Explicit construction of joint multipoint statistics in complex systems

J Friedrich, J Peinke, A Pumir and R Grauer

1 ForWind, Institute of Physics, University of Oldenburg, Küpkersweg 70, D-26129 Oldenburg, Germany
2 Univ. Lyon, ENS de Lyon, Univ. Claude Bernard, CNRS, Laboratoire de Physique, F-69342, Lyon, France
3 Institute for Theoretical Physics I, Ruhr-University Bochum, Universitätsstr. 150, D-44801 Bochum, Germany

E-mail: jan.friedrich@uni-oldenburg.de

Keywords: complex systems, spatial modeling, superstatistics, turbulence models

Abstract
Complex systems often involve random fluctuations for which self-similar properties in space and time play an important role. Fractional Brownian motions, characterized by a single scaling exponent, the Hurst exponent $H$, provide a convenient tool to construct synthetic signals that capture the statistical properties of many processes in the physical sciences and beyond. However, in certain strongly interacting systems, e.g., turbulent flows, stock market indices, or cardiac interbeats, multiscale interactions lead to significant deviations from self-similarity and may therefore require a more elaborate description. In the context of turbulence, the Kolmogorov–Oboukhov model (K62) describes anomalous scaling, albeit explicit constructions of a turbulent signal by this model are not available yet. Here, we derive an explicit formula for the joint multipoint probability density function of a multifractal field. To this end, we consider a scale mixture of fractional Ornstein–Uhlenbeck processes and introduce a fluctuating length scale in the corresponding covariance function. In deriving the complete statistical properties of the field, we are able to systematically model synthetic multifractal phenomena. We conclude by giving a brief outlook on potential applications which range from specific tailoring or stochastic interpolation of wind fields to the modeling of financial data or non-Gaussian features in geophysical or geospatial settings.

1. Introduction

The theory of fractals has provided a unifying view on processes that occur in complex systems [1]. In particular, the concept of self-similarity has numerous applications, ranging from the solutions of ordinary and partial differential equations to the description of complex processes in nature [2–4]. Typical examples include subsurface hydrology [5, 6], turbulence measurements in the solar wind [7], random ameboid motion [8], and heart interbeat fluctuations [9]. In this context, it is convenient to describe fluctuating time series or spatial fields as fractional Brownian motion (fBm) where the roughness of the signal is determined by a single scaling exponent, the Hurst exponent $H$ [10–12]. Nonetheless, many strongly interacting systems exhibit deviations from statistical self-similarity which results in anomalous scaling [13]. Perhaps one of the most emblematic example for such multifractal properties is the phenomenon of intermittency in turbulence [14]. The latter manifests itself by the occurrence of strong velocity fluctuations at small scales which follow a non-Gaussian distribution. Similar phenomena can also be encountered in other systems, e.g., stock market fluctuations [15], hydraulic conductivity measurements in geophysics [16], or urban structures [17–21]. Hence, in contrast to fBm, which is a Gaussian process and thus fully determined by its covariance function, multifractal features are inherently non-Gaussian and, therefore, fundamentally more difficult to describe. Accordingly, numerous scholars have been focusing on random multifractal models, which include multiplicative or hierarchical cascade processes [22–25] and random multifractal walks [26] (see also recent works in references [27–30]).
The corresponding multifractal models are oftentimes formulated as non-Gaussian generalizations of fBm but a statistical characterization of their moments or probability distribution functions is rather difficult to assess.

Furthermore, the multifractal description has been devised in order to infer the scaling behavior of moments of certain quantities, such as velocity or temperature increments and is—with a few exceptions [31, 32]—restricted to a single-scale separation or two-point analysis [14]. Nonetheless, to explore the spatial and temporal structures of the fluctuating field in more detail, it is useful to go beyond considerations of mere scaling aspects. This can be done by determining the probability of field fluctuations at several points $x_i$ [33]. A first step toward such a comprehensive statistical description of a random field $u(x)$ (e.g., a turbulent velocity field) is the definition of the so-called fine-grained $n$-point-probability density function (PDF)

$$f_n(u_1, x_1; \ldots; u_n, x_n) = \prod_{i=1}^{n} \delta(u_i - u(x_i)),$$

where $u_i$ denote sample space variables and the $\delta$-distributions ensure that—given a specific random field realization $u(x)$—the fine-grained PDF is sharply peaked at $u_i = u(x_i)$ for each point $x_i$.

The joint $n$-point PDF can now be obtained by performing ensemble averages, i.e., averaging over different realizations of the random field $u(x)$ according to

$$f_n(u_1, x_1; \ldots; u_n, x_n) = \left\langle \prod_{i=1}^{n} \delta(u_i - u(x_i)) \right\rangle = \left\langle \prod_{i=1}^{n} \delta(u_i - u(x_i)) \right\rangle.$$

The knowledge of the joint $n$-point PDF is particularly useful to unravel the complex behavior resulting from the interplay between spatial scales and is needed in many applications such as time series reconstructions [34–38], for spherical $n$-point correlations in 2D string theory [39], disordered systems [40, 41], many body problems in strongly correlated quantum field theory [42] or solid state physics [43–45], and statistical turbulence theory [37, 46–48]. Nevertheless, explicit determination of the multipoint PDF (2) is rather difficult if the field $u(x)$ exhibits non-Gaussian properties and one typically has to resort to approximations, e.g., the assumption of a Markov process in scale which reduces equation (2) to a product of three-point quantities [35, 36, 49].

We introduce here an explicit construction of the joint multipoint PDF (2) with multifractal scaling that belongs to the class of the Kolmogorov–Oboukhov (K62) model of turbulence [50, 51] (see also [52–55] for further references on the use of such models in turbulence). The method is based on an ensemble of fractional Ornstein–Uhlenbeck processes which have been modified by the introduction of fluctuating length scales. It can thus be considered as a multipoint generalization of the two-point statistics proposed by Kolmogorov and Oboukhov [50, 51]. In the context of the present work, we notice that the K62 model belongs to the class of ‘superstatistics’ put forth by Beck [56], Castaing et al [57–59] and Yakhot [60]. The concept of superstatistics has also been invoked to describe non-Gaussian behavior, e.g., the anomalous diffusion of nanospheres in an F-actin network [61], the spreading process of malignant cancer cells [62], or the modeling of financial data [63] (see also [64] for further reviews on superstatistics).

The main novelty of the presented framework is that it allows for a full statistical characterization (i.e., the determination of the joint $n$-point PDF (2)) of a strongly interacting complex system with multiscale features. The latter knowledge is crucial to accurately model fluctuations in complex systems. Examples include mesoscale atmospheric models [25] or synthetic wind field modeling [65–68]. We will explicitly show how individual realizations of surrogate fields with arbitrary resolution and dimensionality can be drawn from a so-called Gaussian scale mixture. The strength of this rather general modeling ansatz is that it can be readily extended to incorporate other scale-dependent features of complex systems such as anisotropies, skewness, or inhomogeneities, as well as other classes of anomalous scaling. Hence, the proposed multipoint statistics should be applicable to a broad range of problems in the theory and the empirical modeling of complex systems.

The paper is organized as follows: section 2 discusses the ensemble of fractional Ornstein–Uhlenbeck processes and derives their respective correlation functions. Furthermore, it is shown through the example of a turbulent velocity field that a lognormal distribution of the fluctuating scale parameters reproduces the Komogorov–Oboukhov scaling of turbulence. In section 3 we introduce the multipoint statistics of the fractional Ornstein–Uhlenbeck scale mixture and explicitly consider one-, two-, and three-point statistical quantities. These quantities are also compared to empirical findings such as the PDF of velocity multipliers [69]. Further extensions of our method, such as the generalization to a three-dimensional field, are outlined in section 4. Section 5 briefly discusses potential applications of the proposed multi-point statistics in the fields of wind energy, geophysics, and also as a potential ansatz for a multi-point hierarchy in turbulence [46].
2. The Kolmogorov–Oboukhov model of turbulence as a mixture of fractional Ornstein–Uhlenbeck processes

The methodology that is outlined in this section borrows from the superstatistical approach, which describes non-equilibrium systems in terms of a superposition of equilibrium statistics [70]. Hence, we propose a statistical description of a turbulent velocity field \( u(x) \) as an ensemble of Gaussian distributed velocity fields \( w_\xi(x) \). To this end, we use a functional formulation of the velocity field statistics, which was first introduced by Hopf [71]. In the following, we restrict ourselves to the case of a one-dimensional field \( u(x) \) whereas a generalization to multiple dimensions is discussed in section 4. We are thus led to consider the characteristic functional

\[
\varphi[\alpha] = \left( e^{i \int d\alpha(x) u(x)} \right),
\]

in terms of a superposition of characteristic functionals \( \varphi_\xi[\alpha] \) belonging to Gaussian sub-ensembles

\[
\varphi[\alpha] = \int d\xi \varphi_\xi[\alpha].
\]

Here, \( g(\xi) \) denotes the PDF of a parameter \( \xi \) that is arbitrary at this stage. Due to the Gaussianity of the sub-ensemble, the characteristic functional can be calculated explicitly (see for instance [33, 72]) as

\[
\varphi_\xi[\alpha] = e^{-\frac{1}{2} \int dx \int dx' C_\xi(x-x') \rho(x') \alpha(x')},
\]

where we assume that the sub-ensemble possesses no mean and is thus fully characterized by the correlation function \( C_\xi(r) = \langle w_\xi(x+r)w_\xi(x) \rangle \) of a Gaussian random field \( w_\xi(x) \). Furthermore, we make use of the assumption of homogeneity which entails that the two-point correlation function is a function of the spatial separation \( r \) only.

In the next step, we determine the correlation function \( C_\xi(r) \). To this end, we consider a fractional Ornstein–Uhlenbeck process \( w(x) \) which obeys the following ordinary stochastic differential equation

\[
dw(x) = -\frac{1}{L} w(x) dx + dB^H(x),
\]

where \( dB^H(x) \) denotes the increment of fractional Gaussian noise with covariance

\[
\langle B^H(x)B^H(x') \rangle = \frac{\sigma^2}{2} \left( |x|^{2H} + |x'|^{2H} - |x-x'|^{2H} \right).
\]

Here, \( H \) is the Hurst exponent which lies in between 0 and 1. The explicit calculation of the correlation function \( C(r) = \langle w(x+r)w(x) \rangle \) of the stationary fractional Ornstein–Uhlenbeck process can be found in Mardoukhi et al [73] and is reproduced in appendix A. The main idea of this derivation is to solve equation (6) explicitly and to then insert the covariance of the fractional Gaussian noise (7) into \( C(r) = \langle w(x+r)w(x) \rangle \). In order to introduce fluctuations at multiple points of the scale mixture (4), we replace the scale \( r \) in the correlation function \( C(r) \) by a parametrized scale \( \rho(r) \). Hence, we consider a family of Gaussian processes \( w_\xi(x) \) with zero mean, which are characterized by the correlation function

\[
C_\xi(r) = -\frac{\sigma^2}{2} \rho(r)2H + \frac{\sigma^2 L^{2H}}{2} \Gamma(2H+1) \cosh \frac{\rho(r)}{L} + \frac{\sigma^2 \rho(r)^{2H+1}}{4L(2H+1)} \left[ e^{-\frac{\rho(r)}{L} K^{H} \left( \frac{\rho(r)}{L} \right)} - e^{-\frac{\rho(r)}{L} K^{H} \left( -\frac{\rho(r)}{L} \right)} \right],
\]

where we imply that \( r \geq 0 \). Furthermore, we introduce Kummer’s confluent hypergeometric function

\[
\Gamma(a,b,z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 t^{a-1} (1-t)^{b-a-1},
\]

as \( K^{H}(r) = \Gamma(2H+1,2H+2,r) \).

We thus parametrize the correlation function of the fractional Ornstein–Uhlenbeck process by allowing for a more general function of the scale separation \( r \), namely \( \rho(r) \). In the following, we consider an explicit form of this generalized scale

\[
\rho(r) = \xi \sqrt{\frac{\rho}{x^c}} \left( \frac{r + x^c}{x^c} \right)^{\frac{\mu}{r}},
\]

where \( \mu \) denotes the intermittency coefficient, \( x^- \) a small-scale cut-off, and \( x^+ \) a large-scale quantity which does not necessarily have to coincide with the integral length scale \( L \). Hence, by this re-parametrization of the
scale \( r \) to \( \rho_0(r) \), we have introduced a family of Gaussian processes with two-point correlations that are varying functions of the parameter \( \xi \) and thus deviate from the simple inertial range scaling \( C(r) \sim r^{2H} \) of the ordinary fractional Ornstein–Uhlenbeck process \( w(r) \). Moreover, the logarithm in equation (10) is restricted to positive values only

$$
\ln x \begin{cases}
\ln x & \text{for } x \geq 1, \\
0 & \text{for } 0 < x < 1.
\end{cases}
$$

(11)

It should be noted that for \( \mu = 0 \) and \( A = 0 \), \( \rho_0(r) = r \), and the correlation function (8) reduces to the (stationary) correlation function of the fractional Ornstein–Uhlenbeck process [73]. Furthermore, in the limit \( r \to \infty \), the asymptotic behavior of Kummer’s confluent hypergeometric function suggests \( F_1(a, b, z) \sim \Gamma(b) \left[ \frac{\Gamma(a - b)}{\Gamma(a)} \right] \frac{z_{\alpha}^{a - b}}{\Gamma(\alpha)} \) and we obtain \( \lim_{r \to \infty} C_\xi(r) = 0 \).

In the final step, we demand that the parameter \( \xi \) follows a lognormal distribution

$$
g(\xi) = \frac{1}{\sqrt{2\pi}\xi} e^{-\frac{\ln^2 \xi}{2}},
$$

(12)

which results in a scale mixture (4) that exhibits strong correlations between multiple points. As it will be shown below, this particular choice of the mixing distribution is appropriate for the modeling of turbulent flows. Nonetheless, other complex systems may require a different choice of the mixing distribution \( g(\xi) \), which can also be extracted from experimental data (see Beck and Swinney [74] for other examples within the superstatististical formalism).

We will now verify that the increment statistics derived from (4) belongs to the class of the K62 model of turbulence. To this end, we calculate the velocity increment PDF

$$
h(v, r) = \langle \delta(v - u(x + r) + u(x)) \rangle = \frac{1}{2\pi} \int dw e^{-iwv} \langle e^{iw[u'(x+r)-u(x)]} \rangle
$$

$$
= \frac{1}{2\pi} \int dw e^{-iwv} \alpha(w') = w'[\delta(x' - x - r) - \delta(x' - x)]
$$

$$
= \frac{1}{2\pi} \int dw e^{-iwv} \int_0^\infty d\xi g(\xi)e^{ix^0(2C_\xi(0)-2C_\xi(r))}.
$$

(13)

Here, we can introduce the second-order structure function

$$
S_{2\xi}(r) = 2C_\xi(0) - 2C_\xi(r),
$$

(14)

which yields

$$
h(v, r) = \int_0^\infty d\xi g(\xi) \frac{1}{\sqrt{2\pi S_{2\xi}(r)}} \exp \left[-\frac{v^2}{2S_{2\xi}(r)}\right].
$$

(15)

Taking the moments of \( h(v, r) \) yields

$$
\langle [u(x + r) - u(x)]^n \rangle = \int dv v^n h(v, r)
$$

$$
= \int_0^\infty d\xi g(\xi)(n-1)!![S_{2\xi}(r)]^{n/2},
$$

(16)

for even \( n \) and all odd moments are zero. Furthermore, for large \( L \), we can approximate \( S_{2\xi}(r) \approx \sigma^2 \rho_0(r)^{2H} \) and obtain

$$
\langle [u(x + r) - u(x)]^n \rangle = (n-1)!!\sigma^n \left( \frac{r + x^*}{x^*} \right)^{nH} \int_0^{\infty} d\xi g(\xi) \xi^{nH} \sqrt{\left[ \ln \xi + \frac{\ln \xi}{x^*} \right]}
$$

$$
\approx e^{-\frac{r^2H^2}{2}} \left( \frac{\ln \xi}{x^*} \right) \int_{-\infty}^{\infty} \frac{d\xi g(\xi)}{\xi^{nH}}
$$

$$
= C_n \left( \frac{r + x^*}{x^*} \right)^{(nH-n^2H^2)} r^{nH},
$$

(17)

which, for \( H = 1/3 \) and in the limit of \( r \gg x^* \), reduces to the original K62 scaling with scaling exponents \( \zeta_n = \frac{n}{3} - \frac{n}{18}n(n - 3) \). Moreover, for \( r \ll L \), it is possible to transform the one-increment PDF (15) into a form similar to the PDF proposed by Yakhot [60] which can be obtained from a Mellin transform of the structure
As it has been shown in the previous section, the Gaussian scale mixture (4) with correlation function (8) and (iii) we sum over all fluctuations of the parameter \( \xi \). Moreover, as the parameter \( f \) one-, two-, and three-point quantities of the Gaussian scale mixture.

Following sections, we compare the statistics of the simulated velocity field realizations to explicitly derived distributions [75].

In this section, we derive the parameter distribution leads to the K62-scaling of velocity increments (17). In this section, we derive the characteristic functional (4) which contains the K62 increment statistics in equations (17) and (18): (i) we consider a Gaussian random process in form of a fractional Ornstein–Uhlenbeck process \( \nu(x) \) whose monofractal behavior is characterized by the Hurst exponent \( H \) and whose two-point correlation function is given by \( C(r) \), (ii) in this correlation function, we introduce a re-parametrization of the scale parameter \( r \rightarrow \rho_c(r) \). We thus consider a family of Gaussian processes \( \nu_c(x) \) with fluctuating two-point correlation functions (8). We propose an explicit form for the re-parametrization in equation (10), and (iii) we sum over all fluctuations of the parameter \( \xi \) by weighing it according to a lognormal distribution (12). Hence, by virtue of these three steps, we introduce strong correlations which culminate in the non-Gaussian statistics of field increments (18). This will be further elaborated upon in the following section, where we derive the main result of this paper, i.e., the multipoint statistics of the Kolmogorov–Oboukhov-type velocity field.

3. Multipoint statistics of the Gaussian scale mixture

As it has been shown in the previous section, the Gaussian scale mixture (4) with correlation function (8) and the parameter distribution leads to the K62-scaling of velocity increments (17). In this section, we derive the joint \( n \)-point PDF of the ensemble

\[
 f_n(u_1, x_1; \ldots; u_n, x_n) = \left\{ \prod_{i=1}^{n} \delta(u_i - u(x_i)) \right\} .
\]  

From equation (4), we obtain the \( n \)-point PDF

\[
 f_n(u_1, x_1; \ldots; u_n, x_n) = \int d\xi g(\xi) f_{n,\xi}(u_1, x_1; \ldots; u_n, x_n),
\]  

as a superposition of multivariate normal distributions

\[
 f_{n,\xi}(u_1, x_1; \ldots; u_n, x_n) = \frac{1}{\sqrt{(2\pi)^n} \det \sigma_\xi} e^{-\frac{1}{2} \xi T \sigma_\xi^{-1} \xi},
\]  

where the parameter \( \xi \) of the covariance matrices \( \sigma_{\xi,ij} = C_\xi(x_i, x_j) \) in equation (8) is distributed according to a lognormal distribution (12). Hence, the \( n \)-point PDF (20) belongs to the class of compound probability distributions [75]. Moreover, as the parameter \( \xi \) in equation (10) represents a scale parameter, the multipoint model can also be apprehended as a multivariate Gaussian scale mixture which, in this particular case, leads to heavy-tailed behavior (see equation (18)).

In the following, we want to further examine the multipoint statistics governed by equation (20). In the general case (i.e., for \( n > 1 \)), equation (20) has no closed form and one has to resort to numerical treatments such as Markov Chain Monte Carlo or collapsed Gibbs sampling. Here, we draw samples from the joint \( n \)-point PDF (20) by means of a multivariate Gaussian mixture model using TensorFlow [76]. A typical realization is depicted in figure 1(a) with model parameters summarized in table 1. Furthermore, in order to generate the Gaussian scale mixture, we consider 200 realizations of the lognormally distributed parameter \( \xi \). In the following sections, we compare the statistics of the simulated velocity field realizations to explicitly derived one-, two-, and three-point quantities of the Gaussian scale mixture.

3.1. Single-point statistical quantities

Due to the stationarity of the underlying fractional Ornstein–Uhlenbeck process, the single-point PDF \( f_1(u_1, x_1) \) becomes independent of \( x_1 \) and reduces to

\[
 f_1(u_1, x_1) = \int d\xi g(\xi) \frac{1}{\sqrt{2\pi C_\xi(0)}} e^{-\frac{\xi^2}{2C_\xi(0)}}.
\]
Figure 1. (a) Typical realization of the velocity field $u(x)$ drawn from the compound $n$-point PDF (20) with spatial resolution $N = 4096$. (b) Single-point PDF from 5000 realizations of velocity fields similar to the one depicted in (a). Dashed lines correspond to the prediction (23).

Table 1. Characteristic parameters of model simulations in figure 1 from the model equations (20), (8) and (10): spatial resolution $N$, integral length scale $L$ of the fractional Ornstein–Uhlenbeck processes, small-scale cut-off $x^-$, and large-scale cut-off $x^+$ of the re-parametrized scale (10). The monofractal and multifractal features of the field are determined by the Hurst exponent $H$ and the intermittency coefficient $\mu$.

| $N$  | $L$  | $x^-$ | $x^+$ | $H$  | $\mu$ |
|------|------|------|------|------|------|
| 4096 | 1.01 | 0.00025 | 1.01 | 1/3 | 0.227 |

From equation (8) we obtain $C_{\xi}(0) = \frac{\sigma^2 L^2 H}{2} \Gamma(2H + 1)$ which is independent of the parameter $\xi$ and yields a Gaussian single-point PDF

$$f_1(u_1) = \frac{1}{\sqrt{\pi} \Gamma(2H + 1) \sigma L^H} e^{-\frac{u_1^2}{\sigma^2 L^2 \Gamma(2H + 1)}}.$$  (23)

Hence, the correlation function (8) of the fractional Ornstein–Uhlenbeck process ensures that the multi-point statistics is homogeneous. Figure 1(b) depicts the single-point PDF (blue) obtained from 5000 realizations similar to the one in figure 1(a). The dashed line corresponds to equation (23) and agrees well with the numerical simulations. The current framework thus entails a Gaussian single-point velocity PDF and non-Gaussian features emerge in higher-order statistics such as the two-point quantities discussed in the next section.

3.2. Two-point statistical quantities

The correlation function $C(r) = \langle u(x + r)u(x) \rangle$ can readily be derived from equation (20) and yields

$$C(r) = \int d\xi g(\xi) C_{\xi}(r),$$  (24)

which implies that $C(0) = \frac{\sigma^2 L^2 H}{2} \Gamma(2H + 1)$ and thus coincides with the variance of the individual fractional Ornstein–Uhlenbeck process. By the same token, we can calculate the structure functions of the velocity increments $\delta v(x) = u(x + r) - u(x)$

$$S_n(r) = \langle (\delta v)^n \rangle = \int d\xi g(\xi)(n - 1)! [S_{2\xi}(r)]^{n/2},$$  (25)

which obviously agree with the moments of the increment PDF derived in equation (16). The structure functions of the simulated Gaussian scale mixture model for $n = 2, 4, 6$ are depicted in figure 2(a) and agree well with the theoretical predictions of the K62 model (dashed lines) at small scales. At large scales the structure functions saturate which is a consequence of the underlying fractional Ornstein–Uhlenbeck processes. Indeed, due to the asymptotic behavior of Kummer’s confluent hypergeometric function we obtain

$$\lim_{r \to \infty} S_n(r) = (n - 1)!2^{n/2} u_{rms}^n,$$

where $u_{rms}^2 = \frac{\sigma^2 L^2 \Gamma(2H + 1)}{2}$. Furthermore, it has to be stressed that all odd-order structure functions are strictly zero which also entails that the multipoint statistics possesses no third-order structure function. This result is inconsistent with the so-called 4/5-law for the third-order structure function which can be derived from the Navier–Stokes equation.
and suggests $S_3(r) = -\frac{4}{5} \langle \varepsilon \rangle r$, where $r$ lies within the inertial range of scales [14]. Here, $\langle \varepsilon \rangle$ is defined as the average local energy dissipation rate, which for the present one-dimensional velocity field is defined according to

$$\langle \varepsilon \rangle = 2\nu \left\langle \left( \frac{\partial u(x)}{\partial x} \right)^2 \right\rangle.$$  

The statistics of velocity gradients, however, cannot be predicted by the present framework due to the non-differentiability of the velocity field. This is also highlighted by the fact that structure functions in figure 2 exhibit no dissipative range which would imply that $S_n(r) \sim r^n$ for small $r$. In section 5, we will briefly discuss how skewness and differentiability can be incorporated in the current framework.

3.3. Three-point statistical quantities

As it has been discussed in the previous section, the Gaussian scale mixture was devised in a way to ensure that two-point statistical quantities obey the K62 phenomenology in the inertial range. Therefore, structure functions as well as velocity increment PDFs in figures 2(a) and (b) agree with phenomenological predictions. Hence, in this section, we want to investigate the behavior of three-point statistical quantities. Such quantities are quite important in the context of multiscale models of turbulence, e.g., the operator product expansion/fusion rules [77, 78] or a stochastic interpretation of the turbulent energy cascade as a Markov process of velocity increments in scale [35, 36]. In the following, we want to consider the PDF of the ratio of velocity increments

$$p(w, r, r') = \frac{1}{\pi} \left\langle \delta \left( w - \frac{\delta_w(x)}{\delta_v(x)} \right) \right\rangle,$$  

for $r < r'$, which was first discussed by Kolmogorov [50] and is also referred to as the PDF of velocity increment multipliers [69]. On the basis of experimental data, Chen et al [69] argued that the multiplier PDF is reproduced by a Cauchy distribution derived for increments of fBm with Hurst parameter $H$, namely

$$p_{\text{fBm}}(w, r, r') = \frac{1}{\pi} \frac{\gamma(r, r')}{(w - w_0(r, r'))^2 + \gamma(r, r')^2},$$  

with scale and location parameter given by

$$\gamma(r, r') = \sqrt{1 - \frac{[r^{2H} + r'^{2H} - (r' - r)^{2H}]^2}{4r^{2H}r'^{2H}}},$$  

$$w_0(r, r') = \frac{r^{2H} + r'^{2H} - (r' - r)^{2H}}{2r^{2H}}.$$  

In the following, we want to determine the multiplier PDF of the Gaussian scale mixture and estimate intermittency corrections. It will be shown that the multiplier PDF agrees fairly well with the prediction from fBm (28) and that intermittency corrections are largely dominated by self-similar features (the ratio of two Gaussian distributed random variables follows a Cauchy distribution [79]) and thus might not be observable in
turbulence experiments or numerical simulations. Similar arguments have already been put forth by Siefert and Peinke [80] who reproduced the Cauchy distribution by an ordinary Ornstein–Uhlenbeck process.

The multiplier PDF (27) can be obtained from the two-increment PDF

\[ h_2(v, r; v', r') = \langle \delta(v - \delta v(x))\delta(v' - \delta v(x)) \rangle, \]

according to

\[ p(w, r, r') = \int dv \int dv' \delta \left( w - \frac{v'}{r'} \right) h_2(v, r; v', r'). \]

We thus derive the two-increment PDF from the three-point PDF (equation (20) for \( n = 3 \)) according to

\[ h_2(v, r; v', r') = \int du_1 du_2 du_3 \delta(v - u_3 + u_1)\delta(v' - u_2 + u_1) \]

\[ \times \int dx_1 dx_2 dx_3 \delta(r - x_3 + x_1)\delta(r' - x_2 + x_1) f_3(u_1, x_1; u_2, x_2; u_3, x_3), \]

which results in a compound bivariate normal distribution

\[ h_2(v, r; v', r') = \int d\xi \frac{1}{\pi} \frac{\gamma(\xi, r')}{\sqrt{S_{\xi}(r)S_{\xi}(r')}} \exp \left[ -\frac{\gamma(\xi, r')^2}{S_{\xi}(r)S_{\xi}(r')/2} \right], \]

where the correlation \( \rho(r, r') \) between two increments is defined as

\[ \rho(r, r') = \frac{S_{\xi}(r) + S_{\xi}(r') - S_{\xi}(r' - r)}{2\sqrt{S_{\xi}(r)S_{\xi}(r')}}. \]

Inserting the two-increment PDF into equation (32) yields

\[ p(w, r, r') = \int d\xi \frac{1}{\pi} \frac{\gamma(\xi, r')}{\sqrt{\left[ w - w_{0,\xi}(r, r') \right]^2 + \gamma(\xi, r')^2}}. \]

Hence, the multiplier PDF is a compound Cauchy distribution with location and scale parameter defined as

\[ \gamma(\xi, r')^2 = \sqrt{1 - \rho(r, r')^2} \frac{S_{\xi}(r)}{S_{\xi}(r')}, \quad w_{0,\xi}(r, r') = \sqrt{\frac{S_{\xi}(r)}{S_{\xi}(r')}} \rho(r, r'). \]

We have numerically evaluated the integral in equation (36) for fixed \( r' = 0.25L \) and for different values of small-scale \( r \). As shown in figure 3(a), the multiplier PDF of the Gaussian scale mixture is in good agreement with the prediction of the fBm model (dashed lines) given by equation (28). At larger scale separations, however, slight differences in the tails can be perceived. These differences manifest themselves in the cumulative distribution function \( P(w, r, r') \) in figure 3(b) as well. In order to further quantify the intermittency corrections, we have computed the Kullback–Leibler divergence [81]

\[ K(r, r') = \int dw p(w, r, r') \log \left( \frac{p(w, r, r')}{P_{\text{fBm}}(w, r, r')} \right), \]

between the compound Cauchy distribution and the prediction by the fBm model (28) for different intermittency coefficients \( \mu \) and small-scale \( r \) (with fixed large-scale \( r' = 0.25L \)). Although moments of the Cauchy distributions are not defined, it is possible to compute the Kullback–Leibler divergence of two Cauchy distributions [82], which should be sufficient to justify using it for our purpose here. As expected, the Kullback–Leibler divergence depicted in figure 3(c) suggests deviations for large scale separations \( r' - r \), but decreases for smaller intermittency coefficients \( \mu \). For intermediate scale separations, the fBm approximation holds regardless of the magnitude of \( \mu \), whereas deviations reappear for \( r \approx r' \), as the Cauchy distribution approaches the Dirac delta-function. Hence, although the multiplier PDF appears to be rather insensitive to intermittency corrections, the multiplier PDF is not fully monofractal as suggested by experimental studies [69, 80]. Due to the Gaussian self-similar features contained in the velocity increments, the multiplier PDF is already so heavy-tailed that intermittency corrections are barely observable. For a proper multiscale description of the turbulent energy cascade, it might thus be appropriate to consider other observables such as transition PDFs [35] or multipoint moments [77, 78, 83].
velocity field according to temporal fluctuations are distributed, one could thus modify the correlation function in equation (8) accordingly. In the previous sections we have presented a one-dimensional multipoint statistics of a Gaussian scale mixture which exhibits non-Gaussian features. In the following, we focus on multi-dimensional generalizations as further potential improvements. Extensions and generalizations of the Gaussian scale mixture multipoint statistics as further potential improvements. which can be considered as a generalization of the von-Kármán–Howarth relation [84]. In the context of turbulence, it remains unclear whether longitudinal and transverse statistical quantities exhibit different scaling properties [85–91]. Establishing connections between longitudinal and transverse statistics on the basis of the integral constraint (41) will be a task for future work.

Nevertheless, other potential requirements for the statistics such as anisotropies due to the existence of mean fields can be incorporated in the framework by an alternative form of the defining correlation tensor in equation (40) [92, 93]. For the case of other turbulent systems such as passive scalars, Rayleigh–Bénard convection or magnetohydrodynamic (MHD) turbulence, multipoint statistics of different fields can be generated by using cross correlation functions in the Gaussian ansatz, e.g., in the case of MHD, the cross correlation between velocity and magnetic field [94]. Moreover, spatiotemporal statistics of the velocity field \( \mathbf{u}(\mathbf{x}, t) \) could be introduced by generalizing the ansatz for the correlation functions to \( C_{\xi,ij}(\mathbf{r}, \tau) \). Depending on how spatial and temporal fluctuations are distributed, one could thus modify the correlation function in equation (8) accordingly.

4. Extensions and generalizations of the Gaussian scale mixture multipoint statistics

In the previous sections we have presented a one-dimensional multipoint statistics of a Gaussian scale mixture which exhibits non-Gaussian features. In the following, we focus on multi-dimensional generalizations as well as further potential improvements.

The joint \( n \)-point PDF (20) can readily be generalized to the multipoint statistics of a three-dimensional velocity field according to

\[
\mathcal{f}_n(\mathbf{u}_1, \mathbf{x}_1; \ldots; \mathbf{u}_n, \mathbf{x}_n) = \int d\xi g(\xi) \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{1}{2} \sum_{i,j=1}^{n} \sigma_{\xi,ij}(\mathbf{u}_i - \mathbf{x}_i)} ,
\]

where Greek indices indicate spatial components \( \alpha = 1, 2, 3 \), and where we introduce the covariance matrix

\[
\sigma_{\xi,ij,\alpha,\beta} = C_{\xi,ij}(\mathbf{x}_i - \mathbf{x}_j)
\]

with the correlation function

\[
C_{\xi,ij}(\mathbf{r}) = \left( C_{\xi,ij}(\mathbf{r}) - C_{\xi,ij}(0) \right) \frac{\delta_{ij}}{\tau^2} + C_{\xi,ij}(0) \delta_{ij} .
\]

Here, we assume that either the longitudinal \( C_{\xi,ij}(\mathbf{r}) \) or the transverse correlation function \( C_{\xi,ij}(\mathbf{r}) \) obey equation (8). A relation between both functions can then be established by the incompressibility condition \( \nabla \cdot \mathbf{u}(\mathbf{x}) = 0 \) for the velocity field. This leads to the following relation:

\[
\int d\xi g(\xi) \left[ C_{\xi,ij}(\tau) - \frac{1}{2\tau} \frac{\partial}{\partial \tau} (\tau^2 C_{\xi,ij}(\tau)) \right] = 0 ,
\]

which can be considered as a generalization of the von-Kármán–Howarth relation [84]. In the context of turbulence, it remains unclear whether longitudinal and transverse statistical quantities exhibit different scaling properties [85–91]. Establishing connections between longitudinal and transverse statistics on the basis of the integral constraint (41) will be a task for future work.

Nevertheless, other potential requirements for the statistics such as anisotropies due to the existence of mean fields can be incorporated in the framework by an alternative form of the defining correlation tensor in equation (40) [92, 93]. For the case of other turbulent systems such as passive scalars, Rayleigh–Bénard convection or magnetohydrodynamic (MHD) turbulence, multipoint statistics of different fields can be generated by using cross correlation functions in the Gaussian ansatz, e.g., in the case of MHD, the cross correlation between velocity and magnetic field [94]. Moreover, spatiotemporal statistics of the velocity field \( \mathbf{u}(\mathbf{x}, t) \) could be introduced by generalizing the ansatz for the correlation functions to \( C_{\xi,ij}(\tau, \tau') \). Depending on how spatial and temporal fluctuations are distributed, one could thus modify the correlation function in equation (8) accordingly.

![Figure 3](image-url)

Figure 3. (a) Multiplier PDF \( p(w, r, \tau') \) determined from numerical integration of equation (36). Dashed lines correspond to the prediction of the fBm model (28), (b) Cumulative PDF of the multipliers \( P(w, r, \tau') \), dashed lines correspond to the fBm prediction \( P_{\text{lin}}(w, r, \tau') = \frac{1}{2} + \frac{1}{\pi} \arctan \frac{w}{r} \) with scale and location parameter defined in equations (29) and (30). (c) Kullback–Leibler divergence (38) as a function of the scale separation \( r' - r \) for different values of the intermittency coefficient \( \mu \). Deviations between the distributions become visible at small and large scale separations and are increasing with increasing intermittency coefficient \( \mu \).
Furthermore, due to the underlying fractional Ornstein–Uhlenbeck processes, which dictate the monofractal behavior, the framework should also be general enough to include other phenomenological models of turbulence. The She–Leveque model [95], for instance, could be reproduced by choosing a log-Poisson distribution instead of the lognormal distribution (12), possibly in combination with another scale parametrization for $\rho_\xi(r)$ in equation (10). It may also be possible to derive a more general joint multipoint PDF for other complex systems (e.g., spin systems [96], fast and slow solar wind [97], or power grid fluctuations [98]), which have been successfully treated within the superstatistical framework.

5. Outlook

We have proposed an explicit form of the joint multipoint PDF of a multifractal field. The proposed methodology relies on three different elements: (i) the stationary solution of a fractional Ornstein–Uhlenbeck process with Hurst exponent $H$, (ii) a re-parametrization $r \rightarrow \rho_\xi(r)$ of the scale separation in the corresponding correlation function $C_\xi(r)$, and, (iii) introduction of fluctuations of the parameter $\xi$. As highlighted by the joint multipoint PDF (20), these three elements introduce fluctuating correlations which are determined by the parameter or mixing distribution $g(\xi)$. The proposed Gaussian scale mixture (20) has been used to generate synthetic multifractal fields. Statistical quantities such as the one-point PDF (which is Gaussian due to the underlying fractional Ornstein–Uhlenbeck process), the increment PDF (two-point quantity, which exhibits non-Gaussian features at small scales), the PDF of increment multipliers (three-point quantity), and even the statistics of longitudinal and transverse increments have been reproduced and agree well with empirical findings from turbulence research. Hence, the joint multipoint PDF (39) defines an appropriate statistical model of a turbulent flow [33].

Future work should address the inclusion of skewness of the velocity increment distribution into the framework which would thus make it more truthful to the original turbulence problem (i.e., the 4/5-law derived from the Navier–Stokes equation). This can be done by the introduction of a mean in the Gaussian ansatz (5), which has already been discussed in the context of superstatistics [59, 99]. Moreover, dissipative effects might be incorporated by embedding the Ornstein–Uhlenbeck process which effectively leads to the differentiability of the velocity field [100, 101].

Due to its straightforward generalizations to multiple dimensions outlined in section 4, the model is directly applicable to the specific tailoring of turbulent wind fields. A central problem in the field of wind energy is the interpolation of sparse wind field measurements [65]. It would thus be desirable to incorporate multiscale features into a recently developed stochastic interpolation scheme of sparse measurements by fBm [102]. Moreover, a multiscale refinement method [38], which is based on the empirical knowledge of the three-point PDF, could be modeled by the compound distribution (20) for $n = 3$. Instead of drawing velocity field realizations similar to the one in figure 1 from the full $n$-point statistics, latter refinement method could thus be used to generate surrogate data at a relatively low computational cost (although deviations from the K62 scaling at small scales might be expected). On a more basic level, the multipoint statistics (39) could also be used as a closure method for the multipoint hierarchy derived from the Navier–Stokes equation [46, 103]. Other potential applications include the modeling of financial data [15], geophysical settings (non-Gaussian features in the measurements of hydraulic conductivity [16] or subsurface hydrology [5]), as well as the synthesis of astrophysical magnetic fields and their impact on cosmic particle transport [104–106].

Acknowledgments

JF acknowledges funding from the Humboldt Foundation within a Feodor-Lynen fellowship and benefited from financial support through the Project IDEXLYON of the University of Lyon in the framework of the French program ‘Programme Investissements d’Avenir’ (ANR-16-IDEX-0005).

Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.
Appendix A. Derivation of the covariance of the fractional Ornstein–Uhlenbeck process

In this appendix we want to derive the stationary limit of the covariance of the fractional Ornstein–Uhlenbeck process (6), i.e., equation (8) with $\rho_1(r)$ set to $r$. The stationary solution of equation (6) reads

$$w(x) = \int_{-\infty}^{x} dB^H(x') e^{-(x-x')/L}, \quad (A.1)$$

which can be integrated by parts according to

$$w(x) = B^H(x) - \frac{1}{L} \int_{-\infty}^{x} dx' \ e^{-(x-x')/L} \ B^H(x'). \quad (A.2)$$

The covariance of the fractional Ornstein–Uhlenbeck process can thus be calculated from

$$\langle w(x_1) w(x_2) \rangle = \langle B^H(x_1) B^H(x_2) \rangle - \frac{1}{L} \int_{-\infty}^{x_1} dx' \ B^H(x') \langle B^H(x_1) B^H(x_2) \rangle$$

$$- \frac{1}{L} \int_{-\infty}^{x_2} dx' \ B^H(x') \langle B^H(x_1) B^H(x_2) \rangle + \frac{1}{L^2} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} dx' dx'' \ e^{-(x_1-x')/L} \ e^{-(x_2-x'')/L} \ \langle B^H(x'_1) B^H(x''_2) \rangle. \quad (A.3)$$

We start with the second integral in equation (A.3) and insert the covariance of the fractional Gaussian noise (7)

$$\int_{-\infty}^{x_2} dx' \ e^{r/L} \left( |x_1|^{2H} + |x_2^{1/2} - (x_1 - x_2)|^{2H} \right)$$

$$= L e^{r/L} |x_1|^{2H} + \int_{-\infty}^{0} dx' e^{r/L} |x_1|^{2H} + \int_{0}^{x_2} dx' e^{r/L} |x_2^{1/2} - x_1|^{2H}$$

$$- \int_{-\infty}^{x_1} dx' e^{r/L} |x_1 - x_2|^{2H} - \int_{0}^{x_1} dx' e^{r/L} (x_2 - x_1)^{2H}$$

$$= L e^{r/L} |x_1|^{2H} + \int_{-\infty}^{0} dx' e^{r/L} (x_2 - x_1)^{2H} + |x_2^{1/2} - x_1|^{2H+1} \int_{0}^{1} \frac{dr \ e^{r/L}}{L^{2H}}$$

$$- \int_{0}^{x_1} \frac{dr \ e^{r/L}}{L^{2H}} - \int_{0}^{x_1-x_2} \frac{dr \ e^{r/L}}{L^{2H}}$$

$$= L e^{r/L} |x_1|^{2H} + \Gamma(2H+1) L^{2H+1} + \frac{|x_2|^{2H+1}}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_2}{L} \right)$$

$$- e^{r/L} \Gamma(2H+1) L^{2H+1} - e^{r/L} |x_2 - x_1|^{2H+1} \int_{0}^{1} \frac{dr \ e^{r/L}}{L^{2H}}$$

$$= L e^{r/L} |x_2|^{2H} + \Gamma(2H+1) L^{2H+1} \left( 1 - e^{x_2/L} \right) + \frac{|x_2|^{2H+1}}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_2}{L} \right)$$

$$- \frac{|x_2 - x_1|^{2H+1}}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_2 - x_1}{L} \right). \quad (A.4)$$

Similarly, the first integral reads

$$\int_{-\infty}^{x_1} dx' e^{r/L} \left( |x_1^{1/2} + |x_2^{1/2} - (x_1 - x_2)|^{2H} \right)$$

$$= L e^{r/L} |x_2|^{2H} + \Gamma(2H+1) L^{2H+1} \left( 1 - e^{x_2/L} \right) + \frac{|x_1|^{2H+1}}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_1}{L} \right)$$

$$+ \frac{|x_2 - x_1|^{2H+1}}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_2 - x_1}{L} \right). \quad (A.5)$$
The double integral in equation (A.3) is treated according to
\[
\int_{-\infty}^{x_1} dx'_1 \int_{-\infty}^{x_2'} dx'_2 e^{(x'_1 + x'_2)/L} \left( |x'_1|^{2H} + |x'_2|^{2H} - |x'_1 - x'_2|^{2H} \right)
\]
\[
= L e^{x_1/L} \left( \int_{-\infty}^{0} dx'_1 e^{-x'_1/2H} + \int_{0}^{x_1} dx'_1 e^{x'_1/L}(x'_1)^{2H} \right)
+ L e^{x_1/L} \left( \int_{-\infty}^{0} dx'_2 e^{-x'_2/2H} + \int_{0}^{x_2} dx'_2 e^{x'_2/2L}(x'_2)^{2H} \right)
- \int_{-\infty}^{x_1} dx'_1 \int_{-\infty}^{x_2'} dx'_2 e^{(x'_1 + x'_2)/L}(x'_1 - x'_2)^{2H} - \int_{-\infty}^{x_2} dx'_2 \int_{-\infty}^{x_1} dx'_1 e^{(x'_1 + x'_2)/L}(x'_2 - x'_1)^{2H}
\]
\[
= L e^{x_1/L} \left[ \Gamma(2H + 1)L^{2H+1} + \left| x_1 - x_2 \right|^{2H+1} \right] F_1 \left( 2H + 1, 2H + 2, \frac{x_1}{L} \right)
+ L e^{x_1/L} \left[ \Gamma(2H + 1)L^{2H+1} + \left| x_2 - x_1 \right|^{2H+1} \right] F_1 \left( 2H + 1, 2H + 2, \frac{x_2}{L} \right)
- \int_{-\infty}^{x_1} dx'_1 \int_{0}^{x_2 - x'_1} dr e^{(2r + x'_1)/L} r^{2H}.
\]  
(A.6)

Here, the remaining double integral can be evaluated by a partial integration which yields
\[
\int_{-\infty}^{x_2} dx'_1 \int_{0}^{x_2 - x'_1} dr e^{(2r + x'_1)/L} r^{2H}
\]
\[
= L \int_{0}^{x_2} dr e^{x_1/L} r^{2H} + L \int_{-\infty}^{x_2} dx'_1 e^{(2x'_1 + x)/L}(2x'_1 + x)\frac{2H+1}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_1 - x_2}{L} \right)
\]
\[
= L \int_{0}^{x_2} dr e^{x_1/L} r^{2H} + L \int_{-\infty}^{x_2} dx'_1 e^{(2x'_2 - x_2)/L}(x'_2 - x_2)\frac{2H+1}{2H+1} F_1 \left( 2H + 1, 2H + 2, \frac{x_1 - x_2}{L} \right)
\]
\[
= L \frac{2H+2}{2} e^{x_1/L} \left[ \Gamma(2H + 1) + \frac{L}{2} e^{x_2/L} \left| x_2 - x_1 \right|^{2H+1} \right] F_1 \left( 2H + 1, 2H + 2, \frac{x_1 - x_2}{L} \right).
\]  
(A.7)

Inserting the terms (A.4)–(A.6) into the correlation function (A.3) yields
\[
\langle w(x_1)w(x_2) \rangle = \frac{\sigma^2}{2} (x_2 - x_1)^{2H} + \frac{\sigma^2 L^{2H}}{2} \Gamma(2H + 1) \cosh \frac{x_2 - x_1}{L} + \frac{\sigma^2 (x_2 - x_1)^{2H+1}}{4L(2H+1)}
\times \left\{ e^{-\frac{\sigma L x_1}{2H+1}} F_1 \left( 2H + 1, 2H + 2, \frac{x_2 - x_1}{L} \right) - e^{-\frac{\sigma L x_2}{2H+1}} F_1 \left( 2H + 1, 2H + 2, -\frac{x_2 - x_1}{L} \right) \right\},
\]  
(A.8)

where we assumed that $x_1 \leq x_2$.

**ORCID iDs**

J Friedrich 
[https://orcid.org/0000-0002-9862-6268](https://orcid.org/0000-0002-9862-6268)

J Peinke 
[https://orcid.org/0000-0002-0775-7423](https://orcid.org/0000-0002-0775-7423)

A Pumir 
[https://orcid.org/0000-0001-9946-7353](https://orcid.org/0000-0001-9946-7353)

R Grauer 
[https://orcid.org/0000-0003-0622-071X](https://orcid.org/0000-0003-0622-071X)

**References**

[1] Mandelbrot B B 1982 The Fractal Geometry of Nature vol 1 (San Francisco, CA: Freeman)
[70] Beck C and Cohen E G D 2003 Physica A 322 267–75
[71] Hopf E 1952 Indiana Univ. Math. J. 1 87–123
[72] Wilczek M 2016 New J. Phys. 18 125009
[73] Mardoukhi Y, Chechkin A and Metzler R 2020 New J. Phys. 22 073012
[74] Beck C, Cohen E G D and Swinney H L 2005 Phys. Rev. E 72 056133
[75] Feller W 1957 An Introduction to Probability Theory and Its Applications (New York: Wiley)
[76] Dillon J et al 2017 arXiv:1711.10604
[77] Eyink G L 1993 Phys. Lett. A 172 355–60
[78] L’vov V and Procaccia I 1996 Phys. Rev. Lett. 76 2898–901
[79] Rényi A 2007 Foundations of Probability (New York: Courier Corporation)
[80] Siefert M and Peinke J 2007 Phys. Lett. A 371 34–8
[81] Kullback S and Leibler R A 1951 Ann. Math. Stat. 22 79–86
[82] Chyžak F and Nielsen F 2019 arXiv:1903.10965
[83] Friedrich J, Margazoglou G, Biferale L and Grauer R 2018 Phys. Rev. E 98 023104
[84] De Karman T and Howarth L 1938 Proc. R. Soc. A 164 192–215
[85] Gotoh T, Fukayama D and Nakano T 2002 Phys. Fluids 14 1065
[86] Shen X and Warhaft Z 2002 Phys. Fluids 14 370–81
[87] Boratav O N and Pelz R B 1997 Phys. Fluids 9 1400
[88] Ishihara T, Gotoh T and Kaneda Y 2009 Annu. Rev. Fluid Mech. 41 165–80
[89] Grauer R, Homann H and Pinton J-F 2012 New J. Phys. 14 063016
[90] Iyer K P, Sreenivasan K R and Yeung P 2017 Phys. Rev. E 95 021101
[91] Friedrich J, Homann H, Schäfer T and Grauer R 2016 New J. Phys. 18 125008
[92] Robertson H P 1940 Math. Proc. Camb. Phil. Soc. 36 209–23
[93] Chandrasekhar S 1950 Phil. Trans. R. Soc. A 242 557–77
[94] Friedrich J 2020 Atmosphere 11 382
[95] Dubrulle B 1994 Phys. Rev. Lett. 73 959–62
[96] Cheraghizadeh J, Seifi M, Ebadi Z, Mohammadzadeh H and Najafi M 2021 Phys. Rev. E 103 032104
[97] Livadiotis G, Desai M I and Wilson L B 2018 Astrophys. J. 853 142
[98] Goriško L R, Schäfer B, Witthaut D and Beck C 2021 New J. Phys. 23 073016
[99] Sosa-Correa W, Pereira R, Macêdo A, Raposo E, Salazar D and Vasconcelos G 2019 Phys. Rev. Fluids 4 064602
[100] Sawford B L 1991 Phys. Fluids A 3 1577–86
[101] Viggiano B, Friedrich J, Volk R, Bourgoin M, Cal R B and Chevillard L 2020 J. Fluid Mech. 900 F1
[102] Friedrich J, Gallon S, Pumir A and Grauer R 2020 Phys. Rev. Lett. 125 170602
[103] Friedrich J 2020 Non-Perturbative Methods in Statistical Descriptions of Turbulence (Berlin: Springer)
[104] Reichherzer P, Tjus J B, Zweibel E G, Merten L and Pueschel M J 2020 Mon. Not. R. Astron. Soc. 498 5051–64
[105] Giacalone J and Jokipii J R 1999 Astrophys. J. 520 204
[106] Snoidin A P, Shukurov A, Sarson G R, Bushby P J and Rodrigues L F S 2016 Mon. Not. R. Astron. Soc. 457 3975