1 Introduction

An important chapter in differential geometry on holonomy groups is closed by R. Bryant’s work in [9], where the long-standing question of existence of metrics with special holonomy groups $G_2$ and Spin$_7$ is settled (and it is extended for closed manifolds by D. Joyce). This success story has been a locomotive for many other developments such as deformation theory of $G_2$-manifolds (see [2]).

The exceptional Lie group $G_2$, similar to any other Lie group, has different guises depending on the underlying field; it has two real forms and a complex form. All of these incarnations have descriptions as a stabilizer group. We denote these three forms by $G_2^+$, $G_2^-$, and by $G_2$, respectively, where first two are real forms. The first of these is compact, connected, simple, simply connected, of (real) dimension 14, the second group is non-compact, connected, simple, of (real) dimension 14. The complex form of $G_2$ is connected, simple, simply connected, of (complex) dimension 14.

Let $V$ denote either $\mathbb{R}^7$ or $\mathbb{C}^7$, $e_1, \ldots, e_7$ denote its standard basis, and let $x_1 = e_1^*$, $\ldots$, $x_7 = e_7^*$ denote the dual basis. We write $GL_7(\mathbb{R})$ or $GL_7(\mathbb{C})$ instead of $GL(V)$ when there is no danger of confusion. If the underlying field does not play a role, then we write simply $GL_7$.

Consider the fourth fundamental representation $\bigwedge^4 V$ of $GL_7$, which is irreducible. It is peculiar to the numbers 7, 3 that $\bigwedge^3 V^*$ is naturally isomorphic as a $GL_7$ representation to $\bigwedge^4 V$, where $V^*$ denotes the dual space. On the other hand, $\bigwedge^4 V$ and $\bigwedge^3 V$ are dual representations, hence $\bigwedge^4 V \simeq (\bigwedge^3 V)^* \simeq \bigwedge^3 V^*$. We use the shorthand $e^{ijk}$ for $dx_i \wedge dx_j \wedge dx_k \in \bigwedge^3 V^*$. Following [9], we set

$$\phi^+ := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

and

$$\phi^- := e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}.$$

It turns out $G_2^-$ is the stabilizer of $\phi^-$ in $GL_7(\mathbb{R})$, $G_2^+$ is the stabilizer of $\phi^+$ in $GL_7(\mathbb{R})$, and finally, $G_2$ is the stabilizer of $\phi := \phi^+ + \phi^-$ in $GL_7(\mathbb{C})$. In fact, the orbit $GL_7 \cdot \phi$ is Zariski open in $\bigwedge^3 V^*$; over real numbers this orbit splits into two with stabilizers $G_2^+$ and $G_2^-$. A pleasant consequence of this openness is that $GL_7(\mathbb{C})$ has only finitely many orbits in $\bigwedge^3 V^*$. 

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Let $\mathbb{O}_k$ denote the octonion algebra over a field $k$. To ease our notation, when $k = \mathbb{C}$ we write $\mathbb{O}$. Following [21], we view $\mathbb{O}_k$ as an 8-dimensional composition algebra; it is non-associative, unital (with unity $e \in \mathbb{O}_k$), and it is endowed with a non-degenerate quadratic form $N : \mathbb{O}_k \times \mathbb{O}_k \to k$ such that
\[ N(xy) = N(x)N(y) \] for all $x, y \in \mathbb{O}_k$.

Composition algebras exist only in dimensions 1, 2, 4 and 8. Moreover, they are uniquely determined (up to isotopy) by their quadratic form. When $k = \mathbb{C}$, there is a unique isomorphism class of quadratic forms and any member of this class is isotropic, that is to say the norm of the composition algebra vanishes on a nonzero element. When $k = \mathbb{R}$ there are essentially two isomorphism classes of quadratic forms, first of which gives isotropic composition algebras, and the second class gives composition algebras with positive-definite quadratic forms.

It turns out, for any real octonion algebra $\mathbb{O}_R$, the tensor product $\mathbb{C} \otimes \mathbb{O}_R \otimes \mathbb{R}$ is isomorphic to $\mathbb{O}$ (by the uniqueness of the octonion algebra over $\mathbb{C}$). In literature $\mathbb{C} \otimes \mathbb{O}_R \otimes \mathbb{R} = \mathbb{O}$ is known as the “complex bioctonion algebra”. We denote by $G_\mathbb{R}$ the group of algebra automorphisms of $\mathbb{O}_R$, and denote by $G$ the group of algebra automorphisms of $\mathbb{O}$. By Proposition 2.4.6 of [21] we know that the group of $\mathbb{R}$-rational points of $G$, denoted by $G(\mathbb{R})$, is equal to $G_\mathbb{R}$. Of course (!), $G_\mathbb{R}$ is either $G_2^+$ or $G_7^-$ depending on which $\mathbb{O}_k$ we start with. Meanwhile, $G$ is equal to $G_2$. See Theorem 2.3.3, [21].

Let $N_1$ denote the restriction of the norm $N$ to $e_0^\bot$, the (7 dimensional) orthogonal complement of the identity vector $e_0 \in \mathbb{O}_k$. For notational ease we are going to denote $e_0^\bot$ in $\mathbb{O}_k$ by $I_k$ and denote $e_0^\bot$ in $\mathbb{O}$ simply by $I$. The automorphism groups $G_{\mathbb{R}}$ and $G$ preserve the norms on their respective octonion algebras, and obviously any automorphism maps identity to identity. Thus, we know that $G_{\mathbb{R}}$ is contained in $SO(I_k)$ and $G$ is contained in $SO(I)$. Here, $SO(W)$ denotes the group of orthogonal transformations of determinant 1 on a vector space $W$. If there is no danger of confusion, we write $SO_n (n = \dim W)$ in place of $SO(W)$.

An *quaternion algebra*, $\mathbb{D}_k$ over a field $k$ is a 4 dimensional composition algebra over $k$. As it is mentioned earlier, there are essentially (up to isomorphism) two quaternion algebras over $k = \mathbb{R}$, and there is a unique quaternion algebra over $k = \mathbb{C}$, which we denote simply by $\mathbb{D}$, and call it the split quaternion algebra. (More generally, any composition algebra over $\mathbb{C}$ is called split.) Any split quaternion algebra over $\mathbb{C}$ is isomorphic to the algebra of $2 \times 2$ matrices over $\mathbb{C}$ with determinant as its norm.

The split octonion algebra $\mathbb{O}$ over $\mathbb{C}$ has a description which is built on $\mathbb{D}$ by the Cayley-Dickson doubling process: As a vector space, $\mathbb{O}$ is equal to $\mathbb{D} \oplus \mathbb{D}$ and its multiplicative structure is given by
\[ (a, b)(c, d) = (ac + \bar{d}b, da + b\bar{c}), \quad \text{where } a, b, c, d \in \mathbb{D}, \] and its norm is defined by $N((a, b)) = N(a) - N(b) = \det a - \det b$.

\(^1\)A quadratic form $N$ on a vector space $V$ over $k$ is a map $N : V \to k$ such that i) $N(\lambda x) = \lambda^2 N(x)$ for all $\lambda \in k, x \in V$; ii) The mapping $\langle \cdot, \cdot \rangle : V \times V \to k$ defined by $2\langle x, y \rangle = N(x + y) - N(x) - N(y)$ is bilinear.
Let \( Gr_k(3, \mathbb{I}_k) \) denote the Grassmann variety of 3 dimensional subspaces of \( \mathbb{I}_k \). Let \( D_k \subset \mathbb{O}_k \) be the quaternion subalgebra generated by the first four generators \( e_1 = e, e_2, e_3, e_4 \) of \( \mathbb{O}_k \). As usual, if \( k = \mathbb{C} \), then we set \( D = D_k \). Let us denote by \( W_0 \) the intersection \( D_k \cap \mathbb{I}_k \), the span of \( e_2, e_3, e_4 \) in \( \mathbb{I}_k \). By Corollary 2.2.4 [21], when \( k = \mathbb{C} \), we know that \( G_2 \) acts transitively on the set of all quaternion subalgebras of \( \mathbb{O} \). Since an algebra automorphism fixes the identity, under this action, \( W_0 \) is mapped to another 3-plane of the form \( W' = D \cap \mathbb{I} \), for some other split quaternion subalgebra \( D' \subset \mathbb{O} \). Thus, the \( G_2 \)-action on \( Gr(3, \mathbb{I}) \) has at least two orbits, one of which is \( G_2 \cdot W_0 \) and there is at least one other orbit of the form \( G_2 \cdot W \) for some 3-plane \( W \) in \( \mathbb{I} \). The goal of our paper is to obtain an understanding of the geometry of the Zariski closure of the orbit \( G_2 \cdot W_0 \). We achieve this goal by using techniques from calibrated geometries.

After we obtained some of our main results we learned from Michel Brion about an earlier work of Alessandro Ruzzi [19, 20]. Since it is closely related to our work, we briefly mention his results. Let \( G \) be a connected semisimple group over \( \mathbb{C} \), let \( \theta \) be an involution of \( G \), and let \( G^\theta \) denote the subgroup of \( G \) consisting of elements \( g \in G \) that are fixed by \( \theta \). Let \( H \) be a subgroup of \( G \) such that \( (G^\theta)^0 \subset H \subset N_G(G^\theta) \). (Here, the superscript 0 indicates the connected component of the identity element.) The quotient varieties of the form \( G/H \) are called symmetric varieties. In his 2011 paper, Ruzzi classifies all symmetric varieties of Picard number 1 and in Theorem 2 (a) of 2010 paper [19], he shows that the smooth equivariant completion with Picard number one of the symmetric variety \( G_2/SL_2 \times SL_2 \) \(^2\), which we denote by \( X_{min} \), is the intersection of the grassmannian \( Gr(3, \mathbb{I}) \) with a 27 dimensional \( G_2 \)-stable linear space in \( \mathbb{P}(\wedge^3 \mathbb{I}) \).

Here in our paper, we analyze the natural action of the maximal torus of \( G_2 \) on \( X_{min} \). In particular, we compute the Poincaré polynomial of \( X_{min} \) by using the Bialynicki-Birula decomposition. We should mention that, over the field of real numbers the symmetric variety \( ASS := G_2^+/SO_4(\mathbb{R}) \) is already compact, and its geometry is well understood; a complete description of the ring structure of the \( H^*(ASS, \mathbb{Z}) \) is described in [4].

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## 2 Grassmann of 3-planes

For a subset \( S \subset \mathbb{O}_k \), we denote by \( A(S) \) the subalgebra of \( \mathbb{O}_k \) that is generated by \( S \). Let \( W \in Gr_k(3, \mathbb{I}_k) \) be a 3-plane and let \( u_1, u_2, u_3 \in W \) be a basis. Thus, the vector space dimension of \( A(W) \) is either \( \dim A(W) = 4 \) or \( \dim A(W) = 8 \). In latter case, obviously, \( A(W) = \mathbb{O}_k \). In the former case, \( A(W) \) is an associative subalgebra (as follows from Proposition 1.5.2 of [21]) but it does not need to be a composition subalgebra. Our orbit \( G_k \cdot W_0 \) in \( Gr_k(3, \mathbb{I}_k) \) is precisely that set of 3-planes, where \( A(W) \) is a 4 dimensional composition subalgebra, which we state in our next lemma:

\(^2\)Over \( \mathbb{C} \), \( SL_2 \times SL_2 \) is identified with the special orthogonal group \( SO_4 \).
**Remark 2.2.** The element-wise stabilizer of \( \mathbb{D} \) in \( G \) is isomorphic to \( \text{SL}_2 \). (See Proposition 2.2.1 [21]). Heuristically this follows from the fact that \( \mathbb{O} = \mathbb{D} \oplus \mathbb{D} \), and that \( (\mathbb{D}, N) = (\text{Mat}_2, \text{det}) \).

Since \( \mathbb{O} = \mathbb{D} \oplus \mathbb{D} \) and \( \mathbb{D} = \text{Mat}_2 \), we take \( \{(e_{ij}, 0) : i, j = 1, 2\} \cup \{(0, e_{ij}) : i, j = 1, 2\} \) as a basis for \( \mathbb{O} \). Here, \( e_{ij} \) is the \( 2 \times 2 \) matrix with 1 at the \( i,j \)th position and 0’s everywhere else. Recall that \( \mathbb{I} \) is the orthogonal complement of the identity \( e = (e_{11} + e_{22}, 0) \) of \( \mathbb{O} \). A straightforward computation shows that \( (x, y) \in \mathbb{O} \) is in \( \mathbb{I} \) if and only if the trace of \( x \) is 0. Thus, we write \( \mathbb{I} = \mathfrak{sl}_2 \oplus \text{Mat}_2 \) (we are going to make use of Lie algebra structure on \( \mathfrak{sl}_2 \) in the sequel).

Let \( W \in \text{Gr}(3, \mathbb{I}) \) be a 3-plane in \( \mathbb{I} \) and suppose \( u_1, u_2, u_3 \) is a basis for \( W \). The map \( P : \text{Gr}(3, \mathbb{I}) \to \mathbb{P}(\wedge^3 \mathbb{I}) \) defined by \( P(W) = [u_1 \wedge u_2 \wedge u_3] \) is the Plücker embedding of \( \text{Gr}(3, \mathbb{I}) \) into 34 dimensional projective space \( \mathbb{P}(\wedge^3 \mathbb{I}) \). Note that \( \text{GL}(\mathbb{I}) \) acts on both of the varieties \( \text{Gr}(3, \mathbb{I}) \) and \( \wedge^3 \mathbb{I} \) via its natural action on \( \mathbb{I} \). Note also that the Plücker embedding is equivariant with respect to these actions. In particular, it is equivariant with respect to the subgroup \( G_2 \).

We make the identifications
\[
1 \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \leftrightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k \leftrightarrow \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]
and take \( \{(i, 0), (j, 0), (k, 0), (0, 1), (0, i), (0, j), (0, k)\} \) as a basis for \( \mathbb{I} \). Let \( W_0 \) denote the span of \( \{(i, 0), (j, 0), (k, 0)\} \) and let \( W_0^* \) denote the span of \( \{(0, i), (0, j), (0, k)\} \). Thus,
\[
\mathbb{I} = W_0 \oplus W_0^* \oplus \mathbb{C}.
\]

**Remark 2.3.** It is noted earlier that a copy of \( \mathfrak{sl}_2 \) sits in \( \mathbb{I} \):
\[
\mathfrak{sl}_2 = \mathfrak{sl}_2 \oplus 0 \hookrightarrow \mathfrak{sl}_2 \oplus \text{Mat}_2 = \mathbb{I}.
\]

This copy of \( \mathfrak{sl}_2 \) is \( W_0 \) as a vector space.
Remark 2.4. A straightforward calculation shows that if \((0, v) \in W_0^*\), then for all \((x, 0) \in \mathfrak{sl}_2\), \((x, 0)(0, v) = (0, vx)\).

Next, we elaborate on a portion of the discussion given [14], §22.3 and analyze \(\wedge^3 I\) more closely.

Let \(U\) denote \(W_0 \oplus W_0^*\) so that we have \(\wedge^3(W_0 \oplus W_0^* \oplus \mathbb{C}) = \bigoplus_{n=0}^{3} \wedge^n U \otimes \wedge^{3-n} \mathbb{C} = \wedge^3 U \oplus \wedge^2 U\). Since \(\dim W_0 = \dim W_0^* = 3\), we have canonical identifications \(W_0 = \wedge^2 W_0^*\) and \(W_0^* = \wedge^2 W_0\). It follows that

\[
\wedge^3 U = (\mathbb{C} \otimes \mathbb{C}) \oplus (W_0 \otimes \wedge W_0^*) \oplus (\wedge W_0 \otimes W_0^*) \oplus (\mathbb{C} \otimes \mathbb{C})
\]

and that

\[
\wedge^2 U = \wedge W_0 \otimes \mathbb{C} \oplus W_0 \otimes W_0^* \oplus \mathbb{C} \otimes \wedge W_0^*
\]

Putting all of the above together we see that

\[
\wedge^3 I = \mathbb{C} \oplus (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C} \oplus W_0^* \oplus (W_0 \otimes W_0^*) \oplus W_0
\]

Next, we analyze \(\text{Sym}^2 I\) more closely;

\[
\text{Sym}^2 I = \text{Sym}^2 (U \oplus \mathbb{C})
\]

\[
= (\text{Sym}^2 U \otimes \mathbb{C}) \oplus (\text{Sym}^1 U \otimes \mathbb{C}) \oplus (\mathbb{C} \otimes \text{Sym}^2 \mathbb{C})
\]

\[
= \text{Sym}^2 W_0 \oplus (W_0 \otimes W_0^*) \oplus \text{Sym}^2 W_0^* \oplus (W_0 \oplus W_0^*) \oplus \mathbb{C}
\]

\[
= (\text{Sym}^2 W_0 \oplus W_0^*) \oplus (W_0 \otimes W_0^*) \oplus (\text{Sym}^2 W_0^* \oplus W_0) \oplus \mathbb{C}
\]

\[
= (\text{Sym}^2 W_0 \oplus \wedge^2 W_0) \oplus (W_0 \otimes W_0^*) \oplus (\text{Sym}^2 W_0^* \oplus \wedge W_0^*) \oplus \mathbb{C}
\]

\[
= \text{End}(W_0) \oplus (W_0 \otimes W_0^*) \oplus \text{End}(W_0^*) \oplus \mathbb{C}
\]

\[
\simeq (W_0 \otimes W_0) \oplus (W_0 \otimes W_0^*) \oplus (W_0^* \otimes W_0^*) \oplus \mathbb{C}.
\]

Remark 2.5. The last term is only an isomorphism since we are using non-canonical identification of \(W_0\) with \(W_0^*\).

Therefore, we see that

\[
\wedge^3 I \simeq \text{Sym}^2 I \oplus I. \quad (4)
\]
Furthermore, it is true that \( \text{Sym}^2 \mathbb{I} = \Gamma_{2,0} \oplus \mathbb{C} \), where

\[
\Gamma_{2,0} = (W_0 \otimes W_0) \oplus (W_0^* \otimes W_0^*) \oplus (W_0 \otimes W_0^*)
\]

\[
= (W_0 \otimes \bigwedge^2 W_0^*) \oplus (\bigwedge^2 W_0 \otimes W_0^*) \oplus (W_0 \otimes W_0^*).
\]

(5)

is an irreducible representation of \( G_2 \) with highest weight \( 2\omega_1 \), where \( \omega_1 \) is the highest weight of the first fundamental representation \( \mathbb{I} \) of \( G_2 \). See [14], §22.3. Once the root system \( \Phi = \{\alpha_1, \ldots, \alpha_6, \beta_1, \ldots, \beta_6\} \) is chosen as in [14], §22.2 (pg. 347), we see that \( 2\omega_1 = \alpha_1 + \alpha_3 + \alpha_4 \).

The structure of the representation of \( G_2 \) on \( \mathbb{I} \) can be spelled out to a finer degree once we linearize the action. Let \( g_2 \) denote the Lie algebra of \( G_2 \). It is well known that \( g_2 \) contains a copy of \( g_0 = \mathfrak{sl}_3 \), and moreover, as a representation of \( g_0 \) it has the following decomposition:

\[
g_2 = g_0 \oplus W \oplus W^*.
\]

where \( W \) is a suitable 3 dimensional subspace. Moreover, the unique 7 dimensional irreducible representation \( V \) of \( g_2 \) can be identified with \( V = W \oplus W^* \oplus \mathbb{C} \) as an \( \mathfrak{sl}_3 \)-module. In our notation, we are going to take \( W \) as \( W_0 \). Before making this identification we choose a basis for \( W \) using roots system \( \Phi = \{\alpha_1, \ldots, \beta_{12}\} \).

Let \( V_i \subset V \) \((i = 1, \ldots, 6) \) denote the eigenspace (corresponding to the eigenvalue \( \alpha_i \)) for the action of the maximal abelian subalgebra \( \mathfrak{h} \subset g_2 \) corresponding to \( \Phi \). Let \( Y_i \) be the root vector whose eigenvalue is \(-\alpha_i = \beta_i \) for \( i = 1, \ldots, 6 \). Arguing as in pg. 354 of [14], we have the basis \( e_1 = v_4, e_2 = w_1, e_3 = w_3 \) for \( W \), where \( w_i \)’s are found as follows:

\[
v_3 = Y_1(v_4), \quad v_1 = -Y_2(v_3), \quad u = Y_1(v_1), \quad w_1 = \frac{1}{2} Y_1(u), \quad w_3 = Y_2(w_1), \quad w_4 = -Y_1(w_3).
\]

The corresponding dual basis \( e_1^*, e_2^*, e_3^* \) is given by \( w_4, v_1, v_3 \). The upshot of all of these is that we identify \( W_0 \) with \( W \) in such a way that the basis \( v_4, w_1, w_3 \) corresponds (in the given order) to \( (i, 0), (j, 0), (k, 0) \), and the basis \( w_1, v_1, v_3 \) for \( W^* \) corresponds to \( (0, i), (0, j), (0, k) \). With respect to these identifications, we observe that the highest weight vector in \( \text{Sym}^2 V \subset \bigwedge^3 V \) of the highest weight \( 2\omega_1 = \alpha_1 + \alpha_3 + \alpha_4 \) is given by the 3-form \( v_1 \wedge v_3 \wedge v_4 \), or by \( (i, 0) \wedge (0, j) \wedge (0, k) \) in the case of \( \text{Sym}^2 \mathbb{I} \subset \bigwedge^3 \mathbb{I} \). It is clear that the 3-form \( (i, 0) \wedge (0, j) \wedge (0, k) \) is actually an element of \( W_0 \otimes \bigwedge^2 W_0^* \subset \Gamma_{2,0} \) by (5).

It is straightforward to verify that the octonions \( (i, 0), (0, j), \) and \( (0, k) \) generate a quaternion algebra which we denote by \( \mathbb{U} \). By Lemma 2.1 we see that the stabilizer subgroup of \( \mathbb{U} \) is \( \text{SO}_4 \). It is well known that the highest weight vector in \( \Gamma_{2,0} \subset \text{Sym}^2(\mathbb{I}) \) is the direction vector of the line that is stabilized by \( \text{SO}_4 \).
We view the projectivization $\mathbb{P}(\Gamma_{2,0})$ as a (closed) subvariety of $\mathbb{P}(\bigwedge^3 \mathbb{I})$. The image of $\text{Gr}(3, \mathbb{I})$ intersects $\mathbb{P}(\Gamma_{2,0})$ nontrivially. In fact, by the above discussion we know that the image of the 3-plane $U_0 := \mathbb{U} \cap \mathbb{I} \in \text{Gr}(3, \mathbb{I})$ under Plücker embedding is the $\text{SO}_4$-fixed point $[u_0] \in \mathbb{P}(\Gamma_{2,0})$. On one hand, since it is a $G_2$-equivariant isomorphism onto its image, the orbit $G_2 \cdot U_0$ in $\text{Gr}(3, \mathbb{I})$ is mapped isomorphically onto $G_2 \cdot [u_0] \subset \mathbb{P}(\bigwedge^3 \mathbb{I})$. On the other hand, as we are going to see in the sequel, the closure of the orbit $G_2 \cdot U_0$ in $\text{Gr}(3, \mathbb{I})$ is smooth, however, the Zariski closure of the orbit $G_2 \cdot [u_0]$ in $\mathbb{P}(\bigwedge^3 \mathbb{I})$ is not. The latter closure is the smallest “degenerate” compactification of the symmetric variety $G_2/\text{SO}_4$, whereas the former compactification is the smallest, smooth $G_2$-equivariant compactification.

3 More on Octonions

In this section we collect and improve some known facts about alternating forms on (split) composition algebras.

The multiplicative structure of a quaternion algebra (over $\mathbb{R}$ or $\mathbb{C}$) is always associative (but not commutative). To measure how badly the associativity of multiplication fails in $\mathcal{O}$ one looks at the associator, defined by

$$[x, y, z] = (xy)z - x(yz) \text{ for all } x, y, z \in \mathcal{O}_k.$$

(6)

It is well known that the associator is an alternating 3-form (see Section 1.4 of [21]).

There are several other related multiplication laws on the imaginary part $\mathbb{I}_k$ of $\mathcal{O}_k$. For example, the “cross-product” is defined by

$$a \times b = \frac{1}{2}(ab - ba) \text{ for all } a, b \in \mathbb{I}_k.$$

Obviously, the cross-product is alternating.

The “dot product” is defined by

$$a \cdot b = -\frac{1}{2}(ab + ba) \text{ for all } a, b \in \mathbb{I}_k.$$

It is also obvious that $ab = a \times b - a \cdot b$ for $a, b \in \mathbb{I}_k$. With these products on $\mathbb{I}_k$ recovers the product structure on $\mathcal{O}_k$; any element of $\mathcal{O}_k$ has the form $x = \alpha(1,0) + a$, where $\alpha \in k$, $a \in \mathbb{I}_k$, and if $y = \beta(1,0) + b$ is another element from $\mathcal{O}_k$, then

$$xy = (\alpha, a)(\beta, b) = (\alpha\beta - a \cdot b, \alpha a + \beta b + a \times b),$$

(7)

where we use the identification $\alpha(1,0) + a = (\alpha, a)$. In particular, if $x = a$ and $y = b$ are from $\mathbb{I}_k$, then $xy = (-x \cdot y, x \times y)$, hence

$$N(xy) = (x \cdot y)^2 + N(x \times y).$$

(8)

Now we focus on $k = \mathbb{C}$ and extend some results from [16] to our setting. First, we re-label the basis $\{e, e_1, \ldots, e_7\}$ for $\mathcal{O}$ so that $\{e_1 = (i,0), e_2 = (j,0), e_3 = (k,0), e_4 = (0,1), e_5 = (0,i), e_6 = (0,j), e_7 = (0,k)\}$ is the standard basis for $\mathbb{I}$. Consider the trilinear form

$$\varphi(x, y, z) = \langle x \times y, z \rangle, \quad x, y, z \in \mathbb{I}.$$
Lemma 3.1. ϕ is an alternating 3-form on $I$.

Proof. Since both cross-product and the inner product $⟨\cdot, \cdot⟩$ are bilinear, we see it is enough to check the assertion on the basis $\{e_1, \ldots, e_7\}$. We verified this by using software called Maple.

Remark 3.2. It is follows from (7) that
$$\varphi(x, y, z) = ⟨xy, z⟩$$ for $x, y, z \in I$. (9)

It is not difficult to verify (by using Maple, or by hand) that $\varphi$ is equal to the 3-form
$$\varphi = e^{123} - e^{145} + e^{167} - e^{246} - e^{257} - e^{347} + e^{356},$$ (10)
where $e^{ijk} = de_i \wedge de_j \wedge de_k$ as before. In particular, we see that from (9) that $G_2 = \text{Aut}(Ω)$ stabilizes the form (10).

Definition 3.3. For a 3-plane $W \in \text{Gr}(3, I)$ we define $\varphi(W)$ to be the evaluation of $\varphi$ on any orthonormal basis $\{x, y, z\}$ of $W$. We call a 3-plane $W \in \text{Gr}(3, I)$ associative if $W = D \cap I$, where $D$ is a (necessarily split) quaternion subalgebra of $Ω$.

Theorem 3.4. A 3-plane $W \in \text{Gr}(3, I)$ is associative if and only if $\varphi(x, y, z) \in \{-1, +1\}$ for any orthonormal basis $\{x, y, z\}$ of $W$.

Proof. Any two elements $x, y$ of an orthonormal triplet $x, y, z$ from $I$ form a “special (1, 1)-pair” in the sense of [21], Definition 1.7.4. Since $G_2$ acts transitively on special (1, 1)-pairs, and since $((i, 0), (j, 0))$ is such, there exists $g \in G_2$ such that $g(x) = (i, 0), g(y) = (j, 0)$. In particular, it follows that $g(z) = (k, 0)$.

We claim that $x, y, z$ generates a quaternion algebra if and only if $g(z) = \pm(k, 0)$. Indeed, unless $xy$ is a scalar multiple of $z$, the span in $I$ of $e, x, y, z$ and $xy$ is 5 dimensional, hence the composition algebra generated by $x, y, z$ is not a quaternion subalgebra. It follows that $1, x, y, z$ is an orthonormal basis for a quaternion subalgebra if and only if $z$ is a scalar multiple of $xy$. Since the norm of $z$ is 1, we see that $z = \pm xy$, hence $g(z) = \pm(k, 0)$. Then

$$\langle xy, z \rangle = \langle g^{-1}(i, 0), g^{-1}(j, 0), g^{-1}(k, 0) \rangle$$
$$= \langle i, 0, j, 0, k, 0 \rangle$$
$$= \pm 1.$$ 

We define the triple-cross product on $Ω$ as follows
$$x \times y \times z = \frac{1}{2}(x(\bar{y}z) − z(\bar{y}x))$$ for all $x, y, z \in Ω$. (11)
Lemma 3.5. The triple-cross product is trilinear and alternating. Moreover, \( N(x \times y \times z) = N(x)N(y)N(z) \) for all \( x, y, z \in O \).

Proof. The trilinearity is obvious. To prove the second claim we check \( x \times x \times z = 0, \)
\( x \times y \times y = 0, \) and \( x \times y \times x = 0 \). We use \([21], \text{Lemma 1.3.3} \) for the first two:

\[
x \times x \times z = \frac{1}{2}(x(\bar{z}x) - z(\bar{x}x)) = \frac{1}{2}(N(x)z - zN(x)) = 0,
\]
\[
x \times y \times y = \frac{1}{2}(x(\bar{y}y) - y(\bar{y}y)) = \frac{1}{2}(xN(y) - N(y)x) = 0,
\]
\[
x \times y \times x = \frac{1}{2}(x(\bar{y}x) - x(\bar{y}x)) = 0.
\]

Finally, to prove \( N(x \times y \times z) = N(x)N(y)N(z) \) we expand \( x \times y \times z \) in the orthonormal basis \( e = (1,0), e_1, \ldots, e_7 \) of \( O \). In particular, this allows us to assume that \( x, y \) and \( z \) as scalar multiples of standard basis vectors. Now it is straightforward verify (by Maple) that \( x(\bar{y}z) = -z(\bar{y}x) \), hence \( x \times y \times z = x(\bar{y}z) \) and our claim follows by taking norms.

Next, we show that the “associator identity” of Harvey-Lawson (Theorem 1.6 \([16]\)) holds for split octonion algebras.

Theorem 3.6. For all \( x, y, z \in I \), the associator \([x, y, z]\) lies in \( I \), and

\[
x \times y \times z = \langle xy, z \rangle e + [x, y, z],
\]

where \( e \) is the identity element \((1,0)\) of \( O \). Moreover,

\[
|x \wedge y \wedge z| = \varphi(x, y, z)^2 + N([x, y, z]),
\]

where \( |x \wedge y \wedge z| \) is defined as \( N(x)N(y)N(z) \).

Proof. By linearity and alternating property, it is enough to prove our first two assertions for (orthonormal) triplets \((x, y, z)\) from the basis \(\{e_1, \ldots, e_7\} \) and once again this is straightforward to verify by using Maple. To prove (13), we note as in the proof of Lemma 3.5 that \( N(x \times y \times z) = N(x(\bar{y}z)) = N(x)N(y)N(z) = |x \wedge y \wedge z| \). Thus, our claim now follows from the simple fact that \( N(\alpha e + u) = \alpha^2 + N(u) \) whenever \( u \in e^\perp, \alpha \in C \).

Corollary 3.7. Let \( W \in \text{Gr}(3, I) \) be a 3-plane and let \( x, y, z \) be an orthonormal basis for \( W \) (which always exists by Gram-Schmidt process). Then \( W \) is associative if and only if \([x, y, z] = 0\).

Proof. By linearity and alternating property, it is enough to prove the statement “\( \varphi(x, y, z) \in \{-1, +1\} \) if and only if \([x, y, z] = 0\)” on the orthonormal basis \(\{e_1, \ldots, e_7\} \). We verified the cases by using Maple.
Recall that when \( x, y, \) and \( z \) are orthonormal vectors from \( \mathbb{I} \), \( x \times y \times z = x(\overline{yz}) \). We verified by using Maple that if the associator \([x, y, z]\) is nonzero for \( \{x, y, z\} \subset \{e_1, \ldots, e_7\} \), then
\[
x \times y \times z = [x, y, z] \quad \text{and} \quad \varphi(x, y, z) = 0,
\]
and if \([x, y, z] = 0\), then
\[
x \times y \times z = \varphi(x, y, z)e,
\]
which conforms with (12). It is not difficult to show that the form \(*\varphi(u, v, w, z)\) defined by
\[
*\varphi(u, v, w, z) = \langle u \times v \times w, z \rangle
\]
is an alternating 4-form, and it can be expressed as in
\[
*\varphi = -e^{4567} + e^{2367} - e^{2345} + e^{1346} + e^{1256} - e^{1247}.
\]
This can be seen from the fact that the monomials of \( \varphi \) and \(*\varphi\) are complementary in the sense that \( e^{ijkl} \) appears in \(*\varphi\) if and only if there exists unique monomial \( e^{rst} \) in \( \varphi \) such that \( rstijkl = 1234567 \). Also, it can be checked directly on the standard orthogonal basis.

An important consequence of these definitions and the above discussion (specifically, the equation (14)) is that if a 3-plane spanned by \( x, y, z \in \mathbb{I} \) generates a quaternion subalgebra, then \([x, y, z] = 0\), which implies \( \chi(x, y, z) = 0 \). We record this in our next lemma:

**Lemma 3.8.** If the 3-plane \( W \) generated by \( x, y, z \in \mathbb{I} \) is associative, then \( \chi(x, y, z) = 0 \).

The vanishing locus of \( \chi \) on \( \mathbb{P}(\wedge^3 \mathbb{I}) \) can be made more precise since
\[
\chi = (e^{247} - e^{256} - e^{346} - e^{357})e_1 + (e^{156} - e^{147} + e^{345} - e^{367})e_2 + (-e^{245} + e^{267} + e^{146} + e^{157})e_3 + (e^{567} + e^{127} - e^{136} + e^{235})e_4 + (-e^{126} - e^{467} - e^{137} - e^{234})e_5 + (e^{457} + e^{125} + e^{134} - e^{237})e_6 + (e^{135} - e^{124} - e^{456} + e^{236})e_7,
\]
which follows from (16).

**Remark 3.9.** It appears that the idea of using these seven linear equations obtained from \( \chi \) to study associative manifolds is first used in [3].
4 Charts of Gr(3, \mathbb{P})

We recall some standard facts. Let $p_I$ denote a variable indexed by a sequence $I = i_1i_2 \ldots i_d$ of elements from \{1, \ldots, n\}. We assume that if the order of two entries of the sequence $I$ are interchanged, hence we have another sequence $I'$ of the same length, then $p_I = -p_{I'}$. In particular, it follows that if two entries of $I$ are equal, then $p_I = 0$. We are going to view $p_I$'s as coordinate functions on $\mathbb{P}(\wedge^d \mathbb{C}^n)$.

The homogeneous coordinate ring of the Grassmann variety of $d$ dimensional subspaces in $\mathbb{C}^n$ is the quotient of the polynomial ring $\mathbb{C}[p_I : I \subset \{1, \ldots, n\}, |I| = d]$ by the ideal generated by the following quadratic polynomials: $\sum_{d+1}^*(-1)^sp_{i_1i_2\ldots i_{d-1}, j, p_{j_1j_2\ldots j_{d-1}, j_{d+1}}}$, where $i_1, \ldots, i_{d-1}, j_1, \ldots, j_{d+1}$ are arbitrary numbers from \{1, \ldots, n\}. Of course, the case of interest for us is when $d = 3, n = 7$, and (17) is a (at most) 4-term relation:

$$p_{i_1i_2i_3}p_{j_1j_2j_3} - p_{i_1i_2j_3}p_{j_1i_3j_2} + p_{i_1i_3j_2}p_{j_1j_2i_3} - p_{i_1j_2i_3}p_{j_1i_2j_3} = 0. \quad (17)$$

To see the complex manifold structure on $Gr(d, \mathbb{C}^n)$, one looks at the intersections of $Gr(d, \mathbb{C}^n)$ with the standard open charts in $\mathbb{P}(\wedge^d \mathbb{C}^n)$:

$$U_I := Gr(d, \mathbb{C}^n) \cap \{x \in \mathbb{P}(\wedge^d \mathbb{C}^n) : p_I(x) \neq 0\}.$$ 

It is not difficult to show that the coordinate function on $U_I$ are given by $p_J/p_I$, where $J = j_1 \ldots j_d$ is a sequence satisfying $|\{j_1, \ldots, j_d\} \cap \{i_1, \ldots, i_d\}| = d - 1$. Indeed, it is not difficult to verify (by using Plücker relations) that any other rational function of the form $p_K/p_I$ is a polynomial in $p_J/p_I$'s. These are all well known facts. See for example [17].

We compute some examples in our case of interest. Let $J = 123$ and $I = 456$. By (17) we compute $p_Ip_J$ and divide the result by $p_J^2$:

$$\frac{p_{1456}}{p_{123}} = \frac{p_{145}p_{236}}{p_{123}p_{123}} + \frac{p_{245}p_{136}}{p_{123}p_{123}} + \frac{p_{345}p_{126}}{p_{123}p_{123}}. \quad (18)$$

Now, to ease our notation, we denote the coordinate function $p_{abc}/p_{123}$ on $U_{123}$ by $q_{abc}$. Our goal is to describe each $q_{abc}$ by using \{\(q_{124}, q_{125}, q_{126}, q_{127}, q_{134}, q_{135}, q_{136}, q_{137}, q_{234}, q_{235}, q_{236}, q_{237}\}\}, only. We have the following quadratic identities

$$q_{145} = -q_{124}q_{135} - q_{134}q_{125}$$
$$q_{146} = -q_{124}q_{136} - q_{134}q_{126}$$
$$q_{147} = -q_{124}q_{137} - q_{134}q_{127}$$
$$q_{156} = -q_{125}q_{136} - q_{135}q_{126}$$
$$q_{157} = -q_{125}q_{137} - q_{135}q_{127}$$
$$q_{167} = -q_{126}q_{137} - q_{136}q_{127}$$

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\( q_{245} = q_{124}q_{235} - q_{234}q_{125} \)
\( q_{246} = q_{124}q_{236} - q_{234}q_{126} \)
\( q_{247} = q_{124}q_{237} - q_{234}q_{127} \)
\( q_{256} = q_{125}q_{236} - q_{235}q_{126} \)
\( q_{257} = q_{125}q_{237} - q_{235}q_{127} \)
\( q_{267} = q_{126}q_{237} - q_{236}q_{127} \)

\( q_{345} = q_{134}q_{235} + q_{234}q_{135} \)
\( q_{346} = q_{134}q_{236} + q_{234}q_{136} \)
\( q_{347} = q_{134}q_{237} + q_{234}q_{137} \)
\( q_{356} = q_{135}q_{236} + q_{235}q_{136} \)
\( q_{357} = q_{135}q_{237} + q_{235}q_{137} \)
\( q_{367} = q_{136}q_{237} + q_{236}q_{137} \)

Thus we obtain cubic identities

\( q_{456} = (q_{124}q_{135} - q_{134}q_{125})q_{236} + (q_{124}q_{235} - q_{234}q_{125})q_{136} + (q_{134}q_{235} + q_{234}q_{135})q_{126} \)
\( q_{457} = (q_{124}q_{135} - q_{134}q_{125})q_{237} + (q_{124}q_{235} - q_{234}q_{125})q_{137} + (q_{134}q_{235} + q_{234}q_{135})q_{127} \)
\( q_{467} = (q_{124}q_{136} - q_{134}q_{126})q_{237} + (q_{124}q_{236} - q_{234}q_{126})q_{137} + (q_{134}q_{236} + q_{234}q_{136})q_{127} \)
\( q_{567} = (q_{125}q_{136} - q_{135}q_{126})q_{237} + (q_{125}q_{236} - q_{235}q_{126})q_{137} + (q_{135}q_{236} + q_{235}q_{136})q_{127} \)

In the previous section we expressed the vanishing locus of \( \chi \) on \( \mathbb{P}(\bigwedge^3 \mathbb{I}) \) in terms of orthogonal frames, which in turn can be expressed in Plücker coordinates by the following 7 linear equations:

\[ p_{247} - p_{256} - p_{346} - p_{357} = 0 \]  \hspace{1cm} (19)
\[ p_{156} - p_{147} + p_{345} - p_{367} = 0 \]  \hspace{1cm} (20)
\[ -p_{245} + p_{267} + p_{146} + p_{157} = 0 \]  \hspace{1cm} (21)
\[ p_{567} + p_{127} - p_{136} + p_{235} = 0 \]  \hspace{1cm} (22)
\[ -p_{126} - p_{467} - p_{137} - p_{234} = 0 \]  \hspace{1cm} (23)
\[ p_{457} + p_{125} + p_{134} - p_{237} = 0 \]  \hspace{1cm} (24)
\[ p_{135} - p_{124} - p_{456} + p_{236} = 0 \]  \hspace{1cm} (25)

Moreover, it follows from Lemma 3.8 that the variety of all associative 3-planes in \( \mathbb{P}(\bigwedge^3 \mathbb{I}) \), namely the image of the associative Grassmannian \( G_2/\text{SO}_4 \) under the Plücker embedding of \( \text{Gr}(3, \mathbb{I}) \) lies in the intersection of these 7 hyperplanes with \( \text{Gr}(3, \mathbb{I}) \). We express these 7 defining linear forms in terms of the coordinates on \( U_{123} \) (in the above order) and they
become quadratic and cubic polynomials:

\[
\begin{align*}
f_1 := & q_{124}q_{237} - q_{236}q_{125} + q_{126}q_{235} - q_{127}q_{234} \\
& - q_{134}q_{236} - q_{135}q_{237} - q_{136}q_{234} - q_{137}q_{235} \\
f_2 := & q_{124}q_{137} - q_{125}q_{136} - q_{135}q_{126} + q_{134}q_{127} \\
& + q_{134}q_{235} + q_{234}q_{135} - q_{136}q_{237} - q_{236}q_{137} \\
f_3 := & -q_{124}q_{136} - q_{124}q_{235} - q_{125}q_{137} + q_{234}q_{125} \\
& - q_{134}q_{126} + q_{126}q_{237} + q_{135}q_{127} - q_{127}q_{236} \\
f_4 := & -q_{125}q_{136}q_{237} - q_{125}q_{137}q_{236} + q_{126}q_{135}q_{237} \\
& - q_{126}q_{137}q_{235} - q_{127}q_{135}q_{236} - q_{127}q_{136}q_{235} + q_{127} - q_{136} + q_{235} \\
f_5 := & -q_{124}q_{136}q_{237} + q_{124}q_{137}q_{236} - q_{126}q_{134}q_{237} \\
& + q_{126}q_{137}q_{234} + q_{127}q_{134}q_{236} + q_{127}q_{136}q_{234} - q_{126} - q_{137} - q_{234} \\
f_6 := & q_{124}q_{135}q_{237} - q_{124}q_{137}q_{235} + q_{125}q_{134}q_{237} \\
& - q_{125}q_{137}q_{234} - q_{127}q_{134}q_{235} - q_{127}q_{135}q_{234} + q_{125} + q_{134} - q_{237} \\
f_7 := & -q_{124}q_{135}q_{236} + q_{124}q_{136}q_{235} - q_{125}q_{134}q_{236} \\
& + q_{125}q_{136}q_{234} + q_{126}q_{134}q_{235} + q_{126}q_{135}q_{234} - q_{124} + q_{135} + q_{236}
\end{align*}
\]

5 Two $SL_2$ actions

A 2-dimensional maximal torus $T$ of $G_2$ is described by Springer and Veldkamp in Section 2.3 of [21] as the subgroup of automorphisms of $\mathcal{O}$ consisting of the following transformations:

\[ t_{\lambda,\mu} : (x, y) \mapsto (c_\lambda x c_\mu^{-1}, c_\mu y c_\lambda^{-1}), \]

where $(x, y) \in \mathcal{O}$, $\lambda, \mu \in k^*$ and $c_\lambda, c_\mu$ are the diagonal matrices $\text{diag}(\lambda, \lambda^{-1})$, $\text{diag}(\mu, \mu^{-1})$, respectively.

We look more closely at how $T$ acts on the grassmannian, so we express the action in our coordinates. Let $e_{ij}$ denote the elementary $2 \times 2$ matrix which has 1 at $ij$th position and 0’s elsewhere. The set of pairs \{$(e_{11}, 0), (e_{12}, 0), (e_{21}, 0), (e_{22}, 0), (0, e_{11}), (0, e_{12}), (0, e_{21}), (0, e_{22})$\} forms a basis for $\mathcal{O}$. In this basis, $t_{\lambda,\mu}$ is the diagonal matrix

\[ t_{\lambda,\mu} = \text{diag}(1, \lambda^2, \lambda^{-2}, 1, \lambda^{-1} \mu, \lambda \mu, \lambda^{-1} \mu^{-1}, \lambda \mu^{-1}). \]

We are going to switch to the basis \{$e = (e_{11} + e_{22}, 0), e_1 = (i, 0), e_2 = (j, 0), e_3 = (k, 0), e_4 = (0, e_{11} + e_{22}), e_5 = (0, i), e_6 = (0, j), e_7 = (0, k)$\} The proof of the next lemma is straightforward so we skip it

**Lemma 5.1.** The action of maximal torus $T = t_{\lambda,\mu}$ of $G_2$ on the basis \{$e_1, \ldots, e_7$\} of $\mathcal{I}$ is
Note that $I = \text{given by}$

Proposition 5.2. Let $x = (x_1, x_2)$ be an octonion from $I = \mathfrak{sl}_2 \oplus \text{Mat}_2$. The two $\text{SL}_2$ actions on $I$ defined by

1. $g \cdot x = (gx_1g^{-1}, gx_2g^{-1})$ and
2. $g \cdot x = (x_1, gx_2)$

induce $\text{SL}_2$ actions on associative 3-planes.

Proof. Let $W \in \text{Gr}(3, I)$ be an associative 3-plane spanned by the orthogonal basis $\{x, y, z\} \subset I = \mathfrak{sl}_2 \oplus \text{Mat}_2$. We know from that Lemma 3.8 $W$ is associative if $[x, y, z] = 0$. Thus, it suffices to check the vanishing of the associator

$$[g \cdot x, g \cdot y, g \cdot z] = (g \cdot x g \cdot y)g \cdot z - g \cdot x(g \cdot y g \cdot z).$$

Note that $\mathfrak{g} = g^{-1}$ for all $g \in \text{SL}_2$. Note also that for any $x = (x_1, x_2), y = (y_1, y_2)$ from $I = \mathfrak{sl}_2 \oplus \text{Mat}_2$ we have

$$(g \cdot x)(g \cdot y) = (gx_1y_1g^{-1} + y_2g^{-1}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}gy_1g^{-1})$$
$$= (gx_1y_1g^{-1} + y_2g^{-1}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}gy_1g^{-1})$$
$$= (gx_1y_1g^{-1} + y_2g^{-1}gx_2g^{-1}, gy_2x_1g^{-1} + gx_2g^{-1}gy_1g^{-1})$$
$$= g \cdot ((x_1, x_2)(y_1, y_2)).$$
Therefore, if \([x, y, z] = 0\), then

\[
\begin{align*}
[g \cdot x, g \cdot y, g \cdot z] &= (g \cdot x \cdot g \cdot y)g \cdot z - g \cdot x(g \cdot y \cdot g \cdot z) \\
&= (g \cdot (xy))g \cdot z - g \cdot x(g \cdot (yz)) \\
&= g \cdot ((xy)z) - g \cdot (x(yz)) \\
&= g \cdot ([x, y, z] \\
&= 0.
\end{align*}
\]

Next, we check our claim for the second action:

\[
(g \cdot x)(g \cdot y) = (x_1 y_1 + \overline{xy_2} x_2, g y_2 x_1 + g x_2 \overline{y_1}) \\
= (x_1 y_1 + \overline{y_2} x_2, g(y_2 x_1 + x_2 \overline{y_1})) \\
= g \cdot ((x_1, x_2)(y_1, y_2)).
\]

The rest follows as in the previous case. □

As a consequence of Proposition 5.2 we obtain two \(SL_2\) actions on the Zariski closure \(X_{\text{min}}\) of the set of all associative 3-planes in \(Gr(3, \mathbb{I})\). (Our choice of notation \(X_{\text{min}}\) for the Zariski closure in \(Gr(3, \mathbb{I})\) of associative 3-planes will be explained in the Appendix.)

**Remark 5.3.** We denote by \(U\) the following unipotent subgroup:

\[U = \left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{C} \right\} \subset SL_2.\]

The matrices of the actions of a generic element \(g_u := \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in U\) on the ordered basis \(e_1, \ldots, e_7\) of \(\mathbb{I}\) are given by

\begin{align*}
1. \quad [g_u] &= \begin{pmatrix}
1 & -iu & -u & 0 & 0 & 0 & 0 \\
-\overline{u} & 1/2 u^2 + 1 & -i/2 u^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 - 1/2 u^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -iu \\
0 & 0 & 0 & 0 & u & -i/2 u^2 & -u \\
0 & 0 & 0 & 0 & u & -i/2 u^2 & 1 - 1/2 u^2
\end{pmatrix} \\
2. \quad [g_u] &= \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & u/2 & -i/2 u & 0 \\
0 & 0 & 0 & 0 & 1 - i/2 u & -u/2 & 0 \\
0 & 0 & 0 & 0 & -u/2 & i/2 u & 1 & 0 \\
0 & 0 & 0 & 0 & i/2 u & u/2 & 0 & 1
\end{pmatrix}
\end{align*}
For both of these actions of $U$ on $X_{min}$ the fixed point sets are positive dimensional. Indeed, the points $[e_{123}]$ and $[-e_{126} + ie_{127} + ie_{136} + e_{137}]$ of $\mathbb{P}(\wedge^{3} I)$ lie on $X_{min}$ (this can be verified by using eqs. (19)–(25)) and both of these points are fixed by the first action of $U$. By a result of Horrocks [15] we know that the fixed point set of a unipotent group acting on a connected complete variety is connected. Therefore, the fixed point set of $U$ on $X_{min}$ is positive dimensional for the first action. Similarly, the points $[e_{123}]$ and $[-e_{346} + ie_{347} - ie_{356} + e_{357}]$ of $X_{min}$ are fixed by the second action of $U$, hence the fixed point set of this action is also positive dimensional.

6 Torus fixed points

Now we go back to analyzing fixed point set of the maximal torus $t_{\lambda, \mu}$ of $G_{2}$ on $X_{min}$. The action of $t_{\lambda, \mu}$ on the basis $\{e_{1}, \ldots, e_{7}\}$ is computed in the previous section. The eigenvalues are $1, \frac{1}{\lambda}, \lambda^{2}, \frac{1}{\mu}, \lambda, \frac{1}{\mu}, \lambda \mu$ and respective eigenvectors are

$$
\begin{align*}
\tilde{e}_{1} &= e_{1}, \\
\tilde{e}_{2} &= -ie_{2} + e_{3}, \\
\tilde{e}_{3} &= ie_{2} + e_{3}, \\
\tilde{e}_{4} &= -ie_{4} + e_{5}, \\
\tilde{e}_{5} &= ie_{4} + e_{5}, \\
\tilde{e}_{6} &= -ie_{6} + e_{7}, \\
\tilde{e}_{7} &= ie_{6} + e_{7}.
\end{align*}
$$

For $i, j$ and $k$ from $\{1, \ldots, 7\}$ we write $\tilde{e}_{ijk}$ for $\tilde{e}_{i} \wedge \tilde{e}_{j} \wedge \tilde{e}_{k}$. Accordingly, we write $\tilde{p}_{ijk}$ for the transformed Plücker coordinate functions so that

$$
\tilde{p}_{ijk}(\tilde{e}_{rst}) = \begin{cases}
1 & \text{if } i = r, j = s, k = t; \\
0 & \text{otherwise}.
\end{cases}
$$

Our defining equations (19)–(25) become:

$$
\begin{align*}
\tilde{f}_{1} &:= \tilde{p}_{247} + \tilde{p}_{356} = 0 \quad (33) \\
\tilde{f}_{2} &:= 2\tilde{p}_{147} + 2\tilde{p}_{156} + \tilde{p}_{245} + \tilde{p}_{345} - \tilde{p}_{267} - \tilde{p}_{367} = 0 \quad (34) \\
\tilde{f}_{3} &:= 2\tilde{p}_{147} + 2\tilde{p}_{156} + \tilde{p}_{245} - \tilde{p}_{345} - \tilde{p}_{267} + \tilde{p}_{367} = 0 \quad (35) \\
\tilde{f}_{4} &:= 2\tilde{p}_{127} - 2\tilde{p}_{136} + \tilde{p}_{234} + \tilde{p}_{235} + \tilde{p}_{467} + \tilde{p}_{567} = 0 \quad (36) \\
\tilde{f}_{5} &:= -2\tilde{p}_{127} - 2\tilde{p}_{136} + \tilde{p}_{234} - \tilde{p}_{235} + \tilde{p}_{467} - \tilde{p}_{567} = 0 \quad (37) \\
\tilde{f}_{6} &:= 2\tilde{p}_{124} - 2\tilde{p}_{135} - \tilde{p}_{236} - \tilde{p}_{237} + \tilde{p}_{456} + \tilde{p}_{457} = 0 \quad (38) \\
\tilde{f}_{7} &:= 2\tilde{p}_{124} + 2\tilde{p}_{135} - \tilde{p}_{236} + \tilde{p}_{237} + \tilde{p}_{456} - \tilde{p}_{457} = 0 \quad (39)
\end{align*}
$$

It is easily verified that the following 35 vectors are eigenvectors for the action of $t_{\lambda, \mu}$ on $\wedge^{3} I$ (together with the eigenvalues indicated on the left column):
\[ \begin{array}{|c|c|} \hline \frac{1}{\lambda \mu} & \tilde{e}_{126} \\ \hline \lambda^3 \mu & \tilde{e}_{137} \\ \hline \mu^2 & \tilde{e}_{157} \\ \hline \mu^{-2} & \tilde{e}_{146} \\ \hline \frac{\lambda^2}{\mu^2} & \tilde{e}_{346} \\ \hline \frac{\mu^2}{\lambda^2} & \tilde{e}_{257} \\ \hline \frac{\lambda^2}{\mu} & \tilde{e}_{134} \\ \hline \frac{\mu}{\lambda} & \tilde{e}_{125} \\ \hline \frac{1}{\lambda \mu} & \tilde{e}_{246} \\ \hline \lambda^2 \mu^2 & \tilde{e}_{357} \\ \hline \lambda^4 & \tilde{e}_{347} \\ \hline \lambda^{-4} & \tilde{e}_{256} \\ \hline \frac{1}{\lambda} & \tilde{e}_{567}, \tilde{e}_{235}, \tilde{e}_{127} \\ \hline \frac{1}{\mu} & \tilde{e}_{467}, \tilde{e}_{234}, \tilde{e}_{136} \\ \hline \frac{1}{\lambda \mu} & \tilde{e}_{456}, \tilde{e}_{236}, \tilde{e}_{124} \\ \hline \lambda \mu & \tilde{e}_{457}, \tilde{e}_{237}, \tilde{e}_{135} \\ \hline \frac{1}{\lambda^2} & \tilde{e}_{267}, \tilde{e}_{245}, \tilde{e}_{156} \\ \hline \lambda^2 & \tilde{e}_{367}, \tilde{e}_{345}, \tilde{e}_{147} \\ \hline 1 & \tilde{e}_{356}, \tilde{e}_{247}, \tilde{e}_{167}, \tilde{e}_{145}, \tilde{e}_{123} \end{array} \]

**Theorem 6.1.** Among the eigenvectors of \( t_{\lambda, \mu} \) in \( \Lambda^3 \mathbb{I} \), only the images of the following vectors in \( \mathbb{P}(\Lambda^3 \mathbb{I}) \) lie in \( X_{\min} \):

\[ \begin{array}{|c|c|} \hline \lambda^2 \mu^2 & \tilde{e}_{357} \\ \hline \frac{1}{\lambda^2 \mu^2} & \tilde{e}_{246} \\ \hline \frac{1}{\lambda^2 \mu} & \tilde{e}_{126} \\ \hline \lambda^3 \mu & \tilde{e}_{137} \\ \hline \mu^2 & \tilde{e}_{157} \\ \hline \frac{1}{\mu^2} & \tilde{e}_{146} \\ \hline \frac{1}{\lambda^2} & \tilde{e}_{256} \\ \hline \lambda^4 & \tilde{e}_{347} \\ \hline \frac{1}{\lambda^4} & \tilde{e}_{125} \\ \hline \frac{1}{\lambda^2} & \tilde{e}_{134} \\ \hline \frac{1}{\mu^2} & \tilde{e}_{257} \\ \hline \frac{1}{\mu^2} & \tilde{e}_{346} \\ \hline 1 & \tilde{e}_{167}, \tilde{e}_{145}, \tilde{e}_{123} \end{array} \]
Proof. It is easily checked that the points that are given in the hypothesis of the theorem are all torus fixed and all of them lie in $X_{\text{min}}$. The only place we have to be careful is that $X_{\text{min}}$ may intersect eigenspaces of dimension $\geq 2$. Nevertheless, this potential problem does not occur; when we substitute a nontrivial linear combination of eigenvectors belonging to the same eigenvalue into eqns. (33)–(39), we get a contradiction. \hfill \Box

7 Smoothness

Before proving our next result we recall the well-known “Jacobian Criterion for Smoothness.”

Theorem 7.1. Let $I = (f_1, \ldots, f_m)$ be an ideal from $\mathbb{C}[x_1, \ldots, x_n]$ and let $x \in V(I)$ be a point from the vanishing locus of $I$ in $\mathbb{C}^n$. Suppose $d = \dim V(I)$. If the rank of the Jacobian matrix $(\partial f_i/\partial x_j)_{i=1,\ldots,m, \ j=1,\ldots,n}$ at $x$ is equal to $n - d$, then $x$ is a smooth point of $V(I)$.

Theorem 7.2. The variety $X_{\text{min}}$ is smooth.

Proof. If there is a singular point on $X_{\text{min}}$, then it has to occur at the torus fixed points. Thus, it suffices to analyze neighborhoods of fixed points by using affine charts that are described earlier.

We start with the fixed point $m = [\tilde{e}_{123}]$, which lies on the open chart $\tilde{U}_{123}$ as its origin. Here, “tilde” indicates that we are using transformed Plücker coordinates. Recall that $X_{\text{min}}$ is cut-out on $\tilde{U}_{123}$ by the vanishing of the seven linear forms (33)–(39). A straightforward calculation shows that the Jacobian of these polynomials with respect to variables $\tilde{q}_{124}, \tilde{q}_{125}, \tilde{q}_{126}, \tilde{q}_{127}, \tilde{q}_{134}, \tilde{q}_{135}, \tilde{q}_{136}, \tilde{q}_{137}, \tilde{q}_{234}, \tilde{q}_{235}, \tilde{q}_{236}, \tilde{q}_{237}$ (in the written order) evaluated at the origin (which is $\tilde{e}_{123}$ is equal to $\text{Jac}(\tilde{f}_1, \ldots, \tilde{f}_7)|_{\tilde{q}_{ijk}=0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 0 & -1/2 & 0 & 1/4 & 1/4 \\ 0 & 0 & 0 & -1/2 & 0 & 0 & -1/2 & 0 & 1/4 & -1/4 \\ 1/2 & 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 0 & 0 \end{pmatrix}$ which is obviously of rank 4. Hence, $m$ is a smooth point of $X_{\text{min}}$.

We repeat this procedure for other torus fixed points, which is tedious now. We verified this by using Maple. The outcome for each of the torus fixed points that are listed in Theorem 6.1 turns out to be the same. In summary, all of the 15 torus fixed points on $X_{\text{min}}$ are nonsingular points, therefore, $X_{\text{min}}$ is a smooth projective variety of dimension 8. \hfill \Box

8 Tangent space at $[\tilde{e}_{123}]$

In this section we perform a sample calculation of weights of a generic one-parameter $\gamma : \mathbb{C}^* \rightarrow t_{\lambda,\mu}$ subgroup on the tangent space of $X_{\text{min}}$ at the $t_{\lambda,\mu}$-fixed point $m = [\tilde{e}_{123}] \in X_{\text{min}}$. 18
We use the term “generic” in the algebraic geometric sense, which is equivalent to the statement that the pairing between \( \gamma \) and any character \( \alpha : t_{\lambda,\mu} \rightarrow \mathbb{C}^* \) is nonzero. In other words, we choose a regular one-parameter subgroup \( \gamma \) of \( t_{\lambda,\mu} \) so that the fixed point set of \( \gamma \) on \( X_{\text{min}} \) is the same as that of \( t_{\lambda,\mu} \). For example, \( \gamma (s) := t_{sz,s}, s \in \mathbb{C}^* \) is regular.

Recall that the tangent space at \( p = (p_1, \ldots, p_n) \) of an affine variety \( V \subseteq \mathbb{C}^n \) defined by the vanishing of the polynomials \( f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n] \) is the intersection of the hyperplanes
\[
\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} (p) (x_i - p_i) = 0, \quad \text{for } j = 1, \ldots, r.
\]
Equivalently, \( T_p V \) is the kernel of the Jacobian matrix of \( f_1, \ldots, f_r \) (with respect to \( x_i \)'s) evaluated at the point \( p \in V \). In our case, \( p = m = [\tilde{e}_{123}] \) (the origin of the tangent space) and the Jacobian with respect to local coordinates
\[
\bar{q}_{124}, \bar{q}_{125}, \bar{q}_{126}, \bar{q}_{134}, \bar{q}_{135}, \bar{q}_{136}, \bar{q}_{234}, \bar{q}_{235}, \bar{q}_{236}, \bar{q}_{237}
\]
is as given in the proof of Theorem 7.2. It is straightforward to verify that
\[
\left\{ -\frac{1}{2} x_{135} + x_{237}, \frac{1}{2} x_{124} + x_{236}, -\frac{1}{2} x_{127} + x_{235}, \frac{1}{2} x_{136} + x_{234}, x_{137}, x_{134}, x_{126}, x_{125} \right\}
\]
is a basis for the kernel of the Jacobian matrix computed in the proof of Theorem 7.2. Here, \( x_{ijk} \) stands for the tangent vector \( \frac{\partial}{\partial q_{ijk}} \).

Recall from Section 5 that \( t_{\lambda,\mu} \) acts on \( \mathbb{C}^7 \) according to
\[
\begin{align*}
t_{\lambda,\mu}(\tilde{e}_1) &= \tilde{e}_1 \\
t_{\lambda,\mu}(\tilde{e}_2) &= \frac{1}{\lambda^2} \tilde{e}_2 \\
t_{\lambda,\mu}(\tilde{e}_3) &= \lambda^2 \tilde{e}_3 \\
t_{\lambda,\mu}(\tilde{e}_4) &= \frac{\lambda}{\mu} \tilde{e}_4 \\
t_{\lambda,\mu}(\tilde{e}_5) &= \frac{\mu}{\lambda} \tilde{e}_5 \\
t_{\lambda,\mu}(\tilde{e}_6) &= \frac{1}{\lambda \mu} \tilde{e}_6 \\
t_{\lambda,\mu}(\tilde{e}_7) &= \lambda \mu \tilde{e}_7
\end{align*}
\]
Let us denote by \( w_{ijk}(\lambda, \mu) \) the weight (the eigenvalue) of the action \( t_{\lambda,\mu} \cdot \tilde{e}_{ijk} \).

The action of \( t_{\lambda,\mu} \) on a Plücker coordinate \( \tilde{p}_{ijk} \) is given by
\[
t_{\lambda,\mu} \cdot \tilde{p}_{ijk}(x) = \tilde{p}_{ijk}(t_{\lambda^{-1},\mu^{-1}} \cdot x) = w_{ijk}(\lambda^{-1}, \mu^{-1}) \tilde{p}_{ijk}.
\]
\[\text{Here we are abusing the notation. To be precise, } x_i \text{ should be replaced by the vector field } \partial/\partial x_i.\]
and therefore, its action on a local Plücker coordinate function \( \tilde{q}_{rst} \) on \( \tilde{U}_{ijk} \) is given by

\[
t_{\lambda, \mu} \cdot \tilde{q}_{rst}(x) = \frac{\tilde{p}_{rst}(t_{\lambda^{-1}, \mu^{-1}} \cdot x)}{\tilde{p}_{ijk}(t_{\lambda^{-1}, \mu^{-1}} \cdot x)} = \frac{w_{rst}(\lambda^{-1}, \mu^{-1})}{w_{ijk}(\lambda^{-1}, \mu^{-1})} \tilde{q}_{rst}.
\]

Consequently, if \( x = \sum_{r,s,t} a_{rst} \frac{\partial}{\partial \tilde{q}_{rst}} \) is a tangent vector at \( \tilde{e}_{ijk} \in \tilde{U}_{ijk} \), then the action of the one-parameter subgroup \( \gamma(\lambda) = t_{\lambda^{10}, \lambda}, \lambda \in \mathbb{C}^* \) on \( x \) is given by

\[
\gamma \cdot x = \sum_{r,s,t} \lim_{\lambda \to 1} \left( \frac{w_{rst}(\lambda^{10}, \lambda)}{w_{ijk}(\lambda^{10}, \lambda)} \right) \frac{\partial}{\partial \tilde{q}_{rst}}.
\]

(41)

For example, the action of \( \gamma \) on the basis vectors (40), which we denote by \( v_1, \ldots, v_8 \) in the written order, is given by

\[
\begin{align*}
\gamma \cdot v_1 & = 11v_1 \\
\gamma \cdot v_2 & = -11v_2 \\
\gamma \cdot v_3 & = -9v_3 \\
\gamma \cdot v_4 & = 9v_4 \\
\gamma \cdot v_5 & = 31v_5 \\
\gamma \cdot v_6 & = 29v_6 \\
\gamma \cdot v_7 & = -31v_7 \\
\gamma \cdot v_8 & = -29v_8.
\end{align*}
\]

9 Białyńcki-Birula Decomposition

Let \( X \) be a smooth projective variety over \( \mathbb{C} \) on which an algebraic torus \( T \) acts with finitely many fixed points. Let \( T' \) be a 1 dimensional subtorus with \( X^{T'} = X^T \). For \( p \in X^{T'} \) define the sets \( C^+_p = \{ y \in X : \lim_{s \to 0} s \cdot y = p, s \in T' \} \) and \( C^-_p = \{ y \in X : \lim_{s \to \infty} t \cdot y = p, s \in T' \} \), called the plus cell and minus cell of \( p \), respectively.

A well known result of A. Białyńcki-Birula, which we call the BB–theorem, states that

**Theorem 9.1** ([6]). Let \( X, T \) and \( T' \) be as above. Then

1. Both of the sets \( C^+_p \) and \( C^-_p \) are locally closed subvarieties in \( X \), furthermore they are isomorphic to an affine space;

2. if \( T_pX \) is the tangent space of \( X \) at \( p \), then \( C^+_p \) (resp., \( C^-_p \)) is \( T' \)-equivariantly isomorphic to the subspace \( T^+_pX \) (resp., \( T^-_pX \)) of \( T_pX \) spanned by the positive (resp., negative) weight spaces of the action of \( T' \) on \( T_pX \).

As a consequence of Theorem 9.1, there exists a filtration

\[
X^{T'} = V_0 \subset V_1 \subset \cdots \subset V_n = X, \quad n = \dim X,
\]

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of closed subsets such that for each \( i = 1, \ldots, n \), \( V_i - V_{i-1} \) is the disjoint union of the plus (resp., minus) cells in \( X \) of (complex) dimension \( i \). It follows that the odd-dimensional integral cohomology groups of \( X \) vanish, the even-dimensional integral cohomology groups of \( X \) are free, and the Poincaré polynomial \( P_X(t) := \sum_{i=0}^{2n} \dim C^i \cap H^i(X; \mathbb{Z}) t^i \) of \( X \) is given by

\[
P_X(t) = \sum_{p \in X^{\prime \prime}} t^{2 \dim C^+_p} = \sum_{p \in X^{\prime \prime}} t^{2 \dim C^-_p}.
\]

Now, let \( T' \) denote the 1 dimensional subtorus of \( T = t_{\lambda, \mu} \) that is given by the image of the regular one-parameter subgroup \( \gamma(\lambda) = t_{\lambda^{10}, \lambda}, \lambda \in \mathbb{C}^* \). In the rest of this section we are going to compute the weights of \( T' \) on the tangent spaces at the torus fixed points. We have already made a sample calculation of this sort in Section 8.

1. \( p = [\bar{e}_{246}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{-x_{234} + x_{467}, -1/2 x_{124} + x_{456}, x_{346}, x_{245} + x_{267}, x_{256}, 1/2 x_{124} + x_{236}, x_{146}, x_{126}\}
\]

The weights (in the order of the eigenvectors) are 31, 11, 40, 2, -18, 11, 20, -9.

2. \( p = [\bar{e}_{157}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{x_{125}, -1/2 x_{127} + x_{567}, 1/2 x_{135} + x_{457}, x_{357}, x_{257}, x_{167}, x_{145}, x_{137}\}
\]

The weights (in the order of the eigenvectors) are -31, -11, 9, 20, -20, -2, -2, 29.

3. \( p = [\bar{e}_{256}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{-x_{235} + x_{567}, x_{236} + x_{456}, 1/2 x_{156} + x_{267}, x_{257}, x_{246}, -1/2 x_{156} + x_{245}, x_{126}, x_{125}\}
\]

The weights (in the order of the eigenvectors) are 31, 29, 10, 22, 18, 10, 9, 11.

4. \( p = [\bar{e}_{126}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{1/2 x_{156} + x_{267}, x_{256}, x_{246}, 1/2 x_{124} + x_{236}, x_{167}, x_{146}, x_{125}, x_{123}\}
\]

The weights (in the order of the eigenvectors) are 11, -9, 9, 10, 31, 29, 2, 31.

5. \( p = [\bar{e}_{167}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{-1/2 x_{127} + x_{567}, 1/2 x_{136} + x_{467}, -1/2 x_{147} + x_{367}, 1/2 x_{156} + x_{267}, x_{157}, x_{146}, x_{137}, x_{126}\}
\]

The weights (in the order of the eigenvectors) are -9, 9, 10, -10, 2, -2, 31, -31.

6. \( p = [\bar{e}_{145}] \). An eigenbasis for tangent space at \( p \) is given by

\[
\{1/2 x_{135} + x_{457}, -1/2 x_{124} + x_{456}, 1/2 x_{147} + x_{345}, -1/2 x_{156} + x_{245}, x_{157}, x_{146}, x_{134}, x_{125}\}
\]

The weights (in the order of the eigenvectors) are 11, -11, 10, -10, 2, -2, 29, -29.
Theorem 9.2. The Poincaré polynomial of \( X_{\text{min}} \) is
\[
P_X(t^{1/2}) = 1 + t + 2t^2 + 2t^3 + 3t^4 + 2t^5 + 2t^6 + t^7 + t^8.
\]
\[\]
Proof. The proof follows from the discussion at the beginning of this section and the computations made above. \[\]
10 Generalities on Degenerate Compactifications

Let $G$ be a semisimple, simply connected algebraic group defined over $k$. A $G$-variety $X$ is called a regular embedding if it is comprised of finitely many $G$-orbits such that

1) any $G$-orbit closure is smooth and equal to the transversal intersection of codimension one orbit closures that contain it;

2) for any $x \in X$, the normal space $T_x(X)/T_x(G \cdot x)$ contains a dense orbit of the isotropy group $G_x = \{ g \in G : g \cdot x = x\}$.

Since there exists finitely many $G$-orbits, there exists a (generic) point $x \in X$ such that the $G$-orbit $G \cdot x \cong G/G_x$ is open in $X$.

Now suppose we have an involutory automorphism $\sigma : G \to G$. The symmetric variety associated with $\sigma$ is affine algebraic variety $G/H^0$, where $H^0$ is the subgroup of elements fixed by $\sigma$. Let us denote by $H$ the normalizer of $H^0$ in $G$. According to De Concini and Procesi [12], there exists a unique minimal, smooth projective regular embedding having $G/H$ as a dense open $G$-orbit. Appropriately, such embeddings of symmetric varieties are called the wonderful compactifications. For a good introduction to wonderful compactifications (and more generally to “spherical varieties”) we recommend Michel Brion’s lecture notes [7].

An irreducible representation $V$ with highest weight $\lambda$ of $G$ is said to be spherical if there exists a nonzero vector $v$ fixed by $H^0$. In this case, there exists a unique point $[v]$ (the line in direction of $v$) in $\mathbb{P}(V)$ which is fixed by $H$. We call the Zariski closure $Z_\lambda$ of the $G$-orbit $G \cdot [v]$ in $\mathbb{P}(V)$ a special compactification of $G/H$. Similar to wonderful compactifications (which are always special), these are the smallest possible $G$-equivariant compactifications of $G/H$ (see [11]) and first considered by Vust in [22].

In general special compactifications are not normal, let alone being smooth. A normalization for these embeddings are first defined in [11] and it goes as follows:

The map $G/H \to \mathbb{P}(V_\lambda)$ given by $gH \mapsto g[v]$ extends to a map $q : X \to Z_\lambda \subseteq \mathbb{P}(V_\lambda)$, where $X$ is the wonderful compactification of $G/H$. Let $\mathcal{L}_\lambda = q^*(\mathcal{O}_{\mathbb{P}(V_\lambda)}(1))$ denote the pull-back of the hyperplane bundle of $\mathbb{P}(V_\lambda)$ to $X$. The ring $A(\mathcal{L}_\lambda) := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}_\lambda^n)$ is generated in degree 1 (see [10]) hence it can be identified with the projective coordinate ring of the image of $X$ in $\mathbb{P}(H^0(X, \mathcal{L})^*)$. In other words, $A(\mathcal{L}_\lambda)$ is the coordinate ring of the image of $p : X \to \mathbb{P}(H^0(X, \mathcal{L})^*)$, which we denote by $Y_\lambda$. Since $X$ is smooth, $Y_\lambda$ is normal. Furthermore, the subring of $A(\mathcal{L}_\lambda)$ that is generated by $V_\lambda^*$ is isomorphic to the projective coordinate ring of $Z_\lambda$. The corresponding natural projection map $r : Y_\lambda \to Z_\lambda$ satisfies $q = r \circ p$.

Next, following Maffei’s work we describe the $G$-orbits in $Z_\lambda$. To this end we introduce the necessary notation. Let $T \subseteq G$ be a $\sigma$-stable maximal torus (which always exists) such that the dimension of the neutral component of $\{ t \in T : \sigma(t) = t^{-1}\}$ is maximal. These choices give us the root system $\Phi(G, T)$ and the associated “restricted root system” $\tilde{\Phi} := \{ \alpha - \sigma(\alpha) : \alpha \in \Phi\}$. Note that the restricted root system does not need to be irreducible. Next, we choose a Borel subgroup $B \subset G$ such that $B \cap \sigma(B)$ has minimal
possible dimension. Corresponding to this choice, we have the set of simple roots $\Delta$ and the associated restricted simple roots $\tilde{\Delta}$.

Let $\Phi_0 \subset \Phi$ denote the subset of roots that are fixed by $\sigma$, $\Phi_1$ denote $\Phi - \Phi_0$. Let $\Delta_0$ denote $\Delta \cap \Phi_0$ and let $\Delta_1 = \Delta - \Delta_0$. There exists an induced involution $\overline{\sigma}$ on $\Delta_1$ such that $\sigma(\alpha) + \overline{\sigma}(\alpha)$ is in the span of $\Delta_0$ and if $\beta = \overline{\sigma}(\alpha)$, then $\beta = \check{\alpha}$. Furthermore, the action of $\sigma$ on the simple roots can be described explicitly (by using $\overline{\sigma}$): if $\alpha \in \Delta_0$, then $\sigma(\alpha) = \check{\alpha}$, and if $\alpha \in \Delta_1$, then $\sigma(\alpha) = -w_{\Delta_0}\overline{\sigma}(\alpha)$. Here, $w_{\Delta_0}$ is the longest element of the parabolic subgroup $W_{\Delta_0}$ of the Weyl group $W$ of $\Phi(G,T)$. The support of a weight $\lambda$ is the set of simple roots $\alpha \in \Delta$ such that $\kappa(\alpha, \lambda) \neq 0$, where $\kappa(\cdot, \cdot)$ is the Killing form. We denote the support of $\lambda$ by $\text{supp}(\lambda)$.

Recall that a highest weight $\lambda$ of $G$ is called spherical (with respect to $H^0$) if its highest weight vector $v \in V_\lambda$ is fixed by $H^0$. Let us denote by $\Omega$ the set (semigroup) of all highest weights of $G$ relative to $T$ and denote by $\Omega^+ \subset \Omega$ the subsemigroup of all spherical highest weights (of course, $T$ has to be a $\sigma$-stable maximal torus). A highest weight is called special if $\sigma(\lambda) = -\lambda$.

If $I$ and $J$ are subsets of $\tilde{\Delta}$, then define
\[
\Delta_I(J) := \bigcup \{\text{connected components of } \tilde{\Delta} - J \text{ which intersect } I\}.
\]

It is shown in [12] that the boundary of $G/H$ in its wonderful compactification $X$ is a union $\bigcup_{\tilde{\alpha} \in \tilde{\Delta}} X_{\tilde{\alpha}}$ of irreducible $G$-stable divisors which are indexed by the simple roots of the restricted root system. These boundary divisors intersects each other transversally, and moreover, the closure of any $G$-orbit is of the form $X_J := \bigcap_{\alpha \in J} X_{\alpha}$ for some subset $J \subseteq \tilde{\Delta}$. Thus, $X_J = \overline{O_J}$ where $O_J$ is the $G$-orbit determined by the subset $J \subseteq \tilde{\Delta}$. The unique closed $G$-orbit $X_{\tilde{\Delta}}$ is isomorphic to $G/P_{\Delta_0}$, where $P_{\Delta_0}$ is the parabolic subgroup of $G$ corresponding to the set of ($\sigma$-fixed) simple roots $\Delta_0$. It is shown by Maffei in [18] that the $G$-orbits of the special compactification $Z_{\lambda}$ of $G/H$ are precisely the images of the $G$-orbits in $X$ under $q : X \to Z_{\lambda}$ which is described above. Furthermore, for subsets $K$ and $J$ of $\tilde{\Delta}$ we have
\[
q(O_J) = q(O_K) \iff \Delta_I(J) = \Delta_I(K)
\]
where $I \subseteq \tilde{\Delta}$ is the support of $\lambda$ in $\tilde{\Delta}$.

We know from [12] that each subset $J \subseteq \tilde{\Delta}$, the $G$-orbit $O_J$ as well as its closure $X_J$ is a fibration over a suitable partial flag variety. Indeed, let $d(J)$ denote $\Delta_0$ union the set of restricted simple roots $\check{\alpha}$ whose restriction to the maximal $\sigma$-split torus of $G$ lies in $J$. In this case, the parabolic subgroup $P_{d(J)}$ and $\sigma(P_{d(J)})$ are opposite parabolic subgroup, hence $L_J := P_{d(J)} \cap \sigma(P_{d(J)})$ is the Levi subgroup of $P_{d(J)}$. Let $H_J$ denote the fixed subgroup of $\sigma$ on $L_J$. Then
\[
O_J \simeq G \times_{\sigma(P_{d(J)})} (L_J/H_J) \\
X_J \simeq G \times_{\sigma(P_{d(J)})} (L_J/H_J)
\]

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where \((L_J/H_J)\) is the wonderful compactification of \(L_J/H_J\). In other words, the orbit \(O_J\) is a fibration over \(G/P_{d(J)}\) and its fibers are isomorphic to the symmetric variety \(L_J/H_J\), and similarly \(X_J\) is a fibration over \(G/P_{d(J)}\) and its fibers are isomorphic to the wonderful compactification of the symmetric variety \(L_J/H_J\).

11 The wonderful compactification of \(G_2/\text{SO}_4\).

According to Borel and de Siebenthal [5], \(G_2\) has a unique involution \(\sigma : G_2 \to G_2\) whose fixed point subgroup \(H^0\) is one of its two maximal subgroups, namely \(\text{SO}_4\). The standard maximal torus \(T\) of \(G_2\) is split relative to the involution determined by \(\text{SO}_4\). We choose the Borel subgroup appropriately so that the induced involution on \(\Phi(G_2, T)\) is given by \(\sigma(\alpha) = -\alpha\), sending a simple root to its negative. Therefore, \(\tilde{\Delta} = \{\tilde{\alpha} : \alpha \in \Delta\} = \{2\alpha_1, 2\alpha_2\}\), where \(\alpha_1\) (short) and \(\alpha_2\) (long) are the simple roots of \((G_2, T)\). Thus, there are 4 \(G_2\)-orbits in the wonderful compactification of \(G_2/\text{SO}_4\) and these orbits correspond to the subsets of \(\tilde{\Delta}\). The closed orbit is \(G_2/P_0 = G_2/B\), hence the flag variety of \(G_2\). The other two orbits are fibrations over grassmannians \(G_2/P_{\{\alpha_1\}}\) and \(G_2/P_{\{\alpha_2\}}\).

**Theorem 11.1** (De Concini - Springer [13], Example 3.4). The Poincaré polynomial of the wonderful compactification \(X_{\text{won}}\) of \(G_2/\text{SO}_4\) is

\[P_{X_{\text{won}}}(t^{1/2}) = 1 + 2t + 4t^2 + 4t^3 + 5t^4 + 4t^5 + 4t^6 + 2t^7 + t^8.\]

12 Minimal degenerate compactification of \(G_2/\text{SO}_4\).

Recall from Section 2 that \(G_2/\text{SO}_4\) embeds into \(\mathbb{P}(\Gamma_{2,0})\), where \(\Gamma_{2,0}\) is the irreducible representation of \(G_2\) with the highest weight \(2\omega_1\). Since \(\omega_1\) is the Killing dual of \(\alpha_1\), we see that \(\text{supp}_{\Delta}(2\omega_1) = \{2\alpha_1\}\). The set of restricted simple roots has 4 subsets \(\emptyset, \{2\alpha_1\}, \{2\alpha_2\}\) and \(\tilde{\Delta}\) itself. Then

\[
\begin{align*}
\Delta_I(\emptyset) &= \tilde{\Delta}, \\
\Delta_I(\{2\alpha_1\}) &= \emptyset, \\
\Delta_I(\{2\alpha_2\}) &= \{2\alpha_1\}, \\
\Delta_I(\{2\alpha_1, 2\alpha_2\}) &= \emptyset.
\end{align*}
\]

Therefore, we see that our special compactification of \(G_2/\text{SO}_4\) in \(\mathbb{P}(\Gamma_{2,0})\) has 3 \(G_2\)-orbits; the images of \(G_2/\text{SO}_4\), \(G_2/P_{\{2\alpha_2\}}\), and of \(G_2/B\) in \(\mathbb{P}(\Gamma_{2,0})\), where \(P_{\{2\alpha_2\}}\) is the parabolic subgroup of \(G_2\) that corresponds to the weight \(2\alpha_2\). It follows from these observations that the Picard number of this minimal degenerate compactification cannot be one.

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