LOCAL LARGE DEVIATIONS: A McMILLIAN THEOREM FOR COLOURED RANDOM GRAPH PROCESSES

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Abstract. For a finite typed graph on \( n \) nodes and with type law \( \mu \), we define the so-called spectral potential \( \rho_\lambda(\cdot, \mu) \), of the graph. From the \( \rho_\lambda(\cdot, \mu) \) we obtain Kullback action or the deviation function, \( \mathcal{H}_\lambda(\pi \parallel \nu) \), with respect to an empirical pair measure, \( \pi \), as the Legendre dual. For the finite typed random graph conditioned to have an empirical link measure \( \pi \) and empirical type measure \( \mu \), we prove a Local large deviation principle (LLDP), with rate function \( \mathcal{H}_\lambda(\pi \parallel \nu) \) and speed \( n \). We deduce from this LLDP, a full conditional large deviation principle and a weak variant of the classical McMillian Theorem for the typed random graphs. Given the typical empirical link measure, \( \lambda \mu \otimes \mu \), the number of typed random graphs is approximately equal to \( e^{n \parallel \lambda \mu \otimes \mu \parallel H(\lambda \mu \otimes \mu / \parallel \lambda \mu \otimes \mu \parallel)} \). Note that we do not require any topological restrictions on the space of finite graphs for these LLDPs.

1. Background

1.1 Introduction

We consider random graph models, where nodes are assigned types independently according to some type on a finite alphabet and between any two given nodes, a link is present with a probability that depends on the type of the nodes. This random graph model was first proposed and studied extensively by Penman [16] as a generalization of the Erdos-Renyi graphs. Indeed, the random graph model which is known to model fairly well network structured data, has the Erdos-Renyi graph as a special case. See, [4] for an exposition on this random graphs and their applications.

Now, some Large deviation principles (LDPs) and Coding Theorems exists for networked data structures modelled as the typed random graph (TRG) models. See, [15], [2], [9], [3], [7], [5] and the reference therein. O'Connor [15] proved large deviation principle (LDP) for the relative size of the largest connected component in the random graph with small edge probability. Biggin and Penman [2] have found LDPs for the number of edges of TRG models, where the link probabilities are independent of the number of nodes, using the Garten-Ellis Theorem, see [12]. [9] proved LDPs for the empirical measures of the TRG where the link probabilities are dependent on the number of nodes of the graph. In [3], LDP for the empirical neighborhood distribution in sparse random graphs was proved using a technique that relies on the typical behavior within the framework of the local weak convergence of finite graph sequences. Asymptotic Equipartition Properties including the Lossy version have been found in [7] and [5], by the techniques of exponential change of measure and random allocation, respectively.

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We present in this article a LLDP for the TRG models conditioned on the empirical type measure of the graph. Refer to [1] and [8] for similar results for the empirical measure of iid random variables and the empirical offspring measure of multitype Galton-Watson processes, respectively. This article shares similar features as [8], but defers from all the LDPs discussed above, i.e. [15], [2], [9], [3], [7], [5]. The main technique use to prove our main result is rooted in spectral potential theory. See, [8] and the reference therein for similar idea for the LLDP for the multitype Galton-Watson processes. To be specific about this technique, we define the spectral potential of the TRG, and use it to calculate an extended version of the relative entropy, and show that this relative entropy which has all the properties of the classical relative entropy, see [12], is the Legendre dual of our spectral potential of the TRG. From the LLDP for the TRG we deduce the weak variant of the classical McMillian-Breiman Theorem and the full large deviation principle for the TRG conditioned on the empirical link measure and under the conditional law of the TRG given the type law.

1.2 Coloured Random Process. Let $p_n : \mathcal{Y} \times \mathcal{Y} \to [0, 1]$ be a symmetric function and $\mu$ on $\mathcal{Y}$ be a probability measure. We can define the typed random graph $Y$ with $\{1, 2, 3, ..., n\}$ nodes as follows:

- Assign to each vertex $v \in \{1, 2, 3, ..., n\}$ colour $Y(v)$ independently according to the colour law $\mu$.
- Given the colours, we connect any two vertices $u, v \in \{1, 2, 3, ..., n\}$, independently of everything else, with connection probability $p_n(Y(u), Y(v))$.

We always consider $Y = ((Y(v) : v \in \{1, 2, 3, ..., n\}), E)$ under the combine law of the graph and type, and interpret $Y$ as typed random graph.

Denote by $\mathcal{G}(\{n\}, \mathcal{Y})$ the set of all coloured graphs with colour set $\mathcal{Y}$ and $n$ vertices. We shall only study $Y$ with connection probabilities satisfy

$$a_n^{-1}p_n(a, b) \to C(a, b), \quad \forall a, b \in \mathcal{Y}, \text{ where } C : \mathcal{Y} \times \mathcal{Y} \to [0, \infty) \text{ is a nonzero function.} \quad (1.1)$$

If the sequence $a_n$ in (1.1) above satisfies (i) $a_n \to 1$ (ii) $a_n \to 0$ and (iii) $a_n \to \infty$ we call $X$ sparse, subcritical and supercritical respectively.

In this article we also assume that the sequence $(a_n)$ converges to 0 as $n$ approaches $\infty$.

Notation: For any finite or countable set $\mathcal{L}$, we denote by $\mathcal{L}(\mathcal{Y})$ the space of probability measures by $\hat{\mathcal{L}}(\mathcal{Y})$ the space of finite positive measures on $\mathcal{Y}$, by $\hat{\mathcal{L}}_s(\mathcal{Y})$ we denote the subspace of symmetric measures in $\hat{\mathcal{L}}(\mathcal{Y})$. By $\mathcal{P}(\mathcal{Y})$ the space of all real-valued bounded measurable functions on $\mathcal{Y}$, by $\mathcal{P}_s(\mathcal{Y})$ the space of continuous linear functionals on $\mathcal{P}(\mathcal{Y})$ and by $\mathcal{P}_+(\mathcal{Y})$ the collection of all positive linear functionals on $\mathcal{P}(\mathcal{Y})$.

For every coloured random graph $Y$, we define the empirical type distribution $L_1^Y \in \mathcal{L}(\mathcal{Y})$ by,

$$L_1^Y(a) = \frac{1}{n} \sum_{v \in \{1, 2, 3, ..., n\}} \delta_{Y(v)}(a), \quad \text{for } a \in \mathcal{Y}. \quad (1.2)$$

and the empirical link distribution $L_2^Y \in \hat{\mathcal{L}}_s(\mathcal{Y} \times \mathcal{Y})$ is defined by,

$$L_2^Y(a, b) = \frac{1}{a_n n^2} \sum_{(u, v) \in E} \left[ \delta_{Y(v), Y(u)} + \delta_{Y(u), Y(v)} \right](a, b), \quad \text{for } a, b \in \mathcal{Y}. \quad (1.3)$$
Note that \( a_n n^2 \) is the maximum possible number of edges in the graph, and we have that

\[
a_n n^2 L_\lambda^2(a, b) = \begin{cases} 
\#\{\text{number of edges between vertices of colours } a \text{ and } b \} & \text{if } a = b \\
2 \times \#\{\text{number of edges between vertices of colour } a \} & \text{if } a \neq b.
\end{cases}
\] (1.4)

The remaining part of the article is organized in the following manner: Section 2 contain the main results of the article; Theorem 2.1, Corollary 2.2 and Theorem 2.3. In Section 3 this results of the article are proved.

## 2. Statement of main results

We assume through out the remaining part of this article that the typed random graph process is near-critical or sparse. Write \( \langle f , \sigma \rangle := \sum_{y \in Y} \sigma(y) f(y) \) and define the spectral potential \( \rho_\lambda(g, \mu) \) of the near-critical typed random graph process \( Y \) by

\[
\rho_\lambda(g, \mu) = - \left\langle (1 - e^g), \lambda \mu \otimes \mu \right\rangle / 2.
\] (2.1)

Notice, \( \rho_\lambda \) is (i) finite on \( \{ g : Y \times Y \to \mathbb{R} | e^{-\frac{1}{2} \langle (1-e^g), \lambda \mu \otimes \mu \rangle} < \infty \} \) (ii) monotone (iii) additively homogeneous and convex in \( g \). For \( \nu \in \mathcal{P}(Y \times Y) \) we define the Kullback action by a nonlinear functional

\[
\mathcal{H}_\lambda(\pi \| \mu) := \left( \left\langle \pi, \log \frac{\pi}{\lambda \mu \otimes \mu} \right\rangle + \left\| \lambda \mu \otimes \mu \right\| - \left\| \pi \right\| \right) / 2
\] (2.2)

and note that \( \mathcal{H}_\lambda \) above is nonlinear functional.

Let \( P_\mu(y) = \mathbb{P} \{ Y = y | L_y^1 = \mu \} \) be the distribution of the near-critical typed random graph process \( y \) on \([n] \). In Theorem 2.1 below we state our main result, the LLDP for the multitype Galton-Watson tree.

### Theorem 2.1 (LLDP).

Let \( y = (y(v) : v \in [n]) \) be a typed random graph process with type law \( \mu \) and link probabilities that satisfies \( a_n^{-1} p_n(a, b) \to \lambda(a, b) \), for \( a, b \in Y \) and \( na_n \to 1 \). Then,

(i) for any functional \( \omega \in \mathcal{L}_s(Y \times Y) \) and a number \( \varepsilon > 0 \), there exists a weak neighborhood \( B_\omega \) such that

\[
P_\mu \left\{ y \in \mathcal{G}([n], Y) | L_y^2 \in B_\omega \right\} \leq e^{-n \mathcal{H}_\lambda(\pi \| \mu) - n\varepsilon}.
\]

(ii) for any \( \nu \in \mathcal{L}_s(Y \times Y) \), a number \( \varepsilon > 0 \) and a fine neighborhood \( B_\omega \) we have the asymptotic estimate:

\[
P_\mu \left\{ y \in \mathcal{G}([n], Y) | L_y^2 \in B_\omega \right\} \geq e^{-n \mathcal{H}_\lambda(\pi \| \mu) + n\varepsilon}.
\]

Next we state a corollary of Theorem 2.1 the McMillian-Breiman Theorem for the typed random graph process. We define an entropy by

\[
\mathcal{F}_\lambda^\pi (\pi) := \left( \left\| \pi \right\| - \left\| \lambda \mu \otimes \mu \right\| - \left\langle \pi, \log \frac{\pi}{\lambda \mu \otimes \mu} \right\rangle \right) / 2
\] (2.3)

### Corollary 2.2 (McMillian Theorem).

Let \( \mathcal{G}([n], Y) \) be the space of all typed random graph process with type law \( \mu \) and link probabilities that satisfies \( a_n^{-1} p_n(a, b) \to \lambda(a, b) \), for \( a, b \in Y \) and \( na_n \to 1 \).
(i) For any empirical link measure \( \rho \) on \( \mathcal{Y} \times \mathcal{Y} \) and \( \epsilon > 0 \), there exists a neighborhood \( B_\rho \) such that
\[
\text{Card}\left( \{ y \in G([n], \mathcal{Y}) \mid L_y^2 \in B_\rho \} \right) \geq e^{n(\delta(\rho)+\epsilon)}.
\]
(ii) for any neighborhood \( B_\rho \) and \( \epsilon > 0 \), we have
\[
\text{Card}\left( \{ y \in G([n], \mathcal{Y}) \mid L_y^2 \in B_\rho \} \right) \leq e^{n(\delta(\rho)-\epsilon)},
\]
where \( \text{Card}(A) \) means the cardinality of \( A \).

**Remark 1** For \( \rho = \lambda \mu \otimes \mu \), equation 2.3 above reduces to
\[
H(\lambda \mu \otimes \mu, \log \lambda \mu \otimes \mu / \| \lambda \mu \otimes \mu \|)
\]
and therefore,
\[
\text{Card}\left( \{ y \in G([n], \mathcal{Y}) \} \right) \approx e^{n(\delta(\lambda \mu \otimes \mu, \text{dist}(\lambda \mu \otimes \mu)) / \| \lambda \mu \otimes \mu \|)}.
\]

Finally, we state in Theorem 2.3 the full LDP for the typed random graph process.

**Theorem 2.3** (LDP). Let \( y = (y(v) : v \in [n]) \) be a typed random graph process with type law \( \mu \) and link probabilities that satisfies
\[
a^{-1} p_n(a, b) \to \lambda(a, b), \text{ for } a, b \in \mathcal{Y} \text{ and } na_n \to 1.
\]

(i) Let \( F \) be open subset of \( \mathcal{L}(\mathcal{Y} \times \mathcal{Y}) \). Then we have
\[
\lim_{n \to \infty} \frac{1}{n} \log P_\mu \left\{ y \in G([n], \mathcal{Y}) \mid L_y^2 \in F \right\} \geq - \inf_{\nu \in F} H(\lambda \mu \otimes \mu, \nu).
\]
(ii) Let \( G \) be closed subset of \( \mathcal{L}(\mathcal{Y} \times \mathcal{Y}) \). The we have
\[
\lim_{n \to \infty} \frac{1}{n} \log P_\mu \left\{ y \in G([n], \mathcal{Y}) \mid L_y^2 \in G \right\} \leq - \inf_{\nu \in G} H(\lambda \mu \otimes \mu, \nu).
\]

3. **Proof of Main Results**

3.1 **Properties of the Kullback action.** In this subsection Lemma 3.1, which summaries the properties of 2.2 above. This will help us circumvent the topological problems faced in [6] and [11]. Denote by \( \mathcal{C} \) is the space of continuous functions \( g : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R} \).

**Lemma 3.1.** The following holds for the Kullback action or divergence function \( H(\lambda \mu \otimes \mu, \nu) \).

(i) \( H(\lambda \mu \otimes \mu, \nu) = \frac{1}{2} \sup_{g \in \mathcal{C}} \left\{ \langle g, \pi \rangle - \rho_\lambda(g, \mu) \right\} \).
(ii) The function \( H(\lambda \mu \otimes \mu, \nu) \) is lower semi-continuous on the space \( \mathcal{P}_*(\mathcal{Y} \times \mathcal{Y}) \).
(iii) For any real \( c \), the set \( \{ \nu \in \mathcal{P}_*(\mathcal{Y} \times \mathcal{Y}) : H(\lambda \mu \otimes \mu, \nu) \leq c \} \) is weakly compact.

Please we refer to [11] for similar result and proof for the empirical measures on measurable spaces.

The proof below follows similar ideas as the proof of [11] Lemma 2.2] for the empirical measures on measurable spaces.

**Proof.** (i) Let \( g \in \mathcal{C} \) be such that \( \langle g, \pi \rangle \) approximates the functional \( \langle \phi, \pi \rangle \) and \( \rho_\lambda(g, \mu) \) approximates \( \rho_\lambda(\phi, \mu) \) where \( \phi \in \mathcal{B}(\mathcal{Y} \times \mathcal{Y}) \). Suppose \( \pi \) is absolutely continuous with respect to \( \lambda \mu \otimes \mu \). Define the
function $g$ by $g := \log \frac{\pi}{\lambda \mu \otimes \mu}$. For $t > 0$, we define the approximating function $g_t \in B(Y \times Y)$ as follows

$$g_t(a, b) := \begin{cases} 
  g(a, b), & \text{if } -t < g(a, b) < t, \\
  e^t, & \text{if } g(a, b) > t \\
  e^{-t}, & \text{if } g(a, b) < -t
\end{cases}$$

for all $(a, c) \in Y \times Y^*$. Now for $t \to \infty$ we have that

$$\langle 1 - e^{g_t}, \lambda \mu \otimes \mu \rangle = \int (1 - e^{g_t}) \mathbb{1}_{\{g(a, b) < t\}} \lambda \mu \otimes \mu(da, db) + \int t \mathbb{1}_{\{g(a, b) > t\}} \lambda \mu \otimes \mu(da, db) \to \langle 1 - e^g, \lambda \mu \otimes \mu \rangle = \|\lambda \mu \otimes \mu\| - \|\pi\|.$$

Therefore we have $\lim_{t \to \infty} \frac{1}{2} (\langle g, \pi \rangle - \langle 1 - e^{g_t}, \lambda \mu \otimes \mu \rangle) \to H_{\lambda}(\pi || \mu)$ which proves Lemma 3.1 (i).

Suppose $\pi$ is not absolutely continuous with respect to $\lambda \mu \otimes \mu$, i.e. there exists an $\varepsilon > 0$ such that for any $1 > \eta > 0$ there exists $B_\eta \subset Y \times Y^*$ with $\lambda \mu \otimes \mu(B_\eta) \leq \eta/(1 - \eta)$ and at the same time we have $\pi(B_\eta) > \varepsilon$. For this $\eta$ define the function

$$g_\delta(a, b) := \begin{cases} 
  -\log \eta, & \text{if } (a, b) \in B_\eta, \\
  0, & \text{if } (a, b) \notin B_\eta.
\end{cases}$$

Then we have $\lim_{\eta \to 0} \frac{1}{2} (\langle g_\eta, \pi \rangle - \langle 1 - e^{g_\eta}, \lambda \mu \otimes \mu \rangle) \geq \frac{\varepsilon}{2} \log \eta - \frac{1}{2} ((1 - e^{g_\eta}), \lambda \mu \otimes \mu) \geq -\frac{\varepsilon}{2} \log \eta + \frac{1}{2}.$

Taking limit as $\eta \downarrow 0$ we have that $H_{\lambda}(\pi || \mu) = +\infty$, which ends the proof of Lemma 3.1 (i).

(ii) & (iii). Observe from the variational formulation of the relative entropy, see Dembo et al. [12], and Lemma 3.1(i) that $\frac{1}{2} \sup_{g \in C} \{ \langle g, \pi \rangle - \rho_\lambda(g, \mu) \}$ reduces to equation 2.2 above. Now the relative entropy lower semi-continuous, and by [7, Remark 4] all its level sets are compact. Hence it holds $H_{\lambda}(\pi || \mu)$ is lower semi-continuous, and all its level sets are weakly compact in the weak topology which ends the proof of the Lemma.

Note that Lemma 3.1 (i) above implies the so-called variational principle. See, example [14].

3.2 Proof of Theorem 2.1. By Lemma 3.1 for any $\varepsilon > 0$ there exists a function $g \in \mathcal{P}(Y \times Y)$ such that

$$H_{\lambda}(\pi || \mu) - \frac{\varepsilon}{2} < \langle g, \pi \rangle - \rho_\lambda(g, \mu).$$

We define the probability distribution $\tilde{P}_n$ by
\[ \hat{P}_\mu(Y) = \prod_{(u,v) \in E} \tilde{P}_n(Y(u), Y(v)) \prod_{(u,v) \notin E} 1 - \tilde{P}_n(Y(u), Y(v)) \]

\[ = \prod_{(u,v) \in E} \frac{\tilde{P}_n(Y(u), Y(v))}{n-n\tilde{P}_n(Y(u), Y(v))} \prod_{(u,v) \in E} (n-n\tilde{P}_n(Y(u), Y(v))) \]

\[ = \prod_{(u,v) \in E} e^{g(\mu(u),\mu(v))} \frac{\mu_p(Y(u), Y(v))}{n-n\tilde{P}_n(Y(u), Y(v))} \prod_{(u,v) \in E} e^{\frac{1}{n} \mathcal{H}_n(Y(u), Y(v))} (n-n\tilde{P}_n(Y(u), Y(v))) \quad (3.3) \]

Using (3.3) above, we have

\[ \frac{dP_\mu(Y)}{d\hat{P}_\mu(Y)} = \prod_{(u,v) \in E} e^{-g(\mu(u),\mu(v))} \prod_{(u,v) \in E} e^{-\frac{1}{n} \mathcal{H}_n(Y(u), Y(v))} = e^{-n(\frac{1}{2}L^2, \tilde{g})-n(\frac{1}{2}L^1 \otimes L^1, \tilde{h}_n)+\frac{1}{2}L^1_n, \tilde{h}_n}, \quad (3.4) \]

while

\[ L^1_n = \delta(Y(u), Y(u)). \]

Now we define a neighbourhood of the functional \( \nu \) as follows:

\[ B_\pi = \left\{ \varpi \in \mathcal{P}(Y \times Y) : \langle g, \varpi \rangle > \langle g, \pi \rangle - \frac{\varepsilon}{2} \right\}. \]

Therefore, under the condition \( L^1_n \in B_\nu \) we have that

\[ \frac{dP_\mu(x)}{d\hat{P}_\mu(x)} < e^{\frac{1}{2} (\rho_\lambda(g, \mu) - \langle g, \pi \rangle) + \frac{\varepsilon}{2}} < e^{-n\mathcal{H}_\lambda(\pi, \mu) + n\varepsilon}. \]

Hence, we have

\[ P_\mu \left\{ y \in \mathcal{G}([n], Y) | L^2_n \in B_\pi \right\} \leq \int_{\mathcal{G}([n], p_n)} \mathbb{1}_{\{L^2_n \in B_\pi \}} d\hat{P}_\mu(x) \leq \int_{\mathcal{G}([n], Y)} \mathbb{1}_{\{L^2_n \in B_\pi \}} e^{-n\mathcal{H}_\lambda(\pi, \mu) - n\varepsilon} d\hat{P}_\mu(x) \]

\[ \leq e^{-n\mathcal{H}_\lambda(\pi, \mu) - n\varepsilon}. \]

Note that \( \mathcal{H}_\lambda(\pi, \mu) = \infty \) implies Theorem 2.1 ii) and so it suffice to prove that for a probability measure of the form \( \pi = e^\lambda \mu \otimes \mu \), where the Kullback action \( \mathcal{H}_\lambda(\pi, \mu) = \langle g, \pi \rangle + \langle (1 - e^\theta), \lambda \mu \otimes \mu \rangle \) is finite. Fix any number \( \varepsilon > 0 \) and any neighbourhood \( B_\pi \subset \mathcal{L}(Y \times Y) \). We define the sequence of sets

\[ \mathcal{G}(\pi, \mu) := \left\{ y \in \mathcal{G}([n], Y) : L^2_n \in B_\pi \mid \langle g, L^2_n \rangle - \langle g, \pi \rangle \leq \frac{\varepsilon}{2} \right\}. \]

Observe that, for all \( x \in \mathcal{T}_n \) we have

\[ \frac{dP_\mu(x)}{d\hat{P}_\mu(x)} = e^{-n(\frac{1}{2}L^2, \tilde{g})-n(\frac{1}{2}L^1 \otimes L^1, \tilde{h}_n)+\frac{1}{n}L^1_n, \tilde{h}_n} > e^{-n(\frac{1}{2}L^1 \otimes L^1, \tilde{h}_n)+\frac{1}{2}L^1_n, \tilde{h}_n)(1-\frac{\pi}{\lambda \mu \otimes \mu})}. \]

This gives

\[ P_\mu \left( \mathcal{G}(\pi, \mu) \right) = \int_{\mathcal{G}(\pi, \mu)} dP_\mu(x) \geq \int_{\mathcal{G}(\pi, \mu)} e^{-n(\frac{1}{2}L^1 \otimes L^1, \tilde{h}_n)+\frac{1}{2}L^1_n, \tilde{h}_n)(1-\frac{\pi}{\lambda \mu \otimes \mu})} d\hat{P}_\mu(x) \]

\[ = e^{-n\mathcal{H}_\lambda(\pi, \mu) + n\varepsilon} \hat{P}_\mu \left( \mathcal{G}(\pi, \mu) \right). \]

Using the law of large numbers we have \( \lim_{n \to \infty} \hat{P}_\mu \left( \mathcal{G}(\pi, \mu) \right) = 1 \) which completes the proof.
4. Proof of Corollary 2.2 and Theorem 2.3

4.1 Proof of Corollary 2.2. The proof of Corollary 2.2 follows from the definition of the Kullback action and Theorem 2.1 if we set \( \pi = \rho \) and \( \lambda \mu \otimes (a, b) = \| \lambda \mu \otimes \mu \| \), for all \((a, b) \in \mathcal{Y} \times \mathcal{Y} \).

The proof of Theorem 2.3 below, follows from Theorem 2.1 above using similar arguments as in [1, p. 544].

4.2 Proof of Theorem 2.3.

Proof. Note that the empirical link measure is a finite measure and so belongs to some ball in \( B_c(\mathcal{Y} \times \mathcal{Y}) \). Hence, without loss of generality we may assume that the set \( \Gamma \) in Theorem 2.3(ii) is relatively compact. See Lemma 3.1 (iii). Choose any \( \varepsilon > 0 \). Then for every functional \( \pi \in \Gamma \) we can find a weak neighbourhood such that the estimate of Theorem 2.1(i) holds. We choose from all these neighbourhood a finite cover of \( G([n], p_n) \) and sum up over the estimate in Theorem 2.1(i) to obtain

\[
\lim_{n \to \infty} \frac{1}{n} \log P_n \left\{ y \in G([n], \mathcal{Y}) \left| L_y^2 \in \Gamma \right. \right\} \leq - \inf_{\pi \in \Gamma} H_{\lambda}(\pi \| \mu) + \varepsilon.
\]

As \( \varepsilon \) was arbitrarily chosen and the lower bound in Theorem 2.1(ii) implies the lower bound in Theorem 2.3(i) we have the desired results which ends the proof of the Theorem.

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