The finest regular coarsening and recursively-regular subdivisions

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Abstract

We generalize the notion of regular polyhedral subdivision of a point (or vector) configuration in a new direction. This is done after studying some related objects, like the finest regular coarsening and the regularity tree of a subdivision. Properties of these two objects are derived, which confer more structure to the class of non-regular subdivisions, relating them to its (in a sense) closest regular subdivision. We introduce the class of recursively-regular subdivisions and show that it is a proper superclass of the regular subdivisions and a proper subclass of the visibility-acyclic subdivisions. We also show that recursively-regular triangulations of a given configuration are in general not connected by geometric bistellar flips. Finally, some algorithms related to these new concepts are given and applications of the main results of the article are pointed out. In particular, relations to covering by floodlights, covering by homotheties, tensegrity of spider webs and a high-dimensional graph embedding problem are presented.

1 Introduction

Regular subdivisions are quite well understood and they exhibit some properties from which some algorithms take advantage. In addition, the close relation they hold with polytopes makes them attractive for even a wider set of researchers. For instance, the fact that the graph of flips restricted to regular triangulations of a point set is connected has been proven to be extremely useful. Another remarkable result is that regular subdivisions cannot contain cycles in the visibility relation defined on the faces of a complex from a certain point of view, as defined in [10].

On the other hand, not so much is known about non-regular subdivisions. While it is known that the flip-graph is connected for triangulations in the plane [14], this is not the case for dimensions 5 and higher [19]. The connectivity of this graph is still unknown for dimensions 3 and 4. Some generalizations of the concept of regularity have been studied in order to better understand non-regular subdivisions. For instance, the subdivisions induced by the projection of a polytope were introduced in [4] and, together with their variations, have been extensively studied.

In the present note, we introduce a different generalization of regular subdivisions. The initial motivation for this study was the observation that some results exploit regularity in a way that can be transformed into a recursive scheme. The property we investigate is deservedly called recursive regularity. Roughly speaking, a subdivision is recursively regular if it is regular or it has a regular coarsening that splits it into recursively regular parts. The first question we may ask is whether this class is actually different from regular subdivisions. We show that it is the case, although it contains all the regular ones and is disjoint from the set of visibility-cyclic subdivisions. We further show the existence of recursively regular triangulations belonging to different connected components of the flip graph of a point configuration, which suggests that the class is, in some sense, meaningfully bigger that the class of regular ones. One could also suspect that our recursive definition is not necessary and that one level of recursion could be sufficient. We disprove this intuition by means of a simple example.

Finally, we address the problem of easily characterizing the subdivision of this class. We give an algorithm that relies on the concept of the finest regular coarsening of a polyhedral subdivision. This useful object happens to be not well studied, as far as we know.

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With respect to applications, we show how the new results translate through already known relations and introduce some new connections to not well-studied problems. Results related to tensegrity theory and spatial decompositions are of the first type, while the problems concerning floodlights, homotheties and graph embeddings are of the second.

2 Preliminaries

2.1 Regular subdivisions

Let \( A = \{ p_1, ..., p_n \} \) be a set of points in \( \mathbb{R}^d \). A polyhedral subdivision \( S \) of \( A \) is a polytopal complex covering \( \text{conv}(A) \) and whose set of vertices is contained in \( A \). The full-dimensional polytopes of the complex are called cells. The set of all the cells of a subdivision will be denoted by \( \text{cells}(S) \) and \( |S| \) should be understood as the union of the cells of \( S \). \( S \) is regular if it is the projection of a \((d+1)\)-dimensional convex polyhedral surface \( P^* \), i.e., if there exists a height function

\[
\omega : A \to \mathbb{R} \\
p_i \mapsto \omega_i
\]

such that each face of \( S \) is the projection of a face in the lower convex hull of \( A^\omega = \left\{ (\omega_1, ..., \omega_n) \in \mathbb{R}^n \right\} \). The function \( \omega \) will be identified with the vector \( \omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n \). Given a cell \( C \in S \), \( A^\omega|_C \) denotes \( \left\{ (\omega_i) \in A^\omega : p_i \in C \right\} \).

It is known that the regularity of a polyhedral subdivision can be checked by solving a linear program. More precisely, the Theorem 2.3.20 in [9] establishes that a polyhedral subdivision \( S \) is regular if and only if the two following conditions hold:

- For every cell \( C \in \text{cells}(S) \) the lifted subconfiguration \( A^\omega|_C \) lies in a hyperplane of \( \mathbb{R}^{d+1} \) (coplanarity condition).
- For every wall \( W \in S \), \( W = C_1 \cap C_2 \), with \( C_1, C_2 \in \text{cells}(S) \), \( (p_1) \) lies strictly above the hyperplane containing \( A^\omega|_{C_2} \), for any chosen \( p_1 \in C_1 \setminus C_2 \) (local folding condition).

The coplanarity condition is immediately satisfied for any simplicial cell. For the non-simplicial cells, it translates into a set of linear homogeneous equations in the heights of the vertices of the cell. It is usually required that any \((d+2)\)-tuple of vertices of a cell is lifted planar, but these conditions are redundant. It is not hard to see that it is enough to choose an affine basis for each cell and extend this set by each other vertex in the cell. Then, for each of these \((d+2)\)-tuples, the corresponding lifted point set is required to be affinely dependent. Hence, all the coplanarity conditions restrict the set of possible height functions \( \omega \) to a linear subspace of \( \mathbb{R}^n \).

Consider now the local strong folding condition for a wall \( W = C_1 \cap C_2 \). Choose an affine basis \( \left\{ (p_0, \omega_1), ..., (p_{d+1}, \omega_{d+1}) \right\} \) of the hyperplane containing \( C_2 \) and a point \( p_1 \in C_1 \setminus C_2 \). Then the strong local folding condition for \( W \) can be expressed as

\[
\begin{vmatrix}
1 & \ldots & 1 \\
p_{b_1} & \ldots & p_{b_{d+1}} \\
w_{b_1} & \ldots & w_{b_{d+1}}
\end{vmatrix}
= \begin{vmatrix}
1 & \ldots & 1 & 1 \\
p_{b_1} & \ldots & p_{b_{d+1}} & p_1 \\
w_{b_1} & \ldots & w_{b_{d+1}} & w_1
\end{vmatrix} > 0
\]

Developing the last row of the determinant becomes clear that this condition can be stated as a linear homogeneous strict inequality in the heights of the lifted points. Therefore, the local strong folding conditions for all the walls of \( S \) define an open cone in the subspace determined by the coplanarity conditions.

We call the system of equations and inequalities obtained from these conditions the regularity system of \( S \). We call the (maybe empty) set of solutions of this system the regularity cone. If we use weak inequalities instead of strict ones, we get the weak regularity system, whose (always non-empty) set of solution is called weak regularity cone.
Note that most of the notions and results presented here can be easily generalized to the case where the initial object \( A \) is not a set of points but a set of vectors. Instead of polyhedral subdivisions we obtain polyhedral fans whose 1-faces are rays with directions taken from \( A \). We use the term configuration when we do not want to distinguish between a point set or a vector set. More details on these definitions and properties can be found in [9].

### 2.2 Systems of homogeneous linear inequalities

In this section, we proof a basic result of this article. It is a general result on linear algebra that will be used in the following sections.

Consider a set of \( m \) linear homogeneous inequalities on \( n \) variables

\[
\begin{align*}
  & s_1 \cdot x > 0 \\
  & \vdots \\
  & s_m \cdot x > 0 
\end{align*}
\]

where \( x = (x_1, \ldots, x_n) \) is an \( n \) dimensional vector representing the values of the variables. Assume that this system is incompatible. We want to know if we can “relax” the system in a way that it has a non-zero solution. More precisely, we want to replace the strict inequality (\( > \)) by a weak inequality (\( \geq \)) for a minimal set of constraints. However, we consider instead the substitution of “\( > \)” by “\( = \)” for reasons that will become clear later.

The matrix of the system (1) is

\[
M = \begin{pmatrix}
  s_1 \\
  \vdots \\
  s_m
\end{pmatrix}
\]

where \( s_i \) denote here row vectors.

For our purposes it is easier to argue in terms of what we will call the dual system, whose matrix is \( M^\top \), related to the original one by the next lemma.

**Lemma 2.1 (Gordan’s Theorem).** Exactly one of the following systems has a solution:

\[
\begin{align*}
  & M x > 0 \\
  & x \in \mathbb{R}^n \\
  & M^\top y = 0 \\
  & y \in \mathbb{R}^m \\
  & y \geq 0, y \neq 0
\end{align*}
\]

This result is an special case of Farka’s lemma and can be read in the following way. There is no solution for the original system if, and only if, it exists a non-zero positive linear combination \( y_0 \) of its inequalities leading to the contradiction \( 0 > 0 \). This combination \( y_0 \) is precisely a solution to the dual system. Note that imposing equality in one of the inequalities in the primal problem corresponds to unrestricting the associated variable in the dual, i.e., allowing it to be negative. This makes sense also in terms of the contradiction construction: equations can be added or subtracted to a strict inequality and the relation (the sign \( > \)) will be preserved, while strict inequalities can be only added to ensure that the relation does not change.

### 3 Minimal relaxation of a system

In this section, we introduce a notion of relaxation for a (non-compatible) system of strict homogeneous linear inequalities, which will be shown to be useful for the development in the next sections. More precisely, we say that a system is a relaxation of another if it has the same matrix but some of the strict inequalities has been replaced by equations. It is not clear, a priori, that there exists a minimal relaxation. Indeed, it is not obvious that the process is actually a relaxation, i.e., that the set of solutions is bigger when a strict
inequality is replaced by the corresponding equation. Although this is in general not the case, we will show later that it is true when the original system is non-compatible. Formally, we show that the minimal set of strict inequalities we need to convert into equations in order to obtain a compatible system is well defined. This contrasts with the more studied notion of relaxation consisting on “forgetting” some of the inequalities, in which case the minimal relaxation is not well defined.

An intuition on why the the minimal relaxation is unique in our case can be easily obtained if one looks at the complementary problem. That is, given a system of weak homogeneous linear inequalities, decide which is the maximum number of constraints that can be satisfied strictly. The set of solutions of the system is a closed polyhedral cone \( K \). It is not hard to see that if \( x_0 \in \text{rel \, int \, } K \), then the desired maximal set of constraints is exactly those strictly satisfied by \( x_0 \). That is, finding this minimal relaxation is equivalent to find a point in the relative interior of a polyhedral cone. Although stated in this second way this result seems more obvious, we will prove its first version directly, since the intuitions derived from the arguments will be afterwards used in the correctness proofs of the algorithms in Section 7. Moreover, this approach reflects the notion of incompatibility cycles, whose counterpart in the regularity system is of interest, since they hint a possible simplification of the algorithms once the special form of the regularity matrices is observed.

Let \( S \) denote the system

\[
S : \begin{cases}
s_1 \cdot x > 0 \\
\vdots \\
s_m \cdot x > 0.
\end{cases}
\]

Given a subset \( E \subset [m] = \{1, ..., m\} \), \( S(E) \) will denote the system

\[
S(E) : \begin{cases}
s_i \cdot x = 0 \quad \forall i \in E \\
s_j \cdot x > 0 \quad \forall j \in [m] \setminus E
\end{cases}
\]

and will be called the system \( S \) relaxed by \( E \).

From now on, \( S \) will be considered fixed. \( S(E) \) will be eventually referred to as the “relaxed system” if \( E \) is clear by the context.

The main aim of this section is to proof the following theorem.

**Theorem 3.1.** \( S(\hat{E}) \) is compatible, where \( \hat{E} := \bigcap_{E \subset [m] \text{ compatible}} E \).

We will prove first the following lemma, which translates Gordan’s Theorem to the case where also linear homogeneous equations are included in the system. Using it, it is easy to derive Theorem 3.1.

**Lemma 3.2** (Extension of Gordan’s Theorem).
Given \( E \subset [m] \), exactly one of the following systems has a solution:

\[
P : \begin{cases}
s_i \cdot x = 0 \quad \forall i \in E \\
s_j \cdot x > 0 \quad \forall j \in [m] \setminus E \\
x \in \mathbb{R}^n
\end{cases}
\]

\[
D : \begin{cases}
M^T y = 0 \\
y \in \mathbb{R}^m \\
y^j \geq 0 \forall j \in [m] \setminus E \\
\exists k \in [m] \setminus E \text{ such that } y^k > 0
\end{cases}
\]

where \( M = \begin{pmatrix} s_1 & \cdots & s_m \end{pmatrix} \).

**Proof.** Observe that the set of equations in the first system \( P \) (for primal) restricts the possible solutions to an \( l \)-dimensional subspace of \( \mathbb{R}^n \) and the set of solutions is a relatively open cone in that subspace. We can assume without loss of generality, that this subspace is the one obtained by setting the first \( l < |E| \)
coordinates of $x$ to 0. Then, we can consider a new set of inequalities, which also have the first $l$ coefficients set to 0. In other words, we assume

\[
M = \begin{pmatrix}
\vdots & \vdots \\
\cdots & \cdots \\
\vdots & \vdots \\
\end{pmatrix} = \begin{pmatrix}
\text{Id}_{l \times l} & 0_{l \times (n-l)} \\
0_{(m-l) \times l} & N^{\top} \\
\end{pmatrix}
\]

where $N$ is a $(m-l) \times (n-l)$ matrix.

Note that removing redundant equations preserves the set of solutions of $P$. In the dual system, this corresponds to reducing the dimensionality of the space: a redundant equation in the primal obviously correspond to a column in the matrix of the dual, which is linearly dependent with the others. However, this transformation is harmless, since any solution of the original problem can be converted into a solution of the modified one by using the linear relations between the remaining columns and the removed ones.

Now that the system is in this canonical form, we claim some “equivalences” between systems, that should be obvious.

**Claim:** The following systems have both a solution or none has.

\[
P : \begin{cases}
s_i \cdot x &= 0 \quad \forall i \in E \\
s_j \cdot x &> 0 \quad \forall j \in [m] \setminus E \\
x &\in \mathbb{R}^n
\end{cases}
\]

\[
P' : \begin{cases}
N y &> 0 \\
y &\in \mathbb{R}^{n-l}
\end{cases}
\]

On the other hand, we consider the second system $D$ (for dual) associated to $P$ after applying the previous assumptions.

Its matrix is

\[
M^{\top} = \begin{pmatrix}
\text{Id}_{l \times l} & 0_{l \times (m-l)} \\
0_{(m-l) \times l} & N^{\top} \\
\end{pmatrix}
\]

and it is not hard to see that the following claim is true.

**Claim:** The following systems have both a solution or none has.

\[
D : \begin{cases}
M^{\top} y &= 0 \\
y &\in \mathbb{R}^m \\
y^j &\geq 0 \quad \forall j \in [m] \setminus [l] \\
\exists k \in [m] \setminus [l] \text{ such that } y^k > 0
\end{cases}
\]

\[
D' : \begin{cases}
N^{\top} y &= 0 \\
y &\in \mathbb{R}^{m-l} \\
y &\geq 0, y \neq 0
\end{cases}
\]

We can apply Gordan’s Theorem to relate the systems $P'$ and $D'$ and derive finally the desired statement.

\[\square\]

An intuitive explanation of the previous result, in comparison with Gordan’s Theorem, can be read in terms of contradictions in the system. Assume that a system without equations $P$ is not compatible. A solution $y$ of $D$ represents a positive linear combination of the rows of $M$ leading to the contradiction $0 > 0$. If all the active inequalities are included in $E$, then the same combination will lead to the equation $0 = 0$ in the dual of the relaxed system and the contradiction will be avoided. However, the equations we created in the relaxed system can be now used to derive other contradictions, that is, they confer more freedom to the dual system. This can be deduced from the fact that the corresponding dual variables are now allowed to take negative values. The corresponding intuition is that, given an strict inequality, one can only add another inequality if the relation $>$ is to be maintained. However, equations can be either added or subtracted and the relation will be preserved. Note also that if we exclude from $E$ one of the inequalities which is active in the contradiction, the same rows may be combined with it and construct again the contradiction. In other words, if the system contains a contradiction, all the involved inequalities must be relaxed in order to obtain a compatible system. This intuition is developed below in the proof of the theorem.

**Proof.** (Theorem 3.1) If $S$ is compatible, $\hat{E} = \emptyset$ and nothing leaves to be proven. Otherwise, note that $S([m])$ is always compatible, since it is an homogeneous system of linear equations and, therefore, has at least 0 as a solution. This ensures that the intersection has at least one operand. Since $S$ is not compatible,
Lemma 2.1 ensures that there exists a solution \( y_0 \) to the dual system. Let \( E_0 \neq \emptyset \) be the set of positive coordinates of \( y_0 \). We claim that if \( S(E) \) is compatible, then \( E_0 \subset E \). If we assume the contrary, then \( y_0 \) is also a solution of the dual of \( S(E) \) in the sense of Lemma 3.2 and, consequently, \( S(E) \) is not compatible, a contradiction. It follows that any \( E \) making \( S(E) \) compatible should contain \( E_0 \) and, hence, we can focus now on the system \( S(E_0) \). If it is compatible, then obviously \( \hat{E} = E_0 \). Otherwise, Lemma 3.2 ensures the existence of \( y_1 \), a solution of its dual system. If \( E_1 \) is the set of positive coordinates of \( y_1 \), then it has to be \( E_1 \supset E_0 \). In addition, \( \hat{E} \supset E_1 \), since the argument of the previous paragraph applies again.

The iteration of the process leads to an \( E_l \) with \( S(E_l) \) compatible, since \( |E_k \setminus E_{k-1}| \geq 1 \) and \( S([m]) \) is compatible. Then \( \hat{E} = E_l \) and, by construction, \( S(\hat{E}) \) is compatible.

The previous theorem motivates the next definition

**Definition 3.3.** Given a system \( S \), its *minimal relaxation set* is

\[
\hat{E}_S = \bigcap_{E \supset [m]} \bigcap_{E \subset \mathbb{Z}^{|m|}} S(E)
\]

The following proposition justifies the way we have chosen the definition of relaxation of a system. Let \( S^\geq(E) \) denote the system

\[
S^\geq(E) : \begin{cases}
s_i : x &= 0 \quad \forall i \in E \\
 s_j : x &\geq 0 \quad \forall j \in [m] \setminus E.
\end{cases}
\]

**Proposition 3.4.** If \( S \) is incompatible then \( S^\geq(\hat{E}_S) \) and \( S(\hat{E}_S) \) has the same set of solutions.

**Proof.** It is clear that the set of solutions of \( S^\geq(\hat{E}_S) \) is contained in the set of solutions of \( S(\hat{E}_S) \). Assume \( x_0 \) is a solution of \( S^\geq(\hat{E}_S) \) and is not a solution of \( S(\hat{E}_S) \). This means that some of the inequalities in \( \hat{E}_S \) are strictly satisfied by \( x_0 \). If \( E_0 \neq \emptyset \) is the set of such inequalities, the \( x_0 \) is a solution of \( S(\hat{E}_S \setminus E_0) \) and, hence, \( S(\hat{E}_S \setminus E_0) \) is compatible. This contradicts the definition of \( \hat{E}_S \).

The results of this section can be adapted in order to apply also when dealing with non-homogeneous inequalities. The main difference in this case is that \( S([m]) \) is not necessarily compatible and, therefore, there may be no possible relaxation at all. Nevertheless, whenever it exists a compatible relaxed system, the minimal relaxation is well defined and can be computed in the same way as in the homogeneous case.

### 4 The finest regular coarsening

The immediate application for which we needed the algebraic results in the previous section is the definition of the finest regular coarsening of a subdivision.

A *coarsening* of a subdivision \( S \) is a subdivision \( S' \) such that every face \( f \in S \) is contained in some face \( \kappa(f) \in S' \). The function \( \kappa : S \to S' \) is called the *coarsening function* from \( S \) to \( S' \). Given two coarsenings \( S_1 \) and \( S_2 \) of a subdivision \( S \), we say that \( S_1 \) is finer than \( S_2 \) if the second is a coarsening of the first. A coarsening \( \kappa \) of a subdivision \( S \) is **proper** if it has strictly less cells than the original subdivision. The **trivial coarsening** is the one merging all the cells into a single one. This relation induces a partial order on the set of coarsenings of a subdivision. The restriction of this partial order to regular subdivisions is indeed a lattice. Moreover, this lattice is isomorphic to the lattice of faces of the secondary polytope of the point set \([13]\). However, as far as we know, not much work has been done concerning coarsenings of non-regular subdivisions. A reason for that could be that the existence of a finest regular coarsening, to be defined later in this section, was not clear. The aim of this section is to prove that this is the case and derive some consequences of this fact. In particular, we show that all the regular coarsenings of a subdivision are coarsenings of its finest regular coarsening. As a consequence, we can match every non-regular subdivision of a configuration to a unique regular one which is, in a specific sense, the most similar to it.

We introduce next the procedural definition of the object of this Section.

**Definition 4.1.** The *finest regular coarsening* of \( S \) is the subdivision \( S_0 \) obtained by the following procedure:
1. Consider the $n$-dimensional regularity system of $S$ with $m$ constraints, as defined in Section 2.1 and name it $S(E)$, where $E \subset [m]$ is the index set corresponding to the equations of the coplanarity conditions. $E$ will be empty if $S$ is a triangulation.

2. Compute the minimal relaxation set of $S(E)$ and name it $\hat{E}$.

3. For each $i \in \hat{E} \setminus E$, consider the wall $W$ associated to $i$ and merge the two cells sharing it.

The next theorem proves that this definition is actually meaningful.

**Theorem 4.2.**

(i) $S_0$ is a regular coarsening of $S$

(ii) All the regular coarsenings of $S$ are coarsenings of $S_0$

**Proof.** (i) The key observation is that relaxing a constraint in the regularity system of $S$ corresponds to requiring the two cells sharing this wall to be lifted flat into a hyperplane. Therefore, if there exists a solution to the relaxed system, the subdivision associated to it will be the same as the original, except for the fact that the cells sharing the “flattened” wall will be merged. Since there is always a minimum relaxation set, the algorithm will end with a regular polyhedral subdivision: the projection of the lower convex hull of the lifted points. It will be a coarsening of the original because it respects the initial coplanarity equations.

(ii) Translating the Theorem 3.1 to the regularity setting, we can argue that none of the walls which are flattened to obtain $S_0$ can appear as a non-flattened wall in any regular coarsening of $S$. Then, any regular coarsening can be obtained by coarsening $S_0$. $\square$

It will come in handy later to say that a subdivision is completely non-regular if its finest regular coarsening is its trivial coarsening. This implies, in particular, that every wall of the subdivision can appear in a contradiction cycle.

Once we know that the finest regular coarsening is well defined for any subdivision, it makes sense trying to find a less algorithmic way to define it. Regular subdivisions were first studied by Gelfand, Kapranov & Zelevinsky [13], who introduced the secondary fan and the secondary polytope. These objects encode the combinatorics of the refinement poset of regular subdivision of a point set. In addition, hidden in their construction we find a nice insight of our problem for the special case of triangulations. With a little more work, it can be shown that, indeed, the secondary polytope encodes all the information to relate an arbitrary polyhedral subdivision to its finest regular coarsening. However, computing the secondary polytope can be algorithmically involved. Our approach, developed in this section, shows that it is not necessary to construct the secondary polytope to find the finest regular coarsening and leads to an efficient algorithm (see Section 7) to do it. For completeness, we summarize the results of [13] and extend them to arbitrary subdivisions along this section.

Gelfand, Kapranov & Zelevinsky introduced the weak regularity cone of a regular triangulation $T$ (as defined in 2.1), i.e. the height functions for which the convex hull of the lifted point set projects onto $T$ or one of its regular coarsenings. They showed that the set of such cones for all regular triangulations define a complete polyhedral fan (called the secondary fan of $A$). Its cells are in correspondence with the regular triangulations and their faces correspond to their regular coarsenings.

They define the $GKZ$-vector of a triangulation $T$ of a point set $A$ as:

$$\alpha_T = \sum_{a_i \in A} \sum_{\sigma \ni a_i} \text{Vol}(\sigma)e_i,$$

where the summation can be taken to be only over maximal simplices of $T$. They name the convex hull $\Sigma(A)$ of all the vectors $\alpha_T$ the secondary polytope of the point configuration $A$.

Their main result is that the normal fan of $\Sigma(A)$ is precisely the secondary fan of $A$. Therefore, vertices of $\Sigma(A)$ correspond to regular triangulations, edges correspond to “flips” between regular triangulations and the
whole polytope corresponds to the trivial subdivision. This proves, in particular, that regular triangulations are connected in the "graph of flips".

Although it seems that in this setting there is no place for non-regular subdivisions, a closer look at the construction reveals that the $GKZ$-vector are defined for any triangulation, regular or not. Non-regular triangulations have $GKZ$-vectors which are not vertices of $\Sigma(A)$. Observe tha for non-regular coarsenings of a triangulation $T$ corresponds to the subdivision associated to the minimal face in $\Sigma(A)$ containing $\alpha_T$.

It is not explicitly stated in [13] but possibly known by the authors that the weak regularity cone can be similarly defined for non-simplicial subdivisions. In such a case, this cone will be contained in a linear subspace of the height functions space defined by the coplanarity conditions. If we assume that the dimension of this subspace is $k < n$, the subdivision is regular if its weak regularity cone is also $k$-dimensional. Then, any height function in its relative interior will produce the subdivision. If the subdivision is not regular, this cone will have empty relative interior even with respect to the "coplanarity subspace". But a point contained in its relative interior will correspond to the finest regular coarsening of the subdivision. This approach would have certainly lead to a simpler proof for the results in Section 4, but would have hidden some insight into the specific nature of out problem.

5 The regularity tree and recursive regularity

In this section we define a partial order on the power set of cells of a given subdivision $S$, which will be called the regularity tree. This object provides some structure to the set of non-regular subdivisions of a given point set. The definition of a class of polyhedral subdivisions which, as will be shown in the next section, have some applications in different areas motivates the study of this tree. We will also justify the generality of this class and approach the way to identify its elements.

**Definition 5.1.** A polyhedral subdivision $S$ is recursively regular if:

1. $S$ is regular or
2. There exists a proper non-trivial coarsening $S'$ of $S$ with coarsening function $\kappa$ such that:
   - $S'$ is a regular subdivision and
   - For each cell $C \in S'$, $\kappa^{-1}(C)$ is recursively regular.

This definition can be extended to polyhedral fans, as the other results presented in this article. $R(A)$ will denote the set of recursively-regular subdivisions of a point configuration $A$. The class of all recursively-regular subdivisions of any point set will be denoted by $R$.

One could, at a first glance, think that $R$ contains only regular subdivisions. Another naive observation would be that only one level of recursion is needed in the definition to capture $R$. We will disprove both statements exhibiting some 2-dimensional examples. Later, we will give a heavier argument showing that $R$ includes some pathological subdivisions, whose properties are very different from the regular ones.

It is convenient to denote by $2^{\text{cells}(S)}$ the power set of the cells of $S$. Given $C = \{C_1, ..., C_l\} \in 2^{\text{cells}(S)}$, we denote $|C| = \bigcup_{C_i \in C} C_i$. A subdivision tree in $2^{\text{cells}(S)}$ is an acyclic directed and connected graph whose vertices are elements of $2^{\text{cells}(S)}$ and such that if $C$ is connected to $C_1, ..., C_l$, then $|C_1|, ..., |C_l|$ are the cells of a polyhedral subdivision of $A \cap |C|$. In addition, the root (the unique element with in-degree 0) of the subdivision tree will be required to be cells$(S)$.

**Definition 5.2.** The regularity tree of the subdivision $S$ is the subdivision tree created by the following recursion:

1. The regularity tree of a regular subdivision is the union of all its cells.
2. The regularity tree of a non-regular subdivision with trivial finest regular coarsening is the union of all its cells.

3. The regularity tree of a non-regular subdivision \( S \) with a non-trivial finest regular coarsening \( S_0 \) is obtained by appending to its trivial coarsening the regularity tree of \( \kappa^{-1}(C) \) for each cell \( C \in S_0 \).

Figure 1 shows an example of a regularity tree. Note that the leaves of the regularity tree of \( S \) are a partition of cells(\( S \)). The subdivision induced by \( S \) onto each of the leaves is either regular or completely non-regular. We will abuse notation and say that the leaves are regular or completely non-regular if the subdivision induced by \( S \) on them is.

The basic result of this section is the following theorem, which relates the regularity tree and the recursive regularity of a subdivision.

**Theorem 5.3.** A polyhedral subdivision \( S \) is in \( \mathcal{R} \) if and only if the leaves of its regularity tree are regular.

**Proof.** If all the leaves are regular, the regularity tree itself certifies the belonging of \( S \) to \( \mathcal{R} \), proving the if direction.

For the only if, it will be proved that the leaves of the regularity tree of any subdivision in \( \mathcal{R} \) are regular. Let \( S \) be in \( \mathcal{R} \). Then, it has a regular coarsening \( \bar{S} \) with coarsening function \( \bar{\kappa} \) splitting \( S \) into smaller recursively-regular subdivisions. Indeed, by definition, there is a subdivision tree in \( 2^{\text{cells}(S)} \) representing the set of coarsenings certifying its belonging to \( \mathcal{R} \). We want to show that the regularity tree is also a valid certificate. The second part of Theorem 4.2 affirms that \( \bar{S} \) is a coarsening of the finest regular coarsening \( S_0 \) of \( S \). This implies that each cell \( C \in S_0 \) is contained in some cell \( C' \in \bar{S} \), being \( \bar{\kappa}^{-1}(C') \) recursively regular. But regularity (and, hence, recursive regularity) is behaved well with respect to restrictions to convex regions, i.e., \( \kappa^{-1}(C') \subset \bar{\kappa}^{-1}(C) \) is also recursively regular.

We present now some basic observations on \( \mathcal{R} \). Here, a subdivision \( S \) is said to be acyclic if there exists no point \( q \in \mathbb{R}^d \) such that the visibility relation defined on the faces of \( S \) from \( q \) contains a cycle. The point of view \( q \) can also be considered to be a point at infinity. For more details, see [10], where this definition is introduced and it is proved that any regular subdivision is acyclic.

**Proposition 5.4.**

(i) \( \mathcal{R}(A) \) is a superset of the regular subdivisions of \( A \)

(ii) \( \mathcal{R}(A) \) is a subset of the acyclic subdivisions of \( A \)

(iii) The previous inclusions can be proper

**Proof.**

(i) By definition, \( \mathcal{R} \) contains the regular subdivisions.
(ii) First we will proof that any $S$ in $\mathcal{R}$ must be acyclic. We will do it by induction in the number of cells. For the base case, we use that a single-cell subdivision is always acyclic. If $S$ has more than one cell, we distinguish two cases. If $S$ itself is regular, the Acyclicity Theorem from [10] shows that it must be acyclic. Otherwise, there exists a regular coarsening $S'$ of $S$. Assume for the sake of contradiction that $S$ contains a cycle and consider the image by $\kappa$ of the involved faces. If this image contains more than one cell, the cycle induces another one in $S'$ and, therefore, it leads to a contradiction with its assumed regularity. So the cycle must be contained in $\kappa^{-1}(C)$ for a single $C \in S'$. But $\kappa^{-1}(C)$ is a subdivision with strictly less cells than $S$, so it has no cycle by the induction hypothesis.

(iii) Figure 2(a) shows a non-regular triangulation that belongs to $\mathcal{R}$. Another set of triangulations in dimension five will be presented later, which also proves that the inclusion is proper.

For the properness of the second inclusion, see the example shown in Figure 3, which shows an acyclic non-regular subdivision whose non-trivial coarsenings are all cyclic (hence not regular).

In Figure 1 we show a triangulation belonging to $\mathcal{R}$ which needs two levels of recursion to fit the definition. Using the previous proposition we can argue that any regular coarsening of the subdivision needs to merge the cells showed in Figure 1 into a single one. In the second level, when trying to decompose this cell, the coarsening cannot be finer than the one showed in this Figure and, therefore, one more level is needed to fit the definition.

The next proposition illustrates that $\mathcal{R}$ includes some pathological triangulations. To proof so, we will simply show that the non-regular triangulations used by Santos in [19] are indeed in $\mathcal{R}$.

**Proposition 5.5.** There exists a point set $A \subset \mathbb{R}^5$, $|A| = 50$ and a subdivision $S \in \mathcal{R}(A)$ such that $S$ has no flips in the graph of triangulations of $A$. 

Proof. The construction used in [19] is a triangulation of a 5-dimensional prism. We will show that this
triangulation is in $R$.

The construction of the base of the prism starts from the 24-cell. This is a 4-polytope with 24 facets,
which are regular octahedra. Then, the central subdivision of this polytope is considered, which is regular
(see [9], Section 9.5) and consists of 24 pyramids over octahedra. To proof that the base subdivision is
in $R$, we need to proof that each of the pieces is triangulated in a regular way. And they are, since the
triangulation of a pyramid is regular if and only if the triangulation induced on its base is regular (see [9],
Section 4.2) and the bases are regular octahedra and, hence, have only regular triangulations.

Then, the prism of this subdivision is considered (in the sense of definition 4.2.10 in [9]). The prism
over each simplex is triangulated in a specific way, which we do not need to take into account because any
triangulation of a prism over a simplex is regular ([9], Section 6.2). This ends the proof, since merging the
5-simplexes in each of these prisms we get the prism over a recursively-regular triangulation. Then merging
the prisms over the simplexes in each octahedra we get the prism subdivision over the central subdivision
of the 24-cell. This prism subdivision is regular, since we proved that the base is a regular subdivision.

The previous proposition shows indeed that there is a point set $A$ with at least 12 triangulations in $R(A)$
which are pairwise disconnected and disconnected to any regular triangulation in the graph of flips of $A$.

6 Applications

In this section we expose some applications of the theoretical results introduced in the previous sections.

6.1 Spider webs

In this subsection, we present an application of the concept of finest regular coarsening of a subdivision in
$R^2$. This is related to tensegrity theory (see, e.g, [18]), which studies the rigidity of frameworks made
of cables (which support only positive stresses), struts (supporting only negative stresses) and bars (supporting
any stress). We are interested on frameworks made uniquely of cables. Such frameworks are often called
spider webs, for obvious reasons. However, understanding the general case is needed to show the relation of
this constructions with our results.

Formally, an abstract framework $G = (V; B, C, S)$ is a graph on the vertex set $V = \{v_1, \ldots, v_n\}$, where a
partition of the edge set $E = B \cup C \cup S$ is given. A tensegrity framework in $R^d$ is an abstract framework
together with an embedding of vertices $p : V \to R^d, v_i \mapsto p(v_i) = p_i$. It will be denoted $G(p)$ and $p$ will be
thought as a point $(p_1, \ldots, p_n) \in R^{dn}$. We can consider the configuration space of $G(p)$ to be

$$X(p) = \{x = (x_1, \ldots, x_n) \in R^{nd} : \|x_i - x_j\| = \|p_i - p_j\| \forall i, j \in B;$$
$$\|x_i - x_j\| \leq \|p_i - p_j\| \forall i, j \in C;$$
$$\|x_i - x_j\| \geq \|p_i - p_j\| \forall i, j \in S\}.$$

That is, the set of embeddings of the framework preserving the length of the bars, making the lengths
of the cables no longer and the lengths of the struts no shorter than their original lengths. A tensegrity
framework $G(p)$ is rigid in $R^d$ if there exists an open set $p \in U \subset R^{dn}$ such that $X(p) \cap U = M(p) \cap U$,
where

$$M(p) = \{x = (x_1, \ldots, x_n) \in R^{nd} : \|x_i - x_j\| = \|p_i - p_j\|, 1 \leq i, j \leq n\}$$
is the manifold of rigid motions associated to $p$. In other words, the only rigid motions of the tensegrity are
the trivial ones (the Euclidean group).

Since the study of the quadratic constraints in the definition of $X(p)$ can be tricky, the notion of infinitesimal
rigidity was introduced. Intuitively, this captures the rigidity constraints up to first order. Differentiating
the constraints, a linear system of equations and inequalities is obtained and, then, dimension arguments can
be applied to decide if a framework is infinitesimally rigid (the solutions of the system correspond only to
differentials of trivial motions). It is easy to see that infinitesimal rigidity implies rigidity, but the converse is in general not true.

Due to the Maxwell-Cremona relation (studied, for instance, in [8]), we know that there is a bijection between the equilibrium stresses of a non-crossing framework in the plane and its liftings to a polyhedral surface in $\mathbb{R}^3$. In this correspondence, edges with positive stress are lifted to “mountains” while edges with negative stress become “valleys”. Edges with no stress are lifted to flat edges.

Consider a non-crossing framework in the plane $F$ constructed using only cables, i.e., the edges support only positive stresses. In addition, assume that the nodes which are vertices of the convex hull are pinned down (they are, therefore, in equilibrium by definition). This kind of frameworks are also known as spider webs for obvious reasons.

The two following results further connect the recursive regularity concepts to the rigidity of spider webs.

**Lemma 6.1** (Connelly [7]).
*If a spider web has a stress which is positive on every edge and each vertex not in the convex hull is in equilibrium, then it is rigid.*

**Lemma 6.2** (Roth, Whiteley [18]).
*If a tensegrity framework is infinitesimally rigid, then it has an equilibrium stress that is nonzero on all struts and cables.*

From this classical results and the Maxwell-Cremona correspondence introduced before, it is easy to derive the following proposition.

**Proposition 6.3.** Given a planar spider web with associated subdivision $S$.

i The cables corresponding to edges omitted in the finest regular coarsening of $S$ support no stress in any equilibrium stress.

ii If $S$ is recursively regular, the spider web is rigid.

**Proof.** Given a planar spider web with associated subdivision $S$.

i Since we showed that the edges omitted in the finest regular coarsening are lifted flat by any convex lifting, the Maxwell-Cremona correspondence indicates that the corresponding cables will receive no stress in any equilibrium which is non-negative on every cable.

ii The finest regular coarsening of the subdivision corresponds to a set of cables which can be simultaneously assigned strictly positive stresses. Therefore, the spider web they define is rigid, since the vertices on the convex hull are considered fixed. For each of the subdivsion define by the finest regular coarsening, we can assume that the vertices in their convex hull are now fixed and apply the same argument recursively.

Figure 6.1 illustrates the result, displaying two cable frameworks. Dashed edges are not included in the finest regular coarsening and, therefore, they wont support a positive stress in any equilibrium stress.

Moreover, the finest regular coarsening relates to the concept of infinitesimal rigidity in the following way.

**Corollary 6.4.** If the finest regular coarsening of the polyhedral subdivision associated to a framework made of cables omits no edges, the framework is not infinitesimally rigid.

**Proof.** It is know [18] that if a framework is infinitesimally rigid, there exists a proper stress of it. In the case of spider webs, this means that there is an equilibrium stress that is positive on every edge. This implies that none of the edges can be omitted in the finest regular coarsening, since otherwise they would receive no stress in any equilibrium stress.
6.2 Space decompositions

Aurenhammer, Hoffmann and Aronov [2] introduced in 1998 a way to partition a point set (or a continuous non-vanishing measure in \([0,1]^d\)) in \(\mathbb{R}^d\) by a weighted power diagram of a given site set. A power diagram of a site set \(\{q_1, \ldots, q_n\} \subset \mathbb{R}^d\) with assigned weights \(q_i \mapsto w_i \in \mathbb{R}\) is the partition of \(\mathbb{R}^d\) according to the function

\[
R : \mathbb{R}^d \to \{1, \ldots, n\}
\]

\[
R(x) = \arg\min_{i \in \{1, \ldots, n\}} \{\|x - q_i\|^2 - w_i^2\}.
\]

The following lemma compiles the relevant parts from their work for our study.

**Proposition 6.5.** (i) Given points \(P = \{p_1, \ldots, p_n\} \subset \mathbb{R}^d\), sites \(q_1, \ldots, q_m \in \mathbb{R}^d\), and a natural \(n_i\) for each site such that \(\sum n_i = n\), there exist weights \(w_i\) for the sites such that the corresponding power diagram partitions \(P\) in a way that the region corresponding to \(q_i\) contains exactly \(n_i\) points of \(P\).

(ii) Similarly, given a non-negative real \(v_i \in \mathbb{R}\) for each site such that \(\sum v_i = 1\), there is a weight assignment such that the power diagram region partitions the unit cube (or any other polytope of volume one) in a way that the region corresponding to \(q_i\) has volume \(v_i\).

This partition result is an example of a theorem that uses regularity and that can be adapted to work in a recursive way. That is, one could first partition an area with a power diagram and, afterwards, partition the area in one of its cells using a new power diagram. This observation has been exploited to construct the so called Voronoi tree-maps [3, 15]. These approaches, however, do not consider this hierarchies as a single diagram. We want to do so, in order to relate them to recursive regularity. To this end, we consider first the following well known lemma. We reproduce a simple proof of this fact which we think provides some insight.

**Lemma 6.6** (appearing, e.g., in [16]). *Regular subdivisions are exactly power diagrams.*

**Proof.** Power diagrams are regular by definition. To see the converse, one only needs to take the polyhedron which projects onto the subdivision, and regard the hyperplanes supporting its faces as functions \(f_i : \mathbb{R}^d \to \mathbb{R}^{d+1}\),

\[
f_i(x) = q_i \cdot x + b_i = -\left(\|x - q_i\|^2 - \|x\|^2 - \|q_i\|^2\right)/2 + b_i.
\]
Figure 5: Hierarchy of power diagram partitioning a polygon

Note that, without loss of generality, we can assume $b_i > 0$, for all $i \in \{1, ..., n\}$. The minimization diagram of these functions is the same as the maximization diagram of

$$f_i(x) = \|x - q_i\|^2 - 2b_i - \|q_i\|^2,$$

which can be thought as the power of $x$ with respect the circle centred at $q_i$ with radius $\sqrt{2b_i + \|q_i\|^2}$.

We can, then, assign to any regular subdivision $S$ a set of sites $\text{sites}(S)$ for which it is a power diagram. Although this set of sites is in general not unique, we introduce the following definition, which inherits its indetermination.

**Definition 6.7.** A hierarchy of power diagrams of a subdivision tree of a recursively-regular subdivision $S$ is constructed according to the next procedure:

1. Partition $|S|$ using a power diagram of $\text{sites}(S_0)$, where $S_0$ is the subdivision associated to the root of the tree.
2. Recursively partition each of the created cells using a hierarchy of power diagrams of the corresponding children, if any.

We are now in conditions to state our decomposition result.

**Proposition 6.8.** Given a recursively regular subdivision $S$ in $\mathbb{R}^d$, a polyhedron $P \subset \mathbb{R}^d$ and a function $v : \text{cells}(S) \to \mathbb{R}^+$ such that $\sum_{C \in \text{cells}(S)} v(C) = \text{Vol}(P)$, a hierarchy of power diagrams of $S$ can be found such that it partitions $P$ in a way that the region corresponding to a cell $C$ has volume $v(C)$. The discrete counterpart (analogous to Proposition 6.5(ii)) applies as well.

**Proof.** We can use the recursively-regular structure of $S$ in combination with Lemma 6.6 in order to recursively apply the partition result of Proposition 6.5 using a hierarchy of power diagrams of the regularity tree of $S$, leaving exactly the required volume in each region. In order to do so, we first partition the point set using a power diagram of the finest regular coarsening of $S$ in a way that that each region has volume equal to the sum of the volumes that have been prescribed for the regions it contains and repeat this procedure recursively. The discrete analogous follows immediately.

Figure 5 shows an example of such a partition, where the dotted edges correspond to the second level of the hierarchy.

**6.3 Covering by floodlights in high dimensions**

We introduce here a result on visibility in higher dimensions which uses the concept of recursively-regular polyhedral fans. Definitions and basic properties concerning polyhedral fans, also called subdivisions of vector configurations, can be found in [9]. The ground set of a polyhedral fan $\mathcal{F}$ is the union of all its cells and is denote by $|\mathcal{F}|$. We say that a polyhedral fan is complete if its ground set is the whole space and...
that it is conic if the ground set is a pointed (convex) cone. For a set $B \subset \mathbb{R}^d$, we define its reversed set as $B^- = \{x \in \mathbb{R}^d : -x \in B\}$. The reverse fan of a polyhedral fan $\mathcal{F}$ is the fan obtained by reversing all its faces.

Let $\mathcal{F}$ be a $d$-dimensional complete polyhedral fan with $n$ cells and $P$ a set of $n$ points in $\mathbb{R}^d$. From now on, $\sigma : \text{cells}(\mathcal{F}) \to P$ will be assumed to be a one-to-one assignment of cells to points.

**Definition 6.9.** $\sigma$ is a covering assignment if

$$ \bigcup_{i=1..n} (C_i + \sigma(C_i)) \supset |\mathcal{F}|. $$

Let $f \in \mathcal{F}$ be a $(d-1)$-face incident with the cells $C_1$ and $C_2$ and $v_{12}$ be a vector normal to $f$ pointing from $C_1$ to $C_2$. We say that $\sigma$ satisfies the overlapping condition for $f$ if $(\sigma(C_1) - \sigma(C_2)) \cdot v_{12} \geq 0$, i.e., the translated cells overlap.

It is proved in [11] that a covering assignment can be found if the fan is complete and regular, regardless of the given point set. In particular, the problem in the plane has always a solution. This last statement was rediscovered in [5], where an $O(n \log n)$ algorithm for finding a covering assignment in the plane is given as well, with a small variation in the formulation of the problem. The conic case has also been considered with the extra condition of $P$ being contained in the reverse cone. In such a case, a covering assignment can be always found as well. However, if the points are not required to lie in the reverse cone, deciding the existence of a covering assignment becomes NP-hard even for 2-dimensional fans, since the problem is equivalent to the wedge illumination problem studied in [6]. We generalize here the conic case to higher dimensions and to recursively-regular fans, which will be used to prove afterwards the complete case. The alternative proof of the result from Galperin and Galperin appearing in [17] can be generalized, using more strongly the results from [2] after proving that the original result holds also for conic fans, under some additional hypothesis. This generalization is synthesized in the following theorem.

**Theorem 6.10.** A covering assignment for a recursively-regular polyhedral fan $\mathcal{F}$ consisting on $n$ cells and a set of $n$ points $P \subset |\mathcal{F}|^-$, can always be found.

On the other hand, we show that sometimes a covering assignment does not exist (in any dimension bigger than 2). In particular, if a fan is not acyclic in the sense of [10], then there exists a point set in general position for which there is no covering assignment. This statement is easily derived from Theorem 6.16.

Before proving the theorem, we introduce the following lemma and a proposition easily derived from it.

**Lemma 6.11.** A conic fan $\mathcal{F}$, $|\mathcal{F}| = K$, is regular if, and only if, it is the restriction to $K$ of a complete regular fan

**Proof.** $\Rightarrow$: A cone $K^* \subset \mathbb{R}^{d+1}$ can be defined as $K^* = (\bigcap_{i \in I^+} \Pi_i^+ \cap (\bigcap_{i \in I^-} \Pi_i^-))$, where $\Pi_i^+$ refers to the closed halfspace “above” the hyperplane $\Pi$, $\Pi_i^-$ refers to the closed halfspace “below” $\Pi$, $I^+$ is the set of indices such that $K^* \subset \Pi_i^+$ and $I^-$ is the set of indices such that $K^* \subset \Pi_i^-$. By convention, $K^*$ will be considered to lie “bellow” the vertical hyperplanes. If $\mathcal{F}$ is a conic regular fan obtained by projecting $K^*$, then $L^* = \bigcap_{i \in I^+} \Pi_i^+$ is a cone projecting onto a complete fan $\mathcal{G}$ whose restriction to $|\mathcal{F}|$ is $\mathcal{F}$. Note, in addition, that the cells of $\mathcal{G}$ are in correspondence with the ones in $\mathcal{F}$.

$\Leftarrow$: Conversely, if $L^* \subset \mathbb{R}^{d+1}$ is a cone projecting onto a complete fan $\mathcal{G}$ and $K = \bigcap_i (\pi_i^+) \subset \mathbb{R}^d$, then $L^* \cap (\bigcap_i (\pi_i^+) \cap \Pi_i^+)$ is a cone projecting onto $\mathcal{F}$, the restriction of $\mathcal{G}$ to $K$, where $\pi_i^+$ is the vertical plane above $\pi_i$ in $\mathbb{R}^{d+1}$. 

**Proposition 6.12.** A covering assignment for a regular conic fan $\mathcal{F}$, $|\mathcal{F}| = K$, consisting of $n$ cells and a set of $n$ points $P \subset K^-$, can always be found.

**Proof.** Lemma 6.11 provides us with a fan $\mathcal{G}$ whose restriction to $K$ coincides with $\mathcal{F}$. If we look at the construction of this regular fan, we realize that it has the same number of cells as $\mathcal{F}$. In addition, each cells $C_i$ in $\mathcal{F}$ which had a non-trivial intersection with $\partial K$ can be obtained by truncating a cell $C_i'$ from $\mathcal{G}$ by a plane defining the boundary of $K$. Apply then the theorem from [11] to $\mathcal{G}$ and $P$. Then, replacing $C_i'$ with $C_i$ will not uncover any region in $K$, since $P \subset K^-$. 

15
The proof in [17] is based on the Proposition 6.5(ii) and the Lemma 6.13. In order to state this lemma, we need to establish some notation. It is not hard to see that, as in the case of regular subdivisions, one can find a set of sites and a weights for which a regular fan is exactly their power diagram. Formally, for a given regular polyhedral fan $\mathcal{F} \subset \mathbb{R}^d$ with $n$ cells, fix a $(d + 1)$-dimensional cone projecting onto it. This will have one hyperplane $\Pi_i$ for each of the cells $C_i$ of $\mathcal{F}$. Given any vector $w \in \mathbb{R}^n$, we construct the power diagram $\mathcal{F}_w$ as the lower envelope of the plane arrangement obtained by vertically shifting the plane $\Pi_i$ by $w_i$ and without changing its direction. Similarly, we denote by $\mathcal{F}^*_w$ the upper envelope of these hyperplanes. It is easy to see that both power diagrams must have exactly $n$ cells and that all of them must be unbounded. Therefore, we can refer to the cone of a cell in any of these diagrams as the smallest polyhedron containing the cell, i.e., the result of disregarding the halfspaces defining bounded facets of the cell. In addition, the cells in these diagrams can be paired in a natural way by the hyperplane they come from and named $c$ and $c^*$, respectively. These pairs satisfy the following property already observed in [17] and illustrated in Figure 6.

**Lemma 6.13.** Given a regular polyhedral fan $\mathcal{F}$ with $n$ cells and a vector $w \in \mathbb{R}^n$, every cell $c \in \mathcal{F}_w$ is contained in the reverse cone of $c^* \in \mathcal{F}^*_w$.

We can now prove the main theorem of this section, announced at the beginning, which generalizes the result (and the proof) in [17].

**Proof.** (Theorem 6.10)

Let $\mathcal{F}_0$ be the finest regular coarsening of $\mathcal{F}$ and $\mathcal{F}_0^-$ its reversed fan. Applying Lemma 6.5 to $\mathcal{F}_0^-$ and $P$, we can leave in the region corresponding to a cell of $\mathcal{F}_0^-$ as many points as cells of $\mathcal{F}$ is the corresponding cell in $\mathcal{F}_0$ made of. Applying Lemma 6.13, we can claim that we are left with a family of subproblems consisting on illumination a region contained in a cone $K$ using a recursively-regular fan $\mathcal{G}$ with $|\mathcal{G}| = K$ and points in $K^-$. Due to the recursively-regular nature of $\mathcal{F}$, we can solve these supproblems in a recursive way, which would end the proof.

A compact way to say that, for a fixed fan, there exists a covering assignment for any given point set is to say that this fan is universally covering. After showing that all recursively regular fans are universally covering, one could imagine that all the fans are so. We prove that this is not the case by showing that if a fan is cyclic, there is a point set in general position for which there is no covering assignment. A polyhedral fan is said to be cyclic in a direction $v$ (or from a point $q$) if the in-front/behind relation induced by the rays in direction $v$ (or from a point $q$) contains a cycle. This definition was considered in [10], where it is proven that regular cell complexes are acyclic.

**Lemma 6.14.** If $\sigma$ is a covering assignment, the overlapping condition must hold for every $(d - 1)$-face of the fan.

**Proof.** If the condition is not satisfied for the facet $H = C_1 \cap C_2$, we consider a ray in a direction interior to $H$ (for instance, the barycentre of its rays) and placed at the point $\frac{\sigma(C_1) + \sigma(C_2)}{2}$. No cell of $\mathcal{F}$, except for $C_1$ and $C_2$, can cover an unbounded part of this ray because they do not contain its direction. But none of these two cells cover the ray. Since the ray is unbounded and we have finitely many cones, the ray cannot be completely covered. 

16
However, it is easy to show that, even in the plane, this condition is not sufficient in general. An exception is the case where all the points lie on a line, which is studied in the following lemma.

**Lemma 6.15.** Consider an assignment $\sigma : \text{cells}(\mathcal{F}) \to P$ for a polyhedral fan $\mathcal{F} \subset \mathbb{R}^d$ and a point set $P \subset \mathcal{F}^{-} \cap \mathcal{F}$, where $l$ is a line. If $\sigma$ satisfies the overlapping condition, then it is a covering assignment.

**Proof.** We prove only the complete case but the proof can be easily adapted for the conic case. Figure 7 shows a sketch of a covering assignment for the case we are considering. Without loss of generality, we can assume that $l$ is horizontal. Consider now any other horizontal line. It is clear that at the “left infinity” this line is covered by (the translation of) the cell $C_l$ containing the left horizontal direction. Sweeping this line from left to right, at some point, it will leave this cone through some (assume $(d-1)$-dimensional) face $F$. The overlapping condition (and the special position of the point set) ensures that before it happens, it will have entered one of the (translated) cells adjacent to $C_l$ in the fan. If $F$ is not $(d-1)$-dimensional, the overlapping condition on the cells incident to $F$ also ensures that the ray enters a new cell before leaving the previous one. One can continue the argumentation until the ray enters the cell containing the right horizontal direction. Since any horizontal ray is covered, $\mathbb{R}^d$ is completely covered.

We are now in conditions to prove the next theorem, which shows that there are pairs of a fan and a point set for which there is no covering assignment.

**Theorem 6.16.** Given a polyhedral fan $\mathcal{F} \subset \mathbb{R}^d$ with $n$ cells and set of $n$ points $P \subset \mathbb{R}^d$ on a line $l$ and in $\mathcal{F}^{-}$, there exists a covering assignment for $\mathcal{F}$ and $P$ if, and only if, $\mathcal{F}$ is acyclic in the direction of $l$.

**Proof.**

If $\mathcal{F}$ is acyclic in the direction $v$ of $l$, then we can construct a directed acyclic graph having the cells of $\mathcal{F}$ as vertices and an edge from $C_2$ to $C_1$ if the vector $v_{12}$ normal to $f = C_1 \cap C_2$ pointing from $C_1$ to $C_2$ satisfies $v_{12} \cdot v \geq 0$. If the order as the $\sigma(C_i)$ appears on $l$ respects the partial order represented by such a DAG, then the overlapping condition holds for $\sigma$. Lemma 6.15 ensures that this is sufficient.

We prove the other direction by contrapositive. If there is a cycle $\tau = (C_1...C_k)$ in the direction $v$, there is a cycle in the order the points $\sigma(C_1), ..., \sigma(C_k)$ should appear in the line in order to satisfy the overlapping condition, proved to be necessary for covering the space.

Note that we can perturb the point set in order to obtain general position and still have no covering assignment. Indeed, we can give a precise sufficient condition for a point set $P$ to be cyclic with respect to a cyclic fan $\mathcal{F}$. In that case, there will be no covering assignment for $P$ and $\mathcal{F}$. Consider the sphere representing the directions in $\mathbb{R}^d$. We associate to each direction the permutation representing the order in which the points of $P$ appear. This permutations induce a tessellation $\mathcal{D}$ of the sphere. If the directions of the normal vectors involved in a visibility cycle (which lie in an open hemisphere in the sphere of directions, by definition of visibility cycle) lie in the same cell of $\mathcal{D}$, then we can adapt the proof of the previous theorem.
to ensure that there is no covering assignment. Observe also that, if the covering assignment exists, it can be computed in $O(n \log n)$ time and this time is sufficient to decide its existence as well.

After understanding the previous theorem, one is tempted to conjecture that being acyclic is equivalent to being universally covering. We exhibit an example to show that this is not the case.

**Proposition 6.17.** There exist an acyclic polyhedral fan and a set of points for which there is no assignment satisfying the overlapping condition. Consequently, not every acyclic fan is universally covering.

**Proof.** We will provide a 3-dimensional fan $F$ with 5 cells and a point set $P \in \mathbb{R}^3$ for which there is no covering assignment. More precisely, it can be shown that for each of the $5!$ possible assignments, one of the 8 overlapping conditions is violated. To construct $F$, take the tessellation sketched in Figure 8 and embed it in the plane $\{(x, y, z) \in \mathbb{R}^3 : z = -1\}$. Take then the cones from the origin to each of the cells of this tessellation. They form a fan whose ground set is the halfspace with negative $z$ coordinate. It can be easily checked that this fan is acyclic but not recursively regular.

Consider now, for instance, the point set

$p_1 = (29, 95, 89)$  
$p_2 = (55, 19, 92)$  
$p_3 = (54, 10, 82)$  
$p_4 = (78, 2, 68)$  
$p_5 = (15, 40, 92)$.

It is easy to see that there is no covering assignment fulfilling all the overlapping conditions.

This last example motivates conjecture that a fan is covering if and only if it is recursively regular, since a non-recursively-regular fan must have a completely non-regular convex region, fact that could perhaps be used to construct a point set for a counterexample.

### 6.4 Covering by translated homotheties

The problem of covering by floodlights forming a conic fan is closely related to the following problem. Consider a function $t : \mathbb{R}^d \to \mathbb{R}^d$ consisting of the composition of a homothety with respect to the origin and a translation. In this section, the term *transformation* will be reserved to this type of functions. Assume we have a polyhedral decomposition $\mathcal{S} \subset \mathbb{R}^d$ consisting of $n$ cells. Given a set $T$ of $n$ transformations, we may ask whether there is a bijection between $\text{cells}(\mathcal{S})$ and $T$ such that the set of transformed cells covers $|\mathcal{S}|$. Note that the formulation can be extended to polyhedral subdivisions of the whole $\mathbb{R}^d$. Given an instance of the previous problem, we will construct an instance of the floodlight covering problem. We embed the subdivision into the hyperplane $\Pi = \{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\}$ and consider function $s : \text{cells}(\mathcal{S}) \to \mathbb{R}^{d+1}$ assigning to each cell the cone generated by itself and the point $(0, ..., 0, 1) \in \mathbb{R}^{d+1}$. The cones $\{s(C)\}_{C \in \text{cells}(\mathcal{S})}$ form a polyhedral fan denoted by $F[\mathcal{S}]$. Let $p : T \to \mathbb{R}^{d+1}$ be the function assigning to a transformation $t = v \circ h$, where $h$ is an homothety and $v$ is a translation, the point $(v(0, ..., 0), \|h(1, 0, ..., 0)\|)$. We will abuse notation
and say that a set of transformations $T$ is in a cone $K \subset \mathbb{R}^{d+1}$ if the set of associated points $P = \{p(t) : t \in T\}$ is.

**Lemma 6.18.** If there is an assignment $\sigma : \text{cells}(F[S]) \rightarrow P = \{p(t) : t \in T\}$ such that $\bigcup_{C \in \text{cells}(F)} (C + \sigma(C)) \supset |F[S]|$, then there is an assignment $\mu : \text{cells}(S) \rightarrow T$ such that $\bigcup_{C \in \text{cells}(S)} \mu(C)(C) \supset |S|$.

**Proof.** Assume the assignment $\sigma$ is covering. Since $|S| \subset |F|$, the translated cells cover $|S|$. Indeed, it is covered by the intersection of the translated cells with II. Observe now that $(p(t) + \sigma(p(t))) \cap II = v(O) + h(s^{-1}(\sigma(p(t)))) = t(s^{-1}(\sigma(p(t))))$, where $t = v \circ h$. Therefore, $\bigcup_{C \in \text{cells}(S)} \mu(C)(C) \supset |S|$ where $\mu = s^{-1} \circ \sigma \circ p$. \hfill \Box

The converse of the previous lemma is, in general, not true. Nevertheless, adding an extra condition a similar observation can be done in the opposite direction.

**Lemma 6.19.** If there is no assignment for $F$ and $P = \{p(t) : t \in T\}$ satisfying the overlapping condition, then there is no covering assignment for $S = F \cap \Pi$ and $T$, for some hyperplane $\Pi$.

**Proof.** Assume, without loss of generality, that $|S| \subset |F|$, the translated cells cover $|S|$. Indeed, it is covered by the intersection of the translated cells with II. Observe now that $(p(t) + \sigma(p(t))) \cap II = v(O) + h(s^{-1}(\sigma(p(t)))) = t(s^{-1}(\sigma(p(t))))$, where $t = v \circ h$. Therefore, $\bigcup_{C \in \text{cells}(S)} \mu(C)(C) \supset |S|$ where $\mu = s^{-1} \circ \sigma \circ p$. \hfill \Box

We put everything together in the following proposition, where we assume that the set of transformations has exactly the same cardinality as the the set of cells of $S$.

**Proposition 6.20.** (i) Given a recursively-regular tessellation $S \subset \mathbb{R}^d$ and a set of transformations $T \subset |F[S]|^-$, there always exists a covering assignment for $S$ and $T$.

(ii) Given a cyclic tessellation $S \subset \mathbb{R}^d$, there exists a set of transformations $T \subset |F[S]|^-$ such that there is no covering assignment for $S$ and $T$.

(iii) There exists acyclic non-recursively-regular tessellation $S \subset \mathbb{R}^2$ and sets of transformations $T \subset |F[S]|^-$, such that there is no covering assignment for $S$ and $T$. 

![Figure 9: Partition and translations (left) and transformed cells (right).](image)
6.5 Directional graph embeddings

As shown in Section 6.3, the overlapping condition is necessary for an assignment to be covering. We ignore the amount of influence of this condition on the existence of a covering assignment but the counterexamples we found so far, fail even to fulfil this (weaker) condition. Pursuing this intuition, we head to the following problem. A directional graph is a structure \( G = (V, h) \) where \( h : V \times V \rightarrow \mathbb{R}^d \) is a function such that

(i) \( h(v, v) = 0 \), for all \( v \in V \)

(ii) \( h(v, u) = -h(u, v) \), for all \( (v, u) \in V \times V \).

We may regard this structure as a directed graph, with a non-zero direction associated to each edge. This allows us to talk about paths, trees and cycles understanding them as directional graphs. An embedding of a directional graph on a point set \( P \subset \mathbb{R}^d \) is a one-to-one assignment \( \sigma : V \rightarrow P \) such that

\[
\sigma(v) - \sigma(u) = \lambda_{uv} \cdot h(v, u), \quad \text{for all} \; (v, u) \in V \times V.
\]

If such an embedding exists, we say that \( G \) is embeddable in \( P \). The projection of a directional graph \( G \) into a subspace \( S \subset \mathbb{R}^d \) is the directional graph obtained by projecting every vector \( h(u, v) \subset \mathbb{R}^d \) onto \( S \).

We say that a directional graph is universally embeddable if it is embeddable in any point set of the same cardinality as vertex set of the graph. A directional graph \( G \) is drawable if there is a bijection \( \pi : V \rightarrow P \subset \mathbb{R}^d \) such that \( \pi(v) - \pi(u) = \lambda_{uv} \cdot h(v, u) \) with \( \lambda_{uv} \in \mathbb{R}^+ \), for all \( (v, u) \in V \times V \). The definition of directional graph is illustrated in Figure 10, together with a drawing and an embedding of it.

Polytopes and polyhedral fans are related to some directional graphs in a natural way. The directional graph of a polytope is the set of its vertices, together with the function \( h(u, v) = v - u \) if \( u \) and \( v \) share endpoints of an edge of \( P \) and 0 otherwise. Figure 11 shows a projection of the graph of a 3-prism into the plane and embeddings of it into two different point sets. The normal graph of a polyhedral fan \( \mathcal{F} \) is set of its cells with the function \( h(C, C') \) being a vector normal to the facet common to \( C \) and \( C' \) and pointing “from \( C \) to \( C' \)” if they share a facet and 0 otherwise. Note that the directional graph of a polytope and the graph of its normal fan coincide, according to the previous definitions. In fact, this relation is nothing else than a version of the polar reciprocity where only the direction of the vectors is taken into account.

The following proposition shows that there is a surprisingly large family of universally embeddable directional graphs.

**Proposition 6.21.** If a directional graph is drawable, then it is universally embeddable. In particular, paths and trees are universally embeddable.
Proof. Given a drawable directional graph $\overrightarrow{G} = (V, h)$ and an arbitrary point set $P$, $|P| = |V|$ consider a drawing $\pi$ of $\overrightarrow{G}$. Consider the least-squares optimal matching $\sigma$ between $\pi(V)$ and $P$. We will show that $\sigma \circ \pi$ is an embedding. Assume that is not. Then, it must exist a pair $(u, v) \in V \times V$ such that $h(v, u) \cdot (\sigma(\pi(v)) - \sigma(\pi(u))) < 0$, but $\pi(u) - \pi(v) = \lambda_{uv} \cdot h(v, u)$, for some $\lambda_{uv} \in \mathbb{R}^+$. Therefore, $(\pi(u) - \pi(v)) \cdot (\sigma(\pi(v)) - \sigma(\pi(u))) < 0$, which contradicts the optimality of $\sigma$.

It is not hard to see that if there is a sequence of vertices $v_1, ..., v_l, v_{l+1} = v_1$ in $V$ and a vector $\delta \in \mathbb{R}^d$ such that $h(v_i, v_{i+1}) \cdot \delta > 0$, for $i \in \{1, ..., l\}$, then the graph is not drawable. Such a cycle is called a (d-)forcing cycle. However, the converse is not true in general: for instance, the normal graph of the subdivision in Figure 8 has no forcing cycle but it is also non-drawable.

Recursive regularity comes into play when one observes that, since the normal graph of a fan encodes the overlapping conditions for a covering assignment, the normal graph of a recursively-regular fan must be universally embeddable. Indeed, any projection of the normal graph of a recursively-regular fan is universally embeddable as well. In particular, the graph of any polyhedron is, which it could be already derived from Proposition 6.21. Since the normal graph of a fan is drawable, if and only if, the fan is regular (see [1]), drawable and universally embeddable are not equivalent.

The following proposition summarizes the properties mentioned above.

**Proposition 6.22.** (i) Normal graphs of recursively-regular fans and their projections are universally embeddable.

(ii) Universally embeddable graphs are not necessarily drawable.

(iii) Graphs with forcing cycles (in particular, normal graphs of cyclic fans) are not universally embeddable.

(iv) There are non-universally-embeddable graphs with no forcing cycles.

Proof. (i) Theorem 6.10 ensures that there is a covering assignment for $\mathcal{F}$ and any point set $P$ of the required cardinality. This assignment must satisfy the overlapping condition for each facet of the fan, which is equivalent to the embedding condition for the corresponding edge. If we are given the projection onto $S$ of the normal graph of a recursively-regular fan $\mathcal{F} \subset \mathbb{R}^d$, consider the point set $P$ embedded into an affine subspace $A \subset \mathbb{R}^d$ parallel to $S$ and such that $P \subset |\mathcal{F}|^-$ ($A$ can be taken to be $S$ if $\mathcal{F}$ is complete). The overlapping condition in this case is only dependent on the projection of the vectors normal to facets into $S$ and translates precisely into the projected directional graph embedding conditions.

(ii) As mentioned before, the normal graph of a recursively-regular non-regular fan is not drawable and is, however, universally embeddable.

(iii) Consider a $\delta$-forcing cycle $v_1, ..., v_l, v_{l+1} = v_1$. Take a set of different points in a line in the direction of $\delta$ and label them in this direction. For any embedding $\sigma$, $\sigma(v_{l+1})$ must have a label bigger than $\sigma(v_l)$,
for all \( i \in \{1, \ldots, l\} \), which is obviously impossible. Indeed, the point set can be perturbed in a way that general position is attained.

(iv) The normal graph of the fan obtained by taking cones from the subdivision in Figure 3 has no forcing cycle, since it is acyclic (in the visibility sense). However, we gave a set of points for which all the assignments violate an overlapping condition. Hence, there is no embedding of its normal graph into this point set.

The previous theorem suggests the definition of a recursively drawable directional graphs, which would be universally embeddable. However, this is not directly related to recursive regularity and, therefore, we do not develop this approach here.

7 Algorithms

In this section, we expose some details on how some proofs from previous sections can be translated into algorithms to compute the objects studied in them. However, we remain in a high level of description.

**Proposition 7.1.** The minimal relaxation set \( \hat{E}_S \) of system \( S \) of \( m \) linear inequalities can be computed in polynomial time, in any fixed dimension.

**Proof.** In the proof of Theorem 3.1 we show how \( \hat{E}_S \) can be iteratively constructed. We give here an algorithm for this construction that works in polynomial time. At each iteration, we want to figure out if a given system of equations and inequalities is feasible. This translates into a dual system of equations where some variables are restricted to be non-negative and, among them, at least one must be strictly positive. We can check if there is a dual solution by solving a linear program. More precisely, we maximize, under the dual constraints, the function

\[
 f = \sum_{i \in B} y_i,
\]

where \( B \subset [m] \) are the indices of variables that are restricted to be non-negative. For the ease of argumentation, we add a linear inequality in order to make the feasible region bounded, truncating the cone of solutions. If the optimum value is 0, then none of the dual variables in \( B \) can appear with positive values in a dual solution. The converse is also true: if the function takes a positive value, at least one of these variables must be strictly positive. Even more, all the restricted variables that can take simultaneously positive values, will do so in the optimal solution. Once we know which restricted variables can take a positive value, we unrestrict them and repeat the process until no more restricted variables can be positive in a solution. At each step, we either discover a new variable that must be unrestricted or the process ends. Hence, at most \( m \) iterations are needed. Each iteration consists on solving a linear program with \( n = |A| \) variables and as many equations as walls in \( S \), which are bounded in number by \( O(n^d) \). Therefore, the whole algorithm takes polynomial time, if \( d \) is considered to be constant. \[ \square \]

Note that what we actually do when computing the minimum relaxation of a system is finding a point in the relative interior of the set of solutions of the corresponding weak system. Similar approaches appear in the literature in order to do find such a point for a general polytope. However, our algorithm is probably more intuitive and reflects the particularities of our problem, which we comment below, after stating the following corollary motivated by the previous proposition.

**Corollary 7.2.** The finest regular coarsening \( S_0 \) of a subdivision \( S \) can be computed in polynomial time on the number of vertices, in any fixed dimension.

Probably, some improvements can be done when computing the finest regular coarsening taking into account that the regularity system of \( S \) is not a general system. In particular, the matrix \( M \) associated to the system is sparse and its structure is related to the combinatorics of the subdivision. Each row, corresponding to a wall, has at most \( d+2 \) non-zero coefficients. In addition, \( d \) of the vertices involved can be
taken to be an affine basis for the corresponding wall. Then, the corresponding $d$ coefficients will be positive while the other $2$ will be negative. If $S$ is a triangulation, this means that each vertex involved in a folding condition appearing in a contradiction cycle must be involved in another condition of the contradiction cycle. Moreover, for one of them it must be part of the wall involved and for another one it must not be part of the wall. If $S$ is not a triangulation, a similar combinatorial property still holds.

A subdivision can have, even in dimension $2$, a linear number of simultaneous flips [12]. That means that, a priori, an exponential number of coarsenings of a subdivision may need to be checked in order to decide whether it is recursively regular or not. Nevertheless, as a consequence of Theorem 5.3, this can be indeed decided in polynomial time, as announced by the next proposition.

**Proposition 7.3.** Given a subdivision $S$, its belonging to $\mathcal{R}$ can be decided in polynomial time on the number of vertices, in any fixed dimension.

**Proof.** Theorem 5.3 ensures that we only need to compute the regularity tree of $S$ to decide whether it is in $\mathcal{R}$ or not. This is done by computing the finest regular coarsening of subdivisions of some subconfigurations. Each time we go down a level in the tree, there is one wall in the finest regular coarsening that was not in any previous finest regular coarsening. Therefore, if we charge the computation of the finest regular coarsening to this wall, we can conclude that no more finest regular coarsenings will be computed than walls $S$ has, which is polynomial if $d$ is considered to be a constant. 

8 Concluding remarks and open problems

We have shown that the finest regular coarsening of a subdivision, which can be seen as the regular subdivision that is closest to it, can be used to define a structure called the regularity tree of the subdivision. We have observed that this leaves of this tree define a partition of the subdivision in convex subconfigurations which are either regular or completely non-regular. This object gives more structure to non-regular subdivisions measuring, in a way, its degree of regularity. As a consequence, the class of recursively-regular subdivisions arises in a natural way. We have shown that this class goes beyond regular subdivisions but is disjoint from cyclic ones. Because of this, it maintains several good properties of the first ones and avoids some pathologies of the second set. We have reported on some of such features and have also related the new concepts to previously known results.

In addition, we have studied a collection of applications of the notions we have presented. This results belong to a wide range of different areas, ones more theoretical than others. We expect to find even more applications of the theory developed in this note, since any theorem or algorithm based on the regularity of a subdivision and admitting a recursive scheme can be probably extended to apply for this bigger set of subdivisions. We proved the existence of negative instances for the discussed problems, ensuring therefore that it makes sense consider the corresponding decision problems. These problems have all the same flavour: we are given a subdivision (or a graph) and a set of points and we are asked for an assignment of cells to points satisfying some property. If the subdivision happens to be recursively regular, such an assignment exists and can be easily computed regardless of the point set (it is universal). Although it is not clear yet the significance of this class for these problems, it has been useful to show that other criteria were not the right ones. Moreover, the counterexamples we have presented do not exclude the possibility that universal subdivisions for theses problems are exactly recursively-regular subdivisions. Indeed, the regularity tree of a non-recursively-regular subdivision could be crucial to construct a point set for which an assignment cannot be found.

Finally, we proved that the finest regular coarsening and the regularity tree of a subdivision can be computed in polynomial time. We have used this facts to prove that recursive-regularity of a subdivision can be decided in polynomial time as well, which is relevant for the algorithmic version of the aforementioned problems.

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