Bi-interpretability with \( \mathbb{Z} \) and models of the complete elementary theories of \( \text{SL}_n(\mathcal{O}) \), \( \text{T}_n(\mathcal{O}) \) and \( \text{GL}_n(\mathcal{O}) \), \( n \geq 3 \)

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Abstract

Let \( \mathcal{O} \) be the ring of integers of a number field, and let \( n \geq 3 \). This paper studies bi-interpretability of the ring of integers \( \mathbb{Z} \) with the special linear group \( \text{SL}_n(\mathcal{O}) \), the general linear group \( \text{GL}_n(\mathcal{O}) \) and solvable group of all invertible uppertriangular matrices over \( \mathcal{O} \), \( \text{T}_n(\mathcal{O}) \). For each of these groups we provide a complete characterization of arbitrary models of their complete elementary theories.

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1 Introduction and preliminaries

Let \( \mathcal{O} \) be the ring of integers of a number field, and \( n \geq 3 \) be a natural number. The main problem we tackle in this paper is to characterize arbitrary (pure) groups elementarily equivalent to the special linear group \( \text{SL}_n(\mathcal{O}) \) over \( \mathcal{O} \), the general linear group \( \text{GL}_n(\mathcal{O}) \) over \( \mathcal{O} \) and solvable group of all invertible uppertriangular matrices over \( \mathcal{O} \), \( \text{T}_n(\mathcal{O}) \).

Since Tarski and Mal’cev there has been many interesting results about elementary equivalence of finitely generated groups and rings. A question dominating research in this area has been if and when elementary equivalence between finitely generated groups (rings) implies isomorphism. Recently, Lubotzky et. al. \[2\] coined the term first-order rigidity: a finitely generated group (ring) \( A \) is first-order rigid if any other finitely generated group (ring) elementarily equivalent to \( A \) is isomorphic to \( A \). Indeed a stronger version of rigidity, Quasi Finite

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Axiomatizability, QFA (See Definition 1.3), due to A. Nies has been around over the past two decades and has been studied for various classes of groups and rings. Indeed, a large class of nilpotent and polycyclic groups [12, 8], all free metabelian groups of finite rank [7], the ring of integers \( \mathbb{Z} \) (Sabbagh 2004, [11]), polynomial rings \( \mathbb{Z}[x_1, \ldots, x_m] \) [11], and finitely generated fields of characteristic \( \neq 2 \) [13] are known to have the property. First-order rigidity of non-uniform higher dimensional lattices in semi-simple Lie groups, e.g. \( \text{SL}_n(\mathbb{Z}) \), has been addressed in [2], and finitely generated profinite groups are proved to have the property in [6].

A typical technique in studying whether a finitely generated structure \( A \) is QFA or first-order rigid is to study a stronger property: whether \( A \) is bi-interpretable with the ring of integers \( \mathbb{Z} \) (See Definition 1.1 and Theorem 1.4 below). We note that the integral Heisenberg group \( \text{UT}_3(\mathbb{Z}) \) is QFA, but it is not bi-interpretable with \( \mathbb{Z} \) ([7, 11]). It turns out that studying if a finitely generated group \( G \) is bi-interpretable with \( \mathbb{Z} \) or how it fails in that respect is also very useful in studying arbitrary groups (rings) elementarily equivalent to a given one. For example, in many cases arbitrary groups that are elementarily equivalent to \( G \) seem to have a very particular structure: They are kind of “completions” or “closures” of \( G \) with respect to a ring \( R \) elementarily equivalent to \( \mathbb{Z} \). When dealing with classical groups or algebras such notions of completion or closure coincide with the classical ones, where completions have the same “algebraic scheme”, but the points are over the ring \( R \) as above. In this paper we prove that this is the case for the group \( G = \text{SL}_n(\mathbb{Z}) \), it is bi-interpretable with \( \mathbb{Z} \) (Theorem 2.4 below), so it is first-order rigid. Moreover, any other group \( H \) with \( G \equiv H \) is isomorphic to \( \text{SL}_n(R) \) with \( R \equiv \mathbb{Z} \). On the other hand the “extent” to which a group \( G \) fails to be bi-interpretable with \( \mathbb{Z} \) also often seems to affect the structure of arbitrary groups elementarily equivalent to \( G \). Again it seems such groups are “deformations” of “exact completions” or “exact closures” of \( G \) over a ring \( R \) as above. It only seems proper that these deformations can usually be captured by cohomological data (See 1, 10). For example, the group \( \text{T}_n(\mathcal{O}) \) is not bi-interpretable with \( \mathbb{Z} \), if the ring \( \mathcal{O} \) of integers of a number field has an infinite group of units (though \( \text{T}_n(\mathcal{O}) \) and \( \mathcal{O} \) are mutually interpretable in each other). In this case such failure is modulo the infinite center. We prove in this paper that any group \( H \) with \( H \equiv \text{T}_n(\mathcal{O}) \) is an “abelian deformation” of a group \( \text{T}_n(R) \) where \( R \equiv \mathcal{O} \) (See Theorem 3.6 below). The case of \( \text{GL}_n(\mathcal{O}) \), where \( \mathcal{O} \) has infinite group of units is an exception. Even though, it is not bi-interpretable with \( \mathbb{Z} \), in this paper we prove that all its models are of the type \( \text{GL}_n(R) \).

As we remarked previously, [2] contains a proof of the first-order rigidity of \( \text{SL}_n(\mathbb{Z}) \), \( n \geq 3 \). The authors also prove that \( \text{SL}_n(\mathbb{Z}) \) is prime. In the same paper authors announce that in a future work they would present a proof of the QFA property for \( \text{SL}_n(\mathbb{Z}) \). The approach in that paper is not via bi-interpretability. In a sequel to this paper we study the relevant questions, when \( \mathcal{O} \) is replaced by a field \( F \), where \( F \) is say, a number field (or in general a finitely generated field), or algebraically closed field.
As to the organization of this paper, all results on $\text{SL}_n(\mathcal{O})$ are collected in Section 2, the ones on $\text{T}_n(\mathcal{O})$ in Section 3 and those on $\text{GL}_n(\mathcal{O})$ in Section 4. We fix our notation and state basic definitions and results in Section 1.1.

1.1 Preliminaries

1.1.1 Basic group-theoretic and ring-theoretic notation

For a group $G$ by $Z(G) = \{ x \in G : xy = yx, \forall y \in G \}$ we denote the center of $G$. The derived subgroup $G'$ of $G$ is the subgroup of $G$ generated by all commutators $[x,y] = x^{-1}y^{-1}xy$ of elements $x$ and $y$ of $G$. We also occasionally use $x^y$ for $y^{-1}xy$, for $x$ and $y$ in $G$. For any element $g$ of $G$, $C_G(g) = \{ x \in G : [x,g] = 1 \}$ is the centralizer of $g$ in $G$.

All rings in this paper are commutative associative with unit. We denote the ring of rational integers by $\mathbb{Z}$ and the field of rationals by $\mathbb{Q}$. By a number field we mean a finite extension of $\mathbb{Q}$. By the ring of integers $\mathcal{O}$ of a number field $F$ we mean the subring of $F$ consisting of all roots of monic polynomials with integer coefficients. For a ring $R$, by $R^\times$ we mean the multiplicative group of invertible (unit) elements of $R$. By $R^+$ we mean the additive group of $R$.

Consider the general linear group $\text{GL}_n(R)$ over a commutative associative unitary ring $R$ and let $e_{ij}$, $1 \leq i \neq j \leq n$, be the matrix with $ij$'th entry 1 and every other entry 0, and let $t_{ij} = 1 + e_{ij}$, where $1$ is the $n \times n$ identity matrix. Let also $t_{ij}(\alpha) = 1 + \alpha e_{ij}$, for $\alpha \in R$. The $t_{ij}$ as defined above are called transvections. Let

$$T_{ij} \overset{\text{def}}{=} \{ t_{ij}(\alpha) : \alpha \in R \}$$

i.e. the $T_{ij}$ are one parameter subgroups generated by $t_{ij}$ over $R$. The subgroups $T_{ij}$, where $1 \leq i < j \leq n$, generate the subgroup $\text{UT}_n(R)$ of $\text{GL}_n(R)$ consisting of all upper unitriangular matrices.

For a fixed $n \geq 3$ we order all transvections $t_{ij}$, $1 \leq i \neq j \leq n$, in a fixed but arbitrary way and denote the corresponding tuple by $\bar{t}$. If $\beta_1, \ldots, \beta_m$ lists a set of elements of the ring $R$ we put a fixed but arbitrary order on the set of all transvections $t_{ij}(\beta_k)$, $1 \leq i \neq j \leq n$, $k = 1, \ldots, m$, and denote the corresponding tuple by $\bar{t}(\bar{\beta})$.

Let $\text{diag}([\alpha_1, \ldots, \alpha_n])$ be the $n \times n$ diagonal matrix with $ii$'th entry $\alpha_i \in R^\times$. The subgroup $D_n(R)$ consists precisely of these elements as the $\alpha_i$ range over $R^\times$.

Now, consider the following diagonal matrices, the dilations,

$$d_i(\alpha) \overset{\text{def}}{=} \text{diag}[1, \ldots, \alpha_{i^\text{th}}, \ldots, 1],$$

and let us set

$$d_i \overset{\text{def}}{=} d_i(-1).$$

Clearly the $d_i(\alpha)$ generate $D_n(R)$ as $\alpha$ ranges over $R^\times$. The elements $d_k(\alpha)$, $1 \leq k \leq n$, $\alpha \in R^\times$ and $t_{ij}(\beta)$, $1 \leq i < j \leq n$, $\beta \in R$, generate the subgroup $\text{T}_n(R)$ of $\text{GL}_n(R)$ consisting of all invertible upper triangular matrices.
1.1.2 Bi-interpretability, Quasi-finite axiomatizability and Primeness

For basics of Model Theory our reference is [3]. We denote the first-order language of groups by $\mathcal{L}_{\text{groups}}$ and the first-order language of unitary rings is denoted by $\mathcal{L}_{\text{rings}}$. When a structure $\mathfrak{A}$ is definable or interpretable in a structure $\mathfrak{B}$ with parameters $\bar{b} = (b_1, \ldots, b_n) \in |\mathfrak{B}|^n$ we say that $\mathfrak{A}$ is interpretable in $(\mathfrak{B}, \bar{b})$.

**Definition 1.1.** Consider structures $\mathfrak{A}$ and $\mathfrak{B}$ in possibly different signatures. Assume $\mathfrak{A}$ is interpretable in $\mathfrak{B}$ via an interpretation $\Delta$, $\mathfrak{B}$ is interpretable in $\mathfrak{A}$ via an interpretation $\Gamma$, $\mathfrak{A}$ is the isomorphic copy of $\mathfrak{A}$ defined in itself via $\Gamma \circ \Delta$, and $\mathfrak{B}$ is an isomorphic copy of $\mathfrak{B}$ defined in itself via $\Delta \circ \Gamma$. We say that $\mathfrak{A}$ is bi-interpretable with $\mathfrak{B}$ if there exists an isomorphism $\mathfrak{A} \cong \mathfrak{A}$ which is first-order definable in $\mathfrak{A}$ and there exists an isomorphism $\mathfrak{B} \cong \mathfrak{B}$ which is first-order definable in $\mathfrak{B}$.

**Definition 1.2.** A structure $\mathfrak{A}$ is said to be prime if $\mathfrak{A}$ is isomorphic to an elementary submodel of each model $\mathfrak{B}$ of $\text{Th}(\mathfrak{A})$.

**Definition 1.3.** Fix a finite signature. An infinite f.g. structure is Quasi Finitely Axiomatizable (QFA) if there exists a first-order sentence $\Phi$ of the signature such that

- $\mathfrak{A} \models \Phi$
- If $\mathfrak{B}$ is a f.g. structure in the same signature and $\mathfrak{B} \models \Phi$ then $\mathfrak{A} \cong \mathfrak{B}$.

**Theorem 1.4** (See [11], Theorem 7.14). If $\mathfrak{A}$ is a structure with finite signature which is bi-interpretable (possibly with parameters) with $\mathbb{Z}$, then $\mathfrak{A}$ is prime. If in addition, $\mathfrak{A}$ is finitely generated, then it is QFA.

**Theorem 1.5.** If $\mathfrak{A}$ and $\mathfrak{B}$ are structures bi-interpretable with each other, then $\text{Aut}(\mathfrak{A}) \cong \text{Aut}(\mathfrak{B})$, that is, the automorphism groups of the two structures are isomorphic.

1.1.3 Bi-interpretability of rings of integers of number fields and $\mathbb{Z}$

The following is a known result.

**Lemma 1.6.** Assume $\mathcal{O}$ is the ring of integers of a number field $F$ of degree $m$ and $\beta_1, \ldots, \beta_m$ generate it as a $\mathbb{Z}$-module. Then $(\mathcal{O}, \beta)$ and $\mathbb{Z}$ are bi-interpretable.

**Proof.** By ([11], Proposition 7.12) we need to prove that $\mathcal{O}$ is interpretable in $\mathbb{Z}$ and there is a definable copy $M$ of $\mathbb{Z}$ in $\mathcal{O}$ together with an isomorphism $f : \mathcal{O} \to M$ which is definable in $\mathcal{O}$.
The ring $O$ is interpreted in $\mathbb{Z}$ by the $m$-dimensional interpretation $\Delta$:

$$x = \sum_{i=1}^{m} a_i \beta_i \mapsto (a_1, \ldots, a_m)$$

where $\mathbb{Z}^m$ is equipped with the ring structure:

$$e_i \cdot e_j = \mathbb{Z}(c_{ij1}, c_{ij2}, \ldots, c_{ijm}) \iff \beta_i \cdot \beta_j = \mathbb{O} \sum_{k=1}^{m} c_{ijk} \beta_k$$

and $e_i = (0, \ldots, 0, 1_i, 0, \ldots, 0)$, for $i = 1, \ldots, m$. On the other hand $\mathbb{Z}$ is defined in $O$ without parameters as $\mathbb{Z} \cdot 1_O$ by the well-known result of Julia Robinson [15]. So we can take $M = \prod_{i=1}^{m} \mathbb{Z} \cdot 1_O$ with $f(x)$ defined as

$$f(x) = (a_1 \cdot 1_O, \ldots, a_m \cdot 1_O) \iff x = \sum_{i=1}^{m} a_i \beta_i$$

which is obviously definable in $O$.

By Theorem 1.5 we can not get ride of the parameters, since $O$ is not automorphically rigid while $\mathbb{Z}$ is such. $\blacksquare$

2 \hspace{1em} The case of $\text{SL}_n(O)$

2.1 \hspace{1em} Bi-interpretability of $O$ and $\text{SL}_n(O)$

We firstly point out that by [3] the group $\text{SL}_n(O)$ is boundedly generated by the one parameter subgroups $T_{ij}$ generated by the transvections $t_{ij}$. Also, the transvections satisfy the well-known Steinberg relations:

1. $t_{ij}(\alpha)t_{ij}(\beta) = t_{ij}(\alpha + \beta), \forall \alpha, \beta \in R.$

2. $[t_{ij}(\alpha), t_{kl}(\beta)] = \begin{cases} t_{ij}(\alpha \beta) & \text{if } j = k \\ t_{kj}(-\alpha \beta) & \text{if } i = l \\ 1 & \text{if } i \neq l, j \neq k \end{cases}$ for all $\alpha, \beta \in R.$

In addition to these, the $t_{ij}$ satisfy finitely many other relations depending on $O$. But, those are not the concern of this paper and we don’t need referring to them explicitly.

Lemma 2.1. For $G = \text{SL}_n(O)$ or $G = \text{GL}_n(O)$ and any transvection $t_{kl} \in G$, $Z(C_G(t_{kl})) = T_{kl} \times Z(G)$.

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Proof. For \( x = (x_{ij}) \in G \) a direct calculation imposing \( xt_{kl} = t_{kl}x \) shows that every non-diagonal entry of the \( k' \)th column and \( l' \)th row of \( x \) has to be zero, and \( x_{kk} = x_{ll} \), that is,

\[
x \in C_G(t_{kl}) \iff \begin{cases} x_{kk} = x_{ll} \\ x_{ij} = 0 \quad \text{if } i \neq j, \text{ and either } j = k \text{ or } i = l \end{cases}
\]

In particular every \( t_{ii} = t_{kk} \), and every \( t_{ij} \) where \( i \neq k \) and \( j \neq l \) belongs to \( C_G(t_{kl}) \). Therefore,

\[
Z(C_G(t_{kl})) \leq \left( \bigcap_{1 \leq i \neq j \leq n} C_G(t_{ii}) \right) \cap \left( \bigcap_{1 \leq j \neq k \leq n} C_G(t_{kj}) \right) \cap \left( \bigcap_{1 \leq i \neq j, j \neq l \leq n} C_G(t_{ij}) \right)
\]

So, we have

\[
x \in Z(C_G(t_{kl})) \implies \begin{cases} x_{ij} = 0 \quad \text{if } i \neq j \text{ and } (i,j) \neq (k,l) \\ x_{ii} = x_{jj} \quad \text{if } i \neq l \text{ or } j \neq k \end{cases}
\]

To finish the proof we need to show that for all \( i \) and \( j \), \( x_{ii} = x_{jj} \). Note that if \( x_{kk} \neq x_{ii} \) for some \( i \neq k, l \), then without loss of generality we can assume that \( x_{ii} = 1 \) if \( i \neq l, k \) and \( x_{kk} = x_{ll} = -1 \). Then, \( x \notin C_G(t_{ii}) \), \( i \neq k \) even though \( t_{ii} \in C_G(t_{kl}) \) for such choice of \( i \). Therefore, \( x_{ii} = x_{jj} \) for all \( i \) and \( j \). Now, \( x \in Z(C_G(t_{kl})) \) if and only if \( x_{ii} = x_{jj} \) for all \( i \) and \( j \), and \( x_{ij} = 0 \), if \( i \neq j \), and \((i,j) \neq (l,k)\). Indeed, we proved that

\[
Z(C_G(t_{kl})) = T_{kl} \times Z(G).
\]

\[ \square \]

Corollary 2.2. For any \( k \neq l \), \( T_{kl} \) is definable in \((G,T)\) if \( G = GL_n(O) \) or \( G = SL_n(O) \).

Proof. Just note that for \( k \neq l \), there exists \( j \neq k, l \), such that

\[
T_{kl} = [Z(C_G(t_{kj})), t_{ji}] = [t_{kj}, Z(C_G(t_{ji}))].
\]

\[ \square \]

Notation. In the following we view one parameter subgroups \( T_{kl} \) as a cyclic two-sorted module: \( T_{kl}^O \overset{\text{def}}{=} (T_{kl}, O, \delta) \), where \( \delta \) describes the action \( O \) on \( T_{kl} \), that is,

\[
y = \delta(t_{kl}(\alpha), \beta) \iff y = t_{kl}(\alpha \cdot \beta), \alpha, \beta \in O
\]

Lemma 2.3. For any \( O \), The two-sorted module \( T_{kl}^O \) is interpretable in \((G,T)\), \( G = SL_n(O), GL_n(O) \) where the action of \( O \) on the \( t_{kl} \) respects commutation, i.e. for any \( k \neq l \) and \( \alpha \in O \)

\[
[t_{kj}(\alpha), t_{ji}] = [t_{kj}, t_{ji}(\alpha)] = t_{kl}(\alpha).
\]
Proof. Fix $1 \leq k < j < l \leq n$. All subgroups $T_{kj}, T_{jl}, T_{kl}$ are definable, hence is the subgroup $G_{k,j,l} \overset{\text{def}}{=} (T_{kj}, T_{jl}, T_{kl})$. Note that $G_{k,j,l} \cong \text{UT}_3(\mathcal{O})$. So, one can use Mal’cev interpretation of $\mathcal{O}$ in $\text{UT}_3(\mathcal{O})$ to prove the claim for the choice of $k, j, l$. Using Steinberg relations one can easily interpret the actions on all the $T_{ij}, i \neq j$ so that the ring action respects commutation.

Theorem 2.4. The ring of integers $\mathcal{O}$ of a number field of degree $m$ and the group $(\text{SL}_n(\mathcal{O}), \bar{t})$ are bi-interpretable. Hence, the ring $\mathbb{Z}$ of rational integers and $(\text{SL}_n(\mathcal{O}), \bar{t}(\bar{\beta}))$ are bi-interpretable.

Proof. Let us recall the standard absolute interpretation of $G = \text{SL}_n(\mathcal{O})$ in $\mathcal{O}$. Consider the definable subset of $\mathcal{O}^{n^2}$ consisting of $x = (x_{ij}) \in \mathcal{O}^{n^2}$ defined by $\det(x) = R 1$, where $\det(x)$ is a polynomial in the $x_{ij}$ with integer coefficients and $=R$ denotes identity in the language of rings. Group product and inversion are also defined by (coordinate) polynomials in the $x_{ij}$. These polynomials do also have integer coefficients. Let us denote this interpretation by $\Gamma$.

Let us denote the interpretation of $\mathcal{O}$ in $G$ obtained in Lemma 2.3 by $\Gamma$ and for simplicity assume $i = 1$ and $k = n$. With this choice, $\mathcal{O}$ is defined on the definable subset $T_{1n}$ of $(G, \bar{t})$.

We are about to use bounded generation of $G = \text{SL}_n(\mathcal{O})$ by the $T_{ij}$. For what is coming, it is important to have a fixed order on the way we express elements of $G$ as a product of elements from the $T_{ij}$. Indeed, by bounded generation there exists a number $w = w(n, \mathcal{O})$ depending on both $n$ and $\mathcal{O}$ and a function $f : \{1, \ldots w\} \rightarrow \{(i, j) : 1 \leq i \neq j \leq n\}$, such that, identifying $ij$ with $(i, j)$

$$g \in G \iff \exists \bar{\gamma} \in \mathcal{O}^w(g = \prod_{k=1}^w t_{f(k)}(\gamma_{f(k)}))$$

Now, consider the copy $\tilde{G}$ of $G$ defined in $G$ via $G \overset{\Delta}{\rightarrow} \mathcal{O} \overset{\Gamma}{\rightarrow} G$. Indeed, $\tilde{G}$ is a group defined on the (set) Cartesian product $(T_{1n})^{n^2}$, subject to finitely many group theoretic relations. Given any $g \in G$, it is represented as $g = \prod_{k=1}^w g_{f(k)}$, $g_{f(k)} \in T_{f(k)}$. Next, we define a map

$$G \overset{\Phi}{\rightarrow} \tilde{G}, \quad g \mapsto \prod_{k=1}^w h_{f(k)}$$

where $h_{f(k)}$ is the unique element of $T_{1n}$ such that:
This is an isomorphism between $G$ and $\tilde{G}$ definable in $(G, \tilde{t})$.

Next, consider:

$$O \xrightarrow{\Gamma} G \xrightarrow{\Delta} O$$

and denote the copy of $O$ defined in itself via the composition of interpretations by $\tilde{O}$. The copy of $G$ defined in $O$ is of the form of the matrix products $g = \prod_{k=1}^{w} t_{f(k)}(\gamma_{f(k)})$, for $\gamma_{f(k)}$ in $O$. Since there is a fixed order of representation of elements $g$ of $G$ as products of transvections, for each $1 \leq i, j \leq n$, there exists a polynomial $P_{ij}(\bar{y}) \in \mathbb{Z}[^{\bar{y}}]$, where $\bar{y} = (y_1, \ldots, y_w)$, such that for every matrix $g = (\beta_{ij})_{n \times n} \in SL_n(O) \subset O^{n^2}$

$$\beta_{ij} = \beta_{1n} \Leftrightarrow \exists \gamma \in O^{\bar{w}}(\beta = P_{1n}(\gamma)).$$

We note that by our choice of interpretation of $O$ in $G$, $\tilde{O}$ is defined on the subset $T_{1n}$ of $O^{n^2} = (\beta_{ij})_{n \times n}$. So, the following sets up an isomorphism between $O$ and $\tilde{O}$, which is definable in $\mathcal{O}$

$$\beta = \beta_{1n} \Leftrightarrow \exists \gamma \in O^{\bar{w}}(\beta = P_{1n}(\gamma))$$

Bi-interpretability with $\mathbb{Z}$ with the given parameters follows from above and Lemma 1.6.

**Corollary 2.5.** The group $SL_n(O)$ is QFA and prime for any ring of integers $O$.

### 2.2 Models of the complete elementary theory of $SL_n(O)$

**Theorem 2.6.** Assume $H$ is a group and $H \equiv SL_n(O)$ in $\mathcal{L}_{\text{groups}}$. Then $H \cong SL_n(R)$ for some ring $R \equiv O$ as rings.

**Proof.** Assume $\Gamma(\tilde{t}(\tilde{\beta}))$ is the interpretation of $\mathbb{Z}$ in $(SL_n(O), \tilde{t}(\tilde{\beta}))$ introduced in Theorem 2.4. By (11), Theorem 7.14 there is a formula $\Psi_{sl}(\bar{x})$ of $\mathcal{L}_{\text{groups}}$ that is satisfied by $\tilde{t}(\tilde{\beta})$ and if $\bar{s}(\gamma)$ is a sequence of elements of $SL_n(O)$ so that $SL_n(O) \models \Psi_{sl}(\bar{s})$, then $\Gamma(\bar{s}(\gamma))$, and $\Delta$ as in Theorem 2.4 bi-interpret $\mathbb{Z}$ and $(SL_n(O), \bar{s}(\gamma))$. Now assume $\{\Psi^i : i \in I\}$ lists all axioms of $\mathbb{Z}$ and for each $i,$
\( \Psi^i_{\Gamma(i(\bar{u})))} \) is the \( \mathcal{L}_{\text{groups}} \) sentence such that \( \mathbb{Z} \models \Psi^i \Leftrightarrow \text{SL}_n(\mathcal{O}) \models \Psi^i_{\Gamma(i(\bar{u})))}. \) Then, by the above argument for each \( i \in I, \text{SL}_n(\mathcal{O}) \models \forall \bar{x}(\Psi_{st}(\bar{x}) \rightarrow \Psi^i_{\Gamma(i(\bar{u})))}. \)

Therefore, there exists a tuple \( \bar{u} \) of \( H \) such that for each \( i \in I, H \models \Psi^i_{\Gamma(i(\bar{u}))). \) This implies that \( \Gamma(\bar{u}) \) interprets a ring \( R \) in \( H \) where \( R \cong \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}^*, \) where \( \mathbb{Z}^* \) is a model of the theory of \( \mathbb{Z}. \) By Keisler-Shelah’s Theorem there is a non-principal ultrafilter \( \mathcal{D} \) on a set \( I, \) such that the ultraproducts \( \mathbb{Z}^I/\mathcal{D} \cong (\mathbb{Z}^*)^I/\mathcal{D}. \) Now \( \mathcal{O}^I/\mathcal{D} \cong \mathcal{O} \otimes_{\mathbb{Z}} (\mathbb{Z}^I/\mathcal{D}) \cong \mathcal{O} \otimes_{\mathbb{Z}} ((\mathbb{Z}^*)^I/\mathcal{D}) \cong R^I/\mathcal{D}, \) proving that \( \mathcal{O} \models R. \)

Since \( G \) is interpretable in \( \mathcal{O}, \) the isomorphism between \( \text{SL}_n(\mathcal{O}), \) as the group of all matrices \( n \times n \) matrices over \( \mathcal{O} \) of determinant 1, and itself as the group boundedly generated by one-parameter subgroups is definable in \( \mathcal{O}. \) So the same fact holds in \( \bar{G}. \) Since the isomorphism \( \phi : G \rightarrow \bar{G} \) is definable in \( G, \) the same first-order fact is expressible in \( \bar{G}. \) Now, by the above paragraph \( H \) is boundedly generated by some one-parameter subgroups of \( H \) generated over the ring \( R. \) Since all formulas involved in the bi-interpretation of \( G \) and \( \mathcal{O} \) are uniform, the fact that \( H \) is the group of all \( n \times n \) matrices of determinant 1 over the ring \( R \) holds in \( H. \)

## 3 The case of \( T_n(\mathcal{O}) \)

**Lemma 3.1.** Assume \( \beta_1, \ldots, \beta_m \) are free generators of \( \mathcal{O} \) as a \( \mathbb{Z} \)-module. Then for each pair \( 1 \leq k < l \leq n, \) the subgroup \( T_{kl} \) is definable in

\[
(G, t_{kl}(\beta_1), \ldots, t_{kl}(\beta_m), d_k)
\]

where \( G = T_n(\mathcal{O}). \)

**Proof.** Firstly note

\[
C_G(t_{kl}) = \langle d_p(\alpha), d_k(\alpha)d_l(\alpha), T_{kl}, T_{kj}, T_{ij}: \alpha \in \mathcal{O}^* \land (p \neq k, l) \land i \neq l \land j \neq k \rangle
\]

An argument similar to the one in the proof of Lemma 2.1 shows that \( Z(C_G(t_{kl})) = T_{kl} \times Z(G). \) Next note that \( T_{kl}^2 = T_{kl}(2\mathcal{O}) = [d_k, T_{kl}] = [d_k, Z(C_G(T_{kl}))]. \) This shows that \( T_{kl}^2 \) is a definable subgroup. Hence is \( T_{kl} = \langle T_{kl}^2, t_{kl}(\beta_1), \ldots, t_{kl}(\beta_m) \rangle. \)

**Theorem 3.2.** Let \( G = T_n(\mathcal{O}), \) assume \( \mathcal{O} \) has a finite group of units \( \mathcal{O}^* \) with generators \( \alpha_1, \ldots, \alpha_l \) and let \( \beta_1, \ldots, \beta_m \) be free generators of \( \mathcal{O} \) as a \( \mathbb{Z} \)-module. Then the ring \( \mathcal{O} \) is bi-interpretable with \( (G, \bar{I}(\bar{\beta}), \bar{d}(\bar{\alpha})). \)

**Proof.** Recall that the group \( G = T_n(\mathcal{O}) \) is isomorphic to a semi-direct product \( D_n(\mathcal{O}) \rtimes \text{UT}_n(\mathcal{O}). \) There is an ordering on the \( T_{ij} \) where each element \( g \) of
UT\(_n(\mathcal{O})\) has a unique representation as a product of elements of the \(T_{ij}\) in that ordering, i.e. there exists a surjective function
\[
f: \{1, 2, \ldots, n(n-1)/2\} \rightarrow \{(i,j): 1 \leq i < j \leq n\}
\]
and unique elements \(t_{f(k)}(\gamma_{f(k)}(g)) \in T_{f(k)}\) such that
\[
g = \prod_{k=1}^{n(n-1)/2} t_{f(k)}(\gamma_{f(k)}(g))
\]
By assumption all the subgroups \(d_k(\mathcal{O}^\times)\) are definable with parameters since these are all finite and the subgroups \(T_{ij}\) are definable with parameters by Lemma 3.1. Now the result follows with a similar argument as in the proof of Theorem 2.4.

Corollary 3.3. Assume \(H\) is any finite extension of \(UT_n(\mathcal{O})\) in \(T_n(\mathcal{O})\) which includes all the \(d_i, i = 1, \ldots, n\). Then \(H\) is bi-interpretable with \(\mathcal{O}\) and consequently with \(\mathbb{Z}\).

Remark 3.4. We note that if \(\mathcal{O}^\times\) is infinite then by Corollary 3. of [12], \(T_n(\mathcal{O})\) is not QFA, hence it is not bi-interpretable with \(\mathbb{Z}\). However, a group \(H\) as in our Corollary 3.3 is QFA and prime. This also follows from the main theorem of [8].

3.1 Models of the complete theory of \(T_n(\mathcal{O})\) where \(\left|\mathcal{O}^\times\right|<\infty\)

Lemma 3.5. Assume \(G = UT_n(\mathcal{O}) \rtimes A\) where \(A\) is a finite subgroup of \(D_n\) containing all the \(d_i\). If \(H\) is any group such that \(H \equiv G\), then \(G \equiv UT_n(R) \rtimes A\) for some ring \(R \equiv \mathcal{O}\), where the action of \(A\) on \(UT_n(R)\) is the natural extension of the action of \(A\) on \(UT_n(\mathcal{O})\).

Proof. The commutator subgroup \(G'\) is absolutely and uniformly definable in \(G\) since it is of finite width. In general \(G'\) is a finite index subgroup of \(UT_n(\mathcal{O})\) (See proof Lemma 3.9) and it includes \(UT_n(2\mathcal{O})\). Now \(\sqrt{G'} \equiv \{x \in G: x^2 \in G'\}\) is uniformly definable in \(G\) and this subgroup includes both \(UT_n(\mathcal{O})\) and the \(d_i\). Hence, by Corollary 3.3 \(\sqrt{G'}\) is bi-interpretable with \(\mathbb{Z}\). Hence, \(G\) is bi-interpretable with \(\mathbb{Z}\). The rest of the proof is similar to that of Theorem 2.6.

As a corollary we get the following statement:

Theorem 3.6. If \(H\) is any group such that \(H \equiv T_n(\mathcal{O})\), where \(\left|\mathcal{O}^\times\right|<\infty\) then \(H \equiv T_n(R)\) for some \(R \equiv \mathcal{O}\).
3.2 Models of the complete elementary theory of $T_n(\mathcal{O})$ where $\mathcal{O}^\times$ is infinite

We recall a few well-known concepts and facts from the extension theory and its relationship with the second cohomology group in Section 3.2.1. Readers familiar with this material may proceed to Section 3.2.2.

3.2.1 Extensions and 2-cocycles

Assume that $A$ is an abelian group and $B$ is a group, both written multiplicatively. A function $f: B \times B \to A$ satisfying

- $f(xy, z)f(x, y) = f(x, yz)f(y, z)$, $\forall x, y, z \in B$,
- $f(1) = f(x, 1) = 1$, $\forall x \in B$,

is called a 2-cocycle. If $B$ is abelian a 2-cocycle $f: B \times B \to A$ is symmetric if it also satisfies the identity:

$$f(x, y) = f(y, x)$$

$\forall x, y \in B$.

By an extension of $A$ by $B$ we mean a short exact sequence of groups

$$1 \to A \xrightarrow{\mu} E \xrightarrow{\nu} B \to 1,$$

where $\mu$ is the inclusion map. The extension is called abelian if $E$ is abelian and it is called central if $A \leq Z(E)$. A 2-coboundary $g: B \times B \to A$ is a 2-cocycle satisfying:

$$\psi(xy) = g(x, y)\psi(x)\psi(y), \quad \forall x, y \in B,$$

for some function $\psi: B \to A$. One can make the set $Z^2(B, A)$ of all 2-cocycles and the set $B^2(B, A)$ of all 2-coboundaries into abelian groups in an obvious way. Clearly $B^2(B, A)$ is a subgroup of $Z^2(B, A)$. Let us set

$$H^2(B, A) = Z^2(B, A)/B^2(B, A).$$

Assume $f$ is a 2-cocycle. Define a group $E(f)$ by $E(f) = B \times A$ as sets with the multiplication

$$(b_1, a_1)(b_2, a_2) = (b_1b_2, a_1a_2f(b_1, b_2)) \quad \forall a_1, a_2 \in A, \forall b_1, b_2 \in B.$$

The above operation is a group operation and the resulting extension is central. It is a well known fact that there is a bijection between the equivalence classes of central extensions of $A$ by $B$ and elements of the group $H^2(B, A)$ given by assigning $f \cdot B^2(B, A)$ the equivalence class of $E(f)$. We write $f_1 \equiv f_2$ for $f_1, f_2 \in Z^2(B, A)$ if they are cohomologous, i.e., if $f_1 \cdot B^2(B, A) = f_2 \cdot B^2(B, A)$.
If $B$ is abelian $f \in Z^2(B,A)$ is symmetric if and only if it arises from an abelian extension of $A$ by $B$. As it can be easily imagined there is a one to one correspondence between the equivalent classes of abelian extensions and the quotient group

$$Ext(B, A) = S^2(B, A)/(S^2(B, A) \cap B^2(B, A)),$$

where $S^2(B, A)$ denotes the group of symmetric 2-cocycles. For further details we refer the reader to ([14], Chapter 11).

### 3.2.2 CoT 2-cocycles

Assume $A = T \times F$ is an abelian group where $T$ is torsion and $F$ is torsion-free and let $B$ be an abelian group. We know that $Ext(A, B) \cong Ext(T, B) \oplus Ext(F, B)$, so any symmetric 2-cocycle $f \in S^2(A, B)$ can be written as $f \equiv f_1 \cdot f_2$, where $f_1 \in S^2(A, B)$, $f_2 \in S^2(T, B)$. The symmetric 2-cocycle $f$ is said to be a coboundary on torsion or CoT if $f_2$ is a 2-coboundary.

### 3.2.3 Non-split tori and abelian deformations of $T_n(R)$

Consider $T_n(R)$ and the torus $D_n(R)$. The subgroup $D_n(R)$ is a direct product $(R^\times)^n$ of $n$ copies of the multiplicative group of units $R^\times$ of $R$. The center $Z(G)$ of $G$ consists of diagonal scalar matrices $Z(G) = \{ \alpha \cdot 1 : \alpha \in R^\times \} \cong R^\times$, where 1 is the identity matrix. It is standard knowledge that $Z(G)$ is a direct factor of $D_n(R)$, i.e. there is a subgroup $B \leq D_n$ such that $D_n = B \times Z(G)$. Now we define a new group just by deforming the multiplication on $D_n$. Let $E_n = E_n(R)$ be an arbitrary abelian extension of $Z(G) \cong R^\times$ by $D_n/Z(G) \cong (R^\times)^{n-1}$. As it is customary in extension theory we can assume $E_n = D_n = B \times Z(G)$ as sets, while the product on $E_n$ is defined as follows:

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 x_2, y_1 y_2 f(x_1, x_2)),$$

for a symmetric 2-cocycle $f \in S^2(B, Z(G))$.

**Remark 3.7.** Indeed any abelian extension $E_n$ of $R^\times$ by $(R^\times)^{n-1}$ is uniquely determined by some symmetric 2-cocycles $f_i \in S^2(R^\times, R^\times)$, $i = 1, \ldots, n - 1$ up to equivalence of extensions due to the fact that $Ext((R^\times)^{n-1}, R^\times) \cong \prod_{i=1}^{n-1} Ext(R^\times, R^\times)$. So if the $f_i$ are the defining 2-cocycles for $E_n$ above we also denote the group $E_n$ obtained above by $D_n(R, f_1, \ldots, f_{n-1})$ or $D_n(R, \bar{f})$.

We are now ready to define abelian deformations $T_n(R, \bar{f})$ of $T_n(R)$ via generators and relations.

For each $i = 1, \ldots, n - 1$ pick $f_i \in S^2(R^\times, R^\times)$. Then, an abelian deformation $T_n(R, f_1, \ldots, f_{n-1})$ of $T_n(R)$ is defined as follows.

$T_n(R, \bar{f})$ is the group generated by

$$\{d_i(\alpha), t_{kl}(\beta) : 1 \leq i \leq n, 1 \leq k < l \leq n, \alpha \in R^\times, \beta \in R\},$$

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with the defining relations:

1. \( t_{ij}(\alpha)t_{ij}(\beta) = t_{ij}(\alpha + \beta) \).

2. 
\[
[t_{ij}(\alpha), t_{kl}(\beta)] = \begin{cases} 
  t_{il}(\alpha\beta) & \text{if } j = k \\
  t_{kj}(-\alpha\beta) & \text{if } i = l \\
  1 & \text{if } i \neq l, j \neq k 
\end{cases}
\]

3. If \( 1 \leq i \leq n - 1 \), then \( d_i(\alpha)d_i(\beta) = d_i(\alpha\beta)\text{diag}(f_1(\alpha,\beta)) \), where \( \text{diag}(f_1(\alpha,\beta)) \) is defined by \( \text{diag}(f_1(\alpha,\beta)) = d_1(f_1(\alpha,\beta)) \cdots d_n(f_1(\alpha,\beta)) \).

4. \([d_i(\alpha), d_j(\beta)] = 1\)

5. 
\[
d_k(\alpha^{-1})t_{ij}(\beta)d_k(\alpha) = \begin{cases} 
  t_{ij}(\beta) & \text{if } k \neq i, k \neq j \\
  t_{ij}(\alpha^{-1}\beta) & \text{if } k = i \\
  t_{ij}(\alpha\beta) & \text{if } k = j 
\end{cases}
\]

**Lemma 3.8.** The set \( T_n(R, \bar{f}) \) is a group for any choice of \( f_i \in S^2(R^\times, R^\times) \).

**Proof.** The \( t_{ij}(\beta) \) generate a group \( G_u \cong \text{UT}_n(R) \) by relations (1.) and (2.). The \( d_i(\alpha) \) generate an abelian group \( E_n \) by (3.) and (4.). Note that both of the above are closed under group operations and \( G_u \cap E_n = 1 \). By (5.) \( G_u \) is stable under the action of \( E_n \) by conjugation which is described by (5.) itself, i.e. (5.) describes a homomorphism \( \psi_{n,R} : E_n \to \text{Aut}(G_u) \) so that \( T_n(R, \bar{f}) = E_n \ltimes_{\psi_{n,R}} G_u \), as an internal product, and \( \ker(\psi_{n,R}) = Z(G) = \{\text{diag}(\alpha) : \alpha \in R^\times, \beta \in R\} \).

**Lemma 3.9.** Assume \( R \) is a commutative associative ring with unit. Then the derived subgroup \( G' \) of \( G = T_n(R, \bar{f}) \) is the subgroup of \( G \) generated by

\[
X = \{ t_{i,i+1}((1-\alpha)\beta), t_{kl}(\beta) : 1 \leq i \leq n-1, 1 < k + 1 < l \leq n, \alpha \in R^\times, \beta \in R \}.
\]

**Proof.** Let \( N \) denote the subgroup generated by \( X \) and \( G_u \) the subgroup generated by all the \( t_{ij}(\beta), \beta \in R \). Each \( t_{kl}(\beta) \), with \( l-k \geq 2 \) is already a commutator by definition, and

\[
d_i(\alpha^{-1})t_{i,i+1}(-\beta)d_i(\alpha)t_{i,i+1}(\beta) = t_{i,i+1}((1-\alpha^{-1})\beta),
\]

for any \( \alpha \in R^\times \) and \( \beta \in R \), hence \( N \leq G' \). To prove the reverse inclusion firstly note that since \( G/G_u \) is abelian, \( G'/G_u \cong \text{UT}_n(R) \). Now, pick \( x, y \in G \). Then, \( x = x_1x_2 \) and \( y = y_1y_2 \), where \( x_1, y_1 \in D_n(R, \bar{f}) \), and \( x_2, y_2 \in G_u \). Now,

\[
[x, y] = [x_1x_2, y_1y_2] = [x_1, y_1]^z_1 [x_2, y_2]^z_2 [x_1, y_1]^z_3 [x_2, y_2]^z_4 = [x_1y_1]^z_2 [x_2, y_1]^z_1 [x_2, y_2]^z_4
\]
for some \( z_i \in G, i = 1, \ldots, 4 \). The commutator \([x_2, y_2] \in (G_u)'\), where \((G_u)'\) is characteristic in \( G_u \) so normal in \( G \). Therefore, \([x_2, y_2]^{4i}\) is a product of \( t_{ij}(\beta) \), \( i + 1 < j \). The commutators \([x_2, y_1] \) and \([x_1, y_2] \) are of the same type. Let us analyze one of them. Indeed, \( x_2 = d_1(\alpha_1) \cdots d_n(\alpha_n) \), and \( y_1 = t_{12}(\beta_{12}) \cdots t_{1n}(\beta_{1n}) \).

So, \([x_2, y_1]\) is a product of conjugates of commutators of type \([d_k(\alpha), t_{ij}(\beta)]\).

In case that \( j > i + 1 \) this is conjugate of a \( t_{ij}(\beta) \) which was dealt with above and is an element of \( N \). It remains to analyze the conjugates of \( t = t_{i, i + 1}(\alpha (\alpha - 1)(\beta)) \). Consider \( z = xy, x = d_1(\alpha_1) \cdots d_n(\alpha_n) \in E_n, y \in G_u \).

Then \( t^z = t_{i, i + 1}(\alpha (\alpha - 1)\alpha_1^{-1} \alpha_{i + 1}^{-1} \beta) \in N \) and \( N \) is normalized by \( y \). This completes the proof.

**Corollary 3.10.** If \( G = T_n(R, \bar{f}) \), then both \( G' \) and \( \sqrt{G} \) are absolutely definable in \( G \).

**Remark 3.11.** Assume for an abelian group \( A \) we have \( A \cong T \times B \) where \( T \) and \( B \) are some subgroups of \( A \). Consider a symmetric 2-cocycle \( f : A \to A \). By abuse of notation we consider \( f \) as \( f : T \cdot B \to T \times B \). Then \( f \) is cohomologous to \((g_1, g_2, h_1, h_2) \) where \( g_1 \in S^2(T, T) \), \( g_2 \in S^2(T, B) \), \( h_1 \in S^2(B, T) \) and finally \( h_2 \in S^2(B, B) \). We will use this notation in the following.

### 3.2.4 Characterization Theorem

**Theorem 3.12.** Assume \( H \) is a group. If \( H \equiv T_n(\mathcal{O}) \) as groups, then \( H \equiv T_n(\mathcal{O}, \bar{f}) \) for some \( \mathcal{O} \equiv \mathcal{O} \) and \( CoT \) 2-cocycles \( f \).

**Proof.** The subgroup \( \sqrt{\mathcal{O}} \) (See proof of Lemma 3.5) is definable in \( H \) by the same formulas which define \( \sqrt{\mathcal{O}}^G \equiv UT_n(\mathcal{O}) \ltimes A \) in \( G \), where \( A \) is a finite subgroup of \( D_n \) including all the \( d_i \). Therefore, by Lemma 3.5, \( \sqrt{\mathcal{O}}^G \equiv UT_n(\mathcal{O}) \ltimes A \) for some \( \mathcal{O} \equiv \mathcal{O} \).

For each \( k = 1, \ldots, n \) the subgroup \( \Delta_k(G) = d_k(\mathcal{O}^\times) \cdot Z(G) \) is definable in \( G \) as the subgroup of \( D_n \) consisting of all \( x \):

\[
t^{\pm}_{ij} = \begin{cases} 
t_{ij} & \text{if } i \neq k, j \neq k \\
t_{ij}(\alpha), \alpha \in \mathcal{O}^\times & \text{if } i = k \text{ or } j = k 
\end{cases}
\]

Note that the above is expressible by \( L_{\mathcal{O} \times} \)-formulas. Therefore, for each \( k \) there exists an interpretable isomorphism \( \Delta_k/Z(G) \to \mathcal{O}^\times \). One can also express in \( L_{\mathcal{O} \times} \), via the interpretable isomorphisms mentioned above, that \( Z(G) = \{ \prod_{k=1}^n d_k(\alpha) : \alpha \in \mathcal{O}^\times \} \).

The facts that \( D_n \cap UT_n(\mathcal{O}) = 1 \) and that \( UT_n(\mathcal{O}) \) is normal in \( T_n(\mathcal{O}) \) are also expressible using \( L_{\mathcal{O} \times} \)-formulas. Now moving to \( H, \) the same formulas define a subgroup \( E_n = E_n(\bar{e}) \) of \( H \), so that \( E_n(\bar{e})/Z(H) \equiv (R^\times)^{(n-1)} \) and \( Z(H) \equiv R^\times \), \( H_n \cap E_n = 1 \) and imply that the action of \( E_n \) on \( H_n \leq H \) is an extension of the action of \( D_n \) on \( UT_n(\mathcal{O}) \). This proves that \( H \equiv UT_n(\mathcal{O}) \ltimes E_n \), where \( E_n \equiv D_n(R, \bar{f}) \).
The torsion subgroup $T(\Delta_i(G))$ of $\Delta_i(G)$ is finite. Let $N$ be its exponent. For all $n$, the sentences $\forall x \in \Delta_i(G) (x^n = 1 \rightarrow x^N = 1)$ hold in $G$, hence in $H$. So, the formula $x^N = 1$ defines $T(\Delta_i(G))$ in $G$ as well as $T(\Delta_i(H))$ in $H$. Hence, $T(\Delta_i(H)) \cong T(\Delta_i(O^\times)) \cong T(O^\times \times T(O^\times))$. In addition, the following holds in $G$, and consequently in $H$:

$$\forall x \in \Delta_i(G), \exists y \in T(\Delta_i(G)), \exists z \in Z(G) (x^N \in Z(G) \rightarrow (x = yz)).$$

This ensures that in $H \equiv G$ each $f_i$ defining the $\Delta_i(H)$ as an extension of $Z(H) \cong R^\times$ by $\Delta_i(H)/Z(H) \cong R^\times$ is CoT.

3.2.5 Sufficiency of the characterization

In this section we shall prove that the necessary condition proven in Theorem 3.12 is also sufficient.

We will need to state a few well-known definitions and results.

Let $B$ be an abelian group and $A$ a subgroup of $B$. Then $A$ is called a pure subgroup of $B$ if $\forall n \in \mathbb{N}, nA = nB \cap A$.

Lemma 3.13. Let $A \leq B$ be abelian groups such that the quotient group $B/A$ is torsion-free. Then $A$ is a pure subgroup of $B$.

**Proof.** One direction is trivial. For the other direction assume that $g \in nB \cap A$. Then there is $h \in B$ such that $g = nh$. To get a contradiction assume that $h \notin A$. Then $g = nh \notin A$ since $B/A$ is torsion free. A contradiction! So $h \in A$, therefore $g = nh \in nA$.

An abelian group $A$ is called pure-injective if $A$ is a direct summand in any abelian group $B$ that contains $A$ as a pure subgroup.

The following theorem expresses a connection between pure-injective groups and uncountably saturated abelian groups.

Theorem 3.14 ([4], Theorem 1.11). Let $\kappa$ be any uncountable cardinal. Then any $\kappa$-saturated abelian group is pure-injective.

**Remark 3.15.** Assume $A$ and $B$ are abelian group and $f \in S^2(B, A)$. Let $\mathcal{D}$ be an ultrafilter on a set $I$. Let $A^\ast$ and $B^\ast$ denote the ultrapowers of $A$ and $B$, respectively, over $(I, \mathcal{D})$. Then $f$ induces a natural 2-cocycle $f^\ast \in S^2(B^\ast, A^\ast)$ representing an abelian extension of $A^\ast$ by $B^\ast$ (See Lemma 7.1 of [9] for details).

Lemma 3.16. Assume $f \in S^2(O^\times, O^\times)$ is CoT and $(I, \mathcal{D})$ is an ultrafilter so that ultraproduct $(O^\times)^\ast$ of $O^\times$ over $\mathcal{D}$ is $\aleph_1$-saturated. Then the 2-cocycle $f^\ast \in S^2((O^\times)^\ast, (O^\times)^\ast)$ induced by $f$ is a 2-coboundary.
This concludes the proof utilizing Keisler-Shelah’s Theorem. Therefore, 

Lemma 3.16. The fact that \( f \) cocycle induced by \( \aleph \) is either trivial or ultrapower \( C \). 

Proof. Let \( (I, D) \) be an \( \aleph \)-incomplete ultrapower. As usual, by \( C^* \) we mean the ultrapower \( C^D \) of a structure \( C \). Then \( B((R^\times)^*) = B((R^\times)^*) = B^*(R^\times) \) is either trivial or \( \aleph \)-saturated. If \( f_i^* \in S^2((R^\times)^*), (R^\times)^*) \) denotes the 2-cocycle induced by \( f_i \) then for each \( i = 1, \ldots, n - 1 \), \( f_i^* \) is a 2-coboundary by Lemma 3.14. The fact that \( T_n^*(R, \bar{f}) \equiv T_n^*(R, \bar{f}^*) \) and each \( f_i^* \) requires only some routine checking. Therefore, 

\[
T_n^*(R, \bar{f}) \equiv T_n^*(R, \bar{f}^*) \equiv T_n^*(R^\times) \equiv T_n^*(R).
\]

This concludes the proof utilizing Keisler-Shelah’s Theorem. 

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4 The case of \( \text{GL}_n(\mathcal{O}) \)

4.1 Bi-interpretability with \( \mathcal{O} \)

Theorem 4.1. Assume \( \mathcal{O} \) has a finite group of units \( \mathcal{O}^\times = \{ \alpha_i : i = 1, \ldots, m \} \) and let \( d_1(\alpha) = (d_1(\alpha_1), \ldots, d_1(\alpha_m)) \). Then \( (\text{GL}_n(\mathcal{O}), \bar{t}, d_1(\bar{\alpha})) \) and \( \mathcal{O} \) are bi-interpretable. 

Proof. We note \( \text{GL}_n(\mathcal{O}) \) is boundedly generated by all the \( T_{ij} \) and the finite subgroup \( d_1(\mathcal{O}^\times) \triangleq \langle d_1(\alpha_i) : i = 1, \ldots, m \rangle \). All these subgroups are definable with the given parameters. The rest of the proof is similar to that of Theorem 2.3. 

Corollary 4.2. If \( \mathcal{O} \) has a finite group of units, then \( \text{GL}_n(\mathcal{O}) \) is QFA and prime.
Remark 4.3. If $O^X$ is infinite by Corollary 3 of [12] and Theorem 1.4, $GL_n(O)$ is not bi-interpretable with $\mathbb{Z}$. In the following we provide an alternative proof of this fact utilizing Theorem 1.5.

Theorem 4.4. If $O$ has an infinite group of units, then $GL_n(O)$ is not bi-interpretable with $\mathbb{Z}$.

Proof. Indeed we prove that $GL_n(O)$ and $O$ are not bi-interpretable for any finite set of parameters picked in $GL_n(O)$. So to derive a contradiction assume that $(GL_n(O), c)$ and $O$ are bi-interpretable for a tuple $\vec{c}$ of constants in $GL_n(O)$. Since $O$ and $\mathbb{Z}$ are bi-interpretable there are constant $\vec{e}$ in $GL_n(O)$ where $(GL_n(O), c, \vec{e})$ and $\mathbb{Z}$ are bi-interpretable. Now take a countable non-standard model $S$ of $\mathbb{Z}$ with countably many automorphisms and a free $\mathbb{Z}$ basis $\beta$ of $O$. Consider $R = O \otimes_{\mathbb{Z}} S$ and note that since $\mathbb{Z}$ and $(O, \beta)$ are bi-interpretable, $S$ and $R$ are also bi-interpretable, and by Theorem 1.5 there are only countably many automorphisms of $R$ fixing $\beta$. Since $G'$ is uniformly definable in $G$, as the subgroup of products of commutators of width $w(n, O)$, and $G$ is bi-interpretable with $O$, all subgroups $T_{ij}$ are definable in $(GL_n(O), c)$. In particular $O$ is interpreted on $T_{ij}$ with the help of the constants $\vec{e}$. Moreover, for $R$ defined above, $R$ and $GL_n(R)$ are bi-interpretable with the same constants as in the bi-interpretation of $GL_n(O)$ and $O$.

We note that for an element $\alpha$ of infinite order in the group of units $O^\times$, which is finitely generated by Dirichlet’s Units Theorem as an abelian group, the cyclic subgroup $\alpha^\mathbb{Z} \overset{\text{def}}{=} \{a^b : b \in \mathbb{Z}\}$ is definable in $O$ with some parameters. This is because for a $\mathbb{Z}$-basis of $O$ the integer exponentiation of $\alpha$ is computed by recursive functions in each coordinate, therefore it is arithmetic in each coordinate, since $O$ with these parameters is bi-interpretable with $\mathbb{Z}$. This also endows $\alpha^\mathbb{Z}$ with a ring structure isomorphic to $\mathbb{Z}$.

Now, there is a unique element $y$ of $T_{ij}$ such that $d_i(\alpha)t_{ij}d_i(\alpha)^{-1} = y$, and we denote it for obvious reasons by $t_{ij}(\alpha)$. Therefore, the set $t_{ij}(\alpha^\mathbb{Z})$ along with its ring structure are definable in $G$. Hence, $H_i = d_i(\alpha^\mathbb{Z}) \cdot Z(G)$ is definable in $G$ as follows: $h \in H_i$ if and only if $h^{-1}t_{ij}h \in t_{ij}(\alpha^\mathbb{Z})$, for all $i \neq j$, and $h^{-1}t_{ij}h = 1$ if $k \neq i$ and $j \neq i$. So $H_i$ is definable in $G$ via bi-interpretability of $G$ and $\mathbb{Z}$. Moreover, $H_i/Z(G)$ inherits an arithmetic structure, which is interpretable in $G$.

Now, working in $G^* = GL_n(R)$, the same formulas as above define subgroup $H^*_i$ of $G^*$, where $H^*/Z(G^*) \cong (\mathbb{S}, +, \cdot)$. Since $S$ is a countable non-standard model of $\mathbb{Z}$ its additive groups splits as $A \oplus D$, where $D \cong \mathbb{Q}^\omega$, $\mathbb{Q}$, the additive group of rational fractions. The group $\mathbb{Q}^\omega$ is a divisible abelian group and splits from the rest of the abelian subgroup $d_i(R^\times)$. By a similar argument $Z(G^*) = \{\prod_{i=1}^n d_i(\alpha) : \alpha \in R^\times\}$ contains subgroups $A'$ and $D' \cong \mathbb{Q}^\omega$, where $Z(G^*) \cong A' \oplus D'$. Since there are uncountably many distinct homomorphisms from $\mathbb{Q}^\omega$ onto $\mathbb{Q}^\omega$, there exist uncountably many distinct non-trivial homomorphisms $\phi_i : d_i(R^\times) \to Z(G^*)$ fixing the standard copy of $d_1(O^\times)$ in $d_1(R^\times)$. For each
such $i$ define a map:

$$\psi_i : SL_n(R) \rtimes d_1(R^\times) \to SL_n(R) \rtimes d_1(R^\times), \quad (x, y) \mapsto (x, y\psi_i(y)).$$

$\psi_i$'s are pairwise distinct and each is a non-trivial automorphism of $GL_n(R) \cong SL_n(R) \rtimes d_1(R^\times)$, fixing $GL_n(O)$ elementwise, inducing identity on all $T_{ij}$ and hence on $(R, \beta)$. Recall that by our hypothesis and Theorem 1.4, there has to exist a one-one correspondence between automorphisms of $(GL_n(R), \vec{e}, \vec{e})$ and automorphisms of $(R, \beta)$, where the former is uncountable, but the latter is countable. Contradiction!

\[\square\]

### 4.2 Models of the complete theory of $GL_n(O)$

Here we prove that all models of the first-order theory of $GL_n(O)$ are of the type $GL_n$. We invite the reader to compare the result with the case of $T_n(O)$.

**Theorem 4.5.** If $H$ is any group such that $H \equiv GL_n(O)$ then $H \equiv GL_n(R)$ for some ring $R \equiv O$.

**Proof.** Let $G = GL_n(O)$. Since $G' = SL_n(O)$ and $SL_n(O)$ is boundedly generated, $G'$ is uniformly definable in $G$. By Theorem 2.6 if $H \equiv G'$, then $H' \equiv SL_n(R)$ for a ring $R \equiv O$. Subgroups $\Delta_i = d_i(O^\times) \cdot Z(G)$ are definable in $G$ similar to the proof of Theorem 3.12. Indeed the same formulas define $d_i(R^\times) \cdot Z(H)$ in $H$ where $Z(H) \cong R^\times$. The action of $\Delta_i(R)$ on $SL_n(R)$ is also an extension of the action of $\Delta_i(O)$ on $SL_n(O)$. Now to prove the theorem we need to prove that torus $E_n = \Delta_1 \cdots \Delta_n$ is actually a split torus, i.e. $E_n \cong d_1(R^\times) \times \cdots \times d_n(R^\times)$, or equivalently that $f_i \equiv 1$ for all $i$. By Remark 3.7 we note that CoT 2-cocycles $f_i$ defining $\Delta_i(R)$ satisfy

$$f_n \equiv f_1^{-1} \cdots f_{n-1}^{-1} \quad (1)$$

On the other hand the following relations hold in $G$:

$$d_i(\beta)d_j(\beta^{-1}) = t_{ij}(\beta)t_{ji}(-\beta^{-1})t_{ij}(\beta)t_{ji}(-1)t_{ji}(1)t_{ij}(-1), \quad i \neq j$$

Hence, there exists an $L_{\text{groups}}$-sentence which holds in $H$ and implies in $H$ that the set $\Delta_{ij}$ of elements $d_i(\beta)d_j(\beta^{-1})$ as $\beta$ ranges over $R^\times$ is indeed a subgroup of $E_n$ which intersects $Z(H)$ trivially. A couple of routine calculations with symmetric 2-cocycles reveals that

$$f_i(\alpha, \beta)f_j(\alpha^{-1}, \beta^{-1}) \equiv f_i^{-1}(\alpha^{-1}\beta^{-1}, \alpha\beta)f_i(\alpha^{-1}, \alpha)f_i(\beta^{-1}, \beta) \equiv 1$$

This together with the fact that $\Delta_{ij}$ splits over $Z(H)$ implies that

$$f_i \equiv f_j, \quad i \neq j \quad (2)$$

Now, Equations (1) and (2) imply that for any $i$

$$f_i^n \equiv 1$$

implying that $f_i \equiv 1$, since $f_i$'s are CoT. \[\square\]
References

[1] O. V. Belegradek, The model theory of unitriangular groups, Ann. Pure App. Logic, 68 (1994) 225-261.
[2] N. Avni, A. Lubotzky, C. Meiri, First order rigidity of non-uniform higher rank arithmetic groups. Invent. Math. 217(1) (2019), 219-240.
[3] D. Carter, G. Keller, Bounded Elementary Generation of $\text{SL}_n(O)$, Am. J. Math. 105(3) (1983) 673-687.
[4] P. C. Eklof, and R. F. Fischer, The elementary theory of abelian groups, Ann. Math. Logic, 4(2) (1972) 115-171.
[5] W. Hodges, *Model Theory*, Encyclopedia of mathematics and its applications: V. 42, Cambridge University Press, 1993.
[6] M. Jarden, A. Lubotzky, Elementary equivalence of profinite groups, Bulletin of the London Mathematical Society 40(5) (2008), 887-896.
[7] A. Khelif, Bi-interprétabilité et structures QFA: étude des groupes résolubles et des anneaux commutatifs, C. R. Acad. Sci. Paris, Ser. I 345 (2007) 59-61.
[8] C. Lasserre, Polycyclic-by-finite groups and first-order sentences, J. Algebra, 396 (2013), 18-38.
[9] A. G. Myasnikov and M. Sohrabi, Groups elementarily equivalent to a free 2-nilpotent group of finite rank, Alg. and Logic, 48(2) (2009), 115-139.
[10] A. G. Myasnikov, M. Sohrabi, Groups elementarily equivalent to a free nilpotent group of finite rank, Ann. Pure Appl. Logic 162(11) (2011), 916-933.
[11] A. Nies, Describing groups, Bull. Sym. Logic, 13 (3) (2007) 306-339.
[12] F. Oger, G. Sabbagh, Quasi finitely axiomatizable nilpotent groups, J. Group Theory, 9 (2006) 95-106.
[13] F. Pop, Distinguishing every finitely generated field of characteristic $\neq 2$ by a single field axiom, Arxiv 1809.00490, (2019).
[14] D. J. S. Robinson, *A Course in the Theory of Groups*, 2nd edn. Springer-Verlag New York, 1996.
[15] J. Robinson, The undecidability of algebraic rings and fields, Proc. Amer. Math. Soc. 10 (1959) 950-957.