Allocating Indivisible Resources under Price Rigidities in Polynomial Time

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Abstract

In many realistic problems of allocating resources, economy efficiency must be taken into consideration together with social equality, and price rigidities are often made according to some economic and social needs. We study the computational issues of dynamic mechanisms for selling multiple indivisible items under price rigidities. We propose a polynomial algorithm that can be used to find over-demanded sets of items, and then introduce a dynamic mechanism with rationing to discover constrained Walrasian equilibria under price rigidities in polynomial time. We also address the computation of sellers’ expected profits and items’ expected prices, and discuss strategical issues in the sense of expected profits.

1 Introduction

Problem of allocating resources among selfish agents has been a well-established research theme in economics and recently becomes an emerging research topic in AI because AI methodologies can provide computational techniques to the balancing of computation tractability and economic (or societal) needs in these problems.

Dynamic mechanisms for resource allocation are trading mechanisms for discovering market-clearing prices and equilibrium allocations based on price adjustment processes. Assume a seller wishes to sell a set of indivisible items to a number of buyers. The seller announces the current prices of the items and the buyers respond by reporting the set of items they wish to buy at the given prices. The seller then calculates the over-demanded set of items and increases the prices of over-demanded items. This iterative process continues until all the selling items can be sold at the prices at which each buyer is assigned with items that maximize her personal net benefit.

Different from one-shot combinatorial auctions, the main issue of a dynamic mechanism is whether the procedure can lead to an equilibrium state (Walrasian equilibrium) at which all the selling items are effectively allocated to the buyers (equilibrium allocation) and the price of items gives the buyers their best values. Most of the discussions on the issues of dynamic mechanisms are based on market models in which there does not exist price rigidities. In fact, “good” allocations must look after both sides economy efficiency and social equality, and price rigidities may play a key role in some of these problems. For instance, in an estate bubble period, housing cost is unbearable for most of the members of society. The government may need to allocate some housing resources (whose prices are not completely flexible but restricted under some price rigidities) to middle-income earners. On one hand, the lower bound prices can be made according to some basic economic requirements (e.g., construction costs); on the other hand, the upper bound prices should be made according to some realistic social foundation (e.g., average income level or pay ability). It is well-known that a Walarasian equilibrium exists in the economy when there are no price rigidities. In the case of price restrictions, a Walrasian equilibrium may not exist since the equilibrium price vector may not be admissible. Tang, Yang and Talman and Yang studied the equilibrium allocation of heterogeneous indivisible items under price rigidities, and proposed the concept of constrained Walrasian equilibria. A constrained Walrasian equilibrium consists of a price vector $p$, a rationing system $R$, and a (constrained) equilibrium allocation $\pi$ s.t. $p$ obeys the price rigidities, and $\pi$ assigns each buyer an item (permited by $R$) that maximizes her personal net benefit at $p$. They also proposed two dynamic auction procedures that produce constrained Walrasian equilibria. However, the computational issues of these procedures have not been touched.

In this paper, we present a polynomial algorithm that can be used to find over-demanded sets of items, and then introduce a dynamic mechanism (called MAPR) with rationing to discover constrained Walrasian equilibria under price rigidities in polynomial time. In MAPR, buyers compete with each other (with the help of the seller) on prices of items for mul-

¹Note that since upper bound prices are often set for the sake of equality between social members (who have some but limited pay ability), they generally accompany a limit to the number of resources one member can get.
tiple rounds. In each round, the seller announces the current price vector (initially, the lower bound price vector) of the items that remain, then the buyers respond by reporting the set of resources they wish to buy, then the seller computes a minimal over-demanded set \( X_{\text{min}} \) of the items. If \( X_{\text{min}} = \emptyset \) then the final allocation is computed by the RM subroutine and MAPR stops. Otherwise if all the prices of the items in \( X_{\text{min}} \) are less than their upper bounds then the seller increases them; else an item \( a \in X_{\text{min}} \) (whose price is on its upper bound) is picked and the buyers who only demand for this item \( a \) in \( X_{\text{min}} \) draw lots for the right to buy \( a \). Since MAPR’s execution process is nondeterministic, we define the concepts of buyers’ expected profits and items’ expected prices, and consider strategical issues (in the sense of expected profit) in MAPR.

Here are main contributions of our work:

- We address the computational problems of dynamic auction proposed by [Talman and Yang, 2008], where these problems have not been touched.
- [Talman and Yang, 2008] has not finished the proof about the existence of constrained Walrasian equilibrium. We propose an algorithm to get the final allocation and several lemmas to prove the criteria required in constrained Walrasian equilibrium.
- We defined the “expected profits” and “expected prices” and discuss strategical issues.

This paper is structured as follows. First, we review some basic notions that are relevant to our work (see [Talman and Yang, 2008] for further details and examples). Second, we represent demand situations with bipartite graphs. Third, we address the computation of minimal over-demanded sets of items. Fourth, we present MAPR, and prove formally that it yields a constrained Walrasian equilibrium in polynomial time. Fifth, we consider strategical issues in MAPR. Finally, we draw some conclusions.

### 2 Preliminaries

Consider a market situation where a seller wishes to sell a finite set \( X \) of indivisible items to a finite number of buyers \( N = \{1, 2, \ldots, n\} \). The item \( o \in X \) is a dummy item which can be assigned to more than one buyer. Items (e.g., houses or apartments) in \( X \setminus \{o\} \) may be heterogeneous.

A price vector \( p \in \mathbb{Z}_+^N \) assigns a non-negative integer to each \( a \in X \) and \( p_a \) is the price of \( a \) under \( p \). It is required that \( p_a \) is not completely flexible and restricted to an interval \([p_a, \overline{p}_a]\) s.t. \( p_a, \overline{p}_a \in \mathbb{Z}_+, \overline{p}_a \leq \overline{p}_a, \) and \( 0 = \overline{p}_o = \overline{p}_a \). We say \( p \) and \( \overline{p} \) as the lower and upper bound price vectors.

\[ P = \{ p \in \mathbb{Z}_+^N | (\forall a \in X) p_a \leq p_a \leq \overline{p}_a \} \]

is called the set of admissible price vectors. Each \( i \in N \) has an integer value function, i.e., \( u_i : X \to \mathbb{Z}_+ \), \( u_i(a) \) is \( i \)'s valuation to item \( a \). We assume \( u_i \) is \( i \)'s private information, \( u_i(o) = 0 \), and \( i \) can pay \( \max_{a \in X} p_a \) units of money. We say \( E = \langle N, X, \{u_i\}_{i \in N} \rangle \) is an economy.

A rationing system is a function \( R : \mathbb{N} \times \mathbb{N} \to \{0, 1\} \) s.t. \( R(i, o) = 1 \) for every \( i \in N \). \( R(i, a) = 1 \) means that buyer \( i \) is allowed to demand item \( a \), while \( R(i, a) = 0 \) means that \( i \) is not allowed to demand \( a \). At \( p \in P \) and rationing system \( R \), the indirect utility \( V_i(p, R) \) and constrained demand \( D_i(p, R) \) of buyer \( i \) is given by:

\[ V_i(p, R) = \max \{ u_i(a) - p_o | a \in X \land R(i, a) = 1 \}, \text{ and } D_i(p, R) = \{ a \in X | R(i, a) = 1 \land u_i(a) - p_o = V_i(p, R) \}. \]

An allocation of \( X \) is a function \( \pi : N \to X \) s.t. \( \pi(i) \neq \pi(j) \) if \( i \neq j \) and \( \pi(i) \in X \setminus \{o\} \). \( \pi \) is an equilibrium allocation if \( \pi(i) \in D_i(p, R) \) for all \( i \in N \).

\( \langle p, R, \pi \rangle \) is a constrained Walrasian equilibrium if (1) \( p \in P, R \) is a rationing system, (2) \( \pi \) is an equilibrium allocation, (3) \( p_a = \overline{p}_a \) for all \( i \in N \), (4) \( p_o = \overline{p}_o \), and \( \pi(i) = a \) for some \( i \in N \) if \( R(j, a) = 0 \) for some \( j \in N \), and (5) \( a \in D_i(p, R) \) if \( R(i, a) = 0 \), where \( R'(j, b) = R(j, b) \) for \( (j, b) \in N \times \mathbb{R} \) except \( R'(j, a) = 1 \).

We defined the “expected profits” and “expected prices” and discuss strategical issues.

### Table 1: Values, Indirect Utilities, and Constrained Demand

| buyer | \( u_i(o) \) | \( u_i(a) \) | \( u_i(b) \) | \( u_i(c) \) | \( u_i(d) \) | \( V_i(p, R) \) | \( D_i(p, R) \) |
|-------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1     | 0           | 4           | 3           | 5           | 7           | 0           | \{o,d\}     |
| 2     | 0           | 7           | 6           | 8           | 3           | 4           | \{c\}       |
| 3     | 0           | 5           | 5           | 8           | 7           | 1           | \{b\}       |
| 4     | 0           | 9           | 4           | 3           | 2           | 4           | \{a\}       |
| 5     | 0           | 6           | 2           | 4           | 10          | 3           | \{d\}       |

### Example 1

Let \( E = \langle N, X, \{u_i\}_{i \in N} \rangle \) be an economy such that \( N = \{1, 2, 3, 4, 5\}, X = \{a, b, c, d, e\}, \) and buyers’ values are given in Table 1; price vector \( p = (0, 5, 4, 4, 7) \), and \( \pi \) be an allocation of \( X \) such that \( \pi(1) = a, \pi(2) = c, \pi(3) = b, \pi(4) = a, \) and \( \pi(5) = d \). Suppose the lower and upper bound price vectors are \( p = (0, 5, 4, 1, 5) \), and \( \overline{p} = (0, 6, 6, 4, 7) \), respectively. So \( p \) is an admissible price vector. Let \( R \) be a rationing system such that \( R(i, x) = 1 \) for all \( (i, x) \in N \times \mathbb{R} \) except \( R(3, c) = 0 \). For each buyer \( i \in N \), \( V_i(p, R) \) and \( D_i(p, R) \) are also shown in Table 1. Obviously, \( \langle p, R, \pi \rangle \) is a constrained Walrasian equilibrium.

### 3 Demand Situation and Maximum Consistent Allocation

Given an economy \( E = \langle N, X, \{u_i\}_{i \in N} \rangle \), we call \( D = \langle D_i \rangle_{i \in N} \) a demand situation of \( E \) if there is a price vector \( p \) and a rationing system \( R \) such that \( D_i = D_i(p, R) \) for all \( i \in N \). An allocation \( \pi \) is consistent with \( D \) if \( \pi(i) \in D_i \) for all \( i \in N \). \( \pi \) is maximum if \( \{i \in N | o \notin D_i \) and \( \pi(i) \neq o\} \geq \{i \in N | o \notin D_i \) and \( \pi'(i) \neq o\} \) for every allocation \( \pi' \) consistent with \( D \).
This is a key issue that we need to consider.

4 Over-demanded Set of Items

What can lead to non-existence of equilibrium allocations? This is a key issue that we need to consider.

Given a demand situation \( D = (D_i)_{i \in N} \), a set of real items \( X' \subseteq X \setminus \{o\} \) is over-demanded in \( D \), if the number of buyers who demand only items in \( X' \) is strictly greater than the number of items in \( X' \), i.e., \(|\{i \in N\mid |D_i \cap X'| > |X'|\}| > \{i \in N\mid |D_i \cap X'| \neq \emptyset\} \geq |X'|\). An over-demanded set \( X' \) is minimal if no strict subset of \( X' \) is over-demanded. We can get Lemma 1 directly based on these definitions.

**Lemma 1** Let \( X' \subseteq X \setminus \{o\} \) is over-demanded. Then for each \( a \in X' \), either there exists a minimal over-demanded set \( X'' \subseteq X' \) s.t. \( a \notin X'' \), or \( a \in X'' \) for every minimal over-demanded set \( X'' \subseteq X' \).

Theorem 1 answers the question proposed in the beginning of this section.

**Theorem 1** There exists an over-demanded set of items in \( D = (D_i)_{i \in N} \) if and only if there does not exist an equilibrium allocation.

**Proof.** Sufficiency is obvious. Let us prove necessity. Suppose there does not exist an equilibrium allocation. Let \( M = M_D \) and \( N' = \{i \in N\mid o \notin D_i\} \). Then \( |M| = |N \cap \cup_{e \in M} e| < |N'|\).

Pick a buyer \( i \) from \( N' \setminus N \cap \cup_{e \in M} e \). We construct a sequence \((X_0, N_0), (X_1, N_1), \ldots \) as follow:

1. \( X_0 = D_i, N_0 = \{j \in N\mid \exists a \in X_0 \}{i, a} \in M\};
2. \( X_{k+1} = \cup_{N_{k+1}} D_i; \) and \( N_{k+1} = \{j \in N\mid \exists a \in X_{k+1} \}{i, a} \in M\};
3. \( X'' = D_i, X'' = \emptyset;\)
4. \( \text{while}(X'' \neq \emptyset)\)
5. \( N'' = \{j \in N\mid \exists a \in X''\}{i, a} \in M\};
6. \( X'' = X'' \cup X''; X'' = \cup_{e \in M} D_i \setminus X'';\)
7. \( X_{min} = \emptyset, X'' = X'';\)
8. for all \( a \in X''\)
9. \( X'' = X'' \setminus \{a\};\)
10. \( N'' = \{i \in N\mid D_i \subseteq X_{min} \cup X''\};\)
11. \( D'' = (D_i)_{i \in N} M; k = |M_D|;\)
12. \( \text{if} k = |N''|\)
13. \( X_{min} = X_{min} \cup \{a\};\)
14. \( \text{return} X_{min};\)

Figure 1: MODS algorithm.
5 Mechanism for Resource Allocation under Price Rigidities

In this section, we present a polynomial mechanism for resource allocation under price rigidities (MAPR). Its basic idea is to eliminate over-demanded sets of items by increasing the prices of over-demanded items or rationing an over-demanded item whose price has reached its upper bound.

MAPR

1. The seller \( \varphi \) announces the set \( X \) of items to allocate, and sets \( p^0 \) := \( p \). \( M^0 := \emptyset \), \( N^0 := N \). Each buyer \( i \in N \) sets \( R_i[a] := 1 \) for all \( a \in X \). Let \( t := 0 \).
2. \( \varphi \) sends \( p^t \) and “Report your demand.” to each \( i \in N^t \).
3. Each \( i \in N^t \) computes and sends \( D_i^t \) to \( \varphi \).
4. \( \varphi \) computes \( N'' = \{ i \in N^t | D_i^t \cap \bigcup e \in M^t e \neq \emptyset \} \). If \( N'' = \emptyset \) then go to step (6). \( \varphi \) sends “Sorry, items in \( D_i^t \) have been sold. Please report your new demand.” to each \( i \in N'' \), and sets \( N' := N'' \).
5. Each \( i \in N' \) sets \( R_i[a] := 0 \) for all \( a \in D_i^t \). Go to (3).
6. Let \( N^t = N \setminus \bigcup e \in M^t e \) and \( D^t = (D_i)_{i \in N} \). \( \varphi \) computes \( \hat{M}_D \). If \( |\hat{M}_D| = |\{ i \in N^t | a \notin D_i \} \} \) then go to step (9). \( \varphi \) computes \( X_{\min} = \text{MODES}(D^t, \hat{M}_D) \).
7. \( \varphi \) computes \( \overline{X} = \{ a \in X_{\min} | p^*_a = p^t \} \). If \( \overline{X} = \emptyset \) then: \( \varphi \) sets \( N^t := N^t \setminus \{ i \} \), \( M^t+1 := M^t \cup \{ \{ i, a \} \} \), \( N^t := N^t \setminus \{ i \} \) and \( p^t+1 := p^t \). Let \( t := t+1 \). Go to (2).
8. \( \varphi \) picks an item \( a \) from \( \overline{X} \) and asks the buyers in \( \{ i \in N^t | a \notin D_i \} \) to draw lots for the right to buy \( a \). Let \( i \) be the winning buyer. \( \varphi \) sets \( M^t+1 := M^t \cup \{ \{ i, a \} \} \), \( N^t := N^t \setminus \{ i \} \) and \( p^t+1 := p^*_a \). Let \( t := t+1 \). Go to (2).
9. \( \varphi \) computes \( M^t := M^t \cap \text{RM}((D_i)_{i \in N}, M^t, p^t, p) \), and then announces \( p^t \) and \( \pi_M^t \) are the final price vector and allocation. MAPR stops.

[1] Talman and Yang, 2008 provides two dynamic procedures that produce constrained Walrasian equilibrium. But it does not address the computation issues, and the third condition of constrained Walrasian equilibrium cannot be guaranteed either. In order to make sure that all the items whose prices exceed their lower bound prices will be sold (the third criterion of constrained Walrasian equilibrium), the RM subroutine shown in Figure 2 is called in step 9. Given a demand situation \( D = (D_i)_{i \in N} \), a partial matching \( M \) consistent with \( D \), the current price vector \( p \), and the lower bound price vector \( \underline{p} \). RM returns a matching \( \hat{M}_D \) such that \( (1) \pi^{M \cup \hat{M}_D} \) is an equilibrium allocation, \( (2) \) \( M \cap \hat{M}_D \neq \emptyset \), and \( (3) \{ a \in X \setminus \bigcup e \in M' e | p^*_a > p^t \} \subseteq \bigcup e \in M' e \).

Observe MAPR and RM subroutine. We can find that:

• computation of each step is polynomial in \(|N| \) and \(|X| \);

• for each \( t \geq 0 \), the number of the loops consisting of steps 3-5 is not more than \(|X| \); and

• the number of the loops consisting of steps 2-8 is not more than \( \sum_{a \in X} (\underline{p}_a - p^t) \).

Consequently, MAPR always terminates and is polynomial in \(|N| \), \(|X| \), and \( \sum_{a \in X} (\underline{p}_a - p^t) \).

In order to prove the correctness of MAPR and RM, we will first give some definitions and provide three lemmas, then we will prove that MAPR can lead to a constrained Walrasian equilibrium with the help of these three lemmas. In the following discussion, we suppose that MAPR terminates at some time \( T \geq 0 \). \( p^t \), \( M^t \), \( R^t(i, a) = R_i[a] \) for all \( i \in N \times X \), where \( R_i \) is the vector kept by buyer \( i \) at time \( t \), and \( (D^t_i)_{i \in N} \) denote the price vector, partial matching that has been made so far, rationing system, and demand situation at time \( 0 \leq t \leq T \), respectively. Let \( X^t = \{ a \in X \setminus \bigcup e \in M^t e | p^*_a > p^t \} \) and \( N^t = \{ i \in N \setminus \bigcup e \in M^t e | D_i^t \cap X^t \neq \emptyset \} \).

Now we introduce three auxiliary lemmas (in which \( D = (D_i)_{i \in N} \) denotes a demand situation). These three lemmas are closely connected. The proof of Lemma 4 is based on Lemma 2 and Lemma 3, and the proof of Theorem 2 is based on the these three lemmas. Lemma 2 states that, each nonempty subset of a minimal over-demanded set of items is not under-demanded.

Lemma 2 Let \( X' \) be a minimal over-demanded set of items. Then for each \( \emptyset \subseteq X'' \subseteq X', |\{ i \in N | D_i \cap X'' \neq \emptyset \} \setminus X''| \geq |X''| \).

The proof of Lemma 2 is not very hard, and comes from using the reduction to absurdity. Lemma 3 states that, the cardinality of a maximum matching is not less than the cardinality of a set of real items if each subset of the set is not under-demanded.

Lemma 3 Let \( X' \subseteq X \setminus \{ o \} \) and \( |\{ i \in N | D_i \cap X'' \neq \emptyset \} \setminus X''| \geq |X''| \) for each \( X'' \subseteq X' \). If \( M \) is a maximum matching of \( B(G((D_i)_{i \in N})) \), then \( |M| \geq |X'| \).

The proof of Lemma 3 is similar to that of Theorem 1. Due to lack of space, it is omitted.

Lemma 4 states that, all the items in \( X^t \) can be sold. The proof of Lemma 4 is based on Lemma 2 and Lemma 3.
Lemma 4 Let $D_{i}^{t} = (D_{i}^{t} \cap X^{t})_{i \in N}$. Then $|\hat{M}_{D'}| = |X^{t}|$ for each $0 \leq t \leq T$.

PROOF. We first prove that $|\{i \in N'|D_{i}^{t} \cap X' \neq \emptyset\}| \geq |X'|$ for each $\emptyset \subset X' \subset X^{t}$ and $0 \leq t \leq \hat{t}$:

1. It holds at $t = 0$ because $X' = \emptyset$.
2. Suppose MAPR does not stop at $\hat{t}$, $0 \leq t$ and $|\{i \in N'|D_{i}^{t} \cap X' \neq \emptyset\}| \geq |X'|$ for each $\emptyset \subset X' \subset X^{t}$ and $0 \leq t \leq \hat{t}$.
3. Then $X_{\text{min}} \neq \emptyset$ and $X' \setminus X_{\text{min}}$ are computed at time $\hat{t}$ and steps 6-7 of MAPR. Pick any $\emptyset \subset X' \subset X^{t+1}$. Let $N_{1} = \{i \in N'|D_{i}^{t} \subset X_{\text{min}} \text{ and } D_{i}^{t} \cap X' \neq \emptyset\}$ and $N_{2} = \{i \in N'|D_{i}^{t} \subset X' \setminus X_{\text{min}}\}$. There are two possibilities:

Case I : $X' = \emptyset$. So $X_{t+1}^{t+1} = X_{t}^{t} \cup X_{\text{min}}$. According to Lemma 2 and item 2, we have $|N_{1}| > |X' \cap X_{\text{min}}|$ and $|N_{2}| \geq |X' \setminus X_{\text{min}}|$. It is easy to find that $D_{i}^{t+1} \cap X' \neq \emptyset$ for each $i \in N_{1} \cup N_{2} \subset N^{t+1}$ and $N_{1} \cap N_{2} = \emptyset$. So $|\{i \in N'|D_{i}^{t+1} \cap X' \neq \emptyset\}| \geq |N_{1} \cup N_{2}| > |X' \cap X_{\text{min}}| + |X' \setminus X_{\text{min}}| = |X'|$.

Case II : $X' \neq \emptyset$ and some $a \in X'$ is assigned to some buyer $j$ such that $a \in D_{j}^{t} \subset X_{\text{min}}$. So $X_{t+1}^{t+1} = X_{t}^{t} \setminus \{a\}$. According to Lemma 2 and item 2, we have $|N_{1}| > |X' \cap X_{\text{min}}|$ and $|N_{2}| \geq |X' \setminus X_{\text{min}}|$. It is easy to find that $D_{i}^{t+1} \cap X' \neq \emptyset$ for each $i \in (N_{1} \setminus \{j\}) \cup N_{2} \subset N^{t+1}$ and $N_{1} \cap N_{2} = \emptyset$. Consequently, $|\{i \in N'|D_{i}^{t+1} \cap X' \neq \emptyset\}| \geq [(|N_{1} \setminus \{j\}| \cup N_{2}] \geq |N_{1}| - 1 + |N_{2}| \geq |X' \cap X_{\text{min}}| + |X' \setminus X_{\text{min}}| = |X'|$.

Consequently, $|\{i \in N^{t+1}|D_{i}^{t+1} \cap X' \neq \emptyset\}| \geq |X'|$.

According to items 1–3, $|\{i \in N'|D_{i}^{t} \cap X' \neq \emptyset\}| \geq |X'|$ for each $X' \subset X^{t}$ and $0 \leq t \leq T$. It is easy to find that $|\hat{M}_{D'}| \leq |X^{t}|$ for each $0 \leq t \leq T$. According to Lemma 3, we have $|\hat{M}_{D'}| \geq |X^{t}|$ for each $0 \leq t \leq T$. So $|\hat{M}_{D'}| = |X^{t}|$ for each $0 \leq t \leq T$. □

Now we are ready to establish the following correctness theorem for MAPR (and RM subroutine).

Theorem 2 $(p^{T}, R^{T}, \pi^{M^{T}})$ found by MAPR, is a constrained Walrasian equilibrium.

PROOF. (Sketch) $(p^{T}, R^{T}, \pi^{M^{T}})$ is a constrained Walrasian equilibrium iff it satisfies the five conditions shown in page 2.

1. It is easy to find that conditions (1), (4), and (5) are satisfied by $(p^{T}, R^{T}, \pi^{M^{T}})$.

2. For each buyer $i$ and the item assigned to her $a = \pi^{M^{T}}(i)$, there are two possibilities: Case I (step 8), $i$ is the winner of a lottery on item $a$ at some time $T' \leq T$, and Case II (step 6) and (9), $i$ is assigned to $a$ at time $T'$.

(a) In case I, $a \in D_{i}(p^{T}, R^{T})$. So $u_{i}(a) - p_{i}^{T} \geq u_{i}(b) - p_{i}^{T}$ for all $b \in X(R^{T}(i, b) = 1)$. Because $R^{T}(i, a) = R^{T}(i, a) = 1$, $p_{i}^{T} = p_{i}^{T}$, $R^{T}(i, b) \geq R^{T}(i, b)$ and $p_{i}^{T} \leq p_{i}^{T}$ for all $b \in X$, $u_{i}(a) - p_{i}^{T} \geq u_{i}(b) - p_{i}^{T}$ for all $b \in \{i \in X(R^{T}(i, b) = 1)$. So $a \in D_{i}(p^{T}, R^{T})$.

(b) In case II, according to the definition of $\pi^{M^{T}}$ (see RM subroutine and steps (6)–(9)), we have $a \in D_{i}(p^{T}, R^{T})$.

Consequently, $\pi^{M^{T}}$ is an equilibrium allocation.

3. According to Lemma 4, all the items in $X^{T}$ are sold. Consequently, $p_{a}^{T} = p_{a}$ for each $a \in \{b \in X(\forall i \in N)\pi^{M^{T}}(i) \neq b\}$. The correctness of RM subroutine can derive from item 2 and item 3 directly.

So $(p^{T}, R^{T}, \pi^{M^{T}})$ is a constrained Walrasian equilibrium. □

Example 4 See Example 1. Apply MAPR to $(E, p, R)$. The demands, price vectors, rationing system and other relevant data generated by MAPR are illustrated in Table 2, where $U_{i}$, $D_{i}$, $X'$, $N^{T}$, $X_{\text{min}}$ denote $\{a \in X(R^{i}(a) = 0)\}$, $D_{i}(p^{T}, R^{T})$, $X \cap \bigcup_{a \in X}N^{T}$, $\bigcap_{a \in X}$ $\bigcup_{a \in N^{T}}$, and the value of $X_{\text{min}}$ computed by the seller at step (6) and time $t$.

At $t = 3$, the price of $c$ has reached its upper bound. The seller assigns randomly to buyer 2 or buyer 3. So there are two different possible histories of resource allocation from $t = 3$.

6 Expected profits, Expected Prices, and Strategical Issues

Since the history of MAPR is nondeterministic, we need to introduce concepts of buyers’ expected profits and items’ expected prices. Let $R_{i}^{t}$ be a rationing system s.t. $R_{i}^{t}(i, a) = 1$ if $\{i, a\} \in M^{T}$ or $a \notin \bigcup_{a \in N}$ $e$, and 0 otherwise. Because we can induce $M^{T}$ from $R_{i}^{t}$. So $M^{T}$ can be written as $M^{R_{i}}$. We say $(p^{T}, R_{i}^{T})$ is an allocation situation. Assume that the computation of MODS algorithm and the selection of items in step (8) are deterministic, all the lots happening in MAPR are fair.

Then buyer i’s expected profit and item a’s expected price on $(p, R)$ (i.e., $u_{i}^{*}(p, R)$ and $p_{a}^{*}(p, R)$) are:

$$u_{i}^{*}(p, R) = \begin{cases} V_{i}(p, R) & \text{if } X_{\text{min}} = \emptyset \\ \frac{V_{i}(p, R)}{|N|} & \text{if } X = \emptyset \end{cases}$$

$$p_{a}^{*}(p, R) = \begin{cases} p_{a} & \text{if } X_{\text{min}} = \emptyset \\ \frac{p_{a}}{|N|} & \text{if } X = \emptyset \end{cases}$$

where (let $D = (D_{i}(p, R))_{i \in N}$):

- $X_{\text{min}} = \emptyset$ if $|\hat{M}_{D}| = |\{i \in N|a \notin D_{i}(p, R)\}|$ and MODS($D, \hat{M}_{D}$) otherwise; $X = \{a \in X_{\text{min}} \pi_{a} = \emptyset\}$;
- $p_{a}^{*} = p_{a}$ for all $a \notin X_{\text{min}}$ and $p_{a} = p_{a} + 1$ for all $a \in X_{\text{min}}$;
- $b \in X_{\text{min}}$ is the item selected by the seller in step (8);
- $N^{T} = \{i \in N|b \in D_{i}(p, R) \subseteq X_{\text{min}}\}$;
- for all $(i, a) \in X \times X$, $R_{i}(i, a) = R_{i}(i, a)$ if $a \neq b$; $R_{i}(i, b) = 0$ if $i \neq i$; and $R_{i}(i, b) = 1$.\footnote{Suppose there are $k$ buyers drawing lots for the right to buy item $a$. Then the lot is fair if each one of these buyers has $1/k$ chance of winning the lot.}
and (ii) \(|N'|\) branches otherwise. See Table 1 and Table 2. We can find that 
\[ u_1'(p^0, R^0) = 0.5 * u_1'(p^{d,1}, R^{d,1}) + 0.5 * u_1'(p^{d,2}, R^{d,2}) = 0, \]
\[ u_3'(p^0, R^0) = 0.5 * u_3'(p^{d,1}, R^{d,1}) + 0.5 * u_3'(p^{d,2}, R^{d,2}) = 2.5, \]
\[ p_n'(p^0, R^0) = 0.5 * p_n'(p^{d,1}, R^{d,1}) + 0.5 * p_n'(p^{d,2}, R^{d,2}) = 5. \]

As most collective decision mechanisms, MAPR is generally not strategyproof (in the sense of expected profit). For instance, see Example 4. If buyer 1 reports her demands sincerely, then her expected profit is 0. However, if 1 knows other buyers’ valuations and reports strategically, then she reports \(c\) from \(t = 0\) to \(t = 3\) (i.e., as if her valuation to item \(c\) is not less than 7), then reports sincerely, then her expected profit changes to 1/3, which makes her better off.

Now we are interested in two questions: (1) is MAPR strategyproof for some restricted domains? (2) when it is not, how hard is it for an buyer who knows the valuations of the others to compute an optimal strategy?

First, we define reporting strategies and manipulation problems formally. Without loss of generality, let 1 be the manipulator. Note that not every sequence of 1’s demands is reasonable. For instance, see Example 4 and Table 2. The seller can detect 1’s manipulation if 1 reports \(c\), \(c\), \(c, d\), and \(c\) at \(t = 0, 1, 2, 3\), respectively, because there is no value function \(u\) s.t. \(u(c) - p^{c}_n = u(c) - 3 = u(d) - 5 = u(d) - p^{d}_n = u(d) - p^{d}_n < u(c) - p^{d}_n = u(c) - 4\). A strategy for buyer 1 is a value function \(u : X \to [0, 1]\), with \(u(a) = 0\). So 1 can safely manipulate the process of MAPR when she reports her demands according to \(u\) completely (as if \(u\) is her true value function). A manipulation problem \(M\) for buyer 1 is a \(5\)-tuple \(\langle N, X, \{u_i\}_{i \in N}, \bar{p}, \bar{p}\rangle\) where \(\langle N, X, \{u_i\}_{i \in N}\rangle\) is an economy, \(\bar{p}\) and \(\bar{p}\) are the lower and upper bound price vectors on \(X\), respectively. A strategy for \(M\) is optimal if 1 can not strictly increase her expected profit by reporting her demands according to any other strategy.

Now, back to question (1): we show that the answer is positive when there are two buyers.

**Theorem 3.** Let \(M = \langle N, X, \{u_i\}_{i \in N}, \bar{p}, \bar{p}\rangle\) be a manipulation problem s.t. \(N = \{1, 2\}\). Then \(u_1\) is optimal for \(M\).

**Proof.** Suppose that if 1 reports sincerely, then her expected profit is \(\Delta\). Let \(D_1\) and \(D_2\) be 1 and 2’s true demands at \(p\) and \(R\) respectively, where \(R(i, a) = 1\) for each \(i \in N\) and \(a \in X\).

Obviously, if \(D_1 \cup D_2 = \{o\}\) or \(D_1 \cup D_2 = \{d\}\) \(\geq 2\) (i.e., \(X_{\text{min}} = \emptyset\) at \(t = 0\)) then \(\Delta = \max_{a \in X} (u_1(a) - p_n)\), which is the best possible outcome for 1. So \(u_1\) is optimal in these cases.

Now, suppose \(D_1 = D_2 = \{a\}\) s.t. \(a \neq o\). Pick any strategy \(u'\).

Let \(k = p_n - p_{\tilde{a}}\), \(k_l = u_1(a) - p_{\tilde{a}} - \max_{b \in X\setminus\{a\}} (u(b) - p_b)\), \(b_i \in X\setminus\{a\}\) s.t. \(u_1(b_i) - p_{\tilde{a}} > u_1(a) - p_{\tilde{a}} - k_l\), and \(\hat{k} = \min(k, k_1, k_2 - 1)\). Then if 1 applies strategy \(u_1\), then she will report \(D_1\) from \(t = 0\) to \(t = \hat{k}\) and:

1. If \(k = \hat{k}\), then \(\Delta = 0.5 * (u_1(a) - p_n - k) + 0.5 * (u_1(b_1) - p_{\tilde{a}}) = u_1(b_1) - p_{\tilde{a}} + 0.5 * (k_1 - \hat{k}) > u_1(b_1) - p_{\tilde{a}}\). If 1 applies \(u'\) instead, then her expected profit will not be better than \(u_1(b_1) - p_{\tilde{a}}\), and \(\Delta\) will not be better than \(\Delta\) otherwise.

2. If \(k > \hat{k} = k_1 - 1\), then \(\Delta = u_1(b_1) - p_{\tilde{a}}\). Because 2 can insist on \(a\) to \(t = \min(k, k_2 - 1)\), \(\Delta\)’s expected profit can not be better than \(\Delta\).

3. If \(k > \hat{k} = k_2 - 1\), then \(\Delta = u_1(a) - p_n - k_2 > u_1(a) - p_{\tilde{a}}\). Because 2 can insist on \(a\) to \(t = k_2 - 1\), 1’s expected profit can not be better than \(\Delta\).

To sum up, in all cases, 1 can not strictly increase her expected profit by applying strategy \(u'\). So \(u_1\) is optimal for \(M\).

For the cases where there are more than two buyers, we conjecture that the manipulation problem is NP-hard, but we could not find a proof.

**7 Conclusion.**

We have presented a decentralized protocol for allocating indivisible resources under price rigidities, and proved formally that it can discover constrained Walrasian equilibria in polynomial time. We also have studied the protocol from the points of computation of buyers’ expected profits and items’ expected prices, and discussed the manipulation (by one buyer) problem in the sense of buyer’s expected profit. There are several directions for future work. One direction would be to prove the conjecture about the complexity of manipulation (in the sense of expected profits) by one buyer. Another direction would be to study manipulation (in the sense of expected prices) by one or more buyers (whose manipulation motivation is not to buy some resources but to put up the prices of some resources). Furthermore, we plan to study the problems of allocating divisible resources [Brams et al., 2014] and sharable resources [Airiau and Endriss, 2010] under prices rigidities.
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