A Novel Ansatz for the Energy-Momentum Tensor on the Lattice

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Abstract
The comparison of structural analogies between the energy-momentum tensors in general relativity and in a gauge theory of Yang-Mills type is tentatively extended to lattice physics. These considerations are guiding to a new lattice model for the symmetric energy-momentum tensor $\Theta_{\mu\nu}$ of the pure Yang-Mills gauge sector, basing on half powers of the plaquette variable. The concept of non-trivial principal square roots of unitary matrices in lattice gauge theories can be epitomized to reconcile the pretension to a uniform construction principle for the components of $\Theta_{\mu\nu}$ with general qualitative thermodynamic demands concerning arguments in favour of a Wilson form for $\langle \Theta_{\mu} \rangle$ and $\langle \Theta_{44} \rangle$. SU(2) Monte Carlo results for the Euclidean expectation values on a 10\,**\,4 lattice are compared with that of competing hitherto existing lattice models for $\Theta_{\mu\nu}$.

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1 Introduction

An observable which comparatively little attention has so far been paid to is the energy-momentum tensor on the lattice. Apparently, a true deep understanding of the energy-momentum tensor $\Theta_{\mu\nu}$ in general and its loose connection (the puzzle ‘why?’ is still not yet resolved beyond comparisons regarding various applications of Noether’s Theorem, cf. [11]) with general relativistic variational processes in continuum physics is still absent. If we cannot substantially improve this situation by standard classical, quantum, or lattice field theory we can instead look out for similar patterns in physics and try to explain something less understood by an unconventional confrontation with something else being less understood for similar reasons, thereby attempting to "cast out the devil by Beelzebub" in a positive sense. This is exactly what we will do in Section 2, inspiring a novel model for $\Theta_{\mu\nu}$ on the lattice.

With the help of several rather technical details and auxiliary definitions, to be given in Section 3, Section 4 will accurately specify various lattice models for $\Theta_{\mu\nu}$, including hitherto existing ones and the addressed new version. We will discuss results of Monte Carlo simulations in Section 5, regarding the Euclidean expectation values for the components of $\Theta_{\mu\nu}$ in the scope of the disposable lattice models. We will conclude with a final discussion in Section 6, where we will in particular focus upon general relativistic and thermodynamic aspects around $\Theta_{\mu\nu}$ and upon their implications for lattice physics. Throughout the paper, bold type does not indicate vector rank.

2 General Aspects

In Ref. [13], the energy-momentum tensors of general relativity, $\Theta^{(1)}_{\mu\nu}$ (specification by its equations of motion, being the Einstein field equations), and of a continuum Yang-Mills theory, $\Theta^{(2)}_{\mu\nu}$ (representation by its own ingredients), were shown to be analogously reducible to purely metric expressions and a tensor field $\Omega^{(k)}_{\alpha\beta\gamma\delta}$ for both $k \in \{1, 2\}$ (see Eq. (29) in Ref. [13] – $\Theta^{(2)}_{\mu\nu}$ there corresponds to $\Theta_{\mu\nu}$ in Sec. 1 here). It could be argued there that general relativity automatically supplies an SO(4) Yang-Mills tensor field $\Omega^{(2)}_{\alpha\beta\gamma\delta}$ as introduced just above, with

$$\Omega^{(2)}_{\alpha\beta\gamma\delta} \propto \Omega^{(1)}_{\alpha\beta\mu\nu} g^{\mu\rho} g^{\nu\sigma} \Omega^{(1)}_{\rho\sigma\gamma\delta} \quad (2.1)$$

(v. Eq. (36) in Ref. [13]), revealing some kind of quadratic coherency between $\Omega^{(1)}_{\alpha\beta\gamma\delta}$ and $\Omega^{(2)}_{\alpha\beta\gamma\delta}$, both having the same symmetry properties (cf. Eq. (30))
Therefore it is possible to deduce from the vierbein calculus in Ref. [13] that $\Omega^{(1)}_{\alpha\beta\gamma\delta}$ can be viewed in the context of referring to an SO(4) field strength tensor $F_{\mu\nu}$ while $\Omega^{(2)}_{\alpha\beta\gamma\delta}$ (v. Eq. (31), Ref. [13]) can be related to any Yang-Mills type field strength tensor; i.e. more roughly

$$\Omega^{(1)}_{\alpha\beta\gamma\delta}, \quad \Omega^{(2)}_{\alpha\beta\gamma\delta} \quad \text{continuum} \quad F_{\mu\nu} \quad \text{2.2}$$

Regarding Formulae (31) up to (35) in Ref. [13], the allocation (2.2) becomes plausible relative to that context.

In continuum physics, $\Theta^{(k)}_{\mu\nu}$, $\Omega^{(k)}_{\alpha\beta\gamma\delta}$ (we are silently restricting ourselves to $k \in \{1, 2\}$ throughout the entire paper, as done in Ref. [13]), and corresponding Lagrangian densities $L^{(k)}$ are of order "$k$" in $F_{\mu\nu}$. On a spacetime lattice, however, the standard adaptations of $L^{(k)}$ are known to be of order one in the plaquette variable ([15, 20, 22] and v. Eq. (2.25) in Ref. [19]), or in its respective substitute (being the so-called dual plaquette, e.g.–cf. the commentary on Eq. (3.17) in Ref. [24]). In particular, the most transparent lattice transference of $F_{\mu\nu}$ w.r.t. $\Omega^{(1)}_{\alpha\beta\gamma\delta}$ in the framework of quantum gravity (see Ref. [24], focus on Formula (3.11) there and use it as a starting point for introducing finite lattice spacings and constructing the dual plaquette if desired) suggests

$$R_{\mu\nu}^{ab}(J, n) \propto (\mathcal{F}^{(1)}_{\mu\nu})^{ab} \quad \text{2.3}$$

with

$$\mathcal{F}^{(1)}_{\mu\nu} = \frac{1}{2i g_{a} a^{2}} \left( U_{\mu\nu} - U_{\mu\nu}^{\dagger} \right), \quad \text{2.4}$$

being equivalent to Formula (2.11) in Ref. [19] if $g$ (there is of course no relation with the metric tensor $g_{\mu\nu}$) is understood as the formally relevant lattice coupling, "$a$" is the lattice spacing, "$i$" is the imaginary unit, and $U_{\mu\nu}$ as usual denotes the plaquette variable.

Now consider $\Omega^{(2)}_{\alpha\beta\gamma\delta}$. In principle there is no need to continue the precedingly prevailing assignment

$$\Omega^{(1)}_{\alpha\beta\gamma\delta} \quad \text{lattice} \quad \mathcal{F}^{(1)}_{\mu\nu} \quad \text{2.5}$$

to this type of separate situation. If we consistently define the main (–or principal–) root $\sqrt{M}^{\text{main}}$ of any diagonalizable matrix $M$ (including the
special case of scalars given by $1 \times 1$ matrices) to be performed by effectively replacing its eigenvalues by the principal values of their own square roots (v.i. in Sec. 3) then we can introduce the competing lattice expression

$$\mathcal{F}_{\mu \nu}^{(2)} := \frac{1}{ig_o a^2} \left( \sqrt{U_{\mu \nu} \text{main}} - \sqrt{U_{\mu \nu} \text{main}^\dagger} \right)$$

as well, with $\lim_{a \to 0} \mathcal{F}_{\mu \nu}^{(k)} = F_{\mu \nu} \quad \forall \ k \in \{1, 2\}$ and

$$\sqrt{U_{\mu \nu} \text{main}^\dagger} := \left( \sqrt{U_{\mu \nu} \text{main}} \right)^\dagger.$$

Demanding

$$\Omega_{\alpha \beta \gamma \delta}^{(2)} \xrightarrow{\text{lattice}} \mathcal{F}_{\mu \nu}^{(2)},$$

and plugging this prescription into Eqs. (29) and (31) in Ref. [13] is engendering a new version (the concrete formula will be presented in Sec. 4 by (4.2)) for $\Theta_{\mu \nu}^{(2)}$ on the lattice which is gauge invariant, has the correct continuum limit and can—in contrast to corresponding hitherto developed lattice models—simultaneously fulfil the demands for a uniform construction principle and for a Wilson form [12, 25, 26] of the Hamiltonian component $\Theta_{44}^{(2)}$ as well as for the structure of the trace anomaly [1, 4, 6]. With (2.5) and (2.7), the simplest lattice versions of all quantities $\Theta_{\mu \nu}^{(k)}$, $\Omega_{\alpha \beta \gamma \delta}^{(k)}$, and $\mathcal{L}^{(k)}$ are of order one in the plaquette variable or its respective tied dual extension while their lattice field strength tensor representation is in the non-dual case of order “$k$” in $\mathcal{F}_{\mu \nu}^{(k)}$ as required strictly in the continuum limit w.r.t. $F_{\mu \nu}$.

We obtain an elegant lattice counterpart for (2.1):

$$U_{\mu \nu} = \sum_{l=0}^{2} \frac{(ig_o a^2)^l}{l!} \left( \mathcal{F}_{\mu \nu}^{(l)} \right)^l,$$

with $\left( \mathcal{F}_{\mu \nu}^{(0)} \right)^0 \equiv 1$. Although neither (2.8) nor its continuum analogue (2.1) couple gravity to a Yang-Mills theory the proximity of both domains in (2.1) becomes still closer in case of (2.8).

### 3 Square Roots of Matrices and Semi-Uni-\-tarity

Let us recall the expressions (2.4) and (2.6) of the preceding section:
\[ F^{(1)}_{\mu\nu} = \frac{1}{2i\kappa} \left( U_{\mu\nu} - U_{\mu\nu}^\dagger \right), \quad (3.1) \]

\[ F^{(2)}_{\mu\nu} = \frac{1}{i\kappa} \left( \sqrt{U_{\mu\nu}^{\text{main}}} - \sqrt{U_{\mu\nu}}^{\dagger}\text{main} \right), \quad (3.2) \]

with \( \kappa := g(a) a^2 \) being different from the former \( \kappa \) in Ref. \[ \text{[13]} \]. On the other hand, the standard field strength tensor on the lattice, \( F_{\mu\nu} \), is defined implicitly. (3.1) up to (3.3) can be put together, yielding

\[ F^{(l)}_{\mu\nu} = \frac{l}{\kappa} \sin \left( \frac{\kappa}{l} \mod \left( F_{\mu\nu}, \text{period} \right) \right) \quad \forall \; l \in \{1, 2\}. \quad (3.4) \]

(3.4) has to be understood as an expression which is purely symbolic in case of the underlying gauge group being nonabelian and is then motivated by its own reinterpretation as a continuation of the abelian particular case where the most general gauge group is U(1) and (3.4) has the precise significations

\[ F^{(1)}_{\mu\nu} = \frac{1}{\kappa} \sin \vartheta_{\mu\nu} = \frac{1}{\kappa} \sin \bar{\vartheta}_{\mu\nu}, \quad (3.5) \]

\[ F^{(2)}_{\mu\nu} = \frac{2}{\kappa} \sin \lim_{\varepsilon \to 0} \frac{\mod(\vartheta_{\mu\nu} - |\varepsilon|, 2\pi) - \pi}{2} = \frac{2}{\kappa} \sin \frac{\bar{\vartheta}_{\mu\nu}}{2}, \quad (3.6) \]

using the U(1) gauge field decomposition

\[ \vartheta_{\mu\nu} := \kappa F_{\mu\nu} = \sum_{l=1}^{4} \vartheta_{(l)} \in ]-4\pi, 4\pi[ \] in the exponent of the plaquette variable and the improvement of this representation via

\[ \vartheta_{\mu\nu} = \bar{\vartheta}_{\mu\nu} + 2n\pi, \; n \in \{-2, -1, 0, 1, 2\} , \text{with } \bar{\vartheta}_{\mu\nu} \in ]-\pi, \pi[ . \]

We have already given a prescription for the calculation of the matrix main-value roots in the last section, regarding the case of diagonalizable matrices. We will not discuss the scope for an eventual generalization to other square matrices here where the product of the main root of the diagonalizable part and a finite binomial series for the nilpotent matrix-valued remnant factor (up to the highest multiplicity of any original eigenvalue minus one) has to be taken into account. We just remark that the self-consistency of the main root of a matrix \( M \) in case of diagonalizability \( M = ADA^{-1} \) (with \( D \) being a diagonal matrix) can be illustrated via \((\sqrt{ADA^{-1}})^2 = (A\sqrt{D}A^{-1})^2\) and the implementation of unicity by introducing the main-value prescription for the involved square roots. It is clear that any element of any unitary group is diagonalizable. Besides, we would like to indicate following subtle-
ty: in case of the special unitarian gauge group SU(2), e.g., the main-root operation will exceed the group if and only if the tackled group element is unequal to the negative of the identity matrix. But without exception any main (principal) root of an element of any orthogonal, special orthogonal or special unitary group lies in the surrounding, merely unitarian supergroup.

Restoration of rotational invariance on the lattice can be achieved by the substitution of the plaquette variable

\[ U_{\mu\nu}(n) \equiv U(n;\mu,\nu) \]  \hspace{1cm} (3.7)

by the averaging

\[ V_{\mu\nu}(n) := \frac{1}{4} \left[ U(n;\mu,\nu) + U(n;-\mu,\nu) + U(n;\mu,-\nu) + U(n;-\mu,-\nu) \right] \]  \hspace{1cm} (3.8)

of all four plaquettes touching the 4-space lattice site "n" in the \( \mu-\nu \) plane. For reasons of gauge invariance, the respective circulation has to start and to end at the same "n", preserving one scheme of orientation, being counterclockwise according to usual convention. Let us call a 2-by-2 matrix \( U \) semi-unitary here if

\[ U U^\dagger = \text{abs det } U \]  \hspace{1cm} (3.9)

is valid and "special semi-unitary" if both (3.9) and

\[ \text{det } U = \text{Re det } U \geq 0 \]  \hspace{1cm} (3.10)

are fulfilled. Hence the special case of

\[ \text{abs det } U = 1 \]  \hspace{1cm} (3.11)

complementally changes semi-unitarity into unitarity and special semi-unitarity into special unitarity. If \( U_{\mu\nu}(n) \) in (3.7) is an element of the gauge group SU(2) then \( V_{\mu\nu}(n) \) in (3.8) is special semi-unitary.

The main root of such a \( V_{\mu\nu}(n) \) is just semi-unitary if \( V := V_{\mu\nu}(n) \) is a (stretching/scaling down) point-symmetry reflection

\[ V = -\sqrt{(\text{det } V)^{\text{main} \cdot 1}} \]  \hspace{1cm} (3.12)

(the main root is here equal to the habitual principal value of the square root, v.s.) and special semi-unitary else. Eq. (3.12) is the \( \alpha(V) = -1 \) case of the "semi-unitarity signature"

\[ \alpha(V) := 2 \text{ sgn det } \left( V + \sqrt{\text{det } V^{\text{main} \cdot 1}} \right) - 1 \in \{-1,+1\} , \]  \hspace{1cm} (3.13)
erecting the evaluation criterion

$$\alpha(\rho, \sigma; \xi, \eta) := \frac{1}{2} \left( \alpha(V_{\rho\sigma}) + \alpha(V_{\xi\eta}) \right)$$  \hspace{1cm} (3.14)\

for the kernel

$$\langle \rho \sigma \xi \eta \rangle := \text{trace}\left( \mathcal{F}_{\rho\sigma}^{(2)} \mathcal{F}_{\xi\eta}^{(2)} \right),$$  \hspace{1cm} (3.15)\

which is proportional to $\Omega_{\rho\sigma\xi\eta}^{(2)}$ in Sec. 2 if the limit $a \to 0$ is investigated.

There are two possible refinements for $\mathcal{F}_{\mu\nu}^{(l)}$ in (3.1) and (3.2). First, the steps (3.1) and (3.2) may admix U(N) generators to $\mathcal{F}_{\mu\nu}^{(l)}$ which are outside of the original Lie algebra spanning $\mathcal{F}_{\mu\nu}$ in (3.3) if this original Lie algebra refers to the non-trivial subgroups O(N), SO(N), or SU(N). Such artefact generators can be removed by the reprojection

$$\mathcal{F}_{\mu\nu}^{(l)} \rightarrow \mathcal{F}_{\mu\nu}^{(l)[R]} := 2 \text{trace}\left( \mathcal{F}_{\mu\nu}^{(l)} \tilde{\tau}^A \right) \tilde{\tau}^A ,$$  \hspace{1cm} (3.16)\

onto the original Lie algebra. Second, rotational invariance on the lattice can be implemented by

$$\mathcal{F}_{\mu\nu}^{(l)} \left( U_{\mu\nu}(n) \right) \rightarrow \mathcal{F}_{\mu\nu}^{(l)} \left( U_{\mu\nu}(n) \rightarrow V_{\mu\nu}(n) \right),$$  \hspace{1cm} (3.17)\

according to (3.7) and (3.8). If (3.16) is applied definitely and (3.17) may be supplemented optionally then the–admittedly not finally fixed–resulting refinement shall be denoted by $\mathcal{F}_{\mu\nu}^{(l)'}$.

The more general evaluation of (3.14) and (3.15) is rendered possible if the 4-plaquettes mixing in (3.8) and (3.17) is switched on

$$\left( \sqrt{\text{main}} \right):$$

$$\langle \rho \sigma \xi \eta \rangle = \frac{4}{\kappa^2} \frac{\text{Im} V_{\rho\sigma}^{11} \text{Im} V_{\xi\eta}^{11} + \text{Re} V_{\rho\sigma}^{12} \text{Re} V_{\xi\eta}^{12} + \text{Im} V_{\rho\sigma}^{12} \text{Im} V_{\xi\eta}^{12}}{\sqrt{\text{Re} V_{\rho\sigma}^{11}} + \sqrt{\text{det} V_{\rho\sigma}}} \frac{\sqrt{\text{Re} V_{\xi\eta}^{11}} + \sqrt{\text{det} V_{\xi\eta}}}{\text{Re} V_{\rho\sigma}^{11} + \sqrt{\text{det} V_{\rho\sigma}}}$$  \hspace{1cm} (3.18)

for $\alpha(\rho, \sigma; \xi, \eta) = +1$.
\( \langle \rho \sigma \xi \eta \rangle = 0 \) for \( \alpha(\rho, \sigma; \xi, \eta) = 0 \) \hspace{1cm} (3.19)

\[ \langle \rho \sigma \xi \eta \rangle = \frac{8}{\kappa^2} \sqrt{(V_{\rho\sigma} \cdot V_{\xi\eta})^{11}} \]

for \( \alpha(\rho, \sigma; \xi, \eta) = -1 \) \hspace{1cm} (3.20)

The substitution (3.16) will leave (3.18) and (3.19) unchanged while (3.20) then becomes

\[ \langle \rho \sigma \xi \eta \rangle^{[R]} = 0 \] \hspace{1cm} \text{for} \hspace{0.5cm} \alpha(\rho, \sigma; \xi, \eta) = -1 . \] \hspace{1cm} (3.21)

4 Lattice Versions for the Energy-Momentum Tensor

(3.9) up to (3.15) and (3.18) up to (3.21) have been formulated for 2-by-2 matrices whereas all of the other formulae in Sec_3 are destined to the general case of N-by-N matrices to be returned to now. Using the hitherto introduced auxiliary quantities, it will become convenient to compare various versions for the symmetric energy-momentum tensor on the lattice. The concrete realization of the ideas (2.5) and (2.7) in Sec_2 is performed by starting with the continuum version of the energy-momentum tensor and by substituting the continuum field strength tensor \( F_{\mu\nu} \) there by the refinements \( \tilde{F}^{(1)\prime}_{\rho\sigma} \) \( \tilde{F}^{(1)\prime}_{\xi\eta} \) of \( \tilde{F}^{(1)}_{\rho\sigma} \) \( \tilde{F}^{(1)}_{\xi\eta} \), which have been defined by a concatenation of both (3.1) and (3.2) with (3.16) and a contingent application of (3.17). The actual application of (3.17) shall be marked by the label \('HYBRID'\) and its respective omission by the counterassignment \('PURE'\).

A well-known lattice model for the energy-momentum tensor \( \Theta_{\mu\nu}^{\text{cara}} \) can be in this mode described by

\[ \Theta_{\mu\nu}^{\text{cara}} = \left( \frac{1}{2} \delta_{\mu\nu} \delta^{\rho\xi} \delta^{\sigma\eta} + 2 \delta_{\mu}^{\rho} \delta^{\sigma\xi} \delta_{\nu}^{\eta} \right) \cdot \text{trace} \left( \tilde{F}^{(1)\prime}_{\rho\sigma} \tilde{F}^{(1)\prime}_{\xi\eta} \right) \Big|_{\text{HYBRID}} \] \hspace{1cm} (4.1)

so that \( \Theta_{\mu\nu}^{\text{caraPURE}} \) is different from \( \Theta_{\mu\nu}^{\text{cara}} \equiv \Theta_{\mu\nu}^{\text{caraHYBRID}} \). Switching from (2.5) to (2.7), we obtain the totally new lattice model
\[ \Theta_{\mu \nu}^{\text{own}} := \left( \frac{1}{2} \delta_{\mu \nu} \delta^{\rho \xi} \delta^{\sigma \eta} + 2 \delta_{\mu}^{\rho} \delta^{\sigma \xi} \delta_{\nu}^{\eta} \right) \text{trace} \left( \mathcal{F}_{\mu \sigma}^{(2)'} \mathcal{F}_{\xi \eta}^{(2)'} \right), \quad (4.2) \]

leaving undecided initially whether \( \Theta_{\mu \nu}^{\text{ownPURE}} \) or \( \Theta_{\mu \nu}^{\text{ownHYBRID}} \) has to be preferred. A third model with intermediate properties \(^{[16]}\) uses the heterogeneous construction principle

\[ \Theta_{\mu \nu}^{\text{karsch}} = - \frac{2}{\kappa^2} \text{trace} \left( - \sum_{\lambda \neq \mu} U_{\mu \lambda} + \sum_{\sigma, \lambda \neq \mu, \sigma > \lambda} U_{\sigma \lambda} \right) + \mathcal{O}(\kappa^2) \]

for \( \mu = \nu \) \quad (4.3a)

and

\[ \Theta_{\mu \nu}^{\text{karsch}} = - 2 \delta^{\sigma \lambda} \text{trace} \left( \mathcal{F}_{\mu \sigma}^{(1)'} \mathcal{F}_{\nu \lambda}^{(1)'} \right) \quad \text{for} \quad \mu \neq \nu. \quad (4.3b) \]

## 5 Monte Carlo Results

The Euclidean expectation values for the components of the presented lattice models for the symmetric energy-momentum tensor \( \Theta_{\mu \nu} \) have been measured on a \( 10 \times 10 \times 4 \) lattice. For this purpose, a heat bath SU(2) Monte Carlo simulation \(^{[5]}\) has been performed, measuring once every 50 sweepings after a cold start with a transient state of 10 000 lattice updatings. Basically the average concerning all of the lattice sites and every performance of a measurement of the quantity of reference there until a preliminarily most recent lattice sweeping \( X \) is defined as the Monte Carlo output read at sweep \( X \). The Figures 5.1 and 5.2 compare the ground state expectation values for a representative off-diagonal component and for a likewise vicarious (it could be \( \Theta_{44} \) as well, by reason of \( O(4) \) invariance in the regarded system) diagonal component of \( \Theta_{\mu \nu} \), respectively, with \( \hat{\beta} = 2N/g_5^2 \) (the hat of \( \hat{\beta} \) is dropped when paraphrased by the Latin capitals ”BETA” in the plot data captions inside the Figs. 5.1 and 5.2 themselves) for \( N = 2 \) as close as possible to the domain of the SU(2) scaling window.

In four dimensions, the investigated \( \text{abs} \left\langle \Theta_{22} \right\rangle \) has to thermalize towards zero in each model presented above because \( O(4) \) invariance effectively supplies equal numbers of terms and analogous subtractive counterterms then. On the other hand, a model-dependent lattice simulation of \( \text{abs} \left\langle \Theta_{13} \right\rangle \) is the better the more its limiting value lies in vicinity to the corresponding vanishing Euclidean vacuum expectation value for \( \Theta_{13} \) in continuum physics. Since the actually used lattice itself is fairly too coarse.
Figure 5.1: The Euclidean ground state expectation values for an arbitrarily chosen off-diagonal component of the symmetric energy-momentum tensor are plotted as a function of the number of relevant lattice updatings executed so far. The first 10 000 (9999) lattice updatings are discarded and ignored when counting the total number of measurements and the same disregarding is for the sakes of pseudo-decorrelation bound to happen to every 49 of respective 50 subsequent iterations each. There are effectively four diverse graphs, resulting from six different constructions for the whole energy-momentum tensor on the lattice which are discussed in the main part of the text.
Figure 5.2: The same scenario as in Fig. 5.1, but for an arbitrarily selected diagonal component of the energy-momentum tensor this time. Again, six lattice models merge into four effective graphs, but in a manner that is unlike that one concerning Fig. 5.1. Statistical fluctuations, referring to the applied Monte Carlo algorithm, are better visible than in Fig. 5.1 because the scale for the axis of ordinates of the displayed plot is much smaller here.
w.r.t. its spacing and much too small regarding the extension of one periodicity segment Figs. 5.1 and 5.2 reveal that the most convincing results for \( \langle \Theta_{13} \rangle \) and \( \langle \Theta_{22} \rangle \) are obtained if the underlying model is chosen to be of the Caracciolo \([2, 3]\) type \('C A R A '\) (cf. (4.1)) and/or the 4-plaquettes hybridization procedure \('H Y B R I D '\) (3.17) is activated. By construction, the expectation values of \( \Theta_{22} \) are equal for the models \('K A R S C H P U R E '\), \('K A R S C H H Y B R I D '\), and \('O W N P U R E '\), implying that \( \langle \Theta_{22}^{\text{HYBRID}} \rangle \) can thermalize better than \( \langle \Theta_{22}^{\text{HYBRID}} \rangle \).

To the contrary, \( \Theta_{13}^{\text{karschPURE}} = \Theta_{13}^{\text{CARA}} \) and \( \Theta_{13}^{\text{karschHYBRID}} = \Theta_{13}^{\text{HYBRID}} \) are direct consequences of (4.1) and (4.3b). Thus \( \text{abs} \langle \Theta_{13}^{\text{karschHYBRID}} \rangle \) is closer to zero than \( \text{abs} \langle \Theta_{13}^{\text{ownHYBRID}} \rangle \). Nevertheless, there is no real need for \( \langle \Theta_{13} \rangle \) to vanish on the lattice because Euclidean expectation values of the symmetric energy-momentum tensor indeed do not have negative parity—or spacetime 4-parity, respectively. It is intuitively evident that the model \('O W N '\) (4.2) should at least exhibit slightly deteriorated numerical convergence properties relative to \('C A R A '\) (4.1) in the lattice simulation, due to the sensitively more sophisticated algebraic realization of the half-angle concept (3.4) by (3.2). This is in point of fact what is observed. Monte Carlo simulation improvements by 4-plaquettes averaging are throughout not inhibited if the model of reference possesses a uniform construction principle for each component of the energy-momentum tensor, like (4.1) and (4.2).

### 6 Discussion

The mathematical contents of an arbitrary Yang-Mills theory (we do not care about principles of symmetry breaking here) is just the nonabelian generalization of structures that we classically know in the form of electromagnetism. Ref. [13] demonstrates that these structures are pure "whirl" structures basing upon appropriately generalized cross products and curl operations so that we tend to infer from this insight solely that these structures are not suited to "arrive" at the symmetric energy-momentum tensor by a genuine access, like differential forms, e.g.. The same phenomenon is reflected by the circumstance that the only purely special relativistic Noether current representation

\[
\Theta^{\mu \nu} = g^{\mu \nu} \mathcal{L} - g^{\mu \lambda} \frac{\partial \mathcal{L}}{\partial \phi^a} \partial_\lambda \phi^a + \partial_\lambda \Sigma^{\mu \nu \lambda} \tag{6.1}
\]

of the symmetric energy-momentum tensor does not supply D-dimensional
information for the structure of the trace anomaly because $\Sigma^{\mu \nu \lambda} = - \Sigma^{\mu \lambda \nu}$ has to be fitted "by hand" for implying $\Theta^{\mu \nu} = \Theta^{\nu \mu}$. $\Sigma^{\mu \nu \lambda} = 0$ would merely give the canonical energy-momentum tensor, referring to the investigated spacetime translation invariance.

This difficulty can be algebraically overcome by structures with more refined combinations of the Levi-Civita tensor. The continuation of self-similarity arguments to the domain of general relativity in Ref. [13] is associated with the gauge group SO(4), whose structure constants can be in this way described by elementary spacetime tensors exclusively if $\varphi(A) := a$ and $\psi(A) := b$ are the inverse index-index functions relative to

$$A(a, b) := \max(a + 2b - 5, 1) \quad \forall \ a, b \in \{1, \ldots, 4\} \text{ with } b > a, \quad (6.2)$$

obeying $\varphi(A) < \psi(A)$:

$$f_{ABC}^{SO(4)} \equiv -2 \, i \, \text{trace} \left( \left[ \hat{\tau}_A, \hat{\tau}_B \right] - \hat{\tau}_C \right) = \frac{1}{2} \delta^{ag} \delta^{ce} \delta^{df} \sum_{b=1}^{4} \varepsilon_{\varphi(A)\psi(A) a b} \varepsilon_{\varphi(B)\psi(B) c b} \varepsilon_{\varphi(C)\psi(C) d b} \varepsilon_{efgb} \quad \forall \ A, B, C \in \{1, \ldots, 6\}. \quad (6.3)$$

The related $(ict)$-Euclidean Riemann tensor $R_{\alpha \beta \gamma \delta}$ is, again, in contact with a further type of such an ingeniously concatenated combination of Levi-Civita tensors if exterior calculus is utilized for its remodeling into the Einstein tensor $G_{\mu \nu}$, being proportional to the symmetric energy-momentum tensor of general relativity (the $\Theta^{(1)}_{\mu \nu}$ of Sec.2) after evaluation of the Einstein field equations:

$$dx^\alpha \wedge \hat{e}^\beta \left(-\frac{1}{4}\right) \varepsilon_{\alpha \beta \gamma \delta} R^{\gamma \delta}_{\mu \nu} dx^\mu \wedge dx^\nu =$$

$$= \hat{e}^\beta \left(-\frac{1}{4}\right) \varepsilon_{\alpha \beta \gamma \delta} R^{\gamma \delta}_{\mu \nu} \varepsilon^{\alpha \lambda \mu \nu} d^3 \Omega_{\lambda} = \hat{e}^\sigma G^{\rho \sigma} d^3 \Omega_{\rho} \quad (6.4)$$

In contrast to Yang-Mills theories, the geometrodynamical representation of general relativity is spanned by two sorts of base systems, being differential elements $dx^\alpha$ and Cartan base vectors $\hat{e}^\beta$, both kinds of them displayed in (6.4). This peculiarity may be seen in context with (6.3) and genuinely procreates symmetric tensors of the second rank in the framework of exterior calculus. The corresponding Noether current representation uses a variation of the Lagrangian density relative to the metric tensor and this procedure can be extended to Yang-Mills theories, acting as a cryptic (a posteriori invisible) general relativistic transit that specifies the trace anomaly of the symmetric energy-momentum tensor there appropriately.
Therefore a fundamental comprehension of the symmetric energy-momentum tensor $\Theta_{\mu\nu}$ basically has something to do with general relativity. The pattern (2.8) in Sec. 2 is a continuation of this aspect, promoting (4.2) as a new lattice ansatz for $\Theta_{\mu\nu}$. The average expectation value for the trace anomaly of (4.2) would formally be compatible with the description of the trace of the energy-momentum tensor in general relativity referring to the extent of deviation from an ultrarelativistic ideal-gas state of massless gauge bosons (which is comparatively least unrealistic for the non-selfinteracting gauge group $U(1)$—or perhaps for the deconfined high-temperature regime of QCD, as well as for low-temperature gluonium states [8, 10, 17, 18] if the isotropic special case (realizable by gases) is considered for position space, with the stress tensor becoming a product of pressure $p$ and the unit matrix in the comoving system (whereas its trace is of course independent of the choice of the physical reference system—a completely finite $(3 + 1)$-lattice with spacings “$a$” and $a_\tau$ and an inversely defined—anisotropy parameter $\xi = a_\tau/a$ is suitable to the scenario of a field theory at finite physical temperature $T$, where [14, 23], the integration scale for Euclidean “time”, “$y$”, in the action runs from 0 to $1/T$ for $c = \hbar = k_B = 1$):

$$\Theta_{\mu\nu} := \epsilon - 3p = -\frac{xy}{V} \frac{\partial \ln Z(V = \text{const} = x^3 y \propto x^3/T)}{\partial(x y)} = \frac{a}{V} \langle \partial_a S_G \rangle^{\xi = \text{const}} = \langle \partial_a S_G \rangle^{\xi = \text{const}}(6.5)$$

For deriving (6.5), we have used the lattice transfer (thereby suppressing a ground-state energy normalization and integrations over constant field configurations—cf. [9, 10]) of the statistical-physics relationship between the partition function $Z$ and the ground-state expectation value

$$\langle \partial_A S_G \rangle := \langle \Omega | \partial_A S_G | \Omega \rangle = \frac{1}{Z} \int DU (\partial_A S_G) e^{-S_G} = -\partial_A \ln Z \quad (6.6)$$

relative to any suited quantity of reference $A$ ($DU$ is just the appropriate integration measure and does not directly concern the lattice quantity $U_{\mu\nu}$). Repeating this for $A = \xi$ (at fixed couplings and for adjusted ground-state energy normalization, v. [7, 10, 21]) gives

$$\Theta_{44} := \epsilon = -\frac{y}{V} \frac{\partial \ln Z(V/y = \text{const} = x^3)}{\partial y} = \frac{\xi}{V} \langle \partial_\xi S_G \rangle^{a = \text{const}} = \langle \partial_\xi S_G \rangle^{a = \text{const}}, \quad (6.7)$$

whereat $V$ is the 4-volume of one periodicity segment of the regarded anisotropic lattice and $S_G$ is the employed lattice action for the pure gauge field sector.

In lieu of dimensional regularization of the continuum $\Theta_{\mu\nu}$, the (spatial)
lattice spacing "a" is the regulator in (6.5), used in order to prepare the renormalization procedure, which involves the lattice counterpart

$$\beta_L(g_a) = -a \partial g_a / \partial a$$

relative to the positive-sign Callan-Symanzik function in continuum physics. The naïve average formulae (6.5) and (6.7) have a thermodynamic character and compare $\langle \Theta_{\mu \nu} \rangle$ with a renormalization procedure containing the lattice Lagrangian and $\langle \Theta_{44} \rangle$ with the corresponding lattice Hamiltonian, being equal to $\langle \Theta_{44} \rangle$ in the sense pointed out above. The choice (4.2) symbolically attaches standard Wilson form to the mentioned Lagrangian and Hamiltonian!

The same advantage is realized by the $\Theta_{\mu \nu}$ lattice model of the Karsch group, (4.3a) and (4.3b), but at the cost of renouncing a uniform construction principle for its components. On the other hand, the $\Theta_{\mu \nu}$ lattice model of the Pisa group, (4.1), has a uniform construction principle but cannot generically offer a naïve Wilson access to the trace anomaly and the Hamiltonian component $\Theta_{44}$. The new ansatz presented here, (4.2), agrees upon both advantages, but it numerically suffers from a more complicated algebraic construction principle. There are no conservation laws for $\Theta_{\mu \nu}$ on the lattice–except for (4.1), where the SU(N) version of $\Theta_{\mu \nu}^{\text{HYBRID}}$ is conserved perturbatively in 1-loop order.

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