GEOMETRIC CRYSTAL AND TROPICAL $R$ FOR $D_{n}^{(1)}$

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ABSTRACT. We construct a geometric crystal for the affine Lie algebra $D_{n}^{(1)}$ in the sense of Berenstein and Kazhdan. Based on a matrix realization including a spectral parameter, we prove uniqueness and explicit form of the tropical $R$, the birational map that intertwines products of the geometric crystals. The tropical $R$ commutes with geometric Kashiwara operators and satisfies the Yang-Baxter equation. It is subtraction-free and yields a piecewise linear formula of the combinatorial $R$ for crystals upon ultradiscretization.

1. Introduction

In this paper we construct a geometric crystal for $D_{n}^{(1)}$ corresponding to the $D_{n}^{(1)}$-crystal in [KKM], and study the associated tropical $R$. Geometric crystal is a notion introduced by Berenstein and Kazhdan in [BR] as an algebro-geometric analogue of the crystal theory [K1, K2]. The latter is a theory of quantum groups at $q = 0$, which involves combinatorial operations described by various piecewise linear functions. In the geometric setting in [BR], such a structure corresponds to birational maps described by subtraction-free functions.

Let us illustrate these features in the case of $A_{n-1}^{(1)}$ based on the constructions in [BR] and [KN]. Elements $x$ of the crystal $B_l$ for the $l$-symmetric tensor representation is specified by the coordinates $x = (x_1, \ldots, x_n) \in \mathbb{Z}_{\geq 0}^n$ with $x_1 + \cdots + x_n = l$, where all the indices are considered to be in $\mathbb{Z}/n\mathbb{Z}$ [KKM]. The action of the Kashiwara operator $e_i$ up to 2-fold tensor product is given, unless they vanish, by

$$e_i^c(x) = (\ldots, x_{i-1}, x_i + c, x_{i+1} - c, x_{i+2}, \ldots),$$

$$e_i^c(x \otimes y) = e_i^c(x) \otimes e_i^c(y),$$

$$c_1 = \max(x_i + c, y_{i+1}) - \max(x_i, y_{i+1}),$$

$$c_2 = \max(x_i, y_{i+1}) - \max(x_i, y_{i+1} - c).$$

In the geometric crystal, one still has the coordinates $x = (x_1, \ldots, x_n) \in B$ and the corresponding structure looks

$$e_i^c(x) = (\ldots, x_{i-1}, cx_i, c^{-1}x_{i+1}, x_{i+2}, \ldots),$$

$$e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y)),$$

$$c_1 = \frac{cx_i + y_{i+1}}{x_i + y_{i+1}}, \quad c_2 = \frac{x_i + y_{i+1}}{x_i + c^{-1}y_{i+1}}.$$

We call it the geometric Kashiwara operator. Note that the latter $c_1, c_2$ contain no minus sign, i.e., they are subtraction-free. The former $c_1, c_2$ are piecewise linear and obtained from the latter by replacing $+, \times, /$ with $\max, +, -$, respectively. The procedure, which we call ultradiscretization in this paper, is well defined since the two sides of $p(q + r) = pq + pr$ have the coincident image $p + \max(q, r) = \max(p, q) + \max(p, r)$.
The geometric crystal $\mathcal{B}$ is actually associated with the coherent family $\{B_l\}_{l \geq 1}$ [KKM] rather than the single crystal $B_l$.

To each $x \in \mathcal{B}$ we assign the $n$ by $n$ matrix

$$M(x, z) = \left( \sum_{i=1}^{n} \frac{1}{x_i} E_{i,i} - \sum_{i=1}^{n-1} E_{i+1,i} - z E_{1,n} \right)^{-1}$$

containing a spectral parameter $z$. Then on the product $M = M(x^{(1)}, z) \cdots M(x^{(L)}, z)$ corresponding to $(x^{(1)}, \ldots, x^{(L)}) \in \mathcal{B}^\times L$, one can realize the geometric Kashiwara operator as the multiplications of unipotent matrices:

$$G_i \left( \frac{c - 1}{z^{\delta_{i0}} \varepsilon_i(M)} \right) MG_i \left( \frac{c - 1}{z^{\delta_{i0}} \varphi_i(M)} \right) = M(e_i^{\varepsilon_i}(x^{(1)}), z) \cdots M(e_i^{\varepsilon_i}(x^{(L)}), z),$$

where $G_i(a) = E + aE_{i,i+1}$, $(1 \leq i \leq n-1)$, $G_0(a) = E + aE_{n,1}$ and

$$\varepsilon_i(M) = z^{-\delta_{i0}} \frac{M_{i+1,i}}{M_{i,i+1}} \mid_{z=0}, \quad \varphi_i(M) = z^{-\delta_{i0}} \frac{M_{i+1,i}}{M_{i,i+1}} \mid_{z=0}.$$ 

The product $(e_i^{\varepsilon_i}(x^{(1)}), \ldots, e_i^{\varepsilon_i}(x^{(L)})) \in \mathcal{B}^\times L$ corresponds to the element $\tilde{e}_i^{\varepsilon_i}(x^{(1)} \otimes \cdots \otimes x^{(L)})$ in the tensor product crystal under the ultradiscretization mentioned above.

One of the axioms of the geometric crystal is the relation $e_i^d e_j^d e_i^c = e_j^d e_i^c e_j^d$ in the case of $(\alpha_i^v, \alpha_j) = (\alpha_j^v, \alpha_i) = -1$. The matrix realization reduces its proof to a simple manipulation on the matrices $G_i$'s.

In the crystal theory, the isomorphism of tensor products $B_l \otimes B_k \simeq B_k \otimes B_l$ is called the combinatorial $R$. It is the quantum $R$ matrix at $q = 0$, and plays a fundamental role. To incorporate it into the geometric setting we consider the matrix equation $M(x, z)M(y, z) = M(xz, M(y^e, z))$. Explicitly, it is a discrete Toda type equation $x_i y_i = x_i^e y_i^e, \frac{1}{x_i} + \frac{1}{y_i} = \frac{1}{x_i^e} + \frac{1}{y_i^e}$. Given $x = (x_1, \ldots, x_n) \in \mathcal{B}$, call $\ell(x) = x_1 \cdots x_n$ the level. It turns out that the equation with the level constraints $\ell(x) = \ell(y'), \ell(y) = \ell(x')$ characterizes the unique solution:

$$x'_i = y_i \frac{P_i(x, y)}{P_{i-1}(x, y)}, \quad y'_i = x_i \frac{P_{i-1}(x, y)}{P_i(x, y)},$$

$$P_i(x, y) = \sum_{k=1}^{n} \prod_{j=k}^{n} x_{i+j} \prod_{j=1}^{k} y_{i+j}.$$ 

We call the rational map $(x, y) \mapsto (x', y')$ the tropical $R$, and simply denote by $R$. From the characterization one can derive the geometric analogues of most important properties of the combinatorial $R$, e.g., commutativity with geometric Kashiwara operators $e_i^R R = R e_i^c$, the inversion relation $R(R(x, y)) = (x, y)$ showing $R$ is birational, the Yang-Baxter relation and so forth. Moreover the ultradiscretization of the tropical $R$ yields an explicit piecewise linear formula for the combinatorial $R$.

In this paper we show that all the features explained above along type $A$ persist also for $D_n^{(1)}$. What we do is to start from a new geometric crystal associated with the coherent family of $D_n^{(1)}$-crystals in [KKM], and develop an elementary approach based on explicit calculations. It is motivated by several recent works; geometricization of canonical bases [BFZ, BR, L2], tropical combinatorics and birational representations of affine Weyl groups [K, KNY, NoY, N], integrable cellular automata associated to crystals [HKT1, FOY, HKT2, HKOTY], discrete soliton
equations arising as their tropical evolution equation \cite{TTMS,HHIKT1,Y}, combinatorial $R$ in terms of the bumping algorithm \cite{HKOT1,HKOT2}, etc. We hope to report on applications of the present result to these issues in near future.

The paper is arranged as follows. In section 2 we begin with the definition and basic facts on geometric crystals following \cite{BK}. The geometric $D_n^{(1)}$-crystal $\mathcal{B}$ is presented, which is parametrized with the $2n-1$ coordinates $x = (x_1, \ldots, x_n, x_{n-1}, \ldots, x_1)$. See section 4.2 for the precise correspondence to the coordinates in the crystal. In section 5 we introduce the $2n$ by $2n$ matrix $M(x,z)$ that realizes the geometric crystal $\mathcal{B} \times L$. It is quadratic in $z$ in contrast with the essentially linear one for type $A$. At $z = 0$ it admits a factorization (3.12) and belongs to the lower triangular Borel subgroup of $D_n$ as in Proposition 3.6, whereas at $z = \ell(x)^{-1}$ it shrinks to the rank one matrix as in Proposition 3.3. A key in our approach is to make full use of such properties controlled by the spectral parameter of affine nature, which we have been unable to learn in \cite{BK}. In section 6 we study the tropical $R$. From Theorem 5.13 its uniqueness is immediate, hence our main task is to assure the existence. We shall show that the explicit birational map $(x,y) \rightarrow (x',y')$ in Definition 5.4 is the answer. By the ultradiscretization we then derive a piecewise linear formula for the combinatorial $R$ and the associated quantum function. Appendix A provides an alternative proof of the relation $e_i^d e_j^d = e_j^d e_i^d$ on $\mathcal{B} \times L$. Appendix B contains a proof of $M(x,z)M(y,z) = M(x'z)M(y',z)$ at $z = 0$.

We remark that our terminology “tropical” in this paper follows the usage in \cite{Ki, NoY}, which is unfortunately the opposite of \cite{BK}. The phrase “ultradiscretization” has been used in many papers inspired by \cite{TTMS}.

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2. Geometric $D_n^{(1)}$-crystal

2.1. Geometric crystal. Let $\mathfrak{g}$ be an affine Lie algebra of simply laced type, i.e., $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}$ or $E_6^{(1)}, E_7^{(1)}, E_8^{(1)}$. Let $I = \{0, 1, \ldots, n\}$ be the index set of vertices of the Dynkin diagram of $\mathfrak{g}$. We denote by $\{\alpha_i \mid i \in I\}$ (resp. $\alpha_i' \mid i \in I\}$) the set of simple roots (resp. simple coroots). Note that $\langle \alpha_i', \alpha_j \rangle = 0$ or $-1$.

Following \cite{BK} we introduce a geometric crystal associated to $\mathfrak{g}$. Throughout this paper, variable means a real variable unless otherwise stated. Let $x = (x_1, x_2, \ldots, x_N)$ be an $N$-tuple of variables. For $i \in I$ let $\varepsilon_i(x), \varphi_i(x)$ be rational functions in $x$ and set

$$\gamma_i(x) = \varphi_i(x)/\varepsilon_i(x).$$

For $i \in I$ let $\mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N : (c,x) \rightarrow c^i(x)$ be a rational mapping, namely, $c^i(x) = (x'_1, x'_2, \ldots, x'_N)$ with each $x'_j (1 \leq j \leq N)$ being a rational function in $c$ and $x$. In what follows we consider $c$ to be a parameter and $c^i$ to be a rational transformation on $\mathbb{R}^N$. We also abbreviate the symbol $\circ$ of composition. For instance we write $e_i^d e_j^2$ instead of $e_i^d \circ e_j^2$. 


Definition 2.1. A geometric pre-crystal is a family \( B = \{ x, \varepsilon_i, \varphi_i, e_i^c \} \) satisfying

(i) \( e_i^c e_i^{c'}(x) = e_i^{c+1}(x) \), \( e_i^c(x) = x \),

(ii) \( \varepsilon_i(e_i^c(x)) = c^{-1} \varepsilon_i(x), \varphi_i(e_i^c(x)) = c \varphi_i(x) \),

(iii) \( \gamma_i(e_i^c(x)) = c(\alpha_i^\vee, \alpha_i) \gamma_i(x) \).

Definition 2.2. A geometric pre-crystal \( B \) is called geometric crystal, if it further satisfies the following relations:

\[
\begin{align*}
\text{(2.2)} & \quad e_i^c e_j^c = e_j^c e_i^c \\
\text{(2.3)} & \quad e_i^c e_j e_i^c e_j = e_j e_i^{c+1} e_j e_i^c
\end{align*}
\]

if \( \langle \alpha_i^\vee, \alpha_j \rangle = 0 \),

if \( \langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_i^\vee, \alpha_i \rangle = -1 \).

Remark 2.3. The definition of the geometric pre-crystal is slightly different from the original one in [BK]. We adopt it in order to make the formulas parallel with those in crystals.

Remark 2.4. As mentioned in [BK] Remarks after Lemma 2.1, (2.2)–(2.3) can be thought of as multiplicative analogues of the Verma relations in the universal enveloping algebra \( U(g) \). See also [BK] Proposition 39.3.7 for the version of the quantum enveloping algebra \( U_q(g) \). The relations in the cases of \( \langle \alpha_i^\vee, \alpha_j \rangle \langle \alpha_i^\vee, \alpha_i \rangle \geq 2 \) read as follows:

\[
\begin{align*}
\text{if } \langle \alpha_i^\vee, \alpha_j \rangle = -1, \langle \alpha_i^\vee, \alpha_i \rangle = -2, & \quad e_i^c e_j e_i^c e_j = e_j e_i^{c+1} e_j e_i^c \\
\text{if } \langle \alpha_i^\vee, \alpha_j \rangle = -1, \langle \alpha_i^\vee, \alpha_i \rangle = -3 & \quad e_i^c e_j e_i^c e_j = e_j e_i^{c+1} e_j e_i^c.
\end{align*}
\]

Proposition 2.5 ([BK]). Let \( B \) be a geometric crystal. Set \( s_i(x) = e_i^{\gamma_i(x)^{-1}}(x) \). Then \( s_i \) \( (i \in I) \) generates a birational action of the Weyl group corresponding to \( g \).

Proof. By definition, \( s_i^2(x) = e_i^c e_i^c(x) \) with \( c = \gamma_i(x)^{-1}, c' = \gamma_i(e_i^{\gamma_i(x)^{-1}}(x))^{-1} \).

From Definition 2.1 (iii), we have

\[
c' = ((\gamma_i(x)^{-1})^2 \gamma_i(x))^{-1} = \gamma_i(x).
\]

Thus \( s_i^2(x) = x \) by Definition 2.1 (i).

Similarly, if \( \langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = -1, \) we have

\[
\begin{align*}
\text{if } s_i s_j s_i(x) = e_i^{c''} e_j^{c'} e_i^c(x), & \quad s_j s_i s_j(x) = e_i^{c''} e_j^{c''} e_i^d(x), \\
\text{with } c = d'' = \gamma_i(x)^{-1}, c' = \gamma_i(x)^{-1} \gamma_j(x)^{-1}, c'' = d = \gamma_j(x)^{-1}. & \quad \text{Thus } s_i s_j s_i(x) = s_j s_i s_j(x) \text{ by Definition 2.2.}
\end{align*}
\]

If \( \langle \alpha_i^\vee, \alpha_j \rangle = 0, \) one clearly has \( s_i s_j(x) = s_j s_i(x) \).

\( \square \)

Remark 2.6. This Weyl group action is an analogue of the one on crystals given in [K].
Example 2.7. Let \( g = A_n^{(1)}, N = n + 1, x = (x_1, x_2, \ldots, x_{n+1}) \). For \( i \in I = \{0, 1, \ldots, n\} \) we define

\[
\begin{align*}
\varepsilon_i(x) &= x_{i+1}, \quad \varphi_i(x) = x_i, \\
\varepsilon_i^c(x) &= (\ldots, c_{x_{i}}, c^{-1}x_{i+1}, \ldots).
\end{align*}
\]

Here \( x_0 \) should be understood as \( x_{n+1} \). One can check that \( B = \{x, \varepsilon_i, \varphi_i, \varepsilon_i^c\} \) is a geometric crystal. \( s_i(x) \) is given by

\[
s_i(x) = (\ldots, x_{i+1}, i+1x_i, \ldots).
\]

2.2. Geometric \( D_n^{(1)} \)-crystal \( B \). We are interested in a particular geometric crystal \( B \) of type \( D_n^{(1)} \) given below. Let \( N = 2n - 1 \) and consider a set of variables \( x = (x_1, x_2, \ldots, x_n, x_{n-1}, \ldots, x_1) \). \( \varepsilon_i(x), \varphi_i(x), \varepsilon_i^c(x) \) \( (i \in I = \{0, 1, \ldots, n\}) \) are given as follows.

\[
\begin{align*}
\varepsilon_0(x) &= x_1, \quad \varphi_0(x) = \frac{x_2}{x_2} + 1, \\
\varepsilon_i(x) &= x_i, \quad \varphi_i(x) = \frac{x_{i+1}}{x_i} + 1 \quad (i = 1, \ldots, n - 2), \\
\varepsilon_{n-1}(x) &= x_{n-n}, \quad \varphi_{n-1}(x) = x_{n-1}, \\
\varepsilon_i(x) &= x_{n-n}, \quad \varphi_i(x) = x_{n-1-x_n}, \\
\varepsilon_i^c(x) &= (\ldots, c_{x_{i}}, c^{-1}x_{i+1}, \ldots), \quad (i = 1, \ldots, n - 2), \\
\varepsilon_{n-1}^c(x) &= (\ldots, c_{x_{n-1}}, c^{-1}x_{n-1}, \ldots), \\
\varepsilon_i^c(x) &= (\ldots, c_{x_n}, c^{-1}x_{n-1}, \ldots), \\
\text{where} \quad \xi_i &= \frac{x_i + \xi_{i+1}}{x_i + x_i} \quad (i = 1, \ldots, n - 1).
\end{align*}
\]

We set

\[
\ell(x) = x_1 x_2 \cdots x_n x_{n-1} x_{n-2} \cdots x_1
\]

and call it the level. Note that \( \ell(x) \) is invariant under the transformation \( \varepsilon_i^c \) for any \( i \) and \( c \).

Remark 2.8. This \( B \) is obtained as a multiplicative analogue of the coherent family of perfect crystals \( \{B_j\}_{j \geq 1} \) of type \( D_n^{(1)} \) given in [KKM]. Of course, \( e_i^c(x) \) corresponds to \( e_i^c b \).

The functions \( \varepsilon_i(x), \varphi_i(x) \) of our \( B \) satisfy the following good properties, which will be used in the next section.

Lemma 2.9. (a) If \( \langle \alpha_i^\vee, \alpha_j \rangle = 0 \), then

\[
\varepsilon_i(e_j^c(x)) = \varepsilon_i(x), \quad \varphi_i(e_j^c(x)) = \varphi_i(x).
\]

(b) If \( \langle \alpha_i^\vee, \alpha_j \rangle = \langle \alpha_j^\vee, \alpha_i \rangle = -1 \), then there exist rational functions \( \varepsilon_{ij}(x), \varepsilon_{ji}(x) \) such that

\[
\begin{align*}
\varepsilon_i(x) \varepsilon_j(x) &= \varepsilon_{ij}(x) + \varepsilon_{ji}(x), \\
\varepsilon_{ij}(e_j^c(x)) &= \varepsilon_{ij}(x), \quad \varepsilon_{ji}(e_j^c(x)) = \varepsilon_{ji}(x), \\
\varepsilon_{ji}(e_j^c(x)) &= \varepsilon_{ij}(x), \quad \varepsilon_{ji}(e_j^c(x)) = \varepsilon_{ji}(x).
\end{align*}
\]
and \( \varphi_{ij}(x), \varphi_{ji}(x) \) such that
\[
\varphi_i(x)\varphi_j(x) = \varphi_{ij}(x) + \varphi_{ji}(x), \\
\varphi_{ij}(\epsilon_i^r(x)) = c\varphi_{ij}(x), \\
\varphi_{ji}(\epsilon_j^r(x)) = \varphi_{ij}(x), \\
\varphi_{ji}(\epsilon_j^\gamma(x)) = c\varphi_{ji}(x).
\]

**Remark 2.10.** Due to Definition 2.1 (iii) it suffices to check Lemma 2.9 only for \( \varphi_i(x) \) (or \( \varepsilon_i(x) \)). The functions \( \varepsilon_{ij}(x), \varepsilon_{ji}(x) \) in (b) are obtained as
\[
\varepsilon_{ij}(x) = \varphi_{ji}(x)/(\gamma_i(x)\gamma_j(x)), \\
\varepsilon_{ji}(x) = \varphi_{ij}(x)/(\gamma_i(x)\gamma_j(x)),
\]
once \( \varphi_{ij}(x), \varphi_{ji}(x) \) are obtained.

**Proof of Lemma 2.3.** (a) is easy. For (b) it suffices to give a list of \( \varphi_{ij}(x) \) and \( \varphi_{ji}(x) \) when \( \langle \alpha_i^\gamma, \alpha_j^\gamma \rangle = \langle \alpha_i^\gamma, \alpha_i^\gamma \rangle = -1. \)

\[
\varphi_{ij}(x) = \frac{2}{x_i} \left( \frac{3}{3} + 1 \right), \\
\varphi_{ji}(x) = \frac{2}{x_j} \left( \frac{3}{3} + 1 \right), \\
\varphi_{ij}(x) = \frac{2}{x_i} \left( \frac{3}{3} + 1 \right), \\
\varphi_{ji}(x) = \frac{2}{x_j} \left( \frac{3}{3} + 1 \right) \quad (1 \leq i \leq n - 3),
\]

2.3. **Product.** Let \( B_1 = \{x = (x_1, \ldots, x_N) : \varepsilon_i, \varphi_i, \epsilon_i^r \}, B_2 = \{y = (y_1, \ldots, y_N), \varepsilon_i, \varphi_i, \epsilon_i^r \} \) be geometric pre-crystals. For brevity we use the same symbols \( \varepsilon_i, \varphi_i, \epsilon_i^r \) for different geometric pre-crystals, since they are distinguished by the sets of variables. Now consider the set of variables \( (x, y) = (x_1, \ldots, x_N, y_1, \ldots, y_N) \). We define a new structure of a geometric pre-crystal \( B_1 \times B_2 \) on \( (x, y) \) by

\[
\varepsilon_i(x, y) = \varepsilon_i(x) + \varepsilon_i(x)\varepsilon_j(y)/\varphi_i(x), \\
\varphi_i(x, y) = \varphi_i(y) + \varphi_i(x)\varphi_i(y)/\varepsilon_i(y), \\
\epsilon_i^r(x, y) = \epsilon_i^r(x) + \epsilon_i^r(y) , \\
\text{where} \quad c_1 = \frac{c\varphi_i(x) + \varepsilon_i(y)}{\varphi_i(x) + \varepsilon_i(y)} , \\
c_2 = \frac{\varphi_i(x) + \varepsilon_i(y)}{\varphi_i(x) + c_1\varepsilon_i(y)}.
\]

**Remark 2.11.** These are analogues of formulas for the tensor product of crystals \( K \) given by

\[
\varepsilon_i(x \otimes y) = \max(\varepsilon_i(x), \varepsilon_i(x) + \varepsilon_i(y) - \varphi_i(x)), \\
\varphi_i(x \otimes y) = \max(\varphi_i(y), \varphi_i(x) + \varphi_i(y) - \varepsilon_i(y)), \\
\epsilon_i^r(x \otimes y) = \epsilon_i^r(x) \otimes \epsilon_i^r(y), \\
\text{where} \quad c_1 = \max(c + \varphi_i(x), \varepsilon_i(y)) - \max(\varphi_i(x), \varepsilon_i(y)), \\
c_2 = \max(\varphi_i(x), \varepsilon_i(y)) - \max(\varphi_i(x), -c + \varepsilon_i(y)).
\]

In (2.10) \( c \) can be a negative integer, in which case we understand \( \tilde{\epsilon}_i^r = \tilde{f}_i^{-c} \).

**Lemma 2.12.** Assume that the functions \( \varepsilon_i, \varphi_i \) for both \( B_1 \) and \( B_2 \) satisfy the properties in Lemma 2.7. Then those for \( B_1 \times B_2 \) also satisfy the same property.
Proof. If \( (\alpha_1^\gamma, \alpha_2) = 0 \), the statement is clear. Suppose \( (\alpha_1^\gamma, \alpha_2) = (\alpha_1^\gamma, \alpha_1) = -1 \). Due to Remark 2.10 it suffices to construct desired functions \( \varphi_{ij}(x, y), \varphi_{ji}(x, y) \).

Define

\[
\varphi_{ij}(x, y) = \varphi_{ij}(y) + \varphi_i(x)\varphi_j(y)\gamma_i(y) + \varphi_{ij}(x)\gamma_i(y)\gamma_j(y)
\]

and \( \varphi_{ji}(x, y) \) by interchanging \( i \) and \( j \). To illustrate we calculate \( \varphi_{ij}(e_i^c(x, y)) \). Let \( c_1, c_2 \) be as in (2.7). Using Definition 2.1 (iii) and the assumption,

\[
\varphi_{ij}(e_i^c(x, y)) = c_2\varphi_{ij}(y) + \frac{c_1}{c_2}c_1\varphi_i(x)c_2\varphi_{ij}(y) + \frac{c_1}{c_2}c_1\varphi_j(x)c_2\varphi_{ji}(x)
\]

\[
= c_2\varphi_{ij}(y) + \frac{c_1}{c_2}\varphi_i(x)c_1\varphi_j(y) + \frac{c_1}{c_2}\varphi_j(x)c_1\varphi_i(y)
\]

\[
= c\varphi_{ij}(y) + \frac{c_1}{c_2}\varphi_i(x)c_1\varphi_j(y) + \frac{c_1}{c_2}\varphi_j(x)c_1\varphi_i(y)
\]

\[
= c\varphi_{ij}(x, y).
\]

The other cases are similar. \( \square \)

We now consider a multiple product. Let \( B_1, \ldots, B_L \) be geometric pre-crystals such that \( B_i = (x^i, (x_1^i, \ldots, x_N^i)) \) \( (i = 1, \ldots, L) \). Let us set \( x = (x_1, \ldots, x_L) \).

Then \( \varepsilon_i, \varphi_i, e_i^c \) of the \( L \)-fold product of geometric pre-crystals \( B_1 \times \cdots \times B_L \) is given by

(2.13) \[
\varepsilon_i(x) = \frac{\sum_{k=1}^L \left( \prod_{j=1}^k \varepsilon_i(x^j) \right) \left( \prod_{j=k+1}^{L-1} \varphi_i(x^j) \right)}{\prod_{j=1}^{L-1} \varphi_i(x^j)},
\]

(2.14) \[
\varphi_i(x) = \frac{\sum_{k=1}^L \left( \prod_{j=1}^k \varepsilon_i(x^j) \right) \left( \prod_{j=k+1}^L \varphi_i(x^j) \right)}{\prod_{j=2}^L \varepsilon_i(x^j)},
\]

(2.15) \[
e_i^c(x) = (e_i^c(x^1), \ldots, e_i^c(x^L)),
\]

(2.16) \[
\text{with } c_i = \frac{\sum_{k=1}^L e^{\theta(k \leq L)} \left( \prod_{j=2}^k \varepsilon_i(x^j) \right) \left( \prod_{j=k+1}^{L-1} \varphi_i(x^j) \right)}{\sum_{k=1}^L e^{\theta(k \leq L-1)} \left( \prod_{j=2}^k \varepsilon_i(x^j) \right) \left( \prod_{j=k+1}^{L-1} \varphi_i(x^j) \right)}.
\]

Here \( \theta(s) = 1 \) if \( s \) is true and \( = 0 \) otherwise. Note that the product is associative, \( e_i^c(x^1, x^2, x^3) = (e_i^c, e_i^c, e_i^c) \).

3. Realization by unipotent matrices

3.1. Matrix \( M(x, z) \). Let us introduce \( 2n \) by \( 2n \) matrices \( G_i(a) \) for \( 0 \leq i \leq n \) by

\[
G_i(a) = E + \begin{cases} 
    a(E_{i,i+1} + E_{2n-i,2n+1-i}) & \text{for } 1 \leq i \leq n-1, \\
    a(E_{n-1,n+1} + E_{n,n+2}) & \text{for } i = n, \\
    a(E_{2n-1,1} + E_{2n,2}) & \text{for } i = 0.
\end{cases}
\]

Here \( E \) is the identity matrix, \( E_{ij} \) is the \( (i, j) \) matrix unit, and \( a \) is a parameter.

\( G_i(a) \)'s have the following properties.

(3.1) \( G_i(a)G_i(b) = G_i(a+b) \),

(3.2) \( G_i(a)G_j(b)G_i(a) = G_j(b)G_i(a) \) \( \text{if } \langle \alpha^\gamma_i, \alpha_j \rangle = 0 \),

(3.3) \( G_i(a)G_j(b)G_i(c) = G_j(a')G_i(b')G_j(c') \) \( \text{if } \langle \alpha_i^\gamma, \alpha_j \rangle = \langle \alpha_i^\gamma, \alpha_i \rangle = -1 \).
where \( a' = bc/(a + c), b' = a + c, c' = ab/(a + c) \).

Let \( \mathcal{B} = \{x = (x_1, \ldots, x_n), \varepsilon_i, \varphi_i, e_i^c\} \) be the geometric \( D_n^{(1)} \)-crystal given in the previous section. In this subsection we introduce a \( 2n \times 2n \) matrix \( M(x, z) \) that satisfies

\[
G_i \left( \frac{c - 1}{z b_i \varphi_i(x)} \right) M(x, z) G_i \left( \frac{c^{-1} - 1}{z b_i \varepsilon_i(x)} \right) = M(e_i^c(x), z)
\]

for any \( i (0 \leq i \leq n) \) and an extra parameter \( z \). We call \( z \) the spectral parameter.

We give an explicit form of \( M(x, z) \). Matrix elements in the first column \( M(x, z)_{1,1} \) are given as follows.

\[
M(x, z)_{1,1} = \begin{cases} 
 x_1/\overline{x}_1 & \text{for } i = 1, \\
 x_1 \cdots x_{i-1} (1 + x_i/\overline{x}_i) & \text{for } 2 \leq i \leq n-1, \\
 x_1 \cdots x_{n-1} x_n & \text{for } i = n, \\
 \ell(x)/(\overline{x}_1 \cdots \overline{x}_{2n-1}) & \text{for } n + 2 \leq i \leq 2n.
\end{cases}
\]

(3.4)

Given the first column as above we define the other matrix elements of \( M(x, z) \) by the relations (3.5)-(3.10).

\[
M(x, z)_{i,j+1} = \frac{M(x, z)_{i,j}}{x_j} + \begin{cases} 
 (\ell(x)z - 1)/\overline{x}_j & \text{for } i = j, \\
 \ell(x)z - 1 & \text{for } i = j + 1, \\
 0 & \text{for } i \neq j, j + 1,
\end{cases}
\]

(3.5)

for \( 1 \leq j \leq n-2 \).

\[
M(x, z)_{i,n} = \frac{M(x, z)_{i,n-1}}{x_n} + \begin{cases} 
 (\ell(x)z - 1)/\overline{x}_{n-1} & \text{for } i = n-1, \\
 \ell(x)z - 1 & \text{for } i = n, \\
 0 & \text{for } i \neq n-1, n+1.
\end{cases}
\]

(3.6)

\[
M(x, z)_{i,n+1} = \frac{M(x, z)_{i,n+1}}{x_{n-1} x_n} + \begin{cases} 
 (\ell(x)z - 1)/(\overline{x}_{n-1} x_n) & \text{for } i = n-1, \\
 \ell(x)z - 1 & \text{for } i = n, \\
 0 & \text{for } i \neq n-1, n.
\end{cases}
\]

(3.7)

\[
M(x, z)_{i,2n} = z \frac{\overline{x}_1 M(x, z)_{i,1}}{x_1} + \begin{cases} 
 \overline{x}_1 (1 - \ell(x)z) & \text{for } i = 2n, \\
 z(\ell(x)z - 1) & \text{for } i = 1, \\
 0 & \text{for } i \neq 1, 2n.
\end{cases}
\]

(3.8)

Let us denote \( 1 + \overline{x}_i/x_i \) by \( \overline{X}_i \).

\[
M(x, z)_{i,2n-1} = z \overline{x}_1 \overline{X}_2 M(x, z)_{i,1} + \begin{cases} 
 \overline{x}_1 \overline{X}_2 (1 - \ell(x)z) & \text{for } i = 2n, \\
 \overline{x}_2 (1 - \ell(x)z) & \text{for } i = 2n - 1, \\
 0 & \text{for } i \neq 2n - 1, 2n.
\end{cases}
\]

(3.9)
\begin{equation}
M(x, z)_{i, 2n - j} = \begin{cases} 
\frac{X_{j+1}}{X_j} M(x, z)_{i, 2n+1-j} & \text{for } i = 2n + 1 - j, \\
\frac{X_{j+1}}{X_j} (1 - \ell(x) z) & \text{for } i = 2n - j, \\
0 & \text{for } i \neq 2n + 1 - j, 2n - j,
\end{cases}
\end{equation}

for $2 \leq j \leq n - 2$. For later use we also present matrix elements in the last row.

\begin{equation}
M(x, z)_{2n, i} = \begin{cases} 
\ell(x)/(x_1 \cdots x_{i-1}) & \text{for } 1 \leq i \leq n, \\
x_1 \cdots x_{n-1} & \text{for } i = n + 1, \\
x_1 \cdots x_{2n-i}(1 + x_{2n+1-i}/x_{2n+1-i}) & \text{for } n + 2 \leq i \leq 2n - 1, \\
x_1/x_1 & \text{for } i = 2n.
\end{cases}
\end{equation}

It can be seen that each matrix element is a polynomial in $z$ of degree at most 2. We denote the coefficients of $M(x, z)$ of degree 0, 1, 2 by $A(x)$, $B(x)$, $C(x)$. Hence we have

\[ M(x, z) = A(x) + zB(x) + z^2C(x). \]

**Example 3.1** ($n = 4$ case). $x = (x_1, x_2, x_3, x_4, \overline{x}_3, \overline{x}_2, \overline{x}_1)$. We use the notations $[1] = x_1$, $[2] = x_2$, ..., $[1] = \overline{x}_1$, and $[\overline{(2)}] = 1 + x_2/\overline{x}_2$, $[\overline{(3)}] = 1 + \overline{x}_3/x_3$, etc. For instance, $[343] = x_3 x_4 \overline{x}_3$ and $[1(2)] = x_1 (1 + \overline{x}_2/x_2)$. Define $8 \times 8$ matrices $A(x)$, $B(x)$, $C(x)$ as follows.

\[
A(x) = \begin{pmatrix}
[1] & [2] \\
[12] & [23] & [3] & [4] \\
[1234] & [234] & [34] & [4] \\
[123432] & [23432] & [3432] & [432] & [32] & [\overline{(3)}] & [\overline{(2)}] & [1] \\
[1234321] & [234321] & [34321] & [4321] & [321] & [\overline{(3)}] & [\overline{(2)}] & [1]
\end{pmatrix},
\]
The matrix $M(x, z)$ for $D_4^{(1)}$ is given by $M(x, z) = A(x) + zB(x) + z^2C(x)$.

Before going into the main theorem (Theorem 3.11) we present several properties of the matrix $M(x, z) = A(x) + zB(x) + z^2C(x)$. The only nonzero element of $C(x)$ is $C(x)_{1, 2n}$ that is equal to $\ell(x)$. The matrix $A(x)$ is lower triangular. Explicit expressions for the nonzero elements of $A(x)$ are given as follows:

- $A(x)_{i,i} = x_i/x_i(1 \leq i \leq n-1) = x_n(i = n) = 1/x_n(i = n + 1) = x_{2n+1-i}/x_{2n+1-i}(n + 2 \leq i \leq 2n)$.
- For $2 \leq i \leq n - 1, 1 \leq j \leq i - 1$, we have $A(x)_{i,j} = x_j \cdots x_{i-1}(1 + x_i/x_i)$ and $A(x)_{2n+1-j,2n+1-i} = x_j \cdots x_{i-1}(1 + x_i/x_i)$.
- For $1 \leq j \leq n-1$ we have $A(x)_{n,j} = x_j \cdots x_n, A(x)_{n+1,j} = x_j \cdots x_{n-1}, A(x)_{2n+1-j,n} = x_j \cdots x_{n-1}x_n$, and $A(x)_{2n+1-j,n+1} = x_j \cdots x_{n-1}$.
- For $1 \leq i, j \leq n - 1$ we have $A(x)_{2n+1-i,j} = (x_j \cdots x_n) \times (x_i \cdots x_{n-1})$.

**Lemma 3.2.** Let $F_i(a) = ^tG_i(a)$ and

$$d(x) = \text{diag}(x_1/x_1, \ldots, x_{n-1}/x_{n-1}, x_n, 1/x_n, x_{n-1}/x_{n-1}, \ldots, x_1/x_1).$$

Then the matrix $A(x)$ has the following factorization.

\[(3.12) \quad A(x) = F_1(x_1)F_2(x_2) \cdots F_{n-2}(x_{n-2})F_{n}(x_{n-1})d(x) \times F_{n-1}(x_{n-1})F_{n-2}(x_{n-2}) \cdots F_2(x_2)F_1(x_1)\]

**Proof.** $F_1(x_1)^{-1}A(x)F_1(x_1)^{-1}$ is given by deleting the elements in the 1st column and the $2n$-th row of $A(x)$ except the diagonal elements. To illustrate we take the
$n = 4$ case. See Example 3.4.

\[
F_1(\pi_1)^{-1}A(x)F_1(x)^{-1} = \begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
\vdots & \vdots & \vdots & \vdots \\
2(n-1) & 2(n-1) & 2(n-1) & 2(n-1)
\end{pmatrix}.
\]

Similarly, $F_2(\tau_2)^{-1}F_1(\pi_1)^{-1}A(x)F_1(x)^{-1}F_2(x_2)^{-1}$ is given by deleting the elements in the 2nd column and the $(2n-1)$-th row of $F_1(\pi_1)^{-1}A(x)F_1(x)^{-1}$ except the diagonal elements. This process continues until we obtain $d(x)$. For example, the final step for the $n = 4$ case goes as

\[
\begin{pmatrix}
1 & 1 & 1 \\
-3 & -3 & -3
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
3 & 3 & 3
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-3 & -3
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-3 & -3
\end{pmatrix}
\]

**Remark 3.3.** We note that the sequence (1, 2, ..., $n-2$, $n$, $n-1$, ..., 1) extracted from the indices of $F$ in the right hand side of (3.12) is essentially the same as the one given in [KMOTU, HKT2].

**Proposition 3.4.**

(3.13) \[ M(x, \ell(x)^{-1}) = \frac{1}{\ell(x)}D_2(x)PD_1(x), \]

where

\[
D_1(x) = diag(A(x)_{2n,1}, A(x)_{2n,2}, \ldots, A(x)_{2n,2n}),
\]

\[
D_2(x) = diag(A(x)_{1,1}, A(x)_{2,1}, \ldots, A(x)_{2n,1}),
\]

and $P$ is the matrix with all the entries being 1.

**Corollary 3.5.**

\[
B(x) = D_2(x)PD_1(x) - \ell(x)A(x) - \ell(x)^{-1}C(x).
\]

Set $T = (\delta_{i+j,n+1}(-1)^{i-1})_{1 \leq i, j \leq n}$ and define a $2n$ by $2n$ matrix $S$ by

(3.14) \[ S = \begin{pmatrix} T & \text{T} \end{pmatrix}. \]

It is clear that $S^{-1} = S = S$. 

Proposition 3.6. \( M(x, z)S^4M(x, z)S = (1 - z\ell(x))^2E. \)

Proof. For brevity we write \( M(x, z) = M, \ A(x) = A, \ B(x) = B, \ C(x) = C, \ D_1(x) = D_1, \ D_2(x) = D_2, \) and \( \ell(x) = \ell. \) For any \( 2n \) by \( 2n \) matrix \( X \) we denote \( S'X\overline{S} \) by \( \overline{X}. \) Then we have
\[
\overline{MM} = \overline{AA} + z(\overline{AB} + \overline{BA}) + z^2(\overline{AC} + B\overline{B} + C\overline{A}).
\]
First we see that \( \overline{AA} = E \) since \( A \) has the factorization \((3.12)\) and any factor \( X \) thereof satisfies \( X\overline{X} = E. \) By Corollary 3.3 we have \( B = D_2PD_1 - \ell A - \ell^{-1}C. \) Let us denote \( \ell^{-1}C = (\delta_{i, j, 2n})_{i, j, 2n} \) by \( \overline{H}. \) Then we have
\[
\overline{AB} = \overline{A}\overline{D}_1\overline{PD}_2 - \ell \overline{E} - A\overline{H},
\]
\[
\overline{BA} = \overline{D}_2\overline{PD}_1\overline{A} - \ell \overline{E} - H\overline{A}.
\]
Here we have used \( \overline{AA} = E \) and \( \overline{H} = H. \) It is easy to see that \( \overline{AD}_1\overline{PD}_2 = H\overline{A}, \overline{D}_2\overline{PD}_1\overline{A} = A\overline{H}, \) and thereby \( \overline{AB} + \overline{BA} = -2\ell E. \)

It remains to check the desired relation for one special value of \( z \) other than \( z = 0. \) Let us put \( z = \ell^{-1}. \) By Proposition 3.4 we have \( \overline{MM} = \ell^{-2}\overline{D}_2\overline{P}\overline{D}_1\overline{P}\overline{D}_2. \) By using \( \overline{AA} = E \) we see \( \overline{PD}_1\overline{P}\overline{D} = O, \) where \( O \) is the zero matrix. The proof is completed. \( \square \)

Corollary 3.7. \( \det M(x, z) = (1 - z\ell(x))^{2n}. \)

On the set of variables of the geometric crystal \( B \) we introduce involutive automorphisms \( \sigma_1, \sigma_n, \) and \( \tau \) by
\[
(3.15) \quad \sigma_1 : x_1 \mapsto \overline{x}_1,
\]
\[
(3.16) \quad \sigma_n : x_{n-1} \mapsto x_{n-1}x_n, \quad \overline{x}_{n-1} \mapsto \overline{x}_{n-1}x_n, \quad x_n \mapsto 1/x_n,
\]
\[
(3.17) \quad \tau : x_i \mapsto \overline{x}_i, \quad \overline{x}_i \mapsto x_i, \quad (1 \leq i \leq n - 2)
\]
\[
\quad x_{n-1} \mapsto \overline{x}_{n-1}x_n, \quad \overline{x}_{n-1} \mapsto x_{n-1}x_n, \quad x_n \mapsto 1/x_n.
\]
Note that \( \sigma_1, \sigma_n, \) and \( \tau \) are mutually commutative. For simplicity we write \( x_{\sigma_1} \) to mean \( \sigma_1(x), \) etc. We can easily check

Lemma 3.8.

(3.18) \( \varepsilon_1(x^{\sigma_1}) = \varepsilon_0(x), \quad \varphi_1(x^{\sigma_1}) = \varphi_0(x), \quad e_1^{\sigma_1}(x^{\sigma_1}) = (e_0^{\sigma_1}(x))^{\sigma_1}, \)

(3.19) \( \varepsilon_{n-1}(x^{\sigma_n}) = \varepsilon_n(x), \quad \varphi_{n-1}(x^{\sigma_n}) = \varphi_n(x), \quad e_{n-1}^{\sigma_n}(x^{\sigma_n}) = (e_n^{\sigma_n}(x))^{\sigma_n}. \)

The other \( \varepsilon_i \) and \( \varphi_i \) are invariant under \( \sigma_1 \) and \( \sigma_n. \)

Let us introduce the \( 2n \) by \( 2n \) matrices:
\[
(3.20) \quad J_1(z) = zE_{1, 2n} + z^{-1}E_{2n, 1} + \sum_{i=2}^{2n-1}E_{i, i},
\]
\[
(3.21) \quad J_n = E_{n, n+1} + E_{n+1, n} + \sum_{i=1}^{n-1}(E_{i, i} + E_{2n+1-i, 2n+1-i}),
\]
\[
(3.22) \quad J = \sum_{i=1}^{2n}E_{i, 2n+1-i}.
\]

The following lemma is immediate.

Lemma 3.9.

(3.23) \( J_1(z)G_1(a) = G_0(\frac{a}{z})J_1(z), \)

(3.24) \( J_nG_{n-1}(a) = G_n(a)J_n. \)
Lemma 3.10.

(3.25) \[ J_1(z)M(x, z)J_1(z) = M(x^{\sigma_1}, z), \]
(3.26) \[ J_n M(x, z)J_n = M(x^{\sigma_n}, z), \]
(3.27) \[ J^1 M(x, z)J = M(\tau(x), z). \]

**Proof.** First we prove (3.25). In this case, only the first and \(2n\)-th rows and columns are changed. Let us consider \(J_1(z)A(x)J_1(z)\). All the elements in the first column and the \(2n\)-th row of this matrix are zero except the \((1,1)\) and \((2n,2n)\) elements. And we have

\[
(J_1(z)A(x)J_1(z))_{1,i} = \begin{cases} 
\frac{\tau_1}{x_1} & \text{for } i = 1, \\
\ell(x)z^2 & \text{for } i = 2n, \\
A(x)_{2n,i}z & \text{for } i \neq 1,2n,
\end{cases}
\]

\[
(J_1(z)A(x)J_1(z))_{i,2n} = \begin{cases} 
\ell(x)z^2 & \text{for } i = 1, \\
x_1/\tau_1 & \text{for } i = 2n, \\
A(x)_{i,1}z & \text{for } i \neq 1,2n.
\end{cases}
\]

Note that \(\tau_1/x_1 = A(x^{\sigma_1})_{1,1}, A(x)_{2n,i} = B(x^{\sigma_1})_{1,i} (i \neq 1, 2n), x_1/\tau_1 = A(x^{\sigma_1})_{2n,2n}\), and \(A(x)_{i,1} = B(x^{\sigma_1})_{i,2n} (i \neq 1, 2n)\). Here we have used

\[
B(x)_{1,i} = \begin{cases} 
0 & \text{for } i = 1,2n, \\
A(x^{\sigma_1})_{2n,i} & \text{for } i \neq 1,2n,
\end{cases}
\]

\[
B(x)_{i,2n} = \begin{cases} 
0 & \text{for } i = 1,2n, \\
A(x^{\sigma_1})_{i,1} & \text{for } i \neq 1,2n,
\end{cases}
\]

which is obtained from Corollary 3.3. Let us consider \(J_1(z)B(x)J_1(z)\). From Corollary 3.3 we find \(B(x)_{2n,i} = B(x)_{i,1} = 0\) for \(1 \leq i \leq 2n\). Therefore all elements in the first row and the \(2n\)-th column of this matrix are zero. And we have

\[
(J_1(z)B(x)J_1(z))_{2n,i} = \begin{cases} 
0 & \text{for } i = 1,2n, \\
B(x)_{1,i}z^{-1} & \text{for } i \neq 1,2n,
\end{cases}
\]

\[
(J_1(z)B(x)J_1(z))_{i,1} = \begin{cases} 
0 & \text{for } i = 1,2n, \\
B(x)_{i,2n}z^{-1} & \text{for } i \neq 1,2n.
\end{cases}
\]

Note that \(B(x)_{1,i} = A(x^{\sigma_1})_{2n,i}\) and \(B(x)_{i,2n} = A(x^{\sigma_1})_{i,1}\) for \(i \neq 1,2n\). Let us consider \(J_1(z)C(x)J_1(z)\). The only nonzero element of this matrix is the \((2n,1)\) element that takes the value \(\ell(x)z^{-2}\). Putting all the above relations together we obtain (3.25).

Next let us consider (3.26). From (3.12) we have

\[
J_n A(x)J_n = F_1(\tau_1)F_2(\tau_2) \cdots F_{n-2}(\tau_{n-2})F_{n-1}(\tau_{n-1})d(x^{\sigma_n})
\]

\[
\times F_n(x_{n-1})F_{n-2}(x_{n-2}) \cdots F_2(x_2)F_1(x_1).
\]

Since \(F_{n-1}(\tau_{n-1})d(x^{\sigma_n}) = d(x^{\sigma_n})F_{n-1}(x_{n-1}x_n)\) and \(d(x^{\sigma_n})F_n(x_{n-1}) = F_n(\)}
\[ \overline{x}_{n-1} x_n d(x^\sigma) \] we obtain \( J_n A(x) J_n = A(x^\sigma) \). Note that \( F_{n-1} \) and \( F_n \) are commutative. Clearly we have \( J_n C(x) J_n = C(x) = C(x^\sigma) \). Then by Corollary 3.3 we obtain \( J_n B(x) J_n = B(x^\sigma) \). The relation (3.26) is proved.

Finally we show (3.27). From (3.12) we have

\[
J^i A(x) J = F_1(x_1) F_2(x_2) \cdots F_{n-2}(x_{n-2}) F_{n-1}(x_{n-1}) d(\tau(x)) \\
\times \quad F_n(\overline{x}_{n-1}) F_{n-2}(\overline{x}_{n-2}) \cdots F_2(\overline{x}_2) F_1(\overline{x}_1).
\]

Since \( F_{n-1}(x_{n-1}) d(\tau(x)) = d(\tau(x)) F_{n-1}(\overline{x}_{n-1} x_n) \) and \( d(\tau(x)) F_n(\overline{x}_{n-1}) = F_n(x_{n-1} x_n) d(\tau(x)) \) we obtain \( J^i A(x) J = A(\tau(x)) \). Clearly we have \( J^i C(x) J = C(x) = C(\tau(x)) \). Then by Corollary 3.3 we can obtain \( J^i B(x) J = B(\tau(x)) \). The relation (3.27) is proved. \( \square \)

The following theorem states that the transformation \( e_i^c \) of the geometric \( D_n^{(1)} \)-crystal \( B \) is realized as the multiplications of unipotent matrices.

**Theorem 3.11.** For \( 0 \leq i \leq n \) we have

\[
(3.28) \quad G_i \left( \frac{c - 1}{z^i \phi_i(x)} \right) M(x, z) G_i \left( \frac{c^{-1} - 1}{z^i \phi_i(x)} \right) = M(e_i^c(x), z).
\]

**Proof.** First note that the \( i = n \) case can be derived from the \( i = n - 1 \) case as follows.

\[
M(e_n^c(x), z) = J_n M(e_{n-1}^c(x^\sigma), z) J_n \\
= J_n G_{n-1} \left( \frac{c - 1}{z_{n-1}(x^\sigma)} \right) M(x^\sigma, z) G_{n-1} \left( \frac{c^{-1} - 1}{\phi_{n-1}(x^\sigma)} \right) J_n \\
= G_n \left( \frac{c - 1}{z_n(x)} \right) J_n M(x^\sigma, z) J_n G_n \left( \frac{c^{-1} - 1}{\phi_n(x)} \right) \\
= G_n \left( \frac{c - 1}{z_n(x)} \right) M(x, z) G_n \left( \frac{c^{-1} - 1}{\phi_n(x)} \right).
\]

Here we have used (3.19), (3.24), (3.26). In a similar way the \( i = 0 \) case can be derived from the \( i = 1 \) case by using (3.18), (3.23), (3.25). Thus we can assume \( 1 \leq i \leq n - 1 \). The identity to be proved is of degree 2 with respect to \( z \). However, it is easy to see that the coefficients of \( z^2 \) in the both hand sides coincide. Thus it remains to check the identity with two distinct values of \( z \), which we take to be \( \ell(x)^{-1} \) and 0.

First let us put \( z = \ell(x)^{-1} \). From Proposition 3.4 the identity to be proved reads as

\[
(3.29) \quad G_i \left( \frac{c - 1}{z_i(x)} \right) D_2(x) P D_1(x) G_i \left( \frac{c^{-1} - 1}{\phi_i(x)} \right) = D_2(e_i^c(x)) P D_1(e_i^c(x)).
\]

This equality indeed holds since we have

\[
(3.30) \quad G_i \left( \frac{c - 1}{z_i(x)} \right) D_2(x) P = D_2(e_i^c(x)) P,
\]

\[
(3.31) \quad P D_1(x) G_i \left( \frac{c^{-1} - 1}{\phi_i(x)} \right) = P D_1(e_i^c(x)) ,
\]

\[14\]
or equivalently
\[ A(e^c_i(x))_{k,1} = \begin{cases} A(x)_{k,1} + A(x)_{k+1,1} \frac{c-1}{\varphi_1(x)} & \text{for } k = i, 2n - i, \\ A(x)_{k,1} & \text{otherwise}, \end{cases} \]
\[ A(e^c_i(x))_{2n,k} = \begin{cases} A(x)_{2n,k-1} \frac{c-1}{\varphi_1(x)} + A(x)_{2n,k} & \text{for } k = i + 1, 2n + 1 - i, \\ A(x)_{2n,k} & \text{otherwise}. \end{cases} \]

One can check them from (3.14) and (3.11).

Next let us put \( z = 0 \). We consider the \( i = 1 \) case. The other cases are similar. The identity to be proved reads as
\[ G_1 \left( \frac{c-1}{\varepsilon_1(x)} \right) A(x) G_1 \left( \frac{c-1}{\varphi_1(x)} \right) = A(e^c_1(x)). \]

Substitute the factorized form of \( A(x) \) (3.12) into the left hand side. Exchange the leftmost two matrices as
\[ G_1 \left( \frac{c-1}{\varepsilon_1(x)} \right) F_1(\xi_1) = F_1(\xi_1^{-1} \xi_1) G_1 \left( \frac{c-1}{\varepsilon_1(x)} \right). \]

Here \( \tilde{G}_1(a) \) is the matrix whose 1st and \((2n - 1)\)-th (resp. 2nd and 2n-th) diagonal elements are \( \xi_2 \) (resp. \( \xi_2^{-1} \)) and the other matrix elements are the same as those of \( G_1(a) \). Recall \( \xi_2 \) is defined in section 2.2. Then the next step is to exchange the second and third matrices as
\[ \tilde{G}_1 \left( \frac{c-1}{\varepsilon_1(x)} \right) F_2(\xi_2) = F_2(\xi_2 \xi_1) \tilde{G}_1 \left( \frac{c-1}{\varepsilon_1(x)} \right). \]

Similarly, for the rightmost three matrices we have
\[ F_2(x_2) F_1(x_1) G_1 \left( \frac{c-1}{\varphi_1(x)} \right) = \tilde{G}_1 \left( \frac{c-1}{\varphi_1(x)} \right) F_2(c^{-1} x_2) F_1(c^{-1} x_1). \]

Here \( \tilde{G}_1(a) \) is the matrix whose 1st and \((2n - 1)\)-th (resp. 2nd and 2n-th) diagonal elements are \( c \xi_2^{-1} \) (resp. \( c^{-1} \xi_2 \)) and the other matrix elements are the same as those of \( G_1(a) \). The other \( F_1 \)'s commute with \( \tilde{G}_1 \) and \( \tilde{G}_1 \). Thus the left hand side of (3.32) leads to
\[ F_1(\xi_2^{-1} \xi_1) F_2(\xi_2 \xi_1) \cdots \tilde{G}_1 \left( \frac{c-1}{\varepsilon_1(x)} \right) d(x) \tilde{G}_1 \left( \frac{c-1}{\varphi_1(x)} \right) \cdots F_2(c^{-1} x_2) F_1(c^{-1} x_1). \]

Thanks to
\[ d(x) \tilde{G}_1 \left( \frac{c-1}{\varphi_1(x)} \right) = \tilde{G}_1 \left( \frac{c-1}{\varepsilon_1(x)} \right)^{-1} d(e^c_1(x)), \]

we obtain the right hand side of (3.32).

The proof is completed.

3.2. **Product of** \( M(x, z) \). Throughout this subsection we consider the \( L \)-fold product \( B^{\times L} \) of the geometric \( D_n^{(1)} \)-crystal \( B \) given in section 2.2. Our main theorem here is Theorem 3.10 which states that \( B^{\times L} \) also becomes a geometric crystal.

**Theorem 3.12.** Let \( x = (x^1, \cdots, x^L) \) the set of variables of \( B^{\times L} \) and set \( e^c_i(x) = (y^1, \cdots, y^L) \). Then we have
\[ M(y^1, z) \cdots M(y^L, z) = G_1 \left( \frac{c-1}{\varepsilon_1(x)} \right) M(x^1, z) \cdots M(x^L, z) G_1 \left( \frac{c-1}{\varphi_1(x)} \right), \]

where \( G_1 \) is the \( i \)-th \( \varepsilon \)-function for \( B \).
for $0 \leq i \leq n$.

**Proof.** From Theorem 3.11 we have

$$M(y^l, z) = M(c_i^l(x^l), z) = G_i \left( \frac{c_i - 1}{\epsilon_i(x^l)} \right) M(x^l, z) G_i \left( \frac{c_i^{-1} - 1}{\epsilon_i(x^l)} \right),$$

for $1 \leq l \leq L$. Using (2.13)–(2.16) we can check

$$\frac{c_i^{-1} - 1}{\epsilon_i(x^l)} + \frac{c_i - 1}{\epsilon_i(x^l)} = 0 \quad \text{for} \quad 2 \leq l \leq L,$$

$$\frac{c_1 - 1}{\epsilon_i(x^1)} = \frac{c_L - 1}{\epsilon_i(x^L)} = \frac{c^{-1} - 1}{\epsilon_i(x)},$$

which finish the proof.

The following theorem plays a crucial role in the subsequent part of this paper. It asserts that we can retrieve components $x^i$ from the product of matrices $M(x^1, z) \cdots M(x^L, z)$.

**Theorem 3.13.** Let $x = (x^1, \ldots, x^L)$ and $y = (y^1, \ldots, y^L)$ be sets of variables of $B^\times L$. Suppose

$$\ell(x^i) = \ell(y^i) \quad (1 \leq i \leq L),$$

$$\ell(x^i) \neq \ell(x^j) \quad (i \neq j),$$

and

$$M(x^1, z) \cdots M(x^L, z) = M(y^1, z) \cdots M(y^L, z).$$

Then we have $x^i = y^i$ ($1 \leq i \leq L$).

**Proof.** Denote $\ell(x^1) = \ell(y^1)$ by $a^{-1}$. Let us put $z = a$ in (3.33). Using Proposition 3.4 we obtain

$$D_2(y^1)^{-1}D_2(x^1)P = P D_1(y^1) M(y^2, a) \cdots M(y^L, a) M(x^L, a)^{-1} \cdots M(x^2, a)^{-1} D_1(x^1)^{-1}.$$ 

Here we used the property that $\det M(x, z) \neq 0$ unless $z = \ell(x)^{-1}$. See Corollary 3.7. We see that the both sides of this equation should be a scalar multiple of $P$. From the $2n$-th row of the left hand side we see that the scalar should be 1. Hence we have $D_2(x^1) = D_2(y^1)$, that implies $x^1 = y^1$. In the same way we can deduce $x^i = y^i$ for all $i$.

We prepare lemmas. Let $x = (x^1, \ldots, x^L)$ be the set of variables of $B^\times L$. Define $x_{\sigma_a} = ((x^1)^{\sigma_a}, \ldots, (x^L)^{\sigma_a})$ for $a = 1, n$.

**Lemma 3.14.**

$$\epsilon_1(x^{\sigma_1}) = \epsilon_0(x), \quad \phi_1(x^{\sigma_1}) = \phi_0(x),$$

$$\epsilon_{n-1}(x^{\sigma_n}) = \epsilon_n(x), \quad \phi_{n-1}(x^{\sigma_n}) = \phi_n(x).$$

**Proof.** The $L = 1$ case is given in Lemma 3.8. By (2.13) and (2.14) we see that the claim is also true for $L \geq 2$. \hfill \Box
Lemma 3.15.
\[ e_i^e(x^{\sigma_1}) = (e_i^y(x))^{\sigma_1}, \]
\[ e_{n-1}^e(x^{\sigma_n}) = (e_{n-1}^y(x))^{\sigma_n}. \]

Proof. Denote the left (resp. right) hand side of the second relation by \( y = (y^1, \cdots, y^L) \) (resp. \( w = (w^1, \cdots, w^L) \)). We have
\[
M(y^1, z) \cdots M(y^L, z)
= G_{n-1} \left( \frac{c-1}{\varepsilon_{n-1}(y_x^{\sigma_1})} \right) M((x^1)^{\sigma_1}, z) \cdots M((x^L)^{\sigma_1}, z) G_{n-1} \left( \frac{c-1}{\varphi_{n-1}(x^{\sigma_1})} \right)
= G_{n-1} \left( \frac{c-1}{\varepsilon_n(x)} \right) J_n M(x^1, z) \cdots M(x^L, z) J_n G_{n-1} \left( \frac{c-1}{\varphi_n(x)} \right)
= J_n G_n \left( \frac{c-1}{\varepsilon_n(x)} \right) M(x^1, z) \cdots M(x^L, z) G_n \left( \frac{c-1}{\varphi_n(x)} \right) J_n
= M(w^1, z) \cdots M(w^L, z).
\]
Here we have used Lemma [3.9, 3.10, 3.14] and Theorem 3.13. Now Theorem 3.13 tells \( y = w \). The first relation is similar.

Our main theorem in this section is

**Theorem 3.16.** \( B^x \) is a geometric crystal. Namely,
\[ e_i^e(x) = e_i^y(x) \]
for \( \langle \alpha^y_1, \alpha_j \rangle = 0 \), and
\[ e_i^e(x) = e_i^y(x) \]
for \( \langle \alpha^y_1, \alpha_j \rangle = \langle \alpha^y_j, \alpha_i \rangle = -1 \).

Proof. We prove (3.34) first. Denote the left (resp. right) hand side of (3.35) by \( y = (y^1, \cdots, y^L) \) (resp. \( w = (w^1, \cdots, w^L) \)). It suffices to prove \( M(y^1, z) \cdots M(y^L, z) = M(w^1, z) \cdots M(w^L, z) \) by Theorem 3.13. Let us define
\[
P = G_i \left( \frac{d-i}{\varepsilon_i(e_i^d e_i^e(x))} \right) G_j \left( \frac{c-i}{\varepsilon_j(e_j^d e_j^e(x))} \right) G_i \left( \frac{d-i}{\varepsilon_i(x)} \right),
\]
\[
P' = G_j \left( \frac{c-i}{\varepsilon_j(e_i^{cd} e_j^e(x))} \right) G_i \left( \frac{c-i}{\varepsilon_i(e_j^d e_j^e(x))} \right) G_j \left( \frac{d-i}{\varepsilon_j(x)} \right),
\]
\[
Q = G_i \left( \frac{d-i-1}{\varphi_i(e_i^{cd} e_i^e(x))} \right) G_j \left( \frac{(cd)^{-1}-1}{\varphi_j(e_j^d e_j^e(x))} \right) G_i \left( \frac{d-i-1}{\varphi_i(x)} \right),
\]
\[
Q' = G_j \left( \frac{c-i-1}{\varphi_j(e_i^{cd} e_j^e(x))} \right) G_i \left( \frac{(cd)^{-1}-1}{\varphi_i(e_j^d e_j^e(x))} \right) G_j \left( \frac{d-i-1}{\varphi_j(x)} \right).
\]
For notational simplicity we have supposed \( i, j \neq 0 \). Otherwise the associated arguments should be divided by \( z \). From Theorem 3.12 we have
\[
M(y^1, z) \cdots M(y^L, z) = P M(x^1, z) \cdots M(x^L, z) Q^{-1},
\]
\[
M(w^1, z) \cdots M(w^L, z) = P' M(x^1, z) \cdots M(x^L, z) Q'^{-1}.
\]
We show $Q = Q'$ only, for $P = P'$ follows from this by using (2.1) and Definition 2.7 (iii). From (3.3) it suffices to check
\[ \varphi_i(x)\varphi_j(e_i^d(x)) = \varphi_i(e_j^d(x))\varphi_j(e_i^d(x)), \]
\[ \varphi_j(e_i^d(x))\varphi_i(e_j^d(x)) = \varphi_j(x)\varphi_i(e_j^d(x)), \]
\[ \frac{c^{-1} - 1}{\varphi_i(x)} + \frac{d^{-1} - 1}{\varphi_i(e_j^d(x))} = \frac{(cd)^{-1} - 1}{\varphi_i(e_j^d(x))}. \]

Let us prove the first identity. Using Lemma 2.9 (b) for the multiple product case we have
\[ \varphi_j(e_i^d e_j^d(x)) = \frac{\varphi_{ij}(e_i^d e_j^d(x)) + \varphi_{ji}(e_i^d e_j^d(x))}{\varphi_i(e_j^d(x))} = \frac{cd\varphi_{ij}(x) + d\varphi_{ji}(x)}{cd\varphi_i(e_j^d(x))}. \]

Therefore
\[ \varphi_i(e_j^d(x))\varphi_j(e_i^d(x)) = c^{-1}(c\varphi_{ij}(x) + \varphi_{ji}(x)) = c^{-1}\varphi_i(e_i^d(x))\varphi_j(e_i^d(x)) = \varphi_i(x)\varphi_j(e_i^d(x)). \]

The proof of the second identity is similar. Let us consider the third one. By similar calculation we have
\[ \varphi_i(e_i^d(x)) = \frac{\varphi_{ij}(x) + d\varphi_{ji}(x)}{d\varphi_i(x)}, \quad \varphi_i(e_i^d e_i^d(x)) = \frac{\varphi_{ij}(x) + d\varphi_{ji}(x)}{d} \times \frac{c\varphi_i(x)}{c\varphi_{ij}(x) + \varphi_{ji}(x)}. \]

Therefore the identity to be proved is written as
\[ \frac{c^{-1} - 1}{\varphi_i(x)} + \frac{(d^{-1} - 1)d}{\varphi_{ij}(x) + d\varphi_{ji}(x)} \cdot \frac{c\varphi_i(x) + \varphi_{ji}(x)}{c\varphi_i(x)} = \frac{((cd)^{-1} - 1)d\varphi_{ij}(x)}{\varphi_{ij}(x) + d\varphi_{ji}(x)}, \]

or equivalently
\[ (c^{-1} - 1)(\varphi_{ij}(x) + d\varphi_{ji}(x)) + (1 - d)(\varphi_{ij}(x) + c^{-1}\varphi_{ji}(x)) = (c^{-1} - d)\varphi_{ij}(x)\varphi_{ji}(x). \]

This equality holds since $\varphi_i(x)\varphi_j(x) = \varphi_{ij}(x) + \varphi_{ji}(x).$

The proof of (3.3) is much simpler due to Lemma 2.3 (a) for the multiple product case. The proof is completed.

We present an alternative proof of this theorem in appendix A.

4. Tropical $R$

4.1. Definition and basic properties. Let $M(x, z) = A(x) + zB(x) + z^2C(x)$ be the $2n$ by $2n$ matrix defined in the previous section. Consider the relation
\[ M(x, z)M(y, z) = M(x', z)M(y', z). \]

Given $x, y$, we regard it as a system of simultaneous equations for the $4n - 2$ unknowns $x', y'$ containing a generic parameter $z$.

**Theorem 4.1.** There is a unique solution of (4.4) under the constraints $\ell(x) = \ell(y') \neq \ell(y) = \ell(x')$. The map $(x, y) \rightarrow (x', y')$ is a birational transformation.
Lemma 4.3. Then we have
Proof. By noting that

Definition 4.2. The unique solution \((x',y')\) of (4.1) specified in Theorem 4.1 is denoted by \(R(x,y)\), and the birational transformation \((x,y) \to R(x,y)\) is called the tropical \(R\) for the geometric \(D^{(1)}\)-crystal \(B\).

In Theorem 4.1 uniqueness is immediate from Theorem 3.13. To show the existence, we introduce an explicit birational transformation \(\bar{R}\) in section 4.2. We then prove in section 4.3 that \(\bar{R}\) actually solves (4.1), hence \(R = \bar{R}\).

Here we exhibit basic properties of the tropical \(R\) by assuming its existence. Let \(x = (x_1,\ldots,x_t)\) and \(y = (y_1,\ldots,y_t)\) be sets of variables for the geometric \(D^{(1)}\)-crystal \(B\). On the pair \((x,y)\) we introduce mutually commuting involutive automorphisms:

\[
(x,y)^{\sigma_1} = (x^{\sigma_1}, y^{\sigma_1}), \quad (x,y)^{\sigma_n} = (x^{\sigma_n}, y^{\sigma_n}), \quad (x,y)^* = (x^*, y^*)
\]

in terms of \(\sigma_1,\sigma_n\) specified in (3.13) and (3.16). The automorphism \(*\) is the only one that mixes \(x\) and \(y\) and defined as

\[
* : x_i \longleftrightarrow y_i, \quad x_i \longleftrightarrow y_i \quad (1 \leq i \leq n-1), \quad x_n \longleftrightarrow y_n.
\]

Lemma 4.3. Set

\[
J_* = E_{n,n} + E_{n+1,n+1} + \sum_{i=1}^{n-1} (E_{2n+1-i} + E_{2n+1-i,i}).
\]

Then we have

\[
J_* M(y,z)J_* = M(x^*,z), \quad J_* M(x,z)J_* = M(y^*,z),
\]

\[
J_* D_1(y)J_* = D_2(x^*), \quad J_* D_1(x)J_* = D_2(y^*).
\]

Proof. By noting that \(x^* = \tau(y^{\sigma_n})\), \(y^* = \tau(x^{\sigma_n})\), and \(J_* = JJ_n\), the first two relations follow from (3.26) and (3.27). The latter relations are derived by considering the 1st column of the former.

Let us present basic properties of \(R\).

Proposition 4.4.

\[
R((x,y)^{\sigma_1}) = (R(x,y))^{\sigma_1},
\]

\[
R((x,y)^{\sigma_n}) = (R(x,y))^{\sigma_n},
\]

\[
R((x,y)^*) = (R(x,y))^*.
\]

Proof. Write the left (resp. right) side of (4.4) as \(((x^{\sigma_1})',(y^{\sigma_1})')\) (resp. \(((x')^{\sigma_1},(y')^{\sigma_1})\)). Then we have

\[
M((x^{\sigma_1})',z)M((y^{\sigma_1})',z) = M(x^{\sigma_1},z)M(y^{\sigma_1},z)
\]

\[
= J_1(z)M(x,z)M(y,z)J_1(z)
\]

\[
= J_1(z)M(x',z)M(y',z)J_1(z)
\]

\[
= M((x')^{\sigma_1},z)M((y')^{\sigma_1},z).
\]

Here we have used Lemma 3.11. Due to Theorem 3.13 we have (4.1), (4.5) can be shown similarly. Write the left (resp. right) side of (4.6) as \(((x')',(y')^*)\)
(resp. \((x')^*, (y')^*\)). Then we have
\[
M((x)^*, z)M((y)^*, z) = M(x^*, z)M(y^*, z)
\]
\[
= J_* (M(y, z))^i M(x, z)J_*
\]
\[
= J_* (M(x', z))^i M(y', z)J_*
\]
\[
= J_* (M(y', z))^i M(x', z)J_*
\]
\[
= M((x)^*, z)M((y')^*, z).
\]
Here we have used Lemma 4.3. Again Theorem 3.13 implies (4.6). \(\blacksquare\)

**Proposition 4.5.** \(\varepsilon_i(R(x, y)) = \varepsilon_i(x, y), \quad \varphi_i(R(x, y)) = \varphi_i(x, y)\).

**Proof.** For \(1 \leq i \leq n - 1\), Lemma 3.13 states that
\[
\varepsilon_i(x, y) = \frac{(M(x, 0)M(y, 0))_{i+1,i}}{(M(x, 0)M(y, 0))_{i,i}},
\]
\[
\varphi_i(x, y) = \frac{(M(x, 0)M(y, 0))_{i+1,i}}{(M(x, 0)M(y, 0))_{i,i}}.
\]

hence the claim follows. The case \(i = 0\) is reduced to \(i = 1\) as
\[
\varepsilon_0(R(x, y)) = \varepsilon_0((R(x^{0}, y^{0}))^{0})
\]
\[
= \varepsilon_0(R(x^{0}, y^{0}))
\]
\[
= \varepsilon_0(x^{0}, y^{0})
\]
\[
= \varepsilon_0(x, y).
\]

Here we have used Lemma 3.14 and Proposition 4.4. Similarly \(i = n\) case follows from \(i = n - 1\) case. \(\blacksquare\)

**Proposition 4.6.**
\[
e_i^c R = R e_i^c.
\]

**Proof.** For \(e_i^c(x, y) = (e_i^{c_1}(x), e_i^{c_2}(y))\) we write
\[
R(e_i^c(x, y)) = ((e_i^{c_1}(x))', (e_i^{c_2}(y))'),
\]
and for \(R(x, y) = (x', y')\),
\[
e_i^c(R(x, y)) = ((e_i^{c_1}(x'), e_i^{c_2}(y'))),
\]
where \(c_1' = \frac{\varphi_i(x') + \varepsilon_i(y')}{\varphi_i(x') + \varepsilon_i(y')}, c_2' = \frac{\varphi_i(x') + \varepsilon_i(y')}{\varphi_i(x') + \varepsilon_i(y')}\). By using Theorem 3.12 and Proposition 4.3 we have
\[
M((e_i^{c_1}(x))', z)M((e_i^{c_2}(y))', z) = M(e_i^{c_1}(x), z)M(e_i^{c_2}(y), z)
\]
\[
= G_i \left( \frac{c - 1}{z^{\delta_0 \varepsilon_i(x, y)}} \right) M(x, z)M(y, z) G_i \left( \frac{c - 1}{z^{\delta_0 \varphi_i(x, y)}} \right)
\]
\[
= G_i \left( \frac{c - 1}{z^{\delta_0 \varepsilon_i(x', y')}} \right) M(x', z)M(y', z) G_i \left( \frac{c - 1}{z^{\delta_0 \varphi_i(x', y')}} \right)
\]
\[
= M(e_i^{c_1}(x'), z)M(e_i^{c_2}(y'), z).
\]
Then Theorem 3.13 tells \((e_i^{c_1}(x))', (e_i^{c_2}(y))' = (e_i^{c_1}(x'), e_i^{c_2}(y'))\). \(\blacksquare\)
Proposition 4.7. \( R(R(x,y)) = (x,y) \).

Proof. Writing \( R(x,y) = (x',y') \) and \( R(x',y') = (x'',y'') \), we get \( M(x,z)M(y,z) = M(x',z)M(y',z) \) by definition. Then \( (x,y) = (x'',y'') \) owing to Theorem 3.13.

Finally we prove the Yang-Baxter equation. Let \( B \times B \times B \) be the 3-fold product of the geometric \( D_n^{(1)} \)-crystal \( B \). Denote by \( R_{12} \) (resp. \( R_{23} \)) the birational map on \( B \times B \times B \) that acts on the first (resp. last) two components as \( R \) and the other single component trivially.

Proposition 4.8. We have \( R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \) as a birational map on \( B \times B \times B \).

Proof. Let \( (x^1,x^2,x^3) \in B \times B \times B \). Writing \( (u^1,u^2,u^3) = R_{12}R_{23}R_{12}(x^1,x^2,x^3) \) and \( (v^1,v^2,v^3) = R_{23}R_{12}R_{23}(x^1,x^2,x^3) \), one has

\[
M(u^1,z)M(u^2,z)M(u^3,z) = M(x^1,z)M(x^2,z)M(x^3,z) = M(u^1,z)M(v^2,z)M(v^3,z).
\]

Thus \( (u^1,u^2,u^3) = (v^1,v^2,v^3) \) follows from Theorem 3.13.

4.2. Birational map \( \tilde{R} \). Let \( V_0 = (A(x)A(y))_{2n,1} \). Explicitly we have

\[
V_0 = \ell(x) \frac{y_1}{x_1} + \ell(x) \sum_{m=2}^{n-1} \left( \prod_{i=1}^{m-1} \frac{y_i}{x_i} \right) \left( 1 + \frac{y_m}{x_m} \right) x_n y_n + \ell(y) \frac{x_1}{y_1} + \ell(y) \sum_{m=2}^{n-1} \left( \prod_{i=1}^{m-1} \frac{x_i}{y_i} \right) \left( 1 + \frac{x_m}{y_m} \right) y_n x_n.
\]

Define \( V_i (1 \leq i \leq n - 1) \) by

\[
V_i = \frac{y_i}{x_i} V_{i-1} + (\ell(x) - \ell(y)) \left( 1 + \frac{y_i}{x_i} \right), \quad (1 \leq i \leq n - 2)
\]

\[
V_{n-1} = \frac{y_{n-1}}{x_{n-1}} V_{n-2} + (\ell(x) - \ell(y)) \left( 1 + \frac{y_{n-1}}{x_{n-1}} \right).
\]

Define \( W_i (1 \leq i \leq n - 1) \) by

\[
W_i = V_i V_i^* + (\ell(y) - \ell(x)) V_i^* + (\ell(x) - \ell(y)) V_i, \quad (1 \leq i \leq n - 2)
\]

\[
W_{n-1} = V_{n-1} V_{n-1}^*.
\]

Here \( V_i^* = V_i((x,y)^*) \), and similar notations will be used for \( \sigma_1, \sigma_n \) from now on.
Definition 4.9. A rational transformation specified below is denoted by \( \tilde{R}(x, y) = (x', y') \):

\[
x_1' = y_1 \frac{V_0}{V_1}, \quad \overline{x}_1 = y_1 \frac{V_0}{V_1}, \\
x_i' = y_i \frac{V_{i-1}W_i}{V_iW_{i-1}}, \quad \overline{x}_i = y_i \frac{V_{i-1}}{V_i}, \quad (2 \leq i \leq n-1) \\
x_n' = y_n \frac{V_{n-1}}{V_n}, \\
y_1' = x_1 \frac{V_0}{V_1}, \quad \overline{y}_1 = x_1 \frac{V_{0}^{\sigma_1}}{V_1}, \\
y_i' = x_i \frac{V_{i-1}^*W_i}{V_i^*W_{i-1}}, \quad \overline{y}_i = x_i \frac{V_{i-1}^*}{V_i^*}, \quad (2 \leq i \leq n-1) \\
y_n' = x_n \frac{V_{n-1}^*}{V_n}. 
\]

To provide an explicit formula for \( V_i \)'s, we introduce

\[
\theta_{i,j}(x, y) = \begin{cases} 
\ell(x) \prod_{k=j+1}^{i} \frac{y_k}{x_k} & \text{for } 1 \leq j \leq i, \\
\ell(y) \prod_{k=i+1}^{j} \frac{x_k}{y_k} & \text{for } i + 1 \leq j \leq n-2,
\end{cases}
\]

\[
\theta_{i,j}'(x, y) = \ell(x) \left( \prod_{k=1}^{i} \frac{y_k}{x_k} \right) \left( \prod_{k=1}^{j} \frac{y_k}{x_k} \right) \quad \text{for } j = 1, \ldots, n-2,
\]

\[
\eta_{i,j}(x, y) = \begin{cases} 
\ell(x) \left( \prod_{k=j+1}^{i} \frac{y_k}{x_k} \right) \left( \frac{y_j}{x_j} \right) & \text{for } 1 \leq j \leq i, \\
\ell(y) \left( \prod_{k=i+1}^{j} \frac{x_k}{y_k} \right) \frac{y_j}{x_j} & \text{for } i + 1 \leq j \leq n-1, \\
\ell(y) \left( \prod_{k=1}^{n-1} \frac{x_k}{y_k} \right) x_n & \text{for } j = n,
\end{cases}
\]

\[
\eta_{i,j}'(x, y) = \begin{cases} 
\ell(x) \left( \prod_{k=1}^{i} \frac{y_k}{x_k} \right) \left( \prod_{k=k+1}^{j} \frac{y_k}{x_k} \right) \left( \frac{x_j}{y_j} \right) & \text{for } 1 \leq j \leq n-1, \\
\ell(x) \left( \frac{\ell(x)}{\ell(y)} \right) \delta_{i,n-1} \left( \prod_{k=1}^{i} \frac{y_k}{x_k} \right) \left( \prod_{k=1}^{n-1} \frac{y_k}{x_k} \right) \left( \frac{1}{x_n} \right) & \text{for } j = n.
\end{cases}
\]

Proposition 4.10. For \( 0 \leq i \leq n-1 \) one has

\[
V_i = \sum_{j=1}^{n-2} \left( \theta_{i,j}(x, y) + \theta_{i,j}'(x, y) \right) \sum_{j=1}^{n} \left( \eta_{i,j}(x, y) + \eta_{i,j}'(x, y) \right).
\]
Proof. For \( i = 0 \) the above expression agrees with (5.7). Note that \( \eta_0,n(x,y) = \left(\prod_{i=1}^{n-1} \pi_i y_i\right) x_n y_n \) and \( \eta_0^n(x,y) = \left(\prod_{i=1}^{n-1} \pi_i y_i\right) \). For \( 1 \leq i \leq n-2 \) the recursion relation (4.8) is valid since the nonzero contribution to the difference \( V_i - V_{i-1} \) only comes from \( \theta_i(x,y) - \frac{\eta_0}{\pi_i} \theta_{i-1}(x,y) = \ell(x) - \ell(y) \) and \( \eta_i(x,y) - \frac{\eta_0}{\pi_i} \eta_{i-1}(x,y) = (\ell(x) - \ell(y)) \frac{\pi_i}{\pi_y} \). Similarly (4.9) is checked by \( \eta_{n-1,n-1}(x,y) - \frac{\eta_0}{\pi_y} \eta_{n-2,n-1}(x,y) = (\ell(x) - \ell(y)) \frac{\pi_y}{\pi_{n-1}} \) and \( \eta'_{n-1,n}(x,y) - \frac{\eta_0}{\pi_y} \eta'_{n-2,n}(x,y) = (\ell(x) - \ell(y)) \frac{1}{\pi_y} \). \qed

Proposition 4.10 with \( i = n-1 \) reads

\[
V_{n-1} = x_1 y_1 x_n \left(\prod_{m=2}^{n-1} x_m y_m\right) + \ell(y) \sum_{m=2}^{n-1} \left(\prod_{i=m}^{n-1} x_i y_i\right) \left(1 + \frac{y_m}{y_i}\right) + \ell(x) x_n + \left(\prod_{m=2}^{n-1} x_m y_m\right) + \ell(x) \sum_{m=2}^{n-1} \left(\prod_{i=m}^{n-1} x_i y_i\right) \left(1 + \frac{y_m}{x_i}\right),
\]

from which \( V_{n-1} = V_{n-1}^\sigma = (V_{n-1}^*)^\sigma \) is easily seen. Similarly from (4.7) one finds \( V_0 = V_0^{\sigma_0} = V_0^* \). Starting from these properties one can use (4.8)–(4.11) to figure out the transformation property of \( V_i, W_i \) under the commuting automorphisms \( \sigma_1, \sigma_n \) and \(*\). The result is summarized in

| Table 1. Transformation by automorphisms and \( \tilde{R} \). |
|-----------------------------------------------|
| \( \sigma_1 \) | \( V_0^{\sigma_1} \) | \( V_1 \) | \( V_{n-1} \) | \( W_i \) |
| \( \sigma_n \) | \( V_0 \) | \( V_i \) | \( V_{n-1}^* \) | \( W_i \) |
| \( \ast \) | \( V_0 \) | \( V_i^* \) | \( V_{n-1}^* \) | \( W_i \) |
| \( \tilde{R} \) | \( W_i/V_i^* \) | \( V_{n-1} \) | \( W_i \) |

The transformation properties under \( \tilde{R} \) will be shown in Lemma 4.17.

**Proposition 4.11.**

\[
\tilde{R}((x,y)^{\sigma_1}) = (\tilde{R}(x,y))^{\sigma_1},
\]

\[
\tilde{R}((x,y)^{\sigma_n}) = (\tilde{R}(x,y))^{\sigma_n},
\]

\[
\tilde{R}((x,y)^*) = (\tilde{R}(x,y))^*.
\]

**Proof.** Apply Table 1 to Definition 4.9. \qed

**Lemma 4.12.** \( W_1 = V_0 V_0^{\sigma_1} \).

**Proof.** Consider the identities \( V_0 = \frac{\pi_1}{x_1} V_1 + (\ell(y) - \ell(x)) \frac{x_1}{y_1} \left(1 + \frac{x_1}{y_1}\right), V_0^{\sigma_1} = \frac{\pi_1}{x_1} V_1^* + (\ell(x) - \ell(y)) \frac{x_1}{y_1} \left(1 + \frac{x_1}{y_1}\right), \) which are equivalent to (4.8) and \(*\circ \sigma_1\) of (4.8) with \( i = 1 \). Thus we obtain \( V_0 V_0^{\sigma_1} = V_1 V_1^* - (\ell(y) - \ell(x)) V_1^* - (\ell(x) - \ell(y)) V_1 = (\ell(y) - \ell(x)) \frac{\pi_1}{x_1} V_1^* + (\ell(x) - \ell(y)) \frac{y_1}{x_1} V_1 - (\ell(x) - \ell(y))^2 \frac{\pi_1}{x_1} \left(1 + \frac{x_1}{y_1}\right) \left(1 + \frac{x_1}{y_1}\right). \) The right hand side vanishes due to (4.8) and \(*\) of (4.8) for \( i = 1 \). \qed

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Lemma 4.13.

\begin{equation}
W_i = \frac{x_i y_i}{x_i y_i} V_{i-1} V_i^* + (\langle x \rangle - \langle y \rangle) \frac{y_i}{y_i} V_i^* V_{i-1} + (\langle y \rangle - \langle x \rangle) \frac{x_i}{x_i} V_i^* V_{i-1},
\end{equation}

for \(1 \leq i \leq n - 1\).

Proof. For \(i = 1\) this relation reduces to \(V_{0}^{\sigma_1} = \frac{x_1 y_1}{x_1 y_1} + (\langle x \rangle - \langle y \rangle) \frac{y_1}{y_1} + (\langle y \rangle - \langle x \rangle) \frac{x_1}{x_1}\), which follows from (4.8) and \(\sigma_1\) of (4.8) with \(i = 1\). For \(2 \leq i \leq n - 2\) the relation is shown by substituting (4.8) and \(*\) of (4.8) into (1.10). Next consider \(i = n - 1\). From (1.3) and \(*\) of (4.9) one has \(\text{LHS} - \text{RHS} = (\langle x \rangle - \langle y \rangle) \frac{y_{n-1}}{y_{n-1}} V_{n-2}^* + (\langle y \rangle - \langle x \rangle) \frac{x_{n-1}}{x_{n-1}} V_{n-2} - (\langle x \rangle - \langle y \rangle)^2 \left(1 + \frac{x_{n-1}}{y_{n-1}} \right) \left(1 + \frac{y_{n-1}}{x_{n-1}} \right)\). The right hand side vanishes because of (4.9) and \(*\) of (4.9).

Lemma 4.14.

\begin{equation}
\left(1 + \frac{1}{x_i y_i}\right) W_i = \frac{1}{y_i} V_i V_i^* + \frac{1}{x_i} V_{i-1} V_i^* \quad (1 \leq i \leq n - 2).
\end{equation}

Proof. For \(i = 1\) this relation reduces to \(\left(1 + \frac{1}{x_1 y_1}\right) V_0^{\sigma_1} = \frac{V_0^*}{x_1} + \frac{V_0 y_1}{y_1}\), which follows from \(\sigma_1\) of (4.8) and \(\sigma_1 \ast \) of (4.8) with \(i = 1\). To see the other cases, substitute (4.8) and \(*\) of (4.8) into the right hand side of (1.14). The result is the right hand side of (1.16) multiplied by \(\left(1 + \frac{1}{x_i y_i}\right)\).

Remark 4.15. We call a rational function subtraction-free, if its denominator and numerator are polynomials with nonnegative coefficients. From Proposition 4.14, Lemma 4.12, 4.13 and 4.11, all the functions \(V_i, W_i\) appearing in Definition 4.3 are subtraction-free Laurent polynomials in \(x, y\). Thus \(\hat{R}\) is also subtraction-free. This property will be used in section 4.3.

Proposition 4.16. \(\hat{R}(x, y) = (x', y')\) solves the equation \(A(x)A(y) = A(x')A(y')\).

A proof by a direct but lengthy calculation is available in appendix 3.

Lemma 4.17. \(V_0, V_0^{\sigma_1}, V_{n-1}, V_{n-1}^{\sigma_1}, W_i \quad (1 \leq i \leq n - 1)\) are invariant under \(\hat{R}\). \(\hat{R}\) acts as \(V_i \rightarrow W_i/V_i^*\) and \(V_i^* \rightarrow W_i/V_i\) \(\quad (1 \leq i \leq n - 1)\).

Proof. Proposition 4.16 tells that \(V_0 = (A(x)A(y))_{2n, 1}\) is invariant. Then Proposition 4.11 tells \(V_0^{\sigma_1}\) and thereby \(W_1 = V_0 V_0^{\sigma_1}\) are also invariant. Consider \(V_i\) and \(V_i^*\). By applying \(\hat{R}(x, y) = (x', y')\) to (4.8) we have \(\hat{R}(V_i) = \frac{y_i}{x_i} V_0 + (\langle x \rangle - \langle y \rangle) \left(1 + \frac{y_i}{x_i}\right)\). By substituting (4.17) and (1.8) with \(i = 1\) into its right hand side we find \(\hat{R}(V_i) = V_0^{\sigma_1} V_{n-1} V_i^* = W_i / V_i^*\). Hence by Proposition 4.11 we also have \(\hat{R}(V_i^*) = W_i / V_i\).

The claim for \(V_i\) and \(V_i^*\) with \(2 \leq i \leq n - 2\) is proved by induction on \(i\). Suppose that the claim is true for \(V_{i-1}\). By applying \(\hat{R}(x, y) = (x', y')\) to (4.8) we have \(\hat{R}(V_i) = \frac{y_i}{x_i} W_{i-1} + (\langle x \rangle - \langle y \rangle) \left(1 + \frac{y_i}{x_i}\right)\). By substituting (4.17) and (1.8) into the right hand side we obtain \(\hat{R}(V_i) = W_i / V_i^*\). Hence by Proposition 4.11 we also have \(\hat{R}(V_i^*) = W_i / V_i\) for \(2 \leq i \leq n - 2\).
Let us consider $W_i$ for $2 \leq i \leq n - 1$. By applying $\tilde{R}(x, y) = (x', y')$ to (4.16) we have $\tilde{R}(W_i) = \frac{w_i x_i}{y_i} \left( \frac{W_j}{W_{i-1}} \right)^2 + (\ell(x) - \ell(y)) \left( \frac{W_j}{W_{i-1}} \right) + (\ell(x) - \ell(y)) \frac{W_{i-1}}{y_i}$. Dividing both hand sides by $W_i$ and substituting (4.16) into the right hand side we find $\tilde{R}(W_i)/W_i = 1$ for $2 \leq i \leq n - 1$.

From (4.16) and its $\sigma_n \circ *$, the identity $V_{n-1} \left( \frac{1}{x_{n-1}} + \frac{1}{y_{n-1}} \right) = \frac{V_{n-2}}{x_{n-1}} + \frac{V_{n-2}}{y_{n-1}}$ can be derived. Applying $\tilde{R}$ to this and using the identity once again we get $\tilde{R}(V_{n-1}) = V_{n-1}$. The $\tilde{R}$ also leaves $V_{n-1}$ invariant since $\tilde{R}$ and $*$ are commutative. \hfill $\square$

**Remark 4.18.** From the explicit formulas (4.7) and (4.12) we find that $V^\sigma_0 = (A(x)C(y))_{1,2n} + (B(x)B(y))_{1,2n} + (C(x)A(y))_{1,2n}$, $V_0 = (A(x)B(y))_{n,n+1} + (B(x)A(y))_{n,n+1}$, and $V^*_n = (A(x)B(y))_{n+1,n} + (B(x)A(y))_{n+1,n}$. Therefore they are invariant under the tropical $R$, and thus should be so also under $\tilde{R}$.

**Proposition 4.19.** $\tilde{R}(\tilde{R}(x, y)) = (x, y)$.

**Proof.** Let us illustrate the $x_2$ case. The other cases are similar. Under $\tilde{R}(x, y) = (x', y')$ $x_2$ becomes $x_2' = y_2 V_1 W_2/(V_2 W_1)$. By applying it once again it changes into $y_2' \tilde{R}(V_1)\tilde{R}(W_2) = x_2 \tilde{R}(V_1)\tilde{R}(W_2)$. Proposition 4.19 is the inversion relation of $\tilde{R}$, and will play a role in the proof of Lemma 4.23.

Set

(4.18) $\tilde{V}_j = \frac{1}{V_j} \left( W_j + \frac{y_{j+1}}{x_{j+1}} W_{j+1} \right), \quad (1 \leq j \leq n - 2)$.

**Lemma 4.20.**

\[
\tilde{V}_1 = V_0 \frac{y_1}{x_1} \tilde{\tau}_2 + (\ell(x) - \ell(y)) \left[ \frac{y_2}{y_1} \tilde{\tau}_2 + \frac{y_1}{x_1} \tilde{\tau}_2 \right],
\]

\[
\tilde{V}_j = \tilde{V}_{j-1} \frac{y_j}{x_j} \frac{y_{j+1}}{y_j} + (\ell(x) - \ell(y)) \left[ \frac{y_{j+1}}{y_j} \tilde{\tau}_j + \frac{y_j}{x_j} \tilde{\tau}_j \tilde{\tau}_{j+1} \right],
\]

\[
(2 \leq j \leq n - 2)
\]

where $\tilde{\tau}_i = 1 + \tilde{\tau}_i/x_i$, $\tilde{\tau}_i = 1 + \tilde{\tau}_i/y_i$, for $1 \leq i \leq n - 1$.

**Proof.** From (4.10) and (4.16) we get

\[
\tilde{V}_j = \tilde{V}_{j+1} V_j + (\ell(x) - \ell(y)) \left( \frac{\tilde{\tau}_{j+1} \tilde{\tau}_{j+1}}{x_{j+1} y_{j+1}} - 1 \right).
\]

This allows us to write $V_j$ in terms of $\tilde{V}_j$. Substitution of them into (4.18) leads to the desired relations. \hfill $\square$

**Remark 4.21.** The equation (4.18) gives an alternative definition of $W_j$’s in terms of $\tilde{V}_j$’s. Given $W_1$ and $\tilde{V}_j$’s one can determine all the other $W_j$’s from (4.18). This fact will be used in the proof of Lemma 4.23.
4.3. Proof of Theorem [4.1]. As mentioned after Definition [1.2], Theorem [4.1] is established at the same time with the explicit form $R = \bar{R}$ once it is shown that $(x', y') = R(x, y)$ solves the defining equation (4.1) of the tropical $R$. In view of $B(x)C(y) + C(x)B(y) = C(x)C(y) = O$, we have the expansion

$$M(x, z)M(y, z) = A(x)A(y) + z(A(x)B(y) + B(x)A(y)) + z^2(A(x)C(y) + B(x)B(y) + C(x)A(y)).$$

Since the equation (4.1) is quadratic with respect to $z$, it suffices to check it for three distinct values of $z$. We have already done it at $z = 0$ in Proposition [4.10]. In what follows we treat the cases $z = \ell(x)^{-1}$ and $z = \ell(y)^{-1}$.

Let us put $z = \ell(x)^{-1} = \ell(y)^{-1}$ in (4.1). Proposition [3.10] leads to

$$PD_1(x)M(y, \ell(x)^{-1})D_1(y')^{-1} = D_2(x)^{-1}M(x', \ell(x)^{-1})D_2(y')P.$$ (4.19)

Thus the both sides is a scalar multiple of $P$. Denoting the scalar by $\alpha$ we see that (4.19) is equivalent to the simultaneous equations

$$PD_1(x)M(y, \ell(x)^{-1}) = \alpha PD_1(y'),$$ (4.20)

$$M(x', \ell(x)^{-1})D_2(y')P = \alpha D_2(x)P.$$ (4.21)

Equation (4.21) will be considered later (Lemma [4.22]).

Lemma 4.22. The following $y'_i$’s and $\overline{y}_i$’s solve the equation (4.24).

$$y'_i = \frac{Q_i}{Q_{i+1}} (1 \leq i \leq n), \quad \overline{y}_i = \ell(x)\frac{Q_{2n}}{Q_2},$$ (4.22)

$$\overline{y}_i' = -\sum_{k=1}^{i} (-1)^k Q_k Q_{2n+1-k}, \quad (2 \leq i \leq n - 1)$$ (4.23)

where $Q_i = (A(x)M(y, \ell(x)^{-1}))_{2n,i}$.

Proof. All elements in the $i$th column of the LHS (resp. RHS) of (4.20) are equal to $Q_i$ (resp. $\alpha A(y')_{2n,i}$). Considering the first column fixes the scalar as $\alpha = Q_1/A(y')_{2n,1} = Q_1/\ell(x)$. Taking the ratios of the $i$th and the $i + 1$th column for $1 \leq i \leq n$ leads to $y'_i = Q_i/Q_{i+1}$. Similarly the second column and the last column imply $\overline{y}_i = \ell(x)Q_{2n}/Q_2$, and the $(2n - 1)$th column does $Q_{2n-1} = \alpha \left(1 + \frac{\overline{y}_i}{y'_i}\right)\overline{y}_i'$. Therefore we have $1 + \frac{\overline{y}_i}{y'_i} = Q_{i+1}Q_{2n}/Q_{2n}$ or equivalently (4.23) for $i = 2$. Now let us show

$$1 + \frac{\overline{y}_i}{y'_i} = (-1)^{i+1} \frac{Q_i Q_{2n+1-i}}{\sum_{k=1}^{i-1} (-1)^k Q_k Q_{2n+1-k}},$$ (4.24)

or equivalently (4.23) for $3 \leq i \leq n - 1$. This claim is checked by induction on $i$. We have already shown $i = 2$ case. Assume the identity (4.23) with $i$ replaced by $i - 1$. Then we have

$$\frac{Q_{2n+1-i}}{Q_{2n+2-i}} = \left(1 + \frac{\overline{y}_i}{y'_i}\right)\frac{y'_{i-1}}{y'_{i-1}} = \left(1 + \frac{\overline{y}_i}{y'_i}\right)\frac{\sum_{k=1}^{i-1} (-1)^k Q_k Q_{2n+1-k} Q_{i-1}}{(-1)^{i+1} Q_{i-1} Q_{2n+2-i} Q_i},$$

which is equivalent to (4.24). \hfill $\square$

Lemma 4.23. The $y'_i$’s and $\overline{y}_i$’s specified in Lemma [4.22] coincide with those given by $\bar{R}(x, y) = (x', y')$ in Definition [4.2].
\[ \Delta V_0 = Q_1, \quad \Delta V_i^* = x_1 \ldots x_i Q_{i+1} \quad (1 \leq i \leq n - 1), \]
\[ \Delta V_{n-1} = x_1 \ldots x_{n-1} x_n Q_{n+1}, \quad \Delta V_{n}^* = \ell(x) \frac{x_1}{x_1} Q_2, \]
\[ \Delta W_i = \ell(x) \frac{x_1 \ldots x_i}{x_1 \ldots x_i} (-1)^i \sum_{k=1}^{i} (-1)^k Q_k Q_{2n-1-k} \quad (1 \leq i \leq n - 1). \]

We show that these quantities coincide with the same symbols without \( \Delta \).

First note \( \Delta V_0 = (A(x) A(y))_{n+1} = V_0 \) from (1.1). It is not difficult to see that \( \Delta V_0, \Delta V_{n}^*, \Delta V_i^* \), and \( \Delta V_{n-1} \) satisfy the same recursion relations as those without \( \Delta \). For instance consider
\[ \Delta V_i^* - \frac{x_i}{y_i} \Delta V_{i-1}^* = x_1 \ldots x_i \sum_{k=1}^{2n} A(x)_{2n,k} \left[ M(y, \ell(x)^{-1})_{k,i-1} = M(y, \ell(x)^{-1})_{k,i} \right]. \]

By using (4.3) and \( A(x)_{2n,i} = \ell(x)/(x_1 \ldots x_i) \), this becomes
\[ \Delta V_i^* = \frac{x_i}{y_i} \Delta V_{i-1}^* + (\ell(y) - \ell(x)) \left( 1 + \frac{x_i}{y_i} \right), \]
which is \( * \) of (1.8) for some scalar \( \beta \). The other cases are similar. Note that \( V_{0}^* \) is related to \( V_i^* \) by \( * \sigma_i \) of (1.3) and \( V_{n-1} \) to \( V_{n-2}^* \) by \( * \sigma_n \) of (1.3). By repeated use of the recursion relations, \( \Delta V_0^*, \Delta V_i^* \), and \( \Delta V_{n-1} \) are related to \( \Delta V_0 \) in the same way as those without \( \Delta \) are to \( V_0 \), proving the coincidence.

So far we have considered \( \Delta V_0, \Delta V_{n}^*, \Delta V_i^*, \) and \( \Delta V_{n-1} \). Next we treat \( \Delta W_i \). Note \( \Delta W_1 = \Delta V_0 \Delta V_{n}^* = V_0 V_{n}^* = W_1 \). To verify \( \Delta W_i = W_i \) for \( i \geq 2 \) we define \( \Delta V_j = \frac{\ell(x)}{x_1 \ldots x_j} Q_{2n-j} \) for \( 1 \leq j \leq n - 2 \). Then
\[ \Delta V_j = \frac{1}{\Delta V_{j-1}} \left( \Delta W_j + \frac{\bar{y}_j}{y_j} \Delta W_{j+1} \right) \]
holds for \( 1 \leq j \leq n - 2 \). This is (4.18). Since we have already shown \( \Delta W_1 = W_1 \) and \( \Delta V_j = V_j \), it suffices to show \( \Delta V_j = V_j \) to verify \( \Delta W_i = W_i \) for \( i \geq 2 \). (See Remark 4.21) Thus it remains to check that \( \Delta V_0 \) and \( \Delta V_j \)'s satisfy the recursion relations in Lemma 4.20. For instance consider
\[ \Delta V_j - \Delta V_{j-1} \frac{\bar{y}_j}{y_j} \frac{\bar{y}_{j+1}}{y_{j+1}} = \frac{\ell(x)}{x_1 \ldots x_j} \sum_{k=1}^{2n} A(x)_{2n,k} \mathcal{M}(y, \ell(x)^{-1})_{k,j}, \]
where \( \mathcal{M}(y, z)_{k,j} = M(y, z)_{k,2n-j} - \frac{\bar{y}_j}{y_j} \sum_{k=1}^{\infty} M(y, z)_{k,2n+1-j} \). By using (4.10) and \( A(x)_{2n,2n+1-j} = \bar{y}_{j-1} \frac{\ell(x)}{x_1 \ldots x_{j-1} (1 + \bar{y}_j/x_j)} \) we obtain the desired relation. \( \Box \)

Let us proceed to \( z = \ell(y)^{-1} \) case. Setting \( z = \ell(y)^{-1} = \ell(x)^{-1} \) in (4.1) leads to
\[ D_2(x^{'})^{-1} M(x, \ell(y)^{-1}) D_2(y) P = PD_1(x') M(y', \ell(y)^{-1}) D_1(y)^{-1}. \]
As (4.19), (4.23) is equivalent to the simultaneous equations
\[ D_2(x')^{-1} M(x, \ell(y)^{-1}) D_2(y) P = \beta D_2(x') P, \]
\[ PD_1(x') M(y', \ell(y)^{-1}) = \beta PD_1(y) \]
for some scalar \( \beta \). Equation (4.24) will be considered later (Lemma 4.23).
Lemma 4.24. The relation $\tilde{R}(x,y) = (x',y')$ solves the equation (4.20) with $\beta = Q_1^*/\ell(y)$.

Proof. Take the transposition of (4.20) and multiply $J$'s from the both sides. Due to Lemma 4.3 the result becomes

$$M(x^*, \ell(x)^{-1})D_2(y^*)P = \alpha D_2((x')^*)P.$$ 

From Proposition 4.11 we know $(x')^* = (x^*)'$. Thus replacing $(x,y)$ with $(x,y)^*$ in this relation yields (4.24).

Lemma 4.25. The relation $\tilde{R}(x,y) = (x',y')$ also solves (4.21) with $\alpha = V_0/\ell(x)$ and (4.27) with $\beta = V_0/\ell(y)$.

Proof. Recall $Q_1 = Q_1^* = V_0$ and its invariance under $\tilde{R}$ (Lemma 4.11). Due to Proposition 4.19 we can exchange $(x,y)$ with $(x',y')$ in (4.21) and (4.26), which become (4.27) and (4.21), respectively. Then the claim follows from Lemmas 4.23 and 4.24.

Proof of Theorem 4.4. Lemmas 4.22, 4.23, 4.24, and 4.27 establish that $(x',y') = \tilde{R}(x,y)$ solves the equation (4.1) at $z = \ell(x)^{-1}$ and $\ell(y)^{-1}$. With the $z = 0$ case verified in Proposition 4.16 we conclude that it solves (4.1) for any $z$ as argued in the beginning of the present subsection. The level constraints $\ell(x) = \ell(y') \neq \ell(y) = \ell(x')$ are obviously satisfied. Due to Theorem 3.13 there is no other solution.

4.4. Properties of $V_i, W_i$ under $e_i^\prime$. Here we list the transformation property of the functions $V_i, W_i$ under $e_i^\prime$. Set

$$\omega_i = \frac{c_\varphi(x) + e_i(y)}{\varphi(x) + e_i(y)} \quad \psi_i = \frac{c_\varphi(x) + e_i(y)}{\varphi(x) + e_i(y)} \quad (0 \leq i \leq n),$$

$$\Omega_i = \left\{ \begin{array}{ll}
\frac{x_{i+1} + \psi_i + \psi_i + \sigma_i}{x_i + \psi_i + \sigma_i + \sigma_i} & (1 \leq i \leq n - 1), \\
\frac{x_{i+1} + \psi_i + \psi_i + \sigma_i}{x_i + \psi_i + \sigma_i + \sigma_i} & (i = 0).
\end{array} \right.$$ 

For a function $F = F((x,y))$ in the variables $(x,y)$ we write $R(F) = F(R(x,y))$ and $e_i^\prime(F) = F(e_i^\prime(x,y))$.

Proposition 4.26. Let $(x', y') = R(x, y)$. Then we have

$$V_0(e_i^\prime(x, y)) = V_0(x, y) \left( \frac{c_\varphi(x') + e_0(y')}{c_\varphi(x') + e_0(y')} \right)^\delta_0.$$ 

Proof. First suppose $1 \leq i \leq n$. From Theorem 3.12 the relation $(\hat{x}, \hat{y}) = e_i^\prime(x,y)$ implies $M(\hat{x}, z)M(\hat{y}, z) = G_i(a)M(x, z)M(y, z)G_i(b)$ for some $a, b$. But $V_0 = (M(x, z)M(y, z))_{2n,1}$ is unchanged by the multiplication of the $G_i$'s, proving $e_i^\prime V_0 = V_0$. To show the $i = 0$ case, note from Lemma 3.13 that $e_i^\prime((x, y)^\sigma_1) = (e_i^\prime(x, y))^\sigma_1$. Thus we have $V_0(e_i^\prime(V_0^{-1})) = V_0((e_i^\prime(x, y))^\sigma_1) = (e_i^\prime V_0)^\sigma_1$. Therefore, $e_i^\prime R \left( \frac{x_1}{y_1} \right) = e_i^\prime \left( \frac{y_1}{y_1} V_0^{-1} \right) = \frac{y_1}{y_1} e_i^\prime \left( e_i^\prime V_0 \right)$ and $R e_i^\prime \left( \frac{x_1}{y_1} \right) = R \left( \frac{x_1}{y_1} \right) = e_i^\prime V_0^{-1} V_0$. Now the assertion follows from $e_i^\prime R = R e_i^\prime$ (Proposition 4.6).

Proposition 4.26 indicates that $V_0$ is a tropical analogue of the energy function in crystal theory, which will be argued in Remark 4.29.
Similar properties can be derived for the other $V_i, W_i$ with the help of Proposition 4.6 and Definition 4.4. Besides Table 2 the result is summarized as

$$e^c_i(V_j) = \begin{cases} V_j & (j \neq i), \\ V_j^{R(\Omega_i)} \psi & (j = i), \end{cases}$$

$$e^c_i(W_j) = \begin{cases} W_j & (j \neq i), \\ W_j^{R(\Omega_i)} \psi & (j = i), \end{cases}$$

for $2 \leq j \leq n - 2$ and $0 \leq i \leq n$. For instance the $j = i$ case of the former (resp. the latter) relation can be obtained by applying $e^c_i R = Re^c_i$ on $\mathbf{x}_1 \cdots \mathbf{x}_i$ (resp. $(\mathbf{x}_1 \cdots \mathbf{x}_i)/(x_1 \cdots x_i)$).

| $e^c_i$ | $V_0$ | $V_0^{\sigma_1}$ | $V_1$ | $V_n^{*}$ | $V_n$ |
|---|---|---|---|---|---|
| $e^c_0$ | $V_0$ | $V_0^{\sigma_1}$ | $V_1^{R(\Omega_1)} \psi$ | $V_n^{*}$ | $V_n$ |
| $e^c_1$ | $V_0$ | $V_0^{\sigma_1}$ | $V_1^{R(\Omega_1)} \psi$ | $V_n^{*}$ | $V_n$ |
| $e^c_2 \ (2 \leq i \leq n - 2)$ | $V_0^{\sigma_1}$ | $V_0^{\sigma_1}$ | $V_1$ | $V_n^{*}$ | $V_n$ |
| $e^c_{n-1}$ | $V_0$ | $V_0^{\sigma_1}$ | $V_1^{R(\Omega_{n-1})} \psi$ | $V_n^{*}$ | $V_n$ |
| $e^c_n$ | $V_0$ | $V_0^{\sigma_1}$ | $V_1^{R(\Omega_n)} \psi$ | $V_n^{*}$ | $V_n$ |

Table 2. Transformation under $e^c_i$.

4.5. Piecewise linear formula for the combinatorial $R$. Suppose a rational function $F = F(x, y; c)$ is subtraction-free in the sense of Remark 4.13 with respect to the variables $x = (x_1, x_2, \ldots, x_1)$, $y = (y_1, y_2, \ldots, y_1)$ and $c$. By ultradiscretization of $F$ we mean the expression obtained by replacing $+, \times, /$ with max, $+,-,$ respectively in $F$. (It is called “tropicalization” in [BK] on the contrary.) The procedure is well-defined particularly because the both sides of $p(q + r) = pq + pr$ have the equal image $p + max(q, r) = max(p + q, p + r)$. A way to substantiate it is to consider the limit $\lim_{c \to 0} e^c \log F(e^{x/c}, e^{y/c}, e^{c/c})$ after replacing $x, y, c$ by the real positive arrays $e^{x/c} := (e^{x_1/c}, e^{x_2/c}, \ldots, e^{x_1/c})$, $e^{y/c} := (e^{y_1/c}, e^{y_2/c}, \ldots, e^{y_1/c})$ and $e^{c/c}$ [TTMS]. Note that ultradiscretized functions are piecewise linear. In this subsection we use the same symbol to stand for the original (tropical) and the ultradiscretized (piecewise linear) objects, supposing no confusion might arise.

Example 4.27.

$$F = \frac{c\varphi_0(x) + \varepsilon_0(y)}{\varphi_0(x) + \varepsilon_0(y)} = \frac{c\mathbf{x}_1(1 + \mathbf{x}_2/x_2) + y_1(1 + y_2/\mathbf{y}_2)}{\mathbf{x}_1(1 + \mathbf{x}_2/x_2) + y_1(1 + y_2/\mathbf{y}_2)}$$

is subtraction-free. Its ultradiscretization reads

$$F = \max (c + \mathbf{x}_1 + (\mathbf{x}_2 - x_2)_+, y_1 + (y_2 - \mathbf{y}_2)_+) - \max (\mathbf{x}_1 + (\mathbf{x}_2 - x_2)_+, y_1 + (y_2 - \mathbf{y}_2)_+),$$

where $(x)_+ := \max(x, 0)$. Moreover if $c = 1$,

$$F = \theta(\mathbf{x}_1 + (\mathbf{x}_2 - x_2)_+ \geq y_1 + (y_2 - \mathbf{y}_2)_+).$$
Let \((x', y') = R(x, y)\) be the ultradiscretization of our tropical \(R\). Namely,

\[(4.28)\]
\[
x'_1 = y_1 + V_0^* - V_1, \quad \bar{\eta}_1 = \bar{y}_1 + V_0 - V_1, \\
x'_i = y_i + V_{i-1} + W_i - V_i - W_{i-1}, \quad \bar{\eta}_i = \bar{y}_i + V_{i-1} - V_i, \quad (2 \leq i \leq n-1) \\
x'_n = y_n + V_{n-1} - V_{n-1}^*, \\
y'_1 = x_1 + V_0^* - V_1^*, \quad \bar{\eta}_1 = \bar{x}_1 + V_0^* - V_1^*, \\
y'_i = x_i + V_{i-1}^* - V_i^*, \quad \bar{\eta}_i = \bar{x}_i + V_{i-1}^* + W_i - V_i^* - W_{i-1}, \quad (2 \leq i \leq n-1) \\
y'_n = x_n + V_{n-1}^* - V_{n-1}.
\]

Here \(V_i, W_i\) are given by

\[(4.29)\]
\[
V_i = \max \left\{ \{\theta_{i,j}(x, y), \theta'_{i,j}(x, y) | 1 \leq j \leq n-2\} \right\} \\
\cup \{\eta_{i,j}(x, y), \eta'_{i,j}(x, y) | 1 \leq j \leq n\}, \\
W_1 = V_0 + V_0^*, \quad W_{n-1} = V_{n-1} + V_{n-1}^*, \\
W_i = \max \left\{ V_i + V_{i-1}^* - y_i, V_i + V_i^* - \bar{x}_i \right\} + \min(x_i, \bar{y}_i), \quad (2 \leq i \leq n-2),
\]

where

\[
\theta_{i,j}(x, y) = \begin{cases} 
\ell(x) + \sum_{k=j+1}^{i} (\bar{y}_k - \bar{\eta}_k) & \text{for } 1 \leq j \leq i, \\
\ell(y) + \sum_{k=i+1}^{j} (\bar{\eta}_k - \bar{\eta}_k) & \text{for } i+1 \leq j \leq n-2,
\end{cases}
\]

\[
\theta'_{i,j}(x, y) = \begin{cases} 
\ell(x) + \sum_{k=1}^{i} (\bar{y}_k - \bar{\eta}_k) + \sum_{k=1}^{j} (y_k - x_k) & \text{for } j = 1, \ldots, n-2,
\end{cases}
\]

\[
\eta_{i,j}(x, y) = \begin{cases} 
\ell(x) + \sum_{k=j+1}^{i} (\bar{y}_k - \bar{\eta}_k) + (\bar{y}_j - x_j) & \text{for } 1 \leq j \leq i, \\
\ell(y) + \sum_{k=i+1}^{n-1} (\bar{\eta}_k - \bar{\eta}_k) + x_n & \text{for } j = n,
\end{cases}
\]

\[
\eta'_{i,j}(x, y) = \begin{cases} 
\ell(x) + \sum_{k=1}^{i} (\bar{y}_k - \bar{\eta}_k) + \sum_{k=1}^{j} (y_k - x_k) + (x_j - \bar{\eta}_j) & \text{for } 1 \leq j \leq n-1, \\
\ell(x) + \delta_{i,n-1} (\ell(x) - \ell(y)) + \sum_{k=1}^{i} (\bar{y}_k - \bar{\eta}_k) + \sum_{k=1}^{n-1} (y_k - x_k) - x_n & \text{for } j = n,
\end{cases}
\]

\[
\ell(x) = \sum_{k=1}^{n} x_k + \sum_{k=1}^{n-1} \bar{\eta}_k, \quad \ell(y) = \sum_{k=1}^{n} \bar{y}_k + \sum_{k=1}^{n-1} \bar{y}_k.
\]

The operations \(\sigma_1, \sigma_n\) and * in (4.28) are still defined by (4.2) and (4.3).
Let us recall the $D_n^{(1)}$-crystal $B_l$ ($l \in \mathbb{Z}_{>0}$) from [KKM]. It is a finite set

$$B_l = \{ (\zeta_1, \ldots, \zeta_n, \overline{\zeta}_n, \ldots, \overline{\zeta}_1) \in \mathbb{Z}_{\geq 0}^n \mid \sum_{i=1}^{n} (\zeta_i + \overline{\zeta}_i) = l, \quad \zeta_n \overline{\zeta}_n = 0 \}$$

endowed with certain maps $\tilde{e}_i, \tilde{f}_i$ ($0 \leq i \leq n$) as described in section 5.4 of the mentioned paper. The set $B_l$ is in one to one correspondence with another set

$$B_l' = \{ (x_1, \ldots, x_n, \overline{x}_{n-1}, \ldots, \overline{x}_1) \in \mathbb{Z}^{2n-1} \mid x_i, \overline{x}_i \geq 0 \text{ for } 1 \leq i \leq n-1, \quad x_n \geq -\min(x_{n-1}, \overline{x}_{n-1}), \sum_{i=1}^{n-1} (x_i + \overline{x}_i) + x_n = l \}$$

via $x_i = \zeta_i, \overline{x}_i = \overline{\zeta}_i$ ($1 \leq i \leq n-2$), $x_{n-1} = \zeta_{n-1} + \overline{\zeta}_n, x_n = \zeta_n - \overline{\zeta}_n, \overline{x}_{n-1} = \overline{\zeta}_{n-1} + \zeta_n, \zeta_n = \max(0, x_n), \overline{\zeta}_n = \max(0, \overline{x}_n)$, $\zeta_{n-1} = x_{n-1} + \min(0, x_n), \overline{\zeta}_{n-1} = \overline{x}_{n-1} + \min(0, \overline{x}_n)$. Note that $x_n$ can be negative. Below we give the explicit crystal structure of $B_l'$, which is also obtained by ultradiscretizing the geometric crystal structure of $B$ in section 2.2.

$$\begin{align*}
e_0(x) &= x_1 + (x_2 - \overline{x}_2)_+, \quad \varphi_0(x) = \overline{x}_1 + (\overline{x}_2 - x_2)_+, \\
e_1(x) &= \overline{x}_1 + (x_{i+1} - \overline{x}_{i+1})_+, \quad \varphi_1(x) = x_i + (x_{i+1} - x_{i+1})_+ \quad (i = 1, \ldots, n-2), \\
e_{n-1}(x) &= x_{n-1} + \overline{x}_{n-1}, \quad \varphi_{n-1}(x) = x_{n-1}, \\
e_n(x) &= \overline{x}_{n-1}, \quad \varphi_n(x) = x_{n-1} + x_n, \\
c_0(x) &= (x_1 - \xi_2, x_2 + \xi_2 - c, \ldots, x_2 + \xi_2, x_1 - \xi_2 + c), \\
c_i(x) &= (\ldots, x_i - \xi_{i+1} + c, x_{i+1} + \xi_{i+1} + c, \ldots, \overline{x}_{i+1} + \xi_{i+1}, \overline{x}_i - \xi_{i+1}, \ldots) \\
c_{n-1}(x) &= (\ldots, x_{n-1} + c, x_n - c, \ldots), \\
c_n(x) &= (\ldots, x_n + c, \overline{x}_{n-1} - c, \ldots), \\
\text{where} \quad \xi_i = \max(x_i, \overline{x}_i) - \min(x_i, \overline{x}_i) \quad (i = 1, \ldots, n-1).
\end{align*}$$

Here $c$ is an integer. If $c$ is negative, we understand $c$ as $-c$. Also, if $c_i(x)$ does not belong to $B_l'$, one should assume it to be 0.

We are interested in the bijection $R : B_l' \otimes B_k' \rightarrow B_l' \otimes B_k'$ which commutes with $\tilde{e}_i$. It is called the combinatorial $R$. See Remark 4.21 for the action of $\tilde{c}_i$ on the tensor product. Below we write $(x, y)$ for an element $x \otimes y$ of $B_l' \otimes B_k'$. We remark that the combinatorial $R : B_l \otimes B_k \rightarrow B_k \otimes B_l$ has been given in terms of a certain insertion procedure [HKOT2].

**Theorem 4.28.** Restriction of (4.24) to $x \in B_l', y \in B_k'$ yields an explicit formula $(x, y) \mapsto (x', y')$ for the combinatorial $R : B_l' \otimes B_k' \rightarrow B_l' \otimes B_k'$.

**Proof.** Set

$$\tilde{B}_l' = \{ (x_1, \ldots, x_n, \overline{x}_{n-1}, \ldots, \overline{x}_1) \in \mathbb{Z}^{2n-1} \mid \sum_{i=1}^{n-1} (x_i + \overline{x}_i) + x_n = l \}.$$

Let $E_i^c$ be the map on $\tilde{B}_l' \otimes \tilde{B}_l'$ obtained by ultradiscretizing $e_i^c$ on $B \times B$, let $R_0$ be the map $B_l' \otimes B_k' \rightarrow B_l' \otimes B_k'$, given by ultradiscretizing the tropical $R$, and let $R : B_l' \otimes B_k' \rightarrow B_l' \otimes B_k'$ be the combinatorial $R$. We are to show $R(x, y) = R_0(x, y)$ if $(x, y) \in B_l' \otimes B_k'$.

First recall that $R$ is uniquely characterized by
A.1. Let \( B \) be the one in [KKM] associated to the crystal \( G \).

Remark 4.29. By the ultradiscretization of Proposition 4.26, we have

\[
\ell_i = \begin{cases} 
\ell_i & \text{if } (x,y) = (x,0), \frac{e_i}{c_i} \in B_{j}^{F} \otimes B_{j}^{F}, \\
E_{\ell_0} R_0(x,y) = E_0 \frac{e_i}{c_i}(x,y) & \text{for } (x,y) \in B_{j}^{F} \otimes B_{j}^{F}.
\end{cases}
\]

The second equality is obtained by ultradiscretizing Proposition 4.6. Thus we get

\[
R_0(x,y) = E_0 \frac{e_i}{c_i}(x,y) = R(x,y).
\]

The proof is completed.

Appendix A. Alternative proof of Theorem 3.14

Let \( X = \sum_{1 \leq j \leq i \leq 2n} x_{i,j} E_{i,j} \) be a lower triangular matrix satisfying

\[
AXS^T = S^T XS = E,
\]

where \( S \) is the matrix given by (3.14).
Proof. Put \( X' = G_k(u)XG_k(v)^{-1} \). The elements of the matrix \( X' = (x'_{i,j}) \) read
\[
x'_{i,j} = x_{i,j} + u(\delta_{i,k}x_{k+1,j} + \delta_{i,2n-k}x_{2n+1-k,j}) - v(\delta_{j,k}x_{1,i} + \delta_{2n+1-k,j}x_{1,2n-k}) - u(\delta_{i,k}x_{j+1,i} + \delta_{i,2n-k}x_{j+1,2n-k}) + \delta_{i,2n-k}x_{1,k} + \delta_{2n+1-k,j}x_{1,2n-k}.
\]
The only non-zero upper-triangular elements are
\[
x'_{k,k+1} = ux_{k+1,k+1} - vx_{k,k},
x'_{2n-k,2n+1-k} = ux_{2n+1-k,2n+1-k} - vx_{2n-k,2n-k} - uvx_{2n+1-k,2n-k}.
\]
Hence the relations (A.2) and (A.3) follow.

Note that the following parameterization solves the relation (A.2):
\[
u = \frac{x_{k,k}}{x_{k+1,k}} (c - 1), \quad v = \frac{x_{k+1,k+1}}{x_{k+1,k}} (c - 1).
\]
It also solves the relation (A.3) due to (A.1). In view of this, we define the action \(e^c_i\) \((1 \leq i \leq n - 1)\) on \(X\) by
\[
e^c_i(X) = G_i \left( \frac{x_{i,i}}{x_{i+1,i}} (c - 1) \right) X G_i \left( \frac{x_{i+1,i+1}}{x_{i+1,i}} (c - 1) \right)^{-1}.
\]
Note that we have \(e^c_i(X)S e^c_i(X)S = S e^c_i(X)S e^c_i(X) = E\).

Proposition A.2.
\[
e^d_{c} e^d_{i+1} e^c_i(X) = e^c_i e^d_i e^d_{i+1} (X) \quad \text{for} \quad 1 \leq i \leq n - 2.
\]

Proof. Setting \( Y = e^c_i(X) \), \( Z = e^d_i(X) \), \( Y' = e^d_{i+1}(X) \), \( Z' = e^d_{i+1}(Y') \), we introduce
\[
u_1 = \frac{X_{i,i}}{X_{i+1,i}} (c - 1), \quad u_2 = \frac{Y_{i+1,i+1}}{Y_{i+2,i+1}} (cd - 1), \quad u_3 = \frac{Z_{i,i}}{Z_{i+1,i}} (d - 1),
\]
\[
u'_3 = \frac{Z'_{i+1,i+1}}{Z'_{i+2,i+1}} (d - 1), \quad u'_3 = \frac{Z'_{i+1,i+1}}{Z_{i+1,i}} (c - 1),
\]
From (A.3) we know
\[
e^d_{i} e^c_{i+1} e^c_i(X) = G_i(u_3)G_{i+1}(u_2)G_i(u_1)X (G_i(v_3)G_{i+1}(v_2)G_i(v_1))^{-1},
e^c_i e^d_{i} e^d_{i+1}(X) = G_{i+1}(u'_3)G_i(u'_2)G_{i+1}(u'_1)X (G_{i+1}(v'_3)G_i(v'_2)G_{i+1}(v'_1))^{-1}.
\]
A direct calculation leads to
\[
u_1 = \frac{x_{i,i}}{x_{i+1,i}} (c - 1), \quad u_2 = \frac{x_{i+1,i}x_{i+1,i+1}}{w_1} (cd - 1), \quad u_3 = \frac{x_{i,i}w_1}{x_{i+1,i}w_2} (d - 1),
\]
\[
u'_1 = \frac{x_{i+1,i+1}}{x_{i+2,i+1}} (d - 1), \quad u'_2 = \frac{x_{i,i}x_{i+2,i+1}}{w'_2} (cd - 1), \quad u'_3 = \frac{x_{i+1,i+1}w_2}{x_{i+2,i+1}w_1} (c - 1),
\]
\[
w_1 = (1 - c)x_{i+2,i}x_{i+1,i+1} + cx_{i+1,i}x_{i+2,i+1},
w_2 = (d - 1)x_{i+2,i}x_{i+1,i+1} + x_{i+1,i}x_{i+2,i+1}.
\]
Then the relations
\[ u'_1 = \frac{u_2u_3}{u_1 + u_3}, \quad u'_2 = u_1 + u_3, \quad u'_3 = \frac{u_1u_2}{u_1 + u_3}, \]
can be checked, and from (3.3),
\[ G_i(u_3)G_{i+1}(u_2)G_i(u_1) = G_{i+1}(u'_3)G_i(u'_2)G_{i+1}(u'_1). \]
Similarly, one can check
\[ G_i(v_3)G_{i+1}(v_2)G_i(v_1) = G_{i+1}(v'_3)G_i(v'_2)G_{i+1}(v'_1). \]

**Lemma A.3.** Let \( X = (X_{i,j})_{1 \leq i,j \leq 2n} = A(x^1) \cdots A(x^L) \). Then we have
\[ \frac{X_{i+1,i}}{X_{i,i}} = \varepsilon_i(x), \quad \frac{X_{i+1,i}}{X_{i+1,i+1}} = \varphi_i(x) \quad \text{for} \quad 1 \leq i \leq n-1. \]

**Proof.** We prove by induction on \( L \). If \( L = 1 \) the claim can be checked directly. Suppose \( L > 1 \). Let \( \tilde{X} = (\tilde{X}_{i,j})_{1 \leq i,j \leq 2n} = A(x^1) \cdots A(x^{L-1}) \). Since \( \tilde{X} \) and \( A(x^L) \) are lower-triangular we have
\[ X_{i,i} = \tilde{X}_{i,i}A(x^L)_{i,i}, \]
\[ X_{i+1,i} = \tilde{X}_{i+1,i}A(x^L)_{i,i} + \tilde{X}_{i+1,i+1}A(x^L)_{i+1,i}, \]
\[ X_{i+1,i+1} = \tilde{X}_{i+1,i+1}A(x^L)_{i+1,i+1}. \]
Denote \( (x^1, \ldots, x^{L-1}) \) by \( \tilde{x} \). Suppose
\[ \frac{\tilde{X}_{i+1,i}}{X_{i,i}} = \varepsilon_i(\tilde{x}), \quad \frac{\tilde{X}_{i+1,i}}{X_{i+1,i+1}} = \varphi_i(\tilde{x}). \]
Then we have
\[ \frac{X_{i+1,i}}{X_{i,i}} = \frac{\tilde{X}_{i+1,i}}{X_{i,i}} + \frac{\tilde{X}_{i+1,i+1}A(x^L)_{i+1,i}}{X_{i,i}A(x^L)_{i,i}} = \varepsilon_i(\tilde{x}) + \frac{\varepsilon_i(\tilde{x})\varepsilon_i(x^L)}{\varphi_i(x)} = \varepsilon_i(x), \]
\[ \frac{X_{i+1,i}}{X_{i+1,i+1}} = \frac{A(x^L)_{i+1,i}}{A(x^L)_{i+1,i+1}} + \frac{\tilde{X}_{i+1,i}A(x^L)_{i,i}}{X_{i+1,i+1}A(x^L)_{i+1,i+1}} = \frac{\varphi_i(x^L)}{\varepsilon_i(x^L)} = \varphi_i(x). \]

**Proof of Theorem 3.16.** We concentrate on the nontrivial case (3.35) with \( i < j \). Note that \((i,j) = (0,2)\) case reduces to \((i,j) = (1,2)\) as
\[ c_0^d c_2^e c_0^c(x) = (c_1^d c_2^e c_1^c(x^{\sigma_1}))^{\sigma_1} = (c_1^d c_2^e c_1^c(x^{\sigma_1}))^\sigma_i = c_2^e c_0^d c_4^f(x). \]
Similarly \((i,j) = (n-2,n)\) case reduces to \((i,j) = (n-2,n-1)\). Thus it suffices to prove the relations \( P = P' \) and \( Q = Q' \) (1 \leq i \leq n-2) for \( P, P', Q \) and \( Q' \) specified in (3.35)-(3.39).

Let \( x = (x^1, \ldots, x^L) \) and \( X = A(x^1) \cdots A(x^L) \). Since \( X \) is lower triangular and satisfies (A.3), we can apply Lemma A.2 keeping all the notations in its proof. Then (A.7) and (A.8) are nothing but \( P = P' \) and \( Q = Q' \), respectively. Let us show the former for example. By Lemma A.3 we have \( u_1 = \frac{c}{\varepsilon_i(x^i)} \) and \( u'_1 = \frac{d-1}{\varepsilon_{i+1}(x^i)}. \)
Next we are to show $u_2 = \frac{cd - 1}{c_2(x, y)}$ and $u_2' = \frac{cd - 1}{c_2(x, y)}$. We illustrate the way along $u_2$. Setting $e_i^y(x) = (y^1, \ldots, y^L)$, we get

$$Y = e_i^y(X) = G_i(u_t) X G_i(v_t)^{-1}$$

$$= G_i(u_t) A(x^1) \cdots A(x^L) G_i(v_t)^{-1}$$

$$= A(y^1) \cdots A(y^L),$$

where the last equality is due to Theorem 3.12. Now Lemma A.3 tells that $Y_i \varepsilon_{i+1, i+1} = \varepsilon_{i+1}(e_i^y(x))$. The remaining relations are similarly verified.

**APPENDIX B. PROOF OF PROPOSITION 4.16**

Set $I = A(x)A(y)$, which should not be confused with the identity matrix $E$.

**Lemma B.1.**

(B.1) \[ 0 = \sum_{m=i}^{n} I_{m,i}I_{2n+1-m,i}(-1)^m, \quad (1 \leq i \leq n) \]

(B.2) \[ I_{j,i} = y_i \left( 1 + \frac{x_i}{y_i} \right) \left( 1 + \frac{x_j}{y_j} \right) \left( \prod_{m=i+1}^{j-1} x_m \right) + y_i (I_{j,i+1} - \delta_{j,i+1}), \quad (1 \leq i \leq j - 1 \leq n - 2) \]

(B.3) \[ I_{n,n-1} = \frac{y_{n-1}}{y_n} x_{n-1} x_n + y_{n-1} I_{n,n}, \]

(B.4) \[ I_{n,i} = y_i \left( 1 + \frac{x_i}{y_i} \right) \left( \prod_{m=i+1}^{n-1} x_m \right) + y_i I_{n,i+1}, \quad (1 \leq i \leq n - 2) \]

(B.5) \[ I_{j,i} = y_i \left( 1 + \frac{x_i}{y_i} \right) \left( \prod_{m=i+1}^{j-1} x_m \right) \left( \prod_{m=2n+1-j}^{n-1} y_m \right) + y_i I_{j,i+1}, \quad (1 \leq i \leq 2n - j \leq n - 2) \]

(B.6) \[ I_{2n+1-i,i} = y_i x_i \left( 1 + \frac{x_i}{y_i} \right) \left( x_i x_n \prod_{m=i+1}^{n-1} x_m y_m + y_i y_n \prod_{m=i+1}^{n-1} y_m \right) \]

$$+ y_i x_i I_{2n+1-i,1}, \quad (1 \leq i \leq n - 2)$$

**Proof.** From Proposition 3.6, one has $IS'IS = E$. In particular $\sum_{m=i}^{2n+1-i} I_{m,i}I_{2n+1-m,i}$ S_{m, 2n+1-m} = 0, which is equivalent to (B.1).

To show (B.2), (B.4) and (B.5), we consider

$$I_{j,i} - y_i I_{j,i+1} = \sum_{k=1}^{2n} A(x)_{j,k} (A(y)_{k,i} - y_i A(y)_{k,i+1})$$

$$= A(x)_{j,i} y_i + A(x)_{j,i+1} y_i$$

$$= y_i \left( 1 + \frac{x_i}{y_i} \right) A(x)_{j,i} - y_i \delta_{j,i+1},$$

where (B.7) is used twice. Repeated use of (B.7) leads to $A(x)_{j,i} = A(x)_{j,1}/(x_{i-1} \cdots x_1)$. Substituting the explicit form (B.4) of $A(x)_{j,1}$ we obtain the desired relations.
Note that \( I_{n,n-1} = A(x)_{n,n} A(y)_{n,n-1} + A(x)_{n,n-1} A(y)_{n-1,n-1} = x_n y_n - 1 y_n + x_{n-1} x_n \). \((\mathbf{3.3})\) follows from this and \( I_{n,n} = x_n y_n \).

Let us consider \((\mathbf{3.6})\). From \((\mathbf{3.5})\) and \((\mathbf{3.27})\) it follows that

\[
A(x)_{2n+1-i,k} = A(x)_{2n-i,k} x_i + \begin{cases} 
\overline{x}_i / x_i & \text{for } k = 2n + 1 - i, \\
\overline{x}_i & \text{for } k = 2n - i, \\
0 & \text{for } k \neq 2n + 1 - i, 2n - i,
\end{cases}
\]

\[
A(y)_{k,i} = A(y)_{k+1,i} y_i + \begin{cases} 
y_i / y_i & \text{for } k = i, \\
y_i & \text{for } k = i + 1, \\
0 & \text{for } k \neq i, i + 1.
\end{cases}
\]

Substitute these relations into \( I_{2n+1-i,i} = \sum_{k=1}^{2n} A(x)_{2n+1-i,k} A(y)_{k,i} \). The result reads

\[
I_{2n+1-i,i} - y_i \overline{x}_i I_{2n-i+1} = \overline{x}_i y_i \left( A(x)_{2n-i,i} \frac{1}{y_i} + A(x)_{2n-i+1} \right) + \overline{x}_i y_i \left( A(y)_{2n+1-i,i+1} \frac{1}{x_i} + A(y)_{2n-i+1} \right) = \overline{x}_i y_i \left( \frac{1}{x_i} + \frac{1}{y_i} \right) \left( A(x)_{2n-i,i} + A(y)_{2n-i+1} \right),
\]

where \( A(x)_{2n-i,i+1} = A(x)_{2n-i,i} / x_i \) and \( A(y)_{2n-i,i+1} = A(y)_{2n+1-i,i+1} / y_i \) are used in the last equality. It remains to show \( A(x)_{2n-i,i} = x_i x_n \prod_{m=i+1}^{n-1} x_m \overline{x}_m \) and \( A(y)_{2n+1-i,i+1} = y_i y_n \prod_{m=i+1}^{n-1} y_m \overline{y}_m \). Repeated use of \((\mathbf{3.3})\) leads to \( A(x)_{2n-i,i} = A(x)_{2n-i,i} / (x_{i-1} \cdots x_1) \). The former relation is derived by substituting \((\mathbf{3.4})\) into \( A(x)_{2n-i,i} \) here. The latter follows from the former by \((\mathbf{3.27})\).

**Lemma B.2.**

\[
\begin{align*}
I_{i,i} &= I_{2n+1-i,2n+1-j} \quad (1 \leq i,j \leq n), \\
I_{n+1,i} &= I_{n,i} \quad (1 \leq i \leq n - 1).
\end{align*}
\]

**Proof.** From the proof of Lemma \((\mathbf{3.3})\) we know \((x,y) = \tau((x^{\sigma_n},y^{\sigma_n})^*)\). Therefore by using \((\mathbf{3.27})\) we have \( A(x) A(y) = J_j A(y^{\sigma_n}) J_i A(x^{\sigma_n}) J \). \((\mathbf{3.4})\) yields the former relation. The latter relation follows from \( A(x) A(y) = J_n A(x^{\sigma_n}) A(y^{\sigma_n}) J_n \) due to \((\mathbf{3.26})\).

The following expressions can be checked directly:

\[
\begin{align*}
I_{n+1,n} &= 0, \\
I_{i,i} &= \frac{x_i y_i}{x_i y_i} \quad (1 \leq i \leq n - 1), \quad I_{n,n} = x_n y_n, \\
I_{n,n-1} &= x_{n-1} x_n y_n - 1 y_n \left( \frac{1}{x_{n-1}} + \frac{1}{y_n} \right), \\
I_{n+2,n-1} &= y_n - 1 y_n x_n \left( y_{n-1} + x_n \right) \left( \frac{1}{x_n} + \frac{1}{y_n} \right).
\end{align*}
\]

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Lemma B.3. Let \((x', y') = \tilde{R}(x, y)\). Then we have
\[
1 + \frac{x'_j}{x_j} = \frac{V_{j-1}}{W_{j-1}} \left[ \left( 1 + \frac{x_j}{y_j} \right) V^*_j + (\ell(y) - \ell(x)) \left( \frac{x_j y_j}{x_j y_j} - 1 \right) \right],
\]
for \(2 \leq j \leq n - 1\).

Proof. By definition of \(\tilde{R}\) we obtain
\[
(B.13) \quad 1 + \frac{x'_j}{x_j} = \frac{1}{W_{j-1}} \left( W_{j-1} + \frac{y_j}{y_j} W_j \right).
\]
In the parentheses of the right side, apply (4.10) to (4.16) to rewrite \(W_{j-1}\) and \(W_j\).

Lemma B.4. Let \((x', y') = \tilde{R}(x, y)\). Then we have
\[
y'_i \left( 1 + \frac{x'_i}{y'_i} \right) = y_i \left( 1 + \frac{x_i}{y_i} \right) \frac{W_i}{V_i V^*_i},
\]
for \(1 \leq i \leq n - 2\).

Proof. First we suppose \(2 \leq i \leq n - 2\). By definition of \(\tilde{R}\) we have
\[
1 + \frac{x'_i}{y'_i} = \frac{y_i}{V_i V^*_{i-1}} \left( \frac{V_i V^*_{i-1}}{y_i} + \frac{V_{i-1} V^*_i}{x_i} \right)
\]
\[
(B.15) \quad = \frac{y_i}{V_i V^*_{i-1}} \left( \frac{1}{x_i} + \frac{1}{y_i} \right) W_i,
\]
where the last step is due to (4.11). Multiplying the both sides by \(y'_i = x_i \frac{V^*_i}{V_i}\) we get the desired relation. For \(i = 1\), we have
\[
1 + \frac{x'_1}{y'_1} = \frac{y_1}{V_1} \left( \frac{V_1}{y_1} + \frac{V^*_1}{x_1} \right)
\]
\[
(B.16) \quad = \frac{y_1}{V_1} \left( \frac{1}{x_1} + \frac{1}{y_1} \right) V^*_1 \sigma_1,
\]
where the last step is due to \(\sigma_1\) and \(\star \circ \sigma_1\) of (4.8). Now multiply the both sides by \(y'_1 = x_1 \frac{V^*_1}{V_1}\) and simplify the result by Lemma 4.12.

Lemma B.5. Let \((x', y') = \tilde{R}(x, y)\). Then we have
\[
y'_i = y_i \left[ 1 + \frac{\ell(x) - \ell(y)}{V^*_i} \left( 1 + \frac{x_i}{y_i} \right) \right],
\]
for \(1 \leq i \leq n - 2\).

Proof. This identity is a consequence of \(y'_i = x_i \frac{V^*_i}{V_i}\) and of (4.8).
Lemma B.6. Let \((x', y') = \tilde{R}(x, y)\). Then we have

\[
\left( \prod_{m=1}^{n} x_m' \right) \left( \prod_{m=2n+1-j}^{n-1} x_m' \right) = \left( \prod_{m=1}^{j-1} y_m \right) \frac{V_i W_{j-1}}{V_{j-1} W_i}, \quad (1 \leq i \leq j - 1 \leq n - 2)
\]

\[
\left( \prod_{m=i+1}^{n} x_m' \right) \left( \prod_{m=2n+1-j}^{n-1} \bar{x}_m \right) = \left( \prod_{m=1}^{j-1} y_m \right) \frac{V_i V_{2n-j}}{W_i}.
\]

(n + 1 \leq j \leq 2n - 1)

Lemma B.7. Let \(i, j\) be integers such that \(1 \leq i \leq j - 1 \leq n - 2\). Suppose \(I_{j,i+1}\) is invariant under \(\tilde{R}\). Then \(I_{j,i}\) is also invariant under \(\tilde{R}\).

Proof. Let \((x', y') = \tilde{R}(x, y)\). By applying Lemmas B.3, B.4, B.5 and B.6, the relation (B.2) is written as

\[
I_{j,i}(x', y') = y_i \left( 1 + \frac{x_i}{y_i} \right) \frac{1}{V_i} \left[ 1 + \frac{x_j}{y_j} \right] V_{j-1}^* + (\ell(y) - (\ell(x))(I_{j,j} - 1) \right] \left( \prod_{m=1}^{j-1} y_m \right)
\]

(B.18) \[+ y_i \left[ 1 + \frac{\ell(x) - \ell(y)}{V_i^*} \left( 1 + \frac{x_i}{y_i} \right) \right] (I_{j,i+1} - \delta_{j,i+1}).
\]

Thus it reduces to the following Lemma.

Lemma B.8.

\[
\left[ 1 + \frac{x_j}{y_j} \right] V_{j-1}^* + (\ell(y) - (\ell(x))(I_{j,j} - 1) \right] \left( \prod_{m=1}^{j-1} y_m \right)
\]

(B.19) \[
\left( \prod_{m=i+1}^{j-1} x_m \right) \left[ 1 + \frac{x_j}{y_j} \right] V_i^* + (\ell(y) - (\ell(x))(I_{j,i+1} - \delta_{j,i+1}),
\]

for \(1 \leq i \leq j - 1 \leq n - 2\).

Proof. By applying (B.2) and \(^*\) of (4.8) this is transformed into the same relation with \(i\) replaced by \(i - 1\). Thus it suffices to check (B.19) for \(i = j - 1\), which is straightforward.

Lemma B.9. Let \(i\) be an integer such that \(1 \leq i \leq n - 2\). Suppose \(I_{n,i+1}\) is invariant under \(\tilde{R}\). Then \(I_{n,i}\) is also invariant under \(\tilde{R}\).

Proof. Let \((x', y') = \tilde{R}(x, y)\). By applying Lemmas B.4, B.5 and B.6, the relation (B.4) is written as

(B.20) \[I_{n,i}(x', y') = y_i \left( 1 + \frac{x_i}{y_i} \right) \frac{V_{n-1}}{V_i^*} \left( \prod_{m=i+1}^{n} y_m \right) + y_i \left[ 1 + \frac{\ell(x) - \ell(y)}{V_i^*} \right] \left( 1 + \frac{x_i}{y_i} \right) I_{n,i+1}.
\]

Thus it reduces to the following Lemma.

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Lemma B.10. For $1 \leq i \leq n-2$

$$(B.21) \quad \left(\prod_{m=i+1}^{n} y_m \right) V_{n-1} = \left(\prod_{m=i+1}^{n} x_m \right) V_i^* + (\ell(y) - \ell(x)) I_{n,i+1}. $$

Proof. By using (B.4) and $\ast$ of (1.8) this is transformed into the same relation with $i$ replaced by $i-1$. Thus it suffices to check (B.21) for $i = n-2$, which reads

$$y_{n-1}y_n V_{n-1} = x_{n-1}x_n V_{n-2}^* + (\ell(y) - \ell(x)) I_{n,n-1}. $$

This follows from (B.11) and $\ast \circ \sigma_n$ of (1.9).

Lemma B.11. Let $i$, $j$ be integers such that $1 \leq i \leq 2n-j \leq n-2$. Suppose $I_{j,i+1}$ is invariant under $\tilde{R}$. Then $I_{j,i}$ is also invariant under $\tilde{R}$.

Proof. Let $(x', y') = \tilde{R}(x, y)$. By applying Lemmas B.4, B.5 and B.6, the relation (B.2) is written as

$$I_{j,i}(x', y') = y_i \left( 1 + \frac{x_j}{y_i} \right) V_{2n-j} \left( \prod_{m=i+1}^{n} y_m \right) \left( \prod_{m=2n+1-j}^{n} \overline{y}_m \right) + y_i \left[ 1 + \frac{\ell(x) - \ell(y)}{V_i^*} \left( 1 + \frac{x_j}{y_i} \right) \right] I_{j,i+1}. $$

Thus it reduces to the following Lemma.

Lemma B.12.

$$V_{2n-j} \left( \prod_{m=i+1}^{n} y_m \right) \left( \prod_{m=2n+1-j}^{n} \overline{y}_m \right) = V_i^* \left( \prod_{m=i+1}^{n} x_m \right) \left( \prod_{m=2n+1-j}^{n} \overline{x}_m \right) + (\ell(y) - \ell(x)) I_{j,i+1}, $$

for $1 \leq i \leq 2n-j \leq n-2$.

Proof. By applying $\ast$ of (1.8) and (B.5) this is transformed into the same relation with $i$ replaced by $i-1$. Thus it suffices to check (B.23) for $i = 2n-j$, which is done in the next Lemma.

Lemma B.13.

$$(B.24) \quad V_i \left( y_n \prod_{m=i+1}^{n-1} y_m \overline{y}_m \right) = V_i^* \left( x_n \prod_{m=i+1}^{n-1} x_m \overline{x}_m \right) + (\ell(y) - \ell(x)) I_{2n-i,i+1}, $$

for $1 \leq i \leq n-2$.

Proof. By applying (1.8), $\ast$ of (1.8) and (B.6) this is transformed into the same relation with $i$ replaced by $i-1$. Thus it suffices to check (B.24) for $i = 1$. Consider the trivial identity:

$$V_0 \left( y_n \prod_{m=1}^{n-1} y_m \overline{y}_m \right) = V_0 \left( x_n \prod_{m=1}^{n-1} x_m \overline{x}_m \right) + (\ell(y) - \ell(x)) I_{2n,1}. $$

The $i = 1$ case of (B.24) is shown by applying (1.8), $\ast$ of (1.8) and (B.6) to this identity.
Lemma B.14. Let \( i \) be an integer such that \( 1 \leq i \leq n \). Then \( I_{i,i} \) is invariant under \( \tilde{R} \).

Proof. This is clear from (B.10). \( \square \)

Lemma B.15. \( I_{n,n-1} \) is invariant under \( \tilde{R} \).

Proof. Let \( (x',y') = \tilde{R}(x,y) \). From (B.3) and Theorem B.14 we have

\[
I_{n,n-1}(x',y') = \frac{x_{n-1} y_{n-1} V_{n-2}}{V_{n-1}} + x_{n-1} \frac{V_{n-2}^*}{V_{n-1}} I_{n,n}.
\]

By using \( \sigma_n \) and * of (B.9) we recover the right hand side of (B.3). \( \square \)

Proof of Proposition 4.14. It should be shown that all \( I_{i,i} \)'s are invariant under \( \tilde{R} \).

In view of \( I_{n+1,n} = 0, I_{j,j} = 0 (j < i) \), (B.7) and Proposition 4.11, it suffices to check the case \( i + j \leq 2n + 1 \). Suppose \( 1 \leq j \leq n - 1 \). Then the claim follows from Lemmas B.7 and B.14.

Suppose \( j = n \). Then the claim follows from Lemmas B.9 and B.14. Similarly, it suffices to check the case \( i + j \leq 2n + 1 \). Suppose \( 1 \leq j \leq n - 1 \). Then the claim follows from \( j = n \) due to (B.8) and Proposition 4.11.

We have already proved the invariance of \( I_{n-1,n-1}, I_{n,k} \) and \( I_{n+1,n-1} \). Because of the identity (B.1), \( I_{n+2,n-1} \) in (B.13) is also invariant. Then the claim for \( j = n + 2 \) follows from Lemma B.11. Similarly the invariance of \( I_{n-1,n-1}, I_{n,k} \) and \( I_{n+1,n-1} \) (1 \( \leq k \leq n - 1 \)) implies the invariance of \( I_{n+1,n-1} \) by (B.1), and the claim for \( j = n + 2 \) follows from Lemma B.11. By induction on \( a \) the proof is completed. \( \square \)

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