We give a review of classical, thermodynamic and quantum properties of black holes relevant to fundamental physics.

1. Introduction

The term black hole is usually associated with a large astrophysical object that has formed due to huge gravitational fields that can arise in the center of massive concentrations (see, e.g., 1). However, the black hole is an object in itself which should be studied within the domain of physics, irrespective of the interactions with exterior astrophysical plasmas which excite, and are excited by, the strong gravitational fields of the black hole. Here we want to understand a black hole as a physical object. This program was consciously initiated by Wheeler 2 back in the 1950s. We have not yet understood it entirely, but we have come very far, if we think that, back in 1960, Wheeler, Kruskal and others 3 managed to understand, for the first time, the global causal structure of the complete manifold of the simpler black hole, the Schwarzschild black hole. During the 1960s the black hole became well

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understood as a classical object, mainly due to the works of Penrose 4 and then Hawking 5 and Carter 6. But, then in the 1970s due to Bekenstein 7 and Hawking 8, the whole field was revolutionized, the black hole concept entered the quantum arena. Of course, quantum dynamics is the underlying dynamics of the world and black holes have to be understood in this context. Conversely, the simple structure of a black hole, can be used to probe and learn about the quantum structure of gravitation. As such, a black hole is considered by many 9 as the gravitational equivalent of the hydrogen atom in mechanics, in the sense that this atom was used by Bohr, Sommerfeld and others to touch and grasp the novel ideas of quantum mechanics 10.

Hawking’s monumental discovery of 1974, perhaps the most important discovery in theoretical physics of the second half of the twentieth century, that a black hole radiates quantum mechanically, was followed by some interesting developments, but, perhaps, there was no ostensible growth, after that. Then, many physicists from different fields, like particle physics and field theory, moved to string theory. String theory, although a theory subject to criticisms on several grounds, can tackle important problems related to black hole physics. String theorists in taking the black hole problem into their hands back in the 1990s 11, 12, out from the general relativists alone, opened the subject to the physics community overall, and revolutionized it into myriads of new directions. Before string theory attacked the problem, one should obey the general relativity bible, which was strict, allowing one to venture into other fields, such as non asymptotically flat spacetimes or naked singularities, only with extreme care and perhaps permission. String theory opened up the book and the theoretical discussion on black holes and their problems grew exponentially. Of course, general relativity itself benefited from it. For instance, new black hole solutions in general relativity, now called toroidal black holes, living in asymptotically anti-de Sitter spacetimes were found 13, 14, 15, 16, and many other connections were made.

Thus, suddenly in the 1990s, the field of black holes was again lively and growing fast. Five topics of the utmost interest are: (a) the thermodynamics of black holes, (b) the black hole entropy and its degrees of freedom, (c) the information paradox, (d) the holographic principle and its connection to the generalized second law and to the covariant entropy bound, and (e) the inside of a black hole and its singularity. I will report on the first two topics, the others require reviews on their own, and will be left for other opportunities. Due to a large bibliography on these two first topics I cannot be complete in listing references, the ones that are not mentioned will be left to another larger review. I have benefited tremendously from
the reviews of Bekenstein \(^{17,18}\) and Fursaev \(^{19}\), see also the recent thorough reviews by Page \(^{20}\) and Padmanabhan \(^{21}\). Unless otherwise stated we use units in which \(G = c = \hbar = 1\).

2. Black hole thermodynamics and Hawking radiation

2.1. Preliminaries

It is now certain that a black hole can form from the collapse of an old massive star, or from the collapse of a cluster of stars. Many x-ray sources observed in our galaxy contain a black hole of about \(10 M_\odot\). Quasars, the most powerful distant objects, belong to a class which is one of the representatives of active galactic nuclei that are powered by a very massive black hole, with masses as high as \(10^{10} M_\odot\). Our own Galaxy harbors a dead quasar with a mass of \(10^6 M_\odot\) in its core. Mini black holes, with masses lying within a wide range, a typical one could have \(10^{15} \text{gm} \sim 10^{-18} M_\odot\) (with radius \(10^{-13} \text{cm}\)), may have formed in the early universe. Finally, Planck black holes of mass \(10^{-5} \text{gm}\) and radius \(10^{-33} \text{cm}\) may form in an astronomical collider which could provide a center of mass energy of \(10^{19} \text{GeV}\), or perhaps less if the idea of extra large dimensions is correct \(^{22}\).

A black hole is a gravitational object whose interior region is invisible for the outside spacetime world. The boundary of this region is the event horizon of the black hole. To the outside world the black hole is like a tear in the spacetime, which interacts with its environment by attracting and scattering particles and waves in its neighborhood.

Black holes appear naturally, as exact solutions, in the theory of general relativity. The most simple is the Schwarzschild black hole which has only one parameter, the mass, and has a spherical horizon. By adding charge one obtains a black hole with more structure, the Reissner-Nordström black hole \(^{23}\). The theory of black holes received a tremendous boost after Kerr \(^{24}\) found that a rotating black hole is also an exact solution of general relativity, a totally unexpected result at the time, that continues to flabbergast many people up to now. The Kerr black hole provides extra non-trivial dynamics to the spacetime, from which novel ideas sprang. Kerr black holes with charge are called Kerr-Newman black holes, a name also used to designate the whole family. There are now other important families of black holes, such as the family of anti-de Sitter spacetimes, with negative cosmological constant, whose horizons have topologies other than spherical \(^{16}\), or other families in a variety of different theories of gravitation \(^{25}\). The Kerr-Newman family was the first to be thoroughly investigated classically.
Some important properties, generically valid for other families, have been worked out in detail.

First, the event horizon acts as a one way membrane. Due to the strong gravitational field near the black hole, the light cones of the spacetime get tilted, so much so that their exterior boundary lies tangent to the horizon. The horizon thus acts as a one way membrane, i.e., no object, not even a light ray, that crosses it inwards can ever cross it back outwards. As a result, any physical quantity, such as energy, entropy or information, that is damped into the black hole remains permanently trapped inside, classically.

Second, a black hole has no hair. What does this mean? For an exterior observer, placed outside the horizon, the black hole forgets everything that it has swallowed. The black hole can have been formed from baryons alone, or from leptons alone, or from both, or anything else, the exterior observer cannot have access to what formed the black hole. The only thing the observer probes is the black hole mass $M$, electromagnetic charge $Q$, and angular momentum $J$. This is referred to as the baldness of the black hole, or as a black hole has no hair, in the language of Wheeler. In fact, it has three hairs, $M, Q, J$, but the nomenclature is still correct, one usually associates to someone that has three hairs that he is bald, has no hair!

Third, a black hole absorbs and scatters particles and waves. These properties involving scattering and absorption of particles and waves by black holes, specially by rotating black holes, were very important in the later developments. The whole subject started with the Penrose process, which branched into superradiance on one hand and the irreducible area concept on the other, and culminated with Hawking’s theorem on area growth. Let us comment on these features briefly, first with special emphasis in superradiation. When a particle is scattered by a Kerr black hole and broken in two pieces in the process, energy can be extracted from the black hole rotation into the outgoing particle, using the existence of an ergosphere (a region just outside the event horizon), a kind of relativistic sling shot phenomenon. The wave analog of the Penrose process, whereby an incoming wave (scalar, electromagnetic, gravitational or plasma) with positive energy that impinges on the rotating black hole splits up into an absorbed wave with negative energy and a reflected wave with enhanced positive energy, is called superradiance. Consider a wave of the form $e^{-i\omega t + im\phi}$, where $\omega$ is the frequency of the wave, $t$ the time parameter, $m$ the azimuthal angular wavenumber around the axis of rotation of the black hole, and $\phi$ the angular coordinate. Considering then that such a wave collides with the black hole, one concludes that if the frequency $\omega$ of
the incident wave satisfies the superradiant condition \( \omega < m\Omega \), where \( \Omega \) is the angular velocity of the black hole, then the scattered wave is amplified \(^{26,27,28}\). One simple way to get an idea of what is happening is by resorting to the inverse of the characteristic frequencies, i.e., the period of the wave \( \tau = 2\pi/\omega \), and the rotation period of the cylinder \( T = 2\pi/\Omega \). Then the superradiant condition is now \( m\tau > T \), which means that for \( m = 1 \) say, the wave suffers superradiant scattering if it takes a longer time in the neighborhood of the black hole than the time the black hole takes to make one revolution, so that there is enough time for the black hole to transfer part of its rotating energy to the wave. This way of seeing superradiance corresponds to giving a necessary condition, i.e., to exist superradiance there should exist enough time so that the black hole can transmit part of its energy to the wave.

Fourth, the area of a black hole always grows in any physical process. The Penrose process also led to the concept of irreducible mass \(^{29}\), which in turn led Hawking \(^{30}\) to prove a theorem stating that the black hole area always grows in any physical process, classically. This theorem proved to be decisive for further developments. In turn, and in passing, one can use this theorem to prove superradiance. Indeed, following the lines of Zel’dovich \(^{26}\) one roughly finds that the scattering of a wave by a black hole obeys

\[
\frac{\kappa}{\pi} \frac{dA}{dt} = (P_i - P_r) \left( 1 - \frac{m\Omega}{\omega} \right),
\]

where \( A \) is the area of the event horizon, \( \kappa \) is its surface gravity, and \( P_i \) and \( P_r \) are the incident and reflected power of the wave, respectively. From the area law for black holes, which states that the area of the event horizon never decreases, i.e., \( dA \geq 0 \) \(^{30}\), one finds that if the frequency of the incident wave satisfies the superradiant condition, the second factor in the right hand side of the equation is negative. In order to guarantee that the area does not decrease during the scattering process, one must have \( P_r > P_i \). Thus, the energy of the wave that is reflected is higher than the energy of the incident wave, as long has the superradiant condition is satisfied. On other developments on superradiance and how it can be used, along with a mirror, to build a black hole bomb see \(^{31,32}\).

With these four ingredients, i.e., one-way membrane, no hair, scattering properties, and area law, all is set to put the black hole in a thermodynamic context.

### 2.2. Thermodynamics and Hawking radiation

A Kerr-Newman black hole, say, can form from the collapse of an extremely complex distribution of ions, electrons and radiation. But once
it has formed the only parameters we need to specify the system are the parameters that characterize the Kerr-Newman black holes, the mass \( M \), the charge \( Q \) and the angular momentum \( J \). Thus we have a system specified by three parameters only, which hide lots of other parameters. In physics there is another instance of this kind of situation, whereby a system is specified and usefully described by few parameters, but on a closer look there are many more other parameters that are not accounted for in the compact description. This is the case in thermodynamics. For thermodynamical systems one gives the energy \( E \), the volume \( V \), and the number of particles \( N \), say, and one can describe the system in a usefully manner, although the system encloses, and the description hides, a huge number of molecules.

Connected to this, was the question Wheeler was raising in the corridors of Princeton University \(^3\), that in the vicinity of a black hole entropy can be dumped onto it, thus disappearing from the outside world, and grossly violating the second law of thermodynamics. Bekenstein, a Ph.D. student in Princeton at the time, solved part of the problem in one stroke. He postulated, entropy is area, more precisely \(^7\), \( S_{\text{BH}} = \eta \frac{A}{l_{\text{pl}}} k_B \), where one is using full units, \( \eta \) is a number of the order of unity or so, that could not be determined, \( l_{\text{pl}} \equiv \sqrt{\frac{G \hbar c^2}{\kappa}} \) is the Planck length, of the order of \( 10^{-33} \) cm, and \( k_B \) is the Bolztmann constant. This is, of course, aligned with the area's law of Hawking, and the Penrose and superradiance processes. Bekenstein invoked several physical arguments to why the entropy \( S \) should go with \( A \) and not with \( \sqrt{A} \) or \( A^2 \). For instance, it cannot go with \( \sqrt{A} \) (\( A \) itself goes with \( \sim M^2 \)) because when two black holes merge the final mass obeys \( M < M_1 + M_2 \) since there is emission of gravitational radiation. But if \( S_{\text{BH}} \propto M < M_1 + M_2 \) \( \propto S_{\text{BH}1} + S_{\text{BH}2} \) the entropy could decrease, so such a law is no good. The correct option turns out to be \( S \propto A \), the one that Bekenstein took. Also correct, it seems, is to understand that this is a manifestation of quantum gravity, so that one should divide the area by the Planck area, and multiply by the Boltzmann constant to convert from the usual area units into the usual entropy units. There is thus a link between black holes and thermodynamics.

One can then wonder whether there is a relation obeyed by black hole dynamics equivalent to the first law of thermodynamics. For a Schwarzschild black hole one has that the area of the event horizon is given by \( A = 4\pi r_+^2 \). Since \( r_+ = 2M \) one has \( A = 16\pi M^2 \). Then one finds
\[ dM = \frac{1}{(32 \pi M)} dA, \] which can be written as
\[ dM = \frac{\kappa}{8\pi} dA, \tag{1} \]
which is the first law of black hole dynamics \(^{33}\). The surface gravity of the event horizon of the Schwarzschild black hole is \( \kappa = 1/4M \). Equation (1) can be compared with
\[ dE = T dS, \tag{2} \]
which is the first law of thermodynamics. Note that, a priori, the analogy between \( S \) and \( A \), and \( T \) and \( \kappa \), is merely mathematical, whereas the analogy between \( E \) and \( M \), is physical, they are the same quantity \(^{34}\). For a generic Kerr-Newman black hole one has the relation
\[ dM = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ, \]
where \( \Omega \) is the angular velocity of the black hole horizon, and \( \Phi \) the electric potential. Comparing with the thermodynamical relation
\[ dE = T dS + p dV + \mu dN, \]
where the symbols have their usual meanings, it further strengthens the analogy.

Following Bekenstein, this is no mere analogy though, the black hole system is indeed a thermodynamic system with the entropy of this system being proportional to the area. But what is \( \eta \) in the equation proposed by Bekenstein? Thermodynamic arguments alone were not sufficient to determine this number. Using quantum field theory methods in curved spacetime Hawking \(^8\) showed that a Schwarzschild black hole radiates quantically as a black body at temperature
\[ T = \frac{1}{8\pi M}. \tag{3} \]
Since \( \kappa = 1/4M \), the temperature and the surface gravity are essentially the same physical quantity, with \( T = \kappa/2\pi \). Moreover, from equation (2), one obtains \( \eta = 1/4 \), yielding finally
\[ S = \frac{1}{4} A. \tag{4} \]
in geometrical units. Thus the Hawking radiation solved definitely the thermodynamic conundrum. However, it introduced several others puzzles.

The black hole is then a thermodynamic system. Thus, the second law of thermodynamics \( \Delta S \geq 0 \) should be obeyed. Since one does not know for sure the meaning of black hole entropy, it is useful to write the entropy as a sum of the black hole entropy \( S_{\text{BH}} \), and the usual matter entropy \( S_{\text{matter}} \), i.e., \( S = S_{\text{BH}} + S_{\text{matter}} \), allowing one to write the second law as
\[ \Delta S_{\text{BH}} + \Delta S_{\text{matter}} \geq 0, \tag{5} \]
commonly called the generalized second law\textsuperscript{35}. The generalized second law proved important in many developments.

3. Statistical interpretation of black hole entropy

3.1. Preliminaries

In statistical mechanics, the entropy of an ordinary object is a measure of the number of states available to it, i.e., it is the logarithm of the number of quantum states that the object may access given its energy. This is the statistical meaning of the entropy. Since black holes have entropy, one can ask what does the black hole entropy represent? What is the statistical mechanics of a black hole as a thermodynamic object?

Retrieving full units to equation (4) one has

\[ S_{BH} = \frac{1}{4} \frac{A}{l^2_{pl}} k_B, \]

where again, \( l^2_{pl} = \frac{\hbar^2}{Gc^3} \) is the Planck area, and \( k_B \) is the Boltzmann constant. Judging from the four fundamental constants appearing in the formula, namely, \( G, c, \hbar, k_B \), one gets a system where relativistic gravitation, quantum mechanics and thermodynamics, are mixed together, indicating that a statistical interpretation should be in sight. Moreover, the number \( \frac{A}{l^2_{pl}} \) itself suggests that the states of the black hole are some kind or another of quantum states. In addition, the factor \( 1/4 \) became a target for any theory that wants to explain black hole entropy from fundamental principles.

Note that black hole entropy is large. A neutron star with one solar mass has entropy of the order of \( S \sim 10^{57} \) (in units where \( k_B = 1 \)) in a region within a radius of about 10 Km. A solar mass black hole has an entropy of \( 10^{79} \) in a region within a radius of 3 Km. There is a huge difference in entropy for these two objects of about the same size, suggesting, somehow, the black hole harnesses entropy that can be peeled away through the black hole’s lifetime, i.e., the time the black hole takes to radiate its own mass via Hawking radiation.

3.2. Entropy in the volume

Bekenstein\textsuperscript{36} tried first to connect the entropy of a black hole with the logarithm of the number of quantum configurations of any matter that could have served as the black hole origin, in perfect consonance with the no hair theorem. Now, the number of those quantum configurations can
be associated to the number of internal states that a black hole can have, hinting in this way that the entropy of a black hole lies on the volume inside the black hole (see also \cite{37}). This idea of bulk entropy, although interesting, has many drawbacks, see \cite{38,39,40}.

3.3. Entropy in the area

There are now many alternative interpretations that associate the black hole entropy with the its area, the area of the horizon. One can divide these interpretations into those that claim the degrees of freedom are on the quantum matter in the neighborhood of the horizon that gives rise to the Hawking radiation, those that claim that the degrees of freedom are on the gravitational field alone, and those that put the degrees of freedom on both, matter and gravitational fields, like string theory.

3.3.1. Matter entropy

(i) Entropy of quantum fields

One interpretation says that the black hole entropy comes from the entropy of quantum matter fields fluctuations in the vicinity of the horizon. This was first advanced by Gerlach \cite{41} who proposed that the entropy was related to the number of zero-point fluctuations that give rise to the Hawking radiation. So the entropy comes from the all time matter fields surrounding the horizon created by the Hawking process. Later Zurek and Thorne \cite{42} proposed the quantum atmosphere picture and 't Hooft \cite{43} developed the idea in the brick wall model. The advantage of these insights is that the linear dependence of $S_{BH}$ on the horizon area, $S_{BH} = \eta A$ comes automatically, since the matter that gives rise to the entropy is in a thin shell surrounding a surface, the horizon. One great disadvantage, is that the coefficient $\eta$ is infinite since ultraviolet wavelengths, with wavelengths arbitrarily small, also take part in the matter fields surrounding the horizon. One can cure this by imposing a cutoff for these lengths at the Planck length, which, although sensible, is ad hoc and incapable of giving the goal factor $1/4$. Moreover, $\eta$ is proportional to the number of fields existing in nature, making even harder to connect it with the coefficient $1/4$.

(ii) Entropy of entanglement

Another, somehow connected, interpretation comes from Sorkin and collaborators \cite{44}, who suggested that the entropy is related to the entanglement entropy arising from tracing out the degrees of freedom existing beyond the
horizon. In other words, the entropy is generated by dynamical degrees of freedom, excited at a certain time, associated to the matter in the black hole interior near the horizon through non-causal EPR correlations with the external matter. It has been used by many different authors, see e.g., 45,46. This has the advantages and disadvantages of the above interpretation.

(iii) Entropy in induced gravity
Other interpretation that can be mentioned is the one that associates the degrees of freedom to heavy matter fields that when integrated out induce, naturally, general relativity. This way of seeing general relativity was envisaged by Sakharov 47 and the corresponding entropy interpretation was put forward in 48. Of course, this interpretation could be in the gravitational entropy sector, since what is matter and what is gravity is blurred here.

3.3.2. Gravitational entropy
(i) Entropy from boundary conditions
An improved interpretation, perhaps, is that of Solodukhin 49 and Carlip 50,51,52 who, independently, switched from matter field fluctuations to gravitational field fluctuations. They showed that the existence of a horizon, the surface where the fluctuations occur, makes the fluctuations themselves obey the laws of a conformal field theory in two spatial dimensions, this number two is related to the dimensionality of the horizon. Conformal field theory has been thoroughly investigated, yielding for the logarithm of number of states associated with the fluctuations, a value for the entropy that matches exactly the entropy formula for a black hole, with the coefficient 1/4 coming out perfect. The idea is to use the correct boundary conditions at a horizon so as to give rise to new degrees of freedom that do not exist in the bulk spacetime. However interesting it may be, see also 53, it lacks a direct physical interpretation, since the boundary conditions are too formal.

(ii) Heuristic interpretation for the degrees of freedom
A physical interpretation for the gravitational degrees of freedom comes from the intuitive idea of Bekenstein and Mukhanov 54 that the area of the horizon being an adiabatic invariant, should be quantized in Ehrenfest's way. Suppose, then, that the area of the horizon is quantized with uniformly spaced levels of order of the Planck length squared, i.e., $A = \alpha l_{pl}^2 n$ with $\alpha$ a pure number, and $n = 1, 2, \ldots$. Thus a small black hole is constructed from a small number of Planck areas, one can build the next black hole putting an extra Planck area, and so on. The horizon, according with
this view, can be thought of as a patchwork of patches with area $\alpha l^2_{pl}$. If every Planck patch can have two distinct states, say, then a black hole with two Planck areas can be in four different states, a black hole with three Planck areas can be in eight different states, a large black hole with $n$ Planck areas can be in $2^n$ different surface states. Now, degeneracy and entropy are connected in such a way that latter is the logarithm of the former, i.e., $S_{BH} = \ln 2^n = (\ln 2) n = \frac{\ln 2 A}{\alpha l^2_{pl}}$. The area law is then recovered, by default. Further, from Hawking’s work we know that $\frac{\ln 2}{\alpha} = \frac{1}{4}$ so that the quantization law is $A = 4 (\ln 2) l^2_{pl} n$. We can instead think that every area patch has $k$ distinct states instead of two. Then the same reasoning follows, and one has that a black hole with area $A = \alpha l^2_{pl} n$ can be in any of $k^n$ states. The entropy is then $S_{BH} = \ln k^n = (\ln k) n = \frac{\ln 2 A}{\alpha l^2_{pl}}$, and the area quantization law is $A = 4 (\ln k) l^2_{pl} n$, and $\alpha = 4 \ln k$. The question is now, what is $k$? Hod 55 found a way to determine $k$. Inspired by Bohr’s correspondence principle, that transition frequencies at large quantum numbers should equal classical oscillation frequencies, one should associate the classical oscillation frequencies of the black hole with the highly damped quasinormal frequencies, since these take no time, as quantum transitions take no time. So, for instance, the highly damped quasinormal frequencies of the Schwarzschild black hole are found to be $M \omega_n = \frac{\ln 3}{8\pi} - \frac{i}{4} \left( n + \frac{1}{2} \right)$, to leading order. The factor $\frac{\ln 3}{8\pi}$ was first found numerically 56, and much later analytically 57. Then using $\Delta M = \omega$ and so $\Delta A = 32 \pi M \Delta M = 32 \pi M \omega = 4 (\ln 3) l^2_{pl}$, along with, from the very definition of $A$, $\Delta A = 4 (\ln k) l^2_{pl} \Delta n$, constrained by $\Delta n = 1$ as it is required for a single simple area transition, one finds $k = 3$. Then the quantization of the area is given by $A_n = 4 (\ln 3) l^2_{pl} n$. This has been also used by people of loop quantum gravity, and received a boost as the whole idea of Hod fixes the Barbiero-Imirzi parameter, a loose parameter in the theory 58.

The spin-area parameter $k$ was fixed in the case of a Schwarzschild black hole, $k = 3$. What can one say about the other black holes? The subject of quasinormal modes is a subject in which research has been very active since Vishveshwara noticed that the signal from a perturbed black hole is, for most of the time, an exponentially decaying ringing signal, with the ringing frequency and damping timescales being characteristic of the black hole, depending only on its parameters like $M$, $Q$ and $J$, and the cosmological constant $\Lambda$, say. Whereas for astrophysical black holes the most important quasinormal frequencies are the lowest ones, i.e., frequencies with small imaginary part, so that the signal can be detected, for black holes in funda-
mental physics the most important are the highly damped ones, since one is interested in the transition between classical and the quantum physics (see, e.g., 59 and for a review 60). Ultimately, one wants to understand whether the number $k = 3$ depends on the nature of the black hole (does a Kerr black hole give the Schwarzschild number), on the nature of spacetime (asymptotically flat, de Sitter, anti-de Sitter), and on the dimension of spacetime or not. Different spacetimes yield different boundary conditions, and thus completely different behavior for quasinormal modes, whereas one might expect black hole area levels to depend only on local physics near the horizon, so that it is not obvious how to reconcile such locality with the quasinormal mode behavior. This makes it hard to argue that $k$ is universal, as it should be. The study on other different black holes has not been conclusive.

(iii) York’s interpretation
York 61 made a very interesting proposal where the entropy of the black hole comes from the statistical mechanics of zero-point quantum fluctuations of the metric, in the form of quasinormal modes, over the entire time of evaporation. The approach has thus a very physical interpretation for the entropy, and gets the coefficients for the black hole entropy and temperature within the same order of magnitude as the exact ones. York’s idea is the translation of Gerlach’s quantum matter fields fluctuations 41 to fluctuations in the gravitational field, and has been retaken in 62.

(iv) Other methods
Other methods are Euclidean path integral 63, giving $S_{BH} = \frac{1}{4}A$ directly, but it is flawed, since it uses a saddle point approximation at a point that is not a minimum. There is a method of surface fields and Euclidean conical singularities 64. There is the Noether’s charge method 65, a very useful one that has been frequently used. There are also hints that the entropy depends on the gravitational Einstein-Hilbert action alone, and like energy in general relativity, is a global concept 66.

There are other techniques that, although not constructed to yield an interpretation, corroborate that there should be a statistical interpretation. One of these is related to pair creation of black holes. In the Schwinger process of production of charged particles in a background electric field, the total production rate grows as the number of particle species produced. If this is extrapolated to black hole production in a background field then the rate of the number of black hole pairs produced should go as the number of
black hole states. Indeed, one can show that the factor \( \Gamma \sim e^{\frac{1}{4} A_{BH}} = e^{S_{BH}} \) multiplies indeed the pair production amplitude, consistent with interpretation that the entropy counts black hole microstates. To work out these results one has to find the instanton solution, i.e., the solution that gives the transition rates, of the Euclidean C-metric, where the C-Metric is the solution for two black holes accelerating apart. This has been done for asymptotically flat spacetimes\(^67\) and for de Sitter and anti-de Sitter spacetime\(^68,69,70,71\).

The notion of black hole entropy has been extended to higher dimensional spacetimes where one can also have black \(p\)-branes. A black hole is a special case of a black \(p\)-brane, one with \(p = 0\), a black string has \(p = 1\), a black membrane has \(p = 2\), and so on. These black branes suffer from a gravitational instability, the Gregory-Laflamme instability, and entropic arguments suggest that the fate of such a brane is a set of black holes\(^72\).

3.3.3. Entropy in string theory

So far, I have not mentioned what is the contribution of string theory to the interpretation of black hole entropy. String theory has been extremely helpful in the advancement of black hole theory for several reasons. In relation to the calculation of black hole entropy it has given new methods, and many new and different black hole solutions on which one can apply these new methods. On the other hand, in relation to the interpretation of black hole entropy, to answer the question of where are the degrees of freedom, it has come short of a result. Let us see several developments in the context provided by string theory.

(i) Heuristics

First, heuristics\(^73\). String theory is a theory that provides many fields, which can be called matter fields, besides the gravitational field, and so the degrees of freedom for the entropy can come from both, the matter and the gravitational fields, now studied together in a coherent fashion. Since a string is matter, the entropy of a string goes with mass, and one can write \( S_{\text{string}} \sim l_s M \), where \( l_s \) is the fundamental string length (i.e., the string lengthscale), and \( M \) is the mass of the string (here we use string units). The entropy of a Schwarzschild black hole goes as \( S_{BH} \sim G M^2 \sim g^2 l_s^2 M^2 \), where now it was advisable to recover \( G \), which in string units is equal to \( g \) the coupling of the string with the spacetime times the string lengthscale \( l_s \), both squared. The black hole radius goes as \( r_{BH} \sim G M \sim g^2 l_s^2 M \). Now,
the coupling $g$ can be changed. Start from a black hole state, and assume one decreases the coupling reversibly and adiabatically, i.e., maintaining $S_{\text{BH}}$ constant. Then $M \sim 1/g$ increases, and $r_{\text{BH}} \sim g$ decreases. Thus as one puts less coupling, maintaining the entropy, the mass of the black hole increases so as to compensate in the number of states; on the other hand the radius decreases because there is much less gravity, a behavior that is similar to polytropic white dwarf stars. Now, one cannot go on decreasing the radius forever, the process has to stop when the radius of the black hole is of the order of the string scale $r_{\text{BH}} \sim l_s$. So, $l_s \sim g_{\text{crit}}^2 l_s^2 M_{\text{crit}}$ yielding from the black hole side $M_{\text{crit}} \sim 1/(g_{\text{crit}} l_s)$. This, in turn, implies $S_{\text{BH}} \sim 1/g_{\text{crit}}^2$, and from the string side $S_{\text{string}} \sim 1/g_{\text{crit}}^2$. Thus, this heuristic reasoning gives that there is a transition point from the black hole state to the string state, and vice versa, meaning that heavy string states form black holes, a not unexpected result. Unfortunately, there is no control as to where are the degrees of freedom when the black hole forms in this setup. One knows where are the degrees of freedom of the string, in the string itself, the way it curves, wiggles, vibrates, and so on, but then when it collapses and turns into a black hole at the transition point, it is a usual gravitational collapse, leaving us again in the dark. The nice thing about this calculation is that at the transition point the entropy is about the same for string and black hole, but how is the entropy transferred from the string to the black hole, or vice-versa, the calculation leaves us blind. See, however, the fuzzball proposal for black holes $^{74}$, where there is a retrieval of the interpretation that the entropy of a black hole lies on the volume inside the black hole, not in its area.

(ii) Exact calculations for extreme black holes

Extreme black holes allow an exact, though tricky, calculation of the entropy, done for the first time by Strominger and Vafa $^{75}$. A simple extreme black hole has mass $M$ and charge $Q$ that obey the relation $Q = (\sqrt{G})M$. The entropy is then $S_{\text{BH}} = 4\pi G M^2 = 4\pi Q^2$. Since the entropy does not depend on the gravitational constant $G$, the entropy is a measure of the number of the elementary charges of the extreme black hole alone. As we now know, $G = g^2 l_s^2$, and so the entropy does not depend on $g$. One can vary the string coupling $g$ and obtain the same entropy. On the other hand, $r_{\text{BH}} = G M = \sqrt{G} M = g l_s Q$, so $r_{\text{BH}}$ depends on $g$. For weak coupling one has $g << 1$ and so $r_{\text{BH}} << l_s$, the object is a condensed string in an almost flat spacetime, it is actually an intricate condensate of strings and branes, whereas for strong coupling one has $g >> 1$ and so $r_{\text{BH}} >> l_s$, the object
is a black hole. Now, some extreme black holes have the property they are
supersymmetric, i.e., supersymmetric transformations do not change the
black hole, and there are some theorems that say that there are no quan-
tum corrections when going from strong to weak coupling and vice-versa.
So, one can calculate the entropy of the object at weak coupling, where one
has an object in flat spacetime and then extrapolate directly and exactly
this calculation to strong coupling. At weak coupling, one finds that the
dual theory that governs the dynamics of the condensate of branes and
strings is a conformal field theory. One can then use the machinery of
conformal field theory, through the Cardy formula, and get the entropy.
Amazingly, for certain black holes in string theory, with several differ-
ent charges, it gives exactly the black hole entropy. This calculation is very
interesting indeed, but again it leaves us blind to what are and where a-
re the black hole degrees of freedom. Another snag of the calculation,
is that it does not work out for general black holes, it works out only for extreme
black holes, and even so not all extreme black holes.

(iii) What is conformal field theory?
We have been talking about conformal field theory, in various connections,
namely in connection with the degrees of freedom of the horizon related
to the method of Carlip and Solodukhin, and in connection with the
string theory methods. But what is conformal field theory? A way to see
this is to work with massless scalar fields in one spatial dimension, i.e.,
in two spacetime dimensions. The Klein-Gordon equation for each field is
\[
(\partial_t^2 - \partial_x^2) \phi_k(t, x) = 0, \quad k = 1, \ldots, c,
\]
valid in a one dimensional box of length \( b \), i.e, with boundary conditions
given by \( \phi_k(t, 0) = \phi_k(t, b) = 0 \). This has a Planck radiation spectrum
whose free energy is given by \( F(T, b) = cT \Sigma_n \ln (1 - e^{-\omega_n/T}) \), \( \omega_n = \frac{\pi}{b} n, n = 1, 2, \ldots \), where the \( \omega_n \) are the normal frequencies of the fields. In
the thermodynamic limit \( Tb \gg 1 \), one can evaluate the sum to obtain
\( F(T, b) = -\frac{c}{6} b T^2 \), and thus
\[
S(E, b) = 2\pi \sqrt{\frac{c b E}{6 \pi}}.
\]

Now, how can one calculate this entropy using conformal field theory
methods. First, one notes that the theory given in equation (7) is indeed
conformal invariant. Using null coordinates \( x_- = t - x \) and \( x_+ = t + x \) the
Klein-Gordon equation turns into \( \partial_{x_-} \partial_{x_+} \phi_k = 0 \). Indeed, this is invariant
under conformal transformation \( x_- \to x'_- = f(x_-) \) and \( x_+ \to x'_+ = f(x_+) \).
Now, when one has a symmetry, in this case conformal, one has an associated conserved charge. In turn these conserved charges are the generators of the corresponding symmetry transformation (for instance, the Hamiltonian is the generator of time translations, translations being included in conformal transformations). In two dimensions the generators, $L_n$ and $\bar{L}_n$, of conformal transformations are infinite. They give the standard Virasoro algebra, $[L_n, L_m] = (n-m) L_{n+m}$, and the same for the complex conjugate, where the brackets are Poisson brackets. Interesting to note that the algebra of the generators is the same as the algebra of the Fourier components of the infinitesimal vector field that gives the coordinate transformations. The Hamiltonian generator is $\frac{2\pi}{b}(L_0 + \bar{L}_0)$. This is classical, and there is no entropy for the $c$ scalar fields. However, when quantized the generators get an extra term, quantum mechanics yields always a scale which in turn produces an anomaly in the conformal field theory. This gives rise to an extra term for the algebra,

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12} (n^3 - n) \delta_{n+m,0},$$

where the brackets should be viewed as a commutator now, and the generators as operators. The Hamiltonian operator is then $H = \frac{2\pi}{b}(L_0 + \bar{L}_0)$, which when applied in a state $|h, \bar{h}\rangle$ gives the energy $E = \frac{2\pi}{b}(h + \bar{h})$. Now, the state $|h, \bar{h}\rangle$ can be constructed from vacuum, the box without fields, in many different ways, since $|h, \bar{h}\rangle = \Pi_k \alpha_k \Pi_p \alpha_p (L_{-k})^{\alpha_k} (L_{-p})^{\alpha_p} |0\rangle$, with $\Sigma_k \alpha_k = h$, $\Sigma_p \alpha_p = \bar{h}$, and where $|0\rangle$ is the vacuum vector, and $L_{-k}$ are creation operators. The state $|h, \bar{h}\rangle$ is an eigenstate of $L_0$ and $\bar{L}_0$, sure. One can then find the degeneracy, as Cardy did, and show that $D = e^{2\pi \sqrt{\frac{c}{6} b E}}$. Thus the entropy of the $c$ conformal fields in a periodic box is $S = \ln D = 2\pi \sqrt{\frac{c b E}{6 \pi}}$, as in the thermodynamic result.

In possession of these ideas, we can better understand the Strominger-Vafa calculation. For low $g$ one has a condensate of strings and branes instead of a black hole, which obey a conformal field theory. With the theory in hand one finds $c$, $E$ and $b$, then one gets $S_{\text{CFT}}$, and through supersymmetry arguments, extrapolates to high $g$, giving $S_{\text{BH}}$, through $S_{\text{BH}} = S_{\text{CFT}}$. The entropy $S_{\text{BH}}$ obtained in this way gives precisely $S_{\text{BH}} = \frac{1}{4} A$. This is exact, but no interpretation for the entropy.

(iv) The BTZ black hole and the AdS/CFT conjecture

There is another place where these calculations are exact, it is the three dimensional BTZ black hole that lives in a cosmological constant $\Lambda$ background, i.e., in an anti-de Sitter spacetime. The idea came as follows.
Brown and Henneaux \(^{78}\) showed for the first time that the asymptotic group of three-dimensional anti-de Sitter spacetime is the conformal group in two dimensions, stating in addition that any quantum theory of such a type of spacetime should take this into account. At about the same time Cardy gave a formula, now famous, for the entropy of a two-dimensional conformal field theory with central charge \(c^{76}\). Then, later, Strominger \(^{77}\) applied the Cardy formula to Brown and Henneaux results \(^{78}\) and showed that it gave the formula discovered by Bekenstein and Hawking, \(S_{BH} = \frac{1}{4}A\). More precisely, in this spacetime one has an intrinsic length scale, which is \(l = 1/\sqrt{\Lambda}\). One also has the black hole radius \(r_{BH}\). Now, the black hole entropy can be calculated through gravitational methods to give
\[
S_{BH} = \frac{1}{4}A = \frac{1}{4} \frac{2\pi r_{BH}}{G} = 2\pi \sqrt{\frac{l^2 M}{2G}}, \quad \text{with} \quad M = \frac{r_{BH}^2}{8\pi l^2 G}.
\]
Now, compare with the Cardy formula \(S_{CFT} = 2\pi \sqrt{\frac{c b E}{6\pi}}\). For this put \(b = 2\pi l\), forcing the conformal field theory to live on a cylinder of perimeter \(2\pi l\), identify \(M = E\), and then choose the central charge as \(c = \frac{3}{2} \frac{l}{G}\). Then, with these choices \(S_{BH} = S_{CFT}\). This equation relates classical and quantum quantities. The conformal theory is quantum lives on a flat spacetime \(M_2\), one dimensional lower than the black hole, which lives in three dimensional spacetime. The metric on \(M_2\) is a cylindrical flat metric, \(ds^2 = -dt^2 + l^2 d\varphi^2\). On the other hand, the metric for the black hole spacetime at constant large radius is \(ds^2 = \frac{r_{BH}^2}{2}(-dt^2 + l^2 d\varphi^2)\). So \(M_2\) can be seen, apart from a superficial factor, as the asymptotic infinity of \(M_3\), as its asymptotic boundary. Therefore, the black hole entropy (a semiclassical limit of quantum gravity), is determined by a quantum conformal field theory (CFT) defined at the asymptotic infinity of the bulk anti-de Sitter (AdS) spacetime. This is an example of the AdS/CFT conjecture of Maldacena \(^{79}\), which was based in other spacetimes, and also works here. Since this type of computation for the black hole entropy is done at infinity, the infinity of anti-de Sitter spacetime, it does not see the details of the horizon. Thus, more than a direct computation of black hole entropy, this type of computation gives an upper bound for the entropy of anti-de Sitter spacetime in three dimensions. In this case, it is just as good, since the maximum of entropy in a region arises by inserting a black hole in it.

4. Conclusions

Thus we see that we are still far from having a consensus \(^{39}\). Are the degrees of freedom located in the volume or in the area, or in both, or are they complementary descriptions? Are they realized in the matter or in
the gravitational field or in both? The answer still lies ahead. The entropy puzzle does not exhaust the black hole. Other sources of fascinating problems and conundrums are the information paradox \^{80,81}, the holographic principle \^{82,83} (for some developments see \^{84,85,86,87}), and last but not the least the inside of a black hole and the problem of spacetimes singularities \^{88,89}. All of these are problems in fundamental physics whose solutions will help in a better understanding of the connections between quantum theory, statistical and information theory, and gravitation, and ultimately can lead us to the correct quantum theory of gravity.

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