Light-front versus Bethe-Salpeter forms of two nucleon amplitudes*

S.G. Bondarenko¹, V.V. Burov¹, M. Beyer², and S.M. Dorkin³

¹ Bogoliubov Laboratory of Theoretical Physics, JINR Dubna, 141980 Russia
² Department of Physics, University of Rostock, 18051 Rostock, Germany
³ Far Eastern State University of Vladivostok, 690000 Russia

Abstract. We discuss the relation between the two nucleon Bethe-Salpeter amplitude and the light front wave functions. Both approaches provide a covariant description for the deuteron bound state and the two nucleon scattering state. A comparison is done for the spin-orbit functions as well explicit integrals are given on the basis of the Nakanishi integral representation method.

1 Introduction

The description of deuteron properties and reactions involving the deuteron is reflected in a large part of Arenhövel’s work. Relaying also on his analyzes of experimental data the discussion of relativistic issues in reactions involving the deuteron has become more and more important in recent years. There are well known examples of clear experimental evidence, in particular in the electromagnetic disintegration of the deuteron [2, 3].

Meanwhile two relativistic approaches to the nucleon nucleon system have received special attention in the last few years. One is based on the Bethe-Salpeter equation [4] and its various three-dimensional reductions. Based on quantum field theory the Bethe-Salpeter formalism is four-dimensional and explicitly Lorentz covariant. Interactions (e.g. electromagnetic currents) are consistently treated via the Mandelstam formalism leading to Feynman diagrams and the corresponding rules. The second approach considered here is based on light front dynamics [5]. The state vector describing the system is expanded in Fock components defined on a hypersphere in the four-dimensional space time. This approach is intuitively appealing since it is formally close to the nonrelativistic conception in terms of Hamiltonians, and state vectors maybe directly interpreted as wave functions.

The equivalence between these field theoretic and light front approaches has been a subject of recent discussions, see e.g. [6] and references therein. A comparison of both approaches for the deuteron as a two body system clarifies the

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structure of the different components of the amplitude. It is also useful in the context of three particle dynamics where the proper covariant and/or light front construction of the nucleon amplitude in terms of three valence quarks (including spin dependence and configuration mixing) is presently discussed [7, 8]. Although the relation between light front and Bethe-Salpeter amplitudes for the two nucleon amplitude has been spelled out to some extend in a recent report [9] we provide here some useful details and additionally discuss the use of the Nakanishi representation.

To proceed we first present different ways to construct a complete (covariant) Bethe-Salpeter amplitude, see, e.g. Ref. [10]. In particular, we consider the so called direct product and the matrix representation form. Besides the spin structure of the wave function (amplitudes) we also present a comparison of the “radial” part of the amplitude on the basis of the Nakanishi integral representation [11]. This integral representation – well known and elaborated for the scalar case, see, e.g. Ref [12], however, not so frequently used – allows us to establish a connection between the different approaches also for the weight functions (or densities) that has not been done so far and is relevant for a treatment of the Bethe-Salpeter equation in Minkowski space.

In the following we present different ways to construct a complete (covariant) Bethe-Salpeter amplitude. In this context the so called direct product representation used in the rest frame of the nucleon nucleon system using the \( \rho \)-spin notation is close to the nonrelativistic coupling scheme and provides states of definite angular momentum. To construct the covariant basis this form is transformed into a matrix representation which will then be expressed in terms of Dirac matrices. A generalization to arbitrary deuteron momenta finally leads to the covariant representation of the Bethe-Salpeter amplitude. This will be explained in the next section along with an explicit construction of the deuteron (\( J = 1 \)) and the \( J = 0 \) nucleon nucleon state.

The construction of the light front form from the Bethe-Salpeter amplitude will be given in Section 3. Presentation of the light front approach will be kept concise here, since is has been presented at length in a recent report [9]. Again we show the results for \( J = 1 \) and \( J = 0 \) deuteron and scattering states. Finally, we present the analysis in terms of the Nakanishi integral representation.

2 The Bethe-Salpeter approach to the two nucleon system

Commonly, two forms are utilized to describe the Bethe-Salpeter wave functions (amplitudes) known as direct product form and matrix form. They will be explained in the following.

For convenience, we introduce the \( \rho \)-spin notation for Dirac spinors with momentum \( p \), and spin projection \( \mu \),

\[
U^\rho_\mu(p) = \begin{cases} 
  u_\mu(p), & \rho = + \\
  v_{-\mu}(-p), & \rho = - 
\end{cases}
\]  

(1)
The Dirac spinors \( u_\mu(p), v_\mu(p) \) are defined according to Ref. [13], viz.

\[
\begin{align*}
  u_\mu(p) &= \mathcal{L}(p)u_\mu(0), \\
  v_\mu(p) &= \mathcal{L}(p)v_\mu(0).
\end{align*}
\]  

(2)

where the boost of a spin-\( \frac{1}{2} \) particle with mass \( m \) is given by

\[
\mathcal{L}(p) = \frac{m + p \cdot \gamma_0}{\sqrt{2E(m + E)}},
\]  

(3)

where the nucleon four momentum is \( p = (E, p) \), and \( E = \sqrt{p^2 + m^2} \). In the rest frame of the particles the spinors are given by

\[
\begin{align*}
  u_\mu(0) &= \left( \begin{array}{c} \chi_\mu^0 \\ \bar{0} \end{array} \right), \\
  v_\mu(0) &= \left( \begin{array}{c} 0 \\ \bar{\chi}_{-\mu} \end{array} \right).
\end{align*}
\]

In the direct product form the basis of the two particle spinor in the rest frame is represented by

\[
U_{\rho_1}^{\mu_1}(p) U_{\rho_2}^{\mu_2}(-p).
\]  

(4)

Which one of the combinations \( \rho_1 \rho_2 \) are actually present in a particular amplitude depends on the parity and permutation symmetries required. E.g. since \( U^+U^- \) is parity odd, in the deuteron this combination could only appear for \( L = 1 \). Therefore the appearance of \( P \)-states is a typical relativistic effect, because \( U^- \rightarrow 0 \) for the nonrelativistic limit. The spin-angular momentum part \( Y_M^\alpha(p) \) of the two nucleon amplitude is then given by

\[
Y_M^\alpha(p) = i^L \sum_{\mu_1 \mu_2 m_L} \langle L m_L S m_S | J M \rangle \langle \frac{1}{2} \mu_1 \frac{1}{2} \mu_2 | S m_S \rangle Y_{Lm_L}(\hat{p}) U_{\mu_1}^{\rho_1}(p) U_{\mu_2}^{\rho_2}(-p),
\]

(5)

where \( \langle \cdot | \cdot \rangle \) denotes the Clebsch-Gordan coefficient, and \( \hat{p} = p/|p| \). The decomposition is according to the quantum numbers of relative orbital angular momentum \( L \), total spin \( S \), total angular momentum \( J \) with projection \( M \), and \( \rho \)-spin \( \rho_1 \), \( \rho_2 \), collectively denoted by \( \alpha \) [14]. The Bethe-Salpeter amplitude of the deuteron with mass \( M_d \) is then written in the following way (see also [15])

\[
\Phi_{JM}(\vec{P}, p) = \sum_\alpha g_\alpha(p_0, |p|) Y_M^\alpha(p),
\]

(6)

where \( \vec{P} = (M_d, 0) \). The radial parts of the wave function are denoted by \( g_\alpha(p_0, |p|) \).

The matrix representation of the Bethe-Salpeter amplitude [11] is obtained from the above expression Eq. (3) by transposing the spinor of the second particle. In the rest frame of the system this reads for the basis spinors

\[
U_{\mu_1}^{\rho_1}(p) U_{\mu_2}^{\rho_2}(-p) \rightarrow U_{\mu_2}^{\rho_2}(p) U_{\mu_1}^{\rho_1}(-p),
\]

(7)

which is now a \( 4 \times 4 \) matrix in the two particle spinor space. The nucleon nucleon Bethe-Salpeter wave function in this basis is then represented by

\[
\Psi_{JM}(P(0), p) = \sum_\alpha g_\alpha(p_0, |p|) \Gamma_M^\alpha(p) C.
\]

(8)
where $C$ is the charge conjugation matrix, $C = i\gamma_2\gamma_0$, and $\Gamma_{M}^{\alpha}$ is defined as $\mathcal{Y}_{M}^{\alpha}$ where the replacement Eq. (7) is used.

As an illustration we give an example how to calculate the spin-angular momentum part of the vertex function for the $^3S^+_1$ state, where we use the spectroscopic notation $^{2S+1}L_{J}\rho_{J}\rho_{2}$ of Ref. [4]. In this case

$$\sqrt{4\pi} \Gamma_{M}^{^3S^+_1}(\mathbf{p}) = \sum_{\mu_1\mu_2} \langle \frac{1}{2} \mu_1 \frac{1}{2} \mu_2 |1M \rangle u_{\mu_1}(\mathbf{p}) u_{\mu_2}^{\top}(-\mathbf{p})$$

$$= \mathcal{L}(\mathbf{p}) \sum_{\mu_1\mu_2} \langle \frac{1}{2} \mu_1 \frac{1}{2} \mu_2 |1M \rangle \begin{pmatrix} \chi_{\mu_1} \\ 0 \end{pmatrix} \begin{pmatrix} \chi_{\mu_2}^{\top} \\ 0 \end{pmatrix} \mathcal{L}^{\top}(-\mathbf{p})$$

$$= \mathcal{L}(\mathbf{p}) \left( \sum_{\mu_1\mu_2} \langle \frac{1}{2} \mu_1 \frac{1}{2} \mu_2 |1M \rangle \chi_{\mu_1} \chi_{\mu_2}^{\top} 0 0 \right) \mathcal{L}^{\top}(-\mathbf{p})$$

$$= \mathcal{L}(\mathbf{p}) \frac{1 + \gamma_0}{2} \begin{pmatrix} 0 & -\sigma \cdot \epsilon_{M} \\ \sigma \cdot \epsilon_{M} & 0 \end{pmatrix} \begin{pmatrix} 0 & -i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \mathcal{L}^{\top}(-\mathbf{p})$$

$$= \frac{1}{2E(m + E)} \sqrt{2} (m + \gamma \cdot p_1) \frac{1 + \gamma_0}{2} \gamma \cdot \epsilon_{M} (m - \gamma \cdot p_2) C,$$

Here we make use of $\sqrt{2} \sum_{\mu_1\mu_2} \langle \frac{1}{2} \mu_1 \frac{1}{2} \mu_2 |1M \rangle \chi_{\mu_1} \chi_{\mu_2}^{\top} = (\sigma \cdot \epsilon_{M}) (i\sigma_2)$, where $\epsilon_{M}$ is the polarization vector of the spin-1 composite system with the components in the rest frame given by

$$\epsilon_{+1} = (-1, -i, 0)/\sqrt{2}, \quad \epsilon_{-1} = (1, -i, 0)/\sqrt{2}, \quad \epsilon_0 = (0, 0, 1), \quad (9)$$

and the four-vector $\epsilon_{M} = (0, \epsilon_{M})$. This replacement can be done for all Clebsch-Gordan coefficients that in turn allows us to write the basis in terms of Dirac matrices.

To keep the notation short the $\rho$-spin dependence is taken out of the matrices and therefore the spin-angular momentum functions $\Gamma_{M}^{\alpha}(\mathbf{p})$ are replaced in the following way

$$\Gamma_{M}^{\tilde{\alpha},++}(\mathbf{p}) = \frac{\gamma \cdot p_1 + m}{\sqrt{2E(m + E)}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}_{M}^{\tilde{\alpha}}(\mathbf{p}) \frac{\gamma \cdot p_2 - m}{\sqrt{2E(m + E)}},$$

$$\Gamma_{M}^{\tilde{\alpha},--}(\mathbf{p}) = \frac{\gamma \cdot p_2 - m}{\sqrt{2E(m + E)}} \frac{1 - \gamma_0}{2} \tilde{\Gamma}_{M}^{\tilde{\alpha}}(\mathbf{p}) \frac{\gamma \cdot p_1 + m}{\sqrt{2E(m + E)}},$$

$$\Gamma_{M}^{\tilde{\alpha},+-}(\mathbf{p}) = \frac{\gamma \cdot p_1 + m}{\sqrt{2E(m + E)}} \frac{1 + \gamma_0}{2} \tilde{\Gamma}_{M}^{\tilde{\alpha}}(\mathbf{p}) \frac{\gamma \cdot p_2 - m}{\sqrt{2E(m + E)}},$$

$$\Gamma_{M}^{\tilde{\alpha},-+}(\mathbf{p}) = \frac{\gamma \cdot p_2 - m}{\sqrt{2E(m + E)}} \frac{1 - \gamma_0}{2} \tilde{\Gamma}_{M}^{\tilde{\alpha}}(\mathbf{p}) \frac{\gamma \cdot p_1 + m}{\sqrt{2E(m + E)}},$$

(10)

with $\tilde{\alpha} = ^{2S+1}L_{J}$. The matrices $\tilde{\Gamma}_{M}^{\tilde{\alpha}}$ for $J = 0, 1$ states are given later in Tabs. [2.1] and [2.2].

To conclude this paragraph we give the following useful relations. The adjoint functions are defined through

$$\tilde{\Gamma}_{M}^{\alpha}(\mathbf{p}) = \gamma_0 [\Gamma_{M}^{\alpha}(\mathbf{p})]^{\dagger} \gamma_0,$$  

(11)
and the orthogonality condition is given by

$$\int d^2 \hat{p} \ Tr\{\Gamma^\alpha_M(p)\Gamma^\alpha_{M'}(p)\} = \delta_{MM'}\delta_{\alpha\alpha'}.$$  \hfill (12)

In addition, for identical particles the Pauli principle holds, which reads

$$\Psi_{JM}(\vec{P},p) = -P_{12}\Psi_{JM}(\vec{P},p) = (-1)^{I+1}C\left[\Psi_{JM}(\vec{P},-p)\right]^\top C.$$  \hfill (13)

where $I$ denotes the channel isospin. This induces a definite transformation property of the radial functions $g_{\alpha}(p_0,|\vec{p}|)$ on replacing $p_0 \rightarrow -p_0$, which is even or odd, depending on $\alpha$. Also, since the $P^{\rho_1\rho_2}$ amplitudes do not have a definite symmetry we use instead

$$\Gamma^P_{\rho_e} = \frac{1}{\sqrt{2}}(\Gamma^{P,+}_M + \Gamma^{P,-}_M),$$

$$\Gamma^P_{\rho_o} = \frac{1}{\sqrt{2}}(\Gamma^{P,+}_M - \Gamma^{P,-}_M).$$  \hfill (14)

These functions have definite even(e) or odd(o) $\rho$-parity, which allows us to define a definite symmetry behavior under particle exchange.

We now discuss the $^1S_0$ channel and the deuteron channel in some detail.

2.1 The $^1S_0$ channel

For the two nucleon system in the $J = 0$ state the relativistic wave function consists of four states, i.e. $^1S_0^{++}$, $^1S_0^{--}$, $^3P_0^e$, $^3P_0^o$, labeled by 1, ... , 4 in the following. The Dirac matrix representation of the spin structures are shown in Table 2.1. Note the formally covariant relation for $|\vec{p}|$, and also for $p_0$ and $E$,

| $\alpha$ | $\sqrt{8\pi} \tilde{\Gamma}_0^{\alpha}$ |
| --- | --- |
| $^1S_0$ | $-\gamma_5$ |
| $^3P_0$ | $|\vec{p}|^{-1}(\gamma \cdot p_1 - \gamma \cdot p_2)\gamma_5$ |

used in the following,

$$p_0 = \frac{\vec{P} \cdot \vec{p}}{M}, \quad E = \sqrt{\frac{(\vec{P} \cdot \vec{p})^2}{M^2} - p^2 + m^2}, \quad |\vec{p}| = \sqrt{\frac{(\vec{P} \cdot \vec{p})^2}{M^2} - p^2}.$$  \hfill (15)

Eq. (10) along with Table 2.1 may now be used as a guideline to construct covariant expressions for the $J = 0$ nucleon nucleon Bethe-Salpeter amplitude.
This will be achieved by allowing the momenta involved to be off-shell. Introducing then four Lorentz invariant functions $h_i(P \cdot p, p^2)$ this amplitude is given by

$$\Psi_{00}(P,p) = h_1 \gamma_5 + h_2 \frac{1}{m} (\gamma \cdot p_1 \gamma_5 + \gamma_5 \gamma \cdot p_2) + h_3 \left( \frac{\gamma \cdot p_1 - m}{m} \gamma_5 - \frac{\gamma_5 \gamma \cdot p_2 + m}{m} \right) + h_4 \frac{\gamma \cdot p_1 - m}{m} \gamma_5 \frac{\gamma_5 \gamma \cdot p_2 + m}{m}$$

(16)

The connection between the invariant functions $h_i(P \cdot p, p^2)$ and the functions $g_i(p_0, |p|)$ given before is achieved by expanding the Dirac matrices appearing in Eq. (16) into the $\Gamma^\alpha$. The resulting relation is

$$h_1 = -\sqrt{2} D_1 (g_1 + g_2) - \mu p_0 |p|^{-1} g_3 - 4m|p|^{-1} D_0 g_4$$
$$h_2 = \frac{1}{4m} |p|^{-1} g_3$$
$$h_3 = 8a_0 m^2 (g_1 + g_2) - \frac{1}{2} \mu p_0 |p|^{-1} g_3 - 8a_0 m |p|^{-1} \epsilon (m - E) g_4$$
$$h_4 = -4a_0 \sqrt{2} m^2 (g_1 + g_2) + 8a_0 m^3 |p|^{-1} g_4$$

(17)

where $a_0 = 1/(16ME)$, $\epsilon = 2m + E$, $\mu = m/M$, $M = \sqrt{(p_1 + p_2)^2}$, and

$$D_0 = a_0 (4p_0^2 + 16m^2 - 12E^2 - M^2),$$
$$D_1 = a_0 (-M^2/4 + p_0^2 - E^2 + 16m^2 + ME).$$

(18) (19)

Note, that only $h_2$ and $g_3$ are odd with respect to $p_0 \to -p_0$, and all other functions are even.

2.2 $3S_1 - 3D_1$ channel

In the deuteron channel the relativistic wave function consists of eight states, i.e. $3S_1^+, 3S_1^-, 3D_1^+, 3D_1^-, 3P_1^e, 3P_1^o, 1P_1^e, 1P_1^o$, labeled by 1, ..., 8 in the following. There Dirac matrix representation of the spin structures $\tilde{\Gamma}_M^\alpha$ is shown in Table 2.2.

**Table 2.2.** Spin angular momentum parts $\tilde{\Gamma}_M^\alpha$ for the deuteron channel

| $\tilde{\alpha}$ | $\sqrt{8\pi} \tilde{\Gamma}_M^\alpha$ |
|------------------|----------------------------------|
| $3S_1^+$         | $\gamma \cdot \epsilon_M$      |
| $3S_1^-$         | $- \frac{1}{\sqrt{2}} \left[ \gamma \cdot \epsilon_M + \frac{3}{8} (\gamma \cdot p_1 - \gamma \cdot p_2)p \cdot \epsilon_M |p|^{-2} \right]$ |
| $3D_1^+$         | $\sqrt{2} \left[ \frac{3}{2} \gamma \cdot \epsilon_M (\gamma \cdot p_1 - \gamma \cdot p_2) - p \cdot \epsilon_M \right] |p|^{-1}$ |
| $3D_1^-$         | $\sqrt{2} \left[ \frac{3}{2} \gamma \cdot \epsilon_M (\gamma \cdot p_1 - \gamma \cdot p_2) + p \cdot \epsilon_M \right] |p|^{-1}$ |
| $3P_1^e$         | $\sqrt{3} \gamma \cdot \epsilon_M |p|^{-1}$ |
| $3P_1^o$         | $\sqrt{3} \gamma \cdot \epsilon_M |p|^{-1}$ |
| $1P_1^e$         | $\sqrt{3} \gamma \cdot \epsilon_M |p|^{-1}$ |
| $1P_1^o$         | $\sqrt{3} \gamma \cdot \epsilon_M |p|^{-1}$ |

Again, generalizing the Dirac representation it is possible to achieve a covariant form of the Bethe-Salpeter amplitude with eight Lorentz invariant functions.
Note now, that for the deuteron, the functions $h_i(P \cdot p, p^2)$ and $g_i(p_0, |p|)$ are connected via

$$h_1 = D_1^+(g_3 - \sqrt{2}g_1) + D_1^-(g_4 - \sqrt{2}g_2) + \frac{1}{2} \sqrt{6} \mu_0 |p|^{-1} g_5 + \sqrt{6} \mu_0 |p|^{-1} g_6$$

$$h_2 = \sqrt{2}(D_2^- g_1 + D_2^+ g_2) - D_3^+ g_3 - D_3^- g_4 - \frac{1}{2} \sqrt{6} \mu_0 |p|^{-1} g_5 - \sqrt{6} g_6 + \sqrt{3} m^2 |p|^{-1} E^{-1} g_7$$

$$h_3 = -\frac{1}{2} \sqrt{6} m |p|^{-1} g_5$$

$$h_4 = 8 a_1 \sqrt{2} m p_0 (g_1 - g_2) + 8 a_2 \varepsilon m p_0 (g_3 - g_4)$$

$$h_5 = 16 a_0 \sqrt{3} m^2 |p|^{-1} (p_0 g_7 - E g_8)$$

$$h_6 = 4 a_1 \sqrt{2} m [D_6^- g_1 + D_6^+ g_2] - 4 m^2 |p|^{-1} [D_5^+ g_3 + D_5^- g_4]$$

$$h_7 = 4 a_0 m^2 \sqrt{2} (g_1 + g_2) - (g_3 + g_4) - 4 a_0 \sqrt{6} m^3 |p|^{-1} g_6$$

$$h_8 = 4 a_0 m^3 |p|^{-2} \sqrt{2} (m - E)(g_1 + g_2) - (2E + m)(g_3 + g_4)$$

$$+ 4 a_0 \sqrt{6} m^3 |p|^{-1} g_6$$

(21)

with $a_0, \varepsilon, \mu, D_0$ given above, $a_1 = a_0 m/(m + E)$, $a_2 = a_1/(m - E)$, and the dimensionless functions

$$D_1^+ = a_0(4p_0^2 + 16m^2 - M_d^2 - 4E^2 \pm 4M_d E)$$

$$D_2^+ = a_1(16m^2 + 16mE + E^2 + M_d^2 - 4p_0^2 \pm 4M_d \varepsilon)$$

$$D_3^+ = a_2[-12mE^2 + 2M_d^2 E - 8p_0^2 E + 16m^3 + mM_d^2 - 4mp_0^2 + 8E^3$$

$$\pm (16m^2 M_d + 4mM_d E - 8E^2 M_d)]$$

$$D_4 = a_0(16m^2 - 4E^2 - p_0^2 + m^2)$$

$$D_5^+ = a_0(-2E^2 + 4m^2 + 4mE \pm \varepsilon M_d)$$

$$D_6^+ = a_0(2\varepsilon \pm M_d)$$

(22) - (27)

Note now, that $h_3$, $h_4$ and $g_5$, $g_8$ are odd, and all other functions are even under $p_0 \rightarrow -p_0$. 

For the deuteron, the functions $h_i(P \cdot p, p^2)$ and $g_i(p_0, |p|)$ are connected via
3 Construction of the light-front wave function of two nucleon system from the Bethe-Salpeter amplitude

We now compare the above given covariant amplitudes of the Bethe-Salpeter approach to the covariant light front form. The state vector defining the light-front plane is denoted by $\omega$, where $\omega = (1, 0, 0, -1)$ leads to the standard light front formulation defined on the frame $t + z = 0$. The formal relation between the light-front wave functions $\Phi(k_1, k_2, p, \omega \tau)$, depending on the on-shell momenta $k_1$, $k_2$, and $p = k_1 + k_2 - \omega \tau$, and the Bethe-Salpeter amplitude $\Psi(p_1, p_2)$, where $p_1$ and $p_2$ are off-shell momenta has been given in Ref. [9],

$$\Phi(k_1, k_2, p, \omega \tau) = \frac{k_1 \cdot \omega k_2 \cdot \omega}{\pi \omega \cdot p \sqrt{m}} \int_{-\infty}^{+\infty} \Psi(k_1 - \omega \tau/2 + \omega \beta, k_2 - \omega \tau/2 - \omega \beta) d\beta. \quad (28)$$

In the theory on the null plane the integration of Eq. (28) corresponds to an integration over $dk^-$. Since $k_1$ and $k_2$ are on the mass shell it is possible to use the Dirac equation after making the replacement of the arguments indicated in Eq. (28). This will be done explicitly for the $J = 0$ and the deuteron channel in the following.

3.1 $^1S_0$ channel

Using the Dirac equations $\bar{u}(k_1)(\gamma \cdot k_1 - m) = 0$, and $(\gamma \cdot k_2 + m)Cu(k_2)^\top = 0$ one obtains the following form of the light front wave function from the Bethe-Salpeter amplitude using Eq. (23)

$$\Psi_{00} \rightarrow H_1^{(0)} \gamma_5 + 2H_2^{(1)} \frac{\beta \gamma \cdot \omega}{m \omega \cdot P} \gamma_5, \quad (29)$$

The functions $H_1(s, x)$ and $H_2(s, x)$, depending now on $x = \omega \cdot k_1/\omega \cdot P$ and $s = (k_1 + k_2)^2 = 4(q^2 + m^2)$ are obtained from the functions $h_i(P \cdot p, p^2)$ through the remaining integrals over $\beta$ implied in Eq. (28),

$$H_1^{(0)}(s, x) = N \int h_i((1 - 2x)(s - M^2) + \beta \omega \cdot P, -s/4 + m^2 + (2x - 1)\beta) \omega \cdot P d\beta$$
$$\equiv N \int \tilde{h}_i(s, x, \beta') d\beta'$$

$$H_1^{(k)}(s, x) \equiv N \int \tilde{h}_i(s, x, \beta') (\beta')^k d\beta' \quad (30)$$

where the variable $\beta' = \beta \omega \cdot P$ has been introduced, and $N = x(1 - x), 1 - x = \omega \cdot k_2/\omega \cdot P$. The functions $h_3$ and $h_4$ do not contribute. Instead of the four structures appearing in the Bethe-Salpeter wave function, the light front function consists of only two. Note, that the second term in parenthesis is defined by the pure relativistic component of the Bethe-Salpeter amplitude.
3.2 $^3S_1^3D_1$ case

In the deuteron case, starting from formula Eq. (28), replacing the momenta $p_i$, and applying the Dirac equation we arrive at

$$\Psi_{1M} \rightarrow H_1^{(0)} \gamma \cdot \epsilon_M + H_2^{(0)} \frac{k \cdot \epsilon}{m} + [H_2^{(1)} + 2H_5^{(1)}] \frac{\omega \cdot \epsilon}{m \omega \cdot P}$$

$$+ 2H_6^{(1)} \frac{k \cdot \epsilon \omega}{m^2 \omega \cdot P} + 2H_3^{(1)} \frac{\gamma \cdot \epsilon \omega - \gamma \cdot \omega \gamma \cdot \epsilon}{\omega \cdot P}$$

$$+ [2H_6^{(2)} + 2H_7^{(2)}] \frac{\omega \cdot \epsilon \gamma \cdot \omega}{m^2 (\omega \cdot P)^2},$$

(31)

where $H_i^{(k)}$ are defined in eq. (30). In this case the functions $h_4$ and $h_8$ do not contribute. The expression $(\gamma \cdot \epsilon \gamma \cdot \omega - \gamma \cdot \omega \gamma \cdot \epsilon)$ at the term $H_5$ given in Eq. (31) can be transformed to a different one to compare directly to the light front form given in Ref. [9]. Using in addition the on shellness of the momenta $k_1$ and $k_2$ the resulting form is

$$\tilde{u}_1(\gamma \cdot \epsilon \gamma \cdot \omega - \gamma \cdot \omega \gamma \cdot \epsilon)C \tilde{u}_2^\top = \frac{4}{s} \tilde{u}_1[-i\gamma_5 \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu k_1} k_{2\rho} \omega_{\sigma}$$

$$+ k \cdot \epsilon \omega \cdot P - m \gamma \cdot \epsilon \omega \cdot P$$

$$- \frac{1}{2}(s - M^2)(x - \frac{1}{2})\omega \cdot \epsilon$$

$$+ \frac{1}{2}m(s - M^2) \frac{\gamma \cdot \omega \omega \cdot \epsilon}{\omega \cdot P})C \tilde{u}_2^\top$$

(32)

The final form of light front wave function then is

$$\Psi_{1M} \rightarrow H_1' \gamma \cdot \epsilon_M + H_2' \frac{k \cdot \epsilon}{m} + H_3' \frac{\omega \cdot \epsilon}{m \omega \cdot P} + H_4' \frac{k \cdot \epsilon \gamma \cdot \omega}{m^2 \omega \cdot P}$$

$$+ H_5' i\gamma_5 \epsilon_{\mu \nu \rho \sigma} \epsilon_{\mu k_1} k_{2\rho} \omega_{\sigma} + H_6' \frac{\omega \cdot \epsilon \gamma \cdot \omega}{m^2 (\omega \cdot P)^2},$$

(33)

with the functions

$$H_1' = H_1^{(0)} - \frac{4}{s} 2H_3^{(1)},$$

$$H_2' = H_2^{(0)} + \frac{4}{s} 2H_3^{(1)},$$

$$H_3' = [H_2^{(1)} + 2H_5^{(1)}] - \frac{(s - M^2)}{s} (2x - 1)2H_3^{(1)},$$

$$H_4' = 2H_6^{(1)},$$

$$H_5' = \frac{4}{ms} 2H_3^{(1)},$$

$$H_6' = [2H_6^{(2)} + 2H_7^{(2)}] + 2 \frac{s - M^2}{s} m^2 2H_3^{(1)}.$$ 

(34)

Provided the invariant functions $h_i$ are given from a solution of the Bethe-Salpeter equation the above relations allow us to directly calculate the corresponding light front components of the wave functions.
Thus, the projection of the Bethe-Salpeter amplitudes to the light front reduces the number of independent functions to six instead of eight for the $^3S_1 - ^3D_1$ channel and to two instead of four for the $^1S_0$ channel. The reduction is because the nucleon momenta $k_1$ and $k_2$ are on-mass-shell in the light front formalism. The result is based on the application of the Dirac equation and the use of the covariant form. Any other representation (e.g. spin orbital momentum basis) also leads to a reduction of the number of amplitudes for the two nucleon wave function that is however less transparent. For an early consideration compare, e.g. Ref. [16].

4 Integral representation method

A deeper insight into the connection of Bethe-Salpeter amplitudes and light front wave functions will be provided within the integral representation proposed by Nakanishi [11]. This method has recently been fruitfully applied to solve the Bethe-Salpeter equation both in ladder approximation and beyond within scalar theories [12]. In this framework the following ansatz for radial Bethe-Salpeter amplitudes of orbital momentum $\ell$ has been proposed,

$$
\phi_\ell(P \cdot p, p^2) = \int_0^\infty d\alpha \int_{-1}^{+1} dz \frac{g_\ell(\alpha, z)}{(\alpha + \kappa^2 - p^2 - z P \cdot p - i\epsilon)^n},
$$

where $g_\ell(\alpha, z)$ are the densities or weight functions, $\kappa = m^2 - M^2_\ell/4$ and the integer $n \geq 2$. The weight functions $g_\ell(\alpha, z)$ that are continuous in $\alpha$ vanish at the boundary points $z = \pm 1$. The form eq. (35) opens the possibility to solve the Bethe-Salpeter amplitude in the whole Minkowski space while commonly solutions are restricted to the Euclidean space only. In fact the densities could be considered as the main object of the Bethe-Salpeter theory, because knowing them allows one to calculate all relevant amplitudes.

For the realistic deuteron we need to expand to Nakanishi form to the spinor case, which has not been done so far. The key point to do so is choosing the proper spin-angular momentum functions and perform the integration over angles in the Bethe-Salpeter equation. The choice of the covariant form of the amplitude allows us to establish a system of equations for the densities $g_{ij}(\alpha, z)$, suggesting the following general form for the radial functions $h_i(P \cdot p, p^2)$ (even in $P \cdot p$)

$$
\begin{align*}
    h_i(P \cdot p, p^2) &= \int_0^\infty d\alpha \int_{-1}^{+1} dz \left\{ \frac{g_{i1}(\alpha, z)}{(\alpha + \kappa^2 - p^2 - z P \cdot p)^n} \\
    &\quad + \frac{g_{i2}(\alpha, z) p^2}{(\alpha + \kappa^2 - p^2 - z P \cdot p)^{n+1}} \\
    &\quad + \frac{g_{i3}(\alpha, z) (P \cdot p)^2}{(\alpha + \kappa^2 - p^2 - z P \cdot p)^{n+2}} \right\}.
\end{align*}
$$

For the functions that are odd in $P \cdot p$ the whole integrand is multiplied by factor $P \cdot p$. Although now the number of densities is larger the total number of independent functions is still eight. The form given in eq. (37) is valid only
for the deuteron case. The continuum amplitudes of the $^1S_0$ state, e.g., require a different form.

The basic point now is that using this integral representation allows us to perform the integration over $\beta'$ in the expressions of eq. (31). Substituting the arguments of the functions $h_i$ into the integral representation eq. (37) leads to a denominator of the form

$$D^k(\alpha, z; x, s, \beta') = (\alpha + \frac{s}{4}(1 + (2x + 1)z) - \beta'(2x - 1 + z) - i\epsilon)^k$$

(37)

Then, using the identity for an analytic function $F(z)$

$$\int_{-1}^{+1} dz \int_0^{\infty} d\beta' \frac{F(z)}{D^k(\alpha, z; x, s, \beta')} = \frac{i\pi}{k-1} \frac{F(1 - 2x)}{(\alpha + sx(1 - x))^{k-1}}$$

(38)

allows us to express the radial amplitudes in terms of Nakanishi densities. E.g., $H_5(s, x)$ reads

$$H_5(s, x) = \frac{x(1 - x)}{s} \int d\alpha \left\{ \frac{g_{51}(\alpha, 1 - 2x)}{(\alpha + sx(1 - x))} + \frac{g_{52}(\alpha, 1 - 2x)sx(1 - x)}{(\alpha + sx(1 - x))^2} \right\}$$

(39)

Note that the dependence of the amplitude on the light front argument $x$ is fully determined by the dependence of the density on the variable $z = 1 - 2x$, which has also been noted in Ref. [9] for the Wick-Cutkosky model.

This fully completes the connection between the Bethe-Salpeter amplitude and the light front form. Evaluation of the Nakanishi integrals does not lead to cancelations of functions. Although some functions are cancelled for reasons given above all spin orbital momentum functions (or all densities) in principle contribute to the light front wave functions.

Once the Bethe-Salpeter amplitudes are given (or the Nakanishi densities) the light front wave function can explicitly be calculated. The reduction of the number of amplitudes is due to $k_1$ and $k_2$ being on-shell in the light front form. The Nakanishi spectral densities of the Bethe-Salpeter amplitudes lead directly to the light front wave function.

5 Conclusion

Even more than 60 years after its discovery the deuteron is still an object of intense research. In recent years the focus on relativistic aspects has increased, in particular in the context of ed scattering and relativistic pd reactions. Meanwhile a relativistic description of the deuteron (i.e. the nucleon nucleon system) has achieved considerable progress. Two successful relativistic approaches are based on the Bethe-Salpeter equation or the light front dynamics. In this paper we provide a detailed comparison of the different approaches on the basis of the spin-orbital amplitudes and the radial dependence on the basis of the Nakanishi integral representation. In this context the $P$ waves of the deuteron play an important role. To reach this conclusion the covariant form has been given in terms of the partial wave representation using the $\rho$-spin notation.
We would like to stress that the two relativistic approaches have shown qualitatively similar results in the description of the electrodisintegration near threshold. The functions \(f_5\) and \(g_2\) (notation of Ref. [9]) may be related to the pair current in the light front approach whereas the functions \(h_5\) and \(h_2\) play this role in the Bethe-Salpeter approach. The results presented here allow us to specify this relation on a more fundamental level.

References

1. Ritz, F., Arenhövel, H., Wilbois: T. Few Body Syst. 24, 123 (1998); Ritz, F., Göller, H., Wilbois, T., Arenhövel, H.: Phys. Rev. C 55 2214 (1997); Beck, G., Wilbois, T., Arenhövel, H.: Few Body Syst. 17 91 (1994); Wilbois, T., Beck, G., Arenhövel, H.: Few Body Syst. 15 39 (1993); Göller, H., Arenhövel, H.: Few Body Syst. 13 117 (1992); Wilhelm, P., Leidemann, W., Arenhövel, H.: Acta Phys. Austriaca 3 111 (1988); Arenhövel, H.: Z. Naturforsch. 32A 244 (1977) 244

2. Cambi, A., Mosconi, B., Ricci, P.: Phys. Rev. Lett 48, 462 (1982)

3. van der Schaar, M et al.: Phys. Rev. Lett. 68, 776 (1992), Phys. Rev. Lett. 66, 2855 (1991)

4. Salpeter, E.E., Bethe, H.A.: Phys. Rev. 84 1232 (1951)

5. Dirac P.A.M., Phys. Mod. Rev. 21 (1949) 392

6. Schoonderwoerd, N.C.J., Bakker, B.L.G.: Phys. Rev. D 57 4965 (1998); Schoonderwoerd, N.C.J., Bakker, B.L.G., Karmanov, V.A.: VUTH-98-20, submitted to Phys. Rev. C e-Print Archive: hep-ph/9806363; Ligterink, N.E., Bakker, B.L.G.: Phys. Rev. D 52 5954 (1995); de Melo, J.P.B.C., Frederico, T.: Phys. Rev. C 55 2043 (1997)

7. Beyer, M., Kuhrts, C., Weber, H.J.: Ann. Phys. (NY) to be published, eprint archive nucl-th/9804021

8. Karmanov, V.A.: e-Print Archive, nucl-th/9802053

9. Carbonell, J., Desplanques, B., Karmanov, V.A., Mathiot, J.F.: Phys. Rep. 300 215 (1998)

10. Bondarenko, S.G., Burov, V.V., Beyer, M., Dorkin, S.M.: Phys. Rev. C 58, vol. 6 (1998)

11. Nakanishi, N.: Prog. Theor. Phys. (Kyoto) Suppl. 43, 1 (1969); Nakanishi, N.: Graph Theory and Feynman Integrals, Gordon and Breach, New York 1971

12. Kusaka, K., Simpson, K., Williams A.G.: Phys. Rev. D 56 5071 (1997) and references therein
13. Itzikson, C., Zuber, J.-B.: Quantum Field Theory, (McGraw-Hill, 1980, Singapore)

14. Kubis, J.J.: Phys. Rev. 6 (1972) 547

15. Nosava, N., Ishida, S.: Phys. Rev. C45, 47 (1992)

16. Gourdin, M, Tran Thanh Van, J: Nouvo Cim. XVIII, 443 (1960), Annales de Physique 9, 139 (1964)