Conjugation Matters
Bioctonionic Veronese Vectors and Cayley-Rosenfeld Planes

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Abstract
Motivated by the recent interest in Lie algebraic and geometric structures arising from tensor products of
division algebras and their relevance to high energy theoretical physics, we analyze generalized bioctonionic
projective and hyperbolic planes. After giving a Veronese representation of the complexification of the Cayley
plane $\mathbb{O}P^2$, we present a novel, explicit construction of the bioctonionic Cayley-Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O}) P^2$, again by exploiting Veronese coordinates. We discuss the isometry groups of all generalized bioctonionic
planes, recovering all complex and real forms of the exceptional groups $F_4$ and $E_6$, and characterizing such
planes as symmetric and Hermitian symmetric spaces. We conclude by discussing some possible physical
applications.
1 Introduction

Recently, attention has been focused by theoretical physicists on Lie structures arising from tensor products of division algebras, and on their geometrical characterizations. It is well known that Tits-Freudenthal magic square has numerous applications in super Yang-Mills and supergravity theories \cite{GST, BM, CCM}. At the same time, extensions of the Standard Model based on non-division algebras resulting from the Cayley-Dickson and their tensor product was proposed in a recent article by Masi \cite{Ma}. On a different, but convergent, path, Todorov, Dubois-Violette \cite{TDV} and Krasnov \cite{Kra} characterized the Standard Model gauge group $G_{SM}$ as a subgroup of the automorphism group of the exceptional Jordan algebra $J_3(\mathbb{O})$, while Boyle \cite{Bo, Co} pointed to its complexification $J_3(\mathbb{C})$. It is well known that the group of determinant preserving automorphisms (also named reduced structure group) of the complexification of the exceptional Jordan algebra is related to the collineations of the complexification of the Cayley plane $\mathbb{O}P^2_2$ (namely, the complexification of the octonionic projective plane) and to $E_6(\mathbb{C})$, while the group of trace preserving automorphisms of the same Jordan algebra is related with the isometries of the Cayley plane and to $F_4(\mathbb{C})$. While $F_4$ does not have complex representations, the exceptional Lie Group $E_6$ is historically well known as a candidate for Grand Unification Theories \cite{GRS}.

Complex and real forms of $E_6$ are also notoriously related to symmetries of the bioctonionic Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O})P^2$. In a series of seminal papers resumed and completed in \cite{Ros97, Ros98}, Rosenfeld linked all the Lie groups arising from Tits-Freudenthal magic square with the groups of collineations and elliptic motions of generalised projective planes defined over tensor product of division algebras and their split versions. Consequently, he identified what are now known as Rosenfeld planes as symmetric spaces and Hermitian symmetric spaces over compact and non compact form of real Lie groups. However, in his articles Rosenfeld left some ambiguity in the definition of the octonionic part of the construction: due to the lack of associativity of octonions, the usual identification of the plane through the quotient of a module could not indeed be pursued; moreover, when the resulting tensor algebra is not a composition algebra, neither a direct completion of a generalised affine plane could be worked out. A well known identification, due to Jordan, von Neumann and Wigner \cite{JvNW}, relates points of the projective plane $\mathbb{K}P^2$ with rank-1 idempotent elements of the Jordan algebra $J_3(\mathbb{K})$ of Hermitian three by three matrices over the division algebra $\mathbb{K}$. This provides a generalisation, but when defining a Jordan algebra is not possible, the construction breaks down (see e.g. the discussion at the end of Sec. 5 of \cite{ABDN}, and Refs. therein).

This paper is devoted to highlight the crucial role played by conjugation (and related norm) in determining the algebraic-geometric structure of projective plane (in a generalized sense) over tensor products of division algebras. In the present paper we will focus on the bioctonionic algebra $\mathbb{C} \otimes \mathbb{O}$, pointing out how its composition nature strictly depends on the conjugation being considered. For the first time, we will employ Veronese coordinates over the bioctonions in order to describe suitable real forms of the bioctonionic Rosenfeld plane; this will pave the way to the treatment of more complicated generalized projective spaces, which we plan to deal with in future works. We will present an alternative, simple construction, based on the definition of Veronese vectors over the bioctonions, that allows an explicit description of two generalised projective planes over the algebra of bioctonions $\mathbb{C} \otimes \mathbb{O}$ that are of the most interest: the complexification of the octonionic projective plane or Cayley plane $\mathbb{O}P^2_2$, and the bioctonionic Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O})P^2$. In Sec. 2 we introduce the algebra of bioctonions with its linear structure, conjugations and norms over both fields of real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$. In Sec. 3 we explicitly construct $\mathbb{O}P^2_2$ and $(\mathbb{C} \otimes \mathbb{O})P^2$ making use of Veronese coordinates, while in the former case the construction has already appeared in the literature (see e.g. \cite{SBG11}), in the latter case the construction by means of Veronese vectors is new, and it exhibits some non-trivial features. Indeed, on one hand the complexification of the Cayley plane $\mathbb{O}P^2_2$ is derived from a composition algebra, it respects the Moufang identities and can be considered as a completion of a generalised affine plane over the bioctonionic algebra. On the other hand, since bioctonions are not a composition algebra with respect to a suitably defined real norm, the bioctonionic Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O})P^2$ violates the basic axioms of projective geometry, and it cannot be considered as an extension and completion of a would-be affine Rosenfeld plane. Then, in Sec. 4 we exploit the relation between Veronese vectors and simple, rank-3 Jordan algebras, and we thus identify $\mathbb{O}P^2_2$ with the space of rank-1 idempotent elements the complexification of the exceptional Jordan Algebra $J_3(\mathbb{C})$. In Sec. 5 we proceed to analyze the group of motions of the generalised projective planes, recovering $F_4$ as the isometry group of complexification of the Cayley plane $\mathbb{O}P^2_2$, and $E_6$ as the isometry group of the bioctonionic
Figure 1: Multiplication rule of octonions $\mathbb{O}$ (left) and of split-octonions $\mathbb{O}_s$ (right) as real vector space $\mathbb{R}^8$ in the basis $\{i_0 = 1, i_1, ..., i_7\}$. In the case of the octonions $i_0^2 = 1$ and $i_k^2 = -1$ for $k = 1...7$, while in the case of split-octonions $i_0^2 = 1$, for $k = 1, 2, 3$ and $i_k^2 = -1$ for $k \neq 1, 2, 3$. By exchanging the indices $2 \leftrightarrow 3$ and $4 \leftrightarrow 5$, the octonionic multiplication corresponds to the Cayley-Graves’ one [Cay, Gra], recently discussed in [GKLY].

Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O})P^2$. Sec. 6 then deals with a systematic definition of bioctonionic planes as symmetric and Hermitian symmetric spaces, retrieving all the real forms of $F_4$ and $E_6$. Finally, in Sec. 7 we discuss some possible applications of $F_4$ and $E_6$ and the above related geometrical structures to high energy theoretical physics. An outlook and comments on further future developments are given in Sec. 8 which concludes the paper.

2 The Algebra of Bioctonions

Let the octonions $\mathbb{O}$ be the only non-associative normed division algebra with $\mathbb{O}_s$ as its split version, and let $\mathbb{C}$ be the algebra of complex numbers and $\mathbb{C}_s$ its split algebra. We then define the algebra of bioctonions as the complexification of the algebra of Octonions, i.e. as the tensor product $\mathbb{C} \otimes \mathbb{O}$ or, equivalently, as $\mathbb{C} \otimes \mathbb{O}_s$. Since $\mathbb{O}$ is an alternative algebra and $\mathbb{C}$ is a commutative algebra, then $\mathbb{C} \otimes \mathbb{O}$ is an alternative algebra. In the following sections we will work with the $\mathbb{R}^{16}$ and the $\mathbb{C}^8$ decomposition of bioctonions. In the $\mathbb{R}^{16}$ decomposition, an element of the bioctonionic algebra is given by

$$b := \sum_{\alpha=0}^{7} (x^\alpha + iy^\alpha) i_\alpha,$$

where $x^\alpha, y^\alpha \in \mathbb{R}$, the imaginary unit commutes with the octonionic units, i.e. $ii_\alpha = i_\alpha i$, the multiplication rules of $i_\alpha$ are given by the Fano plane in Fig.1 left or right if bioctonions are considered as $\mathbb{C} \otimes \mathbb{O}$ or as $\mathbb{C} \otimes \mathbb{O}_s$ respectively. Rewriting (2.1) we obtain the $\mathbb{C}^8$ decomposition

$$b := \sum_{\alpha=0}^{7} z^\alpha i_\alpha,$$

where $z^\alpha \in \mathbb{C}$. The two decompositions (2.1) and (2.2) highlight two different vector space structures available on the algebra of bioctonions: the first is over the field of the real numbers $\mathbb{R}$ and is of dimension 16, while the second is over the complex field $\mathbb{C}$ and has complex dimension 8. It is worth noting that the algebra of bioctonions is not a division algebra e.g. $(ii_\alpha + 1) (ii_\alpha - 1) = 0$.

Footnote: The treatment of the Veronese vectors over algebras containing zero-divisors has been given e.g. in [Cha], in which a suitably generalized Veronese map is proposed (see Th. 5.2 therein).
2.1 Complex Norm

Considering the bioctonions \( \mathbb{C} \otimes \mathbb{O} \) as a complex vector space, it is natural to define a complex norm. Let \( b = z \otimes w \) a bioctonion with \( z \in \mathbb{C} \) and \( w \in \mathbb{O} \), then its \textit{octonionic conjugate} \( b^* \) is the element \( b^* = z \otimes w^* \), where \( w^* \in \mathbb{O} \) is the conjugate of \( \mathbb{O} \). Applying the complex decomposition \( z = z^0 + \sum_{\alpha=1}^7 z^\alpha i_\alpha \), then the octonionic conjugate of \( b \) has the form

\[
b^* := z^0 - \sum_{\alpha=1}^7 z^\alpha i_\alpha.
\]  

We then define an \textit{octonionic inner product} \( \langle \cdot, \cdot \rangle_\mathbb{O} \) over \( \mathbb{C} \otimes \mathbb{O} \) as

\[
\langle b_1, b_2 \rangle_\mathbb{O} := z_1^0 z_2^0 + \ldots + z_1^7 z_2^7 \in \mathbb{C},
\]

where \( z_1^i, z_2^i \in \mathbb{C} \) are the complex coefficients of \( b_1 \) and \( b_2 \) respectively. The octonionic inner product induces a \textit{complex norm} \( N(\cdot) \) in \( \mathbb{C} \), given as

\[
N(b) := \langle b, b \rangle_\mathbb{O} \in \mathbb{C},
\]

i.e. \( N(b) = (z^0)^2 + \ldots + (z^7)^2 = bb^* = b^*b \). The complex norm \( N \) is a non degenerate quadratic form over the complex vector space \( \mathbb{C} \otimes \mathbb{O} \). Moreover, \( b \) is a zero divisor if and only if \( N(b) = 0 \). In respect to the complex norm \( N \) we also have \( N(\lambda b) = \lambda^2 N(b) \) for every \( \lambda \in \mathbb{C} \), and

\[
N(b_1 b_2) = N(b_1) N(b_2),
\]

and therefore \( \mathbb{C} \otimes \mathbb{O} \) is a \textit{composition algebra} with respect to the complex norm \( N \).

\textbf{Remark 1.} As the octonionic inner product induces an inner product and a complex norm over \( \mathbb{C} \otimes \mathbb{O} \), also its split version gives rise to a split-octonionic inner product with a norm. Even though \( \mathbb{C} \otimes \mathbb{O} \) and \( \mathbb{C} \otimes \mathbb{O}_s \) give rise to the same bioctonionic algebra, we will write \( \mathbb{C} \otimes \mathbb{O}_s \) when we will intend the bioctonionic algebra equipped with the split octonionic inner product and its norm.

2.2 Real Norm

We also define a real norm given by the \textit{bioctonionic conjugation}, i.e. \( b^* = \overline{z} \otimes w^* \), where \( w^* \) is the octonionic conjugate of \( w \) in \( \mathbb{O} \) and \( \overline{z} \) is the complex conjugate of \( z \) in \( \mathbb{C} \). Consequently the inner product \( \langle \cdot, \cdot \rangle_{\mathbb{C} \otimes \mathbb{O}} \) is defined as

\[
\langle b_1, b_2 \rangle_{\mathbb{C} \otimes \mathbb{O}} := \overline{z_1^0 z_2^0} + \ldots + \overline{z_1^7 z_2^7},
\]

and induce a \textit{real norm} \( \|\cdot\| \) that is the sum of the norms of the complex coefficients of \( b \), i.e.

\[
\|b\|^2 := \langle b, b \rangle_{\mathbb{C} \otimes \mathbb{O}} = |z^1|^2 + \ldots + |z^7|^2 \in \mathbb{R}.
\]

Since \( \|ab\|^2 \neq \|a\|^2 \|b\|^2 \) then \( \mathbb{C} \otimes \mathbb{O} \) is not a \textit{composition algebra} in respect to the real norm.

2.3 Automorphisms

Since the automorphisms of \( \mathbb{C} \) are isomorphic to \( \mathbb{Z}_2 \) \cite{Bae}, and the automorphisms of \( \mathbb{O} \) are isomorphic to the exceptional Lie group \( G_2 \), i.e. \( \text{Aut(\mathbb{O})} \cong G_2 \), the group of automorphisms of the algebra of bioctonions is isomorphic to

\[
\text{Aut(\mathbb{C} \otimes \mathbb{O})} = \mathbb{Z}_2 \times G_2,
\]

and consequently the Lie algebra of derivations is isomorphic to that of octonions, i.e. \( \mathfrak{der}(\mathbb{C} \otimes \mathbb{O}) \cong \mathfrak{g}_2 \).

\footnote{We discard the so-called ‘wild’ automorphisms \cite{Ya}.}
3 Veronese Vectors over Bioctonions

In this section we define explicitly two bioctonionic planes making use of Veronese coordinates. In the case of the projective and hyperbolic planes on octonions, i.e. $\mathbb{O}P^2$ and $\mathbb{O}H^2$ respectively, the construction is well known [SRGT]. The projective plane over the octonions with real coefficients $\mathbb{R} \otimes \mathbb{O} \cong \mathbb{R}_8$ has been studied extensively in [Tit53, Fre54]. A rigorous definition of the octonionic projective plane, and the proof that its automorphism group is a simple group of type $E_6$ in all characteristics, can be found in [Cha] (see Th. 5.1 therein).

In the present paper, we study the complexification of the Cayley Plane $\mathbb{C}P^2$ and the bioctonionic Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O}) P^2$. In the former case, we make use of the complex vector space structure of $\mathbb{C} \otimes \mathbb{O}$, and therefore of the octonionic conjugation $b^\ast$, the octonionic inner product $\langle \cdot , \cdot \rangle_\mathbb{O}$ and complex norm $N(\cdot)$. In the latter case, we rely on the real vector space structure of $\mathbb{C} \otimes \mathbb{O}$, and therefore on the bioctonionic conjugation $b^\ast$, the bioctonionic inner product $\langle \cdot , \cdot \rangle_{\mathbb{C} \otimes \mathbb{O}}$ and real norm $\|\cdot\|$; we will provide a Veronese representation of the bioctonionic Rosenfeld plane $(\mathbb{C} \otimes \mathbb{O}) P^2$ even if the bioctonions are not a composition algebra with respect to the real norm $\|\cdot\|$.

3.1 The Complexification of the Cayley Plane $\mathbb{O}P^2_\mathbb{C}$

Let $V \cong (\mathbb{C} \otimes \mathbb{O})^3 \times \mathbb{C}^3$ be a complex vector space, with elements $\omega$ of the form

$$ (b_\nu ; \lambda_\nu) = (b_1, b_2, b_3; \lambda_1, \lambda_2, \lambda_3), \tag{3.1} $$

where $b_\nu \in \mathbb{C} \otimes \mathbb{O}$, $\lambda_\nu \in \mathbb{C}$ for $\nu = 1, 2, 3$. A vector $\omega \in V$ is called Veronese iff

$$ \lambda_1 b_1^* = b_2 b_3, \quad \lambda_2 b_2^* = b_3 b_1, \quad \lambda_3 b_3^* = b_1 b_2, \tag{3.2} $$

$$ N(b_1) = \lambda_2 \lambda_3, \quad N(b_2) = \lambda_3 \lambda_1, \quad N(b_3) = \lambda_1 \lambda_2. \tag{3.3} $$

Let the set $H \subset V$ be the set of Veronese vectors inside $V$. Since $\mathbb{C}$ is commutative and $\lambda^* = \lambda$, since $\lambda b = b \lambda$ and $N(\lambda b) = \lambda^2 N(b)$ when $\lambda \in \mathbb{C}$, then if $\omega$ is a Veronese vector, all complex multiples $\mathbb{C} \omega$ are again Veronese vectors, i.e. if $\omega \in H$ then $\mu \omega \in H$ when $\mu \in \mathbb{C}$. We then define the complexified octonionic plane or complexification of the Cayley plane $\mathbb{O}P^2_\mathbb{C}$ as the set of 1-dimensional complex subspaces $\mathbb{C} \omega$ that we will call points of the plane, i.e.

$$ \mathbb{O}P^2_\mathbb{C} := \{ \mathbb{C} \omega : \omega \in H \setminus \{0\} \}. \tag{3.4} $$

3.1.1 Complexified Cayley Lines

Lines in $\mathbb{O}P^2_\mathbb{C}$ are orthogonal subspaces of a point of the plane. Therefore, let $\mathbb{C} \omega$ be a point in $\mathbb{O}P^2_\mathbb{C}$, we define the line $\ell$ as the orthogonal subspace

$$ \ell := \omega^\perp = \{ v \in V : \beta(v, \omega) = 0 \}, \tag{3.5} $$

where the complex bilinear form $\beta$ is given by

$$ \beta(v, \omega) := \sum_{\nu=1}^3 \left( \langle b_\nu^*, b_\mu \rangle_\mathbb{O} + \lambda_\nu^1 \lambda_\mu^2 \right), \tag{3.6} $$

with $v, \omega \in V$, of coordinates $(b_\nu^*; \lambda_\nu^1)_\nu, (b_\mu^*; \lambda_\mu^2)_\nu$ respectively.

3.1.2 Elliptic and Hyperbolic Polarity on $\mathbb{O}P^2_\mathbb{C}$

Since every point $\mathbb{C} \omega$ of the plane defines an orthogonal line $\omega^\perp \subset \mathbb{O}P^2_\mathbb{C}$ and, as converse, every line defines a point, we call standard elliptic polarity $\pi^+$ the involutive map that corresponds points to lines and lines to points through orthogonality, i.e.

$$ \pi^+ (\omega) := \omega^\perp, \pi^+ (\omega^\perp) := \omega, \tag{3.7} $$

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using the complex bilinear form $\beta(\cdot, \cdot)$ so that
\[
\omega \longrightarrow \{\beta(\cdot, \omega) = 0\}. \tag{3.8}
\]
Explicitly, $\beta(v, \omega) = 0$ when
\[
b_1 b_2^* + b_2 b_3^* + b_3 b_1^* + \lambda_1^1 \lambda_2^2 + \lambda_2^1 \lambda_3^2 + \lambda_3^1 \lambda_1^2 = 0, \tag{3.9}
\]
where, as before, we intended, $(b_1; \lambda_1^1)_\nu$ and $(b_2^*; \lambda_2^2)_\nu$ as the coordinates of $v, \omega \in V$. We also define an hyperbolic polarity $\pi^-$ as the involutive map between points and lines which still has
\[
\pi^- (\omega) := \omega^\perp, \pi^- (\omega^\perp) := \omega, \tag{3.10}
\]
but through the use of the bilinear form $\beta_-$ which has a change of sign in the last coordinate, i.e. $\beta_- (v, \omega) = 0$ when
\[
b_1 b_2^* + b_2 b_3^* - b_3 b_1^* + \lambda_1^1 \lambda_2^2 + \lambda_2^1 \lambda_3^2 - \lambda_3^1 \lambda_1^2 = 0. \tag{3.11}
\]
The projective plane equipped with the bilinear form $\beta_-$ and the hyperbolic polarity $\pi^-$ it will be called the complexified hyperbolic Cayley plane $\mathbb{O}H^2_C$.

### 3.1.3 Complexified Octonionic Affine Plane

In analogy to the classic case, the complexification of the Cayley plane can also be seen as the completion and topological compactification of a bioctonionic affine plane $(\mathbb{C} \otimes \mathbb{O}) A^2$, but, since $\mathbb{C} \otimes \mathbb{O}$ is not a division algebra, the strict set of axioms and results of affine geometry are not valid on this plane. Indeed, the map from $(\mathbb{C} \otimes \mathbb{O})^2$ to $V$ defined as
\[
(x, y) \mapsto \mathbb{C}(x, y^*, y x^*; N(y), N(x), 1), \tag{3.12}
\]
sends elements $(x, y) \in (\mathbb{C} \otimes \mathbb{O})^2$ into Veronese vectors, and therefore to $\mathbb{O}P^2_C$ establishing a correspondence between points of the complexification of the Cayley plane and elements of an affine plane $(\mathbb{C} \otimes \mathbb{O}) A^2$ with
bioctonionic coordinates \((x, y)\). To show that \(\mathbb{O}P^2\) is a completion of the affine plane we add two sets of point of coordinates \((x)\) and \((\infty)\) that will extend the map to cover the whole \(\mathbb{O}P^2\) as follows:

\[
\begin{align*}
(x) & \mapsto \mathbb{C}(0, 0, x; N(x), 1, 0), \\
(\infty) & \mapsto (0, 0, 0; 1, 0, 0),
\end{align*}
\]

where \(x \in \mathbb{C} \otimes \mathbb{O}\). Finally, in order to give a complete picture, a line in the affine plane \((\mathbb{C} \otimes \mathbb{O}) A^2\) will be given by

\[
[s, t] := \{(x, sx + t) : x, t \in \mathbb{C} \otimes \mathbb{O}\},
\]

where \(s \in \mathbb{C} \otimes \mathbb{O}\) is the slope of the line. Vertical lines will be indicated as \([c] := \{c\} \times (\mathbb{C} \otimes \mathbb{O})\). As for the previous case, lines of the affine plane \((\mathbb{C} \otimes \mathbb{O}) A^2\) have a correspondence with lines of the projective plane \(\mathbb{O}P^2\) through the map

\[
[s, t] \mapsto \mathbb{C}(s^*t, -t^*, -s; 1, N(s), N(t))^\perp, \quad (3.16)
\]

\[
[c] \mapsto \mathbb{C}(-c, 0, 0; 0, 1, N(c))^\perp, \quad (3.17)
\]

where \(s, t, c \in \mathbb{C} \otimes \mathbb{O}\). The correspondence is bijective if we add a line \([\infty]\) that will correspond to

\[
[\infty] \mapsto \mathbb{C}(0, 0, 0; 0, 0, 1)^\perp. \quad (3.18)
\]

**Remark 2.** The complexification of the octonionic affine plane does not satisfy the axioms of usual affine geometry. It is critical the existence of adjacent points, i.e. points that are separated by a singular affine vector \((v_1, v_2)\) such that \((av_1, av_2) = 0\), with \(a, v_1, v_2 \in \mathbb{C} \otimes \mathbb{O}\). Between two adjacent points passes more than one line, that are called adjacent themselves. Two lines that are not adjacent but that can be transformed into adjacent through a translation are called diverging lines. So any two given lines in this plane might be incident, parallel, coincident, adjacent, or divergent (cfr. [Ros97]).

**Remark 3.** The counting of (complex) dimensions goes as follows. By construction, \(\dim_{\mathbb{C}} V = 27\). Since \(\mathbb{C} \otimes \mathbb{O}\) is composition with respect to the complex norm \(N(-)\), only 10 relations are independent out of all 27 ones defining Veronese vectors; thus \(\dim_{\mathbb{C}} H = 17\). Finally, from \(\dim_{\mathbb{C}}(\mathbb{O}P^2) = \dim_{\mathbb{C}} H - 1 = 16\), as expected.

### 3.2 The Bioctonionic Rosenfeld Plane \((\mathbb{C} \otimes \mathbb{O}) P^2\)

We now replicate the same construction, considering as a starting point the real vector space \(V \cong (\mathbb{C} \otimes \mathbb{O})^3 \times \mathbb{R}^3\) and the real norm over the bioctonionic algebra. In this case elements of \(V\) are of the form

\[
(b_\nu; \lambda_\nu)_\nu = (b_1, b_2, b_3; \lambda_1, \lambda_2, \lambda_3),
\]

where \(b_\nu \in \mathbb{C} \otimes \mathbb{O}\), \(\lambda_\nu \in \mathbb{R}\) and \(\nu = 1, 2, 3\). A vector \(\omega \in V\) is called Veronese iff

\[
\begin{align*}
\lambda_1 b_1^* & = b_2 b_3, \\
\lambda_2 b_2^* & = b_3 b_1, \\
\lambda_3 b_3^* & = b_1 b_2, \\
\|b_1\|^2 & = \lambda_2 \lambda_3, \\
\|b_2\|^2 & = \lambda_3 \lambda_1, \\
\|b_3\|^2 & = \lambda_1 \lambda_2.
\end{align*}
\]

Let \(H \subset V\) be the set of Veronese vectors. Since \(\mathbb{R}\) is a commutative field, \(\mu^* = \mu\) and \(\|\mu b\|^2 = \mu^2 \|b\|^2\) when \(\mu \in \mathbb{R}\), given a Veronese vector \(\omega \in H\), all real multiples \(\mathbb{R}\omega\) are again Veronese vectors, i.e. \(\mathbb{R}\omega \subset H\). We define the bioctonionic Rosenfeld plane \((\mathbb{C} \otimes \mathbb{O}) P^2\) as the set of 1-dimensional subspaces \(\mathbb{R}\omega\) that we will call points of the plane, i.e.

\[
(\mathbb{C} \otimes \mathbb{O}) P^2 := \{\mathbb{R}\omega : \omega \in H \setminus \{0\}\}.
\]
3.2.1 Rosenfeld Lines

As in the case of the complexification of the Cayley plane, we define the lines of the bioctonionic Rosenfeld plane \((\mathbb{C} \otimes \mathbb{O})P^2\) as orthogonal subspaces of a point through the extension of the bioctonionic inner product. Therefore, let \(R\omega\) be a point in \((\mathbb{C} \otimes \mathbb{O})P^2\), then the line \(\ell\) is the orthogonal subspace

\[ \ell := \omega^\perp = \{ v \in V : \beta_{\mathbb{C}\otimes \mathbb{O}}(v, \omega) = 0 \} , \]

where the bilinear form is defined as

\[ \beta_{\mathbb{C}\otimes \mathbb{O}}(v, \omega) := \sum_{\nu=1}^{3} \left( \langle b^1_{\nu}; b^2_{\nu} \rangle_{\mathbb{C}\otimes \mathbb{O}} + \lambda^1_{\nu} \lambda^2_{\nu} \right) , \]

with \(v, \omega \in V\) of coordinates \((b^1_{\nu}; \lambda^1_{\nu}), (b^2_{\nu}; \lambda^2_{\nu})\) respectively.

3.2.2 Elliptic and Hyperbolic Polarity on \((\mathbb{C} \otimes \mathbb{O})P^2\)

Since every point \(R\omega\) defines an orthogonal line \(\omega^\perp \subset (\mathbb{C} \otimes \mathbb{O})P^2\) and every line defines a point, we call standard elliptic polarity \(\pi^+\) the involutive map that correspond points to lines and lines to points through orthogonality, i.e.

\[ \pi^+(\omega) := \omega^\perp, \pi^+(\omega^\perp) := \omega , \]

making use of the bilinear form \(\beta_{\mathbb{C}\otimes \mathbb{O}}(\cdot, \cdot)\). Explicitly, \(\beta_{\mathbb{C}\otimes \mathbb{O}}(v, \omega) = 0\) when

\[ b^1_1 b^{2*}_1 + b^2_2 b^{2*}_2 + b^3_3 b^{2*}_3 + \lambda^1_1 \lambda^1_2 + \lambda^2_2 \lambda^2_2 + \lambda^3_3 \lambda^3_3 = 0 , \]

where, as before, we intended, \((b^1_{\nu}; \lambda^1_{\nu}), (b^2_{\nu}; \lambda^2_{\nu})\) as the coordinates of \(v, \omega \in V\).

As in the previous case, the elliptic polarity is not the only possible one, indeed we define the \textit{hyperbolic polarity} \(\pi^-\) as the involutive map between points and lines which still has

\[ \pi^-(\omega) := \omega^\perp, \pi^-(\omega^\perp) := \omega , \]

but that uses the bilinear form \(\beta^-_{\mathbb{C}\otimes \mathbb{O}}\) which has a change of sign in the last coordinate, i.e. \(\beta^-_{\mathbb{C}\otimes \mathbb{O}}(v, \omega) = 0\) when

\[ b^1_1 b^{2*}_1 + b^2_2 b^{2*}_2 - b^3_3 b^{2*}_3 + \lambda^1_1 \lambda^1_2 + \lambda^2_2 \lambda^2_2 - \lambda^3_3 \lambda^3_3 = 0 . \]

The projective plane equipped with the bilinear form \(\beta^-_{\mathbb{C}\otimes \mathbb{O}}\) instead of \(\beta_{\mathbb{C}\otimes \mathbb{O}}\) it will be called the bioctonionic Rosenfeld hyperbolic plane \((\mathbb{C} \otimes \mathbb{O})H^2\).

Remark 4. Since \(\mathbb{C} \otimes \mathbb{O}\) is not a composition algebra with respect of the real norm \(||\cdot||\), a map as in \((3.12)\) would not be well defined. Therefore, we cannot consider the Bioctonionic Rosenfeld plane as an extension and completion of an affine Rosenfeld plane.

Remark 5. The counting of (real) dimensions goes as follows. Since \(\mathbb{C} \otimes \mathbb{O}\) is not a composition algebra with respect to the real norm \(||\cdot||\), one must consider \((3.21)\) and, say, the first of \((3.20)\), as independent relations out of the relations \((3.20)-(3.21)\) defining Veronese vectors. This corresponds to \(8 \times 2 + 1 + 1 = 19\) real conditions out of the \(8 \times 2 \times 3 + 1 + 1 + 1 = 51\) real relations \((3.20)-(3.21)\). Thus, the real dimension of \((\mathbb{C} \otimes \mathbb{O})P^2\) is given by \(\dim_{\mathbb{R}} ((\mathbb{C} \otimes \mathbb{O})P^2) = 51 - 19 = 32\), as expected.

4 Veronese Vectors and Jordan Algebras

A well known identification relates rank-1 idempotent elements of the Jordan algebra \(\mathfrak{J}_3(\mathbb{O})\) with points of the octonionic projective plane \(\mathbb{O}P^2\). While this identification still stands for the complexification of the Cayley plane \(\mathbb{O}P^2\), when it is applied to the Rosenfeld plane \((\mathbb{C} \otimes \mathbb{O})P^2\) one obtains that \(\mathfrak{J}_3(\mathbb{C} \otimes \mathbb{O})\) is a simple Jordan algebra, but not a formally real one (cfr. e.g. \[\text{Bae}\]).

As we discussed above, Veronese vectors, defined by conditions in \((3.2)\), are an alternative and useful way to characterize rank-1 idempotent elements of a Jordan algebra \(\mathfrak{J}_3(\mathbb{K})\), with \(\mathbb{K}\) being any tensor product of division algebras.
\section{\(\mathbb{O}P^2_3\)}

In order to show such relation within \(\mathbb{O}P^2_3\), let \(\omega\) be an element of \(V \cong (\mathbb{C} \otimes \mathbb{O})^3 \times \mathbb{C}^3\) with coordinates \((b_1, b_2, b_3; \lambda_1, \lambda_2, \lambda_3)\) and define \(A_\omega\) as the three by three bioctonionic matrix given by

\[
V \ni \omega \rightarrow A_\omega := \begin{pmatrix}
\lambda_1 & b_3 & b_2^* \\
\lambda_2^* & \lambda_3 & b_1 \\
b_2 & \lambda_1^* & \lambda_2
\end{pmatrix} \in \mathbb{C} \otimes \mathbb{J}_3(\mathbb{O}).
\]  

(4.1)

Note that since the scalar field \(\mathbb{C}\) commutes with the coefficient \(b\), then all \(\mathbb{C}\)-multiples of the vector \(\omega\) are sent in multiple of the matrix \(A_\omega\). Therefore the map is well defined and induces a bijective map between points \(\mathbb{C}\omega \in \mathbb{O}P^2_3\) and subspaces of the form \(\mathbb{C}A_\omega\). It is here worth recalling that the cubic norm \(N\) of a non-zero element \(A_\omega \in \mathbb{C} \otimes \mathbb{J}_3(\mathbb{O})\) is defined in terms of the generalization of the determinant for three by three matrices with not necessarily associative elements\(^{[\text{Yok}]}\):

\[
N(A_\omega) \equiv \det (A_\omega) := \lambda_1 \lambda_2 \lambda_3 - \lambda_1 N(b_1) - \lambda_2 N(b_2) - \lambda_3 N(b_3) + 2\Re((b_1b_2)b_3),
\]  

(4.2)

where \(\Re((b_1b_2)b_3) := \frac{1}{2} ((b_1b_2)b_3 + \overline{b_3}(b_2b_1))\), implying that \(\det(aA_\omega) = a^3 \det(A_\omega), \forall a \in \mathbb{C}\). It should be remarked that the determinant is actually well defined, as one can realize by recalling the Hamilton Cayley identity (see e.g. \[\text{Yok}\]), i.e.

\[
A_\omega \circ A_\omega^2 - \tr (A_\omega) A_\omega^2 + \frac{1}{2} \left(\tr(A_\omega)^2 - \tr(A_\omega^2)\right) A_\omega = \det (A_\omega) I,
\]  

(4.3)

where \(I\) is the three by three identity matrix, and \(\circ\) is the Jordan product \(A \circ B := \frac{1}{2}(AB + BA)\).

By further specifying that \(\omega\) is a Veronese vector in \(V\), namely that \(\omega \in H \subset V\), and thus by plugging the Veronese conditions (3.2)-(3.3) into (4.2), one obtains that the norm of the corresponding element \(A_\omega\) vanishes :

\[
\omega \in H \iff N(A_\omega) = 0.
\]  

(4.4)

Moreover, one can consider the image \(A_\omega^2\) of a non-zero element \(A_\omega\), with \(\omega \in H\), under the so-called adjoint (\(\ast\)-)map of \(\mathbb{C} \otimes \mathbb{J}_3(\mathbb{O})\), which is given by (again, cf. e.g. example 5 of \[\text{Krut}\])

\[
A_\omega^2 := \begin{pmatrix}
\lambda_2 \lambda_3 - N(b_1) & b_2 b_1 - \lambda_3 b_3 & b_3 b_1 - \lambda_2 b_2 \\
b_1 b_2 - \lambda_3 b_3 & \lambda_1 \lambda_3 - N(b_2) & b_3 b_1 - \lambda_1 b_2 \\
b_1 b_3 - \lambda_2 b_2 & b_2 b_3 - \lambda_1 b_1 & \lambda_1 \lambda_2 - N(b_3)
\end{pmatrix},
\]  

(4.5)

and which, when recalling (3.2)-(3.3), can be realized to vanish (again!), thus implying that

\[
\omega \in H \iff A_\omega^2 = 0.
\]  

(4.6)

Thus, by the Definition 11 of \[\text{Krut}\] (namely, from the invariant definition of the rank of an element of \(\mathbb{J}_3^\mathbb{O}\) \[\text{Jac61}\]), one obtains that the Veronese conditions (3.2)-(3.3) are an equivalent characterization of the rank-1 elements of the complexification of the exceptional Jordan algebra \(\mathbb{C} \otimes \mathbb{J}_3(\mathbb{O}) \equiv \mathbb{J}_3^\mathbb{C}(\mathbb{O})\) (see e.g. \[\text{Jac6S}\] for an extensive analysis).

\textbf{Remark 6.} The definition (3.4) characterizes the points of \(\mathbb{O}P^2_3\) as complex ‘Veronese rays’, thus obtaining a 16\(\mathbb{C}\)-dimensional subspace of the unique orbit of rank-1 elements of \(\mathbb{J}_3^\mathbb{O}(\mathbb{O})\). Any well defined representative of such a 16-dimensional subspace has a fixed trace.

\footnote{\text{See e.g. the example 5 of \[\text{Krut}\], which actually is a simplified version of the reduced cubic factor example in Sec. I.3.9 of \[\text{McGr}\].}}
4.2 $(\mathbb{C} \otimes \mathbb{O}) \mathbb{P}^2$

The same argument, with a different ending point, applies to the bioctonionic Rosenfeld plane. Given a Veronese vector $\omega$ in $V \cong (\mathbb{C} \otimes \mathbb{O})^3 \times \mathbb{R}^3$ with coordinates $(b_1, b_2, b_3; \lambda_1, \lambda_2, \lambda_3)$ we define $A_\omega$ as the three by three bioctonionic Hermitian matrix as

$$\omega \rightarrow A_\omega := \begin{pmatrix} \lambda_1 & b_3^* & b_2^* \\ b_3 & \lambda_2 & b_1 \\ b_2 & b_1 & \lambda_3 \end{pmatrix}. \quad (4.7)$$

Since the scalar field $\mathbb{R}$ commutes with the coefficient $b_i$ then all $\mathbb{R}$-multiples of the vector $\omega$ are sent in multiple of the matrix $A_\omega$. Therefore, the map is well defined and induces a injective map between points $\mathbb{R}\omega \in (\mathbb{C} \otimes \mathbb{O}) \mathbb{P}^2$ and subspaces $\mathbb{R}A_\omega \subset H_3 (\mathbb{C} \otimes \mathbb{O})$, the algebra of $3 \times 3$ matrices with $(\mathbb{C} \otimes \mathbb{O})$-valued entries and Hermitian with respect to the bi-octonionic conjugation; this time, the bioctonionic conjugation still allows $A_\omega$ to be endowed with the structure of a simple Jordan algebra, but not of a formally real one.

5 Real Forms of $F_4$ and $E_6$

Symmetries of the generalised projective planes over the bioctonionic algebra lead naturally to all the complex and real forms of the exceptional group $E_6$ and $F_4$. To do so we will look to generalised collineations, i.e. automorphisms of generalised planes that sends lines into lines, and elliptic and hyperbolic motions, i.e. collineations over the projective (hyperbolic) plane that preserve the elliptic (hyperbolic) polarity. Sometimes, due to correspondence between idempotent Jordan matrices and points in the projective plane, the collineation group of the octonionic plane $\text{Coll} (\mathbb{O} \mathbb{P}^2)$ is called $SL (3, \mathbb{O})$, while the elliptic polarity preserving group $\text{Iso} (\mathbb{O} \mathbb{P}^2)$ is identified with $SU (3, \mathbb{O})$ [DW].

To recover the collineations group and the polarity preserving group of the octonionic projective and hyperbolic space, one might proceed in a geometric [SBG11], group algebraic [Jac68] and Lie algebraic way [Rosen9]. We will represent the last one following Rosenberg focusing on the Lie algebra of the collineation group $\text{Coll} (\mathbb{O} \mathbb{P}^2)$ that is given by the direct sum the Lie Algebra given by the group of automorphisms of the field, in this case $\text{Aut} (\mathbb{O}) = G_2$, and the algebra $\mathfrak{a}_3 (\mathbb{O})$ of three by three matrices on $\mathbb{O}$ and null trace, i.e. $\text{tr} (A) = 0$. We therefore have

$$\text{coll} (\mathbb{O} \mathbb{P}^2) = \mathfrak{g}_2 \oplus \mathfrak{a}_3 (\mathbb{O}). \quad (5.1)$$

A simple count on the dimension of the generators of the $\mathfrak{a}_3 (\mathbb{O})$ algebra, imposing the null trace condition, leads 8 entries of dimension 8 and therefore $\dim_{\mathbb{R}} \mathfrak{a}_3 = 64$, that brings to

$$\dim_{\mathbb{R}} (\text{coll} (\mathbb{O} \mathbb{P}^2)) \cong 78 = 64 + 14, \quad (5.2)$$

which leads to the group $\text{Coll} (\mathbb{O} \mathbb{P}^2)$ be a $E_6$ type Lie group as expected.

The same argument is applied for the polarity preserving group $\text{Iso} (\mathbb{O} \mathbb{P}^2)$, i.e. collineations that preserve also the elliptic polarity $\pi^+$ or equivalently the form $\beta$. This argument leads to the Lie algebra $\text{iso} (\mathbb{O} \mathbb{P}^2)$ that is given

$$\text{iso} (\mathbb{O} \mathbb{P}^2) = \mathfrak{g}_2 \oplus \mathfrak{sa}_3 (\mathbb{O}), \quad (5.3)$$

where we intended $\mathfrak{sa}_3 (\mathbb{O})$ the anti-Hermitian matrices, i.e. $a_{ij} = -a^*_{ji}$, of null trace. Elements of this algebra are of the form

$$A = \begin{pmatrix} a_1^1 & a_2 & -(a_3^1)^* \\ -(a_1^2)^* & a_2^2 & a_3^2 \\ a_3^1 & -(a_3^2)^* & a_3^3 \end{pmatrix}, \quad (5.4)$$

with $a_3^3 = -(a_1^3 + a_2^3)$ and $\text{Re} (a_1^1) = \text{Re} (a_2^2) = 0$. The dimension count on the generators of the algebra leads to 3 entries of dimension 8, 2 of dimension 7 and therefore $\dim_{\mathbb{R}} \mathfrak{sa}_3 (\mathbb{O}) = 38$ and therefore

$$\dim_{\mathbb{R}} \text{iso} (\mathbb{O} \mathbb{P}^2) \cong 52 = 38 + 14, \quad (5.5)$$
Table 1: The isometry group of the octonionic and split-octonionic projective and hyperbolic planes give rise to complex and real forms of $F_4$.

Table 2: All real forms of $E_6$ arise as isometries of generalised bioctonionic projective and hyperbolic planes.

which points to $\text{Iso} (\mathbb{O} P^2)$ as an $F_4$-type Lie group as expected. With more efforts, following Yokota [Yok], we can recover all isometry groups giving rise to complex and real forms of $F_4$ (Table 1).

The same argument can be applied to the elliptic motion group of the bioctonionic Rosenfeld plane $\text{Iso} ((\mathbb{C} \otimes \mathbb{O}) P^2)$. Since $\mathbb{C}$ and $\mathbb{O}$ are both composition algebras, with $\text{der}(\mathbb{C}) \cong 0$ and $\text{der}(\mathbb{O}) \cong g_2$ respectively, then the Lie algebra of the group of elliptic motion $\text{iso} ((\mathbb{C} \otimes \mathbb{O}) P^2)$ is given by the direct sum

$$\text{iso} ((\mathbb{C} \otimes \mathbb{O}) P^2) \cong \mathfrak{so}_3 (\mathbb{C} \otimes \mathbb{O}) \oplus g_2,$$

(5.6)

where $\mathfrak{so}_3 (\mathbb{C} \otimes \mathbb{O})$ are the anti-hermitian traceless three by three matrices in the bioctonionic algebra, i.e. $a_{ij} = -a_{ji}^*$ and $a_{00} + a_{11} + a_{22} = 0$. Proceeding with the counting on the generators of the algebra we obtain

$$\dim_{\mathbb{R}} (\text{iso} ((\mathbb{C} \otimes \mathbb{O}) P^2)) = 16 \times 3 + 8 \times 2 + 14 = 78,$$

(5.7)

that gives the well known link between $\text{iso} ((\mathbb{C} \otimes \mathbb{O}) P^2)$ and the exceptional Lie group $E_6$. In the next section we will define all generalised bioctonionic projective and hyperbolic planes from their isometry group given as real forms of $E_6$ (see Tab. 2).

### 6 Bioctonionic Planes as Symmetric Spaces

All real forms of rank-3 Magic Squares have been classified and analyzed e.g. in [CCM] (see also Refs. therein). Moreover, in [ABDN] the $D = 3$ layer of the ‘magic pyramid’ of supergravities, containing various isometry Lie algebras of some Rosenfeld projective planes, is identified with the $4 \times 4$ Lorentzian rank-3 Magic Square $\mathfrak{M}_{2,1} (A, B)$, where $A$ and $B$ are the four normed division Hurwitz’s algebras $\mathfrak{H}$. It is here worth remarking that in [BM] a $6 \times 6$ extension of the Magic Square was also discussed, by introducing null extensions of quaternions and complex numbers, respectively given by sextonions and tritonions (see also [BD]).

In the octonionic case, we start from the complexification of the Cayley plane

$$\mathbb{O} P^2 (\mathbb{C}) \simeq \frac{F_4(\mathbb{C})}{S_{\text{pin}_3}(\mathbb{C})},$$

(6.1)

and define four different real forms of the plane: one totally compact of type $(0,16)$ and character $\chi = -16$ identified as $\mathbb{O} P^2$ and that is known as the classical Cayley plane or as the octonionic projective plane; one totally non-compact of type $(16,0)$ and character $\chi = 16$ identified as $\mathbb{O} H^2$ and known as the hyperbolic octonionic.
Figure 3: Satake diagrams of real forms of $F_4$, $E_6$, their character $\chi$ and corresponding projective plane of which they are the isometry group.

plane; and two of type $(8,8)$ and character $\chi = 0$ named $\mathbb{O} \tilde{H}^2$ and $\mathbb{O}_s \tilde{H}^2$. In all cases the type identifies the signature, namely the cardinality of non-compact and compact generators, i.e. $(\#_{nc}, \#_c)$, and the character $\chi$ is given by the difference between the two, i.e. $\chi = \#_{nc} - \#_c$. The four plane are then defined as

$$\mathbb{O} P^2 \simeq F_4(-52)/\text{Spin}^9,$$

$$\mathbb{O} H^2 \simeq F_4(-20)/\text{Spin}^9,$$

$$\mathbb{O} \tilde{H}^2 \simeq F_4(-20)/\text{Spin}_{8,1},$$

$$\mathbb{O}_s \tilde{H}^2 \simeq \mathbb{O}_s P^2 \simeq \mathbb{O}_s H^2 \simeq \frac{F_4(4)}{\text{Spin}_{5,4}}.$$

For the bioctonionic case, things are a little more involved and starting from the complex form of the bioctonionic Rosenfeld plane\footnote{The two semispinors $16_C$ of $\text{Spin}(10)_C$ in the tangent space of $\text{(6.6)}$ are an example of Jordan pair which is not made by a pair of Jordan algebras (see e.g. [McCr]).} i.e.\footnote{\text{(6.6)} has a Kähler structure pertaining to the $\mathbb{C}$ factor in the isotropy/holonomy group.}

$$\mathbb{C} \otimes \mathbb{O} \otimes \mathbb{O} \otimes \mathbb{C} \simeq \frac{E_6(\mathbb{C})}{\text{Spin}_{10}(\mathbb{C}) \otimes \mathbb{C}},$$

we have eight real different forms: one totally compact of type $(0,32)$ and character $\chi = -32$ identified as $(\mathbb{C} \otimes \mathbb{O}) P^2$ and that we define as the \textit{bioctonionic Rosenfeld projective plane}; one totally non-compact of type $(32,0)$ and character $\chi = 32$ identified as $(\mathbb{C} \otimes \mathbb{O}) H^2$ and that we define as the \textit{bioctonionic Rosenfeld hyperbolic plane}; four plane of type $(16,16)$ and character $\chi = 0$ and that are $(\mathbb{C} \otimes \mathbb{O}) \tilde{H}^2$, $\mathbb{C} \otimes \mathbb{O}_s \tilde{H}^2$, $(\mathbb{C}_s \otimes \mathbb{O}) \tilde{H}^2$ and
and (12 systems, such a manifold is related to a pair of octonionic vectors (see [GST2] and Refs. therein).

black hole attractors in \(D\) Riemannian space (\(C\) SO Spaces with \(C\) SO).

Finally, there are other two pseudo-Riemannian real forms of the bioctonionic plane (6.10), of type (20, 12) and (12, 20), both Kähler, respectively with character \(\chi = 8\) and \(\chi = -8\), namely:

\[
\frac{E_{6(2)}}{SO_{10} \otimes U_1},
\]

\[
\frac{E_{6(-14)}}{SO^{*}_{10} \otimes U_1},
\]

this fact can be traced back to the absence of the Lie algebra \(so^{*}_{4n+2}\) in the entries of the real forms of the Magic Square (cfr. [CCM], and Refs. therein), and we leave this intriguing issue for further future work. Here, we only notice that \(6(14)\) appears as the enlarged scalar manifold of \(N = 5, D = 3 + 1\) “pure” supergravity timelike reduced to \(N = 10, D = 3 + 0\) “pure” supergravity (after complete dualization of 1-forms to 0-forms); cf. [BGM].

7 Musings on the Physics of \(E_{6(-78)}\) and \(E_{6(2)}\)

The so-called “exceptional sequence” is given by the Lie algebras \(e_n\) for \(n = 3, \ldots, 8\), which respectively correspond to \(\mathfrak{sl}_3 \oplus \mathfrak{sl}_2\), \(\mathfrak{sl}_5\), \(\mathfrak{so}_{10}\), \(\mathfrak{e}_6\), \(\mathfrak{e}_7\), and \(\mathfrak{e}_8\). The application of exceptional Lie algebras in physics was pioneered by Gürsey. SU5 Grand Unified Theories (GUT) unifies the bosons into a single representation and Spin\(_{10}\) GUT unifies one generation of the fermions, which are contained within \(E_6\) GUT [GRS]. Bars and Günyaydın explored \(E_6\) GUT for three generations of the Standard Model [BG]. While it is commonly thought that \(E_6\) is the only exceptional GUT algebra with complex representations [Wil], Barr investigated the role of \(E_8 \rightarrow E_6 \otimes SU_3\), showing how the \(SU_3\) flavor symmetry leads to three generations with mirror fermions and \(E_6\) GUT [Barr].

More recently, Dubois-Violette and Torodov explored the state space of three generations of fermions via \(\tilde{\mathfrak{g}}(\mathfrak{g})\) in relation to \(F_4\) [TDV]. Boyle elaborated on the role of \(E_6\) via states from \(\tilde{\mathfrak{g}}(\mathfrak{g})\) independent of \(E_6\) GUT [BF] [Bo]. Krasnov has also discussed the role of \(O_7 \otimes \mathfrak{o}\) and \(\text{Spin}_{11,3}\), but did not obtain three generations [Krb]. Two of the authors have previously discussed the role of \(E_{8(-24)}\) with \(\text{Spin}_{12,4}\) and a spinor from \((O_4 \otimes \mathfrak{O}) P^2\) for three generations of matter [CMR].

Wilczek et al. articulate how \(\text{Spin}_3\) flavor symmetry is preferred over \(SU_3\) for anomaly cancellation without mirror fermions [RVW]. \(E_{8(-24)}\) contains \(\text{Spin}_{4,4} \supset \text{Spin}_{4,1} \otimes \text{Spin}(3)\), implying extra time dimensions relate to mass eigenstates, as energy/mass are the time components of energy-momentum in phase space [Ko CMR].
Wilson found a similar interpretation with Spin$_{3,3}$ and geometric algebra [Wil]. Moreover, since Spin$_{6,2} \cong SU_{2,2}(\mathbb{H})_\mathbb{R}$, one can reasonably guess that Spin$_{4,4} \cong SU_{2,2}(\mathbb{H})_\mathbb{R}$. Thus, it holds that

\begin{equation}
\begin{align*}
\text{Spin}_{3,3} & \cong SL_2(\mathbb{C})_\mathbb{R} \\
\text{Spin}_{4,4} & \cong SU_{2,2}(\mathbb{H})_\mathbb{R} \cong SL_4(\mathbb{R})
\end{align*}
\end{equation}

When singling SL$_2(\mathbb{C})_\mathbb{R}$ out from SL$_2(\mathbb{H})_\mathbb{R}$, half of the spinorial degrees of freedom are lost, as $\mathbb{H}$ is made à la Cayley-Dickson from two $\mathbb{C}$, and only contains one $\mathbb{C}$. The triality of Spin$_{4,4}$ leads to three charts of Spin$_{4,4} \otimes U_1$, allowing for three generations for the price of two in a manner that avoids mirror fermions due to only containing a single complex representation in $\mathbb{H}$ via $SL_2(\mathbb{C})_\mathbb{R} \subset SL_2(\mathbb{H})_\mathbb{R}$ or $SU_{2,2} \subset SU_{2,2}(\mathbb{H})_\mathbb{R}$. A unification of classical phase space with spacetime and energy-momentum for one generation can be found in Spin$_{6,4}$ or Spin$_{4,4}$. Therefore, three mass generations leads to two more energy dimensions, resulting in Spin$_{6,4}$, a subalgebra of $E_6(2)$. The Peirce decomposition of 27 w.r.t. $E_6(2)$ is used to identify only 16 representations as fermionic, while $E_6(2)$ itself also contains spinors; in this framework, Spin$_{4,4}$ triality can be proposed to describe three generations of matter.

Recent works discussing $J_3(\mathbb{O})$ and $J_3^C(\mathbb{O})$ motivate three generations within a single 26 or 27 representation of $F_4$ or $E_6$ [1DV 25]. While three charts of Spin$_9$ and Spin$_{10} \otimes U_1$ are in $F_4$ and $E_6$, an appropriate counting of on-shell and off-shell states must be found. For instance, a complex spinor 16 of Spin$_{10}$ is contained within the Peirce decomposition of 27 of $E_6$. However, three sets of 16 as two-component spinors overlap significantly within 27. In other words, we suggest that the triality of Spin$_{4,4}$, rather than Spin$_8$, gives three generations. $J_3^C(\mathbb{O})$ cannot fully contain the field content of three generations of spinors, but Weyl spinors of Spin$_{4,4} \otimes$ Spin$_8$ can, which stems from a single Majorana-Weyl spinor of Spin$_{12,4}$ within $E_{8(-24)}$ [CMR].

The three generations of Standard Model fermions within $(\mathbb{O}_s \otimes \mathbb{O})^2$ allows for three $(\mathbb{C} \otimes \mathbb{O})^2$ spinors, since there are three complex units in $\mathbb{O}_s$. A single $(\mathbb{C} \otimes \mathbb{O})^2$ cannot encode three generations, as three combinations of 16 are needed within the theory. If 27 contains a mix of fermions and bosons, rather than 27 fermions as found in $E_6$ GUT, the 27 of $E_{6(-78)}$, then the 27 as weight vectors does not have the fermionic roots determined. Once one of three Spin$_{10} \otimes U_1$ is chosen, then, this determines which roots are fermions. A key departure from $E_6$ GUT is the interpretation of the 27 representation as purely fermionic; since the fermions of the standard model are assigned to the pseudo-Riemannian space $E_{8(-24)}/$Spin$_{12,4}$, the Peirce decomposition of 27 as 16 + 10 + 1 identifies only the 16 as containing fermions. On the other hand, since Spin$_{10}$ is the largest GUT that contains no additional fermions, the role of $E_{6(-78)}$ is primarily suggested to encode three charts of flavor eigenstates of Spin$_{10} \otimes U_1$. In this manner, the utility of $E_{6(-78)}$ is similar if not identical to the one suggested by Boyle [25]. Our distinction is that only a single generation of fermions comes from a single $16 \in 27$ of Spin$_{10} \subset E_{6(-78)}$; instead, three generations of leptons are found within$^7$ $E_{6(2)} \otimes 27 \oplus 27$. Since $E_{6(2)} \otimes SU_3$ contains the SU3 of the strong force with color charge, Spin$_{6,4} \otimes U_1 \subset E_{6(2)}$ can be identified as a type of “gravi-weak” symmetry [NP Da Al]. On the other hand, the electroweak symmetry SU2 ⊗ U1 is the only subsector of Spin$_{10}$ that is a chiral gauge theory at low energies, which stems back to spinors of spacetime. Thus, by putting color aside, E$_{6(2)}$ can be studied to encode mass eigenstates, which provides a new physical motivation for the non-compact real form(s) of $E_6$ and split octonions. Various paths lead from $E_{8(-24)}$ to $SU_{2,2} \otimes U_1 \otimes SU_3 \otimes U_1$ at low energies, many of which pass through $SU_{2,2} \otimes U_1 \otimes SU_3 \otimes SU_2 \otimes U_1$ [CMR].

8 Conclusion

Making use of the Veronese coordinates, we have explicitly constructed two different bioctonionic planes, namely the complexification of the Cayley plane and the bioctonionic Rosenfeld plane, showing some of their different geometrical features, which yield to different algebraic structures on the three by three bioctonionic Hermitian matrices. We have also discussed the isometry groups of the two planes, and then characterized systematically all possible octonionic and bioctonionic planes as symmetric spaces (of Kähler or pseudo-Kähler type in the bioctonionic case). This approach brought us to single out two pseudo-Riemannian real forms of the bioctonionic

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$^7$Private correspondence with Todorov confirms that the 27 does not contain three generations, but provides the manifold for three generations.
plane that apparently do not have a projective or hyperbolic equivalent on the bioctonionic algebra. One of these spaces appears as enlarged scalar manifold of a “pure” (i.e. not matter-coupled) supergravity theory with 20 supersymmetries when dimensionally reduced from 3 + 1 to 3 + 0 space-time dimensions (namely, when reduced along time); we plan to investigate more on this in the future. Finally, we have briefly commented on the physical relevance of some bioctonionic Rosenfeld plane, hinting to some interesting applications of real forms of $E_6$.

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