On the multiplicities of families of complex hypersurface-germs
with constant Milnor number

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December 3, 2011

Abstract. We show that the possible drop in multiplicity in a polynomial family
$F(z,t)$ of complex analytic hypersurface singularities with constant Milnor number
is controlled by the powers of $t$. We prove equimultiplicity of $\mu$ constant families
of the form $f + tg + t^2h$ if the singular set of the tangent cone of $\{f = 0\}$ is not
contained in the tangent cone of $\{h = 0\}$.

1. Background.

Let $F : \mathbb{C}^n \times \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function in $z_1, \ldots, z_n$ and $t$.
We study the following conjecture, stated implicitly by Teissier in 1974 in his
Arcata survey [18] as well as at the beginning of his Cargèse paper [17], and which
implies a parametrised version of Zariski’s problem [23] about the topological in-
variance of the multiplicity (Conjecture 1.2 below).

Conjecture 1.1 (Teissier 1972 [17]). If for $t$ in some neighbourhood $U$ of $0$
in $\mathbb{C}$, each function $F(,t)$ has an isolated singularity at the origin with the same
Milnor number $\mu$, then the functions $F(,t)$ have the same multiplicity at $0$ for $t \in U$.

Teissier made the stronger conjecture at Cargèse in 1972 that $\mu$-constancy implies
that the Whitney conditions hold for $(F^{-1}(0), 0 \times \mathbb{C})$. (The same conjecture was
made by Lê Dũng Tráng and Ramanujam in [11], published in 1976, although
submitted in June 1973.) This turned out to be false as first illustrated by the
famous examples of Briançon and Speder [5]. Thus Conjecture 1.1 may also be
considered as a conjecture of Teissier which remains open. It has two corollaries,
as follows.

Conjecture 1.2 (Zariski’s problem for families). Families of complex an-
alytic hypersurfaces with isolated singularities of constant topological type are equi-
multiple.

Proof. This would follow from Conjecture 1.1 because the Milnor number is a
topological invariant (Milnor [13], Teissier [17]). □

Conjecture 1.3 Bekka $(C)$-regular families of complex hypersurfaces are equi-
multiple.

Proof. Use the analogue of the Thom-Mather isotopy theorem for $(C)$-regularity
as proved by Bekka in his thesis [1], together with Conjecture 1.2. □

Equimultiplicity was established in the case of Whitney regularity for general
complex analytic varieties by Hironaka [9], and with a different proof, for Whitney
regularity of families of complex analytic hypersurfaces by Briançon and Speder [6].
The proof of Briançon and Speder was first extended to arbitrary complex analytic
varieties by Navarro Aznar in [14], and the result is a special case of Teissier’s
general characterisation [19] of Whitney regularity in terms of equimultiplicity of polar varieties.

Conjecture 1.3 for the stronger hypothesis of weak Whitney regularity (defined by Bekka and Trotman [2], weak Whitney regularity is weaker than Whitney regularity but stronger than $(C)$-regularity [3]) was proved directly in 2010 [22] by the second author and Duco van Straten, i.e. weak Whitney regularity implies equimultiplicity for families of complex hypersurfaces.

Conjecture 1.2 is still unproved, as is Conjecture 1.3. It is also unknown whether constant topological type implies $(C)$-regularity. It was shown recently by Bekka and Trotman [4] that $(C)$-regularity is in general weaker than weak Whitney regularity for the 1975 quasi-homogeneous examples of Briançon and Speder [5]. No example is currently known of a weakly Whitney regular complex analytic stratification not also satisfying Whitney regularity, while the equivalence of the two conditions has only been proved in the classical case of a family of plane curves, using that weak Whitney regularity implies $(C)$-regularity [3], which implies topological triviality by [1], and hence constant Milnor number, and Whitney regularity is equivalent to constancy of the Milnor number for families of plane curves [18].

Lê-Saito-Teissier criterion for $\mu$ constancy.

According to Lê and Saito [10] and Teissier [17], the constancy of the Milnor number of $F(.,t)$ is equivalent to $F$ being a Thom map, i.e. equivalent to the $(a_F)$ condition being satisfied. This can be reformulated as saying that

$$\frac{|F_t|}{\|F_z\|} \to 0$$

as $(z,t) \to (0,0)$

where $F_t$ is notation for $\frac{\partial F}{\partial t}$, $F_z$ is notation for $(\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_n})$, $|.|$ denotes the modulus of a complex number and $\|.|\|$ denotes the hermitian norm on $\mathbb{C}^n$.

In this paper we use this criterion for constancy of the Milnor number to determine some situations when equimultiplicity holds (Propositions 1.1 and 3.2), and to reduce possible jumps in the multiplicity (Propositions 2.1 and 2.2).

Write $F(z,t) = f(z) + \sum_{k \geq 1} t^k g_k(z)$.

Then $F_t = \sum_{k \geq 1} k t^{k-1} g_k$, and $F_z = f_z + \sum_{k \geq 1} t^k (g_k)_z$.

Due to its upper semicontinuity the multiplicity is non-constant iff $m = m(f) > m_1 = \min \limits_{k \geq 1} m(g_k)$ for all $t$ in some punctured neighbourhood of 0.

**Proposition 1.1** If $F(z,t) = f(z) + tg(z)$ is a 1-parameter family of isolated complex analytic hypersurface singularities whose Milnor numbers are constant, then the multiplicity at 0 of $g$ is greater than or equal to the multiplicity at 0 of $f$.

**Proof.**

Suppose that $m(f) = m > m_1 = m(g)$.

Consider analytic arcs $\gamma : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^{n+1}, 0)$, $\gamma(u) = (z(u), t(u))$, such that $\gamma(0) = 0 \in \mathbb{C}^{n+1}$. We must find an arc $\gamma$ such that

$$\frac{|F_t(\gamma(u))|}{\|F_z(\gamma(u))\|} \not\to 0$$

as $u \to 0$. 

For any analytic function $Q : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, write $V(Q)$ for the order in $u$ at 0 of $Q \circ \gamma : \mathbb{C} \to \mathbb{C}$, and for any analytic function $P : \mathbb{C}^n \to \mathbb{C}$ write $v(P)$ for the order in $u$ at 0 of $P \circ \pi_z \circ \gamma$. We must choose an analytic arc $\gamma$ such that $V(F_i) - \min\{v(\partial F/\partial z_i)\} \leq 0$.

Now $V(F_i) - \min\{v(\partial F/\partial z_i)\} = v(g) - \min\{v(\partial f/\partial z_i + t\partial g/\partial z_i)\}$. Let $\gamma(u) = (u_{z_0}, 0)$ where $z_0 \in \mathbb{C}^n - \{0\}$.

Then $V(F_i) - \min\{v(\partial F/\partial z_i)\} = v(g) - \min\{v(\partial f/\partial z_i)\}$. For $z_0$ sufficiently general, the right-hand side is $v(g) - (v(f) - 1) = m_1 - m + 1 \leq 0$, because $m_1 < m$.

Thus $V(F_i) - \min\{v(\partial F/\partial z_i)\} \leq 0$, contradicting the hypothesis that $\mu$ be constant, using the Lê-Saito-Teissier characterisation. □

**Remark 1.2.** The result of Proposition 1.1 was discovered by the second author during the academic year 1976-77 and announced in a talk given in March 1977 [18] as one of the weekly A’Campo-MacPherson singularity seminars at the University of Paris 7. The text of this talk was included in the second author’s Thèse d’Était [21] defended at Orsay in January 1980. The result was rediscovered by Gert-Martin Greuel in 1986 [8], and used to prove Teissier’s conjecture in the case of quasi-homogeneous, and semi-quasihomogeneous functions $f$.

**Remark 1.3.** Parusinski [16] has proved that a $\mu$-constant family of the form $f + tg$ has constant topological type by integrating an appropriate vector field, with an argument which works for all $n$. Lê Dũng Tráng and Ramanujam [11] proved that for a $\mu$ constant family of complex hypersurfaces defined by $\{F(z, t) = 0\}$, the hypersurfaces $\{F(z, t) = 0\} \cap \{\mathbb{C}^n \times \{t\}\}$ have constant topological type when $n \neq 3$.

2. Controlling multiplicity.

**Proposition 2.1.** If $F(z, t) = f(z) + tg(z) + t^2h(z)$ is an analytic 1-parameter family of isolated hypersurface singularities with Milnor number $\mu$ constant, then the multiplicity at the origin $g$ is greater than or equal to the multiplicity $m$ at the origin of $f$, and the multiplicity at $0$ of $h$ is greater than or equal to $m - 1$.

**Proof.** Because

$$\frac{|F_i|}{|F_i|} = \frac{|g + 2th|}{|f_z +tg_z + t^2h_z|},$$

it follows that on a generic curve of the form $(uz_0, 0)$ with $z_0 \neq 0$, $V(F_i) - V(F_z) = m(g) - v(f_z) = m(g) - m + 1$. Hence if $m(g) - m + 1 \leq 0$, i.e. $m(g) < m$, we obtain a contradiction to the hypothesis that the Milnor number remains constant.

Thus we obtain that the coefficient $g$ of $t$ has multiplicity $m(g) \geq m = m(f)$.

Suppose that $m(h) \leq m - 2$.

On a generic curve of the form $(uz_0, ut_0)$, with both $z_0 \neq 0$ and $t_0 \neq 0$, if $\Delta = \frac{|F_i|}{|F_i|}$, then $\Delta \sim \frac{|g + 2th|}{|f_z + tg_z + t^2h_z|}$.

Hence $V(\Delta) = 1 + m(h) - \min\{m - 1, 2 + m(h) - 1\} = 1 + m(h) - (m(h) + 1) = 0$.

This again contradicts the hypothesis that the Milnor number of the family $F(\cdot, t)$ is constant, proving that $m(h)$ is at least $m - 1$. □

Now we generalize to arbitrary deformations of $f$ which are polynomial in $t$.

**Proposition 2.2.** If the family $F(z, t) = f(z) + tg_1(z) + t^2g_2(z) + t^3g_3(z) + \cdots + t^r g_r(z)$ has constant Milnor number at $(0, t)$ as $t$ varies in a neighbourhood of 0,
and $f$ has multiplicity $m$ at the origin, then $m(g_1) \geq m$, $m(g_2) \geq m - 1, \ldots$, and $m(g_r) \geq m - r + 1$.

**Proof.**

Here, $V(F_1) - V(F_z) = \min\{(k-1) + (v(g_k)) - V(f_z + \sum_{k \geq 1} t^k(g_k)z)\}$, assuming that we are on a generic arc for which there is no cancellation of terms in the expression for $F_t$.

In particular, $V(F_1) = \min\{m(g_1), m(g_2) + 1, m(g_r) + r - 1\}$, and $V(F_z) = \min\{m(f) - 1, m(g_1) - 1 + 1, m(g_2) - 1 + 2, \ldots, m(g_r) - 1 + r\} = \min\{m - 1, V(F_1)\}$.

But the family $F(z,t)$ has constant Milnor number, so that $V(F_1) > V(F_z)$, by the Lê-Saito-Teissier theorem. The conclusion follows. □

It is interesting to compare the previous result with the following, proved by Greuel in 1986 [8]. Observe the extra restrictions Greuel imposed on the $\{g_i\}$.

**Proposition 2.3 (Greuel).** Let $\lambda_j : (\mathbb{C},0) \rightarrow (\mathbb{C},0)$ and $g_j : (\mathbb{C}^n,0) \rightarrow (\mathbb{C},0)$, $j = 1, \ldots, r$, be holomorphic functions such that $\lambda_j \neq 0$ and the initial forms of $g_j$ are $\mathbb{C}$-linearly independent. Assume that

$$F(z,t) = f(z) + \sum_{j=1}^r \lambda_j(t)g_j(z)$$

is a $\mu$-constant unfolding of $f$. Then $\nu(\lambda_j) + m(g_j) > m(f)$ for all $j = 1, \ldots, r$, where $\nu(\lambda)$ denotes the order in $t$ of $\lambda(t)$.

### 3. Obtaining equimultiplicity.

**Proposition 3.1.** Let $F(z,t)$ be a $\mu$-constant family of complex hypersurfaces with isolated singularities at $z = 0$ for each $t$ in a neighbourhood of 0, of the form $F(z,t) = f(z) + \sum_{k=1}^r t^k g_k(z)$. Suppose that the tangent cone of $f$ has an isolated singularity at 0. Then the multiplicity $m(F(.,t))$ is constant as $t$ varies in a neighbourhood of 0.

**Proof.** Suppose that the tangent cone $\{f_m = 0\}$ has an isolated singularity. Then $f$ is semi-homogeneous, in particular semi-quasihomogeneous, and by work of Varchenko, Greuel [8] and O'Shea [16] independently proved equimultiplicity. The special case of homogeneous $f$ was previously treated by Gabrielov and Kouchinenko [7]. □

Motivated by the proof of Proposition 2.1, we could study what happens in a family $F(z,t) = f(z) + t q(z) + t^2 h(z)$ with constant Milnor number if we take a more general generic curve of the form $(u^p z_0, u^p t_0)$ with $z_0 \neq 0, t_0 \neq 0$, and where $p \neq q$ and $p$ and $q$ are non-negative integers. This turns out not to be fruitful however.

So we change tactics by choosing an appropriate non-generic line segment, whereas the previous results were obtained by choosing suitable generic line segments.

**Proposition 3.2.** Let $F(z,t)$ be a $\mu$-constant family of complex hypersurfaces with isolated singularities at $z = 0$ for each $t$ in a neighbourhood of 0, of the form $F(z,t) = f(z) + t q(z) + t^2 h(z)$. Suppose that the singular set of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{h = 0\}$. Then the multiplicity $m(F(.,t))$ is constant as $t$ varies in a neighbourhood of 0.
Proof. By Proposition 2.1, we know that $m(g) \geq m = m(f)$, and that $m(h) \geq m - 1$.

Assume that $m(h) = m - 1$.

For a complex line segment $\gamma(u) = (uz_0, ut_0)$, calculating

$$V(\Delta) = V\left(\frac{|g + 2th|}{|f_z + tg_z + t^2h_z|}\right) = V(g + 2th) - \inf \{V(f_z + tg_z + t^2h_z)\},$$

we note that if $\inf \{v(f_z)\} > m - 1$ then the tangent cone of $\{f = 0\}$ in $\mathbb{C}^n$ must have a non isolated singularity at 0 and the line segment $\gamma(u) = (uz_0, ut_0)$ must be such that $uz_0$ lies in the singular locus $\Sigma(f_m)$ of the tangent cone to $\{f = 0\}$. If in addition $\Sigma(f_m)$ is not contained in the tangent cone to $\{h = 0\}$, then $v(h) = m(h)$, which equals $m - 1$ by assumption, and because $m(g) \geq m$ we can choose a generic $t_0$ in $\mathbb{C}$ so that

$$V(g + th) = V(th) = m,$$

and

$$V(f_z + tg_z + t^2h_z) \geq m.$$

It follows that $V(\Delta) \leq m - m = 0$, which implies a contradiction to the hypothesis of constant Milnor number, by the Lê-Saito-Teissier criterion. □

Remark 3.3. Similarly to the argument in the previous proof we can obtain a contradiction to the hypothesis of constant Milnor number by the Lê-Saito-Teissier criterion if $m(g) = m$ and $\Sigma(f_m)$ is not contained in the tangent cone to $\{g = 0\}$. Thus if a family $F(z, t) = f(z) + tg(z) + t^2h(z)$ has constant Milnor number, and the singular locus of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{g = 0\}$, then $m(g) \geq m + 1$.

More generally, the same argument shows that if a family $F(z, t) = f(z) + tg_1(z) + t^2g_2(z) + t^3g_3(z) + \cdots + t^r g_r(z)$ has constant Milnor number, and the singular locus of the tangent cone of $\{f = 0\}$ is not contained in the tangent cone of $\{g_k = 0\}$, then $m(g_k) \geq m - k + 2$.

Bibliography.

1. K. Bekka, C-régularité et trivialité topologique, *Singularity theory and applications, Warwick 1989* (eds. D. M. Q. Mond and J. Montaldi), Springer Lecture Notes 1462 (1991), 42-62.
2. K. Bekka and D. Trotman, Propriété métriques de familles Φ-radiales de sous-variétés différentiables, *C. R. Acad. Sci., Paris* 305 (1987), 389-392.
3. K. Bekka and D. Trotman, Weakly Whitney stratified sets. *Real and complex singularities (Proceedings, Sao Carlos 1998, edited by J. W. Bruce and F. Tari), Chapman and Hall/CRC*, (2000) 1-15.
4. K. Bekka and D. Trotman, Briançon-Speder type examples and the failure of weak Whitney regularity, preprint, 2011.
5. J. Briançon and J.-P. Speder, La trivialité topologique n’implique pas les conditions de Whitney, *C. R. Acad. Sci., Paris* 280 (1975), 365-367.
6. J. Briançon and J.-P. Speder, Les conditions de Whitney impliquent $\mu^*-$ constant, *Annales de l’Institut Fourier, Grenoble*, 26 (2) (1976), 153-163.
7. A. Gabrielov and A. Kouchnirenko, Description of deformations with constant Milnor number for homogeneous functions, *Funct. Analysis* 9 (1975), 67-68.
8. G.-M. Greuel, Constant Milnor number implies constant multiplicity for quasihomogeneous singularities, *Manuscripta Math.* 56 (1986), 159-166.
9. H. Hironaka, Normal cones of analytic Whitney stratifications, *Publ. Math. IHES.* 36 (1969), 127-138.
10. Lê Dung Tráng and K. Saito, La constance du nombre de Milnor donne des bonnes stratifications, *C. R. Acad. Sci. Paris* 277 (1973), 793-795.
11. Lê Dung Tráng and C. P. Ramanujam, The invariance of Milnor’s number implies the invariance of the topological type, *Amer. J. of Math.* 98 (1976), 67-78.
12. Lê Dung Tráng and B. Teissier, Report on the problem session, *Singularities, Part 2 (Arcata, Calif., 1981)*, *Proc. Sympos. Pure Math.*, 40, Amer. Math. Soc., Providence, R.I. (1983), 105-116.
13. J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, Princeton, 1968.
14. V. Navarro Aznar, Conditions de Whitney et sections planes, *Inventiones Math.* 61 (1980), 199-225.
15. D. O’Shea, Topologically trivial deformations of isolated quasihomogeneous hypersurface singularities are equimultiple, *Proc. Amer. Math. Soc.* 101 (1987), 260-262.
16. A. Parusinski, Topological triviality of $\mu$-constant deformations of type $f(x) + tg(x)$, *Bull. London Math. Soc.* 31 (1999), 686-692.
17. B. Teissier, Cycles évanescents, sections planes et conditions de Whitney, *Singularités à Cargèse (ed. F. Pham)*, *Astérisque* 7-8 (1973), 285-362.
18. B. Teissier, Introduction to equisingularity problems, *Algebraic Geometry (ed. F. Pham)*, *Proc. AMS summer symposia* 29 (1975), 593-632.
19. B. Teissier, Variétés polaires II: Multiplicités polaires, sections planes, et conditions de Whitney, *Algebraic Geometry Proceedings, La Rabida 1981*, Lecture Notes in Math. 961, Springer-Verlag, New York(1982), 314-491.
20. D. Trotman, Partial results on the topological invariance of the multiplicity of a complex hypersurface, *Séminaire A’Campo-MacPherson, Université de Paris 7*, 26 March 1977.
21. D. Trotman, *Équisingularité et conditions de Whitney*, Thèse d’État, Université de Paris-Sud, Orsay, 1980.
22. D. Trotman and D. Van Straten, Weak Whitney regularity implies equimultiplicity for families of singular complex analytic hypersurfaces, preprint, 2011.
23. O. Zariski, Some open questions in the theory of singularities, *Bull. Amer. Math. Soc.* 77 (1971), 481-491.

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