Enumeration of labelled 4-regular planar graphs

Marc Noy *  Clément Requilé †  Juanjo Rué ‡

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Abstract

We present the first combinatorial scheme for counting labelled 4-regular planar graphs through a complete recursive decomposition. More precisely, we show that the exponential generating function of labelled 4-regular planar graphs can be computed effectively as the solution of a system of equations, from which the coefficients can be extracted. As a byproduct, we also enumerate labelled 3-connected 4-regular planar graphs, and simple 4-regular rooted maps.

1 Introduction

The enumeration of labelled planar graphs has been recently the subject of much research; see [11] [12] for surveys on the area. The problem of counting planar graphs was first solved by Giménez and Noy [8], while cubic planar graphs where enumerated in Bodirsky, Kang, Löffler and McDiarmid.. [1] (see also [13] for an update). On the other hand, the enumeration of simpler classes of planar graphs, such as series-parallel graphs and, more generally, subcritical classes of graphs is easier and well understood [6].

One of the open problems in this area is the enumeration of labelled 4-regular planar graphs. There are several references on the exhaustive generation of 4-regular planar graphs. Starting with a collection of basic graphs one shows how to generate all graphs in a certain class starting from the basic pieces and applying a sequence of local modifications. This was first done for the class of 4-regular planar graphs by Lehel [9], using as basis the graph of the octahedron. For 3-connected 4-regular planar graphs a similar generation scheme was shown by Boersma, Duijvestijn and Göbel [3]; by removing isomorphic duplicates they were able to compute the numbers of 3-connected 4-regular planar graphs up to 15 vertices. It is also the approach of the more recent work by Brinkmann, Greenberg, Greenhill, McKay, Thomas and Wollan [2] for generating planar quadrangulations of several types. The authors of [2] use several enumerative formulas to check the correctness of their generation procedure. However this does not include the class of 3-connected quadrangulations, which by duality correspond to 3-connected 4-regular planar graphs, a class for which no enumeration scheme was known until now.

*Universitat Politècnica de Catalunya and Barcelona Graduate School of Mathematics, Department of Mathematics, Edifici Omega, 08034 Barcelona, Spain. E-mail: marc.noy@upc.edu. Supported by the Spanish Ministerio de Economía y Competitividad projects MTM2014-54745-P and MDM-2014-0445.
†Freie Universität Berlin, Institut für Mathematik und Informatik, Arnimallee 3, 14195 Berlin, Germany. E-mail: requile@math.fu-berlin.de. Supported by the Berlin Mathematical School.
‡Universitat Politècnica de Catalunya and Barcelona Graduate School of Mathematics, Department of Mathematics, Edifici Omega, 08034 Barcelona, Spain. E-mail: juan.jose.rue@upc.edu. Supported by the Spanish Ministerio de Economía y Competitividad project MTM2014-54745-P.
In the present paper we provide the first scheme for counting 4-regular planar graphs through a complete recursive decomposition. In what follows all graphs are labelled. Let \( C(x) = \sum c_n x^n / n! \) be the exponential generating function of labelled connected 4-regular planar graphs counted according to the number of vertices. We show that the derivative \( C'(x) \) can be computed effectively as the solution of a (rather involved) system of algebraic equations. In particular \( C'(x) \) is an algebraic function. Using a computer algebra system we can extract the coefficients of \( C'(x) \), hence also of \( C(x) \). If \( G(x) = \sum g_n x^n / n! \) is now the generating function of all 4-regular planar graphs, the exponential formula \( G(x) = e^{C(x)} \) allows us to find the coefficients \( g_n \) as well.

Our main result is the following:

**Theorem 1.1.** The generating function \( C'(x) \) is algebraic and is expressible as the solution of a system of algebraic equations from which one can compute effectively their coefficients.

As a corollary we obtain:

**Corollary 1.2.** The sequence \( \{g_n\}_{n \geq 0} \) is \( P \)-recursive, that is, it satisfies a linear recurrence with polynomial coefficients.

We also determine the generating function \( \tau(x) = \sum t_n x^n / n! \) of 3-connected 4-regular graphs and show that its derivative is algebraic.

**Theorem 1.3.** The generating function \( \tau'(x) \) is algebraic and is expressible as the solution of a system of algebraic equations from which one can compute effectively their coefficients.

To obtain our results we follow the classical technique introduced by Tutte: take a graph rooted at a directed edge and classify the possible configurations arising from the removal of the root edge. This produces several combinatorial classes that are further decomposed, typically in a recursive way. The combinatorial decomposition translates into a system of equations for the associated generating functions, which in our case is considerably involved. Next we provide a brief overview of the combinatorial scheme in our solution.

Using a variant of the classical decomposition of 2-connected graphs into 3-connected components in the spirit of [1], we find an equation linking \( C(x) \) to the generating function \( T(u,v) \) of 3-connected 4-regular planar graphs, counted according to the number of simple edges and the number of double edges. Actually, \( T \) will be the generating function of rooted 3-connected maps (a rooted map is an embedding of a planar graph where a directed edge is distinguished) but by Whitney’s theorem, 3-connected planar graphs have a unique embedding in the oriented sphere and in this situation counting graphs is equivalent to counting maps. As will be seen later, it is essential to count 3-connected maps according to simple and double edges, otherwise there is not enough information to obtain \( C(x) \).

Once we have access to \( T(u,v) \) we can compute the coefficients of \( C(x) \) to any order. In order to compute \( T(u,v) \) we apply the reverse procedure working with maps instead of graphs. The starting point is the fact that the number \( M_n \) of rooted 4-regular maps with 2n edges is well known, since they are in bijection with rooted (arbitrary) maps on n edges, and equal to

\[
M_n = \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}.
\]

Using again the decomposition into 3-connected components, one can obtain an equation linking \( T(u,v) \) and \( M(z) = \sum M_n z^n \). However this is not sufficient since
\( T(u, v) \) is a bivariate series and cannot be recovered uniquely from the univariate series \( M(z) \). In order to overcome this situation, we enrich the combinatorial scheme and count maps according to a secondary parameter: the number of isolated faces of degree 2, namely, those not incident with another face of degree 2. Notice that this parameter, when restricted to 3-connected 4-regular planar graphs, is precisely the number of double edges.

If \( M(z, w) \) is the associated series, where \( w \) marks the new parameter, then we can enrich the corresponding equations and obtain an algebraic relation of the form

\[
T(f(z, w, M(z, w)), g(z, w, M(z, w))) = h(z, w, M(z, w)),
\]

where \( f, g \) and \( h \) are explicit functions. The transformation

\[
u = f(z, w, M), \quad v = g(z, w, M)
\]

turns out to have non-zero Jacobian and can be inverted explicitly. This allows us to express \( T(u, v) \) as a power series whose coefficients can be computed in terms of those of \( M(z, w) \) and the inverse mapping \( (z, w) \rightarrow (u, v) \). From here, we can compute the coefficients of \( T(u, v) \) to any order. In particular, the coefficients of \( T(u, 0) \) give the numbers \( T_n \) of simple rooted 3-connected 4-regular planar maps. By double counting, we obtain the number of labelled 3-connected 4-regular planar graphs as \( t_n = T_n (n - 1)! / 8 \).

It remains to compute \( M(z, w) \). To this end, we use the dual bijection between 4-regular maps and quadrangulations, and count quadrangulations according to the number of faces and the number of vertices of degree 2 not adjacent to another vertex of degree 2. This is technically demanding but can be achieved using the decomposition of quadrangulations along faces and edges, refined to take into account the new parameter.

Once we have access to \( T(u, v) \), we can also enumerate simple 4-regular maps, a result of independent interest (the enumeration of simple 3-regular maps can be found in [7]).

**Theorem 1.4.** The generating function of rooted simple 4-regular maps is algebraic, and is expressible as the solution of a system of algebraic equations from which one can compute its coefficients effectively.

The steps towards the proofs of Theorems 1.1 and 1.4 are summarized in the following diagram.

\[
\begin{align*}
\text{Simple quadrangulations (Section 2.1)} & \quad \Downarrow \\
\text{Arbitrary quadrangulations} & \leftrightarrow \text{4-regular maps (Section 2.2)} \\
\text{3-connected 4-regular graphs} & \leftrightarrow \text{3-connected 4-regular maps (Section 3)} \\
\text{4-regular graphs (Section 4)} & \Downarrow \Downarrow \\
\text{Simple 4-regular maps (Section 5)}
\end{align*}
\]

We remark that the final equations relating the power series \( M(z, w) \), \( T(u, v) \) and \( C(x) \) are not written down explicitly. Instead, we work with several intermediate systems of equations that allow us to extract the coefficients of the corresponding series. This is computationally demanding but it is within the capabilities of a computer algebra system such as Maple.
Here is a summary of the paper. In Section 2, we determine the series $M(z, w)$ as the solution of a system of polynomial equations. In Section 3, we obtain $T(u, v)$ as a computable function of $M(z, w)$, thus proving the second part of Theorem 1.1. In Section 4, we find an equation connecting $C(x)$ and $T(u, v)$, which allows us to compute the coefficients of $C(x)$, that is, the number of connected 4-regular planar graphs, and to complete the proof of Theorem 1.1. From the relation $G(x) = \exp(C(x))$, we obtain the coefficients of $G(x)$. Finally, in Section 5, we count simple 4-regular maps.

2 Counting quadrangulations

All maps in this paper are rooted; this means that an edge $uv$ is marked and directed from $u$ to $v$. Then $u$ is the root vertex and the face to the right of $uv$ is the root face, which by convention is taken to be the outer face. A rooted map has no non-trivial automorphism, hence all vertices, edges and faces are distinguishable.

A quadrangulation is a planar map in which all faces have degree 4. A diagonal in a quadrangulation is a path of length 2 joining opposite vertices of the external face. If $uv$ is the root edge, there are two kinds of diagonals, those incident with $u$ and those incident with $v$. By planarity both cannot be present at the same time. A cycle of length 4 which is not the boundary of a face is called a separating quadrangle. Vertices in the outer face are called external vertices. A vertex of degree 2 is isolated if it is not adjacent to another vertex of degree 2. An isolated vertex of degree 2 is called a 2-vertex.

2.1 Simple quadrangulations

All quadrangulations in this section are simple, that is, have no multiple edges. This implies in particular that all faces are quadrangles, that is, simple 4-cycles. Following [10], we consider the following classes of quadrangulations, illustrated in Figure 1:

- $Q$ are all (simple) quadrangulations.
- $S$ are quadrangulations without diagonals or separating quadrangles.
- $N$ are quadrangulations containing a diagonal incident with the root vertex. By symmetry they are in bijection with quadrangulations containing a diagonal not incident with the root vertex.
- $N_i$ are quadrangulations in $N$ with exactly $i$ external 2-vertices, for $i = 0, 1, 2$.
- $R$ are quadrangulations obtained from $S$ by possibly replacing each internal face with a quadrangulation in $Q$.

In the following generating functions, $z$ marks internal faces and $w$ marks 2-vertices. For each class of quadrangulations we have the corresponding generating function written with the same letter. For instance $Q(z, w)$ is associated with the class $Q$, and so on. The exception is $S(z)$, since a quadrangulation in $S$ has no vertex of degree 2, and variable $w$ does not appear.

The generating function $S(z)$ is well-known, since $S$ is in bijection with rooted 3-connected planar maps counted by number of edges [15, 10]. It is given by

$$S(z) = z^2 \left( \frac{2}{1+z} - 1 - \frac{(1+u)^4}{(1+2u)^3} \right), \quad u = z(1+z(1+u)^2)^2.$$
Figure 1: Examples of simple quadrangulations. The first three, from left to right, belong to \(N_0, N_1\) and \(N_2\), respectively. The last one is in \(R\): a quadrangulation from \(S\) in which one face has been replaced with a quadrangulation in \(N_1\). Isolated vertices of degree 2 are shown in white.

The following lemma guarantees the existence of a unique solution to a system of non-negative equations. We refer to Section 2.2.5 of [5] for a proof and discussion. It is further shown in [5] that under certain conditions the solution is analytic, but in this work we consider only formal power series.

**Lemma 2.1.** Let \(y_1(x,u), \ldots, y_m(x,u)\) satisfy the system of equations

\[
\begin{align*}
y_1 &= F_1(x,y_1,\ldots,y_m,u), \\
y_2 &= F_2(x,y_1,\ldots,y_m,u), \\
& \vdots \\
y_m &= F_m(x,y_1,\ldots,y_m,u),
\end{align*}
\]

where the \(F_i\) are power series in the variables indicated.

If the \(F_i\) have non-negative Taylor coefficients then there is a unique solution \(y = (y_1, \ldots, y_m)\) as power series in \(x\) and \(u\) with non-negative coefficients satisfying \(y(0,0) = 0\).

For the various systems of equations that we encounter we have to check that the conditions of the previous result are satisfied, which is usually an easy task. We then compute the unique non-negative solution by iteration: we start with \(y_0(x,u) = 0\) and set \(y_{i+1}(x,u) = F(x,y_i(x,u),u)\), where \(F = (F_1, \ldots, F_m)\).

**Lemma 2.2.** Let

\[
N = N_0 + N_1 + N_2, \quad \tilde{N} = N_0 + \frac{N_1}{w} + \frac{N_2}{w^2}.
\]

Then the following system of equations holds:

\[
\begin{align*}
Q &= z + 2N + R, \\
R &= S(z + 2\tilde{N} + R), \\
N_0 &= (\tilde{N} + R) \left(\tilde{N} + R + N_0 + \frac{N_1}{2w}\right), \\
N_1 &= 2zw \left(\tilde{N} + R + N_0 + \frac{N_1}{2}\right), \\
N_2 &= z^2 w^3 + zw \left(N_2 + \frac{N_1}{2}\right).
\end{align*}
\]

Moreover, the system and has a unique solution with non-negative coefficients.

**Proof.** First we check that the system has non-negative coefficients. As \(S(z)\) has non-negative Taylor coefficients, we only need to argue on the terms \(N_1/w\) and
But by definition, quadrangulations in $N_i$ have at least $i$ 2-vertices, and the generating function $N_i$ has $w^i$ as a factor.

The first equation follows from the fact that a quadrangulation, not reduced to a single quadrangle, either has a diagonal or is obtained from a quadrangulation in $S$ by replacing internal faces with arbitrary quadrangulations in $Q$.

The second equation expresses the recursive nature of the class $R$. Notice that the substitution inside $S$ contain the term $\tilde{N}$ instead of $N$. The reason is that vertices of degree 2 in the outer face become vertices of degree more than two after substitution.

In the rest of the proof, and without loss of generality, the left-hand terms in the last three equations correspond to quadrangulations in $N$ whose diagonal is incident to the root vertex. They are decomposed following their rightmost diagonal, that is, the first diagonal to the left of the root edge. Let $A$ be the quadrangulation consisting of a single quadrangle with a diagonal. It has two inner faces: $f_1$, incident with the root edge, and $f_2$. A quadrangulation in $N$ is obtained by replacing the inner faces of $A$ with simple quadrangulations, such that the quadrangulation replacing $f_1$ does not have a diagonal incident with the root vertex, as otherwise the decomposition would not be unique. In what follows, the top vertex is the external vertex adjacent to the root vertex and not incident with the root edge.

**Equation for $N_0$.** Both $f_1$ and $f_2$ can be replaced either with quadrangulations in $R$ or in $N$ whose diagonals are not adjacent to the root vertex, hence the factor $\tilde{N} + R$. In addition, $f_2$ can be replaced by a quadrangulation with a diagonal adjacent to its root vertex, but only if the top vertex is not of degree 2, that is, any quadrangulation in $N_0$ and half of the ones in $N_1$ (those in which the top vertex is not of degree 2).

**Equation for $N_1$.** The argument is as in the previous equation, except that now either $f_1$ or $f_2$ are not replaced. Notice the term $N_1/2$ instead of $N_1/(2w)$: this is because the middle vertex of the diagonal remains of degree 2.

**Equation for $N_2$.** In this case face $f_1$ is not replaced. If neither is $f_2$, we get $A$, hence the term $z^2w^3$. Else, $f_2$ is replaced with a quadrangulations in $N$ whose top vertex is of degree 2, corresponding to the term $N_2 + N_1/2$, as before.

From the previous system of equations we can compute the coefficients to any order of all the series involved by iteration. As we are going to see, a modified version of the series $Q, N_0, N_1, N_2$ is needed in Section 2.2.

### 2.2 Arbitrary quadrangulations

A quadrangulation of a 2-cycle is a rooted map in which each face is of degree 4 except the outer face which is of degree 2. One of the two edges in the outer face is taken as the root edge, and its tail is the root vertex. An arbitrary quadrangulation is obtained from a simple quadrangulation by replacing edges with quadrangulations of a 2-cycle. Conversely, given an arbitrary quadrangulation, collapsing all maximal 2-cycles, one obtains a simple quadrangulation. Notice that among the simple quadrangulations, we need to include the degenerate case consisting of a path of length 2, which we denote by $P_2$ (see Figure 2 and the corresponding caption).

In an arbitrary quadrangulation there are three possibilities for the shape of the root face: it is either a quadrangle, the result of gluing two 2-cycles through a vertex, or gluing one 2-cycle and one edge (see Figure 2).

We define the following classes of quadrangulations:

- $A = A_0 \cup A_1$ are quadrangulations of a 2-cycle. $A_1$ are those whose root vertex is a 2-vertex (by symmetry, they are in bijection with those in which the other external vertex is a 2-vertex), and $A_0$ are those without external 2-vertices.
Figure 2: The different types of quadrangulations in $\mathcal{B}$, depending on the nature of the root face. Isolated vertices of degree 2 are shown in white.

Figure 3: On the left are the two types of quadrangulations in $\mathcal{B}_0^*$. On the right are two types of substitutions of $P_3$ counted in $\mathcal{B}_1$.

$\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_1$ are arbitrary quadrangulations. $\mathcal{B}_1$ are those in which the root edge is incident to exactly one 2-vertex, and $\mathcal{B}_0 \cup \mathcal{B}_0^*$ are those in which the root edge is not incident to a 2-vertex. Furthermore, $\mathcal{B}_0^*$ are the quadrangulations obtained by replacing one of the two edges incident with the root edge on the single quadrangle, as illustrated in Figure 3. The class $\mathcal{B}_0^*$ is introduced for technical reasons that will become apparent in the next section, when we consider the dual class of 4-regular maps.

In the generating functions $A_0(z, w)$ and $A_1(z, w)$, variables $z$ and $w$ mark, respectively, internal faces and 2-vertices, whereas in $B_0(z, w)$, $B_1(z, w)$ and $B_0^*(z, w)$ variable $z$ marks all faces: it is important to keep in mind this distinction when checking the equations satisfied by the various generating functions. An exception, which again becomes clear when passing to the dual, is the term $2z$ encoding the path $P_3$ (which can be rooted in two different ways), where the middle vertex is not considered to be a 2-vertex.

The next generating function encodes the substitution of edges in simple quadrangulations:

$$\tilde{A} = A_0 + \frac{2A_1}{w}.$$  

The reason for $w$ in the denominator is that, after substitution, the 2-vertex in $A_1$ no longer has degree 2, and the factor 2 is because this vertex can be any of the two endpoints of the root edge.

To encode the substitution of edges by objects from $\mathcal{A}$ we first remark that the number of edges in a quadrangulation is twice the number of faces, hence there is a bijection between faces and pairs of edges. Moreover, since 2-vertices are isolated pairs of edges incident with a 2-vertex are uniquely determined. These considera-
tions justify the following change of variables:

\[ s = s(z, w) = (1 + \tilde{A})^2, \quad t = t(z, w) = \frac{w + 2\tilde{A} + \tilde{A}^2}{(1 + \tilde{A})^2}, \]

whose meaning is the following. The term \((1 + \tilde{A})^2\) in \(s\) encodes pairs of edges that are possibly substituted. Pairs of edges incident with a 2-vertex must be treated differently: if any of them is replaced with an object from \(A\), the vertex no longer has degree 2, hence the term \(w + 2\tilde{A} + \tilde{A}^2\) in \(t\). This must be corrected with the term \((1 + \tilde{A})^2\) in the denominator of \(t\) since those edges where already counted in \(s\).

Consider now the system (1) from the previous section with the change of variables

\[ z = zs, \quad w = t. \]

We remark that the factor \(z\) in \(zs\) encodes faces in the initial simple quadrangulation. We let \(Q_1\) and \(Q_0\) correspond, respectively, to quadrangulations in which the root edge is incident or not with a 2-vertex. Also, let \(E\) correspond to the case where the quadrangulation before the substitution is a single quadrangle. In all these three cases, \(z\) marks inner faces.

**Lemma 2.3.** The following equalities hold:

\[
Q_1 = \frac{1}{t} (N_1(zs, t) + 2N_2(zs, t), \quad Q_0 = s(2N_0(zs, t) + N_1(zs, t) + R(zs, t)) + (2\tilde{A} + \tilde{A}^2)Q_1, \quad E = z(1 + \tilde{A})^4 - 4z\tilde{A}^2 + 4zw\tilde{A}^2,
\]

where \(z\) marks inner faces and \(w\) marks 2-vertices.

**Proof.** Let \(w\) be the root edge. In all three cases we consider the simple quadrangulation obtained when collapsing all 2-cycles. The first equation holds because quadrangulations where there is one 2-vertex incident with the root edge are encoded by \(N_1 + 2N_2\), and the factor \(1/t\) because we do not replace any of its two incident edges.

In the second equation, we need to distinguish whether either \(u\) or \(v\) in the initial simple quadrangulation were of degree 2 or not. If this is the case, we must replace at least one of its two incident edges with a quadrangulation of a 2-cycle, obtaining the correcting factor \((2\tilde{A} + \tilde{A}^2)\), hence the term \((2\tilde{A} + \tilde{A}^2)Q_1\). If neither \(u\) nor \(v\) were of degree 2 we get \(s(2N_0 + N_1 + R)\), where the factor \(s\) is needed to adjust the change of parameters between edges and inner faces.

As for the last equation, vertices of degree 2 in the root face can become isolated after replacing two consecutive edges of a quadrangle by quadrangulations of a 2-cycle. There are four possibilities for this situation, encoded by \(4z\tilde{A}^2\) in the expansion of \(z(1 + \tilde{A})^4\) (see Figure 4). Hence the term \(z(1 + \tilde{A})^4 - 4z\tilde{A}^2 + 4zw\tilde{A}^2\).

We treat separately the generating function encoding the substitution of \(P_3\), the path on 3 vertices, which corresponds to

\[ \tilde{A} = A_0 + A_1 + \frac{A_1}{w}. \]  \(2\)

The difference with \(\hat{A}\) is that the two edges of \(P_3\) have one endpoint of degree 1, and when the external 2-vertex of a quadrangulation in \(A_1\) is identified with one of them, its degree does not increase (see Figures 2 and 5).
Figure 4: The four types of quadrangulations obtained by replacing two consecutive edges of a quadrangle with quadrangulations of a 2-cycle, where a 2-vertex (in white) is created. The first two on the left are in $B_0$, while the two on the right are in $B_1$.

**Lemma 2.4.** Let $Q_0$, $Q_1$ and $E$ be as in Lemma 2.3. Then the following equations hold and have a unique solution with non-negative coefficients:

$$
A_1 = zw(1 + \hat{A}),
A_0 = 2z\tilde{A}(1 + \hat{A}) + z(Q_0 + Q_1 + E + 2z\tilde{A}(w - 1) + 2z\tilde{A}^2(1 - w)).
$$

**Proof.** We first observe that a quadrangulation of a 2-cycle can be thought as an ordinary quadrangulation adding an edge parallel to the root edge. When removing the root vertex of a quadrangulation of a 2-cycle in $A_1$, we obtain either an edge (term $zw$), a quadrangulation of a 2-cycle in $A_0$ (term $zwA_0$), a quadrangulation of a 2-cycle in $A_1$ (term $zA_1$) or its symmetric. In the last case, we create a 2-vertex, hence the factor $w$ in $zwA_1$.

The term $2z\tilde{A}(1 + \hat{A})$ in the equation for $A_0$ encodes quadrangulations of the 2-cycle built from a quadrangulation whose external face is not a 4-cycle. The terms $zQ_0 + zQ_1 + zE$ arise from the corresponding quadrangulations when building the 2-cycle. The term $zE$ has to be adjusted, because either we create a 2-vertex (term $2z\tilde{A}^2(1 - w)$), illustrated by the two graphs on the right-hand side of Figure 4, or we remove a 2-vertex (term $2z\tilde{A}^2(1 - w)$), illustrated by the two graphs on the right-hand side of Figure 4.

Finally, we check that the series on the right-hand side have non-negative coefficients. Since $E = z(1 + \hat{A})^4 - 4zA^2 + 4zw\tilde{A}^2$, the series $E + 2z\tilde{A}(w - 1) + 2z\tilde{A}^2(1 - w)$ has non-negative coefficients.\qed

From the previous lemma we can obtain the generating functions associated to $B_0$, $B_0^*$ and $B_1$.

**Lemma 2.5.** Let $Q_0, Q_1, E, A_0, A_1$ as in the previous two lemmas, and $\hat{A}$ as defined in Equation (2). Then $B_0$, $B_1$ and $B_0^*$ are given by

$$
B_0 = 2z(1 + \hat{A})(1 + \hat{A} - A_1) + z(Q_0 + E - 2zw\tilde{A}^2 - 2z\tilde{A}),
B_1 = 2z(1 + \hat{A})A_1 + zw(Q_1 + 2z\tilde{A}^2),
B_0^* = 2z^2A,
$$

where $z$ marks faces and $w$ marks 2-vertices.

**Proof.** Recall that an arbitrary quadrangulation is obtained by substituting edges by quadrangulations of a 2-cycle in a simple quadrangulation. Factor $z$ in all equations is used to encode the outer face, which has not been considered in the previous generating functions.

The term $2z(1 + \hat{A})(1 + \hat{A} - A_1)$ encodes quadrangulations obtained from $P_3$. Notice that quadrangulations counted by the term $2z(1 + \hat{A})A_1$ are in $B_1$ and
must be removed from $B_0$. The second term $z(Q_0 + E - 2zw\tilde{A}^2 - 2z\tilde{A})$ encodes quadrangulations obtained from simple quadrangulations whose root face is a 4-cycle. In this situation, we have to remove from $zE$ the terms $2z^2w\tilde{A}^2$ and $2z^2\tilde{A}$, which contribute to $B_0'$ and $B_1$, respectively (see the two leftmost maps in Figure 3 and the two rightmost maps in Figure 4, respectively). Finally, there is an extra contribution to $B_1$ with the term $zwQ_1$.

From the Systems (1) and (4) we can compute, by iteration, the coefficients to any order of all the series involved. In particular we can compute the coefficients of the series $B_1$, $B_0$ and $B_0^*$, which are needed in the next section.

3 Rooted 3-connected 4-regular planar maps

In this section we count 3-connected 4-regular planar maps according to the number of simple and double edges. Because they have a unique embedding on the oriented sphere, this is equivalent to counting labelled 3-connected 4-regular planar graphs.

We notice that a 3-connected 4-regular map cannot have triple edges, and double edges must be vertex disjoint. In addition, all maps in this section are rooted.

For brevity, a face of degree 2 not adjacent to another face of degree 2 is called a 2-face. We say that an edge is in a 2-face if it is one of its two boundary edges. An edge is ordinary if it is not in the boundary of a 2-face. Since the number of edges is even, the number of ordinary edges is also even. Maps are counted according to two parameters: the number of 2-faces, marked by variable $w$, and half the number of ordinary edges, marked by $q$. Setting $w = q$ one recovers the enumeration of 4-regular maps according to half the number of edges. Observe that the dual of a quadrangulation with $\ell$ 2-vertices is a 4-regular map with $\ell$ 2-faces.

We need to define the replacement of edges by maps. Let $M$ be a rooted map. Consider a fixed orientation of the edges in $M$; since in a rooted map all vertices and edges are distinguishable we can define such an orientation unambiguously.

Replacement of simple edges. Let let $e = uv$ be an edge of $M$ and $N$ a map whose root edge is simple. The replacement of $e$ with $N$ is the map obtained by the following operation. Subdivide $e$ twice transforming it into the path $uu'v'v$, remove the edge $u'v'$, and identify $u'$ and $v'$ with the end vertices of the root edge of $N$, respecting the orientations. An example is shown in Figure 5, bottom right.

Replacement of double edges. Let $(e, e')$ be a double edge and $N$ a map whose root is a double edge. The replacement of $(e, e')$ with $N$ is the map obtained by identifying $(e, e')$ with the root of $N$, preserving the orientation and the embedding. An example is shown in Figure 5, bottom second map from the right.

Notice that if $M$ and $N$ are 4-regular, the maps obtained by replacement of edges are also 4-regular.

We now consider the following families of (rooted) 4-regular maps.

- $\mathcal{M} = \mathcal{M}_0 \cup \mathcal{M}_0^* \cup \mathcal{M}_1$ are 4-regular maps. $\mathcal{M}_0 \cup \mathcal{M}_0^*$ are 4-regular maps in which the root edge is not incident with a 2-face, and $\mathcal{M}_1$ are those for which the root edge is incident with exactly one 2-face. $\mathcal{M}_0^*$ are maps in which the root is one of the extreme edges of a triple edge, corresponding to dual maps of quadrangulations on the left of Figure 3. These classes are in bijection with the classes $B_0$, $B_0^*$ and $B_1$ from the previous section, and Lemma 2.5 gives access to the associated generating functions.

The next classes are all subclasses of $\mathcal{M}$. Given a map $M$, we let $M^*$ be the map obtained by removing the root edge $st$. In accordance with the terminology introduced in the next section, the poles are the endpoints $s, t$ of the root edge.
• $\mathcal{L}$ are loop maps: the root-edge is a loop.

• $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1$ are series maps: $M^-$ is connected and there is an edge in $M^-$ that separates the poles. As above, the index $i = 0, 1$ refers to the number of 2-faces incident with the root edge.

• $\mathcal{P} = \mathcal{P}_0 \cup \mathcal{P}_1$ are parallel maps: $M^-$ is connected, there is no edge in $M^-$ separating the poles, and either $st$ is an edge of $M^-$ or $M - \{s, t\}$ is disconnected. The index $i = 0, 1$ has the same meaning as in the previous class.

• $\mathcal{F}$ are maps $M$ such that the face to the right of the root-edge is a 2-face, and such that $M - \{s, t\}$ is connected; see Figure 5.

• $\mathcal{H}$ are $h$-maps: they are obtained from a 3-connected 4-regular map $C$ (the core) and possibly replacing every non-root edge of $C$ with a map in $\mathcal{M}$. The root edge of $C$ must be a simple edge.

Figure 5: Root-decomposition of 4-regular maps. On top a loop map (left) and a series map (right). Bottom, from left to right: parallel map, map in $\mathcal{F}$, and two maps in $\mathcal{H}$.

Lemma 3.1. The former classes partition $\mathcal{M}$, that is,

$$\mathcal{M} = \mathcal{L} \cup \mathcal{S} \cup \mathcal{P} \cup \mathcal{H} \cup \mathcal{F}.$$  

Proof. Let $G$ be 4-regular map rooted at $e = st$, and suppose $e$ is not a loop, so we are not in the class $\mathcal{L}$. Consider the 2-connected component $C$ containing $e$. By Tutte’s theory of decomposition of 2-connected graphs into 3-connected components (see [4] for a detailed exposition), $C$ is either a series, parallel or $h$-composition. Series and $h$-compositions correspond, respectively, to classes $\mathcal{S}$ and $\mathcal{H}$. Parallel compositions are either in $\mathcal{P}$ or in $\mathcal{F}$; we need to distinguish between these two classes when replacing a double edge in a map. 

The next step is to determine the algebraic relations among the generating functions of the previous classes. Variable $q$ marks half the number of ordinary edges, and $w$ the number of 2-faces. We need to introduce an auxiliary generating function $D$. It is combinatorially equivalent to the generating function associated
to the class $\mathcal{M}$, but with the following modification. When the root edge it incident to a 2-face, it is not counted by the variable $w$; when it is ordinary, it is counted by the variable $q$. This is a technical device that simplifies the forthcoming equations.

Let $T(u, v)$ be the generating function of 3-connected 4-regular maps in which the root edge is simple, where $u$ marks half the number of simple edges and $v$ marks the number of double edges (let us recall again that the number of edges in a 4-regular graph is even). The next result provides a link between the known series $M_0, M_0^*$ and $M_1$, and the series $T$ we wish to determine.

Lemma 3.2. The following system of equations hold, where $q$ marks half the number of ordinary edges, and $w$ the number of 2-faces.

\begin{align*}
M_0 &= S_0 + P_0 + L + H, \\
M_1 &= \frac{w}{q} (S_1 + P_1 + 2qF), \\
M_0^* &= 2q^2 D, \\
D &= M_0 + \frac{2}{w} M_1 + \frac{w}{q} M_0^*, \\
L &= 2q(1 + D - L) + L(w + q), \\
S_0 &= D(D - S_0 - S_1) - \frac{L^2}{2}, \\
S_1 &= \frac{L^2}{2}, \\
P_0 &= q^2 (1 + D + D^2 + D^3) + 2qDF, \\
P_1 &= 2q^2 D^2, \\
H &= \frac{T (q(1 + D)^2, w + q(2D + D^2) + F)}{1 + D}.
\end{align*}

Proof. The first four equations follow directly from the definitions. The correcting factors $w/q$ and $q/w$ in the second and fourth equations are due to the convention on how to mark the root edge in the class $\mathcal{M}$.

Consider now loop maps. The double-loop is the map consisting of a single vertex and two loops. A loop map is obtained by possibly replacing the non-root loop of the double-loop with a map in $\mathcal{M}$. We have two possibilities illustrated in Figure 6, giving

$$L = 2q(1 + D - L) + L(q + w).$$

A series map is obtained by taking a map in $\mathcal{M}$ with poles $s_1$ and $t_1$, a map in $\mathcal{M} \setminus S$ with poles $s_2$ and $t_2$, and then replacing $s_1t_1$ and $s_2t_2$ with edges $t_1s_2$ and $s_1t_2$, the latter being the new root. When the two maps connected in series are in $L$, a 2-face is created containing the root. (see Figure 6), and we obtain a map in $S_1$. There are four ways to connect two loop maps in series, but only two of them produce a map in $S_1$: when they are both rooted either both on a face of degree one, or both on a face of degree more than one. In terms of generating functions we have

$$S_0 = D(D - S_1) - S_0$$ and $$S_1 = \frac{L^2}{2}.$$

A parallel map can be obtained in two different ways. First, from a map in $\mathcal{F}$ with double root edge $r$ and replacing exactly one edge in $r$ with a map in $\mathcal{M}$. This is encoded by $2qFD$ (see Figure 7). Secondly, we take the 4-bond (an edge of multiplicity 4) and replace any of its three non-root edges with maps in $\mathcal{M}$. This is encoded by $q^2(1 + D)^3$ (see Figure 5), but two particular cases must be considered:

1) When exactly two edges of the 4-bond are substituted, encoded by $P_1$ (see Figure 7), where a double edge is created;
2) When exactly one edge of the 4-bond is substituted.

Again there are two instances, encoded by $M^*_0$ (see Figure 7), where the root edge belongs to exactly one of the two faces of degree 2. Notice that these faces are not isolated and hence are encoded as three ordinary edges. When the root edge is removed, the 2-face containing it is removed and the remaining one becomes isolated. Summing up this gives the expressions for $P_0$ and $P_1$.

A map in $\mathcal{H}$ is obtained by possibly replacing the non-root simple edges of a core by a map in $\mathcal{M}$, and the double edges with either:

1) A map in $\mathcal{M}$ on one of the edges of the double edge;
2) Two maps in $\mathcal{M}$, one on each edge;
3) A map in $\mathcal{F}$ (see Figure 5).

This gives

$$H = \frac{T(1 + D, w + q(2D + D^2) + F)}{1 + D}.$$ 

Proof of Theorem 1.3. From the knowledge of $M$ and the $M_i$ in the previous section we can determine $D = M^*_0/(2q^2)$. Hence we can also determine $L$, then $S_0$ and $S_1$. We also know $P_1 = 2wD^2$ and from $M_1 = S_1 + P_1 + 2wF$ we obtain $F$. This is enough to compute $P_0$, and from $M_0 = S_0 + P_0 + L + H$ we determine $H$. Since $M$ and the $M_i$ were algebraic functions, so are all the functions in the previous system.

We are ready for the final step. Consider the following change of variables

$$u = q(1 + D)^2, \quad v = w + q(2D + D^2) + F,$$
relating $T$ and $H$. The first terms in the expansion of $u$ and $v$ in $q$ and $w$ are

$$u = q + \cdots, \quad v = w + \cdots.$$

It follows that the Jacobian at $(0, 0)$ is equal to 1 and the system can be inverted, in the sense that we can determine uniquely the coefficients of the inverse series. Computationally, this can be explicitly obtained using Gröbner basis (we are grateful to Manuel Kauers for this observation).
Let the inverse of the system be
\[ q = a(u,v), \quad w = b(u,v). \]
Since \( D \) and \( F \) are algebraic functions, so are the inverse functions \( a \) and \( b \). Now we use the last equation in Lemma 3.2 to get
\[ T(u,v) = (1 + D(a(u,v),b(u,v))H(a(u,v),b(u,v)). \]
This equation determines \( T \). Since all the series involved are algebraic, so is \( T \).

Recall that \( t_n \) is the number of labelled 3-connected 4-regular planar graphs. Let \( T_n = [u^k]T(u,0) \) be the number of simple rooted 3-connected 4-regular maps. Then we have the relation
\[ 8nt_n = n!T_n. \]
This follows by double counting. We can label the vertices of a rooted map in \( n! \) different ways, since vertices in a rooted map are distinguishable, and on the other hand, from a labelled graph we obtain 8n rooted maps: 4n choices for the directed root edge, and 2 choices for the root face. As a consequence,
\[ 8xT'(x) = T(u,0). \]
Since \( T \) is algebraic, so is \( T'(x) \).

4 Labelled 4-regular planar graphs

In this section we complete the proof of Theorem 1.1. In the sequel all graphs are labelled. First we define networks. A network is a connected 4-regular multigraph \( G \) with an ordered pair of adjacent vertices \((s,t)\), such that the graph obtained by removing the edge \( st \) is simple. Vertices \( s \) and \( t \) are the poles of the network.
We now define several classes of networks, similar to the classes introduced in the previous section. We use the same letters, but they now represent classes of labelled graphs instead of classes of maps. No confusion should arise since in this section we deal with graphs.

- $\mathcal{D}$ is the class of all networks.
- $\mathcal{L}, \mathcal{S}, \mathcal{P}$ correspond as before to loop, series and parallel networks. We do not need to distinguish between $\mathcal{S}_0$ and $\mathcal{S}_1$ and between $\mathcal{P}_0$ and $\mathcal{P}_1$.
- $\mathcal{F}$ is the class of networks in which the root edge has multiplicity exactly 2 and removing the poles does not disconnect the graph.
- $\mathcal{S}_2$ are networks in $\mathcal{F}$ such that after removing the two poles there is a cut vertex, see Figure 8.
- $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$ are h-networks: in $\mathcal{H}_1$ the root edge is simple and in $\mathcal{H}_2$ it is double, see Figure 8.

**Lemma 4.1.** The two classes $\mathcal{S}_2$ and $\mathcal{H}_2$ partition $\mathcal{F}$, that is

$$\mathcal{F} = \mathcal{S}_2 \cup \mathcal{H}_2.$$ 

**Proof.** Clearly $\mathcal{S}_2$ and $\mathcal{H}_2$ are contained in $\mathcal{F}$. Now take a network in $\mathcal{F}$ and remove its two poles. By definition, the resulting graph must be connected and has either a cut-vertex or is at least 2-connected. If it has a cut-vertex then it belongs to $\mathcal{S}_2$. Otherwise, by the same argument as in Lemma 3.1 it is obtained from a 3-connected core rooted at a double edge, hence if belong to $\mathcal{H}_2$.

The generating functions of networks are of the exponential type, for instance $D(x) = \sum D_n x^n/n!$. We need in addition the generating functions

$$T_i(x, u, v) = \sum T^{(i)}_{n, k, \ell} u^k v^\ell x^n/n!$$

of 3-connected 4-regular planar graphs rooted at a directed edge, where $i = 1, 2$ indicates the multiplicity of the root, $x$ marks vertices and $u, v$ mark, respectively, half the number of simple and the number of double edges. The coefficients $T^{(i)}_{n, k, \ell}$ of $T_i$ are easily obtained from those of $T = \sum t_{k, \ell} u^k v^\ell$, the generating function of 3-connected 4-regular maps computed in the previous section. By double counting we have

$$T^{(1)}_{n, k, \ell} = n! t_{k, \ell}/2, \quad \ell T^{(2)}_{n, k, \ell} = k T^{(1)}_{n, k, \ell}, \quad n = \ell + k/2.$$
Since we can compute the coefficients \( t_{k,\ell} \) as in the previous section, we can compute the coefficients \( T^{(i)}_{n,k,\ell} \) as well.

In terms of generating functions this amounts to

\[
T^{(1)}(x, u, v) = \frac{1}{2} T(ux, vx), \quad u \frac{\partial}{\partial u} T^{(2)}(x, u, v) = v \frac{\partial}{\partial u} T^{(1)}(x, u, v). \tag{6}
\]

Since \( T \) is an algebraic function, \( T^{(1)} \) is algebraic too, but to prove that \( T^{(2)} \) is algebraic needs a separate argument.

Similarly to Lemma 4.1 but in the maps setting, \( F \) (as a class of maps) is partitioned into \( S_2 \) and \( H_2 \). Although redundant for the purpose of extracting the coefficients of \( T \), this decomposition can be added to the system of equations (5) as follows:

\[
F = S_2 + H_2,
\]

\[
S_2 = (F + q(1 + D)^2)(F + q(1 + D)^2 - S_2), \tag{7}
\]

\[
H_2 = \frac{T_2(q(1 + D)^2, w + q(2D + D^2) + F)}{w + q(2D + D^2) + F}
\]

where \( T_2(u,v) \) counts 3-connected 4-regular maps rooted at a double edge. Since \( F \) and \( D \) are algebraic by Lemma 3.2, so are \( S_2 \) and \( H_2 \). And so is \( T_2 \) since it can be derived form \( H_2 \) by the same algebraic inversion as before. Finally we have

\[
T^{(2)}(x, u, v) = \frac{1}{4} T_2(ux, vx), \tag{8}
\]

where the division by four encodes the choice of the root face, and the fact that in a multigraph a double edge has only one rooting instead of two for maps.

**Lemma 4.2.** The following system of equations among the previous series hold:

\[
D = L + S + P + H_1 + F
\]

\[
L = \frac{x}{2}(D - L)
\]

\[
S = D(D - S)
\]

\[
P = x^2 \left( \frac{D^2}{n} + \frac{D^2}{2} \right) + \frac{FD}{2}
\]

\[
F = S_2 + H_2
\]

\[
S_2 = \frac{1}{x} \left( F + x^2 \left( D + \frac{D^2}{2} \right) \right) \left( F + x^2 \left( D + \frac{D^2}{2} \right) - S_2 \right)
\]

\[
H_1 = \frac{T^{(1)}(x, 1 + D, D + \frac{D^2}{2} + \frac{E}{x})}{1 + \frac{D}{2} + \frac{D^2}{x} + \frac{E}{x^2}}
\]

\[
H_2 = \frac{T^{(2)}(x, 1 + D, D + \frac{D^2}{2} + \frac{E}{x})}{D + \frac{D^2}{2} + \frac{E}{x^2}}
\]

**Proof.** The first equation follows from a direct adaptation of Lemma 3.1 to the context of graphs. The remaining equations follow by adapting the proof of Lemma 3.2 form maps to graphs. We briefly indicate the differences.

In the equation for \( L \) we must take into account that graphs are not embedded, hence the division by two, and also that the double loop is not admissible as a network. In the equation for \( S \) the only difference with Lemma 3.2 is that the family \( S_1 \) is no longer needed. Similar considerations apply to the equation for \( P \). The equation for \( F \) follows directly from Lemma 4.1.

The equation for \( S_2 \) describes the decomposition of a network in \( S_2 \). It is essentially a series composition of two networks, one in \( F \) and one in \( F \setminus S_2 \), in which
the second pole of the first one is identified with the first pole of the second one, hence the division by $x$. In addition we have the cases were one of the two networks in the series composition is a fat polygon, that is, a cycle in which each edge is doubled, and every double edge is replaced with one or two networks in $D$. Notice that contrary to double edges of networks counted in $T^{(1)}$ and $T^{(2)}$, double edges of the fat polygon are not replaced with networks in $F$, as this would create a series composition with two networks in $S_2$.

The last two equations are similar to that for $H$ in Lemma 3.2 with the difference that double edges must be replaced with either: a network in $D$; two networks in $D$, encoded by $D^2/2$; or a network in $F$ for which the two poles were removed, encoded by $F/x^2$.

Proof of Theorem 1.1. Equations (9) can be rewritten as a system with non-negative coefficients using the identities $D - L = S + P + H_1 + F$, $D - S = L + P + H_1 + F$, and $F - S_2 = H_2$. Hence it has a unique solution with non-negative terms, which can be computed by iteration from the knowledge of $T^{(1)}$ and $T^{(2)}$. Since all the functions involved are algebraic, the solution consists of algebraic functions.

Let $C(x)$ now be the generating function of labelled 4-regular planar graphs. There is a simple relation between $C(x)$ and the series $D(x)$ of networks, namely

$$4xC'(x) = D(x) - L(x) - L(x)^2 - F(x) - \frac{x^2}{2} D(x).$$

The series on the left corresponds to labelled graphs with a distinguished vertex $v$ in which one of the 4 edges incident with $v$ is selected. These correspond precisely to networks, except for the fact that since we are counting simple graphs we have to remove from $D(x)$ networks containing either loops or double edges, which correspond to the terms subtracted.

Finally, since $D$, $L$ and $F$ are algebraic functions, so is $C'(x)$.

Proof of Corollary 1.2. A series is $D$-finite if it satisfies a linear differential equation with polynomial coefficients. It is well-known (see Chapter 6 in [14]) that $\sum f_n x^n / n!$ is $D$-finite if and only if $\{f_n\}$ is $P$-recursive. Since $G(x) = \exp(C(x))$ and $C'(x)$ is algebraic, it is enough to prove the following:

If $C'(x)$ is algebraic then $\exp(C(x))$ is $D$-finite.

Let $G(x) = e^{C(x)}$. It is easily proved by induction that each derivative $C^{(i)}(x)$ is a rational function of $C'(x)$ and $x$. Using this observation and applying induction we have that $G^{(i)} = R_i(C', x) G$, where $R_i$ is a rational function in $C'$ and $x$, and we set $R_0 = 1$. Since $C'$ is algebraic, $Q(C', x)$ is finite dimensional over $Q(x)$, say of dimension $k$. Hence there are rational functions $S_i(x)$ such that $\sum_{i=0}^k S_i(x) R_i(C', x) = 0$. It follows that

$$S_0(x) G + S_1(x) G' + \cdots + S_k(x) G^{(k)} = 0.$$

proving that $G$ is $D$-finite.

5 Simple 4-regular maps

The enumeration of simple 4-regular maps is obtained by adapting the arguments in the previous section to maps instead of graphs. We define the various classes of simple maps exactly as we did for networks, keeping the same notation. The decomposition scheme starts from the series $T_1(x, u, v)$ and $T_2(x, u, v)$ of 3-connected
4-regular maps, where the indices and variables have the same meaning as in the previous section. As before, they are accessible from the series \( T(u, v) \):
\[
T_1(x, u, v) = T(xu^2, xv) \quad \text{and} \quad u \frac{\partial}{\partial u} T_2(x, u, v) = 2v \frac{\partial}{\partial v} T_1(x, u, v),
\]
where multiplication by 2 on the right-hand side of the second equation is because a double edge can be rooted at any of its two edges.

Lemma 5.1. The following equations hold:
\[
\begin{align*}
D &= L + S + P + H_1 + 2F \\
L &= 2x(D - L) \\
S &= D(D - S) \\
P &= x^2(3D^2 + D^3) + 2FD \\
F &= S_2 + \frac{H_2}{2} \\
S_2 &= \frac{1}{x}(F + x^2(2D + D^2))(F + x^2(2D + D^2) - S_2) \\
H_1 &= \frac{T_1(x, 1 + D, 2D + D^2 + \frac{F}{x})}{1 + D} \\
H_2 &= \frac{T_2(x, 1 + D, 2D + D^2 + \frac{F}{x})}{2D + D^2 + \frac{F}{x}}
\end{align*}
\]

Proof. The proof is essentially the same as that of Lemma 4.2, with the following differences. Maps are embedded, hence there are \( \binom{m}{k} \) ways to substitute \( k \) maps in \( D \) for an edge of multiplicity \( m \). This justifies the term \( 3D^2 \) in the fourth equation and the term \( 2D \) in the last three equations. Finally, the fact that maps have a root face explains the factors 2 in the first, second and fourth equations.

Proof of Theorem 1.4. From the knowledge of \( T(u, v) \) of Section 3 and from Lemma 5.1, we can compute the coefficients of the series \( D, L, S, P, S_2, F, H_1 \) and \( H_2 \) up to any order. By removing maps having a loop or a multiple edge, the series \( M(x) \) of rooted 4-regular simple maps is equal to
\[
M(x) = D(x) - L(x) - L(x)^2 - 3D(x)^2 - 2F(x).
\]
Since \( D, L \) and \( F \) are algebraic, so is \( M \).

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Tables

As an illustration we present in Table 1 the numbers of 4-regular planar graphs up to 24 vertices. We see that the first discrepancy is at \( n = 12 \). For this number of vertices there are 4-regular planar graphs which are either disconnected (the union of two octahedra) or are connected but not 3-connected (gluing two octahedra via two parallel edges). Table 2 and 3 give, respectively, the numbers of rooted 3-connected 4-regular maps, and simple 4-regular maps.
Table 1: Numbers of arbitrary, connected and 3-connected labelled 4-regular planar graphs with $n$ vertices.

| $n$ | $t_n,0$ | $c_n$ | $t_n$ |
|-----|---------|--------|-------|
| 6   | 15      | 15     | 15    |
| 7   | 0       | 0      | 0     |
| 8   | 2520    | 2520   | 2520  |
| 9   | 30240   | 30240  | 30240 |
| 10  | 3991680 | 3991680| 3991680|
| 11  | 1517746359 | 1517746359 | 1517746359|
| 12  | 6536874960 | 6536874960 | 6536874960|
| 13  | 300163485200 | 300163485200 | 300163485200|
| 14  | 148316381413200 | 148316381413200 | 148316381413200|
| 15  | 79377209200000 | 79377209200000 | 79377209200000|
| 16  | 131544000000000 | 131544000000000 | 131544000000000|
| 17  | 30240000000000000 | 30240000000000000 | 30240000000000000|
| 18  | 65368749600000000 | 65368749600000000 | 65368749600000000|
| 19  | 1483163814132000000 | 1483163814132000000 | 1483163814132000000|
| 20  | 79377209200000000000 | 79377209200000000000 | 79377209200000000000|
| 21  | 131544000000000000000 | 131544000000000000000 | 131544000000000000000|
| 22  | 3024000000000000000000 | 3024000000000000000000 | 3024000000000000000000|
| 23  | 65368749600000000000000 | 65368749600000000000000 | 65368749600000000000000|
| 24  | 1483163814132000000000000 | 1483163814132000000000000 | 1483163814132000000000000|

Table 2: Coefficients of $T(u,v) = \sum t_{k,\ell} u^k v^\ell$ where $t_{k,\ell}$ is the number of rooted 3-connected 4-regular maps with $\ell$ double edges and $2k$ simple edges, in which the root edge is simple.

| $\ell \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-------------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 0                 | 1 |   |   |   |   |   |   |   |   |   |    |    |    |
| 1                 |   | 1 |   |   |   |   |   |   |   |   |    |    |    |
| 2                 |   |   | 6 | 28 | 128 | 396 | 1460 | 5148 | 18696 | 70440 | 262896 | 990384 | 3561536 |
| 3                 |   |   |   | 7 | 46 | 510 | 3630 | 21350 | 115440 | 593622 | 3149600 | 15748000 | 73740000 |
| 4                 |   |   |   |   | 8 | 512 | 4608 | 34794 | 2282544 | 13791064 | 78768368 | 431603212 | 2158016064 |
| 5                 |   |   |   |   |   | 9 | 560 | 5100 | 39794 | 26825440 | 166310640 | 9867683680 | 529232160960 | 2719660384000 |
| 6                 |   |   |   |   |   |   | 10 | 612 | 6600 | 51000 | 34794000 | 228254400 | 13791064000 | 787683680000 |
| 7                 |   |   |   |   |   |   |   | 11 | 660 | 7752 | 66000 | 5100000 | 347940000 | 22825440000 | 1379106400000 |
| 8                 |   |   |   |   |   |   |   |   | 12 | 7374 | 89642 | 77520 | 6600000 | 510000000 | 34794000000 | 2282544000000 |
| 9                 |   |   |   |   |   |   |   |   |   | 13 | 89642 | 10728 | 12344 | 92160 | 7752000 | 660000000 | 51000000000 |
| 10                |   |   |   |   |   |   |   |   |   |   | 14 | 12344 | 172374 | 21350 | 21344 | 921600 | 775200000 | 66000000000 |
| 11                |   |   |   |   |   |   |   |   |   |   |   | 15 | 21350 | 302400 | 347940 | 3479400 | 2282544000 | 137910640000 |
| 12                |   |   |   |   |   |   |   |   |   |   |   |   | 347940 | 51000000 | 5100000000 | 510000000000 | 34794000000000 |

Table 3: $t_{n,0}$ is the number of simple rooted 3-connected 4-regular maps; $M_n$ is the number of simple 4-regular maps. As for graphs, the first discrepancy is at $n = 12$. The numbers $t_{n,0}$ match those given in Table 1 from [3] given up to $n = 15$.
References

[1] M. Bodirsky, M. Kang, M. Löffler, and C. McDiarmid. Random cubic planar graphs. *Random Structures & Algorithms*, 30(1–2):78–94, 2007.

[2] G. Brinkmann, S. Greenberg, C. Greenhill, B. D. McKay, R. Thomas, and P. Wollan. Generation of simple quadrangulations of the sphere. *Discrete Mathematics*, 305:33–54, 2005.

[3] H. J. Broersma, A. J. W. Duijvestijn, and F. Göbel. Generating all 3-connected 4-regular planar graphs from the octahedron graph. *Journal of Graph Theory*, 17(5):613–620, 1993.

[4] G. Chapuy, É. Fusy, M. Kang, and B. Shoilekova. A complete grammar for decomposing a family of graphs into 3-connected components. *Electronic Journal of Combinatorics*, 15(1):148, 2008.

[5] M. Drmota. *Random trees. An interplay between combinatorics and probability*. SpringerWienNewYork, Vienna, 2009.

[6] M. Drmota, É. Fusy, M. Kang, V. Kraus, and J. Rué. Asymptotic study of subcritical graph classes. *SIAM Journal on Discrete Mathematics*, 25(4):1615–1651, 2011.

[7] Z. Gao and N. C. Wormald. Enumeration of rooted cubic planar maps. *Annals of Combinatorics*, 6(3-4):313–325, 2002.

[8] O. Giménez and M. Noy. Asymptotic enumeration and limit laws of planar graphs. *Journal of the American Mathematical Society*, 22:309–329, 2009.

[9] J. Lehel. Generating all 4-regular planar graphs from the graph of the octahedron. *Journal of Graph Theory*, 5(4):423–426, 1981.

[10] R. C. Mullin and P. J. Schellenberg. The enumeration of c-nets via quadrangulations. *Journal of Combinatorial Theory*, 4:259–276, 1968.

[11] M. Noy. Random planar graphs and beyond. In *Proceedings of the ICM, Seoul*, volume IV, pages 407–430, 2014.

[12] M. Noy. Graph enumeration. In M. Bóna, editor, *Handbook of Enumerative Combinatorics*, pages 402–442. CRC Press, 2015.

[13] M. Noy, C. Requilé, and J. Rué. Random cubic planar graphs revisited. In preparation.

[14] R. P. Stanley. *Enumerative Combinatorics*, volume 2. Cambridge University Press, 2001.

[15] W. T. Tutte. A census of planar maps. *Canadian Journal of Mathematics*, 15:249–271, 1963.