On the inverse problem of identifying an unknown coefficient in a space-time fractional differential equation

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Abstract

In this study, we focus on identifying solution and an unknown space-dependent coefficient in a space-time fractional differential equation by employing fractional Taylor series method. The substantial advantage of this method is that we don’t take any over-measured data into account. Consequently, we determine the solution and unknown coefficient more precisely. The presented examples illustrate that outcomes of this method are in high agreement with the exact ones of the corresponding problem. Moreover, it can be implemented and applied effectively comparing with other methods.

Keywords: Space-time fractional partial differential equations, Fractional Taylor series method, Inverse problems, Heat equation.

1. Introduction

Inverse problems of identifying unknown coefficients or source terms draw the growing attention of researchers in diverse branches of science, since it has diverse applications in real life \cite{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}. Identifying unknown parameters in fractional differential equations by numerical methods is one of the significant challenges in inverse problems \cite{13, 14, 15, 16} since modelling scientific processes by fractional differential equations becomes one of the most attractive topics in science. One of the main reasons of this result is the nonlocal properties of fractional derivatives, which leads to better modelling.

In this research, inverse problems of unknown coefficients in a space-time fractional differential equation with Neumann like boundary conditions are considered by utilizing fractional Taylor series method. The fractional derivative is in Caputo sense \cite{17}, which is one of the most common fractional...
derivatives. In the establishment of solution and space-dependent unknown coefficient, no need of any over-measured data is one of the significant aspect of this method, since using over-measured data leads to greater error in calculations. Moreover, the Neumann like boundary condition at the final point and initial condition allow us to establish the solution and unknown space-dependent coefficient without any difficulty. The other boundary and initial conditions guarantee uniqueness of the solution and unknown coefficient. As a result, we conclude that having all these properties makes fractional Taylor series method one of the best method in inverse and direct problems of fractional differential equations.

The focus of this article is on the identifying of the unknown space-dependent coefficient of the following governing space-time fractional differential equation with initial and Neumann-like boundary conditions:

\[ D_t^\alpha u(x, t) = D_x^{2\beta} u(x, t) + p(x) f(x, t), 0 < x < 1, 0 < t < 1, 0 < \alpha, \beta \leq 1, \quad (1) \]

\[ u(x, 0) = \varphi(x), 0 \leq x \leq 1, \quad (2) \]

\[ D_x^{\beta} u(0, t) = \mu_1(t), 0 < t \leq 1, \quad (3) \]

\[ D_x^{\beta} u(1, t) = \mu_2(t), 0 < t \leq 1, \quad (4) \]

2. Preliminaries

Essential concepts and features of fractional derivatives are presented in this section \[1, 2, 3, 4\].

**Definition 1.** The Riemann-Liouville fractional integral of order \( \alpha \) (\( \alpha \geq 0 \)) is given as

\[ J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \ x > 0, \quad (5) \]

\[ J^0 f(x) = f(x). \quad (6) \]

**Definition 2.** The Liouville-Caputo fractional derivative of order \( \alpha \) is given as

\[ D^\alpha f(x) = J^{n-\alpha} D^n f(x) = \int_0^x (x - t)^{n-\alpha-1} \frac{d^n}{dt^n} f(t)dt, \quad n - 1 < \alpha < n, \ x > 0, \quad (7) \]
where \( D^n \) denotes the ordinary derivative of order \( n \).

**Definition 3.** The \( \alpha \)-th order derivative of \( u(x,t) \) in Liouville-Caputo sense is given as

\[
D^\alpha_t u(x,t) = \begin{cases} 
\frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\xi)^{n-\alpha-1} \frac{\partial^n u(x,\xi)}{\partial \xi^n} d\xi, & n - 1 < \alpha < n, \\
\partial^n u(x,t) & \alpha = n \in \mathbb{N}.
\end{cases}
\]  
(8)

**Definition 4.** An \((\alpha, \beta)\)-fractional Taylor series is defined as follows [18]:

\[
\sum_{i+j=0}^{\infty} g_{i,j} t^{i\alpha} x^{j\beta} = g_{0,0} + g_{1,0} t^\alpha + g_{0,1} x^\beta + \sum_{k=0}^{n} g_{n-k,k} t^{(n-k)\alpha} x^k + \ldots
\]  
(9)

where \( g_{i,j}, i, j \epsilon \mathbb{N} \) are the coefficients of the series.

**Lemma 5.** Let \( u(x,t) \) has a fractional Taylor series representation as (9) for \((x,t) \epsilon [0, R_x) \times [0, R_t) \). If \( D^r_t D^s_x u(x,t) \epsilon ((0, R_x) \times (0, R_t)) \) for \( r, s \epsilon \mathbb{N} \), then

\[
D^r_t u(x,t) = \sum_{i+j=0}^{\infty} g_{i+r,j} \frac{\Gamma((i+r)\alpha + 1)}{\Gamma(i\alpha + 1)} t^{i\alpha} x^{j\beta}
\]  
(10)

\[
D^s_x u(x,t) = \sum_{i+j=0}^{\infty} g_{i,j+s} \frac{\Gamma((j+s)\beta + 1)}{\Gamma(j\beta + 1)} t^{i\alpha} x^{j\beta}
\]  
(11)

### 3. Fractional Taylor series method

In order to determine the diffusion coefficient \( p(x) \) of time in the space-time fractional diffusion problem (1)-(4), in the series form we plug the fractional Taylor series of \( u = u(x,t) \) and \( p = p(x) \) into (1)-(4) which leads to:

\[
\sum_{i+j=0}^{\infty} g_{i+1,j} \frac{\Gamma((i+1)\alpha + 1)}{\Gamma(i\alpha + 1)} t^{i\alpha} x^{j\beta} = \sum_{i+j=0}^{\infty} g_{i,j+2} \frac{\Gamma((j+2)\beta + 1)}{\Gamma(j\beta + 1)} t^{i\alpha} x^{j\beta}
\]

\[
+ \sum_{k=0}^{\infty} p_k \left\{ \sum_{i+j=0}^{\infty} \frac{t^{i\alpha} x^{(j+k)\beta}}{\Gamma(i\alpha + 1)\Gamma((j+k)\beta + 1)} \right\}
\]  
(12)

Making two series on both sides of above equation equal to each other, the unknown coefficients in the fractional Taylor series of \( p(x) \) are acquired.
4. Illustrative Examples

In this section, we present some examples to illustrate implementation of fractional Taylor series method for the inverse problem of revealing an unknown space-dependent coefficient in space-time fractional differential equations. The following examples are about determination of unknown coefficient depending on $x$.

**Example 1.** Consider the inverse space-dependent coefficient problem involving space-time fractional differential equations:

\[ D^\alpha_t u(x, t) = D^{2\beta}_x u(x, t) + p(x)u(x, t), \quad 0 < x < 1, \quad 0 < t < 1, \quad 0 < \alpha, \beta \leq 1, \quad (13) \]

\[ u(x, 0) = E_\beta(x^{2\beta}), \quad 0 \leq x \leq 1, \quad (14) \]

\[ D_x^{\beta} u(0, t) = 0, \quad 0 < t \leq 1, \quad (15) \]

\[ D_x^{\beta} u(1, t) = E_\alpha(2t^\alpha) D_x^{\beta} E_\beta(x^{2\beta})|_{x=1}, \quad 0 < t \leq 1, \quad (16) \]

where $E_\beta(x^{2\beta}) = \sum_{j=1}^\infty \frac{x^{2j\beta}}{\Gamma(j\beta+1)}$ is the fractional generalization of the function $\exp(x^2)$. We determine the unknown function $p(x)$ in fractional Taylor series form as follows:

\[ p(x) = \sum_{k=0}^\infty p_k \frac{x^{k\beta}}{\Gamma(1+k\beta)}, \quad 0 < \beta \leq 1. \quad (17) \]

which leads to Eq. (12), with the initial coefficients

\[ g_{0,j} = \frac{1}{\Gamma(j\beta+1)}, \quad (18) \]

\[ g_{0,0} = 1. \quad (19) \]

The coefficients $g_{k,j}$ are acquired by equating two series in Eq. (12), which allow us to form the solution of Eq. (13) as follows:

\[ u(x, t) = 1 + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1) \cdot \Gamma(\alpha+1)} \frac{t^\alpha}{\Gamma(\beta+1) \cdot \Gamma(2\alpha+1)} + \frac{\Gamma(2\beta+1)}{\Gamma(\beta+1) \cdot \Gamma(2\alpha+1)} \frac{t^{2\alpha}}{\Gamma(\beta+1) \cdot \Gamma(2\alpha+1)} \]
In order to establish the unknown coefficient $p(x)$, the Neumann-like boundary condition at $x = 1$ taken into account in (16) which produces the coefficients $p_k$ as follows:

$$p_0 = 0,$$

$$p_1 = 0,$$

$$p_2 = 2\frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} - \frac{\Gamma(4\beta + 1)}{\Gamma(2\beta + 1)},$$

$$\vdots$$

As a result, the unknown coefficient $p(x)$ is determined in the series form as follows:

$$p(x) = \left(2\frac{\Gamma(2\beta + 1)}{\Gamma(\beta + 1)} - \frac{\Gamma(4\beta + 1)}{\Gamma(2\beta + 1)}\right) \frac{x^{2\beta}}{\Gamma(2\beta + 1)} + \ldots$$
Table 1: Comparison of absolute errors at \( x = 0.5 \) with \( E(\alpha, \beta) \) of Example 1.

| \( t \) | Exact | \( E(1, 1) \) | \( E(1, 0.9) \) | \( E(0.9, 1) \) | \( E(0.9, 0.9) \) | \( E(0.9, 0.7) \) | \( E(0.7, 1) \) | \( E(0.7, 0.9) \) | \( E(0.7, 0.7) \) |
|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 1.418884 | 4.40e-02 | 4.16e-02 | 4.39e-02 | 3.51e-02 | 2.57e-02 | 4.32e-02 | 1.13e-02 | 3.48e-02 |
| 0.10 | 1.56810 | 4.36e-02 | 2.57e-02 | 1.84e-03 | 4.33e-02 | 1.46e-02 | 2.74e-02 | 7.06e-02 | 2.41e-01 |
| 0.15 | 1.73302 | 4.24e-02 | 2.62e-02 | 7.99e-03 | 3.81e-02 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.20 | 1.91528 | 4.17e-02 | 2.62e-02 | 3.31e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.25 | 2.11672 | 4.14e-02 | 2.62e-02 | 3.81e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.30 | 2.33933 | 4.06e-02 | 2.62e-02 | 4.24e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.35 | 2.58534 | 3.91e-02 | 2.62e-02 | 4.24e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.40 | 2.85721 | 3.70e-02 | 2.62e-02 | 4.24e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.45 | 3.15763 | 3.42e-02 | 2.62e-02 | 4.24e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |
| 0.50 | 3.48959 | 3.06e-02 | 2.62e-02 | 4.24e-02 | 1.48e-01 | 3.81e-02 | 7.06e-02 | 2.41e-01 | 2.41e-01 |

**Example 2.** Consider the inverse coefficient problem involving space-time fractional differential equations:

\[
D_t^\alpha u(x, t) = D_x^{2\beta} u(x, t) + p(x) u(x, t), 0 < x < 1, 0 < t < 1, 0 < \alpha, \beta \leq 1, (22)
\]

\[
u(x, 0) = E_\beta(x^{3\beta}), 0 \leq x \leq 1,
\]

\[
D_x^{\beta} u(0, t) = 0, 0 < t \leq 1,
\]

\[
D_x^{\beta} u(1, t) = E_\alpha(t^{\alpha}) D_x^{\beta} E_\beta(x^{3\beta})|_{x=1}, 0 < t \leq 1,
\]

where \( E_\beta(x^{3\beta}) = \sum_{j=1}^{\infty} x^{3j\beta} \Gamma(j\beta+1) \) is the fractional generalization of the function \( \exp(x^3) \). We determine the unknown function \( p(x) \) in fractional Taylor series form as follows:

\[
p(x) = \sum_{k=0}^{\infty} p_k \frac{x^{k\beta}}{\Gamma(1 + k\beta)}, 0 < \beta \leq 1.
\]
which leads to Eq. (12), with the initial coefficients
\[ g_{0,j} = \frac{1}{\Gamma(j\beta + 1)}, \]  
\[ g_{0,0} = 1. \]  
The coefficients \( g_{i,j} \) are acquired by equating two series in Eq. (12), which allow us to form the solution of Eq. (22) as follows:
\[ u(x, t) = 1 + (1 + \Gamma(2\beta + 1) \frac{t^\alpha}{\Gamma(\beta + 1)} + (1 + \Gamma(2\beta + 1) \frac{t^{2\alpha}}{\Gamma(2\alpha + 1)} \\
+ (1 + \Gamma(2\beta + 1) \frac{t^{3\alpha}}{\Gamma(3\alpha + 1)} + \Gamma(3\beta + 1) \frac{t^\alpha}{\Gamma(3\beta + 1)} x^{3\beta}) + \Gamma(6\beta + 1) \frac{x^{5\beta}}{\Gamma(2\beta + 1) \Gamma(2\beta + 1)^2} + \ldots \]  
In order to establish the unknown coefficient \( p(x) \), the Neumann boundary condition at \( x = 1 \) taken into account in (25) which produces the coefficients \( p_k \) as follows:
\[ p_0 = 1, \]  
\[ p_1 = -\frac{\Gamma(3\beta + 1)}{\Gamma(\beta + 1)}, \]  
\[ p_2 = 0, \]  
\[ p_3 = -\frac{\Gamma(6\beta + 1)}{\Gamma(2\beta + 1)} + \frac{\Gamma(3\beta + 1) \Gamma(4\beta + 1)}{(\Gamma(\beta + 1))^2}, \]  
\[ \vdots \]  
As a result, the unknown coefficient \( p(x) \) is determined in the series form as follows:
\[ p(x) = 1 - \frac{\Gamma(3\beta + 1)}{\Gamma(\beta + 1)} \frac{x^\beta}{\Gamma(\beta + 1)} \\
- \left( \frac{\Gamma(6\beta + 1)}{\Gamma(2\beta + 1)} - \frac{\Gamma(3\beta + 1) \Gamma(4\beta + 1)}{(\Gamma(\beta + 1))^2} \right) \frac{x^{2\beta}}{\Gamma(2\beta + 1)} + \ldots \]
Table 2: Comparison of absolute errors at $x = 1$ with $E(\alpha, \beta)$ of Example 2.

| $t$  | Exact | $E(1, 1)$ | $E(1, 0.9)$ | $E(1, 0.7)$ | $E(0.9, 1)$ | $E(0.9, 0.9)$ | $E(0.9, 0.7)$ | $E(0.7, 1)$ | $E(0.7, 0.9)$ | $E(0.7, 0.7)$ |
|------|-------|---------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.00 | 2.51253 | 7.52e-03 | 5.75e-03 | 2.82e-03 | 1.33e-02 | 1.02e-02 | 4.99e-03 | 4.12e-02 | 3.15e-02 | 1.54e-02 |
| 0.01 | 2.52131 | 1.15e-02 | 5.65e-03 | 2.49e-02 | 1.91e-02 | 1.35e-02 | 9.08e-02 | 6.94e-02 | 3.40e-02 |
| 0.015| 2.53782 | 2.7e-02 | 8.49e-03 | 3.61e-02 | 2.76e-02 | 1.35e-02 | 9.08e-02 | 6.94e-02 | 3.40e-02 |
| 0.02 | 2.55053 | 1.4e-02 | 4.69e-02 | 3.59e-02 | 1.76e-02 | 1.12e-01 | 8.56e-02 | 4.20e-02 |
| 0.025| 2.56329 | 2.9e-02 | 5.76e-02 | 4.40e-02 | 2.16e-02 | 1.32e-01 | 1.01e-01 | 4.95e-02 |
| 0.03 | 2.57614 | 3.49e-02 | 6.81e-02 | 5.21e-02 | 2.55e-02 | 1.51e-01 | 1.16e-01 | 5.66e-02 |
| 0.035| 2.58905 | 4.08e-02 | 7.85e-02 | 6.00e-02 | 2.94e-02 | 1.70e-01 | 1.30e-01 | 6.35e-02 |
| 0.04 | 2.60203 | 4.68e-02 | 8.89e-02 | 6.79e-02 | 3.33e-02 | 1.88e-01 | 1.43e-01 | 7.03e-02 |
| 0.045| 2.61507 | 5.28e-02 | 9.92e-02 | 7.58e-02 | 3.71e-02 | 2.05e-01 | 1.57e-01 | 7.68e-02 |
| 0.05 | 2.62818 | 5.88e-02 | 1.09e-01 | 8.36e-02 | 4.10e-02 | 2.22e-01 | 1.70e-01 | 8.32e-02 |

5. Conclusion

In this research, the identification of an unknown space-time coefficient in a space-time fractional differential equation with initial and Neumann-like boundary conditions by fractional Taylor series method is investigated. First, the solution of the equation is established by employing fractional Taylor series method with the help of initial condition and Neumann-like boundary condition at the final point. Later, the unknown space-dependent coefficient is identified in the series form without using any over-measured data which is substantial advantage of this method. Future work will be on the construction of unknown coefficient or source term in space-dependent fractional differential equations with various boundary conditions.

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