Multiply Robust Causal Mediation Analysis with Continuous Treatments

Numair Sani∗,1, Yizhen Xu∗,2, AmirEmad Ghassami∗,3, Ilya Shpitser1

1. Department of Computer Science, Johns Hopkins University, USA
2. Division of Biostatistics, Department of Population Health Sciences, University of Utah, USA
3. Department of Mathematics and Statistics, Boston University, USA

Abstract

In many applications, researchers are interested in the direct and indirect causal effects of a treatment or exposure on an outcome of interest. Mediation analysis offers a rigorous framework for identifying and estimating these causal effects. For binary treatments, efficient estimators for the direct and indirect effects are presented in Tchetgen Tchetgen and Shpitser (2012) based on the influence function of the parameter of interest. These estimators possess desirable properties, such as multiple-robustness and asymptotic normality, while allowing for slower than root-n rates of convergence for the nuisance parameters. However, in settings involving continuous treatments, these influence function-based estimators are not readily applicable without making strong parametric assumptions. In this work, utilizing a kernel-smoothing approach, we propose an estimator suitable for settings with continuous treatments inspired by the influence function-based estimator of Tchetgen Tchetgen and Shpitser (2012). Our proposed approach employs cross-fitting, relaxing the smoothness requirements on the nuisance functions, and allowing them to be estimated at slower rates than the target parameter. Additionally, similar to influence function-based estimators, our proposed estimator is multiply robust and asymptotically normal, making it applicable for inference in settings where a parametric model cannot be assumed.

1 Introduction

Estimating the effect of a treatment, policy, or intervention is of interest in various fields, such as epidemiology, economics, medicine, and sociology. A common estimate is the Average Causal Effect (ACE), which has been studied extensively (Hernán and Robins, 2020). However, in addition to estimating the treatment effect, one can also be interested in the pathways and mechanisms through which the treatment affects the outcome of interest. Causal mediation analysis offers a precise and rigorous mathematical framework to answer such questions (Robins and Greenland, 1992; Tchetgen Tchetgen and Shpitser, 2012; Pearl, 2001; VanderWeele, 2009; Goetgeluk et al., 2008; Imai et al., 2010; van der Laan and Petersen, 2008; Lange and Hansen, 2011; Lange et al., 2012).

Much of the literature on mediation analysis assumes that the treatment of interest is binary. However, interventions involving the dosage of a drug, duration or frequency of activity are better described with continuous treatments. In such cases, mediation effects are naturally described by a multidimensional surface rather than a scalar parameter. This scenario is challenging because it involves learning a multidimensional surface without imposing a priori shape constraints. Additionally, in the presence of high-dimensional nuisance parameters, the estimator may inherit the slow rates of our nuisance estimators, adversely affecting the inference of the target parameter.

∗ denotes equal contribution.
The challenges related to estimating ACE in the continuous treatment setting have been addressed by Kennedy et al. (2017), Ai et al. (2021), Hirano and Imbens (2004), Kreif et al. (2015), Imbens (2000), Su et al. (2019), Kallus and Zhou (2018), Colangelo and Lee (2020), and Hill (2011). A common method for dealing with continuous treatments involves using Bayesian Additive Regression Trees (BART), as used by Hill (2011). However, this requires correct specification of the relevant models and inherits the rate of the outcome regression estimation. An alternative approach that leverages semiparametric theory involves specifying a parametric form for the dose-response curve or projecting the true curve onto a parametric model, as presented in Robins (2000), Van Der Laan and Robins (1998), Neugebauer and van der Laan (2007). However, these methods may suffer from bias when the dose-response curve is misspecified.

In contrast to approaches involving parametric assumptions on the dose-response curve, flexible approaches to modeling the dose-response curve have also been proposed. For example, Kennedy et al. (2017) utilize a two-stage estimator that first constructs a doubly robust pseudo-outcome in the first stage, and then regresses the pseudo-outcome on the treatment in the second stage using non-parametric regression methods. Colangelo and Lee (2020) utilize double machine learning along with applying kernel smoothing to the Augmented Inverse Propensity Weighted (AIPW) score (Robins and Rotnitzky, 1995). This allows for slower estimation of nuisance parameters, while still obtaining fast rates for the target parameter. However, estimating mediation effects in the presence of continuous treatments has not been studied to the same extent.

In this paper, we propose a kernel smoothing approach combined with influence function-based estimators (Tsiatis, 2007; Newey, 1994; Bickel et al., 1993; Ichimura and Newey, 2015) to deal with continuous treatment for causal mediation analysis. We propose an estimator that, under mild regularity conditions, is consistent, asymptotically linear, and asymptotically normal. Our work aims to extend the results for the continuous treatment ACE to the case of mediation analysis involving continuous treatments in the presence of high-dimensional covariates. Huber et al. (2020) tackle this problem by weighting the observations by a generalized propensity score that is given as either the conditional density of treatment given (1) the covariates or (2) the covariates and the mediator. The authors estimate the generalized propensity score either parametrically or non-parametrically, and establish asymptotic normality. However, this method inherits the rate of estimation of the generalized propensity score, which can be slow. In contrast, we propose an approach motivated by influence functions and hence obtain many of the desirable properties of influence functions, namely allowing for slower estimation of nuisance parameters, as well as robustness properties. Our work draws heavily on the existing causal mediation literature that discusses the identification and estimation of such effects (Pearl, 2001; Imai et al., 2010; Tchetgen Tchetgen and Shpitser, 2012). Additionally, we utilize the double machine learning paradigm from Chernozhukov et al. (2018).

The remainder of this paper is organized as follows. Section 2 introduces the formal mediation framework, describes its identifying assumptions, and discusses an influence function-based estimator of mediation effects for binary treatments. Section 3 extends the influence function-based approach to continuous treatment settings and describes the sample-splitting and smoothing procedures. In Section 4, we provide our main results along with the requisite regularity conditions.

2 Mediation Analysis

Let $A$ be the continuous treatment variable taking values in $\mathcal{A}$, $Y$ be the outcome variable with values in $\mathcal{Y}$, and $M$ be a mediator variable with values in $\mathcal{M}$ that relays part of the causal effect of $A$ on $Y$. In addition, let $X$ denote the observed pre-treatment covariates in the system, taking values in $\mathcal{X}$. To describe the causal effect of the treatment on the outcome, we use the potential outcome framework (Pearl, 2001). Let $Y(A=a)$ be the potential outcome variable representing the outcome when (possibly contrary to the fact) the treatment is set to value $a$. Suppose we are interested in changing the value of the treatment from $a$ to $a'$. A popular way to measure the causal effect of this change of treatment is to use the average causal effect
(ACE), which captures the difference in the expected value of the counterfactual outcome variables, that is

\[ \text{ACE} = \mathbb{E}[Y^{(a)} - Y^{(a')}], \]

where \( \mathbb{E}[\cdot] \) denotes the population level expectation operator.

The total average causal effect of the treatment on outcome \( Y \) can be partitioned into the part mediated by variable \( M \) and the part directly affecting outcome \( Y \). To formally define this partitioning, let \( Y^{(a,m)} \) denote the potential outcome variable corresponding to the outcome when the treatment is set to value \( a \) and the mediator is set to value \( m \), and \( M^{(a)} \) denote the mediator variable when the treatment is set to value \( a \). \[ \text{Pearl} \ (2001) \] proposed the following partitioning of the average causal effect into the natural direct and indirect effects:

\[
\text{ACE} = \mathbb{E}[Y^{(a)} - Y^{(a')}] = \mathbb{E}[Y^{(a,M^{(a')})} - Y^{(a',M^{(a')})}] + \mathbb{E}[Y^{(a,M^{(a')})} - Y^{(a',M^{(a'})}}].
\]

In words, the natural direct effect (NDE) and natural indirect effect (NIE) can be described as follows. NDE captures the change in the expectation of the outcome if the value of the treatment variable is switched between the two arms of the experiment, while the mediator behaves as if the treatment has not changed. NIE captures the change in the expectation of the outcome if the value of the treatment variable is fixed, while the mediator behaves as if the treatment has been switched between the two arms of the experiment. In the following subsection, we discuss the estimation of NDE and NIE from observational data.

2.1 Estimating Natural Direct and Indirect Effects

To estimate the natural direct and indirect effects, from the partitioning in display (1) it suffices to focus on estimating quantities of the form

\[ \psi_0(a,a') = \mathbb{E}[Y^{(a,M^{(a')})}], \]

for \( a, a' \in A \). Suppose i.i.d. data from the distribution \( P \) on variables \( O = \{A, X, M, Y\} \) are given. In general, the estimand \( \psi_0 \) is not identified from observational data, and identification assumptions are required to relate the distribution of the observational data to that of counterfactual variables. We require the following assumptions for the identification of \( \psi_0 \) from the observed distribution on variables \( P(O) \).

**Assumption 1** (Identification Assumptions).

- **Consistency.** For all \( a \in A \) and \( m \in M \),
  \[
  Y^{(a,m)} = Y \quad \text{if } A = a \text{ and } M = m,
  M^{(a)} = M \quad \text{if } A = a.
  \]

- **Sequential Exchangeability.** For all \( a, a' \in A \),
  \[
  (Y^{(a,m)}, M^{(a')}) \perp A \mid X,
  Y^{(a,m)} \perp M \mid A = a', X.
  \]
• **Positivity.** For all \( a \in \mathcal{A}, m \in \mathcal{M} \) and \( x \in \mathcal{X} \),

\[
\begin{align*}
    f_{M|A,X}(m|A, X) &> 0, \\
    f_{A|X}(a|X) &> 0,
\end{align*}
\]

where \( f_{M|A,X} \) and \( f_{A|X} \) are the conditional density of \( M \) given \( A \) and \( X \), and the conditional density of \( A \) given \( X \), respectively.

Under Assumption 1 the estimand \( \psi_0(a, a') \) can be identified from the observed distribution \( P(O) \) using the following expression called the mediation formula, originally proposed in \cite{pearl2001}.

\[
\psi_0(a, a') = \int_{\mathcal{M}} \int_{\mathcal{X}} \mathbb{E}[Y|A = a, M = m, X = x]f_{M|A,X}(m|A = a', X = x)f_X(x)dm dx,
\]  

(2)

where \( f_X \) is the marginal distribution of \( X \).

Using Equation (2), one can estimate the parameter of interest \( \psi_0(a, a') \) by first estimating the nuisance functions \( \mathbb{E}[Y|A, M, X] \) and \( f_{M|A,X} \), and then using a plug-in estimator to estimate \( \psi_0 \) as follows

\[
\hat{\psi}_0^{MF}(a, a') = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathcal{M}} \int_{\mathcal{X}} \mathbb{E}[Y_i|A = a, M = m, X_i]f_{M|A,X}(m|A = a', X_i)dm dx.
\]

Unfortunately, this estimator is sensitive to bias in the estimation of nuisance functions; that is, the misspecification of either of the nuisance functions induces bias in the estimation of the parameter of interest.

As an alternative approach, in the case of binary treatment, that is, \( \mathcal{A} = \{0,1\} \), \cite{tchetgen2012} develop a general semiparametric framework for inference on parameter \( \psi \). The authors derive the efficient influence function for \( \psi_0(a, a') \)

\[
\begin{align*}
    IF_{\psi_0}(O) &= \frac{I(A = a)f_{M|A,X}(M|A = a', X)}{f_{A|X}(a|X)f_{M|A,X}(M|A = a, X)}\{Y - \mathbb{E}[Y|A = a, M, X]\} \\
                     &+ \frac{I(A = a')}{f_{A|X}(a'|X)}\{\mathbb{E}[Y|A = a, M, X] - \eta(a, a', X)\} + \eta(a, a', X) - \psi_0(a, a'),
\end{align*}
\]  

(3)

where \( a, a' \in \{0,1\} \), \( I(\cdot) \) denotes the indicator function, and

\[
\eta(a, a', X) = \int_{\mathcal{M}} \mathbb{E}[Y|A = a, M = m, X]f_{M|A,X}(m|A = a', X)dm.
\]

Note that \( IF_{\psi_0} \) is a function of three nuisance functions: \( \mathbb{E}[Y|A, M, X] \), \( f_{M|A,X} \), and \( f_{A|X} \). \cite{tchetgen2012} show that the estimator based on this influence function has the triple robustness property, that is, it is consistent even if the model for one (but not more than one) nuisance function is misspecified. Formally, let

• \( \mathcal{M}_{ym} \) be the sub-model in which the model for \( \mathbb{E}[Y|A, M, X] \) and \( f_{M|A,X} \) are correctly specified.

• \( \mathcal{M}_{ya} \) be the sub-model in which the model for \( \mathbb{E}[Y|A, M, X] \) and \( f_{A|X} \) are correctly specified.

• \( \mathcal{M}_{ma} \) be the sub-model in which the model for \( f_{M|A,X} \) and \( f_{M|A,X} \) are correctly specified.

Subsequently, the estimator for \( \psi_0 \) based on the influence function \( IF_{\psi_0} \) is consistent in the submodel \( \mathcal{M}_{ym} \cup \mathcal{M}_{ya} \cup \mathcal{M}_{ma} \). In the following section, we extend the theory of \cite{tchetgen2012} to deal with continuous treatment variables.
3 Continuous Mediation Analysis

In this section, we provide estimation results closely related to those of Tchetgen Tchetgen and Shpitser (2012) for estimating mediation effects. In the case of continuous treatments, the parameter is no longer regular (Bickel et al., 1993); therefore, the results of Tchetgen Tchetgen and Shpitser (2012) cannot be applied straightforwardly. However, the estimator can be modified to be suitable for inference in the case of continuous treatments, while still obtaining desirable properties such as asymptotic normality, robustness to misspecification of nuisance models, and valid inference while allowing for slower than $\sqrt{n}$-rates of convergence of nuisance parameters. Specifically, we modify Equation (3) by utilizing a kernel smoothing technique, in which the data with treatment value in a neighborhood (defined by the bandwidth parameter $h$) of $a$ and $a'$ are assigned a weight that is used to calculate the final estimate of $\psi(a, a')$.

Equation (3) is modified by employing a kernel smoothing technique, wherein data points with treatment values falling within a neighborhood defined by the bandwidth parameter $h$, centered around $a$ and $a'$, are utilized to assign weights to each data point. These weights are then used to compute the estimate of $\psi(a, a')$.

Let $d_A$ denote the dimension of the treatment variable, and let

$$K_h(a) := \frac{1}{h^{d_A}} \prod_{j=1}^{d_A} k\left(\frac{a_j}{h}\right),$$

where $k(\cdot)$ is a kernel function, and $h$ denotes the bandwidth parameter. We propose the following function for estimating $\psi_0(a, a')$.

$$m(O; \alpha, \lambda, \gamma, \psi(a, a')) = K_h(A - a)\lambda(a, X) \frac{\alpha(a', M, X)}{\alpha(a, M, X)} \left\{Y - \gamma(X, M, a)\right\} + K_h(A - a')\lambda(a', X)\left\{\gamma(X, M, a) - \eta(a, a', X)\right\} + \eta(a, a', X) - \psi(a, a'),$$

(4)

where $\lambda(a, X) := 1/f(a|X)$, $\alpha(a, M, X) := f(M|a, X)$, and $\gamma(X, M, a) := \mathbb{E}[Y|A = a, M, X]$ are the nuisance functions. We require the kernel function to satisfy the following conditions.

**Assumption 2** (Kernel & Bandwidth Assumptions). The kernel function $k(\cdot)$ satisfies

1. $\int k(u)du = 1$
2. $\int uk(u)du = 0$
3. $0 < \int u^6k(u)du < \infty$
4. $\int k^2(u)du < \infty$
5. $0 < \int u^2k^2(u)du < \infty$

Additionally, the kernel bandwidth is assumed to be a function of the sample size $n$ which satisfies $h \to 0$, $nh^{d_A} \to \infty$ and $nh^{d_A+4} \to C_h$, for a constant $C_h$, as $n$ goes to infinity.

These assumptions are satisfied by common kernels such as the Gaussian kernel and Epanechnikov kernel. Note that in the moment function Equation (4), the nuisance parameters are not functions of the parameter of interest $\psi$. Therefore, having estimators for nuisance functions suffices for obtaining an estimator for the parameter of interest. Next, we describe the estimation procedure for utilizing Equation (4) to obtain an estimate for $\psi(a, a')$. 

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3.1 Estimation Procedure

We use the cross-fitting estimation approach of Chernozhukov et al. (2018) for separating the estimation of the nuisance functions from the parameter of interest. This approach is beneficial since weaker smoothness requirements are needed for the estimation of nuisance functions. In the cross-fitting approach, we partition the samples into \( L \) equal size parts \( \{ O_{I_1}, ..., O_{I_L} \} \), where data from the \( \ell \)-th partition is denoted as \( O_{I_\ell} \), and the data in the rest of the partitions is denoted as \( O_{C_\ell} \). For \( \ell \in \{1, ..., L\} \), we estimate the nuisance functions \( \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell \) on data from \( O_{C_\ell} \). For all \( \ell \), let \( \hat{\psi}_\ell \) be the estimation of \( \psi_0 \) obtained by solving

\[
\frac{1}{|I_\ell|} \sum_{i \in I_\ell} m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \hat{\psi}_\ell(a, a')) = 0.
\]

Then, our final estimator of \( \psi_0 \) is obtained by

\[
\hat{\psi}_{TR}(a, a') = \frac{1}{L} \sum_{\ell=1}^{L} \hat{\psi}_\ell(a, a').
\] (5)

Next, we present the asymptotic properties of our proposed estimator, along with the required regularity conditions.

4 Asymptotic Analysis

In this section, we study the asymptotic properties of the cross-fitting estimator in Equation (5), and provide pointwise results for the convergence of \( \hat{\psi}_{TR}(a, a') \). We start by stating the required regularity conditions.

**Assumption 3 (Regularity Conditions).**

1. For all \( Y, M, X \), the functions \( f(a | Y, M, X) \), \( f(a | M, X) \), \( f(a | X) \), \( \gamma(X, M, a) \) as a function of \( a \) are three times continuously differentiable, and the functions and their first, second and third derivatives are bounded.

2. The ground truth nuisance functions \( \alpha, \lambda, \gamma \) and the estimations \( \hat{\alpha}, \hat{\lambda}, \hat{\gamma} \) are bounded. Additionally, \( \alpha \), \( \lambda \) and their estimation \( \hat{\alpha}, \hat{\lambda} \) are bounded away from zero.

3. \( Y_i \)’s conditional variance \( \text{var}(Y_i | a, m, x) \) and its first and second derivative are bounded over \( a \in A_0 \) for any \( m \in M \) and \( x \in X \).

In addition to the regularity conditions, we require the following conditions regarding the convergence of the estimators of the nuisance functions.

**Assumption 4 (Consistency).**

For any value \( a \in A \), the estimators \( \hat{\alpha}(a, M, X), \hat{\lambda}(a, X), \text{ and } \hat{\gamma}(X, M, a) \) are consistent, that is,

1. \( \int \left( \hat{\lambda}(a, x) - \lambda(a, x) \right)^2 f_X(x) dx \xrightarrow{P} 0 \)

2. \( \int_X \int_{M} (\hat{\alpha}(a, m, x) - \alpha(a, m, x))^2 f_{M,X}(m, x) dmdx \xrightarrow{P} 0 \)

3. \( \int_X \int_{M} (\hat{\gamma}(x, m, a) - \gamma(x, m, a))^2 f_{M,X}(m, x) dmdx \xrightarrow{P} 0 \)

Where \( \xrightarrow{P} \) indicates convergence in probability.
Assumption 5 (Nuisance Convergence Rates).
For any value \(a, a' \in \mathcal{A}\), the estimators \(\hat{\alpha}(a, M, X), \hat{\lambda}(a, X),\) and \(\hat{\gamma}(X, M, a)\) are rate doubly robust, that is,

1. 
\[
\sqrt{n h^{dA}} \left( \int (\hat{\alpha}(a', m, x) - \alpha(a', m, x))^2 f_{M, X}(m, x) \, dm \, dx \right)^{\frac{1}{2}}
\times \left( \int (\hat{\gamma}(x, m, a) - \gamma(x, m, a))^2 f_{M, X}(m, x) \, dm \, dx \right)^{\frac{1}{2}} \xrightarrow{P} 0
\]

2. 
\[
\sqrt{n h^{dA}} \left( \int (\hat{\lambda}(a', x) - \lambda(a', x))^2 f_{X}(X) \, dx \right)^{\frac{1}{2}} \left( \int (\hat{\gamma}(x, m, a) - \gamma(x, m, a))^2 f_{M, X}(m, x) \, dm \, dx \right)^{\frac{1}{2}} \xrightarrow{P} 0
\]

3. 
\[
\sqrt{n h^{dA}} \left( \int (\hat{\lambda}(a', x) - \lambda(a', x))^2 f_{X}(X) \, dx \right)^{\frac{1}{2}} \left( \int (\hat{\alpha}(a, m, x) - \alpha(a, m, x))^2 f_{M, X}(m, x) \, dm \, dx \right)^{\frac{1}{2}} \xrightarrow{P} 0
\]

Note that as per Assumption 5, we do not require any convergence rates on individual nuisance functions. Rather, the rate double-robustness condition implies that our requirements on the convergence rate of nuisance function estimators are on the product of the rates of the individual nuisance function estimators. Therefore, if one of the estimators converges at a slow rate, the other estimator can compensate. This is a desirable property when working with non-parametric estimators since these typically have slow rates of convergence.

Although the existing assumptions are sufficient to prove the consistency of the proposed estimator, additional assumptions are required to ensure that the distribution of the estimator converges to a Gaussian distribution. Asymptotic normality is necessary to construct Wald-style confidence intervals. The Central Limit Theorem cannot be applied because bandwidth \(h\) varies as a function of the sample size, implying that the data are no longer i.i.d. In this case, asymptotic normality is proved using the Lyapunov Central Limit Theorem, and we state the additional assumptions required.

Assumption 6 (Conditions for Asymptotic Normality).

1. \(\mathbb{E} \left[ |Y - \gamma(X, M, a)|^3 \right] \) is bounded uniformly over any \((a, a', m, x) \in \mathcal{A}^2 \times \mathcal{M} \times \mathcal{X}\).

2. \(\int_{-\infty}^{\infty} k(u)^{c_1} k(u + \tilde{c})^{c_2} \, du < \infty\) and \(\int_{-\infty}^{\infty} u^2 k(u)^{c_1} k(u + \tilde{c})^{c_2} \, du < \infty\) for \(\tilde{c} \in \mathcal{R}\) and \(c_1, c_2 \in \{2, 3\}\).

We now provide the following result regarding the convergence of the cross-fitting estimator in Equation 5.

Theorem 1. Suppose Assumptions 2-5 hold. Then for any value pair \(a, a' \in \mathcal{A}\),

\[
\sqrt{n h^{dA}} (\hat{\psi}^T \hat{\psi} (a, a') - \psi_0(a, a')) = \sqrt{\frac{h^{dA}}{n} \sum_{i=1}^{n} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))} + o_p(1).
\]
Assumption 7

We present the assumptions necessary for the consistency of $\hat{\psi}_{\text{TR}}(a, a')$. Additionally, if Assumption 6 holds, then $\sqrt{nh^{d_X}}(\hat{\psi}_{\text{TR}}(a, a') - \psi_0(a, a') - h^2 B(a, a'))$ converges to the Gaussian distribution $N(0, V(a, a'))$, where $V(a, a')$ and $B(a, a')$ are defined as

$$B(a, a') = \int u^2 k(u)du \times \mathbb{E}\left\{ \frac{\alpha(a', M, X)}{\alpha(a, M, X)} \left( \frac{\partial \alpha(X, M, a)}{\partial a} f(a | X) - \frac{1}{2} f(a | X) \frac{\partial^2 \alpha(X, M, a)}{\partial a^2} \right) \right\} + \{ \gamma(X, M, a) - \eta(a, a', X) \} \frac{1}{2} \frac{\partial^2 f(a' | X, M)}{\partial (a')^2} + O(h),$$

and

$$V(a, a') = \int k(u)^2 du \times \mathbb{E}\left\{ \frac{\alpha^2(a', M, X) f(a | X, M)}{\alpha^2(a, M, X) f(a | X)} \text{var}(Y | X, M, a) + \frac{1}{f(a' | X)} \text{var}[E(Y | X, M, a) | X, a'] \right\}$$

Theorem 1 provides results on the point-wise convergence of $\hat{\psi}_{\text{TR}}(a, a')$, and establishes the asymptotic normality of our estimator. Additionally, $\hat{\psi}_{\text{TR}}$ shares the multiple robustness property of the estimator based on $IF_{\psi_0}$, formally stated in Lemma 1.

**Lemma 1.** Under Assumptions 2, 3 and 6 the proposed $\hat{\psi}_{\text{TR}}(a, a')$ will be a consistent estimator for $\psi(a, a')$ as long as any two out of three conditions in Assumption 4 hold.

While Theorem 1 and Lemma 1 establish properties of $\hat{\psi}_{\text{TR}}$ that are desirable for point estimation, uncertainty quantification through the calculation of valid confidence intervals requires the estimation of $V(a, a')$ and $B(a, a')$. However, these are hard to estimate well because of their complicated analytical form. Nevertheless, by choosing an undersmoothing bandwidth $h$ that satisfies $\sqrt{nh^{d_X+4}} \to 0$, valid confidence intervals can still be constructed without estimating $\hat{B}(a, a')$. In this case, we only need to focus on estimating $V(a, a')$. A valid estimator for $V(a, a')$, under additional assumptions, can be obtained as follows.

$$\hat{V}(a, a') = \frac{1}{L} \sum_{l=1}^{L} \frac{1}{|I_l|} \sum_{i \in I_l} m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))$$

We present the assumptions necessary for the consistency of $\hat{V}(a, a')$ below.

**Assumption 7** (Consistency of $\hat{V}(a, a')$).

1. $\mathbb{E}[\mathbb{I}_{\{Y - \gamma(X, M, a) \geq 0\}} Y | A = a', M = m, X = x]$ is bounded uniformly over any $(a', a', m, x) \in \mathcal{A}^2 \times \mathcal{M} \times \mathcal{X}$.
2. $\int_{-\infty}^{\infty} k(u)^{c_1} k(u + \tilde{c})^{c_2} du < \infty$ and $\int_{-\infty}^{\infty} u^2 k(u)^{c_1} k(u + \tilde{c})^{c_2} du < \infty$ for $\tilde{c} \in \mathcal{R}$ and $c_1 + c_2 \in \{2, 3, 4\}$ for $c_1, c_2 \in \{0, 1, 2, 3, 4\}$.
3. 

$$\int (\gamma(X, M, a) - \hat{\gamma}(X, M, a))^2 \left( \frac{1}{f(a | X)} - \frac{1}{f(a' | X)} \right)^2 f(X, M) dX dM \xrightarrow{P} 0$$

$$\int \left( \frac{f(M | A = a', X)}{f(M | A = a, X)} \frac{f(M | A = a', X)}{f(M | A = a, X)} \right)^2 (\gamma(X, M, a) - \hat{\gamma}(X, M, a))^2 f(X, M) dM dX \xrightarrow{P} 0$$

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Next, we prove that \( \hat{V}(a,a') \) is a consistent estimator of \( V(a,a') \) in Lemma 2.

**Lemma 2.** Suppose Assumptions 2-7 hold. Then for any value pair \( a,a' \in A \), \( \hat{V}(a,a') \) is a consistent estimator for \( V(a,a') \).

Using the result of Lemma 2, \( \hat{V}(a,a') \) can be used to construct asymptotically valid confidence intervals as follows. Choose an undersmoothing bandwidth \( h (\sqrt{n h^d A + 4} \to 0) \) such that the \( \sqrt{n h^d A + 4 B(a,a')} \) is asymptotically negligible. Then, a \((1-\alpha)\) confidence interval is given as

\[
\hat{\psi}_{TR}(a,a') \pm \Phi^{-1}(1-\alpha/2) \frac{\sqrt{\hat{V}(a,a')}}{nh^{dA}}
\]

where \( \Phi \) is the CDF of \( \mathcal{N}(0,1) \).

5 Simulation Study

We now present a simulation study that demonstrates the proposed estimator is multiple robust, and we also demonstrate the undersmoothing procedure used to construct confidence intervals. The data generating process for this simulation is modified from the one used by Kennedy et al. (2017). The data generating process is as follows:

\[
\begin{align*}
X &= (X_1, X_2, X_3, X_4) \sim \mathcal{N}(0, I_4) \\
\mu(X) &= -0.8 + 0.1X_1 + 0.1X_2 - 0.1X_3 + 0.2X_4 \\
A/20 \mid X &\sim \text{Beta}(\mu(X) \times 5, (1-\mu(X)) \times 5) \\
\delta(A, X) &= 1 + (0.2, 0.2, 0.3, -0.1)X + A(0.1 - 0.1X_1 + 0.1X_3 - 0.0169A^2) \\
M &\sim \text{Bernoulli}(\delta(A, X)) \\
Y &\sim -0.646A + 0.539MX_1 + X_2 + \epsilon \\
\epsilon &\sim \mathcal{N}(0, 1)
\end{align*}
\]

Based on the simulation above, the parameter of interest when \( a = 2.5 \) and \( a' = 5 \) is \( \psi_0(2.5, 5) \) and based on Monte Carlo integration using 10000 samples is equal to \(-1.647\). To demonstrate the multiple robustness property, datasets of size 1000 are used along with 1000 Monte Carlo replicates. The bandwidth is chosen using the Silverman rule of thumb (Silverman, 2018), and a Gaussian kernel is used. The number of folds used
in cross fitting is equal to 2. The results are seen in Figure 1(a), where \( \mathcal{M}_{y,m,a} \) denotes the correctly specified models for \( \mathbb{E}[Y|A,M,X] \), \( f(M|A,X) \), and \( p(A|X) \), \( \mathcal{M}_{y,m,a}^* \) denotes the model where only \( \mathbb{E}[Y|A,M,X] \) and \( f(M|A,X) \) are specified correctly, \( \mathcal{M}_{y,m,a}^{**} \) denotes the model where only \( p(A|X) \) and \( f(M|A,X) \) are specified correctly, \( \mathcal{M}_{y,m,a}^{***} \) denotes the model where only \( \mathbb{E}[Y|A,M,X] \) and \( f(M|A,X) \) are specified incorrectly. As expected, the estimates show minimal bias for \( \mathcal{M}_{y,m,a} \), \( \mathcal{M}_{y,m,a}^* \), \( \mathcal{M}_{y,m,a}^{**} \), and \( \mathcal{M}_{y,m,a}^{***} \) while bias is introduced in the model \( \mathcal{M}_{y,m,a}^{****} \).

The construction of confidence intervals by utilizing the proposed undersmoothing approach is difficult, since choosing an undersmoothing bandwidth is more challenging. To this end, in Figure 1(b) we plot the coverage of 95% confidence intervals against various choices of undersmoothing bandwidths. To do this, we define an undersmoothing parameter \( \delta \) related to the bandwidth as \( h = h_{srt} + \delta \), where \( h_{srt} \) is the bandwidth chosen using the Silverman rule of thumb. As \( \delta \) increases, the coverage improves and the bias decreases, however this comes at the cost of increasing the width of the confidence intervals and increasing variance. Regardless of the choice of the undersmoothing bandwidth, the coverage is theoretically guaranteed as long as \( n \) goes to infinity and \( \sqrt{nh^{\delta A + 1}} \to 0 \). For example, when \( d_A = 1 \), our choice of the bandwidth under \( \delta > 0 \) satisfies the requirement for an undersmoothing bandwidth. However, we deal with finite sample size in practice, therefore the choice of bandwidth influences the coverage and bias of our estimator, e.g., in our simulation as \( \delta \) increases the coverage increases and bias decreases. In practice, we conservatively choose a larger value of \( \delta \) to guarantee that the coverage is sufficient in finite sample settings.

6 Conclusion

In this paper, we present an estimator for the direct and indirect components of the causal effect in the presence of continuous treatments, while allowing for complex data generating processes. Our estimator is inspired by influence function based estimation strategies from the literature on semiparametric statistics (Bickel et al., 1993; Tsiatis, 2007). We provide results on the convergence rate and asymptotic normality of our proposed estimator, which allows for the construction of confidence intervals and hypothesis tests. Extending the point-wise results to uniform results and providing optimal rates is left for future work.

Acknowledgment

We are grateful to Prof. Eric Tchetgen Tchetgen for helpful discussions and comments.
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7 Appendix

Before we start with the proofs, we establish some lemmas that will help us with the proofs in the rest of the appendix.

**Lemma 3.** Let \( \{X_m\} \) and \( \{Y_m\} \) be a sequence of random variables. Then under conditions outlined in Lemma 6.1 in Chernozhukov et al. (2018), \( \mathbb{E}[|X_m| \mid Y_m] = o_p(1) \) implies \( X_m = o_p(1) \).

**Proof.** By the Conditional Markov Inequality, for any \( \epsilon > 0 \),

\[
p(\{|X_m| \geq \epsilon \mid Y_m\}) \leq \frac{\mathbb{E}[|X_m| \mid Y_m]}{\epsilon}
\]

By \( \mathbb{E}[|X_m| \mid Y_m] = o_p(1) \), there is \( p(\{|X_m| \geq \epsilon \mid Y_m\}) = o_p(1) \). An application of Lemma 6.1 then yields \( p(|X_m| > \epsilon) \to 0 \), therefore \( X_m = o_p(1) \).

**Lemma 4.** Under Assumption 2, for a twice continuously differentiable function \( f \) with bounded first and second derivative, we have

\[
\int_A K_h(A - a)f(A)dA = f(a) + O(h^2).
\]

**Proof.**

\[
\int_A K_h(A - a)f(A)dA = \int \left[ \prod_{j=1}^{d_A} k(u_j) \right] f(uh + a)du_1 \ldots du_{d_A} \\
= \int \left[ \prod_{j=1}^{d_A} k(u_j) \right] \left\{ f(a) + \sum_{j=1}^{d_A} u_j h \partial_{u_j} f(a) + \frac{1}{2} \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \partial_{u_j, u_{j'}} f(a) \right\} du_1 \ldots du_{d_A} \\
= f(a) + O(h^2),
\]

Where \( \bar{a} \) is in between \( A \) and \( a \).

Remark: We assume the second derivative is bounded over the support of the function \( f(a) \), which is a stronger assumption than \( O(1) \) since the bound holds everywhere as opposed to only for \( a \geq c \) where \( c \) is a constant. If \( \nu(x) \) and \( \omega(x) \) are two arbitrary functions, then \( \int \nu(x)\omega(x)dx = O(1) \int |\omega(x)|dx \) is true when \( \nu(x) \) is bounded, but not when \( \nu(x) = O(1) \), e.g. when \( \nu(x) = 1/x \) and \( \omega(x) = \mathbf{1}\{0 \leq x \leq 1\} \).

7.1 Proof for Theorem 1

We follow a similar outline as Colangelo and Lee (2020) and Chernozhukov et al. (2018). The proof for this theorem is split into two parts. The first part establishes that the proposed estimator satisfies

\[
\sqrt{n} \frac{h^{d_A}}{n} \sum_{\ell=1}^{L} \sum_{\alpha \in \mathcal{I}_\ell} \left\{ m(O_i; \hat{\alpha} \ell, \hat{\lambda} \ell, \hat{\gamma} \ell, \psi_0(a, a')) - m(O_i; a, \lambda, \gamma, \psi_0(a, a')) \right\} = o_p(1),
\]

and the second part establishes that \( \sqrt{n} h^{d_A} (\hat{\psi}^{TR}(a, a') - \psi_0(a, a') - B(a, a')) \) converges to the Gaussian distribution \( \mathcal{N}(0, \Sigma(a, a')) \).
Starting with the first part of the proof, note that

\[
\sqrt{n} h^{d_A} \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \hat{\psi}_\ell(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\} 
= \sqrt{h^{d_A}} \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \hat{\psi}_\ell(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\}
+ m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))
= - \sqrt{n} h^{d_A} (\hat{\psi}^{TR}(a, a') - \psi_0(a, a'))
+ \sqrt{h^{d_A}} \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\}.
\]

Since \( \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \hat{\psi}_\ell(a, a')) = 0 \), we have

\[
\sqrt{n} h^{d_A} (\hat{\psi}^{TR}(a, a') - \psi_0(a, a'))
= \sqrt{h^{d_A}} \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\}
+ \sqrt{h^{d_A}} \frac{1}{n} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\}.
\]

In order to establish an asymptotically linear representation for our proposed estimator, it suffices to to show that for all \( 1 \leq \ell \leq L \) we have

\[
\sqrt{h^{d_A}} \frac{1}{n} \sum_{i \in I_{\ell}} \left\{ m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\} = o_p(1).
\]

Next, we expand \( m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \) into multiple terms and bound each term individually. Note that

\[
m(O_i; \hat{\alpha}_\ell, \hat{\lambda}_\ell, \hat{\gamma}_\ell, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))
= K_h(A_i - a) \left\{ \hat{\lambda}(a, X_i) \frac{\hat{\alpha}(a', M_i, X_i)}{\hat{\alpha}(a, M_i, X_i)} \{ Y_i - \hat{\gamma}(a, M_i, X_i) \} - \lambda(a, X_i) \frac{\alpha(a', M_i, X_i)}{\alpha(a, M_i, X_i)} \{ Y_i - \gamma(X_i, M_i, a) \} \right\}
+ K_h(A_i - a') \left\{ \hat{\lambda}(a', X_i) \{ \hat{\gamma}(a, M_i, X_i) - \hat{\eta}(a, a', X_i) \} - \lambda(a', X_i) \{ \gamma(X_i, M_i, a) - \eta(a, a', X_i) \} \right\}
+ \hat{\eta}(a, a', X_i) - \eta(a, a', X_i).
\]

Defining \( R(M_i, X_i) := \frac{\alpha(a', M_i, X_i)}{\alpha(a, M_i, X_i)} \), terms (6) and (7) can be expanded additionally. Expanding term (6), we
get

\[ K_h(A_i - a) \left\{ \hat{\lambda}(a, X_i) \hat{R}(M_i, X_i) \{ Y_i - \hat{\gamma}(X_i, M_i, a) \} - \lambda(a, X_i) R(M_i, X_i) \{ Y_i - \gamma(X_i, M_i, a) \} \right\} \]

\[ = -K_h(A_i - a) (\hat{R}(M_i, X_i) - R(M_i, X_i)) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \]  
(CS1)

\[ + K_h(A_i - a) (\hat{R}(M_i, X_i) - R(M_i, X_i)) (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \hat{\lambda}(a, X_i) \]  
(CS2)

\[ - K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) R(M_i, X_i) \]  
(CS3)

\[ + \left\{ K_h(A_i - a) (\hat{R}(M_i, X_i) - R(M_i, X_i)) \hat{\lambda}(a, X_i) (Y_i - \gamma(X_i, M_i, a)) \right\} \]  
\[ - \mathbb{E} \left[ K_h(A_i - a) (\hat{R}(M_i, X_i) - R(M_i, X_i)) \hat{\lambda}(a, X_i) (Y_i - \gamma(X_i, M_i, a)) \mid O_{i}^{c} \right] \} \]  
(E1)

\[ + \mathbb{E} \left[ K_h(A_i - a) (\hat{R}(M_i, X_i) - R(M_i, X_i)) \hat{\lambda}(a, X_i) (Y_i - \gamma(X_i, M_i, a)) \mid O_{i}^{c} \right] \]  
(TR1)

\[ + \left\{ K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i) (Y_i - \gamma(X_i, M_i, a)) \right\} \]  
\[ - \mathbb{E} \left[ K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i) (Y_i - \gamma(X_i, M_i, a)) \mid O_{i}^{c} \right] \} \]  
(E2)

\[ + \mathbb{E} \left[ K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i) (Y_i - \gamma(X_i, M_i, a)) \mid O_{i}^{c} \right] \]  
(TR2)

\[ - K_h(A_i - a) (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a, X_i) R(M_i, X_i) \]  
(R2)

For term \(7\), note that

\[ K_h(A_i - a') \{ \hat{\lambda}(a', X_i) (\hat{\gamma}(X_i, M_i, a) - \eta(a, a', X_i)) - \lambda(a', X_i) (\gamma(X_i, M_i, a) - \eta(a, a', X_i)) \} \]

\[ = K_h(A_i - a') (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \]  
(CS5)

\[ - K_h(A_i - a') (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) (\eta(a, a', X_i) - \hat{\eta}(a, a', X_i)) \]  
(CS6)

\[ + K_h(A_i - a') (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \gamma(X_i, M_i, a) \]  
(R3)

\[ + K_h(A_i - a') (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) \]  
(R4)

\[ - K_h(A_i - a') (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \hat{\eta}(a, a', X_i) \]  
(R5)

\[ - K_h(A_i - a') (\hat{\eta}(a, a', X_i) - \hat{\eta}(a, a', X_i)) \lambda(a', X_i). \]  
(R6)

Next, we group terms (R1)-(R6) as follows. We pair (R1) with (R6), (R2) with (R4), and (R3) with (R5). Note that every expectation introduced here is only over \(O_i\), conditional on \(O_{i}^{c}\), i.e., \(\mathbb{E}(\cdot \mid O_{i}^{c})\), and hence all the terms are random variables. For (R1)+(R6) we have

\[ (R1) + (R6) \]

\[ = (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) - K_h(A_i - a') (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) \lambda(a', X_i) \]

\[ = (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) - \mathbb{E} [\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)] \]  
(E3)

\[ - \mathbb{E} [K_h(A_i - a') (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) \lambda(a', X_i) \mid O_{i}^{c} \} \]  
(E4)

\[ + \mathbb{E} [\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)] (1 - K_h(A_i - a') \lambda(a', X_i)) \mid O_{i}^{c} \} \]  
(TR3)
For (R2)+(R4) we have

\[(R2) + (R4)\]

\[-K_h(A_i - a)\left(\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\right)\lambda(a, X_i)R(M_i, X_i) +
K_h(A_i - a')\left(\hat{\gamma}_a(M_i, X_i) - \gamma(X_i, M_i, a)\right)\lambda(a', X_i)\]

\[-\mathbb{E}\left[K_h(A_i - a)\left(\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\right)\lambda(a, X_i)R(M_i, X_i) \mid O_{i_1}^c\right]\}

\[\left\{K_h(A_i - a')\left(\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\right)\lambda(a', X_i)\right\}\]

\[-\mathbb{E}\left[K_h(A_i - a')\left(\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\right)\lambda(a', X_i) \mid O_{i_1}^c\right]\}

\[\mathbb{E}\left[\left(K_h(A_i - a')\lambda(a', X_i) - K_h(A_i - a)\lambda(a, X_i)R(M_i, X_i)\right) \mid O_{i_1}^c\right].\]

For (R3)+(R5) we have

\[(R3) + (R5)\]

\[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\gamma(X_i, M_i, a) - K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\eta(a, a', X_i)\]

\[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\gamma(X_i, M_i, a)\]

\[\mathbb{E}\left[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\gamma(X_i, M_i, a) \mid O_{i_1}^c\right]\}

\[\mathbb{E}\left[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\eta(a, a', X_i) \mid O_{i_1}^c\right]\]

\[\mathbb{E}\left[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\eta(a, a', X_i) \mid O_{i_1}^c\right]\]

\[+\mathbb{E}\left[K_h(A_i - a')\left(\hat{\lambda}(a', X_i) - \lambda(a', X_i)\right)\{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} \mid O_{i_1}^c\right].\]

And so, to prove \(\sqrt{\frac{h^d\ell}{n}} \sum_{\ell \in I_{\ell}} \left\{m(O; \hat{\alpha}, I_{\ell}, \hat{\gamma}, \psi_0(a, a')) - m(O; \alpha, I_{\ell}, \gamma, \psi_0(a, a'))\right\} = o_p(1)\), we provide proofs for the convergence of the terms (CS1) - (CS6), (E1) - (E8) and (TR1) - (TR5) in the following sub-sections.

### 7.1.1 Proof for Terms (CS1)-(CS6)

All of these terms contain the product of two or more errors and can be treated similarly. We provide a detailed proof for (CS2), and a similar method can be followed for the rest of the terms.

For (CS2), write \(\Delta_{\ell} = K_h(A_i - a)\left[\hat{R}(M_i, X_i) - R(M_i, X_i)\right]\left[\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\right]\left[Y_i - \gamma(X_i, M_i, a)\right].\)

Following Lemma 3, it suffices to bound \(\mathbb{E}\left[\sqrt{\frac{h^d\ell}{n}} \sum_{i \in I_{\ell}} \Delta_{\ell} \mid O_{i_1}^c\right]\) as \(o_p(1)\) in order to show that

\(\sqrt{\frac{h^d\ell}{n}} \sum_{i \in I_{\ell}} \Delta_{\ell} = o_p(1).\)

First, from the triangle inequality, \(\mathbb{E}\left[\sqrt{\frac{h^d\ell}{n}} \sum_{i \in I_{\ell}} \Delta_{\ell} \mid O_{i_1}^c\right] \leq \frac{1}{2} \sqrt{\frac{h^d\ell}{n}} \mathbb{E}\left[\Delta_{\ell} \mid O_{i_1}^c\right],\) and so it suffices to bound \(\sqrt{h^d\ell} \mathbb{E}\left[\Delta_{\ell} \mid O_{i_1}^c\right].\) In the interest of space, we introduce the following notation \(\hat{k}(u) = \prod_{j=1}^{d} k(u_j),\) where \(u\) is a vector in \(\mathbb{R}^d.\)
\[
\sqrt{n h^d \lambda} \mathbb{E} \left[ \Delta_{i_{t}} \left| O_{i_{t}} \right. \right] \\
= \sqrt{n h^d} \int K_{h}(A, a) \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] [Y_{i} \gamma(X_{i}, M_{i}, a)] f(Y_{i}, A_{i}, M_{i}, X_{i}) dO_{i} \\
= \sqrt{n h^d} \int \hat{k}(u) \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] [Y_{i} \gamma(X_{i}, M_{i}, a)] f(Y_{i}, uh + a, M_{i}, X_{i}) dudY_{i} dM_{i} dX_{i} \\
= \sqrt{n h^d} \int \left\{ \int \hat{k}(u) f(uh + a|M_{i}, X_{i}) \left\{ \int \left| Y_{i} - \gamma(X_{i}, M_{i}, a) \right| f(Y_{i}|uh + a, M_{i}, X_{i}) dY_{i} \right\} du \right\} \\
\left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] f(M_{i}, X_{i}) dM_{i} dX_{i}
\]

Next, Assumption 3.1 on the boundedness of \( \gamma(X, M, a) \) and Assumption 3.3 on the boundedness of \( \text{var}(Y_{i}|a, m, x) \), along with an application of Lemma 3 on \( f(a | M, X) \), we get

\[
= O\left(\sqrt{n h^d} \right) \int \left\{ f(a | M_{i}, X_{i}) + O(h^2) \right\} \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] f(M_{i}, X_{i}) dM_{i} dX_{i} \\
+ O\left(\sqrt{n h^d} \right) \int \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] f(M_{i}, X_{i}) dM_{i} dX_{i} \\
\leq O\left(\sqrt{n h^d} \right) \int \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] f(M_{i}, X_{i}) dM_{i} dX_{i} \\
+ O\left(\sqrt{n h^d} \right) \int \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] \left[ \hat{\gamma}(X_{i}, M_{i}, a) - \gamma(X_{i}, M_{i}, a) \right] f(M_{i}, X_{i}) dM_{i} dX_{i} \\
\leq O\left(\sqrt{n h^d} \right) \left\{ \int \left[ \hat{R}(M_{i}, X_{i}) - R(M_{i}, X_{i}) \right] f(M_{i}, X_{i}) dM_{i} dX_{i} \right\}^{1/2} + o_{p}(1)
\]

\[
= o_{p}(1).
\]

Where (a) follows from an application of Holder’s inequality combined with Assumption 3.1 on the boundedness of \( f(a | M, X) \), and (b) and the last equality follows from an application of Cauchy-Schwartz, combined with Assumption 5.1 and \( nh^{d+4} \rightarrow C_{h} \) by Assumption 2.

### 7.1.2 Proof for Terms (E1)-(E8)

Terms (E1)-(E8) are normalized terms of the form of a bias times a bounded quantity; they can all be treated similarly. We only provide the proof of the convergence in probability to zero for the term (E2). (E2) is given as
\[ K_h(A_i - a)(\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i)(Y_i - \gamma(X_i, M_i, a)) \]
\[ - \mathbb{E}[K_h(A_i - a)(\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i)(Y_i - \gamma(X_i, M_i, a)) | O_{i\ell}] \]

To prove \( \sqrt{nh^d} \times \) times (E2) is \( o_p(1) \), we set \( \hat{\Delta}_{i\ell} \) as (E2).

By construction, \( O_{i\ell}^c \) and \( O_i \) are independent, \( i \in I_t \), and consequently \( \mathbb{E}[\hat{\Delta}_{i\ell}|O_{i\ell}^c] = 0 \) and \( \mathbb{E}[\hat{\Delta}_{i\ell}\hat{\Delta}_{j\ell}|O_{i\ell}] = 0 \) for \( i, j \in I_t \) and all \( a', a \in A_0 \). Next we note that

\[ h^{dA}\mathbb{E}[\hat{\Delta}_{i\ell}^2|O_{i\ell}^c] \]
\[ = h^{dA} \int K_h^2(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R^2_f(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i, A_i, M_i, X_i) dO_i \]
\[ = \int \hat{k}(u)^2 \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R^2_f(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i, uh + a, M_i, X_i) dudY_i dM_i dX_i \]
\[ = \int \int \hat{k}(u)^2 f(uh + a|M_i, X_i) \left\{ \int [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i|uh + a, M_i, X_i) dY_i \right\} du \]
\[ \overset{(a)}{=} O \left( \int \hat{k}(u)^2 dud \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R^2_f(M_i, X_i) f(M_i, X_i) dM_i dX_i \right) \]
\[ \overset{(b)}{=} O(1) \int \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R^2_f(M_i, X_i) f(M_i, X_i) dM_i dX_i \]
\[ \overset{(c)}{=} o_p(1) \]

Where (a) follows from Assumption 3.1 on the boundedness of \( f(a | M, X) \), along with Assumption 3.1 and Assumption 3.3 combined with the derivation provided below

\[ \int \mathbb{E}[Y_i^2|uh + a, M_i, X_i] dY_i \]
\[ = \mathbb{E}[Y_i^2|uh + a, M_i, X_i] - 2\gamma(X_i, M_i, a) \int Y_i f(Y_i|uh + a, M_i, X_i) dY_i \]
\[ = \mathbb{E}[Y_i^2|uh + a, M_i, X_i] + \gamma^2_a(M_i, X_i) - 2\gamma(X_i, M_i, a) \int Y_i f(Y_i|uh + a, M_i, X_i) dY_i \]
\[ = \mathbb{E}[Y_i^2|uh + a, M_i, X_i] + \gamma^2_a(M_i, X_i) - 2\gamma(X_i, M_i, a) - 2\gamma(X_i, M_i, a) \gamma_{uh+a}(M_i, X_i) \]
\[ = O(1). \]

Next, (b) follows from Assumption 3.4, and finally, (c) follows Assumption 3.2 along with Assumption 3.1.

Then \( \mathbb{E} \left[ \left( \sqrt{h^{dA}/n} \sum_{i=1}^L \sum_{\ell\in I_t} \hat{\Delta}_{i\ell} \right)^2 | O_{i\ell}^c \right] = h^{dA}/n \sum_{i=1}^L \sum_{\ell\in I_t} \mathbb{E} \left[ \hat{\Delta}_{i\ell}^2 | O_{i\ell}^c \right] = h^{dA} \mathbb{E} \left[ \hat{\Delta}_{i\ell}^2 | O_{i\ell}^c \right] = o_p(1). \)

Applying Lemma 1 to the above gives \( \sqrt{h^{dA}/n} \sum_{i=1}^L \sum_{\ell\in I_t} \hat{\Delta}_{i\ell} \overset{P}{\to} 0 \), i.e. \( \sqrt{nh^{dA}} \times \) times (E2) being \( o_p(1) \).
7.1.3 Proof for Terms (TR1)-(TR5)

The proofs of the convergence in probability to zero for the terms (TR1)-(TR5) require extra considerations, and we prove them on a case by case basis below.

**Proof for Terms TR1 and TR2**

Terms (TR1) and (TR2) are similar; we only provide the proof of the convergence in probability to zero for the term (TR2).

To bound TR2, first set \( \hat{\Delta} = K_h(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] \). Bounding (TR2) amounts to showing \( \forall h \in \mathbb{R}^+ \),

\[
\sqrt{nh^{d_A}} \mathbb{E} \left[ \left| \hat{\Delta} \right| O^c_i \right] = o_p(1).
\]

Applying Lemma \( \text{Lemma 4} \) under Assumption \( 3 \) and Assumption \( 4.1 \), Cauchy-Schwartz combined with the boundedness of \( \int |Y_i - \gamma(X_i, M_i, a)| f(Y_i) \),

\[
\sqrt{nh^{d_A}} \mathbb{E} \left[ \left| \hat{\Delta} \right| O^c_i \right] = \sqrt{nh^{d_A}} \mathbb{E} \left[ K_h(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] \right] O^c_i
\]

\[
= \sqrt{nh^{d_A}} \int K_h(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, A_i, M_i, X_i) dO_i
\]

\[
= \sqrt{nh^{d_A}} \int \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i dM_i dX_i
\]

Applying Lemma \( \text{Lemma 4} \) under Assumption \( 3 \)

\[
= \sqrt{nh^{d_A}} \int f(a \mid Y_i, M_i, X_i) + O(h^2)
\]

\[
\left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i dM_i dX_i
\]

\[
= \sqrt{nh^{d_A}} \int O(h^2)
\left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i dM_i dX_i
\]

\[
\overset{(a)}{=} O(\sqrt{nh^{d_A+4}}) 
\left[ \int \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) \right]
\int [Y_i - \gamma(X_i, M_i, a)] f(Y_i \mid M_i, X_i) dY_i f(M_i, X_i) dM_i dX_i
\]

\[
\overset{(c)}{=} o_p(1)
\]

where (a) follows from

\[
\int [Y_i - \gamma(X_i, M_i, a)] f(Y_i \mid a, M_i, X_i) dY_i = \int Y_i f(Y_i \mid a, M_i, X_i) dY_i - \gamma(X_i, M_i, a) = 0
\]

(b) is from the exchange of \( O(\cdot) \) and integration, (c) follows from Assumption 2 \( (nh^{d_A+4} \to C_h, h \to 0) \), Assumption 3 and Assumption 4.1, Cauchy-Schwartz combined with the boundedness of \( \int |Y_i - \gamma(X_i, M_i, a)| f(Y_i) \)
where the last line also comes from the Cauchy-Schwartz inequality.

**Proof for TR3**

For Term (TR3), we have

$$
\sqrt{nh^{d_A}} \mathbb{E} \left[ (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) \{ 1 - K_h(A_i - a') \lambda(a', X_i) \} | O_{10}^i \right]
$$

$$
= \sqrt{nh^{d_A}} \int (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) (1 - K_h(A_i - a') \lambda(a', X_i)) f(A_i, X_i) dA_i dX_i
$$

$$
= \sqrt{nh^{d_A}} \int (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) (1 - \int K_h(A_i - a') f(A_i | X_i) dA_i) \lambda(a', X_i) f(X_i) dX_i
$$

$$
\overset{(a)}{=} \sqrt{nh^{d_A}} \int (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) (1 - f(a' | X_i) \lambda_w(X_i)) f(X_i) dX_i
$$

$$
\overset{(b)}{=} o_p(1).
$$

where (a) follows from Lemma [4] and (b) follows from the definition of $\lambda_w(X_i)$, $\sqrt{nh^{d_A} + 4} \to C_h$, (consistency of $\hat{\eta}(X_i)$, Assumption 3 (boundedness of $\lambda$) combined with an application of Cauchy-Schwartz inequality.

**Proof For TR4**

Demonstrating the bound for (TR4), we have

$$
\sqrt{nh^{d_A}} \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \{ K_h(A_i - a') \lambda(a', X_i) - K_h(A_i - a) \lambda(a, X_i) R(M_i, X_i) \} \right]
$$

$\overset{\text{TR4-1}}{=} \sqrt{nh^{d_A}} \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \{ K_h(A_i - a') \lambda(a', X_i) \} \right]
$$

$\overset{\text{TR4-2}}{=} \sqrt{nh^{d_A}} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) \left\{ \int K_h(A_i - a') f(A_i | M_i, X_i) dA_i \right\} f(M_i, X_i) dM_i dX_i$

TR-4.1 can be written as

$$
\sqrt{nh^{d_A}} \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \{ K_h(A_i - a') \lambda(a', X_i) \} \right]
$$

$$
= \sqrt{nh^{d_A}} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) f(a' | M_i, X_i) f(M_i, X_i) dM_i dX_i
$$

An application of Lemma [4] to TR-4.1 gives

$$
\sqrt{nh^{d_A}} \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \{ K_h(A_i - a') \lambda(a', X_i) \} \right]
$$

$$
= \sqrt{nh^{d_A}} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) f(a' | M_i, X_i) f(M_i, X_i) dM_i dX_i
$$

$$
+ \sqrt{nh^{d_A}} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) O(h^2) f(M_i, X_i) dM_i dX_i
$$

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A similar approach applied to TR-4-2 gives
\[
\sqrt{n h^d A} \mathbb{E}\left[\{\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)\} \{K_h(A_i - a)\lambda(a, X_i)R(M_i, X_i)\}\right]
\]
\[
= \sqrt{n h^d A} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a))\lambda(a, X_i)R(M_i, X_i)f(a \mid M, X)f(M_i, X_i) dM_i dX_i
\]
\[
+ \sqrt{n h^d A} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a))\lambda(a, X_i)R(M_i, X_i)O(h^2)f(M_i, X_i) dM_i dX_i
\]

Now, the first terms of TR-4-1 and TR-4-2 cancel out with each other, shown below
\[
\int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \{\lambda(a', X_i)f(a' \mid M, X) - \lambda(a, X_i)R(M_i, X_i)f(a \mid M, X)\} f(M_i, X_i) dM_i dX_i = 0
\]

This can be seen from
\[
\lambda(a', X_i)f(a' \mid M_i, X_i) = \frac{f(X_i)}{f(a', X_i)} \frac{f(a', M_i, X_i)}{f(M_i, X_i)}
\]

Along with
\[
\lambda(a, X_i)R(M_i, X_i)f(a \mid M_i, X_i) = \frac{f(X_i)}{f(a', X_i)} \frac{f(a, M_i, X_i)}{f(M_i, X_i)} \frac{f(a, M_i, X_i)}{f(a', M_i, X_i)}\]
\[
= \frac{f(X_i)}{f(a', X_i)} \frac{f(a, M_i, X_i)}{f(M_i, X_i)}
\]

Consequently the first terms in TR4-1 and TR4-2 cancel each other out, and this leaves us to bound the remaining terms.
\[
\sqrt{n h^d A} \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a))\lambda(a', X_i)O(h^2)f(M, X) dM_i dX_i = o_p(1)
\]

The second term in TR-4-1 and TR-4-2 can be bounded by an application of Cauchy-Schwartz, combined with Assumption 2 (consistency of \(\hat{\gamma}\)) and boundedness of \(\lambda\) in Assumption 3.1.

**Proof For TR5**

Finally, for term (TR5), we note that
\[
\sqrt{n h^d A} \mathbb{E}\left[K_h(A_i - a')(\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \{\gamma(X_i, M_i, a) - \eta(X_i)\} \mid O_f^i\right]
\]
\[
= \sqrt{n h^d A} \int K_h(A_i - a')(\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} f(A_i, M_i, X_i) dA_i dM_i dX_i
\]
\[
= \sqrt{n h^d A} \int \left\{\int K_h(A_i - a') f(A_i \mid M_i, X_i) dA_i\right\} (\hat{\lambda}(a', X_i) - \lambda(a', X_i))
\]
\[
\times \{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} f(M_i, X_i) dM_i dX_i
\]
\[
\overset{(a)}{=} \sqrt{n h^d A} \int (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} f(a', M_i, X_i) dM_i dX_i
\]
\[
+ \sqrt{n h^d A} \int (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} O(h^2)f(M_i, X_i) dM_i dX_i
\]
\[
\overset{(b)}{=} 0 + O(\sqrt{n h^d A + 4}) \int (\hat{\lambda}(a', X_i) - \lambda(a', X_i)) \{\gamma(X_i, M_i, a) - \eta(a, a', X_i)\} f(M_i, X_i) dM_i dX_i
\]
\[
\overset{(c)}{=} o_p(1)
\]
Given

\[
\sqrt{n}h^{d_{A}n^{-1}}m(O; \alpha, \lambda, \gamma, \psi_{0}(a, a')) = \sum_{i=1}^{n} \left\{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \right\}
\]

Since \( \lambda \), \( \gamma \), \( \psi \) are constants, we have the following:

\[
\sqrt{n}h^{d_{A}n^{-1}}m(O; \alpha, \lambda, \gamma, \psi_{0}(a, a'))
\]

and \( s_{n}^{2} \) is the variance of \( \sqrt{n}h^{d_{A}n^{-1}}m(O; \alpha, \lambda, \gamma, \psi_{0}(a, a')) \). To prove the Lyapunov condition holds, we first derive \( \mu_{i} \) and \( \sigma_{i}^{2} \).

**7.2 Calculation for \( B(a, a') \) and \( \mu_{i} \)**

Given

\[
m(O; \alpha, \lambda, \gamma, \psi_{0}(a, a')) = \frac{K_{h}(A_{i} - a)f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} \{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \}
\]

\[
+ \frac{K_{h}(A_{i} - a')}{f(a' | X_{i})} \{ E[Y | X_{i}, M_{i}, A = a] - \eta(a, a', X_{i}) \} + \eta(a, a', X_{i}) - \psi_{0}(a, a')
\]

Since \( E[\eta(a, a', X_{i}) - \psi_{0}(a, a')] = 0 \), we focus on

\[
\frac{K_{h}(A_{i} - a)f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} \{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \}
\]

We start by computing the expectation of the first term.

**Expectation Part 1**

\[
E \left\{ \frac{K_{h}(A_{i} - a)f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} \{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \} \right\}
\]

From \( E\{E[\gamma(X, M, a)]X, M) \} = E\{E[\gamma(X, M)] \} \) and \( E[\gamma(X, M, a)] = E[\gamma(X, M)] \), expectation of the first term

\[
E \left\{ \frac{K_{h}(A_{i} - a)f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} \{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \} \right\}
\]

\[
= E \left\{ \frac{K_{h}(A_{i} - a)f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} \{ Y_{i} - E[Y | X_{i}, M_{i}, A = a] \} \right\}
\]

\[
= E \left\{ \frac{f(M_{i} | A = a', X_{i})}{f(M_{i} | A = a, X_{i})f(a | X_{i})} E \left[ K_{h}(A_{i} - a)(\gamma(X, M, a) - \gamma(X, M, a)) \right] \right\}
\]
The inner product further expands as follows,

\[
\mathbb{E}\left[K_h(A-a)(\gamma(X,M,a) - \gamma(X,M,a))\right|X,M]
\]

\[
= \int K_h(A-a)(\gamma(X,M,a) - \gamma(X,M,a))f(A|X,M)dA
\]

\[
=\int \left[\prod_{j=1}^{d_A} \frac{1}{h}\left(\frac{A_j - a}{h}\right)\right] (\gamma(X,M,a) - \gamma(X,M,a))f(A|X,M)dA
\]

\[
= \int \left[\prod_{j=1}^{d_A} k(u_j)\right](\gamma(a + uh, X,M) - \gamma(X, M, a))f(a + uh|X, M)du
\]

\[
= \int \left[\prod_{j=1}^{d_A} k(u_j)\right]\left(\sum_{j=1}^{d_A} u_j h \partial_{u_j} \gamma(X, M, a) + \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \partial_{u_j} \partial_{u_{j'}} \gamma(X, M, a)\right)
\]

\[
\times \left( f(a|X,M) + \sum_{j=1}^{d_A} u_j h \partial_{u_j} f(a|X,M) + \frac{u_j^2 h^2}{2} \partial_{u_j}^2 f(a|X,M)\right)du_1 \cdots du_{d_A} + O(h^3)
\]

\[
= h^2 \int u^2 k(u)du \left(\sum_{j=1}^{d_A} \partial_{u_j} \gamma(X, M, a) \partial_{u_j} f(a|X,M) + \frac{1}{2} \sum_{j=1}^{d_A} \partial_{u_j}^2 \gamma(X, M, a)\right) f(a|X,M) + O(h^3)
\]

for all \(X, M\) in respective range. Inserting this back into the original expectation we get,

\[
\mathbb{E}\left\{\frac{f(M | A = a', X)}{f(M | A = a, X)} \mathbb{E}\left[K_h(A-a)(\gamma(X,M,a) - \gamma(X,M,a))\right|X,M]\right\}
\]

\[
= h^2 \int u^2 k(u)du \times \mathbb{E}\left[\frac{f(M | A = a', X)}{f(M | A = a, X)} \left(\sum_{j=1}^{d_A} \partial_{u_j} \gamma(X, M, a) \partial_{u_j} f(a|X,M) + \frac{1}{2} \sum_{j=1}^{d_A} \partial_{u_j}^2 \gamma(X, M, a)\right) f(a|X,M)\right] + O(h^3)
\]

**Expectation Part 2**

\[
\frac{K_h(A-a')}{f(a' | X)} \{\gamma(X,M,a) - \gamma(a,a',X)\}
\]

\[
\mathbb{E}\left[\frac{K_h(A-a')}{f(a' | X)} \{\gamma(X,M,a) - \gamma(a,a',X)\}\right|X,M\}
\]

\[
= \mathbb{E}\left\{\frac{1}{f(a' | X)} \mathbb{E}\left[K_h(A-a')\{\gamma(X,M,a) - \gamma(a,a',X)\}\right|X,M\}\right\}
\]

\[
= \mathbb{E}\left\{\frac{\gamma(X,M,a) - \gamma(a,a',X)}{f(a' | X)} \mathbb{E}[K_h(A-a')|X,M]\right\}
\]
The inner expectation can be written as

$$
E \left[ \frac{K_h(A - a')}{X, M} \right] = \int \left[ \prod_{j=1}^{d_A} \frac{k(A_j - a'_j)}{h} \right] f(A|X, M) dA
$$

$$
= \int k(u_1) \cdot \cdot \cdot k(u_{d_A}) \left( f(a'|X, M) + \sum_{j=1}^{d_A} u_j h \partial_{a_j} f(a'|X, M) + \frac{1}{2} \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_j' h^2 \partial_{a_j} \partial_{a_{j'}} f(a'|X, M) \right) du_1 \cdot \cdot \cdot du_{d_A}
$$

$$
= f(a'|X, M) + \frac{1}{2} h^2 \int u^2 k(u) du \sum_{j=1}^{d_A} \partial^2_{a_j} f(a'|X, M) + O(h^3)
$$

Plugging this back into the above expectation

$$
E \left\{ \frac{1}{f(a'|X)} \{ \gamma(X, M, a) - \eta(a, a', X) \} \times \left( f(a'|X, M) + \frac{1}{2} h^2 \int u^2 k(u) du \sum_{j=1}^{d_A} \partial^2_{a_j} f(a'|X, M) \right) \right\} + O(h^3)
$$

$$
= E \left\{ \gamma(X, M, a) - \eta(a, a', X) \left( \frac{f(a'|X, M)}{f(a'|X)} + \frac{1}{2} h^2 \int u^2 k(u) du \sum_{j=1}^{d_A} \partial^2_{a_j} f(a'|X, M) \right) \right\} + O(h^3)
$$

$$
= h^2 \int u^2 k(u) du \left[ \gamma(X, M, a) - \eta(a, a', X) \frac{1}{2} \sum_{j=1}^{d_A} \partial^2_{a_j} f(a'|X, M) \right] + O(h^3)
$$

from having the first term in this expectation equal to zero, which we prove below

$$
E \left\{ \gamma(X, M, a) - \eta(a, a', X) \right\} \frac{f(a'|X, M)}{f(a'|X)}
$$

$$
= \int \left\{ \gamma(X, M, a) - \eta(a, a', X) \right\} \frac{f(a'|X, M)}{f(a'|X)} f(M|X) dMdX
$$

$$
= \int \left\{ \gamma(X, M, a) - \eta(a, a', X) \right\} \frac{f(A = a', X, M)}{f(A = a', X)} f(X) dMdX
$$

$$
= \int \left\{ \gamma(X, M, a) - \eta(a, a', X) \right\} f(M|A = a', X) dM f(X) dX
$$

$$
= \int \left\{ \gamma(a, a', X) - \eta(a, a', X) \right\} f(X) dX = 0
$$
Hence, letting

\[ B(a, a') = \left[ \int u^2 k(u) du \right] \]

\[ \mathbb{E} \left[ \frac{f(M | A = a', X)}{f(M | A = a, X)} f(a | X) \right] \left( \sum_{j=1}^{d_A} \partial_{a_j} \gamma(X, M, a) \partial_{a_j} f(a | X, M) + \frac{1}{2} \left[ \sum_{j=1}^{d_A} \partial_{a_j}^2 \gamma(X, M, a) \right] f(a | X, M) \right) \]

\[ + \{ \gamma(X, M, a) - \eta(a, a', X) \} \frac{1}{2} \sum_{j=1}^{d_A} \partial_{a_j}^2 f(a' | X, M) \]

\[ + O(h) \]

we have \( \mathbb{E} [m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))] = h^2 B(a, a'). \) Additionally from this derivation \( \mathbb{E} \left[ \sqrt{nh^{d_A} n^{-1} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))} \right] = O(\sqrt{\frac{h^{d_A+4}}{n}}). \) Next, we prove the properties of variance.

\[ 7.2.2 \quad \text{Calculation for } V(a, a') \text{ and } s_n^2 \]

From the definition of \( s_n^2, \) we have

\[ s_n^2 = \sum_{i=1}^{n} \sigma_i^2 = \sum_{i=1}^{n} \text{var} \left( \sqrt{nh^{d_A} n^{-1} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a'))} \right) = h^{d_A} \text{var} \left( m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right) \]

Consequently, we calculate

\[ h^{d_A} \times \text{var} \left\{ \frac{K_h(A - a)f(M | A = a', X)}{f(M | A = a, X)f(a | X)} \right\} \]

\[ \{ Y - \mathbb{E}[Y | X, M, A = a] \} \]

\[ + \frac{K_h(A - a')}{f(a' | X)} \{ \mathbb{E}[Y | X, M, A = a] - \eta(a, a', X) \} \]

\[ + \eta(a, a', X) - \psi_0(a, a') \}

Using the property that \( \text{var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 \) and constant values do not contribute to the variance, the variance term above can be re-written as

\[ h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a)f(M | A = a', X)}{f(M | A = a, X)f(a | X)} \right] \{ Y - \mathbb{E}[Y | X, M, A = a] \} \right\} \]

\[ + \frac{K_h(A - a')}{f(a' | X)} \{ \mathbb{E}[Y | X, M, A = a] - \eta(a, a', X) \} \]

\[ + \eta(a, a', X) \}

\[- h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a)f(M | A = a', X)}{f(M | A = a, X)f(a | X)} \right] \{ Y - \mathbb{E}[Y | X, M, A = a] \} \right\} \]

\[ + \frac{K_h(A - a')}{f(a' | X)} \{ \mathbb{E}[Y | X, M, A = a] - \eta(a, a', X) \} \]

\[ + \eta(a, a', X) \}^2 \]
Examining each of the terms above one by one, the first term can be expanded as

\[
= h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right]^2 \right\}
\]

\[
+ h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a')}{f(a' \mid X)} \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} \right]^2 \right\} + h^{d_A} \mathbb{E} \left\{ \eta^2(a, a', X) \right\}
\]

\[
+ 2h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right] \times \left[ \frac{K_h(A - a')}{f(a' \mid X)} \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} \right] \right\}
\]

\[
+ 2h^{d_A} \mathbb{E} \left\{ \eta(a, a', X) \left[ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right] \right\}
\]

\[
+ 2h^{d_A} \mathbb{E} \left\{ \eta(a, a', X) \left[ \frac{K_h(A - a')}{f(a' \mid X)} \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} \right] \right\}
\]

We analyze each of these terms part by part

**Variance Part 1**

\[
h^{d_A} \mathbb{E} \left\{ \left[ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right]^2 \right\}
\]

\[
= h^{d_A} \mathbb{E} \left\{ \mathbb{E} \left[ \left[ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right]^2 \mid X, M \right] \right\}
\]

\[
= h^{d_A} \mathbb{E} \left\{ \frac{f(M \mid A = a', X)^2}{f(M \mid A = a, X)^2f(a \mid X)^2} \mathbb{E} \left[ K_h(A - a)^2(Y - \mathbb{E}[Y \mid X, M, A = a])^2 \mid X, M \right] \right\}
\]

\[
= h^{d_A} \mathbb{E} \left\{ \frac{f(M \mid A = a', X)^2}{f(M \mid A = a, X)^2f(a \mid X)^2} \times \mathbb{E} \left[ K_h(A - a)^2 \mathbb{E} \left\{ (Y - \mathbb{E}[Y \mid X, M, A = a])^2 \mid X, M, A \right\} \mid X, M \right] \right\}
\]

\[
= h^{d_A} \mathbb{E} \left\{ \frac{f(M \mid A = a', X)^2}{f(M \mid A = a, X)^2f(a \mid X)^2} \times \mathbb{E} \left[ K_h(A - a)^2 \left[ \text{var}(Y \mid X, M, A) + \gamma(X, M, a)^2 - 2\gamma(X, M, a)\gamma(X, M, a) + \gamma(X, M, a)^2 \right] \right] \mid X, M \right\}
\]

Because \(0 < \int u^6 k(u)du < \infty\) from Assumption 2 (3), we also have boundedness of \(\int u^6 k^2(u)du\). The inner
expectation can be written as
\[
h^{da} \mathbb{E} \left\{ K_h(A - a)^2 \left[ \text{var}(Y | X, M, A) + (\gamma(X, M, a) - \gamma(X, M, a))^2 \right] \bigg| X, M \right\}
\]
\[
= h^{da} \int \frac{1}{h^2} \prod_{k=1}^{da} \left( \frac{A_k - a_k}{h} \right)^2 \left\{ \text{var}(Y | X, M, A) + (\gamma(X, M, a) - \gamma(X, M, a))^2 \right\} f(A | X, M) dA
\]
\[
= \int \hat{k}(u)^2 \times \left\{ \text{var}(Y | X, M, a + uh) + (\gamma(a + uh, M, X) - \gamma(X, M, a))^2 \right\} f(a + uh | X, M) du
\]
\[
= \int k(u_1)^2 \cdots k(u_{da})^2 \times \left\{ \text{var}(Y | X, M, a) + \sum_{j=1}^{da} u_j h \partial a_j \text{var}(Y | X, M, a) + \sum_{j=1}^{da} \sum_{j'=1}^{da} u_j u_{j'} h^2 \partial a_j \partial a_{j'} \text{var}(Y | X, M, \bar{a}_v) \right. \\
+ \left. \left[ \sum_{j=1}^{da} u_j h \partial a_j \gamma(X, M, a) + \sum_{j=1}^{da} \sum_{j'=1}^{da} u_j u_{j'} h^2 \partial a_j \partial a_{j'} \gamma(\bar{a}_\gamma, M, X) \right]^2 \right\} \times \left[ f(a | X, M) + \sum_{j=1}^{da} u_j h \partial a_j f(a | X, M) + \sum_{j=1}^{da} \sum_{j'=1}^{da} u_j u_{j'} h^2 \partial a_j \partial a_{j'} f(\bar{a}_f | X, M) \right] du_1 \cdots du_{da}
\]
\[
= \left[ \int \hat{k}(u)^2 du \right] \times \text{var}(Y | X, M, a) f(a | X, M) + O(h^2)
\]
where $\bar{a}_v, \bar{a}_\gamma$, and $\bar{a}_f$ are between $a$ and $a + uh$. Hence, part 1 of the variance
\[
h^{da} \mathbb{E} \left\{ \left[ \frac{K_h(A - a) f(M | a', X)}{f(M | a, X) f(a | X)} \{ Y - \mathbb{E}[Y | X, M, a] \} \right]^2 \right\}
\]
\[
= \left[ \int k(u)^2 du \right]^{da} \mathbb{E} \left\{ \frac{f(M | a', X)^2}{f(M | a, X)^2 f(a | X)^2} \text{var}(Y | X, M, a) f(a | X, M) \right\} + O(h^2)
\]

**Variance Part 2**
\[
h^{da} \mathbb{E} \left\{ \left[ \frac{K_h(A - a')}{f(a' | X)} \left( \mathbb{E}(Y | X, M, A = a) - \eta(a, a', X) \right) \right]^2 \right\}
\]
\[
= h^{da} \mathbb{E} \left\{ \frac{1}{f(a' | X)^2} \left[ K_h(A - a')^2 \left( \gamma(X, M, a) - \eta(a, a', X) \right) \right]^2 \right\}
\]
The inner expectation can be written as

\[ h^d \mathbb{E} \left[ K_h(A - a')^2 \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 | X \right] \]

\[ = h^d \int \left[ \prod_{j=1}^d \frac{1}{h^2} k \left( \frac{A_j - a'_j}{h} \right)^2 \right] \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 f(A|M, X)f(M | X)dAdM \]

\[ = \int \left[ \prod_{j=1}^d k(u_j)^2 \right] \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 f(a' + u\hbar|M, X)f(M | X)dudM \]

\[ = \int \left[ \prod_{j=1}^d k(u_j)^2 \right] \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 \left\{ f(a'|M, X) + \sum_{j=1}^d u_j h \partial_a_j f(a | X, M) \right\} f(M | X)dudM \]

\[ = \int k^2(u_1) \cdots k^2(u_d\hbar) \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 f(a'|X, M)f(M | X)du_1 \cdots du_d\hbar dM + O(h^2) \]

\[ = \left[ \int k(u)^2 du \right]^d \times \text{var}[E(Y|X, M, a)|X, a']f(a'|X) + O(h^2) \]

the last equation is from

\[ \text{var}[E(Y|X, M, a)|X, a'] = \mathbb{E} \left\{ \left[ E(Y|X, M, a) - \eta(a, a', X) \right]^2 | X, a' \right\} \]

\[ = \int \left[ E(Y|X, M, a) - \eta(a, a', X) \right]^2 f(M|X, a')dM. \]

Hence, the part 2 of variance

\[ h^d \mathbb{E} \left\{ \frac{1}{f(a'|X)^2} \mathbb{E} \left[ K_h(A - a')^2 \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 | X \right] \right\} \]

\[ = \left[ \int k(u)^2 du \right]^d \times \mathbb{E} \left\{ \frac{1}{f(a'|X)^2} \text{var}[E(Y|X, M, a)|X, a'] \right\} + O(h^2) \]

**Variance Part 3**

\[ h^d \mathbb{E} \left[ \eta^2(a, a', X) \right] = O(h^d) \]

This holds because we assume \( \eta \) is bounded.
Variance Part 4

\[
\begin{align*}
& h^d A \mathbb{E} \left\{ \frac{K_h(A - a) f(M | A = a', X)}{f(M | A = a, X) f(a | X)} \{ Y - \mathbb{E}[Y | X, M, A = a] \} \right. \\
& \quad \times \left\{ \frac{K_h(A - a') f(M | a', X)}{f(a' | X)} \{ \mathbb{E}[Y | X, M, A = a] - \eta(a, a', X) \} \right\} \\
& = h^d A \mathbb{E} \left\{ \frac{K_h(A - a) K_h(A - a') f(M | a', X)}{f(M | a, X) f(a' | X)} \left[ Y - \gamma(X, M, a) \right] \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right\} \\
& = h^d A \mathbb{E} \left\{ \frac{1}{f(a | X) f(a' | X)} \frac{f(M | a', X)}{f(M | a, X)} \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right\} \\
& \quad \times \mathbb{E} \left\{ K_h(A - a) K_h(A - a') \left[ Y - \gamma(X, M, a) \right] | X, M \right\} \\
& = h^d A \mathbb{E} \left\{ \frac{1}{f(a | X) f(a' | X)} \frac{f(M | a', X)}{f(M | a, X)} \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right\} \\
& \quad \times \mathbb{E} \left\{ K_h(A - a) K_h(A - a') \left[ \gamma(X, M, a) - \gamma(X, M, a) \right] | X, M \right\} \\
& \quad \times \mathbb{E} \left\{ K_h(A - a) K_h(A - a') \left[ \gamma(X, M, a) - \gamma(X, M, a) \right] | X, M \right\}
\end{align*}
\]

The inner expectation

\[
\begin{align*}
& h^d A \mathbb{E} \left\{ K_h(A - a) K_h(A - a') \left[ \gamma(X, M, a) - \gamma(X, M, a) \right] | X, M \right\} \\
& = h^d A \int \left[ \prod_{j=1}^{d_A} k\left( \frac{A_j - a}{h} \right) k\left( \frac{A_j - a'}{h} \right) \right] \left[ \gamma(X, M, a) - \gamma(X, M, a) \right] f(A | X, M) dA \\
& = \int k(u_1) \cdots k(u_{d_A}) k(u_1 + \frac{a - a'}{h}) \cdots k(u_{d_A} + \frac{a - a'}{h}) \left[ \gamma(uh + a, M, X) - \gamma(X, M, a) \right] f(uh + a | X, M) dA \\
& = \int k(u_1) \cdots k(u_{d_A}) k(u_1 + \frac{a - a'}{h}) \cdots k(u_{d_A} + \frac{a - a'}{h}) \\
& \quad \times \left[ \sum_{j=1}^{d_A} u_j h \partial_{a_j} \gamma(X, M, a) + \frac{u_j^2 h^2}{2} \partial_{a_j}^2 \gamma(X, M, a) + \frac{u_j^3 h^3}{6} \partial_{a_j}^3 \gamma(a, M, X) \right] \\
& \quad \times f(a | X, M) + \sum_{j=1}^{d_A} u_j h \partial_{a_j} f(a | X, M) + \frac{u_j^2 h^2}{2} \partial_{a_j}^2 f(a | X, M) \right] du_1 \cdots du_{d_A} \\
& = O(h)
\end{align*}
\]

Hence, the part 4 of variance

\[
\begin{align*}
& \mathbb{E} \left\{ \frac{1}{f(a | X) f(a' | X)} \frac{f(M | a', X)}{f(M | a, X)} \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right. \\
& \quad \times \left\{ K_h(A - a) K_h(A - a') \left[ \gamma(X, M, a) - \gamma(X, M, a) \right] | X, M \right\} \\
& = O(h)
\end{align*}
\]
Variance Part 5

\[
2h^d A \left\{ \eta(a, a', \mathbf{X}) \left[ \frac{K_h(A - a)f(M \mid A = a', \mathbf{X})}{f(M \mid A = a, \mathbf{X})f(a \mid \mathbf{X})} \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right] \right\}
= 2h^d A \left\{ \eta(a, a', \mathbf{X}) \frac{f(M \mid A = a, \mathbf{X})f(a \mid \mathbf{X})}{f(M \mid A = a, \mathbf{X})} \mathbb{E} \left[ \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \mid X, M \right] \right\}
\]

Applying the same expansion in Expectation Part 1, we can write the inner expectation as

\[
h^2 \int u^2 k(u) du \left( \sum_{j=1}^d \partial_{a_j} \gamma(X, M, a) \partial_{a_j} f(a \mid X, M) + \frac{1}{2} \sum_{j=1}^d \partial_{a_j}^2 \gamma(X, M, a) \right) f(a \mid X, M) + O(h^3)
\]

Inserting this back into the full expectation, combined with the boundedness of \( \eta \), \( f(M \mid a, X) \) and \( f(a \mid X) \), we get

\[
2h^d A \left\{ \eta(a, a', \mathbf{X}) \left[ K_h(A - a)f(M \mid A = a', \mathbf{X}) \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right] \right\} = O(h^{d+2})
\]

Variance Part 6

\[
2h^d A \left\{ \eta(a, a', \mathbf{X}) \left[ K_h(A - a')f(M \mid A = a, \mathbf{X}) \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} \right] \right\}
= 2h^d A \left\{ \mathbb{E} \left[ K_h(A - a')\eta(a, a', \mathbf{X}) \frac{\gamma(X, M, a) - \eta(a, a', X)}{f(a' \mid X)} \mid X, M \right] \right\}
= 2h^d A \left\{ \frac{\gamma(X, M, a) - \eta(a, a', X)}{f(a' \mid X)} \eta(a, a', \mathbf{X}) \mathbb{E}[K_h(A - a') \mid X, M] \right\}
\]

Using the expansion from Part 2 of the expectation on \( \mathbb{E}[K_h(A - a') \mid X, M] \), we get

\[
\mathbb{E}[K_h(A - a') \mid X, M] = f(a' \mid X, M) + \frac{1}{2} h^2 \int u^2 k(u) du \sum_{j=1}^d \partial_{a_j}^2 f(a' \mid X, M) + O(h^3)
\]

Plugging this back into the full expectation, we get

\[
2h^d A \left\{ \eta(a, a', \mathbf{X}) \left[ K_h(A - a')f(M \mid A = a', \mathbf{X}) \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} \right] \right\} = O(h^{d+2})
\]

And using the calculation for the bias,

\[
K^d A \left\{ \left[ K_h(A - a)f(M \mid A = a', \mathbf{X}) \frac{f(M \mid A = a, \mathbf{X})f(a \mid \mathbf{X})}{f(M \mid A = a, \mathbf{X})f(a \mid \mathbf{X})} \right] \{Y - \mathbb{E}[Y \mid X, M, A = a]\} \right\}^2
+ \left[ K_h(A - a')f(a' \mid X) \{\mathbb{E}[Y \mid X, M, A = a] - \eta(a, a', X)\} + \eta(a, a', X) \right]^2
= O(h^{2d+4})
\]
Finally, putting the pieces of the variance together, we have

\[ h^{dA} \times \text{var} \left\{ \frac{K_h(A-a) f(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} [Y - E[Y \mid X, M, A = a]] \right\} \]

\[ + \frac{K_h(A-a')}{f(a' \mid X)} \left[ E[Y \mid X, M, A = a] - \eta(a, a', X) \right] \]

\[ + \eta(a, a', X) - \psi_0(a, a') \} = V(a, a') + O(h) \]

where the term converges to \( V(a, a') \) as \( h \to 0 \) and

\[ V(a, a') = \left[ \int k(u)^2 du \right]^{dA} \times E \left\{ \frac{f(M \mid a', X)^2}{f(M \mid a, X)^2} \text{var}(Y \mid X, M, a) f(a \mid X, M) + \frac{1}{f(a' \mid X)} \text{var}[E(Y \mid X, M, a) \mid X, a'] \right\}. \]

Having derived the bias and variance terms, we now prove the Lyapunov condition for \( \delta = 1 \).

### 7.2.3 Proof for Lyapunov Condition

We now prove the Lyapunov condition

\[ \lim_{n \to \infty} \frac{1}{n^3} \sum_{i=1}^{n} E \left[ \left| \sqrt{nh^{dA}n^{-1}} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) - \mu_i \right|^3 \right] = 0 \]

Note that

\[ \left| \sqrt{nh^{dA}n^{-1}} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) - \mu_i \right|^3 \leq \left| \sqrt{nh^{dA}n^{-1}} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right|^3 + \left| \mu_i \right|^3 \]

Since both sides are positive,

\[ \left| \sqrt{nh^{dA}n^{-1}} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) - \mu_i \right|^3 \leq (h^{dA}n^{-1})^{3/2} \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right|^3 + \left| \mu_i \right|^3 \]

\[ + 3(h^{dA}n^{-1}) \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right|^2 \left| \mu_i \right| \]

\[ + 3(h^{dA}n^{-1})^{1/2} \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right| \left| \mu_i \right|^2 \]

From the monotonicity of the expected value, have

\[ \sum_{i=1}^{n} E \left[ \left| \sqrt{nh^{dA}n^{-1}} m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) - \mu_i \right|^3 \right] \leq \sum_{i=1}^{n} E \left[ (h^{dA}n^{-1})^{3/2} \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right|^3 \right] + \sum_{i=1}^{n} \left| \mu_i \right|^3 \]

\[ + \sum_{i=1}^{n} 3(h^{dA}n^{-1}) \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right|^2 \left| \mu_i \right| \]

\[ + \sum_{i=1}^{n} 3(h^{dA}n^{-1})^{1/2} \left| m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right| \left| \mu_i \right|^2 \]

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Since $\sum_{i=1}^{n} |\mu_i|^3 = O(h^{(d_x+4)/2}n^{1/2}) = o(1)$, $\sum_{i=1}^{n} 3h^{d_x}n^{-1}E\left[|m(O_i; \alpha, \gamma, \psi_0(a,a'))|^2\right] |\mu_i | = O\left(\frac{h^{d_x+4}}{n}\right) = o(1)$, and $\sum_{i=1}^{n} 3(h^{d_x}n^{-1})^{1/2}E \left[|m(O_i; \alpha, \gamma, \psi_0(a,a'))|\right] |\mu_i |^2 = o(1)$, it suffices to prove the following condition

$$\lim_{n \to \infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \left( \sqrt{h^{d_x}} \frac{m(O_i; \alpha, \gamma, \psi_0(a,a'))}{n} - \mu_i \right) \xrightarrow{d} N(0, 1)$$

Combining this with $s_n^2 = V(a, a') + o(1)$ proves the Lyapunov condition. Hence,

$$\frac{1}{s_n^2} \sum_{i=1}^{n} \left( \sqrt{h^{d_x}} \frac{m(O_i; \alpha, \gamma, \psi_0(a,a'))}{n} - \mu_i \right) \xrightarrow{d} \mathcal{N}(0, 1)$$

An application of Slutsky’s theorem provides the desired result that

$$\sqrt{nh^{d_x}}(\hat{\psi}^{TR}(a,a') - \psi_0(a,a') - B(a,a')) \xrightarrow{d} \mathcal{N}(0, V(a, a'))$$

### 7.3 Proof for Multiple Robustness

**Lemma 5.** Under Assumptions 2, 3, and 6 the proposed $\hat{\psi}^{TR}(a,a')$ will be a consistent estimator for $\psi(a,a')$ as long as any two out of three conditions in Assumption 4 hold.

Following a similar breakdown as that in Theorem 1, $\hat{\psi}^{TR}(a,a') - \psi_0(a,a')$ can be expanded as

$$\hat{\psi}^{TR}(a,a') - \psi_0(a,a') = \frac{1}{\sqrt{nh^{d_x}}} \times \sqrt{\frac{h^{d_x}}{n}} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} m(O_i; \alpha, \gamma, \psi_0(a,a'))$$

$$+ \frac{1}{\sqrt{nh^{d_x}}} \times \sqrt{\frac{h^{d_x}}{n}} \sum_{\ell=1}^{L} \sum_{i \in I_{\ell}} \left\{ m(O_i; \hat{\alpha}_{\ell}, \hat{\lambda}_{\ell}, \hat{\gamma}_{\ell}, \psi_0(a,a')) - m(O_i; \alpha, \gamma, \psi_0(a,a')) \right\}.$$
The remainder of the proof demonstrates the remaining remaining terms are \( o_p(1) \), i.e.

\[
\frac{1}{\sqrt{n h}^{d_{a}}} \times \sqrt{\frac{h_{d_{a}}}{n}} \sum_{i=1}^{L} \sum_{i \in I_t} \left\{ m(O_i; \hat{\alpha}_t, \hat{\lambda}_t, \gamma_t, \psi_0(a, a')) - m(O_i; \alpha, \lambda, \gamma, \psi_0(a, a')) \right\}.
\]

To see this, we first expand these terms identically as the proof of Theorem 1, and provide proofs for the convergence of the terms (CS1) - (CS6), (E1) - (E8) and (TR1) - (TR5) under the assumption that any two out of three nuisance models are correctly specified in the following sub-sections.

### 7.3.1 Proof for Terms (CS1)-(CS6)

All of these terms contain the product of two or more errors and can be treated similarly. We provide a detailed proof for (CS2), and a similar method can be followed for the rest of the terms.

For (CS2), write \( \Delta_{it} = K_h(A_t - a) [\hat{R}(M_t, X_t) - R(M_t, X_t)] [\hat{\gamma}(X_t, M_t, a) - \gamma(X_t, M_t, a)] [Y_i - \gamma(X_t, M_t, a)] \).

Following Lemma 3, it suffices to bound \( E \left[ \frac{1}{\sqrt{n h}^{d_{a}}} \times \sqrt{\frac{h_{d_{a}}}{n}} \sum_{i \in I_t} \Delta_{it} \right| O_{it}^f \] as \( o_p(1) \) in order to show that

\[
\frac{1}{\sqrt{n h}^{d_{a}}} \times \sqrt{\frac{h_{d_{a}}}{n}} \sum_{i \in I_t} \Delta_{it} = o_p(1).
\]

First, from the triangle inequality, \( E \left[ \frac{1}{\sqrt{n h}^{d_{a}}} \times \sqrt{\frac{h_{d_{a}}}{n}} \sum_{i \in I_t} \Delta_{it} \right| O_{it}^f \] \( \leq \) \( E \left[ \left| \Delta_{it} \right| \right| O_{it}^f \] and so it suffices to bound \( E \left[ \left| \Delta_{it} \right| \right| O_{it}^f \] .

\[
E \left[ \left| \Delta_{it} \right| \right| O_{it}^f \\
= \int K_h(A_t - a) [\hat{R}(M_t, X_t) - R(M_t, X_t)] [\hat{\gamma}(X_t, M_t, a) - \gamma(X_t, M_t, a)] [Y_i - \gamma(X_t, M_t, a)] f(Y_i, A_t, M_t, X_t) dO_t \\
= \int \tilde{k}(u) [\hat{R}(M_t, X_t) - R(M_t, X_t)] [\hat{\gamma}(X_t, M_t, a) - \gamma(X_t, M_t, a)] [Y_i - \gamma(X_t, M_t, a)] f(Y_i, uh + a, M_t, X_t) dudY_i dM_t dX_t \\
= \int \left\{ \int \tilde{k}(u) f(uh + a|M_t, X_t) \right\} \left[ \int [Y_i - \gamma(X_t, M_t, a)] f(Y_i|uh + a, M_t, X_t) dY_i \right] dudX_t \\
= \int \left[ \hat{R}(M_t, X_t) - R(M_t, X_t) \right] [\hat{\gamma}(X_t, M_t, a) - \gamma(X_t, M_t, a)] f(M_t, X_t) dM_t dX_t
\]

Next, Assumption \[3\] on the boundedness of \( \gamma(X, M, a) \) and Assumption \[3\] on the boundedness of
\( \text{var}(Y_i | a, m, x) \), along with an application of Lemma 4 on \( f(a | M, X) \), we get

\[
= O(1) \int \{ f(a | M_i, X_i) + O(h^2) \} \left[ \hat{R}(M_i, X_i) - R(M_i, X_i) \right] \left[ \hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a) \right] f(M_i, X_i) dM_i dX_i
\]

\[
= O(1) \int f(a | M_i, X_i) \left[ \hat{R}(M_i, X_i) - R(M_i, X_i) \right] \left[ \hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a) \right] f(M_i, X_i) dM_i dX_i
\]

\[
+ O(h^2) \int \left[ \hat{R}(M_i, X_i) - R(M_i, X_i) \right] \left[ \hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a) \right] f(M_i, X_i) dM_i dX_i
\]

\[
(0) \leq O(1) \int \left[ \hat{R}(M, X_i) - R(M, X_i) \right] \left[ \hat{\gamma}(X_i, M, a) - \gamma(X_i, M, a) \right] f(M, X_i) dM_i dX_i
\]

\[
+ O(h^2) \int \left[ \hat{R}(M, X_i) - R(M, X_i) \right] \left[ \hat{\gamma}(X_i, M, a) - \gamma(X_i, M, a) \right] f(M, X_i) dM_i dX_i
\]

As long as either Assumption 3.2 or Assumption 3.3 hold, then combined with Assumption 3.2, (CS2) will be \( o_p(1) \). A similar approach can be used to bound the remaining CS terms.

7.3.2 Proof for Terms (E1)-(E8)

Terms (E1)-(E8) are normalized terms of the form of a bias times a bounded quantity; they can all be treated similarly. We only provide the proof of the convergence in probability to zero for the term (E2). (E2) is given as

\[
K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i) (Y_i - \gamma(X_i, M_i, a))
\]

\[
- \mathbb{E} [K_h(A_i - a) (\hat{\lambda}(a, X_i) - \lambda(a, X_i)) R(M_i, X_i) (Y_i - \gamma(X_i, M_i, a)) | O_i]
\]

To prove this, we set \( \hat{\Delta}_{i\ell} \) as (E2). By construction, \( O_i^\ell \) and \( O_i \) are independent, \( i \in I_\ell \), and consequently

\[
\mathbb{E} [\hat{\Delta}_{i\ell} | O_i^\ell] = 0 \quad \text{and} \quad \mathbb{E} [\hat{\Delta}_{i\ell} \hat{\Delta}_{j\ell} | O_i^\ell] = 0 \quad \text{for} \quad i, j \in I_\ell \quad \text{and} \quad a', a \in A_0.
\]

Next we note that

\[
\mathbb{E} \left[ \hat{\Delta}_{i\ell}^2 | O_i^\ell \right] = \int \hat{R}_h^2(a_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R_i^2(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i, A_i, M_i, X_i) dO_i
\]

\[
= \frac{1}{h^d a} \int \hat{k}(u)^2 \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right]^2 R_i^2(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i, a + u, M_i, X_i) du
\]

\[
\text{where (a) follows from Assumption 3.1 on the boundedness of } f(a | M, X), \text{ along with Assumption 3.1 and}
\]

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Assumption 3.3 combined with the derivation provided below

\[ \int_{y} [Y_i - \gamma(X_i, M_i, a)]^2 f(Y_i|uh + a, M_i, X_i) dY_i \]

\[ = \int_{y} [Y_i^2 + \gamma^2(X_i, M_i, X_i) - 2\gamma(X_i, M_i, a)Y_i] f(Y_i|uh + a, M_i, X_i) dY_i \]

\[ = \mathbb{E}[Y_i^2|uh + a, M_i, X_i] + \gamma^2(X_i, M_i, X_i) - 2\gamma(X_i, M_i, a) \int_{y} Y_i f(Y_i|uh + a, M_i, X_i) dY_i \]

\[ = \mathbb{E}[Y_i^2|uh + a, M_i, X_i] + \gamma^2(X_i, M_i, X_i) - 2\gamma(X_i, M_i, a) - 2\gamma(X_i, M_i, a) \gamma_{uh+a}(M_i, X_i) \]

\[ = O(1). \]

Next, (b) follows from Assumption 2.4, and finally, (c) follows Assumption 3.2 along with Assumption 4.1. Then

\[ \mathbb{E} \left( \frac{1}{\sqrt{nh^d/n}} \times \sqrt{h^d/n} \sum_{i \in t_i} \Delta_i t \right)^2 \mathbb{E}_{O_{t_i}} = \frac{1}{h} \sum_{i \in t_i} \mathbb{E} \left[ \Delta_i t | O_{t_i} \right] = O \left( \frac{1}{h} \right) \mathbb{E} \left[ \Delta_i t | O_{t_i} \right] = O_p \left( \frac{1}{nh^d/n} \right) = o_p(1). \]

Applying Lemma 1 to the above gives \( \frac{1}{\sqrt{nh^d/n}} \times \sqrt{h^d/n} \sum_{i \in t_i} \Delta_i t \overset{p}{\to} 0. \)

### 7.3.3 Proof for Terms (TR1)-(TR5)

The proofs of the convergence in probability to zero for the terms (TR1)-(TR5) follows a similar outline as Theorem 1, and we prove convergence on a case by case below.

**Proof for Terms TR1 and TR2**

Terms (TR1) and (TR2) are similar; we only provide the proof of the convergence in probability to zero for the term (TR2).

To bound TR2, first set \( \hat{\Delta}_t = K_h(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)]. \) Bounding (TR2) amounts to showing \( \mathbb{E}[\hat{\Delta}_t | O_{t_i}] = o_p(1). \)

\[ \mathbb{E} \left[ \hat{\Delta}_t | O_{t_i} \right] \]

\[ = \mathbb{E} \left[ K_h(A_i - a) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] | O_{t_i} \right] \]

Following identical steps as the proof for Theorem 1, gives

\[ = \int [f(a | Y_i, M_i, X_i) + O(h^2)] \]

\[ = \int O(h^2) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i dM_i dX_i \]

\[ \overset{(a)}{=} \int O(h^2) \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i dM_i dX_i \]

\[ \overset{(b)}{=} O(h^2) \int \left[ \hat{\lambda}(a, X_i) - \lambda(a, X_i) \right] R(M_i, X_i) \]

\[ \int [Y_i - \gamma(X_i, M_i, a)] f(Y_i, M_i, X_i) dY_i f(M_i, X_i) dM_i dX_i \]

\[ \overset{(c)}{=} o_p(1) \]
where the equalities follow identically as in the proof of Theorem 1, and the final equality follows from $h \to 0$ along with the boundedness assumptions in Assumption 3.

**Proof for TR3**

For Term (TR3), we have

$$\mathbb{E} \left[ (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) \left( 1 - K_h(A_i - a')\lambda(a', X_i) \right) \bigg| \mathcal{F}_t \right]$$

Following a similar approach as used in the proof for Theorem 1, we have

$$= \int (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) (1 - f(a' | X_i)\lambda(a', X_i)) f(X_i) dX_i$$

$$+ \int (\hat{\eta}(a, a', X_i) - \eta(a, a', X_i)) O(h^2) \lambda(a', X_i) f(X_i) dX_i$$

$$(b) \overset{}{=} a_p(1),$$

where (b) follows from the definition of $\lambda_{a'}(X_i)$, $h \to 0$, Assumption 3 (boundedness of $\lambda$, $\hat{\eta}$ and $\eta$).

**Proof for TR4**

Demonstrating the bound for (TR4), we have

$$\mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a')\lambda(a', X_i) - K_h(A_i - a)\lambda(a, X_i) R(M_i, X_i) \right\} \right]$$

$$= \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a')\lambda(a', X_i) \right\} \right] - \mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a)\lambda(a, X_i) R(M_i, X_i) \right\} \right]$$

(TR-4-1) (TR-4-2)

TR-4-1 can be written as

$$\mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a')\lambda(a', X_i) \right\} \right]$$

$$= \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) \left\{ \int K_h(A_i - a') f(A_i | M_i, X_i) dA_i \right\} f(M_i, X_i) dM_i dX_i$$

An application of Lemma 3 to TR-4-1 gives

$$\mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a')\lambda(a', X_i) \right\} \right]$$

$$= \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) f(a' | M_i, X_i) f(M_i, X_i) dM_i dX_i$$

$$+ \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a', X_i) O(h^2) f(M_i, X_i) dM_i dX_i$$

A similar approach applied to TR-4-2 gives

$$\mathbb{E} \left[ (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \left\{ K_h(A_i - a)\lambda(a, X_i) R(M_i, X_i) \right\} \right]$$

$$= \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a, X_i) R(M_i, X_i) f(a | M, X) f(M_i, X_i) dM_i dX_i$$

$$+ \int (\hat{\gamma}(X_i, M_i, a) - \gamma(X_i, M_i, a)) \lambda(a, X_i) R(M_i, X_i) O(h^2) f(M_i, X_i) dM_i dX_i$$

Now, the first terms of TR-4-1 and TR-4-2 cancel out with each other, with an identical proof to that used in the proof of Theorem 1.
Consequently only the remaining terms must be bounded.

\[ \int (\gamma(X_i, M_i, a) - \gamma(X_i, M_i, a))\lambda(a', X_i)O(h^2)f(M, X)dM_i dX_i = o_p(1) \]

The second term in TR-4-1 and TR-4-2 can be bounded from \( h \rightarrow 0 \), combined with the boundedness assumptions in Assumption 3.2.

**Proof For TR5**

Finally, for term (TR5), we note that

\[ \sqrt{nh^d} \mathbb{E} \left[ K_h(A_i - a') \left( \hat{\lambda}(a', X_i) - \lambda(a', X_i) \right) \gamma(X_i, M_i, a) - \eta(X_i) \right] \]

\[ = \sqrt{nh^d} \int K_h(A_i - a') \left( \hat{\lambda}(a', X_i) - \lambda(a', X_i) \right) \gamma(X_i, M_i, a) - \eta(a, a', X_i) \right] f(A_i, M_i, X_i) dA_i dM_i dX_i \]

\[ = \sqrt{nh^d} \int \left\{ \int K_h(A_i - a') f(A_i | M_i, X_i) dA_i \right\} \left( \hat{\lambda}(a', X_i) - \lambda(a', X_i) \right) \gamma(X_i, M_i, a) - \eta(a, a', X_i) \right] f(M_i, X_i) dM_i dX_i \]

\[ \overset{(a)}{=} \sqrt{nh^d} \int \left( \hat{\lambda}(a', X_i) - \lambda(a', X_i) \right) \gamma(X_i, M_i, a) - \eta(a, a', X_i) \right] f(M_i, X_i) dM_i dX_i \]

\[ \overset{(b)}{=} 0 + O(\sqrt{nh^d} + 4) \int \left( \hat{\lambda}(a', X_i) - \lambda(a', X_i) \right) \gamma(X_i, M_i, a) - \eta(a, a', X_i) \right] f(M_i, X_i) dM_i dX_i \]

\[ \overset{(c)}{=} o_p(1) \]

Where (a) follows from an application of Lemma 4, (b) follows from the definition of \( \eta_i \), and (c) follows from an application of Cauchy-Schwartz combined with the consistency of \( \hat{\lambda} \).

### 7.4 Proof for Consistency of \( \hat{V}(a, a') \)

Recall that

\[ m(O; a, \lambda, \gamma, \psi(a, a')) = K_h(A - a)\lambda(a, X)R(M, X)\{Y - \gamma(X, M, a)\} \]

\[ + K_h(A - a')\lambda(a', X)\{\gamma(X, M, a) - \eta(a, a', X)\} + \eta(a, a', X) - \psi(a, a'), \]

To prove consistency of \( \hat{V}(a, a') \), we first prove propositions (I), (II) and (III), which together prove the desired result.

**I** \( h^d n^{-1} \sum_{i \in I} m^2(O_i; a, \lambda, \gamma, \psi(a, a')) - V(a, a') = o_p(1) \)

To simplify notation, denote \( m(O_i; a, \lambda, \gamma, \psi(a, a')) \) as \( m_i(a, a') \). From the proof of Theorem 1, we have \( h^d n^{-1} E[m_i^2(a, a')] = V(a, a') + o_p(1) \).

We write

\[ U_1(a, a') = K_h(A - a)\lambda(a, X)R(M, X)\{Y - \gamma(X, M, a)\}, \]

\[ U_2(a, a') = K_h(A - a')\lambda(a', X)\{\gamma(X, M, a) - \eta(a, a', X)\}, \]

\[ U_3(a, a') = \eta(a, a', X) - \psi(a, a'). \]
Then,
\[ E(m^4) = E[(U_1 + U_2 + U_3)^4] \]
\[ = E(U_1^4) + 4E(U_1^3U_2) + 4E(U_1^3U_3) + 6E(U_1^2U_2^2) + 12E(U_1^2U_2U_3) + \]
\[ 6E(U_1U_2^3) + 4E(U_1U_2^2) + 12E(U_1U_2U_3) + 12E(U_1U_2U_3) + 4E(U_1U_3^2) + \]
\[ E(U_2^4) + 4E(U_2^3U_3) + 6E(U_2^2U_3^2) + 4E(U_2U_3^3) + E(U_3^4) \]

We only need to investigate the terms \( E(U_1^4) \), \( E(U_2^4) \), \( E(U_1^2U_2^2) \), \( E(U_1^2U_3^2) \), \( E(U_1U_2U_3^2) \), and \( E(U_1U_2U_3^2) \) for positive \( c_1 \), \( c_2 \), and \( c_3 \).

1. \( E(U_1^4) \): By the assumed boundedness of \( \lambda(a, X), R(M, X), \) and \( E\{[Y - \gamma(X, M, a)]^4|A = a', M = m, X = x\} \) over any \( (a, a', m, x) \) from Assumption \( 7 \)

\[ E(U_1^4) = \int \left\{ K_h(A - a)\lambda(a, X)R(M, X)[Y - \gamma(X, M, a)] \right\}^{c_1} f(Y, A, M, X)dO \]
\[ = O\left( \frac{1}{h^{(c_1-1)d_A}} \right) \int \tilde{k}(u)^{c_1} \int [Y - \gamma(X, M, a)]^{c_1} f(Y|A = uh + a, M, X)dY \int f(uh + a, M, X)dudMdX \]
\[ = O\left( \frac{1}{h^{(c_1-1)d_A}} \right) \int \tilde{k}(u)^{c_1} \int f_{MX}(M, X) \int \left\{ f(a|M, X) + \sum_{j=1}^{d_A} u_j h \frac{\partial}{\partial a} f(a|M, X) + \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \frac{\partial^2}{\partial a_j \partial a_{j'}} f(\bar{a}|M, X) \right\} dudMdX \]
\[ = O\left( \frac{1}{h^{(c_1-1)d_A}} \right) \int \tilde{k}(u)^{c_1} du + o\left( \frac{1}{h^{(c_1-1)d_A}} \right) = O\left( \frac{1}{h^{(c_1-1)d_A}} \right). \]

where \( \bar{a} \) is between \( a \) and \( a + uh \).

2. \( E(U_2^4) \): From the boundedness of \( \lambda(a', X), \gamma(X, M, a) \) and \( \eta(a, a', X) \) over any \( (a, a', a'', m, x) \) from \( A^3 \times M \times X \),

\[ E(U_2^4) = \int \left\{ K_h(A - a')\lambda(a', X)[\gamma(X, M, a) - \eta(a, a', X)] \right\}^{c_2} f(A, M, X)dO \]
\[ = O\left( \frac{1}{h^{(c_2-1)d_A}} \right) \int \tilde{k}(u)^{c_2} f_{MX}(M, X) f(uh + a'|M, X)dudMdX \]
\[ = O\left( \frac{1}{h^{(c_2-1)d_A}} \right) \int \tilde{k}(u)^{c_2} f_{MX}(M, X) \int \left\{ f(a'|M, X) + \sum_{j=1}^{d_A} u_j h \frac{\partial}{\partial a'} f(a'|M, X) + \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \frac{\partial^2}{\partial a'_j \partial a'_{j'}} f(\bar{a}|M, X) \right\} dudMdX \]
\[ = O\left( \frac{1}{h^{(c_2-1)d_A}} \right) \int \tilde{k}(u)^{c_2} du + o\left( \frac{1}{h^{(c_2-1)d_A}} \right) = O\left( \frac{1}{h^{(c_2-1)d_A}} \right). \]

where \( \bar{a} \) is between \( a' \) and \( a' + uh \).
3. \( \mathbb{E}(U_1^n U_2^n) \)

\[
\mathbb{E}(U_1^n U_2^n) = \int \left\{ K_h(A - a) \lambda(a, X) \frac{\alpha(a', M, X)}{\alpha(a, M, X)} [Y - \gamma(X, M, a)] \right\}^{c_1} \left\{ K_h(A - a') \lambda(a', X) [\gamma(X, M, a) - \eta(a, a', X)] \right\}^{c_2} f(Y, A, M, X) dO
\]

\[ = O\left( \frac{1}{h(c_1 + c_2 - 1) d_A} \right) \left\{ \int |Y - \gamma(X, M, a)|^{c_1} f(Y | A = uh + a, M, X) dY \right\}
\]

\[
\tilde{k}(u) = f_{M,X}(M, X) f(uh + a | M, X) dudMdX
\]

\[ = O\left( \frac{1}{h(c_1 + c_2 - 1) d_A} \right) \left\{ \int \prod_{j=1}^{d_A} k(u_j) \frac{a_j - a'_j}{h} \right\}^{c_2} f_{M,X}(M, X) dudMdX
\]

\[ = O\left( \frac{1}{h(c_1 + c_2 - 1) d_A} \right) \left\{ \sum_{j=1}^{d_A} u_j h \frac{\partial}{\partial a} f(a | M, X) + \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \frac{\partial^2}{\partial a_j \partial a_{j'}} f(\bar{a} | M, X) \right\} dudMdX
\]

\[ = O\left( \frac{1}{h(c_1 + c_2 - 1) d_A} \right)
\]

where \( \bar{a} \) is between \( a \) and \( a + uh \).

4. \( \mathbb{E}(U_1^n U_3^n) \)

\[
\mathbb{E}(U_1^n U_3^n) = \int \left\{ K_h(A - a) \lambda(a, X) \frac{\alpha(a', M, X)}{\alpha(a, M, X)} [Y - \gamma(X, M, a)] \right\}^{c_1} \left\{ \eta(a, a', X) - \psi(a, a') \right\}^{c_2} f(Y, A, M, X) dO
\]

\[ = O(1) \left\{ \int K_h(A - a)^{c_1} [Y - \gamma(X, M, a)]^{c_1} f(Y, A, M, X) dO \right\}
\]

\[ = O(1) \left\{ \int K_h(A - a)^{c_1} \mathbb{E} [|Y - \gamma(X, M, a)|^{c_1} | A, M, X] f(A, M, X) dO \right\}
\]

\[ = O(1) \left\{ \int K_h(A - a)^{c_1} f(A | M, X) dAf_{M,X}(M, X) dMdX \right\}
\]

\[ = O\left( \frac{1}{h(c_1 - 1) d_A} \right) \left\{ \sum_{j=1}^{d_A} u_j h \frac{\partial}{\partial a} f(a | M, X) + \sum_{j=1}^{d_A} \sum_{j'=1}^{d_A} u_j u_{j'} h^2 \frac{\partial^2}{\partial a_j \partial a_{j'}} f(\bar{a} | M, X) \right\} dudMdX
\]

\[ = O\left( \frac{1}{h(c_1 - 1) d_A} \right)
\]

where \( \bar{a} \) is between \( a \) and \( a + uh \), the second equality is from from the boundedness of \( \lambda, \eta, \alpha \) and \( \psi \), and the fourth equality comes from the assumed boundedness of \( \mathbb{E} [\|Y - \gamma\|^4 | A, M, X] \).
5. $E(U_{2}^{c}U_{3}^{g})$

$$E(U_{2}^{c}U_{3}^{g}) = \int \left\{ K_{h}(A - a') \frac{\lambda(a', X)\gamma(X, M, a) - \eta(a, a', X)}{\alpha(a, M, X)} \right\} c_{2} \int \left\{ \eta(a, a', X) - \psi(a, a') \right\} c_{3} f(Y, A, M, X)dO \quad \text{(c)}$$

$$= O(\frac{1}{h^{(c_{1} - 1)d_{A}}}) \int \hat{k}(u + \frac{a - a'}{h}) c_{2} f_{M}(M, X)$$

$$= O(\frac{1}{h^{(c_{1} - 1)d_{A}}}) \int \hat{k}(u + \frac{a - a'}{h}) c_{2} f_{M}(M, X)$$

$$\left\{ f(a'|M, X) + \sum_{j=1}^{d_{A}} u_{j} h \frac{\partial}{\partial a'} f(a'|M, X) + \sum_{j=1}^{d_{A}} u_{j} u_{j'} h^{2} \frac{\partial^{2}}{\partial a' \partial a''} f(a'|M, X) \right\} dudMdx$$

$$= O(\frac{1}{h^{(c_{1} - 1)d_{A}}}) \int \hat{k}(u + \frac{a - a'}{h}) c_{2} du + o(\frac{1}{h^{(c_{1} - 1)d_{A}}})$$

where $\hat{a}$ is between $a'$ and $a' + uh$.

6. $E(U_{1}^{c}U_{2}^{c}U_{3}^{g})$

$$E(U_{1}^{c}U_{2}^{c}U_{3}^{g}) = \int \left\{ K_{h}(A - a) \frac{\alpha(a', M, X)\gamma(X, M, a) - \eta(a, a', X)}{\alpha(a, M, X)} \right\} c_{2} \int \left\{ \eta(a, a', X) - \psi(a, a') \right\} c_{3} f(Y, A, M, X)dO \quad \text{(d)}$$

$$= O(\frac{1}{h^{(c_{1} + c_{2} - 1)d_{A}}}) \int \hat{k}(u) c_{2} \int \left\{ Y - \gamma(X, M, a) \right\} c_{3} f(Y|A = uh + a, M, X)dY$$

$$\hat{k}(u + \frac{a - a'}{h}) c_{2} f_{M}(M, X) f(uh + a|M, X) dudMdx$$

where the last equality is obtained as in the calculation for $E(U_{1}^{c}U_{2}^{c})$.

Combining all the terms, we obtain $E(m_{1}^{2}) = O(h^{-2d_{A}})$. Then by Markov inequality, for any $\epsilon > 0$,

$$P(\left| h^{d_{A}n^{-1}} \sum_{i \in I_{t}} m_{i}^{2}(a, a') - V(a, a') \right| > \epsilon) \leq \frac{1}{\epsilon^{2}} E \left\{ \left[ h^{d_{A}n^{-1}} \sum_{i \in I_{t}} m_{i}^{2}(a, a') - V(a, a') \right]^{2} \right\}$$

$$= \frac{1}{\epsilon^{2}} E \left\{ \left[ h^{d_{A}n^{-1}} \sum_{i \in I_{t}} m_{i}^{2}(a, a') - h^{d_{A}} E \left[ m_{i}^{2}(a, a') \right] + o_{p}(1) \right]^{2} \right\}$$

$$= \frac{h^{2d_{A}} n^{-2} \epsilon^{2}}{\epsilon^{2}} E \left\{ \left[ \sum_{i \in I_{t}} m_{i}^{2} - E(\sum_{i \in I_{t}} m_{i}^{2}) \right]^{2} \right\} + o_{p}(1)$$

$$= \frac{h^{2d_{A}} n^{-2} \epsilon^{2}}{\epsilon^{2}} var(\sum_{i \in I_{t}} m_{i}^{2}) + o_{p}(1)$$

$$= \frac{h^{2d_{A}} n^{-2} \epsilon^{2}}{\epsilon^{2}} var(m_{1}^{2}) + o_{p}(1)$$

$$= O\left( \frac{1}{nh^{d_{A}}} \right) = o_{p}(1),$$

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where the equality in the last row comes from $\text{var}(m_i^2) = O(\mathbb{E}(m_i^4)) = O(h^{-3d_A})$.

(II): $h^{d_A}|I_c|^{-1} \sum_{t \in I_c} \mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) - m^2(O_i; \alpha, \lambda, \gamma, \psi(a, a')) | O_{i_t}^c] = o_p(1)$

For simplicity in notation, we ignore the subscripts $\ell$ below for nuisance parameters estimated from $O_{i_t}^c$.

First, we analyze $h^{d_A}\mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) | O_{i_t}^c]$ as follows. We write

\[
\begin{align*}
\hat{U}_1(a, a') &= K_h(A - a) \hat{\lambda}(a, X) \hat{R}(M, X) \{Y - \hat{\gamma}(X, M, a)\}, \\
\hat{U}_2(a, a') &= K_h(A - a') \hat{\lambda}(a', X) \{\hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X)\}, \\
\hat{U}_3(a, a') &= \hat{\eta}(a, a', X) - \hat{\psi}(a, a').
\end{align*}
\]

Denote $m(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))$ as $\hat{m}_i$. Then,

\[
\begin{align*}
\mathbb{E}(\hat{m}_i^2 | O_{i_t}^c) &= \mathbb{E}[\hat{U}_1^2 + \hat{U}_2^2 + \hat{U}_3^2 | O_{i_t}^c] \\
&= \mathbb{E}(\hat{U}_1^2 | O_{i_t}^c) + \mathbb{E}(\hat{U}_2^2 | O_{i_t}^c) + \mathbb{E}(\hat{U}_3^2 | O_{i_t}^c) + 2\mathbb{E}(\hat{U}_1 \hat{U}_2 | O_{i_t}^c) + 2\mathbb{E}(\hat{U}_1 \hat{U}_3 | O_{i_t}^c) + 2\mathbb{E}(\hat{U}_2 \hat{U}_3 | O_{i_t}^c)
\end{align*}
\]

1. $h^{d_A}\mathbb{E}(\hat{U}_1^2 | O_{i_t}^c)$

\[
\begin{align*}
&= h^{d_A}\mathbb{E} \left( \left\{ \left. \frac{K_h(A - a) \hat{f}(M | A = a', X) \{Y - \hat{\gamma}(X, M, a)\}^2}{\hat{f}(M | A = a, X) \hat{f}(a | X)} \right| \right\} \right) | O_{i_t}^c \\
&= h^{d_A}\mathbb{E} \left( \left\{ \left. \frac{K_h(A - a) \hat{f}(M | A = a', X) \{Y - \hat{\gamma}(X, M, a)\}^2}{\hat{f}(M | A = a, X) \hat{f}(a | X)} \right| \right\} \right) | X, M, O_{i_t}^c \\
&= h^{d_A}\mathbb{E} \left( \left\{ \left. \frac{\hat{f}(M | A = a', X)^2}{\hat{f}(M | A = a, X) \hat{f}(a | X)^2} \right| \right\} \right) \times \mathbb{E} \left( \left\{ \left. K_h(A - a)^2 \mathbb{E}\left\{ \{Y - \hat{\gamma}(X, M, a)\}^2 | X, M, A, O_{i_t}^c \right\} \right| \right\} \right) | X, M, O_{i_t}^c \\
&\mathbb{E}(\hat{m}_i^2 | O_{i_t}^c)
\end{align*}
\]

After adding and subtracting $\mathbb{E}[Y | X, M, A]$, the middle expectation can be written as

\[
\begin{align*}
&= \int \left[ k(u_{j_1}) \cdots k(u_{j_A}) \right] \mathbb{E}[\text{var}(Y | X, M, A) + \{\gamma(X, M, a) - \hat{\gamma}(X, M, a)\}^2 | X, M, O_{i_t}^c] \\
&\mathbb{E}(\hat{m}_i^2 | O_{i_t}^c)
\end{align*}
\]

\[
\begin{align*}
&\mathbb{E}(\hat{m}_i^2 | O_{i_t}^c)
\end{align*}
\]

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where $\bar{a}_v$, $a_r$, and $\bar{a}_f$ are between $a$ and $a+h$. Equality (a) comes from the boundedness of $\int u^6k^2(u)du$, which is true because we assume $0 < \int u^6k(u)du < \infty$. Plugging this back into the original expectation gives
\[
\mathbb{E}\left\{ \frac{\hat{f}(M \mid A = a', X)^2}{\hat{f}(M \mid A = a, X)^2f(a \mid X)} \left[ \int \tilde{k}(u)^2du \right] \times \mathbb{E}\{[Y - \hat{\gamma}(X, M, a)]^2 \mid X, M, a, O_{It}^c \} \bigg| O_{It}^c \right\} + o_p(1)
\]

2. $h^{d_x} \mathbb{E}(\hat{U}_2^2 \mid O_{It}^c)$

\[
h^{d_x} \mathbb{E}\left\{ \frac{K_h(A - a')}{\hat{f}(a' \mid X)} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right) \bigg| O_{It}^c \right\}
\]

\[
= h^{d_x} \mathbb{E}\left\{ \frac{1}{\hat{f}(a' \mid X)^2} \mathbb{E}\left[ K_h(A - a')^2 \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right) \bigg| X, O_{It}^c \right] \bigg| O_{It}^c \right\}
\]

The inner expectation can be written as
\[
h^{d_x} \mathbb{E}\left[ K_h(A - a')^2 \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right) \bigg| X, O_{It}^c \right]
\]

Following a similar kernel expansion as before
\[
= \int k^2(u_1) \cdots k^2(u_{d_x}) \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 f(a' \mid X) f(M \mid X) du_1 \cdots du_{d_x} dM + o(h^2)
\]

Plugging this back into the original expectation leads to
\[
\int \tilde{k}(u)^2du \times \mathbb{E}\left\{ \frac{1}{\hat{f}(a' \mid X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \bigg| O_{It}^c \right\} + o_p(1)
\]

3. $h^{d_x} \mathbb{E}(\hat{U}_2^2 \mid O_{It}^c)$

\[
h^{d_x} \mathbb{E}\left\{ [\eta(a, a', X) - \hat{\psi}(a, a')]^2 \bigg| O_{It}^c \right\} = o_p(1)
\]

This holds because we assume the nuisance estimators are bounded, and following a similar calculation as the variance it can be seen that $h^{d_x} \mathbb{E}[\hat{\psi}^2(a, a') \mid O_{It}^c] = o_p(1)$, which combined with Jensen’s inequality can be used to obtain the desired result.

4. $h^{d_x} \mathbb{E}(\hat{U}_1\hat{U}_2|O_{It}^c)$

\[
h^{d_x} \mathbb{E}\left\{ \frac{K_h(A - a)f(M \mid A = a', X)}{f(M \mid A = a, X)f(a \mid X)} \{Y - \hat{\gamma}(X, M, a)\} \times \frac{K_h(A - a')}{\hat{f}(a' \mid X)} \{\hat{\gamma}(X, M, a) - \eta(a, a', X)\} \bigg| O_{It}^c \right\}
\]

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5. Assumption 4 bounds this term as

\[ h^d A \mathbb{E} \left\{ \left[ \frac{K_h(A-a)\hat{f}(M \mid A = a', X)}{\hat{f}(M \mid A = a, X)\hat{f}(a \mid X)} \{ Y - \hat{\gamma}(X, M, a) \} \right] \times \left[ \frac{K_h(A-a')}{\hat{f}(a' \mid X)} \{ \hat{\gamma}(X, M, a) - \hat{\gamma}(a, a', X) \} \right] \right\} \]

\[ = h^d A \mathbb{E} \left\{ \frac{K_h(A-a)K_h(A-a')\hat{f}(M|a', X)}{\hat{f}(a|X)\hat{f}(a'\mid X)} \frac{\hat{f}(M|a, X)}{\hat{f}(M|a, X)} \{ Y - \hat{\gamma}(X, M, a) \} \left[ \hat{\gamma}(X, M, a) - \hat{\gamma}(a, a', X) \right] \right\} \]

\[ = h^d A \mathbb{E} \left\{ \frac{1}{\hat{f}(a|X)\hat{f}(a'\mid X)} \frac{\hat{f}(M|a', X)}{\hat{f}(M|a, X)} \{ \hat{\gamma}(X, M, a) - \hat{\gamma}(a, a', X) \} \right\} \times \mathbb{E} \left\{ K_h(A-a)K_h(A-a') \left[ \hat{\gamma}(X, M, A) - \hat{\gamma}(X, M, a) \right] \left| X, M \right\} \right\} \]

The inner expectation

\[ h^d A \mathbb{E} \left\{ K_h(A-a)K_h(A-a') \left[ \hat{\gamma}(X, M, A) - \hat{\gamma}(X, M, a) \right] \left| X, M \right\} \right\} \]

\[ = h^d A \int \left[ \prod_{j=1}^{d_A} \frac{1}{h^2} k\left( \frac{A_j - a_j}{h} \right) k\left( \frac{A_j - a_j'}{h} \right) \right] \left[ \hat{\gamma}(X, M, A) - \hat{\gamma}(X, M, a) \right] f(A|X)\hat{f}(A)dA \]

\[ = \int \tilde{k}(u)\tilde{k}(u + \frac{a - a'}{h}) \left[ \hat{\gamma}(X, M, uh + a) - \hat{\gamma}(X, M, a) \right] f(uh + a|X, M)du \]

\[ = \int \tilde{k}(u_{1}) \cdots k(u_{d_A}) k(u_{1} + \frac{a - a'}{h}) \cdots k(u_{d_A} + \frac{a - a'}{h}) \times \left[ \left( \hat{\gamma}(X, M, a) - \hat{\gamma}(X, M, a) \right) + \sum_{j=1}^{d_A} u_{j} h \partial_{a_{j}} \hat{\gamma}(X, M, a) + \frac{u_{j}^{2} h^{2}}{2} \partial_{a_{j}}^{2} \hat{\gamma}(X, M, a) + \frac{u_{j}^{3} h^{3}}{6} \partial_{a_{j}}^{3} \hat{\gamma}(X, M, a) \right] \times \left[ f(a|X, M) + \sum_{j=1}^{d_A} u_{j} h \partial_{a_{j}} f(a|X, M) + \frac{u_{j}^{2} h^{2}}{2} \partial_{a_{j}}^{2} f(\bar{a} \mid X, M) \right] du_{1} \cdots du_{d_A} \]

where \( \bar{a}_{\gamma} \) and \( \bar{a}_{f} \) are between \( a \) and \( a + h \). Inserting this back into the full expectation combined with Assumption 4 bounds this term as \( o_{p}(1) \).

5. \( h^d A \mathbb{E} (\hat{U}_i \hat{U}_a | O^{c}_{l_i}) \)

\[ 2h^d A \mathbb{E} \left\{ \left[ \hat{\gamma}(a, a', X) - \psi(a, a') \right] \left[ \frac{K_h(A-a)\hat{f}(M \mid A = a', X)}{\hat{f}(M \mid A = a, X)\hat{f}(a \mid X)} \{ Y - \hat{\gamma}(X, M, a) \} \right] \right\} = o_{p}(1) \]
Expanding this into two terms,

\[ 2h^{d A} \mathbb{E}\left\{ \hat{\phi}(a, a', X) \left[ K_h(A - a) \hat{f}(M | A = a', X) \right] \right\} O_{\ell_t}^c \]

\[- 2h^{d A} \mathbb{E}\left\{ \hat{\psi}(a, a') \left[ K_h(A - a) \hat{f}(M | A = a', X) \right] \right\} O_{\ell_t}^c \]

The first term can be bounded as \( o_p(1) \) using a similar approach used above, and for the second term, from the i.i.d assumption on the data we can re-write it as

\[ 2h^{d A} \mathbb{E}\left\{ \hat{\psi}(a, a') \left[ K_h(A - a) \hat{f}(M | A = a', X) \right] \right\} O_{\ell_t}^c \]

\[ = 2h^{d A} |I_{\ell}|^{-1} \left( \sum_{i \in I_{\ell}} \mathbb{E}\left\{ \left[ K_h(A_i - a) \hat{f}(M_i | A = a', X_i) \right] \right\} \right)^2 O_{\ell_t}^c \]

\[ + \sum_{i \in I_{\ell}} \mathbb{E}\left\{ \left[ K_h(A_i - a) \hat{f}(M_i | A = a', X_i) \right] \right\} \left( \frac{K_h(A - a')}{\hat{f}(a' | X)} \left( \hat{\gamma}(X, M, a) - \hat{\gamma}(a, a', X) \right) \right) O_{\ell_t}^c \]

By the boundedness of

\[ \mathbb{E}\{ Y - \hat{\gamma}(X, M, a) \}^2 | X, M, A, O_{\ell_t}^c \} = var(Y | X, M, A) + [\hat{\gamma}(X, M, a) - \hat{\gamma}(X, M, a)]^2 \]

from Assumption 3, and following the results in the first part of (II), we know that \( h^{d A} \mathbb{E}\left[ K_h(A - a) \right] \] is bounded. Thus, the first term is \( O(|I_{\ell}|^{-1}) = o_p(1) \) from the law of total expectation. Because \( \hat{f}(a' | X) \), \( \hat{\gamma} \), and \( \hat{\gamma} \) are bounded by assumptions, the boundedness of \( h^{d A} \mathbb{E}\left[ K_h(A_i - a) K_h(A_j - a') \{ Y - \hat{\gamma}(X, M, a) \} | X, M, O_{\ell_t}^c \} \] can be obtained similar to the third part of (I). Hence, the second term also has \( O(|I_{\ell}|^{-1}) = o_p(1) \). From the boundedness of \( h^{d A/2} \mathbb{E}\left[ K_h(A - a) \right] \) based on Jensen’s inequality and the boundedness of \( \hat{\gamma} \), the third term satisfies \( O(h^{d A/2} |I_{\ell}|^{-1}) = o_p(1) \). As a result, \( h^{d A} \mathbb{E}(\hat{U}_1 \hat{U}_2 | O_{\ell_t}^c) \) is also \( o_p(1) \).

6. \( h^{d A} \mathbb{E}(\hat{U}_2 \hat{U}_2 | O_{\ell_t}^c) \)

\[ h^{d A} \mathbb{E}\left\{ \left[ \hat{\gamma}(a, a', X) - \hat{\psi}(a, a') \right] \left[ K_h(A - a') \right] \right\} O_{\ell_t}^c \]

From the boundedness of \( \hat{\gamma} \), \( \hat{\psi} \), and \( \hat{f}(a' | X) \), there is

\[ h^{d A} \mathbb{E}\left\{ \left[ \hat{\gamma}(a, a', X) \left[ K_h(A - a') \right] \right] \right\} O_{\ell_t}^c = O(h^{d A}) = o_p(1). \]

A similar proof as the fifth part of (II) above can show that the second term is also \( o_p(1) \).
Combining all the six parts, we have $h^{d,\alpha}\mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))]$ equal to

$$\left[ \int \tilde{k}(u)^2 du \right] \mathbb{E}\left\{ \frac{\tilde{f}(M | A = a', X)^2}{\tilde{f}(M | A = a, X)^2} \cdot \tilde{f}(a | X)^2 \cdot \mathbb{E}\{(Y - \hat{\gamma}(X, M, a))^2 | X, M, a, O_i^c \} \right\} \mathbb{E}[O_i]$$

$$+ \left[ \int \tilde{k}(u)^2 du \right] \mathbb{E}\left\{ \frac{\tilde{f}(M | A = a', X)^2}{\tilde{f}(M | A = a, X)^2} \cdot \tilde{f}(a | X)^2 \cdot \mathbb{E}\{(Y - \hat{\gamma}(X, M, a))^2 | X, M, a, O_i^c \} \right\} \mathbb{E}[O_i]$$

Next, $h^{d,\alpha}\mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))]$ can be written as

$$\left[ \int \tilde{k}(u)^2 du \right] \mathbb{E}\left\{ \frac{f(M | A = a', X)^2}{f(M | A = a, X)^2} \cdot \mathbb{E}\{(Y - \gamma(X, M, a))^2 | X, M, a, O_i^c \} \right\} \mathbb{E}[O_i]$$

$$+ \left[ \int \tilde{k}(u)^2 du \right] \mathbb{E}\left\{ \frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \right\} \mathbb{E}[O_i]$$

Define $\int \tilde{k}(u)^2 du = R_{d,\alpha}^2$,

$$\omega_1 = R_{d,\alpha}^2 \mathbb{E}\left\{ \frac{f(M | A = a', X)^2}{f(M | A = a, X)^2} \cdot \mathbb{E}\{(Y - \gamma(X, M, a))^2 | X, M, a, O_i^c \} \right\} \mathbb{E}[O_i]$$

$$- R_{d,\alpha}^2 \mathbb{E}\left\{ \frac{f(M | A = a', X)^2}{f(M | A = a, X)^2} \cdot \mathbb{E}\{(Y - \gamma(X, M, a))^2 | X, M, a, O_i^c \} \right\}, \text{ and}$$

$$\omega_2 = R_{d,\alpha}^2 \mathbb{E}\left\{ \frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \right\} \mathbb{E}[O_i]$$

$$- R_{d,\alpha}^2 \mathbb{E}\left\{ \frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \right\} \mathbb{E}[O_i].$$

Then $h^{d,\alpha}\mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))] - h^{d,\alpha}\mathbb{E}[m^2(O_i; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))] = \omega_1 + \omega_2 + o_p(1)$. First, we focus on simplifying $\omega_2$, which equals

$$R_{d,\alpha}^2 \mathbb{E}\left\{ \frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \right\} - \frac{1}{f(a'|X)^2} \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 \mathbb{E}[O_i].$$

From expressing $\frac{1}{f(a'|X)^2} (\hat{\gamma}(X, M, a) - \eta(a, a', X))$ as

$$\frac{1}{f(a'|X)} \left( \gamma(X, M, a) - \eta(a, a', X) \right) + \frac{1}{f(a'|X)} \left( \gamma(X, M, a) - \gamma(X, M, a) \right)$$

$$+ \frac{1}{f(a'|X)} (\eta(a, a', X) - \eta(a, a', X)) + \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right) \frac{1}{f(a'|X)} - \frac{1}{f(a'|X)},$$

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\[\frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right)^2 \]
there is
\[
\omega_2 = R_{da}^2 \mathbb{E} \left\{ \frac{1}{f(a'|X)^2} \left( \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right)^2 - \frac{1}{f(a'|X)^2} \left( \gamma(X, M, a) - \eta(a, a', X) \right)^2 \right\} O_{f_i}^c
\]
\[
= R_{da}^2 \mathbb{E} \left\{ \frac{1}{f^2(a'|X)} \left( \hat{\gamma}(X, M, a) - \gamma(X, M, a) \right)^2 + \frac{1}{f^2(a'|X)} \left( \eta(a, a', X)^2 - \hat{\eta}(a, a', X)^2 \right) \right. \\
+ \left( \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right)^2 \left( \frac{1}{f(a'|X)} - \frac{1}{f(a'|X)} \right)^2 \\
+ 2 \frac{1}{f(a'|X)} \left( \gamma(X, M, a) - \eta(a, a', X) \right) \frac{1}{f(a'|X)} \left( \hat{\gamma}(X, M, a) - \gamma(X, M, a) \right) \\
+ 2 \frac{1}{f(a'|X)} \left( \gamma(X, M, a) - \eta(a, a', X) \right) \frac{1}{f(a'|X)} \left( \eta(a, a', X) - \hat{\eta}(a, a', X) \right) \\
+ 2 \frac{1}{f(a'|X)} \left( \hat{\gamma}(X, M, a) - \eta(a, a', X) \right) \left( \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right) \left( \frac{1}{f(a'|X)} - \frac{1}{f(a'|X)} \right) \\
+ 2 \frac{1}{f(a'|X)} \left( \hat{\gamma}(X, M, a) - \gamma(X, M, a) \right) \frac{1}{f(a'|X)} \left( \eta(a, a', X) - \hat{\eta}(a, a', X) \right) \\
+ 2 \frac{1}{f(a'|X)} \left( \hat{\gamma}(X, M, a) - \gamma(X, M, a) \right) \left( \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right) \left( \frac{1}{f(a'|X)} - \frac{1}{f(a'|X)} \right) \\
\left. \right\} O_{f_i}^c
\]

We show each of these terms are $o_p(1)$ as follows. Because $f^2(a'|X)$ is bounded away from 0 based on Assumption 3 (ii) and the consistency of $\hat{\gamma}$ from Assumption 4(iii), there is
\[
\mathbb{E} \left\{ \frac{1}{f^2(a'|X)} \left[ \hat{\gamma}(X, M, a) - \gamma(X, M, a) \right]^2 \right\} O_{f_i}^c = o_p(1).
\]

Under a similar argument and from Assumption 4(iv),
\[
\mathbb{E} \left\{ \frac{1}{f^2(a'|X)} \left[ \eta(a, a', X) - \hat{\eta}(a, a', X) \right]^2 \right\} O_{f_i}^c = o_p(1).
\]

Based on the boundedness of nuisance estimators from Assumption 3(ii) and the consistency of $\hat{f}(a'|X)$ from Assumption 4(i), there is
\[
\mathbb{E} \left\{ \left[ \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right]^2 \left( \frac{1}{f(a'|X)} - \frac{1}{f(a'|X)} \right)^2 \right\} O_{f_i}^c = o_p(1).
\]

Each of the remaining cross terms is a product of a term that is $o_p(1)$ from the estimator’s consistency and a term that is bounded. Hence, we have $\omega_2 = o_p(1)$.

Next, we employ a similar derivation to simplify $\omega_1$. 

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Note that
\[
\frac{\hat{f}(M \mid A = a', X)}{\hat{f}(M \mid A = a, X) f(a \mid X)} (Y - \hat{\gamma}(X, M, a))
\]
\[
= \frac{f(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (Y - \gamma(X, M, a))
\]
\[
+ \left( \frac{\hat{f}(M \mid A = a', X)}{\hat{f}(M \mid A = a, X)} - \frac{f(M \mid A = a', X)}{f(M \mid A = a, X)} \right) \frac{1}{f(a \mid X)} (Y - \gamma(X, M, a))
\]
\[
+ \frac{\hat{f}(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (Y - \gamma(X, M, a))
\]
\[
+ \frac{\hat{f}(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (\gamma(X, M, a) - \hat{\gamma}(X, M, a))
\]

Hence,
\[
\omega_1 = R^2_{4,4} \mathbb{E} \left\{ \mathbb{E} \left[ \frac{\hat{f}(M \mid A = a', X)^2}{\hat{f}(M \mid A = a, X)^2 f(a \mid X)^2} (Y - \hat{\gamma}(X, M, a))^2 \right] \right\}
\]
\[
- \mathbb{E} \left\{ \mathbb{E} \left[ \frac{f(M \mid A = a', X)^2}{f(M \mid A = a, X)^2 f(a \mid X)^2} (Y - \gamma(X, M, a))^2 \right] \right\}
\]
\[
R^2_{4,4} \mathbb{E} \left\{ \left[ \frac{\hat{f}(M \mid A = a', X)}{\hat{f}(M \mid A = a, X)} - \frac{f(M \mid A = a', X)}{f(M \mid A = a, X)} \right] \frac{1}{f(a \mid X)} (Y - \gamma(X, M, a))
\]
\[
+ \frac{\hat{f}(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (\gamma(X, M, a) - \hat{\gamma}(X, M, a)) \right\}
\]
\[
+ 2R^2_{4,4} \mathbb{E} \left\{ \left[ \frac{f(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (Y - \gamma(X, M, a)) \right] \right\}
\]
\[
+ \frac{\hat{f}(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (\gamma(X, M, a) - \hat{\gamma}(X, M, a)) \right\}
\]
\[
\left[ \frac{f(M \mid A = a', X)}{f(M \mid A = a, X) f(a \mid X)} (Y - \gamma(X, M, a)) \right] \right\}
\]
\[
\mathbb{E} \left\{ \mathbb{E} \left[ m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) \right] \right\}
\]
\[
\mathbb{E} \left\{ \mathbb{E} \left[ m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) \right] \right\}
\]

After further expansions, we can show that the squared terms contain a component that is bounded based on Assumption 3 and another component that is \( o_p(1) \) from Assumption 4. The \( (Y - \gamma(X, M, a))^2 \) in some squared terms is integrated out as a bounded component due to var\((Y\mid X, M, a)\) being bounded as assumed in Assumption 3(3). For interaction terms, those containing \((Y - \gamma(X, M, a))\) equals zero because \( \int (Y - \gamma(X, M, a)) f(Y\mid X, M, a) dY = 0 \). All of the interaction terms contain a bounded component and a \( o_p(1) \) component. Consequently, \( \omega_1 = o_p(1) \), leading to \( h^d A \mathbb{E}[m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a'))] - h^d A \mathbb{E}[m^2(O; \alpha, \lambda, \gamma, \psi(a, a'))] = o_p(1) \).

\( (III) \) \( h^d A \mid I \mid \sum_{i \in I} \Delta_i = o_p(1) \), where
\[
\Delta_i = m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) - m^2(O; \alpha, \lambda, \gamma, \psi(a, a')) - \mathbb{E} \left\{ m^2(O; \hat{\alpha}, \hat{\lambda}, \hat{\gamma}, \hat{\psi}(a, a')) - m^2(O; \alpha, \lambda, \gamma, \psi(a, a')) \right\}
\]
By Lemma 3, it suffices to bound $\mathbb{E}\left[ h^{d_A} | I_\ell |^{-1} \sum_{i \in I_\ell} \Delta_i \right]^2 | O_i^c ] = h^{2d_A} | I_\ell |^{-1} \mathbb{E} \left[ \Delta_i^2 | O_i^c \right] = o_p(1)$. Note that $\mathbb{E}[\Delta_i] = 0$ and interaction terms are zero due to conditional independence. We start with analyzing $\mathbb{E}[\Delta_i^2 | O_i^c]$ as follows. For simplicity of notation, we adopt the notation definitions in parts (I) and (II), ignoring the subscripts $\ell$ for nuisance estimators. We have

$$
\mathbb{E}[\Delta_i^2 | O_i^c] = \mathbb{E}\left[ (\hat{m}_i^2 - m_i^2 - \mathbb{E}(\hat{m}_i^2 - m_i^2 | O_i^c)]^2 | O_i^c \right]
= \mathbb{E} \left[ (\hat{m}_i^2 - m_i^2)^2 | O_i^c \right] - \mathbb{E} \left[ (\hat{m}_i^2 - m_i^2 | O_i^c)^2 \right]
$$

From (II), we know that $\mathbb{E}(\hat{m}_i^2 - m_i^2 | O_i^c)^2 = o_p(h^{-2d_A})$. To bound $\mathbb{E} \left[ (\hat{m}_i^2 - m_i^2)^2 | O_i^c \right]$, by $\hat{m}_i = \hat{U}_1 + \hat{U}_2 + \hat{U}_3$ and $m_i = U_1 + U_2 + U_3$, we can rewrite the term as

$$
\mathbb{E} \left[ (\hat{m}_i^2 - m_i^2)^2 | O_i^c \right] = \mathbb{E} \left\{ (\hat{U}_1 + \hat{U}_2 + \hat{U}_3)^2 - (U_1 + U_2 + U_3)^2)^2 \right\} | O_i^c
= \mathbb{E} \left\{ (U_1^2 + U_2^2 + U_3^2 + 2\hat{U}_1 \hat{U}_2 + 2\hat{U}_2 \hat{U}_3 + 2\hat{U}_1 \hat{U}_3 - (U_1^2 + U_2^2 + U_3^2 + 2U_1 U_2 + 2U_2 U_3 + 2U_1 U_3))^2 \right\} | O_i^c
= \mathbb{E} \left\{ (U_1^2 - U_2^2 + U_2^2 - U_3^2 + (U_3^2 - U_3^2) + 2(\hat{U}_1 \hat{U}_2 - U_1 U_2 + \hat{U}_2 \hat{U}_3 - U_2 U_3 + \hat{U}_1 \hat{U}_3 - U_1 U_3))^2 \right\} | O_i^c
= \mathbb{E} \left\{ (U_1^2 - U_2^2)^2 + (U_2^2 - U_3^2)^2 + (U_3^2 - U_3^2)^2 \right\} + \sum_{\bar{c} \in \mathcal{W}} c_{\bar{c}} \hat{U}_1^{\bar{c}_1} \hat{U}_2^{\bar{c}_2} \hat{U}_3^{\bar{c}_3} U_1^{\bar{c}_4} U_2^{\bar{c}_5} U_3^{\bar{c}_6} | O_i^c
$$

where $\bar{c} = (c_1, \ldots, c_6)$ and $\mathcal{W}$ represents the possible combinations of $\bar{c}$ from the decomposition. We will prove that $\mathbb{E} \left\{ \hat{U}_1^{c_1} \hat{U}_2^{c_2} \hat{U}_3^{c_3} U_1^{c_4} U_2^{c_5} U_3^{c_6} | O_i^c \right\} = O(h^{-(c_1 + c_2 + c_4 + c_5 + 1)d_A})$. Note that

$$
\mathbb{E} \left\{ \hat{U}_1^{c_1} \hat{U}_2^{c_2} \hat{U}_3^{c_3} U_1^{c_4} U_2^{c_5} U_3^{c_6} | O_i^c \right\} = \int \int \hat{U}_1^{c_1} \hat{U}_2^{c_2} \hat{U}_3^{c_3} U_1^{c_4} U_2^{c_5} U_3^{c_6} f(Y, A, M, X | O_i^c) dY dMdAdX.
$$

By the boundedness of nuisance parameters and their estimates (Assumption 3(ii)), the above term equals

$$
O \left( \int \int K_h(A-a)^{c_1+c_4} K_h(A-a')^{c_2+c_5} |Y - \tilde{\gamma}(X, M, a)^{c_1} [Y - \gamma(X, M, a)]^{c_4} | f(Y, A, M, X | O_i^c) dY dMdAdX \right).
$$

The possible combinations of $c_1, c_4$ in $\bar{c}$ are $\{(c_1, c_4) : (1, 1), (2, 0), (0, 2), (2, 1), (1, 2), (3, 0), (0, 3)\}$. Similar to the derivation in part (I), we will prove that the rate is $O(h^{-(c_1 + c_2 + c_4 + c_5 - 1)d_A})$ case-by-case. For the terms with $c_1 = 0$, the boundedness of $\mathbb{E}[|Y - \gamma|^4 | X, M, A]$ from Assumption $[E]$ provides the boundedness of lower moments by separately considering the regions on which $|Y - \gamma|^4$ is $\geq$ or $< 1$. Next, we prove for the remaining terms.

1. $c_1 > 0$ and $c_4 = 0$. The integral can be written as

$$
\int \int K_h^{c_1+c_4} (A-a) K_h^{c_2+c_5} (A-a') |Y - \tilde{\gamma}(X, M, a)|^{c_1} f(Y, A, M, X | O_i^c) dY dMdAdX
= \int \int K_h^{c_1+c_4} (A-a) K_h^{c_2+c_5} (A-a') \mathbb{E} \left[ |Y - \tilde{\gamma}(X, M, a)|^{c_1} | f(A, M, X | O_i^c) dM dAdX.
$$
1.
The remaining terms to bound are \( E \left[ (\hat{Y} - \gamma(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right] \) can be bounded as follows,

\[
E \left[ (\hat{Y} - \gamma(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right] = E \left[ (\hat{Y} - \gamma(X, M, a)) \mid A, M, X \right] + E[(\gamma(X, M, a) - \hat{\gamma}(X, M, a))^2 \mid A, M, X, O_{i_k}^c] \\
\leq E\left| (\hat{Y} - \gamma(X, M, a))^2 \mid A, M, X \right| + \sum_{k=1}^{c_1-1} \left( \frac{c_1}{k} \right) \mathbb{E}(\hat{\gamma}(X, M, a) - \hat{\gamma}(X, M, a)^{c_1-1} \mid A, M, X, O_{i_k}^c] 
\]

Each of the terms in the expansion can be bounded from Assumption 3(ii) combined with the boundedness of \( E \left[ (\hat{Y} - \gamma(X, M, a)) \mid A, M, X \right] \) from Assumption 7. Hence, the original integral equals

\[
O \left( \int \int K_h(a - a')^2 f(A, M, X \mid O_{i_k}^c) dM dAdX \right) = O(h^{-c_1+c_2+c_4-1}d_4)
\]

where the last equality holds from the boundedness of the integrals of the kernels.

2. \( c_1 > 0 \) and \( c_4 > 0 \). The integral is

\[
\int \int K_h(a - a')^2 K_h(a - a')^{c_2+c_5} f(A, M, X \mid O_{i_k}^c) dM dAdX = \int \int K_h(a - a')^2 K_h(a - a')^{c_2+c_5} f(A, M, X \mid O_{i_k}^c) dM dAdX
\]

The inner expectation \( E \left[ (\hat{Y} - \gamma(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right] \) can be bounded with

\[
E \left[ (\hat{Y} - \gamma(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right] = E \left\{ (\hat{Y} - \gamma(X, M, a) + \gamma(X, M, a) - \hat{\gamma}(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right\} \\
\leq E \left[ (\gamma(X, M, a) - \hat{\gamma}(X, M, a))^2 \mid A, M, X, O_{i_k}^c \right] + \sum_{k=1}^{c_1-1} \left( \frac{c_1}{k} \right) \mathbb{E}(\gamma(X, M, a) - \hat{\gamma}(X, M, a)^{c_1-1} \mid A, M, X, O_{i_k}^c]
\]

The first term is the conditional variance, which is bounded by Assumption 3(3). The second and third terms can be bounded from Assumption 3(ii) and Assumption 7. The bound of the integral follows similarly as before.

The remaining terms to bound are \( E[\hat{U}_1^2 - U_1^2)^2 \mid O_{i_k}^c] \), \( E[(\hat{U}_2^2 - U_2^2)^2 \mid O_{i_k}^c] \), and \( E[(\hat{U}_3^2 - U_3^2)^2 \mid O_{i_k}^c] \). First, \( E[(\hat{U}_2^2 - U_2^2)^2 \mid O_{i_k}^c] \) can be bounded from Assumption 3(ii). Next, we demonstrate the boundedness of \( E[(\hat{U}_2^2 - U_2^2)^2 \mid O_{i_k}^c] \); a similar derivation applies to \( E[\hat{U}_1^2 - U_1^2)^2 \mid O_{i_k}^c] \). To start with, we re-express the term

\[
E[(\hat{U}_2^2 - U_2^2)^2 \mid O_{i_k}^c] = E(\hat{K}_h(a - a')^2 \left( \frac{1}{f(a'|X)^2} [\gamma(X, M, a) - \hat{\eta}(a, a', X)]^2 - \frac{1}{f(a'|X)^2} [\gamma(X, M, a) - \hat{\eta}(a, a', X)]^2 \right)^2 \mid O_{i_k}^c]
\]

\[
= E\left( \frac{1}{f(a'|X)^2} [\gamma(X, M, a) - \hat{\eta}(a, a', X)]^2 - \frac{1}{f(a'|X)^2} [\gamma(X, M, a) - \hat{\eta}(a, a', X)]^2 \right)^2 \\
\times E[K_h(a - a')^2 \mid X, M] \mid O_{i_k}^c]
\]

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From Assumption 7,

\[ \mathbb{E}[K_h^2(A - a') | X, M] = \sum_{j=1}^{d_A} \frac{1}{h^4} k(\frac{A_j - a_j'}{h})^4 f(A | X, M) dA \]

\[ = h^{-3d_A} \int \hat{k}(u)^4 f(uh + a' | X, M) du = O(h^{-3d_A}). \]

Hence,

\[ \mathbb{E}[(\hat{U}_1^2 - U_1^2)^2 | O_{i_t}] = O(h^{-3d_A}) \mathbb{E} \left( \frac{1}{f(a' | X)^2} \left[ \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right]^2 - \frac{1}{f(a' | X)^2} [\gamma(X, M, a) - \eta(a, a', X)]^2 \right)^2 | O_{i_t} \right). \]

From the expansion of \( \omega_2 \) in proving 6 of the part (II), we can express \( \frac{1}{f(a' | X)^2} \left[ \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right]^2 - \frac{1}{f(a' | X)^2} [\gamma(X, M, a) - \eta(a, a', X)]^2 \) as a summation of 9 components, i.e.

\[ \frac{1}{f(a' | X)^2} \left( \frac{\hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X)}{f(a' | X)^2} \left[ \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right]^2 - \frac{1}{f(a' | X)^2} \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right)^2 + \left( \frac{\hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X)}{f(a' | X)^2} \left[ \hat{\gamma}(X, M, a) - \hat{\eta}(a, a', X) \right]^2 - \frac{1}{f(a' | X)^2} \left[ \gamma(X, M, a) - \eta(a, a', X) \right] \right)^2 \]

For the multiplication of any two of the nine components chosen with replacement, the corresponding conditional expectation \( \mathbb{E}[(\hat{U}_1^2 - U_1^2)^2 | O_{i_t}] \) is a construct of a subcomponent that is \( o_p(1) \) from the consistency of nuisance parameters multiplied by other subcomponents that are bounded from Assumption 3. As a consequence, we obtained that \( \mathbb{E}[(\hat{U}_1^2 - U_1^2)^2 | O_{i_t}] = o_p(h^{-3d_A}) \). A similar argument can be used to prove \( \mathbb{E}[\hat{U}_1^2 - U_1^2]^2 | O_{i_t}] = o_p(h^{-3d_A}) \) by utilizing the boundedness of \( \mathbb{E}[(Y - \gamma)^4 | X, M, A] \) from Assumption 7(i).

Because \( c_1 + c_2 + c_3 + c_5 \leq 4, O(1) \leq O(h^{-c_1+c_2+c_3+c_5-1}) \leq O(h^{-3d_A}) \). We conclude that \( \mathbb{E} \left[ \Delta_{i_t}^2 | O_{i_t} \right] = O(h^{-3d_A}) \) and

\[ \mathbb{E} \left[ \left( h^{d_A} | I_t |^{-1} \sum_{t \in I_t} \Delta_{i_t} \right)^2 | O_{i_t} \right] = h^{2d_A} | I_t |^{-1} \mathbb{E} \left[ \Delta_{i_t}^2 | O_{i_t} \right] = O \left( \frac{\ln(h^{d_A})}{1} \right) = o_p(1). \]

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