Quantum metric dipole and non-reciprocal bulk plasmons in parity-violating magnets

Arpit Arora\textsuperscript{1}, Mark S. Rudner\textsuperscript{2}, and Justin C. W. Song\textsuperscript{1,2}\textsuperscript{*}

\textsuperscript{1}Division of Physics and Applied Physics, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371 and
\textsuperscript{2}Department of Physics, University of Washington, Seattle WA 98195, USA

The optical response of metals is often dominated by plasmonic resonances—the collective oscillations of an interacting electron liquid. Here we unveil a new class of plasmons that arise in a wide range of parity violating magnetic metals. In these materials, broken time reversal and parity symmetries work together with electron-electron interactions to produce intrinsically non-reciprocal bulk plasmons wherein the frequencies of forward and backward moving modes are split. Strikingly, when interactions are strong we find plasmonic non-reciprocity is dominated by the quantum metric dipole that characterizes the real-space profile of Bloch electron wavepackets; in this regime, the quantum metric dipole can give rise to strongly non-reciprocal collective dynamics even when the electronic group velocity is only weakly asymmetric. We anticipate these intrinsic non-reciprocal bulk plasmons can be realized in a wide range of parity violating magnets including twisted bilayer graphene heterostructures where quantum geometric quantities can achieve large values.

In reciprocal materials, the forward and backward propagation of light along any direction cannot be differentiated. Non-reciprocal materials, on the other hand, possess the ability to differentiate between forward and backward flows\cite{1}. A striking venue for observing non-reciprocity in light-matter interaction\cite{2} is plasmonic materials wherein the collective motion of electrons strongly couples with light\cite{3,4}. In this context, methods for achieving non-reciprocity have largely focused on Hall effect schemes\cite{5–9} or out-of-equilibrium protocols\cite{10–14}. In the former, chiral plasmonic modes appear along the boundaries of a magnetic material while its bulk plasmonic excitations remain reciprocal; in the latter, the active driving of a current flow through the device is required to achieve non-reciprocity. Means for realizing bulk plasmonic non-reciprocity intrinsically in equilibrium systems (i.e., passively) remain elusive.

Here we argue that parity-violating metallic magnets can host non-reciprocal bulk plasmons in equilibrium. This non-reciprocity arises from a bulk directional current (BDC) that emerges in the presence of inhomogeneous electric fields when both inversion/parity (\(P\)) and time-reversal (\(T\)) symmetries are broken. Crucially, the spatial pattern of the BDC (black arrows, Fig. 1) does not change when the direction of plasmon propagation is reversed; its configuration is determined by the intrinsic broken \(P\) and \(T\) symmetries and electron density profile. This allows to stiffen (soften) the frequency for forward (backward) propagating plasmonic modes (Fig. 1), thus enabling non-reciprocity.

Strikingly, in the presence of strong Coulomb interactions, we find that bulk plasmonic non-reciprocity and BDC can be dominated by the asymmetry of the quantum metric across the Fermi surface (“quantum metric dipole”). For Bloch bands, the quantum metric, \(g_{ab}(\mathbf{k})\), quantifies the spatial distance between Bloch states \(\psi(\mathbf{k})\) with nearby wavevectors \(\mathbf{k}\) and \(\mathbf{k} + \mathbf{e}\mathbf{k}\)\cite{15,16}: \(g_{ab}(\mathbf{k})\) plays a role in diverse physical phenomena ranging from the shapes of Wannier states\cite{17}, to superfluidity\cite{18}, and Landau-level structure\cite{19}. Here and below the Roman indices \(\{a, b\}\) denote Cartesian coordinates. As we explain below, when \(g_{ab}(\mathbf{k})\) is asymmetric in \(\mathbf{k}\), a non-reciprocal coupling between electromagnetic waves and electrons emerges, inducing a large quantum geometric boost to the BDC for strong Coulomb interactions.

We anticipate a wide range of currently available parity-violating magnets\cite{20,21} can host bulk non-reciprocal plasmons. Of particular interest are narrow bands moiré systems which possess parity-violating magnetic states\cite{22,23}, as well as exposed surface...
states that allow direct access via scanning near-field techniques \[26, 27\]. In particular, in twisted bilayer graphene heterostructures close to \(3/4\) filling, we find strong Coulomb interactions in the narrow bands may enable the quantum metric (dipole) to dominate plasmonic non-reciprocity. Another platform of interest is parity violating magnetic systems \[21, 23\] (e.g., MnBi\(_2\)Te\(_4\) layers or CuMnAs) where the composite \(PT\) symmetry is preserved (while \(P\) and \(T\) are individually broken). In such systems the Hall effect vanishes, while BDC and plasmonic non-reciprocity persists, distinguishing it from other \(T\)-breaking phenomena.

**Semiclassical picture of non-reciprocal bulk plasmons:** To illustrate the origins of intrinsic non-reciprocal plasmons, we first analyze the semiclassical collective dynamics of an electron liquid in a slowly varying electric field \(E(r, t)\). A full quantum mechanical treatment via the random phase approximation (RPA) follows thereafter.

Semiclassically, the velocity of an electron with wavevector \(k\) is given by \[28, 29\]
\[
\mathbf{v}(t) = \frac{e}{\hbar} \mathbf{E}(r, t) \times \mathbf{\Omega}(k).
\]
(1)

Here \(\epsilon(k)\) is the Bloch band energy, \(\delta\epsilon_{QM}(k) = \frac{e}{\hbar} \mathbf{\nabla}_r \cdot [g(k) \mathbf{E}(r, t)]\), where the rank-2 tensor \(g(k)\) is the quantum metric, and \(\mathbf{\Omega}(k)\) is the Berry curvature. The first and last terms of Eq. (1) are the group velocity \([v_k = (1/h)\mathbf{\nabla}_k \epsilon(k)]\) and anomalous velocity, respectively. These contributions arise even under uniform electric fields. In contrast, the second term (a “quantum metric velocity” contribution) only appears for non-uniform electric fields and originates from the coupling of spatially inhomogeneous electric fields to the \((k\)-dependent part\) of the electric quadrupole moment \[28, 29\].

Plasmons arise as self-sustained collective density oscillations: \(\partial_t \delta n(r, t) + \nabla_r \cdot \mathbf{j}(r, t) = 0\), where \(\delta n(r, t)\) is the deviation of the particle density from equilibrium, and \(\mathbf{j}(r, t) = \int \mathbf{d}k \mathbf{v}(k, t) f(k, r, t)\) is the particle current density. Here \(f(k, r, t)\) is the distribution function. We use the short hand \(\int \mathbf{d}k = \int \mathbf{d}k / (2\pi)^d\) to denote the \(d\)-dimensional \(k\)-space integral throughout. The evolution of \(f(r, k, t)\) in the collisionless limit is determined by the kinetic equation, \(\partial_t f + \mathbf{v} \cdot \nabla_r f + k \cdot \nabla_k f = 0\), with \(\hbar k = -eE\). Linearizing the kinetic equation and decomposing into Fourier harmonics \(e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}')}\), we find the divergence of the particle current density
\[
\mathbf{v} \cdot \mathbf{j} = i \int \mathbf{d}k \left[ \frac{i \mathbf{v}_k \mathbf{v}_k'}{\omega - \mathbf{q} \cdot \mathbf{v}_k} - \frac{i \mathbf{v}_k \mathbf{v}_k'}{2} \right] \frac{\partial f_0}{\partial \epsilon} \tilde{E}_0(k),
\]
(2)

where \(\tilde{O}\) denotes the amplitude of the Fourier harmonic \(\tilde{O}(\mathbf{q}, \omega)\) and repeated indices are implicitly summed. In obtaining the second term, we integrated by parts and noted \([\mathbf{g}_{abc}(k)] f_0(\epsilon)\)_{BZ} vanishes. Here \(f_0(\epsilon)\) is the Fermi function.

The current is driven by the plasmon’s electric field \(eE(r, t) = \nabla_r \int V(r, r') \delta n(r', t) \mathbf{d}r'\) generated by \(\delta n(r, t)\), where \(V(r, r')\) is the Coulomb interaction. In Fourier space, this corresponds to \(e\tilde{E}(\mathbf{q}, \omega) = i \mathbf{q} \tilde{V}(\mathbf{q}) \delta n(\mathbf{q}, \omega)\).

Solving the continuity equation with this relation and the current density in Eq. (2) produces plasmonic modes.

To illustrate this, it is instructive to analyze the behavior of the particle current density in the curly parentheses of Eq. (2) \[30\] for forward \(e^{i\mathbf{q} \cdot \mathbf{r} - i\omega t}\) and backward \(e^{-i\mathbf{q} \cdot \mathbf{r} - i\omega t}\) propagating waves. In the absence of the quantum metric and in the long-wavelength limit \((\omega \gg v_k \cdot \mathbf{q})\), Eq. (2) is dominated by the normal Drude contribution that goes as \(\omega^{-1}\). This captures a reciprocal particle current density that switches sign when the propagation direction is reversed \((\mathbf{q} \to -\mathbf{q})\). These currents feed the build-up of charge density to produce the usual bulk plasmon dispersion relation \(\omega(\mathbf{q}) = \omega_0(\mathbf{q})\), with bare plasmon frequency
\[
\omega_0(\mathbf{q}) = \sqrt{V(\mathbf{q})q^2 D\mathbf{q}}, \quad [\mathbf{D}]_{ab} = -\int \mathbf{d}k \frac{\partial f_0}{\partial \epsilon} \mathbf{v}_k^a \mathbf{v}_k^b/k,
\]
(3)

where \(\mathbf{D}\) is the Drude weight. Since the currents that lead to \(\omega_0(\mathbf{q})\) are reciprocal, we have \(\omega_0(\mathbf{q}) = \omega_0(-\mathbf{q})\).

In contrast, contributions of even powers in \(1/\omega\) in the curly parentheses of Eq. (2) are non-reciprocal (e.g., the metric term, \(g_{abc}(k)\), as well as other even power contributions in the expansion of the first term). When the propagation direction is reversed, the particle current density pattern described by these contributions maintains its direction (black arrows, Fig. 1). We term these contributions bulk directional currents (BDCs); BDCs only arise when both \(P\) and \(T\) symmetry are broken \[31\].

Importantly, BDCs induce non-reciprocal bulk plasmon dispersion relations. To see this explicitly, we first make the simple replacement \(\mathbf{q} \to -\mathbf{q}\) \(\approx \omega\) in Eq. (2) \[31\]; see below for a systematic derivation and expansion (in \(\mathbf{q}\)) of the plasmon dispersion. This produces a (quantum metric dominated) non-reciprocal bulk plasmon dispersion:
\[
\omega(\mathbf{q}) = [\omega_0^2(\mathbf{q}) + \omega_{QM}^2(\mathbf{q})]^{1/2} + \omega_{QM}(\mathbf{q}),
\]
where the non-reciprocity \(\Delta\omega(\mathbf{q}) = \omega(\mathbf{q}) - \omega(-\mathbf{q})\) reads as
\[
\Delta\omega(\mathbf{q}) = 2\omega_{QM}(\mathbf{q}), \quad \omega_{QM}(\mathbf{q}) = \xi \tilde{V}(\mathbf{q}) G^{abc} g_{ab} g_{c}/2.
\]
(4)

Here \(\xi\) is a dimensionless constant; for the semiclassical treatment we find \(\xi = 1/2\), see discussion below for full quantum treatment. The quantum metric dipole
\[
G^{abc} = \int \mathbf{d}k \frac{\partial f_0}{\partial \epsilon} \mathbf{v}_k^a \mathbf{v}_k^b \mathbf{v}_k^c
\]
(5)

measures the asymmetry of the quantum metric \(g_{abc}(k)\) around the Fermi surface. Crucially, because \(\omega_{QM}(\mathbf{q})\) is odd in \(\mathbf{q}\), the bulk plasmon dispersion is non-reciprocal: \(\omega(\mathbf{q}) \neq \omega(-\mathbf{q})\) with a non-reciprocity controlled by the strength of the quantum metric dipole, Eq. (4). While in Eq. (4) we have focussed on non-reciprocity dominated by \(G^{abc}\), we note that non-reciprocity can also be induced by other types of BDCs so long as \(P\) and \(T\) are broken (see below).
Non-reciprocal bulk plasmons in RPA: To see non-reciprocal bulk plasmons from a systematic quantum mechanical perspective, we examine screening in a metal, where plasmons appear as zero modes of the dielectric function $\varepsilon(q, \omega) = 1 - V(q) \Pi(q, \omega)$. In the random phase approximation (RPA), the polarization function is [32]

$$\Pi(q, \omega) = \sum_{\alpha, \beta, \mathbf{k}} \frac{f_0[\epsilon_{\alpha}(\mathbf{k} + \mathbf{q})] - f_0[\epsilon_{\beta}(\mathbf{k})]}{\epsilon_{\alpha}(\mathbf{k} + \mathbf{q}) - \epsilon_{\beta}(\mathbf{k}) - i\omega} \mathcal{F}_{\alpha\beta}(\mathbf{k}, q),$$

where $\mathcal{F}_{\alpha\beta}(\mathbf{k}, q) = \langle |\psi_{\alpha}(\mathbf{k} + \mathbf{q})| |\psi_{\beta}(\mathbf{k})| \rangle^2$ is the coherence factor. We use Greek letters to denote band indices throughout this work. The electronic states $\{ |\psi_{\alpha}(\mathbf{k})| \}$ have corresponding energies $\{ \epsilon_{\alpha}(\mathbf{k}) \}$. Non-reciprocal plasmons can be obtained as solutions to $\text{Re}[\varepsilon(q, \omega)] = 0$, see full numerical implementation below.

To get better intuition about the role of the quantum metric and terms higher order in $q$ in the polarization function, we first analyze the intraband polarization function focussing on $\alpha = \beta$ in Eq. (6). Expanding in powers of $q = |\mathbf{q}|$, and using $\langle |\psi(\mathbf{k} + \mathbf{q})| |\psi(\mathbf{k})| \rangle^2 = 1 - g_{ab}(\mathbf{k}) q_a q_b - i\frac{1}{2} \partial_{\mathbf{k}a} \partial_{\mathbf{k}b} g_{bc}(\mathbf{k}) q_a q_b q_c + \mathcal{O}(q^3)$ (see SI), gives

$$\Pi_{\text{intra}}(q, \omega) \approx \frac{g_{ab} q_a q_b}{\omega^2} [\mathcal{D}]_{ab} + \frac{g_{ab} q_a q_b}{\omega} \left[ C^{abc} + \frac{C^{abc}}{\omega^2} \right].$$

Here $C^{abc}$ is a Drude-like weight

$$C^{abc} = -\int d\mathbf{k} v_a(\mathbf{k}) v_b(\mathbf{k}) v_c(\mathbf{k}) \frac{\partial f_0}{\partial \epsilon},$$

that measures the velocity asymmetry of electrons on the Fermi surface. In obtaining Eq. (7), we have kept all non-vanishing terms to lowest order up to $\mathcal{O}(q^3)$. We note that just as the quantum metric term $G^{abc}$ leads to non-reciprocity, the $C^{abc}$ term (being odd in $q$) is similarly non-reciprocal. We note the origin of $C^{abc}$, however, is essentially classical. Both $G$ and $C$ terms contribute to the BDC and lead to non-reciprocal bulk plasmons. Indeed, they arise under the same symmetry conditions: both $C$ and $G$ tensors vanish in the presence of either $\mathcal{T}$ or $\mathcal{P}$ symmetries.

For large frequencies ($\omega^2 \gg C/G$), quantum metric induced non-reciprocity can dominate over $C^{abc}$. Indeed, in this regime approximating the dielectric function with its intraband contribution in Eq. (7) produces the non-reciprocal bulk plasmon dispersion in Eq. (4) with $\xi \to \xi_{\text{RPA}} = 1$. As we will see below, in the presence of strong interactions, the quantum metric regime is dominant; we will focus on this regime in what follows.

Illustration of quantum metric and non-reciprocal bulk plasmons in one dimension: To elucidate the physical nature of non-reciprocal bulk plasmons and their connection with the quantum metric, we now demonstrate their appearance in a simple one-dimensional (1D) model. After using this model to develop a transparent conceptual picture of the physics of (quantum metric dipole dominated) non-reciprocal plasmons, we will turn to numerical simulations of non-reciprocal bulk plasmons in twisted bilayer graphene heterostructures – a promising candidate material for their experimental observation.

As depicted in Fig. 2a, we consider a bipartite 1D lattice with uniform on-site energies (set to zero without loss of generality), and intra- and inter-cell nearest-neighbor hopping amplitudes $u$ and $v$, respectively. We additionally include a complex third-neighbor hopping amplitude $w = |w|e^{i\phi}$, with phase convention as indicated by the arrows on the figure. A nonzero value of the phase $\phi$ breaks $\mathcal{T}$ and $\mathcal{P}$ symmetries, as required for the emergence of non-reciprocal plasmons. In the ordered basis $\{|k, A|, |k, B|\}$, the system’s Bloch Hamiltonian is

$$H_{1D}(k) = \begin{pmatrix} 0 & J^*(k) \\ J(k) & 0 \end{pmatrix},$$

with $J(k) = ue^{-ika/2} + ve^{ika/2} + |w|e^{-i(3ka/2 - \phi)}$, where $a$ is the lattice constant. The dispersion relations of...
is dramatically suppressed (orange curve). We note that setting \( G \) that in the absence of the quantum metric [explored by the quantum metric dipole. To see this, note that plasmons in Fig. 2d (black) with non-reciprocity dominate: \( \tilde{V}^{1d} \) of the upper band; for simplicity, we approximate the polarization for the case where the chemical potential is in the upper band. For the chiral symmetric model considered here, the system’s Bloch states are constrained to lie in the equatorial plane of the Bloch sphere (see Fig. 2b), where the system’s Bloch states are constrained to lie in the equatorial plane of the Bloch sphere (see Fig. 2b), and that it is not symmetric with respect to the dispersion relation. Consequently, the relative shapes of the electronic wave packets at opposite Fermi points can differ significantly \( 28 \). This asymmetry directly leads to non-reciprocal coupling to electromagnetic fields.

To demonstrate non-reciprocity, we obtain for plasmon dispersion relation as zero modes of the dielectric function for the case where the chemical potential is in the upper band; for simplicity, we approximate the polarization function with Eq. (7) and use a contact interaction, \( V_{1d} \). For large contact interactions (dimensionless ratio: \( V_0 = V_{1d}/ua \gg 1 \)), this produces the non-reciprocal plasmons in Fig. 2a (black) with non-reciprocity dominated by the quantum metric dipole. To see this, note that in the absence of the quantum metric [explored by setting \( G = 0 \) in Eq. (7)] we find plasmon non-reciprocity is dramatically suppressed (orange curve). We note that for weaker interactions (smaller \( V_0 \)), nonreciprocity is no longer solely dominated by \( \tilde{V} \). Instead, classical contributions from \( \tilde{C} \) can also play a role, see SI.

Quantum metric plasmons in twisted bilayer graphene heterostructures: We now examine quantum metric plasmons [i.e. non-reciprocal plasmons dominated by \( \tilde{G}_{\alpha \beta} \)] in a promising candidate system: twisted bilayer graphene (TBG) heterostructures. TBG is a particularly attractive venue for quantum metric plasmons due to its strong Coulomb interactions with typical values that can exceed the bandwidth of the narrow bands \( 34 \). Additionally, when combined with hBN, TBG heterostructures break inversion symmetry to display large values of quantum geometric quantities \( 35-38 \) in its narrow bands (Fig. 3a). Importantly, in such heterostructures, a time-reversal symmetry (TRS) broken ferromagnetic phase emerges close to 3/4 filling \( 24,25 \).

Guided by this, we focus on the plasmonic excitations of TBG/hBN heterostructures in the magnetic phase near 3/4 filling (see schematic in Fig. 3a). We model the electrons in this system using the continuum model \( 39 \), including the effect of hBN alignment \( 35 \) as well as a modest heterostrain \( 40 \) (found in many TBG devices \( 41,44 \)). This model produces the narrow bands shown in Fig. 3, see SI for simulation details.

We numerically identify the plasmonic dispersion by plotting the loss function \( \text{Im}[-e^{-1}(q, \omega + i\eta)] \) in Fig. 3, including both intraband (\( \alpha = \beta \)) as well as interband (\( \alpha \neq \beta \)) contributions; \( \hbar \eta \) is a lifetime parameter \( 32 \). Here we have employed an effective 2D Coulomb potential \( V^{2D} = 2\pi e^2/(|k|q) \) with \( k = 3 \) the background dielectric constant \( 34 \), and have concentrated on electrons in the narrow bands \( 40 \).
Strikingly, a pronounced non-reciprocal TBG plasmon dispersion emerges in Fig. 3: with plasmon energies that rise above the bandwidth of the narrow bands. Interestingly, the bulk plasmon non-reciprocity is dominated by the quantum metric. To see this, when the Bloch overlaps are set artificially to \( F_{\alpha \beta} = \delta_{\alpha \beta} \) in Eq. (6) (i.e., only classical terms are allowed), an almost reciprocal plasmon dispersion relation is observed (see Fig. 3). The dichotomy between Fig. 3 (full polarization function) and Fig. 3 (only classical) highlights the crucial role that the quantum metric plays in the non-reciprocity of TBG plasmons.

The unusual non-reciprocal collective plasmon dynamics that we have unveiled is a new addition to a growing class of magneto-chiral effects \([1, 29, 47–51]\), the superconducting diode effect \([51]\). Given the fundamental origin of non-reciprocal plasmons (including the important role of the quantum metric), we anticipate a range of electronic processes that rely on Bloch wavefunction overlaps (e.g., scattering) can also be morphed to take on a chiral character.

**Acknowledgements:** This work was supported by Singapore MOE Academic Research Fund Tier 3 Grant MOE2018-T3-1-002 and a Nanyang Technological University start-up grant (NTU-SUG).
[33] C. Kane, L. Balents and M. P. A. Fisher, Phys. Rev. Lett. 79, 5086 (1997).
[34] C. Lewandowski and L. S. Levitov, Proceedings of the National Academy of Sciences 116, 20869–20874 (2019).
[35] P. A. Pantaleón, T. Low, and F. Guinea, Phys. Rev. B 103, 205403 (2021).
[36] D. Kaplan, T. Holder, B. Yan, arXiv:2101.07539 (2021).
[37] A. Arora, J. F. Kong and J. C. W. Song, Phys. Rev. B 104, L241404 (2021).
[38] S. Chaudhary, C. Lewandowski and G. Refael, arXiv:2107.09090 (2021).
[39] M. Koshino, N. F. Q. Yuan, T. Koretsune, M. Ochi, K. Kuroki, and L. Fu, Phys. Rev. X 8, 031087 (2018).
[40] Z. Bi, N. F. Q. Yuan, and L. Fu, Phys. Rev. B 100, 035448 (2019).
[41] A. Kerelsky, L. J. McGilly, D. M. Kennes, L. Xian, M. Yankowitz, S. Chen, K. Watanabe, T. Taniguchi, J. Hone, C. Dean, A. Rubio, and A. N. Pasupathy, Nature (London) 572, 95 (2019).
[42] Y. Xie, B. Lian, B. Jäck, X. Liu, C.-L. Chiu, K. Watanabe, T. Taniguchi, B. A. Bernevig, and A. Yazdani, Nature (London) 572, 101 (2019).
[43] Y. Choi, J. Kemmer, Y. Peng, A. Thomson, H. Arora, R. Polški, Y. Zhang, H. Ren, J. Alicea, G. Refael, F. von Oppen, K. Watanabe, T. Taniguchi, and S. Nadj-Perge, Nat. Phys. 15, 1174 (2019).
[44] N. P. Kazmierczak, M. Van Winkle, C. Ophus, K. C. Bustillo, S. Carr, H. G. Brown, J. Ciston, T. Taniguchi, K. Watanabe, and D. K. Bediako, Nat. Mater. (2021).
[45] N. C. H. Hesp, I. Torre, D. Rodan-Legrain, P. Novelli, Y. Cao, S. Carr, S. Fang, P. Stepanov, D. Barcons-Ruiz, H. Herzig-Sheinfux, K. Watanabe, T. Taniguchi, D. K. Efetov, E. Kaxiras, P. Jarillo-Herrero, M. Polini and F. H. L. Koppens, Nat. Phys. 17, 1162-1168 (2021).
[46] For the TBG parameters we used, excitations from remote bands have energies far larger than the plasmon energies considered.
[47] G. L. J. Rikken and E. Raupach, Nature 390, 493-494 (1997).
[48] Y. Iguchi, S. Uemura, K. Ueno and Y. Onose, Phys. Rev. B 92, 184419 (2015).
[49] S. Seki, Y. Okamura, K. Kondou, K. Shibata, M. Kubota, R. Takagi, F. Kagawa, M. Kawasaki, G. Tatara, Y. Otani, et al., Phys. Rev. B 93, 235131 (2016).
[50] T. Nomura, X.-X. Zhang, S. Zherlitsyn, J. Wosnitza, Y. Tokura, N. Nagaosa, S. Seki, Phys. Rev. Lett. 122, 145901 (2019).
[51] F. Ando, Y. Miyasaka, T. Li, J. Ishizuka, T. Arakawa, Y. Shiota, T. Moriyama, Y. Yanase, T. Ono, Nature 584, 373-376 (2020).
**Supplementary Information for “Quantum metric dipole and non-reciprocal bulk plasmons in parity-violating magnets”**

**Identity for expansion of coherence factor**

Here we summarize a set of identities which we use to expand the coherence factor $|\langle \psi(k+q) | \psi(k) \rangle|^2$ up to $\mathcal{O}(q^4)$. We begin with the Taylor expand for small $q$: $|\langle \psi(k+q) | \psi(k) \rangle| = |\langle \psi(k) \rangle| + q \langle \partial_k \psi(k) \rangle + \frac{q^2}{2} \langle \partial_k \partial_k \psi(k) \rangle + \frac{q^3}{6} \langle \partial_k \partial_k \partial_k \psi(k) \rangle + \mathcal{O}(q^4)$. Collecting terms, the coherence factor takes the form

$$|\langle \psi(k+q) | \psi(k) \rangle|^2 = 1 + q \langle \partial_k \psi(k) \rangle + \frac{q^2}{2} \langle \partial_k \partial_k \psi(k) \rangle + \frac{q^3}{6} \langle \partial_k \partial_k \partial_k \psi(k) \rangle + \mathcal{O}(q^4)$$

(S1)

where, as in the main text, repeated indices are implicitly summed. The functions $F^{(2,3)}_{ab}$ can be explicitly written as

$$F^{(2)}_{ab} = \frac{1}{2} [\langle \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle + \langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle] + \frac{1}{2} [\langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle + \langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle].$$

(S2)

and

$$F^{(3)}_{abc} = \frac{1}{6} [\langle \partial_k \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle + \langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle]$$

(S3)

We first examine $F^{(2)}_{ab}$. Recalling the normalization condition $\langle \psi(k) | \psi(k) \rangle = 1$ yields $\partial_k \partial_k \langle \psi(k) | \psi(k) \rangle = 0$ which helps us express the first term in Eq. (S2) as $-\text{Re}[\langle \partial_k \psi(k) | \partial_k \psi(k) \rangle]$. Similarly, the second term in Eq. (S3) can be instantly written in terms of the Berry connection: $\langle \psi(k) | \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle = A_k A_k(k)$ where $A_k = i \langle \psi(k) | \partial_k \psi(k) \rangle$. Putting these together, enables us to identify $F^{(2)}_{ab}$ in terms of the quantum metric

$$g_{ab}(k) = \text{Re}[\langle \partial_k \psi(k) | \partial_k \psi(k) \rangle] - A_k A_k(k) = -F^{(2)}_{ab}.$$  

(S4)

Next, we examine $q_a q_b q_c F^{(3)}_{abc}$. We again use $\langle \psi(k) | \psi(k) \rangle = 1$ so that $\partial_k \partial_k \partial_k \langle \psi(k) | \psi(k) \rangle = 0$ which helps us express the first term in Eq. (S3) as:

$$\frac{q_a q_b q_c}{6} [\langle \partial_k \partial_k \partial_k \psi(k) \rangle |\psi(k)\rangle + \langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle] = -\frac{q_a q_b q_c}{2} \partial_k \partial_k \partial_k \text{Re}[\langle \partial_k \partial_k \psi(k) | \partial_k \psi(k) \rangle].$$

(S5)

Here we have repeatedly summed across dummy indices. In the same fashion, the second term in Eq. (S3) can be expressed in terms of the derivative of products of Berry connections:

$$\frac{q_a q_b q_c}{2} [\langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle |\partial_k \partial_k \partial_k \psi(k) \rangle + \langle \psi(k) | \partial_k \partial_k \partial_k \psi(k) \rangle |\partial_k \partial_k \partial_k \psi(k) \rangle] = \frac{q_a q_b q_c}{2} \partial_k \partial_k \partial_k [A_k A_k A_k].$$

(S6)

By combining the identities in Eq. (S5) and (S6) we get

$$q_a q_b q_c F^{(3)}_{abc} = -\frac{q_a q_b q_c}{2} \partial_k \partial_k \partial_k \text{Re}[\langle \partial_k \partial_k \psi(k) | \partial_k \psi(k) \rangle] - A_k A_k A_k(k) = -\frac{q_a q_b q_c}{2} \partial_k \partial_k g_{abc}(k).$$

(S7)

Applying Eq. (S7) and Eq. (S4) into Eq. (S1) we obtain the identity

$$|\langle \psi(k+q) | \psi(k) \rangle|^2 = 1 - g_{ab}(k) q_a q_b - \frac{1}{2} \partial_k g_{bc}(k) q_a q_b q_c + \mathcal{O}(q^4)$$

(S8)

used in the main text.

**Non-reciprocal plasmons and dependence on interaction strength**

In the main text, we described how non-reciprocal quantum metric plasmons arise in a 1D magnetic bipartite model in presence of contact potential for $V_0 = 20$. In this section, we highlight the impact of the (contact) interaction strength $V_0$. In Fig. [S1] we display plasmon dispersion relations obtained from the zeroes of the dielectric function and
FIG. S1: Interaction strength dependence of non-reciprocal bulk plasmons. Panel (a)-(f) show plasmon dispersion for different values of $\tilde{V}_0$ as indicated. Here, black solid line shows plasmon dispersion obtained from zeros of the dielectric function approximating $\Pi(q, \omega)$ with Eq. (7) of the main text. To highlight the competition between $G$ and $C$, we also draw guidelines where quantum metric contribution is set to zero (i.e. $G = 0$) as the orange dashed curve, and where the classical velocity asymmetry contribution is set to zero (i.e. $C = 0$) as the dashed green line. Model parameters: $v = 0.5u$, $|w| = 0.4u$ and $\phi = 1$ along with $\mu = 0.8u$.

approximating the polarization function with Eq. (7) in the main text (black curves). To highlight the competition between $C$ and $G$, in addition to the plasmon dispersion (black curves) we plot guide lines when QM contribution is set to zero (orange dashed curve), i.e. $G = 0$, as well as when the classical velocity asymmetry contribution is set to zero (green dashed curve), i.e. $C = 0$.

As shown in Fig. S1, when interaction strength $\tilde{V}_0$ is large, the plasmon dispersion (black curve) adheres closer to the green guideline (as compared to the orange curve). This demonstrates that at large $\tilde{V}_0$, plasmon non-reciprocity is dominated by the quantum metric. In contrast, for intermediate values of $\tilde{V}_0$ the plasmon dispersion (black curve) lies in between orange and green curves; it adheres closer to the orange curve when $\tilde{V}_0$ is small. This demonstrates how $\tilde{V}_0$ controls the competition between quantum metric dominated non-reciprocity (from the $G$ contribution) prevalent at large interaction strength, and classical velocity asymmetry dominated non-reciprocity (from the $C$ contribution) prevalent at low interaction strength.

Continuum model for strained TBG-hBN heterostructure

In this section, we detail how we simulated the electronic structure of TBG using the continuum model. For TBG, we define the lattice structure as in Ref. [39]. In each graphene layer the primitive (original) lattice vectors are $\mathbf{a}_1 = a_G(1, 0)$ and $\mathbf{a}_2 = a_G(1/2, \sqrt{3}/2)$ with $a_G = 0.246$ nm being the lattice constant. The corresponding reciprocal space lattice vectors are $\mathbf{b}_1 = (2\pi/a_G)(1, -1/\sqrt{3})$ and $\mathbf{b}_2 = (2\pi/a_G)(0, 2/\sqrt{3})$, and Dirac points are located at $K_\zeta = -\zeta(2\mathbf{b}_1 + \mathbf{b}_2)/3$. For a twist angle $\theta$ (accounting for the rotation of layers), the lattice vectors of layer $l$ are given by $\mathbf{a}_{l,i} = R(\mp\theta/2)\mathbf{a}_i$, $\mp$ for $l = 1, 2$ respectively, and $R(\theta)$ represents rotation by an angle $\theta$ about the
normal. Also, from \( a_i, b_{i,j} = 2 \pi \delta_{i,j} \) we can check that the reciprocal lattice vectors become \( \mathbf{b}_{i,j} = R(\pi \theta/2) \mathbf{b}_i \) with corresponding Dirac points now located at \( \mathbf{K}_{i,\ell} = -\zeta(2 \mathbf{b}_{1,i} + \mathbf{b}_{1,2})/3 \).

At small angles, the slight mismatch of the lattice period between two layers gives rise to long range moiré superlattices. The reciprocal lattice vectors for these moiré superlattices are given as \( \mathbf{g}_i = \mathbf{b}_{1,i} - \mathbf{b}_{2,i} \). The superlattice vectors \( \mathbf{L} \), can then be found using \( \mathbf{g}_i, \mathbf{L}_j = 2 \pi \delta_{i,j} \), where \( \mathbf{L}_1 \) and \( \mathbf{L}_2 \) span the moiré unit cell with lattice constant \( L = L_1 = L_2 = a_G/[2 \sin \theta/2] \).

Next, when the moiré superlattice constant is much longer than the atomic scale, the electronic structure can be described using an effective continuum model for each valley \( \zeta = \pm \). The total Hamiltonian is block diagonal in the valley index, and for each valley, the effective Hamiltonian in the continuum model is written in terms of the sublattice and layer basis \((A_1, B_1, A_2, B_2)\) \[39\]

\[
H_{\zeta} = \left( \begin{array}{cc}
H_{1,\zeta}(\mathbf{p}) & T_{\zeta}^f \\
T_{\zeta} & H_{2,\zeta}(\mathbf{p})
\end{array} \right)
\]

where \( H_{1,\zeta} = -\hbar v_F R(\pi \theta/2) \mathbf{p} \cdot (\zeta \sigma_x, \sigma_y) \) is the Hamiltonian for each layer with \( \hbar v_F/a_G = 2135.4 \) meV, and

\[
T_{\zeta} = \left( \begin{array}{cc}
t_{AA} & t_{AB}^\ell e^{-i \zeta \pi/2} \\
t_{AB}^\ell e^{i \zeta \pi/2} & t_{AA}
\end{array} \right) e^{i \mathbf{g}_1 \cdot \mathbf{r}} + \left( \begin{array}{cc}
t_{AA} & t_{AB}^\ell e^{-i \zeta \pi/2} \\
t_{AB}^\ell e^{i \zeta \pi/2} & t_{AA}
\end{array} \right) e^{i \mathbf{g}_2 \cdot \mathbf{r}}
\]

where \( \mathbf{g}_\ell \) is the reciprocal lattice vector of mBZ. In what follows, we use the tunnelling parameters \( t_{AB} = 20 \) meV and \( t_{AB}^\ell = 97.5 \) meV. These parameters were used to simulate the optical properties of TBG in good agreement with a recent TBG experiment \[35\]. When hBN is aligned with the graphene layers, \( C_2 \) symmetry is broken modifying the layer Hamiltonians \( H_{1,\zeta} \). This can be described by introducing a sublattice staggered potential \( \Delta_l \) so that the Hamiltonian for each layer \( H_{1,\zeta}(\mathbf{p}) \rightarrow H_{1,\zeta}(\mathbf{p}) + \Delta_l \sigma_z \).

Finally, the presence of a uniaxial heterostrain in TBG of magnitude \( \chi \) can be described by the linear strain tensor \[40\]

\[
\mathcal{E}_l = \mathcal{F}(l) \chi \left( \begin{array}{cc}
-\cos^2 \varphi + \nu \sin^2 \varphi & (1 + \nu) \cos \varphi \sin \varphi \\
(1 + \nu) \cos \varphi \sin \varphi & \nu \cos^2 \varphi - \sin^2 \varphi
\end{array} \right)
\]

where \( \mathcal{F}(l = 1, 2) = \pi/2, \nu = 0.165 \) is the Poisson ratio of graphene and \( \varphi \) gives direction of the applied strain. The strain tensor satisfies general transformations in each layer, \( \mathbf{a}_i \rightarrow \mathbf{a}_i' = [1 + \mathcal{E}_l] \mathbf{a}_i \) and \( \mathbf{b}_i \rightarrow \mathbf{b}_i' = [1 - \mathcal{E}_l] \mathbf{b}_i \) for real and reciprocal lattice vectors respectively \[40\]. The strain induced geometric deformations affect the interlayer coupling and further changes the electron motion via gauge field \( \mathbf{A}_l = \sqrt{3/2} a_G (\mathcal{E}_l^{xx} + \mathcal{E}_l^{yy}, -2 \mathcal{E}_l^{xy}) \), where \( \beta = 3.14 \). As a result, we have \( \mathbf{p} \rightarrow \mathbf{p}_l = [1 + \mathcal{E}_l] \mathbf{k} - \mathbf{K}_{l,\zeta} \) with \( \mathbf{K}_{l,\zeta} = [1 - \mathcal{E}_l] \mathbf{K}_{l,\zeta} - \zeta \mathbf{A}_l \).

The effective TBG Hamiltonian modified by the effects of strain and hBN alignment with graphene layers via sublattice staggered potential, can be re-written as

\[
H_{\zeta} = \left( \begin{array}{cc}
H_{1,\zeta}(\mathbf{p}_1, \chi) + \Delta_1 \sigma_z & T_{\zeta}^f \\
T_{\zeta} & H_{2,\zeta}(\mathbf{p}_2, \chi) + \Delta_2 \sigma_z
\end{array} \right)
\]

Note that for a given \( \mathbf{q} \) in the mBZ, the \( 4 \times 4 \) Hamiltonian in Eq. \[S12\] is cast into a multiband eigensystem problem as the interlayer coupling leads to hybridisation of the eigenstates at Bloch vectors \( \mathbf{q} \) and \( \mathbf{q}' = \mathbf{p} + \mathbf{g} \), where \( \mathbf{g} = m_1 \mathbf{g}_1 + m_2 \mathbf{g}_2 + m_{1,2} \mathbf{g}_3 \) and \( m_{1,2} \in \mathbb{Z} \) \[39\]. We truncate the size of the matrix by defining a circular cut-off \( |\mathbf{p} - \mathbf{p}'| < 4 |\mathbf{g}_3| \). For a given Bloch vector \( \mathbf{p} \), this gives us 61 sites in reciprocal space, and a corresponding matrix of size \( 244 \times 244 \) which is then diagonalized to obtain eigenvalues and eigenvectors.

**Numerical Evaluation of plasmon dispersion in TBG**

The polarization function in Eq. \[6\] is calculated numerically for TBG on hBN with a heterostrain of \( \chi = 0.05\% \) and hBN staggered potential due to alignment between graphene and hBN layers as \( \Delta_1 = 0 \) and \( \Delta_2 = 0.5 \) meV. The \( k \)-space integral over the mBZ was performed as a Riemann sum on a grid of \( 400 \times 400 \) points with \( h \eta = 0.3 \) meV for each value of \( \mathbf{q} \) and \( \omega \). We include all intra- and inter-band contributions between the flat bands, and neglect the effect of remote bands as our parameters the flat and remote bands separation is nearly \( 80 \) meV which is much larger than the energy of plasmons of our interest [see fig. \[S2\]]. The corresponding dielectric function is obtained as:

\[
\varepsilon(q, \omega + i\eta) = 1 - \frac{2\pi e^2}{\kappa |q|} \Pi(q, \omega + i\eta)
\]

which is finally inverted to get the loss function. Here we have used the effective 2D Coulomb potential \( \hat{V}^{2D}(\mathbf{q}) = 2\pi e^2/(\kappa |q|) \) with \( \kappa = 3 \) the background dielectric constant. The plasmons appear as highly peaked values in the loss function as shown in the main text.
FIG. S2: TBG band structure along the specified path in mBZ for interlayer couplings $t_{AA} = 20$ meV, $t'_{AB} = 97.5$ meV with heterostrain of $\chi = 0.05\%$. We take sublattice symmetry breaking potentials $\Delta_1 = 0$ and $\Delta_2 = 0.5$ meV.