Radiatively-induced Magnetic moment in four-dimensional anisotropic QED in an external magnetic field

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We discuss one-loop radiatively-induced magnetic moment in four-dimensional quantum electrodynamics (QED) with anisotropic coupling, and examine various cases which may be of interest in effective gauge theories of antiferromagnets, whose planar limit corresponds to highly anisotropic QED couplings. We find a different scaling with the magnetic field intensity in case there are extra statistical gauge interactions in the model with spontaneous symmetry breaking. Such a case is encountered in the $CP^1$ $\sigma$-model sector of effective spin-charge separated gauge theories of antiferromagnetic systems. Our work provides therefore additional ways of possible experimental probing of the gauge nature of such systems.
1 Introduction

In a recent work [1] we have examined dynamical mass generation for fermionic fields of four-dimensional quantum electrodynamics (QED) with spatially anisotropic couplings across planar two-dimensional surfaces in the $xy$ planes, in the presence of an external homogeneous strong magnetic field along the $z$ direction of space. The phenomenon of magnetically induced fermionic mass is known as magnetic catalysis [2], and has wide applications ranging from condensed matter [3] to early universe [4]. It was found in [1] that the magnetically induced mass depended on the anisotropy parameter, and it was maximum for strong anisotropic (effectively planar QED) situations. In the strong anisotropic limit, the induced mass looks as if it is a parity-invariant three-dimensional mass among the effectively induced three-dimensional fermion species on the plane, as a result of appropriate dimensional reduction of the four-dimensional spinors.

The presence of an external magnetic field, however, breaks parity explicitly, and this should be somehow seen in the effective theory. Indeed, this is what happens as a result of the radiatively-induced magnetic moment of the fermions. At one (and higher) loop there will be induced a Pauli-type coupling $F_{\text{ext}}^{\mu\nu} \sigma_{\mu\nu}$, where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu}, \gamma_{\nu}]$ in the effective one-loop lagrangian, where $F_{\text{ext}}$ denotes the field strength of the external gauge potential, corresponding to the applied magnetic field. From a physical point of view, such a situation may be encountered in effective gauge theories of (doped) antiferromagnets, involving spin-charge separation [5, 6], in the phase where the spin degrees of freedom (spinons) acquire a mass gap, and thus have been integrated out from the effective path-integral of the low-energy (massless) degrees of freedom, such as holons (fermion fields electrically charged). Indeed, in such a case, the low-energy effective lagrangian near the nodes of the fermi surface of such systems, which are known to be $d$-wave superconductors, consists of relativistic electrically-charged fermionic matter coupled to two-kinds of gauge fields: statistical ones, which is an effective way of describing the magnetic Heisenberg interactions in the underlying microscopic condensed-matter model, and real electromagnetic ones.

As we shall show below, the scaling of the induced magnetic moment with the magnetic-field intensity is different in case there is spontaneous breaking in the statistical gauge sector, compared to the situation where the gauge field is massless. This will allow one to discuss experimental probing of the nature of the gauge interactions in such systems, with obvious phenomenological significance.

The structure of the article is as follows: in section 2 we discuss the anisotropic QED case setting up notations and conventions for completeness. In section 3 we calculate the one-loop induced magnetic moment for strong external fields, by resorting to lowest-Landau level computation, which is the only approximation that allows analytic treatment. We discuss quantum fluctuations assuming a generic anisotropic coupling to the statistical gauge field. The computation is done for a massless gauge field as well as for a massive one (e.g. in case of spontaneous gauge symmetry breaking). The computation demonstrates clearly a different scaling of the radiatively-induced magnetic moment with the magnetic-field intensity between the two cases. In section 4 we discuss a specific physical application of interest to condensed matter physics, namely we present a model where the spontaneous
symmetry breaking occurs. The model is a four-dimensional \( CP^1 \) \( \sigma \)-model coupled to fermions, with a \( SU(2) \) gauge symmetry. We show that the integration over the bosonic field induces the non-Abelian kinetic term for the gauge field. The model may be linked to the effective low-energy theory of doped antiferromagnets in the spin-charge separated phase, and we discuss how it is connected to the QED case discussed in previous sections. Finally, conclusions and outlook are presented in section 5. Technical aspects of our approach are presented in an Appendix.

2 Anisotropic four-dimensional QED in an external magnetic field

Our starting point will be the following four-dimensional space-time Lagrangian density, which includes the anisotropy [1]:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} \left[ i \partial^\mu - g \gamma^\mu A - x(i\partial_3 - eA_3)\gamma^3 \right] \psi,
\]

where \( \mu, \nu = 0, \ldots, 3 \), and the parameter \( x \) controls the anisotropy: \( x = 0 \) is the usual isotropic QED and \( x = 1 \) the completely anisotropic case, i.e. the fermions live effectively in 2+1 dimensions, whereas the gauge field still lives in 3+1 dimensions. In the absence of an external field, the bare fermion propagator is (the terminology ‘bare’ is intended to imply quantities without interaction with the dynamical field)

\[
iS^{-1}(p) = \frac{1}{p - xp_3\gamma^3 - m},
\]

and the bare vertices

\[
\Lambda^\mu = \gamma^\mu \quad \text{if} \quad \mu \neq 3 \\
\Lambda^3 = (1 - x)\gamma^3.
\]

In the presence of a constant external field \( A^\mu_{\text{ext}} \), it is known [7] that the bare fermion propagator acquires a phase and becomes

\[
S(y, z) = e^{i e g A^\mu_{\text{ext}}(z)} \tilde{S}(y - z),
\]

which has also been shown to be true via a non-perturbative method for the full dressed propagator [8]. In the propagator [7], the coupling \( e \) to the electromagnetic field is different from the coupling \( g \) to the statistical gauge field which represents an effective interaction in the condensed-matter system, as will be discussed in section 4.

In the case of a constant magnetic field, we will use the so-called lowest Landau level (LLL) approximation [9] valid for strong fields. In this approximation the Fourier transform of the translational invariant part \( \tilde{S} \) of the fermion propagator is
where the magnetic field is in the direction 3 and \( \perp \) denotes the transverse directions 1 and 2 in the plane where the fermions are localized in the fully anisotropic case \[1\].

The one-loop vertex was computed in the isotropic case \[9\] and its general expression is

\[
\Gamma^\lambda(r; y, z) = -e^2 \Lambda^\mu S(y, r) \Lambda^\nu S(r, z) \Lambda^\nu D_{\mu \nu}(z - y)
\]

\[
= e^{iey^\mu A_{\text{ext}}(z)} \tilde{\Gamma}^\lambda(y - r, z - r).
\]

We note that the expression (6) involves the Green functions in coordinate space, which is the necessary starting point so as to take into account the phase factors that appear in Eq.(4). The Fourier transform of the one-loop vertex is then

\[
\tilde{\Gamma}^\lambda(k_1, k_2) = -g^2 \int_p \int_z e^{iz(p - q - k_2)} D_{\mu \nu}(q) \times \Lambda^\mu \tilde{S}(k_1 + eA_{\text{ext}}(z) - q) \Lambda^\nu \tilde{S}(p) \Lambda^\nu,
\]

where for simplicity we used the notation \( \int_p = \int d^4p/(2\pi)^4 \) and \( \int_z = \int d^4z \).

### 3 Radiatively-induced magnetic moment

In this section, we compute the anomalous magnetic moment, induced radiatively at one loop in the model of the previous section. We assume the following configuration for the external gauge field: \( A_{\text{ext}}(z) = (0, -Bz_2/2, Bz_1/2, 0) \), which corresponds to the physically relevant case of a constant magnetic field along the \( z \) axis (direction 3).

Consider the full vertex function in isotropic four-dimensional QED without external field \[11\]:

\[
\Gamma^\lambda(0, k) = \gamma^\lambda F_1(k) + \frac{i}{2m} \sigma^{\lambda \rho} k_\rho F_2(k).
\]

The anomalous magnetic moment \( \mu_0 \) of the fermions is given by \( F_2(0) \).

We now note that, in the presence of an external magnetic field, the corrections along the transverse directions 1,2, \( \Gamma^\perp \) are zero in the LLL approximation \[11\]. To see this, let us introduce the projectors

\[
P^{(\pm)} = (1 \pm i\gamma^1\gamma^2)/\sqrt{2}.
\]

Because we have \( P^{(-)} \gamma^\perp = \gamma^\perp P^{(+)} \) and \( P^{(-)} P^{(+)} = 0 \), the components \( \Gamma^\perp \) must vanish since \( \tilde{S}(p) \) is proportional to \( P^{(-)} \).
To determine the induced magnetic moment, we will then look for the behaviour of the component $\Gamma^3$ and consider its part which is proportional to $\sigma^{30} k_0 = \gamma^3 \gamma^0 k_0$. For the photon propagator, we will take the Feynman gauge and we introduce the Euclidean $\gamma$ matrices $\gamma^\mu$, $\mu = 1,\ldots,4$ satisfying $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$.

Starting from (7), the integration over $z_\parallel$ can be performed and leads to a $\delta^2(q_\parallel - p_\parallel + k_\parallel)$. The integrations over $p_\perp$ and $z_\perp$ are Gaussian and give

$$\tilde{\Gamma}^3(0,k) = m \Lambda^\mu \Lambda^3 \sqrt{2} P^{(-)} \Lambda^4 \Lambda_\mu \times g^2 \int_{p_\parallel q_\perp} \frac{1}{(p_4 - k_4)^2 + (1 - x)^2(p_3 - k_3)^2 + m^2} \times \frac{1}{p_4^2 + (1 - x)^2 p_3^2 + m^2} \times \frac{1}{q_\perp^2 + (p_\parallel - k_\parallel)^2} + \text{other terms} \quad (10)$$

where ‘other terms’ denote the terms which are not proportional to the expected $\Lambda^3 \Lambda^4$ contribution obtained here as:

$$\Lambda^\mu \Lambda^3 \sqrt{2} P^{(-)} \Lambda^4 \Lambda_\mu = (x^2 + 2x - 2) \Lambda^3 \Lambda^4 \sqrt{2} P^{(-)} + 2 \Lambda^3 \Lambda^4 \sqrt{2} P^{(+)} \quad (11)$$

We finally define the anomalous magnetic moment $\mu_B$ by the expression

$$\tilde{\Gamma}^3(0,k) = \sqrt{2} P^{(+)} \frac{\mu_B}{2m} \Lambda^3 \Lambda^4 k_4 + \mathcal{O}(k^2) + \text{other terms}. \quad (12)$$

After integration over $p_\parallel$ in (10) we find:

$$\mu_B(x) = \frac{g^2}{4\pi^2} \int_0^1 \frac{d\eta \eta(1 - \eta)}{\sqrt{\eta + (1 - \eta)(1 - x)^2}} \int_0^{\infty} \frac{du e^{-\frac{m^2}{|eB|}}}{[1 + \eta(u - 1)]^2} \quad (13)$$

where $\eta$ is a Feynman parameter. The integration on $u$ can be approximated in the following way for $|eB| >> m^2$:

$$\int_0^{\infty} \frac{du e^{-\frac{m^2}{|eB|}}}{[1 + \eta(u - 1)]^2} \approx \int_0^{\frac{|eB|}{m^2}} \frac{du}{[1 + \eta(u - 1)]^2} \approx \frac{1}{\eta(1 - \eta)} \quad (14)$$

such that the integration over $\eta$ in Eq.(13) gives finally:

$$\mu_B(x) \approx \frac{g^2}{2\pi^2} \frac{1}{2 - x} \quad (15)$$

Let us discuss Eq.(13). The first new result is the following: in the isotropic case, $x = 0$, the anomalous magnetic moment $\mu_B(0)$ is twice the well-known result obtained in the absence
of magnetic field \[\mu_0 = g^2/8\pi^2\]. Then it increases with the anisotropy, and reaches its maximum value in the fully anisotropic regime, \(x = 1\), which is again twice its isotropic value, in other words, four times the value without an external field, \(4\mu_0\).

Thus the anisotropy has the same impact on the anomalous magnetic moment than it had on the dynamical mass \([1]\), i.e. it provides an enhancement, which however, in this case is not exponential as it was for the dynamical mass. Note that \(\mu_B(x)\) does not depend on the fermion mass in the LLL approximation. We will study now the case of a massive gauge field, where it will be shown that the anomalous magnetic moment depends on the fermion dynamical mass, and therefore will imply a more complicated scaling with the magnetic field, given the associated magnetic-field dependence of the fermion mass \([1]\).

Let us suppose that the gauge field has acquired a mass \(M\) via some Higgs mechanism that we will discuss in the next section. In this case, Eq.(13) becomes

\[
\mu_B(x) = \frac{g^2}{4\pi^2} \int_0^1 \frac{d\eta \eta(1-\eta)}{\sqrt{\eta + (1-\eta)(1-x)^2}} \int_0^{\infty} \frac{du e^{-u m^2}}{[1 + \eta(u + \epsilon \phi^2)]^2},
\]

where \(\phi = |(M/m)^2 - 1|\) and \(\epsilon = \text{sign}\{(M/m)^2 - 1\}\). The approximation \((14)\) then gives:

\[
\mu_B(x) \approx \frac{g^2}{4\pi^2} \int_0^1 \frac{d\eta (1-\eta)}{(1 + \epsilon \eta \phi^2)\sqrt{\eta + (1-\eta)(1-x)^2}}.
\]

We are interested in the anisotropic case \(x = 1\). The integration over \(\eta\) gives then

\[
\mu_B(1) = \frac{g^2}{2\pi^2} \left[ \tan^{-1} \frac{\phi}{\phi^2} \right] \left( 1 + \frac{1}{\phi^2} \right) - \frac{1}{\phi^2} \quad \text{if } M \geq m
\]

\[
\mu_B(1) = \frac{g^2}{2\pi^2} \left[ \tanh^{-1} \frac{\phi}{\phi^2} \right] \left( 1 - \frac{1}{\phi^2} \right) + \frac{1}{\phi^2} \quad \text{if } M \leq m
\]

where \(\tan^{-1}(z) = \arctan(z)\) and \(\tanh^{-1}(z) = 1/2 \ln ((1 + z)/(1 - z))\). When \(M \ll m\), we recover the result \((15)\) from Eq.\((18)\) since

\[
\mu_B(1) \to \frac{g^2}{2\pi^2} \quad \text{when } \phi \to 1.
\]

In the other limit, when \(m \to 0\) or \(\phi \to \infty\), we also recover the fact that \(\mu_B\) vanishes.

The magnetic field dependence becomes non-trivial in the case where \(M \gg m\). Our motivation to study this case is given in the next section. \(m\) is the magnetically induced (dynamically generated) fermion mass gap \([1]\)

\[
m_{\text{dyn}}(x = 1) \approx \frac{g^2}{4\pi} \sqrt{|eB|}.
\]

Furthermore, as will be discussed in the next section, \(M\) does not depend on the magnetic field. Therefore we have:
\[ \mu_B(1) \simeq \frac{g^4 \sqrt{|eB|}}{8\pi^3 M} \]  when \( \phi >> 1 \),

i.e. the anomalous magnetic moment increases with the applied field. We stress that this result is valid as long as \( m << M \), which can be achieved in a wide region of magnetic field, due to the relation (20).

The results in this section, then, may be summarized as follows: in the case of a massive gauge field the anomalous magnetic moment scales as the square root of \( |eB| \), whereas it is independent of the magnetic field if the gauge field is massless.

4 An application: spin-charge separating effective theories of antiferromagnets

In this section we discuss a physically interesting potential application of the above phenomenon, of relevance to condensed-matter physics. Namely, we shall discuss a (continuum) model related to the low-energy physics of planar antiferromagnets in the spin-charge separated phase. We shall link this model with the case of the anomalous-magnetic moment induced in the spontaneously broken gauge symmetry situation (21) encountered in the previous section.

In the condensed-matter model of doped antiferromagnets discussed in ref. [11], we made use of an (approximate for low-doping) particle-hole symmetry in effective microscopic models of doped antiferromagnets, to arrive at the following spin-charge separation ansatz which had a manifest \( SU(2) \) local gauge symmetry even away from half-filling (doped case)

\[
\begin{pmatrix}
  c_1 & c_2 \\
  \bar{c}_2 & -c_1^\dagger
\end{pmatrix} =
\begin{pmatrix}
  \psi_1 & \psi_2 \\
  \bar{\psi}_2 & -\bar{\psi}_1^\dagger
\end{pmatrix}
\begin{pmatrix}
  z_1 & -\bar{z}_2 \\
  \bar{z}_2 & z_1
\end{pmatrix}
\]  (22)

Here, \( c_\alpha \), with \( \alpha \in \{1, 2\} \) are electron operators with spin up or down, and \( \psi_\alpha, z_\alpha \) with \( \alpha \in \{1, 2\} \) are fermions (Grassmann) and bosons respectively. The \( \psi_\alpha \) describe electrically-charged degrees of freedom (holons) and the \( z_\alpha \) describe the spin degrees of freedom (magnons). The index \( \alpha \) is related to the underlying bipartite (antiferromagnetic) lattice structure.

In [11] we have restricted ourselves to the planar case for such a separation, in which all the degrees of freedom are confined on the \( xy \) plane. It is only in that case that the fermionic matrix in (22) can be transformed, at a continuum effective low-energy lagrangian level, to appropriate Nambu-Dirac two-spatial-dimensional spinors \( \Psi_c \), where \( c = 1, 2 \) is a colour \( SU(2) \) index of the fundamental representation of \( SU(2) \). The effective lagrangian describes nodal excitations around zeroes of the fermi surface of the microscopic model.

For our purposes here we shall assume that a local \( SU(2) \) symmetry also characterizes a fully four-dimensional case of nodal excitations, in which an appropriately modified spin-charge separation (22) is valid, but the \( SU(2) \) gauge coupling to be spatially anisotropic.
The effective planar case, then, corresponds to the highly anisotropic limit \( x = 1 \) for the coupling. In that case, the electrically-charged nodal excitations are viewed as fully-fledged four-dimensional relativistic Dirac spinors, while the spin (boson) parts retains its \( CP^1 \) \( \sigma \)-model form. The assumed continuum effective lagrangian of the nodal liquid then of excitations around the fermi-surface nodes of this four-dimensional problem has the form:

\[
L_{\text{eff}} = \frac{1}{\gamma} |(\partial_{\mu} - igA_{\mu})z|^2 + \lambda(|z|^2 - M_z^2)
- e\bar{\psi} A^{ext}\psi + \bar{\psi} \left[ i\frac{\partial}{\partial t} - g A - x(i\partial_3 - gA_3)\gamma^3 \right] \psi,
\]

(23)

where \( \gamma \) is a dimensionless coupling constant (in four space-time dimensions), and \( |z|^2 = z^\dagger z \). In the Lagrangian (23), \( A_{\mu} = a_{\mu}^c T^c \), \( c = 1, 2, 3 \), and \( a_{\mu}^c \) are the gauge bosons of the statistical gauge group \( SU(2) \) with generators \( T^c \). \( \lambda \) is a Lagrange multiplier field implementing the \( CP^1 \) constraint \( |z|^2 = M_z^2 \). We did not write the anisotropic coupling of the holons to the external field since the latter has no component in the direction 3. Finally, our model also takes into account the spin-charge separation at the level of the coupling to the statistical gauge field, since only the holons are coupled anisotropically whereas the spinons are coupled isotropically. This is a consequence of different hopping of the holons and spinons between the antiferromagnetic planes.

We will now discuss the emergence of a dynamical, and massive, \( SU(2) \) gauge field, after integrating out the \( z \) degrees of freedom. We first assume that the field \( z \) has a non zero expectation value

\[
< z > = z_0 \neq 0
\]

(24)

where \( z_0 \) is assumed constant to a first approximation, spatial inhomogeneities are suppressed for our purposes here. The bosonic contribution to the Lagrangian (23) reads then

\[
\frac{1}{\gamma} |(\partial_{\mu} - igA_{\mu})z|^2 + \lambda(|z|^2 - M_z^2)
- \frac{g^2}{\gamma} z_0 A_{\mu} A^\mu z_0 + \frac{1}{\gamma} |(\partial_{\mu} - igA_{\mu})\bar{z}|^2 + \lambda|\bar{z}|^2
+ 2Re \left( z_0^\dagger \lambda \bar{z} + \frac{ig}{\gamma} z_0^\dagger A_{\mu} (\partial^\mu - igA^\mu) \bar{z} \right) + \lambda(|z_0|^2 - M_z^2),
\]

(25)

where we used \( A_{\mu}^\dagger = A_{\mu} \). In Eq.(23), we see the appearence of a mass term for the three gauge fields. In order of magnitude, the gauge-boson mass is

\[
M^2 \approx \frac{g^2}{\gamma} z_0^\dagger z_0,
\]

(26)

i.e. it is linked to the expectation value of the spinon field, as is usual in a Higgs mechanism. Since the spinon field is neutral, i.e. not coupled to the electromagnetic field, \( M \) will be
independent of the magnetic field, depending only on the vacuum expectation value of $z$ only.

Some comments on the order of magnitude of $M$, as compared to the dynamical fermion mass $m$ are in order. In the physical situation we have in mind, the relativistic fermions encountered in the model represent the continuum limit of excitations of a microscopic condensed-matter system near the nodes of a $d$-wave superconducting gap. Experimentally, the disappearance of the nodes, and therefore the opening up of a gap for the quasi-particle excitations at those points on the fermi surface, is observed indirectly [12], through plateaux in the thermal conductivity of the high-temperature $d$-wave superconducting materials, below a certain temperature. The later is much lower than the bulk critical temperature of the superconductor.

In certain models [11, 3], the magnetically induced fermion (holon) mass gap $m$ is found to be much smaller than the spinon gap $M_z$ and the gauge boson mass gap $M$, both related to spin degrees of freedom in the problem. In such cases one has $M \gg m$, a situation encountered in section 3, which implies a non-trivial scaling (21) of the induced magnetic moment with the external field intensity. Hence, by measuring experimentally such a scaling, one can probe deeper into the possible gauge structure of such spin-charge separated systems. Recall from our discussion in section 3, that in the opposite situation, where the gauge field mass is smaller than the holon mass, there will be no appreciable scaling (19) of the induced magnetic moment with the external field.

The gauge kinetic term will be obtained via the integration over the field $\tilde{z}$. This integration was done in 2 dimensions in the Abelian case [13], without spontaneous symmetry breaking for the field $z$. We show in the appendix that one can also recover the gauge kinetic term in 4 dimensions, with a non-Abelian gauge symmetry, taking into account the non-vanishing vacuum expectation value of $z$. The path integration over $z$ fields, then, yields the following gauge-field contributions to the effective lagrangian:

$$L_{\text{gauge}} = -\frac{g^2}{96\pi^2\varepsilon} F_{\mu\nu}^a F^{a\mu\nu} + \frac{g^2}{\gamma z_0} A_\mu A^\mu z_0,$$  \hspace{1cm} (27)

where we used a dimensional regularization ($d = 4 - \varepsilon$) and $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$.

In the framework of the condensed-matter effective study that we make, the pole $1/\varepsilon$ is understood as a natural U.V. cut-off of the theory, depending on the specific microscopic lattice system considered (e.g. $1/\varepsilon \sim \ln(L/a)$, where $L$ is an I.R. scale, and $a$ is an U.V. one).

Thus, starting from the Lagrangian (23), we have derived the effective theory containing holons, which in turn are anisotropically coupled to a massive dynamical $SU(2)$ gauge field. In this way we have arrived at the situations described in section 3.

## 5 Conclusion

In this paper we have derived the radiatively-induced magnetic moment in the case of a four-dimensional gauge field theory model with an anisotropic coupling of the fermions to
the gauge fields. We saw that the resulting anomalous magnetic moment is independent of
the magnetic field if the gauge field is massless, whereas it scales as the square root of the
magnetic field if the gauge field acquires a mass via some (spontaneous gauge symmetry
breaking) mechanism. This situation has to be distinguished from that discussed in [14],
where the anomalous magnetic moment in the presence of an external magnetic field was
computed in strictly 2+1 dimensions, and therefore did not lead to the same scaling. We
believe that the present anisotropic situation is more suitable for the effective description
of the planar high-temperature superconductors, which notably are quasi three-dimensional
systems, involving a small but finite electron hopping across the superconducting planes.

The scaling of the induced magnetic moment with the magnetic field intensity con-
stitutes another interesting experimental probe of the spin-charge separation ansatz and
the presence of spontaneous symmetry breaking for the spinon degrees of freedom, in the
way explained in this article. The associated parity violation of the effective theory, which
accompanies the apperance of the external magnetic field, and manifests itself through
the induced magnetic moment, may result in edge (parity-violating) currents in the super-
conducting materials, whose intensity would depend on the applied magnetic field. Such
effects should be directly measurable. We plan to return to a systematic analysis of such
tests in a forthcoming publication.

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Appendix: Generation of a dynamical gauge field in a
four-dimensional $CP^1$ $\sigma$-model

The purpose of this appendix is to perform explicitly, and discuss technical details of, the
path integration over the field $\tilde{z}$ in the Lagrangian (23), so as to generate dynamically the
kinetic term of the statistical gauge field.

In this integration, the linear terms in $\tilde{z}$ can be omitted in Eq.(25) for the following
reason: let us consider the integral

$$\int dz \exp \left( -az - f(z^2) \right)$$

$$= \int dz \cosh(az) \exp \left( -f(z^2) \right)$$

$$= \int dz \exp \left( \ln \cosh(az) - f(z^2) \right)$$

$$= \int dz \exp \left( -f(z^2) + \frac{(az)^2}{2} + ... \right),$$

(28)
such that the term which was originally linear in $\tilde{z}$ actually appears only with even powers and therefore has irrelevant contributions, since its square has already mass dimension 5 (the field $\lambda$ has mass dimension 2). We have

$$
\int D[z]D[z^\dagger] \exp \left( - \int \frac{1}{\gamma}(\partial_\mu - ig A_\mu)z^\dagger z + \lambda(|z|^2 - m^2) \right)
$$

$$
= \text{cte} \times \exp \left( - \int \frac{g^2}{\gamma} z_0^\dagger A_\mu A^\mu z_0 \right)
$$

$$
\times \int D[\tilde{z}]D[\tilde{z}^\dagger] \exp \left( - \int_{pq} \tilde{z}^\dagger (p) \mathcal{O}(p, q) \tilde{z}(-q) + \text{irrelevant} \right),
$$

where the operator $\mathcal{O}$ is

$$
\mathcal{O}(p, q) = (\tilde{\lambda} + p^2) \delta(p + q) - (p - q)^\mu g A_\mu (p + q) + g^2 \int_k A_\mu(k) A^\mu(p + q - k)
$$

and $\tilde{\lambda} = \gamma \lambda$. The integration over $\tilde{z}$ leads to the trace of the logarithm of $\mathcal{O}$ that we expand up to the forth order in the gauge field so as to recover the non-Abelian gauge kinetic term:

$$
\text{Tr} \ln \left[ \frac{\mathcal{O}(p, q)}{\tilde{\lambda} + p^2} \right]
$$

$$
= g^2 \text{tr} \int_k \Pi^{\mu\nu}_{(2)}(k) A_\mu(k) A_\nu(-k)
$$

$$
+ g^3 \text{tr} \int_{kp} \Pi^{\mu\nu\rho}_{(3)}(k, p) A_\mu(k) A_\nu(p) A_\rho(-k - p)
$$

$$
+ g^4 \text{tr} \int_{kpq} \Pi^{\mu\nu\rho\sigma}_{(4)}(k, p, q) A_\mu(k) A_\nu(p) A_\rho(q) A_\sigma(-k - p - q)
$$

+ \text{higher orders},
$$
\text{(31)}
$$

where ‘Tr’ denotes the trace over momenta and indices of $SU(2)$ generators, and

$$
\Pi^{\mu\nu}_{(2)}(k) = \int_r \frac{1}{\lambda + r^2} \left( g^{\mu\nu} - \frac{1}{2} \frac{(2r - k)^\mu(2r - k)^\nu}{\lambda + (k - r)^2} \right)
$$

$$
\Pi^{\mu\nu\rho}_{(3)}(k, p) = \int_r \frac{p^\mu - 2r^\mu}{\tilde{\lambda} + (p - r)^2[\tilde{\lambda} + r^2]}
$$

$$
\times \left( g^{\nu\rho} + \frac{1}{3} (2r^\nu + k^\nu)(p^\rho - k^\rho - 2r^\rho) \right) \frac{1}{\lambda + (k + r)^2}
$$

$$
\Pi^{\mu\nu\rho\sigma}_{(4)}(k, p, q) = \int_r \frac{1}{\lambda + (p + k - r)^2[\lambda + r^2]}
$$
The integrals (32) are computed within dimensional regularization \((d = 4 - \varepsilon)\), and we obtain

\[
\Pi_{(2)}^{\mu\nu}(k) = \frac{1}{48\pi^2 \varepsilon} (g^{\mu\nu} k^2 - k^\mu k^\nu) + \text{finite}
\]

\[
\Pi_{(3)}^{\mu\nu\rho}(k, p) = \frac{1}{72\pi^2 \varepsilon} [g^{\mu\nu} (2p^\rho - k^\rho) - g^{\nu\rho} (p^\mu + 2k^\mu) + g^{\mu\nu} (k^\rho - p^\rho)] + \text{finite}
\]

\[
\Pi_{(4)}^{\mu\nu\rho\sigma}(k, p, q) = \frac{1}{96\pi^2 \varepsilon} [2g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}] + \text{finite (33)}
\]

We use then the following traces

\[
\begin{align*}
\text{tr} \left( T^a T^b \right) &= \delta^{ab} \\
\text{tr} \left( T^a \{ T^b, T^c \} \right) &= 0 \\
\text{tr} \left( T^a [ T^b, T^c ] \right) &= if^{abc} \\
\text{tr} \left( [ T^a, T^b ] [ T^c, T^d ] \right) &= -f^{abc} f^{cde}
\end{align*}
\]

(34)

where \( f^{abc} \) are the \( SU(2) \) structure constants, and we find

**quadratic term:**

\[
\begin{align*}
\text{tr} \int_k \Pi_{(2)}^{\mu\nu}(k) A_\mu(k) A_\nu(-k) &= -\frac{g^2}{48\pi^2 \varepsilon} \int_k (g^{\mu\nu} k^2 - k^\mu k^\nu) A^\alpha_\mu(k) A^\alpha_\nu(-k) \\
&= -\frac{g^2}{96\pi^2 \varepsilon} \int_z \left( \partial_\mu A^\alpha_\mu(z) - \partial_\nu A^\alpha_\mu(z) \right) \left( \partial^\mu A^\nu_\mu(z) - \partial^\nu A^\nu_\mu(z) \right)
\end{align*}
\]

**cubic term:**

\[
\begin{align*}
\text{tr} \int_{kp} \Pi_{(3)}^{\mu\nu\rho}(k, p) A_\mu(k) A_\nu(p) A_\rho(-k - p) &= -\frac{g^3}{72\pi^2 \varepsilon} \int_k 3 if^{abc} p^\mu A^a_\mu(k) A^b_\nu(p) A^c_\rho(-k - p) \\
&= -\frac{g^3}{48\pi^2 \varepsilon} \int_k if^{abc} A^a_\mu(k) A^b_\nu(-k - p) \left( p^\mu A^\nu_\rho(p) - p^\nu A^\nu_\rho(p) \right) \\
&= -\frac{g^3}{96\pi^2 \varepsilon} \int_z 2 f^{abc} A^a_\mu(z) A^b_\nu(z) \left( \partial^\mu A^\nu_\rho(z) - \partial^\nu A^\nu_\rho(z) \right)
\end{align*}
\]

**quartic term:**

\[
\begin{align*}
\text{tr} \int_{kpq} \Pi_{(4)}^{\mu\nu\rho\sigma}(k, p, q) A_\mu(k) A_\nu(p) A_\rho(q) A_\sigma(-k - p - q)
\end{align*}
\]
\[ = \frac{g^4}{96\pi^2\varepsilon} \int_{kpq} f^{abc} f^{ade} A^b_\mu(k) A^e_\nu(p) A^\mu_d(q) A^\nu_e(-k-p-q) \]

\[ = \frac{g^4}{96\pi^2\varepsilon} \int_z f^{abc} f^{ade} A^b_\mu(z) A^e_\nu(z) A^\mu_d(z) A^\nu_e(z) \]
such that the integration over \( \tilde{z} \) gives finally the expected kinetic term

\[ \text{Tr} \ln \left[ \frac{O(p,q)}{\lambda + p^2} \right] = -\frac{g^2}{96\pi^2\varepsilon} \int_z F^a_{\mu\nu} F^{a\mu\nu} + \text{higher orders}, \quad (36) \]

where \( F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc} A^b_\mu A^c_\nu \). In Eq. (36), ‘higher orders’ denotes the irrelevant operators (mass dimension greater than 5) that are not taken into account in the low-energy derivative expansion we study here. We note that the field \( \tilde{\lambda} \) does not appear in Eq. (36) (it actually only appears in the irrelevant terms) and thus its integration can be omitted in the final path integral defining the effective model.

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