{Hom}(TM, \mathcal{J}^* TZ) \).
Proposition 1. For every \( p \in M \) and every \( X, Y \in T_pM \),
\[
(\tilde{D}_X \mathcal{J}_*)(Y) = \frac{1}{2} [D_{XY}^2 \mathcal{J} - g(D_{XY}^2 \mathcal{J}, \mathcal{J})(p) + D_Y^2 X \mathcal{J} - g(D_Y^2 X \mathcal{J}, \mathcal{J})(p) \]
\[
- D_{T(X,Y)} \mathcal{J} - (T(X,Y))_{\mathcal{J}(p)}^h \]
\[
+ (\tilde{D}_{DX} Y^h)_{\mathcal{J}(p)} + (\tilde{D}_{DY} X^h)_{\mathcal{J}(p)},
\]
where \( D_{XY}^2 \mathcal{J} = D_X D_Y \mathcal{J} - D_{DX} Y \mathcal{J} \) is the second covariant derivative of the section \( \mathcal{J} \) of \( \Lambda^2_t TM \) and \( T \) is the torsion of \( D \).

Proof. Extend \( X \) and \( Y \) to vector fields in a neighbourhood of the point \( p \). Take an oriented orthonormal frame \( E_1, ..., E_4 \) near \( p \) such that \( E_3 = JE_2, E_4 = JE_1 \), so \( \mathcal{J} = s_3 \). Introduce coordinates \((\tilde{x}_\alpha, y_j)\) as above by means of this frame and a coordinate system of \( M \) at \( p \). Let \( V_1, V_2 \) be the \( g \)-orthonormal frame of vertical vector fields in a neighbourhood of the point \( \sigma = \mathcal{J}(p) \) defined by (7). For this frame, \( V_1 \circ \mathcal{J} = s_1, V_2 \circ \mathcal{J} = s_2 \). Note also that \([V_1, V_2]_\sigma = 0\). This and the Koszul formula imply \((\tilde{D}_{V_k} V_l)_\sigma = 0 \) since \( \tilde{D}_{V_k} V_l \) are vertical vector fields, \( k, l = 1, 2 \). Thus, \( \tilde{D}_W V_l = 0, l = 1, 2 \), for every vertical vector \( W \) at \( \sigma \). We have
\[
\mathcal{J}_* \circ Y = Y^h \circ \mathcal{J} + D_Y \mathcal{J} = Y^h \circ \mathcal{J} + \sum_{k=1}^2 g(D_Y \mathcal{J}, s_k)(V_k \circ \mathcal{J}),
\]
hence
\[
\tilde{D}_X^* (\mathcal{J}_* \circ Y) = (\tilde{D}_X \mathcal{J}_*) \circ Y + \sum_{k=1}^2 g(D_X \mathcal{J}, s_k)(\tilde{D}_X \mathcal{J}_* V_k) \circ \mathcal{J}
\]
\[
+ \sum_{k=1}^2 [g(D_X D_Y \mathcal{J}, s_k) + g(D_Y \mathcal{J}, D_X s_k)](V_k \circ \mathcal{J}).
\]
This, in view of Lemma 3, implies
\[
\tilde{D}_{X^p}^* (\mathcal{J}_* \circ Y) = (\nabla X Y)_{\mathcal{J}(p)}^h + \frac{1}{2} R^D(X,Y) \mathcal{J}(p)
\]
\[
+ (\tilde{D}_{DX} Y^h)_{\mathcal{J}(p)} + (\tilde{D}_{DY} X^h)_{\mathcal{J}(p)}
\]
\[
+ \sum_{k=1}^2 g(D_{X^p} D_Y \mathcal{J}, s_k(p))s_k(p)
\]
\[
+ \sum_{k=1}^2 [g(D_{Y^p} \mathcal{J}, s_k(p))[X^h, V_k]_{\mathcal{J}(p)}
\]
\[
+ g(D_{Y^p} \mathcal{J}, D_{X^p} s_k(p))s_k(p)].
\]
By the convention for the curvature tensor,
\[
R^D(X,Y) \mathcal{J}(p) = D_{DX_p Y} \mathcal{J} - D_{DY_p X} \mathcal{J} - D_{T(X,Y)_p} \mathcal{J} - D_{X^p D_Y \mathcal{J}} + D_{Y^p D_X \mathcal{J}}.
\]
Identities (8) imply
\[
\sum_{k=1}^2 [g(D_{Y^p} \mathcal{J}, s_k(p))[X^h, V_k]_{\mathcal{J}(p)} + g(D_{Y^p} \mathcal{J}, D_{X^p} s_k(p))s_k(p)] = 0.
\]
Since \( g(D_X \mathfrak{J}, \mathfrak{J}) = 0 \) for every \( X \), we have
\[
g(D_Y D_X \mathfrak{J}, \mathfrak{J}) = -g(D_X \mathfrak{J}, D_Y \mathfrak{J}) = g(D_X D_Y \mathfrak{J}, \mathfrak{J}).
\]

Hence,
\[
\sum_{k=1}^{2} g(D_{X_k} D_Y \mathfrak{J}, s_k(p)) s_k(p) = D_{X_p} D_Y \mathfrak{J} - \frac{1}{2} g(D_{X_p} D_Y \mathfrak{J} + D_{Y_p} D_X \mathfrak{J}, \mathfrak{J}(p)) \mathfrak{J}(p).
\]

The identities above give
\[
\mathcal{D}_p^X (\mathfrak{J}_* \circ Y) = (\nabla_X Y)^h_{\mathfrak{J}(p)}
\]
\[
+ \frac{1}{2} [(D_{X_p} D_Y \mathfrak{J} + D_{Y_p} D_X \mathfrak{J}) - \frac{1}{2} g(D_{X_p} D_Y \mathfrak{J} + D_{Y_p} D_X \mathfrak{J}, \mathfrak{J}(p))] \mathfrak{J}(p)
\]
\[
+ \frac{1}{2} [D_{D_{X_p} Y} \mathfrak{J} - D_{D_{Y_p} X} \mathfrak{J} - D_{T(X,Y)_p} \mathfrak{J}] + (\mathcal{D}_{D_X \mathfrak{J}} X^h_{\mathfrak{J}})_{\mathfrak{J}(p)} + (\mathcal{D}_{D_Y \mathfrak{J}} X^h_{\mathfrak{J}})_{\mathfrak{J}(p)}.
\]

This and (1) imply the desired formula for
\[
(\mathcal{D}_X \mathfrak{J}_*)(Y) = \mathcal{D}_p^X (\mathfrak{J}_* \circ Y) - (D_{X_p} Y)^h_{\mathfrak{J}(p)} - D_{D_{X_p} Y} \mathfrak{J}.
\]

Proposition 1 and Lemma 3 imply

**Corollary 2.** For every \( p \in M \),
\[
\mathcal{V}(Trace_g \mathcal{D}_p^X \mathfrak{J}_*) = Trace_g \{ T_p M \ni X \rightarrow \mathcal{V} D_{X_p}^2 \mathfrak{J} \}
\]
\[
\mathcal{H}(Trace_g \mathcal{D}_p^X \mathfrak{J}_*) = 2 Trace_g \{ T_p M \ni X \rightarrow (\mathcal{D}_{D_X \mathfrak{J}} X^h_{\mathfrak{J}})_{\mathfrak{J}(p)} \},
\]
where \( \mathcal{V} \) and \( \mathcal{H} \) mean the horizontal and the vertical component, respectively.

Note that
\[
g(D_{X_p}^2 \mathfrak{J}, Z \wedge U) = \frac{1}{2} g(D_{X_p}^2 J)(Z), U), \quad X, Y, Z, U \in TM.
\]

Every vector of the vertical space \( \mathcal{V}_{\mathfrak{J}(p)}, p \in M \), is a linear combination of vectors of the type \( Z \wedge U - JZ \wedge JU, Z, U \in T_p M \). Moreover, by (4), each vector \( Z \wedge U - JZ \wedge JU \) is orthogonal to every \( a \in \Lambda^2 TM \) since the endomorphisms \( J \) and \( K_a \) of \( T_p M \) commute. Also, \( Z \wedge U - JZ \wedge JU \) is orthogonal to \( \mathfrak{J}(p) \). Hence \( Z \wedge U - JZ \wedge JU \in \mathcal{V}_{\mathfrak{J}(p)} \).

**Assumption.** Suppose that the metric connection \( D \) has skew-symmetric torsion \( T \), so \( T(X, Y, Z) = g(T(X, Y), Z) \) is a skew-symmetric 3-form.

**Notation.** It is convenient to set
\[
Q(X, Z \wedge U - JZ \wedge JU) = T(X, Z, (\nabla_X J)(U)) - T(X, U, (\nabla_J X)(Z))
\]
\[
-T(X, JZ, (\nabla_X J)(JU)) + T(X, JU, (\nabla_X J)(JZ))
\]

**Lemma 4.** For every tangent vectors \( X, Y, Z, U \in T_p M \),
\[
2g(D_{X_p}^2 \mathfrak{J}, Z \wedge U - JZ \wedge JU) = 2g(D_{X_p}^2 \mathfrak{J}, Z \wedge U - JZ \wedge JU)
\]
\[
+ (\nabla_X T)(X \wedge (Z \wedge U + JZ \wedge U)) + Q(X, Z \wedge U - JZ \wedge JU)
\]
\[
- \frac{1}{2} T(X, JU(X, Z), U) + \frac{1}{2} T(X, JU(X, JZ), JU).
\]
Proof. Extend $X, Y, Z, U$ to vector fields. Then an easy computation gives

$$
2g(D_X^2 \mathfrak{J}, Z \wedge U - JZ \wedge JU) = 2g(\nabla^2_X \mathfrak{J}, Z \wedge U - JZ \wedge JU)
- g(\nabla_X JT(X, Z), U) - g(T(X, J\nabla_X Z), U) - \frac{1}{2}g(T(X, JT(X, Z)), U)
+ g(\nabla_X JT(X, JZ), JU) + g(T(X, J\nabla_X JZ), JU)
+ \frac{1}{2}g(T(X, JT(X, JZ)), JU)
- g(T(\nabla_X X, JZ), U) - g(T(\nabla_X X, Z), JU)
= -g((\nabla_X J)(T(X, Z)), U) + g(\nabla_X T)(X, Z), JU)
+ g(T(X, (\nabla_X J)(Z)), U)
+ g((\nabla_X J)(T(X, JZ)), JU) + g((\nabla_X T)(X, JZ), U)
- g(T(X, (\nabla_X J)(JZ)), JU)
- \frac{1}{2}g(T(X, JT(X, Z)), U) + \frac{1}{2}g(T(X, JT(X, JZ)), JU).
$$

This proves the lemma.

We also need some properties of the curvature of the connection $D$. Denote by $R^\nabla$ the curvature tensor of the Levi-Civita connections $\nabla$.

A simple computation gives

$$
g(R^D(X, Y)Z, U) = g(R^\nabla(X, Y)Z, U)
- \frac{1}{2}[(\nabla_X T)(Y, Z, U) - (\nabla_Y T)(X, Z, U)]
+ \frac{1}{4} \sum_{i=1}^{4} [T(X, U, E_i)T(Y, Z, E_i) - T(X, Z, E_i)T(Y, U, E_i)],
$$

where $X, Y, Z, U \in T_p M$ and $\{E_1, ..., E_4\}$ is an orthonormal basis of $T_p M$. This implies

$$
g(R^D(X, Y)Z, U) = -g(R^D(X, Y)U, Z) \tag{12}
$$

Notation. The 1-form $\ast T$ will be denoted by $\tau$.

Clearly, the form $\tau$ uniquely determines the 3-form $T$, hence the connection $D$.

Convention. We identify $TM$ and $T^* M$ be means of the metric $g$ and extend this isomorphism to identification of $\Lambda^k TM$ and $\Lambda^k T^* M$. The exterior product on $\Lambda^k T^* M$ is so that $(\alpha_1 \wedge ... \wedge \alpha_k)(v_1, ..., v_k) = det[\alpha_i(v_j)]$ for $\alpha_i \in T^* M$ and $v_j \in TM$.

For a given orthonormal frame $E_1, ..., E_4$, it is convenient to set

$$
E_{ijk} = E_i \wedge E_j \wedge E_k, \quad T_{ijk} = T(E_{ijk}).
$$

We consider $\Lambda^3 TM$ with the metric for which $E_{ijk}, 1 \leq i < j < k \leq 4$, is an orthonormal basis. Then we have

$$
T = T_{123}E_{123} + T_{124}E_{124} + T_{134}E_{134} + T_{234}E_{234}
\tau = -T_{234}E_1 + T_{134}E_2 - T_{124}E_3 + T_{123}E_4. \tag{13}
$$
It follows that
\[
\begin{align*}
(\nabla_X T)(E_{123}) &= (\nabla_X \tau)(E_4), \\
(\nabla_X T)(E_{124}) &= -((\nabla_X \tau)(E_3), \\
(\nabla_X T)(E_{134}) &= (\nabla_X \tau)(E_2), \\
(\nabla_X T)(E_{234}) &= -((\nabla_X \tau)(E_1).
\end{align*}
\]

(14)
The latter identities imply that for every \(a \in \Lambda^2_+ TM\)
\[
d\tau(a) = -\delta T(a).
\]

(15)

Notation. The Ricci tensor and \(*\)-Ricci tensor are defined by
\[
\begin{align*}
\rho_D(X, Y) &= \text{Trace}\{Z \to g(R^D(X, Z)Y, Z)\}, \\
\rho^*_D(X, Y) &= \text{Trace}\{Z \to g(R^D(JZ, X)JY, Z)\}.
\end{align*}
\]

Note that \(\rho^*_D(X, Y) = \rho_D(JY, JX)\) by the identity \(g(R^D(JZ, X)JY, Z) = -g(R^D(Z, JY)J(JX), JZ)\). Equivalently, \(\rho^*_D(X, JX) = 0\).

Proposition 2.
\[
\begin{align*}
\rho_D(X, Y) &= \rho^*_D(X, Y) - \frac{1}{2} \delta T(X, Y) - 2g(\iota_X T, \iota_Y T) \\
\rho^*_D(X, Y) &= \rho^*_D(X, Y) + dT(3 \wedge X \wedge JY) + (\nabla_{JY} T)(3 \wedge X) \\
&\quad - \frac{1}{4}[2g(T(\mathfrak{J}), T(X \wedge JY)) + \chi(X, Y)],
\end{align*}
\]

where \(\chi(X, Y) = \text{Trace}\{\Lambda^2_+ T_p M \ni a \to T(a \wedge X)T((\mathfrak{J} \times a) \wedge JY)\}\). Moreover
\[
\chi(X, Y) = \chi(Y, X), \quad \chi(X, Y) = \chi(JY, JX).
\]

Proof. The first formula follows directly from \((11)\).

Let \(E_1, \ldots, E_4\) be an oriented orthonormal basis of a tangent space \(T_p M\) such that \(E_2 = JE_1, E_4 = JE_3.\) By \((11)\)
\[
\rho^*_D(X, Y) = \rho^*_D(X, Y) - \frac{1}{2} \sum_{k=1}^{4} \left[ (\nabla_X \tau)(X, JY, E_k) - (\nabla_X T)(JE_k, JY, E_k) \right] \\
+ \frac{1}{4} \sum_{i,k=1}^{4} \left[ T(JE_k, E_k, E_i) T(X, JY, E_i) \\
- T(JE_k, JY, E_i) T(X, E_k, E_i) \right].
\]

To compute the second summand in the right-hand side, we apply the identity
\[
dT(JE_k, X, JY, E_k) = (\nabla_X \tau)(X, JY, E_k) - (\nabla_X T)(JE_k, JY, E_k) \\
+ (\nabla_{JY} T)(JE_k, X, E_k) - (\nabla_{E_k} T)(JE_k, X, JY).
\]

Summing up, we get
\[
-2dT(3 \wedge X \wedge JY) = \sum_{k=1}^{4} \left[ (\nabla_X \tau)(X, JY, E_k) - (\nabla_X T)(JE_k, JY, E_k) \right] \\
+ 2(\nabla_{JY} T)(3 \wedge X).
\]

Next,
\[
\sum_{i,k=1}^{4} T(JE_k, E_k, E_i) T(X, JY, E_i) = -2g(T(3), T(X \wedge JY)).
\]
Finally, a direct computation gives
\[
\sum_{i,k=1}^{4} T(JE_k, JY, E_i)T(X, E_k, E_i) \\
= -T(s_2 \wedge X)T(s_3 \wedge JY) + T(s_3 \wedge X)T(s_2 \wedge JY) \\
= -\text{Trace}\{\Lambda_+^2 T_p M \ni a \rightarrow T(a \wedge X)T((J \times a) \wedge JY)\} = -\chi(X, Y).
\]
We have \(\chi(X, Y) = \chi(JY, JX)\) since
\[
\sum_{k=1}^{n} T(JE_k, JY, E_i)T(X, E_k, E_i) = -\sum_{k=1}^{n} T(E_k, J(JX), E_i)T(JY, JE_k, E_i).
\]
A direct computation shows that \(\chi(E_i, E_j) = \chi(E_j, E_i), i, j = 1, ..., 4.\) \(\square\)

5. Harmonicity of \(J\) in the Case of a Hermitian Structure

Let \(\Omega(X, Y) = g(JX, Y)\) be the fundamental 2-form of the almost Hermitian manifold \((M, g, J)\). Denote by \(N\) the Nijenhuis tensor of \(J\):
\[
N(Y, Z) = -[Y, Z] + [JY, JZ] - J[Y, JZ] - J[JY, Z].
\]
Clearly \(N(Y, Z) = -N(Z, Y), \quad N(JX, Y) = N(X, JY) = -JN(X, Y).\)

It is well-known (and easy to check) that
\[
2g(\nabla X J)(Y, Z) = d\Omega(X, Y, Z) - d\Omega(X, JY, JZ) + g(N(Y, Z), JX), \quad (16)
\]
for all \(X, Y, Z \in TM\). Note also that
\[
g(\nabla X J)(Y, Z) = (\nabla X \Omega)(Y, Z).
\]

Suppose that the almost complex structure \(J\) is integrable. This is equivalent to
\[
(\nabla X J)(Y) = (\nabla JX J)(JY), \quad X, Y \in TM, \quad (17)
\]
[11, Corollary 4.2]. Let \(B\) be the vector field on \(M\) dual to the Lee form \(\theta = -\delta \Omega \circ J\) with respect to the metric \(g\). Then \(16\) and the identity \(d\Omega = \theta \wedge \Omega\) imply the following well-known formula
\[
2(\nabla X J)(Y) = g(JX, Y)B - g(B, Y)JX + g(X, Y)JB - g(JB, Y)X. \quad (18)
\]
Since \(\nabla\) and \(D\) are metric connections,
\[
g(\nabla X \mathfrak{J}, Y \wedge Z) = \frac{1}{2}g((\nabla X J)(Y), Z), \quad g(DX \mathfrak{J}, Y \wedge Z) = \frac{1}{2}g((DX J)(Y), Z).
\]
Hence,
\[
\nabla X \mathfrak{J} = \frac{1}{2}(JX \wedge B + X \wedge JB).
\]
To compute the vertical part of \((\text{Trace}_{\varphi} \mathcal{D}\mathcal{J}_*)_p\), we apply Corollary 2 and Lemma 4. First, identities (18) and (19) imply ([9])

\[
(\nabla^2_{XY} J)(Z) = \frac{1}{2} \left[ g((\nabla_X J)(Y), Z)B - g(B, Z)(\nabla_X J)(Y) + g(Y, Z)(\nabla_Y J)(B) - g((\nabla_Y J)(B), Z)Y \\
+ g(JY, Z)\nabla_X B - g(\nabla_X B, Z)JY \\
+ g(Y, Z)J\nabla_X B - g(J\nabla_X B, Z)Y \right].
\]

It follows that

\[
2g(\text{Trace}\{X \to (\nabla^2_{XX} J)(Z)\}, U) = ||B||^2 g(Z, JU) - d\theta(JZ, U) - d\theta(Z, JU).
\]

(20)

Let \(E_1, \ldots, E_4\) be an orthonormal basis of a tangent space \(T_p M\) with \(E_2 = JE_1\), \(E_4 = JE_3\). Define \(s_1, s_2, s_3\) by means of this basis via (3). Thus, \(\mathcal{J} = s_1\) and \(s_2, s_3\) is a \(g\)-orthonormal basis of the vertical space \(\mathcal{V}_{\mathcal{J}_p}\).

Identity (20) implies

\[
g(\text{Trace}\{((\nabla^2_{XX} J)(E_1)\}, E_3) + g(\text{Trace}\{((\nabla^2_{XX} J)(E_4)\}, E_2) = -d\theta(s_3),
\]

\[
g(\text{Trace}\{((\nabla^2_{XX} J)(E_1)\}, E_4) + g(\text{Trace}\{((\nabla^2_{XX} J)(E_2)\}, E_3) = d\theta(s_2).
\]

Also, it follows from (15)

\[
\text{Trace}\{((\nabla X T)(X, s_2)\} = d\tau(s_2), \quad \text{Trace}\{((\nabla X T)(X, s_3)\} = d\tau(s_3).
\]

Applying (18), we obtain

\[
\text{Trace}\{Q(X, s_2)\} = \sum_{k=1}^{4} Q(E_k, E_1 \wedge E_3 - E_2 \wedge E_4)
\]

\[
= -g(B, E_1)T_{123} + g(B, E_2)T_{124} + g(B, E_3)T_{134} \\
- g(B, E_4)T_{234} - T(E_1, E_3, JB) + T(E_1, E_4, B) \\
+ T(E_2, E_3, B) + T(E_2, E_4, JB) \\
= g(B, E_1)T_{123} - g(B, E_3)T_{134} \\
+ g(B, E_2)T_{234} - g(B, E_4)T_{124} \\
= g(T(E_1 \wedge E_4 + E_2 \wedge E_3), B) = g(T(s_3), B).
\]

Also,

\[
\text{Trace}\{X \to -T(X, JT(X, E_1), E_3) + T(X, JT(X, E_2), E_4)\}
\]

\[
= -T_{213}T_{243} + T_{412}T_{413} - T_{124}T_{134} + T_{321}T_{324} = 0.
\]

Similarly,

\[
\text{Trace}\{Q(X, s_3)\} = \sum_{k=1}^{4} Q(E_k, E_1 \wedge E_4 + E_2 \wedge E_3)
\]

\[
= -g(T(E_1 \wedge E_3 + E_4 \wedge E_2), B) = -g(T(s_2), B)
\]

and

\[
\text{Trace}\{X \to -T(X, JT(X, E_1), E_4) - T(X, JT(X, E_2), E_3)\} = 0.
\]
Thus, by Lemma 4,
\[
2g(\text{Trace} D^2_X J, s_2) = -d\theta(s_3) + d\tau(s_3) + g(T(s_3), B)
\]
\[
2g(\text{Trace} D^2_X J, s_3) = d\theta(s_2) - d\tau(s_2) - g(T(s_2), B).
\]
Finally, note that for every \(X, Y \in TM\), the 2-vector \(X \wedge Y - JX \wedge JY\) is a linear combination of \(s_2\) and \(s_3\). Thus, by Corollary 2, \(\forall \text{Trace}_g \tilde{\mathcal{D}}_s = 0\) if and only if the 2-form \(d\theta - d\tau - \nu B T\) is of type \((1,1)\) with respect to \(J\).

Now, suppose \(B_p \neq 0\) at a point \(p \in M\). Take an oriented orthonormal basis \(E_1, \ldots, E_4 \) of \(T_p M\) such that \(E_2 = JE_1, E_3 = \frac{B_p}{\|B_p\|}, E_4 = JE_3\). Using this basis, define \(s_1, s_2, s_3\) via (3) so that \(J = s_1\). By (10),
\[
h_t(\text{Trace}\{T_p M \ni X \rightarrow D_{DX} J X^h\}, Z^h)_{J(p)} = -\frac{t}{2} \sum_{k=1}^{4} g(\mathcal{R}^D(E_k, Z) J(p), D_{E_k} J).
\]
for every \(Z \in T_p M\). Note that
\[
(D_X J)(Y) = (\nabla_X J)(Y) + \frac{1}{2}[T(X, JY) - JT(X, Y)].
\]
This identity implies
\[
D_X J = g(D_X J, s_2)s_2 + g(D_X J, s_3)s_3
= \frac{1}{2}[g((D_X J)(E_1), E_3) + g((D_X J)(E_4), E_2)]s_2
+ \frac{1}{2}[g((D_X J)(E_1), E_4) + g((D_X J)(E_2), E_3)]s_3
= \nabla_X J + \frac{1}{2}[T(X, s_3)s_2 - T(X, s_2)s_3].
\]
Thus, by (19),
\[
D_X J = \frac{1}{2}[JX \wedge B + X \wedge JB + T(X, s_3)s_2 - T(X, s_2)s_3].
\]
Then, applying Lemma 2, we compute:
\[
\sum_{k=1}^{4} g(\mathcal{R}^D(E_k, Z) J(p), D_{E_k} J)
= -\frac{1}{2} ||B_p||[g(\mathcal{R}^D(E_1 \wedge Z), s_2) + g(\mathcal{R}^D(E_2 \wedge Z), s_3)]
+ \frac{1}{2} \sum_{k=1}^{4} [T(E_k, s_2)g(\mathcal{R}^D(E_k \wedge Z), s_2) + T(E_k, s_3)g(\mathcal{R}^D(E_k \wedge Z), s_3)]
= -\frac{1}{2} [||B_p||g(\mathcal{R}^D(E_1 \wedge Z), s_2) + ||B_p||g(\mathcal{R}^D(E_2 \wedge Z), s_3)
- g(\mathcal{R}^D(T(s_2) \wedge Z), s_2) - g(\mathcal{R}^D(T(s_3) \wedge Z), s_2)].
\]
Using (12), it is easy to see that
\[
||B_p||[g(\mathcal{R}^D(E_1 \wedge Z), s_2) + g(\mathcal{R}^D(E_2 \wedge Z), s_3)] = \rho_D(Z, B) - \rho_D^s(Z, B).
\]
Next,
\[ T(s_2) = -T_{124}E_1 - T_{123}E_2 + T_{234}E_3 + T_{134}E_4, \]
\[ T(s_3) = T_{123}E_1 - T_{124}E_2 - T_{134}E_3 + T_{234}E_4. \]

Then a direct computation gives
\[ g(\mathcal{R}^D(T(s_2) \wedge Z), s_2) + g(\mathcal{R}^D(T(s_3) \wedge Z), s_3) = \rho_D(Z, \tau) - \rho_D^*(Z, \tau). \] (21)

Thus, if \( B_p \neq 0 \),
\[ h_t(\text{Trace}\{T_pM \ni X \to D_{D^*X}X^h\}, Z^h)_{\mathfrak{J}(p)} = \frac{t}{4}[\rho_D(Z, B) - \rho_D^*(Z, B) - \rho_D(Z, \tau) + \rho_D^*(Z, \tau)]. \] (22)

If \( B_p = 0 \) for a point \( p \in M \), then \( \nabla_X \mathfrak{J} = 0 \) for every \( X \in T_pM \) by (19). Now, the computation above holds for every oriented orthonormal basis \( E_1, ..., E_4 \) of \( T_pM \) with \( E_2 = JE_1 \) and \( E_4 = JE_3 \), and we get identity (22) with vanishing first two summands on the right hand side. Hence, \( \mathcal{H}\text{Trace}_g \tilde{\mathfrak{J}} = 0 \) if and only if for every tangent vector \( Z \in TM \rho_D(Z, B) - \rho_D^*(Z, B) - \rho_D(Z, \tau) + \rho_D^*(Z, \tau) = 0. \)

By Corollary 1, the considerations above prove the following.

**Theorem 1.** An integrable almost Hermitian structure \( J \) determines a harmonic map from \((M, g)\) into \((Z, h_t)\) if and only if 2-form \( d\theta - d\tau - \iota_B T \) is of type \((1, 1)\) with respect to \( J \) and for every \( Z \in TM \rho_D(Z, B) - \rho_D^*(Z, B) - \rho_D(Z, \tau) + \rho_D^*(Z, \tau) = 0. \)

**Remark.** If \( D \) is the Levi-Civita connection, i.e., \( \tau = 0 \), this theorem coincides with [9, Theorem 1].

Let \( D \) be the metric connection with totally skew-symmetric torsion determined by the 1-form \( \tau = \theta \). Then the conditions in Theorem 1 are trivially satisfied. To specify the torsion 3-form \( \mathcal{T} = -\star \theta \), recall that
\[ \theta = (\star d \star \Omega) \circ J = (\star d\Omega) \circ J. \]

Hence, \( \star(\theta \circ J) = -\star^2 d\Omega = d\Omega. \) For any 1-form \( \alpha \),
\[ (\star\alpha)(X, Y, Z) = -(\star(\alpha \circ J))(JX, JY, JZ). \]

Then \( \mathcal{T}(X, Y, Z) = (\star(\alpha \circ J))(JX, JY, JZ) = d\Omega(JX, JY, JZ). \) The latter identity, (17) and (16) imply \( (D_X J)(Y) = 0. \) Thus \( D \) is the Bismut–Strominger connection. Of course, the fact that \( \mathfrak{J} \) is harmonic when we endow \( M \) with the Bismut–Strominger connection follows directly from Corollary 2.

**Example 1.** It has been observed in Ref. [20] that every Inoue surface \( M \) of type \( S^0 \) admits a locally conformal Kähler metric \( g \) for which the Lee form \( \theta \) is nowhere vanishing, see below. Define the metric \( h_t \) by means of the Levi-Civita connection of the metric \( g \). It is shown in Ref. [9] that in this case the map \( \mathfrak{J} : (M, g) \to (Z, h_t) \) is not harmonic. But, \( \mathfrak{J} \) is harmonic if we use the Bismut–Strominger connection to define \( h_t \). It is natural to ask if there are other metric connections with (totally) skew-symmetric torsion on \( M \) for which \( \mathfrak{J} \) is harmonic.
First, recall the construction of the Inoue surfaces of type $S^0$ ([14]). Let $A \in SL(3, \mathbb{Z})$ be a matrix with a real eigenvalue $\alpha > 1$ and two complex eigenvalues $\beta$ and $\overline{\beta}$, $\beta \neq \overline{\beta}$. Choose eigenvectors $(a_1, a_2, a_3) \in \mathbb{R}^3$ and $(b_1, b_2, b_3) \in \mathbb{C}^3$ of $A$ corresponding to $\alpha$ and $\beta$, respectively. Then the vectors $(a_1, a_2, a_3), (b_1, b_2, b_3), (\overline{b_1}, \overline{b_2}, \overline{b_3})$ are $\mathbb{C}$-linearly independent. Denote the upper-half plane in $\mathbb{C}$ by $H$, and let $\Gamma$ be the group of holomorphic automorphisms of $H \times \mathbb{C}$ generated by

$$g_0: (w, z) \to (\alpha w, \beta z), \quad g_i: (w, z) \to (w + a_i, z + b_i), \ i = 1, 2, 3.$$ 

The group $\Gamma$ acts on $H \times \mathbb{C}$ freely and properly discontinuously. Then $M = (H \times \mathbb{C})/\Gamma$ is a compact complex surface known as Inoue surface of type $S^0$.

Following [20], consider on $H \times \mathbb{C}$ the Hermitian metric

$$g = \frac{1}{v^2}(du \otimes du + dv \otimes dv) + v(dx \otimes dx + dy \otimes dy), \quad u + iv \in H, \quad x + iy \in \mathbb{C}.$$ 

This metric is invariant under the action of the group $\Gamma$, so it descends to a Hermitian metric on $M$ which we denote again by $g$. Instead on $M$, we work with $\Gamma$-invariant objects on $H \times \mathbb{C}$. Let $\Omega$ be the fundamental 2-form of the Hermitian structure $(g, J)$ on $H \times \mathbb{C}$, $J$ being the standard complex structure. Then

$$d\Omega = \frac{1}{v} dv \wedge \Omega.$$ 

Hence, the Lee form is $\theta = d \ln v$. In particular, $d\theta = 0$, i.e. $(g, J)$ is a locally conformal Kähler structure. Set

$$E_1 = v \frac{\partial}{\partial u}, \quad E_2 = v \frac{\partial}{\partial v}, \quad E_3 = \frac{1}{\sqrt{v}} \frac{\partial}{\partial x}, \quad E_4 = \frac{1}{\sqrt{v}} \frac{\partial}{\partial y}.$$ 

These are $\Gamma$-invariant vector fields constituting an orthonormal basis such that $JE_1 = E_2$, $JE_3 = E_4$. Note that the vector field dual to the Lee form is $B = E_2$. The non-zero Lie brackets of $E_1, ..., E_4$ are

$$[E_1, E_2] = -E_1, \quad [E_2, E_3] = -\frac{1}{2} E_3, \quad [E_2, E_4] = -\frac{1}{2} E_4.$$ 

Then we have the following table for the Levi-Civita connection $\nabla$ of $g$ ([9]):

$$\nabla_{E_1} E_1 = E_2, \quad \nabla_{E_2} E_2 = -E_1,$$

$$\nabla_{E_3} E_2 = \frac{1}{2} E_3, \quad \nabla_{E_3} E_3 = -\frac{1}{2} E_2, \quad \nabla_{E_4} E_2 = \frac{1}{2} E_4, \quad \nabla_{E_4} E_3 = -\frac{1}{2} E_2$$

all other $\nabla_{E_i} E_j = 0$. (23)

Denote the dual basis of $E_1, ..., E_4$ by $\eta_1, ..., \eta_4$. Take a $\Gamma$-invariant 1-form $\tau = a_1 \eta_1 + ... + a_4 \eta_4$ where $a_1, ..., a_4$ are real constants. Denote by $D$ the connection which is metric w.r.t. $g$ and has skew-symmetric torsion determined by $\tau$. Note that the 3-form $T$ corresponding to $\tau$ is $T = a_4 E_{123} - a_3 E_{124} - a_2 E_{134} - a_1 E_{234}$.

We have $d\eta_1 = \eta_1 \wedge \eta_2, d\eta_2 = 0, d\eta_3 = \frac{1}{2} \eta_2 \wedge \eta_3, d\eta_4 = \frac{1}{2} \eta_2 \wedge \eta_4$. It follows that $d\theta - d\tau - i_B \mathcal{T}$ is of type $(1, 1)$ exactly when $a_3 = a_4 = 0$. Hence, $\rho_D(Z, B) - \rho_D^*(Z, B) - \rho_D(Z, \tau) + \rho_D^*(Z, \tau) = 0$ for every $Z$ exactly when

$$-a_1 \rho_D(Z, E_1) + (1 - a_2) \rho_D(Z, E_2) + a_1 \rho_D^*(Z, E_1) - (1 - a_2) \rho_D^*(Z, E_2) = 0.$$ 

(24)
The torsion $T$ corresponding to $\tau$ is given by

$$T(E_1, E_2) = 0 \quad T(E_1, E_3) = a_2 E_4, \quad T(E_1, E_4) = -a_2 E_3,$$
$$T(E_2, E_3) = -a_1 E_4, \quad T(E_2, E_4) = a_1 E_3, \quad T(E_3, E_4) = a_2 E_1 - a_1 E_2.$$  

Then, using (11) and (23), we get the following table after a tedious computation.

$$\rho_D(E_1, E_1) = -\frac{1}{2} a_2^2, \quad \rho_D(E_2, E_1) = \frac{1}{2} a_1 a_2, \quad \rho_D(E_3, E_1) = \rho_D(E_4, E_1) = 0$$
$$\rho_D(E_1, E_2) = \frac{1}{2} a_1 a_2, \quad \rho_D(E_2, E_2) = -\frac{3}{2} - \frac{1}{2} a_1^2, \quad \rho_D(E_3, E_2) = \rho_D(E_4, E_2) = 0$$
$$\rho_D^*(E_1, E_1) = -1 + \frac{1}{2} a_2, \quad \rho_D^*(E_2, E_1) = \rho_D^*(E_3, E_1) = \rho_D^*(E_4, E_1) = 0$$
$$\rho_D^*(E_1, E_2) = -a_1, \quad \rho_D^*(E_2, E_2) = -1 - \frac{1}{2} a_2, \quad \rho_D^*(E_3, E_2) = \rho_D^*(E_4, E_2) = 0.$$  

(25)

It follows that identity (24) holds for every $Z$ only for $a_1 = 0$, $a_2 = 1$, i.e. $\tau = \theta$. Thus, the complex structure of an Inoue surface $M$ of type $S^0$ is a harmonic map from $(M, g)$ into its twistor space $(\mathcal{Z}, h_D)$ only when $D$ is the Bismut-Strominger connection.

**Example 2.** Recall that a primary Kodaira surface $M$ is the quotient of $\mathbb{C}^2$ by a group of transformations acting freely and properly discontinuously [15, p.787]. This group is generated by the affine transformations $\varphi_k(z, w) = (z + a_k, w + akz + b_k)$ where $a_k$, $b_k$, $k = 1, 2, 3, 4$, are complex numbers such that $a_1 = a_2 = 0$, $b_2 \neq 0$, $Im(a_3\overline{a}_4) = mb_1 \neq 0$ for some integer $m > 0$. The quotient space is compact.

It is well-known that $M$ can also be described as the quotient of $\mathbb{C}^2$ endowed with a group structure by a discrete subgroup $\Gamma$. The multiplication on $\mathbb{C}^2$ is defined by

$$(a, b), (z, w) = (z + a, w + \overline{a}z + b), \quad (a, b), (z, w) \in \mathbb{C}^2,$$
and $\Gamma$ is the subgroup generated by $(a_k, b_k)$, $k = 1, \ldots, 4$ (see, for example, [4]). Considering $M$ as the quotient $\mathbb{C}^2/\Gamma$, every left-invariant object on $\mathbb{C}^2$ descends to a globally defined object on $M$.

We identify $\mathbb{C}^2$ with $\mathbb{R}^4$ by $(z = x + iy, w = u + iv) \rightarrow (x, y, u, v)$ and set

$$A_1 = -\frac{\partial}{\partial x} - x \frac{\partial}{\partial u} + y \frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} + y \frac{\partial}{\partial u} + x \frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v}.$$  

These form a basis for the space of left-invariant vector fields on $\mathbb{C}^2$. We note that their Lie brackets are

$$[A_1, A_2] = -2A_4, \quad [A_i, A_j] = 0$$  

for all other $i, j$. It follows that the group $\mathbb{C}^2$ defined above is solvable.

Denote by $g$ the left-invariant Riemannian metric on $M$ for which the basis $A_1, \ldots, A_4$ is orthonormal.

By [13], every complex structure on $M$ is induced by a left-invariant complex structure on $\mathbb{C}^2$. It is easy to see that every such a structure is given by ([5,16])

$$JA_1 = \varepsilon_1 A_2, \quad JA_3 = \varepsilon_2 A_4, \quad \varepsilon_1, \varepsilon_2 = \pm 1.$$  


Let $J = J_{\varepsilon_1, \varepsilon_2}$ be the complex structure defined by these identities for fixed $\varepsilon_1, \varepsilon_2$.

The non-zero covariant derivatives $\nabla A_i A_j$ are ([9])

\[ \nabla A_1 A_2 = -\nabla A_2 A_1 = -A_4, \quad \nabla A_1 A_4 = \nabla A_4 A_1 = A_2, \quad \nabla A_2 A_4 = \nabla A_4 A_2 = -A_1. \]

This implies that the Lie form is

\[ \theta(X) = -2\varepsilon_1 g(X, A_3). \]

Therefore

\[ B = -2\varepsilon_1 A_3, \quad \nabla \theta = 0. \]

Denote the dual basis of $A_1, \ldots, A_4$ by $\alpha_1, \ldots, \alpha_4$. Thus $\theta = -2\varepsilon_1 \alpha_3$. Any left-invariant 1-form $\tau$ is of the form $\alpha = a_1 \alpha_1 + \ldots + a_4 \alpha_4$, where $a_1, \ldots, a_4$ are real constants. We have $d\alpha_1 = d\alpha_2 = d\alpha_3 = 0$, $d\alpha_4 = 2\alpha_1 \wedge \alpha_2$. The 3-form $T$ corresponding to $\tau$ is $T = a_4 \alpha_1 \alpha_2 \alpha_3 + a_2 \alpha_1 \alpha_2 \alpha_4 - a_1 \alpha_2 \alpha_3$ where $\alpha_{ijk} = \alpha_i \wedge \alpha_j \wedge \alpha_k$. Hence the form $d\theta - d\tau - \nu_B T = 2(\varepsilon_1 - 1) a_4 \alpha_1 \wedge \alpha_2 - 2\varepsilon_1 (a_2 \alpha_1 \wedge \alpha_4 - a_1 \alpha_2 \wedge \alpha_4)$ is of type $(1, 1)$ w.r.t. $J$ if and only if $a_1 = a_2 = 0$.

Let $D$ be the metric connection with skew-symmetric torsion $T$ determined by the form $\tau$. Then the non-zero covariant derivatives $D A_i A_j$ are given in the following table

\[
\begin{align*}
D A_1 A_2 &= \frac{1}{2} a_4 A_3 - (1 + \frac{1}{2} a_3) A_4, & D A_1 A_3 &= -\frac{1}{2} a_4 A_2, & D A_1 A_4 &= (1 + \frac{1}{2} a_3) A_2 \\
D A_2 A_1 &= -\frac{1}{2} a_4 A_3 + (1 + \frac{1}{2} a_3) A_4, & D A_2 A_3 &= \frac{1}{2} a_4 A_1, & D A_2 A_4 &= -(1 + \frac{1}{2} a_3) A_1 \\
D A_3 A_1 &= \frac{1}{2} a_4 A_2, & D A_3 A_2 &= -\frac{1}{2} a_4 A_1, & D A_3 A_4 &= (1 - \frac{1}{2} a_3) A_2, & D A_4 A_2 &= -(1 - \frac{1}{2} a_3) A_1
\end{align*}
\]

Using this table, one can compute the values $\rho_{D ij} = \rho_D (A_i, A_j)$ of the Ricci tensor. The non-zero ones are

\[ \rho_{D 11} = \rho_{D 22} = -\frac{a_4^2 + a_4^2 + 4}{2}, \quad \rho_{D 22} = -\frac{a_4^2}{2}, \quad \rho_{D 33} = a_4(1 + \frac{1}{2} a_3), \quad \rho_{D 44} = 2 - \frac{1}{2} a_3^2. \]

Also, the non-zero values of $\rho^{*}_{D ij} = \rho^{*}_D (A_i, A_j)$ of the *-Ricci tensor are

\[ \rho^{*}_{D 11} = -\rho^{*}_{D 22} = \frac{a_4^2 + a_4^2 + 12}{4}, \quad \rho^{*}_{D 12} = -\rho^{*}_{D 21} = -\varepsilon_1 \varepsilon_2 a_4. \]

It follows that the identity

\[ \rho_D (Z, B) - \rho^{*}_D (Z, B) - \rho_D (Z, \tau) + \rho^{*}_D (Z, \tau) = 0. \]

is satisfied for every $Z \in TM$ if and only if

\[ (1 - \varepsilon_1) a_4 = 0, \quad (1 - \varepsilon_1)(2 - a_3) a_4 = 0. \]

Clearly, if $\varepsilon_1 = 1$, the system is satisfied for any $a_3$ and $a_4$, and if $\varepsilon_1 = -1$, the solution of the system is $a_4 = 0$, $a_3$-arbitrary. Thus, on a Kodaira surface, there are many metric connections with totally skew-symmetric torsion for which the complex structures $J_{\varepsilon_1, \varepsilon_2}$ are harmonic maps from $(M, g)$ into its twistor space $(\mathcal{Z}, h^D)$. The next statement is a weaker version of [17, Proposition 1.3].
**Corollary 3.** Let $J_1$ and $J_2$ be two Hermitian structures on a (connected) Riemannian four-manifold $(M, g)$. Suppose that $J_1$ and $J_2$ have the same Lee form. If $J_1$ and $J_2$ coincide on an open subset or, more-generally, if they coincide to infinite order at a point, they coincide on the whole manifold $M$.

**Proof.** The complex structures $J_1$ and $J_2$ determine the same orientation on $M$, hence the same positive twistor space $Z$. Also, $J_1$ and $J_2$ determine the same Bismut-Strominger connection $D$. If $h_t$ is the metric on $Z$ defined by means of $D$, the maps $\mathfrak{J}_1$ and $\mathfrak{J}_2$ of $(M, g)$ into $(Z, h_t)$ are harmonic. Thus, the result follows from the uniqueness theorem for harmonic maps [18].

**Remark.** In fact, M. Pontecorvo [17] has proved that the conclusion of the above corollary holds without the assumption on the Lee forms. The idea of his proof is inspired by the theory of pseudo-holomorphic curves.

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