EXPLICIT SOLUTION BY RADICALS, GONAL MAPS AND PLANE MODELS OF ALGEBRAIC CURVES OF GENUS 5 OR 6

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Abstract. We give explicit computational algorithms to construct minimal degree (always \( \leq 4 \)) ramified covers of \( \mathbb{P}^1 \) for algebraic curves of genus 5 and 6. This completes the work of Schicho and Sevilla (who dealt with the \( g \leq 4 \) case) on constructing radical parametrisations of arbitrary genus \( g \) curves. Zariski showed that this is impossible for the general curve of genus \( \geq 7 \). We also construct minimal degree birational plane models and show how the existence of degree 6 plane models for genus 6 curves is related to the gonality and geometric type of a certain auxiliary surface.

1. Introduction

A famous result of Zariski (see, eg, [Fri89]) states that the general algebraic curve \( C \) over \( \mathbb{C} \) of genus \( g > 6 \) is not soluble by radicals over a rational function field. That is, it is not possible to write a set of coordinate functions of \( C \) as radical expressions of a single parameter \( x \) that is itself a rational function on \( C \): the function field of \( C \), \( \mathbb{C}(C) \), cannot be expressed as a radical extension of a \( \mathbb{C}(x) \) subfield.

In contrast to this, Brill-Noether theory tells us that a curve of genus \( g \) over an algebraically-closed field \( k \) (any characteristic) has a map to \( \mathbb{P}^1 \) of degree \( \leq \lceil g/2 \rceil + 1 \) [KL74]. Thus the function field of any \( C \) of genus \( \leq 6 \) is an extension of degree \( \leq 4 \) of a rational function field and so \( C \) is soluble by radicals.

Once we have found \( k(x) \) in \( k(C) \) of index \( \leq 4 \), the standard formulae for the roots of a polynomial of degree \( \leq 4 \) give radical expressions for the generators of \( k(C) \). The main computational problem is the construction of the smallest degree \( d \) map from \( C \) to \( \mathbb{P}^1 \) (\( d \) is traditionally referred to as the gonality of \( C \)).

For genus \( \leq 4 \), when \( d \leq 3 \), explicit algorithms for this have been described in [SS]. Here, we will fill the gap by giving explicit algorithms for the genus 5 and 6 cases. When \( C \) is not hyperelliptic, the algorithms are based on the theory of rational scrolls containing the canonical embedding \( C_{can} \) of \( C \) as described in [Sch86]. The case that requires the most work to produce an algorithm from the general theory is for the general curve of genus 6 (with gonality 4). There is also the added complication of distinguishing between the gonality 3 case and the \( g_5^2 \) case (gonality 4) for genus 6. The general genus 5 case (gonality 4 also) is fairly straightforward both theoretically and computationally. We include it for completeness. The majority of the paper, though, deals with genus 6.

The construction of minimal degree covers of \( \mathbb{P}^1 \) (we sometimes refer to these as gonal maps) is an interesting and important computational problem in its own right, with applications outside of radical parametrisation. For example, it is useful to be able to express the function field of a curve as a finite extension of the rational function field \( k(x) \) of as small a degree as possible when applying function field algorithms like those of Florian Hess or Mark van Hoeij to the computation of Riemann-Roch spaces and the like. Of course, we assume that such algorithms are available for some of the operations on our initial curve here. However, the possibility
of reexpressing the curve as a smaller degree extension of $k(x)$ for more general analysis is a valuable one.

Of related interest is the construction of a smallest degree birational plane model of the curve in $\mathbb{P}^2$ (singular, in general). Fortunately, in a number of cases, these can be constructed directly from the same data generated from the algorithm to construct gonal maps. In particular, this is true in the generic genus 6 case. In addition, in the genus 6, 4-gonal case, we will show that there are only finitely many gonal maps up to equivalence precisely when the curve has a degree 6 plane model. Because of these relations and the intrinsic interest, we also show how our methods allow us to construct these minimal degree models and analyse the question of the existence of plane models of certain degrees in some detail.

The contents of the paper are as follows. In the next section, we introduce the general setup and discuss the 3-gonal cases and the plane quintic genus 6 case. The third section gives the algorithm for 4-gonal, genus 5 curves. The fourth section covers the general (Clifford-index 2) genus 6 case. There, we review Schreyer’s analysis of the minimal free polynomial resolution of the canonical coordinate ring, present the algorithms for finding gonal maps and plane models and, finally, prove some results about the existence of degree 6 plane models. We also consider the example of the genus 6 modular curve $X_0(58)$, using our algorithm to construct a degree 4 rational function, a degree 6 plane model and a radical parametrisation.

The algorithms are applied to $C$ defined over a non-algebraically closed base field $k$ and construct degree $d$ maps to $\mathbb{P}^1$ over a finite extension of $k$. In general, no degree $d$ map exists over $k$.

I would like to thank Josef Schicho and David Sevilla for introducing me to the problem of computing radical parametrisations.

2. Generalities and the non-generic cases

The curve $C$ of geometric genus $g$ is assumed to be initially given in a general geometric realisation: by a birational (maybe singular) model in affine or projective space of arbitrary dimension or as an algebraic function field. There are known algorithms to compute the canonical image $C_{can}$ in $\mathbb{P}^{g-1}$ along with an explicit birational map from $C$ to $C_{can}$ or an isomorphism of their function fields. For example, the function field methods of Hess [Hes02]. We also need to explicitly compute the minimal free resolution of the defining ideal $I_{C_{can}}$ of $C_{can}$ as a graded $R$-module, where $R = k[x_1, \ldots, x_g]$ is the coordinate ring of $\mathbb{P}^{g-1}$. There are also well-known algorithms for this, computing syzygies via Gröbner bases.

Throughout the paper, we use the language of linear systems. $g_d^r$ will denote a (not necessarily complete) linear system on $C$ of degree $d$ and projective dimension $r$. Such a system gives a morphism of $C$ into $\mathbb{P}^r$ up to a linear change of coordinates, as described in Chapter II, Section 7 of [Har77]. If the $g_d^r$ is basepoint free, this map will be a finite map of degree $d$ onto its image. If the system has a base locus of degree $e$, then the map will be of degree $d - e$. The image lies in no hyperplane. Thus, a degree $d$ cover of $\mathbb{P}^1$, up to automorphisms of $\mathbb{P}^1$, is equivalent to a basepoint free $g_d^1$ on $C$. This is also equivalent to a rational function of degree $d$ on $C$ up to equivalence, where $f$ and $g$ are equivalent if $g = (af + b)/(cf + d)$ for a non-singular matrix \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}.
\]

The results in [Sch86] (or [Eis05]) tell us that a $g_d^1$ on $C$ will give rise to a rational scroll $X$ in $\mathbb{P}^{g-1}$ of dimension $d - 1$ containing $C_{can}$ such that the ruling of $X$ cuts out the $g_d^1$ on $C_{can}$. Furthermore, the minimal free resolution of $I_X$, the defining ideal of $X$, occurs as direct summand of the quadratic strand of the minimal free resolution of $I_{C_{can}}$ (see Section 6C and
Appendix A2H of [Eis05]). Given $X$, the map to the $\mathbb{P}^1$ base of the ruling, which induces the degree $d$ map on $C_{\text{can}}$, can be computed in a number of ways - the Lie algebra method, for example [SS]. When the scroll is of codimension 2 in $\mathbb{P}^{g-1}$, the map can be read off directly from the resolution of $I_X$. This occurs for a $g_3^3$ for genus 5 or a $g_4^4$ for genus 6.

The problem of constructing gonal maps can thus be reduced to explicitly constructing direct sum subcomplexes of the canonical resolution $F$ for $I_{C_{\text{can}}}$ that will give the resolution of such a scroll. The gonality $d$ of $C$ can be read off from the shape of $F$ ( [Sch86]). The hyperelliptic case ($d = 2$) can be recognised by the arithmetic genus of the canonical image $C_{\text{can}}$ being zero. Then, $C_{\text{can}}$ is a rational normal curve, $C \rightarrow C_{\text{can}}$ is of degree 2, and a parametrisation of $C_{\text{can}}$ can be constructed in a number of ways: the Lie algebra method again, repeated adjunction mappings or using function field methods ( [Hes02]).

The classical $d = 3$ case (Petri’s Theorem) is dealt with in [SS] using the Lie algebra method. This is characterised by $I_{C_{\text{can}}}$ having a minimal basis that includes cubic forms as well as quadrics (for $g \geq 4$). The minimal resolution of $I_X$ is then given by the full quadratic strand of the canonical resolution. No extra computation is required to find $X : I_X$ is just the ideal generated by the space of quadric forms in $I_{C_{\text{can}}}$. There is the extra complication in the genus 6 case where $X$ may be the Veronese surface in $\mathbb{P}^5$ rather than a rational scroll. Here $C$ is isomorphic to a non-singular plane curve of degree 5 and the gonality is 4 rather than 3. In our algorithm, this possibility is discovered when the Lie algebra method is applied to $X$ giving the sub-Lie algebra of trace 0 $6 \times 6$ matrices corresponding to the closed subgroup of automorphisms of $\mathbb{P}^5$ that leave $X$ invariant. When $X$ is Veronese, this Lie algebra is simple of dimension 8, isomorphic to $\mathfrak{sl}_3$ over $\hat{k}$. In the scroll cases, the algebra is of dimension 7 or 9 or is soluble of dimension 8.

When $X$ is a Veronese surface, the Lie algebra method gives an isomorphism $\phi$ of $X$ to $\mathbb{P}^2$ over a finite extension of $\hat{k}$. This comes from computing a linear isomorphism from $X$ onto the standard Veronese surface $X_0$ in $\mathbb{P}^5$ by determining a linear isomorphism within $\mathfrak{sl}_3$ taking the Lie algebra of $X$ to that of $X_0$. Under $\phi$, $C_{\text{can}}$ is mapped to a non-singular degree 5 plane curve $C_5$. Then, the degree 4 maps to $\mathbb{P}^1$ are precisely the projections from points on $C_5$.

This just leaves the generic cases for genus 5 and 6 curves where the Betti diagrams for the minimal free resolutions of the canonical coordinate rings are as follows ( [Sch86]).

| 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 0 | 1 | - | - | - |
| 1 | - | 3 | - | - |
| 2 | - | - | 3 | - |
| 3 | - | - | - | 1 |

The algorithms for these cases will be given in the two following sections.

We should note here some situations when it is easy to find gonal maps. If we have a plane model of $C$ of degree $d$ and $P$ is a point on $C$ of multiplicity $m$, then projection from $P$ gives a degree $d - m$ map to $\mathbb{P}^1$. A small degree (singular) plane model with a singular point of sufficiently high multiplicity can, therefore, give us gonal maps by simple projection.

Assume that we have computed a gonal map for $C$ which gives us a rational function $t$ on $C$ of degree $\leq 4$. We can then compute a radical parametrisation as follows. Firstly, we choose a second rational function $x$ that generates the function field $\hat{k}(C)$ over $\hat{k}(t)$, preferably a coordinate function of $C$ in an affinisation of its given algebraic representation (N.B.: $\hat{k}(C)/\hat{k}(t)$ is separable for gonal maps). We then express all coordinate functions on $C$ as rational functions in $x$ and $t$. This can be achieved with standard elimination techniques using Gröbner bases or resultants. Finally, we find a radical expression for $x$ in terms of $t$ using the minimal polynomial
of $x$ over $\bar{k}(t)$. This polynomial is of degree $\leq 4$ and we can just use the standard formulæ for its roots as radical expressions in the coefficients.

3. Genus 5

In the generic (no $g^1_d$ for $d \leq 3$) case, the $g^1_d$s correspond to degree 2, dimension 3 rational scrolls in $\mathbb{P}^4$ containing $C_{\text{can}}$ ((6.2) [Sch86]). These are defined by singular quadrics $Q$ in the 3-dimensional space of quadrics in $I_{\text{can}}$.

Such singular quadrics are given by 5-variable quadric forms of rank 4 or 3 (no quadrics of rank $\leq 2$ can occur in $I_{\text{can}}$ since such quadrics are geometrically reducible and $C_{\text{can}}$ isn’t contained in any hyperplane). These correspond to rational scrolls of type $S(1,1,0)$ or $S(2,0,0)$ in Schreyer’s notation.

The first type are cones over non-singular quadric surfaces $S$ in $\mathbb{P}^3$. The projection(s) to $\mathbb{P}^1$ are given by projection from the unique singular point followed by one of the 2 classes of fibre projection from $S$ to $\mathbb{P}^1$.

The second type are cones over non-singular conics $C_0$ in $\mathbb{P}^2$. For these, the projections to $\mathbb{P}^1$ are given by projection from the singular locus (a line) followed by a birational map of $C_0$ to $\mathbb{P}^1$.

In either case, the projections may be defined over a quadratic extension of the base field $k$ and it is well-known how to compute them explicitly.

It remains to find such singular quadrics and this is also computationally fairly straightforward.

If $Q_1, Q_2, Q_3$ form a basis for the space of quadrics in $I_{\text{can}}$, we can write the general quadric as $Q_{x,y,z} = xQ_1 + yQ_2 + zQ_3$ for variable $x, y, z$. The condition that $Q_{x,y,z}$ is singular leads to a single degree 5 homogeneous equation $F(x,y,z) = 0$ as follows.

The 5 partial derivatives of $\partial Q_{x,y,z}/\partial x_i$ with respect to the 5 coordinate variables $x_i$ of $\mathbb{P}^4$ are linear forms in the $x_i$ with coefficients given by linear forms in $x, y, z$. The condition that all partial derivatives vanish at a point of $\mathbb{P}^4$ is that the 5x5 matrix of coefficients of the partials w.r.t the $x_i$ is singular: i.e., that its determinant vanishes. $F(x,y,z)$ is just given by this determinant.

Thus we explicitly find a one-dimensional projective family of singular quadrics. This accords with Brill-Noether theory which tells us that the dimension of the family of $g^1_d$s is at least one.

We can now choose a non-trivial solution of $F(x,y,z) = 0$ over a finite extension of $k$ of degree $\leq 5$. In general, $F$ will be non-singular and irreducible, as is easily checked in random examples. It is a number-theoretically very difficult question to check whether a solution exists over $k$ when $k = \bar{Q}$.

Plane models. If $C$ is birationally equivalent to a plane curve of degree $d$ in $\mathbb{P}^2$, $d \geq 5$ by the arithmetic genus formula. A plane model of degree 5 exists if and only if $C$ has a basepoint-free $g^3_2$. If $K$ denotes the canonical divisor class on $C$, then $g^3_2$s correspond to $g^3_1$s by Riemann-Roch under $D \leftrightarrow K - D$. So, for a plane model of degree 5, the gonality of $C$ must be less than 4. A degree 5 plane model would have to have a node or cusp as the unique singularity for its normalisation $C$ to have genus 5. It is then easy to see that the canonical linear system would separate points on $C$ because, by adjunction, it is given by the system of plane quadrics passing through the singular point. Hence, $C$ cannot be hyperelliptic so must have gonality exactly 3.

Conversely, if $C$ has gonality 3, it has a degree 5 plane model. $C_{\text{can}}$ is contained in a rational scroll $X$ which is isomorphic to the Hirzebruch surface $X_1$ embedded in $\mathbb{P}^4$ via the very ample divisor $C_0 + 2f$, in the notation of Chapter V, Section 2 of [Har77]. $C_{\text{can}}$ has divisor class $3C_0 + 5f$ on $X_1$. Contraction of the unique (-1)-curve $C_0$ on $X_1$ gives a birational morphism of
$X_1$ onto $\mathbb{P}^2$ that maps $C_{\text{can}}$ birationally onto a degree 5 curve (the self-intersection of the image of $C_{\text{can}}$ is 25).

Computationally, a rational map from $X$ to $\mathbb{P}^2$ that contracts $C_0$ can be constructed by projecting from any line in the ruling of $X$. Lines in the ruling of $X$ can be located by the Lie algebra method again, just as the fibration $X \to \mathbb{P}^1$ is computed in [SS].

It is generally easy to find a degree 6 model for 4-gonal genus 5 curve. If $D$ is an effective divisor of degree 6 on $C$, then, by Riemann-Roch, $\dim |D| \geq 2$ if and only if $[K-D]$ is non-empty, i.e., $D$ is the complement of a subcanonical degree 2 divisor. Any effective divisor of degree 2 is subcanonical. If $C$ is not hyperelliptic, then we can take $D$ equal to the canonical complement of any two distinct points, $P, Q$ on $C$. $|D| = |K - (P + Q)|$ is given by the hyperplane sections of $C_{\text{can}}$ in $\mathbb{P}^4$ that contain the line $L$ joining the images of $P$ and $Q$. So $\dim |D|$ is exactly 2 and the rational map $C \to \mathbb{P}^2$ is just given by projection from $L$. If this rational map is not birational on $C$, then it must give a 2-1 map onto a smooth cubic (if it were 3-1 onto a conic or the cubic was singular then $C$ would have gonality $\leq 3$), so $C$ is a 2-1 cover of a genus 1 curve $E$.

In this special case, $E$ is the unique genus 1 curve that $C$ is a double cover of (c.f. Remark 2 of Section 1.222). I think that, in this case, all $g_1^1$s on $C$ are pullbacks of $g_1^1$s on $E$. This is true in the genus 6 case - as I show in the next section - but I haven’t checked it properly for genus 5. It would imply that $\dim |D| \geq 2$ in the genus 6 case - as I show in the next section - but I haven’t checked it properly for genus 5. If $C$ is not hyperelliptic, then we can take $D$ equal to the canonical complement of any two distinct points, $P, Q$ on $C$. $|D| = |K - (P + Q)|$ is given by the hyperplane sections of $C_{\text{can}}$ in $\mathbb{P}^4$ that contain the line $L$ joining the images of $P$ and $Q$. So $\dim |D|$ is exactly 2 and the rational map $C \to \mathbb{P}^2$ is just given by projection from $L$. If this rational map is not birational on $C$, then it must give a 2-1 map onto a smooth cubic (if it were 3-1 onto a conic or the cubic was singular then $C$ would have gonality $\leq 3$), so $C$ is a 2-1 cover of a genus 1 curve $E$.

Throughout this section, $R$ will denote the coordinate ring $k[x_1, \ldots, x_6]$ of the $\mathbb{P}^5$ containing $C_{\text{can}}$ and we will write $I_Z$ for the saturated ideal of $R$ that corresponds to a closed subscheme $Z$ of $\mathbb{P}^5$. As usual, $R(n), n \in \mathbb{Z}$, will denote the graded $R$-module equal to $R$ as a plain $R$-module but having a shift in the grading so that $R(n)_m = R_{n+m}, m \in \mathbb{Z}$. All $R$-module homomorphisms between graded $R$-modules preserve the grading. Elements of modules of the form $R(a_1)^{m_1} \oplus \cdots \oplus R(a_r)^{m_r}$ will be thought of as row vectors of length $m_1 + \cdots + m_r$ with entries in $R$. Thus, a homomorphism from $R(a_1) \oplus \cdots \oplus R(a_r)$ to $R(b_1) \oplus \cdots \oplus R(b_s)$ will be represented by an $r$-by-$s$ $R$-matrix whose $(i, j)$-th entry will be a homogeneous polynomial of degree $b_j - a_i$ (necessarily zero, if $a_i > b_j$) acting on row vectors by right multiplication.

The minimal resolution of $I_X$ as a graded $R$-module is of the form $R(-2)^3 \leftarrow R(-3)^2 \leftarrow 0$. This complex must occur as a graded direct summand (see Prop. 6.13 of [Eis05]) of a minimal
resolution of $R/I_{\text{Can}}$

$$\text{res} : \quad R \xrightarrow{\psi} R(-2)^6 \xrightarrow{\phi} R(-3)^5 \oplus R(-4)^5 \leftarrow R(-5)^6 \leftarrow R(-6) \leftarrow 0$$

the $R(-2)^3$ being a direct summand of $R(-2)^6$ and the $R(-3)^2$ a direct summand of $R(-3)^5$.

We fix a particular such resolution and $\text{res}$ will refer to it throughout this section.

Firstly, we have the following

**Lemma 4.1.** There is a 1-1 equivalence between $g^1_d$'s on $C$ and the degree 3, dimension 3 rational scrolls in $\mathbb{P}^5$ containing $C_{\text{Can}}$.

**Proof.** A $g^1_d$ gives a unique rational scroll of the correct type that is the union of the degree 2 linear spans of the $D \in g^1_d$ on $C_{\text{Can}}$. Note that the $g^1_d$ pencil is a complete, basepoint-free linear system as otherwise there would exist an $E \in g^1_d$ giving a $g^1_d$ for $d < 4$. This would contradict the gonality 4 assumption on $C$.

On the other hand a degree 3, dimension 3 rational scroll containing $C_{\text{Can}}$ cuts out a $g^1_d$ with $\dim (K - D) = 2$ for $D \in g^1_d$ (Section 2, [Sch86]). $K$ denotes the canonical divisor class on $C$. It remains to show that $d$ must be 4 here. But $d$ must be at least 4, as $C$ has gonality 4, $\deg(K - D)$ is 10 − $d$, so if $d \geq 5$ then we would have a $g^2_e$ or a $g^3_e$ for $e \leq 4$. The latter would lead to a $g^1_f$ for $f < 4$. Neither of these is possible by assumption. \qed

Our approach here is to explicitly compute such scrolls by a direct, brute-force method. This does appear to work quite efficiently in practise. An alternative would be to try to work with the easily computable surface $Y$ discussed in the next section. The next section contains a summary of more detailed structure of $\text{res}$ that we need to refer to. Our algorithm will be given in the following section.

### 4.1. The Canonical Resolution of $C_{\text{Can}}$. Fixing a $g^1_3$ on $C$ (which exist by [KL74]) and corresponding rational scroll $X$ containing $C_{\text{Can}}$, Sections (6.2)-(6.5) of [Sch86] give the following more detailed description of the way that the canonical resolution $\text{res}$ is built up.

$C_{\text{Can}}$ is the complete intersection of two divisors $Y$ and $Z$ of $X$. The surfaces $Y$ and $Z$ are (absolutely) irreducible since $C_{\text{Can}}$ is. Writing $H$ for the hyperplane class on $X$ and $R$ for the class of the ruling, we have the following equivalence of rational divisor classes on $X$

$$Y \sim 2H - R \quad Z \sim 2H$$

In Schreyer’s notation, $f = 3$, $b_1 = 1$ and $b_2 = 0$ here. The case $(b_1, b_2) = (2, -1)$ cannot occur, as it would lead to $C$ having a $g^1_3$ or $g^2_3$ as Schreyer notes in his (6.3).

In fact, if $Y \sim 2H - 2R$, $Y$ would be a degree 4 ($\deg(Y) = 6 - b_1$) surface in $\mathbb{P}^5$ and so would be a Veronese surface or a 2-dimensional rational scroll. The Veronese case would give a $g^2_3$, an isomorphism of $Y$ to $\mathbb{P}^2$ mapping $C_{\text{Can}}$ isomorphically to a non-singular degree 5 plane curve. In the scroll case, $Y$ would have to be a Hirzebruch surface $X_0$, $X_2$ or $X_4$ embedded in $\mathbb{P}^5$ by the invertible sheaves with divisor class $H_1 = C_0 + 2f$, $C_0 + 3f$ or $C_0 + 4f$ respectively, in the notation of Chapter V, Section 2 of [Har77] ($S(2,2)$, $S(3,1)$ or $S(4,0)$ in Schreyer’s notation). $C_{\text{Can}}$ pulls back to a smooth genus 6 curve which has intersection 10 with $H_1$. Thus, the divisor class of this pullback would be $3C_0 + mf$ with $m$ equal to 4, 7 or 10, and the ruling of $Y$ would induce a $g^1_3$ on $C$.

Sections (6.4) and (6.5) of [Sch86] give two possibilities for $Y$ depending upon whether the generic fibre over $\mathbb{P}^1$ is an irreducible conic or not.
In the general case (6.4), Y is a (possibly singular) degree 5 Del Pezzo surface, anticanonically embedded in \( \mathbb{P}^5 \). In (6.4), \( k = 1 \) and \( \delta = 3 \), so Y is the Hirzebruch surface \( X_1 \) blown up at 3 points and \( X_1 \) is itself \( \mathbb{P}^2 \) blown up at a point.

In case (6.5), Y is a (singular) cone over a projective normal elliptic curve E of a hyperplane of \( \mathbb{P}^5 \).

These possibilities also follow from the classification of surfaces of degree \( d \) in \( \mathbb{P}^d \) [Nag60] and the fact that Y is arithmetically Cohen-Macaulay, so linearly normal and not a projection from \( \mathbb{P}^6 \) (the minimal free resolution of its defining ideal \( I_Y \) is given below).

Replacing coherent sheaves \( \mathcal{F} \) on \( \mathbb{P}^5 \) by their corresponding maximal graded \( R \)-modules \( \mathcal{F} \to \bigoplus_{n=-\infty}^{\infty} H^0(\mathbb{P}^5, \mathcal{F}(n)) \), so \( \mathcal{O}_{\mathbb{P}^5}(n) \) gives \( R(n) \), \( \mathcal{O}_{\text{can}} \) gives \( R/I_{\text{can}} \) etc., Schreyer shows that Serre twists of the Buchsbaum-Eisenbud exact sequences (see Section 1, [Sch86]), \( \mathcal{E}^0 \) and \( \mathcal{E}^1 \)

\[
\mathcal{E}^0 : \quad R \to R(-2)^3 \to R(-3)^2 \to 0
\]
\[
\mathcal{E}^1 : \quad R^2 \to R(-1)^3 \to R(-3) \to 0
\]
determine the minimal free resolutions (up to isomorphism) of the coordinate rings of \( C_{\text{can}} \), \( Y \) and \( Z \) in the following way.

\( \mathcal{E}^0, \mathcal{E}^0(-2), \mathcal{E}^1(-2), \mathcal{E}^1(-4) \) are respectively minimal free resolutions of the maximal graded modules corresponding to the sheaves \( \mathcal{O}_X, \mathcal{O}_X(-Y)^n \mathcal{O}_X(-Z), \mathcal{O}_X(R-4H) \).

The morphisms of the Koszul-complex resolution of \( \mathcal{O}_{\text{can}} \) by locally free \( \mathcal{O}_X \) sheaves

\[
\mathcal{O}_X \to \mathcal{O}_X(-Y) \oplus \mathcal{O}_X(-Z) \to \mathcal{O}_X(R-4H) \to 0
\]
extends to morphisms between complexes of graded modules \( \mathcal{E}^1(-4) \xrightarrow{(f_1,f_2)} \mathcal{E}^0(-2) \oplus \mathcal{E}^1(-2) \) and \( \mathcal{E}^0(-2) \xrightarrow{\sigma_1} \mathcal{E}^0, \mathcal{E}^1(-2) \xrightarrow{\sigma_2} \mathcal{E}^0 \) such that the mapping cones of the latter two give minimal free resolutions of the coordinate rings of \( Z \) and \( Y \) respectively and the iterated mapping cone of

\[
\left[ \mathcal{E}^1(-4) \xrightarrow{(f_1,f_2)} \mathcal{E}^0(-2) \oplus \mathcal{E}^1(-2) \right] \xrightarrow{\sigma_1+\sigma_2} \mathcal{E}^0
\]
gives a minimal free resolution of \( R/I_{\text{can}} \).

Explicitly we get

\[
\begin{align*}
R(-7) \overset{(\alpha_3,\beta_3)}{\longrightarrow} & \quad R(-5)^2 \oplus R(-5) \overset{\chi_1+\phi_3}{\longrightarrow} R(-3)^2 \\
R(-5)^3 \overset{(\alpha_2,\beta_2)}{\longrightarrow} & \quad R(-4)^3 \oplus R(-3)^3 \overset{\chi_2+\phi_2}{\longrightarrow} R(-2)^3 \\
R(-4)^2 \overset{(\alpha_1,\beta_1)}{\longrightarrow} & \quad R(-2)^2 \oplus R(-2) \overset{\chi_1+\phi_1}{\longrightarrow} R
\end{align*}
\]

\[\text{This isn’t quite correct if } Y \text{ intersects the singular locus of } X \text{ and is non-Cartier. } \mathcal{O}_X(-Y) \text{ then actually means the pushforward of the corresponding sheaf on the projective bundle of which } X \text{ is an image.}\]
The columns are the twisted $\mathscr{C}$ exact sequences, the middle one being the direct sum of $\mathscr{C}^0(-2)$ and $\mathscr{C}^1(-2)$. $f_1, f_2, g_1, g_2$ are given by the $\alpha, \beta, \chi$ and $\phi$ maps. The $\sigma$ maps are those that arise from iterating the mapping cone. Apart from these diagonal maps, the other maps in the diagram give commutative squares.

**Uniqueness of $\mathcal{Y}$**

A resolution of $R/I_{\mathcal{Y}}$ is given by the mapping cone of the morphism between complexes defined by the $\phi$ maps. $I_{\mathcal{Y}} \subset I_{C_{can}}$ is the sum of the images of $\phi_1$ and $\gamma_1$ in $R$ and is resolved by the direct summand subcomplex arising from the $\mathcal{C}^1(-2)$ and $\mathcal{C}^0$ terms (apart from $R$). Our chosen $\text{res}$ is isomorphic to this iterated mapping cone as a complex of graded $R$-modules, but different choices of $\mathcal{X}$ will give different decompositions of it.

However, the direct summand subresolution of $I_{\mathcal{Y}}$ is always the same and so we get the same unique $\mathcal{Y}$ for any $X$ (this was noted by Schreyer in [Sch86]).

The point is that in the resolution of $R/I_{\mathcal{Y}}$

$$R \xrightarrow{-\phi_1+\gamma_1} R(-2)^3 \oplus R(-2)^3 \xrightarrow{c_1 \oplus \phi_2 \oplus \gamma_1} R(-3)^3 \oplus R(-3)^2 \xrightarrow{c_2 - \phi_3} R(-5) \leftarrow 0$$

the weight -3 part must correspond precisely to the $R(-3)^3$ summand in the third term of $\text{res}$ and the -2 part must then correspond to the unique dimension 5 summand $F_5$ of the $R(-2)^6$ second term that contains the image of $R(-3)^5$. $I_{\mathcal{Y}}$ is generated by the 5 quadrics which are the images of the generators of $F_5$ under $\psi$. $Z$, which varies with $X$, is the intersection of $X$ with a single quadric hypersurface and $C_{can}$ is the intersection of that hypersurface with $Y$.

The computational approach taken here is to actually compute scrolls and this seems to be fairly efficient, algorithmically. However, it may be possible to just work with $Y$, which is very easy to determine from $\text{res}$ by computing $F_5$. When $Y$ is a cone over an elliptic curve $E$, there are an infinite number of $g_1$ on $C$, given by projecting to $E$ (a degree 2 map on $C_{can}$) and then taking any degree 2 map to $\mathbb{P}^1$. We do use this projection when we recognise that we are in this case. In the general case, $Y$ is a degree 5 Del Pezzo and an alternative is to try to parametrise $Y$ by plane cubics over an extension of the base field. The inverse parametrisation gives a birational map of $Y$ to a degree 6 plane curve and we can then just project from a singular point. We will briefly return to this later. In any case, $F_5$ is used in the algorithm for computing $X$.

### 4.2. The Algorithm

Our method consists of constructing scrolls $X$ by determining the graded direct summand subcomplexes of $\text{res}$ that resolve their defining ideals. For this purpose, we want computational conditions that are as simple as possible for characterising these subcomplexes. One possible potential problem is that the restriction of $\phi$ to the $R(-3)^5$ term in $\text{res}$ has a nontrivial kernel that must be avoided. The next proposition, however, shows that we don’t have to worry about this and that only the most obvious restriction needs to be checked. This reduces the computation to a reasonably straightforward Grassmannian-type algebra problem.

**Proposition 4.2.** If $F$ is any rank 3 direct summand of the $R(-2)^6$ term of $\text{res}$ and $G$ is any rank 2 direct summand of the $R(-3)^5$ term such that $G$ maps into $F$ under $\phi$, then $F \leftarrow G \leftarrow 0$ is a minimal free resolution of $I_X$, the image of $F$ in $R$ under $\psi$, which is the (maximal) defining ideal of a degree 3, dimension 3 scroll $X$ containing $C_{can}$.

**Proof.** The key result that is needed is the

**Claim:** For any $F$ and $G$ as in the statement of the proposition, $G \rightarrow F$ is injective.

We will prove this below. We now show that the proposition follows from the claim.
Let $I_X$ denote the ideal image of $F$ under $\psi$. $I_X$ is generated by three quadric forms, $Q_1, Q_2, Q_3$ which are $k$-linearly independent in the space of all quadric (degree 2) forms in $R$, where $\psi : F \to R$ is given by the $1 \times 3$ matrix $[Q_1 \ Q_2 \ Q_3]$.

We want to show that the complex

$$0 \leftarrow R/I_X \leftarrow R \xleftarrow{\psi} F \xleftarrow{\phi} G \leftarrow 0$$

is exact. $\phi$ here is given by a $2 \times 3$ matrix $M$ which can be written as

$$\begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$$

where the $\lambda_i, \mu_i$ are linear forms in $R$.

Exactness follows from the Hilbert-Burch theorem (Thm. 20.15, [Eis94]) once we show that there exists a non-zero constant $a$ such that the three minors $Q'_i$ of $M$ ($Q'_i$ is the minor obtained by leaving out the $i$th column of $M$) satisfy $Q'_i = (-1)^i a Q_i$, $1 \leq i \leq 3$, and that the ideal $I_X$ (which they then generate) has depth $\geq 2$.

First note that each $Q_i$ is irreducible in $R \otimes_k \bar{k}$ since if one of them decomposed into a product of 2 linear forms then $C_{can}$ would lie in a hyperplane, which it doesn’t. As $R$ is a UFD, the principal ideal $(Q_1)$ is therefore prime and, as the $Q_i$ are $k$-linearly independent, $Q_2$ (or $Q_3$) doesn’t lie in $(Q_1)$ and $I_X$ has depth $\geq 2$.

Now, if $E$ denotes the field of fractions of $R$, then $M$ has rank 2 over $E$, as $\phi$ is injective on $G$, so the $Q'_i$ are not all zero and the right kernel of $M$ in $E^5$ is generated by the column vector $v' := (Q'_1, -Q'_2, Q'_3)^t$. From our complex, $v := (Q_1, Q_2, Q_3)^t$ lies in the $R^3$-kernel.

The $Q'_i$ are quadrics. Let $b$ be a GCD of them in $R$ and write $U_i$ for $Q'_i/b$. Then the right $R^3$-kernel of $M$ is free of rank 1 over $R$ generated by $w := (U_1, U_2, U_3)^t$. Let the degree of $b$ be $r$. Then $v$ is equal to $aw$ with a non-zero homogeneous polynomial $a$ of degree $r$. This implies that $r = 0$ and that $a, b$ are constants, since $\text{GCD} (Q_1, Q_2, Q_3) = 1$. So $v' = (a/b)v$, as required.

From the minimal free $R$-resolution of $S := R/I_X$, it is immediate that $S$ has Hilbert series $(1 + 2t)/(1 - t)^4$ and Hilbert polynomial $H(n) = (1/2)(n + 2)(n + 1)^2$, so the closed subscheme $X$ of $\mathbb{P}^5$ corresponding to $I_X$ has degree 3 and dimension 3.

We claim that $M$ is 1-generic (over $k$), as defined in Chapter 6 of [Eis05]. Any non-zero quadric in the $k$-linear span $(Q_1, Q_2, Q_3)$ of its minors is irreducible over $k$, since no irreducible quadric vanishes on $C_{can}$ as noted above. If $M$ were not 1-generic, we could find invertible 2x2 and 3x3 matrices $U, V$ with coefficients in $k$ giving $UMV$ a zero in the top left position. But then at least two of the minors of $UMV$ would be reducible and they are non-zero linear combinations of the linearly independent $Q_i$.

Then, Theorems 6.4 and A2.64 of [Eis05] show that $X$ is a rational scroll.

**Proof of claim**

It remains to prove that $G \to F$ must be injective. This follows from the more detailed description of $\text{res}$ given in Section 4 of [Eis05] after choosing a particular rational scroll $X_0$ containing $C_{can}$.

The results there show that the restriction of $\phi$ to the $R(-3)^5$ summand does indeed have a nontrivial kernel which is the image of the map $(e_2, -\phi_3)$ in (1). It is generated by a single element $m$. It suffices to show that a non-zero multiple of $m$ cannot lie in a direct summand of $R(-3)^5$ of rank less than 3.

Take an $R$-basis for $R(-3)^5$ such that the first three elements are a basis for the $R(-3)^3$ summand of the $R(-3)^3 \oplus R(-3)^2$ direct sum decomposition appearing in (1). Then $m$ is given
by a 5-vector with quadric entries, the first three of which are the image of the generator of $R(-5)$ under $\epsilon_2$.

$\epsilon_2$ is the last map of the twisted Buchsbaum-Eisenbud complex $\mathcal{C}^1(-2)$. The explicit description of these complexes (see (1.5) [Sch86], for example) shows that the first three entries in the vector for $m$ can be taken as the 2x2 minors of the 2x3 matrix which define $X_0$. These three quadrics are linearly independent in the space of all quadrics on $\mathbb{P}^5$. However, if $am$ lay in a direct summand of $R(-3)^5$ of rank 2 for a non-zero homogeneous polynomial $a$, the vector for $am$ would be equal to $a_1v_1 + a_2v_2$ for homogenous polynomials $a_1, a_2$ and vectors $v_1, v_2$ with entries in $k$ which generate the summand. Then we would arrive at the contradiction that the five entries of the $am$ vector lay in the space of polynomials spanned by $a_1$ and $a_2$, which has dimension at most two. □

A rank $f$ direct summand of $R(\epsilon)^d$ is generated by $f$ linearly-independent $d$-vectors with coefficients in the base field $k$. The idea is to simply represent $F$ and $G$ as summands generated by the rows of variable matrices. The condition that $\phi$ maps $G$ into $F$ is then easily translatable into a system of (degree 1 and 2) polynomial equations in these variables. The solutions of these over $\bar{k}$ will give us precisely the scrolls $X$ that we are interested in.

Before doing this however, we compute the rank 5 direct summand $F_5$ of $R(-2)^6$ that $R(-3)^5$ maps into under $\phi$ and the surface $Y$. These are used in two ways.

- When $Y$ is an elliptic cone, we will use projection to the genus one curve to construct $g_1$'s.
- $F$ must lie in $F_5$, and it reduces the number of variables in the problem to consider $F$ as a direct summand of $F_5$, rather than $R(-2)^6$.

We will prove in Section 4.2.3 that then there are only finitely many $g_1$'s on $C$ when $Y$ is not an elliptic cone. Therefore our system of polynomial equations for the scrolls will be zero-dimensional in that case.

4.2.1. Computing $F_5$ and $Y$. These come from basic linear algebra. Let $M_5$ be the 5x6 matrix of linear forms giving the map $\phi$ restricted to $R(-3)^5$. $F_5$ is generated by 5 linearly independent 6-vectors $v_i$ with coefficients in $k$ such that each row of $M_5$ is a $R$-linear combination of the $v_i$. The one-dimensional right $k$-kernel of the 5x6 matrix with rows $v_i$ is precisely the one-dimensional right $k$-kernel of $M_5$. This kernel has to exist and be one-dimensional for the image of $\phi$ to lie in a direct summand of rank 5 of $R(-2)^6$ but to lie in no smaller rank direct summand. $Y$ is defined by the 5 quadrics that are the images of the $v_i$ under $\psi$. Thus, we have the following simple algorithm, Algorithm 1, that also gives the matrix $M_5$ for the restricted map $\phi$ from $R(-3)^5$ into $F_5$.

4.2.2. The elliptic cone case. Algorithm 2 determines whether $Y$ is a cone over a projectively normal genus 1 curve $E$ and returns $E$ and the degree 2 projection from $C_{can}$ to $E$ in the affirmative case.

Computing the singular locus $S$ of $Y$ is a standard procedure. We apply the Jacobi criterion: $S$ being defined by the 3x3 minors of the 5x6 Jacobi matrix of partial derivatives of the 5 quadric generators of $I_Y$. A Gröbner basis computation tells us whether $S$ is zero-dimensional and consists of one point or not. Note that the singular locus coincides with the locus of non-smoothness over $\bar{k}$ for the possible $Y$ here.

To find a degree 4 map of $C_{can}$ to $\mathbb{P}^1$ over $\bar{k}$, it remains to compute a degree 2 rational function on $E$. One way to do this computationally is to choose any degree 2 effective divisor $D$ of $E$.
Algorithm 1 Computation of $F_5$, $Y$ and $M_5$

**Step 0:** Input $M_6$.

**Step 1:** Let $M_1$ be the $30 \times 6$ $k$-matrix obtained from $M_6$ by replacing each linear form by a 6-element column vector containing its coefficients with respect to the variables of $R$. Compute $v$, a basis for the right kernel of $M_1$.

**Step 2:** Compute an echeloned basis $B$ for the 5-dimensional $k$-subspace of $k^6$ orthogonal (with respect to the usual scalar product) to $v$. $F_5$ is the direct summand of $R(-2)^6$ with $R$-basis given by the vectors in $B$. Set $I_Y$ equal to the ideal of $R$ generated by the five quadrics $\psi(b)$ for $b \in B$. $I_Y$ is the defining ideal of $Y$.

**Step 3:** Let $i$ be a complementary index for the echeloned basis $B$. That is, if we remove column $i$ from the $5 \times 6$ matrix whose rows are the elements of $B$, we get the identity matrix. Then, set $M_5$ equal to the $5 \times 5$ matrix given by removing the $i$th column of $M_6$.

over $\bar{k}$ (e.g., twice a $\bar{k}$ point) and compute the Riemann-Roch space $L(D)$ using Florian Hess’ function field package or something similar. Any non-constant function in $L(D)$ is a degree 2 function on $E$. Whether a degree 2 $k$-rational function exists or not on $E$ is trickier. $E$ is a principal homogeneous space over its elliptic curve Jacobian $J(E)$ of order 1 or 5 in the Tate-Shafarevich group of $J(E)$ over $k$. If it is of order 5, then the only $k$-rational divisors on $E$ have degree divisible by 5 and we must go to an extension of degree at least 5 over $k$ to find effective divisors of degree 2. If it is of order 1, then $E$ is isomorphic to $J(E)$ over $k$ and there is at least one $k$-rational point. Over $\mathbb{Q}$, there is no known effective procedure for determining whether a $k$-rational point exists, though in practice, if there are $k$-rational points, we can often find one by a point search over small height projective points.

**Remarks:**
1) It is easy to see that any $g^1_4$ on $C_{can}$ is the pullback of a $g^1_2$ on $E$. In fact, (6.4) and (6.5) of [Sch86] show that the rulings on a scroll $X$ corresponding to the $g^1_4$ intersect the cone $Y$ in the union of two lines.

2) $E$ is the unique genus 1 curve up to $k$-isomorphism which has a degree 2 covering by $C$ over $k$. If $E_1$ were another such, consider the product map of $C$ into $E \times E_1$ and let $C_1$ be its image. $C \rightarrow C_1$ is of degree 1 or 2. If it were degree 1, then $C_1$ would be birationally equivalent to $C$ and an irreducible divisor of $E \times E_1$ of arithmetic genus $\geq 6$ with a degree 2 projection to $E$ and $E_1$. Divisor theory on $E \times E_1$ shows that this is impossible. Thus $C \rightarrow C_1$ is of degree 2 and $C_1$ projects $k$-isomorphically onto both $E$ and $E_1$.

Algorithm 2 Testing for the elliptic cone case

**Step 0:** Input $I_Y$.

**Step 1:** Compute the singular locus $S$ of $Y$. If $S$ is empty or contains more than one point, return false. Otherwise, let $P$ be the unique ($k$-rational) point in $S$.

**Step 2:** Find a linear change of coordinates of $\mathbb{P}_5$ such that $P$ becomes the point $(0 : 0 : 0 : 0 : 0 : 1)$. If $x_1, \ldots, x_6$ are the new homogeneous coordinates functions, set $B$ equal to the set of 5 quadrics given by expressing the 5 generators of $I_Y$ in the $x_i$.

**Step 3:** If $B$ contains an element that is not a quadric in $x_1, \ldots, x_5$ only, return false. Otherwise, identifying $\mathbb{P}^4$ with the hyperplane of $\mathbb{P}^5$ defined by $x_6 = 0$, set $E$ equal to the subvariety of $\mathbb{P}^4$ with defining ideal generated by $B$ and $pr_j$ equal to the map from $C_{can}$ to $E$ given by composing the linear transformation from Step 2 with the projection from $\mathbb{P}^5$ to $\mathbb{P}^4$. Return true, $E, pr_j$. 
4.2.3. The general case. We now consider the general case when $Y$ is a Del Pezzo. In this case, as mentioned earlier, there are only finitely many $g_1$'s and scrolls $X$ (at most 5, in fact). We represent direct summands $F$ of $F_5$ and $G$ of $R(-3)^5$ by rows of variable matrices giving their generators and translate the condition that $\phi$ maps $G$ into $F$ into a system of polynomial equations in the variables.

We can work projectively but it is computationally simpler (and involves fewer variables) to work on affine patches of the Grassmannian representations of the $F$, $G$ summands. For example, one case is where the first minor of each of the matrices giving the generators of $F$ and $G$ is non-zero. Then we can choose unique linear combinations of the generators of the two submodules such that $F$ (resp. $G$) is generated by the rows of the matrix

$$M_F = \begin{pmatrix} 1 & 0 & 0 & u_1 & u_2 \\ 0 & 1 & 0 & v_1 & v_2 \\ 0 & 0 & 1 & w_1 & w_2 \end{pmatrix}$$

resp.

$$M_G = \begin{pmatrix} 1 & 0 & r_1 & r_2 & r_3 \\ 0 & 1 & s_1 & s_2 & s_3 \end{pmatrix}$$

where $u_i, v_i, w_i, r_i, s_i$ will lie in $\tilde{k}$. Considering these as 12 independent variables, the condition that $G$ maps into $F$ produces a zero-dimensional ideal of relations $J$ in $k[u_i, v_i, w_i, r_i, s_i]$. The procedure for computing solutions when the generators of $F$ and $G$ can be put into the above form is given explicitly in Algorithm 3 below. The input is the matrix $5\times5$ matrix of linear forms $M_5$ that represents $\phi : R(-3)^5 \to F_5$ and was computed earlier in Algorithm 1.

For each set of return values $F_{sol}, G_{sol}, T_{sol}$ of Algorithm 3, $R$-bases for $F$ as a submodule of $F_5$ (resp. $G$ as a submodule of $R(-3)^5$) are given by the rows of $F_{sol}$ (resp. $G_{sol}$) and the $2\times3$ matrix representing $\phi : G \to F$ with respect to these bases is $T_{sol}$.

The corresponding scroll $X$ is defined by the three quadrics that are the images in $R$ of the basis of $F$ under the inclusion map of $F_5$ into $R(-2)^6$ followed by $\psi$, but equivalently it is defined by the three quadrics which are the $2\times2$ minors of $T_{sol}$ (see proof of Prop. 4.2).

$T_{sol}$ is a 1-generic matrix of homogeneous linear forms in $x_i$

$$\begin{pmatrix} L_0 & L_2 & L_4 \\ L_1 & L_3 & L_5 \end{pmatrix}$$

As $X$ is defined by the maximal minors of $T_{sol}$, the rational function $L_0/L_1 = L_2/L_3 = L_4/L_5$ on $X$ gives a rational map to $\mathbb{P}^1$ which induces the linear pencil of the ruling. Since the associated $g_1^4$ on $C_{can}$ is the restriction of this ruling, we can therefore read off the desired degree 4 rational function directly from $T_{sol}$:

A degree 4 rational function on $C_{can}$ that induces the $g_1^4$ corresponding to $X$

is just $L_0/L_1$ (or $L_2/L_3$, $L_4/L_5$) restricted to $C_{can}$

We have shown how to compute scrolls $X$ and the associated $g_1^4$'s and rational maps that come from $F$ and $G$ summands with a particular affine Grassmannian representation. The same procedure works in the other cases (e.g., $M_G = \begin{pmatrix} 1 & r_1 & r_2 & 0 & r_3 \\ 0 & s_1 & s_2 & 1 & s_3 \end{pmatrix}$). There are $\binom{5}{2} \times \binom{5}{3} = 100$ possibilities in total, though many will lead to the same $F, G$ pairs in general.

It follows from [KL74] that a gonality 4, genus 6 curve with only finitely many $g_1^4$'s has at most 5 $g_1^4$'s. We will also prove this in Prop. 4.3, Section 4.3 This means that the system of equations that we have to solve in any particular case give a zero-dimensional ideal $J$ of degree at most 5 generated by polynomials of degree at most 2. Thus, computing a lex Gröbner of $J$
Algorithm 3 Computation of scrolls $X$ for the first pair of Grassmannian affine patches

Step 0: Input $M_5$.

Step 1: Compute $M_G \ast M_5$. The result is of the form

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 & \mu_4 & \mu_5 \\ \nu_1 & \nu_2 & \nu_3 & \nu_4 & \nu_5 \end{pmatrix}$$

The $\mu_i$ and $\nu_i$ are homogeneous linear forms in the $x_i$ with coefficients non-homogeneous linear forms in $r_i$ (for the $\mu_i$) and $s_i$ (for the $\nu_i$). The rows of this matrix generate the image of $G$ under $\phi$.

Step 2: The condition that $\phi$ maps $G$ into $F$ translates into 24 equations of degree $\leq 2$ in the 12 variables given by taking the 4 equations

$$\begin{align*}
\mu_4 &= u_1\mu_1 + v_1\mu_2 + w_1\mu_3 & \mu_5 &= u_2\mu_1 + v_2\mu_2 + w_2\mu_3 \\
\nu_4 &= u_1\nu_1 + v_1\nu_2 + w_1\nu_3 & \nu_5 &= u_2\nu_1 + v_2\nu_2 + w_2\nu_3
\end{align*}$$

between linear forms in $x_1, \ldots, x_6$ and equating the coefficients of each $x_i$ on the LHS and RHS. Let $J$ denote the ideal generated by the 24 equations.

Step 3: Compute a lex Gröbner basis of the zero-dimensional ideal $J$. From this, we can read off the solutions in $\bar{k}$ for the $u_i, \ldots, s_i$ to the system of 24 equations. For each solution, set $F_{sol}$ and $G_{sol}$ equal to $M_F$ and $M_G$ evaluated at the $u_i, \ldots, s_i$ values and set $T_{sol}$ equal to the 2x3 matrix of linear forms in $\bar{k}[x_1, \ldots, x_6]$ given by evaluating

$$\begin{pmatrix} \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix}$$

at the $r_i, s_i$ values. Return $F_{sol}, G_{sol}, T_{sol}$.

and even decomposing into prime components is generally quite fast even though there are 12 variables.

We can adopt the following overall procedure. If we are only interested in finding a degree 4 function over $\bar{k}$, we can search for solutions over affine patches until we find one and then stop. If we would like a function over $k$ (if one exists), we can proceed through solutions over different patches, stopping if we find a $k$-rational one or if we have already found five. If we want to find all solutions, we can keep searching through patches, stopping if we have found five solutions at any point. In practise, we compute the prime components of the ideals $J$ while searching and only specialise into some field extension of $k$ when we have chosen our desired solution.

4.2.4. Degree six plane models. We will show in Section [13] that $C$ is not birationally equivalent to a degree 6 plane curve in the special case where $Y$ is an elliptic cone.

In the general case, on the other hand, a nice feature of the above computation of scrolls is that a degree 6 singular plane model for $C$ is also easily constructed from the $T_{sol}$ matrix.

Note that if $|D|$ is a $g^1_4$ then $|K - D|$ is a $g^2_6$ and vice-versa by Riemann-Roch. Explicitly, $L_0$ and $L_1$ in $T_{sol}$ are two hyperplanes that intersect $C_{can}$ in degree 10 divisors with a degree 6 common factor $K - D$ where $D$ is the divisor of points lying on $L_0$ but not $L_1$. The sections of $|K - D|$ as subsections of $|K|$ correspond to the space of hyperplanes through $D$. This space is precisely the degree 1 part of the saturation ideal $(I_{C_{can}} + (L_0) : (L_1)^\infty)$. Standard Gröbner basis algorithms to compute saturation ideals are well-known (see [GP02]). A 3-element basis of the space defines the map from $C_{can}$ to $\mathbb{P}^2$ which is birational onto a degree 6 image $C_1$.

The fact that the map corresponding to $|K - D|$ is birational follows easily from the proof of Prop. [13] in Section [13] where the $g^1_4$'s on $C_{can}$ are identified with the restrictions of particular
linear pencils on the Del Pezzo $Y$. Furthermore, we can also deduce from the analysis there that $|D|$ corresponds to the pencil of conics in $\mathbb{P}^2$ that pass through the 4 singular points of $C_1$ when $C_1$ only has nodes and simple cusps as singularities.

When $Y$ is an elliptic cone over genus one curve $E$, the $|K - D|$ for $g_1^4$'s $D$ must give degree 2 maps onto nonsingular degree 3 plane curves (if the image were singular or the map 3-1 onto a plane conic then $C$ would have gonality at most 3). These images must be isomorphic to $E$ after Remark 2 of 4.2.2.

4.2.5. Example: $X_0(58)$. As an example, we take the case of the genus 6 modular curve $X_0(58)$.

With projective coordinate variables $x, y, z, u, v, w$ a canonical model has defining equations given by the following polynomials

\begin{align*}
  &x^2 - xz + yu + yv - xw + uw + vw, \\
  &x^2 - xy - y^2 - xz - xu + zu + yv + zv, \\
  &-x^2 + xy + y^2 + u^2 + uw - yw, \\
  &-y^2 - xy + yz - xu + yu + zu + u^2 - vw, \\
  &y^2 - xz + yz + z^2 - xu + zu + yv + zw, \\
  &xy - yz - xu + zu + yv + zw + uv - xw + yw + zw + uw
\end{align*}

This can be derived by computing relations between the $q$-expansions of a basis of weight 2 cusp forms. The canonical minimal resolution is of general type.

Working with Magma (BCP97) on a 2.2 GHz dual core AMD Opteron machine, it took less than a second in total to compute the Gröbner basis and degree of each of the $J$s corresponding to different Grassmannian affine patches. As expected there were 5 points in total. Decomposing a suitable $J$, we found that there was 1 rational solution and 2 pairs of conjugate solutions over quadratic fields.

Taking the rational solution, it took a fraction of a second to compute that the $g_1^4$ is determined by the linear pencil generated by the hyperplanes

\[
2y - v + w \quad \text{and} \quad 2x + 2z - v + w
\]

and that the complementary $g_2^4$ is determined by the 3 hyperplanes

\[
y + u + v \quad z - v + w \quad x - u - v
\]

which gives a degree 6 plane model with 4 nodes. The affine version of this has defining polynomial

\[
\begin{align*}
Y^5 - Y^4X^2 - 2Y^4X - 10Y^4 + Y^3X^3 + 11Y^3X^2 + 9Y^3X + 33Y^3 - Y^2X^4 - 2Y^2X^3 - \\
36Y^2X^2 - 10Y^2X - 37Y^2 + 3YX^5 + 22YX^4 + 12YX^3 + 36YX^2 + 6YX + 17Y - \\
2X^6 + 2X^5 - 3X^4 + 4X^3 - 3X^2 + 2X - 2 = 0
\end{align*}
\]

where $X$ and $Y$ are the rational functions $(y + v + w)/(x - u - v)$ and $(z - v + w)/(x - u - v)$ on $C_{can}$.

We will now derive radical expressions for $X$ and $Y$ using the degree 4 rational function on this affine plane curve that comes from the $g_4^1$ on $C_{can}$.

As noted above, the $g_4^1$ on the projective plane model is given by the pencil of quadrics through the 4 nodes. This was simple to compute in Magma. The result is that the degree 4 function $t$ is given by

\[
t = (3X^2 + Y^2 - 6Y + 3)/(XY - Y^2 + 3X + 4Y)
\]
Eliminating $Y$ by a resultant computation, we find that the function field is generated by $t$ and $X$ and that $X$ satisfies the degree 4 relation over $\mathbb{Q}(t)$

$$(8t^5 + 12t^2 + 8t + 4)X^4 - (4t^5 + 6t^4 - 12t^3 - 32t^2 - 22t - 12)X^3 - (12t^5 + 17t^4 - 4t^3 - 40t^2 - 27t - 14)X^2 - (12t^5 + 9t^4 - 9t^3 - 48t^2 - 34t - 12)X - 4t^5 + 3t^4 + 13t^3 + 32t^2 + 25t + 10 = 0$$

A further Gröbner basis computation shows that

$$Y = (2t^2 - 3t + 3)X^2 + (t^2 - 6t)X + 2t^2 - 3t + 3)/((2t^2 + 5t + 3)X + 5t^2 + 5t + 9)$$

It remains to use the formulæ for the roots of a degree 4 polynomial to give a radical expression for $X$ in $t$. This is rather messy but we include it for completeness. Let $\zeta$ be a primitive cube root of unity. Define

$$\begin{align*}
P_1 &= (3t^5 + 10t^4 + 31t^3 + 40t^2 + 27t + 14)(72t^{10} + 42t^9 + 475t^8 + 662t^7 + 1770t^6 + 2420t^5 + 2654t^4 + 2092t^3 + 1093t^2 + 324t + 52) \\
P_2 &= (t^2 + 2)(16t^4 + 26t^3 + 79t^2 + 42t + 45)(t^6 + 2t^5 + 11t^4 + 22t^3 + 21t^2 + 12t + 4) \\
P_3 &= 3(7t^6 - 2t^5 - 7t^4 - 14t^3 - 17t^2 - 12t - 4) \\
A &= 12t^5 + 17t^4 - 4t^3 - 40t^2 - 27t - 14 \\
B &= 12t^{10} - 66t^9 - 223t^8 - 926t^7 - 1974t^6 - 3332t^5 - 3854t^4 - 3340t^3 - 2101t^2 - 900t - 244 \\
C &= t^5 + (3/2)t^4 - 3t^3 - 8t^2 - (11/2)t - 3 \\
D &= (t + 1)(2t^2 + t + 1)
\end{align*}$$

and let $R_1$ be the radical expression

$$R_1 = \sqrt[3]{-P_1 + 3P_2\sqrt{-P_3}}$$

and

$$\begin{align*}
a_1 &= (-2A + (R_1 - (B/R_1)))/3D \\
a_2 &= (-2A - (1/2)((R_1 - (B/R_1)) + \zeta(R_1 + (B/R_1)))/3D \\
a_3 &= (-2A - (1/2)((R_1 - (B/R_1)) - \zeta(R_1 + (B/R_1)))/3D
\end{align*}$$

Then, we get the following radical expression for $X$

$$X = (1/4)((C/D) + \sqrt{(C/D)^2 - a_1 + \sqrt{(C/D)^2 - a_2 + \sqrt{(C/D)^2 - a_3}}}}$$

where the third square root should actually be written as a certain rational function in $t$ divided by the product of the other two square roots.

4.3. Plane Models of Genus 6 Curves. In this section, we prove some results about which types of genus 6 curve have degree 6 singular plane models and relate the existence of such a model to the finiteness of the number of distinct $g^2_3$s in the gonality 4 case. As well as having a bearing on our algorithm, these results also give some interesting information about the stratification of the moduli space of genus 6 curves into the following parts: hyperelliptic, gonality 3, plane quintic, Clifford-index 2 with $Y$ an elliptic cone and Clifford-index 2 with $Y$ a Del Pezzo. $Y$ refers to the surface described in Section 4.1 that contains the canonical curve.

In order to discuss double points more easily, we assume that the characteristic of $k$ is not 2.

**Proposition 4.3.** For any genus 6 curve $C$ the following are equivalent

(a) $C$ is of gonality 4 with no $g^2_3$ and $Y$ is a Del Pezzo surface.
b) $C$ has only finitely many $g_1^1$s.

c) $C$ is birational over $\bar{k}$ to a degree 6 plane curve with only double points.

In this case, $C$ has at most 5 $g_1^1$s.

Proof. (b)$\Rightarrow$(a)

We show that there are infinitely many $g_1^1$s in every other case.

Firstly, assume $C$ has gonality at most 3. We choose a $g_1^1$ containing the effective divisor $D$. Then for each $P$ in $C(\bar{k})$, $\dim |D + P| \geq 1$, so we can find a $g_1^1$ containing $D + P$. These are all distinct as $P$ is not linearly equivalent to $Q$ for $P \neq Q$, so $D + P$ is also not equivalent to $D + Q$.

If $C$ is isomorphic to a non-singular degree 5 plane curve ($C$ has a $g_5^2$), then the linear system of lines through each $P \in C(\bar{k})$ gives a $g_1^1$ after removing the base point $P$. These are clearly distinct for different $P$.

If $C$ has gonality 4 and $Y$ is a cone over an elliptic curve $E$, then the pullback under the degree 2 projection of $C$ to $E$ of the infinitely many distinct $g_2^1$s on $E$ lead to infinitely many distinct $g_1^1$s on $C$ (the pullbacks of $|P + Q|$ and $|R + S|$ can be the same on $C$ only if $P + Q = R + S$ in the group law of $E$).

(a)$\Rightarrow$(c)

As noted before, the inverse of a birational parametrisation $\mathbb{P}^2 \to Y$ by cubic polynomials (over $\bar{k}$) maps $C_{\text{can}}$ birationally to a plane sextic $C_1$. If $C_1$ had a singularity of multiplicity at least 3, projection from it would give lead to a map from $C$ to $\mathbb{P}^1$ of degree at most 3, contradicting the gonality 4 assumption. Thus, $C_1$ only has double points.

(c)$\Rightarrow$(b)

Let $C_1$ be a plane sextic birational to $C$ with only double points. Since we are assuming that char($k$) $\neq 2$, any singularity is of type $A_n$ for $n \geq 1$ (analytically isomorphic to $y^2 = x^{n+1} + \text{higher powers of } x$). Such a singularity is resolved by sequence of $[(n+1)/2]$ blowings up: at each stage, if we have an $A_n$ singularity $P$ on the transformed curve in the blown-up plane, after blowing up at $P$, the strict transform of the curve has a single $A_{n-2}$ singularity over $P$ if $n > 2$, a single nonsingular point over $P$ if $n = 2$ (simple cusp), or 2 nonsingular points over $P$ if $n = 1$ (node). We continue getting singularities of multiplicity 2 until the singularity is resolved. By [Har77], the arithmetic genus of the strict transform of $C_1$ drops by one at each blow up until it reaches 6, when it must be isomorphic to $C$ and non-singular. Thus we have to blow up exactly 4 times and then the blow-up of $\mathbb{P}^2$ is $Y_1$, a degree 5 Del Pezzo surface. $Y_1$ is a degenerate Del Pezzo if there is an $A_n$ singularity with $n > 2$, when we have to blow up infinitely near points. Note that for all possible singularity types of $C_1$ ($4(A_1/A_2)$, $2(A_1/A_2) + 1(A_3/A_4)$, $2(A_3/A_4)$, $1(A_7/A_8)$), the sequence of blow-up points is “almost general” as per Definition 1, Section III.2 of [Dem80], so $Y_1$ is a (possibly degenerate) Del Pezzo. In particular, if a line went through all 4 blow-up points (including infinitely near ones), then its intersection number with $C_1$ would be at least 8, which is impossible as $C_1$ is irreducible of degree 6. The strict transform of $C_1$ is isomorphic to $C$, so we will identify it with $C$.

Let $H$ denote the divisor class of $Y_1$ that is the total transform of the class of a line in $\mathbb{P}^2$ and $E_1, E_2, E_3, E_4$ the total transforms of the exceptional curves generated in the blow-ups (so $E_i^2 = 1$ and $E_i \cdot E_j = 0$ for $i \neq j$). $C$ is an irreducible divisor of $Y_1$ rationally equivalent to $6H - 2E_1 - 2E_2 - 2E_3 - 2E_4 = -2K_{Y_1}$ where $K_{Y_1}$ is the canonical class on $Y_1$. By adjunction (Prop. 8.20, Chap. II, [Har77]), the canonical class $K_C$ on $C$ is the restriction of the class $-K_{Y_1}$ of $Y_1$. Also as $H^1(Y_1, O(K_{Y_1})) = 0$ ($Y_1$ is rational), the standard cohomology sequence shows that the restriction map on global sections $H^0(Y_1, O_Y(-K_{Y_1})) \to H^0(C, O_C(K_C))$ is an isomorphism. Thus, the anticanonical map of $Y_1$ into $\mathbb{P}^5$ restricts to the canonical embedding
of $C$ onto $C_{can}$. Note that $-K_{Y_1}$ is birational on $Y_1$ and $1 - 1$ outside of some possible (-2)-curves, so the canonical map on $C$ is birational. Hence, $C$ is not hyperelliptic and its canonical map gives an embedding. Let the image of $Y_1$ be $Y_0 \supset C_{can}$. Note that $Y_0$ is singular when $Y_1$ is degenerate: fundamental cycles of (-2)-curves are contracted to (simple) singular points (see [Denn80]).

Now, the one-dimensional linear system $|D|$ on $Y_1$, where $D = 2H - E_1 - E_2 - E_3 - E_4$, restricts to a complete $g^1_4$ on $C$. It consists of the strict transforms on the pencil of conics through the 4 blow-up points (if there are repeated blowups, some of the divisors in the pencil may be the sum of the strict transform of a conic and some irreducible (-1)-curves and/or (-2)-curves). It is of degree 4 on $C$ as $D \cdot (-2K_{Y_1}) = 4$. It gives a complete $g^1_4$ because $-D - K_{Y_1} = H$ and so, by Serre duality, $h^1(Y_1, O_{Y_1}(D - C)) = h^1(Y_1, O_{Y_1}(D + 2K_{Y_1})) = h^1(Y_1, O_{Y_1}(H)) = 0$. The last equality holds by applying Riemann-Roch to $h$. However, each such divisor is the intersection of all of the divisors in $|D|$ is either empty or consists of a union of (-2)-curves. However, $C$ is irreducible and $C \cdot E = (-2K_{Y_1}) \cdot E = 0$, so $C$ doesn’t intersect any (-2)-curve $E$. Thus, the $g^1_4$ is also basepoint free.

This $g^1_4$ corresponds to the rulings of a degree 3, dimension 3 scroll $X$ containing $C_{can}$ (this is true for any complete, basepoint free $g^1_4$). Further, Schreyer’s derivation of the canonical resolution using $X$ applies to give the description in section 4.1 (see sections 4 and 6.2 of [Sch86]) and show that the minimal free resolution of $R/I_{C_{can}}$ is of shape res. This means that $C$ must be of gonality 4 with no $g^2_1$.

We now show that the $Y$ associated to $X$ is precisely $Y_0$. Firstly, $Y_0 \subset X$. The two-dimensional linear spaces comprising the ruling of $X$ are the linear spans of the supports of the divisors in the $g^1_4$. However, each such divisor is the intersection of $C_{can}$ with the image of a divisor $E$ in $|D|$. Since, $D \cdot (-K_{Y_1}) = 2$, these images are of degree 2 and so lie in a plane which must be the span of the corresponding divisor on $C_{can}$ except for a finite number of degenerate cases. As the divisors in $|D|$ cover $Y_1$, its image lies in $X$.

Now $X$ is a scroll of type $S(1, 1, 1)$ or $S(2, 1, 0)$ in Schreyer’s notation from [Sch86]. The first is non-singular and the second is a cone with a single singular point $P$. In the second case, if $Y_0$ contained $P$ then it would be a singular point of $Y_0$. This is because $P$ is the intersection of all of the ruling planes and so would lie in the intersection of all of the images of divisors in $|D|$. As noted above, this intersection, if non-empty, consists of some irreducible (-2)-curves, so must map into singular points under $-K_{Y_1}$. Thus, in either case, $Y_1 \rightarrow Y_0 \subset X$ factors through a map $Y_1 \rightarrow \mathbb{P}(\mathcal{E})$, where $\mathcal{E}$ is a 3-dimensional locally-free sheaf on $\mathbb{P}^3$ and $\mathbb{P}(\mathcal{E})$ is the projective bundle of which $X$ is the image. We abuse notation slightly by denoting the strict transforms of $Y_0$ and $Y$ in $\mathbb{P}(\mathcal{E})$ by the same symbols. Let $H'$ denote the divisor class of the pullback of the Cartier hyperplane class of $X$ and $R$ the class of the ruling. By standard theory, $Pic(\mathbb{P}(\mathcal{E}))$ is generated by $H'$ and $R$ (see, e.g., Ex. 12.3, Chapter III of [Har77]). We have that $H'^3 = 3$ (the degree of $X$), $H'^2 \cdot R = 1$ (as the rulings map to linear spaces in $\mathbb{P}^5$) and $R^2 = 0$ in the cycle class group of $\mathbb{P}(\mathcal{E})$. We know that $Y$ is equivalent to $2H' - R$. We want to show the same for $Y_0$. Let $Y_0$ be equivalent to $aH' + bR$ with $a, b \in \mathbb{Z}$. $Y_0$ of degree 5 in $\mathbb{P}^5$ implies that $H'^2 \cdot Y_0 = 5$. Also, $R$ pulls back to $D$ in $Y_1$, which means that $Y_0 \cdot R \cdot H' = D \cdot (-K_{Y_1}) = 2$. Thus we get $a = 2$ and $b = -1$ as required.

If $Y \neq Y_0$, since both are irreducible, $U = Y \cap Y_0$ would be a sum (with multiplicities) of irreducible curves in $\mathbb{P}(\mathcal{E})$ with $U \cdot H' = (2H' - R) \cdot (2H' - R) \cdot H' = 8$. But $C_{can} \subset U$ and $C_{can} \cdot H' = 10$. Thus $Y$ is $Y_0$ as claimed. We can also note that $C_{can}$ misses any singular locus of $X$, as it is non-singular and a complete intersection in $X$.

Now we know that any $g^1_4$ on $C$ comes from a ruling on some $X$ which restricts to a pencil of divisors on the unique $Y$, which is $Y_0$. This pulls back to a pencil, contained in $|D_1|$ say, on
Y₁. Since the ruling induces a degree 4 divisor on C_{can} and C_{can} pulls back isomorphically to C which is equivalent to −2K₁₁ on Y₁, D₁ · K₁₁ = −2. If X is nonsingular, or Y₀ misses its unique singular point P, D₁ = 0. Otherwise, any two rulings meet transversally at P, which must be singular point of Y₀, so D₁ = D₀ + F where F is the fixed part of the pencil - a fundamental cycle of (-2)-curves - and D₀ is the variable part with D₀ = 0. As C is disjoint from all irreducible (-2)-curves E and K₁₁ · E = 0, we can replace D₁ by D₀ in this case. Thus any g₁ comes from the restriction of some pencil of divisors within |D₁| where D₁ = 0 and D₁ · K₁₁ = −2. Since a g₁ on C is complete, it is entirely determined by the class of any divisor within it, and so any pencil induced from D₁ will only depend on the equivalence class of D₁ in Pic(Y₁).

A simple computation shows that the only classes with self intersection 0 and intersection −2= 14 of (-2)-curves - and that this case, therefore, the five pencils only restrict to 3 distinct classes on C. Thus we have the stronger result (as expected from Brill-Noether) that there are at most five distinct g₁'s on C. □

Remarks: 1) In some cases, several members of the five classes listed at the end of the last proof can restrict to the same g₁ on C. This can occur when the plane model of C has higher order Aₙ singularities and one of the classes differs from another by a fixed component consisting of (-2)-curves [C = −2K₁₁ on Y₁ means that (-2)-curves are the only effective divisors which don’t intersect C].

2) When the plane model C₁ only has nodes and simple cusps, the five g₁'s correspond to the four pencils of lines through a singular point and the pencil of conics through all four singular points.

Example: Let C be birational to the plane curve C₀ with defining polynomial (w.r.t. x, y, z projective coordinates)

\[ x^2y^4 + 2x^3y^2z - xy^4z + x^4z^2 + 2x^3yz^2 - x^2y^2z^2 - xy^3z^2 - 2y^4z^2 - x^3z^3 + 2x^2yz^3 - xy^2z^3 + y^3z^3 - x^2z^4 + 2xyz^2 + y^2z^4 \]

C₀ has nodes (A₁ singularities) at P₁ := (0 : 0 : 1) and P₃ := (0 : 1 : 0) and a type A₃ singularity at P₂ := (1 : 0 : 0). P₄ will denote the infinitely near point above P₁ that needs to be blown up to resolve the A₂ singularity. We write Lᵢⱼ for the line through Pᵢ and Pⱼ (L₁₄ is the line through P₁ whose tangent direction corresponds to P₄). L₁₂ doesn’t pass through P₄ but L₁₃ does, so L₁₃ = L₁₄ = L₃₄ and L₂₄ doesn’t exist. Blowing up the points in the order P₁, P₂, P₃, P₄, we see that E₁ = E₁ + E₄ with E₁ an irreducible (-2)-curve and E₂, E₃, E₄ irreducible (-1)-curves.

\[ \hat{L} \], the strict transform of L₁₃, is an irreducible (-2)-curve. Letting Eᵢⱼ denote the usual (-1)-classes H − Eᵢ − Eⱼ on Y₁, E₁₂ and E₂₃ give irreducible (-1)-curves, but E₁₃ = \hat{L} + E₄, E₁₄ = \hat{L} + E₃, E₃₄ = \hat{L} + \hat{E}_₁ + E₄ and E₂₄ = E₁₂ + \hat{E}_₁.

We easily find that |H − E₁|, |H − E₂| and |H − E₃| are distinct pencils without fixed points or components on Y₁, but (H − E₁) · \hat{E}_₁ = D · \hat{L} = −1 means that \hat{E}_₁ and \hat{L} are fixed components of the two respective pencils and that |H − E₁| = |H − E₁| + \hat{E}_₁ and |D| = |H − E₂| + \hat{L}. In this case, therefore, the five pencils only restrict to 3 distinct g₁'s on C.

Taking the canonical map C₀ → ℙ⁵ defined by the polynomials (x²z : xy² : xyz : xz² : y²z : yz²) to give C_{can}, we find explicitly that Y is defined by the 5 quadrics

\[ z^2 − xt, \ ys − xt, \ zs − xu, \ zt − yu, \ st − zu \]
and that there are indeed only 3 scrolls containing \( C_{\text{can}} \):

\[
\begin{align*}
X_1 &: \quad ys - xt = zs - xu = zt - yu = 0 \\
X_2 &: \quad z^2 - ys = zt - yu = st - zu = 0 \\
X_3 &: \quad z^2 - xt = zs - xu = st - zu = 0
\end{align*}
\]

\( Y \) has singularities at \( p_1 := (1 : 0 : 0 : 0 : 0 : 0) \) and \( p_2 := (0 : 1 : 0 : 0 : 0 : 0) \), the images of \( E_1 \) and \( L \). \( X_1 \) is non-singular, \( X_2 \) has singular point \( p_1 \) and \( X_3 \) has singular point \( p_2 \). This illustrates that the \( X \)'s for a given \( C \) may be of different type and that \( Y \) may pass through the singular point of an \( X \).

**Proposition 4.4.** A genus 6 curve \( C \) has gonality 3 if and only if it is birationally equivalent over \( k \) to a degree 6 plane curve with at least one singularity of multiplicity greater than 2.

**Proof.** Let \( C \) be birationally equivalent to \( C_1 \), a degree 6 plane curve with a singularity \( P \) of multiplicity of at least 3. Projection from \( P \) gives a map to \( \mathbb{P}^1 \) of degree at most 3 and \( C \) is hyperelliptic or has gonality 3. Since \( C \) has genus 6, there are only two possibilities for the singularities of \( C_1 \). Either there is an triple point that is resolved by a single blow-up and a cusp or node, or there is a single triple point with a single infinitely-near node or cusp. In either case, it is easy to see that the canonical linear system on \( C_1 \), which is given by the subsystem of plane cubics that satisfy the adjoint conditions at the singularities, separates the non-singular points. Therefore, \( C \) cannot be hyperelliptic. For example, in the first case we can take reducible cubics consisting of two lines through the triple point and one line through the other singularity.

Conversely, let \( C \) have gonality three. Then \( C \) is isomorphic to a 3-section in a Hirzebruch surface, which is swept out in the canonical embedding by the line spans of the divisors of a \( g^1_3 \). There are two possibilities for the surface and the divisor class of \( C \) within it: \( X_0 \) with \( C \sim 3C_0 + 4f \) or \( X_2 \) with \( C \sim 3C_0 + 7f \) in the notation of Chapter V, Section 2 of [Har77].

In either case, we can find a birational isomorphism of \( X_1 \) onto \( \mathbb{P}^2 \) that maps \( C \) onto a degree 6 curve.

In the \( X_2 \) case, let \( P \) be any point on \( C \) which doesn’t lie on the distinguished (-2)-curve \( C_0 \). We begin by blowing up \( P \) and blowing down the strict transform of the fibre \( f \) containing \( P \). This gives a birational isomorphism of \( X_2 \) to \( X_1 \) where the unique (-1)-curve is the strict transform of \( C_0 \). We then blow down this (-1)-curve to get to \( \mathbb{P}^2 \). Tracing through the intersections of transforms of \( C \) with the curves that are blown down, it is easy to see that the self-intersection of the overall strict transform of \( C \) is 36 (\( C^2 = 24 \)), so that it is a degree 6 plane curve. Furthermore it has a unique singular point that is a triple point (which resolves to the intersection of \( C \) with \( f \) on \( X_2 \)) with a single node or cusp above it on \( X_1 \), so we have the second case above for singularities.

In the \( X_0 \) case, we proceed similarly. Here, the class of \( C_0 \) does not contain a unique curve but gives a second ruling of \( X_0 \). The rulings \( C_0 \) and \( f \) are distinguished here by \( C \cdot f = 3 \) and \( C \cdot C_0 = 4 \). We choose any point \( P \) on \( C \) to blow up but, this time, we will let \( C_0 \) denote the unique curve in its class passing through \( P \). Then, we blow down the strict transform of \( f \) followed by the strict transform of \( C_0 \) as above. Again we see that the overall strict transform \( C_1 \) of \( C \) has degree 6 in the plane. If \( f \) meets \( C \) transversally at \( P \), we get a triple point and a node/cusp on \( C_1 \), which resolve to the intersection of the strict transforms of \( C \) and \( C_0 \) and the strict transforms of \( C \) and \( f \) after the blowing up of \( P \). \( X_0 \) is isomorphic to a non-singular quadric surface in \( \mathbb{P}^3 \) and the birational map to \( \mathbb{P}^2 \) corresponds to projection from the point \( P \) on that model. \( \square \)
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