ON THE PERIODICITY OF A CLASS OF ARITHMETIC FUNCTIONS ASSOCIATED WITH MULTIPLICATIVE FUNCTIONS

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Abstract. Let \( k \geq 1, a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers. Let \( f \) be a multiplicative function with \( f(n) \neq 0 \) for all positive integers \( n \). We define the arithmetic function \( g_{k,f} \) for any positive integer \( n \) by

\[
g_{k,f}(n) := \prod_{i=0}^{k} \left( b + a(n + ic) \right) \text{lcm}_{0 \leq i \leq k} \{ b + a(n + ic) \}.
\]

We first show that \( g_{k,f} \) is periodic and \( \text{lcm}(1, \ldots, k) \) is its period. Consequently, we provide a detailed local analysis to the periodic function \( g_{k,\varphi} \), and determine the smallest period of \( g_{k,\varphi} \), where \( \varphi \) is the Euler phi function.

1. Introduction

Chebyshev [3] initiated the study of the least common multiple of consecutive positive integers for the first significant attempt to prove prime number theorem. An equivalent of prime number theorem says that \( \log \text{lcm}(1, \ldots, n) \sim n \) as \( n \) goes to infinity. Hanson [6] and Nair [12] got the upper bound and lower bound of \( \text{lcm}_{1 \leq i \leq n} \{i\} \) respectively. Bateman, Kalb and Stenger [2] obtained an asymptotic estimate for the least common multiple of arithmetic progressions. Hong, Qian and Tan [8] obtained an asymptotic estimate for the least common multiple of a sequence of products of linear polynomials.

On the other hand, the study of periodic arithmetic function has been a common topic in number theory for a long time. For the related background information, we refer the readers to [1] and [11]. Recently, this topic is still active. When studying the arithmetic properties of the least common multiple of finitely many consecutive positive integers, Farhi [4] defined the arithmetic function \( g_k \) for any positive integer \( n \) by

\[
g_k(n) := \prod_{i=0}^{k} (n + i) \text{lcm}_{0 \leq i \leq k} \{n + i\}.
\]

In the same paper, Farhi showed that \( g_k \) is periodic of \( k! \) and posed an open problem of determining the smallest period of \( g_k \). Let \( P_k \) be the smallest period of \( g_k \). Define \( L_0 := 1 \) and for any integer \( k \geq 1 \), we define \( L_k := \text{lcm}(1, \ldots, k) \). Subsequently, Hong and Yang [9] improved the period \( k! \) to \( L_k \) and produced a conjecture stating that \( \frac{L_k + 1}{k+1} \) divides \( P_k \) for all nonnegative integers \( k \). By proving the Hong-Yang conjecture, Farhi and Kane [5] determined the smallest period of \( g_k \) and finally solved the open problem posed by Farhi [4]. Let \( k \geq 1, a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers. Let \( \mathbb{Q} \) and \( \mathbb{N} \) denote the field of rational numbers and the set of nonnegative integers. Define \( \mathbb{N}^* := \mathbb{N} \setminus \{0\} \). In order to investigate the least common multiple of any \( k+1 \) consecutive terms in the arithmetic progression \( \{b + am\}_{m \in \mathbb{N}^*} \), Hong and Qian [7] introduced the arithmetic function \( g_{k,a,b} \) defined for any positive integer \( n \) by

\[
g_{k,a,b}(n) := \prod_{i=0}^{k} (b + a(n + i)) \text{lcm}_{0 \leq i \leq k} \{b + a(n + i)\}.
\]

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showed that $g_{k,a,b}$ is periodic and obtained the formula of the smallest period of $g_{k,a,b}$, which extends the Farhi-Kane theorem to the general arithmetic progression case.

Let $f$ be a multiplicative function with $f(n) \neq 0$ for all $n \in \mathbb{N}^*$. To measure the difference between $\prod_{i=0}^{k} f(b + a(n + ic))$ and $f(lcm_{0 \leq i \leq k}\{b + a(n + ic)\})$, we define the arithmetic function $g_{k,f}$ for any positive integer $n$ by

$$g_{k,f}(n) := \frac{\prod_{i=0}^{k} f(b + a(n + ic))}{f(lcm_{0 \leq i \leq k}\{b + a(n + ic)\})}.$$  

One naturally asks the following interesting question.

**Problem 1.1.** Let $f$ be a multiplicative function such that $f(n) \neq 0$ for all positive integers $n$. Is $g_{k,f}$ periodic, and if so, what is the smallest period of $g_{k,f}$?

As usual, for any prime number $p$, we let $v_p$ be the normalized $p$-adic valuation of $\mathbb{Q}$, i.e., $v_p(a) = s$ if $p^s \parallel a$. For any real number $x$, by $\lfloor x \rfloor$ we denote the largest integer no more than $x$. Evidently, $v_p(L_k) = \max_{1 \leq i \leq k}\{v_p(i)\} = \lfloor \log_p k \rfloor$ is the largest exponent of a power of $p$ that is at most $k$. We have the first main result of this paper which answers the first part of Problem 1.1.

**Theorem 1.2.** Let $k \geq 1$, $a \geq 1$, $b \geq 0$ and $c \geq 1$ be integers. If $f$ is a multiplicative function so that $f(n) \neq 0$ for all $n \in \mathbb{N}^*$, then the arithmetic function $g_{k,f}$ is periodic and $cL_k$ is its period.

It seems to be difficult to answer completely the second part of Problem 1.1. We here are able to answer it for the Euler phi function $\varphi$ case. In fact, we first prove a generalization of Hua’s identity and then use it to show that the arithmetic function $g_{k,\varphi}$ is periodic. Subsequently, we develop $p$-adic techniques to determine the exact value of the smallest period of $g_{k,\varphi}$. Note that it was proved by Farhi and Kane [5] that there is at most one prime $p \leq k$ such that $v_p(k + 1) \geq v_p(L_k) \geq 1$. We can now state the second main result of this paper as follows.

**Theorem 1.3.** Let $k \geq 1$, $a \geq 1$, $b \geq 0$ and $c \geq 1$ be integers. Let $d := \gcd(a,b)$ and $a' := a/d$. Then $g_{k,\varphi}$ is periodic, and its smallest period equals $Q_{k,a',c}$ except that $v_p(k + 1) \geq v_p(L_k) \geq 1$ for at most one odd prime $p \nmid a'$, in which case its smallest period is equal to $\frac{cL_k}{\eta_{2,k,a',c}\prod_{\text{prime } q|a'} q^{v_q(cL_k)}}$, where

$$Q_{k,a',c} := \frac{cL_k}{\eta_{2,k,a',c}\prod_{\text{prime } q|a'} q^{v_q(cL_k)}},$$

and

$$\eta_{2,k,a',c} := \begin{cases} 2^{v_2(L_k)}, & \text{if } 2 \nmid a' \text{ and } v_2(k + 1) \geq v_2(L_k) \geq 2, \\ 2, & \text{if } 2 \nmid a \text{ and } v_2(cL_k) = 1, \text{or } k = 3, 2 \nmid a \text{ and } 2 \nmid c, \text{or } k = 3, 2 \nmid a' \text{ and } 2 \nmid d, \\ 1, & \text{otherwise}. \end{cases}$$

So we answer the second part of Problem 1.1 for the Euler phi function.

The paper is organized as follows. In Section 2, we show that $g_{k,f}$ is periodic and $cL_k$ is its period. In Section 3, we provide a detailed $p$-adic analysis to the periodic arithmetic function $g_{k,\varphi}$, and finally we determine the smallest period of $g_{k,\varphi}$. The final section is devoted to the proof of Theorem 1.3.
2. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. We begin with the following lemma.

**Lemma 2.1.** Let $A$ be any given totally ordered set, and $a_1, \ldots, a_n$ be any $n$ nonzero elements of $A$ (not necessarily different). If we can define formal multiplication and formal division for the set $A$, then we have

$$\max(a_1, \ldots, a_n) = a_1 \cdots a_n \prod_{r=2}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (\min(a_{i_1}, \ldots, a_{i_r}))^{(-1)^{r-1}}.$$ 

**Proof.** Rearrange these $n$ elements $a_1, \ldots, a_n$ such that $a_{j_1} \geq \cdots \geq a_{j_n}$. For convenience, we let $b_i = a_{j_i}$, $i = 1, 2, \ldots, n$. Then the desired result in Lemma 2.1 becomes

$$b_1 = b_1 \cdots b_n \prod_{r=2}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (\min(b_{i_1}, \ldots, b_{i_r}))^{(-1)^{r-1}}. \quad (2.1)$$

To prove the result, it suffices to prove that for each $b_i$ the number of times that $b_i$ occurs on the left side of (2.1) equals the number of times that $b_i$ occurs on the right side of (2.1). We distinguish the following two cases.

**Case 1.** If $b_1 = b_2 = \cdots = b_n$, then the number of times that $b_1$ occurs on the right side of (2.1) is

$$n - \binom{n}{2} + \cdots + (-1)^{n-1} \binom{n}{n} = -1 + \sum_{r=1}^{n} (-1)^{r-1} \binom{n}{r} + 1 = -(1-1)^n + 1 = 1.$$

Whereas, 1 is just the number of times that $b_1$ occurs on the left side of (2.1).

**Case 2.** If there exists a positive integer $s < n$ such that $b_1 = b_2 = \cdots = b_s > b_{s+1}$, then the number of times $b_1$ occurs on the right side of (2.1) is: $s - \binom{2}{2} + \cdots + (-1)^{s-1} \binom{s}{s} = 1$. For any $j > s$, we can always assume that $b_{t+1}, b_{t+2}, \ldots, b_{s+t}$ are just the $t$ terms of the sequence $\{b_i\}_{i=1}^{n}$ such that $b_s > b_{s+1} = \cdots = b_j = \cdots = b_{s+t}$ for some $t \geq s$. Thus, the number of times that $b_j$ occurs on the right side of (2.1) is

$$l - (\binom{t+2}{2} - \binom{t}{2}) + \cdots + (-1)^{t-1} \left(\binom{t+l}{t} - \binom{t}{t}\right) + (t+l) \binom{t+l}{t+1} + \cdots + (-1)^{t+l-1} \binom{t+l}{t+l} = l + \sum_{r=2}^{t} (-1)^{r+1} \binom{t}{r} \sum_{i=2}^{l} (-1)^{i-1} \binom{t+l}{i} = l + (1-1)t + \binom{t}{0} - (1-1)^{t+l} + \binom{t+l}{0} - \binom{t+l}{1} = 0.$$

This completes the proof of Lemma 2.1. \hfill \Box

In [10], Hua gave the following beautiful identity

$$\text{lcm}(a_1, \ldots, a_n) = a_1 \cdots a_n \prod_{r=2}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (\gcd(a_{i_1}, \ldots, a_{i_r}))^{(-1)^{r-1}},$$

where $a_1, \ldots, a_n$ are any given $n$ positive integers. In what follows, using Lemma 2.1, we generalize the above Hua’s identity to the multiplicative function case.
Lemma 2.2. Let \( f \) be a multiplicative function, and \( a_1, a_2, \ldots, a_n \) be any \( n \) positive integers. If \( f(m) \neq 0 \) for each \( m \in \mathbb{N}^* \), then

\[
f(\text{lcm}(a_1, a_2, \ldots, a_n)) = f(1) \cdots f(a_n) \prod_{r=2}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (f(\gcd(a_{i_1}, \ldots, a_{i_r}))^{(-1)^{r-1}}).
\]

Proof. Since \( f \) is a multiplicative function, we have

\[
f(\text{lcm}(a_1, a_2, \ldots, a_n)) = \prod_{p \text{ prime}} f(p^{\max(v_p(a_1), v_p(a_2), \ldots, v_p(a_n))})
\]

and

\[
f(\gcd(a_{i_1}, \ldots, a_{i_r})) = \prod_{p \text{ prime}} f(p^{\min(v_p(a_{i_1}), \ldots, v_p(a_{i_r}))}).
\]

Thus it suffices to prove that

\[
f(p^{\max(v_p(a_1), v_p(a_2), \ldots, v_p(a_n))}) = \prod_{r=1}^{n} \prod_{1 \leq i_1 < \cdots < i_r \leq n} (f(p^{\min(v_p(a_{i_1}), \ldots, v_p(a_{i_r}))})^{(-1)^{r-1}} (2.2)
\]

for every prime \( p \). Now we define an order \( \geq \) for the set \( S = \{ f(p^m) : m \in \mathbb{N} \} \) according to the size of the power \( m \) of the prime \( p \). That is, \( f(p^i) \geq f(p^j) \) if \( i \geq j \) and \( f(p^i) \succ f(p^j) \) if \( i > j \). It is easy to check that \( S \) is a totally ordered set for the order \( \geq \). So the equality (2.2) follows immediately from Lemma 2.1 by letting \( a_i = f(p^{v_p(a_i)}) \) for \( 1 \leq i \leq n \) in Lemma 2.1. The proof of Lemma 2.2 is complete. \( \square \)

If \( \gcd(a_i, a_j) = \gcd(b_i, b_j) \) for any \( 1 \leq i < j \leq n \), then for any \( t \geq 3 \), one has \( \gcd(a_{i_1}, a_{i_2}, \ldots, a_{i_t}) = \gcd(b_{i_1}, b_{i_2}, \ldots, b_{i_t}) \) for any \( 1 \leq i_1 < \cdots < i_t \leq n \). Therefore we immediately derive the following result from Lemma 2.2.

Lemma 2.3. Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be any \( 2n \) positive integers. Let \( f \) be a multiplicative function with \( f(n) \neq 0 \) for all \( n \in \mathbb{N}^* \). If \( \gcd(a_i, a_j) = \gcd(b_i, b_j) \) for any \( 1 \leq i < j \leq n \), then we have

\[
\frac{\prod_{1 \leq i \leq n} f(a_i)}{f(\text{lcm}_{1 \leq i \leq n}(a_i))} = \frac{\prod_{1 \leq i \leq n} f(b_i)}{f(\text{lcm}_{1 \leq i \leq n}(b_i))}.
\]

We are now in a position to show Theorem 1.2.

Proof of Theorem 1.2. Let \( n \) be a given positive integer. For any \( 0 \leq i < j \leq k \), we have

\[
\gcd(b + a(n + ic + cL_k), b + a(n + jc + cL_k)) = \gcd(b + a(n + ic + cL_k), (j-i)ac)
\]

\[
= \gcd(b + a(n + ic), (j-i)ac)
\]

\[
= \gcd(b + a(n + ic), b + a(n + jc)).
\]

Thus by Lemma 2.3, we obtain that \( g_{k,f}(n + cL_k) = g_{k,f}(n) \) for any positive integer \( n \). Therefore \( g_{k,f} \) is periodic and \( cL_k \) is its period. \( \square \)

Obviously, by Theorem 1.2, the arithmetic function \( g_{k,\varphi} \) is periodic and \( cL_k \) is its period. In the next section, we will provide detailed \( p \)-adic analysis to the arithmetic function \( g_{k,\varphi} \) which leads us to determine the exact value of the smallest period of \( g_{k,\varphi} \).
3. Local analysis of \( g_{k,\varphi} \)

Throughout this section, we let \( a' = a/d \) and \( b' = b/d \) with \( d = \gcd(a, b) \). Then \( \gcd(a', b') = 1 \). Let

\[
S_{k, a', b', c}(n) := \{ b' + a'n, b' + a'(n + c), \ldots , b' + a'(n + kc) \}
\]

be the set consisting of \( k + 1 \) consecutive jumping terms with gap \( c \) in the arithmetic progression \( \{ b' + a'm \}_{m \in \mathbb{N}} \). For any given prime number \( p \), define \( g_{p, k, \varphi} \) for any \( n \in \mathbb{N}^* \) by \( g_{p, k, \varphi}(n) := v_p(g_{k, \varphi}(n)) \). Let \( P_{k, \varphi} \) be the smallest period of \( g_{k, \varphi} \). Then \( g_{p, k, \varphi} \) is a periodic function for each prime \( p \) and \( P_{k, \varphi} \) is a period of \( g_{p, k, \varphi} \). Let \( P_{p, k, \varphi} \) be the smallest period of \( g_{p, k, \varphi} \). Since

\[
\varphi(b + a(n + ic)) = \varphi(d(b' + a'(n + ic))) = \varphi\left( \prod_{p|d} p^{v_p(b' + a'(n + ic))} \prod_{p|d} p^{v_p(b' + a'(n + ic))} \right)
\]

we have

\[
g_{k,\varphi}(n) = \frac{\prod_{i=0}^{k} \varphi(b + a(n + ic))}{\varphi(\text{lcm}_{a' \leq k} \{ b + a(n + ic) \})} = \frac{\prod_{i=0}^{k} \varphi(d(b' + a'(n + ic)))}{\varphi(d \cdot \text{lcm}_{a' \leq k} \{ b' + a'(n + ic) \})}
\]

\[
= \frac{\prod_{i=0}^{k} \varphi(d)\left( \prod_{p|d} p^{v_p(b' + a'(n + ic))} \right)\left( \prod_{p|d} \varphi(p^{v_p(b' + a'(n + ic))}) \right)}{\varphi(d)\left( \prod_{p|d} p^{\max_{a' \leq k} \{ v_p(b' + a'(n + ic)) \}} \right)\left( \prod_{p|d} \varphi(p^{\max_{a' \leq k} \{ v_p(b' + a'(n + ic)) \}}) \right)}
\]

\[
= (\varphi(d))^{-k} \frac{\prod_{i=0}^{k} \left( \prod_{p|d} p^{v_p(b' + a'(n + ic))} \right)\left( \prod_{p|d} \varphi(p^{v_p(b' + a'(n + ic))}) \right)}{\prod_{p|d} \varphi(p^{\max_{a' \leq k} \{ v_p(b' + a'(n + ic)) \}})\left( \prod_{p|d} \varphi(p^{\max_{a' \leq k} \{ v_p(b' + a'(n + ic)) \}}) \right)}.
\]

Note that for any prime \( q \), we have that for any positive integer \( e \), \( \varphi(q^e) = q^{e-1}(q - 1) \). So when computing \( p \)-adic valuation of \( g_{k,\varphi}(n) \), we not only need to compute \( v_p(\varphi(p^a)) \) for \( a \geq 2 \), but also need to consider \( p \)-adic valuation of \( q - 1 \) for those primes \( q \) with \( p|q(q - 1) \). By some computations, we obtain the following two equalities.

If \( p \nmid d \), then

\[
g_{p, k, \varphi}(n) = \sum_{m \in S_{k, a', b', c}(n)} \max(0, \# \{ m \in S_{k, a', b', c}(n) : q|m \} - 1) \cdot v_p(q - 1) + k v_p(\varphi(d)) + \sum_{c \geq 2} \max(0, \# \{ m \in S_{k, a', b', c}(n) : p^c|m \} - 1)
\]

\[
+ \sum_{\text{prime } q : q|d, p|(q - 1)} \max(0, \# \{ m \in S_{k, a', b', c}(n) : q|m \} - 1) \cdot v_p(q - 1).
\]
If \( p \mid d \), then
\[
g_{p,k,\varphi}(n) = kv_p(\varphi(d)) + \sum_{i=0}^{k} v_p(b' + a'(n + ic)) - \max_{0 \leq i \leq k} \{ v_p(b' + a'(n + ic)) \}
\]
\[
+ \sum_{\text{prime } q \mid d, q \neq p} \max(0, \#\{m \in S_{k,a',b',c}(n) : q \mid m\} - 1) \cdot v_p(q - 1)
\]
\[
= kv_p(\varphi(d)) + \sum_{e \geq 1} \max(0, \#\{m \in S_{k,a',b',c}(n) : p^e \mid m \} - 1)
\]
\[
+ \sum_{\text{prime } q \mid d, p \mid (q - 1)} \max(0, \#\{m \in S_{k,a',b',c}(n) : q \mid m\} - 1) \cdot v_p(q - 1).
\]

(3.2)

In order to analyze the function \( g_{p,k,\varphi} \) in detail, we need the following results.

**Lemma 3.1.** Let \( e \) and \( m \) be positive integers. If \( p \mid a' \), then any \( p^e \) consecutive terms in the arithmetic progression \( \{b' + a'(m + ic)\}_{i \in \mathbb{N}} \) are pairwise incongruent modulo \( p^e \varphi(c) + c \). In particular, there is at most one term divisible by \( p \) for each prime \( p \) is periodic for each prime \( p \).

**Proof.** Suppose that there are two integers \( i, j \) such that \( 0 < j - i \leq p^e - 1 \) and \( b' + (m + ic) a' \equiv b' + (m + jc) a' \pmod{p^e \varphi(c) + c} \). Then \( p^e \mid (j - i) a' \). Since \( \gcd(p, a') = 1 \), we have \( p^e \mid (j - i) \). This is a contradiction. \( \Box \)

**Lemma 3.2.** Let \( F \) be a positive rational-valued arithmetic function. For any prime \( p \), define \( F_p \) by \( F_p(n) := v_p(F(n)) \) for \( n \in \mathbb{N}^* \). Then \( F \) is periodic if and only if \( F_p \) is periodic for each prime \( p \) and \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \) is finite, where \( T_{p,F} \) is the smallest period of \( F_p \). Furthermore, if \( F \) is periodic, then the smallest period \( T_F \) of \( F \) is equal to \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \).

**Proof.** \( \Rightarrow \) Since \( F \) is periodic and \( T_F \) is its smallest period, we have \( F(n + T_F) = F(n) \) for any \( n \in \mathbb{N}^* \), and hence \( F_p(n + T_F) = v_p(F(n + T_F)) = v_p(F(n)) = F_p(n) \). In other words, \( F_p \) is periodic and \( T_F \) is a period of \( F_p \) for every prime \( p \). So we have \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \mid T_F \) and \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \) is finite.

\( \Leftarrow \) Since for any \( n \in \mathbb{N}^* \), we have that \( v_q(F(n + \operatorname{lcm}_{p \mid F} \{T_{p,F} \}) = v_q(F(n)) \) for each prime \( q \). Thus \( F(n + \operatorname{lcm}_{p \mid F} \{T_{p,F} \}) = F(n) \) for any \( n \in \mathbb{N}^* \). So \( F \) is periodic and \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \) is a period of it. Hence \( T_F \) divides \( \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \).

From the above discussion, we immediately derive that \( T_F = \operatorname{lcm}_{p \mid F} \{T_{p,F} \} \) if \( F \) is periodic. \( \Box \)

For any prime \( p \geq cL_k + 1 \), we have by Lemma 3.1 that there is at most one term divisible by \( p \) in \( S_{k,a',b',c}(n) \) and there is at most one element divisible by the prime \( q \) satisfying \( p \mid (q - 1) \) in \( S_{k,a',b',c}(n) \). Thus for any prime \( p \geq cL_k + 1 \), we can get from (3.1) and (3.2) that \( g_{p,k,\varphi}(n) = kv_p(\varphi(d)) \) for every positive integer \( n \). Namely, we have \( g_{p,k,\varphi} = 1 \) for each prime \( p \) such that \( p \geq cL_k + 1 \). Thus by Lemma 3.2, we immediately have the following.

**Lemma 3.3.** We have
\[
P_{k,\varphi} = \operatorname{lcm}_{p \leq cL_k} \{P_{p,k,\varphi} \}.
\]
In what follows it is enough to compute $P_{p, k, \varphi}$ for every prime $p$ with $p \leq cL_k$. First we need to simplify $g_{p, k, \varphi}$ for $p \leq cL_k$. For any prime $q$ satisfying $q \nmid cL_k$, we obtain by Lemma 3.1 that there is at most one term divisible by $q$ in $S_{k, a', b', c}(n)$. On the other hand, for any prime $q$ satisfying $q|a'$, we have $\gcd(q, b') = 1$ since $\gcd(a', b') = 1$. Thus for $0 \leq i \leq k$, we have that $\gcd(q, b' + a'(n + ic)) = 1$ for all $n \in \mathbb{N}$. So there is no term divisible by any prime factor $q$ of $a'$ in $S_{k, a', b', c}(n)$. Thus from (3.1) and (3.2), we derive the following equality:

$$g_{p, k, \varphi}(n) = k\nu_p(\varphi(d)) + \sum_{e=1}^{v_p(cL_k)} f_e(n) + \sum_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)} \cdot q^n} h_q(n),$$

(3.3)

where

$$f_e(n) := \begin{cases} \max(0, \# \{m \in S_{k, a', b', c}(n) : p^e|m\} - 1), & \text{if } p \nmid q \text{ and } e = 1, \\ 0, & \text{otherwise} \end{cases}$$

and

$$h_q(n) := \max(0, \# \{m \in S_{k, a', b', c}(n) : q|m\} - 1) \cdot v_p(q - 1).$$

For any positive integer $n$, it is easy to check that $f_e(n + p^{\nu_p(cL_k)}) = f_e(n)$ for each $1 \leq e \leq v_p(cL_k)$ and $h_q(n + q) = h_q(n)$ for each prime $q$ such that $q \nmid a$, $q|cL_k$ and $p|(q - 1)$. Consequently, we obtain that $p^{\nu_p(cL_k)} \prod_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)}} q$ is a period of the function $g_{p, k, \varphi}$. To get the smallest period of $g_{p, k, \varphi}$ for each prime $p \leq cL_k$, we need to make more detailed $p$-adic analysis about $g_{p, k, \varphi}$. We divide it into the following four cases.

**Lemma 3.4.** Let $p$ be a prime such that $p \leq cL_k$ and $p \nmid cL_k$. Then

$$P_{p, k, \varphi} = \prod_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)}} q.$$  

**Proof.** Since $p \leq cL_k$ and $p \nmid cL_k$, we have $k < p \leq cL_k$ and $v_p(cL_k) = 0$. Hence we have that $g_{p, k, \varphi}(n) = k\nu_p(\varphi(d)) + \sum_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)}} h_q(n)$ by (3.3). If there is no prime $q$ satisfying $q|cL_k$, $p|(q - 1)$ and $q \nmid a$, then we have $g_{p, k, \varphi}(n) = k\nu_p(\varphi(d))$ for any positive integer $n$, and hence $P_{p, k, \varphi} = 1$ for such primes $p$. If there is a prime $q$ satisfying $q|cL_k$, $p|(q - 1)$ and $q \nmid a$, then we must have $q|c$, since such $q$ must satisfy $q \geq p + 1 > k$. So by the argument before Lemma 3.4, we have that $A := \prod_{q \nmid cL_k, \, p \nmid (q-1)} q$ is a period of $g_{p, k, \varphi}$.

Now it remains to prove that $A$ is the smallest period of $g_{p, k, \varphi}$. For any prime factor $q$ of $A$, we can choose a positive integer $n_0$ such that $v_q(b' + a'n_0) \geq 1$ because $q \nmid a$. Since $q > k$ and $q|c$, we have that $v_q(b' + a'(n_0 + ic)) \geq 1$ and $v_q(b' + a'(n_0 + ic + A/q)) = v_q(a'A/q) = 0$ for each $0 \leq i \leq k$. Hence there is no term divisible by $q$ in $S_{k, a', b', c}(n_0 + A/q)$. Thus $h_q(n_0) = k\nu_q(p(q - 1)) \geq k > 0 = h_q(n_0 + A/q)$. On the other hand, $h_q(n_0) = h_q(n_0 + A/q)$ for any other prime factors $q' \neq q$ of $A$. It then follows that $g_{p, k, \varphi}(n_0) \neq g_{p, k, \varphi}(n_0 + A/q)$. Therefore $A$ is the smallest period of $g_{p, k, \varphi}$. This completes the proof of Lemma 3.4. \qed

**Lemma 3.5.** Let $p$ be a prime such that $p|cL_k$ and $p|a'$. Then

$$P_{p, k, \varphi} = \left( \prod_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)}, \, k+1 \not\equiv 0 \pmod{q}} q \right) \left( \prod_{\text{prime } q \mid cL_k, \, q \nmid p^{(q-1)}} q \right).$$
Lemma 3.6. Let \( p \) be a prime such that \( p | cL_k \), \( p \nmid a' \) and \( p \nmid d \). Then

\[
P_{p,k,\varphi} = p^{c(p,k)} \left( \prod_{\text{prime } q \mid \varphi(a), \ q \nmid cL_k, \ q \mid p \mid q \mid p(q-1), \ k+1 \not\equiv 0 \ (\text{mod } q)} q \right) \left( \prod_{\text{prime } q \mid \varphi(a), \ q \nmid d, \ q \mid p \mid q \mid p(q-1)} q \right),
\]

where

\[
c(p,k) := \begin{cases} 
0, & \text{if } v_p(cL_k) = 1, \\
v_p(c), & \text{if } v_p(k+1) \geq v_p(L_k) \text{ and } v_p(cL_k) \geq 2, \\
v_p(cL_k), & \text{if } v_p(k+1) < v_p(L_k) \text{ and } v_p(cL_k) \geq 2.
\end{cases}
\]
Proof. From (3.3), we get that
\[ g_p(n) = k v_p(\varphi(d)) + \sum_{e=2}^{v_p(cL_k)} f_e(n) + \sum_{\text{prime } q \mid n} h_q(n). \] (3.4)

Let \( A \) denote the number \( p^{e(p,k)} \prod_{\text{prime } q \mid n} q \) for every prime factor \( p, k, \varphi \). We distinguish the following cases.

Case 1. \( v_p(cL_k) = 1 \). Since \( v_p(cL_k) = 1 \), we have
\[ g_p(n) = k v_p(\varphi(d)) + \sum_{\text{prime } q \mid n} h_q(n) \]
for every positive integer \( n \) by (3.4). The process of proving that \( A \) is the smallest period of \( g_p, \varphi \) is the same as the proof of Lemma 3.5, one can easily check it.

Case 2. \( v_p(cL_k) = 2 \). If \( v_p(cL_k) = 2 \), we obtain that \( p^{e(p,c)} \) is a period of \( f_1 \) for each \( 2 \leq e \leq v_p(cL_k) = v_p(c) \). By the same method as in the proof of Lemma 3.5, we can derive that \( A / p^{e(p,c)} \) is a period of \( \sum_{\text{prime } q \mid n} h_q(n) \), and hence by (3.4) \( A \) is a period of \( g_p, \varphi \).

Now it suffices to prove that \( A / p \) is not the period of \( g_p, \varphi \) for any prime factor \( P \) of \( A \).

For the prime \( p \), we have by (3.4) that \( A / p \) is a period of \( f_1 \) for each \( 2 \leq e \leq v_p(c) - 1 \) and also a period of \( h_q \) for each prime \( q \) such that \( q \not\mid a \), \( q \mid cL_k \) and \( p, q \not\mid (q - 1) \). It is easy to see that \( p^{e(p,c)} | b' + a'(n_0 + ic) \) and \( p^{e(p,c)} \not\mid b' + a'(n_0 + A/p + ic) \) for each \( 0 \leq i \leq k \). Thus comparing the two sets \( S_{k,a',b',c}(n_0) = \{ b' + a'(n_0 + ic) \} \) and \( S_{k,a',b',c}(n_0 + A/p) = \{ b' + a'(n_0 + A/p + ic) \} \), we obtain that
\[ f_{v_p(c)}(n_0 + A/p) = 0, \# \{ m \in S_{k,a',b',c}(n_0 + A/p) : p^{e(p,c)} | m \} = 0. \]
Therefore, \( A / p \) is not the period of \( g_p, \varphi \).

For any prime factor \( q \) of \( A \) such that \( q \mid cL_k, p, q \not\mid (q - 1) \) and \( q \not\mid a \). It is easy to see that \( A / q \) is a period of \( \sum_{e=2}^{v_p(cL_k)} f_e(n) + \sum_{\text{prime } q \mid n} h_q(n) \) for each prime factor \( q \) of \( A \) satisfying \( q/e(\varphi(cL_k), p) \) and \( q/e \). Similarly to the proof of Lemma 3.5, we can deduce that \( A / q \) is not the period of \( g_p \), and hence \( A / q \) is not the period of \( g_p, \varphi \). Therefore, \( A \) is the smallest period of \( g_p, \varphi \) in this subcase.

Subcase 2.1. \( p > k \). To prove that \( A \) is a period of \( g_p, \varphi \), it suffices to prove that \( p^{e(p,c)} \) is a period of \( f_1 \) for each \( 1 \leq e \leq v_p(cL_k) \) by the argument in Subcase 2.1 of this proof. For each positive integer \( n \), comparing the two sets \( S_{k,a',b',c}(n) = \{ b' + a'(n + ic) \} \) and \( S_{k,a',b',c}(n + c) = \{ b' + a'(n + ic) \} \), we find that their distinct terms are \( b' + a'n \) and \( b' + a'(n + (k+1)c) \). From \( v_p(k+1) \geq v_p(L_k) \) we deduce that
\[ b' + a'n \equiv b' + a'(n + (k+1)c) \pmod{p^{e(p,c)}}. \]
Therefore we obtain that \( f_e(n) = f_e(n + c) \) for each \( e \in \{ 2, \ldots, v_p(cL_k) \} \). Since \( \gcd(c/p^{e(p,c)}, p) = 1 \), we can always find two integers \( r, r_1 \) such that \( r/c/p^{e(p,c)} = r_1/p^{e(p,c)} + 1 \). Note that \( p^{e(p,c)} \) is a period of \( f_e \) for each \( 2 \leq e \leq v_p(cL_k) \). Therefore, we have
f_e(n + p^v_p(c)) = f_e(n + p^v_p(c) + t_1 p^v_p(cL_k)) = f_e(n + t_1 p^v_p(cL_k)) = f_e(n + tc) = f_e(n)
for each positive integer n and each 2 ≤ e ≤ v_p(cL_k). Thus A is a period of g_{p,k,ϕ} as required.

Now we only need to prove that A/P is not the period of g_{p,k,ϕ} for any prime factor P of A. For any prime factor q of A such that q \mid A and q \nmid a, the proof is similar to Subcase 2.1. If v_p(c) = 0, then p is not a prime factor of A, and the proof of this case is complete. In the following, we need to prove that A/p is not the period of g_{p,k,ϕ} if v_p(c) ≥ 1. Since A/p is a period of h_q for each q such that q \nmid a, q\mid cL_k and p|(q - 1), it is enough to prove that A/p is not the period of the function \sum_{e=2}^{v_p(cL_k)} f_e(n).

If v_p(c) ≥ 2, we choose n_0 \in \mathbb{N^*} such that v_p(b' + a'n_0) = v_p(c). Comparing S_{k,a',b',c}(n_0) = \{b' + a'(n_0 + ic)\}_{0 \leq i \leq k} with S_{k,a',b',c}(n_0 + A/p) = \{b' + a'(n_0 + A/p + ic)\}_{0 \leq i \leq k}, we obtain that each term of S_{k,a',b',c}(n_0) is divisible by p^v_p(c), while there is no term divisible by p^v_p(c) in S_{k,a',b',c}(n_0 + A/p) since v_p(b' + a'(n_0 + A/p + ic)) = \min(v_p(b' + a'(n_0 + ic)), v_p(a'A/p)) = v_p(c) - 1 for each 0 \leq i \leq k. So

\[ \sum_{e=2}^{v_p(cL_k)} f_e(n_0) \geq k(v_p(c) - 1) > k(v_p(c) - 2) = \sum_{e=2}^{v_p(cL_k)} f_e(n_0 + A/p). \]

If v_p(c) = 1, then v_p(A/p) = 0. Choosing n_0 \in \mathbb{N^*} such that v_p(b' + a'n_0) = v_p(cL_k), we have that there is at least two terms b' + a'n_0 and b' + a'(n_0 + p^v_p(cL_k) + ic) divisible by p^v_p(cL_k) in S_{k,a',b',c}(n_0) but no term is divisible by p in S_{k,a',b',c}(n_0 + A/p) since v_p(b' + a'(n_0 + A/p + ic)) = 0 for all 0 \leq i \leq k. Therefore, we have

\[ \sum_{e=2}^{v_p(cL_k)} f_e(n_0) \geq v_p(cL_k) - 1 > 0 = \sum_{e=2}^{v_p(cL_k)} f_e(n_0 + A/p). \]

Thus A/p is not the period of g_{p,k,ϕ} in this case.

Case 3. p ≤ k, v_p(k + 1) < v_p(cL_k) and v_p(cL_k) ≥ 2. By the discussion before Lemma 3.4, it is easy to get that A is a period of g_{p,k,ϕ}. As above, it suffices to prove that A/P is not the period of g_{p,k,ϕ} for any prime factor P of A in the following. By a similar argument as in Subcase 2.1, we now only need to show that A/p is not the period of f_{v_p(cL_k)}. Since v_p(A/p) = v_p(cL_k) - 1, we can select a positive integer r_0 such that r_0 A/p \equiv p^v_p(cL_k) - 1 (mod p^v_p(cL_k)). In the following, we prove that p^v_p(cL_k) - 1 is not the period of f_{v_p(cL_k)}, from which we can deduce that A/p is not the period of g_{p,k,ϕ}. Since v_p(k + 1) < v_p(cL_k), we can always suppose that k + 1 ≡ r (mod p^v_p(cL_k)) for some 1 ≤ r ≤ p^v_p(cL_k) - 1. We distinguish the following two subcases.

Subcase 3.1. 1 ≤ r ≤ p^v_p(cL_k) - p^v_p(cL_k) - 1. Choose a positive integer n_0 such that v_p(b' + a'n_0) ≥ v_p(cL_k). Compare the number of terms divisible by p^v_p(cL_k) in the two sets S_{k,a',b',c}(n_0) = \{b' + a'(n_0 + kc)\}_{0 \leq i \leq k} and S_{k,a',b',c}(n_0 + p^v_p(cL_k) - 1) = \{b' + a'(n_0 + p^v_p(cL_k) - 1) + ic\}_{0 \leq i \leq k}. Since \{b' + a'(n_0 + p^v_p(cL_k) - 1) + ic\}_{0 \leq i \leq k} is the intersection of S_{k,a',b',c}(n_0) and S_{k,a',b',c}(n_0 + p^v_p(cL_k) - 1), it suffices to compare the set \{b' + a'n_0, \ldots , b' + a'(n_0 + p^v_p(cL_k) - 1)\} with the set \{b' + a'(n_0 + (k + 1)c), \ldots , b' + a'(n_0 + (k + p^v_p(cL_k) - 1)c)\}. By Lemma 3.1, we know that the terms divisible by p^v_p(cL_k) in the arithmetic progression \{b' + a'(n_0 + ic)\}_{i \in \mathbb{N}} are of the form b' + a'(n_0 + tp^v_p(cL_k)) \in \mathbb{N}. Since k + 1 ≡ r (mod p^v_p(cL_k)) and 1 ≤ r ≤ p^v_p(cL_k) - p^v_p(cL_k) - 1, we have k + j + r + j - 1 ≠ 0 (mod p^v_p(cL_k)) for all 1 ≤ j ≤ p^v_p(cL_k) - 1. Hence p^v_p(cL_k) \nmid \{b' + a'(n_0 + (k + j)c)\} for all 1 ≤ j ≤ p^v_p(cL_k) - 1. Whereas, b' + a'n_0 is the only term in the set \{b' + a'n_0, b' + a'(n_0 +
is not the period of $f_{\text{subcases}}$, we deduce that $p$ is the smallest period of $f_{\text{subcases}}$, which is divisible by $p^{v_p(cL_k)}$. Therefore we have

\[ f_{p^{v_p(cL_k)}}(n_0 + p^{v_p(cL_k)} - 1) = f_{p^{v_p(cL_k)}}(n_0) - 1. \]

**Case 3.2.** $p^{v_p(cL_k)} - p^{v_p(cL_k) - 1} < r \leq p^{v_p(cL_k) - 1}$. Pick a positive integer $n_0$ such that $v_p(b' + a' + (n_0 + (p^{v_p(cL_k) - 1} - 1)c)) \geq v_p(cL_k)$. Then the terms divisible by $p^{v_p(cL_k)}$ in the arithmetic progression $\{b' + a'(n_0 + (p^{v_p(cL_k) - 1} - 1)c) \mid t \in \mathbb{N} \}$ be in the form of $b' + a'(n_0 + (k + 1)c), \ldots, b' + a'(n_0 + (k + p^{v_p(cL_k)} - 1)c)$. By comparison, we obtain that $p^{v_p(cL_k)} \mid \{b' + a'(n_0 + (k + j)c) \}$ for all $1 \leq j \leq \Delta p^{v_p(cL_k) - 1}$, while the term $b' + a'(n_0 + (p^{v_p(cL_k) - 1} - 1)c)$ is the only term divisible by $p^{v_p(cL_k)}$ in the set $\{b' + a'(n_0 + (p^{v_p(cL_k) - 1} - 1)c) \}$. Hence we have $f_{p^{v_p(cL_k)}}(n_0 + p^{v_p(cL_k) - 1}) = f_{p^{v_p(cL_k)}}(n_0) - 1$. From the argument in the above two subcases, we deduce that $p^{v_p(cL_k) - 1}$ is not the period of $f_{p^{v_p(cL_k)}}$, which shows that $A/p$ is not the period of $f_{p^{v_p(cL_k)}}$.

Thus $A$ is the smallest period of $g_{p,k,\varphi}$ as desired. This completes the proof of Lemma 3.6.

**Lemma 3.7.** Let $p$ be a prime such that $p \mid cL_k$, $p \nmid a'$ and $p \nmid d$. Then

\[ P_{p,k,\varphi} = p^{e(p,k)} \left( \prod_{p \mid (q-1), \atop \gcd(q, \varphi) = 1} q \right) \left( \prod_{p \mid (q-1), \atop \gcd(q, \varphi) = 1} q \right), \]

where

\[ e(p,k) := \begin{cases} v_p(c), & \text{if } v_p(k + 1) \geq v_p(L_k), \\ v_p(cL_k), & \text{if } v_p(k + 1) < v_p(L_k). \end{cases} \]

**Proof.** Similarly to the proof of Lemma 3.6, it is enough to show that $p^{e(p,k)}$ is the smallest period of $\sum_{e=1}^{v_p(cL_k)} f_e(n)$ by (3.3). We divide the proof into the following two cases.

**Case 1.** $v_p(k + 1) \geq v_p(L_k)$. As in the proof of Subcase 2.2 in Lemma 3.6, since $b' + a'n \equiv b' + a'(n + (k + 1)c) \pmod{p^{v_p(cL_k)}}$, we can obtain that $p^{v_p(c)}$ is a period of $\sum_{e=1}^{v_p(cL_k)} f_e(n)$. If $v_p(c) = 0$, it is complete. If $v_p(c) \geq 1$, then choosing a positive integer $n_0$ such that $v_p(b' + a'n_0) = v_p(c)$, we can show that $p^{v_p(c) - 1}$ is not the period of $\sum_{e=1}^{v_p(cL_k)} f_e(n)$ using a similar method as in the proof of Case 2 in Lemma 3.6.

**Case 2.** $v_p(k + 1) < v_p(L_k)$. Using the same way as the proof of Case 3 in Lemma 3.6, one can easily check that $p^{v_p(cL_k) - 1}$ is not the period of $\sum_{e=1}^{v_p(cL_k)} f_e(n)$. The proof of Lemma 3.7 is complete.

**4. Proof of Theorem 1.3 and examples**

In this section, we first use the results presented in the previous section to show Theorem 1.3.

**Proof of Theorem 1.3.** By Theorem 1.2, we know that $g_{k,\varphi}$ is periodic and $P_{k,\varphi}|cL_k$. To determine the exact value of $P_{k,\varphi}$, it is sufficient to determine the $p$-adic valuation of $P_{k,\varphi}$ for each prime $p$. By Lemma 3.3, we have $P_{k,\varphi} = \text{lcm}_{p \leq L_k} \{P_{p,k,\varphi} \}$. So it is enough to compute $\max_{q \leq cL_k} \{v_p(P_{q,k,\varphi})\}$ for each prime $p$. We consider the following four cases.
CASE 1. $p \nmid cL_k$. Since $P_{k,\varphi}|cL_k$, it is clear that $v_p(P_{k,\varphi}) = v_p(cL_k) = 0$.

CASE 2. $p|cL_k$ and $p|a'$. Observe from Lemmas 3.4-3.7 that $v_p(P_{q,k,\varphi}) = 0$ for each prime $q \leq cL_k$. So we have $v_p(P_{k,\varphi}) = 0$.

CASE 3. $p = 2$. From the discussion in Case 1 and Case 2, we know that $v_2(P_{k,\varphi}) = 0$ if $2|cL_k$ and $2|a'$ or if $2 \nmid cL_k$. It remains to consider the case $2|cL_k$ and $2 \nmid a'$. By Lemmas 3.4-3.7, we know that $v_2(P_{p,k,\varphi}) = 0$ for all odd primes $p$. So we only need to compute $v_2(P_{2,k,\varphi})$. We now distinguish the following four subcases.

SUBCASE 3.1. $2 \nmid a$ and $v_2(cL_k) = 1$. In this case, by Lemma 3.6, one has $v_2(P_{k,\varphi}) = 0 = v_2(cL_k) - 1$.

SUBCASE 3.2. $2 \mid a$, $v_2(cL_k) \geq 2$ and $v_2(cL_k) = v_2(k+1) = v_2(L_k) = 1$, or $2 \mid a', 2|d$ and $v_2(cL_k) = v_2(k+1) = v_2(L_k) = 1$. Since $v_2(k+1) \geq 2v_2(L_k)$ and $v_2(L_k) = 1$, we get $k = 3$. Thus by Lemmas 3.6 and 3.7, we have that if $k = 3$, $2 \mid a$ and $v_2(cL_k) = v_2(cL_k) - v_2(L_k) \geq 2 - 1 = 1$, or if $k = 3, 2 \mid a'$ and $2|d$, then $v_2(P_{k,\varphi}) = v_2(cL_k) - v_2(L_k) = 1$.

SUBCASE 3.3. $v_2(k+1) \geq v_2(L_k) \geq 2$. Using Lemmas 3.6 and 3.7, we obtain that $v_2(P_{k,\varphi}) = v_2(cL_k) - v_2(L_k)$.

SUBCASE 3.4. $2 \nmid a$, $v_2(cL_k) = 0$ and $v_2(cL_k) \geq 2$, or $2 \mid a', 2|d$ and $v_2(cL_k) = 0$, or $2 \nmid a$, $v_2(cL_k) \geq 2$ and $v_2(k+1) < v_2(cL_k)$, or $2 \nmid a', 2|d$ and $v_2(k+1) < v_2(cL_k)$. If $2 \nmid a$, $v_2(L_k) = 0$ and $v_2(cL_k) \geq 2$, or if $2 \nmid a', 2|d$ and $v_2(cL_k) = 0$, we get $v_2(P_{2,k,\varphi}) = v_2(cL_k)$. So we have $v_2(P_{k,\varphi}) = v_2(cL_k)$ in this case.

Combining all the above information on $v_2(P_{k,\varphi})$, we have

$$v_2(P_{k,\varphi}) = \begin{cases} 
0, & \text{if } 2|a', \\
v_2(cL_k) - v_2(L_k), & \text{if } 2 \nmid a' \text{ and } v_2(k+1) \geq v_2(L_k) \geq 2, \\
v_2(cL_k) - 1, & \text{if } 2 \mid a \text{ and } v_2(cL_k) = 1, \text{ or } k = 3, 2 \nmid a \text{ and } 2|d, \\
& \text{or } k = 3, 2 \mid a' \text{ and } 2|d, \\
v_2(cL_k), & \text{otherwise.}
\end{cases} \quad (4.1)$$

CASE 4. $p \neq 2, p|cL_k$ and $p \nmid a'$. Note that $2|(p-1)$ for each odd prime $p$. Evidently, if $2 \nmid cL_k$, then $k = 1$. So there is no odd prime $p$ so that $p|cL_k$ if $2 \nmid cL_k$. Thus by Lemmas 3.4-3.7, for all odd prime factors $p$ of $cL_k$, we obtain that $v_p(P_{2,\varphi}) = 1$ except that either $p \nmid ac, p|L_k$ and $k+1 \equiv 0 \pmod{p}$ or $p|d$, in which case $v_p(P_{2,\varphi}) = 0$. On the other hand, for all odd primes $q$ such that $q \neq p$ and $q \leq cL_k$, we have by Lemmas 3.4-3.7 that $v_p(P_{q,k,\varphi}) = 0$ if $|d|$ or if $p \nmid ac, p|L_k$ and $k+1 \equiv 0 \pmod{p}$, and $v_p(P_{q,k,\varphi}) \leq 1$ otherwise. Hence $v_p(P_{2,k,\varphi}) \geq v_p(P_{q,k,\varphi})$ for all odd primes $q$ such that $q \neq p$ and $q \leq cL_k$. Therefore we deduce immediately that

$$v_p(P_{k,\varphi}) = \max_{\text{prime } q \leq cL_k} \{v_p(P_{q,k,\varphi})\} = \max(v_p(P_{2,k,\varphi}), v_p(P_{p,k,\varphi})). \quad (4.2)$$

Using Lemmas 3.6 and 3.7 to compute $v_p(P_{p,k,\varphi})$, we get that

$$v_p(P_{p,k,\varphi}) = \begin{cases} 
0, & \text{if } v_p(cL_k) = 1 \text{ and } p \nmid d, \\
v_p(c), & \text{if } v_p(k+1) \geq v_p(L_k), p \nmid d \text{ and } v_p(cL_k) \geq 2, \\
v_p(cL_k), & \text{if } v_p(k+1) < v_p(L_k), p \nmid d \text{ and } v_p(cL_k) \geq 2, \\
or \text{ if } v_p(k+1) < v_p(L_k) \text{ and } p|d.
\end{cases} \quad (4.3)$$

For all the primes $p$ such that $p \nmid d$ and $v_p(cL_k) = 1$, we have by the above discussion that $v_p(P_{k,\varphi}) = 0$ only if $v_p(c) = 0$, $v_p(L_k) = 1$ and $k+1 \equiv 0 \pmod{p}$. Equivalently, $v_p(P_{2,k,\varphi}) = v_p(c) = v_p(cL_k) - v_p(L_k)$ if $v_p(k+1) \geq v_p(L_k) \geq 1$, and $v_p(P_{2,k,\varphi}) = v_p(cL_k)$ otherwise. Therefore, for all the odd primes $p$ satisfying $p|cL_k$ and $p \nmid a'$, we derive from
Thus it follows from (4.5) and (4.6) that
\[ v_p(P_{k,\varphi}) = \begin{cases} 
 v_p(cL_k) - v_p(L_k), & \text{if } v_p(k+1) \geq v_p(L_k) \geq 1, \\
 v_p(cL_k), & \text{otherwise.} 
\end{cases} \tag{4.4} \]

Now putting all the above cases together, we get
\[
P_{k,\varphi} = 2^{v_2(P_{k,\varphi})} \left( \prod_{\text{prime } p, \; p \neq 2} p^{v_p(P_{k,\varphi})} \right) \left( \prod_{\text{prime } p, \; p \neq 2} p^{v_p(cL_k) - v_p(L_k)} \right)
\]
\[
= \frac{2^{v_2(cL_k) - v_2(L_k)}}{cL_k} \prod_{\text{prime } p, \; p \neq 2} p^{v_p(cL_k) - v_p(L_k)} - v_p(P_{k,\varphi})
\]
\[
= \frac{2^{v_2(cL_k) - v_2(L_k)}}{cL_k} \left( \prod_{\text{prime } q, \; q \neq 2} q^{v_q(cL_k) - v_q(L_k)} \right)
\]
\[
= \frac{2^{\delta_{2,k,\varphi}}}{cL_k} \left( \prod_{\text{prime } q, \; q \neq 2} q^{v_q(cL_k) - v_q(L_k)} \right), \tag{4.5}
\]
where
\[ \delta_{2,k,\varphi} := \begin{cases} 
 v_2(cL_k) - v_2(P_{k,\varphi}), & \text{if } 2 \nmid a', \\
 0, & \text{if } 2 | a'. 
\end{cases} \]

It then follows from (4.1) that
\[ \delta_{2,k,\varphi} = \begin{cases} 
 v_2(L_k), & \text{if } 2 \nmid a' \text{ and } v_2(k+1) \geq v_2(L_k) \geq 2, \\
 1, & \text{if } 2 | a \text{ and } v_2(cL_k) = 1, \text{or } k = 3, 2 | a \text{ and } 2 | c, \text{or } k = 3, 2 \nmid a' \text{ and } 2 | d, \\
 0, & \text{otherwise}, 
\end{cases} \]
which implies that \( \eta_{2,k,a',c} = 2^{\delta_{2,k,\varphi}} \). Hence by (1.2) we get
\[
Q_{k,a',c} = \frac{cL_k}{2^{\delta_{2,k,\varphi}} \prod_{\text{prime } q | a'} q^{v_q(cL_k)}}. \tag{4.6}
\]

Since there is at most one odd prime \( p \leq k \) such that \( v_p(k+1) \geq v_p(L_k) \geq 1 \) (see [5]), we derive from (4.4) that
\[
\prod_{\text{prime } q, \; q \neq 2, \; q | a', \; q | cL_k} q^{v_q(cL_k) - v_q(L_k)} = \begin{cases} 
 p^{v_p(L_k)}, & \text{if } v_p(k+1) \geq v_p(L_k) \geq 1 \text{ for an odd prime } p \nmid a', \\
 1, & \text{otherwise.}
\end{cases}
\]

Thus it follows from (4.5) and (4.6) that \( P_{k,\varphi} \) is equal to \( Q_{k,a',c} \) except that \( v_p(k+1) \geq v_p(L_k) \geq 1 \) for at most one odd prime \( p \nmid a' \), in which case \( P_{k,\varphi} \) equals \( Q_{k,a',c} \).

The proof of Theorem 1.3 is complete. \( \square \)

Now we give some examples to illustrate Theorem 1.3.

**Example 4.1.** Let \( a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers, and let \( a' := a / \gcd(a, b) \) be odd. Let \( k = 2^t - 1 \), where \( t \in \mathbb{N} \) and \( t \geq 3 \). Since \( v_2(k+1) = t > v_2(L_k) = t - 1 \geq 2 \), we obtain by Theorem 1.3 that \( \eta_{2,k,a',c} = 2^{v_2(L_k)} \). On the other hand, there is no odd prime \( p \) satisfying \( v_p(k+1) \geq v_p(L_k) \geq 1 \). Thus we have
\[
P_{k,\varphi} = \frac{cL_k}{2^{v_2(L_k)} \prod_{\text{prime } q | a'} q^{v_q(cL_k)}}.
\]
Proposition 4.4. Let \( k \geq 1, a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers, and let \( a' := a/gcd(a,b) \). Let \( p \) be any given odd prime with \( p \nmid a' \), and let \( k = p^\alpha - 1 \) for some integer \( \alpha \geq 2 \). Since \( k = p^\alpha - 1 > 3 \) and \( v_2(k+1) = v_2(p^\alpha) = 0 \), we have \( \eta_{2,k,a',c} = 1 \). The odd prime \( p \) satisfies that \( v_p(k+1) = \alpha > \alpha - 1 = v_p(L_k) \geq 1 \). Hence we get by Theorem 1.3 that

\[
P_k,\varphi = \frac{cL_k}{p^{v_p(L_k)} \prod_{\text{prime } q} q^{v_q(cL_k)}}.
\]

Example 4.3. Let \( a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers, and let \( a' := a/gcd(a,b) \). If \( k \) is an integer of the form \( 35^\alpha - 1 \) with \( \alpha \geq 2 \) and \( \alpha \in \mathbb{N} \), then

\[
P_k,\varphi = \frac{cL_k}{\prod_{\text{prime } q | a'} q^{v_q(cL_k)}}. \tag{4.7}
\]

Actually, since \( 35^\alpha - 1 \geq 3 \) and \( v_2(35^\alpha) = 0 < v_2(L_k) \), we obtain by Theorem 1.3 that \( \eta_{2,k,a',c} = 1 \). On the other hand, we have that \( v_p(k+1) = \alpha < v_p(L_k) \), \( v_q(k+1) = \alpha < v_q(L_k) \) and \( v_q(k+1) = 0 \) for any other odd prime \( q \). Hence we get \( P_{k,\varphi} \) as in (4.7).

Furthermore, if \( a | b \) or \( a' \) is a prime greater than \( cL_k \), then there is no prime factor of \( a' \) dividing \( cL_k \). Therefore for any \( k = 35^\alpha - 1 \) with \( \alpha \geq 2 \) and \( \alpha \in \mathbb{N} \), one has \( P_{k,\varphi} = cL_k \).

Finally, by Theorem 1.3, we only need to compute the first \( P_{k,\varphi} \) values of \( g_{k,\varphi} \) so that we can estimate the difference between \( \prod_{0 \leq i \leq k} \varphi(b+a(n+ic)) \) and \( \varphi(\text{lcm}_{0 \leq i \leq k}(b+a(n+ic))) \) for large \( n \). In other words, we have

\[
\min_{1 \leq m \leq P_{k,\varphi}} \{ g_{k,\varphi}(m) \} \leq \frac{\prod_{0 \leq i \leq k} \varphi(b+a(n+ic))}{\varphi(\text{lcm}_{0 \leq i \leq k}(b+a(n+ic)))} = g_{k,\varphi}(\langle n \rangle_{P_{k,\varphi}}) \leq \max_{1 \leq m \leq P_{k,\varphi}} \{ g_{k,\varphi}(m) \},
\]

where \( \langle n \rangle_{P_{k,\varphi}} \) means the integer between 1 and \( P_{k,\varphi} \) such that \( n \equiv \langle n \rangle_{P_{k,\varphi}} \text{ (mod } P_{k,\varphi}) \).

On the other hand, estimating the difference between \( \prod_{0 \leq i \leq k} \varphi(b+a(n+ic)) \) and \( \text{lcm}_{0 \leq i \leq k}(\varphi(b+a(n+ic))) \) is also an interesting problem. For this purpose, we define the arithmetic function \( G_{k,\varphi} \) for any positive integer \( n \) by

\[
G_{k,\varphi}(n) := \frac{\prod_{0 \leq i \leq k} \varphi(b+a(n+ic))}{\text{lcm}_{0 \leq i \leq k}(\varphi(b+a(n+ic)))}.
\]

Unfortunately, \( G_{k,\varphi} \) may not be periodic. For instance, taking \( a = 1, b = 0 \) and \( c = 1 \), then the arithmetic function \( \bar{G}_{k,\varphi} \) defined by \( \bar{G}_{k,\varphi}(n) := \frac{\prod_{0 \leq i \leq k} \varphi(n+1)}{\text{lcm}_{0 \leq i \leq k}(\varphi(n+1))} \) for \( n \in \mathbb{N}^* \) is not periodic. Indeed, for any given positive integer \( M \), we can always choose a prime \( p > M \) since there are infinitely many primes. By Dirichlet's theorem, we know that there exists a positive integer \( m \) such that the term \( mp^2 + 1 \) is a prime in the arithmetic progression \{\( np^2 + 1 \)\}_{n \in \mathbb{N}}. \) Letting \( n_0 = mp^2 \) gives us that \( p | \varphi(n_0) \) and \( p | \varphi(n_0 + 1) = \varphi(mp^2 + 1) = mp^2 \). Thus \( p | G_{k,\varphi}(n_0) \) and \( \bar{G}_{k,\varphi}(n_0) \geq p > M \). That is, \( \bar{G}_{k,\varphi} \) is unbounded, which implies that \( G_{k,\varphi} \) is not periodic. Applying Theorem 1.3, we can give a nontrivial upper bound about the integer \( \text{lcm}_{0 \leq i \leq k}(\varphi(b+a(n+ic))) \) as follows.

Proposition 4.4. Let \( k \geq 1, a \geq 1, b \geq 0 \) and \( c \geq 1 \) be integers. Then for any
positive integer \( n \), we have

\[
\text{lcm}_{0 \leq i \leq k} \{ \varphi(b + a(n + ic)) \} \leq \prod_{i=0}^{k} \frac{\varphi(b + a(n + ic))}{g_{k,\varphi} \langle n \rangle_{P_{k,\varphi}}}
\]

with \( \langle n \rangle_{P_{k,\varphi}} \) being defined as above.

**Proof.** For each \( 0 \leq i \leq k \), since \( \varphi \) is multiplicative and \( b + a(n + ic) \mid \text{lcm}_{0 \leq j \leq k} \{ b + a(n + je) \} \), we have

\[
\varphi(b + a(n + ic)) \mid \varphi(\text{lcm}_{0 \leq j \leq k} \{ b + a(n + ic) \}).
\]

So we get

\[
\text{lcm}_{0 \leq i \leq k} \{ \varphi(b + a(n + ic)) \} \mid \varphi(\text{lcm}_{0 \leq j \leq k} \{ b + a(n + ic) \}).
\]

Thereby

\[
g_{k,\varphi} \langle n \rangle_{P_{k,\varphi}} = g_{k,\varphi}(n) \leq \prod_{0 \leq i \leq k} \frac{\varphi(b + a(n + ic))}{\text{lcm}_{0 \leq i \leq k} \{ \varphi(b + a(n + ic)) \}}
\]

for any positive integer \( n \). The desired result then follows immediately. \( \square \)

Theorem 1.3 answers the second part of Problem 1.1 for the Euler phi function. However, the smallest period problem is still kept open for all other multiplicative functions \( f \) with \( f(n) \neq 0 \) for all positive integers \( n \). For example, if one picks \( f = \sigma_{\alpha} \) with \( \sigma_{\alpha}(n) := \sum_{d \mid n} d^\alpha \) for \( \alpha \in \mathbb{N} \), then what is the smallest period of \( g_{k,f} \)? If \( f = \xi_{\varepsilon} \) with \( \xi_{\varepsilon}(n) := n^\varepsilon \) for \( \varepsilon \in \mathbb{R} \), then what is the smallest period of \( g_{k,f} \)?

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