Large entropy implies existence of a maximal entropy measure for interval maps

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Abstract

We give a new type of sufficient condition for the existence of measures with maximal entropy for an interval map \( f \), using some non-uniform hyperbolicity to compensate for a lack of smoothness of \( f \). More precisely, if the topological entropy of a \( C^1 \) interval map is greater than the sum of the local entropy and the entropy of the critical points, then there exists at least one measure with maximal entropy. As a corollary, we obtain that any \( C^r \) interval map \( f \) such that \( h_{\text{top}}(f) > 2 \log \|f'\|_{\infty}/r \) possesses measures with maximal entropy.

1 Introduction

Let \( f : X \to X \) be a continuous map, where \( X \) is a compact metric space with distance denoted by \( d \). The \((\epsilon, n)\)-ball around \( x \) is the set

\[ B(x, n, \epsilon) = \{ y \in X \mid \forall 0 \leq k < n, \ d(f^k y, f^k x) \leq \epsilon \}, \]

and, if \( S \subset X \), \( r(\epsilon, n, S) \) is the minimum number of \((\epsilon, n)\)-balls the union of which covers \( S \). Recall that the entropy is a measure of dynamical complexity (see [11] for background). Namely, the topological entropy of \( f : X \to X \) counts the number of orbits in the following way, according to Bowen’s definition [4]:

\[ h_{\text{top}}(f) = h_{\text{top}}(X, f) = \lim_{\epsilon \to 0} h_{\text{top}}(X, f, \epsilon) \]

with

\[ h_{\text{top}}(X, f, \epsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log r(\epsilon, n, X). \]

The entropy \( h_{\text{top}}(S, f) \) of a (not necessarily invariant) subset \( S \subset X \) is defined in the same way.

The entropy of an invariant and ergodic probability measure \( \mu \) of \( f \) is similarly defined, according to Katok’s formula [17]:

\[ h(\mu, f) = \lim_{\epsilon \to 0} h(\mu, f, \epsilon), \]

with

\[ h(\mu, f, \epsilon) = \limsup_{n \to +\infty} \frac{1}{n} \log \inf_{\mu(Y) \geq \lambda} r(\epsilon, n, Y) \]

where \( \lambda \) is any number in \((0, 1]\).

In this continuous and compact setting, it is well-known that the variational principle holds (see, e.g., [25]): the topological entropy \( h_{\text{top}}(f) \) of \( f \) is equal to the supremum of the metric entropies \( h(\mu, f) \) taken over all \( f \)-invariant probability measures. A measure \( \mu \) such that \( h(\mu, f) = h_{\text{top}}(f) \) is called a maximal measure. Such measures, when they exist, are particularly interesting because they reflect the whole topological complexity of the system, and they enable to see where this complexity concentrates.

However maximal measures do not always exist. Continuity and even mild differentiability are insufficient to ensure their existence in contrast to the generality of the variational principle. In fact, given \( r < +\infty \) arbitrary large, there exist \( C^r \) diffeomorphisms of compact 4-dimensional manifolds (constructed by M. Misiurewicz [18]) as well as \( C^r \) interval maps [7, 22] (necessarily with an infinite critical set, see below) which have no maximal measure.

There are mainly two types of situation when existence is known: i) when the dynamics has some expansiveness (for example if the system is expansive, see e.g., [11], or in the case of uniform hyperbolicity,
see e.g., [5]); ii) when the map is $C^\infty$ [20]. In both cases, and in fact in all existence results we know of, one proves that the metric entropy $\mu \mapsto h(\mu, f)$ is upper semicontinuous and therefore reaches its supremum by compactness (see e.g., [11]). The only exceptions are the abstract characterizations of existence due to M. Denker [10] in a topological setting and to S. Newhouse (Theorem 8 of [20]) for diffeomorphisms. These results are obtained by establishing upper semicontinuity of the entropy on an appropriate compact subset of measures.

Our goal is to show that non-uniform hyperbolicity and finite order differentiability can be combined to get a criterion of existence of maximal measures. In this paper, we focus on continuous interval maps $f: [0, 1] \to [0, 1]$ with non-zero topological entropy.

Let $C(f)$ denote the critical set of $f$, that is, the set of points which have no neighbourhood on which $f$ is monotonic (if $f$ is $C^1$ then $C(f)$ is contained in the zeroes of $f'$). If $C(f)$ is finite, $f$ is a (continuous) piecewise monotonic map and for such maps existence is known at least since Hofbauer’s paper [16]. The uniqueness of the maximal measure was first shown for $\beta$-transformations by Takahashi [23] (see also [15]), then Hofbauer extended the method – association of a Markov shift, the Markov diagram, to the initial system – to general piecewise monotonic maps [16]. He showed that these maps have a finite non zero number of ergodic maximal measures as soon as their topological entropy is positive, and that the maximal measure is even unique if in addition $f$ is topologically transitive.

For continuous interval maps with an infinite critical set the situation is more complex. Neither existence nor finite multiplicity of ergodic maximal measures are guaranteed but we are going to give a sufficient condition in the form of a lower bound on the topological entropy.

Two quantities play an important role. The first one is the topological entropy of the critical set: its “smallness” can replace the finiteness of the critical set (which was the required assumption in Hofbauer’s work). Namely, it was shown in [6, 7] that a Markov diagram can be associated to any interval map $f$ and, if $h_{top}(f) > h_{top}(C(f), f)$, then there is a bijection between the maximal measures of $f$ and those of its Markov diagram (see section 3). This is a decisive step because Gurevich gave an equivalent condition for existence and uniqueness of maximal measures for transitive Markov shifts [13]. In many cases the condition $h_{top}(f) > h_{top}(C(f), f)$ can be checked using the fact that the topological entropy of the critical set is bounded by $\log \|f^{\prime}\|_\infty/r$ for a $C^r$ interval map $f$ (if $r$ is not an integer, this means that $f$ is $C^r$ and the $[r]$-th derivative is $(r-[r])$-Hölder); in particular $h_{top}(C(f), f)$ is equal to zero for a $C^\infty$ map [6, 7]. Actually, for an interval map that is $C^{1+\alpha}$, $\alpha > 0$, the condition $h_{top}(f) > 0$ is enough to have a bijection between the maximal measures of $f$ and those of a Markov shift provided one uses a variant of the Markov diagram (see [9]).

The second key notion is that of local entropy. The notion of $\epsilon$-local entropy was introduced by Bowen [3] to bound the difference between $h(\mu, f)$ and the entropy of a partition with diameter less than $\epsilon$, then Misiurewicz showed that local entropy (that he called conditional topological entropy) bounds the defect in upper semicontinuity of the metric entropy $\mu \mapsto h(\mu, f)$ [19]. Hence if the local entropy is zero then there exists some maximal measure.

We shall work with the following (equivalent) definition:

\textbf{Definition} The local entropy of a continuous self-map $f: X \to X$ of a compact metric space is $h_{loc}(f) = \lim_{\epsilon \to 0} h_{loc}(f, \epsilon)$ where

$$h_{loc}(f, \epsilon) = \lim_{\epsilon \to 0} \limsup_{n \to +\infty} \frac{1}{n} \log \sup_{x \in X} r(\delta, n, B(x, n, \epsilon)).$$

The local entropy is bounded by $\frac{d}{2} \log \sup \|f^{\prime}\|_\infty$ for a $C^r$ map on a compact manifold of dimension $d$. This was proved for a (slightly weaker) measure-theoretic local entropy by Newhouse [20] and then for exactly the above notion by one of us [7]. In particular it is zero for $C^\infty$ maps; notice that it is zero for piecewise monotonic maps too.

\textbf{Remark.} Using Blokh’s spectral decomposition [1, 2] for a continuous interval map $f$, it was shown in [6, 7] that there are only finitely many connected components in the Markov diagram with entropy close to $h_{top}(f)$ if $h_{top}(f) > h_{loc}(f)$, each of which supporting at most one ergodic maximal measure.

\textsuperscript{*}Let us notice that in this definition the supremum over all points $x$ can be moved outside of the limits on $\epsilon, \delta$ and $n$ (see [9] or [11]).
Let Theorem 7 in the C infinite critical set and finite smoothness. Indeed, existence had only been proved when $h_{\text{top}}(f) > h_{\text{loc}}(f)$ for $C^{1+\alpha}$ maps, $h_{\text{top}}(f) > h_{\text{loc}}(f)$ is in fact sufficient using [9]).

Recalling that $\log \|f'\|^r_\infty/r$ bounds both $h_{\text{loc}}(f)$ and $h_{\text{top}}(C(f), f)$ for a $C^r$ interval map $f$, we see that as soon as $f$ satisfies $\log \|f'\|^r_\infty/r < h_{\text{top}}(f)$, then there are only finitely many ergodic maximal measures. This condition is optimal in the sense that there exist $C^r$ interval maps $f$ with infinitely many ergodic maximal measures satisfying the equality: $h_{\text{top}}(f) = \log \|f'\|^r_\infty/r$ (see [7]).

The remaining open question therefore is that of existence of maximal measures for interval maps with infinite critical set and finite smoothness. Indeed, existence had only been proved when $h_{\text{loc}}(f) = 0$, which is known to be the case for piecewise monotonic and $C^\infty$ maps. We give an answer to this question in the $C^1$ case:

**Theorem 7** Let $f : [0, 1] \to [0, 1]$ be a $C^1$ map and $C(f)$ the critical set of $f$. Assume that $h_{\text{top}}(f) > h_{\text{top}}(C(f), f) + h_{\text{loc}}(f)$. Then $f$ admits a maximal measure. Moreover, the number of ergodic maximal measures is finite and, if $f$ is transitive, the maximal measure is unique.

Using the previously mentioned bounds on local entropy and entropy of the critical set in terms of the differentiability of the map, we get a condition that is easier to compute:

**Corollary** Let $f : [0, 1] \to [0, 1]$ be a $C^r$ map of the interval with $r \geq 1$.

If $h_{\text{top}}(f) > 2 \log \|f'\|^r_\infty/r$, then $f$ has a finite non-zero number of maximal measures.

**Remark.** This Corollary is relevant only for $r > 2$ because $h_{\text{top}}(f) \leq \log \|f'\|^r_\infty$ (see, e.g., [11]).

The finiteness result in Theorem 7 was already proved in [6, 7] under weaker hypothesis. We nevertheless include it for completeness and also because it is obtained in a completely different way here, in fact as a slight variation of the proof of existence.

For interval maps such that $h_{\text{top}}(C(f), f) = 0$, Theorem 7 is sharp: for all $1 \leq r < +\infty$ there exist $C^r$ interval maps that have no maximal measure and such that $h_{\text{top}}(C(f), f) = 0$ and $h_{\text{top}}(f) = h_{\text{loc}}(f) = \frac{1}{r} \log \|f'\|^r_\infty$ (see [7, 22]). These examples can be adapted to show that this Theorem indeed applies to maps such that the metric entropy $\mu \mapsto h(\mu, f)$ is not upper semicontinuous, in contrast to all other existence results for interval maps. In fact, we get examples with the defect in upper semicontinuity as large as $h_{\text{loc}}(f) = \frac{1}{r} \log \|f'\|^r_\infty$.

For interval maps with $h_{\text{top}}(C(f), f) > 0$ we do not know whether the Theorem is optimal. Actually, the above Corollary was conjectured, without the factor of 2 in [7]. We still believe in this conjecture. In fact, we make the bolder

**Conjecture** If $f : [0, 1] \to [0, 1]$ is a continuous interval map such that $h_{\text{top}}(f) > h_{\text{loc}}(f)$ then $f$ admits measures with maximal entropy.

A way to prove this, would be to establish that for interval maps, if $\mu_1, \mu_2, \ldots$ is a sequence of invariant probability measures vaguely converging to some $\mu_*$, then:

$$\limsup_{n \to \infty} h(\mu_n, f) \leq h(\mu_*, f, h_{\text{loc}}(f)).$$

That is, the obvious bound with a *sum* could be replaced for interval maps by a *maximum* (this is obviously false in higher dimensions).

**Remark.** For interval maps with $h_{\text{top}}(C(f), f) = h_{\text{top}}(f)$ which are not $C^{1+\alpha}$, $\alpha > 0$, the relevant dynamics may be completely missed by the Markov diagram. Hence proof of the above conjecture in its full generality probably requires a different method from the one used in this paper.

**Outline of the paper**

We begin by recalling the relevant theory of countable Markov shifts. In the second section, we introduce the Markov diagram, i.e., a countable Markov shift representing the interval map. In the third section we prove that measures escaping to infinity in this Markov diagram have small entropy. Finally we deduce the Main Theorem from the previous results.
2 Background on Markov shifts

2.1 Graphs and Markov shifts

Let $G$ be an oriented graph with a countable set of vertices. If $u,v$ are two vertices, there is at most one arrow $u \to v$. A path of length $n$ is a sequence of vertices $(u_0, \cdots, u_n)$ such that $u_i \to u_{i+1}$ is an arrow in $G$ for $0 \leq i < n$. This path is called a loop if $u_0 = u_n$. The graph $G$ is called strongly connected, if for all vertices $u,v$ there exists a path in $G$ from $u$ to $v$. A connected component $G'$ is a strongly connected subgraph which is maximal for inclusion; two connected components are equal or disjoint.

Let $u$ be a vertex. In [24] Vere-Jones defines the following quantities.

- $p_u^G(n)$ is the number of loops $(u_0, \cdots, u_n)$ such that $u_0 = u_n = u$; $R_u(G)$ is the radius of convergence of the series $\sum p_u^G(n)z^n$.
- $f_u^G(n)$ is the number of loops $(u_0, \cdots, u_n)$ such that $u_0 = u_n = u$ and $u_i \neq u$ for $0 < i < n$; $L_u(G)$ is the radius of convergence of the series $\sum f_u^G(n)z^n$.

If $G$ is strongly connected, then $R_u(G)$ does not depend on $u$; in this case it is denoted by $R(G)$.

Let $G$ be an oriented graph. $\Sigma_+(G)$ is the set of one-sided infinite paths in $G$, that is,

$$\Sigma_+(G) = \{(u_n)_{n\in\mathbb{N}} \mid \forall n \in \mathbb{N}, u_n \rightarrow v_{n+1} \text{ in } G\}.$$ 

$\sigma$ is the shift on $\Sigma_+(G)$, $\sigma((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}$. The Markov shift on the graph $G$ is the system $(\Sigma_+(G), \sigma)$.

The set $G$ is endowed with the discrete topology and $\Sigma_+(G)$ is endowed with the induced topology of $G^\mathbb{N}$, which has the product topology. The space $\Sigma_+(G)$ is not compact unless $G$ is finite. The system $(\Sigma_+(G), \sigma)$ is transitive if and only if the graph $G$ is strongly connected.

If $S \subset G$, the cylinder $[S]$ is defined as

$$[S] = \{(u_n)_{n\in\mathbb{N}} \in \Sigma_+(G) \mid u_0 \in S\}.$$ 

2.2 Entropy and maximal measures

If $G$ is an oriented graph, the Gurevich entropy of $G$ is defined as

$$h(G) = \sup_{u \in G} -\log R_u(G).$$

If $G'$ is a connected component of $G$, then $R_u(G) = R(G')$ for all $u \in G'$, hence

$$h(G) = \sup\{h(G') \mid G' \text{ connected component of } G\}.$$ 

Moreover, the variational principle is still valid for the Gurevich entropy.

**Theorem 1 (Gurevich [12])** Let $G$ be an oriented graph. Then

$$h(G) = \sup\{h(\mu, \sigma) \mid \mu \text{ } \sigma\text{-invariant probability measure on } \Sigma_+(G)\}.$$ 

Moreover, the supremum can be taken on ergodic Markov measures only.

A maximal measure is a $\sigma$-invariant probability measure $\mu$ on $\Sigma_+(G)$ whose entropy is maximal, that is, $h(\mu, \sigma) = h(G)$.

An ergodic measure on $\Sigma_+(G)$ is necessarily supported by some $\Sigma_+(G')$, where $G'$ is a connected component of $G$. Therefore an ergodic maximal measure on $\Sigma_+(G)$ is a maximal measure for a connected component $G'$ with $h(G') = h(G)$.

A strongly connected oriented graph $G$ is called transient, null recurrent or positive recurrent depending on the values of the series $\sum f_u^G(n)z^n$ and its derivative at point $z = R(G)$ (see Table 1). This classification is due to Vere-Jones [24]. In [13] Gurevich shows that, if $G$ is strongly connected, the Markov shift $(\Sigma_+(G), \sigma)$ admits a maximal measure if and only if $G$ is positive recurrent, and in this case this measure is unique and it is an ergodic Markov measure.

We sum up the results above in the next Theorem.
Table 1: classification of strongly connected graphs into transient, null recurrent and positive recurrent graphs (it does not depend on the vertex $v$).

| $\sum_{n>0} f^G_v(n) R(G)^n$ | transient | null recurrent | positive recurrent |
|-------------------------------|-----------|----------------|--------------------|
| $<1$                          | $1$       | $1$            | $<+\infty$        |
| $\leq +\infty$               | $+\infty$ | $+\infty$     | $<+\infty$        |

Remark 1 Two-sided infinite paths (i.e. paths indexed by $\mathbb{Z}$) are often considered instead of one-sided infinite paths. Gurevich stated his results for such invertible Markov shifts. However they are still valid in the non-invertible case that interests us.

2.3 Almost maximal measures escaping to infinity

Let $G$ be an oriented graph and $G \cup \{\infty\}$ its one-point compactification. The set $\overline{\Sigma(G)} \subset (G \cup \{\infty\})^N$ is compact and so is the set of $\sigma$-invariant measures on $\overline{\Sigma(G)}$ [11]. Gurevich and Savchenko showed that if $G$ is either transient or null recurrent then any sequence of ergodic measures $(\nu_n)_{n \geq 1}$ whose entropy tends to $h(G)$ converges to the Dirac measure $\delta_{\infty}$ on $\overline{\Sigma(G)}$ (for the weak-* topology). This is Theorem 6.3(1) in [14] for a null potential, we restate the measures convergence in term of cylinders then we generalise this result to all oriented graphs with no maximal measure.

Theorem 3 (Gurevich-Savchenko [14]) Let $G$ be a strongly connected graph of finite entropy which is not positive recurrent. If $(\nu_n)_{n \geq 1}$ is a sequence of ergodic measures such that $\lim_{n \to +\infty} h(\nu_n, \sigma) = h(G)$ then for all finite subsets of vertices $F$ one has $\lim_{n \to +\infty} \nu_n([F]) = 0$.

Proposition 1 Let $G$ be an oriented graph of finite entropy. Suppose that $\Sigma(G)$ admits no maximal measure. Then there exists a sequence of ergodic Markov measures $(\nu_n)_{n \geq 1}$ such that $\lim_{n \to +\infty} h(\nu_n, \sigma) = h(G)$ and for all finite subsets of vertices $F$, $\lim_{n \to +\infty} \nu_n([F]) = 0$.

Proof. Suppose first that $G$ has a connected component $G'$ with $h(G) = h(G')$. By Theorem 1 there exists a sequence of ergodic Markov measures $(\nu_n)_{n \geq 1}$ on $\Sigma(G')$ such that $\lim_{n \to +\infty} h(\nu_n, \sigma) = h(G')$. The measures $\nu_n$ can be seen as measures on $\Sigma(G)$. By assumption $\Sigma(G')$ admits no maximal measure thus $G'$ is not positive recurrent by Theorem 2 and Theorem 3 applies.

Now suppose inversely that $G$ has no connected component of entropy equal to $h(G)$. This assumption implies that there exists a sequence of distinct connected components $(G_n)_{n \geq 0}$ such that $\lim_{n \to +\infty} h(G_n) = h(G)$. According to Theorem 1 there exists an ergodic Markov measure $\nu_k$ on $\Sigma(G_k)$ such that $h(\nu_k, \sigma) \geq h(G_k) - \frac{1}{k}$. This implies that $\lim_{k \to +\infty} h(\nu_k, \sigma) = h(G)$. Moreover, if $F$ is a finite subset of vertices of $G$, there exists $n$ such that $F \cap \bigcup_{k \geq n} G_k = \emptyset$, thus $\nu_n([F]) = 0$ for all $k \geq n$. \hfill \Box

Proposition 2 Let $G$ be an oriented graph of finite non-zero entropy. Suppose that $(\nu_k)_{k \geq 1}$ is a sequence of distinct ergodic maximal measures for $\Sigma(G)$. Then for all finite subsets of vertices $F$, one has $\lim_{n \to +\infty} \nu_n([F]) = 0$.

Proof. By Theorem 2, $\nu_n$ is supported by a connected component $G_n$ and all the graphs $G_n$ are disjoint. Let $F$ be a finite subset of vertices. There exists an integer $N$ such that $F \cap G_n = \emptyset$ for all $n \geq N$, thus $\nu_n([F]) = 0$ for all $n \geq N$. \hfill \Box
3 The Markov diagram

This section is devoted to the reduction of the map on the interval to a Markov shift.

This reduction was introduced by Hofbauer [16] for piecewise monotonic maps (see also Takahashi
for a special case [23]). We need the variant introduced in [6, 7] for general interval maps. Let us recall
its definition.

Consider \( f: [0, 1] \to [0, 1] \) a \( C^1 \) map and let \( C(f) \) be the critical set of \( f \), that is, the set of points
in a neighbourhood of which \( f \) is not monotonic. Let \( C_* \) be a finite subset of \([0,1]\) and \( C = C(f) \cup C_* \).
The additional set \( C_* \) will be needed in the proof of Theorem 6. It does not change anything to the
construction and does not affect the entropy of the critical set. Indeed,

\[
h_{\text{top}}(C, f) = \max(h_{\text{top}}(C(f), f), h_{\text{top}}(C_*, f)) = h_{\text{top}}(C(f), f).
\]

Let \( \mathcal{P} \) be the collection of the connected components of \([0,1] \setminus C \) and let \( \mathcal{P}^* \) be the set of finite
sequences \( A_{-n} \ldots A_0 \), where \( A_i \in \mathcal{P} \).

The set \( [A_0 \ldots A_n]_f \) is defined as

\[
[A_0 \ldots A_n]_f = \{ x \in [0, 1] \mid f^i(x) \in A_i, 0 \leq i \leq n \} = \bigcap_{i=0}^n f^{-i}(A_i).
\]

**Lemma 1** Observe that:

- \( [A_0 \ldots A_n]_f \) is an open interval.
- \( f^n \) restricted to \( [A_0 \ldots A_n]_f \) is a homeomorphism on its image.
- \( [A_0 \ldots A_n]_f \) is the intersection of the maximal connected components of \( [0,1] \) for \( f^n \).

Say that \( A_{-n} \ldots A_0 \) and \( B_{-m} \ldots B_0 \) are equivalent if and only if there exists \( 0 \leq k \leq \min(n, m) \) such that:

\[
A_{-k} \ldots A_0 = B_{-k} \ldots B_0
\]

\[
f^k([A_{-k} \ldots A_0]_f) = f^n([A_{-n} \ldots A_0]_f)
\]

\[
f^k([B_{-k} \ldots B_0]_f) = f^m([B_{-m} \ldots B_0]_f).
\]

We write in this situation \( A_{-n} \ldots A_0 \approx B_{-m} \ldots B_0 \).

If \( k \) is minimal with the properties above, then \( A_{-k} \ldots A_0 \) is called the significant part of \( A_{-n} \ldots A_0 \).

Two elements of \( \mathcal{P}^* \) are equivalent if and only if they have the same significant part. If \( \alpha \) is the equivalence
class of \( A_{-n} \ldots A_0 \), we define

\[
\langle \alpha \rangle = f^n([A_{-n} \ldots A_0]_f) = \bigcap_{i=0}^n f^i(A_{-i}).
\]

Let \( \mathcal{D} \) be the set of the equivalence classes \( \alpha \in \mathcal{P}^*/ \approx \) with \( \langle \alpha \rangle \neq \emptyset \). If \( \alpha, \beta \in \mathcal{D} \), there is an arrow
\( \alpha \to \beta \) if and only if there exist \( A_{-n}, \ldots, A_0, A_1 \in \mathcal{P} \) such that \( \alpha \) is the equivalence class of \( A_{-n} \ldots A_0 \)
and \( \beta \) is that of \( A_{-n} \ldots A_0 A_1 \). The countable oriented graph \( \mathcal{D} \) is called the Markov diagram associated
to \( f \) with respect to \( C \). It defines a Markov shift \( (\Sigma_+(D), \sigma) \) (see Section 2).

It is convenient to let \( \mathcal{D}_n \) be the collection of equivalence classes generated by words of length at most \( n + 1 \). We say that an element \( D \) of \( \mathcal{D}_n \) has level or height \( H(D) = n \).

For \( \alpha = (\alpha_n)_{n \geq 0} \in \Sigma_+(\mathcal{D}) \), let \( A_n \) be the element of \( \mathcal{P} \) containing \( \langle \alpha_n \rangle \). The sequence \( A \) is the
projection or the itinerary of \( \alpha \). Define

\[
\pi(\alpha) \in \bigcap_{n \geq 0} [A_0 \ldots A_n]_f = \bigcap_{n \geq 0} f^{-n}(A_n).
\]

There is an arbitrary choice involved in the definition of \( \pi(\alpha) \) when this intersection is a non-trivial
interval. Notice that this occurs only for countably many \( \alpha \)'s.

If \( \nu \) is an atomless \( \sigma \)-invariant probability measure on \( \Sigma_+(\mathcal{D}) \) then \( \mu = \pi_* (\nu) \) is a \( f \)-invariant probability
measure on \([0,1]\), defined by \( \mu(B) = \nu(\pi^{-1}B) \). Moreover, \( \mu \) is ergodic if \( \nu \) is ergodic.
\textbf{Theorem 4} [7] Let \( f : [0, 1] \to [0, 1] \) be a \( C^1 \) map that satisfies \( h_{\text{top}}(f) > h_{\text{top}}(C, f) \) and let \( \Sigma_+(D), \pi \) be defined as above.

Then the map \( \nu \to \mu = \pi_*(\nu) \) is a bijection preserving entropy between the \( \sigma \)-ergodic measures \( \nu \) and the \( f \)-ergodic measures \( \mu \) such that \( h(\nu, \sigma) > h_{\text{top}}(C, f) \) and \( h(\mu, f) > h_{\text{top}}(C, f) \).

In particular, \( h(D) = h_{\text{top}}(f) \) and \( \pi \) induces a bijection between the maximal measures of \( f \) and \( \Sigma_+(D) \).

We shall need the following facts:

\textbf{Lemma 2} If \( \alpha_0 \ldots \alpha_n \) is a path on \( D \) and if \( A_k \) is the element of \( \mathcal{P} \) containing \( \langle \alpha_k \rangle \), then

\[ \alpha_n \text{ is equivalent to } B_{-m} \ldots B_0 A_1 \ldots A_n \]

for any \( B_{-m} \ldots B_0 \) which is equivalent to \( \alpha_0 \).

This is a rephrasing of Lemma 5.4 of [7]. We give a proof for completeness.

\textit{Proof.} Suppose that \( B_{-m} \ldots B_0 \) is the significant part of \( \alpha_0 \). Since \( \alpha_0 \to \alpha_1 \), there exist \( A_{-k}, \ldots, A_0, A_1 \) in \( \mathcal{P} \) such that \( \alpha_0 \) is equivalent to \( A_{-k} \ldots A_0 \) and \( \alpha_1 \) is equivalent to \( A_{-k} \ldots A_0 A_1 \). Thus, \( A_{-k} \ldots A_0 \approx B_{-m} \ldots B_0 \). This implies \( k \geq m \), \( A_{-m} \ldots A_0 = B_{-m} \ldots B_0 \) and:

\[ \langle \alpha_0 \rangle = f^k([A_{-k} \ldots A_0]_f) = f^m([B_{-m} \ldots B_0]_f). \]

It follows immediately that \( A_{-m} \ldots A_0 A_1 = B_{-m} \ldots B_0 A_1 \) and:

\[ f^{m+1}([B_{-m} \ldots B_0 A_1]_f) = A_1 \cap \bigcap_{i=0}^m f^{i+1}(B_{-i}) = A_1 \cap f([\alpha_0]) \]

i.e., \( A_{-k} \ldots A_0 A_1 \approx B_{-m} \ldots B_0 A_1 \). Moreover \( \langle \alpha_1 \rangle \subset A_1 \).

The rest of the proof follows by induction. \( \square \)

\textbf{Lemma 3} Let \( \alpha = (\alpha_n)_{n \geq 0} \in \Sigma_+(D) \) and \( x = \pi(\alpha) \). If the significant part of \( \alpha_n \) is \( A_{-k} \ldots A_0 \) and if \( k \leq n \), then \( f^{n-j}x \in A_{-j} \) for \( 0 \leq j \leq k \).

\textit{Proof.} Let \( 0 \leq j \leq k \). If \( \alpha_{-j} \) is the equivalence class of some \( B_{-q} \ldots B_0 \) then there exist \( B_1, \ldots, B_j \in \mathcal{P} \) such that \( \alpha_n \) is the equivalence class of \( B_{-q} \ldots B_0 B_1 \ldots B_j \) (see Lemma 2). Therefore \( B_0 \ldots B_j = A_{-j} \ldots A_0 \). By definition of \( \pi \), this implies \( f^{n-j}(x) \in A_{-j} \) and proves the Lemma. \( \square \)

\textbf{Remark 2} Lemma 3 would be false if we had used Hofbauer’s Markov diagram. Indeed, in Hofbauer’s Markov diagram, the vertices of the graph are not the sequences \( \alpha \in D \) as above but the intervals \( \langle \alpha \rangle \). But completely different words \( \alpha \) (sharing only their last symbol) may by coincidence give the same interval. These words will give disjoint paths ending at the same vertex, in contradiction with the Lemma.

Finally, we need that the transitivity of \( f \) implies that the Markov diagram is essentially irreducible.

\textbf{Lemma 4} If \( f \) is transitive then its Markov diagram contains at most one connected component with entropy larger than \( h_{\text{top}}(C, f) \).

\textit{Proof.} Let \( G_1, G_2 \subset D \) be two connected components with entropy larger than \( h_{\text{top}}(C, f) \). By symmetry, it is enough to build a path from \( G_1 \to G_2 \) to prove that \( G_1 = G_2 \).

Define \( \mathcal{P}^n \) as the collection of disjoint open intervals \( [A_0 \ldots A_{n-1}]_f \) with \( A_i \in \mathcal{P} \). If \( x \in [0, 1] \), let \( \mathcal{P}^n(x) \) denote the element of \( \mathcal{P}^n \) that contains \( x \) when such an element exists.

Let \( \alpha_0 \in G_1 \). Let \( I \) be the open, non-empty interval \( \langle \alpha_0 \rangle \). The set \( K = \bigcup_{n \geq 0} f^n(I) \) is a union of intervals. By transitivity, \( f^k(I) \cap I \neq \emptyset \) for some \( k \) so that \( K \) is a finite union of intervals. Again by transitivity, \( K \) is dense in \([0, 1] \). Hence \([0, 1] \setminus K \) is reduced to finitely many points.

Fix \( \nu_2 \) an ergodic and invariant probability measure on \( \Sigma_+(G_2) \) with \( h(\nu_2, \sigma) > h_{\text{top}}(C, f) \). Let \( \mu_2 = \pi_*(\nu_2) \). Let us observe a number of generic properties:
• \( \mu_2 \) is non-atomic so that \( \mu_2(K) = 1 \).
• \( \mu_2(\pi(\Sigma_+(G_2))) = 1 \).
• \( \mu_2(\bigcup_{n,m \geq 0} f^{-n} f^m C) = 0 \). Otherwise \( \mu_2(f^m C) = \mu_2(f^{-n} f^m C) > 0 \) for some \( n,m \geq 0 \) but this would imply \( h(\mu_2, f) \leq h_{\text{top}}(f^m C, f) = h_{\text{top}}(C, f) \). But \( h(\mu_2, f) = h(\nu_2, f) \) by Theorem 4, which leads to a contradiction.

• for \( \mu_2 \text{-a.e.} \ x \), \( \mathcal{P}_n(x) \) is well-defined for all \( n \geq 1 \) and \( \lim_{n \to +\infty} \text{diam} \mathcal{P}_n(x) = 0 \).

From the properties above we deduce that there exists \( y \in K \cap \pi(\Sigma_+(G_2)) \) such that \( y \notin \bigcup_{n,m \geq 0} f^{-n} f^m C \) and \( \lim_{n \to +\infty} \text{diam} \mathcal{P}_n(y) = 0 \). Let \( \beta \in \Sigma_+(G_2) \) such that \( y = \pi(\beta) \), \( \beta_0 \) being the equivalence class of some \( B_{-q} \ldots B_0 \). Define \( J = \langle \beta_0 \rangle \); this is an open interval containing \( y \). Since \( y \in K \), there exist \( x \in I \) and \( k \geq 0 \) such that \( y = f^k(x) \). Moreover, for all \( n \geq 0 \) there exists \( A_n \in \mathcal{P} \) such that \( f^n(x) \in A_n \). Let \( \alpha_n \) be the equivalence class of \( A_{-p} \ldots A_n \) for all \( n \geq 0 \), where \( A_{-p} \ldots A_0 \approx \alpha_0 \). The set \( J' = \langle \alpha_k \rangle \) is an open interval containing \( y \).

By Lemma 2, \( \beta_n \) is the equivalence class of \( B_{-q} \ldots B_0 A_{k+1} \ldots A_{k+n} \), with \( B_0 = A_k \). One has \( \mathcal{P}_n(y) = [A_k \ldots A_{k+n}]f \), and its diameter tends to 0 by the choice of \( y \). Therefore there exists \( n \geq 0 \) such that \( [A_k \ldots A_{k+n}]f \subset J' \cap J \). One has

\[
\langle \alpha_{n+k} \rangle = f^{n+k+p}([A_{-p} \ldots A_{n+k}]f) = f^n(f^{k+p}([A_{-p} \ldots A_{k}]f) \cap [A_k \ldots A_{n+k}]f) = f^n(J' \cap [A_k \ldots A_{n+k}]f) = f^n([A_k \ldots A_{n+k}]f).
\]

The same computation gives

\[
\langle \beta_n \rangle = f^n(J \cap [A_k \ldots A_{n+k}]f) = f^n([A_k \ldots A_{n+k}]f).
\]

Therefore \( \alpha_{n+k} = \beta_n \), and \( \alpha_0 \rightarrow \cdots \rightarrow \alpha_{n+k} \) is a path between \( \alpha_0 \in G_1 \) and \( \beta_n \in G_2 \). \qed

4 Entropy at infinity in \( D \)

In this section, we consider a sequence of ergodic measures on \( \Sigma_+(D) \) which charge less and less any finite set of vertices and whose entropy is bounded from below by \( h_{\text{top}}(C(f), f) \). We prove (Proposition 3) that these measures escape to the high levels of the diagram. Then we show in Theorem 6 that, if \( C_* = \{ k\delta \mid k = 1, \ldots, [\delta^{-1}] \} \) for small \( \delta > 0 \), such a sequence of measures cannot have a large entropy.

To prove the Proposition we need the following result (which we restrict to interval maps and ergodic measures).

**Theorem 5 (Ruelle-Margulis inequality [21])** Let \( f : [0,1] \to [0,1] \) be a \( C^1 \) map and \( \mu \) a \( f \)-ergodic measure. The quantity

\[
\lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \log |f^n'(x)| = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(f^k(x))|
\]

exists almost everywhere in \([ -\infty, +\infty ] \) and is almost constant; let \( \lambda \) be this constant.

Then \( h(\mu, f) \leq \max(\lambda, 0) \).

**Proposition 3** Let \( f : [0,1] \to [0,1] \) be a \( C^1 \) map of the interval. Let \( C_* \) be a finite subset of \([0,1]\) and consider the Markov diagram \( D \) associated to \( f \) with respect to \( C = C(f) \cup C_* \). Let \((\nu_m)_{m \geq 1}\) be a sequence of invariant, ergodic measures on \( \Sigma_+(D) \) such that \( h(\nu_m, \sigma) > h_{\text{top}}(C(f), f) \) and suppose that for all finite subsets \( F \subset D \), \( \lim_{m \to +\infty} \nu_m([F]) = 0 \). Then for all integers \( N \), one has \( \lim_{m \to +\infty} \nu_m([D_N]) = 0 \).

Let us remark that in the cases that are of interest to us, the sets \( D_N \) are not finite.
Proof. Fix an integer $N$. If $r$ is a positive number, we define the following subset of the Markov diagram:

$$F_r = \{ A_{-n} \ldots A_0 \in \mathcal{D} \mid n \leq N \text{ and diam } A_{-k} > r \text{ for all } 0 \leq k \leq n \} \subset \mathcal{D}_N.$$ 

The set $F_r$ is finite because only finitely many elements $A \in \mathcal{P}$ satisfy diam $A > r$. Therefore $\lim_{m \to +\infty} \nu_m([F_r]) = 0$ by assumption. By definition, $C = C(f) \cup C_*$, where $C_*$ is a finite set, and $C(f) \subset (f')^{-1}\{0\}$ thus there exists $r_0 > 0$ such that for all $r \leq r_0$ and $A \in \mathcal{P}$,

$$\text{diam } A \leq r \Rightarrow \forall x \in \mathcal{A}, d(x, (f')^{-1}\{0\}) \leq r.$$ 

Let $\epsilon > 0$. Fix $0 < \beta < 1$ such that $\frac{\log |f'|}{|\log \beta|} < \frac{\epsilon}{N+1}$. By continuity of $f'$ one can choose $r > 0$ such that for all $A \in \mathcal{P}$ with diam $A \leq r$,

$$\forall x \in \mathcal{A}, |f'(x)| < \beta.$$ 

Choose $m_0$ such that for all $m \geq m_0$, $\nu_m([F_r]) < \epsilon$ and put $\mu_m = \pi_*(\nu_m)$; $\mu_m$ is ergodic and, according to Theorem 4, $h(\mu_m, f) = h(\nu_m, \sigma) > 0$. Let $\lambda(x) = \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(f^k(x))|$. Applying Theorem 5 we get that $0 < h(\mu_m, f) \leq \lambda(x)$ for $\mu_m$-a.e. $x$. Consequently, there exists a $\nu_m$-generic point $\alpha = (\alpha_n)_{n \geq 0} \in \Sigma_+(\mathcal{D})$ such that, for $x = \pi(\alpha)$,

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log |f'(f^k(x))| = 0.$$ 

Let $n$ be large enough so that $\sum_{k=0}^{n-1} \log |f'(f^k(x))| > 0$. Let

$$J = \{ 0 \leq k < n \mid f^k(x) \in \mathcal{A} \text{ with } A \in \mathcal{P} \text{ and diam } A \leq r \}.$$ 

Using (2), one has

$$0 < \sum_{k=0}^{n-1} \log |f'(f^k(x))| = \sum_{k \in J} \log |f'(f^k(x))| + \sum_{k \in [0, n] \setminus J} \log |f'(f^k(x))|$$

$$\leq -\#J \cdot |\log \beta| + n \log \|f'\|_{\infty}.$$ 

Thus

$$\#J < \frac{n \log \|f'\|_{\infty}}{|\log \beta|} < \frac{ne}{N+1}.$$ 

Let $N \leq k < n$ be such that $\alpha_k \in \mathcal{D}_N \setminus F_r$, i.e., the significant part of $\alpha_k$ is of the form $A_{-q} \ldots A_0$ with $0 \leq q \leq N$ with diam $A_{-p} \leq r$ for some $0 \leq p \leq q$. Since $p \leq k$, Lemma 3 applies and $f^{k-p}(x) \in \mathcal{A}_{-p}$, thus $k-p \in J$. Observe that for a given $k$ there are at most $N+1$ indices $p$ as above, thus

$$\frac{1}{N+1} \#\{N \leq k < n \mid \alpha_k \in \mathcal{D}_N \setminus F_r\} \leq \#J.$$ 

This implies that

$$\frac{1}{n} \#\{0 \leq k < n \mid \alpha_k \in \mathcal{D}_N \setminus F_r\} \leq \frac{N}{n} + \frac{(N+1)\#J}{n} \leq \frac{N}{n} + \epsilon$$

and this inequality is valid for all integers $n$ large enough. Moreover, the point $\alpha$ is generic for $\nu_m$, therefore

$$\nu_m([\mathcal{D}_N \setminus F_r]) = \lim_{n \to +\infty} \frac{1}{n} \#\{0 \leq k < n \mid \alpha_k \in \mathcal{D}_N \setminus F_r\} \leq \epsilon.$$ 

For $m \geq m_0$, one obtains that $\nu_m([\mathcal{D}_N]) = \nu_m([\mathcal{D}_N \setminus F_r]) + \nu_m([F_r]) \leq 2\epsilon$. This concludes the proof. □

We now turn to the
**Theorem 6** Let $f: [0, 1] \rightarrow [0, 1]$ be a $C^1$ map. Let $\gamma > 0$. Then there exists $\delta > 0$ satisfying the following property. Define $C_\ast = \{k\delta | k = 1, \ldots, [\delta^{-1}]\}$ and consider the Markov diagram $\mathcal{D}$ associated to $f$ with respect to $C = C(f) \cup C_\ast$.

Let $(\nu_m)_{m \geq 1}$ be a sequence of ergodic measures on $\Sigma_+(\mathcal{D})$ such that, for all finite subsets $F \subset \mathcal{D}$ one has
\[
\lim_{m \rightarrow +\infty} \nu_m([F]) = 0.
\]

Then
\[
\limsup_{m \rightarrow +\infty} h(\nu_m, \sigma) \leq h_{\text{top}}(C(f), f) + h_{\text{loc}}(f) + \gamma.
\]

For the proof of this Theorem, we need two more facts.

The first is a standard estimate. It derives from the Stirling formula.

**Lemma 5** Let $0 < \alpha < 1/2$ and $\epsilon > 0$. Define $\phi(\alpha) = -\alpha \log \alpha - (1-\alpha) \log(1-\alpha)$. Then for all integers $n$ large enough one has
\[
n \left( \frac{n}{\alpha n} \right) \leq e^{(\phi(\alpha)+\epsilon)n},
\]
and $\lim_{\alpha \rightarrow 0} \phi(\alpha) = 0$.

The second fact follows from the definition of the local entropy and Katok’s entropy formula.

**Lemma 6** Let $f: X \rightarrow X$ be a continuous self-map of a compact metric space and $\mu$ an ergodic invariant Borel measure for $f$. Then for all $\epsilon > 0$,
\[
h(\mu, f) \leq h(\mu, f, \epsilon) + h_{\text{loc}}(f, \epsilon).
\]

*Proof:* [of the Theorem] Let $\epsilon = \gamma/(2 + \log(4\|f\|_\infty + 5))$. One can choose $\delta > 0$ such that:
\[
\begin{align*}
&h_{\text{top}}(C(f), f, \delta) < h_{\text{top}}(C(f), f) + \epsilon \\
&h_{\text{loc}}(f, 4\delta) < h_{\text{loc}}(f) + \epsilon.
\end{align*}
\]

Let $C_\ast = \{k\delta | 1 \leq k \leq [\delta^{-1}]\}$ and $C = C(f) \cup C_\ast$. One has $r(\delta, n, C) \leq r(\delta, n, C(f)) + \#C_\ast$ thus
\[
h_{\text{top}}(C, f, \delta) < h_{\text{top}}(C(f), f, \delta) < h_{\text{top}}(C(f), f) + \epsilon.
\]

There exists an integer $N_0$ such that, for all $n \geq N_0$, $r(\delta, n, C) \leq e^{(h_{\text{top}}(C(f), f) + \epsilon)n}$. Let $C_n$ be a $(\delta, n)$-cover of $C$ of cardinality $r(\delta, n, C)$.

According to Lemma 5, there exist two integers $M, N$ such that $N \geq N_0$ and
\[
\forall n \geq M, \quad \frac{n}{N} \left( \frac{n}{2n/N} \right) < e^{\epsilon n}. \quad (3)
\]

Let $(\nu_m)_{m \geq 1}$ be a sequence of ergodic measures satisfying the assumption of the Theorem. Observe that we can assume that $h(\nu_m, \sigma) > h_{\text{top}}(C(f), f)$ for all integers $m \geq 1$. By Proposition 3, $\lim_{m \rightarrow +\infty} \nu_m([\mathcal{D}_N]) = 0$. Fix $m \geq M$ such that $\nu_m([\mathcal{D}_N]) < \epsilon$ and define $\nu = \nu_m$ and $\mu = \pi_*(\nu)$.

Theorem 4 says that $h(\mu, f) = h(\nu, \sigma)$.

By the ergodic Theorem, for $\nu$-almost every $(\alpha_n)_{n \geq 0} \in \Sigma_+(\mathcal{D})$, one has
\[
\lim_{n \rightarrow +\infty} \frac{1}{n} \# \{0 \leq k < n | \alpha_k \in \mathcal{D}_N \} = \nu([\mathcal{D}_N]) < \epsilon.
\]

Consequently, there exist a set $S_0 \subset \Sigma_+(\mathcal{D})$ and an integer $T \geq M$ such that $\nu(S_0) > 0$ and for all $(\alpha_n)_{n \geq 0} \in S_0$ and $n \geq T$,
\[
\frac{1}{n} \# \{0 \leq k < n | \alpha_k \in \mathcal{D}_N \} < \epsilon. \quad (4)
\]

Let $D \in \mathcal{D}$ such that $\nu(S_0 \cap [D]) > 0$ and define $S = S_0 \cap [D]$. One has $\mu(\pi(S)) \geq \nu(S) > 0$. We are going to bound $r(4\delta, n, \pi(S))$, which will give a bound on $h(\nu, \sigma)$.

Let $\alpha = (\alpha_n)_{n \geq 0} \in S$, $x = \pi(\alpha)$ and $n \geq T$. We define a finite set $I = \{1, \ldots, j\}$ and disjoint integer subintervals $[a_i, b_i], i \in I$ satisfying the following properties.
1. \([a_i, b_i] \subset [−H(D), n]\) for all \(i \in I\) (recall that \(H(D)\) is the height of \(D\) in the graph \(D\), see section 3).
2. \(n_i = b_i − a_i > N\) for all \(i \in I\).
3. \(#((0, n) \setminus \bigcup_{i \in I}[a_i, b_i]) < \epsilon n\).
4. There exists \(z_i \in C_{n_i}\) such that \(f^{n_i}(x) \in B(z_i, n_i, 2\delta)\) for all \(i \in I\).

To define \(I = \{1, \ldots, j\}\) and the subintervals \([a_i, b_i]\), we set \(a_0 = n\) and proceed inductively. Assume that \(a_{i−1}\) is already defined. Let \(k\) be the largest integer such that \(0 < k ≤ a_i−1\) and \(a_k \not∈ D_N\). If there is no such \(k\) then we stop here setting \(j = i − 1\). Otherwise, we let \(b_i = k\) and \(a_i = b_i − H(\alpha_{b_i})\). Since \(H(\alpha_{b_i}) > N\) by choice of \(k\), the induction ultimately ends.

We prove that these intervals have the stated properties. The significant part of \(\alpha_0 = D\) is some \(A_{−H(D)} \ldots A_0\). By Lemma 2 there exist \(A_1, A_2, \ldots \in \mathcal{P}\), such that \(\alpha_k\) is the equivalence class of \(A_{−H(D)} \ldots A_0 A_1 \ldots A_k\) for each \(k ≥ 0\). This implies that \(H(\alpha_k) ≤ H(D) + k\). Therefore \(a_i = b_i − H(\alpha_{b_i}) ≥ −H(D);\) this is property (i).

By definition, \(\alpha_{b_i} \not∈ D_N\), that is, \(H(\alpha_{b_i}) > N\). Since \(n_i = b_i − a_i = H(\alpha_{b_i})\), property (ii) holds.

Let \(I = \{0 ≤ k < n \mid \alpha_k \in D_N\}\). Equation (4) says that \(#I < \epsilon n\). If \(k\) satisfies \(0 ≤ k ≤ a_j \) or \(b_i < k ≤ a_i−1\) for some \(i \in I\), then \(\alpha_k \in D_N\) by definition of \((b_i)_{i \in I}\). Therefore

\[\bigcup_{i \in I}(b_i, a_i−1) \subset J.\]

One has

\[\#((0, n) \setminus \bigcup_{i \in I}(a_i, b_i)) = \#([0, a_j] \cup \bigcup_{i \in I}(b_i, a_i−1)).\]

Moreover, \(#(a, b) = \#(a, b],\) hence

\[\#((0, n) \setminus \bigcup_{i \in I}(a_i, b_i)) = \#([0, a_j] \cup \bigcup_{i \in I}(b_i, a_i−1)) ≤ \#I < \epsilon n.\]

This is property (iii).

Finally, we show that property (iv) holds. Let \(i \in I\) and let \(A_{−p} \ldots A_0\) be the significant part of \(\alpha_{a_i}\).

Using Lemma 2 there exist \(A_1, \ldots, A_{n_i} \in \mathcal{P}\) such that \(\alpha_{b_i}\) is the equivalence class of \(A_{−p} \ldots A_0 A_1 \ldots A_{n_i}\). But the significant part of \(\alpha_{b_i}\) is some \(B_{−n_i} \ldots B_0\) (recall that \(n_i = H(\alpha_{b_i})\)). Therefore, by definition of the equivalence, \(A_0 \ldots A_{n_i} = B_{−n_i} \ldots B_0\). By definition of the significant part, we have

\[f^{n_i−1}([B_{−n_i+1} \ldots B_0]) \supseteq f^{n_i}([B_{−n_i} \ldots B_0]).\]

By definition,

\[f^{n_i}([B_{−n_i} \ldots B_0]) = \bigcap_{k=0}^{n_i} f^k(B_{−k}) = f^{n_i}(B_{−n_i}) \cap f^{n_i−1}([B_{−n_i+1} \ldots B_0]),\]

thus \(f^{n_i}(B_{−n_i}) \not\supset f^{n_i−1}([B_{−n_i+1} \ldots B_0])\), which implies that

\[f(B_{−n_i}) \not\supset [B_{−n_i+1} \ldots B_0].\]

In addition, \([B_{−n_i} \ldots B_0] \neq \emptyset\) so that

\[f(B_{−n_i}) \cap [B_{−n_i+1} \ldots B_0] \neq \emptyset.\]

\(f\) is monotonic on \(\overline{B_{−n_i}}\) and by Lemma 1 the set \([B_{−n_i+1} \ldots B_0]\) is an interval; combining this with (5) and (6), it follows that there exists \(z \in \partial B_{−n_i}\) such that \(f(z) \in [B_{−n_i+1} \ldots B_0]\). In other words, \(f^{k}(z) \in \overline{B_{−n_i+k}}\) for \(k = 0, \ldots, n_i\).

\(H(\alpha_{b_i}) = n_i\) and \(b_i = a_i+n_i\), hence \(f^{n_i−(n_i−k)}(x) = f^{a_i+k}(x) \in \overline{B_{−n_i+k}}\) for \(k = 0, \ldots, n_i\) according to Lemma 3. Moreover the diameter of \(\mathcal{P}\) is at most \(\delta\) by the definition of \(C\). Therefore \(f^{n_i}(x) \in B(z, n_i, \delta)\). Since \(z \in \partial B_{−n_i} \subset C\), there exists \(z_i \in C_{n_i}\) such that \(z \in B(z_i, n_i, \delta)\). Thus \(f^{n_i}(x) \in B(z_i, n_i, 2\delta)\) and property (iv) is satisfied.
A description of $x$ up to time $n$ is a sequence of points $(x_k)_{0 \leq k < n}$ such that

- $x_{a_i+k} = f^k(z_i)$ if $i \in I$ and $0 \leq k < n_i$,
- $x_k \in C_*$ and $|f^k(x) - x_k| \leq 2\delta$ if $k \not\in \bigcup_{i \in I} [a_i, b_i]$.

Notice that these conditions imply that $|f^k(x) - x_k| \leq 2\delta$ for $0 \leq k < n$. Let us bound the number of distinct possible descriptions.

Firstly, $\#I \leq \frac{n + H(D)}{N}$ and, when $j = \#I$ is already fixed, there are at most $\binom{n + H(D)}{2j}$ choices for the positions of the integers $a_i, b_i \ (i \in I)$ in $[-H(D), n)$. Hence the total number of choices of the intervals $[a_i, b_i]$ is bounded by

$$\frac{n + H(D)}{N} \binom{n + H(D)}{2(n + H(D))/N} < e^{cn},$$

the inequality being implied by (3).

Secondly, for each $i \in I$, there are at most $\#C_{n_i} \leq e^{(h_{top}(C(f), f) + c)n_i}$ choices of $z_i \in C_{n_i}$ because $n_i > N \geq N_0$. Thus the number of choices of $(z_i)_{i \in I}$ is bounded by

$$\prod_{i \in I} e^{(h_{top}(C(f), f) + c)n_i} \leq e^{(h_{top}(C(f), f) + c)n}.$$

Thirdly, consider $k \in [0, n) \setminus \bigcup_{i \in I} [a_i, b_i]$. If $k = 0$ then the number of choices of $x_0$ is at most $\#C_* \leq \delta^{-1}$. If $k > 0$ then

$$|x_k - f(x_{k-1})| \leq |x_k - f^k(x)| + |f^k(x) - f(x_{k-1})| \leq 2\delta + |f^{k-1}(x) - x_{k-1}||f'||\infty \leq \delta (2||f'||\infty + 2).$$

Thus the number of possible $x_k \in C_*$ is at most $4||f'||\infty + 5$ if the points $x_0, \ldots, x_{k-1}$ are already chosen. Moreover, $\#([0, n) \setminus \bigcup_{i \in I} [a_i, b_i]) < cn$ because the intervals $[a_i, b_i], i \in I$ satisfy property (iii). Therefore, the number of choices of $x_k, k \in [0, n) \setminus \bigcup_{i \in I} [a_i, b_i]$, is bounded by

$$\delta^{-1} (4||f'||\infty + 5)^cn.$$

Finally, the number of distinct descriptions is at most

$$N_d = \delta^{-1} e^{(h_{top}(C(f), f) + c + \epsilon \log(4||f'||\infty + 5))n}.$$

If $x, y$ admit the same description then $|f^k(x) - f^k(y)| \leq 4\delta$ for all $0 \leq k \leq n$. Therefore there exists a $(4\delta, n)$-cover of $\pi(S)$ of cardinality at most $N_d$, that is, $r(4\delta, n, \pi(S)) \leq N_d$. But $h(\mu, f, 4\delta) \leq \limsup_n \frac{1}{n} \log r(4\delta, n, \pi(S))$ because $\mu(\pi(S)) > 0$, hence

$$h(\mu, f, 4\delta) \leq h_{top}(C(f), f) + c + \epsilon \log(4||f'||\infty + 5).$$

According to Lemma 6 and the choice of $\delta$, one has

$$h(\mu, f) \leq h(\mu, f, 4\delta) + h_{loc}(f, 4\delta) \leq h_{top}(C(f), f) + h_{loc}(f) + \epsilon (2 + \log(4||f'||\infty + 5)) \leq h_{top}(C(f), f) + h_{loc}(f) + \gamma.$$

Since $h(\nu, \sigma) = h(\mu, f)$, this concludes the proof. \hfill $\square$

5 Existence of maximal measures

In this section, we prove the main Theorem by combining the results of the previous sections.
Theorem 7  Let \( f : [0,1] \to [0,1] \) be a \( C^1 \) map and \( C(f) \) the critical set of \( f \). Assume that \( h_{\text{top}}(f) > h_{\text{top}}(C(f), f) + h_{\text{loc}}(f) \). Then \( f \) admits a maximal measure. Moreover, the number of ergodic maximal measures is finite and, if \( f \) is transitive, the maximal measure is unique.

Proof. Let \( \epsilon > 0 \) such that \( h_{\text{top}}(f) > h_{\text{top}}(C(f), f) + h_{\text{loc}}(f) + \epsilon \). Let \( \delta > 0 \) be given by Theorem 6 with \( D \) the corresponding Markov diagram. By Theorem 4, one has \( h(D) = h_{\text{top}}(f) \). Suppose that \( \Sigma_+(D) \) has no maximal measure. By Proposition 1, there exists a sequence of ergodic measures \( (\nu_n)_{n \geq 1} \) such that \( h(\nu_n, \sigma) \to h(D) = h_{\text{top}}(f) \) and for all finite subsets \( F \subset D, \nu_n([F]) \to 0 \). By Theorem 6, one has

\[
\limsup_{n \to +\infty} h(\nu_n, \sigma) \leq h_{\text{top}}(C(f), f) + h_{\text{loc}}(f) + \epsilon < h_{\text{top}}(f),
\]

which is a contradiction. Consequently \( \Sigma_+(D) \) has a maximal measure, and so has \( f \) by Theorem 4, proving the first claim of the Theorem.

Suppose now that there is a sequence \( (\mu_n)_{n \geq 1} \) of distinct ergodic maximal measures for \( f \). Let \( \nu_n \) be the ergodic measure on \( \Sigma_+(D) \) that corresponds to \( \mu_n \) by \( \pi \) (Theorem 4). The \( \nu_n \) are distinct ergodic maximal measures, thus for all finite subsets \( F \subset D \), one has \( \nu_n([F]) \to 0 \) by Proposition 2. As previously, this leads to a contradiction by Theorem 6, proving the finiteness claim.

Finally, suppose that \( f \) is transitive. Then by Lemma 4, \( D \) has a unique connected component of large entropy and therefore admits at most one maximal measure by Theorem 2. Thus, \( f \) has at most one maximal measure by Theorem 4. \( \square \)

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